DISTRIBUTED CONTROL OF MULTI-AGENT SYSTEMS AND MANAGEMENT OF NETWORKED BATTERY UNITS

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Abstract

In this dissertation, distributed control problems of multi-agent systems are studied. The applications of distributed control algorithms to the management of networked battery units are also investigated.

The leader-following almost output consensus problem for both continuous-time and discrete-time linear heterogeneous multi-agent systems is considered, in which the unstable zero dynamics of the follower agents are affected by disturbance. Due to the inapplicability of high gain feedback to the discrete-time setting, different conditions on the way the agents are affected by the disturbance in the two cases have to be identified. Low-and-high gain-based state feedback and output feedback consensus protocols are proposed for continuous-time multi-agent systems. State feedback and output feedback consensus protocols for discrete-time multi-agent systems are constructed based on low gain feedback and a modified discrete-time Riccati equation. The proposed consensus protocols are shown to achieve leader-following output consensus to an arbitrarily high level of accuracy, and attenuate the effect of the disturbance on the consensus errors to an arbitrarily low level.

The almost output consensus problem of nonlinear multi-agent systems is then considered. Conditions on the nonlinear systems are established under which distributed consensus protocols are designed in a recursive manner. The protocols are shown to achieve almost output consensus, that is, output consensus of the system is achieved in the absence of the disturbances, and the L_2-gain from the disturbances to the output consensus error of agents when the system is operating in output consensus can be made arbitrarily small.

The suboptimal output consensus problem for discrete-time heterogeneous linear multi-agent systems with unstable zero dynamics is also studied, where each agent possesses its own objective function, and the sum of all these private objective functions, called the overall objective function, is to be minimized. Mild assumptions on the communication topology and the agent dynamics are made under which a parameterized distributed consensus protocol based on low gain feedback is proposed for each agent. The multi-agent system is shown to achieve suboptimal output consensus under the proposed protocols in the sense that the states of all agents remain bounded while their outputs converge to a pre-specified arbitrarily small neighborhood of the optimal point as long as the design parameter is chosen small enough.

Finally, the management problem of networked battery units in DC microgrids is studied. Specifically, the control problem of balancing the state-of-charge (SoC) among the networked battery units while satisfying the total charging/discharging power demand is considered. Power allocation

algorithms for the battery units that make use of distributed estimators for the average desired power and the average unit state and the adaptive parameter estimators are proposed. Power allocation algorithms are also proposed based on adaptive parameter estimations for battery units with unknown parameters. Algorithms that make use of SoC observers based on equivalent circuit models of the batteries are also constructed for networked battery systems with unknown SoC. The algorithms are shown to achieve SoC balancing among all battery units while satisfying the power demand.

APPROVAL SHEET

This

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Chapter 1

Introduction

The consensus problem of multi-agent systems has been an active research topic in recent years due to its applications in various research and engineering areas such as distributed sensor fusion in sensor networks, management and control of power systems, and formation control of multi-vehicle systems (see, *e.g.*, [10, 14, 67, 72, 68, 98, 40, 12]). Consensus or output consensus of a multi-agent system entails that the states or outputs of all its agents reach agreement at some value, under the influence of consensus protocols that specify interaction among neighbor agents.

Applications of consensus protocols to real-world systems inevitably lead to practical considerations of the consensus protocols such as their robustness to disturbances, wind gusts affecting the formation flight control of unmanned aerial vehicles [46, 32], as an example. Disturbance rejection in individual systems has been an active research topic for several decades (see, *e.g.*, [50, 28, 29, 42, 44]). There have also been many results on the design of consensus protocols that eliminate or reduce the effect of the disturbances on the consensus errors (see, *e.g.*, [25, 47, 95, 79, 3, 13, 78, 18, 76, 81]). Most works on the design of consensus protocols for multi-agent systems subject to disturbances have implicitly assumed that the agent dynamics is of minimum phase, or nonminimum phase systems with disturbances not affecting the unstable zero dynamics. However, many actual dynamical systems are known to exhibit non-minimum phase characteristics. For example, the inverted pendulum on a cart [21], and the V/STOL aircraft [23] are both nonminimum phase systems. The disturbances that affect the unstable zero dynamics of a system incur additional difficulty in meeting the control objectives. Therefore, we are motivated to consider the consensus problem of multi-agent systems whose agents dynamics are of nonminimum phase and are affected by the disturbance. Besides the disturbance, nonlinearity in the dynamics of agents is also commonly seen in real-world multi-agent systems. Therefore, in this thesis, we design consensus protocols that achieve leader-following almost output consensus for both continuous-time and discrete-time multi-agent systems with disturbance-affected unstable zero dynamics. The "almost output consensus" is borrowed from the terminology "almost disturbance decoupling" for single systems. Here by "almost" we mean that the output consensus error caused by the disturbance can be attenuated to an arbitrarily low level. We also design consensus protocols for nonlinear multi-agent systems that are affected by the disturbance.

The optimal consensus of a multi-agent system, on the other hand, entails that the states or outputs of its agents not only reach an agreement, but such an agreement also minimizes the sum of the individual objective functions of all agents, called the overall objective function. Originally investigated in [65] for discrete-time systems, the optimal consensus problem has been further studied in [48, 37, 22, 80]. The optimal output consensus is studied in [82, 83]. The global optimal consensus problem is studied for multi-agent systems with bounded controls in [88], [89] and [96], where the agent dynamics is represented by first-order and second-order integrators, continuous-time general high order systems, and first-order discrete-time systems, respectively. In [90], the suboptimal output consensus problem is considered for general continuous-time multi-agent systems, whose agent dynamics are described by a weakly nonminimum phase linear system. In this thesis, we solve the suboptimal output consensus problem for a discrete-time multi-agent system, whose agents may possess polynomial unstable zero dynamics (*i.e.*, the agents may be of weakly nonminimum phase). In order to prevent the states of the unstable zero dynamics from growing unbounded as the output approaches a nonzero constant value, the proposed design achieves suboptimal output consensus, instead of optimal output consensus, by allowing the output to vary in a neighborhood of the optimal point, whose size can be pre-specified to be arbitrarily small.

Battery energy storage systems (BESSs) have seen rapid growth recently in microgrid applications as their performance and durability continue to improve and their costs decrease [38]. BESSs can be used to compensate the peak power demand and absorb excess power in off-peak time [39], therefore enabling the generation equipment to operate near its optimal efficiency. They can also be combined with renewable energy resources, which are of intermittent and stochastic characteristics, to enhance supply reliability [63, 97]. Compared with other energy storage systems, BESSs have their advantages, such as high efficiency, high energy density, and versatility [11].

Control of BESSs is an important but challenging problem (see, *e.g.*, [86, 100, 64, 77, 99, 101]). Every battery unit in a BESS has to be judiciously controlled so that safe and efficient operation can be ensured while the charging and discharging requirements from the microgrid applications are fulfilled. An important control problem for a BESS is to balance the state-of-charge (SoC) of all its battery units while delivering the total desired charging/discharging power. Because of the differences between the units' intrinsic characteristics and their ambient variations, the SoC of the units tends to differ from each other under normal operations. Maintaining SoC balancing can not only avoid overcharge/overdischarge of battery units but also maximize the available energy storage capacity and charging/discharging rates (see, *e.g.*, [85, 49, 27, 62]). In real-world applications, the parameters such as capacities and terminal voltages of the battery units may not be precisely known. Their true values may deviate from their nominal values due to aging and/or variations in their manufacturing. In addition, the units' parameters may vary under severe operating conditions such as heavy load or low/high-temperature [102]. In this thesis, we consider the control design for networked heterogeneous battery units in a BESS with both known and unknown parameters. We design both non-adaptive and adaptive distributed power-allocating algorithms, under which the SoC of all the units achieves balancing and the power demand is satisfied. Unlike voltage and current, the SoC of battery units is not directly measurable. Therefore, SoC estimation is necessary and is a challenging problem due to the complex chemical process in batteries. In this thesis, we also consider the SoC observer design for the battery units based on an equivalent circuit model of battery dynamics.

This thesis is organized as follows. In Chapter 2, we formulate and solve the almost output consensus problem for linear continuous-time multi-agent systems with disturbance-affected unstable zero dynamics. In Chapter 3, we formulate and solve the almost output consensus problem for linear discrete-time multi-agent systems with disturbance-affected unstable zero dynamics. In Chapter 4, we formulate and solve the almost output consensus problem for nonlinear multi-agent systems in the presence of external disturbance. In Chapter 5, we formulate and solve the suboptimal output consensus problem for linear discrete-time multi-agent systems with unstable zero dynamics. In Chapter 6, we formulate and solve the control problem of balancing the state-of-charge among battery units in networked battery systems while satisfying the total charging/discharging power demand.

We use graphs to represent the communication networks in multi-agent systems. Specifically, for a leaderless multi-agent system consisting of N agents, its communication topology is described by a directed graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$, where $\mathcal{V} = \{v_1, v_2, \cdots, v_N\}$ is the set of nodes representing the agents, and $\mathcal{E}$ is the set of edges representing the communication channels among agents. An edge in $\mathcal{G}$ is an ordered pair $(v_i, v_j) \in \mathcal{E}$, in which v_i is said to be the parent of v_j. A path in the directed graph $\mathcal{G}$ from node v_{i_1} to node v_{i_k} is a ordered sequence of edges $(v_{i_1}, v_{i_2}), (v_{i_2}, v_{i_3}), \cdots, (v_{i_{k-1}}, v_{i_k})$. A direct graph is called a directed tree if every node in the graph has one parent, except the root which has no parent, and there is a directed path from the root to every other node. A directed tree is called a directed spanning tree of graph $\mathcal{G}$ if it contains all the nodes in $\mathcal{G}$. The adjacency matrix $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N\times N}$ of the graph $\mathcal{G}$ is defined as $a_{ij} = 1$ if agent i can receive information directly from follower agent j, otherwise $a_{ij} = 0$. In addition, we assume that $a_{ii} = 0$, $i \in \{1, 2, \cdots, N\}$. The Laplacian matrix associated with the graph $\mathcal{G}$ is defined as $\mathcal{L} = [l_{ij}] \in \mathbb{R}^{N\times N}$, where $l_{ij} = -a_{ij}$ if $i \neq j$ and $l_{ii} = \sum_{k=1,k\neq i}^{N} a_{ik}$. The graph $\mathcal{G}$ is undirected if $a_{ij} = a_{ji}$, for all $i, j, \in \{1, 2, \cdots, N\}$.

The communication topology of a leader-following multi-agent system is described by a directed graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$, where $\mathcal{V} = \{v_0, v_1, \cdots, v_N\}$ is the set of nodes with v_0 representing the leader agent and $v_1, v_2, \cdots, v_N$ representing the N follower agents, and $\mathcal{E}$ is the set of edges representing the communication channels among agents. Let $\bar{\mathcal{G}} = \{\bar{\mathcal{V}}, \bar{\mathcal{E}}\}$ be a subgraph of $\mathcal{G}$ that is only associated with the communication topology of the N followers, *i.e.*, $\bar{\mathcal{V}} = \{v_1, v_2, \cdots, v_N\}$. The adjacency matrix $\bar{\mathcal{A}} = [a_{ij}] \in \mathbb{R}^{N\times N}$ and the Laplacian matrix $\bar{\mathcal{L}} = [l_{ij}] \in \mathbb{R}^{N\times N}$ are defined with the subgraph $\bar{\mathcal{G}}$. The matrix $\bar{\mathcal{B}} = \mathrm{diag}\{b_1, b_2, \cdots, b_N\} \in \mathbb{R}^{N\times N}$ is defined to indicate the direct accessibility of the leader agent's information by the follower agents, where $b_i = 1$ means that follower agent i has direct knowledge of the leader agent, otherwise $b_i = 0$.

The following notations will be used. Let $\mathbb{R}^n$ denote n-dimensional Euclidean space. For a matrix A, let $\mathrm{Im}(A)$ denote its image space, $\|A\|$ denote its norm and $\lambda(A)$ denote its eigenvalues. For a vector v, let $\|v\|$ denote its norm. Let $\mathbb{C}$, $\mathbb{C}^{\odot}$, and $\mathbb{C}^{\bigcirc}$ denote the entire complex plane, the set of complex numbers inside the unit circle, and the set of complex numbers on the unit circle, respectively.

Chapter 2

Almost Output Consensus of Linear Continuous-Time Multi-Agent Systems

2.1 Introduction

In this chapter, we design both state feedback and output feedback consensus protocols for the continuous-time multi-agent systems with disturbance-affected unstable zero dynamics over a directed communication topology. The poles of the zero dynamics are allowed to be anywhere in the closed left half-plane. The condition of the way the disturbance affects the zero dynamics of each follower agent is identified. The leader agent's output to be followed can be any bounded signal that does not contain the frequency components of the $j\omega$-axis invariant zeros of the follower agents. Novel state feedback and output feedback consensus protocols are constructed of a low-and-high gain feedback structure in which the low gain feedback design technique [43] is utilized to stabilize the zero dynamics of each follower agent by allowing its output to vary within a small neighborhood of the desired output, and therefore maintains the boundedness of its state in the absence of the disturbances. A state observer that facilitates the output feedback design is constructed for each agent under the assumption that the agent dynamics are detectable. The observer error is shown to be in the same order as the disturbance affecting the dynamics of the agent in terms of the L_2-gain. We show that, these state feedback and output feedback protocols achieve leader-following almost output consensus, as long as the communication topology of the multi-agent system contains a directed spanning tree with the leader as the root node. More specifically, the leader-following output consensus is achieved to any pre-specified degree of accuracy while the states remain bounded in the absence of the disturbances, and when the system is operating in output consensus within the desired level of accuracy, the L_2-gain from the disturbances to the difference between each follower agent's output with and without the disturbances from the same initial condition is attenuated to any desired level of accuracy. We note that, compared to the results on individual systems, where the output converges toward zero precisely in the absence of the disturbances, the output of the multiple agents under our proposed protocols reaches consensus to a time-varying desired signal only with a pre-specified accuracy. In order to analyze the effect of the disturbance on the output consensus of the system, we have to compare the output in the absence and in the presence of the disturbance.

The remainder of this chapter is organized as follows. Section 2.2 formulates the leader-following almost output consensus problem for linear continuous-time multi-agent systems with disturbance-affected unstable zero dynamics. Section 2.3 establishes the state feedback results. Section 2.4 presents the simulation for state feedback design. Section 2.5 establishes the output feedback results. Section 2.6 presents the simulation for output feedback design. Section 2.7 concludes this chapter.

2.2 Problem Statement

In this section, we will formulate the leader-following almost output consensus problem for linear continuous-time multi-agent systems with disturbance-affected unstable zero dynamics. Consider a linear heterogeneous multi-agent system consisting of N follower agents and one leader agent. The communication network among these agents is represented by a directed graph $\mathcal{G}$ that satisfies the following assumption.

Assumption 2.1. The directed graph $\mathcal{G}$ representing the communication network of the leader-following multi-agent system contains a directed spanning tree with the leader as the root node.

The dynamics of follower agents are described by

$$\begin{cases} \dot{x}_{i,0} = A_{i,0}x_{i,0} + B_{i,0}x_{i,1} + D_{i,0}w_i, \\ \dot{x}_{i,m} = x_{i,m+1} + d_{i,m}w_i, \ m = 1, 2, \cdots, \rho - 1, \\ \dot{x}_{i,\rho} = E_{i,0}x_{i,0} + \beta_{i,1}x_{i,1} + \beta_{i,2}x_{i,2} + \cdots + \beta_{i,\rho}x_{i,\rho} + u_i + d_{i,\rho}w_i, \\ y_i = x_{i,1}, \ i \in \{1, 2, \cdots, N\}, \end{cases} \tag{2.1}$$

where $x_{i,0} = [x_{i,0,1} \ x_{i,0,2} \ \cdots \ x_{i,0,r_i}]^{\mathrm{T}} \in \mathbb{R}^{r_i}$ and $x_i = [x_{i,1} \ x_{i,2} \ \cdots \ x_{i,\rho}]^{\mathrm{T}} \in \mathbb{R}^{\rho}$ are the states, $u_i \in \mathbb{R}$ is the control input, $y_i \in \mathbb{R}$ is the output, and $w_i \in \mathbb{R}$ is the disturbance. Let $D_{i,0} = [d_{i,0,1} \ d_{i,0,2} \ \cdots \ d_{i,0,r_i}]^{\mathrm{T}} \in \mathbb{R}^{r_i}$. The zero dynamics of the ith follower agent is the dynamics when the output is restricted to zero and is represented by

$$\dot{x}_{i,0} = A_{i,0}x_{i,0}, \ i \in \{1, 2, \cdots, N\}.$$

It is noted that any single input single output linear system can be transformed into the form of (2.1), where ρ is the relative degree of the system, through a state transformation [5]. The follower agents in the form of (2.1) are heterogeneous except they share the same relative degree ρ.

The leader agent provides the desired output for the follower agents to follow and thus its dynamics is assumed to have the same relative degree as the follower agents, as given by

$$\begin{cases} \dot{x}_{0,m} = x_{0,m+1}, \ m = 1, 2, \cdots, \rho - 1, \\ \dot{x}_{0,\rho} = u_0, \\ \quad y_0 = x_{0,1}, \end{cases} \tag{2.2}$$

where $x_0 = [x_{0,1} \ x_{0,2} \ \cdots \ x_{0,\rho}]^{\mathrm{T}} \in \mathbb{R}^\rho$ is the state, $u_0 \in \mathbb{R}$ is any control input that produces the desired bounded leader output $y_0 \in \mathbb{R}$.

Assumptions 2.2 and 2.3 below, respectively, ensure that the dynamics of the follower agents is stabilizable and detectable (for output feedback design).

Assumption 2.2. The pair $(A_{i,0}, B_{i,0})$ that represents the zero dynamics is stabilizable, and all eigenvalues of $A_{i,0}$ are in the closed left half-plane.

Assumption 2.3. (For output feedback) The pair $(A_{i,0}, E_{i,0})$ is detectable.

Assumption 2.4. The vector $D_{i,0}$ satisfies

$$D_{i,0} \in \cap_{\omega \in \lambda^0(A_{i,0})} \mathrm{Im}(\omega I - A_{i,0}), \ i \in \{1, 2, \cdots, N\},$$

where $\lambda^0(A_{i,0})$ are the set of all imaginary eigenvalues of $A_{i,0}$.

Assumption 2.5. The leader agent's output $y_0(t)$ does not contain frequencies corresponding to the imaginary eigenvalues of $A_{i,0}$, $i \in \{1, 2, \cdots, N\}$.

The leader-following almost output consensus problem for linear multi-agent systems with disturbance-affected unstable zero dynamics is stated as follows.

Problem 2.1. Consider the linear multi-agent system described by (2.1) and (2.2). For any pre-specified $\eta > 0$ and $\gamma > 0$, design distributed state feedback and output feedback consensus protocols under which leader-following almost output consensus is achieved in the following sense:

(i) In the absence of the disturbance, the states of all follower agents are bounded.

(ii) In the absence of the disturbance, leader-following output consensus is achieved within an accuracy specified by η, *i.e.*,

$$\limsup_{t \to \infty} |y_{i,w=0}(t) - y_0(t)| \leq \eta, \ i \in \{1, 2, \cdots, N\},$$

where $y_{i,w=0}$ is the ith follower agent's output.

(iii) In steady state, the effect of the disturbance $w = [w_1\ w_2\ \cdots\ w_N]^{\mathrm{T}}$ on the leader-following output consensus, measured by the L_2-gain, is attenuated to a level specified by γ, *i.e.*,

$$\int_0^{\infty} \big(y_i(t) - y_{i,w=0}(t)\big)^2 \mathrm{d}t \le \gamma^2 \int_0^{\infty} \|w(t)\|^2 \mathrm{d}t.$$

where $y_i(t)$ and $y_{i,w=0}(t)$ are, respectively, the output of the ith follower agent in the presence and in the absence of the disturbance.

2.3 State Feedback Results

In this section, state feedback consensus protocols are designed in the following three steps. First, a low gain feedback law is designed for each follower agent that stabilizes its unstable zero dynamics. Second, a new output is renamed and a new set of states is defined for each follower agent, based on the low gain feedback law. Last, a high gain feedback law utilizing the new states is designed to attenuate the effect of the disturbance.

Step 1: Low Gain Feedback

For each follower agent i , $i \in \{1, 2, \cdots, N\}$, find a nonsingular transformation $T_{i,0} \in \mathbb{R}^{r_i \times r_i}$ such that the pair $\big(A_{i,0}, B_{i,0}\big)$ is transformed into

$$T_{i,0}^{-1} A_{i,0} T_{i,0} = \begin{bmatrix} A_{i,0}^0 & 0 \\ 0 & A_{i,0}^- \end{bmatrix},\ T_{i,0}^{-1} B_{i,0} = \begin{bmatrix} B_{i,0}^0 \\ B_{i,0}^- \end{bmatrix}, \tag{2.3}$$

where $A_{i,0}^0 \in \mathbb{R}^{r_i^0 \times r_i^0}$ contains all the purely imaginary eigenvalues of $A_{i,0}$, and $A_{i,0}^- \in \mathbb{R}^{r_i^- \times r_i^-}$ contains all the open left half-plane eigenvalues of $A_{i,0}$. Since $A_{i,0}^-$ is Hurwitz, only the dynamics associated with $A_{i,0}^0$ is needed to be stabilized. Under Assumption 2.2, $\big(A_{i,0}^0, B_{i,0}^0\big)$ is controllable and, without loss of generality, can be assumed to be in the following canonical form,

$$A_{i,0}^0 = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ \alpha_{i,1} & \alpha_{i,2} & \cdots & \alpha_{i,r_i^0} \end{bmatrix},\ B_{i,0}^0 = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}. \tag{2.4}$$

For each $\left(A_{i,0}^0, B_{i,0}^0\right)$, let $F_{i,0}^0(\varepsilon) \in \mathbb{R}^{1\times r_i^0}$ be such that

$$\lambda\big(A_{i,0}^0 - B_{i,0}^0 F_{i,0}^0(\varepsilon)\big) = -\varepsilon + \lambda\big(A_{i,0}^0\big). \tag{2.5}$$

Such design is referred to as low gain feedback since $F_{i,0}^0(\varepsilon)$ tends to zero as the value of the design parameter ε tends to zero.

Let

$$u_{\mathrm{L},i} = -F_{i,0}(\varepsilon)x_{i,0},\ \varepsilon \in (0,1],\ i \in \{1,2,\cdots,N\},$$

where $F_{i,0}(\varepsilon) = \begin{bmatrix} F_{i,0}^0(\varepsilon) & 0 \end{bmatrix} T_{i,0}^{-1}$.

The following lemmas are recalled since they are useful in the later analysis of the closed-loop system.

Lemma 2.1. [43] Consider $\left(A_{i,0}^0, B_{i,0}^0\right)$ in the form of (2.4) and $F_{i,0}^0(\varepsilon)$ in (2.5). There exists constant $\bar{f}_{i,0} > 0$ such that

$$\|F_{i,0}^0(\varepsilon)\| \le \bar{f}_{i,0}\varepsilon,\ \varepsilon \in (0,1].$$

Lemma 2.2. [45] Consider $(A_{i,0}^0, B_{i,0}^0)$ in the form of (2.4) and $F_{i,0}^0(\varepsilon)$ in (2.5). Let

$$\det\big(sI - A_{i,0}^0 + B_{i,0}^0 F_{i,0}^0(\varepsilon)\big) = (s+\varepsilon)^{r_{i,0}^0} \prod_{l=1}^{l_i} (s+\varepsilon - j\omega_{i,l})^{r_{i,l}^0} (s+\varepsilon + j\omega_{i,l})^{r_{i,l}^0}.$$

Then, there exist $\varepsilon^* > 0$ and constant $\gamma_i > 0$ such that

$$\left\| F_{i,0}^0(\varepsilon)\big(j\omega I - A_{i,0}^0 + B_{i,0}^0 F_{i,0}^0(\varepsilon)\big)^{-1} \right\| \le \gamma_i \varepsilon \sum_{l=0}^{l_i} \sum_{r=1}^{r_{i,l}^0} \left| \frac{1}{(j\omega - j\omega_{i,l} + \varepsilon)^r} \right|,\ \varepsilon \in (0,\varepsilon^*],$$

where $\omega_{i,0} = 0$.

Lemma 2.3. [43] Consider $(A_{i,0}^0, B_{i,0}^0)$ in the form of (2.4) and $F_{i,0}^0(\varepsilon)$ in (2.5). There exists a nonsingular transformation matrix $Q_i(\varepsilon) \in \mathbb{R}^{r_i^0 \times r_i^0}$ such that

$$\begin{aligned} Q_i^{-1}(\varepsilon)\big(A_{i,0}^0 - B_{i,0}^0 F_{i,0}^0(\varepsilon)\big)Q_i(\varepsilon) &= J_i(\varepsilon) \\ &= \mathrm{blkdiag}\{J_{i,0}(\varepsilon), J_{i,1}(\varepsilon), \cdots, J_{i,l_i}(\varepsilon)\}, \end{aligned}$$

where

$$J_{i,0} = \begin{bmatrix} -\varepsilon & 1 & & \\ & \ddots & \ddots & \\ & & -\varepsilon & 1 \\ & & & -\varepsilon \end{bmatrix}_{r_{i,0}^0 \times r_{i,0}^0}$$

and for each $l = 1$ to l_i,

$$J_{i,l}(\varepsilon) = \begin{bmatrix} J_{i,l}^\star(\varepsilon) & I_2 & & \\ & \ddots & \ddots & \\ & & J_{i,l}^\star(\varepsilon) & I_2 \\ & & & J_{i,l}^\star(\varepsilon) \end{bmatrix}_{2r_{i,l}^0 \times 2r_{i,l}^0}, \quad J_{i,l}^\star(\varepsilon) = \begin{bmatrix} -\varepsilon & \omega_{i,l} \\ -\omega_{i,l} & -\varepsilon \end{bmatrix},$$

with $\omega_{i,l} > 0$ for all $l = 1$ to l_i and $\omega_{i,l} \neq \omega_{i,k}$ for $l \neq k$. In addition, there exists constant $\bar{q}_i \geq 0$ such that

$$\|Q_i(\varepsilon)\| \leq \bar{q}_i, \ \|Q_i^{-1}(\varepsilon)\| \leq \bar{q}_i, \ \varepsilon \in (0,1].$$

Lemma 2.4. [43] Consider $(A_{i,0}^0, B_{i,0}^0)$ in the form of (2.4) and $F_{i,0}^0(\varepsilon)$ in (2.5). Let $Q_i(\varepsilon)$, l_i, $r_{i,l}^0$, and $l \in I[0, l_i]$, be as defined in Lemma 2.3. Let $S_i(\varepsilon) = \text{blkdiag}\{S_{i,0}(\varepsilon), S_{i,1}(\varepsilon), S_{i,2}(\varepsilon), \cdots, S_{i,l_i}(\varepsilon)\}$, with $S_{i,0}(\varepsilon) = \text{diag}\Big\{\varepsilon^{r_{i,0}^0 - 1}, \varepsilon^{r_{i,0}^0 - 2}, \cdots, \varepsilon, 1\Big\}$ and $S_{i,l}(\varepsilon) = \text{diag}\Big\{\varepsilon^{r_{i,l}^0 - 1} I_2, \varepsilon^{r_{i,l}^0 - 2} I_2, \cdots, \varepsilon I_2, I_2\Big\}$, $l \in \{1, 2, \cdots, l_i\}$. Then,

1. $$S_i(\varepsilon) J_i(\varepsilon) S_i^{-1}(\varepsilon) = \varepsilon \tilde{J}_i(\varepsilon),$$

   where $\tilde{J}_i(\varepsilon) = \text{blkdiag}\{\tilde{J}_{i,0}, \ \tilde{J}_{i,1}(\varepsilon), \cdots, \ \tilde{J}_{i,l_i}\}$,

$$\tilde{J}_{i,0} = \begin{bmatrix} -1 & 1 & & \\ & \ddots & \ddots & \\ & & -1 & 1 \\ & & & -1 \end{bmatrix}_{r_{i,0}^0 \times r_{i,0}^0}$$

and for each $l = 1$ to l_i,

$$\tilde{J}_{i,l}(\varepsilon) = \begin{bmatrix} \tilde{J}^{\star}_{i,l}(\varepsilon) & I_2 & & & \\ & \ddots & \ddots & & \\ & & J^{\star}_{i,l}(\varepsilon) & I_2 & \\ & & & J^{\star}_{i,l}(\varepsilon) \end{bmatrix}_{2r^0_{i,l}\times 2r^0_{i,l}}, \quad \tilde{J}^{\star}_{i,l}(\varepsilon) = \begin{bmatrix} -1 & \omega_{i,l}/\varepsilon \\ -\omega_{i,l}/\varepsilon & -1 \end{bmatrix},$$

with $\omega_{i,l} > 0$ for all $l = 1$ to l_i and $\omega_{i,l} \neq \omega_{i,k}$ for $l \neq k$.

2. The unique positive-definite solution $\tilde{P}_{i,0}$ to the Lyapunov equation

$$\tilde{J}_i^{\mathrm{T}}(\varepsilon)\tilde{P}_{i,0} + \tilde{P}_{i,0}\tilde{J}_i(\varepsilon) = -I$$

is independent of ε.

3. There exists positive constant $\kappa_i > 0$ such that

$$\left\| F^0_{i,0}(\varepsilon)Q_i(\varepsilon)S_i^{-1}(\varepsilon) \right\| \leq \kappa_i\varepsilon, \ \varepsilon \in (0,1].$$

Lemma 2.5. [43] Let $D_{i,0}$, $i \in I[1,N]$, satisfy Assumption 2.4. Let $Q_i(\varepsilon)$ be as given in Lemma 2.3. Partition $Q_i^{-1}(\varepsilon)D^0_{i,0}$ according to that of $J_i(\varepsilon)$ in Lemma 2.3 as

$$Q_i^{-1}(\varepsilon)D^0_{i,0} = \begin{bmatrix} D^0_{i,0,0}(\varepsilon) \\ D^0_{i,0,1}(\varepsilon) \\ \vdots \\ D^0_{i,0,l_i}(\varepsilon) \end{bmatrix},$$

with

$$D^0_{i,0,0} = \begin{bmatrix} D^0_{i,0,0,1}(\varepsilon) \\ D^0_{i,0,0,2}(\varepsilon) \\ \vdots \\ D^0_{i,0,0,r^0_{i,0}}(\varepsilon) \end{bmatrix}_{r^0_{i,0}\times 1}, \text{ and } D^0_{i,0,l} = \begin{bmatrix} D^0_{i,0,l,1}(\varepsilon) \\ D^0_{i,0,l,2}(\varepsilon) \\ \vdots \\ D^0_{i,0,l,r^0_{i,l}}(\varepsilon) \end{bmatrix}_{r^0_{i,l}\times 1}, \ l \in \{1,2,\cdots,l_i\}.$$

Then, there exists constant $\beta_i \geq 0$ such that

$$\left\| D^0_{i,0,l,r^0_{i,l}}(\varepsilon) \right\| \leq \beta_i\varepsilon, \ \varepsilon \in (0,1], l \in \{0,1,2,\cdots,l_i\}.$$

Step 2: Output Renaming

Define a new output for each follower agent i as

$$\tilde{y}_i = y_i - u_{\mathrm{L},i} = x_{i,1} + F_{i,0}(\varepsilon)x_{i,0}.$$

For each $i \in \{1, 2, \cdots, N\}$, also define the state transformation

$$z_{i,0} = \begin{bmatrix} z_{i,0}^{0\ \mathrm{T}} & z_{i,0}^{-\ \mathrm{T}} \end{bmatrix}^{\mathrm{T}} = T_{i,0}^{-1} x_{i,0}.$$

Correspondingly, denote $\begin{bmatrix} D_{i,0}^{0\ \mathrm{T}} & D_{i,0}^{-\ \mathrm{T}} \end{bmatrix}^{\mathrm{T}} = T_{i,0}^{-1} D_{i,0}$. Then, the dynamics of $z_{i,0}$ is given as

$$\begin{bmatrix} \dot{z}_{i,0}^0 \\ \dot{z}_{i,0}^- \end{bmatrix} = \begin{bmatrix} A_{i,0}^0 - B_{i,0}^0 F_{i,0}^0 & 0 \\ B_{i,0}^- F_{i,0}^0 & A_{i,0}^- \end{bmatrix} \begin{bmatrix} z_{i,0}^0 \\ z_{i,0}^- \end{bmatrix} + \begin{bmatrix} B_{i,0}^0 \\ B_{i,0}^- \end{bmatrix} \tilde{y}_i + \begin{bmatrix} D_{i,0}^0 \\ D_{i,0}^- \end{bmatrix} w_i. \tag{2.6}$$

Based on the new output $\tilde{y}_i$ and the state transformation $z_{i,0} = \begin{bmatrix} z_{i,0}^{0\ \mathrm{T}} & z_{i,0}^{-\ \mathrm{T}} \end{bmatrix}^{\mathrm{T}}$, a set of new states $\tilde{z}_{i,0}^0$ and $\tilde{x}_i = \begin{bmatrix} \tilde{x}_{i,1} & \tilde{x}_{i,2} & \cdots & \tilde{x}_{i,\rho} \end{bmatrix}^{\mathrm{T}}$ are defined as

$$\begin{cases} \tilde{z}_{i,0}^0 = S_i(\varepsilon) Q_i^{-1}(\varepsilon) z_{i,0}^0, \\ \tilde{x}_{i,1} = \tilde{y}_i = x_{i,1} + F_{i,0}(\varepsilon) x_{i,0}, \\ \tilde{x}_{i,2} = x_{i,2} + F_{i,0}(\varepsilon) A_{i,0} x_{i,0} + F_{i,0}(\varepsilon) B_{i,0} x_{i,1}, \\ \qquad \vdots \\ \tilde{x}_{i,\rho} = x_{i,\rho} + F_{0,i}(\varepsilon) A_{0,i}^{\rho-1} x_{i,0} + F_{i,0}(\varepsilon) A_{i,0}^{\rho-2} B_{i,0} x_{i,1} + F_{i,0}(\varepsilon) A_{i,0}^{\rho-3} B_{i,0} x_{i,2} + \cdots + F_{i,0}(\varepsilon) B_{i,0} x_{i,\rho-1}. \end{cases}$$

Design a pre-feedback law as

$$\begin{aligned} u_i = & -E_{i,0} x_{i,0} - \beta_{i,1} x_{i,1} - \beta_{i,2} x_{i,2} - \cdots - \beta_{i,\rho} x_{i,\rho} - F_{i,0}(\varepsilon) A_{i,0}^{\rho} x_{i,0} \\ & - F_{i,0}(\varepsilon) A_{i,0}^{\rho-1} B_{i,0} x_{i,1} - F_{i,0}(\varepsilon) A_{i,0}^{\rho-2} B_{i,0} x_{i,2} - \cdots - F_{i,0}(\varepsilon) B_{i,0} x_{i,\rho} + \tilde{u}_i, \end{aligned} \tag{2.7}$$

where $\tilde{u}_i$ is to be designed later. Then, the dynamics of follower agent i in the new states is written as

$$\begin{cases} \dot{\tilde{z}}_{i,0}^0 = \varepsilon \tilde{J}_i(\varepsilon)\tilde{z}_{i,0}^0 + \tilde{B}_{i,0}^0 \tilde{y}_i + \tilde{D}_{i,0}^0 w_i, \\ \dot{\tilde{x}}_{i,1} = \tilde{x}_{i,2} + \tilde{d}_{i,1} w_i, \\ \dot{\tilde{x}}_{i,2} = \tilde{x}_{i,3} + \tilde{d}_{i,2} w_i, \\ \qquad \vdots \\ \dot{\tilde{x}}_{i,\rho-1} = \tilde{x}_{i,\rho} + \tilde{d}_{i,\rho-1} w_i, \\ \dot{\tilde{x}}_{i,\rho} = \tilde{u}_i + \tilde{d}_{i,\rho} w_i, \end{cases}$$

where $\tilde{J}_i(\varepsilon)$ is defined as in Lemma 2.4, $\tilde{B}_{i,0}^0 = S_i(\varepsilon)Q_i^{-1}B_{i,0}^0$, $\tilde{D}_{i,0}^0 = S_i(\varepsilon)Q_i^{-1}D_{i,0}^0$, and $\tilde{d}_{i,m}$, $m = 1, 2, \cdots, \rho$, are defined in a straightforward way. Note that $\|\tilde{B}_{i,0}^0\| \le \bar{b}$ for some constant $\bar{b} > 0$ by Lemma 2.3, and $\|\tilde{D}_{i,0}^0\| \le \bar{d}\varepsilon$ for some constant $\bar{d} > 0$, by Lemma 2.5.

Step 3: Protocol Design

Define the consensus error $\xi_i = [\xi_{i,1}\ \xi_{i,2}\ \cdots\ \xi_{i,\rho}]^{\mathrm{T}}$ as

$$\xi_{i,m} = \sum_{j=1}^{N} a_{ij}(\tilde{x}_{i,m} - \tilde{x}_{j,m}) + b_i(\tilde{x}_{i,m} - x_{0,m}) \tag{2.8}$$

where $m = 1, 2, \cdots, \rho$, $i \in \{1, 2 \cdots, N\}$.

Then, the consensus error dynamics is given as

$$\begin{cases} \dot{\xi}_{i,m} = \sum_{j=1}^{N} a_{ij}\big((\tilde{x}_{i,m+1} + \tilde{d}_{i,m}w_i) - (\tilde{x}_{j,m+1} + \tilde{d}_{j,m}w_j)\big) + b_i\big((\tilde{x}_{i,m+1} + \tilde{d}_{i,m}w_i) - x_{0,m+1}\big), \\ \qquad m = 1, 2, \cdots, \rho - 1, \\ \dot{\xi}_{i,\rho} = \sum_{j=1}^{N} a_{ij}\big((\tilde{u}_i + \tilde{d}_{i,\rho}w_i) - (\tilde{u}_j + \tilde{d}_{j,\rho}w_j)\big) + b_i\big((\tilde{u}_i + \tilde{d}_{i,\rho}w_i) - u_0\big), \end{cases}$$

or in a compact form,

$$\begin{cases} \dot{\xi}_{i,m} = \xi_{i,m+1} + \bar{d}_{i,m}^{\mathrm{T}} w,\ \ m = 1, 2, \cdots, \rho - 1, \\ \dot{\xi}_{i,\rho} = \bar{u}_i + \bar{d}_{i,\rho}^{\mathrm{T}} w, \end{cases} \tag{2.9}$$

where we have denoted

$$\bar{d}_{i,m} = \big[-a_{i1}\tilde{d}_{1,m}\ \ -a_{i2}\tilde{d}_{2,m}\ \ \cdots\ \ -a_{i(i-1)}\tilde{d}_{i-1,m}\ \ (|\mathcal{N}_i| + b_i)\tilde{d}_{i,m}\ \ -a_{i(i+1)}\tilde{d}_{i+1,m}\ \ \cdots\ \ -a_{iN}\tilde{d}_{N,m}\big]^{\mathrm{T}},$$
$$m = 1, 2, \cdots, \rho,$$

$$
\begin{aligned}
w &= [w_1 \ w_2 \ \cdots \ w_N]^{\mathrm{T}}, \\
\bar{u}_i &= \sum_{j=1}^{N} a_{ij}(\tilde{u}_i - \tilde{u}_j) + b_i(\tilde{u}_i - u_0),
\end{aligned}
$$

where $\mathcal{N}_i$ is the set of neighbors of follower agent i. Design $\tilde{u}_i$ such that

$$
\bar{u}_i = -\frac{1}{\varepsilon^{\rho}} f_1 \xi_{i,1} - \frac{1}{\varepsilon^{\rho-1}} f_2 \xi_{i,2} - \cdots - \frac{1}{\varepsilon} f_{\rho} \xi_{i,\rho}, \tag{2.10}
$$

where $f_1, f_2, \cdots, f_{\rho}$ are the coefficients of any Hurwitz polynomial $s^{\rho} + f_{\rho} s^{\rho-1} + f_{\rho-1} s^{\rho-2} + \cdots + f_1$. It is clear that such $\tilde{u}_i$ is given by

$$
\tilde{u}_i = \left(\sum_{i=1}^{N} a_{ij} + b_i \right)^{-1} \left(\sum_{i=1}^{N} a_{ij} \tilde{u}_j + b_i u_0 + \bar{u}_i \right). \tag{2.11}
$$

It then follows from (2.7) and (2.10) that the consensus protocol for each follower agent in the original coordinates is written as

$$
\begin{aligned}
u_i = & -E_{i,0} x_{i,0} - \beta_{i,1} x_{i,1} - \beta_{i,2} x_{i,2} - \cdots - \beta_{i,\rho} x_{i,\rho} - F_{i,0}(\varepsilon) A_{i,0}^{\rho} x_{i,0} - F_{i,0}(\varepsilon) A_{i,0}^{\rho-1} B_{i,0} x_{i,1} \\
& - F_{i,0}(\varepsilon) A_{i,0}^{\rho-2} B_{i,0} x_{i,2} - \cdots - F_{i,0}(\varepsilon) B_{i,0} x_{i,\rho} \\
& + \left(\sum_{i=1}^{N} a_{ij} + b_i \right)^{-1} \left(\sum_{i=1}^{N} a_{ij} \tilde{u}_j + b_i u_0 - \frac{1}{\varepsilon^{\rho}} f_1 \xi_{i,1} - \frac{1}{\varepsilon^{\rho-1}} f_2 \xi_{i,2} - \cdots - \frac{1}{\varepsilon} f_{\rho} \xi_{i,\rho} \right).
\end{aligned} \tag{2.12}
$$

Remark 2.1. The construction of each consensus protocol u_i only involves a state transformation that block diagonalizes matrix A_i, separating the open left-half plane eigenvalues from the $j\omega$-axis eigenvalues, a state transformation that transforms the pair (A_i, B_i) into its controllable canonical form, and a pole placement algorithm.

Theorem 2.1. [55] Consider the multi-agent system with agent dynamics described by (2.1) and (2.2). Let the communication topology satisfy Assumption 2.1. Let the follower agents' dynamics satisfy Assumptions 2.2 and 2.4. Let the leader agent's output to be followed satisfy Assumption 2.5. Then, for any given $\eta > 0$ and $\gamma > 0$, there exists $\varepsilon^* \in (0, 1]$ such that, with any $\varepsilon \in (0, \varepsilon^*]$, the parameterized state feedback consensus protocols (2.12) solve Problem 2.1:

(i) The state of each follower agent is bounded in the absence of the disturbances.

(ii) The leader-following output consensus is achieved to the given accuracy η in the absence of the disturbances, *i.e.*, there exists finite time $T \geq 0$ such that

$$|y_{i,w=0}(t) - y_0(t)| \leq \eta,\ t \geq T,\ i \in \{1, 2, \cdots, N\}.$$

(iii) In steady state, the effect of the disturbance $w = [w_1\ w_2\ \cdots\ w_N]^{\mathrm{T}}$ on the the leader-following output consensus, measured by the L_2-gain, is attenuated to the level specified by γ, *i.e.*,

$$\int_0^\infty \big(y_i(t) - y_{i,w=0}(t)\big)^2 \mathrm{d}t \leq \gamma^2 \int_0^\infty \|w(t)\|^2 \mathrm{d}t.$$

Proof: In the proof, we will first show that the proposed consensus protocols stabilize the unstable zero dynamics and guarantee the boundedness of all follower agents' states, in the absence of the disturbances. We will then show that the leader-following output consensus can be achieved with a specified accuracy $\eta > 0$ by showing that all follower agents' renamed outputs $\tilde{y}_i(t)$, $i \in \{1, 2, \cdots, N\}$, will converge to the leader's output $y_0(t)$, and the difference between each follower agent's output $y_i(t)$ and its renamed output $\tilde{y}_i(t)$ can be made less than η in steady state. Finally, we will evaluate the effect of the disturbances on the consensus errors through Lyapunov analysis.

Consider the closed-loop disturbance-affected $\tilde{z}_{i,0}^0$ dynamics

$$\dot{\tilde{z}}_{i,0}^0 = \varepsilon \tilde{J}_i(\varepsilon) \tilde{z}_{i,0}^0 + \tilde{B}_{i,0}^0 \tilde{y}_i + \tilde{D}_{i,0}^0 w_i,$$

under the control

$$\begin{aligned} u_{\mathrm{L},i} &= -F_{i,0}(\varepsilon) x_{i,0} \\ &= F_{i,0}^0(\varepsilon) z_{i,0}^0 \\ &= -F_{i,0}^0(\varepsilon) Q_i(\varepsilon) S_i^{-1}(\varepsilon) \tilde{z}_{i,0}^0 \\ &\triangleq \tilde{u}_{\mathrm{L},i}(\tilde{z}_{i,0}^0). \end{aligned}$$

Let $\tilde{y}_{i,w=0}(t)$ and $\tilde{y}_{i,w}(t)$ with $\tilde{y}_{i,w}(0) = 0$ be, respectively, the zero input response and the zero state response of $\tilde{y}_i(t)$, that is,

$$\tilde{y}_i(t) = \tilde{y}_{i,w=0}(t) + \tilde{y}_{i,w}(t),$$

by view the disturbance w as the input. Accordingly, we decompose $z_{i,0}^0$ and $\tilde{z}_{i,0}^0$ as

$$\begin{aligned} z_{i,0}^0(t) &= z_{i,0,w=0}^0(t) + z_{i,0,w}^0(t) \\ \tilde{z}_{i,0}^0(t) &= \tilde{z}_{i,0,w=0}^0(t) + \tilde{z}_{i,0,w}^0(t), \end{aligned}$$

with $z^0_{i,0,w}(0)=0$, $\tilde{z}^0_{i,0,w}(0)=0$, and

$$\dot{\tilde{z}}^0_{i,0,w=0} = \varepsilon\tilde{J}_i(\varepsilon)\tilde{z}^0_{i,0,w=0} + \tilde{B}^0_{i,0}\tilde{y}_{i,w=0}, \tag{2.13}$$

$$\dot{\tilde{z}}^0_{i,0,w} = \varepsilon\tilde{J}_i(\varepsilon)\tilde{z}^0_{i,0,w} + B^0_{i,0}\tilde{y}_{i,w} + \tilde{D}^0_{i,0}w_i. \tag{2.14}$$

It is noted that, in the absence of the disturbances, the response of each follower agent's $\tilde{z}^0_{i,0}$ dynamics is governed by (2.13). In the presence of the disturbances, the response of each follower agent's $\tilde{z}^0_{i,0}$ dynamics is the superposition of (2.13) and (2.14).

Consider the closed-loop system of the consensus error dynamics (2.9) with $\bar{u}_i$ being designed as (2.10). Let $\tilde{\xi}_i = [\tilde{\xi}_{i,1}\ \tilde{\xi}_{i,2}\ \cdots\ \tilde{\xi}_{i,\rho}]^{\mathrm{T}} = [\xi_{i,1}\ \varepsilon\xi_{i,2}\ \cdots\ \varepsilon^{\rho-1}\xi_{i,\rho}]^{\mathrm{T}}$. Then, the dynamics in state $\tilde{\xi}_i$ is rewritten as

$$\begin{cases} \varepsilon\dot{\tilde{\xi}}_{i,1} = \tilde{\xi}_{i,2} + \varepsilon\bar{d}^{\mathrm{T}}_{i,1}w, \\ \varepsilon\dot{\tilde{\xi}}_{i,2} = \tilde{\xi}_{i,3} + \varepsilon^2\bar{d}^{\mathrm{T}}_{i,2}w, \\ \qquad\vdots \\ \varepsilon\dot{\tilde{\xi}}_{i,\rho-1} = \tilde{\xi}_{i,\rho} + \varepsilon^{\rho-1}\bar{d}^{\mathrm{T}}_{i,\rho-1}w, \\ \varepsilon\dot{\tilde{\xi}}_{i,\rho} = -f_1\tilde{\xi}_{i,1} - f_2\tilde{\xi}_{i,2} - \cdots - f_\rho\tilde{\xi}_{i,\rho} + \varepsilon^\rho\bar{d}^{\mathrm{T}}_{i,\rho}w. \end{cases} \tag{2.15}$$

It is obvious that

$$\lim_{t\to\infty}\tilde{\xi}_{i,m}(t) = 0,\ m=1,2,\cdots,\rho,\ i\in\{1,2,\cdots,N\},$$

in the absence of the disturbances because of the choice of the parameters $f_1, f_2, \cdots, f_\rho$.

Under Assumption 2.1, we have $\mathcal{H} = \bar{\mathcal{L}} + \bar{\mathcal{B}} > 0$ [26] and $\bar{\mathcal{L}}\mathbf{1} = \mathbf{0}$. It follows from (2.8) that

$$\begin{aligned} [\xi_{1,m}\ \xi_{2,m}\ \cdots\ \xi_{N,m}]^{\mathrm{T}} &= \mathcal{H}[\tilde{x}_{1,m}\ \tilde{x}_{2,m}\ \cdots\ \tilde{x}_{N,m}]^{\mathrm{T}} - \bar{\mathcal{B}}x_{0,m}\mathbf{1} \\ &= \mathcal{H}[\tilde{x}_{1,m}\ \tilde{x}_{2,m}\ \cdots\ \tilde{x}_{N,m}]^{\mathrm{T}} - (\bar{\mathcal{B}}+\bar{\mathcal{L}})x_{0,m}\mathbf{1} \\ &= \mathcal{H}\big([\tilde{x}_{1,m}\ \tilde{x}_{2,m}\ \cdots\ \tilde{x}_{N,m}]^{\mathrm{T}} - x_{0,m}\mathbf{1}\big), \end{aligned} \tag{2.16}$$

$m=1,2,\cdots,\rho$, and hence for $i\in\{1,2,\cdots,N\}$, we have

$$\begin{aligned} &\lim_{t\to\infty}\big(\tilde{y}_{i,w=\mathbf{0}}(t) - y_0(t)\big) = 0, \\ &\lim_{t\to\infty}\big(\tilde{x}_{i,m}(t) - x_{0,m}(t)\big) = 0,\ m=2,3,\cdots,\rho. \end{aligned}$$

Note that the leader's output y_0 is bounded by the assumption on the leader agent. Under the consensus protocols, $\tilde{y}_{i,w=0}$ converges toward y_0 exponentially with bounded steady state error.

Thus, $\tilde{y}_{i,w=0}$ is bounded. On the other hand, $\tilde{z}^0_{i,w=0}(0) = S_i(\varepsilon)Q_i^{-1}(\varepsilon)z^0_{i,0}(0)$ by definition and is therefore finite at $t = 0$. Then, in view of equation (2.14), $\tilde{z}^0_{i,w=0}$ remains bounded. Thus, it follows from the fact that $\lim_{t\to\infty}\tilde{\xi}_i = 0$, $i \in \{1, 2, \cdots, N\}$, that the states of the follower agents are bounded in the absence of the disturbances.

Recall that

$$u_{\mathrm{L},i,w=0} = -F^0_{i,0}(\varepsilon)z^0_{i,w=0},$$

In the absence of the disturbance. Then, under Assumption 2.5, Lemma 2.2 implies that, for any given η, there exists $\varepsilon_i^* > 0$ such that for all $\varepsilon \in (0, \varepsilon_i^*]$, $u_{\mathrm{L},i,w=0}(t)$ satisfies

$$\limsup_{t\to\infty} |u_{\mathrm{L},i,w=0}(t)| \le \frac{1}{2}\eta,\ i \in \{1, 2, \cdots, N\}. \tag{2.17}$$

Let $\varepsilon^*_\eta = \min\{\varepsilon^*_1, \varepsilon^*_2, \cdots, \varepsilon^*_N\}$. Then, it follows from

$$y_{i,w=0}(t) = \tilde{y}_{i,w=0}(t) + u_{\mathrm{L},i,w=0}(t)$$

that for any given η,

$$\begin{aligned}\limsup_{t\to\infty} \left|y_{i,w=0}(t) - y_0(t)\right| &= \limsup_{t\to\infty} \left|\left(\tilde{y}_{i,w=0}(t) + u_{\mathrm{L},i,w=0}(t)\right) - y_0(t)\right| \\ &\le \limsup_{t\to\infty} \left|\tilde{y}_{i,w=0}(t) - y_0(t)\right| + \limsup_{t\to\infty} |u_{\mathrm{L},i,w=0}(t)| \\ &\le \eta,\ i \in \{1, 2, \cdots, N\}.\end{aligned}$$

for all $\varepsilon \in (0, \varepsilon^*_z]$,

In the presence of the disturbances, we consider the Lyapunov function $V_i(\tilde{\xi}_i) = \tilde{\xi}_i^{\mathrm{T}}\tilde{P}\tilde{\xi}_i$, where $\tilde{P} \in \mathbb{R}^{\rho\times\rho}$ is the unique positive definite solution to the following Lyapunov equation,

$$\tilde{P}\tilde{A} + \tilde{A}^{\mathrm{T}}\tilde{P} = -I,$$

with

$$\tilde{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -f_1 & -f_2 & -f_3 & \cdots & -f_\rho \end{bmatrix}$$

being Hurwitz.

The time derivative of V_i along the trajectory of (2.15) can be written as

$$\begin{aligned}\dot{V}_i &= \frac{1}{\varepsilon}\tilde{\xi}_i^{\mathrm{T}}\tilde{P}\Big(\tilde{A}\tilde{\xi}_i + \tilde{D}_i(\varepsilon)w\Big) + \frac{1}{\varepsilon}\Big(\tilde{A}\tilde{\xi}_i + \tilde{D}_i(\varepsilon)w\Big)^{\mathrm{T}}\tilde{P}\tilde{\xi}_i \\ &= \frac{1}{\varepsilon}\tilde{\xi}_i^{\mathrm{T}}\big(\tilde{P}\tilde{A} + \tilde{A}^{\mathrm{T}}\tilde{P}\big)\tilde{\xi}_i + \frac{1}{\varepsilon}\tilde{\xi}_i^{\mathrm{T}}\tilde{P}\tilde{D}_i(\varepsilon)w + \frac{1}{\varepsilon}w^{\mathrm{T}}\tilde{D}_i^{\mathrm{T}}(\varepsilon)\tilde{P}\tilde{\xi}_i \\ &\le -\frac{1}{\varepsilon}\tilde{\xi}_i^{\mathrm{T}}\tilde{\xi}_i + \frac{2}{\varepsilon}\|\tilde{\xi}_i\|\big\|\tilde{P}\tilde{D}_i(\varepsilon)\big\|\|w\| \\ &\le -\frac{1}{\varepsilon}\tilde{\xi}_i^{\mathrm{T}}\tilde{\xi}_i + \frac{1}{\varepsilon}\Big(\frac{1}{4}\|\tilde{\xi}_i\|^2 + 4\big\|\tilde{P}\tilde{D}_i(\varepsilon)\big\|^2\|w\|^2\Big) \\ &\le -\frac{3}{4\varepsilon}\|\tilde{\xi}_i\|^2 + \frac{4}{\varepsilon}\big\|\tilde{P}\big\|^2\big\|\tilde{D}_i(\varepsilon)\big\|^2\|w\|^2,\end{aligned} \tag{2.18}$$

where $\tilde{D}_i(\varepsilon) = [\varepsilon\bar{d}_{i,1}\ \varepsilon^2\bar{d}_{i,2}\ \cdots\ \varepsilon^\rho\bar{d}_{i,\rho}]^{\mathrm{T}}$ and

$$\lim_{\varepsilon\to 0}\big\|\tilde{D}_i(\varepsilon)\big\| = 0.$$

When the system is operating in steady state, we have $\tilde{\xi}_{i,m}(0) = 0$, $m = 1, 2, \cdots, \rho$, $i \in \{1, 2, \cdots, N\}$, which implies that $V_i\big(\xi_i(0)\big) = 0$. By integrating both sides of inequality (2.18) and using $V_i\big(\xi_i(0)\big) = 0$, we obtain that

$$\int_0^\infty \|\tilde{\xi}_i(t)\|^2\mathrm{d}t \le \frac{16}{3}\big\|\tilde{P}\big\|^2\big\|\tilde{D}_i(\varepsilon)\big\|^2\int_0^\infty \|w(t)\|^2\mathrm{d}t,$$

and hence

$$\int_0^\infty \|\tilde{\xi}_{i,1}(t)\|^2\mathrm{d}t \le \frac{16}{3}\big\|\tilde{P}\big\|^2\big\|\tilde{D}_i(\varepsilon)\big\|^2\int_0^\infty \|w(t)\|^2\mathrm{d}t.$$

The L_2-gain from the disturbance w to the difference between $\tilde{y} = [\tilde{y}_1\ \tilde{y}_2\ \cdots\ \tilde{y}_N]^{\mathrm{T}}$, the renamed output of follower agents, and $y_0\mathbf{1}$, the output of the leader, can be obtained from (2.16) as

$$\begin{aligned}\int_0^\infty \|\tilde{y}(t) - y_0(t)\mathbf{1}\|^2\mathrm{d}t &= \int_0^\infty \big\|\mathcal{H}^{-1}\big\|^2\big\|[\xi_{1,1}(t)\ \xi_{2,1}(t)\ \cdots\ \xi_{N,1}(t)]^{\mathrm{T}}\big\|^2\mathrm{d}t \\ &\le \frac{1}{\lambda_{\min}^2(\mathcal{H})}\int_0^\infty\sum_{i=1}^N \|\tilde{\xi}_{i,1}(t)\|^2\mathrm{d}t \\ &\le \frac{16\|\tilde{P}\|^2}{3\lambda_{\min}^2(\mathcal{H})}\left(\sum_{i=1}^N\big\|\tilde{D}_i(\varepsilon)\big\|^2\right)\int_0^\infty \|w(t)\|^2\mathrm{d}t.\end{aligned} \tag{2.19}$$

Since $\tilde{\xi}_{i,1}(0) = 0$, we have

$$\tilde{y}_{i,w=0}(t) = y_0(t),\ t \ge 0, i \in \{1, 2, \cdots, N\}.$$

Replace $y_0(t)$ by $\tilde{y}_{i,w=0}(t)$ in (2.19), we have

$$\int_0^\infty \|\tilde{y}_i(t) - \tilde{y}_{i,w=0}(t)\|^2 \mathrm{d}t \leq \frac{16\|\tilde{P}\|^2}{3\lambda_{\min}^2(\mathcal{H})} \left(\sum_{i=1}^N \|\tilde{D}_i(\varepsilon)\|^2 \right) \int_0^\infty \|w(t)\|^2 \mathrm{d}t. \tag{2.20}$$

Recall that (2.14) can be rewritten as

$$\dot{\tilde{z}}_{i,0,w}^0 = \varepsilon \tilde{J}_i(\varepsilon) \tilde{z}_{i,0,w}^0 + \tilde{B}_{i,0}^0 (\tilde{y}_i - \tilde{y}_{i,w=0}) + \tilde{D}_{i,0}^0 w_i. \tag{2.21}$$

Consider the Lyapunov function

$$V_{i,0}\big(\tilde{z}_{i,0,w}^0\big) = \tilde{z}_{i,0,w}^0{}^{\mathrm{T}} \tilde{P}_{i,0} \tilde{z}_{i,0,w}^0$$

with $\tilde{P}_{i,0}$ being as defined in Lemma 2.4. The time derivative of $V_{i,0}$ along the trajectory of (2.21) can be written as

$$\begin{aligned}
\dot{V}_{i,0} &= \Big(\varepsilon \tilde{J}_i(\varepsilon) \tilde{z}_{i,0,w}^0 + \tilde{B}_{i,0}^0 (\tilde{y}_i - \tilde{y}_{i,w=0}) + \tilde{D}_{i,0}^0 w_i\Big)^{\mathrm{T}} \tilde{P}_{i,0} \tilde{z}_{i,0,w}^0 \\
&\quad + \tilde{z}_{i,0,w}^0{}^{\mathrm{T}} \tilde{P}_{i,0} \Big(\varepsilon \tilde{J}_i(\varepsilon) \tilde{z}_{i,0,w}^0 + \tilde{B}_{i,0}^0 (\tilde{y}_i - \tilde{y}_{i,w=0}) + \tilde{D}_{i,0}^0 w_i\Big) \\
&\leq -\varepsilon \|\tilde{z}_{i,0,w}^0\|^2 + 2\tilde{z}_{i,0,w}^0{}^{\mathrm{T}} \tilde{P}_{i,0} \tilde{B}_{i,0}^0 (\tilde{y}_i - \tilde{y}_{i,w=0}) + 2\tilde{z}_{i,0,w}^0{}^{\mathrm{T}} \tilde{P}_{i,0} \tilde{D}_{i,0}^0 w_i \\
&\leq -\frac{\varepsilon}{2} \|\tilde{z}_{i,0,w}^0\|^2 + \frac{4}{\varepsilon} \|\tilde{P}_{i,0}\|^2 \|\tilde{B}_{i,0}^0\|^2 (\tilde{y}_i - \tilde{y}_{i,w=0})^2 + 4\varepsilon \bar{d}^2 \|\tilde{P}_{i,0}\|^2 \|w_i\|^2.
\end{aligned}$$

Integrating both sides of the above inequality and using $V_{i,0}\big(\tilde{z}_{i,0,w}^0(0)\big) = 0$ and (2.20), we obtain that

$$\int_0^\infty \|\tilde{z}_{i,0,w}^0(t)\|^2 \mathrm{d}t \leq 8\|\tilde{P}_{i,0}\|^2 \left(\frac{16 \sum_{i=1}^N \|\tilde{D}_i(\varepsilon)\|^2}{3\varepsilon^2 \lambda_{\min}^2(\mathcal{H})} \|\tilde{P}_{i,0}\|^2 \|\tilde{B}_{i,0}^0\|^2 + \bar{d}^2 \right) \int_0^\infty \|w(t)\|^2 \mathrm{d}t.$$

Recall that $u_{\mathrm{L},i,w} = -F_{i,0}^0(\varepsilon) Q_i(\varepsilon) S_i^{-1}(\varepsilon) \tilde{z}_{i,0,w}^0$ and the fact $|F_{i,0}^0(\varepsilon) Q_i(\varepsilon) S_i^{-1}(\varepsilon)| \leq \kappa_i \varepsilon$ by Lemma 2.4, we have

$$\int_0^\infty u_{\mathrm{L},i,w}^2(t) \mathrm{d}t \leq 8\kappa_i^2 \|\tilde{P}_{i,0}\|^2 \left(\frac{16 \sum_{i=1}^N \|\tilde{D}_i(\varepsilon)\|^2}{3\lambda_{\min}^2(\mathcal{H})} \|\tilde{P}_{i,0}\|^2 \|\tilde{B}_{i,0}^0\|^2 + \varepsilon^2 \bar{d}^2 \right) \int_0^\infty \|w(t)\|^2 \mathrm{d}t. \tag{2.22}$$

Consider the difference between $y_i(t)$, the output of each follower agent i in the presence of the disturbances, and $y_{i,w=0}(t)$, the output of each follower agent i in the absence of the disturbances from the same initial condition. Since $y_i = \tilde{y}_i + u_{\mathrm{L},i} = \tilde{y}_i + (u_{\mathrm{L},i,w=0} + u_{\mathrm{L},i,w})$, we have,

$$\begin{aligned} y_i - y_{i,w=0} &= (\tilde{y}_{i,w\neq 0} + u_{\mathrm{L},i,w=0} + u_{\mathrm{L},i,w}) - (\tilde{y}_{i,w=0} + u_{\mathrm{L},i,w=0}) \\ &= \tilde{y}_i - \tilde{y}_{i,w=0} + u_{\mathrm{L},i,w}. \end{aligned} \tag{2.23}$$

Integrate the squares of both sides of (2.23) and use inequality (2.22), we have

$$\begin{aligned} &\int_0^\infty \big(y_i(t) - y_{i,w=0}(t)\big)^2 \mathrm{d}t \\ &\leq \int_0^\infty \Big(2\big(\tilde{y}_i(t) - \tilde{y}_{i,w=0}(t)\big)^2 + 2u^2_{\mathrm{L},i,w}(t)\Big)\mathrm{d}t \\ &\leq \left(\frac{32\big\|\tilde{P}\big\|^2\big(\sum_{i=1}^N \big\|\tilde{D}_i(\varepsilon)\big\|^2\big)}{3\lambda^2_{\min}(\mathcal{H})} + 16\kappa_i^2\big\|\tilde{P}_{i,0}\big\|^2\left(\frac{16\sum_{i=1}^N \|\tilde{D}_i(\varepsilon)\|^2}{3\lambda^2_{\min}(\mathcal{H})}\|\tilde{P}_{i,0}\|^2\bar{b}^2 + \varepsilon^2\bar{d}^2\right)\right)\int_0^\infty \|w(t)\|^2\mathrm{d}t. \end{aligned}$$

For any given $\gamma > 0$, let $\varepsilon^*_\gamma \in (0,1]$ be such that

$$\left(\frac{32\big\|\tilde{P}\big\|^2\big(\sum_{i=1}^N \big\|\tilde{D}_i(\varepsilon^*_\gamma)\big\|^2\big)}{3\lambda^2_{\min}(\mathcal{H})} + 16\kappa_i^2\big\|\tilde{P}_i\big\|^2\left(\frac{16\sum_{i=1}^N \|\tilde{D}_i(\varepsilon^*_\gamma)\|^2}{3\lambda^2_{\min}(\mathcal{H})}\|\tilde{P}_{i,0}\|^2\bar{b}^2 + {\varepsilon^*_\gamma}^2\bar{d}^2\right)\right) < \gamma^2$$

and let $\varepsilon^* = \min\{\varepsilon^*_\eta, \varepsilon^*_\gamma\}$ to complete the proof. □

2.4 Simulation For State Feedback

Consider a group of five agents, including one leader agent, labeled as 0, and four follower agents. The follower agent dynamics are described by

$$\begin{cases} \dot{z}_{i,1} = z_{i,2} + w_i, \\ \dot{z}_{i,2} = x_{i,1}, \\ \dot{x}_{i,1} = x_{i,2} + w_i, \\ \dot{x}_{i,2} = u_i + w_i, \\ \quad y_i = x_{i,1}, \ \ i \in \{1,2,3,4\}. \end{cases}$$

The desired output is generated by the leader agent, whose dynamics is described by

$$\begin{cases} \dot{x}_{0,1} = x_{0,2}, \\ \dot{x}_{0,2} = u_0, \\ \quad y_0 = x_{0,1}, \end{cases}$$

with $u_0 = -x_{0,1} - 2x_{0,2} + 30\sin(0.6t)$.

