NATURALITY OF THE CONTACT INVARIANT IN MONOPOLE FLOER HOMOLOGY UNDER STRONG SYMPLECTIC COBORDISMS

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B.S. in Mathematics, Universidad de Costa Rica, 2013

A Dissertation presented to the Graduate Faculty
of the University of Virginia in Candidacy for the Degree of
Doctor of Philosophy

Department of Mathematics

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March, 2019

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Abstract

The contact invariant is an element in the monopole Floer homology groups of an oriented closed three manifold canonically associated to a given contact structure. A non-vanishing contact invariant implies that the original contact structure is tight, so understanding its behavior under symplectic cobordisms is of interest if one wants to further exploit this property.

Under a suitable reinterpretation of work by Mrowka and Rollin, we will show that the contact invariant behaves naturally under a strong symplectic cobordism.

As quick applications of the naturality property, we give alternative proofs for the vanishing of the contact invariant in the case of an overtwisted contact structure, its non-vanishing in the case of strongly fillable contact structures and its vanishing in the reduced part of the monopole Floer homology group in the case of a planar contact structure. We also prove that a strong filling of a contact manifold which is an L space must be negative definite.

Acknowledgments

I would like to thank my advisor Tom Mark for suggesting this problem to me and for all of his indispensable help and support, as well as trying to keep my feet on the ground throughout the years.

I am also very grateful to professor Tom Mrowka for allowing me to visit MIT during 2018 and helping me with many technical aspects of this work. I would like to thank professor Cliff Taubes for allowing me to attend his student seminar, as well as answering extremely quickly all the emails I ever sent him. I am also deeply indebted to Jianfeng Lin and Boyu Zhang for many useful conversations of both direct and indirect importance to this work.

I am also very thankful to Michael Hutchings, Danny Ruberman, Nikolai Saveliev, Paul Feehan, Tom Leness, Peter Kronheimer, Matthew Stoffregen and many others for sharing their mathematical expertise with me throughout the years. I would also like to thank Sara Maloni, Slava Krushkal and Jeffrey Teo for serving on my thesis committee.

Finally, I would like to thank my parents and my brother for all of their support throughout the years.

CONTENTS

1. STATING THE RESULT AND SOME APPLICATIONS

Monopole Floer Homology associates to a closed, oriented, connected 3-manifold Y three abelian groups $\widetilde{HM}_\bullet(Y)$, $\widehat{HM}_\bullet(Y)$, $\overline{HM}_\bullet(Y)$, pronounced HM-to, HM-from and HM-bar respectively. They admit a direct sum decomposition over spin-c structures of Y, in the sense that

$$\widetilde{HM}_\bullet(Y) = \bigoplus_{\mathfrak{s}} \widetilde{HM}_\bullet(Y,\mathfrak{s})$$
$$\widehat{HM}_\bullet(Y) = \bigoplus_{\mathfrak{s}} \widehat{HM}_\bullet(Y,\mathfrak{s})$$
$$\overline{HM}_\bullet(Y) = \bigoplus_{\mathfrak{s}} \overline{HM}_\bullet(Y,\mathfrak{s})$$

In fact, the previous decomposition is finite [32, Proposition 3.1.1]. The chain complexes whose homology are the previous groups are built using solutions of a perturbed version of the three dimensional Seiberg-Witten equations, which are at the same time critical points of a perturbed Chern-Simons-Dirac functional [32, section 4]. There are three different types of solutions (the boundary stable, boundary unstable and irreducible solutions) and each group uses two of the three types in their corresponding construction.

Now suppose that Y is equipped with a co-orientable contact structure ξ compatible with the orientation of the manifold. In practice this means that there exists a globally defined one form θ on Y for which $\xi = \ker\theta$ and $\theta \wedge d\theta$ is positive everywhere [21, lemma 1.1.1]. As we will review momentarily, ξ determines a spin-c structure $\mathfrak{s}_\xi$ and one can exploit the additional structure provided by ξ in order to construct an element $\mathbf{c}(\xi) \in \widetilde{HM}_\bullet(-Y,\mathfrak{s}_\xi)$ known as the contact invariant of (Y,ξ).

It is important to observe that $\mathbf{c}(\xi)$ belongs to the monopole Floer homology groups of the manifold $-Y$, that is, Y with the opposite orientation. This is because the contact invariant $\mathbf{c}(\xi)$ should actually be regarded as a cohomology element $\mathbf{c}(\xi) \in \widehat{HM}^\bullet(Y,\mathfrak{s}_\xi)$, and there is a natural isomorphism between $\widehat{HM}^\bullet(Y,\mathfrak{s}_\xi)$ and $\widetilde{HM}_\bullet(-Y,\mathfrak{s}_\xi)$ [32, section 22.5]. However, we will work with the homology version of the contact invariant since most of the formulas in [32] are given explicitly for the homology groups.

Monopole Floer homology also has TFQT-like features, which concretely means that given a cobordism $W : Y \to Y'$ between two three manifolds, there are group homomorphisms between the corresponding homology groups

$$
\begin{aligned}
\widecheck{HM}_\bullet(W, \mathfrak{s}_W) &: \widecheck{HM}_\bullet(Y, \mathfrak{s}_Y) \to \widecheck{HM}_\bullet(Y', \mathfrak{s}_{Y'}) \\
\widehat{HM}_\bullet(W, \mathfrak{s}_W) &: \widehat{HM}_\bullet(Y, \mathfrak{s}_Y) \to \widehat{HM}_\bullet(Y', \mathfrak{s}_{Y'}) \\
\overline{HM}_\bullet(W, \mathfrak{s}_W) &: \overline{HM}_\bullet(Y, \mathfrak{s}_Y) \to \overline{HM}_\bullet(Y', \mathfrak{s}_{Y'})
\end{aligned}
$$

Here $\mathfrak{s}_W$ denotes a spin-c structure which restricts in an appropriate sense to the given spin-c structures on Y and Y'. Just as in the contact case, if $(W, \omega) : (Y, \xi) \to (Y', \xi')$ is equipped with a symplectic form ω, it determines a spin-c structure $\mathfrak{s}_\omega$, and so it makes sense to ask the *naturality question*, that is, whether or not

$$
\widecheck{HM}_\bullet(W^\dagger, \mathfrak{s}_\omega)\mathbf{c}(\xi') \stackrel{?}{=} \mathbf{c}(\xi) \tag{1}
$$

where $W^\dagger : -Y' \to -Y$ denotes the cobordism turned "upside-down". The main result of this work is that the answer to the previous question is positive in the case of a strong symplectic cobordism:

Theorem 1. *Let $(W, \omega) : (Y, \xi) \to (Y', \xi')$ be a strong symplectic cobordism between two contact manifolds (Y, ξ) and (Y', ξ'). Then*

$$
\widecheck{HM}_\bullet(W^\dagger, \mathfrak{s}_\omega)\mathbf{c}(\xi') = \mathbf{c}(\xi)
$$

At this point it is important to specify that our notion of a **strong symplectic cobordism** is that of a symplectic cobordism for which the symplectic form is given in collar neighborhoods of the concave *and* convex boundaries by symplectizations of the corresponding contact structures.

To give some context it is important to point out that this theorem appears stated as Theorem 2.4 in [50], though the reference given is a paper by Mrowka and Rollin in preparation that was never published. Also, as will be discussed later in this paper the "special condition" imposed on the cobordism in [50] and [40] can be removed.

One can also ask what is known in the twin versions of monopole Floer homology, namely, embedded contact homology and Heegaard Floer homology. It is not by any means obvious that the corresponding homology groups from Heegaard Floer and ECH are isomorphic to the ones coming from monopole Floer homology and the proof can be found in [5, 6, 7, 8, 9, 61, 62, 60, 55, 56, 57, 58, 59]. Also, the corresponding contact invariants in each version are isomorphic to each other.

In Heegaard Floer Homology naturality holds (for example) if (Y', ξ') is obtained from (Y, ξ) by Legendrian surgery along a Legendrian knot L [35, Theorem 2.3]. This is an interesting case because a 1-handle surgery, or a 2-handle surgery along a Legendrian knot K with framing -1 relative to the canonical framing gives rise to a strong symplectic cobordism. On the ECH side the contact invariant is known to be natural with respect to weakly exact symplectic cobordisms [27, Remark 1.11]. Moreover, Michael Hutchings has communicated to the author that he can improve this result to the case of a strong symplectic cobordism, with the additional advantage that the contact manifolds can be disconnected [24].

Implicitly we have used the coefficient field $\mathbb{F} = \mathbb{Z}/2$ so that we can ignore orientations issues. Clearly one can also ask whether or not one there is an analogous statement in the case of integer coefficients. Unfortunately, Theorem H in [28] shows that there is no canonical choice of sign in the definition of the contact invariant, so the best naturality statement one could hope for in this case is one given up to a sign.

In any case, the contact invariant with mod-2 coefficients is still a useful tool for understanding contact structures and the naturality result is good enough to find properties of this invariant, though the properties we discuss in this work were previously known by other means. Before we discuss these applications, however, we will give some brief history that puts into perspective the construction of the contact invariant and why the following applications were natural things to look for.

In [31] Kronheimer and Mrowka used the contact structure of Y to extend the definition of the Seiberg-Witten invariants to the case of a compact oriented four manifold X bounding it.

More precisely, one considers the non-compact four manifold $X^+ = X \cup_Y ([1,\infty) \times Y)$, where $[1,\infty) \times Y$ is given the structure of an almost Kähler cone using a symplectization ω of a contact form θ defining ξ. In particular, the symplectic form induced by θ determines a canonical spin-c structure $\mathfrak{s}_\omega$ on $[1,\infty) \times Y$, which we can think of as a complex vector bundle $S = S^+ \oplus S^-$ together with a Clifford multiplication $\rho : T^*([1,\infty) \times Y) \to \hom_{\mathbb{C}}(S,S)$ satisfying certain conditions.

The canonical spin-c structure $\mathfrak{s}_\omega$ identifies a canonical section Φ_0 of S^+ together with a canonical spin-c connection A_0 on the spinor bundle. Kronheimer and Mrowka then study solutions of the Seiberg Witten equations on X^+ which are asymptotic to (A_0,Φ_0) on the conical end. These solutions end up having uniform exponential decay with respect to the canonical configuration (A_0,Φ_0) (Proposition 3.15 in [31] or Propositions 5.7 and 5.10 in [65] for a similar situation), which means that the Seiberg Witten equations on X^+ behave very similar to how they would if the manifold were compact, more specifically, the moduli spaces of gauge equivalence classes of such solutions are compact. This allows as in the closed manifold case to define a map

$$SW_{(X,\xi)} : \mathrm{Spin}^c(X,\xi) \to \mathbb{Z}$$

where $\mathrm{Spin}^c(X,\xi)$ denotes the set of isomorphism classes of relative spin-c structures on X that restrict to the spin-c structure $\mathfrak{s}_\xi$ on Y determined by the contact structure ξ. This map can be used to detect properties of contact structures on three manifolds. For example, Theorem 1.3 in [31] shows that for any closed three manifold Y there are only finitely many homotopy classes of 2-plane fields which are realized as semi-fillable contact structures. In section 1.3 of the same paper they mention as well that if (X,ξ) is a 4-manifold with an overtwisted contact structure on its boundary, then $SW_{(X,\xi)}$ vanishes identically.

The latter result is Corollary B in a different paper [40] by Mrowka and Rollin, where they analyzed how the map $SW_{(X,\xi)}$ behaves under a symplectic cobordism $(W,\omega) : (Y,\xi) \to (Y',\xi')$ which they called a **special symplectic cobordism** [40, page 4]. Theorem D in [40] shows that

$$SW_{(X,\xi)} = \pm SW_{(X\cup W,\xi')} \circ \jmath \tag{2}$$

where $\jmath : \mathrm{Spin}^c(X,\xi) \to \mathrm{Spin}^c(X \cup W, \xi')$ is a canonical map that extends the spin-c structure of X across the cobordism W. With respect to $\mathbb{Z}/2\mathbb{Z}$ coefficients, the previous theorem can be interpreted as saying that the mod 2 Seiberg-Witten invariants are the same.

In order to detect more properties of the contact structure, we need to use the machinery of Monopole Floer Homology, whose canonical reference is [32].

As first defined in section 6.3 of [30] , one constructs the contact invariant $\mathbf{c}(\xi) \in \widetilde{HM}_\bullet(-Y, \mathfrak{s}_\xi)$ by studying the Seiberg-Witten equations on $(\mathbb{R}^+ \times -Y) \cup ([1,\infty) \times Y)$ which are asymptotic on the symplectic cone to the canonical configuration (A_0, Φ_0) mentioned before and asymptotic on the half-cylinder to a solution of the (perturbed) three dimensional Seiberg-Witten equations. We will give more details about this construction in the next section, as well as an interpretation of it as a "relative invariant" from the TQFT-perspective. However, it should be clear that based on the analogy with the numerical Seiberg-Witten invariants $SW_{(X,\xi)}$, one would expect the naturality property (our main theorem 1) as well as the vanishing of the contact invariant for an overtwisted structure. It is the latter which we now indicate how to prove.

Corollary 2. *Let (Y,ξ) be an overtwisted contact 3 manifold. Then the contact invariant of ξ vanishes, that is, $\mathbf{c}(\xi) = 0$.*

Proof. First we show that the 3-sphere S^3 admits an overtwisted structure ξ_{ot} for which $\mathbf{c}(\xi_{ot}) = 0$. For this we will use Eliashberg's theorem [15, Theorem 1.6.1] on the existence of an overtwisted contact structure in every homotopy class of oriented plane field and the fact that the Floer groups of any three manifold Y are graded by the set of homotopy classes of oriented plane fields [32, Section 3.1].

Thanks to Proposition 3.3.1 in [32], which describes the Floer homology groups of S^3, we can find a homotopy class of plane field $[\xi]$ for which $\widetilde{HM}_{[\xi]}(S^3) = 0$. Notice that in this case we are not specifying the spin-c structure because S^3 has only one up to isomorphism. By Eliashberg's theorem we can choose an overtwisted structure ξ_{ot} in the homotopy class $[\xi]$. Now, $\mathbf{c}(\xi_{ot})$ is supported in $\widetilde{HM}_{[\xi]}(-S^3) \simeq \widetilde{HM}_{[\xi]}(S^3) = 0$ (see for example Theorem (46) in this paper) and so it will automatically vanish, i.e, $\mathbf{c}(\xi_{ot}) = 0$.

Now, if (Y, ξ) is an arbitrary overtwisted contact 3 manifold, using Theorem 1.2 in [17], we can find a Stein cobordism $(W, \omega) : (Y, \xi) \to (S^3, \xi_{ot})$. Such cobordisms are in fact strong cobordisms so we can conclude that

$$\mathbf{c}(\xi) = \widetilde{HM}_\bullet(W^\dagger, \mathfrak{s}_\omega)\mathbf{c}(\xi_{ot}) = \widetilde{HM}_\bullet(W^\dagger, \mathfrak{s}_\omega)(0) = 0$$

and therefore $\mathbf{c}(\xi)$ vanishes. □

Remark 3. For a proof that does not use the naturality property see Theorem 4.2 in [54]. The vanishing of the contact invariant for overtwisted contact structures is also known on the Heegaard Floer side [46, Theorem 1.4]. For a proof on the ECH side see Michael Hutchings' blog ([25]). In fact, in the case of ECH one can show that the contact invariant vanishes in the case of planar torsion ([63]). The same is also true in the monopole Floer homology side thanks to our naturality result and Theorem 1 in [64].

Corollary 4. *Let (X, ω) be a strong filling of (Y, ξ). Then the contact invariant of ξ is non-vanishing, that is, $\mathbf{c}(\xi) \neq 0$.*

Proof. By Darboux's theorem we can remove a standard small ball B of X to obtain a strong cobordism $(W, \omega) : (S^3, \xi_{tight}) \to (Y, \xi)$. Naturality says that $\mathbf{c}(\xi_{tight}) = \widetilde{HM}_\bullet(W^\dagger, \mathfrak{s}_\omega)\mathbf{c}(\xi)$ but the left hand side is non-vanishing (see for example the appending in this paper) and so we conclude that $\mathbf{c}(\xi)$ is non-vanishing as well. □

Remark 5. The Heegaard Floer version of this fact appears as Theorem 2.13 in [22]. That same paper contains an example of a weak filling where the contact invariant vanishes.

To explain the next corollary we do a quick review of some of the properties of the monopole Floer homology Groups. Formally they behave like the ordinary homology groups $H_*(Z)$, $H_*(Z, A)$ and $H_*(A)$ for a pair of spaces in that they are related by a long exact sequence [32, section 3.1]

$$\cdots \xrightarrow{i_*} \widetilde{HM}_\bullet(Y, \mathfrak{s}) \xrightarrow{j_*} \widehat{HM}_\bullet(Y, \mathfrak{s}) \xrightarrow{p_*} \overline{HM}_\bullet(Y, \mathfrak{s}) \xrightarrow{i_*} \widetilde{HM}_\bullet(Y, \mathfrak{s}) \xrightarrow{j_*} \cdots \tag{3}$$

An important subgroup of $\widehat{HM}_\bullet(Y, \mathfrak{s})$ is the image of $j_* : \widetilde{HM}_\bullet(Y, \mathfrak{s}) \to \widehat{HM}_\bullet(Y, \mathfrak{s})$ which is known as the **reduced Floer homology group** $HM_\bullet(Y, \mathfrak{s})$, and in general it is of great interest to determine whether or not a particular element belongs to it. For example, if $j_* = 0$ we say that Y is an L **space** in analogy with the terminology from Heegaard Floer [32, section 42.6]. To relate this question to the naturality of the contact invariant, we need to use the fact that for a cobordism $(W^\dagger, \mathfrak{s}_W) : (-Y', \mathfrak{s}_Y) \to (-Y, \mathfrak{s}_{Y'})$ there is a commutative diagram

(4)

$$\begin{array}{ccccccccc}
\cdots & \widetilde{HM}_\bullet(-Y', \mathfrak{s}_{Y'}) & \xrightarrow{j_*} & \widehat{HM}_\bullet(-Y', \mathfrak{s}_{Y'}) & \xrightarrow{p_*} & \overline{HM}_\bullet(-Y', \mathfrak{s}_{Y'}) & \xrightarrow{i_*} & \widetilde{HM}_\bullet(-Y', \mathfrak{s}_{Y'}) & \cdots \\
& \downarrow_{\check{HM}_\bullet(W^\dagger, \mathfrak{s}_W)} & & \downarrow_{\widehat{HM}_\bullet(W^\dagger, \mathfrak{s}_W)} & & \downarrow_{\overline{HM}_\bullet(W^\dagger, \mathfrak{s}_W)} & & \downarrow_{\check{HM}_\bullet(W^\dagger, \mathfrak{s}_W)} & \\
\cdots & \widetilde{HM}_\bullet(-Y, \mathfrak{s}_Y) & \xrightarrow{j_*} & \widehat{HM}_\bullet(-Y, \mathfrak{s}_Y) & \xrightarrow{p_*} & \overline{HM}_\bullet(-Y, \mathfrak{s}_Y) & \xrightarrow{i_*} & \widetilde{HM}_\bullet(-Y, \mathfrak{s}_Y) & \cdots
\end{array}$$

Corollary 6. *Let (X, ω) be a strong filling of (Y, ξ). Assume in addition that Y is an L space. Then X must be negative definite.*

Proof. Suppose by contradiction that $b^+(X) \geq 1$. Remove a Darboux ball as before to obtain a cobordism $(W, \omega) : (S^3, \xi_{tight}) \to (Y, \xi)$. By proposition 3.5.2 in [32] we have that $\overline{HM}_\bullet(W^\dagger, \mathfrak{s}_\omega) = 0$. By the commutative diagram and the fact that j_* vanishes for Y we have that $\mathbf{c}(\xi) \in \ker j_* = \mathrm{im} i_*$. Hence $\mathbf{c}(\xi) = i_*([\Psi])$ for some $[\Psi] \in \overline{HM}_\bullet(-Y', \mathfrak{s}_{\xi'})$ and the commutativity together with the naturality says that

$$0 = i_* \overline{HM}(W^\dagger, \mathfrak{s}_\omega)([\Psi]) = \widetilde{HM}_\bullet(W^\dagger, \mathfrak{s}_\omega)\mathbf{c}(\xi) = \mathbf{c}(\xi_{tight})$$

which is a contradiction. □

Remark 7. This result appears as Theorem 1.4 in [45].

Corollary 8. *Suppose that (Y, ξ) is a planar contact manifold. Then $j_* \mathbf{c}(\xi) = 0$ and in particular any strong filling of a planar contact manifold must be negative definite.*

Proof. Observe that the last statement is exactly the proof of the previous corollary, which only used the fact that $\mathbf{c}(\xi) \in \ker j_*$. If (Y, ξ) is a planar contact manifold Theorem 4 in [64] (and the remarks after it) shows that there is a strong symplectic cobordism $(W, \omega) : (Y, \xi) \to (S^3, \xi_{tight})$. The result follows using the commutative diagram 4 and the fact that j_* vanishes on S^3 because it admits a metric of positive scalar curvature [32, Proposition 36.1.3]. □

Remark 9. Theorem 1.2 in [44] shows that if the contact structure ξ on Y is compatible with a planar open book decomposition then its contact invariant vanishes when regarded as an element of the quotient group $HF_{red}(-Y, \mathfrak{s}_\xi)$. The second part of our corollary should be compared with Theorem 1.2 in [16], where it is shown (among other things) that any symplectic filling of a planar contact manifold is negative definite.

The proof of the previous corollary can be extended to the case when Y' admits a metric with positive scalar curvature. First of all, it should be pointed out that this class of manifolds is not very large. Thanks to results of Schoen and Yau an orientable 3-manifold with positive scalar curvature can always be obtained from a manifold with $b_1 = 0$ by making a connected sum of a number of copies of $S^1 \times S^2$.

Corollary 10. *Suppose that* $(W, \omega) : (Y, \xi) \to (Y', \xi')$ *is a strong symplectic cobordism with* Y' *(hence* $-Y'$*) admitting a metric with positive scalar curvature. Then*

a) If $c_1(\mathfrak{s}_{\xi'})$ *is not torsion, then the contact invariant* $\mathbf{c}(\xi')$ *vanishes automatically and by naturality so will the contact invariant* $\mathbf{c}(\xi)$.

b) If $c_1(\mathfrak{s}_{\xi'})$ *is torsion, then* $j_*\mathbf{c}(\xi') = 0$ *and so by naturality* $j_*\mathbf{c}(\xi) = 0$. *In particular, for a strong cobordism* $(W, \omega) : (Y, \xi) \to (S^3, \xi_{tight})$ *we must have that* $j_*\mathbf{c}(\xi) = 0$.

Proof. Proposition 36.1.3 in [32] shows that j_* vanishes when $c_1(\mathfrak{s})$ is torsion and that the Floer groups are zero when $c_1(\mathfrak{s})$ is not torsion, from which the corollary follows immediately. □

In the next section we will sketch the main argument in the proof of Theorem (1). It is our hope that this summary captures the essential ideas of the proof of our main theorem, since the remaining (and more technical) part of the paper will follow very closely the paper [40], which is "required reading" for someone interested in understanding why the naturality theorem will be true.

2. Summary of the Proof

As stated before, we now give a brief summary of the main ideas involved in the proof of Theorem 1. In a nutshell, to show that $\mathbf{c}(\xi)$ equals $\widetilde{HM}_\bullet(W^\dagger, \mathfrak{s}_\omega)\mathbf{c}(\xi')$, we will define an intermediate "hybrid" invariant $\mathbf{c}(\xi', Y) \in \widetilde{HM}_\bullet(-Y, \mathfrak{s}_\xi)$ which will work as bridge between $\mathbf{c}(\xi)$ and $\widetilde{HM}_\bullet(W^\dagger, \mathfrak{s}_\omega)\mathbf{c}(\xi')$. Namely, using a "stretching the neck" argument we will show that

$$\widetilde{HM}_\bullet(W^\dagger, \mathfrak{s}_\omega)\mathbf{c}(\xi') = \mathbf{c}(\xi', Y)$$

while adapting the strategy of [40] (which as we will explain momentarily involves a "dilating the cone" argument) we will show that

$$\mathbf{c}(\xi', Y) = \mathbf{c}(\xi)$$

giving us the desired naturality result.

First we review the definition of the contact invariant, following section 6.2 in [30] (in their paper the contact invariant was denoted $[\check{\psi}_{Y,\xi}]$ but we have decided to switch to the more standard notation used in Heegaard Floer homology). As mentioned in the introduction, given a contact manifold (Y, ξ) we construct the manifold

$$Z^+_{Y,\xi} = \left(\mathbb{R}^+ \times (-Y)\right) \cup ([1, \infty) \times Y)$$

and study the Seiberg-Witten equations which are asymptotic to the canonical solution (A_0, Φ_0) on the conical end $[1, \infty) \times Y$ and to a critical point $\mathfrak{c}$ of the three dimensional Seiberg Witten equations on the cylindrical end $\mathbb{R}^+ \times (-Y)$. To write the Seiberg Witten equations a choice of spin-c structure needs to be made, and in this case the contact structure ξ determines a canonical spin-c structure $\mathfrak{s}$ on $Z^+_{Y,\xi}$ which we will describe later.

There is a gauge group action on such solutions and we define the moduli space $\mathcal{M}(Z^+_{Y,\xi}, \mathfrak{s}, [\mathfrak{c}])$ as the gauge equivalence classes of the solutions to the Seiberg-Witten equations on $Z^+_{Y,\xi}$. As a matter of notation, $[\cdot]$ will represent the gauge-equivalence class of a configuration so $[\mathfrak{c}]$ in this case denotes the gauge equivalence class of the critical point $\mathfrak{c}$. The moduli space $\mathcal{M}(Z^+_{Y,\xi}, \mathfrak{s}, [\mathfrak{c}])$ is not equidimensional, in fact, it admits a partition into components of different topological type

$$\mathcal{M}(Z^+_{Y,\xi}, \mathfrak{s}, [\mathfrak{c}]) = \bigcup_z \mathcal{M}_z(Z^+_{Y,\xi}, \mathfrak{s}, [\mathfrak{c}])$$

where z indexes the different connected components of the previous moduli space. We count points in the zero dimensional moduli spaces (which will be compact, hence finite) and define

$$m_z(Z^+_{Y,\xi}, \mathfrak{s}, [\mathfrak{c}]) = \begin{cases} |\mathcal{M}_z(Z^+_{Y,\xi}, \mathfrak{s}, [\mathfrak{c}])| \mod 2 & \text{if } \dim \mathcal{M}_z(Z^+_{Y,\xi}, \mathfrak{s}, [\mathfrak{c}]) = 0 \\ 0 & \text{otherwise} \end{cases}$$

The contact invariant is then defined at the chain level as

$$c(\xi) = (c^o(\xi), c^s(\xi)) \in \check{C}_*(-Y, \mathfrak{s}_\xi) = \mathfrak{C}^o(-Y, \mathfrak{s}_\xi) \oplus \mathfrak{C}^s(-Y, \mathfrak{s}_\xi) \tag{5}$$

by

$$c^o(\xi) = \sum_{[\mathfrak{a}] \in \mathfrak{C}^o(-Y, \mathfrak{s}_\xi)} \sum_z m_z(Z^+_{Y,\xi}, \mathfrak{s}, [\mathfrak{a}]) e_{[\mathfrak{a}]}$$
$$c^s(\xi) = \sum_{[\mathfrak{a}] \in \mathfrak{C}^s(-Y, \mathfrak{s}_\xi)} \sum_z m_z(Z^+_{Y,\xi}, \mathfrak{s}, [\mathfrak{a}]) e_{[\mathfrak{a}]}$$

In the previous notation $\check{C}_*(-Y, \mathfrak{s}_\xi)$ is the free abelian group generated by the irreducible and boundary stable critical points $[\mathfrak{a}]$ and $e_{[\mathfrak{a}]}$ is a bookkeeping device for each critical point considered as a generator in the group. Lemma 6.6 in [30] then shows that $c(\xi)$ is a cycle, that is, it defines an element $\mathbf{c}(\xi)$ of the Monopole Floer Homology group $\widetilde{HM}_\bullet(-Y, \mathfrak{s}_\xi)$.

The previous construction of the contact invariant $\mathbf{c}(\xi)$ can be regarded as coming from the recipe for defining relative invariants in the following sense. When we have a closed manifold X, we can use the Seiberg-Witten equations together with a choice of spin-c structure $\mathfrak{s}_X$ to produce numerical invariants

$$SW_X : \text{Spin}^c(X) \to \mathbb{Z}$$

Now, what happens if X is not closed, but rather bounds a 3-manifold Y? The TFQT-philosophy dictates that in this case we should try to produce an element

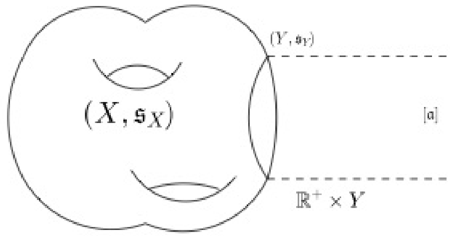

FIGURE 1. Constructing relative invariants.

$[\varphi_X]$ living in the Monopole Floer Homology groups of Y. We should remember that there are different flavors of Monopole Floer homology, so this philosophy needs to be implemented somewhat carefully. In any case, the basic idea is that $\varphi_{X,\mathfrak{s}_X}$ should be defined at the chain level as

$$\varphi_{X,\mathfrak{s}_X} = \sum_{[\mathfrak{a}]} n_{[\mathfrak{a}]}[\mathfrak{a}] = \sum_{[\mathfrak{a}]} \left(\sum \#\mathcal{M}_0(X^*;[\mathfrak{a}])\right)[\mathfrak{a}]$$

where $\#\mathcal{M}_0(X^*;[\mathfrak{a}])$ denotes the 0-dimensional moduli space of solutions to the Seiberg-Witten equations asymptotic to the critical point $[\mathfrak{a}]$.

The contact invariant $\mathbf{c}(\xi)$ should then be regarded as the relative invariant $\varphi_{X,\mathfrak{s}_X}$ associated to a canonical 4-manifold determined by the contact manifold (Y, ξ). The natural choice is to use the symplectization of (Y, ξ), that is, $(X, \omega) = \left(\mathbb{R} \times Y, \frac{1}{2}d(t^2\theta)\right)$ (other choices of symplectic form are usually used). In any case, in order to make up for the fact that X has no boundary we chop it off somewhere, say, we just work with $X = [1, \infty) \times Y$, in this case the natural boundary of X is $\partial X = -Y$, which is why we think of the contact invariant as an element in the monopole Floer homology of $-Y$, rather than Y. The astute reader may just ask in this case why not simply take $X = (-\infty, 1] \times Y$ so that we get a manifold with boundary Y. The problem is that once we introduce riemannian metrics compatible with the symplectic structure, then the first choice will lead to something with *bounded geometry*, while the second does not (we will explain what bounded geometry is in the next section).

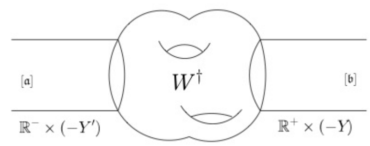

FIGURE 2. Manifold $W^\dagger_*$ with two cylindrical ends used to define the cobordism maps.

Returning to the naturality question, suppose we have a symplectic cobordism $(W,\omega) : (Y,\xi) \to (Y',\xi')$ and we want to decide whether or not $\widetilde{HM}_\bullet(W^\dagger, \mathfrak{s}_\omega)\mathbf{c}(\xi') = \mathbf{c}(\xi)$. Clearly this is equivalent to showing that at the *chain* level

$$\check{m}c(\xi') - c(\xi) \in \mathrm{im}\check{\partial}_{-Y}$$

where $\check{\partial}_{-Y} : \check{C}_*(-Y,\mathfrak{s}_\xi) \to \check{C}_*(-Y,\mathfrak{s}_\xi)$ is the differential that generates $\widetilde{HM}_\bullet(-Y,\mathfrak{s}_\xi)$. Here $\check{m}$ is the chain map (definition 25.3.3 [32])

$$\check{m} = \begin{pmatrix} m^o_o & -m^u_o\bar{\partial}^s_u - \partial^u_o\bar{m}^s_u \\ m^o_s & \bar{m}^s_s - m^u_s\bar{\partial}^s_u - \partial^u_s\bar{m}^s_u \end{pmatrix} : \check{C}_\bullet(-Y',\mathfrak{s}_{\xi'}) \to \check{C}_\bullet(-Y,\mathfrak{s}_\xi)$$

To see what $\check{m}$ does, we will explain the meaning of m^o_s and $\bar{\partial}^s_u$, since the action of the remaining terms can be inferred easily from these two examples. The map m^o_s counts solutions on $W^\dagger : -Y' \to -Y$ with a half-cylinder attached on each end:

$$W^\dagger_* = \left(\mathbb{R}^- \times -Y'\right) \cup W^\dagger \cup \left(\mathbb{R}^+ \times -Y\right)$$

which are asymptotic on $\mathbb{R}^- \times -Y'$ to an irreducible critical point $[\mathfrak{a}] \in \mathfrak{C}^o(-Y',\mathfrak{s}_{\xi'})$ and asymptotic on $\mathbb{R}^+ \times -Y$ to a boundary stable critical point $[\mathfrak{b}] \in \mathfrak{C}^s(-Y,\mathfrak{s}_\xi)$. On the other hand, the map $\bar{\partial}^s_u$ counts solutions on $\mathbb{R} \times (-Y')$ which are asymptotic to a boundary stable critical point $[\mathfrak{a}] \in \mathfrak{C}^s(-Y',\mathfrak{s}_{\xi'})$ as $t \to -\infty$ and to a boundary unstable critical point $[\mathfrak{b}] \in \mathfrak{C}^u(-Y',\mathfrak{s}_{\xi'})$ as $t \to \infty$ (in our context a map like ∂^u_o would count solutions on $\mathbb{R} \times -Y$ instead). The bar indicates that we are only considering reducible solutions, i.e, solutions where the spinor vanishes identically.

Again, we obtain a moduli space $\mathcal{M}([\mathfrak{a}], W_*^\dagger, \mathfrak{s}_\omega, [\mathfrak{b}])$ and as before we can define

$$n_z([\mathfrak{a}], W_*^\dagger, \mathfrak{s}_\omega, [\mathfrak{b}]) = \begin{cases} |\mathcal{M}_z([\mathfrak{a}], W_*^\dagger, \mathfrak{s}_\omega, [\mathfrak{b}])| \mod 2 & \text{if } \dim \mathcal{M}_z([\mathfrak{a}], W_*^\dagger, \mathfrak{s}_\omega, [\mathfrak{b}]) = 0 \\ 0 & \text{otherwise} \end{cases}$$

In the case of a cylinder there is a natural $\mathbb{R}$ action and the corresponding moduli space after we quotient out by this action is denoted $\check{\mathcal{M}}([\mathfrak{a}], \mathfrak{s}_{\xi'}, [\mathfrak{b}])$ (the notation in [32] for this moduli space is $\check{M}_z([\mathfrak{a}], \mathfrak{s}_{\xi'}, [\mathfrak{b}])$). In this case we define

$$n_z([\mathfrak{a}], \mathfrak{s}_{\xi'}, [\mathfrak{b}]) = \begin{cases} |\check{\mathcal{M}}_z([\mathfrak{a}], \mathfrak{s}_{\xi'}, [\mathfrak{b}])| \mod 2 & \text{if } \dim \check{\mathcal{M}}_z([\mathfrak{a}], \mathfrak{s}_{\xi'}, [\mathfrak{b}]) = 0 \\ 0 & \text{otherwise} \end{cases}$$

Under this notation, $\check{m}c(\xi')$ has two terms, and since are working $\mod 2$ we will write them without the signs to simplify the expression. The term corresponding to

$$m_o^o c^o(\xi') + m_o^u \bar{\partial}_u^s c^s(\xi') + \partial_o^u \bar{m}_u^s c^s(\xi')$$

is equivalent to

$$\begin{aligned} &\sum_{[\mathfrak{a}]\in\mathfrak{C}^o(-Y'),[\mathfrak{c}]\in\mathfrak{C}^o(-Y)} \sum_{z_1,z_2} m_{z_1}(Z^+_{Y',\xi'}, \mathfrak{s}', [\mathfrak{a}]) n_{z_2}([\mathfrak{a}], W_*^\dagger, \mathfrak{s}_\omega, [\mathfrak{c}]) e_{[\mathfrak{c}]} \\ + &\sum_{[\mathfrak{a}]\in\mathfrak{C}^s(-Y'),[\mathfrak{b}]\in\mathfrak{C}^u(-Y'),[\mathfrak{c}]\in\mathfrak{C}^o(-Y)} \sum_{z_1,z_2,z_3} m_{z_1}(Z^+_{Y',\xi'}, \mathfrak{s}', [\mathfrak{a}]) \bar{n}_{z_2}([\mathfrak{a}], \mathfrak{s}_{\xi'}, [\mathfrak{b}]) n_{z_3}([\mathfrak{b}], W_*^\dagger, \mathfrak{s}_\omega, [\mathfrak{c}]) e_{[\mathfrak{c}]} \\ + &\sum_{[\mathfrak{a}]\in\mathfrak{C}^s(-Y'),[\mathfrak{b}]\in\mathfrak{C}^u(-Y),[\mathfrak{c}]\in\mathfrak{C}^o(-Y)} \sum_{z_1,z_2,z_3} m_{z_1}(Z^+_{Y',\xi'}, \mathfrak{s}', [\mathfrak{a}]) \bar{n}_{z_2}([\mathfrak{a}], W_*^\dagger, \mathfrak{s}_\omega, [\mathfrak{b}]) n_{z_3}([\mathfrak{b}], \mathfrak{s}_\xi, [\mathfrak{c}]) e_{[\mathfrak{c}]} \end{aligned}$$

Notice that if we fix a critical point $[\mathfrak{c}] \in \mathfrak{C}^o(-Y, \mathfrak{s}_\xi)$ we can consider the coefficient

$$\begin{aligned} &\sum_{[\mathfrak{a}]\in\mathfrak{C}^o(-Y')} \sum_{z_1,z_2} m_{z_1}(Z^+_{Y',\xi'}, \mathfrak{s}', [\mathfrak{a}]) n_{z_2}([\mathfrak{a}], W_*^\dagger, \mathfrak{s}_\omega, [\mathfrak{c}]) \\ + &\sum_{[\mathfrak{a}]\in\mathfrak{C}^s(-Y'),[\mathfrak{b}]\in\mathfrak{C}^u(-Y')} \sum_{z_1,z_2,z_3} m_{z_1}(Z^+_{Y',\xi'}, \mathfrak{s}', [\mathfrak{a}]) \bar{n}_{z_2}([\mathfrak{a}], \mathfrak{s}_{\xi'}, [\mathfrak{b}]) n_{z_3}([\mathfrak{b}], W_*^\dagger, \mathfrak{s}_\omega, [\mathfrak{c}]) \\ + &\sum_{[\mathfrak{a}]\in\mathfrak{C}^s(-Y'),[\mathfrak{b}]\in\mathfrak{C}^u(-Y)} \sum_{z_1,z_2,z_3} m_{z_1}(Z^+_{Y',\xi'}, \mathfrak{s}', [\mathfrak{a}]) \bar{n}_{z_2}([\mathfrak{a}], W_*^\dagger, \mathfrak{s}_\omega, [\mathfrak{b}]) n_{z_3}([\mathfrak{b}], \mathfrak{s}_\xi, [\mathfrak{c}]) \end{aligned} \tag{6}$$

Similarly, the second term of $\check{m}c(\xi')$, i.e,

$$m^o_s c^o(\xi') + \bar{m}^s_s c^s(\xi') + m^u_s \bar{\partial}^s_u c^s(\xi') + \partial^u_s \bar{m}^s_u c^s(\xi')$$

is equivalent for each critical point $[\mathfrak{c}] \in \mathfrak{C}^s(-Y, \mathfrak{s}_\xi)$ to

$$\begin{aligned}
&\sum_{[\mathfrak{a}]\in\mathfrak{C}^o(-Y')} \sum_{z_1,z_2} m_{z_1}(Z^+_{Y',\xi'}, \mathfrak{s}', [\mathfrak{a}]) n_{z_2}([\mathfrak{a}], W^\dagger_*, \mathfrak{s}_\omega, [\mathfrak{c}]) \\
&+ \sum_{[\mathfrak{a}]\in\mathfrak{C}^s(-Y')} \sum_{z_1,z_2} m_z(Z^+_{Y',\xi'}, \mathfrak{s}', [\mathfrak{a}]) \bar{n}_{z_2}([\mathfrak{a}], W^\dagger_*, \mathfrak{s}_\omega, [\mathfrak{c}]) \\
&+ \sum_{[\mathfrak{a}]\in\mathfrak{C}^s(-Y'),[\mathfrak{b}]\in\mathfrak{C}^u(-Y')} \sum_{z_1,z_2,z_3} m_{z_1}(Z^+_{Y',\xi'}, \mathfrak{s}', [\mathfrak{a}]) \bar{n}_{z_2}([\mathfrak{a}], \mathfrak{s}_{\xi'}, [\mathfrak{b}]) n_{z_3}([\mathfrak{b}], W^\dagger_*, \mathfrak{s}_\omega, [\mathfrak{c}]) \\
&+ \sum_{[\mathfrak{a}]\in\mathfrak{C}^s(-Y'),[\mathfrak{b}]\in\mathfrak{C}^u(-Y)} \sum_{z_1,z_2,z_3} m_{z_1}(Z^+_{Y',\xi'}, \mathfrak{s}', [\mathfrak{a}]) \bar{n}_{z_2}([\mathfrak{a}], W^\dagger_*, \mathfrak{s}_\omega, [\mathfrak{b}]) n_{z_3}([\mathfrak{b}], \mathfrak{s}_\xi, [\mathfrak{c}])
\end{aligned} \tag{7}$$

Therefore, we want to show that up to a boundary term, $\sum_z m_z(Z^+_{Y,\xi}, \mathfrak{s}, [\mathfrak{c}])$ is equal to (6) (if $[\mathfrak{c}]$ is irreducible) or (7) (if $[\mathfrak{c}]$ is boundary stable).

If there is any hope of showing the equality between these two quantities we need to find a geometric interpretation to the sums (6), (7). In order to do this we will consider the Seiberg-Witten equations on a slightly more general scenario, one that combines the construction of the contact invariant with the cobordism. More precisely, we will study the Seiberg Witten equations on

$$W^+_{\xi',Y} = ([1,\infty) \times Y') \cup W^\dagger \cup \left(\mathbb{R}^+ \times -Y\right)$$

which are asymptotic on $[1,\infty) \times Y'$ using the canonical solution coming from the contact structure ξ' and asymptotic on $\mathbb{R}^+ \times -Y$ to a critical point $[\mathfrak{c}] \in \check{C}_*(-Y, \mathfrak{s}_\xi)$. The moduli space of such solutions will naturally be denoted $\mathcal{M}(W^+_{\xi',Y}, \mathfrak{s}_\omega, [\mathfrak{c}])$.

Thanks to the compactness arguments in [31, 32] and [65] (which guarantee uniform exponential decay along the conical end) we can proceed as before and define

$$m_z(W^+_{\xi',Y}, \mathfrak{s}_\omega, [\mathfrak{c}]) = \begin{cases} |\mathcal{M}_z(W^+_{\xi',Y}, \mathfrak{s}_\omega, [\mathfrak{c}])| \mod 2 & \text{if } \dim \mathcal{M}_z(W^+_{\xi',Y}, \mathfrak{s}_\omega, [\mathfrak{c}]) = 0 \\ 0 & \text{otherwise} \end{cases}$$

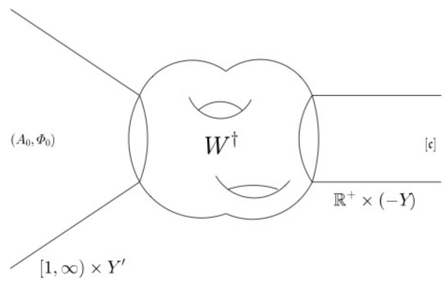

FIGURE 3. Manifold $W^+_{\xi',Y}$ used to define the "hybrid" invariant $\mathbf{c}(\xi', Y)$.

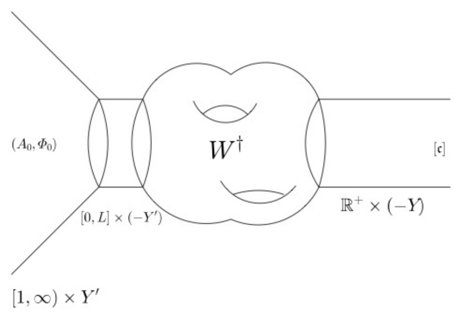

FIGURE 4. $\check{HM}(W^\dagger, \mathfrak{s}_\omega)\mathbf{c}(\xi') = \mathbf{c}(\xi', Y)$ via a "stretching the neck" argument.

