Decomposing the classifying diagram in terms of classifying spaces of groups

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Abstract

The classifying diagram was defined by Rezk and is a generalization of the nerve of a category; in contrast to the nerve, the classifying diagram of two categories is equivalent if and only if the categories are equivalent. In this thesis we prove that the classifying diagram of any category is characterized in terms of classifying spaces of stabilizers of groups. We also prove explicit decompositions of the classifying diagrams for the categories of finite ordered sets, finite dimensional vector spaces, and finite sets in terms of classifying spaces of groups. For the classification diagram, which was defined by Rezk to work with categories with weak equivalences, we prove analogous results that were previously known for the classifying diagram. We close by comparing the classifying and classification diagrams, highlighting the differences and challenges of working with categories that have weak equivalences.

To my parents
in honor of their 40th wedding anniversary

Contents

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1. Introduction

A topological space can be built from a category using the machinery of the nerve, which takes objects in the category to points and chains of n-composable morphisms to n-cells. The resulting space is referred to as the "classifying space" of the category. However, two categories that are not equivalent can produce equivalent classifying spaces because the nerve does not place any value on the data that comes from a morphism being an isomorphism. (See Example 2.15 below).

The classifying diagram, which is a generalization of the nerve, is an alternative machine that can be used to study categories. The classifying diagram respects the data that isomorphisms provide, and as a result, the classifying diagram of two categories are equivalent if and only if the categories are equivalent [2, 3.3.4]. The classifying diagram was defined by Rezk in [13]. Rezk uses complete Segal spaces as a model for homotopy theory. The classifying diagram of a category is a natural occurrence of a complete Segal space. (See Proposition 3.2 below.)

The purpose of this thesis is to provide a deeper understanding of the classifying diagram. We consider specific well-known categories, such as the category of finite ordered sets, finite dimensional vector spaces, and finite sets; we describe the resulting classifying diagrams in terms of classifying spaces of groups. In the process, we prove the valuable result that for a general category we can characterize the classifying diagram in terms of classifying spaces of stabilizers.

We also compare the classifying diagram with the classification diagram, which was defined by Rezk [13]. The classification diagram respects the weak equivalences in the category in the same way the classifying diagram respects isomorphisms. The classification diagram has results that are analogous to the classifying diagram. (Compare Propositions 3.4 and 3.5 with Propositions 6.4 and 6.5.) We also show that there is a subtlety between respecting isomorphisms and weak equivalences.

1.1. **Future work.** There are some results presented in this thesis we wish to expound on in the future. In particular, we believe that the results for the classifying diagram of the category of finite vector spaces in Section 5 can be furthered. In Section 6.3 we work with the classification diagram for the connected model structure on graphs. There are several other model structures on the category of graphs in [4]. Some future work ideas are to compare the classification diagrams for the different models and see if a model leads to a new natural approach for defining homotopy automorphisms.

Biedermann and Röndigs' use model categories and localization as the foundational tools in their perspective on functor calculus [3]. Another research goal is to use the classification diagram to study and compare Biedermann and Röndigs' model categories. In particular we can consider the following questions.

Questions 1.1. Are there nice descriptions of the classification diagrams of Biedermann and Röndigs' model categories? How does Biedermann and Röndigs' use of localization affect the classification diagram?

The classification diagram is also used as a machine that takes categories into complete Segal spaces. Unlike model categories, there is a nice notion of convergence in complete

Segal spaces. Unveiling answers to the above questions will help guide the way to approaching the following deeper questions.

Questions 1.2. Does implementing the classification diagram on the Biedermann and Röndigs' model categories give a nice notion of functor calculus in complete Segal spaces? What is the appropriate notion of the cross effect? What is the correct notion of a derivative? Is there a Taylor tower? If so, what do the layers look like? Does the Taylor tower converge?

1.2. **Organization of the paper.** We begin in Section 2 by recalling relevant category theory tools, the definition and basic structure of simplicial sets, the nerve of a category, and the definition of complete Segal spaces. In Section 3, we provide Rezk's definition for the classifying diagram of a category, explain why the classifying diagram is a complete Segal space, provide some preliminary examples of the classifying diagram, and prove a characterization of the classifying diagram using stabilizers of groups. In Sections 4 and 5 we address the classifying diagram of the categories of finite dimensional vector spaces and finite sets, respectively. We close the paper in Section 6 by providing Rezk's definition of the classification diagram, proving results analogous to what is known about the classifying diagram, and comparing the classification diagram's behavior with the classifying diagram.

2. Background

In this section we review some relevant category theory tools and provide an overview of simplicial sets, simplicial spaces, and complete Segal spaces.

2.1. **Category theory tools.** We recall some of the category theory tools that will be used throughout this paper. In particular, we review definitions and relevant results for natural transformations, equivalent categories, and diagram categories.

The data from a natural transformation $\eta : F \Rightarrow G$ between two functors $F, G : \mathcal{C} \to \mathcal{D}$ can be equivalently packaged as a functor $\eta : \mathcal{C} \times \{0 \to 1\} \to \mathcal{D}$. First, recall the definition of a natural transformation.

Definition 2.1. A *natural transformation* $\eta : F \Rightarrow G$ is a function that assigns to each object c in $\mathcal{C}$ a morphism $\eta_c : F(c) \to G(c)$ of $\mathcal{D}$ in such a way that for every morphism $f : c \to c'$ of $\mathcal{C}$, the diagram

$$\begin{array}{ccc} F(c) & \xrightarrow{\eta_c} & G(c) \\ {\scriptstyle F(f)}\big\downarrow & & \big\downarrow{\scriptstyle G(f)} \\ F(c') & \xrightarrow{\eta_{c'}} & G(c') \end{array}$$

commutes.

Instead of using the labeling for the category with two objects and one nontrivial morphism $\{0 \to 1\}$, we suggestively use $\{F \xrightarrow{\eta} G\}$. First we show that given a natural transformation $\eta : F \Rightarrow G$, we can obtain a functor $\{F \xrightarrow{\eta} G\} \times \mathcal{C} \to \mathcal{D}$. We define the evaluation functor $\mathrm{ev} : \{F \xrightarrow{\eta} G\} \times \mathcal{C} \to \mathcal{D}$ by $\mathrm{ev}(F, c) = F(c)$ and $\mathrm{ev}(G, c) = G(c)$. The diagram

$$\begin{array}{ccc} (F, c) & \xrightarrow{(\eta, \mathrm{id}_c)} & (G, c) \\ {\scriptstyle (\mathrm{id}_F, f)}\big\downarrow & & \big\downarrow{\scriptstyle (\mathrm{id}_G, f)} \\ (F, c') & \xrightarrow{(\eta, \mathrm{id}_{c'})} & (G, c') \end{array}$$

commutes in the category $\{F \xrightarrow{\eta} G\} \times C$. Since functors preserve composition, applying the evaluation functor ev to the above diagram gives us the same diagram from the definition of natural transformations. In this manner, we obtain the functor ev : $\{F \xrightarrow{\eta} G\} \times C \to \mathcal{D}$ from a natural transformation $\eta : F \Rightarrow G$.

The converse is also true. Meaning given functors $F, G : C \to \mathcal{D}$ and ev: $\{F \xrightarrow{\eta} G\} \times C \to \mathcal{D}$ where $\text{ev}(F, c) = F(c)$ and $\text{ev}(G, c) = G(c)$, then we obtain a natural transformation $\eta : F \Rightarrow G$. To see that the converse is true, let $f : c \to c'$ be a morphism in C. Then the square

$$\begin{array}{ccc} (F, c) & \xrightarrow{(\eta, \text{id}_c)} & (G, c) \\ {\scriptstyle (\text{id}_F, f)}\downarrow & & \downarrow{\scriptstyle (\text{id}_G, f)} \\ (F, c') & \xrightarrow{(\eta, \text{id}_{c'})} & (G, c') \end{array}$$

commutes in the category $\{F \xrightarrow{\eta} G\} \times C$. Applying the functor ev to the above diagram gives the commutative diagram

$$\begin{array}{ccc} F(c) & \xrightarrow{\text{ev}(\eta, \text{id}_c)} & (G, c) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ (F, c') & \xrightarrow{\text{ev}(\eta, \text{id}_{c'})} & (G, c') \end{array}$$

in $\mathcal{D}$. Thus if we define a function that assigns to each object c of (C) the morphism $\eta_c := \text{ev}(\eta, id_c) : F(c) \to G(c)$ in $\mathcal{D}$, we construct a natural transformation $\eta : F \Rightarrow G$.

For functors $F, G : C \to \mathcal{D}$, we say that a natural transformation $\eta : F \Rightarrow G$ is a *natural isomorphism* if the morphism $\eta_c : F(c) \to G(c)$ in $\mathcal{D}$ is an isomorphism for every object c of C.

Definition 2.2. Categories C and $\mathcal{D}$ are *equivalent categories* if there exist functors $F : C \to \mathcal{D}$ and $G : \mathcal{D} \to C$ as well as natural isomorphisms $G \circ F \cong id_C$ and $F \circ G \cong id_{\mathcal{D}}$.

We can alternatively determine if two categories are equivalent using the following definitions.

Definition 2.3. Let $F : C \to \mathcal{D}$ be a functor.

(i) If the function between hom-sets $F : \text{Hom}_C(c, c') \to \text{Hom}_{\mathcal{D}}(F(c), F(c'))$ is injective for any objects $c, c' \in C$, then F is *faithful*.
(ii) If the function between hom-sets $F : \text{Hom}_C(c, c') \to \text{Hom}_{\mathcal{D}}(F(c), F(c'))$ is surjective for any objects $c, c' \in C$, then F is *full*.
(iii) If for any object $d \in \mathcal{D}$ there exists an object $c \in C$ such that there is an isomorphism $F(c) \xrightarrow{\cong} d$ in $\mathcal{D}$, then F is *essentially surjective*.

The following result says that there are necessary and sufficient requirements for a functor to define an equivalence of categories.

Proposition 2.4. [11, IV.4.1] *The categories C and $\mathcal{D}$ are equivalent if and only if there exists a functor $F : C \to \mathcal{D}$ that is faithful, full, and essentially surjective.*

We can use categories to define new categories. If the collection of objects in the category $\mathcal{D}$ form a set, then we say that $\mathcal{D}$ is a *small* category.

Example 2.5. Let C be a category and let $\mathcal{D}$ be a small category.

(i) The *opposite category* of C, denoted C^{op}, is the category where $\text{ob}(C) = \text{ob}(C^{op})$ and $f^{op} : b \to a$ is a morphism in C^{op} if and only if $f : a \to b$ is a morphism in C.
(ii) The *functor category* $C^{\mathcal{D}}$ is the category whose objects are functors $F : \mathcal{D} \to C$ and the morphisms are natural transformations. We also refer to $C^{\mathcal{D}}$ as a *diagram category*.

Let $[n]$ be the category consisting of a chain of n composable morphisms

$$0 \to 1 \to 2 \to \cdots \to n.$$

The diagram category $C^{[n]}$ is of particular importance throughout this paper. An object of $C^{[n]}$ is a chain of n composable morphisms in C

$$x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} x_2 \xrightarrow{f_3} \cdots \xrightarrow{f_n} x_n$$

and a morphism from $(f_1, \ldots, f_n)$ to $(g_1, \ldots, g_n)$ is an $(n+1)$-tuple of morphisms $(\alpha_0, \alpha_1, \ldots, \alpha_n)$, where each α_i is a morphism in C, making the diagram

$$\begin{array}{ccccccccc}
x_0 & \xrightarrow{f_1} & x_1 & \xrightarrow{f_2} & x_2 & \xrightarrow{f_3} & \cdots & \xrightarrow{f_n} & x_n \\
\downarrow{\scriptstyle \alpha_0} & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_2} & & & & \downarrow{\scriptstyle \alpha_n} \\
y_0 & \xrightarrow{f_1} & y_1 & \xrightarrow{f_2} & y_2 & \xrightarrow{f_3} & \cdots & \xrightarrow{f_n} & y_n
\end{array}$$

commute in C. If each α_i is an isomorphism in C, then $(\alpha_0, \ldots, \alpha_n)$ is an isomorphism in $C^{[n]}$.

Proposition 2.6. *Let $\mathcal{D}$ and $\mathcal{E}$ be equivalent small categories. Also let C be a category. Then the functor categories $C^{\mathcal{D}}$ and $C^{\mathcal{E}}$ are equivalent.*

Proof. Let $F : \mathcal{D} \to \mathcal{E}$ and $G : \mathcal{E} \to \mathcal{D}$ be functors such that $G \circ F \cong \text{id}_{\mathcal{D}}$ and $F \circ G \cong \text{id}_{C}$. Define $\overline{F} : C^{\mathcal{E}} \to C^{\mathcal{D}}$ by $\overline{F}(f) = f \circ F$, and define $\overline{G} : C^{\mathcal{D}} \to C^{\mathcal{E}}$ by $\overline{G}(g) = g \circ G$. Then

$$\overline{G} \circ \overline{F}(f) = \overline{G}(f \circ F) = (f \circ F) \circ G = f \circ (F \circ G) \cong g \circ \text{id}_{\mathcal{E}} = g$$

and similarly $\overline{F} \circ \overline{G}(g) \cong g$. Thus $\overline{F}$ and $\overline{G}$ define the desired equivalence of categories. □

Proposition 2.7. *Let C and $\mathcal{D}$ be equivalent categories and let $\mathcal{E}$ be a small category. Then the functor categories $C^{\mathcal{E}}$ and $\mathcal{D}^{\mathcal{E}}$ are equivalent.*

The proof of the above proposition is similar to the proof of Proposition 2.6.