The underlying communication topology is described by $\bar{\mathcal{L}}$ and $\bar{\mathcal{B}}$ as

$$\bar{\mathcal{L}} = \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}, \ \bar{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The distributed consensus protocol for each follower agent i is designed as

$$u_i = -\varepsilon^2 x_{i,1} - 2\varepsilon x_{i,2} + \tilde{u}_i, \ i \in \{1,2,3,4\}, \tag{2.24}$$

with $\tilde{u}_i$ as defined in (2.11).

It is noted that for each agent i, $i \in \{1,2,3,4\}$, the implementation of $\tilde{u}_i$ requires the signal $\tilde{u}_j$'s from its neighbors. This requirement is due to the presence of the control input u_0 of the leader agent. A similar requirement can be found in [35]. In the simulation, we set $\tilde{u}_i(0) = 0$ and the signal u_i's are updated in a sequential manner.

Simulation is performed with the initial conditions of the agents $(z_1^{\mathrm{T}}(0), x_1^{\mathrm{T}}(0), z_2^{\mathrm{T}}(0), x_2^{\mathrm{T}}(0), z_3^{\mathrm{T}}(0), x_3^{\mathrm{T}}(0), z_4^{\mathrm{T}}(0), x_4^{\mathrm{T}}(0), x_0^{\mathrm{T}}(0))^{\mathrm{T}} = (0,0,50,30,0,0,100,30,0,0,-100,-20,\ 0,0,0,-20,50,50)^{\mathrm{T}}$.

Figs. 2.1 and 2.2 show the evolution of the output tracking errors between the follower agents and the leader agent as well as the states of the zero dynamics under protocols (2.24) with $\varepsilon = 0.5$ and $\varepsilon = 0.1$, respectively, in the absence of the disturbances. It is obvious that all states remain bounded and as the parameter ε becomes smaller, the output tracking errors become smaller.

Figs. 2.3 and 2.4 show the evolution of the output tracking errors between the follower agents and the leader agent as well as the states of the zero dynamics under protocols (2.24) with $\varepsilon = 0.5$ and $\varepsilon = 0.1$, respectively, in the presence of the disturbances, where the disturbance signals are realized

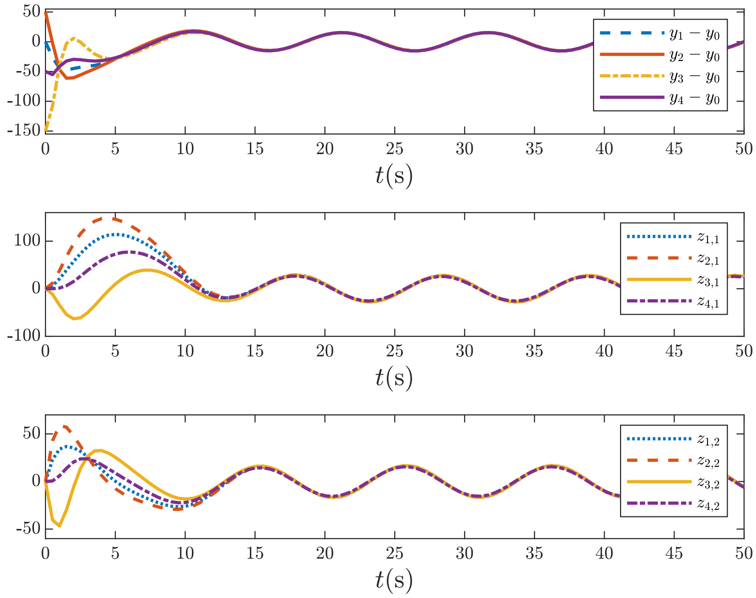

Figure 2.1: Evolution of the output tracking errors between the follower agents and the leader agent as well as the states of the zero dynamics under protocols (2.24) with $\varepsilon = 0.5$ in the absence of the disturbances.

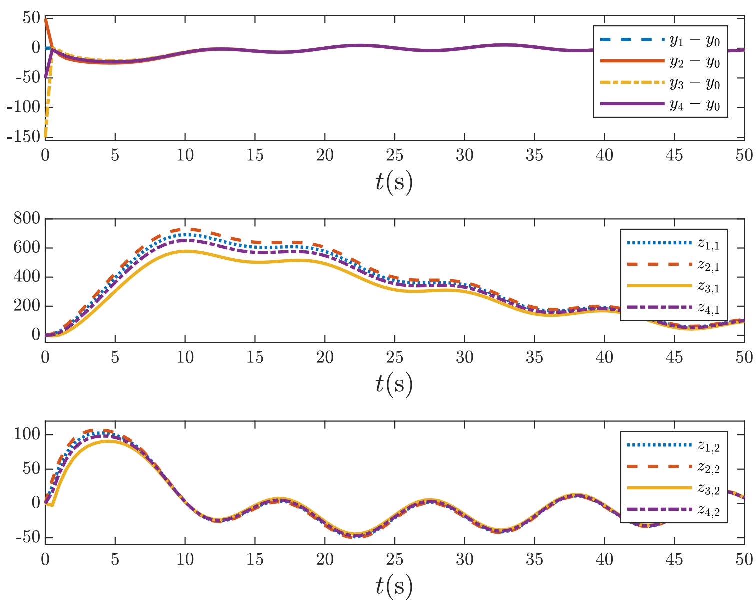

Figure 2.2: Evolution of the output tracking errors between the follower agents and the leader agent as well as the states of the zero dynamics under protocols (2.24) with $\varepsilon = 0.1$ in the absence of the disturbances.

as

$$w_1(t) = 10\sin(t) + 10\cos(5t),$$

$$w_2(t) = 10\sin(2t) + 10\cos(6t),$$
$$w_3(t) = 10\sin(3t) + 10\cos(7t),$$
$$w_4(t) = 10\sin(4t) + 10\cos(8t).$$

It is obvious that all states remain bounded and as the parameter ε becomes smaller, the output tracking errors become smaller.

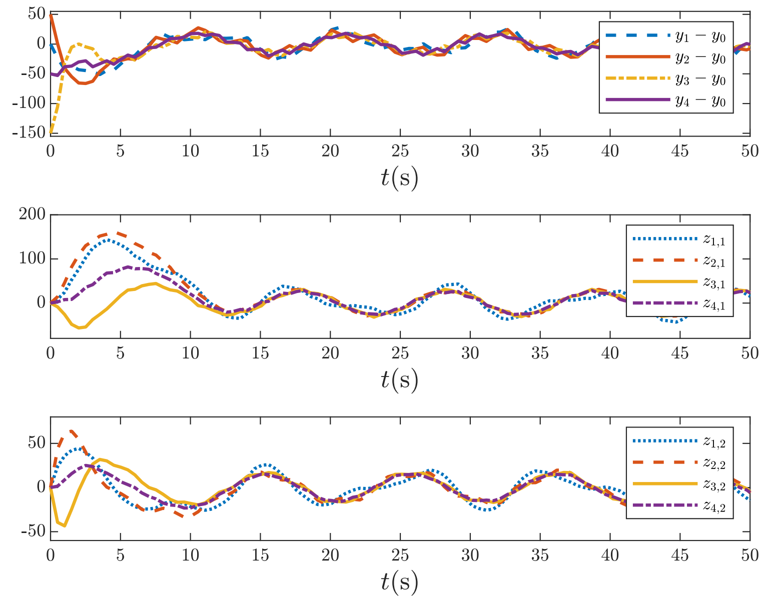

Figure 2.3: Evolution of the output tracking errors between the follower agents and the leader agent as well as the states of the zero dynamics under protocols (2.24) with $\epsilon = 0.5$, in the presence of the disturbances.

2.5 Output Feedback Results

We will design the output feedback consensus protocols in four steps. First, state observers will be constructed for all follower agents and the leader agent. Second, an observer-based low gain feedback law will be designed for each follower agent that stabilizes its unstable zero dynamics. Third, a new output will be renamed for each follower agent based on the low gain feedback law. Last, a high gain feedback law that uses a combination of the observer states and the measured output will be designed that attenuates the effect of the disturbances on the leader-following output consensus.

Step 1: Observer Design

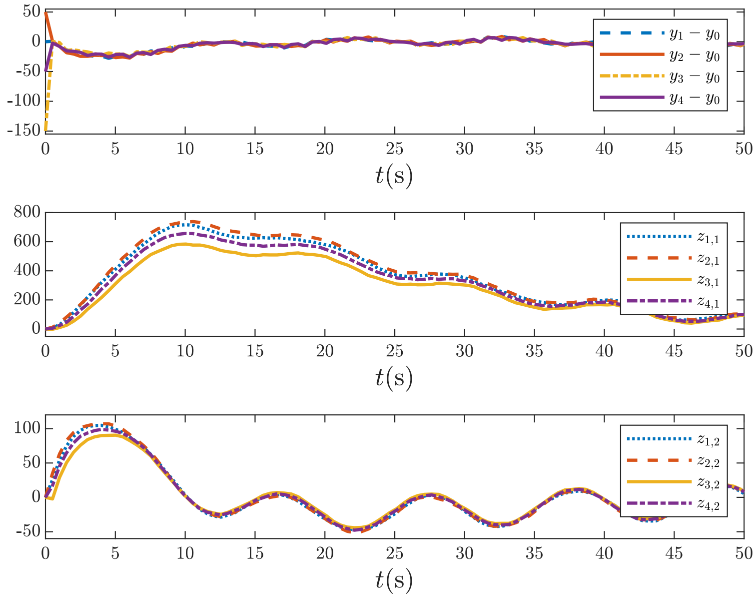

Figure 2.4: Evolution of the output tracking errors between the follower agents and the leader agent as well as the states of the zero dynamics under protocols (2.24) with $\varepsilon = 0.1$, in the presence of the disturbances.

For each follower agent i, $i \in \{1, 2, \cdots, N\}$, denote

$$A_i = \left[\begin{array}{c|cccc} A_{i,0} & B_{i,0} & 0_{r_i \times (\rho-1)} & & \\ \hline & 0 & 1 & \cdots & 0 \\ 0_{(\rho-1)\times r_i} & \vdots & \vdots & \ddots & \vdots \\ & 0 & 0 & \cdots & 1 \\ E_{i,0} & \beta_{i,1} & \beta_{i,2} & \cdots & \beta_{i,\rho} \end{array}\right],$$

$$C_i = \left[\begin{array}{ccc} 0_{1\times r_i} & 1 & 0_{1\times(\rho-1)} \end{array}\right].$$

Choose the observer gain

$$L_i = [L_{i,0}^{\mathrm{T}} \ l_{i,1} \ l_{i,2} \ \cdots \ l_{i,\rho}]^{\mathrm{T}} \in \mathbb{R}^{r_i+\rho}, \ L_{i,0} \in \mathbb{R}^{r_i},$$

be such that $A_i - L_iC_i$ is Hurwitz. Such L_i exists since the pair (A_i, C_i) is detectable, implied by

Assumption 2.3. We construct a standard Leunberger observer as follows,

$$\begin{cases} \dot{\hat{x}}_{i,0} = A_{i,0}\hat{x}_{i,0} + B_{i,0}\hat{x}_{i,1} - L_{i,0}(\hat{x}_{i,1} - y_i), \\ \dot{\hat{x}}_{i,m} = \hat{x}_{i,m+1} - l_{i,m}(\hat{x}_{i,1} - y_i), \ m = 1, 2 \cdots, \rho - 1, \\ \dot{\hat{x}}_{i,\rho} = E_{i,0}\hat{x}_{i,0} + \beta_{i,1}\hat{x}_{i,1} + \beta_{i,2}\hat{x}_{i,2} + \cdots + \beta_{i,\rho}\hat{x}_{i,\rho} + u_i - l_{i,\rho}(\hat{x}_{i,1} - y_i), \end{cases} \tag{2.25}$$

where $\hat{x}_{i,0}$ and $\hat{x}_i = [\hat{x}_{i,1} \ \hat{x}_{i,2} \ \cdots \ \hat{x}_{i,\rho}]^{\mathrm{T}}$ are the estimates of $x_{i,0}$ and x_i, respectively.

Denote the observer errors as

$$e_{i,0} = \hat{x}_{i,0} - x_{i,0},$$
$$e_i = \begin{bmatrix} e_{i,1} \\ e_{i,2} \\ \vdots \\ e_{i,\rho} \end{bmatrix} = \begin{bmatrix} \hat{x}_{i,1} - x_{i,1} \\ \hat{x}_{i,2} - x_{i,2} \\ \vdots \\ \hat{x}_{i,\rho} - x_{i,\rho} \end{bmatrix}.$$

Then, the observer error dynamics is given as

$$\begin{cases} \dot{e}_{i,0} = A_{i,0}e_{i,0} + B_{i,0}e_{i,1} - L_{i,0}e_{i,1} - D_{i,0}w_i, \\ \dot{e}_{i,m} = e_{i,m+1} - l_{i,m}e_{i,1} - d_{i,m}w_i, \ m = 1, 2, \cdots, \rho - 1, \\ \dot{e}_{i,\rho} = B_{i,0}e_{i,0} + \beta_{i,1}e_{i,1} + \beta_{i,2}e_{i,2} + \cdots + \beta_{i,\rho}e_{i,\rho} - l_{i,\rho}e_{i,1} - d_{i,\rho}w_i, \end{cases}$$

or in a compact form,

$$\begin{bmatrix} \dot{e}_{i,0} \\ \dot{e}_i \end{bmatrix} = (A_i - L_iC_i) \begin{bmatrix} e_{i,0} \\ e_i \end{bmatrix} - D_iw_i, \tag{2.26}$$

where $D_i = [D_{i,0}^{\mathrm{T}} \ d_{i,1} \ d_{i,2} \ \cdots \ d_{i,\rho}]^{\mathrm{T}}$.

We also construct an observer for the leader agent as

$$\begin{cases} \dot{\hat{x}}_{0,m} = \hat{x}_{0,m+1} - l_{0,m}(\hat{x}_{0,1} - y_0), \ m = 1, 2, \cdots, \rho - 1, \\ \dot{\hat{x}}_{0,\rho} = u_0 - l_{i,\rho}(\hat{x}_{0,1} - y_0), \end{cases} \tag{2.27}$$

where $l_{0,1}, l_{0,2}, \cdots, l_{0,\rho}$ are such that the polynomial

$$s^\rho + l_{0,\rho}s^{\rho-1} + l_{0,\rho-1}s^{\rho-2} + \cdots + l_{0,2}s + l_{0,1}$$

is Hurwitz. The observer error is defined as $e_0 = [\hat{x}_{0,1} - x_{0,1} \ \hat{x}_{0,2} - x_{0,2} \ \cdots \ \hat{x}_{0,\rho} - x_{0,\rho}]^{\mathrm{T}}$.

Step 2: Low Gain Feedback

A low gain feedback law is designed to stabilize the zero dynamics of each follower agent i, $i \in \{1, 2, \cdots, N\}$. Find the nonsingular transformation $T_{i,0} \in \mathbb{R}^{r_i \times r_i}$ such that the pair $\big(A_{i,0}, B_{i,0}\big)$ is transformed into the form in (2.3).

For each $\big(A_{i,0}^0, B_{i,0}^0\big)$, let $F_{i,0}^0(\varepsilon) \in \mathbb{R}^{1\times r_i^0}$ be such that

$$\lambda\big(A_{i,0}^0 - B_{i,0}^0 F_{i,0}^0(\varepsilon)\big) = -\varepsilon + \lambda\big(A_{i,0}^0\big).$$

Let

$$u_{\mathrm{L}i} = -F_{i,0}(\varepsilon)\hat{x}_{i,0},\ \varepsilon \in (0,1],\ i \in \{1, 2, \cdots, N\},$$

where $F_{i,0}(\varepsilon) = \begin{bmatrix} F_{i,0}^0(\varepsilon) & 0 \end{bmatrix} T_{i,0}^{-1}$.

For each $i \in \{1, 2, \cdots, N\}$, define the state transformation

$$z_{i,0} = \big[{z_{i,0}^0}^{\mathrm{T}}\ {z_{i,0}^-}^{\mathrm{T}}\big]^{\mathrm{T}} = T_{i,0}^{-1} x_{i,0}.$$

Correspondingly, denote $\hat{z}_{i,0} = \big[{\hat{z}_{i,0}^0}^{\mathrm{T}}\ {\hat{z}_{i,0}^-}^{\mathrm{T}}\big]^{\mathrm{T}} = T_{i,0}^{-1}\hat{x}_{i,0}$, $\big[{D_{i,0}^0}^{\mathrm{T}}\ {D_{i,0}^-}^{\mathrm{T}}\big]^{\mathrm{T}} = T_{i,0}^{-1} D_{i,0}$, and $e_{zi,0} = \big[{e_{zi,0}^0}^{\mathrm{T}}\ {e_{zi,0}^-}^{\mathrm{T}}\big]^{\mathrm{T}} = T_{i,0}^{-1} e_{i,0}$. Then, the dynamics of $z_{i,0}$ is given as

$$\begin{bmatrix} \dot{z}_{i,0}^0 \\ \dot{z}_{i,0}^- \end{bmatrix} = \begin{bmatrix} A_{i,0}^0 - B_{i,0}^0 F_{i,0}^0 & 0 \\ B_{i,0}^- F_{i,0}^0 & A_{i,0}^- \end{bmatrix} \begin{bmatrix} z_{i,0}^0 \\ z_{i,0}^- \end{bmatrix} + \begin{bmatrix} B_{i,0}^0 \\ B_{i,0}^- \end{bmatrix} \tilde{y}_i - \begin{bmatrix} B_{i,0}^0 F_{i,0}^0 \\ B_{i,0}^- F_{i,0}^0 \end{bmatrix} e_{zi,0}^0 + \begin{bmatrix} D_{i,0}^0 \\ D_{i,0}^- \end{bmatrix} w_i. \tag{2.28}$$

Step 3: Output Renaming

For each follower agent i, define a new output as

$$\tilde{y}_i = y_i - u_{\mathrm{L}i} = x_{i,1} + F_{i,0}(\varepsilon)(x_{i,0} + e_{i,0}),\ i \in \{1, 2, \cdots, N\}.$$

Based on the definition of the new output $\tilde{y}_i$ and the state transformation $z_{i,0} = \big[{z_{i,0}^0}^{\mathrm{T}}\ {z_{i,0}^-}^{\mathrm{T}}\big]^{\mathrm{T}}$, we define new states $\tilde{z}_{i,0}^0$ and $\tilde{x}_i = \begin{bmatrix}\tilde{x}_{i,1} & \tilde{x}_{i,2} & \cdots & \tilde{x}_{i,\rho}\end{bmatrix}^{\mathrm{T}}$ as

$$\begin{cases} \tilde{z}_{i,0}^0 = S_i(\varepsilon)Q_i^{-1}(\varepsilon)z_{i,0}^0, \\ \tilde{x}_{i,1} = \tilde{y}_i = x_{i,1} + F_{i,0}\hat{x}_{i,0}, \\ \tilde{x}_{i,2} = \hat{x}_{i,2} + F_{i,0}A_{i,0}\hat{x}_{i,0} + F_{i,0}B_{i,0}\hat{x}_{i,1}, \\ \qquad \vdots \\ \tilde{x}_{i,\rho} = \hat{x}_{i,\rho} + F_{i,0}A_{i,0}^{\rho-1}\hat{x}_{i,0} + F_{i,0}A_{i,0}^{\rho-2}B_{i,0}\hat{x}_{i,1} + F_{i,0}A_{i,0}^{\rho-3}B_{i,0}\hat{x}_{i,2} + \cdots + F_{i,0}B_{i,0}\hat{x}_{i,\rho-1}. \end{cases}$$

These new states will be utilized for feedback in the consensus protocols. It is noted that the new state $\tilde{x}_{i,1}$ is a combination of the output $x_{i,1} = y_i$ and the observed state $\hat{x}_{i,0}$. The rest of them, $\tilde{x}_{i,m}$, $m = 2, 3, \cdots, \rho$, are combinations of the observed states $\hat{x}_{i,m}$, $m = 1, 2 \cdots, \rho$.

We design the consensus protocol u_i as follows,

$$\begin{aligned} u_i = & -B_{i,0}\hat{x}_{i,0} - \beta_{i,1}\hat{x}_{i,1} - \beta_{i,2}\hat{x}_{i,2} - \cdots - \beta_{i,\rho}\hat{x}_{i,\rho} \\ & - F_{i,0}A_{i,0}^{\rho}\hat{x}_{i,0} - F_{i,0}A_{i,0}^{\rho-1}B_{i,0}\hat{x}_{i,1} - F_{i,0}A_{i,0}^{\rho-2}B_{i,0}\hat{x}_{i,2} - \cdots - F_{i,0}B_{i,0}\hat{x}_{i,\rho} + \tilde{u}_i, \end{aligned} \tag{2.29}$$

where $\tilde{u}_i$ is to be designed later. The dynamics of follower agent i in the new states is then given as

$$\begin{cases} \dot{\tilde{z}}_{i,0}^0 = \varepsilon\tilde{J}_i\tilde{z}_{i,0}^0 + S_iQ_i^{-1}B_{i,0}^0\tilde{y}_i + S_iQ_i^{-1}D_{i,0}^0w_i - S_iQ_i^{-1}B_{i,0}^0F_{i,0}^0e_{zi,0}, \\ \dot{\tilde{x}}_{i,1} = \tilde{x}_{i,2} - F_{i,0}L_{i,0}e_{i,1} - e_{i,2} + d_{i,1}w_i, \\ \dot{\tilde{x}}_{i,2} = \tilde{x}_{i,3} - \Big(l_{i,2} + F_{i,0}A_{i,0}L_{i,0} + F_{i,0}B_{i,0}l_{i,1}\Big)e_{i,1}, \\ \qquad \vdots \\ \dot{\tilde{x}}_{i,\rho-1} = \tilde{x}_{i,\rho} - \big(l_{i,\rho-1} + F_{i,0}A_{i,0}^{\rho-2}L_{i,0} + F_{i,0}A_{i,0}^{\rho-3}B_{i,0}l_{i,1} + F_{i,0}A_{i,0}^{\rho-4}B_{i,0}l_{i,2} + \cdots + F_{i,0}B_{i,0}l_{i,\rho-2}\big)e_{i,1}, \\ \dot{\tilde{x}}_{i,\rho} = -\big(l_{i,\rho} + F_{i,0}A_{i,0}^{\rho-1}L_{i,0} + F_{i,0}A_{i,0}^{\rho-2}B_{i,0}l_{i,1} + F_{i,0}A_{i,0}^{\rho-3}B_{i,0}l_{i,2} + \cdots + F_{i,0}B_{i,0}l_{i,\rho-1}\big)e_{i,1} + \tilde{u}_i. \end{cases}$$

Denoting

$$\begin{aligned} \tilde{B}_{i,0}^0 &= S_iQ_i^{-1}B_{i,0}^0, \\ \tilde{D}_{i,0}^0 &= S_iQ_i^{-1}D_{i,0}^0, \\ \tilde{F}_{i,0}^0 &= S_iQ_i^{-1}B_{i,0}^0F_{i,0}^0, \\ g_{i,1} &= [-F_{i,0}L_{i,0} \;\; -1 \;\; 0 \;\; \cdots \;\; 0], \\ g_{i,m} &= [-(l_i, m + F_{i,0}A_{i,0}^{m-1}L_{i,0} + F_{i,0}A_{i,0}^{m-2}B_{i,0}l_{i,1} + F_{i,0}A_{i,0}^{m-3}B_{i,0}l_{i,2} + \cdots \end{aligned}$$

$$+ F_{i,0}B_{i,0}l_{i,m-1})\ 0\ \cdots\ 0],\ m = 2,3\cdots,\rho,$$

we have

$$\begin{cases} \dot{\tilde{z}}^0_{i,0} = \varepsilon \tilde{J}_i \tilde{z}^0_{i,0} + \tilde{B}^0_{i,0}\tilde{x}_{i,1} + \tilde{D}^0_{i,0} w_i - \tilde{F}^0_{i,0} e^0_{zi,0}, \\ \dot{\tilde{x}}_{i,1} = \tilde{x}_{i,2} + g_{i,1} e_i + d_{i,1} w_i, \\ \dot{\tilde{x}}_{i,2} = \tilde{x}_{i,3} + g_{i,2} e_i, \\ \qquad \vdots \\ \dot{\tilde{x}}_{i,\rho} = \tilde{u}_i + g_{i,\rho} e_i. \end{cases}$$

Note that $\left\|\tilde{B}^0_{i,0}\right\| \le \bar{b}$, $\left\|\tilde{F}^0_{i,0}\right\| \le \bar{f}\varepsilon$ for some constants $\bar{b} > 0$ and $\bar{f} > 0$, by Lemma 2.3, and $\left\|\tilde{D}^0_{i,0}\right\| \le \bar{d}\varepsilon$ for some constant $\bar{d} > 0$, by Lemma 2.5. Hence, $\|g_{i,m}\| \le \bar{g}$, $m \in I[1,\rho]$, for some constant $\bar{g} > 0$.

Step 4: High Gain Feedback

Let the consensus error be defined as $\xi_i = [\xi_{i,1}\ \xi_{i,2}\ \cdots\ \xi_{i,\rho}]^{\mathrm{T}}$, with

$$\xi_{i,m} = \sum_{j=1}^{N} a_{ij}(\tilde{x}_{i,m} - \tilde{x}_{j,m}) + b_i(\tilde{x}_{i,m} - \hat{x}_{0,m}),\ m = 1,2\cdots,\rho. \tag{2.30}$$

Then, the dynamics of the consensus error is given as

$$\begin{cases} \dot{\xi}_{i,1} = \sum_{j=1}^{N} a_{ij}\big((\tilde{x}_{i,2} + g_{i,1}e_i + d_{i,1}w_i) - (\tilde{x}_{j,2} + g_{j,1}e_j + d_{j,1}w_j)\big) \\ \qquad\quad + b_i\big((\tilde{x}_{i,2} + g_{i,1}e_i + d_{i,1}w_i) - (\hat{x}_{0,1} - l_{0,1}e_{0,1})\big), \\ \dot{\xi}_{i,m} = \sum_{j=1}^{N} a_{ij}\big((\tilde{x}_{i,m+1} + g_{i,m}e_i) - (\tilde{x}_{j,m+1} + g_{j,m}e_j)\big) \\ \qquad\quad + b_i\big((\tilde{x}_{i,m+1} + g_{i,m}e_i) - (\hat{x}_{0,m+1} - l_{0,m}e_{0,1})\big),\ m = 2,3,\cdots,\rho-1, \\ \dot{\xi}_{i,\rho} = \sum_{j=1}^{N} a_{ij}\big((\tilde{u}_i + g_{i,\rho}e_i) - (\tilde{u}_j + g_{j,\rho}e_j)\big) + b_i\big((\tilde{u}_i + g_{i,\rho}e_i) - (u_0 - l_{0,\rho}e_{0,1})\big), \end{cases}$$

or in a compact form,

$$\begin{cases} \dot{\xi}_{i,1} = \xi_{i,2} + \tilde{g}^{\mathrm{T}}_{i,1} e + b_i l_{0,1} e_{0,1} + \tilde{d}^{\mathrm{T}}_{i,1} w, \\ \dot{\xi}_{i,m} = \xi_{i,m+1} + \tilde{g}^{\mathrm{T}}_{i,m} e + b_i l_{0,m} e_{0,m},\ m = 2,3,\cdots,\rho-1, \\ \dot{\xi}_{i,\rho} = \tilde{v}_i + \tilde{g}^{\mathrm{T}}_{i,\rho} e + b_i l_{0,\rho} e_{0,\rho}, \end{cases}$$

where $e = [e_1 \ e_2 \ \cdots \ e_N]^{\mathrm{T}}$,

$$\tilde{v}_i = \sum_{j=1}^{N} a_{ij}(\tilde{u}_i - \tilde{u}_j) + b_i(\tilde{u}_i - u_0),$$

and $\tilde{g}_{i,1}, \tilde{g}_{i,2}, \cdots, \tilde{g}_{i,\rho}$ and $\tilde{d}_{i,1}$ are defined in an obvious way.

Let $\tilde{v}_i$ be designed as

$$\tilde{v}_i = -\frac{1}{\varepsilon^{\rho}} f_{i,1}\xi_{i,1} - \frac{1}{\varepsilon^{\rho-1}} f_{i,2}\xi_{i,2} - \cdots - \frac{1}{\varepsilon} f_{i,\rho}\xi_{i,\rho}, \tag{2.31}$$

where $f_{i,1}, f_{i,2}, \cdots, f_{i,\rho}$ are such that the polynomial

$$s^{\rho} + f_{i,\rho}s^{\rho-1} + f_{i,\rho-1}s^{\rho-2} + \cdots + f_{i,1}$$

is Hurwitz. Then, we have

$$\tilde{u}_i = \left(\sum_{j=1}^{N} a_{ij} + b_i\right)^{-1} \left(\sum_{j=1}^{N} a_{ij}\tilde{u}_j + b_i u_0 + \tilde{v}_i\right).$$

We note that Assumption 2.1 implies that agent i is connected with either another follower or the leader agent and hence $\sum_{j=1}^{N} a_{ij} + b_i > 0$.

In view of (2.29), the consensus protocols are designed as

$$\begin{aligned} u_i = &-B_{i,0}\hat{x}_{i,0} - \beta_{i,1}\hat{x}_{i,1} - \beta_{i,2}\hat{x}_{i,2} - \cdots - \beta_{i,\rho}\hat{x}_{i,\rho} - F_{i,0}A_{i,0}^{\rho}\hat{x}_{i,0} - F_{i,0}A_{i,0}^{\rho-1}B_{i,0}\hat{x}_{i,1} \\ &- F_{i,0}A_{i,0}^{\rho-2}B_{i,0}\hat{x}_{i,2} - \cdots - F_{i,0}B_{i,0}\hat{x}_{i,\rho} + \left(\sum_{j=1}^{N} a_{ij} + b_i\right)^{-1} \left(\sum_{j=1}^{N} a_{ij}\tilde{u}_j + b_i u_0 + \tilde{v}_i\right). \end{aligned} \tag{2.32}$$

The following theorem establishes that the observer-based distributed consensus protocols we have constructed solve Problem 2.1.

Theorem 2.2. [57] Consider the multi-agent system with agent dynamics described by (2.1) and (2.2). Let the communication topology satisfy Assumption 2.1. Let the follower agents' dynamics satisfy Assumptions 2.2, 2.3 and 2.4. Let the leader agent's output to be followed satisfy Assumption 2.5. Then, for any given $\eta > 0$ and $\gamma > 0$, there exists $\varepsilon^* \in (0, 1]$ such that, for any $\varepsilon \in (0, \varepsilon^*]$, the parameterized output feedback consensus protocols (2.32) solve Problem 2.1:

(i) The state of each follower agent is bounded in the absence of the disturbances.

(ii) The leader-following output consensus is achieved to the given accuracy η in the absence of the disturbances, *i.e.*, there exists finite time $T \geq 0$ such that

$$|y_{i,w=0}(t) - y_0(t)| \leq \eta,\ t \geq T,\ i \in \{1, 2, \cdots, N\}.$$

(iii) In steady state, the effect of the disturbance $w = [w_1\ w_2\ \cdots\ w_N]^{\mathrm{T}}$ on the the leader-following output consensus, measured by the L_2-gain, is attenuated to the level specified by γ, *i.e.*,

$$\int_0^{\infty} \big(y_i(t) - y_{i,w=0}(t)\big)^2 \mathrm{d}t \leq \gamma^2 \int_0^{\infty} \|w(t)\|^2 \mathrm{d}t.$$

Proof of Theorem 2.2: In the proof, we will first show that the proposed consensus protocols stabilize the unstable zero dynamics and guarantee the boundedness of all follower agents' states, in the absence of the disturbances. We will then show that the leader-following output consensus can be achieved with a specified accuracy $\eta > 0$ by showing that all follower agents' renamed outputs $\tilde{y}_i(t)$, $i \in \{1, 2, \cdots, N\}$, will converge to the leader's output $y_0(t)$, and the difference between each follower agent's output $y_i(t)$ and its renamed output $\tilde{y}_i(t)$ can be made less than η in steady state. Finally, we will evaluate the effect of the disturbances on the consensus errors through Lyapunov analysis.

Note that, in the absence of the disturbances, the errors of the observers (2.25) and (2.27) satisfy

$$\lim_{t \to \infty} e_{i,0}(t) = 0, \tag{2.33}$$
$$\lim_{t \to \infty} e_i(t) = 0, \tag{2.34}$$
$$\lim_{t \to \infty} e_0(t) = 0, \tag{2.35}$$

because of the choice of L_i and $l_{0,1}, l_{0,2}, \cdots, l_{0,\rho}$.

Define $\tilde{\xi}_i = \big[\tilde{\xi}_{i,1}\ \tilde{\xi}_{i,2}\ \cdots\ \tilde{\xi}_{i,\rho}\big]^{\mathrm{T}} = \big[\xi_{i,1}\ \varepsilon\xi_{i,2}\ \cdots\ \varepsilon^{\rho-1}\xi_{i,\rho}\big]^{\mathrm{T}}$. Then, the $\tilde{\xi}_i$ dynamics is given as

$$\varepsilon\dot{\tilde{\xi}}_i = \tilde{A}_i\tilde{\xi}_i + \varepsilon\tilde{G}_i e + \varepsilon\tilde{G}_{i,0}e_0 + \varepsilon\tilde{D}_i w, \tag{2.36}$$

where

$$\tilde{A}_i = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -f_{i,1} & -f_{i,2} & \cdots & -f_{i,\rho} \end{bmatrix},$$

$$
\tilde{G}_i = \left[\tilde{g}_{i,1} \quad \varepsilon\tilde{g}_{i,2} \quad \cdots \quad \varepsilon^{\rho-2}\tilde{g}_{i,\rho-1} \quad \varepsilon^{\rho-1}\tilde{g}_{i,\rho}\right]^{\mathrm{T}},\ \tilde{D}_i = \left[\tilde{d}_{i,1} \quad 0 \quad \cdots \quad 0 \quad 0\right]^{\mathrm{T}},
$$
$$
\tilde{G}_{i,0} = \mathrm{diag}\left\{b_i l_{0,1}, \varepsilon b_i l_{0,2}, \cdots, \varepsilon^{\rho-1} b_i l_{0,\rho}\right\}.
$$

It is obvious that, in the absence of the disturbances,

$$
\lim_{t\to\infty} \tilde{\xi}_i(t) = 0,\ i \in \{1, 2, \cdots, N\},
$$

because of the choice of the parameters $f_{i,1}, f_{i,2}, \cdots, f_{i,\rho}$ and (2.33)-(2.35). Therefore,

$$
\lim_{t\to\infty} \xi_i(t) = 0,\ i \in \{1, 2, \cdots, N\}.
$$

It follows from (2.30) and $\bar{\mathcal{L}}\mathbf{1} = 0$ that

$$
\begin{aligned}
&\left[\xi_{1,m}\ \xi_{2,m}\ \cdots\ \xi_{N,m}\right]^{\mathrm{T}} \\
&= (\bar{\mathcal{L}} + \bar{\mathcal{B}})\left[\tilde{x}_{1,m}\ \tilde{x}_{2,m}\ \cdots\ \tilde{x}_{N,m}\right]^{\mathrm{T}} - \bar{\mathcal{B}}\hat{x}_{0,m}\mathbf{1} \\
&= (\bar{\mathcal{L}} + \bar{\mathcal{B}})\left[\tilde{x}_{1,m}\ \tilde{x}_{2,m}\ \cdots\ \tilde{x}_{N,m}\right]^{\mathrm{T}} - (\bar{\mathcal{L}} + \bar{\mathcal{B}})\hat{x}_{0,m}\mathbf{1} \\
&= (\bar{\mathcal{L}} + \bar{\mathcal{B}})\left(\left[\tilde{x}_{1,m}\ \tilde{x}_{2,m}\ \cdots\ \tilde{x}_{N,m}\right]^{\mathrm{T}} - \hat{x}_{0,m}\mathbf{1}\right),\ m = 1, 2, \cdots, \rho.
\end{aligned}
$$

Recalling that $\bar{\mathcal{L}} + \bar{\mathcal{B}}$ has rank N, we have,

$$
\begin{aligned}
&\lim_{t\to\infty} \left(\tilde{y}_{i,w=0}(t) - y_0(t)\right) = 0, \\
&\lim_{t\to\infty} \left(\tilde{x}_{i,m}(t) - x_{0,m}(t)\right) = 0,\ m = 2, 3, \cdots, \rho.
\end{aligned}
$$

Since $y_0(t)$ is bounded, the follower agents' renamed output $\tilde{y}_{i,w=0}(t)$ are also bounded in the absence of the disturbances.

Since $\mathrm{col}\left(e^0_{\mathrm{z}i,0}, e^-_{\mathrm{z}i,0}\right) = T^{-1}_{i,0} e_{i,0}$, the boundedness of $e_{i,0}$ implies the boundedness of $e^0_{\mathrm{z}i,0}$. In view of (2.28), the boundedness of $\tilde{y}_{i,w=0}$ guarantee the boundedness of $z_{i,0}$. Since $x_{i,0} = Tz_{i,0}$, we can conclude that $x_{i,0}$ is also bounded. That is, in the absence of the disturbances, the states of each follower agent are bounded.

We will next show that, in the absence of the disturbances, the leader-following output consensus is achieved within the specified accuracy η. Consider the $z^0_{i,0}$ dynamics of the closed-loop system

$$
\dot{z}^0_{i,0} = \left(A^0_{i,0} - B^0_{i,0}F^0_{i,0}(\varepsilon)\right)z^0_{i,0} + B^0_{i,0}\tilde{y}_i - B^0_{i,0}F^0_{i,0}(\varepsilon)e^0_{\mathrm{z}i,0}
$$

Since

$$\lim_{t\to\infty} \mathrm{col}\big(e^0_{\mathrm{z}i,0}(t), e^-_{\mathrm{z}i,0}(t)\big) = \lim_{t\to\infty} T^{-1}_{i,0} e_{i,0}(t) = 0$$

and

$$\lim_{t\to\infty} (\tilde{y}_i(t) - y_0(t)) = 0,$$

the steady state trajectory of $z^0_{i,0}(t)$ is all due to the desired output $y_0(t)$. Consider the transfer function from y_0 to $u_{\mathrm{L}i}$. Under Assumption 2.5, Lemma 2.2 implies that, for the given η, there is $\varepsilon^*_\eta \in (0,1]$ such that, for all $\varepsilon \in (0, \varepsilon^*_\eta]$, $u_{\mathrm{L}i}$ satisfies

$$\limsup_{t\to\infty} |u_{\mathrm{L}i}(t)| \leq \frac{1}{2}\eta.$$

Recall that in steady state, $\tilde{y}_i = y_i - u_{\mathrm{L}i} = y_i + F^0_{i,0}(\varepsilon) z^0_{i,0}$. Then,

$$\begin{aligned}
\limsup_{t\to\infty} |y_i(t) - y_0(t)| &= \limsup_{t\to\infty} |(\tilde{y}_i + u_{\mathrm{L}i}(t)) - y_0(t)| \\
&\leq \limsup_{t\to\infty} |\tilde{y}_i(t) - y_0(t)| + |u_{\mathrm{L}i}(t)| \\
&\leq \eta.
\end{aligned}$$

We will now show that, in steady state, the L_2-gain from the disturbances to the difference between $y_i(t)$ and $y_{i,w=0}(t)$, the outputs of each follower agent i in the presence and in the absence of the disturbances, can be made less than or equal to the given γ.

We assume, without loss of generality, that the system is operating in steady state at $t = 0$, that is, $e_{i,0}(0) = 0$, $e_i(0) = 0$, and $\xi_{i,m}(0) = 0$, $m = 1, 2, \cdots, \rho$. Denote $e_{\mathrm{ob}i} = [e^{\mathrm{T}}_{i,0}\ e^{\mathrm{T}}_i]^{\mathrm{T}}$ and consider the Lyapunov function candidate

$$V_{\mathrm{ob}i}(e_{\mathrm{ob}i}) = e^{\mathrm{T}}_{\mathrm{ob}i} P_{\mathrm{ob}i} e_{\mathrm{ob}i},$$

where $P_{\mathrm{ob}i} > 0$ is the unique positive definite solution to the Lyapunov equation

$$P_{\mathrm{ob}i}(A_i - L_i C_i) + (A_i - L_i C_i)^{\mathrm{T}} P_{\mathrm{ob}i} = -I.$$

Such solution exists since $A_i - L_i C_i$ is a Hurwitz matrix.

The time derivative of $V_{\mathrm{ob}i}(e_{\mathrm{ob}i})$ along the trajectory of (2.26) is evaluated as

$$\begin{aligned}
\dot{V}_{\mathrm{ob}i} &= e^{\mathrm{T}}_{\mathrm{ob}i} P_{\mathrm{ob}i}\big((A_i - L_i C_i) e_{\mathrm{ob}i} - D_i w_i\big) + \big((A_i - L_i C_i) e_{\mathrm{ob}i} - D_i w_i\big)^{\mathrm{T}} P_{\mathrm{ob}i} e_{\mathrm{ob}i} \\
&= -e^{\mathrm{T}}_{\mathrm{ob}i} e_{\mathrm{ob}i} - 2 e^{\mathrm{T}}_{\mathrm{ob}i} P_{\mathrm{ob}i} D_i w_i
\end{aligned}$$

$$
\begin{aligned}
&\leq -e_{\mathrm{ob}i}^{\mathrm{T}} e_{\mathrm{ob}i} + \frac{1}{2}\|e_{\mathrm{ob}i}\|^2 + 2\|P_{\mathrm{ob}i} D_i\|^2 w_i^2 \\
&\leq -\frac{1}{2}\|e_{\mathrm{ob}i}\|^2 + 2\|P_{\mathrm{ob}i} D_i\|^2 w_i^2. \qquad (2.37)
\end{aligned}
$$

Integrating (2.37) and noting that $V_{\mathrm{ob}i}(e_{\mathrm{ob}i}(0)) = 0$, we have

$$
\int_0^\infty \|e_{\mathrm{ob}i}(t)\|^2 \mathrm{d}t \leq \gamma_{\mathrm{ob}i}^2 \int_0^\infty w_i^2(t)\mathrm{d}t, \qquad (2.38)
$$

where $\gamma_{\mathrm{ob}i}^2 = 4\|P_{\mathrm{ob}i} D_i\|^2$.

Consider the Lyapunov function candidate

$$
V_i(\tilde{\xi}_i) = \tilde{\xi}_i^{\mathrm{T}} \tilde{P}_i \tilde{\xi}_i,
$$

where $\tilde{P}_i > 0$ is the unique positive definite solution to the Lyapunov equation

$$
\tilde{P}_i \tilde{A}_i + \tilde{A}_i^{\mathrm{T}} \tilde{P}_i = -I.
$$

Such solution exists since matrix $\tilde{A}_i$ is Hurwitz.

We note that $e_0(t) = 0$, for all $t \geq 0$, under the assumption that the system is operating in steady state. Then, the time derivative of $V_i(\tilde{\xi}_i)$ along the trajectory of (2.36) is given as

$$
\begin{aligned}
\dot{V}_i &= \frac{1}{\varepsilon}\tilde{\xi}_i^{\mathrm{T}} \tilde{P}_i\big(\tilde{A}_i\tilde{\xi}_i + \varepsilon\tilde{G}_i e + \varepsilon\tilde{G}_{i,0} e_0 + \varepsilon\tilde{D}_i w\big) + \frac{1}{\varepsilon}\big(\tilde{A}_i\tilde{\xi}_i + \varepsilon\tilde{G}_i e + \varepsilon\tilde{G}_{i,0} e_0 + \varepsilon\tilde{D}_i w\big)^{\mathrm{T}} \tilde{P}_i\tilde{\xi}_i \\
&= -\frac{1}{\varepsilon}\tilde{\xi}_i^{\mathrm{T}}\tilde{\xi}_i + 2\tilde{\xi}_i^{\mathrm{T}}\tilde{P}_i\tilde{G}_i e + 2\tilde{\xi}_i^{\mathrm{T}}\tilde{P}_i\tilde{D}_i w \\
&\leq -\frac{1}{\varepsilon}\big\|\tilde{\xi}_i\big\|^2 + \frac{1}{3\varepsilon}\big\|\tilde{\xi}_i\big\|^2 + 3\varepsilon\big\|\tilde{P}_i\tilde{G}_i\big\|^2\|e\|^2 + \frac{1}{3\varepsilon}\big\|\tilde{\xi}_i\big\|^2 + 3\varepsilon\big\|\tilde{P}_i\tilde{D}_i\big\|^2\|w\|^2 \\
&= -\frac{1}{3\varepsilon}\big\|\tilde{\xi}_i\big\|^2 + 3\varepsilon\big\|\tilde{P}_i\tilde{G}_i\big\|^2\|e\|^2 + 3\varepsilon\big\|\tilde{P}_i\tilde{D}_i\big\|^2\|w\|^2. \qquad (2.39)
\end{aligned}
$$

Integrating(2.39) and noting that $V_i(\tilde{\xi}_i(0)) = 0$, we have,

$$
\begin{aligned}
\int_0^\infty \big\|\tilde{\xi}_i(t)\big\|^2 \mathrm{d}t &\leq 9\varepsilon^2\big\|\tilde{P}_i\tilde{G}_i\big\|^2 \int_0^\infty \|e(t)\|^2\mathrm{d}t + 9\varepsilon^2\big\|\tilde{P}_i\tilde{D}_i\big\|^2 \int_0^\infty \|w(t)\|^2\mathrm{d}t \\
&\leq \varepsilon^2\gamma_{\tilde{\xi}_i}^2 \int_0^\infty \|w(t)\|^2\mathrm{d}t,
\end{aligned}
$$

where

$$
\gamma_{\tilde{\xi}_i}^2 = 9\big\|\tilde{P}_i\tilde{G}_i\big\|^2\Big(\sum_{j=1}^N 4\|P_{\mathrm{ob}j} D_j\|^2\Big) + 9\big\|\tilde{P}_i\tilde{D}_i\big\|^2.
$$

Recalling that $(\bar{\mathcal{L}}+\bar{\mathcal{B}})(\tilde{y}-y_0\mathbf{1}) = \begin{bmatrix}\tilde{\xi}_{1,1} & \tilde{\xi}_{2,1} & \cdots & \tilde{\xi}_{N,1}\end{bmatrix}^{\mathrm{T}}$ and $\tilde{y} = \begin{bmatrix}\tilde{y}_1 & \tilde{y}_2 & \cdots & \tilde{y}_n\end{bmatrix}^{\mathrm{T}}$, we have

$$\begin{aligned}\int_0^\infty |\tilde{y}(t)-y_0(t)\mathbf{1}|^2\mathrm{d}t &\le \left\|(\bar{\mathcal{L}}+\bar{\mathcal{B}})^{-1}\right\|^2 \sum_{i=1}^{N}\int_0^\infty \left|\tilde{\xi}_{i,1}(t)\right|^2\mathrm{d}t \\ &\le \varepsilon^2\gamma_{\tilde{y}}^2\int_0^\infty \|w(t)\|^2\mathrm{d}t,\end{aligned}$$

where

$$\gamma_{\tilde{y}}^2 = \left\|(\bar{\mathcal{L}}+\bar{\mathcal{B}})^{-1}\right\|\sum_{i=1}^{N}\gamma_{\tilde{\xi}_i}^2.$$

Let $\tilde{y}_{i,w=0}(t)$ and $\tilde{y}_{i,w}(t)$ with $\tilde{y}_{i,w}(0)=0$ be respectively the zero input response and the zero state response of $\tilde{y}_i(t)$. By viewing the disturbance w as the input, we have

$$\tilde{y}_i(t) = \tilde{y}_{i,w=0}(t) + \tilde{y}_{i,w}(t).$$

Accordingly, we decompose $z_{i,0}^0$ and $\tilde{z}_{i,0}^0$ as

$$\begin{aligned}z_{i,0}^0(t) &= z_{i,0,w=0}^0(t) + z_{i,0,w}^0(t),\\ \tilde{z}_{i,0}^0(t) &= \tilde{z}_{i,0,w=0}^0(t) + \tilde{z}_{i,0,w}^0(t),\end{aligned}$$

with $z_{i,0,w}^0(0)=0$, $\tilde{z}_{i,0,w}^0(0)=0$ and

$$\begin{aligned}\dot{\tilde{z}}_{i,0,w=0}^0 &= \varepsilon\tilde{J}_i\tilde{z}_{i,0,w=0}^0 + \tilde{B}_{i,0}^0\tilde{y}_{i,w=0},\\ \dot{\tilde{z}}_{i,0,w}^0 &= \varepsilon\tilde{J}_i\tilde{z}_{i,0,w}^0 + \tilde{B}_{i,0}^0\tilde{y}_{i,w} + \tilde{D}_{i,0}^0 w_i - \tilde{F}_{i,0}^0 e_{\mathrm{z}i,0}^0.\end{aligned} \tag{2.40}$$

Since $\xi_{i,m}(0)=0$, $m=1,2\cdots,\rho$, in view of (2.36), we have

$$\tilde{y}_{i,w=0}(t) = y_0(t), t\ge 0.$$

Then,

$$\begin{aligned}\int_0^\infty \tilde{y}_{i,w}^2(t)\mathrm{d}t &= \int_0^\infty \left(\tilde{y}_i(t)-\tilde{y}_{i,w=0}(t)\right)^2\mathrm{d}t \\ &\le \varepsilon^2\gamma_{\tilde{y}}^2\int_0^\infty \|w(t)\|^2\mathrm{d}t.\end{aligned} \tag{2.41}$$

Consider the Lyapunov function candidate

$$V_{i,0}(\tilde{z}^0_{i,0,w}) = \tilde{z}^0_{i,0,w}{}^{\mathrm{T}}\tilde{P}_{i,0}\tilde{z}^0_{i,0,w},$$

where $\tilde{P}_{i,0}$ is given in Lemma 2.4. Then, $\dot{V}_{i,0}$ along the trajectory of (2.40) is evaluated as

$$\begin{aligned}\dot{V}_{i,0} &= \left(\varepsilon\tilde{J}_i\tilde{z}^0_{i,0,w} + \tilde{B}^0_{i,0}\tilde{y}_{i,w} + \tilde{D}^0_{i,0}w_i - \tilde{F}^0_{i,0}e^0_{zi,0}\right)^{\mathrm{T}}\tilde{P}_{i,0}\tilde{z}^0_{i,0,w}\\ &\quad + \tilde{z}^0_{i,0,w}{}^{\mathrm{T}}\tilde{P}_{i,0}\left(\varepsilon\tilde{J}_i\tilde{z}^0_{i,0,w} + \tilde{B}^0_{i,0}\tilde{y}_{i,w} + \tilde{D}^0_{i,0}w_i - \tilde{F}^0_{i,0}e^0_{\mathrm{z}i,0}\right)\\ &\le -\varepsilon\|\tilde{z}^0_{i,0,w}\|^2 + 2\tilde{z}^0_{i,0,w}{}^{\mathrm{T}}\tilde{P}_{i,0}\tilde{B}^0_{i,0}\tilde{y}_{i,w} + 2\tilde{z}^0_{i,0,w}{}^{\mathrm{T}}\tilde{P}_{i,0}\tilde{D}^0_{i,0}w_i - 2\tilde{z}^0_{i,0,w}{}^{\mathrm{T}}\tilde{P}_{i,0}\tilde{F}^0_{i,0}e^0_{\mathrm{z}i,0}\\ &\le -\frac{1}{4}\varepsilon\|\tilde{z}^0_{i,0,w}\|^2 + \frac{4}{\varepsilon}\big\|\tilde{P}_{i,0}\tilde{B}^0_{i,0}\big\|^2\tilde{y}^2_{i,w} + \frac{4}{\varepsilon}\big\|\tilde{P}_{i,0}\tilde{D}^0_{i,0}\big\|^2\|w\|^2 + \frac{4}{\varepsilon}\big\|\tilde{P}_{i,0}\tilde{F}^0_{i,0}\big\|^2\|e^0_{\mathrm{z}i,0}\|^2. \end{aligned} \tag{2.42}$$

Integrating (2.42) and noting that $V_{i,0}(0) = 0$, $\big\|\tilde{F}^0_{i,0}\big\| \le \bar{f}\varepsilon$ and $\big\|\tilde{D}^0_{i,0}\big\| \le \bar{d}\varepsilon$, we have,

$$\begin{aligned}\int_0^\infty \|\tilde{z}^0_{i,0,w}(t)\|^2\mathrm{d}t \le \frac{16}{\varepsilon^2}\big\|\tilde{P}_{i,0}\big\|^2\bar{b}^2\int_0^\infty \tilde{y}^2_{i,w}(t)\mathrm{d}t + 16\big\|\tilde{P}_{i,0}\big\|^2\bar{d}^2\int_0^\infty \|w(t)\|^2\mathrm{d}t\\ + 16\bar{f}^2\big\|\tilde{P}_{i,0}\big\|^2\int_0^\infty \|e_{\mathrm{z}i,0}(t)\|^2\mathrm{d}t.\end{aligned}$$

In view of (2.38) and (2.41), we have,

$$\int_0^\infty \|\tilde{z}^0_{i,0,w}(t)\|^2\mathrm{d}t \le \gamma^2_{\tilde{z}^0_{i,0,w}}\int_0^\infty \|w(t)\|^2\mathrm{d}t, \tag{2.43}$$

where

$$\gamma^2_{\tilde{z}^0_{i,0,w}} = 16\big\|\tilde{P}_{i,0}\big\|^2\big(\bar{b}^2\gamma^2_{\tilde{y}} + \bar{d}^2 + 4\bar{f}^2\sum_{j=1}^N\big\|P_{\mathrm{obj}}D_jT^{-1}_{i,0}\big\|^2\big).$$

When the system is operating in steady state, we have,

$$\begin{aligned}y_i(t) - y_{i,w=0}(t) &= \big(\tilde{y}_i(t) + u_{\mathrm{L}i,w=0}(t) + u_{\mathrm{L}i,w}(t)\big) - \big(\tilde{y}_{i,w=0}(t) + u_{\mathrm{L}i,w=0}(t)\big)\\ &= \tilde{y}_{i,w}(t) + u_{\mathrm{L}i,w}(t),\end{aligned}$$

where $u_{\mathrm{L}i,w=0} = F^0_{i,0}z^0_{i,0,w=0}$ and $u_{\mathrm{L}i,w} = F^0_{i,0}(z^0_{i,0,w} + e^0_{\mathrm{z}i,0})$.

Recalling that

$$u_{\mathrm{L}i,w} = F^0_{i,0}Q_i(\varepsilon)S_i^{-1}(\varepsilon)\tilde{z}^0_{i,0,w} + F^0_{i,0}e^0_{\mathrm{z}i,0},$$

$\|F_{i,0}^0(\varepsilon)\| \leq \bar{f}_0\varepsilon$ by Lemma 2.1, and $\left\|F_{i,0}^0(\varepsilon)Q_i(\varepsilon)S_i^{-1}(\varepsilon)\right\| \leq \kappa_i\varepsilon$ by Lemma 2.4, we have,

$$\begin{aligned}
&\int_0^\infty \big(y_i(t) - y_{i,w=0}(t)\big)^2 \mathrm{d}t \\
&\leq 2\int_0^\infty \tilde{y}_{i,w}^2(t)\mathrm{d}t + 2\int_0^\infty u_{\mathrm{L}i,w}^2(t)\mathrm{d}t \\
&\leq 2\int_0^\infty \tilde{y}_{i,w}^2(t)\mathrm{d}t + 4\kappa_i^2\varepsilon^2\int_0^\infty \|\tilde{z}_{i,0,w}^0(t)\|^2\mathrm{d}t + 4\varepsilon^2\bar{f}_0^2\left\|T_{i,0}^{-1}\right\|^2\int_0^\infty \|e_{i,0}(t)\|^2\mathrm{d}t,
\end{aligned}$$

which, together with (2.41), (2.43), and (2.38), gives that

$$\int_0^\infty \big(y_i(t) - y_{i,w=0}(t)\big)^2\mathrm{d}t \leq \varepsilon^2\gamma_i^2\int_0^\infty \|w(t)\|^2\mathrm{d}t,$$

where

$$\gamma_i^2 = \left(2\gamma_{\tilde{y}}^2 + 4\kappa_i^2\gamma_{\tilde{z}_{i,0,w}^0} + 4\bar{f}_0^2\left\|T_{i,0}^{-1}\right\|^2\gamma_{\mathrm{ob}i}^2\right).$$

For any given γ, choose $\varepsilon^* \leq \min\left\{\varepsilon_\eta^*, \frac{\gamma}{\gamma_i}\right\}$. Then, for any $\varepsilon \in (0, \varepsilon^*]$, the consensus protocols (2.32) achieve the three objectives of Problem 2.1.

2.6 Simulation For Output Feedback

Consider five follower agents, described by

$$\begin{cases}
\dot{x}_{i,0,1} = x_{i,0,2} + w_i, \\
\dot{x}_{i,0,2} = x_{i,1}, \\
\dot{x}_{i,1} = x_{i,2} + 2w_i, \\
\dot{x}_{i,2} = x_{i,0,1} + x_{i,0,2} - 2x_{i,1} + x_{i,2} + u_i + 3w_i, \\
y_i = x_{i,1},\ i \in \{1, 2, \cdots, 5\},
\end{cases}$$

and one leader agent, described by

$$\begin{cases}
\dot{x}_{0,1} = x_{0,2}, \\
\dot{x}_{0,2} = u_0, \\
y_0 = x_{0,1},
\end{cases}$$

with $u_0(t) = -x_{0,1}(t) - 2x_{0,2}(t) + 10\sin(3.14t)$.

