Combination Theorems for Discrete Convergence Groups

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B.S., University of Virginia (2018)

A Dissertation Presented to the Graduate Faculty
of the University of Virginia in Candidacy for the Degree of
Doctor of Philosophy

Department of Mathematics

University of Virginia

May, 2024

Abstract

The goal of this thesis is to give an overview on *combination theorems* for Kleinian groups in 3-dimensional real hyperbolic space, $\mathbb{H}^3_{\mathbf{R}}$, and then generalize these to *discrete convergence groups.* A combination theorem, for us, will be a criterion under which two subgroups G_1 and G_2 can be combined to form a new subgroup G with a prescribed presentation while preserving geometrical properties of the initial subgroups. The inspiration for the generalizations are the two combination theorems by Maskit, which deal with *amalgamated free products* and *HNN extensions.*

The majority of the arguments Maskit uses involve the dynamics of a Kleinian group acting on the boundary of $\mathbb{H}^3_{\mathbf{R}}$. In particular, he leverages a property of Kleinian groups called *convergence dynamics.* The more general setting we will introduce involves forgetting about the geometry inside $\mathbb{H}^3_{\mathbf{R}}$ and just retaining these convergence dynamics on the boundary. This gives rise to *discrete convergence groups*, which includes Kleinian groups, along with discrete subgroups of the isometries of any rank 1 symmetric space, and many more examples.

These new combination theorems constitute joint work with Theodore Weisman.

Acknowledgments

Thank you to Mikhail Ershov, Thomas Koberda, and Jeffrey Woo for agreeing to be part of my committee, and for their comments on this thesis and for their feedback after the defense. I also need to thank Theodore Weisman, whose collaboration helped bring the theorems in this thesis to fruition. I would also like to thank the National Science Foundation for recognizing me in their Graduate Research Fellowship Program. The funding provided by this fellowship greatly eased the stress of graduate school while I worked on this thesis.

I also give my thanks to the UVA math department as a whole for welcoming me and supporting me over the last seven years, for part of my undergraduate career and my entire graduate career. In particular, the many late nights of math with my good friend Trent Lucas helped me decide to pursue graduate school in the first place.

One faculty member I would like to single out is Daniel James (DJ), from whom I learned how to be a more effective, empathetic, and inclusive teacher. I learned so much from DJ, and I almost decided to pursue teaching as a career path after working with him. I also worked with several postdocs, and of these, Filippo Mazzoli helped me immensely. He was an excellent instructor for multiple courses I took, and was both a mentor and good friend during his time at UVA.

I also need to thank my family, whose support has been unending and very appreciated. My entire academic journey was made possible by them. That journey started in the local community college, where my first math course was precalculus. Without the infectious enthusiasm of the instructor, Justin Storer, I would not have made the swap from an engineering major to a math

major. So, I also thank him for showing me just how fun math can be.

Last but not least, to my strongest supporter and ever available mentor; my incredible advisor and now good friend, Sara Maloni: thank you. It is not an exaggeration to say that none of this would have happened if you were not a part of it. As I got to know you as an undergrad, I knew if I was going to do a PhD, I wanted to do it with you. I will always be proud that I had the privilege of being your first PhD student in what will undoubtedly be a remarkable career, with many more to come.

Contents

Chapter 1

Introduction

In this chapter, we start with a discussion on some of the history of combination theorems and motivate our main theorems. We then give a sketch of the main ideas behind the proof, followed by a brief overview of the structure of the thesis.

1.1 Motivation

First introduced over a century ago, *hyperbolic geometry* has been a rich area of research ever since. As opposed to Euclidean geometry, where the spaces in question are *flat* (have zero curvature), hyperbolic spaces are *negatively curved*, and so locally look like a saddle point. We will mostly consider the real 3-dimensional hyperbolic space, $\mathbb{H}^3_{\mathbf{R}}$, in this thesis. Such a space does not sit nicely in our 3-dimensional world, so we instead study it using various *models*. We will introduce both the upper half-space model, and the ball model. The boundary $\partial\mathbb{H}^3_{\mathbf{R}}$ is topologically a sphere, which can be seen in either model.

Named in 1883 by Klein and Poincaré, *Kleinian groups* are discrete subgroups of $\mathrm{Isom}^+(\mathbb{H}^3_{\mathbf{R}}) \cong \mathrm{PSL}(2, \mathbf{C})$, the orientation preserving isometries of $\mathbb{H}^3_{\mathbf{R}}$. These groups have deep connections to hyperbolic 3-manifolds, and Teichmüller theory. The action of a Kleinian group on $\mathbb{H}^3_{\mathbf{R}}$ extends to the boundary sphere, and the theorems we consider here are phrased in terms of this action on the boundary. In this thesis, we generalize the classical *Klein-Maskit combination theorems* for Kleinian groups to the setting of *discrete convergence groups.*

These theorems involve *geometrical finiteness* of the groups being combined. For a Kleinian group G, this notion captures the idea that the geometry of the quotient $\mathbb{H}^3_{\mathbf{R}}/G$ is finite in a precise sense. An intuitive definition is that $\mathbb{H}^3_{\mathbf{R}}/G$ is obtained from gluing the faces of a hyperbolic polyhedron with finitely many faces. Remarkably, this turns out to be equivalent to a dynamical definition in terms of the action of G on $\partial\mathbb{H}^3_{\mathbf{R}}$, and Maskit leverages this to prove his combination theorems respect this property.

Klein and Maskit's combination theorems give sufficient dynamical conditions for combining two Kleinian groups into a new one. The first one was due to Klein [Kle83], and gave a "ping-pong" like setup to combine two Kleinian groups G_1 and G_2 into a free product $G_1 * G_2$. Maskit [Mas88] generalized this theorem to Kleinian groups G_1 and G_2 with a common subgroup $J = G_1 \cap G_2$, in order to form a new Kleinian group as the *amalgamated free product* $G_1 *_J G_2$, which is a sort of free product while gluing along a common subgroup. He also proved a similar theorem for *HNN extensions*, which is a way to extend a group so two isomorphic subgroups become conjugate. Maskit showed that when combining geometrically finite groups, the result is again geometrically finite. Originally, this was very useful for building new and interesting examples of (geometrically finite) Kleinian groups. More generally, any Kleinian group can be built by combining some finite collection of elementary groups, web groups, and totally degenerate groups, as shown by Abikoff and Maskit [AM77].

Our new combination theorems deal with *discrete convergence groups* instead, a generalization of Kleinian groups. A discrete convergence group is a group G which acts on a compact, infinite, metrizable space M with *convergence dynamics*. We will define this precisely soon, but for now know that these dynamics were first observed for Kleinian groups acting on the sphere, so they give an example of discrete convergence groups with $M = \partial\mathbb{H}^3_{\mathbf{R}}$. This definition forgets about the space $\mathbb{H}^3_{\mathbf{R}}$, though, and abstracting the dynamics from Kleinian groups and replacing the boundary with an arbitrary M gives us our new setting.

It has been shown that discrete subgroups of the isometries of any rank 1 symmetric space is a discrete convergence group by considering the action on the boundary. Even more generally, any discrete subgroup of the isometries of a proper geodesic δ-hyperbolic metric space is a discrete convergence group acting on the Gromov boundary, as shown by Tukia [Tuk94]. So for even the most flexible definitions of hyperbolicity, the discrete subgroups of the corresponding isometry group act with convergence dynamics on the boundary.

In joint work with Theodore Weisman, we prove combination theorems for discrete convergence groups analogous to Maskit's theorems, which also involve a version of geometrical finiteness. Again, our hypotheses involve dynamical assumptions about the action on the compact space M, which should be thought of as the boundary of some hyperbolic space, even though we do not need to reference said space in the theorem. We then explore a possible application of these theorems to investigate different types of convergence for sequences of representations.

1.2 Summary

Now we explore Maskit's first combination theorem in more detail. Let $G_1, G_2 < \mathrm{PSL}(2, \mathbf{C})$ be Kleinian groups, where $G_1 \cap G_2 = J$ is a geometrically finite proper nontrivial common subgroup of G_1 and G_2. Set $G = \langle G_1, G_2 \rangle$, the subgroup generated by G_1 and G_2. In Maskit's first combination theorem, we consider the action of each group on the Riemann sphere $\partial \mathbb{H}^3_{\mathbf{R}} \cong \widehat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$. The main hypothesis is that we can find a J-invariant curve $W \subset \widehat{\mathbf{C}}$ splitting $\widehat{\mathbf{C}}$ into two closed J-inveriant discs, B_1 and B_2, so that

$$gB_1 \subset \mathrm{Int}(B_2), \forall g \in G_1 \setminus J,$$

and

$$gB_2 \subset \mathrm{Int}(B_1), \forall g \in G_2 \setminus J.$$

There are some other more technical hypotheses which will be covered in detail later. Part of the conclusion is that $G = G_1 *_J G_2$, the *amalgamated free product* of G_1 and G_2 amalgamated over J. This setup is reminiscent of the *ping-pong lemma* of geometric group theory, which gives a dynamical condition for two groups acting on some space to generate a free product of the two groups.

Maskit also concludes that G is geometrically finite if and only if G_1 and G_2 are geometrically finite. This is the hardest part of the theorem to prove, and part of the difficulty stems from the fact that geometrical finiteness is about the geometry of the quotient $\mathbb{H}^3_{\mathbf{R}}/G$, while Maskit's hypotheses only reference the action on the boundary.

This difference is reconciled by a characterization of geometrical finiteness purely in terms of the group's action on $\partial \mathbb{H}^3_{\mathbf{R}}$, due to Beardon and Maskit

[BM74]. They prove that a Kleinian group G is geometrically finite if and only if every point $x \in \Lambda(G)$, the *limit set* of G, is either a *conical limit point* or a *bounded parabolic fixed point*. The limit set will be discussed in more detail later, but for now, what is important is these conditions are checked entirely in the boundary. Encoding interesting geometry with the dynamics at infinity is a key theme in this thesis, and is the primary motivation for the definition of discrete convergence groups.

A natural way to generalize these theorems is to consider the minimal necessary assumptions on the dynamics at infinity. As alluded to earlier, Gehring and Martin [GM87] showed that isometries of $\mathbb{H}^n_{\mathbf{R}}$ always act on the boundary $\partial\mathbb{H}^n_{\mathbf{R}} \cong \mathbb{S}^{n-1}$ with *convergence dynamics*. We define this now. We say G acts on a compact metrizable space M with discrete convergence dynamics if, given any sequence of distinct elements $g_k \in G$, we can find a subsequence and a pair of points z_+, z_-, so that the maps g_k restricted to $M \setminus \{z_-\}$ converge to the constant map $z \mapsto z_+$ uniformly on compacts. We call G a *discrete convergence group*. These north-south dynamics imitate the behavior of powers of a single loxodromic isometry acting on the boundary of hyperbolic space.

As mentioned earlier, this setting includes isometries of essentially any space with hyperbolic properties. With minor modifications, there is again a notion of a limit set $\Lambda(G) \subset M$, and the dynamical version of geometrical finiteness can be restated verbatim with M replacing the sphere. In fact, Tukia [Tuk98] showed that geometrical finiteness of a discrete convergence group as stated here is equivalent to having a "nice" geometric description for the quotient $\Theta(M)/G$, where $\Theta(M)$ is the space of distinct triples in M.

If M is playing the role of the sphere, one should think of $\Theta(M)$ as playing the role of $\mathbb{H}^3_{\mathbf{R}}$. Indeed, $\Theta(\partial\mathbb{H}^3_{\mathbf{R}})$ has a natural projection onto $\mathbb{H}^3_{\mathbf{R}}$ obtained by sending (x, y, z) to the orthogonal projection of z onto the geodesic from

x to y. This is not a bijection, but it gives a quasi-isometry between the two spaces, which is a sort of coarse equivalence.

This is the setting where our new theorems hold, and we state them now:

Theorem A (T, Weisman)**.** *Let G_1 and G_2 be discrete convergence groups acting on a compact metrizable space M. Suppose that $J = G_1 \cap G_2$ is geometrically finite, and G_1 and G_2 are in AFP ping-pong position with respect to J. Let $G = \langle G_1, G_2 \rangle < \mathrm{Homeo}(M)$, and suppose G acts as a convergence group. Then the following hold:*

(i) $G = G_1 *_J G_2$.

(ii) *G is discrete.*

(iii) *Elements of G not conjugate into G_1 nor G_2 are loxodromic.*

(iv) *G is geometrically finite if and only if both G_1 and G_2 are geometrically finite.*

And, secondly, the one for HNN extensions:

Theorem B (T, Weisman)**.** *Let G_0 be a discrete convergence group acting on a compact metrizable space M, and suppose that $J_1, J_{-1} < G_0$ are both geometrically finite. Let $G_1 = \langle f \rangle$ be an infinite cyclic discrete convergence group also acting on M, where $fJ_{-1}f^{-1} = J_1$ in $\mathrm{Homeo}(M)$. Suppose G_0 is in HNN ping-pong position with respect to f, J_1 and J_{-1}. Let $G = \langle G_0, G_1 \rangle <$ $\mathrm{Homeo}(M)$, and suppose G acts as a convergence group. Then the following hold:*

(i) $G = G_0 *_f$.

(ii) *G is discrete.*

(iii) Elements of G not conjugate into G_0 are loxodromic.

(iv) G is geometrically finite if and only if G_0 is geometrically finite.

See Section 3.2 and Section 3.3 for the definitions of AFP ping-pong position and HNN ping-pong position respectively. They resemble the ping-pong assumptions made in Maskit's two theorems. The most challenging part of these theorems is again the geometric conclusion involving geometrical finiteness. We outline the main ideas for proving Theorem A below, and Theorem B has a similar type of argument.

The thesis ends with a discussion of a possible application of these new theorems. There are multiple notions of convergence when discussing sequences of representations into $\mathrm{PSL}(2, \mathbf{C})$ (or the isometries of a different hyperbolic space), and we will consider *algebraic convergence* and *geometric convergence*. Algebraic convergence is where one looks at the limits of a generating set and check what those limits generate. Geometric convergence instead arises from considering the Hausdorff distance on the space of closed subgroups, and is directly related to the corresponding sequence of quotient orbifolds. Sometimes this results in a limiting group strictly containing the algebraic limit. When these two limits coincide, the convergence is said to be *strong*.

There are examples of sequences where the convergence is not strong. In particular, Jørgensen produced examples of representations of $\mathbb{Z}$ with a geometric limit isomorphic to $\mathbb{Z}^2$. One place to read about these is [CEG86], but we will list the explicit elements in the last chapter of this thesis. Maloni and Pozzetti [MP22] discuss a generalization to real and complex hyperbolic spaces of all dimensions, and further generalized these examples to include representations of F_2, the free group on two letters, which do not converge strongly. One might study these notions when considering different topologies on the representation spaces, and the existence of these examples imply that these two notions of convergence give very different pictures.

There are also examples of surface group representations which do not converge strongly, due to Kerckhoff and Thurston [KT90]. These examples are constructed using deep results about Kleinian groups, such as the *simultaneous uniformization* theorem of Bers [Ber60], and so there is no clear way to generalize their methods to other settings. Our goal was to use our new combination theorems to combine free groups from Maloni and Pozzetti's work into a sequence of representations of a genus 2 surface group which does not converge strongly. This could give a path to finding such examples in other settings, such as complex hyperbolic geometry. We did not have time to complete this application, but some pictures of limit sets generated from Python code give some evidence that one could find a simple closed curve in the sphere determining two discs B_1 and B_2 so that Theorem A applies.

1.3 Theorem A Proof Ideas

Parts (i) - (iii) are standard arguments relying on the ping-pong setup, and so we focus on part (iv), which the majority of the proof is dedicated towards. Supposing G_1 and G_2 are geometrically finite, we will discuss how one shows G is geometrically finite. The other direction is similar, but less complex.

First, we will sketch how Maskit proves this, and then we will consider the changes necessary to adapt his strategy to the setting of discrete convergence groups. The goal is to show that every $x \in \Lambda(G)$, the limit set of G, is a conical limit point or a bounded parabolic fixed point. One way to define the limit set $\Lambda(G)$ is by taking the accumulation points of any infinite G-orbit in $\widehat{\mathbf{C}}$. To simplify this discussion, assume we have no parabolic elements, and so we wish to show every limit point is a conical limit point.

A limit point x is a *conical limit point* if there is a sequence (g_k) in G of distinct elements such that for every $z \in \widehat{\mathbf{C}} \setminus \{x\}$, the pair $(g_k x, g_k z)$ stays inside a compact subset of $(\widehat{\mathbf{C}} \times \widehat{\mathbf{C}}) \setminus \Delta$, where $\Delta \subset \widehat{\mathbf{C}} \times \widehat{\mathbf{C}}$ is the diagonal subspace. We will call the sequence (g_k) a *conical limiting sequence* for the point x. So, roughly speaking, we wish to find a sequence (g_k) so that $g_k z$ stays far away from $g_k x$ for any $z \neq x$.

The first observation is the following. Since we are assuming G_1 and G_2 are geometrically finite, it follows that $\Lambda(G_1) \cup \Lambda(G_2)$ consists of conical limit points. Even better, G-translates of these points will still be conical limit points for G, since we simply need to alter the given sequences by a fixed element of G. So, we must show points in $\Lambda(G) \setminus G(\Lambda(G_1) \cup \Lambda(G_2))$ are conical limit points for G, where the notation $G(U)$ denotes the union of the G-translates of U.

Elements of $G \setminus J$ can be expressed using *normal forms*, words $g = g_1 \cdots g_n$ where the g_i alternate between $G_1 \setminus J$ and $G_2 \setminus J$. We define the *length* of a normal form to be n (length 0 elements are in J), and we say g is an (i,j)-form if $g_1 \in G_i \setminus J$ and $g_n \in G_j \setminus J$. By results in the combinatorial group theory section, the ping-pong dynamics imply that if g is an (i,j)-form, then $gB_j \subset B_i$. We define certain ping-pong sets recursively, as follows. We let $T_{1,i}$ be the union of $G_i \setminus J$ translates of B_i, and set $T_1 = T_{1,1} \cup T_{1,2}$. Then we let $T_{2,i}$ be the $G_i \setminus J$ translates of $T_1 \cap B_i$, and set $T_2 = T_{2,1} \cup T_{2,2}$. We continue this process, and get a sequence of nested sets. See Figure 1.3.1.

Let $T = \bigcap T_n$. It turns out that points of $\Lambda(G) \setminus G(\Lambda(G_1) \cup \Lambda(G_2))$ lie in T. By construction, the points $x \in T$ correspond to a sequence of elements (g_k) whose lengths go to infinity, where $x \in g_k B_j$ for one of $j = 1, 2$. The sequence (g_k^{-1}) is our initial candidate for a conical limiting sequence for the point x.

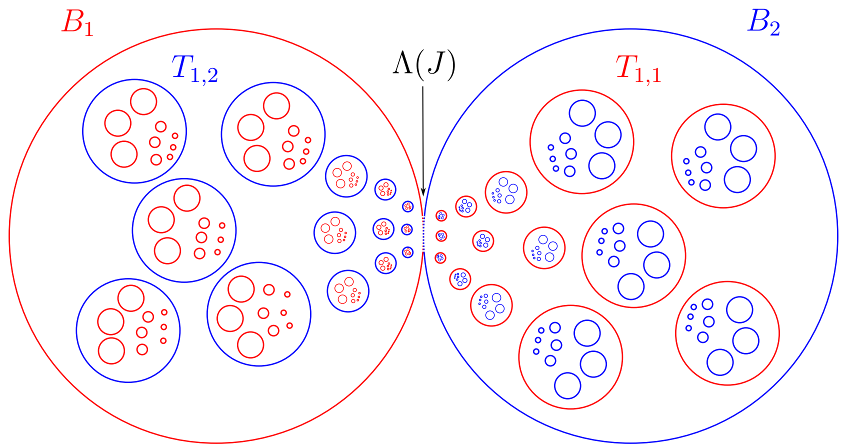

Figure 1.3.1: Part of the sets T_1 and T_2.

The key technical lemma that Maskit uses now is a *contraction property* for such sequences. Namely, anytime there is a sequence $g_k \in G$ of distinct (i, j)-forms, the translates $g_k B_j$ converge to a singleton. To prove this, Maskit uses a geometric argument inside $\mathbb{H}^3_{\mathbf{R}}$.

With the contraction property proved, we have that $g_k B_j$ necessarily converges to x, and so the sequence (g_k^{-1}) maps increasingly smaller discs around x to B_j. We modify this sequence so that $g_k^{-1}x \in B_j$ does not accumulate on ∂B_j. Now, given any $z \neq x$, eventually $z \notin g_k B_j$, and so $g_k^{-1} z \notin B_j$. Since $g_k^{-1} x \in B_j$ does not accumulate on ∂B_j, we find that $(g_k^{-1} z)$ cannot accumulate on the sequence $(g_k^{-1} x)$. This is precisely what it meant for x to be a conical limit point.

In the setting of discrete convergence groups, we needed to replace the geometric argument in $\mathbb{H}^3_{\mathbf{R}}$ with an argument using a *cusped space* for M, a hyperbolic metric space whose boundary naturally identifies with $\Lambda(G)$. Theorem 2.2.17 of Yaman [Yam04] states that such a cusped space can be found for any discrete convergence group G acting geometrically finitely on its limit set. This allows the necessary geometric argument to prove the technical Lemma 2.2.24, which is then specialized to the necessary contraction property for Theorem A in Lemma 3.2.4. This was the biggest change from Maskit's argument, and is the only time we need to appeal to the geometry of some space, rather than using topological arguments in M.

The proof for Theorem B uses a lot of similar ideas, but is somewhat more technical since the normal forms for HNN extensions are more complicated.

1.4 Overview

In Chapter 2, we develop the necessary background for the various combination theorems. This starts with a brief treatment of hyperbolic geometry and Kleinian groups, that is, discrete subgroups of $\mathrm{Isom}^+(\mathbb{H}^3_{\mathbf{R}})$, in Section 2.1. We define and prove several equivalent characterizations for a Kleinian group to be geometrically finite in Section 2.1.4. In Section 2.2, we introduce discrete convergence groups. This is followed by a brief discussion on relatively hyperbolic groups and some important results on these, which we use to prove Lemma 2.2.24, the "technical lemma." We end this background chapter with a discussion of combinatorial group theory in Section 2.3.1 and Section 2.3.2. There are two versions of the combination theorem we prove, one involving *amalgamated free products*, and the other involving *HNN extensions*. These sections discuss the structure of these groups algebraically, and give dynamical conditions ensuring a given pair of groups generates a group isomorphic to either an amalgamated free product or an HNN extension involving the original two groups.

In Chapter 3, we start discussing combination theorems, starting with the classical ones of Klein and Maskit in Section 3.1. After some examples of these classical theorems, we state and prove our main results, starting with the case of amalgamated free products in Section 3.2, and following with the case of HNN extensions in Section 3.3.

Finally, in Chapter 4, we consider an application of these combination theorems. We start with a discussion of *algebraic* and *geometric convergence*, along with some examples of sequences of representations where the convergence is not strong. We conclude with various pictures of limit sets generated via Python scripts, which provide reasons to believe that one can find a simple closed curve in $\partial\mathbb{H}^3_{\mathbf{R}}$ separating the sphere into two discs satisfying the hypotheses of the combination theorem for amalgamated free products.

Chapter 2

Background

This chapter introduces the necessary background for the rest of the thesis. In particular, we review some basic facts from the theory of Kleinian groups. Then we examine different ways of ensuring the geometry of a given Kleinian group is 'finite' in some sense: this gives rise to the notion of geometrical finiteness. This notion has has many equivalent characterizations. The first that we will introduce is the most intuitive, and the last one is less intuitive but is the most useful for our generalizations. Next, we introduce discrete convergence groups, which is the setting where our new theorems hold. Finally, we establish the notation and basic facts for amalgamated free products and HNN extensions, which are a key ingredient in the combination theorems.

2.1 Kleinian Groups

We start by compiling necessary results about Kleinian groups. The main reference for this is Maskit's book [Mas88].

2.1.1 Hyperbolic Geometry

The setting for Maskit's combination theorems is the 3-dimensional real hyperbolic space, $\mathbb{H}^3_{\mathbf{R}}$. This is the unique simply connected 3-dimensional Riemannian manifold with constant real sectional curvature equal to -1. We start by defining the model for $\mathbb{H}^3_{\mathbf{R}}$ which we will primarily use.

Definition 2.1.1. The *upper half-space model* for $\mathbb{H}^3_{\mathbf{R}}$ is given by the set $\{(z,t) \mid z \in \mathbf{C}, t > 0\}$, equipped with the metric

$$ds^2 = \frac{|dz|^2 + dt^2}{t^2}.$$

The boundary $\partial\mathbb{H}^3_{\mathbf{R}}$ is then given by $\widehat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$.

In this model, the geodesics are circular arcs perpendicular to the boundary and vertical lines. Given any circle or line $C \subset \widehat{\mathbf{C}}$, the collection of geodesics connecting points of C form a totally geodesic copy of $\mathbb{H}^2_{\mathbf{R}} \subset \mathbb{H}^3_{\mathbf{R}}$. All totally geodesic planes arise in this way. We will oftentimes think of t as the 'height' of a point, and points in $\mathbf{C}$ as 'finite points' in the boundary. We will refer to the hyperbolic distance as d, regardless of the model we are working in.

We now introduce the other model of $\mathbb{H}^3_{\mathbf{R}}$ which we will occasionally need.

Definition 2.1.2. The *ball model* for $\mathbb{H}^3_{\mathbf{R}}$ is $\{(x,y,z) \mid x^2+y^2+z^2 < 1\}$, equipped with the metric

$$ds^2 = \frac{4|dr|^2}{(1-r^2)^2}, \quad r^2 = x^2+y^2+z^2$$

The boundary $\partial\mathbb{H}^3_{\mathbf{R}}$ is then given by $\mathbb{S}^2$.

It will sometimes be convenient to work in the ball model so we can use the Euclidean distance on the closed ball, which identifies with $\overline{\mathbb{H}^3_{\mathbf{R}}} = \mathbb{H}^3_{\mathbf{R}} \cup \partial\mathbb{H}^3_{\mathbf{R}}$.

We will denote this distance by d_E. The geodesics in this model are circular arcs perpendicular to the boundary and diameters.

Recall that $\mathrm{PSL}(2,\mathbf{C})$ is the projectivization of complex 2×2 matrices with determinant 1. There is a natural action of $\mathrm{PSL}(2,\mathbf{C})$ on $\widehat{\mathbf{C}}$ by Möbius maps. Given

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

we act by

$$g(z) = \frac{az+b}{cz+d}.$$

It is natural to define $g(\infty) = a/c$ and $g(-d/c) = \infty$. This gives a well-defined map from $\mathrm{PSL}(2,\mathbf{C})$ into the conformal automorphisms of the Riemann sphere, and in fact, this map is an isomorphism. We will oftentimes describe elements of $\mathrm{PSL}(2,\mathbf{C})$ in terms of the corresponding Möbius map acting on $\widehat{\mathbf{C}}$. Any such conformal automorphism of $\partial\mathbb{H}^3_{\mathbf{R}}$ extends uniquely to an isometry of $\mathbb{H}^3_{\mathbf{R}}$, and this map is again an isomorphism.

Theorem 2.1.3 ([Mas88] IV.B.7)**.** *We have an isomorphism* $\mathrm{PSL}(2,\mathbf{C}) \cong \mathrm{Isom}^+(\mathbb{H}^3_{\mathbf{R}})$.

There is a natural topology on $\mathrm{PSL}(2,\mathbf{C})$ as a subset of $\mathbf{R}^4$, which allows us to consider discrete subgroups. We note some equivalent characterizations of discrete subgroups now.

Proposition 2.1.4 ([Mas88] II.C.2)**.** *Let* $G < \mathrm{PSL}(2,\mathbf{C})$ *be a subgroup. Then the following are equivalent:*

(i) G *is discrete.*

(ii) G *has no accumulation points.*

(iii) The identity is not an accumulation point of G.

Note that the only implication requiring an argument is $(iii) \Rightarrow (i)$, which is provided in the citation. We give a special name to these discrete subgroups.

Definition 2.1.5. A *Kleinian group* G is a discrete subgroup of $\mathrm{PSL}(2, \mathbf{C})$. A Kleinian group which is conjugate into $\mathrm{PSL}(2, \mathbf{R})$ is called a *Fuchsian group.*

Note that a Fuchsian group necessarily preserves a totally geodesic $\mathbb{H}^2_{\mathbf{R}}$ inside $\mathbb{H}^3_{\mathbf{R}}$.

These isometries satisfy a trichotomy in terms of their traces and/or fixed points. Note that the trace of an element in $\mathrm{PSL}(2, \mathbf{C})$ is only defined up to its sign, which we can fix by taking the square.

Proposition 2.1.6 ([Mas88] I.B.5)**.** *Let $g \in \mathrm{PSL}(2, \mathbf{C})$ be different from the identity. Then g falls into exactly one of the following categories:*

1. *Elliptic: g fixes a point in $\mathbb{H}^3_{\mathbf{R}}$ and $(\mathrm{tr}\ g)^2 \in (0, 4)$.*
2. *Parabolic: g fixes exactly one point in $\partial\mathbb{H}^3_{\mathbf{R}}$ and $(\mathrm{tr}\ g)^2 = 4$.*
3. *Loxodromic: g fixes exactly two points in $\partial\mathbb{H}^3_{\mathbf{R}}$ and none in $\mathbb{H}^3_{\mathbf{R}}$ and $(\mathrm{tr}\ g)^2 \in \mathbf{C} \setminus (0, 4]$.*

We typically only care about properties of Kleinian groups preserved under conjugation. It is a fact that $\mathrm{PSL}(2, \mathbf{C})$ acts simply transitively on ordered triples of distinct points in $\widehat{\mathbf{C}}$, and so, by conjugating appropriately, we can choose the fixed points for a given element in our group G. We will usually use 0, 1, or ∞. This process is called *normalization.*

As a consequence of this trichotomy, we have the following.

Proposition 2.1.7 ([Mas88] I.D.4)**.** *If f has exactly two fixed points and f and g share exactly one fixed point, then the commutator $[f, g]$ is parabolic.*

Proof. Normalize $\langle f, g\rangle$ so that the common fixed point is ∞, and the other fixed point of f is 0. Then

$$f = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}, \quad g = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix},$$

and

$$[f, g] = \begin{pmatrix} 1 & -ab + t^2ab \\ 0 & 1 \end{pmatrix}.$$

Since $a, b \neq 0$ and $|t| \neq 1$, we have that $ab(t^2 - 1) \neq 0$, so the commutator is parabolic. $\square$

When we want to work with explicit matrices, we will always use the upper half-space model, since it is easiest to think about the action on $\widehat{\mathbf{C}}$. We next prove two propositions related to discreteness which we will need later. First is a condition we will use when considering the stabilizer of ∞. This is sometimes called the Shimizu-Leutbecher lemma.

Proposition 2.1.8 ([Mas88] II.C.5)**.** *Let G be Kleinian, where G contains $fz = z + 1$. Then for every*

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G,$$

either $c = 0$, or $|c| \geq 1$.

Proof. Assume by contradiction that we have a $g \in G$ with $0 < |c| < 1$. Let $g_0 = g$, and inductively define g_m by $g_{m+1} = g_m f g_m^{-1}$. Write

$$g_m = \begin{pmatrix} a_m & b_m \\ c_m & d_m \end{pmatrix},$$

then

$$\begin{aligned} a_{m+1} &= 1 - a_m c_m \\ b_{m+1} &= a_m^2 \\ c_{m+1} &= -c_m^2 \\ d_{m+1} &= 1 + a_m c_m. \end{aligned}$$

It follows that $|c_m| = |c|^{2m}$; hence $c_m \to 0$. An induction argument gives us that $|a_m|$ and $|d_m|$ are both bounded by

$$K \sum_{j=0}^{2^m} |c|^j,$$

where $K = \max(|a|, 1)$. Since this is sum is geometric and $b_{m+1} = a_m^2$, we now see that all entries of g_m are bounded. This gives us a convergent subsequence, which contradicts discreteness; in fact $g_m \to f$, although the result holds independent of the value of this limit. □

Lastly, a criterion for non-discreteness which will allow us to rule out certain types of elements in stabilizers.

Proposition 2.1.9 ([Mas88] II.C.6)**.** *If $f, g \in \mathrm{PSL}(2, \mathbf{C})$ are nontrivial, where f is loxodromic and f and g have exactly one fixed point in common, then $\langle f, g \rangle$ is not discrete.*

Proof. By Proposition 2.1.7, we can assume g is parabolic by replacing it with $[f, g]$ if necessary. Normalize so that the common fixed point is at ∞, and the second fixed point of f is at 0. We can replace f by f^{-1} if necessary so that

∞ is the attracting fixed point of f. Write

$$f = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}, \quad |t| > 1, \quad g = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix},$$

then

$$f^{-m} g f^m = \begin{pmatrix} 1 & bk^{-2m} \\ 0 & 1 \end{pmatrix} \to 1.$$

Hence $\langle f, g \rangle$ is not discrete. $\square$

Isometric Circles

We next list some facts about isometric circles for Möbius transformations which we will need later. These tools allow us to prove nice convergence results for sequences of elements (which will motivate the definitions of discrete convergence groups). Let

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

and assume that $g(\infty) \neq \infty$, or equivalently, that $c \neq 0$. Then $g'(z) = (cz + d)^{-2}$.

Definition 2.1.10. Let $g \in \mathrm{PSL}(2, \mathbf{C})$ so that $g(\infty) \neq \infty$. Then the *isometric circle* $I(g)$ of g is the set of points where $|g'(z)| = 1$.

Möbius transformations preserve cicles and lines, and g^{-1} has isometric circle $I(g^{-1}) = g(I(g))$. The radius of both circles is $\rho = |c|^{-1}$. Note that the center is $\alpha = -d/c = g^{-1}(\infty)$, and the center of $I(g^{-1})$ is $a/c = g(\infty)$. There is a nice decomposition of Möbius maps in terms of these isometric circles.

Proposition 2.1.11 ([Mas88] I.C.2)**.** *Every $g \in \mathrm{PSL}(2, \mathbf{C})$ can be written in the form $g = rqp$, where p is reflection in the isometric circle of g, and r and q are Euclidean motions of $\mathbf{C}$.*

Using this decomposition, we can prove an estimate which will be used later when studying dynamics on the boundary.

Proposition 2.1.12 ([Mas88] I.C.7)**.** *Let $g \in \mathrm{PSL}(2, \mathbf{C})$ so that $g(\infty) \neq \infty$, and let T be a closed set which may contain ∞, but does not contain $\alpha = g^{-1}(\infty)$. Let δ be the distance from α to T, and let ρ be the radius of the isometric circle I of g. Then*

$$\mathrm{diam}(gT) \leq 2\rho^2/\delta,$$

where we are using the Euclidean diameter in $\mathbf{C}$, and if $\infty \in T$, then

$$\rho^2/\delta \leq \mathrm{diam}(gT).$$

Proof. Let x be the point of T closest to α. Then $\delta = |x - \alpha|$, and T lies outside the circle of radius δ centered at α. If p is reflection in $I(g)$, then this implies pT lies inside the circle of radius ρ^2/δ centered at α, from which the first inequality follows since $g = rqp$ for r, q Euclidean motions. The second inequality follows from the fact that $\alpha \in pT$, since $|px - \alpha| = \rho^2/\delta$. □

2.1.2 Dynamics on the Boundary

We now turn our attention to the dynamics on the boundary of $\mathbb{H}^3_{\mathbf{R}}$, which identifies with $\widehat{\mathbf{C}}$. A recurring theme in this thesis will be forgetting about the geometry and just working in the boundary.

Definition 2.1.13. Let G be a Kleinian group. We say G acts *discontinuously* at $z \in \mathbb{H}^3_{\mathbf{R}} \cup \partial\mathbb{H}^3_{\mathbf{R}}$ if there is an open neighborhood U of z so that

$$|\{g \in G \mid gU \cap U \neq \varnothing\}| < \infty.$$

Let $\Omega(G)$ be the set of points in $\partial\mathbb{H}^3_{\mathbf{R}}$ where G acts discontinuously. This is called the *domain of discontinuity* for G. Given $x \in \Omega(G)$, if there is a neighborhood U of x so that $gU \cap U = \varnothing$ for every nontrivial $g \in G$, then we further say G acts *freely discontinuously* at x.

It is a fact that discreteness of G is equivalent to G acting discontinuously on all of $\mathbb{H}^3_{\mathbf{R}}$. This is a consequence of the following theorem.

Theorem 2.1.14 ([Mas88] IV.E.3)**.** *Let $x \in \mathbb{H}^3_{\mathbf{R}}$, and let $G < \mathrm{PSL}(2, \mathbf{C})$. Then G acts discontinuously at x if and only if G is discrete.*

The same does not hold in the boundary, where the action may not be discontinuous anywhere. In any case, the set $\Omega(G)$ is open and G-invariant, hence its complement is closed and also G-invariant. We will introduce an alternative definition for the complement later. The set of points in $\Omega(G)$ where G acts discontinuously but not freely discontinuously is precisely the fixed points of elliptic elements of G in the boundary.

Suppose G acts freely discontinuously at some point z, and let U be the corresponding neighborhood. Using stereographic projection, we can identify $\widehat{\mathbf{C}}$ with $\mathbb{S}^2$, inducing a spherical metric, diameter, and measure, which we denote d_S, diam_S, and meas_S, respectively. These are all equivalent to the corresponding Euclidean notion on $\widehat{\mathbf{C}}$ when restricted to any bounded subset of $\mathbf{C}$. Now, since the G-translates of U are all disjoint, the following proposition follows since the sphere has finite area.