2.2. **Simplicial sets.**

A brief account of the definition of simplicial sets and a description of its model category structure are included here.

Before we can provide the definition of a simplicial set, we define the categories `Set` and Δ. Let `Set` denote the category whose objects are sets and morphisms are functions. Let Δ be the category whose objects are finite ordered sets, denoted as $[n] = \{0 \leq 1 \leq 2 \leq \cdots \leq n\}$, and the morphisms are order-preserving functions.

Definition 2.8. A *simplicial set* is a functor from $\Delta^{op} \to$ `Set`.

If X is is a simplicial set, we denote its geometric realization by $|X|$ [8, §I.2]. The category of simplicial sets, which we denote by `SSet`, has simplicial sets as objects and natural transformations for morphisms.

Example 2.9. The *standard n-simplex* is the simplicial set $\Delta[n] := \mathrm{Hom}_\Delta(-, [n])$. The geometric realization $|\Delta[n]|$ is a n-cell.

The data from a simplicial set X can be rewritten in terms of sets $X([n]) =: X_n$ along with maps

$$\begin{array}{lll} d_i : X_n \to X_{n-1}, & 0 \le i \le n & \text{(face maps)} \\ s_j : X_n \to X_{n+1}, & 0 \le j \le n & \text{(degenercy maps)} \end{array}$$

which satisfy simplicial identities [8, §I.1]. We say that X_n is the nth level of the simplicial set X. By the Yoneda Lemma, the set $X_n \cong \mathrm{Hom}_{\mathtt{SSet}}(\Delta[n], X)$.

There is a model structure for SSet where a map $f : X \to Y$ is a weak equivalence if the induced map from geometric realization, $|f| : |X| \to |Y|$, is a weak homotopy equivalence [8, Ch. 1]. An expository account for model categories may be found in [6]. The following proposition says that the hom-sets in SSet have the structure of a simplicial set.

Proposition 2.10. [8, II.2.2] *The category SSet is enriched in SSet. That is, given any two simplicial sets X and Y, the hom-set* $Hom_{\mathtt{SSet}}(X, Y)$ *is a simplicial set.*

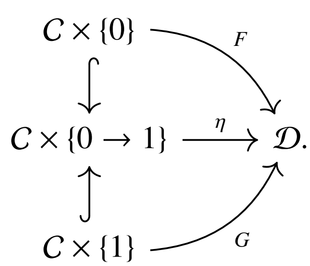
2.3. **The nerve of a category.** In this section, we see that the nerve of a category gives a simplicial set. Using the data that a natural transformation encodes, we show that nerves of categories are weakly equivalent if there exists functors between the categories with appropriate natural transformations with the identity functors.

Definition 2.11. The *nerve* of a category C is the simplicial set defined levelwise by

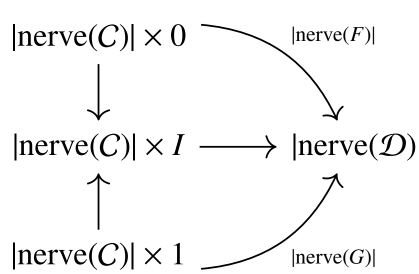
$$\mathrm{nerve}(C)_n := \mathrm{Hom}_{Cat}([n], C).$$

Proposition 2.12. *Given two functors* $F, G : C \to \mathcal{D}$ *and a natural transformation* $\eta : F \Rightarrow G$*, there exists an induced homotopy* $|nerve(F)| \simeq |nerve(G)|$.

Proof. As described in Section 2.1, the notion of a natural transformation $\eta : F \Rightarrow G$ equivalent to a functor $\eta : C \times \{0 \to 1\} \to \mathcal{D}$. More specifically, we can think of $\eta : F \Rightarrow G$ as giving the commutative diagram



The functor $|\mathrm{nerve}(-)|$ can now be applied to this diagram. Note that $|\mathrm{nerve}(\{0 \to 1\})| \simeq I$ where I is the unit interval $[0, 1]$. Also note that $|\mathrm{nerve}(C \times \{0 \to 1\})| \simeq |\mathrm{nerve}(C)| \times I$ because $|\mathrm{nerve}(-)|$ preserves products [9, 14.1.5, 13.1.12]. As a result, the diagram



commutes and hence $|\text{nerve}(\eta)|$ induces the desired homotopy, $|\text{nerve}(F)| \simeq |\text{nerve}(G)|$. □

Corollary 2.13. *If the categories C and $\mathcal{D}$ are equivalent, then the simplicial sets nerve(C) and nerve$(\mathcal{D})$ are weakly equivalent in the model structure for* `SSet`.

Proof. Suppose C and $\mathcal{D}$ are equivalent categories. Then there exist functors $F : C \to \mathcal{D}$ and $G : \mathcal{D} \to C$ such that $G \circ F \cong \text{id}_C$ and $F \circ G \cong \text{id}_{\mathcal{D}}$. By Proposition 2.12, the diagrams

$$\begin{array}{ccc} |\text{nerve}(C)| \times \{0\} & \xrightarrow{|\text{nerve}(G \circ F)|} & \\ \downarrow & & \searrow \\ |\text{nerve}(C)| \times I & \longrightarrow & |\text{nerve}(C)| \\ \uparrow & & \nearrow \\ |\text{nerve}(C)| \times \{1\} & \xrightarrow{|\text{nerve}(\text{id}_C)|} & \end{array} \qquad \begin{array}{ccc} |\text{nerve}(\mathcal{D})| \times \{0\} & \xrightarrow{|\text{nerve}(F \circ G)|} & \\ \downarrow & & \searrow \\ |\text{nerve}(\mathcal{D})| \times I & \longrightarrow & |\text{nerve}(\mathcal{D})| \\ \uparrow & & \nearrow \\ |\text{nerve}(\mathcal{D})| \times \{1\} & \xrightarrow{|\text{nerve}(\text{id}_{\mathcal{D}})|} & \end{array}$$

commute. It should be noted that, for example, $|\text{nerve}(G \circ F)| = |\text{nerve}(G)| \circ |\text{nerve}(F)|$ and $|\text{nerve}(id_C)| = id_{|\text{nerve}(C)|}$. So we have homotopies $|\text{nerve}(G \circ F)| \simeq id_{|\text{nerve}(C)|}$ and $|\text{nerve}(F \circ G)| \simeq id_{|\text{nerve}(\mathcal{D})|}$ and hence we have a homotopy equivalence between $|\text{nerve}(C)|$ and $|\text{nerve}(\mathcal{D})|$. Thus nerve$(C)$ is weakly equivalent to nerve$(\mathcal{D})$ in the model structure for `SSet`. □

There is a more general result.

Proposition 2.14. *Given functors $F : C \to \mathcal{D}$ and $G : \mathcal{D} \to C$ with natural transformations $\eta : G \circ F \Rightarrow \text{id}_C$ and $\theta : F \circ G \Rightarrow \text{id}_{\mathcal{D}}$, nerve$(C)$ is weakly equivalent to nerve$(\mathcal{D})$ in the model structure for* `SSet`.

Proof. By Proposition 2.12, we have homotopies $|\text{nerve}(G \circ F)| \simeq \text{id}_{|\text{nerve}(C)|}$ and $|\text{nerve}(F \circ G)| \simeq \text{id}_{|\text{nerve}(\mathcal{D})|}$. Thus $|\text{nerve}(C)|$ is homotopy equivalent to $|\text{nerve}(\mathcal{D})|$ and hence nerve(C) is weakly equivalent to nerve$(\mathcal{D})$ in `SSet`. □

Note that the direction of the natural transformations η and θ had no effects on the proof for Proposition 2.14.

2.4. **A motivating example.** In Proposition 2.14 the categories C and $\mathcal{D}$ are not required to be equivalent in order for nerve(C) and nerve$(\mathcal{D})$ to be weakly equivalent in `SSet`. In fact, as we see in the following example, two categories can have weakly equivalent nerves even if the categories are not equivalent.

Example 2.15. Let C be the category with one nontrivial morphism between two objects, $f : x \to y$, and let $\mathcal{D}$ be the subcategory containing just the object x and its identity morphism. Let $F : C \to \mathcal{D}$ be the functor sending every object of C to x and every morphism to the identity morphism on x. Let $G : \mathcal{D} \to C$ be the natural inclusion. By construction, $F \circ G$ is the identity functor $\text{id}_{\mathcal{D}}$. Now we want to construct a natural transformation between $G \circ F$ and the identity functor id_C. Note that $G \circ F(f) = \text{id}_x$. We can construct a natural transformation $\eta : G \circ F \Rightarrow \text{id}_C$ by letting $\eta_x := \text{id}_x$ and $\eta_y := f$. Thus, by Proposition 2.14, the nerves of C and $\mathcal{D}$ are weakly equivalent. But we claim that the categories C and $\mathcal{D}$ are not equivalent.

To see that C and $\mathcal{D}$ are not equivalent, note that the functor $F : C \to \mathcal{D}$ is the only possible functor we can construct going from C to $\mathcal{D}$. Note that $\text{Hom}_C(y, x)$ is the empty set, but $\mathrm{Hom}_{\mathcal{D}}(F(y), F(x)) = \mathrm{Hom}_{\mathcal{D}}(x, x) = \{\mathrm{id}_x\}$. Thus F is not faithful and hence the categories C and $\mathcal{D}$ are not equivalent.

This example highlights the necessity for a finer tool than the nerve in order to distinguish the difference between categories that may not have been equivalent. The classifying diagram, which is defined in Section 3, is a generalization of the nerve and it can distinguish the difference between the categories described in the above example. The purpose of this thesis is to provide a deeper investigation into the classifying diagram.

2.5. **Simplicial spaces and the Reedy model structure.** We provide the definition of a simplicial space, explain two different ways to build a simplicial space from a simplicial set, and describe what it means for simplicial spaces to be weakly equivalent.

Definition 2.16. A *simplicial space* is a functor $\Delta^{\mathrm{op}} \to \mathtt{SSet}$.

We denote the category of simplicial spaces by $\mathtt{SSpace}$. That is, $\mathtt{SSpace}$ has simplicial spaces as objects and natural transformations as morphisms. Because $\mathtt{SSpace}$ is enriched in $\mathtt{SSet}$, we let $\mathrm{Map}(X, Y)$ denote the the mapping space between X and Y.

The data for a simplicial space X can be rewritten in terms of simplicial sets $X([n]) =: X_n$ along with face and degeneracy maps.

Given a set X, we define the constant simplicial set by applying the functor const : $\mathtt{Set} \to \mathtt{SSet}$ which maps the set X to the simplicial set defined levelwise $\mathrm{const}(X)_n := X$ where the face and degeneracy maps are identity maps.

If instead we have a simplicial set X, we define two different simplicial spaces by applying two different functors $\mathtt{SSet} \to \mathtt{SSpace}$.

(i) We define a constant simplicial space by applying the functor $\mathtt{SSet} \to \mathtt{SSpace}$ which maps the simplicial set X to the simplicial space where each level is the simplicial set X and the face and degeneracy maps are identity maps. We also denote the resulting constant simplicial space by X.
(ii) Let X_n be the nth level of the simplicial set X. We define a discrete simplicial space, denoted by X^t, by applying the functor $\mathtt{SSet} \to \mathtt{SSpace}$ levelwise; it maps the set X_n to the simplicial set $X^t_n := \mathrm{const}(X_n)$.

The "t" in the notation of the simplicial space X^t is used because X^t is an analog for the "transpose" of the constant simplicial space X.

The category $\mathtt{SSpace}$ is enriched in $\mathtt{SSet}$. The simplicial space $\Delta[n]^t$ is representable and hence we have

$$X_n \cong \mathrm{Map}(\Delta[n]^t, X)$$

using the enriched version of the Yoneda Lemma [13, §2.3].

In this paper we use the Reedy model category structure on $\mathtt{SSpace}$. The weak equivalences in the Reedy model structure are the levelwise weak equivalences of simplicial sets [12, A].

2.6. **Segal spaces.** In this section we see that a Segal space is a simplicial space with additional structure. In particular, if X is a Segal space, then X_n can be written in terms of X_0 and X_1.

In the category Δ, define maps $\alpha^i : [1] \to [n]$ for $0 \leq i < n$ where $\alpha^i(0) = i$ and $\alpha^i(1) = i+1$. Using the construction of the α^i's, we construct the simplicial set

$$G(n) := \bigcup_{i=0}^{n-1} \alpha^i \Delta[1] \subset \Delta[n].$$

For a simplicial set X and $n \geq 2$,

$$\begin{aligned}\mathrm{Hom}_{\mathtt{sSet}}(G(n), X) &\cong \underbrace{X_1 \times_{X_0} \cdots \times_{X_0} X_1}_{n} \\ &= \lim\left(X_1 \xrightarrow{d_0} X_0 \xleftarrow{d_1} X_1 \xrightarrow{d_0} \cdots \xleftarrow{d_1} X_1\right).\end{aligned}$$

The inclusion $G(n) \subset \Delta[n]$ of simplicial sets induces an inclusion $G(n)^t \hookrightarrow \Delta[n]^t$ of simplicial spaces. For a fixed simplicial space X, this inclusion of simplicial spaces induces a map

$$\begin{array}{ccc}\mathrm{Map}(\Delta[n]^t, X) & \longrightarrow & \mathrm{Map}(G(n)^t, X) \\ \| & & \| \\ X_n & \xrightarrow{\varphi_n} & \underbrace{X_1 \times_{X_0} \cdots \times_{X_0} X_1}_{n}\end{array}$$

between simplicial sets for $n \geq 2$. We call φ_n a *Segal map*.

