# Relative braid group symmetries on ıquantum groups

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**Abstract**

In the theory of quantum groups, Lusztig's braid group symmetries, associated to the Weyl group of the underlying Lie algebra, have played a fundamental role. $\imath$Quantum groups, which are quantum algebras arising from the theory of quantum symmetric pairs, can be viewed as generalizations of quantum groups.

In this dissertation, we initiate a general approach to the relative braid group symmetries, associated to relative Weyl group of the underlying symmetric pair, on (universal) $\imath$quantum groups and their modules. We construct such symmetries for $\imath$quantum groups of arbitrary finite type and quasi-split Kac-Moody type. Our approach is built on new intertwining properties of quasi $K$-matrices which we develop and braid group symmetries on (Drinfeld double) quantum groups. Explicit formulas for these new symmetries on $\imath$quantum groups are obtained.

We establish a number of fundamental properties for these symmetries on $\imath$quantum groups, strikingly parallel to their well-known quantum group counterparts. We apply these symmetries to fully establish rank one factorizations of quasi $K$-matrices, and this factorization property in turn helps to show that the new symmetries satisfy relative braid relations. As a consequence, conjectures of Kolb-Pellegrini and Dobson-Kolb are settled affirmatively.

Finally, the above approach allows us to construct compatible relative braid group actions on modules over quantum groups for the first time. Explicit formula for the relative braid group actions on modules are obtained, in terms of elements in $\imath$quantum groups.

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# Contents

# 1 Introduction

## 1.1 Background

Braid group symmetries have played an essential role in understanding the structures of Drinfeld-Jimbo quantum groups $\mathbf{U}$ and have found applications in geometric representation theory and categorification among others. These symmetries were constructed by Lusztig and used in first constructions of PBW bases and canonical bases in ADE type [Lus90a]. They have further been generalized to non-simply-laced types and beyond [Lus90b, Lus93]. Another crucial property is that there exists a compatible braid group action on integrable $\mathbf{U}$-modules. A systematic exposition on the braid group actions on quantum groups and their modules forms a significant portion of Lusztig's book [Lus93, Ch. 5, Part VI].

Let $\widetilde{\mathbf{U}} = \langle E_i, F_i, K_i, K_i' \mid i \in \mathbb{I}\rangle$ be the Drinfeld double quantum group, where $K_iK_i'$ are central. The quantum group $\mathbf{U} = \langle E_i, F_i, K_i^{\pm 1} \mid i \in \mathbb{I}\rangle$ is recovered from $\widetilde{\mathbf{U}}$ by a central reduction:

$$\mathbf{U} = \widetilde{\mathbf{U}}/(K_iK_i' - 1 \mid i \in \mathbb{I}).$$

The Drinfeld doubles naturally arise from the Hall algebra construction of Bridgeland [Br13], and it is shown in [LW22a] that reflection functors provide braid group actions on the Drinfeld doubles; see Proposition 2.6. As a straightforward generalization for Lusztig's symmetries on $\mathbf{U}$ [Lus93, 37.2.4], there are 4 variants of braid group operators $\widetilde{T}'_{i,e}, \widetilde{T}''_{i,e}$ on $\widetilde{\mathbf{U}}$, for $e \in \{\pm 1\}$ and $i \in \mathbb{I}$, which are related to each other by conjugations of certain (anti-) involutions [LW22a]; see (2.14):

$$\widetilde{T}'_{i,-e} = \sigma \circ \widetilde{T}''_{i,+e} \circ \sigma, \qquad \widetilde{T}''_{i,-e} := \psi \circ \widetilde{T}''_{i,+e} \circ \psi, \qquad \widetilde{T}'_{i,+e} := \psi \circ \widetilde{T}'_{i,-e} \circ \psi. \tag{1.1}$$

Here $\psi$ is the bar involution and $\sigma$ is an anti-involution on $\widetilde{\mathbf{U}}$; see Proposition 2.3.

Associated to any Satake diagram $(\mathbb{I} = \mathbb{I}_\bullet \cup \mathbb{I}_\circ, \tau)$, a quantum symmetric pair $(\mathbf{U}, \mathbf{U}^\imath_{\boldsymbol{\varsigma}})$ was introduced by Gail Letzter in finite type [Let99, Let02] as a $q$-deformation of the usual symmetric pair; here $\mathbf{U}^\imath_{\boldsymbol{\varsigma}}$ is a coideal subalgebra of $\mathbf{U}$ depending on parameters $\boldsymbol{\varsigma} = (\varsigma_i)_{i\in\mathbb{I}_\circ}$. Letzter's construction of quantum symmetric pairs was generalized to Kac-Moody type by Kolb [Ko14].

*Universal* quantum symmetric pairs $(\widetilde{\mathbf{U}}, \widetilde{\mathbf{U}}^\imath)$ (of quasi-split type) were formulated in [LW22b], where the parameters are replaced by suitable central elements in $\widetilde{\mathbf{U}}^\imath$, and $\mathbf{U}^\imath_{\boldsymbol{\varsigma}}$ is recovered from $\widetilde{\mathbf{U}}^\imath$ by a central reduction. $(\mathbf{U}^\imath_{\boldsymbol{\varsigma}}, \widetilde{\mathbf{U}}^\imath$ will be referred to as $\imath$quantum groups, and they are called *quasi-split* if $\mathbb{I}_\bullet = \emptyset$ and *split* if in addition $\tau = \mathrm{Id}$.)

According to the $\imath$program proposed by Bao-Wang in [BW18a], various constructions in the theory of quantum groups should admit generalizations to $\imath$quantum groups. Remarkably, a number of fundamental constructions have been generalized successfully to $\imath$quantum groups in recent years; see the ICM lecture [W22]. In fact, a quantum group $\mathbf{U}$ can be regarded as the quantum symmetric pair of diagonal type $(\mathbf{U} \otimes \mathbf{U}, \mathbf{U})$, and hence $\imath$quantum groups can be viewed as a vast generalization of quantum groups. This simple observation has been applied successfully in various $\imath$-generalizations. For instance, this view was instrumental in the development of $\imath$canonical bases in [BW18b, BW21] for quantum symmetric pairs, which generalize canonical bases for quantum groups; it was also used in the construction of $\imath$Hall algebras in [LW22b], generalizing the Hall algebra realization of quantum groups; see also the recent development of $\imath$crystal bases by Watanabe [W21b].

Quasi $K$-matrices were originally formulated in [BW18a] as an intertwiner between the embedding $\imath : \mathbf{U}^\imath_{\boldsymbol{\varsigma}} \to \mathbf{U}$ and a bar-involution conjugated embedding (for parameters $\boldsymbol{\varsigma}$ satisfying strong constraints); a proof in greater generality was given in [BK19] under a technical assumption (which was removed later in [BW21]). A reformulation by Appel and Vlaar [AV22] (also see [KY20]) bypassed a direct use of the bar involutions, allowing more general parameters $\boldsymbol{\varsigma}$.

Lusztig's braid group actions on $\mathbf{U}$ do not preserve the subalgebra $\mathbf{U}^\imath_{\boldsymbol{\varsigma}}$ in general. Kolb-Pellegrini [KP11] proposed that there should be relative braid group symmetries on $\imath$quantum groups corresponding to the relative (or restricted) Weyl groups for the underlying symmetric pairs. For a class of $\imath$quantum groups of finite type (including all quasi-split types and type AII) with some specific parameters, formulas for such braid group actions were found and verified *loc. cit.* via computer computation. The relative braid group action for type AI appeared earlier in [Ch07] and [MR08].

There has been some limited progress on relative braid group action on $\mathbf{U}^\imath_{\boldsymbol{\varsigma}}$ in the last decade; for type AIII see Dobson [Dob20]. An $\imath$Hall algebra approach has been developed to realize the universal *quasi-split* $\imath$quantum groups $\widetilde{\mathbf{U}}^\imath$ [LW22b]. As a generalization of Ringel's construction [Rin96], reflection functors [LW21a, LW22a] are used to construct relative braid group actions on $\widetilde{\mathbf{U}}^\imath$ of quasi-split type, where the braid group operators act on the central elements in $\widetilde{\mathbf{U}}^\imath$ non-trivially. For $\widetilde{\mathbf{U}}^\imath$ or $\mathbf{U}^\imath_{\boldsymbol{\varsigma}}$ in general beyond quasi-split type, no conjectural formulas or conceptual explanations for relative braid group actions were available.

There are braid group actions on $\mathbf{U}$-modules which are compatible with braid group actions on quantum groups; cf. [Lus93]. In contrast, no relative braid group action on $\mathbf{U}^\imath_{\boldsymbol{\varsigma}}$-modules has been known to date. The Hall algebra approach does not help providing any clue on such action at the module level.

## 1.2 Goal

In this dissertation we develop a conceptual and general approach to relative braid group actions on $\imath$quantum groups, arising from (universal) quantum symmetric pairs, and on their modules for the first time. This in particular settles the longstanding conjecture of Kolb and Pellegrini [KP11] in a constructive manner.

It is crucial for us to work with universal $\imath$quantum groups. We shall formulate relative braid group symmetries $\widetilde{\mathbf{T}}'_{i,e}, \widetilde{\mathbf{T}}''_{i,e}$ on $\widetilde{\mathbf{U}}^\imath$, for $e \in \{\pm 1\}$ and $i \in \mathbb{I}_{\circ,\tau}$, which are related to each other via conjugations by a bar involution $\psi^\imath$ and an anti-involution $\sigma^\imath$ on $\widetilde{\mathbf{U}}^\imath$; compare (1.1):

$$\widetilde{\mathbf{T}}'_{i,-e} = \sigma^\imath \circ \widetilde{\mathbf{T}}''_{i,+e} \circ \sigma^\imath, \qquad \widetilde{\mathbf{T}}''_{i,-e} := \psi^\imath \circ \widetilde{\mathbf{T}}''_{i,+e} \circ \psi^\imath, \qquad \widetilde{\mathbf{T}}'_{i,+e} := \psi^\imath \circ \widetilde{\mathbf{T}}'_{i,-e} \circ \psi^\imath.$$

By central reductions and rescaling automorphisms, these symmetries descend to relative braid group actions on $\imath$quantum groups with parameters $\mathbf{U}^\imath_{\boldsymbol{\varsigma}}$. Moreover, we are able to formulate compatible relative braid group actions on integrable $\mathbf{U}^\imath_{\boldsymbol{\varsigma}}$-modules. We further establish a number of basic properties of these new symmetries which are natural $\imath$-counterparts of well-known properties for Lusztig's braid group symmetries.

This dissertation is largely based on two papers: [WZ22] which constructed the relative braid group action for $\imath$quantum groups of arbitrary finite type, and [Z22a] which generalized the constructions in the previous paper to $\imath$quantum groups of arbitrary quasi-split Kac-Moody type.

## 1.3 The basic idea

Following the view that quantum groups are $\imath$quantum groups of diagonal type, a starting point of this dissertation is to review the relation between braid group symmetries on a quantum group $\mathbf{U}$ and $\mathbf{U}\otimes\mathbf{U}$. Denote by $\mathbf{L}_i''$ the rank 1 quasi $R$-matrix associated to $i\in\mathbb{I}$, and let $\mathbf{L}_i'$ be its inverse. The following formula in [Lus93, 37.3.2]:

$$(T'_{i,-1}\otimes T'_{i,-1})\Delta(T''_{i,+1}u)=\mathbf{L}_i'\Delta(u)\mathbf{L}_i'' \tag{1.2}$$

provides a relation between braid group actions on $\mathbf{U}$ and $\mathbf{U}\otimes\mathbf{U}$; a formula similar to (1.2) via a different formulation of braid operators appeared in [LS90] and [KR90]. The quasi $R$-matrices for $\mathbf{U}$, up to a suitable twist, can be identified with the quasi $K$-matrices for $(\mathbf{U}\otimes\mathbf{U},\mathbf{U})$. A variant of the identity (1.2) holds in the setting of universal quantum symmetric pairs of diagonal type $(\widetilde{\mathbf{U}}\otimes\widetilde{\mathbf{U}},\widetilde{\mathbf{U}})$; see §4.4.

Now, let $(\widetilde{\mathbf{U}},\widetilde{\mathbf{U}}^\imath)$ be a general universal quantum symmetric pair. We upgrade the constructions of the quasi $K$-matrix to the universal level. Inspired by the relation (1.2), we aim at formulating a relation between braid group action on the Drinfeld double $\widetilde{\mathbf{U}}$ and the desired relative braid group action on the universal $\imath$quantum group $\widetilde{\mathbf{U}}^\imath$ through conjugations of rank 1 quasi $K$-matrices.

Dobson and Kolb [DK19] proposed (conjectural) factorizations of quasi $K$-matrices in finite types into products of rank 1 quasi $K$-matrices, analogous to factorizations of quasi $R$-matrices [LS90, KR90]. In their formulation, a certain scaling twist shows up. In this dissertation, we upgrade the formulation of the factorization together with the corresponding scaling twist to quasi $K$-matrices $\widetilde{\Upsilon}$ in the universal setting.

Examples indicate that our basic idea of constructing the desired relative braid group actions on $\widetilde{\mathbf{U}}^\imath$ via quasi $K$-matrices and braid group actions on $\widetilde{\mathbf{U}}$ (viewed as a generalization of (1.2)) basically works — up to a simple twist: it is necessary to use *suitably rescaled* braid group operators on $\widetilde{\mathbf{U}}$. Remarkably, this scaling turns out to coincide with the aforementioned scaling which appears in the factorizations of a quasi $K$-matrix $\widetilde{\Upsilon}$. We are able to explore this compatibility to draw strong consequences on the seemingly unrelated topics: relative braid group actions and factorizations of quasi $K$-matrices.

## 1.4 Main results for Part I

In Part I, we formulate new intertwining properties of quasi $K$-matrices. We construct symmetries $\widetilde{\mathbf{T}}'_{i,e}, \widetilde{\mathbf{T}}''_{i,e}$, using quasi $K$-matrices, on the universal $\imath$quantum groups of arbitrary finite type. We will show that these symmetries satisfy the braid relations in the relative braid group in Part III. Part I is largely based on [WZ22, §2-§6].

### New intertwining properties of quasi $K$-matrices

We formulate universal quantum symmetric pairs $(\widetilde{\mathbf{U}}, \widetilde{\mathbf{U}}^\imath)$ associated to arbitrary Satake diagrams and their basic properties in Section 2.6, following and generalizing the quasi-split setting in [LW22b]. The algebra $\widetilde{\mathbf{U}}^\imath$ contains $\widetilde{\mathbf{U}}^{\imath 0}$ and $\widetilde{\mathbf{U}}_\bullet$ naturally as subalgebras, where $\widetilde{\mathbf{U}}_\bullet$ is the Drinfeld double associated to $\mathbb{I}_\bullet$ and $\widetilde{\mathbf{U}}^{\imath 0}$ is a Cartan subalgebra generated by $\widetilde{k}_i = K_i K'_{\tau i}$, for $i \in \mathbb{I}_\circ$.

We recall the more recent formulation of a quasi $K$-matrix $\Upsilon_\varsigma$ for $(\mathbf{U}, \mathbf{U}^\imath_\varsigma)$ from [AV22] (cf. [BW18a, BK19, BW18b] for earlier constructions) in Theorem 3.1 and upgrade it to a universal version $\widetilde{\Upsilon}$ for $(\widetilde{\mathbf{U}}, \widetilde{\mathbf{U}}^\imath)$ in Theorem 3.2. It turns out that $\widetilde{\Upsilon}$ admits a more conceptual and simpler characterization in terms of the anti-involution

$\sigma$ on $\widetilde{\mathbf{U}}$ as follows.

**Theorem A** (Theorem 3.6). The quasi $K$-matrix $\widetilde{\Upsilon} = \sum_{\mu\in\mathbb{N}\mathbb{I}} \widetilde{\Upsilon}^\mu$, for $\widetilde{\Upsilon}^\mu \in \widetilde{\mathbf{U}}^+_\mu$, is uniquely characterized by $\widetilde{\Upsilon}^0 = 1$ and the following intertwining relations:

$$B_i\widetilde{\Upsilon} = \widetilde{\Upsilon}B_i^\sigma \quad (i \in \mathbb{I}_\circ), \qquad\qquad x\widetilde{\Upsilon} = \widetilde{\Upsilon}x \quad (x \in \widetilde{\mathbf{U}}^{\imath 0}\widetilde{\mathbf{U}}_\bullet).$$

This characterization of $\widetilde{\Upsilon}$ plays a basic role in producing explicit formulas for relative braid group actions on $\widetilde{\mathbf{U}}^\imath$; see the proof of Theorem 5.5 in §5.4. There is a similar simple characterization of $\Upsilon_{\boldsymbol{\varsigma}}$ for $\mathbf{U}^\imath_{\boldsymbol{\varsigma}}$ in terms of the anti-involution $\sigma\tau$ on $\mathbf{U}$; see Remark 3.16. (It is tempting to regard this as a new definition of $\Upsilon_{\boldsymbol{\varsigma}}$.)

We use a distinguished scaling automorphism $\widetilde{\Psi}_{\boldsymbol{\varsigma}_\star}$ to define a rescaled bar involution $\psi_\star$ on $\widetilde{\mathbf{U}}$ (by twisting the bar involution $\psi$ on $\widetilde{\mathbf{U}}$). By exploring further intertwining properties via $\widetilde{\Upsilon}$ as in [Ko21], we establish in the Kac-Moody generality a bar involution $\psi^\imath$ (see Proposition 3.4) and an anti-involution $\sigma^\imath$ (see Proposition 3.12) from $\psi_\star$ and $\sigma$, respectively. These (anti-)involutions $\psi^\imath$ and $\sigma^\imath$ were known in some quasi-split cases; see [CLW23].

Denote by $\widetilde{\Upsilon}_i$, for $i \in \mathbb{I}_\circ$, the quasi $K$-matrix associated to the rank one Satake subdiagram $(\mathbb{I}_\bullet \cup \{i, \tau i\}, \tau)$.

## New symmetries $\widetilde{\mathbf{T}}'_{i,e}, \widetilde{\mathbf{T}}''_{i,e}$

Associated to a Satake diagram $(\mathbb{I} = \mathbb{I}_\bullet \cup \mathbb{I}_\circ, \tau)$, one has the (absolute) Weyl group $W$ generated by the simple reflections $s_i$, for $i \in \mathbb{I}$, and a finite parabolic subgroup $W_\bullet = \langle s_i \mid i \in \mathbb{I}_\bullet\rangle$ with longest element $w_\bullet$. Given $i \in \mathbb{I}_\circ$, one has a rank 1 Satake subdiagram $(\mathbb{I}_{\bullet,i} = \mathbb{I}_\bullet \cup \{i, \tau i\}, \tau)$. For each rank 1 Satake subdiagram $\mathbb{I}_{\bullet,i}$ of finite type, one defines an element $\mathbf{r}_i \in W$ as in (2.21). The relative Weyl group $W^\circ$ is a subgroup of $W$ generated by $\mathbf{r}_i$ for $i \in \mathbb{I}_{\circ,\tau}$ (the set $\mathbb{I}_{\circ,\tau}$ parametrizes finite-type rank 1 Satake subdiagrams; see (2.20) for the definition). Abstractly, $W^\circ$ is a Weyl group with $\mathbf{r}_i$ ($i \in \mathbb{I}_{\circ,\tau}$) as simple reflections [Lus76]; also see [OV90, Lus03, DK19].

Let $\widetilde{T}''_{i,+1}$ and $\widetilde{T}'_{i,-1}$, for $i \in \mathbb{I}$, be the braid group operators on $\widetilde{\mathbf{U}}$ [LW22a]; see Proposition 2.6. Let $\widetilde{\mathcal{T}}''_{i,+1}$ and $\widetilde{\mathcal{T}}'_{i,-1}$ be the rescaled version of $\widetilde{T}''_{i,+1}$ and $\widetilde{T}'_{i,-1}$ via conjugation by a scaling automorphism $\widetilde{\Psi}_{\varsigma_\diamond}$; see (4.2)–(4.3). As $\widetilde{\mathcal{T}}'_{j,-1}$, for $j \in \mathbb{I}$, satisfy the braid relations, we can make sense of $\widetilde{\mathcal{T}}'_{w,-1}$, for $w \in W$, and in particular $\widetilde{\mathcal{T}}'_{\mathbf{r}_i,-1}$, for $i \in \mathbb{I}_\circ$, as automorphisms of $\widetilde{\mathbf{U}}$.

**Theorem B** (Theorem 4.7, Proposition 4.11, Theorem 4.14, Theorem 5.5). Let $(\mathbb{I} = \mathbb{I}_\bullet \cup \mathbb{I}_\circ, \tau)$ be a symmetric pair of finite type and $i \in \mathbb{I}_{\circ,\tau}$. There exists a unique automorphism $\widetilde{\mathbf{T}}'_{i,-1}$ of $\widetilde{\mathbf{U}}^\imath$ such that the following intertwining relation holds:

$$\widetilde{\mathbf{T}}'_{i,-1}(x)\widetilde{\Upsilon}_i = \widetilde{\Upsilon}_i \widetilde{\mathcal{T}}'_{\mathbf{r}_i,-1}(x), \qquad \text{for all } x \in \widetilde{\mathbf{U}}^\imath. \tag{1.3}$$

More precisely, the action of $\widetilde{\mathbf{T}}'_{i,-1}$ on $\widetilde{\mathbf{U}}^\imath$ is given as follows:

1. $\widetilde{\mathbf{T}}'_{i,-1}(x) = (\widehat{\tau}_{\bullet,i} \circ \widehat{\tau})(x)$, and $\widetilde{\mathbf{T}}'_{i,-1}(\widetilde{k}_{j,\diamond}) = \widetilde{k}_{\mathbf{r}_i\alpha_j,\diamond}$, for all $x \in \widetilde{\mathbf{U}}_\bullet$, $j \in \mathbb{I}_\circ$.
2. $\widetilde{\mathbf{T}}'_{i,-1}(B_i) = -q^{-(\alpha_i, w_\bullet \alpha_{\tau i})} \widetilde{\mathcal{T}}^2_{w_\bullet}(B_{\tau_{\bullet,i}\tau i}) \mathcal{K}^{-1}_{\tau_{\bullet,i}\tau i}$.
3. The formulas for $\widetilde{\mathbf{T}}'_{i,-1}(B_j)$ ($i \neq j \in \mathbb{I}_{\circ,\tau}$) are listed in Table 3.

See (2.22) and (4.12) for notation $\tau_{\bullet,i}$ and $\widetilde{k}_{\lambda,\diamond}$. By definition, we have $\mathbf{r}_i = \mathbf{r}_{\tau i}$, $\widetilde{\Upsilon}_i = \widetilde{\Upsilon}_{\tau i}$, and $\widetilde{\mathbf{T}}'_{i,-1} = \widetilde{\mathbf{T}}'_{\tau i,-1}$; thus, we only need to consider $\widetilde{\mathbf{T}}'_{i,-1}$, for $i \in \mathbb{I}_{\circ,\tau}$.

In the same spirit of (1.3) in Theorem **B**, the identity (1.2) for the Drinfeld double quantum group $\widetilde{\mathbf{U}}$ can be reformulated as the intertwining relation (4.7) for quantum symmetric pair $(\widetilde{\mathbf{U}} \otimes \widetilde{\mathbf{U}}, \widetilde{\mathbf{U}})$ of diagonal type.

Another symmetry $\widetilde{\mathbf{T}}''_{i,+1}$ on $\widetilde{\mathbf{U}}^\imath$, for $i \in \mathbb{I}_\circ$, is formulated in Theorem 6.1 which satisfies the following intertwining relation in (6.1), similar to (1.3):

$$\widetilde{\mathbf{T}}''_{i,+1}(x)\,\widetilde{\mathfrak{T}}''_{\mathbf{r}_i,+1}(\widetilde{\Upsilon}_i^{-1}) = \widetilde{\mathfrak{T}}''_{\mathbf{r}_i,+1}(\widetilde{\Upsilon}_i^{-1})\,\widetilde{\mathfrak{T}}''_{\mathbf{r}_i,+1}(x), \qquad \text{for all } x \in \widetilde{\mathbf{U}}^\imath.$$

We further define 2 more symmetries $\widetilde{\mathbf{T}}'_{i,+1}$ and $\widetilde{\mathbf{T}}''_{i,-1}$ on $\widetilde{\mathbf{U}}^\imath$ by conjugating $\widetilde{\mathbf{T}}'_{i,-1}$ and $\widetilde{\mathbf{T}}''_{i,+1}$ via the involution $\psi^\imath$; see (6.11). These symmetries are related to each other as follows; compare [Lus93, Chap. 37].

**Theorem C** (Theorem 6.7). *Let $e = \pm 1$ and $i \in \mathbb{I}_\circ$. The symmetries $\widetilde{\mathbf{T}}'_{i,e}$ and $\widetilde{\mathbf{T}}''_{i,-e}$ are mutual inverses. Moreover, we have $\widetilde{\mathbf{T}}'_{i,e} = \sigma^\imath \circ \widetilde{\mathbf{T}}''_{i,-e} \circ \sigma^\imath$.*

Actually, part of the proof of Theorem **B** (i.e., the invertibility of $\widetilde{\mathbf{T}}'_{i,-1}$) is completed only when it is established in Theorem **C** that $\widetilde{\mathbf{T}}'_{i,-1}$ and $\widetilde{\mathbf{T}}''_{i,+1}$ are mutual inverses. This is one main reason why we have formulated $\widetilde{\mathbf{T}}''_{i,+1}$ separately in spite of its many similarities with the properties for $\widetilde{\mathbf{T}}'_{i,-1}$ which we already established.

Here is an outline of proofs of Theorems **B**–**C**. We first establish the existence of an endomorphism $\widetilde{\mathbf{T}}'_{i,-1}$ on $\widetilde{\mathbf{U}}^\imath$ which satisfies the intertwining relation (1.3), by proving Properties (1)-(3) in Theorem **B** one-by-one. Properties (1)-(2) are established uniformly in Proposition 4.11 and Theorem 4.14. We formulate a structural result in Proposition 5.11 as a main step toward a uniform proof of the rank 2 formulas in (3) (see Theorem 5.5); Proposition 5.11 is then verified by a type-by-type computation in Appendix A. In order to prove the invertibility of $\widetilde{\mathbf{T}}'_{i,-1}$, we establish another endomorphism $\widetilde{\mathbf{T}}''_{i,+1}$ on $\widetilde{\mathbf{U}}^\imath$ which satisfies the intertwining relation (6.1) in Theorem 6.1; the existence for $\widetilde{\mathbf{T}}''_{i,+1}$ is proved by a strategy similar to the one for $\widetilde{\mathbf{T}}'_{i,-1}$. Finally, we show in Theorem 6.7 that $\widetilde{\mathbf{T}}'_{i,-1}$ and $\widetilde{\mathbf{T}}''_{i,+1}$ are mutual inverses by invoking the uniqueness of elements satisfying an intertwining relation.

The formulas for actions of $\widetilde{\mathbf{T}}'_{i,-1}$ and $\widetilde{\mathbf{T}}''_{i,+1}$ on generators of $\widetilde{\mathbf{U}}^\imath$ are mostly new. In quasi-split types, up to some twistings, we recover the formulas obtained by Hall algebra computation in [LW21a], and by central reductions to $\mathbf{U}^\imath_{\boldsymbol{\varsigma}}$, we recover formulas obtained by computer computation in [KP11].

## 1.5 Main results for Part II

In Part II, we generalize the construction of symmetries $\widetilde{\mathbf{T}}'_{i,e}, \widetilde{\mathbf{T}}''_{i,e}$ in Part I to a class of Kac-Moody type, including all quasi-split Kac-Moody type. In the Kac-Moody type, simple reflections in the relative Weyl group are parameterized by $\mathbb{I}_{\circ,\tau}$, where $\mathbb{I}_{\circ,\tau}$ is the set of vertices $i$ such that the corresponding rank one Satake diagram is of finite type; see (2.20). In Theorem 7.1, we construct symmetries $\widetilde{\mathbf{T}}'_{i,e}, \widetilde{\mathbf{T}}''_{i,e}$ on $\widetilde{\mathbf{U}}^\imath$ associated to the following three type of vertices $i \in \mathbb{I}_{\circ,\tau}$

(i) $i = \tau i = w_\bullet i$,

(ii) $c_{i,\tau i} = 0, i = w_\bullet i$,

(iii) $c_{i,\tau i} = -1, i = w_\bullet i$.

When the symmetric pair is of quasi-split type, every vertex $i \in \mathbb{I}_{\circ,\tau}$ belongs to one of the three types (i)-(iii); then we have an action of the (whole) relative braid group on $\imath$quantum groups, once the relative braid relations are verified in Part III. As a byproduct, we construct root vectors in $\imath$quantum groups and show that those symmetries send root vectors to root vectors. Part II is largely based on [Z22a].

The major difficulty in such generalizations is to establish higher rank relative braid group formulas of $\widetilde{\mathbf{T}}'_{i,-1}(B_j), \widetilde{\mathbf{T}}''_{i,+1}(B_j)$ for $j \neq i, \tau i$. Those higher rank formulas in Tables 3-4 for finite type were established by lengthy case-by-case computations, using a complete list of finite type rank two Satake diagrams. We will provide a uniform approach of the higher rank formulas in the Kac-Moody setting, independent of the rank two Satake diagrams.

**Higher rank formulas in the quasi-split Kac-Moody case**

In [Lus93, 37.2.1], Lusztig introduced (rank two) elements including $x_{i,j;1,m,e}, x'_{i,j;1,m,e}$, $y_{i,j;1,m,e}, y'_{i,j;1,m,e}$ in quantum groups. Alternatively, these root vectors are determined by recursive relations, according to Lusztig; see Lemma 2.8. These root vectors are used in rank two formulas for braid group symmetries $\widetilde{T}'_{i,e}, \widetilde{T}''_{i,e}$ on $\widetilde{\mathbf{U}}$; see (2.17).

We shall extend this picture from quantum groups to ıquantum groups. Since $\mathbf{r}_i$ are not simple reflections in $W$ when $i$ are of types (ii)-(iii), we will also need some rank 3 root vectors to describe the action of $\widetilde{T}'_{\mathbf{r}_i,e}$ in $\widetilde{\mathbf{U}}$, e.g., elements $y_{i,\tau i,j;m_1,m_2}$ in Definition 9.1 and elements $y_{i,\tau i,j;a,b,c}$ in Definition 10.1.

It turns out the recursive definitions for these (rank 2 or 3) root vectors in $\widetilde{\mathbf{U}}$ admit nontrivial generalizations to $\widetilde{\mathbf{U}}^\imath$, which allows us to define the following root vectors

(i) $b_{i,j;m}, \underline{b}_{i,j;m} \in \widetilde{\mathbf{U}}^\imath$ in Definition 8.1-8.2 when $i = \tau i = w_\bullet i$,

(ii) $b_{i,\tau i,j;m_1,m_2}, \underline{b}_{i,\tau i,j;m_1,m_2} \in \widetilde{\mathbf{U}}^\imath$ in Definition 9.6-9.7 when $c_{i,\tau i} = 0, i = w_\bullet i$,

(iii) $b_{i,\tau i,j;a,b,c}, \underline{b}_{i,\tau i,j;a,b,c} \in \widetilde{\mathbf{U}}^\imath$ in Definition 10.6-10.7 when $c_{i,\tau i} = -1, i = w_\bullet i$.

Our root vectors admit explicit closed formulas. For type (i), the ıdivided powers $B^{(m)}_{i,\overline{p}}$ for $\overline{p} \in \mathbb{Z}/2\mathbb{Z}$ were formulated in [BW18a, BeW18, CLW21] (see (7.7)), arising from the theory of ıcanonical bases. Closed formulas of $b_{i,j;m}, \underline{b}_{i,j;m}$ are given in terms of $\imath$divided powers in Proposition 8.6; that is, the elements $b_{i,j;m}, \underline{b}_{i,j;m}$ coincide with elements $\widetilde{y}'_{i,j;1,m,\overline{p},\overline{t},1}, \widetilde{y}_{i,j;1,m,\overline{p},\overline{t},1}$ introduced in [CLW21, §6.1].

For type (ii)-(iii), these root vectors in $\widetilde{\mathbf{U}}^\imath$ are constructed for the first time and their closed formulas are given in terms of usual divided powers $B_i^{(m)}$. The divided power formulations for $b_{i,\tau i,j;m_1,m_2}, \underline{b}_{i,\tau i,j;m_1,m_2}$ and $b_{i,\tau i,j;a,b,c}, \underline{b}_{i,\tau i,j;a,b,c}$ are respectively provided in Proposition 9.11 and Theorem 10.16.

The desired higher rank formulas for $\widetilde{\mathbf{T}}'_{i,-1}(B_j), \widetilde{\mathbf{T}}''_{i,+1}(B_j)$ are given by root vectors in $\widetilde{\mathbf{U}}^\imath$, as formulated below.

**Theorem D**. [Theorem 5.5] Let $i \in \mathbb{I}_{\circ,\tau}, j \in \mathbb{I}_\circ$ such that $j \neq i, \tau i$. Write $\alpha = -c_{ij}, \beta = -c_{\tau i,j}$.

(i) If $i = \tau i = w_\bullet i$, then $\widetilde{\mathbf{T}}'_{i,-1}(B_j) = b_{i,j;\alpha}$ and $\widetilde{\mathbf{T}}''_{i,+1}(B_j) = \underline{b}_{i,j;\alpha}$. Explicitly, $\widetilde{\mathbf{T}}'_{i,-1}(B_j), \widetilde{\mathbf{T}}''_{i,+1}(B_j)$ are respectively given by formulas (7.8)-(7.9).

(ii) If $c_{i,\tau i} = 0, i = w_\bullet i$, then $\widetilde{\mathbf{T}}'_{i,-1}(B_j) = b_{i,\tau i,j;\alpha,\beta}$ and $\widetilde{\mathbf{T}}''_{i,+1}(B_j) = \underline{b}_{i,\tau i,j;\alpha,\beta}$. Explicitly, $\widetilde{\mathbf{T}}'_{i,-1}(B_j), \widetilde{\mathbf{T}}''_{i,+1}(B_j)$ are respectively given by formulas (7.10)-(7.11).

(iii) If $c_{i,\tau i} = -1, i = w_\bullet i$, then we have $\widetilde{\mathbf{T}}'_{i,-1}(B_j) = b_{i,\tau i,j;\beta,\beta+\alpha,\alpha}$ and $\widetilde{\mathbf{T}}''_{i,+1}(B_j) = \underline{b}_{i,\tau i,j;\beta,\beta+\alpha,\alpha}$. Explicitly, $\widetilde{\mathbf{T}}'_{i,-1}(B_j), \widetilde{\mathbf{T}}''_{i,+1}(B_j)$ are respectively given by formulas (5.9)-(7.13).

The conjectural formulas in [CLW21, Conjecture 6.5] and [CLW23, Conjecture 3.7] for relative braid group actions are verified in full generality by Theorem **D**(i)-(ii).

The recursive definitions of root vectors play an indispensable role in the proof of Theorem **D**. The situations are similar for all types (i)-(iii) and we explain for type (ii). By definition, the root vectors $b_{i,\tau i,j;m_1,m_2}$ in $\widetilde{\mathbf{U}}^\imath$ naturally split into halves

$$b_{i,\tau i,j;m_1,m_2} = b^-_{i,\tau i,j;m_1,m_2} + b^+_{i,\tau i,j;m_1,m_2},$$

and the halves satisfy the same recursions as $b_{i,\tau i,j;m_1,m_2}$ but have different initial terms; see Definition 9.6-9.7. Thanks to the recursions for $b^\pm_{i,\tau i,j;m_1,m_2}$, we can establish relations between $b^\pm_{i,\tau i,j;m_1,m_2}$ and (rank 3) root vectors $y_{i,\tau i,j;m_1,m_2}, x_{i,\tau i,j;m_1,m_2} \in \widetilde{\mathbf{U}}$ via suitable intertwiners in Proposition 9.13-9.14. These new intertwining relations imply that the element $\widetilde{\mathbf{T}}'_{i,-1}(B_j) := b_{i,\tau i,j;-c_{ij},-c_{\tau i,j}}$ satisfies the desired identity (1.3) and then Theorem **D**(ii) is proved; see Theorem 9.17.

### $\widetilde{\mathbf{T}}'_{i,-1}$ and root vectors

The braid group symmetries $\widetilde{T}'_{i,e}, \widetilde{T}''_{i,e}$ on $\widetilde{\mathbf{U}}$ send root vectors to root vectors; cf. [Lus93, Proposition 37.2.5]. We formulate the $\imath$analog of this property for the relative braid group symmetry $\widetilde{\mathbf{T}}'_{i,-1}$ and root vectors in $\widetilde{\mathbf{U}}^\imath$, when the corresponding vertex $i$ is of type (i)-(ii).

**Theorem E** (Theorem 11.1, Theorem 11.3). Let $i \in \mathbb{I}_{\circ,\tau}, j \in \mathbb{I}_\circ$ such that $j \neq i, \tau i$.

(i) If $i = \tau i = w_\bullet i,$, then we have, for any $m \geq 0$,

$$\widetilde{\mathbf{T}}'_{i,-1}(\underline{b}_{i,j;m}) = b_{i,j;-c_{ij}-m}. \tag{1.4}$$

(ii) If $c_{i,\tau i} = 0, i = w_\bullet i,$, then we have, for any $m_1, m_2 \geq 0$,

$$\widetilde{\mathbf{T}}'_{i,-1}(\underline{b}_{i,\tau i,j;m_1,m_2}) = b_{i,\tau i,j;-c_{ij}-m_1,-c_{\tau i,j}-m_2}. \tag{1.5}$$

Theorem **E**(i) in the $\imath$divided power formulation was conjectured by Lu-Wang in a private communication. We conjecture that the symmetry $\widetilde{\mathbf{T}}'_{i,-1}$ in type (iii) also preserves root vectors.

## 1.6 Main results for Part III

In Part III, we establish several properties for the symmetries $\widetilde{\mathbf{T}}'_{i,e}, \widetilde{\mathbf{T}}''_{i,e}$ constructed in Part I-II, which are parallel to those for Lusztig's braid group symmetries on quantum groups. To that end, we show that our symmetries satisfy relative braid relations (i.e., braid relations in the relative Weyl group) and hence our symmetries lead to relative braid group actions on $\imath$quantum groups. Part III is based on [WZ22, §7- §9].

#### A basic property of braid symmetries

The following theorem is a generalization of a well-known basic property of braid group action on quantum groups; see [Lus93].

**Theorem F** (see Theorem 12.13)*.* Suppose that $wi \in \mathbb{I}_\circ$, for $w \in W^\circ$ and $i \in \mathbb{I}_\circ$. Then we have $\widetilde{\mathbf{T}}''_{w,+1}(B_i) = B_{wi}$.

The dependence in the formulation of Theorem 12.13 on reduced expressions $\underline{w}$ of $w$ can be removed, once Theorem **H** on braid relations for $\widetilde{\mathbf{T}}''_{j,+1}$ is established. We reduce the proof of Theorem **F** to the rank 2 cases. The proofs in rank 2 cases are largely uniform (avoiding type-by-type computation), based on the counterpart results in quantum group setting, the defining intertwining property of $\widetilde{\mathbf{T}}''_{w,+1}$, and some weight arguments.

### Factorization of a quasi $K$-matrix

It is well known that a quasi $R$-matrix of finite type admits a factorization into a product of rank 1 $R$-matrices parametrized by positive roots; see [KR90, LS90]; also cf. [Ja95].

Dobson and Kolb [DK19] proposed a conjecture on an analogous factorization for a quasi $K$-matrix of finite type into a product, denoted by $\widetilde{\Upsilon}_{\boldsymbol{w}_\circ}$, of rank 1 factors parametrized by restricted positive roots; see (13.1) for notation. They established a reduction from a general finite type to the rank 2 Satake diagrams. In addition, they established the rank 2 cases of *split* types and type AII/AIII, via a type-by-type lengthy computation based on several explicit formulas for rank 1 quasi $K$-matrices which they computed.

Exploring (the rank 2 cases of) Theorem **F** and some of its consequences, we provide a uniform and concise proof that $\widetilde{\Upsilon}_{\boldsymbol{w}_\circ}$ satisfies the same defining intertwining relations for $\widetilde{\Upsilon}$. Then the factorization property for arbitrary finite types follows by the uniqueness of $\widetilde{\Upsilon}$.

**Theorem G** (Dobson-Kolb Conjecture, Theorem 13.1). The quasi $K$-matrix $\widetilde{\Upsilon}$ for $\widetilde{\mathbf{U}}^\imath$ of finite type admits a factorization $\widetilde{\Upsilon} = \widetilde{\Upsilon}_{\boldsymbol{w}_\circ}$.

### Relative braid group relations

Recall Lusztig's symmetries $T'_{i,e}, T''_{i,e}$ on a quantum group $\mathbf{U}$ satisfy braid group relations associated to the (absolute) Weyl group $W$ [Lus93]; see [LW22a] for analogous statements on a Drinfeld double $\widetilde{\mathbf{U}}$. We have the following generalization in the setting of $\imath$quantum groups. Denote by $\mathrm{Br}(W^\circ)$ the braid group associated to $W^\circ$.

**Theorem H** (Theorem 14.1). Fix $e \in \{\pm 1\}$. The symmetries $\widetilde{\mathbf{T}}'_{i,e}$ (and respectively,

$\widetilde{\mathbf{T}}''_{i,e}$) of $\widetilde{\mathbf{U}}^\imath$, for $i \in \mathbb{I}_{\circ,\tau}$, satisfy the relative braid group relations in $\mathrm{Br}(W^\circ)$.

With the help of the intertwining relation (1.3), the proof of Theorem **H** is built on the braid group relations for $\widetilde{\mathcal{T}}_i$ $(i \in \mathbb{I})$ and the factorization properties of quasi $K$-matrices established in Theorem 13.1 (1).

It was shown in [BW18b] that Lusztig's symmetries $T'_{i,e}$ and $T''_{i,e}$ on $\mathbf{U}$, for $i \in \mathbb{I}_\bullet$, preserve the subalgebra $\mathbf{U}^\imath_{\boldsymbol{\varsigma}}$ (under some constraints on $\boldsymbol{\varsigma}$). We easily upgrade this statement to the universal quantum symmetric pair $(\widetilde{\mathbf{U}}, \widetilde{\mathbf{U}}^\imath)$, providing a braid group action of $\mathrm{Br}(W_\bullet)$ on $\widetilde{\mathbf{U}}^\imath$; see Proposition 4.5. Actually, we obtain 4 variants of actions of $\mathrm{Br}(W_\bullet)$ on $\widetilde{\mathbf{U}}^\imath$ generated by $\widetilde{\mathcal{T}}'_{j,e}$ or $\widetilde{\mathcal{T}}''_{j,e}$, for $j \in \mathbb{I}_\bullet$, respectively.

It is further established that the two ("black and white") braid group actions on $\widetilde{\mathbf{U}}^\imath$ combine neatly into an action of a semi-direct product $\mathrm{Br}(W_\bullet) \rtimes \mathrm{Br}(W^\circ)$ on $\widetilde{\mathbf{U}}^\imath$.

**Theorem I** (Theorem 14.3, Corollary 14.7). Let $e = \pm 1$.

1. There exists a braid group action of $\mathrm{Br}(W_\bullet) \rtimes \mathrm{Br}(W^\circ)$ on $\widetilde{\mathbf{U}}^\imath$ as automorphisms of algebras generated by $\widetilde{\mathcal{T}}'_{j,e}$ $(j \in \mathbb{I}_\bullet)$ and $\widetilde{\mathbf{T}}'_{i,e}$ $(i \in \mathbb{I}_{\circ,\tau})$.
2. There exists a braid group action of $\mathrm{Br}(W_\bullet) \rtimes \mathrm{Br}(W^\circ)$ on $\widetilde{\mathbf{U}}^\imath$ as automorphisms of algebras generated by $\widetilde{\mathcal{T}}''_{j,e}$ $(j \in \mathbb{I}_\bullet)$ and $\widetilde{\mathbf{T}}''_{i,e}$ $(i \in \mathbb{I}_{\circ,\tau})$.

Theorem **I** (or more precisely, its $\mathbf{U}^\imath_{\boldsymbol{\varsigma}}$-counterpart in Theorem 14.10; see §1.6 below) confirms an old conjecture of Kolb and Pellegrini [KP11, Conjecture 1.2] in full generality, and moreover, we have provided precise formulas for the braid group actions.

#### Relative braid group symmetries on $\mathbf{U}^\imath_{\boldsymbol{\varsigma}}$

By central reductions, the symmetries $\widetilde{\mathbf{T}}'_{i,-1}, \widetilde{\mathbf{T}}''_{i,+1}$ on the universal $\imath$quantum group $\widetilde{\mathbf{U}}^\imath$, for $i \in \mathbb{I}_\circ$, descend naturally to the $\imath$quantum group $\mathbf{U}^\imath_{\boldsymbol{\varsigma}_\diamond}$ with the distinguished parameter $\boldsymbol{\varsigma}_\diamond$. On the other hand, the symmetries $\widetilde{\mathbf{T}}'_{i,+1}, \widetilde{\mathbf{T}}''_{i,-1}$ naturally descend to $\mathbf{U}^\imath_{\overline{\boldsymbol{\varsigma}}_{\star\diamond}}$; see the commutative diagrams in §14.4. We then transport the relative braid group symmetries from $\mathbf{U}^\imath_{\boldsymbol{\varsigma}_\diamond}$ and $\mathbf{U}^\imath_{\overline{\boldsymbol{\varsigma}}_{\star\diamond}}$ to the $\imath$quantum groups $\mathbf{U}^\imath_{\boldsymbol{\varsigma}}$ (see Theorems 14.9–14.10), for an arbitrary parameter $\boldsymbol{\varsigma}$, thanks to the isomorphism $\mathbf{U}^\imath_{\boldsymbol{\varsigma}_\diamond} \cong \mathbf{U}^\imath_{\boldsymbol{\varsigma}}$ given in Proposition 2.14.

## 1.7 Main results for Part IV

In Part IV, we construct relative braid group symmetries on modules, which are compatible with symmetries on the $\imath$quantum group $\mathbf{U}^\imath$. Section 15 is based on [WZ22, §10].

#### Relative braid group actions on finite-dimensional U-modules

Let $\boldsymbol{\varsigma}$ be a balanced parameter, $i \in \mathbb{I}_\circ$ and $e = \pm 1$. We show that the symmetries $\mathbf{T}'_{i,e}, \mathbf{T}''_{i,e}$ on the $\imath$quantum group $\mathbf{U}^\imath_{\boldsymbol{\varsigma}}$ (defined by central reductions) satisfy natural intertwining relations with the usual braid group symmetries on $\mathbf{U}$. These intertwining properties allow us to formulate automorphisms (denoted again by the same notations $\mathbf{T}'_{i,e}, \mathbf{T}''_{i,e}$) on any integrable $\mathbf{U}$-module $M$, whose weights are bounded above; see (15.9). These operators on $M$ admit favorable properties parallel to those satisfied by Lusztig's braid group actions on modules.

**Theorem J** (Theorem 15.4, Theorem 15.5)*. Let $i \in \mathbb{I}_\circ$ and $e = \pm 1$, and let $M$ be* any integrable $\mathbf{U}$-module, whose weights are bounded above. The automorphisms $\mathbf{T}'_{i,e}, \mathbf{T}''_{i,e}$ on $M$ are compatible with the corresponding automorphisms on $\mathbf{U}^\imath_{\boldsymbol{\varsigma}}$, i.e.,

$$\mathbf{T}'_{i,e}(xv) = \mathbf{T}'_{i,e}(x)\mathbf{T}'_{i,e}(v), \qquad \mathbf{T}''_{i,e}(xv) = \mathbf{T}''_{i,e}(x)\mathbf{T}''_{i,e}(v),$$

for any $x \in \mathbf{U}^\imath_{\boldsymbol{\varsigma}}, v \in M$. Moreover, the operators $\mathbf{T}'_{i,e}$ (respectively, $\mathbf{T}''_{i,e}$) on $M$, for $i \in \mathbb{I}_\circ$, satisfy the relative braid group relations in $\mathrm{Br}(W^\circ)$.

### Relative braid group actions on integrable $\mathbf{U}^\imath$-modules

The operators $\mathbf{T}'_{i,e}, \mathbf{T}''_{i,e}$ formulated in Theorem **J** are constructed using the quasi $K$-matrices and the compatibility is proved via the intertwining relation (1.3). As a consequence, actions for these operators are given in terms of elements in quantum groups, and for this reason, we need to restrict to certain $\mathbf{U}$-modules instead of general integrable $\mathbf{U}^\imath$-modules; see § 16.1 for the definition of integrable $\mathbf{U}^\imath$-modules.

To remove such a restriction, we rewrite the action of operators $\mathbf{T}'_{i,e}, \mathbf{T}''_{i,e}$ using elements of $\mathbf{U}^\imath$ in the split Kac-Moody case. We recall from [BeW18] the transition matrix between the canonical basis and the $\imath$canonical basis on $\mathbf{U}(\mathfrak{sl}_2)$ modules, and compute its inverse matrix in Proposition 16.4. Using the inverse matrix, we obtain rank one formulas for operators $\mathbf{T}'_{i,e}, \mathbf{T}''_{i,e}$ with arbitrary parameters, in terms of $\imath$divided powers of generators in $\mathbf{U}^\imath$; see § 16.3. The precise formulas are given as follows (see (16.23))

$$\begin{aligned}
\mathbf{T}'_{i,-1}v &= \sum_{\substack{k\geq 0\\ \overline{k}=\overline{p}}} (-q_i^2\varsigma_i)^{-k/2} B^{(k)}_{i,\overline{k},\boldsymbol{\varsigma}} v, &\qquad \mathbf{T}''_{i,-1}v &= \sum_{\substack{k\geq 0\\ \overline{k}=\overline{p}}} (-1)^{k/2}(q_i^2\varsigma_i)^{-k/2} B^{(k)}_{i,\overline{k},\boldsymbol{\varsigma}} v,\\
\mathbf{T}''_{i,+1}v &= \sum_{\substack{k\geq 0\\ \overline{k}=\overline{p}}} (-1)^{k/2}\varsigma_i^{-k/2} B^{(k)}_{i,\overline{k},\boldsymbol{\varsigma}} v, &\qquad \mathbf{T}'_{i,+1}v &= \sum_{\substack{k\geq 0\\ \overline{k}=\overline{p}}} (-\varsigma_i)^{-k/2} B^{(k)}_{i,\overline{k},\boldsymbol{\varsigma}} v,
\end{aligned} \tag{1.6}$$

where $v$ is an $\imath$weight vector of $\imath$weight $\overline{p} \in \mathbb{Z}/2\mathbb{Z}$ with respect to $B_i$.

Using this formula, we define linear operators $\mathbf{T}'_{i,e}, \mathbf{T}''_{i,e}$ on any integrable $\mathbf{U}^\imath$ modules; see (16.23). These operators coincide with those operators in Theorem **J** when acting on $\mathbf{U}$-modules. We show that these operators are compatible with corresponding relative braid group symmetries on the $\imath$quantum groups in Theorem 16.9.

This construction of compatible relative braid group symmetries on integrable $\mathbf{U}^\imath$-modules can be generalized beyond the split type and we will study all of these in a forthcoming paper.

In this dissertation, we have assumed that a ground field $\mathbb{F}$ is the algebraic closure of $\mathbb{Q}(q)$ partly due to uses of rescaling automorphisms, though often it suffices to work with the field $\mathbb{Q}(q^{\frac{1}{2}})$ if we choose the parameters $\boldsymbol{\varsigma}$ suitably. There is a $\mathbb{Q}(q)$-form ${}_{\mathbb{Q}}\widetilde{\mathbf{U}}^\imath$ of $\widetilde{\mathbf{U}}^\imath$ such that $\widetilde{\mathbf{U}}^\imath = \mathbb{F} \otimes_{\mathbb{Q}(q)} {}_{\mathbb{Q}}\widetilde{\mathbf{U}}^\imath$; see (5.16). The symmetries $\widetilde{\mathbf{T}}'_{i,e}, \widetilde{\mathbf{T}}''_{i,e}$ indeed preserve the $\mathbb{Q}(q)$-subalgebra ${}_{\mathbb{Q}}\widetilde{\mathbf{U}}^\imath$; see Proposition 5.9. Theorems **A**–**I** remain valid for ${}_{\mathbb{Q}}\widetilde{\mathbf{U}}^\imath$.

## 1.8 Future works and applications

The formulations of the main results (Theorems **A**–**J**), up to some reasonable rephrasing, make sense for universal quantum symmetric pairs of arbitrary Kac-Moody type, and we conjecture they are valid in this great generality. For example, the symmetries $\widetilde{\mathbf{T}}'_{i,-1}$, for $i \in \mathbb{I}_\circ$, for $\widetilde{\mathbf{U}}^\imath$ of Kac-Moody type can be constructed in a similar way once Conjecture 5.13 is confirmed.

The relative braid group symmetries of affine type has been used crucially in the Drinfeld type presentation of affine $\imath$quantum groups of quasi-split type; cf. [LW21b, Z22b, LWZ22]. It is expected that they will continue to play a key role for Drinfeld type presentations of affine $\imath$quantum groups in general.

One may hope that these new braid group symmetries preserve the integral $\mathbb{Z}[q, q^{-1}]$-form on (modified) $\imath$quantum groups in [BW18b, BW21]. (This will be highly nontrivial to verify, as the $\imath$divided powers are much more sophisticated than the divided powers.) It will be interesting to develop further connections among relative braid group actions, PBW bases and $\imath$canonical bases; compare [Lus93]. They may help to stimulate further KLR type categorification of $\imath$quantum groups as well as $\imath$Hall algebra realization of $\imath$quantum groups beyond quasi-split type.

Kolb and Yakimov [KY20] extended the construction of quantum symmetric pairs to the setting of Nichols algebras of diagonal type. The new intertwining properties of quasi $K$-matrices and the relative braid group actions established in this dissertation seem well suited for generalizations in this direction.

The notion of relative Coexter groups, which is valid in a more general setting than symmetric pairs, admits a geometric interpretation [Lus76, Lus03]. It will be exciting to realize relative braid group action in geometric and categorical frameworks, and develop possible connections to the representation theory of real groups (cf. [BV21] and references therein). It will be very interesting to explore more general braid group actions associated to relative Coxeter groups.

## 1.9 Notations

We list the notations which are often used throughout the dissertation.

▷ $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{C}$ – sets of nonnegative integers, integers, rational and complex numbers

▷ $\mathcal{R}$ – systems of roots with simple systems $\Pi = \{\alpha_i | i \in \mathbb{I}\}$

▷ $\mathcal{R}^\vee$ – systems of coroots with simple systems $\Pi^\vee = \{\alpha_i^\vee | i \in \mathbb{I}\}$

▷ $W, \ell(\cdot)$ – the Weyl group and its length function

▷ $w_0, \tau_0$ – the longest element in $W$ and its associated diagram involution

▷ $\widetilde{T}'_{i,e}, \widetilde{T}''_{i,e}$ – braid group symmetries on $\widetilde{\mathbf{U}}$

▷ $(\mathbb{I} = \mathbb{I}_\bullet \cup \mathbb{I}_\circ, \tau)$ – admissible pairs (aka Satake diagrams)

▷ $\mathbb{I}_{\circ,\tau}^{\mathrm{KM}}$ – a fixed set of representatives for $\tau$ orbits on $\mathbb{I}_\circ$.

▷ $\mathbb{I}_{\circ,\tau}$ – the subset of $\mathbb{I}_{\circ,\tau}^{\mathrm{KM}}$ defined in (2.20).

▷ $W_\bullet, \mathcal{R}_\bullet$ – the Weyl group and root system associated to the subdiagram $\mathbb{I}_\bullet$

▷ $w_\bullet$ – the longest element in $W_\bullet$

▷ $W_{\bullet,i}$ – the parabolic subgroup of $W$ generated by $s_k$, for $k \in \mathbb{I}_{\bullet,i} := \mathbb{I}_\bullet \cup \{i, \tau i\}$

▷ $w_{\bullet,i}$, $\tau_{\bullet,i}$ – the longest element of $W_{\bullet,i}$ and its associated diagram involution

▷ $W^\circ, \ell_\circ(\cdot)$ – the relative Weyl group generated by $\mathbf{r}_i := w_{\bullet,i} w_\bullet$, for $i \in \mathbb{I}_{\circ,\tau}$, and its length function such that $\ell_\circ(\mathbf{r}_i) = 1$

▷ $\boldsymbol{w}_\circ$ – the longest element in $W^\circ$

▷ $\mathbf{U}, \widetilde{\mathbf{U}}$ – quantum group and Drinfeld double

▷ $\widehat{\tau}, \widehat{\tau}_0$ – involutions on $\widetilde{\mathbf{U}}$ induced by the diagram involutions $\tau, \tau_0$

▷ $\widetilde{\mathbf{U}}^\imath, \mathbf{U}^\imath_{\boldsymbol{\varsigma}}$ – universal $\imath$quantum group and $\imath$quantum group with parameter $\boldsymbol{\varsigma}$

- $\triangleright$ $\widetilde{\Upsilon}$ – quasi $K$-matrix for universal quantum symmetric pair $(\widetilde{\mathbf{U}}, \widetilde{\mathbf{U}}^\imath)$

- $\triangleright$ $\boldsymbol{\varsigma}_\diamond, \boldsymbol{\varsigma}_\star$ – two distinguished parameters; see (2.28) and (3.8)

- $\triangleright$ $\widetilde{\Psi}_{\mathbf{a}}$ – a rescaling automorphism of $\widetilde{\mathbf{U}}$; see (2.8)

- $\triangleright$ $\Phi_{\mathbf{a}}$ – a rescaling automorphism of $\mathbf{U}$; see (2.9)

- $\triangleright$ $\pi_{\boldsymbol{\varsigma}}$ – a central reduction from $\widetilde{\mathbf{U}}$ to $\mathbf{U}$; see (2.7)

- $\triangleright$ $\pi^\imath_{\boldsymbol{\varsigma}}$ – a central reduction from $\widetilde{\mathbf{U}}^\imath$ to $\mathbf{U}^\imath_{\boldsymbol{\varsigma}}$; see Proposition 2.12

- $\triangleright$ $\psi^\imath$ – a bar involution on $\widetilde{\mathbf{U}}^\imath$; see (3.10)

- $\triangleright$ $\sigma^\imath$ – an anti-involution on $\widetilde{\mathbf{U}}^\imath$; see (3.24)

- $\triangleright$ $\sigma_\tau$ – an anti-involution on $\mathbf{U}^\imath_{\boldsymbol{\varsigma}}$; see (3.26)

- $\triangleright$ $\widetilde{\mathcal{T}}'_{i,e}, \widetilde{\mathcal{T}}''_{i,e}$ – rescaled (via $\widetilde{\Psi}_{\mathbf{a}}$) braid group symmetries on $\widetilde{\mathbf{U}}$; see (4.2)–(4.3)

- $\triangleright$ $\widetilde{\mathbf{T}}'_{i,e}, \widetilde{\mathbf{T}}''_{i,e}$ – braid group symmetries on $\widetilde{\mathbf{U}}^\imath$

- $\triangleright$ $\widetilde{\mathcal{T}}_i, \widetilde{\mathcal{T}}_i^{-1}, \widetilde{\mathbf{T}}_i, \widetilde{\mathbf{T}}_i^{-1}$ – shorthand notations for $\widetilde{\mathcal{T}}''_{i,+1}, \widetilde{\mathcal{T}}'_{i,-1}, \widetilde{\mathbf{T}}''_{i,+1}, \widetilde{\mathbf{T}}'_{i,-1}$

- $\triangleright$ $\mathcal{T}'_{i,e;\boldsymbol{\varsigma}}, \mathcal{T}''_{i,e;\boldsymbol{\varsigma}}$ – rescaled braid group symmetries on $\mathbf{U}$; see (15.1), (15.7)

# 2 Preliminaries

In this section, we set up notations for quantum groups, Drinfeld doubles, and quantum symmetric pairs in the Kac-Moody setting. We review the braid group action, introduced by Lusztig, on quantum groups. We review the relative Weyl and braid groups associated to Satake diagrams. Several basic properties of (universal) $\imath$quantum groups are presented.

## 2.1 Quantum groups and Drinfeld doubles

We set up notations for a quantum group $\mathbf{U}$ of Kac-Moody type and its Drinfeld double $\widetilde{\mathbf{U}}$.

Let $\mathfrak{g}$ be a symmetrizable Kac-Moody algebra over $\mathbb{C}$ with a generalised Cartan matrix $C = (c_{ij})_{i,j\in\mathbb{I}}$. Let $D = \mathrm{diag}(\epsilon_i \mid \epsilon_i \in \mathbb{Z}_{\geq 1},\ i \in \mathbb{I})$ be a symmetrizer, i.e., $DC$ is symmetric, such that $\gcd\{\epsilon_i \mid i \in \mathbb{I}\} = 1$. Fix a simple system $\Pi = \{\alpha_i | i \in \mathbb{I}\}$ of $\mathfrak{g}$ and a set of simple coroots $\Pi^\vee = \{\alpha_i^\vee | i \in \mathbb{I}\}$. Let $\mathcal{R}$ and $\mathcal{R}^\vee$ be the corresponding root and coroot systems. Denote the root lattice by $\mathbb{Z}\mathbb{I} := \oplus_{i\in\mathbb{I}}\mathbb{Z}\alpha_i$. Let $(\cdot,\cdot)$ be the normalized Killing form on $\mathbb{Z}\mathbb{I}$ so that the short roots have squared length 2. The Weyl group $W$ is generated by the simple reflections $s_i : \mathbb{Z}\mathbb{I} \to \mathbb{Z}\mathbb{I}$, for $i \in \mathbb{I}$, such that $s_i(\alpha_j) = \alpha_j - c_{ij}\alpha_i$. Set $w_0$ to be the longest element of $W$.

Let $q$ be an indeterminate and $\mathbb{Q}(q)$ be the field of rational functions in $q$ with coefficients in $\mathbb{Q}$, the field of rational numbers. Set $\mathbb{F}$ to be the algebraic closure of $\mathbb{Q}(q)$ and $\mathbb{F}^\times := \mathbb{F} \setminus \{0\}$. We denote

$$q_i := q^{\epsilon_i}, \qquad \forall i \in \mathbb{I}.$$

Denote, for $r, m \in \mathbb{N}$,

$$[r]_t = \frac{t^r - t^{-r}}{t - t^{-1}}, \quad [r]_t! = \prod_{i=1}^{r} [i]_t, \quad \begin{bmatrix} m \\ r \end{bmatrix}_t = \frac{[m]_t[m-1]_t \ldots [m-r+1]_t}{[r]_t!}.$$

We mainly take $t = q, q_i$.

Let $(Y, X, \langle \cdot, \cdot \rangle, \cdots)$ be a root datum of type $(\mathbb{I}, \cdot)$; cf. [Lus93, §2.2]. By definition, there are embeddings denoted by $\mathbb{I} \to X, i \mapsto i'$ and $\mathbb{I} \to Y, i \mapsto i$. The relation between $\langle \cdot, \cdot \rangle$ and $(\cdot, \cdot)$ is $\langle i, j' \rangle = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}$.

We define a partial order on the lattice $X$: for $\lambda, \lambda' \in X$,

$$\lambda \leq \lambda' \text{ if and only if } \lambda' - \lambda \in \mathbb{N}\mathbb{I}. \tag{2.1}$$

We always assume that the root datum $(Y, X, \langle \cdot, \cdot \rangle, \cdots)$ is $Y$-regular; that is, the image of $\mathbb{I} \to Y$ is linearly independent in $Y$.

The Drinfeld double quantum group $\widetilde{\mathbf{U}} := \widetilde{\mathbf{U}}_q(\mathfrak{g})$ is defined to be the $\mathbb{F}$-algebra generated by $E_i, F_i, K_i, K_i'$, $i \in \mathbb{I}$, where $K_i, K_i'$ are invertible, subject to the following relations: $K_i, K_j'$ commute with each other, for all $i, j \in \mathbb{I}$,

$$[E_i, F_j] = \delta_{ij} \frac{K_i - K_i'}{q - q^{-1}}, \quad K_i E_j = q_i^{c_{ij}} E_j K_i, \quad K_i F_j = q_i^{-c_{ij}} F_j K_i, \tag{2.2}$$

$$K_i' E_j = q_i^{-c_{ij}} E_j K_i', \qquad K_i' F_j = q_i^{c_{ij}} F_j K_i', \tag{2.3}$$

and the quantum Serre relations, for $i \neq j \in \mathbb{I}$,

$$\sum_{s=0}^{1-c_{ij}} (-1)^s \begin{bmatrix} 1 - c_{ij} \\ s \end{bmatrix}_{q_i} E_i^s E_j E_i^{1-c_{ij}-s} = 0, \tag{2.4}$$

$$\sum_{s=0}^{1-c_{ij}} (-1)^s \begin{bmatrix} 1-c_{ij} \\ s \end{bmatrix}_{q_i} F_i^s F_j F_i^{1-c_{ij}-s} = 0. \tag{2.5}$$

Note that $K_i K_i'$ are central in $\widetilde{\mathbf{U}}$, for all $i \in \mathbb{I}$.

For $\mu = \sum_{i\in\mathbb{I}} c_i \alpha_i \in \mathbb{Z}\mathbb{I}$, set $K_\mu = \prod_{i\in\mathbb{I}} K_i^{c_i}$ and set $K'_\mu$ similarly. The Cartan part $\widetilde{\mathbf{U}}^0 := \langle K_i, K_i', i \in \mathbb{I}\rangle$ is a commutative subalgebra and there is an isomorphism $\mathbb{Z}\mathbb{I} \oplus \mathbb{Z}\mathbb{I} \to \widetilde{\mathbf{U}}^0, (\mu,\nu) \mapsto K_\mu K'_\nu$.

*Remark* 2.1. A more standard version of the Drinfeld double quantum group $\widetilde{\mathbf{U}}^{\rm std} = \langle E_i, F_i, \underline{K}_\mu, \underline{K}'_\mu | i \in \mathbb{I}, \mu \in Y\rangle$ can be formulated where the Cartan part is parametrized by $Y$; this is a "double" version of the quantum group in [Lus93, §3.1.1]. The $\widetilde{\mathbf{U}}$ defined above is identified with a natural subalgebra of $\widetilde{\mathbf{U}}^{\rm std}$ via the following embedding

$$E_i \mapsto E_i, \quad F_i \mapsto F_i, \quad K_i \mapsto \underline{K}_i^{\epsilon_i}, \quad K_i' \mapsto (\underline{K}_i')^{\epsilon_i}.$$

The difference between these two versions of Drinfeld doubles only lies on the Cartan part. As shown in [Lus93, §37.1.3], actions of braid group symmetries on the Cartan part is the same as the Weyl group action on the corresponding lattice. So there is not much difference which versions to use for considering the braid group action, and we choose to wrok with $\widetilde{\mathbf{U}}$ for the sake of simplicity of notations.

The comultiplication $\Delta : \widetilde{\mathbf{U}} \to \widetilde{\mathbf{U}} \otimes \widetilde{\mathbf{U}}$ is defined as follows:

$$\begin{aligned} \Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i, \quad \Delta(F_i) = 1 \otimes F_i + F_i \otimes K_i', \\ \Delta(K_i) = K_i \otimes K_i, \quad \Delta(K_i') = K_i' \otimes K_i'. \end{aligned} \tag{2.6}$$

Let $\mathbf{U} = \mathbf{U}(\mathfrak{g})$ be the Drinfeld-Jimbo quantum group associated to $\mathfrak{g}$ over $\mathbb{F}$ with

Chevalley generators $\{E_i, F_i, K_i^{\pm 1} | i \in \mathbb{I}\}$, whose relations can be obtained from $\widetilde{\mathbf{U}}$ above by simply replacing $K_i'$ by $K_i^{-1}$, for all $i$; that is, one identifies $\mathbf{U} = \widetilde{\mathbf{U}}/(K_iK_i' - 1 \mid i \in \mathbb{I})$. Both $\widetilde{\mathbf{U}}$ and $\mathbf{U}$ admit standard triangular decompositions, $\widetilde{\mathbf{U}} = \widetilde{\mathbf{U}}^- \widetilde{\mathbf{U}}^0 \widetilde{\mathbf{U}}^+$ and $\mathbf{U} = \mathbf{U}^- \mathbf{U}^0 \mathbf{U}^+$; we identify $\widetilde{\mathbf{U}}^+ = \mathbf{U}^+ = \langle E_i \mid i \in \mathbb{I}\rangle$ and $\widetilde{\mathbf{U}}^- = \mathbf{U}^-$.

For any scalars $\mathbf{a} = (a_i)_{i\in\mathbb{I}} \in \mathbb{F}^{\times,\mathbb{I}}$, one has an isomorphism

$$\widetilde{\mathbf{U}}/(K_iK_i' - a_i \mid i \in \mathbb{I}) \xrightarrow{\cong} \mathbf{U}$$

through the central reduction

$$\begin{aligned} \pi_{\mathbf{a}} : \widetilde{\mathbf{U}} &\longrightarrow \mathbf{U}, \\ F_i \mapsto F_i, \quad E_i \mapsto \sqrt{a_i} E_i, &\quad K_i \mapsto \sqrt{a_i} K_i, \quad K_i' \mapsto \sqrt{a_i} K_i^{-1}. \end{aligned} \tag{2.7}$$

The canonical identification uses $\pi_{\mathbf{1}}$, for $\mathbf{1} = \{1\}_{i\in\mathbb{I}}$.

**Proposition 2.2.** *Let $\mathbf{a} = (a_i)_{i\in\mathbb{I}} \in (\mathbb{F}^\times)^{\mathbb{I}}$. We have an automorphism $\widetilde{\Psi}_{\mathbf{a}}$ on the $\mathbb{F}$-algebra $\widetilde{\mathbf{U}}$ such that*

$$\widetilde{\Psi}_{\mathbf{a}} : K_i \mapsto a_i^{1/2} K_i, \quad K_i' \mapsto a_i^{1/2} K_i', \quad E_i \mapsto a_i^{1/2} E_i, \quad F_i \mapsto F_i. \tag{2.8}$$

*We have an automorphism $\Phi_{\mathbf{a}}$ on the $\mathbb{F}$-algebra $\mathbf{U}$ such that*

$$\Phi_{\mathbf{a}} : K_i \mapsto K_i, \quad E_i \mapsto a_i^{1/2} E_i, \quad F_i \mapsto a_i^{-1/2} F_i. \tag{2.9}$$

We have

$$\pi_{\mathbf{a}} = \pi_{\mathbf{1}} \circ \widetilde{\Psi}_{\mathbf{a}}. \tag{2.10}$$

A $\mathbb{Q}$-linear operator on a $\mathbb{F}$-algebra is *anti-linear* if it sends $q^m \mapsto q^{-m}$, for $m \in \mathbb{Z}$.

**Proposition 2.3.**

1. *There exists an anti-linear involution* $\psi$ *on* $\widetilde{\mathbf{U}}$*, which fixes* $E_i, F_i$ *and swaps* $K_i \leftrightarrow K_i'$*, for* $i \in \mathbb{I}$*. There exists an anti-linear involution on* $\mathbf{U}$*, also denoted by* $\psi$*, which fixes* $E_i, F_i$ *and swaps* $K_i \leftrightarrow K_i^{-1}$*, for* $i \in \mathbb{I}$*;*

2. *There exists an anti-involution* $\sigma$ *on* $\widetilde{\mathbf{U}}$ *which fixes* $E_i, F_i$ *and swaps* $K_i \leftrightarrow K_i'$*, for* $i \in \mathbb{I}$*. There exists an anti-involution on* $\mathbf{U}$*, also denoted by* $\sigma$*, which fixes* $E_i, F_i$ *and swaps* $K_i \leftrightarrow K_i^{-1}$*, for* $i \in \mathbb{I}$*;*

3. *There exists a Chevalley involution* $\omega$ *on* $\widetilde{\mathbf{U}}$ *which swaps* $E_i$ *and* $F_i$ *and swaps* $K_i \leftrightarrow K_i'$*, for* $i \in \mathbb{I}$*. There exists a Chevalley involution on* $\mathbf{U}$*, also denoted by* $\omega$*, which swaps* $E_i$ *and* $F_i$ *and swaps* $K_i \leftrightarrow K_i^{-1}$*, for* $i \in \mathbb{I}$*.*

Let $\widetilde{\mathbf{U}} = \bigoplus_{\nu \in \mathbb{Z}\mathbb{I}} \widetilde{\mathbf{U}}_\nu$ be the weight decomposition of $\widetilde{\mathbf{U}}$ such that $E_i \in \widetilde{\mathbf{U}}_{\alpha_i}, F_i \in \widetilde{\mathbf{U}}_{-\alpha_i}, K_i, K_i' \in \widetilde{\mathbf{U}}_0$. Write $\widetilde{\mathbf{U}}^+_\nu := \widetilde{\mathbf{U}}_\nu \cap \widetilde{\mathbf{U}}^+$.