Proposition 2.1.15 ([Mas88] II.B.3)**.** *Let G be a Kleinian group acting freely discontinuously at $z \in \widehat{\mathbf{C}}$, and let U be the neighborhood from the definition. Then*

$$\sum_{g \in G} \mathrm{meas}_S(gU) < \infty.$$

We quickly note that, by standard facts about uncountable sums of positive numbers, this implies that any Kleinian group G acting freely discontinuously at some point $z \in \widehat{\mathbf{C}}$ is countable. Now, as always, writing g with bottom left entry c, if we assume G acts freely discontinuously at ∞, then no $g \in G$ fixes ∞, hence $c \neq 0$ for every $g \in G$.

Proposition 2.1.16 ([Mas88] II.B.5)**.** *Let G be a Kleinian group acting freely discontinuously at ∞, then*

$$\sum_{g \in G \setminus \{1\}} |c|^{-4} < \infty.$$

Proof. Choose a neighborhood U of ∞ of the form $\{z \mid |z| > \rho\} \cup \{\infty\}$, where $gU \cap U = \varnothing$ for all nontrivial $g \in G$. Let α be the center of the isometric circle I of some nontrivial $g \in G$; recall that its radius is $|c|^{-1}$. By definition, $\alpha = g^{-1}(\infty) \notin U$, the center of I. We can further assume that the Euclidean distance δ from α to U is positive. Note that $\delta \leq \rho$. From Proposition 2.1.12, we have

$$\mathrm{diam}(gU) \geq |c|^{-2}\delta^{-1}.$$

Since gU is a bounded circular disc in $\mathbf{C}$, we can use the equivalence of the spherical and Euclidean metrics to find a constant $K > 0$, so that

$$\mathrm{meas}_S(gU) \geq K^{-1}\mathrm{diam}^2(gU).$$

Combining both of these with Proposition 2.1.15, we get

$$\sum_{g \in G \setminus \{1\}} |c|^{-4} \leq \sum_{g \in G \setminus \{1\}} \delta^2 \mathrm{diam}^2(gU) \leq K\rho^2 \sum_{g \in G \setminus \{1\}} \mathrm{meas}_S(gU) < \infty.$$

□

A useful corollary of this proposition is the following.

Corollary 2.1.17 ([Mas88] II.B.6)**.** *Let G be a Kleinian group which acts freely discontinuously at ∞, and let (g_m) be a sequence of distinct elements. If ρ_m is the radius of the isometric circle for g_m, then $\rho_m \to 0$.*

The complement of the domain of discontinuity has a nice description in terms of accumulation points of a given G-orbit in $\mathbb{H}_3^{\mathbf{R}}$.

Definition 2.1.18. Let G be Kleinian. A point $x \in \partial \mathbb{H}^3_{\mathbf{R}}$ is called a *limit point* for G if there is a point $z \in \mathbb{H}^3_{\mathbf{R}}$, and a sequence of distinct elements $g_m \in G$, so that $g_m z \to x$. The set of all limit points is denoted $\Lambda(G)$, and is called the *limit set* for G.

Since every neighborhood of $x \in \Lambda(G)$ has infinitely many translates of some point, we have $\Lambda(G) \cap \Omega(G) = \varnothing$. In fact, we also have $\widehat{\mathbf{C}} = \Lambda(G) \cup \Omega(G)$.

The next result requires the notion of uniform convergence.

Definition 2.1.19. Let (X, d) be a metric space. A sequence $f_n : X \to X$ *converges to f uniformly on compact subsets of X* if, for every compact set $K \subset X$, the maps $f_n|_K$ converge uniformly to $f|_K$. That is, for every $\varepsilon > 0$, there is $N \in \mathbb{N}$, so that, for every $x \in K$, and for every $n \geq N$, we have

$$d_X(f_n(x), f(x)) < \varepsilon.$$

If the map f is the constant function $(x \mapsto z)$ for some $z \in X$, we will say that $f_n x \to z$ uniformly on compact subsets of X.

Later, we will work with metrizable spaces with no specific metric, but this definition does not depend on the choice of metric generating the given topology. We can now state a theorem which illustrates the notion of convergence dynamics, which motivates the definition of discrete convergence groups later in the thesis.

Theorem 2.1.20 ([Mas88] II.D.2)**.** *Let $x \in \Lambda(G)$. Then there is $y \in \Lambda(G)$, and there is a sequence of distinct elements g_m of G, so that $g_m z \to x$ uniformly on compact subsets of $\widehat{\mathbf{C}} \setminus \{y\}$.*

Proof. Since x is a limit point, we can find $z \in \mathbb{H}^3_{\mathbf{R}}$, and $g_m \in G$ distinct, so that $g_m z \to x$. Choose a hyperbolic geodesic L passing through z, with endpoints $z_1, z_2 \in \partial\mathbb{H}^3_{\mathbf{R}} \setminus \{x\}$. Then since $g_m z \to x$, we must have $g_m z_1 \to x$ or $g_m z_2 \to x$. Suppose the former holds. Normalize so $z_1 = \infty$, and pass to a subsequence so that $g_m^{-1}(\infty) \to y$. Then $y \in \Lambda(G)$ since $g_m^{-1} z \to y$.

As in Proposition 2.1.11, write $g_m = r_m q_m p_m$. The result now follows from the following observations: the center of the isometric circle of g_m tends to y; g_m maps the outside of its isometric circle onto the inside of the isometric circle of its inverse; the common radius of the isometric circles of g_m and g_m^{-1} tends to 0 by Corollary 2.1.17; the center of the isometric circle of g_m^{-1} tends to x. $\square$

There is a slightly more general version of this that we will need in the next section. The proof is very similar to what we have done so far, but is one dimension higher and uses *isometric spheres.*

Proposition 2.1.21 ([Mas88] IV.G.9). *Let G be Kleinian, with (g_m) a sequence of distinct elements in G, with $g_m z \to x \in \partial\mathbb{H}^3_{\mathbf{R}}$ for some $z \in \mathbb{H}^3_{\mathbf{R}}$. Then there is a subsequence, and $y \in \partial\mathbb{H}^3_{\mathbf{R}}$, so that $g_m w \to x$ uniformly on compact subsets of $\overline{\mathbb{H}^3_{\mathbf{R}}} \setminus \{y\}$.*

Note that these propositions give us no information on y. If $y \neq x$, then this will be what we call a point of approximation, which we will define later.

2.1.3 Fundamental Domains and Polyhedra

When considering the action of a Kleinian group G on $\Omega(G)$, it is helpful to work with fundamental sets and fundamental domains. First, we record a useful definition which will recur throughout the rest of this section and later in the thesis.

Definition 2.1.22. Suppose G acts on a space X, and $J < G$. We say a subset $B \subset X$ is *precisely invariant* under J in G if B is J-invariant, and for every $g \in G \setminus J$, we have $gB \cap B = \varnothing$.

More generally, given subgroups $J_1, \cdots, J_n < G$, we say a tuple of subsets $(B_1, \cdots, B_n)$ is *precisely invariant* under $(J_1, \cdots, J_n)$ in G if each B_i is precisely invariant under J_i in G, and if for $i \neq j$ and for every $g \in G$, we have $gB_i \cap B_j = \varnothing$.

We can now define fundamental domains.

Definition 2.1.23. A *fundamental domain* D for the Kleinian group G is an open subset of $\Omega(G)$ satisfying the following.

(i) D is precisely invariant under the identity in G.

(ii) For every $z \in \Omega(G)$, there is a $g \in G$, with $gz \in \overline{D}$.

(iii) The boundary of D consists of limit points of G, and a finite or countable collection of curves; each curve lies, except perhaps for endpoints, in Ω. The intersection of the curve with $\Omega(G)$ is called an *edge* of D.

(iv) The edges are paired by G; that is, if E is an edge of D, then there is an edge E', not necessarily distinct from E, and a non-trivial $g_E \in G$, called an *edge pairing transformation*, with $g_E E = E'$. Also $(E')' = E$, and $g_{E'} = g_E^{-1}$.

(v) If E_m is a sequence of edges of D, then $\mathrm{diam}_S(E_m) \to 0$; the edges of D accumulate only at limit points.

(vi) Only finitely many translates of D meet any compact subset of $\Omega(G)$.

It is a fact that any Kleinian group has a fundamental domain. One can use, for example, the Ford region, which is defined as the intersection of the outsides of all the isometric circles of elements of G ([Mas88] II.H.3). This notion is mostly needed for constructing examples of Maskit's combination theorems. We now go up one dimension to consider the corresponding object inside $\mathbb{H}^3_{\mathbf{R}}$.

Our first notion of geometrical finiteness will involve the existence of a fundamental polyhedron with finitely many faces. We now define these objects and prove they exist for any Kleinian group G, using a standard construction called the Dirichlet region. A *hyperplane* in $\mathbb{H}^3_{\mathbf{R}}$ is any totally geodesic copy of $\mathbb{H}^2_{\mathbf{R}}$, and such a hyperplane splits $\mathbb{H}^3_{\mathbf{R}}$ into two half-spaces. A (convex) *polyhedron* D is the intersection of countably many open half-spaces, where only finitely many of the corresponding hyperplanes meet any compact subset of $\mathbb{H}^3_{\mathbf{R}}$. There is a natural cell structure on $\overline{D}$ given by the intersections of the hyperplanes. The 0-cells, 1-cells, and 2-cells are called vertices, edges, and faces, respectively.

Definition 2.1.24. Let G be a Kleinian group. A polyhedron D is a *fundamental polyhedron* for G if the following hold.

(i) D is precisely invariant under the identity in G.

(ii) For every $x \in \mathbb{H}^3_{\mathbf{R}}$, there is $g \in G$, with $gx \in \overline{D}$.

(iii) The faces of D are paired by elements of G; that is, for every face F there is a face F', and there is $g_F \in G$ with $g_F F = F'$. These satisfy: $g_{F'} = g_F^{-1}$ and $(F')' = F$. The element g_F is called a *face pairing transformation.*

(iv) Any compact set meets only finitely many G-translates of D.

The final condition may also be expressed by saying that the tessellation of $\mathbb{H}^3_{\mathbf{R}}$ by translates of D is locally finite.

Now fix a Kleinian group G, and a point $x_0 \in \mathbb{H}^3_{\mathbf{R}}$ not fixed by any non-trivial $g \in G$. For each non-trivial $g \in G$, the perpendicular bisector of the line joining x_0 to gx_0 is a hyperplane. Let D_g be the half-space consisting of points closer to x_0 than gx_0, that is, points such that $d(x, x_0) < d(x, gx_0)$. The *Dirichlet region* D, centered at x_0, is the intersection of all such D_g.

Since any compact subset contains only finitely many points of the form gx_0, D is a convex polyhedron. Since x_0 is only fixed by the identity, each face of D corresponds to a unique element of G.

Theorem 2.1.25 ([Mas88] IV.G.2)**.** *Let G be a Kleinian group, and $x_0 \in \mathbb{H}^3_{\mathbf{R}}$ be a point not fixed by any non-trivial $g \in G$. Then the Dirichlet region D centered at x_0 is a fundamental polyhedron for G.*

Proof. We first check (i). If g is a non-trivial element of G, and $x \in D$, then

$$d(gx, gx_0) = d(x, x_0) < d(x, g^{-1}x_0) = d(gx, x_0).$$

Hence gx is not in D_g, and also not in D, as desired.

We next check condition (ii). Let $x \in \mathbb{H}^3_{\mathbf{R}}$. There is a (not necessarily unique) element $g \in G$ with $d(x, gx_0) \leq d(x, hx_0)$ for every $h \in G$. This is using the fact that G acts discontinuously on $\mathbb{H}^3_{\mathbf{R}}$. Then, writing an arbitrary element of G as gh, we have

$$d(g^{-1}x, x_0) = d(x, gx_0) \leq d(x, ghx_0) = d(g^{-1}x, hx_0)$$

for every $h \in G$. Hence $g^{-1}x \in \overline{D}_h$ for every h, implying $g^{-1}x \in \overline{D}$.

Next, we check condition (iii). Let x be a point of the relative interior of a face F of D. In this case, there is a unique $g \in G$ with $x \in \overline{D}_g$; that is, $d(x, x_0) < d(x, hx_0)$ for every $h \neq g$, and $d(x, x_0) = d(x, gx_0)$. We will show $g_F = g^{-1}$. We can write

$$d(g^{-1}x, x_0) = d(x, x_0) = d(g^{-1}x, g^{-1}x_0),$$

and then for any $h \neq g^{-1}$,

$$d(g^{-1}x, hx_0) = d(x, ghx_0) > d(x, x_0) = d(g^{-1}x, x_0).$$

So, $g^{-1}x$ also lies on a side F' of D. It quickly follows that $g^{-1}F = F'$.

Lastly, for condition (iv), let K be compact; it suffices to assume K is the closed ball of radius ρ centered at x_0. Again, since the action on $\mathbb{H}^3_{\mathbf{R}}$ is discontinuous, only finitely many translates of x_0 can be in the closed ball of radius 2ρ. But when $d(g^{-1}x_0, x_0) > 2\rho$, it follows that $gD \cap K = \varnothing$, so we are done. $\square$

Example 2.1.26. Let

$$g = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \ h = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.$$

Set $G = \langle g, h \rangle$. If we compute the Dirichlet region D centered at i, then we can first apply g and g^{-1} to i which produces two vertical planes as bisectors. Similarly, applying h and h^{-1}, we get two hemispheres as our bisectors. Then D is the intersection of the half-spaces containing i, which is depicted in Figure 2.1.1. Specifically, the region is contained between the two vertical planes and above the two hemispheres.

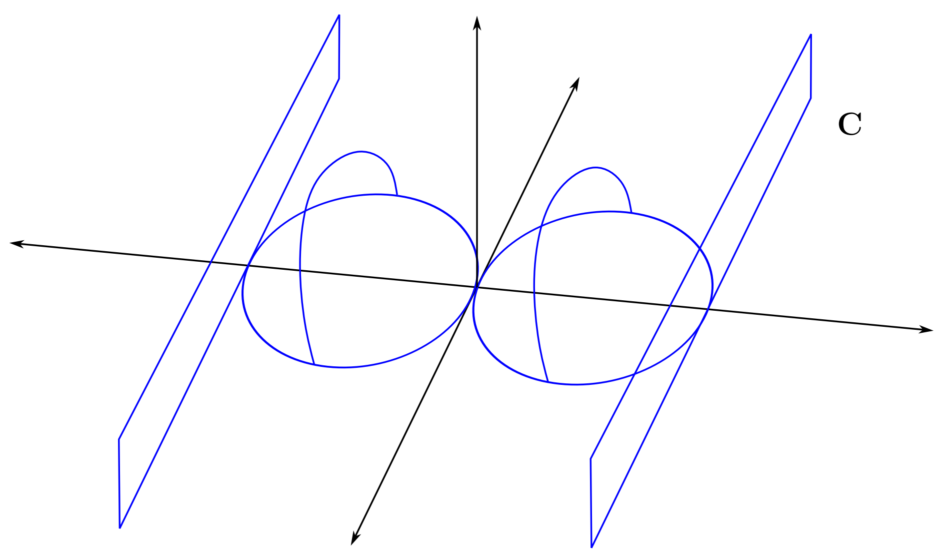

Figure 2.1.1: The fundamental polyhedron for this example.

If one takes the intersection of $\overline{D}$ with $\partial\mathbb{H}^3_{\mathbf{R}}$, we get a fundamental domain for the action of G on $\widehat{\mathbf{C}}$. This is true in general, although we will not need that fact.

Parabolic Subgroups

We will call a Kleinian group G *parabolic* if $\Lambda(G)$ consists of a single point. Then G has no loxodromic elements, and must have parabolic elements. Normalize so $\Lambda(G) = \{\infty\}$, then we can normalize further so that $jz = z + 1$ is in G, and all other parabolic elements are translations $z \mapsto z + \tau$ where $|\tau| > 1$.

Letting J be the subgroup of G consisting of all parabolic elements, we have that J is a discrete subgroup of translations of $\mathbf{C}$. Hence J is free abelian and has rank 1 or 2.

Definition 2.1.27. Given a parabolic group G, we define the *rank* of G to be the rank of its maximal parabolic subgroup.

If $J = G$, then G is either cyclic and generated by $jz = z + 1$ in the case that G has rank 1, or $G \cong \mathbb{Z}^2$ and is generated by $jz = z + 1$ and $hz = z + \tau$ for $\tau \in \mathbf{C} \setminus \mathbf{R}$.

If $G \neq J$, and J has rank 1, then G is necessarily an infinite dihedral group. That is, after normalizing, G is generated by $jz = z+1$ and an order 2 elliptic element fixing 0 and ∞ (after normalization). We call order 2 elliptic elements *half-turns*. This classification of rank 1 parabolic groups will be important in some proofs in the next section. One can also classify the rank 2 parabolic groups where $G \neq J$, but we will not need that classification.

2.1.4 Geometrical Finiteness

In this section we will develop the notion of geometrical finiteness, a criterion for a Kleinian group G ensuring the quotient manifold $\mathbb{H}^3_{\mathbf{R}}/G$ is 'nice' in some sense. Again, our main reference is [Mas88]. Recall that any Kleinian group admits a convex fundamental polyhedron by Theorem 2.1.25. The definition we will start with is very natural in light of this fact.

Definition 2.1.28. We say a Kleinian group G is *geometrically finite* if G has a convex fundamental polyhedron with finitely many faces.

The main goal in this section is to prove an equivalent characterization of geometrical finiteness just in terms of the action of G on $\partial\mathbb{H}^3_{\mathbf{R}}$, which will then be adapted to the more general situation of discrete convergence groups.

Theorem 2.1.29 ([Mas88] VI.C.7)**.** *Let G be a Kleinian group. Then the following are equivalent:*

1. *G is geometrically finite.*
2. *G has an essentially finite convex fundamental polyhedron.*
3. *Every limit point of G is a point of approximation, a rank 2 parabolic fixed point, or a doubly cusped rank 1 parabolic fixed point.*

These other notions will be defined in this section. First, we work on showing that any fundamental polyhedron D with finitely many faces for a Kleinan group G is also essentially finite. This roughly means that, after deleting a finite collection of horoballs about parabolic fixed points, that D is bounded away from $\Lambda(G)$. We will make this more precise soon. First, we fix some notations used throughout this section.

For a polyhedron $D \subset \mathbb{H}^3_{\mathbf{R}}$, we denote the relative boundary of D in $\mathbb{H}^3_{\mathbf{R}}$ by ∂D; we denote the intersection of the Euclidean boundary of D with $\partial\mathbb{H}^3_{\mathbf{R}}$ by

$$\overline{\partial} D = \partial D \cap \partial\mathbb{H}^3_{\mathbf{R}}.$$

The relative interior of $\overline{\partial}D$ in $\partial\mathbb{H}^3_{\mathbf{R}}$ is denoted by $\mathrm{Int}(\overline{\partial}D)$. See Figure 2.1.2 for an example.

If $x \in \overline{\partial}D$, and x lies on the boundary of the face F of D, then we say that F *abuts* x.

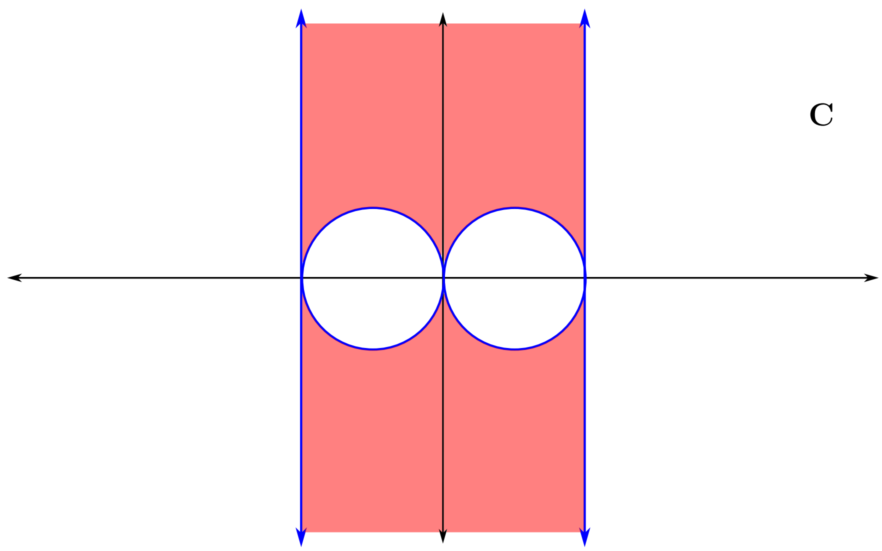

Figure 2.1.2: The fundamental domain determined by the fundamental polyhedron from Figure 2.1.1. The blue arcs bounding the red region are $\overline{\partial}D$, and the red regions are $\mathrm{Int}(\overline{\partial}D)$

.

We will need the notion of a horosphere and horoball.

Definition 2.1.30. A *horosphere* S in $\mathbb{H}^3_{\mathbf{R}}$ is a Euclidean $(n-1)$-sphere tangent to $\partial\mathbb{H}^3_{\mathbf{R}}$ which, aside from the point of tangency, lies in $\mathbb{H}^3_{\mathbf{R}}$. The corresponding open ball tangent to $\partial\mathbb{H}^3_{\mathbf{R}}$ is called a *horoball*. The point of tangency is the *center* or *vertex* of the horosphere and horoball.

Note that a horosphere tangent to a point in $\mathbf{C}$ in the upper half-space model is just a Euclidean sphere, while one tangent to ∞ is a Euclidean plane parallel to $\mathbf{C}$.

Proposition 2.1.31 ([Mas88] VI.A.5)**.** *Let G be Kleinian, where G contains $jz = z + 1$. Then the horoball $T = \{(z, t) \in \mathbb{H}^3_{\mathbf{R}} \mid t > 1\}$ is precisely invariant under* $\mathrm{Stab}(\infty)$.

Proof. Let $J = \mathrm{Stab}(\infty)$. Then no element of J is loxodromic by Proposition 2.1.9. Hence every element of J is parabolic or elliptic and preserves every horosphere centered at ∞ invariant.

If

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G,$$

then by Proposition 2.1.8, either $c = 0$, in which case $g \in J$, or $|c| \geq 1$. In the latter case, the radius of the isometric circle of g is at most one. Write $g = qr$, where r is reflection in the isometric circle of g, and q is a Euclidean motion. Extending to $\mathbb{H}^3_{\mathbf{R}}$, we have that r is reflection in the hyperbolic plane bounded by g's isometric circle. Since the radius of said circle is at most one, $rT \cap T = \varnothing$. On the other hand, q is a Euclidean motion preserving horospheres centered at ∞, hence $qT = T$ and $qrT \cap T = \varnothing$ as desired. $\square$

We now define cusped regions for parabolic fixed points.

Definition 2.1.32. Let G be a Kleinian group, and let J be a rank 1 parabolic subgroup of G. J, or the fixed point x of J, is *cusped* if there is an open circular disc $B \subset \widehat{\mathbf{C}}$ which is precisely invariant under J in G. B is called a *cusped region* for J, or for x. The point x is called the *center* of B.

Similarly, J, or x, is *doubly cusped* if there are two disjoint open circular discs B_1 and B_2, so that $B = B_1 \cup B_2$ is precisely invariant under J in G. Call B a *doubly cusped region.*

Note that the individual discs in a doubly cusped region need not be precisely invariant on their own. If $J = G = \langle z \mapsto z + 1, z \mapsto -z \rangle$, then the discs

$\{z \mid \mathrm{Im}(z) < -1\} \cup \{z \mid \mathrm{Im}(z) > 1\}$ are precisely invariant under J in G, but neither one is invariant on its own.

The existence of doubly cusped regions for rank 1 parabolic fixed points when G is geometrically finite is the first step towards encoding geometrical finiteness in terms of the action on the boundary.

Proposition 2.1.33 ([Mas88] VI.A.10)**.** *Let D be a fundamental polyhedron with finitely many faces for the Kleinian group G, and let $x \in \overline{\partial} D$. Then either $x \in \Omega(G)$, or $J = \mathrm{Stab}(x)$ is a parabolic subgroup of G. Further, if J has rank 1, then x is doubly cusped.*

Proof. This proof is quite long, so we break it down into steps.

Step 1: Show that either $x \in \Omega(G)$, or at least two faces abut x.

Step 2: Modify D inductively to obtain a new object D' whose faces abutting ∞ are all paired with each other.

Step 3: Use D' to show $\mathrm{Stab}(x)$ is a parabolic subgroup if $x \in \Lambda(G)$.

Step 4: Construct a doubly cusped region in the case that $\mathrm{Stab}(x)$ has rank 1.

Step 1: If $x \in \mathrm{Int}(\overline{\partial} D)$, then $x \in \Omega(G)$ since the interior of D is mapped off itself by any $g \in G$. If x is an interior point of a face F of $\mathrm{Int}(\overline{\partial} D)$, then there is another face F' and a face pairing transformation g with $gF = F'$. Again we have that $x \in \Omega(G)$, since a neighborhood of x consists of points in $D, g^{-1}D$ and F. Since D has finitely many faces, it now follows that at least two faces of D abut x. This also applies to isolated points of $\overline{\partial} D$.

Step 2: Normalize so $x = \infty$. Since D has finitely many faces, we can find a horoball T, centered at ∞, so that T meets only those faces of D abutting ∞. These faces need not be paired with each other, so we will modify D inductively as follows. If any faces abutting ∞ are not paired, then there is a face pairing transformation g_1 mapping a face abutting $x_1 \neq \infty$ on $\overline{\partial} D$ to a face abutting ∞. Replace $C_1 = D \cap g_1^{-1} T$ by $g_1 C_1$. We are essentially cutting a horoball centered at x_1 out of D and pasting it back at ∞.

If the resulting object has a face abutting ∞ paired with a face abutting some point $x_2 = g_2^{-1}\infty \neq \infty$, then set $C_2 = D \cap g_2^{-1}(T)$ and replace C_2 by $g_2 C_2$. This can only happen a finite number of times, and we eventually get a set D' with some properties we now list.

This D' is a (not necessarily convex) 'polyhedron' bounded by a finite number of faces, where some of the faces lie on horospheres instead of hyperplanes; D' is still precisely invariant under the identity in G; there is a horoball T', centered at ∞, that meets only those faces of D' abutting ∞; the faces of D' abutting ∞ are now paired with each other.

Step 3: The faces of D' abutting ∞ (if there are any) are pairwise identified by elements of G. Since the tessellation by D is locally finite and D has finitely many sides, $\mathbb{H}^3_{\mathbf{R}}/G$ is complete. When discussing Poincaré's polyhedron theorem, Maskit shows that the completeness condition for a finite sided polyhedron implies these 'infinite cycle transformations' are never hyperbolic ([Mas88], IV.I.6). We now want to show one of the face pairings is parabolic. If no faces of D' abut ∞, then $\partial D'$ is contained inside some Euclidean ball, hence $\infty \in \Omega(G)$ as in the first paragraph. If there is exactly one pair of faces abutting ∞ (including the possibility of one face paired with itself), and the corresponding face pairing j is elliptic, then ∞ is an elliptic fixed point since j sends a vertical plane to a vertical plane. It follows that $\infty \in \Omega(G)$.

Lastly, if more than two faces abut ∞, then either one identification is parabolic or we have at least two elliptic identifications. In the latter case, the commutator of the elliptics is a parabolic fixing ∞ by Proposition 2.1.7 as desired. This implies J has no loxodromic element by Proposition 2.1.9, and so we have established that any $x \in \overline{\partial} D$ satisfies $x \in \Omega(G)$ or $\mathrm{Stab}(x)$ is parabolic.

Step 4: For the second part of the proposition, we assume J is rank 1 and normalized so that $jz = z + 1$ generates the parabolic subgroup. The faces of D' abutting ∞ are paired by elements of J, and J is either cyclic or infinite dihedral if there are any elliptic elements. Either way, if F and F' are faces of D' paired by $j \in J$, then we have $F = F'$ or F is parallel to F'.

Let Σ be the 'polyhedron' bounded by the faces of D' abutting ∞. This is a disjoint union of convex polyhedra, which are convex in both the hyperbolic and Euclidean sense, since all their faces abut ∞. Let $E = \mathrm{Int}(\overline{\partial}\Sigma)$, then E is a union of finitely many Euclidean convex polygons in the plane, with edges paired by elements of J.

Since J has rank 1, there is at least one pair of parallel edges on the boundary of E identified by j. If E has exactly one such pair of edges, then these cannot be parallel to the real axis. So, we can find $b > 0$, so that $B = \{z \mid |\mathrm{Im}(z)| > b\}$ intersects only that pair of edges. Then B is precisely invariant under J in G by construction, and so x is doubly cusped.

In the more general case, consider $\mathrm{Im}(z)$ as height. If there is no highest point in E, then, since E has finitely many edges, we see that at sufficient height, we see only one pair of edges, necessarily identified by j. As above, we can find a cusped region of the form $\{z \mid \mathrm{Im}(z) > b\}$. Similarly, if E has no lowest point, we can find a complementary region $\{z \mid \mathrm{Im}(z) < -b\}$, and so x is doubly cusped.

So, we now suppose E has a highest point z_1. If this point is unique, then z_1 is the fixed point of a half-turn. More generally, these highest points are discrete since E only extends downwards from them. For the same reason, the edges abutting these highest points cannot all be identified by translations.

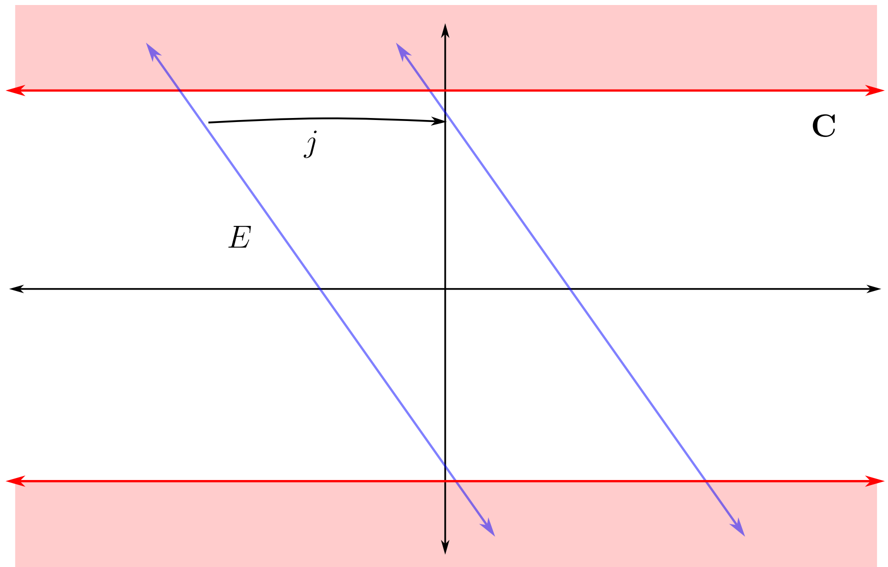

Figure 2.1.3: The simplest case for E when there is no highest nor lowest point. The red region is precisely invariant under J in G.

So, J is not cyclic. This also holds if E has a lowest point. Now assume E has a highest point, and normalize so the fixed points of all half-turns in J have height 0.

Cut and paste using these half-turns to obtain a 'polygon' E' contained in the lower half-plane. Aside from edges lying on the real axis, all edge pairing transformations are now translations. It follows that E' has no lowest point. Using a half-turn, the reflected E' has no highest point, and so we can find a doubly cusped region as above. $\square$

The set D' will be used again in a later proof. Now, we said above that an essentially finite fundamental polyhedron will be one which is bounded away from $\Lambda(G)$ aside from a collection of horoballs. We can now show that each individual parabolic fixed point has a such a horoball.

Proposition 2.1.34 ([Mas88] VI.A.13)**.** *Let D be a fundamental polyhedron* *with finitely many faces for the Kleinian group G, and let $x \in \overline{\partial} D$ be a rank 2 parabolic fixed point. Then there is a precisely invariant horoball T, centered at x, so that $D \setminus T$ is bounded away from x.*

Proof. Let T be any precisely invariant horoball centered at x, whose existence is guaranteed by Proposition 2.1.31. Normalize so that $x = \infty$. Then a sequence of points $(z_m, t_m) \in D \setminus T$ approaches ∞ only if t_m is bounded and $|z_m| \to \infty$. But $\partial T / \mathrm{Stab}(\infty)$ has finite area since $\mathrm{Stab}(\infty)$ has rank 2, hence $|z_m|$ is bounded for $(z_m, t_m) \in D \setminus T$. □

If $J = \mathrm{Stab}(x)$ has rank 1, then the above fails in general. Indeed, if G is generated by $z \mapsto z + 1$, then a fundamental polyhedron D is given by the region between two parallel vertical planes. But then when we delete a horoball centered at ∞, we can still approach ∞ from inside D. There is a natural modification in light of Proposition 2.1.33, though. We now have a doubly cusped region $B = B_1 \cup B_2$. Let H_m be the half space bounded by B_m, and let $T' = T \cup H_1 \cup H_2$, where T is a precisely invariant horoball. Then T' is precisely invariant under J in G, and we call this an *extended horoball* centered at x. If $z \mapsto z + 1$ is our parabolic generator, then we can take T' to be a set of the form $\{(z,t) \in \mathbb{H}^3_{\mathbf{R}} \mid t > 1\} \cup \{(z,t) \in \mathbb{H}^3_{\mathbf{R}} \mid |\mathrm{Im}(z)| > a\}$, for some $a > 0$.

Proposition 2.1.35 ([Mas88] VI.A.14)**.** *Let D be a fundamental polyhedron with finitely many faces for the Kleinian group G, and let $x \in \overline{\partial} D$, where $J = \mathrm{Stab}(x)$ is a rank 1 parabolic subgroup of G. Then there is a precisely invariant extended horoball T', centered at x, so that $D \setminus T'$ is bounded away from x.*

Proof. Normalize once again so that $x = \infty$, and so that $j(z) = z+1$ generates the parabolic subgroup of J. Since D is convex and has finitely many faces, all the faces not abutting ∞ are contained in a large Euclidean ball. Construct the set E as in Proposition 2.1.33, recalling that the edges which extend to ∞ do so in an infinite strip not parallel to the real axis. So, outside of a large Euclidean ball, all points of D either have large height or large imaginary part. Since our extended horoball T' consists of points with either large height or large imaginary part, $D \setminus T'$ stays bounded away from $x = \infty$. $\square$

Now that we have the notion of an extended horoball, we can define essentially finite fundamental polyhedra.

Definition 2.1.36. Suppose $x_1, \cdots, x_n$ are points of $\overline{\partial} D$, and T_m is a horoball or extended horoball at x_m. We say $\{T_1, \cdots, T_n\}$ is *precisely invariant relative to D* if each T_m is precisely invariant under $\mathrm{Stab}(x_m)$, and, whenever there is a $g \in G$ with $gx_i = x_j$, then $gT_i = T_j$.

We say a fundamental polyhedron D for a Kleinian group G is *essentially finite* if there is a finite set of horoballs or extended horoballs $\{T_1, \cdots, T_n\}$, where these are precisely invariant relative to D, and $D \setminus \bigcup T_m$ is bounded away from $\Lambda(G)$.

We are ready to prove the first implication of Theorem 2.1.29. Most of the work is already done.

Proposition 2.1.37 ([Mas88] VI.A.15)**.** *Let G be a Kleinian group, and D a fundamental polyhedron with finitely many faces. Then D is essentially finite.*

Proof. Let $x_1, \cdots, x_n$ be the parabolic fixed points in $\overline{\partial} D$. There are only finitely many by Proposition 2.1.33. From Proposition 2.1.34 we can find precisely invariant horoballs T_j for any rank 2 parabolic fixed points, and by Proposition 2.1.33 and Proposition 2.1.35, we can find precisely invariant extended horoballs T_j for any rank 1 parabolic fixed points, so that $D \setminus \bigcup T_m$

is bounded away from $\Lambda(G)$. By shrinking these horoballs as necessary, we can also ensure $gT_i = T_j$ whenever $gx_i = x_j$. □

For the second implication of Theorem 2.1.29, we have already seen what it means for parabolic points to be cusped or doubly cusped. We now introduce points of approximation. These will turn out to be equivalent to conical limit points, which will play an important role when we switch to discrete convergence groups. Recall that d_E is the Euclidean metric coming from the identification of $\mathbb{H}^3_{\mathbf{R}} \cup \partial\mathbb{H}^3_{\mathbf{R}}$ with the closed unit ball.

Definition 2.1.38. Let G be Kleinian. A point $x \in \partial\mathbb{H}^3_{\mathbf{R}}$ is a *point of approximation* for G if there is a sequence (g_m) of distinct elements of G so that $d_E(g_m x, g_m z) \geq \delta > 0$ on compact subsets of $\partial\mathbb{H}^3_{\mathbf{R}} \setminus \{x\}$.

If $|\Lambda(G)| = 1$, then this one point is a parabolic fixed point and cannot be a point of approximation. In general, parabolic fixed points are never points of approximation. If $|\Lambda(G)| = 2$, then both points are loxodromic fixed points and points of approximation. More generally, fixed points of loxodromic elements are always points of approximation, with g_m being the sequence of powers of the corresponding loxodromic element.

We will need the following characterization of points of approximation.

Proposition 2.1.39 ([Mas88] VI.B.3)**.** *A point $x \in \partial\mathbb{H}^3_{\mathbf{R}} \cong \mathbb{S}^2$ is a point of approximation if and only if there is a point $z \in \mathbb{H}^3_{\mathbf{R}}$, and there is a sequence (g_m) of distinct elements of G so that $d_E(g_m x, g_m z) \geq \delta > 0$.*

Proof. Assume first that x is a point of approximation and let (g_m) be the corresponding sequence of distinct elements of G. Let y_1, y_2 be points of $\mathbb{S}^2$ different from x, and let A be the geodesic connecting y_1 to y_2. Choose a subsequence so that $g_m x \to x', g_m y_1 \to y_1' \neq x'$, and $g_m y_2 \to y_2' \neq x'$. Let $z \in A$. Then either $g_m z \to y_1'$, or $g_m z \to y_2'$, as desired.