Definition 2.17. [13, §4.1] A simplicial space W is a *Segal space* if W is Reedy fibrant and the Segal maps φ_n are weak equivalences for $n \geq 2$.

Generally speaking, in a model structure, every object is weakly equivalent to a fibrant object. In the definition of a Segal space, we require W to be Reedy fibrant in order to guarantee pullbacks and homotopy pullbacks coincide. We also note that the Segal maps φ_n in a Segal space are acyclic fibrations [13, §4.1].

2.7. **Complete Segal spaces.** In order to arrive at the definition of a complete Segal space, we observe that a Segal space mimics the structure of a category.

Definition 2.18. [13, §5.1] The *set of objects* of a Segal space W is $\mathrm{ob}(W) := W_{0,0}$, which is the zeroth level of the simplicial set W_0.

Now that a Segal space has objects like a category, we need to define the analog of the hom-space between two objects.

Definition 2.19. [13, §5.1] Let W be a Segal space and $x, y \in \mathrm{ob}(W)$. The *mapping space* $\mathrm{map}_W(x, y)$ is defined by the pullback square

$$\begin{array}{ccc}\mathrm{map}_W(x, y) & \longrightarrow & W_1 \\ \downarrow & & \downarrow{\scriptstyle (d_1, d_0)} \\ \{(x, y)\} & \longrightarrow & W_0 \times W_0\end{array}$$

of simplicial sets.

The requirement that W is Reedy fibrant means that the the map $(d_1, d_0) : W_1 \to W_0 \times W_0$ is a fibration [13, §5.1]. Hence $\mathrm{map}_W(x, y)$ is also a homotopy pullback in the above diagram.

Given a Segal space W, note that if $x \in W_0$, then $s_0 x \in W_1$ and $d_1 s_0 x = x = d_0 s_0 x$. So if x is in the set of objects of W, then $s_0 x \in \mathrm{map}(x, x)_0$, which leads to the following definition.

Definition 2.20. [13, §5.1] Given a Segal space W and $x \in \mathrm{ob}(W)$, the *identity map* of x is defined to be $id_x := s_0 x \in \mathrm{map}_W(x, x)_0$.

In a Segal space, so far we have objects, mapping spaces between objects, and the identity map. In a category, composition is unique. However, if $f \in \mathrm{map}_W(x, y)_0$ and $g \in \mathrm{map}_W(y, z)_0$, what does "$g \circ f$" mean, and is it in $\mathrm{map}_W(x, z)_0$? To see how Rezk answered this question, we first define what it means to be homotopic in the mapping space, and then generalize the definition of the mapping space.

Definition 2.21. [13, §5.3] Let $f, g \in \mathrm{map}_W(x, y)_0$ where x and y are objects in a Segal space W. We say f and g are *homotopic*, denoted $f \simeq g$, if they lie in the same component of $\mathrm{map}_W(x, y)$.

In order to get to the point where we can talk about composition, we need to be able to relate, for example, $\mathrm{map}_W(x, y) \times \mathrm{map}_W(y, z)$ with $\mathrm{map}_W(x, z)$. To accomplish this relationship, we generalize the definition of the mapping space between two objects and define a mapping space between a finite collection of objects. In particular, given $x_0, \ldots, x_n \in \mathrm{ob}(W)$, $\mathrm{map}_W(x_0, \ldots, x_n)$ is defined by the pullback square

$$\begin{array}{ccc} \mathrm{map}_W(x_0, \ldots, x_n) & \longrightarrow & W_n \\ \downarrow & & \downarrow \\ \{(x_0, \ldots, x_n)\} & \longrightarrow & \underbrace{W_0 \times \cdots \times W_0}_{n+1}. \end{array}$$

Let $V := \mathrm{map}_W(x_0, \ldots, x_n)$ where W is a Segal space. Then the Segal map

$$\varphi_n : V_n \xrightarrow{\simeq} \underbrace{V_1 \times_{V_0} \cdots \times_{V_0} V_1}_{n}$$

is actually the map

$$\varphi_n : \mathrm{map}_W(x_0, \ldots, x_n) \xrightarrow{\simeq} \mathrm{map}_W(x_0, x_1) \times \mathrm{map}_W(x_1, x_2) \times \cdots \times \mathrm{map}_W(x_{n-1}, x_n).$$

Let $(f, g) \in \mathrm{map}_W(x, y)_0 \times \mathrm{map}_W(y, z)_0$ where W is a Segal space. We want to define what it means to "compose" g and f. Recall that the Segal map $\mathrm{map}_W(x, y, z) \to \mathrm{map}_W(x, y) \times \mathrm{map}_W(y, z)$ is an acyclic fibration and the map $\emptyset \to \{(f, g)\}$ is a cofibration in the model structure for `SSet`. Thus, by the fourth model category axiom [6, 3.3], there exists a lift $\{(f, g)\} \to \mathrm{map}_W(x, y, z)$ making the diagram

$$\begin{array}{ccc} \emptyset & \longrightarrow & \mathrm{map}_W(x, y, z) \\ \big\downarrow & \nearrow & \Big\downarrow \simeq\, \varphi_2 \\ \{(x, y, z)\} & \longrightarrow & \mathrm{map}_W(x, y) \times \mathrm{map}_W(y, z) \end{array}$$

commute. Thus we can define a *composition* of g and f as a lift $k \in \mathrm{map}_W(x, y, z)$ of (f, g) along the Segal map φ_2. The *result* of a composition k is $d_1 k \in \mathrm{map}_W(x, z)_0$. A result is not unique, but a result is unique up to homotopy. We let $g \circ f$ denote a result of a composition. In particular, as seen in the following proposition, a Segal space has a category theory structure up to homotopy.

Proposition 2.22. [13, 5.4] *In a Segal space W, let $w, x, y, z \in ob(W)$ and $(f, g, h) \in map_W(w, x)_0 \times map_W(x, y)_0 \times map_W(y, z)_0$. Then*
(i) $(h \circ g) \circ f \simeq h \circ (g \circ f)$ and
(ii) $f \circ \mathrm{id}_w \simeq f \simeq \mathrm{id}_x \circ f$.

We can use the up-to-homotopy category structure in a Segal space to define a category.

Definition 2.23. [13, §5.5] The *homotopy category* of a Segal space W, denoted as $\mathrm{Ho}W$, has $\mathrm{ob}(W)$ as objects and $\mathrm{Hom}_{\mathrm{Ho}(W)}(x, y) = \pi_0 \mathrm{map}_W(x, y)$.

If $f \in \mathrm{map}_W(x, y)_0$, let $[f] \in \mathrm{Hom}_{\mathrm{Ho}(W)}(x, y)$ denote its associated equivalence class.

Definition 2.24. [13, §5.5] Let W be a Segal space with objects x and y. We say $f \in \mathrm{map}_W(x, y)_0$ is a *homotopy equivalence* if $[f] \in \mathrm{Hom}_{\mathrm{Ho}(W)}(x, y)$ is an isomorphism.

In other words, $f \in \mathrm{map}_W(x, y)_0$ is a homotopy equivalence if there exists $g, h \in \mathrm{map}_W(y, x)_0$ such that $f \circ g \simeq id_y$ and $h \circ f \simeq \mathrm{id}_x$. Observe that Proposition 2.22 implies $g \simeq h$. Also note that $\mathrm{id}_x \in \mathrm{map}_W(x, x)_0$ is a homotopy equivalence.

The following result shows us that we can define a subspace using the homotopy equivalences in a Segal space.

Proposition 2.25. [13, 5.8] *If $[f] = [g] \in Hom_{\mathrm{Ho}(W)}(x, y)$, then f is a homotopy equivalence if and only if g is a homotopy equivalence in the Segal space W.*

So in a Segal space W, the *space of homotopy equivalences* is defined as the subspace $W_{\mathrm{heq}} \subseteq W_1$ which consists of the components of W_1 whose 0-simplices are homotopy equivalences.

Note that for any object x in W, $s_0 x := \mathrm{id}_x$ is a homotopy equivalence and hence the degeneracy map $s_0 : W_0 \to W_1$ factors through W_{heq}.

Definition 2.26. [13, §6] A *complete Segal space* W is a Segal space such that the map $s_0 : W_0 \to W_{\mathrm{heq}}$ is a weak equivalence.

To see the importance of requiring s_0 to be a weak equivalence, consider the category $I[1]$, which is given by

$$0 \underset{\cong}{\overset{\cong}{\rightleftarrows}} 1.$$

We let ij denote the isomorphism $i \to j$ in $I[1]$ where $i, j \in \{0, 1\}$. Let $E[1] := \mathrm{nerve}(I[1])$. Then $E[1]_0 = \{0, 1\}$ and $E[1]_1 = \{00, 11, 01, 10\}$. Note that the categories $[0]$ and $I[1]$ are equivalent. Now compare the Segal space $\Delta[0]^t$ and $E[1]^t$. Levelwise $\Delta[0]^t$ is contractible, but $E[1]^t$ is not levelwise contractible. For example, $E[1]^t_0$ is the constant simplicial set given by the set $\{0, 1\}$. By definition, $\mathrm{ob}(E[1]^t) := E[1]^t_{0,0} = \{0, 1\}$. Note that $01 \in \mathrm{map}_{E[1]^t}(0, 1)$ and $10 \in \mathrm{map}_{E[1]^t}(1, 0)$ are homotopy equivalences. So the objects 0 and 1 have homotopy equivalences going between them. However the simplicial set $E[1]^t_0$ is discrete; there is no path in $E[1]^t_0$ between 0 and 1. So in the definition of complete Segal spaces, the whole point of requiring $s_0 : W_0 \to W_{\mathrm{heq}}$ to be a weak equivalence is to guarantee that if there is a homotopy between two objects, then there is also a path between the objects.

In the next section we show that the classifying diagram is a complete Segal space.

3. The classifying diagram

In this section we see that Rezk's classifying diagram, which is a generalization of the nerve, is a simplicial space that naturally has the structure of a complete Segal space. In Example 2.15 we saw that the nerve fails at distinguishing between categories that are not equivalent; we revisit the categories from this example and show that the classifying diagrams are in fact not equivalent. We compute the classifying diagram of two preliminary examples: a finite ordered set $[n]$, and the category of finite ordered sets Δ. Additionally we show that the classifying diagram of a category C is equivalent to the discrete simplicial space $\text{nerve}(C)^t$ if and only if the identities are the only isomorphisms in C. We close the section by proving for a general category that the levels of the classifying diagram can be written in terms of classifying spaces of stabilizers. Every classifying diagram description we provide in this paper is decomposed into classifying spaces of groups.

3.1. The definition of the classifying diagram.

Before we can define the classifying diagram, we need to specify the notation used in the definition. If $\mathcal{D}$ is a category, let $\text{iso}(\mathcal{D})$ denote the subcategory where the objects are the same as in $\mathcal{D}$, but the morphisms are only the isomorphisms of $\mathcal{D}$. In the literature, $\text{iso}(\mathcal{D})$ is sometimes called the maximal subgroupoid of $\mathcal{D}$.

Definition 3.1. [13, §3.5] The *classifying diagram* of a category C is denoted by NC and is the simplicial space defined levelwise by

$$(NC)_n := \text{nerve}(\text{iso}(C^{[n]})).$$

Proposition 3.2. [13, 6.1] *The classifying diagram of a small category C is a complete Segal space.*

Proof. We refer the reader to [2, 9.1.1] for the proof that NC is Reedy fibrant. By construction, the Segal map

$$\varphi_n : (NC)_n \to \underbrace{(NC)_1 \times_{(NC)_0} \cdots \times_{(NC)_0} (NC)_1}_{n}$$

is an isomorphism for all $n \geq 2$, and hence NC is a Segal space. Now we need to show that NC is complete. Note that for any $x, y \in \text{ob}(C) \cong \text{ob}(NC)$, there is a natural bijection between the sets $\text{map}_{NC}(x,y)_0$ and $\text{Hom}_C(x,y)$. So if $f \in \left((NC)_{\text{heq}}\right)_0$ where $f \in \text{map}_{NC}(x,y)_0$, then there exists $g \in \left((NC)_{\text{heq}}\right)_0$ where $f \circ g = \text{id}_y$, $g \circ f = \text{id}_x$, and $g \in \text{map}_{NC}(y,x)_0$. To see that $(NC)_{\text{heq}} \simeq \text{nerve}(\text{iso}(C^{I[1]}))$, note that $f \in \left((NC)_{\text{heq}}\right)_0$ with inverse $g \in \text{map}_{NC}(y,x)_0$ if and only if $(f,g) \in \text{ob}\left(\text{iso}(C^{I[1]})\right) \cong \text{nerve}\left(\text{iso}(C^{I[1]})\right)_0$. Since the categories $I[1]$ and $[0]$ are equivalent, we have that $C^{I[1]} \simeq C^{[0]}$ by Proposition 2.6. Thus, using Corollary 2.13, we have

$$(NC)_0 := \text{nerve}\left(\text{iso}(C^{[0]})\right) \simeq \text{nerve}\left(\text{iso}(C^{I[1]})\right) \simeq (NC)_{\text{heq}}$$

and hence NC is complete. □

The fact that the Segal maps

$$\varphi_n : (NC)_n \to \underbrace{(NC)_1 \times_{(NC)_0} \cdots \times_{(NC)_0} (NC)_1}_{n}$$

are isomorphisms means that if we want to describe the nth level of the classifying diagram of a category C, it suffices to describe $(NC)_0$ and $(NC)_1$ because $(NC)_n$ is n copies of the $(NC)_1$ glued together along $(NC)_0$. Thus in many examples presented in this paper, we only provide a description for the 0th and 1st levels.