## 2.2 Braid group action on quantum groups

Let $\mathrm{Br}(W)$ be the braid group associated to the Weyl group $W$. Lusztig introduced braid group symmetries $T'_{i,e}, T''_{i,e}$, for $i \in \mathbb{I}$ and $e = \pm 1$, on a quantum group $\mathbf{U}$ [Lus93, §37.1.3]. These symmetries lead to a braid group $\mathrm{Br}(W)$-action on $\mathbf{U}$, which is a quantization of the classical braid group $\mathrm{Br}(W)$-action on the enveloping algebra of $\mathfrak{g}$. We recall the formulation of $T''_{i,e}$.

**Proposition 2.4** ([Lus93, §37.1.3])**.** *Set* $r = -c_{ij}$*. There exist an automorphism* $T'_{i,e}, T''_{i,e}$*, for* $i \in \mathbb{I}, e = \pm 1$*, on* $\mathbf{U}$ *such that*

$$T'_{i,e}(K_j) = T''_{i,e}(K_j) = K_j K_i^{-c_{ij}}, \qquad T'_{i,e}(E_i) = -K_i^e F_i, \qquad T'_{i,+1}(F_i) = -E_i K_i^{-e},$$

$$T''_{i,e}(E_i) = -F_iK_i^e, \qquad T''_{i,+1}(F_i) = -K_i^{-e}E_i,$$

$$T''_{i,e}(E_j) = \sum_{s=0}^{r}(-1)^s q_i^{-es} E_i^{(r-s)}E_jE_i^{(s)}, \quad T'_{i,e}(E_j) = \sum_{s=0}^{r}(-1)^s q_i^{es} E_i^{(s)}E_jE_i^{(r-s)}, \quad j \neq i,$$

$$T''_{i,e}(F_j) = \sum_{s=0}^{r}(-1)^s q_i^{es} F_i^{(s)}F_jF_i^{(r-s)}, \quad T'_{i,e}(F_j) = \sum_{s=0}^{r}(-1)^s q_i^{-es} F_i^{(r-s)}F_jF_i^{(s)}, \quad j \neq i.$$

*Moreover, the automorphisms* $T'_{i,e}, T''_{i,e}$, *for* $i \in \mathbb{I}, e = \pm 1$, *satisfy the braid relations.*

It follows by their definitions that $T'_{i,e}, T''_{i,-e}$ are mutually inverses cf. [Lus93, §5.2.3]. Symmetries $T'_{i,e}, T''_{i,e}$ are related to each other via the following identities

$$T'_{i,-1} = \sigma \circ T''_{i,+1} \circ \sigma,$$

$$T''_{i,-e} = \psi \circ T''_{i,+e} \circ \psi, \qquad T'_{i,+e} = \psi \circ T'_{i,-e} \circ \psi,$$

where $\psi, \sigma$ are the bar involution and anti-involution on $\mathbf{U}$ given in Proposition 2.2.

Denote by $E_i^{(n)}, F_i^{(n)}$ the divided powers $\frac{E_i^n}{[n]_i!}, \frac{F_i^n}{[n]_i!}$ in $\mathbf{U}$, for $n \in \mathbb{N}$. Let $M$ be an integrable $\mathbf{U}$-module (of type $\mathbf{1}$); cf. [Lus93, §5]. By definition, $E_i, F_i$ act locally nilpotentlty on $M$ and $M$ has a weight space decomposition of $M$ with respect to a fixed $i \in \mathbb{I}$

$$M = \bigoplus_{n \in \mathbb{Z}} M_n, \qquad M_n = \{v \in M | K_iv = q_i^n v\}.$$

Following [Lus93, §5.2.1], we define linear operators $T'_{i,e}, T''_{i,e}, e = \pm 1$ on $M$ by

$$T'_{i,e}(v) = \sum_{\substack{a,b,c \geq 0; \\ a-b+c=m}} (-1)^b q_i^{e(b-ac)} F_i^{(a)}E_i^{(b)}F_i^{(c)}v, \qquad v \in M_m, \tag{2.11}$$

$$T''_{i,e}(v) = \sum_{\substack{a,b,c \geq 0; \\ -a+b-c=m}} (-1)^b q_i^{e(b-ac)} E_i^{(a)}F_i^{(b)}E_i^{(c)}v, \qquad v \in M_m. \tag{2.12}$$

**Proposition 2.5** ([Lus93, 39.4.3])**.** *Let* $M$ *be an integrable* $\mathbf{U}$*-module. Then, for any*

*$u \in \mathbf{U}, v \in M, e = \pm 1$, we have*

$$T'_{i,e}(uv) = T'_{i,e}(u)T'_{i,e}(v), \qquad T''_{i,e}(uv) = T''_{i,e}(u)T''_{i,e}(v). \tag{2.13}$$

## 2.3 Braid group action on the Drinfeld double $\widetilde{\mathbf{U}}$

Analogous braid group symmetries $\widetilde{T}'_{i,e}, \widetilde{T}''_{i,e}$, for $i \in \mathbb{I}$ and $e = \pm 1$, exist on the Drinfeld double $\widetilde{\mathbf{U}}$; see [LW22a, Propositions 6.20–6.21]. (Our notations $\widetilde{T}'_{i,e}$, $\widetilde{T}''_{i,e}$ here correspond to $\mathbf{T}'_{i,e}$, $\mathbf{T}''_{i,e}$ therein.) We recall the formulation of $\widetilde{T}''_{i,+1}$ below.

**Proposition 2.6** ([LW22a, Proposition 6.21]). *Set $r = -c_{ij}$. There exist an automorphism $\widetilde{T}''_{i,+1}$, for $i \in \mathbb{I}$, on $\widetilde{\mathbf{U}}$ such that*

$$\begin{aligned}
&\widetilde{T}''_{i,+1}(K_j) = K_j K_i^{-c_{ij}}, \qquad \widetilde{T}''_{i,+1}(K'_j) = K'_j K_i'^{-c_{ij}},\\
&\widetilde{T}''_{i,+1}(E_i) = -F_i K_i'^{-1}, \qquad \widetilde{T}''_{i,+1}(F_i) = -K_i^{-1} E_i,\\
&\widetilde{T}''_{i,+1}(E_j) = \sum_{s=0}^{r} (-1)^s q_i^{-s} E_i^{(r-s)} E_j E_i^{(s)}, \qquad j \neq i,\\
&\widetilde{T}''_{i,+1}(F_j) = \sum_{s=0}^{r} (-1)^s q_i^{s} F_i^{(s)} F_j F_i^{(r-s)}, \qquad j \neq i.
\end{aligned}$$

*Moreover, the automorphisms $\widetilde{T}''_{i,+1}$, for $i \in \mathbb{I}$, satisfy the braid relations.*

We sometimes use the following conventional short notations

$$\widetilde{T}_i := \widetilde{T}''_{i,+1}, \quad \widetilde{T}_i^{-1} := \widetilde{T}'_{i,-1}, \quad T_i := T''_{i,+1}, \quad T_i^{-1} := T'_{i,-1}.$$

Hence, we can define

$$\widetilde{T}_w \equiv \widetilde{T}''_{w,+1} := \widetilde{T}_{i_1} \cdots \widetilde{T}_{i_r} \in \mathrm{Aut}(\widetilde{\mathbf{U}}),$$

where $w = s_{i_1} \cdots s_{i_r}$ is any reduced expression of $w \in W$. Similarly, one defines $T_w$ for $w \in W$.

The symmetries $\widetilde{T}'_{i,e}$ and $\widetilde{T}''_{i,e}$, for $i \in \mathbb{I}$, satisfy the following identities in $\widetilde{\mathbf{U}}$ [LW22a] (analogous to [Lus93, 37.2.4] in $\mathbf{U}$)

$$\begin{gathered}\widetilde{T}'_{i,-1} = \sigma \circ \widetilde{T}''_{i,+1} \circ \sigma, \\ \widetilde{T}''_{i,-e} = \psi \circ \widetilde{T}''_{i,+e} \circ \psi, \qquad \widetilde{T}'_{i,+e} = \psi \circ \widetilde{T}'_{i,-e} \circ \psi.\end{gathered} \tag{2.14}$$

The automorphism $\widetilde{T}''_{i,+1}$ descends to Lusztig's automorphisms $T''_{i,+1}$ on $\mathbf{U}$:

$$\pi_{\mathbf{1}} \circ \widetilde{T}''_{i,+1} = T''_{i,+1} \circ \pi_{\mathbf{1}}. \tag{2.15}$$

## 2.4 Braid group action and root vectors

Denote the divided power $\frac{F}{[r]_i!}$ by $F_i^{(r)}$. Define

$$\begin{aligned} y_{i,j;m,e} &= \sum_{r+s=m} (-1)^r q_i^{er(m+c_{ij}-1)} F_i^{(s)} F_j F_i^{(r)}, \qquad y'_{i,j;m,e} = \sigma(y_{i,j;m,e}), \\ x_{i,j;m,e} &= \sum_{r+s=m} (-1)^r q_i^{er(-m-c_{ij}+1)} E_i^{(r)} E_j E_i^{(s)}, \qquad x'_{i,j;m,e} = \sigma(x_{i,j;m,e}). \end{aligned} \tag{2.16}$$

Note that $y_{i,j;m,e} = \sigma\omega\psi(x_{i,j;m,e})$ and $y_{i,j;m,e} = \psi(y_{i,j;m,-e})$.

*Remark* 2.7. In Lusztig's conventions [Lus93, §37.2.1], our $y_{i,j;m,e}$ is identified with his $y_{i,j;1,m,e}$ and $x_{i,j;m,e}$ is identified with his $x_{i,j;1,m,e}$.

**Lemma 2.8** (cf. [Lus93, Lemma 7.1.2]). *We have, for $i \neq j \in \mathbb{I}$,*

*(1)* $-q_i^{-c_{ij}-2m} y_{i,j;m,-1} F_i + F_i y_{i,j;m,-1} = [m+1]_i y_{i,j;m+1,-1}$,

(2) $-q_i^{c_{ij}+2m} E_i x_{i,j;m,-1} + x_{i,j;m,-1} E_i = [m+1]_i x_{i,j;m+1,-1}$,

(3) $-y_{i,j;m,-1} E_i + E_i y_{i,j;m,-1} = [-c_{ij}-m+1]_i y_{i,j;m-1,-1} K_i'$,

(4) $-F_i x_{i,j;m,-1} + x_{i,j;m,-1} F_i = [-c_{ij}-m+1]_i K_i x_{i,j;m-1,-1}$.

The recursive relations for $y_{i,j;m,+1}, x_{i,j;m,+1}$ are obtained by applying $\psi$ to above relations. The recursive relations for $y'_{i,j;m,e}, x'_{i,j;m,e}$ are obtained by applying $\sigma$ to above relations.

Recall that $\widetilde{T}'_{i,e}, \widetilde{T}''_{i,e}$ are the braid group symmetries on $\widetilde{\mathbf{U}}$ formulated in [LW22a, Propositions 6.20-6.21]. We recall the (rank two) actions of $\widetilde{T}'_{i,e}, \widetilde{T}''_{i,e}$ on generators $E_j, F_j$ of $\widetilde{\mathbf{U}}$ for $j \neq i \in \mathbb{I}$

$$\begin{aligned} \widetilde{T}'_{i,e}(F_j) &= y_{i,j;-c_{ij},e}, \qquad \widetilde{T}'_{i,e}(E_j) = x_{i,j;-c_{ij},e}, \\ \widetilde{T}''_{i,e}(F_j) &= y'_{i,j;-c_{ij},-e}, \qquad \widetilde{T}''_{i,e}(E_j) = x'_{i,j;-c_{ij},-e}. \end{aligned} \tag{2.17}$$

cf. [Lus93, Lemma 37.2.2].

## 2.5 Satake diagrams and relative Weyl/braid groups

Given a subset $\mathbb{I}_\bullet \subset \mathbb{I}$ of finite type, denote by $W_\bullet$ the parabolic subgroup of $W$ generated by $s_i, i \in \mathbb{I}_\bullet$. Set $w_\bullet$ to be the longest element of $W_\bullet$. Let $\mathcal{R}_\bullet$ be the set of roots which lie in the span of $\alpha_i, i \in \mathbb{I}_\bullet$. Similarly, $\mathcal{R}^\vee_\bullet$ is the set of coroots which lie in the span of $\alpha_i^\vee, i \in \mathbb{I}_\bullet$. Let $\rho_\bullet$ be the half sum of positive roots in the root system $\mathcal{R}_\bullet$, and $\rho^\vee_\bullet$ be the half sum of positive coroots in $\mathcal{R}^\vee_\bullet$.

An *admissible pair* $(\mathbb{I} = \mathbb{I}_\bullet \cup \mathbb{I}_\circ, \tau)$ (cf. [BBMR, Ko14]) consists of a partition $\mathbb{I}_\bullet \cup \mathbb{I}_\circ$ of $\mathbb{I}$, and a Dynkin diagram involution $\tau$ of $\mathfrak{g}$ (where $\tau = \mathrm{Id}$ is allowed) such that

(1) $\mathbb{I}_\bullet$ is of finite type,

(2) $w_\bullet(\alpha_j) = -\alpha_{\tau j}$ for $j \in \mathbb{I}_\bullet$,

(3) If $j \in \mathbb{I}_\circ$ and $\tau j = j$, then $\alpha_j(\rho_\bullet^\vee) \in \mathbb{Z}$.

The diagrams associated to admissible pairs are known as (generalised) Satake diagrams. We shall use the terms between admissible pairs and Satake diagrams interchangeably. Throughout the dissertation, we shall always work with admissible pairs $(\mathbb{I} = \mathbb{I}_\bullet \cup \mathbb{I}_\circ, \tau)$. A symmetric pair $(\mathfrak{g}, \theta)$ (of Kac-Moody type) consists of a symmetrizable Kac-Moody algebra $\mathfrak{g}$ and an involution $\theta$ on $\mathfrak{g}$ of the second kind; involutions of the second kind on Kac-Moody algebras are classified by Satake diagrams [Ko14, Theorem 2.7].

An admissible pair is called of finite type if the underlying Dynkin diagram is of finite type. An admissible pair $(\mathbb{I} = \mathbb{I}_\bullet \cup \mathbb{I}_\circ, \tau)$ is called quasi-split if $\mathbb{I}_\bullet = \varnothing$, and split if in addition $\tau = \mathrm{Id}$.

Given an admissible pair $(\mathbb{I} = \mathbb{I}_\bullet \cup \mathbb{I}_\circ, \tau)$, the corresponding involution $\theta$ (acting on the weight lattice) is recovered as

$$\theta = -w_\bullet \circ \tau. \tag{2.18}$$

Set $\mathbb{I}_{\circ,\tau}^{\mathrm{KM}}$ to be a (fixed) set of representatives of $\tau$-orbits in $\mathbb{I}_\circ$. The (real) rank of a Satake diagram is the cardinality of $\mathbb{I}_{\circ,\tau}^{\mathrm{KM}}$. We call a Satake diagram $(\mathbb{I}^1 = \mathbb{I}_\bullet^1 \cup \mathbb{I}_\circ^1, \tau^1)$ a subdiagram of another Satake diagram $(\mathbb{I} = \mathbb{I}_\bullet \cup \mathbb{I}_\circ, \tau)$, if $\mathbb{I}^1 \subset \mathbb{I}$, $\mathbb{I}_\bullet^1 = \mathbb{I}^1 \cap \mathbb{I}_\bullet$, and $\tau^1 = \tau|_{\mathbb{I}^1}$.

Given a Satake diagram $(\mathbb{I} = \mathbb{I}_\bullet \cup \mathbb{I}_\circ, \tau)$, the rank one subdiagram associated to

$i \in \mathbb{I}_{\circ,\tau}^{\mathrm{KM}}$ consists of vertices

$$\mathbb{I}_{\bullet,i} := \mathbb{I}_{\bullet} \cup \{i, \tau i\}. \tag{2.19}$$

Let $\mathbb{I}_{\circ,\tau}$ be the subset of $\mathbb{I}_{\circ,\tau}^{\mathrm{KM}}$ given by

$$\mathbb{I}_{\circ,\tau} = \{i \in \mathbb{I}_{\circ,\tau}^{\mathrm{KM}} | \ \mathbb{I}_{\bullet,i} \text{ is of finite type}\}. \tag{2.20}$$

Then $\mathbb{I}_{\circ,\tau}$ parametrizes finite type rank one Satake subdiagrams of $(\mathbb{I} = \mathbb{I}_{\circ} \cup \mathbb{I}_{\bullet}, \tau)$. Let $W_{\bullet,i}$ be the parabolic subgroup of $W$ generated by $s_j, j \in \mathbb{I}_{\bullet,i}$ for $i \in \mathbb{I}_{\circ,\tau}$. Let $w_{\bullet,i}$ the longest element of $W_{\bullet,i}$. The following constructions are a special case of those by Lusztig [Lus76]; also cf. [Lus03, DK19]. Define $\mathbf{r}_i \in W_{\bullet,i}$ such that

$$w_{\bullet,i} = \mathbf{r}_i w_{\bullet} \,(= w_{\bullet}\mathbf{r}_i), \qquad \text{where } \ell(w_{\bullet,i}) = \ell(\mathbf{r}_i) + \ell(w_{\bullet}). \tag{2.21}$$

(It follows from the admissible pair requirement that $w_{\bullet,i}, \mathbf{r}_i$, and $w_{\bullet}$ commute with each other.) Then the subgroup of $W$,

$$W^{\circ} := \langle \mathbf{r}_i | i \in \mathbb{I}_{\circ,\tau} \rangle,$$

is a Weyl group by itself with its simple reflections identified with $\{\mathbf{r}_i \mid i \in \mathbb{I}_{\circ,\tau}\}$. Denote by $\ell_{\circ}$ the length function of the Coxeter system $(W^{\circ}, \mathbb{I}_{\circ,\tau})$ and by $\boldsymbol{w}_{\circ}$ its longest element.

**Proposition 2.9** ([Lus76]). *Let $w_1, w_2 \in W^{\circ}$. Then $\ell(w_1 w_2) = \ell(w_1) + \ell(w_2)$ if and only if $\ell_{\circ}(w_1 w_2) = \ell_{\circ}(w_1) + \ell_{\circ}(w_2)$.*

Hence there is no ambiguity to refer to the Coxeter system $W^{\circ}$ or $W$ when we talk about reduced expressions of an element $w \in W^\circ \subset W$. By definition, we have identifications $\mathbb{I}_{\bullet,i} = \mathbb{I}_{\bullet,\tau i}, W_{\bullet,i} = W_{\bullet,\tau i}, w_{\bullet,i} = w_{\bullet,\tau i}$, and $\mathbf{r}_i = \mathbf{r}_{\tau i}$. Denote by $\tau_{\bullet,i}$ the diagram involution on $\mathbb{I}_{\bullet,i}$ such that

$$w_{\bullet,i}(\alpha_j) = -\alpha_{\tau_{\bullet,i}j}, \qquad \forall j \in \mathbb{I}_{\bullet,i}. \tag{2.22}$$

The *relative Weyl group* associated to the Satake diagram $(\mathbb{I} = \mathbb{I}_\bullet \cup \mathbb{I}_\circ, \tau)$ can be identified with $W^\circ$. Let $\{\overline{\alpha}_i | i \in \mathbb{I}_{\circ,\tau}\}$ be the simple system of the relative (or restricted) root system, where $\overline{\alpha}_i$ is identified with the following element (cf. [DK19, §2.3])

$$\overline{\alpha}_i := \frac{\alpha_i - \theta(\alpha_i)}{2}, \qquad (i \in \mathbb{I}_\circ). \tag{2.23}$$

Note that $\overline{\alpha}_i = \overline{\alpha}_{\tau i}$.

We introduce a subgroup of $W$:

$$W^\theta = \{w \in W \mid w\theta = \theta w\}.$$

It is well known that (see, e.g., [DK19, §2.2])

$$W_\bullet \rtimes W^\circ \cong W^\theta.$$

We shall refer to the braid group associated to the relative Weyl group $W^\circ$ the *relative braid group* and denote it by $\mathrm{Br}(W^\circ)$. Accordingly, we denote the braid group associated to $W_\bullet$ by $\mathrm{Br}(W_\bullet)$.

## 2.6 Universal ıquantum groups

We set up some basics for the universal quantum symmetric pair $(\widetilde{\mathbf{U}}, \widetilde{\mathbf{U}}^\imath)$, following and somewhat generalizing [LW22b].

Let $(\mathbb{I} = \mathbb{I}_\bullet \cup \mathbb{I}_\circ, \tau)$ be a Satake diagram. Define $\widetilde{\mathbf{U}}_\bullet$ to be the subalgebra of $\widetilde{\mathbf{U}}$ with the set of Chevalley generators

$$\widetilde{\mathcal{G}}_\bullet := \{E_j, F_j, K_j, K_j' \mid j \in \mathbb{I}_\bullet\}.$$

The universal ıquantum group associated to the Satake diagram $(\mathbb{I} = \mathbb{I}_\bullet \cup \mathbb{I}_\circ, \tau)$ is defined to be the $\mathbb{F}$-subalgebra of $\widetilde{\mathbf{U}}$

$$\widetilde{\mathbf{U}}^\imath = \langle B_i, \widetilde{k}_i, g \mid i \in \mathbb{I}_\circ, g \in \widetilde{\mathcal{G}}_\bullet \rangle$$

via the embedding $\imath : \widetilde{\mathbf{U}}^\imath \to \widetilde{\mathbf{U}}$, $u \mapsto u^\imath$, with

$$B_i \mapsto F_i + \widetilde{T}_{w_\bullet}(E_{\tau i})K_i', \quad \widetilde{k}_i \mapsto K_i K_{\tau i}', \quad g \mapsto g, \quad \text{for } \ i \in \mathbb{I}_\circ, \ g \in \widetilde{\mathcal{G}}_\bullet. \tag{2.24}$$

By definition, $\widetilde{\mathbf{U}}^\imath$ contains the Drinfeld double $\widetilde{\mathbf{U}}_\bullet$ associated to $\mathbb{I}_\bullet$ as a subalgebra.

Let $\widetilde{\mathbf{U}}^{\imath 0}$ denote the subalgebra of $\widetilde{\mathbf{U}}^\imath$ generated by $\widetilde{k}_i, K_j, K_j'$, for $i \in \mathbb{I}_\circ, j \in \mathbb{I}_\bullet$. The following lemma is clear.

**Lemma 2.10.** *If $i = \tau i, i \in \mathbb{I}_\circ$, then $\widetilde{k}_i$ is central in $\widetilde{\mathbf{U}}^\imath$. If $\tau i \neq i \in \mathbb{I}_\circ$, then $\widetilde{k}_i \widetilde{k}_{\tau i}$ is central in $\widetilde{\mathbf{U}}^\imath$.*

Following [Let99] and [Ko14, §6.2], we formulate a monomial basis for $\widetilde{\mathbf{U}}^\imath$. Denote $B_j = F_j$, for $j \in \mathbb{I}_\bullet$. For a multi-index $J = (j_1, j_2, \cdots, j_n) \in \mathbb{I}^n$, we define $F_J := F_{j_1} F_{j_2} \cdots F_{j_n}$ and $B_J := B_{j_1} B_{j_2} \cdots B_{j_n}$. Let $\mathcal{J}$ be a fixed subset of $\bigcup_{n \geq 0} \mathbb{I}^n$ such that

$\{F_J | J \in \mathcal{J}\}$ forms a basis of $\widetilde{\mathbf{U}}$ as a $\widetilde{\mathbf{U}}^+\widetilde{\mathbf{U}}^0$-module.

**Proposition 2.11.** *(cf. [Ko14, Proposition 6.2]) The set $\{B_J | J \in \mathcal{J}\}$ is a basis of the left (or right) $\widetilde{\mathbf{U}}^+_\bullet \widetilde{\mathbf{U}}^{\imath 0}$-modules $\widetilde{\mathbf{U}}^\imath$.*

## 2.7 $\imath$Quantum group $\mathbf{U}^\imath_{\boldsymbol{\varsigma}}$ via central reduction

We recall some basics for quantum symmetric pairs $(\mathbf{U}, \mathbf{U}^\imath_{\boldsymbol{\varsigma}})$, cf. [Let99, Ko14], where the parameter $\boldsymbol{\varsigma} = (\varsigma_i)_{i \in \mathbb{I}_\circ} \in \mathbb{F}^{\times, \mathbb{I}_\circ}$ is always assume to satisfy the following conditions (cf. [Let99] [Ko14, Section 5.1])

$$\varsigma_i = \varsigma_{\tau i}, \qquad \text{if } \tau i \neq i \text{ and } (\alpha_i, w_\bullet \alpha_{\tau i}) = 0. \tag{2.25}$$

We call $\boldsymbol{\varsigma}$ a *balanced parameter*, if $\varsigma_i = \varsigma_{\tau i}$ for any $i \in \mathbb{I}_\circ$. For an arbitrary parameter $\boldsymbol{\varsigma}$, we define an associated balanced parameter $\boldsymbol{\varsigma}^e$ such that

$$\varsigma^e_i = \varsigma^e_{\tau i} = \sqrt{\varsigma_i \varsigma_{\tau i}}. \tag{2.26}$$

Define $\mathbf{U}_\bullet$ to be the subalgebra of $\mathbf{U}$ with the set of Chevalley generators

$$\mathcal{G}_\bullet := \{E_j, F_j, K_j^{\pm 1} \mid j \in \mathbb{I}_\bullet\}.$$

The $\imath$quantum group associated to the Satake diagram $(\mathbb{I} = \mathbb{I}_\bullet \cup \mathbb{I}_\circ, \tau)$ with parameter $\boldsymbol{\varsigma}$ is defined to be the $\mathbb{F}$-subalgebra of $\mathbf{U}$

$$\mathbf{U}^\imath_{\boldsymbol{\varsigma}} = \langle B_i, k_j, g \mid i \in \mathbb{I}_\circ, j \in \mathbb{I} \setminus \mathbb{I}_{\circ,\tau}, g \in \mathcal{G}_\bullet \rangle$$

via the embedding $\imath : \mathbf{U}^\imath_{\boldsymbol{\varsigma}} \rightarrow \mathbf{U}$ with

$$B_i \mapsto F_i + \varsigma_i T_{w_\bullet}(E_{\tau i}) K_i^{-1}, \qquad k_j \mapsto K_j K_{\tau j}^{-1}, \qquad \forall i \in \mathbb{I}_\circ, j \in \mathbb{I} \setminus \mathbb{I}_{\circ,\tau}. \tag{2.27}$$

We sometimes write $u^\imath \in \mathbf{U}$, for $u \in \mathbf{U}^\imath$. Note that $\mathbf{U}^\imath$ contains $\mathbf{U}_\bullet$ as a subalgebra. For $i \in \mathbb{I}_{\circ,\tau}$, we set $k_i = 1$ if $i = \tau i$ and $k_i = k_{\tau i}^{-1}$ if $i \neq \tau i$. Similarly, we denote by $\mathbf{U}^{\imath 0}$ the subalgebra of $\mathbf{U}^\imath$ generated by $k_i, K_j$, for $i \in \mathbb{I}_\circ, j \in \mathbb{I}_\bullet$.

We have the following central reduction $\pi^\imath_{\boldsymbol{\varsigma}} : \widetilde{\mathbf{U}}^\imath \rightarrow \mathbf{U}^\imath_{\boldsymbol{\varsigma}}$, generalizing [LW22b, Proposition 6.2] in the quasi-split setting.

**Proposition 2.12.** *There exists a quotient morphism* $\pi^\imath_{\boldsymbol{\varsigma}} : \widetilde{\mathbf{U}}^\imath \rightarrow \mathbf{U}^\imath_{\boldsymbol{\varsigma}}$ *sending*

$$B_i \mapsto B_i, \quad \widetilde{k}_j \mapsto \varsigma_{\tau j} k_j, \quad \widetilde{k}_{\tau j} \mapsto \varsigma_j k_{\tau j}, \qquad (i \in \mathbb{I}_\circ, j \in \mathbb{I}_{\circ,\tau}),$$

*and* $\pi^\imath_{\boldsymbol{\varsigma}}|_{\widetilde{\mathbf{U}}_\bullet} = \pi_{\mathbf{1}}|_{\widetilde{\mathbf{U}}_\bullet}$. *The kernel of* $\pi^\imath_{\boldsymbol{\varsigma}}$ *is generated by*

$$\widetilde{k}_i - \varsigma_i \quad (i = \tau i, i \in \mathbb{I}_\circ), \qquad \widetilde{k}_i \widetilde{k}_{\tau i} - \varsigma_i \varsigma_{\tau i} \quad (i \neq \tau i, i \in \mathbb{I}_\circ), \quad K_j K_j' - 1 \quad (j \in \mathbb{I}_\bullet).$$

*Remark* 2.13. For a balanced parameter $\boldsymbol{\varsigma}$, $\pi^\imath_{\boldsymbol{\varsigma}}$ coincides with the restriction of $\pi_{\boldsymbol{\varsigma}}$ on $\widetilde{\mathbf{U}}^\imath$. However, this is not the case for an unbalanced parameter.

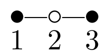
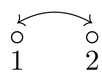
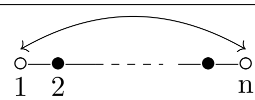
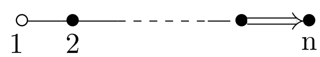
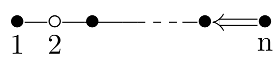
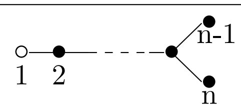
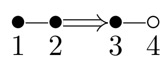
## 2.8 Distinguished parameter $\boldsymbol{\varsigma}_\diamond$ and Table 1

Recall from (2.23) that $\overline{\alpha}_i = (\alpha_i + w_\bullet \alpha_{\tau i})/2$. Define a distinguished balanced parameter $\boldsymbol{\varsigma}_\diamond = (\varsigma_{i,\diamond})_{i \in \mathbb{I}_\circ}$ such that

$$\varsigma_{i,\diamond} = -q^{-(\alpha_i, \alpha_i + w_\bullet \alpha_{\tau i})/2} = -q^{-(\overline{\alpha}_i, \overline{\alpha}_i)}, \quad \text{for } i \in \mathbb{I}_\circ. \tag{2.28}$$

The parameter $\boldsymbol{\varsigma}_\diamond$ will play a basic role in this dissertation; also cf. [DK19]. In fact, only $\varsigma_{i,\diamond}$ for $i \in fwItau$ matter in our later construction. We summarize the parameter $\varsigma_{i,\diamond}$ for finite-type rank one Satake diagrams in the table below.

Table 1: Rank one Satake diagrams of finite types and local datum

| Type | Satake diagram | $\varsigma_{i,\diamond}$ | $\mathbf{r}_i$ |
|---|---|---|---|
| $\mathrm{AI}_1$ |  | $\varsigma_{1,\diamond} = -q^{-2}$ | $\mathbf{r}_1 = s_1$ |
| $\mathrm{AII}_3$ |  | $\varsigma_{2,\diamond} = -q^{-1}$ | $\mathbf{r}_2 = s_{2132}$ |
| $\mathrm{AIII}_{11}$ |  | $\varsigma_{1,\diamond} = -q^{-1}$ | $\mathbf{r}_1 = s_1 s_2$ |
| AIV, n≥2 |  | $\varsigma_{1,\diamond} = -q^{-1/2}$ | $\mathbf{r}_1 = s_{1\cdots n \cdots 1}$ |
| BII, n≥ 2 |  | $\varsigma_{1,\diamond} = -q_1^{-1}$ | $\mathbf{r}_1 = s_{1\cdots n \cdots 1}$ |
| CII, n≥3 |  | $\varsigma_{2,\diamond} = -q_2^{-1/2}$ | $\mathbf{r}_2 = s_{2\cdots n \cdots 212 \cdots n \cdots 2}$ |
| DII, n≥4 |  | $\varsigma_{1,\diamond} = -q^{-1}$ | $\mathbf{r}_1 =$ $s_{1\cdots n-2 \cdot n-1 \cdot n \cdot n-2 \cdots 1}$ |
| FII |  | $\varsigma_{4,\diamond} = -q_4^{-1/2}$ | $\mathbf{r}_4 = s_{432312343231234}$ |

Watanabe [W21a, Lemma 2.5.1] showed that the ıquantum groups for *arbitrarily* different parameters are isomorphic, improving earlier partial results in [Let02] and [Ko14, Proposition 9.2, Theorem 9.7]; Universal ıquantum groups provide an alternative proof for this fact.

**Proposition 2.14** ([W21a, Lemma 2.5.1]). *For any parameter $\boldsymbol{\varsigma}$, there exists an algebra isomorphism $\phi_{\boldsymbol{\varsigma}} : \mathbf{U}^\imath_{\boldsymbol{\varsigma}_\diamond} \to \mathbf{U}^\imath_{\boldsymbol{\varsigma}}$.*

*Proof.* Denote by $\widetilde{\mathbf{a}} = (\widetilde{a}_i)_{i \in \mathbb{I}}$ the scalars given by $\widetilde{a}_i := \varsigma_{i,\diamond}^{-1} \varsigma_i$, $\widetilde{a}_j := 1$, for $i \in \mathbb{I}_\circ, j \in \mathbb{I}_\bullet$.

The corresponding scaling map $\Phi_{\widetilde{a}}$ in Proposition 2.2 acts on $\imath(\mathbf{U}^\imath_{\boldsymbol{\varsigma}_\diamond})$ by sending

$$B_i \mapsto (\varsigma_{i,\diamond}\varsigma_i^{-1})^{1/2}\big(F_i + \sqrt{\varsigma_i\varsigma_{\tau i}}\widetilde{T}_{w_\bullet}(E_{\tau i})K_i^{-1}\big), \qquad k_i \mapsto k_i. \tag{2.29}$$

Hence, $\Phi_{\widetilde{a}}$ restricts to an isomorphism $\mathbf{U}^\imath_{\boldsymbol{\varsigma}_\diamond} \to \mathbf{U}^\imath_{\boldsymbol{\varsigma}^e}$; see (2.26) for $\boldsymbol{\varsigma}^e$.

It remains to show that $\mathbf{U}^\imath_{\boldsymbol{\varsigma}^e}$ is isomorphic to $\mathbf{U}^\imath_{\boldsymbol{\varsigma}}$. Note that, by Proposition 2.12, $\ker \pi^\imath_{\boldsymbol{\varsigma}} = \ker \pi^\imath_{\boldsymbol{\varsigma}^e}$. Hence, we have $\mathbf{U}^\imath_{\boldsymbol{\varsigma}} \cong \widetilde{\mathbf{U}}^\imath / \ker \pi^\imath_{\boldsymbol{\varsigma}} = \widetilde{\mathbf{U}}^\imath / \ker \pi^\imath_{\boldsymbol{\varsigma}^e} \cong \mathbf{U}^\imath_{\boldsymbol{\varsigma}^e}$. □

# Part I

# Relative braid group symmetries for finite type

The goal of Part I is to construct symmetries $\widetilde{\mathbf{T}}'_{i,e}, \widetilde{\mathbf{T}}''_{i,e}$ on $\imath$quantum groups of arbitrary finite type. We will show that these symmetries satisfy the relative braid relations in Part III.

Let $(\mathbb{I} = \mathbb{I}_\circ \cup \mathbb{I}_\bullet, \tau)$ be a Satake diagram of finite type.

# 3 Quasi $K$-matrix and intertwining properties

In this section, we establish the quasi $K$-matrix $\widetilde{\Upsilon}$ for the universal quantum symmetric pair $(\widetilde{\mathbf{U}}, \widetilde{\mathbf{U}}^\imath)$, and a new characterization of $\widetilde{\Upsilon}$ in terms of an anti-involution $\sigma$. Then using suitable intertwining properties with the quasi $K$-matrix, we establish an anti-involution $\sigma^\imath$ and a bar involution $\psi^\imath$ on $\widetilde{\mathbf{U}}^\imath$ from the anti-involution $\sigma$ and a rescaled bar involution $\psi_\star$ on $\widetilde{\mathbf{U}}$. We also establish an anti-involution $\sigma_\tau$ on $\mathbf{U}^\imath_{\boldsymbol{\varsigma}}$ for an arbitrary parameter $\boldsymbol{\varsigma}$.

## 3.1 Quasi $K$-matrix

The quasi $K$-matrix was introduced in [BW18a, §2.3] as the intertwiner between the embedding $\imath : \mathbf{U}^\imath_{\boldsymbol{\varsigma}} \to \mathbf{U}$ and its bar-conjugated embedding (where some constraints on $\boldsymbol{\varsigma}$ are imposed); this was expected to be valid for general quantum symmetric pairs early on. A proof for the existence of the quasi $K$-matrix was given in [BK19] in greater generality (modulo a technical assumption, which was later removed in [BW21]). Appel-Vlaar [AV22, Theorem 7.4] reformulated the definition of quasi $K$-matrix $\Upsilon_{\boldsymbol{\varsigma}}$ associated to $(\mathbf{U}, \mathbf{U}^\imath_{\boldsymbol{\varsigma}})$ without reference to the bar involution on $\mathbf{U}^\imath_{\boldsymbol{\varsigma}}$, and this reformulation removes constraints on the parameter $\boldsymbol{\varsigma}$ for quasi $K$-matrix. Recall the bar involution $\psi$ on $\mathbf{U}$.

**Theorem 3.1** (cf. [AV22])**.** *There exists a unique element* $\Upsilon_\varsigma = \sum_{\mu\in\mathbb{N}\mathbb{I}} \Upsilon_\varsigma^\mu$*, for* $\Upsilon_\varsigma^\mu \in \mathbf{U}_\mu^+$*, such that* $\Upsilon_\varsigma^0 = 1$ *and the following identities hold:*

$$B_i\Upsilon_\varsigma = \Upsilon_\varsigma\Big(F_i + (-1)^{\alpha_i(2\rho_\bullet^\vee)}q^{(\alpha_i, w_\bullet(\alpha_{\tau i})+2\rho_\bullet)}\varsigma_{\tau i}\psi\big(T_{w_\bullet}E_{\tau i}\big)K_i\Big), \tag{3.1}$$

$$x\Upsilon_\varsigma = \Upsilon_\varsigma x, \tag{3.2}$$

*for* $i \in \mathbb{I}_\circ$ *and* $x \in \mathbf{U}^{\imath 0}\mathbf{U}_\bullet$*. Moreover,* $\widetilde{\Upsilon}^\mu = 0$ *unless* $\theta(\mu) = -\mu$*.*

Recall the bar involution $\psi$ on $\widetilde{\mathbf{U}}$ from Proposition 2.3. The quasi $K$-matrix $\widetilde{\Upsilon}$ associated to $(\widetilde{\mathbf{U}}, \widetilde{\mathbf{U}}^\imath)$ is defined in a similar way as Theorem 3.6.

**Theorem 3.2.** *There exists a unique element* $\widetilde{\Upsilon} = \sum_{\mu\in\mathbb{N}\mathbb{I}} \widetilde{\Upsilon}^\mu$ *such that* $\widetilde{\Upsilon}^0 = 1, \widetilde{\Upsilon}^\mu \in \widetilde{\mathbf{U}}_\mu^+$ *and the following identities hold:*

$$B_i\widetilde{\Upsilon} = \widetilde{\Upsilon}\Big(F_i + (-1)^{\alpha_i(2\rho_\bullet^\vee)}q^{(\alpha_i, w_\bullet\alpha_{\tau i}+2\rho_\bullet)}\psi\big(\widetilde{T}_{w_\bullet}E_{\tau i}\big)K_i\Big), \tag{3.3}$$

$$x\widetilde{\Upsilon} = \widetilde{\Upsilon}x, \tag{3.4}$$

*for* $i \in \mathbb{I}_\circ$ *and* $x \in \widetilde{\mathbf{U}}^{\imath 0}\widetilde{\mathbf{U}}_\bullet$*. Moreover,* $\widetilde{\Upsilon}^\mu = 0$ *unless* $\theta(\mu) = -\mu$*.*

*Proof.* Follows by a rerun of the proof of Theorem 3.1 as in [AV22] or in [Ko21]. (The strategy of the proof does not differ substantially from the one given in [BW18a].) □

*Remark* 3.3. Applying the central reduction $\pi_\varsigma$ in (2.7) to (3.3) gives us

$$\begin{aligned}\big(F_i+&\sqrt{\varsigma_i\varsigma_{\tau i}}T_{w_\bullet}(E_{\tau i})K_i^{-1}\big)\pi_\varsigma(\widetilde{\Upsilon}) \\ &= \pi_\varsigma(\widetilde{\Upsilon})\big(F_i + (-1)^{\alpha_i(2\rho_\bullet^\vee)}q^{(\alpha_i, w_\bullet(\alpha_{\tau i})+2\rho_\bullet)}\sqrt{\varsigma_i\varsigma_{\tau i}}\psi\big(T_{w_\bullet}(E_{\tau i})\big)K_i\big),\end{aligned} \tag{3.5}$$

$$x\pi_\varsigma(\widetilde{\Upsilon}) = \pi_\varsigma(\widetilde{\Upsilon})x, \tag{3.6}$$

for $i \in \mathbb{I}_\circ, x \in \mathbf{U}^{\imath 0}\mathbf{U}_\bullet$. Comparing (3.5) with (3.1), we obtain by the uniqueness of the quasi $K$-matrix that (see (2.26) for $\boldsymbol{\varsigma}^e$)

$$\pi_{\boldsymbol{\varsigma}}(\widetilde{\Upsilon}) = \Upsilon_{\boldsymbol{\varsigma}^e}. \tag{3.7}$$

In particular, $\pi_{\boldsymbol{\varsigma}}(\widetilde{\Upsilon}) = \Upsilon_{\boldsymbol{\varsigma}}$ if and only if $\boldsymbol{\varsigma}$ is a balanced parameter.

## 3.2 A bar involution $\psi^\imath$ on $\widetilde{\mathbf{U}}^\imath$

Introduce a balanced parameter $\boldsymbol{\varsigma}_\star = (\varsigma_{i,\star})_{i \in \mathbb{I}_\circ}$ by letting

$$\varsigma_{i,\star} = (-1)^{\alpha_i(2\rho_\bullet^\vee)} q^{(\alpha_i, w_\bullet \alpha_{\tau i} + 2\rho_\bullet)}, \qquad (i \in \mathbb{I}_\circ). \tag{3.8}$$

Note that $\varsigma_{i,\star}$ are exactly the scalars appearing on the RHS (3.3). We extend $\boldsymbol{\varsigma}_\star$ trivially to an $\mathbb{I}$-tuple, again denoted by $\boldsymbol{\varsigma}_\star$ by abuse of notation, by setting

$$\varsigma_{j,\star} = 1 \qquad (j \in \mathbb{I}_\bullet).$$

Recall the scaling automorphism $\widetilde{\Psi}_{\boldsymbol{\varsigma}_\star}$ from (2.8) and the bar involution $\psi$ on $\widetilde{\mathbf{U}}$ from Proposition 2.3. The composition

$$\psi_\star := \widetilde{\Psi}_{\boldsymbol{\varsigma}_\star} \circ \psi \tag{3.9}$$

is an anti-linear involutive automorphism of $\widetilde{\mathbf{U}}$.

**Proposition 3.4.** *There exists a unique anti-linear involution $\psi^\imath$ of $\widetilde{\mathbf{U}}^\imath$ such that*

$$\psi^\imath(B_i) = B_i, \qquad \psi^\imath(x) = \psi_\star(x), \qquad \textit{for } i \in \mathbb{I}_\circ, x \in \widetilde{\mathbf{U}}^{\imath 0}\widetilde{\mathbf{U}}_\bullet. \tag{3.10}$$

*Moreover, $\psi^\imath$ satisfies the following intertwining relation,*

$$\psi^\imath(x)\widetilde{\Upsilon} = \widetilde{\Upsilon}\psi_\star(x), \qquad \textit{for all } x \in \widetilde{\mathbf{U}}^\imath. \tag{3.11}$$

*($\psi^\imath$ is called a bar involution on $\widetilde{\mathbf{U}}^\imath$.)*

*Proof.* We follow the same strategy in [Ko21] who established a bar involution on $\mathbf{U}^\imath_{\boldsymbol{\varsigma}}$ (for suitable $\boldsymbol{\varsigma}$) without using a Serre presentation.

By definition of $\psi_\star$, we have, for $i \in \mathbb{I}_\circ, x \in \widetilde{\mathbf{U}}^{\imath 0}\widetilde{\mathbf{U}}_\bullet$,

$$\begin{aligned} &\psi_\star(B_i) = F_i + (-1)^{\alpha_i(2\rho_\bullet^\vee)} q^{(\alpha_i, w_\bullet \alpha_{\tau i} + 2\rho_\bullet)} \psi\big(\widetilde{T}_{w_\bullet} E_{\tau i}\big) K_i, \\ &\psi_\star(x) \in \widetilde{\mathbf{U}}^{\imath 0}\widetilde{\mathbf{U}}_\bullet. \end{aligned} \tag{3.12}$$

The composition $\mathrm{Ad}_{\widetilde{\Upsilon}} \circ \psi_\star$ is an anti-linear homomorphism from $\widetilde{\mathbf{U}}$ to a completion of $\widetilde{\mathbf{U}}$. Then the image of $\widetilde{\mathbf{U}}^\imath$ under $\mathrm{Ad}_{\widetilde{\Upsilon}} \circ \psi_\star$ is a subalgebra generated by

$$(\mathrm{Ad}_{\widetilde{\Upsilon}} \circ \psi_\star)(B_i), \quad (\mathrm{Ad}_{\widetilde{\Upsilon}} \circ \psi_\star)(x), \qquad \text{for } i \in \mathbb{I}_\circ, x \in \widetilde{\mathbf{U}}^{\imath 0}\widetilde{\mathbf{U}}_\bullet.$$

By Theorem 3.2 and the identities (3.12), we have, for $i \in \mathbb{I}_\circ, x \in \widetilde{\mathbf{U}}^{\imath 0}\widetilde{\mathbf{U}}_\bullet$,

$$(\mathrm{Ad}_{\widetilde{\Upsilon}} \circ \psi_\star)(B_i) = B_i, \quad (\mathrm{Ad}_{\widetilde{\Upsilon}} \circ \psi_\star)(x) = \psi_\star(x). \tag{3.13}$$

Since each element in (3.13) lies in $\widetilde{\mathbf{U}}^\imath$, $\mathrm{Ad}_{\widetilde{\Upsilon}} \circ \psi_\star$ restricts to an anti-linear endomorphism on $\widetilde{\mathbf{U}}^\imath$, which we shall denote by $\psi^\imath : \widetilde{\mathbf{U}}^\imath \to \widetilde{\mathbf{U}}^\imath$.

By construction, $\psi^\imath$ satisfies (3.10)–(3.11). Finally, $\psi^\imath$ is unique and is an involutive automorphism of $\widetilde{\mathbf{U}}^\imath$ since it satisfies (3.10). □

**Proposition 3.5.** *We have*

$$\psi_\star(\widetilde{\Upsilon})\widetilde{\Upsilon} = 1. \tag{3.14}$$

*Proof.* Applying $\psi_\star$ to (3.11) results the identity $\psi_\star(y)\psi_\star(\widetilde{\Upsilon}) = \psi_\star(\widetilde{\Upsilon})\psi^\imath(y)$, for $y \in \widetilde{\mathbf{U}}^\imath$. We rewrite this identity as

$$\psi^\imath(y)\psi_\star(\widetilde{\Upsilon})^{-1} = \psi_\star(\widetilde{\Upsilon})^{-1}\psi_\star(y). \tag{3.15}$$

Using (3.12) and Proposition 3.4, the above identity (3.15) implies following relations

$$\begin{aligned} B_i\psi_\star(\widetilde{\Upsilon})^{-1} &= \psi_\star(\widetilde{\Upsilon})^{-1}\Big(F_i + (-1)^{\alpha_i(2\rho_\bullet^\vee)}q^{(\alpha_i, w_\bullet\alpha_{\tau i}+2\rho_\bullet)}\psi\big(\widetilde{T}_{w_\bullet}E_{\tau i}\big)K_i\Big), \\ x\psi_\star(\widetilde{\Upsilon})^{-1} &= \psi_\star(\widetilde{\Upsilon})^{-1}x, \end{aligned} \tag{3.16}$$

for $i \in \mathbb{I}_\circ, x \in \widetilde{\mathbf{U}}^{\imath 0}\widetilde{\mathbf{U}}_\bullet$. Hence, $\psi_\star(\widetilde{\Upsilon})^{-1}$ satisfies (3.3)–(3.4) as well. Clearly, $\psi_\star(\widetilde{\Upsilon})^{-1}$ has constant term 1. Thanks to the uniqueness of $\widetilde{\Upsilon}$ in Theorem 3.2, we have $\psi_\star(\widetilde{\Upsilon})^{-1} = \widetilde{\Upsilon}$. □

## 3.3 Quasi $K$-matrix and anti-involution $\sigma$

We provide a new characterization for $\widetilde{\Upsilon}$ in terms of the anti-involution $\sigma$ (see Proposition 2.3), which turns out to be much cleaner than Theorem 3.2. Denote

$$B_i^\sigma = \sigma(B_i) = F_i + K_i\widetilde{T}_{w_\bullet}^{-1}(E_{\tau i}), \tag{3.17}$$

where the second identity above follows by noting $\widetilde{T}_{w_\bullet}^{-1} = \sigma\widetilde{T}_{w_\bullet}\sigma$; see (2.14). The following characterization of a quasi $K$-matrix $\widetilde{\Upsilon}$ is valid for $\widetilde{\mathbf{U}}^\imath$ of arbitrary Kac-

Moody type.

**Theorem 3.6.** *The quasi $K$-matrix $\widetilde{\Upsilon}$ is uniquely characterized by $\widetilde{\Upsilon}^0 = 1$ and the following intertwining relations*

$$\begin{aligned} B_i\widetilde{\Upsilon} &= \widetilde{\Upsilon}B_i^{\sigma}, \qquad (i \in \mathbb{I}_\circ),\\ x\widetilde{\Upsilon} &= \widetilde{\Upsilon}x, \qquad (x \in \widetilde{\mathbf{U}}^{\imath 0}\widetilde{\mathbf{U}}_\bullet). \end{aligned} \tag{3.18}$$

*Moreover, $\widetilde{\Upsilon}^\mu = 0$ unless $\theta(\mu) = -\mu$.*

*Proof.* We show that the identity (3.18) is equivalent to (3.3), for any fixed $i \in \mathbb{I}_\circ$. Since $\psi\big(\widetilde{T}_{w_\bullet}(E_{\tau i})\big)$ has weight $w_\bullet\alpha_{\tau i}$, the identity (3.3) is equivalent to

$$B_i\widetilde{\Upsilon} = \widetilde{\Upsilon}\Big(F_i + (-1)^{\alpha_i(2\rho_\bullet^\vee)}q^{(\alpha_i,2\rho_\bullet)}K_i\psi\big(\widetilde{T}_{w_\bullet}(E_{\tau i})\big)\Big). \tag{3.19}$$

Moreover, by [BW18b, Lemma 4.17] and $\widetilde{\mathbf{U}}^+ = \mathbf{U}^+$, we have

$$(-1)^{\alpha_i(2\rho_\bullet^\vee)}q^{(\alpha_i,2\rho_\bullet^\vee)}\psi\big(\widetilde{T}_{w_\bullet}(E_{\tau i})\big) = \widetilde{T}_{w_\bullet}^{-1}(E_{\tau i}),$$

and hence, the identity (3.19) is equivalent to (3.18) as desired. □

*Remark* 3.7. By abuse of notation, we denote again by $\sigma$ the anti-involution on $\mathbf{U}$ which fixes $E_i, F_i$ and sends $K_i \mapsto K_i^{-1}$ for $i \in \mathbb{I}$. For a balanced parameter $\boldsymbol{\varsigma}$, we obtain the intertwining relation for $\mathbf{U}^\imath_{\boldsymbol{\varsigma}}$, $B_i\Upsilon_{\boldsymbol{\varsigma}} = \Upsilon_{\boldsymbol{\varsigma}}B_i^\sigma$ ($i \in \mathbb{I}_\circ$), by applying the central reduction $\pi_{\boldsymbol{\varsigma}}$ to (3.18), thanks to (3.7). Here $B_i^\sigma = \sigma(B_i) = F_i + \varsigma_i K_i T_{w_\bullet}^{-1}(E_{\tau i})$.

On the other hand, for (not necessarily balanced) parameter $\boldsymbol{\varsigma}$, we have

$$B_i\Upsilon_{\boldsymbol{\varsigma}} = \Upsilon_{\boldsymbol{\varsigma}}B_{\tau i}^{\sigma\tau}. \tag{3.20}$$

Let $i \in \mathbb{I}_\circ$. The rank one quasi $K$-matrix

$$\widetilde{\Upsilon}_i \in \widetilde{\mathbf{U}}^+_{\mathbb{I}_{\bullet,i}} (\subset \widetilde{\mathbf{U}}^+)$$

is defined to be the quasi $K$-matrix associated to the rank one Satake subdiagram $(\mathbb{I}_\bullet \cup \{i, \tau i\}, \tau)$; cf. (2.19). Clearly, we have $\widetilde{\Upsilon}_i = \widetilde{\Upsilon}_{\tau i}$.

**Proposition 3.8.** *We have $\sigma(\widetilde{\Upsilon}) = \widetilde{\Upsilon}$ and $\widehat{\tau}(\widetilde{\Upsilon}) = \widetilde{\Upsilon}$, In addition, for $i \in \mathbb{I}_\circ$, we have*

$$\sigma(\widetilde{\Upsilon}_i) = \widetilde{\Upsilon}_i, \qquad \widehat{\tau}(\widetilde{\Upsilon}_i) = \widetilde{\Upsilon}_i.$$

*In addition, $\widehat{\tau}_{\bullet,i}(\widetilde{\Upsilon}_i) = \widetilde{\Upsilon}_i$.*

*Proof.* By applying the anti-involution $\sigma$ to the identities in Theorem 3.6, we have

$$\sigma(\widetilde{\Upsilon}) B_i^\sigma = B_i \sigma(\widetilde{\Upsilon}), \qquad (i \in \mathbb{I}_\circ), \tag{3.21}$$

$$\sigma(\widetilde{\Upsilon}) y = y \sigma(\widetilde{\Upsilon}), \qquad (x \in \widetilde{\mathbf{U}}^{\imath 0} \widetilde{\mathbf{U}}_\bullet), \tag{3.22}$$

where $y = \sigma(x) \in \widetilde{\mathbf{U}}^{\imath 0} \widetilde{\mathbf{U}}_\bullet$. This means that $\sigma(\widetilde{\Upsilon})$ satisfies the same characterization in Theorem 3.6 as $\widetilde{\Upsilon}$, and hence by uniqueness, we have $\sigma(\widetilde{\Upsilon}) = \widetilde{\Upsilon}$.

Noting that $\sigma \widehat{\tau} = \widehat{\tau} \sigma$ and $\widehat{\tau}$ preserves $\widetilde{\mathbf{U}}^{\imath 0} \widetilde{\mathbf{U}}_\bullet$, then the identity $\widehat{\tau}(\widetilde{\Upsilon}) = \widetilde{\Upsilon}$ follows by the same type argument as above.

The identities $\sigma(\widetilde{\Upsilon}_i) = \widetilde{\Upsilon}_i$ and $\widehat{\tau}(\widetilde{\Upsilon}_i) = \widetilde{\Upsilon}_i$ are immediate by restricting $\sigma$ and $\widehat{\tau}$ to the Drinfeld double associated to rank 1 Satake subdiagram $(\mathbb{I}_{\bullet,i}, \mathbb{I}_\bullet, \tau|_{\mathbb{I}_{\bullet,i}})$.

According to the rank one Table 1, $\tau_{\bullet,i} = 1$ except in type AIV when $\tau_{\bullet,i}$ coincides with the restriction of $\tau$ to the rank 1 Satake diagram. In either case, we have

$\widehat{\tau}_{\bullet,i}(\widetilde{\Upsilon}_i) = \widetilde{\Upsilon}_i$. $\square$

*Remark* 3.9. For balanced parameters $\boldsymbol{\varsigma}$, by taking a central reduction $\pi_{\boldsymbol{\varsigma}}$, the property $\tau(\Upsilon_{i,\boldsymbol{\varsigma}}) = \Upsilon_{i,\boldsymbol{\varsigma}}$ remains valid. However, for unbalanced parameters $\boldsymbol{\varsigma}$, we do not necessarily have $\tau(\Upsilon_{i,\boldsymbol{\varsigma}}) = \Upsilon_{i,\boldsymbol{\varsigma}}$; instead, we have $\tau(\Upsilon_{i,\boldsymbol{\varsigma}}) = \Upsilon_{i,\tau\boldsymbol{\varsigma}}$, which can be proved by Theorem 3.1. The property $\Upsilon_{i,\boldsymbol{\varsigma}} = \Upsilon_{\tau i,\boldsymbol{\varsigma}}$ is true, regardless of balanced or unbalanced parameters.

*Remark* 3.10. It follows by Theorem 3.2 that the rank one quasi $K$-matrix $\widetilde{\Upsilon}_i$ has the form $\widetilde{\Upsilon}_i = \sum_{m\geq 0} \widetilde{\Upsilon}_{i,m}$, for $\widetilde{\Upsilon}_{i,m} \in \widetilde{\mathbf{U}}_{m(\alpha_i + w_\bullet \alpha_{\tau i})}$.

## 3.4 An anti-involution $\sigma^\imath$ on $\widetilde{\mathbf{U}}^\imath$

Define $\mathcal{K}_i \in \widetilde{\mathbf{U}}^\imath$ by

$$\mathcal{K}_i = K_i K'_{w_\bullet \alpha_{\tau i}}, \qquad \text{for } i \in \mathbb{I}_\circ. \tag{3.23}$$

**Lemma 3.11.** *Let $i \in \mathbb{I}_\circ$. We have $\mathcal{K}_i \in \widetilde{\mathbf{U}}^{\imath 0}$.*

*Proof.* By definition, the element $\mathcal{K}_i$ is a product of $\widetilde{k}_i = K_i K'_{\tau i} \in \widetilde{\mathbf{U}}^{\imath 0}$ and an element in $\widetilde{\mathbf{U}}^0_\bullet$, and hence $\mathcal{K}_i \in \widetilde{\mathbf{U}}^{\imath 0}$. $\square$

Recall the anti-involution $\sigma$ on $\widetilde{\mathbf{U}}$ from Proposition 2.3.

**Proposition 3.12.** *There exists a unique anti-involution $\sigma^\imath$ of $\widetilde{\mathbf{U}}^\imath$ such that*

$$\sigma^\imath(B_i) = B_i, \qquad \sigma^\imath(x) = \sigma(x), \qquad \textit{for } i \in \mathbb{I}_\circ, x \in \widetilde{\mathbf{U}}^{\imath 0}\widetilde{\mathbf{U}}_\bullet. \tag{3.24}$$

*Moreover, $\sigma^\imath$ satisfies the following intertwining relation:*

$$\sigma^\imath(x)\widetilde{\Upsilon} = \widetilde{\Upsilon}\sigma(x), \qquad \textit{for all } x \in \widetilde{\mathbf{U}}^\imath. \tag{3.25}$$

*Proof.* Given $x \in \widetilde{\mathbf{U}}^\imath$, an element $\widehat{x} \in \widetilde{\mathbf{U}}^\imath$ (if it exists) such that $\widehat{x}\widetilde{\Upsilon} = \widetilde{\Upsilon}\sigma(x)$ must be unique due to the invertibility of $\widetilde{\Upsilon}$.

**Claim** $(*)$**.** Suppose that there exist $\widehat{x}, \widehat{y} \in \widetilde{\mathbf{U}}^\imath$ that $\widehat{x}\widetilde{\Upsilon} = \widetilde{\Upsilon}\sigma(x)$ and $\widehat{y}\widetilde{\Upsilon} = \widetilde{\Upsilon}\sigma(y)$, for given $x, y \in \widetilde{\mathbf{U}}^\imath$. Then we have

$$\widehat{y}\widehat{x}\widetilde{\Upsilon} = \widetilde{\Upsilon}\sigma(xy).$$

Indeed, the Claim holds since $\widehat{y}\widehat{x}\widetilde{\Upsilon} = \widehat{y}\widetilde{\Upsilon}\sigma(x) = \widetilde{\Upsilon}\sigma(y)\sigma(x) = \widetilde{\Upsilon}\sigma(xy)$.

Observe that $\sigma$ preserves the subalgebra $\widetilde{\mathbf{U}}^{\imath 0}\widetilde{\mathbf{U}}_\bullet$ of $\widetilde{\mathbf{U}}^\imath$. Hence by Theorem 3.6, we have $\sigma(x)\widetilde{\Upsilon} = \widetilde{\Upsilon}\sigma(x)$, for all $x \in \widetilde{\mathbf{U}}^{\imath 0}\widetilde{\mathbf{U}}_\bullet$. By Theorem 3.6 again, we have $B_i\widetilde{\Upsilon} = \widetilde{\Upsilon}\sigma(B_i)$, for all $i \in \mathbb{I}_\circ$. Since the assumption for Claim $(*)$ holds for a generating set $\widetilde{\mathbf{U}}^{\imath 0}\widetilde{\mathbf{U}}_\bullet \cup \{B_i | i \in \mathbb{I}_\circ\}$ of $\widetilde{\mathbf{U}}^\imath$, we conclude by Claim $(*)$ that there exists a (unique) $\widehat{x} \in \widetilde{\mathbf{U}}^\imath$ such that $\widehat{x}\widetilde{\Upsilon} = \widetilde{\Upsilon}\sigma(x)$, for any $x \in \widetilde{\mathbf{U}}^\imath$, and moreover, sending $x \mapsto \widehat{x}$ defines an anti-endmorphism of $\widetilde{\mathbf{U}}^\imath$ (which will be denoted by $\sigma^\imath$).

Clearly, by construction $\sigma^\imath$ satisfies (3.24) and the identity (3.25). Finally, $\sigma^\imath$ is an involutive anti-automorphism of $\widetilde{\mathbf{U}}^\imath$ since it satisfies (3.24). □

*Remark* 3.13. The strategy in establishing a bar involution on $\mathbf{U}^\imath_\varsigma$ without use of a Serre presentations appeared first in [Ko21]. For quasi-split $\imath$quantum groups, i.e., $\mathbb{I}_\bullet = \varnothing$, our $\psi^\imath$ coincides with the bar involution in [CLW23, Lemma 2.4(a)] (see also [LW22a, Lemma 6.9]). Unlike the proof *loc. cit.*, our proofs of Proposition 3.4 and Proposition 3.12 do not use a Serre presentation of $\widetilde{\mathbf{U}}^\imath$. Hence, the (anti-) involutions

$\sigma^\imath$ and $\psi^\imath$ are valid for $\widetilde{\mathbf{U}}^\imath$ of arbitrary Kac-Moody type.

## 3.5 An anti-involution $\sigma_\tau$ on $\mathbf{U}^\imath_{\boldsymbol{\varsigma}}$

The anti-involution $\sigma^\imath$ on $\widetilde{\mathbf{U}}^\imath$ in Proposition 3.12 can descend to an $\imath$quantum group $\mathbf{U}^\imath_{\boldsymbol{\varsigma}}$, only for any *balanced* parameter $\boldsymbol{\varsigma}$. It turns out that the anti-involution $\sigma^\imath\tau$ on $\widetilde{\mathbf{U}}^\imath$ can descend to an $\imath$quantum group $\mathbf{U}^\imath_{\boldsymbol{\varsigma}}$, for an *arbitrary* parameter $\boldsymbol{\varsigma}$.

**Proposition 3.14.** *Let $\boldsymbol{\varsigma}$ be an arbitrary parameter. There exists a unique anti-involution $\sigma_\tau$ of $\mathbf{U}^\imath_{\boldsymbol{\varsigma}}$ such that*

$$\sigma_\tau(B_i) = B_{\tau i}, \qquad \sigma_\tau(x) = \sigma\tau(x), \qquad \textit{for } i \in \mathbb{I}_\circ, x \in \mathbf{U}^{\imath 0}\mathbf{U}_\bullet. \tag{3.26}$$

*Moreover, $\sigma_\tau$ satisfies the following intertwining relation:*

$$\sigma_\tau(x)\Upsilon_{\boldsymbol{\varsigma}} = \Upsilon_{\boldsymbol{\varsigma}}\sigma\tau(x), \qquad \textit{for all } x \in \mathbf{U}^\imath_{\boldsymbol{\varsigma}}. \tag{3.27}$$

*Proof.* A proof similar to the one for Proposition 3.12 works here, and we outline it.

We claim that, for any $x \in \mathbf{U}^\imath_{\boldsymbol{\varsigma}}$, there exists $\widehat{x} \in \mathbf{U}^\imath_{\boldsymbol{\varsigma}}$ such that

$$\widehat{x}\Upsilon_{\boldsymbol{\varsigma}} = \Upsilon_{\boldsymbol{\varsigma}}\sigma\tau(x). \tag{3.28}$$

As argued in the proof of Proposition 3.12, it suffices to show that (3.28) holds for $x$ in a generating set $\{B_i | i \in \mathbb{I}_\circ\} \cup \mathbf{U}^{\imath 0}\mathbf{U}_\bullet$ of $\mathbf{U}^\imath_{\boldsymbol{\varsigma}}$. Indeed, by (3.20), we have $B_{\tau i}\Upsilon_{\boldsymbol{\varsigma}} = \Upsilon_{\boldsymbol{\varsigma}}\sigma\tau(B_i)$. For $x \in \mathbf{U}^{\imath 0}\mathbf{U}_\bullet$, note that $\sigma\tau(x) \in \mathbf{U}^{\imath 0}\mathbf{U}_\bullet$, and then by Theorem 3.1, we have $\sigma\tau(x)\Upsilon_{\boldsymbol{\varsigma}} = \Upsilon_{\boldsymbol{\varsigma}}\sigma\tau(x)$. This proves (3.28).

Now sending $x \mapsto \widehat{x}$ defines an anti-endomorphism $\sigma_\tau$, which satisfies (3.26) and

(3.27) by construction above. Finally, $\sigma_\tau$ is involutive since it satisfies (3.26). $\square$

*Remark* 3.15. Our construction of $\sigma_\tau$ generalizes the $\sigma_\imath$ in [BW21, Proposition 3.13], which is constructed via bar involutions under certain restrictions on parameters.

*Remark* 3.16. Thanks to Proposition 3.14, one can formulate a $\mathbf{U}^\imath_{\boldsymbol{\varsigma}}$-variant of Theorem 3.6, which characterizes the quasi $K$-matrix $\Upsilon_{\boldsymbol{\varsigma}}$ via the intertwining property

$$B_{\tau i}\widetilde{\Upsilon} = \widetilde{\Upsilon}\sigma\tau(B_i), \qquad (i \in \mathbb{I}_\circ),$$
$$x\widetilde{\Upsilon} = \widetilde{\Upsilon}x, \qquad (x \in \mathbf{U}^{\imath 0}\mathbf{U}_\bullet).$$

This can also be proved directly. This seems more conceptual than the formulation of Theorem 3.1 (see [AV22]).

# 4 New symmetries $\widetilde{\mathbf{T}}'_{i,-1}$ on $\widetilde{\mathbf{U}}^\imath$

In this section, we define explicitly certain rescaled braid group actions $\widetilde{\mathcal{T}}'_{j,-1}$ on a Drinfeld double $\widetilde{\mathbf{U}}$. We then formulate the new symmetries $\widetilde{\mathbf{T}}'_{i,-1}$ on $\widetilde{\mathbf{U}}^\imath$, for $i \in \mathbb{I}_\circ$, via an intertwining property using the quasi $K$-matrix $\widetilde{\Upsilon}$ and a rescaled braid automorphism $\widetilde{\mathcal{T}}'_{\mathbf{r}_i,-1}$; the proof will be completed in the coming sections. We show that $\widetilde{\mathcal{T}}'_{\mathbf{r}_i,-1}$ on $\widetilde{\mathbf{U}}$ preserves the subalgebra $\widetilde{\mathbf{U}}^{\imath 0}\widetilde{\mathbf{U}}_\bullet$, and that the actions of $\widetilde{\mathbf{T}}'_{i,-1}$ and $\widetilde{\mathcal{T}}'_{\mathbf{r}_i,-1}$ on $\widetilde{\mathbf{U}}^{\imath 0}\widetilde{\mathbf{U}}_\bullet$ coincide. Explicit formulas for the action of $\widetilde{\mathbf{T}}'_{i,-1}$ on $\widetilde{\mathbf{U}}^{\imath 0}\widetilde{\mathbf{U}}_\bullet$ are presented. Then we obtain a compact close rank one formula for $\widetilde{\mathbf{T}}'_{i,-1}(B_i)$.

## 4.1 Rescaled braid group action on $\widetilde{\mathbf{U}}$

Recall the distinguished parameter $\boldsymbol{\varsigma}_\diamond$ from (2.28). Extend $\boldsymbol{\varsigma}_\diamond$ trivially to an $\mathbb{I}$-tuple of scalars $(\varsigma_{i,\diamond})_{i\in\mathbb{I}}$ by setting

$$\varsigma_{j,\diamond} = 1, \qquad \text{for } j \in \mathbb{I}_\bullet. \tag{4.1}$$

Then we have the scaling automorphism $\widetilde{\Psi}_{\boldsymbol{\varsigma}_\diamond}$ on $\widetilde{\mathbf{U}}$ by Proposition 2.2. We define symmetries $\widetilde{\mathcal{T}}''_{i,+1}$ and $\widetilde{\mathcal{T}}'_{i,-1}$ on $\widetilde{\mathbf{U}}$ by rescaling $\widetilde{T}''_{i,+1}$ and $\widetilde{T}'_{i,-1}$ in Proposition 2.6 and (2.14) via the rescaling automorphism $\widetilde{\Psi}_{\boldsymbol{\varsigma}_\diamond}$:

$$\widetilde{\mathcal{T}}''_{i,+1} := \widetilde{\Psi}^{-1}_{\boldsymbol{\varsigma}_\diamond} \circ \widetilde{T}''_{i,+1} \circ \widetilde{\Psi}_{\boldsymbol{\varsigma}_\diamond} \tag{4.2}$$

$$\widetilde{\mathcal{T}}'_{i,-1} := \widetilde{\Psi}^{-1}_{\boldsymbol{\varsigma}_\diamond} \circ \widetilde{T}'_{i,-1} \circ \widetilde{\Psi}_{\boldsymbol{\varsigma}_\diamond}. \tag{4.3}$$

Since $\widetilde{T}''_{i,+1}, \widetilde{T}'_{i,-1}$ are mutually inverses, $\widetilde{\mathcal{T}}''_{i,+1}, \widetilde{\mathcal{T}}'_{i,-1}$ are also mutually inverses. We shall often use the shorthand notation

$$\widetilde{\mathcal{T}}_i = \widetilde{\mathcal{T}}''_{i,+1}, \qquad \widetilde{\mathcal{T}}_i^{-1} = \widetilde{\mathcal{T}}'_{i,-1}.$$

*Remark* 4.1. These rescaled symmetries $\widetilde{\mathcal{T}}_i^{-1}$ will play a central role in our construction of symmetries on $\widetilde{\mathbf{U}}^\imath$; see Theorem 4.7. Our rescaling twist using $\widetilde{\Psi}_{\boldsymbol{\varsigma}_\diamond}$ is compatible with the rescaling twist in [DK19, (3.45), Remark 3.16].

We write down the explicit actions for $\widetilde{\mathcal{T}}_i$ and $\widetilde{\mathcal{T}}_i^{-1}$ for later use.