The communication topology is shown in Fig. 2.5, in which the directed information flow is represented by the arrows.

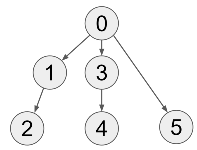

Figure 2.5: The communication topology.

The initial conditions of the agents were generated randomly as

$$
\begin{aligned}
x_0(0) &= \mathrm{col}(3.8974, 7.8025),\\
\mathrm{col}(x_{1,0}(0), x_1(0)) &= \mathrm{col}(2.4169, 4.0391, 0.9645, 1.3197),\\
\mathrm{col}(x_{2,0}(0), x_2(0)) &= \mathrm{col}(9.4205, 9.5613, 5.7521, 0.5978),\\
\mathrm{col}(x_{3,0}(0), x_3(0)) &= \mathrm{col}(2.3478, 3.5316, 8.2119, 0.1540),\\
\mathrm{col}(x_{4,0}(0), x_4(0)) &= \mathrm{col}(0.4302, 1.6899, 6.4912, 7.3172),\\
\mathrm{col}(x_{5,0}(0), x_5(0)) &= \mathrm{col}(6.4775, 4.5092, 5.4701, 2.9632).
\end{aligned}
$$

The initial conditions of the observers were all chosen as zero. The disturbances were realized as

$$
\begin{bmatrix} w_1(t)\\ w_2(t)\\ w_3(t)\\ w_4(t)\\ w_5(t) \end{bmatrix} = \begin{bmatrix} 2\sin(11.1t)+\cos(2.1t)\\ 2\sin(5.8t)+\cos(0.2t)\\ 2\sin(0.6t)+\cos(7.2t)\\ 2\sin(0.1t)+\cos(9.6t)\\ 2\sin(3.6t)+\cos(0.9t) \end{bmatrix}.
$$

Figure 2.6 shows the output consensus errors of the multi-agent system in the absence of the disturbances, with the design parameter $\varepsilon = 0.5$. Figure 2.8 shows the same quantities with the design parameter $\varepsilon = 0.1$. It is observed that the output consensus errors are ultimately bounded and the bounds become smaller as the value of ε becomes smaller. Figures. 2.7 and 2.9 show the states of the follower agents, which remain bounded.

Figure 2.10 shows the output consensus errors of the multi-agent system in the presence of the disturbances, with the design parameter $\varepsilon = 0.5$. Figure 2.11 shows the same quantities with the design parameter $\varepsilon = 0.1$. It is observed that the effect of the disturbances become smaller as the value of ε becomes smaller.

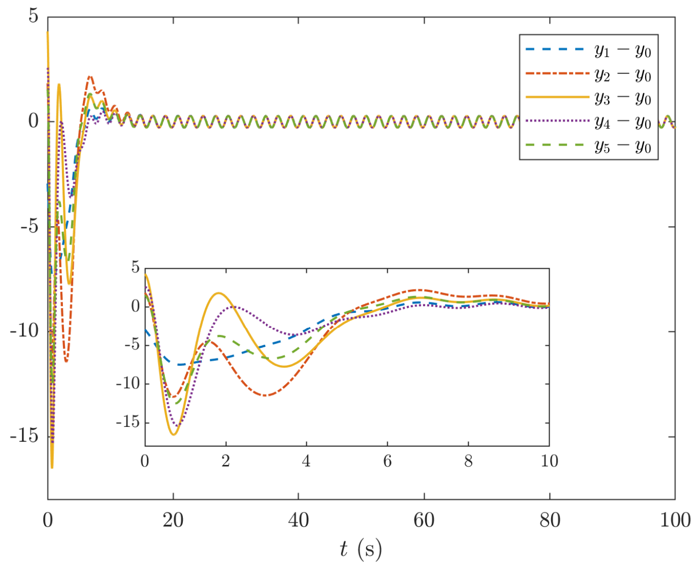

Figure 2.6: The output consensus errors in the absence of the disturbances: $\varepsilon = 0.5$.

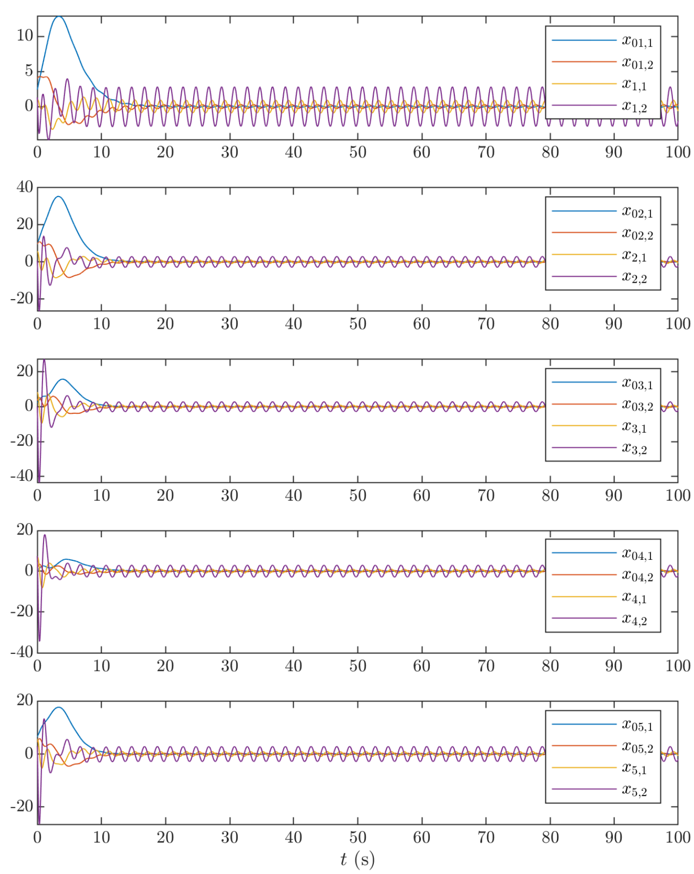

Figure 2.7: The states of the follower agents in the absence of the disturbances: $\varepsilon = 0.5$.

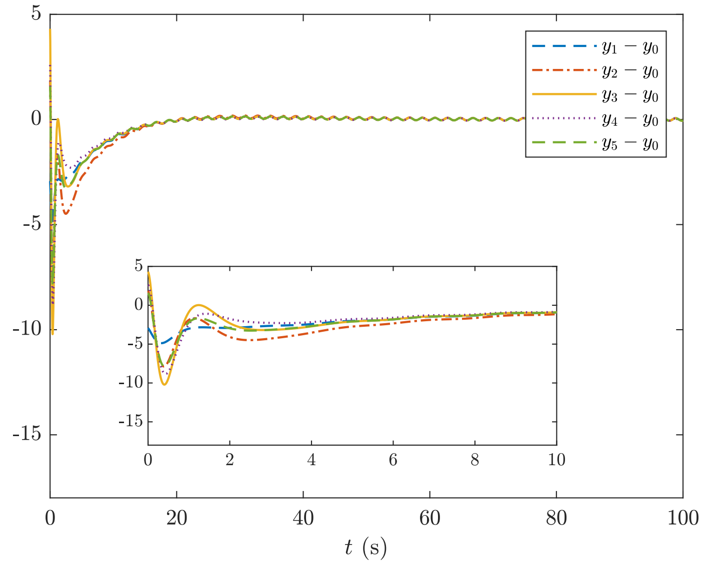

Figure 2.8: The output consensus errors in the absence of the disturbances: $\varepsilon = 0.1$.

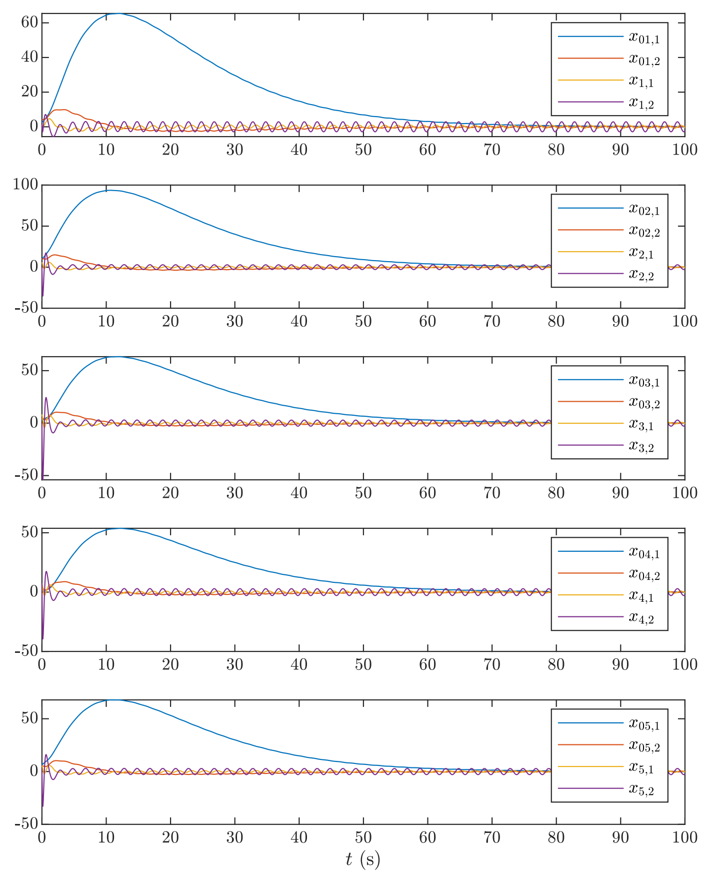

Figure 2.9: The states of the follower agents in the absence of the disturbances: $\varepsilon = 0.1$.

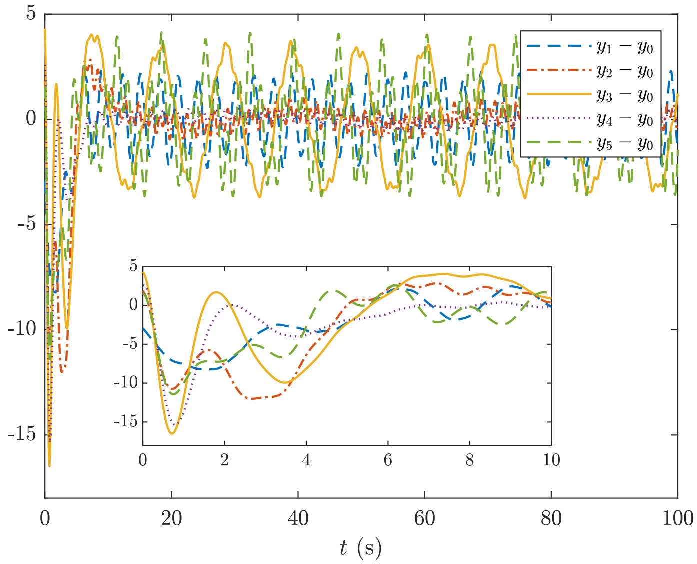

Figure 2.10: The output consensus errors in the presence of the disturbances: $\varepsilon = 0.5$.

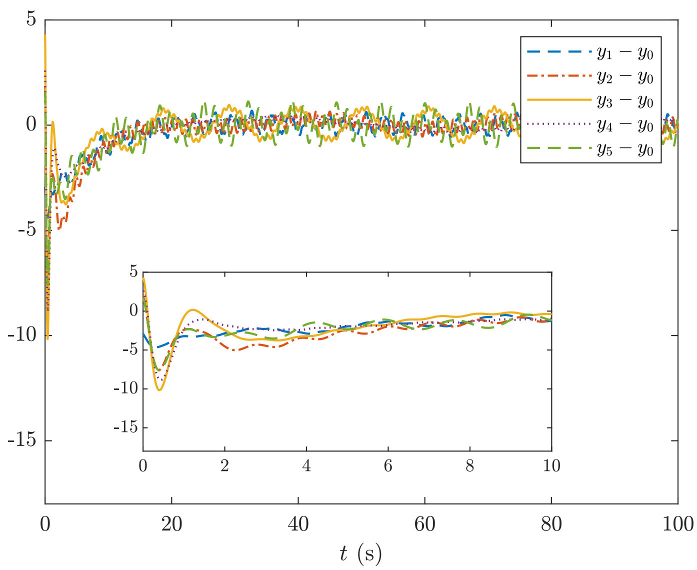

Figure 2.11: The output consensus errors in the presence of the disturbances: $\varepsilon = 0.1$.

2.7 Conclusions

In this chapter, the leader-following almost output consensus problem of linear continuous-time multi-agent systems with disturbance-affected unstable zero dynamics was studied. Under conditions on the agent dynamics and the way the disturbances affect the zero dynamics, we constructed low-and-high gain based state feedback and output feedback consensus protocols for the follower agents. These conditions are the same as those necessary for achieving almost disturbance decoupling for individual systems and are thus mild. The protocols we constructed were shown to achieve almost output consensus as long as the communication topology of the system contains a directed spanning tree with the leader as the root node. Simulations were carried out to validate the established results.

This chapter is based on the following publications:

- Tingyang Meng, and Zongli Lin, "Leader-following almost output consensus for linear heterogeneous multi-agent systems with disturbance-affected unstable zero dynamics by output feedback." *IEEE Transactions on Control of Network Systems* 9.3 (2022): 1281-1293.
- Tingyang Meng, and Zongli Lin, "Leader-following almost output consensus for linear multi-agent systems with disturbance-affected unstable zero dynamics." *Systems & Control Letters* 145 (2020): 104787.

Chapter 3

Almost Output Consensus of Linear Discrete-Time Multi-Agent Systems

3.1 Introduction

In the previous chapter, we considered the leader-following almost output consensus problem for continuous-time linear multi-agent systems in the presence of disturbance-affected unstable zero dynamics. State feedback and output feedback consensus protocols were proposed based on the low-and-high gain feedback design technique.

In this chapter, the problem of leader-following almost output consensus for discrete-time heterogeneous multi-agent systems is formulated and solved. The zero dynamics of the agents may be unstable and subject to external disturbances. In particular, the zero dynamics of the follower agents are allowed to have poles on the closed unit disc and are therefore allowed to be polynomially unstable. When the outputs of follower agents are tracking the non-zero bounded output of the leader agent, the follower agents' states corresponding to their unstable zero dynamics may grow to infinity. It is therefore important to guarantee the internal stability of the follower agents' dynamics while reaching leader-following output consensus. State feedback consensus protocols are designed based on the low gain feedback design technique [43] and the solution of a modified discrete-time algebraic Riccati equation [73]. The unstable part of the zero dynamics is stabilized through a small perturbation, dictated by a low gain feedback law, in the output of each follower agent. The closed-loop system under the proposed consensus protocols is shown to achieve leader-following consensus with a pre-specified arbitrarily high accuracy, and the L_2-gain from the disturbance to the consensus error is shown to be suppressed to a pre-specified arbitrarily low level. Additional conditions are identified under which output feedback results are established.

A key feature of the work in this chapter is the consideration of the consensus and tracking of the outputs of non-minimum phase agents in the presence of external disturbances. Given that precise output tracking with internal stability is not possible for non-minimum phase systems, we identify a class of nonlinear phase agent dynamics for which low gain feedback can be applied to achieve leader-following output consensus to an arbitrarily high level of accuracy. Compared to the results in the continuous-time setting in the previous chapter, which utilize both the low gain and high gain design techniques, the absence of high gain action in the discrete-time setting limits how the disturbance can enter the agent dynamics and our ability to design the consensus protocols and analyze the properties of the resulting closed-loop system, including its stability. The use of a newly established result on a modified discrete-time Riccati equation [73] is instrumental in helping our analysis under the proposed consensus protocols.

The remainder of this chapter is organized as follows. Section 3.2 formulates the leader-following almost output consensus problem for linear discrete-time multi-agent system with disturbance-affected unstable zero dynamics. Section 3.3 establishes the state feedback results. Section 3.4 presents the simulation for state feedback design. Section 3.5 establishes the output feedback results. Section 3.6 presents the simulation for output feedback design. Section 3.7 concludes this chapter.

3.2 Problem Statement

Consider a linear discrete-time heterogeneous multi-agent system consisting of N follower agents, indexed as $1, 2, \cdots, N$, and one leader agent, indexed as 0. The communication network among these agents is represented by a directed graph $\mathcal{G}$ that satisfies the following assumption.

Assumption 3.1. The directed graph $\mathcal{G}$ that represents the communication topology among the $N+1$ agents contains a directed spanning tree with the leader agent as its root.

The dynamics of the ith follower agent, $i \in \{1, 2, \cdots, N\}$, is described by a discrete-time linear system,

$$\begin{cases} x_{i,0}(k+1) = A_{i,0}x_{i,0}(k) + B_{i,0}y_i(k) + D_{i,0}w_i(k), \\ x_{i,m}(k+1) = x_{i,m+1}(k) + d_{i,m}w_i(k), \ m = 1, 2, \cdots, \rho - 1, \\ x_{i,\rho}(k+1) = E_{i,0}x_{i,0}(k) + \alpha_1 x_{i,1}(k) + \alpha_2 x_{i,2}(k) + \cdots + \alpha_\rho x_{i,\rho}(k) + u_i(k) + d_{i,\rho}w_i(k), \\ y_i(k) = x_{i,1}(k), \end{cases} \tag{3.1}$$

where $x_{i,0} \in \mathbb{R}^{n_{i,0}}$ and $x_i = [x_{i,1} \ x_{i,2} \ \cdots \ x_{i,\rho}]^{\mathrm{T}} \in \mathbb{R}^{\rho}$ are the states, $u_i \in \mathbb{R}$ is the input, $y_i \in \mathbb{R}$ is the output, and $w_i \in \mathbb{R}$ is the disturbance. The $x_{i,0}$ dynamics, which are allowed to be heterogeneous for different agents, are referred to as the zero dynamics. The relative degree ρ is assumed to be the same for all follower agents.

The dynamics of the leader agent is described by the following discrete-time linear system,

$$\begin{cases} x_{0,m}(k+1) = x_{i,m+1}(k), \ m = 1, 2, \cdots, \rho - 1, \\ x_{0,\rho}(k+1) = \alpha_1 x_{0,1}(k) + \alpha_2 x_{0,2}(k) + \cdots + \alpha_\rho x_{0,\rho}(k), \\ y_0(k) = x_{0,1}(k), \end{cases} \tag{3.2}$$

where $x_0 = [x_{0,1} \ x_{0,2} \ \cdots \ x_{0,\rho}]^{\mathrm{T}} \in \mathbb{R}^\rho$ is the states, and $y_0 \in \mathbb{R}$ is the output.

We make the following assumptions on the dynamics of the agents.

Assumption 3.2. The pair $(A_{i,0}, B_{i,0})$ is stabilizable, and all eigenvalues of $A_{i,0}$ are on the closed unit disc, *i.e.*, the zero dynamics, which is governed by $A_{i,0}$, may be be polynomially unstable.

Assumption 3.3. Consider system (3.1), that is,

$$\Sigma_i : \begin{cases} x(k+1) = A_i x(k) + B_i u(k) + D_i w_i(k), \\ y(k) = C_i x(k), \end{cases}$$

where

$$A_i = \left[\begin{array}{c|cccc} A_{i,0} & B_{i,0} & 0_{n_{i,0}\times(\rho-1)} & & \\ \hline & 0 & 1 & \cdots & 0 \\ 0_{(\rho-1)\times n_{i,0}} & \vdots & \vdots & \ddots & \vdots \\ & 0 & 0 & \cdots & 1 \\ E_{i,0} & \alpha_1 & \alpha_2 & \cdots & \alpha_\rho \end{array}\right], \quad B_i = \begin{bmatrix} 0_{(n_{i,0}+\rho-1)\times 1} \\ 1 \end{bmatrix},$$

$$C_i = \begin{bmatrix} 0_{1\times n_{i,0}} & 1 & 0_{1\times(\rho-1)} \end{bmatrix}, \quad D_i = \begin{bmatrix} D_{i,0} \\ d_{i,1} \\ d_{i,2} \\ \vdots \\ d_{i,\rho} \end{bmatrix}.$$

Then, the vector D_i satisfies

$$\mathrm{Im}(D_i) \subset \mathcal{V}^{\odot}(\Sigma_i) \cap \{ \cap_{|\lambda=1|} S_\lambda(\Sigma_i) \},$$

where $\mathcal{V}^{\odot}(\Sigma_i)$ is the maximal subspace of $\mathbb{R}^{n_{i,0}+\rho}$, which is $(A_i + B_i F)$-invariant and contained in $\mathrm{Ker}(C_i)$ such that the eigenvalues of $(A_i + B_i F)|\mathcal{V}^{\odot}$ are contained in $\mathbb{C}^{\bigcirc} \cup \mathbb{C}^{\odot}$ for some constant matrix F, and

$$S_\lambda(\Sigma_i) = \left\{ x \in \mathbb{C}^{n_{i,0}+\rho} \middle| \exists u \in \mathbb{C}^{n_{i,0}+\rho+1} : \begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{bmatrix} A_i - \lambda I & B_i \\ C_i & 0 \end{bmatrix} u \right\}.$$

Remark 3.1. By reference [4], conditions in Assumption 3.3 are necessary and sufficient for the solvability of the almost disturbance decoupling problem with stability (ADDPS) for agent i operating as an independent system. The ADDPS entails the design of a state feedback law, under which the closed-loop system is asymptotically stable in the absence of the disturbance and the L_2 gain from the disturbance to the output of the closed-loop system with zero initial condition is smaller than or equal to a pre-specified arbitrarily small value.

Assumption 3.4. The leader's output y_0 is bounded by a constant $\bar{y} > 0$, and it does not contain frequency components corresponding to $\lambda^{\bigcirc}\big(A_{i,0}\big)$, $i \in \{1, 2, \cdots, N\}$, where $\lambda^{\bigcirc}\big(A_{i,0}\big)$ denotes the set of eigenvalues of $A_{i,0}$ that are on the unit circle.

Problem 3.1. Consider the discrete-time leader-following multi-agent system described by (3.1) and (3.2). Let the communication topology satisfy Assumption 3.1. Let the dynamics of the follower agents satisfy Assumptions 3.2 and 3.3. Let the output of the leader satisfy Assumption 3.4. For any pre-specified, arbitrarily small, scalars $\eta > 0$ and $\gamma > 0$, design distributed consensus protocols u_i, $i \in \{1, 2, \cdots, N\}$, under which the following are satisfied.

(i) In the absence of disturbance, the states of all follower agents remain bounded, and the leader-following output consensus is reached within the pre-specified accuracy $\eta > 0$, *i.e.*,

$$\limsup_{k\to\infty} \big| y_{i,w=0}(k) - y_0(k) \big| \le \eta, \ i \in \{1, 2, \cdots, N\},$$

where $y_{i,w=0}(k)$, $i \in \{1, 2, \cdots, N\}$, are the outputs of the follower agents, with the subscript $w = 0$ indicating the absence of the disturbance.

(ii) When the multi-agent system is operating in steady state, the effect of the disturbance $w = [w_1 \ w_2 \ \cdots \ w_N]^{\mathrm{T}}$ on the leader-following output consensus in terms of the L_2-gain, is less than or equal to the pre-specified $\gamma > 0$, *i.e.*,

$$\sum_{k=0}^{\infty} \big| y_{i,w\neq 0}(k) - y_{i,w=0}(k) \big|^2 \le \gamma^2 \sum_{k=0}^{\infty} \|w(k)\|^2,$$

where $y_{i,w\neq 0}(k)$, $i \in \{1, 2, \cdots, N\}$, are the outputs of the follower agents in the presence of the disturbance.

3.3 State Feedback Results

We first present in three steps our design of leader-following almost output consensus protocols. In the first step, a virtual controller is designed by utilizing the low gain feedback design technique for each follower agent to stabilize its zero dynamics. In the second step, a new output is defined by shifting the original output by the low gain feedback designed in Step 1. Consensus protocols are then constructed in Step 3.

Step 1: Low Gain Feedback

For each follower agent i, $i \in \{1, 2, \cdots, N\}$, find a nonsingular transformation matrix $T_{i,0} \in \mathbb{R}^{n_{i,0} \times n_{i,0}}$ for the pair $\big(A_{i,0}, B_{i,0}\big)$ such that,

$$T_{i,0}^{-1} A_{i,0} T_{i,0} = \begin{bmatrix} A_{i,0}^{\bigcirc} & 0 \\ 0 & A_{i,0}^{\odot} \end{bmatrix}, \; T_{i,0}^{-1} B_{i,0} = \begin{bmatrix} B_{i,0}^{\bigcirc} \\ B_{i,0}^{\odot} \end{bmatrix}, \tag{3.3}$$

where $A_{i,0}^{\bigcirc} \in \mathbb{R}^{n_{i,0}^{\bigcirc} \times n_{i,0}^{\bigcirc}}$ and $A_{i,0}^{\odot} \in \mathbb{R}^{n_{i,0}^{\odot} \times n_{i,0}^{\odot}}$ are such that $\lambda\big(A_{i,0}^{\bigcirc}\big) \subset \mathbb{C}^{\bigcirc}$ and $\lambda\big(A_{i,0}^{\odot}\big) \subset \mathbb{C}^{\odot}$.

Under Assumption 3.2, $\big(A_{i,0}^{\bigcirc}, B_{i,0}^{\bigcirc}\big)$ is controllable and, without loss of generality, is assumed to be in its controller canonical form,

$$A_{i,0}^{\bigcirc} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ \alpha_{i,1}^{\bigcirc} & \alpha_{i,2}^{\bigcirc} & \cdots & \alpha_{i,n_{i,0}^{\bigcirc}}^{\bigcirc} \end{bmatrix}, \; B_{i,0}^{\bigcirc} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}. \tag{3.4}$$

Define a low gain feedback [43] virtual control for its unstable zero dynamics as

$$u_{i,0}(k) = F_{i,0}(\varepsilon) x_{i,0}, \; \varepsilon \in (0, 1], \tag{3.5}$$

where $F_{i,0}(\varepsilon) = \big[F_{i,0}^{\bigcirc} \;\; 0\big] T_{i,0}^{-1} \in \mathbb{R}^{1 \times n_{i,0}}$, in which $F_{i,0}^{\bigcirc} \in \mathbb{R}^{1 \times n_{i,0}^{\bigcirc}}$ is such that

$$\lambda\big(A_{i,0}^{\bigcirc} + B_{i,0}^{\bigcirc} F_{i,0}^{\bigcirc}(\varepsilon)\big) = (1 - \varepsilon) \lambda\big(A_{i,0}^{\bigcirc}\big) \subset \mathbb{C}^{\odot}, \tag{3.6}$$

and ε is a design parameter.

Step 2: Output Redefinition

A new output $\check{y}_i(k)$ of each follower agent i, $i \in \{1, 2, \cdots, N\}$, is defined based on its output and the virtual control as

$$\begin{aligned}\check{y}_i &= y_i(k) - u_{i,0}(k)\\ &= x_{i,1} - F_{i,0}(\varepsilon)x_{i,0}(k).\end{aligned}$$

As will be shown later, consensus protocols will be designed such that the new output $\check{y}_i$ will be driven towards the leader agent's output y_0, and the magnitude of $u_{i,0}$ will be made small enough. As such, the unstable zero dynamics is stabilized while the leader-following output consensus is being achieved within the pre-specified accuracy.

A new set of states, $\check{x}_i = \begin{bmatrix}\check{x}_{i,1} & \check{x}_{i,2} & \cdots & \check{x}_{i,\rho}\end{bmatrix}^{\mathrm{T}}$, is also defined based on the new output $\check{y}_i$ as

$$\begin{cases}\check{x}_{i,1}(k) = \check{y}_i = x_{i,1} - F_{i,0}(\varepsilon)x_{i,0}(k),\\ \check{x}_{i,m}(k) = x_{i,m}(k) - F_{i,0}(\varepsilon)A_{i,0}^{m-1}x_{i,0}(k) - \displaystyle\sum_{l=1}^{m-1} F_{i,0}(\varepsilon)A_{i,0}^{m-1-l}B_{i,0}x_{i,l}(k),\ m = 2,3,\cdots,\rho.\end{cases} \tag{3.7}$$

The dynamics of the follower agent i, $i \in \{1,2,\cdots,N\}$, can be written in the new states as

$$\begin{cases}x_{i,0}(k+1) = A_{\mathrm{c}i,0}(\varepsilon)x_{i,0}(k) + B_{i,0}\check{y}_i(k) + D_{i,0}w_i(k),\\ \check{x}_{i,m}(k+1) = \check{x}_{i,m+1}(k) + \check{d}_{i,m}w_i(k),\ r = 1,2,\cdots,\rho-1,\\ \check{x}_{i,\rho}(k+1) = \check{E}_{i,0}x_{i,0}(k) + \check{\alpha}_{i,1}\check{x}_{i,1}(k) + \check{\alpha}_{i,2}\check{x}_{i,2}(k) + \cdots + \check{\alpha}_{i,\rho}\check{x}_{i,\rho}(k) + u_i(k) + \check{d}_{i,\rho}w_i(k),\\ \qquad \check{y}_i(k) = \check{x}_{i,1}(k),\end{cases}$$

where $A_{\mathrm{c}i,0}(\varepsilon) = A_{i,0} + B_{i,0}F_{i,0}(\varepsilon)$, and

$$\begin{aligned}\check{d}_{i,1} &= d_{i,1} - F_{i,0}(\varepsilon)D_{i,0},\\ \check{d}_{i,m} &= d_{i,m} - F_{i,0}(\varepsilon)A_{i,0}^{m-1}D_{i,0} - \sum_{l=1}^{m-1} F_{i,0}(\varepsilon)A_{i,0}^{m-1-l}B_{i,0}d_{i,l},\ m = 2,3,\cdots,\rho,\\ \check{E}_{i,0} &= E_{i,0} - F_{i,0}(\varepsilon)A_{i,0}^{\rho} + \sum_{l=1}^{\rho}\big(\alpha_{i,l} - F_{i,0}(\varepsilon)A_{i,0}^{\rho-l}B_{i,0}\big)F_{i,0}(\varepsilon)A_{\mathrm{c}i,0}^{l-1}(\varepsilon),\\ \check{\alpha}_{i,m} &= \alpha_{i,m} - F_{i,0}(\varepsilon)A_{i,0}^{\rho-m}(\varepsilon)B_{i,0} + \sum_{l=m+1}^{\rho}\big(\alpha_{i,l} - F_{i,0}(\varepsilon)A_{i,0}^{\rho-l}B_{i,0}\big)F_{i,0}(\varepsilon)A_{\mathrm{c}i,0}^{l-1-m}(\varepsilon)B_{i,0},\\ &\qquad m = 1,2,\cdots,\rho-1,\\ \check{\alpha}_{i,\rho} &= \alpha_{i,\rho} - F_{i,0}(\varepsilon)B_{i,0}.\end{aligned}$$

A pre-feedback law is design as

$$u_i(k) = -\check{E}_{i,0}x_{i,0}(k) + (\alpha_1 - \check{\alpha}_{i,1})\check{x}_{i,1}(k) + (\alpha_2 - \check{\alpha}_{i,2})\check{x}_{i,2}(k) + \cdots + (\alpha_\rho - \check{\alpha}_{i,\rho})\check{x}_{i,\rho}(k) + \check{u}_i(k) \quad (3.8)$$

with $\check{u}_i(k)$ to be designed later. Then, the state equation of the ith follower agent is rewritten as

$$\begin{cases} x_{i,0}(k+1) = A_{\mathrm{c}i,0}(\varepsilon)x_{i,0}(k) + B_{i,0}\check{y}_i(k) + D_{i,0}w_i(k), \\ \check{x}_i(k+1) = A_{\mathrm{r}}\check{x}_i(k) + B_{\mathrm{r}}\check{u}_i(k) + \check{d}_i w_i, \\ \check{y}_i(k) = \check{x}_{i,1}(k), \end{cases} \quad (3.9)$$

where

$$A_{\mathrm{r}} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ \alpha_1 & \alpha_2 & \cdots & \alpha_\rho \end{bmatrix}_{\rho\times\rho}, \quad B_{\mathrm{r}} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}_{\rho\times 1}, \quad \text{and } \check{d}_i = \begin{bmatrix} \check{d}_{i,1} \\ \check{d}_{i,2} \\ \vdots \\ \check{d}_{i,\rho} \end{bmatrix}. \quad (3.10)$$

And the dynamics of the leader agent can be written as

$$\begin{cases} x_0(k+1) = A_{\mathrm{r}}x_0(k), \\ y_0(k) = x_{0,1}(k). \end{cases}$$

Step 3: Consensus Protocol

We will first design the feedback control $\check{u}_i(k)$ of follower agent i, $i \in \{1, 2, \cdots, N\}$, in the new states as

$$\check{u}_i(k) = -\kappa K\Big(\sum_{j=1}^{N} a_{ij}\big(\check{x}_i(k) - \check{x}_j(k)\big) + h_i\big(\check{x}_i(k) - x_0(k)\big)\Big),$$

where $K = \big(R + B_{\mathrm{r}}^{\mathrm{T}}PB_{\mathrm{r}}\big)^{-1}B_{\mathrm{r}}^{\mathrm{T}}PA_{\mathrm{r}}$, and P is the unique positive definite solution to the modified discrete-time algebraic Riccati equation,

$$A_{\mathrm{r}}^{\mathrm{T}}PA_{\mathrm{r}} - (1-\delta)\big(A_{\mathrm{r}}^{\mathrm{T}}PB_{\mathrm{r}}\big)\big(R + B_{\mathrm{r}}^{\mathrm{T}}PB_{\mathrm{r}}\big)^{-1}\big(B_{\mathrm{r}}^{\mathrm{T}}PA_{\mathrm{r}}\big) - P = -Q, \quad (3.11)$$

for some positive definite matrices R and Q, and $\sqrt{\delta} \in \Big[\frac{\lambda_N - \lambda_1}{\lambda_N + \lambda_1}, 1\Big)$, with $\lambda_1 > 0$ and $\lambda_N > 0$ being the smallest and the largest eigenvalues of $\bar{\mathcal{L}} + \bar{\mathcal{H}}$, respectively. The scalar κ satisfies

$$\kappa \in \left[\frac{1-\sqrt{\delta}}{\lambda_1}, \frac{1+\sqrt{\delta}}{\lambda_N}\right].$$

We note that, by Assumption 3.1, matrix $\bar{\mathcal{L}} + \bar{\mathcal{H}}$ is positive definite and all its eigenvalues are positive.

The following Lemma guarantees the existence of the solution P to the modified discrete-time algebraic Riccati equation (3.11).

Lemma 3.1. [73] The modified discrete-time algebraic Riccati equation (3.11) converges to a solution if

$$\delta < \frac{1}{\Pi_{i=1}^{n_u} \left|\tilde{\lambda}_i\right|^2},$$

where $\left\{\tilde{\lambda}_1, \tilde{\lambda}_2, \cdots, \tilde{\lambda}_{n_u}\right\}$ is the set of all the n_u unstable eigenvalues of A_{r}.

The consensus protocol $u_i(k)$ is given by (3.8) and $\check{u}_i(k)$ as

$$\begin{aligned} u_i(k) = & - \check{E}_{i,0} x_{i,0} + \sum_{m=1}^{\rho} a_{ij}(\alpha_m - \alpha_{i,m}) \check{x}_{i,m} \\ & - \kappa\big(R + B_{\mathrm{r}}^{\mathrm{T}} P B_{\mathrm{r}}\big)^{-1} B_{\mathrm{r}}^{\mathrm{T}} P A_{\mathrm{r}} \Big(\sum_{j=1}^{N} \big(\check{x}_i(k) - \check{x}_j(k)\big) + h_i\big(\check{x}_i(k) - x_0(k)\big)\Big), \ \varepsilon \in (0,1]. \end{aligned} \tag{3.12}$$

Theorem 3.1. [56] Consider the discrete-time heterogeneous multi-agent system described by (3.1) and (3.2). Let the communication topology satisfy Assumption 3.1. Let the dynamics of the follower agents satisfy Assumptions 3.2 and 3.3. Let the output the leader agent satisfy Assumption 3.4. Then, the family of state feedback consensus protocols (3.12) solve Problem 3.1.

To prove Theorem 3.1, we need to recall some technical lemmas, which will be used in the Lyapunov analysis of the closed-loop system.

Lemma 3.2. [43] Consider the pair $\big(A_{i,0}^{\bigcirc}, B_{i,0}^{\bigcirc}\big)$ as given in (3.4) and $F_{i,0}^{\bigcirc}(\varepsilon)$ as given in (3.6). There exists constant $\bar{F}_{i,0} > 0$ such that

$$\big\|F_{i,0}^{\bigcirc}(\varepsilon)\big\| \leq \bar{F}_{i,0}\varepsilon, \ \varepsilon \in (0,1].$$

Lemma 3.3. [43] Consider the pair $\big(A_{i,0}^{\bigcirc}, B_{i,0}^{\bigcirc}\big)$ as given in (3.4) and $F_{i,0}^{\bigcirc}(\varepsilon)$ as given in (3.6). There exists nonsingular transformation matrix $Q_{i,0}^{\bigcirc}(\varepsilon) \in \mathbb{R}^{n_{i,0}^{\bigcirc} \times n_{i,0}^{\bigcirc}}$ such that

$$\begin{aligned} \big(Q_{i,0}^{\bigcirc}\big)^{-1}(\varepsilon)\big(A_{i,0}^{\bigcirc} + B_{i,0}^{\bigcirc} F_{i,0}^{\bigcirc}(\varepsilon)\big) Q_{i,0}^{\bigcirc}(\varepsilon) &= J_{i,0}^{\bigcirc}(\varepsilon) \\ &= \mathrm{blkdiag}\Big\{J_{i,0,-1}^{\bigcirc}(\varepsilon), J_{i,0,+1}^{\bigcirc}(\varepsilon), J_{i,0,1}^{\bigcirc}(\varepsilon), J_{i,0,2}^{\bigcirc}(\varepsilon), \cdots, J_{i,0,l_i}^{\bigcirc}(\varepsilon)\Big\}, \end{aligned}$$

where

$$J^{\bigcirc}_{i,0,-1} = \begin{bmatrix} -(1-\varepsilon) & 1 & & \\ & \ddots & \ddots & \\ & & -(1-\varepsilon) & 1 \\ & & & -(1-\varepsilon) \end{bmatrix}_{n^{\bigcirc}_{i,0,-1} \times n^{\bigcirc}_{i,0,-1}},$$

$$J^{\bigcirc}_{i,0,+1} = \begin{bmatrix} 1-\varepsilon & 1 & & \\ & \ddots & \ddots & \\ & & 1-\varepsilon & 1 \\ & & & 1-\varepsilon \end{bmatrix}_{n^{\bigcirc}_{i,0,+1} \times n^{\bigcirc}_{i,0,+1}},$$

and for each $l = 1$ to l_i,

$$J^{\bigcirc}_{i,0,l}(\varepsilon) = \begin{bmatrix} J^{\star}_{i,0,l}(\varepsilon) & I_2 & & \\ & \ddots & \ddots & \\ & & J^{\star}_{i,0,l}(\varepsilon) & I_2 \\ & & & J^{\star}_{i,0,l}(\varepsilon) \end{bmatrix}_{2n^{\bigcirc}_{i,0,l} \times 2n^{\bigcirc}_{i,0,l}}, \quad J^{\star}_{i,0,l}(\varepsilon) = (1-\varepsilon) \begin{bmatrix} \alpha_{i,0,l} & \beta_{i,0,l} \\ -\beta_{i,0,l} & \alpha_{i,0,l} \end{bmatrix},$$

with $\alpha^2_{i,0,l} + \beta^2_{i,0,l} = 1$ for all $l = 1$ to l_i and $\alpha_{i,0,j} \neq \omega_{i,k}$ for $j \neq k$. Furthermore,

$$\left\| \left(Q^{\bigcirc}_{i,0}\right)^{-1}(\varepsilon) \right\| \leq \bar{q}_i, \ \left\| Q^{\bigcirc}_{i,0}(\varepsilon) \right\| \leq \bar{q}_i, \ \varepsilon \in (0,1],$$

for some constant $\bar{q}_i \geq 0$.

Lemma 3.4. [43] Consider the pair $\left(A^{\bigcirc}_{i,0}, B^{\bigcirc}_{i,0}\right)$ as given in (3.4) and $F^{\bigcirc}_{i,0}(\varepsilon)$ as given in (3.6). Let $Q^{\bigcirc}_{i,0}(\varepsilon)$ be as defined in Lemma 3.3. Let

$$S^{\bigcirc}_{i,0}(\varepsilon) = \text{blkdiag}\Big\{ S^{\bigcirc}_{i,0,-1}(\varepsilon), S^{\bigcirc}_{i,0,+1}(\varepsilon), S^{\bigcirc}_{i,0,1}(\varepsilon), S^{\bigcirc}_{i,0,2}(\varepsilon), \cdots, S^{\bigcirc}_{i,0,l_i}(\varepsilon) \Big\},$$

where

$$S^{\bigcirc}_{i,0,-1}(\varepsilon) = \text{diag}\Big\{ \varepsilon^{n^{\bigcirc}_{i,0,-1}-1}, \varepsilon^{n^{\bigcirc}_{i,0,-1}-2}, \cdots, \varepsilon, 1 \Big\}, \ S^{\bigcirc}_{i,0,+1}(\varepsilon) = \text{diag}\Big\{ \varepsilon^{n^{\bigcirc}_{i,0,+1}-1}, \varepsilon^{n^{\bigcirc}_{i,0,+1}-2}, \cdots, \varepsilon, 1 \Big\},$$

and for each $l = 1$ to l_i,

$$S_{i,l}(\varepsilon) = \text{diag}\Big\{ \varepsilon^{n^{\bigcirc}_{i,0,l}-1} I_2, \varepsilon^{n^{\bigcirc}_{i,0,l}-2} I_2, \cdots, \varepsilon I_2, I_2 \Big\}.$$

Then,

1.
$$
\begin{aligned}
S_{i,0}^{\bigcirc}(\varepsilon)J_{i,0}^{\bigcirc}(\varepsilon)\left(S_{i,0}^{\bigcirc}\right)^{-1}(\varepsilon) &= \tilde{J}_{i,0}^{\bigcirc}(\varepsilon) \\
&= \mathrm{blkdiag}\Big\{\tilde{J}_{i,0,-1}^{\bigcirc}(\varepsilon), \tilde{J}_{i,0,+1}^{\bigcirc}(\varepsilon), \tilde{J}_{i,0,1}^{\bigcirc}(\varepsilon), \tilde{J}_{i,0,2}^{\bigcirc}\varepsilon), \cdots, \tilde{J}_{i,0,l_i}^{\bigcirc}\Big\},
\end{aligned}
$$

where

$$
\tilde{J}_{i,0,-1}^{\bigcirc} = \begin{bmatrix} -(1-\varepsilon) & \varepsilon & & \\ & \ddots & \ddots & \\ & & -(1-\varepsilon) & \varepsilon \\ & & & -(1-\varepsilon) \end{bmatrix}_{n_{i,0,-1}^{\bigcirc}\times n_{i,0,-1}^{\bigcirc}},
$$

$$
\tilde{J}_{i,0,+1}^{\bigcirc} = \begin{bmatrix} 1-\varepsilon & \varepsilon & & \\ & \ddots & \ddots & \\ & & 1-\varepsilon & \varepsilon \\ & & & 1-\varepsilon \end{bmatrix}_{n_{i,0,+1}^{\bigcirc}\times n_{i,0,+1}^{\bigcirc}},
$$

and for each $l = 1$ to l_i,

$$
\tilde{J}_{i,0,l}^{\bigcirc}(\varepsilon) = \begin{bmatrix} J_{i,0,l}^{\star}(\varepsilon) & \varepsilon I_2 & & \\ & \ddots & \ddots & \\ & & J_{i,0,l}^{\star}(\varepsilon) & \varepsilon I_2 \\ & & & J_{i,0,l}^{\star}(\varepsilon) \end{bmatrix}_{2n_{i,0,l}^{\bigcirc}\times 2n_{i,0,l}^{\bigcirc}}, \quad J_{i,0,l}^{\star}(\varepsilon) = (1-\varepsilon)\begin{bmatrix} \alpha_{i,0,l} & \beta_{i,0,l} \\ -\beta_{i,0,l} & \alpha_{i,0,l} \end{bmatrix},
$$

with $\beta_{i,0,l} \geq 0$ for all $l = 1$ to l_i and $\alpha_{i,0,j} \neq \omega_{i,k}$ for $j \neq k$.

2. There is $\varepsilon^* \in (0,1]$ such that the unique positive definite solution $\tilde{P}_{i,0}^{\bigcirc}(\varepsilon)$ to the Lyapunov equation
$$
\left(\tilde{J}_{i,0}^{\bigcirc}\right)^{\mathrm{T}}(\varepsilon)\tilde{P}_{i,0}^{\bigcirc}(\varepsilon)\tilde{J}_{i,0}^{\bigcirc}(\varepsilon) - \tilde{P}_{i,0}^{\bigcirc}(\varepsilon) = -\varepsilon I
$$
is bounded over $\varepsilon \in (0,\varepsilon^*]$, *i.e.*, there exist positive definite matrices $\tilde{P}_1$ and $\tilde{P}_2$ such that
$$
\tilde{P}_1 \leq \tilde{P}_{i,0}^{\bigcirc}(\varepsilon) \leq \tilde{P}_2, \ \varepsilon \in (0,\varepsilon^*].
$$

Lemma 3.5. [43] Let $D_{i,0}$ satisfy Assumption 3.3. Let $Q_{i,0}^{\bigcirc}(\varepsilon)$ be as given in Lemma 3.3. Denote $\left[\left(D_{i,0}^{\bigcirc}\right)^{\mathrm{T}}\ \left(D_{i,0}^{\odot}\right)^{\mathrm{T}}\right] = T_{i,0}^{-1}D_{i,0}$ and partition $\left(Q_{i,0}^{\bigcirc}\right)^{-1}(\varepsilon)D_{i,0}^{\bigcirc}$ according to that of $J_{i,0}^{\bigcirc}(\varepsilon)$ in Lemma

3.3 as

$$\left(Q_{i,0}^{\bigcirc}\right)^{-1}(\varepsilon)D_{i,0}^{\bigcirc}\begin{bmatrix} D_{i,0,-1}^{\bigcirc}(\varepsilon) \\ D_{i,0,+1}^{\bigcirc}(\varepsilon) \\ D_{i,0,1}^{\bigcirc}(\varepsilon) \\ \vdots \\ D_{i,0,l_i}^{\bigcirc}(\varepsilon) \end{bmatrix},$$

with

$$D_{i,0,-1}^{\bigcirc} = \begin{bmatrix} D_{i,0,-1,1}^{\bigcirc}(\varepsilon) \\ D_{i,0,-1,2}^{\bigcirc}(\varepsilon) \\ \vdots \\ D_{i,0,-1,n_{i,0,-1}^{\bigcirc}}^{\bigcirc}(\varepsilon) \end{bmatrix}_{n_{i,0,-1}^{\bigcirc}\times 1}, \quad D_{i,0,+1}^{\bigcirc} = \begin{bmatrix} D_{i,0,+1,1}^{\bigcirc}(\varepsilon) \\ D_{i,0,+1,2}^{\bigcirc}(\varepsilon) \\ \vdots \\ D_{i,0,+1,n_{i,0,+1}^{\bigcirc}}^{\bigcirc}(\varepsilon) \end{bmatrix}_{n_{i,0,+1}^{\bigcirc}\times 1},$$

$$\text{and } D_{i,0,l}^{\bigcirc} = \begin{bmatrix} D_{i,0,l,1}^{\bigcirc}(\varepsilon) \\ D_{i,0,l,2}^{\bigcirc}(\varepsilon) \\ \vdots \\ D_{i,0,l,n_{i,0,l}^{\bigcirc}}^{\bigcirc}(\varepsilon) \end{bmatrix}_{2n_{i,0,l}^{\bigcirc}\times 1}, \quad l = 1, 2, \cdots, l_i.$$

Then, there exists constant $\bar{D}_i \geq 0$ such that

$$\left\|D_{i,0,-1,n_{i,0,-1}^{\bigcirc}}^{\bigcirc}(\varepsilon)\right\| \leq \bar{D}_i\varepsilon, \ \left\|D_{i,0,+1,n_{i,0,+1}^{\bigcirc}}^{\bigcirc}(\varepsilon)\right\| \leq \bar{D}_i\varepsilon, \text{ and } \left\|D_{i,0,l,n_{i,0,l}^{\bigcirc}}^{\bigcirc}(\varepsilon)\right\| \leq \bar{D}_i\varepsilon, \ l = 1, 2, \cdots, l_i,$$

for all $\varepsilon \in (0, 1]$.

We also recall the following frequency domain property of the closed-loop system under low gain feedback. It will be used to ensure that the actual output of each follower agent stays within the neighborhood of the renamed output in the steady state.

Lemma 3.6. [60] Consider the pair $\left(A_{i,0}^{\bigcirc}, B_{i,0}^{\bigcirc}\right)$ as given in (3.4) and $F_{i,0}^{\bigcirc}(\varepsilon)$ as given in (3.6). Let $\lambda_{\mathrm{c}i,l}^{\bigcirc}$, $l = 1, 2\cdots, m_i$, be the eigenvalues of $A_{\mathrm{c}i,0}^{\bigcirc}(\varepsilon) = A_{i,0}^{\bigcirc} + B_{i,0}^{\bigcirc}F_{i,0}^{\bigcirc}(\varepsilon)$ with multiplicity $n_{i,l}$, *i.e.*, $\det\left(zI - A_{\mathrm{c}i,0}^{\bigcirc}\right) = \prod_{l=1}^{m_i}\left(z - \lambda_{\mathrm{c}i,l}^{\bigcirc}\right)^{n_{i,l}}$. Then, there exists $\varepsilon^* \in (0, \frac{1}{2}]$ such that, for all $\varepsilon \in (0, \varepsilon^*]$,

$$\left|F_{i,0}^{\bigcirc}(\varepsilon)\left(zI - A_{i,0}^{\bigcirc} - B_{i,0}^{\bigcirc}F_{i,0}^{\bigcirc}(\varepsilon)\right)^{-1}\right| \leq \delta_i\varepsilon\sum_{l=1}^{m_i}\sum_{j=1}^{n_{i,l}}(n_{i,l} - j + 1)\left|\frac{1}{\left(z - \lambda_{\mathrm{c}i,l}^{\bigcirc}\right)^j}\right|, \ z \in \mathbb{C},$$

where δ_i is some positive constant independent of ε.

Proof of Theorem 3.1: The proof consists of two parts, corresponding to the two objectives in Problem 3.1. We will first establish the boundedness of the states of all follower agents in the absence of the disturbance, and for a sufficiently small value of ε, the leader-following output consensus within the pre-specified accuracy η.

Define a state transformation for the zero dynamics according to (3.4) as

$$\left[\left(x_{i,0}^{\bigcirc}\right)^{\mathrm{T}} \ \left(x_{i,0}^{\odot}\right)^{\mathrm{T}}\right]^{\mathrm{T}} = T_{i,0}^{-1} x_{i,0}.$$

Then, the dynamics of $x_{i,0}^{\bigcirc}(k)$ and $x_{i,0}^{\odot}(k)$ are given as

$$\begin{bmatrix} x_{i,0}^{\bigcirc}(k+1) \\ x_{i,0}^{\odot}(k+1) \end{bmatrix} = \begin{bmatrix} A_{i,0}^{\bigcirc} + B_{i,0}^{\bigcirc} F_{i,0}^{\bigcirc} & 0 \\ B_{i,0}^{\odot} F_{i,0}^{\bigcirc} & A_{i,0}^{\odot} \end{bmatrix} \begin{bmatrix} x_{i,0}^{\bigcirc}(k) \\ x_{i,0}^{\odot}(k) \end{bmatrix} + \begin{bmatrix} B_{i,0}^{\bigcirc} \\ B_{i,0}^{\odot} \end{bmatrix} \check{y}_i(k) + \begin{bmatrix} D_{i,0}^{\bigcirc} \\ D_{i,0}^{\odot} \end{bmatrix} w_i(k). \tag{3.13}$$

It is obvious that the above system is stable and the states will remain bounded as long as $\check{y}_i$ is bounded, in the absence of the disturbance.

Denote the error between the new states of follower agent i, $i \in \{1, 2, \cdots, N\}$, and that of the leader agent as

$$\tilde{x}_i(k) = \check{x}_i(k) - x_0(k). \tag{3.14}$$

Then, under the consensus protocol (3.12), we have

$$\tilde{x}_i(k+1) = A_{\mathrm{r}} \tilde{x}_i(k) - B_{\mathrm{r}} \kappa K \left(\sum_{j=1}^{N} a_{ij} \big(\tilde{x}_i(k) - \tilde{x}_j(k)\big) + h_i \tilde{x}_i(k) \right).$$

By denoting $\tilde{x}(k) = [\tilde{x}_1(k) \ \tilde{x}_2(k) \ \cdots \ \tilde{x}_N(k)]^{\mathrm{T}}$, the closed-loop system for the error dynamics of all follower agents can be written in the following compact form,

$$\tilde{x}(k+1) = (I_N \otimes A_{\mathrm{r}})\tilde{x}(k) - (\bar{\mathcal{L}} + \bar{\mathcal{H}}) \otimes (B_{\mathrm{r}} \kappa K) \tilde{x}(k). \tag{3.15}$$

Consider the Lyapunov function

$$V\big(\tilde{x}(k)\big) = \tilde{x}^{\mathrm{T}}(k)(I_N \otimes P)\tilde{x}(k).$$

$V\big(\tilde{x}(k+1)\big)$ is evaluated based on (3.15) as

$$V\big(\tilde{x}(k+1)\big) = \tilde{x}^{\mathrm{T}}(k)\big(I_N \otimes A_{\mathrm{r}} - (\bar{\mathcal{L}} + \bar{\mathcal{H}}) \otimes (B_{\mathrm{r}} \kappa K)\big)^{\mathrm{T}} (I_N \otimes P) \big(I_N \otimes A_{\mathrm{r}} - (\bar{\mathcal{L}} + \bar{\mathcal{H}}) \otimes (B_{\mathrm{r}} \kappa K)\big)\tilde{x}(k)$$

$$
\begin{aligned}
&= \tilde{x}^{\mathrm{T}}(k)\big(I_N \otimes A_{\mathrm{r}}^{\mathrm{T}} P A_{\mathrm{r}}\big)\tilde{x}(k) + \tilde{x}^{\mathrm{T}}(k)\big((\bar{\mathcal{L}}+\bar{\mathcal{H}})^2 \otimes (B_{\mathrm{r}}\kappa K)^{\mathrm{T}} P (B_{\mathrm{r}}\kappa K)\big)\tilde{x}(k) \\
&\quad - 2\tilde{x}^{\mathrm{T}}(k)(\bar{\mathcal{L}}+\bar{\mathcal{H}}) \otimes (B_{\mathrm{r}}\kappa K)^{\mathrm{T}} P A_{\mathrm{r}} \tilde{x}(k).
\end{aligned}
$$

Note that

$$
\begin{aligned}
(B_{\mathrm{r}}\kappa K)^{\mathrm{T}} P (B_{\mathrm{r}}\kappa K) &= \kappa^2 A_{\mathrm{r}}^{\mathrm{T}} P B_{\mathrm{r}}\big(R + B_{\mathrm{r}}^{\mathrm{T}} P B_{\mathrm{r}}\big)^{-1} B_{\mathrm{r}}^{\mathrm{T}} P B_{\mathrm{r}}\big(R + B_{\mathrm{r}}^{\mathrm{T}} P B_{\mathrm{r}}\big)^{-1} B_{\mathrm{r}}^{\mathrm{T}} P A_{\mathrm{r}} \\
&= \kappa^2 A_{\mathrm{r}}^{\mathrm{T}} P B_{\mathrm{r}}\big(R + B_{\mathrm{r}}^{\mathrm{T}} P B_{\mathrm{r}}\big)^{-1} B_{\mathrm{r}}^{\mathrm{T}} P A_{\mathrm{r}} \\
&\quad - \kappa^2 A_{\mathrm{r}}^{\mathrm{T}} P B_{\mathrm{r}}\big(R + B_{\mathrm{r}}^{\mathrm{T}} P B_{\mathrm{r}}\big)^{-1} R \big(R + B_{\mathrm{r}}^{\mathrm{T}} P B_{\mathrm{r}}\big)^{-1} B_{\mathrm{r}}^{\mathrm{T}} P A_{\mathrm{r}} \\
&\leq \kappa^2 A_{\mathrm{r}}^{\mathrm{T}} P B_{\mathrm{r}}\big(R + B_{\mathrm{r}}^{\mathrm{T}} P B_{\mathrm{r}}\big)^{-1} B_{\mathrm{r}}^{\mathrm{T}} P A_{\mathrm{r}},
\end{aligned}
$$

and

$$
(B_{\mathrm{r}}\kappa K)^{\mathrm{T}} P A_{\mathrm{r}} = \kappa A_{\mathrm{r}}^{\mathrm{T}} P B_{\mathrm{r}}\big(R + B_{\mathrm{r}}^{\mathrm{T}} P B_{\mathrm{r}}\big)^{-1} B_{\mathrm{r}}^{\mathrm{T}} P A_{\mathrm{r}}.
$$

Therefore,

$$
\begin{aligned}
V\big(\tilde{x}(k+1)\big) \leq \tilde{x}^{\mathrm{T}}(k)\big(I_N \otimes A_{\mathrm{r}}^{\mathrm{T}} P A_{\mathrm{r}}\big)\tilde{x}(k) + \tilde{x}^{\mathrm{T}}(k)\Big(\big(\kappa^2(\bar{\mathcal{L}}+\bar{\mathcal{H}})^2 \\
- 2\kappa(\bar{\mathcal{L}}+\bar{\mathcal{H}})\big) \otimes \big(A_{\mathrm{r}}^{\mathrm{T}} P B_{\mathrm{r}}\big(R + B_{\mathrm{r}}^{\mathrm{T}} P B_{\mathrm{r}}\big)^{-1} B_{\mathrm{r}}^{\mathrm{T}} P A_{\mathrm{r}}\big)\Big)\tilde{x}(k). \qquad (3.16)
\end{aligned}
$$

Since $\bar{\mathcal{L}}+\bar{\mathcal{H}} > 0$ under Assumption 3.1, there exists orthogonal transformation matrix $T \in \mathbb{R}^{N\times N}$ such that

$$
\bar{\mathcal{L}}+\bar{\mathcal{H}} = T^{\mathrm{T}} \Lambda T,
$$

where $\Lambda = \mathrm{diag}\{\lambda_1, \lambda_2, \cdots, \lambda_N\}$, with $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N$ being the eigenvalues of $\bar{\mathcal{L}}+\bar{\mathcal{H}}$. Define a state transformation as

$$
\bar{x}(k) = \big[\bar{x}_1^{\mathrm{T}}(k)\ \bar{x}_2^{\mathrm{T}}(k)\ \cdots\ \bar{x}_N^{\mathrm{T}}(k)\big]^{\mathrm{T}} = (T \otimes I_\rho)\tilde{x}(k),
$$

then, the second term in (3.16) is evaluated as

$$
\begin{aligned}
&\tilde{x}^{\mathrm{T}}(k)\Big(\big(\kappa^2(\bar{\mathcal{L}}+\bar{\mathcal{H}})^2 - 2\kappa(\bar{\mathcal{L}}+\bar{\mathcal{H}})\big) \otimes \big(A_{\mathrm{r}}^{\mathrm{T}} P B_{\mathrm{r}}\big(R + B_{\mathrm{r}}^{\mathrm{T}} P B_{\mathrm{r}}\big)^{-1} B_{\mathrm{r}}^{\mathrm{T}} P A_{\mathrm{r}}\big)\Big)\tilde{x}(k) \\
&= \tilde{x}^{\mathrm{T}}(k)\big(\kappa^2 T^{\mathrm{T}}\Lambda^2 T - 2\kappa T^{\mathrm{T}}\Lambda T\big) \otimes \big(A_{\mathrm{r}}^{\mathrm{T}} P B_{\mathrm{r}}\big(R + B_{\mathrm{r}}^{\mathrm{T}} P B_{\mathrm{r}}\big)^{-1} B_{\mathrm{r}}^{\mathrm{T}} P A_{\mathrm{r}}\big)\tilde{x}(k) \\
&= \tilde{x}^{\mathrm{T}}(k)\big(T^{\mathrm{T}} \otimes I_\rho\big)\Big(\big(\kappa^2\Lambda^2 - 2\kappa\Lambda\big) \otimes \big(A_{\mathrm{r}}^{\mathrm{T}} P B_{\mathrm{r}}\big(R + B_{\mathrm{r}}^{\mathrm{T}} P B_{\mathrm{r}}\big)^{-1} B_{\mathrm{r}}^{\mathrm{T}} P A_{\mathrm{r}}\big)\Big)\big(T \otimes I_\rho\big)\tilde{x}(k) \\
&= \sum_{i=1}^{N} \bar{x}_i^{\mathrm{T}}(k)\big(\kappa^2\lambda_i^2 - 2\kappa\lambda_i\big)\big(A_{\mathrm{r}}^{\mathrm{T}} P B_{\mathrm{r}}\big(R + B_{\mathrm{r}}^{\mathrm{T}} P B_{\mathrm{r}}\big)^{-1} B_{\mathrm{r}}^{\mathrm{T}} P A_{\mathrm{r}}\big)\bar{x}_i(k)
\end{aligned}
$$

$$\leq -(1-\delta)\sum_{i=1}^{N}\bar{x}_i^{\mathrm{T}}(k)\big(A_{\mathrm{r}}^{\mathrm{T}}PB_{\mathrm{r}}\big(R+B_{\mathrm{r}}^{\mathrm{T}}PB_{\mathrm{r}}\big)^{-1}B_{\mathrm{r}}^{\mathrm{T}}PA_{\mathrm{r}}\big)\bar{x}_i(k),$$

where we have used $\kappa \in \big[\frac{1-\sqrt{\delta}}{\lambda_1}, \frac{1+\sqrt{\delta}}{\lambda_N}\big]$ for the last inequality.