Let us revisit the motivating example from Section 2.4.

Example 3.3. Let us consider the categories that were discussed in Example 2.15 and justify the claim that the classifying diagram of these two categories are different. In order to show that the classifying diagrams of these two categories are not weakly equivalent in the Reedy model structure, it suffices to find one level in which the respective simplicial sets are not weakly equivalent in the model structure on for simplicial sets. Let us begin with the subcategory $\mathcal{D}$, which just has one object, x, and its identity morphism. Note that $\mathrm{iso}(\mathcal{D}^{[0]}) \simeq \mathrm{iso}(\mathcal{D}) = \mathcal{D}$, and hence

$$(N\mathcal{D})_0 \simeq \mathrm{nerve}(\mathcal{D}).$$

Now, for the category C, note that the morphism $f : x \to y$ is not an isomorphism and hence $\mathrm{iso}(C)$ only has the two objects, x and y, along with their identity morphisms. So we have $\mathrm{iso}(C^{[0]}) \simeq \mathrm{iso}(C) \simeq \mathcal{D} \coprod \mathcal{D}$, and hence

$$(NC)_0 \simeq \mathrm{nerve}(\mathcal{D}) \amalg \mathrm{nerve}(\mathcal{D}).$$

Therefore the classifying diagrams NC and $N\mathcal{D}$ are not weakly equivalent.

3.2. **Categories with only isomorphisms.** In contrast to the category C in the previous example, we only consider categories whose morphisms are only isomorphisms in this section. We show that the classifying diagram of a category with only isomorphisms is levelwise equivalent to its classifying space.

Any group G can be thought of as a category. Namely, G is a category with one object whose morphisms are given by each element in the group. Since each group element has an inverse, all of the morphisms in the category G are isomorphisms.

Proposition 3.4. [13, §3.5] *Let G be a group thought of as a category with one object. Then*

$$(NG)_n \simeq \mathit{nerve}(G)$$

for any $n \geq 0$.

In other words, the classifying diagram of a group G is levelwise equivalent to the classifying space of G. The proof of the above proposition follows from Proposition 3.5. When describing the classifying diagram in the following sections, we write the levels in terms of classifying spaces of groups.

Recall that a groupoid is a category in which every morphism is an isomorphism. We acquire the following general result.

Proposition 3.5. [13, §3.5] *Let G be a groupoid. Then the classifying diagram of G is levelwise equivalent to the classifying space of G. That is,*

$$(NG)_n \simeq \mathit{nerve}(G)$$

for any $n \geq 0$.

Proof. Note that $\mathrm{iso}(G^{[n]}) = G^{[n]}$ because every morphism in the category G and hence $G^{[n]}$ is an isomorphism. By Corollary 2.13, it suffices to prove the categories G and $G^{[n]}$ are equivalent. Here we prove that G and $G^{[2]}$ are equivalent, which can be extended to the more general result. Define a functor $\iota : G \to G^{[2]}$ on objects by

$$x \longmapsto \left(x \xrightarrow{id} x \xrightarrow{id} x \right)$$

and on morphisms by

$$\begin{array}{c} x \\ \downarrow f \\ y \end{array} \longmapsto \begin{array}{ccccc} x & \xrightarrow{id} & x & \xrightarrow{id} & x \\ \downarrow f & & \downarrow f & & \downarrow f \\ y & \xrightarrow{id} & y & \xrightarrow{id} & y. \end{array}$$

It is a straightforward observation that ι is full and faithful, so we only show that ι is essentially surjective. Let $x \xrightarrow{g} y \xrightarrow{h} z$ be an arbitrary object in $G^{[2]}$. Note that the diagram

$$\begin{array}{ccccc} x & \xrightarrow{id} & x & \xrightarrow{id} & x \\ \downarrow id & & \downarrow g & & \downarrow h\circ g \\ x & \xrightarrow{g} & y & \xrightarrow{h} & z. \end{array}$$

commutes and hence the triple $(id, g, h \circ g)$ defines a morphism from $\iota(x)$ to $x \xrightarrow{g} y \xrightarrow{h} z$ in $G^{[2]}$. In fact, $(id, g, h \circ g)$ is an isomorphism with inverse $(id, g^{-1}, g^{-1} \circ h^{-1})$ because G is groupoid. Thus ι is essentially surjective. Therefore the categories G and $G^{[n]}$ are equivalent. □

In the following sections we use the following notation.

Notation 3.6. (i) We use classifying space notation, BG, instead of writing $\mathrm{nerve}(G)$.
(ii) Let G and H be categories. We use $G \coprod H$ to denote the category with subcategories G and H such that an object (resp. a morphism) is in $G \coprod H$ if and only if it is an object (resp. a morphism) of G or H.

Observation 3.7. Let G_i be a category for each i. Then

$$B\left(\amalg_i G_i\right) \simeq \amalg_i B(G_i).$$

The above observation follows from the construction of the nerve because if we have a chain of n composable morphisms in the category $\amalg_i G_i$, then the chain of morphisms lies only in G_i for some i.

3.3. **Preliminary examples.** In this section we consider two categories in which the classifying diagram is decomposed into disjoint unions of the classifying space of the trivial group.

A finite ordered set. Consider the category $[m]$, which consists of $(m+1)$ objects $0, 1, \ldots, m$ and there exists one morphism $j \to k$ if and only if $j \leq k$.

Proposition 3.8. *The* 0*th level of the classifying diagram of the category* $[m]$ *is given by*

$$N([m])_0 \simeq \coprod_{m+1} B(\{e\})$$

where $\{e\}$ is the trivial group, and the 1st level of the classifying diagram is given by

$$N([m])_1 \simeq \coprod_{\frac{1}{2}(m+1)(m+2)} B(\{e\}).$$

Proof. To compute the 0th level, we need to consider iso($[m]$). Since the only isomorphisms in $[m]$ are identities, iso($[m]$) consists of $m+1$ objects and only the identity morphisms. Thus

$$N([m])_0 \simeq \coprod_{m+1} B(\{e\}).$$

Now to compute the 1st level, consider the functor category iso($[m]^{[1]}$). The number of objects in this functor category is the same as the number of morphisms in the category $[m]$. Observe that each object k in $[m]$ has $k+1$ morphisms mapping into it, and hence the number of morphisms in the category $[m]$ is given by

$$\sum_{k=0}^{m}(k+1) = \frac{(m+1)(m+2)}{2}.$$

Since the only isomorphisms in $[m]$ are identities, the only isomorphisms in $[m]^{[1]}$ are also only given by identities. Thus

$$N([m])_0 \simeq \coprod_{\frac{1}{2}(m+1)(m+2)} B(\{e\}).$$

□

The category of finite ordered sets. Consider the category of finite ordered sets Δ. That is, the objects are given by finite ordered sets $[m] = \{0 \leq 1 \leq 2 \leq \cdots \leq m\}$ and the morphisms are order preserving functions.

Proposition 3.9. *The 0th and 1st levels of the classifying diagram of Δ are both weakly equivalent to a disjoint union of countably many contractible spaces, but $N\Delta$ is not weakly equivalent to the constant simplicial space $\coprod_{\mathbb{N}} B(\{e\})$.*

Proof. Note that the only isomorphisms in Δ are the identity morphisms, hence, since the objects of Δ are in bijection with the natural numbers, we get

$$N(\Delta)_0 \simeq \coprod_{\mathbb{N}} B(\{e\}).$$

For the first level of the classifying diagram, the objects in iso($\Delta^{[1]}$) are morphisms $f : [n] \to [m]$ in Δ, and the morphisms in the category iso($\Delta^{[1]}$) are given by pairs of isomorphisms (α, β) making the diagram

$$\begin{array}{ccc} [n] & \xrightarrow{\alpha} & [n] \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ [m] & \xrightarrow{\beta} & [m] \end{array}$$

commute in Δ. Since the only isomorphisms in Δ are identities, the only morphisms in the category iso($\Delta^{[1]}$) are given by pairs of identities. For any given pair of natural numbers, n and m, there is only a finite number of morphisms $f : [n] \to [m]$ in Δ. Also, we have already observed that there are a countable number of objects in Δ. Thus there is a countable number of morphisms in Δ and hence we get our desired result

$$N(\Delta)_1 \simeq \coprod_{\mathbb{N}} B(\{e\}).$$

It remains to show that $N\Delta$ is not the constant simplicial space $\coprod_{\mathbb{N}} B(\{e\})$. For each $[n]$, let $B(\{e\})_{[n]}$ denote the copy of $B(\{e\})$ in $N(\Delta)_0$ corresponding to $[n]$. Similarly, let $B(\{e\})_{f:[n]\to[m]}$ denote the copy of $B(\{e\})$ in $N(\Delta)_1$ corresponding to the morphism $f : [n] \to [m]$ in Δ. The diagram

$$B(\{e\})_{[n]} \xleftarrow{\;d_1\;} B(\{e\})_{f:[n]\to[m]} \xrightarrow{\;d_0\;} B(\{e\})_{[m]}$$

shows how the face maps $d_0, d_1 : (N\Delta)_1 \to (N\Delta)_0$ interact for a given morphism $f : [n] \to [m]$. In particular, since m can be any nonnegative integer, we have a countable (not finite) collection of copies of $B(\{e\})$ in $(N\Delta)_1$ that map via d_1 to $B(\{e\})_{[n]}$ for each n. Also note that $s_0(B(\{e\})_{[n]}) = B(\{e\})_{\mathrm{id}:[n]\to[n]}$. Thus $N\Delta$ is not weakly equivalent to the constant simplicial space. □

In Proposition 3.5, we saw that the classifying space of a groupoid G is the weakly equivalent to the constant simplicial space BG; in particular $(NG)_0 \simeq (NG)_1 \simeq BG$ and the face/degeneracy maps are essentially identities. In contrast, we just proved that the 0th and 1st levels of $N\Delta$ are both countable disjoint unions of contractible spaces, but the face/degeneracy maps are not essentially identities. It is not a surprise that $N\Delta$ is not a constant simplicial space because Δ is not a groupoid.

3.4. **The classifying diagram and the discrete simplicial space given by the nerve.** In Proposition 3.9, the structures of 0th and 1st levels of $N\Delta$ are reminiscent of the 0th and 1st levels of the discrete simplicial space nerve$(\Delta)^t$. In the following proposition, we provide the necessary and sufficient conditions on a category C to guarantee NC and nerve$(C)^t$ are isomorphic simplicial spaces; as a consequence $N\Delta$ is isomorphic to nerve$(\Delta)^t$.

Proposition 3.10. *The classifying diagram of a category C is isomorphic to the discrete simplicial space nerve$(C)^t$ if and only if iso(C) is discrete.*

Proof. Recall that $\mathrm{nerve}(C)^t_n := \mathrm{const}(\mathrm{nerve}(C)_n)$ is the simplicial set given by the set $\mathrm{nerve}(C)_n$ at each level in which the face and degeneracy maps are identities. The only morphisms in $\mathrm{iso}(C^{[n]})$ are identities if and only if the only isomorphisms in C are identities. Thus any functor $[m] \to \mathrm{iso}(C^{[n]})$ maps $[m]$ to a chain of length m of identity morphisms for an object in $\mathrm{iso}(C^{[n]})$ if and only if the only isomorphisms in C are identities; hence $\mathrm{Hom}_{Cat}([m], \mathrm{iso}(C^{[n]})) \cong \mathrm{ob}(\mathrm{iso}(C^{[n]}))$ if and only if the only isomorphisms in C are identities. Therefore

$$\begin{aligned}
N(C)_{n,m} &:= \text{nerve}(\text{iso}(C^{[n]}))_m \\
&= \text{Hom}_{Cat}([m], \text{iso}(C^{[n]})) \\
&\cong \text{ob}(\text{iso}(C^{[n]})) \\
&\cong \text{ob}(C^{[n]}) \\
&\cong \text{Hom}_{Cat}([n], C) \\
&= \text{const}(\text{Hom}_{Cat}([n], C))_m \\
&=: \text{const}(\text{nerve}(C)_n)_m
\end{aligned}$$

if and only the only isomorphisms in C are identities, which gives the desired result. □

3.5. **The stabilizer characterization of the classifying diagram.** In Proposition 3.13, the 0th level of the classifying diagram is written in terms of classifying spaces of automorphism classes of the category. One way to extend Proposition 3.13 to the higher levels of the classifying diagram is to describe the higher levels in terms of stabilizers of products of automorphism groups. We begin the section by recalling the definition of a stabilizer and the fact that it is a group.

Proposition 3.11. [10, II.4.2] *Let G be a group that acts on a set S.*

(i) The relation on S defined by

$$x \sim x' \iff g \cdot x = x' \text{ for some } g \in G$$

is an equivalence relation.