**Proposition 4.2.** *Set $r = -c_{ij}$, for $i, j \in \mathbb{I}$. The automorphism $\widetilde{\mathcal{T}}_i \in \mathrm{Aut}(\widetilde{\mathbf{U}})$ defined* *in* (4.2) *is given by*

$$\begin{aligned}
&\widetilde{\mathfrak{T}}_i(K_j) = \varsigma_{i,\diamond}^{c_{ij}/2} K_j K_i^{-c_{ij}}, \qquad \widetilde{\mathfrak{T}}_i(K_j') = \varsigma_{i,\diamond}^{c_{ij}/2} K_j' K_i'^{-c_{ij}},\\
&\widetilde{\mathfrak{T}}_i(E_i) = -\varsigma_{i,\diamond} F_i K_i'^{-1}, \quad \widetilde{\mathfrak{T}}_i(F_i) = -K_i^{-1} E_i,\\
&\widetilde{\mathfrak{T}}_i(E_j) = \varsigma_{i,\diamond}^{-r/2} \sum_{s=0}^{r} (-1)^s q_i^{-s} E_i^{(r-s)} E_j E_i^{(s)}, \qquad j \neq i,\\
&\widetilde{\mathfrak{T}}_i(F_j) = \sum_{s=0}^{r} (-1)^s q_i^{s} F_i^{(s)} F_j F_i^{(r-s)}, \qquad j \neq i.
\end{aligned}$$

*The inverse of* $\widetilde{\mathfrak{T}}_i$ *(see* (4.3)*) is given by*

$$\begin{aligned}
&\widetilde{\mathfrak{T}}_i^{-1}(K_j) = \varsigma_{i,\diamond}^{c_{ij}/2} K_j K_i^{-c_{ij}}, \qquad \widetilde{\mathfrak{T}}_i^{-1}(K_j') = \varsigma_{i,\diamond}^{c_{ij}/2} K_j' K_i'^{-c_{ij}},\\
&\widetilde{\mathfrak{T}}_i^{-1}(E_i) = -\varsigma_{i,\diamond} K_i^{-1} F_i, \qquad \widetilde{\mathfrak{T}}_i^{-1}(F_i) = -E_i K_i'^{-1},\\
&\widetilde{\mathfrak{T}}_i^{-1}(E_j) = \varsigma_{i,\diamond}^{-r/2} \sum_{s=0}^{r} (-1)^s q_i^{-s} E_i^{(s)} E_j E_i^{(r-s)}, \qquad j \neq i,\\
&\widetilde{\mathfrak{T}}_i^{-1}(F_j) = \sum_{s=0}^{r} (-1)^s q_i^{s} F_i^{(r-s)} F_j F_i^{(s)}, \qquad j \neq i.
\end{aligned}$$

*Moreover,* $\widetilde{\mathfrak{T}}_i$*, for* $i \in \mathbb{I}$*, satisfy the braid group relations.*

Hence, we obtain

$$\widetilde{\mathfrak{T}}_w = \widetilde{\mathfrak{T}}''_{w,+1} := \widetilde{\mathfrak{T}}_{i_1} \cdots \widetilde{\mathfrak{T}}_{i_r} \in \mathrm{Aut}(\widetilde{\mathbf{U}}), \qquad \text{for } w \in W, \tag{4.4}$$

where $w = s_{i_1} \cdots s_{i_r}$ is any reduced expression. Similarly, we have $\widetilde{\mathfrak{T}}'_{w,-1} \in \mathrm{Aut}(\widetilde{\mathbf{U}})$.

*Remark* 4.3. Let $i \in \mathbb{I}_\bullet$. The rescaling for $\widetilde{\mathfrak{T}}_i^{\pm 1}$ is trivial, thanks to $\varsigma_{\diamond,i} = 1$; that is, $\widetilde{\mathfrak{T}}_i = \widetilde{T}_i$. In particular, $\widetilde{\mathfrak{T}}_{w_\bullet} = \widetilde{T}_{w_\bullet}$. Moreover, $\widetilde{T}_{w_\bullet}(E_{\tau i}) = \widetilde{\mathfrak{T}}_{w_\bullet}(E_{\tau i}) = T_{w_\bullet}(E_{\tau i})$ in $\widetilde{\mathbf{U}}^+ = \mathbf{U}^+$; cf. the formula for $B_i$ in (2.24).

Let $\tau_0$ be the diagram automorphism associated to the longest element $w_0$ of the Weyl group $W$. The following fact is well known (up to the rescaling via $\boldsymbol{\varsigma}_\diamond$); cf., e.g., [Ko14, Lemma 3.4].

**Lemma 4.4.** *We have, for $j \in \mathbb{I}$,*

$$\widetilde{\mathcal{T}}_{w_0}(F_j) = -K_{\tau_0 j}^{-1} E_{\tau_0 j}, \qquad \widetilde{\mathcal{T}}_{w_0}(E_j) = -\varsigma_{j,\diamond} F_{\tau_0 j} K_{\tau_0 j}^{\prime -1},$$
$$\widetilde{\mathcal{T}}_{w_0}^{-1}(E_j) = -\varsigma_{j,\diamond} K_{\tau_0 j}^{-1} F_{\tau_0 j}, \qquad \widetilde{\mathcal{T}}_{w_0}^{-1}(F_j) = -E_{\tau_0 j} K_{\tau_0 j}^{\prime -1}.$$

## 4.2 Symmetries $\widetilde{\mathcal{T}}''_{j,+1}$, for $j \in \mathbb{I}_\bullet$

It is known [BW18b] that Lusztig's operators $T'_{j,\pm 1}, T''_{j,\pm 1}$ on $\mathbf{U}$, for $j \in \mathbb{I}_\bullet$, restrict to automorphisms of $\mathbf{U}^\imath_{\boldsymbol{\varsigma}}$ (where the $\boldsymbol{\varsigma}$ satisfies certain constraints); moreover, these operators fix $\Upsilon$. In this subsection, we formulate analogous statements for the universal quantum symmetric pair $(\widetilde{\mathbf{U}}, \widetilde{\mathbf{U}}^\imath)$ while skipping the identical proofs.

Recall the automorphisms $\widetilde{T}''_{i,+1}$ on the Drinfeld double $\widetilde{\mathbf{U}}$, for $i \in \mathbb{I}$, from Proposition 2.6, and recall Remark 4.3.

**Proposition 4.5** (cf. [BW18b, Theorem 4.2])**.** *Let $j \in \mathbb{I}_\bullet$. The automorphism $\widetilde{\mathcal{T}}''_{j,+1} = \widetilde{T}''_{j,+1}$ on $\widetilde{\mathbf{U}}$ restricts to an automorphism of $\widetilde{\mathbf{U}}^\imath$. Moreover, the action of $\widetilde{\mathcal{T}}''_{j,+1}$ on $B_i$ ($i \in \mathbb{I}_\circ$) is given by*

$$\widetilde{\mathcal{T}}''_{j,+1}(B_i) = \sum_{s=0}^{r} (-1)^s q_j^s F_j^{(s)} B_i F_j^{(r-s)}, \qquad \text{for } r = -c_{ij}. \tag{4.5}$$

**Proposition 4.6** (cf. [BW18b, Proposition 4.13])**.** *Let $j \in \mathbb{I}_\bullet$. Then $\widetilde{\mathcal{T}}''_{j,+1}(\widetilde{\Upsilon}) = \widetilde{\Upsilon}$, and $\widetilde{\mathcal{T}}''_{j,+1}(\widetilde{\Upsilon}_i) = \widetilde{\Upsilon}_i$, for $i \in \mathbb{I}_\circ$.*

## 4.3 Characterization of $\widetilde{\mathbf{T}}'_{i,-1}$

Let $(\widetilde{\mathbf{U}}, \widetilde{\mathbf{U}}^\imath)$ be the quantum symmetric pair associated to an arbitrary Satake diagram $(\mathbb{I} = \mathbb{I}_\bullet \cup \mathbb{I}_\circ, \tau)$. Recall that $\widetilde{\Upsilon}_i$, for $i \in \mathbb{I}_\circ$, are the quasi $K$-matrix associated to the rank one Satake subdiagram $(\mathbb{I}_\bullet \cup \{i, \tau i\}, \tau|_{\mathbb{I}_\bullet \cup \{i,\tau i\}})$. Recall $\mathbf{r}_i \in W$ from (2.21) and $\widetilde{\mathcal{T}}'_{\mathbf{r}_i,-1} \in \mathrm{Aut}(\widetilde{\mathbf{U}})$ from (4.4) whose definition uses (4.2). We now formulate our first main result.

**Theorem 4.7.** *Let $i \in \mathbb{I}_\circ$.*

1. *For any $x \in \widetilde{\mathbf{U}}^\imath$, there is a unique element $x' \in \widetilde{\mathbf{U}}^\imath$ such that $x'\widetilde{\Upsilon}_i = \widetilde{\Upsilon}_i \widetilde{\mathcal{T}}'_{\mathbf{r}_i,-1}(x^\imath)$.*
2. *The map $x \mapsto x'$ is an automorphism of the algebra $\widetilde{\mathbf{U}}^\imath$, denoted by $\widetilde{\mathbf{T}}'_{i,-1}$.*

Therefore, we have

$$\widetilde{\mathbf{T}}'_{i,-1}(x)\widetilde{\Upsilon}_i = \widetilde{\Upsilon}_i \widetilde{\mathcal{T}}'_{\mathbf{r}_i,-1}(x^\imath), \qquad \text{for all } x \in \widetilde{\mathbf{U}}^\imath. \tag{4.6}$$

*Proof.* A complete proof of this theorem requires the developments in the coming Sections 4–6. Let us outline the main steps below.

For a given $x \in \widetilde{\mathbf{U}}^\imath$, $x' \in \widetilde{\mathbf{U}}^\imath$ satisfying the identity in (1) is clearly unique (if it exists) since $\widetilde{\Upsilon}_i$ is invertible.

The explicit formulas of $x'$ associated to generators $x$ of $\widetilde{\mathbf{U}}^\imath$, for each of (rank one and two) Satake diagrams in the forthcoming Sections 4–5. The formulas therein show manifestly that $x' \in \widetilde{\mathbf{U}}^\imath$; see Proposition 4.11 on $\widetilde{\mathbf{U}}^{\imath 0}\widetilde{\mathbf{U}}_\bullet$, Theorem 4.14 for rank 1, and Theorem 5.5 for rank 2.

Assume that $x', y' \in \widetilde{\mathbf{U}}^\imath$ satisfy (1), for $x, y \in \widetilde{\mathbf{U}}^\imath$; that is, $x'\widetilde{\Upsilon}_i = \widetilde{\Upsilon}_i \widetilde{\mathcal{T}}'_{\mathbf{r}_i,-1}(x^\imath)$, and $y'\widetilde{\Upsilon}_i = \widetilde{\Upsilon}_i \widetilde{\mathcal{T}}'_{\mathbf{r}_i,-1}(y^\imath)$. Then it follows readily that $x'y' \in \widetilde{\mathbf{U}}^\imath$ satisfies the identity in

(1) for $xy$; that is, $x'y'\widetilde{\Upsilon}_i = \widetilde{\Upsilon}_i \widetilde{\mathcal{T}}'_{\mathbf{r}_i,-1}((xy)^\imath)$. Hence we have obtained a well-defined endomorphism $\widetilde{\mathbf{T}}'_{i,-1}$ on $\widetilde{\mathbf{U}}^\imath$ which sends $x \mapsto x'$.

To complete the proof of the theorem, it remains to show that $\widetilde{\mathbf{T}}'_{i,-1}$ is surjective. To this end, we introduce and study in depth a variant of $\widetilde{\mathbf{T}}'_{i,-1}$, a second endomorphism $\widetilde{\mathbf{T}}''_{i,+1}$ on $\widetilde{\mathbf{U}}^\imath$ in Section 6. The bijectivity of $\widetilde{\mathbf{T}}'_{i,-1}$ follows by Theorem 6.7 which shows that $\widetilde{\mathbf{T}}'_{i,-1}$ and $\widetilde{\mathbf{T}}''_{i,+1}$ are mutual inverses. □

*Remark* 4.8. By Proposition 3.8 and the definition (2.21) of $\mathbf{r}_i$, we have $\widetilde{\Upsilon}_i = \widetilde{\Upsilon}_{\tau i}$, $\mathbf{r}_i = \mathbf{r}_{\tau i}$, and hence $\widetilde{\mathbf{T}}'_{i,-1} = \widetilde{\mathbf{T}}'_{\tau i,-1}$. Thus, we may label $\widetilde{\mathbf{T}}'_{i,-1}$ by $\mathbb{I}_{\circ,\tau}$ instead of $\mathbb{I}_\circ$.

In this and later sections, we shall construct 4 variants of symmetries of $\widetilde{\mathbf{U}}^\imath$ (denoted by $\widetilde{\mathbf{T}}'_{i,e}$, $\widetilde{\mathbf{T}}''_{i,e}$) via (4.6) and 3 additional intertwining relations and the rescaled braid group symmetries $\widetilde{\mathcal{T}}'_{\mathbf{r}_i,\pm1}, \widetilde{\mathcal{T}}''_{\mathbf{r}_i,\pm1}$ of $\widetilde{\mathbf{U}}^\imath$. We choose to start with the (simplest) intertwining relation (4.6) for $\widetilde{\mathcal{T}}'_{\mathbf{r}_i,-1}$. From now on, we often write

$$\widetilde{\mathcal{T}}_{\mathbf{r}_i}^{-1} = \widetilde{\mathcal{T}}'_{\mathbf{r}_i,-1}, \qquad \widetilde{\mathcal{T}}_{\mathbf{r}_i} = \widetilde{\mathcal{T}}''_{\mathbf{r}_i,+1}.$$

## 4.4 Quantum symmetric pairs of diagonal type

Recall from Proposition 2.3 the Chevalley involution $\omega$ and the comultiplication $\Delta$ (2.6) on $\widetilde{\mathbf{U}}$. Denote ${}^\omega\mathbf{L}''_i := (\omega \otimes 1)\mathbf{L}''_i$ for $i \in \mathbb{I}$, where $\mathbf{L}''_i, i \in \mathbb{I}$ is the rank one quasi $R$-matrix for $\widetilde{\mathbf{U}}$ (same as for $\mathbf{U}$); see [Lus93]. We regard $\widetilde{\mathbf{U}}$ as a coideal subalgebra of $\widetilde{\mathbf{U}} \otimes \widetilde{\mathbf{U}}$ via the embedding ${}^\omega\Delta := (\omega \otimes 1)\Delta$ and then $(\widetilde{\mathbf{U}} \otimes \widetilde{\mathbf{U}}, \widetilde{\mathbf{U}})$ is a universal quantum symmetric pair of diagonal type; cf. [BW18b, Remark 4.10]. In this way, the rank one quasi $K$-matrices for quantum symmetric pairs of diagonal type are given by ${}^\omega\mathbf{L}''_i$.

In this subsection, we shall reformulate the identity [Lus93, 37.3.2] or (1.2), as an intertwining relation in the framework of quantum symmetric pairs similar to (4.6).

**Proposition 4.9.** *For the quantum symmetric pair of diagonal type* $(\widetilde{\mathbf{U}}\otimes\widetilde{\mathbf{U}},\widetilde{\mathbf{U}})$*, the following intertwining relation holds:*

$$
{}^{\omega}\Delta(\widetilde{T}'_{i,-1}u)\,{}^{\omega}\mathbf{L}''_i = {}^{\omega}\mathbf{L}''_i\,(\widetilde{\mathcal{T}}''_{i,-1}\otimes\widetilde{\mathcal{T}}'_{i,-1})\,{}^{\omega}\Delta(u),\qquad \forall u\in\widetilde{\mathbf{U}}. \tag{4.7}
$$

*Proof.* Recall from [Lus93, 37.2.4] that $\omega\circ\widetilde{T}'_{i,-1}\circ\omega=\widetilde{T}''_{i,-1}$. The identity (1.2) for $\mathbf{U}$ admits an identical version for $\widetilde{\mathbf{U}}$. Applying $\omega\otimes 1$ to this identity, we obtain

$$
{}^{\omega}\Delta(\widetilde{T}'_{i,-1}u)\,{}^{\omega}\mathbf{L}''_i = {}^{\omega}\mathbf{L}''_i\,(\widetilde{T}''_{i,-1}\otimes\widetilde{T}'_{i,-1})\,{}^{\omega}\Delta(u),\qquad \forall u\in\widetilde{\mathbf{U}}.
$$

To prove (4.7), it suffices to prove the following identity

$$
(\widetilde{\mathcal{T}}''_{j,-1}\otimes\widetilde{\mathcal{T}}'_{j,-1})\,{}^{\omega}\Delta(u) = (\widetilde{T}''_{j,-1}\otimes\widetilde{T}'_{j,-1})\,{}^{\omega}\Delta(u),\qquad \forall u\in\widetilde{\mathbf{U}}. \tag{4.8}
$$

Clearly, it suffices to prove (4.8) when $u$ is the generator of $\widetilde{\mathbf{U}}$. We have the following formulas:

$$
\begin{aligned}
{}^{\omega}\Delta(E_j) = F_j\otimes 1 + K'_j\otimes E_j,\qquad & {}^{\omega}\Delta(F_j) = 1\otimes F_j + E_j\otimes K'_j,\\
{}^{\omega}\Delta(K_j) = K'_j\otimes K_j,\qquad & {}^{\omega}\Delta(K'_j) = K_j\otimes K'_j.
\end{aligned} \tag{4.9}
$$

Recall $\widetilde{\mathcal{T}}'_{j,-1}=\widetilde{\Psi}_{\varsigma_\diamond}^{-1}\widetilde{T}'_{j,-1}\widetilde{\Psi}_{\varsigma_\diamond}$ from (4.3). By Lemma 14.5 and noting that $\varsigma_{\star\diamond}=\varsigma_\diamond$ in our case, the twisting for $\widetilde{\mathcal{T}}''_{j,-1}$ is opposite to the one on $\widetilde{\mathcal{T}}'_{j,-1}$, i.e., $\widetilde{\mathcal{T}}''_{j,-1}=\widetilde{\Psi}_{\varsigma_\diamond}\widetilde{T}''_{j,-1}\widetilde{\Psi}_{\varsigma_\diamond}^{-1}$. By Proposition 2.2, we see that the RHS of each formula in (4.9) is fixed by $\widetilde{\Psi}_{\varsigma_\diamond}^{-1}\otimes\widetilde{\Psi}_{\varsigma_\diamond}$. The formulas for $\widetilde{T}''_{i,+1}$ is given in Proposition 2.6, and the formulas for $\widetilde{T}'_{i,-1},\widetilde{T}''_{i,-1}$ can be obtained from there by suitable twisting; using these formulas, we observe that $(\widetilde{T}''_{j,-1} \otimes \widetilde{T}'_{j,-1})\ {}^{\omega}\Delta(u)$ is fixed by $\widetilde{\Psi}_{\varsigma_\diamond} \otimes \widetilde{\Psi}_{\varsigma_\diamond}^{-1}$ for $u = E_j, F_j, K_j, K'_j$. Hence, for $u = E_j, F_j, K_j, K'_j, j \in \mathbb{I}$,

$$
\begin{aligned}
(\widetilde{\mathfrak{T}}''_{j,-1} \otimes \widetilde{\mathfrak{T}}'_{j,-1})\ {}^{\omega}\Delta(u) &= (\widetilde{\Psi}_{\varsigma_\diamond} \otimes \widetilde{\Psi}_{\varsigma_\diamond}^{-1})(\widetilde{T}''_{j,-1} \otimes \widetilde{T}'_{j,-1})(\widetilde{\Psi}_{\varsigma_\diamond}^{-1} \otimes \widetilde{\Psi}_{\varsigma_\diamond})^{\omega}\Delta(u) \\
&= (\widetilde{T}''_{j,-1} \otimes \widetilde{T}'_{j,-1})\ {}^{\omega}\Delta(u),
\end{aligned}
$$

which implies the desired identity (4.8). □

## 4.5 Action of $\widetilde{\mathbf{T}}'_{i,-1}$ on $\widetilde{\mathbf{U}}^{\imath 0}\widetilde{\mathbf{U}}_{\bullet}$

We formulate $\widetilde{\mathbf{T}}'_{i,-1}(x)$, for $i \in \mathbb{I}_\circ, x \in \widetilde{\mathbf{U}}^{\imath 0}\widetilde{\mathbf{U}}_{\bullet}$ in this subsection. We will show that $\widetilde{\mathfrak{T}}_{\mathbf{r}_i}^{-1}$ preserves both $\widetilde{\mathbf{U}}^{\imath 0}$ and $\widetilde{\mathbf{U}}_{\bullet}$; hence, by Theorem 3.6, the element $\widetilde{\mathbf{T}}'_{i,-1}(x) := \widetilde{\mathfrak{T}}_{\mathbf{r}_i}^{-1}(x)$ satisfies (4.6) for $x \in \widetilde{\mathbf{U}}^{\imath 0}\widetilde{\mathbf{U}}_{\bullet}$.

Recall that the diagram involution associated to $w_{\bullet,i}$ is denoted by $\tau_{\bullet,i}$. By definition of admissible pairs, the diagram involution associated to $w_\bullet$ is $\tau|_{\mathbb{I}_\bullet}$. The involution $\tau$ induces an involutive automorphism, denoted by $\widehat{\tau}$, on $\widetilde{\mathbf{U}}$. Both $\tau_{\bullet,i}$ and $\tau$ induce (commuting) involutive automorphisms, denoted by $\widehat{\tau}_{\bullet,i}$ and $\widehat{\tau}$, on $\widetilde{\mathbf{U}}_{\bullet}$.

We first calculated $\widetilde{\mathfrak{T}}_{\mathbf{r}_i}^{-1}(x)$ for $x \in \widetilde{\mathbf{U}}_{\bullet}$. By applying Lemma 4.4 twice, we obtain

$$
\widetilde{\mathfrak{T}}_{w_\bullet}^{-1}\widehat{\tau}(x) = \widetilde{\mathfrak{T}}_{w_{\bullet,i}}^{-1}\widehat{\tau}_{\bullet,i}(x) = \widetilde{\mathfrak{T}}_{w_\bullet}^{-1}\widetilde{\mathfrak{T}}_{\mathbf{r}_i}^{-1}\widehat{\tau}_{\bullet,i}(x);
$$

note that the second identity above holds since $\widetilde{\mathfrak{T}}_{w_{\bullet,i}}^{-1} = \widetilde{\mathfrak{T}}_{w_\bullet}^{-1}\widetilde{\mathfrak{T}}_{\mathbf{r}_i}^{-1}$ by (2.21). Hence, we have $\widehat{\tau}(x) = \widetilde{\mathfrak{T}}_{\mathbf{r}_i}^{-1}\widehat{\tau}_{\bullet,i}(x)$, which implies that

$$
\widetilde{\mathfrak{T}}_{\mathbf{r}_i}^{-1}(x) = \widehat{\tau}_{\bullet,i} \circ \widehat{\tau}(x) \in \widetilde{\mathbf{U}}_{\bullet}, \qquad \text{for all } x \in \widetilde{\mathbf{U}}_{\bullet}. \tag{4.10}
$$

We next formulate the actions of $\widetilde{\mathcal{T}}_{\mathbf{r}_i}^{-1}$ on $\widetilde{\mathbf{U}}^{\imath 0}$, for $i \in \mathbb{I}_\circ$. Recall $\boldsymbol{\varsigma}_\diamond$ from (2.28) and (4.1), and $\widetilde{\Psi}_{\boldsymbol{\varsigma}_\diamond}$ from Proposition 2.2. Denote

$$\widetilde{k}_{i,\diamond} := \widetilde{\Psi}_{\boldsymbol{\varsigma}_\diamond}^{-1}(\widetilde{k}_i) = \varsigma_{i,\diamond}^{-1} K_i K'_{\tau i} \in \widetilde{\mathbf{U}}^{\imath 0}. \tag{4.11}$$

Note that $\widetilde{k}_{j,\diamond} = \widetilde{k}_j = K_j K'_{\tau j}$, for $j \in \mathbb{I}_\bullet$. We shall denote

$$\widetilde{k}_{\lambda,\diamond} := \prod_{i\in\mathbb{I}} \widetilde{k}_{i,\diamond}^{m_i} \in \widetilde{\mathbf{U}}^{\imath 0}, \qquad \text{for } \lambda = \sum_{i\in\mathbb{I}} m_i \alpha_i \in \mathbb{Z}\mathbb{I}. \tag{4.12}$$

**Lemma 4.10.** *Let $w \in W$ be such that $w\tau = \tau w$. Then $\widetilde{\mathcal{T}}'_{w,-1}(\widetilde{k}_{j,\diamond}) = \widetilde{k}_{w\alpha_j,\diamond}$, for $j \in \mathbb{I}_\circ$.*

*Proof.* By Proposition 2.6, we have

$$\widetilde{T}'_{w,-1}(\widetilde{k}_j) = \widetilde{T}'_{w,-1}(K_j K'_{\tau j}) = K_{w\alpha_j} K'_{w\alpha_{\tau j}} = K_{w\alpha_j} K'_{\tau w\alpha_j} = \widetilde{k}_{w\alpha_j}.$$

By (4.11)–(4.12), we have $\widetilde{k}_{\lambda,\diamond} = \widetilde{\Psi}_{\boldsymbol{\varsigma}_\diamond}^{-1}(\widetilde{k}_\lambda)$, for $\lambda \in \mathbb{Z}\mathbb{I}$. By (4.2) and (4.4), we have $\widetilde{\mathcal{T}}'_{w,-1} = \widetilde{\Psi}_{\boldsymbol{\varsigma}_\diamond}^{-1} \circ \widetilde{T}'_{w,-1} \circ \widetilde{\Psi}_{\boldsymbol{\varsigma}_\diamond}$, and hence

$$\widetilde{\mathcal{T}}'_{w,-1}(\widetilde{k}_{j,\diamond}) = (\widetilde{\Psi}_{\boldsymbol{\varsigma}_\diamond}^{-1} \circ \widetilde{T}'_{w,-1})(\widetilde{k}_j) = \widetilde{\Psi}_{\boldsymbol{\varsigma}_\diamond}^{-1}(\widetilde{k}_{w\alpha_j}) = \widetilde{k}_{w\alpha_j,\diamond}.$$

The lemma is proved. □

In particular, setting $w = \mathbf{r}_i$ $(i \in \mathbb{I}_\circ)$ in Lemma 4.10 gives us

$$\widetilde{\mathcal{T}}_{\mathbf{r}_i}^{-1}(\widetilde{k}_{j,\diamond}) = \widetilde{k}_{\mathbf{r}_i\alpha_j,\diamond}.$$

Summarizing the above discussion, we have obtained the following.

**Proposition 4.11.** *Let $i \in \mathbb{I}_\circ$. There exists element $\widetilde{\mathbf{T}}'_{i,-1}(x) := \widetilde{\mathcal{T}}^{-1}_{\mathbf{r}_i}(x)$, which satisfies the intertwining relation (4.6), for $x \in \widetilde{\mathbf{U}}^{\imath 0}\widetilde{\mathbf{U}}_\bullet$. More explicitly, we have*

$$\widetilde{\mathbf{T}}'_{i,-1}(u) = (\widehat{\tau}_{\bullet,i} \circ \widehat{\tau})(u), \qquad \text{for } u \in \widetilde{\mathbf{U}}_\bullet, \tag{4.13}$$

$$\widetilde{\mathbf{T}}'_{i,-1}(\widetilde{k}_{j,\diamond}) = \widetilde{k}_{\mathbf{r}_i\alpha_j,\diamond}, \qquad \text{for } j \in \mathbb{I}_\circ. \tag{4.14}$$

## 4.6 Integrality of $\widetilde{\mathbf{T}}'_{i,-1}$

The formula (4.13) clearly preserves the Lusztig integral $\mathbb{Z}[q,q^{-1}]$-form on $\widetilde{\mathbf{U}}_\bullet$. We shall explain below that our braid group action is also integral on the Cartan part, even though the definition (4.11) of $\widetilde{k}_{j,\diamond}$ may involve $q^{1/2}$.

**Lemma 4.12.** *We have*

$$\widetilde{\mathbf{T}}'_{i,-1}(\widetilde{k}_j) = \varsigma^{-1}_{\mathbf{r}_i\alpha_j-\alpha_j,\diamond}\widetilde{k}_{\mathbf{r}_i\alpha_j}, \tag{4.15}$$

*where $\varsigma^{-1}_{\mathbf{r}_i\alpha_j-\alpha_j,\diamond} \in \mathbb{Z}[q,q^{-1}]$, for all $i,j \in \mathbb{I}_\circ$.*

*Proof.* Formula (4.15) follows from (4.14) by unraveling the notation $\widetilde{k}_{j,\diamond}, \widetilde{k}_{\mathbf{r}_i\alpha_j,\diamond}$ in (4.11)–(4.12).

It remains to show that $\varsigma^{-1}_{\mathbf{r}_i\alpha_j-\alpha_j,\diamond} \in \mathbb{Z}[q,q^{-1}]$. Recall from the definition (2.28), we have $\varsigma_{j,\diamond} \in -q^{\mathbb{Z}/2}$, for all $j \in \mathbb{I}_\circ$.

For $j = i$, since $\mathbf{r}_i(\alpha_i) = -\alpha_i + \alpha_\bullet$ for some $\alpha_\bullet \in \mathbb{Z}\mathbb{I}_\bullet$, we have

$$\widetilde{\mathbf{T}}'_{i,-1}(\widetilde{k}_i) = \varsigma^2_{i,\diamond}\widetilde{k}_{\mathbf{r}_i\alpha_i}.$$

where $\varsigma^2_{i,\diamond} \in q^{\mathbb{Z}}$. The integrality for $\widetilde{\mathbf{T}}'_{i,-1}(\widetilde{k}_{\tau i})$ can be then obtained by applying $\widehat{\tau}$ to the above formula.

For $j \neq i, \tau i$, we only need to consider the case $\varsigma_{i,\diamond} = -q^{-1/2}$. In this case, by (2.28), $\overline{\alpha}_i$ is a short root. Moreover, due to the classification of Satake diagrams and the corresponding restricted root systems [Ar62], we have $\frac{(\overline{\alpha}_j,\overline{\alpha}_i)}{(\overline{\alpha}_i,\overline{\alpha}_i)} = -2$ or 0. It remains to consider the nontrivial case $\frac{(\overline{\alpha}_j,\overline{\alpha}_i)}{(\overline{\alpha}_i,\overline{\alpha}_i)} = -2$. It follows that $\mathbf{r}_i\overline{\alpha}_j - \overline{\alpha}_j = 2\overline{\alpha}_i$, which implies $\mathbf{r}_i\alpha_j \in \alpha_j + k\alpha_i + l\alpha_{\tau i} + \mathbb{Z}\mathbb{I}_\bullet$, for some $k, l \geq 0, k+l = 2$. Since $\varsigma_{i,\diamond} = \varsigma_{\tau i,\diamond}$, the formula (4.15) is unraveled as the following integral formula $\widetilde{\mathbf{T}}'_{i,-1}(\widetilde{k}_j) = \varsigma_{i,\diamond}^{-2}\widetilde{k}_{\mathbf{r}_i\alpha_j}$.

Therefore, the integrality of (4.15) holds in all cases. □

## 4.7 A uniform formula for $\widetilde{\mathbf{T}}'_{i,-1}(B_i)$

In this subsection, we introduce a uniform method to calculate $\widetilde{\mathbf{T}}'_{i,-1}(B_i)$. Note that $\widetilde{\mathbf{T}}_i = \widetilde{\mathbf{T}}_{\tau i}$ and this takes care of $\widetilde{\mathbf{T}}'_{i,-1}(B_{\tau i})$. To that end, without loss of generality, we can restrict ourselves to a Satake diagram $(\mathbb{I} = \mathbb{I}_\bullet \cup \mathbb{I}_\circ, \tau)$ of real rank one; that is, for some $i \in \mathbb{I}_\circ$,

$$\mathbb{I}_\circ = \{i, \tau i\} \text{ if } \tau \neq \mathrm{Id}, \qquad \text{and} \qquad \mathbb{I}_\circ = \{i\} \text{ if } \tau = \mathrm{Id}.$$

Recall the diagram involution $\tau_{\bullet,i}$ associated to the longest element $w_{\bullet,i}$ in the Weyl group $W_{\mathbb{I}_\bullet \cup \{i,\tau i\}}$. By definition of admissible pairs, the diagram involution associated to $w_\bullet$ is $\tau$. Observe that $\tau_{\bullet,i}\tau i \in \{i, \tau i\}$, by Table 1 on rank one Satake diagrams.

Recall $\mathcal{K}_i, \mathcal{K}_{\tau i} \in \widetilde{\mathbf{U}}^{\imath 0}$ from (3.23).

**Lemma 4.13.** *We have*

$$\widetilde{\mathcal{T}}_{\mathbf{r}_i}^{-1}(B_i) = -q^{-(\alpha_i, w_\bullet \alpha_{\tau i})}\widetilde{\mathcal{T}}_{w_\bullet}^2(B^\sigma_{\tau_{\bullet,i}\tau i})\mathcal{K}^{-1}_{\tau_{\bullet,i}\tau i}, \tag{4.16}$$

*where $B_i^\sigma$ is given in* (3.17).

*Proof.* Recall from (2.28) and (4.1) that $\varsigma_{i,\diamond} = -q^{-(\alpha_i,\alpha_i+w_\bullet\alpha_{\tau i})/2}$, for $i \in \mathbb{I}_\circ$, and $\varsigma_{j,\diamond} = 1$, for $j \in \mathbb{I}_\bullet$. By (2.21), we have $\widetilde{\mathcal{T}}_{w_{\bullet,i}} = \widetilde{\mathcal{T}}_{\mathbf{r}_i}\widetilde{\mathcal{T}}_{w_\bullet}$. By Lemma 4.4, we compute

$$\begin{aligned}
\widetilde{\mathcal{T}}_{\mathbf{r}_i}^{-1}(B_i) &= \widetilde{\mathcal{T}}_{\mathbf{r}_i}^{-1}\big(F_i + \widetilde{\mathcal{T}}_{w_\bullet}(E_{\tau i})K_i'\big)\\
&= \widetilde{\mathcal{T}}_{w_\bullet}\widetilde{\mathcal{T}}_{w_{\bullet,i}}^{-1}\big(F_i + \widetilde{\mathcal{T}}_{w_\bullet}(E_{\tau i})K_i'\big)\\
&= \widetilde{\mathcal{T}}_{w_\bullet}^2\big(\widetilde{\mathcal{T}}_{w_\bullet}^{-1}\widetilde{\mathcal{T}}_{w_{\bullet,i}}^{-1}(F_i) + \widetilde{\mathcal{T}}_{w_{\bullet,i}}^{-1}(E_{\tau i})\widetilde{\mathcal{T}}_{w_\bullet}^{-1}\widetilde{\mathcal{T}}_{w_{\bullet,i}}^{-1}(K_i')\big)\\
&= \widetilde{\mathcal{T}}_{w_\bullet}^2\big(-\widetilde{\mathcal{T}}_{w_\bullet}^{-1}(E_{\tau_{\bullet,i}i}K_{\tau_{\bullet,i}i}'^{-1}) - q^{-(\alpha_i,\alpha_i+w_\bullet\alpha_{\tau i})}K_{\tau_{\bullet,i}\tau i}^{-1}F_{\tau_{\bullet,i}\tau i}\widetilde{\mathcal{T}}_{w_\bullet}^{-1}(K_{\tau_{\bullet,i}i}'^{-1})\big)\\
&= -\widetilde{\mathcal{T}}_{w_\bullet}^2\big(\widetilde{\mathcal{T}}_{w_\bullet}^{-1}(E_{\tau_{\bullet,i}i})K_{\tau_{\bullet,i}\tau i} + q^{-(\alpha_i,w_\bullet\alpha_{\tau i})}F_{\tau_{\bullet,i}\tau i}\big)K_{\tau_{\bullet,i}\tau i}^{-1}\widetilde{\mathcal{T}}_{w_\bullet}(K_{\tau_{\bullet,i}i}')^{-1}\\
&= -q^{-(\alpha_i,w_\bullet\alpha_{\tau i})}\widetilde{\mathcal{T}}_{w_\bullet}^2\big(K_{\tau_{\bullet,i}\tau i}\widetilde{\mathcal{T}}_{w_\bullet}^{-1}(E_{\tau_{\bullet,i}i}) + F_{\tau_{\bullet,i}\tau i}\big)\mathcal{K}_{\tau_{\bullet,i}\tau i}^{-1}\\
&= -q^{-(\alpha_i,w_\bullet\alpha_{\tau i})}\widetilde{\mathcal{T}}_{w_\bullet}^2(B_{\tau_{\bullet,i}\tau i}^\sigma)\mathcal{K}_{\tau_{\bullet,i}\tau i}^{-1}.
\end{aligned}$$

This proves the lemma. □

**Theorem 4.14.** *Let $i \in \mathbb{I}_\circ$. There exists a unique element $\widetilde{\mathbf{T}}'_{i,-1}(B_i) \in \widetilde{\mathbf{U}}^\imath$ which satisfies the following intertwining relation (see* (4.6)*)*

$$\widetilde{\mathbf{T}}'_{i,-1}(B_i)\widetilde{\Upsilon}_i = \widetilde{\Upsilon}_i\widetilde{\mathcal{T}}_{\mathbf{r}_i}^{-1}(B_i).$$

*More explicitly, we have*

$$\widetilde{\mathbf{T}}'_{i,-1}(B_i) = -q^{-(\alpha_i,w_\bullet\alpha_{\tau i})}\widetilde{\mathcal{T}}_{w_\bullet}^2(B_{\tau_{\bullet,i}\tau i})\mathcal{K}_{\tau_{\bullet,i}\tau i}^{-1}. \tag{4.17}$$

*Proof.* Recall $\tau_{\bullet,i}\tau i \in \{i, \tau i\}$; see Table 1. By Theorem 3.6, we have $\widetilde{\Upsilon}_i B_{\tau_{\bullet,i}\tau i}^\sigma = B_{\tau_{\bullet,i}\tau i}\widetilde{\Upsilon}_i$. By Proposition 4.6, we have $\widetilde{\mathcal{T}}_{w_\bullet}(\widetilde{\Upsilon}_i) = \widetilde{\Upsilon}_i$, and hence $\widetilde{\Upsilon}_i\widetilde{\mathcal{T}}_{w_\bullet}^2(B_{\tau_{\bullet,i}\tau i}^\sigma) = \widetilde{\mathcal{T}}_{w_\bullet}^2(B_{\tau_{\bullet,i}\tau i})\widetilde{\Upsilon}_i$. By Lemma 3.11, we have $\mathcal{K}_{\tau_{\bullet,i}\tau i} \in \widetilde{\mathbf{U}}^{\imath 0}$, and hence $\mathcal{K}_{\tau_{\bullet,i}\tau i}$ commutes with $\widetilde{\Upsilon}_i$. Putting these together with (4.16), we have

$$-q^{-(\alpha_i,w_\bullet(\alpha_{\tau i}))}\widetilde{\mathcal{T}}^2_{w_\bullet}(B_{\tau_{\bullet,i}\tau i})\mathcal{K}^{-1}_{\tau_{\bullet,i}\tau i}\widetilde{\Upsilon}_i = \widetilde{\Upsilon}_i\widetilde{\mathcal{T}}^{-1}_{\mathbf{r}_i}(B_i). \tag{4.18}$$

It follows by Proposition 4.5 that $-q^{-(\alpha_i,w_\bullet(\alpha_{\tau i}))}\widetilde{\mathcal{T}}^2_{w_\bullet}(B_{\tau_{\bullet,i}\tau i})\mathcal{K}^{-1}_{\tau_{\bullet,i}\tau i} \in \widetilde{\mathbf{U}}^\imath$. Hence, setting $\widetilde{\mathbf{T}}'_{i,-1}(B_i) = -q^{-(\alpha_i,w_\bullet\alpha_{\tau i})}\widetilde{\mathcal{T}}^2_{w_\bullet}(B_{\tau_{\bullet,i}\tau i})\mathcal{K}^{-1}_{\tau_{\bullet,i}\tau i}$, we have proved the theorem. □

# 5 Rank two formulas for $\widetilde{\mathbf{T}}'_{i,-1}(B_j)$

Let $(\mathbb{I} = \mathbb{I}_\bullet \cup \mathbb{I}_\circ, \tau)$ be a rank two irreducible Satake diagram. Fix $i, j \in \mathbb{I}_{\circ,\tau}$ such that $i \neq j$, such that $\mathbb{I}_\circ = \{i, \tau i, j, \tau j\}$. A complete list of formulas for $\widetilde{\mathbf{T}}'_{i,-1}(B_j)$ is formulated in Table 3. We show that the formulas for $\widetilde{\mathbf{T}}'_{i,-1}(B_j)$ in Table 3 satisfy the intertwining relation (4.6); see Theorem 5.5. Together with the formulas in the previous section, we have established the existence of an endomorphism $\widetilde{\mathbf{T}}'_{i,-1}$ on $\widetilde{\mathbf{U}}^\imath$ satisfying (4.6).

## 5.1 Some commutator relations with $\widetilde{\Upsilon}$

For $w \in W$, let $\mathbf{U}^+[w]$ be the well-known subalgebra of $\mathbf{U}^+$ spanned by PBW basis elements generated by certain $q$-root vectors so that $\mathbf{U}^+[w_0] = \mathbf{U}^+$; see [Ja95, 8.24]. As we identify $\widetilde{\mathbf{U}}^+ = \mathbf{U}^+$, we denote by $\widetilde{\mathbf{U}}^+[w]$ the subalgebra of $\widetilde{\mathbf{U}}^+$ corresponding to $\mathbf{U}^+[w]$. The next lemma is valid for all Satake diagrams.

**Lemma 5.1.** *For $i \neq j \in \mathbb{I}_{\circ,\tau}$, we have*

$$F_j\widetilde{\Upsilon}_i = \widetilde{\Upsilon}_i F_j, \tag{5.1}$$

$$\widetilde{\mathcal{T}}^{-1}_{\mathbf{r}_i}\big(\widetilde{\mathcal{T}}_{w_\bullet}(E_{\tau j})K'_j\big)\cdot\widetilde{\Upsilon}_i = \widetilde{\Upsilon}_i\cdot\widetilde{\mathcal{T}}^{-1}_{\mathbf{r}_i}\big(\widetilde{\mathcal{T}}_{w_\bullet}(E_{\tau j})K'_j\big). \tag{5.2}$$

*Proof.* Write $\widetilde{\Upsilon}_i = \sum_{m\geq 0} \widetilde{\Upsilon}_{i,m}$, where $\widetilde{\Upsilon}_{i,m} \in \widetilde{\mathbf{U}}^+_{m(\alpha_i+w_\bullet\alpha_{\tau i})}$. By [BW18b, Proposition 4.5], we have $\widetilde{\Upsilon}_{i,m} \in \widetilde{\mathbf{U}}^+[\mathbf{r}_i]$, for $m \geq 0$. Since the simple reflection $s_j$ does not appear in any reduced expression of $\mathbf{r}_i$, $F_j$ commutes with any element in $\widetilde{\mathbf{U}}^+[\mathbf{r}_i]$; in particular, $F_j$ commutes with $\widetilde{\Upsilon}_i$. This proves the identity (5.1).

By Proposition 3.8, $\widetilde{\Upsilon}$ is fixed by $\widehat{\tau}_{\bullet,i}$ (which is equal to either Id or $\widehat{\tau}$). Hence, by Lemma 4.4 and the fact that $\widetilde{\Upsilon}_{i,m} \in \widetilde{\mathbf{U}}^+_{m(\alpha_i+w_\bullet\alpha_{\tau i})}$, we have

$$\widetilde{\mathcal{T}}_{w_\bullet,i}(\widetilde{\Upsilon}_{i,m}) = \widetilde{\mathcal{T}}_{w_\bullet,i}\widehat{\tau}_{\bullet,i}(\widetilde{\Upsilon}_{i,m}) \in \widetilde{\mathbf{U}}^-_{-m(\alpha_i+w_\bullet\alpha_{\tau i})} K'^{-m}_{\alpha_i+w_\bullet\alpha_{\tau i}},$$

or equivalently,

$$\mathcal{F} := \widetilde{\mathcal{T}}_{w_\bullet,i}(\widetilde{\Upsilon}_{i,m}) K'^{m}_{\alpha_i+w_\bullet\alpha_{\tau i}} \in \widetilde{\mathbf{U}}^-_{-m(\alpha_i+w_\bullet\alpha_{\tau i})} \subset \widetilde{\mathbf{U}}^-[\mathbf{r}_i]. \tag{5.3}$$

Since the simple reflection $s_{\tau j}$ does not appear in any reduced expression of $\mathbf{r}_i$, $E_{\tau j}$ commutes with any element in $\widetilde{\mathbf{U}}^-[\mathbf{r}_i]$; in particular, we have by (5.3) that $[E_{\tau j}, \mathcal{F}] = 0$. For each $m$, we compute

$$\begin{aligned}
&\big[E_{\tau j}\widetilde{\mathcal{T}}_{w_\bullet}(K'_j), \widetilde{\mathcal{T}}_{w_\bullet,i}(\widetilde{\Upsilon}_{i,m})\big] \\
&= \big[E_{\tau j}\widetilde{\mathcal{T}}_{w_\bullet}(K'_j), \mathcal{F}K'^{-m}_{\alpha_i+w_\bullet(\alpha_{\tau i})}\big] \\
&= q^{(w_\bullet\alpha_j,\alpha_i+w_\bullet\alpha_{\tau i})} E_{\tau j}\mathcal{F}\widetilde{\mathcal{T}}_{w_\bullet}(K'_j)K'^{-m}_{\alpha_i+w_\bullet\alpha_i} - q^{(\alpha_{\tau j},\alpha_i+w_\bullet\alpha_{\tau i})}\mathcal{F}E_{\tau j}\widetilde{\mathcal{T}}_{w_\bullet}(K'_j)K'^{-m}_{\alpha_i+w_\bullet\alpha_i} \\
&= q^{(\alpha_{\tau j},\alpha_i+w_\bullet\alpha_{\tau i})}[E_{\tau j},\mathcal{F}] \cdot \widetilde{\mathcal{T}}_{w_\bullet}(K'_j)K'^{-m}_{\alpha_i+w_\bullet\alpha_i} = 0.
\end{aligned}$$

Hence we obtain an identity

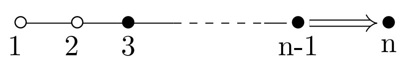
$$E_{\tau j}\widetilde{\mathcal{T}}_{w_\bullet}(K'_j) \cdot \widetilde{\mathcal{T}}_{w_\bullet,i}(\widetilde{\Upsilon}_i) = \widetilde{\mathcal{T}}_{w_\bullet,i}(\widetilde{\Upsilon}_i) \cdot E_{\tau j}\widetilde{\mathcal{T}}_{w_\bullet}(K'_j). \tag{5.4}$$

The desired identity (5.2) follows by applying $\widetilde{\mathcal{T}}_{\mathbf{r}_i}^{-1}\widetilde{\mathcal{T}}_{w_\bullet}$ to (5.4). $\square$

## 5.2 Motivating examples: types BI, DI, $\mathrm{DIII}_4$

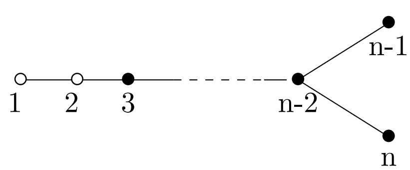
We provide examples in this subsection to motivate how we obtain the general rank two formulas $\widetilde{\mathbf{T}}'_{i,-1}(B_j)$ in Theorem 5.5 below. The three examples are of types $\mathrm{BI}_n$ $(n \geq 3)$, $\mathrm{DI}_n$ $(n \geq 5)$, $\mathrm{DIII}_4$, and they will be treated uniformly.

The Satake diagrams of these types are listed below. For each type, we define elements $t_j \in W_\bullet$ for $j \in \mathbb{I}_\bullet$ following each diiagram; these notations $t_j$ allow a uniform proof of Lemma 5.2.



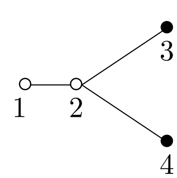

$\mathrm{BI}_n, n \geq 3$

$$t_a = s_a \cdots s_n \cdots s_a, \qquad (3 \leq a \leq n).$$




$\mathrm{DI}_n, n \geq 5$

$$t_a = s_{a\cdots n-2} s_{n-1} s_n s_{n-2\cdots a}, \qquad (3 \leq a \leq n-2), \qquad t_{n-1} = t_n = s_{n-1}s_n.$$




$\mathrm{DIII}_4$

$$t_3 = t_4 = s_3 s_4.$$

Note that, for each of the three types, we always have

$$\mathbf{r}_2 = s_2 t_3 s_2, \qquad \ell(\mathbf{r}_2) = \ell(t_3) + 2, \qquad B_1 = F_1 + E_1 K_1'.$$

Recall the notation $B_i^\sigma$ from (3.17).

**Lemma 5.2.** *We have*

$$\widetilde{\mathcal{T}}_{\mathbf{r}_2}^{-1}(F_1) = \big[\widetilde{\mathcal{T}}_{w_\bullet}(B_2^\sigma), [B_2^\sigma, F_1]_{q_2}\big]_{q_2} - q_2 F_1 \widetilde{\mathcal{T}}_{w_\bullet}(K_2) K_2'. \tag{5.5}$$

*Proof.* By Lemma 1.1, $[B_2^\sigma, F_1]_{q_2} = [F_2, F_1]_{q_2}$, and RHS (5.5) is simplified as follows:

$$\begin{aligned}
\big[\widetilde{\mathcal{T}}_{w_\bullet}(B_2^\sigma), [B_2^\sigma, F_1]_{q_2}\big]_{q_2} &= \big[\widetilde{\mathcal{T}}_{w_\bullet}(B_2^\sigma), [F_2, F_1]_{q_2}\big]_{q_2} \\
&= \big[\widetilde{\mathcal{T}}_{w_\bullet}(F_2), [F_2, F_1]_{q_2}\big]_{q_2} + \big[E_2\widetilde{\mathcal{T}}_{w_\bullet}(K_2), [F_2, F_1]_{q_2}\big]_{q_2} \\
&= \big[\widetilde{\mathcal{T}}_{w_\bullet}(F_2), [F_2, F_1]_{q_2}\big]_{q_2} + q_2 F_1 \widetilde{\mathcal{T}}_{w_\bullet}(K_2) K_2'.
\end{aligned} \tag{5.6}$$

On the other hand, by a direct computation using Proposition 4.2, we have

$$\begin{aligned}
\widetilde{\mathcal{T}}_{\mathbf{r}_2}^{-1}(F_1) &= \widetilde{\mathcal{T}}_2^{-1}\widetilde{\mathcal{T}}_{t_3}^{-1}([F_2, F_1]_{q_2}) \\
&= \big[\widetilde{\mathcal{T}}_2^{-1}\widetilde{\mathcal{T}}_{t_3}^{-1}(F_2), [F_2, F_1]_{q_2}\big]_{q_2} \\
&= \big[\widetilde{\mathcal{T}}_{t_3}(F_2), [F_2, F_1]_{q_2}\big]_{q_2}.
\end{aligned} \tag{5.7}$$

The desired formula (5.5) follows from (5.6)–(5.7) by noting that $\widetilde{\mathcal{T}}_{w_\bullet}(F_2) = \widetilde{\mathcal{T}}_{t_3}(F_2)$.

□

Note that $q_1 = q_2$ in all three types.

**Lemma 5.3.** *We have*

$$\widetilde{\mathcal{T}}_{\mathbf{r}_2}^{-1}(E_1K_1') = \big[\widetilde{\mathcal{T}}_{w_\bullet}(B_2), [B_2, E_1K_1']_{q_2}\big]_{q_2} - q_2 E_1 K_1' \widetilde{\mathcal{T}}_{w_\bullet}(K_2) K_2'. \tag{5.8}$$

*Proof.* We shall establish the identity (5.8) by applying the operator $\mathcal{D} := \widetilde{\mathcal{T}}_{w_\bullet}\widetilde{\mathcal{T}}_{w_0}$ to (5.5) as follows.

Recall $\mathcal{K}_i \in \widetilde{\mathbf{U}}^{\imath 0}$ from (3.23). By the formula (4.17) in Theorem 4.14 and noting $(\alpha_2, w_\bullet \alpha_{\tau 2}) = 0$ in each of the three types, we have $\widetilde{\mathcal{T}}_{\mathbf{r}_2}^{-1}(B_2^\imath) = -\widetilde{\mathcal{T}}_{w_\bullet}^2(B_2^\sigma)\mathcal{K}_2^{-1}$, or equivalently,

$$\widetilde{\mathcal{T}}_{\mathbf{r}_2}^{-1}(B_2^\imath)\mathcal{K}_2 = -\widetilde{\mathcal{T}}_{w_\bullet}^2(B_2^\sigma). \tag{5.9}$$

By Lemma 4.4, we have $\widetilde{\mathcal{T}}_{w_0}(B_2^\sigma) = \widetilde{\mathcal{T}}_{w_{\bullet,2}}(B_2^\sigma)$. Hence, applying $\widetilde{\mathcal{T}}_{\mathbf{r}_2}$ to both sides of (5.9) we obtain

$$B_2^\imath \widetilde{\mathcal{T}}_{\mathbf{r}_2}(\mathcal{K}_2^\imath) = -\widetilde{\mathcal{T}}_{w_\bullet}\widetilde{\mathcal{T}}_{w_{\bullet,2}}(B_2^\sigma) = -\widetilde{\mathcal{T}}_{w_\bullet}\widetilde{\mathcal{T}}_{w_0}(B_2^\sigma). \tag{5.10}$$

Moreover, by Lemma 4.4, we have $\mathcal{D}(F_1) = -K_1^{-1}E_1 = -q_2{}^{-2}E_1K_1'\widetilde{k}_1^{-1}$. Note also that $\mathcal{D}$ commutes with both $\widetilde{\mathcal{T}}_{w_\bullet}$ and $\widetilde{\mathcal{T}}_{\mathbf{r}_2}$. Hence, by applying $\mathcal{D}$ to (5.5) and then using (5.10), we have

$$\begin{aligned}\widetilde{\mathcal{T}}_{\mathbf{r}_2}^{-1}(E_1K_1')\widetilde{\mathcal{T}}_{\mathbf{r}_2}(\widetilde{k}_1^{-1}) = \big[&\widetilde{\mathcal{T}}_{w_\bullet}(B_2)\widetilde{\mathcal{T}}_{w_{\bullet,2}}(\mathcal{K}_2), [B_2\widetilde{\mathcal{T}}_{\mathbf{r}_2}(\mathcal{K}_2), E_1K_1'\widetilde{k}_1^{-1}]_{q_2}\big]_{q_2} \\ &- q_2 E_1 K_1' \widetilde{k}_1^{-1} \widetilde{\mathcal{T}}_{w_\bullet}\mathcal{D}(\mathcal{K}_2).\end{aligned} \tag{5.11}$$

For weight reason, (5.11) is simplified as

$$\widetilde{\mathcal{T}}_{\mathbf{r}_2}^{-1}(E_1K_1')\widetilde{\mathcal{T}}_{\mathbf{r}_2}(\widetilde{k}_1^{-1}) = q_2{}^2\big[\widetilde{\mathcal{T}}_{w_\bullet}(B_2), [B_2, E_1K_1']_{q_2}\big]_{q_2}\widetilde{\mathcal{T}}_{w_{\bullet,2}}(\mathcal{K}_2)\widetilde{\mathcal{T}}_{\mathbf{r}_2}(\mathcal{K}_2)\widetilde{k}_1^{-1}$$

$$- q_2 E_1 K_1' \widetilde{k}_1^{-1} \widetilde{\mathfrak{T}}_{\mathbf{r}_2}(\widetilde{k}_2) K_{w_\bullet \alpha_2 - \alpha_2}. \tag{5.12}$$

By definition (3.23), we have $\mathcal{K}_2 = \widetilde{k}_2 K'_{w_\bullet \alpha_2 - \alpha_2}$; in addition, by (4.10), $K'_{w_\bullet \alpha_2 - \alpha_2}$ is fixed by $\widetilde{\mathfrak{T}}_{\mathbf{r}_2}$. We also have $\widetilde{\mathfrak{T}}_{w_{\bullet,2}}(\mathcal{K}_2) = q_2{}^{-2} \mathcal{K}_2^{-1}$. Hence, (5.12) is further simplified as

$$\begin{aligned}\widetilde{\mathfrak{T}}_{\mathbf{r}_2}^{-1}(E_1 K_1') \widetilde{\mathfrak{T}}_{\mathbf{r}_2}(\widetilde{k}_1^{-1}) = \big[\widetilde{\mathfrak{T}}_{w_\bullet}(B_2), [B_2, E_1 K_1']_{q_2}\big]_{q_2} \widetilde{k}_2^{-1} \widetilde{\mathfrak{T}}_{\mathbf{r}_2}(\widetilde{k}_2) \widetilde{k}_1^{-1} \\ - q_2 E_1 K_1' K_{w_\bullet(\alpha_2)} K_2' \widetilde{k}_1^{-1} \widetilde{\mathfrak{T}}_{\mathbf{r}_2}(\widetilde{k}_2) \widetilde{k}_2^{-1}.\end{aligned} \tag{5.13}$$

Finally, by Lemma 4.10, we have $\widetilde{\mathfrak{T}}_{\mathbf{r}_2}(\widetilde{k}_1^{-1}) = \widetilde{k}_1^{-1} \widetilde{\mathfrak{T}}_{\mathbf{r}_2}(\widetilde{k}_2) \widetilde{k}_2^{-1}$, and then the identity (5.13) can be transformed into an equivalent form (5.8). □

**Proposition 5.4.** *The following element*

$$\widetilde{\mathbf{T}}'_{2,-1}(B_1) := \big[\widetilde{\mathfrak{T}}_{w_\bullet}(B_2), [B_2, B_1]_{q_2}\big]_{q_2} - q_2 B_1 \widetilde{\mathfrak{T}}_{w_\bullet}(\mathcal{K}_2) \in \widetilde{\mathbf{U}}^\imath \tag{5.14}$$

*satisfies the intertwining relation* $\widetilde{\mathbf{T}}'_{2,-1}(B_1) \widetilde{\Upsilon}_2 = \widetilde{\Upsilon}_2 \widetilde{\mathfrak{T}}_{\mathbf{r}_2}^{-1}(B_1^\imath)$ *(i.e., (4.6), for* $i = 2, x = B_1$*).*

*Proof.* The intertwining relation follows by the following computation:

$$\begin{aligned}
&\widetilde{\Upsilon}_2 \widetilde{\mathfrak{T}}_{\mathbf{r}_2}^{-1}(B_1^\imath) \widetilde{\Upsilon}_2^{-1} \\
&= \widetilde{\Upsilon}_2 \big(\widetilde{\mathfrak{T}}_{\mathbf{r}_2}^{-1}(F_1) + \widetilde{\mathfrak{T}}_{\mathbf{r}_2}^{-1}(E_1 K_1')\big) \widetilde{\Upsilon}_2^{-1} \\
&\overset{(5.1)}{=} \widetilde{\Upsilon}_2 \widetilde{\mathfrak{T}}_{\mathbf{r}_2}^{-1}(F_1) \widetilde{\Upsilon}_2^{-1} + \widetilde{\mathfrak{T}}_{\mathbf{r}_2}^{-1}(E_1 K_1') \\
&\overset{(5.5)}{=} \widetilde{\Upsilon}_2 \Big( \big[\widetilde{\mathfrak{T}}_{w_\bullet}(B_2^\sigma), [B_2^\sigma, F_1]_{q_2}\big]_{q_2} - q_2 F_1 \widetilde{\mathfrak{T}}_{w_\bullet}(K_2) K_2' \Big) \widetilde{\Upsilon}_2^{-1} + \widetilde{\mathfrak{T}}_{\mathbf{r}_2}^{-1}(E_1 K_1') \\
&\overset{(*)}{=} \big[\widetilde{\mathfrak{T}}_{w_\bullet}(B_2), [B_2, F_1]_{q_2}\big]_{q_2} - q_2 F_1 \widetilde{\mathfrak{T}}_{w_\bullet}(K_2) K_2' + \widetilde{\mathfrak{T}}_{\mathbf{r}_2}^{-1}(E_1 K_1') \\
&\overset{(5.8)}{=} \big[\widetilde{\mathfrak{T}}_{w_\bullet}(B_2), [B_2, F_1]_{q_2}\big]_{q_2} - q_2 F_1 \widetilde{\mathfrak{T}}_{w_\bullet}(K_2) K_2'
\end{aligned}$$

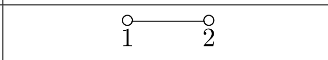
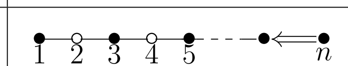
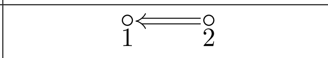
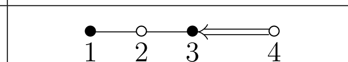
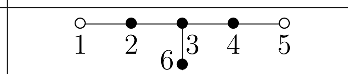
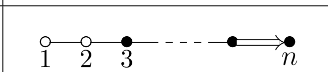
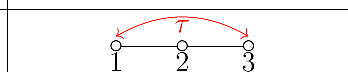
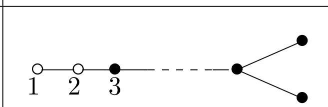
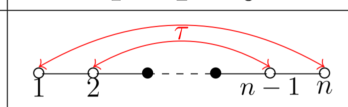
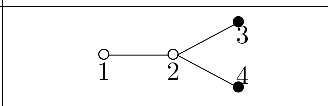
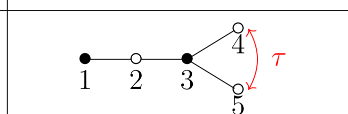
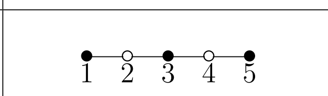
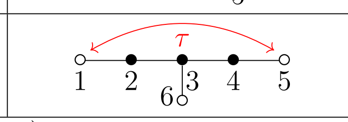
$$+\left[\widetilde{\mathcal{T}}_{w_\bullet}(B_2), [B_2, E_1 K_1']_{q_2}\right]_{q_2} - q_2 E_1 K_1' \widetilde{\mathcal{T}}_{w_\bullet}(K_2) K_2'$$

$$= \left[\widetilde{\mathcal{T}}_{w_\bullet}(B_2), [B_2, B_1]_{q_2}\right]_{q_2} - q_2 B_1 \widetilde{\mathcal{T}}_{w_\bullet}(K_2) K_2' = \widetilde{\mathbf{T}}'_{2,-1}(B_1),$$

where the equality (*) follows from Theorem 3.6 and Lemma 5.1. □

## 5.3 Formulation for $\widetilde{\mathbf{T}}'_{i,-1}(B_j)$

Table 2: Rank two Satake diagrams

| SP | Satake diagrams | RS | SP | Satake diagrams | RS |
|---|---|---|---|---|---|
| $AI_2$ |  | $A_2$ | $CII_n$ |  | $BC_2$ |
| $CI_2$ |  | $C_2$ | $CII_4$ |  | $C_2$ |
| $G_2$ |  | $G_2$ | EIV |  | $A_2$ |
| $BI_n$ |  | $B_2$ | $AIII_3$ |  | $C_2$ |
| $DI_n$ |  | $B_2$ | $AIII_n$ |  | $BC_2$ |
| $DIII_4$ |  | $C_2$ | $DIII_5$ |  | $BC_2$ |
| $AII_5$ |  | $A_2$ | EIII |  | $BC_2$ |

(SP=symmetric pair, RS=relative root system)

**Theorem 5.5.** *The elements $\widetilde{\mathbf{T}}'_{i,-1}(B_j) \in \widetilde{\mathbf{U}}^\imath$ listed in Table 3 satisfy the following intertwining relation (see (4.6)):*

$$\widetilde{\mathbf{T}}'_{i,-1}(B_j)\widetilde{\Upsilon}_i = \widetilde{\Upsilon}_i \widetilde{\mathcal{T}}'_{\mathbf{r}_i,-1}(B_j). \tag{5.15}$$

We clarify a few points regarding Table 3 in the following remarks.

*Remark* 5.6. Recall that $\widetilde{\mathcal{T}}_s$ ($s \in \mathbb{I}_\bullet$) restrict to automorphisms on $\widetilde{\mathbf{U}}^\imath$ by Proposition 4.5; hence, the use of $\widetilde{\mathcal{T}}_s$ ($s \in \mathbb{I}_\bullet$) in the formulas of $\widetilde{\mathbf{T}}'_{i,-1}(B_j)$ is legitimate; see (4.5).

*Remark* 5.7. Let $\rho$ be a diagram involution on the underlying Dynkin diagram ($\rho$ is not necessarily equal to $\tau$). By the intertwining relation (4.6), the formula of $\widetilde{\mathbf{T}}'_{\rho i,-1}(B_{\rho j})$ can be obtained from $\widetilde{\mathbf{T}}'_{i,-1}(B_j)$ via

$$\widetilde{\mathbf{T}}'_{\rho i,-1}(B_{\rho j}) = \rho\big(\widetilde{\mathbf{T}}'_{i,-1}(B_j)\big).$$

In particular, when $\rho = \tau$, we have $\widetilde{\mathbf{T}}'_{i,-1}(B_{\tau j}) = \widehat{\tau}\big(\widetilde{\mathbf{T}}'_{i,-1}(B_j)\big)$ by Remark 4.8. Accordingly, only one formula of $\widetilde{\mathbf{T}}'_{\rho i,-1}(B_{\rho j})$ and $\widetilde{\mathbf{T}}'_{i,-1}(B_j)$ is included in the table; see types $\mathrm{AII}_5$, EIV, and all types with $\tau \neq \mathrm{Id}$.

*Remark* 5.8. The formulas of $\widetilde{\mathbf{T}}'_{i,-1}(B_j)$ only depends on the subdiagram generated by vertices $i, \tau i, j$ and the component of black nodes which is connected to either $i$ or $\tau i$. For example, the formula for $\widetilde{\mathbf{T}}'_{2,-1}(B_4)$ in type $\mathrm{DIII}_5$ is formally identical to the formula for $\widetilde{\mathbf{T}}'_{2,-1}(B_4)$ in type $\mathrm{AII}_5$. (Note that such a subdiagram may not be a Satake subdiagram as the vertex $\tau j$ is not included.)

Recall $\widetilde{\mathbf{U}}^\imath$ is defined over an extension field $\mathbb{F}$ of $\mathbb{Q}(q)$. Denote

$$_{\mathbb{Q}}\widetilde{\mathbf{U}}^\imath := \ \mathbb{Q}(q)\text{-subalgebra of } \widetilde{\mathbf{U}}^\imath \text{ generated by } B_i, \widetilde{k}_i, x \text{ for } i \in \mathbb{I}_\circ, x \in \widetilde{\mathcal{G}}_\bullet. \tag{5.16}$$

**Proposition 5.9.** *The symmetries $\widetilde{\mathbf{T}}'_{i,-1}$ ($i \in \mathbb{I}_\circ$) preserve the $\mathbb{Q}(q)$-algebra $_{\mathbb{Q}}\widetilde{\mathbf{U}}^\imath$.*

*Proof.* This follows by the formula for $\widetilde{\mathbf{T}}'_{i,-1}$ acting on the Cartan part in Proposition 4.11 (see Lemma 4.12), the rank one formulas in (4.17), and the rank 2 formulas in Table 3. □

*Remark* 5.10. It would cause no difficulty if we have replaced $\widetilde{\mathbf{U}}^\imath$ (over $\mathbb{F}$) by ${}_{\mathbb{Q}}\widetilde{\mathbf{U}}^\imath$ over $\mathbb{Q}(q)$ throughout the dissertation. We need to work with $\widetilde{\mathbf{U}}$ over $\mathbb{Q}(q^{\frac{1}{2}})$ in several places. The results for $\mathbf{U}^\imath_{\boldsymbol{\varsigma}_\diamond}$ will be valid over $\mathbb{Q}(q)$, while some results over $\mathbf{U}^\imath_{\boldsymbol{\varsigma}}$, for $\boldsymbol{\varsigma}$ over $\mathbb{Q}(q)$, are valid over $\mathbb{Q}(q^{\frac{1}{2}})$.

## 5.4 Proof of Theorem 5.5

**Proposition 5.11.** *Let $i \neq j \in \mathbb{I}_{\circ,\tau}$ be such that $j \notin \{i, \tau i\}$. Then there exists a non-commutative polynomial $R_{ij}(x_i, x_{\tau i}, y_i, y_{\tau i}, z; \widetilde{\mathcal{G}}_\bullet)$, which is linear in $z$, such that*

1. $\widetilde{\mathfrak{T}}_{\mathbf{r}_i}^{-1}(F_j) = R_{ij}(B_i^\sigma, B_{\tau i}^\sigma, \mathcal{K}_i, \mathcal{K}_{\tau i}, F_j; \widetilde{\mathcal{G}}_\bullet)$;
2. $\widetilde{\mathfrak{T}}_{\mathbf{r}_i}^{-1}\big(\widetilde{\mathfrak{T}}_{w_\bullet}(E_{\tau j})K_j'\big) = R_{ij}(B_i, B_{\tau i}, \mathcal{K}_i, \mathcal{K}_{\tau i}, \widetilde{\mathfrak{T}}_{w_\bullet}(E_{\tau j})K_j'; \widetilde{\mathcal{G}}_\bullet)$.

*Remark* 5.12. In case $\tau i = i$, the polynomials $R_{ij}$ depend only on $x_i, y_i, z$ and $\widetilde{\mathcal{G}}_\bullet$. In this case, it is understood in Proposition 5.11 that $R_{ij}(x_i, x_{\tau i}, y_i, y_{\tau i}, z; \widetilde{\mathcal{G}}_\bullet)$ is replaced by $R_{ij}(x_i, y_i, z; \widetilde{\mathcal{G}}_\bullet)$ (which is linear in $z$), and $R_{ij}(B_i^\sigma, B_{\tau i}^\sigma, \mathcal{K}_i, \mathcal{K}_{\tau i}, F_j; \widetilde{\mathcal{G}}_\bullet)$ is replaced by $R_{ij}(B_i^\sigma, \mathcal{K}_i, F_j; \widetilde{\mathcal{G}}_\bullet)$, and so on.

The proof of Proposition 5.11 will be carried out through type-by-type computation in Appendix A.

We define

$$\widetilde{\mathbf{T}}'_{i,-1}(B_j) := \begin{cases} R_{ij}(B_i, \mathcal{K}_i, B_j; \widetilde{\mathcal{G}}_\bullet), & \text{if } i = \tau i, \\ R_{ij}(B_i, B_{\tau i}, \mathcal{K}_i, \mathcal{K}_{\tau i}, B_j; \widetilde{\mathcal{G}}_\bullet) & \text{if } i \neq \tau i. \end{cases} \tag{5.17}$$

Clearly, we have $\widetilde{\mathbf{T}}'_{i,-1}(B_j) \in \widetilde{\mathbf{U}}^\imath$; see Table 3.

The polynomials $R_{ij}$ in all types can be read off from Table 3. For instance, in type $\mathrm{AII}_5$ it reads as follows:

$$R_{ij}(x,y,z;\widetilde{\mathcal{G}}_\bullet) = \big[[x,F_3],z\big]_q.$$

In order to read $R_{ij}$ off from Table 3, one first needs to unravel $\widetilde{\mathfrak{T}}_w$, for $w \in W_\bullet$, appearing in those formulas in terms of $E_j, F_j, K_j, K_j', j \in \mathbb{I}_\bullet$.

*Proof of Theorem 5.5.* We start with a general comment. Originally, we computed the explicit formulas in Table 3 type by type; see §5.2 for examples in types BI, DI, and $\mathrm{DIII}_4$. In the process, we observed that parts of the arguments can be streamlined a uniform formulation in Proposition 5.11, even though its proof requires quite some computations. We hope this uniform formulation helps to conceptualize the structures of the formulas for $\widetilde{\mathfrak{T}}_{\mathbf{r}_i}^{-1}(B_j)$.

We now prove Theorem 5.5 using Proposition 5.11. Recall by Theorem 3.6 that $\widetilde{\Upsilon}_i B_i^\sigma \widetilde{\Upsilon}_i^{-1} = B_i$ and $\widetilde{\Upsilon}_i x \widetilde{\Upsilon}_i^{-1} = x$ for $x \in \widetilde{\mathbf{U}}^{\imath 0}\widetilde{\mathbf{U}}_\bullet$.