Now, we can evaluate

$$\begin{aligned}
\Delta V(k+1) &:= V\big(\tilde{x}(k+1)\big) - V\big(\tilde{x}(k)\big)\\
&\leq \tilde{x}^{\mathrm{T}}(k)\big(I_N\otimes A_{\mathrm{r}}^{\mathrm{T}}PA_{\mathrm{r}}\big)\tilde{x}(k) - (1-\delta)\sum_{i=1}^{N}\bar{x}_i^{\mathrm{T}}(k)\big(A_{\mathrm{r}}^{\mathrm{T}}PB_{\mathrm{r}}\big(R+B_{\mathrm{r}}^{\mathrm{T}}PB_{\mathrm{r}}\big)^{-1}B_{\mathrm{r}}^{\mathrm{T}}PA_{\mathrm{r}}\big)\bar{x}_i(k)\\
&\quad - \tilde{x}^{\mathrm{T}}(k)(I_N\otimes P)\tilde{x}(k)\\
&= \bar{x}^{\mathrm{T}}(k)\big(I_N\otimes A_{\mathrm{r}}^{\mathrm{T}}PA_{\mathrm{r}}\big)\bar{x}(k) - (1-\delta)\sum_{i=1}^{N}\bar{x}_i^{\mathrm{T}}(k)\big(A_{\mathrm{r}}^{\mathrm{T}}PB_{\mathrm{r}}\big(R+B_{\mathrm{r}}^{\mathrm{T}}PB_{\mathrm{r}}\big)^{-1}B_{\mathrm{r}}^{\mathrm{T}}PA_{\mathrm{r}}\big)\bar{x}_i(k)\\
&\quad - \bar{x}^{\mathrm{T}}(k)(I_N\otimes P)\bar{x}(k)\\
&= \sum_{i=1}^{N}\bar{x}_i^{\mathrm{T}}(k)\Big(A_{\mathrm{r}}^{\mathrm{T}}PA_{\mathrm{r}} - (1-\delta)\big(A_{\mathrm{r}}^{\mathrm{T}}PB_{\mathrm{r}}\big(R+B_{\mathrm{r}}^{\mathrm{T}}PB_{\mathrm{r}}\big)^{-1}B_{\mathrm{r}}^{\mathrm{T}}PA_{\mathrm{r}}\big) - P\Big)\bar{x}_i(k)\\
&= -\tilde{x}^{\mathrm{T}}(k)(I_N\otimes Q)\tilde{x}(k).
\end{aligned}\tag{3.17}$$

Since $-\tilde{x}^{\mathrm{T}}(k)(I_N\otimes Q)\tilde{x}(k) < 0$ for $\tilde{x}(k)\neq 0$, we have

$$\lim_{k\to\infty}\tilde{x}(k) = 0,$$

which is equivalent to

$$\lim_{k\to\infty}\big(\check{x}_i(k) - x_0(k)\big) = 0,\ i\in\{1,2,\cdots,N\}.\tag{3.18}$$

Therefore, we have

$$\lim_{k\to\infty}\big(\check{y}_i(k) - y_0(k)\big) = 0,\ i\in\{1,2,\cdots,N\},\tag{3.19}$$

where $\check{y}_i(k)$ is the new output of follower agent i. By recalling Assumption 3.4 and (3.13), it can be concluded that the states of all follower agents remain bounded in the absence of the disturbance.

Consider the $x_{i,0}^{\bigcirc}$ dynamics in (3.13) of the follower agents. The steady state trajectory of $x_{i,0}^{\bigcirc}(k)$, in the absence of the disturbance, is all due to $y_0(k)$ according to (3.19). By viewing $\check{y}_i(k)$ as the input and $u_{i,0}(k) = F_{i,0}(\varepsilon)x_{i,0}(k) = F_{i,0}^{\bigcirc}(\varepsilon)x_{i,0}^{\bigcirc}(k)$ as the output, the transfer function from $\check{y}_i$ to $u_{i,0}$ is given as $F_{i,0}^{\bigcirc}(\varepsilon)\big(zI - A_{i,0}^{\bigcirc} - B_{i,0}^{\bigcirc}F_{i,0}^{\bigcirc}(\varepsilon)\big)^{-1}B_{i,0}^{\bigcirc}$. Then, under Assumption 3.4, Lemma 3.6

implies that, for the given η, there exists $\varepsilon_\eta^* \in (0, \frac{1}{2}]$ such that for all $\varepsilon \in (0, \varepsilon_\eta^*]$,

$$\lim_{k\to\infty} \sup |u_{i,0}(k)| \le \eta,\ \ i \in \{1, 2, \cdots, N\}.$$

By recalling that $y_i(k) = \check{y}_i(k) + u_{i,0}(k)$, we have, in the absence of disturbance,

$$\begin{aligned}\lim_{k\to\infty} \sup |y_{i,w=0}(k) - y_0(k)| &= \lim_{k\to\infty} \sup |\check{y}_{i,w=0}(k) + u_{i,0}(k) - y_0(k)| \\ &\le \lim_{k\to\infty} \sup |\check{y}_{i,w=0}(k) - y_0(k)| + \lim_{k\to\infty} \sup |u_{i,0}(k)| \\ &\le \eta,\end{aligned}$$

where $y_{i,w=0}(k)$ denotes the output of follower agent i in the absence of the disturbance.

We will next show that, when the multi-agent system is operating in steady state, for a sufficiently small value of ε, the effect of disturbance $w = [w_1\ w_2\ \cdots\ w_N]^{\mathrm{T}}$ on the leader-following output consensus in terms of the L_2-gain, is less than or equal to any pre-specified $\gamma > 0$. In the following analysis, we will assume, without loss of generality, that the multi-agent system is operating in steady state at $k = 0$, *i.e.*, $\tilde{x}(0) = 0$.

To analyze the influence of the disturbance, we recall that

$$y_{i,w\neq 0}(k) = \check{y}_{i,w\neq 0}(k) + F_{i,0}(\varepsilon) x_{i,0,w\neq 0}(k).$$

With the disturbance viewed as the input to the system, the zero input and the zero state responses are respectively given as

$$y_{i,w=0}(k) = \check{y}_{i,w=0}(k) + F_{i,0}(\varepsilon) x_{i,0,w=0}(k), \tag{3.20}$$

$$y_{i,w}(k) = \check{y}_{i,w}(k) + F_{i,0}(\varepsilon) x_{i,0,w}(k), \tag{3.21}$$

in which the subscripts $w = 0$ and w represent zero input and zero state, respectively.

It is noted that the difference $\left|y_{i,w\neq 0}(k) - y_{i,w=0}(k)\right|$ is equivalent to $|y_{i,w}(k)|$. Therefore, in what follows the effect of the disturbance on both $\check{y}_{i,w}(k)$ and $F_{i,0}(\varepsilon) x_{i,0,w}(k)$ with be analyzed.

In the presence of the disturbance, the dynamics (3.15) becomes

$$\tilde{x}(k+1) = (I_N \otimes A_{\mathrm{r}})\tilde{x}(k) - (\bar{\mathcal{L}} + \bar{\mathcal{H}}) \otimes (B_{\mathrm{r}} \kappa K)\tilde{x}(k) + \check{d} w(k),$$

where $\check{d} = \mathrm{blkdiag}\{\check{d}_1, \check{d}_2, \cdots, \check{d}_N\}$ with $\check{d}_i$ given in (3.10). Note that Assumption 3.3 implies that

$d_{i,m}=0$, for all $m=1,2,\cdots,\rho$, and the vector $D_{i,0}$ satisfies $D_{i,0}\in\cap_{\omega\in\lambda^{\bigcirc}A_{i,0}}\mathrm{Im}\big(\omega I-A_{i,0}\big),\ i\in\{1,2,\cdots,N\}$, where $\lambda^{\bigcirc}\big(A_{i,0}\big)$ are the set of all the eigenvalues of $A_{i,0}$ that are on the unit circle [43]. Then, Lemma 3.6 implies that for each $i\in\{1,2,\cdots,N\}$, there exists constant $\bar{d}_i>0$ independent of ε such that

$$\big\|\check{d}_{i,m}\big\|\leq\bar{d}_i\varepsilon,\ m=1,2,\cdots,\rho. \tag{3.22}$$

Then, it is obvious that there exists constant $\bar{d}>0$ independent of ε such that

$$\big\|\check{d}\big\|\leq\bar{d}\varepsilon.$$

The increment of $V\big(\tilde{x}(k)\big)$ in (3.17) will become

$$\begin{aligned}
&\Delta V(k+1):=V(\tilde{x}(k+1))-V(\tilde{x}(k))\\
&\leq-\tilde{x}^{\mathrm{T}}(k)(I_N\otimes Q)\tilde{x}(k)+\big(\check{d}w(k)\big)^{\mathrm{T}}(I_N\otimes P)\check{d}w(k)\\
&\quad+2\tilde{x}^{\mathrm{T}}(k)\big(I_N\otimes A_{\mathrm{r}}-(\bar{\mathcal{L}}+\bar{\mathcal{H}})\otimes(B_{\mathrm{r}}\kappa K)\big)(I_N\otimes P)\check{d}w(k)\\
&=-\tilde{x}^{\mathrm{T}}(k)(I_N\otimes Q)\tilde{x}(k)+\big(\check{d}w(k)\big)^{\mathrm{T}}(I_N\otimes P)\check{d}w(k)\\
&\quad+2\Big(\frac{\sqrt{2}}{2}\tilde{x}^{\mathrm{T}}(k)(I_N\otimes Q)^{\frac{1}{2}}\Big)\Big(\sqrt{2}(I_N\otimes Q)^{-\frac{1}{2}}\big(I_N\otimes A_{\mathrm{r}}-(\bar{\mathcal{L}}+\bar{\mathcal{H}})\otimes(B_{\mathrm{r}}\kappa K)\big)(I_N\otimes P)\check{d}w(k)\Big)\\
&\leq-\tilde{x}^{\mathrm{T}}(k)(I_N\otimes Q)\tilde{x}(k)+\big(\check{d}w(k)\big)^{\mathrm{T}}(I_N\otimes P)\check{d}w(k)+\frac{1}{2}\tilde{x}^{\mathrm{T}}(k)(I_N\otimes Q)\tilde{x}(k)+2\big(\check{d}w(k)\big)^{\mathrm{T}}\check{M}\check{d}w(k)\\
&\leq-\frac{1}{2}\tilde{x}^{\mathrm{T}}(k)(I_N\otimes Q)\tilde{x}(k)+\big(\check{d}w(k)\big)^{\mathrm{T}}M\check{d}w(k),
\end{aligned} \tag{3.23}$$

where $M=(I_N\otimes P)+2\check{M}$ and $\check{M}=(I_N\otimes P)\big(I_N\otimes A_{\mathrm{r}}-(\bar{\mathcal{L}}+\bar{\mathcal{H}})\otimes(B_{\mathrm{r}}\kappa K)\big)^{\mathrm{T}}(I_N\otimes Q)^{-1}\big(I_N\otimes A_{\mathrm{r}}-(\bar{\mathcal{L}}+\bar{\mathcal{H}})\otimes(B_{\mathrm{r}}\kappa K)\big)(I_N\otimes P)$ is independent of ε.

By summing both sides of (3.23), using $V\big(\tilde{x}(0)\big)=0$, and $\|\check{d}\|\leq\bar{d}\varepsilon$, we have

$$\sum_{k=0}^{\infty}\big\|\tilde{x}(k)\big\|^2\leq\varepsilon^2\gamma_x^2\sum_{k=0}^{\infty}\|w(k)\|^2,$$

where $\gamma_x^2=\frac{2\bar{d}^2\lambda_{\max}(M)}{\lambda_{\min}(Q)}$ with $\lambda_{\max}(M)$ being the largest eigenvalue of M and $\lambda_{\min}(Q)$ being the smallest eigenvalue of Q. Since $\check{y}_{i,w}=\tilde{x}_{i,1}$, we have

$$\sum_{k=0}^{\infty}\big|\check{y}_{i,w}(k)\big|^2\leq\gamma_x^2\sum_{k=0}^{\infty}\|w(k)\|^2. \tag{3.24}$$

To see the effect of the disturbance on $F_{i,0}(\varepsilon)x_{i,0,w}(k)$, we define $\tilde{x}_{i,0}^{\bigcirc}=S_{i,0}^{\bigcirc}(\varepsilon)\big(Q_{i,0}^{\bigcirc}\big)^{-1}(\varepsilon)x_{i,0}^{\bigcirc}$. Then, the $\tilde{x}^{\bigcirc}_{i,0}$ dynamics is written as

$$\tilde{x}^{\bigcirc}_{i,0}(k+1) = \tilde{J}^{\bigcirc}_{i,0}(\varepsilon)\tilde{x}^{\bigcirc}_{i,0}(k) + \tilde{B}^{\bigcirc}_{i,0}\breve{y}_i(k) + \tilde{D}^{\bigcirc}_{i,0}w_i(k),$$

where $\tilde{B}^{\bigcirc}_{i,0} = S^{\bigcirc}_{i,0}\big(Q^{\bigcirc}_{i,0}\big)^{-1}B^{\bigcirc}_{i,0}$ and $\tilde{D}^{\bigcirc}_{i,0} = S^{\bigcirc}_{i,0}\big(Q^{\bigcirc}_{i,0}\big)^{-1}D^{\bigcirc}_{i,0}$.

Consider the zero input and the state responses of $\tilde{x}^{\bigcirc}_{i,0}(k)$ according to (3.20) and (3.21) as

$$\begin{aligned}
\tilde{x}^{\bigcirc}_{i,0,w=0}(k+1) &= \tilde{J}^{\bigcirc}_{i,0}(\varepsilon)\tilde{x}^{\bigcirc}_{i,0,w=0}(k) + \tilde{B}^{\bigcirc}_{i,0}\breve{y}_{i,w=0}(k)\\
\tilde{x}^{\bigcirc}_{i,0,w}(k+1) &= \tilde{J}^{\bigcirc}_{i,0}(\varepsilon)\tilde{x}^{\bigcirc}_{i,0,w}(k) + \tilde{B}^{\bigcirc}_{i,0}\breve{y}_{i,w}(k) + \tilde{D}^{\bigcirc}_{i,0}w_i(k).
\end{aligned}$$

Then, the effect of the disturbance is all exhibited by $\tilde{x}^{\bigcirc}_{i,0,w}(k)$.

Consider the Lynapunov function

$$V^{\bigcirc}_{i,0}\big(\tilde{x}^{\bigcirc}_{i,0,w}(k)\big) = \big(\tilde{x}^{\bigcirc}_{i,0,w}\big)^{\mathrm{T}}(k)\tilde{P}^{\bigcirc}_{i,0}(\varepsilon)\tilde{x}^{\bigcirc}_{i,0,w}(k).$$

Then,

$$V^{\bigcirc}_{i,0}\big(\tilde{x}^{\bigcirc}_{i,0,w}(k+1)\big) = \big(\tilde{J}^{\bigcirc}_{i,0}\tilde{x}^{\bigcirc}_{i,0,w}(k) + \tilde{D}^{\bigcirc}_{i,0}w_i(k)\big)^{\mathrm{T}}\tilde{P}^{\bigcirc}_{i,0}(\varepsilon)\big(\tilde{J}^{\bigcirc}_{i,0}\tilde{x}^{\bigcirc}_{i,0,w}(k) + \tilde{D}^{\bigcirc}_{i,0}w_i(k)\big).$$

We have,

$$\begin{aligned}
\Delta V^{\bigcirc}_{i,0}(k+1) :&= V^{\bigcirc}_{i,0}\big(\tilde{x}^{\bigcirc}_{i,0,w}(k+1)\big) - V^{\bigcirc}_{i,0}\big(\tilde{x}^{\bigcirc}_{i,0,w}(k)\big)\\
&= \varepsilon\big(\tilde{x}^{\bigcirc}_{i,0,w}\big)^{\mathrm{T}}(k)\tilde{x}^{\bigcirc}_{i,0,w}(k) + 2\big(\tilde{x}^{\bigcirc}_{i,0,w}\big)^{\mathrm{T}}(k)\big(\tilde{J}^{\bigcirc}_{i,0}\big)^{\mathrm{T}}\tilde{P}^{\bigcirc}_{i,0}\tilde{D}^{\bigcirc}_{i,0}w_i(k)\\
&\quad + \big(\tilde{D}^{\bigcirc}_{i,0}w_i(k)\big)^{\mathrm{T}}\tilde{P}^{\bigcirc}_{i,0}\tilde{D}^{\bigcirc}_{i,0}w_i(k)\\
&= \varepsilon\big(\tilde{x}^{\bigcirc}_{i,0,w}\big)^{\mathrm{T}}(k)\tilde{x}^{\bigcirc}_{i,0,w}(k) + 2\Big(\frac{\sqrt{\varepsilon}}{\sqrt{2}}\big(\tilde{x}^{\bigcirc}_{i,0,w}\big)^{\mathrm{T}}(k)\Big)\Big(\frac{\sqrt{2}}{\sqrt{\varepsilon}}\big(\tilde{J}^{\bigcirc}_{i,0}\big)^{\mathrm{T}}\tilde{P}^{\bigcirc}_{i,0}\tilde{D}^{\bigcirc}_{i,0}w_i(k)\Big)\\
&\quad + \big(\tilde{D}^{\bigcirc}_{i,0}w_i(k)\big)^{\mathrm{T}}\tilde{P}^{\bigcirc}_{i,0}\tilde{D}^{\bigcirc}_{i,0}w_i(k)\\
&\leq -\frac{1}{2}\varepsilon\big\|\tilde{x}^{\bigcirc}_{i,0,w}(k)\big\|^2 + \Big(\frac{2}{\varepsilon}\big\|\big(\tilde{J}^{\bigcirc}_{i,0}\big)^{\mathrm{T}}\tilde{P}^{\bigcirc}_{i,0}\big\|^2 + \big\|\tilde{P}^{\bigcirc}_{i,0}\big\|\Big)\big\|\tilde{D}^{\bigcirc}_{i,0}\big\|^2\|w_i(k)\|^2.
\end{aligned} \tag{3.25}$$

Note that Lemma 3.5 implies that there exists constant $\bar{D} > 0$ such that $\|\tilde{D}^{\bigcirc}_{i,0}\| \leq \bar{D}\varepsilon$, for all $\varepsilon \in (0,1]$. By summing both sides of (3.25) and using $V^{\bigcirc}_{i,0}\big(\tilde{x}^{\bigcirc}_{i,0,w}(0)\big) = 0$, we have

$$\sum_{k=0}^{\infty}\big\|\tilde{x}^{\bigcirc}_{i,0,w}(k)\big\|^2 \leq \gamma_{i,0}^2\sum_{k=0}^{\infty}\|w_i(k)\|^2, \tag{3.26}$$

where $\gamma_{i,0}^2 = \big(4\tilde{J}^2 + 2\big)\big\|\tilde{P}_2\big\|\bar{D}^2$ independent of ε, with $\tilde{J} = \sup_{\varepsilon\in(0,1]} \big\|J_{i,0}^{\bigcirc}(\varepsilon)\big\|$.

By recalling $\big|y_{i,w\neq 0}(k) - y_{i,w=0}(k)\big| \leq \big|\breve{y}_{i,w}(k)\big| + \big|F_{i,0}(\varepsilon)x_{i,0,w}(k)\big|$, $F_{i,0}(\varepsilon)x_{i,0,w}(k) = F_{i,0}^{\bigcirc}(\varepsilon)\tilde{x}_{i,0}^{\bigcirc}(k)$, and using $\|F_{i,0}^{\bigcirc}\| \leq \bar{F}_0\varepsilon$ with a constant $\bar{F}_0 > 0$, (3.24) and (3.26), we have

$$\sum_{k=0}^{\infty} \big|y_{i,w\neq 0}(k) - y_{i,w=0}(k)\big|^2 \leq \varepsilon^2\big(\gamma_x^2 + \bar{F}_0^2\gamma_{i,0}^2\big)\sum_{k=0}^{\infty} |w(k)|^2.$$

Let $\varepsilon_\gamma^* = \frac{\gamma}{\sqrt{\gamma_x^2+\bar{F}_0^2\gamma_{i,0}^2}}$. Then, for any $\varepsilon \in (0, \varepsilon_\gamma^*]$, we have

$$\sum_{k=0}^{\infty} \big|y_{i,w\neq 0}(k) - y_{i,w=0}(k)\big|^2 \leq \gamma^2\sum_{k=0}^{\infty} |w(k)|^2.$$

Finally, choose $\varepsilon \in \big(0, \min\{\varepsilon_\eta^*, \varepsilon_\gamma^*\}\big]$ for the consensus protocol and we have completed the proof. $\square$

3.4 Simulation for State Feedback

To illustrate our state feedback result, we will perform simulation with a discrete-time heterogeneous multi-agent system consisting of four follower agents, labeled as $1, 2, 3, 4$, and one leader agent, labeled as 0. The communication topology is shown in Fig. 3.1, where the arrows indicate the direction of information flow.

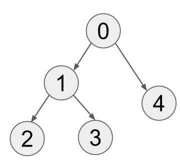

Figure 3.1: The communication topology.

The follower agents are heterogeneous and are described as

$$\begin{cases} x_{i,0}(k+1) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} x_{i,0}(k) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} y_i(k) + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} w_i(k), \\ x_{i,1}(k+1) = x_{i,2}(k), \\ x_{i,2}(k+1) = x_{i,0,1}(k) - x_{i,1}(k) + \sqrt{2}x_{i,2}(k) + u_i(k) \\ \qquad\quad y_i(k) = x_{i,1}(k), \ \ i \in \{1, 2\}, \end{cases}$$

and

$$\begin{cases} x_{i,0}(k+1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x_{i,0}(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} y_i(k) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} w_i(k), \\ x_{i,1}(k+1) = x_{i,2}(k), \\ x_{i,2}(k+1) = x_{i,0,1}(k) - x_{i,1}(k) + \sqrt{2}x_{i,2}(k) + u_i(k) \\ \qquad y_i(k) = x_{i,1}(k), \;\; i \in \{3,4\}. \end{cases}$$

The leader agent is given as

$$\begin{cases} x_0(k+1) = \begin{bmatrix} 0 & 1 \\ -1 & \sqrt{2} \end{bmatrix} x_0(k), \\ \qquad y_0(k) = x_{0,1}(k). \end{cases}$$

Solving the modified discrete-time Riccati equation (3.11) with $\delta = 0.9$ by the iterative algorithm proposed in reference [73], we obtain

$$P = \begin{bmatrix} 37.8816 & -27.4173 \\ -27.4173 & 40.8711 \end{bmatrix},$$

and

$$K = \begin{bmatrix} -0.9761 & 0.7256 \end{bmatrix}.$$

With this K and $\kappa = 0.2$, we implement the family of consensus protocols (3.12) and simulate the closed-loop system. In the simulation, the initial conditions of both the follower agents and the leader agent are chosen randomly as

$$\begin{aligned} [x_{1,0}^{\mathrm{T}}(0) \;\; x_1^{\mathrm{T}}(0)] &= [2.4169 \;\; 9.4205 \;\; 2.3478 \;\; 0.4302 \;\; 6.4775], \\ [x_{2,0}^{\mathrm{T}}(0) \;\; x_2^{\mathrm{T}}(0)] &= [-3.4239 \;\; 4.7059 \;\; 4.5717 \;\; -0.1462 \;\; 3.0028], \\ [x_{3,0}^{\mathrm{T}}(0) \;\; x_3^{\mathrm{T}}(0)] &= [-3.5811 \;\; -0.7824 \;\; 4.1574 \;\; 2.9221], \\ [x_{4,0}^{\mathrm{T}}(0) \;\; x_4^{\mathrm{T}}(0)] &= [1.5574 \;\; -4.6429 \;\; 3.4913 \;\; 4.3399], \\ x_0^{\mathrm{T}}(0) &= [1.5548 \;\; -3.2881], \end{aligned}$$

and the disturbances are realized as

$$\begin{aligned} w_1(k) &= 10\sin(0.7060k + 0.0318), \\ w_2(k) &= 10\sin(0.2769k + 0.0462), \\ w_3(k) &= 10\sin(0.1869k + 0.7572), \end{aligned}$$

$$w_4(k) = 10\sin(0.3804k + 0.0844).$$

Shown in Fig. 3.2 are the output consensus errors in the absence of the disturbance, with $\varepsilon = 0.1$. Shown in Fig. 3.3 are the states of the follower agents in the absence of the disturbance, with $\varepsilon = 0.1$.

Shown in Figs. 3.4 and 3.5 are respectively the output consensus errors and the states in the absence of the disturbance, with $\varepsilon = 0.02$. As can be seen in these figures, the states of all follower agents remain bounded, and the output consensus errors decrease as the value of ε decreases.

Shown in Figs. 3.6 and 3.7 are the output consensus errors in the presence of the disturbance, with $\varepsilon = 0.1$ and $\varepsilon = 0.02$, respectively. It is observed that the effect of the disturbance on the output consensus errors weakens as the value of ε decreases.

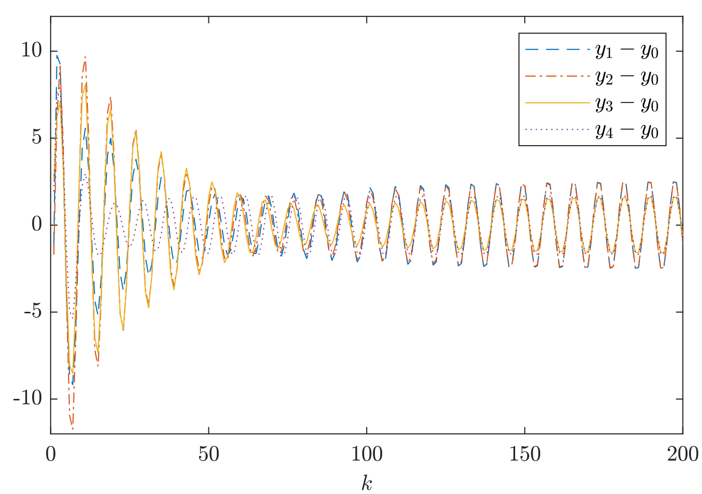

Figure 3.2: The output consensus errors in the absence of the disturbance, with $\varepsilon = 0.1$.

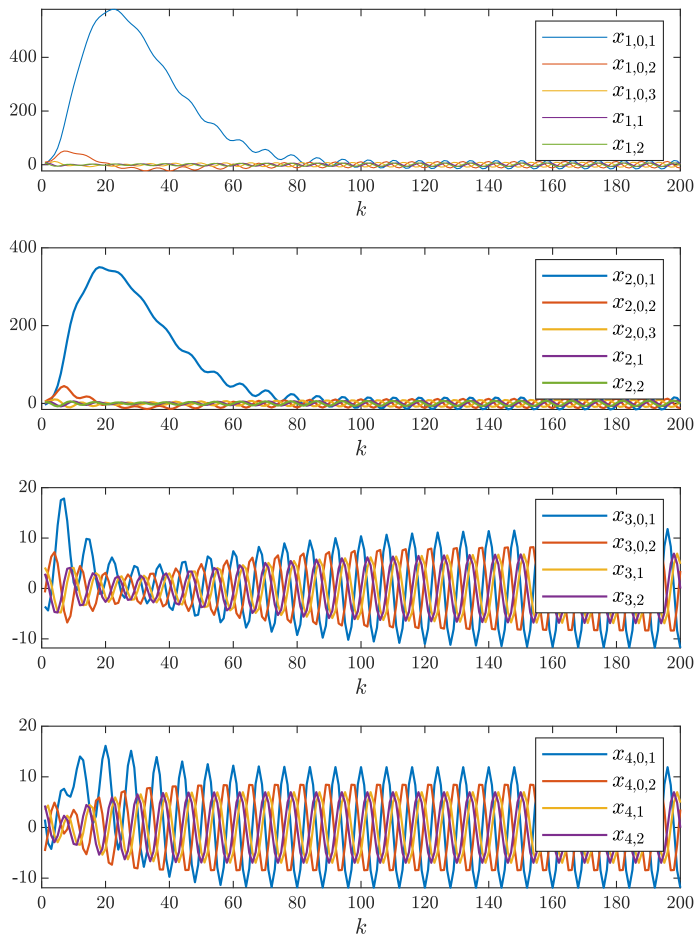

Figure 3.3: The states of the follower agents in the absence of the disturbance, with $\varepsilon = 0.1$.

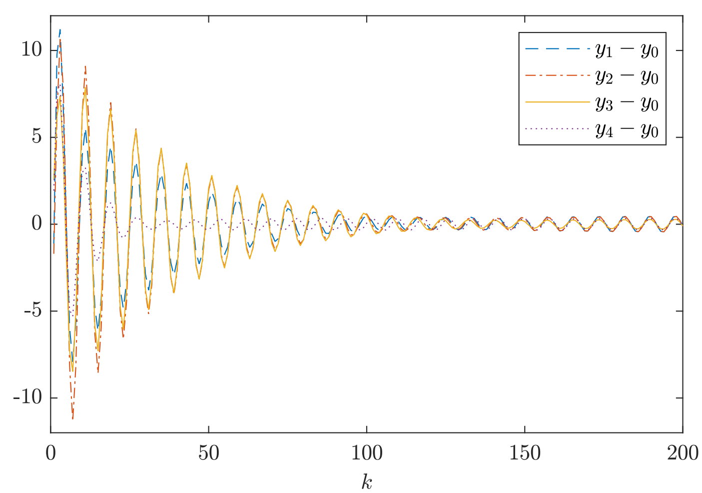

Figure 3.4: The output consensus errors in the absence of the disturbance, with $\varepsilon = 0.02$.

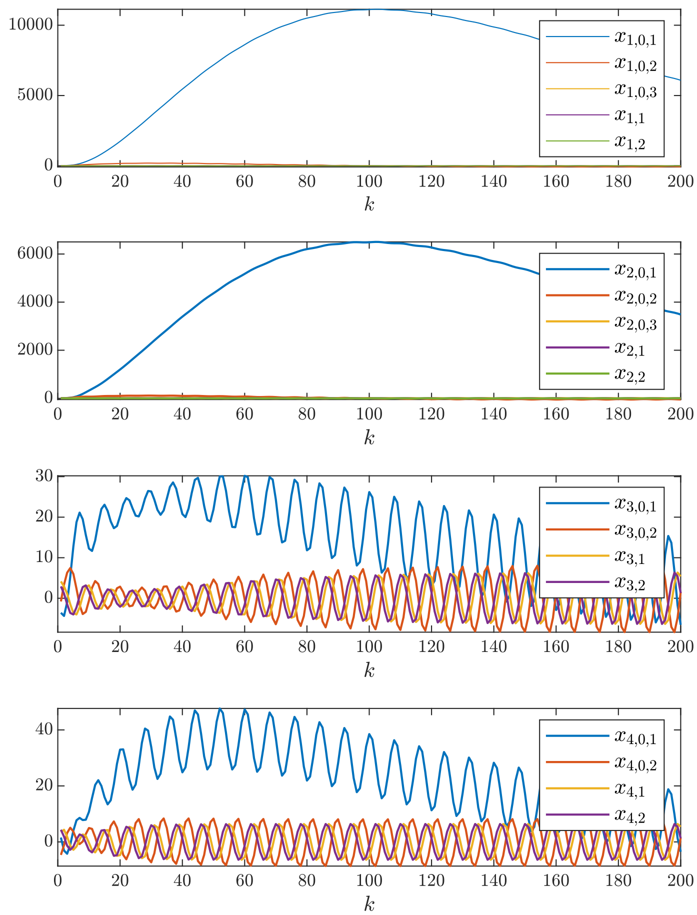

Figure 3.5: The states of the follower agents in the absence of the disturbance, with $\varepsilon = 0.02$.

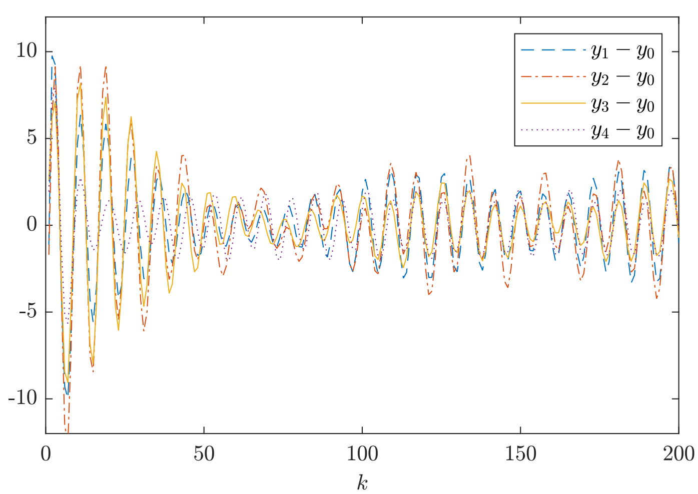

Figure 3.6: The output consensus errors in the presence of the disturbance, with $\varepsilon = 0.1$.

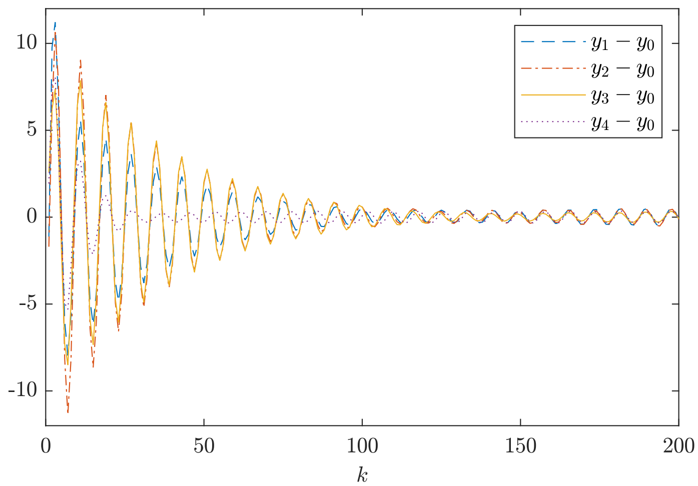

Figure 3.7: The output consensus errors in the presence of the disturbance, with $\varepsilon = 0.02$.

3.5 Output Feedback Results

Consider the follower agent dynamics (3.1) with its zero dynamics transformed in the form of (3.3), that is,

$$\begin{cases} x_{i,0}^{\bigcirc}(k+1) = A_{i,0}^{\bigcirc} x_{i,0}^{\bigcirc}(k) + B_{i,0}^{\bigcirc} y_i(k) + D_{i,0}^{\bigcirc} w_i(k), \\ x_{i,0}^{\odot}(k+1) = A_{i,0}^{\odot} x_{i,0}^{\odot}(k) + B_{i,0}^{\odot} y_i(k) + D_{i,0}^{\odot} w_i(k), \\ x_{i,m}(k+1) = x_{i,m+1}(k) + d_{i,m} w_i(k),\ m = 1, 2, \cdots, \rho - 1, \\ x_{i,\rho}(k+1) = E_{i,0}^{\bigcirc} x_{i,0}^{\bigcirc}(k) + E_{i,0}^{\odot} x_{i,0}^{\odot} + \alpha_1 x_{i,1}(k) + \alpha_2 x_{i,2}(k) + \cdots + \alpha_\rho x_{i,\rho}(k) + u_i(k) + d_{i,\rho} w_i(k), \\ y_i(k) = x_{i,1}(k). \end{cases} \tag{3.27}$$

Beside Assumptions 3.2, 3.3 and 3.4, we made the following additional assumptions on the follower agent dynamics (3.27).

Assumption 3.5. The pair $(A_{i,0}^{\bigcirc}, E_{i,0}^{\bigcirc})$ is observable.

Assumption 3.6. The unstable part of the zero dynamics is not affected by the disturbance, *i.e.*, $D_{i,0}^{\bigcirc} = 0$.

Assumption 3.7. The stable part of the zero dynamics does not contribute the output, *i.e.*, $E_{i,0}^{\odot} = 0$.

Remark 3.2. As seen in Section 3.3, because of the lack of high gain feedback in the discrete-time setting, the effect of any disturbance in the $x_{i,m}$ dynamics on the output y_i cannot be reduced to an arbitrarily high degree, which entails $d_{i,m} = 0$, implied by Assumption 3.3. In the state feedback design, the term $E_{i,0}^{\bigcirc} x_{i,0}^{\bigcirc}(k) + E_{i,0}^{\odot} x_{i,0}^{\odot}$, which carries the effect of the disturbance from the zero dynamics into the $x_{i,m}$ dynamics, is canceled by feedback. In the output feedback case, such a cancellation is carried out by the estimation of states $x_{i,0}^{\bigcirc}$ and $x_{i,0}^{\bigcirc}$. Assumptions 3.6 and 3.7 prevent the disturbances in the zero dynamics from being carried into the $x_{i,m}$ dynamics through the state estimation errors.

Under the above assumptions, we construct a state observer for each follower agent as

$$\begin{cases} \hat{x}_{i,0}^{\bigcirc}(k+1) = A_{i,0}^{\bigcirc} \hat{x}_{i,0}^{\bigcirc}(k) + B_{i,0}^{\bigcirc} \hat{x}_{i,1} + L_{i,1}^{\bigcirc}(\hat{x}_{i,1} - y_i), \\ \hat{x}_{i,m}(k+1) = \hat{x}_{i,m+1}(k) + l_{i,m}(\hat{x}_{i,1} - y_i), \\ \hat{x}_{i,\rho}(k+1) = E_{i,0}^{\bigcirc} \hat{x}_0^{\bigcirc} + \alpha_1 \hat{x}_{i,1}(k) + \alpha_2 \hat{x}_{i,2} + \cdots + \alpha_\rho \hat{x}_{i,\rho} + u_i(k) + l_{i,\rho}(\hat{x}_{i,1} - y_i), \end{cases}$$

where $\hat{x}_{i,0}^{\bigcirc}(k)$ is the estimate of $x_{i,0}^{\bigcirc}(k)$ and $\hat{x}_{i,m}(k)$ is the estimate of $x_{i,m}(k)$, $m = 1, 2, \cdots, \rho$. The observer gain

$$L_i^{\bigcirc} = \mathrm{col}\big(L_{i,0}^{\bigcirc}, l_{i,1}, l_{i,2}, \cdots, l_{i,\rho}\big) \in \mathbb{R}^{n_i^{\bigcirc}+\rho}, \ L_{i,0} \in \mathbb{R}^{n_i^{\bigcirc}},$$

is chosen such that all eigenvalues of $A_i^{\bigcirc} + L_i^{\bigcirc} C_i^{\bigcirc}$ are inside the unit circle, where

$$A_i^{\bigcirc} = \left[\begin{array}{c|cccc} A_{i,0}^{\bigcirc} & B_{i,0}^{\bigcirc} & 0_{n_i^{\bigcirc}\times(\rho-1)} & & \\ \hline & 0 & 1 & \cdots & 0 \\ 0_{(\rho-1)\times n_i^{\bigcirc}} & \vdots & \vdots & \ddots & \vdots \\ & 0 & 0 & \cdots & 1 \\ E_{i,0}^{\bigcirc} & \alpha_1 & \alpha_2 & \cdots & \alpha_\rho \end{array}\right], \ C_i^{\bigcirc} = \left[\begin{array}{ccc} 0_{1\times n_i^{\bigcirc}} & 1 & 0_{1\times(\rho-1)} \end{array}\right].$$

Define the observer errors as $e_{i,0}^{\bigcirc} = \hat{x}_{i,0}^{\bigcirc} - x_{i,0}^{\bigcirc}$ and $e_i = [e_{i,1} \ e_{i,2} \ \cdots \ e_{i,\rho}]^{\mathrm{T}} = [\hat{x}_{i,1} - x_{i,1} \ \hat{x}_{i,2} - x_{i,2} \ \cdots \hat{x}_{i,\rho} - x_{i,\rho}]^{\mathrm{T}}$, $i \in \{1, 2, \cdots, N\}$. The error dynamics is then given as

$$\begin{bmatrix} e_{i,0}^{\bigcirc}(k+1) \\ e_i(k+1) \end{bmatrix} = \big(A_i^{\bigcirc} + L_i^{\bigcirc} C_i^{\bigcirc}\big) \begin{bmatrix} e_{i,0}^{\bigcirc}(k) \\ e_i(k) \end{bmatrix}.$$

With the estimated states $\hat{x}_{i,0}^{\bigcirc}(k)$, $m = 1, 2, \cdots \rho$, the low gain feedback virtual control $u_{i,0} = F_{i,0}(\varepsilon) x_{i,0} = F_{i,0}^{\bigcirc}(\varepsilon) x_{i,0}^{\bigcirc}$ for the zero dynamics is now implemented as

$$u_{i,0}(k) = F_{i,0}^{\bigcirc}(k) \hat{x}_{i,0}^{\bigcirc}(k).$$

Accordingly, the definition of the new output and the new state $\check{x}_{i,1}$ in (3.7) becomes

$$\check{y}_i(k) = \check{x}_{i,1}(k) = x_{i,1}(k) - F_{i,0}^{\bigcirc}(\varepsilon)\hat{x}_{i,0}^{\bigcirc}(k),$$

and the definition of the rest of the new states, $\check{x}_{i,m}$, $m = 2, 3, \cdots, \rho$, remains the same as in the state feedback case.

The dynamics of the follower agents in the new states become

$$\begin{cases} x_{i,0}^{\bigcirc}(k+1) = A_{\mathrm{c}i,0}^{\bigcirc}(\varepsilon)x_{i,0}^{\bigcirc}(k) + B_{i,0}^{0}\check{y}_i(k) + B_{i,0}^{\bigcirc}F_{i,0}^{\bigcirc}e_{i,0}^{\bigcirc}(k), \\ x_{i,0}^{\odot}(k+1) = A_{i,0}^{\odot}x_{i,0}^{\odot}(k) + B_{i,0}^{\odot}y_i(k) + D_{i,0}^{\odot}w_i(k), \\ \check{x}_{i,m}(k+1) = \check{x}_{i,m+1}(k) + \check{d}_{i,m}w_i(k) + \check{g}_{i,m}^{\bigcirc}e_{i,0}^{\bigcirc}(k) + \check{g}_{i,m}e_i(k), \;\; r = 1, 2, \cdots, \rho - 1, \\ \check{x}_{i,\rho}(k+1) = \check{E}_{i,0}^{\bigcirc}x_{i,0}^{\bigcirc}(k) + \check{\alpha}_{i,1}\check{x}_{i,1}(k) + \check{\alpha}_{i,2}\check{x}_{i,2}(k) + \cdots + \check{\alpha}_{i,\rho}\check{x}_{i,\rho}(k) + u_i(k) + \check{d}_{i,\rho}w_i(k) \\ \qquad\qquad + \check{g}_{i,\rho}e_{i,0}^{\bigcirc}(k) + \check{g}_{i,\rho}e_i(k), \\ \check{y}_i(k) = \check{x}_{i,1}(k), \end{cases}$$

where

$$A_{\mathrm{c}i,0}^{\bigcirc}(\varepsilon) = A_{i,0}^{\bigcirc} + B_{i,0}^{\bigcirc}F_{i,0}^{\bigcirc}(\varepsilon),$$
$$\check{E}_{i,0}^{\bigcirc} = E_{i,0}^{\bigcirc} - F_{i,0}^{\bigcirc}(\varepsilon){A_{i,0}^{\bigcirc}}^{\rho} + \sum_{l=1}^{\rho}\big(\alpha_{i,l} - F_{i,0}^{\bigcirc}(\varepsilon){A_{i,0}^{\bigcirc}}^{\rho-l}B_{i,0}^{\bigcirc}\big)F_{i,0}^{\bigcirc}(\varepsilon){A_{\mathrm{c}i,0}^{\bigcirc}}^{l-1}(\varepsilon),$$

and $g_{i,m}^{\bigcirc}$ and $\check{g}_{i,m}$. $m = 1, 2, \cdots, \rho$, are defined in an obvious way.

The consensus protocols are implemented as

$$\begin{aligned} u_i(k) = &- \check{E}_{i,0}^{\bigcirc}\hat{x}_{i,0}^{\bigcirc} + \sum_{m=1}^{\rho} a_{ij}(\alpha_m - \alpha_{i,m})\hat{\check{x}}_{i,m} \\ &- \kappa\big(R + B_{\mathrm{r}}^{\mathrm{T}}PB_{\mathrm{r}}\big)^{-1}B_{\mathrm{r}}^{\mathrm{T}}PA_{\mathrm{r}}\Big(\sum_{j=1}^{N}\big(\hat{\check{x}}_i(k) - \hat{\check{x}}_j(k)\big) + h_i\big(\hat{\check{x}}_i(k) - x_0(k)\big)\Big), \;\; m = 1, 2, \cdots, \rho, \end{aligned} \tag{3.28}$$

where $\hat{\check{x}}_{i,m}$ is $\check{x}_{i,m}$ calculated using the estimated states, and $P > 0$ is as given in (3.11).

Theorem 3.2. [56] Consider the discrete-time heterogeneous multi-agent system described by (3.27) and (3.2). Let the communication topology satisfy Assumption 3.1. Let the dynamics of the follower agents satisfy Assumptions 3.2, 3.3, 3.5, 3.6 and 3.7. Let the output the leader agent satisfy Assumption 3.4. Then, the family of output feedback consensus protocols (3.28) solve Problem 3.1.

Proof of Theorem 3.2: Under the output feedback consensus protocols (3.28), the closed-loop system (3.15) for the error dynamics of all follower agents becomes

$$\tilde{x}(k+1) = (I_N \otimes A_{\mathrm{r}})\tilde{x}(k) - (\bar{\mathcal{L}} + \bar{\mathcal{H}}) \otimes (B_{\mathrm{r}}K)\tilde{x}(k) + Ge(k), \tag{3.29}$$

where $G = [G_1^{\mathrm{T}}\ G_2^{\mathrm{T}}\ \cdots\ G_N^{\mathrm{T}}]^{\mathrm{T}}$ are defined in an obvious way, with $\|G_i\| \leq \bar{g}$, for some $\bar{g} > 0$, and for all $i \in \{1, 2, \cdots, N\}$, and $e(k) = \left[\left[\left(e_{1,0}^{\bigcirc}\right)^{\mathrm{T}}\ e_1\right]^{\mathrm{T}}\ \left[\left(e_{2,0}^{\bigcirc}\right)^{\mathrm{T}}\ e_2\right]^{\mathrm{T}} \cdots\ \left[\left(e_{N,0}^{\bigcirc}\right)^{\mathrm{T}}\ e_N\right]^{\mathrm{T}}\right]^{\mathrm{T}}$.

Let P_{ob} be the unique positive definite solution of the Lyapunov equation

$$\big(A_i^{\bigcirc} + L_i^{\bigcirc} C_i^{\bigcirc}\big)^{\mathrm{T}} P_{\mathrm{ob}} \big(A_i^{\bigcirc} + L_i^{\bigcirc} C_i^{\bigcirc}\big) - P_{\mathrm{ob}} = -Q_{\mathrm{ob}},$$

for some $Q_{\mathrm{ob}} > 0$. Consider the Lyapunov function

$$V_{\mathrm{of}}\big(\tilde{x}(k), e(k)\big) = \tilde{x}^{\mathrm{T}}(k)(I_N \otimes P)\tilde{x}(k) + e^{\mathrm{T}}(k)P_{\mathrm{ob}}e(k).$$

Then, we have

$$\begin{aligned}
&\Delta V_{\mathrm{of}}(k+1) \\
&:= V_{\mathrm{of}}\big(\tilde{x}(k+1), e(k+1)\big) - V_{\mathrm{of}}\big(\tilde{x}(k), e(k)\big) \\
&\leq -\tilde{x}^{\mathrm{T}}(k)(I_N \otimes Q)\tilde{x}(k) - e^{\mathrm{T}}(k)Q_{\mathrm{ob}}e(k) \\
&\quad + 2\tilde{x}^{\mathrm{T}}(k)\big((I_N \otimes A_{\mathrm{r}}) - (\bar{\mathcal{L}} + \bar{\mathcal{H}}) \otimes (B_{\mathrm{r}}K)\big)^{\mathrm{T}}(I_N \otimes P)Ge(k) + \big(Ge(k)\big)^{\mathrm{T}}(I_N \otimes P)Ge(k).
\end{aligned}$$

By choosing Q_{ob} such that $Q_{\mathrm{ob}} - G^{\mathrm{T}}(I_N \otimes P)G - M^{\mathrm{T}}(I_N \otimes Q)^{-1}M$ is positive definite, where $M = \big((I_N \otimes A_{\mathrm{r}}) - (\bar{\mathcal{L}} + \bar{\mathcal{H}}) \otimes (B_{\mathrm{r}}K)\big)^{\mathrm{T}}(I_N \otimes P)G$, we have $\Delta V_{\mathrm{of}}(k+1) < 0$ for $\tilde{x}(k) \neq 0$ or $e(k) \neq 0$. Thus,

$$\begin{aligned}
\lim_{k\to\infty} \tilde{x}(k) &= 0, \\
\lim_{k\to\infty} e(k) &= 0.
\end{aligned}$$

Therefore, as in the proof of Theorem 3.1, we have

$$\lim_{k\to\infty} \big(\check{y}_i(k) - y_0(k)\big) = 0,\ \ i \in \{1, 2, \cdots, N\},$$

Since all eigenvalues of $A_{\mathrm{c}i,0}^{\bigcirc}(\varepsilon)$ and $A_{i,0}^{\odot}$ are inside the unit circle, and $\check{y}_i$, $e_{i,0}^{\bigcirc}$ and y_i are bounded, the states of all follower agents will remain bounded in the absence of the disturbance.

By recalling that $y_i(k) = \breve{y}_i(k) + u_{i,0}(k)$, we have, as in the proof of Theorem 3.1, there exists $\varepsilon_\eta^* \in (0, \frac{1}{2}]$ such that, for all $\varepsilon \in (0, \varepsilon_\eta^*]$,

$$\lim_{k\to\infty} \sup |y_{i,w}(k) - y_0(k)| \leq \eta.$$

That is, the leader-following output consensus is reached within the pre-specified accuracy $\eta > 0$.

In the presence of the disturbance, the $x_{i,0}^{\odot}$ dynamics is affected. By equation (3.29), we have that the effect of the disturbance is decoupled from the leader-following consensus error. Therefore, the L_2-gain from the disturbance to the consensus error is zero and thus less than any pre-specified $\gamma > 0$. $\square$

3.6 Simulation for Output Feedback

To illustrate our output feedback result, we will perform simulation with four follower agents and one leader agent. The follower agents are described as

$$\begin{cases} x_{i,0}(k+1) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0.5 \end{bmatrix} x_{i,0}(k) + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} y_i(k) + \begin{bmatrix} 0 \\ 0 \\ 0.8 \end{bmatrix} w_i(k), \\ x_{i,1}(k+1) = x_{i,2}(k), \\ x_{i,2}(k+1) = x_{i,0,1}(k) - x_{i,1}(k) + \sqrt{2}x_{i,2}(k) + u_i(k), \\ \qquad y_i(k) = x_{i,1}(k), \ i \in \{1, 2\}, \end{cases}$$

and

$$\begin{cases} x_{i,0}(k+1) = \begin{bmatrix} 1 & 0 \\ 0 & 0.3 \end{bmatrix} x_{i,0}(k) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} y_i(k) + \begin{bmatrix} 0 \\ 1.2 \end{bmatrix} w_i(k), \\ x_{i,1}(k+1) = x_{i,2}(k), \\ x_{i,2}(k+1) = x_{i,0,1}(k) - x_{i,1}(k) + \sqrt{2}x_{i,2}(k) + u_i(k), \\ \qquad y_i(k) = x_{i,1}(k), \ i \in \{3, 4\}. \end{cases}$$

The leader agent, the communication topology and the design parameters K and κ are the same as in the state feedback example.

Shown in Figs. 3.8 and 3.9 are respectively the output consensus errors and the states in the presence of the disturbance, with $\varepsilon = 0.1$. Shown in Figs. 3.10 and 3.11 are respectively the output

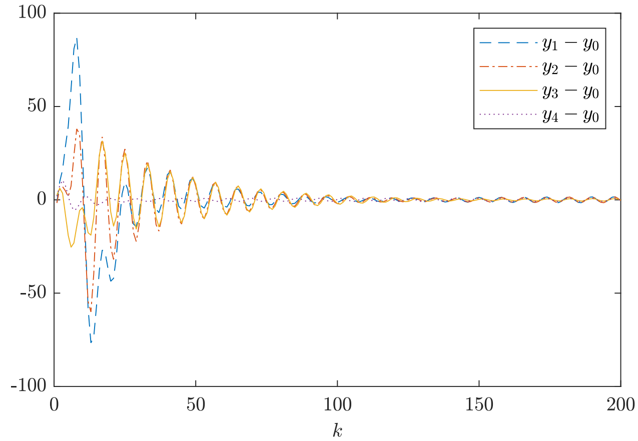

Figure 3.8: The output consensus errors in the presence of the disturbance under output feedback, with $\varepsilon = 0.1$.

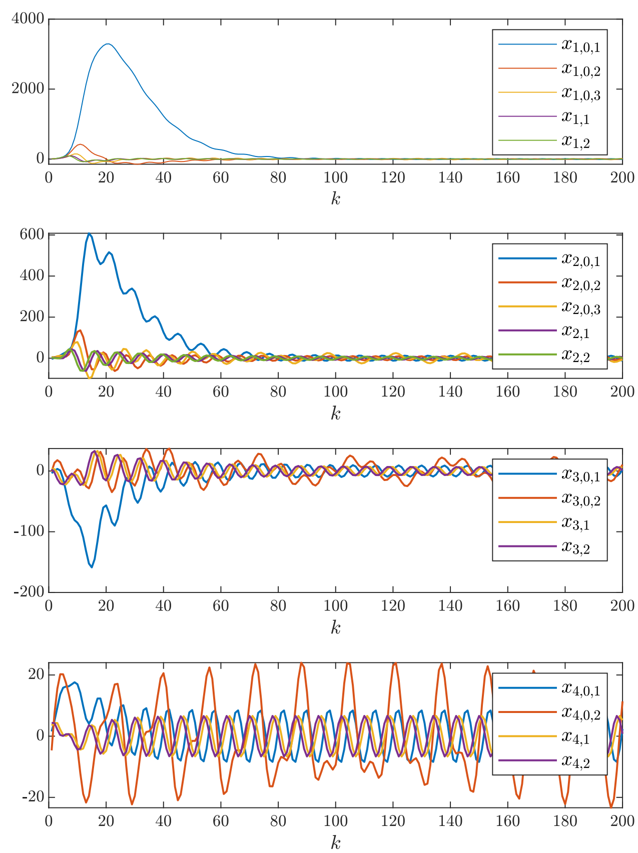

Figure 3.9: The states of the follower agents in the presence of the disturbance under output feedback, with $\varepsilon = 0.1$.

consensus errors and the states in the presence of the disturbance, with $\varepsilon = 0.02$. As can be seen

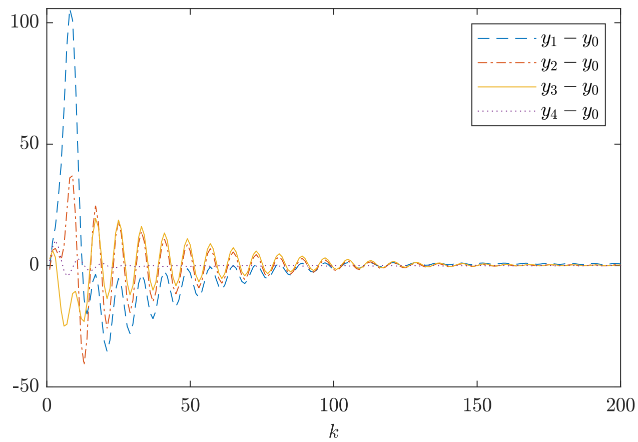

Figure 3.10: The output consensus errors in the presence of the disturbance under output feedback, with $\varepsilon = 0.02$.

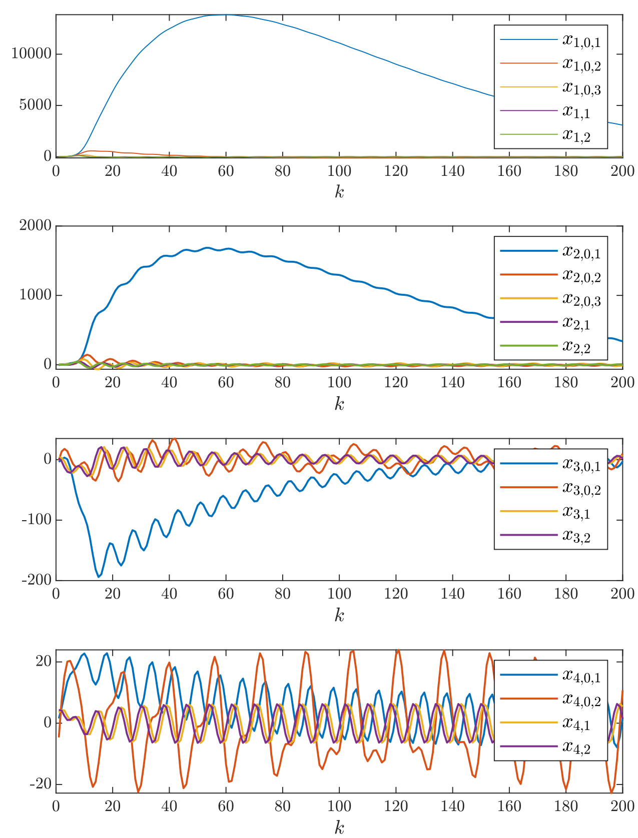

Figure 3.11: The states of the follower agents in the presence of the disturbance under output feedback, with $\varepsilon = 0.02$.

in theses figures, the output consensus errors decrease as the value of ε decreases.

3.7 Conclusions

In this chapter, the leader-following almost output consensus problem of discrete-time multi-agent systems with disturbance-affected unstable zero dynamics was studied. Under conditions on the agent dynamics and the way the disturbance affect the zero dynamics, we constructed state feedback and output feedback consensus protocols based on the low gain feedback design technique and the solution of a modified discrete-time algebraic Riccati equation for the follower agents. The protocols were shown to solve the problem as long as the communication topology of the system contains a directed spanning tree with the leader agent as its root. Simulations were carried out to validate the established results.

This chapter is based on the following publication:

- Tingyang Meng, and Zongli Lin, "Leader-following almost output consensus for discrete-time heterogeneous multi-agent systems in the presence of external disturbances", *Systems & Control Letters*, 169 (2022): 105380.

Chapter 4

Almost Output Consensus of Nonlinear Multi-Agent Systems

4.1 Introduction

Motivated by the fact that nonlinear dynamics and disturbances extensively exist in real-world multi-agent systems, in this chapter we consider the almost output consensus problem for nonlinear multi-agent systems subject to mismatched disturbances. Moreover, our design allows the effect of disturbances to be state-dependent and nonlinear and thus can be applied to more general dynamics. Our approach is motivated by the results of the almost disturbance decoupling problem for individual nonlinear systems (see, for example, [51, 52, 28, 29, 42]). In particular, we resort to a recursive procedure to design parameterized distributed high gain consensus protocols with state-dependent gains. Output consensus can be achieved in the absence of disturbance under our proposed consensus protocols. In addition, the L_2-gain from the disturbances to the output consensus error of agents when the system is operating in output consensus, is inversely proportional to the parameter, and thus can be attenuated to any desired degree of accuracy.

The remainder of this chapter is organized as follows. Section 4.2 formulates the almost output consensus problem for nonlinear multi-agent systems in the presence of external disturbances. Section 4.3 presents the consensus protocols as well as the analysis of the closed-loop system. Section 4.4 presents a simulation example to verify the theoretical results. Section 4.5 concludes this chapter.

4.2 Problem Statement

Consider a multi-agent system consisting of N agents that are subject to external disturbances. The nonlinear dynamics of each agent i, $i \in \{1, 2, \ldots, N\}$, is described by

$$\begin{cases} \dot{x}_i = f(x_i) + g(x_i)u_i + \sum_{l=1}^{p} q_l(x_i)\theta_{i,l}(t), \\ y_i = h(x_i), \end{cases} \tag{4.1}$$

where $x_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}$ and $y_i \in \mathbb{R}$ are the state, control input and output, respectively, $f : \mathbb{R}^n \to \mathbb{R}^n,\ g : \mathbb{R}^n \to \mathbb{R}^n,\ q_l : \mathbb{R}^n \to \mathbb{R}^n,\ h : \mathbb{R}^n \to \mathbb{R}$ are smooth functions, p is a positive integer, and $\theta_{i,l}(t),\ l = 1, 2, \ldots, p$, are the disturbances.

We recall the definition of the control characteristic index and the disturbance characteristic index as follows [52].

Definition 4.1. The control characteristic index ρ of each agent with dynamics (4.1) is defined such that

$$
\begin{aligned}
&L_g L_f^i h(x) = 0,\ 0 \le i \le \rho - 2, \forall x \in \mathbb{R}^n,\\
&L_g L_f^{\rho-1} h(x) \neq 0,\ \forall x \in \mathbb{R}^n.
\end{aligned}
$$

If $L_g L_f^i h(x) = 0, \forall i, \forall x \in \mathbb{R}^n$, then $\rho = \infty$.