(ii) The stabilizer for some $x \in S$, $G_x = \{g \in G | g \cdot x = x\}$, is a subgroup of G.

In an category C, the automorphisms on an object form a group. The following proposition uses automorphisms and hom-sets in a category to define a group action.

Proposition 3.12. *Let $x_0, x_1 \ldots, x_n$ be objects in the category C. Given the $(n+1)$-tuple $\underline{x} = (x_0, \ldots, x_n)$, consider the group*

$$Aut(\underline{x}) := Aut(x_0) \times Aut(x_1) \times \cdots Aut(x_n)$$

and the set

$$Hom(\underline{x}) := Hom_C(x_0, x_1) \times Hom_C(x_1, x_2) \times \cdots \times Hom_C(x_{n-1}, x_n).$$

The map

$$\bullet : Aut(\underline{x}) \times Hom(\underline{x}) \longrightarrow Hom(\underline{x})$$

$$((\alpha_0, \ldots, \alpha_n), (f_1, \ldots, f_n)) \longmapsto \left(\alpha_1 f_1 \alpha_0^{-1}, \ldots, \alpha_n f_n \alpha_{n-1}^{-1}\right),$$

where each α_i is an automorphism on x_i and $f_i : x_{i-1} \to x_i$ is a morphism in C, defines a group action.

Proof. It suffices to check that $\bullet$ is a group action on each coordinate of $\mathrm{Hom}(\underline{x})$. In other words, we show that the map

$$\cdot : (Aut(x_{i-1}) \times Aut(x_i)) \times \mathrm{Hom}_C(x_{i-1}, x_i) \longrightarrow \mathrm{Hom}_C(x_{i-1}, x_i)$$

$$((\alpha_{i-1}, \alpha_i), f_i) \longmapsto \alpha_i f_i \alpha_{i-1}^{-1}$$

defines a group action for $1 \leq i \leq n$. Note that $(\mathrm{id}_{x_{i-1}}, \mathrm{id}_{x_i}) \cdot f_i = \mathrm{id}_{x_i} f_i \mathrm{id}_{x_{i-1}} = f_i$. So it remains to show the compatibility of the action. To see that the action is compatible, observe that

$$\begin{aligned}((\alpha_{i-1}, \alpha_i)(\beta_{i-1}, \beta_i)) \cdot f_i &= (\alpha_{i-1}\beta_{i-1}, \alpha_i\beta_i) \cdot f_i \\ &= (\alpha_i\beta_i)\, f_i\, (\alpha_{i-1}\beta_{i-1})^{-1} \\ &= \alpha_i\beta_i f_i \beta_{i-1}^{-1}\alpha_{i-1}^{-1} \\ &= \alpha_i \left[(\beta_{i-1}, \beta_i) \cdot f_i\right] \alpha_{i-1}^{i-1} \\ &= (\alpha_{i-1}, \alpha_i) \cdot \left[(\beta_{i-1}, \beta_i) \cdot f_i\right].\end{aligned}$$

Thus, since $\bullet$ is defined coordinate-wise and we have shown that $\cdot$ defines the action on each coordinate, $\bullet$ defines an action of $Aut(\underline{x})$ on $\mathrm{Hom}(\underline{x})$. □

Before we use $\bullet$ to describe the higher levels of the classifying diagram, we first provide the known description of the 0th level the served as inspiration.

Proposition 3.13. [1, §7.2] *Let C be a category. For a given object x, let $\langle x \rangle$ denote its isomorphism equivalence class. Then*

$$N(C)_0 \simeq \coprod_{\langle x \rangle} B(Aut(x)).$$

Proof. Note the categories $\mathrm{iso}(C^{[0]})$ and $\coprod_{\langle x \rangle} Aut(x)$ are equivalent. By Corollary 2.13, $N(C)_0 \simeq B\left(\coprod_{\langle x \rangle} Aut(x)\right)$. Applying Observation 3.7 gives the desired result.

□

Moving onto to higher levels, it was previously shown that the 1st level of the classifying diagram can be written in terms of automorphisms of morphisms.

Proposition 3.14. [1, §7.2] *Let C be a category. Then*

$$N(C)_1 \simeq \coprod_{\langle x \rangle, \langle y \rangle} \coprod_{\langle \alpha : x \to y \rangle} B(Aut(\alpha))$$

where $\langle \alpha : x \to y \rangle$ is the automorphism class of the morphism $\alpha : x \to y$ in C.

We use the action $\bullet$ to form a new characterization of the higher levels for the classifying diagram by using stabilizers under the action.

Theorem 3.15. *Let $\langle f_1, \ldots, f_n \rangle$ denote the equivalence class of $(f_1, \ldots, f_n) \in Hom(\underline{x})$ defined by the group action $\bullet$. Then for $n \geq 1$,*

$$N(C)_n \simeq \coprod_{\langle f_1, \ldots, f_n \rangle} B\left[Aut(\underline{x})_{(f_1, \ldots, f_n)}\right]$$

where $Aut(\underline{x})_{(f_1, \ldots, f_n)}$ is the stabilizer of $(f_1, \ldots, f_n)$.

Proof. It suffices to argue that the categories

$$\mathrm{iso}(C^{[n]}) \quad \text{and} \quad \coprod_{\langle f_1,\dots,f_n\rangle} Aut(\underline{x})_{(f_1,\dots,f_n)}$$

are equivalent. Note that the objects in $\mathrm{iso}(C^{[n]})$ are given by chains of n composable morphisms $f_1, f_2, \dots, f_n$ between objects $x_0, x_1, \dots, x_n$ in C:

$$x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} x_n.$$

We denote this object in $\mathrm{iso}(C^{[n]})$ by the n-tuple $(f_1, f_2, \dots, f_n)$. A morphism between objects $(f_1, f_2, \dots, f_n)$ and $(g_1, g_2, \dots, g_n)$ in $\mathrm{iso}(C^{[n]})$ is given by an n-tuple $(\alpha_1, \alpha_2, \dots, \alpha_n)$ where each α_i is an isomorphism in C making the diagram

$$\begin{array}{ccccccc} x_0 & \xrightarrow{f_1} & x_1 & \xrightarrow{f_2} & \cdots & \xrightarrow{f_n} & x_n \\ \cong\downarrow\alpha_1 & & \cong\downarrow\alpha_2 & & & & \cong\downarrow\alpha_n \\ y_0 & \xrightarrow{g_1} & y_1 & \xrightarrow{g_2} & \cdots & \xrightarrow{g_n} & y_n \end{array}$$

commute in C. As we saw in Proposition 3.13, it suffices to consider the automorphism classes of objects in $\mathrm{iso}(C^{[n]})$. So, in other words, for a given $(f_1, \dots, f_n)$, we need to describe all possible morphisms $(\alpha_1, \dots, \alpha_n)$ that fix $(f_1, \dots, f_n)$. But this is exactly what the stabilizer $Aut(\underline{x})_{(f_1,\dots,f_n)}$ does. Thus, $\mathrm{iso}(C^{[n]})$ is equivalent to

$$\coprod_{\langle f_1,\dots,f_n\rangle} Aut(\underline{x})_{(f_1,\dots,f_n)}.$$

□

4. The classifying diagram for the category of vector spaces

In this section prove that the classifying diagram of finite vector spaces can be written in terms of classifying spaces general linear groups; we use the group action defined in Proposition 3.12 as well as the other results from Section 3.5. We then produce a more detailed description if we work over the field $\mathbb{F}_2$ and restrict the dimension of the matrices.

The category of finite dimensional vector spaces over the field $\mathbb{F}$ has finite vector spaces as objects and linear maps as morphisms. Recall that every finite dimensional vector space is isomorphic to $\mathbb{F}^n$. Let $\mathtt{Vect}(\mathbb{F})$ denote the subcategory of finite vector spaces where the objects are $\mathbb{F}^n$ and the morphisms are matrices with entries in $\mathbb{F}$. Note that $\mathtt{Vect}(\mathbb{F})$ is equivalent to the category of finite dimensional vector spaces over $\mathbb{F}$. We let $\mathrm{Mat}_{n\times m}(\mathbb{F})$ be the set of $n \times m$ matrices. Also let $\mathrm{GL}_n(\mathbb{F})$ denote the general linear group of dimension n.The following proposition is a well-known result in linear algebra and is a specific case of the action $\bullet$ from Proposition 3.12, but it is used to prove Corollary 4.2.

Proposition 4.1. *The map*

$$\bullet : (\mathrm{GL}_n(\mathbb{F}) \times \mathrm{GL}_m(\mathbb{F})) \times \mathrm{Mat}_{n\times m}(\mathbb{F}) \longrightarrow \mathrm{Mat}_{n\times m}(\mathbb{F})$$

$$((f, g), A) \longmapsto gAf^{-1}$$

defines a group action.

Proof. For simplicity, we refrain from referencing the underlying field $\mathbb{F}$ in this proof. Specifically, we write $\mathrm{GL}_n := \mathrm{GL}_n(\mathbb{F})$ and $\mathrm{Mat}_{n\times m} := \mathrm{Mat}_{n\times m}(\mathbb{F})$. We need to verify two things to verify that $\bullet$ is a group action. First we need to verify that the action of the identity element in $\mathrm{GL}_n \times \mathrm{GL}_m$ preserves any element of the set $\mathrm{Mat}_{n\times m}$. Let $A \in \mathrm{Mat}_{n\times m}$ and let I_n denote the identity $n \times n$ matrix. So the identity element in the group $\mathrm{GL}_n \times \mathrm{GL}_m$ is given by (I_n, I_m). Thus

$$\begin{aligned}(I_n, I_m) \bullet A &= I_m A I_n^{-1} \\ &= I_m A I_n \\ &= A.\end{aligned}$$

Next, the compatibility of the action must be verified. Let (f_1, g_1) and (f_2, g_2) be elements in $\mathrm{GL}_n \times \mathrm{GL}_m$. Therefore

$$\begin{aligned}((f_1, g_1)(f_2, g_2)) \bullet A &= (f_1 f_2, g_1 g_2) \bullet A \\ &= (g_1 g_2) A (f_1 f_2)^{-1} \\ &= g_1 g_2 A f_2^{-1} f_1^{-1} \\ &= g_1 \left((f_2, g_2) \bullet A\right) f_1^{-1} \\ &= (f_1, g_1) \bullet ((f_2, g_2) \bullet A)\end{aligned}$$

and hence $\bullet$ defines a group action. □

Let $A, B \in \mathrm{Mat}_{n\times m}(\mathbb{F})$. The matrices A and B are in the same equivalence class under the group action $\bullet$ if and only if there exists some $g \in \mathrm{GL}_n(\mathbb{F}) \times \mathrm{GL}_m(\mathbb{F})$ such that $g \bullet A = B$. We let $\langle A \rangle$ denote the equivalence class of A under this equivalence relation. Note that the stabilizer of A, $(\mathrm{GL}_n(\mathbb{F}) \times \mathrm{GL}_m(\mathbb{F}))_A$, is a subgroup of $\mathrm{GL}_n(\mathbb{F}) \times \mathrm{GL}_m(\mathbb{F})$.

Now we have enough background to state the result for the classifying diagram of the category of finite vector spaces over a field.

Corollary 4.2. *The* 0*th level of the classifying diagram of* $\mathtt{Vect}(\mathbb{F})$ *is given by*

$$N(\mathtt{Vect}(\mathbb{F}))_0 \simeq \coprod_{n\in\mathbb{N}} B(\mathrm{GL}_n(\mathbb{F}))$$

and the 1*st level is given by*

$$N(\mathtt{Vect}(\mathbb{F}))_1 \simeq \coprod_{n,m\in\mathbb{N}} \left[\coprod_{\langle A\rangle \in \mathrm{Mat}_{n\times m}} B\left((\mathrm{GL}_n \times \mathrm{GL}_m)_A\right) \right]$$

where $\langle A \rangle$ *is the equivalence class of A under the relation defined by the group action* $\bullet$ *of* $\mathrm{GL}_n \times \mathrm{GL}_m$ *acting on* $\mathrm{Mat}_{n\times m}(\mathbb{F})$*, and* $(\mathrm{GL}_n \times \mathrm{GL}_m)_A$ *is the stabilizer of A.*

Proof. The characterization of the 0th level follows from Proposition 3.13 and the fact that the isomorphisms in $\mathtt{Vect}(\mathbb{F})$ are given by $\mathrm{GL}_n(\mathbb{F})$ for all non negative integers n. Since morphisms from $\mathbb{F}^n$ to $\mathbb{F}^m$ are given by $A \in \mathrm{Mat}_{n\times m}$ and isomorphisms between two $n \times m$ matrices are given by pairs $(f, g) \in \mathrm{GL}_n \times \mathrm{GL}_m$, the desired decomposition of the 1st level of $N(\mathtt{Vect}(\mathbb{F}))$ follows from Theorem 3.15. □

We now turn our attention to finding the classifying diagram of a subcategory of $\mathtt{Vect}(\mathbb{F}_2)$, where $\mathbb{F}_2$ is the finite group with two elements.

Example 4.3. Let $\texttt{Vect}_{\leq 2}(\mathbb{F}_2)$ be the subcategory of $\texttt{Vect}(\mathbb{F}_2)$ where the objects are $\mathbb{F}_2$ and $\mathbb{F}_2^2$, and the morphisms are given by linear maps. For the 0th level of the classifying diagram, we get $B(\mathrm{GL}_1(\mathbb{F}_2)) \coprod B(\mathrm{GL}_2(\mathbb{F}_2))$. Observe that $B(\mathrm{GL}_1(\mathbb{F}_2)) = B(\{e\})$.