For definiteness, let us assume that $i \neq \tau i$. (The case when $i = \tau i$ is similar using the interpretation of notation in Remark 5.12.) By Lemma 5.1, Proposition 5.11 and definition of $\widetilde{\mathbf{T}}'_{i,-1}(B_j)$ in (5.17), we have

$$\begin{aligned}
\widetilde{\Upsilon}_i\widetilde{\mathfrak{T}}_{\mathbf{r}_i}^{-1}(B_j) &= \widetilde{\Upsilon}_i\widetilde{\mathfrak{T}}_{\mathbf{r}_i}^{-1}(F_j) + \widetilde{\Upsilon}_i\widetilde{\mathfrak{T}}_{\mathbf{r}_i}^{-1}(\widetilde{\mathfrak{T}}_{w_\bullet}(E_{\tau j})K_j') \\
&= \widetilde{\Upsilon}_i\widetilde{\mathfrak{T}}_{\mathbf{r}_i}^{-1}(F_j) + \widetilde{\mathfrak{T}}_{\mathbf{r}_i}^{-1}(\widetilde{\mathfrak{T}}_{w_\bullet}(E_{\tau j})K_j')\widetilde{\Upsilon}_i \\
&= \widetilde{\Upsilon}_i R_{ij}(B_i^\sigma, B_{\tau i}^\sigma, \mathcal{K}_i, \mathcal{K}_{\tau i}, F_j; \widetilde{\mathcal{G}}_\bullet) + R_{ij}(B_i, B_{\tau i}, \mathcal{K}_i, \mathcal{K}_{\tau i}, \widetilde{\mathfrak{T}}_{w_\bullet}(E_{\tau j})K_j'; \widetilde{\mathcal{G}}_\bullet)\widetilde{\Upsilon}_i \\
&= R_{ij}(B_i, B_{\tau i}, \mathcal{K}_i, \mathcal{K}_{\tau i}, F_j; \widetilde{\mathcal{G}}_\bullet)\widetilde{\Upsilon}_i + R_{ij}(B_i, B_{\tau i}, \mathcal{K}_i, \mathcal{K}_{\tau i}, \widetilde{\mathfrak{T}}_{w_\bullet}(E_{\tau j})K_j'; \widetilde{\mathcal{G}}_\bullet)\widetilde{\Upsilon}_i \\
&= R_{ij}(B_i, B_{\tau i}, \mathcal{K}_i, \mathcal{K}_{\tau i}, B_j; \widetilde{\mathcal{G}}_\bullet)\widetilde{\Upsilon}_i = \widetilde{\mathbf{T}}'_{i,-1}(B_j)\widetilde{\Upsilon}_i,
\end{aligned}$$

where the second last step follows form the linearity of $R_{ij}$ in its fifth component.

This proves the desired identity (5.15), whence the theorem. □

**Conjecture 5.13.** For $\widetilde{\mathbf{U}}^\imath$ of Kac-Moody type, Proposition 5.11 remains valid.

Assume Conjecture 5.13 holds. Then $\widetilde{\mathbf{T}}'_{i,-1}(B_j) \in \widetilde{\mathbf{U}}^\imath$ defined in (5.17) satisfies the intertwining relation (5.15), and hence, $\widetilde{\mathbf{T}}'_{i,-1}$ is a symmetry of $\widetilde{\mathbf{U}}^\imath$ of Kac-Moody type.

## 5.5 A comparison with earlier results

We compare our formulas with some special cases obtained in the literature.

By choosing a reduced expression of $w_\bullet$, we can write out the formula (4.17) explicitly for rank one Satake diagrams in Table 1. We list some explicit formulas of $\widetilde{\mathbf{T}}'_{i,-1}(B_i)$ and compare them with braid group actions obtained earlier in [LW21a],[Dob20],[KP11]. (The index $i$ is specified in each case.) In some rank 2 cases, our formulas differ from those in [LW21a] and they can be matched by some twisting. As noted in [LW21a, Remark 7.4], the formulas for braid operators in [KP11] may involve $\sqrt{v}$ and are related to those in [LW21a] by some other twisting.

### Type $\mathrm{AI}_1$

We shall label the single white node in rank 1 type AI by 1. In this case, the formula (4.17) reads as follows:

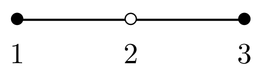
$$\widetilde{\mathbf{T}}'_{1,-1}(B_1) = -q^{-2} B_1 \mathcal{K}_1^{-1} = -q^{-2} B_1 \widetilde{k}_1^{-1}. \tag{5.18}$$

Note also that $\varsigma_{1,\diamond} = -q^{-2}$. Applying the central reduction $\pi^\imath_{\boldsymbol{\varsigma}_\diamond}$ to (5.18), we have $\mathbf{T}^{-1}_{1,\diamond}(B_1) = B_1 \in \mathbf{U}^\imath_{\boldsymbol{\varsigma}_\diamond}$. Our formula (5.18) of $\widetilde{\mathbf{T}}'_{1,-1}(B_1)$ coincides with the formula

$\mathrm{T}_i^{-1}(B_i)$ in [LW21a, Lemma 5.1]. Our formulation of $\mathbf{T}_{1,\diamond}^{-1}(B_1)$ coincides with the formula $\tau_i^{-1}(B_i)$ given in [KP11, (3.1)] for $(\mathbf{U}, \mathbf{U}^\imath_{q^{-2}})$.

### Type $\mathrm{AII}_3$

The rank 1 Satake diagram of type AII is given by

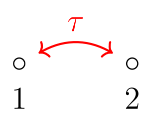



By Table 1, $\mathbf{r}_2 = s_{2132}$, and the formula (4.17) reads as follows

$$\begin{aligned}\widetilde{\mathbf{T}}'_{2,-1}(B_2) &= -q^{-2}(q-q^{-1})^2\big[[B_2,F_3]_q,F_1\big]_q E_3E_1\widetilde{k}_2^{-1}\\ &\quad + (q-q^{-1})\big([B_2,F_3]_qK_1E_3 + [B_2,F_1]_qK_3E_1\big)\widetilde{k}_2^{-1} - q^2B_2K_3K_1\widetilde{k}_2^{-1}.\end{aligned}$$

### Type $\mathrm{AIII}_{11}$

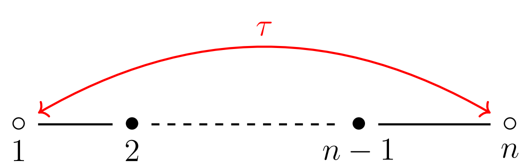
The $\mathrm{AIII}_{11}$ Satake diagram is given by




In this case, the formula (4.17) reads as $\widetilde{\mathbf{T}}'_{1,-1}(B_1) = -B_2\mathcal{K}_2^{-1} = -B_2\widetilde{k}_2^{-1}$.

### Type $\mathrm{AIII}_{11}$

The rank 1 AIV Satake diagram is given by



In this case, the formula (4.17) reads as $\widetilde{\mathbf{T}}'_{1,-1}(B_1) = -q\widetilde{\mathcal{T}}^2_{w_\bullet}(B_1)\mathcal{K}_1^{-1}\prod_{j\in\mathbb{I}_\bullet} K_j'^{-1}$.

*Remark* 5.14. For type AIV, Dobson [Dob20, Theorem 3.4] obtained a different automorphism $\mathcal{T}_1$ on $\mathbf{U}^\imath_{\boldsymbol{\varsigma}}$ such that $\mathcal{T}_1^{-1}(B_1) = qB_1k_nK_{\varpi_{n-1}-\varpi_2}$. Here $\varpi_j$ are the fundamental weights and $k_i$ is denoted by $L_i$ *loc. cit.*

### Split type

The formulas of $\widetilde{\mathbf{T}}'_{i,-1}(B_j)$ in the split types $\mathrm{AI}_2$, $\mathrm{CI}_2$ and $\mathrm{G}_2$ are identical to the braid group operators obtained using the $\imath$Hall algebra approach, cf. [LW21a, Lemma 5.1].

### Formulas on $\mathbf{U}^\imath_{\boldsymbol{\varsigma}_\diamond}$

Applying central reductions and isomorphisms $\phi_{\boldsymbol{\varsigma}} : \mathbf{U}^\imath_{\boldsymbol{\varsigma}_\diamond} \cong \mathbf{U}^\imath_{\boldsymbol{\varsigma}}$ (see §14.4 below) to our formulas, we recover various formulas obtained for $\mathbf{U}^\imath_{\boldsymbol{\varsigma}}$ in [KP11] in split types and type AII.

## 6 New symmetries $\widetilde{\mathbf{T}}''_{i,+1}$ on $\widetilde{\mathbf{U}}^\imath$

In this section, we introduce new symmetries $\widetilde{\mathbf{T}}''_{i,+1}$ on $\widetilde{\mathbf{U}}^\imath$, for $i \in \mathbb{I}_\circ$, via a new intertwining property using the quasi $K$-matrix, and establish explicit formulas of $\widetilde{\mathbf{T}}''_{i,+1}$ acting on the generators of $\widetilde{\mathbf{U}}^\imath$. Then we show that $\widetilde{\mathbf{T}}'_{i,-1}$ and $\widetilde{\mathbf{T}}''_{i,+1}$ are mutual inverses. (This in particular completes the proof of Theorem 4.7 that $\widetilde{\mathbf{T}}'_{i,-1}$ is an automorphism.)

## 6.1 Characterization of $\widetilde{\mathbf{T}}''_{i,+1}$

We formulate $\widetilde{\mathbf{T}}''_{i,+1}$ below, as a variant of $\widetilde{\mathbf{T}}'_{i,-1}$ introduced in Theorem 4.7.

**Theorem 6.1.** *Let $i \in \mathbb{I}_\circ$.*

1. *For any $x \in \widetilde{\mathbf{U}}^\imath$, there is a unique element $x'' \in \widetilde{\mathbf{U}}^\imath$ such that $x''\widetilde{\mathfrak{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_i^{-1}) = \widetilde{\mathfrak{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_i^{-1})\widetilde{\mathfrak{T}}_{\mathbf{r}_i}(x)$.*

2. *The map $x \mapsto x''$ defines an automorphism of the algebra $\widetilde{\mathbf{U}}^\imath$, denoted by $\widetilde{\mathbf{T}}''_{i,+1}$.*

The strategy of proving Theorem 6.1 is largely parallel to that of Theorem 4.7 given in the previous sections. We shall prove Theorem 6.1(1) and a weaker version of Part (2) that $x \mapsto x''$ defines an endomorphism $\widetilde{\mathbf{T}}''_{i,+1}$ of the algebra $\widetilde{\mathbf{U}}^\imath$, by combining Proposition 6.2, Proposition 6.3, and Theorem 6.6. Finally, we show that $\widetilde{\mathbf{T}}''_{i,+1}$ is an automorphism of $\widetilde{\mathbf{U}}^\imath$ in Theorem 6.7.

Hence $\widetilde{\mathbf{T}}''_{i,+1}$ satisfies the following intertwining relation:

$$\widetilde{\mathbf{T}}''_{i,+1}(x)\widetilde{\mathfrak{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_i^{-1}) = \widetilde{\mathfrak{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_i^{-1})\widetilde{\mathfrak{T}}_{\mathbf{r}_i}(x), \qquad \text{for all } x \in \widetilde{\mathbf{U}}^\imath. \tag{6.1}$$

## 6.2 Action of $\widetilde{\mathbf{T}}''_{i,+1}$ on $\widetilde{\mathbf{U}}^{\imath 0}\widetilde{\mathbf{U}}_\bullet$

Just as for Proposition 4.11, we can prove the following.

**Proposition 6.2.** *Let $i \in \mathbb{I}_\circ$. For each $x \in \widetilde{\mathbf{U}}^{\imath 0}\widetilde{\mathbf{U}}_\bullet$, there exists a unique element $\widetilde{\mathbf{T}}''_{i,+1}(x) \in \widetilde{\mathbf{U}}^{\imath 0}\widetilde{\mathbf{U}}_\bullet$ such that the intertwining relation $\widetilde{\mathbf{T}}''_{i,+1}(x)\widetilde{\mathfrak{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_i^{-1}) = \widetilde{\mathfrak{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_i^{-1})\widetilde{\mathfrak{T}}_{\mathbf{r}_i}(x)$ holds; see (6.1). More explicitly,*

$$\widetilde{\mathbf{T}}''_{i,+1}(u) = (\widehat{\tau}_{\bullet,i} \circ \widehat{\tau})(u), \quad \widetilde{\mathbf{T}}''_{i,+1}(\widetilde{k}_{j,\diamond}) = \widetilde{k}_{\mathbf{r}_i\alpha_j,\diamond}, \qquad \textit{for } u \in \widetilde{\mathbf{U}}_\bullet \textit{ and } j \in \mathbb{I}_\circ.$$

It follows by Proposition 4.11 and Proposition 6.2 that $\widetilde{\mathbf{T}}'_{i,-1}, \widetilde{\mathbf{T}}''_{i,+1}$, and $\widetilde{\mathfrak{T}}^{\pm 1}_{\mathbf{r}_i}$ coincide on $\widetilde{\mathbf{U}}^{\imath 0}\widetilde{\mathbf{U}}_{\bullet}$. In particular, we have

$$\widetilde{\mathbf{T}}''_{i,+1}(x) = (\sigma^{\imath} \circ \widetilde{\mathbf{T}}'_{i,-1} \circ \sigma^{\imath})(x), \qquad \text{for } x \in \widetilde{\mathbf{U}}^{\imath 0}\widetilde{\mathbf{U}}_{\bullet}. \tag{6.2}$$

## 6.3 Rank one formula for $\widetilde{\mathbf{T}}''_{i,+1}(B_i)$

We shall establish a uniform formula for $\widetilde{\mathbf{T}}''_{i,+1}(B_i)$, for $i \in \mathbb{I}_{\circ}$, a counterpart of Theorem 4.14. Recall the anti-involution $\sigma^{\imath}$ of $\widetilde{\mathbf{U}}^{\imath}$ from Proposition 3.12.

**Proposition 6.3.** *Let $i \in \mathbb{I}_{\circ}$. There exists a unique element $\widetilde{\mathbf{T}}''_{i,+1}(B_i) \in \widetilde{\mathbf{U}}^{\imath}$ which satisfies the following intertwining relation (see (6.1))*

$$\widetilde{\mathbf{T}}''_{i,+1}(B_i)\, \widetilde{\mathfrak{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_i)^{-1} = \widetilde{\mathfrak{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_i)^{-1}\, \widetilde{\mathfrak{T}}_{\mathbf{r}_i}(B_i). \tag{6.3}$$

*More explicitly, we have*

$$\widetilde{\mathbf{T}}''_{i,+1}(B_i) = -q^{-(\alpha_i,\alpha_i)} \widetilde{\mathfrak{T}}^{-2}_{w_\bullet}(B_{\tau_{\bullet,i}\tau i}) \widetilde{\mathfrak{T}}_{w_\bullet}(\mathcal{K}^{-1}_{\tau_{\bullet,i} i}). \tag{6.4}$$

*In particular, we have $\widetilde{\mathbf{T}}''_{i,+1}(B_i) = (\sigma^{\imath} \circ \widetilde{\mathbf{T}}'_{i,-1})(B_i)$.*

*Proof.* By Theorem 3.6, we have $B_i \widetilde{\Upsilon} = \widetilde{\Upsilon} B_i^{\sigma}$, which can be rewritten as

$$\widetilde{\mathfrak{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_i) \widetilde{\mathfrak{T}}_{\mathbf{r}_i}(B_i^{\sigma}) = \widetilde{\mathfrak{T}}_{\mathbf{r}_i}(B_i) \widetilde{\mathfrak{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_i). \tag{6.5}$$

Hence, by comparing (6.3) and (6.5) and then applying (2.14), we obtain that

$$\widetilde{\mathbf{T}}''_{i,+1}(B_i) = \widetilde{\mathfrak{T}}_{\mathbf{r}_i}(B_i^{\sigma}) = (\sigma \circ \widetilde{\mathfrak{T}}^{-1}_{\mathbf{r}_i})(B_i). \tag{6.6}$$

We now convert the formula (6.6) to the desired formula (6.4) for $\widetilde{\mathbf{T}}''_{i,+1}(B_i)$, which particularly shows that $\widetilde{\mathbf{T}}''_{i,+1}(B_i) \in \widetilde{\mathbf{U}}^\imath$. To that end, note that $\sigma(\mathcal{K}_{\tau_{\bullet,i}\tau i}) = \widetilde{\mathcal{T}}_{w_\bullet}(\mathcal{K}_{\tau_{\bullet,i}i})$, by Proposition 2.3 and definition (3.23) of $\mathcal{K}_i$. Applying $\sigma$ to the identity $\widetilde{\mathcal{T}}^{-1}_{\mathbf{r}_i}(B_i) = -q^{-(\alpha_i, w_\bullet \alpha_{\tau i})}\widetilde{\mathcal{T}}^2_{w_\bullet}(B^\sigma_{\tau_{\bullet,i}\tau i})\mathcal{K}^{-1}_{\tau_{\bullet,i}\tau i}$ in (4.16) and using (6.6), we have established the formula (6.4) for $\widetilde{\mathbf{T}}''_{i,+1}(B_i)$.

It remains to show that $\widetilde{\mathbf{T}}''_{i,+1}(B_i) = (\sigma^\imath \circ \widetilde{\mathbf{T}}'_{i,-1})(B_i)$. Since $\sigma^\imath$ satisfies (3.24), we have, for $i \in \mathbb{I}_\circ$,

$$\sigma^\imath(\mathcal{K}_i) = \sigma(\mathcal{K}_i), \qquad \sigma^\imath(B_i) = (B_i).$$

It follows by definition (3.23) of $\mathcal{K}_i$ that $\sigma^\imath(\mathcal{K}^{-1}_{\tau_{\bullet,i}\tau i}) = \widetilde{\mathcal{T}}_{w_\bullet}(\mathcal{K}^{-1}_{\tau_{\bullet,i}i})$.

Moreover, by Proposition 4.6, $\mathrm{Ad}_{\widetilde{\Upsilon}}$ commutes with $\widetilde{\mathcal{T}}_j$ for $j \in \mathbb{I}_\bullet$. By (3.25), $\sigma^\imath = \mathrm{Ad}_{\widetilde{\Upsilon}} \circ \sigma$ and hence we have

$$\sigma^\imath \widetilde{\mathcal{T}}_{w_\bullet} = \mathrm{Ad}_{\widetilde{\Upsilon}} \circ \sigma \circ \widetilde{\mathcal{T}}_{w_\bullet} = \widetilde{\mathcal{T}}^{-1}_{w_\bullet} \circ \mathrm{Ad}_{\widetilde{\Upsilon}} \circ \sigma = \widetilde{\mathcal{T}}^{-1}_{w_\bullet}\sigma^\imath.$$

Using the formula (4.17) for $\widetilde{\mathbf{T}}'_{i,-1}(B_i)$ and the formula (6.4) for $\widetilde{\mathbf{T}}''_{i,+1}(B_i)$, we compute

$$\begin{aligned}(\sigma^\imath \circ \widetilde{\mathbf{T}}'_{i,-1})(B_i) &= -q^{-(\alpha_i, w_\bullet \alpha_{\tau i})}\sigma^\imath(\mathcal{K}^{-1}_{\tau_{\bullet,i}\tau i})\sigma^\imath \widetilde{\mathcal{T}}^2_{w_\bullet}(B_{\tau_{\bullet,i}\tau i}) \\ &= -q^{-(\alpha_i,\alpha_i)}\widetilde{\mathcal{T}}^{-2}_{w_\bullet}(B_{\tau_{\bullet,i}\tau i})\widetilde{\mathcal{T}}_{w_\bullet}(\mathcal{K}^{-1}_{\tau_{\bullet,i}i}) = \widetilde{\mathbf{T}}''_{i,+1}(B_i).\end{aligned}$$

This completes the proof of the proposition. □

## 6.4 Rank two formulas for $\widetilde{\mathbf{T}}''_{i,+1}(B_j)$

The next lemma is a reformulation of Lemma 5.1.

**Lemma 6.4.** *We have*

*(1)* $\widetilde{\mathfrak{T}}_{\mathbf{r}_i}(F_j)$ *commutes with* $\widetilde{\mathfrak{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_i)$.

*(2)* $\widetilde{\mathfrak{T}}_{w_\bullet}(E_{\tau j})K'_j$ *commutes with* $\widetilde{\mathfrak{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_i)$.

Introduce a shorthand notation

$$\widehat{B}_i := \widetilde{\mathfrak{T}}_{\mathbf{r}_i}\big(\widetilde{\mathbf{T}}'_{i,-1}(B_i)\big). \tag{6.7}$$

We reformulate the intertwining relation (5.15) as

$$\widehat{B}_i \cdot \widetilde{\mathfrak{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_i) = \widetilde{\mathfrak{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_i) \cdot B_i. \tag{6.8}$$

**Proposition 6.5.** *Let* $i \neq j \in \mathbb{I}_{\circ,\tau}$ *be such that* $j \notin \{i, \tau i\}$. *Then there exists a non-commutative polynomial* $P_{ij}(x_i, x_{\tau i}, y_i, y_{\tau i}, z; \widetilde{\mathcal{G}}_\bullet)$, *which is linear in* $z$, *such that*

1. $\widetilde{\mathfrak{T}}_{\mathbf{r}_i}(F_j) = P(B_i, B_{\tau i}, \widetilde{k}_i, \widetilde{k}_{\tau i}, F_j; \widetilde{\mathcal{G}}_\bullet)$,
2. $\widetilde{\mathfrak{T}}_{\mathbf{r}_i}\big(\widetilde{\mathfrak{T}}_{w_\bullet}(E_{\tau j})K'_j\big) = P(\widehat{B}_i, \widehat{B}_{\tau i}, \widetilde{k}_i, \widetilde{k}_{\tau i}, \widetilde{\mathfrak{T}}_{w_\bullet}(E_{\tau j})K'_j; \widetilde{\mathcal{G}}_\bullet)$.

The proof of Proposition 6.5 is carried out through a type-by-type computation similar to Appendix A and will be postponed to Appendix B.

We set

$$\widetilde{\mathbf{T}}''_{i,+1}(B_j) := P(B_i, B_{\tau i}, \widetilde{k}_i, \widetilde{k}_{\tau i}, B_j; \widetilde{\mathcal{G}}_\bullet). \tag{6.9}$$

Clearly, we have $\widetilde{\mathbf{T}}''_{i,+1}(B_j) \in \widetilde{\mathbf{U}}^\imath$.

**Theorem 6.6.** *Let $i \neq j \in \mathbb{I}_{\circ,\tau}$. The elements $\widetilde{\mathbf{T}}''_{i,+1}(B_j)$ listed in Table 4 satisfy the following intertwining relation (see (6.1))*

$$\widetilde{\mathbf{T}}''_{i,+1}(B_j)\widetilde{\mathcal{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_i)^{-1} = \widetilde{\mathcal{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_i)^{-1}\widetilde{\mathcal{T}}_{\mathbf{r}_i}(B_j). \tag{6.10}$$

*Proof.* Recall $B_j = F_j + \widetilde{\mathcal{T}}_{w_\bullet}(E_{\tau j})K'_j$. By Lemma 6.4, (6.8) and (6.9), we have

$$\begin{aligned}
&\widetilde{\mathcal{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_i)^{-1} \cdot \widetilde{\mathcal{T}}_{\mathbf{r}_i}(B_j) \cdot \widetilde{\mathcal{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_i)\\
&= \widetilde{\mathcal{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_i)^{-1}\Big(\widetilde{\mathcal{T}}_{\mathbf{r}_i}(F_j) + \widetilde{\mathcal{T}}_{\mathbf{r}_i}\big(\widetilde{\mathcal{T}}_{w_\bullet}(E_{\tau j})K'_j\big)\Big)\widetilde{\mathcal{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_i)\\
&= \widetilde{\mathcal{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_i)^{-1}\widetilde{\mathcal{T}}_{\mathbf{r}_i}(F_j)\widetilde{\mathcal{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_i) + \widetilde{\mathcal{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_i)^{-1}\widetilde{\mathcal{T}}_{\mathbf{r}_i}\big(\widetilde{\mathcal{T}}_{w_\bullet}(E_{\tau j})K'_j\big)\widetilde{\mathcal{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_i)\\
&= P(B_i, B_{\tau i}, \widetilde{k}_i, \widetilde{k}_{\tau i}, F_j; \widetilde{\mathcal{G}}_\bullet) + P(B_i, B_{\tau i}, \widetilde{k}_i, \widetilde{k}_{\tau i}, \widetilde{\mathcal{T}}_{w_\bullet}(E_j)K'_j; \widetilde{\mathcal{G}}_\bullet)\\
&= P(B_i, B_{\tau i}, \widetilde{k}_i, \widetilde{k}_{\tau i}, B_j; \widetilde{\mathcal{G}}_\bullet)\\
&= \widetilde{\mathbf{T}}''_{i,+1}(B_j),
\end{aligned}$$

the linearity of the polynomial $P$ with respect to the fifth variable is used in the last step. This proves the desired intertwining property (6.10) and whence the theorem. □

## 6.5 $\widetilde{\mathbf{T}}'_{i,e}$ and $\widetilde{\mathbf{T}}''_{i,-e}$ as inverses

Recall the automorphisms $\widetilde{\mathbf{T}}'_{i,-1} \in \mathrm{Aut}(\widetilde{\mathbf{U}}^\imath)$ by Theorem 4.7. Recalling the bar involution $\psi^\imath$ on $\widetilde{\mathbf{U}}^\imath$ from Proposition 3.4, we define two more automorphisms $\widetilde{\mathbf{T}}''_{i,-1}, \widetilde{\mathbf{T}}'_{i,+1} \in \mathrm{Aut}(\widetilde{\mathbf{U}}^\imath)$ via

$$\widetilde{\mathbf{T}}''_{i,-1} := \psi^\imath \circ \widetilde{\mathbf{T}}''_{i,+1} \circ \psi^\imath, \qquad \widetilde{\mathbf{T}}'_{i,+1} := \psi^\imath \circ \widetilde{\mathbf{T}}'_{i,-1} \circ \psi^\imath. \tag{6.11}$$

Recall that Lusztig's symmetries $\widetilde{T}'_{i,e}$ and $\widetilde{T}''_{i,-e}$ are mutually inverses, for $i \in \mathbb{I}, e = \pm 1$; see [Lus93, 37.1.2]. They in addition satisfy the relation $\widetilde{T}'_{i,-1} = \sigma \circ \widetilde{T}''_{i,+1} \circ \sigma$; see (2.14). We shall prove the following $\imath$-analog.

**Theorem 6.7.** *$\widetilde{\mathbf{T}}'_{i,e}$ and $\widetilde{\mathbf{T}}''_{i,-e}$ are mutually inverse automorphisms on $\widetilde{\mathbf{U}}^\imath$, for $e = \pm 1, i \in \mathbb{I}_\circ$. Moreover, we have*

$$\widetilde{\mathbf{T}}'_{i,e} = \sigma^\imath \circ \widetilde{\mathbf{T}}''_{i,-e} \circ \sigma^\imath. \tag{6.12}$$

*Proof.* By definition (6.11), $\widetilde{\mathbf{T}}''_{i,-1} = \psi^\imath \widetilde{\mathbf{T}}''_{i,+1} \psi^\imath$, and $\widetilde{\mathbf{T}}'_{i,+1} = \psi^\imath \widetilde{\mathbf{T}}'_{i,-1} \psi^\imath$. Hence, it suffices to show that $\widetilde{\mathbf{T}}'_{i,-1}$ and $\widetilde{\mathbf{T}}''_{i,+1}$ are mutually inverses.

We already knew that $\widetilde{\mathbf{T}}'_{i,-1} : \widetilde{\mathbf{U}}^\imath \rightarrow \widetilde{\mathbf{U}}^\imath$ is an injective endomorphism. Let us now prove that this endomorphism $\widetilde{\mathbf{T}}'_{i,-1}$ is surjective. More precisely, we shall show that

**Claim.** For any $z \in \widetilde{\mathbf{U}}^\imath$, set $y := \widetilde{\mathbf{T}}''_{i,+1}(z)$. Then we have $z = \widetilde{\mathbf{T}}'_{i,-1}(y)$.

Let us prove the Claim. The identity (6.1) reads in our setting as $y^\imath\, \widetilde{\mathfrak{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_i^{-1}) = \widetilde{\mathfrak{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_i^{-1}) \widetilde{\mathfrak{T}}_{\mathbf{r}_i}(z)$. Applying $\widetilde{\mathfrak{T}}_{\mathbf{r}_i}^{-1}$ to both sides of this identity, we obtain $\widetilde{\mathfrak{T}}_{\mathbf{r}_i}^{-1}(y^\imath) \widetilde{\Upsilon}_i^{-1} = \widetilde{\Upsilon}_i^{-1} z$, which can be rewritten as $z \widetilde{\Upsilon}_i = \widetilde{\Upsilon}_i \widetilde{\mathfrak{T}}_{\mathbf{r}_i}^{-1}(y^\imath)$. By (4.6) and the uniqueness in Theorem 4.7(1), we conclude that $z = \widetilde{\mathbf{T}}'_{i,-1}(y)$.

By an entirely similar argument as above (switching the role of $\widetilde{\mathbf{T}}'_{i,-1}$ and $\widetilde{\mathbf{T}}''_{i,+1}$) and using the uniqueness in Theorem 6.1(1), we show that, for any $y_1 \in \widetilde{\mathbf{U}}^\imath$, we have $y_1 = \widetilde{\mathbf{T}}''_{i,+1}(z_1)$, where $z_1 := \widetilde{\mathbf{T}}'_{i,-1}(y_1)$.

Hence $\widetilde{\mathbf{T}}'_{i,-1}$ and $\widetilde{\mathbf{T}}''_{i,+1}$ are mutually inverses. As $\widetilde{\mathbf{T}}'_{i,-1}$ is an endomorphism, we see that both $\widetilde{\mathbf{T}}'_{i,-1}$ and $\widetilde{\mathbf{T}}''_{i,+1}$ are automorphisms of $\widetilde{\mathbf{U}}^\imath$.

Recall the anti-involuton $\sigma^\imath$ on $\widetilde{\mathbf{U}}^\imath$ from Proposition 3.12. It remains to prove that $\widetilde{\mathbf{T}}''_{i,+1} = \sigma^\imath \circ \widetilde{\mathbf{T}}'_{i,-1} \circ \sigma^\imath$. This follows from the identity (6.2), the identity $\widetilde{\mathbf{T}}''_{i,+1}(B_i) =$

$(\sigma^\imath \circ \widetilde{\mathbf{T}}'_{i,-1})(B_i)$ from Proposition 6.3, and $\widetilde{\mathbf{T}}''_{i,+1}(B_j) = (\sigma^\imath \circ \widetilde{\mathbf{T}}'_{i,-1})(B_j)$, for $i \neq j \in \mathbb{I}_{\circ,\tau}$; the last identity follows by comparing the rank 2 formulas for $\widetilde{\mathbf{T}}'_{i,-1}(B_j)$ in Table 3 and for $\widetilde{\mathbf{T}}''_{i,+1}(B_j)$ in Table 4. □

In particular, Theorem 6.7 above completes the proof of Theorem 4.7 that $\widetilde{\mathbf{T}}'_{i,-1}$ are automorphisms of $\widetilde{\mathbf{U}}^\imath$. From now on, thanks to Theorem 6.7, we shall denote

$$\widetilde{\mathbf{T}}_i := \widetilde{\mathbf{T}}''_{i,+1}, \qquad \widetilde{\mathbf{T}}_i^{-1} := \widetilde{\mathbf{T}}'_{i,-1}.$$

Table 3: Rank two formulas for $\widetilde{\mathbf{T}}'_{i,-1}(B_j)$ $\quad(i \neq j \in \mathbb{I}_{\circ,\tau})$

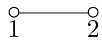
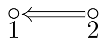
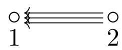
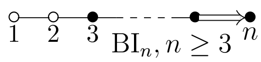
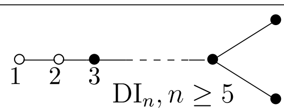
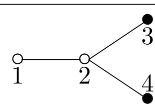
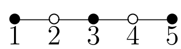
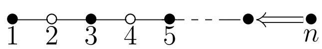
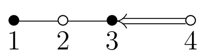
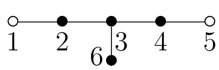
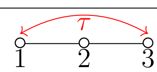
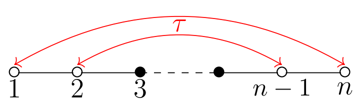
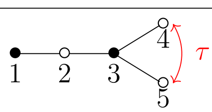
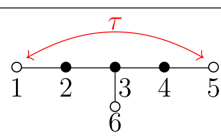
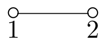
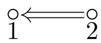
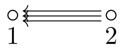
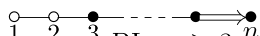
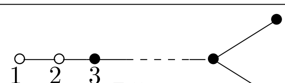
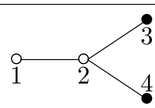
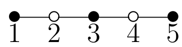
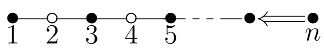
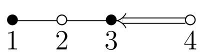
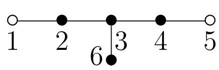
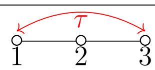
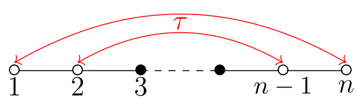
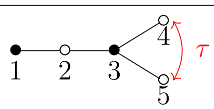
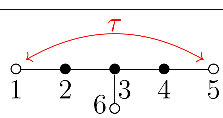
| Rank two Satake diagrams | Formulas for $\widetilde{\mathbf{T}}'_{i,-1}(B_j)$ |
|---|---|
| $\mathrm{AI}_2$  | $\widetilde{\mathbf{T}}'_{1,-1}(B_2) = [B_1, B_2]_q$ |
| $\mathrm{CI}_2$  | $\widetilde{\mathbf{T}}'_{1,-1}(B_2) = \frac{1}{[2]_{q_1}}\big[B_1, [B_1, B_2]_{q_1^2}\big] - q_1^2 B_2 \mathcal{K}_1$ |
| $\mathrm{G}_2$  | $\widetilde{\mathbf{T}}'_{1,-1}(B_2) = \frac{1}{[3]!}\Big[B_1, \big[B_1, [B_1, B_2]_{q^3}\big]_q\Big]_{q^{-1}}$ $-\frac{1}{[3]!}\big(q(1+[3])[B_1, B_2]_{q^3} + q^3[3][B_1, B_2]_{q^{-1}}\big)\widetilde{k}_1$ |
|  $\mathrm{BI}_n, n \geq 3$ | $\widetilde{\mathbf{T}}'_{2,-1}(B_1) = \big[\widetilde{\mathfrak{T}}_{w_\bullet}(B_2), [B_2, B_1]_{q_2}\big]_{q_2} - q_2 B_1 \widetilde{\mathfrak{T}}_{w_\bullet}(\mathcal{K}_2)$ |
|  $\mathrm{DI}_n, n \geq 5$ | $\widetilde{\mathbf{T}}'_{2,-1}(B_1) = \big[\widetilde{\mathfrak{T}}_{w_\bullet}(B_2), [B_2, B_1]_q\big]_q - q B_1 \widetilde{\mathfrak{T}}_{w_\bullet}(\mathcal{K}_2)$ |
| $\mathrm{DIII}_4$  | $\widetilde{\mathbf{T}}'_{2,-1}(B_1) = \big[\widetilde{\mathfrak{T}}_{w_\bullet}(B_2), [B_2, B_1]_q\big]_q - q B_1 \widetilde{\mathfrak{T}}_{w_\bullet}(\mathcal{K}_2)$ |
| $\mathrm{AII}_5$  | $\widetilde{\mathbf{T}}'_{4,-1}(B_2) = [\widetilde{\mathfrak{T}}_3(B_4), B_2]_q$ |
|  $\mathrm{CII}_n, n \geq 5$ | $\widetilde{\mathbf{T}}'_{4,-1}(B_2) = \big[[\widetilde{\mathfrak{T}}_{5\cdots n\cdots 5}(B_4), \widetilde{\mathfrak{T}}_3(B_4)]_{q_2}, B_2\big]_{q_2}$ $-q_2 \widetilde{\mathfrak{T}}_3^{-2}(B_2)\widetilde{\mathfrak{T}}_{5\cdots n\cdots 5}(\mathcal{K}_4)$ |
| $\mathrm{CII}_4$  | $\widetilde{\mathbf{T}}'_{4,-1}(B_2) = \big[[B_4, F_3]_{q_4}, B_2\big]_{q_3}$ <br> $\widetilde{\mathbf{T}}'_{2,-1}(B_4) = \big[\widetilde{\mathfrak{T}}_3(B_2), [\widetilde{\mathfrak{T}}_3(B_2), B_4]_{q_3^2}\big]$ $-(q_3 - q_3^{-1})[F_3, B_4]_{q_3^2} E_1 \widetilde{\mathfrak{T}}_3(\mathcal{K}_2) K_1'^{-1}$ |
| EIV  | $\widetilde{\mathbf{T}}'_{1,-1}(B_5) = \big[\widetilde{\mathfrak{T}}_4\widetilde{\mathfrak{T}}_3\widetilde{\mathfrak{T}}_2(B_1), B_5\big]_q$ |
| $\mathrm{AIII}_3$  | $\widetilde{\mathbf{T}}'_{1,-1}(B_2) = \big[B_3, [B_1, B_2]_q\big]_q - q B_2 \mathcal{K}_3$ |
|  $\mathrm{AIII}_n, n \geq 4$ | $\widetilde{\mathbf{T}}'_{1,-1}(B_2) = [B_1, B_2]_q$ <br> $\widetilde{\mathbf{T}}'_{2,-1}(B_1) = \big[\widetilde{\mathfrak{T}}_{w_\bullet}(B_{n-1}), [B_2, B_1]_q\big]_q - B_1 \widetilde{\mathfrak{T}}_{w_\bullet}(\mathcal{K}_{n-1})$ |
| $\mathrm{DIII}_5$  | $\widetilde{\mathbf{T}}'_{2,-1}(B_4) = [\widetilde{\mathfrak{T}}_3(B_2), B_4]_q$ <br> $\widetilde{\mathbf{T}}'_{4,-1}(B_2) = \big[B_4, [\widetilde{\mathfrak{T}}_3(B_5), B_2]_q\big]_q - \widetilde{\mathfrak{T}}_3^{-2}(B_2)\mathcal{K}_4$ |
| EIII  | $\widetilde{\mathbf{T}}'_{6,-1}(B_1) = [\widetilde{\mathfrak{T}}_{23}(B_6), B_1]_q$ <br> $\widetilde{\mathbf{T}}'_{1,-1}(B_6) = \big[\widetilde{\mathfrak{T}}_4(B_5), [\widetilde{\mathfrak{T}}_{32}(B_1), B_6]_q\big]_q$ $-\widetilde{\mathfrak{T}}^{-1}_{32323}(B_6)\widetilde{\mathfrak{T}}_4(\mathcal{K}_5)$ |

Table 4: Rank two formulas for $\widetilde{\mathbf{T}}''_{i,+1}(B_j)$ $\quad(i \neq j \in \mathbb{I}_{\circ,\tau})$

| Rank two Satake diagrams | Formulas for $\widetilde{\mathbf{T}}''_{i,+1}(B_j)$ |
|---|---|
| $\mathrm{AI}_2$  | $\widetilde{\mathbf{T}}''_{1,+1}(B_2) = [B_2, B_1]_q$ |
| $\mathrm{CI}_2$  | $\widetilde{\mathbf{T}}''_{1,+1}(B_2) = \frac{1}{[2]_{q_1}}\big[[B_2, B_1]_{q_1^2}, B_1\big] - q_1^2 B_2 \mathcal{K}_1$ |
| $\mathrm{G}_2$  | $\widetilde{\mathbf{T}}''_{1,+1}(B_2) = \frac{1}{[3]_1!}\Big[\big[[B_2, B_1]_{q_1^3}, B_1\big]_{q_1}, B_1\Big]_{q_1^{-1}}$ <br> $-\frac{1}{[3]_1!}\big(q_1(1+[3]_1)[B_2, B_1]_{q_1^3} + q_1^3[3]_1[B_2, B_1]_{q_1^{-1}}\big)\widetilde{k}_1$ |
|  $\mathrm{BI}_n, n \geq 3$ | $\widetilde{\mathbf{T}}''_{2,+1}(B_1) = \big[[B_1, B_2]_{q_2}, \widetilde{\mathcal{T}}^{-1}_{w_\bullet}(B_2)\big]_{q_2} - q_2 B_1 \mathcal{K}_2$ |
|  $\mathrm{DI}_n, n \geq 5$ | $\widetilde{\mathbf{T}}''_{2,+1}(B_1) = \big[[B_1, B_2]_q, \widetilde{\mathcal{T}}^{-1}_{w_\bullet}(B_2)\big]_q - q B_1 \mathcal{K}_2$ |
| $\mathrm{DIII}_4$  | $\widetilde{\mathbf{T}}''_{2,+1}(B_1) = \big[[B_1, B_2]_q, \widetilde{\mathcal{T}}^{-1}_{w_\bullet}(B_2)\big]_q - q B_1 \mathcal{K}_2$ |
| $\mathrm{AII}_5$  | $\widetilde{\mathbf{T}}''_{4,+1}(B_2) = [B_2, \widetilde{\mathcal{T}}_3^{-1}(B_4)]_q$ |
|  $\mathrm{CII}_n, n \geq 5$ | $\widetilde{\mathbf{T}}''_{4,+1}(B_2) = \big[B_2, [\widetilde{\mathcal{T}}_3^{-1}(B_4), \widetilde{\mathcal{T}}^{-1}_{5\cdots n\cdots 5}(B_4)]_{q_2}\big]_{q_2}$ <br> $-q_2^2 \widetilde{\mathcal{T}}_3^2(B_2)\widetilde{\mathcal{T}}_3(\mathcal{K}_4)$ |
| $\mathrm{CII}_4$  | $\widetilde{\mathbf{T}}''_{4,+1}(B_2) = \big[B_2, [F_3, B_4]_{q_4}\big]_{q_3}$ <br> $\widetilde{\mathbf{T}}''_{2,+1}(B_4) = \big[[B_4, \widetilde{\mathcal{T}}_3^{-1}(B_2)]_{q_3^2}, \widetilde{\mathcal{T}}_3^{-1}(B_2)\big]$ <br> $-(q_3 - q_3^{-1})[B_4, F_3]_{q_3^2} E_1 \mathcal{K}_2 K_1'^{-1}$ |
| $\mathrm{EIV}$  | $\widetilde{\mathbf{T}}''_{1,+1}(B_5) = \big[B_5, \widetilde{\mathcal{T}}_4^{-1}\widetilde{\mathcal{T}}_3^{-1}\widetilde{\mathcal{T}}_2^{-1}(B_1)\big]_q$ |
| $\mathrm{AIII}_3$  | $\widetilde{\mathbf{T}}''_{1,+1}(B_2) = \big[[B_2, B_1]_q, B_3\big]_q - q B_2 \mathcal{K}_1$ |
|  $\mathrm{AIII}_n, n \geq 4$ | $\widetilde{\mathbf{T}}''_{1,+1}(B_2) = [B_2, B_1]_q$ <br> $\widetilde{\mathbf{T}}''_{2,+1}(B_1) = \big[[B_1, B_2]_q, \widetilde{\mathcal{T}}^{-1}_{w_\bullet}(B_{n-1})\big]_q - \mathcal{K}_2 B_1$ |
| $\mathrm{DIII}_5$  | $\widetilde{\mathbf{T}}''_{2,+1}(B_4) = [B_4, \widetilde{\mathcal{T}}_3^{-1}(B_2)]_q$ <br> $\widetilde{\mathbf{T}}''_{4,+1}(B_2) = \big[[B_2, \widetilde{\mathcal{T}}_3^{-1}(B_5)]_q, B_4\big]_q - q\widetilde{\mathcal{T}}_3^2(B_2)\widetilde{\mathcal{T}}_3(\mathcal{K}_5)$ |
| $\mathrm{EIII}$  | $\widetilde{\mathbf{T}}''_{6,+1}(B_1) = [B_1, \widetilde{\mathcal{T}}_2^{-1}\widetilde{\mathcal{T}}_3^{-1}(B_6)]_q$ <br> $\widetilde{\mathbf{T}}''_{1,+1}(B_6) = \big[[B_6, \widetilde{\mathcal{T}}_3^{-1}\widetilde{\mathcal{T}}_2^{-1}(B_1)]_q, \widetilde{\mathcal{T}}_4^{-1}(B_5)\big]_q$ <br> $-q\widetilde{\mathcal{T}}_{32323}(B_6)\widetilde{\mathcal{T}}_{s_4 w_\bullet}(\mathcal{K}_1)$ |

# Part II

# Relative braid group symmetries for Kac-Moody type

# 7 Construction of symmetries $\widetilde{\mathbf{T}}'_{i,+1}, \widetilde{\mathbf{T}}''_{i,+1}$

We construct relative braid group symmetries on $\imath$quantum groups $\widetilde{\mathbf{U}}^\imath$ of Kac-Moody type in Theorem 7.1, generalizing the finite-type construction in Part I. The higher rank formulas of these symmetries are presented in § 7.2, whose proofs occupy the coming sections.

## 7.1 Main Theorem

Let $(\mathbb{I} = \mathbb{I}_\circ \cup \mathbb{I}_\bullet, \tau)$ be a symmetric pair of Kac-Moody type. We shall construct relative braid group symmetries associated to the following three types vertices $i \in \mathbb{I}_{\circ,\tau}$

(i) $i = \tau i = w_\bullet i$,

(ii) $c_{i,\tau i} = 0, i = w_\bullet i$,

(iii) $c_{i,\tau i} = -1, i = w_\bullet i$.

Let $\boldsymbol{\varsigma}_\diamond = (\varsigma_{\diamond,i})_{i \in \mathbb{I}_\circ}$ be the distinguished parameters such that

$$\varsigma_{\diamond,i} = -q^{-(\alpha_i, \alpha_i + w_\bullet \alpha_{\tau i})/2}. \tag{7.1}$$

We extend $\boldsymbol{\varsigma}_\diamond$ to an $\mathbb{I}$-tuple of scalars by setting $\varsigma_{\diamond,j} = 1$ for $j \in \mathbb{I}_\bullet$.

The definition (4.3) of rescaled braid group symmetries $\widetilde{\mathscr{T}}''_{i,+1}, \widetilde{\mathscr{T}}'_{i,-1}$ on $\widetilde{\mathbf{U}}$ can be easily generalized to the Kac-Moody type. We still use the short notations $\widetilde{\mathscr{T}}_i$ for $\widetilde{\mathscr{T}}''_{i,+1}$, and hence $\widetilde{\mathscr{T}}_i^{-1} = \widetilde{\mathscr{T}}'_{i,-1}$.

The rank one quasi $K$-matrices $\widetilde{\Upsilon}_i$ for $i \in \mathbb{I}_{\circ,\tau}$ are defined to be the quasi $K$-matrix associated to the (finite-type) rank one subdiagram $(\mathbb{I}_{\bullet,i} = \{i, \tau i\} \cup \mathbb{I}_\bullet, \tau|_{\mathbb{I}_{\bullet,i}})$.

**Theorem 7.1.** *Let $i \in \mathbb{I}_{\circ,\tau}$ be a vertex of type (i)-(iii).*

(1) *For any $x \in \widetilde{\mathbf{U}}^\imath$, there exists an element $x' \in \widetilde{\mathbf{U}}^\imath$ such that*

$$x'\widetilde{\Upsilon}_i = \widetilde{\Upsilon}_i \widetilde{\mathcal{T}}'_{\mathbf{r}_i,-1}(x). \tag{7.2}$$

*Moreover, the map $x \mapsto x'$ is an automorphism of $\widetilde{\mathbf{U}}^\imath$, denoted by $\widetilde{\mathbf{T}}'_{i,-1}$.*

(2) *For any $x \in \widetilde{\mathbf{U}}^\imath$, there exists an element $x'' \in \widetilde{\mathbf{U}}^\imath$ such that*

$$x''\widetilde{\mathcal{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_i)^{-1} = \widetilde{\mathcal{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_i)^{-1} \widetilde{\mathcal{T}}''_{\mathbf{r}_i,+1}(x). \tag{7.3}$$

*Moreover, the map $x \mapsto x''$ is an automorphism of $\widetilde{\mathbf{U}}^\imath$, denoted by $\widetilde{\mathbf{T}}''_{i,+1}$.*

(3) *Automorphisms $\widetilde{\mathbf{T}}'_{i,-1}$ and $\widetilde{\mathbf{T}}''_{i,+1}$ are mutually inverses and they satisfy $\widetilde{\mathbf{T}}'_{i,-1} = \sigma^\imath \circ \widetilde{\mathbf{T}}''_{i,+1} \circ \sigma^\imath$.*

*Remark* 7.2. It is worth noting that, when the symmetric pair is of quasi-split type (i.e., $\mathbb{I}_\bullet = \varnothing$), every vertex $i \in \mathbb{I}_{\circ,\tau}$ belongs to one of the three types (i)-(iii).

Using the bar involution $\psi^\imath$ on $\widetilde{\mathbf{U}}^\imath$ (see Proposition 3.4), we define other two variants of the symmetries

$$\widetilde{\mathbf{T}}'_{i,+1} := \psi^\imath \circ \widetilde{\mathbf{T}}'_{i,-1} \circ \psi^\imath, \qquad \widetilde{\mathbf{T}}''_{i,-1} := \psi^\imath \circ \widetilde{\mathbf{T}}''_{i,+1} \circ \psi^\imath. \tag{7.4}$$

In order to prove Theorem 7.1, it suffices to construct elements $\widetilde{\mathbf{T}}'_{i,-1}(x), \widetilde{\mathbf{T}}''_{i,+1}(x)$ satisfying (7.2)-(7.3) for each generator $x$ of $\mathbf{U}^\imath$. For $x \in \widetilde{\mathbf{U}}^{\imath 0}\widetilde{\mathbf{U}}_\bullet$, the elements $\widetilde{\mathbf{T}}'_{i,-1}(x), \widetilde{\mathbf{T}}''_{i,+1}(x)$ are obtained in the same way as the finite-type case in Proposition 4.11. For $x = B_i, B_{\tau i}$, the rank one formulas and their proofs in Theorem 4.14

remain valid in the Kac-Moody setting. We will provide the construction of elements $\widetilde{\mathbf{T}}'_{i,-1}(B_j), \widetilde{\mathbf{T}}''_{i,+1}(B_j) \in \widetilde{\mathbf{U}}^\imath$ for $j \neq i, \tau i$ in the next subsection.

## 7.2 Higher rank formulas for new symmetries

We sketch the proof of the existence of $x', x''$ in Theorem 7.1 for $x = B_j, j \neq i, \tau i, j \in \mathbb{I}_\circ$ in this subsection. We need to find elements $\widetilde{\mathbf{T}}'_{i,-1}(B_j), \widetilde{\mathbf{T}}''_{i,+1}(B_j) \in \widetilde{\mathbf{U}}^\imath$ such that

$$\widetilde{\mathbf{T}}'_{i,-1}(B_j)\widetilde{\Upsilon}_i = \widetilde{\Upsilon}_i \widetilde{\mathfrak{T}}'_{\mathbf{r}_i,-1}(B_j), \tag{7.5}$$

$$\widetilde{\mathbf{T}}''_{i,+1}(B_j)\widetilde{\mathfrak{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_i)^{-1} = \widetilde{\mathfrak{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_i)^{-1}\widetilde{\mathfrak{T}}''_{\mathbf{r}_i,+1}(B_j). \tag{7.6}$$

The proofs depend on which of types (i)-(iii) the vertex $i \in \mathbb{I}_{\circ,\tau}$ belongs to. We will construct root vectors in $\widetilde{\mathbf{U}}^\imath$ for each of these three types in Sections 8-10:

(i) For $i = \tau i = w_\bullet i$, we define root vectors $b_{i,j;m}, \underline{b}_{i,j;m} \in \widetilde{\mathbf{U}}^\imath$ for $m \geq 0$ in Definition 8.1-8.2.

(ii) For $c_{i,\tau i} = 0, i = w_\bullet i$, we define root vectors $b_{i,\tau i,j;m_1,m_2}, \underline{b}_{i,\tau i,j;m_1,m_2} \in \widetilde{\mathbf{U}}^\imath$ in Definition 9.6-9.7.

(iii) For $c_{i,\tau i} = -1, i = w_\bullet i$, we define root vectors $b_{i,\tau i,j;a,b,c}, \underline{b}_{i,\tau i,j;a,b,c} \in \widetilde{\mathbf{U}}^\imath$ in Definition 10.6-10.7.

It turns out that the desired elements $\widetilde{\mathbf{T}}'_{i,-1}(B_j), \widetilde{\mathbf{T}}''_{i,+1}(B_j)$ are given by these root vectors, as formulated in the Theorem 7.3 below.

For type (i), the $\imath$divided powers were formulated in [CLW21], generalizing [BW18a, BeW18]. We recall definitions *loc. cit.* of $\imath$divided powers $B^{(m)}_{i,\overline{p}} \in \widetilde{\mathbf{U}}^\imath$ for $m \geq 0, p \in$

$\mathbb{Z}/2\mathbb{Z}$ as follows

$$B_{i,\bar{0}}^{(m)} = \frac{1}{[m]_i!} \begin{cases} B_i \prod_{r=1}^{k} (B_i^2 - q_i \widetilde{k}_i [2r]_i^2), & \text{if } m = 2k+1, \\ \prod_{r=1}^{k} (B_i^2 - q_i \widetilde{k}_i [2r-2]_i^2), & \text{if } m = 2k; \end{cases}$$
$$B_{i,\bar{1}}^{(m)} = \frac{1}{[m]_i!} \begin{cases} B_i \prod_{r=1}^{k} (B_i^2 - q_i \widetilde{k}_i [2r-1]_i^2), & \text{if } m = 2k+1, \\ \prod_{r=1}^{k} (B_i^2 - q_i \widetilde{k}_i [2r-1]_i^2), & \text{if } m = 2k. \end{cases} \tag{7.7}$$

For type (ii)-(iii), the ıdivided powers are the same as usual divided powers

$$B_i^{(m)} = \frac{B_i^m}{[m]_i!}.$$

These ıdivided powers have appeared in conjectural formulas for relative braid group actions cf. [CLW21, Conjecture 6.5] for type (i) and in [CLW23, Conjecture 3.7] for type (ii). These conjectures were confirmed via Hall algebras in [LW22a] under assumptions that $\mathbb{I}_\bullet = \varnothing$ and $c_{j,\tau j}$ are even for all $j \in \mathbb{I}$. We shall prove these conjectures in full generality in Theorem 7.3(i)(ii) respectively.

**Theorem 7.3.** *Let $i \in \mathbb{I}_{\circ,\tau}, j \in \mathbb{I}_\circ$ such that $j \neq i, \tau i$. Write $\alpha = -c_{ij}, \beta = -c_{\tau i,j}$.*

*(i) If $i = \tau i = w_\bullet i$, then the element $\widetilde{\mathbf{T}}'_{i,-1}(B_j) := b_{i,j;\alpha}$ satisfies (7.5) and the element $\widetilde{\mathbf{T}}''_{i,+1}(B_j) := \underline{b}_{i,j;\alpha}$ satisfies (7.6). Explicitly, we have*

$$\widetilde{\mathbf{T}}'_{i,-1}(B_j) = \sum_{u \geq 0} \sum_{\substack{r+s+2u=\alpha \\ \overline{r}=\overline{p}}} (-1)^{r+u} q_i^{r+2u} B_{i,\overline{p+\alpha}}^{(s)} B_j B_{i,\overline{p}}^{(r)} \widetilde{k}_i^u, \tag{7.8}$$
$$\widetilde{\mathbf{T}}''_{i,+1}(B_j) = \sum_{u \geq 0} \sum_{\substack{r+s+2u=\alpha \\ \overline{r}=\overline{p}}} (-1)^{r+u} q_i^{r+2u} B_{i,\overline{p}}^{(r)} B_j B_{i,\overline{p+\alpha}}^{(s)} \widetilde{k}_i^u. \tag{7.9}$$

*(ii) If $c_{i,\tau i} = 0, i = w_\bullet i$, then the element $\widetilde{\mathbf{T}}'_{i,-1}(B_j) := b_{i,\tau i,j;\alpha,\beta}$ satisfies (7.5) and*

*the element* $\widetilde{\mathbf{T}}''_{i,+1}(B_j) := \underline{b}_{i,\tau i,j;\alpha,\beta}$ *satisfies* (7.6). *Explicitly, we have*

$$\widetilde{\mathbf{T}}'_{i,-1}(B_j) = \sum_{u=0}^{\min(\alpha,\beta)} \sum_{r=0}^{\alpha-u} \sum_{s=0}^{\beta-u} (-1)^{r+s+u} q_i^{r(-u+1)+s(u+1)+u} \times B_i^{(\alpha-r-u)} B_{\tau i}^{(\beta-s-u)} B_j B_{\tau i}^{(s)} B_i^{(r)} \widetilde{k}_i^u, \tag{7.10}$$

$$\widetilde{\mathbf{T}}''_{i,+1}(B_j) = \sum_{u=0}^{\min(\alpha,\beta)} \sum_{r=0}^{\alpha-u} \sum_{s=0}^{\beta-u} (-1)^{r+s+u} q_i^{r(-u+1)+s(u+1)+u} \times \widetilde{k}_{\tau i}^u B_i^{(r)} B_{\tau i}^{(s)} B_j B_{\tau i}^{(\beta-s-u)} B_i^{(\alpha-r-u)}. \tag{7.11}$$

*(iii) If* $c_{i,\tau i} = -1, i = w_\bullet i$, *then the element* $\widetilde{\mathbf{T}}'_{i,-1}(B_j) := b_{i,\tau i,j;\beta,\beta+\alpha,\alpha}$ *satisfies* (7.5) *and the element* $\widetilde{\mathbf{T}}''_{i,+1}(B_j) := \underline{b}_{i,\tau i,j;\beta,\beta+\alpha,\alpha}$ *satisfies* (7.6). *Explicitly, we have*

$$\widetilde{\mathbf{T}}'_{i,-1}(B_j) = \sum_{u,v\geq 0} \sum_{t=0}^{\beta-v} \sum_{s=0}^{\beta+\alpha-v-u} \sum_{r=0}^{\alpha-u} (-1)^{t+v+r+s+u} q_i^{t(-2v+1)+r(u+1)+s(v-2u+1)+uv} \times q_i^{-\frac{u(u-1)+v(v-1)}{2}} B_i^{(\beta-v-t)} B_{\tau i}^{(\alpha+\beta-v-u-s)} B_i^{(\alpha-u-r)} B_j B_i^{(r)} B_{\tau i}^{(s)} \widetilde{k}_{\tau i}^u B_i^{(t)} \widetilde{k}_i^v, \tag{7.12}$$

$$\widetilde{\mathbf{T}}''_{i,+1}(B_j) = \sum_{u,v\geq 0} \sum_{t=0}^{\beta-v} \sum_{s=0}^{\beta+\alpha-v-u} \sum_{r=0}^{\alpha-u} (-1)^{t+v+r+s+u} q_i^{t(-2v+1)+r(u+1)+s(v-2u+1)+uv} \times q_i^{-\frac{u(u-1)+v(v-1)}{2}} \widetilde{k}_{\tau i}^v B_i^{(t)} \widetilde{k}_i^u B_{\tau i}^{(s)} B_i^{(r)} B_j B_i^{(\alpha-u-r)} B_{\tau i}^{(\alpha+\beta-v-u-s)} B_i^{(\beta-v-t)}. \tag{7.13}$$

*Proof.* We only outline the proof here; details will be included in later Sections 8-10. The first statements in (i)-(iii) are respectively proved in Theorem 8.11, Theorem 9.17, and Theorem 10.14. The explicit formulas in (i) are obtained by specializing ıdivided power formulations for $b_{i,j;m}, \underline{b}_{i,j;m}$ in Proposition 8.6 at $m = \alpha$. The explicit formulas in (ii) are obtained by specializing ıdivided power formulations for $b_{i,\tau i,j;m_1,m_2}, \underline{b}_{i,\tau i,j;m_1,m_2}$ in Proposition 9.11 at $m_1 = \alpha, m_2 = \beta$. The explicit formulas in (iii) are obtained by specializing ıdivided power formulations for $b_{i,\tau i,j;a,b,c}, \underline{b}_{i,\tau i,j;a,b,c}$ in Theorem 10.16 at $a = \beta, b = \alpha + \beta, c = \alpha$. □

*Remark* 7.4. In [LW22a, Theorem 6.10-6.11], Lu-Wang formulated four (relative) braid group symmetries $\mathbf{T}'_{i,e}, \mathbf{T}''_{i,e}$ on $\widetilde{\mathbf{U}}^\imath$ for $c_{i,\tau i} = 0$. In fact, our symmetries can be related to theirs via a rescaling automorphism $\Phi$ on $\widetilde{\mathbf{U}}^\imath$, which sends

$$\Phi : B_i \mapsto -q_i^{-1} B_i, B_{\tau i} \mapsto B_{\tau i}, \quad \widetilde{k}_i \mapsto -q_i^{-1} \widetilde{k}_i, \quad \widetilde{k}_{\tau i} \mapsto -q_i^{-1} \widetilde{k}_{\tau i},$$

and fixes $B_j, \widetilde{k}_j$ if $j \neq i, \tau i$.

One can then show that $\mathbf{T}''_{i,-1} = \Phi \widetilde{\mathbf{T}}'_{i,-1} \Phi^{-1}$ and $\mathbf{T}'_{i,+1} = \Phi \widetilde{\mathbf{T}}''_{i,+1} \Phi^{-1}$.

# 8 Higher rank formulas for $\tau i = i = w_\bullet i$

We fix an $i \in \mathbb{I}_{\circ,\tau}$ such that $\tau i = i = w_\bullet i$ in this section. In this case, $B_i = F_i + E_i K'_i, \widetilde{k}_i = K_i K'_i$, and $\mathbf{r}_i = s_i$.

We define root vectors $b_{i,j;m}, \underline{b}_{i,j;m} \in \widetilde{\mathbf{U}}^\imath$ for $j \in \mathbb{I}_\circ, j \neq i$ in Definitions 8.1-8.2 via recursive relations. The ıdivided power formulations for these elements are obtained in Proposition 8.6. We show that $b_{i,j;-c_{ij}}, \underline{b}_{i,j;-c_{ij}}$ provide the higher rank formulas for $\widetilde{\mathbf{T}}'_{i,-1}(B_j), \widetilde{\mathbf{T}}''_{i,+1}(B_j)$ in Theorem 8.11 and complete the proof for Theorem 7.3(i).

## 8.1 Definitions of root vectors

Note that, in this case, $c_{ij} = c_{i,\tau j}$.

**Definition 8.1.** Let $j \in \mathbb{I}_\circ, j \neq i$. Define $b^{\pm}_{i,j;m}$ be the elements in $\widetilde{\mathbf{U}}$ defined recursively for $m \geq -1$ as follows

$$\begin{aligned}
&b^-_{i,j;0} = F_j, \quad b^+_{i,j;0} = \widetilde{\mathfrak{T}}_{w_\bullet}(E_{\tau j})K'_j, \quad b^\pm_{i,j;-1} = 0,\\
&- q_i^{-(c_{ij}+2m)} b^\pm_{i,j;m} B_i + B_i b^\pm_{i,j;m}\\
=&[m+1]_i b^\pm_{i,j;m+1} + [-c_{ij} - m + 1]_i q_i^{-2m-c_{ij}+2} b^\pm_{i,j;m-1} \widetilde{k}_i, \qquad m \geq 0.
\end{aligned} \tag{8.1}$$

Set $b_{i,j;m} := b^-_{i,j;m} + b^+_{i,j;m}$. Since $b_{i,j;0} = B_j \in \widetilde{\mathbf{U}}^\imath$, one can recursively show that $b_{i,j;m} \in \widetilde{\mathbf{U}}^\imath$ for any $m \geq -1$.

**Definition 8.2.** Let $j \in \mathbb{I}_\circ, j \neq i$. Define $\underline{b}^\pm_{i,j;m}$ be the elements in $\widetilde{\mathbf{U}}$ defined recursively for $m \geq -1$ as follows

$$\begin{aligned}
&\underline{b}^-_{i,j;0} = F_j, \quad \underline{b}^+_{i,j;0} = \widetilde{\mathfrak{T}}_{w_\bullet}(E_{\tau j})K'_j, \quad \underline{b}^\pm_{i,j;-1,-1} = 0,\\
&- q_i^{-(c_{ij}+2m)} B_i \underline{b}^\pm_{i,j;m} + \underline{b}^\pm_{i,j;m} B_i\\
=&[m+1]_i \underline{b}^\pm_{i,j;m+1} + [-c_{ij} - m + 1]_i q_i^{-2m-c_{ij}+2} \underline{b}^\pm_{i,j;m-1} \widetilde{k}_i, \qquad m \geq 0.
\end{aligned} \tag{8.2}$$

Set $\underline{b}_{i,j;m} := \underline{b}^-_{i,j;m} + \underline{b}^+_{i,j;m}$. One can recursively show that $\underline{b}_{i,j;m} \in \widetilde{\mathbf{U}}^\imath$ for any $m \geq -1$.

Recall the anti-involution $\sigma^\imath$ on $\widetilde{\mathbf{U}}^\imath$ from Proposition 3.12.

**Proposition 8.3.** *Let $j \in \mathbb{I}_\circ, j \neq i$. Then $\underline{b}_{i,j;m} = \sigma^\imath(b_{i,j;m})$ for $m \geq 0$.*

*Proof.* The recursive relation (8.1) implies that $\sigma^\imath(b_{i,j;m})$ satisfies the same relation (8.2) as $\underline{b}_{i,j;m}$. Since $\underline{b}_{i,j;0} = B_j = \sigma^\imath(b_{i,j;0})$, this proposition follows by induction. □

**Lemma 8.4.** *We have, for $m \geq 0, , j \in \mathbb{I}_\circ, j \neq i$,*

$$\underline{b}^-_{i,j;m} = \sigma\big(\widetilde{\Upsilon}_i^{-1} b^-_{i,j;m} \widetilde{\Upsilon}_i\big). \tag{8.3}$$

*Proof.* Consider the subalgebra $\widetilde{\mathbf{U}}^-_{[i;j]}$ of $\widetilde{\mathbf{U}}$ generated by $B_i, \widetilde{k}_i, F_j$. It is clear from the above definitions that $b^-_{i,j;m}, \underline{b}^-_{i,j;m} \in \widetilde{\mathbf{U}}^-_{[i;j]}$. By Theorem 3.6 and Lemma 5.1, there is a well-defined anti-automorphism $\sigma_{ij}$ on $\widetilde{\mathbf{U}}^-_{[i;j]}$, which is given by

$$\sigma_{ij} : x \mapsto \sigma\big(\widetilde{\Upsilon}_i^{-1} x \widetilde{\Upsilon}_i\big).$$

Moreover, $\sigma_{ij}$ fixes $B_i, F_j, \widetilde{k}_i$. Applying $\sigma_{ij}$ to (8.1), it is clear that $\sigma_{ij}(b^-_{i,j;m})$ satisfies the same recursive relation as $\underline{b}^-_{i,j;m}$. Then the desired identity follows by induction. □

*Remark* 8.5. One can also formulate the relation between $\underline{b}^+_{i,j;m}, b^+_{i,j;m}$. However, it is much more complicated than (8.3) and hard to prove directly. We do not need the relation between $\underline{b}^+_{i,j;m}, b^+_{i,j;m}$ in this dissertation. The situation is similar when the vertex $i$ is of type (ii)-(iii).

## 8.2 An ıdivided power formulation

In [CLW21, §6.1], elements $\widetilde{y}_{i,j;n,m,\overline{p},\overline{t},e}, \widetilde{y}'_{i,j;n,m,\overline{p},\overline{t},e}$ in $\widetilde{\mathbf{U}}^\imath$ were defined via ıdivided powers. Comparing Definition 8.1 and [CLW21, Theorem 6.2], it is clear that our $b_{i,j;m}$ (*resp.* $\underline{b}_{i,j;m}$) and their $\widetilde{y}'_{i,j;1,m,\overline{p},\overline{t},1}$ (*resp.* $\widetilde{y}_{i,j;1,m,\overline{p},\overline{t},1}$) satisfy the same recursive relations, which implies that

$$b_{i,j;m} = \widetilde{y}'_{i,j;1,m,\overline{p},\overline{t},1}, \qquad \underline{b}_{i,j;m} = \widetilde{y}_{i,j;1,m,\overline{p},\overline{t},1}, \qquad (m \geq 0, \overline{p}, \overline{t} \in \mathbb{Z}/2\mathbb{Z}). \tag{8.4}$$

We thus obtained ıdivided power formulations for $b_{i,j;m}, \underline{b}_{i,j;m}$ in the next proposition, by specializing ıdivided power formulas of $\widetilde{y}_{i,j;n,m,\overline{p},\overline{t},e}, \widetilde{y}'_{i,j;n,m,\overline{p},\overline{t},e}$ at $n = 1, e = 1$.

**Proposition 8.6.** *Let $i \in \mathbb{I}_{\circ,\tau}, j \in \mathbb{I}_\circ$ such that $\tau i = i = w_\bullet i, j \neq i$.*

(1) *The element $b_{i,j;m}$ in Definition 8.1 admits an ıdivided power formulation: for $m + c_{ij}$ odd,*

$$b_{i,j;m} = \sum_{u\geq 0}(q_i\widetilde{k}_i)^u\Big\{\sum_{\substack{r+s+2u=m\\ \overline{r}=\overline{p}+\overline{1}}}(-1)^r q_i^{-(m+c_{ij})(r+u)+r}\begin{bmatrix}\frac{m+c_{ij}-1}{2}\\ u\end{bmatrix}_{q_i^2} B^{(s)}_{i,\overline{p}} B_j B^{(r)}_{i,\overline{p+c_{ij}}}$$
$$+\sum_{\substack{r+s+2u=m\\ \overline{r}=\overline{p}}}(-1)^r q_i^{-(m+c_{ij}-2)(r+u)-r}\begin{bmatrix}\frac{m+c_{ij}-1}{2}\\ u\end{bmatrix}_{q_i^2} B^{(s)}_{i,\overline{p}} B_j B^{(r)}_{i,\overline{p+c_{ij}}}\Big\},$$

*and for $m + c_{ij}$ even,*

$$b_{i,j;m} = \sum_{u\geq 0}(q_i\widetilde{k}_i)^u\Big\{\sum_{\substack{r+s+2u=m\\ \overline{r}=\overline{p}+\overline{1}}}(-1)^r q_i^{-(m+c_{ij}-1)(r+u)}\begin{bmatrix}\frac{m+c_{ij}}{2}\\ u\end{bmatrix}_{q_i^2} B^{(s)}_{i,\overline{p}} B_j B^{(r)}_{i,\overline{p+c_{ij}}}$$
$$+\sum_{\substack{r+s+2u=m\\ \overline{r}=\overline{p}}}(-1)^r q_i^{-(m+c_{ij}-1)(r+u)}\begin{bmatrix}\frac{m+c_{ij}-2}{2}\\ u\end{bmatrix}_{q_i^2} B^{(s)}_{i,\overline{p}} B_j B^{(r)}_{i,\overline{p+c_{ij}}}\Big\}.$$

(2) *The element $\underline{b}_{i,j;m}$ in Definition 8.2 admits an ıdivided power formulation: for $m + c_{ij}$ odd,*

$$\underline{b}_{i,j;m} = \sum_{u\geq 0}(q_i\widetilde{k}_i)^u\Big\{\sum_{\substack{r+s+2u=m\\ \overline{r}=\overline{p}+\overline{1}}}(-1)^r q_i^{-(m+c_{ij})(r+u)+r}\begin{bmatrix}\frac{m+c_{ij}-1}{2}\\ u\end{bmatrix}_{q_i^2} B^{(r)}_{i,\overline{p}} B_j B^{(s)}_{i,\overline{p+c_{ij}}}$$
$$+\sum_{\substack{r+s+2u=m\\ \overline{r}=\overline{p}}}(-1)^r q_i^{-(m+c_{ij}-2)(r+u)-r}\begin{bmatrix}\frac{m+c_{ij}-1}{2}\\ u\end{bmatrix}_{q_i^2} B^{(r)}_{i,\overline{p}} B_j B^{(s)}_{i,\overline{p+c_{ij}}}\Big\},$$

*and for $m + c_{ij}$ even,*

$$b_{i,j;m} = \sum_{u\geq 0}(q_i\widetilde{k}_i)^u \Big\{ \sum_{\substack{r+s+2u=m \\ \overline{r}=\overline{p}+\overline{1}}} (-1)^r q_i^{-(m+c_{ij}-1)(r+u)} \begin{bmatrix} \frac{m+c_{ij}}{2} \\ u \end{bmatrix}_{q_i^2} B^{(r)}_{i,\overline{p}} B_j B^{(s)}_{i,\overline{p+c_{ij}}}$$

$$+ \sum_{\substack{r+s+2u=m \\ \overline{r}=\overline{p}}} (-1)^r q_i^{-(m+c_{ij}-1)(r+u)} \begin{bmatrix} \frac{m+c_{ij}-2}{2} \\ u \end{bmatrix}_{q_i^2} B^{(r)}_{i,\overline{p}} B_j B^{(s)}_{i,\overline{p+c_{ij}}} \Big\}.$$

## 8.3 Intertwining properties

We write $y_{i,j;m}, x_{i,j;m}$ for $y_{i,j;m,-1}, x_{i,j;m,-1}$ respectively.