These numbers give rise to the hybrid invariant $\mathbf{c}(\xi', Y)$ mentioned at the beginning of this section. In order to show the equality $\widetilde{HM}_\bullet(W^\dagger, \mathfrak{s}_\omega)\mathbf{c}(\xi') = \mathbf{c}(\xi', Y)$ we must consider the parametrized moduli space

$$\bigcup_{L\in[0,\infty)} \{L\} \times \mathcal{M}(W^+_{\xi',Y}(L), \mathfrak{s}_\omega, [\mathfrak{c}]) \tag{8}$$

where $\mathcal{M}(W^+_{\xi',Y}(L), \mathfrak{s}_\omega, [\mathfrak{c}])$ denotes the moduli space of solutions to the Seiberg-Witten equations on the manifold

$$W^+_{\xi',Y}(L) = ([1,\infty) \times Y') \cup ([0,L] \times -Y') \cup W^\dagger \cup \left(\mathbb{R}^+ \times -Y\right)$$

The parametrized moduli space (8) is not compact; its compactification will be denoted

$$\bigcup_{L\in[0,\infty]} \{L\} \times \mathcal{M}^+(W^+_{\xi',Y}(L), \mathfrak{s}_\omega, [\mathfrak{c}]) \tag{9}$$

where the definition of $\mathcal{M}^+(W^+_{\xi',Y}(\infty), \mathfrak{s}_\omega, [\mathfrak{c}])$ is given in (34). For now, it suffices to say that when we count the endpoints of all one dimensional moduli spaces inside (9) we will get 0.

The count coming from the fiber over $L = 0$ will give the term $\sum_z m_z(W^+_{\xi',Y}, \mathfrak{s}_\omega, [\mathfrak{c}])$ while the count coming from the fiber over $L = \infty$ will give one of the sums (6) or (7) depending on whether $[\mathfrak{c}]$ is irreducible or boundary stable. Finally, the count coming from the other fibers will contribute a boundary term (see Theorem (27) for the precise statement). At the level of homology, this means that $\widetilde{HM}_\bullet(W^\dagger, \mathfrak{s}_\omega)\mathbf{c}(\xi') = \mathbf{c}(\xi', Y)$ so the naturality proof has been reduced to showing that $\mathbf{c}(\xi', Y) = \mathbf{c}(\xi)$. Again, from the chain level perspective this means that up to boundary terms, for each critical point $[\mathfrak{c}]$, the numbers $\sum_z m_z(W^+_{\xi',Y}, \mathfrak{s}_\omega, [\mathfrak{c}])$ must equal $\sum_{z'} m_{z'}(Z^+_{Y,\xi}, \mathfrak{s}, [\mathfrak{c}])$.

If one were to replace the half-cylindrical end $\mathbb{R}^+ \times (-Y)$ with a compact piece X so that we could work with numbers instead of homology classes, the previous quantities would be the same due to Theorem D in [40] (i.e, equation (2) in our paper). Therefore, it becomes clear at this point that what we need to do is adapt the Mrowka-Rollin theorem to the case in which we have a half-infinite cylinder.

Two things that change in this new setup are the certain inclusions of Sobolev spaces are no longer compact and in order to achieve transversality (i.e, obtain unobstructed moduli spaces in the terminology of [40]) one must use the "abstract perturbations" defined by Kronheimer and Mrowka in [32]. In particular, these perturbations introduce new terms that do not appear in the usual linearizations of the Seiberg-Witten equations, so for the gluing argument we will employ one needs to check that the new contributions do not mess up the desired behavior of the linearized Seiberg Witten equations. Namely, we will see that the contributions have leading terms which are quadratic in a appropriate sense. Had the leading term been linear, the gluing argument would not have worked.

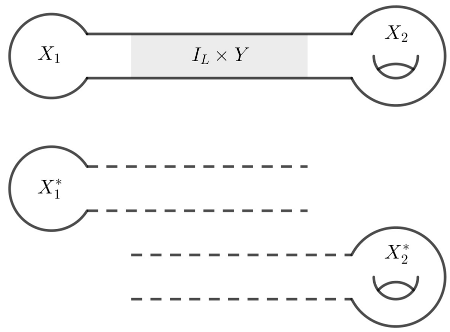

Figure 5. Gluing technique for the "stretching the neck" argument.

Our gluing argument and the proof of Theorem D [40] morally follows the same basic ideas as the other gluing arguments in gauge theory but as expected differs in the specific details (a few references include [32, 42, 48, 12, 39, 18, 19]). Perhaps the most common gluing argument is gauge theory is the one involving the "*stretching the neck*" operation on a closed oriented Riemannian 4 manifold X which has a separating hypersurface Y inside it.

Namely, one writes X as $X = X_1 \cup X_2$ and after choosing a metric which is cylindrical near Y one can stretch the metric along Y in order to have a cylinder $I_L \times Y$ of length L inserted between X_1 and X_2 as shown in the picture. The point is that as L increases, the Seiberg Witten equations on $X_L = X_1 \cup (I_L \times Y) \cup X_2$ start behaving more like the solutions on the manifolds with cylindrical ends X_1^* and X_2^*. More precisely, one can start from solutions on X_1^* and X_2^* which agree on their respective ends in order to construct a pre-solution on X_L , that is, a configuration on X_L which is a solution to the Seiberg Witten equations on X_L, except perhaps for a region supported on $I_L \times Y$. The main point of the gluing argument, is that one can find an L_0 sufficiently large, so that for all L bigger than L_0 we can obtain an actual solution to the Seiberg Witten equations on X_L thanks to an application of the implicit function theorem for Banach spaces. In order for this to work it is imperative to have estimates that become independent of L.

Likewise, in our situation we want to take advantage of the fact that for a strong symplectic cobordism the symplectic structure is given near the boundary by the

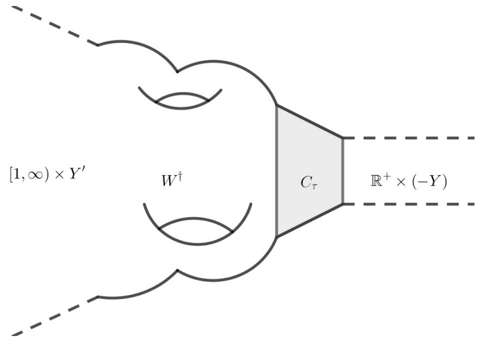

FIGURE 6. "Dilating the cone" argument used to show that $\mathbf{c}(\xi', Y) = \mathbf{c}(\xi)$.

symplectization of the contact structure, so that in analogy with the cylindrical case we can perform a "*dilating the cone*" operation, where now the key parameter is a dilation parameter τ , which determines the size of the cone C_τ determined by the symplectization of the contact structure near the boundary. As in the cylindrical case, the main idea is that once τ is sufficiently large, the moduli space of solutions to the Seiberg Witten equations on the manifold shown below can be described in terms of the moduli space used to define the contact invariant of (Y, ξ). Again, this will rely on an application of the implicit function theorem, which requires guaranteeing that certain estimates become independent of τ (once it becomes sufficiently large). As we will explain near the end of the paper this gluing theorem will establish that $\mathbf{c}(\xi', Y) = \mathbf{c}(\xi)$.

3. Setting Up The Equations

3.1 The Seiberg-Witten Equations and the Configuration Space. As explained in the previous section, we will analyze first the equations on $W^+_{\xi',Y}$. In particular, we begin by stating some basic geometric properties of the manifolds we are going to be working with.

Suppose we have a closed oriented three manifold Y with contact structure ξ. We assume that $\xi = \ker\theta$ and choose the unique Riemannian metric g_θ such that [31, Section 2.3]:

- The contact form θ has unit length.
- $d\theta = 2 *_Y \theta$
- If J is a fixed choice of an almost complex structure on ξ, then for any $v, w \in \xi$, $g_\theta(v,w) = d\theta(v, Jw)$.

The contact structure ξ determines a canonical spin-c structure $\mathfrak{s}_\xi$: define the spinor bundle S as the rank-2 vector bundle $S = \underline{\mathbb{C}} \oplus \xi$ where $\underline{\mathbb{C}}$ is the trivial vector bundle and we are considering ξ as a complex line bundle. Moreover, there is a Clifford map $\rho_Y : TY \to \hom(S,S)$ which identifies TY isometrically with the subbundle $\mathfrak{su}(S)$ of traceless, skew-adjoint endomorphisms equipped with the inner product $\frac{1}{2}\mathrm{tr}(a^*b)$ [32, Section 1.1]. Using $(Y, g_\theta, \mathfrak{s}_\xi)$ we can write the configuration space on which the Seiberg-Witten equations are defined [32, Section 9.1]: for any integer or half integer $k \geq 0$ define

$$\mathcal{C}_k(Y, \mathfrak{s}_\xi) = (B_{ref}, 0) + L^2_k(M; iT^*Y \oplus S) = \mathcal{A}_k(Y, \mathfrak{s}_\xi) \times L^2_k(Y; S)$$

where B_{ref} is a reference smooth connection on the spinor bundle S compatible with the Levi-Civita connection defined on TY and $\mathcal{A}_k(Y, \mathfrak{s}_\xi)$ denotes the (affine) space of spin-c connections of S with Sobolev regularity L^2_k. We will always assume whenever needed that $k \geq 5$, but by elliptic regularity the constructions end up being independent of k because one can always find a smooth representative in each gauge equivalence class of solutions to the Seiberg-Witten equations so will not dwell a lot on the actual value of k being used.

The gauge group $\mathcal{G}_{k+1}(Y)$ is

$$\mathcal{G}_{k+1}(Y) = \{u \in L^2_{k+1}(Y; \mathbb{C}) \mid |u| = 1 \text{ pointwise}\}$$

It acts on the configuration space via

$$u \cdot (B, \Psi) = (B - u^{-1}du, u\Psi)$$

The action is not free at the **reducible configurations**, that is, the configurations $(B, 0)$ with the spinor component identically zero. The stabilizer at those configurations consists of the constant maps $u : Y \to S^1$ which we can identify with S^1. To handle reducible configurations Kronheimer and Mrowka introduced the **blown-up configuration space** [32, Section 6.1]

$$\mathcal{C}_k^\sigma(Y, \mathfrak{s}_\xi) = \{(B, s, \phi) \mid \|\phi\|_{L^2(Y)=1}, s \geq 0\} = \mathcal{A}_k(Y, \mathfrak{s}_\xi) \times \mathbb{R}^{\geq} \times \mathbb{S}(L_k^2(Y; S))$$

Here $\mathbb{S}(L_k^2(Y; S))$ denotes those elements ϕ in $L_k^2(Y; S)$ whose L^2 norm (not L_k^2 norm!) is equal to 1. In this case the gauge action is

$$u \cdot (B, s, \phi) = (B - u^{-1}du, s, u\phi)$$

and it is easy to check that the gauge group acts freely on this space. In fact, Lemma 9.1.1 in [32] shows that the space $\mathcal{C}_k^\sigma(Y, \mathfrak{s}_\xi)$ is naturally a Hilbert manifold with boundary and when $k \geq 1$, the space $\mathcal{G}_{k+1}(Y)$ is a Hilbert Lie group which acts smoothly and freely on $\mathcal{C}_k^\sigma(Y, \mathfrak{s}_\xi)$.

We are interested in triples (B, s, ϕ) which satisfy a perturbed version of the Seiberg-Witten equations. At this point the nature of the perturbations is not that important. For now it suffices to say that we will take them to be **strongly tame perturbations** as in definition 3.6 of [65]. As a technical point it is useful to note that the cylindrical functions constructed in section 11.1 of [32] are strongly tame perturbations so the theorems from [32] which used this class of perturbations continue to work in this context. We will denote such a perturbation by $\mathfrak{q}_{Y, g_\theta, \mathfrak{s}_\xi}$. In general a strongly tame perturbation $\mathfrak{q}$ can be regarded as a map $\mathfrak{q} : \mathcal{C}_k(Y, \mathfrak{s}_\xi) \to L_k^2(Y; iT^*Y \oplus S)$, where one thinks of the codomain as a copy of the tangent space $T_{(B,\Psi)}\mathcal{C}_k(Y, \mathfrak{s}_\xi)$ for each configuration $(B, \Psi) \in \mathcal{C}_k(Y, \mathfrak{s}_\xi)$. Since the codomain naturally splits one can write $\mathfrak{q} = (\mathfrak{q}^0, \mathfrak{q}^1)$ and in section 10.2 of [32] it is explained how $\mathfrak{q}$ gives rise to a perturbation on the blown-up configuration space $\mathfrak{q}^\sigma = (\mathfrak{q}^0, \hat{\mathfrak{q}}^{1,\sigma})$ (notice that only the second component is modified).

The corresponding equations (B, s, ϕ) satisfy are (section 10.3 [32])

$$
\begin{cases} \frac{1}{2} * F_{B^t} + s^2 \rho_Y^{-1}(\phi\phi^*)_0 + \mathfrak{q}^0_{Y,g_\theta,\mathfrak{s}_\xi}(B, s\phi) = 0 \\ \Lambda_{\mathfrak{q}_{Y,g_\theta,\mathfrak{s}_\xi}}(B, s, \phi)s = 0 \\ D_B\phi - \Lambda_{\mathfrak{q}_{Y,g_\theta,\mathfrak{s}_\xi}}(B, s, \phi)\phi + \tilde{\mathfrak{q}}^1_{Y,g_\theta,\mathfrak{s}_\xi}(B, s, \phi) = 0 \end{cases} \tag{10}
$$

where:

- F_{B^t} denotes the curvature of the connection B^t on $\det(S)$.
- $(\phi\phi^*)_0$ denotes the trace-free part of the hermitian endomorphism $\phi\phi^*$: $(\phi\phi^*)_0 = \phi\phi^* - \frac{1}{2}|\phi|^2 1_S$.
- D_B is the Dirac operator corresponding to the connection B.
- $\Lambda_{\mathfrak{q}_{Y,g_\theta,\mathfrak{s}_\xi}}(B, s, \phi) = \mathrm{Re}\left\langle \phi, D_B\phi + \tilde{\mathfrak{q}}^1_{Y,g_\theta,\mathfrak{s}_\xi}(B, s, \phi) \right\rangle_{L^2(Y)}$ and $\tilde{\mathfrak{q}}^1(B, r, \psi) = \int_0^1 \mathcal{D}_{(B,sr\psi)}\mathfrak{q}^1(0, \psi)ds$

(here $\mathcal{D}$ denotes the linearization of the map $\mathfrak{q}^1$).

Using the equations (10) we can distinguish three types of solutions (or critical points) $\mathfrak{c} = (B, s, \phi)$ [30, Definition 4.4], the **irreducible critical point,** the **boundary stable reducible critical point** and the **boundary unstable reducible critical point.** What is important about this classification for us is that solutions of the four dimensional Seiberg Witten equations on $\mathbb{R} \times Y$ for which the spinor does not vanish identically can only be asymptotic as $t \to \infty$ to irreducible critical points or boundary stable reducible critical points. The gauge equivalence class of any of these points will be denoted as $[\mathfrak{c}]$.

The triple $(Y, g_\theta, \mathfrak{s}_\xi)$ induces a spin-c structure on $(-Y, g_\theta)$ given by the same spinor bundle S_ξ and changing the Clifford multiplication from ρ_ξ to $-\rho_\xi$ [32, Section 22.5]. We will continue to denote this spin-c structure by $\mathfrak{s}_\xi$. Given this structure we can use the cylindrical metric and the spin-c structure induced by $-Y$ on the cylinder $\mathbb{R}^+ \times -Y$ [32, Section 4.3]. We use the perturbation $-\mathfrak{q}_{Y,g_\theta,\mathfrak{s}_\xi}$ on $-Y$.

Consider now the manifold

$$W^+_{\xi',Y} = ([1, \infty) \times Y') \cup W^\dagger \cup \left(\mathbb{R}^+ \times -Y\right)$$

We will define the appropriate geometric structures needed on each piece together with the perturbations we will be using.

- On $\mathbb{R}^+ \times -Y$, we use the cylindrical metric and the canonical spin-c structure induced by $\mathfrak{s}_\xi$ on the cylinder. As explained on section 10.1 of [32], we have a four dimensional perturbation $-\hat{\mathfrak{q}}_{Y,g_\theta,\mathfrak{s}_\xi}: \mathcal{C}_k(\mathbb{R}^+\times -Y,\mathfrak{s}_\xi)\to L^2_k(\mathbb{R}^+\times -Y; iT^*(\mathbb{R}^+\times -Y)\oplus S)$ on the half-cylinder $\mathbb{R}^+\times -Y$, defined by restriction to each slice.

• On $W^\dagger$ we choose a metric g_W on $W^\dagger$ such that the metric g_W is cylindrical in collar neighborhoods of the boundary components. To define the perturbation on $W^\dagger$ we follow section 24.1 in [32]. Since the Riemannian metric is cylindrical in the neighborhood of the boundary it contains on each boundary component an isometric copy of $I_1\times -Y$ and $I_2\times Y'$ where $I_1=(-C_1,0]$, $I_2=(-C_2,0]$. Since the argument is the same for both ends we will use generic notation. Let β be a cut-off function, equal to 1 near $t=0$ and equal to 0 near $t=-C$. Let β_0 be a bump function with compact support in $(-C,0)$, equal to one on a compact subset inside $(-C,0)$. Choose another perturbation $\mathfrak{p}_0$ of the three dimensional equations and consider the perturbation

$$\hat{\mathfrak{p}}_W=\beta\hat{\mathfrak{q}}+\beta_0\hat{\mathfrak{p}}_0$$

It is useful to note that the reason why we use two perturbations is so that one can be varied when we use a transversality argument.

• On $[1,\infty)\times Y'$ we assume that the metric is cylindrical in a collar neighborhood $[1,C_K)\times Y'$ and on a complement of this neighborhood (like $N_K=[C_K+1,\infty)\times Y'$ for instance) it is given by the metric

$$g_{K,\theta'}=dt\otimes dt+t^2g_{\theta'}$$

with symplectic form

$$\omega_{\theta'}=\frac{1}{2}d(t^2\theta')$$

Here K stands for Kahler, although in most cases the cone will not be a Kahler manifold (in fact occurs only when (Y,ξ) is a Sasakian manifold [3].The form is self dual with respect to $g_{K,\theta'}$ and $|\omega_{\theta'}|_{g_{K,\theta'}}=\sqrt{2}$ pointwise. By Lemma 2.1 in [31], on the symplectic cone we have a unit length section Φ_0 associated to the canonical spinor bundle $S_{\omega_{\theta'}}$. For this section Φ_0 we have a corresponding connection A_0 such that $D_{A_0}\Phi_0=0$. Choose a smooth extension of (A_0,Φ_0) to all of $W^+_{\xi',Y}$ in such a way that (A_0,Φ_0) is translation invariant on the cylindrical end $\mathbb{R}^+\times -Y$. Define

$$\mathfrak{p}_K=\left(-\frac{1}{2}\rho(F^+_{A_0^t})+(\Phi_0\Phi_0^*)_0,0\right)$$

and choose a bump function β_K which is supported on N_K and identically equal to 1 on $[C_K+2,\infty)$. Choose also a bump function β_{N_K} which is supported on $[1,C_K)\times Y'$ and identically equal to 1 near the boundary $\partial([1,C_K)\times Y')$.

Our global perturbation will be

(11)

$$\mathfrak{p}_{W^+_{\xi',Y}} = -\hat{\mathfrak{q}}_{Y,g_\theta,\mathfrak{s}_\xi} + \left(\beta\hat{\mathfrak{q}}_{Y,g_\theta,\mathfrak{s}_\xi} + \beta_0'\hat{\mathfrak{p}}_0\right) + \left(\beta_0'\hat{\mathfrak{p}}_0' + \beta'\hat{\mathfrak{q}}_{Y',g_{\theta'},\mathfrak{s}_{\xi'}}\right) + \left(\beta_{N_K}\hat{\mathfrak{q}}_{Y',g_{\theta'},\mathfrak{s}_{\xi'}} + \beta_K\mathfrak{p}_K\right)$$

where β_0', β' are cutoff functions defined analogously for the other cylindrical neighborhood $I_2\times Y'$.

In words the previous perturbation behaves as follows: if we start on the cylindrical end $\mathbb{R}^+\times -Y$ we will see the translation invariant perturbation $-\hat{\mathfrak{q}}_{Y,g_\theta,\mathfrak{s}_\xi}$. As we enter the cobordism through the boundary $-Y\subset W^\dagger$ (recall that $\partial W^\dagger = -Y\sqcup Y'$) this perturbation is modified into a combined perturbation $\beta\hat{\mathfrak{q}}_{Y,g_\theta,\mathfrak{s}_\xi} + \beta_0'\hat{\mathfrak{p}}_0$, which is supported on a collar neighborhood of this end. After we exit this collar neighborhood we will see no perturbations until we reach again the collar neighborhood of the end $Y'\subset W^\dagger$, where the perturbation is $\beta_0'\hat{\mathfrak{p}}_0' + \beta'\hat{\mathfrak{q}}_{Y',g_{\theta'},\mathfrak{s}_{\xi'}}$. Finally, as we exit the cobordism we will see a perturbation identically equal to $\hat{\mathfrak{q}}_{Y',g_{\theta'},\mathfrak{s}_{\xi'}}$ for a small time until it becomes zero again and then it will eventually be changed into the perturbation identically equal to $\mathfrak{p}_K$. We will explain the reason why the perturbations were chosen in this way near the end of this section.

Now we must define the corresponding configuration space that we want to use in order to analyze the Seiberg-Witten equations. In general one needs to define the ordinary configuration space and its blow-up (see sections 13 and 24.2 for some motivation behind this construction). Due to the asymptotic condition we will impose, our solutions will always be irreducible so the gauge group action will be free without having to blow up the configuration space. Therefore, most of the time we will simply use the ordinary configuration space. However, if one wants to describe the compactification of the moduli spaces in terms of the space of broken trajectories then the blow up model is more convenient so for completeness sake we will write the equations in the blow up model (but we will switch to the ordinary configuration space when some computations become more transparent there).

We are interested in the configurations that solve the following perturbed version of the Seiberg-Witten equations:

$$\mathfrak{F}_{\mathfrak{p}} = \mathfrak{F} + \mathfrak{p}_{W^+_{\xi',Y}} = 0 \tag{12}$$

where the **unperturbed Seiberg Witten map** is [32, eq. 4.12]

$$\mathfrak{F}(A,\Phi) = \left(\frac{1}{2}\rho(F^+_{A^t}) - (\Phi\Phi^*)_0, D_A\Phi\right)$$

Both the perturbed and unperturbed maps are defined on elements of the following configuration space (def. 3.5 in [65] and def. 13.1 in [32]):

Definition 11. Define the **configuration space** (without blow-up) $\mathcal{C}_{k,loc}(W^+_{\xi',Y},\mathfrak{s}_\omega)$ as follows. It will consist of pairs (A,Φ) such that:

1) A is a locally L^2_k spin-c connection for S and Φ is a locally L^2_k section of S^+.

2) It is L^2_k close to the canonical solution on the conical end, that is,

$$A - A_0 \in L^2_k([1,\infty)\times Y', iT^*([1,\infty)\times Y'))$$
$$\Phi - \Phi_0 \in L^2_{k,A_0}([1,\infty)\times Y', S^+)$$

Remark 12. a) Recall that we chose an extension of A_0 to the cylindrical end in such a way that it was translation invariant so the condition that A is a locally L^2_k spin-c connection means that $A - A_0 \in L^2_{k,loc}(W^+_{\xi',Y}; iT^*W^+_{\xi',Y})$.

b) Notice that the second condition implies that Φ cannot be identically 0, i.e, $\mathcal{C}_{k,loc}(W^+_{\xi',Y},\mathfrak{s})$ contains no reducible configurations. In the notation of [32], we would write $\mathcal{C}_{k,loc}(W^+_{\xi',Y},\mathfrak{s}) = \mathcal{C}^*_{k,loc}(W^+_{\xi',Y'},\mathfrak{s})$.

c) Due to the lack of a norm the space $\mathcal{C}_{k,loc}(W^+_{\xi',Y},\mathfrak{s})$ is not a Banach space unless we impose some asymptotic condition on the cylindrical end.

The **blown-up configuration space** $\mathcal{C}^\sigma_{k,loc}(W^+_{\xi',Y},\mathfrak{s}_\omega)$ is defined as follows:

Definition 13. If S denotes the spinor bundle, define the sphere $\mathbb{S}$ as the topological quotient of $L^2_{k,loc}(W^+_{\xi',Y};S^+)\backslash 0$ by the action of $\mathbb{R}^+$ [32, section 6.1]. The blown-up configuration space associated to $\mathcal{C}_{k,loc}(W^+_{\xi',Y},\mathfrak{s}_\omega)$ is

$$\mathcal{C}^\sigma_{k,loc}(W^+_{\xi',Y},\mathfrak{s}_\omega) = \{(A,\mathbb{R}^+\phi,\Phi) \mid \Phi \in \mathbb{R}^{\geq 0}\phi,\ \ \phi \in \mathbb{S} \text{ and } (A,\Phi) \in \mathcal{C}_{k,loc}(W^+_{\xi',Y},\mathfrak{s}_\omega)\}$$

Just as its blown-down version, $\mathcal{C}^\sigma_{k,loc}(W^+_{\xi',Y},\mathfrak{s}_\omega)$ is not a Banach manifold, much less a Hilbert manifold, so we will not try to find useful slices on this space. These slices would have been "orthogonal" in some suitable sense to the gauge group action, which begs the question, what is the gauge group in this situation? A provisional definition for the gauge group is to use

$$\mathcal{G}_{k+1,loc}(W^+_{\xi',Y}) = \{u : W^+_{\xi',Y} \to \mathbb{C} \mid |u| = 1 \text{ and } u \in L^2_{k+1,loc}(W^+_{\xi',Y})\} \tag{13}$$

where the action of $u \in \mathcal{G}_{k+1,loc}$ on a triple $(A, \mathbb{R}^+\phi, \Phi) \in \mathcal{C}^\sigma_{k,loc}(W^+_{\xi',Y}, \mathfrak{s})$ is given by

$$u \cdot (A, \mathbb{R}^+\phi, \Phi) = (A - u^{-1}du, \mathbb{R}^+(u\phi), u\Phi) \tag{14}$$

The topology we will give to it is the topology of L^2_{k+1} convergence on compact subsets. At this point it is not clear that this is indeed a group nor that 14 defines an action. We will see that in fact we will need to impose further conditions on our gauge group. In any case, to check these properties we actually need the help of the Sobolev multiplication theorems. Since our manifold is not compact this adds a further complication because the consequences of the Sobolev theorems on open manifolds are not as powerful as for closed manifolds. However, our situation is not that terrible given that we are still within the realm of bounded geometry, which we now proceed to describe.

3.2 Bounded Geometry and the Gauge Group. As we just said, since our manifold $W^+_{\xi',Y}$ is non-compact some care is required when verifying certain properties of our configuration/moduli spaces. For example, the Fredholm property of elliptic operators, the closed range property of differential operators with injective symbol, the Rellich lemma all fail on general open manifolds [13] . Fortunately, most of what is needed remains true in the setting of **bounded geometry** [14, pages 5-14]:

Definition 14. Let (M^n, g) be a smooth n dimensional Riemannian manifold with metric g. We say that it has **bounded geometry up to order** l if one can find constants $C_1, \cdots, C_l$ such that conditions (I) and $(B_i(M, g))$ are satisfied:

$$\begin{cases} (I): \quad \mathrm{r}_{inj}(M, g) > 0 \\ (B_i(M, g): \quad |\nabla^i_{LC} R| \leq C_i \quad \forall i = 1, 2, \cdots l \end{cases}$$

where is r_{inj} the injectivity radius of M, ∇_{LC} denotes the Levi-Civita connection and R the curvature tensor.

If $(E, h, \nabla^h) \to (M^n, g)$ is a Riemannian vector bundle we can use the Levi-Civita connection ∇_{LC} and the connection ∇^h to define metric connections ∇ in all tensor bundles $T^u_v(M) \otimes E = (TM)^{\otimes u} \otimes (T^*M)^{\otimes v} \otimes E$. If $\varphi \in \Gamma(E)$ we can regard it as a E-valued zero form so we define for k a non-negative integer (notice that the case when k is a half-integer is only being used in our context for the case of compact three manifolds) and $p \in \mathbb{R}$ the quantity

$$\|\varphi\|_{L^p_k(M)} = \left(\int \sum_{i=0}^{k} |\nabla^i \varphi|^p_x \mathrm{vol}_x(g) \right)^{1/p}$$

together with the spaces

$$\begin{aligned} \Omega^p_k(E) &= \{\varphi \in C^\infty(E) \mid \|\varphi\|_{L^p_k(M)} < \infty\} \\ \bar{\Omega}^p_k(E) &= \{\text{completion of } \Omega^p_k(E) \text{ with respect to } \|\cdot\|_{L^p_k}\} \\ \overset{\circ}{\Omega}{}^p_k(E) &= \{\text{completion of } C^\infty_c(E) \text{ with respect to } \|\cdot\|_{L^p_k}\} \\ \Omega^p_k(E) &= \{\varphi \mid \varphi \text{ is a measurable distribution section with } \|\varphi\|_{L^p_k(M)} < \infty\} \end{aligned}$$

Likewise, we can define a pointwise norm

$$|\varphi|_{b,k} = \sup_{x \in M} \quad \sum_{i=0}^{k} |\nabla^i \varphi|_x$$

with corresponding spaces

$$\begin{aligned} \Omega_{b,k}(E) &= \{\varphi \mid \varphi \text{ is a } C^k - \text{section and } |\varphi|_{b,k} < \infty\} \\ \overset{\circ}{\Omega}_{b,k}(E) &= \{\text{completion of } C^\infty_c(E) \text{ with respect to } |\cdot|_{b,k}\} \end{aligned}$$

Fortunately our manifold $W^+_{\xi',Y}$ satisfies the condition $(B_l(M,g))$ for all l and in this case we do not have to worry about this plethora of spaces thanks to the following theorem, which actually corresponds to a combination of Proposition 3.1, Proposition 3.2, Theorem 3.3 , Theorem 3.4, Theorem 3.12 and Corollary 3.14 in [14]. Also notice that unlike the case of a compact manifold, we are not claiming that any of these inclusions are compact.

Proposition 15. *i) The spaces* $\overset{\circ}{\Omega}{}^p_k(E) \subset \bar{\Omega}^p_k(E) \subset \Omega^p_k(E)$ *are Banach spaces (Hilbert for* $p = 2$*) and* $\overset{\circ}{\Omega}_{b,k}(E) \subset \Omega_{b,k}(E)$ *are Banach spaces.*

ii) If (M^n, g) *satisfies* (I) *and* $(B_l(M,g))$ *then* $\overset{\circ}{\Omega}{}^p_r(E) = \bar{\Omega}^p_r(E) = \Omega^p_r(E)$ *for* $0 \leq r \leq l+2$.

iii) If for $k \geq 1$ *we have* $(B_l(E, \nabla^h))$, (I) *and* $(B_l(M,g))$ *then we have the continuous inclusions:*

$$(15) \qquad \begin{aligned} &(l \geq r, \quad r - \tfrac{n}{p} \geq s - \tfrac{n}{q}, \quad r \geq s, \quad q \geq p) \implies \Omega^p_r(E) \hookrightarrow \Omega^q_s(E) \\ &(r - \tfrac{n}{p} > s) \quad \implies \Omega^p_r(E) \hookrightarrow \Omega_{b,s}(E) \end{aligned}$$

iv) If $(E_i, h_i, \nabla^h_i) \to (M^n, g)$ *are vector bundles with* (I), $(B_l(M^n, g))$, $(B_l(E_i, \nabla_i))$ *for* $i = 1, 2$ *and*

a) $0 < r \leq r_1, r_2 \leq l$, $\frac{1}{p} \leq \frac{1}{p_1} + \frac{1}{p_2}$ *, then we have a continuous bilinear map*

$$(16) \qquad \left.\begin{array}{r} r - \frac{n}{p} < r_1 - \frac{n}{p_1} \\ r - \frac{n}{p} < r_2 - \frac{n}{p_2} \\ r - \frac{n}{p} \leq r_1 - \frac{n}{p_1} + r_2 - \frac{n}{p_2} \\ \\ \text{or} \\ \\ r - \frac{n}{p} \leq r_1 - \frac{n}{p_1} \\ r - \frac{n}{p} \leq r_2 - \frac{n}{p_2} \\ r - \frac{n}{p} < r_1 - \frac{n}{p_1} + r_2 - \frac{n}{p_2} \end{array}\right\} \implies \Omega^{p_1}_{r_1}(E_1, \nabla^h_1) \times \Omega^{p_2}_{r_2}(E_2, \nabla^h_2) \to \Omega^p_r(E_1 \otimes E_2, \nabla^h_1 \otimes \nabla^h_2)$$

b) $0 = r \leq r_1, r_2 \leq l$ *then we have a continuous bilinear map*

(17)
$$\left.\begin{array}{c}
\begin{aligned}
r-\tfrac{n}{p} &< r_1-\tfrac{n}{p_1}\\
r-\tfrac{n}{p} &< r_2-\tfrac{n}{p_2}\\
r-\tfrac{n}{p} &\le r_1-\tfrac{n}{p_1}+r_2-\tfrac{n}{p_2}\\
\tfrac{1}{p} &\le \tfrac{1}{p_1}+\tfrac{1}{p_2}
\end{aligned}\\[2ex]
\text{or}\\[2ex]
\begin{aligned}
r-\tfrac{n}{p} &\le r_1-\tfrac{n}{p_1}\\
0 &< r_2-\tfrac{n}{p_2}\\
\tfrac{1}{p} &\le \tfrac{1}{p_1}
\end{aligned}\\[2ex]
\text{or}\\[2ex]
\begin{aligned}
0 &< r_1-\tfrac{n}{p_1}\\
r-\tfrac{n}{p} &\le r_2-\tfrac{n}{p_2}\\
\tfrac{1}{p} &\le \tfrac{1}{p_2}
\end{aligned}
\end{array}\right\}
\Longrightarrow \quad \Omega^{p_1}_{r_1}(E_1,\nabla^h_1)\times\Omega^{p_2}_{r_2}(E_2,\nabla^h_2)\to\Omega^{p}_{r}(E_1\otimes E_2,\nabla^h_1\otimes\nabla^h_2)$$

Remark 16. a) Because of ii) in the previous theorem, we can safely write $L^2_r(E)$ to represent $\Omega^2_r(E)$.

b) It is not terribly difficult to see from our the Sobolev Multiplication Theorem that $\mathcal{G}_{k+1,loc}(W^+_{\xi',Y})$ will be a group.

Now we return to the question of what is the appropriate definition for the gauge group. As we mentioned earlier, we clearly want 14 to define an action, which means that if $(A,\mathbb{R}^+\phi,\Phi)\in\mathcal{C}^\sigma_{k,loc}(W^+_{\xi',Y},\mathfrak{s})$ then it should also be the case that $(A-u^{-1}du,\mathbb{R}^+(u\phi),u\Phi)\in\mathcal{C}^\sigma_{k,loc}(W^+_{\xi',Y},\mathfrak{s})$. If we look at the definition of $\mathcal{C}^\sigma_{k,loc}(W^+_{\xi',Y},\mathfrak{s})$, this means that $A-u^{-1}du$ should be L^2_k close from A_0 and $u\Phi$ should be L^2_k close from Φ_0 on the conical end. As we will see in the next lemma, this imposes the following condition on u:

$$\text{condition on } u\in\mathcal{G}_{k+1,loc}: \quad 1-u\in L^2_{k+1}([1,\infty)\times Y') \tag{18}$$

In other words, we will take the gauge group to be

$$\mathcal{G}_{k+1}(W^+_{\xi',Y}) = \{u : W^+_{\xi',Y} \to \mathbb{C}^* \mid |u| = 1 \text{ and } 1-u \in L^2_{k+1}([1,\infty)\times Y')\} \tag{19}$$

where (again) the action of $u \in \mathcal{G}_{k+1}$ on a triple $(A, \mathbb{R}^+\phi, \Phi) \in \mathcal{C}^{\sigma}_{k,loc}(W^+_{\xi',Y}, \mathfrak{s}_\omega)$ is given by

$$u \cdot (A, \mathbb{R}^+\phi, \Phi) = (A - u^{-1}du, \mathbb{R}^+(u\phi), u\Phi) \tag{20}$$

We will regard $\mathcal{G}_{k+1}(W^+_{\xi',Y})$ as a subset of $\mathcal{G}_{k+1,loc}(W^+_{\xi',Y})$ and correspondingly **we will give $\mathcal{G}_{k+1}(W^+_{\xi',Y})$ the subspace topology**. Moreover, observe that for $u \in \mathcal{G}_{k+1}(W^+_{\xi',Y})$ the condition $1-u \in L^2_{k+1}([1,\infty)\times Y')$ can be rewritten as

$$\begin{aligned}
\|1-u\|^2_{L^2([1,\infty)\times Y')} &< \infty \\
\|du\|^2_{L^2([1,\infty)\times Y')} &< \infty \\
\|\nabla_{LC}(du)\|^2_{L^2([1,\infty)\times Y')} = \|\nabla^2_{LC}u\|^2_{L^2([1,\infty)\times Y')} &< \infty \\
&\vdots \\
\|\nabla^k_{LC}(du)\|^2_{L^2([1,\infty)\times Y')} = \|\nabla^{k+1}_{LC}u\|^2_{L^2([1,\infty)\times Y')} &< \infty
\end{aligned} \tag{21}$$

At the same time, by part iii) in Proposition (15) we can use using the embedding $L^2_{k+1}(E) \hookrightarrow L^4_k(E)$ (take $n=4$, $p=2$, $q=4$, $r=k+1$, $s=k$ and $l=\infty$) to conclude that $1-u \in L^4_k([1,\infty)\times Y')$, that is,

$$\begin{aligned}
\|1-u\|^4_{L^4([1,\infty)\times Y')} &< \infty \\
\|du\|^4_{L^4([1,\infty)\times Y')} &< \infty \\
\|\nabla_{LC}(du)\|^4_{L^4([1,\infty)\times Y')} = \|\nabla^2_{LC}u\|^2_{L^4([1,\infty)\times Y')} &< \infty \\
&\vdots \\
\|\nabla^{k-1}_{LC}(du)\|^4_{L^2([1,\infty)\times Y')} = \|\nabla^{k}_{LC}u\|^4_{L^4([1,\infty)\times Y')} &< \infty
\end{aligned} \tag{22}$$

Another natural thing we might want from our gauge group is that if $(\tilde{A}, \mathbb{R}^+\tilde{\phi}, \tilde{\Phi})$, $(A, \mathbb{R}^+\phi, \Phi)$ both belong to $\mathcal{C}^{\sigma}_{k,loc}(W^+_{\xi',Y}, \mathfrak{s})$ and $u \in \mathcal{G}_{k+1,loc}(W^+_{\xi',Y})$, then in fact $u \in \mathcal{G}_{k+1}(W^+_{\xi',Y})$. As we will see in the next lemma, this is indeed possible.

Lemma 17. *Suppose that $k \geq 4$. Then $\mathcal{G}_{k+1}(W^+_{\xi',Y})$ is a group. Moreover, the action of $\mathcal{G}_{k+1}(W^+_{\xi',Y})$ on $\mathcal{C}_{k,loc}(W^+_{\xi',Y}, \mathfrak{s}_\omega)$ is well defined in that:*

i) if $(A, \Phi) \in \mathcal{C}_k(W^+_{\xi',Y}, \mathfrak{s}_\omega)$ and $u \in \mathcal{G}_{k+1}(W^+_{\xi',Y})$ then $u \cdot (A, \Phi) \in \mathcal{C}_k(W^+_{\xi',Y}, \mathfrak{s}_\omega)$ and similarly, *ii) if* $u \cdot (A, \Phi) = (\tilde{A}, \tilde{\Phi})$ *for two configurations* $(A, \Phi), (\tilde{A}, \tilde{\Phi}) \in \mathcal{C}_k(W^+_{\xi',Y}, \mathfrak{s}_\omega)$ *and* u *is a* $L^2_{k+1,loc}(W^+_{\xi',Y})$ *gauge transformation, then* $1 - u \in L^2_{k+1}([1, \infty) \times Y')$.

Proof. First of all, since $k \geq 4$ by part *iii)* in Theorem (15) an element $u \in \mathcal{G}_{k+1}(W^+_{\xi',Y})$ can be regarded as a continuous map so that the first condition $|u| = 1$ makes sense. For $u, v \in \mathcal{G}_{k+1}(W^+_{\xi',Y})$ observe that

$$1 - uv = (1 - u) + (1 - v) - (1 - u)(1 - v)$$

Because of the Sobolev Multiplication Theorem, i.e, part *iv)* in Theorem (namely, take $n = 4$, $p_1 = p_2 = p = 2$, $r_1 = r_2 = r = k + 1$, $l = \infty$ in part iv. a)) (15) , we have that $(1-u)(1-v) \in L^2_{k+1}([1, \infty) \times Y')$ and so it is clear that $uv \in \mathcal{G}_{k+1}(W^+_{\xi',Y})$. To check that it is closed under the inverse operation we will verify that u^{-1} satisfies the properties in (21). From

$$1 - u^{-1} = uu^{-1} - u^{-1} = -(1 - u)u^{-1}$$

it is clear that

$$\|1 - u^{-1}\|_{L^2} = \|1 - u\|_{L^2}$$

so the first condition (i.e, $1 - u^{-1} \in L^2$) has been verified. For the second condition observe that

$$\|du^{-1}\|^2_{L^2} = \| - u^{-2}du\|^2_{L^2} = \|du\|^2_{L^2}$$

which is finite because of (21). The other inequalities that u^{-1} must satisfy, namely, the properties in (21) are obtained by similar (recursive) arguments. For example,

$$\|\nabla(du^{-1})\|^2_{L^2} = \|\nabla(u^{-2}du)\|^2_{L^2} = \| - 2u^{-3}du + u^{-2}\nabla(du)\|^2_{L^2} \leq \|du\|^2_{L^2} + \|\nabla^2 u\|^2_{L^2}$$

and we already know that both terms are finite. Verifying the smoothness conditions is essentially the same as what needs to be done on a compact manifold [18, Appendix A]. To verify *i)* observe that if $(\tilde{A}, \tilde{\Phi}) = u \cdot (A, \Phi) = (A - u^{-1}du, u\Phi)$ for some $u \in \mathcal{G}_{k+1}(W^+_{\xi',Y})$ then we know first of all that

$$A - A_0 \in L^2_k([1, \infty) \times Y'; iT^*[1, \infty) \times Y') \quad \Phi - \Phi_0 \in L^2_{k,A_0}([1, \infty) \times Y'; S^+)$$

In this case we will show that $(\tilde{A}, \tilde{\Phi})$ satisfies the same asymptotic conditions, namely

$$\tilde{A} - A_0 \in L^2_k([1, \infty) \times Y'; iT^*[1, \infty) \times Y') \quad \tilde{\Phi} - \Phi_0 \in L^2_{k,A_0}([1, \infty) \times Y'; S^+)$$

Observe that

$$\begin{cases} (\bullet) \quad A - u^{-1}du - A_0 = (A - A_0) - u^{-1}du \\ (\bullet\bullet) \quad u\Phi - \Phi_0 = (u-1)\Phi + (\Phi - \Phi_0) \end{cases}$$

To deal with $(\bullet)$ since $A - A_0 \in L^2_k([1,\infty) \times Y'; iT^*[1,\infty) \times Y')$ we just need to control $u^{-1}du$ to guarantee that $\tilde{A} - A_0 \in L^2_k([1,\infty) \times Y'; iT^*[1,\infty) \times Y')$. Clearly

$$\|u^{-1}du\|^2_{L^2} = \|du\|^2_{L^2}$$

so we just need to control $\|\nabla(u^{-1}du)\|_{L^2}, \cdots \|\nabla^k(u^{-1}du)\|_{L^2}$. Because of Leibniz's rule

$$\begin{aligned} &\nabla(u^{-1}du) \\ &= (du^{-1}) \otimes du + u^{-1}\nabla(du) \\ &= -u^{-2}du \otimes du + u^{-1}\nabla(du) \\ &= u^{-1}\left[-(u^{-1}du) \otimes du + \nabla(du)\right] \end{aligned}$$

In particular

$$\begin{aligned} &\|\nabla(u^{-1}du)\|^2_{L^2} \\ &= \| - (u^{-1}du) \otimes du + \nabla(du)\|^2_{L^2} \\ &\leq \|u^{-1}du \otimes du\|^2_{L^2} + \|\nabla(du)\|^2_{L^2} \\ &= \|du \otimes du\|^2_{L^2} + \|\nabla(du)\|^2_{L^2} \end{aligned}$$

Because of (21) the second term will bounded. To bound the first term we use the first version of (17) with $r = r_1 = r_2 = 0$, $n = 4$, $p = 2$, $p_1 = p_2 = 4$ and the bounds we have for $\|du\|_{L^4}$ coming from (22). We conclude that $\|\nabla(u^{-1}du)\|_{L^2}$ is finite and applying a similar procedure we can control the higher derivatives $\nabla^j(u^{-1}du)$ which means that $u^{-1}du \in L^2_k([1,\infty) \times Y'; iT^*[1,\infty) \times Y')$.