The 1st level of the classifying diagram is more interesting. We break the functor category $\mathrm{iso}\left(\texttt{Vect}_{\leq 2}(\mathbb{F}_2)^{[1]}\right)$ into four types of objects:

(i) $\mathbb{F}_2 \to \mathbb{F}_2$,
(ii) $\mathbb{F}_2 \to \mathbb{F}_2^2$,
(iii) $\mathbb{F}_2^2 \to \mathbb{F}_2$, and
(iv) $\mathbb{F}_2^2 \to \mathbb{F}_2^2$.

If two objects $\mathbb{F}_2^i \to \mathbb{F}_2^j$ and $\mathbb{F}_2^n \to \mathbb{F}_2^m$ are isomorphic, then $i = n$ and $j = m$. Note that an isomorphism between two objects $A, B : \mathbb{F}_2^n \to \mathbb{F}_2^m$ is given by a pair of matrices (C, D) in $\mathrm{GL}_n(\mathbb{F}_2) \times \mathrm{GL}_m(\mathbb{F}_2)$ such that the diagram

$$\begin{array}{ccc} \mathbb{F}_2^n & \xrightarrow[\cong]{C} & \mathbb{F}_2^n \\ \downarrow{\scriptstyle A} & & \downarrow{\scriptstyle B} \\ \mathbb{F}_2^m & \xrightarrow[\cong]{D} & \mathbb{F}_2^m \end{array}$$

commutes. Observe that the objects A and B in the category $\mathrm{iso}\left(\texttt{Vect}_{\leq 2}(\mathbb{F}_2)^{[1]}\right)$ are $m \times n$ matrices with entries in $\mathbb{F}_2$.

Type (i). There are only two 1×1 matrices with entries in $\mathbb{F}_2$, namely, $[0]$ and $[1]$. Thus $[0]$ and $[1]$ are the only objects of this type. Both objects have $([1], [1])$ as the only automorphism and there is no isomorphism between these objects. So type (i) objects contribute $B(\mathrm{GL}_1) \coprod B(\mathrm{GL}_1)$ to the first level of the classifying diagram.

Type (ii). There are four objects of this type:

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \text{ and } \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Under the action $\bullet$, we have the two equivalence classes

$$\left\langle \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\rangle = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\} = \{A \in \mathrm{Mat}_{2\times 1}(\mathbb{F}_2) : rank(A) = 0\}$$

and

$$\left\langle \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} = \{A \in \mathrm{Mat}_{2\times 1}(\mathbb{F}_2) : rank(A) = 1\}.$$

To verify that

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \in \left\langle \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle,$$

observe that

$$\left(\begin{bmatrix} 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right) \bullet \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix}^{-1} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{4.4}$$

By Corollary 4.2, it suffices to consider the stabilizer of a representative from each equivalence class. Using matrix multiplication, one can check that

$$(\mathrm{GL}_1 \times \mathrm{GL}_2)_{\begin{bmatrix} 0 \\ 0 \end{bmatrix}} = \mathrm{GL}_1 \times \mathrm{GL}_2 \cong \mathrm{GL}_2$$

and

$$(\mathrm{GL}_1 \times \mathrm{GL}_2)_{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} = \left\{ \left([1], \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right), \left([1], \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right) \right\} \cong \mathbb{F}_2.$$

Thus type (ii) objects contribute $B(\mathrm{GL}_2) \coprod B(\mathbb{F}_2)$ to the first level of the classifying diagram.

Type (iii). Using properties of the transpose of matrices, we get that $(C, D) \in (\mathrm{GL}_1 \times \mathrm{GL}_2)_A$ if and only if $(D^T, C^T) \in (\mathrm{GL}_2 \times \mathrm{GL}_1)_{A^T}$. Thus, just like type (ii), type (iii) also contributes $B(\mathrm{GL}_2) \coprod B(\mathbb{F}_2)$ to the first level of the classifying diagram.

Type (iv). There are 16 objects of this type, which, under the group action $\bullet$, fall into three equivalence classes:

$$\left\langle \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\rangle = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\} = \{A \in \mathrm{Mat}_{2\times 2}(\mathbb{F}_2) : rank(A) = 0\},$$

$$\left\langle \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right\rangle = \{A \in \mathrm{Mat}_{2\times 2}(\mathbb{F}_2) : rank(A) = 1\},$$

and

$$\left\langle \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\rangle = \mathrm{GL}_2(\mathbb{F}_2) = \{A \in \mathrm{Mat}_{2\times 2}(\mathbb{F}_2) : rank(A) = 2\}.$$

Note that to verify the above equivalence classes, we use a similar process as (4.4). Using matrix multiplication, one can check that the stabilizers of representatives from each equivalence class are given by

$$(\mathrm{GL}_2 \times \mathrm{GL}_2)_{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}} = \mathrm{GL}_2 \times \mathrm{GL}_2,$$

$$\begin{aligned} &(\mathrm{GL}_2 \times \mathrm{GL}_2)_{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}} \\ &= \left\{ \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right), \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right), \left(\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right), \left(\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right) \right\} \\ &\cong \mathbb{F}_2 \times \mathbb{F}_2, \end{aligned}$$

and

$$(\mathrm{GL}_2 \times \mathrm{GL}_2)_{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} = \left\{ (C, D) \in \mathrm{GL}_2 \times \mathrm{GL}_2 : D = C^{-1} \right\} \cong \mathrm{GL}_2.$$

Thus type (iv) contributes $B(\mathrm{GL}_2 \times \mathrm{GL}_2) \coprod B(\mathbb{F}_2 \times \mathbb{F}_2) \coprod B(\mathrm{GL}_2)$ to the first level of the classifying diagram.

Putting together what we obtain from all four types of objects in $\mathrm{iso}\left(\mathtt{Vect}_{\leq 2}(\mathbb{F}_2)^{[1]}\right)$, we can describe the first level of the classifying diagram as

$$\begin{aligned} &N(\mathtt{Vect}_{\leq 2}(\mathbb{F}_2))_1 \simeq \\ &\left(\coprod_2 \left[B(\mathrm{GL}_1) \coprod B(\mathrm{GL}_2) \coprod B(\mathbb{F}_2) \right] \right) \coprod B(\mathrm{GL}_2 \times \mathrm{GL}_2) \coprod B(\mathbb{F}_2 \times \mathbb{F}_2) \coprod B(\mathrm{GL}_2). \end{aligned}$$

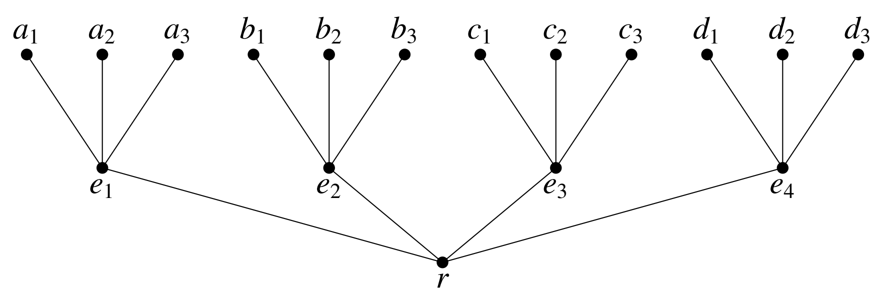
5. The classifying diagram for the category of finite sets

In this section we prove that the classifying diagram for the category of finite sets, denoted by `FinSet`, can be decomposed into the classifying spaces of products of wreath products. As a consequence, we also prove decompositions of the subcategories $\texttt{FinSet}_{inj}$ and $\texttt{FinSet}_{surj}$, the subcategories consisting of injective and surjective functions, respectively. In order to work with the 1st level of the classifying diagram, as we have seen previously, we need to understand automorphisms of morphisms; each function between finite sets can be depicted by a tree. The section begins with the definition of a wreath product and recalling the relationship between wreath products and automorphisms of trees.

5.1. **Wreath products and trees.** Let us recall the definition of the wreath product. Let K and L be two groups and $\rho : K \to \Sigma_n$ be a homomorphism where Σ_n is the nth symmetric group. Let $H := L^n$; an injective homomorphism $\phi : \Sigma_n \to Aut(H)$ can be constructed by letting the elements of Σ_n permute the n factors of H. The *wreath product* of L by K, denoted by $L \wr K$, is the semidirect product $H \rtimes K$ with respect to the homomorphism $\phi \circ \rho : K \to Aut(H)$ [5, §5.5, Ex. 23]. Wreath products are nice tools for describing the group of automorphisms of specific types of rooted trees. We recall definitions relevant to trees.

Definition 5.1.
(i) A *rooted tree* is a connected simple graph without cycles and with a distinguished vertex called the *root*.
(ii) A vertex u is *adjacent* to a vertex v in a tree if there is an edge between u and v.
(iii) The *level* of a vertex v in a rooted tree is the length of the unique path from the root to this vertex.
(iv) If u is a vertex at level j that is adjacent to a vertex v at level $j+1$, then v is said to be a *child* of u and u is the *parent* of v.
(v) The *height* of a rooted tree is the length of the longest path from the root to any vertex.
(vi) Let V be the set of vertices for a rooted tree Γ. An *automorphism* of Γ is a bijection $\phi : V \to V$ such that u and v are adjacent if and only if $\phi(u)$ and $\phi(v)$ are adjacent.

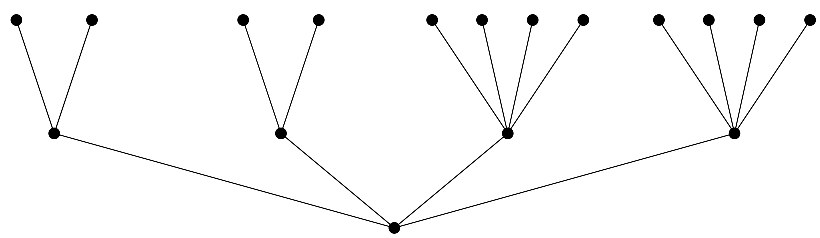
We only consider trees of height 2. To see how wreath products are used to describe automorphisms on rooted trees of height 2, consider the tree



with root r; each level 1 vertex is the parent of 3 children. An automorphism on this tree has two different group actions on the vertices. First, we have a Σ_4 action occurring on the vertices $\{e_1, e_2, e_3, e_4\}$. We also have four different Σ_3 actions occurring; Σ_3 acts on the children of each e_i. For example, we have a Σ_3 action on $\{a_1, a_2, a_3\}$. This is a good example for what a wreath product captures. The automorphisms on the above tree are described be the wreath product of Σ_3 by Σ_4, or using the wreath notation, it is $\Sigma_3 \wr \Sigma_4$. We denote the above tree as $\Gamma_{3,4}$.

The more general result also holds. Let $\Gamma_{n,m}$ denote the rooted tree where the root has children $e_1, \ldots, e_m$ and each e_i has n children. The group of automorphisms $Aut(\Gamma_{n,m})$ is isomorphic to $\Sigma_n \wr \Sigma_m$.

5.2. **The classifying diagram.** So what is the purpose of talking about automorphism on trees like $\Gamma_{n,m}$? We can use these trees to describe a morphism from a set of order n to a set of order m; in the first level of the classifying diagram, we need to describe the group of automorphisms for a morphism. The above pictured tree, $\Gamma_{3,4}$, is an example of how a set of order 12 can map to a set of order 4. But there are many other ways for a set of order 12 to map to a set of order 4. For example, the tree



is another way for a set of order 12 to map to a set of 4; the group of automorphisms is now given by a product of wreath products: $(\Sigma_2 \wr \Sigma_2) \times (\Sigma_4 \wr \Sigma_2)$. We denote this latter tree as $\Gamma_{2,2} \cup \Gamma_{4,2}$.

In general, given two rooted trees Γ_1 and Γ_2, the rooted tree $\Gamma_1 \cup \Gamma_2$ is given by the trees Γ_1 and Γ_2 identified at the root.

Let us revisit the trees $\Gamma_{3,4}$ and $\Gamma_{2,2} \cup \Gamma_{4,2}$; these trees depict functions from a set of order 12 to a set of order 4. Let k_i be the number of children of the root that has i children. For the tree $\Gamma_{3,4}$, $(k_0, k_1, k_2, k_3, k_4) = (0, 0, 0, 4, 0)$, and for the tree $\Gamma_{2,2} \cup \Gamma_{4,2}$ $(k_0, k_1, k_2, k_3, k_4) = (0, 0, 2, 0, 2)$. In either case, and in general for the tree associated with an arbitrary morphism going from a set of order 12 to a set of order 4, the equations $k_1 + 2k_2 + 3k_3 + 4k_4 = 12$ and $k_0 + k_1 + k_2 + k_3 + k_4 = 4$ are satisfied. We get an analogous system of equations if we instead considered functions from a set of order n to a set of order m.

Now that we have set up the desired notation for trees and their correspondence with functions between finite sets, we can state the result for the classifying diagram of `FinSet`.