**Proposition 8.7.** *We have for any $m \geq 0, j \in \mathbb{I}_\circ, j \neq i$,*

$$b^-_{i,j;m}\widetilde{\Upsilon}_i = \widetilde{\Upsilon}_i y_{i,j;m} \tag{8.5}$$

*Proof.* We use an induction on $m$. The base cases $m = 0$ is a consequence of Lemma 5.1.

Suppose that (8.5) is true for $1, 2, \cdots, m$. Note that $B_i^\sigma = F_i + K_iE_i$. We have, by induction hypothesis and Theorem 3.6,

$$\begin{aligned} &\widetilde{\Upsilon}_i^{-1}(-q_i^{-(c_{ij}+2m)} b^-_{i,j;m}B_i + B_ib^-_{i,j;m})\widetilde{\Upsilon}_i \\ =& - q_i^{-(c_{ij}+2m)} y_{i,j;m}B_i^\sigma + B_i^\sigma y_{i,j;m} \\ =& - q_i^{-(c_{ij}+2m)} y_{i,j;m}F_i + F_iy_{i,j;m} - q_i^{-(c_{ij}+2m)} y_{i,j;m}K_iE_i + K_iE_iy_{i,j;m} \\ =& - q_i^{-(c_{ij}+2m)} y_{i,j;m}F_i + F_iy_{i,j;m} + K_i(-y_{i,j;m}E_i + E_iy_{i,j;m}). \end{aligned}$$

Using Lemma 2.8 to simplify the RHS of above formula, we obtain

$$
\begin{aligned}
\text{RHS} =&[m+1]_i y_{i,j;m+1} + [-c_{ij}-m+1]_i K_i y_{i,j;m-1} K_i' \\
=&[m+1]_i y_{i,j;m+1} + [-c_{ij}-m+1]_i q_i^{-2m-c_{ij}+2} y_{i,j;m-1} K_i K_i'.
\end{aligned}
$$

Combining the above two computations, we have the following identity

$$
\begin{aligned}
&-q_i^{-(c_{ij}+2m)} b^-_{i,j;m} B_i + B_i b^-_{i,j;m} \\
=&\widetilde{\Upsilon}_i([m+1]_i y_{i,j;m+1} + [-c_{ij}-m+1]_i q_i^{-2m-c_{ij}+2} y_{i,j;m-1} K_i K_i') \widetilde{\Upsilon}_i^{-1}.
\end{aligned}
\tag{8.6}
$$

On the other hand, by definition (8.1), we have

$$
\begin{aligned}
&-q_i^{-(c_{ij}+2m)} b^-_{i,j;m} B_i + B_i b^-_{i,j;m} \\
=&[m+1]_i b^-_{i,j;m+1} + [-c_{ij}-m+1]_i q_i^{-2m-c_{ij}+2} b^-_{i,j;m-1} \widetilde{k}_i.
\end{aligned}
\tag{8.7}
$$

Comparing (8.6) with (8.7) and then using the induction hypothesis, we deduce that (8.5) is true for $m+1$ as desired. □

**Proposition 8.8.** *We have for any $m \geq 0, j \in \mathbb{I}_\circ, j \neq i$,*

$$
b^+_{i,j;m} = (-1)^m q_i^{-2m(c_{ij}+m-1)} \widetilde{\mathcal{T}}_{w_\bullet}(x_{i,\tau j;m}) K_j' (K_i')^m. \tag{8.8}
$$

*Proof.* We use an induction on $m$. The base cases $m = 0, 1$ are verified by straightforward computations.

We denote $\widetilde{\mathcal{T}}_{w_\bullet}(x_{i,\tau j;m})$ by $x_{i,w_\bullet \tau j;m}$ in the proof. Let $R_m$ denote the RHS (8.8), i.e., $R_m = (-1)^m q_i^{-2m(c_{ij}+m-1)} x_{i,w_\bullet \tau j;m} K_j' (K_i')^m$. It suffices to show that $R_m$ satisfies the same recursive relation as $b^+_{i,j;m}$.

Let $Q_m$ denote the element $Q_m = x_{i,w_\bullet\tau j;m}K'_j(K'_i)^m$. We first obtain the recursive relation for $Q_m$ as follows

$$\begin{aligned}
&-q_i^{-(c_{ij}+2m)}Q_mB_i+B_iQ_m\\
=&-q_i^{-(c_{ij}+2m)}x_{i,w_\bullet\tau j;m}K'_j(K'_i)^m(F_i+E_iK'_i)+(F_i+E_iK'_i)x_{i,w_\bullet\tau j;m}K'_j(K'_i)^m\\
=&-\big(x_{i,w_\bullet\tau j;m}F_i-F_ix_{i,w_\bullet\tau j;m}\big)K'_j(K'_i)^m\\
&-q_i^{-2(c_{ij}+2m)}(x_{i,w_\bullet\tau j;m}E_i-q_i^{c_{ij}+2m}E_ix_{i,w_\bullet\tau j;m})K'_j(K'_i)^{m+1}.
\end{aligned}$$

Recall that $c_{i,\tau j}=c_{ij}$ in this case. Using Lemma 2.8 to simplify the RHS of the above formula, we have

$$\begin{aligned}
RHS=&-[-c_{ij}-m+1]_iK_ix_{i,w_\bullet\tau j;m-1}K'_j(K'_i)^m\\
&-q_i^{-2(c_{ij}+2m)}[m+1]_ix_{i,w_\bullet\tau j;m+1}K'_j(K'_i)^{m+1}\\
=&-[-c_{ij}-m+1]_iq_i^{c_{ij}+2m-2}x_{i,w_\bullet\tau j;m-1}K'_j(K'_i)^{m-1}\widetilde{k}_i\\
&-q_i^{-2(c_{ij}+2m)}[m+1]_ix_{i,w_\bullet\tau j;m+1}K'_j(K'_i)^{m+1}.
\end{aligned}$$

Combining the above two formulas, we have

$$\begin{aligned}
-q_i^{-(c_{ij}+2m)}Q_mB_i+B_iQ_m=&-[-c_{ij}-m+1]_iq_i^{c_{ij}+2m-2}Q_{m-1}\widetilde{k}_i\\
&-q_i^{-2(c_{ij}+2m)}[m+1]_iQ_{m+1}
\end{aligned}\tag{8.9}$$

Note that $R_m=(-1)^mq_i^{-2m(c_{ij}+m-1)}Q_m$. Then, by (8.9), we have

$$-q_i^{-(c_{ij}+2m)}R_mB_i+B_iR_m=[m+1]_iR_{m+1}+[-c_{ij}-m+1]_iq_i^{-(c_{ij}+2m-2)}R_{m-1}\widetilde{k}_i.\tag{8.10}$$

Comparing (8.10) with (8.1), it is clear that $R_m$ satisfies the same recursive relation as $b^+_{i,j;m}$.

Therefore, $b^+_{i,j;m} = R_m$ for $m \geq 0$. $\square$

We next formulate the relations between $\underline{b}^{\pm}_{i,j;m}$ and $y'_{i,j;m}, x'_{i,j;m}$.

**Proposition 8.9.** *We have, for $m \geq 0, j \in \mathbb{I}_\circ, j \neq i$,*

$$\underline{b}^-_{i,j;m} = y'_{i,j;m}. \tag{8.11}$$

*Proof.* This proposition is a consequence of Proposition 8.7 and Lemma 8.4. $\square$

Set $\widehat{B}_i := -q_i^{-2}\widetilde{\mathfrak{T}}_i(B_i\widetilde{k}_i^{-1}) = q_i^{-2}F_iK_iK_i'^{-1} + E_iK_i'$, and then we have $\widehat{B}_i\widetilde{\mathfrak{T}}_i(\widetilde{\Upsilon}_i) = \widetilde{\mathfrak{T}}_i(\widetilde{\Upsilon}_i)B_i$, following [WZ22, §6.4]. .

**Proposition 8.10.** *We have, for $m \geq 0, j \in \mathbb{I}_\circ, j \neq i$,*

$$\underline{b}^+_{i,j;m}\widetilde{\mathfrak{T}}_i(\widetilde{\Upsilon}_i)^{-1} = (-1)^m q_i^{-2m(c_{ij}+m-1)}\widetilde{\mathfrak{T}}_i(\widetilde{\Upsilon}_i)^{-1}\widetilde{\mathfrak{T}}_{w_\bullet}(x'_{i,\tau j;m})K'_j(K'_i)^m. \tag{8.12}$$

*Proof.* We denote $\widetilde{\mathfrak{T}}_{w_\bullet}(x'_{i,\tau j;m})$ by $x'_{i,w_\bullet\tau j;m}$ in the proof. Let $R_m$ denote RHS (8.12) i.e.,

$$R_m = (-1)^m q_i^{-2m(c_{ij}+m-1)}\widetilde{\mathfrak{T}}_i(\widetilde{\Upsilon}_i)^{-1}x'_{i,w_\bullet\tau j;m}K'_j(K'_i)^m\widetilde{\mathfrak{T}}_i(\widetilde{\Upsilon}_i).$$

We use an induction on $m$. By definition, $\underline{b}^+_{i,j;0} = \widetilde{\mathfrak{T}}_{w_\bullet}(E_{\tau j})K'_j = R_0$ and $\underline{b}^+_{i,j;-1} = 0 = R_{-1}$. Hence, it suffices to show that $R_m$ satisfies the same recursive relation as $\underline{b}^+_{i,j;m}$.

The recursive relation for $x'_{i,\tau j;m}$ is obtained by applying $\sigma$ to Lemma 2.8(2)(4). Since $w_\bullet i = i$, both $E_i, F_i$ are fixed by $\widetilde{\mathfrak{T}}_{w_\bullet}$ and hence $x'_{i,w_\bullet\tau j;m}$ satisfy the same recursive relation as $x'_{i,\tau j;m}$. Thus, we have

$$\begin{aligned} -q_i^{c_{ij}+2m} x'_{i,w_\bullet \tau j;m} E_i + E_i x'_{i,w_\bullet \tau j;m} &= [m+1]_i x'_{i,w_\bullet \tau j;m+1}, \\ -x'_{i,w_\bullet \tau j;m} F_i + F_i x'_{i,w_\bullet \tau j;m} &= [-c_{ij}-m+1]_i x'_{i,w_\bullet \tau j;m-1} K'_i. \end{aligned} \tag{8.13}$$

Let $Q_m := \widetilde{\mathfrak{T}}_i(\widetilde{\Upsilon}_i)^{-1} x'_{i,w_\bullet \tau j;m} K'_j (K'_i)^m \widetilde{\mathfrak{T}}_i(\widetilde{\Upsilon}_i)$. We first formulate the recursive relation for $Q_m$ as follows

$$\begin{aligned} &\widetilde{\mathfrak{T}}_i(\widetilde{\Upsilon}_i)\big(-q_i^{-(c_{ij}+2m)} B_i Q_m + Q_m B_i\big)\widetilde{\mathfrak{T}}_i(\widetilde{\Upsilon}_i)^{-1} \\ =& -q_i^{-(c_{ij}+2m)} \widehat{B}_i x'_{i,w_\bullet \tau j;m} K'_j (K'_i)^m + x'_{i,w_\bullet \tau j;m} K'_j (K'_i)^m \widehat{B}_i \\ =& -q_i^{-(c_{ij}+2m)} (q_i^{-2} F_i K_i K_i'^{-1} + E_i K'_i) x'_{i,w_\bullet \tau j;m} K'_j (K'_i)^m \\ &+ x'_{i,w_\bullet \tau j;m} K'_j (K'_i)^m (q_i^{-2} F_i K_i K_i'^{-1} + E_i K'_i) \\ =& -q_i^{c_{ij}+2m-2} (-F_i x'_{i,w_\bullet \tau j;m} + x'_{i,w_\bullet \tau j;m} F_i) K_i K'_j (K'_i)^{m-1} \\ &- q_i^{-2c_{ij}-4m} (E_i x'_{i,w_\bullet \tau j;m} - q_i^{c_{ij}+2m} x'_{i,w_\bullet \tau j;m} E_i) K'_j (K'_i)^{m+1}. \end{aligned}$$

Now applying (8.13) to the RHS, we obtain

$$\begin{aligned} \text{RHS } &= -q_i^{c_{ij}+2m-2} [-c_{ij}-m+1]_i x'_{i,w_\bullet \tau j;m-1} K'_j (K'_i)^{m-1} \widetilde{k}_i \\ &\quad + -q_i^{-2c_{ij}-4m} [m+1]_i x'_{i,w_\bullet \tau j;m+1} K'_j (K'_i)^{m+1}. \end{aligned}$$

Combining the above two formulas, we conclude that $Q_m$ satisfies

$$\begin{aligned} -q_i^{-(c_{ij}+2m)} B_i Q_m + Q_m B_i =& -q_i^{c_{ij}+2m-2} [-c_{ij}-m+1]_i Q_{m-1} \widetilde{k}_i \\ &- q_i^{-2c_{ij}-4m} [m+1]_i Q_{m+1}. \end{aligned} \tag{8.14}$$

Hence, by definition, $R_m$ satisfies recursive relation

$$-q_i^{-(c_{ij}+2m)} B_i R_m + R_m B_i = q_i^{-c_{ij}-2m+2} [-c_{ij}-m+1]_i R_{m-1} \widetilde{k}_i + [m+1]_i R_{m+1}, \tag{8.15}$$

which is the same as the defining recursive relation for $\underline{b}^+_{i,j;m}$. Therefore, $\underline{b}^+_{i,j;m} = R_m$ for $m \geq 0$. $\square$

## 8.4 Proof of Theorem 7.3(i)

By (2.17), the actions for $\widetilde{T}'_{i,-1}$ are given by

$$\widetilde{T}'_{i,-1}(F_j) = y_{i,j;-c_{ij}}, \qquad \widetilde{T}'_{i,-1}(E_{\tau j}) = x_{i,\tau j;-c_{ij}} \qquad \widetilde{T}'_{i,e}(K'_j) = K'_j (K'_i)^{-c_{ij}}. \tag{8.16}$$

Recall the rescaled symmetries $\widetilde{\mathcal{T}}'_{i,-1}$ from (4.2). In this case, since $\tau i = i = w_\bullet i$, $\varsigma_{i,\diamond} = -q_i^{-2}$. Then we have

$$\begin{aligned} \widetilde{\mathcal{T}}'_{i,-1}(F_j) &= y_{i,j;-c_{ij}}, \\ \widetilde{\mathcal{T}}'_{i,-1}\big(\widetilde{\mathcal{T}}_{w_\bullet}(E_{\tau j}) K'_j\big) &= (-1)^{c_{ij}} q_i^{-2c_{ij}} \widetilde{\mathcal{T}}_{w_\bullet}(x_{i,\tau j;-c_{ij}}) K'_j (K'_i)^{-c_{ij}}, \end{aligned} \tag{8.17}$$

where the second formula follows from that $\widetilde{\mathcal{T}}_{w_\bullet} \widetilde{\mathcal{T}}'_{i,-1} = \widetilde{\mathcal{T}}'_{i,-1} \widetilde{\mathcal{T}}_{w_\bullet}$.

We also have analogous formulas for $\widetilde{\mathcal{T}}''_{i,+1}$

$$\begin{aligned} \widetilde{\mathcal{T}}''_{i,+1}(F_j) &= y'_{i,j;-c_{ij}}, \\ \widetilde{\mathcal{T}}''_{i,+1}\big(\widetilde{\mathcal{T}}_{w_\bullet}(E_{\tau j}) K'_j\big) &= (-1)^{c_{ij}} q_i^{-2c_{ij}} \widetilde{\mathcal{T}}_{w_\bullet}(x'_{i,\tau j;-c_{ij}}) K'_j (K'_i)^{-c_{ij}}. \end{aligned} \tag{8.18}$$

Recall elements $b_{i,j;m}, \underline{b}_{i,j;m}$ defined in Definitions 8.1-8.2.

**Theorem 8.11.** *Let $j \neq i \in \mathbb{I}_\circ$.*

*(1) The element $b_{i,j;-c_{ij}} \in \widetilde{\mathbf{U}}^\imath$ satisfies*

$$b_{i,j;-c_{ij}} \widetilde{\Upsilon}_i = \widetilde{\Upsilon}_i \widetilde{\mathcal{T}}'_{i,-1}(B_j). \tag{8.19}$$

*(2) The element $\underline{b}_{i,j;-c_{ij}} \in \widetilde{\mathbf{U}}^\imath$ satisfies*

$$\underline{b}_{i,j;-c_{ij}} \widetilde{\mathcal{T}}_i (\widetilde{\Upsilon}_i)^{-1} = \widetilde{\mathcal{T}}_i (\widetilde{\Upsilon}_i)^{-1} \widetilde{\mathcal{T}}''_{i,+1}(B_j). \tag{8.20}$$

*(3)* $\underline{b}_{i,j;-c_{ij}} = \sigma^\imath(b_{i,j;-c_{ij}})$.

In other word, the element $\widetilde{\mathbf{T}}'_{i,-1}(B_j) := b_{i,j;-c_{ij}}$ satisfies (7.5) and $\widetilde{\mathbf{T}}''_{i,+1}(B_j) := \underline{b}_{i,j;-c_{ij}}$ satisfies (7.6). Hence, we have proved the first statement in Theorem 7.3(i).

*Proof.* We prove (1). By Lemma 5.1 and (8.17), we have

$$\begin{aligned}
\widetilde{\Upsilon}_i \widetilde{\mathcal{T}}'_{i,-1}(B_j) =& \widetilde{\Upsilon}_i \widetilde{\mathcal{T}}'_{i,-1}(F_j) + \widetilde{\Upsilon}_i \widetilde{\mathcal{T}}'_{i,-1}(E_j K'_j) \\
=& \widetilde{\Upsilon}_i \widetilde{\mathcal{T}}'_{i,-1}(F_j) + \widetilde{\mathcal{T}}'_{i,-1}(E_j K'_j) \widetilde{\Upsilon}_i \\
=& \widetilde{\Upsilon}_i y_{i,j;-c_{ij}} + (-1)^{c_{ij}} q_i^{-2c_{ij}} \widetilde{\mathcal{T}}_{w_\bullet}(x_{i,\tau j;-c_{ij}}) K'_j (K'_i)^{-c_{ij}} \widetilde{\Upsilon}_i.
\end{aligned}$$

On the other hand, setting $m = -c_{ij}$ in Proposition 8.7-8.8, we have

$$b^-_{i,j;-c_{ij}} \widetilde{\Upsilon}_i = \widetilde{\Upsilon}_i y_{i,j;-c_{ij}}, \qquad b^+_{i,j;-c_{ij}} = (-1)^{c_{ij}} q_i^{-2c_{ij}} \widetilde{\mathcal{T}}_{w_\bullet}(x_{i,\tau j;-c_{ij}}) K'_j (K'_i)^{-c_{ij}}.$$

Therefore, by the above two formulas, we have

$$\widetilde{\Upsilon}_i \widetilde{\mathcal{T}}'_{i,-1}(B_j) = b^-_{i,j;-c_{ij}} \widetilde{\Upsilon}_i + b^+_{i,j;-c_{ij}} \widetilde{\Upsilon}_i = b_{i,j;-c_{ij}} \widetilde{\Upsilon}_i.$$

(2) is proved by similar arguments above, using intertwining relations in Proposition 8.9-8.10 and (8.18).

(3) is a consequence of Proposition 8.3. □

# 9 Higher rank formulas for $c_{i,\tau i} = 0, w_\bullet i = i$

Fix $i \in \mathbb{I}_{\circ,\tau}$ such that $c_{i,\tau i} = 0, w_\bullet i = i$ throughout this section. Since $\tau$ commutes with $w_\bullet$, we also have $w_\bullet \tau i = \tau i$. In this case, we have $q_i = q_{\tau i}, B_i = F_i + E_{\tau i} K_i', \mathbf{r}_i = s_i s_{\tau i}$.

We define higher rank root vectors $b_{i,\tau i,j;m_1,m_2}, \underline{b}_{i,\tau i,j;m_1,m_2} \in \widetilde{\mathbf{U}}^\imath$ in Definitions 9.6-9.7 via recursive relations. The divided power formulations for these elements are obtained in Proposition 9.11. We show that $b_{i,\tau i,j;-c_{ij},-c_{\tau i,j}}, \underline{b}_{i,\tau i,j;-c_{ij},-c_{\tau i,j}}$ provide the higher rank formulas for $\widetilde{\mathbf{T}}'_{i,-1}(B_j), \widetilde{\mathbf{T}}''_{i,+1}(B_j)$ in Theorem 9.17 and complete the proof for Theorem 7.3(ii).

## 9.1 Definitions of root vectors

**Definition 9.1.** Define elements $y_{i,\tau i,j;m_1,m_2}, x_{i,\tau i,j;m_1,m_2}$ for $m_1, m_2 \geq 0, j \neq i, \tau i, j \in \mathbb{I}_\circ$ as follows

$$y_{i,\tau i,j;m_1,m_2} = \sum_{r=0}^{m_1} \sum_{s=0}^{m_2} (-1)^{r+s} q_i^{-r(m_1+c_{ij}-1)} q_{\tau i}^{-s(m_2+c_{\tau i,j}-1)} F_i^{(m_1-r)} F_{\tau_i}^{(m_2-s)} F_j F_{\tau i}^{(s)} F_i^{(r)}$$

$$x_{i,\tau i,j;m_1,m_2} = \sum_{r=0}^{m_1} \sum_{s=0}^{m_2} (-1)^{r+s} q_i^{r(m_1+c_{ij}-1)} q_{\tau i}^{s(m_2+c_{\tau i,j}-1)} E_i^{(r)} E_{\tau i}^{(s)} E_j E_{\tau i}^{(m_2-s)} E_i^{(m_1-r)}.$$

*Remark* 9.2. Recall the elements $y_{i,j;m}, x_{i,j;m}$ in Section 8. We have

$$\begin{aligned} y_{i,\tau i,j;m_1,m_2} &= \sum_{r=0}^{m_1} (-1)^r q_i^{-r(m_1+c_{ij}-1)} F_i^{(m_1-r)} y_{\tau i,j;m_2} F_i^{(r)} \\ &= \sum_{s=0}^{m_2} (-1)^s q_i^{-s(m_2+c_{\tau i,j}-1)} F_{\tau i}^{(m_2-s)} y_{i,j;m_1} F_{\tau i}^{(s)}. \end{aligned}$$

In particular, $y_{i,\tau i,j;m,0} = y_{i,j;m}$ and $y_{i,\tau i,j;0,m} = y_{\tau i,j;m}$. Similarly, we have $x_{i,\tau i,j;m,0} = x_{i,j;m}$ and $x_{i,\tau i,j;0,m} = x_{\tau i,j;m}$.

**Definition 9.3.** Define elements $y'_{i,\tau i,j;m_1,m_2}, x'_{i,\tau i,j;m_1,m_2}$ for $m_1, m_2 \geq 0, j \neq i, \tau i, j \in \mathbb{I}_\circ$ as follows

$$y'_{i,\tau i,j;m_1,m_2} = \sigma(y_{i,\tau i,j;m_1,m_2}), \qquad x'_{i,\tau i,j;m_1,m_2} = \sigma(x_{i,\tau i,j;m_1,m_2}).$$

Set $y_{i,\tau i,j;m_1,m_2} = 0$ and $x_{i,\tau i,j;m_1,m_2} = 0$, if $m_1 < 0$ or $m_2 < 0$. Similar for $y'_{i,\tau i,j;m_1,m_2}, x'_{i,\tau i,j;m_1,m_2}$.

**Lemma 9.4.** *We have, for $j \neq i, \tau i, j \in \mathbb{I}_\circ, m_1, m_2 \in \mathbb{Z}$*

*(1)* $-q_i^{-(c_{ij}+2m_1)} y_{i,\tau i,j;m_1,m_2} F_i + F_i y_{i,\tau i,j;m_1,m_2} = [m_1+1]_i \, y_{i,\tau i,j;m_1+1,m_2}$.

*(2)* $-q_{\tau i}^{-(c_{\tau i,j}+2m_2)} y_{i,\tau i,j;m_1,m_2} F_{\tau i} + F_{\tau i} y_{i,\tau i,j;m_1,m_2} = [m_2+1]_{\tau i} \, y_{i,\tau i,j;m_1,m_2+1}$.

*(3)* $-y_{i,\tau i,j;m_1,m_2} E_i + E_i y_{i,\tau i,j;m_1,m_2} = [-c_{ij}-m_1+1]_i \, y_{i,\tau i,j;m_1-1,m_2} K'_i$.

*(4)* $-y_{i,\tau i,j;m_1,m_2} E_{\tau i} + E_{\tau i} y_{i,\tau i,j;m_1,m_2} = [-c_{\tau i,j}-m_2+1]_{\tau i} \, y_{i,\tau i,j;m_1,m_2-1} K'_{\tau i}$.

*Proof.* Clearly, $[F_{\tau i}, E_i] = [F_i, E_{\tau i}] = 0$. Since $c_{i,\tau i} = 0$, we have $[F_i, F_{\tau i}] = 0$. Then these four identities are immediate consequences of Lemma 2.8 and Remark 9.2. □

**Lemma 9.5.** *We have, for $j \neq i, \tau i, j \in \mathbb{I}_\circ, m_1, m_2 \in \mathbb{Z}$*

(1) $-q_i^{c_{ij}+2m_1} E_i x_{i,\tau i,j;m_1,m_2} + x_{i,\tau i,j;m_1,m_2} E_i = [m_1+1]_i x_{i,\tau i,j;m_1+1,m_2}$.

(2) $-q_{\tau i}^{c_{\tau i,j}+2m_2} E_{\tau i} x_{i,\tau i,j;m_1,m_2} + x_{i,\tau i,j;m_1,m_2} E_{\tau i} = [m_2+1]_{\tau i} x_{i,\tau i,j;m_1,m_2+1}$.

(3) $-F_i x_{i,\tau i,j;m_1,m_2} + x_{i,\tau i,j;m_1,m_2} F_i = [-c_{ij}-m_1+1]_i \; K_i x_{i,\tau i,j;m_1-1,m_2}$.

(4) $-F_{\tau i} x_{i,\tau i,j;m_1,m_2} + x_{i,\tau i,j;m_1,m_2} F_{\tau i} = [-c_{\tau i,j}-m_2+1]_i \; K_{\tau i} x_{i,\tau i,j;m_1,m_2-1}$.

*Proof.* By definition $x_{i,\tau i,j;m_1,m_2} = \sigma\omega\psi(y_{i,\tau i,j;m_1,m_2})$. These four identities are obtained by applying $\sigma\omega\psi$ to those four identities in Lemma 9.4. □

**Definition 9.6.** Let $j \neq i, \tau i, j \in \mathbb{I}_\circ$. Define $b^{\pm}_{i,\tau i,j;m_1,m_2}$ for $m_1, m_2 \geq -1$ to be the elements in $\widetilde{\mathbf{U}}$ determined by the following two recursive relations,

$$\begin{aligned}
&- q_i^{-(c_{ij}+2m_1)} b^{\pm}_{i,\tau i,j;m_1,m_2} B_i + B_i b^{\pm}_{i,\tau i,j;m_1,m_2} \\
=&[m_1+1]_i \; b^{\pm}_{i,\tau i,j;m_1+1,m_2} + q_i^{-(c_{ij}+2m_1)} [-c_{\tau i,j}-m_2+1]_i \; b^{\pm}_{i,\tau i,j;m_1,m_2-1} \widetilde{k}_i, \qquad (9.1)\\
&- q_i^{-(c_{\tau i,j}+2m_2)} b^{\pm}_{i,\tau i,j;m_1,m_2} B_{\tau i} + B_{\tau i} b^{\pm}_{i,\tau i,j;m_1,m_2} \\
=&[m_2+1]_i \; b^{\pm}_{i,\tau i,j;m_1,m_2+1} + q_i^{-(c_{\tau i,j}+2m_2)} [-c_{i,j}-m_1+1]_i \; b^{\pm}_{i,\tau i,j;m_1-1,m_2} \widetilde{k}_{\tau i}, \qquad (9.2)
\end{aligned}$$

where we set

$$b^{\pm}_{i,\tau i,j;m_1,-1} = b^{\pm}_{i,\tau i,j;-1,m_2} = 0, \qquad b^{-}_{i,\tau i,j;0,0} = F_j, \qquad b^{+}_{i,\tau i,j;0,0} = \widetilde{\mathfrak{T}}_{w_\bullet}(E_{\tau j}) K'_j. \qquad (9.3)$$

Set $b_{i,\tau i,j;m_1,m_2} = b^{-}_{i,\tau i,j;m_1,m_2} + b^{+}_{i,\tau i,j;m_1,m_2}$. Since $b_{i,\tau i,j,0,0} = B_j \in \widetilde{\mathbf{U}}^\imath$, one can inductively show that $b_{i,\tau i,j;m_1,m_2} \in \widetilde{\mathbf{U}}^\imath$ for $m_1, m_2 \geq 0$.

**Definition 9.7.** Let $j \neq i, \tau i, j \in \mathbb{I}_\circ$. Define $\underline{b}^{\pm}_{i,\tau i,j;m_1,m_2}$ for $m_1, m_2 \geq -1$ to be the elements in $\widetilde{\mathbf{U}}$ determined by the following two recursive relations,

$$- q_i^{-(c_{ij}+2m_1)} B_i \underline{b}^{\pm}_{i,\tau i,j;m_1,m_2} + \underline{b}^{\pm}_{i,\tau i,j;m_1,m_2} B_i$$

$$=[m_1+1]_i\, \underline{b}^{\pm}_{i,\tau i,j;m_1+1,m_2} + q_i^{-(c_{\tau i,j}+2m_2-2)}[-c_{\tau i,j}-m_2+1]_i\, \underline{b}^{\pm}_{i,\tau i,j;m_1,m_2-1}\widetilde{k}_{\tau i}, \tag{9.4}$$

$$- q_i^{-(c_{\tau i,j}+2m_2)} B_{\tau i}\underline{b}^{\pm}_{i,\tau i,j;m_1,m_2} + \underline{b}^{\pm}_{i,\tau i,j;m_1,m_2} B_{\tau i}$$

$$=[m_2+1]_i\, \underline{b}^{\pm}_{i,\tau i,j;m_1,m_2+1} + q_i^{-(c_{ij}+2m_1-2)}[-c_{i,j}-m_1+1]_i\, \underline{b}^{\pm}_{i,\tau i,j;m_1-1,m_2}\widetilde{k}_i, \tag{9.5}$$

where we set

$$\underline{b}^{\pm}_{i,\tau i,j;m_1,-1} = \underline{b}^{\pm}_{i,\tau i,j;-1,m_2} = 0, \qquad \underline{b}^{-}_{i,\tau i,j;0,0} = F_j, \qquad \underline{b}^{+}_{i,\tau i,j;0,0} = \widetilde{\mathcal{T}}_{w_\bullet}(E_{\tau j})K'_j. \tag{9.6}$$

Set $\underline{b}_{i,\tau i,j;m_1,m_2} = \underline{b}^{-}_{i,\tau i,j;m_1,m_2} + \underline{b}^{+}_{i,\tau i,j;m_1,m_2}$. One can also inductively show that $\underline{b}_{i,\tau i,j;m_1,m_2} \in \widetilde{\mathbf{U}}^\imath$ for $m_1, m_2 \geq 0$.

Recall the anti-involution $\sigma^\imath$ on $\widetilde{\mathbf{U}}^\imath$ from Proposition 3.12.

**Proposition 9.8.** *Let $j \in \mathbb{I}_\circ, j \neq i, \tau i$. Then $\underline{b}_{i,j;m_1,m_2} = \sigma^\imath(b_{i,j;m_1,m_2})$ for $m_1, m_2 \geq 0$.*

*Proof.* By Definition 9.6-9.7, it is clear that $\sigma^\imath(b_{i,j;m_1,m_2})$ satisfies the same recursive relations as $\underline{b}_{i,j;m_1,m_2}$. Since $\underline{b}_{i,j;0,0} = B_j = \sigma^\imath(b_{i,j;0,0})$, this proposition follows by induction. □

**Lemma 9.9.** *Let $j \in \mathbb{I}_\circ, j \neq i, \tau i$. We have, for $m_1, m_2 \geq 0$,*

$$\underline{b}^{-}_{i,\tau i,j;m_1,m_2} = \sigma\big(\widetilde{\Upsilon}_i^{-1} b^{-}_{i,\tau i,j;m_1,m_2} \widetilde{\Upsilon}_i\big).$$

*Proof.* Consider the subalgebra $\widetilde{\mathbf{U}}^-_{[i;j]}$ of $\widetilde{\mathbf{U}}$ generated by $B_i, B_{\tau i}, \widetilde{k}_i, \widetilde{k}_{\tau i}, F_j$. It is clear from the above definitions that $b^{-}_{i,\tau i,j;m_1,m_2}, \underline{b}^{-}_{i,\tau i,j;m_1,m_2} \in \widetilde{\mathbf{U}}^-_{[i;j]}$. By Theorem 3.6 and Lemma 5.1, there is a well-defined anti-automorphism $\sigma_{ij}$ on $\widetilde{\mathbf{U}}^-_{[i;j]}$, which is given by

$$\sigma_{ij} : x \mapsto \sigma\big(\widetilde{\Upsilon}_i^{-1} x \widetilde{\Upsilon}_i\big).$$

Moreover, $\sigma_{ij}$ fixes $B_i, B_{\tau i}, F_j$ and sends $\widetilde{k}_i \leftrightarrow \widetilde{k}_{\tau i}$. Applying $\sigma_{ij}$ to (9.1)-(9.2), it is clear that $\sigma_{ij}(b^-_{i,\tau i,j;m_1,m_2})$ satisfies the same recursive relations as $\underline{b}^-_{i,\tau i,j;m_1,m_2}$. Then the desired identity follows by induction. □

## 9.2 A divided power formulation

In this subsection, we derive the formulas of root vectors $b_{i,\tau i,j;m_1,m_2}, \underline{b}_{i,\tau i,j;m_1,m_2}$, in terms of divided powers of generators of $\widetilde{\mathbf{U}}^\imath$, from their recursive Definitions 9.6-9.7. Denote

$$[\widetilde{k}_i; a] := \frac{\widetilde{k}_i q_i^a - \widetilde{k}_{\tau i} q_i^{-a}}{q_i - q_i^{-1}}.$$

**Lemma 9.10.** *We have for any $m \geq 0$,*

$$B_{\tau i} B_i^{(m)} - B_i^{(m)} B_{\tau i} = B_i^{(m-1)} [\widetilde{k}_i; 1-m].$$

*Proof.* For $c_{i,\tau i} = 0$, $B_i, B_{\tau i}$ satisfy the following relation

$$[B_{\tau i}, B_i] = \frac{\widetilde{k}_i - \widetilde{k}_{\tau i}}{q_i - q_i^{-1}}.$$

Using this relation, one can then prove this lemma by induction on $m$ c.f. [Ja95, §1.3]. □

**Proposition 9.11.** *Let $j \in \mathbb{I}_\circ, j \neq i, \tau i$. The elements $b_{i,\tau i,j;m_1,m_2}, \underline{b}_{i,\tau i,j;m_1,m_2}$ defined in Definitions 9.6-9.7 admit following formulas*

$$b_{i,\tau i,j;m_1,m_2} = \sum_{u=0}^{\min(m_1,m_2)} \sum_{r=0}^{m_1-u} \sum_{s=0}^{m_2-u} (-1)^{r+s+u} q_i^{r(\alpha-m_1-u+1)+s(\beta-m_2+u+1)+u(\alpha-m_1+1)}$$

$$\times \begin{bmatrix} \beta - m_2 + u \\ u \end{bmatrix}_i B_i^{(m_1-r-u)} B_{\tau i}^{(m_2-s-u)} B_j B_{\tau i}^{(s)} B_i^{(r)} \widetilde{k}_i^u. \tag{9.7}$$

$$\underline{b}_{i,\tau i,j;m_1,m_2} = \sum_{u=0}^{\min(m_1,m_2)} \sum_{r=0}^{m_1-u} \sum_{s=0}^{m_2-u} (-1)^{r+s+u} q_i^{r(\alpha-m_1-u+1)+s(\beta-m_2+u+1)+u(\alpha-m_1+1)}$$

$$\times \begin{bmatrix} \beta - m_2 + u \\ u \end{bmatrix}_i \widetilde{k}_{\tau i}^u B_i^{(r)} B_{\tau i}^{(s)} B_j B_{\tau i}^{(m_2-s-u)} B_i^{(m_1-r-u)}. \tag{9.8}$$

*Proof.* The second formula (9.8) is obtained by applying $\sigma_\imath$ to the first formula. Hence, it suffices to show that the elements $b_{i,\tau i,j;m_1,m_2}$ defined by (9.7) satisfy the recursive relations (9.1)-(9.2).

(1) Indeed, we have

$$\begin{aligned}
&\text{LHS (9.1)} - \text{ RHS (9.1)} \\
&= - q_i^{\alpha-2m_1} b_{i,\tau i,j;m_1,m_2} B_i + B_i b_{i,\tau i,j;m_1,m_2} \\
&\quad - [m_1+1]_i \, b_{i,\tau i,j;m_1+1,m_2} - q_i^{\alpha-2m_1} [\beta - m_2 + 1]_i \, b_{i,\tau i,j;m_1,m_2-1} \widetilde{k}_i \\
&= \sum_{u=0}^{\min(m_1,m_2)} \sum_{r=0}^{m_1-u} \sum_{s=0}^{m_2-u} (-1)^{r+s+u} q_i^{r(\alpha-m_1-u+1)+s(\beta-m_2+u+1)+u(\alpha-m_1+1)} \varepsilon_{r,s,u} \\
&\quad \times \begin{bmatrix} \beta - m_2 + u \\ u \end{bmatrix}_i B_i^{(m_1-r-u)} B_{\tau i}^{(m_2-s-u)} B_j B_{\tau i}^{(s)} B_i^{(r)} \widetilde{k}_i^u.
\end{aligned}$$

where the scalar $\varepsilon_{r,s,u}$ is given by

$$\begin{aligned}
\varepsilon_{r,s,u} =& [r]_i q_i^{-2u-(\alpha-m_1-u+1)+\alpha-2m_1} + [m_1 - r - u + 1]_i - [m_1+1]_i q_i^{-r-u} \\
&+ [u]_i q_i^{-(\alpha-m_1+1)+r} \\
=& 0.
\end{aligned}$$

Hence, we have proved that $b_{i,\tau i,j;m_1,m_2}$ defined by (9.7) satisfy (9.1).

(2) We next show that $b_{i,\tau i,j;m_1,m_2}$ satisfy (9.2). By Lemma 9.10, we have

$$\begin{aligned}
&\text{LHS } (9.2) - \text{ RHS } (9.2) \\
=& -q_i^{\beta-2m_2} b_{i,\tau i,j;m_1,m_2} B_{\tau i} + B_{\tau i} b_{i,\tau i,j;m_1,m_2} \\
& - [m_2+1]_i \, b_{i,\tau i,j;m_1,m_2+1} - q_i^{\beta-2m_2}[\alpha-m_1+1]_i \, b_{i,\tau i,j;m_1-1,m_2} \widetilde{k}_{\tau i} \\
=& \sum_{u\geq 0} \sum_{r=0}^{m_1-u} \sum_{s=0}^{m_2+1-u} (-1)^{r+s+u} q_i^{r(\alpha-m_1-u+1)+s(\beta-m_2+u+1)+u(\alpha-m_1+1)} \xi_{r,s,u} \\
& \qquad \times \begin{bmatrix} \beta-m_2+u \\ u \end{bmatrix}_i B_i^{(m_1-r-u)} B_{\tau i}^{(m_2+1-s-u)} B_j B_{\tau i}^{(s)} B_i^{(r)} \widetilde{k}_i^u \\
& + \sum_{u\geq 0} \sum_{r=0}^{m_1-1-u} \sum_{s=0}^{m_2-u} (-1)^{r+s+u} q_i^{r(\alpha-m_1-u+1)+s(\beta-m_2+u+1)+u(\alpha-m_1+1)} \xi'_{r,s,u} \\
& \qquad \times \begin{bmatrix} \beta-m_2+u \\ u \end{bmatrix}_i B_i^{(m_1-1-r-u)} B_{\tau i}^{(m_2-s-u)} B_j B_{\tau i}^{(s)} B_i^{(r)} \widetilde{k}_i^u \widetilde{k}_{\tau i}.
\end{aligned}$$

where the scalars $\xi_{r,s,u}, \xi'_{r,s,u}$ are given by

$$\begin{aligned}
\xi_{r,s,u} =& [s]_i q_i^{-(\beta-m_2+u+1)+2u+\beta-2m_2} + [m_2+1-s-u]_i - q_i^{-s}[m_2+1]_i \frac{[\beta-m_2]_i}{[\beta-m_2+u]_i} \\
& + \frac{q_i^{\beta+u-2m_2-s-1} - q_i^{-\beta-u+2m_2-s+1}}{q_i-q_i^{-1}} \frac{[u]_i}{[\beta-m_2+u]_i} \\
=& q_i^{-s} \frac{[m_2+1-u]_i[\beta-m_2+u]_i - [m_2+1]_i[\beta-m_2]_i + [\beta+u-2m_2-1]_i[u]_i}{[\beta-m_2+u]_i} \\
=& 0, \\
\xi'_{r,s,u} =& -q_i^{\beta-2m_2+r+u}[\alpha-m_1+1]_i + \frac{q_i^{\beta+\alpha+r+u-m_1-2m_2+1} - q_i^{\beta-\alpha+r+u+m_1-2m_2-1}}{q_i-q_i^{-1}} \\
=& 0.
\end{aligned}$$

Hence, we have proved that $b_{i,\tau i,j;m_1,m_2}$ defined by (9.7) satisfy (9.2). $\square$

*Remark* 9.12. By Remark 9.2 and Propositions 9.13-9.14,9.15-9.16, we have

$$b_{i,\tau i,j;m_1,m_2}=0,\quad \underline{b}_{i,\tau i,j;m_1,m_2}=0,\quad \text{ if } m_1>-c_{ij},\quad \text{ or } m_2>-c_{\tau i,j}.$$

Furthermore, according to the divided power formulations, the Serre relations in $\widetilde{\mathbf{U}}^\imath$ are given by

$$b_{i,\tau i,j;-c_{ij}+1,0}=0,\qquad b_{i,\tau i,j;0,-c_{\tau i,j}+1}=0,\qquad j\neq i,\tau i. \tag{9.9}$$

## 9.3 Intertwining properties

We establish precise intertwining relations between those elements $b^{\pm}_{i,\tau i,j;m_1,m_2}$ (*resp.* $\underline{b}^{\pm}_{i,\tau i,j;m_1,m_2}$) and elements $y_{i,\tau i,j;m_1,m_2}, x_{i,\tau i,j;m_1,m_2}$ (*resp.* $y'_{i,\tau i,j;m_1,m_2}, x'_{i,\tau i,j;m_1,m_2}$). These relations will be the key for the construction of relative braid group action on $\widetilde{\mathbf{U}}^\imath$.

**Proposition 9.13.** *Let $j\in\mathbb{I}_\circ$ such that $j\neq i,\tau i$. We have, for $m_1,m_2\geq 0$,*

$$b^-_{i,\tau i,j;m_1,m_2}\widetilde{\Upsilon}_i=\widetilde{\Upsilon}_i y_{i,\tau i,j;m_1,m_2}. \tag{9.10}$$

*Proof.* Let $R_{m_1,m_2}$ denote $\widetilde{\Upsilon}_i y_{i,\tau i,j;m_1,m_2}\widetilde{\Upsilon}_i^{-1}$. By Lemma 5.1, $R_{0,0}=F_j=b^-_{i,\tau i,j;0,0}$. Moreover, by definition, $y_{i,\tau i,j;m_1,-1}=y_{i,\tau i,j;-1,m_2}=0=R_{m_1,-1}=R_{-1,m_2}$. Hence, it suffices to prove that $R_{m_1,m_2}$ satisfies the same recursive relations as $b^-_{i,\tau i,j;m_1,m_2}$.

Recall that $B_i^\sigma=F_i+K_iE_{\tau i}$ . We have, by Theorem 3.6,

$$\begin{aligned}&\widetilde{\Upsilon}_i^{-1}\big(-q_i^{-(c_{ij}+2m_1)}R_{m_1,m_2}B_i+B_iR_{m_1,m_2}\big)\widetilde{\Upsilon}_i\\=&-q_i^{-(c_{ij}+2m_1)}y_{i,\tau i,j;m_1,m_2}B_i^\sigma+B_i^\sigma y_{i,\tau i,j;m_1,m_2}\end{aligned}$$

$$
\begin{aligned}
=& - q_i^{-(c_{ij}+2m_1)} y_{i,\tau i,j;m_1,m_2}(F_i + K_i E_{\tau i}) + (F_i + K_i E_{\tau i}) y_{i,\tau i,j;m_1,m_2} \\
=& - q_i^{-(c_{ij}+2m_1)} y_{i,\tau i,j;m_1,m_2} F_i + F_i y_{i,\tau i,j;m_1,m_2} \\
& + q_i^{-(c_{ij}+2m_1)} (-y_{i,\tau i,j;m_1,m_2} E_{\tau i} + E_{\tau i} y_{i,\tau i,j;m_1,m_2}) K_i.
\end{aligned}
$$

Now using Lemma 9.4 to simplify the RHS of above formula, we have

$$
\begin{aligned}
\text{RHS} =& [m_1+1]_i \; y_{i,\tau i,j;m_1+1,m_2} \\
& + q_i^{-(c_{ij}+2m_1)} [-c_{\tau i,j} - m_2 + 1]_{\tau i} \; y_{i,\tau i,j;m_1,m_2-1} K_i K'_{\tau i}.
\end{aligned}
$$

Combining the above two formulas, we have

$$
\begin{aligned}
& - q_i^{-(c_{ij}+2m_1)} R_{m_1,m_2} B_i + B_i R_{m_1,m_2} \\
=& [m_1+1]_i \; R_{m_1+1,m_2} + q_i^{-(c_{ij}+2m_1)} [-c_{\tau i,j} - m_2 + 1]_i \; R_{m_1,m_2-1} K_i K'_{\tau i}.
\end{aligned} \tag{9.11}
$$

The following variant of (9.11) can be obtained by a similar strategy

$$
\begin{aligned}
& - q_i^{-(c_{\tau i,j}+2m_2)} R_{m_1,m_2} B_{\tau i} + B_{\tau i} R_{m_1,m_2} \\
=& [m_2+1]_i \; R_{m_1,m_2+1} + q_i^{-(c_{\tau i,j}+2m_2)} [-c_{i,j} - m_1 + 1]_i \; R_{m_1-1,m_2} K'_i K_{\tau i}.
\end{aligned} \tag{9.12}
$$

Comparing (9.11)-(9.12) with (9.1)-(9.2), it is clear that $R_{m_1,m_2}$ satisfies the same recursive relations as $b^-_{i,\tau i,j;m_1,m_2}$. Therefore, we have proved (9.10). □

**Proposition 9.14.** *Let $j \in \mathbb{I}_\circ$ such that $j \neq i, \tau i$. We have, for $m_1, m_2 \geq 0$,*

$$
\begin{aligned}
b^+_{i,\tau i,j;m_1,m_2} =& (-1)^{m_1+m_2} q_i^{-(m_1+m_2)(c_{ij}+c_{\tau i,j}+m_1+m_2-1)} \times \\
& \times \widetilde{\mathcal{T}}_{w_\bullet}(x_{\tau i,i,\tau j;m_1,m_2}) K'_j (K'_i)^{m_1} (K'_{\tau i})^{m_2}.
\end{aligned} \tag{9.13}
$$

*(Note the shift of indices on the right-hand side.)*

*Proof.* Let $P_{m_1,m_2}$ denote the RHS (9.13) and $x_{\tau i,i,w_\bullet \tau j;m_1,m_2}$ denote $\widetilde{\mathcal{T}}_{w_\bullet}(x_{\tau i,i,\tau j;m_1,m_2})$. By definition (9.3), $b^+_{i,\tau i,j;0,0} = \widetilde{\mathcal{T}}_{w_\bullet}(E_{\tau j})K'_j = x_{\tau i,i,w_\bullet\tau j;0,0}K'_j = P_{0,0}$. Moreover, by definition, $x_{\tau i,i,\tau j;-1,m_2} = x_{\tau i,i,\tau j;m_1,-1} = 0$ and $b^+_{i,\tau i,j;-1,m_2} = b^+_{i,\tau i,j;m_1,-1} = 0$. Thus, it suffices to show that $P_{m_1,m_2}$ satisfies the same recursive relations as $b^+_{i,\tau i,j;m_1,m_2}$.

Let $Q_{m_1,m_2}$ denote $x_{\tau i,i,w_\bullet\tau j;m_1,m_2}K'_j(K'_i)^{m_1}(K'_{\tau i})^{m_2}$. We first formulate the recursive relations for $Q_{m_1,m_2}$. We have

$$\begin{aligned}
&- q_i^{-(c_{ij}+2m_1)}Q_{m_1,m_2}B_i + B_iQ_{m_1,m_2}\\
&= -q_i^{-(c_{ij}+2m_1)}x_{\tau i,i,w_\bullet\tau j;m_1,m_2}K'_j(K'_i)^{m_1}(K'_{\tau i})^{m_2}(F_i+E_{\tau i}K'_i)\\
&\quad + (F_i+E_{\tau i}K'_i)x_{\tau i,i,w_\bullet\tau j;m_1,m_2}K'_j(K'_i)^{m_1}(K'_{\tau i})^{m_2}\\
&= -(x_{\tau i,i,w_\bullet\tau j;m_1,m_2}F_i - F_i x_{\tau i,i,w_\bullet\tau j;m_1,m_2})K'_j(K'_i)^{m_1}(K'_{\tau i})^{m_2}\\
&\quad - q_i^{-(c_{ij}+c_{\tau i,j}+2m_1+2m_2)}\big[x_{\tau i,i,w_\bullet\tau j;m_1,m_2}, E_{\tau i}\big]_{q_i^{c_{i,j}+2m_1}}K'_j(K'_i)^{m_1+1}(K'_{\tau i})^{m_2}.
\end{aligned}$$

Since $w_\bullet i = i$, both $F_i, E_{\tau i}$ are fixed by $\widetilde{\mathcal{T}}_{w_\bullet}$. Then the recursion involving $x_{\tau i,i,w_\bullet\tau j;m_1,m_2}$ and $F_i$ (*resp.* $E_{\tau i}$) is the same as the recursion involving $x_{\tau i,i,\tau j;m_1,m_2}$ and $F_i$ (*resp.* $E_{\tau i}$). By Lemma 9.5, one can obtain those recursions for $x_{\tau i,i,w_\bullet\tau j;m_1,m_2}$ and then the RHS of the above formula is simplified as below

$$\begin{aligned}
\text{RHS} &= -[-c_{\tau i,j}-m_2+1]_i K_i x_{\tau i,i,w_\bullet\tau j;m_1,m_2-1}K'_j(K'_i)^{m_1}(K'_{\tau i})^{m_2}\\
&\quad - q_i^{-(c_{ij}+c_{\tau i,j}+2m_1+2m_2)}[m_1+1]_i x_{\tau i,i,w_\bullet\tau j;m_1+1,m_2}K'_j(K'_i)^{m_1+1}(K'_{\tau i})^{m_2}\\
&= -q_i^{2m_2-2+c_{\tau i,j}}[-c_{\tau i,j}-m_2+1]_i x_{\tau i,i,w_\bullet\tau j;m_1,m_2-1}K'_j(K'_i)^{m_1}(K'_{\tau i})^{m_2-1}\widetilde{k}_i\\
&\quad - q_i^{-(c_{ij}+c_{\tau i,j}+2m_1+2m_2)}[m_1+1]_i x_{\tau i,i,w_\bullet\tau j;m_1+1,m_2}K'_j(K'_i)^{m_1+1}(K'_{\tau i})^{m_2}.
\end{aligned}$$

Combining the above two formulas, we have

$$\begin{aligned}
&- q_i^{-(c_{ij}+2m_1)} Q_{m_1,m_2} B_i + B_i Q_{m_1,m_2} \\
=&- q_i^{-(c_{ij}+c_{\tau i,j}+2m_1+2m_2)} [m_1+1]_i Q_{m_1+1,m_2} + q_i^{2m_2-2+c_{\tau i,j}} [c_{\tau i,j}+m_2-1]_i Q_{m_1,m_2-1} \widetilde{k}_i.
\end{aligned}$$

Hence, $P_{m_1,m_2}$ satisfies the following recursive relation,

$$\begin{aligned}
&- q_i^{-(c_{ij}+2m_1)} P_{m_1,m_2} B_i + B_i P_{m_1,m_2} \\
=&[m_1+1]_i P_{m_1+1,m_2} + q_i^{-(c_{ij}+2m_1)} [-c_{\tau i,j}-m_2+1]_i P_{m_1,m_2-1} \widetilde{k}_i.
\end{aligned} \tag{9.14}$$

Comparing (9.14) and (9.1), it is clear that $P_{m_1,m_2}$ satisfies the defining recursive relation (9.1) for $b^+_{i,\tau i,j;m_1,m_2}$. Using a similar strategy, one can show that $P_{m_1,m_2}$ satisfies the other defining recursive relation (9.2) for $b^+_{i,\tau i,j;m_1,m_2}$. Therefore, $b^+_{i,\tau i,j;m_1,m_2} = P_{m_1,m_2}$ for any $m_1, m_2 \geq 0$. □

We next formulate the relation between $y'_{i,\tau i,j;m_1,m_2}, x'_{i,\tau i,j;m_1,m_2}$ in Definition 9.3 and $\underline{b}^{\pm}_{i,\tau i,j;m_1,m_2}$ in Definition 9.7.

**Proposition 9.15.** *Let $j \in \mathbb{I}_\circ$ such that $j \neq i, \tau i$. We have, for $m_1, m_2 \geq 0$,*

$$\underline{b}^-_{i,\tau i,j;m_1,m_2} = y'_{i,\tau i,j;m_1,m_2}. \tag{9.15}$$

*Proof.* This proposition is a consequence of Proposition 9.13 and Lemma 9.9. □

**Proposition 9.16.** *Let $j \in \mathbb{I}_\circ$ such that $j \neq i, \tau i$. We have, for $m_1, m_2 \geq 0$,*

$$\begin{aligned}
\underline{b}^+_{i,\tau i,j;m_1,m_2} =& (-1)^{m_1+m_2} q_i^{-(m_1+m_2)(c_{ij}+c_{\tau i,j}+m_1+m_2-1)} \times \\
& \times \widetilde{\mathcal{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_i)^{-1} \widetilde{\mathcal{T}}_{w_\bullet}(x'_{\tau i,i,\tau j;m_1,m_2}) K'_j (K'_i)^{m_1} (K'_{\tau i})^{m_2} \widetilde{\mathcal{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_i).
\end{aligned} \tag{9.16}$$

*Proof.* Let $P_{m_1,m_2}$ denote the RHS (9.16) and $x'_{\tau i,i,w_\bullet \tau j;m_1,m_2}$ denote $\widetilde{\mathcal{T}}_{w_\bullet}(x'_{\tau i,i,\tau j;m_1,m_2})$. By Lemma 5.1, $P_{0,0} = \widetilde{\mathcal{T}}_{w_\bullet}(E_{\tau j})K'_j = \underline{b}^+_{i,\tau i,j;0,0}$. Moreover, by definition, we have $P_{-1,m} = P_{m,-1} = 0$ and $\underline{b}^+_{i,\tau i,j;-1,m_2} = \underline{b}^+_{i,\tau i,j;m_1,-1} = 0$. Hence, it suffices to show that $P_{m_1,m_2}$ satisfies the defining recursive relations for $\underline{b}^+_{i,\tau i,j;m_1,m_2}$.

Applying $\sigma$ to Lemma 9.5(1)(4) and then shifting the indices $i, j$ to $\tau i, \tau j$, we obtain the recursions for $x'_{\tau i,i,\tau j;m_1,m_2}$. Since $F_i, E_{\tau i}$ are fixed by $\widetilde{\mathcal{T}}_{w_\bullet}$, recursions for $x'_{\tau i,i,w_\bullet \tau j;m_1,m_2}$ are the same as recursions for $x'_{\tau i,i,\tau j;m_1,m_2}$. Thus, we have

$$
\begin{aligned}
&- x'_{\tau i,i,w_\bullet \tau j;m_1,m_2} F_i + F_i x'_{\tau i,i,w_\bullet \tau j;m_1,m_2} = [-c_{\tau i,j} - m_2 + 1]_i \, x'_{\tau i,i,w_\bullet \tau j;m_1,m_2-1} K'_i,\\
&- q_i^{c_{ij}+2m_1} x'_{\tau i,i,w_\bullet \tau j;m_1,m_2} E_{\tau i} + E_{\tau i} x'_{\tau i,i,w_\bullet \tau j;m_1,m_2} = [m_1+1]_i x'_{\tau i,i,w_\bullet \tau j;m_1+1,m_2}.
\end{aligned}
\tag{9.17}
$$

Let $Q_{m_1,m_2}$ denote $\widetilde{\mathcal{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_i)^{-1} x'_{\tau i,i,w_\bullet \tau j;m_1,m_2} K'_j (K'_i)^{m_1} (K'_{\tau i})^{m_2} \widetilde{\mathcal{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_i)$. We first formulate the recursive relation for $Q_{m_1,m_2}$.

In the case $c_{i,\tau i} = 0$, set

$$
\widehat{B}_i = -\widetilde{\mathcal{T}}_{\mathbf{r}_i}(B_{\tau i} \widetilde{k}_{\tau i}^{-1}) = F_i K_{\tau i} K'^{-1}_{\tau i} + E_{\tau i} K'_i. \tag{9.18}
$$

Then, due to [WZ22, §6.4], $\widehat{B}_i$ satisfies $\widehat{B}_i \widetilde{\mathcal{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_i) = \widetilde{\mathcal{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_i) B_i$.

We compute

$$
\begin{aligned}
&\widetilde{\mathcal{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_i)\big(-q_i^{-(c_{ij}+2m_1)} B_i Q_{m_1,m_2} + Q_{m_1,m_2} B_i\big) \widetilde{\mathcal{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_i)^{-1}\\
=& - q_i^{-(c_{ij}+2m_1)} \widehat{B}_i x'_{\tau i,i,w_\bullet \tau j;m_1,m_2} K'_j (K'_i)^{m_1} (K'_{\tau i})^{m_2}\\
&+ x'_{\tau i,i,w_\bullet \tau j;m_1,m_2} K'_j (K'_i)^{m_1} (K'_{\tau i})^{m_2} \widehat{B}_i\\
=& - q_i^{c_{ij}+2m_1} (F_i x'_{\tau i,i,w_\bullet \tau j;m_1,m_2} - x'_{\tau i,i,w_\bullet \tau j;m_1,m_2} F_i) K'_j (K'_i)^{m_1-1} (K'_{\tau i})^{m_2-1} \widetilde{k}_{\tau i}
\end{aligned}
$$

$$
\begin{aligned}
& - q_i^{-c_{\tau i,j}-2m_2}(q_i^{-c_{ij}-2m_1} E_{\tau i} x'_{\tau i,i,w_\bullet \tau j;m_1,m_2} - x'_{\tau i,i,w_\bullet \tau j;m_1,m_2} E_{\tau i}) K'_j (K'_i)^{m_1+1} (K'_{\tau i})^{m_2} \\
= & - q_i^{c_{ij}+2m_1}[-c_{\tau i,j} - m_2 + 1]_i \, x'_{\tau i,i,w_\bullet \tau j;m_1,m_2-1} K'_j (K'_i)^{m_1} (K'_{\tau i})^{m_2-1} \widetilde{k}_{\tau i} \\
& - q_i^{-c_{ij}-c_{\tau i,j}-2m_1-2m_2}[m+1]_i x'_{\tau i,i,w_\bullet \tau j;m_1+1,m_2} K'_j (K'_i)^{m_1+1} (K'_{\tau i})^{m_2},
\end{aligned}
$$

where the last equality follows by applying (9.17).

The above computation shows that $Q_{m_1,m_2}$ satisfies the following recursive relation

$$
\begin{aligned}
& - q_i^{-(c_{ij}+2m_1)} B_i Q_{m_1,m_2} + Q_{m_1,m_2} B_i \\
& = -q_i^{c_{ij}+2m_1}[-c_{\tau i,j} - m_2 + 1]_i Q_{m_1,m_2-1} \widetilde{k}_{\tau i} - q_i^{-c_{ij}-c_{\tau i,j}-2m_1-2m_2}[m+1]_i Q_{m_1+1,m_2}.
\end{aligned}
\tag{9.19}
$$

By definition, $P_{m_1,m_2} = (-1)^{m_1+m_2} q_i^{-(m_1+m_2)(c_{ij}+c_{\tau i,j}+m_1+m_2-1)} Q_{m_1,m_2}$. Hence, $P_{m_1,m_2}$ satisfies the following relation

$$
\begin{aligned}
& - q_i^{-(c_{ij}+2m_1)} B_i P_{m_1,m_2} + P_{m_1,m_2} B_i \\
& = [m+1]_i P_{m_1+1,m_2} + q_i^{-(c_{\tau i,j}+2m_2-2)}[-c_{\tau i,j} - m_2 + 1]_i P_{m_1,m_2-1} \widetilde{k}_{\tau i}.
\end{aligned}
\tag{9.20}
$$

Comparing (9.20) with (9.4), it is clear that $P_{m_1,m_2}$ satisfies the defining recursive relation (9.4) for $\underline{b}^+_{i,\tau i,j;m_1,m_2}$. Using a similar strategy, one can show that $P_{m_1,m_2}$ satisfies the other defining recursive relation (9.5) for $\underline{b}^+_{i,\tau i,j;m_1,m_2}$. Therefore, we conclude that $\underline{b}^+_{i,\tau i,j;m_1,m_2} = P_{m_1,m_2}$ for $m_1, m_2 \geq 0$. □

## 9.4 Proof of Theorem 7.3(ii)

In the case $c_{i,\tau i} = 0$, $\mathbf{r}_i = s_i s_{\tau i}$. It follows by [Lus93, §37.2] that, for $j \neq i, \tau i$,

$$
\widetilde{T}'_{\mathbf{r}_i,-1}(F_j) = y_{i,\tau i,j;-c_{ij},-c_{\tau i,j}}, \qquad \widetilde{T}'_{\mathbf{r}_i,-1}(E_{\tau j}) = x_{\tau i,i,\tau j;-c_{ij},-c_{\tau i,j}}, \tag{9.21}
$$

$$\widetilde{T}''_{\mathbf{r}_i,+1}(F_j) = y'_{i,\tau i,j;-c_{ij},-c_{\tau i,j}}, \qquad \widetilde{T}''_{\mathbf{r}_i,+1}(E_{\tau j}) = x'_{\tau i,i,\tau j;-c_{ij},-c_{\tau i,j}}. \tag{9.22}$$

Recall the rescaled symmetries $\widetilde{\mathcal{T}}'_{i,-1}, \widetilde{\mathcal{T}}''_{i,+1}$ from (4.2). In the case $c_{i,\tau i} = 0$, $\varsigma_{i,\diamond} = -q_i^{-1}$. By (9.21), we have

$$\begin{aligned}
&\widetilde{\mathcal{T}}'_{\mathbf{r}_i,-1}(F_j) = y_{i,\tau i,j;-c_{ij},-c_{\tau i,j}},\\
&\widetilde{\mathcal{T}}'_{\mathbf{r}_i,-1}(\widetilde{\mathcal{T}}_{w_\bullet}(E_{\tau j})K'_j) = (-q_i)^{-c_{ij}-c_{\tau i,j}}\widetilde{\mathcal{T}}_{w_\bullet}(x_{\tau i,i,\tau j;-c_{ij},-c_{\tau i,j}})K'_j(K'_i)^{-c_{ij}}(K'_{\tau i})^{-c_{\tau i,j}},
\end{aligned} \tag{9.23}$$

where the second formula follows from $\widetilde{\mathcal{T}}'_{\mathbf{r}_i,-1}\widetilde{\mathcal{T}}_{w_\bullet} = \widetilde{\mathcal{T}}_{w_\bullet}\widetilde{\mathcal{T}}'_{\mathbf{r}_i,-1}$.

By (9.22), we have analogous formulas for the symmetry $\widetilde{\mathcal{T}}''_{\mathbf{r}_i,+1}$

$$\begin{aligned}
&\widetilde{\mathcal{T}}''_{\mathbf{r}_i,+1}(F_j) = y'_{i,\tau i,j;-c_{ij},-c_{\tau i,j}},\\
&\widetilde{\mathcal{T}}''_{\mathbf{r}_i,+1}(\widetilde{\mathcal{T}}_{w_\bullet}(E_{\tau j})K'_j) = (-q_i)^{-c_{ij}-c_{\tau i,j}}\widetilde{\mathcal{T}}_{w_\bullet}(x'_{\tau i,i,\tau j;-c_{ij},-c_{\tau i,j}})K'_j(K'_i)^{-c_{ij}}(K'_{\tau i})^{-c_{\tau i,j}}.
\end{aligned} \tag{9.24}$$

Recall elements $b_{i,\tau i,j;m_1,m_2}, \underline{b}_{i,\tau i,j;m_1,m_2}$ defined in Definitions 9.6-9.7.

**Theorem 9.17.** *Let $j \in \mathbb{I}_\circ, j \neq i, \tau i$.*

*(1) The element $b_{i,\tau i,j;-c_{ij},-c_{\tau i,j}} \in \widetilde{\mathbf{U}}^\imath$ satisfies*

$$b_{i,\tau i,j;-c_{ij},-c_{\tau i,j}}\widetilde{\Upsilon}_i = \widetilde{\Upsilon}_i\widetilde{\mathcal{T}}'_{\mathbf{r}_i,-1}(B_j). \tag{9.25}$$

*(2) The element $\underline{b}_{i,\tau i,j;-c_{ij},-c_{\tau i,j}} \in \widetilde{\mathbf{U}}^\imath$ satisfies*

$$\underline{b}_{i,\tau i,j;-c_{ij},-c_{\tau i,j}}\widetilde{\mathcal{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_i)^{-1} = \widetilde{\mathcal{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_i)^{-1}\widetilde{\mathcal{T}}''_{\mathbf{r}_i,+1}(B_j). \tag{9.26}$$

(3) $\underline{b}_{i,\tau i,j;-c_{ij},-c_{\tau i,j}} = \sigma^{\imath}(b_{i,\tau i,j;-c_{ij},-c_{\tau i,j}})$.

In other word, the element $\widetilde{\mathbf{T}}'_{i,-1}(B_j) := b_{i,\tau i,j;-c_{ij},-c_{\tau i,j}}$ satisfies (7.5) and the element $\widetilde{\mathbf{T}}''_{i,+1}(B_j) := \underline{b}_{i,\tau i,j;-c_{ij},-c_{\tau i,j}}$ satisfies (7.6). Hence, we have proved the first statement in Theorem 7.3(ii).

*Proof.* We prove (1). By Lemma 5.1 and (9.23), we have

$$\begin{aligned}
\widetilde{\Upsilon}_i\widetilde{\mathcal{T}}'_{\mathbf{r}_i,-1}(B_j) =& \widetilde{\Upsilon}_i\widetilde{\mathcal{T}}'_{\mathbf{r}_i,-1}(F_j) + \widetilde{\Upsilon}_i\widetilde{\mathcal{T}}'_{\mathbf{r}_i,-1}\big(\widetilde{\mathcal{T}}_{w_\bullet}(E_j)K'_j\big)\\
=& \widetilde{\Upsilon}_i\widetilde{\mathcal{T}}'_{\mathbf{r}_i,-1}(F_j) + \widetilde{\mathcal{T}}'_{\mathbf{r}_i,-1}\big(\widetilde{\mathcal{T}}_{w_\bullet}(E_j)K'_j\big)\widetilde{\Upsilon}_i\\
=& \widetilde{\Upsilon}_i y_{i,\tau i,j;-c_{ij},-c_{\tau i,j}}\\
&+ (-q_i)^{-c_{ij}-c_{\tau i,j}}\widetilde{\mathcal{T}}_{w_\bullet}(x_{\tau i,i,\tau j;-c_{ij},-c_{\tau i,j}})K'_j(K'_i)^{-c_{ij}}(K'_{\tau i})^{-c_{\tau i,j}}\widetilde{\Upsilon}_i.
\end{aligned}$$

On the other hand, setting $m_1 = -c_{ij}, m_2 = -c_{\tau i,j}$ in Proposition 9.13-9.14, we have

$$\begin{aligned}
&b^-_{i,\tau i,j;-c_{ij},-c_{\tau i,j}}\widetilde{\Upsilon}_i = \widetilde{\Upsilon}_i y_{i,\tau i,j;-c_{ij},-c_{\tau i,j}},\\
&b^+_{i,\tau i,j;-c_{ij},-c_{\tau i,j}} = (-q_i)^{-c_{ij}-c_{\tau i,j}}\widetilde{\mathcal{T}}_{w_\bullet}(x_{\tau i,i,\tau j;-c_{ij},-c_{\tau i,j}})K'_j(K'_i)^{-c_{ij}}(K'_{\tau i})^{-c_{\tau i,j}}.
\end{aligned}$$

Therefore, by the above formulas, we obtain the desired identity

$$\widetilde{\Upsilon}_i\widetilde{\mathcal{T}}'_{\mathbf{r}_i,-1}(B_j) = b^-_{i,\tau i,j;-c_{ij},-c_{\tau i,j}}\widetilde{\Upsilon}_i + b^+_{i,\tau i,j;-c_{ij},-c_{\tau i,j}}\widetilde{\Upsilon}_i = b_{i,\tau i,j;-c_{ij},-c_{\tau i,j}}\widetilde{\Upsilon}_i.$$

We prove (2). By Lemma 5.1 and (9.24), we have

$$\begin{aligned}
&\widetilde{\mathcal{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_i)^{-1}\widetilde{\mathcal{T}}''_{\mathbf{r}_i,+1}(B_j)\widetilde{\mathcal{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_i)\\
=& \widetilde{\mathcal{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_i)^{-1}\widetilde{\mathcal{T}}_{\mathbf{r}_i}(F_j)\widetilde{\mathcal{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_i) + \widetilde{\mathcal{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_i)^{-1}\widetilde{\mathcal{T}}_{\mathbf{r}_i}\big(\widetilde{\mathcal{T}}_{w_\bullet}(E_j)K'_j\big)\widetilde{\mathcal{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_i)\\
=& \widetilde{\mathcal{T}}_{\mathbf{r}_i}(F_j) + \widetilde{\mathcal{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_i)^{-1}\widetilde{\mathcal{T}}_{\mathbf{r}_i}\big(\widetilde{\mathcal{T}}_{w_\bullet}(E_j)K'_j\big)\widetilde{\mathcal{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_i)
\end{aligned}$$

$$=y'_{i,\tau i,j;-c_{ij},-c_{\tau i,j}}$$
$$+(-q_i)^{-c_{ij}-c_{\tau i,j}}\widetilde{\mathfrak{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_i)^{-1}\widetilde{\mathfrak{T}}_{w_\bullet}(x'_{\tau i,i,\tau j;-c_{ij},-c_{\tau i,j}})K'_j(K'_i)^{-c_{ij}}(K'_{\tau i})^{-c_{\tau i,j}}\widetilde{\mathfrak{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_i).$$

On the other hand, setting $m_1=-c_{ij}, m_2=-c_{\tau i,j}$ in Proposition 9.15-9.16, we have

$$\underline{b}^-_{i,\tau i,j;-c_{ij},-c_{\tau i,j}}=y'_{i,\tau i,j;-c_{ij},-c_{\tau i,j}},$$
$$\underline{b}^+_{i,\tau i,j;-c_{ij},-c_{\tau i,j}}$$
$$=(-q_i)^{-c_{ij}-c_{\tau i,j}}\widetilde{\mathfrak{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_i)^{-1}\widetilde{\mathfrak{T}}_{w_\bullet}(x'_{\tau i,i,\tau j;-c_{ij},-c_{\tau i,j}})K'_j(K'_i)^{-c_{ij}}(K'_{\tau i})^{-c_{\tau i,j}}\widetilde{\mathfrak{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_i).$$

Therefore, by above two formulas, we obtain the desired identity

$$\widetilde{\mathfrak{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_i)^{-1}\widetilde{\mathfrak{T}}''_{\mathbf{r}_i,+1}(B_j)\widetilde{\mathfrak{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_i)=\underline{b}^-_{i,\tau i,j;-c_{ij},-c_{\tau i,j}}+\underline{b}^+_{i,\tau i,j;-c_{ij},-c_{\tau i,j}}=\underline{b}_{i,\tau i,j;-c_{ij},-c_{\tau i,j}}.$$

The statement (3) is a consequence of Proposition 9.8. □

# 10 Higher rank formulas for $c_{i,\tau i}=-1, w_\bullet i=i$

Fix $i\in\mathbb{I}_{\circ,\tau}$ such that $c_{i,\tau i}=-1, w_\bullet i=i$ throughout this section. Since $w_\bullet$ commutes with $\tau$, $w_\bullet\tau i=\tau i$. In this case, we have $B_i=F_i+E_{\tau i}K'_i$ and $\mathbf{r}_i=s_is_{\tau i}s_i=s_{\tau i}s_is_{\tau i}$.