Now assume we have $z \in \mathbb{H}^3_{\mathbf{R}}$ so that $d_E(g_m x, g_m z) \geq \delta > 0$. Choose a subsequence so that $g_m z \to y$. Then $g_m w \to y$ uniformly in compact subsets of the complement of some point y' by Proposition 2.1.21. Since $d_E(g_m x, g_m z) \geq \delta$, we must have $y' = x$, and so x is a point of approximation. $\square$

We can now show boundary points of a fundamental polyhedron are never points of approximation.

Proposition 2.1.40 ([Mas88] VI.B.5)**.** *Let D be a fundamental polyhedron for the Kleinian group G, and let $x \in \overline{\partial} D$. Then x is not a point of approximation.*

Proof. Let (g_m) be any sequence of distinct elements of G. Up to subsequence, we may assume $g_m x \to x'$. Let L be a semi-infinite hyperbolic line segment lying entirely inside D with one endpoint at x. Since the Euclidean diameter of $g_m D$ in the ball model goes to 0, so does the Euclidean diameter of $g_m L$. This implies $g_m z \to x'$ for all $z \in L$, and then Proposition 2.1.21 implies there can be no point $z \in \mathbb{H}^3_{\mathbf{R}}$ so that $d_E(g_m x, g_m z)$ stays bounded away from 0. So x is not a point of approximation by Proposition 2.1.39. $\square$

Sometimes, when we have points accumulating to ∞, we will want to modify by $\mathrm{Stab}(\infty)$ so that the new set of points no longer accumulates to ∞. This type of result will appear again when we switch to discrete convergence groups, and play an essential role there.

Proposition 2.1.41 ([Mas88] VI.C.1)**.** *Let D be an essentially finite fundamental polyhedron for the Kleinian group G, where ∞ is a parabolic fixed point on $\overline{\partial} D$, and let (z_m) be a sequence of points in $\mathbf{C}$ which are G-translates of* *some point y. Then there are elements $j_m \in J = \mathrm{Stab}(\infty)$, with $j_m z_m$ contained in a bounded subset of* $\mathbf{C}$.

Proof. If J has rank 2, then a fundamental domain for J acting on $\mathbf{C}$ is compact, so we are done. If J has rank 1, then let T be a precisely invariant extended horoball centered at ∞, and let $B = \overline{\partial} T$. Modulo J, there is at most one translate of y in B, hence we can assume the points z_m lie outside B. This is an infinite horizontal strip after normalizing so that $z \mapsto z+1$ generates the parabolic subgroup of J, and again, a fundamental domain for the action of J on this strip is compact. $\square$

We can now prove the main part of the second implication of Theorem 2.1.29.

Proposition 2.1.42 ([Mas88] VI.C.2)**.** *Let D be an essentially finite fundamental polyhedron for the Kleinian group G. Then every limit point of G, which is not a translate of a point of $\overline{\partial} D$, is a point of approximation.*

Proof. Let x be such a limit point, and let L be a hyperbolic line passing through D with one endpoint at x. Let y be the other endpoint of L. We have enough degrees of freedom to ensure y is not a parabolic fixed point, and that L does not lie in any translate of a face of D. Since x is not on a translate of a face of D, L cannot be in any one translate of D. So, we find a linearly ordered sequence (x_m) on L so that $x_m \to x$, and there is a sequence of distinct elements (g_m) of G, where $z_m = g_m x_m \in D$. Pass to a subsequence so that $z_m \to z'$, $g_m x \to x'$, and $g_m y \to y'$.

The lines $L_m = g_m L$ converge to a line, or to a point. If $L_m \to M$, a line, then the endpoints of M are x' and $y' \neq x'$, and $z' \in \overline{M}$. Otherwise, L_m converges to $x' = y' = z' \in \partial \mathbb{H}^3_{\mathbf{R}}$. Suppose first that $z' \in \mathbb{H}^3_{\mathbf{R}}$. On L_m, z_m separates $g_m x$ from $g_m x_1$. It follows that in the ball model, $d_E(g_m x, g_m x_1)$ is

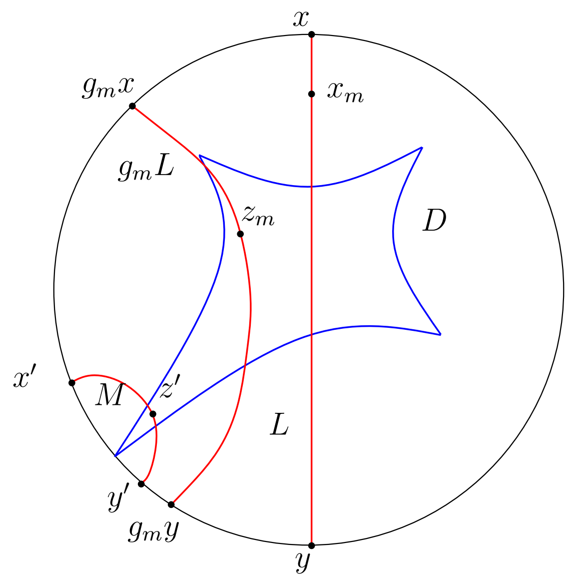

Figure 2.1.4: A schematic picture showing the various points and lines under consideration.

bounded away from 0, and so by Proposition 2.1.39, x is a point of approximation.

Similarly, if $z' = y' \neq x'$, then $g_m x_1 \to y'$, so x is a point of approximation.

If $z' = x'$, then since every $z_m \in D$, we have $x' \in \overline{\partial} D$, so x' is a parabolic fixed point. Let T be a precisely invariant horoball or extended horoball at x'. Since $D \setminus T$ is bounded away from x', the lines L_m all pass through T. If L_m converges to a line M, then M also passes through T.

We briefly step back to consider the case L_m converges to x'. We will reduce this to the case in the previous paragraph. Since y is not a parabolic fixed point, x' is not a translate of y. So, we can pick $j_m \in J = \mathrm{Stab}(x')$ so that $j_m g_m y$ is bounded away from x' by Proposition 2.1.41. Up to subsequence, $j_m g_m y \to y'$, and since each L_m passes through T, so does each $j_m L_m$.

In either case, under the assumption $z' = x'$, we have built a sequence of lines $h_m L$, where $h_m = j_m g_m$ (if L_m converged to a line M then $j_m = 1$), so that each $h_m L$ passes through T; $h_m x \to x'$, the center of T; and $h_m y \to y' \neq x'$.

Consider the points $h_m z_1$. If these all lie outside T (up to subsequence), then $d_E(h_m x, h_m z_1)$ is bounded away from zero, and so x is a point of approximation with sequence (h_m). So, we now assume that (up to subsequence) all the points $h_m z_1 \in T$, and we will find a contradiction. Notice that, for $m \neq n$, we have $z_1 \in g_n^{-1} T \cap g_m^{-1} T$, and so this intersection is nonempty. Precise invariance of T under J in G then implies that g_n, g_m are in the same left J-coset of G, which further implies that $h_m z_1$ and $h_n z_1$ are J-equivalent.

So, we can find $k_m \in J$ so that $k_m h_m z_1 = h_1 z_1$. Since z_1 is equivalent to a point of D, only the identity can fix z_1, so in fact $k_m h_m = h_1$. Rewriting things, we see that $g_m = \hat{j}_m g_1$, where $\hat{j}_m = j_m^{-1} k_m^{-1} j_1 \in J$. Since the g_m are all distinct, so are the $\hat{j}_m$. We know $g_m L = \hat{j}_m g_1 L$ converges to the line M which has an endpoint at x'; since $\hat{j}_m$ are elements of the parabolic subgroup J, this happens only if an endpoint of $g_1 L$ lies at the fixed point x' as well. Hence x' is equivalent to either x or y. But then either x is a translate of a point of $\overline{\partial} D$, or y is parabolic, which is a contradiction. □

Using Proposition 2.1.42 along with the definition of an essentially finite fundamental polyhedron, we establish the second part of Theorem 2.1.29.

Corollary 2.1.43 ([Mas88] VI.C.3)**.** *Let G be Kleinian, with an essentially finite fundamental polyhedron D. Then every limit point of G is a point of approximation, or a rank 2 parabolic fixed point, or a doubly cusped rank 1 parabolic fixed point.*

We need one quick result about uniqueness of face pairings before we finish the proof of the theorem.

Proposition 2.1.44 ([Mas88] VI.A.1). *Let $F_1 \neq F_2$ be faces of a fundamental polyhedron D for the Kleinian group G, and let g_1 and g_2 be the corresponding face pairing transformations; i.e., there are faces F_1' and F_2', so that $g_m F_m = F_m'$. Then $g_1 \neq g_2$.*

Proof. The hyperplane on which F_1' lies separates D from $g_1 D$; in particular, it separates F_2' from $g_1 F_2$. Hence $g_1 F_2$ is not a face of D, while $g_2 F_2 = F_2'$ is. □

And finally, we prove the final part of Theorem 2.1.29.

Theorem 2.1.45 ([Mas88] VI.C.4). *Let D be a convex fundamental polyhedron for the Kleinian group G. If every limit point of G is either a point of approximation, or a rank 2 parabolic fixed point, or a doubly cusped rank 1 parabolic fixed point, then D has finitely many faces.*

Proof. We assume D has infinitely many faces. Then we can find a sequence of faces (F_m) which accumulate at some point $x \in \overline{\partial} D$. Let Q_m be the hyperplane on which F_m lies. In the ball model, we necessarily have that the Euclidean diameters of Q_m are going to 0. Since there is a translate of D on either side of Q_m, we see that $x \notin \Omega(G)$, hence $x \in \Lambda(G)$, and is not a point of approximation by Proposition 2.1.40. This forces x to be a parabolic fixed point.

Normalize so $x = \infty$, and so that $J = \mathrm{Stab}(\infty)$ contains $jz = z + 1$ as a primitive element. Since D is convex, the Euclidean closure of D is also hyperbolically convex; in particular, if $(z, t) \in \overline{D} \setminus \{\infty\}$, then the hyperbolic line between (z, t) and ∞ is contained in D.

Let A be the set of points $z \in \mathbf{C}$ for which there is $t > 0$ so that $(z, t) \in D$. This is the straight line projection of D onto $\mathbf{C}$. If $z_1, z_2 \in A$, then for large enough t, the points $(z_1, t), (z_2, t) \in D$; hence z_1 cannot be J-equivalent to z_2, since this would give a J-equivalence of two points in D. This implies A is precisely invariant under the identity in J.

Claim: A is convex in $\mathbf{C}$.

Let $z_1, z_2 \in A$, then there are numbers t_1, t_2 so $(z_1, t_1), (z_2, t_2) \in D$. Let L be the hyperbolic line segment joining these points. Since D is convex, L is in D. Projecting L onto A, we find a line segment contained in A from z_1 to z_2 as desired.

The sequence (F_m) converges to ∞. Up to subsequence, none of the faces abut ∞, or they all do. We need to consider these two cases, with subcases where J has rank 1 or rank 2.

Case 1: We assume none of the faces abut ∞, so ∞ does not lie on the boundary of any Q_m. Take $(z_m, t_m) \in F_m$ so that $(z_m, t_m) \to \infty$. Since none of the F_m abut ∞, we see that $z_m \in A$. There is enough freedom in these choices to ensure the points z_m do not all lie on the same line in $\mathbf{C}$.

Subcase 1: First suppose J has rank 2. Then since A is precisely invariant under the identity in J, A must have finite area. Since A also has non-empty interior, we see that A is bounded. Hence $|z_m|$ is bounded and $t_m \to \infty$. This means the Q_m approach the vertical, that is, they all pass through a compact subset of $\mathbb{H}^3_{\mathbf{R}}$. This violates local finiteness of the tessellation by D.

Subcase 2: Now we assume J has rank 1. Normalize further so that any half-turns of J have their finite fixed point on the real axis, and so that A is contained in the strip between some line L and $L' = L + 1$, where L is not parallel to the real axis. For each F_m there is a face F'_m, and a face pairing transformation g_m so that $g_m F_m = F'_m$.

Assume that t_m is bounded. If $\mathrm{Im}(z_m)$ is also bounded, then since L is not parallel to the real axis, $\mathrm{Re}(z_m)$ is also bounded. This cannot happen, since

$(z_m, t_m) \to \infty$. We conclude $t_m \to \infty$, or $|\mathrm{Im}(z_m)| \to \infty$.

Consider the semi-infinite line segment L_m from $g_m(z_m, t_m)$ to ∞; note that $L_m \subset D$. The endpoints of $g_m^{-1}L_m$ are at (z_m, t_m) and $g_m^{-1}(\infty) \neq \infty$. Points with sufficiently large imaginary part lie in an extended horoball T which is precisely invariant under J in G. If the points $g_m^{-1}(\infty)$ had arbitrarily large imaginary part, we could find two such points $g_{m_1}^{-1}(\infty)$ and $g_{m_2}^{-1}(\infty)$ in T where one has larger imaginary part. Then $g_{m_2}^{-1}g_{m_1} \in J$ since it maps $g_{m_1}^{-1}(\infty)$ to $g_{m_2}^{-1}(\infty)$ which are both in T. But elements of J are purely real translations or half-turns about points on the real axis, and preserve the magnitude of the imaginary part, giving a contradiction. So, $g_m^{-1}(\infty)$ has bounded imaginary part.

There is $j_m \in J$ so that $j_m g_m^{-1}(\infty)$ also has bounded real part, and again elements of J leave $|\mathrm{Im}(z_m)|$ unchanged. The height t_m is also unchanged by elements of J. So, $j_m g_m^{-1} L_m$ has one endpoint in a bounded part of $\mathbf{C}$, and the other endpoint has either unbounded imaginary part or unbounded height. In either case, these lines pass through a compact subset of $\mathbb{H}^3_{\mathbf{R}}$. Hence by local finiteness of the tessellation by D, there are only finitely many distinct elements of the form $j_m g_m^{-1}$.

Note that if $j_m g_m^{-1} = j_k g_k^{-1}$, then $g_m(\infty) = g_k(\infty)$. Hence we can assume there is a subsequence (g_m) where $g_m = g_1 j_m$ (we can assume $j_1 = 1$). Since F_1 does not abut ∞, the point $g_1(\infty) = g_m(\infty) \notin \overline{\partial} D$. Now consider the line M_m from (z_m, t_m) to ∞. We look at $g_m(M_m)$, which has one endpoint at $g_m(\infty) = g_1(\infty)$. Choose a subsequence so that $g_m(z_m, t_m) \to y$, then $y \in \overline{\partial} D$ since $(z_m, t_m) \in \partial D$. It follows that the endpoints of $g_m(M_m)$ converge to distinct points, so that $g_m(M_m)$ converges to a line M. By Proposition 2.1.44, the elements g_m are all distinct. It follows that the tessellation of $\mathbb{H}^3_{\mathbf{R}}$ by D is not locally finite near M, a contradiction.

Case 2: What remains is the case when each s_m abuts ∞. Again, we have faces F'_m and face pairing transformations g_m, with $g_m F_m = F'_m$.

The points $g_m(\infty)$ all lie in $\overline{\partial} D$. If there were infinitely many distinct such points, then there would be an accumulation point $z \in \overline{\partial} D$ which is necessarily a parabolic fixed point since $z \in \Lambda(G)$. Temporarily re-normalize so $z = \infty$, so we have a sequence of limit points $z_m \in \overline{\partial} D$ with $z_m \to \infty$. These points are all equivalent under G, they all lie in A, and they all lie in the complement of the boundary of the possibly extended horoball at ∞. We have previously observed such a set is always bounded, giving a contradiction. So, there are only finitely many points of the form $g_m(\infty)$. This will allow us to construct the set D' from Proposition 2.1.33 soon.

We already know that there cannot be infinitely many faces that accumulate to ∞ without abutting ∞. Therefore, for t_0 sufficiently large, the horoball $T_0 = \{(z,t) \mid t > t_0\}$ meets only those faces of D abutting ∞.

Subcase 1: Suppose J has rank 2. The Euclidean area of $\partial T_0 / J$ is finite; hence $D \cap \partial T_0$ is bounded; hence A, the vertical projection of $D \cap \partial T_0$, is also bounded. If there were infinitely many faces abutting ∞, it now follows that they would all pass through a bounded part of ∂T_0, which again contradicts local finiteness of the tessellation.

Subcase 2: We now suppose J has rank 1, which will split into two more cases. Suppose first that no points of $\overline{\partial} D$ are equivalent to ∞, other than ∞ itself. The faces F_m all project to lines on ∂T_0, which is convex and precisely invariant under J in G. Since J has rank 1, there are at most a finite number of elements of J that can identify the faces of $D \cap \partial T_0$. Since distinct faces of D are identified by distinct elements of G, and elements of $G \setminus J$ cannot identify faces of $D \cap \partial T_0$, we see that D has only finitely many faces abutting ∞, which is a contradiction.

The other possibility is that we have a finite number of points equivalent to ∞ on $\overline{\partial}D$. As in Proposition 2.1.33, we construct the set D', which is no longer a polyhedron, since some of its 'faces' lie on horospheres. Recall that D' is precisely invariant under the identity in G still. The faces of D' are paired by elements of G, and while these elements need not all be distinct (like the case for D), there are at most finitely many of them equal to any given one (our cut and paste operations to construct D' are performed a finite number of times, and they correspond to conjugating some of the face pairing transformations).

Since the faces of D' abutting ∞ are paired with each other by elements of J, we can now proceed as in the previous case, and show that only finitely many elements of J pair such faces. This will again imply only finitely many faces abut ∞.

We know faces of D cannot accumulate to ∞ without abutting ∞; this implies the same statement for points $x \in \overline{\partial}D$ equivalent to D. Hence we can find a horosphere $T_1 = \{(z,t) \mid t = t_1\}$, so that every face of D' intersecting T_1 abuts ∞. Then T_1 is precisely invariant under J in G, and $T_1 \cap D'$ is precisely invariant under the identity in J.

Let $\widehat{D}$ be the vertical projection of $T_1 \cap D'$ to $\mathbf{C}$. Then $\widehat{D}$ is also precisely invariant under the identity in J. This is a connected finite union of Euclidean convex polygons, but need not be convex itself. The edges of $\widehat{D}$ are paired by J. Each polygon is either bounded, in which case it has finitely many edges, or is contained in a strip between parallel lines at horizontal distance 1 from each other. For each strip, there can be at most finitely many distinct edges that are paired with edges of the same strip. Indeed, if there were infinitely many, then there would be infinitely many paired by the same element of J.

There is only one way we could have infinitely many distinct edge pairing transformations in J at this point. This is when we have two strips S and S', where S and S' both contain infinitely many edges of $\widehat{D}$, there is a sequence of edges E_m in S paired with E'_m in S', and the side pairing transformations $j_m \in J$, satisfying $jE_m = E'_m$, are all distinct.

Since $\widehat{D} \cap S$ is convex, the edges (E_m) have a limiting direction; similarly, the (E'_m) have a limiting direction. These limiting directions are those of the parallel lines bounding the respective strips. Since $j_m E_m = E'_m$, the edges E_m and E'_m are also parallel, hence the limiting directions are parallel. So, we can find a point in the intersection of infinitely many of the edges E_m being identified to a point in the intersection of infinitely edges E'_m. But there are at most a finite number of elements of J that can identify a point of a (non-horizontal) infinite strip of bounded width with a point of a parallel strip, which is also of bounded width. So once again, only finitely many faces can abut ∞. $\square$

We end this section by noting another characterization of points of approximation and a characterization of parabolic fixed points which are either rank 2 or rank 1 and doubly cusped. These then form the basis for generalizing geometrical finiteness to discrete convergence groups.

Proposition 2.1.46. *Let G be a Kleinian group, then $x \in \partial\mathbb{H}^3_{\mathbf{R}}$ is a point of approximation if and only if there is a sequence (g_m) of distinct elements of G, so that, for every $z \in \partial\mathbb{H}^3_{\mathbf{R}}$, $(g_m x, g_m z)$ stays in a compact subset of $(\partial\mathbb{H}^3_{\mathbf{R}} \times \partial\mathbb{H}^3_{\mathbf{R}}) \setminus \Delta$, where Δ is the diagonal subspace.*

Proof. For this, we note that $d_E(g_m x, g_m z) \geq \delta > 0$ is equivalent to $(g_m x, g_m z)$ staying in a compact subset of $(\partial\mathbb{H}^3_{\mathbf{R}} \times \partial\mathbb{H}^3_{\mathbf{R}}) \setminus \Delta$. $\square$

Lastly, we establish a different finiteness condition for parabolic fixed points that more readily generalizes to the setting of discrete convergence groups. We will later call these bounded parabolic fixed points.

Proposition 2.1.47. *Let G be a Kleinian group. A parabolic fixed point $x \in \partial\mathbb{H}^3_{\mathbf{R}}$ is rank 2 or rank 1 and doubly cusped if and only if $(\Lambda(G) \setminus \{x\})/\mathrm{Stab}(x)$ is compact.*

Proof. Normalize so $x = \infty$ in the upper half-space model and contains $jz = z + 1$. First suppose x is rank 2 or rank 1 and doubly cusped. If x is rank 2, then we have a compact fundamental domain for the action of $\mathrm{Stab}(x)$ on $\mathbf{C}$, and hence $\mathbf{C}/\mathrm{Stab}(x)$ is compact and so is $(\Lambda(G) \setminus \{x\})/\mathrm{Stab}(x)$. If x is rank 1 and doubly cusped, then $\Lambda(G) \setminus \{x\}$ is necessarily contained in the complement of the cusped regions. This complement is an infinite horizontal strip, and there is a compact fundamental domain for the action of $\mathrm{Stab}(x)$ on this strip, so once again $(\Lambda(G) \setminus \{x\})/\mathrm{Stab}(x)$ is compact.

Conversely, suppose $(\Lambda(G) \setminus \{x\})/\mathrm{Stab}(x)$ is compact. If x has rank 2, then we are done, so suppose x has rank 1. We first argue that $\Lambda(G) \setminus \{x\}$ is contained in an infinite horizontal strip. Indeed, since our only parabolic elements translate horizontally, and the only other possible elements of $\mathrm{Stab}(x)$ are half-turns (by the discussion following Definition 2.1.27), if $\Lambda(G) \setminus \{x\}$ were not contained in a horizontal strip, then we could not have a compact subset of $\Lambda(G) \setminus \{x\}$ whose translates by $\mathrm{Stab}(x)$ cover $\Lambda(G) \setminus \{x\}$. This contradicts the fact that this action is cocompact. Then, for sufficiently large b, we have that $\{z \mid |\mathrm{Im}(z)| > b\}$ is a doubly cusped region for x, and we are done. □

In the language of discrete convergence groups, geometrical finiteness will directly generalize the dynamical characterization from Theorem 2.1.29 which only involves information about the action of G on the ideal boundary. These last two propositions give us a reasonable definition to start with which will make sense in the new context while being equivalent to the definitions for Kleinian groups.

2.2 Discrete Convergence Groups

This section is mostly devoted to background related to convergence group actions. We start by defining convergence groups and geometrical finiteness in Section 2.2.1 and Section 2.2.2. In Section 2.2.3 we recall some background on relatively hyperbolic groups. At the end of this section, we state and prove a key proposition (Proposition 2.2.25) about relatively quasi-convex subgroups of geometrically finite convergence groups which we will need in Chapter 3.

2.2.1 Convergence Groups

We refer to Tukia [Tuk94], [Tuk98], and Bowditch [Bow99] for further background on the material in this section.

Definition 2.2.1. Let G be a group acting on a compact metrizable space M. We say the action is a *convergence action* and call G a *convergence group* if, whenever (g_k) is a sequence of pairwise distinct elements in G, we can take a subsequence so that one of the following two conditions is satisfied:

1. The sequence (g_k) converges to a homeomorphism g in the compact-open topology on $\mathrm{Homeo}(M)$.

2. There are points $z_+, z_- \in M$ (not necessarily distinct) so that the maps $g_k|_{M\setminus\{z_-\}}$ and $g_k^{-1}|_{M\setminus\{z_+\}}$ converge to the constant maps $z \mapsto z_+$ and $z \mapsto z_-$ uniformly on compacts, respectively..

If G is a convergence group such that only the second condition occurs, we call G a *discrete convergence group* and the action a *discrete convergence action.*

Remark 2.2.2. Note that when $G < \mathrm{Homeo}(M)$ is a convergence group, G is a discrete convergence group if and only if G is a discrete subgroup of

Homeo(M) with respect to the compact-open topology on Homeo(M). So, if (g_k) is a divergent sequence (that is, a sequence which leaves every compact subset of Homeo(M)) in a discrete convergence group G, then we can extract a subsequence so that the second condition above holds.

When M is a topological n-sphere, the definition of a convergence group is due to Gehring and Martin [GM87], who observed that the isometry group of $\mathbb{H}^n_{\mathbf{R}}$ always acts as a convergence group on $\partial\mathbb{H}^n_{\mathbf{R}}$. Gehring and Martin also showed (again when M is an n-sphere) that a group G is a discrete convergence group if and only if the induced action of G on the space of distinct triples in M is properly discontinuous; later Bowditch [Bow99] observed that the same holds when M is an arbitrary compact Hausdorff space.

In the setting where M is compact metrizable, convergence groups were studied systematically by Tukia [Tuk94]. In particular Tukia showed that any group of isometries acting properly discontinuously on a proper geodesic Gromov-hyperbolic metric space X acts as a discrete convergence group on both the boundary ∂X and the compactification $\overline{X} = X \sqcup \partial X$ (see also Freden [Fre95]).

Definition 2.2.3. Following Tukia [Tuk94], if (g_k) is a sequence in $G <$ Homeo(M) such that the second condition of Definition 2.2.1 holds *without* extracting a subsequence, then we say that (g_k) is a *convergence sequence*.

In this case, the (uniquely defined) points z_+ and z_- are respectively called the *attracting point* and *repelling point* of the sequence (g_k).

When G is a discrete convergence group, an *arbitrary* sequence of pairwise distinct elements in G is not necessarily a convergence sequence, but it always has a subsequence which is.

One consequence of the definitions above is the following:

Proposition 2.2.4. *Let (g_k) be a convergence sequence in a discrete convergence group G acting on a compact metrizable space M containing at least 3 points. If U is any open neighborhood of the repelling point z_- of (g_k), then $(g_k\overline{U})$ converges to M in the topology on closed subsets of M induced by Hausdorff distance.*

Proof. If $\overline{U} = M$ the result is immediate, so assume that $M \setminus U$ is nonempty. Then, since $M \setminus U$ is a nonempty compact subset of $M \setminus \{z_-\}$, the set $g_k(M \setminus U)$ converges to a singleton $\{z_+\}$. So $g_k\overline{U}$ eventually contains every compact in the complement of $\{z_+\}$, and must converge to the closure of $M \setminus \{z_+\}$. In addition, since M contains at least 3 points, there are distinct points $x, y \in M$ so that $(g_k x)$ and $(g_k y)$ both converge to z_+. This implies z_+ is not an isolated point of M and so the closure of $M \setminus \{z_+\}$ is M. □

Given a discrete convergence group G acting on M, we can again define $\Omega(G)$ as the set of points in M where G acts discontinuously (recall Definition 2.1.13). We again call this the *domain of discontinuity* for G. This set is open, since given $x \in \Omega(G)$ and a neighborhood U from the definition, we have $U \subset \Omega(G)$. We set $\Lambda(G) = M \setminus \Omega(G)$, which is again called the *limit set* for G, and whose points are called *limit points* for G. Note that $\Lambda(G)$ is closed. We see that $\Omega(G)$ is G-invariant since we are acting by homeomorphisms, hence $\Lambda(G)$ is also G-invariant. We can again give a characterization of $\Lambda(G)$.

Lemma 2.2.5 (Tukia [Tuk94], Lemma 2M)**.** *A point $x \in M$ is a limit point of the discrete convergence group G if and only if x is the attracting point of a convergence sequence.*

Proof. First, if $x \in M$ is the attracting point of a convergence sequence, then $x \notin \Omega(G)$, so $x \in \Lambda(G)$, proving one direction.

Conversely, suppose $x \in \Lambda(G)$. Since $x \notin \Omega(G)$, we can find $x_k \to x$ and distinct $g_k \in G$ so that $g_k x_k \to x$. Pass to a subsequence so that (g_k) is a convergence sequence. Then x must be either the attracting or repelling point for (g_k). In the latter case, x is the attracting point for (g_k^{-1}). $\square$

We will call a discrete convergence group G *elementary* if $|\Lambda(G)|$ is finite. The next result shows that, in this case, we actually have $|\Lambda(G)| \leq 2$.

Theorem 2.2.6 (Tukia [Tuk94] Theorem 2S)**.** *Let G be a discrete convergence group acting on M. If $\Lambda(G)$ contains more than two points, then $\Lambda(G)$ is an infinite perfect set (hence uncountable). If G is non-elementary, then $\Lambda(G)$ is in the accumulation set of any orbit Gx for $x \in M$, and so, if $x \in \Lambda(G)$, then $\overline{Gx} = \Lambda(G)$.*

Proof. For the first claim, we must show $\Lambda(G)$ is perfect, i.e. $\Lambda(G)$ has no isolated points. If $a \in \Lambda(G)$, we can take a convergence sequence (g_k) so that a is the attracting point of the sequence, that is, $z_+ = a$. Let z_- be the repelling point. Since $\Lambda(G)$ contains more than two points, we have distinct $x, y \in \Lambda(G) \setminus \{z_-\}$. Then $g_k x$ and $g_k y$ are distinct and tend to z_+ as $k \to \infty$, so z_+ is an accumulation point of $G\{x, y\} \subset \Lambda(G)$, showing $z_+ = a$ is not isolated. Standard topological arguments now imply $\Lambda(G)$ is uncountable.

This argument also shows that, for any $a \in \Lambda(G)$, then $a \in \overline{Gx}$, as long as $x \neq z_-$. If G is non-elementary, then $hz_- \neq z_-$ for some $h \in G$, and so $a \in \overline{Gx}$ even if $x = z_-$. $\square$

The classification of isometries in hyperbolic geometry also generalizes to a classification of the elements of a group G acting as a convergence group on M, as shown by Tukia:

Proposition 2.2.7 ([Tuk94] Theorem 2B)**.** *Let G act as a convergence group on a compact metrizable space M. Every $g \in G$ satisfies exactly one of the following:*

- *The closure of the cyclic group $\langle g \rangle$ is compact in* $\mathrm{Homeo}(M)$, *in which case we say g is elliptic.*
- *g is not elliptic and g fixes exactly one point in M, in which case we say g is parabolic.*
- *g is not elliptic and g fixes exactly two points in M, in which case we say g is loxodromic.*

Moreover, if g is parabolic or loxodromic, then (g^n) is a convergence sequence, and the set of attracting and repelling points $\{z_\pm\}$ of (g^n) is precisely the set of fixed points of g.

Proof. Suppose $g \in G$ is not elliptic. Then the sequence (g^n) is divergent, so we can extract a convergence subsequence (g_n) with unique attracting and repelling points z_+, z_-. We claim these are both fixed points of g. Indeed, take $z \in M \setminus \{z_-\}$ satisfying $gz \neq z_-$. Then $z_+ = \lim_{n\to\infty} g_n z$, hence

$$gz_+ = \lim_{n\to\infty} g g_n z = \lim_{n\to\infty} g_n g z = z_+.$$

An identical argument involving powers of g^{-1} shows z_- is also a fixed point of g. The convergence dynamics imply g cannot have more than two fixed points, since then (g_n) would not be a convergence sequence, and so we see that g is parabolic if $z_- = z_+$ and loxodromic otherwise. $\square$

If G is a discrete convergence group, then the elliptic elements of G are precisely those with finite order. The classification also implies that if G is a virtually cyclic discrete convergence group, then G is elementary (but note that the converse need not hold).

2.2.2 Geometrical Finiteness

Geometrical finiteness was originally defined in real hyperbolic spaces of dimension 2 and 3, where the definition concerned the existence of a well-behaved fundamental domain for the action of G on X, as in Section 2.1.4. However, this definition proved to be unsatisfactory in hyperbolic spaces of higher dimension and in other negatively curved spaces.

In [Bow95], Bowditch gave several different definitions of geometrical finiteness for groups of isometries of a Hadamard manifold X with pinched negative curvature, and proved that they are all equivalent. One of Bowditch's definitions (Definition GF5), based on work of Beardon and Maskit [BM74], can be expressed entirely in terms of the convergence action of G on its limit set in ∂X, and therefore generalizes readily to the situation where G is a convergence group acting on an arbitrary compact metrizable space M. We also saw some of these equivalences in the previous section for $X = \mathbb{H}^3_{\mathbf{R}}$.

Before giving the definition we recall some essential terminology, which have been motivated by Proposition 2.1.46 and Proposition 2.1.47.

Definition 2.2.8. Let G be a discrete convergence group acting on a compact metrizable space M.

i) A point $x \in \Lambda(G)$ is a *conical limit point* if there is a sequence (g_k) in G of distinct elements such that for every $z \in M \setminus \{x\}$, the pair $(g_k x, g_k z)$ stays inside a compact subset of $(M \times M) \setminus \Delta$, where $\Delta \subset M \times M$ is the diagonal subspace. We will call the sequence (g_k) a *conical limiting sequence* for the point x.

ii) A point $x \in \Lambda(G)$ is a *parabolic point* if it is the fixed point of a parabolic isometry in G. A *parabolic subgroup* of G is the stabilizer in G of a parabolic point in $\Lambda(G)$. A parabolic point x is *bounded* if the quotient $(\Lambda(G) \setminus \{x\})/\mathrm{Stab}_G(x)$ is compact.

Remark 2.2.9. Tukia [Tuk98] showed that no point in M can be both a parabolic point and a conical limit point. By using the convergence group condition and extracting subsequences, one can also see that a point $x \in M$ is a conical limit point if and only if there are distinct points $a, b \in M$ and a conical limiting sequence (g_k) in G such that $(g_k x)$ converges to a and $(g_k y)$ converges to b for all $y \neq x$. The sequence (g_k) is then a convergence sequence, with $z_+ = b$ and $z_- = x$.

Furthermore, we could just as well ask that the defining condition for a conical limiting sequence holds only for $z \in \Lambda(G) \setminus \{x\}$, and then the discrete convergence dynamics imply this also holds in $\Omega(G)$.

Definition 2.2.10. Let G be a discrete convergence group acting on a compact metrizable space M. We say that G is *geometrically finite* if every point of $\Lambda(G)$ is either a conical limit point or a bounded parabolic point.

Remark 2.2.11. Unfortunately, the standard definitions of 'geometrically finite' in the geometric and dynamical contexts do not exactly agree. According to the definitions in e.g. Bowditch [Bow12], or Dahmani [Dah03], a convergence group G acting on M is 'geometrically finite' if every point of M (not just of $\Lambda(G)$) is a conical limit point or bounded parabolic point. With this convention, if X is a Hadamard manifold with pinched negative curvature, and G is a geometrically finite subgroup of $\mathrm{Isom}(X)$ (according to the definitions in Bowditch [Bow93], [Bow95]), then the action of G on ∂X is not a 'geometrically finite convergence action' if $\Lambda(G)$ is a proper subset of ∂X.

In this thesis, we adopt the convention that a convergence group acting on M is geometrically finite if and only if it acts geometrically finitely (in the sense of [Bow12], [Dah03]) on its limit set in M. So for us, when G acts by isometries on a hyperbolic space X, 'geometrically finite' means the same thing regardless of whether we consider the isometric action on X or the induced action by homeomorphisms on ∂X.

When G is geometrically finite, Tukia [Tuk98] showed that, if X is the set of distinct triples in $\Lambda(G)$, then X/G is *cusp-uniform* - that is, X/G consists of a compact piece with finitely many parabolic ends. In particular, if G has no parabolic elements, then X/G is compact. This set of distinct triples plays the role of $\mathbb{H}^3_{\mathbf{R}}$ in the case that $M = \mathbb{S}^2$. This result requires a lot of work and is beyond the scope of this thesis.

We conclude this subsection with another simple but useful criterion which can be used to guarantee that a point $x \in M$ is a conical limit point.

Lemma 2.2.12. *Let G be a discrete convergence group acting on a compact metrizable space M. Let Y be a subset of M containing at least two points, let K_1, K_2 be disjoint compact subsets of M, and let $x \in M$. If there exists a sequence (g_k) of pairwise distinct elements of G such that for all $k \in \mathbb{N}$ we have $g_k x \in K_2$ and $g_k Y \subset K_1$, then x is a conical limit point for G.*

Proof. Since G is a discrete convergence group we can extract a subsequence so that, for points $z_\pm \in M$, the sequence (g_k) converges in $\mathrm{Homeo}(M)$ to the constant map z_+ uniformly on compacts. In particular, for any $y \neq z_-$, the sequence $(g_k y)$ converges to z_+. Since Y contains at least two points, it contains at least one point y not equal to z_-. Then since $g_k y \in g_k Y \subset K_1$ we must have $z_+ \in K_1$. Since $g_k x \in K_2$, $(g_k x)$ cannot converge to z_+, hence $x = z_-$. Then for any $y \in M$ with $y \neq x$, $(g_k y)$ converges to z_+. The characterization of conical limit points described in Remark 2.2.9 implies that x is a conical limit point. $\square$

2.2.3 Relatively Hyperbolic Groups

For most of this thesis, we will only ever need to work with the dynamical definition of geometrical finiteness given above. However, our proof of one key technical lemma (Proposition 2.2.25) does rely on a geometric interpretation of the definition, which is best understood via the connection between geometrically finite groups and *relative hyperbolicity*. We refer to [Bow12], [Hru10] for further background on relatively hyperbolic groups.

Recall that a geodesic in a metric space (X, d) is an isometrically embedded copy of $\mathbf{R}$ or an interval. X is *proper* if X is complete and locally compact (equivalently, closed balls are compact), and X is *geodesic* if any $x, y \in X$ has a geodesic from x to y. A *geodesic triangle* in X is a triple of geodesic segments which pairwise share exactly one endpoint.

Definition 2.2.13. Let (X, d) be a proper geodesic metric space, and let $\delta > 0$. We say X is *δ-hyperbolic* if for any geodesic triangle with sides s_1, s_2, s_3, we have s_1 is contained in the uniform δ-neighborhood of $s_2 \cup s_3$.