For $(\bullet\bullet)$ notice that $\Phi - \Phi_0 \in L^2_k([1,\infty) \times Y'; S^+)$ and so we just need to control $\|(u-1)\Phi\|_{L^2_k}$. On the other hand,

$$\begin{aligned} &\|(u-1)\Phi\|^2_{L^2_k} \\ \leq \|(u-1)(\Phi-\Phi_0)\|^2_{L^2_k} + &\|(u-1)(\Phi_0)\|^2_{L^2_k} \end{aligned}$$

The first term will be controlled thanks to the Sobolev multiplication theorems, namely, use $n=4$, $p=p_1=p_2=2$, $r=k$, $r_1=k+1$, $r_2=k$ in the second version of (16) . The second term will be controlled since we have control on of $u-1$ and the derivatives of Φ_0 (since the covariant derivatives $\nabla^{\bullet}_{A_0}\Phi_0$ are pointwise bounded given that the ends of our manifold are cylindrical and asymptotically flat). In this way we have that $\|(u-1)\Phi\|_{L^2_k} < \infty$ so $u \cdot (A,\Phi) \in \mathcal{C}_k(W^+_{\xi',Y}, \mathfrak{s})$ as we wanted to show.

For $ii)$ suppose that both (A,Φ) and $(\tilde{A},\tilde{\Phi})$ are elements in the configuration space, which means that

$$\begin{cases} -u^{-1}du = \tilde{A} - A = (\tilde{A}-A_0) - (A-A_0) \in L^2_k([1,\infty)\times Y'; iT^*([1,\infty)\times Y')) \\ (u-1)\Phi = (\tilde{\Phi}-\Phi) = (\tilde{\Phi}-\Phi_0) - (\Phi-\Phi_0) \in L^2_{k,A_0}([1,\infty)\times Y'; S^+) \end{cases}$$

First of all, observe that for any $T>0$

$$\|1-u\|^2_{L_{k+1}([1,\infty)\times Y')} = \|1-u\|^2_{L_{k+1}([1,T]\times Y')} + \|1-u\|^2_{L_{k+1}([T,\infty)\times Y')}$$

and the norm in the middle is finite because the submanifold is compact. Therefore, we just need to show that $\|1-u\|^2_{L_{k+1}([T,\infty)\times Y')}$ is finite, i.e, that u satisfies the inequalities in (21) on the conical end $[T,\infty)\times Y'$. Observe that by (15) the C^0 norm is controlled by the L^2_k norm. In particular, since $\Phi-\Phi_0 \in L^2_{k,A_0}$ (we can also write $L^2_{k,A}$ for the second factor if we wish it, e.g [14, Theorem 3.22]) and $|\Phi_0| = 1$ on the cone we can find T sufficiently large so that

$$||\Phi|-1|_{C^0} = ||\Phi|-|\Phi_0||_{C^0} \leq |\Phi-\Phi_0|_{C^0} < \frac{1}{2} \quad \text{on} \quad [T,\infty)\times Y'$$

that is, $|\Phi| \geq \frac{1}{2}$ on $[T,\infty)\times Y'$. In this way

$$\begin{aligned}
&\|1-u\|^2_{L^2([T,\infty)\times Y')} \\
&= \int_{[T,\infty)\times Y'} (1-u)^2 \\
&\leq 4\int_{[T,\infty)\times Y'} (1-u)^2|\Phi|^2 \\
&= 4\|(1-u)\Phi\|^2_{L^2} < \infty
\end{aligned}$$

Now we want to show that $\|du\|^2_{L^2([T,\infty)\times Y')} < \infty$, for a potentially different value of T. Notice that

$$\begin{aligned}
&\|du\|^2_{L^2} \\
&= \|u^{-1}du\|^2_{L^2} \\
&= \|\tilde{A}-A\|^2_{L^2} < \infty
\end{aligned}$$

The other norms can be controlled using similar (recursive) arguments. $\square$

Therefore it makes sense to define

$$\mathcal{B}^{\sigma}_{k,loc}(W^+_{\xi',Y}, \mathfrak{s}_\omega) = \mathcal{C}^{\sigma}_{k,loc}(W^+_{\xi',Y}, \mathfrak{s}_\omega)/\mathcal{G}_{k+1}(W^+_{\xi',Y})$$

Again, since the original space $\mathcal{C}^{\sigma}_{k,loc}(W^+_{\xi',Y}, \mathfrak{s}_\omega)$ is not a Banach manifold, we won't be interested in studying directly $\mathcal{B}^{\sigma}_{k,loc}(W^+_{\xi',Y}, \mathfrak{s}_\omega)$, although this is the space where the solutions to the Seiberg-Witten equations live.

To define the moduli space to the Seiberg-Witten equations, we need to introduce the τ model first. Let

$$[\mathfrak{c}] \in \mathfrak{C}^o(-Y, g_\theta, \mathfrak{s}_\xi, -\mathfrak{q}_{Y,g_\theta,\mathfrak{s}_\xi}) \cup \mathfrak{C}^s(-Y, g_\theta, \mathfrak{s}_\xi, -\mathfrak{q}_{Y,g_\theta,\mathfrak{s}_\xi})$$

be a critical point [32, Proposition 12.2.5] to the blown -up three dimensional Seiberg Witten equations on $-Y$ (10). Write $[\mathfrak{c}] = [(B, s, \phi)]$ and let $\mathfrak{c} = (B, s, \phi)$ be a smooth representative in $\mathcal{C}^{\sigma}_k(-Y, \mathfrak{s}_\xi)$. The critical point $\mathfrak{c}$ gives rise to a translation invariant configuration $\gamma_{\mathfrak{c}}$ on the half-infinite cylinder $\mathbb{R}^+ \times -Y$.

Definition 18. Define on $\mathbb{R}^+ \times -Y$ the τ model $\mathcal{C}^\tau_k(\mathbb{R}^+ \times -Y, \mathfrak{s}_\xi, \mathfrak{c})$ associated to $\mathfrak{c}$ as the space of triples [32, Section 13.3]

$$\gamma = (A, r(t), \phi(t)) \in \mathcal{A}_{k,loc}(\mathbb{R}^+ \times -Y, \mathfrak{s}_\xi) \times L^2_{k,loc}(\mathbb{R}^+; \mathbb{R}) \times L^2_{k,loc}(\mathbb{R}^+ \times -Y; S^+)$$

such that

i) $\gamma - \gamma_\mathfrak{c} \in L^2_{k,loc}(iT^*(\mathbb{R}^+ \times -Y)) \times L^2_{k,loc}(\mathbb{R}^+; \mathbb{R}) \times L^2_{k,loc}(\mathbb{R}^+ \times -Y; S^+)$, i.e, γ is $L^2_{k,loc}$ close from $\gamma_\mathfrak{c}$.

ii) For all $t \in \mathbb{R}^+$, we have that $r(t) \geq 0$.

iii) For all $t \in \mathbb{R}^+$, we have that $\|\phi(t)\|_{L^2(-Y)} = 1$, i.e, on each slice the L^2 norm (not the L^2_k norm) is one.

There is a natural restriction of the gauge group $\mathcal{G}_{k+1}(W^+_{\xi',Y})$ to $\mathbb{R}^+ \times -Y$ which acts on $\tilde{\mathcal{C}}^\tau_{k,loc}(\mathbb{R}^+ \times -Y, \mathfrak{s}_\xi, \mathfrak{c})$ via

$$u \cdot (A, r(t), \phi(t)) = (A - u^{-1}du, r(t), u\phi(t))$$

The gauge equivalence classes of configurations under this gauge group action will be denoted as

$$\mathcal{B}^\tau_{k,loc}(\mathbb{R}^+ \times -Y, \mathfrak{s}_\xi, [\mathfrak{c}]) = \mathcal{C}^\tau_{k,loc}(\mathbb{R}^+ \times -Y, \mathfrak{s}_\xi, \mathfrak{c}) / \mathcal{G}_{k+1,loc}(\mathbb{R}^+ \times -Y)$$

We will also use the **unique continuation principle,** which will essentially allow us for the most part to avoid working with the blow-up model. The versions most convenient to us are Proposition 7.1.4 and Proposition 10.8.1 in [32].

These imply that if a solution of the perturbed Dirac equation vanishes on a slice $\{t\} \times -Y$ of the cylindrical end $\mathbb{R}^+ \times -Y$, then it would have to vanish on the entire half-cylinder $\mathbb{R}^+ \times -Y$ and then on the entire four manifold $W^+_{\xi',Y}$. However, since we will be interested in solutions which are asymptotic on the conical end to the spinor Φ_0 (which is non-vanishing), this cannot be the case so we can safely conclude that no such solutions will exist, that is, our spinor Φ will never vanish on an open set or a cylindrical slice. Thanks to this, the following definition makes sense (compare with definition 24.2.1 of [32]):

Definition 19. The moduli space $\mathcal{M}(W^+_{\xi',Y}, \mathfrak{s}_\omega, [\mathfrak{c}])$ for a critical point

$$[\mathfrak{c}] \in \mathfrak{C}^o(-Y, g_\theta, \mathfrak{s}_\xi, -\mathfrak{q}_{Y,g_\theta,\mathfrak{s}_\xi}) \cup \mathfrak{C}^s(-Y, g_\theta, \mathfrak{s}_\xi, -\mathfrak{q}_{Y,g_\theta,\mathfrak{s}_\xi})$$

consists gauge equivalence classes of triples

$$[A, \mathbb{R}^+\phi, \Phi] \in \mathcal{B}^\sigma_{k,loc}(W^+_{\xi',Y}, \mathfrak{s}_\omega)$$

such that:

1) $(A, \mathbb{R}^+\phi, \Phi) \in \mathcal{C}^\sigma_{k,loc}(W^+_{\xi',Y}, \mathfrak{s}_\omega)$ and (A, Φ) satisfies the perturbed Seiberg-Witten equations $\mathfrak{F}_\mathfrak{p}(A, \Phi) = 0$ on $W^+_{\xi',Y}$. Here $\mathfrak{p}$ refers to the perturbation explained before equation (12).

2) Because of the unique continuation principle, Φ can not be identically zero on each of the cylindrical slices. Therefore we can define for each t [32, Sections 6.2 and 13.1]:

$$(r(t), \psi(t)) = \left(\|\check{\Phi}(t)\|_{L^2(-Y)}, \frac{\check{\Phi}(t)}{\|\check{\Phi}(t)\|_{L^2(-Y)}} \right)$$

Also, if we decompose the covariant derivative ∇_A in the $\frac{d}{dt}$ direction as

$$\nabla_{A,\frac{d}{dt}} = \frac{d}{dt} + a_t \otimes 1_S$$

we require that $\gamma = (A, r(t), \psi(t))$ be an element of $\mathcal{C}^\tau_{k,loc}(\mathbb{R}^+ \times -Y, \mathfrak{s}_\xi, \mathfrak{c})$ and that it solves the following Seiberg-Witten equations on the cylinder [32, eq 10.9]

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\check{A}^t = -\frac{1}{2} *_{-Y} F_{\check{A}^t} + da_t - r^2\rho^{-1}(\psi\psi^*)_0 - \mathfrak{q}^0(\check{A}, r\psi) \\
\frac{d}{dt}r = -\Lambda_\mathfrak{q}(\check{A}, r, \psi)r \\
\frac{d}{dt}\psi = -D_{\check{A}}\psi - a_t\psi - \tilde{\mathfrak{q}}^1(\check{A}, r, \psi) + \Lambda_\mathfrak{q}(\check{A}, r, \psi)\psi
\end{aligned}$$

where $\check{A}(t)$ denotes the restriction of A to the t slice. Moreover, we require that the gauge equivalence class $[\gamma]$ of γ be asymptotic as $t \to \infty$ to $[\mathfrak{c}]$ in the sense of Definition 13.1.1 in [32].

The moduli space $\mathcal{M}(W^+_{\xi',Y}, \mathfrak{s}_\omega, [\mathfrak{c}])$ is naturally a subset of $\mathcal{B}^\sigma_{k,loc}(W^+_{\xi',Y}, \mathfrak{s}_\omega)$. However, since the latter space is not in any natural way a Hilbert manifold we will use a fiber product description of $\mathcal{M}(W^+_{\xi',Y}, \mathfrak{s}_\omega, [\mathfrak{c}])$ instead [32, Lemma 24.2.2, Lemma 19.1.1]. The idea is that we can "break" the moduli space $\mathcal{M}(W^+_{\xi',Y}, \mathfrak{s}_\omega, [\mathfrak{c}])$ into three moduli spaces which we will show are Hilbert manifolds. These moduli spaces are the moduli space on the cobordism $\mathcal{M}(W^\dagger, \mathfrak{s}_\omega)$, the moduli space on the half cylinder $\mathcal{M}(\mathbb{R}^+ \times -Y, \mathfrak{s}_\xi)$ and the moduli space on the conical end $\mathcal{M}([1,\infty) \times Y', \mathfrak{s}')$. The fiber product description will then allow us to show that $\mathcal{M}(W^+_{\xi',Y}, \mathfrak{s}_\omega, [\mathfrak{c}])$ has a Hilbert manifold structure but in order to explain this we need to introduce some additional notation and explanations.

First, we need to describe each individual piece in the fiber product: the piece corresponding to the moduli space $\mathcal{M}(W^\dagger, \mathfrak{s}_\omega)$, is described in [32, Proposition 24.3.1] where it is shown to be a Hilbert manifold. Likewise, the second moduli space $\mathcal{M}(\mathbb{R}^+ \times -Y, \mathfrak{s}_\xi)$ is described in [32, Sections 13 and 14] where it is shown that it is a Hilbert manifold. Strictly speaking, they analyzed an entire cylinder $\mathbb{R} \times Y$ rather than a half cylinder $\mathbb{R}^+ \times -Y$ but the analysis is essentially the same if one is only concerned with showing that the moduli space is a Hilbert manifold, the main difference between the two cases is that for a half-cylinder the moduli space will be infinite dimensional while for the entire cylinder it will be finite dimensional. Therefore, we will start the next section showing that $\mathcal{M}([1,\infty) \times Y', \mathfrak{s}')$ is a Hilbert manifold, following the arguments in section 24 of [32]. At the end of the day, we obtain restriction (or trace) maps

$$\begin{aligned}
R_\tau : \mathcal{M}(\mathbb{R}^+ \times -Y, \mathfrak{s}_\xi, [\mathfrak{c}]) &\to \mathcal{B}^\sigma_{k-1/2}(Y, \mathfrak{s}_\xi) \\
R^-_W : \mathcal{M}(W^\dagger, \mathfrak{s}_\omega) &\to \mathcal{B}^\sigma_{k-1/2}(-Y, \mathfrak{s}_\xi) \\
R^+_W : \mathcal{M}(W^\dagger, \mathfrak{s}_\omega) &\to \mathcal{B}^\sigma_{k-1/2}(Y', \mathfrak{s}_{\xi'}) \\
R_K : \mathcal{M}([1,\infty) \times Y', \mathfrak{s}') &\to \mathcal{B}^\sigma_{k-1/2}(-Y', \mathfrak{s}_\xi)
\end{aligned}$$

given by restricting the (gauge equivalence class of a) solution to the boundary of each of the corresponding manifolds. We should point out that there is an identification between $\mathcal{B}^\sigma_k(-Y, \mathfrak{s})$ and $\mathcal{B}^\sigma_k(Y, \mathfrak{s})$ [32, Section 22.5] and we can identify $\mathcal{M}(W^+_{\xi',Y}, \mathfrak{s}, [\mathfrak{c}])$ with the fiber product $\mathrm{Fib}(R_\tau, R^-_W, R^+_W, R_K)$ given by

(23)

$$\left\{ \left([\gamma_{\mathbb{R}^+\times -Y}], [\gamma_W], [\gamma_{[1,\infty)\times Y'}]\right) \;\middle|\; R_\tau[\gamma_{\mathbb{R}^+\times -Y}] = R^-_W[\gamma_W] \text{ and } R^+_W[\gamma_W] = R_K[\gamma_{[1,\infty)\times Y'}] \right\}$$

Now we can explain how to give $\mathcal{M}(W^+_{\xi',Y}, \mathfrak{s}, [\mathfrak{c}])$ a Hilbert manifold structure (the precise definitions as well as the domains of the following maps appear in the next section). For convenience write $R = (R_\tau, R_W^-, R_W^+, R_K)$ and suppose that $[\gamma] = \big([\gamma_{\mathbb{R}^+\times -Y}], [\gamma_W], [\gamma_{[1,\infty)\times Y'}]\big) \in \mathrm{Fib}(R_\tau, R_W^-, R_W^+, R_K)$ is such that the map R is transverse at $([\mathfrak{b}], [\mathfrak{b}'])$, where $[\mathfrak{b}] = R_\tau([\gamma_{\mathbb{R}^+\times -Y}]) = R_W^-([\gamma_W])$ and $[\mathfrak{b}'] = R_W^+([\gamma_W]) = R_K([\gamma_{[1,\infty)\times Y'}])$. In other words, we want the linearized map $\mathcal{D}_{[\gamma]}R$ to be Fredholm and surjective. If this can be achieved, then near $\mathcal{M}(W^+_{\xi',Y}, \mathfrak{s}_\omega, [\mathfrak{c}])$ will have the structure of smooth manifold of dimension $\dim\ker \mathcal{D}_{[\gamma]}R$. The Fredholm property is proven in Lemma 26 of our paper in the next section. The surjectivity of the map $\mathcal{D}_{[\gamma]}R$ may not be true for an arbitrary perturbation of the form described in equation 11 earlier, however, an application of Sard's theorem shows that one can choose generic perturbations such that the surjectivity is achieved as well (this is stated precisely in Theorem 25 of our paper). In fact, achieving the surjectivity is essentially the same as the proof Kronheimer and Mrowka gave for the case of a manifold X^* with cylindrical ends [32, Proposition 24.4.7]. By choosing a perturbation from this generic set, one can then guarantee that $\mathrm{Fib}(R_\tau, R_W^-, R_W^+, R_K) = \mathcal{M}(W^+_{\xi',Y}, \mathfrak{s}_\omega, [\mathfrak{c}])$ has the structure of a smooth manifold (possibly disconnected with components of different dimensions).

4. TRANSVERSALITY AND FIBER PRODUCTS

4.1 The moduli space on the cobordism $M(W^\dagger, \mathfrak{s}_\omega)$**:** We will start by describing each of the individual moduli spaces. The easiest to begin with is $M(W^\dagger, \mathfrak{s}_\omega)$. By proposition 24.3.1 in [32], the moduli space $M(W^\dagger, \mathfrak{s}_\omega)$ is a smooth Hilbert manifold regardless of the choice of perturbation. To understand the restriction (trace) maps we review first some basic ideas and notation involving the construction of $M(W^\dagger, \mathfrak{s}_\omega)$.

Observe that since our solutions will always be irreducible, we don't need to worry about the boundary of the moduli space $M(W^\dagger, \mathfrak{s}_\omega)$, which corresponds to reducible solutions. Moreover, since $W^\dagger$ is a compact manifold, we can use the standard model for the blowup, that is, we can define [32, p. 113]

$$\begin{aligned}\mathcal{C}_k^\sigma(W^\dagger, \mathfrak{s}_\omega) &= \{(A, s, \phi) \mid \|\phi\|_{L^2(W^\dagger)} = 1, s \geq 0\} \\ &\subset \mathcal{A}_k \times \mathbb{R}^{\geq 0} \times L_k^2(W^\dagger; S^+)\end{aligned}$$

Due to the lack of reducible solutions we can also describe the elements of $M(W^\dagger, \mathfrak{s}_\omega)$ as gauge equivalence classes $[A, s\phi]$. We will use both descriptions depending on which is more convenient at a given moment.

A tangent vector to $\mathcal{C}_k^\sigma(W^\dagger, \mathfrak{s}_\omega)$ at a configuration $\gamma = (A, s, \phi)$ can be written as [32, p. 136]

$$\mathcal{T}^\sigma_{k,(A,s,\phi)} = \{(a, t, \psi) \in L_k^2(W^\dagger; iT^*W^\dagger) \oplus \mathbb{R} \oplus L_k^2(W^\dagger; S^+) \mid \operatorname{Re}\langle \psi, \phi \rangle_{W^\dagger} = 0\}$$

If $\langle \phi \rangle^\perp$ is the real orthogonal complement of ϕ, the linearization of the unperturbed Seiberg Witten map [32, p. 114]

$$\mathfrak{F}^\sigma(A, s, \phi) = \left(\frac{1}{2}\rho_W(F^+_{A^t}) - s^2(\phi\phi^*)_0, D_A\phi\right)$$

is

$$\begin{aligned}\mathcal{D}_{(A,s,\phi)}\mathfrak{F}^\sigma &: L_k^2(W^\dagger; iT^*W^\dagger) \oplus \mathbb{R} \oplus \langle \phi \rangle^\perp \to L_{k-1}^2(W^\dagger; i\mathfrak{su}(S^+) \oplus S^-) \\ (a, t, \psi) &\to \big(\rho(d^+a) - 2ts(\phi\phi^*)_0 - s^2(\phi\psi^* + \psi\phi^*)_0, D_A\psi + \rho(a)\phi\big)\end{aligned}$$

Remark 20. See page 467 of [32]. To compute the linearization we compute the difference $\mathfrak{F}^\sigma(A+at', s+t't, \phi+t'\psi)-\mathfrak{F}^\sigma(A,s,\phi)$. Since the corresponding connection on the determinant line bundle is $(A+t'a)^t = A^t + 2t'a$ the formula on book has an extra factor of $\frac{1}{2}$ in the first term which actually gets cancelled.

The derivative of the perturbation can be regarded as an operator

$$\mathcal{D}_{(A,s,\phi)}\hat{\mathfrak{p}}_W : L^2_k(W^\dagger; iT^*W^\dagger) \times \mathbb{R} \times L^2_k(W^\dagger; S^+) \to L^2_{k-1}(W^\dagger; i\mathfrak{su}(S^+) \oplus S^-)$$

with formal adjoint

$$\left(\mathcal{D}_{(A,s,\phi)}\hat{\mathfrak{p}}_W\right)^* = (\mathfrak{r}_1, \mathfrak{r}_2, \mathfrak{r}_3) : L^2_k(W^\dagger; i\mathfrak{su}(S^+) \oplus S^-) \to L^2_{k-1}(W^\dagger; iT^*W^\dagger) \times \mathbb{R} \times L^2_{k-1}(W^\dagger; S^+)$$

Instead of working with the equivalence classes, one can work directly at the level of the configuration space if one chooses a particular slice, the so called Coulomb Neumann gauge. Namely, we can define the subspace $\mathcal{K}^\sigma_{k,(A,s,\phi)} \subset \mathcal{T}^\sigma_{k,(A,s,\phi)}$ consisting of triples (a,t,ψ) satisfying [32, eq 9.11]

$$\begin{cases} \langle a, n\rangle = 0 & \text{at } -Y \cup Y' \\ -d^*a + is^2 \mathrm{Re}\,\langle i\phi, \psi\rangle = 0 & \\ \mathrm{Re}\left[\langle i\phi, \psi\rangle_{L^2(W^\dagger)}\right] = 0 & \end{cases}$$

where n is the unit outward normal vector and the definition of the derivative of the gauge group action [32, eq 9.10]

$$\begin{aligned} \mathbf{d}^\sigma_{(A,s,\phi)} : T_e\mathcal{G}_{k+1}(W^\dagger) &\to T_{(A,s,\phi)}\mathcal{C}^\sigma_k(W^\dagger, \mathfrak{s}_\omega) \\ \xi &\to (-d\xi, 0, \xi\phi) \end{aligned}$$

Then one defines [32, formula 24.8]

$$\begin{aligned} \mathbf{d}^{\sigma,\dagger}_{(A,s,\phi)} : \mathcal{T}^\sigma_{k,(A,s,\phi)} &\to L^2_{k-1}(W^\dagger; i\mathbb{R}) \\ (a,t,\psi) &\to -d^*a + is^2\mathrm{Re}\,\langle i\phi,\psi\rangle + i|\phi|^2 \mathrm{Re}\mu_{W^\dagger}\left(\langle i\phi,\psi\rangle\right) \end{aligned}$$

where $\mu_{W^\dagger}$ is the average value of $\langle i\phi,\psi\rangle$, i.e, $\mu_{W^\dagger} = \frac{\int_W \langle i\phi,\psi\rangle}{\mathrm{vol}(W)}$. Notice that the kernel of $\mathbf{d}^{\sigma,\dagger}_{(A,s,\phi)}$ captures the last two conditions defining $\mathcal{K}^\sigma_{k,(A,s,\phi)}$. In order to work with unconstrained sections of the spinor bundle [32, p.208] define

$$\tilde{\psi} = \psi + t\phi$$

The operator

$$Q^{\sigma}_{(A,s,\phi)} = \mathcal{D}_{(A,s,\phi)}\mathfrak{F}^{\sigma}_{\mathfrak{p}} \oplus \mathbf{d}^{\sigma,\dagger}_{(A,s,\phi)}$$

has domain which we identify with $L^2_k(W^\dagger; iT^*W^\dagger \oplus S^+)$. The book then shows the surjectivity of $Q^{\sigma}_{(A,s,\phi)}$ [32, p. 467], thus showing that $M(W^\dagger, \mathfrak{s}_\omega)$ is a manifold.

Moreover, if we use the restriction maps $R^-_W : M(W^\dagger, \mathfrak{s}_\omega) \to \mathcal{B}^{\sigma}_{k-1/2}(-Y, \mathfrak{s}_\xi)$ and $R^+_W : M(W^\dagger, \mathfrak{s}_\omega) \to \mathcal{B}^{\sigma}_{k-1/2}(Y', \mathfrak{s}_{\xi'})$ we can identify the tangent space to $\mathcal{B}^{\sigma}_{k-1/2}(-Y, \mathfrak{s}_\xi)$ and $\mathcal{B}^{\sigma}_{k-1/2}(Y', \mathfrak{s}_{\xi'})$ at $[\mathfrak{b}] = R^-_W[A, s, \phi]$, $[\mathfrak{b}'] = R^+_W[A, s, \phi]$ with the linear spaces $\mathcal{K}^{\sigma}_{k-1/2,\mathfrak{b}}$, $\mathcal{K}^{\sigma}_{k-1/2,\mathfrak{b}'}$. Therefore, we obtain derivatives [32, p.470]

$$\mathcal{D}R^-_W : T_{[A,s,\phi]}M(W^\dagger, \mathfrak{s}_\omega) \to \mathcal{K}^{\sigma}_{k-1/2,\mathfrak{b}}(-Y, \mathfrak{s}_\xi)$$
$$\mathcal{D}R^+_W : T_{[A,s,\phi]}M(W^\dagger, \mathfrak{s}_\omega) \to \mathcal{K}^{\sigma}_{k-1/2,\mathfrak{b}'}(Y', \mathfrak{s}_{\xi'})$$

This review covers the basics that we need to know for now about the moduli space on the cobordism $W^\dagger$. Now we proceed to analyze the equations on the cylindrical end.

4.2 The moduli space on the cylindrical end $M(\mathbb{R}^+ \times -Y, \mathfrak{s}; [\mathfrak{c}])$: Now we analyze the moduli space $M(\mathbb{R}^+ \times -Y, \mathfrak{s}, [\mathfrak{c}])$ following section 14 in [32] Again, they analyze the moduli space $M([\mathfrak{a}], [\mathfrak{b}])$ of trajectories on the cylinders asymptotic to the critical points $[\mathfrak{a}], [\mathfrak{b}]$ on each end but the analysis works just as well in this case.

At a configuration $\gamma = (A, s(t), \phi(t)) \in \mathcal{C}^{\tau}_{k,loc}(\mathbb{R}^+ \times -Y, \mathfrak{s}_\xi)$ the tangent space is [32, Eq 14.5].

$$\mathcal{T}^{\tau}_{k,(A,s,\phi)} = \{(a, \tilde{t}, \psi) \mid \mathrm{Re}\,\langle \phi\,|_t, \psi\,|_t\rangle_{L^2(-Y)} = 0 \text{ for all } t\}$$
$$\subset L^2_k(\mathbb{R}^+ \times -Y, iT^*(\mathbb{R}^+ \times -Y)) \oplus L^2_k(\mathbb{R}; \mathbb{R}) \oplus L^2_{k,A}(\mathbb{R}^+ \times -Y; S^+)$$

The derivative of the action of the gauge group is then [32, p. 248]

$$\mathbf{d}^{\tau}_{(A,s,\phi)}(\xi) = (-d\xi, 0, \xi\phi)$$

In this case the gauge group slice at (A, s, ϕ) consists of triples $(A+a, t, \psi)$ satisfying [32, Eq 14.6]

$$-d^* a + ist\mathrm{Re}\,\langle i\phi, \psi\rangle + i|\phi|^2\mathrm{Re}\mu_{-Y}\left(\langle i\phi, \psi\rangle\right) = 0$$

where μ_{-Y} is the averaging function on $-Y$. We denote by $\mathcal{S}^\tau_{k,(A,s,\phi)} \subset \mathcal{C}^\tau_k(\mathbb{R}^+ \times -Y, \mathfrak{s}, \mathfrak{c})$ the subset of triples satisfying this condition and

$$\mathrm{Coul}^\tau_{(A,s,\phi)} : \mathcal{C}^\tau_k(\mathbb{R}^+ \times -Y, \mathfrak{s}, \mathfrak{c}) \to L^2_{k-1}(\mathbb{R}^+ \times -Y, i\mathbb{R})$$

is the map defined by the left hand side of this equation. We linearize the Coulomb map $\mathrm{Coul}^\tau_{(A,s,\phi)}$ to obtain an operator [32, p. 248]

$$\begin{aligned} \mathbf{d}^{\tau,\dagger}_{(A,s,\phi)} &: \mathcal{T}^\tau_k \to L^2_{k-1}(\mathbb{R}^+ \times -Y, i\mathbb{R}) \\ (a, t, \psi) &\to -d^* a + is^2\mathrm{Re}\,\langle i\phi, \psi\rangle + i|\phi|^2\mathrm{Re}\mu_{-Y}\,\langle i\phi, \psi\rangle \end{aligned}$$

and let $\mathcal{K}^\tau_{k,(A,s,\phi)} \subset \mathcal{T}^\tau_{k,(A,s,\phi)}$ be the kernel of $\mathbf{d}^{\tau,\dagger}_{(A,s,\phi)}$. The Seiberg-Witten map is in this case [32, Eq 9.19]

$$\mathfrak{F}^\tau(A, s, \phi) = \left(\frac{1}{2}\rho(F^+_{A^t}) - s^2(\phi\phi^*)_0, \frac{d}{dt}s + \mathrm{Re}\,\langle D_A\phi, \phi\rangle_{L^2(-Y)}\, s, D_A\phi - \mathrm{Re}\,\langle D_A\phi, \phi\rangle_{L^2(-Y)}\,\phi\right)$$

Because $\mathcal{V}^\tau_{k-1}$ is not a trivial vector bundle, the definition of the derivative of $\mathfrak{F}^\tau_\mathfrak{q}$ as a bundle map

$$\mathcal{D}\mathfrak{F}^\tau_\mathfrak{q} : \mathcal{T}\,{}^\tau_k \to \mathcal{V}^\tau_{k-1}$$

requires a projection. Let [32, p. 252]

$$\begin{aligned} \Pi^\tau_{(A,s,\phi)} : L^2_k(\mathbb{R}^+ \times -Y, i\mathfrak{su}(S^+)) \oplus L^2_k(\mathbb{R};\mathbb{R}) \oplus L^2_{k,A}(\mathbb{R}^+ \times -Y, S^-) &\to \mathcal{V}^\tau_{k,(A,s,\phi)} \\ (a, r, \psi) &\to (a, r, \Pi^\perp_{\phi(t)}\psi) \end{aligned}$$

$$\Pi^\perp_{\phi(t)}\psi = \psi - \mathrm{Re}\,\left\langle \check{\phi}(t), \psi(t)\right\rangle_{L^2(-Y)}\phi$$

be defined by applying the L^2 projection on each slice $\{t\} \times -Y$. The derivative is then defined as the derivative in the ambient space, followed by the projection. The configuration (A, s, ϕ) defines a path $\check{\gamma}(t) = (B(t), r(t), \phi(t))$. We decompose the elements (a, r, ψ) on the domain of $\mathcal{D}\mathfrak{F}^\tau_\mathfrak{q}$ as $a = b + cdt$ where b is in temporal gauge and c is a 0 form. In this way the domain $\mathcal{T}^{\tau}_{(A,s,\phi)}$ can be written as sections along (A, s, ϕ) of the bundle $\mathcal{T}^{\sigma}(-Y) \oplus L^2(-Y, i\mathbb{R})$. Write the section as (V, c), where $V(t) = (b(t), r(t), \psi(t))$ defines an element of $\mathcal{T}^{\sigma}_{\check{\gamma}(t)}(-Y)$ and $c(t)$ is in $L^2(-Y, i\mathbb{R})$. Set

$$\frac{D}{Dt}V = \left(\frac{db}{dt}, \frac{dr}{dt}, \Pi^{\perp}_{\phi(t)}\frac{d\psi}{dt}\right)$$

Under this notation, $\mathcal{D}\mathfrak{F}^{\tau}_{\mathfrak{q}}$ is given by [32, p. 254]

$$(V, c) \to \frac{D}{Dt}V + \mathcal{D}(\text{grad}\mathcal{L})^{\sigma}(V) + \mathbf{d}^{\sigma}_{\check{\gamma}(t)}c$$

As in theorem 14.4.2 of [32] one can show that $M(\mathbb{R}^+ \times -Y, \mathfrak{s}, [\mathfrak{c}])$ is a Hilbert manifold by studying the surjectivity of the operator (Theorem 14.4.2 , Proposition 14.4.3 and Lemma 14.5.4 in [32])

$$Q^{\tau}_{(A,s,\phi)} = \mathcal{D}_{(A,s,\phi)}(\mathfrak{F}^{\tau}_{\mathfrak{q}}) \oplus \mathbf{d}^{\tau,\dagger}_{(A,s,\phi)}$$

Finally, we also have the restriction map

$$R_{\tau} : M(\mathbb{R}^+ \times -Y, \mathfrak{s}_{\xi}, [\mathfrak{c}]) \to \mathcal{B}^{\sigma}_{k-1/2}(Y, \mathfrak{s}_{\xi})$$

with derivative

$$\mathcal{D}R^{\tau}_{[\gamma_{\mathbb{R}^+ \times -Y}]} : T_{[\gamma_{\mathbb{R}^+ \times -Y}]}M(\mathbb{R}^+ \times -Y, \mathfrak{s}_{\xi}, [\mathfrak{a}]) \to \mathcal{K}^{\sigma}_{k-1/2,\mathfrak{b}}(Y, \mathfrak{s}_{\xi})$$

4.3 The moduli space on the conical end $\mathcal{M}([1,\infty) \times Y', \mathfrak{s}')$: We want to regard $\mathcal{M}([1,\infty) \times Y', \mathfrak{s}')$ as a Hilbert submanifold of a Hilbert manifold $\mathcal{B}_k([1,\infty) \times Y', \mathfrak{s}')$. Denote for simplicity $K_{Y'} = [1,\infty) \times Y'$ and define

$$\mathcal{C}_k(K_{Y'}, \mathfrak{s}) = \{(A, \Phi) \mid A - A_0 \in L^2_k(iT^*K_{Y'}) \ , \ \Phi - \Phi_0 \in L^2_{k,A_0}(S^+)\}$$

We take the gauge group to be

$$\mathcal{G}_{k+1}(K_{Y'}) = \{u : K_{Y'} \to \mathbb{C} \mid |u| = 1, \ \ u \in L^2_{k+1,loc}(K_{Y'}) \ \ , 1 - u \in L^2_{k+1}(K_{Y'})\}$$

Clearly $\mathcal{C}_k(K_{Y'}, \mathfrak{s}')$ will be a Hilbert manifold because of the L^2_k asymptotic conditions. It is also easy to see that $\mathcal{G}_{k+1}(K_{Y'})$ will be a Hilbert Lie group. Therefore, to show that

$$\mathcal{B}_k(K_{Y'}, \mathfrak{s}') = \mathcal{C}_k(K_{Y'}, \mathfrak{s}')/\mathcal{G}_{k+1}(K_{Y'})$$

we can use Lemma 9.3.2 in [32] which we quote for convenience:

Lemma 21. *Suppose we have a Hilbert Lie group G acting smoothly and freely on a Hilbert manifold C with Hausdorff quotient. Suppose that at each $c \in C$, the map $d_0 : T_eG \to T_cC$ (obtained from the derivative of the action) has closed range. Then the quotient C/G is also a Hilbert manifold.*

The Hilbert manifold structure is given as follows. If $S \subset C$ is any locally closed submanifold containing c, satisfying

$$T_cC = \operatorname{im}(d_0) \oplus T_cS$$

then the restriction of the quotient map $S \to C/G$ is a diffeomorphism from a neighborhood of c in S to a neighborhood of Gc in C/G. Therefore, we need to verify first that $\mathcal{B}_k(K_{Y'}, \mathfrak{s})$ is a Hausdorff space which is the content of the next lemma.

Lemma 22. *The quotient space $\mathcal{B}_k(K_{Y'}, \mathfrak{s}')$ is Hausdorff.*

Proof. We follow the proof of Proposition 9.3.1 in [32]. Namely, suppose that $\gamma_n = (A_n, \Phi_n)$ and $\tilde{\gamma}_n = (\tilde{A}_n, \tilde{\Phi}_n)$ are two sequences converging to $\gamma = (A_\infty, \Phi_\infty)$ and $\tilde{\gamma} = (\tilde{A}_\infty, \tilde{\Phi}_\infty)$. Suppose that $u_n \in \mathcal{G}_{k+1}(K_{Y'})$ is a sequence of gauge transformations such that $u_n \cdot \gamma_n = \tilde{\gamma}_n$. For each compact subset C of $K_{Y'}$, we can use the proof of Proposition 9.3.1 in [32] to conclude that there is a gauge transformation u_C^∞ such that $u_C^\infty \cdot (A_\infty, \Phi_\infty) \mid_C = (\tilde{A}_\infty, \tilde{\Phi}_\infty) \mid_C$, that is

$$\begin{cases} A_\infty - (u_C^\infty)^{-1} du_C^\infty = \tilde{A}_\infty \\ u_C^\infty \Phi_\infty = \tilde{\Phi}_\infty \end{cases} \tag{24}$$

Notice that because of the condition $\Phi_\infty - \Phi_0 \in L^2_{k,A_0}(S^+)$ there must exist a compact set C such that $\|\Phi_\infty\|_{L^2_{k,A_0,C}} \neq 0$, where the subscript C means that we are taking the norm restricted to the compact subset C . For k large enough we can use the Sobolev embedding $L^2_{k+1} \hookrightarrow C^0$ on C (Theorem 1.2.15 [42]) to assume that our section Φ_∞ is continuous on C. In particular, there must exist a point $x_0 \in C$ for which $\Phi_\infty(x_0) \neq 0$. If $C' \supset C$ is another compact subset containing C then we obtain similar relations to (24). In particular, we must have $u_C^\infty(x) = u_{C'}^\infty(x) \quad \forall x \in C$. To see why this is true notice than when restricted to C

$$\begin{aligned} & d((u_C^\infty)^{-1}u_{C'}^\infty) \\ &= -(u_C^\infty)^{-2}(du_C^\infty)u_{C'}^\infty + (u_C^\infty)^{-1}du_{C'}^\infty \\ &= (u_C^\infty)^{-1}u_{C'}^\infty\left[-(u_C^\infty)^{-1}du_C^\infty + (u_{C'}^\infty)^{-1}du_{C'}^\infty\right] \\ &= (u_C^\infty)^{-1}u_{C'}^\infty\left[A - \tilde{A} + (\tilde{A} - A)\right] \\ &= 0 \end{aligned}$$

Since the cone $K_{Y'}$ is connected we conclude that $(u_C^\infty)^{-1}u_{C'}^\infty = c$ for some constant $c \in S^1$. Since both C, C' contain x_0 evaluating the spinor at this point we see that

$$u_C^\infty(x_0)\Phi_\infty(x_0) = \tilde{\Phi}_\infty(x_0) = u_{C'}^\infty(x_0)\Phi_\infty(x_0) = cu_C^\infty(x_0)\Phi_\infty(x_0)$$

from which we can see that $c = 1$.

Therefore it makes sense to define $u^\infty : K_{Y'} \to S^1$ without any reference to a compact subset. It is also clear that $u^\infty \cdot (A_\infty, \Phi_\infty) = (\tilde{A}, \tilde{\Phi})$ and that $u^\infty \in L^2_{k+1,loc}(K_{Y'})$. To show that $1 - u^\infty \in L^2_{k+1}(K_{Y'})$ we can now apply condition $ii)$ in Lemma (17). □

As is usually the case for Seiberg Witten or Yang Mills moduli spaces, we do not want any random slice to the gauge group action. Rather, we want to use the so-called Coulomb-Neumann slice [32, Section 9.3]. A tangent vector to $\mathcal{C}_k(K_{Y'}, \mathfrak{s}')$ at $\gamma = (A, \Phi)$ can be written as

$$(a, \Psi) \in L^2_k(K_{Y'}, iT^*K_{Y'}) \oplus L^2_{k,A_0}(K_{Y'}, S^+)$$

and the derivative of the gauge group action is

$$\begin{aligned} \mathbf{d}_\gamma : L^2_{k+1}(K_{Y'}; i\mathbb{R}) \to \mathcal{T}_k = L^2_k(K_{Y'}, iT^*K_{Y'}) \oplus L^2_{k,A_0}(K_{Y'}, S^+) \\ \mathbf{d}_{(A,\Phi)}\ (\zeta) = (-d\zeta, \zeta\Phi) \end{aligned} \tag{25}$$

We use the inner product

$$\langle (a_1, \Psi_1), (a_2, \Psi_2) \rangle_{L^2} = \int \langle a_1, a_2 \rangle + \mathrm{Re}\,\langle \Psi_1, \Psi_2 \rangle$$

to define the formal adjoint of $\mathbf{d}_{(A,\Phi)}$ and it is given by [32, Lemma 9.3.3]

$$\mathbf{d}^*_{(A,\Phi)}(a,\Psi) = -d^*a + i\mathrm{Re}\,\langle i\Phi, \Psi\rangle \tag{26}$$

To use Lemma (21) we just need to show that $\mathbf{d}_\gamma$ has closed range. In order to this we will rely on Theorem 3.3 in [31] and Proposition 4.1 in [65].

First we need to define another map which will be used soon to show that $\mathcal{M}([1,\infty)\times Y', \mathfrak{s}')$ is a Hilbert manifold. The linearization of the unperturbed Seiberg-Witten map is [31, Eq. 8]

$$\begin{aligned}\mathcal{D}_{(A,\Phi)}\mathfrak{F} : L^2_k(K_{Y'}, iT^*K_{Y'}) \oplus L^2_{k,A_0}(K_{Y'}, S^+) &\to L^2_{k-1}(K_{Y'}, i\mathfrak{su}(S^+)) \oplus L^2_{k-1,A_0}(K_{Y'}, S^-)\\ (a,\Psi) &\to \big(\rho(d^+a) - \{\Phi\Psi^* + \Psi\Phi^*\}_0, D_A\Psi + \rho(a)\Phi\big)\end{aligned}$$

where

$$\{\Phi\Psi^* + \Psi\Phi^*\}_0 = \Phi\Psi^* + \Psi\Phi^* - \frac{1}{2}\langle\Phi,\Psi\rangle - \frac{1}{2}\langle\Psi,\Phi\rangle = \Phi\Psi^* + \Psi\Phi^* - \mathrm{Re}\,\langle\Phi,\Psi\rangle$$

Define as before (in [65, 40, 31] this is the operator $\mathcal{D}$)

$$\begin{aligned} Q_{(A,\Phi)} &= \mathcal{D}_{(A,\Phi)}\mathfrak{F} \oplus \mathbf{d}^*_{(A,\Phi)}\\ (a,\Psi) &\to (\rho(d^+a) - \{\Phi\Psi^* + \Psi\Phi^*\}_0, D_A\Psi + \rho(a)\Phi, -d^*a + i\mathrm{Re}\,\langle i\Phi,\Psi\rangle)\end{aligned} \tag{27}$$

We also want a formula for the formal adjoint: $Q^*_{(A,\Phi)}$: this is essentially eq. 24.10 in [32]. Modulo notational differences, for we obtain

$$Q^*_{(A,\Phi)}(\eta,\psi,\vartheta) = \big((d^+)^*\rho^*\eta + \rho^*(\psi\Phi^*) - d\vartheta, D^*_A\psi - \eta\Phi + \vartheta\Phi\big) \tag{28}$$

In particular, taking $\eta = 0$ and $\psi = 0$ one obtains sees that:

$$Q^*_{(A,\Phi)}(0,0,\vartheta) = (-d\vartheta, \vartheta\Phi) = \mathbf{d}_{(A,\Phi)}(\vartheta) \tag{29}$$

Now we are finally ready to finish showing that $\mathcal{B}_k(K_{Y'}, \mathfrak{s})$ is Hausdorff.