Definition 4.2. The disturbance characteristic index ν of each agent with dynamics (4.1) is defined such that

$$
\begin{aligned}
&L_{q_l} L_f^i h(x) = 0,\ 1 \le l \le p, 0 \le i \le \nu - 2, \forall x \in \mathbb{R}^n,\\
&L_{q_l} L_f^{\nu-1} h(x) \neq 0,\ \text{for some } x \in \mathbb{R}^n, \text{some } l, 1 \le l \le p.
\end{aligned}
$$

Assumption 4.1. For the multi-agent system described by (4.1), we make the following assumptions on the agent dynamics.

(i) the relative degree ρ is well defined,

(ii) $\beta_{\rho-1} = \mathrm{span}\{g, \mathrm{ad}_f g, \ldots, \mathrm{ad}_f^{\rho-1} g\}$ is involutive and of constant rank ρ in $\mathbb{R}^n$,

(iii) $\mathrm{ad}_{q_l} \beta_m \subset \beta_m, l = 1, 2, \ldots, p, m = 0, 1, \ldots, \rho - 2$, with $\beta_m = \mathrm{span}\{g, \ldots, \mathrm{ad}_f^m g\}$, and

(iv) the vector fields

$$
\tilde{f} = f - \frac{1}{L_g L_f^{\rho-1} h} L_f^{\rho} h,\ \tilde{g} = \frac{1}{L_g L_f^{\rho-1} h} g
$$

are complete.

It is known that exact disturbance decoupling for individual systems can be achieved [52, 30, 66] if and only if $\nu > \rho$. Therefore here we assume that $\nu \le \rho$.

We also make the following assumption on the communication topology.

Assumption 4.2. The graph associated with the communication topology is undirected and connected.

Problem 4.1. Consider the nonlinear multi-agent system described by (4.1). For any given $\gamma > 0$, design smooth distributed consensus protocols under which almost output consensus can be achieved in the following sense:

(i) In the absence of the disturbances, that is, $\theta_{i,l} = 0,\ l = 1, 2, \ldots, p,\ i \in \{1, 2, \ldots, N\}$, the output consensus can be achieved, *i.e.*,

$$\lim_{t\to\infty} \big(y(t) - \alpha(t)\mathbf{1}\big) = \mathbf{0},$$

for some $\alpha(t) \in \mathbb{R}$, $t \geq 0$, where $y(t) = [y_1(t)\ y_2(t)\ \ldots\ y_N(t)]^{\mathrm{T}} \in \mathbb{R}^N$.

(ii) When the system is operating in output consensus, the L_2-gain from the disturbances to the output consensus error of the agents is less than or equal to γ, *i.e.*,

$$\int_0^t \big\|e(\tau)\big\|_2^2 \mathrm{d}\tau \leq \gamma^2 \int_0^t \|\theta(\tau)\|_2^2 \mathrm{d}\tau,\ t \in [0, T),$$

for any open interval $[0, T)$ in which the corresponding solution exists, where $e(t) = y(t) - \alpha(t)\mathbf{1}$ for some $\alpha(t) \in \mathbb{R}$, $t \geq 0$, $\theta = [\theta_1^{\mathrm{T}}\ \theta_2^{\mathrm{T}}\ \ldots\ \theta_N^{\mathrm{T}}]^{\mathrm{T}}$ and $\theta_i = [\theta_{i,1}\ \theta_{i,2}\ \ldots\ \theta_{i,p}]^{\mathrm{T}}$, $i \in \{1, 2, \ldots, N\}$.

4.3 Main Results

We first recall the Barbalat's lemma as follows.

Lemma 4.1 (Barbalat's Lemma)**.** [34] Let $f : \mathbb{R} \to \mathbb{R}$ be a uniformly continuous function on $[0, \infty)$. Suppose that $\lim_{t\to\infty} \int_0^t f(\tau)\mathrm{d}\tau$ exists and is finite. Then,

$$\lim_{t\to\infty} f(t) = 0.$$

Lemma 4.2. Let $\mathcal{L}$ be the graph Laplacian associated with an undirected and connected graph. Let $\lambda_2 > 0$ be the second smallest eigenvalue of $\mathcal{L}$.

(a) For any $y(t) \in \mathbb{R}^N$ and $d(t) \in \mathbb{R}^M$, if $\|\mathcal{L}y(t)\|_2 \leq c\|d(t)\|_2$, $t \geq 0$, for some constant $c > 0$, then we have

$$\|e(t)\|_2 \leq \frac{c}{\lambda_2}\|d(t)\|_2,$$

where $e(t) = y(t) - \alpha(t)\mathbf{1}$ for some $\alpha(t) \in \mathbb{R}$, $t \geq 0$.

(b) If

$$\lim_{t\to\infty} \mathcal{L}y(t) = \mathbf{0},$$

then,

$$\lim_{t\to\infty} (y(t) - \alpha(t)\mathbf{1}) = \mathbf{0},$$

for some $\alpha(t) \in \mathbb{R}, t \geq 0$.

Proof: For any symmetric matrix $\mathcal{L} \in \mathbb{R}^{N\times N}$, there is an orthogonal matrix $\Sigma = [\sigma_1\ \sigma_2\ \dots \sigma_N] \in \mathbb{R}^{N\times N}$, whose columns are eigenvectors of $\mathcal{L}$ such that [53]

$$\Sigma^{\mathrm{T}}\mathcal{L}\Sigma = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_N \end{bmatrix},$$

or

$$\mathcal{L} = \Sigma \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_N \end{bmatrix} \Sigma^{\mathrm{T}},$$

where $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N$ are the corresponding eigenvalues of $\mathcal{L}$.

Since $\mathcal{L}$ is the Laplacian of an undirected and connected graph, $\lambda_1 = 0$ with the corresponding eigenvector $\sigma_1 = \frac{1}{\sqrt{N}}\mathbf{1}$, and $\lambda_2 > 0$ [68].

It is obvious that, for any $y(t) \in \mathbb{R}^N$, there exist $\alpha_1(t), \alpha_2(t), \dots, \alpha_N(t) \in \mathbb{R}$, such that

$$y(t) = \alpha_1(t)\sigma_1 + \alpha_2(t)\sigma_2 + \cdots + \alpha_N(t)\sigma_N.$$

Let $e(t) = y(t) - \alpha_1(t)\sigma_1 = y(t) - \alpha(t)\mathbf{1}$, where $\alpha(t) = \frac{1}{\sqrt{N}}\alpha_1(t)$. Then,

$$e(t) = \alpha_2(t)\sigma_2 + \alpha_3(t)\sigma_3 + \cdots + \alpha_N(t)\sigma_N,$$
$$e^{\mathrm{T}}(t)\sigma_1 = 0,$$

$$
\begin{aligned}
\mathcal{L}y(t) &= \mathcal{L}e(t) + \alpha(t)\mathcal{L}\mathbf{1} \\
&= \mathcal{L}e(t).
\end{aligned}
$$

Consequently, we have

$$
\begin{aligned}
\|\mathcal{L}y(t)\|_2^2 &= y^{\mathrm{T}}(t)\mathcal{L}^{\mathrm{T}}\mathcal{L}y(t) \\
&= e^{\mathrm{T}}(t)\mathcal{L}\mathcal{L}e(t) \\
&= e^{\mathrm{T}}(t)[\sigma_1\ \sigma_2 \ldots \sigma_N] \begin{bmatrix} 0 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_N \end{bmatrix} \Sigma^{\mathrm{T}}\Sigma \begin{bmatrix} 0 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_N \end{bmatrix} \begin{bmatrix} \sigma_1^{\mathrm{T}} \\ \sigma_2^{\mathrm{T}} \\ \vdots \\ \sigma_N^{\mathrm{T}} \end{bmatrix} e(t) \\
&= e^{\mathrm{T}}(t)[\sigma_1\ \sigma_2 \ldots \sigma_N] \begin{bmatrix} 0 & 0 & \ldots & 0 \\ 0 & \lambda_2^2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_N^2 \end{bmatrix} \begin{bmatrix} \sigma_1^{\mathrm{T}} \\ \sigma_2^{\mathrm{T}} \\ \vdots \\ \sigma_N^{\mathrm{T}} \end{bmatrix} e(t) \\
&= e^{\mathrm{T}}(t)(\lambda_2^2 + \lambda_3^2 + \cdots + \lambda_N^2)e(t) \\
&\geq \lambda_2^2\|e(t)\|_2^2, \qquad (4.2)
\end{aligned}
$$

which implies that

$$
\|e(t)\|_2 \leq \frac{c}{\lambda_2}\|d(t)\|_2,
$$

from which Item (a) follows.

To show Item (b), we note from (4.2) that

$$
\lim_{t\to\infty} \mathcal{L}y(t) = \mathbf{0}
$$

implies that

$$
\lim_{t\to\infty} e(t) = \mathbf{0},
$$

that is,

$$
\lim_{t\to\infty} (y(t) - \alpha(t)\mathbf{1}) = \mathbf{0},
$$

for some $\alpha \in \mathbb{R}, t \geq 0$.

Under Assumption 4.1, we can define a change of coordinates globally in $\mathbb{R}^n$ for each agent i as[52]

$$
\begin{aligned}
z_{i,1} &= h(x_i),\\
z_{i,2} &= L_f h(x_i),\\
&\vdots\\
z_{i,\rho} &= L_f^{\rho-1} h(x_i),\\
z_{i,\rho+1} &= \phi_{\rho+1}(x_i),\\
z_{i,\rho+2} &= \phi_{\rho+2}(x_i),\\
&\vdots\\
z_{i,n} &= \phi_n(x_i),
\end{aligned}
$$

with $\phi_m(x_i)$, $m = \rho+1, \rho+2, \ldots, n$, $\phi_m(0) = 0$, such that $L_{\beta_{\rho-1}}\phi_m(x_i) = 0$ and the state feedback

$$v_i = L_g L_f^{\rho-1} h(x_i) u_i + L_f^{\rho} h(x_i)$$

transforms (4.1) into

$$
\begin{cases}
\dot{z}_{i,m} = z_{i,m+1} + w_{i,m}^{\mathrm{T}}(z_{i,1}, z_{i,2}, \ldots, z_{i,m}, z_{i,r})\theta_i, \; m = 1, 2, \ldots, \rho-1,\\
\dot{z}_{i,\rho} = v_i + w_{i,\rho}^{\mathrm{T}}(z_{i,1}, z_{i,2}, \ldots, z_{i,\rho}, z_{i,r})\theta_i,\\
\dot{z}_{i,r} = \varphi_i(z_{i,1}, z_{i,r}) + \psi_i^{\mathrm{T}}(z_{i,1}, z_{i,r})\theta_i,\\
y_i = z_{i,1},
\end{cases}
$$

where $z_{i,r} = (z_{i,\rho+1}, z_{i,\rho+2}, \ldots, z_{i,n})^{\mathrm{T}} \in \mathbb{R}^{n-\rho}$, $\varphi_i \in \mathbb{R}^{n-\rho}$, $\psi_i \in \mathbb{R}^{p\times(n-\rho)}$, $\theta_i = (\theta_{i,1}, \theta_{i,2}, \ldots, \theta_{i,p})^{\mathrm{T}} \in \mathbb{R}^p$ and $w_{i,m} \in \mathbb{R}^p$ is defined as

$$
\begin{aligned}
\sum_{l=1}^{p} L_{q_l} h(x_i)\theta_{i,l} &= w_{i,1}^{\mathrm{T}}(z_{i,1}, z_{i,r})\theta_i,\\
\sum_{l=1}^{p} L_{q_l} L_f^{m-1} h(x_i)\theta_{i,l} &= w_{i,m}^{\mathrm{T}}(z_{i,1}, z_{i,2}, \ldots, z_{i,m}, z_{i,r})\theta_i, \; m = 2, 3, \ldots, \rho.
\end{aligned}
$$

We note here that if $\nu > \rho$, $w_{i,m}(z_{i,1}, z_{i,2}, \cdots, z_{i,m}, z_{i,r}) \equiv \mathbf{0}$, $m = 1, 2, \ldots, \rho$, $i \in \{1, 2, \ldots, N\}$.

Theorem 4.1. [54] Consider the nonlinear multi-agent system with agent dynamics described by (4.1). There exist consensus protocols that solve the almost output consensus problem.

Proof: We design distributed consensus protocols for each agent in a recursive manner. Consider the following dynamics for each agent i,

$$\dot{z}_{i,1} = z_{i,2} + w_{i,1}(z_{i,1}, z_{i,r})^{\mathrm{T}}\theta_i,\ i \in \{1, 2, \dots, N\}.$$

Define

$$z^*_{i,2} = -\Big(1 + \frac{1}{4}k\big(1 + w^{\mathrm{T}}_{i,1}w_{i,1}\big)\Big)\sum_{j=1}^{N} a_{ij}\big(z_{i,1} - z_{j,1}\big), \tag{4.3}$$

where the constant $k > 0$ is a design parameter to be specified later. Denote $z_m = (z_{1,m}, z_{2,m}, \dots, z_{N,m})^{\mathrm{T}}$, $W_m = \mathrm{blkdiag}\{w_{1,m}, w_{2,m}, \dots, w_{N,m}\}$, $m = 1, 2, \dots, \rho$, and $\theta = (\theta^{\mathrm{T}}_1, \theta^{\mathrm{T}}_2, \dots, \theta^{\mathrm{T}}_N)^{\mathrm{T}}$. Let $z_{i,2} = z^*_{i,2}$, $i \in \{1, 2, \dots, N\}$. Then,

$$\dot{z}_1 = -\mathcal{L}z_1 - \frac{1}{4}k(I + W^{\mathrm{T}}_1 W_1)\mathcal{L}z_1 + W^{\mathrm{T}}_1\theta. \tag{4.4}$$

Consider the function

$$V_1\big(z_1(t)\big) = \frac{1}{2}z^{\mathrm{T}}_1\mathcal{L}z_1 \geq 0.$$

Its time derivative along the trajectory of (4.4) can be evaluated as

$$\begin{aligned}
\dot{V}_1 &= z^{\mathrm{T}}_1\mathcal{L}\dot{z}_1\\
&= -(\mathcal{L}z_1)^{\mathrm{T}}(\mathcal{L}z_1) - \frac{1}{4}k(\mathcal{L}z_1)^{\mathrm{T}}(I + W^{\mathrm{T}}_1W_1)(\mathcal{L}z_1) + (\mathcal{L}z_1)^{\mathrm{T}}W^{\mathrm{T}}_1\theta\\
&= -(\mathcal{L}z_1)^{\mathrm{T}}(\mathcal{L}z_1) - k\Big(\frac{1}{4}(\mathcal{L}z_1)^{\mathrm{T}}(I + W^{\mathrm{T}}_1W_1)(\mathcal{L}z_1) - \frac{1}{k}(\mathcal{L}z_1)^{\mathrm{T}}W^{\mathrm{T}}_1\theta\\
&\quad + \frac{1}{k^2}(W_1\theta)^{\mathrm{T}}(I + W^{\mathrm{T}}_1W_1)^{-1}(W^{\mathrm{T}}_1\theta)\Big) + \frac{1}{k}(W_1\theta)^{\mathrm{T}}(I + W^{\mathrm{T}}_1W_1)^{-1}(W^{\mathrm{T}}_1\theta)\\
&= -\|\mathcal{L}z_1\|^2_2 - k\Big\|(\frac{1}{2}(I + W^{\mathrm{T}}_1W_1)^{\frac{1}{2}}(\mathcal{L}z_1) - \frac{1}{k}(I + W^{\mathrm{T}}_1W_1)^{-\frac{1}{2}}(W^{\mathrm{T}}_1\theta)\Big\|^2_2\\
&\quad + \frac{1}{k}(W_1\theta)^{\mathrm{T}}(I + W^{\mathrm{T}}_1W_1)^{-1}(W^{\mathrm{T}}_1\theta)\\
&\leq -\|\mathcal{L}z_1\|^2_2 + \frac{1}{k}\sum_{i=1}^{N}\frac{\|w^{\mathrm{T}}_{i,1}\theta_i\|^2_2}{1 + \|w_{i,1}\|^2_2}\\
&\leq -\|\mathcal{L}z_1\|^2_2 + \frac{1}{k}\|\theta\|^2_2.
\end{aligned} \tag{4.5}$$

Integrating both sides of inequality (4.5), we have

$$V_1\big(z_1(t)\big) - V_1\big(z_1(0)\big) \leq -\int_0^t \big\|\mathcal{L}z_1(\tau)\big\|^2_2\mathrm{d}\tau + \frac{1}{k}\int_0^t \big\|\theta(\tau)\big\|^2_2\mathrm{d}\tau. \tag{4.6}$$

The consensus protocols for $\rho = 1$ can then be designed in the following way.

Case 1: $\rho = 1$

Design $v_i = z^*_{i,2}$. In the original coordinates, the consensus protocol for each agent i can be written as

$$u_i = \frac{z^*_{i,2} - L_f h(x_i)}{L_g h(x_i)} \tag{4.7}$$

with $z^*_{i,2}$ as given in (4.3).

Conclusions can then be drawn from (4.6) as follows.

(i) In the absence of the disturbances, i.e., $\theta(t) = 0$, for all $t \geq 0$, we have

$$\int_0^t \|\mathcal{L}z_1(\tau)\|_2^2 \mathrm{d}\tau \leq V_1\big(z_1(0)\big) - V_1\big(z_1(t)\big) \leq V_1\big(z_1(0)\big).$$

By Lemma 4.1, we have

$$\lim_{t\to\infty} \|\mathcal{L}z_1(t)\|_2^2 = 0,$$

which in turn, by Lemma 4.2, implies that

$$\lim_{t\to\infty} (y(t) - \alpha(t)\mathbf{1}) = 0$$

for some $\alpha(t) \in \mathbb{R}, t \geq 0$.

(ii) When the system is operating in output consensus, i.e., $z_{1,1}(0) = z_{2,1}(0) \cdots = z_{N,1}(0)$, with $\theta(t) \neq \mathbf{0}$, we have

$$V_1(z_1(0)) = 0,$$

and, from (4.6),

$$\int_0^t \big\|\mathcal{L}z_1(\tau)\big\|_2^2 \mathrm{d}\tau \leq \frac{1}{k}\int_0^t \big\|\theta(\tau)\big\|_2^2 \mathrm{d}\tau,$$

since $V_1(z_1(t)) \geq 0$. Noting that $y(t) = z_1(t)$, we have, by Lemma 4.2,

$$\int_0^t \big\|e(\tau)\|_2^2 \mathrm{d}\tau \leq \frac{1}{k\lambda_2^2}\int_0^t \big\|\theta(\tau)\big\|_2^2 \mathrm{d}\tau,$$

where $e(t) = y(t) - \alpha(t)\mathbf{1}$ for some $\alpha(t) \in \mathbb{R}, t \geq 0$. Thus, the L_2-gain from the disturbances to the output consensus error is less than or equal to any given $\gamma > 0$ by setting $k \geq \frac{1}{\gamma^2\lambda_2^2}$.

Case 2: $\rho > 1$

Denote $z_r = (z_{1,r}, z_{2,r}, \ldots, z_{N,r})^{\mathrm{T}}$, $\Phi = (\varphi_1^{\mathrm{T}}, \varphi_2^{\mathrm{T}}, \ldots, \varphi_N^{\mathrm{T}})^{\mathrm{T}}$ and $\Psi = \mathrm{blkdiag}\{\psi_1, \psi_2, \ldots, \psi_N\}$. Assume that for a given index m, $1 \le m \le \rho$, and the dynamics

$$\begin{cases} \dot{z}_1 = z_2 + W_1^{\mathrm{T}}\theta, \\ \dot{z}_2 = z_3 + W_2^{\mathrm{T}}\theta, \\ \quad\vdots \\ \dot{z}_m = z_{m+1} + W_m^{\mathrm{T}}\theta, \end{cases}$$

with $z_{m+1} = z_{m+1}^*$, there exist m functions

$$z_\mu^* = z_\mu^*(z_1, z_2, \ldots, z_{\mu-1}, z_r, k),\ \ \mu = 2, 3, \ldots, m+1,$$

such that in new the coordinates

$$\begin{aligned} \tilde{z}_1 &= z_1, \\ \tilde{z}_\mu &= z_\mu - z_\mu^*(z_1, z_2, \ldots, z_{\mu-1}, z_r, k),\ \ \mu = 2, 3, \ldots, m, \end{aligned}$$

the function

$$V_m(\tilde{z}_1, \tilde{z}_2, \ldots, \tilde{z}_m) = \frac{1}{2}\sum_{\mu=1}^{m} \tilde{z}_\mu^{\mathrm{T}} \mathcal{L} \tilde{z}_\mu \ge 0$$

has time derivative along the trajectory

$$\begin{cases} \dot{\tilde{z}}_1 = \tilde{z}_2 + z_2^* + W_1^{\mathrm{T}}\theta, \\ \dot{\tilde{z}}_2 = \tilde{z}_3 + z_3^* + W_2^{\mathrm{T}}\theta - \left(\dfrac{\partial z_2^*}{\partial z_1}\big(z_2 + W_1^{\mathrm{T}}\theta\big) + \dfrac{\partial z_2^*}{\partial z_r}\big(\Phi + \Psi^{\mathrm{T}}\theta\big)\right), \\ \quad\vdots \\ \dot{\tilde{z}}_{m-1} = \tilde{z}_m + z_m^* + W_{m-1}^{\mathrm{T}}\theta - \left(\displaystyle\sum_{\mu=1}^{m-2} \dfrac{\partial z_{m-1}^*}{\partial z_\mu}\big(z_{\mu+1} + W_\mu^{\mathrm{T}}\theta\big) + \dfrac{\partial z_{m-1}^*}{\partial z_r}\big(\Phi + \Psi^{\mathrm{T}}\theta\big)\right), \\ \dot{\tilde{z}}_m = z_{m+1}^* + W_m^{\mathrm{T}}\theta - \left(\displaystyle\sum_{\mu=1}^{m-1} \dfrac{\partial z_m^*}{\partial z_\mu}\big(z_{\mu+1} + W_\mu^{\mathrm{T}}\theta\big) + \dfrac{\partial z_m^*}{\partial z_r}\big(\Phi + \Psi^{\mathrm{T}}\theta\big)\right) \end{cases}$$

that satisfies

$$\dot{V}_m \le -\sum_{\mu=1}^{m} \|\mathcal{L}\tilde{z}_\mu\|_2^2 + \frac{m}{k}\|\theta\|_2^2.$$

Then, for index $m+1$, consider the system

$$\begin{cases} \dot{z}_1 = z_2 + W_1^{\mathrm{T}}\theta, \\ \dot{z}_2 = z_3 + W_2^{\mathrm{T}}\theta, \\ \quad\vdots \\ \dot{z}_{m+1} = z_{m+2} + W_{m+1}^{\mathrm{T}}\theta, \end{cases}$$

with

$$z_{m+2} = z_{m+2}^*(z_1, z_2, \ldots, z_{m+1}, z_r, k),$$

in the new coordinates

$$\begin{aligned} &\tilde{z}_\mu,\ \ \mu = 1, 2, \ldots, m, \\ &\tilde{z}_{m+1} = z_{m+1} - z_{m+1}^*(z_1, z_2, \ldots, z_m, z_r, k). \end{aligned}$$

The time derivative of the function

$$V_{m+1}(\tilde{z}_1, \tilde{z}_2, \ldots, \tilde{z}_{m+1}) = \frac{1}{2}\sum_{\mu=1}^{m+1} \tilde{z}_\mu^{\mathrm{T}} \mathcal{L} \tilde{z}_\mu$$

along the trajectory of

$$\begin{cases} \dot{\tilde{z}}_1 = \tilde{z}_2 + z_2^* + W_1^{\mathrm{T}}\theta, \\ \dot{\tilde{z}}_2 = \tilde{z}_3 + z_3^* + W_2^{\mathrm{T}}\theta - \left(\dfrac{\partial z_2^*}{\partial z_1}\big(z_2 + W_1^{\mathrm{T}}\theta\big) + \dfrac{\partial z_2^*}{\partial z_r}\big(\Phi + \Psi^{\mathrm{T}}\theta\big)\right), \\ \quad\vdots \\ \dot{\tilde{z}}_m = \tilde{z}_{m+1} + z_{m+1}^* + W_m^{\mathrm{T}}\theta - \left(\displaystyle\sum_{\mu=1}^{m-1} \dfrac{\partial z_m^*}{\partial z_\mu}\big(z_{\mu+1} + W_\mu^{\mathrm{T}}\theta\big) + \dfrac{\partial z_m^*}{\partial z_r}\big(\Phi + \Psi^{\mathrm{T}}\theta\big)\right) \\ \dot{\tilde{z}}_{m+1} = z_{m+2}^* + W_{m+1}^{\mathrm{T}}\theta - \left(\displaystyle\sum_{\mu=1}^{m} \dfrac{\partial z_{m+1}^*}{\partial z_\mu}\big(z_{\mu+1} + W_\mu^{\mathrm{T}}\theta\big) + \dfrac{\partial z_{m+1}^*}{\partial z_r}\big(\Phi + \Psi^{\mathrm{T}}\theta\big)\right) \end{cases}$$

can be evaluated as

$$\dot{V}_{m+1} \leq -\sum_{\mu=1}^{m} \|\mathcal{L}\tilde{z}_\mu\|^2 + \frac{m}{k}\|\theta\|^2$$

$$+ \tilde{z}_{m+1}^{\mathrm{T}}\mathcal{L}\Bigg(\tilde{z}_m + W_{m+1}^{\mathrm{T}}\theta - \sum_{\mu=1}^{m}\frac{\partial z_{m+1}^*}{\partial z_\mu}\big(z_{\mu+1} + W_\mu^{\mathrm{T}}\theta\big) - \frac{\partial z_{m+1}^*}{\partial z_r}\big(\Phi + \Psi^{\mathrm{T}}\theta\big) + z_{m+2}^*\Bigg). \quad (4.8)$$

Define

$$\alpha(z_1, z_2, \ldots, z_{m+1}, z_r) = \tilde{z}_m - \sum_{\mu=1}^{m}\frac{\partial z_{m+1}^*}{\partial z_\mu}z_{\mu+1} - \frac{\partial z_{m+1}^*}{\partial z_r}\Phi,$$

$$\tilde{W}_{m+1}(z_1, z_2, \ldots, z_{m+1}, z_r) = W_{m+1} - \sum_{\mu=1}^{m}W_\mu\bigg(\frac{\partial z_{m+1}^*}{\partial z_\mu}\bigg)^{\mathrm{T}} - \Psi\bigg(\frac{\partial z_{m+1}^*}{\partial z_r}\bigg)^{\mathrm{T}},$$

$$z_{m+2}^* = -\alpha - \mathcal{L}\tilde{z}_{m+1} - \frac{1}{4}k\big(I + \tilde{W}_{m+1}^{\mathrm{T}}\tilde{W}_{m+1}\big)\mathcal{L}\tilde{z}_{m+1}.$$

Then, inequality (4.8) becomes

$$\begin{aligned}
\dot{V}_{m+1} \leq& -\sum_{\mu=1}^{m+1}\|\mathcal{L}\tilde{z}_\mu\|_2^2 + \frac{m}{k}\|\theta\|_2^2 + (\mathcal{L}\tilde{z}_{m+1})^{\mathrm{T}}\tilde{W}_{m+1}^{\mathrm{T}}\theta - \frac{1}{4}k(\mathcal{L}\tilde{z}_{m+1})^{\mathrm{T}}\big(I + \tilde{W}_{m+1}^{\mathrm{T}}\tilde{W}_{m+1}\big)\mathcal{L}\tilde{z}_{m+1}\\
=& -\sum_{\mu=1}^{m+1}\|\mathcal{L}\tilde{z}_\mu\|^2 + \frac{m}{k}\|\theta_i\|^2 - k\Bigg(\frac{1}{4}(\mathcal{L}\tilde{z}_{m+1})^{\mathrm{T}}\big(I + \tilde{W}_{m+1}^{\mathrm{T}}\tilde{W}_{m+1}\big)(\mathcal{L}\tilde{z}_{m+1}) - \frac{1}{k}(\mathcal{L}\tilde{z}_{m+1})^{\mathrm{T}}\tilde{W}_{m+1}^{\mathrm{T}}\theta\\
&+ \frac{1}{k^2}(\tilde{W}_{m+1}^{\mathrm{T}}\theta)^{\mathrm{T}}\big(I + \tilde{W}_{m+1}^{\mathrm{T}}\tilde{W}_{m+1}\big)^{-1}(\tilde{W}_{m+1}^{\mathrm{T}}\theta)\Bigg)\\
&+ \frac{1}{k}(\tilde{W}_{m+1}^{\mathrm{T}}\theta)^{\mathrm{T}}\big(I + \tilde{W}_{m+1}^{\mathrm{T}}\tilde{W}_{m+1}\big)^{-1}(\tilde{W}_{m+1}^{\mathrm{T}}\theta)\\
=& -\sum_{\mu=1}^{m+1}\|\mathcal{L}\tilde{z}_\mu\|_2^2 + \frac{m}{k}\|\theta\|_2^2\\
&- k\left\|\frac{1}{2}\big(I + \tilde{W}_{m+1}^{\mathrm{T}}\tilde{W}_{m+1}\big)^{\frac{1}{2}}(\mathcal{L}\tilde{z}_{m+1}) - \frac{1}{k}\big(I + \tilde{W}_{m+1}^{\mathrm{T}}\tilde{W}_{m+1}\big)^{-\frac{1}{2}}(\tilde{W}_{m+1}^{\mathrm{T}}\theta)\right\|_2^2\\
&+ \frac{1}{k}(\tilde{W}_{m+1}^{\mathrm{T}}\theta)^{\mathrm{T}}\big(I + \tilde{W}_{m+1}^{\mathrm{T}}\tilde{W}_{m+1}\big)^{-1}(\tilde{W}_{m+1}^{\mathrm{T}}\theta)\\
\leq& -\sum_{\mu=1}^{m+1}\|\mathcal{L}\tilde{z}_\mu\|_2^2 + \frac{m}{k}\|\theta\|_2^2 + \frac{1}{k}\|\theta\|_2^2 = -\sum_{\mu=1}^{m+1}\|\mathcal{L}\tilde{z}_\mu\|_2^2 + \frac{m+1}{k}\|\theta\|_2^2.
\end{aligned}$$

Now consider the system

$$\begin{cases}\dot{z}_m = z_{m+1} + W_m^{\mathrm{T}}\theta, \ m = 1, 2, \ldots, \rho - 1,\\ \dot{z}_\rho = v + W_\rho^{\mathrm{T}}\theta.\end{cases}$$

in the new coordinates

$$\tilde{z}_1 = z_1,$$
$$\tilde{z}_\mu = z_\mu - z_\mu^*(z_1, z_2, \ldots, z_\mu, z_r, k),\ \mu = 2, 3, \ldots, \rho,$$

with $v = (v_1, v_2, \ldots, v_N)^{\mathrm{T}}$ and $z_2^*, z_3^*, \ldots, z_{\rho+1}^*$ designed recursively according to the previous process.

Define the consensus protocols as

$$v_i = z_{i,\rho+1}^*,\ i \in \{1, 2, \ldots, N\},$$

which, in the original coordinates, can be written as

$$u_i = \frac{z_{i,\rho+1}^* - L_f^\rho h(x_i)}{L_g L_f^{\rho-1} h(x_i)},\ i \in \{1, 2, \ldots, N\}.$$

Then, under these consensus protocols, the time derivative of the function

$$V_\rho(\tilde{z}_1, \tilde{z}_2, \ldots, \tilde{z}_\rho) = \frac{1}{2} \sum_{\mu=1}^{\rho} \tilde{z}_\mu^{\mathrm{T}} \mathcal{L} \tilde{z}_\mu$$

along the trajectories of the closed-loop system satisfies

$$\dot{V}_\rho \leq -\sum_{\mu=1}^{\rho} \|\mathcal{L}\tilde{z}_\mu\|_2^2 + \frac{\rho}{k}\|\theta\|_2^2. \tag{4.9}$$

By integrating both sides of inequality (4.9), the following conclusion can be made:

(i) In the absence of the disturbances, i.e., $\theta(t) = \mathbf{0}$, for all $t \geq 0$, we have

$$\int_0^t \sum_{\mu=1}^{\rho} \|\mathcal{L}\tilde{z}_\mu(\tau)\|_2^2 \mathrm{d}\tau \leq V_\rho(\tilde{z}_1(0), \tilde{z}_2(0), \ldots, \tilde{z}_\rho(0)).$$

By Lemma 4.1, we have

$$\lim_{t\to\infty} \sum_{\mu=1}^{\rho} \|\mathcal{L}\tilde{z}_\mu(t)\|_2^2 = 0,$$

and therefore

$$\lim_{t\to\infty} \|\mathcal{L}z_1(t)\|_2^2 = 0,$$

which in turn, by Lemma 4.2, implies that,

$$\lim_{t\to\infty}(y(t)-\alpha(t)\mathbf{1})=\mathbf{0},$$

for some $\alpha(t)\in\mathbb{R}, t\geq 0$.

(ii) When the system is operating in output consensus, i.e., $z_{1,1}(0)=z_{2,1}(0)=\cdots=z_{N,1}(0)$, which also implies that $z_{1,m}(0)=z_{2,m}(0)=\cdots=z_{N,m}(0)$, $m=2,3,\ldots,\rho$, and hence

$$V_\rho(\tilde{z}_1(0),\tilde{z}_2(0),\ldots\tilde{z}_\rho(0))=0.$$

Then, we have,

$$\int_0^t\|e(\tau)\|_2^2\mathrm{d}\tau\leq\frac{\rho}{k\lambda_2^2}\int_0^t\|\theta(\tau)\|_2^2\mathrm{d}\tau,$$

where $e(t)=y(t)-\alpha(t)\mathbf{1}$ for some $\alpha(t)\in\mathbb{R}, t\geq 0$. Thus, the L_2-gain from the disturbances to the output consensus error is less than or equal to any given $\gamma>0$ by setting $k\geq\frac{\rho}{\gamma^2\lambda_2^2}$.

4.4 Simulation

Consider a group of four agents, labeled as $1,2,3$, and 4. The dynamics of each agent i is described by

$$\begin{cases}\dot{x}_{i,1}=u_i+\sin x_{i,2}\theta_{i,1},\\ \dot{x}_{i,2}=-0.1x_{i,2}^3+x_{i,1}(x_{i,2}+\theta_{i,2}),\\ \quad y_i=x_{i,1}.\end{cases}$$

The communication topology is described by $\mathcal{L}$ as

$$\mathcal{L}=\begin{bmatrix}2&-1&0&-1\\-1&2&-1&0\\0&-1&2&-1\\-1&0&-1&2\end{bmatrix}.$$

The distributed consensus protocols are designed as

$$u_i=-\Big(1+\frac{1}{4}k\big(1+x_{i,2}^2\big)\Big)\sum_{j=1}^N a_{ij}\big(x_{i,1}-x_{j,1}\big),\ i\in\{1,2,3,4\}. \tag{4.10}$$

Figure 4.1 shows the evolution of the outputs of all agents with initial conditions $[0\ 0\ 5\ 0\ 10\ 0\ 15\ 0]^{\mathrm{T}}$ in the absence of external disturbances. It is obvious that output consensus of the system is achieved under protocols (4.10).

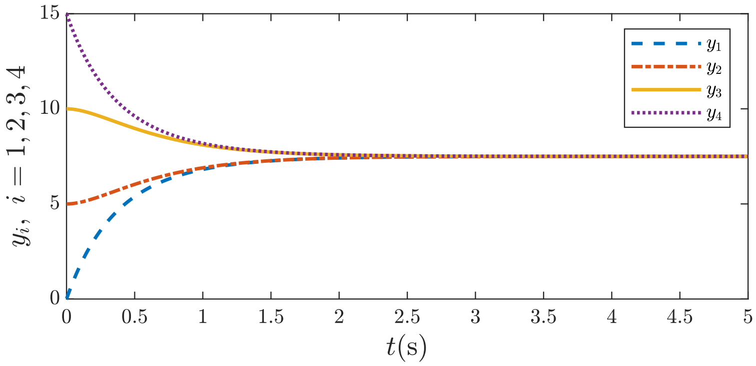

Figure 4.1: Evolution of the outputs of all agents under the consensus protocols (4.10) with $k = 0.1$ in the absence of the external disturbances.

We next consider the presence of external disturbances. The disturbance signals are realized as $\theta_{1,1}(t) = \theta_{4,2}(t) = \theta_1(t)$, $\theta_{1,2}(t) = \theta_{2,2}(t) = \theta_2(t)$, $\theta_{2,1} = \theta_3(t)$, $\theta_{3,2} = \theta_4(t)$, $\theta_{3,1} = \theta_{4,1} = \theta_6(t)$, where the signals $\theta_1(t), \theta_2(t), \ldots, \theta_6(t)$ are shown in Fig. 4.2.

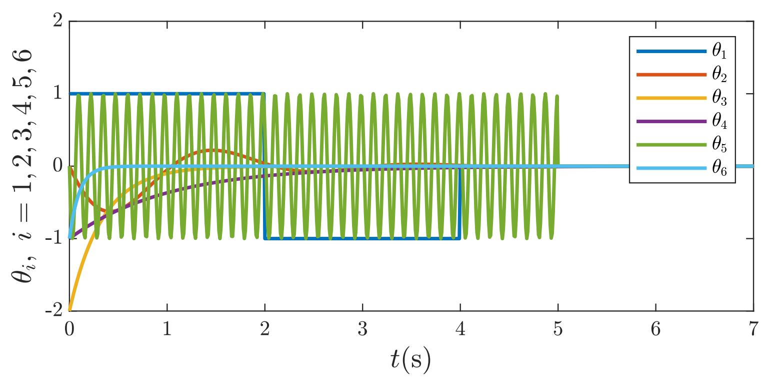

Figure 4.2: External disturbances.

Fig. 4.3 shows the evolution of the output differences between agents from the initial condition $(10, 10, 10, 10,\ 10, 10, 10, 10)^{\mathrm{T}}$, i.e., the system is operating in output consensus at $t = 0$, in the presence of the external disturbances. It is observed that the outputs of agents approach closer as the parameter k becomes larger.

Fig. 4.4 shows the evolution of the output differences between agents with non-identical initial conditions $(0, 0, 5, 0,\ 10, 0, 15, 0)^{\mathrm{T}}$ in the presence of external disturbances. It is observed that the outputs of all agents converged to a common value and the effect of the disturbances is attenuated

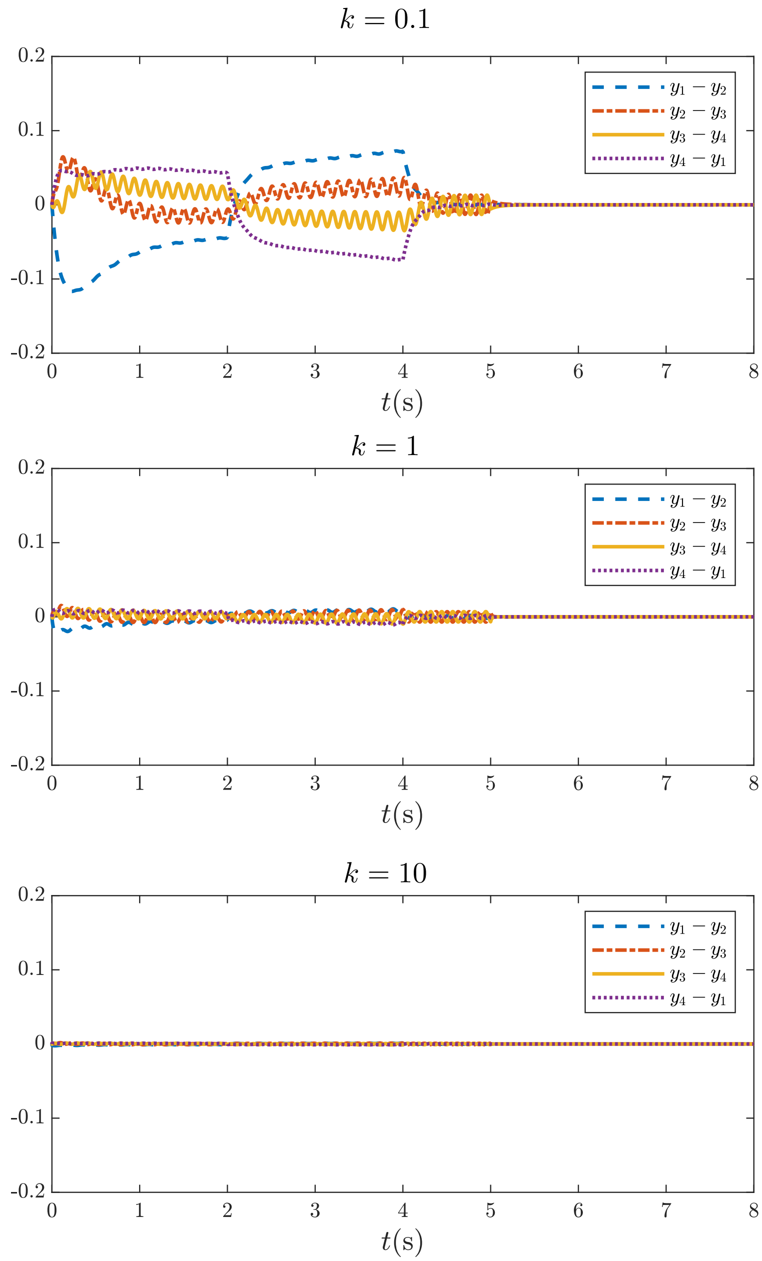

Figure 4.3: Evolution of the output differences between agents in the presence of the external disturbances with the initial condition $(10, 10, 10, 10,\ 10, 10, 10, 10)^{\mathrm{T}}$ under the consensus protocols (4.10), with $k = 0.1$, $k = 1$ and $k = 10$, respectively.

as the value of the parameter k is increased.

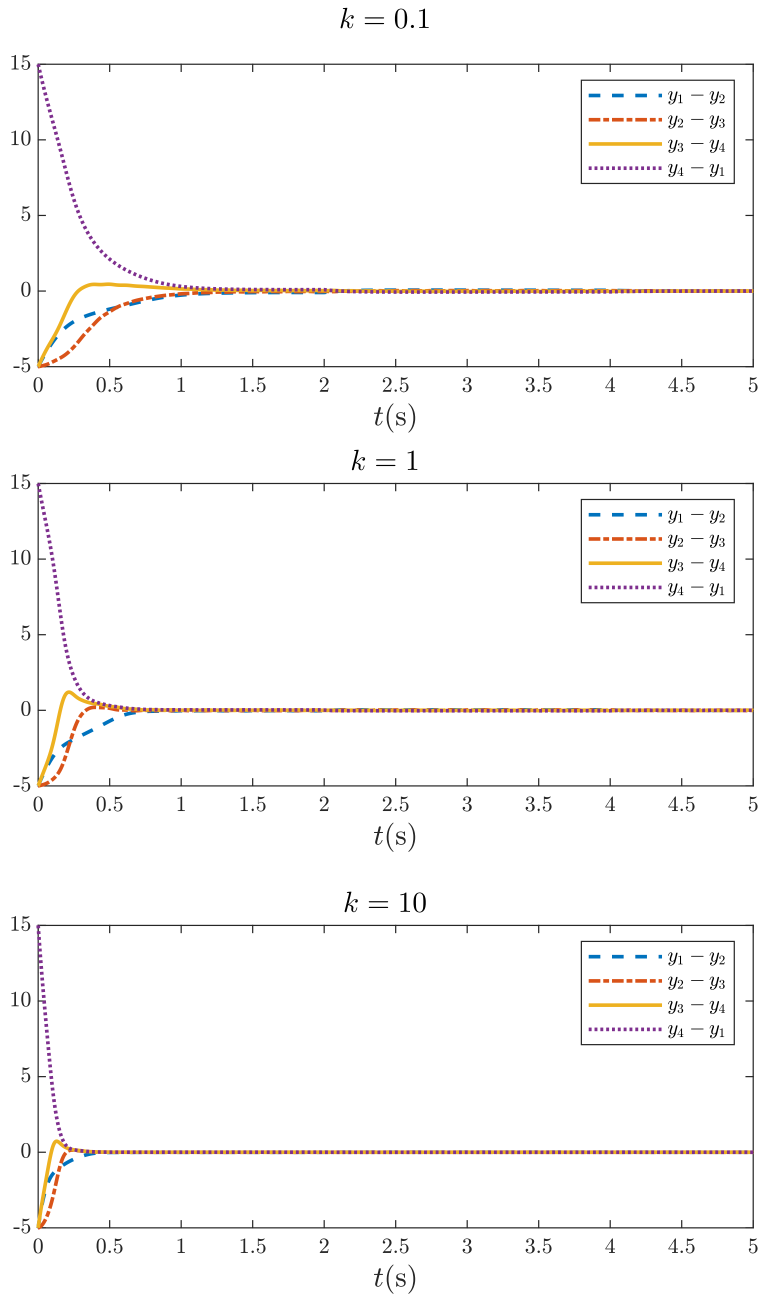

Figure 4.4: Evolution of the output differences between agents in the presence of the external disturbances with the initial condition $(0, 0, 5, 0, 10, 0, 15, 0)^{\mathrm{T}}$ under the consensus protocols (4.10) with $k = 0.1$, $k = 1$ and $k = 10$, respectively.

4.5 Conclusions

In this chapter, the almost output consensus problem of nonlinear multi-agent systems in the presence of external disturbances was studied. Under conditions on the agent dynamics, which are considered mild in the literature on disturbance rejection for individual nonlinear systems, we constructed distributed consensus protocols for the agents. These consensus protocols were shown to achieve almost output consensus when the communication topology is undirected and connected. The simulation was carried out to validate the established results.

This chapter is based on the following publication:

- Tingyang Meng, and Zongli Lin, "Almost output consensus of nonlinear multiagent systems in the presence of external disturbances." *International Journal of Robust and Nonlinear Control* 30.17 (2020): 7355-7369.

Chapter 5

Suboptimal Output Consensus of Linear Discrete-Time Multi-Agent Systems

5.1 Introduction

In this chapter, we consider the suboptimal output consensus problem for a discrete-time multi-agent system, whose agents may possess polynomial unstable zero dynamics (*i.e.*, the agents may be of weakly nonminimum phase). The motivation for considering such agent dynamics is twofold. First, many real-world systems evolve in discrete-time or are discretized for digital control. Second, non-minimum phase systems are frequently found in real-world applications (see, for example, [23]), for which a nonzero steady-state value of the output will cause the states of the zero dynamics to grow unbounded. In order to prevent this from happening, we aim to achieve suboptimal output consensus, instead of optimal output consensus, by explicitly taking the unstable zero dynamics into consideration in the protocol design. Rather than aiming to reach perfect consensus at the exact optimal point, we aim to design consensus protocols that stabilize the zero dynamics of each agent by allowing its output to vary in a neighborhood of the optimal value, whose size can be pre-specified to be arbitrarily small.

In order to prevent the states of the unstable zero dynamics from growing unbounded as the output approaches a nonzero constant value, we aim to achieve suboptimal output consensus, instead of optimal output consensus, by explicitly taking the unstable zero dynamics into consideration in the protocol design. Rather than aiming to reach a perfect consensus at the exact optimal point, we stabilize the zero dynamics of each agent by allowing its output to vary in a neighborhood of the optimal point, whose size can be pre-specified to be arbitrarily small. More specifically, we propose for each agent a parameterized distributed protocol based on the low gain feedback design technique under which the states of all agents remain bounded, and suboptimal output consensus of the system is achieved, i.e., the outputs of all agents converge to a pre-specified arbitrarily small neighborhood of the optimal point that minimizes the overall objective function of the multi-agent system, as long as the value of the low gain feedback parameter is chosen small enough.

Unlike in the continuous-time setting [90], a state transformation is required to transform the agent dynamics into a form conducive for the construction of the consensus protocol. Furthermore, a frequency response property of a discrete-time closed-loop system under low gain feedback, which is not available in the literature, also needs to be established. Such a property is expected to be useful in other contexts in a discrete-time setting.

The remainder of this chapter is organized as follows. Section 5.2 formulates the suboptimal output consensus problem for discrete-time linear multi-agent systems. Section 5.3 presents the design of the consensus protocols and establishes that they solve the problem formulated in Section 5.2. Section 5.4 provides a simulation example to demonstrate the proposed protocols. Section 5.5 concludes this chapter.

5.2 Problem Statement

Consider a discrete-time linear multi-agent system consisting of N agents, with possibly heterogeneous unstable zero dynamics.

The dynamics of the ith agent, $i \in \{1, 2, \cdots, N\}$, is described by the following discrete-time system,

$$\begin{cases} x_{i,0}(k+1) = A_{i,0}x_{i,0}(k) + B_{i,0}x_{i,1}(k), \\ x_{i,r}(k+1) = x_{i,r+1}(k), \; r = 1, 2, \cdots, \rho - 1, \\ x_{i,\rho}(k+1) = E_{i,0}x_{i,0}(k) + \alpha_{i,1}x_{i,1}(k) + \alpha_{i,2}x_{i,2}(k) + \cdots + \alpha_{i,\rho}x_{i,\rho}(k) + u_i(k), \\ \qquad y_i(k) = x_{i,1}(k), \end{cases} \tag{5.1}$$

where $x_{i,0} \in \mathbb{R}^{n_{i,0}}$ and $x_i = [x_{i,1} \; x_{i,2} \; \cdots \; x_{i,\rho}]^{\mathrm{T}} \in \mathbb{R}^{\rho}$ are the states, $u_i \in \mathbb{R}$ is the input, and $y_i \in \mathbb{R}$ is the output. The dynamics of $x_{i,0}$ is the zero dynamics of the system. The relative degree ρ is assumed to be the same for all agents. The agents are allowed to have different zero dynamics. We note that any single input single output linear system can be transformed into the form (5.1) by a state transformation.

Assumption 5.1. The zero dynamics governed by $A_{i,0}$ is allowed to be polynomial unstable but not exponentially unstable, and the eigenvalues of $A_{i,0}$ are away from $z = 1$. Without loss of generality, assume that all eigenvalues of $A_{i,0}$ are located on the unit circle.

Assumption 5.2. The pair $(A_{i,0}, B_{i,0})$ is controllable, and is in the following canonical form,

$$A_{i,0} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ a_{i,0,1} & a_{i,0,2} & \cdots & a_{i,0,n_{i,0}} \end{bmatrix}, \ B_{i,0} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

Each agent i, $i \in \{1, 2, \cdots, N\}$, has its local objective function $f_i(y_i) : \mathbb{R} \to \mathbb{R}$ that is only known to itself. The overall objective function of the multi-agent system is $\sum_{i=1}^{N} f_i(y_i)$ subject to $y_1 = y_2 = \cdots = y_N$.

Assumption 5.3. The objective function $f_i : \mathbb{R} \to \mathbb{R}$, $i \in \{1, 2 \cdots, N\}$, satisfies the following conditions.

(i) It is differentiable and its gradient is Lipschitz with constant $M_i > 0$ in $\mathbb{R}$, *i.e.*,

$$|\nabla f_i(x) - \nabla f_i(y)| \leq M_i |x - y|, \ \forall x, y \in \mathbb{R}.$$

(ii) It is strongly convex with constant $m_i > 0$ in $\mathbb{R}$, *i.e.*,

$$(x - y)(\nabla f_i(x) - \nabla f_i(y)) \geq m_i (x - y)^2, \ \forall x, y \in \mathbb{R}.$$

We make the following assumption on the communication network.

Assumption 5.4. The graph that describes the communication network of the multi-agent system is undirected and connected.

The suboptimal output consensus problem we are to study is formulated as follows.

Problem 5.1. Consider the discrete-time multi-agent system described by (5.1). Let the agent dynamics satisfy Assumptions 5.1 and 5.2. Let the objective functions satisfy Assumption 5.3. Let the communication network satisfy Assumption 5.4. For any given arbitrarily small scalar $\gamma > 0$, design a distributed suboptimal output consensus protocol u_i for each agent i, $i \in \{1, 2, \cdots, N\}$, under which the states of all agents are bounded and

$$\lim_{k \to \infty} \left| y_i(k) - y^* \right| \leq \gamma,$$

where $y^* \in \mathbb{R}$ is an optimal point that minimizes the overall objective function

$$\sum_{i=1}^{N} f_i(y_i), \text{ subject to } y_1 = y_2 = \cdots = y_N,$$

with $y_1 = y_2 = \cdots = y_N = y^*$.

Remark 5.1. Under Assumption 5.3, the optimal point y^* is unique and satisfies the following condition,

$$\sum_{i=1}^{N} \nabla f_i(y^*) = 0.$$

5.3 Main Results

We will present the protocol design in the following few steps. We will then show that the resulting protocols solve Problem 5.1.

For each agent $i \in \{1, 2, \cdots, N\}$, define

$$u_{i,0} = F_{i,0}(\varepsilon) x_{i,0},$$

where $F_{i,0}(\varepsilon) \in \mathbb{R}^{1 \times n_{i,0}}$ is the unique matrix such that

$$\lambda(A_{i,0} + B_{i,0} F_{i,0}(\varepsilon)) = (1 - \varepsilon)\lambda(A_{i,0}) \in \mathbb{C}^{\odot} \tag{5.2}$$

with $\varepsilon \in (0, 1]$ being a design parameter. Note that $|F_{i,0}(\varepsilon)|$ tends to zero as ε tends to zero and such feedback is referred to as the low gain feedback [43].

Define a new output for each agent $i \in \{1, 2, \cdots, N\}$ as

$$\begin{aligned} \check{y}_i(k) &= y_i(k) - u_{i,0}(k) \\ &= x_{i,1}(k) - F_{i,0}(\varepsilon) x_{i,0}(k). \end{aligned}$$

Consensus protocols will be constructed later, under which the new output $\check{y}_i$ is driven to the optimal point y^*, and the real output y_i is allowed to vary within a small vicinity of y^*, whose size is specified by γ. In this way, the zero dynamics is stabilized and the suboptimal output consensus of the system is achieved under the designed protocols.

Define a set of new states $[\check{x}_{i,1} \ \check{x}_{i,2} \ \cdots \ \check{x}_{i,N}]^{\mathrm{T}}$ for each agent $i \in \{1, 2, \cdots, N\}$ based on the renamed output $\check{y}_i$ as

$$\begin{cases} \check{x}_{i,1}(k) = \check{y}_i(k) = x_{i,1}(k) - F_{i,0}(\varepsilon)x_{i,0}(k), \\ \check{x}_{i,r}(k) = x_{i,r}(k) - F_{i,0}(\varepsilon)A_{i,0}^{r-1}x_{i,0}(k) \\ \qquad\qquad - \displaystyle\sum_{l=1}^{r-1} F_{i,0}(\varepsilon)A_{i,0}^{r-1-l}B_{i,0}x_{i,l}(k),\ r = 2, 3, \cdots, \rho. \end{cases} \tag{5.3}$$

Then, the agent's dynamics (5.1) can be rewritten in these states as

$$\begin{cases} x_{i,0}(k+1) = A_{\mathrm{c}i,0}(\varepsilon)x_{i,0}(k) + B_{i,0}\check{x}_{i,1}(k), \\ \check{x}_{i,r}(k+1) = \check{x}_{i,r+1}(k),\ r \in I[1, \rho-1], \\ \check{x}_{i,\rho}(k+1) = \check{E}_{i,0}x_{i,0}(k) + \check{\alpha}_{i,1}\check{x}_{i,1}(k) + \check{\alpha}_{i,2}\check{x}_{i,2}(k) + \cdots \\ \qquad\qquad + \check{\alpha}_{i,\rho}\check{x}_{i,\rho}(k) + u_i(k), \\ \qquad \check{y}_i(k) = \check{x}_{i,1}(k), \end{cases} \tag{5.4}$$

where $A_{\mathrm{c}i,0}(\varepsilon) = A_{i,0} + B_{i,0}F_{i,0}(\varepsilon)$, and

$$\begin{aligned} \check{E}_{i,0} &= E_{i,0} - F_{i,0}(\varepsilon)A_{i,0}^{\rho} + \sum_{r=1}^{\rho}\big(\alpha_{i,r} - F_{i,0}(\varepsilon)A_{i,0}^{\rho-r}B_{i,0}\big)F_{i,0}(\varepsilon)A_{\mathrm{c}i,0}^{r-1}(\varepsilon), \\ \check{\alpha}_{i,r} &= \alpha_{i,r} - F_{i,0}(\varepsilon)A_{i,0}^{\rho-r}(\varepsilon)B_{i,0} \\ &\quad + \sum_{l=r+1}^{\rho}\big(\alpha_{i,l} - F_{i,0}(\varepsilon)A_{i,0}^{\rho-l}B_{i,0}\big)F_{i,0}(\varepsilon)A_{\mathrm{c}i,0}^{l-1-r}(\varepsilon)B_{i,0}, \\ &\quad r = 1, 2, \cdots, \rho-1, \\ \check{\alpha}_{i,\rho} &= \alpha_{i,\rho} - F_{i,0}(\varepsilon)B_{i,0}. \end{aligned}$$

Let

$$u_i(k) = \check{u}_i(k) - \check{E}_{i,0}x_{i,0}(k) - \check{\alpha}_{i,1}\check{x}_{i,1}(k) - \check{\alpha}_{i,2}\check{x}_{i,2}(k) - \cdots - \check{\alpha}_{i,\rho}\check{x}_{i,\rho}(k), \tag{5.5}$$

where $\check{u}_i(k)$ is to be designed later. Then, the dynamics (5.4) can be rewritten as

$$\begin{cases} x_{i,0}(k+1) = A_{\mathrm{c}i,0}(\varepsilon)x_{i,0}(k) + B_{i,0}\check{x}_{i,1}(k), \\ \check{x}_i(k+1) = \check{A}\check{x}_i(k) + \check{B}\check{u}_i(k), \\ \check{y}_i(k) = \check{x}_{i,1}(k), \end{cases}$$

where

$$\breve{A} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{bmatrix}_{\rho\times\rho}, \ \breve{B} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}_{\rho\times 1}.$$

We now define a state transformation $\breve{x}_i = \breve{T}\breve{x}_i$ for each agent $i \in \{1, 2, \cdots, N\}$ that transforms the agent dynamics into the following form,

$$\breve{x}_i(k+1) = \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 1 \\ -\mathrm{C}_\rho^\rho & -\mathrm{C}_\rho^{\rho-1} & -\mathrm{C}_\rho^{\rho-2} & \cdots & -\mathrm{C}_\rho^2 & -\mathrm{C}_\rho^1 + 1 \end{bmatrix} \breve{x}_i(k) + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \breve{u}_i(k), \tag{5.6}$$

where $\breve{x}_i = [\breve{x}_{i,1} \ \breve{x}_{i,2} \ \cdots \ \breve{x}_{i,\rho}]^{\mathrm{T}}$. Such a state transformation is explicitly given as

$$\breve{T} = \begin{bmatrix} \mathrm{C}_0^0 & 0 & \cdots & 0 & 0 \\ -\mathrm{C}_1^1 & \mathrm{C}_1^0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ (-1)^{\rho-2}\mathrm{C}_{\rho-2}^{\rho-2} & (-1)^{\rho-3}\mathrm{C}_{\rho-2}^{\rho-3} & \cdots & \mathrm{C}_{\rho-2}^0 & 0 \\ (-1)^{\rho-1}\mathrm{C}_{\rho-1}^{\rho-1} & (-1)^{\rho-2}\mathrm{C}_{\rho-1}^{\rho-2} & \cdots & -\mathrm{C}_{\rho-1}^{\rho-2} & \mathrm{C}_{\rho-1}^0 \end{bmatrix}.$$

We further define a state transformation $\bar{x}_i = \bar{T}\breve{x}_i$ that transforms the dynamics (5.6) into the following form,

$$\bar{x}_i(k+1) = \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ 0 & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \bar{x}_i(k) + \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{bmatrix} \bar{u}_i(k),$$

where $\bar{x}_i = [\bar{x}_{i,1} \ \bar{x}_{i,2} \ \cdots \ \bar{x}_{i,\rho}]^{\mathrm{T}}$ and

$$\bar{u}_i = [-\mathrm{C}_\rho^\rho \ -\mathrm{C}_\rho^{\rho-1} \ \cdots \ -\mathrm{C}_\rho^1]\breve{x}_i + \breve{u}_i. \tag{5.7}$$

Such a state transformation is explicitly given as

$$\bar{T} = \begin{bmatrix} \mathrm{C}_{\rho-1}^{\rho-1} & \mathrm{C}_{\rho-1}^{\rho-2} & \cdots & \mathrm{C}_{\rho-1}^{0} \\ 0 & \mathrm{C}_{\rho-2}^{\rho-2} & \cdots & \mathrm{C}_{\rho-2}^{0} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathrm{C}_{0}^{0} \end{bmatrix}.$$

As will be seen later, the above state transformation will facilitate the protocol design and the analysis of the closed-loop system.

We now construct $\bar{u}_i$, $i \in \{1, 2, \cdots, N\}$, based on the transformed state $\bar{x}_i$ as follows,

$$\begin{cases} v_i(k+1) = v_i(k) + \alpha\beta \displaystyle\sum_{j=1}^{N} a_{ij}\big(\bar{x}_{i,1}(k) - \bar{x}_{j,1}(k)\big), \ v_i(0) = 0, \\ \quad \bar{u}_i(k) = -\displaystyle\sum_{r=2}^{\rho} \bar{x}_{i,r}(k) - v_i(k) - \beta \sum_{j=1}^{N} a_{ij}\big(\bar{x}_{i,1}(k) - \bar{x}_{j,1}(k)\big) \\ \qquad\qquad - \alpha \nabla f_i\big(\bar{x}_{i,1}(k)\big), \end{cases} \tag{5.8}$$

where $\bar{x}_i = \bar{T}\breve{T}\breve{x}_i$ with $\breve{x}_i$ being defined in (5.3).