We will say X is *hyperbolic* when such a δ exists. Note that the spaces defined earlier in this thesis are all hyperbolic in this sense as well, but this definition is more general since we are only asking X to be a metric space. We can associate a boundary to any such space, and in the case of $\mathbb{H}^3_{\mathbf{R}}$, this recovers the sphere, as one would expect.

Definition 2.2.14. Let (X, d) be a hyperbolic metric space. Define $\partial X = \{\text{geodesic rays in } X\}/\sim$, the *Gromov boundary* of X, where two rays are equivalent if points in one ray are at a uniformly bounded distance from the other ray, and vice versa.

Given a discrete subgroup $G < \mathrm{Isom}(X)$ (using the compact-open topology), we get an induced action on ∂X in the natural way, and in fact G acts on ∂X as a discrete convergence group by earlier remarks. This allows us to define geometrical finiteness of such subgroups exactly as we did in Section 2.1.4 using the third dynamical characterization, with $\Lambda(G)$ defined the same way as well. Equivalently, we can forget about X and define geometrical finiteness as we did in Definition 2.2.10. Our definition of relative hyperbolicity includes the data of the hyperbolic space X, though, and remarkably, that turns out to be the same thing as geometrical finiteness of the G-action just on ∂X.

The definitions of relative hyperbolicity we will use are given in Proposition 2.2.16 below. As in the classical (Kleinian) case of geometrical finiteness, the proposition says that, if G is as above and acts geometrically finitely on ∂X, then an appropriately defined 'convex core' for the G-action has a 'thick-thin' decomposition into a compact piece and some standard 'cusps.' This 'convex core' can be defined via the following. For any closed subset Z of ∂X, we let $\mathrm{join}(Z)$ denote the union of all bi-infinite geodesics in X joining distinct points in Z.

Note that a horoball in this more general setting is defined as the preimage of $\mathbf{R}_{\geq 0}$ under what is called a *horofunction.* They behave similarly to horoballs in the Kleinian setting, and our proofs will not rely on the full technical definition.

Proposition 2.2.15 (see e.g. Section 5 of Bowditch [Bow12])**.** *Suppose that X is a proper geodesic δ-hyperbolic metric space, and $Z \subset \partial X$ is a closed subset containing at least two points. Then* $\mathrm{join}(Z)$ *(with the metric induced* *by X) is the image of a quasi-isometrically embedded proper geodesic metric space, and its ideal boundary is precisely Z.*

In the above proposition, a *quasi-isometric embedding* is a map which is bi-Lipschitz with an additive error term. This notion will not occur beyond this section.

When G is a Kleinian group, $\mathrm{join}(\Lambda(G))$ is within uniformly bounded Hausdorff distance of the convex hull of the limit set of G, i.e. the minimal closed G-invariant convex subset of $\mathbb{H}^3_{\mathbf{R}}$ whose closure in $\overline{\mathbb{H}^3_{\mathbf{R}}}$ contains $\Lambda(G)$. So in the general setting, we can think of the quotient $\mathrm{join}(\Lambda(G))/G$ as a 'convex core' for X/G.

Proposition 2.2.16 (see Section 6 of Bowditch [Bow12])**.** *Let X be a proper geodesic δ-hyperbolic metric space and let G be an infinite discrete subgroup of* $\mathrm{Isom}(X)$*. Then the following are equivalent:*

- *The induced action of G on ∂X is geometrically finite in the sense of Definition 2.2.10.*
- *There exists a G-invariant system of pairwise disjoint horoballs $\mathcal{B}$ in X, such that the stabilizer in G of each $B \in \mathcal{B}$ is a parabolic subgroup, and G acts cocompactly on the set*

$$C(G, \mathcal{B}) := \mathrm{join}(\Lambda(G)) \setminus \bigcup_{B \in \mathcal{B}} B.$$

Moreover, if $|\Lambda(G)| > 1$, then for any *G-invariant system of pairwise disjoint horoballs $\mathcal{B}$ in X, the action of G on $C(G, \mathcal{B})$ is cocompact if and only if the set of centers of horoballs in $\mathcal{B}$ is precisely the set of parabolic points in $\Lambda(G)$.*

Proof. Since G is infinite and discrete, $\Lambda(G)$ cannot be empty. If $|\Lambda(G)| = 1$, then the first bullet point is trivial because the unique point in $\Lambda(G)$ is trivially bounded parabolic, and the second bullet point is trivial because $\mathrm{join}(\Lambda(G))$ is empty. So we assume $|\Lambda(G)| > 1$.

The space $Y = \mathrm{join}(\Lambda(G))$ is a *taut* hyperbolic metric space (i.e. every point in Y lies within uniformly bounded distance of a bi-infinite geodesic in Y). Furthermore, horoballs in Y (which can be viewed as a proper geodesic hyperbolic metric space via Proposition 2.2.15) are at a uniformly bounded Hausdorff distance away from horoballs in X intersected with Y. We need to replace X with Y so that $\Lambda(G) = \partial Y$, because of different conventions for geometrical finiteness (recall Remark 2.2.11). The result now follows from two results of Bowditch [Bow12] applied to Y. Proposition 6.12 of this paper gives the backwards direction, and Proposition 6.13 gives the forwards direction. $\square$

If $|\Lambda(G)| > 1$ in the situation above, then we say G is a *relatively hyperbolic group*, and the stabilizers of horoballs in $\mathcal{B}$ are called the *peripheral subgroups*. We say G is *hyperbolic relative to* the collection $\mathcal{P}$ of peripheral subgroups. We also say that any countably infinite group G is hyperbolic relative to $\{G\}$, and that any finite group is hyperbolic relative to an empty collection of peripheral subgroups.

In the special case where X is taut and $\Lambda(G) = \partial X$, we say that X is a *cusped space* for the data of the relatively hyperbolic group G and the peripheral subgroups $\mathcal{P}$. If $|\Lambda(G)| > 1$ we can always find a cusped space by replacing X with $\mathrm{join}(\Lambda(G))$.

The cusped space is in general *not* uniquely determined, even up to quasi-isometry. However, its ideal boundary is a well-defined G-space once the peripheral subgroups of G have been specified (see section 9 of Bowditch

[Bow12]). This space is called the *Bowditch boundary* of G and we denote it ∂G (the notation ignores the dependence on $\mathcal{P}$). When $\mathcal{P} = \{G\}$, then the Bowditch boundary of G is defined to be a singleton, and when G is finite its Bowditch boundary is empty.

When $|\partial G| \leq 2$, then we say G is *elementary*. The Bowditch boundary of a non-elementary relatively hyperbolic group is always *perfect*, i.e. it contains no isolated points. In particular if $|\partial G| \geq 3$, then ∂G is infinite.

A result of Yaman shows that the action of G on its Bowditch boundary can actually be used to completely recover the definition of G as a relatively hyperbolic group:

Theorem 2.2.17 (Yaman [Yam04])**.** *Let G be a discrete convergence group acting on a perfect compact metrizable space M. If every point of M is either a conical limit point or a bounded parabolic point (equivalently, if G is geometrically finite and $\Lambda(G) = M$), then there is a proper geodesic δ-hyperbolic metric space X, an embedding $G \to \mathrm{Isom}(X)$, and a G-equivariant homeomorphism from M to ∂X.*

The theorem implies in particular that a geometrically finite convergence group is the same thing as a relatively hyperbolic group. The proof of this result is beyond the scope of this thesis, but we will give a sketch of the key ideas. The main steps are are follows. First, Yaman constructs a collection of 'anuli' satisfying certain properties which allows her to define a quasimetric. This quasimetric is used to build an action of the group on a hyperbolic graph, and this is then used to prove G is relatively hyperbolic. The final step is showing ∂G identifies naturally with M. Some of the details are given below, starting with the definition of an annulus.

Definition 2.2.18. An *annulus*, A, is an ordered pair, (A^-, A^+), of disjoint closed subsets of M such that $M \setminus (A^- \cup A^+) \neq \varnothing$. A *system of annuli* $\mathcal{A}$ is a set of such annuli. Given $A = (A^-, A^+)$, we define $-A = (A^+, A^-)$, and say the system $\mathcal{A}$ is *symmetric* if $-A \in \mathcal{A}$ for every $A \in \mathcal{A}$.

Given $K \subset M$ closed, write $K < A$ if $K \subset \mathrm{Int}(A^-)$, and $K > A$ if $K \subset \mathrm{Int}(A^+)$. Given two annuli A, B, write $A < B$ if $M = \mathrm{Int}(A^+) \cup \mathrm{Int}(B^-)$. Note that $A < B$ implies $-B < -A$, and $A < B < C$ implies $A < C$.

Definition 2.2.19. Let $\mathcal{A}$ be a system of annuli. For closed $K, L \subset M$, we define $(K|L) \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ as the maximal number n such that we can find nested annuli, $A_1, \cdots, A_n \in \mathcal{A}$ that separate K and L, that is, $K < A_1 < \cdots < A_n < L$. If the maximum is not attained we define $(K|L) = \infty$.

Given a system of annuli, one can define a map from the space of distinct quadruples in M by sending (x, y, z, w) to $(\{x, y\}|\{z, w\})$. When $\mathcal{A}$ is symmetric, this defines a *crossratio* on M. This is similar to the classical crossratio in hyperbolic geometry. Then, if $\Theta(M)$ is the space of distinct triples in M and Π is the set of bounded parabolic points in M, they define a *quasimetric* ρ on $\Theta(M) \cup \Pi$ via

$$\begin{aligned}
\rho(a, b) &= (\{a\}|\{b\}), \\
\rho(a, X) = \rho(X, a) &= \max\{(\{a\}|\{x_i, x_j\}), i \neq j\}, \\
\rho(X, Y) &= \max\{(\{x_i, x_j\}|\{y_k, y_l\}), i \neq j, k \neq l\},
\end{aligned}$$

where $a, b \in \Pi, X = (x_1, x_2, x_3), Y = (y_1, y_2, y_3) \in \Theta(M)$. A quasimetric is a metric which only satisfies the triangle inequality up to some uniform additive constant, and where two distinct points can be at distance 0 from each other. Yaman constructs a specific system of annuli satisfying certain additional properties, and then considers the corresponding quasimetric. The next step is to construct a graph $\mathcal{K}$ which G acts on using this quasimetric, and then the following alternative definition of relative hyperbolicity is applied.

Proposition 2.2.20 ([Bow12]). *G is hyperbolic relative to $\mathcal{G}$ if and only if G admits an action on a connected graph, $\mathcal{K}$, with the following properties:*

1. *$\mathcal{K}$ is δ-hyperbolic and each edge of $\mathcal{K}$ is contained in only finitely many circuits of length n for any integer n.*

2. *There are finitely many G-orbits of edges, and each edge stabiliser is finite.*

3. *The elements of $\mathcal{G}$ are precisely the vertex stabilisers ot infinite valence of $\mathcal{K}$.*

Once they establish that G is relatively hyperbolic, they show the Bowditch boundary ∂G naturally identifies with M. The space X then comes from the first equivalent definition of relative hyperbolicity described in Proposition 2.2.16, namely, any relatively hyperbolic group acts on a δ-hyperbolic proper geodesic X so that ∂X identifies with ∂G, which in turn identifies with M.

Remark 2.2.21. Some definitions of relative hyperbolicity explicitly require either the group G or the peripheral subgroups in $\mathcal{P}$ to be finitely generated. We do not make this assumption, since both Proposition 2.2.16 and Theorem 2.2.17 hold without it. Our setup does always force the groups in $\mathcal{P}$ to be infinite, since they are parabolic subgroups of a convergence group.

Accumulation in Geometrically Finite Subgroups

Yaman's theorem means that we can always understand a non-elementary discrete convergence group G which is geometrically finite in the sense of Definition 2.2.10 using its isometric action on a cusped space X. In the case $|\partial G| = 0$ or $|\partial G| = 2$, we can also find a cusped space by taking X to be either a point or a line; if $|\partial G| = 1$ and G is finitely generated, then we can take the cusped space to be a 'horoball' modeled on G (see Gehring and Martin [GM08], Hruska [Hru10]).

We take advantage of the existence of the cusped space to prove some properties of subgroups of G which act geometrically finitely on $\Lambda(G)$. A convenient notation we will use here and many times later whenever we have a group G acting on M is

$$H(U) = \bigcup_{g \in H} gU$$

for some $U \subset M, H \subset G$. For the orbit of a point, we will just write Hx.

Definition 2.2.22. Let G be a relatively hyperbolic group, with Bowditch boundary ∂G. A subgroup $H \leq G$ is *relatively quasi-convex* if H acts geometrically finitely on ∂G (i.e. if every point of $\Lambda(H) \subseteq \partial G$ is either a conical limit point or a bounded parabolic point for the H-action).

Following Dahmani [Dah03], we say that a relatively quasi-convex subgroup H is *fully quasi-convex* if for all but finitely many left cosets gH, we have $gH(\Lambda(H)) \cap \Lambda(H) = \varnothing$.

Observe that, if G is elementary, then any fully quasi-convex subgroup of G is either finite or has finite index in G.

Lemma 2.2.23. *Let G be a non-elementary relatively hyperbolic group with associated cusped space $X = X(G)$, and let $H \leq G$ be a fully quasi-convex* *subgroup of G.*

Fix $x \in X$, and suppose that (g_k) is an infinite sequence in $G \setminus H$ such that

$$d_X(g_k x, x) = d_X(g_k x, Hx) \tag{2.2.1}$$

for all k. Then no attracting point of g_k in ∂X lies in $\Lambda(H)$.

Proof. Suppose for a contradiction that g_k has an attracting point $z \in \Lambda(H) \subset \partial X$. It follows that H is infinite, since $\Lambda(H)$ is nonempty. Since G acts as a convergence group on both ∂X and $X \sqcup \partial X$, we see that $(g_k x)$ converges to z in $X \sqcup \partial X$.

For each k, we let $c_k : [0, r_k] \to X$ be a geodesic ray in X from x to $g_k x$; since (g_k) is divergent we have $r_k \to \infty$. We may extend each c_k to a map $[0, \infty) \to X$ by setting $c_k(t) = c_k(r_k)$ for all $t \geq r_k$. Up to subsequence, these maps converge uniformly on compacts to a geodesic ray $c_z : [0, \infty) \to X$, whose ideal endpoint must be z.

By Proposition 2.2.16, there is a G-invariant family $\mathcal{B}_G$ of pairwise disjoint horoballs in X such that the parabolic subgroups of G are precisely the stabilizers of the horoballs in $\mathcal{B}_G$, and the quotient of

$$C(G, \mathcal{B}_G) = X \setminus \bigcup_{B \in \mathcal{B}_G} B \tag{2.2.2}$$

by the action of G is compact. By shrinking the horoballs in $\mathcal{B}_G$ if necessary, we can also assume that $x \in C(G, \mathcal{B}_G)$.

We claim that z is the center of some horoball $B \in \mathcal{B}_G$. If H is an infinite subgroup of a parabolic subgroup P in G, this is immediate, because then the unique point in $\Lambda(H)$ is the center of the unique horoball in $\mathcal{B}_G$ fixed by P. Otherwise, $\Lambda(H)$ contains at least two points, and we can consider the space

$\mathrm{join}(\Lambda(H)) \subset X$.

Let $\mathcal{B}_H$ be the horoballs in $\mathcal{B}_G$ whose centers are parabolic points in $\Lambda(H)$. By Proposition 2.2.16 again, H acts cocompactly on the set

$$C(H, \mathcal{B}_H) := \mathrm{join}(\Lambda(H)) \setminus \bigcup_{B \in \mathcal{B}_H} B.$$

Since the endpoint of the geodesic c_z lies in $\Lambda(H)$, there is some uniform $R > 0$ so that every point in the image of c_z lies within distance R of $\mathrm{join}(\Lambda(H))$.

Now, suppose that, for arbitrarily large t, the point $c_z(t)$ lies in an open R-neighborhood of the set $C(H, \mathcal{B}_H)$. Then, for some $k = k(t)$, the point $c_k(t)$ also lies in an R-neighborhood of $C(H, \mathcal{B}_H)$. Since H acts cocompactly on $C(H, \mathcal{B}_H)$, this means that $c_k(t)$ is within uniform distance of hx for some $h \in H$, which contradicts assumption (2.2.1).

So, for all sufficiently large times t, $c_z(t)$ must lie in some horoball in $\mathcal{B}_H$. Since the horoballs in $\mathcal{B}_H$ are pairwise disjoint, there is in fact a single horoball $B \in \mathcal{B}_H$ so that $c_z(t)$ is in the interior of B for all large enough t. The center of this horoball must be z.

Since (c_k) converges to c_z, for all sufficiently large k, the geodesic c_k enters B. However, since we have assumed $x \in C(G, \mathcal{B}_G)$, we know that $g_k x \in C(G, \mathcal{B}_G)$, and thus c_k must also leave the horoball B after it enters it. So, let w_k denote the last point where c_k leaves B. The distances $d_X(x, w_k)$ must tend to infinity as $k \to \infty$, since c_z never leaves B. See Figure 2.2.1.

Since c_k is a geodesic we know that $d_X(x, g_k x) = d_X(x, w_k) + d_X(w_k, g_k x)$. Then, because $d_X(x, w_k)$ tends to infinity, assumption (2.2.1) implies that $d_X(Hx, w_k)$ tends to infinity as well. On the other hand, we also know that the stabilizer of B in G acts cocompactly on ∂B. Then, since $w_k \in \partial B$, there is some constant $D > 0$ so that, for every k, we have $s_k \in G$ preserving B such

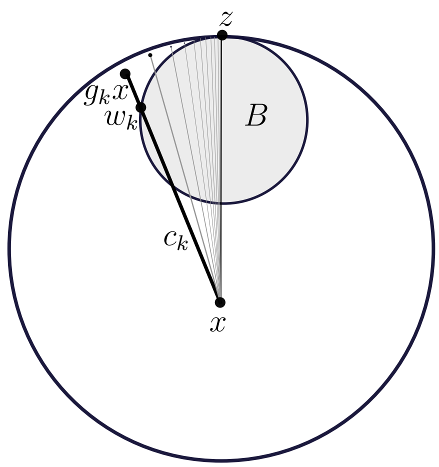

Figure 2.2.1: Illustration for the proof of Lemma 2.2.23. The geodesic c_k from x to $g_k x$ must enter B, and leave B far from x.

that $d_X(x, s_k^{-1} w_k) < D$. Hence $d_X(s_k x, w_k) < D$. It follows that the elements in the sequence (s_k) cannot lie in finitely many left cosets of H. However, since s_k preserves B, each s_k also fixes the point $z \in \Lambda(H)$, which contradicts the full quasi-convexity of H. $\square$

The geometric statement of the lemma above has the following (completely dynamical) consequence:

Lemma 2.2.24. *Let G be a relatively hyperbolic group with Bowditch boundary ∂G, and let J_1, J_2 be fully quasi-convex subgroups of G.*

For any sequence (g_k) in G, there exists $j_k \in J_1, j'_k \in J_2$ such that the sequence $(j_k g_k j'_k)$ has no attracting points in $\Lambda(J_1) \subset \partial G$ and no repelling points in $\Lambda(J_2) \subset \partial G$.

Proof. If $g_k \in J_1 \cup J_2$, then we can choose $j_k \in J_1$ and $j'_k \in J_2$ so that $j_k g_k j'_k$ is the identity. A bounded sequence has no attracting or repelling points. So, we may assume $g_k \in G \setminus (J_1 \cup J_2)$ for all k.

If G is elementary, then J_1 and J_2 are both either finite or finite-index subgroups of G. In this case the result is immediate, so we can assume G is non-elementary and let X be a cusped space for G. Fix $x \in X$. For each k, we choose $j_k \in J_1$, $j'_k \in J_2$ so that

$$d_X(g_k(J_2 x), J_1 x) = d_X(g_k j'_k x, j_k^{-1} x).$$

We know such elements $j_k \in J_1, j'_k \in J_2$ exist because $J_i x$ are discrete subsets of X for $i = 1, 2$. Let $g'_k = j_k g_k j'_k$. We will show that g'_k has no repelling points in $\Lambda(J_2)$. The argument that g'_k has no attracting points in $\Lambda(J_1)$ is completely symmetric, after replacing g'_k with its inverse.

Since $j J_i x = J_i x$ for any $j \in J_i$, we know that for all $k \in \mathbb{N}$ we have

$$d_X(g'_k x, x) = d_X(g_k J_2 x, J_1 x) = d_X(g'_k J_2 x, J_1 x).$$

By definition of g'_k, we know that

$$d_X(g'_k J_2 x, J_1 x) \leq d_X(g'_k J_2 x, x) = d_X(J_2 x, (g'_k)^{-1} x),$$

so combining this with the previous equality we conclude

$$d_X(x, (g'_k)^{-1} x) = d_X(g'_k x, x) \leq d_X(J_2 x, (g'_k)^{-1} x).$$

So in fact $d_X(x, (g'_k)^{-1} x) = d_X(J_2 x, (g'_k)^{-1} x)$ for every k. Then Lemma 2.2.23 implies that $((g'_k)^{-1})$ has no attracting points in $\Lambda(J_2)$, or equivalently (g'_k) has no repelling points in $\Lambda(J_2)$. $\square$

The main application of these lemmas is the technical proposition below. Roughly, this proposition tells us that, in certain circumstances, it is possible to strengthen the 'ping-pong' combinatorics of geometrically finite convergence groups. That is, the proposition gives us a way to modify a 'ping-pong' element $g \in \mathrm{Homeo}(M)$, so that instead of nesting the closure of an open subset $U \subset M$ inside of another open subset $V \subset M$, g takes the closure of U inside of a fixed compact subset $K \subset V$. This 'strong nesting' property will be useful throughout Chapter 3.

Proposition 2.2.25. *Let G be a geometrically finite convergence group acting on a compact metrizable space M, let H be a subgroup of G, and let $J_1, J_2 \leq H$ be fully quasi-convex subgroups of G. Let U_1, U_2 be open subsets of M such that, for $i \in \{1, 2\}$, we have $J_i(U_i) = U_i$ and $\Lambda(H) \setminus \Lambda(J_i) \subset U_i$. Suppose that for every $g \in H \setminus J_2$, we have $g(M \setminus U_2) \subset U_1$.*

Then, there exists a compact set $K \subset U_1$ such that for all $g \in H \setminus J_2$, we can find $j \in J_1$ such that $jg(M \setminus U_2) \subset K$.

Proof. Suppose that the claim does not hold. This means that we can find a sequence of group elements (g_k) in $H \setminus J_2$ such that for any sequence (j_k) in J_1, there is a sequence (x_k) in $M \setminus U_2$ such that the sequence $(j_k g_k x_k)$ accumulates in $M \setminus U_1$.

Fix this sequence (g_k). Lemma 2.2.24 gives a pair of sequences (j_k) in J_1 and (j'_k) in J_2 so that any attracting points of the sequence $(g'_k) = (j_k g_k j'_k)$ do not lie in $\Lambda(J_2)$, and any repelling points do not lie in $\Lambda(J_1)$. Then, since U_2 is J_2-invariant, there is a sequence (x_k) in $M \setminus U_2$ so that $(j_k g_k j'_k x_k)$ accumulates in $M \setminus U_1$. After taking a subsequence, we may assume that $(j_k g_k j'_k x_k)$ has a unique limit $z \in M \setminus U_1$.

Again using the fact that U_1 and U_2 are invariant under J_1 and J_2 respectively, we know that for every k, we have $g'_k(M \setminus U_2) \subset U_1$. So, if only finitely many different elements appear in the sequence (g'_k), we can find a fixed compact set $K \subset U_1$ so that $g'_k(M \setminus U_2) \subset K$ for every k, hence $g'_k x_k \in K$ for every k. This is impossible if $g'_k x_k \to z \in M \setminus U_1$.

So, we may extract a subsequence so that the elements in (g'_k) are pairwise distinct. After taking a further subsequence, we can find a pair of points $z_+ \in M \setminus \Lambda(J_1)$ and $z_- \in M \setminus \Lambda(J_2)$ so that (g'_k) converges uniformly to the constant map z_+, uniformly on compacts in $M \setminus \{z_-\}$. Both of $z_\pm$ lie in $\Lambda(H)$, and in fact $z_+ \in U_1$ and $z_- \in U_2$.

Since $M \setminus U_2$ is closed, x_k cannot accumulate on z_-, which means $(g'_k x_k)$ converges to $z_+ \in U_1$, which contradicts the fact that $g'_k x_k \to z$. $\square$

2.3 Combinatorial Group Theory

In this section we establish the notation and basic facts for amalgamated free products (AFP) and HNN extensions (HNN), which will play an important role in both the classical and new combination theorems.

2.3.1 AFP Combinatorial Group Theory

We first deal with *amalgamated free products*. The abstract group-theoretic definition is as follows.

Definition 2.3.1. Let G_1, G_2, J be groups, and let $\varphi_i : J \to G_i$ be a pair of injective homomorphisms. We define $G_1 *_J G_2$, the *amalgamated free product of* G_1 *and* G_2 *over* J, as

$$G_1 *_J G_2 = (G_1 * G_2)/N,$$

where N is the normal closure of $\{\varphi_1(j)\varphi_2(j)^{-1} \mid j \in J\}$.

One can think of this construction as gluing the two groups along a common subgroup J. We will next introduce an equivalent way to view amalgamated free products. Our reference throughout is [Mas88]. In this section, M will denote a compact metrizable space, although the results in this section are purely set-theoretic. We further assume throughout this section that G_1, G_2 are subgroups of $\mathrm{Homeo}(M)$, and $G_1 \cap G_2 = J$, where J is a proper subgroup of both G_1 and G_2. We let G denote $\langle G_1, G_2 \rangle$, the subgroup generated by G_1 and G_2. When we discuss the classical combination theorems, we will have $M = \partial \mathbb{H}^3_{\mathbf{R}}$ and our subgroups will live inside $\mathrm{PSL}(2, \mathbf{C})$, but this more general framework will not change any arguments and is needed later in the thesis.

Given a word $g = g_1 \cdots g_n$ in the elements of G_1 and G_2, we call g a *normal form* when the elements g_i alternate between $G_1 \setminus J$ and $G_2 \setminus J$. We say two normal forms $g = g_1 \cdots g_n$ and $h = h_1 \cdots h_n$ are *equivalent* if g can be obtained from h by inserting or deleting finitely many words of the form jj^{-1} for $j \in J$. We can then form the amalgamated free product as

$$G_1 *_J G_2 = J \cup \{\text{equivalence classes of normal forms}\}.$$

We have a group operation on $G_1 *_J G_2$ given by concatenation, which is well-defined on normal forms up to equivalence. Up to isomorphism, this produces the same group as Definition 2.3.1, with the inclusion maps playing the role of the injective homomorphisms.

The normal form $g = g_1 \cdots g_n$ is called an *(i, j)-form* if $g_1 \in G_i$ and $g_n \in G_j$. The *length* of the normal form is defined as $|g| = n$. By convention, we will say that elements of J have length 0. Note that if g is an (i, j)-form, then its formal inverse g^{-1} is a (j, i)-form.

There is a group homomorphism

$$\begin{aligned} \varphi : G_1 *_J G_2 &\to G \\ g_1 \cdots g_n &\mapsto g_1 \circ \cdots \circ g_n, \end{aligned}$$

where on the right we are just composing the corresponding elements in $\mathrm{Homeo}(M)$. This map is always surjective, but its kernel need not be trivial. When φ is an isomorphism, we will abuse notation and leave it implicit, writing $G = G_1 *_J G_2$; then we can view elements of the subgroup G as (equivalence classes of) normal forms in the abstract amalgamated free product $G_1 *_J G_2$.

Using a ping-pong technique (Proposition 2.3.6 below), we can give a sufficient condition which guarantees that φ is actually an isomorphism.

Definition 2.3.2. A pair of disjoint nonempty J-invariant sets $U_1, U_2 \subset M$ is called an *interactive pair* for G_1 and G_2 if for every $g \in G_i \setminus J$, we have $gU_i \subset U_{3-i}$.

If, in addition, $gU_i \subset U_{3-i}$ is a proper inclusion for every $g \in G_i \setminus J$ for at least one of $i \in \{1, 2\}$, then we call (U_1, U_2) a *proper interactive pair.*

Remark 2.3.3. Maskit's convention is to call an interactive pair U_1, U_2 *proper* if the G_i-translates of U_i do not cover U_{3-i} for at least one $i \in \{1, 2\}$. Our assumption is slightly weaker, but does not change any of the standard arguments.

It is immediate that if (U_1, U_2) is an interactive pair, then U_i is precisely invariant under J in G_i for $i = 1, 2$. We observe the following:

Proposition 2.3.4. *If (U_1, U_2) is a proper interactive pair for G_1 and G_2, then both U_1 and U_2 are infinite sets.*

Proof. Since J is a proper subgroup of G_i for $i = 1, 2$, there is at least one element $g_1 \in G_1 \setminus J$ and at least one element $g_2 \in G_2 \setminus J$. We know that at least one inclusion $g_1 U_1 \subset U_2$ or $g_2 U_2 \subset U_1$ is proper, so $g_2 g_1 U_1$ is a proper subset of U_1. Therefore U_1 is infinite, and since $g_1 U_1 \subset U_2$, so is U_2. $\square$

Via the map φ, normal forms in $G_1 *_J G_2$ act in a 'ping-pong' manner on the sets in an interactive pair.

Lemma 2.3.5 ([Mas88] VII.A.9)**.** *Let (U_1, U_2) be an interactive pair. Then if $g \in G_1 *_J G_2$ is an (i, j)-form, we have $\varphi(g)U_j \subset U_{3-i}$. Further, this inclusion is proper if (U_1, U_2) is proper and $|g| \geq 2$.*

The lemma can be proved via a straightforward combinatorial argument; see the reference for details. To illustrate the idea, suppose the G_1-translates of U_1 are all properly contained in U_2, and that g has length 2. If $g = g_1 g_2$ is a $(2, 1)$-form, then $g_2(U_1) \subset U_2$ is already proper, and hence $\varphi(g)U_1 \subset U_1$ is also a proper inclusion. If g is a $(1, 2)$-form, then $g_2 U_2 \subset U_1$ need not be a proper inclusion, but then applying g_1 will cause the next inclusion $\varphi(g)U_2 = g_1 g_2 U_2 \subset U_2$ to be proper.

Proposition 2.3.6 (Ping-pong for amalgamated free products; see [Mas88] VII.A.10)**.** *Suppose (U_1, U_2) is a proper interactive pair for G_1 and G_2. Set $G = \langle G_1, G_2 \rangle$. Then $G = G_1 *_J G_2$.*

Proof. We will show the surjective group homomorphism $\varphi : G_1 *_J G_2 \to G$ has trivial kernel. The only length 0 element sent to the identity is the identity, and length 1 elements are all nontrivial in G_1 or G_2, so it suffices to show $\varphi(g) \neq 1$ when $|g| \geq 2$. Suppose g is an (i, j)-form. We now note that because we have a proper interactive pair, $\varphi(g)U_j \subset U_{3-i}$ is a proper inclusion by Lemma 2.3.5, and so $\varphi(g)$ cannot be the identity. The result follows. $\square$

2.3.2 HNN Combinatorial Group Theory

In this section we establish notation and give some basic facts about *HNN extensions*. Again, we start with the group-theoretic definition.

Definition 2.3.7. Let $G = \langle S \mid R \rangle$ be a group defined via generators and relations, and $H, K < G$ subgroups with an isomorphism $\varphi : H \to K$. Define $G*_\alpha$, the *HNN extension of G relative to α*, as

$$G*_\alpha = \langle S, t \mid R, tht^{-1} = \alpha(h), \forall h \in H \rangle.$$

This time, we are constructing a larger group where two isomorphic subgroups will now be conjugate to each other. There is again a version using normal forms which we will develop next. Our main reference is again [Mas88]. In this section, M is again an arbitrary compact metrizable space, but as in Section 2.3.1, these results are purely set-theoretic. We further assume throughout this section that G_0, G_1 are subgroups of $\mathrm{Homeo}(M)$, where $G_1 = \langle f \rangle$ is infinite cyclic, and J_1, J_{-1} are subgroups of G_0 with $fJ_{-1}f^{-1} = J_1$. Technically, we think of f as t from Definition 2.3.7, and conjugation by f as α. We let G denote $\langle G_0, G_1 \rangle$, the subgroup of $\mathrm{Homeo}(M)$ generated by G_0 and G_1. The abstract isomorphism $J_{-1} \to J_1$ induced by conjugating by f will be denoted by f_* here, instead of α. The indices are chosen to make notation more convenient later.

As was the case for amalgamated free products, we can define HNN extensions using equivalence classes of *normal forms*.

Definition 2.3.8. A word $g = f^{\alpha_1} g_1 \cdots f^{\alpha_n} g_n$ in f and elements g_k of G_0 is a *normal form* if:

(1) Each $g_k \in G_0$ is nontrivial for $k < n$;

(2) Each α_k is an integer, with $\alpha_k \neq 0$ whenever $k > 1$;

(3) If $\alpha_k < 0$ and $g_{k-1} \in J_{-1} \setminus \{1\}$, then $\alpha_{k-1} < 0$;

(4) If $\alpha_k > 0$ and $g_{k-1} \in J_1 \setminus \{1\}$, then $\alpha_{k-1} > 0$.

Two words $g = f^{\alpha_1} g_1 \cdots f^{\alpha_n} g_n$ and $h = f^{\beta_1} h_1 \cdots f^{\beta_n} h_n$ are *equivalent* if we can obtain g from h by inserting finitely many conjugates and inverses of words of the form $fjf^{-1}(f_*(j))^{-1}$ for $j \in J_{-1}$ (words of this form are the identity in G). Every word of the form $f^{\alpha_1} g_1 \cdots f^{\alpha_n} g_n$ is equivalent either to a normal form or to the identity, which means that every word in f and elements of G_0 is equivalent to either a normal form or the identity.

The *length* of a normal form $g = f^{\alpha_1} g_1 \cdots f^{\alpha_n} g_n$ is defined to be $|g| = \sum_{i=1}^{n} |\alpha_i|$. Note that, in contrast to normal forms for amalgamated free products, the length of a normal form $f^{\alpha_1} g_1 \cdots f^{\alpha_n} g_n$ is *not* necessarily n. Length-0 normal forms correspond by definition to elements of G_0.

If a normal form g has positive length, $i \in \{0, \pm 1\}$ and $j \in \{\pm 1\}$, then we say $g = f^{\alpha_1} g_1 \cdots f^{\alpha_n} g_n$ is an (i, j)-form if α_1 is positive (resp. negative, zero) and $i = 1$ (resp. -1, 0), and α_n is positive (resp. negative) and $j = 1$ (resp. -1). Our notation differs slightly from Maskit's, which will make some of our later arguments less cumbersome.

We set

$$G_0 *_f = \{\mathrm{id}\} \cup \{\text{equivalence classes of normal forms}\}.$$

This set forms a group, with operation given by concatenation followed by reduction to a normal form. It is called the *HNN extension of* G_0 *by* f. Note that it is *not* in general true that the formal inverse of a normal form g is also a normal form (see Lemma 2.3.13 below), but it is a formal product of normal forms, which tells us that $G_0 *_f$ contains inverses.

We again have a natural surjective homomorphism

$$\varphi : G_0 *_f \to G$$
$$f^{\alpha_1} g_1 \cdots f^{\alpha_n} g_n \mapsto f^{\alpha_1} \circ g_1 \circ \cdots \circ f^{\alpha_n} \circ g_n.$$

The map φ may or may not be an isomorphism. As was the case for amalagamated free products, if φ is an isomorphism, we will abuse notation and say that $G = G_0 *_f$. In this situation, we implicitly identify elements of G with equivalence classes of normal forms in $G_0 *_f$.

As in Section 2.3.1, we want a 'ping-pong' condition ensuring that φ actually is an isomorphism.

Definition 2.3.9. Let $U_1, U_{-1} \subset M$ be nonempty disjoint sets, with $A = M \setminus (U_1 \cup U_{-1})$ nonempty. We call (A, U_1, U_{-1}) an *interactive triple* for G_0 and G_1 if the following hold:

1. The pair (U_1, U_{-1}) is precisely invariant under (J_1, J_{-1}) in G_0.

2. For $i \in \{\pm 1\}$, and for every $g \in G_0$, $gU_i \subset A \cup U_i$.

3. We have $f(A \cup U_1) \subset U_1$ and $f^{-1}(A \cup U_{-1}) \subset U_{-1}$.

We say an interactive triple is *proper* if the set $A \setminus (G_0(U_1 \cup U_{-1}))$ is nonempty.

Note that these conditions imply that in particular $gU_i \subset A$ for $g \in G_0 \setminus J_i$. Similarly to Section 2.3.1, we can observe:

Proposition 2.3.10. *If (A, U_1, U_{-1}) is a proper interactive triple for G_0 and G_1, then A, U, and U_{-1} are all infinite sets.*

Proof. Since J_1 is a proper subgroup G_0, there is some element $g \in G_0 \setminus J_1$, and by precise invariance we have $gU_1 \subset A$. By properness of the triple, the inclusion is proper, which means that fgU_1 is a proper subset of U_1. We conclude that U_1 is infinite. Since $gU_1 \subset A$ and $f^{-1}gU_1 \subset U_{-1}$ the other two sets are infinite as well. □

We have a description of the way normal forms in $G_0 *_f$ act on certain sets in the interactive triple, in analogy to the way normal forms in an amalgamated free product act on sets in an interactive pair.

Lemma 2.3.11 ([Mas88] VII.D.11)**.** *Let (A, U_1, U_{-1}) be an interactive triple for G_0 and G_1, and set $A_0 = A \setminus G_0(U_1 \cup U_2)$. Let $g = f^{\alpha_1} g_1 \cdots f^{\alpha_n} g_n \in G_0 *_f$ be a normal form with $|g| > 0$. Then the following hold.*

i) If g is an (i, j)-form for $i, j \in \{\pm 1\}$, then $\varphi(g)(A_0 \cup U_j) \subset U_i$.

ii) If g is a $(0, j)$-form for $j \in \{\pm 1\}$, then there is $h \in G_0$ so $\varphi(g)(A_0 \cup U_j) \subset hU \subset A$, where $U = U_{-1}$ if $\alpha_2 < 0$ and $U = U_1$ if $\alpha_2 > 0$.