Lemma 23. *Define at a configuration $\gamma = (A, \Phi)$ the subspaces*

$$\mathcal{K}_{k,\gamma} = \{(a, \Psi) \mid \mathbf{d}^*_{(A,\Phi)}(a, \Psi) = 0, \quad \left\langle a \mid_{\partial K_{Y'}}, n \right\rangle = 0 \quad at\ \partial K_{Y'}\}$$

$$\mathcal{J}_{k,\gamma} = im\ \mathbf{d}_\gamma$$

As γ varies over $\mathcal{C}_k(K_{Y'}, \mathfrak{s})$, the subspaces $\mathcal{J}_{k,\gamma}$ and $\mathcal{K}_{k,\gamma}$ define complementary closed subbundles of $\mathcal{T}_{k,\gamma}$ and we have a smooth decomposition

$$T\mathcal{C}_k(K_{Y'}, \mathfrak{s}) = \mathcal{J}_k \oplus \mathcal{K}_k$$

Proof. First of all, Theorem 3.3 in [31] shows that the operator $Q^*_{(A,\Phi)}$ has closed range whenever it is defined on a manifold without boundary which has a conical end except on a compact subset. In particular, equation 29 says that $\mathbf{d}_{(A,\Phi)}$ has closed range. Likewise, we know that on a compact manifold with boundary the operator $\mathbf{d}_{(A,\Phi)}$ has closed range as well (this is implicit in the proof of Proposition 9.3.4 in [32]). Observe that we are working on a manifold with boundary which has a conical end so the closed range property follows from a patching argument from the previous two situations.

In order to show the smooth decomposition $T\mathcal{C}_k(K_{Y'}, \mathfrak{s}) = \mathcal{J}_k \oplus \mathcal{K}_k$ we can follow again the proof of Proposition 9.3.4 in [32] and reduce this to the invertibility of the "Laplacian"

$$\vartheta \to \triangle \vartheta + |\Phi|^2 \vartheta \tag{30}$$

This property can be proved using a parametrix argument (which is essentially the same as Lemma 26 and Theorem 40 in this paper): choosing a compact subset large enough for which $|\Phi|^2$ is not identically zero, one knows from Proposition 9.3.4 in [32] that the operator (30) is invertible. On the other hand, Lemma 2.3.2 in [40] says that on any four manifold with conical end (like the manifold X^+ we just used), the operator 30 is invertible. Notice that their lemma requires a solution to the Seiberg Witten equations but this is only because this section was trying to find uniform bounds (independent of the solution used). At this stage this is not our concern so the proof they give near the end of that section can be adapted to any configuration. Therefore, we can splice these two inverses to get an approximate inverse to 30 on our domain of interest $K_{Y'}$. By choosing appropriate cutoff functions one can then guarantee that 30 will be invertible (again, the proof of Theorem 40 provides more details). □

Continuing with our analysis of our moduli space $\mathcal{M}(K_{Y'}, \mathfrak{s}')$, to show that it is a Hilbert submanifold of $\mathcal{B}_k(K_{Y'}, \mathfrak{s}')$ we seek for an analogue of proposition 24.3.1 in [32]. The main point in that proof was to show that the operator $Q_{(A,\Phi)}$ introduced before in (29) is surjective. To show surjectivity, the idea in the book was to apply Corollary 17.1.5 in [32]. We will not use directly the corollary but rather its proof.

Namely, using the same argument as in the proof of the previous lemma, we can see that $Q_{(A,\Phi)}$ has closed range. Therefore we just need to show that $Q^*_{(A,\Phi)}$ has the property that every non-zero solution of $Q^*_{(A,\Phi)} v = 0$ for $v = (\eta, \psi, \vartheta)$ has non-zero restriction to the boundary $\partial K_{Y'}$.

Using the equation (29) for the adjoint $Q^*_{(A,\Phi)}$, we can see that the equation $Q^*_{(A,\Phi)}(\eta, \psi, \vartheta) = 0$ becomes in the coordinates (a, Ψ) of $Q_{(A,\Phi)}$ (compare with eq 24.10 in [32])

$$\begin{aligned} (d^+)^* \rho^* \eta + \rho^*(\psi \Phi^*) - d\vartheta &= 0 \\ D_A^* \psi - \eta \Phi + \vartheta \Phi &= 0 \end{aligned} \tag{31}$$

As in eq. (24.15) of [32], the equations in the last form have the shape

$$\frac{d}{dt} v + (L_0 + h(t)) v = 0$$

where L_0 is a self-adjoint elliptic operator on Y' and h is a time dependent operator on Y' satisfying the conditions of the unique continuation lemma. Since v vanishes on the boundary, it vanishes on the collar too and therefore on the cone $K_{Y'}$. Therefore the moduli space $\mathcal{M}([1,\infty) \times Y', \mathfrak{s}')$ is a Hilbert sub-manifold of $\mathcal{B}_k(K_{Y'}, \mathfrak{s}')$. Moreover, as in the other cases we have a restriction map

$$R_K : \mathcal{M}([1,\infty) \times Y', \mathfrak{s}') \to \mathcal{B}^\sigma_{k-1/2}(-Y', \mathfrak{s}_{\xi'})$$

4.4 Gluing the Moduli Spaces. Now that we know that each moduli space appearing in the fiber product description is a Hilbert manifold, we need to show that their fiber product is a finite dimensional manifold, possibly with components of different dimensions. As mentioned before, we have the restrictions maps

$$\begin{aligned} R_\tau : \mathcal{M}(\mathbb{R}^+ \times -Y, \mathfrak{s}_\xi, [\mathfrak{c}]) &\to \mathcal{B}^\sigma_{k-1/2}(Y, \mathfrak{s}_\xi) \\ R^-_W : \mathcal{M}(W^\dagger, \mathfrak{s}_\omega) &\to \mathcal{B}^\sigma_{k-1/2}(-Y, \mathfrak{s}_\xi) \\ R^+_W : \mathcal{M}(W^\dagger, \mathfrak{s}_\omega) &\to \mathcal{B}^\sigma_{k-1/2}(Y', \mathfrak{s}_{\xi'}) \\ R_K : \mathcal{M}([1,\infty) \times Y', \mathfrak{s}') &\to \mathcal{B}^\sigma_{k-1/2}(-Y', \mathfrak{s}_\xi) \end{aligned}$$

If we write as before an element $[\gamma] \in \mathcal{M}(W^+_{\xi',Y}, \mathfrak{s}_\omega, [\mathfrak{c}])$ as

$$\mathrm{Fib}(R_\tau, R^-_W, R^+_W, R_K) \ni [\gamma] = ([\gamma_{\mathbb{R}^+\times -Y}], [\gamma_W], [\gamma_{[1,\infty)\times Y'}])$$

and define

$$\begin{cases} \mathcal{B}^\sigma_{k-1/2}(-Y, \mathfrak{s}_\xi) \ni \mathfrak{b} = R^-_W(\gamma_W) \\ \mathcal{B}^\sigma_{k-1/2}(Y', \mathfrak{s}_{\xi'}) \ni \mathfrak{b}' = R^+_W(\gamma_W) \end{cases}$$

then the derivatives of our restriction maps can be written as

$$\begin{aligned} \mathcal{D}R^\tau_{[\gamma_{\mathbb{R}^+\times -Y}]} : T_{[\gamma_{\mathbb{R}^+\times -Y}]}\mathcal{M}(\mathbb{R}^+ \times -Y, \mathfrak{s}_\xi, [\mathfrak{c}]) &\to \mathcal{K}^\sigma_{k-1/2,\mathfrak{b}}(Y, \mathfrak{s}_\xi) \\ \mathcal{D}R^-_{W,[\gamma_W]} : T_{[\gamma_W]}\mathcal{M}(W^\dagger, \mathfrak{s}_\omega) &\to \mathcal{K}^\sigma_{k-1/2,\mathfrak{b}}(-Y, \mathfrak{s}_\xi) \\ \mathcal{D}R^+_{W,[\gamma_W]} : T_{[\gamma_W]}\mathcal{M}(W^\dagger, \mathfrak{s}_\omega) &\to \mathcal{K}^\sigma_{k-1/2,\mathfrak{b}'}(Y', \mathfrak{s}_{\xi'}) \\ \mathcal{D}R_{K,[\gamma_{[1,\infty)\times Y'}]} : T\mathcal{M}_{[\gamma_{[1,\infty)\times Y'}]}([1,\infty) \times Y', \mathfrak{s}) &\to \mathcal{K}^\sigma_{k-1/2,\mathfrak{b}'}(-Y', \mathfrak{s}_{\xi'}) \end{aligned}$$

where the right hand side is the corresponding Couloumb slice at each configuration $\mathfrak{b}, \mathfrak{b}'$. The next definition is the analogue of definition 24.4.2 in [32]:

Definition 24. Let $[\gamma] \in \mathcal{M}(W^+_{\xi',Y}, \mathfrak{s}_\omega, [\mathfrak{c}])$ and

$$\rho : \mathcal{M}(W^+_{\xi',Y}, \mathfrak{s}_\omega, [\mathfrak{c}]) \to \mathcal{M}(\mathbb{R}^+ \times -Y, \mathfrak{s}_\xi, [\mathfrak{c}]) \times \mathcal{M}(W^\dagger, \mathfrak{s}_\omega) \times \mathcal{M}([1,\infty) \times Y', \mathfrak{s}')$$

the restriction map. Write

$$\rho([\gamma]) = ([\gamma_1], [\gamma_2], [\gamma_3]) = \big([\gamma_{\mathbb{R}^+\times -Y}], [\gamma_W], [\gamma_{[1,\infty)\times Y'}]\big) \in \mathrm{Fib}(R_\tau, R^-_W, R^+_W, R_K)$$

and

$$[\mathfrak{b}] = R_\tau([\gamma_{\mathbb{R}^+\times -Y}]) = R_W^-([\gamma_W]) \in \mathcal{B}^\sigma_{k-1/2}(-Y, \mathfrak{s}_\xi)$$
$$[\mathfrak{b}'] = R_W^+([\gamma_W]) = R_K([\gamma_{[1,\infty)\times Y'}]) \in \mathcal{B}^\sigma_{k-1/2}(Y', \mathfrak{s}_{\xi'})$$

We say that the moduli space $\mathcal{M}(W^+_{\xi',Y}, \mathfrak{s}_\omega, [\mathfrak{c}])$ is **regular** at $[\gamma]$ if the map

$$R = \left(\left(R_\tau, R_W^-\right), \left(R_W^+, R_K\right)\right) : \mathrm{Fib}(R_\tau, R_W^-, R_W^+, R_K) \to \mathcal{B}^\sigma_{k-1/2}(-Y, \mathfrak{s}_\xi) \times \mathcal{B}^\sigma_{k-1/2}(Y', \mathfrak{s}_{\xi'})$$

is transverse at $\rho[\gamma]$. That is, (R_τ, R_W^-) is transverse at $[\mathfrak{b}]$ while $\left(R_W^+, R_K\right)$ is transverse at $[\mathfrak{b}']$.

Following the strategy in section 24.4 of [32], to show regularity what we really need is an analogue of Lemma 24.4.1 (which is our next lemma). The other pieces used by [32] do not change so we can conclude the following transversality result (compare with Proposition 24.4.7 [32]):

Theorem 25. *Let $\mathfrak{q}_{-Y}, \mathfrak{q}_{Y'}$ be fixed perturbations for $-Y, Y'$ respectively such that for all critical points $[\mathfrak{a}], [\mathfrak{b}] \in \mathcal{B}^\sigma_{k-1/2}(-Y, \mathfrak{s}_\xi)$ and $[\mathfrak{a}'], [\mathfrak{b}'] \in \mathcal{B}^\sigma_{k-1/2}(Y', \mathfrak{s}_\xi)$, the moduli spaces $\mathcal{M}([\mathfrak{a}], \mathbb{R}\times -Y, \mathfrak{s}_\xi, [\mathfrak{b}])$ and $\mathcal{M}([\mathfrak{a}'], \mathbb{R}\times Y, \mathfrak{s}_{\xi'}, [\mathfrak{b}'])$ are cut out transversely. Then there is a residual subset $\mathcal{P}_0$ of the large space of perturbations $\mathcal{P}(-Y, \mathfrak{s}_\xi) \times \mathcal{P}(Y', \mathfrak{s}_{\xi'})$ defined in section 11.6 of [32] for which the following holds: if for any $(\mathfrak{p}_0, \mathfrak{p}'_0) \in \mathcal{P}_0 \subset \mathcal{P}(-Y, \mathfrak{s}_\xi) \times \mathcal{P}(Y', \mathfrak{s}_{\xi'})$ one forms perturbation*

$$\mathfrak{p}_{W^+_{\xi',Y}} = -\hat{\mathfrak{q}}_{Y,g_\theta,\mathfrak{s}_\xi} + \left(\beta\hat{\mathfrak{q}}_{Y,g_\theta,\mathfrak{s}_\xi} + \beta'_0\hat{\mathfrak{p}}_0\right) + \left(\beta'_0\hat{\mathfrak{p}}'_0 + \beta'\hat{\mathfrak{q}}_{Y',g_{\theta'},\mathfrak{s}_{\xi'}}\right) + \left(\beta_{N_K}\hat{\mathfrak{q}}_{Y',g_{\theta'},\mathfrak{s}_{\xi'}} + \beta_K\mathfrak{p}_K\right)$$

described in equation (11) , then the moduli space $\mathcal{M}(W^+_{\xi',Y}, \mathfrak{s}_\omega, [\mathfrak{c}], \mathfrak{p}_{W^+_{\xi',Y}})$ defined using the perturbation $\mathfrak{p}_{W^+_{\xi',Y}}$ is regular, in other words, we have transversality at $\rho[\gamma]$ for all $[\gamma] \in \mathcal{M}(W^+_{\xi',Y}, \mathfrak{s}_\omega, [\mathfrak{c}], \mathfrak{p}_{W^+_{\xi',Y}})$.

In particular, for any perturbation belonging to this residual set, the moduli space $\mathcal{M}(W^+_{\xi',Y}, \mathfrak{s}_\omega, [\mathfrak{c}], \mathfrak{p}_{W^+_{\xi',Y}})$ will be a manifold whose components have dimensions equal to $\mathrm{ind}\mathcal{D}_{\rho[\gamma]}R = \dim\ker\mathcal{D}_{\rho[\gamma]}R$.

Again, the proof of this theorem is a consequence of the following lemma:

Lemma 26. *Let $[\gamma] \in \mathcal{M}(W^+_{\xi',Y}, \mathfrak{s}_\omega, [\mathfrak{c}])$. Then the sum of the derivatives*

$$\begin{aligned}\mathcal{D}_{\rho[\gamma]}R = \big(\mathcal{D}_{[\gamma_1]}R_\tau + \mathcal{D}_{[\gamma_2]}R_W^-\big) \oplus \big(\mathcal{D}_{[\gamma_2]}R_W^+ + \mathcal{D}_{[\gamma_3]}R_K\big) :\\ T_{[\gamma_1]}\mathcal{M}(\mathbb{R}^+ \times -Y, \mathfrak{s}_\xi, [\mathfrak{c}]) \oplus T_{[\gamma_2]}\mathcal{M}(W^\dagger, \mathfrak{s}_\omega) \oplus T_{[\gamma_3]}\mathcal{M}([1,\infty) \times Y', \mathfrak{s})\\ \to \mathcal{K}^\sigma_{k-1/2,\mathfrak{b}}(-Y) \oplus \mathcal{K}_{k-1/2,\mathfrak{b}'}(Y')\end{aligned}$$

is a Fredholm map.

Proof. We will begin showing that the following maps are Fredholm and compact:
(32)

$\pi_{\mathfrak{b}} \circ \mathcal{D}_{[\gamma_1]}R_\tau$	is compact	(1)	$(1-\pi_{\mathfrak{b}}) \circ \mathcal{D}_{[\gamma_2]}R_W^-$	is compact	(5)
$(1-\pi_{\mathfrak{b}}) \circ \mathcal{D}_{[\gamma_1]}R_\tau$	is Fredholm	(2)	$\pi_{\mathfrak{b}} \circ \mathcal{D}_{[\gamma_2]}R_W^-$	is Fredholm	(6)
$(1-\pi_{\mathfrak{b}'}) \circ \mathcal{D}_{[\gamma_2]}R_W^+$	is compact	(3)	$\pi_{\mathfrak{b}'} \circ \mathcal{D}_{[\gamma_3]}R_K$	is compact	(7)
$\pi_{\mathfrak{b}'} \circ \mathcal{D}_{[\gamma_2]}R_W^+$	is Fredholm	(4)	$(1-\pi_{\mathfrak{b}'}) \circ \mathcal{D}_{[\gamma_3]}R_K$	is Fredholm	(8)

Here $\pi_{\mathfrak{b}}, \pi_{\mathfrak{b}'}$ are defined as follows [32, Sections 12.4, 17.3]. We have a Hessian operator $\text{Hess}^\sigma_{\mathfrak{q}} : \mathcal{K}^\sigma_k \to \mathcal{K}^\sigma_{k-1}$ obtained by projecting $\mathcal{D}(\text{grad}\mathcal{L})^\sigma$ onto the subspace $\mathcal{K}^\sigma_{k-1}$. The spectrum of $\text{Hess}^\sigma_{\mathfrak{q}}$ is real, discrete and with finite dimensional generalized eigenspaces. If the operator is hyperbolic (that is, zero is not an eigenvalue) we have a spectral decomposition

$$\mathcal{K}^\sigma_{k-1/2,\mathfrak{b}} = \mathcal{K}^+_{\mathfrak{b}} \oplus \mathcal{K}^-_{\mathfrak{b}}$$

where $\mathcal{K}^+_{\mathfrak{b}}$ is the closure of the span of the positive eigenspaces and $\mathcal{K}^-_{\mathfrak{b}}$ of the negative eigenspaces. In the non-hyperbolic case, we choose ϵ sufficiently small that there are no eigenvalues in $(0,\epsilon)$ and then define $\mathcal{K}^\pm_{k-1/2,\mathfrak{b}}$ using the spectral decomposition of the operator $\text{Hess}^\sigma_{\mathfrak{q},\mathfrak{b}} - \epsilon$. The effect is that the generalized 0 eigenspace belongs to $\mathcal{K}^-_{\mathfrak{b}}$.

Also, notice that the roles of the different operators are sometimes opposite because of the different orientations on the manifolds, namely

$$\begin{aligned}\mathcal{K}^-_{\mathfrak{b}}(-Y) &= \mathcal{K}^+_{\mathfrak{b}}(Y)\\ \mathcal{K}^-_{\mathfrak{b}'}(-Y') &= \mathcal{K}^+_{\mathfrak{b}'}(Y')\end{aligned}$$

•By Proposition 24.3.2 in [32], $(3),(4),(5),(6)$ are true (remember that in this section of the book the boundary is the compact four manifold is allowed to be disconnected. In our case the boundary is simply $-Y \cup Y'$).

•By the discussion in Lemma 24.4.1 in [32], (1) and (2) are true. So really the only thing left to verify are (7) and (8). To explain what we need to do we will chase through some theorems of [32] (and Proposition 2.18 , Lemma 3.17 in [34]).

•Assertions (7) and (8) are the "conical" versions of Proposition 24.3.2 in the book. The proof of this theorem in turn refers to Theorem 17.3.2, which at the same time requires Proposition 17.2.6, which depends at the same time on Proposition 17.2.5. The latter uses essentially Theorem 17.1.3 and the only part that is not proven explicitly is part a) , which depends on a parametrix argument (modeled on Proposition 14.2.1) of Theorem 17.1.4.

In a nutshell, we must do the following. Decompose $Q_{(A,\Phi)}$ as

$$\begin{aligned}Q_{(A,\Phi)} &= D_0 + K \\ D_0(a,\Psi) &= \big(\rho(d^+a), D_{A_0}\Psi, -d^*a\big) \\ K(a,\Psi) = (-\{\Phi\Psi^* + \Psi\Phi^*\}_0, \rho(A-A_0)\Psi &+ \rho(a)\Phi + i\mathrm{Re}\,\langle i\Phi,\Psi\rangle)\end{aligned}$$

On the collar of $\partial K_{Y'}$, D_0 can be written in the form

$$\frac{d}{dt} + L_0$$

where $L_0 : C^\infty(-Y';E_0) \to C^\infty(-Y';E_0)$ is a first order, self-adjoint elliptic differential operator. We will not write the exact formula for the domain and codomain since they would rather cumbersome. Rather we will denote the bundles involved by the letter E_0 when referring to the three manifolds and by E for the four manifolds just as the book does.

If H_0^+ and H_0^- are the closures in $L^2_{1/2}(Y;E_0)$ of the spans of the eigenvectors belonging to positive and non-positive eigenvalues of L_0 and

$$\Pi_0 : L^2_{1/2}(Y;E_0) \to L^2_{1/2}(Y;E_0)$$

is the projection with image H_0^- and kernel H_0^+, we need to show that the operator

$$Q_{(A,\Phi)} \oplus (\Pi_0 \circ r_{-Y'}) : L^2_k(K_{Y'}; E) \to L^2_{k-1}(K_{Y'}; E) \oplus (H_0^- \cap L^2_{k-1/2})$$

is Fredholm. First, for notational purposes take the collar neighborhood of $\partial K_{Y'}$ to be $(-5, 0] \times -Y'$, where $\partial K_{Y'}$ has now been identified with $\{0\} \times -Y'$. Also denote for simplicity

$$Q_{K_{Y'}} = Q_{(A,\Phi)} : L^2_k(K_{Y'}; E) \to L^2_{k-1}(K_{Y'}; E)$$

To see this we will give a parametrix argument, which is essentially the same as the one used in Proposition 14.2.1 of [32]. Namely, we modify the manifold $K_{Y'}$ in two different ways. For the first modification we close up $K_{Y'}$ first by extending the collar neighborhood a little bit (to the left in our figure) and then finding a four manifold X (dots on the left side of the figure) bounding Y'. For the second modification, we forget about the part of the cone $K_{Y'}$ which does not have a product structure, in other words, we take the collar neighborhood of $K_{Y'}$ and extend it into a half-infinite cylinder which extends indefinitely to the right in our figure (see next page). In particular, notice that we superimposed both modifications in our image to save some space but they do not interact with each other. Each modification provides a parametrix as follows.

Regarding the first modification, we can define the manifold $X^+ = X \cup \text{cylinder} \cup K_{Y'}$ and extend $Q_{K'_Y}$ to an operator

$$Q_{X^+} : L^2_k(X^+; E) \to L^2_{k-1}(X^+; E)$$

and by theorem Theorem 3.3 in [32] there is a parametrix (that is, $Q_{X^+}P_{X^+} - I$ and $P_{X^+}Q_{X^+} - I$ are compact operators) which we denote

$$P_{X^+} : L^2_{k-1}(X^+; E) \to L^2_k(X^+; E)$$

Similarly, we define the half-cylinder $Z = (-\infty, 0] \times -Y'$. By Theorem 17.1.4 in [32], the operator

$$Q_Z \oplus (\Pi_0 \circ r_{-Y'}) : L^2_k(Z; E) \to L^2_{k-1}(Z; E) \oplus (H_0^- \cap L^2_{k-1/2}(-Y'; E_0))$$

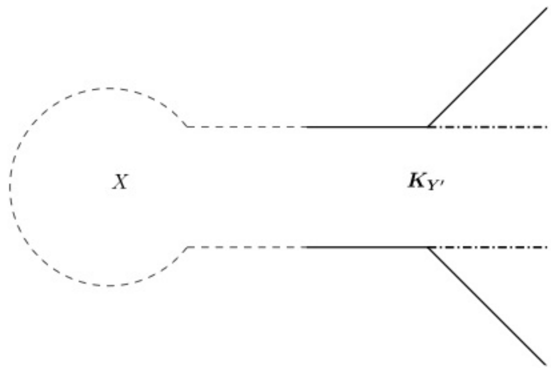

FIGURE 7. Closing up the cone $K_{Y'}$ into the manifold $X \cup K_{Y'}$. Simultaneously, we extend the product neighborhood $(-5, 0] \times -Y'$ of $K_{Y'}$ into a half-infinite cylinder $Z = (-\infty, 0] \times -Y'$.

has a parametrix

$$P_Z : L^2_{k-1}(Z; E) \oplus (H_0^- \cap L^2_{k-1/2}(-Y'; E_0)) \to L^2_k(Z; E)$$

Finally, to define the parametrix corresponding to $Q_{K_{Y'}} \oplus (\Pi_0 \circ r_{-Y'})$, let $1 = \eta_1 + \eta_2$ be a partition of unity subordinate to a covering of $K_{Y'}$ by the open sets $U_1 = K_{Y'} \backslash ([-2, 0] \times -Y')$ and $U_2 = (-3, 0] \times -Y'$. Let γ_1 be a function which is 1 on the support of η_1 and vanishes on $(-1, 0] \times -Y'$. Similarly, let γ_2 be 1 on the support of η_2 and vanishing outside $[-4, 0] \times Y'$. Define

$$\begin{aligned} P_{K_{Y'}} : L^2_{k-1}(K_{Y'}; E) \oplus (H_0^- \cap L^2_{k-1/2}) &\to L^2_k(K_{Y'}; E) \\ e &\to \gamma_1 P_{X^+}(\eta_1 e) + \gamma_2 P_Z(\eta_2 e) \end{aligned}$$

Notice that thanks to how the supports of the functions where chosen, the function is actually well defined. A similar computation to Proposition 14.2.1 in [32] shows that $P_{K_{Y'}}$ is a parametrix for $Q_{K_{Y'}} \oplus (\Pi_0 \circ r_{-Y'})$.

Returning back to the proof of the lemma, thanks to the eight identities (32) we can see that

$$\begin{aligned} &\mathcal{D}_{[\gamma_1]}R_\tau + \mathcal{D}_{[\gamma_2]}R_W^- \\ = \underbrace{(1-\pi_{\mathfrak{b}}) \circ \mathcal{D}_{[\gamma_1]}R_\tau}_{\text{Fredholm}} + \underbrace{\pi_{\mathfrak{b}} \circ \mathcal{D}_{[\gamma_2]}R_W^-}_{\text{Fredholm}} + &\underbrace{\pi_{\mathfrak{b}} \circ \mathcal{D}_{[\gamma_1]}R_\tau}_{\text{compact}} + \underbrace{(1-\pi_{\mathfrak{b}}) \circ \mathcal{D}_{[\gamma_2]}R_W^-}_{\text{compact}} \end{aligned}$$

Likewise,

$$\begin{aligned} &\mathcal{D}_{[\gamma_3]}R_K + \mathcal{D}_{[\gamma_2]}R_W^+ \\ = \underbrace{(1-\pi_{\mathfrak{b}'}) \circ \mathcal{D}_{[\gamma_3]}R_K}_{\text{Fredholm}} + \underbrace{\pi_{\mathfrak{b}'} \circ \mathcal{D}_{[\gamma_2]}R_W^+}_{\text{Fredholm}} + &\underbrace{\pi_{\mathfrak{b}'} \circ \mathcal{D}_{[\gamma_3]}R_K}_{\text{compact}} + \underbrace{(1-\pi_{\mathfrak{b}'}) \circ \mathcal{D}_{[\gamma_2]}R_W^+}_{\text{compact}} \end{aligned}$$

Therefore,

$$\left(\mathcal{D}_{[\gamma_1]}R_\tau + \mathcal{D}_{[\gamma_2]}R_W^-\right) \oplus \left(\mathcal{D}_{[\gamma_2]}R_W^+ + \mathcal{D}_{[\gamma_3]}R_K\right)$$

differs by the compact operator

$$\left(\pi_{\mathfrak{b}} \circ \mathcal{D}_{[\gamma_1]}R_\tau + (1-\pi_{\mathfrak{b}}) \circ \mathcal{D}_{[\gamma_2]}R_W^-\right) \oplus \left(\pi_{\mathfrak{b}'} \circ \mathcal{D}_{[\gamma_3]}R_K + (1-\pi_{\mathfrak{b}'}) \circ \mathcal{D}_{[\gamma_2]}R_W^+\right)$$

from the direct sum of the Fredholm operators

$$\left((1-\pi_{\mathfrak{b}}) \circ \mathcal{D}_{[\gamma_1]}R_\tau \oplus \pi_{\mathfrak{b}} \circ \mathcal{D}_{[\gamma_2]}R_W^-\right) \oplus \left((1-\pi_{\mathfrak{b}'}) \circ \mathcal{D}_{[\gamma_3]}R_K \oplus \pi_{\mathfrak{b}'} \circ \mathcal{D}_{[\gamma_2]}R_W^+\right)$$

and so the result follows. $\square$

5. Stretching the Neck

As promised when we explained our strategy for proving naturality, we will consider a parametrized moduli space following the ideas used in sections 4.9, 4.10, 6.3 of [30] and sections 24.6, 26.1 and 27.4 of [32]. Thanks to the computations done in sections 5.5 and 6 of [65], formally our situation cylinder+compact+cone behaves in the same way as if we were working in the context of cylinder+compact, which is where the theorems just mentioned strictly speaking apply.

Recall that we want to show that $\widetilde{HM}_\bullet(W^\dagger, \mathfrak{s}_\omega)\mathbf{c}(\xi') = \mathbf{c}(\xi', Y)$, in other words, at the chain-level we must have

$$\check{m}c(\xi') - c(\xi', Y) \in \mathrm{im}\check{\partial}_{-Y}$$

The strategy we spelled out consisted in attaching a cylinder of length L to $W^+_{\xi',Y}$ and studying the Seiberg-Witten equations on

$$W^+_{\xi',Y}(L) = ([1,\infty) \times Y') \cup ([0,L] \times -Y') \cup W^\dagger \cup (\mathbb{R}^+ \times -Y)$$

Equivalently, as explained in section 24.6 of [32], we can consider a family of metrics g_L and perturbations on $W^\dagger$, all of which are equal near Y'. For example, we can choose a fraction of the collar neighborhood near Y' and instead of using the product metric $dt \otimes dt + g_{Y'}$, we use a smoothed out version of the metric $g_L = L^2 dt \otimes dt + g_{Y'}$, which agrees with the old metric outside this region. In any case, we obtain a parametrized configuration space

$$\mathfrak{M}_z(W^+_{\xi',Y}, \mathfrak{s}_\omega, [\mathfrak{c}]) = \bigcup_{L\in[0,\infty)} \{L\} \times \mathcal{M}_z(W^+_{\xi',Y}(L), \mathfrak{s}_\omega, [\mathfrak{c}])$$

which we can identify with a subset of $[0,\infty) \times \mathcal{B}^\sigma_{k,loc}(W^+_{\xi',Y}, \mathfrak{s}_\omega)$ as follows (see the remark before definition 24.4.9 in [32] and section 2.3 in [41]):

For any $t \in [0,\infty)$ there is a unique automorphism $b_t : TW^+_{\xi',Y} \to TW^+_{\xi',Y}$ that is positive, symmetric with respect to g_0 and has the property that $g_0(u,v) = g_t(b_t(u), b_t(v))$. The map induced by b_t on orthonormal frames gives rise to a map of spinor bundles $\bar{b}_t : S^\pm_0 \to S^\pm_t$ associated to the metrics g_0 and g_t . This map is an isomorphism preserving the fiberwise length of spinors. The identification

$$[0,\infty)\times\mathcal{B}^{\sigma}_{k,loc}(W^{+}_{\xi',Y},\mathfrak{s}_{\omega})\to\bigcup_{L\in[0,\infty)}\{L\}\times\mathcal{B}^{\sigma}_{k,loc}(W^{+}_{\xi',Y}(L),\mathfrak{s}_{\omega})$$

is then given by

$$(L,A,\mathbb{R}^{+}\phi,\Phi)\to(L,A,\mathbb{R}^{+}\bar{b}_{L}(\phi),\bar{b}_{L}(\Phi)) \tag{33}$$

Just as in proposition 26.1.3 in [32], the moduli space $\mathfrak{M}_z(W^{+}_{\xi',Y},\mathfrak{s}_{\omega},[\mathfrak{c}])$ is a smooth manifold with boundary. The boundary is the fiber over $L=0$, that is, the original moduli space $\mathcal{M}_z(W^{+}_{\xi',Y},\mathfrak{s},[\mathfrak{c}])$. Each individual moduli space $\mathcal{M}_z(W^{+}_{\xi',Y}(L),\mathfrak{s}_{\omega},[\mathfrak{c}])$ can be compactified into $\mathcal{M}^{+}_z(W^{+}_{\xi',Y}(L),\mathfrak{s}_{\omega},[\mathfrak{c}])$ by adding broken trajectories as in definition 24.6.1 of [32][1] and to compactify

$$\bigcup_{L\in[0,\infty)}\{L\}\times\mathcal{M}^{+}_z(W^{+}_{\xi',Y}(L),\mathfrak{s},[\mathfrak{c}])$$

we add a fiber over $L=\infty$, which is denoted $\mathcal{M}^{+}_z(W^{+}_{\xi',Y}(\infty),\mathfrak{s},[\mathfrak{c}])$, where

$$W^{+}_{\xi',Y}(\infty)=\left(K_{Y'}\cup\left[\mathbb{R}^{+}\times -Y'\right]\right)\cup\left(\left[\mathbb{R}^{-}\times -Y'\right]\cup W^{\dagger}\cup\left[\mathbb{R}^{+}\times -Y\right]\right) \tag{34}$$

An element in this space consists (at most) of a quadruple $([\gamma_{K'}],[\breve{\boldsymbol{\gamma}}_{Y'}],[\gamma_{W^{\dagger}}],[\breve{\boldsymbol{\gamma}}_{Y}])$ where :

- $[\gamma_{K'}]\in\mathcal{M}(Z^{+}_{Y',\xi'},\mathfrak{s}',[\mathfrak{a}_{Y'}])$ is a solution on $[\mathbb{R}^{+}\times -Y]\cup K_{Y'}$.
- $[\breve{\boldsymbol{\gamma}}_{Y'}]\in\breve{\mathcal{M}}^{+}([\mathfrak{a}_{Y'}],\mathfrak{s}_{\xi'},[\mathfrak{b}_{Y'}])$ is an unparametrized trajectory on the cylinder $\mathbb{R}\times -Y'$.
- $[\gamma_{W^{\dagger}}]\in\mathcal{M}([\mathfrak{b}_{Y'}],W^{\dagger}_{*},\mathfrak{s}_{\omega},[\mathfrak{b}_{Y}])$ is a solution on $W^{\dagger}_{*}$, that is, $W^{\dagger}$ with two cylindrical ends attached to it.
- $[\breve{\boldsymbol{\gamma}}_{Y}]\in\breve{M}^{+}([\mathfrak{b}_{Y}],\mathfrak{s}_{\xi},[\mathfrak{c}])$ is an unparametrized trajectory on the cylinder $\mathbb{R}\times -Y$.

Just as in proposition 26.1.4 in [32], the space

$$\mathfrak{M}^{+}_z(W^{+}_{\xi',Y},\mathfrak{s},[\mathfrak{c}])=\bigcup_{L\in[0,\infty]}\{L\}\times\mathcal{M}^{+}_z(W^{+}_{\xi',Y}(L),\mathfrak{s},[\mathfrak{c}])$$

[1]More precisely, for us a broken trajectory asymptotic to $[\mathfrak{c}]$ consists of an element $[\gamma_0]$ in a moduli space $\mathcal{M}_{z_0}(W^{+}_{\xi',Y}(L),\mathfrak{s}_{\omega},[\mathfrak{c}])$ and an unparametrized broken trajectory $[\breve{\boldsymbol{\gamma}}]$ in a moduli space $\breve{\mathcal{M}}_z([\mathfrak{c}_0],\mathfrak{s}_{\xi},[\mathfrak{c}])$.

is compact and when it is of dimension 1 the 0 dimensional strata over $L = \infty$ are of the following forms (compare with proposition 26.1.6 [32]):

i) $\mathcal{M}_{Z^+_{Y',\xi'}} \times \mathcal{M}_{W^\dagger_*}$

ii) $\mathcal{M}_{Z^+_{Y',\xi'}} \times \mathcal{M}_{W^\dagger_*} \times \check{\mathcal{M}}_{-Y}$

iii) $\mathcal{M}_{Z^+_{Y',\xi'}} \times \check{\mathcal{M}}_{-Y'} \times \mathcal{M}_{W^\dagger_*}$

Here $\check{\mathcal{M}}$ denotes an unparametrized moduli space. Also, in the last two cases the middle space denotes a boundary-obstructed moduli space, i.e, it denotes trajectories which connect a boundary stable point (as $t \to -\infty$) with a boundary unstable point (as $t \to \infty$).

The following theorem shows that up to a boundary term, $\sum_z m_z(W^+_{\xi',Y}, \mathfrak{s}_\omega, [\mathfrak{c}])$ equals either of the sums (6), (7). It can be seen as the analogue of Lemma 4.15 in [30] and Proposition 24.6.10 in [32] (in fact, it was implicitly used in the proof of the pairing formula in Proposition 6.8 of [30] and Theorem 6.2 in [65]):

Proposition 27. *If $\mathfrak{M}_z(W^+_{\xi',Y}, \mathfrak{s}, [\mathfrak{c}])$ is zero-dimensional, it is compact. If $\mathfrak{M}_z(W^+_{\xi',Y}, \mathfrak{s}_\omega, [\mathfrak{c}])$ is one-dimensional and contains irreducible trajectories, then the compactification $\mathfrak{M}^+_z(W^+_{\xi',Y}, \mathfrak{s}_\omega, [\mathfrak{c}])$ is a 1-dimensional manifold whose boundary points are of the following types:*

1) The fiber over $L = 0$, namely the space $\mathcal{M}_z(W^+_{\xi',Y}, \mathfrak{s}_\omega, [\mathfrak{c}])$.

2) The fiber over $L = \infty$, namely the three products described previously.

3) Products of the form $\mathfrak{M}(W^+_{\xi',Y}, \mathfrak{s}_\omega, [\mathfrak{b}]) \times \check{\mathcal{M}}([\mathfrak{b}], \mathfrak{s}_\xi, [\mathfrak{c}])$ or $\mathfrak{M}(W^+_{\xi',Y}, \mathfrak{s}_\omega, [\mathfrak{a}]) \times \check{\mathcal{M}}([\mathfrak{a}], \mathfrak{s}_\xi, [\mathfrak{b}]) \times \check{\mathcal{M}}([\mathfrak{b}], \mathfrak{s}_\xi, [\mathfrak{c}])$ where the middle one is boundary obstructed.

In order to apply the proposition define $P = [0, \infty)$ and the numbers

$$m_z(W^+_{\xi',Y}, \mathfrak{s}_\omega, [\mathfrak{a}])_P = \begin{cases} |\mathfrak{M}_z(W^+_{\xi',Y}, \mathfrak{s}_\omega, [\mathfrak{a}])| \mod 2 & \text{if } \dim \mathfrak{M}_z(W^+_{\xi',Y}, \mathfrak{s}, [\mathfrak{a}]) = 0 \\ 0 & \text{otherwise} \end{cases}$$

Recall also that the differential on $\check{C}_\bullet(-Y, \mathfrak{s}_\xi) = \mathfrak{C}^o(-Y, \mathfrak{s}_\xi) \oplus \mathfrak{C}^s(-Y, \mathfrak{s}_\xi)$ is [32, Definition 22.1.3]

$$\check{\partial} = \begin{pmatrix} \partial^o_o & -\partial^u_o \bar{\partial}^s_u \\ \partial^o_s & \bar{\partial}^s_s - \partial^u_u \bar{\partial}^s_u \end{pmatrix}$$

Suppose now that $\mathfrak{M}_z(W^+_{\xi',Y}, \mathfrak{s}_\omega, [\mathfrak{c}])$ is one dimensional. We use the previous proposition to count the endpoints of $\mathfrak{M}^+_z(W^+_{\xi',Y}, \mathfrak{s}_\omega, [\mathfrak{c}])$ by making cases on $[\mathfrak{c}]$.

Case $[\mathfrak{c}] \in \mathfrak{C}^o(-Y, \mathfrak{s}_\xi)$ *[irreducible critical point].*

(1) The fiber over $L = 0$, gives the contributions

$$\sum_z m_z(W^+_{\xi',Y}, \mathfrak{s}_\omega, [\mathfrak{c}]) \tag{35}$$

These numbers were used in the chain-level definition of $c(\xi', Y)$.

(2) The fiber over $L = \infty$ gives the contributions (6)

$$\begin{gathered}\sum_{[\mathfrak{a}]\in\mathfrak{C}^o(-Y')} \sum_{z_1,z_2} m_{z_1}(Z^+_{Y',\xi'}, \mathfrak{s}', [\mathfrak{a}]) n_{z_2}([\mathfrak{a}], W^\dagger_*, \mathfrak{s}_\omega, [\mathfrak{c}]) \\ + \sum_{[\mathfrak{a}]\in\mathfrak{C}^s(-Y'),[\mathfrak{b}]\in\mathfrak{C}^u(-Y')} \sum_{z_1,z_2,z_3} m_{z_1}(Z^+_{Y',\xi'}, \mathfrak{s}', [\mathfrak{a}]) \bar{n}_{z_2}([\mathfrak{a}], \mathfrak{s}_{\xi'}, [\mathfrak{b}]) n_{z_3}([\mathfrak{b}], W^\dagger_*, \mathfrak{s}_\omega, [\mathfrak{c}]) \\ + \sum_{[\mathfrak{a}]\in\mathfrak{C}^s(-Y'),[\mathfrak{b}]\in\mathfrak{C}^u(-Y)} \sum_{z_1,z_2,z_3} m_{z_1}(Z^+_{Y',\xi'}, \mathfrak{s}', [\mathfrak{a}]) \bar{n}_{z_2}([\mathfrak{a}], W^\dagger_*, \mathfrak{s}_\omega, [\mathfrak{b}]) n_{z_3}([\mathfrak{b}], \mathfrak{s}_\xi, [\mathfrak{c}])\end{gathered} \tag{36}$$

These numbers were used in the chain-level definition of $\check{m}c(\xi')$.

(3) We obtain contributions of the form

$$\begin{gathered}\sum_{[\mathfrak{a}]\in\mathfrak{C}^o(-Y)} \sum_{w_1,w_2} m_{w_1}(W^+_{\xi',Y}, \mathfrak{s}_\omega, [\mathfrak{a}])_P n_{w_2}([\mathfrak{a}], \mathfrak{s}_\xi, [\mathfrak{c}]) \\ + \sum_{[\mathfrak{a}]\in\mathfrak{C}^s(-Y),[\mathfrak{b}]\in\mathfrak{C}^u(-Y')} \sum_{w_1,w_2,w_3} m_{w_1}(W^+_{\xi',Y}, \mathfrak{s}_\omega, [\mathfrak{a}])_P \bar{n}_{w_2}([\mathfrak{a}], \mathfrak{s}_\xi, [\mathfrak{b}]) n_{w_3}([\mathfrak{b}], \mathfrak{s}_\xi, [\mathfrak{c}])\end{gathered} \tag{37}$$

These numbers will be used momentarily to define the boundary term.

The proposition tells us that the sum of (35), (36) and (37) equals 0.

Case $[\mathfrak{c}] \in \mathfrak{C}^s(-Y, \mathfrak{s}_\xi)$ [boundary stable critical point].

(1) The fiber over $L = 0$, gives the contributions

$$\sum_z m_z(W^+_{\xi',Y}, \mathfrak{s}_\omega, [\mathfrak{c}]) \tag{38}$$

These numbers were used in the chain-level definition of $c(\xi', Y)$.

(2) The fiber over $L = \infty$ gives the contributions (6)

$$\begin{gathered}\sum_{[\mathfrak{a}]\in\mathfrak{C}^o(-Y')} \sum_{z_1,z_2} m_{z_1}(Z^+_{Y',\xi'}, \mathfrak{s}', [\mathfrak{a}]) n_{z_2}([\mathfrak{a}], W^\dagger_*, \mathfrak{s}_\omega, [\mathfrak{c}]) \\ + \sum_{[\mathfrak{a}]\in\mathfrak{C}^s(-Y')} \sum_{z_1,z_2} m_z(Z^+_{Y',\xi'}, \mathfrak{s}', [\mathfrak{a}]) \bar{n}_{z_2}([\mathfrak{a}], W^\dagger_*, \mathfrak{s}_\omega, [\mathfrak{c}]) \\ + \sum_{[\mathfrak{a}]\in\mathfrak{C}^s(-Y'),[\mathfrak{b}]\in\mathfrak{C}^u(-Y')} \sum_{z_1,z_2,z_3} m_{z_1}(Z^+_{Y',\xi'}, \mathfrak{s}', [\mathfrak{a}]) \bar{n}_{z_2}([\mathfrak{a}], \mathfrak{s}_\xi, [\mathfrak{b}]) n_{z_3}([\mathfrak{b}], W^\dagger_*, \mathfrak{s}_\omega, [\mathfrak{c}]) \\ + \sum_{[\mathfrak{a}]\in\mathfrak{C}^s(-Y'),[\mathfrak{b}]\in\mathfrak{C}^u(-Y)} \sum_{z_1,z_2,z_3} m_{z_1}(Z^+_{Y',\xi'}, \mathfrak{s}', [\mathfrak{a}]) \bar{n}_{z_2}([\mathfrak{a}], W^\dagger_*, \mathfrak{s}_\omega, [\mathfrak{b}]) n_{z_3}([\mathfrak{b}], \mathfrak{s}_\xi, [\mathfrak{c}])\end{gathered} \tag{39}$$

These numbers were used in the chain-level definition of $\check{m}c(\xi')$.