The closed-loop system, under the protocol (5.8), is given as

$$\begin{cases} x_{i,0}(k+1) = A_{\mathrm{c}i,0}(\varepsilon)x_{i,0}(k) + B_{i,0}\breve{x}_{i,1}(k), \\ \quad v_i(k+1) = v_i(k) + \alpha\beta \displaystyle\sum_{j=1}^{N} a_{ij}\big(\bar{x}_{i,1}(k) - \bar{x}_{j,1}(k)\big), \\ \bar{x}_{i,1}(k+1) = \bar{x}_{i,1}(k) - v_i(k) - \beta \displaystyle\sum_{j=1}^{N} a_{ij}\big(\bar{x}_{i,1}(k) - \bar{x}_{j,1}(k)\big) \\ \qquad\qquad - \alpha \nabla f_i\big(\bar{x}_{i,1}(k)\big), \\ \bar{x}_{i,2}(k+1) = -v_i(k) - \beta \displaystyle\sum_{j=1}^{N} a_{ij}\big(\bar{x}_{i,1}(k) - \bar{x}_{j,1}(k)\big) - \alpha \nabla f_i\big(\bar{x}_{i,1}(k)\big), \\ \bar{x}_{i,r}(k+1) = -\displaystyle\sum_{l=2}^{r-1} \bar{x}_{i,l}(k) - v_i(k) - \beta \sum_{j=1}^{N} a_{ij}\big(\bar{x}_{i,1}(k) - \bar{x}_{j,1}(k)\big) \\ \qquad\qquad - \alpha \nabla f_i\big(\bar{x}_{i,1}(k)\big), \ r = 3, 4, \cdots, \rho. \end{cases} \tag{5.9}$$

It is noted that the unstable $x_{i,0}$ dynamics is now stabilized and governed by $A_{\mathrm{c}i,0}$. The rest of the system can be viewed as two cascaded subsystems, namely, the v_i, $\bar{x}_{i,1}$ subsystem and the $\bar{x}_{i,2}, \bar{x}_{i,3}, \cdots, \bar{x}_{i,r}$ subsystem. As will be seen in the proof of Theorem 5.1, $\bar{x}_{i,1}$ will be driven to y^* and $\bar{x}_{i,2}, \bar{x}_{i,3}, \cdots, \bar{x}_{i,r}$ will converge to zero.

We have the following result on the solution of Problem 5.1.

Theorem 5.1. [60] Consider the multi-agent system (5.1). Let the agent dynamics satisfy Assumptions 5.1 and 5.2. Let the objective functions satisfy Assumption 5.3. Let the communication network satisfy Assumption 5.4. For any given $\gamma > 0$, there exists $\varepsilon^* \in (0, 1]$ such that, for all $\varepsilon \in (0, \varepsilon^*]$, the proposed distributed consensus protocols (5.5), (5.7) and (5.8) solve Problem 5.1.

Remark 5.2. Compared to the result in [90] for suboptimal output consensus of weakly nonminimum phase linear systems in the continuous-time setting, our result focuses on discrete-time weakly nonminimum phase agents and thus a different protocol design needs to be carried out and a new analytical tool needs to be established. Compared to [96], in which the result is for discrete-time first-order dynamics, our result is applicable to more general discrete-time dynamics.

Remark 5.3. The proposed suboptimal consensus protocol for each agent i, $i \in \{1, 2, \cdots, N\}$, is based on its own state, the gradient of its own objective function, and the states of its neighbors. A standard Luenberger observer can be employed as long as system (5.1) representing the agent dynamics is observable, i.e., the pair $(A_{i,0}, E_{i,0})$, $i \in \{1, 2, \cdots, N\}$, is observable, and the protocol design will remain the same. We only present the state feedback design here for brevity.

To prove Theorem 5.1, we first present the following lemma.

Lemma 5.1. Consider the pair $(A_{i,0}, B_{i,0})$ as given in (5.2) and $F_{i,0}(\varepsilon)$ as given in (5.2). Let $\lambda_{\mathrm{c}i,l}$, $l \in \{1, 2 \cdots, m_i\}$, be the eigenvalues of $A_{\mathrm{c}i,0} = A_{i,0} + B_{i,0}F_{i,0}(\varepsilon)$ with multiplicity $n_{i,l}$, i.e., $\det(zI - A_{\mathrm{c}i,0}) = \prod_{l=1}^{m_i}(z - \lambda_{\mathrm{c}i,l})^{n_{i,l}}$. Then, there exists an $\varepsilon^* \in (0, \frac{1}{2}]$ such that, for all $\varepsilon \in (0, \varepsilon^*]$,

$$\begin{aligned}
&\left|F_{i,0}(\varepsilon)\big(zI - A_{i,0} - B_{i,0}F_{i,0}(\varepsilon)\big)^{-1}\right| \\
&\leq \delta_i\varepsilon \sum_{l=1}^{m_i}\sum_{j=1}^{n_{i,l}}(n_{i,l} - j + 1)\left|\frac{1}{(z - \lambda_{\mathrm{c}i,l})^j}\right|, \ z \in \mathbb{C},
\end{aligned}$$

where δ_i is some positive constant independent of ε.

Proof of Lemma 5.1: Recall that the state transition matrix $\Phi_i(k)$ of the zero dynamics

$$x_{i,0}(k) = A_{\mathrm{c}i,0}x_{i,0}(k) + B_{i,0}\check{y}_i(k)$$

is given as

$$\begin{aligned}\Phi_i(k) &= (A_{i,0} + B_{i,0}F_{i,0}(\varepsilon))^k \\ &= \mathcal{Z}^{-1}\{z(zI - A_{i,0} - B_{i,0}F_{i,0}(\varepsilon))^{-1}\}.\end{aligned}$$

From the above equation, we have,

$$(zI - A_{i,0} - B_{i,0}F_{i,0}(\varepsilon))^{-1} = \frac{1}{z}\mathcal{Z}\{\Phi_i(k)\}.$$

Let $P_i = [p_1^1\ p_2^1\ \cdots\ p_{n_{i,1}}^1\ \cdots\ p_1^{m_i}\ p_2^{m_i}\ \cdots\ p_{n_{i,m_i}}^{m_i}]$, where, for each $l = 1$ to m_i, $p_1^l, p_2^l, \cdots, p_{n_{i,l}}^l$ are the $n_{i,l}$ generalized eigenvectors of $A_{i,0}$. Let $Q_i(\varepsilon) = [q_1^1\ q_2^1\ \cdots\ q_{n_{i,1}}^1\ \cdots\ q_1^{m_i}\ q_2^{m_i}\ \cdots\ q_{n_{i,m_i}}^{m_i}]$, where, for each $l = 1$ to m_i, $q_1^l, q_2^l, \cdots, q_{n_{i,l}}^l$ are the $n_{i,l}$ generalized eigenvectors of $A_{\mathrm{c}i,0}$. Then, there exists an $\varepsilon^* \in (0, \frac{1}{2}]$ such that, for all $\varepsilon \in (0, \varepsilon^*]$ ([43]),

$$|Q_i(\varepsilon)| \leq |P_i| + 1,\ \left|Q_i^{-1}(\varepsilon)\right| \leq \left|P_i^{-1}\right| + 1.$$

In addition,

$$Q_i^{-1}(\varepsilon)\big(A_{i,0} + B_{i,0}F_{i,0}(\varepsilon)\big)Q_i(\varepsilon) = \begin{bmatrix} J_{i,1} & 0 & \cdots & 0 \\ 0 & J_{i,2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_{i,m_i} \end{bmatrix},$$

where, for $l = 1$ to m_i,

$$J_{i,l} = \begin{bmatrix} \lambda_{\mathrm{c}i,l} & 1 & 0 & \cdots & 0 \\ 0 & \lambda_{\mathrm{c}i,l} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_{\mathrm{c}i,l} \end{bmatrix}.$$

Then, we have

$$\begin{aligned}&\left|F_{i,0}(\varepsilon)\big(zI - A_{i,0} - B_{i,0}F_{i,0}(\varepsilon)\big)^{-1}\right| \\ &= \left|\frac{1}{z}F_{i,0}(\varepsilon)\mathcal{Z}\big\{(A_{i,0} + B_{i,0}F_{i,0}(\varepsilon))^k\big\}\right| \\ &= \left|\frac{1}{z}F_{i,0}(\varepsilon)\mathcal{Z}\Big\{Q_i(\varepsilon)\big(Q_i^{-1}(\varepsilon)(A_{i,0} + B_{i,0}F_{i,0}(\varepsilon))Q_i(\varepsilon)\big)^k Q_i^{-1}(\varepsilon)\Big\}\right| \\ &\leq \big(|P_i| + 1\big)\big(|P_i^{-1}| + 1\big)\sum_{l=1}^{m_i}\left|F_{i,0}(\varepsilon)\right|\left|\frac{1}{z}\mathcal{Z}\big\{J_{i,l}^k\big\}\right|\end{aligned}$$

$$
\begin{aligned}
&\le \delta_i\varepsilon\sum_{l=1}^{m_i}\left|\frac{1}{z}\mathcal{Z}\left\{\begin{bmatrix}\lambda_{\mathrm{ci},l}^{k} & k\lambda_{\mathrm{ci},l}^{k-1} & \frac{k(k-1)}{2!}\lambda_{\mathrm{ci},l}^{k-2} & \cdots & \frac{k(k-1)\cdots(k-n_{i,l}+2)}{(n_{i,l}-1)!}\lambda_{\mathrm{ci},l}^{k-n_{i,l}+1}\\ 0 & \lambda_{\mathrm{ci},l}^{k} & k\lambda_{\mathrm{ci},l}^{k-1} & \cdots & \frac{k(k-1)\cdots(k-n_{i,l}+1)}{(n_{i,l}-2)!}\lambda_{\mathrm{ci},l}^{k-n_{i,l}+2}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & \lambda_{\mathrm{ci},l}^{k}\end{bmatrix}\right\}\right|\\
&= \delta_i\varepsilon\sum_{l=1}^{m_i}\sum_{j=1}^{n_{i,l}}(n_{i,l}-j+1)\left|\frac{1}{(z-\lambda_{\mathrm{ci},l})^j}\right|,
\end{aligned}
$$

for some $\delta_i > 0$ independent of ε, where we have used the fact that $|F_{i,0}(\varepsilon)| \le \alpha_i\varepsilon$ for some constant α_i independent of ε [43].

We next present the proof of the theorem.

Proof of Theorem 5.1: Denote $\boldsymbol{v} = [v_1\ v_2\ \cdots\ v_N]^{\mathrm{T}}$ and $\bar{\boldsymbol{x}}_r = [\bar{x}_{1,r}\ \bar{x}_{2,r}\ \cdots\ \bar{x}_{N,r}]^{\mathrm{T}}$, $r \in I[1,\rho]$. Correspondingly, denote the equilibrium point of the closed-loop system (5.9) as $\boldsymbol{v}^{\mathrm{e}}$, $\bar{\boldsymbol{x}}_1^{\mathrm{e}}$, $\bar{\boldsymbol{x}}_2^{\mathrm{e}}$, $\cdots$, $\bar{\boldsymbol{x}}_\rho^{\mathrm{e}}$. Denote $\nabla F(\bar{\boldsymbol{x}}_1) = [\nabla f_1(\bar{x}_{1,1})\ \nabla f_2(\bar{x}_{2,1})\ \cdots\ \nabla f_N(\bar{x}_{N,1})]^{\mathrm{T}}$. Partition the states as $\chi_1 = [\boldsymbol{v}^{\mathrm{T}}\ \bar{\boldsymbol{x}}_1^{\mathrm{T}}]^{\mathrm{T}}$ and $\chi_2 = [\bar{\boldsymbol{x}}_2^{\mathrm{T}}\ \bar{\boldsymbol{x}}_3^{\mathrm{T}}\ \cdots \bar{\boldsymbol{x}}_\rho^{\mathrm{T}}]^{\mathrm{T}}$, and the closed-loop system (5.9) without the zero dynamics takes the following cascade form,

$$
\begin{cases}\chi_1(k+1) = g_1(\chi_1(k)),\\ \chi_2(k+1) = A_2\chi_2(k) + B_2 g_2(\chi_1(k)),\end{cases} \tag{5.10}
$$

where the χ_1 dynamics is given as

$$
\boldsymbol{v}(k+1) = \boldsymbol{v}(k) + \alpha\beta\boldsymbol{\mathcal{L}}\bar{\boldsymbol{x}}_1(k), \tag{5.11}
$$

$$
\bar{\boldsymbol{x}}_1(k+1) = \bar{\boldsymbol{x}}_1(k) - \boldsymbol{v}(k) - \beta\boldsymbol{\mathcal{L}}\bar{\boldsymbol{x}}_1(k) - \alpha\nabla F(\bar{\boldsymbol{x}}_1(k)), \tag{5.12}
$$

and the χ_2 dynamics is described by

$$
A_2 = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0\\ -I_N & 0 & \cdots & 0 & 0\\ -I_N & -I_N & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ -I_N & -I_N & \cdots & -I_N & 0\end{bmatrix},\ B_2 = \begin{bmatrix} I_N\\ I_N\\ \vdots\\ I_N\end{bmatrix},
$$

$$
g_2(\chi_1) = -\bar{\boldsymbol{v}} - \beta\boldsymbol{\mathcal{L}}\bar{\boldsymbol{x}}_1 - \alpha\nabla F(\bar{\boldsymbol{x}}_1).
$$

In what follows, we will prove the result of Theorem 5.1 in three steps. First, we will show that the equilibrium point $\bar{\boldsymbol{x}}_1^{\mathrm{e}}$ takes the form $\bar{\boldsymbol{x}}_1^{\mathrm{e}} = x^{\mathrm{c}}\mathbf{1}$, where x^{c} is some scalar that minimizes the overall objective function. Second, we will show that the closed-loop system is globally asymptotically stable at the equilibrium point $\boldsymbol{v}^{\mathrm{e}}$, $\bar{\boldsymbol{x}}_1^{\mathrm{e}}$, $\bar{\boldsymbol{x}}_2^{\mathrm{e}}$, $\cdots$, $\bar{\boldsymbol{x}}_\rho^{\mathrm{e}}$, where $[\bar{\boldsymbol{x}}_1^{\mathrm{e}}\ \bar{\boldsymbol{x}}_2^{\mathrm{e}}\ \cdots\ \bar{\boldsymbol{x}}_\rho^{\mathrm{e}}] = [x^{\mathrm{c}}\mathbf{1}\ 0\ \cdots\ 0]$. Finally, we will show that the proposed protocols solve Problem 5.1.

Step 1: From (5.11) and (5.12), we see that the equilibrium point must satisfy

$$\alpha\beta\mathcal{L}\bar{\boldsymbol{x}}_1^{\mathrm{e}} = 0, \tag{5.13}$$

$$-\boldsymbol{v}^{\mathrm{e}} - \beta\mathcal{L}\bar{\boldsymbol{x}}_1^{\mathrm{e}} - \alpha\nabla F(\bar{\boldsymbol{x}}_1^{\mathrm{e}}) = 0. \tag{5.14}$$

Under Assumption 5.4, equation (5.13) implies that $\bar{\boldsymbol{x}}_1^{\mathrm{e}} = x^{\mathrm{c}}\mathbf{1}$ for some constant x^{c}.

Left multiplying (5.11) by $\mathbf{1}^{\mathrm{T}}$ gives

$$\mathbf{1}^{\mathrm{T}}\boldsymbol{v}(k+1) = \mathbf{1}^{\mathrm{T}}\boldsymbol{v}(k) + \alpha\beta\mathbf{1}^{\mathrm{T}}\mathcal{L}\bar{\boldsymbol{x}}_1(k),$$

which, together with $\mathbf{1}^{\mathrm{T}}\mathcal{L} = 0$, implies that the term $\mathbf{1}^{\mathrm{T}}\boldsymbol{v}(k)$ remains constant for all k. Therefore,

$$\mathbf{1}^{\mathrm{T}}\boldsymbol{v}^{\mathrm{e}} = \mathbf{1}^{\mathrm{T}}\boldsymbol{v}(0) = \sum_{i=1}^{N} v_i(0) = 0,\ \forall k = 0, 1, \cdots .$$

By left multiplying (5.14) by $\mathbf{1}^{\mathrm{T}}$, we have,

$$-\mathbf{1}^{\mathrm{T}}\boldsymbol{v}^{\mathrm{e}} - \beta\mathbf{1}^{\mathrm{T}}\mathcal{L}\bar{\boldsymbol{x}}_1^{\mathrm{e}} - \alpha\mathbf{1}^{\mathrm{T}}\nabla F(\bar{\boldsymbol{x}}_1^{\mathrm{e}}) = 0,$$

from which we can conclude that

$$\mathbf{1}^{\mathrm{T}}\nabla F(\bar{\boldsymbol{x}}_1^{\mathrm{e}}) = \sum_{i=1}^{N} \nabla f_i(x^{\mathrm{c}}) = 0,$$

i.e., $y^* = x^{\mathrm{c}}$ is the unique optimal point that minimizes the overall objective function $f(y) = \sum_{i=1}^{N} f_i(y)$.

Step 2: Define error terms as $\tilde{\boldsymbol{x}}_1(k) = \bar{\boldsymbol{x}}_1(k) - \bar{\boldsymbol{x}}_1^{\mathrm{e}}$ and $\tilde{\boldsymbol{v}}(k) = \boldsymbol{v}(k) - \boldsymbol{v}^{\mathrm{e}}$. Then, the dynamics of $\tilde{\boldsymbol{x}}_1(k)$ and $\tilde{\boldsymbol{v}}(k)$ is given as

$$\begin{cases} \tilde{\boldsymbol{x}}_1(k+1) = \tilde{\boldsymbol{x}}_1(k) - \tilde{\boldsymbol{v}}(k) - \beta\mathcal{L}\tilde{\boldsymbol{x}}_1(k) \\ \qquad\qquad - \alpha\big(\nabla F(\tilde{\boldsymbol{x}}_1(k) + \bar{\boldsymbol{x}}_1^{\mathrm{e}}) - \nabla F(\bar{\boldsymbol{x}}_1^{\mathrm{e}})\big), \\ \tilde{\boldsymbol{v}}(k+1) = \tilde{\boldsymbol{v}}(k) + \alpha\beta\mathcal{L}\tilde{\boldsymbol{x}}_1(k). \end{cases} \tag{5.15}$$

We will show that the origin of the system (5.15) is globally asymptotically stable in the subspace $\{[\boldsymbol{v}^{\mathrm{T}}\ \bar{\boldsymbol{x}}_1^{\mathrm{T}}]^{\mathrm{T}}:\ \mathbf{1}^{\mathrm{T}}\tilde{\boldsymbol{v}}=0\}$.

To this end, consider the Lyapunov function candidate

$$V=\tilde{\boldsymbol{\eta}}^{\mathrm{T}}\tilde{\boldsymbol{\eta}}+2\alpha\beta\tilde{\boldsymbol{\eta}}^{\mathrm{T}}\mathcal{L}\tilde{\boldsymbol{x}}_1+\alpha\beta\tilde{\boldsymbol{x}}_1^{\mathrm{T}}\mathcal{L}\tilde{\boldsymbol{x}}_1,$$

where

$$\tilde{\boldsymbol{\eta}}(k):=-\tilde{\boldsymbol{v}}(k)-\beta\mathcal{L}\tilde{\boldsymbol{x}}_1(k)-\alpha\big(\nabla F(\tilde{\boldsymbol{x}}_1(k)+\bar{\boldsymbol{x}}_1^{\mathrm{e}})-\nabla F(\bar{\boldsymbol{x}}_1^{\mathrm{e}})\big).$$

Note that V can be rewritten as

$$V=\begin{bmatrix}\tilde{\boldsymbol{\eta}}^{\mathrm{T}} & \tilde{\boldsymbol{x}}_1^{\mathrm{T}}\mathcal{L}\end{bmatrix}\begin{bmatrix}1 & \alpha\beta\\ \alpha\beta & \kappa\end{bmatrix}\begin{bmatrix}\tilde{\boldsymbol{\eta}}\\ \mathcal{L}\tilde{\boldsymbol{x}}_1\end{bmatrix}+\tilde{\boldsymbol{x}}_1^{\mathrm{T}}\big(\alpha\beta\mathcal{L}-\kappa\mathcal{L}^2\big)\tilde{\boldsymbol{x}}_1. \tag{5.16}$$

It is observed that the first term in (5.16) is non-negative and is equal to zero if and only if $\tilde{\boldsymbol{\eta}}=0$ and $\mathcal{L}\tilde{\boldsymbol{x}}_1=0$ when $\kappa>\alpha^2\beta^2$. It is also observed that the second term is non-negative and is equal to zero if and only if $\mathcal{L}\tilde{\boldsymbol{x}}_1=0$ when $\kappa<\frac{\alpha\beta}{\lambda_{\max}(\mathcal{L})}$. Therefore, $V\geq 0$ and is radially unbounded in the subspace $\{[\boldsymbol{v}^{\mathrm{T}}\ \bar{\boldsymbol{x}}_1^{\mathrm{T}}]^{\mathrm{T}}:\ \mathbf{1}^{\mathrm{T}}\tilde{\boldsymbol{v}}=0\}$ if κ, α and β satisfy the above conditions.

We will next show that $V>0$, that is, $V=0$ if and only if $\tilde{\boldsymbol{x}}_1=0$ and $\tilde{\boldsymbol{v}}=0$. It is noted that $V=0$ if and only if $\mathcal{L}\tilde{\boldsymbol{x}}_1=0$ and $\tilde{\boldsymbol{\eta}}=0$. By recalling the definition of $\tilde{\boldsymbol{\eta}}$, we have,

$$-\tilde{\boldsymbol{v}}(k)-\alpha\big(\nabla F(\tilde{\boldsymbol{x}}_1(k)+\bar{\boldsymbol{x}}_1^{\mathrm{e}})-\nabla F(\bar{\boldsymbol{x}}_1^{\mathrm{e}})\big)=0. \tag{5.17}$$

By left multiplying (5.17) with $\mathbf{1}^{\mathrm{T}}$ and recalling $\mathbf{1}^{\mathrm{T}}\tilde{\boldsymbol{v}}(k)=0$, $\forall k=0,1,\cdots$, and $\mathbf{1}^{\mathrm{T}}\nabla F(\bar{\boldsymbol{x}}_1^{\mathrm{e}})=0$, we have,

$$\mathbf{1}^{\mathrm{T}}\nabla F(\tilde{\boldsymbol{x}}_1(k)+\bar{\boldsymbol{x}}_1^{\mathrm{e}})=0. \tag{5.18}$$

Since $\bar{\boldsymbol{x}}_1^{\mathrm{e}}=x^{\mathrm{c}}\mathbf{1}$ with x^{c} being the unique optimal point such that $\sum_{i=1}^{N}\nabla f_i(x^{\mathrm{c}})=0$, it follows from (5.18) that $\tilde{\boldsymbol{x}}_1(k)=0$ if $V=0$. Since the second term in (5.17) is equal to zero as $\tilde{\boldsymbol{x}}_1(k)=0$, we conclude that $\tilde{\boldsymbol{v}}(k)=0$ if $V=0$.

We will now consider

$$\Delta V\big(\tilde{\boldsymbol{x}}_1(k),\tilde{\boldsymbol{v}}(k)\big):=V\big(\tilde{\boldsymbol{x}}_1(k+1),\tilde{\boldsymbol{v}}(k+1)\big)-V\big(\tilde{\boldsymbol{x}}_1(k),\tilde{\boldsymbol{v}}(k)\big).$$

Denote $V(k+1) := V\big(\tilde{\boldsymbol{x}}_1(k+1), \tilde{\boldsymbol{v}}(k+1)\big)$ and it is computed as

$$V(k+1) = \tilde{\boldsymbol{\eta}}^{\mathrm{T}}(k+1)\tilde{\boldsymbol{\eta}}(k+1) + 2\alpha\beta\tilde{\boldsymbol{\eta}}^{\mathrm{T}}(k+1)\boldsymbol{\mathcal{L}}\tilde{\boldsymbol{x}}_1(k+1) + \alpha\beta\tilde{\boldsymbol{x}}_1^{\mathrm{T}}(k+1)\boldsymbol{\mathcal{L}}\tilde{\boldsymbol{x}}_1(k+1),$$

with

$$\begin{aligned}
\tilde{\boldsymbol{\eta}}(k+1) &= -\tilde{\boldsymbol{v}}(k+1) - \alpha\big(\nabla F(\tilde{\boldsymbol{x}}_1(k+1) + \bar{\boldsymbol{x}}_1^{\mathrm{e}}) - \nabla F(\bar{\boldsymbol{x}}_1^{\mathrm{e}})\big) - \beta\boldsymbol{\mathcal{L}}\tilde{\boldsymbol{x}}_1(k+1) \\
&= -\big(\tilde{\boldsymbol{v}}(k) + \alpha\beta\boldsymbol{\mathcal{L}}\tilde{\boldsymbol{x}}_1(k)\big) - \alpha\big(\nabla F(\tilde{\boldsymbol{x}}_1(k) + \tilde{\boldsymbol{\eta}}(k) + \bar{\boldsymbol{x}}_1^{\mathrm{e}}) - \nabla F(\bar{\boldsymbol{x}}_1^{\mathrm{e}})\big) \\
&\quad - \beta\boldsymbol{\mathcal{L}}(\tilde{\boldsymbol{x}}_1(k) + \tilde{\boldsymbol{\eta}}(k)) \\
&= \tilde{\boldsymbol{\eta}} - \alpha\beta\boldsymbol{\mathcal{L}}\tilde{\boldsymbol{x}}_1 - \alpha\Delta F - \beta\boldsymbol{\mathcal{L}}\tilde{\boldsymbol{\eta}},
\end{aligned}$$

where $\Delta F := \nabla F(\tilde{\boldsymbol{x}}_1(k+1) + \bar{\boldsymbol{x}}_1^{\mathrm{e}}) - \nabla F(\tilde{\boldsymbol{x}}_1(k) + \bar{\boldsymbol{x}}_1^{\mathrm{e}})$.

Let us now compute

$$\begin{aligned}
&\Delta V\big(\tilde{\boldsymbol{x}}_1(k), \tilde{\boldsymbol{v}}(k)\big) \\
&= \tilde{\boldsymbol{\eta}}^{\mathrm{T}}(k+1)\tilde{\boldsymbol{\eta}}(k+1) - \tilde{\boldsymbol{\eta}}^{\mathrm{T}}(k)\tilde{\boldsymbol{\eta}}(k) + 2\alpha\beta\tilde{\boldsymbol{\eta}}^{\mathrm{T}}(k+1)\boldsymbol{\mathcal{L}}\tilde{\boldsymbol{x}}_1(k) \\
&\quad + 2\alpha\beta\tilde{\boldsymbol{\eta}}^{\mathrm{T}}(k+1)\boldsymbol{\mathcal{L}}\tilde{\boldsymbol{\eta}}(k) + \alpha\beta\tilde{\boldsymbol{\eta}}^{\mathrm{T}}(k)\boldsymbol{\mathcal{L}}\tilde{\boldsymbol{\eta}}(k) \\
&= -3\alpha^2\beta^2(\boldsymbol{\mathcal{L}}\tilde{\boldsymbol{x}}_1)^{\mathrm{T}}(\boldsymbol{\mathcal{L}}\tilde{\boldsymbol{x}}_1) - (2\alpha-1)\beta^2(\boldsymbol{\mathcal{L}}\tilde{\boldsymbol{\eta}})^{\mathrm{T}}(\boldsymbol{\mathcal{L}}\tilde{\boldsymbol{\eta}}) - (2-3\alpha)\beta\tilde{\boldsymbol{\eta}}^{\mathrm{T}}\boldsymbol{\mathcal{L}}\tilde{\boldsymbol{\eta}} \\
&\quad + \alpha^2(\Delta F)^{\mathrm{T}}\Delta F - 4\alpha^2\beta^2(\boldsymbol{\mathcal{L}}\tilde{\boldsymbol{x}}_1)^{\mathrm{T}}(\boldsymbol{\mathcal{L}}\tilde{\boldsymbol{\eta}}) - 2\alpha\tilde{\boldsymbol{\eta}}^{\mathrm{T}}\Delta F + 2\alpha\beta(1-\alpha)(\Delta F)^{\mathrm{T}}\boldsymbol{\mathcal{L}}\tilde{\boldsymbol{\eta}}.
\end{aligned}$$

Under Assumption 5.3, we have

$$\begin{aligned}
(\Delta F)^{\mathrm{T}}\Delta F &= \sum_{i=1}^{N}\big(\nabla f_i(\tilde{x}_{i,1}(k+1) + \bar{x}_{i,1}^{\mathrm{e}}) - \nabla f_i(\tilde{x}_{i,1}(k) + \bar{x}_{i,1}^{\mathrm{e}})\big)^2 \\
&\le \sum_{i=1}^{n} M_i^2\big(\tilde{x}_{i,1}(k+1) - \tilde{x}_{i,1}(k)\big)^2 \\
&\le \bar{M}\tilde{\boldsymbol{\eta}}^{\mathrm{T}}\tilde{\boldsymbol{\eta}} \qquad (5.19)
\end{aligned}$$

where $\bar{M} = \max_{i\in\{1,2,\cdots,N\}}\{M_i^2\}$. We also have

$$\begin{aligned}
&\big(\tilde{x}_{i,1}(k+1) - \tilde{x}_{i,1}(k)\big)\big(\nabla f_i(\tilde{x}_{i,1}(k+1) + \bar{x}_{i,1}^{\mathrm{e}}) - \nabla f_i(\tilde{x}_{i,1}(k) + \bar{x}_{i,1}^{\mathrm{e}})\big) \\
&\ge m_i\big(\tilde{x}_{i,1}(k+1) - \tilde{x}_{i,1}(k)\big)^2,
\end{aligned}$$

i.e.,

$$\tilde{\boldsymbol{\eta}}^{\mathrm{T}}\Delta F \ge \underline{m}\tilde{\boldsymbol{\eta}}^{\mathrm{T}}\tilde{\boldsymbol{\eta}}, \qquad (5.20)$$

where $\underline{m} = \min_{i\in\{1,2,\cdots,N\}}\{m_i\}$.

In view of (5.19) and (5.20), ΔV can be further evaluated as

$$\begin{aligned}
&\Delta V(\tilde{\boldsymbol{x}}_1(k), \tilde{\boldsymbol{v}}(k)) \\
= &-3\alpha^2\beta^2(\boldsymbol{\mathcal{L}}\tilde{\boldsymbol{x}}_1)^{\mathrm{T}}(\boldsymbol{\mathcal{L}}\tilde{\boldsymbol{x}}_1) - (2\alpha-1)\beta^2(\boldsymbol{\mathcal{L}}\tilde{\boldsymbol{\eta}})^{\mathrm{T}}(\boldsymbol{\mathcal{L}}\tilde{\boldsymbol{\eta}}) - (2-3\alpha)\beta\tilde{\boldsymbol{\eta}}^{\mathrm{T}}\boldsymbol{\mathcal{L}}\tilde{\boldsymbol{\eta}} \\
&+ \alpha^2\Delta F^{\mathrm{T}}\Delta F - 2\big(\sqrt{2}\alpha\beta(\boldsymbol{\mathcal{L}}\tilde{\boldsymbol{x}}_1)^{\mathrm{T}}\big)\big(\sqrt{2}\alpha\beta(\boldsymbol{\mathcal{L}}\tilde{\boldsymbol{\eta}})\big) - 2\alpha\tilde{\boldsymbol{\eta}}\Delta F \\
&+ 2(\alpha\Delta F^{\mathrm{T}})\big((1-\alpha)\beta(\boldsymbol{\mathcal{L}}\tilde{\boldsymbol{\eta}})\big) \\
\leq &-\alpha^2\beta^2(\boldsymbol{\mathcal{L}}\tilde{\boldsymbol{x}}_1)^{\mathrm{T}}(\boldsymbol{\mathcal{L}}\tilde{\boldsymbol{x}}_1) - \big((2\alpha-1) - 2\alpha^2 - (1-\alpha)^2\big)\beta^2(\boldsymbol{\mathcal{L}}\tilde{\boldsymbol{\eta}})^{\mathrm{T}}(\boldsymbol{\mathcal{L}}\tilde{\boldsymbol{\eta}}) \\
&-(2-3\alpha)\beta\tilde{\boldsymbol{\eta}}^{\mathrm{T}}\boldsymbol{\mathcal{L}}\tilde{\boldsymbol{\eta}} - 2(\alpha\underline{m} - \alpha^2\bar{M})\tilde{\boldsymbol{\eta}}^{\mathrm{T}}\tilde{\boldsymbol{\eta}}.
\end{aligned}$$

By choosing $\alpha < \min\{\underline{m}/\bar{M}, 3/2\}$, and $\beta < \sqrt{\frac{2(\alpha\underline{m}-\alpha^2\bar{M})}{\phi\lambda_{\max}(\mathcal{L}^2)}}$, where $\phi = -((2\alpha-1)-2\alpha^2-(1-\alpha)^2) > 0$, we have, $\Delta V(\tilde{\boldsymbol{x}}_1(k), \tilde{\boldsymbol{v}}(k)) \leq 0$, and the equality holds if and only if $\boldsymbol{\mathcal{L}}\tilde{\boldsymbol{x}}_1(k) = 0$ and $\tilde{\boldsymbol{\eta}}(k) = 0$. By recalling the previous analysis, we can further conclude that the equality holds if and only if $\tilde{\boldsymbol{x}}_1(k) = 0$ and $\tilde{\boldsymbol{v}}(k) = 0$. Therefore, the origin of system (5.15) is globally asymptotically stable in the subspace $\{[\tilde{\boldsymbol{v}}^{\mathrm{T}}, \tilde{\boldsymbol{x}}_1^{\mathrm{T}}]^{\mathrm{T}} : \mathbf{1}^{\mathrm{T}}\tilde{\boldsymbol{v}} = 0\}$, i.e., the equilibrium point $\boldsymbol{v}^{\mathrm{e}}$, $\bar{\boldsymbol{x}}_1^{\mathrm{e}}$ of the system (5.11)-(5.12) is globally asymptotically stable in the subspace $\{[\boldsymbol{v}^{\mathrm{T}}\ \bar{\boldsymbol{x}}_1^{\mathrm{T}}]^{\mathrm{T}} :\ \mathbf{1}^{\mathrm{T}}\boldsymbol{v} = 0\}$. Since $g_2(\bar{\boldsymbol{x}}^{\mathrm{e}}, \boldsymbol{v}^{\mathrm{e}}) = 0$ by (5.14) and all eigenvalues of A_2 are inside the unit circle, the χ_2 subsystem in (5.10) is globally asymptotically stable at its equilibrium point $\bar{\boldsymbol{x}}_2^{\mathrm{e}} = \bar{\boldsymbol{x}}_3^{\mathrm{e}} = \cdots = \bar{\boldsymbol{x}}_\rho^{\mathrm{e}} = 0$.

Therefore, the cascade system (5.10) is globally asymptotically stable at its equilibrium point $\boldsymbol{v}^{\mathrm{e}}$, $\bar{\boldsymbol{x}}_1^{\mathrm{e}} = x^{\mathrm{c}}\mathbf{1}$, $\bar{\boldsymbol{x}}_2^{\mathrm{e}} = \bar{\boldsymbol{x}}_3^{\mathrm{e}} = \cdots = \bar{\boldsymbol{x}}_\rho^{\mathrm{e}} = 0$.

Step 3: By recalling the state transformations $\breve{x}_i = \breve{T}\check{x}_i$ and $\bar{x}_i = \bar{T}\breve{x}_i$, we have, $\check{x}_i = \breve{T}^{-1}\bar{T}^{-1}\bar{x}_i$. Since $\lim_{k\to\infty}\bar{x}_{i,1}(k) = x^{\mathrm{c}} = y^*$ and $\lim_{k\to\infty}\bar{x}_{i,r}(k) = 0$, for all $r = 2, 3, \cdots, \rho$ and $i \in \{1, 2, \cdots, N\}$, we have

$$\lim_{k\to\infty}\check{y}_i(k) = \lim_{k\to\infty}\check{x}_{i,1}(k) = x^{\mathrm{c}} = y^*,$$

i.e., the new output $\check{y}_i$ converges to the optimal point y^*.

We will now show that there exists $\varepsilon^* \in (0, 1]$ such that, for all $\varepsilon \in (0, \varepsilon^*]$,

$$\lim_{k\to\infty}\left|F_{i,0}(\varepsilon)x_{i,0}(k)\right| \leq \gamma,\ \ i \in \{1, 2, \cdots, N\}.$$

Recall that the zero dynamics $x_{i,0}$ in (5.4) can be written as

$$x_{i,0}(k+1) = A_{\mathrm{c}i,0}(\varepsilon)x_{i,0}(k) + B_{i,0}(\check{y}_i(k) - y^*) + B_{i,0}y^*,$$

with $A_{ci,0}$ being Hurwitz and $\lim_{k\to\infty} \check{y}_i(k) = y^*$. Thus, the steady state trajectory of $x_{i,0}$ is all due to y^*. Consider the stable transfer function from y^* to $u_{i,0} = F_{i,0}(\varepsilon)x_{i,0}$. By applying the final value theorem, we have,

$$|u_{i,0}|_{\text{ss}} = \lim_{z\to 1}\left|(z-1)\Big(F_{i,0}(\varepsilon)(zI - A_{i,0} + B_{i,0}F_{i,0}(\varepsilon))^{-1}B_{i,0}\Big)\Big(\frac{z}{z-1}y^*\Big)\right|$$

Under Assumption 5.1, Lemma 5.1 implies that,

$$|u_{i,0}|_{\text{ss}} \le \bar{\delta}_i\varepsilon,$$

for some positive constant $\bar{\delta}_i$ independent of ε.

By choosing $0 < \varepsilon^* \le \frac{\gamma}{\max_{i\in\{1,2,\cdots,N\}}\bar{\delta}_i}$, we have, for all $\varepsilon \in (0, \varepsilon^*]$,

$$\lim_{k\to\infty}\big|F_{i,0}(\varepsilon)x_{i,0}(k)\big| \le \gamma,\ \ i \in \{1,2,\cdots,N\},$$

i.e., for all $\varepsilon \in (0,\varepsilon^*]$,

$$\begin{aligned}\lim_{k\to\infty}\big|y_i(k) - y^*\big| &\le \lim_{k\to\infty}\big|y_i(k) - \check{y}_i(k)\big| + \lim_{k\to\infty}\big|\check{y}_i(k) - y^*\big| \\ &= \lim_{k\to\infty}\big|F_{i,0}(\varepsilon)x_{i,0}(k)\big| + 0 \\ &\le \gamma.\end{aligned}$$

This completes the proof of Theorem 5.1. □

5.4 Simulation

The simulation is performed with a multi-agent system consisting of three agents. The communication network is shown in Fig 5.1.

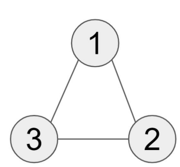

Figure 5.1: The communication network.

The agent dynamics is described by

$$\begin{cases} x_{i,0}(k+1) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 2\sqrt{2} & -4 & 2\sqrt{2} \end{bmatrix} x_{i,0}(k) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} x_{i,1}(k), \\ x_{i,1}(k+1) = x_{i,2}(k), \\ x_{i,2}(k+1) = -x_{i,1}(k) + 2x_{i,2}(k) + u_i(k), \\ \qquad y_i(k) = x_{i,1}(k), \;\; i \in \{1,2,3\}. \end{cases}$$

The zero dynamics is polynomially unstable with poles at $\{\frac{\sqrt{2}}{2}+j\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}+j\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}-j\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}-j\frac{\sqrt{2}}{2}\}$. The low gain feedback gain matrix is constructed for each agent i, $i \in \{1,2,3\}$, as

$$F_{i,0}(\varepsilon) = \begin{bmatrix} -\varepsilon^4+4\varepsilon^3-6\varepsilon^2+4\varepsilon & -2\sqrt{2}\varepsilon^3+6\sqrt{2}\varepsilon^2-6\sqrt{2}\varepsilon & -4\varepsilon^2+8\varepsilon & -2\sqrt{2}\varepsilon \end{bmatrix},$$

where $\varepsilon \in (0,1]$ is the design parameter. The individual objective functions are $f_1(y_1) = (y_1-1)^2$, $f_2(y_2) = y_2^2$, and $f_3(y_3) = (y_3-5)^2$, respectively, with $M_1 = M_2 = M_3 = 2$, and $m_1 = m_2 = m_3 = 2$. The optimal point is $y^* = 2$. The suboptimal consensus protocols are constructed according to (5.5), (5.7) and (5.8).

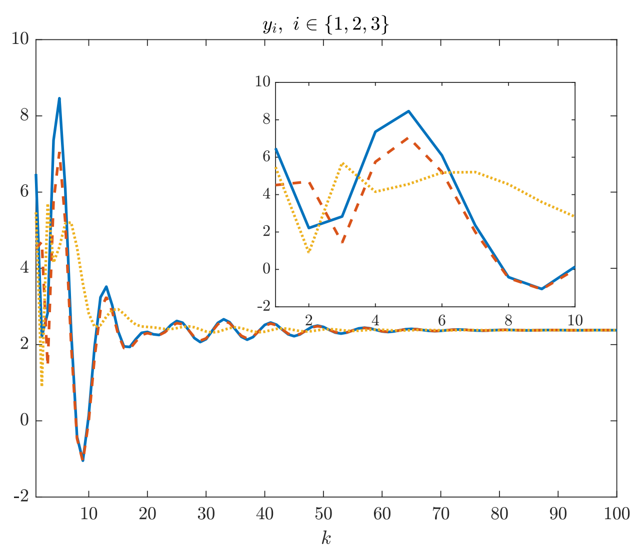

Figure 5.2: The outputs of all agents: $\varepsilon = 0.1$.

Shown in Figs. 5.2 and 5.3 are the outputs of all agents and the evolution of their zero dynamics,

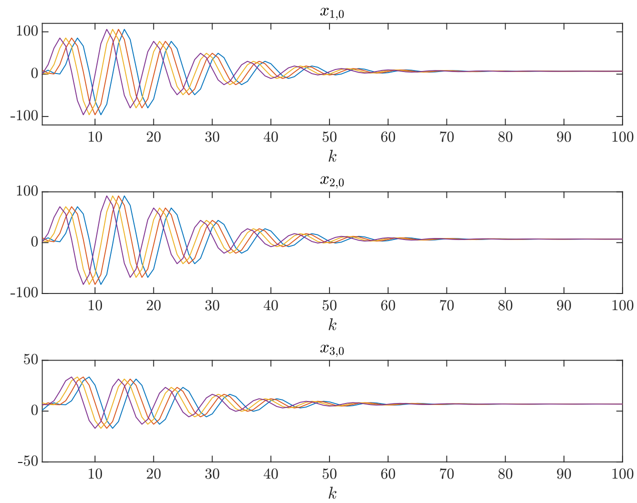

Figure 5.3: The evolution of $x_{i,0}$, $i \in \{1, 2, 3\}$: $\varepsilon = 0.1$.

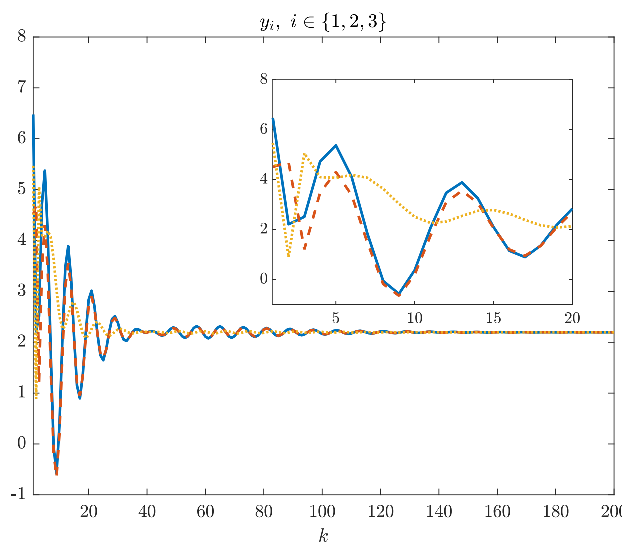

Figure 5.4: The outputs of all agents: $\varepsilon = 0.05$.

respectively, with the design parameter $\varepsilon = 0.1$. It is observed that the outputs of all agents satisfy $\lim_{k\to\infty} |y_i(k) - y^*| \leq 0.38$, $i \in \{1, 2, 3\}$, and all their states remain bounded.

Shown in Figs. 5.4 and 5.5, Figs. 5.6 and 5.6 are the simulation results with the design parameter $\varepsilon = 0.05$ and $\varepsilon = 0.01$, respectively. It is observed that $\lim_{k\to\infty} |y_i(k) - y^*| \leq 0.20$ when $\varepsilon = 0.05$, and $\lim_{k\to\infty} |y_i(k) - y^*| \leq 0.04$ when $\varepsilon = 0.01$, and the states remain bounded.

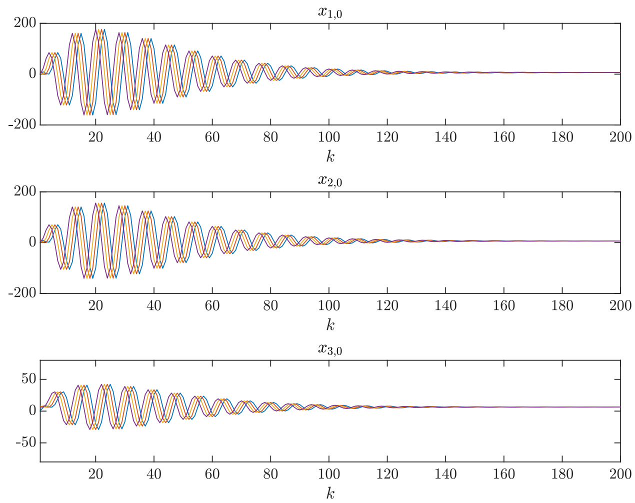

Figure 5.5: The evolution of $x_{i,0}$, $i \in \{1, 2, 3\}$: $\varepsilon = 0.05$.

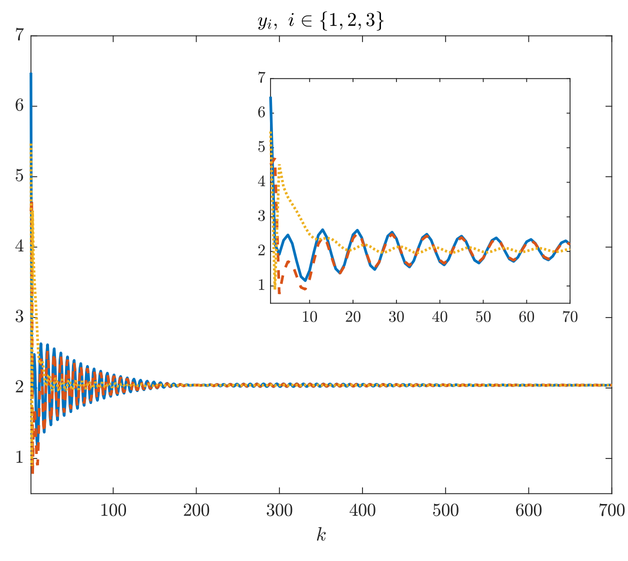

Figure 5.6: The outputs of all agents: $\varepsilon = 0.01$.

As shown in the simulation, the output of all agents converges closer to the optimal point as the value of the design parameter ε decrease, which demonstrates the effectiveness of the proposed protocols.

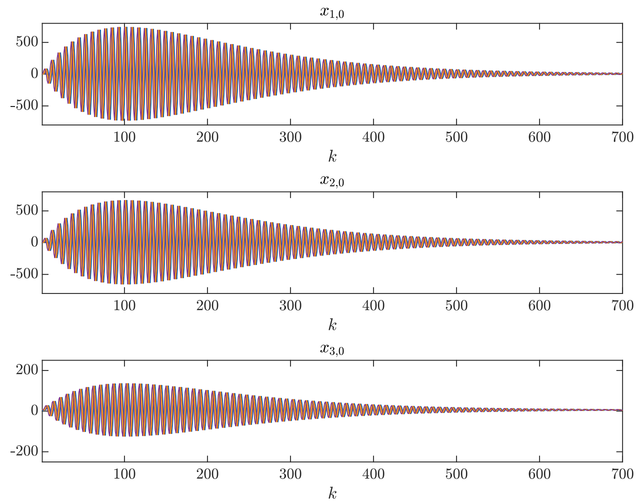

Figure 5.7: The evolution of $x_{i,0}$, $i \in \{1, 2, 3\}$: $\varepsilon = 0.01$.

5.5 Conclusions

In this chapter, the suboptimal output consensus problem of discrete-time linear multi-agent systems with unstable zero dynamics was studied, where each agent possesses a private objective function. A low gain feedback based parameterized consensus protocol is proposed for each agent utilizing its own objective function and the states of its neighbors. Suboptimal output consensus of the system is shown to be achieved when the design parameter is chosen properly and the communication topology is undirected and connected. The simulation was carried out to validate the established results.

This chapter is based on the following publication:

- Tingyang Meng, Yijing Xie, and Zongli Lin, "Suboptimal output consensus of a group of discrete-time heterogeneous linear non-minimum phase systems." *Systems & Control Letters* 161 (2022): 105134.

Chapter 6

Management of Networked Battery Units

6.1 Introduction

Energy storage systems are essential components in microgrids [6]. They not only ensure power quality and reliability but also reduce energy loss in microgrids. Among various energy storage technologies, battery energy storage systems (BESSs) have emerged as an appealing technology due to their versatility, rapid response, high energy density, and efficiency [38]. By absorbing power from the grid during the off-peak time and supplying power to the grid in peak time, BESSs enable the grid to have the ability of peak-shaving/shifting, power quality enhancement, and congestion relief [24]. As a result, BESSs of various types are increasingly being integrated into modern energy systems [20, 17]. Despite the technological advancements in electrochemistry, management/control of BESSs remains a challenging problem [70].

In general, a BESS may be composed of multiple battery units. Each unit monitors its own state and controls its own charging/discharging power while communicating with nearby units. The fundamental control objective of a BESS is to satisfy the charging/discharging power desired by the grid while balancing the state-of-charge (SoC) of all its units. Due to variations in their manufacturing process and in their operating conditions, battery units may exhibit different characteristics even if they have the same nominal specifications. As a result, the SoC of all battery units may diverge with the same charging/discharging power. The divergence of their SoC can eventually lead to overcharge/overdischarge of the units, which significantly reduces not only the efficiency of the system but also the lifetime of battery units, and may even cause dangerous situations. Therefore, the design of coordinated control of charging/discharging power for the units in a BESS with SoC balancing has been an active research problem (see, for example, [1, 62, 94, 91, 92]).

The control scheme of a BESS consisting of networked battery units can be either centralized or distributed. A centralized controller monitors all battery units' SoC and other critical states, and coordinates the charging/discharging power of each individual unit by external balancing circuits [2]. Such centralized control schemes for large-scale systems are costly to implement and can introduce single-point failures. In addition, the balancing circuits may introduce circulating currents that cause energy loss. Distributed control schemes, on the other hand, provide advantages such as robustness and reconfigurability. Models such as multi-agent systems match the networked structure of battery units in BESSs. Thus, the control problem of a BESS can be reformulated in the framework of multi-agent systems (see, for example, [68, 69, 16, 71, 41, 15, 9]), in which the battery units are regarded as a group of locally interacting agents. Distributed controllers and estimators can be designed by using local information and communication among neighbors so that the desired power is satisfied while SoC balancing of all units is achieved.

State-of-charge (SoC)[74] balancing control of BESSs has recently attracted considerable research interest. For example, in [27], an energy-sharing controller was proposed based on a redesigned DC–DC power stage to achieve SoC balancing among the battery cells while providing DC bus voltage regulation. In [1], a distributed SoC balancing control scheme based on the local reference power calculated from individual energy coordinators and distributed average desired power observers was proposed. In [62], a multi-agent sliding mode SoC balancing control algorithm was proposed, which can be integrated with multi-agent secondary control for average voltage regulation and current sharing. In [91], with the desired total charging/discharging power known to each BESS in a group of networked BESSs, each representing a battery unit, the charging/discharging power of each BESS is assigned based on the average of the states of the BESSs that contain both intrinsic properties as well as the SoC of the BESSs. Event-triggered design is adopted to reduce unnecessary communication loads. Simulation results demonstrate that, in the event that the average of the states of the BESSs is not accessible to all individual BESSs and needs to be estimated by distributed average state estimators, the controllers are able to achieve the control objective approximately despite the estimation errors.

In this chapter, we first design distributed power allocation algorithms for a BESS consisting of N networked heterogeneous battery units. A novel control framework based on distributed estimators is proposed that only requires the, possibly time-varying, power demand of the system to be known by at least one of the battery units. The SoC of all battery units is balanced through the charging/discharging process by allocating the power demand of the entire BESS among its units. A distributed power allocation algorithm is designed for each battery unit based on the distributed estimators built for the battery unit that estimates the average power demand and the average battery unit state. These estimators are referred to as the average power demand estimators and the average unit state estimators, respectively. Compared to the existing results in [91], we provide an explicit analysis of the effects of estimation errors on control accuracy. We show that, by setting the design parameters properly, the charging/discharging power demand can be satisfied and SoC balancing can be achieved by any pre-specified accuracy. In addition, we propose alternative finite-time estimators that can achieve perfect power supply and SoC balancing.

In real-world applications, the parameters such as capacities and terminal voltages of the battery units may not be precisely known. Their true values may deviate from their nominal values due to aging and/or variations in their manufacturing. In addition, the units' parameters may vary under severe operating conditions such as heavy load or low/high temperature [102]. In this chapter, we also design distributed adaptive power allocation algorithms for the BESS. In particular, the adaptive estimation of the battery parameters facilitates us to develop algorithms without direct knowledge of battery units' parameters. The SoC balancing is achieved by actively allocating the power demand based on distributed estimations of the average unit state and the average desired power, and the adaptive estimations of the units' unknown parameters. We show that, under the proposed distributed adaptive power allocating algorithms, the BESS achieves SoC balancing among its battery units and satisfies the power demand. Our ability to precisely estimate the battery parameters can also help us to monitor the health of the battery units. For example, a capacity drop may indicate a degradation or failure of the battery unit [75].

Lastly, we consider a BESS consisting of N battery units, whose SoC are assumed to be unknown. The battery characteristic is captured by a second-order Thevenin equivalent circuit model. A distributed observer-based SoC balancing control law is proposed for each battery unit in the BESS. In particular, a nonlinear state observer is constructed based on the equivalent circuit model of the battery dynamics and a nonlinear map from the SoC to the equivalent open circuit voltage (OCV) of the battery unit. Distributed estimation algorithms for the average battery state and the average power demand are also constructed. The power demand of the system is then distributed among individual battery units based on their estimated SoC, the estimated average state of the battery units, and the estimated average power demand. Analysis of the closed-loop system establishes that, under mild conditions on the communication network and the power demand, the BESS achieves SoC balancing among all its battery units while satisfying the power demand. The use of an electrical equivalent circuit model of the battery dynamics is appealing in real-time applications because of their concise structure and low implementation cost, compared to physics-based electrochemical models that are based on the physical and chemical processes inside the batteries, and data-driven models that are usually based on neural networks [87]. Compared to the look-up table-based SoC estimation method that requires measurements of the battery at rest for a sufficiently long period, and the coulomb counting SoC estimation method in which errors may accumulate from open-loop integration [93], the design of an equivalent circuit model-based SoC observer facilitates us to estimate the SoC while the battery unit is in a charging or discharging transient process. The observer's initial condition can be set close to the true state according to the look-up table established for the battery units at rest. During a dynamic process of a battery, the observer provides a real-time estimate of the SoC based on the measurement of the terminal voltage.

The remainder of this chapter is organized as follows. Section 6.2 formulates the distributed control problem of balancing the SoC of networked battery units in a BESS while satisfying the total charging/discharging power demand. Section 6.3 presents power allocation algorithms based on average state estimators and average power demand estimators. Section 6.4 provides simulation examples to demonstrate the design in 6.3. Section 6.5 extends the results in Section 6.3 to BESSs with unknown battery parameters by deploying an adaptive parameter estimator for each battery unit. Section 6.6 provides a simulation example for the design in 6.3. Section 6.7 presents the design of a power allocation algorithm based on a nonlinear observer and distributed average estimators. Section 6.8 presents a simulation example to illustrate the effectiveness of the design in 6.7. Section 6.9 concludes this chapter.

6.2 Problem Statement

In this section, we will formulate the distributed control problem of balancing the SoC of networked battery units in a BESS while satisfying the total charging/discharging power demand. We will consider a BESS consisting of N heterogeneous networked battery units. Each battery unit, with its own distributed controller, is able to communicate with nearby battery units and exchange information so that the overall control objective of the networked system can be achieved.

Assumption 6.1. The graph associated with the communication topology among the battery units is undirected and connected, and the power demand is known to at least one battery unit.

Assumption 6.2. The power demand $P^*(t)$ is known to at least one battery unit.

The SoC $S_i(t)$ of battery unit i is a dimensionless quantity, and it is theoretically defined by Coulomb counting as

$$S_i(t) = S_{i,\text{init}} - \int_0^t \frac{I_i(\tau)}{Q_{\text{nom},i}} \mathrm{d}\tau, \tag{6.1}$$

where $S_{i,\text{init}}$ is the initial SoC, $Q_{\text{nom},i}$ is the nominal capacity of the battery unit, and $I_i(t)$ is the output current. The output current $I_i > 0$ indicates discharging and $I_i < 0$ indicates charging.

The SoC dynamics of each battery unit is given as

$$\dot{S}_i = -\frac{1}{Q_{\mathrm{nom},i}} I_i,\ i \in \{1, 2, \cdots, N\}. \tag{6.2}$$

The output power $P_i(t)$ of each battery unit is calculated as

$$P_i(t) = V_i I_i(t),\ i \in \{1, 2, \cdots, N\},$$

where V_i is the constant terminal voltage of the DC-to-DC bidirectional converter connected to the battery unit. The output power $P_i > 0$ indicates discharging and $P_i < 0$ indicates charging.

The relationship between the SoC $S_i(t)$ and the output power $P_i(t)$ of the ith battery unit is written as

$$\dot{S}_i = -\frac{1}{Q_{\mathrm{nom},i} V_i} P_i,\ i \in \{1, 2, \cdots, N\}. \tag{6.3}$$

Since the state of the BESS will remain unchanged with zero charging/discharging power demand, we will, without loss of generality, make the following assumption on the power demand.

Assumption 6.3. The desired total charging/discharging power $P^*(t)$ of the BESS satisfies

$$\begin{aligned} &\underline{P} \le |P^*(t)| \le \bar{P},\ t \ge 0, \\ &|\dot{P}^*(t)| \le \bar{Q},\ t \ge 0 \end{aligned}$$

for some positive constants $\underline{P}$, $\bar{P}$ and $\bar{Q}$. $P^* > 0$ indicates discharging, and $P^* < 0$ indicates charging.

The state-of-charge balancing and power delivery problem of the battery energy storage system is formulated as follows.

Problem 6.1. Consider a BESS consisting of N networked battery units with their SoC dynamics modeled by (6.2). Let the communication network satisfy Assumptions 6.1 and 6.2. Let the power demand satisfy Assumption 6.3. For each battery unit i, design a power allocation law P_i such that

(i) all units achieve SoC balancing with any pre-specified accuracy $\varepsilon_s \ge 0$ in steady state, *i.e.*,

$$\lim_{t\to\infty} \left|S_i(t) - S_j(t)\right| \le \varepsilon_s,\ \text{for all } i, j \in \{1, 2, \cdots, N\},$$

(ii) the total charging/discharging power of the BESS tracks the power demand with any pre-specified accuracy $\varepsilon_p \geq 0$, *i.e.*,

$$\lim_{t\to\infty} \left| P_\Sigma(t) - P^*(t) \right| \leq \varepsilon_p,$$

where $P_\Sigma(t) = \sum_{i=1}^{N} P_i(t)$ is the total charging/discharging power.

6.3 Average Estimation based State-of-Charge Balancing and Power Delivery

In this section, we will first recall the SoC variation rules for the battery units based on their SoC, and power allocation laws based on the average state of all units [91]. We will then design average battery unit state estimators and average power demand estimators to facilitate the final design of power allocation laws.

It is noted that, by properly allocating the power demand $P^*(t)$ among individual battery units based on their SoC, capacity, and voltage, the SoC balancing can be achieved during the charging/discharging process.

In [91] the authors proposed the following rules to regulate the SoC variation during the charging/discharging process:

(i) in discharging mode

$$\frac{\dot{S}_1}{S_1} = \frac{\dot{S}_2}{S_2} = \cdots = \frac{\dot{S}_N}{S_N} = -k_{\rm d}(t) < 0, \tag{6.4}$$

(ii) in charging mode

$$\frac{\dot{S}_1}{1 - S_1} = \frac{\dot{S}_2}{1 - S_2} = \cdots = \frac{\dot{S}_N}{1 - S_N} = k_{\rm c}(t) > 0, \tag{6.5}$$

where $k_{\rm d}(t)$ and $k_{\rm c}(t)$ are some functions lower bounded by a positive number and satisfying the power demand. In other words, during the discharging process, a battery unit with higher SoC discharges at a higher rate while a unit with lower SoC discharges at a lower rate. A similar principle holds for the charging process.

Lemma 6.1. [91] Consider a BESS consisting of N networked battery units with SoC dynamics described by (6.2). Under discharging/charging rules (6.4) and (6.5), the SoC of all battery units will converge to the same value asymptotically.

Next, we recall power allocation laws that specify the charging/discharging power of individual battery units [91].