The combinatorics in this case are slightly more complicated than for amalgamated free products, but the basic idea is the same. To illustrate the idea, consider a $(1, 1)$-form of length 2, for example $g = fg_1fg_2$. Then $g_2(A_0 \cup U_1) \subset A \cup U_1$ by definition (in fact A_0 is G_0-invariant by our conditions). Then we have

$$\begin{aligned} g(A_0 \cup U_1) &\subset fg_1f(A \cup U_1) \\ &\subset fg_1(U_1) \\ &\subset f(A \cup U_1) \\ &\subset U_1. \end{aligned}$$

Conditions (3) and (4) in Definition 2.3.8 ensure that when we iteratively apply a normal form to $A_0 \cup U_i$, we always can say where each set is mapped to next. We have chosen our notation so that if g is an (i, j)-form with $i \neq 0$, then $gU_j \subset U_i$. This is consistent with the convention for amalgamated free products.

The proposition below gives the combinatorial condition we need to ensure that φ is actually an isomorphism.

Proposition 2.3.12 (Ping-pong for HNN extensions; [Mas88] VII.D.12)**.** *Suppose (A, U_1, U_{-1}) is a proper interactive triple for G_0 and G_1. Then $G = G_0 *_f$.*

Proof. We just need to show that $\varphi : G_0 *_f \to G$ is injective. This map is already injective on G_0, so suppose $g \in G_0 *_f$ has $|g| > 0$. Then by Lemma 2.3.11 we have $\varphi(g)x \neq x$ for any $x \in A_0 = A \setminus G_0(U_1 \cup U_{-1})$, showing that $\varphi(g)$ is not the identity. $\square$

Normal forms in an HNN extension are slightly more complicated than normal forms for an amalgamated free product, so now we collect some results which will later make working with these normal forms a little easier.

Formal Inverses for Words in an HNN Extension

In several situations later in the thesis, we will want to work with formal inverses of (i, j)-forms. These inverses may not themselves be normal forms, as we will see in the proof, but it is still useful to work with them directly, rather than with an equivalent normal form. To that end, we prove:

Lemma 2.3.13. *Let $g = f^{\alpha_1} g_1 \cdots f^{\alpha_n} g_n$ be an (i, j)-form for $i, j \in \{\pm 1\}$ (so in particular, g has positive length). Then the formal inverse*

$$g^{-1} = g_n^{-1} f^{-\alpha_n} \cdots g_1^{-1} f^{-\alpha_1}$$

is a $(0,-i)$*-form if* $g_n \in G_0 \setminus (J_1 \cup J_{-1})$. *The word* $f^{-\alpha_n} \cdots g_1^{-1} f^{-\alpha_1}$ *is a* $(-j,-i)$*-form, regardless of* g_n.

Proof. We set $\beta_0 = 0$ and $\beta_k = -\alpha_{n+1-k}$ for $1 \le k < n$, so that g^{-1} is the word

$$f^{\beta_0} g_n^{-1} f^{\beta_1} g_{n-1}^{-1} \cdots f^{\beta_{n-1}}.$$

We need to verify that this word is a normal form. The only conditions in Definition 2.3.8 which could possibly fail are the technical requirements (3) and (4).

For (3), we must show that, for $k \ge 0$, if $\beta_{k+1} < 0$ and $g_{n-k}^{-1} \in J_{-1} \setminus \{\mathrm{id}\}$, then $\beta_k < 0$. Equivalently, we need to show that if $\alpha_{n+1-k} > 0$ and $g_{n-k} \in J_{-1}$, then $\alpha_{n-k} < 0$. When $k \ge 1$ this follows from condition (4) on our original normal form g, and when $k = 0$ the condition is vacuous because we assume $g_n \notin J_{-1}$. The argument for condition (4) is nearly identical.

The same reasoning implies that $f^{\beta_1} g_{n-1}^{-1} \cdots f^{\beta_{n-1}}$ is a normal form, with $\beta_1 = -\alpha_n$ and $\beta_{n-1} = -\alpha_1$. $\square$

Ping-pong for HNN Normal Forms

When we have an interactive triple (A, U_1, U_{-1}) for an HNN extension G, Lemma 2.3.11 above gives us a way to locate sets of the form gU_i when g is a normal form in G. However, the statement of the lemma is often a little unwieldy to work with directly, so to simplify some arguments later on, we introduce some additional terminology.

Definition 2.3.14. Let $G = G_0 *_f$ be the HNN extension of G_0 along $J_1 = f^{-1} J_{-1} f$. We say that a normal form

$$g = f^{\alpha_1} g_1 \cdots f^{\alpha_n} g_n$$

is an *HNN ping-pong form of type 1* (or just a *type-1 form*) if either $g_n \in G_0 \backslash J_1$, or $\alpha_n > 0$. Similarly a normal form is an *HNN ping-pong form of type −1* if either $\alpha_n < 0$ or $g_n \in G_0 \setminus J_{-1}$.

Note that if $|g| = 0$, then g has type i if and only if $g \in G_0 \setminus J_i$. An (i, j)-form is always type j, and it may or may not also be type $-j$. If (A, U_1, U_{-1}) is an interactive triple for G_0, $\langle f \rangle$, then a normal form g has type k when the dynamics of the triple allow us to locate the set gU_k. That is, we have the following immediate consequence of Lemma 2.3.11:

Lemma 2.3.15. *Let (A, U_1, U_{-1}) be an interactive triple for G_0 and $\langle f \rangle$. If g is an (i, j)-form of type k, and $i \neq 0$, then $gU_k \subset U_i$.*

Frequently we will want to apply inductive arguments to normal forms, which means that we want some control over the ping-pong behavior of a prefix of an (i, j)-form. The lemma below gives one way to do this. Here (and elsewhere), a 'prefix' h' of a normal form h is a normal form which appears as an initial subword of h. That is, if h is a normal form $f^{\alpha_1} g_1 \cdots f^{\alpha_n} g_n$, then a prefix h' is a normal form $f^{\alpha_1} g_1 \cdots f^{\alpha_k} g_k$ for some $1 \leq k \leq n$.

Lemma 2.3.16. *Let (A, U_1, U_{-1}) be an interactive triple for G_0 and $\langle f \rangle$, and let g be a type-i normal form of length $m \geq 1$. Then for some $j \in \{-1, 1\}$, there is a length-$(m-1)$ prefix g' of g and $g_0 \in G_0$ so that $g = g' f^j g_0$ and $f^j g_0 U_i \subset U_j$. If $|g'| \geq 1$, then g' is type j.*

Proof. When $m = 1$ we can just take $g' = \mathrm{id}$, so assume $m > 1$. We let $g = f^{\alpha_1} g_1 \cdots f^{\alpha_n} g_n$ be a type-i normal form. Without loss of generality assume $\alpha_n > 0$, and consider the normal form

$$g' = g g_n^{-1} f^{-1} = f^{\alpha_1} g_1 \cdots f^{\alpha_{n-1}} g_{n-1} f^{\alpha_n - 1}.$$

This is a normal form with positive length $m-1$. It is also type 1: if $\alpha_n > 1$ or $g_{n-1} \in G \setminus J_1$, then this follows directly from the definition; if $\alpha_n = 1$ and $g_{n-1} \in J_1$, then, since $f^{\alpha_1} g_1 \cdots f^{\alpha_{n-1}} g_{n-1} f^{\alpha_n} g_n$ is a normal form, we must have $\alpha_{n-1} > 0$, which again means the above form has type 1.

We need to verify that $f g_n U_i \subset U_1$, which will show the lemma holds with $g_0 = g_n$. If $i = 1$, then $f g_n U_i = f g_n U_1 \subset U_1$. On the other hand, if $i = -1$, then since $\alpha_n > 0$ and g has type i, we must have $g_n \in G \setminus J_{-1}$. Then $f g_n U_i = f g_n U_{-1} \subset U_1$. $\square$

We will also sometimes want to characterize elements in G via their action on sets in an interactive triple (A, U_1, U_{-1}). This can again be expressed using the ping-pong type of normal forms for these elements. The lemma below is a precise statement of this form, and generalizes the fact that in G_0, (U_1, U_{-1}) is precisely invariant under (J_1, J_{-1}):

Lemma 2.3.17. *Let (A, U_1, U_{-1}) be an interactive triple for G_0 and $\langle f \rangle$. Let g be a ping-pong form of type i and let h be a ping-pong form of type k.*

Suppose that $|g| = |h|$. Then either $i = k$, $gU_i = hU_i$, and $g = hj$ for $j \in J_i$, or $gU_i \cap hU_k = \varnothing$.

Proof. We proceed by induction on the length m of g and h; the main idea is to use the previous lemma to find prefixes of g and h where we can assume that the statement holds, and then apply precise invariance of (U_1, U_{-1}) under (J_1, J_{-1}) for the inductive step.

First observe that, if g, h are elements of G_0, and if $gU_i \cap hU_k \neq \varnothing$, then the fact that (U_1, U_{-1}) is precisely invariant under (J_1, J_{-1}) implies $i = k$ and $g = hj$ for $j \in J_i$. Now, let $m \geq 1$, let g, h be normal forms with $|g| = |h| = m$, and suppose that g has type i, h has type k, and $gU_i \cap hU_k \neq \varnothing$.

By Lemma 2.3.16 we can find prefixes g', h' of type i', k' respectively, with $|g'| = |h'| = m - 1$ and $g = g'f^{i'}g_0$, $h = h'f^{k'}h_0$ for $g_0, h_0 \in G_0$ satisfying $f^{i'}g_0U_i \subset U_{i'}$ and $f^{k'}h_0U_k \subset U_{k'}$. Then we know that both $gU_i \subset g'U_{i'}$ and $hU_k \subset h'U_{k'}$ hold, which means that $h'U_{k'} \cap g'U_{i'} \neq \varnothing$. By induction (or by precise invariance if $n = 1$), we know that $i' = k'$ and $h' = g'j'$ for $j' \in J_{i'}$. Without loss of generality take $i' = k' = 1$, so $j' \in J_1$.

Since $gU_i = g'fg_0U_i$ has nonempty intersection with $hU_k = h'fh_0U_k = g'j'fh_0U_k$, the intersection $fg_0U_i \cap j'fh_0U_k$ is also nonempty. Since f conjugates J_1 to J_{-1}, for some $j'' \in J_{-1}$ we have $j'fh_0 = fj''h_0$. Then we have that $fg_0U_i \cap fj''h_0U_k$ is nonempty as well, hence $g_0U_i \cap j''h_0U_k \neq \varnothing$. Then by precise invariance we know $i = k$ and $g_0 = j''h_0j$ for $j \in J_i$.

Finally, we see that

$$g = g'fg_0 = g'fj''h_0j = g'j'fh_0j = h'fh_0j = hj,$$

and we are done. $\square$

Chapter 3

Combination Theorems

In this chapter, we first discuss the classical combination theorems of Klein and Maskit in Section 3.1, which were the inspiration for our combination theorems for discrete convergence groups. We then have one section devoted to each of the new theorems, Section 3.2 for amalgamated free products and Section 3.3 for HNN extensions. The new theorems in this chapter constitute joint work with Theodore Weisman.

3.1 Classical Combination Theorems

In this section, we state versions of the classical Klein and Maskit Combination Theorems. These theorems start with two Kleinian groups, and give sufficient dynamical conditions ensuring we can describe the group they generate in terms of the starting groups. The latter two theorems also guarantee geometrical finiteness is preserved when we combine two groups. The first of the combination theorems is the one by Klein, which is closely related to the *ping-pong lemma* and does not refer to geometrical finiteness.

Theorem 3.1.1 ([Mas88] VII.A.13, Klein Combination Theorem)**.** *Let G_1, G_2 be Kleinian groups. Suppose we have two disjoint nonempty open sets $B_1, B_2 \subset \widehat{\mathbf{C}}$ such that $B_1 \cup B_2 = \widehat{\mathbf{C}}, B_1 \cap B_2 \neq \varnothing$, and*

$$gB_i \cap B_i = \varnothing,$$

for every $g \in G \setminus \{1\}$. Set $G = \langle G_1, G_2 \rangle$, the subgroup of $\mathrm{PSL}(2, \mathbf{C})$ generated by G_1 and G_2. Then $G = G_1 * G_2$, and G is Kleinian.

The proof is nearly identical to several arguments we give later.

Example 3.1.2. Let $g_1, g_2 \in \mathrm{PSL}(2, \mathbf{C})$ be loxodromic elements with disjoint fixed point sets. Let $G_i = \langle g_i \rangle$. Then, by replacing g_i with a sufficiently high power if necessary, we can assume there are four disjoint open discs $B_{i,\pm}$ around each of the fixed points $z_{i,\pm}$ of g_1 and g_2 (here the $+$ indicates the attracting point, while the $-$ indicates the repelling point), so that g_i maps the outside of $B_{i,-}$ into $B_{i,+}$. If we then let $B_i = \widehat{\mathbf{C}} \setminus (B_{i,+} \cup B_{i,-})$, we satisfy all the conditions of Theorem 3.1.1, and so $\langle G_1, G_2 \rangle = G_1 * G_2 \cong F_2$, the free group on two letters.

This example allows one to prove that Kleinian groups satisfy the *Tits alternative*, that is, any subgroup of the Kleinian G is either virtually solvable (contains a finite index solvable subgroup) or contains a non-abelian free group. Indeed, the only virtually solvable Kleinian groups are elementary, that is their limit sets have cardinality 1 or 2, and any non-elementary Kleinian group contains at least two loxodromic elements with distinct fixed point sets.

We move onto the first of Maskit's combination theorems now, which allows for more interesting combinations than just free products, as well as interacting nicely with geometrical finiteness.

Theorem 3.1.3 ([Mas88] VII.C.2, Maskit's First Combination Theorem)**.** *Let G_1, G_2 be Kleinian groups, with $G_1 \cap G_2 = J$, where $J \neq G_i$ and J is geometrically finite. Suppose $W \subset \widehat{\mathbf{C}}$ is a topological circle dividing $\widehat{\mathbf{C}}$ into two closed J-invariant discs, B_1 and B_2. Suppose*

$$gB_i \subset \mathrm{Int}(B_{3-i}),$$

for every $g \in G_i \setminus J$, and

$$\Lambda(G_i) \setminus \Lambda(J) \subset \mathrm{Int}(B_{3-i}).$$

Set $G = \langle G_1, G_2 \rangle$. Then:

(i) $G = G_1 *_J G_2$.

(ii) *G is discrete.*

(iii) *Elements of G not conjugate into G_1 nor G_2 are loxodromic.*

(iv) *G is geometrically finite if and only if both G_1 and G_2 are geometrically finite.*

Remark 3.1.4. We have stated a version of Maskit's original theorem which is slightly weaker, but which suffices for every example Maskit presents in his book [Mas88]. Specifically, Maskit allows the translates of B_i by $G_i \setminus J$ to intersect ∂B_i. Maskit also has several conclusions about how fundamental domains for G_1 and G_2 can be used to build a fundamental domain for G. The slightly weaker version stated here will be directly implied by the new results later in this chapter.

We will illustrate this theorem with a couple of examples.

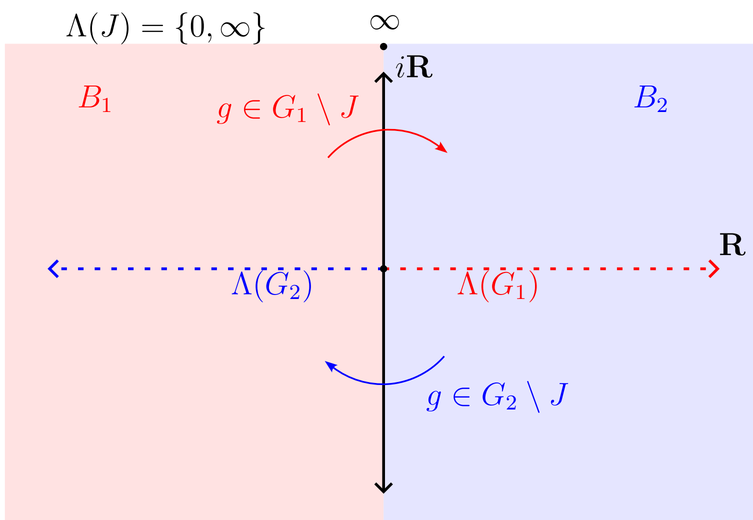

Figure 3.1.1: Illustration for the example. The limit sets $\Lambda(G_i)$ are Cantor sets.

Example 3.1.5. Let G be a Fuchsian genus 2 surface group, embedded in$\mathrm{PSL}_2(\mathbf{C}) \cong \mathrm{Isom}(\mathbb{H}^3)$ via the inclusion $\mathrm{PSL}_2(\mathbf{R}) < \mathrm{PSL}_2(\mathbf{C})$.

Then G has the presentation $\langle a, b, c, d \mid [a, b][c, d] = 1\rangle$. Set $j = [a, b] = [c, d]^{-1}$. We then take $G_1 = \langle a, b\rangle \cong F_2$, and $G_2 = \langle c, d\rangle \cong F_2$, the free group on 2 letters, and $J = G_1 \cap G_2 = \langle j\rangle \cong \mathbb{Z}$. We can arrange our generators so $\Lambda(G_1) \subset \mathbf{R}_{\geq 0} \cup \{\infty\}$ and $\Lambda(G_2) \subset \mathbf{R}_{\leq 0} \cup \{\infty\}$, where $\Lambda(J) = \{0, \infty\}$. These will be the fixed points of $j = (x \mapsto \lambda x)$ where $\lambda > 1$. See Figure 3.1.1.

This is precisely the picture one gets when gluing the sides of an octagon in $\mathbb{H}^2_{\mathbf{R}}$ to form a surface of genus 2, and then isometrically embedding this picture into the standard $\mathbb{H}^2_{\mathbf{R}}$ sitting inside $\mathbb{H}^3_{\mathbf{R}}$ whose boundary is $\mathbf{R} \cup \{\infty\}$. The limit set of the surface group coincides with $\partial\mathbb{H}^2_{\mathbf{R}}$, and then the octagon with faces identified appears once in each connected component of $\partial\mathbb{H}^3_{\mathbf{R}} \setminus \partial\mathbb{H}^2_{\mathbf{R}}$. Our J-invariant sets B_1 and B_2 are then the closed left and right closed half-planes respectively, including ∞, and their common intersection is $i\mathbf{R} \cup \infty$. If we identify $\widehat{\mathbf{C}}$ with S^2, then B_1 and B_2 are complementary hemispheres.

For the other examples in this section, we will need two brief definitions. A *fundamental set* for the Kleinian G will be a set containing one point from every G-orbit in $\widehat{\mathbf{C}}$. No assumptions are made about the toplogy of a fundamental set. One can obtain a fundamental set from a fundamental domain by adding some boundary points. A *constrained* fundamental set is a fundamental set whose interior is a fundamental domain.

Example 3.1.6 ([Mas88] VIII.C.3)**.** Let G_1 be a finitely generated Fuchsian group, that is, a subgroup of $\mathrm{PSL}(2, \mathbf{R}) \subset \mathrm{PSL}(2, \mathbf{C})$, which contains an elliptic element of order ν. Normalize so that $J = \mathrm{Stab}(0) = \mathrm{Stab}(\infty)$ has order ν. For $\delta > 0$ sufficiently small, we know that $B_1 = \{|z| \geq \delta\}$ is preserved by J and otherwise mapped off of it self, using that $0 \in \Omega(G)$ which is open. Hence B_1 is precisely invariant under J in G_1. Let D_1 be a constrained fundamental set for G_1 contained inside the 'natural' constrained fundamental set for J: $E = \{z \mid 0 \leq \arg z \leq \nu\}$. We may also choose D_1 so that $D_1 \cap B_1$ is a fundamental set for the action of J on B_1.

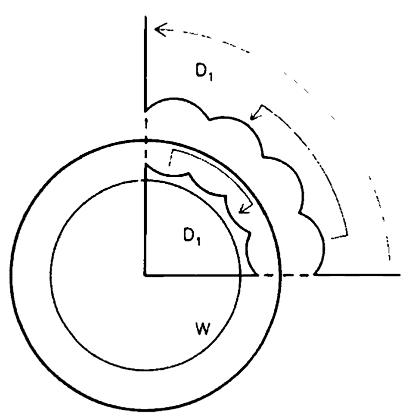

Figure 3.1.2: [Mas88] Fig. VIII.C.2. Illustration depicting D_1. The circle W is the boundary of B_1.

Let G_2 be another finitely generated Fuchsian group with a maximal elliptic subgroup of order ν fixing 0 and ∞. This implies $G_1 \cap G_2 = J$. Like above, we can find precisely invariant neighborhoods of either fixed point. If we conjugate by a dilation of the form $z \mapsto tz, 0 < t < 1$ (which preserves J), we can assume $B_2 = \{|z| \geq \delta\} \cup \{\infty\}$ is precisely invariant under J in G_2. Let D_2 be a constrained fundamental set for G_2 again contained in E, and choose D_2 so that $D_2 \cap B_2$ is a fundamental set for J acting on B_2. We claim Theorem 3.1.3 holds here. Indeed, we can take $W = B_1 \cap B_2 = \partial B_1$, and then W is J-invariant, and the precise invariance of B_1 and B_2 ensures $G_i \setminus J$ translates of B_i are contained in $\mathrm{Int}(B_{3-i})$.

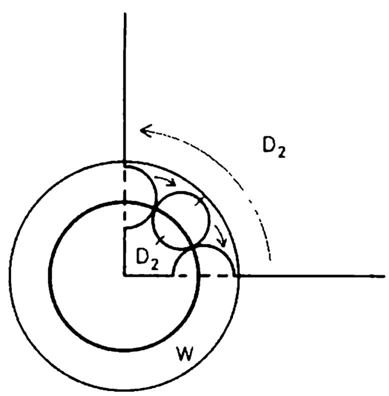

Figure 3.1.3: [Mas88] Fig. VIII.C.3. Illustration depicting D_2.

So, setting $G = \langle G_1, G_2 \rangle$, we conclude $G = G_1 *_J G_2$. Since finitely generated Fuchsian groups are geometrically finite, we find that G is also geometrically finite.

The second combination theorem deals with HNN extensions instead.

Theorem 3.1.7 ([Mas88] VII.E.5, Maskit's Second Combination Theorem)**.** *Let G_0 be a Kleinian group, with $J_1, J_{-1} < G_0$ proper subgroups which are* *both geometrically finite. Let $G_1 = \langle f \rangle$ be infinite cyclic, where $fJ_{-1}f^{-1} = J_1$. Suppose we can find disjoint closed discs $B_1, B_{-1} \subset \widehat{\mathbf{C}}$ satisfying the following:*

1. *(B_1, B_{-1}) is precisely invariant under (J_1, J_{-1}) in G_0.*

2. *If $A = \widehat{\mathbf{C}} \setminus (B_1 \cup B_{-1})$, then $f(A \cup B_1) = \mathrm{Int}(B_1)$.*

3. *For $i \in \{\pm 1\}$, $\Lambda(G_0) \cap B_i = \Lambda(J_i)$.*

4. *The set $A_0 = \widehat{\mathbf{C}} \setminus G_0(B_1 \cup B_{-1})$ is nonempty.*

Set $G = \langle G_0, G_1 \rangle$. Then the following hold.

(i) $G = G_0 *_f$.

(ii) *G is Kleinian.*

(iii) *Elements of G not conjugate into G_0 are loxodromic.*

(iv) *G is geometrically finite if and only if G_0 is geometrically finite.*

Remark 3.1.8. Like for the previous theorem, this version is slightly weaker again. Maskit allows the two discs to intersect along their boundary, so that the conjugating element could be parabolic, and Maskit also has conclusions about fundamental domains for the combination. The version we have stated will be directly implied by the new results later in this chapter.

Example 3.1.9. To illustrate the second combination theorem, we will work with very explicit groups. Let G_0 be the double dihedral group generated by $jz = -z, gz = 16/z$, and $hz = 400/z$. One possible fundamental domain for G_0 is $D_0 = \{z \mid 4 < |z| < 20, -\pi/2 < \arg z < \pi/2\}$, and by adding some points to the boundary we can assume D_0 is a constrained fundamental set. Let $B_1 = \{z \mid |z - 25| \leq 15\}$ and $B_{-1} = \{z \mid |z - 5| \leq 3\}$, and let

$fz = (25z - 80)/(z - 5)$. Then f maps the outside of B_{-1} onto the inside of B_1.

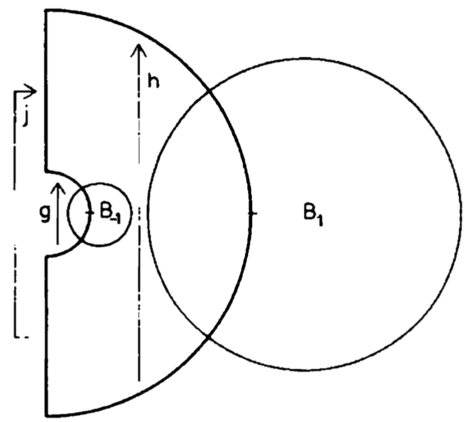

Figure 3.1.4: [Mas88] Fig. VIII.C.7. Illustration depicting D, B_1 and B_{-1}.

Letting $J_1 = \langle h \rangle$ and $J_{-1} = \langle g \rangle$, we have $fJ_{-1}f^{-1} = J_1$, and (B_1, B_{-1}) are precisely invariant under (J_1, J_{-1}) in G_0. The second combination theorem applies in this case, and so setting $G = \langle G_0, f \rangle$, we find that $G = G*_f$, and G is geometrically finite since G_0, J_1 and J_{-1} are.

3.2 New Combination Theorems: Amalgamated Free Products

We now introduce the main definition for Theorem A.

Definition A (AFP ping-pong position)**.** Let G_1 and G_2 act as discrete convergence groups on a compact metrizable space M, and suppose that $G_1 \cap G_2 = J$ is a geometrically finite group distinct from both G_1 and G_2. We say G_1 and G_2 are in *AFP ping-pong position* (with respect to J) if there exist closed sets $B_1, B_2 \subset M$ with nonempty disjoint interiors satisfying the following:

1. For $i \in \{1, 2\}$, B_i is J-invariant.

2. For $i \in \{1, 2\}$, and for each $g \in G_i \setminus J$, $gB_i \subset \mathrm{Int}(B_{3-i})$.

3. For $i \in \{1, 2\}$, $\Lambda(G_i) \setminus \Lambda(J) \subset \mathrm{Int}(B_{3-i})$.

The definition above for the most part mimics the setup in Maskit's original combination theorem for amalgamated free products of Kleinian groups. Referring to Figure 3.1.1 can be helpful. We can now state and begin proving Theorem A.

Theorem A. *Let G_1 and G_2 be discrete convergence groups acting on a compact metrizable space M. Suppose that $J = G_1 \cap G_2$ is geometrically finite, and G_1 and G_2 are in AFP ping-pong position with respect to J. Let $G = \langle G_1, G_2 \rangle < \mathrm{Homeo}(M)$, and suppose G acts as a convergence group. Then the following hold:*

(i) $G = G_1 *_J G_2$.

(ii) *G is discrete.*

(iii) *Elements of G not conjugate into G_1 nor G_2 are loxodromic.*

(iv) *G is geometrically finite if and only if both G_1 and G_2 are geometrically finite.*

Remark 3.2.1. The hypotheses for Theorem A are different from the hypotheses for Maskit's original combination theorems in $\mathbb{H}^3_{\mathbf{R}}$ in two respects. First, Maskit insists that the sets B_1, B_2 in Definition A are topological balls in

$M = \partial \mathbb{H}^3_{\mathbf{R}}$, satisfying $\partial B_1 = \partial B_2$ and $B_1 \cup B_2 = M$. This requirement is unnatural in our setting, since M may not even be a manifold, and it is not needed in any of our arguments.

Second, and more significantly, Maskit's version of condition (2) in Definition A is weaker than what we have given here. Our condition implies in particular that if $g \in G_i \setminus J$, then $gB_i \cap B_i = \varnothing$. This means that if $P < J$ is a maximal parabolic subgroup in J, then P must also be a maximal parabolic subgroup in G_i.

Maskit's original statement in $\mathbb{H}^3_{\mathbf{R}}$ allows gB_i to intersect B_i in limit points of J, which means his theorem allows for amalgamations along subgroups $J < G_i$ whose parabolic subgroups are *not* maximal in G_i. This means our theorem is not strong enough to recover Maskit's original result in the case $M = \partial \mathbb{H}^3_{\mathbf{R}}$. However, most of the examples constructed in Maskit's book satisfy the stronger hypothesis we have given above.

Below, we give a quick proof of the first three parts of Theorem A.

Proof of (i) - (iii) in Theorem A. (i) Let B_1, B_2 be the closed subsets of M from Definition A. We note that since $gB_i \subset \mathrm{Int}(B_{3-i})$ for any $g \in G_i \setminus J$, it follows that $g\mathrm{Int}(B_i) \subset \mathrm{Int}(B_i)$ is a proper inclusion for every $g \in G_i \setminus J$. Hence $(\mathrm{Int}(B_1), \mathrm{Int}(B_2))$ form a proper interactive pair for G_1 and G_2 by conditions (1) and (2), so we are done by Proposition 2.3.6.

(ii) It suffices to show no sequence in G accumulates at the identity. Let (g_k) be a sequence of distinct elements in G. Since G_1, G_2 are discrete we can assume $|g_k| > 1$. If the length of g_k is odd, then g_k maps one of the sets B_1, B_2 into the interior of the other and hence is far from the identity, so assume the lengths are all even. Without loss of generality, we may assume every g_k is a $(2, 1)$-form. We have $g_k B_1 \subset \mathrm{Int}(B_1)$ for every k.

Suppose for a contradiction that (g_k) converges to the identity. Then $g_k B_1$ converges to B_1. Write $g_k = h_k g'_k$ where $|g'_k| = |g_k| - 1$ and $h_k \in G_2 \setminus J$. Then $g_k B_1 \subset h_k B_2 \subset \mathrm{Int}(B_1)$ for every k since $g'_k B_1 \subset B_2$. It now also follows that $(h_k B_2)$ converges to B_1. Now, in general, when $g, h \in G_2 \setminus J$, we will have gB_2 and hB_2 either disjoint or equal. Indeed, if $gB_2 \cap hB_2 \neq \varnothing$, then $h^{-1}g$ sends a point in B_2 back into B_2, hence $h^{-1}g = j \in J$. Then $gB_2 = hjB_2 = hB_2$ as desired.

Since $h_1 B_2 \subset \mathrm{Int}(B_1)$ has nonempty interior and $h_k B_2 \subset \mathrm{Int}(B_1)$ converges to B_1, it follows that for some fixed large k, we will have $h_k B_2 \cap h_1 B_2 \neq \varnothing$, and also $h_k B_2 \neq h_1 B_2$. This gives our contradiction, so we conclude G is discrete.

(iii) Assume $g \in G$ is not conjugate into G_1 nor G_2. Take g to have minimal length in its conjugacy class. If g is an (i,i)-form (that is, $|g|$ is odd) then we can conjugate by an element of G_i to reduce its length, hence g has even length. Without loss of generality suppose g is a $(2,1)$-form. Since $g^n B_1 \subset \mathrm{Int}(B_1)$ is a proper inclusion for every n, we see that g has infinite order, hence is parabolic or loxodromic since G is discrete. At least one fixed point of g is an attracting point z_+ for the convergence sequence (g^n) (see Proposition 2.2.7). Since B_1 has nonempty interior, there is some $w \in B_1$ so that $g^n w \to z_+$. But for every $n \geq 1$, the set $g^n B_1$ is a subset of the fixed compact $gB_1 \subset \mathrm{Int}(B_1)$, so we must have $z_+ \in \mathrm{Int}(B_1)$. An identical argument applied to g^{-1} (a $(1,2)$-form) gives a fixed point for g in $\mathrm{Int}(B_2)$, hence g is loxodromic by Proposition 2.2.7. $\square$

3.2.1 Limit Sets of Amalgamated Free Products

The rest of the section is devoted to the proof of part (iv) of Theorem A, so for the rest of the section, we fix groups G_1, G_2, J, G and sets $B_1, B_2 \subset M$ satisfying the conditions of Definition A. We will prove each direction of the theorem separately, but we start by making some general observations about the positioning of the limit sets of subgroups of G.

Proposition 3.2.2. *Each of the following holds.*

(i) $\Lambda(J) \subset \partial B_1 \cap \partial B_2$. *In particular, if J is infinite, then $\partial B_1 \cap \partial B_2$ is nonempty.*

(ii) *For $i \in \{1, 2\}$, $\Lambda(G_i) \subset B_{3-i}$.*

(iii) *For $i \in \{1, 2\}$, and any $g \in G \setminus G_i$, we have $g(\Lambda(G_i)) \cap \Lambda(G_i) = \varnothing$.*

Proof. (i) Note that since J preserves the closed set B_1 and $\mathrm{Int}(B_1)$ is an infinite set by Proposition 2.3.4, we have $\Lambda(J) \subset B_1$. Similarly, $\Lambda(J) \subset B_2$, hence $\Lambda(J) \subset B_1 \cap B_2 = \partial B_1 \cap \partial B_2$ since these sets have disjoint interiors.

(ii) This is an immediate consequence of condition (3) in Definition A along with (i) above.

(iii) For concreteness, take $i = 1$, and let $g \in G \setminus G_1$. In particular $g \notin J$, so g has a normal form with positive length. We can always find some $h, h' \in G_1$ so that $g' = hgh'$ is a $(1, 2)$-form. Then, applying (ii), we know that $g'\Lambda(G_1) \subset \mathrm{Int}(B_1)$ and so

$$g'\Lambda(G_1) \cap \Lambda(G_1) = \varnothing.$$

Now, since $\Lambda(G_1)$ is invariant under G_1, we see that $g'\Lambda(G_1) = hg\Lambda(G_1)$, and therefore

$$hg\Lambda(G_1) \cap \Lambda(G_1) = \varnothing.$$

But then $h^{-1}(hg\Lambda(G_1) \cap \Lambda(G_1)) = g\Lambda(G_1) \cap h^{-1}\Lambda(G_1) = g\Lambda(G_1) \cap \Lambda(G_1)$ is empty as well. $\square$

3.2.2 AFP Ping-Pong and Contraction

Both directions of the proof of Theorem A rely crucially on a key contraction property of the ping-pong action of G on the sets B_1 and B_2, stated as Lemma 3.2.5 below. This contraction lemma gives a sufficient condition for a sequence of sets $(g_k B_i)$ to converge to a singleton in M.

The proof of the contraction lemma relies on an application of Proposition 2.2.25 to the subgroups we are currently considering. Recall that this proposition gives us control over the topological behavior of the action of fully quasi-convex subgroups on certain subsets of M. So, in order to apply the proposition, we first need to check:

Lemma 3.2.3. *Let H be one of G, G_1, or G_2. If H is geometrically finite, then J is a fully quasi-convex subgroup of H.*

Proof. We know J is relatively quasi-convex since it is a geometrically finite subgroup of M, so we just need to prove that for all but finitely many $h \in H \setminus J$ we have $h\Lambda(J) \cap \Lambda(J) = \varnothing$. In fact, we will see that this is true for *all* $h \in H \setminus J$.

First, if $H = G_i$ for $i = 1$ or 2, by assumption we know that for any $h \in H \setminus J$ we have $h\Lambda(J) \subset hB_i \subset \mathrm{Int}(B_{3-i})$, hence $h\Lambda(J) \cap \Lambda(J) = \varnothing$ by part (i) of Proposition 3.2.2. If $H = G$, then any $h \in H \setminus J$ is an (i, j)-form, so that $h\Lambda(J) \subset hB_j \subset \mathrm{Int}(B_{3-i})$ and again $h\Lambda(J) \cap \Lambda(J) = \varnothing$. $\square$

Now, we can specialize Proposition 2.2.25 to the current setting.

Lemma 3.2.4. *Suppose that either G is geometrically finite, or both G_1 and G_2 are geometrically finite. For $i \in \{1, 2\}$, there exists a compact $K_i \subset \mathrm{Int}(B_{3-i})$ so that for any $g \in G_i \setminus J$, there is $j \in J$ so that $jgB_i \subset K_i$.*

Proof. This follows directly from Proposition 2.2.25, taking the ambient geometrically finite group G to be either G or G_i for $i \in \{1, 2\}$, H to be G_i,

$J_1 = J_2 = J$, U_1 to be $\mathrm{Int}(B_{3-i})$, and U_2 to be $M \setminus B_i$. By assumption we know that $\Lambda(G_i) \setminus \Lambda(J) \subset \mathrm{Int}(B_{3-i}) \subset M \setminus B_i$, so in fact $\Lambda(G_i) \setminus \Lambda(J) \subset U_1 \cap U_2$ and the hypotheses of the proposition are satisfied. $\square$

Finally, we can establish the contraction property for sequences in G.

Lemma 3.2.5 (Contraction for amalgamated free products). *Suppose that either G is geometrically finite, or both G_1 and G_2 are geometrically finite. If (h_k) is a sequence of (i, j)-forms (for fixed i and j) lying in distinct left cosets of J, then, up to subsequence, $(h_k B_j)$ converges to a singleton $\{x\}$.*

It is not hard to verify directly that the subgroup $\{g \in G : gB_j = B_j\}$ is exactly J. So, asking for the sequence of cosets $(h_k J)$ to be pairwise distinct is equivalent to asking for the sequence of translates $(h_k B_j)$ to be pairwise distinct.