Theorem 5.2. *The* 0*th level of the classifying diagram of* `FinSet` *is given by*

$$N(\mathtt{FinSet})_0 \simeq \coprod_{n \in \mathbb{N}} B(\Sigma_n).$$

And the 1*st level is described as*

$$N(\mathtt{FinSet})_1 \simeq \coprod B\left(\Sigma_{k_1} \times (\Sigma_2 \wr \Sigma_{k_2}) \times \cdots \times (\Sigma_n \wr \Sigma_{k_n})\right)$$

where the disjoint union is over solutions to the equations

$$k_1 + 2k_2 + \cdots + nk_n = n \quad \text{and} \quad k_0 + k_1 + \cdots + k_n = m$$

given that $n, m, k_0, \ldots, k_n$ *are non negative integers.*

Proof. We begin by starting with the 0th level of the classifying diagram. The functor

$$\iota : \coprod_n \Sigma_n \to \mathrm{iso}(\mathtt{FinSet}),$$

which is given by sending, for each n, the subcategory Σ_n to a set of order n defines an equivalence of categories. Applying the nerve functor gives the desired result.

Now we consider the 1st level of the classifying diagram. Note that the objects in $\mathrm{iso}(\mathtt{FinSet}^{[1]})$ are functions between finite sets, $f : A \to B$, where A is a set of order n and B is a set of order m. Let k_i be the number of elements in B whose preimage under f has cardinality i. The function $f : A \to B$ can be identified with the tree $\Gamma_{0,k_0} \cup \Gamma_{1,k_1} \cup \cdots \cup \Gamma_{n,k_n}$. The tree Γ_{0,k_0} represents a function from the empty set to a set of order k_0. There are no functions from the empty set to a set of order k_0, so we cannot include the tree Γ_{0,k_0} when computing the automorphisms of the tree that is identified with $f : A \to B$. Thus, in order to find the group of automorphisms on $f : A \to B$, we consider instead the tree $\Gamma_{1,k_2} \cup \cdots \cup \Gamma_{n,k_n}$, which has automorphism group $\Sigma_{k_1} \times (\Sigma_2 \wr \Sigma_{k_2}) \times \cdots \times (\Sigma_n \wr \Sigma_{k_n})$. For a fixed n and m, note that there is a bijective correspondence between automorphism classes of $f : A \to B$ and nonnegative integer solutions to the equations

$$k_1 + 2k_2 + \cdots + nk_n = n \quad \text{and} \quad k_0 + k_1 + \cdots + k_n = m.$$

Thus for a fixed n and m,

$$\coprod B\left(\Sigma_{k_1} \times (\Sigma_2 \wr \Sigma_{k_2}) \times \cdots \times (\Sigma_n \wr \Sigma_{k_n})\right),$$

where the disjoint union is over solutions to the above equations, captures the contributions to the first level of the classifying diagram of functions from sets of order n to sets of order m. Ranging over all possible n and m gives the desired result. □

Observe that the diagram

$$\begin{array}{ccc} & B\left(\Sigma_{k_1} \times (\Sigma_2 \wr \Sigma_{k_2}) \times \cdots \times (\Sigma_n \wr \Sigma_{k_n})\right) & \\ \swarrow^{d_1} & & \searrow^{d_0} \\ B(\Sigma_n) & & B(\Sigma_m) \end{array}$$

shows how the face maps $d_0, d_1 : N(\mathtt{FinSet})_1 \to N(\mathtt{FinSet})_0$ interact where the equations

$$k_1 + 2k_2 + \cdots + nk_n = n \quad \text{and} \quad k_0 + k_1 + \cdots + k_n = m$$

are satisfied.

Let $\mathtt{FinSet}_{inj}$ denote the subcategory of $\mathtt{FinSet}$ where the objects are the same as $\mathtt{FinSet}$, but the morphisms are restricted to the injective functions.

Corollary 5.3. *The zeroth level of the classifying diagram of* $\mathtt{FinSet}_{inj}$ *is given by*

$$N\left(\mathtt{FinSet}_{inj}\right)_0 \simeq \coprod_{n\in\mathbb{N}} B(\Sigma_n).$$

And the first level is described as

$$N\left(\mathtt{FinSet}_{inj}\right)_1 \simeq \coprod_{n,m\in\mathbb{N}, n\leq m} B(\Sigma_n).$$

Proof. The proof for the 0th level is identical to Theorem 5.2. So it remains to justify the description of the 1st level. Using the same setup at the proof of 5.2 with the exception that we require $f : A \to B$ to be injective and hence $n \leq m$, the function $f : A \to B$ is identified with the tree $\Gamma_{0,k_0} \cup \Gamma_{1,k_1} \cup \cdots \cup \Gamma_{n,k_n}$ where $k_i = 0$ if $i \geq 2$ since f is injective. Following the argument of the proof of 5.2, the system of equations is simplified to just

$$k_0 + n = m.$$

Thus for a fixed n and m where $n \leq m$,

$$B(\Sigma_n)$$

captures the contributions to the first level of the classifying diagram of injective functions from sets of order n to sets of order m. Ranging over all possible n and m, where $n \leq m$, gives the desired result. □

Let $\texttt{FinSet}_{surj}$ denote the subcategory of $\texttt{FinSet}$ where the objects are the same as $\texttt{FinSet}$, but the morphisms are restricted to surjective functions.

Corollary 5.4. *The zeroth level of the classifying diagram of* $\texttt{FinSet}$ *is given by*

$$N\left(\texttt{FinSet}_{surj}\right)_0 \simeq \coprod_{n \in \mathbb{N}} B(\Sigma_n).$$

And the first level is described as

$$N\left(\texttt{FinSet}_{surj}\right)_1 \simeq \coprod B\left(\Sigma_{k_1} \times (\Sigma_2 \wr \Sigma_{k_2}) \times \cdots \times (\Sigma_n \wr \Sigma_{k_n})\right)$$

where the disjoint union is over solutions to the equations

$$k_1 + 2k_2 + \cdots + nk_n = n \quad \text{and} \quad k_1 + \cdots + k_n = m$$

given that $n, m, k_0, \ldots, k_n$ *are non negative integers and* $n \geq m$.

Proof. The proof of the 0th level is identical to Theorem 5.2. To get the desired result for the 1st level, we implement the same setup as the proof of 5.2 with the exception that we require $f : A \to B$ to be a surjective function. Since f is surjective, $n \geq m$. Hence $f : A \to B$ is identified with the tree $\Gamma_{0,k_0} \cup \Gamma_{1,k_1} \cup \cdots \cup \Gamma_{n,k_n}$ where $k_0 = 0$ since f is surjective. Following the same argument of the proof of 5.2 with the added restrictions that $k_0 = 0$ and $n \geq m$ gives the desired result. □

6. The classification diagram

In Section 3, we saw that the classifying diagram of a category is a simplicial space; the isomorphisms in the category impact the structure of the classifying diagram. However, in homotopy theory we work with categories that have a weak notion of equivalences, such as weak homotopy equivalences in the category of topological spaces; the classifying diagram treats weak homotopy equivalences that are not homeomorphisms like ordinary morphisms. The classification diagram has a similar structure as the classifying diagram, but the weak equivalences are the morphisms that impact the structure of the resulting simplicial space instead of solely the isomorphisms.

Definition 6.1. The pair $(\mathcal{M}, \mathcal{W})$ denotes a category $\mathcal{M}$ along with a subcategory $\mathcal{W}$ where $\text{ob}(C) = \text{ob}(\mathcal{W})$ and the morphisms in $\mathcal{W}$ are called *weak equivalences*.

- Suppose $f : x \to y$ and $g : y \to z$ are morphisms in $\mathcal{M}$. The category $(\mathcal{M}, \mathcal{W})$ satisfies the *2-of-3 axiom* if two of f, g, and $g \circ f$ belong to $\mathcal{W}$ implies the third also belongs to $\mathcal{W}$ [6, 3.3].
- The pair $(\mathcal{M}, \mathcal{W})$ is a *category with weak equivalences* if $\mathrm{iso}(\mathcal{M}) \subseteq \mathcal{W}$ and the 2-of-3 axiom is satisfied.

Notation 6.2. Let $(\mathcal{M}, \mathcal{W})$ be a category with weak equivalences. We sometimes let $\mathrm{we}(\mathcal{M})$ denote the subcategory $\mathcal{W}$.

The letter choice of "$\mathcal{M}$" for the underlying category hints at the fact that the classification diagram is used to study model categories. But, in general we do not need the full power of a model category.

Definition 6.3. [13, 1.2] Let $(\mathcal{M}, \mathcal{W})$ be a category with weak equivalences. The *classification diagram* of $(\mathcal{M}, \mathcal{W})$ is denoted as $N(\mathcal{M}, \mathcal{W})$ and is the simplicial space defined levelwise as

$$N(\mathcal{M}, \mathcal{W})_n = \mathrm{nerve}(\mathrm{we}(\mathcal{M}^{[n]})).$$

Note that in the case where $\mathcal{W} = \mathrm{iso}(\mathcal{M})$, then the classification diagram and the classifying diagram are the same. In general, the classification diagram is not a complete Segal space because it fails to be Reedy fibrant. However, the classification diagram of a category is weakly equivalent to a complete Segal space [1, 6.2][13, 8.3].

6.1. **Categories with only weak equivalences.** Let us consider the analog to the classifying diagram of a group G. Since $\mathrm{iso}(G) \subseteq \mathrm{we}(G)$ and every morphism in G is an isomorphism, $(G, \mathrm{iso}(G))$ is a category with weak equivalences and $N(C) = N(C, \mathrm{iso}(G))$. So an appropriate analog for G is a category C with one object and every morphism a weak equivalence.

Proposition 6.4. *Let C be a category with one object and in which every morphism is a weak equivalence. Then the classification diagram $N(C, \mathcal{W})$ of C is levelwise weakly equivalent to the nerve of C. In other words $N(C, \mathcal{W})_n \simeq BC$ for any $n \geq 0$.*

Proof. Note that $N(C, \mathcal{W})_0 = BC$ since every morphism in C is a weak equivalence. For $n > 0$, by Proposition 2.14, it suffices to show that there exist functors $\iota : C \to C^{[n]}$ and $\rho : C^{[n]} \to C$ along with natural transformations $\alpha : \rho \circ \iota \Rightarrow id_C$ and $\beta : \iota \circ \rho \Rightarrow id_{C^{[n]}}$. Here we prove the case when $n = 2$; the proof can be extended to an arbitrary n.

Let $\bullet$ be the object of C. Define $\iota : C \to C^{[2]}$ on objects by

$$\bullet \longmapsto \left(\bullet \xrightarrow{id} \bullet \xrightarrow{id} \bullet \right)$$

and on morphisms by

$$\begin{array}{ccc} \bullet & & \bullet \xrightarrow{id} \bullet \xrightarrow{id} \bullet \\ \downarrow f & \longmapsto & \downarrow f \qquad \downarrow f \qquad \downarrow f \\ \bullet & & \bullet \xrightarrow{id} \bullet \xrightarrow{id} \bullet. \end{array}$$

Also define $\rho : C^{[2]} \to C$ on objects by

$$\left(\bullet \xrightarrow{g_1} \bullet \xrightarrow{g_2} \bullet \right) \longmapsto \bullet$$

and on morphisms by

$$\begin{array}{ccccc} \bullet & \xrightarrow{g_1} & \bullet & \xrightarrow{g_2} & \bullet \\ \Big\downarrow{\scriptstyle f_0} && \Big\downarrow{\scriptstyle f_1} && \Big\downarrow{\scriptstyle f_3} \\ \bullet & \xrightarrow{h_1} & \bullet & \xrightarrow{h_2} & \bullet \end{array} \quad\longmapsto\quad \begin{array}{c}\bullet\\ \Big\downarrow{\scriptstyle f_0}\\ \bullet.\end{array}$$

Note that $\rho \circ \iota = id_C : C \to C$ and hence $\alpha : \rho \circ \iota \Rightarrow id_C$ can be taken to be the identity natural transformation. Also note that $\iota \circ \rho : C^{[2]} \to C^{[2]}$ on morphisms is given by

$$\begin{array}{ccccc} \bullet & \xrightarrow{g_1} & \bullet & \xrightarrow{g_2} & \bullet \\ \Big\downarrow{\scriptstyle f_0} && \Big\downarrow{\scriptstyle f_1} && \Big\downarrow{\scriptstyle f_3} \\ \bullet & \xrightarrow{h_1} & \bullet & \xrightarrow{h_2} & \bullet \end{array} \quad\longmapsto\quad \begin{array}{ccccc} \bullet & \xrightarrow{id} & \bullet & \xrightarrow{id} & \bullet \\ \Big\downarrow{\scriptstyle f_0} && \Big\downarrow{\scriptstyle f_0} && \Big\downarrow{\scriptstyle f_0} \\ \bullet & \xrightarrow{id} & \bullet & \xrightarrow{id} & \bullet. \end{array}$$

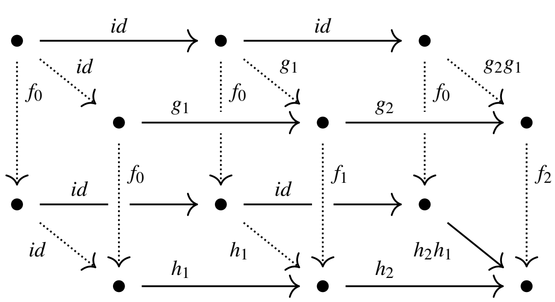
We need to define a natural transformation $\beta : \iota \circ \rho \Rightarrow id_{C^{[2]}}$. Let $x := \left(\bullet \xrightarrow{g_1} \bullet \xrightarrow{g_2} \bullet\right)$ and $y := \left(\bullet \xrightarrow{h_1} \bullet \xrightarrow{h_2} \bullet\right)$ be objects in $C^{[2]}$. Let $\beta_x : \iota \circ \rho(x) \to id_{C^{[2]}}(x)$ be the morphism in $C^{[2]}$ defined by the triple $(id, g_1, g_2 g_1)$. In other words, β_x is given by the vertical arrows in the commutative diagram

$$\begin{array}{ccccc} \bullet & \xrightarrow{id} & \bullet & \xrightarrow{id} & \bullet \\ \Big\downarrow{\scriptstyle id} && \Big\downarrow{\scriptstyle g_1} && \Big\downarrow{\scriptstyle g_2 g_1} \\ \bullet & \xrightarrow{g_1} & \bullet & \xrightarrow{g_2} & \bullet. \end{array}$$

In order to show β is a natural transformation, it needs to be checked that for any morphism $f := (f_0, f_1, f_2) : x \to y$ in $C^{[2]}$, the diagram

$$\begin{array}{ccc} \iota \circ \rho(x) & \xrightarrow{\beta_x} & x \\ \Big\downarrow{\scriptstyle \iota\circ\rho(f)} && \Big\downarrow{\scriptstyle f} \\ \iota \circ \rho(y) & \xrightarrow{\beta_x} & y \end{array}$$

commutes. This diagram can be rewritten as



where the front, back, top and bottom faces commute. (Note that the front faces commute by the definition of a morphism $f : x \to y$ in $C^{[2]}$.) In the above diagram, there are three squares formed by only dotted arrows; it remains to show that these three squares commute. The commutativity of these three squares follows from the commutativity of the front faces.