We define higher rank root vectors $b_{i,\tau i,j;a,b,c},\underline{b}_{i,\tau i,j;a,b,c}\in\widetilde{\mathbf{U}}^\imath$ in Definitions 10.6-10.7 via recursive relations. We show that the higher rank formulas $\widetilde{\mathbf{T}}'_{i,-1}(B_j),\widetilde{\mathbf{T}}''_{i,+1}(B_j)$ are given by these root vectors in Theorem 10.14 and complete the proof for Theorem 7.3(iii). The divided power formulations for $b_{i,\tau i,j;a,b,c},\underline{b}_{i,\tau i,j;a,b,c}$ are obtained in Theorem 10.16.

## 10.1 Definitions of root vectors

Let Ad be the adjoint action on $\widetilde{\mathbf{U}}$, explicitly given by

$$\begin{aligned}\mathrm{Ad}(E_i)u &= E_i u - K_i u K_i^{-1} E_i,\\ \mathrm{Ad}(F_i)u &= (F_i u - u F_i) K_i'^{-1},\\ \mathrm{Ad}(K_i)u &= K_i u K_i^{-1}.\end{aligned}$$

Set ${}^{\omega\psi}\mathrm{Ad} := \omega\psi \circ \mathrm{Ad} \circ \omega\psi$ and ${}^{\sigma}\mathrm{Ad} := \sigma \circ \mathrm{Ad} \circ \sigma$, where $\omega, \psi, \sigma$ are the Chevalley involution, the bar involution, and the anti-involution on $\widetilde{\mathbf{U}}$.

**Definition 10.1.** Define elements $y_{i,\tau i,j;a,b,c}, x_{i,\tau i,j;a,b,c}, y'_{i,\tau i,j;a,b,c}, x'_{i,\tau i,j;a,b,c}$ for $a,b,c \geq 0, j \neq i, \tau i$ as follows

$$\begin{aligned}y_{i,\tau i,j;a,b,c} &= {}^{\omega\psi}\mathrm{Ad}(E_i^{(a)} E_{\tau i}^{(b)} E_i^{(c)}) F_j, \qquad & y'_{i,\tau i,j;a,b,c} &= {}^{\sigma\omega\psi}\mathrm{Ad}(E_i^{(a)} E_{\tau i}^{(b)} E_i^{(c)}) F_j,\\ x_{i,\tau i,j;a,b,c} &= {}^{\sigma}\mathrm{Ad}(E_i^{(a)} E_{\tau i}^{(b)} E_i^{(c)}) E_j, \qquad & x'_{i,\tau i,j;a,b,c} &= \mathrm{Ad}(E_i^{(a)} E_{\tau i}^{(b)} E_i^{(c)}) E_j.\end{aligned}$$

Denote $E_{\tau i}^{(a,b,c)} := E_i^{(a)} E_{\tau_i}^{(b)} E_i^{(c)}$ and $[K_i; x] := \frac{K_i q_i^x - K_i^{-1} q_i^{-x}}{q_i - q_i^{-1}}$.

**Lemma 10.2.** *We have, for $a,b,c \geq 0$,*

$$\begin{aligned}E_i E_{\tau i}^{(a,b,c)} &= [a+1] E_{\tau i}^{(a+1,b,c)},\\ E_{\tau i} E_{\tau i}^{(a,b,c)} &= [b-a+1] E_{\tau i}^{(a,b+1,c)} + [c+1] E_{\tau i}^{(a-1,b+1,c+1)}.\end{aligned} \tag{10.1}$$

*Proof.* The first identity is obvious. The second identity follows from a standard though lengthy induction. □

**Lemma 10.3.** *We have, for $a, b, c \geq 0$,*

$$\begin{aligned} [F_i, E_{\tau i}^{(a,b,c)}] &= -E_{\tau i}^{(a-1,b,c)}[K_i; a-b+2c-1] - E_{\tau i}^{(a,b,c-1)}[K_i; c-1], \\ [F_{\tau i}, E_{\tau i}^{(a,b,c)}] &= -E_{\tau i}^{(a,b-1,c)}[K_{\tau i}; b-c-1]. \end{aligned} \tag{10.2}$$

*Proof.* The first identity follows from the following relations

$$[F_i, E_{\tau i}] = 0, \qquad [F_i, E_i^{(m)}] = -[m]E_i^{(m-1)}[K_i; m-1], \qquad [K_i; m]E_i = E_i[K_i; m+2],$$

cf. [Ja95, 1.3,1.6]. One can prove the second identity via similar relations. □

We write $[A, B]_x := AB - xBA$ for scalars $x$.

**Lemma 10.4.** *We have, for $a, b, c \geq 0$,*

*(1)* $\big[F_i, y_{i,\tau i,j;a,b,c}\big]_{q_i^{b-2a-2c-c_{ij}}} = [a+1]_i y_{i,\tau i,j;a+1,b,c}$.

*(2)* $\big[F_{\tau i}, y_{i,\tau i,j;a,b,c}\big]_{q_i^{-2b+a+c-c_{\tau i,j}}} = [b-a+1]_i y_{i,\tau i,j;a,b+1,c} + [c+1]_i y_{i,\tau i,j;a-1,b+1,c+1}$.

*(3)* $[E_i, y_{i,\tau i,j;a,b,c}] = [-c_{ij}-a+b-2c+1]_i y_{i,\tau i,j;a-1,b,c} K_i' + [-c_{ij}-c+1]_i y_{i,\tau i,j;a,b,c-1} K_i'$.

*(4)* $[E_{\tau i}, y_{i,\tau i,j;a,b,c}] = [-c_{\tau i,j} - b + c + 1]_i y_{i,\tau i,j;a,b-1,c} K_{\tau i}'$.

*Proof.* We give a detailed proof for (2). On one hand, for any $u \in \widetilde{\mathbf{U}}$, we have ${}^{\omega\psi}\mathrm{Ad}(E_{\tau i})u = F_{\tau i}u - K_{\tau i} u K_{\tau i}^{-1} F_{\tau i}$, which implies that

$${}^{\omega\psi}\mathrm{Ad}(E_{\tau i}) y_{i,\tau i,j;a,b,c} = \big[F_{\tau i}, y_{i,\tau i,j;a,b,c}\big]_{q_i^{-2b+a+c-c_{\tau i,j}}}.$$

On the other hand, by definition of $y_{i,\tau i,j;a,b,c}$ and Lemma 10.2, we have

$${}^{\omega\psi}\mathrm{Ad}(E_{\tau i}) y_{i,\tau i,j;a,b,c}$$

$$
\begin{aligned}
&= {}^{\omega\psi}\mathrm{Ad}(E_{\tau i}) {}^{\omega\psi}\mathrm{Ad}(E_i^{(a)} E_{\tau i}^{(b)} E_i^{(c)}) F_j \\
&= [b-a+1]_i {}^{\omega\psi}\mathrm{Ad}(E_i^{(a)} E_{\tau i}^{(b+1)} E_i^{(c)}) F_j + [c+1]_i {}^{\omega\psi}\mathrm{Ad}(E_i^{(a-1)} E_{\tau i}^{(b+1)} E_i^{(c+1)}) F_j \\
&= [b-a+1]_i y_{i,\tau i,j;a,b+1,c} + [c+1]_i y_{i,\tau i,j;a-1,b+1,c+1}.
\end{aligned}
$$

The identity (2) follows by above two formulas.

The identity (1) is obtained by considering the action of ${}^{\omega\psi}\mathrm{Ad}(E_i)$ on $y_{i,\tau i,j;a,b,c}$ via similar arguments. Identities (3)-(4) are obtained by respectively considering the action of ${}^{\omega\psi}\mathrm{Ad}(F_i), {}^{\omega\psi}\mathrm{Ad}(F_{\tau i})$ on $y_{i,\tau i,j;a,b,c}$ and using Lemma 10.3. We omit details for them. □

**Lemma 10.5.** *We have, for* $a, b, c \geq 0$*,*

*(1)* $x_{i,\tau i,j;a,b,c} E_i - q_i^{-b+2a+2c+c_{ij}} E_i x_{i,\tau i,j;a,b,c} = [a+1]_i x_{i,\tau i,j;a+1,b,c}.$

*(2)* $\big[x_{i,\tau i,j;a,b,c}, E_{\tau i}\big]_{q_i^{2b-a-c+c_{\tau i,j}}} = [b-a+1]_i x_{i,\tau i,j;a,b+1,c} + [c+1]_i x_{i,\tau i,j;a-1,b+1,c+1}.$

*(3)* $[x_{i,\tau i,j;a,b,c}, F_i] = [-c_{ij}-a+b-2c+1]_i K_i x_{i,\tau i,j;a-1,b,c} + [-c_{ij}-c+1]_i K_i x_{i,\tau i,j;a,b,c-1}.$

*(4)* $[x_{i,\tau i,j;a,b,c}, F_{\tau i}] = [-c_{\tau i,j} - b + c + 1]_i K_{\tau i} x_{i,\tau i,j;a,b-1,c}.$

*Proof.* By definition, we have

$$
x_{i,\tau i,j;a,b,c} = \sigma\omega\psi(y_{i,\tau i,j;a,b,c}).
$$

Then these four identities are obtained by applying $\sigma\omega\psi$ to those four identities in Lemma 10.4. □

**Definition 10.6.** Let $j \in \mathbb{I}_\circ$ such that $j \neq i, \tau i$. Define $b^{\pm}_{i,\tau i,j;a,b,c}$ to be elements in

$\widetilde{\mathbf{U}}$ determined by the following recursive relations

$$\begin{aligned}&B_i b^{\pm}_{i,\tau i,j;a,b,c} - q_i^{b-2a-2c-c_{ij}} b^{\pm}_{i,\tau i,j;a,b,c} B_i \\ =&[a+1]_i b^{\pm}_{i,\tau i,j;a+1,b,c} + q_i^{b-2a-2c-c_{ij}-1}[-c_{\tau i,j}-b+c+1]_i b^{\pm}_{i,\tau i,j;a,b-1,c}\widetilde{k}_i,\end{aligned} \tag{10.3}$$

and

$$\begin{aligned}&B_{\tau i} b^{\pm}_{i,\tau i,j;a,b,c} - q_i^{-2b+a+c-c_{\tau i,j}} b^{\pm}_{i,\tau i,j;a,b,c} B_{\tau i} \\ &= [b-a+1]_i b^{\pm}_{i,\tau i,j;a,b+1,c} + [c+1]_i b^{\pm}_{i,\tau i,j;a-1,b+1,c+1} \\ &+ q_i^{-2b+c+a-c_{\tau i,j}-1}\big([-c_{ij}-a+b-2c+1]_i b^{\pm}_{i,\tau i,j;a-1,b,c} + [-c_{ij}-c+1]_i b^{\pm}_{i,\tau i,j;a,b,c-1}\big)\widetilde{k}_{\tau i},\end{aligned} \tag{10.4}$$

where we set $b^{\pm}_{i,\tau i,j;a,b,c} = 0$ if either one of $a,b,c$ is negative, and set

$$b^{-}_{i,\tau i,j;0,0,0} = F_j, \qquad b^{+}_{i,\tau i,j;0,0,0} = \widetilde{\mathcal{T}}_{w_\bullet}(E_{\tau j})K'_j. \tag{10.5}$$

**Definition 10.7.** Let $j \in \mathbb{I}_\circ$ such that $j \neq i, \tau i$. Define $\underline{b}^{\pm}_{i,\tau i,j;a,b,c}$ to be elements in $\widetilde{\mathbf{U}}$ determined by the following recursive relations

$$\begin{aligned}&\underline{b}^{\pm}_{i,\tau i,j;a,b,c} B_i - q_i^{b-2a-2c-c_{ij}} B_i \underline{b}^{\pm}_{i,\tau i,j;a,b,c} \\ =&[a+1]_i \underline{b}^{\pm}_{i,\tau i,j;a+1,b,c} + q_i^{b-2a-2c-c_{ij}-1}[-c_{\tau i,j}-b+c+1]_i \widetilde{k}_{\tau i} \underline{b}^{\pm}_{i,\tau i,j;a,b-1,c},\end{aligned} \tag{10.6}$$

and

$$\begin{aligned}&\underline{b}^{\pm}_{i,\tau i,j;a,b,c} B_{\tau i} - q_i^{-2b+a+c-c_{\tau i,j}} B_{\tau i} \underline{b}^{\pm}_{i,\tau i,j;a,b,c} \\ =&[b-a+1]_i \underline{b}^{\pm}_{i,\tau i,j;a,b+1,c} + [c+1]_i \underline{b}^{\pm}_{i,\tau i,j;a-1,b+1,c+1} \\ &+ q_i^{-2b+c+a-c_{\tau i,j}-1}\widetilde{k}_i\big([-c_{ij}-a+b-2c+1]_i \underline{b}^{\pm}_{i,\tau i,j;a-1,b,c} + [-c_{ij}-c+1]_i \underline{b}^{\pm}_{i,\tau i,j;a,b,c-1}\big),\end{aligned} \tag{10.7}$$

where we set $\underline{b}^{\pm}_{i,\tau i,j;a,b,c} = 0$ if either one of $a, b, c$ is negative, and set

$$\underline{b}^{-}_{i,\tau i,j;0,0,0} = F_j, \qquad \underline{b}^{+}_{i,\tau i,j;0,0,0} = \widetilde{\mathcal{T}}_{w_\bullet}(E_{\tau j})K'_j. \tag{10.8}$$

Define $b_{i,\tau i,j;a,b,c} := b^{-}_{i,\tau i,j;a,b,c} + b^{+}_{i,\tau i,j;a,b,c}$. Similarly, define $\underline{b}_{i,\tau i,j;a,b,c}$. Since

$$b_{i,\tau i,j;0,0,0} = \underline{b}_{i,\tau i,j;0,0,0} = B_j \in \widetilde{\mathbf{U}}^\imath,$$

it follows from the above recursive definitions that $b_{i,\tau i,j;a,b,c}, \underline{b}_{i,\tau i,j;a,b,c} \in \widetilde{\mathbf{U}}^\imath$ for any $a, b, c$.

Recall the anti-involution $\sigma^\imath$ on $\widetilde{\mathbf{U}}^\imath$ from Proposition 3.12.

**Proposition 10.8.** *Let $j \in \mathbb{I}_\circ, j \neq i, \tau i$. Then $\underline{b}_{i,j;a,b,c} = \sigma^\imath(b_{i,j;a,b,c})$ for $a, b, c \geq 0$.*

*Proof.* By Definition 10.6-10.7, $\sigma^\imath(b_{i,j;a,b,c})$ satisfies the same recursive relations as $\underline{b}_{i,j;a,b,c}$. Since $\underline{b}_{i,j;0,0,0} = B_j = \sigma^\imath(b_{i,j;0,0,0})$, this proposition follows. □

**Lemma 10.9.** *We have, for $a, b, c \geq 0, j \neq i, \tau i, j \in \mathbb{I}_\circ$,*

$$\underline{b}^{-}_{i,\tau i,j;a,b,c} = \sigma\big(\widetilde{\Upsilon}_i^{-1} b^{-}_{i,\tau i,j;a,b,c} \widetilde{\Upsilon}_i\big).$$

*Proof.* Consider the subalgebra $\widetilde{\mathbf{U}}^{-}_{[i;j]}$ of $\widetilde{\mathbf{U}}$ generated by $B_i, B_{\tau i}, \widetilde{k}_i, \widetilde{k}_{\tau i}, F_j$. It is clear from the above definitions that $b^{-}_{i,\tau i,j;a,b,c}, \underline{b}^{-}_{i,\tau i,j;a,b,c} \in \widetilde{\mathbf{U}}^{-}_{[i;j]}$. By Theorem 3.6 and Lemma 5.1, there is a well-defined anti-automorphism $\sigma_{ij}$ on $\widetilde{\mathbf{U}}^{-}_{[i;j]}$, which is given by

$$\sigma_{ij} : x \mapsto \sigma\big(\widetilde{\Upsilon}_i^{-1} x \widetilde{\Upsilon}_i\big).$$

Moreover, $\sigma_{ij}$ fixes $B_i, B_{\tau i}, F_j$ and sends $\widetilde{k}_i \leftrightarrow \widetilde{k}_{\tau i}$. Applying $\sigma_{ij}$ to (10.3)-(10.4), it is clear that $\sigma_{ij}(b^-_{i,\tau i,j;a,b,c})$ satisfies the same recursive relations as $\underline{b}^-_{i,\tau i,j;a,b,c}$. Then the desired identity follows by induction. □

## 10.2 Intertwining properties

We formulate the intertwining relations between elements $b^{\pm}_{i,\tau i,j;a,b,c}$ and $y_{i,\tau i,j;a,b,c}, x_{\tau i,i,\tau j;a,b,c}$.

**Proposition 10.10.** *We have, for $a,b,c \geq 0, j \neq i, \tau i, j \in \mathbb{I}_\circ$,*

$$b^-_{i,\tau i,j;a,b,c}\widetilde{\Upsilon}_i = \widetilde{\Upsilon}_i y_{i,\tau i,j;a,b,c}. \tag{10.9}$$

*Proof.* Let $R_{a,b,c}$ denote $\widetilde{\Upsilon}_i y_{i,\tau i,j;a,b,c}\widetilde{\Upsilon}_i^{-1}$. By Lemma 5.1, $R_{0,0,0} = F_j = b^-_{i,\tau i,j;0,0,0}$. Moreover, by definition, if either one of $a,b,c$ is negative, then $y_{i,\tau i,j;a,b,c} = y_{i,\tau i,j;a,b,c} = 0$. Hence, it suffices to prove that $R_{a,b,c}$ satisfies the same recursive relations as $b^-_{i,\tau i,j;a,b,c}$.

Recall that $B_i^\sigma = F_i + K_i E_{\tau i}$. We have, by Theorem 3.6,

$$\begin{aligned}
&\widetilde{\Upsilon}_i^{-1}\big(B_i R_{a,b,c} - q_i^{b-2a-2c-c_{ij}} R_{a,b,c} B_i\big)\widetilde{\Upsilon}_i \\
=&B_i^\sigma y_{i,\tau i,j;a,b,c} - q_i^{b-2a-2c-c_{ij}} y_{i,\tau i,j;a,b,c} B_i^\sigma \\
=&F_i y_{i,\tau i,j;a,b,c} - q_i^{b-2a-2c-c_{ij}} y_{i,\tau i,j;a,b,c} F_i \\
&+ q_i^{b-2a-2c-c_{ij}-1}(E_{\tau i} y_{i,\tau i,j;a,b,c} - y_{i,\tau i,j;a,b,c} E_{\tau i}) K_i \\
=&[a+1]_i y_{i,\tau i,j;a+1,b,c} + q_i^{b-2a-2c-c_{ij}-1}[-c_{\tau i,j} - b + c + 1]_i y_{i,\tau i,j;a,b-1,c}\widetilde{k}_i,
\end{aligned}$$

where the last step follow from Lemma 10.4(1)(4). This computation shows that the element $R_{a,b,c}$ satisfies (10.3).

For $B^{\sigma}_{\tau i} = F_{\tau i} + K_{\tau i} E_i$, by Theorem 3.6, we similarly have

$$
\begin{aligned}
&\widetilde{\Upsilon}_i^{-1}\big(B_{\tau i} R_{a,b,c} - q_i^{-2b+c+a-c_{\tau i,j}} R_{a,b,c} B_{\tau i}\big)\widetilde{\Upsilon}_i \\
&= B^{\sigma}_{\tau i} y_{i,\tau i,j;a,b,c} - q_i^{-2b+c+a-c_{\tau i,j}} y_{i,\tau i,j;a,b,c} B^{\sigma}_{\tau i} \\
&= F_{\tau i} y_{i,\tau i,j;a,b,c} - q_i^{-2b+c+a-c_{\tau i,j}} y_{i,\tau i,j;a,b,c} F_{\tau i} \\
&\quad + q_i^{c+a-2b-c_{\tau i,j}-1}(E_i y_{i,\tau i,j;a,b,c} - y_{i,\tau i,j;a,b,c} E_i) K_{\tau i} \\
&= [b-a+1]_i y_{i,\tau i,j;a,b+1,c} + [c+1]_i y_{i,\tau i,j;a-1,b+1,c+1} \\
&+ q_i^{c+a-2b-c_{\tau i,j}-1}\big([-c_{ij}-a+b-2c+1]_i y_{i,\tau i,j;a-1,b,c} + [-c_{ij}-c+1]_i y_{i,\tau i,j;a,b,c-1}\big)\widetilde{k}_{\tau i}.
\end{aligned}
$$

This computation shows that the element $R_{a,b,c}$ satisfies (10.4). Therefore, we have proved (10.9) for any $a,b,c \geq 0$. □

**Proposition 10.11.** *We have, for $a,b,c \geq 0, j \neq i, \tau i, j \in \mathbb{I}_\circ$,*

$$
b^{+}_{i,\tau i,j;a,b,c} = (-1)^{a+b+c} q_i^{-\frac{1}{2}(a+b+c)(a+b+c-1+2c_{ij}+2c_{\tau i,j})} \widetilde{\mathcal{T}}_{w_\bullet}(x_{\tau i,i,\tau j;a,b,c}) K'_j (K'_i)^{a+c} (K'_{\tau i})^{b}. \tag{10.10}
$$

*Proof.* Let $P_{a,b,c}$ denote RHS (10.10) and $x_{\tau i,i,w_\bullet \tau j;a,b,c}$ denote $\widetilde{\mathcal{T}}_{w_\bullet}(x_{\tau i,i,\tau j;a,b,c})$. It is clear that $P_{0,0,0} = \widetilde{\mathcal{T}}_{w_\bullet}(E_{\tau j})K'_j = b^{+}_{i,\tau i,j;0,0,0}$. Moreover, if either one of $a,b,c$ is negative, then $P_{a,b,c} = 0 = b^{+}_{i,\tau i,j;a,b,c}$. Hence, it suffices to show that $P_{a,b,c}$ also satisfies the defining recursive relations (10.3)-(10.4) for $B^{+}_{i,\tau i,j;a,b,c}$.

Let $Q_{a,b,c}$ denote $x_{\tau i,i,w_\bullet \tau j;a,b,c} K'_j (K'_i)^{a+c} (K'_{\tau i})^{b}$. Applying $\tau$ to Lemma 10.5(1)(4), we obtain two recursions for $x_{\tau i,i,\tau j;a,b,c}$. Since $F_i, E_{\tau i}$ are fixed by $\widetilde{\mathcal{T}}_{w_\bullet}$, $x_{\tau i,i,w_\bullet \tau j;a,b,c}$ also satisfies the same recursions. To this end, we have

$$
\begin{aligned}
& x_{\tau i,i,w_\bullet \tau j;a,b,c} E_{\tau i} - q_i^{-b+2a+2c+c_{ij}} E_{\tau i} x_{\tau i,i,w_\bullet \tau j;a,b,c} = [a+1]_i x_{i,\tau i,w_\bullet \tau j;a+1,b,c}, \\
& [x_{\tau i,i,w_\bullet \tau j;a,b,c}, F_i] = [-c_{\tau i,j} - b + c + 1]_i K_i x_{\tau i,i,w_\bullet \tau j;a,b-1,c}.
\end{aligned} \tag{10.11}
$$

We formulate the recursive relation for $Q_{a,b,c}$ as follows

$$
\begin{aligned}
&B_i Q_{a,b,c} - q_i^{b-2a-2c-c_{ij}} Q_{a,b,c} B_i \\
&= (F_i x_{\tau i,i,w_\bullet \tau j;a,b,c} - x_{\tau i,i,w_\bullet \tau j;a,b,c} F_i) K_j' (K_i')^{a+c} (K_{\tau i}')^b \\
&+ q_i^{-a-c-b-c_{\tau i,j}-c_{ij}} (q_i^{2a+2c-b+c_{ij}} E_{\tau i} x_{\tau i,i,w_\bullet \tau j;a,b,c} - x_{\tau i,i,w_\bullet \tau j;a,b,c} E_{\tau i}) K_j' (K_i')^{a+c+1} (K_{\tau i}')^b \\
&= -[-c_{\tau i,j} - b + c + 1]_i q_i^{-a-c+2b-2+c_{\tau i,j}} x_{\tau i,i,w_\bullet \tau j;a,b-1,c} K_j' (K_i')^{a+c} (K_{\tau i}')^{b-1} \widetilde{k}_i \\
&\quad - q_i^{-a-c-b-c_{\tau i,j}-c_{ij}} [a+1]_i x_{i,w_\bullet \tau i,j;a+1,b,c} E_{\tau i}) K_j' (K_i')^{a+c+1} (K_{\tau i}')^b,
\end{aligned}
$$

where we used (10.11) in the last step.

The above computation implies that $Q_{a,b,c}$ satisfies the following recursive relation

$$
\begin{aligned}
&B_i Q_{a,b,c} - q_i^{b-2a-2c-c_{ij}} Q_{a,b,c} B_i \\
=& - q_i^{-a-c-b-c_{\tau i,j}-c_{ij}} [a+1]_i Q_{a+1,b,c} - q_i^{-a-c+2b-2+c_{\tau i,j}} [-c_{\tau i,j} - b + c + 1]_i Q_{a,b-1,c} \widetilde{k}_i.
\end{aligned} \tag{10.12}
$$

Similarly, one can show that $Q_{a,b,c}$ also satisfies the following recursive relation

$$
\begin{aligned}
&B_{\tau i} Q_{a,b,c} - q_i^{-2b+a+c-c_{\tau i,j}} Q_{a,b,c} B_{\tau i} \\
=& - q_i^{2a+2c-2-b+c_{ij}} \big( [-c_{ij} - a + b - 2c + 1]_i Q_{a-1,b,c} + [-c_{ij} - c + 1]_i Q_{a,b,c-1} \big) \widetilde{k}_{\tau i} \\
& - q_i^{-b-a-c-c_{ij}-c_{\tau i,j}} \big( [b-a+1]_i Q_{a,b+1,c} + [c+1]_i Q_{a-1,b+1,c+1} \big).
\end{aligned} \tag{10.13}
$$

Since $P_{a,b,c} = (-1)^{a+b+c} q_i^{-\frac{1}{2}(a+b+c)(a+b+c-1+2c_{ij}+2c_{\tau i,j})} Q_{a,b,c}$, we obtain that $P_{a,b,c}$ satisfy the following two relations

$$
B_i P_{a,b,c} - q_i^{b-2a-2c-c_{ij}} P_{a,b,c} B_i
$$

$$=[a+1]_i P_{a+1,b,c} + q_i^{b-2a-2c-c_{ij}-1}[-c_{\tau i,j}-b+c+1]_i P_{a,b-1,c}\widetilde{k}_i, \tag{10.14}$$

and

$$\begin{aligned} & B_{\tau i} P_{a,b,c} - q_i^{-2b+a+c-c_{\tau i,j}} P_{a,b,c} B_{\tau i} \\ =& [b-a+1]_i P_{a,b+1,c} + [c+1]_i P_{a-1,b+1,c+1} \\ & + q_i^{-2b+c+a-c_{\tau i,j}-1}\big([-c_{ij}-a+b-2c+1]_i P_{a-1,b,c} + [-c_{ij}-c+1]_i P_{a,b,c-1}\big)\widetilde{k}_{\tau i}. \end{aligned} \tag{10.15}$$

These two relations tell that $P_{a,b,c}$ satisfy the recursive relations (10.3)-(10.4). Therefore, $P_{a,b,c} = b^+_{i,\tau i,\tau j;a,b,c}$ for any $a,b,c \geq 0$. □

We next formulate intertwining relations between elements $\underline{b}^{\pm}_{i,\tau i,j;a,b,c}$ and $y'_{i,\tau i,j;a,b,c}$, $x'_{\tau i,i,\tau j;a,b,c}$.

**Proposition 10.12.** *We have, for $a,b,c \geq 0, j \neq i, \tau i, j \in \mathbb{I}_\circ$,*

$$\underline{b}^-_{i,\tau i,j;a,b,c} = y'_{i,\tau i,j;a,b,c}, \tag{10.16}$$

$$\begin{aligned} \underline{b}^+_{i,\tau i,j;a,b,c} =& (-1)^{a+b+c} q_i^{-\frac{1}{2}(a+b+c)(a+b+c-1+2c_{ij}+2c_{\tau i,j})} \times \\ & \times \widetilde{\mathcal{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_i)^{-1} \widetilde{\mathcal{T}}_{w_\bullet}(x_{\tau i,i,\tau j;a,b,c}) K'_j (K'_i)^{a+c} (K'_{\tau i})^b \widetilde{\mathcal{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_i). \end{aligned} \tag{10.17}$$

*Proof.* The firs identity is obtained by applying Lemma 10.9 to Proposition 10.10. We prove the second identity.

Let $P_{a,b,c}$ denote the RHS (10.17) and $x'_{\tau i,i,w_\bullet \tau j;a,b,c}$ denote $\widetilde{\mathcal{T}}_{w_\bullet}(x_{\tau i,i,\tau j;a,b,c})$. By Lemma 5.1, $P_{0,0,0} = \widetilde{\mathcal{T}}_{w_\bullet}(E_{\tau j})K'_j = \underline{b}^+_{i,\tau i,j;0,0,0}$. Moreover, $P_{a,b,c} = 0 = \underline{b}^+_{i,\tau i,j;a,b,c}$ if either one of $a,b,c$ is negative. Hence, it suffices to show that $P_{a,b,c}$ satisfies the same recursive relations for $\underline{b}^+_{i,\tau i,j;a,b,c}$.

Applying $\sigma$ to Lemma 10.5(1)(4) and then shifting the indices $i, j$ to $\tau i, \tau j$, we have two recursions for $x'_{\tau i,i,\tau j;a,b,c}$. Since $F_i, E_{\tau i}$ are fixed by $\widetilde{\mathfrak{T}}_{w_\bullet}$, $x'_{\tau i,i,w_\bullet\tau j;a,b,c}$ satisfies the same recursions. To this end, we obtain

$$\begin{aligned}
&- x'_{\tau i,i,w_\bullet\tau j;a,b,c}F_i + F_i x'_{\tau i,i,w_\bullet\tau j;a,b,c} = [-c_{\tau i,j} - b + c + 1]_i \; x'_{\tau i,i,w_\bullet\tau j;a,b-1,c}K'_i,\\
&- q_i^{-b+2a+2c+c_{ij}} x'_{\tau i,i,w_\bullet\tau j;a,b,c}E_{\tau i} + E_{\tau i}x'_{\tau i,i,w_\bullet\tau j;a,b,c} = [a+1]_i x'_{\tau i,i,w_\bullet\tau j;a+1,b,c}.
\end{aligned} \tag{10.18}$$

In the case $c_{i,\tau i} = -1$, set

$$\widehat{B}_i := -q_i \widetilde{\mathfrak{T}}_{\mathbf{r}_i}(B_i \widetilde{k}_i^{-1}) = q_i F_i K_{\tau i} K'^{-1}_{\tau i} + E_{\tau i} K'_i.$$

It follows from [WZ22, §6.4] that $\widehat{B}_i \widetilde{\mathfrak{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_i) = \widetilde{\mathfrak{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_i) B_i$.

Let $Q_{a,b,c}$ denote $\widetilde{\mathfrak{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_i)^{-1} x'_{\tau i,i,w_\bullet\tau j;a,b,c} K'_j (K'_i)^{a+c} (K'_{\tau i})^b \widetilde{\mathfrak{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_i)$. We first formulate the recursive relations for $Q_{m_1,m_2}$ as follows

$$\begin{aligned}
&\widetilde{\mathfrak{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_i)\big(Q_{a,b,c}B_i - q_i^{b-2a-2c-c_{ij}} B_i Q_{a,b,c}\big)\widetilde{\mathfrak{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_i)^{-1}\\
&= x'_{\tau i,i,w_\bullet\tau j;a,b,c} K'_j (K'_i)^{a+c} (K'_{\tau i})^b \widehat{B}_i\\
&\quad - q_i^{b-2a-2c-c_{ij}} \widehat{B}_i x'_{\tau i,i,w_\bullet\tau j;a,b,c} K'_j (K'_i)^{a+c} (K'_{\tau i})^b\\
&= q_i^{2a+2c-b+c_{ij}+1}\big(x'_{\tau i,i,w_\bullet\tau j;a,b,c}F_i - F_i x'_{\tau i,i,w_\bullet\tau j;a,b,c}\big) K'_j (K'_i)^{a+c-1} (K'_{\tau i})^{b-1} \widetilde{k}_{\tau i}\\
&+ q_i^{-a-b-c-c_{ij}-c_{\tau i,j}}\big(q_i^{2a+2c-b+c_{ij}} x'_{\tau i,i,w_\bullet\tau j;a,b,c}E_{\tau i} - E_{\tau i} x'_{\tau i,i,w_\bullet\tau j;a,b,c}\big) K'_j (K'_i)^{a+c+1} (K'_{\tau i})^b\\
&= q_i^{2a+2c-b+c_{ij}+1}[-c_{\tau i,j} - b + c + 1]_i \; x'_{\tau i,i,w_\bullet\tau j;a,b-1,c} K'_j (K'_i)^{a+c} (K'_{\tau i})^{b-1} \widetilde{k}_{\tau i}\\
&\quad - q_i^{-a-b-c-c_{ij}-c_{\tau i,j}}[a+1]_i \; x'_{\tau i,i,w_\bullet\tau j;a+1,b,c} K'_j (K'_i)^{a+c+1} (K'_{\tau i})^b,
\end{aligned}$$

where the last step follows by applying (10.18).

The above computation shows that $Q_{a,b,c}$ satisfies the next recursive relation

$$
\begin{aligned}
&Q_{a,b,c}B_i - q_i^{b-2a-2c-c_{ij}} B_i Q_{a,b,c}\\
=&q_i^{2a+2c-b+c_{ij}+1}[-c_{\tau i,j} - b + c + 1]_i Q_{a,b-1,c}\widetilde{k}_{\tau i} - q_i^{-a-b-c-c_{ij}-c_{\tau i,j}}[a+1]_i Q_{a+1,b,c}.
\end{aligned}
$$

Similarly, one can show that $Q_{a,b,c}$ also satisfies

$$
\begin{aligned}
&Q_{a,b,c}B_{\tau i} - q_i^{-2b+a+c-c_{\tau i,j}} B_{\tau i} Q_{a,b,c}\\
=& -q_i^{-a-b-c-c_{ij}-c_{\tau i,j}}\big([b-a+1]_i Q_{a,b+1,c} + [c+1]_i Q_{a-1,b+1,c+1}\big)\\
&- q_i^{-b+2c+2a+c_{i,j}-2}\widetilde{k}_i\big([-c_{ij} - a + b - 2c + 1]_i Q_{i,\tau i,j;a-1,b,c} + [-c_{ij} - c + 1]_i Q_{i,\tau i,j;a,b,c-1}\big).
\end{aligned}
$$

By definition, $P_{a,b,c} = (-1)^{a+b+c} q_i^{-\frac{1}{2}(a+b+c)(a+b+c-1+2c_{ij}+2c_{\tau i,j})} Q_{a,b,c}$. Then $P_{a,b,c}$ satisfies the following recursive relations

$$
\begin{aligned}
&P_{a,b,c}B_i - q_i^{b-2a-2c-c_{ij}} B_i P_{a,b,c}\\
=&[a+1]_i P_{a+1,b,c} - q_i^{a+c-2b-c_{\tau i,j}+2}[-c_{\tau i,j} - b + c + 1]_i P_{a,b-1,c}\widetilde{k}_{\tau i}.
\end{aligned}
\tag{10.19}
$$

and

$$
\begin{aligned}
&P_{a,b,c}B_{\tau i} - q_i^{-2b+a+c-c_{\tau i,j}} B_{\tau i} P_{a,b,c}\\
=&[b-a+1]_i P_{i,\tau i,j;a,b+1,c} + [c+1]_i P_{a-1,b+1,c+1}\\
&+ q_i^{-2b+c+a-c_{\tau i,j}-1}\widetilde{k}_i\big([-c_{ij} - a + b - 2c + 1]_i P_{a-1,b,c} + [-c_{ij} - c + 1]_i P_{a,b,c-1}\big).
\end{aligned}
\tag{10.20}
$$

These two relations (10.19)-(10.20) indicate that $P_{a,b,c}$ also satisfies the defining recursive relations (10.6)-(10.7) for $b^+_{i,\tau i,j;a,b,c}$. Therefore, we have proved $P_{a,b,c} = b^+_{i,\tau i,j;a,b,c}$ for arbitrary $a, b, c \geq 0$. □

## 10.3 Proof of Theorem 7.3(iii)

Recall $\mathbf{r}_i = s_i s_{\tau i} s_i = s_{\tau i} s_i s_{\tau i}$ when $c_{i,\tau i} = -1$. We first formulate the relation between elements $y_{i,\tau i,j;a,b,c}, x_{i,\tau i,j;a,b,c}$ in Definition 10.1 and (unrescaled) Lusztig symmetries $\widetilde{T}'_{i,-1}$ on $\widetilde{\mathbf{U}}$.

For a subset $J \subset I$, denote $\widetilde{\mathbf{U}}_J$ to be the subalgebra of $\widetilde{\mathbf{U}}$ generated by $E_j, F_j, K_j, K'_j$, $j \in J$.

**Lemma 10.13.** *Let $w \in W$ with a reduced expression $w = s_{i_1} \cdots s_{i_k}$. For any $j \notin \{i_1 \cdots i_k\}$, we have*

$$\widetilde{T}'_{w,-1}(E_j) = {}^{\sigma}Ad(E_{i_1}^{(a_1)} \cdots E_{i_k}^{(a_k)})E_j \tag{10.21}$$

*where $a_s = -\langle s_{i_k} s_{i_{k-1}} \cdots s_{i_{s+1}}(\alpha_{i_s}^\vee), \alpha_j\rangle$ for $1 \le s \le k$.*

*Proof.* We prove this lemma by induction on $k = l(w)$. For $k = 1$, this result is well-known; see [Ja95, 8.14(6)].

Suppose that $k > 1$. Set $w' = s_{i_2} \cdots s_{i_k}$ and $x := {}^{\sigma}\mathrm{Ad}(E_{i_2}^{(a_2)} \cdots E_{i_k}^{(a_k)})E_j$. By induction hypothesis, we have $\widetilde{T}'_{w',-1}(E_j) = x$. It remains to show that

$$\widetilde{T}'_{i_1,-1}(x) = {}^{\sigma}\mathrm{Ad}(E_{i_1}^{(a_1)})x. \tag{10.22}$$

We consider the $\widetilde{\mathbf{U}}_{i_1,\cdots,i_k}$-module ${}^{\sigma}\mathrm{Ad}(\widetilde{\mathbf{U}}_{i_1,\cdots,i_k})E_j$ and denote its irreducible submodule containing $E_j$ by $M_j$. Since $j \notin \{i_1 \cdots i_k\}$, we have ${}^{\sigma}\mathrm{Ad}(F_{i_s})E_j = 0$ and then $E_j$ is the lowest weight vector for the $M_j$. Note that both $x$ and ${}^{\sigma}\mathrm{Ad}(E_{i_1}^{(a_1)})x$ are extremal weight vectors in $M_j$. Then we must have

$$ {}^{\sigma}\mathrm{Ad}(F_{i_1})x = 0, \qquad {}^{\sigma}\mathrm{Ad}(E_{i_1}^{(a_1+1)})x = 0. \tag{10.23} $$

On the other hand, for any integrable $\widetilde{\mathbf{U}}$-module $V$, recall that

$$ \widetilde{T}'_{i,-1}v = \sum_{a-b+c=m} (-1)^b q_i^{ac-b} F_{i_1}^{(a)} E_{i_1}^{(b)} F_{i_1}^{(c)} v, \qquad \forall v \in V_\lambda, $$

where $m = \langle \alpha_i^\vee, \lambda \rangle$. Using similar arguments in [Ja95, 8.9-8.10] and (10.23), we obtain

$$ \widetilde{T}'_{i_1,-1}(xv) = {}^{\sigma}\mathrm{Ad}(E_{i_1}^{(a_1)})x\widetilde{T}'_{i_1,-1}(v) $$

for any vector $v$ in any integrable $\widetilde{\mathbf{U}}$-module. Therefore, we have proved this lemma by induction. □

Write $\alpha, \beta$ for $-c_{ij}, -c_{\tau i,j}$ respectively. By Lemma 10.13, we have

$$ \widetilde{T}'_{\mathbf{r}_i,-1}(F_j) = y_{i,\tau i,j;\beta,\alpha+\beta,\alpha}, \qquad \widetilde{T}'_{\mathbf{r}_i,-1}(E_{\tau j}) = x_{\tau i,i,\tau j;\beta,\alpha+\beta,\alpha}. \tag{10.24} $$

$$ \widetilde{T}''_{\mathbf{r}_i,+1}(F_j) = y'_{i,\tau i,j;\beta,\alpha+\beta,\alpha}, \qquad \widetilde{T}''_{\mathbf{r}_i,+1}(E_{\tau j}) = x'_{\tau i,i,\tau j;\beta,\alpha+\beta,\alpha}. \tag{10.25} $$

Since $c_{i,\tau i} = -1$, $\varsigma_{i,\diamond} = -q_i^{-1/2}$. By (10.24), the action of (rescaled) symmetries $\widetilde{\mathfrak{T}}'_{\mathbf{r}_i,-1}$ is given by

$$ \begin{aligned} \widetilde{\mathfrak{T}}'_{\mathbf{r}_i,-1}(F_j) &= y_{i,\tau i,j;\beta,\alpha+\beta,\alpha}, \\ \widetilde{\mathfrak{T}}'_{\mathbf{r}_i,-1}\big(\widetilde{\mathfrak{T}}_{w_\bullet}(E_{\tau j})K'_j\big) &= q_i^{\alpha+\beta}\widetilde{\mathfrak{T}}_{w_\bullet}(x_{\tau i,i,\tau j;\beta,\alpha+\beta,\alpha})K'_j(K'_iK'_{\tau i})^{\alpha+\beta}, \end{aligned} \tag{10.26} $$

where the second formula follows from $\widetilde{\mathfrak{T}}'_{\mathbf{r}_i,-1}\widetilde{\mathfrak{T}}_{w_\bullet} = \widetilde{\mathfrak{T}}_{w_\bullet}\widetilde{\mathfrak{T}}'_{\mathbf{r}_i,-1}$.

By (10.25), we obtain analogous formulas for $\widetilde{\mathcal{T}}''_{\mathbf{r}_i,+1}$ below

$$\begin{aligned} \widetilde{\mathcal{T}}''_{\mathbf{r}_i,+1}(F_j) &= y'_{i,\tau i,j;\beta,\alpha+\beta,\alpha}, \\ \widetilde{\mathcal{T}}''_{\mathbf{r}_i,+1}\big(\widetilde{\mathcal{T}}_{w_\bullet}(E_{\tau j})K'_j\big) &= q_i^{\alpha+\beta}\widetilde{\mathcal{T}}_{w_\bullet}(x'_{\tau i,i,\tau j;\beta,\alpha+\beta,\alpha})K'_j(K'_iK'_{\tau i})^{\alpha+\beta}, \end{aligned} \tag{10.27}$$

Recall elements $b_{i,\tau i,j;a,b,c}, \underline{b}_{i,\tau i,j;a,b,c}$ defined in Definitions 10.6-10.7.

**Theorem 10.14.** *Let $j \in \mathbb{I}_\circ, j \neq i, \tau i$.*

*(1) The element $b_{i,\tau i,j;\beta,\alpha+\beta,\alpha}$ satisfies*

$$b_{i,\tau i,j;\beta,\alpha+\beta,\alpha}\widetilde{\Upsilon}_i = \widetilde{\Upsilon}_i\widetilde{\mathcal{T}}'_{\mathbf{r}_i,-1}(B_j). \tag{10.28}$$

*(2) The element $\underline{b}_{i,\tau i,j;\beta,\alpha+\beta,\alpha}$ satisfies*

$$\underline{b}_{i,\tau i,j;\beta,\alpha+\beta,\alpha}\widetilde{\mathcal{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_i)^{-1} = \widetilde{\mathcal{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_i)^{-1}\widetilde{\mathcal{T}}''_{\mathbf{r}_i,+1}(B_j). \tag{10.29}$$

*(3) $\underline{b}_{i,\tau i,j;\beta,\alpha\beta,\alpha} = \sigma^\imath(b_{i,\tau i,j;\beta,\alpha\beta,\alpha})$.*

In other word, the element $\widetilde{\mathbf{T}}'_{i,-1}(B_j) := b_{i,\tau i,j;\beta,\alpha\beta,\alpha}$ satisfies (7.5) and the element $\widetilde{\mathbf{T}}''_{i,+1}(B_j) := \underline{b}_{i,\tau i,j;\beta,\alpha\beta,\alpha}$ satisfies (7.6). Hence, we have proved Theorem 7.3(iii).

*Proof.* We prove (1). By Lemma 5.1 and (10.26), we have

$$\begin{aligned} \widetilde{\Upsilon}_i\widetilde{\mathcal{T}}'_{\mathbf{r}_i,-1}(B_j) =& \widetilde{\Upsilon}_i\widetilde{\mathcal{T}}'_{\mathbf{r}_i,-1}(F_j) + \widetilde{\Upsilon}_i\widetilde{\mathcal{T}}'_{\mathbf{r}_i,-1}(E_jK'_j) \\ =& \widetilde{\Upsilon}_i\widetilde{\mathcal{T}}'_{\mathbf{r}_i,-1}(F_j) + \widetilde{\mathcal{T}}'_{\mathbf{r}_i,-1}(E_jK'_j)\widetilde{\Upsilon}_i \\ =& \widetilde{\Upsilon}_i y_{i,\tau i,j;\beta,\beta+\alpha,\alpha} + q_i^{\alpha+\beta}\widetilde{\mathcal{T}}_{w_\bullet}(x_{\tau i,i,j;\beta,\beta+\alpha,\alpha})K'_j(K'_iK'_{\tau i})^{\alpha+\beta}\widetilde{\Upsilon}_i \end{aligned}$$

On the other hand, setting $a = \beta, b = \beta + \alpha, c = \alpha$ in Proposition 10.10-10.11, we have

$$b^{-}_{i,\tau i,j;\beta,\beta+\alpha,\alpha}\widetilde{\Upsilon}_i = \widetilde{\Upsilon}_i y_{i,\tau i,j;\beta,\beta+\alpha,\alpha},$$
$$b^{+}_{i,\tau i,j;\beta,\beta+\alpha,\alpha} = q_i^{\alpha+\beta}\widetilde{\mathcal{T}}_{w_\bullet}(x_{\tau i,i,j;\beta,\beta+\alpha,\alpha})K'_j(K'_iK'_{\tau i})^{\alpha+\beta}.$$

Therefore, by the above formulas, we have

$$\widetilde{\Upsilon}_i\widetilde{\mathcal{T}}'_{\mathbf{r}_i,-1}(B_j) = b^{-}_{i,\tau i,j;\beta,\beta+\alpha,\alpha}\widetilde{\Upsilon}_i + b^{+}_{i,\tau i,j;\beta,\beta+\alpha,\alpha}\widetilde{\Upsilon}_i = b_{i,\tau i,j;\beta,\beta+\alpha,\alpha}\widetilde{\Upsilon}_i.$$

Similarly, one can obtain (2) using Proposition 10.12 and (10.27).

The statement (3) is a consequence of Proposition 10.8. □

## 10.4 A divided power formulation

In this subsection, we derive divided power formulations for root vectors $b_{i,\tau i,j;a,b,c}$, $\underline{b}_{i,\tau i,j;a,b,c}$ from their recursive Definitions 10.6-10.7.

**Lemma 10.15.** *We have for any $a \geq 0$,*

$$b_{i,\tau i,j;a,0,0} = b_{i,\tau i,j;0,0,a}.$$

*Proof.* It suffices to show that $b^{\pm}_{i,\tau i,j;a,0,0} = b^{\pm}_{i,\tau i,j;0,0,a}$. By Definition 10.1, we have

$$y_{i,\tau i,j;a,0,0} = y_{i,\tau i,j;0,0,a}, \qquad x_{i,\tau i,j;a,0,0} = x_{i,\tau i,j;0,0,a}.$$

Then the desired identities follow from Proposition 10.10-10.11. □

**Theorem 10.16.** *The elements $b_{i,\tau i,j;a,b,c}, \underline{b}_{i,\tau i,j;a,b,c}$ for $a,b,c \geq 0$ admit the following divided power formulations*

$$b_{i,\tau i,j;a,b,c} = \sum_{u,v\geq 0}\sum_{t=0}^{a-v}\sum_{s=0}^{b-v-u}\sum_{r=0}^{c-u}(-1)^{t+v+r+s+u} \\ \times q_i^{t(b-2c+\alpha-a-2v+1)+v(b-2c+\alpha-a-\frac{v-1}{2})+r(\alpha-c+u+1)+s(c+\beta-b+v-2u+1)+u(c+\beta-b+v-\frac{u-1}{2})} \\ \times \begin{bmatrix} \beta-b+c+v \\ v \end{bmatrix} \begin{bmatrix} \alpha-c+u \\ u \end{bmatrix} B_i^{(a-v-t)}B_{\tau i}^{(b-v-u-s)}B_i^{(c-u-r)}B_jB_i^{(r)}B_{\tau i}^{(s)}\widetilde{k}_{\tau i}^{u}B_i^{(t)}\widetilde{k}_i^{v}, \tag{10.30}$$

$$\underline{b}_{i,\tau i,j;a,b,c} = \sum_{u,v\geq 0}\sum_{t=0}^{a-v}\sum_{s=0}^{b-v-u}\sum_{r=0}^{c-u}(-1)^{t+v+r+s+u} \\ \times q_i^{t(b-2c+\alpha-a-2v+1)+v(b-2c+\alpha-a-\frac{v-1}{2})+r(\alpha-c+u+1)+s(c+\beta-b+v-2u+1)+u(c+\beta-b+v-\frac{u-1}{2})} \\ \times \begin{bmatrix} \beta-b+c+v \\ v \end{bmatrix} \begin{bmatrix} \alpha-c+u \\ u \end{bmatrix} \widetilde{k}_{\tau i}^{v}B_i^{(t)}\widetilde{k}_i^{u}B_{\tau i}^{(s)}B_i^{(r)}B_jB_i^{(c-u-r)}B_{\tau i}^{(b-v-u-s)}B_i^{(a-v-t)}. \tag{10.31}$$

*Proof.* Recall from Proposition 10.8 that $\underline{b}_{i,\tau i,j;a,b,c} = \sigma^\imath(b_{i,\tau i,j;a,b,c})$. The second formula is obtained by applying $\sigma^\imath$ to the first one.

The first formula for $b_{i,\tau i,j;a,b,c}$ is derived from recursive relations (10.3)-(10.4) in Definition 10.6 via three steps.

1. Recall that $b_{i,\tau i,j;0,0,0} = B_j$. Setting $b = c = 0$ in (10.3), we have a recursive relation for $b_{i,\tau i,j;a,0,0}$. Using this relation and an induction on $a$, we obtain the formula of $b_{i,\tau i,j;a,0,0}$ as follows

$$b_{i,\tau i,j;a,0,0} = \sum_{r=0}^{a}(-1)^r q_i^{r(\alpha-c+1)}B_i^{(a-r)}B_jB_i^{(r)}.$$

By Lemma 10.15, $b_{i,\tau i,j;0,0,a} = b_{i,\tau i,j;a,0,0}$ is given by the same formula.

2. Setting $a = 0$ in (10.4), we have a recursive relation for $b_{i,\tau i,j;0,b,c}$. Using this relation and an induction on $b$, we can write $b_{i,\tau i,j;0,b,c}$ in terms of $b_{i,\tau i,j;0,0,c-u}$ for $0 \le u \le \min(b,c)$ as follows

$$b_{i,\tau i,j;0,b,c} = \sum_{u=0}^{\min(b,c)} \sum_{s=0}^{b-u} (-1)^{s+u} q_i^{s(c+\beta-b-2u+1)+u(c+\beta-b-\frac{u-1}{2})} \times \begin{bmatrix} \alpha - c + u \\ u \end{bmatrix} B_{\tau i}^{(b-u-s)} b_{i,\tau i,j;0,0,c-u} B_{\tau i}^{(s)} \widetilde{k}_{\tau i}^{u}.$$

3. Using (10.3) and an induction on $a$, we can write $b_{i,\tau i,j;a,b,c}$ in terms of $b_{i,\tau i,j;0,b-v,c}$ for $0 \le v \le \min(a,b)$ as follows

$$b_{i,\tau i,j;a,b,c} = \sum_{v=0}^{\min(a,b)} \sum_{t=0}^{a-v} (-1)^{t+v} q_i^{t(b-2c+\alpha-a-2v+1)+v(b-2c+\alpha-a-\frac{v-1}{2})} \times \begin{bmatrix} \beta - b + c + v \\ v \end{bmatrix} B_i^{(a-v-t)} b_{i,\tau i,j;0,b-v,c} B_i^{(t)} \widetilde{k}_i^{v}.$$

Now the desired formula (10.30) is obtained by combining formulas in steps (1)-(3). □

# 11 Symmetry $\widetilde{\mathbf{T}}'_{i,-1}$ and root vectors

The braid group symmetries $\widetilde{T}'_{i,e}, \widetilde{T}''_{i,e}$ on $\widetilde{\mathbf{U}}$ send root vectors to root vectors

$$
\begin{aligned}
\widetilde{T}'_{i,e}(x'_{i,j;m,e}) &= x_{i,j;-m-c_{ij},e}, \qquad \widetilde{T}''_{i,-e}(x_{i,j;m,e}) = x'_{i,j;-m-c_{ij},e}.\\
\widetilde{T}'_{i,e}(y'_{i,j;m,e}) &= y_{i,j;-m-c_{ij},e}, \qquad \widetilde{T}''_{i,-e}(y_{i,j;m,e}) = y'_{i,j;-m-c_{ij},e}.
\end{aligned}
\tag{11.1}
$$

cf. [Lus93, Proposition 37.2.5].

We will show that our symmetry $\widetilde{\mathbf{T}}'_{i,-1}$ analogously sends root vectors in $\widetilde{\mathbf{U}}^\imath$ to root vectors in $\widetilde{\mathbf{U}}^\imath$. Precisely, when $i = \tau i = w_\bullet i$, the actions of $\widetilde{\mathbf{T}}'_{i,-1}$ on root vectors $\underline{b}_{i,j;m}$ are formulated in Theorem 11.1; when $c_{i,\tau i} = 0, i = w_\bullet i$, the actions of $\widetilde{\mathbf{T}}'_{i,-1}$ on root vectors $\underline{b}_{i,\tau i,j;m_1,m_2}$ are formulated in Theorem 11.3.

## 11.1 Case $i = \tau i = w_\bullet i$

Note that $c_{ij} = c_{i,\tau j}$ when $i = \tau i$. Recall that $\widetilde{\mathcal{T}}'_{i,-1} = \widetilde{\Psi}^{-1}_{\varsigma_\diamond} \widetilde{T}'_{i,-1} \widetilde{\Psi}_{\varsigma_\diamond}$ and $\varsigma_{i,\diamond} = -q_i^{-2}$ in this case. Then we have

$$
\begin{aligned}
\widetilde{\mathcal{T}}'_{i,-1}(y'_{i,j;m}) &= y_{i,j;-c_{ij}-m},\\
\widetilde{\mathcal{T}}'_{i,-1}\Big(\widetilde{\mathcal{T}}_{w_\bullet}(x'_{i,\tau j;m}) K'_j (K'_i)^m\Big) &= \varsigma_{i,\diamond}^{c_{ij}+2m} \widetilde{\mathcal{T}}_{w_\bullet}(x_{i,\tau j;-c_{ij}-m}) K'_j (K'_i)^{-c_{ij}-m},
\end{aligned}
\tag{11.2}
$$

where the second identity uses the commutativity of $\widetilde{\mathcal{T}}'_{i,-1}, \widetilde{\mathcal{T}}_{w_\bullet}$.

**Theorem 11.1.** *We have, for $m \geq 0, j \neq i \in \mathbb{I}_\circ$,*

$$
\widetilde{\mathbf{T}}'_{i,-1}(\underline{b}_{i,j;m}) = b_{i,j;-c_{ij}-m}. \tag{11.3}
$$

*Proof.* By Proposition 8.10 and (11.2), we have

$$\begin{aligned}
&\widetilde{\Upsilon}_i\widetilde{\mathcal{T}}'_{i,-1}(\underline{b}^+_{i,j;m})\widetilde{\Upsilon}_i^{-1}\\
&=(-1)^m q_i^{-2m(c_{ij}+m-1)}\widetilde{\mathcal{T}}'_{i,-1}\Big(\widetilde{\mathcal{T}}_{w_\bullet}(x'_{i,\tau j;m})K'_j(K'_i)^m\Big)\\
&=(-1)^{c_{ij}+m}q_i^{-2(c_{ij}+m)(m+1)}\widetilde{\mathcal{T}}_{w_\bullet}(x_{i,\tau j;-c_{ij}-m})K'_j(K'_i)^{-c_{ij}-m}. \qquad (11.4)
\end{aligned}$$

By the definition of $\widetilde{\mathbf{T}}'_{i,-1}$, we have

$$\begin{aligned}
\widetilde{\mathbf{T}}'_{i,-1}(\underline{b}_{i,j;m}) &= \widetilde{\Upsilon}_i\big(\widetilde{\mathcal{T}}'_{i,-1}(\underline{b}^+_{i,j;m})+\widetilde{\mathcal{T}}'_{i,-1}(\underline{b}^-_{i,j;m})\big)\widetilde{\Upsilon}_i^{-1}\\
&\overset{(\star)}{=} \widetilde{\Upsilon}_i\big(\widetilde{\mathcal{T}}'_{i,-1}(\underline{b}^+_{i,j;m})+\widetilde{\mathcal{T}}'_{i,-1}(y'_{i,j;m})\big)\widetilde{\Upsilon}_i^{-1}\\
&\overset{(\dagger)}{=} (-1)^{c_{ij}+m}q_i^{-2(c_{ij}+m)(m+1)}x_{i,\tau j;-c_{ij}-m}K'_j(K'_i)^{-c_{ij}-m}\\
&\quad+\widetilde{\Upsilon}_i y_{i,j;-c_{ij}-m}\widetilde{\Upsilon}_i^{-1}\\
&\overset{(\ddagger)}{=} b^+_{i,j;-c_{ij}-m}+b^-_{i,j;-c_{ij}-m}\\
&= b_{i,j;-c_{ij}-m},
\end{aligned}$$

where the equality $(\star)$ follows from Proposition 8.9; the equality $(\dagger)$ follows from (11.4) and (11.2); the equality $(\ddagger)$ follows form Propositions 8.7-8.8. $\square$

## 11.2 Case $c_{i,\tau i}=0, w_\bullet i=i$

Recall the root vectors $y'_{i,\tau i,j;m_1,m_2}, x'_{\tau i,i,\tau j;m_1,m_2}$ in Definition 9.3. We first show that the symmetry $\widetilde{\mathcal{T}}'_{\mathbf{r}_i,-1}$ on $\widetilde{\mathbf{U}}$ sends these root vectors to root vectors, as a generalization of [Lus93, Proposition 37.2.5]. We then formulate the ıanalog of this property for the $\widetilde{\mathbf{T}}'_{i,-1}$-action on $\widetilde{\mathbf{U}}^\imath$ in Theorem 11.3.

**Proposition 11.2.** *Let $i, j \in \mathbb{I}, j \neq i, \tau i$. We have*

$$\widetilde{\mathfrak{T}}'_{\mathbf{r}_i,-1}(y'_{i,\tau i,j;m_1,m_2}) = y_{i,\tau i,j;-c_{ij}-m_1,-c_{\tau i,j}-m_2}, \tag{11.5}$$

$$\widetilde{\mathfrak{T}}'_{\mathbf{r}_i,-1}(x'_{\tau i,i,\tau j;m_1,m_2}) = \varsigma_{i,\diamond}^{(c_{ij}+c_{\tau i,j})/2+m_1+m_2} x_{\tau i,i,\tau j;-c_{ij}-m_1,-c_{\tau i,j}-m_2}, \tag{11.6}$$

$$\widetilde{\mathfrak{T}}'_{\mathbf{r}_i,-1}(K'_j(K'_i)^{m_1}(K'_{\tau i})^{m_2}) = \varsigma_{i,\diamond}^{(c_{ij}+c_{\tau i,j})/2+m_1+m_2} K'_j(K'_i)^{-c_{ij}-m_1}(K'_{\tau i})^{-c_{\tau i,j}-m_2}. \tag{11.7}$$

*Proof.* All three formulas are proved by similar computations; we only give the proof for the first formula here. Recall that, since $c_{i,\tau i} = 0$, we have $\mathbf{r}_i = s_i s_{\tau i}$ and $\varsigma_{i,\diamond} = -q_i^{-1}$. The element $y'_{i,\tau i,j;m_1,m_2}$ admits a reformulation similar to Remark 9.2

$$y'_{i,\tau i,j;m_1,m_2} = \sum_{s=0}^{m_2} (-1)^s q_i^{-s(m_2+c_{\tau i,j}-1)} F_{\tau i}^{(s)} y'_{i,j;m_1,-1} F_{\tau i}^{(m_2-s)} \tag{11.8}$$

By [Lus93, Proposition 37.2.5], we have $\widetilde{\mathfrak{T}}'_{i,-1}(y'_{i,j;m_1,-1}) = y_{i,j;-c_{ij}-m_1,-1}$. By (11.8), we have

$$\begin{aligned}
\widetilde{\mathfrak{T}}'_{i,-1}(y'_{i,\tau i,j;m_1,m_2}) &= \sum_{s=0}^{m_2} (-1)^s q_i^{-s(m_2+c_{\tau i,j}-1)} F_{\tau i}^{(s)} \widetilde{\mathfrak{T}}'_{i,-1}(y'_{i,j;m_1,-1}) F_{\tau i}^{(m_2-s)} \\
&= \sum_{s=0}^{m_2} (-1)^s q_i^{-s(m_2+c_{\tau i,j}-1)} F_{\tau i}^{(s)} y_{i,j;-c_{ij}-m_1,-1} F_{\tau i}^{(m_2-s)} \\
&= \sum_{r=0}^{-c_{ij}-m_1} (-1)^r q_i^{r(m_1+1)} F_i^{(-c_{ij}-m_1-r)} y'_{\tau i,j;m_2,-1} F_i^{(r)}.
\end{aligned}$$

Now applying $\widetilde{\mathfrak{T}}'_{\tau i,-1}$ to the above formula and using [Lus93, Proposition 37.2.5] again, we obtain

$$\begin{aligned}
\widetilde{\mathfrak{T}}'_{\mathbf{r}_i,-1}(y'_{i,\tau i,j;m_1,m_2}) &= \sum_{r=0}^{-c_{ij}-m_1} (-1)^r q_i^{r(m_1+1)} F_i^{(-c_{ij}-m_1-r)} \widetilde{\mathfrak{T}}'_{\tau i,-1}(y'_{\tau i,j;m_2,-1}) F_i^{(r)} \\
&= \sum_{r=0}^{-c_{ij}-m_1} (-1)^r q_i^{r(m_1+1)} F_i^{(-c_{ij}-m_1-r)} y_{\tau i,j;-c_{\tau i,j}-m_2,-1} F_i^{(r)}
\end{aligned}$$

$$= y_{i,\tau i,j;-c_{ij}-m_1,-c_{\tau i,j}-m_2},$$

where the last equality follows by Remark 9.2. □

Using $\widetilde{\mathcal{T}}'_{\mathbf{r}_i,-1}\widetilde{\mathcal{T}}_{w_\bullet} = \widetilde{\mathcal{T}}_{w_\bullet}\widetilde{\mathcal{T}}'_{\mathbf{r}_i,-1}$ and Proposition 11.2, we have the following formula

$$\begin{aligned}
&\widetilde{\mathcal{T}}'_{\mathbf{r}_i,-1}(\widetilde{\mathcal{T}}_{w_\bullet}(x'_{\tau i,i,\tau j;m_1,m_2})K'_j(K'_i)^{m_1}(K'_{\tau i})^{m_2})\\
=&\varsigma_{i,\diamond}^{c_{ij}+c_{\tau i,j}+2m_1+2m_2}\widetilde{\mathcal{T}}_{w_\bullet}(x_{\tau i,i,\tau j;-c_{ij}-m_1,-c_{\tau i,j}-m_2})K'_j(K'_i)^{-c_{ij}-m_1}(K'_{\tau i})^{-c_{\tau i,j}-m_2}.
\end{aligned} \tag{11.9}$$

**Theorem 11.3.** *Let $i, j \in \mathbb{I}, j \neq i, \tau i$. We have*

$$\widetilde{\mathbf{T}}'_{i,-1}(\underline{b}_{i,\tau i,j;m_1,m_2}) = b_{i,\tau i,j;-c_{ij}-m_1,-c_{\tau i,j}-m_2}. \tag{11.10}$$

*Proof.* Recall that $\widetilde{\mathbf{T}}'_{i,-1}(x) = \widetilde{\Upsilon}_i\widetilde{\mathcal{T}}'_{\mathbf{r}_i,-1}(x)\widetilde{\Upsilon}_i^{-1}$ for any $x \in \widetilde{\mathbf{U}}^\imath$. By Proposition 9.16 and (11.9), we have

$$\begin{aligned}
&\widetilde{\Upsilon}_i\widetilde{\mathcal{T}}'_{\mathbf{r}_i,-1}(\underline{b}^+_{i,\tau i,j;m_1,m_2})\widetilde{\Upsilon}_i^{-1}\\
=&(-1)^{m_1+m_2}q_i^{-(m_1+m_2)(c_{ij}+c_{\tau i,j}+m_1+m_2-1)}\widetilde{\mathcal{T}}'_{\mathbf{r}_i,-1}\big(\widetilde{\mathcal{T}}_{w_\bullet}(x'_{\tau i,i,\tau j;m_1,m_2})K'_j(K'_i)^{m_1}(K'_{\tau i})^{m_2}\big)\\
=&(-1)^{m_1+m_2+c_{ij}+c_{\tau i,j}}q_i^{-(m_1+m_2+1)(c_{ij}+c_{\tau i,j}+m_1+m_2)}\times\\
&\times\widetilde{\mathcal{T}}_{w_\bullet}(x_{\tau i,i,\tau j;-c_{ij}-m_1,-c_{\tau i,j}-m_2})K'_j(K'_i)^{-c_{ij}-m_1}(K'_{\tau i})^{-c_{\tau i,j}-m_2}\\
=&b^+_{i,\tau i,j;-c_{ij}-m_1,-c_{\tau i,j}-m_2},
\end{aligned}$$

where the last step follows from Proposition 9.14.

On the other hand, by Proposition 9.15, (11.5) and Proposition 9.13, we have

$$\widetilde{\Upsilon}_i\widetilde{\mathcal{T}}'_{\mathbf{r}_i,-1}(\underline{b}^-_{i,\tau i,j;m_1,m_2})\widetilde{\Upsilon}_i^{-1}$$

$$
\begin{aligned}
&= \widetilde{\Upsilon}_i \widetilde{\mathfrak{T}}'_{\mathbf{r}_i,-1}(y'_{i,\tau i,j;m_1,m_2})\widetilde{\Upsilon}_i^{-1} \\
&= \widetilde{\Upsilon}_i y_{i,\tau i,j;-c_{ij}-m_1,-c_{\tau i,j}-m_2}\widetilde{\Upsilon}_i^{-1} \\
&= b^-_{i,\tau i,j;-c_{ij}-m_1,-c_{\tau i,j}-m_2}.
\end{aligned}
$$

Using the above two formulas, we have

$$
\begin{aligned}
\widetilde{\mathbf{T}}'_{i,-1}(\underline{b}_{i,\tau i,j;m_1,m_2}) &= \widetilde{\Upsilon}_i \widetilde{\mathfrak{T}}'_{\mathbf{r}_i,-1}(\underline{b}^-_{i,\tau i,j;m_1,m_2} + \underline{b}^+_{i,\tau i,j;m_1,m_2})\widetilde{\Upsilon}_i^{-1} \\
&= b^-_{i,\tau i,j;-c_{ij}-m_1,-c_{\tau i,j}-m_2} + b^+_{i,\tau i,j;-c_{ij}-m_1,-c_{\tau i,j}-m_2} \\
&= b_{i,\tau i,j;-c_{ij}-m_1,-c_{\tau i,j}-m_2},
\end{aligned}
$$

as desired. □

# Part III

# Properties for relative braid group symmetries

# 12 A basic property of new symmetries

In this section, we establish a basic property that $\widetilde{\mathbf{T}}_{\underline{w}}$, for $w \in W^{\circ}$, sends $B_i$ to $B_j$, if $w\alpha_i = \alpha_j$; see Theorem 12.13. This is a generalization of a well-known property of braid group action on Chevalley generators in the setting of quantum groups.

We shall first study the rank 2 cases separately, depending on whether $\ell_{\circ}(\boldsymbol{w}_{\circ}) = 3$, 4, or 6. Then we deal with the general cases.

## 12.1 Rank 2 cases with $\ell_{\circ}(\boldsymbol{w}_{\circ}) = 3$

Assume that $\mathbb{I}_{\circ,\tau} = \{i, j\}$ such that $\ell_{\circ}(\boldsymbol{w}_{\circ}) = 3$; in this case, according to Table 2, we must have $\tau = \mathrm{Id}$ and hence we identify $\mathbb{I}_{\circ} = \{i, j\}$ as well.

**Lemma 12.1.** *We have* $\widetilde{\mathcal{T}}_{\mathbf{r}_i}\widetilde{\mathcal{T}}_{\mathbf{r}_j}(B_i) = B_j$.

*Proof.* Noting that $\ell(\mathbf{r}_i\mathbf{r}_j) = \ell(\mathbf{r}_i) + \ell(\mathbf{r}_j)$, we have $\widetilde{\mathcal{T}}_{\mathbf{r}_i}\widetilde{\mathcal{T}}_{\mathbf{r}_j} = \widetilde{\mathcal{T}}_{\mathbf{r}_i\mathbf{r}_j}$. Noting that $\mathbf{r}_i\mathbf{r}_j(\alpha_i) = \alpha_j$, we have that $\widetilde{\mathcal{T}}_{\mathbf{r}_i\mathbf{r}_j}(X_i) = X_j$, for $X = F, E$ or $K'$; cf. [Lus93, 39.2] or [Ja95, Proposition 8.20].

Recall $\tau = \mathrm{Id}$, and $B_i = F_i + \widetilde{\mathcal{T}}_{w_\bullet}(E_i)K_i'$. Thanks to (2.21), $\widetilde{\mathcal{T}}_{w_\bullet}$ commutes with both $\widetilde{\mathcal{T}}_{\mathbf{r}_i}$ and $\widetilde{\mathcal{T}}_{\mathbf{r}_j}$. Therefore, we have

$$\widetilde{\mathcal{T}}_{\mathbf{r}_i}\widetilde{\mathcal{T}}_{\mathbf{r}_j}(B_i) = \widetilde{\mathcal{T}}_{\mathbf{r}_i\mathbf{r}_j}(F_i + \widetilde{\mathcal{T}}_{w_\bullet}(E_i)K_i') = F_j + \widetilde{\mathcal{T}}_{w_\bullet}(E_j)K_j' = B_j.$$

The lemma is proved. □

**Proposition 12.2.** *We have* $\widetilde{\mathbf{T}}_i^{-1}\widetilde{\mathbf{T}}_j^{-1}(B_i) = B_j$*; ot equivalently,* $\widetilde{\mathbf{T}}_j\widetilde{\mathbf{T}}_i(B_j) = B_i$.