Proof. We first prove the following:

Claim. *There exists a compact subset $K \subset \mathrm{Int}(B_{3-j})$ and a sequence (j_k) in J such that $j_k h_k^{-1} B_i \subset K$ for all k.*

To prove the claim, first observe that if $|h_k| = 1$ for every k, then $i = j$ and $h_k \in G_i \setminus J$ for all k. Then the claim follows directly from Lemma 3.2.4. Otherwise, suppose that $|h_k| > 1$, and write a normal form for h_k:

$$h_k = g_{k,1} \cdots g_{k,n}.$$

Although n can depend on k, we ignore this in the notation. The word $h_k^{-1} = g_{k,n}^{-1} \cdots g_{k,1}^{-1}$ is a (j, i)-form, and the word

$$g_{k,n} h_k^{-1} = g_{k,n-1}^{-1} \cdots g_{k,1}^{-1}$$

is a $(3-j,i)$-form. This means that $g_{k,n}h_k^{-1}B_i \subset B_j$. Then, we can apply Lemma 3.2.4 again to find a fixed compact $K \subset \mathrm{Int}(B_{3-j})$ and $j_k \in J$ so that $j_k g_{k,n}^{-1} B_j \subset K$ for every k, and therefore

$$j_k h_k^{-1} B_i = j_k g_{k,n}^{-1} g_{k,n} h_k^{-1} B_i \subset j_k g_{k,n}^{-1} B_j \subset K.$$

This proves the claim, so now we consider the sequence $(h_k j_k^{-1})$. Since the left cosets $h_k J$ are all distinct, it follows that the sequence of group elements $(h_k j_k^{-1})$ is divergent in G, and therefore we can extract a convergence subsequence: we can find attracting and repelling points $z_+, z_- \in M$ so that $(h_k j_k^{-1} y)$ converges to z_+ whenever $y \neq z_-$. Equivalently, $(j_k h_k^{-1} y)$ converges to z_- whenever $y \neq z_+$.

By Proposition 2.3.4, the set B_j is infinite, so there is at least one point $y \in B_j \setminus \{z_+\}$. Since $j_k h_k^{-1} B_j \subset K$, we must have $z_- \in K$. In particular, z_- must lie in $\mathrm{Int}(B_{3-j})$, which means that B_j is a compact subset of $M \setminus \{z_-\}$. Thus, $(h_k j_k^{-1} B_j) = (h_k B_j)$ converges to the singleton $\{z_+\}$ as desired. □

3.2.3 Geometrical Finiteness of the Product

We now turn to the proof of the implication (G_1 and G_2 geometrically finite) $\implies$ (G geometrically finite), which is one of the directions of Theorem A (iv).

The proof of this direction of the theorem relies on the fact that limit points of G fall into one of two classes: either they are G-translates of limit points of G_1 or G_2, or else they are limit points of sequences of (i,j)-forms in G whose length tends to infinity. The essential step in the proof is to show that any limit point x of the latter form can be "coded" by a sequence of nested translates of B_1 or B_2.

Precisely, we prove the following:

Proposition 3.2.6 (AFP coding for G-limit points)**.** *Suppose that G_1 and G_2 are geometrically finite, and let x be a point in $\Lambda(G) \setminus G(\Lambda(G_1) \cup \Lambda(G_2))$. Then there exists a sequence (g_k) in $(G_1 \cup G_2) \setminus J$ so that for every k,*

$$h_k = g_1 \cdots g_k$$

has length k, and if $g_k \in G_j$, then $x \in h_k B_j$.

To prove this proposition, we follow Maskit's strategy, and consider a sequence of "ping-pong" sets in M, defined inductively as follows. We let $T_0 = B_1 \cup B_2$. Then, for every $n > 0$, and $i \in \{1, 2\}$, we define

$$T_{n,i} = \bigcup_{g \in G_i \setminus J} g(B_i \cap T_{n-1}).$$

Then we define

$$T_n = T_{n,1} \cup T_{n,2}.$$

The set T_1 is just the union of the $G_1 \setminus J$ translates of B_1 and the $G_2 \setminus J$ translates of B_2. More generally, T_n is the union of translates of B_1 by $(i, 1)$-forms of length n and the translates of B_2 by $(i, 2)$-forms of length n. See Figure 3.2.1 for a depiction of T_1 and T_2. We see that these sets are decreasing, so let

$$T = \bigcap_{n=0}^{\infty} T_n.$$

We will see that limit points of G which are not translates of limit points of G_1 nor G_2 are in T, which allows us to construct the sequence given by the conclusion of the proposition above.

We observe:

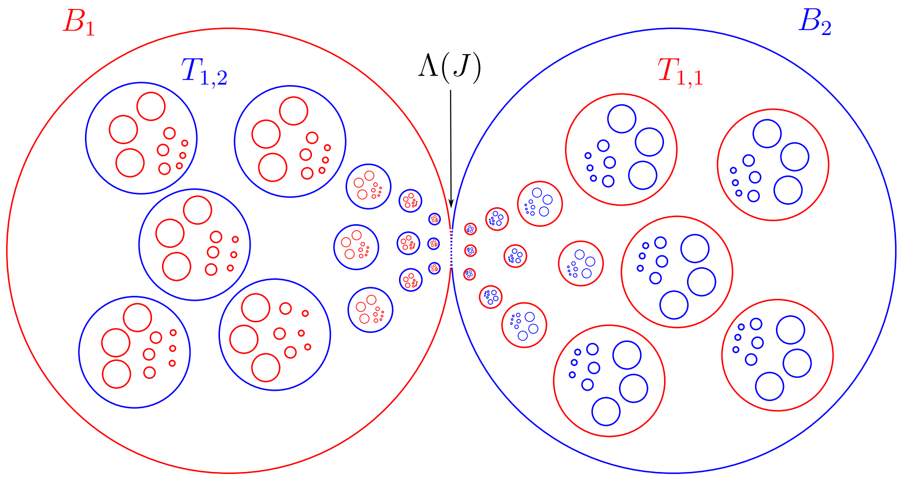

Figure 3.2.1: Part of the sets T_1 and T_2.

Lemma 3.2.7. *The set T is G-invariant and nonempty. In particular, since G is non-elementary, we have $\Lambda(G) \subset \overline{T} \subset B_1 \cup B_2$.*

Proof. We know T is nonempty because it is the intersection of a decreasing sequence of nonempty subsets of the compact space M. The definition of T_n implies that if $x \in T_n$ and $g \in G_1 \cup G_2$, then $gx \in T_{n-1}$. Inductively, we see that if $g \in G$ has $|g| = k$, and $x \in T_{n+k}$, then $gx \in T_n$. It follows that if $x \in T$ then $gx \in T$ for any $g \in G$. □

Lemma 3.2.8. *If G_1 and G_2 are geometrically finite, we have $\Lambda(G) \setminus (\Lambda(G_1) \cup \Lambda(G_2)) \subset T_1$.*

Proof. We will prove that if $y \in \Lambda(G) \setminus T_1$, then $y \in \Lambda(G_1) \cup \Lambda(G_2)$, so suppose that $y \in \Lambda(G)$ does not lie in T_1. Using the above lemma, we know $y \in B_1 \cup B_2$, so without loss of generality assume $y \in B_2$. Since y is in the limit set of G, we can find a sequence (g_k) in G so that $(g_k w)$ converges to y for all but a single point in M. If $y \in \Lambda(J)$ we are done, so we can assume that $g_k \notin J$ for infinitely many k.

Then, after extracting a subsequence, we can assume that for every k, g_k is an (i,j)-form for i,j fixed, and then find $w \in B_j$ so that $g_k w \to y$.

Since g_k is an (i,j)-form and $w \in B_j$, we have $g_k w \in G_i(B_i)$ for every k. So, we may write $g_k w = g'_k z_k$ for $g'_k \in G_i \setminus J$ and $z_k \in B_i$. Note that g'_k is just the first letter in the (i,j)-form g_k. In particular, we know $g'_k z_k \in T_1$ for every k, so $g'_k z_k$ is never equal to y. If, up to subsequence, there are only finitely many distinct translates $g'_k B_i$, then we would have $g'_k z_k \in \bigcup g'_k B_i$, a compact set in the complement of T_1, which contradicts the fact that $g'_k z_k \to y \in T_1$. Hence we may assume that the translates $g'_k B_i$ are all distinct, which means that the left cosets $g'_k J$ are all distinct.

Now, Lemma 3.2.5 implies that $(g'_k B_i)$ converges to a singleton. This singleton must be y since $g'_k z_k \to y$. It follows that $g'_k z \to y$ for any $z \in B_i$, and since B_i is an infinite set it follows that $y \in \Lambda(G_i)$ as desired. $\square$

Proof of Proposition 3.2.6. We first claim that $\Lambda(G) \setminus G(\Lambda(G_1) \cup \Lambda(G_2))$ is a subset of T. So, fix $z \in \Lambda(G)$, and suppose $z \notin T$. We will show $z \in G(\Lambda(G_1) \cup \Lambda(G_2))$.

By Lemma 3.2.7 we know that $\Lambda(G) \subset B_1 \cup B_2 = T_0$, so there is some $n > 0$ such that $z \in T_{n-1} \setminus T_n$. In particular, because $z \in T_{n-1}$, there is an (i,j)-form $g \in G$, with $|g| = n-1$, such that $gy = z$ for $y \in B_j$. We must have $y \notin T_1$, since otherwise we would have $y = hw$ for $w \in B_{3-j}$ and $h \in G_{3-j} \setminus J$, and then $z = ghw$ would lie in T_n. Then, since $\Lambda(G)$ is G-invariant we see that $y \in \Lambda(G)$ but $y \notin T_1$, so by the previous lemma we have $y \in \Lambda(G_1) \cup \Lambda(G_2)$, hence $z \in G(\Lambda(G_1) \cup \Lambda(G_2))$.

We have now seen that $\Lambda(G) \setminus G(\Lambda(G_1) \cup \Lambda(G_2))$ is a subset of T, so we just need to show that for any $x \in T$, there is a sequence of (i, j)-forms (h_k) satisfying the conclusions of the proposition. We construct this sequence inductively. Take h_0 to be the identity. For $k > 0$, assume that $x \in h_{k-1}B_j$ for an (i, j)-form

$$h_{k-1} = g_1 \cdots g_{k-1}.$$

By Lemma 3.2.7, T is G-invariant, so $h_{k-1}^{-1}x \in B_j \cap T$. In particular, $h_{k-1}^{-1}x$ lies in $T_1 \cap B_j = T_{1,j}$, so there is some $g_k \in G_{3-j} \setminus J$ so that $h_{k-1}^{-1}x \in g_k B_{3-j}$. Then if h_k is the $(i, 3-j)$-form

$$g_1 \cdots g_k,$$

we have $x \in h_k(B_{3-j})$ and $|h_k| = |h_{k-1}| + 1$, as required. □

The next step is to use the "coding" of limit points given by Proposition 3.2.6 to prove that there is a conical limit sequence for every point in $\Lambda(G) \setminus G(\Lambda(G_1) \cup \Lambda(G_2))$.

Lemma 3.2.9. *If G_1 and G_2 are geometrically finite, every point in $\Lambda(G) \setminus G(\Lambda(G_1) \cup \Lambda(G_2))$ is a conical limit point for G.*

Proof. Let $x \in \Lambda(G) \setminus G(\Lambda(G_1) \cup \Lambda(G_2))$. We know $x \in B_1 \cup B_2$ from Lemma 3.2.7, so to simplify notation assume $x \in B_2$. We let (g_k) be the sequence in $(G_1 \cup G_2) \setminus J$ from Proposition 3.2.6, so that, for every k, we have $|g_1 \cdots g_k| = k$ and if $g_k \in G_j$, then $x \in g_1 \cdots g_k B_j$.

For each k, we let $h_k = g_1 \cdots g_{2k}$, so that h_k is an (i, j)-form for fixed $i \neq j$. Since $h_k B_j \subset \mathrm{Int}(B_{3-i})$, and $x \in B_2$, we have $i = 1$ and thus h_k is a $(1, 2)$-form for every k. This means that (g_{2k}) is a sequence in $G_2 \setminus J$. So, using Lemma 3.2.4, we find a fixed compact subset $K \subset \mathrm{Int}(B_1)$ and a sequence (j_k) in J so that $j_k g_{2k}^{-1} B_2 \subset K$.

Consider the sequence (f_k) given by $f_k = h_k j_k^{-1}$. Since $|f_k| \to \infty$, a subsequence of (f_k^{-1}) consists of pairwise distinct elements of G. Since f_k^{-1} is a $(2,1)$-form, we know that

$$f_k^{-1} B_1 = j_k g_{2k}^{-1} \cdots g_1^{-1} B_1 \subset K.$$

On the other hand, by construction, we know that

$$h_k^{-1} x = g_{2k}^{-1} \cdots g_1^{-1} x \in B_2.$$

Since B_2 is J-invariant, we also see that $f_k^{-1} x = j_k h_k^{-1} x \in B_2$ for every k. By Proposition 2.3.4, $\mathrm{Int}(B_1)$ is an infinite set. Then, since B_2 and K are disjoint compact subsets of M, we can apply Lemma 2.2.12 (with $Y = \mathrm{Int}(B_1)$, $K_1 = K$, and $K_2 = B_2$) to complete the proof. □

Next we deal with parabolic points.

Lemma 3.2.10. *If both G_1 and G_2 are geometrically finite, then every parabolic point of G in $\Lambda(G_1) \cup \Lambda(G_2)$ is a bounded parabolic point for the action of G on $\Lambda(G)$.*

Proof. Fix a parabolic point $p \in \Lambda(G_1)$, and let $P < G$ be the parabolic subgroup stabilizing p. We will show that there is a compact set $K \subset \Lambda(G) \setminus \{p\}$ so that $P(K) = \Lambda(G) \setminus \{p\}$, which implies the action is cocompact. The main idea here is to apply Proposition 2.2.25 to the parabolic subgroup P, which gives us a way to use elements of P to position certain points in M far away from $\Lambda(P) = \{p\}$. Our strategy is to decompose the set $\Lambda(G) \setminus \{p\}$ into pieces. We will show that every point in $\Lambda(G) \setminus \{p\}$ is either far away from p to begin with, or else it is in a piece of $\Lambda(G) \setminus \{p\}$ which can be translated far away from p using either Proposition 2.2.25 or the boundedness of p in $\Lambda(G_1)$.

We consider two cases. For both cases, in order to apply Proposition 2.2.25, we need to know that J and P are fully quasi-convex subgroups of G_1; for J this follows from Lemma 3.2.3, and for P this is true because P is exactly the stabilizer of its limit set $\{p\} \subset \Lambda(G_1)$ in G_1.

Case 1: $p \in \Lambda(G_1) \setminus G_1(\Lambda(J))$

Using Lemma 3.2.8, we can see that every point in $\Lambda(G) \setminus \{p\}$ lies in one of the sets $\Lambda(G_1), \Lambda(G_2)$, or T_1. Since $\Lambda(G_2) \subset B_1$, and $T_1 \subset B_1 \cup B_2$, this means that every point in $\Lambda(G)$ lies in one of the sets

$$L_1 = \Lambda(G_1), \quad L_2 = B_1, \quad L_3 = T_1 \cap B_2.$$

Now, for each i, we will find a compact set $K_i \subset M \setminus \{p\}$ so that $P(K_i)$ contains $(\Lambda(G) \setminus \{p\}) \cap L_i$. Then we can define $K = (K_1 \cup K_2 \cup K_3) \cap \Lambda(G)$, so that $P(K) = \Lambda(G) \setminus \{p\}$.

Since p is a bounded parabolic point for the action of G_1 on $\Lambda(G_1)$, and $\Lambda(G_1)$ is locally compact, we already know that there is a compact $K_1 \subset \Lambda(G_1) \setminus \{p\}$ so that $P(K_1) = \Lambda(G_1) - \{p\}$. And, by part 3 of Definition A, we know $p \in \mathrm{Int}(B_2)$, so B_1 is already a compact subset of $M \setminus \{p\}$ and we can take $K_2 = B_1$. So, we just need to construct the compact set K_3.

For this, we apply Proposition 2.2.25, with $G = H = G_1$, $J_1 = P$, $J_2 = J$, $U_1 = M \setminus \{p\}$, and $U_2 = M \setminus B_1$. To verify that the hypotheses of the proposition are satisfied, we need to check that $gB_1 \subset M \setminus \{p\}$ for every $g \in G_1 \setminus J$. But, since $\Lambda(G_1)$ is G_1-invariant we can only have $p \in gB_1$ if $g^{-1}p \in B_1 \cap \Lambda(G_1) = \Lambda(J)$, which is impossible since we assume $p \in \Lambda(G_1) \setminus G_1(\Lambda(J))$.

So, we know there is a compact subset $K' \subset M \setminus \{p\}$ so that for any $g \in G_1 \setminus J$, we can find $h \in P$ so that $hgB_1 \subset K'$. But by definition, any

$y \in T_1 \cap B_2$ lies in $(G_1 \setminus J)(B_1)$, so we can take $K_3 = K'$ and we are done.

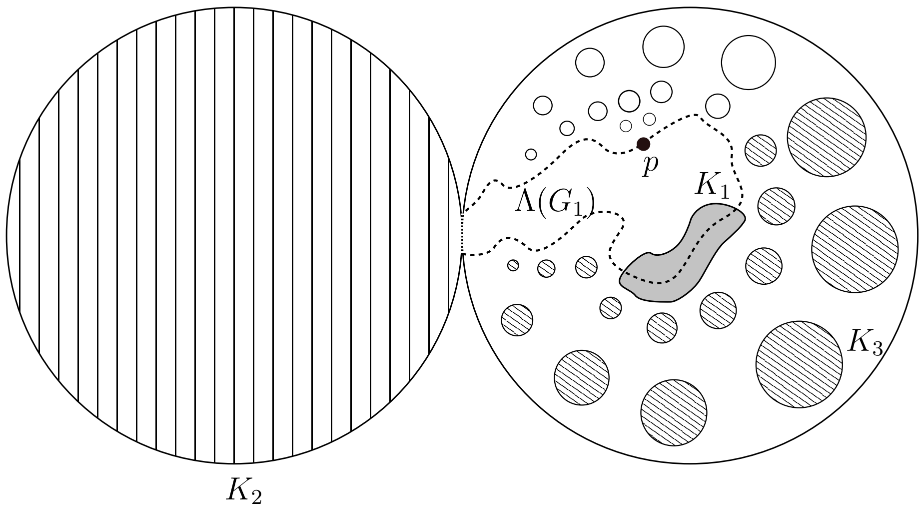

Figure 3.2.2: The sets K_1, K_2, and K_3 proving that $p \in \Lambda(G_1)$ is a bounded parabolic point (Case 1).

Case 2: $p \in G_1(\Lambda(J))$

Since G acts by homeomorphisms on $\Lambda(G)$ it suffices to consider the case $p \in \Lambda(J)$. For this case, we again use Lemma 3.2.8 to see that every point in $\Lambda(G)$ lies in one of the three sets

$$L_1 = \Lambda(G_1), \quad L_2 = \Lambda(G_2), \quad L_3 = T_1.$$

As in the previous case, for each of these sets, we will find a compact set $K_i \subset M \setminus \{p\}$ so that $P(K_i)$ contains $(\Lambda(G) \setminus \{p\}) \cap L_i$.

For $i = 1, 2$, as in Case 1, we can use the fact that p is a bounded parabolic point for the G_i-action on $\Lambda(G_i)$, to find compact sets $K_i \subset \Lambda(G_i) \setminus \{p\}$ such that $P(K_i) = \Lambda(G_i) \setminus \{p\}$.

To find K_3, we apply Proposition 2.2.25 twice: for $i = 1, 2$, we take $G =$

$H = G_i$, $J_1 = P$, $J_2 = J$, $U_1 = M \setminus \{p\}$, and $U_2 = M \setminus B_i$. As in the previous case we need to verify that $gB_i \subset M \setminus \{p\}$ for every $g \in G_i \setminus J$, but this follows because $gB_i \subset \mathrm{Int}(B_{3-i})$, which is disjoint from $\Lambda(J)$ and hence does not contain p.

This gives us a pair of compact set $K_{3,1}$ and $K_{3,2}$, such that for any $g \in G_i \setminus J$, we can find $h \in P$ so that $hgB_i \subset K_{3,i}$. Then, since any $y \in T_1$ lies in $(G_1 \setminus J)(B_1) \cup (G_2 \setminus J)(B_2)$ by definition, we can take $K_3 = K_{3,1} \cup K_{3,2}$ and we are done. $\square$

Finally, we can complete the proof of this direction of Theorem A part (iv).

Proposition 3.2.11. *If G_1 and G_2 are geometrically finite, then G is geometrically finite.*

Proof. Let $x \in \Lambda(G)$. We must show x is either a conical limit point or a bounded parabolic point for G. First, if x is not a translate of a limit point of G_1 nor G_2, then x is a conical limit point by Lemma 3.2.9. So, assume $x \in G(\Lambda(G_1) \cup \Lambda(G_2))$. Acting by elements of G preserves the properties we are trying to show, so in fact we may assume $x \in \Lambda(G_1) \cup \Lambda(G_2)$. If x is a parabolic point of G, we are done by Lemma 3.2.10. Otherwise, x is necessarily a conical limit point for G_1 or G_2 since these are geometrically finite, and again we are done since x will also be a conical limit point for G. $\square$

3.2.4 Geometrical Finiteness of the Factors

The last thing to do in this section is prove the other direction of Theorem A part (iv), and show that G_1 and G_2 are geometrically finite if G is geometrically finite. The first step is the following lemma, which makes use of the contraction property proved earlier in this section.

Lemma 3.2.12. *Assume that G is geometrically finite. Let $x \in \Lambda(G_i)$ for $i \in \{1,2\}$, and suppose that (h_k) is a conical limit sequence in G for x. Then, after extracting a subsequence, we can find some $h \in G$ so that $h_k \in hG_i$ for every k.*

Proof. Without loss of generality take $x \in \Lambda(G_1)$. Let (h_k) be a conical limit sequence for x. This means that there are distinct points $a, b \in M$ such that $h_k x \to a$ and $h_k z \to b$ for any $z \in M \setminus \{x\}$.

If there is some $h \in G$ so that $h_k \in hJ$ for infinitely many k, then we are done. So we may assume that, after taking a subsequence, each h_k is an (i,j)-form for i,j fixed, and each h_k represents a different left J-coset in G. There are two cases to consider: either every h_k is an $(i,1)$-form or every h_k is an $(i,2)$-form.

First suppose that h_k is an $(i,2)$-form. By Lemma 3.2.5, after extraction the sets $(h_k B_2)$ converge to a singleton. Since $x \in \Lambda(G_1) \subset B_2$, and $h_k x \to a$, we must have $h_k B_2 \to \{a\}$. Since B_2 is an infinite set by Proposition 2.3.4, there is some point $z \in B_2 \setminus \{x\}$, which must satisfy $h_k z \to a$. But this is impossible if (h_k) is a conical limit sequence for x.

We conclude that each h_k must be $(i,1)$-form. If $h_k \in G_1$ for infinitely many k then we are done, so assume that this is not the case. Then after taking a subsequence we have $|h_k| > 1$ for every k. We write h_k as an (i,j)-form of length $n \geq 2$:

$$h_k = g_{k,1} \cdots g_{k,n}.$$

Note that although n can depend on k, we omit this from the notation. Since h_k is an $(i,1)$-form, we have $g_{k,n} \in G_1$, and since $\Lambda(G_1)$ is G_1-invariant, $g_{k,n}x$

lies in $\Lambda(G_1) \subset B_2$. Then

$$(h_k g_{k,n}^{-1}) = (g_{k,1} \cdots g_{k,n-1})$$

is a sequence of $(i,2)$-forms. If the elements in this sequence lie in infinitely many different left J-cosets in G, then we extract a subsequence and apply Lemma 3.2.5 to see that $(h_k g_{k,n}^{-1} B_2)$ again converges to a singleton. This singleton contains the limit of $(h_k g_{k,n}^{-1} g_{k,n} x) = (h_k x)$, so it is again equal to $\{a\}$. But then for any $z \in B_1 \setminus \{x\}$, we have $g_{k,n} z \in B_2$ and thus $h_k z = h_k g_{k,n}^{-1} g_{k,n} z \to a$, again giving a contradiction. We conclude that a subsequence of $(h_k g_{k,n}^{-1})$ lies in a single coset hJ for $h \in G$, hence $h_k \in hJg_{k,n} \subset hG_1$. $\square$

Proposition 3.2.13. *If G is geometrically finite, then G_1 and G_2 are geometrically finite.*

Proof. Let $x \in \Lambda(G_1)$. We will show that x is either a conical limit point or a bounded parabolic point for G_1. Since G is geometrically finite, we know x is either a conical limit point for G or a bounded parabolic point for G.

If x is a conical limit point for G, then it has a conical limit sequence (h_k) in G, i.e. a sequence such that $(h_k x, h_k z)$ lies in a compact subset of $(M \times M) \setminus \Delta$ for any $z \neq x$ in M. By Lemma 3.2.12, we know that, up to subsequence, $h_k = hg_k$ for $g_k \in G_1$ and h fixed. Then (g_k) is a conical limit sequence for x in G_1 and we are done.

Otherwise, suppose x is a bounded parabolic point for G. Let P be the stabilizer of x in G. As we observed in the proof of Lemma 3.2.10, part (iii) of Proposition 3.2.2 implies that P is a subgroup of G_1.

Since x is a bounded parabolic point for G, again applying local compactness of $\Lambda(G) \setminus \{x\}$, there is a compact $K \subset \Lambda(G) \setminus \{x\}$ so that $P(K) =$

$\Lambda(G) \setminus \{x\}$. Let $K_1 = K \cap \Lambda(G_1)$. Since $\Lambda(G_1)$ is closed, K_1 is compact, and since $\Lambda(G_1)$ is G_1-invariant (hence P-invariant), we have

$$P(K_1) = P(K \cap \Lambda(G_1)) = P(K) \cap \Lambda(G_1) = \Lambda(G_1) \setminus \{x\}.$$

Thus x is bounded parabolic for G_1 and we are done. $\square$

3.3 New Combination Theorems: HNN Extensions

In this section we prove Theorem B. The proof is very similar in spirit and structure to the proof of Theorem A, but the details are different. Where possible, we have tried to imitate the structure of Section 3.2, and have indicated the analogies between the proofs.

We start (as in Section 3.2) by setting up the general ping-pong framework.

Definition B (HNN Ping-Pong Position)**.** Let G_0 be a discrete convergence group acting on a compact metrizable space M, and suppose that $J_1, J_{-1} < G_0$ are both geometrically finite. Let $G_1 = \langle f \rangle$ be an infinite discrete convergence group also acting on M, where $fJ_{-1}f^{-1} = J_1$ in $\mathrm{Homeo}(M)$. We will say G_0 is in *HNN ping-pong position* (with respect to f, J_1 and J_{-1}) if there exists closed sets $B_1, B_{-1} \subset M$ with nonempty disjoint interiors satisfying the following:

(1) (B_1, B_{-1}) is precisely invariant under (J_1, J_{-1}) in G_0 (recall Definition 2.1.22).

(2) If $A = M \setminus (B_1 \cup B_{-1})$, then $f(A \cup B_1) = \mathrm{Int}(B_1)$.

(3) For $i \in \{\pm 1\}$, we have $\Lambda(G_0) \cap B_i = \Lambda(J_i)$.

(4) The set $A_0 = M \setminus G_0(B_1 \cup B_{-1})$ is nonempty.

Remark 3.3.1. Note that our precise invariance assumption forces $B_1 \cap B_{-1} = \varnothing$.

We can now state and begin proving Theorem B.

Theorem B. *Let G_0 be a discrete convergence group acting on a compact metrizable space M, and suppose that $J_1, J_{-1} < G_0$ are both geometrically finite. Let $G_1 = \langle f \rangle$ be an infinite cyclic discrete convergence group also acting on M, where $fJ_{-1}f^{-1} = J_1$ in $\mathrm{Homeo}(M)$. Suppose G_0 is in HNN ping-pong position with respect to f, J_1 and J_{-1}. Let $G = \langle G_0, G_1 \rangle < \mathrm{Homeo}(M)$, and suppose G acts as a convergence group. Then the following hold:*

(i) $G = G_0 *_f$.

(ii) *G is discrete.*

(iii) *Elements of G not conjugate into G_0 are loxodromic.*

(iv) *G is geometrically finite if and only if G_0 is geometrically finite.*

Remark 3.3.2. As was the case for amalgamated free products, when $M = \partial \mathbb{H}^3_{\mathbf{R}}$, this theorem is not strong enough to recover Maskit's full result, since we ask for stronger hypotheses on our ping-pong configuration. Specifically, we do not allow B_1 and B_{-1} to intersect, and consequently f cannot be parabolic. This condition ensures that our subgroups are fully quasi-convex, and allows us to apply Proposition 2.2.25.

Before proving the first three parts of Theorem B, we give the following slightly stronger version of Lemma 2.3.15, which will be useful throughout this section.

Lemma 3.3.3. *Suppose that g is an (i,j)-form of type k. If $i \neq 0$, then $gB_k \subsetneq \mathrm{Int}(B_i)$, and if $i = 0$, then $gB_k \subsetneq A$.*

Proof. First suppose that $i \neq 0$. For concreteness, assume g is a $(1, j)$-form. We first suppose that $|g| = 1$, so that $g = fg_1$ for $g_1 \in G_0$. If $k = 1$, then $g_1 B_1$ is a subset of $M \setminus B_{-1} = \mathrm{Int}(A \cup B_1)$ by precise invariance. In fact it is a proper subset by properness of the interactive triple, so $fg_1 B_1 \subsetneq \mathrm{Int}(B_1)$ by condition (2) in Definition B. If $k = -1$, then since g has type k, we must have $g_1 \in G_0 \setminus J_{-1}$ and $g_1 B_{-1} \subset M \setminus B_{-1} = \mathrm{Int}(A \cup B_1)$. Again, the inclusion is proper by properness of the interactive triple, so again we have $fg_1 B_{-1} \subsetneq \mathrm{Int}(B_1)$.

When $|g| > 1$, we can apply Lemma 2.3.16 and induction: we write $g = g' f^j g_n$, where g' is a type-j normal form with length $|g| - 1$, and $f^j g_n B_k \subset B_j$. Via induction we know that $g' B_j \subsetneq \mathrm{Int}(B_1)$, which means $gB_k \subsetneq \mathrm{Int}(B_1)$.

The case $i = 0$ follows from the first case and precise invariance of B_i under J_i, since any $(0, j)$-form g can be written $g = g_1 g'$, where g' is an (i, j)-form and $g_1 \in G_0 \setminus J_i$. □

We now prove the first three parts of Theorem B.

Proof of (i) - (iii) in Theorem B. (i) Let B_1 and B_{-1} be the sets given by our conditions, and set $A = M \setminus (B_1 \cup B_{-1})$. Note that condition (2) of Definition B implies $f^{-1}(A \cup B_{-1}) = \mathrm{Int}(B_{-1})$. The result now follows from Proposition 2.3.12 since $(A, \mathrm{Int}(B_1), \mathrm{Int}(B_{-1}))$ form an interactive triple which is proper by condition (4) of Definition B.

(ii) It suffices to show no sequence (g_k) in G can accumulate at the identity. If $|g_k| = 0$, then g_k lies in the discrete group G_0, so assume $|g_k| \geq 1$ for all k. We can consider several cases. If a normal form for g_k ends in a power of f, then $g_k A_0 \subset B_1 \cup B_{-1}$. Otherwise, g_k is either a $(0, 1)$ or a $(0, -1)$-form. In the former case, $g_k B_1 \subset A$, and in the latter case $g_k B_{-1} \subset A$. In each of these cases, g_k takes a fixed set with nonempty interior into another fixed disjoint set, which means g_k cannot accumulate on the identity.

(iii) Let $g = f^{\alpha_1} g_1 \cdots f^{\alpha_n} g_n$ be a normal form not conjugate into G_0. Conjugating and replacing g with g^{-1} if necessary, we can assume that $\alpha_1 > 0$ (so g is a $(1, j)$-form) and that $|g|$ is minimal in its conjugacy class. Note that if $\alpha_n < 0$ and $g_n \in J_1$, then $f^{-1} g f = f^{\alpha_1 - 1} g_1 \cdots f^{\alpha_n + 1} f^{-1} g_n f$ has a strictly smaller length than g since $f^{-1} g_n f \in J_{-1}$, so we know that either $\alpha_n > 0$ or $g_n \in G_0 \setminus J_1$. That is, g is a $(1, j)$-form of type 1, so by Lemma 3.3.3, gB_1 is a proper subset of $\mathrm{Int}(B_1)$. Then the same argument as in Theorem A part (iii) implies that g has infinite order, and a fixed point in $\mathrm{Int}(B_1)$.

On the other hand if gB_1 is a proper subset of B_1, then $g^{-1}(M \setminus gB_1)$ is a proper subset of $M \setminus B_1$, so the same argument again shows that g^{-1} has a fixed point in the closure of $M \setminus B_1$. Thus g has two distinct fixed points and is loxodromic. □

3.3.1 Limit Sets of HNN Extensions

The remainder of the section is meant to prove part (iv) of Theorem B, so for the rest of the paper we fix the space M and groups $G_0, J, \langle f \rangle, G$ in $\mathrm{Homeo}(M)$ satisfying the conditions of Definition B. As for Theorem A, we start by establishing some properties of the limit points of G under these assumptions.

Proposition 3.3.4. *With the above conditions and notation, each of the following holds.*

(i) $B_1 \cap B_{-1} = \varnothing$, *and f is loxodromic with attracting fixed point in* $\mathrm{Int}(B_1)$ *and repelling fixed point in* $\mathrm{Int}(B_{-1})$.

(ii) $\Lambda(J_{\pm 1}) \subset \partial B_{\pm 1}$.

(iii) $\Lambda(G_0) \setminus G_0(\Lambda(J_1) \cup \Lambda(J_{-1})) \subset A_0$.

Proof. (i) The fact that $B_1 \cap B_{-1} = \varnothing$ follows from precise invariance of (B_1, B_{-1}) under (J_1, J_{-1}) in G_0. Now, since $f(A \cup B_1) = \mathrm{Int}(B_1)$, we have $f(\partial B_{-1}) = \partial B_1$, and so $fB_1 \subset \mathrm{Int}(B_1)$. Arguing as in the proof of Theorem A part (iii), we know this implies f has a fixed point in $\mathrm{Int}(B_1)$ which is necessarily attracting. The same argument applied to f^{-1} gives a fixed point in $\mathrm{Int}(B_{-1})$ which is necessarily a repelling fixed point for f.

(ii) Since B_i is closed and J_1-invariant, and $\mathrm{Int}(B_i)$ is infinite by Proposition 2.3.10, it follows that $\Lambda(J_i) \subset B_i$. Further, since f conjugates J_{-1} to J_1, f maps $\Lambda(J_{-1})$ bijectively onto $\Lambda(J_1)$. Hence

$$\Lambda(J_1) = f\Lambda(J_{-1}) \subset fB_{-1} = B_{-1} \cup A \cup \partial B_1.$$

So we conclude $\Lambda(J_1) \subset (B_{-1} \cup A \cup \partial B_1) \cap B_1 = \partial B_1$ as desired. Applying an identical argument using f^{-1} gives $\Lambda(J_{-1}) \subset \partial B_{-1}$.

(iii) Fix $x \in \Lambda(G_0)$, and suppose that $x \notin A_0$, i.e. that $x = gy$ for $g \in G_0$ and $y \in B_i$. Then $y = g^{-1}x \in \Lambda(G_0) \cap B_i$, which means $y \in \Lambda(J_i)$ by condition (3) in Definition B, and therefore $x \in G_0(\Lambda(J_i))$. It follows that $\Lambda(G_0) \setminus A_0$ is contained in $G_0(\Lambda(J_1) \cup \Lambda(J_{-1}))$, which is equivalent to the desired claim. $\square$

3.3.2 HNN Ping-Pong and Contraction

Next, we will establish a contraction lemma for HNN ping-pong sequences, similar to Lemma 3.2.5 for amalgamated free products. As in the earlier case, the key tool is Proposition 2.2.25, so we start by establishing that the subgroups J_1 and J_{-1} are fully quasi-convex in some ambient geometrically finite group.

First, we show:

Lemma 3.3.5. *Fix $i, j \in \{\pm 1\}$ and $g \in G$. Then $g\partial B_i \cap \partial B_j \neq \varnothing$ if and only if either:*

1. *$i = j$ and $g \in J_i$, or*

2. *$i = -j$ and $g = f^j h$ for $h \in J_i$.*

Proof. We induct on the length of g. If $|g| = 0$, then the claim follows from precise invariance of (B_1, B_{-1}) under (J_1, J_{-1}) in G_0, so suppose $|g| \geq 1$, and for concreteness, assume $i = 1$. We write a normal form $f^{\alpha_1} g_1 \cdots f^{\alpha_n} g_n$ for g. If this normal form has type 1, then Lemma 3.3.3 implies that $gB_1 \subset \mathrm{Int}(B_1) \cup \mathrm{Int}(B_{-1}) \cup A$, hence $g\partial B_1 \cap \partial B_j = \varnothing$. So we can assume that this normal form does *not* have type 1, which means that $\alpha_n < 0$ and $g_n \in J_1$. In this case, $f^{-1} g_n \partial B_1 = \partial B_{-1}$.

The group element $g' = g g_n^{-1} f$ has strictly smaller length than g, so if $g' \partial B_{-1} \cap \partial B_j \neq \varnothing$ then by induction we know that either $j = -1$ and $g' \in J_{-1}$, or $j = 1$ and $g' = fh$ for $h \in J_{-1}$. In the former case we can rewrite $g = g' f^{-1} g_n = f^{-1} g'' g_n$ for $g'' \in J_1$, and in the latter case we can rewrite $g = g' f^{-1} g_n = f h f^{-1} g_n = g'' g_n$ for $g'' \in J_1$. Since $g_n \in J_1$ the conclusion follows. $\square$

The lemma above implies in particular that ∂B_i is precisely invariant under J_i in both G and G_0. Then, after applying part (ii) of Proposition 3.3.4, we see:

Corollary 3.3.6. *Let H be one of G or G_0. If H is geometrically finite, then J_1 and J_{-1} are fully quasi-convex subgroups of H.*

Now, we can apply Proposition 2.2.25 to the present setting:

Lemma 3.3.7. *Suppose that either G or G_0 is geometrically finite. For $i \in \{\pm 1\}$, we can find a compact $K \subset A \cup B_{-i}$ so that both of the following hold:*

(i) For any $g \in G_0 \setminus J_i$, we have $j \in J_i$ so $jgB_i \subset K$.