□

We generalize the previous result by considering the analog to the classifying diagram of a groupoid G. More specifically, we consider a category C where each morphism is a weak equivalence.

Proposition 6.5. *Let C be a category where every morphism is a weak equivalence. Then the classification diagram $N(C, \mathcal{W})$ of C is levelwise weakly equivalent to the nerve of C. In other words $N(C, \mathcal{W})_n \simeq BC$ for any $n \geq 0$.*

Proof. Note that $N(C, \mathcal{W})_0 = BC$. By Proposition 2.14, it suffices to show that there exist functors $\iota : C \to C^{[n]}$ and $\rho : C^{[n]} \to C$ along with natural transformations $\alpha : \rho \circ \iota \Rightarrow id_C$ and $\beta : \iota \circ \rho \Rightarrow id_{C^{[n]}}$. Here we outline the case when $n = 2$.

Define $\iota : C \to C^{[2]}$ on morphisms by

$$\begin{array}{ccc} c & & \begin{array}{ccccc} c & \xrightarrow{id} & c & \xrightarrow{id} & c \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} \\ d & \xrightarrow{id} & d & \xrightarrow{id} & d. \end{array} \\ \downarrow{\scriptstyle f} & \longmapsto & \\ d & & \end{array}$$

Also define $\rho : C^{[2]} \to C$ on morphisms by

$$\begin{array}{ccccccccc} c_0 & \xrightarrow{g_1} & c_1 & \xrightarrow{g_2} & c_2 & & c_0 \\ \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_3} & \longmapsto & \downarrow{\scriptstyle f_0} \\ d_0 & \xrightarrow{h_1} & d_1 & \xrightarrow{h_2} & d_2 & & d_0. \end{array}$$

Observe that $\rho \circ \iota = id_C$ and hence we let α be the identity natural transformation. It remains to define a natural transformation $\beta : \iota \circ \rho \Rightarrow id_{C^{[2]}}$. Let $x := \left(c_0 \xrightarrow{g_1} c_1 \xrightarrow{g_2} c_2\right)$ and $y := \left(d_0 \xrightarrow{h_1} d_1 \xrightarrow{h_2} d_2\right)$ be objects in $C^{[2]}$; let the triple (f_0, f_1, f_2) define a morphism $x \to y$. Note that $\iota \circ \rho : C^{[2]} \to C^{[2]}$ on morphisms is given by

$$\begin{array}{ccccccccccc} c_0 & \xrightarrow{g_1} & c_1 & \xrightarrow{g_2} & c_2 & & c_0 & \xrightarrow{id} & c_0 & \xrightarrow{id} & c_0 \\ \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_3} & \longmapsto & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_0} \\ d_0 & \xrightarrow{h_1} & d_1 & \xrightarrow{h_2} & d_2 & & d_0 & \xrightarrow{id} & d_0 & \xrightarrow{id} & d_0. \end{array}$$

If we let $\beta_x : \iota \circ \rho(x) \to id_{C^{[2]}}(x)$ be the morphism in $C^{[2]}$ defined by the triple $(id, g_1, g_2 g_1)$, it follows that β defines the desired natural transformation using a similar argument as the proof of Proposition 6.4.

□

If C is a category that has morphisms that are not weak equivalences, describing the classification diagram is a much more difficult task.

Here we say a category C is *weakly connected* if there exists a zigzag of morphisms between any two objects x and y in C. Also, a subcategory $\mathcal{D}$ is a *maximal weakly connected subcategory* of C if there does not exist a weakly connected subcategory $\mathcal{D}'$ of C such that $\mathcal{D}$ is a proper subcategory of $\mathcal{D}'$.

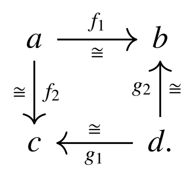
Remark 6.6. If $C = \coprod_i \mathcal{D}_i$, where each $\mathcal{D}_i$ is a distinct maximal weakly connected subcategory of C, then observe that levelwise the classification diagram of C is given by

$$N(C, \mathcal{W})_n \simeq BC \simeq \coprod_i B(\mathcal{D}_i).$$

Unfortunately, it is harder to classify the classification diagram in a more enlightening way. We saw in Theorem 3.15 that the classifying diagram can be written in terms of classifying spaces of stabilizers of automorphism groups. Perhaps having the capabilities to describe the classification diagram in terms of classifying spaces of groups is too much to wish for because we(C) does not naturally have a group structure. In the next two sections we see some of the difficulties that arise when a naive approach to a notion of homotopy automorphisms is used.

6.2. **Comparing the classifying diagram and the classification diagram.** It is tempting to naively define a homotopy automorphism by treating the weak equivalences like isomorphisms. There are categories where the classifying diagram and the classification diagram may appear to be levelwise weakly equivalent, but they give surprisingly different results. We observe in this section that the direction of the weak equivalences actually matters.

For example, consider the category C depicted as



For any object in C, observe that the only automorphism is the identity. Let $(\mathcal{D}, \mathcal{W})$ be the analogous category where the morphisms f_1, f_2, g_1, g_2 are weak equivalences instead of isomorphisms. It is tempting to think that $N(C)$ is equivalent to $N(\mathcal{D}, \mathcal{W})$. However, we show that the zeroth level of the classifying diagram of C is not weakly equivalent to the zeroth level of the classification diagram of $(\mathcal{D}, \mathcal{W})$.

Let us consider the zeroth level of the classifying diagram of C. Since C is a groupoid, we can apply Proposition 3.5 which gives us $N(C)_0 \simeq \text{nerve}(C) \simeq B(Aut(a))$. Note that $id_a = f_1 g_2^{-1} g_1 f_2^{-1}$, and hence $Aut(a) = \{id_a\}$. Thus we have that $|N(C)_0| \simeq *$. Now we turn our attention to the zeroth level of the classification diagram of $(\mathcal{D}, \mathcal{W})$. Note that there are only degenerate simplices in $\text{nerve}(\text{we}(\mathcal{D}))_n$ for $n \geq 2$. Thus we get that $|N(\mathcal{D}, \mathcal{W})_0| \simeq S^1$. Therefore we have shown that $N(C)$ and $N(\mathcal{D}, \mathcal{W})$ are not levelwise weakly equivalent.

In the next section we see a more in-depth example.

6.3. **The connected model structure on the category of graphs.** When describing the classifying diagram, we relied heavily on automorphism classes of objects. More specifically, if x is an object in a groupoid G and there exists an isomorphism between x and any other object of G, then the subcategory $\langle x \rangle$ is equivalent to G. In this section we use a model structure on the category of graphs to highlight the fact that a naive approach to defining homotopy automorphisms does not work. We work with undirected finite graphs that may have loops but have at most one edge between any two (not necessarily different) vertices.

Definition 6.7. A *graph* G is a symmetric binary relation on a finite set. We write $G = (V_G, E_G)$ where V_G is the underlying set of *vertices* and E_G is a set of unordered pairs of vertices called *edges*.

Definition 6.8. A *homomorphism* f between the graphs G and H is a map $f : V_G \to V_H$ such that for any vertices x and y in V_G such that (x, y) is an edge in E_G, then $(f(x), f(y))$ is an edge in E_H.

Let $\mathcal{G}_{all}$ denote the category where the objects are given by graphs, and the morphisms are given by graph homomorphism. Let $\mathcal{G}$ be the full subcategory of $\mathcal{G}_{all}$ with set of objects obtained by choosing one representative for each isomorphism type of graph. Thus $\mathcal{G}_{all}$ and $\mathcal{G}$ are equivalent categories. We choose to work with the category $\mathcal{G}$ because it is small.

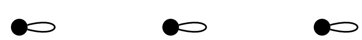
Definition 6.9. [4, 4.2] Let $\mathbb{CC}$ be the connected component model structure on $\mathcal{G}$ where the morphisms inducing isomorphisms between the sets of connected components of the two graphs are the weak equivalences.

This model structure is interested to consider because of the following theorem.

Theorem 6.10. [4, 4.2] *The homotopy category of* $\mathbb{CC}$ *is isomorphic to the category of finite sets.*

With this theorem in mind and since the classification diagram is a generalization of the classifying diagram, a natural question arises. Is the classifying diagram for the category `FinSet` the same as the classification diagram for the category $\mathbb{CC}$? More specifically, are $N(\texttt{FinSet})$ and $N(\mathbb{CC}, \mathcal{W})$ levelwise weakly equivalent?

Notation 6.11.
- (i) Let $n_\bullet$ denote the graph with n vertices and no edges.
- (ii) Let $n_\circ$ denote the graph with n vertices where each vertex has a loop. For example, the graph



  is $3_\circ$.

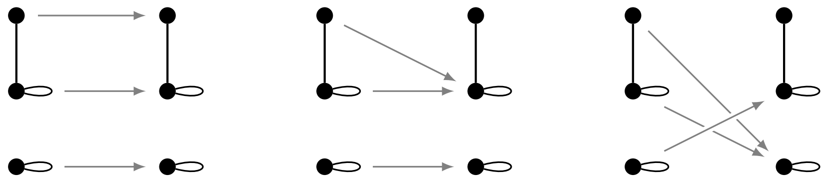
Let G be a graph with n connected components. In general the nth symmetric group Σ_n describes the collection of weak equivalences
- (i) $n_\bullet \to n_\bullet$,
- (ii) $n_\circ \to n_\circ$,
- (iii) $n_\bullet \to n_\circ$, and
- (iv) $G \to n_\circ$.

There are no homomorphisms (and hence no weak equivalences) $n_\circ \to n_\bullet$ because there are no edges to map the loops to in $n_\bullet$. We can always find weak equivalences $n_\bullet \to G$; there are at least $|\Sigma_n| = n!$ such weak equivalences, but the exact number of weak equivalences depends on the number of vertices in each connected component of G. Also the weak equivalences $G \to G$ are not necessarily described by Σ_n. Consider the following examples.

Example 6.12. We consider a couple of graphs with 2 connected components.
- (i) For the graph G depicted below

  the only weak equivalence $G \to G$ is the identity map.
- (ii) Consider the graph G

There are 3 weak equivalences $G \to G$, which are depicted below



Note that $2_\bullet$ is the initial graph in $\mathcal{G}$ that has 2 connected components, and $2_\circ$ is the terminal graph with 2 connected components; the group Σ_2 describes the collections of weak equivalences $\{2_\bullet \xrightarrow{\simeq} 2_\bullet\}$, $\{2_\circ \xrightarrow{\simeq} 2_\circ\}$, and $\{2_\bullet \xrightarrow{\simeq} 2_\circ\}$. Let G be a graph with two connected components. We can find a chain of weak equivalences $2_\bullet \xrightarrow{\simeq} G \xrightarrow{\simeq} 2_\circ$ and we only have two options for the result of the composition. However, as we saw in the above examples, the collection of weak equivalences $\{G \xrightarrow{\simeq} G\}$ are not necessarily described by Σ_2.

So since $n_\bullet$ is initial and $n_\circ$ is terminal in the collection of graphs with n connected components, it is tempting to naively want to define the homotopy equivalences classes based of the weak equivalences $n_\bullet \xrightarrow{\simeq} n_\circ$, but this fails to consider what happens with an arbitrary graph with n connected components.

Remark 6.13. When we gave characterizations of classifying diagrams in Section 3, we relied heavily on automorphism classes of objects and morphisms. It would be nice if we could use a similar approach for classification diagrams, but we showed above that a naive approach for handling weak equivalences to get a notion of a homotopy automorphism does not work.

In [1, §6], a notion of homotopy automorphisms for simplicial model categories is defined.It is shown that the classification diagram is weakly equivalent to a complete Segal space that is decomposed in terms of classifying spaces of homotopy automorphisms of the appropriate simplicial category [1, 7.3].

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