(3) We obtain contributions of the form

(40)
$$\begin{gathered}\textstyle\sum_{[\mathfrak{a}]\in\mathfrak{C}^o(-Y)}\sum_{w_1,w_2} m_{w_1}(W^+_{\xi',Y},\mathfrak{s}_\omega,[\mathfrak{a}])_P n_{w_2}([\mathfrak{a}],\mathfrak{s}_\xi,[\mathfrak{c}])\\ \textstyle+\sum_{[\mathfrak{a}]\in\mathfrak{C}^s(-Y)}\sum_{w_1,w_2} m_{w_1}(W^+_{\xi',Y},\mathfrak{s}_\omega,[\mathfrak{a}])_P \bar{n}_{w_2}([\mathfrak{a}],\mathfrak{s}_\xi,[\mathfrak{c}])\\ \textstyle+\sum_{[\mathfrak{a}]\in\mathfrak{C}^s(-Y),[\mathfrak{b}]\in\mathfrak{C}^u(-Y)}\sum_{w_1,w_2,w_3} m_{w_1}(W^+_{\xi',Y},\mathfrak{s}_\omega,[\mathfrak{a}])_P \bar{n}_{w_2}([\mathfrak{a}],\mathfrak{s}_\xi,[\mathfrak{b}]) n_{w_3}([\mathfrak{b}],\mathfrak{s}_\xi,[\mathfrak{c}])\end{gathered}$$

These numbers will be used momentarily to define the boundary term.

The proposition tells us that the sum of (38), (39) and (40) equals 0.

Define the chain element $\psi \in \mathfrak{C}^o(-Y,\mathfrak{s}_\xi)\oplus\mathfrak{C}^s(-Y,\mathfrak{s}_\xi)$ via the formula

$$\psi = \left(\sum_{[\mathfrak{a}]\in\mathfrak{C}^o(-Y)}\sum_{w_1} m_{w_1}(W^+_{\xi',Y},\mathfrak{s}_\omega,[\mathfrak{a}])_P e_{[\mathfrak{a}]},\ \sum_{[\mathfrak{a}]\in\mathfrak{C}^s(-Y)}\sum_{w_1} m_{w_1}(W^+_{\xi',Y},\mathfrak{s}_\omega,[\mathfrak{a}])_P e_{[\mathfrak{a}]}\right)$$

It is not hard to see that

$$\check{\partial}\psi = \left(\sum_{[\mathfrak{c}]\in\mathfrak{C}^o(-Y)} C_o e_{[\mathfrak{c}]},\ \sum_{[\mathfrak{c}]\in\mathfrak{C}^s(-Y)} C_s e_{[\mathfrak{c}]}\right)$$

where C_o equals (37) and C_s equals (40). In other words, we have the chain-level identity

$$\check{m}c(\xi') - c(\xi',Y) = \check{\partial}\psi$$

which gives us the desired identity

$$\widetilde{HM}_\bullet(W^\dagger,\mathfrak{s}_\omega)\mathbf{c}(\xi') = \mathbf{c}(\xi',Y)$$

concluding the first phase in the proof for the naturality of the contact invariant under strong symplectic cobordisms. Now we proceed to address the second part of the proof (as explained at the beginning of the paper). Namely, we will show that $\mathbf{c}(\xi',Y)$ equals $\mathbf{c}(\xi)$ by adapting Mrowka and Rollin's "dilating the cone" technique to the case of a manifold with cylindrical end.

6. Generalized Gluing-Excision Theorem

6.1 Gluing and Identifying Spin-c Structures. Our next objective in this section is to modify the arguments in [40] to the case where there is a cylindrical end instead of a compact manifold. The results in that paper apply only under the assumption that we are working with a special symplectic cobordism $W : Y \to Y'$. This appears near formula (1.1) of [40], and it was defined as follows:

Definition 28. A cobordism $(W, \omega) : (Y, \xi) \to (Y', \xi')$ is said to be a **special symplectic cobordism** if:

1) With the symplectic orientation, $\partial W = -Y \sqcup Y'$ and ω is strictly positive on ξ and ξ' with their induced orientations.

2) The symplectic form is given in a collar neighborhood of the concave boundary by a symplectization of (Y, ξ).

3) The map induced by the inclusion $i^* : H^1(W, Y'; \mathbb{Z}) \to H^1(Y; \mathbb{Z})$ is the zero map.

Notice that it is the last condition the one that makes the symplectic cobordism "special". We want to work with strong cobordisms, which in particular means that the convex end is also given by a symplectization of (Y, ξ) and that the special condition does not appear. To explain why we can ultimately drop this condition, we will now say a quick words on how to identify (relative) spin-c structures, since this is the place where Mrowka and Rollin used it. It is useful to think of the special condition in the following way [40, Remark 1.2.2]: if $u \in \mathrm{Map}(W, S^1)$ is homotopic to the identity along Y' (so $[u] \in H^1(W, Y')$), then u must be homotopic to the identity along Y (so $[u \mid_Y] = 0 \in H^1(Y)$). We will explain how the special assumption is related to the uniqueness of certain gluing operations involving spin-c structures.

It is well known that there are several different ways to think about spin-c structures, but from the perspective of monopole Floer homology, it is very convenient to think of it as being realized concretely by a spinor bundle. Therefore, we will start by discussing what is meant by a spinor bundle. The following is based on the exposition by Salamon in [49].

Definition 29. Suppose (M, g) is an oriented Riemannian manifold of dimension $2n$ or $2n + 1$. A **spin-c structure** on the vector bundle $TM \to M$ is a pair (S, ρ)

where $S \to M$ is a Hermitian vector bundle of rank 2^n and a Clifford multiplication map

$$\rho : TM \to \hom_{\mathbb{C}}(S, S)$$

which is a homomorphism satisfying for all $v \in TM$

$$\rho(v) \circ \rho(v) = -|v|_g^2 \mathrm{Id}_S \tag{41}$$

and for all $\Phi, \Phi' \in \Gamma(S)$

$$\langle \rho(v)\Phi, \Phi' \rangle = - \langle \Phi, \rho(v)\Phi' \rangle \tag{42}$$

where $\langle \cdot, \cdot \rangle$ is the hermitian inner product on S.

If (S_1, ρ_1) and (S_2, ρ_2) are two spin-c structures associated to TM, a **spin-c isomorphism** from (S_1, ρ_1) to (S_2, ρ_2) is a unitary bundle morphism

$$\mathfrak{S} : S_1 \to S_2 \tag{43}$$

such that for all $v \in TM$, $\Phi_1 \in S_1$ we have

$$\mathfrak{S}(\rho_1(v)\Phi_1) = \rho_2(v)\mathfrak{S}(\Phi_1) \tag{44}$$

In the case that $(S_1, \rho_1) = (S_2, \rho_2)$ we will call $\mathfrak{S}$ an **automorphism of the spinor bundle** (S, ρ)**.**

The definition of spin-c isomorphism basically says that $\mathfrak{S}$ intertwines the two Clifford actions. Recall that using the metric duality ρ can be defined on the cotangent bundle T^*M and every three or four manifold admits a spin-c structure. We will denote by $\mathrm{Spin}^c(M)$ the isomorphism classes of spin-c structures under the previous relation and $\mathfrak{s} = [(S, \rho)] \in \mathrm{Spin}^c(M)$ will denote an isomorphism class of a spin-c structure. We will sometimes say that $\mathfrak{s}$ is an **abstract spin-c structure** and the pair (S, ρ) is an **instantiation of the abstract spin-c structure.**

Observe that given a particular instantiation (S, ρ) it is quite easy to produce automorphisms associated to it. For example, choose a map $u : M \to S^1$ and define

$$\mathfrak{S}_u(\Phi) = u\Phi$$

where just as in the case of a gauge transformation the right hand side means fiberwise multiplication of the spinor by a complex number. Since all automorphisms of an instantiation (S,ρ) arise in this way, we can think of the gauge group as the group of automorphisms of an abstract spin-c structure $\mathfrak{s}$ [32, Section 1.1].

In general it is useful to think of a spin-c structure $\mathfrak{s}$ as being independent of the Riemannian metric g used on the manifold M. Therefore, we need to generalize our definition of isomorphic spin-c structures to include the case in which the spinor bundles are built from different metrics on the manifold.

Suppose that g_0, g_1 are two metrics on M and that $x \in M$. Since the metrics are symmetric and positive definite bilinear forms, there is a a unique automorphism [2]

$$b_{g_0,g_1} : TM \to TM$$

that is positive, symmetric with respect to g_0 and satisfies for all $v, w \in TM$

$$g_0(v,w) = g_1(b_{g_0,g_1}(v), b_{g_0,g_1}(w))$$

If we have an abstract spin-c structure $\mathfrak{s}_0$ with respect to (M, g_0) instantiated by the spinor bundle (S_0, ρ_0), using b_{g_0,g_1} we will define a canonical spinor bundle (S_1, ρ_1) associated to (M, g_1) and we will define $\mathfrak{s}_1$ as its corresponding abstract spin-c structure. Under this construction we would say that $\mathfrak{s}_0$ is equivalent to $\mathfrak{s}_1$. More concretely:

(1) As a vector bundle, we define $S_1 \equiv S_0$.
(2) To define ρ_1, observe that it must satisfy condition (41), namely, we want for $v \in TM$, $\rho_1(v) \circ \rho_1(v) = -|v|^2_{g_1} Id_{S_1}$. Since

$$|v|^2_{g_1} = g_1(v,v) = g_0(b^{-1}_{g_0,g_1}(v), b^{-1}_{g_0,g_1}(v)) = \left|b^{-1}_{g_0,g_1}(v)\right|^2_{g_0}$$

we can see that the natural definition for ρ_1 should be

$$\rho_1(v) \equiv \rho_0\left(b^{-1}_{g_0,g_1}(v)\right)$$

(3) If $\langle \cdot,\cdot \rangle_0$ represents the hermitian inner product on S_0 with respect to g_0 then we can also define $\langle \cdot,\cdot \rangle_1 \equiv \langle \cdot,\cdot \rangle_0$ as the hermitian inner product on $S_1 = S_0$ with respect to g_1, that is, the inner product remains unchanged. It is not difficult to see that ρ_1 satisfies property (42).

[2]This construction can be done fiberwise so it is a consequence of linear algebra [37, Section 2]. Namely, if V is an m dimensional real vector space and $g_0, g_1 \in \mathrm{Sym}(V^* \otimes V^*)$ are two metrics then there is a unique positive endomorphism H of V such that $g_1(\cdot,\cdot) = g_0(H\cdot,\cdot)$. One then takes $b_{g_0,g_1} = H^{-1/2}$.

(4) We denote this construction as the map $\bar{b}_{S_0,S_1} : (S_0, \rho_0) \to (S_1, \rho_1)$.

Definition 30. Identification of abstract spin-c structures: suppose that M is an oriented manifold with metric g_0 and abstract spin-c structure $\mathfrak{s}_0$ associated to g_0. If g_1 is another metric and $\mathfrak{s}_1$ is an abstract spin-c structure we will say that the **abstract spin-c structures $\mathfrak{s}_0$ and $\mathfrak{s}_1$ are equivalent** if the following happens: one can find an instantiation (S_0, ρ_0) of $\mathfrak{s}_0$ such that the spinor bundle (S_1, ρ_1) as described before is an instantiation of $\mathfrak{s}_1$.

Remark 31. *i*) Observe that if (S_1', ρ_1') is another instantiation of (S_1, ρ_1) then by definition we have a unitary isomorphism $\mathfrak{S} : (S_1, \rho_1) \to (S_1', \rho_1')$ which intertwines ρ_1 and ρ_1' . If we define

$$\bar{b}_{S_0,S_1'} \equiv \mathfrak{S} \circ \bar{b}_{S_0,S_1} : (S_0, \rho_0) \to (S_1', \rho_1')$$

we obtain a map of spinor bundles which preserves the pointwise norms of spinors. It is not difficult to check that $\bar{b}_{S_1',S_0} \circ \bar{b}_{S_0,S_1'} = \mathrm{Id}_{S_0}$.

ii) Thanks to the previous definition we can talk of an abstract spin-c structure $\mathfrak{s}$ in a way that is metric independent. For example, it is well known the isomorphism classes of spin-c structures $\mathrm{Spin}^c(M)$ on M is a $H^2(M;\mathbb{Z})$ torsor, i.e, if we fix a spin-c structure $\mathfrak{s}_0$ as the "basepoint", any other spin-c structure $\mathfrak{s}_1$ differs from $\mathfrak{s}_0$ by an element of $H^2(M;\mathbb{Z})$ [32, Proposition 1.1.1].

We briefly repeat the proof this fact to show how it fits in our current framework. Let (S_0, ρ_0) be an instantiation of $\mathfrak{s}_0$ and start with an element $\tilde{e} \in H^2(M;\mathbb{Z})$. Choose a hermitian line bundle $L_{\tilde{e}}$ such that $c_1(L_{\tilde{e}}) = \tilde{e}$ where $c_1(\bullet)$ denotes the first Chern class. Define $\mathfrak{s}_{\tilde{e}} \equiv \mathfrak{s}_0 + \tilde{e}$ as the abstract spin-c structure which has a particular instantiation the spinor bundle $S_{\tilde{e}} = S_0 \otimes L_{\tilde{e}}$, $\rho_{\tilde{e}} = \rho_0 \otimes 1_{L_{\tilde{e}}}$. The inner product on $S_{\tilde{e}}$ is the usual inner product on the tensor product of two vector spaces:

$$\langle \Phi \otimes \sigma_{\tilde{e}}, \Phi' \otimes \sigma_{\tilde{e}}' \rangle = \langle \Phi, \Phi' \rangle \langle \sigma_{\tilde{e}}, \sigma_{\tilde{e}}' \rangle$$

Conversely, if we start with instantiations (S_0, ρ_0) and (S_1, ρ_1) of $\mathfrak{s}_0, \mathfrak{s}_1$ respectively, then we can define the difference line bundle L_{S_0,S_1} as the subbundle of $\hom(S_0, S_1)$ consisting of homomorphisms $\mathfrak{S} : S_0 \to S_1$ that intertwine ρ_0, ρ_1 in the sense of equation (44) . This subbundle has rank 1 because the only endomorphisms of S_0 that commute with the image of ρ_0 are the scalar endomorphisms (Schur's Lemma).

Define $\tilde{e} \equiv c_1(L_{S_0,S_1}) \in H^2(M;\mathbb{Z})$. It can be checked that $(S_{\tilde{e}}, \rho_{\tilde{e}})$ and (S_1, ρ_1) represent the same abstract spin-c structure $\mathfrak{s}_1$ via the "tautological" map $\mathfrak{S}_{\tilde{e}} : S_{\tilde{e}} \to S_1$, $(\Phi, \mathfrak{S}) \to \mathfrak{S}(\Phi)$ (which is clearly unitary) and that if $\tilde{e} = 0$, i.e, the line bundle L_{S_0,S_1} admits a global section, then $\mathfrak{s}_0 = \mathfrak{s}_1$.

Our main objective is to analyze the spin-c structure associated to a symplectic form on a compact four manifold and to a contact structure on a three manifold. More generally, we want to see to what extent are these different spin-c structures related. To make this precise, we first need to address how a spin-c structure on manifold X with boundary ∂X induces a spin-c structure on its boundary and how a spin-c structure on a three manifold Y induces a spin-c structure on the cylinder $\mathbb{R} \times Y$ [32, Sections 4.3 and 4.5].

Definition 32. 1) Suppose that Y is a closed 3 manifold with abstract spin-c structure $\mathfrak{s}_Y$. It induces an abstract spin-c structure $\mathfrak{s}_Z$ on $Z = \mathbb{R} \times Y$ as follows. Let (S_Y, ρ_Y) be an instantiation of $\mathfrak{s}_Y$. Use the product metric on Z and define the spinor bundle

$$S_Z = S_Y \oplus S_Y$$

and the Clifford multiplication to be

$$\rho_Z(\partial_t) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\rho_Z(v) = \begin{pmatrix} 0 & -\rho_Y(v)^* \\ \rho_Y(v) & 0 \end{pmatrix} \quad v \in TY$$

Define $\mathfrak{s}_Z$ as the abstract spin-c structure whose instantiation is S_Z.

2) Suppose that X is a compact, oriented Riemannian 4 manifold X with boundary $Y = \partial X$ and abstract spin-c structure $\mathfrak{s}_X$. It induces an abstract spin-c structure $\mathfrak{s}_Y$ on Y as follows. Let $(S_X = S^+ \oplus S^-, \rho_X)$ be an instantiation of $\mathfrak{s}_X$ Here the splitting comes from the eigenspaces of $\rho(\mathrm{vol}_X)$ as described in Section 1.1 in [32]. Use an outward normal n (which we take to be of unit norm and orthogonal to TY) to identify S^+ and S^- at the boundary $\rho_X(n) : S^+ |_Y \simeq S^- |_Y$, i.e, $(\Phi_+, 0) \in S^+ |_Y$

is identified with $\rho_X(n)(\Phi_+, 0) \in S^- |_Y$. Define the spinor bundle as

$$S_Y = S^+ |_Y \simeq S^- |_Y$$

and the Clifford multiplication is

$$\rho_Y(v) = \rho_X(n)^{-1}\rho_X(v)$$

Define $\mathfrak{s}_Y$ as the abstract spin-c structure whose instantiation is (S_Y, ρ_Y).

3) Suppose that Y is a closed oriented three manifold with abstract spin-c structure $\mathfrak{s}_Y$. It induces an abstract spin-c structure $\mathfrak{s}_{-Y}$ as follows. Let (S_Y, ρ_Y) be an instantiation of $\mathfrak{s}_Y$. Define

$$S_{-Y} \equiv S_Y$$
$$\rho_{-Y} \equiv -\rho_Y$$

We reversed the sign of the Clifford map to continue to have $\rho(vol) = \mathrm{id}$. Define $\mathfrak{s}_{-Y}$ as the abstract spin-c structure whose instantiation is (S_{-Y}, ρ_{-Y}).

Now we will define abstract spin-c structures canonically associated to a symplectic form on a symplectic manifold and on a contact manifold. Some references for the symplectic case can be found in Lemma 4.3 of [26] or 3.1.4 [20]. For the contact case see section 2 in [47] and section 2.1 in [53].

Definition 33. 1) Suppose that (W, ω) is a symplectic four manifold. We construct an abstract spin-c structure $\mathfrak{s}_\omega$ as follows. Choose a metric g and an almost complex structure so that (g, J, ω) is a **compatible triple**. That is, for all $v_1, v_2 \in TW$, $\omega(v, Jv) > 0$, $\omega(Jv_1, Jv_2) = \omega(v_1, v_2)$ and $g(v_1, v_2) = \omega(v_1, Jv_2)$. See section 12.3 in [10] for more information.

There is an induced almost complex structure $J^\flat : T^*W \to T^*W$ on T^*W via the rule $(J^\flat\eta)(v) = -\eta(Jv)$. This is an awkward convention but agrees with the one used by [36, 20].

We can extend $J^\flat$ to all the exterior algebra of W so we can define $\Lambda_J^{0,0}$ as the $-i$ eigenbundle of $J^\flat$ (when acting of 0 forms) and $\Lambda_J^{0,2}$ for the $+i$ eigenbundle of $J^\flat$ when acting on two-forms. We define the spinor bundle

$$S_\omega = \Lambda_J^{0,0} \oplus \Lambda_J^{0,2}$$

and the Clifford multiplication

$$\rho_\omega : \Omega^1(M) \otimes \mathbb{C} \to \hom_{\mathbb{C}}(S, S)$$
$$\rho_\omega(\eta)\gamma = \sqrt{2}\left(\eta^{0,1} \wedge \gamma - \imath(\eta_{0,1})\gamma\right) \tag{45}$$

Here η is a one form and we are decomposing it into types $\eta = \eta^{1,0} + \eta^{0,1}$. Under the $\mathbb{C}$ linear metric duality $\eta_{0,1}$ represents the corresponding $(0,1)$ vector field. The abstract spin-c structure $\mathfrak{s}_\omega$ is the one whose instantiation is (S_ω, ρ_ω).

2) Suppose that (Y, ξ) is a contact 3 manifold. We construct an abstract spin-c structure $\mathfrak{s}_\xi$ as follows. Choose a one form θ such that $\xi = \ker\theta$ and $\theta \wedge d\theta > 0$. Choose the metric g on Y for which $|\theta|_g = 1$, $d\theta = 2 *_{Y,g} \theta$ and J is an isometry with respect to ξ, where J is a choice of an almost complex structure on ξ such that for any $v \in \xi$, (v, Jv) is a positively oriented basis for ξ. Define the spinor bundle

$$S_\xi = \mathbb{C} \oplus \mathcal{K}^{-1}$$

where $\mathcal{K}^{-1}$ denotes the complex line bundle $(\langle\theta\rangle^\perp, J^\flat)$. Any form $\eta \in \Gamma(\mathcal{K}^{-1})$ can be decomposed according to type as $\eta = \eta^{1,0} + \eta^{0,1}$ and a generic one form can be written as

$$\eta = \eta(R)\theta + \eta^{1,0} + \eta^{0,1}$$

where R is the Reeb vector field of θ, i.e, the metric dual of θ. The corresponding Clifford multiplication map is

$$\rho_\xi : T^*Y \otimes \mathbb{C} \to \hom_{\mathbb{C}}(S, S) \tag{46}$$
$$\rho_\xi(\eta)(\alpha, \beta) = (i\eta(R)\alpha, -i\eta(R)\beta) - \sqrt{2}(\eta^{0,1} \wedge \Phi - \imath(\eta_{0,1})\Phi)$$

where we use the $\mathbb{C}$ linear metric duality. The abstract spin-c structure $\mathfrak{s}_\xi$ is the one whose instantiation is (S_ξ, ρ_ξ).

Remark 34. As a small digression (which will be useful in the next section) we can decompose S^+_ω as $S^+_\omega = \mathbb{C}\Phi_0 \oplus \langle\Phi_0\rangle^\perp$ where denotes the orthogonal complex line bundle. The canonical connection A_0 is then the unique spin-c connection for which $D_{A_0}\Phi_0 = 0$ [26, Section 4.3]. The Seiberg-Witten equations (with respect to Taubes' perturbations from [52]) can then be written as

$$SW_{Taubes}:\begin{cases}\frac{1}{2}\rho(F_A^+)-(\Phi\Phi^*)_0=\frac{1}{2}\rho(F_{A_0}^+)-\frac{i}{4}\rho(\omega)\\ D_A\Phi=0\end{cases}$$

Since $\frac{i}{4}\rho(\omega) = (\Phi_0\Phi_0^*)_0$ then (A_0,Φ_0) is tautologically a solution of the previous equations. Now suppose that we dilate the metric, that is, for $\tau>0$ a constant we define $g_\tau=\tau^2 g$. Since τ is constant then $\omega_\tau=\tau^2\omega$ will continue to be a symplectic form compatible with the metric g_τ (in the sense that it has pointwise norm $\sqrt{2}$). Also, because of our recipe for identifying spin-c structures for different metrics on the same manifolds, we take $S_{\omega,\tau}\equiv S_\omega$ and $\rho_\tau=\frac{\rho}{\tau}$, where **we are assuming that the Clifford map is defined on one forms.** When we extend ρ_τ to the rest of the exterior algebra we have that on two forms $\rho_\tau=\frac{\rho}{\tau^2}$ so in particular $\rho_\tau(\omega_\tau)=\rho(\omega)$. Moreover, a dilation is a very trivial case of a conformal change of metric so using the formula for how the Dirac operator changes with respect to it [11, eq. D.1] it is not difficult to see that $D_{A,g_\tau}\Phi=\tau^{-1}D_{A,g}\Phi$ so in particular being a harmonic spinor is independent of the metric g_τ and moreover the canonical connection A_0 is preserved under dilations, that is, $A_{0,\tau}=A_0$. Since the notion of self duality is also conformally invariant the system of equations

$$SW^\tau_{Taubes}:\begin{cases}\frac{1}{2}\rho_\tau(F_A^+)-(\Phi\Phi^*)_0=\frac{1}{2}\rho_\tau(F_{A_{0,\tau}}^+)-\frac{i}{4}\rho_\tau(\omega_\tau)\\ D_{A,g_\tau}\Phi=0\end{cases}$$

can be rewritten as [recall that $\rho_\tau(F_A^+)=\frac{\rho(F_A^+)}{\tau^2}$]

$$SW^\tau_{Taubes}:\begin{cases}\frac{1}{2}\rho(F_A^+)-\tau^2(\Phi\Phi^*)_0=\frac{1}{2}\rho(F_{A_0}^+)-\tau^2\frac{i}{4}\rho(\omega)\\ D_{A,g}\Phi=0\end{cases}$$

setting $\tau=\sqrt{r}$ this suggests how one could find the class of perturbations used by Taubes [51, eq. 1.20].

Back to our main topic of gluing and identifying spin-c structures, suppose that $(W,\omega):(Y,\xi)\to(Y',\xi')$ is a strong symplectic cobordism between (Y,ξ) and (Y',ξ'). The next lemma addresses how is $\mathfrak{s}_\omega$ related to $\mathfrak{s}_\xi$ and $\mathfrak{s}_{\xi'}$.

Lemma 35. *If* $(W,\omega) : (Y,\xi) \to (Y',\xi')$ *is a strong symplectic cobordism then* $\mathfrak{s}_\omega |_{Y'} = \mathfrak{s}_\xi$ *and* $\mathfrak{s}_\omega |_{-Y} = \mathfrak{s}_{-Y}$ *as abstract spin-c structures where* $\mathfrak{s}_Y = \mathfrak{s}_\xi$.

Proof. We start with the convex end. Choose a contact form θ' and metric $g_{\theta'}$ as explained before. We choose the canonical spinor bundle $(S_{\theta'}, \rho_{\theta'})$ constructed above. Being a convex end, we can choose a collar neighborhood $(0,1] \times Y' \subset W$ such that the symplectic form ω on this collar neighborhood looks like the symplectization $\frac{1}{2}d(t^2\theta')$.

Choose the conical metric $dt \otimes dt + t^2 g_{\theta'}$ on this collar neighborhood. Then ∂_t is the outward normal vector and if $e^1_{Y'}, e^2_{Y'}, e^3_{Y'}$ is a coframe of Y' then $(e^0, e^1, e^2, e^3) = (dt, e^1_{Y'}, e^2_{Y'}, e^3_{Y'})$ is a coframe of W at the boundary $\partial[(0,1] \times Y'] = \{1\} \times Y' \simeq Y'$. For the canonical spinor bundle (S_ω, ρ_ω) we have that a section $\Phi \in \Gamma(S^+_\omega |_{Y'})$ can locally be written as

$$\Phi = \alpha_W + \beta_W \bar{\epsilon}^{01} \wedge \bar{\epsilon}^{23}$$

where $\bar{\epsilon}^{01} = \frac{1}{\sqrt{2}}(e^0 - ie^1)$ and $\bar{\epsilon}^{23} = \frac{1}{\sqrt{2}}(e^2 - ie^3)$. Likewise, a section $\Psi \in \Gamma(S_{\theta'})$ can be written as

$$\Psi = \alpha_{Y'} + \beta_{Y'} \bar{\epsilon}^{23}$$

The isomorphism between $(S_\omega |_{Y'}, \rho_\omega |_{Y'})$ and $(S_{\theta'}, \rho_{\theta'})$ is given by

$$\begin{aligned} \mathfrak{S} : (S_\omega |_{Y'}, \rho_\omega |_{Y'}) &\to (S_{\theta'}, \rho_{\theta'}) \\ \Phi = \alpha_W + \beta_W \bar{\epsilon}^{01} \wedge \bar{\epsilon}^{23} &\to \Psi = \alpha_W + \beta_W \bar{\epsilon}^{23} \end{aligned}$$

Clearly $\mathfrak{S}$ will be a unitary isomorphism. The only thing that needs to be checked is that $\mathfrak{S}$ intertwines the Clifford multiplication, that is, for all $v \in TY'$ we have

$$\mathfrak{S}(\rho_W^{-1}(\partial_t)\rho_W(v)\Phi) = \rho_{\theta'}(v)\mathfrak{S}(\Phi)$$

This can be verified using the definition given for ρ_ω and ρ_ξ in formulas 45 , 46 and also that $\rho_W^{-1}(\partial_t) = -\rho_W(\partial_t)$. The concave case can be dealt with in a similar manner.

□

Now we return to the problem of gluing spin-c structures. Following [31], suppose that X is a compact four manifold with boundary $Y = \partial X$ (potentially disconnected) which will ultimately be equipped with a contact structure ξ, though for the moment we won't use this.

A **relative spinor bundle** is a triple $(S_X, \rho_X, \mathfrak{S}_Y)$ where (S_X, ρ_X) is the instantiation of some abstract spin-c structure $\mathfrak{s}_X \in \mathrm{Spin}^c(X)$ and

$$\mathfrak{S}_Y : (S_X, \rho_X)\mid_{\partial X} \to (S_Y, \rho_Y)$$

is an isomorphism to some instantiation (S_Y, ρ_Y) of some abstract spin-c structure $\mathfrak{s}_Y \in \mathrm{Spin}^c(Y)$. In other words,

$$\forall v \in TY, \forall \Phi \in S_X^+\mid_Y \qquad \mathfrak{S}_Y\left[\rho_X(n_X)^{-1}\rho_X(v)\Phi\right] = \rho_Y(v)\mathfrak{S}_Y(\Phi)$$

Observe that if (S'_X, ρ'_X) is a different instantiation of $\mathfrak{s}_X$ we have a map

$$\mathfrak{S}'_X : (S'_X, \rho'_X) \to (S_X, \rho_X)$$

which satisfies

$$\forall v \in TX, \forall \Phi' \in S'_X \qquad \mathfrak{S}'_X(\rho'_X(v)\Phi') = \rho_X(v)\mathfrak{S}'_X(\Phi')$$

In particular, if we define $\mathfrak{S}'_Y \equiv \mathfrak{S}_Y \circ (\mathfrak{S}'_X\mid_Y) : (S'_X, \rho'_X)\mid_{\partial X} \to (S_Y, \rho_Y)$ it is easy to see that $\forall v \in TY, \forall \Phi' \in S_X^{+\prime}\mid_Y$

$$\begin{aligned}
&\mathfrak{S}'_Y\left[\rho'_X(n_X)^{-1}\left(\rho'_X(v)\Phi'\right)\right]\\
&= \mathfrak{S}_Y \circ (\mathfrak{S}'_X\mid_Y)\left[\rho'_X(n_X)^{-1}\left(\rho'_X(v)\Phi'\right)\right]\\
&= \mathfrak{S}_Y \circ (\mathfrak{S}'_X\mid_Y)\left(\mathfrak{S}'_X\right)^{-1}\left(\rho_X(n_X)^{-1}\mathfrak{S}'_X\left(\rho'_X(v)\Phi'\right)\right)\\
&= \mathfrak{S}_Y\left(\rho_X(n_X)^{-1}\mathfrak{S}'_X\left(\rho'_X(v)\Phi'\right)\right)\\
&= \mathfrak{S}_Y(\rho_X(n_X)^{-1}\rho_X(v)\mathfrak{S}'_X(\Phi'))\\
&= \rho_Y(v)\mathfrak{S}_Y \circ \mathfrak{S}'_X(\Phi')\\
&= \rho_Y(v)\mathfrak{S}'_Y(\Phi')
\end{aligned}$$

In other words, the triple $(S'_X, \rho'_X, \mathfrak{S}'_Y = \mathfrak{S}_Y \circ (\mathfrak{S}'_X\mid_Y))$ is also a relative spinor bundle. Similarly, if (S''_Y, ρ''_Y) is another instantiation of the same abstract spin-c structure $\mathfrak{s}_Y$ then we have a map

$$\mathfrak{S} : (S_Y, \rho_Y) \to (S''_Y, \rho''_Y)$$

which satisfies

$$\forall v \in TY, \forall \Psi \in S_Y \qquad \mathfrak{S}(\rho_Y(v)\Psi) = \rho''_Y(v)\mathfrak{S}(\Psi)$$

and if we define $\mathfrak{S}''_Y = \mathfrak{S} \circ \mathfrak{S}_Y : (S_X, \rho_X) \mid_{\partial X} \to (S''_Y, \rho''_Y)$ it is not hard to see that $\forall v \in TY, \forall \Phi \in S^+_X \mid_Y$

$$\begin{aligned} &\mathfrak{S}''_Y \left[\rho_X(n_X)^{-1}\rho_X(v)\Phi\right] \\ &= \mathfrak{S} \circ \mathfrak{S}_Y \left[\rho_X(n_X)^{-1}\rho_X(v)\Phi\right] \\ &= \mathfrak{S}\left[\rho_Y(v)\mathfrak{S}_Y(\Phi)\right] \\ &= \rho''_Y(v)\mathfrak{S} \circ \mathfrak{S}_Y(\Phi) \\ &= \rho''_Y \mathfrak{S}''_Y(\Phi) \end{aligned}$$

In other words, the definition of relative spinor bundle $(S_X, \rho_X, \mathfrak{S})$ depends on (S_Y, ρ_Y) only in terms of the abstract spin-c structure $\mathfrak{s}_Y$. Therefore, it makes sense to talk about **a relative spinor bundle** $(S_X, \rho_X, \mathfrak{S}_Y)$ **associated to an abstract spin-c structure** $\mathfrak{s}_Y$ and we can now define a **relative spin-c structure** as follows:

Definition 36. Let X be a four compact with boundary Y and choose an abstract spin-c structure $\mathfrak{s}_Y$. A **relative spin-c structure** $(\mathfrak{s}_X, [\mathfrak{S}_Y])$ **associated to** $\mathfrak{s}_Y$ is an equivalence class of triples $(S_X, \rho_X, \mathfrak{S}_Y)$ where $(S_X, \rho_X, \mathfrak{S}_Y) \sim (S_X,' \rho'_X, \mathfrak{S}'_Y)$ if the following are satisfied:

i) $(S_X, \rho_X, \mathfrak{S}_Y)$ is a relative spinor bundle associated to $\mathfrak{s}_Y$.

ii) (S_X, ρ_X) and (S'_X, ρ'_X) are instantiations of the same abstract spin-c structure $\mathfrak{s}_X$.

iii) The isomorphisms $\mathfrak{S}'_Y$ and $\mathfrak{S}_Y$ are related by $\mathfrak{S}'_Y \equiv \mathfrak{S}_Y \circ (\mathfrak{S}'_X \mid_Y)$ where $\mathfrak{S}'_X : (S'_X, \rho'_X) \to (S_X, \rho_X)$ is the usual inter-twiner map.

In the case that Y is equipped with a contact structure ξ we define

$$\text{Spin}^c(X, \xi) = \{\text{relative spin-c structures } (\mathfrak{s}_X, [\mathfrak{S}_Y]) \text{ associated to } \mathfrak{s}_\xi)\}$$

In practice, we will think of an element $(\mathfrak{s}_X, [\mathfrak{S}_Y]) \in \text{Spin}^c(X, \xi)$ as being represented by an instantiation $(S_X, \rho_X, \mathfrak{S}_Y)$ where $\mathfrak{S}_Y : (S_X, \rho_X) \mid_{\partial X} \to (S_\theta, \rho_\theta)$ is an isomorphism between the spinor bundle (S_X, ρ_X) induces on the boundary and the instantiation (S_θ, ρ_θ) associated to $\mathfrak{s}_\xi$ which is determined by the choice of a contact form θ for ξ. The following lemma clarifies the structure of $\mathrm{Spin}^c(X, \xi)$.

Lemma 37. *The set Spin$^c(X, \xi)$ is a principal homogeneous space for the relative cohomology $H^2(X, \partial X; \mathbb{Z})$.*

Proof. We follow a similar strategy to the one used in determining $\mathrm{Spin}^c(X)$. Let $(S_X, \rho_X, \mathfrak{S}_Y)$ be a relative spinor bundle associated to $(\mathfrak{s}_X, [\mathfrak{S}_Y])$ and choose $e \in H^2(X; \partial X; \mathbb{Z})$. In particular we can choose a line bundle L_e over X such that L_e is trivial over ∂X, i.e, there is a trivialization $\Upsilon_e : L_e \to \partial X \times \mathbb{C}$ which we can take to be unitary.

Using the long-exact sequence in cohomology for the pair $(X, \partial X)$ we have (section 3.1 [23])

$$\begin{array}{ccccccccc} \cdots \to & H^1(X) \to & H^1(\partial X) \xrightarrow{\delta} & H^2(X; \partial X) \xrightarrow{j^*} & H^2(X) \xrightarrow{i^*} & H^2(\partial X) \to \cdots \\ & & & e & \tilde{e} & 0 \end{array}$$

where we take $\tilde{e} = j^*(e)$.

Therefore we define $(\mathfrak{s}_e, [\mathfrak{S}_e])$ as the equivalence class of the relative spinor bundle $(S_e, \rho_e, \mathfrak{S}_e)$ where $S_e = S_X \otimes L_e$, $\rho_e = \rho_X \otimes 1_{L_e}$ and $\mathfrak{S}_e : (S_e, \rho_e) \mid_{\partial X} \to (S_Y, \rho_Y)$ is given for all

$$\Phi \otimes \sigma_e \in \Gamma(S_e) = \Gamma(S_X^+ \mid_Y) \otimes \Gamma(L_e \mid_Y)$$

by

$$\mathfrak{S}_e(\Phi \otimes \sigma_e) \equiv \Upsilon_e(\sigma_e)\mathfrak{S}_Y(\Phi)$$

where we are using the trivialization Υ_e so that the previous multiplication makes sense globally on ∂X. We need to verify that

$$\forall v \in TY, \qquad \mathfrak{S}_e\left[\rho_e(n_X)^{-1}\rho_e(v)(\Phi \otimes \sigma_e)\right] = \rho_Y(v)\mathfrak{S}_e(\Phi \otimes \sigma_e)$$

For this we simply start with the left hand side

$$\begin{aligned}
&\mathfrak{S}_e\left[\rho_e(n_X)^{-1}\rho_e(v)(\Phi\otimes\sigma_e)\right]\\
&=\mathfrak{S}_e\left[\rho_e(n_X)^{-1}(\rho_X(v)\Phi\otimes\sigma_e)\right]\\
&=\mathfrak{S}_e\left(\rho_X(n_X)^{-1}\rho_X(v)\Phi\otimes\sigma_e\right)\\
&=\Upsilon_e(\sigma_e)\mathfrak{S}_Y\left(\rho_X(n_X)^{-1}\rho_X(v)\Phi\right)\\
&=\Upsilon_e(\sigma_e)\rho_Y(v)\mathfrak{S}_Y(\Phi)\\
&=\rho_Y(v)\mathfrak{S}_e\left(\Phi\otimes\sigma_e\right)
\end{aligned}$$

which is what we wanted to show.

Conversely, suppose that we start with two relative spinor bundles $(S_X, \rho_X, \mathfrak{S}_Y)$ and $(S'_X, \rho'_X, \mathfrak{S}'_Y)$ associated to $\mathfrak{s}_Y$, which is instantiated by (S_Y, ρ_Y), so that we regard $\mathfrak{S}_Y$ and $\mathfrak{S}'_Y$ as maps $\mathfrak{S}_Y : (S_X, \rho_X) \mid_{\partial X} \rightarrow (S_Y, \rho_Y)$, $\mathfrak{S}'_Y : (S'_X, \rho'_X) \mid_{\partial X} \rightarrow (S_Y, \rho_Y)$. Consider the subbundle L_{S_X, S'_X} of $\hom(S_X, S'_X)$ of homomorphisms $\mathfrak{S} : S_X \rightarrow S'_X$ which intertwine ρ_X, ρ'_X. Our construction is set up in such a way that L_{S_X, S'_X} is trivializable along the boundary. Therefore we obtain a well defined element $e \equiv c_1(L_{S_X, S'_X}) \in H^2(X, \partial X; \mathbb{Z})$ and it is not difficult to see that $(S_e, \rho_e, \mathfrak{S}_e)$ will be in the same equivalence class as $(S'_X, \rho'_X, \mathfrak{S}'_Y)$. $\square$

If $(W, \omega) : (Y, \xi) \rightarrow (Y', \xi')$ is a strong cobordism we can define a map

$$\jmath : \mathrm{Spin}^c(X, \xi) \rightarrow \mathrm{Spin}^c(X \cup W, \xi')$$

as follows [40, Section 1.2] : if $(S_X, \rho_X, \mathfrak{S}_Y)$ is a relative spinor bundle associated to $(\mathfrak{s}_X, [\mathfrak{S}_Y]) \in \mathrm{Spin}^c(X, \xi)$ there is an isomorphism

$$\mathfrak{S}_Y : (S_X, \rho_X) \mid_Y \rightarrow (S_\theta, \rho_\theta)$$

as explained previously. It satisfies in particular that

$$\forall v \in TY, \forall \Phi \in S_X^+ \mid_Y \qquad \mathfrak{S}_Y\left[\rho_X(n_X)^{-1}\rho_X(v)\Phi\right] = \rho_\theta(v)\mathfrak{S}_Y(\Phi)$$

The spinor bundle (S_θ, ρ_θ) on Y induces the natural spinor bundle $(S_\theta, -\rho_\theta)$ on $-Y$ which we now we can identify with $(S_\omega, \rho_\omega) \mid_{-Y}$ via a map

$$\mathfrak{S}_\theta : (S_\theta, -\rho_\theta) \rightarrow (S_\omega, \rho_\omega) \mid_{-Y}$$

which satisfies (here we are identifying TY with $T(-Y)$ for notational convenience):

$$\forall v \in TY, \forall \Phi \in S_\theta \qquad \mathfrak{S}_\theta\left[-\rho_\theta(v)\Phi\right] = \rho_W(n_W)^{-1}\rho_W(v)\mathfrak{S}_\theta(\Phi)$$

Define the relative spinor bundle $(S_{X\cup W}, \rho_{X\cup W}, \mathfrak{S}_{Y'})$ as the one obtained from S_X and S_ω using the transition map

$$\mathfrak{S}_\theta \circ \mathfrak{S}_Y : (S_X, \rho_X) \mid_Y \to (S_\omega, \rho_\omega) \mid_{-Y}$$

We take $\jmath(\mathfrak{s}_X, [\mathfrak{S}_Y])$ as the equivalence class defined by the relative spinor bundle $(S_{X\cup W}, \rho_{X\cup W}, \mathfrak{S}_{Y'})$.

To consider the injectivity of the map $\jmath$, suppose that $\jmath(\mathfrak{s}_X, [\mathfrak{S}_Y]) = \jmath(\mathfrak{s}'_X, [\mathfrak{S}'_Y]) \in \mathrm{Spin}^c(X\cup W, \xi')$. According to our definition this means that if we choose relative spinor bundles $(S_{X\cup W}, \rho_{X\cup W}, \mathfrak{S}_{Y'})$ and $(S'_{X\cup W}, \rho'_{X\cup W}, \mathfrak{S}'_{Y'})$ then

• $(S_{X\cup W}, \rho_{X\cup W})$ and $(S'_{X\cup W}, \rho'_{X\cup W})$ are instantiations of the same abstract spin-c structure $\mathfrak{s}_{X\cup W}$. Restricting the inter-twining map to X this means that $\mathfrak{s}_X = \mathfrak{s}'_X$ as abstract spin-c structures over X. Notice we still can't conclude that they define the same relative spin-c structure because it is not clear that condition $iii)$ in our definition is already satisfied.