*Proof.* Since $\widetilde{\mathbf{T}}_i^{-1}$ and $\widetilde{\mathbf{T}}_j^{-1}$ are automorphism of $\widetilde{\mathbf{U}}^\imath$, we have $\widetilde{\mathbf{T}}_i^{-1}\widetilde{\mathbf{T}}_j^{-1}(B_i) - B_j \in \widetilde{\mathbf{U}}^\imath$.

Then we can write this element in terms of monomial basis of $\widetilde{\mathbf{U}}^\imath$ (see Proposition 2.11):

$$\widetilde{\mathbf{T}}_i^{-1}\widetilde{\mathbf{T}}_j^{-1}(B_i) - B_j = \sum_{J\in\mathcal{J}} A_J B_J, \qquad \text{for some } A_J \in \widetilde{\mathbf{U}}_\bullet^+ \widetilde{\mathbf{U}}^{\imath 0}. \tag{12.1}$$

On the other hand, using the intertwining relation (4.6) twice, we have

$$\widetilde{\mathbf{T}}_i^{-1}\widetilde{\mathbf{T}}_j^{-1}(B_i) = \widetilde{\Upsilon}_i \widetilde{\mathcal{T}}_{\mathbf{r}_i}^{-1}(\widetilde{\Upsilon}_j) \cdot \widetilde{\mathcal{T}}_{\mathbf{r}_i}^{-1}\widetilde{\mathcal{T}}_{\mathbf{r}_j}^{-1}(B_i) \cdot \widetilde{\mathcal{T}}_{\mathbf{r}_i}^{-1}(\widetilde{\Upsilon}_j^{-1})\widetilde{\Upsilon}_i^{-1}$$

By Lemma 12.1, we rewrite the above identity as

$$\widetilde{\mathbf{T}}_i^{-1}\widetilde{\mathbf{T}}_j^{-1}(B_i) = \widetilde{\Upsilon}_i \widetilde{\mathcal{T}}_{\mathbf{r}_i}^{-1}(\widetilde{\Upsilon}_j) \cdot B_j \cdot \widetilde{\mathcal{T}}_{\mathbf{r}_i}^{-1}(\widetilde{\Upsilon}_j^{-1})\widetilde{\Upsilon}_i^{-1}. \tag{12.2}$$

By the equality (12.2), we rewrite (12.1) in the following form

$$\widetilde{\Upsilon}_i \widetilde{\mathcal{T}}_{\mathbf{r}_i}^{-1}(\widetilde{\Upsilon}_j) \cdot B_j \cdot \widetilde{\mathcal{T}}_{\mathbf{r}_i}^{-1}(\widetilde{\Upsilon}_j^{-1})\widetilde{\Upsilon}_i^{-1} - B_j = \sum_{J\in\mathcal{J}} A_J B_J. \tag{12.3}$$

Now we claim $A_J B_J = 0$, for each $J \in \mathcal{J}$, by comparing the weights in $\mathbb{Z}\mathbb{I}$. Recall from Remark 3.10 that $\widetilde{\Upsilon}_i = \sum_{m\geq 0} \widetilde{\Upsilon}_i^m$ where $\mathrm{wt}(\widetilde{\Upsilon}_i^m) = m(\alpha_i + w_\bullet \alpha_{\tau i})$ and then weights of $\widetilde{\mathcal{T}}_{\mathbf{r}_i}^{-1}(\widetilde{\Upsilon}_j)$ lie in $\mathbb{N}(\mathbf{r}_i\alpha_j + \mathbf{r}_i w_\bullet \alpha_{\tau j})$. Hence, the weights appearing on LHS (12.3) must belong to the set $Q_{ij}$, where

$$\begin{aligned}
Q_{ij} &= Q_{ij}^- \cup Q_{ij}^+,\\
Q_{ij}^- &:= -\alpha_j + \mathbb{N}(\alpha_i + w_\bullet \alpha_{\tau i}) + \mathbb{N}(\mathbf{r}_i\alpha_j + \mathbf{r}_i w_\bullet \alpha_{\tau j}),\\
Q_{ij}^+ &:= w_\bullet(\alpha_j) + \mathbb{N}(\alpha_i + w_\bullet \alpha_{\tau i}) + \mathbb{N}(\mathbf{r}_i\alpha_j + \mathbf{r}_i w_\bullet \alpha_{\tau j}).
\end{aligned}$$

On the other hand, note that the weight of the lowest weight component of $A_J B_J$ lies in $Q_J := -\mathrm{wt}(J) + \mathbb{N}\mathbb{I}_\bullet$. Then $A_J B_J \neq 0$ only if $Q_J \cap Q_{ij} \neq \varnothing$. It immediately follows that $A_J B_J = 0$ unless $\mathrm{wt}(J) \in \alpha_j + \mathbb{N}\mathbb{I}_\bullet$. Moreover, when $\mathrm{wt}(J) \in \alpha_j + \mathbb{N}\mathbb{I}_\bullet$, the only possible element in the intersection $Q_J \cap Q_{ij}$ is $-\alpha_j$.

However, since $\widetilde{\Upsilon}_i \widetilde{\mathcal{T}}_{\mathbf{r}_i}^{-1}(\widetilde{\Upsilon}_j)$ has constant term 1, the weight $(-\alpha_j)$ component for LHS (12.3) is 0. This implies that $A_J B_J = 0$, for each $J \in \mathcal{J}$, and then the desired identity follows by (12.1). □

**Corollary 12.3.** *We have*

$$\widetilde{\Upsilon}_i \widetilde{\mathcal{T}}_{\mathbf{r}_i}^{-1}(\widetilde{\Upsilon}_j) B_j = B_j \widetilde{\Upsilon}_i \widetilde{\mathcal{T}}_{\mathbf{r}_i}^{-1}(\widetilde{\Upsilon}_j). \tag{12.4}$$

*Proof.* One reads off from the proof of Proposition 12.2 that $A_J B_J = 0$, for $J \in \mathcal{J}$, and hence the corollary follows from the relation (12.3). □

**Corollary 12.4.** *We have*

$$B_i \widetilde{\Upsilon}_j \widetilde{\mathcal{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_j) = \widetilde{\Upsilon}_j \widetilde{\mathcal{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_j) B_i, \tag{12.5}$$

$$B_j^\sigma \widetilde{\mathcal{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_j) \widetilde{\Upsilon}_i = \widetilde{\mathcal{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_j) \widetilde{\Upsilon}_i B_j^\sigma. \tag{12.6}$$

*Proof.* Switching $i, j$ in (12.4), we obtain

$$\widetilde{\Upsilon}_j \widetilde{\mathcal{T}}_{\mathbf{r}_j}^{-1}(\widetilde{\Upsilon}_i) B_i = B_i \widetilde{\Upsilon}_j \widetilde{\mathcal{T}}_{\mathbf{r}_j}^{-1}(\widetilde{\Upsilon}_i). \tag{12.7}$$

By Proposition 13.3, we have $\widetilde{\mathcal{T}}_{\mathbf{r}_j}^{-1}(\widetilde{\Upsilon}_i) = \widetilde{\mathcal{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_j)$. Hence, (12.7) implies the desired identity (12.5).

Recall from Proposition 3.8 that $\widetilde{\Upsilon}_i, \widetilde{\Upsilon}_j$ are both fixed by the anti-involution $\sigma$. Recall also that $\sigma \widetilde{\mathcal{T}}_{\mathbf{r}_i}^{-1} \sigma = \widetilde{\mathcal{T}}_{\mathbf{r}_i}$. Applying the anti-involution $\sigma$ to the identity (12.4), we have proved (12.6). ☐

## 12.2 Rank 2 cases with $\ell_\circ(\boldsymbol{w}_\circ) = 4$

In this subsection, we assume that $\mathbb{I}_{\circ,\tau} = \{i, j\}$ such that $\ell_\circ(\boldsymbol{w}_\circ) = 4$. Let $\{i, \tau i\}$ and $\{j, \tau j\}$ be the corresponding two distinct $\tau$-orbits of $\mathbb{I}_\circ$.

**Lemma 12.5.** *Denote the diagram involution $\varrho := \tau_0 \tau_{\bullet,i}$. Then we have*

$$\mathbf{r}_j \mathbf{r}_i \mathbf{r}_j(\alpha_i) = \alpha_{\varrho i}, \quad \text{and} \quad \widetilde{\mathcal{T}}_{\mathbf{r}_j} \widetilde{\mathcal{T}}_{\mathbf{r}_i} \widetilde{\mathcal{T}}_{\mathbf{r}_j}(B_i) = B_{\varrho i}.$$

*(Moreover, a nontrivial $\varrho$ can occur only in type AIII, and in this case, $\varrho = \tau$.)*

*Proof.* As before, set $w_0$ to be the longest element of the Weyl group $W$ and $w_{\bullet,i} = \mathbf{r}_i w_\bullet$; set $\tau_0$ and $\tau_{\bullet,i}$ to be the diagram automorphisms corresponding to $w_0$ and $w_{\bullet,i}$, respectively. In this case, $w_0$ satisfies the relation $w_0 = \boldsymbol{w}_\circ w_\bullet = \mathbf{r}_j \mathbf{r}_i \mathbf{r}_j \mathbf{r}_i w_\bullet = \mathbf{r}_j \mathbf{r}_i \mathbf{r}_j w_{\bullet,i}$. Then we have

$$\tau_0(\alpha_i) = -w_0(\alpha_i) = -\mathbf{r}_j \mathbf{r}_i \mathbf{r}_j w_{\bullet,i}(\alpha_i) = \mathbf{r}_j \mathbf{r}_i \mathbf{r}_j \tau_{\bullet,i}(\alpha_i).$$

Setting $\rho := \tau_0 \tau_{\bullet,i}$, we have obtained $\mathbf{r}_j \mathbf{r}_i \mathbf{r}_j(\alpha_i) = \alpha_{\varrho i}$. (We thank Stefan Kolb for providing the above conceptual argument which replaces our earlier case-by-case proof of the existence of $\rho$; moreover, his argument produces a precise formula for $\rho$.)

In particular, we observe that a nontrivial $\varrho$ occurs only in type AIII (for some particular $i$), and in this case, $\varrho = \tau$.

Recalling $\mathbf{r}_i = \mathbf{r}_{\tau i}$, we also have $\mathbf{r}_j \mathbf{r}_i \mathbf{r}_j(\alpha_{\tau i}) = \alpha_{\varrho \tau i}$.

We have $\ell(\mathbf{r}_j \mathbf{r}_i \mathbf{r}_j) = \ell(\mathbf{r}_j) + \ell(\mathbf{r}_i) + \ell(\mathbf{r}_j)$, by Proposition 2.9. Therefore, it follows from $\mathbf{r}_j\mathbf{r}_i\mathbf{r}_j(\alpha_i) = \alpha_{\varrho i}$ that $\widetilde{\mathcal{T}}_{\mathbf{r}_j}\widetilde{\mathcal{T}}_{\mathbf{r}_i}\widetilde{\mathcal{T}}_{\mathbf{r}_j}(X_i) = X_{\varrho i}$, for $X = F, K'$; cf. [Lus93, 39.2] or [Ja95, Proposition 8.20]. Similarly, we have $\widetilde{\mathcal{T}}_{\mathbf{r}_j}\widetilde{\mathcal{T}}_{\mathbf{r}_i}\widetilde{\mathcal{T}}_{\mathbf{r}_j}(E_{\tau i}) = E_{\varrho\tau i}$.

Recall $B_i = F_i + \widetilde{\mathcal{T}}_{w_\bullet}(E_{\tau i})K_i'$. Thanks to (2.21), $\widetilde{\mathcal{T}}_{w_\bullet}$ commutes with both $\widetilde{\mathcal{T}}_{\mathbf{r}_i}$ and $\widetilde{\mathcal{T}}_{\mathbf{r}_j}$. Therefore, we have

$$\widetilde{\mathcal{T}}_{\mathbf{r}_j}\widetilde{\mathcal{T}}_{\mathbf{r}_i}\widetilde{\mathcal{T}}_{\mathbf{r}_j}(B_i) = \widetilde{\mathcal{T}}_{\mathbf{r}_j\mathbf{r}_i\mathbf{r}_j}(F_i + \widetilde{\mathcal{T}}_{w_\bullet}(E_{\tau i})K_i') = F_{\varrho i} + \widetilde{\mathcal{T}}_{w_\bullet}(E_{\varrho\tau i})K_{\varrho i}' = B_{\varrho i}.$$

The lemma is proved. □

**Proposition 12.6.** *Retain the notation in Lemma 12.5. Then* $\widetilde{\mathbf{T}}_j^{-1}\widetilde{\mathbf{T}}_i^{-1}\widetilde{\mathbf{T}}_j^{-1}(B_i) = B_{\varrho i}$*; or equivalently,* $\widetilde{\mathbf{T}}_j\widetilde{\mathbf{T}}_i\widetilde{\mathbf{T}}_j(B_i) = B_{\varrho i}$.

*Proof.* Since $\widetilde{\mathbf{T}}_i^{-1}$ and $\widetilde{\mathbf{T}}_j^{-1}$ are automorphism of $\widetilde{\mathbf{U}}^\imath$, we have $\widetilde{\mathbf{T}}_j^{-1}\widetilde{\mathbf{T}}_i^{-1}\widetilde{\mathbf{T}}_j^{-1}(B_i) - B_{\varrho i} \in \widetilde{\mathbf{U}}^\imath$. Then we can write this element in terms of monomial basis of $\widetilde{\mathbf{U}}^\imath$ (see Proposition 2.11):

$$\widetilde{\mathbf{T}}_j^{-1}\widetilde{\mathbf{T}}_i^{-1}\widetilde{\mathbf{T}}_j^{-1}(B_i) - B_{\varrho i} = \sum_{J\in\mathcal{J}} A_J B_J, \qquad \text{for some } A_J \in \widetilde{\mathbf{U}}_\bullet^+\widetilde{\mathbf{U}}^{\imath 0}. \tag{12.8}$$

On the other hand, using the intertwining relation (4.6) of $\widetilde{\mathbf{T}}_i^{-1}$, we have

$$\begin{aligned}&\widetilde{\mathbf{T}}_j^{-1}\widetilde{\mathbf{T}}_i^{-1}\widetilde{\mathbf{T}}_j^{-1}(B_i)\\ &= \widetilde{\Upsilon}_j\,\widetilde{\mathcal{T}}_{\mathbf{r}_j}^{-1}(\widetilde{\Upsilon}_i)\,\widetilde{\mathcal{T}}_{\mathbf{r}_j}^{-1}\widetilde{\mathcal{T}}_{\mathbf{r}_i}^{-1}(\widetilde{\Upsilon}_j)\cdot\widetilde{\mathcal{T}}_{\mathbf{r}_j}^{-1}\widetilde{\mathcal{T}}_{\mathbf{r}_i}^{-1}\widetilde{\mathcal{T}}_{\mathbf{r}_j}^{-1}(B_i)\cdot\widetilde{\mathcal{T}}_{\mathbf{r}_j}^{-1}\widetilde{\mathcal{T}}_{\mathbf{r}_i}^{-1}(\widetilde{\Upsilon}_j^{-1})\,\widetilde{\mathcal{T}}_{\mathbf{r}_j}^{-1}(\widetilde{\Upsilon}_i^{-1})\,\widetilde{\Upsilon}_j^{-1}.\end{aligned}$$

Since $\widetilde{\mathcal{T}}_{\mathbf{r}_j}^{-1}\widetilde{\mathcal{T}}_{\mathbf{r}_i}^{-1}\widetilde{\mathcal{T}}_{\mathbf{r}_j}^{-1}(B_i) = B_{\varrho i}$ by Lemma 12.5, we rewrite the above identity as

$$\widetilde{\mathbf{T}}_j^{-1}\widetilde{\mathbf{T}}_i^{-1}\widetilde{\mathbf{T}}_j^{-1}(B_i) = \widetilde{\Upsilon}_j\,\widetilde{\mathcal{T}}_{\mathbf{r}_j}^{-1}(\widetilde{\Upsilon}_i)\,\widetilde{\mathcal{T}}_{\mathbf{r}_j}^{-1}\widetilde{\mathcal{T}}_{\mathbf{r}_i}^{-1}(\widetilde{\Upsilon}_j)\cdot B_{\varrho i}\cdot\widetilde{\mathcal{T}}_{\mathbf{r}_j}^{-1}\widetilde{\mathcal{T}}_{\mathbf{r}_i}^{-1}(\widetilde{\Upsilon}_j^{-1})\,\widetilde{\mathcal{T}}_{\mathbf{r}_j}^{-1}(\widetilde{\Upsilon}_i^{-1})\,\widetilde{\Upsilon}_j^{-1}. \tag{12.9}$$

By the identity (12.9), we rewrite (12.8) in the following form

$$\widetilde{\Upsilon}_j\,\widetilde{\mathcal{T}}_{\mathbf{r}_j}^{-1}(\widetilde{\Upsilon}_i)\,\widetilde{\mathcal{T}}_{\mathbf{r}_j}^{-1}\widetilde{\mathcal{T}}_{\mathbf{r}_i}^{-1}(\widetilde{\Upsilon}_j)\cdot B_{\varrho i}\cdot \widetilde{\mathcal{T}}_{\mathbf{r}_j}^{-1}\widetilde{\mathcal{T}}_{\mathbf{r}_i}^{-1}(\widetilde{\Upsilon}_j^{-1})\,\widetilde{\mathcal{T}}_{\mathbf{r}_j}^{-1}(\widetilde{\Upsilon}_i^{-1})\,\widetilde{\Upsilon}_j^{-1} - B_{\varrho i} = \sum_{J\in\mathcal{J}} A_J B_J. \tag{12.10}$$

By a weight argument entirely similar to the proof of Proposition 12.2, we obtain $\sum_{J\in\mathcal{J}} A_J B_J = 0$. Thus, the proposition follows by (12.8). □

**Corollary 12.7.** *We have*

$$B_{\varrho i}\widetilde{\Upsilon}_j\,\widetilde{\mathcal{T}}_{\mathbf{r}_j}^{-1}(\widetilde{\Upsilon}_i)\,\widetilde{\mathcal{T}}_{\mathbf{r}_j}^{-1}\widetilde{\mathcal{T}}_{\mathbf{r}_i}^{-1}(\widetilde{\Upsilon}_j) = \widetilde{\Upsilon}_j\,\widetilde{\mathcal{T}}_{\mathbf{r}_j}^{-1}(\widetilde{\Upsilon}_i)\,\widetilde{\mathcal{T}}_{\mathbf{r}_j}^{-1}\widetilde{\mathcal{T}}_{\mathbf{r}_i}^{-1}(\widetilde{\Upsilon}_j)B_{\varrho i}. \tag{12.11}$$

*Proof.* Since $\sum_{J\in\mathcal{J}} A_J B_J = 0$, as shown in the proof of Proposition 12.6, the corollary follows from the relation (12.10). □

**Corollary 12.8.** *We have*

$$B_i\widetilde{\Upsilon}_j\,\widetilde{\mathcal{T}}_{\mathbf{r}_j}^{-1}(\widetilde{\Upsilon}_i)\,\widetilde{\mathcal{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_j) = \widetilde{\Upsilon}_j\,\widetilde{\mathcal{T}}_{\mathbf{r}_j}^{-1}(\widetilde{\Upsilon}_i)\,\widetilde{\mathcal{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_j)B_i, \tag{12.12}$$

$$B_j^{\sigma}\widetilde{\mathcal{T}}_{\mathbf{r}_j}^{-1}(\widetilde{\Upsilon}_i)\,\widetilde{\mathcal{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_j)\,\widetilde{\Upsilon}_i = \widetilde{\mathcal{T}}_{\mathbf{r}_j}^{-1}(\widetilde{\Upsilon}_i)\,\widetilde{\mathcal{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_j)\,\widetilde{\Upsilon}_i B_j^{\sigma}. \tag{12.13}$$

*Proof.* We prove (12.12). Noting that $\varrho$ equals either Id or $\tau$, we have by Remark 4.8 that $\varrho$ commutes with $\widetilde{\mathcal{T}}_{\mathbf{r}_i}, \widetilde{\mathcal{T}}_{\mathbf{r}_j}$, and by Proposition 3.8 that $\varrho$ fixes $\widetilde{\Upsilon}_i, \widetilde{\Upsilon}_j$. Hence, applying $\varrho$ to both sides of (12.11), we have

$$B_i\widetilde{\Upsilon}_j\,\widetilde{\mathcal{T}}_{\mathbf{r}_j}^{-1}(\widetilde{\Upsilon}_i)\,\widetilde{\mathcal{T}}_{\mathbf{r}_j}^{-1}\widetilde{\mathcal{T}}_{\mathbf{r}_i}^{-1}(\widetilde{\Upsilon}_j) = \widetilde{\Upsilon}_j\,\widetilde{\mathcal{T}}_{\mathbf{r}_j}^{-1}(\widetilde{\Upsilon}_i)\,\widetilde{\mathcal{T}}_{\mathbf{r}_j}^{-1}\widetilde{\mathcal{T}}_{\mathbf{r}_i}^{-1}(\widetilde{\Upsilon}_j)B_i. \tag{12.14}$$

By Proposition 13.3, we have $\widetilde{\mathcal{T}}_{\mathbf{r}_j}^{-1}\widetilde{\mathcal{T}}_{\mathbf{r}_i}^{-1}(\widetilde{\Upsilon}_j) = \widetilde{\mathcal{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_j)$. Hence, the desired relation

(12.12) follows by (12.14).

We next show (12.13). Recall from Proposition 3.8 that $\widetilde{\Upsilon}_i, \widetilde{\Upsilon}_j$ are both fixed by the anti-involution $\sigma$. Recall also that $\sigma\widetilde{\mathcal{T}}_{\mathbf{r}_i}^{-1}\sigma = \widetilde{\mathcal{T}}_{\mathbf{r}_i}$. Switching $i, j$ in (12.12) and then applying $\sigma$ to it, we obtain (12.13). □

## 12.3 Rank 2 case with $\ell_\circ(\boldsymbol{w}_\circ) = 6$

The rank 2 case with $\ell_\circ(\boldsymbol{w}_\circ) = 6$ occurs only in split $G_2$ type. Let $(\mathbb{I} = \mathbb{I}_\circ, \mathrm{Id})$ be a Sakate diagram of split type $G_2$. In this case, the relative Weyl group $W^\circ$ is identified with $W$ and $\mathbf{r}_a = s_a$ for $a \in \mathbb{I} = \mathbb{I}_\circ = \{i, j\}$. We do not specify which root $i$ or $j$ is long.

Set $\underline{w_i} = s_j s_i s_j s_i s_j$ and $\widetilde{\mathcal{T}}_{\underline{w_i}} = \widetilde{\mathcal{T}}_j \widetilde{\mathcal{T}}_i \widetilde{\mathcal{T}}_j \widetilde{\mathcal{T}}_i \widetilde{\mathcal{T}}_j$. Then we have $\underline{w_i}(\alpha_i) = \alpha_i$.

**Lemma 12.9.** *We have* $\widetilde{\mathcal{T}}_{\underline{w_i}}^{-1}(B_i) = B_i$.

*Proof.* Follows by [Lus93, 39.2] and the same type of arguments as for Lemma 12.1 and Lemma 12.5. □

**Proposition 12.10.** *We have* $\widetilde{\mathbf{T}}_{\underline{w_i}}^{-1}(B_i) = B_i$*; or equivalently,* $\widetilde{\mathbf{T}}_{\underline{w_i}}(B_i) = B_i$.

*Proof.* Since $\widetilde{\mathbf{T}}_i^{-1}$ and $\widetilde{\mathbf{T}}_j^{-1}$ are automorphism of $\widetilde{\mathbf{U}}^\imath$, we have $\widetilde{\mathbf{T}}_{\underline{w_i}}^{-1}(B_i) - B_i \in \widetilde{\mathbf{U}}^\imath$. Then we can write this element in terms of monomial basis of $\widetilde{\mathbf{U}}^\imath$ (see Proposition 2.11):

$$\widetilde{\mathbf{T}}_{\underline{w_i}}^{-1}(B_i) - B_i = \sum_{J \in \mathcal{J}} A_J B_J, \qquad \text{for some } A_J \in \widetilde{\mathbf{U}}_\bullet^+ \widetilde{\mathbf{U}}^{\imath 0}. \tag{12.15}$$

On the other hand, using the intertwining relation (4.6) of $\widetilde{\mathbf{T}}_i^{-1}$, we have

$$\widetilde{\mathbf{T}}_{\underline{w_i}}^{-1}(B_i) = \Omega_i \widetilde{\mathcal{T}}_{\underline{w_i}}^{-1}(B_i) \Omega_i^{-1}, \tag{12.16}$$

where

$$\Omega_i = \widetilde{\Upsilon}_j \, \widetilde{\mathcal{T}}_j^{-1}(\widetilde{\Upsilon}_i) \, \widetilde{\mathcal{T}}_j^{-1}\widetilde{\mathcal{T}}_i^{-1}(\widetilde{\Upsilon}_j) \, \widetilde{\mathcal{T}}_j^{-1}\widetilde{\mathcal{T}}_i^{-1}\widetilde{\mathcal{T}}_j^{-1}(\widetilde{\Upsilon}_i) \, \widetilde{\mathcal{T}}_j^{-1}\widetilde{\mathcal{T}}_i^{-1}\widetilde{\mathcal{T}}_j^{-1}\widetilde{\mathcal{T}}_i^{-1}(\widetilde{\Upsilon}_j). \tag{12.17}$$

By Lemma 12.9, we rewrite the identity (12.16) as

$$\widetilde{\mathbf{T}}_{\underline{w_i}}^{-1}(B_i) = \Omega_i B_i \Omega_i^{-1}. \tag{12.18}$$

By the identity (12.18), we rewrite (12.15) in the following form

$$\Omega_i B_i \Omega_i^{-1} - B_i = \sum_{J \in \mathcal{J}} A_J B_J. \tag{12.19}$$

By a weight argument entirely similar to the proof of Proposition 12.2, we obtain $\sum_{J\in\mathcal{J}} A_J B_J = 0$. Thus, the proposition follows by (12.15). □

**Corollary 12.11.** *Let $\Omega_i$ be as in (12.17). We have*

$$B_i \Omega_i = \Omega_i B_i. \tag{12.20}$$

*Proof.* Since $\sum_{J\in\mathcal{J}} A_J B_J = 0$, as shown in the proof of Proposition 12.10, the corollary follows from the formula (12.19). □

**Corollary 12.12.** *We have the following intertwining relations:*

$$\begin{aligned}
&B_i \widetilde{\Upsilon}_j \widetilde{\mathcal{T}}_{s_i s_j s_i s_j}(\widetilde{\Upsilon}_i) \widetilde{\mathcal{T}}_{s_i s_j s_i}(\widetilde{\Upsilon}_j) \widetilde{\mathcal{T}}_{s_i s_j}(\widetilde{\Upsilon}_i) \widetilde{\mathcal{T}}_i(\widetilde{\Upsilon}_j) \\
&\quad = \widetilde{\Upsilon}_j \widetilde{\mathcal{T}}_{s_i s_j s_i s_j}(\widetilde{\Upsilon}_i) \widetilde{\mathcal{T}}_{s_i s_j s_i}(\widetilde{\Upsilon}_j) \widetilde{\mathcal{T}}_{s_i s_j}(\widetilde{\Upsilon}_i) \widetilde{\mathcal{T}}_i(\widetilde{\Upsilon}_j) B_i, \qquad (12.21) \\
&B_j^\sigma \widetilde{\mathcal{T}}_{s_i s_j s_i s_j}(\widetilde{\Upsilon}_i) \widetilde{\mathcal{T}}_{s_i s_j s_i}(\widetilde{\Upsilon}_j) \widetilde{\mathcal{T}}_{s_i s_j}(\widetilde{\Upsilon}_i) \widetilde{\mathcal{T}}_i(\widetilde{\Upsilon}_j) \widetilde{\Upsilon}_i \\
&\quad = \widetilde{\mathcal{T}}_{s_i s_j s_i s_j}(\widetilde{\Upsilon}_i) \widetilde{\mathcal{T}}_{s_i s_j s_i}(\widetilde{\Upsilon}_j) \widetilde{\mathcal{T}}_{s_i s_j}(\widetilde{\Upsilon}_i) \widetilde{\mathcal{T}}_i(\widetilde{\Upsilon}_j) \widetilde{\Upsilon}_i B_j^\sigma. \qquad (12.22)
\end{aligned}$$

*Proof.* By Proposition 13.3, we have $\widetilde{\mathcal{T}}_{s_js_is_js_is_j}(\widetilde{\Upsilon}_i) = \widetilde{\Upsilon}_i$ and $\widetilde{\mathcal{T}}_{s_is_js_is_js_i}(\widetilde{\Upsilon}_j) = \widetilde{\Upsilon}_j$. Then we have

$$\Omega_i = \widetilde{\Upsilon}_j \widetilde{\mathcal{T}}_{s_is_js_is_j}(\widetilde{\Upsilon}_i)\widetilde{\mathcal{T}}_{s_is_js_i}(\widetilde{\Upsilon}_j)\widetilde{\mathcal{T}}_{s_is_j}(\widetilde{\Upsilon}_i)\widetilde{\mathcal{T}}_i(\widetilde{\Upsilon}_j).$$

Hence, the desired identity (12.21) follows by (12.20).

We next prove (12.22). Switching $i, j$ in (12.20), we have

$$B_j\Omega_j = \Omega_j B_j, \tag{12.23}$$

where $\Omega_j$ is defined by switching $i, j$ in (12.17).

Recall from Proposition 3.8 that $\widetilde{\Upsilon}_i$, $\widetilde{\Upsilon}_j$ are both fixed by $\sigma$. Then by the definition of $\Omega_j$, we have

$$\sigma(\Omega_j) = \widetilde{\mathcal{T}}_{s_is_js_is_j}(\widetilde{\Upsilon}_i)\widetilde{\mathcal{T}}_{s_is_js_i}(\widetilde{\Upsilon}_j)\widetilde{\mathcal{T}}_{s_is_j}(\widetilde{\Upsilon}_i)\widetilde{\mathcal{T}}_i(\widetilde{\Upsilon}_j)\widetilde{\Upsilon}_i.$$

Hence, applying $\sigma$ to (12.23) and then using this formula of $\sigma(\Omega_j)$, we obtain (12.22). □

## 12.4 The general identity $\widetilde{\mathbf{T}}_w(B_i) = B_{wi}$

Let $w \in W^\circ$. Given a reduced expression $\underline{w} = \mathbf{r}_{i_1}\mathbf{r}_{i_2}\ldots\mathbf{r}_{i_k}$ for $w$, we shall denote $\widetilde{\mathbf{T}}_{\underline{w}} = \widetilde{\mathbf{T}}_{i_1}\widetilde{\mathbf{T}}_{i_2}\ldots\widetilde{\mathbf{T}}_{i_k}$.

**Theorem 12.13.** *Suppose that $wi \in \mathbb{I}_\circ$, for $w \in W^\circ$ and $i \in \mathbb{I}_\circ$. Then $\widetilde{\mathbf{T}}_{\underline{w}}(B_i) = B_{wi}$, for* some *reduced expression $\underline{w}$ of $w$.*

(Once Theorem 14.1 on braid relation for $\widetilde{\mathbf{T}}_i$ is proved, we can replace $\widetilde{\mathbf{T}}_{\underline{w}}$ in

Theorem 12.13 by $\widetilde{\mathbf{T}}_w$, which depends only on $w$, not on a reduced expression $\underline{w}$ of $w$.)

*Proof.* The strategy of the proof is modified from a well-known quantum group counterpart, cf. [Ja95, Lemma 8.20]. We shall reduce the proof to the rank 2 cases which were established earlier and finish the proof by induction on $\ell_\circ(w)$.

The statement holds for arbitrary rank 2 Satake (sub-)diagrams $(\mathbb{I}_\bullet \cup \{i, \tau i, j, \tau j\}, \tau)$. Indeed in the case when $\ell(\boldsymbol{w}_\circ) = 2$, the claim is trivial. In the case when $\ell(\boldsymbol{w}_\circ) = 3, 4$ or 6, the claim has been established in Proposition 12.2, Proposition 12.6, and Proposition 12.10 respectively. In the case when $\ell(\boldsymbol{w}_\circ) = \infty$, there do not exist elements $w \in W^\circ, i \in \mathbb{I}_\circ$ such that $wi \in \mathbb{I}_\circ$ and then the claim is trivial.

In general, we use an induction on $l_\circ(w)$, for $w \in W^\circ$, where $l_\circ$ is the length function for the relative Weyl group $W^\circ$. Recall the simple system $\{\overline{\alpha}_i | i \in \mathbb{I}_{\circ,\tau}\}$ for the relative root system from (2.23). Since $w\theta = \theta w$ and $wi \in \mathbb{I}_\circ$ by assumption, we have $w(\overline{\alpha}_i) = \overline{\alpha}_{wi}$. We denote a positive (and negative) root in the relative root system by $\beta > 0$ (and respectively, $\beta < 0$).

Suppose that $l_\circ(w) > 0$. Then there exists $j \in \mathbb{I}_\circ$ such that $w(\overline{\alpha}_j) < 0$; clearly $j \neq i, \tau i$ since $w(\overline{\alpha}_i) > 0$. Consider the minimal length representatives of $W^\circ$ with respect to the rank 2 parabolic subgroup $\langle \mathbf{r}_i, \mathbf{r}_j \rangle$. We have a decomposition $w = w'w''$ in $W^\circ$ such that $w'(\overline{\alpha}_i) > 0, w'(\overline{\alpha}_j) > 0$ and $w''$ lies in the subgroup $\langle \mathbf{r}_i, \mathbf{r}_j \rangle$; moreover, $l_\circ(w) = l_\circ(w') + l_\circ(w'')$. Now $w(\overline{\alpha}_i) > 0$ and $w(\overline{\alpha}_j) < 0$ implies that $w''(\overline{\alpha}_i) > 0$ and $w''(\overline{\alpha}_j) < 0$ (since $w'$ preserves the signs of the roots $w''(\overline{\alpha}_i)$ and $w''(\overline{\alpha}_j)$). It follows that

$$w''(\alpha_i) > 0, \qquad w''(\alpha_j) < 0, \qquad w'(\alpha_i) > 0, \qquad w'(\alpha_j) > 0.$$

(The positive system of the restricted root system is compatible with the positive system of $\mathcal{R}$.) Moreover since $\mathbf{r}_s$, for any $s \in \mathbb{I}_{\circ,\tau}$, acts on $\mathbb{I}_\bullet$ as the involution $\tau_{\bullet,s}\tau$, we must have $w'(\alpha_a) > 0$, for any $a \in \mathbb{I}_\bullet$; see also Proposition 4.11.

We show that $w''i \in \mathbb{I}_\circ$. Since $w''(\alpha_i) > 0$ and $w''(\alpha_i) \in \mathcal{R} \cap (\mathbb{Z}\alpha_i + \mathbb{Z}\alpha_j + \mathbb{Z}\mathbb{I}_\bullet)$, we can write $w''(\alpha_i) \in \mathcal{R}$ in the following form

$$w''(\alpha_i) = r\alpha_i + s\alpha_j + \alpha_\bullet$$

for some $r, s \geq 0, \alpha_\bullet \in \mathbb{N}\mathbb{I}_\bullet$. We consider the following cases:

1. At least two of $r, s, \alpha_\bullet$ are nonzero. Then $w'w''(\alpha_i) = rw'(\alpha_i) + sw'(\alpha_j) + w'(\alpha_\bullet)$ cannot be simple for $w'(\alpha_i) > 0, w'(\alpha_j) > 0, w'(\alpha_\bullet) > 0$; this contradicts that $w(\alpha_i) = w'w''(\alpha_i)$ is simple.

2. $r = 0, \alpha_\bullet = 0$ and $s > 0$. Then $s = 1$ and $w''(\alpha_i) = \alpha_j$ is simple. A similar argument applying to the case $s = 0, \alpha_\bullet = 0$ and $r > 0$ shows that $w''(\alpha_i) = \alpha_i$ is simple.

3. $r = s = 0, \alpha_\bullet \neq 0$. We show that this case cannot occur. Indeed, we have $\theta w''(\alpha_i) = \theta(\alpha_\bullet) = \alpha_\bullet = w''(\alpha_i)$. Since $w''\theta = \theta w''$, the above identity implies that $\alpha_i$ is fixed by $\theta$, which is impossible for $i \in \mathbb{I}_\circ$.

Therefore, we have shown $w''i \in \mathbb{I}_\circ$ and $w''(\alpha_i) = \alpha_{w''i}$. By the rank two results in Proposition 12.2 and Proposition 12.6, we have $\widetilde{\mathbf{T}}_{\underline{w''}}(B_i) = B_{w''i}$, for any reduced expression $\underline{w''}$ of $w''$. Now using the induction hypothesis, there exists a reduced expression $\underline{w'}$ such that $\underline{w} = \underline{w'} \cdot \underline{w''}$ is a reduced expression for $w$ and

$$\widetilde{\mathbf{T}}_{\underline{w}}(B_i) = \widetilde{\mathbf{T}}_{\underline{w'}}\widetilde{\mathbf{T}}_{\underline{w''}}(B_i) = \widetilde{\mathbf{T}}_{\underline{w'}}(B_{w''i}) = B_{wi}.$$

The theorem is proved. □

# 13 Factorization of quasi $K$-matrices

It is conjectured by Dobson and Kolb [DK19] that quasi $K$-matrices admit factorization into products of rank 1 quasi $K$-matrices analogous to the factorization properties of quasi $R$-matrices. They showed that the factorization of quasi $K$-matrices for arbitrary finite types reduces to the rank two cases. In this section, using (the rank 2 cases of) Theorem 12.13 we provide a uniform proof of the factorization of quasi $K$-matrices for all rank two Satake diagrams, hence completing the proof of Dobson-Kolb conjecture in all finite types.

## 13.1 Factorization of $\widetilde{\Upsilon}$

Let $(\mathbb{I} = \mathbb{I}_\bullet \cup \mathbb{I}_\circ, \tau)$ be a Satake diagram of arbitrary Kac-Moody type. Let $w$ be any element in the relative Weyl group $W^\circ$ with a reduced expression

$$\underline{w} = \mathbf{r}_{i_1}\mathbf{r}_{i_2}\cdots\mathbf{r}_{i_m};$$

here $m = \ell_\circ(w)$, the length of $w \in W^\circ$ (not to be confused as the length $\ell(w)$ in $W$).

Following [DK19] (who worked in the setting of $\mathbf{U}^\imath_{\boldsymbol{\varsigma}}$), we define, for $1 \le k \le m$,

$$\begin{aligned} \widetilde{\Upsilon}^{[k]} &= \widetilde{\mathcal{T}}_{\mathbf{r}_{i_1}}\widetilde{\mathcal{T}}_{\mathbf{r}_{i_2}}\cdots\widetilde{\mathcal{T}}_{\mathbf{r}_{i_{k-1}}}(\widetilde{\Upsilon}_{i_k}), \\ \widetilde{\Upsilon}_{\underline{w}} &= \widetilde{\Upsilon}^{[m]}\widetilde{\Upsilon}^{[m-1]}\cdots\widetilde{\Upsilon}^{[1]}. \end{aligned} \tag{13.1}$$

(In the notation $\widetilde{\Upsilon}^{[k]}$ above, we have suppressed the dependence on $\underline{w}$.)

The goal of this section is to establish Theorem 13.1, which is a $\widetilde{\mathbf{U}}^\imath$-variant of (and implies) [DK19, Conjecture 3.22] for $\mathbf{U}^\imath_{\boldsymbol{\varsigma}}$ with general parameters $\boldsymbol{\varsigma}$. The restriction on parameters $\boldsymbol{\varsigma}$ in [DK19] can be removed in light of the development in [AV22, KY20] which allows more general parameters in quasi $K$-matrices.

**Theorem 13.1.**

*(1) For any $w \in W^\circ$, the partial quasi $K$-matrix $\widetilde{\Upsilon}_{\underline{w}}$ is independent of the choice of reduced expressions of $w$ (and hence can be denoted by $\widetilde{\Upsilon}_w$).*

*(2) The quasi $K$-matrix $\widetilde{\Upsilon}$ for $\widetilde{\mathbf{U}}^\imath$ of any finite type admits a factorization $\widetilde{\Upsilon} = \widetilde{\Upsilon}_{\boldsymbol{w}_\circ}$, where $\boldsymbol{w}_\circ$ is the longest element in the relative Weyl group $W^\circ$.*

## 13.2 Reduction to rank 2

Let us recall some partial results from [DK19] in this direction (which can be adapted from $\mathbf{U}^\imath_{\boldsymbol{\varsigma}}$ to $\widetilde{\mathbf{U}}^\imath$ without difficulties).

**Theorem 13.2** (cf. [DK19, Theorems 3.17 and 3.20])**.** *(a) Theorem 13.1 (1) holds for $\widetilde{\mathbf{U}}^\imath$ of a given Kac-Moody type if it holds for all its finite-type rank 2 Satake subdiagrams.*

*(b) Theorem 13.1 (2) holds for $\widetilde{\mathbf{U}}^\imath$ of a given finite type if it holds for all its rank 2 Satake subdiagrams.*

The proof for Theorem 13.2 (a) is essentially the same as [DK19, Theorem 3.17] (Even though only finite type Satake diagrams are considered in that paper, the proof therein can be adopted for our proposition with no difficulty).

The arguments for Theorem 13.2 (b) are largely formal once the following crucial result (see [DK19, Proposition 3.18]) is in place. We provide a short new proof below. Recall $\boldsymbol{w}_\circ$ is the longest element in $W^\circ$. Recall also the diagram involution $\tau_0$ such that $w_0(\alpha_i) = -\alpha_{\tau_0 \alpha_i}$, for all $i$, where $w_0$ is the longest element in $W$.

**Proposition 13.3** (cf. [DK19, Proposition 3.18]). *Let $\boldsymbol{w}_\circ = \mathbf{r}_{i_1}\mathbf{r}_{i_2}\cdots\mathbf{r}_{i_m}$ be a reduced expression of $\boldsymbol{w}_\circ$. Then we have $\widetilde{\mathcal{T}}_{\mathbf{r}_{i_1}}\widetilde{\mathcal{T}}_{\mathbf{r}_{i_2}}\cdots\widetilde{\mathcal{T}}_{\mathbf{r}_{i_{m-1}}}(\widetilde{\Upsilon}_{i_m}) = \widetilde{\Upsilon}_{\tau_0 i_m}$.*

*Proof.* We have $w_0 = \boldsymbol{w}_\circ w_\bullet$, and hence, $\widetilde{\mathcal{T}}_{w_0} = \widetilde{\mathcal{T}}_{\boldsymbol{w}_\circ}\widetilde{\mathcal{T}}_{w_\bullet}$. It follows by Lemma 4.4 that $\widetilde{\mathcal{T}}_{w_0}^{-1}\widehat{\tau}_0 = \widetilde{\mathcal{T}}^{-1}_{w_{\bullet,i_m}}\widehat{\tau}_{\bullet,i_m}$ when acting on $\widetilde{\mathbf{U}}_{\mathbb{I}_{\bullet,i_m}}$. Thus,

$$\widetilde{\mathcal{T}}_{w_0}^{-1}\widehat{\tau}_0(\widetilde{\Upsilon}_{i_m}) = \widetilde{\mathcal{T}}^{-1}_{w_{\bullet,i_m}}\widehat{\tau}_{\bullet,i_m}(\widetilde{\Upsilon}_{i_m}) = \widetilde{\mathcal{T}}^{-1}_{w_{\bullet,i_m}}(\widetilde{\Upsilon}_{i_m}),$$

since the quasi $K$-matrix $\widetilde{\Upsilon}_{i_m}$ lies in a completion of $\widetilde{\mathbf{U}}^+_{\mathbb{I}_{\bullet,i_m}}$ and $\widehat{\tau}_{\bullet,i_m}(\widetilde{\Upsilon}_{i_m}) = \widetilde{\Upsilon}_{i_m}$ (see Proposition 3.8). Then we obtain

$$\widetilde{\mathcal{T}}_{w_0}^{-1}(\widetilde{\Upsilon}_{\tau_0 i_m}) = \widetilde{\mathcal{T}}_{w_0}^{-1}\widehat{\tau}_0(\widetilde{\Upsilon}_{i_m}) = \widetilde{\mathcal{T}}^{-1}_{w_{\bullet,i_m}}(\widetilde{\Upsilon}_{i_m}) = \widetilde{\mathcal{T}}^{-1}_{\mathbf{r}_{i_m}}\widetilde{\mathcal{T}}^{-1}_{w_\bullet}(\widetilde{\Upsilon}_{i_m}) = \widetilde{\mathcal{T}}^{-1}_{\mathbf{r}_{i_m}}(\widetilde{\Upsilon}_{i_m}),$$

where the last equality follows by Proposition 4.6. By Proposition 4.6 again we have

$$\widetilde{\mathcal{T}}^{-1}_{\boldsymbol{w}_\circ}(\widetilde{\Upsilon}_{\tau_0 i_m}) = \widetilde{\mathcal{T}}_{w_0}^{-1}\widetilde{\mathcal{T}}_{w_\bullet}(\widetilde{\Upsilon}_{\tau_0 i_m}) = \widetilde{\mathcal{T}}_{w_0}^{-1}(\widetilde{\Upsilon}_{\tau_0 i_m}) = \widetilde{\mathcal{T}}^{-1}_{\mathbf{r}_{i_m}}(\widetilde{\Upsilon}_{i_m}).$$

Hence, $\widetilde{\mathcal{T}}_{\mathbf{r}_{i_1}}\widetilde{\mathcal{T}}_{\mathbf{r}_{i_2}}\cdots\widetilde{\mathcal{T}}_{\mathbf{r}_{i_{m-1}}}(\widetilde{\Upsilon}_{i_m}) = \widetilde{\mathcal{T}}_{\boldsymbol{w}_\circ}\widetilde{\mathcal{T}}^{-1}_{\mathbf{r}_{i_m}}(\widetilde{\Upsilon}_{i_m}) = \widetilde{\mathcal{T}}_{\boldsymbol{w}_\circ}\widetilde{\mathcal{T}}^{-1}_{\boldsymbol{w}_\circ}(\widetilde{\Upsilon}_{\tau_0 i_m}) = \widetilde{\Upsilon}_{\tau_0 i_m}$. □

*Remark* 13.4. It was verified in [DK19] that Theorem 13.1 holds in all type A rank 2 and all split rank 2 cases. The long computational proof therein is carried out case-by-case based on several explicit rank 1 formulas which they also computed.

We note that in the finite-type rank 2 setting the first statement in Theorem 13.1

is nontrivial only when $w = \boldsymbol{w}_\circ$, the longest element in $W^\circ$. Hence, in the remainder of this section, to prove Theorem 13.1 we can and shall assume that

$$(\mathbb{I} = \mathbb{I}_\bullet \cup \mathbb{I}_\circ, \tau) \text{ is any } \textit{rank two} \text{ Satake diagram of finite type, and } w = \boldsymbol{w}_\circ.$$

Moreover, we denote $\mathbb{I}_\circ = \{i, \tau i, j, \tau j\}$.

Let $\boldsymbol{w}_\circ = \mathbf{r}_{i_1}\mathbf{r}_{i_2}\cdots\mathbf{r}_{i_m}$ be a reduced expression. Theorem 13.1 in the case for $\ell_\circ(\boldsymbol{w}_\circ) = 2$, i.e., $\boldsymbol{w}_\circ = \mathbf{r}_i\mathbf{r}_j = \mathbf{r}_j\mathbf{r}_i$, trivially holds. The next proposition reduces the proof of Theorem 13.1 in the remaining nontrivial cases into verifying its assumption.

**Proposition 13.5.** *Assume that $B_p\widetilde{\Upsilon}_{\boldsymbol{w}_\circ} = \widetilde{\Upsilon}_{\boldsymbol{w}_\circ}B_p^\sigma$, for $p = i, j$. Then we have $\widetilde{\Upsilon} = \widetilde{\Upsilon}_{\boldsymbol{w}_\circ}$, for any reduced expression of $\boldsymbol{w}_\circ$.*

*Proof.* The identity $x\widetilde{\Upsilon}_{\boldsymbol{w}_\circ} = \widetilde{\Upsilon}_{\boldsymbol{w}_\circ}x$, for $x \in \widetilde{\mathbf{U}}^{\imath 0}\widetilde{\mathbf{U}}_\bullet$, holds by (3.4), Proposition 4.11, and (13.1). Together with the assumption that $B_p\widetilde{\Upsilon}_{\boldsymbol{w}_\circ} = \widetilde{\Upsilon}_{\boldsymbol{w}_\circ}B_p^\sigma$ $(p = i, j)$, we conclude that $\widetilde{\Upsilon}_{\boldsymbol{w}_\circ}$ satisfies the same intertwining relations in Theorem 3.6 as for $\widetilde{\Upsilon}$. Note also that clearly we have the constant term $(\widetilde{\Upsilon}_{\boldsymbol{w}_\circ})^0 = 1$. Therefore, the desired identity $\widetilde{\Upsilon} = \widetilde{\Upsilon}_{\boldsymbol{w}_\circ}$ follows by the uniqueness in Theorem 3.6. □

## 13.3 Factorizations in rank 2

The verification that $B_p\widetilde{\Upsilon}_{\boldsymbol{w}_\circ} = \widetilde{\Upsilon}_{\boldsymbol{w}_\circ}B_p^\sigma$ in the three cases $\ell_\circ(\boldsymbol{w}_\circ) = 3, 4$, or $6$, is based on the same idea, though the notations are a little different. In the subsections below, we shall consider the three cases separately.

**Factorization for $\ell_\circ(\boldsymbol{w}_\circ) = 3$**

In this subsection, we deal with the rank 2 cases for $\ell_\circ(\boldsymbol{w}_\circ) = 3$, with the help of Proposition 12.2 and Corollary 12.4.

Assume that $\mathbb{I}_{\circ,\tau} = \{i, j\}$ such that $\ell_\circ(\boldsymbol{w}_\circ) = 3$; in this case only $\tau = \mathrm{Id}$ and hence we identify $\mathbb{I}_\circ = \{i, j\}$ as well. The longest element $\boldsymbol{w}_\circ$ of the relative Weyl group has a reduced expression

$$\boldsymbol{w}_\circ = \mathbf{r}_i \mathbf{r}_j \mathbf{r}_i. \tag{13.2}$$

By definition (13.1) of $\widetilde{\Upsilon}^{[k]}$ and $\widetilde{\Upsilon}_{\boldsymbol{w}_\circ}$, we have

$$\widetilde{\Upsilon}_{\boldsymbol{w}_\circ} = \widetilde{\Upsilon}^{[3]} \widetilde{\Upsilon}^{[2]} \widetilde{\Upsilon}^{[1]}. \tag{13.3}$$

where by Proposition 13.3, $\widetilde{\mathcal{T}}_{\mathbf{r}_i} \widetilde{\mathcal{T}}_{\mathbf{r}_j}(\widetilde{\Upsilon}_i) = \widetilde{\Upsilon}_j$ and $\widetilde{\mathcal{T}}_{\mathbf{r}_j} \widetilde{\mathcal{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_j) = \widetilde{\Upsilon}_i$, and hence,

$$\widetilde{\Upsilon}^{[3]} = \widetilde{\Upsilon}_j, \qquad \widetilde{\Upsilon}^{[2]} = \widetilde{\mathcal{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_j), \qquad \widetilde{\Upsilon}^{[1]} = \widetilde{\Upsilon}_i. \tag{13.4}$$

By Corollary 12.4, we have

$$B_i \widetilde{\Upsilon}^{[3]} \widetilde{\Upsilon}^{[2]} = \widetilde{\Upsilon}^{[3]} \widetilde{\Upsilon}^{[2]} B_i, \tag{13.5}$$

$$B_j^\sigma \widetilde{\Upsilon}^{[2]} \widetilde{\Upsilon}^{[1]} = \widetilde{\Upsilon}^{[2]} \widetilde{\Upsilon}^{[1]} B_j^\sigma. \tag{13.6}$$

It follows by Theorem 3.6 that, for $p = i, j$,

$$B_p \widetilde{\Upsilon}_p = \widetilde{\Upsilon}_p B_p^\sigma. \tag{13.7}$$

Now we show that $\widetilde{\Upsilon}_{\boldsymbol{w}_\circ}$ satisfies the following intertwining relations

$$B_p \widetilde{\Upsilon}_{\boldsymbol{w}_\circ} = \widetilde{\Upsilon}_{\boldsymbol{w}_\circ} B_p^\sigma, \qquad (p = i, j).$$

Indeed, $B_i \widetilde{\Upsilon}_{\boldsymbol{w}_\circ} = B_i \widetilde{\Upsilon}^{[3]} \widetilde{\Upsilon}^{[2]} \widetilde{\Upsilon}^{[1]} = \widetilde{\Upsilon}^{[3]} \widetilde{\Upsilon}^{[2]} B_i \widetilde{\Upsilon}^{[1]} = \widetilde{\Upsilon}^{[3]} \widetilde{\Upsilon}^{[2]} \widetilde{\Upsilon}^{[1]} B_i^\sigma$, by (13.3), (13.5) and (13.7). Also, $B_j \widetilde{\Upsilon}_{\boldsymbol{w}_\circ} = B_j \widetilde{\Upsilon}^{[3]} \widetilde{\Upsilon}^{[2]} \widetilde{\Upsilon}^{[1]} = \widetilde{\Upsilon}^{[3]} B_j^\sigma \widetilde{\Upsilon}^{[2]} \widetilde{\Upsilon}^{[1]} = \widetilde{\Upsilon}^{[3]} \widetilde{\Upsilon}^{[2]} \widetilde{\Upsilon}^{[1]} B_j^\sigma$, by (13.4), (13.7), and (13.6).

It follows by Proposition 13.5 (whose assumption is verified above), we have $\widetilde{\Upsilon} = \widetilde{\Upsilon}_{\boldsymbol{w}_\circ}$. Using the other reduced expression for $\boldsymbol{w}_\circ$ amounts to switching notations $i, j$ above. Hence, $\widetilde{\Upsilon} = \widetilde{\Upsilon}_{\boldsymbol{w}_\circ}$ is independent of the choice of a reduced expression for $\boldsymbol{w}_\circ$.

**Factorization for $\ell_\circ(\boldsymbol{w}_\circ) = 4$**

In this subsection, we deal with the rank 2 cases for $\ell_\circ(\boldsymbol{w}_\circ) = 4$, with the help of Proposition 12.6 and Corollary 12.8.

Assume that $\mathbb{I}_{\circ,\tau} = \{i, j\}$ such that $\ell_\circ(\boldsymbol{w}_\circ) = 4$. Let $\{i, \tau i\}$ and $\{j, \tau j\}$ be the corresponding two distinct $\tau$-orbits of $\mathbb{I}_\circ$. The longest element $\boldsymbol{w}_\circ$ of the relative Weyl group has a reduced expression

$$\boldsymbol{w}_\circ = \mathbf{r}_i \mathbf{r}_j \mathbf{r}_i \mathbf{r}_j. \tag{13.8}$$

By definition (13.1) of $\widetilde{\Upsilon}^{[k]}$ and $\widetilde{\Upsilon}_{\boldsymbol{w}_\circ}$, we have

$$\widetilde{\Upsilon}_{\boldsymbol{w}_\circ} = \widetilde{\Upsilon}^{[4]} \widetilde{\Upsilon}^{[3]} \widetilde{\Upsilon}^{[2]} \widetilde{\Upsilon}^{[1]}. \tag{13.9}$$

where by Proposition 13.3, $\widetilde{\mathcal{T}}_{\mathbf{r}_i}\widetilde{\mathcal{T}}_{\mathbf{r}_j}\widetilde{\mathcal{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_j) = \widetilde{\Upsilon}_j$ and $\widetilde{\mathcal{T}}_{\mathbf{r}_j}\widetilde{\mathcal{T}}_{\mathbf{r}_i}\widetilde{\mathcal{T}}_{\mathbf{r}_j}(\widetilde{\Upsilon}_i) = \widetilde{\Upsilon}_i$, and hence

$$\widetilde{\Upsilon}^{[4]} = \widetilde{\Upsilon}_j, \quad \widetilde{\Upsilon}^{[3]} = \widetilde{\mathcal{T}}_{\mathbf{r}_i}\widetilde{\mathcal{T}}_{\mathbf{r}_j}(\widetilde{\Upsilon}_i) = \widetilde{\mathcal{T}}_{\mathbf{r}_j}^{-1}(\widetilde{\Upsilon}_i), \quad \widetilde{\Upsilon}^{[2]} = \widetilde{\mathcal{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_j), \quad \widetilde{\Upsilon}^{[1]} = \widetilde{\Upsilon}_i. \tag{13.10}$$

By Corollary 12.8, we have

$$B_i\widetilde{\Upsilon}^{[4]}\widetilde{\Upsilon}^{[3]}\widetilde{\Upsilon}^{[2]} = \widetilde{\Upsilon}^{[4]}\widetilde{\Upsilon}^{[3]}\widetilde{\Upsilon}^{[2]}B_i, \tag{13.11}$$

$$B_j^{\sigma}\widetilde{\Upsilon}^{[3]}\widetilde{\Upsilon}^{[2]}\widetilde{\Upsilon}^{[1]} = \widetilde{\Upsilon}^{[3]}\widetilde{\Upsilon}^{[2]}\widetilde{\Upsilon}^{[1]}B_j^{\sigma}. \tag{13.12}$$

Just as in §13.3, using the identities (13.11)–(13.12) we can show that $\widetilde{\Upsilon}_{\boldsymbol{w}_\circ}$ satisfies the following intertwining relations $B_p\widetilde{\Upsilon}_{\boldsymbol{w}_\circ} = \widetilde{\Upsilon}_{\boldsymbol{w}_\circ}B_p^{\sigma}$, for $p = i, j$. It follows by Proposition 13.5 (which assumption is verified above), we have $\widetilde{\Upsilon} = \widetilde{\Upsilon}_{\boldsymbol{w}_\circ}$, which is independent of the choice of a reduced expression for $\boldsymbol{w}_\circ$.

**Factorization for $\ell_\circ(\boldsymbol{w}_\circ) = 6$**

The case for $\ell_\circ(\boldsymbol{w}_\circ) = 6$ occurs only in split $G_2$ type. We shall prove this using Proposition 12.10 and Corollary 12.12.

Let $(\mathbb{I} = \mathbb{I}_\circ, \tau = \mathrm{Id})$ be the Satake diagram of split type $G_2$. In this case, $W^\circ = W$ and $\mathbf{r}_a = s_a$. Assume that $\mathbb{I} = \{i, j\}$ such that $\ell_\circ(\boldsymbol{w}_\circ) = 6$. The longest element $\boldsymbol{w}_\circ$ of the relative Weyl group has a reduced expression

$$\boldsymbol{w}_\circ = s_is_js_is_js_is_j. \tag{13.13}$$

By definition (13.1) of $\widetilde{\Upsilon}^{[k]}$ and $\widetilde{\Upsilon}_{\boldsymbol{w}_\circ}$, we have

$$\widetilde{\Upsilon}_{\boldsymbol{w}_\circ} = \widetilde{\Upsilon}^{[6]}\widetilde{\Upsilon}^{[5]}\widetilde{\Upsilon}^{[4]}\widetilde{\Upsilon}^{[3]}\widetilde{\Upsilon}^{[2]}\widetilde{\Upsilon}^{[1]}. \tag{13.14}$$

where by Proposition 13.3, $\widetilde{\mathfrak{T}}_{s_is_js_is_js_i}(\widetilde{\Upsilon}_j) = \widetilde{\Upsilon}_j$, and hence

$$\begin{aligned}
&\widetilde{\Upsilon}^{[6]} = \widetilde{\Upsilon}_j, \quad \widetilde{\Upsilon}^{[5]} = \widetilde{\mathfrak{T}}_{s_is_js_is_j}(\widetilde{\Upsilon}_i), \quad \widetilde{\Upsilon}^{[4]} = \widetilde{\mathfrak{T}}_{s_is_js_i}(\widetilde{\Upsilon}_j),\\
&\widetilde{\Upsilon}^{[3]} = \widetilde{\mathfrak{T}}_{s_is_j}(\widetilde{\Upsilon}_i), \quad \widetilde{\Upsilon}^{[2]} = \widetilde{\mathfrak{T}}_{s_i}(\widetilde{\Upsilon}_j), \quad \widetilde{\Upsilon}^{[1]} = \widetilde{\Upsilon}_i.
\end{aligned} \tag{13.15}$$

By Corollary 12.12, we have

$$B_i\widetilde{\Upsilon}^{[6]}\widetilde{\Upsilon}^{[5]}\widetilde{\Upsilon}^{[4]}\widetilde{\Upsilon}^{[3]}\widetilde{\Upsilon}^{[2]} = \widetilde{\Upsilon}^{[6]}\widetilde{\Upsilon}^{[5]}\widetilde{\Upsilon}^{[4]}\widetilde{\Upsilon}^{[3]}\widetilde{\Upsilon}^{[2]}B_i, \tag{13.16}$$

$$B_j^\sigma\widetilde{\Upsilon}^{[5]}\widetilde{\Upsilon}^{[4]}\widetilde{\Upsilon}^{[3]}\widetilde{\Upsilon}^{[2]}\widetilde{\Upsilon}^{[1]} = \widetilde{\Upsilon}^{[5]}\widetilde{\Upsilon}^{[4]}\widetilde{\Upsilon}^{[3]}\widetilde{\Upsilon}^{[2]}\widetilde{\Upsilon}^{[1]}B_j^\sigma. \tag{13.17}$$

Just as in §13.3, using the identities (13.16)–(13.17) we can show that $\widetilde{\Upsilon}_{\boldsymbol{w}_\circ}$ satisfies the following intertwining relations $B_p\widetilde{\Upsilon}_{\boldsymbol{w}_\circ} = \widetilde{\Upsilon}_{\boldsymbol{w}_\circ}B_p^\sigma$, for $p = i, j$. It follows by Proposition 13.5 (which assumption is verified above), we have $\widetilde{\Upsilon} = \widetilde{\Upsilon}_{\boldsymbol{w}_\circ}$, which is independent of the choice of a reduced expression for $\boldsymbol{w}_\circ$.

*Remark* 13.6. A different and more computational proof of the factorization of the quasi $K$-matrix in split type $G_2$ was given earlier in Dobson's thesis [Dob19].

# 14 Relative braid group actions on ıquantum groups

Let ($\mathbb{I} = \mathbb{I}_\circ \cup \mathbb{I}_\bullet$) be a symmetric pair of arbitrary finite type or quasi-split Kac-Moody type. In this section, we show that $\widetilde{\mathbf{T}}'_{i,e}, \widetilde{\mathbf{T}}''_{i,e}$, where $e = \pm 1$ and $i \in \mathbb{I}_{\circ,\tau}$, satisfy the relative braid group relations in $\mathrm{Br}(W^\circ)$. An action of $\mathrm{Br}(W_\bullet)\rtimes\mathrm{Br}(W^\circ)$ on $\widetilde{\mathbf{U}}^\imath$ is then established. Moreover we show that, by central reductions and isomorphisms among $\imath$quantum groups with different parameters, the symmetries $\widetilde{\mathbf{T}}'_{i,e}, \widetilde{\mathbf{T}}''_{i,e}$ on $\widetilde{\mathbf{U}}^\imath$ descend to $\mathbf{T}'_{i,e}, \mathbf{T}''_{i,e}$ on the $\imath$quantum groups $\mathbf{U}^\imath_{\boldsymbol{\varsigma}}$, inducing relative braid group actions on $\mathbf{U}^\imath_{\boldsymbol{\varsigma}}$, for an arbitrary parameter $\boldsymbol{\varsigma}$.

## 14.1 Braid group relations among $\widetilde{\mathbf{T}}_i$

Let $\mathrm{Ad}_y$ be the operator such that $\mathrm{Ad}_y(u) := yuy^{-1}$ for $y$ invertible. For $i \neq j \in \mathbb{I}_{\circ,\tau}$, let $m_{ij}$ be the order of $\mathbf{r}_i\mathbf{r}_j$ in $W^\circ$, with $m_{ij} \in \{2,3,4,6,\infty\}$. Then the following braid relation is satisfied in $\mathrm{Br}(W^\circ)$:

$$\underbrace{\mathbf{r}_i\mathbf{r}_j\mathbf{r}_i\cdots}_{m_{ij}} = \underbrace{\mathbf{r}_j\mathbf{r}_i\mathbf{r}_j\cdots}_{m_{ij}}. \tag{14.1}$$

**Theorem 14.1.** *For $i \neq j \in \mathbb{I}_{\circ,\tau}, e = \pm 1,$ we have*

$$\begin{aligned} \underbrace{\widetilde{\mathbf{T}}'_{i,e}\widetilde{\mathbf{T}}'_{j,e}\widetilde{\mathbf{T}}'_{i,e}\cdots}_{m_{ij}} &= \underbrace{\widetilde{\mathbf{T}}'_{j,e}\widetilde{\mathbf{T}}'_{i,e}\widetilde{\mathbf{T}}'_{j,e}\cdots}_{m_{ij}}, \\ \underbrace{\widetilde{\mathbf{T}}''_{i,e}\widetilde{\mathbf{T}}''_{j,e}\widetilde{\mathbf{T}}''_{i,e}\cdots}_{m_{ij}} &= \underbrace{\widetilde{\mathbf{T}}''_{j,e}\widetilde{\mathbf{T}}''_{i,e}\widetilde{\mathbf{T}}''_{j,e}\cdots}_{m_{ij}}. \end{aligned} \tag{14.2}$$

*Proof.* By Theorem 6.7, $\widetilde{\mathbf{T}}''_{i,+1}$ is the inverse of $\widetilde{\mathbf{T}}'_{i,-1}$. Moreover, by definition (6.11), $\widetilde{\mathbf{T}}'_{i,+1}, \widetilde{\mathbf{T}}''_{i,-1}$ are conjugations of $\widetilde{\mathbf{T}}'_{i,-1}, \widetilde{\mathbf{T}}''_{i,+1}$ respectively. When $m_{ij}=\infty$, the identity (14.2) is trivial. Hence, it suffices to prove the identity (14.2) for $\widetilde{\mathbf{T}}'_{i,-1}$ and $m_{ij}<\infty$. We shall write $\widetilde{\mathbf{T}}_i^{-1}$ for $\widetilde{\mathbf{T}}'_{i,-1}$ in the proof.

Set $m = m_{ij}$. Following Theorem 4.7, when acting on $\widetilde{\mathbf{U}}$, we have

$$\widetilde{\mathbf{T}}_i^{-1} = \mathrm{Ad}_{\widetilde{\Upsilon}_i} \circ \widetilde{\mathfrak{T}}_{\mathbf{r}_i}^{-1}, \qquad \widetilde{\mathbf{T}}_j^{-1} = \mathrm{Ad}_{\widetilde{\Upsilon}_j} \circ \widetilde{\mathfrak{T}}_{\mathbf{r}_j}^{-1}. \tag{14.3}$$

In addition, we have

$$\widetilde{\mathfrak{T}}_{\mathbf{r}_i}^{-1} \circ \mathrm{Ad}_y = \mathrm{Ad}_{\widetilde{\mathfrak{T}}_{\mathbf{r}_i}^{-1}(y)} \circ \widetilde{\mathfrak{T}}_{\mathbf{r}_i}^{-1}, \tag{14.4}$$

$$\mathrm{Ad}_{y_1} \circ \mathrm{Ad}_{y_2} = \mathrm{Ad}_{y_1 y_2}. \tag{14.5}$$

Let $\boldsymbol{w}_\circ = \underbrace{\mathbf{r}_i \mathbf{r}_j \mathbf{r}_i \cdots}_{m}$ be a reduced expression. Define $\boldsymbol{w}_k$, for $1 \leq k \leq m$, to be

$$\boldsymbol{w}_1 = \mathbf{r}_i, \quad \boldsymbol{w}_2 = \mathbf{r}_i \mathbf{r}_j, \quad \boldsymbol{w}_3 = \mathbf{r}_i \mathbf{r}_j \mathbf{r}_i, \quad \ldots, \quad \boldsymbol{w}_m = \boldsymbol{w}_\circ.$$

Write $\boldsymbol{w}'_\circ$ for the other reduced expression $\underbrace{\mathbf{r}_j \mathbf{r}_i \cdots}_{m}$, and define $\boldsymbol{w}'_k$, for $1 \leq k \leq m$, accordingly. Let $r$ denote the last index in the reduced expression of $\boldsymbol{w}_\circ$; that is, $r = i$ if $m = 2, 4, 6$ and $r = j$ if $m = 3$. Similarly, we define $r'$ for $\boldsymbol{w}'_\circ$.

We rewrite LHS (14.2) as follows:

$$\begin{aligned}
\underbrace{\widetilde{\mathbf{T}}_i^{-1} \widetilde{\mathbf{T}}_j^{-1} \widetilde{\mathbf{T}}_i^{-1} \cdots}_{m} &\overset{(14.3)}{=} \underbrace{(\mathrm{Ad}_{\widetilde{\Upsilon}_i} \circ \widetilde{\mathfrak{T}}_{\mathbf{r}_i}^{-1}) \circ (\mathrm{Ad}_{\widetilde{\Upsilon}_j} \circ \widetilde{\mathfrak{T}}_{\mathbf{r}_j}^{-1}) \cdots}_{m} \\
&\overset{(14.4)}{=} \big(\mathrm{Ad}_{\widetilde{\Upsilon}_i} \circ \mathrm{Ad}_{\widetilde{\mathfrak{T}}_{\boldsymbol{w}_1}^{-1}(\widetilde{\Upsilon}_j)} \circ \cdots \circ \mathrm{Ad}_{\widetilde{\mathfrak{T}}_{\boldsymbol{w}_{m-1}}^{-1}(\widetilde{\Upsilon}_r)}\big) \circ (\underbrace{\widetilde{\mathfrak{T}}_{\mathbf{r}_i}^{-1} \widetilde{\mathfrak{T}}_{\mathbf{r}_j}^{-1} \cdots}_{m}) \\
&\overset{(14.5)}{=} \big(\mathrm{Ad}_{\widetilde{\Upsilon}_i \cdot \widetilde{\mathfrak{T}}_{\boldsymbol{w}_1}^{-1}(\widetilde{\Upsilon}_j) \cdots \widetilde{\mathfrak{T}}_{\boldsymbol{w}_{m-1}}^{-1}(\widetilde{\Upsilon}_r)}\big) \circ (\underbrace{\widetilde{\mathfrak{T}}_{\mathbf{r}_i}^{-1} \widetilde{\mathfrak{T}}_{\mathbf{r}_j}^{-1} \cdots}_{m}).
\end{aligned}$$

Similarly, RHS (14.2) can be rewritten as

$$\underbrace{\widetilde{\mathbf{T}}_j^{-1}\widetilde{\mathbf{T}}_i^{-1}\widetilde{\mathbf{T}}_j^{-1}\cdots}_{m} = (\mathrm{Ad}_{\widetilde{\Upsilon}_j\cdot\widetilde{\mathcal{T}}^{-1}_{\boldsymbol{w}'_1}(\widetilde{\Upsilon}_i)\cdots\widetilde{\mathcal{T}}^{-1}_{\boldsymbol{w}'_{m-1}}(\widetilde{\Upsilon}_{r'})}) \circ (\underbrace{\widetilde{\mathcal{T}}_{\mathbf{r}_j}^{-1}\widetilde{\mathcal{T}}_{\mathbf{r}_i}^{-1}\cdots}_{m}).$$

By Proposition 4.2, $\widetilde{\mathcal{T}}_i$ satisfies braid relations. As $\ell(\mathbf{r}_i\mathbf{r}_j\mathbf{r}_i\cdots) = \ell(\boldsymbol{w}_\circ) = \ell(\mathbf{r}_j\mathbf{r}_i\mathbf{r}_j\cdots)$, we have

$$\underbrace{\widetilde{\mathcal{T}}_{\mathbf{r}_i}^{-1}\widetilde{\mathcal{T}}_{\mathbf{r}_j}^{-1}\widetilde{\mathcal{T}}_{\mathbf{r}_i}^{-1}\cdots}_{m} = \underbrace{\widetilde{\mathcal{T}}_{\mathbf{r}_j}^{-1}\widetilde{\mathcal{T}}_{\mathbf{r}_i}^{-1}\widetilde{\mathcal{T}}_{\mathbf{r}_j}^{-1}\cdots}_{m}. \tag{14.6}$$

Hence to prove (14.2) it remains to show that

$$\widetilde{\Upsilon}_i\cdot\widetilde{\mathcal{T}}^{-1}_{\boldsymbol{w}_1}(\widetilde{\Upsilon}_j)\cdots\widetilde{\mathcal{T}}^{-1}_{\boldsymbol{w}_{m-1}}(\widetilde{\Upsilon}_r) = \widetilde{\Upsilon}_j\cdot\widetilde{\mathcal{T}}^{-1}_{\boldsymbol{w}'_1}(\widetilde{\Upsilon}_i)\cdots\widetilde{\mathcal{T}}^{-1}_{\boldsymbol{w}'_{m-1}}(\widetilde{\Upsilon}_{r'}). \tag{14.7}$$

By definition (13.1), $\widetilde{\Upsilon}_{\boldsymbol{w}_\circ} = \widetilde{\mathcal{T}}_{\boldsymbol{w}_{m-1}}(\widetilde{\Upsilon}_r)\cdots\widetilde{\mathcal{T}}_{\boldsymbol{w}_1}(\widetilde{\Upsilon}_j)\widetilde{\Upsilon}_i$. Applying $\sigma$ to this identity and then using Proposition 3.8, we obtain

$$\sigma(\widetilde{\Upsilon}_{\boldsymbol{w}_\circ}) = \widetilde{\Upsilon}_i\cdot\widetilde{\mathcal{T}}^{-1}_{\boldsymbol{w}_1}(\widetilde{\Upsilon}_j)\cdots\widetilde{\mathcal{T}}^{-1}_{\boldsymbol{w}_{m-1}}(\widetilde{\Upsilon}_r). \tag{14.8}$$

We have a similar formula for $\sigma(\widetilde{\Upsilon}_{\boldsymbol{w}'_\circ})$ as well. It follows by Theorem 13.1 that $\sigma(\widetilde{\Upsilon}_{\boldsymbol{w}_\circ}) = \sigma(\widetilde{\Upsilon}_{\boldsymbol{w}'_\circ})$. The identity (14.7) now follows by the formula (14.8) and its $\boldsymbol{w}'_\circ$-counterpart. □

For $w \in W^\circ$, take a reduced expression $w = \mathbf{r}_{i_1}\mathbf{r}_{i_2}\cdots\mathbf{r}_{i_k}$ and define

$$\widetilde{\mathbf{T}}'_{w,e} := \widetilde{\mathbf{T}}'_{i_1,e}\widetilde{\mathbf{T}}'_{i_2,e}\cdots\widetilde{\mathbf{T}}'_{i_k,e}, \qquad \widetilde{\mathbf{T}}'_{w,e} := \widetilde{\mathbf{T}}''_{i_1,e}\widetilde{\mathbf{T}}''_{i_2,e}\cdots\widetilde{\mathbf{T}}''_{i_k,e}. \tag{14.9}$$

By Theorem 14.1, these are independent of the choice of reduced expressions for $w$.