(ii) For any $g \in G_0$, we have $j \in J_i$ so that $jgB_{-i} \subset K$.

Proof. Take $i = 1$ to simplify notation. We can find a compact for each claim separately and take their union. First, we focus on (i). We can assume $B_1 \subset K$, so the statement follows immediately for $g \in J_1$ by taking j to be the identity. Otherwise, we apply Proposition 2.2.25 with the ambient geometrically finite group as G or G_0 (depending on which one is geometrically finite) and $H = G_0$ in both cases, and our two fully quasi-convex subgroups J_1 and J_{-1} with corresponding invariant open sets $U_1 = M \setminus B_1$ and $U_{-1} = M \setminus B_{-1}$. Then if $g \in G_0 \setminus J_1$, we have $g(M \setminus U_1) = g(B_1) \subset A \subset U_{-1}$, and so the proposition gives our desired compact subset $K \subset U_{-1} = A \cup B_1$.

For (ii), the proof is identical with J_1 playing the role of both fully quasi-convex subgroups in the statement of Proposition 2.2.25, and both open sets being $M \setminus B_1 = A \cup B_{-1}$. □

We can now establish the HNN contraction property:

Lemma 3.3.8 (Contraction for HNN extensions)**.** *Suppose that either G or G_0 is geometrically finite, and let (h_k) be a sequence of type-i forms such that the left cosets $h_k J_i$ are all distinct. Then up to subsequence, $(h_k B_i)$ converges to a singleton $\{x\}$.*

It follows from Lemma 3.3.5 that any group element $g \in G$ satisfying $gB_i = B_i$ must lie in J_i. So, asking for the left cosets $(h_k J_i)$ to be distinct is the same as asking for the translates $(h_k B_i)$ to be distinct.

Proof. To simplify notation, assume $i = 1$. The proof is very similar to the proof of Lemma 3.2.5. The first step is to show the following:

Claim. *After extracting a subsequence, there is a fixed $\ell = \pm 1$, a compact set $K \subset A \cup B_{-1}$, and a sequence (j_k) in J_1 so that $j_k h_k^{-1} B_\ell \subset K$ for all k.*

To prove the claim, we first suppose that $|h_k| = 0$. Then, since h_k has type 1, we know $h_k \in G_0 \setminus J_1$. Then we take $\ell = 1$, and apply Lemma 3.3.7 to find the required set K and elements j_k. Otherwise, suppose that $|h_k| \geq 1$, and write out a normal form for h_k:

$$h_k = f^{\alpha_{k,1}} g_{k,1} \cdots f^{\alpha_{k,n}} g_{k,n}.$$

We consider the inverse word

$$g_{k,n}^{-1} f^{-\alpha_{k,n}} \cdots g_{k,1}^{-1} f^{-\alpha_{k,1}}.$$

By Lemma 2.3.13, the sub-word $g_{k,n} h_k^{-1} = f^{-\alpha_{k,n}} \cdots g_{k,1}^{-1} f^{-\alpha_{k,1}}$ is a normal form, which must have length at least 1. Up to subsequence, for every k this normal form is type ℓ for some fixed $\ell = \pm 1$, meaning that $g_{k,n} h_k^{-1} B_\ell \subset B_1$ if $-\alpha_{k,n} > 0$ and $g_{k,n} h_k^{-1} B_\ell \subset B_{-1}$ if $-\alpha_{k,n} < 0$. After extracting another subsequence we can assume one of these conditions holds for every k.

In the case where $-\alpha_{k,n} < 0$ for every k, we can use Lemma 3.3.7 to find elements $j_k \in J_1$ and a compact $K \subset A \cup B_{-1}$ so that $j_k g_{k,n}^{-1} B_{-1} \subset K$ for every k. Then, we know that for every k, we have

$$j_k h_k^{-1} B_\ell = j_k g_{k,n}^{-1} g_{k,n} h_k^{-1} B_\ell \subset j_k g_{k,n}^{-1} B_{-1} \subset K.$$

On the other hand, if $-\alpha_{k,n} > 0$, then since h_k has type 1 we know that

$g_{k,n} \in G_0 \setminus J_1$. Then again by Lemma 3.3.7 we can find a compact $K \subset A \cup B_{-1}$ and $j_k \in J_1$ so that $j_k g_{k,n}^{-1} B_1 \subset K$. Thus, we have

$$j_k h_k^{-1} B_\ell = j_k g_{k,n}^{-1} g_{k,n} h_k^{-1} B_\ell \subset j_k g_{k,n}^{-1} B_1 \subset K.$$

We have shown the claim above, so now consider the sequence $(h_k j_k^{-1})$. Since all the translates $h_k B_1$ are distinct, the group elements h_k lie in infinitely many left J_1-cosets, hence so do the group elements $h_k j_k^{-1}$. In particular, the sequence $h_k j_k^{-1}$ is divergent in G, so we can extract a convergence subsequence and assume that there are points $z_+, z_- \in M$ so that $(h_k j_k^{-1} y)$ converges to z_+ whenever $y \neq z_-$. Equivalently, $(j_k h_k^{-1} y)$ converges to z_- whenever $y \neq z_+$.

Proposition 2.3.10 tells us that the set B_1 is infinite, so in particular there must be some $y \in B_1 \setminus \{z_+\}$. Then, since $j_k h_k^{-1} B_1 \subset K$ we conclude that $z_- \in K$. Finally, since B_1 is a compact set in the complement of K, we see that $(h_k j_k^{-1} B_1) = (h_k B_1)$ must converge to $\{z_+\}$. $\square$

3.3.3 Geometrical Finiteness of the Extension

We now prove that (G_0 geometrically finite) $\implies$ (G geometrically finite). This gives one of the directions of Theorem B part (iv).

As in the proof of the analogous direction of Theorem A, the key for this direction of theorem is to show that limit points in $\Lambda(G) \setminus \Lambda(G_0)$ can be "coded" by sequences of (i, j)-forms in G. The precise statement is:

Proposition 3.3.9 (HNN coding for G-limit points)**.** *Suppose that either G or G_0 is geometrically finite, and let $x \in B_1 \cup B_{-1}$ be a point in $\Lambda(G) \setminus G(\Lambda(G_0))$. Then for fixed ℓ, there is a sequence of type-ℓ forms (h_k) in G so that $|h_k| \to \infty$, each h_k is a prefix of h_{k+1}, and $x \in h_k B_\ell$ for every k.*

We can think of this proposition as a less explicit version of Proposition 3.2.6 in the amalgamated free product case. The construction in this case is slightly more involved, and we need a little more information about the location of certain points in $\Lambda(G)$. So, we start by showing the following:

Lemma 3.3.10. *Suppose that either G or G_0 is geometrically finite. Then the only limit points of G in $\partial B_{\pm 1}$ are limit points of $J_{\pm 1}$. That is, $\Lambda(G) \cap \partial B_{\pm 1} = \Lambda(J_{\pm 1})$.*

Proof. We will show that the intersection $\Lambda(G) \cap (\partial B_1 \cup \partial B_{-1})$ is a subset of $\Lambda(G_0)$; then we will be done by condition (3) in Definition B. So, let $x \in \Lambda(G) \cap \partial B_1$. We can find a sequence (g_k) in G so that $g_k z \to x$ for all but perhaps a single $z \in M$. Now, if $g_k \in G_0$ for infinitely many k, the conclusion immediately follows. So we may assume $|g_k| \geq 1$ for every k.

Up to subsequence, the g_k are all (i, j)-forms for fixed $i \in \{0, \pm 1\}$ and $j \in \{\pm 1\}$. If $i = -1$, then $g_k B_j \subset \mathrm{Int}(B_{-1})$. But for some $z \in B_j$, the sequence $(g_k z)$ converges to $x \in \partial B_1$. So, we know that either $i = 0$ or $i = 1$.

If $i = 0$, then by Lemma 2.3.11, for some $\ell = \pm 1$ and some $h_k \in G_0 \setminus J_i$, we have $g_k B_j \subset h_k B_\ell \subset A$. There must be infinitely many distinct translates $h_k B_\ell$, since otherwise each $g_k z$ would lie in a fixed compact subset of A, and $(g_k z)$ could not converge to $x \in B_1$. So, the left cosets $h_k J_\ell$ are all distinct. Then by Lemma 3.3.8, the sequence of sets $(h_k B_\ell)$ converges to a singleton, which must be x. But since $h_k \in G_0$ this again implies that $x \in \Lambda(G_0)$.

Finally, we consider the case $i = 1$. We write out a normal form for g_k:

$$g_k = f^{\alpha_{k,1}} g_{k,1} \cdots f^{\alpha_{k,n}} g_{k,n}.$$

First observe that if $\alpha_{k,1} > 1$ for infinitely many k, then the word

$$g'_k = f^{-1}g_k = f^{\alpha_{k,1}-1}g_{k,1}\cdots f^{\alpha_{k,n}}g_{k,n}$$

is still a $(1,j)$-form, which means that $f^{-1}g_kB_j \subset B_1$ for infinitely many k. But then for infinitely many k, the point g_kz lies in the compact subset $fB_1 \subset \mathrm{Int}(B_1)$, which is impossible if $g_kz \to x \in \partial B_1$.

We conclude that after extraction, we have $\alpha_{k,1} = 1$ for every k. After further extraction, we can assume that one of the the three conditions below holds for every k:

(a) The length of g'_k is zero;

(b) g'_k is a $(0,j)$-form;

(c) g'_k is not a normal form, hence $g_{k,1} \in J_1$ and $\alpha_{k,2} > 0$.

If either (a) or (b) holds, we can use the first two cases of this proof to see that that $f^{-1}x$ lies in $\Lambda(G_0) \cap \partial B_{-1} = \Lambda(J_{-1})$, and thus $x \in \Lambda(J_1)$. And, (c) cannot occur: if $\alpha_{k,2} > 0$, then the word $f^{\alpha_{k,2}}g_{k,2}\cdots f^{\alpha_{k,n}}g_{k,n}$ is a $(1,j)$-form, which means $g'_kz \in g_{k,1}B_1$, and if $g_{k,1} \in J_1$ then $g_kz = fg'_kz \in fB_1$ and again (g_kz) cannot converge to $x \in \partial B_1$. $\square$

We now set about proving Proposition 3.3.9. As in the analogous situation in the amalgamated free product case, we follow Maskit's strategy by defining certain "ping-pong" sets in M. Let $T_{0,i} = G_0(B_i)$, and $T_0 = T_{0,1} \cup T_{0,-1} = G_0(B_1 \cup B_{-1})$, the union of all G_0 translates of B_1 and B_{-1}.

More generally, let

$$T_{m,-1} = \bigcup gB_{-1},$$

where the union is taken over length-m normal forms g of type -1. Similarly, let

$$T_{m,1} = \bigcup gB_1,$$

where the union is taken over length-m normal forms of type 1. Let

$$T_m = T_{m,1} \cup T_{m,-1}.$$

Lemma 2.3.16 implies that the sets T_m are decreasing: for any length-m normal form g with type i, we can use the lemma to find a length-$(m-1)$ normal form g' with type j, and $g_0 \in G_0$ so that $gB_i = g'f^j g_0 B_i \subset g'B_j$. So, we can now consider the set

$$T = \bigcap_{m=0}^{\infty} T_m.$$

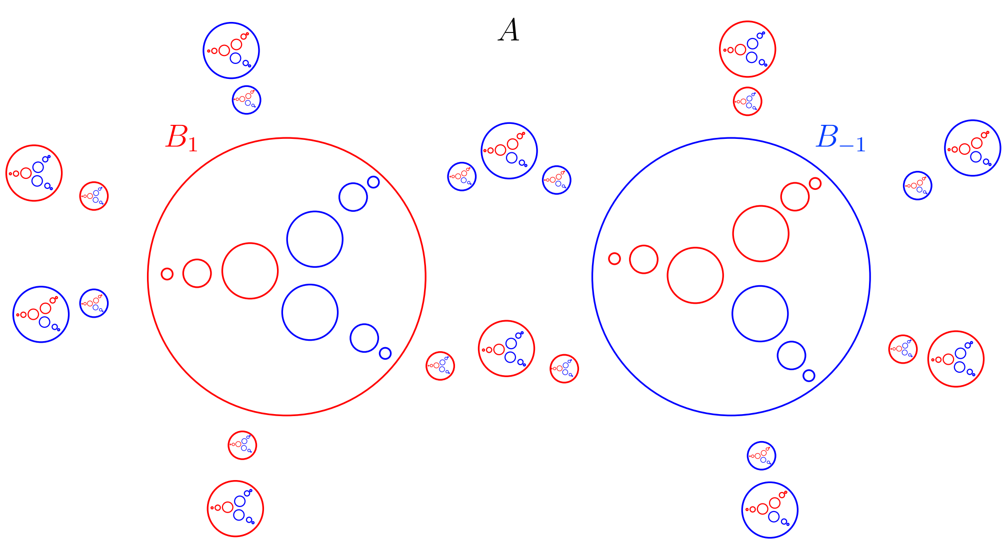

Figure 3.3.1: Part of the sets T_0 and T_1.

The proof of Proposition 3.3.9 mainly involves showing that $\Lambda(G)\backslash G(\Lambda(G_0)) \subset$

T. Then, we construct the desired sequence using the definition for T. The first step is:

Lemma 3.3.11. *Suppose either G or G_0 is geometrically finite. Then $\Lambda(G) \setminus \Lambda(G_0) \subset T_0$.*

Proof. This argument is similar to the proof of Lemma 3.3.10. Suppose $y \in \Lambda(G)$ does not lie in T_0 (that is, $y \in A_0$). We must show $y \in \Lambda(G_0)$. Since $y \in \Lambda(G)$, we can find a sequence (g_k) in G so that $g_k w \to y$ for all but a single $w \in M$. If $g_k \in G_0$ for infinitely many k we will have $y \in \Lambda(G_0)$ as desired, so now suppose that $|g_k| \geq 1$ for infinitely many k. After extracting a subsequence we can assume that each g_k is an (i, j)-form for i, j fixed. Since B_j is an infinite set, we can fix some $w \in B_j$ so that $(g_k w)$ converges to y.

By definition, we know that $g_k w \in T_n$, so in particular $g_k w \in T_0$ for every k. Then, we can extract a further subsequence so that $g_k w \in G_0(B_i)$ for fixed i and write $g_k w = g_k' z_k$ for $g_k' \in G_0$ and $z_k \in B_i$.

As $y \notin T_0$, there must be infinitely many distinct translates $g_k' B_i$, because otherwise every $g_k' z_k$ would lie in a fixed compact subset of T_0. Thus there are infinitely many distinct cosets $g_k' J_i$, and Lemma 3.3.8 tells us that after extraction, $(g_k' B_i)$ must converge to a singleton. Since $g_k' z_k \to y$, it follows that this singleton is y. Hence for any choice of $z \in B_i$ we have $g_k' z \to y$, so $y \in \Lambda(G_0)$. $\square$

Proof of Proposition 3.3.9. We first prove that $\Lambda(G) \setminus G(\Lambda(G_0)) \subset T$. So, fix $z \in \Lambda(G)$, and suppose $z \notin T$. We will show $z \in G(\Lambda(G_0))$.

If $z \in \Lambda(G_0)$ we are done, hence by Lemma 3.3.11 we can assume $z \in T_0$. Then we can find $m > 0$ so that $z \in T_{m-1} \setminus T_m$ since these sets are decreasing. Without loss of generality, we have $z \in gB_{-1}$ for $g = f^{\alpha_1} g_1 \cdots f^{\alpha_n} g_n$ a normal form with length $m - 1$ and type -1. If $g^{-1} z \in \partial B_{-1}$, then since $\Lambda(G)$ is

G-invariant we have $g^{-1}z \in \Lambda(G) \cap \partial B_1 = \Lambda(J_{-1})$ by Lemma 3.3.10 and we are done. So suppose $g^{-1}z \in \mathrm{Int}(B_{-1})$.

Since $z \notin T_m$, we have $fg^{-1}z \notin B_{-1}$ since gf^{-1} has length m. Also, $fg^{-1}z \notin B_1$ since f does not map any points of $\mathrm{Int}(B_{-1})$ into B_1. It follows that $fg^{-1}z \notin B_1 \cup B_2$, but also, $fg^{-1}z$ cannot be in a translate of B_1 nor B_2. Indeed, if $fg^{-1}z = hy$ for $y \in B_i$ and $h \in G_0$, then $h \in G_0 \setminus J_i$ since $hy \notin B_i$ and B_i is J_i-invariant. Hence $z = gf^{-1}hy \in T_m$, a contradiction. Hence $fg^{-1}z \notin T_0$, and so by Lemma 3.3.11 we have $fg^{-1}z \in \Lambda(G_0)$ and $z \in G(\Lambda(G_0))$ as desired.

We have now shown that $\Lambda(G) \setminus G(\Lambda(G_0)) \subset T$, so consider $z \in T$. We will construct our sequence (h_k) of normal forms inductively. We know that $z \in T_1$, so we can find some normal form h_1 with type i_1 so that $z \in h_1 B_{i_1}$. Now, assume that we have constructed a normal form h_k of type i so that $z \in h_k B_i$. Since $z \in T_{k+1}$, we can find a normal form h'_{k+1} with length $k+1$ and type ℓ so that $x \in h'_{k+1} B_\ell$. Then, by Lemma 2.3.16, there is a type-i' form h'_k with length k and $g_0 \in G_0$ so that $h'_{k+1} = h'_k f^{i'} g_0$ and $h'_{k+1} B_\ell \subset h'_k B_{i'}$. Then $h'_k B_{i'}$ has nonempty intersection with $h_k B_i$, so by Lemma 2.3.17 we have $i = i'$ and $h_k j = h'_k$ for $j \in J_i$. We can write $jf^i = f^i j'$ for $j' \in J_{-i}$. Then since h_k has type i, h_k is a prefix of the type-ℓ form $h_{k+1} = h_k f^i j' g_0$. This form is equivalent in G to the type-ℓ form h'_{k+1}, hence $z \in h_{k+1} B_\ell$.

Finally, by taking a subsequence, we can assume that each h_k is a form of type ℓ for ℓ fixed, and we are done. □

As for amalgamated free products, we can use the coding given by Proposition 3.3.9 to construct conical limit sequences for points in $\Lambda(G) \setminus G(\Lambda(G_0))$:

Lemma 3.3.12. *If G_0 is geometrically finite, then every point of $\Lambda(G) \setminus$*

$G(\Lambda(G_0))$ *is a conical limit point for* G.

Proof. Let $x \in \Lambda(G) \setminus G(\Lambda(G_0))$. Proposition 3.3.9 says that for i fixed, we can find a sequence (h_k) of ping-pong forms of type i, with $|h_k| \to \infty$, so that each h_k is a prefix of h_{k+1}, and $x \in h_k B_i$ for all k. Possibly after relabeling we may assume $i = 1$.

We write h_k in a normal form:

$$f^{\alpha_1} g_1 \cdots f^{\alpha_{n_k}} g_{n_k}.$$

If $\alpha_1 = 0$, then $\alpha_2 \neq 0$, in which case $h'_k = f^{\alpha_2} g_2 \cdots f^{\alpha_{n_k}} g_{n_k}$ is a ping-pong form of type 1 such that $h'_k B_1$ contains $x' = g_1^{-1} x$. Since x' is a conical limit point if and only if x is, if necessary we can replace x with x' and h_k with h'_k, and assume that $\alpha_1 \neq 0$. That is, h_k is an (ℓ, j)-form for $\ell \neq 0$, so $h_k B_1 \subset B_\ell$.

Further, since $x \in h_1 B_1$, by replacing x with $h_1^{-1} x$ and h_k with $h_1^{-1} h_k$, we can assume that also $x \in B_1$, hence $h_k B_1 \cap B_1 \neq \varnothing$. Since $h_k B_1 \subset B_\ell$ we have $\ell = 1$, meaning $\alpha_1 > 0$.

Now, consider the sequence of sets

$$(h_k^{-1} B_{-1}) = (g_{n_k}^{-1} f^{-\alpha_{n_k}} \cdots g_1^{-1} f^{-\alpha_1} B_{-1}).$$

By Lemma 2.3.13, the word $f^{-\alpha_{n_k}} \cdots g_1^{-1} f^{-\alpha_1}$ is a normal form; since $\alpha_1 > 0$ it is a form of type -1, implying that $f^{-\alpha_{n_k}} \cdots g_1^{-1} f^{-\alpha_1} B_{-1}$ is a subset of B_1 if $\alpha_{n_k} < 0$ and a subset of B_{-1} if $\alpha_{n_k} > 0$. And, since h_k is a form of type 1, we know that either $g_{n_k} \in G_0 \setminus J_1$ or $\alpha_{n_k} > 0$.

To prove that x is conical, we want to apply Lemma 2.2.12, which means we need to produce distinct elements g_k, a set Y with at least two points, and disjoint compact sets K_1 and K_2 so that $g_k x \in K_2$ and $g_k Y \subset K_1$. Let

$K \subset A \cup B_{-1} = M \setminus B_1$ be the compact from Lemma 3.3.7, and take $Y = B_{-1}, K_1 = K$ and $K_2 = B_1$. We know Y contains at least two points from Proposition 2.3.10, so we just need to produce the sequence (g_k) by modifying h_k^{-1}.

For each fixed k, we already have $h_k^{-1}x \in B_1$ as desired. If $\alpha_{n_k} > 0$, then $h_k^{-1}B_{-1} \subset g_{n_k}^{-1}B_{-1}$. From the definition of K, we can find $j_k \in J_1$ so that $j_k g_{n_k}^{-1}B_{-1} \subset K$, hence $j_k h_k^{-1}B_{-1} \subset K$.

On the other hand, if $\alpha_{n_k} < 0$, then we necessarily have $g_{n_k} \in G_0 \setminus J_1$, and $h_k^{-1}B_{-1} \subset g_{n_k}^{-1}B_1 \subset A$. Again using the definition of K, we can find $j_k \in J_1$ so that $j_k g_{n_k}^{-1}B_1 \subset K$, hence $j_k h_k^{-1}B_{-1} \subset K$.

In either of these cases, we have $j_k h_k^{-1}B_{-1} \subset K$ and $j_k h_k^{-1}x \in j_k B_1 = B_1$, which means we can take $g_k = j_k h_k^{-1}$ to complete the proof. □

We next consider parabolic points.

Lemma 3.3.13. *Suppose that G_0 is geometrically finite. If $p \in \Lambda(G_0)$ is a parabolic point for the action of G_0 on $\Lambda(G_0)$, then p is a bounded parabolic point for the action of G on $\Lambda(G)$.*

Proof. Let $p \in \Lambda(G_0)$ be a parabolic point for G, and let P be the stabilizer of p in G. Since p is a bounded parabolic point, and P contains the stabilizer of p in G_0, we know that there is a compact $\widehat{K} \subset \Lambda(G_0) \setminus \{p\}$ so that $P(\widehat{K}) = \Lambda(G_0) \setminus \{p\}$. We want to find a compact $K \subset \Lambda(G) \setminus \{p\}$ so that $P(K) = \Lambda(G) \setminus \{p\}$.

As in the proof of Lemma 3.2.10, our strategy is to show that $\Lambda(G) \setminus \{p\}$ can be decomposed into several pieces, such that each piece is either far away from p to begin with, or can be pushed uniformly far away from p using either the boundedness of p in $\Lambda(G_0)$ or an application of Proposition 2.2.25. We consider two cases:

Case 1: $p \in \Lambda(G_0) \setminus G_0(\Lambda(J_1) \cup \Lambda(J_{-1}))$

In this case, Lemma 3.3.11 tells us that each point in $\Lambda(G) \setminus \{p\}$ lies in the union $\Lambda(G_0) \cup T_0$. We can further decompose T_0 by intersecting it with $(B_1 \cup B_{-1})$ and its complement A, meaning we decompose $\Lambda(G) \setminus \{p\}$ into three pieces lying in

$$L_1 = \Lambda(G_0), \quad L_2 = (B_1 \cup B_{-1}), \quad L_3 = T_0 \cap A.$$

For each L_i, we need to find a compact set $K_i \subset M \setminus \{p\}$ so that if $y \in \Lambda(G) \cap L_i$, then we can find $h \in P$ so that $hy \in K_i$. Then we can take $K = K_1 \cup K_2 \cup K_3$.

We know we can take $K_1 = \widehat{K}$ from the boundedness of p in $\Lambda(G_0)$, and from part (3) of Definition B we know that $B_1 \cup B_{-1}$ is already a compact subset of $M \setminus \{p\}$. So, we just need to find the compact set K_3.

We apply Proposition 2.2.25, taking $H = G_0$, $J_1 = P$, $U_1 = M \setminus \{p\}$, $J_2 = J_{\pm 1}$, and $U_2 = M \setminus B_{\pm 1}$, to see that there are sets $K_+, K_- \subset M \setminus \{p\}$ such that for any $g \in G_0 \setminus J_{\pm 1}$, we can find $h \in P$ so that $hgB_{\pm 1} \subset K_{\pm}$. To justify the application of the proposition, we need to check that for every $g \in G_0 \setminus J_{\pm 1}$, we have $gB_{\pm 1} \subset M \setminus \{p\}$, but this follows from part (iii) of Proposition 3.3.4. Then, we take $K_3 = K_+ \cup K_-$.

Now, if $y \in T_0 \cap A$, then by definition we know that $y \in (G_0 \setminus J_1)(B_1) \cup (G_0 \setminus J_{-1})(B_{-1})$. But then by definition of $K_{\pm}$ we know that we can find $h \in P$ so that $hy \in K_+ \cup K_- = K_3$ and we are done.

Case 2: $p \in G_0(\Lambda(J_1) \cup \Lambda(J_{-1}))$

Since G acts by homeomorphisms it suffices to consider $p \in \Lambda(J_1) \cup \Lambda(J_{-1})$. Without loss of generality take $p \in \Lambda(J_1)$. As in the previous case, we decompose $\Lambda(G) \setminus \{p\}$ into several different pieces, by writing M as the union

$$M = fB_1 \cup fA \cup \partial B_1 \cup A \cup B_{-1}.$$

Since $p \in \partial B_1$, the sets $fB_1 \subset \mathrm{Int}(B_1)$ and B_{-1} are compact sets in the complement of p.

So, we only need to consider the three pieces of $\Lambda(G) \setminus \{p\}$ contained in the three sets

$$\partial B_1, \quad A, \quad fA.$$

We can further decompose these pieces by intersecting each of them with the sets $\Lambda(G_0)$, $f\Lambda(G_0)$ and their complements in M. By Lemma 3.3.10, we know that $\partial B_1 \cap \Lambda(G) \subset \Lambda(G_0)$. Also, from Lemma 3.3.11, we know that $\Lambda(G) \setminus \Lambda(G_0)$ lies in T_0, which means we now only need to consider the pieces of $\Lambda(G) \setminus \{p\}$ contained in the four sets

$$L_1 = \Lambda(G_0), \quad L_2 = T_0 \cap A, \quad L_3 = f\Lambda(G_0), \quad L_4 = f(T_0 \cap A).$$

We want to find compact sets $K_1, K_2, K_3, K_4 \subset M \setminus \{p\}$ so that for each $y \in L_i \cap (\Lambda(G) \setminus \{p\})$, we can find $h \in P$ so that $hy \in K_i$.

We already know that we can take $K_1 = \widehat{K}$, and to find K_2, we can use the exact same construction we used for K_3 in Case 1. To justify the application of Proposition 2.2.25 in this situtation, we again need to check that for any $g \in G_0 \setminus J_{\pm 1}$, we have $gB_{\pm 1} \subset M \setminus \{p\}$. This time, the desired inclusion follows from precise invariance of (B_1, B_{-1}) under (J_1, J_{-1}) and the fact that $p \in B_1$.

Finally, to find K_3 and K_4, we just apply the same exact arguments to the parabolic point $f^{-1}p \in \Lambda(J_{-1})$ and its stabilizer $f^{-1}Pf$, to obtain a pair of compact sets $K_3', K_4' \subset M \setminus \{f^{-1}p\}$ such that for any $z \in (\Lambda(G) \setminus \{f^{-1}p\}) \cap (L_1 \cup$

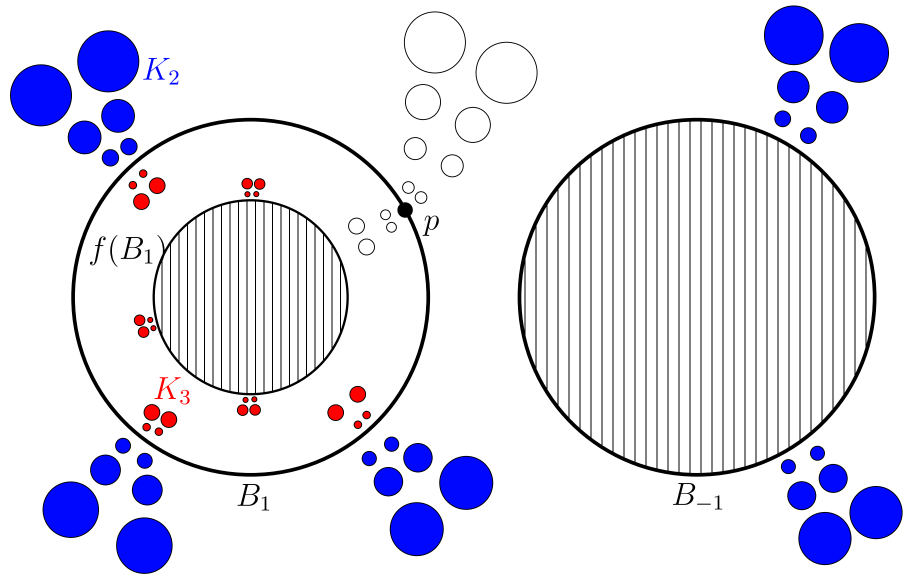

Figure 3.3.2: Illustration for Case 2 of Lemma 3.3.13. The sets B_{-1} and $f(B_1)$ are already compact subsets of $M \setminus \{p\}$, so we need to divide the rest of $\Lambda(G)$ into pieces. The sets K_1 and K_4 (not pictured) lie in $\Lambda(G_0) \setminus \{p\}$ and $f(\Lambda(G_0)) \setminus \{p\}$.

L_2), we can find $h \in P$ so that $f^{-1}hfz \in K'_3 \cup K'_4$. We can take $K_3 = fK'_3$ and $K_4 = fK'_4$ (see Figure 3.3.2). Then if $y \in (\Lambda(G) \setminus \{p\}) \cap (L_3 \cup L_4) = (\Lambda(G) \setminus \{p\}) \cap (fL_1 \cup fL_2)$, we have $y = fz$ for $z \in (\Lambda(G) \setminus \{f^{-1}p\}) \cap (L_1 \cup L_2)$, and we can find $h \in P$ so that $f^{-1}hfz \in f^{-1}K_3 \cup f^{-1}K_4$, hence $hy \in K_3 \cup K_4$. $\square$

Finally, we complete the proof of this direction of Theorem B part (iv):

Proposition 3.3.14. *If G_0 is geometrically finite, then G is geometrically finite.*

Proof. We must show that any $x \in \Lambda(G)$ is a conical limit point or bounded parabolic point. By Lemma 3.3.12, we may assume $x \in G(\Lambda(G_0))$. Since G acts by homeomorphisms, in fact we can assume that $x \in \Lambda(G_0)$. Since G_0 is geometrically finite, x is either a conical limit point or bounded parabolic point for the action of G_0 on $\Lambda(G_0)$. In the former case, x is also a conical limit point for G acting on $\Lambda(G)$, and in the latter case x is a bounded parabolic point for G acting on $\Lambda(G)$ by Lemma 3.3.13. Hence G is geometrically finite. $\square$

3.3.4 Geometrical Finiteness of G_0

Finally, we prove the other direction of Theorem B part (iv), and show that if G is geometrically finite, then so is G_0. As for the amalgamated free product case, the first step is the following:

Lemma 3.3.15. *Assume that G is geometrically finite. Let $x \in \Lambda(G_0) \setminus G_0(\Lambda(J_1) \cup \Lambda(J_{-1}))$, and suppose that $h_k \in G$ is a conical limit sequence for x. Then, after extracting a subsequence, we can find some $h \in G$ so that $h_k \in hG_0$ for every k.*

Proof. By Proposition 3.3.4 part (iii), we know $x \in A_0 \subset A$. As x is a conical limit point, we can find a conical limit sequence (h_k) for x, so that for distinct points $a, b \in M$, we have $h_k x \to a$ and $h_k z \to b$ for any $z \in M \setminus \{x\}$.

If $h_k \in G_0$ for infinitely many k then we are done, so we may assume $|h_k| \geq 1$ for every k. Suppose we can write $h_k = h'_k f g_k$ where $|h'_k| = |h_k| - 1$ (the case where $h_k = h_k f^{-1} g_k$ is similar). We note that $g_k x \in A_0 \subset A$ still since A_0 is G_0-invariant.

Consider the sequence $(h_k g_k^{-1}) = (h'_k f)$. We know that $f g_k x \in fA \subset B_1$, so $h_k x$ lies in $h'_k B_1$ for every k. If the h'_k are all in distinct left J_1-cosets in G, the sequence $(h'_k B_1)$ converges to a singleton by Lemma 3.3.8. The limit of $(h_k x)$ is contained in this singleton, so the singleton is $\{a\}$. On the other hand, since A is infinite, the set $g_k^{-1} A$ is also infinite, so there is at least one point z in $g_k^{-1} A \setminus \{x\}$. But then $h_k z \in h_k g_k^{-1} A$, so we have

$$h_k z \in h'_k f g_k g_k^{-1} A \subset h'_k f A \subset h'_k B_1.$$

This means that $(h_k z)$ converges to a, which contradicts the fact that (h_k) is a conical limit sequence for x.

So, after taking a subsequence, we must have $h'_k \in h'J_1$ for some fixed $h' \in G$. Then for every k, we have $h_k \in h'J_1 f g_k = h' f J_{-1} g_k \subset h' f G_0$, and we are done. $\square$

Proposition 3.3.16. *If G is geometrically finite, then G_0 is geometrically finite.*

Proof. We must show that any $x \in \Lambda(G_0)$ is a conical limit point or bounded parabolic point for the G_0-action. Since G is geometrically finite, x is either a conical limit point or bounded parabolic point for the G-action. In the former case, by Lemma 3.3.15 we conclude that there is a conical limit sequence of the form (hg_k) for x, where $h \in G$ and $g_k \in G_0$. Then (g_k) is a conical limit sequence for x in G_0 and we are done.

In the latter case, let $P < G$ be the stabilizer of x, a parabolic subgroup of G. We claim that in fact P is a subgroup of G_0. If $x \in \Lambda(J_1) \cup \Lambda(J_{-1})$, then x lies in either ∂B_1 or ∂B_{-1}, and then this follows from Lemma 3.3.5. And, if $x = gy$ for $y \in \Lambda(J_1) \cup \Lambda(J_{-1})$ and $g \in G_0$, then the stabilizer of x lies in $gG_0g^{-1} = G_0$. Finally, if $x \in \Lambda(G_0) \setminus G_0(\Lambda(J_1) \cup \Lambda(J_{-1}))$, then part (iii) of Proposition 3.3.4 says that $x \in A_0$, and Lemma 2.3.11 implies that no element of G with positive length can fix a point in A_0.

Now, since x is a bounded parabolic point for the G-action on $\Lambda(G)$, local compactness of $\Lambda(G) \setminus \{x\}$ implies that there is some compact $K \subset \Lambda(G)$ so that $P(K) = \Lambda(G) \setminus \{x\}$. We let $K_0 = K \cap \Lambda(G_0)$, which is a compact in $\Lambda(G_0) \setminus \{x\}$.

Using G_0-invariance (and hence P-invariance) of $\Lambda(G_0)$, we now have that

$$P(K_0) = P(K \cap \Lambda(G_0)) = P(K) \cap \Lambda(G_0) = \Lambda(G_0) \setminus \{x\}$$

as desired. $\square$

Chapter 4

An Application

In this chapter, we explore an application of the combination theorems to Kleinian groups which would produce new examples of geometric limits which are strictly larger than the corresponding algebraic limit. First, we introduce these two types of convergence, as well as the classical Jørgensen example of limits of cyclic groups. We then discuss the application and some evidence why we believe this should work, along with possible ideas for future exploration.

4.1 Algebraic and Geometric Convergence

We first define the two types of convergence we need. This exposition follows a paper by Maloni and Pozzetti [MP22], and one can also read more in Thurston [Thu79], Marden [Mar07], or Kapovich [Kap09]. The first notion is the one that is most sensible from an algebraic standpoint, hence its name.

Definition 4.1.1. Let $\rho_n : \Gamma \to \mathrm{PSL}(2, \mathbf{C})$ be a sequence of representations of a fixed group Γ. The sequence ρ_n *converges algebraically* to $\rho_\infty : \Gamma \to \mathrm{PSL}(2, \mathbf{C})$ if, for all $\gamma \in \Gamma$, the sequence $(\rho_n(\gamma))$ converges to $\rho_\infty(\gamma)$ in the topology on $\mathrm{PSL}(2, \mathbf{C})$. The representation ρ_∞ is called the *algebraic limit* of (ρ_n).

The other type of convergence we will consider is about closed subgroups of $\mathrm{PSL}(2, \mathbf{C})$, but we will only be interested in the case when said subgroups are the images of a sequence of representations.