• However, we still do have that $\mathfrak{S}'_{Y'} \equiv \mathfrak{S}_{Y'} \circ (\mathfrak{S}'_{X\cup W} \mid_{Y'})$ where $\mathfrak{S}'_{X\cup W} : (S'_{X\cup W}, \rho'_{X\cup W}) \to (S_{X\cup W}, \rho_{X\cup W})$ is the inter-twining map. Since the restrictions $(S_{X\cup W}, \rho_{X\cup W}) \mid_W = (S_\omega, \rho_\omega)$ and $(S'_{X\cup W}, \rho'_{X\cup W}) \mid_W = (S_\omega, \rho_\omega)$ differ by an automorphism of the same concrete spinor bundle S_ω we obtain a map $u : W \to S^1$. Along Y, if $v \in TY$ and n_X is the outward normal vector we the trivialization provided by u is such that we may regard $S^+_X \mid_Y = S^{+\prime}_X \mid_Y$ and $\rho_X \mid_Y$, $\rho'_X \mid_Y$ as related by

$$\rho_X(n_X)^{-1}\rho_X(v)\Phi = \rho_X(n_X)^{-1}\rho_X(v)(u\Phi)$$

If it could extend $u \mid_Y$ to the rest of X then we could take S_X and S'_X as representing the same concrete spinor bundle and in this way, $(S_X, \rho_X, \mathfrak{S}_Y)$ and $(S'_X, \rho'_X, \mathfrak{S}'_Y)$ would give rise to the same relative spin-c structure, i.e, $(\mathfrak{s}_X, [\mathfrak{S}_Y]) = (\mathfrak{s}'_X, [\mathfrak{S}'_Y])$. From the long exact sequence

$$\begin{array}{ccccccccccc} \cdots \to & H^1(X) & \xrightarrow{i^*} & H^1(Y) & \xrightarrow{\delta} & H^2(X;Y) & \xrightarrow{j^*} & H^2(X) & \xrightarrow{i^*} & H^2(Y) & \to \cdots \\ & & & [u \mid_Y] & & & & & & & \end{array}$$

we see that it suffices for $\delta([u\mid_Y])$ to vanish. Notice that in general this does need to happen. However, for the proof of naturality we will take $X = \mathbb{R}^+ \times -Y$ and since $H^2(\mathbb{R}^+ \times -Y; Y) = 0$ in particular this means that $\delta([u\mid_Y])$ vanishes automatically so the problem of extending $u\mid_Y$ disappears, hence the map $\jmath$ is automatically injective. In particular, all the arguments in [40] where the injectivity of $\jmath$ was invoked can be used for our case without any concerns and this explains why the "special" condition can be dropped.

6.2 Connected Sum Along Y. We will now adapt the gluing/ excision theorem in [40] to our situation. More precisely we want an analogue of their corollary 3.2.2. The following construction is based on sections 4.1 and 2.1.5 from that paper. There they proved a gluing result for a class of manifolds with a so called *AFAK end Z,* that is, an asymptotically flat almost Kahler end, the idea being that this class of manifolds behave sufficiently nice near the symplectic end that all the analysis goes through. Their definition of an AFAK end is given in Definition 2.1.2 of [40]. The important things that we need to point out regarding this definition is that:

- Their last condition regarding the vanishing of map between de Rham cohomologies mimics the special condition for a symplectic cobordism that we already discussed before. Therefore, in our context this condition is not needed.
- To our cobordism (W, ω) one can associated an AFAK end (Z, ω_Z) as explained in section 4.1 of [40] . We can simplify their construction in our case because our cobordism is strong so in fact we can exploit the fact that near the convex end ω is also determined by a symplectization of the contact structure. We can use a collar neighborhood $[T_0, T_1) \times Y$ of $Y \subset \partial W$ (with $T_0 > 1$) and a contact form θ such that the symplectic form ω near the concave end of that neighborhood is given by $\frac{1}{2}d(t^2\theta)$. Glue a sharp cone on the boundary Y by extending the collar neighborhood into $(0, T_1) \times Y$ with its symplectic form. Likewise, we have a similar collar neighborhood near the convex end and we can therefore glue (after some reparameterizations) the half-infinite cone $[1, \infty) \times Y'$ with the symplectic form $\frac{1}{2}d(t'^2\theta')$ where t' denotes the time coordinate on $[1, \infty) \times Y'$. Therefore

$$Z = ((0, T_0) \times Y) \cup W \cup ([1, \infty) \times Y')$$

and we can find a "time coordinate" σ_Z on Z as they described in that section of [40] (in fact, after reparametrization in can be taken to agree with the natural time coordinate on the third factor $[1, \infty) \times Y'$ of Z). Among other properties:

- On $(0, T_0) \times Y$, σ_Z agrees with the time coordinate.
- For all $\epsilon > 0$, the function $e^{-\epsilon \sigma_Z}$ is integrable on Z.
- There is a constant $\kappa > 0$ such that the injectivity radius satisfies $\kappa \mathrm{inj}(x) > \sigma_Z(x)$ for all $x \in Z$.
- For each $x \in Z$, if e_x is the map $e_x : v \to \exp_x(\sigma_Z(x)v/\kappa)$ and γ_x the metric on the unit ball in T_xZ defined as $e_x^* \gamma_x / \sigma_Z(x)^2$, then these metrics have bounded geometry in the sense that all covariant derivatives of the curvature are bounded by some constants independent of x.
- For each $x \in Z$, if o_x denotes the symplectic form $e_x^* \omega_Z / \sigma(x)^2$ on the unit ball, then o_x similarly approximates the translation invariant symplectic form, with all its derivatives.

Notice in particular that our symplectic form ω_Z has the property that it is exact except for a compact set (which is contained in W). Hence the class of manifolds we are using could be called *AFAKAE ends* (where *AE* stands for almost exact) but for convenience we will keep calling this manifold an AFAK end. After choosing a metric g_Z and almost complex structure J_Z on Z so that ω_Z is self-dual and of pointwise norm $\sqrt{2}$ the data $(Z, \omega_Z, J_Z, g_Z, \sigma_Z)$ will represent an AFAK end with the caveats mentioned above.

This is the class of manifolds to which the generalized excision/gluing theorem will apply, though the theorem will only be used for this particular Z. The idea will be to glue Z to the cylindrical end $\mathbb{R}^+ \times -Y$ using an operation that Mrowka and Rollin named **connected sum along** Y.

To be more precise, consider as before the symplectic cone $[1, \infty) \times Y$ for the contact form θ with metric

$$g_{K,\theta} = dt \otimes dt + t^2 g_\theta$$

and symplectic form

$$\omega_\theta = \frac{1}{2} d(t^2 \theta)$$

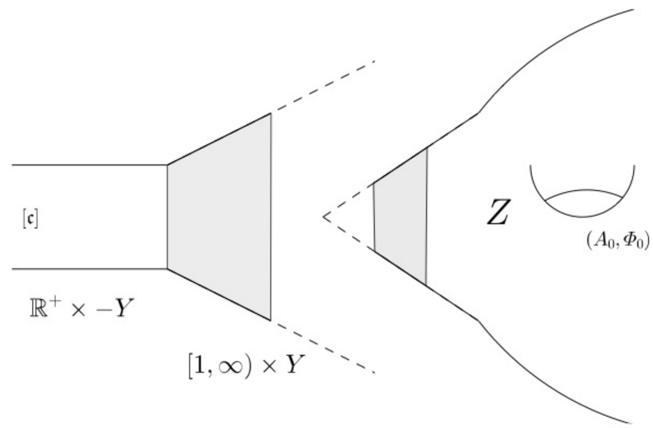

FIGURE 8. Using the "connected sum along Y" operation to obtain the family of manifolds M_τ.

Choose a number $\tau > 1$ [3] and identify an annulus $(1,\tau)\times Y$ in $[1,\infty)\times Y$ with an annulus $(1/\tau, 1)\times Y \subset Z$ using the dilation map

$$\begin{aligned} (1,\tau)\times Y &\xrightarrow{\nu_\tau} (1/\tau,1)\times Y \\ (t,y) &\to (t/\tau, y) \end{aligned}$$

Define M_τ as the union of $(\mathbb{R}^+ \times -Y)\cup[1,\tau)\times Y$ and $Z\cap\{\sigma_Z > 1/\tau\}$

$$M_\tau = \left(\left(\mathbb{R}^+ \times -Y\right)\cup[1,\tau)\times Y)\right)\cup\left(Z\cap\{\sigma_Z>1/\tau\}\right)$$

glued along the previous annuli via the dilation map ν_τ.

In the figure, the gray regions represent the annuli that are identified and the dashed regions are the parts of the cone and Z that are taken off in the construction. We need to say how to redefine the geometric structures we had in place (metric, symplectic form, etc) so that they agree under the identification operation (this is why we discussed the effect of a dilation on the canonical spin-c structure in the

[3]It goes without saying that the τ is completely unrelated to the τ used in the τ-model of the configuration space.

previous section). The symplectic form can be taken as

$$\omega_{Z,\tau} = \tau^2 \omega_Z$$

and the new "time coordinate" becomes

$$\sigma_{\tau,Z} = \tau \sigma_Z$$

The metric is a dilation of the original metric, that is,

$$g_{\tau,Z} = \tau^2 g_Z$$

In this way, with respect to $g_{\tau,Z}$, $\omega_{Z,\tau}$ is self-dual with norm $\sqrt{2}$. As usual, g_Z and ω_Z determine a compatible almost complex structure $J_{Z,\tau}$ which in fact can be taken to be independent of τ, i.e,

$$J_{Z,\tau} = J_Z$$

The natural Clifford multiplication is

$$\rho_{\tau,Z}(\eta) = \frac{\rho_Z(\eta)}{\tau} \quad \eta \text{ a one form}$$

while the spinor bundle remains the same, i.e

$$S_{\tau,Z} = S_Z$$

We will specify the abstract spin-c structure $\mathfrak{s}_\tau$ on M_τ as the equivalence class of the following spinor bundle (S_τ, ρ_τ):

• On $\mathbb{R}^+ \times -Y$ we use the spinor bundle $(S_{\mathbb{R}^+\times -Y}, \rho_{\mathbb{R}^+\times -Y})$ that the canonical spinor bundle (S_θ, ρ_θ) on Y induces on $\mathbb{R}^+ \times -Y$.

• Along the boundary, we identify $(S_{\mathbb{R}^+\times -Y,\theta}, \rho_{\mathbb{R}^+\times -Y,\theta})\,|_Y$ with $(S_{K_Y}, \rho_{K_Y})\,|_{\{1\}\times Y}$ where (S_{K_Y}, ρ_{K_Y}) denotes the canonical spinor bundle associated to the symplectic cone $K_Y = [1,\infty) \times Y$.

• Over $M_\tau \cap \{\sigma_{\tau,Z} < \tau\} = M_\tau \cap \{\sigma_Z < 1\} = M_\tau \cap \{(1,\tau) \times Y\}$ we use the spinor bundle (S_{K_Y}, ρ_{K_Y}) .

• Over $M_\tau \cap \{\sigma_{\tau,Z} > 1\} = M_\tau \cap \{\sigma_Z > 1/\tau\}$ we use the spinor bundle $(S_{\tau,Z}, \rho_{\tau,Z}) = \left(S_{\tau,Z}, \frac{\rho_Z}{\tau}\right)$.

To write the transition map from (S_{K_Y}, ρ_{K_Y}) to $(S_{\tau,Z}, \rho_{\tau,Z})$ over $M_\tau \cap \{1/\tau < \sigma_Z < 1\}$ observe that if e^1_Y, e^2_Y, e^3_Y is a coframe at the slice $\{1\} \times Y \simeq Y$ then

$dt, te^1_Y, te^2_Y, te^3_Y$ is a coframe on $(1,\tau)\times Y \subset K_Y$ while $\tau dt, \tau te^1_Y, \tau te^2_Y, \tau te^3_Y$ is a coframe on $\{1/\tau < \sigma_Z < 1\} \subset Z$. Therefore we can define as before $\bar{\epsilon}^{01}_t = \frac{1}{\sqrt{2}}(dt - ite^1_Y)$, $\bar{\epsilon}^{23}_t = \frac{t}{\sqrt{2}}(e^2_Y - ie^3_Y)$ and the identification map

$$\begin{aligned} \mathfrak{G}_\tau : S_{K_Y} &\to S_{\tau,Z} \\ \alpha_{K_Y} + \beta_{K_Y}\bar{\epsilon}^{01}_t \wedge \bar{\epsilon}^{23}_t &\to \alpha_{K_Y} + \tau^2\beta_{K_Y}\bar{\epsilon}^{01}_t \wedge \bar{\epsilon}^{23}_t \end{aligned}$$

Remark 38. In the case of [40] , their construction required (in their notation) the choice of an element $(\mathfrak{s}, h) \in \mathrm{Spin}^c(M,\omega)$ (see section 2.1.7). As we explained before, by using a half infinite cylinder instead of a compact piece, all of our constructions can be done in a canonical way, which is why our construction is more simple in a sense and we can drop the explicit reference to h.

Our (unperturbed) Seiberg Witten map continues to be

$$\mathfrak{F}(A,\Phi) = \left(\frac{1}{2}\rho(F^+_{A^t}) - (\Phi\Phi^*)_0, D_A\Phi\right)$$

To define the perturbations, write the half-infinite cylinder as

$$\mathbb{R}^+ \times -Y = ([0,1]\times -Y) \cup ([1,\infty)\times -Y)$$

where $[0,1]\times -Y$ is going to play the role of a trivial cobordism. By that we simply mean that the perturbations we use on $[0,1]\times -Y$ are of the form $\hat{\mathfrak{p}} = \beta\hat{\mathfrak{q}} + \beta_0\hat{\mathfrak{p}}_0$ where $\hat{\mathfrak{p}}$ coincides near $\{1\}\times -Y$ with a strongly tame perturbation $-\hat{\mathfrak{q}}_{Y,g_\theta,\mathfrak{s}_\xi}$ on $[1,\infty)\times -Y$ and near $\{0\}\times -Y$ it vanishes. On

$$Z_\tau = [1,\tau)\times Y \cup (Z \cap \{\sigma_Z > 1/\tau\})$$

consider the perturbation

$$\mathfrak{p}_{Z_\tau} = -\frac{1}{2}\rho_\tau(F^+_{A^t_{0,\tau}}) + (\Phi_{\tau,0}\Phi^*_{\tau,0})_0$$

where $(A^t_{0,\tau}, \Phi_{\tau,0})$ denotes the canonical solution. Again, similar to the perturbation $\mathfrak{p}_{W^+_{\xi',Y}}$ defined in equation (11) we can produce a perturbation

$$\mathfrak{p}_{M_\tau} = -\hat{\mathfrak{q}}_{Y,g_\theta,\mathfrak{s}_\xi} + (\beta\hat{\mathfrak{q}}_{Y,g_\theta,\mathfrak{s}_\xi} + \beta_0'\hat{\mathfrak{p}}_0) + \beta_K\mathfrak{p}_{Z_\tau} \tag{47}$$

It is also useful to think of the manifold $Z^+_{Y,\xi}$ [where the contact invariant $\mathbf{c}(\xi)$ of (Y,ξ) is defined] as the manifold M_τ obtained by taking '$\tau = \infty$'. In other words, we will write $M_\infty \equiv Z^+_{Y,\xi}$. Notice that on this manifold we can also define a perturbation $\mathfrak{p}_{M_\infty}$ in exactly the same way as for $\mathfrak{p}_{M_\tau}$ (so it agrees with $-\hat{\mathfrak{q}}_{Y,g_\theta,\mathfrak{s}_\xi}$ on half-cylinder $[1,\infty)\times -Y$, it agrees with $\mathfrak{p}_K$ on the cone $[1,\infty)\times Y$ and it is interpolated between these two perturbations on the finite cylinder $[0,1]\times -Y$ through a perturbation $\beta\hat{\mathfrak{q}}_{Y,g_\theta,\mathfrak{s}_\xi} + \beta_0'\hat{\mathfrak{p}}_0$).

Our previous transversality Theorem (25) now reads as follows: for all critical points $[\mathfrak{c}] \in \mathfrak{C}^o(-Y,\mathfrak{s}_\xi) \oplus \mathfrak{C}^s(-Y,\mathfrak{s}_\xi)$ and for each $0 < \tau \leq \infty$ there is a residual subset $\mathcal{P}_\tau$ of the large space of perturbations $\mathcal{P}(Y,\mathfrak{s}_\xi)$ such that for any $\mathfrak{p}_\tau \in \mathcal{P}_\tau$ the corresponding perturbation $\mathfrak{p}_{M_\tau}$ satisfies the property that all the moduli spaces $\mathcal{M}(M_\tau,\mathfrak{s}_\tau,[\mathfrak{c}],\mathfrak{p}_{M_\tau})$ are cut out transversely.

When we study the properties of the gluing map it will become clear that we want to be able to choose a single perturbation $\mathfrak{p}_{all}$ such that when we plug it in the formula for $\mathfrak{p}_{M_\tau}$ it guarantees transversality *simultaneously* for all moduli spaces $\mathcal{M}(M_\tau,\mathfrak{s}_\tau,[\mathfrak{c}],\mathfrak{p}_{M_\tau})$. In other words, we would like to be able to choose a perturbation $\mathfrak{p}_{all} \in \bigcap_{0<\tau\leq\infty}\mathcal{P}_\tau$. Since we are ultimately interested in the case when τ is sufficiently large we can choose an increasing sequence τ_n with $\tau_n \to \infty$ and then use the fact that the countable intersection of residual sets is residual [43, Theorem 1.4] so that $(\cap_n \mathcal{P}_{\tau_n}) \cap \mathcal{P}_\infty$ is residual as well. In particular this means that we can take $\mathfrak{p}_{all} \in (\cap_n \mathcal{P}_{\tau_n}) \cap \mathcal{P}_\infty$, which we will assume from now on.

Strictly speaking, since we will work with an additional family M'_τ obtained by using another connected operation with another AFAK end Z' we should really take $\mathfrak{p}_{all} \in (\cap_n \mathcal{P}_{\tau_n}) \cap \mathcal{P}_\infty \cap \big(\cap_n \mathcal{P}'_{\tau_n}\big)$, where $\mathcal{P}'_{\tau_n}$ denotes the residual space of perturbations for the manifold M'_τ. However, for the proof of the gluing theorem we will end up taking $Z' = (0,\infty)\times Y$ (as in section 4.1 of [40]), in which case one can check that *all* the M'_τ end up coinciding with $Z^+_{Y,\xi} = M_\infty$. Hence, this subtle points does not make much a difference. Also, for notational convenience, we will keep writing the moduli spaces typically as $\mathcal{M}(M_\tau,\mathfrak{s}_\tau,[\mathfrak{c}])$ instead of $\mathcal{M}(M_{\tau_n},\mathfrak{s}_{\tau_n},[\mathfrak{c}])$.

6.3 Gluing Map. Our main objective in this section is to adapt Theorem 3.1.9 in [40] to our situation. First we need to define a pregluing map that allows us to compare solutions in the moduli spaces corresponding to the manifolds M_τ and

M'_τ. This will then be promoted to an actual gluing map which basically says that once τ becomes sufficiently large the Seiberg-Witten solutions on M_τ are in bijective correspondence with the Seiberg-Witten solutions on M'_τ (the precise statement is Theorem 44).

As can be seen from Figure (8) and the definition of the manifold Z, one should think of the manifolds M_τ as being diffeomorphic versions of the manifold $W^+_{\xi',Y}$ described in Figure (2). The moduli space of Seiberg-Witten equations over each of the M_τ gives rise to a "τ-hybrid" invariant $c(\xi', Y, \tau) \in \check{C}_*(-Y, \mathfrak{s}_\xi)$, but a standard deformation of metrics and perturbations argument which is explained at the end of the paper tells us that in fact they all define the same homology class $\mathbf{c}(\xi', Y, \tau) = \mathbf{c}(\xi', Y)$, where the right hand side denotes our original "hybrid" invariant. On the other hand, when we take $Z' = (0, \infty) \times Y$, the resulting manifolds M'_τ agree with $Z^+_{Y,\xi}$ as mentioned at the end of the previous section. Therefore, from the moduli space of Seiberg-Witten equations over M'_τ we obtain the ordinary contact invariant $\mathbf{c}(\xi)$ and then the our gluing argument will imply that these two invariants agree.

We write $(M_\tau, g_\tau, \omega_\tau, J_\tau, \sigma_\tau)$ and $(M'_\tau, g'_\tau, \omega'_\tau, J'_\tau, \sigma'_\tau)$ to make explicit the data required in our construction. Notice that on the domains $\{\sigma_\tau \leq \tau\} \subset M_\tau$ and $\{\sigma'_\tau \leq \tau\} \subset M'_\tau$ all the previous structures agree (including the spinor bundles and the canonical solutions). In fact, we can regard these regions as subsets of $Z^+_{Y,\xi}$.

Let (A, Φ) be a solution of the Seiberg-Witten equations on M_τ. We want to transport (A, Φ) into an approximate solution on M'_τ.

First we need to construct a spinor bundle $S_{(A,\Phi)}$ associated to (A, Φ) on M'_τ. To be more precise, as an abstract spin-c structure the construction that we provide is independent of the solution (A, Φ) that we use, but the particular instantiation will depend on the solution since it will be used to define a transition function.

Using Lemma 2.2.8 in [40] we can find a compact set C with the following significance: for every τ large enough and for every solution to the Seiberg-Witten equations on M_τ we have $|\alpha| \geq \frac{1}{2}$ on $M_\tau \backslash [(\mathbb{R}^+ \times -Y) \cup C]$ (recall that $\Phi = (\alpha, \beta)$ and that the paper [40] writes it instead as (β, γ)). We may write C as $C = [1, T] \times Y \subset Z^+_{Y,\xi}$ where T is large enough and independent of τ and the solution (A, Φ). **From now on we will assume that τ is chosen so that it is larger than T.**

For $\tau > T$, we construct the spinor bundle $S'_{(A,\Phi)}$ on M'_τ as follows ([40] named this spinor bundle $S_{(A,\Phi)}$):

(1) Over the region $M'_\tau \cap \{\sigma'_\tau \leq \tau\} \subset Z^+_{Y,\xi}$, we use the spinor bundle S_ξ determined by ξ. Over the region $M'_\tau \cap \{\sigma'_\tau \geq T\}$, we use the spinor bundle determined by the almost complex structure J'_τ, i.e, S'_{J_τ}. In other words

$$S'_{(A,\Phi)} = \begin{cases} S_\xi & \text{over } M'_\tau \cap \{\sigma'_\tau \leq \tau\} \\ S_{J'_\tau} & \text{over } M'_\tau \cap \{\sigma'_\tau \geq T\} \end{cases}$$

(2) To specify what happens over the annulus $\{T \leq \sigma'_\tau \leq \tau\}$ define the map (gauge transformation) [4]

$$h_{(A,\Phi)} : M_\tau \backslash [(\mathbb{R}^+ \times -Y) \cup C] \to S^1$$
$$h_{(A,\Phi)} = \tfrac{|\alpha|}{\alpha}$$

If $\tilde{\Phi} \in \Gamma(S_{\xi | M'_\tau \cap \{\sigma'_\tau \leq \tau\}})$, then over the annulus $\{T \leq \sigma'_\tau \leq \tau\}$ we can write with respect to a coframe

$$\tilde{\Phi} = \tilde{\alpha} + \tilde{\beta}\bar{\epsilon}^{01} \wedge \bar{\epsilon}^{23}$$

and if we write $h_{(A,\Phi)} \cdot \tilde{\Phi} = \left(\frac{|\alpha|}{\alpha}\right) \tilde{\Phi}$ as

$$h_{(A,\Phi)} \cdot \tilde{\Phi} = \tilde{\alpha}_\Phi + \tilde{\beta}_\Phi \bar{\epsilon}^{01} \wedge \bar{\epsilon}^{23}$$

then we will consider $h_{(A,\Phi)}$ as the transition map from S_ξ to $S_{J'_\tau}$ over the annulus. That is, the section $\tilde{\Phi}$ over $S_\xi \mid_{\{T \leq \sigma'_\tau \leq \tau\}}$ is identified with the section $h_{(A,\Phi)} \cdot \tilde{\Phi}$ over $S_{J'_\tau} \mid_{\{T \leq \sigma'_\tau \leq \tau\}}$ as the next image indicates.

Notice that if $u \in \mathcal{G}(M_\tau)$ then $h_{u \cdot (A,\Phi)} = u^{-1} h_{(A,\Phi)}$ and so we can easily build an isomorphism

$$u^\# : S_{(A,\Phi)} \to S_{u \cdot (A,\Phi)}$$

Our next job is to construct a configuration on the spinor bundle $S'_{(A,\Phi)}$ over M'_τ. For this we recall a family of cut-off functions described in section 2.2.1 of [40] . Let

[4]Here we do not use the notation $h'_{(A,\Phi)}$ that can be found in [40] , since our isomorphism h is already canonical and so there is no need to distinguish $h'_{(A,\Phi)}$ from $h_{(A,\Phi)}$.

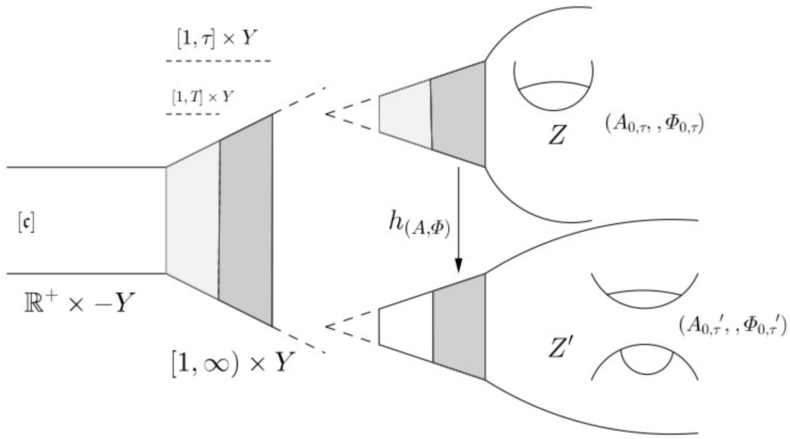

FIGURE 9. Defining the spinor bundle $S'_{(A,\Phi)}$ over M'_τ .

$\chi(t)$ be a smooth decreasing function such that

$$\chi(t)=\begin{cases}0 & t\geq 1\\ 1 & t\leq 0\end{cases}$$

and define

$$\chi_\tau(t)=\chi\left(\frac{t-\tau}{N_0}+1\right)=\begin{cases}0 & t\geq \tau\\ 1 & t\leq \tau-N_0\end{cases}$$

where N_0 is a number that is fixed later to control the derivatives of χ_τ. With the help of this function define $S'_{(A,\Phi)}$ as follows:

- On the region $M'_\tau\cap\{\sigma'_\tau<\tau\}$ we can identify the structures on M_τ with M'_τ and so $(A,\Phi)\mid_{M_\tau\cap\{\sigma_\tau<\tau\}}$ defines a configuration on $S'_{(A,\Phi)}\mid_{M'_\tau\cap\{\sigma'_\tau\leq\tau\}}$.

• On the region $M'_\tau \cap \{\sigma'_\tau \geq T\}$ we can write Φ as a pair (α, β) and A as $A = A_{0,\tau} + a$ so if we regard $h_{(A,\Phi)}$ as a gauge transformation then

$$\begin{aligned} &h_{(A,\Phi)} \cdot (A, \Phi) \\ &= h_{(A,\Phi)} \cdot (A_{0,\tau} + a, (\alpha, \beta)) \\ &= \left(A_{0,\tau} + a - \frac{\alpha}{|\alpha|} d\left(\frac{|\alpha|}{\alpha}\right), \left(|\alpha|, \frac{|\alpha|}{\alpha}\beta\right)\right) \\ &= \left(A_{0,\tau} + \hat{a}, (\hat{\alpha}, \hat{\beta})\right) \end{aligned}$$

Notice that $\hat{\alpha}$ is a real function with $\hat{\alpha} \geq \frac{1}{2}$. Therefore we define on $M'_\tau \cap \{\sigma'_\tau \geq T\}$

$$(A, \Phi)^\# \equiv (A'_{0,\tau} + (\chi_\tau \circ \sigma'_\tau)\hat{a}, (\hat{\alpha}^{\chi_\tau \circ \sigma'_\tau}, (\chi_\tau \circ \sigma'_\tau)\hat{\beta}))$$

• On the end $\{\sigma'_\tau \geq \tau\}$ we set

$$(A, \Phi)^\# = (A'_{0,\tau}, \Phi'_{0,\tau})$$

Since the construction is compatible with the gauge group action in the sense that

$$u^\# \cdot (A, \Phi)^\# = (u \cdot (A, \Phi))^\#$$

we have constructed our pregluing map

$$\# : \mathcal{M}(M_\tau, \mathfrak{s}_\tau, [\mathfrak{c}]) \to (\mathcal{C}/\mathcal{G})(M'_\tau)$$

It is easy to see that Lemma 2.5.4 in [40] still holds. That is, there is a $\delta > 0$ and T large enough such that for every $N_0 \geq 1, k \in \mathbb{N}$, τ satisfying $\tau \geq T + N_0$ and every solution (A, Φ) of the Seiberg-Witten equations on M_τ, we have that $(A, \Phi)^\#$ satisfies the Seiberg-Witten equations on $\{\sigma'_\tau \leq T\} \subset M'_\tau$ and

$$|\mathfrak{F}_{\mathfrak{p}_{M'_\tau}}(A, \Phi)^\#|_{C^k(g'_\tau, A^\#)} \leq c_k e^{-\delta \sigma_\tau} \tag{48}$$

on $\{\sigma'_\tau \geq T\} \subset M'_\tau$.

Our objective now is to modify the pre-gluing map $\# : M(M_\tau, \mathfrak{s}_\tau, [\mathfrak{c}]) \to (\mathcal{C}/\mathcal{G})(M'_\tau)$ to obtain a gluing map (Theorem 3.1.9 [40])

$$\mathfrak{G}_\tau : \mathcal{M}(M_\tau, \mathfrak{s}_\tau, [\mathfrak{c}]) \to \mathcal{M}(M'_\tau, \mathfrak{s}'_\tau, [\mathfrak{c}])$$

We want to define $\mathfrak{G}_\tau$ at the level of configuration spaces in such a way that is gauge equivariant. Our proposal is that this map should decompose as

$$\mathfrak{G}_\tau(A,\Phi) = (A,\Phi)^{\#} + \left(\mathcal{D}_{(A,\Phi)^{\#}}\mathfrak{F}_{\mathfrak{p}_{M'_\tau}}\right)^*(b',\psi') \tag{49}$$

where $(b',\psi') \in L^2_{k,A}(i\mathfrak{su}(S'^{+}_\tau)\oplus S'^{-}_\tau)$. Here $\mathcal{D}_{(A,\Phi)^{\#}}\mathfrak{F}_{\mathfrak{p}_{M'_\tau}}$ denotes the linearization of the perturbed Seiberg-Witten map $\mathfrak{F}_{\mathfrak{p}_{M'_\tau}}$. In the old days of Seiberg-Witten theory where only the curvature equation was perturbed by some imaginary-valued self dual two form, this linearized map $\mathcal{D}_{(A,\Phi)^{\#}}\mathfrak{F}_{\mathfrak{p}_{M'_\tau}}$ would coincide with the linearization of the *unperturbed* Seiberg-Witten map $\mathcal{D}_{(A,\Phi)^{\#}}\mathfrak{F}$, since the perturbations where independent of the configuration $(A,\Phi)^{\#}$ being used. In fact, analyzing the formula (47), we can see that the discrepancy between these two maps is due to the (abstract) perturbations used on the cylindrical end, so to emphasize this point we may sometimes write $\mathcal{D}_{\mathfrak{q},(A,\Phi)^{\#}}\mathfrak{F}$ instead of the more precise notation $\mathcal{D}_{(A,\Phi)^{\#}}\mathfrak{F}_{\mathfrak{p}_{M'_\tau}}$.

Recall that the perturbed Seiberg-Witten equation is

$$\mathfrak{F}_{\mathfrak{p}_{M'_\tau}}(A,\Phi) = \mathfrak{F}(A,\Phi) + \mathfrak{p}_{M'_\tau}(A,\Phi) = 0$$

By definition, we want $\mathfrak{G}_\tau(A,\Phi)$ to solve the previous equation which means that

$$\mathfrak{F}\mathfrak{G}_\tau(A,\Phi) + \mathfrak{p}_{M'_\tau}\mathfrak{G}_\tau(A,\Phi) = 0$$

and we want to think of the previous equation as depending on (b',ψ') when we write $\mathfrak{G}_\tau(A,\Phi)$ in an explicit way as in (49). At this point one needs to write this expression in a very explicit way to see many cancellations occur. There is nothing particularly difficult with this, so the reader may prefer to skip to the statement of Theorem 39.

We start by computing $\mathfrak{F}\mathfrak{G}_\mathfrak{q}(A,\Phi)$. First we need some notation. The derivative of the perturbation can be regarded as an operator

$$\mathcal{D}_{(A,\Phi)}\hat{\mathfrak{q}} : L^2_k(\mathbb{R}^+\times -Y; iT^*(\mathbb{R}^+\times -Y)\oplus S^+) \to L^2_{k-1}(\mathbb{R}^+\times -Y; i\mathfrak{su}(S^+)\oplus S^-)$$

and so the formal adjoint is simply

$$\left(\mathcal{D}_{(A,\Phi)}\hat{\mathfrak{q}}\right)^* = (\mathfrak{Q}_1,\mathfrak{Q}_2) : L^2_k(\mathbb{R}^+\times -Y; i\mathfrak{su}(S^+)\oplus S^-) \to L^2_{k-1}(\mathbb{R}^+\times -Y; iT^*(\mathbb{R}^+\times -Y)\oplus S^+))$$

Therefore we will define

$$\begin{aligned} a_q &= \mathfrak{Q}_1(b', \psi') \\ \psi_q &= \mathfrak{Q}_2(b', \psi') \end{aligned}$$

Observe that

$$\begin{aligned} \mathfrak{G}_{\mathfrak{q}}(A, \Phi) &= \mathfrak{G}(A, \Phi) + \left(\mathcal{D}_{(A,\Phi)}\hat{\mathfrak{q}}\right)^* (b', \psi') \\ &= (A, \Phi)^{\#} + \left(\mathcal{D}_{(A,\Phi)^{\#}}\mathfrak{F}\right)^* (b', \psi') + (a_q, \psi_q) \\ &= \left(A^{\#} + (d^+)^* \rho^* b' + \rho^*(\psi'(\Phi^{\#})^*), \Phi^{\#} + D^*_{A^{\#}}\psi' - b'\Phi^{\#}\right) + (a_q, \psi_q) \\ &= (A', \Phi') + (a_q, \psi_q) \\ &= (A'_q, \Phi'_q) \end{aligned}$$

By definition

$$\mathfrak{F}\mathfrak{G}_{\mathfrak{q}}(A, \Phi) = \mathfrak{F}(A'_q, \Phi'_q) = \left(\frac{1}{2}\rho(F^+_{A'^t_q}) - (\Phi'_q\Phi'^*_q)_0, D_{A'_q}\Phi'_q\right)$$

The spinor term is

$$\begin{aligned} &D_{A'_q}\Phi'_q \\ &= D_{A'+a_q}(\Phi' + \psi_q) \\ &= D_{A'}\Phi' + D_{A'}\psi_q + \rho(a_q)\Phi' + \rho(a_q)\psi_q \end{aligned}$$

while the curvature term is

$$\begin{aligned} &\frac{1}{2}\rho(F^+_{A'^t_q}) \\ &= \frac{1}{2}\rho(F^+_{A'^t} + 2d^+a_q) \\ &= \frac{1}{2}\rho\left(F^+_{A'^t}\right) + \rho(d^+a_q) \end{aligned}$$

The quadratic term $(\Phi'_q\Phi'^*_q)_0 = \Phi'_q\Phi'^*_q - \frac{1}{2}\left\langle \Phi'_q, \Phi'_q \right\rangle$ equals

$$\begin{aligned}(\Phi' + \psi_q)\left(\Phi'^* + \psi_q^*\right) - \frac{1}{2}\left\langle \Phi' + \psi_q, \Phi' + \psi_q \right\rangle \\ = \Phi'\Phi'^* + \Phi'\psi_q^* + \psi_q\Phi'^* + \psi_q\psi_q^* \\ -\frac{1}{2}\left\langle \Phi', \Phi' \right\rangle - \frac{1}{2}\left\langle \Phi', \psi_q \right\rangle - \frac{1}{2}\left\langle \psi_q, \Phi' \right\rangle - \left\langle \psi_q, \psi_q \right\rangle \\ = (\Phi'\Phi'^*)_0 + (\psi_q\psi_q^*)_0 + \{\Phi'\psi_q^* + \psi_q\Phi'^*\}_0\end{aligned}$$

Therefore, the equation $\mathfrak{F}\mathfrak{G}_{\mathfrak{q}}(A,\Phi) + \mathfrak{p}_{M'_\tau}\mathfrak{G}_{\mathfrak{q}}(A,\Phi) = 0$ is equivalent to

(50)

$$\begin{pmatrix} \frac{1}{2}\rho\left(F^+_{A'^t}\right) + \rho(d^+a_q) - (\Phi'\Phi'^*)_0 - (\psi_q\psi_q^*)_0 - \{\Phi'\psi_q^* + \psi_q\Phi'^*\}_0 \\ D_{A'}\Phi' + D_{A'}\psi_q + \rho(a_q)\Phi' + \rho(a_q)\psi_q \end{pmatrix} = -\mathfrak{p}_{M'_\tau}\mathfrak{G}_{\mathfrak{q}}(A,\Phi)$$

Now we define the "**perturbed Seiberg-Witten Laplacian**"

$$\triangle_{2,\mathfrak{q},(A,\Phi)^\#} = \left(\mathcal{D}_{(A,\Phi)^\#}\mathfrak{F}_{\mathfrak{q}}\right) \circ \left(\mathcal{D}_{(A,\Phi)^\#}\mathfrak{F}_{\mathfrak{q}}\right)^*$$

We can relate this operator to the "**unperturbed Seiberg-Witten Laplacian**"

$$\triangle_{2,(A,\Phi)^\#} = \left(\mathcal{D}_{(A,\Phi)^\#}\mathfrak{F}\right) \circ \left(\mathcal{D}_{(A,\Phi)^\#}\mathfrak{F}\right)^*$$

as follows:

$$\begin{aligned}\left(\mathcal{D}_{(A,\Phi)^\#}\mathfrak{F}_{\mathfrak{q}}\right) \circ \left(\mathcal{D}_{(A,\Phi)^\#}\mathfrak{F}_{\mathfrak{q}}\right)^* \\ = \left(\mathcal{D}_{(A,\Phi)^\#}\mathfrak{F} + \mathcal{D}_{(A,\Phi)}\hat{\mathfrak{q}}\right) \circ \left(\left(\mathcal{D}_{(A,\Phi)^\#}\mathfrak{F}\right)^* + \left(\mathcal{D}_{(A,\Phi)}\hat{\mathfrak{q}}\right)^*\right) \\ = \triangle_{2,(A,\Phi)^\#} + \left(\mathcal{D}_{(A,\Phi)^\#}\mathfrak{F}\right) \circ \left(\mathcal{D}_{(A,\Phi)}\hat{\mathfrak{q}}\right)^* \\ + \left(\mathcal{D}_{(A,\Phi)}\hat{\mathfrak{q}}\right) \circ \left(\mathcal{D}_{(A,\Phi)^\#}\mathfrak{F}\right)^* + \left(\mathcal{D}_{(A,\Phi)}\hat{\mathfrak{q}}\right) \circ \left(\mathcal{D}_{(A,\Phi)}\hat{\mathfrak{q}}\right)^*\end{aligned}$$

To continue analyzing the gluing equation we will write A' and Φ' in terms of $A^\#$ and $\Phi^\#$.

$$\begin{cases} A' = A^\# + (d^+)^*\rho^* b' + \rho^*(\psi'(\Phi^\#)^*) \\ \Phi' = \Phi^\# + D^*_{A^\#}\psi' - b'\Phi^\# \end{cases}$$

Therefore we can write the left hand side of 50 as follows:

$$\begin{aligned}&\frac{1}{2}\rho\left(F^+_{A't}\right)\\ =\frac{1}{2}\rho(F^+_{A\#}) + \rho\left(d^+\left((d^+)^*\rho^*b' + \rho^*(\psi'(\Phi^\#)^*)\right)\right)\end{aligned}$$

$$\begin{aligned}&(\Phi'\Phi'^*)_0\\ =\left(\Phi^\#\Phi^{\#*}\right)_0 + \left(\left(D^*_{A\#}\psi' - b'\Phi^\#\right)\left(D^*_{A\#}\psi' - b'\Phi^\#\right)^*\right)_0\\ +\left\{\Phi^\#\left(D^*_{A\#}\psi' - b'\Phi^\#\right)^* + \left(D^*_{A\#}\psi' - b'\Phi^\#\right)(\Phi^\#)^*\right\}_0\end{aligned}$$

$$\begin{aligned}&\{\Phi'\psi^*_q + \psi_q\Phi'^*\}_0\\ =\left\{\left(\Phi^\# + D^*_{A\#}\psi' - b'\Phi^\#\right)\psi^*_q + \psi_q\left(\Phi^\# + D^*_{A\#}\psi' - b'\Phi^\#\right)^*\right\}_0\end{aligned}$$

Therefore the first row on the left hand side of 50 is the same as

$$\begin{aligned}&\frac{1}{2}\rho(F^+_{A\#}) + \rho\left(d^+\left((d^+)^*\rho^*b' + \rho^*(\psi'(\Phi^\#)^*)\right)\right) + \rho(d^+a_q) - (\psi_q\psi^*_q)_0\\ &- \left(\Phi^\#\Phi^{\#*}\right)_0 - \left(\left(D^*_{A\#}\psi' - b'\Phi^\#\right)\left(D^*_{A\#}\psi' - b'\Phi^\#\right)^*\right)_0\\ &- \left\{\Phi^\#\left(D^*_{A\#}\psi' - b'\Phi^\#\right)^* + \left(D^*_{A\#}\psi' - b'\Phi^\#\right)(\Phi^\#)^*\right\}_0\\ &- \left\{\left(\Phi^\# + D^*_{A\#}\psi' - b'\Phi^\#\right)\psi^*_q + \psi_q\left(\Phi^\# + D^*_{A\#}\psi' - b'\Phi^\#\right)^*\right\}_0\end{aligned}$$

For the spinor part we have

$$\begin{aligned}&D_{A'}\Phi'\\ &= D_{A\#}\Phi^\# + D_{A\#}\left(D^*_{A\#}\psi' - b'\Phi^\#\right)\\ &+ \left[\rho\left((d^+)^*\rho^*b' + \rho^*(\psi'(\Phi^\#)^*\right)\Phi^\#\right] + \left[\rho\left((d^+)^*\rho^*b' + \rho^*(\psi'(\Phi^\#)^*\right)\left(D^*_{A\#}\psi' - b'\Phi^\#\right)\right]\end{aligned}$$

$$\begin{aligned} &D_{A'}\psi_q \\ &= D_{A^\#}\psi_q + \rho\left[(d^+)^*\rho^* b'\right]\psi_q + \rho\left[\rho^*(\psi'(\Phi^\#)^*)\right]\psi_q \end{aligned}$$

$$\begin{aligned} &\rho(a_q)\Phi' \\ &= \rho(a_q)\Phi^\# + \rho(a_q) D^*_{A^\#}\psi' - \rho(a_q)(b'\Phi^\#) \end{aligned}$$

Therefore the second row on the left hand side of 50 is the same as

$$\begin{aligned} D_{A^\#}\Phi^\# + D_{A^\#}\left(D^*_{A^\#}\psi' - b'\Phi^\#\right) + \left[\rho\left((d^+)^*\rho^* b' + \rho^*(\psi'(\Phi^\#)^*\right)\Phi^\#\right] \\ + \left[\rho\left((d^+)^*\rho^* b' + \rho^*(\psi'(\Phi^\#)^*\right)\left(D^*_{A^\#}\psi' - b'\Phi^\#\right)\right] \\ + D_{A^\#}\psi_q + \rho\left[(d^+)^*\rho^* b'\right]\psi_q + \rho\left[\rho^*(\psi'(\Phi^\#)^*)\right]\psi_q \\ + \rho(a_q)\Phi^\# + \rho(a_q) D^*_{A^\#}\psi' - \rho(a_q)(b'\Phi^\#) + \rho(a_q)\psi_q \end{aligned}$$

We compute first

$$\begin{aligned} &\triangle_{2,\mathfrak{q},(A,\Phi)^\#}(b',\psi') \\ &= \triangle_{2,(A,\Phi)^\#}(b',\psi') + \left(\mathcal{D}_{(A,\Phi)^\#}\mathfrak{F}\right)\circ\left(\mathcal{D}_{(A,\Phi)}\hat{\mathfrak{q}}\right)^*(b',\psi') \\ &+ \left(\mathcal{D}_{(A,\Phi)}\hat{\mathfrak{q}}\right)\circ\left(\mathcal{D}_{(A,\Phi)^\#}\mathfrak{F}\right)^*(b',\psi') + \left(\mathcal{D}_{(A,\Phi)}\hat{\mathfrak{q}}\right)\circ\left(\mathcal{D}_{(A,\Phi)}\hat{\mathfrak{q}}\right)^*(b',\psi') \end{aligned}$$