For each battery unit i, $i \in \{1, 2, \cdots, N\}$, define the state

$$x_{\mathrm{d},i}(t) = Q_{\mathrm{nom},i} V_i S_i(t),$$

for the discharging mode, and

$$x_{\mathrm{c},i}(t) = Q_{\mathrm{nom},i} V_i \big(1 - S_i(t)\big),$$

for the charging mode. The overall state of the battery unit i is denoted as

$$x_i(t) = \begin{cases} x_{\mathrm{d},i}(t) & \text{in discharging mode,} \\ x_{\mathrm{c},i}(t) & \text{in charging mode.} \end{cases}$$

In view of (6.3), we have

$$\dot{x}_{\mathrm{d},i} = Q_{\mathrm{nom},i} V_i \dot{S}_i = -P_i(t),$$

for the discharging mode, and

$$\dot{x}_{\mathrm{c},i} = -Q_{\mathrm{nom},i} V_i \dot{S}_i = P_i(t),$$

for the charging mode.

Since

$$\frac{\dot{S}_i}{S_i} = -\frac{P_i}{x_{\mathrm{d},i}}, \ \ i \in \{1, 2, \cdots, N\},$$

for the discharging mode, and

$$\frac{\dot{S}_i}{1 - S_i} = -\frac{P_i}{x_{\mathrm{c},i}}, \ \ i \in \{1, 2, \cdots, N\},$$

for the charging mode, the charging/discharging power of all battery units must satisfy

$$\frac{P_1}{x_{\mathrm{d},1}} = \frac{P_2}{x_{\mathrm{d},2}} = \cdots = \frac{P_N}{x_{\mathrm{d},N}},$$

for the discharging mode, and

$$\frac{P_1}{x_{\mathrm{c},1}} = \frac{P_2}{x_{\mathrm{c},2}} = \cdots = \frac{P_N}{x_{\mathrm{c},N}},$$

for the charging mode.

Design the charging/discharging power of each battery unit as

$$P_i(t) = \frac{x_{\mathrm{d},i}(t)}{\sum_{j=1}^{N} x_{\mathrm{d},j}(t)} P^*(t), \ \ i \in \{1, 2, \cdots, N\}, \tag{6.6}$$

for the discharging mode, and

$$P_i(t) = \frac{x_{\mathrm{c},i}(t)}{\sum_{j=1}^{N} x_{\mathrm{c},j}(t)} P^*(t),\ i \in \{1, 2, \cdots, N\}, \tag{6.7}$$

for the charging mode.

Then, it is straightforward to verify that both the SoC balancing and the desired total power is satisfied. Denote the average unit state as

$$x_{\mathrm{a}}(t) = \frac{1}{N} \sum_{i=1}^{N} x_{\mathrm{d},i}(t),$$

for the discharging mode, and

$$x_{\mathrm{a}}(t) = \frac{1}{N} \sum_{i=1}^{N} x_{\mathrm{c},i}(t),$$

for the charging mode. Denote the average power demand as

$$P_{\mathrm{a}}(t) = \frac{1}{N} P^*(t).$$

Then, (6.6) and (6.7) can be written as

$$P_i(t) = \frac{x_i(t)}{x_{\mathrm{a}}(t)} P_{\mathrm{a}}(t).$$

Assumption 6.4. There exist constants a_1, $a_2 > 0$ such that

$$a_1 \leq x_i \leq a_2,\ t \geq 0,\ i \in \{1, 2, \cdots, N\}.$$

Since the signals $x_{\mathrm{a}}(t)$ and $P_{\mathrm{a}}(t)$ are global information that might not be available for all battery units, we will design distributed estimators for such quantities.

In order to estimate the average desired power $P_{\mathrm{a}}(t) = \frac{1}{N} P^*(t)$, we define the diagonal matrix $\mathcal{B} = \mathrm{diag}\{b_1, b_2, \cdots, b_N\}$, where $b_i = 1$ if the ith battery unit has access to the desired power $P^*(t)$ and $b_i = 0$ otherwise. Under Assumptions 6.1 and 6.2, the matrix $\mathcal{H} = \mathcal{L} + \mathcal{B} > 0$ [26].

For each battery unit i, $i \in \{1, 2, \cdots, N\}$, we design the following average unit state estimator that estimates the signal $x_{\mathrm{a}}(t)$ asymptotically,

$$\begin{cases} \dot{\hat{x}}_{\mathrm{a},i} = \dot{x}_i - \alpha(\hat{x}_{\mathrm{a},i} - x_i) - \beta \sum_{j=1}^{N} a_{ij}(\hat{x}_{\mathrm{a},i} - \hat{x}_{\mathrm{a},j}) - \nu_i, \\ \dot{\nu}_i = \alpha\beta \sum_{j=1}^{N} a_{ij}(\hat{x}_{a,i} - \hat{x}_{a,j}), \end{cases} \tag{6.8}$$

where $\hat{x}_{\mathrm{a},i}$ is the ith battery unit's estimate of $x_{\mathrm{a}}(t)$ and ν_i is the internal state. The initial conditions of the estimators are chosen as $\hat{x}_{\mathrm{a},i}(0) = x_i(0)$ and $\nu_i(0) = 0$, $i \in \{1, 2, \cdots, N\}$. The positive constants α and β are design parameters.

We also design the following estimator adopted from leader-following consensus protocols as the average power demand estimator to estimate the signal $P_{\mathrm{a}}(t)$,

$$\dot{\hat{p}}_{\mathrm{a},i} = -\kappa\left(\sum_{j=1}^{N} a_{ij}(\hat{p}_{\mathrm{a},i} - \hat{p}_{\mathrm{a},j}) + b_i(\hat{p}_{\mathrm{a},i} - P_{\mathrm{a}})\right), \tag{6.9}$$

where $\hat{p}_{\mathrm{a},i}$ is the ith battery unit's estimate of $P_{\mathrm{a}}(t)$ and $\kappa > 0$ is a design parameter. that control the convergence rate of the estimator. The initial condition of the estimator is chosen as $\hat{p}_{\mathrm{a},i}(0) = 0$, $i \in \{1, 2, \cdots, N\}$.

The following two lemmas establish the convergence of the average unit state estimators and the average power demand estimators, respectively.

Lemma 6.2. [36] There exists a constant $\gamma_x > 0$ such that, for any α, $\beta > 0$, the estimator state $\hat{x}_{\mathrm{a},i}(t)$ with initial conditions $\hat{x}_{\mathrm{a},i}(0)$, $\nu_i(0)$ such that $\sum_{i=1}^{N} \nu_i(0) = 0$ converges toward $x_{\mathrm{a}}(t)$ exponentially fast with a steady state error

$$\lim_{t\to\infty} \sup\left|\hat{x}_{\mathrm{a},i}(t) - x_{\mathrm{a}}(t)\right| \leq \frac{\gamma_x}{\beta\lambda_2}.$$

Lemma 6.3. [1] There exists a constant $\gamma_p > 0$ such that, for any $\kappa > 0$, the estimator state $\hat{p}_{\mathrm{a},i}(t)$ converges toward $P_{\mathrm{a}}(t)$ exponentially fast with a steady-state error

$$\lim_{t\to\infty} \sup|\hat{p}_{\mathrm{a},i}(t) - p_{\mathrm{a}}(t)| \leq \frac{\psi\gamma_p}{\kappa}.$$

It is noted that a larger value of the design parameter κ in the estimator (6.9) results in a smaller steady state error.

By using the average unit state estimators and the average power demand estimators, the charging/discharging power of each battery unit i, (6.6) and (6.7), can be implemented as

$$P_i(t) = \frac{x_{\mathrm{d},i}(t)}{\max\{\frac{a_1}{2}, \hat{x}_{\mathrm{a},i}(t)\}}\hat{p}_{\mathrm{a},i}(t), \tag{6.10}$$

for the discharging mode, and

$$P_i(t) = \frac{x_{\mathrm{c},i}(t)}{\max\{\frac{a_1}{2}, \hat{x}_{\mathrm{a},i}(t)\}}\hat{p}_{\mathrm{a},i}(t), \tag{6.11}$$

for the charging mode. Note that $\max\{\frac{a_1}{2}, \hat{x}_{\mathrm{a},i}(t)\} \geq \frac{a_1}{2} > 0$, where a_1 is defined in Assumption 6.4.

We have the following result on the performance of the distributed power allocation algorithms (6.10) and (6.11) with estimators (6.8) and (6.9).

Theorem 6.1. [58] Consider a BESS consisting of N networked battery units. The relation between the dynamics of SoC and the charging/discharging power of each battery unit is described by (6.3). Let the communication topology satisfy Assumptions 6.1 and 6.2. Let the power demand satisfy Assumption 6.3. Then, for any given ε_s, $\varepsilon_p > 0$, there exist β, $\kappa > 0$ such that the distributed power allocation algorithms (6.10) and (6.11) with the average unit state estimator (6.8) and the average power demand estimator (6.9) solve Problem 6.1.

Proof: We consider the discharge mode in the following analysis. The analysis for the charging mode can be carried out in a similar way.

Define

$$\begin{aligned}\tilde{x}_i(t) &= \hat{x}_{\mathrm{a},i}(t) - x_{\mathrm{a}}(t),\\ \tilde{p}_i(t) &= \hat{p}_{\mathrm{a},i}(t) - P_{\mathrm{a}}(t),\end{aligned}$$

as the estimator errors. Choose $\beta \geq \frac{2\gamma_x}{\alpha_1\lambda_2}$, where γ_x is as defined in Lemma 6.2. Then, under Assumption 6.4, we have $\lim_{t\to\infty}\sup|\tilde{x}_i(t)| < \frac{1}{2}a_1$ by Lemma 6.2. Since $x_{\mathrm{a}}(t) \geq a_1$, $t \geq 0$, by Assumption 6.4, in steady state, we have

$$\hat{x}_{a,i} \geq \frac{1}{2}a_1.$$

By Lemma 6.3, $\tilde{p}_i(t)$ converges toward zero exponentially fast with $\lim_{t\to\infty}\sup|\tilde{p}_i(t)| \leq \frac{\bar{Q}\gamma_p}{\kappa}$, we have

$$\lim_{t\to\infty}\sup\left|\frac{\tilde{p}_i(t)}{P_\mathrm{a}(t)}\right| \le \frac{N\bar{Q}\gamma_p}{\kappa\underline{P}}.$$

The discharging power of each battery unit i in steady state is written as

$$\begin{aligned} P_i(t) &= \frac{x_{\mathrm{d},i}(t)}{x_\mathrm{a}(t)+\tilde{x}_i(t)}(P_\mathrm{a}+\tilde{p}_i) \\ &= \frac{1+\frac{\tilde{p}_i}{P_\mathrm{a}}}{1+\frac{\tilde{x}_i}{x_\mathrm{a}}}\frac{x_{\mathrm{d},i}}{x_\mathrm{a}}P_\mathrm{a}. \end{aligned}$$

Then, in steady state, we have,

$$\frac{1-\frac{N\bar{Q}\gamma_p}{\kappa\underline{P}}}{1+\frac{\gamma_x}{a_1\beta\lambda_2}} \le \frac{1+\frac{\tilde{p}_i(t)}{P_\mathrm{a}(t)}}{1+\frac{\tilde{x}_i(t)}{x_\mathrm{a}(t)}} \le \frac{1+\frac{N\bar{Q}\gamma_p}{\kappa\underline{P}}}{1-\frac{\gamma_x}{a_1\beta\lambda_2}}. \tag{6.12}$$

Define

$$\begin{aligned} \delta_i(t) &= \frac{1+\frac{\tilde{p}_i(t)}{P_\mathrm{a}(t)}}{1+\frac{\tilde{x}_i(t)}{x_\mathrm{a}(t)}} - 1, \\ \delta^- &= \frac{1-\frac{N\bar{Q}\gamma_p}{\kappa\underline{P}}}{1+\frac{\gamma_x}{a_1\beta\lambda_2}} - 1 < 0, \\ \delta^+ &= \frac{1+\frac{N\bar{Q}\gamma_p}{\kappa\underline{P}}}{1-\frac{\gamma_x}{a_1\beta\lambda_2}} - 1 > 0. \end{aligned}$$

Then, in steady state, we have

$$\delta^- \le \delta_i(t) \le \delta^+.$$

In addition, the bound can be made arbitrarily tight by choosing the parameters β and κ sufficiently large.

Define

$$k_i(t) = \frac{x_{\mathrm{d},i}(t)}{x_\mathrm{a}(t)}.$$

Then, equation (6.10) can be rewritten in terms of $k(t)$ and $\delta_i(t)$ as

$$P_i = k(1+\delta_i)\hat{p}_{\mathrm{a},i}.$$

Note that $k(t) \ge \frac{a_1}{a_2} > 0$ since $x_{\mathrm{d},i}(t) \ge a_1$ and $x_\mathrm{a}(t) \le a_2$, $t \ge 0$.

Recall that $\dot{S}_i = -\frac{1}{Q_{\mathrm{nom},i} V_i} P_i$ and $x_{\mathrm{d},i} = Q_{\mathrm{nom},i} V_i S_i$, we have

$$\dot{S}_i = -k(1+\delta_i)S_i. \tag{6.13}$$

Consider the function

$$V_{ij} = \frac{1}{2}\big(S_i(t) - S_j(t)\big)^2,\ i \neq j,\ i,j \in \{1,2,\cdots,N\}.$$

The time derivative of V_{ij} along the trajectory of (6.13) is written as

$$\begin{aligned}\dot{V}_{ij} &= (S_i - S_j)(\dot{S}_i - \dot{S}_j)\\ &= -k(t)(S_i - S_j)\Big((1+\delta_i)S_i - (1+\delta_j)S_j\Big).\end{aligned}$$

It is obvious that $\dot{V}_{ij} < 0$ if

$$(S_i - S_j)\Big((1+\delta_i)S_i - (1+\delta_j)S_j\Big) > 0,$$

i.e.,

$$S_i > S_j \text{ and } \frac{S_i}{S_j} > \frac{1+\delta_j}{1+\delta_i}$$

or

$$S_j < S_i \text{ and } \frac{S_i}{S_j} < \frac{1+\delta_j}{1+\delta_i}.$$

Since $\delta^- \leq \delta_i(t) \leq \delta^+$, from the above analysis, we have $\dot{V}_{ij} < 0$ if

$$\frac{S_i}{S_j} > \frac{1+\delta^+}{1+\delta^-} \text{ or } \frac{S_i}{S_j} < \frac{1+\delta^-}{1+\delta^+}.$$

That is, in steady state,

$$\frac{1+\delta^-}{1+\delta^+} \leq \frac{S_i}{S_j} \leq \frac{1+\delta^+}{1+\delta^-}$$

for any pair of S_i and S_j.

Since $0 < S_i(t) \leq 1$, $i \in \{1,2,\cdots,N\}$, for all $t \geq 0$, the SoC of any pair of battery units i and j will converge to the set $\{(S_i,S_j) : |S_i - S_j| \leq \frac{\delta^+ - \delta^-}{1+\delta^-}\}$.

The total output power of the system is calculated as

$$P_\Sigma = \sum_{i=1}^{N} \frac{1+\frac{\tilde{p}_i}{P_\mathrm{a}}}{1+\frac{\tilde{x}_i}{x_\mathrm{a}}} \frac{x_i}{x_\mathrm{a}} P_\mathrm{a}.$$

Given (6.12), the output power at steady state can be bounded as

$$(1+\delta^-)P^* \le P_\Sigma \le (1+\delta^+)P^*.$$

For any given $\varepsilon_s > 0$ and $\varepsilon_p > 0$, choose β and κ such that

$$\begin{aligned}\frac{\delta^+ - \delta^-}{1+\delta^-} &\le \varepsilon_s,\\ \delta^+ &\le \varepsilon_p,\end{aligned}$$

and the control objectives are achieved. $\square$

We will next present power allocation laws based on finite-time average battery state estimators and finite-time average power demand estimators. Motivated by [19], where a robust dynamic average consensus algorithm for signals with bounded derivatives was proposed, we design, for each battery unit i, $i \in \{1, 2, \cdots, N\}$, the following finite-time average unit state estimator,

$$\begin{cases} \dot{q}_i = -\alpha \mathrm{sign}\Big(\sum_{j=1}^{N} a_{ij}(\hat{x}_{\mathrm{a},i} - \hat{x}_{\mathrm{a},j})\Big), \\ \hat{x}_{\mathrm{a},i} = \sum_{j=1}^{N} a_{ij}(q_i - q_j) + x_i, \end{cases} \tag{6.14}$$

where $\hat{x}_{\mathrm{a},i}$ is the ith battery unit's estimate of $x_{\mathrm{a}}(t)$ and q_i is the internal state, and $\alpha > 0$ is a design parameter. The initial condition of the estimator is chosen as $\hat{x}_{\mathrm{a},i}(0) = x_i(0)$ and $q_i(0) = 0$. The following lemma establishes the property of the average unit state estimators (6.14).

Lemma 6.4. Consider a BESS consisting of N networked battery units with estimators designed as (6.14). Let the SoC dynamics of each battery unit satisfy Assumption 6.3. Then, there exists a finite $T_x^* \ge 0$ such that

$$\hat{x}_{\mathrm{a},i}(t) = x_{\mathrm{a}}(t),\ t \ge T_x^*,\ i \in \{1, 2, \cdots, N\},$$

if the design parameter α is chosen such that

$$\alpha \ge \frac{\sqrt{N}\bar{P}}{\lambda_2} + 1,$$

where $\bar{P}$ is as defined in Assumption 6.3 and λ_2 is the second smallest eigenvalue of $\mathcal{L}$.

Next we design, for each battery unit i, $i \in \{1, 2, \cdots, N\}$, the following average power demand estimator,

$$\begin{cases} \dot{\hat{p}}_{a,i} = -\beta \mathrm{sign}(\nu_{\mathrm{i}}), \\ \nu_i = \sum_{j\in\mathcal{N}_i} (\hat{p}_{a,i} - \hat{p}_{a,j}) + b_i(\hat{p}_{a,i} - P_a), \end{cases} \tag{6.15}$$

where $\hat{p}_{\mathrm{a},i}$ is the ith battery unit's estimate of $P_{\mathrm{a}}(t)$ and $\beta > 0$ is a design parameter.

The following lemma establishes the property of the average power demand estimators (6.15).

Lemma 6.5. Consider a BESS consisting of N networked battery units, each with its average power demand estimator designed as (6.15). Let the communication topology satisfy Assumptions 6.1 and 6.2. Then, there exists a finite $T_p^* \geq 0$ such that

$$\hat{p}_{a,i}(t) = \frac{1}{N} P^*(t),\ t \geq T_p^*,\ i \in \{1, 2, \cdots, N\},$$

if the design parameter β is chosen such that

$$\beta \geq \frac{\bar{Q}}{N} + 1,$$

where $\bar{Q}$ is as defined in Assumption 6.3.

Proof: Denote

$$\begin{aligned} \hat{p} &= [\hat{p}_{\mathrm{a},1}\ \hat{p}_{\mathrm{a},2}\ \cdots\ \hat{p}_{\mathrm{a},N}]^{\mathrm{T}}, \\ \tilde{p} &= [\tilde{p}_1\ \tilde{p}_2\ \cdots\ \tilde{p}_N]^{\mathrm{T}} \\ &= \hat{p} - \mathbf{1}_N P_{\mathrm{a}}, \\ \nu &= [\nu_1\ \nu_2\ \cdots\ \nu_N)^{\mathrm{T}}. \end{aligned}$$

Then,

$$\begin{aligned} \nu &= \mathcal{H}\hat{p} - \mathcal{B}\mathbf{1}_N P_{\mathrm{a}} \\ &= \mathcal{H}\tilde{p}, \end{aligned}$$

where $\mathcal{H} = \mathcal{L} + \mathcal{B} > 0$ and

$$\begin{aligned} \dot{\nu} &= \mathcal{H}\dot{\tilde{p}} \\ &= \mathcal{H}\Big(\dot{\hat{p}} - \frac{1}{N}\mathbf{1}_N \dot{P}^*\Big). \end{aligned} \tag{6.16}$$

Consider the Lyapunov function

$$V(\nu) = \frac{1}{2}\nu^{\mathrm{T}}\mathcal{H}^{-1}\nu.$$

The time derivative of V along the trajectory of (6.16) is evaluated as

$$\begin{aligned}
\dot{V} &= \nu^{\mathrm{T}}\mathcal{H}^{-1}\mathcal{H}\dot{\hat{p}} \\
&= \nu^{\mathrm{T}}\Big(\dot{\hat{p}} - \frac{1}{N}\mathbf{1}_N\dot{P}^*\Big) \\
&\le -\beta\sum_{i=1}^{N}|\nu_i| + \frac{\bar{Q}}{N}\|\nu\|_1 \\
&\le \Big(\frac{\bar{Q}}{N} - \beta\Big)\|\nu\|_1,
\end{aligned}$$

where $\bar{Q}$ is as given in Assumption 6.3.

Thus, for

$$\beta \ge \frac{\bar{Q}}{N} + 1,$$

we have

$$\begin{aligned}
\dot{V} &\le -\|\nu\|_1 \\
&\le -\|\nu\|_2.
\end{aligned}$$

which, in view of $V \le \frac{1}{2}\lambda_{\max}(\mathcal{H}^{-1})\|\nu\|_2^2$, implies that

$$\dot{V} \le -\sqrt{\frac{2}{\lambda_{\max}(\mathcal{H}^{-1})}}V^{\frac{1}{2}}.$$

Let $W = V^{\frac{1}{2}}$, we have

$$\dot{W} \le -\sqrt{\frac{2}{\lambda_{\max}(\mathcal{H}^{-1})}},$$

which in turn implies that W and hence $\tilde{p}$ will reach zero in a finite time, $T_p^* \ge 0$ and remain at zero thereafter. □

It is noted that the estimated average unit state and the estimated average power demand are equal to the true values for all $t \ge \max\{T_x^*, T_p^*\}$. Therefore, these estimators are referred to as finite-time estimators.

We have the following result on the performance of the distributed power allocation algorithms

(6.10) and (6.11) with estimators (6.14) and (6.15).

Theorem 6.2. [58] Consider a BESS consisting of N networked battery units. The relation between the dynamics of SoC and the charging/discharging power of each battery unit is described by (6.3). Let the communication topology satisfy Assumptions 6.1 and 6.2. Let the power demand satisfy Assumption 6.3. Then, the distributed power allocation algorithms (6.10) and (6.11) with the average unit state estimators (6.14) and the average power demand estimator (6.15) solve Problem 6.1.

Proof: It is obvious that, with the proposed estimators (6.14) and (6.15), we have $\hat{x}_{\mathrm{a},i}(t) = x_{\mathrm{a}}(t)$, $\hat{p}_{\mathrm{a},i}(t) = P_{\mathrm{a}}(t)$, $i \in \{1, 2 \ldots, N\}$, for all $t \geq \max\{T_x^*, T_p^*\}$. After that, controllers (6.10) and (6.11) with such estimators become the desired controllers (6.6) and (6.7). From the previous analysis, we know that SoC balancing will be achieved among all battery units and the power demand will be satisfied. $\square$

6.4 Simulation for Average Estimation Based Algorithms

Consider a BESS consisting of six networked battery units. The parameters of the battery units are

$$
\begin{aligned}
(Q_{\mathrm{nom},1}, Q_{\mathrm{nom},2}, Q_{\mathrm{nom},3}, Q_{\mathrm{nom},4}, Q_{\mathrm{nom},5}, Q_{\mathrm{nom},6}) &= (180, 190, 200, 210, 220, 230)\mathrm{Ah}, \\
(V_1, V_2, V_3, V_4, V_5, V_6) &= (20, 20, 20, 20, 20, 20)\mathrm{V}.
\end{aligned}
$$

The communication topology of the system is shown in Fig. 6.1. In addition, only battery unit 1 has access to the charging/discharging power demand, i.e., $b_1 = 1$ and $b_i = 0, i = 2, 3, \cdots, 6$.

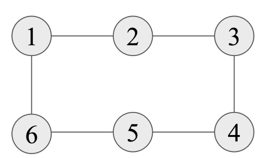

Figure 6.1: The communication topology.

The distributed allocated power for each battery unit i is designed as,

$$P_i = \frac{x_i}{\hat{x}_i}\hat{p}_i, \;\; i = 1, 2, \ldots, 6, \tag{6.17}$$

where $x_i = Q_{\mathrm{nom},i} V_i s_i$ for discharging mode and $x_i = Q_{\mathrm{nom},i} V_i (1 - s_i)$ for charging mode, $\hat{x}_i$ and $\hat{p}_i$ are the estimate of $x_a = \frac{1}{6} \sum_{j=1}^{N} x_i$ and $P_a = \frac{1}{6} P^*$ by the ith battery unit, respectively.

6.4.1 Asymptotic Estimation

The estimates $\hat{x}_i$ and $\hat{p}_i$ for each battery unit i are given by,

$$\begin{cases} \dot{\hat{x}}_i = \dot{x}_i - \alpha(\hat{x}_i - x_i) - \beta \sum_{j \in \mathcal{N}_i} (\hat{x}_i - \hat{x}_j) - \nu_i, \\ \dot{\nu}_i = \alpha\beta \sum_{j \in \mathcal{N}_i} (\hat{x}_i - \hat{x}_j), \end{cases} \tag{6.18}$$

with $\hat{x}_i(0) = x_i(0)$, $\nu_i(0) = 0$, for $i = 1, 2, \cdots, 6$, and

$$\dot{\hat{p}}_i = -\kappa \left(\sum_{j=1}^{N} a_{ij} (\hat{p}_i - \hat{p}_j) + b_i (\hat{p}_i - P_a) \right), \tag{6.19}$$

with $\hat{p}_i(0) = 0$ for $i = 1, 2, \cdots, 6$.

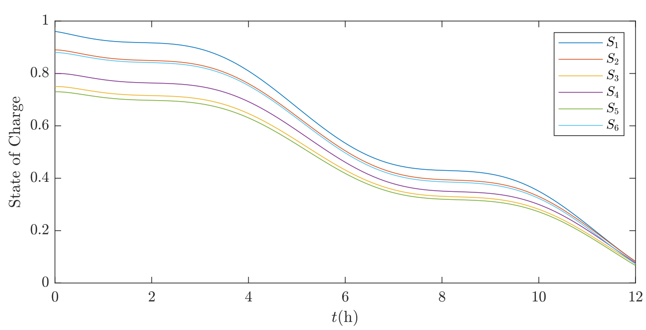

Figure 6.2: The evolution of SoC of all battery units under power allocation (6.17) and estimators (6.18), (6.19) with $\alpha = 1000$, $\beta = 2$ and $\kappa = 20$.

Shown in Figs. 6.2-6.9 are simulation results performed with the discharging power demand $P^*(t) = (-4200 \sin(t) + 4200)$W under power allocation (6.17) and estimators (6.18), (6.19) for two different sets of design parameters in the estimators. The initial SoC of the battery units is $(0.96, 0.89, 0.75, 0.8, 0.73, 0.88)$. Note that $P_\Sigma > 0$ indicates that the BESS is discharging and providing power to the grid.

Shown in Figs. 6.2-6.5 are the SoC evolution of all battery units during the discharging process, the total power of the BESS and the power demand and the state evolution of the estimators, respectively, with the parameters in the estimators chosen as $\alpha = 1000$, $\beta = 2$ and $\kappa = 20$. Shown

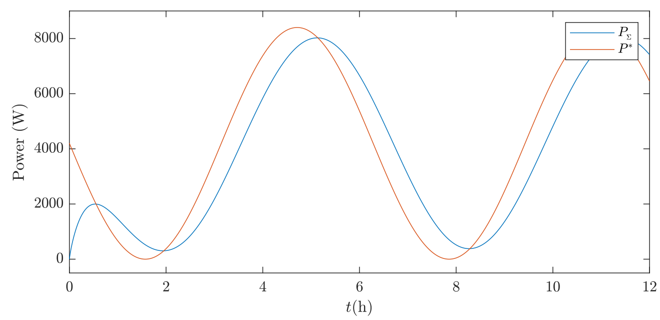

Figure 6.3: The total output power of all battery units and the power demand under power allocation (6.17) and estimators (6.18), (6.19) with $\alpha = 1000$, $\beta = 2$ and $\kappa = 20$.

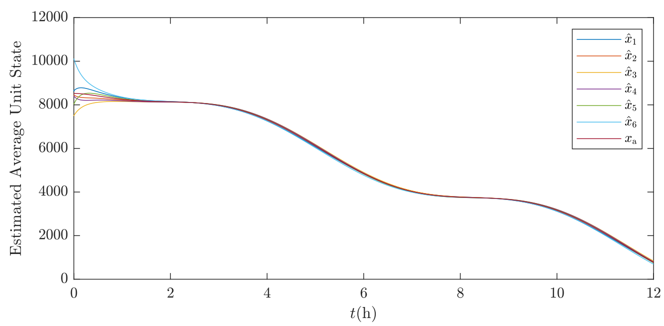

Figure 6.4: The estimated value by all battery units and the true value of x_a under estimators (6.18) with $\alpha = 1000$ and $\beta = 2$.

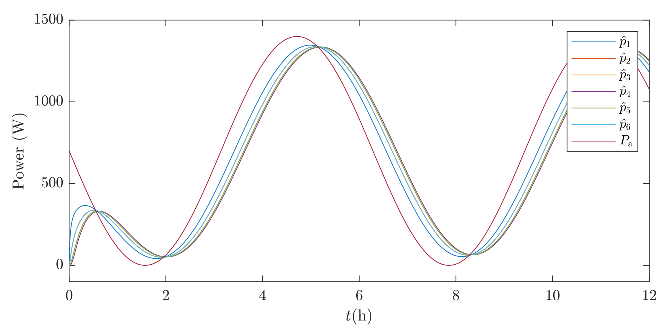

Figure 6.5: The estimated value by all battery units and the true value of P_a under estimators (6.19) with $\kappa = 20$.

in Figs. 6.6-6.9 are the same quantities with the parameters in the estimators chosen as $\alpha = 1000$, $\beta = 0.3$ and $\kappa = 3$. It is observed that with larger values of β and κ, the SoC of all battery units converge closer to each other during the discharging process and the total output power of the BESS tracks the power demand more accurately.

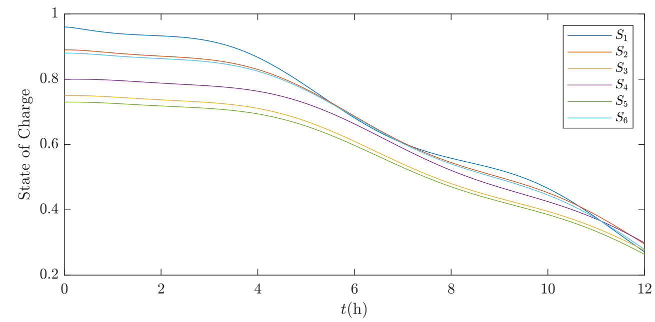

Figure 6.6: The evolution of SoC of all battery units under power allocation (6.17) and estimators (6.18), (6.19) with $\alpha = 1000$, $\beta = 0.3$ and $\kappa = 3$.

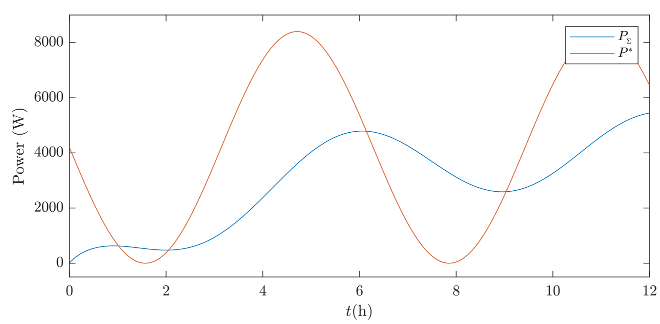

Figure 6.7: The total output power of all battery units and the power demand under power allocation (6.17) and estimators (6.18), (6.19) with $\alpha = 1000$, $\beta = 0.3$ and $\kappa = 3$.

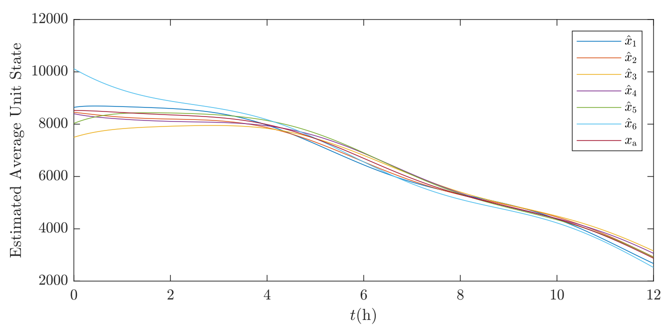

Figure 6.8: The estimated value by all battery units and the true value of x_a under estimators (6.18) with $\alpha = 1000$ and $\beta = 0.3$.

Shown in Figs. 6.10-6.17 are simulation results performed with the charging power demand $P^*(t) = (4200 \sin(t) - 4200)$W under power allocation (6.17) and estimators (6.18), (6.19), again for two different sets of design parameters in the estimators. The initial SoC of the battery units is $(0.04, 0.11, 0.25, 0.2, 0.27, 0.12)$. Note that $p_\Sigma < 0$ indicates that the BESS is charging and absorbing power from the grid.

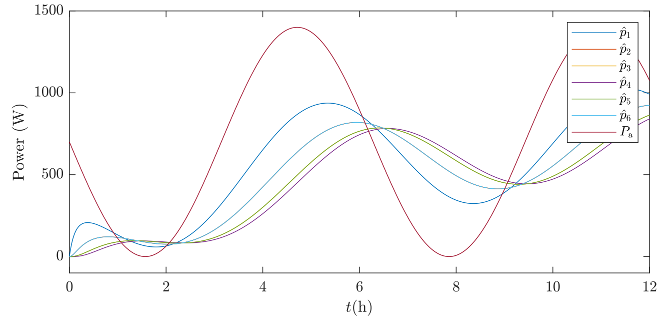

Figure 6.9: The estimated value by all battery units and the true value of P_a under estimators (6.19) with $\kappa = 3$.

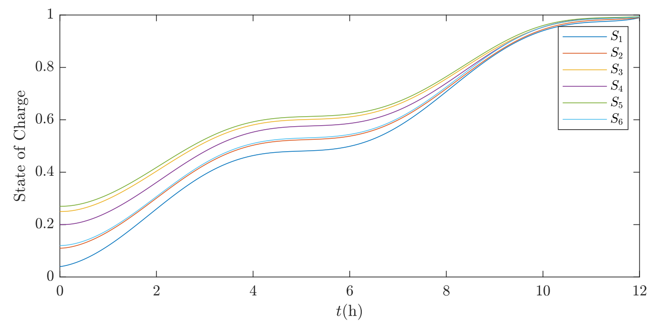

Figure 6.10: The evolution of SoC of all battery units under power allocation (6.17) and estimators (6.18), (6.19) with $\alpha = 1000$, $\beta = 2$ and $\kappa = 20$.

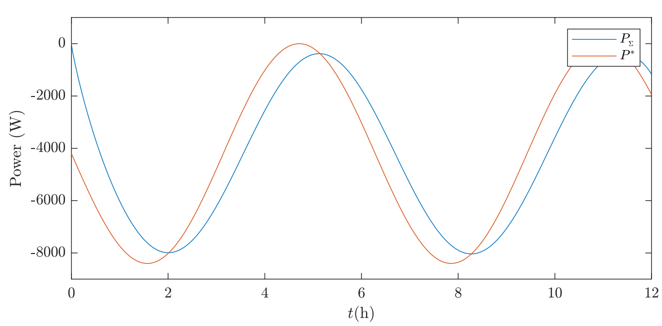

Figure 6.11: The total output power of all battery units and the power demand under power allocation (6.17) and estimators (6.18), (6.19) with $\alpha = 1000$, $\beta = 2$ and $\kappa = 20$.

Shown in Figs. 6.10-6.13 are the SoC evolution of all battery units during the charging process, the total power of the BESS and the power demand and the state evolution of the estimators, respectively, with the parameters in the estimators chosen as $\alpha = 1000$, $\beta = 2$ and $\kappa = 20$. Shown in Figs. 6.14-6.17 are the same quantities with the parameters in the estimators chosen as $\alpha = 1000$, $\beta = 0.3$ and $\kappa = 3$. Again, with larger β and κ, the SoC of all battery units converge closer to

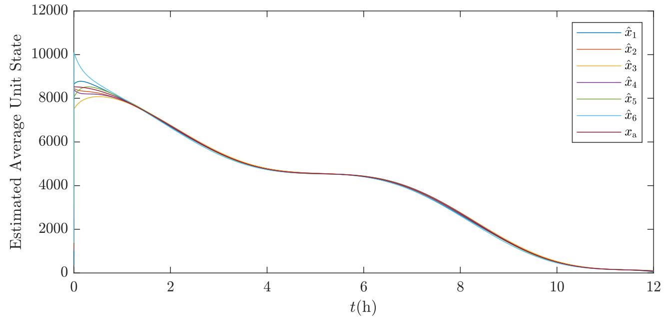

Figure 6.12: The estimated value by all battery units and the true value of x_a under estimators (6.18) with $\alpha = 1000$ and $\beta = 2$.

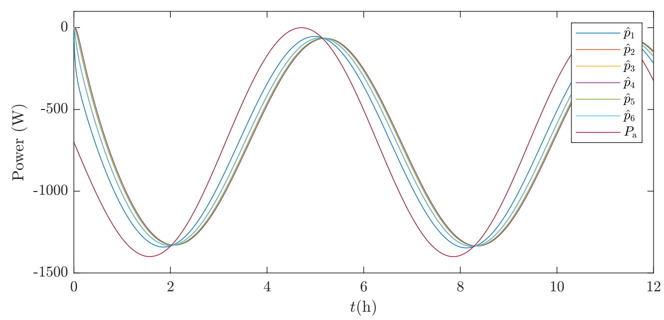

Figure 6.13: The estimated value by all battery units and the true value of P_a under estimators (6.19) with $\kappa = 20$.

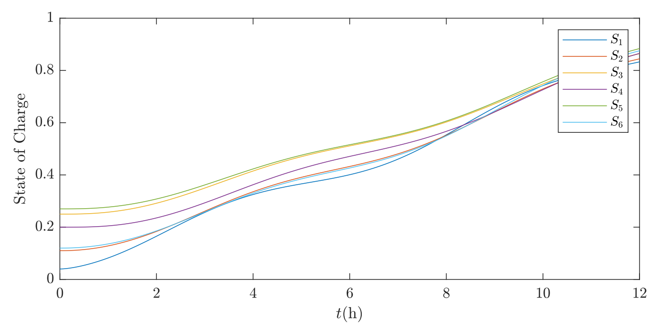

Figure 6.14: The evolution of SoC of all battery units under power allocation (6.17) and estimators (6.18), (6.19) with $\alpha = 1000$, $\beta = 0.3$ and $\kappa = 3$.

each other during the charging process and the total output power of the BESS tracks the power demand more accurately.

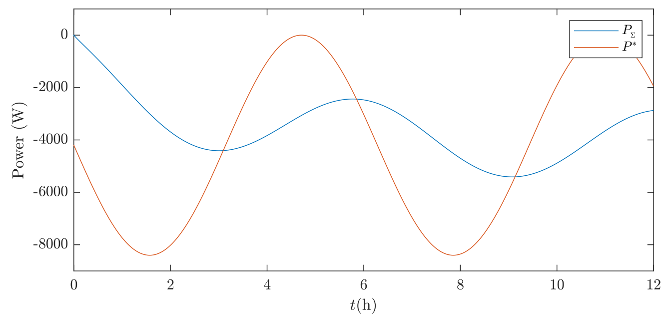

Figure 6.15: The total output power of all battery units and the power demand under power allocation (6.17) and estimators (6.18), (6.19) with $\alpha = 1000$, $\beta = 0.3$ and $\kappa = 3$.

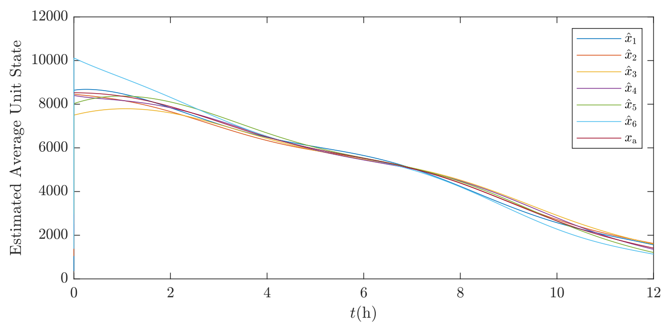

Figure 6.16: The estimated value by all battery units and the true value of x_a under estimators (6.18) with $\alpha = 1000$ and $\beta = 0.3$.

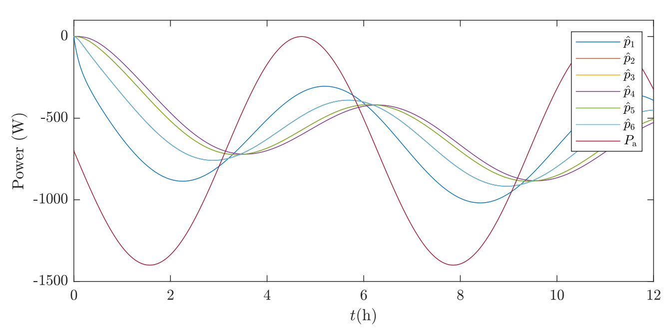

Figure 6.17: The estimated value by all battery units and the true value of P_a under estimators (6.19) with $\kappa = 3$.

6.4.2 Finite-Time Estimation

The estimates $\hat{x}_i$ and $\hat{p}_i$ for each battery unit i are given by

$$\begin{cases} \dot{q}_i = -\alpha \text{sign}\Big(\sum_{j=1}^{N} a_{ij}(\hat{x}_i - \hat{x}_j)\Big), \\ \hat{x}_i = \sum_{j=1}^{N} a_{ij}(q_i - q_j) + x_i, \end{cases}$$

with $q_i(0) = 0$, $\hat{x}_i(0) = x_i(0)$, $i = 1, 2, \cdots, 6$, and

$$\dot{\hat{p}}_i = -\beta \text{sign}\Big(\sum_{j=1}^{N} a_{ij}(\hat{p}_i - \hat{p}_j) + b_i(\hat{p}_i - P_a)\Big), \tag{6.20}$$

with $\hat{p}_i(0) = 0$, $i = 1, 2, \cdots, 6$.

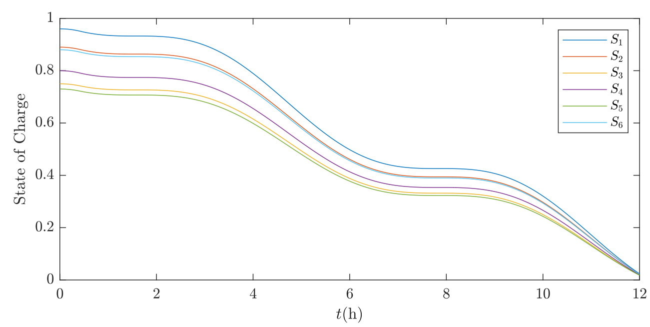

Figure 6.18: The evolution of SoC of all battery units under power allocation (6.17) and estimators (6.4.2), (6.20).

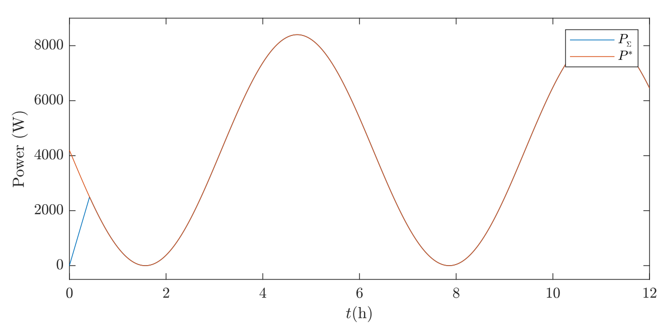

Figure 6.19: The total output power of all battery units and the power demand under power allocation (6.17) and estimators (6.4.2), (6.20).

Shown in Figs. 6.18-6.21 are simulation results performed with the discharging power demand

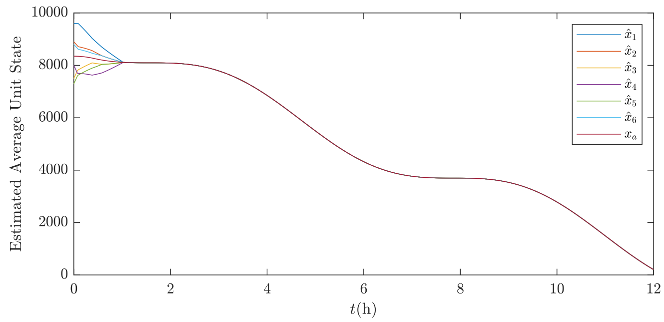

Figure 6.20: The estimated value by all battery units and the true value of x_a under estimators (6.4.2).

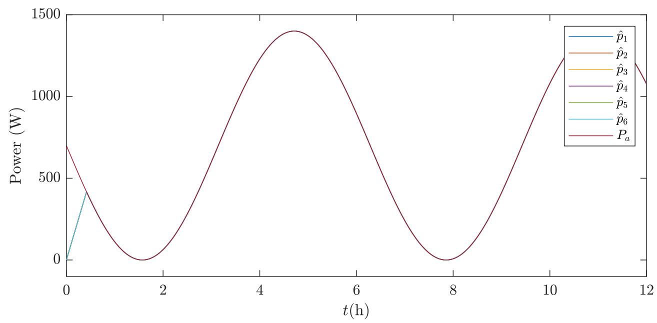

Figure 6.21: The estimated value by all battery units and the true value of P_a under estimators (6.20).

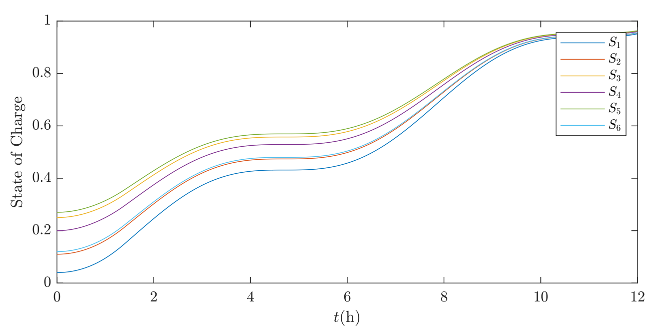

Figure 6.22: The evolution of SoC of all battery units under power allocation (6.17) and estimators (6.4.2), (6.20).

$P^*(t) = (4200\sin(t) + 4200)$W under power allocation (6.17) and estimators (6.4.2), (6.20). The initial SoC of the battery units are $(0.96, 0.89, 0.75, 0.8, 0.73, 0.88)$.

Shown in Figs. 6.22-6.25 are the SoC evolution of all battery units during the charging process, the total power of the BESS and the power demand and the state evolution of the estimators,

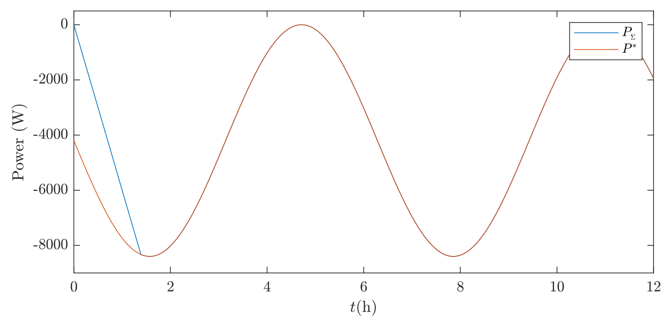

Figure 6.23: The total output power of all battery units and the power demand under power allocation (6.17) and estimators (6.4.2), (6.20).

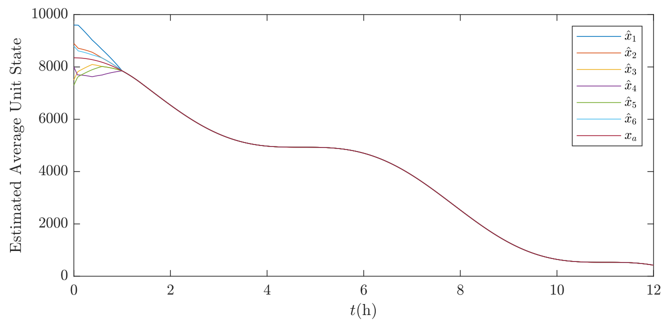

Figure 6.24: The estimated value by all battery units and the true value of x_a under estimators (6.4.2).

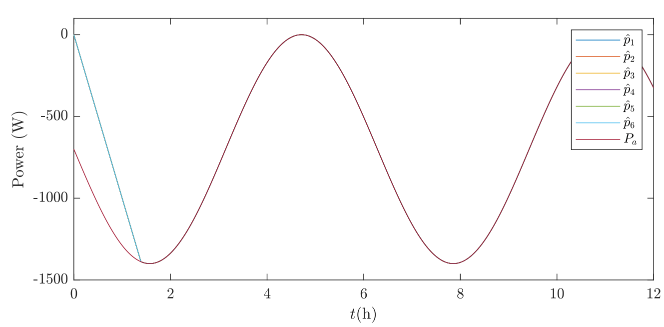

Figure 6.25: The estimated value by all battery units and the true value of P_a under estimators (6.20).

respectively, with the parameters in the estimators chosen as $\alpha = 1000$ and $\beta = 1000$. It is observed that the SoC of all battery units converges closer to each other during the charging/discharging process and the total charging/discharging power of the BESS tracks the power demand. The estimates of x_a and p_a by all battery units converge to the true value accurately in finite time.

6.5 Adaptive Parameter Estimation Based State-of-Charge Balancing and Power Delivery

In this section, we consider the control design for networked heterogeneous battery units in a BESS with unknown parameters. A novel design of adaptive power distribution algorithms is presented, followed by a rigorous analysis of the performance of the closed-loop system. In particular, the adaptive estimation of the battery parameters facilitates us to develop distributed control algorithms for the battery units without direct knowledge of their parameters. This design strengthens the design in the previous section, which requires precise knowledge of the capacities and the terminal voltages of the battery units. The SoC balancing is achieved by adaptively allocating the total power demand based on distributed estimations of the average unit state and the average desired power, and the adaptive estimations of the units' unknown parameters. We show that, under the proposed distributed adaptive power allocating algorithms, the BESS achieves SoC balancing among its battery units and delivers the desired total power to any pre-specified level of accuracy. Our ability to precisely estimate the battery parameters can also help us to monitor the health of the battery units. For example, a capacity drop may indicate a degradation or failure of the battery unit [75].

We will first design adaptive parameter estimators for the battery units. We rewrite the SoC dynamics of battery unit i as

$$s[S_i](t) = \theta_i^* P_i(t), \tag{6.21}$$

where $\theta_i^* = -\frac{1}{Q_{\text{nom},i} V_i}$ is the true value of the unknown constant parameter and s is the differentiation operator $s[S_i](t) = \dot{s}_i$.

For each battery unit i, choose $\Lambda_i(s) = s + \lambda_{0,i}$ with $\lambda_{0,i} > 0$. By operating on both sides of (6.21) with the stable filter $\frac{1}{\Lambda_i(s)}$, we have

$$\frac{s}{s+\lambda_{0,i}}[S_i](t) = \frac{\theta_i^*}{s+\lambda_{0,i}}[P_i](t),$$

which can be rewritten as

$$S_i(t) - \frac{\lambda_{0,i}}{s+\lambda_{0,i}}[S_i](t) = \frac{\theta_i^*}{s+\lambda_{0,i}}[P_i](t). \tag{6.22}$$

We note the presence of the unknown parameter θ_i^* in the right-hand side of (6.22).

Let $\theta_i(t)$ be the estimate of the parameter θ_i^* and define the estimation error as $\epsilon_i(t) = \big(\theta_i(t) -$

$\theta_i^*)\frac{1}{s+\lambda_{0,i}}[P_i](t)$. Then, in view of (6.22), we have

$$\epsilon_i(t) = \frac{\theta_i(t)}{s+\lambda_{0,i}}[P_i](t) - \left(S_i(t) - \frac{\lambda_{0,i}}{s+\lambda_{0,i}}[S_i](t)\right).$$

To generate the signals $\omega_{i,1}(t) = \frac{1}{s+\lambda_{0,i}}[P_i](t)$ and $\omega_{i,2}(t) = \frac{1}{s+\lambda_{0,i}}[S_i](t)$, we construct the following dynamic systems,

$$\begin{cases} \dot{\omega}_{i,1}(t) = -\lambda_{0,i}\omega_{i,1}(t) + P_i(t), \\ \dot{\omega}_{i,2}(t) = -\lambda_{0,i}\omega_{i,2}(t) + S_i(t). \end{cases}$$

Then, the estimation error is given as

$$\epsilon_i(t) = \theta_i(t)\omega_{i,1}(t) - \big(S_i(t) - \lambda_{0,i}\omega_{i,2}(t)\big).$$

We will use the normalized gradient algorithm [84] for updating the parameter estimation $\theta_i(t)$. Consider the quadratic cost function $J_i(\theta_i) = \frac{\epsilon_i^2(t)}{2m_i^2(t)}$, with $m_i(t)$ being the normalizing signal. The steepest decent direction of $J_i(\theta_i(t))$ is

$$-\frac{\partial J_i}{\partial \theta_i} = \frac{\epsilon_i(t)w_{i,1}(t)}{m_i^2(t)}.$$

Given the knowledge of the parameter region $[\theta_{\min}, \theta_{\max}]$, an adaptive parameter update law for $\theta_i(t)$ is designed as

$$\dot{\theta}_i(t) = g_i(t) + f_i(t),\ \theta_i(0) = \theta_{i,0},\ t \geq 0, \tag{6.23}$$

where $\theta_{i,0} \in [\theta_{\min}, \theta_{\max}]$ is the initial estimate of θ_i^* and is usually set to the nominal value of the parameter,

$$g_i(t) = -\frac{\Gamma_i\omega_{i,1}(t)\epsilon_i(t)}{m_i^2(t)}, \tag{6.24}$$

with the gain $\Gamma_i > 0$ and $m_i(t) = \sqrt{1+\gamma_i\omega_{i,1}^2}$, $\gamma_i > 0$, and

$$f_i(t) = \begin{cases} 0, & \text{if } \theta_i(t) \in (\theta_{\min}, \theta_{\max}), \text{ or} \\ & \text{if } \theta_i(t) = \theta_{\min},\ g_i(t) \geq 0, \text{ or} \\ & \text{if } \theta_i(t) = \theta_{\max},\ g_i(t) \leq 0, \\ -g_i(t), & \text{otherwise.} \end{cases} \tag{6.25}$$

Theorem 6.3. The adaptive parameter update law (6.23)-(6.25) guarantees that $\theta_i(t)$, $\dot{\theta}_i(t)$ and

$\frac{\epsilon_i(t)}{m_i(t)}$ are bounded, and $\theta_i(t) \in [\theta_{\min}, \theta_{\max}]$. In addition, $\lim_{t\to\infty} \theta_i(t) = \theta_i^*$ exponentially.

Proof: Consider the positive definite function $\tilde{V}_i(\tilde{\theta}_i) = \frac{1}{2\Gamma_i}\tilde{\theta}_i^2(t)$, where $\tilde{\theta}_i(t) = \theta_i(t) - \theta_i^*$. Note that $\dot{\tilde{\theta}}_i(t) = \dot{\theta}_i(t)$ as θ_i^* is constant. Recalling that $\epsilon_i(t) = \tilde{\theta}_i(t)\omega_{i,1}(t)$, we have

$$\dot{\tilde{V}}_i = -\frac{\epsilon_i^2(t)}{m_i^2(t)} + \frac{1}{\Gamma_i}\tilde{\theta}_i(t)f_i(t), \ t \geq 0. \tag{6.26}$$

It is noted that $\frac{1}{\Gamma}\tilde{\theta}_i(t)f_i(t) \leq 0$ because of the definition of $f_i(t)$. Hence, $\tilde{V}_i(\tilde{\theta}_i)$, $\tilde{\theta}_i(t)$ and $\theta_i(t)$ are bounded. It can also be observed from the definition of $f_i(t)$ that the update law ensures that $\theta_i(t) \in [\theta_{\min}, \theta_{\max}]$, $t \geq 0$. By the boundedness of $\tilde{\theta}_i(t)$ and the relation $\frac{|\epsilon_i(t)|}{m_i(t)} = \frac{|\tilde{\theta}_i(t)||\omega_{i,1}(t)|}{\sqrt{1+\gamma_i\omega_{i,1}^2(t)}}$, we have the boundedness of $\frac{\epsilon_i(t)}{m_i(t)}$, which, together with the inequality

$$|\dot{\theta}_i(t)| \leq \frac{\Gamma_i|\omega_{i,1}(t)|}{\sqrt{1+\gamma_i\omega_{i,1}^2(t)}}\frac{|\epsilon_i(t)|}{m_i(t)}, \tag{6.27}$$

implies that $\dot{\theta}_i(t)$ is bounded.

Note that, under Assumption 6.3, both signals $\omega_{i,1}(t)$ and $\frac{\omega_{i,1}(t)}{m_i(t)}$ are persistently exciting [84]. Thus, there exist constants $\delta > 0$ and $c > 0$ such that $\int_\sigma^{\sigma+\delta} \frac{\omega_{i,1}^2(\tau)}{m_i^2(\tau)}\mathrm{d}\tau \geq c$ for any $\sigma \geq 0$. It follows from (6.26) that

$$\dot{\tilde{V}}_i \leq -2\Gamma_i\frac{\omega_{i,1}^2(t)}{m_i^2(t)}\tilde{V}_i, \tag{6.28}$$

which, by the comparison theorem, implies that,

$$\begin{aligned}\tilde{V}_i(\sigma+\delta) &\leq e^{-2\Gamma_i\int_\sigma^{\sigma+\delta}\frac{\omega_{i,1}^2(\tau)}{m_i^2(\tau)}\mathrm{d}\tau}\tilde{V}_i(\sigma)\\ &\leq e^{-2c\Gamma_i\delta}\tilde{V}_i(\sigma), \ \sigma \geq 0,\end{aligned}$$

or

$$|\tilde{\theta}_i(\sigma+\delta)| \leq e^{-c\Gamma_i\delta}|\tilde{\theta}_i(\sigma)|, \ \sigma \geq 0.$$

Consequently, we have $|\tilde{\theta}(\sigma+k\delta)| \leq e^{-kc\Gamma_i\delta}|\tilde{\theta}(\sigma)|$, for any $\sigma \geq 0$ and any integer k. Since $e^{-c\Gamma_i} < 1$, we conclude that $\lim_{t\to\infty}\tilde{\theta}_i(t) = 0$ exponentially. $\square$

With the adaptive parameter estimations, the distributed average desired power estimators (6.9) remain unchanged and the distributed average unit state estimators (6.8) are implemented as

$$\begin{cases} \dot{\hat{x}}_{\mathrm{a},i} = \dot{\bar{x}}_i - \alpha(\hat{x}_{\mathrm{a},i} - \bar{x}_i) - \beta \sum_{j=1}^{N} a_{ij}(\hat{x}_{\mathrm{a},i} - \hat{x}_{\mathrm{a},j}) - \nu_i, \ \hat{x}_{\mathrm{a},i}(0) = x_i(0), \\ \dot{\nu}_i = \alpha\beta \sum_{j=1}^{N} a_{ij}(\hat{x}_{\mathrm{a},i} - \hat{x}_{\mathrm{a},j}), \ v_i(0) = 0, \end{cases} \tag{6.29}$$

where

$$\bar{x}_i(t) = \begin{cases} \bar{x}_{\mathrm{d},i}(t) = -S_i(t)/\theta_i(t) \text{ (in discharging mode)}, \\ \bar{x}_{\mathrm{c},i}(t) = (S_i(t) - 1)/\theta_i(t) \text{ (in charging mode)}. \end{cases}$$

The power distribution algorithms (6.10) and (6.11) for battery unit i is implemented with the adaptive parameter estimation as,

$$P_i(t) = \begin{cases} \frac{\bar{x}_{\mathrm{d},i}(t)}{\max\{\frac{a_1}{2}, \hat{x}_{\mathrm{a},i}(t)\}} \hat{p}_{\mathrm{a},i}(t) \text{ (in discharging mode)}, \\ \frac{\bar{x}_{\mathrm{c},i}(t)}{\max\{\frac{a_1}{2}, \hat{x}_{\mathrm{a},i}(t)\}} \hat{p}_{\mathrm{a},i}(t) \text{ (in charging mode)}, \end{cases} \tag{6.30}$$

where $\hat{x}_{\mathrm{a},i}$ is given by (6.29).

The following result pertains to the proposed adaptive power distribution algorithms (6.30).

Theorem 6.4. [61] Consider a BESS consisting of N networked battery units with unknown parameters. The relation between the dynamics of SoC and the charging/discharging power of each battery unit is described by (6.3). Let the communication topology satisfy Assumptions 6.1 and 6.2. Let the power demand satisfy Assumption 6.3. Then, for any given ε_s, $\varepsilon_p > 0$, there exist β, $\kappa > 0$ such that the distributed battery unit control algorithms (6.30) solve Problem 6.1.

Proof: We will prove only the case of discharging. The case of charging can be proven similarly.

Define $\tilde{x}_{\mathrm{d},i} = \bar{x}_{\mathrm{d},i} - x_{\mathrm{d},i}$, $\tilde{x}_{\mathrm{a},i} = \hat{x}_{\mathrm{a},i} - x_{\mathrm{a},i}$, and $\tilde{p}_{\mathrm{a},i} = \hat{p}_{\mathrm{a},i} - P_{\mathrm{a}}$. Choose $\beta > \frac{2\gamma_{\mathrm{s}}}{\alpha_1 \lambda_2}$. Then, there exists $T > 0$ such that $\hat{x}_{\mathrm{a},i}(t) \geq \frac{1}{2} a_1$, $t \geq T$.