Definition 4.1.2. Let $\rho_n : \Gamma \to \mathrm{PSL}(2, \mathbf{C})$ be a sequence of representations of a fixed group Γ. The sequence $\rho_n(\Gamma)$ of images *converges geometrically* to $\Gamma^\rho_{\mathrm{geo}}$ if the following two conditions hold:

1. for every $\gamma \in \Gamma^\rho_{\mathrm{geo}}$, there exists a sequence $\gamma_n \in \rho_n(\Gamma)$, so that $\gamma_n \to \gamma$;

2. for every subsequence $\gamma_{n_i} \in \rho_{n_i}(\Gamma)$, if $\gamma_{n_i} \to \gamma'$, then $\gamma' \in \Gamma^\rho_{\mathrm{geo}}$.

Remark 4.1.3. There is a reason for the name geometric convergence. This definition corresponds to the pointed quotient orbifolds $\mathbb{H}^3_{\mathbf{R}}/\rho_n(\Gamma)$ of the sequence converging to the pointed quotient orbifold of the geometric limit (with respect to the Gromov-Hausdorff convergence). The base-point is determined by identifying $\mathbb{H}^3$ with $\mathrm{PSL}(2, \mathbf{C})/PSU(2)$, and then considering the orbit of $[PSU(2)]$ under ρ. See Canary, Epstein, and Green [CEG86] for more details on this. Alternatively, Marden proved this is also equivalent to *polyhedral convergence*, which refers to convergence of fundamental polyhedra for the sequence of Kleinian groups. See Section 4.3 of Marden [Mar07].

Applying condition 2 above and the definition of algebraic convergence, we see that any element in $\rho_\infty(\Gamma)$ must also be contained in $\Gamma^\rho_{\mathrm{geo}}$. This gives us the following proposition.

Proposition 4.1.4. *If $\rho_n : \Gamma \to \mathrm{PSL}(2, \mathbf{C})$ converges algebraically to ρ_∞ and $\rho_n(\Gamma)$ converges geometrically to Γ^ρ_{geo}, then $\rho_\infty(\Gamma) \subset \Gamma^\rho_{geo}$.*

The other containment is not true in general, and the following definition addresses this.

Definition 4.1.5. Let $\rho_n : \Gamma \to \mathrm{PSL}(2, \mathbf{C})$ be a sequence of representations of a fixed group Γ, with algebraic limit ρ_∞, such that the images $\rho_n(\Gamma)$ converge geometrically to Γ^ρ_{geo}. We say that ρ_n *converges strongly* to ρ_∞ if $\rho_\infty(\Gamma) = \Gamma^\rho_{\text{geo}}$.

4.1.1 Examples

We now give some examples of sequences of representations where the convergence is not strong. The following example is due to Jørgensen [Jør73], and one can read more details in the notes of Canary, Epstein, and Green [CEG86] or the MSRI lecture notes of Brock. The work of Maloni and Pozzetti [MP22] generalizes this example to real and complex hyperbolic spaces of all dimensions.

Example 4.1.6 (Jørgensen [Jør73], cyclic group example)**.** Let $\omega = \frac{1}{n^2} + i\frac{\pi}{n}$. Define a representation $\rho_n : \mathbb{Z} \to \mathrm{PSL}(2, \mathbf{C})$ by

$$\rho_n(1) = g_n = \begin{pmatrix} e^{\omega_n} & n\sinh(\omega_n) \\ 0 & e^{-\omega_n} \end{pmatrix}.$$

Then

$$\lim_{n\to\infty} \rho_n(1) = g_\infty = \begin{pmatrix} 1 & i\pi \\ 0 & 1 \end{pmatrix},$$

and so this matrix generates the algebraic limit $\rho_\infty(\mathbb{Z}) = \langle g_\infty \rangle$. On the other hand, we have

$$\lim_{n\to\infty} \rho_n(n) = \lim_{n\to\infty} g_n^n = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

which shows that the convergence is not strong. This behavior arises from forcing the fixed points of $\rho_n(1)$ to both converge to ∞, while also tweaking the translation distance and rotational part of $\rho_n(1)$ in a very specific way. For a fixed $x \in \mathbb{H}^3_{\mathbf{R}}$, the elements $\rho_n(1)$ and $\rho_n(n)$ both translate x along a cone centered at the axis of $\rho_n(1)$, but in different directions. Specifically, $\rho_n(1)$ rotates x slightly around the cone (and slightly up the cone) while $\rho_n(n)$ rotates x entirely around the cone while translating upward. See Example 4.1.6

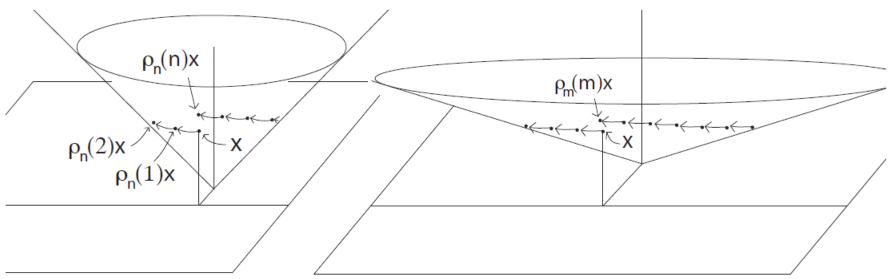

Figure 4.1.1: Illustration from lecture notes of Brock depicting the different translation directions along the cone, with $m > n$. As n increases, the cone flattens out to a horosphere through x.

As n goes to infinity, the finite fixed point of g_n approaches infinity, and the cone degenerates to a horosphere through x, where g_∞ and $\lim g_n^n$ translate in different directions, giving a geometric limit isomorphic to $\mathbb{Z}^2$.

Maloni and Pozzetti [MP22] generalize the above to construct examples of representations of the cyclic group into the groups $\mathrm{Isom}^+(\mathbb{H}^n_{\mathbf{R}})$ and $\mathrm{Isom}^+(\mathbb{H}^n_{\mathbf{C}})$ of isometries of $\mathbb{H}^n_{\mathbf{R}}$ and $\mathbb{H}^n_{\mathbf{C}}$ (complex hyperbolic spaces). They also use these examples to construct representations of the rank two free group into $\mathrm{Isom}^+(\mathbb{H}^n_{\mathbf{R}})$ and $\mathrm{Isom}^+(\mathbb{H}^n_{\mathbf{C}})$ which do not converge strongly. These generalize ideas of Thurston [Thu79], also discussed in Kapovich [Kap09], for $\mathrm{PSL}(2, \mathbf{C})$. They use the notion of a *Schottky pair*. A Schottky pair is a pair of loxodromic elements satisfying the conditions of the Klein combination theorem, as stated in Theorem 3.1.1. In Example 3.1.2, we described how to produce Schottky pairs out of any pair of loxodromic elements with distinct fixed point sets by taking large enough powers.

Example 4.1.7 (Thurston [Thu79], rank 2 free group example)**.** A Schottky pair can be constructed from $\rho_n(1) = g_n$ from Example 4.1.6 and a new element h by first showing that, for sufficiently large n, there is a disc B of radius $\frac{1}{3}$ centered at the origin which is contained in a fundamental domain for the action of $\rho_n(\mathbb{Z})$. Then, choose disjoint open discs $B_+, B_- \subset B$, and a hyperbolic element $h \in \mathrm{PSL}(2, \mathbf{C})$ with attracting (respectively repelling) fixed point in B_+ (respectively B_-), so that $h(\mathbf{C} \setminus B_-) \subset B_+$. Since B is in a fundamental domain for $\rho_n(\mathbb{Z})$, we can also find open discs for the fixed points of $\rho_n(1)$ satisfying the hypotheses in Example 3.1.2 which are disjoint from B_+ and B_-. Namely, the two boundary components of the fundamental domain containing B determine two such discs.

To construct this element explicitly, we can take B_- to be the disc of radius $\frac{1}{24}$ centered at the origin, and B_+ to be the disc of radius $\frac{1}{12}$ centered at $\frac{1}{6}$. Then one can check that the element

$$h = \begin{pmatrix} 4 & 0 \\ \frac{45}{2} & \frac{1}{4} \end{pmatrix},$$

which has fixed points 0 and $\frac{1}{6}$, maps the complement of B_- into B_+, as desired. Defining $\varphi_n : F_2 = \langle a, b \rangle \to \mathrm{PSL}(2, \mathbf{C})$ by $\varphi_n(a) = \rho_n(1)$ and $\varphi_n(b) = h$, we produce a sequence of representations with geometric limit isomorphic to $\mathbb{Z}^2 * \mathbb{Z}$, since one of the free factors becomes a $\mathbb{Z}^2$. The algebraic limit is given by $\langle \rho_\infty(1), h \rangle \cong F_2$.

We hope to apply our new combination theorems to produce examples of surface group representations with strictly larger geometric limit. In $\mathrm{PSL}(2, \mathbf{C})$, Kerckhoff and Thurston [KT90] produced such examples. We sketch their methods now. First, we need some background material.

Let S be a closed oriented surface. The *mapping class group*

$$\mathrm{Mod}(S) = \mathrm{Homeo}^+(S)/\mathrm{Homeo}_0(S)$$

of S is the group of orientation preserving homeomorphisms of S up to isotopy, where $\mathrm{Homeo}_0(S)$ is the identity component of $\mathrm{Homeo}^+(S)$. These isotopy classes are called *mapping classes*. A standard example of a mapping class is a *Dehn twist*, which we now describe. Let a be a simple closed curve on S. A representative of the Dehn twist T_a is then obtained by cutting S along a, twisting a neighborhood of one side of the curve one full rotation, and then re-gluing the surface. It turns out these mapping classes finitely generate $\mathrm{Mod}(S)$. See Farb and Margalit [FM11] for more details.

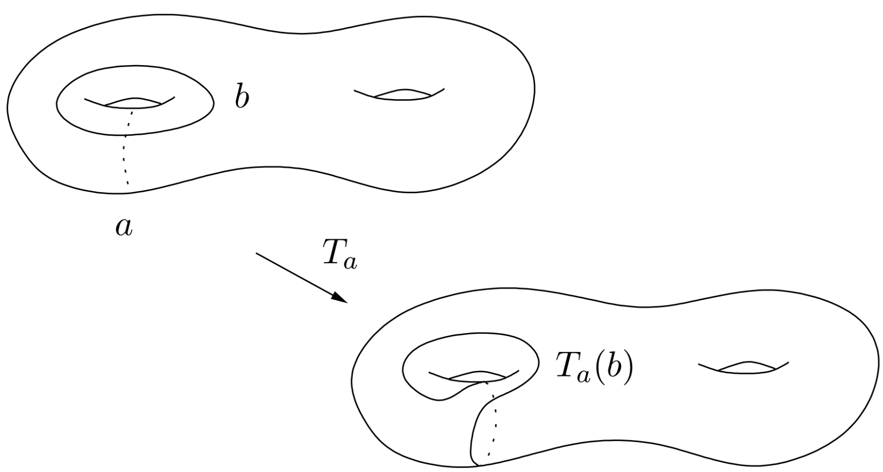

Figure 4.1.2: [FM11] Figure 3.2. Depicts the action of T_a on another simple closed curve b.

The next notion we need is that of the *Teichmüller space* $\mathcal{T}(S)$ of S. Recall that a Riemann surface X is a manifold with charts into $\mathbf{C}$ such that the transition maps are biholomorphisms. Teichmüller space is the set of pairs $\mathfrak{X} = (X, f)$, where X is a Riemann surface and $f : S \to X$ is a diffeomorphism, up to isotopy. More precisely, $(X, f) \sim (X', f')$ if $f' \circ f^{-1} : X \to X'$ is isotopic to a biholomorphism. Such pairs are called *marked Riemann surface structures* or *marked complex structures*.

The mapping class group acts on $\mathcal{T}(S)$ by changing the marking. Specifically, given the isotopy class $[g] \in \mathrm{Mod}(S)$ of a homeomorphism $g : S \to S$, we have

$$[g] \cdot [(X, f)] = [(X, f \circ g^{-1})].$$

This action is well defined since everything is defined only up to isotopy. Taking the quotient $\mathcal{T}(S)/\mathrm{Mod}(S)$ produces the set of *unmarked complex structures* on S, also known as the *moduli space* of S. Again, Farb and Margalit [FM11] is a possible reference to learn more about these topics.

The final notion we will need is that of a *quasi-Fuchsian* group. A Kleinian group G is quasi-Fuchsian if $\Lambda(G)$ is a simple closed curve in $\widehat{\mathbf{C}}$. Note that some authors only ask that $\Lambda(G)$ is contained in a simple closed curve, but we do not need this more general definition. We will let $\mathcal{QF}(S)$ be the set of discrete and faithful representations $\rho : \pi_1(S) \to \mathrm{PSL}(2, \mathbf{C})$, so that $\rho(\pi_1(S))$ is quasi-Fuchsian, up to conjugation by $\mathrm{PSL}(2, \mathbf{C})$.

Given a discrete and faithful representation $\rho : \pi_1(S) \to \mathrm{PSL}(2, \mathbf{C})$ with a quasi-Fuchsian image $G = \rho(\pi_1(S))$, the domain of discontinuity $\Omega(G) = \widehat{\mathbf{C}} \setminus \Lambda(G)$ consists of two G-invariant topological discs, B_1 and B_2. Since G acts freely and properly discontinuously on B_i, it follows that $\pi_1(B_i/G) \cong G$, and hence $B_i/G \cong S$. Note that the orientation on S induces an orientation on B_1, and the opposite orientation on B_2. Refer to S with the opposite orientation as $\overline{S}$.

Since G also acts by biholomorphisms, we get two induced marked Riemann surface structures, one on S and the other on $\overline{S}$, which we will denote $(\mathfrak{X}, \mathfrak{X}')$. Conjugating ρ does not change the class of $\mathfrak{X}$ nor $\mathfrak{X}'$, so we get a map $\mathcal{QF}(S) \to \mathcal{T}(S) \times \mathcal{T}(\overline{S})$. Bers [Ber60] proved that any pair of Riemann surface structures $(\mathfrak{X}, \mathfrak{X}') \in \mathcal{T}(S) \times \mathcal{T}(\overline{S})$ uniquely determines a point $[\rho] \in \mathcal{QF}(S)$ inducing said structures as above. This is called *Bers simultaneous uniformization*. This produces another map $\mathfrak{B} : \mathcal{T}(S) \times \mathcal{T}(\overline{S}) \to \mathcal{QF}(S)$. When one of the structures is fixed and the other is varied, we get what is called a *Bers slice*.

Example 4.1.8 (Kerckhoff, Thurston [KT90], surface group example)**.** Let $S = S_2$, the genus 2 surface. Let a be a separating curve on S_2, and T_a the corresponding Dehn twist. Fix a structure $\mathfrak{X} \in \mathcal{T}(S_2)$. Kerckhoff and Thurston considered a sequence of pairs of structures $(\mathfrak{X}, \mathfrak{X}_n)$ in the Bers slice determined by $\mathfrak{X}$, where $\mathfrak{X}_n = T_a^n \cdot \mathfrak{X}$. Bers simultaneous uniformization produces a sequence of points $\mathfrak{B}(\mathfrak{X}, \mathfrak{X}_n) \in \mathcal{QF}(S_2)$, and by taking representatives and passing to a subsequence, they show this sequence has a geometric limit strictly containing the algebraic limit. They prove this by considering the geometry and topology of the corresponding quotient 3-manifolds. In the limit, a curve in the 3-manifolds corresponding to a gets pinched (its length tends to 0), and the quotients by the algebraic and geometric limits have different topologies.

A downside to this approach is that there is no clear way to generalize this method to other spaces, such as higher dimensional real hyperbolic spaces, or complex or even quaternionic hyperbolic spaces. An approach using combination theorems to construct such examples using existing examples of representations of free groups has a chance to generalize, though.

4.2 A Conjecture

Let φ_n be as in Example 4.1.7, the general strategy is to obtain another sequence of representations by conjugating φ_n by something which commutes with the commutator, $C_n = [\varphi_n(a), \varphi_n(b)]$. This will produce representations ψ_n, where $\psi_n(F_2) \cap \varphi_n(F_2) = \langle C_n \rangle$ in $\mathrm{PSL}(2, \mathbf{C})$. The hope is that the combination theorems will then apply, allowing us to form the combination $\varphi_n(F_2) *_{\langle C_n \rangle} \psi_n(F_2)$. By standard facts from algebraic topology, this group will be isomorphic to a genus 2 surface group $\pi_1(S_2)$, and by construction, the geometric limit of these representations will strictly contain the algebraic limit.

So, we make the following conjecture.

Conjecture 4.2.1. *There is a sequence $\xi_n : \pi_1(S_2) \to \mathrm{PSL}(2, \mathbf{C})$ of representations constructed via applying a combination theorem to φ_n and ψ_n from above, such that the images $\xi_n(\pi_1(S_2))$ converge geometrically to a group Γ^{ρ}_{geo} strictly containing the algebraic limit $\xi_\infty(\pi_1(S_2))$.*

We now sketch a possible plan for proving this conjecture. We need only perform this combination for sufficiently large n, since we can then pass to a subsequence. Let $G_{1,n} = \varphi_n(F_2)$, and let $J_n = \langle C_n \rangle$, the subgroup generated by the commutator. We need a J_n-invariant simple closed curve W_n, dividing $\widehat{\mathbf{C}}$ into two closed discs $B_{1,n}$ and $B_{2,n}$, so that $\Lambda(G_{1,n}) \setminus \Lambda(J_n) \subset \mathrm{Int}(B_{2,n})$, with $\Lambda(J_n) \subset \partial B_{2,n} = W_n$. Further, we then need to show that $gB_{1,n} \subset \mathrm{Int}(B_{2,n})$ for $g \in G_{1,n} \setminus J_n$.

Given such sets, we can construct $\psi_n : F_2 \to \mathrm{PSL}(2, \mathbf{C})$ by conjugating $G_{1,n}$ by reflection in W_n. This reflection commutes with the commutator C_n, which preserves W_n, and so this conjugation preserves J_n. Let $G_{2,n} = \psi_n(F_2)$. Then, by construction, we have $\Lambda(G_{2,n}) \setminus \Lambda(J_n) \subset \mathrm{Int}(B_{1,n})$, and $gB_{2,n} \subset \mathrm{Int}(B_{1,n})$ for

$g \in G_{2,n} \setminus J_n$. This suffices to apply the combination theorem for amalgamated free products.

Now, a natural way to search for the set W_n is by considering the algebraic limits first, and finding a curve W for this group. Let $G_1 = \varphi_\infty(F_2)$, with $J = \langle C_\infty \rangle$, the subgroup generated by the commutator in the algebraic limit. A calculation shows $(\mathrm{tr}\, C_\infty)^2 \in \mathbf{R}_{>4}$, implying that C_∞ is not only loxodromic, but *hyperbolic*, i.e, C_∞ is conjugate into $\mathrm{PSL}(2, \mathbf{R})$. We can normalize so that C_∞ has repelling fixed point at -1 and attracting fixed point at 1. Then $W = \widehat{\mathbf{R}} = \mathbf{R} \cup \{\infty\}$ is J-invariant. Let B_1 be the lower half-plane in $\widehat{\mathbf{C}}$, and B_2 the upper half-plane.

We now show that the sequence can be adjusted so that we may take $W_n = \widehat{\mathbf{R}}$ for every n, and hence $B_{1,n} = B_1$ and $B_{2,n} = B_2$ for every n. When choosing the element h from before, we selected a hyperbolic element fixing 0 and $\frac{1}{6}$ which conjugates to

$$E(4) = \begin{pmatrix} 4 & 0 \\ 0 & \frac{1}{4} \end{pmatrix}.$$

In general, loxodromic elements are conjugate to

$$E(\lambda) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$$

with $|\lambda| > 1$. Hyperbolic elements correspond to $\mathrm{Im}(\lambda) = 0$. Conjugating by

$$\begin{pmatrix} 1 & 0 \\ 6 & -1 \end{pmatrix},$$

which is its own inverse and sends ∞ to $\frac{1}{6}$ and fixes 0, produces

$$h(\lambda) = \begin{pmatrix} \lambda & 0 \\ 6\lambda - 6\lambda^{-1} & \lambda^{-1} \end{pmatrix}.$$

So $h = h(4)$ with this notation, the element constructed in Example 4.1.7. Recall that $C_n = [\varphi_n(a), \varphi_n(b)]$, where

$$\varphi_n(a) = \rho_n(1) = \begin{pmatrix} e^{\omega_n} & n\sinh(\omega_n) \\ 0 & e^{-\omega_n} \end{pmatrix}$$

and $\varphi_n(b) = h$. Let $g_n = \rho_n(1)$ again to simplify notation. Our goal now is to find a sequence $\lambda_n \in \mathbf{C}$, where $\mathrm{Re}(\lambda_n) = 4$ for every n, and $\mathrm{Im}(\lambda_n) \to 0$, so that $(\mathrm{tr}[g_n, h(\lambda_n)])^2 \in \mathbf{R}$. Then, setting $\varphi_n(b) = h(\lambda_n)$ instead, we still have that $\langle g_n, h(\lambda_n) \cong F_2$ for sufficiently large n since the ping-pong dynamics are an open condition. But now, after normalizing so that C_n fixes -1 and 1, we find that C_n fixes $\widehat{\mathbf{R}}$, so in fact we can take $W_n = \widehat{\mathbf{R}}$ as desired.

These computations are messy but straightforward, but constructing an explicit λ_n seems hard. Existence is enough here, however, and we can establish this using the intermediate value theorem on the imaginary part of the trace. We should also note that the elements we conjugate by to normalize the representations are not all the same, but since they are determined by the fixed points of C_n which converge to the fixed points of C_∞, this process does not affect the limiting group either.

If we can now show that $\Lambda(G_{1,n}) \setminus \Lambda(J_n) \subset \mathrm{Int}(B_2)$, and $gB_1 \subset \mathrm{Int}(B_2)$ for $g \in G_{1,n} \setminus J$ for sufficiently large n, then letting $G_{2,n}$ be obtained from $G_{1,n}$ by conjugating by reflection in $\widehat{\mathbf{R}}$ (complex conjugation), the combination theorem will apply just as above. Alternatively, one could show these statements for just G_1, and then try to argue that the corresponding statements will hold for $G_{1,n}$ for sufficiently large n.

There was not enough time to finish this final step, but some pictures of limit sets generated in Python give promising evidence that the desired properties hold. We now show these pictures and discuss how they were generated.

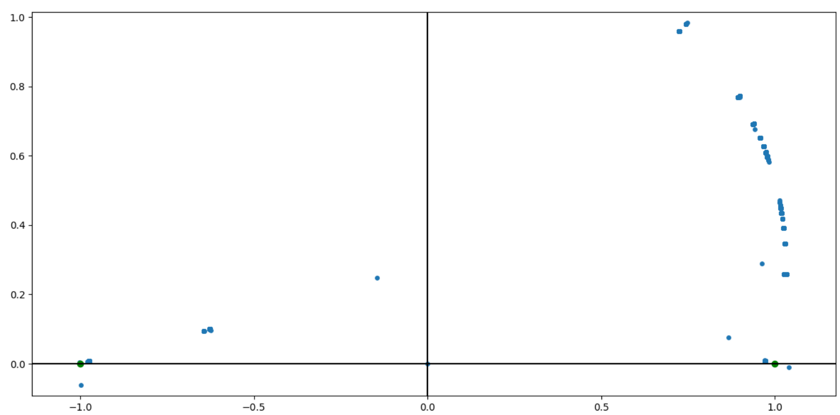

Figure 4.2.1: Approximation of $\Lambda(G_1)$, obtained by applying elements to fixed points of the commutator.

Figure 4.2.1 depicts an approximation of the limit set of G_1, the limit group isomorphic to F_2. The picture was produced by first normalizing so the commutator fixes 1 and -1, and then applying a large number of elements to these fixed points and plotting the results. Since translates of limit points are again limit points, and the closure of any infinite G_1-orbit gives $\Lambda(G_1)$, we can approximate the limit set by plotting a large number of such translates. The fixed points of the commutator at 1 and -1 are plotted in green.

The code first produces all possible words in F_2 up to a given length using a form of recursion called backtracking, and then retrieves the corresponding elements of G_1 by multiplying the generators together. Finally, these are applied to the commutator's fixed points and displayed in a scatter plot using the popular Python package MatPlotLib.

There appear to be some isolated points beneath the real axis, but these are believed to be errors arising from the large number of numerical computations. No limit points can be isolated in all of $\Lambda(G_1)$, and these points remain isolated even when increasing the length of the words as high as 12. The clusters of points which accumulate in various places all seem to be happening above the real axis, as desired. The following was obtained by zooming in to the commutator's fixed point at 1.

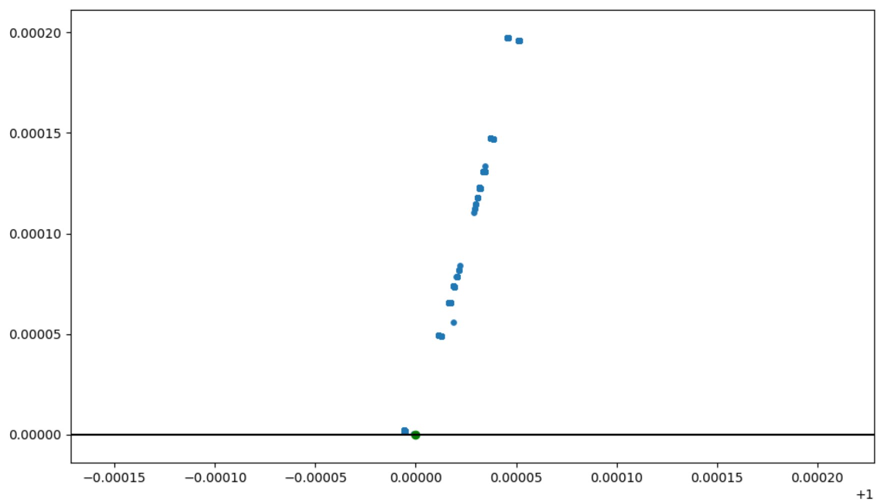

Figure 4.2.2: $\Lambda(G_1)$ approximated via translates of a fixed point, near the fixed point 1 of the commutator.

Another way to approximate the limit set is by plotting the centers of isometric circles of elements. Recall Definition 2.1.10 for the definition of the isometric circle of an element in $\mathrm{PSL}(2, \mathbf{C})$, and see the short paper of Ford [For29] for more details on why this method works. An upside to this method is that we need fewer computations in the code, reducing the approximation error. These pictures do not have the isolated points occurring beneath $W = \widehat{\mathbf{R}}$, which is another positive sign. The second figure is again zoomed in quite a bit. We also get a natural "depth" associated to each center of an isometric circle, depending on the length of the corresponding word. Different colors are used to depict this.

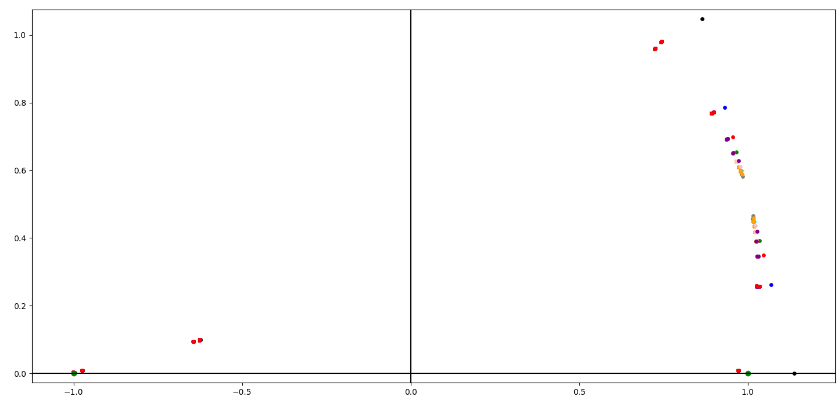

Figure 4.2.3: Approximation of $\Lambda(G_1)$, obtained by plotting centers of isometric circles. Each color corresponds to a fixed word length.

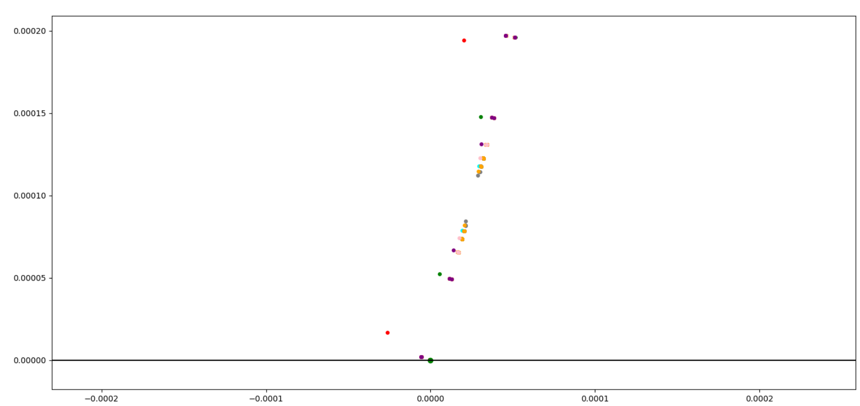

Figure 4.2.4: $\Lambda(G_1)$ approximated via centers of isometric circles, near the fixed point 1 of the commutator.

With some additional work, this construction could be performed in $\mathbb{H}^n_{\mathbf{R}}$ as well, which would produce truly new examples. With the right computational tools, a similar construction could work in $\mathbb{H}^2_{\mathbf{C}}$, the complex hyperbolic plane, for which no such examples exist yet.

Chapter 5

Appendix

In this appendix we provide the Python code used to generate the pictures in Chapter 4. To use the widget at the start for MatPlotLib, one should run this code in a Jupyter notebook. This first code is for translating the fixed points of the commutator via all elements up to a given length.

```
import numpy as np
import matplotlib.pyplot as plt

def apply_map(mat,z):
    """Applies a matrix to a complex number and returns the result as a pair."""
    return (np.real((mat[0,0] * z + mat[0,1]) / (mat[1,0] * z + mat[1,1])),
            np.imag((mat[0,0] * z + mat[0,1]) / (mat[1,0] * z + mat[1,1])))

def get_fixed_points(mat):
    """Returns the fixed points of an element."""
    return [(mat[0,0] - mat[1,1] + np.sqrt((mat[0,0] + mat[1,1]) ** 2 - 4)) / (2 * mat[1,0]),
            (mat[0,0] - mat[1,1] - np.sqrt((mat[0,0] + mat[1,1]) ** 2 - 4)) / (2 * mat[1,0])]

def get_words(n, letters):
    """Returns words of length n in a list of lists. Works for free group of rank 2."""
    results = []

    def backtrack(path, index):
        if len(path) == n:
            results.append(path.copy())
            return

        for letter in letters:
            if len(path) == 0:
                path.append(letter.copy())
                backtrack(path, index + 1)
```

```python
                path.pop()
            elif (
                (np.array_equal(path[index-1], A) and np.array_equal(letter, A_inv))
                or (np.array_equal(path[index-1], A_inv) and np.array_equal(letter, A))
                or (np.array_equal(path[index-1], B) and np.array_equal(letter, B_inv))
                or (np.array_equal(path[index-1], B_inv) and np.array_equal(letter, B))
                ):
                continue
            else:
                path.append(letter.copy())
                backtrack(path, index + 1)
                path.pop()

    backtrack([], 0)
    return results

# Swaps 1 and infinity, is its own inverse
J = np.array([[1,0],[1,-1]])

# First generator
G = np.array([[1, np.pi * 1j], [0,1]])

# Second generator
H = np.array([[4,0],[45/2,1/4]])

# Commutator
Comm = np.linalg.multi_dot([G,H,np.linalg.inv(G),np.linalg.inv(H)])

# Normalized elements so no fixed points are at infinity
NG = np.linalg.multi_dot([J,G,J])
NH = np.linalg.multi_dot([J,H,J])
NComm = np.linalg.multi_dot([J,Comm,J])

# Normalized so commutator fixes 1, -1
J1 = np.array([[1,-get_fixed_points(NComm)[0]],[1,-get_fixed_points(NComm)[1]]])
J2 = np.array([[1,1],[1,-1]])
N = np.linalg.multi_dot([np.linalg.inv(J2),J1])

NNG = np.linalg.multi_dot([N,NG,np.linalg.inv(N)])
NNH = np.linalg.multi_dot([N,NH,np.linalg.inv(N)])
NNComm = np.linalg.multi_dot([N,NComm,np.linalg.inv(N)])

# Names for generators
A = NNG
A_inv = np.linalg.inv(A)
B = NNH
B_inv = np.linalg.inv(B)

letters = [A,A_inv,B,B_inv]

# Initializes list for words
word_list = []

# Adds words of a given length
```

```python
for i in range(2,10):
    word_list += get_words(i, letters)

# Initializes lists for x and y coords
x_coords = []
y_coords = []

# Applies words to fixed points of the commutator
lp1 = get_fixed_points(NNComm_lim)[0]
lp2 = get_fixed_points(NNComm_lim)[1]

for i in range(len(letters)):
    x, y = apply_map(letters[i], lp1)
    x_coords.append(x)
    y_coords.append(y)

for i in range(len(letters)):
    x, y = apply_map(letters[i], lp2)
    x_coords.append(x)
    y_coords.append(y)

for i in range(len(word_list)):
    x, y = apply_map(np.linalg.multi_dot(word_list[i]), lp1)
    x_coords.append(x)
    y_coords.append(y)

for i in range(len(word_list)):
    x, y = apply_map(np.linalg.multi_dot(word_list[i]), lp2)
    x_coords.append(x)
    y_coords.append(y)

fig, ax = plt.subplots()

# Plots limit points
ax.scatter(x_coords, y_coords, s=15)

# Plot commutator fixed points
ax.plot(np.real(get_fixed_points(NNComm_lim)[0]), np.imag(get_fixed_points(NNComm_lim)[0]),
        'go', label='marker_only', markersize=6)

ax.plot(np.real(get_fixed_points(NNComm_lim)[1]), np.imag(get_fixed_points(NNComm_lim)[1]),
        'go', label='marker_only', markersize=6)

plt.axis("equal")
plt.axvline(0, color='black')
plt.axhline(0, color='black')
fig.tight_layout()
plt.show()
```

The following code is for plotting centers of isometric circles, with colors determined by the length of the corresponding word.

```python
import numpy as np
import matplotlib.pyplot as plt
```

```python
import pandas as pd

def get_center(mat):
"""Returns center of isometric circle."""
return (np.real(-mat[1,1] / mat[1,0]), np.imag(-mat[1,1] / mat[1,0]))

def get_words(n, letters):
    """Returns words of length n in a list of lists. Works for free group of rank 2."""
    results = []

    def backtrack(path, index):
        if len(path) == n:
            results.append(path.copy())
            return

        for letter in letters:
            if len(path) == 0:
                path.append(letter.copy())
                backtrack(path, index + 1)
                path.pop()
            elif (
                (np.array_equal(path[index-1], A) and np.array_equal(letter, A_inv))
                or (np.array_equal(path[index-1], A_inv) and np.array_equal(letter, A))
                or (np.array_equal(path[index-1], B) and np.array_equal(letter, B_inv))
                or (np.array_equal(path[index-1], B_inv) and np.array_equal(letter, B))
                ):
                continue
            else:
                path.append(letter.copy())
                backtrack(path, index + 1)
                path.pop()

    backtrack([], 0)
    return results

# Swaps 1 and infinity, is its own inverse
J = np.array([[1,0],[1,-1]])

# First generator
G = np.array([[1, np.pi * 1j], [0,1]])

# Second generator
H = np.array([[4,0],[45/2,1/4]])

# Commutator
Comm = np.linalg.multi_dot([G,H,np.linalg.inv(G),np.linalg.inv(H)])

# Normalized elements so no fixed points are at infinity
NG = np.linalg.multi_dot([J,G,J])
NH = np.linalg.multi_dot([J,H,J])
NComm = np.linalg.multi_dot([J,Comm,J])

# Normalized so commutator fixes 1, -1
J1 = np.array([[1,-get_fixed_points(NComm)[0]],[1,-get_fixed_points(NComm)[1]]])
```

```
J2 = np.array([[1,1],[1,-1]])
N = np.linalg.multi_dot([np.linalg.inv(J2),J1])

NNG = np.linalg.multi_dot([N,NG,np.linalg.inv(N)])
NNH = np.linalg.multi_dot([N,NH,np.linalg.inv(N)])
NNComm = np.linalg.multi_dot([N,NComm,np.linalg.inv(N)])

# Names for generators
A = NNG
A_inv = np.linalg.inv(A)
B = NNH
B_inv = np.linalg.inv(B)

letters = [A,A_inv,B,B_inv]

# Initializes list for words
word_list = []

# Adds words of a given length
for i in range(2,10):
    word_list += get_words(i, letters)

# Initializes a discrete color map
color_map = ['black', 'blue', 'red',
'green', 'purple', 'pink', 'cyan', 'orange', 'grey']

# Initializes lists for x and y coords
x_coords = []
y_coords = []

# Gets centers of starting letters
for i in range(len(letters)):
    x, y = get_center(letters[i])
    x_coords.append(x)
    y_coords.append(y)
    colors.append(color_map[0])

# Gets centers of words
for i in range(len(word_list)):
    x, y = get_center(np.linalg.multi_dot(word_list[i]))
    x_coords.append(x)
    y_coords.append(y)
    colors.append(color_map[len(word_list[i])-1])

# Dataframe with points to scatter and colors corresponding to word length
df = pd.DataFrame(data={'x':x_coords, 'y':y_coords, 'c':colors})

fig, ax = plt.subplots()

# Plots points with their corresponding color
for g,b in df.groupby(by='c'):
    plt.scatter(b['x'],b['y'],color=g,s=15)

# Plot commutator fixed points
```

```
ax.plot(np.real(get_fixed_points(NNComm_lim)[0]), np.imag(get_fixed_points(NNComm_lim)[0]),
        'go', label='marker only', markersize=6)

ax.plot(np.real(get_fixed_points(NNComm_lim)[1]), np.imag(get_fixed_points(NNComm_lim)[1]),
        'go', label='marker only', markersize=6)

plt.axis("equal")
plt.axvline(0, color='black')
plt.axhline(0, color='black')
fig.tight_layout()
plt.show()
```

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