The first term is

$$\begin{aligned} &\triangle_{2,(A,\Phi)^\#}(b',\psi') \\ &= [\rho(d^+\left[(d^+)^*\rho^* b' + \rho^*(\psi'(\Phi^\#)^*)\right] - \left\{\Phi^\#\left(D^*_{A^\#}\psi' - b'\Phi^\#\right)^* + \left(D^*_{A^\#}\psi' - b'\Phi^\#\right)(\Phi^\#)^*\right\}_0, \\ &D_{A^\#}\left(D^*_{A^\#}\psi' - b'\Phi^\#\right) + \rho\left((d^+)^*\rho^* b' + \rho^*(\psi'(\Phi^\#)^*)\right)\Phi^\#] \end{aligned}$$

$$\begin{aligned}
&\left(\mathcal{D}_{(A,\Phi)^\#}\mathfrak{F}\right)\circ\left(\mathcal{D}_{(A,\Phi)}\hat{\mathfrak{q}}\right)^*(b',\psi')\\
&=\left(\mathcal{D}_{(A,\Phi)^\#}\mathfrak{F}\right)\circ(a_q,\psi_q)\\
&=\left(\rho(d^+a_q)-\left\{\Phi^\#\psi_q^*+\psi_q\left(\Phi^\#\right)^*\right\}_0, D_{A^\#}\psi_q+\rho(a_q)\Phi^\#\right)
\end{aligned}$$

We won't simplify the remaining two terms since we haven't used any special notation for $\mathcal{D}_{(A,\Phi)}\hat{\mathfrak{q}}$. For the map

$$\mathcal{Q}(a,\phi)=(-(\phi\phi^*)_0,\rho(a)\phi)$$

we also compute

$$\begin{aligned}
&\mathcal{Q}\circ\left[\left(\mathcal{D}_{(A,\Phi)^\#}\mathfrak{F}\right)^*+\left(\mathcal{D}_{(A,\Phi)}\hat{\mathfrak{q}}\right)^*\right](b',\psi')\\
&=\mathcal{Q}\begin{pmatrix}(d^+)^*\rho^*b'+\rho^*\left(\psi'\left(\Phi^\#\right)^*\right)+a_q\\ D^*_{A^\#}\psi'-b'\Phi^\#+\psi_q\end{pmatrix}\\
&=\begin{pmatrix}-\left(\left(D^*_{A^\#}\psi'-b'\Phi^\#+\psi_q\right)\left(D^*_{A^\#}\psi'-b'\Phi^\#+\psi_q\right)^*\right)_0\\ \rho\left[(d^+)^*\rho^*b'+\rho^*\left(\psi'\left(\Phi^\#\right)^*\right)+a_q\right]\left(D^*_{A^\#}\psi'-b'\Phi^\#+\psi_q\right)\end{pmatrix}
\end{aligned}$$

The first term can be simplified as

$$\begin{aligned}
&-\left(D^*_{A^\#}\psi'-b'\Phi^\#+\psi_q\right)\left(D^*_{A^\#}\psi'-b'\Phi^\#+\psi_q\right)^*\\
&+\frac{1}{2}\left\langle\left(D^*_{A^\#}\psi'-b'\Phi^\#+\psi_q\right),\left(D^*_{A^\#}\psi'-b'\Phi^\#+\psi_q\right)\right\rangle\\
&=-\left(D^*_{A^\#}\psi'-b'\Phi^\#\right)\left(D^*_{A^\#}\psi'-b'\Phi^\#\right)^*-\left(D^*_{A^\#}\psi'-b'\Phi^\#\right)\psi_q^*\\
&\quad-\psi_q\left(D^*_{A^\#}\psi'-b'\Phi^\#\right)^*-\psi_q\psi_q^*\\
&+\frac{1}{2}\left\langle D^*_{A^\#}\psi'-b'\Phi^\#,D^*_{A^\#}\psi'-b'\Phi^\#\right\rangle+\frac{1}{2}\left\langle D^*_{A^\#}\psi'-b'\Phi^\#,\psi_q\right\rangle\\
&\quad+\frac{1}{2}\left\langle\psi_q,D^*_{A^\#}\psi'-b'\Phi^\#\right\rangle+\frac{1}{2}\langle\psi_q,\psi_q\rangle\\
&=-\left(\left(D^*_{A^\#}\psi'-b'\Phi^\#\right)\left(D^*_{A^\#}\psi'-b'\Phi^\#\right)^*\right)_0-\left(\psi_q\psi_q^*\right)_0\\
&\quad-\left\{\left(D^*_{A^\#}\psi'-b'\Phi^\#\right)\psi_q^*+\psi_q\left(D^*_{A^\#}\psi'-b'\Phi^\#\right)^*\right\}_0
\end{aligned}$$

The second term can be written as

$$\begin{aligned}\rho\left[\left(d^{+}\right)^{*}\rho^{*}b'\right]\left(D^{*}_{A^{\#}}\psi'-b'\Phi^{\#}\right)+\rho\left[\left(d^{+}\right)^{*}\rho^{*}b'\right]\psi_{q}\\+\rho\left[\rho^{*}\left(\psi'\left(\Phi^{\#}\right)^{*}\right)\right]\left(D^{*}_{A^{\#}}\psi'-b'\Phi^{\#}\right)+\rho\left[\rho^{*}\left(\psi'\left(\Phi^{\#}\right)^{*}\right)\right]\psi_{q}\\+\rho(a_{q})\left(D^{*}_{A^{\#}}\psi'-b'\Phi^{\#}\right)+\rho(a_{q})\psi_{q}\end{aligned}$$

Therefore, the first row of $\triangle_{2,\mathfrak{q},(A,\Phi)^{\#}}(b',\psi')+\mathcal{Q}\circ\left[\left(\mathcal{D}_{(A,\Phi)^{\#}}\mathfrak{F}\right)^{*}+\left(\mathcal{D}_{(A,\Phi)}\hat{\mathfrak{q}}\right)^{*}\right](b',\psi')$ is equal to

$$\begin{aligned}\left[\rho(d^{+}\left[(d^{+})^{*}\rho^{*}b'+\rho^{*}(\psi'(\Phi^{\#})^{*})\right]\right.\\-\left\{\Phi^{\#}\left(D^{*}_{A^{\#}}\psi'-b'\Phi^{\#}\right)^{*}+\left(D^{*}_{A^{\#}}\psi'-b'\Phi^{\#}\right)(\Phi^{\#})^{*}\right\}_{0}\\+\rho(d^{+}a_{q})-\left\{\Phi^{\#}\psi_{q}^{*}+\psi_{q}\left(\Phi^{\#}\right)^{*}\right\}_{0}\\\left[\left(\mathcal{D}_{(A,\Phi)}\hat{\mathfrak{q}}\right)\circ\left(\mathcal{D}_{(A,\Phi)^{\#}}\mathfrak{F}\right)^{*}(b',\psi')+\left(\mathcal{D}_{(A,\Phi)}\hat{\mathfrak{q}}\right)\circ\left(\mathcal{D}_{(A,\Phi)}\hat{\mathfrak{q}}\right)^{*}(b',\psi')\right]^{1}\\-\left(\left(D^{*}_{A^{\#}}\psi'-b'\Phi^{\#}\right)\left(D^{*}_{A^{\#}}\psi'-b'\Phi^{\#}\right)^{*}\right)_{0}-\left(\psi_{q}\psi_{q}^{*}\right)_{0}\\-\left\{\left(D^{*}_{A^{\#}}\psi'-b'\Phi^{\#}\right)\psi_{q}^{*}+\psi_{q}\left(D^{*}_{A^{\#}}\psi'-b'\Phi^{\#}\right)^{*}\right\}_{0}\end{aligned}$$

while the second row is equal to

$$\begin{aligned}D_{A^{\#}}\left(D^{*}_{A^{\#}}\psi'-b'\Phi^{\#}\right)+\rho\left((d^{+})^{*}\rho^{*}b'+\rho^{*}(\psi'(\Phi^{\#})^{*})\right)\Phi^{\#}]\\+D_{A^{\#}}\psi_{q}+\rho(a_{q})\Phi^{\#}\\+\left[\left(\mathcal{D}_{(A,\Phi)}\hat{\mathfrak{q}}\right)\circ\left(\mathcal{D}_{(A,\Phi)^{\#}}\mathfrak{F}\right)^{*}(b',\psi')+\left(\mathcal{D}_{(A,\Phi)}\hat{\mathfrak{q}}\right)\circ\left(\mathcal{D}_{(A,\Phi)}\hat{\mathfrak{q}}\right)^{*}(b',\psi')\right]^{2}\\\rho\left[\left(d^{+}\right)^{*}\rho^{*}b'\right]\left(D^{*}_{A^{\#}}\psi'-b'\Phi^{\#}\right)+\rho\left[\left(d^{+}\right)^{*}\rho^{*}b'\right]\psi_{q}\\+\rho\left[\rho^{*}\left(\psi'\left(\Phi^{\#}\right)^{*}\right)\right]\left(D^{*}_{A^{\#}}\psi'-b'\Phi^{\#}\right)+\rho\left[\rho^{*}\left(\psi'\left(\Phi^{\#}\right)^{*}\right)\right]\psi_{q}\\+\rho(a_{q})\left(D^{*}_{A^{\#}}\psi'-b'\Phi^{\#}\right)+\rho(a_{q})\psi_{q}\end{aligned}$$

A simple comparison shows that

$$
\begin{aligned}
&[\mathfrak{F}\mathfrak{G}_{\mathfrak{q}}(A,\Phi)]^1 \\
&= \triangle_{2,\mathfrak{q},(A,\Phi)^\#}(b',\psi') + \mathcal{Q} \circ \left[\left(\mathcal{D}_{(A,\Phi)^\#}\mathfrak{F}\right)^* + (\mathcal{D}_{(A,\Phi)}\hat{\mathfrak{q}})^* \right](b',\psi') \\
&= \frac{1}{2}\rho(F^+_{A^\#}) - (\Phi^\#\Phi^{\#*})_0 - \left\{ (\Phi^\# + D^*_{A^\#}\psi' - b'\Phi^\#)\psi^*_q + \psi_q(\Phi^\# + D^*_{A^\#}\psi' - b'\Phi^\#)^* \right\}_0 \\
&\quad + \left\{ \Phi^\#\psi^*_q + \psi_q(\Phi^\#)^* \right\}_0 + \left\{ \left(D^*_{A^\#}\psi' - b'\Phi^\#\right)\psi^*_q + \psi_q\left(D^*_{A^\#}\psi' - b'\Phi^\#\right)^* \right\}_0 \\
&\quad - \left[\left(\mathcal{D}_{(A,\Phi)}\hat{\mathfrak{q}}\right) \circ \left(\mathcal{D}_{(A,\Phi)^\#}\mathfrak{F}\right)^*(b',\psi') + \left(\mathcal{D}_{(A,\Phi)}\hat{\mathfrak{q}}\right) \circ \left(\mathcal{D}_{(A,\Phi)}\hat{\mathfrak{q}}\right)^*(b',\psi') \right]^1
\end{aligned}
$$

To simplify this further notice that

$$
\begin{aligned}
&\left\{ \Phi^\#\psi^*_q + \psi_q(\Phi^\#)^* \right\}_0 + \left\{ \left(D^*_{A^\#}\psi' - b'\Phi^\#\right)\psi^*_q + \psi_q\left(D^*_{A^\#}\psi' - b'\Phi^\#\right)^* \right\}_0 \\
&\quad - \left\{ (\Phi^\# + D^*_{A^\#}\psi' - b'\Phi^\#)\psi^*_q + \psi_q(\Phi^\# + D^*_{A^\#}\psi' - b'\Phi^\#)^* \right\}_0 \\
&= \Phi^\#\psi^*_q + \psi_q(\Phi^\#)^* - \mathrm{Re}\left\langle \Phi^\#, \psi_q \right\rangle \\
&\quad + \left(D^*_{A^\#}\psi' - b'\Phi^\#\right)\psi^*_q + \psi_q\left(D^*_{A^\#}\psi' - b'\Phi^\#\right)^* - \mathrm{Re}\left\langle D^*_{A^\#}\psi' - b'\Phi^\#, \psi_q \right\rangle \\
&\quad - (\Phi^\# + D^*_{A^\#}\psi' - b'\Phi^\#)\psi^*_q - \psi_q(\Phi^\# + D^*_{A^\#}\psi' - b'\Phi^\#)^* \\
&\quad + \mathrm{Re}\left\langle \Phi^\# + D^*_{A^\#}\psi' - b'\Phi^\#, \psi_q \right\rangle \\
&= 0
\end{aligned}
$$

In other words, we have that

$$
\begin{aligned}
&[\mathfrak{F}\mathfrak{G}_{\mathfrak{q}}(A,\Phi)]^1 \\
&= \triangle_{2,\mathfrak{q},(A,\Phi)^\#}(b',\psi') + \mathcal{Q} \circ \left[\left(\mathcal{D}_{(A,\Phi)^\#}\mathfrak{F}\right)^* + (\mathcal{D}_{(A,\Phi)}\hat{\mathfrak{q}})^* \right](b',\psi') \\
&\quad + \frac{1}{2}\rho(F^+_{A^\#}) - (\Phi^\#\Phi^{\#*})_0 \\
&\quad - \left[\left(\mathcal{D}_{(A,\Phi)}\hat{\mathfrak{q}}\right) \circ \left(\mathcal{D}_{(A,\Phi)^\#}\mathfrak{F}\right)^*(b',\psi') + \left(\mathcal{D}_{(A,\Phi)}\hat{\mathfrak{q}}\right) \circ \left(\mathcal{D}_{(A,\Phi)}\hat{\mathfrak{q}}\right)^*(b',\psi') \right]^1
\end{aligned}
$$

At the same time

$$\begin{aligned}&[\mathfrak{F}\mathfrak{G}_{\mathfrak{q}}(A,\Phi)]^2\\ &= D_{A^\#}\Phi^\# - \left[\left(\mathcal{D}_{(A,\Phi)}\hat{\mathfrak{q}}\right)\circ\left(\mathcal{D}_{(A,\Phi)^\#}\mathfrak{F}\right)^*(b',\psi') + \left(\mathcal{D}_{(A,\Phi)}\hat{\mathfrak{q}}\right)\circ\left(\mathcal{D}_{(A,\Phi)}\hat{\mathfrak{q}}\right)^*(b',\psi')\right]^2\end{aligned}$$

Therefore, we just found the following: solving

$$\mathfrak{F}\mathfrak{G}_{\mathfrak{q}}(A,\Phi) + \mathfrak{p}_{M'_\tau}\mathfrak{G}_{\mathfrak{q}}(A,\Phi) = 0$$

is equivalent to solving

$$\begin{aligned}&\triangle_{2,\mathfrak{q},(A,\Phi)^\#}(b',\psi') + \mathcal{Q}\circ\left[\left(\mathcal{D}_{(A,\Phi)^\#}\mathfrak{F}\right)^* + (\mathcal{D}_{(A,\Phi)}\hat{\mathfrak{q}})^*\right](b',\psi')\\ &- \left[\left(\mathcal{D}_{(A,\Phi)}\hat{\mathfrak{q}}\right)\circ\left(\mathcal{D}_{(A,\Phi)^\#}\mathfrak{F}\right)^*(b',\psi') + \left(\mathcal{D}_{(A,\Phi)}\hat{\mathfrak{q}}\right)\circ\left(\mathcal{D}_{(A,\Phi)}\hat{\mathfrak{q}}\right)^*(b',\psi')\right]\\ &= -\mathfrak{F}(A,\Phi)^\# - \mathfrak{p}_{M'_\tau}(A,\Phi)^\# + \mathfrak{p}_{M'_\tau}(A,\Phi)^\# - \mathfrak{p}_{M'_\tau}\mathfrak{G}_{\mathfrak{q}}(A,\Phi)\end{aligned}$$

To make this even more compact define

$$P(b',\psi') = \mathfrak{p}_{M'_\tau}(\mathfrak{G}(A,\Phi)) - \mathfrak{p}_{M'_\tau}(A,\Phi)^\# - \left(\mathcal{D}_{(A,\Phi)}\hat{\mathfrak{q}}\right)\circ\left(\mathcal{D}_{(A,\Phi)^\#}\mathfrak{F}_{\mathfrak{q}}\right)^*$$

and we have found the following (this should be compared with equation 3.2 in [40]):

Theorem 39. *The configuration $\mathfrak{G}_{\mathfrak{q}}(A,\Phi) = (A,\Phi)^\# + \left(\mathcal{D}_{\mathfrak{q},(A,\Phi)^\#}\mathfrak{F}_{\mathfrak{q}}\right)^*(b',\psi')$ is a solution to the perturbed Seiberg-Witten equations $\mathfrak{F}_{\mathfrak{p}_{M'_\tau}}\mathfrak{G}_{\mathfrak{q}}(A,\Phi) = 0$ if and only if*

$$\triangle_{2,\mathfrak{q},(A,\Phi)^\#}(b',\psi') + \mathcal{Q}\circ\left(\mathcal{D}_{(A,\Phi)^\#}\mathfrak{F}_{\mathfrak{q}}\right)^*(b',\psi') + P(b',\psi') = -\mathfrak{F}_{\mathfrak{p}_{M'_\tau}}(A,\Phi)^\# \tag{51}$$

Notice that the term $P(b',\psi')$ is a new term that does not appear in the usual linearization of the Seiberg Witten equations. This appears solely due to the presence of the abstract perturbations used in [32]. To solve this equation we will need a sharp version of the contraction mapping theorem.

Namely, the basic idea is to define

$$V_{\mathfrak{q}} = \triangle_{2,\mathfrak{q},(A,\Phi)^\#}(b',\psi')$$

Our intention is to show that $\triangle_{2,\mathfrak{q},(A,\Phi)^\#}$ is invertible so that if we define $S_{\mathfrak{q},(A,\Phi)^\#}(V_{\mathfrak{q}})$ as

$$S_{\mathfrak{q},(A,\Phi)^\#}(V_{\mathfrak{q}}) \equiv -\mathcal{Q}\circ\left[\left(\mathcal{D}_{(A,\Phi)^\#}\mathfrak{F}_{\mathfrak{q}}\right)^*\right]\left(\triangle^{-1}_{2,\mathfrak{q},(A,\Phi)^\#}V_{\mathfrak{q}}\right) - P\left(\triangle^{-1}_{2,\mathfrak{q},(A,\Phi)^\#}V_{\mathfrak{q}}\right)$$

the gluing equation (51) that we need to solve can be written as

$$V_{\mathfrak{q}} = S_{\mathfrak{q},(A,\Phi)^{\#}}(V_{\mathfrak{q}}) - \mathfrak{F}_{\mathfrak{p}_{M'_\tau}}(A,\Phi)^{\#}$$

The solution of this equation will be guaranteed once we shows the hypothesis of Proposition 2.3.5 in [40] are satisfied. Therefore, we will show first that $\triangle_{2,\mathfrak{q},(A,\Phi)^{\#}}$ is indeed invertible.

6.4 Invertibility of $\triangle_{2,\mathfrak{q},(A,\Phi)^{\#}}$. In this section we seek a version of Proposition 3.1.2 and Corollary 3.1.6 in [40] , namely, we want to show that:

Theorem 40. *For each $k \geq 0$ there exists a constant $c_k > 0$ such that for every τ large enough, every $N_0 \geq 1$ and every solution (A,Φ) of the Seiberg-Witten equations on M_τ belonging to the zero dimensional strata of $\mathcal{M}(M_\tau, \mathfrak{s}_\tau, [\mathfrak{c}])$, the operator*

$$\begin{aligned}
\triangle_{2,\mathfrak{q},(A,\Phi)^{\#}} &: L^2_{k+2,A^{\#}}(M'_\tau, g'_\tau) \to L^2_{k,A^{\#}}(M'_\tau, g'_\tau) \\
(b',\psi') &\to \left(\mathcal{D}_{\mathfrak{q},(A,\Phi)^{\#}}\mathfrak{F}\right) \circ (\mathcal{D}_{\mathfrak{q},(A,\Phi)^{\#}}\mathfrak{F})^*(b',\psi')
\end{aligned}$$

is an isomorphism, and moreover, its inverse $\triangle^{-1}_{2,\mathfrak{q},(A,\Phi)^{\#}}$ satisfies for all (b',ψ')

$$c_k \|(b',\psi')\|_{L^2_{k+1}(g'_\tau, A^{\#})} \geq \|\triangle^{-1}_{2,\mathfrak{q},(A,\Phi)^{\#}}(b',\psi')\|_{L^2_{k+3}(g'_\tau, A^{\#})}$$

Before proceeding we make a few clarifications:

(1) The norms used for the gluing arguments are gauge equivariant norms, which depend on the configuration $(A,\Phi)^{\#}$ being used, as can be seen from our use of subscript in the formulas for the Sobolev spaces.

(2) Our hypothesis regarding the fact that the solution $[(A,\Phi)]$ belongs to the 0 dimensional strata of the moduli space $\mathcal{M}(M_\tau, \mathfrak{s}_\tau, [\mathfrak{c}])$ has to do with the fact that we will need to find uniform bounds which we will depend (partly) on the norms of these solutions. Since we are using gauge equivariant norms and the zero dimensional moduli spaces $\mathcal{M}_0(M_\tau, \mathfrak{s}_\tau, [\mathfrak{c}])$ are compact, for a fixed τ there can only be finitely many terms to worry about. Clearly, the a priori the bounds that we get still depend on the value of τ chosen, but we will see that a transversality argument will help us control these quantities in a way that is τ-independent. It should be pointed out that this assumption regarding

the zero dimensional strata is not that different from the hypothesis used in other gluing arguments. See for example Theorem 4.17 in [12] (which uses a compactness assumption as well) or Theorem 18.3.5 in [32] (which describes all small solutions of a moduli space).

(3) The strategy that we will use to prove the invertibility of $\triangle_{2,\mathfrak{q},(A,\Phi)^{\#}}$ differs from the one employed by [40] mainly because of the following reasons. The way [40] controlled the norm $\triangle_{2,\mathfrak{q},(A,\Phi)^{\#}}$ was by first controlling the norm a different operator $\square_{(A,\Phi)^{\#}} = Q_{\mathfrak{q},(A,\Phi)} \circ Q^*_{\mathfrak{q},(A,\Phi)}$ (defined in the proof of Proposition 3.1.2) and then relating the norms of these two operators through equation (3.6) in their paper. However, these norms were only comparable because their equation (3.7), which uses the fact that $D_{A^{\#}}\Phi^{\#}$ is almost zero. This was true in their case because the usual Seiberg-Witten equations do not perturb the Dirac equation and since $(A,\Phi)^{\#}$ is very close to being a solution this means that $\Phi^{\#}$ is very close to being a harmonic spinor with respect to $D_{A^{\#}}$. However, the abstract perturbations $\mathfrak{q}$ used in [32] do modify the Dirac equation, so any clear relationship between $\square_{(A,\Phi)^{\#}}$ and $D_{A^{\#}}$ is lost. Despite this, we will see momentarily that part of their strategy can be salvaged and turns out being useful for our purposes.

The proof of this theorem will follow a splicing argument similar to the one used in section 4.2.2 of [38] (or section 4.4 in [12]). Namely, we can separate the manifold M'_τ into two pieces (see figure (9)):

• **The unperturbed region**: this refers to the region where $(A,\Phi)^{\#} = (A,\Phi)$, that is, the solution (A,Φ) was not modified. Notice that this includes the cylinder $\mathbb{R}^+ \times -Y$ and the section of the cone $[1,T]\times Y$ and we can use the fact that the moduli space on (M_τ, g_τ) is regular to conclude that $Q_{\mathfrak{q},(A,\Phi)}$ is surjective on M_τ [32, Def 14.5.6]. Using that $Q_{\mathfrak{q},(A,\Phi)} = \mathcal{D}_{(A,\Phi)}\mathfrak{F}_{\mathfrak{q}} \oplus \mathbf{d}^*_{(A,\Phi)}$ and the fact $\mathbf{d}^*_{(A,\Phi)}$ has trivial cokernel [32, Proposition 14.4.3] we conclude that $\mathcal{D}_{(A,\Phi)}\mathfrak{F}_{\mathfrak{q}}$ must be surjective as well. Now, since $Q_{\mathfrak{q},(A,\Phi)}$ is a Fredholm operator we can easily see that $Q^*_{\mathfrak{q},(A,\Phi)}$ must be injective. Moreover $Q^*_{\mathfrak{q},(A,\Phi)} = \left(\mathcal{D}_{(A,\Phi)}\mathfrak{F}_{\mathfrak{q}}\right)^* \oplus \mathbf{d}_{(A,\Phi)}$ where as before $\mathbf{d}_{(A,\Phi)}(f) = (-df, f\Phi)$. The fact that Φ is irreducible implies that $\mathbf{d}_{(A,\Phi)}$ is injective and so $\left(\mathcal{D}_{(A,\Phi)}\mathfrak{F}_{\mathfrak{q}}\right)^*$ must be injective as well since $\ker Q^*_{\mathfrak{q},(A,\Phi)} = \ker\left(\left(\mathcal{D}_{(A,\Phi)}\mathfrak{F}_{\mathfrak{q}}\right)^* |_{\ker(\mathbf{d}_{(A,\Phi)})}\right)$ [29, Eq. 3.4]. In particular, it is not difficult to check that because of this $\triangle_{2,\mathfrak{q},(A,\Phi)}$ will be invertible as an operator on M_τ. To emphasize that we care about this operator when applied to sections supported on the unperturbed region (which contains the cylinder) we will write the inverse as a map

$$\triangle^{-1}_{2,\mathfrak{q},(A,\Phi),cyl} : L^2_A(M_\tau) \to L^2_{2,A}(M_\tau)$$

• **The perturbed region:** this refers to the region where $(A,\Phi)^\#$ and (A,Φ) do not necessarily agree. Notice that this includes the region $[1,\tau] \times Y$ together with remaining piece of the AFAK end Z'. Moreover $\triangle_{2,\mathfrak{q},(A,\Phi)^\#} = \triangle_{2,(A,\Phi)^\#}$ on this part of the manifold. In particular, as long as we work with sections (b',ψ') supported on the perturbed region (vanishing for example on $[1,T/2] \times Y]$), we can follow the strategy used by Mrowka and Rollin described in point 3. of the previous Remark. Namely, in this case we can indeed bound the norm of $\|\triangle_{2,(A,\Phi)^\#}(b',\psi')\|^2_{L^2(g_{\tau'},Z_\tau)}$ using the norm of $\Box_{(A,\Phi)^\#}(0,b',\psi')$ where the domain of this operator is triples and we set the first entry equal to 0. We would then get the analogue of equation 3.8 in [40] to obtain the following: there is a constant c_0 such that for all τ sufficiently large, for all solutions (A,Φ) on M_τ and all sections (b',ψ') on M'_τ supported on the perturbed region and vanishing on $[1,T/2] \times Y$, we have that

$$\|\triangle_{2,(A,\Phi)^\#}(b',\psi')\|_{L^2(g_{\tau'},Z_\tau)} \geq c_0\|(b',\psi')\|_{L^2_2(g'_\tau,A^\#,Z_\tau)}$$

Notice that because of the support condition we can write this as:

$$\|\triangle_{2,(A,\Phi)^\#}(b',\psi')\|_{L^2(g_{\tau'},Z_\tau\setminus[1,T/2)\times Y)} \geq c_0\|(b',\psi')\|_{L^2_2(g'_\tau,A^\#,Z_\tau\setminus[1,T/2)\times Y)}$$

Also, similar inequalities hold for the $L^2_{k,A^\#}$ norms with a constant c_k instead of c_0. This means that $\triangle_{2,(A,\Phi)^\#}$ is an operator bounded from below on this domain and hence it is injective with closed range [1, Theorem 2.5]. In particular

$$U = \mathrm{im}\triangle_{2,(A,\Phi)^\#}\,|_{Z_\tau\setminus[1,T/2)\times Y}$$

will be a closed subspace and as a consequence of the open mapping theorem for Banach spaces [4, Corollary 2.7] it follows that

$$\triangle_{2,(A,\Phi)^\#} : L^2_{2,A^\#}(g_{\tau'}, Z_\tau\setminus[1,T/2) \times Y) \to U$$

is bijective with continuous inverse [the bounds for higher regularity in the Sobolev spaces are essentially the same]. Therefore we have an inverse which for convenience we will denote

$$\triangle^{-1}_{2,(A,\Phi)^{\#},end} : U \to L^2_{2,A^{\#}}(g_{\tau'}, Z_\tau \backslash [1,T/2)\times Y)$$

Now we will introduce some cutoff functions that will allow us to splice these two operators: these will be denoted η_{cyl} and η_{end}. They satisfy the following properties:

• $0 \leq \eta_{cyl}, \eta_{end} \leq 1$ and $\eta^2_{cyl} + \eta^2_{end} = 1$.

• η_{cyl} is supported on the unperturbed region. Moreover, $\eta_{cyl} \equiv 1$ on a small neighborhood of the region $\mathbb{R}^+ \times -Y \cup [1,T/2]\times Y$. In particular, **the gradient of η_{cyl} is supported on the fixed region $[1,T]\times Y$.**

• η_{end} is supported on the perturbed region. Moreover, $\eta_{end} \equiv 1$ on a small neighborhood of $([T,\tau]\times Y)\cup\{Z'\cap\{\sigma_{Z'} > 1/\tau\}$. In particular, **the gradient of η_{end} is supported on the fixed region $[1,T]\times Y$.**

• For $\eta = \eta_{cyl}, \eta_{end}$ we have $|\nabla^n \eta| \leq \left(\frac{2}{T}\right)^n$.

Our proto-inverse will be the operator

$$\begin{aligned}\tilde{\triangle}^{-1}_{2,\mathfrak{q},(A,\Phi)^{\#}} &: L^2(M'_\tau, g'_\tau) \to L^2_2(M'_\tau, g'_\tau)\\
(b',\psi') \to \eta_{cyl}\triangle^{-1}_{2,\mathfrak{q},(A,\Phi),cyl}&[\eta_{cyl}(b',\psi')] + \eta_{end}\triangle^{-1}_{2,(A,\Phi)^{\#},end}[\eta_{end}(b',\psi')]\end{aligned}$$

First of all, notice that this operator provides a parametrix for $\triangle_{2,\mathfrak{q},(A,\Phi)^{\#}}$: this is because

$$\begin{aligned}&\triangle_{2,\mathfrak{q},(A,\Phi)^{\#}}\left[\tilde{\triangle}^{-1}_{2,\mathfrak{q},(A,\Phi)^{\#}}(b',\psi')\right]\\
&= \triangle_{2,\mathfrak{q},(A,\Phi)^{\#}}\left[\eta_{cyl}\triangle^{-1}_{2,\mathfrak{q},(A,\Phi),cyl}[\eta_{cyl}(b',\psi')] + \eta_{end}\triangle^{-1}_{2,(A,\Phi)^{\#},end}[\eta_{end}(b',\psi')]\right]\\
&= \left\{\mathcal{D}_{(A,\Phi)}\eta_{cyl}, \triangle^{-1}_{2,\mathfrak{q},(A,\Phi),cyl}[\eta_{cyl}(b',\psi')]\right\} + \eta^2_{cyl}(b',\psi')\\
&+ \left\{\mathcal{D}_{(A,\Phi)}\eta_{end}, \triangle^{-1}_{2,(A,\Phi)^{\#},end}[\eta_{end}(b',\psi')]\right\} + \eta^2_{end}(b',\psi')\\
&= \left\{\mathcal{D}_{(A,\Phi)}\eta_{cyl}, \triangle^{-1}_{2,\mathfrak{q},(A,\Phi),cyl}[\eta_{cyl}(b',\psi')]\right\} + \left\{\mathcal{D}_{(A,\Phi)}\eta_{end}, \triangle^{-1}_{2,(A,\Phi)^{\#},end}[\eta_{end}(b',\psi')]\right\} + (b',\psi')\end{aligned}$$

Here the notation $\{\cdot,\cdot\}$ is used to indicate a bilinear pointwise multiplication between some (higher order) derivatives of the cutoff functions and the elements in the domain. Also, the notation $\mathcal{D}_{(A,\Phi)}\eta_{cyl}$ means that this expression involves (higher order) derivatives of the perturbation (and a priori the configuration (A,Φ), but whose precise form is not important to use. Notice that the first two terms are supported on the compact subset $[1,T]\times Y$, where $(A,\Phi)^{\#}=(A,\Phi)$. Also, we dropped the dependence on $\mathfrak{q}$ for the derivatives $\mathcal{D}_{(A,\Phi)}\eta_{\bullet}$ since this perturbation affects only the cylindrical region. To analyze if there is any dependence of $\mathcal{D}_{(A,\Phi)}\eta_{\bullet}$ on (A,Φ), we will study $\mathcal{D}_{(A,\Phi)}\eta_{cyl}$ since the other case is exactly the same. We need to compute

$$\triangle_{2,\mathfrak{q},(A,\Phi)^{\#}}\left(\eta_{cyl}(b_{cyl},\psi_{cyl})\right) \tag{52}$$

where we defined

$$(b_{cyl},\psi_{cyl})\equiv\triangle^{-1}_{2,\mathfrak{q},(A,\Phi),cyl}[\eta_{cyl}(b',\psi')]$$

Notice that we may write

$$\triangle_{2,\mathfrak{q},(A,\Phi)^{\#}}=\left(\mathcal{D}_{(A,\Phi)^{\#}}\mathfrak{F}_{\mathfrak{q}}\right)\circ\left(\mathcal{D}_{(A,\Phi)^{\#}}\mathfrak{F}_{\mathfrak{q}}\right)^{*}=\left(\mathcal{D}_{(A,\Phi)^{\#}}\mathfrak{F}\right)\circ\left(\mathcal{D}_{(A,\Phi)^{\#}}\mathfrak{F}\right)^{*}=\triangle_{2,,(A,\Phi)^{\#}}$$

since we are only interested in computing (52) on the region $[1,T]\times Y$, where η_{cyl} is not constant. The advantage of using this unperturbed Seiberg-Witten 'Laplacian' is that we can give an explicit formula for it based on the equations (27) and (28). We find that for arbitrary (b',ψ')

$$\begin{aligned}
&\triangle_{2,(A,\Phi)^{\#}}(b',\psi')\\
&=\left(\mathcal{D}_{(A,\Phi)^{\#}}\mathfrak{F}\right)\left((d^{+})^{*}\rho^{*}b'+\rho^{*}(\psi'(\Phi^{\#})^{*}),D^{*}_{A^{\#}}\psi'-b'\Phi^{\#}\right)\\
&=[\rho(d^{+}\left[(d^{+})^{*}\rho^{*}b'+\rho^{*}(\psi'(\Phi^{\#})^{*})\right]-\left\{\Phi^{\#}\left(D^{*}_{A^{\#}}\psi'-b'\Phi^{\#}\right)^{*}+\left(D^{*}_{A^{\#}}\psi'-b'\Phi^{\#}\right)(\Phi^{\#})^{*}\right\}_{0},\\
&\qquad D_{A^{\#}}\left(D^{*}_{A^{\#}}\psi'-b'\Phi^{\#}\right)+\rho\left((d^{+})^{*}\rho^{*}b'+\rho^{*}(\psi'(\Phi^{\#})^{*})\right)\Phi^{\#}]
\end{aligned}$$

therefore $\triangle_{2,(A,\Phi)^{\#}}\left[(\eta_{cyl}b_{cyl},\eta_{cyl}\psi_{cyl})\right]$ becomes

$$
\begin{aligned}
&[\rho(d^+\left[(d^+)^*\rho^*\left(\eta_{cyl}b_{cyl}\right)+\rho^*(\eta_{cyl}\psi_{cyl}(\Phi^{\#})^*)\right]\\
&-\left\{\Phi^{\#}\left(D^*_{A^{\#}}\left(\eta_{cyl}\psi_{cyl}\right)-(\eta_{cyl}b_{cyl})\Phi^{\#}\right)^*+\left(D^*_{A^{\#}}\left(\eta_{cyl}\psi_{cyl}\right)-(\eta_{cyl}b_{cyl})\Phi^{\#}\right)(\Phi^{\#})^*\right\}_0\\
&+[\rho(d^+\left[(d^+)^*\rho^*(\eta_{cyl}b_{cyl})+\rho^*((\eta_{cyl}\psi_{cyl})\,(\Phi^{\#})^*)\right]\\
&-\left\{\Phi^{\#}\left(D^*_{A^{\#}}\left(\eta_{cyl}\psi_{cyl}\right)-(\eta_{cyl}b_{cyl})\Phi^{\#}\right)^*+\left(D^*_{A^{\#}}\left(\eta_{cyl}\psi_{cyl}\right)-(\eta_{cyl}b_{cyl})\Phi^{\#}\right)(\Phi^{\#})^*\right\}_0
\end{aligned}
$$

Since the Dirac operator D satisfies the Leibniz Rule [2, Prop. 3.38]

$$D(\eta\psi)=\rho(d\eta)\psi+\eta D\psi$$

it is not difficult to see from the previous expression that η_{cyl} is being differentiate only through quantities which depend on the Riemannian metric and other structures of the manifold of $[1,T]\times Y$, which are fixed, i.e, independent of τ, but not on the specific cofigurations $(A,\Phi)^{\#}$. A similar story will be true for η_{end}. Since $\triangle^{-1}_{2,\mathfrak{q},(A,\Phi),cyl}[\eta_{cyl}(b',\psi')]$ and $\triangle^{-1}_{2,(A,\Phi)^{\#},end}[\eta_{end}(b',\psi')]$ are elements of $L^2_{2,A^{\#}}(M'_\tau)$, our previous discussion in fact tells us that the operator

$$
\begin{aligned}
&K_{(A,\Phi)}:L^2([1,T]\times Y)\to L^2([1,T]\times Y)\\
&(b',\psi')\to\left\{\mathcal{D}\eta_{cyl},\triangle^{-1}_{2,\mathfrak{q},(A,\Phi),cyl}[\eta_{cyl}(b',\psi')]\right\}+\left\{\mathcal{D}\eta_{end},\triangle^{-1}_{2,(A,\Phi)^{\#},end}[\eta_{end}(b',\psi')]\right\}
\end{aligned}
$$

can in fact be regarded as an operator

$$K_{(A,\Phi)}:L^2([1,T]\times Y)\to L^2_2([1,T]\times Y)$$

and using the compact inclusion $L^2_2([1,T]\times Y)\hookrightarrow L^2([1,T]\times Y)$ on a compact manifold we conclude that $K_{(A,\Phi)}$ is a compact operator. This provides the desired parametrix and considering the composition of the operators $\triangle_{2,\mathfrak{q},(A,\Phi)^{\#}}$ and $\tilde{\triangle}^{-1}_{2,\mathfrak{q},(A,\Phi)^{\#}}$ in the opposite order it is not difficult to see we just proved the following:

Lemma 41. *For any solution (A,Φ) to the Seiberg Witten equations on the manifold M_τ, the operator $\triangle_{2,\mathfrak{q},(A,\Phi)^{\#}}$ is a Fredholm operator on M'_τ.*

It is possible to show that the norms of the parametrices are uniform, that is, that there exist a constant C_T so that for all solutions (A,Φ) we have that $\|K_{(A,\Phi)}\|\le\frac{C_T}{T}$.

In fact, we will show something better, which is that one could have chosen a constant c_∞ which is independent of T , in other words, $\|K_{(A,\Phi)}\| \leq \frac{C_\infty}{T}$.

Notice that a priori the only term that may not seem controllable in terms of T is

$$\left\{\mathcal{D}\eta_{cyl}, \triangle^{-1}_{2,\mathfrak{q},(A,\Phi),cyl}[\eta_{cyl}(b',\psi')]\right\}$$

However, if we take a sequence of solutions (A_n,Φ_n) on M_{τ_n} then on $[1,T]$ it will converge strongly to a solution (A_∞,Φ_∞) on $Z^+_{Y,\xi}$ [this is because of the compactness theorem 2.2.11 in [40]] and hence for all (b',ψ')

$$\left\{\mathcal{D}\eta_{cyl}, \triangle^{-1}_{2,\mathfrak{q},(A_n,\Phi_n),cyl}[\eta_{cyl}(b',\psi')]\right\}$$

converges to

$$\left\{\mathcal{D}\eta_{cyl}, \triangle^{-1}_{2,\mathfrak{q},(A_\infty,\Phi_\infty),cyl}[\eta_{cyl}(b',\psi')]\right\}$$

It is clear then that it would be enough to have a uniform bound on the operator norms

$$\left\|\triangle^{-1}_{2,\mathfrak{q},(A_\infty,\Phi_\infty)}\right\|_{L^2(Z^+_{Y,\xi})\to L^2_{2,A_\infty}(Z^+_{Y,\xi})}$$

As we will make more explicitly in the next proof, since we are taking a sequence of solutions (A_n,Φ_n) which belong to the zero dimensional strata of the moduli spaces $\mathcal{M}(M_{\tau_n},\mathfrak{s}_{\tau_n},[\mathfrak{c}])$, the limiting solution (A_∞,Φ_∞) must belong to the zero dimensional strata of $\mathcal{M}(Z^+_{Y,\xi},\mathfrak{s},[\mathfrak{c}])$, and since we are using gauge equivariant norms, there are only finitely many values the previous operator norm can take (this is related to the second point in the remarks we made after stating the invertibility of the Laplacian). Therefore, we will have the uniform bound for the operator $K_{(A,\Phi)}$, that is, $\|K_{(A,\Phi)}\| \leq \frac{C_\infty}{T}$ where C_∞ is independent of τ, T and the solutions (A,Φ) used.

Therefore, there is no loss of generality in assuming that T was chosen from the beginning so that it would also satisfy the condition

$$\|K_{(A,\Phi)}\|_{L^2([1,T]\times Y)\to L^2_2([1,T]\times Y)} \leq \frac{C_\infty}{T} \leq \frac{1}{2}$$

for all the solutions of the Seiberg Witten equations on M_τ. In particular, from the identity

$$\triangle_{2,\mathfrak{q},(A,\Phi)^\#}\left[\tilde{\triangle}^{-1}_{2,\mathfrak{q},(A,\Phi)^\#}(b',\psi')\right] = K_{(A,\Phi)}(b',\psi') + (b',\psi')$$

we see that the operator norms satisfy

$$\|\triangle_{2,\mathfrak{q},(A,\Phi)^\#}\tilde{\triangle}^{-1}_{2,\mathfrak{q},(A,\Phi)^\#} - \mathrm{Id}\|_{L^2_{2,A^\#}(M'_\tau,g'_\tau)\to L^2(M'_\tau,g_{\tau'})} \leq \frac{1}{2}$$

In particular we conclude that $\triangle_{2,\mathfrak{q},(A,\Phi)^\#}\tilde{\triangle}^{-1}_{2,\mathfrak{q},(A,\Phi)^\#}$ is invertible and they (and their inverses) are uniformly bounded since

$$\frac{1}{2} \leq \|\triangle_{2,\mathfrak{q},(A,\Phi)^\#}\tilde{\triangle}^{-1}_{2,\mathfrak{q},(A,\Phi)^\#}\|_{L^2_{2,A^\#}(M'_\tau,g'_\tau)\to L^2(M'_\tau,g_{\tau'})} \leq \frac{3}{2}$$

Therefore the inverse of $\triangle_{2,\mathfrak{q},(A,\Phi)^\#}$ is $\tilde{\triangle}^{-1}_{2,\mathfrak{q},(A,\Phi)^\#}\left(\triangle_{2,\mathfrak{q},(A,\Phi)^\#}\tilde{\triangle}^{-1}_{2,\mathfrak{q},(A,\Phi)^\#}\right)^{-1}$.

Returning to our proof of Theorem (40), since $\left(\triangle_{2,\mathfrak{q},(A,\Phi)^\#}\tilde{\triangle}^{-1}_{2,\mathfrak{q},(A,\Phi)^\#}\right)^{-1}$ is uniformly bounded we just need to check that $\tilde{\triangle}^{-1}_{2,\mathfrak{q},(A,\Phi)^\#}$ is uniformly bounded to conclude that $\triangle_{2,\mathfrak{q},(A,\Phi)^\#}$ is uniformly bounded [a similar argument would work to give uniform bounds on $\triangle^{-1}_{2,\mathfrak{q},(A,\Phi)^\#}$] . Looking at the definition of $\tilde{\triangle}^{-1}_{2,\mathfrak{q},(A,\Phi)^\#}$ it becomes clear that it suffices to show that $\eta_{cyl}\triangle^{-1}_{2,\mathfrak{q},(A,\Phi),cyl}[\eta_{cyl}(b',\psi')]$ is uniformly bounded.