The discharging power (6.30) can be rewritten as

$$\begin{aligned} P_i(t) &= \frac{x_{\mathrm{d},i}(t) + \tilde{x}_{\mathrm{d},i}(t)}{x_{\mathrm{a}}(t) + \tilde{x}_i(t)} (P_{\mathrm{a}}(t) + \tilde{p}_i(t)) \\ &= \frac{\left(1 + \frac{\tilde{x}_{\mathrm{d},i}(t)}{x_{\mathrm{d},i}(t)}\right)\left(1 + \frac{\tilde{p}_i(t)}{P_{\mathrm{a}}(t)}\right)}{1 + \frac{\tilde{x}_i(t)}{x_{\mathrm{a}}(t)}} \frac{P_{\mathrm{a}}(t)}{x_{\mathrm{a}}(t)} x_{\mathrm{d},i}(t) \\ &= k(t)(1 + \delta_i(t)) P_{\mathrm{a}}(t), \ t \geq T, \end{aligned}$$

where $k(t) = \frac{P_{\mathrm{a}}(t)}{x_{\mathrm{a}}(t)} \geq \frac{P}{Na_2}$ and $\delta_i(t) = \frac{\left(1+\frac{\tilde{x}_{\mathrm{d},i}(t)}{x_{\mathrm{d},i}(t)}\right)\left(1+\frac{\tilde{p}_i(t)}{P_{\mathrm{a}}(t)}\right)}{1+\frac{\tilde{x}_i(t)}{x_{\mathrm{a}}(t)}} - 1$. Then, recalling that $\dot{S}_i = -\frac{1}{Q_{\mathrm{nom},i}V_i}P_i$ and $x_{\mathrm{d},i} = Q_{\mathrm{nom},i}V_iS_i$, we have

$$\dot{S}_i = -k(t)(1+\delta_i(t))S_i. \tag{6.31}$$

The time derivative of the function

$$V_{ij} = \frac{1}{2}\left(S_i - S_j\right)^2,\ i \neq j,\ i,j \in \{1,2,\cdots,N\},$$

along the trajectory of (6.31) is given as

$$\dot{V}_{ij} = -k(t)(S_i - S_j)\Big((1+\delta_i(t))S_i - (1+\delta_j(t))S_j\Big).$$

It follows that $\dot{V}_{ij} < 0$ if $S_i > S_j$ and $\frac{S_i}{S_j} > \frac{1+\delta_j(t)}{1+\delta_i(t)}$, or if $S_j < S_i$ and $\frac{S_i}{S_j} < \frac{1+\delta_j(t)}{1+\delta_i(t)}$. Then, in steady state, we have

$$\frac{1+\underline{\delta}}{1+\bar{\delta}} \leq \frac{S_i}{S_j} \leq \frac{1+\bar{\delta}}{1+\underline{\delta}}$$

where $\underline{\delta} = \frac{1-\frac{N\bar{Q}\gamma_{\mathrm{p}}}{\kappa p}}{1+\frac{\gamma_{\mathrm{s}}}{a_1\beta\lambda_2}} - 1 < 0$ and $\bar{\delta} = \frac{1+\frac{N\bar{Q}\gamma_{\mathrm{p}}}{\kappa p}}{1-\frac{\gamma_{\mathrm{s}}}{a_1\beta\lambda_2}} - 1 > 0$ can be rendered arbitrarily small by selecting sufficiently large values of the design parameters β and κ.

The total discharging power of the BESS is given as

$$P_{\Sigma}(t) = \sum_{i=1}^{N} \frac{\left(1+\frac{\tilde{x}_{\mathrm{d},i}(t)}{x_{\mathrm{d},i}(t)}\right)\left(1+\frac{\tilde{p}_i(t)}{P_{\mathrm{a}}(t)}\right)}{1+\frac{\tilde{x}_i(t)}{x_{\mathrm{a}}(t)}} \frac{x_{\mathrm{d},i}(t)}{x_{\mathrm{a}}(t)} P_{\mathrm{a}}(t).$$

In steady state, $P_{\Sigma}(t)$ satisfies

$$(1+\underline{\delta})P^* \leq P_{\Sigma}(t) \leq (1+\bar{\delta})P^*,$$

and thus can be made arbitrarily close to $P^*(t)$ by selecting sufficiently large values of the parameters β and κ. □

6.6 Simulation For Adaptive Parameter Estimation Based Algorithm

Consider a BESS with six battery units. The communication network is as shown in Fig. 6.26. Let battery unit 1 be the only battery unit with the knowledge of the power demand, *i.e.*, $b_1 = 1$ and

$b_i = 0, i \neq 1$.

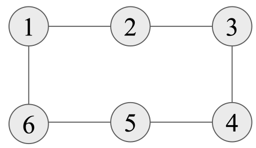

Figure 6.26: The communication topology.

Let the true values of the parameters of the battery units be $(Q_{\text{nom},1}, Q_{\text{nom},2}, Q_{\text{nom},3}, Q_{\text{nom},4}, Q_{\text{nom},5}, Q_{\text{nom},6}) = (100, 190, 200, 210, 220, 230)$ Ah and $(V_1,\ V_2, V_3, V_4, V_5, V_6) = (50, 50, 50, 50, 50, 50)$ V, which are unknown to the controllers. The capacity of the 4th battery unit drops to 70% at $t = 15$ h.

Let the initial SoC be $(S_1(0), S_2(0), S_3(0), S_4(0), S_5(0), S_6(0)) = (0.96, 0.89, 0.75, 0.8, 0.73, 0.88)$. Let the discharging power demand be $P^*(t) = 1680$ W. Let the parameters in the distributed estimators be $\alpha_i = 2000$, $\beta_i = 20$ and $\kappa_i = 50$, $i = 1, 2, \cdots, 6$. The parameters in the adaptive update laws are chosen as $\Gamma_i = 1$, $\gamma_i = 1$ and $\lambda_{0,i} = 100$, $i = 1, 2, \cdots, 6$. Let $\omega_{i,1}(0) = \omega_{i,2}(0) = 0$, $i = 1, 2, \cdots, 6$. The region of the parameter is chosen as $\left[-\frac{1}{0.2Q_{\text{nom},i}V_i}, -\frac{1}{1.5Q_{\text{nom},i}V_i}\right]$, $i = 1, 2, \cdots, 6$. In the simulation, we adopt the nominal value $\theta_{i,0} = -\frac{1}{10000}$ $(\text{Wh})^{-1}$, $i = 1, 2, \cdots, 6$.

Shown in Fig. 6.27 is the evolution of the SoC of all battery units. Clearly, the six battery units achieve SoC balancing. Fig. 6.28 shows that the total power of all batter units tracks the discharging power demand.

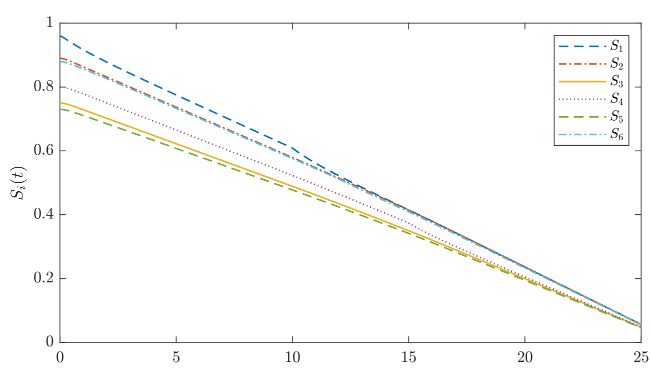

Figure 6.27: The SoC of all battery units.

Fig. 6.29 shows the power of individual battery units. It is observed that units with a larger value of their state have higher power. After $t = 15$ h, the powers of all units adapt to the capacity drop of the 4th battery unit and satisfy the power demand.

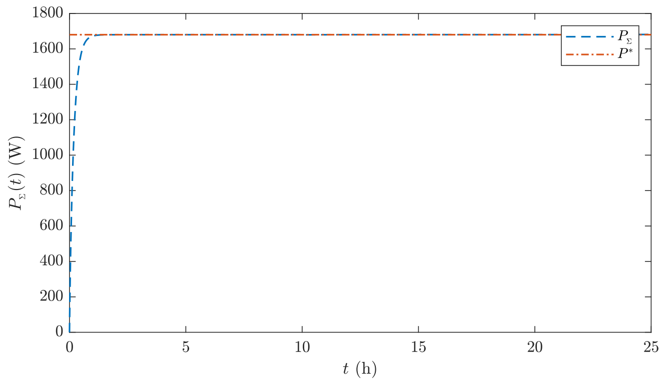

Figure 6.28: The total power and the power demand.

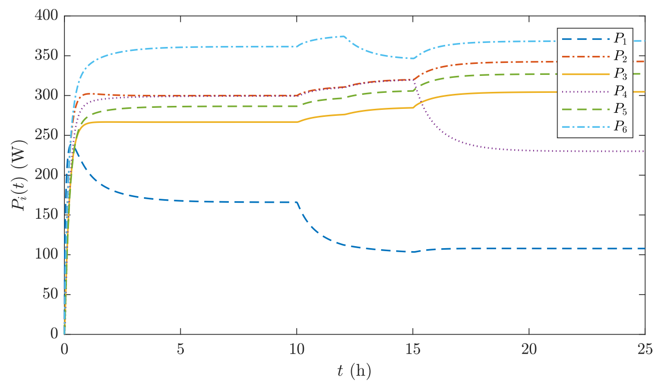

Figure 6.29: The discharging powers of individual battery units.

Fig. 6.30 shows that the estimated average power demand tracks the true average power demand. Fig. 6.31 shows that the estimated average unit state tracks the true average unit state. Fig. 6.32 shows the estimated parameters of all battery units. It is observed that the estimated parameters stay in the pre-specified region and converge to their true values $(-0.2000 \times 10^{-3}, -0.1053 \times 10^{-3}, -0.1000 \times 10^{-3}, -0.0952 \times 10^{-3}, -0.0909 \times 10^{-3}, -0.0870 \times 10^{-3})$ $(\mathrm{Wh})^{-1}$ in $t = 0$ to $t = 15$ h. After $t = 15$ h, the estimate $\theta_4(t)$ converges to its new true value.

6.7 Nonlinear Observer Based State-of-Charge Balancing and Power Delivery

The second-order Thevenin equivalent circuit model is used to characterize the internal dynamics of the battery units. The equivalent circuit model is shown in Fig. 6.33, in which $U_{\mathrm{OC},i}$ is the equivalent OCV of the ith battery unit, $U_{\mathrm{B},i}$ is the terminal voltage and I_i is the current. The resistor $R_{0,i}$ represents the internal ohmic resistance. The two resistor-capacitor circuit components,

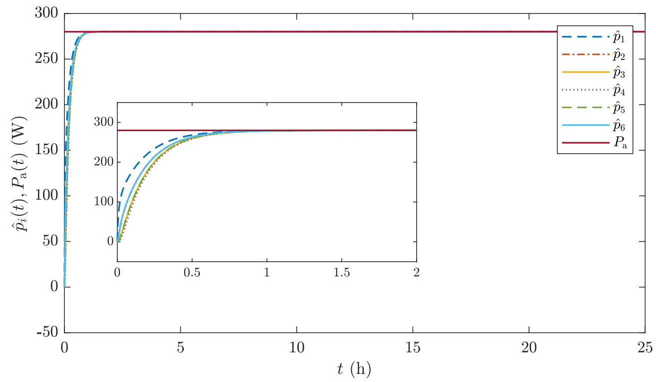

Figure 6.30: The estimated average power demand versus the true average power demand.

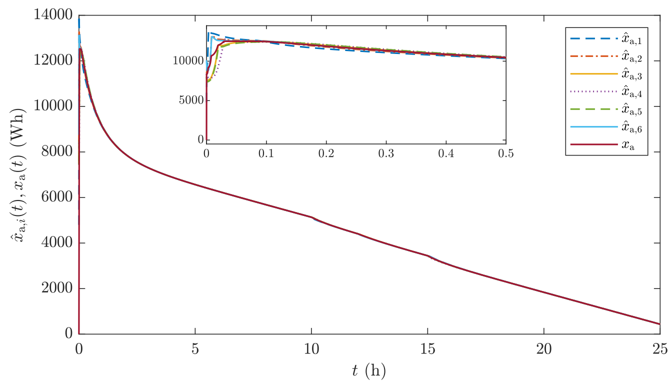

Figure 6.31: The estimated average unit state versus the true average unit state.

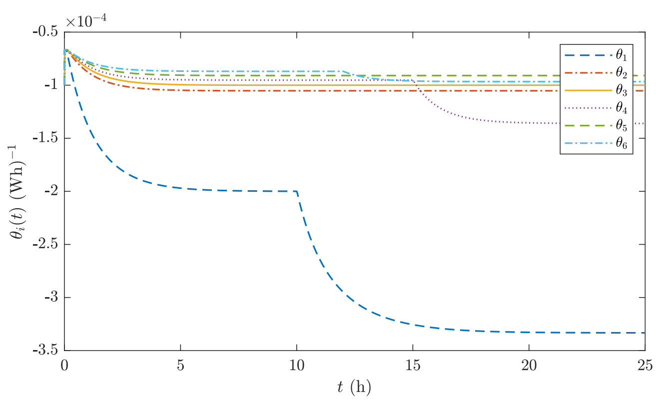

Figure 6.32: The adaptively estimated battery unit parameters.

respectively characterized by $(R_{1,i}, C_{1,i})$ and $(R_{2,i}, C_{2,i})$, are used to simulate the physical processes inside the battery units. These parameters can be identified by experimental results and are assumed to be known constants in this work.

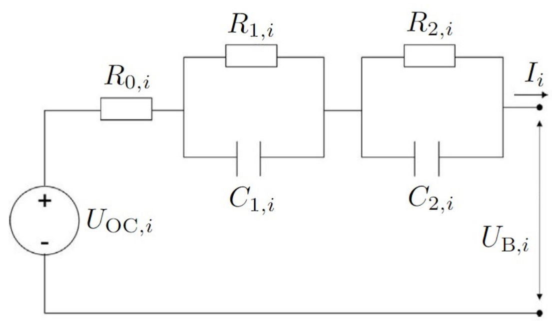

Figure 6.33: The second-order Thevenin model of a battery unit.

For a battery unit in its steady state, its OCV $U_{\mathrm{OC},i}$ is directly correlated to its SoC and other operating conditions such as ambient temperature. We assume that the operating condition remains constant in this work, under which the $U_{\mathrm{OC},i}$ is known to be monotonically and nonlinearly related to the SoC S_i. Such a nonlinear SoC-OCV map can be approximated as (see, for example, [7, 33])

$$U_{\mathrm{OC},i}(S_i) = \sum_{k=0}^{\bar{k}} c_k S_i^k, \;\; S_i \in [0,1],$$

with $\frac{\mathrm{d}U_{\mathrm{OC},i}}{\mathrm{d}S_i} \geq \underline{d}_i > 0$ for some constant $\underline{d}_i$, in which $\bar{k} > 0$ is the degree of the polynomial and c_k are the coefficients that can be derived by curve fitting of the SoC-OCV relation.

By applying Kirchoff's voltage and current laws to the above equivalent circuit model, we obtain the following equations that describe the dynamics inside each battery unit i,

$$\begin{cases} U_{\mathrm{OC},i}(S_i) = I_i R_{0,i} + U_{1,i} + U_{2,i} + U_{\mathrm{B},i}, \\ \qquad\quad I_i = \dfrac{U_{1,i}}{R_{1,i}} + C_{1,i}\dot{U}_{1,i}, \\ \qquad\quad I_i = \dfrac{U_{2,i}}{R_{2,i}} + C_{2,i}\dot{U}_{2,i}, \end{cases}$$

where $U_{1,i}$ and $U_{2,i}$ are the voltage across the capacitors $C_{1,i}$ and $C_{2,i}$, respectively. Meanwhile, recall the equation of the dynamics of the SoC related to the current as

$$\dot{S}_i = -\frac{1}{Q_{\mathrm{nom},i}} I_i. \tag{6.32}$$

By rearranging the above equations, we can rewrite the dynamical model of the ith battery unit as

$$\begin{cases} \dot{U}_{1,i} = -\dfrac{1}{R_{1,i}C_{1,i}}U_{1,i} + \dfrac{1}{C_{1,i}}I_i, \\ \dot{U}_{2,i} = -\dfrac{1}{R_{2,i}C_{2,i}}U_{2,i} + \dfrac{1}{C_{2,i}}I_i, \\ \dot{S}_i = -\dfrac{1}{Q_{\mathrm{nom},i}}I_i, \\ U_{\mathrm{B},i} = U_{\mathrm{OC},i}(S_i) - U_{1,i} - U_{2,i} - R_{0,i}I_i. \end{cases} \tag{6.33}$$

The power of the ith battery unit is calculated as

$$P_{\mathrm{B},i} = U_{\mathrm{B},i}I_i.$$

Given the direction of I_i as indicated in Fig. 6.33, $P_{\mathrm{B},i} > 0$ when the ith battery unit is discharging, and $P_{\mathrm{B},i} < 0$ when the unit is charging. The total power of the system is given by

$$P_\Sigma = \sum_{i=1}^{N} P_{\mathrm{B},i}.$$

In what follows, a nonlinear observer is first constructed for each battery unit i based on the equivalent circuit model and the nonlinear SoC-OCV map. Then, distributed estimation algorithms for the average battery state and the average power demand are designed, based on which a power distribution law is proposed for each battery unit i. Finally, the analysis of the closed-loop system is provided to show that the proposed design meets the objectives of SoC balancing and power delivery.

The dynamical model (6.33) of the battery units can be written in a compact state-space form as

$$\begin{cases} \dot{x}_i = A_i x_i + B_i u_i, \\ y_i = h_i(x_i, u_i), \end{cases} \tag{6.34}$$

where

$$A_i = \begin{bmatrix} -\frac{1}{R_{1,i}C_{1,i}} & 0 & 0 \\ 0 & -\frac{1}{R_{2,i}C_{2,i}} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; B_i = \begin{bmatrix} \frac{1}{C_{1,i}} \\ \frac{1}{C_{2,i}} \\ -\frac{1}{Q_{\mathrm{nom},i}} \end{bmatrix},$$

$$h_i(x_i, u_i) = -x_{i,1} - x_{i,2} - U_{\mathrm{OC},i}(x_{i,3}) - R_{0,i}u_i,$$

and $x_i = [x_{i,1}\ x_{i,2}\ x_{i,3}]^{\mathrm{T}} = [U_{1,i}\ U_{2,i}\ S_i]^{\mathrm{T}}$ is the state, $u_i = I_i$ is the input, and $y_i = U_{\mathrm{B},i}$ is the output. It is noted that a nonlinear relationship exists from the state x_i to the output y_i because of the nonlinear function $U_{\mathrm{OC},i}(x_{i,3})$.

Since the SoC of each battery unit is modeled as an unknown internal state, we will design an observer to estimate the SoC from the battery terminal voltage and the current. Motivated by [31], we construct a nonlinear SoC observer for battery unit i based on its model (6.34) as follows,

$$\dot{\hat{x}}_i = A_i\hat{x}_i + B_iu_i + K_i\Big(\frac{\partial h_i}{\partial x_i}\Big)^{\mathrm{T}}(\hat{x}_i, u_i)\big(y - h_i(\hat{x}_i, u_i)\big), \tag{6.35}$$

where $\hat{x}_i = [\hat{x}_{i,1}\ \hat{x}_{i,2}\ \hat{x}_{i,3}]^{\mathrm{T}}$ is the estimate of $x_i = [x_{i,1}\ x_{i,2}\ x_{i,3}]^{\mathrm{T}} = [U_{1,i}\ U_{2,i}\ S_i]^{\mathrm{T}}$. The initial condition of the observer is set according to the measurement of the battery unit at rest, which can be assumed to be close to the true state. The gain matrix K_i is designed to be a symmetric and positive definite solution to the Lyapunov equation

$$-D_i = A_i^{\mathrm{T}}K_i^{-1} + K_i^{-1}A_i, \tag{6.36}$$

with the matrix $D_i = \mathrm{diag}\{d_1, d_2, 0\}$, in which $d_1 > 0$, $d_2 > 0$. Such a K_i exists due to the negative semidefiniteness and the structure of A_i.

From the estimated state $\hat{x}_i$, the estimated SoC is given as

$$\hat{S}_i = \hat{x}_{i,3},\ \ i = 1, 2, \cdots, N.$$

The power distribution law for the ith battery unit for discharging is designed as

$$P_{\mathrm{B},i} = \frac{Q_{\mathrm{nom},i}U_{\mathrm{B},i}S_i}{\frac{1}{N}\sum_{j=1}^{N} Q_{\mathrm{nom},j}U_{\mathrm{B},j}S_j}P_{\mathrm{a}}^*,\ \ i = 1, 2, \cdots, N,$$

and the power distribution law for the ith battery unit for charging is designed as

$$P_{\mathrm{B},i} = \frac{Q_{\mathrm{nom},i}U_{\mathrm{B},i}(1 - S_i)}{\frac{1}{N}\sum_{j=1}^{N} Q_{\mathrm{nom},j}U_{\mathrm{B},j}(1 - S_j)}P_{\mathrm{a}}^*,\ \ i = 1, 2, \cdots, N,$$

where $P_{\mathrm{a}}^* = \frac{1}{N}P^*$ is the average power demand per battery unit, and the signals $\frac{1}{N}\sum_{j=1}^{N} Q_{\mathrm{nom},j}U_{\mathrm{B},j}S_j$ and $\frac{1}{N}\sum_{j=1}^{N} Q_{\mathrm{nom},j}U_{\mathrm{B},j}(1 - S_j)$ are the average of signals $Q_{\mathrm{nom},j}U_{\mathrm{B},j}S_j$ and $Q_{\mathrm{nom},j}U_{\mathrm{B},j}(1 - S_j)$ of all batteries, respectively. It is noted that the power of the ith battery unit is based on P_{a}^*, and adjusted by the signal $Q_{\mathrm{nom},i}U_{\mathrm{B},i}S_i$ over the average of all battery units for discharging, or by

$Q_{\mathrm{nom},i}U_{\mathrm{B},i}(1-S_i)$ over the corresponding average for charging.

Using the estimated SoC $\hat{S}_i$ from the observer (6.35), the above power distribution laws are given as

$$P_{\mathrm{B},i} = \frac{Q_{\mathrm{nom},i}U_{\mathrm{B},i}\hat{S}_i}{\frac{1}{N}\sum_{j=1}^{N} Q_{\mathrm{nom},j}U_{\mathrm{B},j}\hat{S}_j} P_{\mathrm{a}}^*, \ i = 1, 2, \cdots, N, \tag{6.37}$$

for discharging, and

$$P_{\mathrm{B},i} = \frac{Q_{\mathrm{nom},i}U_{\mathrm{B},i}(1-\hat{S}_i)}{\frac{1}{N}\sum_{j=1}^{N} Q_{\mathrm{nom},j}U_{\mathrm{B},j}(1-\hat{S}_j)} P_{\mathrm{a}}^*, \ i = 1, 2, \cdots, N, \tag{6.38}$$

for charging.

Since the average signals $\frac{1}{N}\sum_{j=1}^{N} Q_{\mathrm{nom},j}U_{\mathrm{B},j}\hat{S}_j$ and $\frac{1}{N}\sum_{j=1}^{N} Q_{\mathrm{nom},j}U_{\mathrm{B},j}(1-\hat{S}_j)$ are global information of the system, we design the following distributed finite-time average estimation algorithm [19],

$$\begin{cases} \dot{\mu}_i = -\alpha \mathrm{sign}\Big(\sum_{j=1}^{N} a_{ij}\big(\hat{\xi}_{\mathrm{a},i} - \hat{\xi}_{\mathrm{a},j}\big)\Big), \\ \hat{\xi}_{\mathrm{a},i} = \sum_{j=1}^{N} a_{ij}(\mu_i - \mu_j) + \xi_i, \end{cases} \tag{6.39}$$

where $\alpha > 0$ is a design parameter, and $\hat{\xi}_{\mathrm{a},i}$ is the ditributed estimation of the average signal $\frac{1}{N}\sum_{j=1}^{N}\xi_j$ by the ith battery unit, with $\xi_i = Q_{\mathrm{nom},i}U_{\mathrm{B},i}\hat{S}_i$ for discharging and $\xi_i = Q_{\mathrm{nom},i}U_{\mathrm{B},i}(1-\hat{S}_i)$ for charging.

Since the average power demand is only known to a portion of the battery units, we design the following distributed finite-time average power estimation algorithm [59],

$$\begin{cases} \dot{\hat{P}}_{\mathrm{a},i} = -\beta \mathrm{sign}(\nu_{\mathrm{i}}), \\ \nu_i = \sum_{j=1}^{N} a_{ij}(\hat{P}_{\mathrm{a},i} - \hat{P}_{\mathrm{a},j}) + b_i(\hat{P}_{\mathrm{a},i} - P_{\mathrm{a}}^*), \end{cases} \tag{6.40}$$

where $\beta > 0$ is a design parameter and $\hat{P}_{\mathrm{a},i}$ is the distributed estimation of P_{a}^* by the ith battery unit.

The proposed estimation algorithms (6.39) and (6.40) have the following properties.

Lemma 6.6. [59, 19] Under Assumptions 6.1 and 6.2, there exist finite $T_1 \geq 0$ and $T_2 \geq 0$, and

$\underline{\alpha} > 0$ and $\underline{\beta} > 0$ such that, for all $i = 1, 2, \cdots, N$,

$$\hat{\xi}_{\mathrm{a},i}(t) = \begin{cases} \dfrac{1}{N} \displaystyle\sum_{j=1}^{N} Q_{\mathrm{nom},j} U_{\mathrm{B},j} \hat{S}_j(t), & \text{for discharging,} \\ \dfrac{1}{N} \displaystyle\sum_{j=1}^{N} Q_{\mathrm{nom},j} U_{\mathrm{B},j} \big(1 - \hat{S}_j(t)\big), & \text{for charging,} \end{cases}$$

for $t \geq T_1$ and $\alpha \geq \underline{\alpha}$, and

$$\hat{P}_{\mathrm{a},i}(t) = \frac{1}{N} P^*,$$

for $t \geq T_2$ and $\beta \geq \underline{\beta}$.

With the proposed distributed estimation algorithms, the power distribution laws for discharging are implemented as

$$P_{\mathrm{B},i} = \frac{Q_{\mathrm{nom},i} U_{\mathrm{B},i} \hat{S}_i}{\hat{\xi}_{\mathrm{a},i}} \hat{P}_{\mathrm{a},i}, \ i = 1, 2, \cdots, N, \tag{6.41}$$

and for charging are implemented as

$$P_{\mathrm{B},i} = \frac{Q_{\mathrm{nom},i} U_{\mathrm{B},i} (1 - \hat{S}_i)}{\hat{\xi}_{\mathrm{a},i}} \hat{P}_{\mathrm{a},i}, \ i = 1, 2, \cdots, N. \tag{6.42}$$

As will be shown in the next subsection, under such power distribution laws, the BESS achieves SoC balancing among all its battery units while satisfying the power demand.

We now establish the local stability of the observer error. Then, we show that SoC balancing is achieved among all the battery units. Finally, we show that the total power of the system satisfies the power demand.

Define the estimation error of the observer for the ith battery unit as $\tilde{x}_i = x_i - \hat{x}_i$. Then, the estimation error dynamics is obtained from (6.33) and (6.35) as

$$\dot{\tilde{x}}_i = A_i \tilde{x}_i + K_i \big(h_i'\big)^{\mathrm{T}} (x_i + \tilde{x}_i) \big(h_i(x_i, u_i) - h_i(x_i + \tilde{x}_i, u_i)\big), \tag{6.43}$$

where

$$\begin{aligned} h_i'(x_i + \tilde{x}_i) &\triangleq \frac{\partial h_i}{\partial x_i} \big(x_i + \tilde{x}_i, u_i\big) \\ &= \begin{bmatrix} -1 & -1 & -U_{\mathrm{OC},i}' \end{bmatrix}, \end{aligned}$$

with $U'_{\mathrm{OC},i} = \frac{\mathrm{d}U_{\mathrm{OC},i}}{\mathrm{d}S_i}(\hat{S}_i)$ for notational simplicity.

Applying the mean value theorem yields

$$\begin{aligned}
&h_i(x_i, u_i) - h_i(x_i + \tilde{x}_i, u_i)\\
&= -h'_i(z_i)\tilde{x}_i\\
&= -h'_i(x_i + \tilde{x}_i)\tilde{x}_i - \big(h'_i(z_i) - h'_i(x_i + \tilde{x}_i)\big)\tilde{x}_i,\\
&\triangleq -h'_i(x_i + \tilde{x}_i)\tilde{x}_i - g_i(x_i + \tilde{x}_i)\tilde{x}_i,
\end{aligned}$$

where z_i is a point on the line segment connecting x_i to $x_i + \tilde{x}_i$, and the function $g_i(x_i + \tilde{x}_i) \triangleq h'_i(z_i) - h'_i(x_i + \tilde{x}_i)$ satisfies $\|g_i(x_i + \tilde{x}_i)\| \to 0$ as $\|\tilde{x}_i\| \to 0$ because of the continuity of $\partial h_i/\partial x_i$.

The estimation error dynamics (6.43) is then further written as

$$\dot{\tilde{x}}_i = \big(A_i - K_i(h'_i)^{\mathrm{T}}h'_i\big)\tilde{x}_i - K_i(h'_i)^{\mathrm{T}}g_i\tilde{x}_i, \tag{6.44}$$

where $h'_i = h'_i(x_i + \tilde{x}_i)$ and $g_i = g_i(x_i + \tilde{x}_i)$.

Consider the following function of the estimation error,

$$V_{\mathrm{ob},i}(\tilde{x}_i) = \tilde{x}_i^{\mathrm{T}}K_i^{-1}\tilde{x}_i.$$

Define a level set of $V_{\mathrm{ob},i}$ as $\bar{\Omega}_{\mathrm{ob},i} = \{\tilde{x}_i \,|\, V_{\mathrm{ob},i} \leq \bar{c}_i\}$ for some constant $\bar{c}_i > 0$. The time derivative of $V_{\mathrm{ob},i}$ along the trajectory of (6.44) inside $\bar{\Omega}_{\mathrm{ob},i}$ is evaluated as

$$\begin{aligned}
\dot{V}_{\mathrm{ob},i} &= \tilde{x}_i^{\mathrm{T}}\big(K_i^{-1}A_i + A_i^{\mathrm{T}}K_i^{-1}\big)\tilde{x}_i - 2\tilde{x}_i^{\mathrm{T}}(h'_i)^{\mathrm{T}}h'_i\tilde{x}_i - 2\tilde{x}_i^{\mathrm{T}}(h'_i)^{\mathrm{T}}g_i\tilde{x}_i\\
&= -\tilde{x}_i^{\mathrm{T}}\big(D_i + 2(h'_i)^{\mathrm{T}}h'_i\big)\tilde{x}_i - 2\tilde{x}_i^{\mathrm{T}}(h'_i)^{\mathrm{T}}g_i\tilde{x}_i,
\end{aligned}$$

where

$$D_i + 2(h'_i)^{\mathrm{T}}h'_i = \begin{bmatrix} d_1 + 2 & 2 & 2U'_{\mathrm{OC},i} \\ 2 & d_2 + 2 & 2U'_{\mathrm{OC},i} \\ 2U'_{\mathrm{OC},i} & 2U'_{\mathrm{OC},i} & 2\big(U'_{\mathrm{OC},i}\big)^2 \end{bmatrix}. \tag{6.45}$$

It is obvious that $\|h'_i\| \leq \bar{h}_i$ for some constant $\bar{h}_i > 0$, for all $\tilde{x}_i \in \bar{\Omega}_{\mathrm{ob},i}$. It is also straightforward to verify that $D_i + 2(h'_i)^{\mathrm{T}}h'_i > 0$, given that $U'_{\mathrm{OC},i} \geq \underline{d}_i$.

Let $\lambda_{i,m} > 0$, $m = 1, 2, 3$, be the eigenvalues of $D_i + 2(h'_i)^{\mathrm{T}}h'_i$. Then,

$$\lambda_{i,1}\lambda_{i,2}\lambda_{i,3} = \det\big(D_i + 2(h'_i)^{\mathrm{T}}h'_i\big)$$

$$= 2d_1 d_2 \big(U'_{\mathrm{OC},i}\big)^2$$
$$\geq 2d_1 d_2 \underline{d}_i^2. \tag{6.46}$$

Since for $\tilde{x}_i \in \bar{\Omega}_{\mathrm{ob},i}$, there exists constant $\bar{d}_i$ such that $U'_{\mathrm{OC},i} \leq \bar{d}_i$, we have

$$\begin{aligned}\lambda_{i,1} + \lambda_{i,2} + \lambda_{i,3} &= \mathrm{trace}\big(D_i + 2(h'_i)^{\mathrm{T}} h'_i\big)\\ &= 4 + d_1 + d_2 + 2\big(U'_{\mathrm{OC},i}\big)^2\\ &\leq 4 + d_1 + d_2 + 2\bar{d}_i^2.\end{aligned} \tag{6.47}$$

Inequality (6.47) and the fact that $\lambda_{i,m} > 0$ imply that $\lambda_{i,m} < 4 + d_1 + d_2 + 2\bar{d}_i^2$, $m = 1, 2, 3$, which, along with (6.46), further imply that there exists constant $\underline{\lambda}_i > 0$, such that $\lambda_{i,\min} = \min\{\lambda_{i,1},\, \lambda_{i,2},\, \lambda_{i,3}\} \geq \underline{\lambda}_i$, for all $\tilde{x}_i \in \bar{\Omega}_{\mathrm{ob},i}$.

Recall that $\|g_i(x_i + \tilde{x}_i)\| \to 0$ as $\|\tilde{x}_i\| \to 0$. We have, for any $\gamma_i > 0$, there exists positive constant $c_i \leq \bar{c}_i$ such that $\|g_i(x_i + \tilde{x}_i)\tilde{x}_i\| \leq \gamma_i \|\tilde{x}_i\|$, for all $\tilde{x}_i \in \Omega_{\mathrm{ob},i} = \{\tilde{x}_i \,|\, V_{\mathrm{ob},i} \leq c_i\} \subseteq \bar{\Omega}_{\mathrm{ob},i}$. Then, $\dot{V}_{\mathrm{ob},i}$ is further evaluated as

$$\begin{aligned}\dot{V}_{\mathrm{ob},i} &\leq -\lambda_{i,\min}\|\tilde{x}_i\|^2 + 2\gamma_i \|h'_i\| \|\tilde{x}_i\|^2\\ &\leq -\big(\underline{\lambda}_i - 2\gamma_i \bar{h}_i\big)\|\tilde{x}_i\|^2.\end{aligned} \tag{6.48}$$

Let γ_i satisfy $\gamma_i < \underline{\lambda}_i / 2\bar{h}_i$. Then, we have, $\dot{V}_{\mathrm{ob},i} < 0$, for all $\tilde{x}_i \in \Omega_{\mathrm{ob},i}$. That is, any trajectory of the observer error that starts in $\Omega_{\mathrm{ob},i}$ will stay in it and converge to zero.

We will next establish that the SoC balancing is achieved among the battery units. It is noted that under Assumptions 6.1, 6.2, and 6.3, for $t \geq \max\{T_1, T_2\}$, the implemented power distribution laws (6.41) and (6.42) become the desired power distribution laws (6.37) and (6.38).

Recall that the equation of $\dot{S}_i$ can be rewritten in terms of the battery power $P_{\mathrm{B},i}$ as

$$\dot{S}_i = -\frac{1}{Q_{\mathrm{nom},i} U_{\mathrm{B},i}} P_{\mathrm{B},i}.$$

Then, consider the following quantities associated with the evolution of the SoC under (6.37) and (6.38),

$$\frac{\dot{S}_i}{S_i} = -\frac{P_{\mathrm{B},i}}{Q_{\mathrm{nom},i} U_{\mathrm{B},i} S_i}$$

$$
\begin{aligned}
&= -\frac{\hat{S}_i}{S_i}\frac{P_{\mathrm{a}}^*}{\frac{1}{N}\sum_{j=1}^{N} Q_{\mathrm{nom},j}U_{\mathrm{B},j}\hat{S}_j} \\
&\triangleq -\frac{\hat{S}_i}{S_i}k_{\mathrm{d}},
\end{aligned} \tag{6.49}
$$

in which $k_{\mathrm{d}} > 0$ satisfies $\underline{k}_{\mathrm{d}} \le k_{\mathrm{d}} \le \bar{k}_{\mathrm{d}}$ for positive constants $\underline{k}_{\mathrm{d}}$ and $\bar{k}_{\mathrm{d}}$ under Assumption 6.3 for discharging mode, and

$$
\begin{aligned}
\frac{\dot{S}_i}{1-S_i} &= -\frac{P_{\mathrm{B},i}}{Q_{\mathrm{nom},i}U_{\mathrm{B},i}(1-S_i)} \\
&= -\frac{1-\hat{S}_i}{1-S_i}\frac{P_{\mathrm{a}}^*}{\frac{1}{N}\sum_{j=1}^{N} Q_{\mathrm{nom},j}U_{\mathrm{B},j}(1-\hat{S}_j)} \\
&\triangleq -\frac{1-\hat{S}_i}{1-S_i}k_{\mathrm{c}},
\end{aligned} \tag{6.50}
$$

in which $k_{\mathrm{c}} < 0$ satisfies $\underline{k}_{\mathrm{c}} \le k_{\mathrm{c}} \le \bar{k}_{\mathrm{c}}$ for negative constants $\underline{k}_{\mathrm{c}}$ and $\bar{k}_{\mathrm{c}}$ under Assumption 6.3 for charging mode. Equations (6.49) and (6.50) will be used in the later analysis of the closed-loop system.

Consider the following function for the discharging mode,

$$
V_{ij}\big(\Delta S_{ij}, \tilde{x}_i, \tilde{x}_j\big) = \frac{1}{2}\Delta S_{ij}^2 + k_i\tilde{x}_i^{\mathrm{T}}K_i^{-1}\tilde{x}_i + k_j\tilde{x}_j^{\mathrm{T}}K_j^{-1}\tilde{x}_j,
$$

where $\Delta S_{ij} \triangleq S_i - S_j$, and $k_i > 0$, $k_j > 0$ are to be determined. It is obvious that $V_{ij} > 0$ for all $(\Delta S_{ij},\, \tilde{x}_i,\, \tilde{x}_j) \neq 0$. Assume that the initial conditions of the observers are set sufficiently close to the true states, that is, the observer errors satisfy $\tilde{x}_i(0) \in \Omega_{\mathrm{ob},i}$ and $\tilde{x}_j(0) \in \Omega_{\mathrm{ob},j}$. This implies that $\tilde{x}_l$ stays in $\Omega_{\mathrm{ob},l}$ and converges to zero, $l = i, j$.

The time derivative of V_{ij} along the trajectories of (6.44) and (6.49) is evaluated as

$$
\begin{aligned}
\dot{V}_{ij} &= \Delta S_{ij}\big((-\hat{S}_i)k_{\mathrm{d}} - (-\hat{S}_j)k_{\mathrm{d}}\big) - k_i\tilde{x}_i^{\mathrm{T}}\big(D_i + 2(h_i')^{\mathrm{T}}h_i' + 2(h_i')^{\mathrm{T}}g_i\big)\tilde{x}_i \\
&\quad - k_j\tilde{x}_j^{\mathrm{T}}\big(D_j + 2(h_j')^{\mathrm{T}}h_j' + 2(h_j')^{\mathrm{T}}g_j\big)\tilde{x}_j \\
&= -\Delta S_{ij}\big(\Delta S_{ij} + (\tilde{S}_i - \tilde{S}_j)\big)k_{\mathrm{d}} - k_i\lambda_{i,\min}\|\tilde{x}_i\|^2 - k_j\lambda_{j,\min}\|\tilde{x}_j\|^2 \\
&\quad + 2k_i\gamma_i\bar{h}_i\|\tilde{x}_i\|^2 + 2k_j\gamma_j\bar{h}_j\|\tilde{x}_j\|^2 \\
&\le -k_{\mathrm{d}}\Delta S_{ij}^2 - k_{\mathrm{d}}\Delta S_{ij}(\tilde{S}_i - \tilde{S}_j) - k_i\big(\underline{\lambda}_i - 2\gamma_i\bar{h}_i\big)\|\tilde{x}_i\|^2 \\
&\quad - k_j\big(\underline{\lambda}_j - 2\gamma_j\bar{h}_j\big)\|\tilde{x}_j\|^2,
\end{aligned} \tag{6.51}
$$

where $\tilde{S}_i = \hat{S}_i - S_i$, $i = 1, 2, \cdots, N$. Recall that $\tilde{S}_i = \tilde{x}_{i,3}$. Then, $\dot{V}_{ij}$ in (6.51) is further evaluated as

$$
\begin{aligned}
\dot{V}_{ij} &\leq -k_{\mathrm{d}}\Delta S_{ij}^2 + k_{\mathrm{d}}|\Delta S_{ij}|\big(\|\tilde{x}_i\| + \|\tilde{x}_j\|\big) - k_i\big(\underline{\lambda}_i - 2\gamma_i\bar{h}_i\big)\|\tilde{x}_i\|^2 \\
&\quad - k_j\big(\underline{\lambda}_j - 2\gamma_j\bar{h}_j\big)\|\tilde{x}_j\|^2 \\
&= -k_{\mathrm{d}}\Delta S_{ij}^2 + 2\Big(\frac{1}{2}\sqrt{k_{\mathrm{d}}}|\Delta S_{ij}|\Big)\big(\sqrt{k_{\mathrm{d}}}\|\tilde{x}_i\|\big) + 2\Big(\frac{1}{2}\sqrt{k_{\mathrm{d}}}|\Delta S_{ij}|\Big)\big(\sqrt{k_{\mathrm{d}}}\|\tilde{x}_j\|\big) \\
&\quad - k_i\big(\underline{\lambda}_i - 2\gamma_i\bar{h}_i\big)\|\tilde{x}_i\|^2 - k_j\big(\underline{\lambda}_j - 2\gamma_j\bar{h}_j\big)\|\tilde{x}_j\|^2 \\
&\leq -k_{\mathrm{d}}\Delta S_{ij}^2 + \frac{1}{4}k_{\mathrm{d}}\Delta S_{ij}^2 + k_{\mathrm{d}}\|\tilde{x}_i\|^2 + \frac{1}{4}k_{\mathrm{d}}\Delta S_{ij}^2 + k_{\mathrm{d}}\|\tilde{x}_j\|^2 \\
&\quad - k_i\big(\underline{\lambda}_i - 2\gamma_i\bar{h}_i\big)\|\tilde{x}_i\|^2 - k_j\big(\underline{\lambda}_j - 2\gamma_j\bar{h}_j\big)\|\tilde{x}_j\|^2 \\
&\leq -\frac{1}{2}\underline{k}_{\mathrm{d}}\Delta S_{ij}^2 - \big(k_i(\underline{\lambda}_i - 2\gamma_i\bar{h}_i) - \bar{k}_{\mathrm{d}}\big)\|\tilde{x}_i\|^2 - \big(k_j(\underline{\lambda}_j - 2\gamma_j\bar{h}_j) - \bar{k}_{\mathrm{d}}\big)\|\tilde{x}_j\|^2.
\end{aligned}
$$

Let $k_i > \bar{k}_{\mathrm{d}}/(\underline{\lambda}-2\gamma_i\bar{h})$ and $k_j > \bar{k}_{\mathrm{d}}/(\underline{\lambda}-2\gamma_j\bar{h})$. Then, it follows that $\dot{V}_{ij} < 0$, for all $(\Delta S_{ij},\, \tilde{x}_i,\, \tilde{x}_j) \neq 0$, with $\tilde{x}_i \in \Omega_{\mathrm{ob},i}$ and $\tilde{x}_j \in \Omega_{\mathrm{ob},j}$. Since it has been shown that any $\tilde{x}_l$, $l = i, j$, with $\tilde{x}_l(0) \in \Omega_{\mathrm{ob},l}$, stays in $\Omega_{\mathrm{ob},l}$ and converges to zero, ΔS_{ij} will also converge to zero. Thus, SoC balancing is achieved among the battery units, as long as the initial estimates of the states are close enough. A similar analysis can be carried out for the charging mode.

We will now consider the total power of the system. For $t \geq \max\{T_1, T_2\}$, the total power during a discharging process is calculated as

$$
\begin{aligned}
P_\Sigma &= \sum_{i=1}^{N} \frac{Q_{\mathrm{nom},i}U_{\mathrm{B},i}\hat{S}_i}{\frac{1}{N}\sum_{j=1}^{N} Q_{\mathrm{nom},j}U_{\mathrm{B},j}\hat{S}_j} P_{\mathrm{a}}^* \\
&= \frac{\sum_{i=1}^{N} Q_{\mathrm{nom},i}U_{\mathrm{B},i}\hat{S}_i}{\sum_{j=1}^{N} Q_{\mathrm{nom},j}U_{\mathrm{B},j}\hat{S}_j} P^* \\
&= P^*,
\end{aligned}
$$

and the total power of the system during a charging process is calculated as

$$
\begin{aligned}
P_\Sigma &= \sum_{i=1}^{N} \frac{Q_{\mathrm{nom},i}U_{\mathrm{B},i}(1-\hat{S}_i)}{\frac{1}{N}\sum_{j=1}^{N} Q_{\mathrm{nom},j}U_{\mathrm{B},j}(1-\hat{S}_j)} P_{\mathrm{a}}^* \\
&= \frac{\sum_{i=1}^{N} Q_{\mathrm{nom},i}U_{\mathrm{B},i}(1-\hat{S}_i)}{\sum_{j=1}^{N} Q_{\mathrm{nom},j}U_{\mathrm{B},j}(1-\hat{S}_j)} P^* \\
&= P^*.
\end{aligned}
$$

Thus, the total power of the system P_Σ will be equal to the power demand P^* for $t \geq \max\{T_1, T_2\}$.

Summarizing the above analysis, the following theorem is stated.

Theorem 6.5. Consider a BESS consisting of N battery units. Let the communication network satisfy Assumptions 6.1 and 6.2. Let the power demand satisfy Assumption 6.3. Assume that the initial conditions of the observers are set sufficiently close to the true states. Then, the power distribution laws (6.41) and (6.42), based on the observer (6.35) and the average estimation algorithms (6.39) and (6.40), solve Problem 6.2. That is, under (6.41) and (6.42), the BESS achieves SoC balancing among all its battery units and satisfies the power demand.

6.8 Simulation For Observer Based Algorithms

A battery energy storage system consisting of four battery units is considered in the simulation. The communication topology among the battery units is shown in Fig. 6.34. The power demand $P^*(t)$ is only known to battery unit 1.

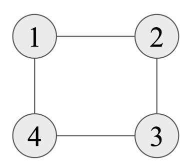

Figure 6.34: The communication topology.

The parameters in the models of the battery units are [8]

$$\begin{aligned}
&R_{0,i} = 1 \times 10^{-3}\Omega, \\
&R_{1,i} = 0.7 \times 10^{-3}\Omega,\ R_{2,i} = 2.5 \times 10^{-3}\Omega, \\
&C_{1,i} = 6.5 \times 10^{3}\text{F},\ C_{2,i} = 2.2 \times 10^{5}\text{F},
\end{aligned}$$

for $i = 1, 2, 3, 4$, and

$$(Q_{\text{nom},1}, Q_{\text{nom},2}, Q_{\text{nom},3}, Q_{\text{nom},4}) = (20, 17, 22, 19)\text{AH}.$$

The polynomial approximation of the SoC-OCV relation, $U_{\text{OC},i}(S_i)$, $i = 1, 2, 3, 4$, used in the simulation is [33]

$$U_{\text{OC},i} = 11.65S_i^7 - 35.01S_i^6 + 40.4S_i^5 - 24.87S_i^4 + 11.73S_i^3 - 4.439S_i^2 + 1.28S_i + 3.42.$$

Given the above parameters and quantities, the matrices representing the state-space models are calculated as

$$A_i = \begin{bmatrix} -0.2198 & 0 & 0 \\ 0 & -0.0018 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; i = 1, 2, 3, 4$$

$$B_1 = 10^{-3} \begin{bmatrix} 0.1538 \\ 0.0045 \\ -0.0136 \end{bmatrix}, \; B_2 = 10^{-3} \begin{bmatrix} 0.1538 \\ 0.0045 \\ -0.0160 \end{bmatrix},$$

$$B_3 = 10^{-3} \begin{bmatrix} 0.1538 \\ 0.0045 \\ -0.0124 \end{bmatrix}, \; B_4 = 10^{-3} \begin{bmatrix} 0.1538 \\ 0.0045 \\ -0.0135 \end{bmatrix}.$$

The observer gain matrices are designed as

$$K_i = \begin{bmatrix} 0.001 & 0 & 0 \\ 0 & 0.001 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; i = 1, 2, 3, 4,$$

with

$$D_i = \begin{bmatrix} 4.396 \times 10^{-4} & 0 & 0 \\ 0 & 3.6 \times 10^{-6} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; i = 1, 2, 3, 4.$$

The parameters in the average battery state estimators and the average power estimators are chosen as $\alpha = 300$ and $\beta = 30$.

We first demonstrate the discharging process. Let the power demand be $P^*(t) = 1200\text{W}$. The initial SoC for the simulation is

$$(S_{1,\text{init}},\, S_{2,\text{init}},\, S_{3,\text{init}},\, S_{4,\text{init}}) = (0.99,\, 0.92,\, 0.83,\, 0.75).$$

The initial condition of the observers are chosen as

$$\hat{x}_i(0) = [0 \; 0 \; 0.70]^{\mathrm{T}}, \; i = 1, 2, 3, 4,$$

that is, the initial estimates of the SoC are chosen as $\hat{S}_i(0) = 0.70$, $i = 1, 2, 3, 4$.

Figure 6.35 shows the evolution of the estimated SoC $\hat{S}_i$, and the true SoC S_i of all battery units in discharging mode. It is seen that the estimated SoC from the observers converges to the true SoC. Figure 6.36 shows the individual power $P_{\mathrm{B},i}$, the total power of the system P_Σ, and the power demand P^* in discharging mode. It is seen that the power demand is satisfied. Figures 6.37 and 6.38 show the estimated average state and the estimated average power demand by the battery units converge to their true values.

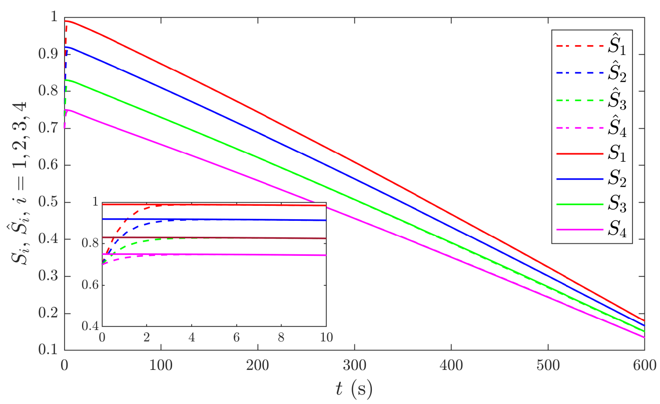

Figure 6.35: The estimated SoC and the true SoC of all battery units for discharging.

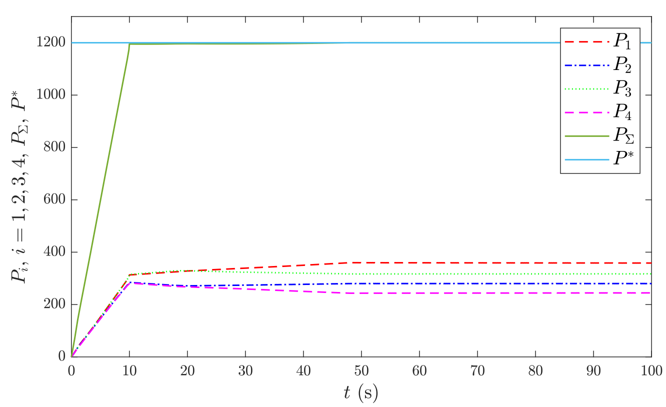

Figure 6.36: The powers of the individual battery units, the total power, and the power demand for discharging.

We next demonstrate the charging process. Let the power demand be $P^*(t) = -1200$W. The initial SoC for the simulation is

$$(S_{1,\mathrm{init}},\ S_{2,\mathrm{init}},\ S_{3,\mathrm{init}},\ S_{4,\mathrm{init}}) = (0.27,\ 0.08,\ 0.17,\ 0.25).$$

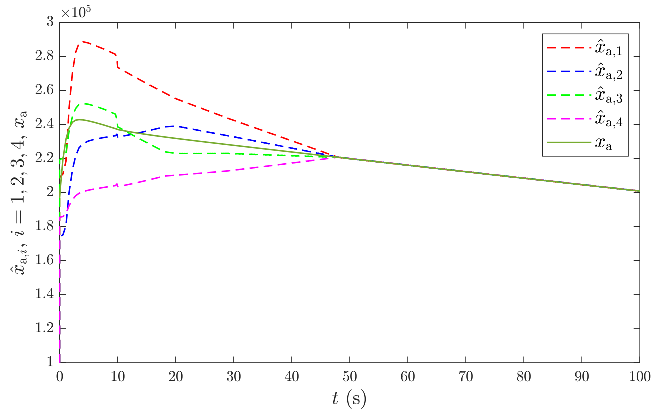

Figure 6.37: The estimated average state of the battery units, and the true average state for discharging.

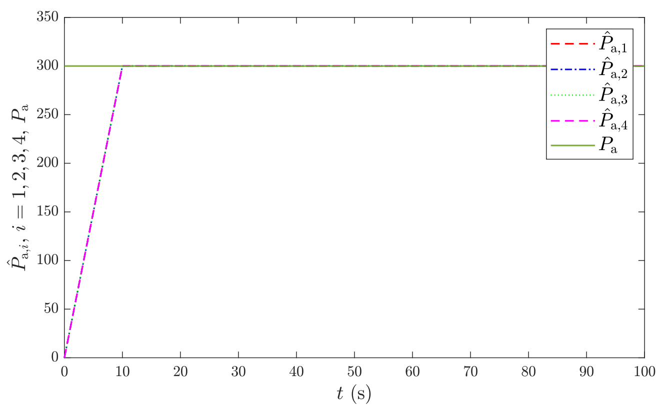

Figure 6.38: The estimated average power demand by the battery units, and the true average power demand for discharging.

The initial condition of the observers are chosen as

$$\hat{x}_i(0) = [0\ 0\ 0.40]^{\mathrm{T}},\ i = 1, 2, 3, 4,$$

that is, the initial estimates of the SoC are chosen as $\hat{S}_i(0) = 0.40$, $i = 1, 2, 3, 4$.

Figure 6.39 shows the evolution of the estimated SoC $\hat{S}_i$, and the true SoC S_i of all battery units in charging mode. It is seen that the estimated SoC from the observers converges to the true SoC. Figure 6.40 shows the individual power $P_{\mathrm{B},i}$, the total power of the system P_Σ, and the power demand P^* in charging mode. It is seen that the power demand is satisfied. Figures 6.41 and 6.42 show the estimated average state and the estimated average power demand by the battery units converge to their true values.

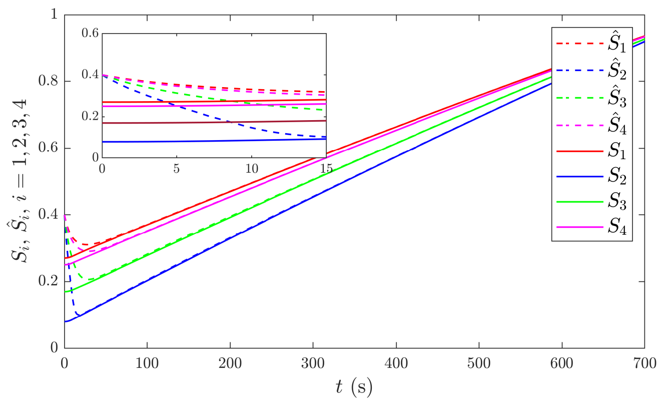

Figure 6.39: The estimated SoC and the true SoC of all battery units for charging.

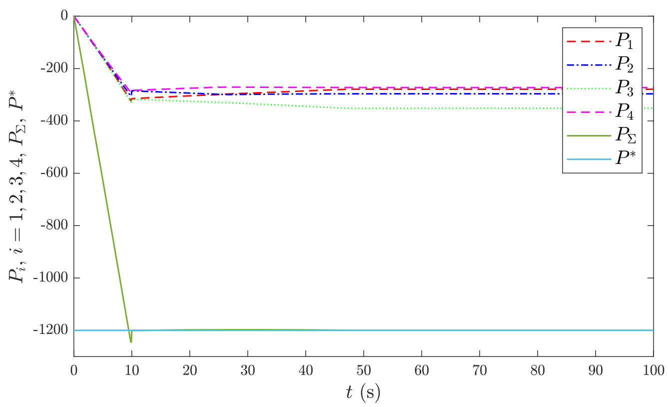

Figure 6.40: The powers of the individual battery units, the total power, and the power demand for charging.

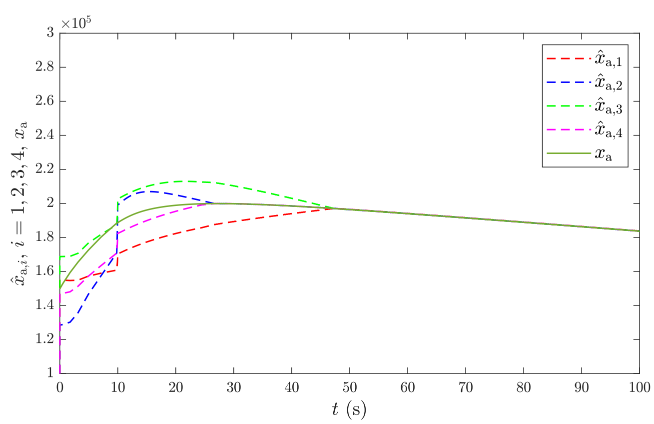

Figure 6.41: The estimated average state of the battery units, and the true average state for charging.

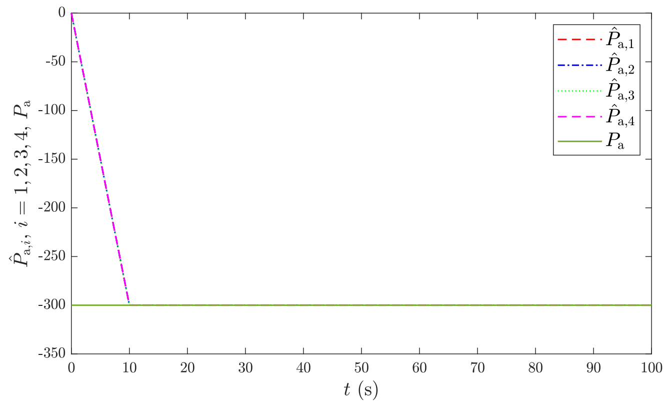

Figure 6.42: The estimated average power demand by the battery units, and the true average power demand for charging.

In summary, it is shown by the simulation that during the operation of the system both in discharging and charging, the SoC is correctly estimated, the system reaches SoC balancing among its battery units, and the power demand is satisfied.

6.9 Conclusions

In this chapter, the distributed control problem of a BESS consisting of networked battery units is studied. In particular, the problem of balancing the SoC and satisfying the power demand is considered. A distributed power allocation algorithm is first designed based on distributed average unit state estimators and distributed average power demand estimators. Then, a distributed and adaptive power-allocating algorithm is designed for each battery unit with unknown parameters, based on adaptive parameter estimation and average estimation. Last, a nonlinear state-of-charge observer-based power allocation algorithm is designed for each battery unit whose SoC is unknown, based on the state-of-charge as estimated by the observer, as well as the estimated battery average state and the estimated average power demand. We show that, by choosing the design parameters properly, the proposed power allocation algorithms achieve SoC balancing of all battery units and satisfy the power demand, as long as the communication topology among battery units is undirected and connected, and the power demand is under certain constraints and is known to at least one battery unit.

This chapter is based on the following publications:

- Tingyang Meng, Zongli Lin, Yan Wan, and Yacov A. Shamash, "Nonlinear observer-based state-of-charge balancing of networked battery energy storage systems”, *Systems & Control Letters*, (2023, under review).

- Tingyang Meng, Zongli Lin, Yan Wan, and Yacov A. Shamash, “State-of-charge balancing for battery energy storage systems in DC microgrids by distributed adaptive power distribution.” *IEEE Control Systems Letters* 6 (2021): 512-517.

- Tingyang Meng, Zongli Lin, and Yacov A. Shamash, “Distributed cooperative control of battery energy storage systems in DC microgrids.” *IEEE/CAA Journal of Automatica Sinica* 8.3 (2021): 606-616.

Chapter 7

Summary and Future Work

In this dissertation, distributed control problems of multi-agent systems are studied, and consensus algorithms for solving these problems are proposed. Applications of consensus algorithms to the management problems of networked battery units are also investigated.

The following are potential topics for future work. First, the distributed control problems of multi-agent systems can be studied under time-varying communication topologies.

Second, the power distribution algorithms based on equivalent circuit models can be improved by introducing online battery model parameter identification. Currently, the power distribution based on the equivalent circuit model of batteries relies on off-line generated parameters such as the capacities and resistances in the model. Online parameter estimation will enhance the adaptivity of the algorithms.

Third, models of battery dynamics and battery aging can be constructed using neural networks. In the current work, the battery dynamics is approximated by an equivalent circuit model, and the aging of batteries is not considered. Neural networks can be adopted to model the complex and nonlinear behaviors of battery dynamics and battery aging. In order to construct and train the networks, data sets need to be generated by conducting data acquisition experiments with battery systems.

Fourth, modeling, and analysis of the power conversion systems associated with networked battery systems can be considered. In the current work, the power distribution algorithms are proposed at a high level, without consideration of the effect of the dynamics of power conversion systems on the design.

Fifth, the proposed power distribution laws can be implemented and experimented with real hardware and battery systems.

Last, extra safety features of the management of networked battery systems such as fault avoidance, fault detection, and fault handling of battery packs can be considered in future design.

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