Here we will use again the assumption we mentioned at the end of the previous proof. Namely, we are now assuming that the gauge equivalence classes of our solutions $(A,\Phi) \in \mathcal{M}(M_\tau,\mathfrak{s}_\tau,[\mathfrak{c}])$ all belong to the zero dimensional strata of the moduli spaces. Since the Laplacians are gauge equivariant in the sense that

$$\triangle_{2,\mathfrak{q},u\cdot(A,\Phi)}[u\cdot(b,\psi)] = u\cdot\triangle_{2,\mathfrak{q},(A,\Phi)}(b,\psi)$$

and we are using the gauge equivariant norms $\|\cdot\|_{L^2_{k,A}}$, then for each τ there are only finitely gauge equivalence classes we need to worry about, which immediately implies that for each τ we have a control on the Laplacians (and their inverses). Clearly we still need to see what happens if as we change τ. Let K be a subset of $(\mathbb{R}^+\times -Y)\cup([1,\infty)\times Y)$ and use $\|\triangle_{2,\mathfrak{q},(A,\Phi)}\|_{A,K}$ or $\|\triangle^{-1}_{2,\mathfrak{q},(A,\Phi)}\|_{A,K}$ to denote the operator norms of $\triangle_{2,\mathfrak{q},(A,\Phi)}$ and $\triangle^{-1}_{2,\mathfrak{q},(A,\Phi)}$ when restricted to sections supported on K. Clearly if $K\subset K'$ then $\|\triangle^{-1}_{2,\mathfrak{q},(A,\Phi)}\|_{A,K} \leq \|\triangle^{-1}_{2,\mathfrak{q},(A,\Phi)}\|_{A,K'}$.

Now, recall that we are actually working with a sequence τ_n increasing to ∞ so for each τ_n let $[(A_n,\Phi_n)] \in \mathcal{M}_0(M_{\tau_n},\mathfrak{s}_\tau,[\mathfrak{c}])$ be a (gauge equivalence class of) solution belonging to the zero dimensional strata. Notice that each compact subset $K\subset(\mathbb{R}^+\times -Y)\cup([1,\infty)\times Y)$ eventually belongs to all M_{τ_n} (once τ_n is sufficiently large) so the compactness theorem in this case says that we can choose representatives (A_n,Φ_n)

which converge to a solution (A_∞, Φ_∞) which solves the equations on $Z^+_{\xi,Y}$ and this convergence is strong when restricted to the compact subset K. In particular, it is clear from this that

$$\|\triangle^{-1}_{2,\mathfrak{q},(A_n,\Phi_n)}\|_{A_n,K} \to \|\triangle^{-1}_{2,\mathfrak{q},(A_\infty,\Phi_\infty)}\|_{A_\infty,K} \tag{53}$$

In fact, we must also have that the limiting solution (A_∞, Φ_∞) belongs to the zero dimensional strata because the different strata are labeled by the index of the operator $Q_{\mathfrak{q},(A,\Phi)}$ and this index can only decrease (this is how the broken trajectories appear). However, since the index of each element in the sequence was already zero then the index of the limiting configuration would need to be negative if it were to decrease but transversality rules this out, since we do not have negative dimensional moduli spaces. Therefore the convergence is without broken trajectories, that is, $[(A_\infty, \Phi_\infty)] \in \mathcal{M}_0(Z^+_{Y,\xi}, \mathfrak{s}, [\mathfrak{c}])$. In particular, the fact that no energy is lost along the half-cylinder allows us to improve the convergence in (53) to (we will say more about this in a moment)

$$\|\triangle^{-1}_{2,\mathfrak{q},(A_n,\Phi_n)}\|_{A_n,K_t} \to \|\triangle^{-1}_{2,\mathfrak{q},(A_\infty,\Phi_\infty)}\|_{A_\infty,K_t} \tag{54}$$

where now $K_t = (\mathbb{R}^+ \times -Y) \cup ([1,t] \times Y)$ ($t > 1$ is arbitrary). In particular,

$$\|\triangle^{-1}_{2,\mathfrak{q},(A_\infty,\Phi_\infty)}\|_{A_\infty,K_t} \leq \|\triangle^{-1}_{2,\mathfrak{q},(A_\infty,\Phi_\infty)}\|_{A_\infty,Z^+_{Y,\xi}} \leq C$$

where

$$C = \max\left\{\|\triangle^{-1}_{2,\mathfrak{q},(A_\infty,\Phi_\infty)}\|_{A_\infty,Z^+_{Y,\xi}} \mid [(A_\infty,\Phi_\infty)] \in \mathcal{M}_0(Z^+_{Y,\xi},\mathfrak{s},[\mathfrak{c}])\right\}$$

Since t and the sequence was arbitrary this clearly gives us the uniform bound that we were after so we have proven Theorem (40).

We will now say more about why the convergence (54) is true. For this we need to recall that thanks to the fiber product description of our moduli spaces, we can restrict each solution $[(A_n, \Phi_n)]$ to a solution on the cylindrical end moduli space $\mathcal{M}(\mathbb{R}^+ \times -Y, \mathfrak{s}_\xi, [\mathfrak{c}])$, which we will denote as $[(A_n, \Phi_n)]_{cyl} \in \mathcal{M}(\mathbb{R}^+ \times -Y, \mathfrak{s}_\xi, [\mathfrak{c}])$. Likewise, the limiting solution $[(A_\infty, \Phi_\infty)]$ can also be restricted to this moduli space so we have as well that $[(A_\infty, \Phi_\infty)]_{cyl} \in \mathcal{M}(\mathbb{R}^+ \times -Y, \mathfrak{s}_\xi, [\mathfrak{c}])$. When we described the configuration spaces at the beginning of the paper we used the topology of strong convergence on compact subsets $L^2_{k,loc}$ to define the moduli spaces. However, as explained in Theorem 13.3.5 of [32], the same moduli space $\mathcal{M}(\mathbb{R}^+ \times -Y, \mathfrak{s}_\xi, [\mathfrak{c}])$ can also be obtained if we had used the stronger topology of L^2_k convergence along the entire half-cylinder $\mathbb{R}^+ \times -Y$ [they really did this for the moduli space on the cylinder $\mathbb{R} \times Y$ but it does not affect our claim]. Therefore, the convergence of $[(A_n, \Phi_n)]_{cyl}$ towards $[(A_\infty, \Phi_\infty)]_{cyl}$ can be regarded as a strong convergence with respect to the $L^2_{k,A_\mathfrak{c}}$ norm, where $A_\mathfrak{c}$ represents the translation invariant connection associated to a smooth representative $\mathfrak{c}$ of the critical point $[\mathfrak{c}]$. In other words, we can choose representatives of $[(A_n, \Phi_n)]_{cyl}$ and $[(A_\infty, \Phi_\infty)]_{cyl}$ so that

$$\begin{cases} A_n = A_\mathfrak{c} + a_n \\ A_\infty = A_\mathfrak{c} + a_\infty \\ \Phi_n = \Phi_\mathfrak{c} + \phi_n \\ \Phi_\infty = \Phi_\mathfrak{c} + \phi_\infty \end{cases}$$

where $\Phi_\mathfrak{c}$ is a translation invariant representative of $\mathfrak{c}$ and we have that

$$\lim_{n\to\infty} \|A_n - A_\infty\|_{L^2_k(\mathbb{R}^+\times -Y)} = \lim_{n\to\infty} \|a_n - a_\infty\|_{L^2_k(\mathbb{R}^+\times -Y)} = 0$$
$$\lim_{n\to\infty} \|\Phi_n - \Phi_\infty\|_{L^2_{k,A_\mathfrak{c}}(\mathbb{R}^+\times -Y)} = \lim_{n\to\infty} \|\phi_n - \phi_\infty\|_{L^2_{k,A_\mathfrak{c}}(\mathbb{R}^+\times -Y)} = 0$$

The norms $\|\cdot\|_{L^2_{k,A_n}}$ and $\|\cdot\|_{L^2_{k,A_\mathfrak{c}}}$ can now be compared thanks to the Sobolev multiplication theorems (since for example $\nabla_{A_n}\bullet = \nabla_{A_\infty}\bullet + (a_n - a_\infty)\otimes\bullet$ with similar formulas for the higher derivatives] and the previous limits make it clear that the operator norm convergence (53) on compact subsets K can be improved to the operator norm convergence (54) on sets of the form “half-cylinder +compact”.

Our next step is to explain the properties of the gluing map one obtains using the invertibility of the Laplacian.

Remark 42. Many of the following arguments will have a similar structure to the one before. Namely, because we are taking solutions belonging to the zero dimensional strata for an individual τ we will find a bound, but a priori this may depend on τ. However, as we take τ_n sufficiently large the bounds end up being controlled by limiting case '$\tau = \infty$', since we can invoke the strong convergence on the half-cylindrical end. Since the arguments are essentially the same in each case we will not repeat the strategy so we will just say that it "follows by similar arguments".

6.5 Definition and some Properties of the Gluing Map: As explained before, if we write $V_{\mathfrak{q}} = \triangle_{2,\mathfrak{q},(A,\Phi)^{\#}}(b', \psi')$ then the gluing equation 51 is equivalent to solving the equation

$$V_{\mathfrak{q}} = S_{\mathfrak{q},(A,\Phi)^{\#}}(V_{\mathfrak{q}}) - \mathfrak{F}_{\mathfrak{p}_{M'_\tau}}(A,\Phi)^{\#}$$

The solution of this equation requires an application of the contraction mapping theorem, which requires us to first show that the map $S_{\mathfrak{q}}$ is a uniform contraction in the following sense (this is analogue of lemma 3.1.8 in [40]):

Theorem 43. *For every k large enough there exist constants $\alpha_k > 0$, $\kappa_k \in (0, 1/2)$ such that for every τ large enough, every $N_0 \geq 1$ and every approximate solution of the Seiberg Witten equations $(A,\Phi)^{\#}$ on M'_τ , which comes from an actual solution (A,Φ) on M_τ whose gauge equivalence class $[A,\Phi]$ belongs to the zero dimensional strata of the moduli space $\mathcal{M}_0(M_\tau;\mathfrak{s}_\tau;[\mathfrak{c}])$,we have for all $V_1, V_2 \in L^2_k(M'_\tau; i\mathfrak{su}(S^+) \oplus S^-, g'_\tau; A^{\#})$*

$$\|V_1\|_{L^2_k(g'_\tau,A^{\#})}, \|V_2\|_{L^2_k(g'_\tau;A^{\#})} \leq \alpha_k \implies \left\|S_{\mathfrak{q},(A,\Phi)^{\#}}(V_2) - S_{\mathfrak{q},(A,\Phi)^{\#}}(V_1)\right\|_{L^2_k(g'_\tau,A^{\#})} \leq \kappa_k \|V_2 - V_1\|_{L^2_k(g'_\tau,A^{\#})}$$

Proof. Recall that

$$S_{\mathfrak{q},(A,\Phi)^{\#}}(V_{\mathfrak{q}}) = -\mathcal{Q} \circ \left[\left(\mathcal{D}_{(A,\Phi)^{\#}}\mathfrak{F}_{\mathfrak{q}}\right)^*\right] \left(\triangle^{-1}_{2,\mathfrak{q},(A,\Phi)^{\#}} V_{\mathfrak{q}}\right) - P\left(\triangle^{-1}_{2,\mathfrak{q},(A,\Phi)^{\#}} V_{\mathfrak{q}}\right)$$

We will mention the main differences compared with the proof given in [40]. First of all, we need the bounds in proposition 11.4.1 in [32], which say that for $k \geq 2$

$$\|\mathcal{D}^l_{(A,\Phi)}\hat{\mathfrak{q}}\| \leq C(1 + \|a\|_{L^2_k(Z)})^{2k(l+1)}(1 + \|\Phi\|)_{L^2_{k,A}(Z)})^{l+1} \tag{55}$$

Here C is a constant independent of the configuration and in this theorem Z denotes a finite cylinder, while $A = A_0 + a \otimes 1$ for some reference configuration. First of all these bounds can be used on the half-cylinder $\mathbb{R}^+ \times (-Y)$ as well. Simply decompose it as

$$\mathbb{R}^+ \times (-Y) = \bigcup_{n \geq 0} \underbrace{[n, n+1] \times (-Y)}_{Z_n}$$

If $\bullet$ denotes an element in the domain of $\mathcal{D}^l_{(A,\Phi)}\hat{\mathfrak{q}}$ then we have

$$\begin{aligned} &\|\mathcal{D}^l_{(A,\Phi)}\hat{\mathfrak{q}}(\bullet)\|_{\mathbb{R}^+\times -Y} \\ &= \sum_{n=0}^{\infty}\|\mathcal{D}^l_{(A,\Phi)}\hat{\mathfrak{q}}(\bullet)\|_{Z_n} \\ \leq C\sum_{n=0}^{\infty}&(1+\|a\|_{L^2_k(Z_n)})^{2k(l+1)}(1+\|\Phi\|)_{L^2_{k,A}(Z_n)})^{l+1}\|\bullet\|_{Z_n} \end{aligned}$$

where in the last step we used the bounds coming from the operator norm (55). If we define

$$C_{n,(A,\Phi)} = (1+\|a\|_{L^2_k(Z_n)})^{2k(l+1)}(1+\|\Phi\|)_{L^2_{k,A}(Z_n)})^{l+1}$$

then it is not too difficult to see that

$$C_{\max,(A,\Phi)} = \max_n C_{n,(A,\Phi)} < \infty$$

One way to see this is the the previous quantities $C_{n,(A,\Phi)}$ do not differ too much from those for the translation invariant solution $C_{n,(A_{\mathfrak{c}},\Phi_{\mathfrak{c}})}$, which are independent of n. In any case we end up with

$$\|\mathcal{D}^l_{(A,\Phi)}\hat{\mathfrak{q}}(\bullet)\|_{\mathbb{R}^+\times -Y} \leq CC_{\max,(A,\Phi)}\sum_{n=0}^{\infty}\|\bullet\|_{Z_n} = CC_{\max,(A,\Phi)}\|\bullet\|_{\mathbb{R}^+\times -Y}$$

Since $\bullet$ was arbitrary this says that each $\mathcal{D}^l_{(A,\Phi)}\hat{\mathfrak{q}}$ is a bounded operator on the half-cylinder. For each τ , we are only dealing with finitely many gauge equivalence classes of solutions because of our assumption on the strata so the bounds are once again controlled for a fixed τ. By analogous arguments, one can find bounds which actually become independent of τ so that $\|\mathcal{D}^l_{(A,\Phi)}\hat{\mathfrak{q}}\| \leq C_l$ for some constant C_l on the half-infinite cylinder.

The other ingredient is that the leading term of $P\left(\triangle^{-1}_{2,\mathfrak{q},(A,\Phi)^\#}V_{\mathfrak{q}}\right)$ is quadratic in the following sense. To emphasize its dependence on V, we will write $P\left(\triangle^{-1}_{2,\mathfrak{q},(A,\Phi)^\#}V_{\mathfrak{q}}\right)$ as

$$f(V) = \mathfrak{q}\left((A,\Phi)^\# + (\mathcal{D}_{(A,\Phi)^\#}\mathfrak{F}_{\mathfrak{q}})^*\triangle^{-1}_{2,\mathfrak{q},(A,\Phi)^\#}V\right) - \mathfrak{q}(A,\Phi)^\# - \left(\mathcal{D}_{(A,\Phi)^\#}\mathfrak{q}\right)\circ(\mathcal{D}_{(A,\Phi)^\#}\mathfrak{F}_{\mathfrak{q}})^*\triangle^{-1}_{2,\mathfrak{q},(A,\Phi)^\#}V$$

We want to compute $f'(V)$ and $f''(V)$, that is, the Banach spaces derivatives with respect to V . For this define the functions

$$\begin{cases} f_1(V) = \mathfrak{q}\left((A,\Phi)^{\#} + (\mathcal{D}_{(A,\Phi)^{\#}}\mathfrak{F}_{\mathfrak{q}})^{*}\triangle^{-1}_{2,\mathfrak{q},(A,\Phi)^{\#}}V\right) - \mathfrak{q}(A,\Phi)^{\#} \\ f_2(V) = \left(\mathcal{D}_{(A,\Phi)^{\#}}\mathfrak{q}\right)\circ(\mathcal{D}_{(A,\Phi)^{\#}}\mathfrak{F}_{\mathfrak{q}})^{*}\triangle^{-1}_{2,\mathfrak{q},(A,\Phi)^{\#}}V \end{cases}$$

so that

$$f(V) = f_1(V) - f_2(V)$$

Since $f_2(V)$ is linear in V it is easy to determine that

$$f_2'(V) = \left(\mathcal{D}_{(A,\Phi)^{\#}}\mathfrak{q}\right)\circ(\mathcal{D}_{(A,\Phi)^{\#}}\mathfrak{F}_{\mathfrak{q}})^{*}\triangle^{-1}_{2,\mathfrak{q},(A,\Phi)^{\#}}(V)$$

Clearly f_2' is independent as a linear transformation of the "basepoint" (which is hidden in our notation) so we will have that $f_2^{(n)} = 0$ for $n \geq 2$. To compute the derivative of $f_1(V)$ think of the Taylor expansion of $\mathfrak{q}$ about $(A,\Phi)^{\#}$ (which plays the role of 0 in our affine space interpretation for the domain of $\mathfrak{q}$ so we can use corollary 4.4 in Chapter 1 from [33]). In this way

$$f_1(V) = \left(\mathcal{D}_{(A,\Phi)^{\#}}\mathfrak{q}\right)\circ\left((\mathcal{D}_{(A,\Phi)^{\#}}\mathfrak{F}_{\mathfrak{q}})^{*}\triangle^{-1}_{2,\mathfrak{q},(A,\Phi)^{\#}}\right)V + \frac{1}{2}\left(\mathcal{D}^2_{(A,\Phi)^{\#}}\mathfrak{q}\right)V^{(2)} + \cdots +$$

where $V^{(2)} = (V,V)$. Notice that the first term is exactly $f_2(V)$! Therefore

$$\begin{cases} f_1'(V) = f_2(V) \\ f_2'' = \left(\mathcal{D}^2_{(A,\Phi)^{\#}}\mathfrak{q}\right) \end{cases}$$

This means that the leading term for the Taylor expansion of $f(V)$ will be quadratic, that is

$$f(V) = \frac{1}{2}\left(\mathcal{D}^2_{(A,\Phi)^{\#}}\mathfrak{q}\right)V^{(2)} + \cdots + \tag{56}$$

. To see why this is important notice that in the case of $-\mathcal{Q}\circ\left[\left(\mathcal{D}_{(A,\Phi)^{\#}}\mathfrak{F}_{\mathfrak{q}}\right)^{*}\right]\left(\triangle^{-1}_{2,\mathfrak{q},(A,\Phi)^{\#}}V_{\mathfrak{q}}\right)$ Mrowka and Rollin found a bound (after eq. 3.14 [40]) which can be adapted to our case to read

$$\left\| \mathcal{Q} \circ \left[\left(\mathcal{D}_{(A,\Phi)^{\#}} \mathfrak{F}_{\mathfrak{q}} \right)^{*} \right] \left(\triangle^{-1}_{2,\mathfrak{q},(A,\Phi)^{\#}} V_2 \right) - \mathcal{Q} \circ \left[\left(\mathcal{D}_{(A,\Phi)^{\#}} \mathfrak{F}_{\mathfrak{q}} \right)^{*} \right] \left(\triangle^{-1}_{2,\mathfrak{q},(A,\Phi)^{\#}} V_1 \right) \right\|_{L^2_k(g'_\tau, A^{\#})}$$
$$\leq C'_k \|V_2 + V_1\|_{L^2_k(g_\tau, A^{\#})} \|V_2 - V_1\|_{L^2_k(g'_\tau, A^{\#})} \tag{57}$$

where C'_k is a constant which is independent of τ (once it is large enough) , the approximate solution (A, Φ) and the constant $N_0 \geq 1$ used in the perturbations defining the connected sum along Y operation. Since we are assuming that $\|V_1\|_{L^2_k}, \|V_2\|_{L^2_k} \leq \alpha_k$, we can use the triangle inequality to obtain that

$$\|V_2 + V_1\|_{L^2_k(g_\tau, A^{\#})} \leq \|V_2\|_{L^2_k(g_\tau, A^{\#})} + \|V_1\|_{L^2_k(g_\tau, A^{\#})} \leq 2\alpha_k$$

so the inequality (57) reads

$$\left\| \mathcal{Q} \circ \left[\left(\mathcal{D}_{(A,\Phi)^{\#}} \mathfrak{F}_{\mathfrak{q}} \right)^{*} \right] \left(\triangle^{-1}_{2,\mathfrak{q},(A,\Phi)^{\#}} V_2 \right) - \mathcal{Q} \circ \left[\left(\mathcal{D}_{(A,\Phi)^{\#}} \mathfrak{F}_{\mathfrak{q}} \right)^{*} \right] \left(\triangle^{-1}_{2,\mathfrak{q},(A,\Phi)^{\#}} V_1 \right) \right\|_{L^2_k(g'_\tau, A^{\#})}$$
$$\leq 2\alpha_k C'_k \|V_2 - V_1\|_{L^2_k(g'_\tau, A^{\#})}$$

Hence to make this contribution less than $\frac{\kappa_k}{2} \|V_2 - V_1\|_{L^2_k(g'_\tau, A^{\#})}$ we just need to take $\alpha_k < \frac{\kappa}{4C'_k}$.

Likewise, since

$$P\left(\triangle^{-1}_{2,\mathfrak{q},(A,\Phi)^{\#}} V_2 \right) - P\left(\triangle^{-1}_{2,\mathfrak{q},(A,\Phi)^{\#}} V_1 \right)$$

is the same as

$$f(V_2) - f(V_1)$$

and each has quadratic leading terms according to equation (56) , the norm

$$\left\| P\left(\triangle^{-1}_{2,\mathfrak{q},(A,\Phi)^{\#}} V_2 \right) - P\left(\triangle^{-1}_{2,\mathfrak{q},(A,\Phi)^{\#}} V_1 \right) \right\|_{L^2_k(g'_\tau, A^{\#})}$$

can now be bounded by and expression of the form

$$f(\alpha_k, C''_k) \|V_2 - V_1\|_{L^2_k(g'_\tau, A^{\#})}$$

where $f(\alpha_k, C''_k)$ will be some expression in α_k whose particular details do not interest us and C''_k denotes constants that do not depend on τ or the solution used. In any case, the important thing is that we can again choose α_k so that $f(\alpha_k, C''_k) < \frac{\kappa_k}{2}$ and so combining both inequalities the result follows. $\square$

At this point we can use the Contraction Mapping Theorem (proposition 2.3.5 [40]) to obtain our definition of the **gluing map** (Theorem 3.1.9 [40]):

Theorem 44. *There exists constants $\alpha_k, c_k > 0$ such that for every τ large enough, every solution (A, Φ) of the Seiberg-Witten equations on M_τ whose gauge equivalence class belongs to the zero dimensional strata of the moduli space $\mathcal{M}(M_\tau; \mathfrak{s}_\tau; [\mathfrak{c}])$ and every constant $N_0 \geq 1$, there is a unique section (b', ψ') on M'_τ such that*

$$\mathfrak{G}_\tau(A, \Phi) = (A, \Phi)^{\#} + (\mathcal{D}_{\mathfrak{q},(A,\Phi)^{\#}}\mathfrak{F}_{\mathfrak{q}})^*(b', \psi')$$

is a solution of the Seiberg-Witten equations with $\|(b', \psi)\|_{L^2_{k+2}(g_{\tau'}, A^{\#})} \leq \alpha_k$. Furthermore, the map is gauge equivariant and induces a map

$$\mathfrak{G}_\tau : \mathcal{M}_0(M_\tau; \mathfrak{s}_\tau; [\mathfrak{c}]) \to \mathcal{M}_0(M'_\tau; \mathfrak{s}'_\tau; [\mathfrak{c}])$$

where $\mathcal{M}_0(M_\tau; \mathfrak{s}_\tau; [\mathfrak{c}])$ denotes the zero dimensional strata of $\mathcal{M}(M_\tau; \mathfrak{s}_\tau; [\mathfrak{c}])$. Moreover both $\|(b', \psi')\|_{L^2_{k+2}(g'_\tau, A^{\#})}$ and $\|\mathfrak{G}_\tau(A, \Phi) - (A, \Phi)^{\#}\|_{L^2_{k+1}(g'_\tau, A^{\#})}$ are bounded by
(58)
$$\|(b', \psi')\|_{L^2_{k+2}(g'_\tau, A^{\#})}, \quad \|\mathfrak{G}_\tau(A, \Phi) - (A, \Phi)^{\#}\|_{L^2_{k+1}(g'_\tau, A^{\#})} \leq c_k \|\mathfrak{F}_{\mathfrak{p}_{M'_\tau}}(A, \Phi)^{\#}\|_{L^2_k(g'_\tau, A^{\#})}$$

Furthermore, this map is an injection and since the construction is reversible it is a bijection. Hence the $\bmod 2$ *cardinality of $\mathcal{M}_0(M_\tau; \mathfrak{s}_\tau; [\mathfrak{c}])$ and $\mathcal{M}_0(M'_\tau; \mathfrak{s}'_\tau; [\mathfrak{c}])$ is the same.*

Proof. We need to verify that the gluing map preserves the dimensionality of the zero dimensional strata. For this recall that if $[(A, \Phi)]$ belongs to $\mathcal{M}_0(M_\tau; \mathfrak{s}_\tau; [\mathfrak{c}])$, then the index of the operator

$$Q_{\mathfrak{q},(A,\Phi)} = \mathbf{d}^*_{(A,\Phi)} \oplus \mathcal{D}_{(A,\Phi)}\mathfrak{F}_{\mathfrak{p}_{M_\tau}}$$

is precisely the dimension of the strata to which $[(A_n, \Phi_n)]$ belongs. Since the transversality condition already implied that $Q_{\mathfrak{q},(A,\Phi)}$ was surjective we conclude that in fact $Q_{\mathfrak{q},(A,\Phi)}$ is an invertible operator.

Now we use the same splicing procedure as in the case of finding the inverse for the Seiberg Witten Laplacians $\triangle_{2,\mathfrak{q},(A,\Phi)^{\#}}$. Namely, the operator $\eta_{end} Q_{(A'_{0,\tau}, \Phi'_\tau)}(\eta_{end} \cdot)$ associated to the canonical solution $(A_{0,\tau'}, \Phi'_\tau)$ on the AFAK end Z' will be invertible on a suitable domain using Lemma 3.1.4 in [40]. Therefore, we can patch together

$\eta_{cyl} Q^{-1}_{\mathfrak{q},(A,\Phi)}(\eta_{cyl}\cdot)$ and $\eta_{end} Q^{-1}_{(A_{0,\tau'},\Phi'_\tau)}(\eta_{end}\cdot)$ to show that $Q_{\mathfrak{q},(A,\Phi)^\#}$ will become invertible.

To compare $Q_{\mathfrak{q},(A,\Phi)^\#}$ and $Q_{\mathfrak{q},\mathfrak{G}_{\mathfrak{q}}(A,\Phi)}$ notice that inequality (48) and the bound in (58) allow us to conclude that the operator norms of $Q_{\mathfrak{q},(A,\Phi)^\#}$ and $Q_{\mathfrak{q},\mathfrak{G}_{\mathfrak{q},\tau}(A,\Phi)}$ are very close to each other. Since being an invertible operator is an open condition it follows that $Q_{\mathfrak{q},\mathfrak{G}_{\mathfrak{q},\tau}(A,\Phi)}$ will have to be invertible as well.

Now we must address the injectivity of our map. It is essentially the same as the proof of Corollary 3.2.2 in [40]. If the injectivity of the map is not true for τ large enough then we obtain a sequence $\tau_j \to \infty$ and solutions to the Seiberg Witten equations (A_j,Φ_j) and $(\tilde{A}_j,\tilde{\Phi}_j)$ on M_{τ_j} such that for all j, $[A_j,\Phi_j] \neq [\tilde{A}_j,\tilde{\Phi}_j]$ while $[\mathfrak{G}_{\tau_j}(A_j,\Phi_j)] = [\mathfrak{G}_{\tau_j}(\tilde{A}_j,\tilde{\Phi}_j)]$. Moreover, after taking gauge transformation we can assume that they have exponential decay and converge on every compact subset of $Z^+_{Y,\xi}$ to some solutions (A_∞,Φ_∞) and $(\tilde{A}_\infty,\tilde{\Phi}_\infty)$. Moreover, for all j we have $[A_j,\Phi_j] \neq [\tilde{A}_j,\tilde{\Phi}_j]$ as gauge equivalence classes . We want to show that if $(A_\infty,\Phi_\infty) = (\tilde{A}_\infty,\tilde{\Phi}_\infty)$ then

$$\|(A_j,\Phi_j) - (\tilde{A}_j,\tilde{\Phi}_j)\|_{L^2_{k+1}(g_\tau,A_j)} \to 0 \tag{59}$$

First of all, from (58) and (48) we already know that have that

$$\|\mathfrak{G}_{\tau_j}(A_j,\Phi_j) - (A_j,\Phi_j)^\#\|_{L^2_{k+1}(g'_\tau,A^\#)} \to 0 \tag{60}$$

hence $\mathfrak{G}_{\tau_j}(A_j,\Phi_j)$ converges on every compact towards (A_∞,Φ_∞) since (A_j,Φ_j) does. Similarly $\mathfrak{G}_{\tau_j}(\tilde{A}_j,\tilde{\Phi}_j)$ converges to $(\tilde{A}_\infty,\tilde{\Phi}_\infty)$. The fact that $\mathfrak{G}(A_j,\Phi_j)$ and $\mathfrak{G}(\tilde{A}_j,\tilde{\Phi}_j)$ are gauge equivalent for each j implies that the limits are also gauge equivalent. Hence the limits of (A_j,Φ_j) and $(\tilde{A}_j,\tilde{\Phi}_j)$ are gauge equivalent. After making further gauge transformations, we can then assume that (A_j,Φ_j) and $(\tilde{A}_j,\tilde{\Phi}_j)$ converge toward the same limit (A_∞,Φ_∞) on $Z^+_{Y,\xi}$. In principle, this would be weak convergence along the cylindrical end $\mathbb{R}^+ \times Y$. However, by the discussion from before when we analyzed the restriction of a solution to the cylindrical moduli space $\mathcal{M}(\mathbb{R}^+ \times -Y, \mathfrak{s}_\xi, [\mathfrak{c}])$, we can actually assume that the convergence is strong along the entire cylindrical end, in other words, (A_j,Φ_j) and $(\tilde{A}_j,\tilde{\Phi}_j)$ are converging strongly towards (A_∞,Φ_∞) on the cylindrical end as well. This allows us to conclude that (59) is true.

Since we now have strong convergence along the cylinder then the estimates in [40] continue to hold in that we can find a "radius" r small enough [independent of τ] for which whenever there is j such that $\|(A_j, \Phi_j) - (\tilde{A}_j, \tilde{\Phi}_j)\|_{L^2_{k+1}(g_\tau, A_j)} < r$ then (A_j, Φ_j) and $(\tilde{A}_j, \tilde{\Phi}_j)$ are gauge equivalent [this is a much weaker version of their proposition 3.2.1]. From (59) it is clear that such j will exist and hence we are done. □

We have reached the proof of the naturality property for the contact invariant under strong symplectic cobordisms, that is, Theorem (1). To see why this is the case recall that in the first part of this paper (section 5 to be more specific) we showed that

$$\widetilde{HM}_\bullet(W^\dagger, \mathfrak{s}_\omega)\mathbf{c}(\xi') = \mathbf{c}(\xi', Y)$$

The gluing theorem we just proved was aimed at showing that

$$\mathbf{c}(\xi', Y) = \mathbf{c}(\xi) \tag{61}$$

To see why this the case we want to apply Theorem (44) to the case in which the second AFAK end is $Z' = (0, \infty) \times Y$. It is not difficult to see that in this case the corresponding manifolds M'_τ in fact all agree with each other in the sense that their metrics, spinor bundles, symplectic forms, etc are the same, and in fact coincide with the manifold $Z^+_{Y,\xi}$ used to define the contact invariant of (Y, ξ). In particular, we have that for all $\tau > 0$ that

$$|\mathcal{M}_0(M'_\tau, \mathfrak{s}', [\mathfrak{c}])| \mod 2 = |\mathcal{M}_0(Z^+_{Y,\xi}, \mathfrak{s}, [\mathfrak{c}])| \mod 2$$

Now choose τ_{large} such that

$$|\mathcal{M}_0(M_\tau; \mathfrak{s}_{\tau_{large}}; [\mathfrak{c}])| \mod 2 = |\mathcal{M}_0(M'_\tau, \mathfrak{s}', [\mathfrak{c}])| \mod 2 = |\mathcal{M}_0(Z^+_{Y,\xi}, \mathfrak{s}, [\mathfrak{c}])| \mod 2$$

If we think of using the numbers $|\mathcal{M}_0(M_\tau; \mathfrak{s}_{\tau_{large}}; [\mathfrak{c}])| \mod 2$ in order to define a chain-level element $c(\xi', Y, \tau_{large}) \in \check{C}_*(-Y, \mathfrak{s}_\xi)$ as in formula (5), then the previous identity says that at the chain level

$$c(\xi', Y, \tau_{large}) = c(\xi)$$

which in particular gives the identity of homology classes

$$\mathbf{c}(\xi', Y, \tau_{large}) = \mathbf{c}(\xi) \tag{62}$$

Now, $c(\xi', Y, \tau_{large})$ is not the same chain-level element as the element $c(\xi', Y)$ we used during the initial sections of this paper. However, it is not difficult to see that use a one parameter family of metrics $g(t)$ and perturbations $\mathfrak{p}_0(t)$ on $M_{\tau_{large}}$ (which is diffeomorphic to $W^{\dagger}_{\xi', Y}$) to go from one element to the other. Therefore, one can consider a parameterized moduli space and use the same argument as in section 5 to conclude that $c(\xi', Y, \tau_{large})$ and $c(\xi', Y)$ do define the same homology element in $\widetilde{HM}_\bullet(-Y, \mathfrak{s}_\xi)$, in other words

$$\mathbf{c}(\xi', Y, \tau_{large}) = \mathbf{c}(\xi', Y) \tag{63}$$

Combining the identities (61), (62) and (63) the naturality result follows, i.e, we have shown that for a strong symplectic cobordism $(W, \omega) : (Y, \xi) \to (Y', \xi')$ one has

$$\widetilde{HM}_\bullet(W^{\dagger}, \mathfrak{s}_\omega)\mathbf{c}(\xi') = \mathbf{c}(\xi', Y)$$

7. Appendix

7.1 The Grading of the Contact Invariant. For sake of completeness we will show in this section that the contact invariant $\mathbf{c}(\xi)$ is supported in the homotopy class of the plane field ξ. The proof follows from a simple application of the excision theorem, and since we used this property to show the vanishing of the contact invariant for overtwisted structures we felt it should at least be explained why it is true.

As we mentioned at the beginning of the paper, the Floer homology groups are graded by the homotopy classes of oriented 2-plane fields. Now we will give a more precise statement [32, section 28.2].

Let Y be a closed oriented 3 manifold with a spin-c structure $\mathfrak{s}$. After making a choice of metric and suitable perturbations, we can assign to each critical point $[\mathfrak{a}]$ a homotopy class of non-vanishing sections Φ_0 as follows.

Fix a compact manifold X with oriented boundary Y carrying a spin-c structure $\mathfrak{s}_X$ extending $\mathfrak{s}$. Consider the solutions on X^* which are asymptotic to $[\mathfrak{a}]$. Pick a component z and consider $\mathrm{gr}_z(X, \mathfrak{s}_X, [\mathfrak{a}])$, which is the dimension of $M_z(X^*, \mathfrak{s}_X, [\mathfrak{a}])$ if the moduli space is non-empty and regular. Choose a section Φ_0 of $S = S^+ \mid_Y$ such that the relative Euler class satisfies

$$e(S^+, \Phi_0)[X, \partial X] = \mathrm{gr}_z(X, \mathfrak{s}_X; [\mathfrak{a}])$$

Proposition 28.2.2 in [32] shows that the isomorphism class of (S, Φ_0) is, up to homotopy of Φ_0 , independent of X and depends only on $Y, \mathfrak{s}, [\mathfrak{a}]$. Using Lemma 28.1.1 in [32] such isomorphism class gives an oriented plane plane field ξ. Moreover, such an element can be regarded by Lemma 28.2.1 as defining an element in $\mathbb{Z}/(d\mathbb{Z})$, where d is the divisibilityof $c_1(\mathfrak{s})$. Recall that if $\mathfrak{s}$ is torsion then the divisibility is 0 by definition. If it is not torsion, $d(\mathfrak{s}) = \text{g.c.d}\{\langle c_1(\mathfrak{s}), \sigma\rangle \mid \sigma \in H_2(Y, \mathbb{Z})\}$.

In order to show that the grading of the contact invariant $[\check{\psi}_{Y,\xi}]$ is represented by the homotopy class of ξ we need the following version of the excision theorem [41, section 5.3]:

Proposition 45. *Let A_1, B_1, A_2, B_2 be (not necessarily compact) oriented 4 manifolds such that $\partial A_1 = \partial A_2 = Y$ and $\partial B_1 = \partial B_2 = -Y$ where Y is a compact oriented*

3 manifold. Let operators

$$D_1 : L^2(A_1 \cup B_1) \to L^2(A_1 \cup B_1)$$
$$D_2 : L^2(A_2 \cup B_2) \to L^2(A_2 \cup B_2)$$

be (unbounded) Fredholm differential operators such that $D_1 = D_2$ on Y. Suppose that

$$\bar{D}_1 : L^2(A_1 \cup B_2) \to L^2(A_1 \cup B_2)$$
$$\bar{D}_2 : L^2(A_2 \cup B_1) \to L^2(A_2 \cup B_1)$$

be defined as

$$\bar{D}_1 = \begin{cases} D_1 & \text{on } A_1 \\ D_2 & \text{on } B_2 \end{cases}$$

$$\bar{D}_2 = \begin{cases} D_2 & \text{on } A_2 \\ D_1 & \text{on } A_1 \end{cases}$$

are (unbounded) Fredholm differential operators. Then

$$indD_1 + indD_2 = ind\bar{D}_1 + ind\bar{D}_2$$

Theorem 46. *Suppose that Y is a closed oriented 3 manifold with contact structure ξ. Let $d = div(\mathfrak{c}_1(\mathfrak{s}_\xi))$. If the critical point $[\mathfrak{a}]$ makes a non-trivial contribution to the contact invariant $\mathbf{c}(\xi)$, that is, for some z we have $m_z(Z^+_{Y,\xi}, \mathfrak{s}, [\mathfrak{a}]) \neq 0$, then the element in $\mathbb{Z}/d\mathbb{Z}$ determined by the homotopy class of oriented plane field $\xi_{[\mathfrak{a}]}$ defined by $[\mathfrak{a}]$ is the same as the element in $\mathbb{Z}/d\mathbb{Z}$ determined by the homotopy class of the oriented plane field defined by ξ.*

Proof. Recall that if $m_z(Z^+_{Y,\xi}, \mathfrak{s}, [\mathfrak{a}]) \neq 0$ then the moduli space $\mathcal{M}_z(Z^+_{Y,\xi}, \mathfrak{s}, [\mathfrak{a}])$ must be 0 dimensional. At the same time, this is the index of a certain operator D_1 defined on the manifold

$$\underbrace{\left(\mathbb{R}^+ \times -Y\right)}_{A_1} \cup \underbrace{\left([1,\infty) \times Y\right)}_{B_1}$$

On the other hand, $\xi_{[\mathfrak{a}]}$ is determined by choosing a pair $(X, \mathfrak{s}_X)$ together with a section $\Phi_{[\mathfrak{a}]}$ of $S = S^+ \mid_Y$ satisfying

$$e(S^+, \Phi_{[\mathfrak{a}]})[X, \partial X] = \mathrm{gr}_z(X, \mathfrak{s}_X; [\mathfrak{a}])$$

where $\mathrm{gr}_z(X, \mathfrak{s}_X; [\mathfrak{a}])$ is the index of an operator D_2 defined on

$$\underbrace{X}_{A_2} \cup \underbrace{\left(\mathbb{R}^+ \times Y\right)}_{B_2}$$

By the excision principle,

$$\mathrm{ind} D_1 + \mathrm{ind} D_2 = \mathrm{ind} \bar{D}_1 + \mathrm{ind} \bar{D}_2 \tag{64}$$

where $\bar{D}_1$,$\bar{D}_2$ are defined on the manifolds

$$\bar{D}_1 : \quad (\mathbb{R}^+ \times -Y) \cup (\mathbb{R}^+ \times Y) \simeq \mathbb{R} \times Y$$

$$\bar{D}_2 : \quad X \cup [1, \infty) \times Y$$

By hypothesis

$$\mathrm{ind} D_1 = \dim \mathcal{M}_z(Z^+_{Y,\xi}, \mathfrak{s}, [\mathfrak{a}]) = 0$$

Also, by theorem 3.3 in [31]

$$\mathrm{ind} \bar{D}_2 = e(S^+, \Phi_\xi)[X, \partial X]$$

where Φ_ξ is the section defined by the contact structure ξ. By lemma 14.4.6 in [32],

$$\mathrm{ind} \bar{D}_1 = \mathrm{gr}_{z_u}([\mathfrak{a}], \mathfrak{s}_\xi, [\mathfrak{a}]) = ([u] \cup c_1(S))[Y]$$

where $[u]$ denotes the homotopy class of $u : Y \to S^1$ corresponding to the loop z_μ. Therefore 64 becomes

$$e(S^+, \Phi_{[\mathfrak{a}]})[X, \partial X] = ([u] \cup c_1(S))[Y] + e(S^+, \Phi_\xi)[X, \partial X]$$

The statement now follows once we reduce the last equation mod $\mathrm{div}(c_1(\mathfrak{s}_\xi))$. ☐

7.2 Non vanishing of the Contact Invariant for (S^3, ξ_{tight})**.** We also quickly mention a proof that $\mathbf{c}(\xi_{tight}) \neq 0$, that is, the contact invariant of the 3-sphere for the unique tight contact structure is non-zero. One basically copies the idea behind the proof of Proposition 6.8 in [30], so we will just remind the reader what this consisted of. Clearly there is an exact filling of (S^3, ξ_{tight}), which is the four ball with the standard symplectic structure (B^4, ω_{std}).

We can equip B^4 with a metric with positive scalar curvature containing a collar region $[0, 1] \times S^3$ in which the metric is cylindrical and $Y = S^3$ is given the round metric, we let $[B_0, 0] \in \mathcal{B}(S^3, \mathfrak{s}_0)$ be the unique critical point for the unperturbed three dimensional Seiberg-Witten equations [32, eq 4.4]. We can choose a perturbation so that there is still a unique critical point $[\beta] = [B, 0]$ and the perturbed Dirac operator has simple spectrum. We can label the eigenvalues in increasing order as λ_i, with λ_0 the first positive eigenvalue and the corresponding critical point in $\mathcal{B}^\sigma_k(S^3, \mathfrak{s}_0)$ as $[\mathfrak{b}_i]$. Then from lemma 27.4.2 in [32] we conclude that

$$n_z((B^4)^*, [\mathfrak{b}_i]) = 0 \text{ unless } [\mathfrak{b}_i] = [\mathfrak{b}_{-1}]$$

Moreover, since B^4 has only one spin-c structure and $\mathcal{M}(B^4, [\mathfrak{b}_{-1}])$ has only one component [32, section 24.4] we deduce that

$$n(B^4, [\mathfrak{b}]) = \begin{cases} 1 & \text{if } [\mathfrak{b}] = [\mathfrak{b}_{-1}] \\ 0 & \text{otherwise} \end{cases}$$

In the case of the sphere S^3 the Floer groups are isomorphic to the chain groups [32, section 22.7] so it makes sense to define the relative invariant $[\hat{\varphi}_{B^4}] \in \widehat{HM}_\bullet(S^3)$ as

$$[\hat{\varphi}_{B^4}] = [\mathfrak{b}_{-1}]$$

Now, we can also have considered attaching a conical end to B^4 which produces the numerical Seiberg-Witten invariants $SW_{(B^4, \xi_{tight})} : \text{Spin}^c(B^4, \xi_{tight}) \to \mathbb{Z}$ explained at the beginning of the paper. In fact, there is only one relative spin-c structure in the domain of the previous map and since we are dealing with a filling of (S^3, ξ_{tight}) Theorem 1.1 in [31] says that this invariant is identically one.

At the same time, we can consider insering a neck of length L between the four ball and the cone

$$B^+_{S^3,\xi}(L) = B^4 \cup \big([0, L] \times S^3\big) \cup \big([1, \infty) \times S^3\big)$$

and considering the parameterized moduli space $\bigcup_{L \in [0,\infty]} \{L\} \times \mathcal{M}^+(B^+_{S^3,\xi}(L))$ [here $B^+_{S^3,\xi}(\infty) = (B^4 \cup \mathbb{R}^+ \times S^3) \cup (\mathbb{R}^+ \times -S^3 \cup [1, \infty) \times S^3)$] the usual count tells us that

$$\sum_{\mathfrak{s} \in \text{Spin}^c(B^4, \xi_{tight})} SW_{(B^4, \xi_{tight})}(\mathfrak{s}) = \langle \check{\omega}\mathbf{c}(\xi_{tight}), [\hat{\varphi}_{B^4}] \rangle$$

where $\breve{\omega} : \widecheck{HM}_\bullet(-S^3) \to \widehat{HM}^\bullet(S^3)$ is the map described in section 3.1 of [32]. The left hand side is identically one, which means that $\mathbf{c}(\xi_{tight})$ had to be non-vanishing.

LIST OF FIGURES

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