## 14.2 Action of the braid group $\mathrm{Br}(W_\bullet) \rtimes \mathrm{Br}(W^\circ)$ on $\widetilde{\mathbf{U}}^\imath$

We first establish a commutator relation between $\widetilde{\mathbf{T}}'_{i,-1}$ $(i \in \mathbb{I}_\circ)$ and $\widetilde{\mathcal{T}}_j^{-1} \equiv \widetilde{\mathcal{T}}'_{j,-1}$ $(j \in \mathbb{I}_\bullet)$.

**Lemma 14.2.** *We have* $\widetilde{\mathcal{T}}_j^{-1}\widetilde{\mathbf{T}}'_{i,-1}(x) = \widetilde{\mathbf{T}}'_{i,-1}\widetilde{\mathcal{T}}^{-1}_{\tau_{\bullet,i}\tau j}(x)$, *for* $i \in \mathbb{I}_{\circ,\tau}$, $j \in \mathbb{I}_\bullet$, *and* $x \in \widetilde{\mathbf{U}}^\imath$.

*Proof.* Note that $\tau(j), \tau_{\bullet,i}(j), \tau_{\bullet,i}\tau(j) \in \mathbb{I}_\bullet$, for $j \in \mathbb{I}_\bullet$. Since $w_\bullet s_j = s_{\tau j} w_\bullet$, for $j \in \mathbb{I}_\bullet$, and $w_{\bullet,i} s_j = s_{\tau_{\bullet,i} j} w_{\bullet,i}$, for $i \in \mathbb{I}_\circ$, we have

$$\mathbf{r}_i s_j = w_{\bullet,i} w_\bullet^{-1} s_j = s_{\tau_{\bullet,i}\tau j} w_{\bullet,i} w_\bullet^{-1} = s_{\tau_{\bullet,i}\tau j}\mathbf{r}_i. \tag{14.10}$$

Since $\ell(\mathbf{r}_i s_j) = \ell(\mathbf{r}_i) + 1$, it follows by (14.10) that

$$\widetilde{\mathcal{T}}_{\tau_{\bullet,i}\tau(j)}\widetilde{\mathcal{T}}_{\mathbf{r}_i} = \widetilde{\mathcal{T}}_{\mathbf{r}_i}\widetilde{\mathcal{T}}_j. \tag{14.11}$$

By Proposition 4.6, $\widetilde{\Upsilon}_i$ is fixed by $\widetilde{\mathcal{T}}_j^{-1}$. Hence, applying $\widetilde{\mathcal{T}}_j^{-1}$ to the intertwining relation (4.6) in Theorem 4.7 and then using (14.11), we obtain, for $x \in \widetilde{\mathbf{U}}^\imath$,

$$\begin{aligned}\widetilde{\mathcal{T}}_j^{-1}\widetilde{\mathbf{T}}'_{i,-1}(x)\widetilde{\Upsilon}_i &= \widetilde{\Upsilon}_i\widetilde{\mathcal{T}}_j^{-1}\widetilde{\mathcal{T}}_{\mathbf{r}_i}^{-1}(x)\\ &= \widetilde{\Upsilon}_i\widetilde{\mathcal{T}}_{\mathbf{r}_i}^{-1}\widetilde{\mathcal{T}}^{-1}_{\tau_{\bullet,i}\tau j}(x) = \widetilde{\mathbf{T}}'_{i,-1}\widetilde{\mathcal{T}}^{-1}_{\tau_{\bullet,i}\tau j}(x)\widetilde{\Upsilon}_i,\end{aligned} \tag{14.12}$$

where the last step uses Theorem 4.7 and the fact that $\widetilde{\mathcal{T}}^{-1}_{\tau_{\bullet,i}\tau j}(x) \in \widetilde{\mathbf{U}}^\imath$ by Proposition 4.5. The identity (14.12) clearly implies the identity in the lemma. □

Let $\mathrm{Br}(W_\bullet)$ and $\mathrm{Br}(W^\circ)$ be the braid groups associated to $W_\bullet$ and $W^\circ$ respectively.

**Theorem 14.3.** *There exists a braid group action of* $\mathrm{Br}(W_\bullet) \rtimes \mathrm{Br}(W^\circ)$ *on* $\widetilde{\mathbf{U}}^\imath$ *as automorphisms of algebras generated by* $\widetilde{\mathcal{T}}'_{j,-1}$ $(j \in \mathbb{I}_\bullet)$ *and* $\widetilde{\mathbf{T}}'_{i,-1}$ $(i \in \mathbb{I}_{\circ,\tau})$.

*Proof.* By Remark 4.8, $\widetilde{\mathbf{T}}'_{i,-1}$ is independent of the choice of representatives in a $\tau$-orbit. The defining relations of $\mathrm{Br}(W_\bullet) \rtimes \mathrm{Br}(W^\circ)$ consist of braid relations for $\mathrm{Br}(W_\bullet)$, the braid relations for $\mathrm{Br}(W^\circ)$, and relations (14.10). The braid relations for $\widetilde{\mathcal{T}}'_{j,-1}, j \in \mathbb{I}_\bullet$ is verified in Proposition 4.2. The braid relations for $\widetilde{\mathbf{T}}'_{i,-1}, i \in \mathbb{I}_{\circ,\tau}$ is verified in Theorem 14.1. The commutator relation for $\widetilde{\mathcal{T}}'_{j,-1}, \widetilde{\mathbf{T}}'_{i,-1}$ corresponding to (14.10) is verified in Lemma 14.2. □

*Remark* 14.4. Since $\widetilde{\mathbf{T}}'_{i,-1}, \widetilde{\mathbf{T}}''_{i,+1}$ are mutually inverses and $\widetilde{\mathcal{T}}'_{j,-1}, \widetilde{\mathcal{T}}''_{j,+1}$ are mutually inverses, there also exists a braid group action of $\mathrm{Br}(W_\bullet) \rtimes \mathrm{Br}(W^\circ)$ on $\widetilde{\mathbf{U}}^\imath$ as automorphisms of algebras generated by $\widetilde{\mathcal{T}}''_{j,+1}$ ($j \in \mathbb{I}_\bullet$) and $\widetilde{\mathbf{T}}''_{i,+1}$ ($i \in \mathbb{I}_{\circ,\tau}$).

Recall the remaining two symmetries $\widetilde{\mathbf{T}}'_{i,+1}, \widetilde{\mathbf{T}}''_{i,-1}$ from (6.11). We shall establish a variant of Theorem 14.3 for $\widetilde{\mathbf{T}}'_{i,e}$ and $\widetilde{\mathcal{T}}'_{j,e}$ (and respectively, $\widetilde{\mathbf{T}}''_{i,e}$ and $\widetilde{\mathcal{T}}''_{j,e}$).

Let $j \in \mathbb{I}$. Recall $\widetilde{\mathcal{T}}''_{j,+1}$ and $\widetilde{\mathcal{T}}'_{j,-1}$ from (4.2)–(4.3). Recalling $\psi_\star = \widetilde{\Psi}_{\varsigma_\star} \circ \psi$ from (3.9), we define

$$\widetilde{\mathcal{T}}''_{j,-1} := \psi_\star \circ \widetilde{\mathcal{T}}''_{j,+1} \circ \psi_\star, \qquad \widetilde{\mathcal{T}}'_{j,+1} := \psi_\star \circ \widetilde{\mathcal{T}}'_{j,-1} \circ \psi_\star. \tag{14.13}$$

Let $\varsigma_{\star\diamond} := (\varsigma_{j,\star}\varsigma_{j,\diamond})_{j\in\mathbb{I}_\circ}$ be the parameter obtained as the componentwise product of parameters $\varsigma_\diamond$ and $\varsigma_\star$ from (2.28) and (3.8).

**Lemma 14.5.** *The $\widetilde{\mathcal{T}}''_{j,-1}, \widetilde{\mathcal{T}}'_{j,+1}$ are related to $\widetilde{T}''_{j,-1}, \widetilde{T}'_{j,+1}$ via a rescaling automorphism:*

$$\widetilde{\mathcal{T}}''_{j,-1} = \widetilde{\Psi}_{\varsigma_{\star\diamond}} \widetilde{T}''_{j,-1} \widetilde{\Psi}^{-1}_{\varsigma_{\star\diamond}}, \qquad \widetilde{\mathcal{T}}'_{j,+1} = \widetilde{\Psi}_{\varsigma_{\star\diamond}} \widetilde{T}'_{j,+1} \widetilde{\Psi}^{-1}_{\varsigma_{\star\diamond}}.$$

*Proof.* Recall $\widetilde{\mathcal{T}}''_{j,+1} = \widetilde{\Psi}^{-1}_{\varsigma_\diamond} \circ \widetilde{T}''_{j,+1} \circ \widetilde{\Psi}_{\varsigma_\diamond}$ and $\widetilde{\mathcal{T}}'_{j,-1} = \widetilde{\Psi}^{-1}_{\varsigma_\diamond} \circ \widetilde{T}'_{j,-1} \circ \widetilde{\Psi}_{\varsigma_\diamond}$ from (4.2)–(4.3).

Recall from (2.14) that $\widetilde{T}''_{i,-1} = \psi \circ \widetilde{T}''_{i,+1} \circ \psi$ and $\widetilde{T}'_{i,+1} = \psi \circ \widetilde{T}'_{i,-1} \circ \psi$. Then we have

$$\widetilde{\mathfrak{T}}''_{j,-1} = \psi_\star \circ \widetilde{\mathfrak{T}}''_{j,+1} \circ \psi_\star$$
$$= \widetilde{\Psi}_{\varsigma_\star}\psi \circ \widetilde{\Psi}^{-1}_{\varsigma_\diamond}\widetilde{T}''_{j,+1}\widetilde{\Psi}_{\varsigma_\diamond} \circ \widetilde{\Psi}_{\varsigma_\star}\psi = \widetilde{\Psi}_{\varsigma_{\star\diamond}}\widetilde{T}''_{j,-1}\widetilde{\Psi}^{-1}_{\varsigma_{\star\diamond}},$$

where we used $\psi \circ \widetilde{\Psi}^{-1}_{\varsigma_\diamond} = \widetilde{\Psi}_{\varsigma_\diamond} \circ \psi$. The proof for the other formula is similar. □

By Proposition 4.5, the automorphisms $\widetilde{\mathfrak{T}}''_{j,+1}, \widetilde{\mathfrak{T}}'_{j,-1}$ for $j \in \mathbb{I}_\bullet$ restrict to automorphisms on $\widetilde{\mathbf{U}}^\imath$.

**Lemma 14.6.** *The automorphisms $\widetilde{\mathfrak{T}}''_{j,e}, \widetilde{\mathfrak{T}}'_{j,e}$, for $j \in \mathbb{I}_\bullet$ and $e = \pm 1$, restrict to automorphisms on $\widetilde{\mathbf{U}}^\imath$. Moreover, the following identities hold:*

$$\widetilde{\mathfrak{T}}''_{j,-1} := \psi^\imath \circ \widetilde{\mathfrak{T}}''_{j,+1} \circ \psi^\imath, \qquad \widetilde{\mathfrak{T}}'_{j,+1} := \psi^\imath \circ \widetilde{\mathfrak{T}}'_{j,-1} \circ \psi^\imath. \tag{14.14}$$

*Proof.* As $\widetilde{\mathfrak{T}}_j \equiv \widetilde{\mathfrak{T}}''_{j,+1}$ restricts to an automorphism on $\widetilde{\mathbf{U}}^\imath$ by Proposition 4.5, it suffices to prove (14.14).

By Proposition 3.4, we have $\psi_\star = \mathrm{Ad}_{\widetilde{\Upsilon}^{-1}} \circ \psi^\imath$ when acting on $\widetilde{\mathbf{U}}^\imath$. By Proposition 4.6, $\mathrm{Ad}_{\widetilde{\Upsilon}^{-1}}$ commutes with $\widetilde{\mathfrak{T}}_j$. By Proposition 3.5, we have $\psi_\star \circ \mathrm{Ad}_{\widetilde{\Upsilon}^{-1}} = \mathrm{Ad}_{\widetilde{\Upsilon}} \circ \psi_\star$. Using these properties and (14.13), we have, for $x \in \widetilde{\mathbf{U}}^\imath$,

$$\widetilde{\mathfrak{T}}''_{j,-1}(x) = \psi_\star \circ \widetilde{\mathfrak{T}}''_{j,+1} \circ \psi_\star(x) = \psi_\star \circ \widetilde{\mathfrak{T}}''_{j,+1} \circ \mathrm{Ad}_{\widetilde{\Upsilon}^{-1}} \circ \psi^\imath(x)$$
$$= \psi_\star \circ \mathrm{Ad}_{\widetilde{\Upsilon}^{-1}} \circ \widetilde{\mathfrak{T}}''_{j,+1} \circ \psi^\imath(x) = \mathrm{Ad}_{\widetilde{\Upsilon}} \circ \psi_\star \circ \widetilde{\mathfrak{T}}''_{j,+1} \circ \psi^\imath(x) = \psi^\imath \circ \widetilde{\mathfrak{T}}''_{j,+1} \circ \psi^\imath(x),$$

where the last equality uses (3.18).

The proof of the other formula for $\widetilde{\mathfrak{T}}'_{j,+1}$ is similar and hence skipped. □

The next result follows from (6.11), Theorem 14.3, Remark 14.4, and Lemma 14.6.

**Corollary 14.7.** *Let $e = \pm 1$.*

1. *There exists a braid group action of* $\mathrm{Br}(W_\bullet) \rtimes \mathrm{Br}(W^\circ)$ *on* $\widetilde{\mathbf{U}}^\imath$ *as automorphisms of algebras generated by* $\widetilde{\mathcal{T}}'_{j,e}$ $(j \in \mathbb{I}_\bullet)$ *and* $\widetilde{\mathbf{T}}'_{i,e}$ $(i \in \mathbb{I}_{\circ,\tau})$.

2. *There exists a braid group action of* $\mathrm{Br}(W_\bullet) \rtimes \mathrm{Br}(W^\circ)$ *on* $\widetilde{\mathbf{U}}^\imath$ *as automorphisms of algebras generated by* $\widetilde{\mathcal{T}}''_{j,e}$ $(j \in \mathbb{I}_\bullet)$ *and* $\widetilde{\mathbf{T}}''_{i,e}$ $(i \in \mathbb{I}_{\circ,\tau})$.

## 14.3 Intertwining properties of $\widetilde{\mathbf{T}}'_{i,+1}, \widetilde{\mathbf{T}}''_{i,-1}$

The automorphisms $\widetilde{\mathbf{T}}'_{i,+1}, \widetilde{\mathbf{T}}''_{i,-1}$ on $\widetilde{\mathbf{U}}^\imath$ also satisfy intertwining relations similar to those satisfied by $\widetilde{\mathbf{T}}'_{i,-1}$ in (4.6) and $\widetilde{\mathbf{T}}''_{i,+1}$ in (6.1). These relations on $\widetilde{\mathbf{U}}^\imath$ will descend to $\mathbf{U}^\imath_{\boldsymbol{\varsigma}}$ (see Proposition 15.2) and will then be used to define the relative braid operators on module level (see Definition 15.3).

**Proposition 14.8.** *The automorphisms* $\widetilde{\mathbf{T}}'_{i,+1}, \widetilde{\mathbf{T}}''_{i,-1}$ *satisfy the following intertwining relations*

$$\widetilde{\mathbf{T}}'_{i,+1}(x)\widetilde{\mathcal{T}}'_{\mathbf{r}_i,+1}(\widetilde{\Upsilon}_i^{-1}) = \widetilde{\mathcal{T}}'_{\mathbf{r}_i,+1}(\widetilde{\Upsilon}_i^{-1})\widetilde{\mathcal{T}}'_{\mathbf{r}_i,+1}(x), \tag{14.15}$$

$$\widetilde{\mathbf{T}}''_{i,-1}(x)\widetilde{\Upsilon}_i = \widetilde{\Upsilon}_i\widetilde{\mathcal{T}}''_{\mathbf{r}_i,-1}(x). \tag{14.16}$$

*Proof.* We prove the first identity (14.15); the second identity (14.16) can be derived from the first one by noting that $\widetilde{\mathbf{T}}'_{i,+1}, \widetilde{\mathbf{T}}''_{i,-1}$ are inverses and $\widetilde{\mathcal{T}}'_{\mathbf{r}_i,+1}, \widetilde{\mathcal{T}}''_{\mathbf{r}_i,-1}$ are inverses.

We claim the following identity holds:

$$\widetilde{\mathbf{T}}'_{i,+1}(x) \cdot \widetilde{\Upsilon}\psi_\star(\widetilde{\Upsilon}_i)\widetilde{\mathcal{T}}'_{\mathbf{r}_i,+1}(\widetilde{\Upsilon}^{-1}) = \widetilde{\Upsilon}\psi_\star(\widetilde{\Upsilon}_i)\widetilde{\mathcal{T}}'_{\mathbf{r}_i,+1}(\widetilde{\Upsilon}^{-1}) \cdot \widetilde{\mathcal{T}}'_{\mathbf{r}_i,+1}(x). \tag{14.17}$$

Let us prove (14.17). Recall from (6.11) that $\widetilde{\mathbf{T}}'_{i,+1} = \psi^\imath \widetilde{\mathbf{T}}'_{i,-1} \psi^\imath$ and from (3.11) that $\widetilde{\Upsilon}^{-1}\psi^\imath(u)\widetilde{\Upsilon} = \psi_\star(u)$. Hence,

$$\widetilde{\Upsilon}^{-1}\widetilde{\mathbf{T}}'_{i,+1}(x)\widetilde{\Upsilon} = \widetilde{\Upsilon}^{-1}\psi^\imath(\widetilde{\mathbf{T}}'_{i,-1}(\psi^\imath x))\widetilde{\Upsilon} = \psi_\star(\widetilde{\mathbf{T}}'_{i,-1}(\psi^\imath x)).$$

By (4.6), $\widetilde{\Upsilon}_i^{-1}\widetilde{\mathbf{T}}'_{i,-1}(\psi^\imath x)\widetilde{\Upsilon}_i = \widetilde{\mathcal{T}}'_{\mathbf{r}_i,-1}(\psi^\imath x)$. Hence

$$\psi_\star(\widetilde{\Upsilon}_i)^{-1}\widetilde{\Upsilon}^{-1}\widetilde{\mathbf{T}}'_{i,+1}(x)\widetilde{\Upsilon}\psi_\star(\widetilde{\Upsilon}_i) = \psi_\star(\widetilde{\mathcal{T}}'_{\mathbf{r}_i,-1}(\psi^\imath(x))).$$

This allows us to write (14.17) as an equivalent identity

$$\psi_\star(\widetilde{\mathcal{T}}'_{\mathbf{r}_i,-1}(\psi^\imath(x)))\widetilde{\mathcal{T}}'_{\mathbf{r}_i,+1}(\widetilde{\Upsilon}^{-1}) = \widetilde{\mathcal{T}}'_{\mathbf{r}_i,+1}(\widetilde{\Upsilon}^{-1})\widetilde{\mathcal{T}}'_{\mathbf{r}_i,+1}(x). \tag{14.18}$$

Recalling by (14.13) that $\widetilde{\mathcal{T}}'_{\mathbf{r}_i,+1} = \psi_\star\widetilde{\mathcal{T}}'_{\mathbf{r}_i,-1}\psi_\star$, we reduce the proof of (14.18) to verifying that $\psi^\imath(x)\psi_\star(\widetilde{\Upsilon})^{-1} = \psi_\star(\widetilde{\Upsilon})^{-1}\psi_\star(x)$, which by Proposition 3.5 is equivalent to $\psi^\imath(x)\widetilde{\Upsilon} = \widetilde{\Upsilon}\psi_\star(x)$. This last identity holds by (3.11). Therefore, (14.17) is proved.

Observe that if we define $\widetilde{\Upsilon}_{[w]}$ by replacing $\widetilde{\mathcal{T}}_{\mathbf{r}_i} \equiv \widetilde{\mathcal{T}}''_{\mathbf{r}_i,+1}$ in the definition (13.1) of $\widetilde{\Upsilon}_w$ by $\widetilde{\mathcal{T}}'_{\mathbf{r}_i,+1}$, then we still have a factorization $\widetilde{\Upsilon} = \widetilde{\Upsilon}_{[\boldsymbol{w}_\circ]}$, for any reduced expression of $\boldsymbol{w}_\circ$. Below we shall use this version of factorization.

Let $\boldsymbol{w}'_\circ$ be a reduced expression of $\boldsymbol{w}_\circ$ starting with $\mathbf{r}_i$, and $\boldsymbol{w}''_\circ (= w_0\boldsymbol{w}'_\circ w_0)$ be a reduced expression of $\boldsymbol{w}_\circ$ ending with $\mathbf{r}_{\tau_0 i}$. It follows by definition that

$$\widetilde{\Upsilon} = \widetilde{\Upsilon}_{[\boldsymbol{w}'_\circ]} = \widetilde{\mathcal{T}}'_{\mathbf{r}_i,+1}(\widetilde{\Upsilon}_{[\mathbf{r}_i\boldsymbol{w}'_\circ]})\widetilde{\Upsilon}_i. \tag{14.19}$$

Since $w_0\mathbf{r}_{\tau_0 i} = \mathbf{r}_i w_0$ and $w_0 = \boldsymbol{w}_\circ w_\bullet$, we have $\boldsymbol{w}_\circ\mathbf{r}_{\tau_0 i} = \mathbf{r}_i\boldsymbol{w}_\circ$. By definition and

Proposition 13.3, we obtain

$$\widetilde{\Upsilon} = \widetilde{\Upsilon}_{[\boldsymbol{w}''_\circ]} = \widetilde{\Upsilon}_i \widetilde{\Upsilon}_{[\boldsymbol{w}_\circ \mathbf{r}_{\tau_0 i}]} = \widetilde{\Upsilon}_i \widetilde{\Upsilon}_{[\mathbf{r}_i \boldsymbol{w}_\circ]}. \tag{14.20}$$

Now, using (14.19)-(14.20), we can simplify a key component appearing in (14.17) as follows:

$$\begin{aligned}\widetilde{\Upsilon}\psi_\star(\widetilde{\Upsilon}_i)\widetilde{\mathcal{T}}'_{\mathbf{r}_i,+1}(\widetilde{\Upsilon}^{-1}) &= \widetilde{\Upsilon}\widetilde{\Upsilon}_i^{-1}\widetilde{\mathcal{T}}'_{\mathbf{r}_i,+1}(\widetilde{\Upsilon}^{-1}) \\ &= \widetilde{\mathcal{T}}'_{\mathbf{r}_i,+1}(\widetilde{\Upsilon}_{[\mathbf{r}_i \boldsymbol{w}_\circ]}\widetilde{\Upsilon}^{-1}) = \widetilde{\mathcal{T}}'_{\mathbf{r}_i,+1}(\widetilde{\Upsilon}_i^{-1}).\end{aligned}$$

Hence, the identity (14.15) follows from (14.17). □

## 14.4 Braid group action on $\mathbf{U}^\imath_{\boldsymbol{\varsigma}}$

Recall from (2.27) the $\imath$quantum group $\mathbf{U}^\imath_{\boldsymbol{\varsigma}}$ with parameter $\boldsymbol{\varsigma}$ satisfying (2.25) (à la Letzter), and recall a central reduction $\pi^\imath_{\boldsymbol{\varsigma}} : \widetilde{\mathbf{U}}^\imath \to \mathbf{U}^\imath_{\boldsymbol{\varsigma}}$ from Proposition 2.12.

We first construct the braid group action on $\mathbf{U}^\imath_{\boldsymbol{\varsigma}_\diamond}$ for the distinguished parameter $\boldsymbol{\varsigma}_\diamond$ (2.28). By the definition (4.11) of $\widetilde{k}_{j,\diamond}$ and Proposition 2.12, the kernel $\ker \pi^\imath_{\boldsymbol{\varsigma}_\diamond}$ is generated by

$$\widetilde{k}_{j,\diamond} - 1 \quad (\tau j = j \in \mathbb{I}_\circ), \quad \widetilde{k}_{j,\diamond}\widetilde{k}_{\tau j,\diamond} - 1 \quad (\tau j \neq j \in \mathbb{I}_\circ), \quad K_j K'_j - 1 \quad (j \in \mathbb{I}_\bullet).$$

In addition, by Proposition 4.11, we have $\widetilde{\mathbf{T}}''_{i,+1}(\widetilde{k}_{j,\diamond}) = \widetilde{k}_{\mathbf{r}_i\alpha_j,\diamond}$. Hence, the kernel of $\pi_{\boldsymbol{\varsigma}_\diamond}$ is preserved by $\widetilde{\mathbf{T}}''_{i,+1}$. Therefore, $\widetilde{\mathbf{T}}''_{i,+1}$ induces a automorphism $\mathbf{T}''_{i,+1;\boldsymbol{\varsigma}_\diamond}$ on $\mathbf{U}^\imath_{\boldsymbol{\varsigma}_\diamond}$ such that the following diagram commutes:

$$\begin{array}{ccc} \widetilde{\mathbf{U}}^\imath & \xrightarrow{\widetilde{\mathbf{T}}''_{i,+1}} & \widetilde{\mathbf{U}}^\imath \\ \downarrow{\scriptstyle \pi^\imath_{\varsigma_\diamond}} & & \downarrow{\scriptstyle \pi^\imath_{\varsigma_\diamond}} \\ \mathbf{U}^\imath_{\varsigma_\diamond} & \xrightarrow{\mathbf{T}''_{i,+1;\varsigma_\diamond}} & \mathbf{U}^\imath_{\varsigma_\diamond} \end{array}$$

It follows from Theorem 14.1 that $\mathbf{T}''_{i,+1;\varsigma_\diamond}$ satisfy the braid relations. By definition, $\widetilde{\mathcal{T}}_j$ $(j \in \mathbb{I}_\bullet)$ descends to Lusztig's automorphism $T_j$ under the central reduction $\pi^\imath_{\varsigma_\diamond}$. It then follows by Theorem 14.3 and Remark 14.4 that there exists an action of the braid group $\mathrm{Br}(W_\bullet) \rtimes \mathrm{Br}(W^\circ)$ on $\mathbf{U}^\imath_{\varsigma_\diamond}$ generated by $T_j, \mathbf{T}''_{i,+1;\varsigma_\diamond}$, for $j \in \mathbb{I}_\bullet, i \in \mathbb{I}_{\circ,\tau}$.

We now consider the symmetries on $\mathbf{U}^\imath_\varsigma$, for an arbitrary parameter $\varsigma$ satisfying (2.25).

Via the isomorphism $\phi_\varsigma : \mathbf{U}^\imath_{\varsigma_\diamond} \to \mathbf{U}^\imath_\varsigma$ constructed in Proposition 2.14, we transport the relative braid group action on $\mathbf{U}^\imath_{\varsigma_\diamond}$ to a relative braid group action on $\mathbf{U}^\imath_\varsigma$. More precisely, there exist automorphisms $\mathbf{T}''_{i,+1;\varsigma}$ on $\mathbf{U}^\imath_\varsigma$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathbf{U}^\imath_{\varsigma_\diamond} & \xrightarrow{\mathbf{T}''_{i,+1;\varsigma_\diamond}} & \mathbf{U}^\imath_{\varsigma_\diamond} \\ \downarrow{\scriptstyle \phi_\varsigma} & & \downarrow{\scriptstyle \phi_\varsigma} \\ \mathbf{U}^\imath_{\varsigma} & \xrightarrow{\mathbf{T}''_{i,+1}} & \mathbf{U}^\imath_{\varsigma} \end{array}$$

**Our convention here and below is that we suppress the dependence on a general parameter $\varsigma$ for the symmetries $\mathbf{T}''_{i,+1}$ (and $\mathbf{T}'_{i,-1}$, $\mathbf{T}''_{i,-1}$ and $\mathbf{T}'_{i,+1}$ below) on $\mathbf{U}^\imath_\varsigma$.**

In addition, $T_j$ commutes with $\phi_\varsigma$ for $j \in \mathbb{I}_\bullet$. Summarizing we have obtained the following braid group action on $\mathbf{U}^\imath_\varsigma$ (from Theorem 14.1, Theorem 14.3 and Remark 14.4).

**Theorem 14.9.** *For an arbitrary parameter $\boldsymbol{\varsigma}$ satisfying (2.25), there exists a braid group action of* $\mathrm{Br}(W_\bullet) \rtimes \mathrm{Br}(W^\circ)$ *on* $\mathbf{U}^\imath_{\boldsymbol{\varsigma}}$ *as automorphisms of algebras generated by* $T_j$ $(j \in \mathbb{I}_\bullet)$ *and* $\mathbf{T}''_{i,+1}$ $(i \in \mathbb{I}_{\circ,\tau})$.

We next construct $\mathbf{T}'_{i,+1}$ on $\mathbf{U}^\imath_{\boldsymbol{\varsigma}}$ for general parameters $\boldsymbol{\varsigma}$. By a similar argument as in §4.5, we have $\widetilde{\mathbf{T}}'_{i,+1} = \widetilde{\mathfrak{T}}'_{\mathbf{r}_i,+1}$ on $\widetilde{\mathbf{U}}^{\imath 0}$ and both are given by

$$\varsigma_{j,\star\diamond}\widetilde{k}_j \mapsto \varsigma_{\mathbf{r}_i\alpha_j,\star\diamond}\widetilde{k}_{\mathbf{r}_i\alpha_j}. \tag{14.21}$$

Denote the parameter $\overline{\boldsymbol{\varsigma}}_{\star\diamond} := (\overline{\varsigma}_{j,\star\diamond})_{j\in\mathbb{I}_\circ}$. Then by (14.21), $\widetilde{\mathbf{T}}'_{i,+1}$ preserves the kernel of $\pi^\imath_{\overline{\boldsymbol{\varsigma}}_{\star\diamond}}$ and hence it induces an automorphism $\mathbf{T}'_{i,+1;\overline{\boldsymbol{\varsigma}}_{\star\diamond}}$ on $\mathbf{U}^\imath_{\overline{\boldsymbol{\varsigma}}_{\star\diamond}}$ such that the following diagram commutes:

$$\begin{array}{ccc}
\widetilde{\mathbf{U}}^\imath & \xrightarrow{\ \widetilde{\mathbf{T}}'_{i,+1}\ } & \widetilde{\mathbf{U}}^\imath \\
\downarrow{\scriptstyle \pi^\imath_{\overline{\boldsymbol{\varsigma}}_{\star\diamond}}} & & \downarrow{\scriptstyle \pi^\imath_{\overline{\boldsymbol{\varsigma}}_{\star\diamond}}} \\
\mathbf{U}^\imath_{\overline{\boldsymbol{\varsigma}}_{\star\diamond}} & \xrightarrow{\ \mathbf{T}'_{i,+1;\overline{\boldsymbol{\varsigma}}_{\star\diamond}}\ } & \mathbf{U}^\imath_{\overline{\boldsymbol{\varsigma}}_{\star\diamond}}
\end{array}$$

On the other hand, by Lemma 14.5, $\widetilde{\mathfrak{T}}'_{j,+1}$ descends to Lusztig's automorphism $T'_{j,+1}$ under the central reduction $\pi^\imath_{\overline{\boldsymbol{\varsigma}}_{\star\diamond}}$. Hence, by Corollary 14.7, there exits an action of the braid group $\mathrm{Br}(W_\bullet) \rtimes \mathrm{Br}(W^\circ)$ on $\mathbf{U}^\imath_{\overline{\boldsymbol{\varsigma}}_{\star\diamond}}$ generated by $T'_{j,+1}$ $(j \in \mathbb{I}_\bullet)$ and $\mathbf{T}'_{i,+1;\overline{\boldsymbol{\varsigma}}_{\star\diamond}}$ $(i \in \mathbb{I}_{\circ,\tau})$.

Now, for an arbitrary parameter $\boldsymbol{\varsigma}$, we can use the isomorphism $\phi_{\boldsymbol{\varsigma}}\phi^{-1}_{\overline{\boldsymbol{\varsigma}}_{\star\diamond}}$ to translate this action on $\mathbf{U}^\imath_{\overline{\boldsymbol{\varsigma}}_{\star\diamond}}$ to an action on $\mathbf{U}^\imath_{\boldsymbol{\varsigma}}$, i.e., there exists automorphisms $\mathbf{T}'_{i,+1}$ on $\mathbf{U}^\imath_{\boldsymbol{\varsigma}}$ such that

$$\mathbf{T}'_{i,+1} \circ \phi_{\boldsymbol{\varsigma}}\phi^{-1}_{\overline{\boldsymbol{\varsigma}}_{\star\diamond}} = \phi_{\boldsymbol{\varsigma}}\phi^{-1}_{\overline{\boldsymbol{\varsigma}}_{\star\diamond}} \circ \mathbf{T}'_{i,+1;\boldsymbol{\varsigma}_{\star\diamond}}.$$

In addition, $\widetilde{\mathcal{T}}'_{j,+1}$ commutes with $\phi_{\boldsymbol{\varsigma}}\phi^{-1}_{\overline{\boldsymbol{\varsigma}}_{\star\diamond}}$.

Similarly, we can formulate the automorphisms $\mathbf{T}'_{i,-1}, \mathbf{T}''_{i,-1}$ on $\mathbf{U}^{\imath}_{\boldsymbol{\varsigma}}$, which are inverses to $\mathbf{T}''_{i,+1}, \mathbf{T}'_{i,+1}$; the detail is skipped. Summarizing, we have established the following theorem, which was conjectured in [KP11, Conjecture 1.2].

**Theorem 14.10.** *Let $e = \pm 1$, and $\boldsymbol{\varsigma}$ be an arbitrary parameter satisfying (2.25).*

1. *There exists a braid group action of $\mathrm{Br}(W_\bullet) \rtimes \mathrm{Br}(W^\circ)$ on $\mathbf{U}^{\imath}_{\boldsymbol{\varsigma}}$ as automorphisms of algebras generated by $T'_{j,e}$ $(j \in \mathbb{I}_\bullet)$ and $\mathbf{T}'_{i,e}$ $(i \in \mathbb{I}_{\circ,\tau})$.*
2. *There exists a braid group action of $\mathrm{Br}(W_\bullet) \rtimes \mathrm{Br}(W^\circ)$ on $\mathbf{U}^{\imath}_{\boldsymbol{\varsigma}}$ as automorphisms of algebras generated by $T''_{j,e}$ $(j \in \mathbb{I}_\bullet)$ and $\mathbf{T}''_{i,e}$ $(i \in \mathbb{I}_{\circ,\tau})$.*

# Part IV

# Relative braid group symmetries on modules

# 15 Relative braid group actions on U-modules

Let $(\mathbb{I} = \mathbb{I}_\bullet \cup \mathbb{I}_\circ, \tau)$ be a Satake diagram of arbitrary finite type or quasi-split Kac-Moody type, and $(\mathbf{U}, \mathbf{U}^\imath_{\boldsymbol{\varsigma}})$ be the associated quantum symmetric pair. We set $\boldsymbol{\varsigma}$ to be a balanced parameter throughout this section. Based on the intertwining properties of $\mathbf{T}'_{i,e}, \mathbf{T}''_{i,e}$ on $\mathbf{U}^\imath_{\boldsymbol{\varsigma}}$, we formulate the compatible action of corresponding operators on any integrable $\mathbf{U}$-module $M$, whose weights are bounded above. We then show that these operators on $M$ satisfy relative braid group relations.

## 15.1 Intertwining relations on $\mathbf{U}^\imath_{\boldsymbol{\varsigma}}$

Recall that the symmetries $\mathbf{T}'_{i,e}$ and $\mathbf{T}''_{i,e}$ on $\mathbf{U}^\imath_{\boldsymbol{\varsigma}}$, for $e = \pm 1$, were defined in §14.4. In this subsection we formulate the intertwining properties of these symmetries.

Recall $\phi_{\boldsymbol{\varsigma}}$ from Proposition 2.14. Since $\boldsymbol{\varsigma}$ is a balanced parameter, $\phi_{\boldsymbol{\varsigma}}$ is the restriction of $\Phi_{\overline{\boldsymbol{\varsigma}}_\diamond \boldsymbol{\varsigma}}$, where $\overline{\boldsymbol{\varsigma}}_\diamond \boldsymbol{\varsigma}$ is defined by componentwise multiplication with $\overline{\boldsymbol{\varsigma}}_\diamond = (\overline{\varsigma_{j,\diamond}})_{j \in \mathbb{I}_\circ}$; see the proof of Proposition 2.14. Define

$$\mathcal{T}''_{i,+1;\boldsymbol{\varsigma}} := \Phi_{\overline{\boldsymbol{\varsigma}}_\diamond \boldsymbol{\varsigma}} T''_{i,+1} \Phi^{-1}_{\overline{\boldsymbol{\varsigma}}_\diamond \boldsymbol{\varsigma}}, \qquad \mathcal{T}'_{i,-1;\boldsymbol{\varsigma}} := \Phi_{\overline{\boldsymbol{\varsigma}}_\diamond \boldsymbol{\varsigma}} T'_{i,-1} \Phi^{-1}_{\overline{\boldsymbol{\varsigma}}_\diamond \boldsymbol{\varsigma}}. \tag{15.1}$$

**Proposition 15.1.** *Let $\boldsymbol{\varsigma}$ be a balanced parameter. The automorphisms $\mathbf{T}'_{i,-1}$ and $\mathbf{T}''_{i,+1}$ on $\mathbf{U}^\imath_{\boldsymbol{\varsigma}}$ satisfy the following intertwining relations:*

$$\mathbf{T}'_{i,-1}(x) \Upsilon_{i,\boldsymbol{\varsigma}} = \Upsilon_{i,\boldsymbol{\varsigma}} \mathcal{T}'_{\mathbf{r}_i,-1;\boldsymbol{\varsigma}}(x), \tag{15.2}$$

$$\mathbf{T}''_{i,+1}(x)\, \mathcal{T}''_{\mathbf{r}_i,+1;\boldsymbol{\varsigma}}(\Upsilon^{-1}_{i,\boldsymbol{\varsigma}}) = \mathcal{T}''_{\mathbf{r}_i,+1;\boldsymbol{\varsigma}}(\Upsilon^{-1}_{i,\boldsymbol{\varsigma}})\, \mathcal{T}''_{\mathbf{r}_i,+1;\boldsymbol{\varsigma}}(x), \tag{15.3}$$

*for $x \in \mathbf{U}^\imath_{\boldsymbol{\varsigma}}$.*

*Proof.* By Theorem 4.7 and Theorem 6.1, we have, for any $x \in \widetilde{\mathbf{U}}^\imath$,

$$\begin{aligned} \widetilde{\mathbf{T}}'_{i,-1}(x)\, \widetilde{\Upsilon}_i &= \widetilde{\Upsilon}_i\, \widetilde{\mathcal{T}}'_{\mathbf{r}_i,-1}(x), \\ \widetilde{\mathbf{T}}''_{i,+1}(x)\, \widetilde{\mathcal{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_i^{-1}) &= \widetilde{\mathcal{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_i^{-1})\, \widetilde{\mathcal{T}}''_{\mathbf{r}_i,+1}(x). \end{aligned} \tag{15.4}$$

Let $T'_{i,e}, T''_{i,e}$ be Lusztig's automorphisms on $\mathbf{U}$. Recall the central reduction $\pi_{\boldsymbol{\varsigma}_\diamond}: \widetilde{\mathbf{U}} \to \mathbf{U}$ from (2.7). By (2.10) (with $\mathbf{a} = \boldsymbol{\varsigma}_\diamond$) and (2.15), we have

$$\pi_{\boldsymbol{\varsigma}_\diamond} \circ \widetilde{\mathcal{T}}''_{i,+1} = T''_{i,+1} \circ \pi_{\boldsymbol{\varsigma}_\diamond}, \qquad \pi_{\boldsymbol{\varsigma}_\diamond} \circ \widetilde{\mathcal{T}}'_{i,-1} = T'_{i,-1} \circ \pi_{\boldsymbol{\varsigma}_\diamond}.$$

Hence, $\pi^\imath_{\boldsymbol{\varsigma}_\diamond} \circ \widetilde{\mathbf{T}}''_{i,+1} = \mathbf{T}''_{i,+1;\boldsymbol{\varsigma}_\diamond} \circ \pi^\imath_{\boldsymbol{\varsigma}_\diamond}$. Since the parameter $\boldsymbol{\varsigma}_\diamond$ is balanced, $\pi^\imath_{\boldsymbol{\varsigma}_\diamond}$ is the restriction of $\pi_{\boldsymbol{\varsigma}_\diamond}$ to $\mathbf{U}^\imath_{\boldsymbol{\varsigma}_\diamond}$. Applying $\pi_{\boldsymbol{\varsigma}_\diamond}$ to the intertwining relations (15.4), we obtain, for any $x \in \mathbf{U}^\imath_{\boldsymbol{\varsigma}_\diamond}$,

$$\begin{aligned} \mathbf{T}'_{i,-1;\boldsymbol{\varsigma}_\diamond}(x)\, \Upsilon_{i,\boldsymbol{\varsigma}_\diamond} &= \Upsilon_{i,\boldsymbol{\varsigma}_\diamond}\, T'_{\mathbf{r}_i,-1}(x), \\ \mathbf{T}''_{i,+1;\boldsymbol{\varsigma}_\diamond}(x)\, T''_{\mathbf{r}_i,+1}(\Upsilon_{i,\boldsymbol{\varsigma}_\diamond}^{-1}) &= T''_{\mathbf{r}_i,+1}(\Upsilon_{i,\boldsymbol{\varsigma}_\diamond}^{-1})\, T''_{\mathbf{r}_i,+1}(x). \end{aligned} \tag{15.5}$$

Recall $\phi_{\boldsymbol{\varsigma}}$ from Proposition 2.14. As we have seen in §14.4, we have $\phi_{\boldsymbol{\varsigma}} \circ \mathbf{T}''_{i,+1;\boldsymbol{\varsigma}_\diamond} = \mathbf{T}''_{i,+1} \circ \phi_{\boldsymbol{\varsigma}}$, and $\phi_{\boldsymbol{\varsigma}} \circ \mathbf{T}'_{i,-1;\boldsymbol{\varsigma}_\diamond} = \mathbf{T}'_{i,-1} \circ \phi_{\boldsymbol{\varsigma}}$. Therefore, applying $\phi_{\boldsymbol{\varsigma}}$ to the identities (15.5) gives us the desired intertwining relations in the proposition. $\square$

We next formulate intertwining relations for the other two automorphisms $\mathbf{T}'_{i,+1}$ and $\mathbf{T}''_{i,-1}$.

Recall the central reductions $\pi_{\boldsymbol{\varsigma}}: \widetilde{\mathbf{U}} \to \mathbf{U}$ from (2.7) and $\pi^\imath_{\boldsymbol{\varsigma}}: \widetilde{\mathbf{U}}^\imath \to \mathbf{U}^\imath_{\boldsymbol{\varsigma}}$ from Proposition 2.12. By Lemma 14.5, we have $\pi_{\overline{\boldsymbol{\varsigma}}_{\star\diamond}} \circ \widetilde{\mathcal{T}}'_{i,+1} = T'_{i,+1} \circ \pi_{\overline{\boldsymbol{\varsigma}}_{\star\diamond}}$ and $\pi^\imath_{\overline{\boldsymbol{\varsigma}}_{\star\diamond}} \circ \widetilde{\mathbf{T}}'_{i,+1} = \mathbf{T}'_{i,+1;\overline{\boldsymbol{\varsigma}}_{\star\diamond}} \circ \pi^\imath_{\overline{\boldsymbol{\varsigma}}_{\star\diamond}}$. Since the parameter $\overline{\boldsymbol{\varsigma}}_{\star\diamond}$ is balanced, $\pi^\imath_{\overline{\boldsymbol{\varsigma}}_{\star\diamond}}$ is the restriction of $\pi_{\overline{\boldsymbol{\varsigma}}_{\star\diamond}}$ to

$\widetilde{\mathbf{U}}^\imath$. Applying $\pi_{\bar{\boldsymbol{\varsigma}}_{\star\diamond}}$ to (14.15)-(14.16), we have, for any $x \in \mathbf{U}^\imath_{\bar{\boldsymbol{\varsigma}}_{\star\diamond}}$,

$$\begin{aligned} \mathbf{T}'_{i,+1;\bar{\boldsymbol{\varsigma}}_{\star\diamond}}(x)\, T'_{\mathbf{r}_i,+1}(\Upsilon^{-1}_{i,\bar{\boldsymbol{\varsigma}}_{\star\diamond}}) &= T'_{\mathbf{r}_i,+1}(\Upsilon^{-1}_{i,\bar{\boldsymbol{\varsigma}}_{\star\diamond}})\, T'_{\mathbf{r}_i,+1}(x), \\ \mathbf{T}''_{i,-1;\bar{\boldsymbol{\varsigma}}_{\star\diamond}}(x)\Upsilon_{i,\bar{\boldsymbol{\varsigma}}_{\star\diamond}} &= \Upsilon_{i,\bar{\boldsymbol{\varsigma}}_{\star\diamond}} T''_{\mathbf{r}_i,-1}(x). \end{aligned} \tag{15.6}$$

Since $\boldsymbol{\varsigma}$ is a balanced parameter, by the proof of Proposition 2.14, $\phi_{\boldsymbol{\varsigma}}\phi^{-1}_{\bar{\boldsymbol{\varsigma}}_{\star\diamond}}$ is the restriction of $\Phi_{\bar{\boldsymbol{\varsigma}}^{-1}_{\star\diamond}\boldsymbol{\varsigma}} = \Phi_{\boldsymbol{\varsigma}_{\star\diamond}\boldsymbol{\varsigma}}$. Define

$$\mathcal{T}''_{i,-1;\boldsymbol{\varsigma}} := \Phi_{\boldsymbol{\varsigma}_{\star\diamond}\boldsymbol{\varsigma}} T''_{i,-1} \Phi^{-1}_{\boldsymbol{\varsigma}_{\star\diamond}\boldsymbol{\varsigma}}, \qquad \mathcal{T}'_{i,+1;\boldsymbol{\varsigma}} := \Phi_{\boldsymbol{\varsigma}_{\star\diamond}\boldsymbol{\varsigma}} T'_{i,+1} \Phi^{-1}_{\boldsymbol{\varsigma}_{\star\diamond}\boldsymbol{\varsigma}}. \tag{15.7}$$

Applying $\phi_{\boldsymbol{\varsigma}}\phi^{-1}_{\bar{\boldsymbol{\varsigma}}_{\star\diamond}}$ to (15.6), we have established the following.

**Proposition 15.2.** *Let $\boldsymbol{\varsigma}$ be a balanced parameter. The automorphisms $\mathbf{T}'_{i,+1;\boldsymbol{\varsigma}}$ and $\mathbf{T}''_{i,-1;\boldsymbol{\varsigma}}$ on $\mathbf{U}^\imath_{\boldsymbol{\varsigma}}$ satisfy the following intertwining relations, for all $x \in \mathbf{U}^\imath_{\boldsymbol{\varsigma}}$:*

$$\begin{aligned} \mathbf{T}'_{i,+1}(x)\; \mathcal{T}'_{\mathbf{r}_i,+1;\boldsymbol{\varsigma}}(\Upsilon^{-1}_{i,\boldsymbol{\varsigma}}) &= \mathcal{T}'_{\mathbf{r}_i,+1;\boldsymbol{\varsigma}}(\Upsilon^{-1}_{i,\boldsymbol{\varsigma}})\; \mathcal{T}'_{\mathbf{r}_i,+1;\boldsymbol{\varsigma}}(x), \\ \mathbf{T}''_{i,-1}(x)\Upsilon_{i,\boldsymbol{\varsigma}} &= \Upsilon_{i,\boldsymbol{\varsigma}} \mathcal{T}''_{\mathbf{r}_i,-1;\boldsymbol{\varsigma}}(x). \end{aligned}$$

## 15.2 Compatible actions of $\mathbf{T}'_{i,e}, \mathbf{T}''_{i,e}$ on U-modules

Recall from Proposition 2.5 that Lusztig's symmetries $T'_{i,e}, T''_{i,e}$ admit compatible actions on an integrable **U**-module $M$. Symmetries $\mathcal{T}'_{i,e;\boldsymbol{\varsigma}}, \mathcal{T}''_{i,e;\boldsymbol{\varsigma}}$, defined in (15.1) and (15.7), are merely rescalings of $T'_{i,e}, T''_{i,e}$. Applying exactly the same rescalings to the operators on modules (2.11)–(2.12), we obtain operators $\mathcal{T}'_{i,e;\boldsymbol{\varsigma}}, \mathcal{T}''_{i,e;\boldsymbol{\varsigma}}$ on $M$ which satisfy

$$\mathcal{T}'_{i,e;\boldsymbol{\varsigma}}(uv) = \mathcal{T}'_{i,e;\boldsymbol{\varsigma}}(u)\mathcal{T}'_{i,e;\boldsymbol{\varsigma}}(v), \qquad \mathcal{T}''_{i,e;\boldsymbol{\varsigma}}(uv) = \mathcal{T}''_{i,e;\boldsymbol{\varsigma}}(u)\mathcal{T}''_{i,e;\boldsymbol{\varsigma}}(v). \tag{15.8}$$

for any $u \in \mathbf{U}, v \in M$.

Recall from (2.1) the partial order on the weight lattice $X$. We assume that weights of the $\mathbf{U}$-module $M$ are bounded above, and then the quasi $K$-matrix admits a well-defined action on $M$. We regard $M$ as an $\mathbf{U}^\imath$-module by restriction.

**Definition 15.3.** Define linear operators $\mathbf{T}'_{i,e}, \mathbf{T}''_{i,e}$ on $M$, for $i \in \mathbb{I}_\circ$ and $e = \pm 1$, by

$$\begin{aligned}
\mathbf{T}'_{i,-1}(v) &:= \Upsilon_{i,\varsigma} \mathcal{T}'_{\mathbf{r}_i,-1;\varsigma}(v), \\
\mathbf{T}''_{i,+1}(v) &:= \mathcal{T}''_{\mathbf{r}_i,+1;\varsigma}(\Upsilon_{i,\varsigma}^{-1}) \mathcal{T}''_{\mathbf{r}_i,+1;\varsigma}(v), \\
\mathbf{T}'_{i,+1}(v) &:= \mathcal{T}'_{\mathbf{r}_i,+1;\varsigma}(\Upsilon_{i,\varsigma}^{-1}) \mathcal{T}'_{\mathbf{r}_i,+1;\varsigma}(v), \\
\mathbf{T}''_{i,-1}(v) &:= \Upsilon_{i,\varsigma} \mathcal{T}''_{\mathbf{r}_i,-1;\varsigma}(v),
\end{aligned} \tag{15.9}$$

for any $v \in M$.

(In these notations, we have suppressed the dependence on $\varsigma$ on these operators.)

The automorphisms $\mathbf{T}'_{i,e}, \mathbf{T}''_{i,e}$ on $M$ in (15.9) are compatible with the corresponding automorphisms on $\mathbf{U}^\imath_\varsigma$.

**Theorem 15.4.** *Let $M$ be an integrable $\mathbf{U}$-module, whose weights are bounded above. Fix $i \in \mathbb{I}_\circ$ and $e = \pm 1$. Then we have*

$$\mathbf{T}'_{i,e}(xv) = \mathbf{T}'_{i,e}(x)\mathbf{T}'_{i,e}(v), \qquad \mathbf{T}''_{i,e}(xv) = \mathbf{T}''_{i,e}(x)\mathbf{T}''_{i,e}(v), \tag{15.10}$$

*for any $x \in \mathbf{U}^\imath_\varsigma, v \in M$.*

*Proof.* We prove the identity for $\mathbf{T}'_{i,-1}$ ; the proofs for the remaining ones are similar. In the proof, we omit the subindex $\varsigma$ for $\Upsilon_{i,\varsigma}$ and $\mathcal{T}'_{\mathbf{r}_i,-1;\varsigma}$ as there is no confusion.

Since $\mathcal{T}'_{\mathbf{r}_i,-1}(xv) = \mathcal{T}'_{\mathbf{r}_i,-1}(x)\mathcal{T}'_{\mathbf{r}_i,-1}(v)$, we have

$$\Upsilon_i \mathcal{T}'_{\mathbf{r}_i,-1}(xv) = \big(\Upsilon_i \mathcal{T}'_{\mathbf{r}_i,-1}(x)\Upsilon_i^{-1}\big)\Upsilon_i \mathcal{T}'_{\mathbf{r}_i,-1}(v), \tag{15.11}$$

By Proposition 15.1, we have $\Upsilon_i \mathcal{T}_{\mathbf{r}_i,-1}(x)\Upsilon_i^{-1} = \mathbf{T}'_{i,-1}(x)$. Hence, using the definition (15.9), the identity (15.11) implies that $\mathbf{T}'_{i,-1}(xv) = \mathbf{T}'_{i,-1}(x)\mathbf{T}'_{i,-1}(v)$ as desired. $\square$

## 15.3 Relative braid relations on U-modules

Let $m_{ij}$ denotes the order of $\mathbf{r}_i\mathbf{r}_j$ in $W^\circ$.

**Theorem 15.5.** *Let $M$ be an integrable $\mathbf{U}$-module, whose weights are bounded above. The relative braid relations hold for the linear operators $\mathbf{T}'_{i,e}$ (and respectively, $\mathbf{T}''_{i,e}$) on $M$; that is, for any $i \neq j \in \mathbb{I}_{\circ,\tau}$ and for any $v \in M$, we have*

$$\underbrace{\mathbf{T}'_{i,e}\mathbf{T}'_{j,e}\mathbf{T}'_{i,e}\cdots}_{m_{ij}}(v) = \underbrace{\mathbf{T}'_{j,e}\mathbf{T}'_{i,e}\mathbf{T}'_{j,e}\cdots}_{m_{ij}}(v). \tag{15.12}$$

$$\underbrace{\mathbf{T}''_{i,e}\mathbf{T}''_{j,e}\mathbf{T}''_{i,e}\cdots}_{m_{ij}}(v) = \underbrace{\mathbf{T}''_{j,e}\mathbf{T}''_{i,e}\mathbf{T}''_{j,e}\cdots}_{m_{ij}}(v). \tag{15.13}$$

*Proof.* We prove the first identity for $e = -1$ ; the proofs for the remaining ones are similar and skipped.

Set $m = m_{ij}$. We keep the notations $\boldsymbol{w}_\circ, \boldsymbol{w}'_\circ, \boldsymbol{w}_k, \boldsymbol{w}'_k$ for $1 \leq k \leq m$ from the proof of Theorem 14.1. We shall write $\mathcal{T}_{\mathbf{r}_i}^{-1}$ for $\mathcal{T}'_{\mathbf{r}_i,-1,\varsigma}$ and omit the subindex $\varsigma$ for $\Upsilon_{i,\varsigma}$ in the proof, since there is no confusion.

By definition (15.9), for any $v \in M$, we have

$$\underbrace{\mathbf{T}'_{i,-1}\mathbf{T}'_{j,-1}\mathbf{T}'_{i,-1}\cdots}_{m}(v) = (\Upsilon_i \mathcal{T}_{\boldsymbol{w}_1}^{-1}\Upsilon_j \mathcal{T}_{\boldsymbol{w}_2}^{-1}\Upsilon_i \cdots)\underbrace{\mathcal{T}_{\mathbf{r}_i}^{-1}\mathcal{T}_{\mathbf{r}_j}^{-1}\mathcal{T}_{\mathbf{r}_i}^{-1}\cdots}_{m}(v) \tag{15.14}$$

By taking a central to (14.8), the first factor on RHS (15.14) is $\sigma(\widetilde{\Upsilon}_{\boldsymbol{w}_\circ})$. Hence, we have

$$\underbrace{\mathbf{T}'_{i,-1}\mathbf{T}'_{j,-1}\mathbf{T}'_{i,-1}\cdots}_{m}(v) = \sigma(\widetilde{\Upsilon}_{\boldsymbol{w}_\circ})\underbrace{\mathcal{T}_{\mathbf{r}_i}^{-1}\mathcal{T}_{\mathbf{r}_j}^{-1}\mathcal{T}_{\mathbf{r}_i}^{-1}\cdots}_{m}(v). \tag{15.15}$$

Similarly, by switching $i, j$ in (15.15), we obtain

$$\underbrace{\mathbf{T}'_{j,-1}\mathbf{T}'_{i,-1}\mathbf{T}'_{j,-1}\cdots}_{m}(v) = \sigma(\widetilde{\Upsilon}_{\boldsymbol{w}'_\circ})\underbrace{\mathcal{T}_{\mathbf{r}_j}^{-1}\mathcal{T}_{\mathbf{r}_i}^{-1}\mathcal{T}_{\mathbf{r}_j}^{-1}\cdots}_{m}(v). \tag{15.16}$$

Applying a central reduction to Theorem 13.1, we have $\Upsilon_{\boldsymbol{w}_\circ} = \Upsilon_{\boldsymbol{w}'_\circ}$. Since $\mathcal{T}_i$ are defined by rescaling $T''_{i,+1}$ in (15.1), they satisfy the braid relations. Hence, we have

$$\underbrace{\mathcal{T}_{\mathbf{r}_i}^{-1}\mathcal{T}_{\mathbf{r}_j}^{-1}\mathcal{T}_{\mathbf{r}_i}^{-1}\cdots}_{m}(v) = \underbrace{\mathcal{T}_{\mathbf{r}_j}^{-1}\mathcal{T}_{\mathbf{r}_i}^{-1}\mathcal{T}_{\mathbf{r}_j}^{-1}\cdots}_{m}(v). \tag{15.17}$$

Combining (15.15)–(15.17), we have proved the first identity for $e = -1$. □

# 16 Relative braid group symmetries on $\mathbf{U}^\imath$-modules: split type

In this section, we consider a quantum symmetric pair $(\mathbf{U}, \mathbf{U}^\imath_{\boldsymbol{\varsigma}})$ of split Kac-Moody type with an arbitrary parameter $\boldsymbol{\varsigma}$. We introduce the notation of integrable $\mathbf{U}^\imath_{\boldsymbol{\varsigma}}$-

modules. We reformulate the linear operators $\mathbf{T}'_{i,e}, \mathbf{T}''_{i,e}$ introduced in Definition 15.3, in terms of elements in $\mathbf{U}^\imath_{\boldsymbol{\varsigma}}$. This reformulation allows us to define compatible relative braid group symmetries on any integrable $\mathbf{U}^\imath_{\boldsymbol{\varsigma}}$-modules; see Theorem 16.9.

Our construction of compatible relative braid group symmetries on integrable $\mathbf{U}^\imath$-modules can be generalized beyond the split type and we will study all of these in a forthcoming paper.

## 16.1 Integrable $\mathbf{U}^\imath_{\boldsymbol{\varsigma}}$-modules

Recall the lattice $X$ from the root datum $(Y, X, \langle\cdot,\cdot\rangle, \cdots)$; see § 2.1. We assume that $\tau$ extends to an involution on $X$ and an involution on $Y$ such that the bilinear pairing $\langle\cdot,\cdot\rangle$ is invariant under $\tau$. Then $\theta$ acts on $X, Y$ via (2.18).

Following [BW18b], we define

$$X_\imath = X/\breve{X}, \quad \text{where } \breve{X} = \{\lambda - \theta(\lambda) | \lambda \in X\}. \tag{16.1}$$

We shall call $X_\imath$ the $\imath$-weight lattice (even though $X_\imath$ is *not* a lattice). For any $\lambda \in X$, write $\overline{\lambda}$ to be its image in $X_\imath$. In the split rank one case, $X_\imath \cong \mathbb{Z}/2\mathbb{Z}$.

We define an $X_\imath$-grading on $\mathbf{U}^\imath_{\boldsymbol{\varsigma}}$ by setting $\deg B_i = -\overline{i'}$ following [BW21, §3.5]. A $\mathbf{U}^\imath_{\boldsymbol{\varsigma}}$-module $M$ is called an $X_\imath$-weight module if it is $X_\imath$-graded. By definition, $M$ is equipped with a decomposition $M = \bigoplus_{\overline{\lambda}\in X_\imath} M_{\overline{\lambda}}$ such that $B_i M_{\overline{\lambda}} \subset M_{\overline{\lambda - i'}}$ for any $i \in \mathbb{I}$. Elements in $M_{\overline{\lambda}}$ are called *ıweight vectors*.

Let $M$ be an $X_\imath$-weight modules and write $M = M_{\overline{0}} \oplus M_{\overline{1}}$ with respect to $B_i$ for a fixed $i \in \mathbb{I}$. We say $B_i$ acts *locally nilpotently* on $M$ if the following two conditions are both satisfied:

1. for any vector $v \in M_{\bar{0}}$, $B_{i,\bar{0}}^{(n)} v = 0$ for $n$ sufficiently large;

2. for any vector $v \in M_{\bar{1}}$, $B_{i,\bar{1}}^{(n)} v = 0$ for $n$ sufficiently large.

Here $B_{i,\bar{0}}^{(n)}, B_{i,\bar{1}}^{(n)}$ are $\imath$divided powers formulated in (7.7) descending to $\mathbf{U}^\imath_{\boldsymbol{\varsigma}}$ by central reductions.

**Definition 16.1.** An $X_\imath$-weight $\mathbf{U}^\imath_{\boldsymbol{\varsigma}}$-module $M$ is called *integrable* if and only if $B_i$ acts locally nilpotently on $M$ for any $i \in \mathbb{I}$.

For example, any integrable $\mathbf{U}$-module is an integrable $\mathbf{U}^\imath$-module via restriction.

## 16.2 Transition matrices between canonical and $\imath$canonical bases

Consider the Satake diagram of type $\mathrm{AI}_1$. Set $\mathbb{I} = \mathbb{I}_\circ = \{i\}$. We then omit the first subindex $i$ for notations $T_i, T'_{i,e}, T''_{i,e}, \Upsilon_{i,\boldsymbol{\varsigma}}$ in this subsection. We also write $\varsigma$ for $\boldsymbol{\varsigma}$.

Denote by $\varpi$ the fundamental weight for $\mathfrak{sl}_2$. Let $L(n)$ be the irreducible highest weight $\mathbf{U}(\mathfrak{sl}_2)$-module with highest weight $n\varpi$ and highest weight vector $\eta$. Set $v_k = F^{(k)}\eta$ if $1 \le k \le n$ and $v_k = 0$ if $k < 0$ or $k > n$. We have

$$Fv_k = \begin{cases} [k+1]v_{k+1}, & \text{if } k < n, \\ 0, & \text{if } k = n, \end{cases} \qquad Ev_k = \begin{cases} 0, & \text{if } k = 0, \\ [n+1-k]v_{k-1}, & \text{if } k > 0. \end{cases} \tag{16.2}$$

By induction, we have for $m \ge 0$, (cf. [Ja95, Section 8.3])

$$F^{(m)}v_k = \begin{bmatrix} m+k \\ m \end{bmatrix} v_{k+m}, \qquad E^{(m)}v_k = \begin{bmatrix} n+m-k \\ m \end{bmatrix} v_{k-m}. \tag{16.3}$$

**Lemma 16.2** ([Lus93, Proposition 5.2.2]). *We have for* $0 \le k \le n, e = \pm 1$,

$$T'_e(v_k) = (-1)^k q^{ek(n-k+1)} v_{n-k}, \qquad T''_e(v_k) = (-1)^{n-k} q^{e(n-k)(k+1)} v_{n-k}.$$

Set the parameter $\varsigma = q^{-1}$. Then there exists a bar involution $\psi^\imath$ on $\mathbf{U}^\imath_\varsigma$ which fixes $B$. Explicitly,

$$B = F + q^{-1} E K^{-1}.$$

The $\mathbf{U}$-module $L(n)$ is equipped with the anti-linear involution $\psi$ which fixes $v_k \eta$ for $0 \le k \le n$. Then $v_k$, for $0 \le k \le n$, is the canonical basis for $L(n)$. Following [BW18b, Proposition 5.1], $(L(n), \psi^\imath)$ is an involutive $\mathbf{U}^\imath_\varsigma$-module where $\psi^\imath$ acts by $\psi^\imath := \Upsilon_\varsigma \cdot \psi$. The $\imath$canonical basis for $L(n)$ is $B^{(m)}_{\overline{n}} \eta$, for $0 \le m \le n$. (see [BeW18, Theorem 2.10,3.6][BW18b, Theorem 5.7])

We first recall the formulas for the $\imath$canonical basis elements from [BeW18]. Note that a notation $\begin{Bmatrix} m-\lambda-c \\ c \end{Bmatrix}$ was used in those formulas in [BeW18], which is replaced here by a standard notation thanks to

$$\begin{Bmatrix} m-\lambda-c \\ c \end{Bmatrix} = q^{2(m-\lambda)c} \begin{bmatrix} c+\lambda-m \\ c \end{bmatrix}_{q^2}.$$

**Lemma 16.3** ([BeW18, (2.16)-(2.17),(3.8)-(3.9)]). *(1) For* $n = 2\lambda \in 2\mathbb{N}$ *and* $0 \le$

$m \leq \lambda$, *we have*

$$B_{\bar{0}}^{(2m)}\eta = \sum_{c=0}^{m} q^{-2c^2+c+2(m-\lambda)c} \begin{bmatrix} c+\lambda-m \\ c \end{bmatrix}_{q^2} F^{(2m-2c)}\eta, \tag{16.4}$$

$$B_{\bar{0}}^{(2m-1)}\eta = \sum_{c=0}^{m-1} q^{-2c^2-c+2(m-\lambda)c} \begin{bmatrix} c+\lambda-m \\ c \end{bmatrix}_{q^2} F^{(2m-1-2c)}\eta. \tag{16.5}$$

*(2) For* $n = 2\lambda + 1 \in 2\mathbb{N} + 1$ *and* $0 \leq m \leq \lambda$, *we have*

$$B_{\bar{1}}^{(2m)}\eta = \sum_{c=0}^{m} q^{-2c^2-c+2(m-\lambda)c} \begin{bmatrix} c+\lambda-m \\ c \end{bmatrix}_{q^2} F^{(2m-2c)}\eta,$$

$$B_{\bar{1}}^{(2m+1)}\eta = \sum_{c=0}^{m} q^{-2c^2+c+2(m-\lambda)c} \begin{bmatrix} c+\lambda-m \\ c \end{bmatrix}_{q^2} F^{(2m+1-2c)}\eta.$$

We obtain the inverse formulas to those in Lemma 16.3 below.

**Proposition 16.4.** *(1) For* $n = 2\lambda \in 2\mathbb{N}$ *and* $0 \leq m \leq \lambda$, *we have*

$$F^{(2m)}\eta = \sum_{c=0}^{m} (-1)^c q^{-c+2(m-\lambda)c} \begin{bmatrix} c+\lambda-m \\ c \end{bmatrix}_{q^2} B_{\bar{0}}^{(2m-2c)}\eta, \tag{16.6}$$

$$F^{(2m-1)}\eta = \sum_{c=0}^{m-1} (-1)^c q^{-3c+2(m-\lambda)c} \begin{bmatrix} c+\lambda-m \\ c \end{bmatrix}_{q^2} B_{\bar{0}}^{(2m-1-2c)}\eta. \tag{16.7}$$

*(2) For* $n = 2\lambda + 1 \in 2\mathbb{N} + 1$ *and* $0 \leq m \leq \lambda$, *we have*

$$F^{(2m)}\eta = \sum_{c=0}^{m} (-1)^c q^{-3c+2(m-\lambda)c} \begin{bmatrix} c+\lambda-m \\ c \end{bmatrix}_{q^2} B_{\bar{1}}^{(2m-2c)}\eta, \tag{16.8}$$

$$F^{(2m+1)}\eta = \sum_{c=0}^{m}(-1)^c q^{-c+2(m-\lambda)c} \begin{bmatrix} c+\lambda-m \\ c \end{bmatrix}_{q^2} B_{\bar{1}}^{(2m+1-2c)}\eta. \tag{16.9}$$

*Proof.* The proofs of all 4 formulas (16.6)–(16.9) are entirely similar, and we shall only provide the details for the proof of (16.6). Recall a standard $q$-binomial identity [Lus93, 1.3.1(e)], for $k \in \mathbb{N}$,

$$\sum_{a+c=k} v^{ax-cy} \begin{bmatrix} x \\ c \end{bmatrix}_v \begin{bmatrix} y \\ a \end{bmatrix}_v = \begin{bmatrix} x+y \\ k \end{bmatrix}_v. \tag{16.10}$$

Thanks to to (16.4), the 2 sets

$$\mathcal{CB} := \{F^{(2m)}\eta | 0 \le m \le \lambda\}, \qquad \imath\mathcal{CB} := \{B_{\bar{0}}^{(2m)}\eta | 0 \le m \le \lambda\}$$

are bases for the same subspace of $L(2\lambda)$; moreover, the transition matrix from $\mathcal{CB}$ to $\imath\mathcal{CB}$ is uni-triangular. We shall show that (16.6) provides the inverse transition matrix from $\imath\mathcal{CB}$ to $\mathcal{CB}$. To that end, plugging the formula (16.6) into RHS(16.4), we obtain

$$\begin{aligned} RHS(16.4) &= \sum_{c=0}^{m}\sum_{a=0}^{m-c} q^{-2c^2+c+2(m-\lambda)c} \begin{bmatrix} c+\lambda-m \\ c \end{bmatrix}_{q^2} \\ &\qquad \times (-1)^a q^{-a+2(m-\lambda-c)a} \begin{bmatrix} a+c+\lambda-m \\ a \end{bmatrix}_{q^2} B_{\bar{0}}^{(2m-2c-2a)}\eta \\ &\overset{(*)}{=} \sum_{k=0}^{m}\sum_{a+c=k} (-1)^{a+c} q^{-2c^2+c-a+2(m-\lambda)c+2(m-\lambda-c)a} \end{aligned}$$

$$\times \begin{bmatrix} m-\lambda-1 \\ c \end{bmatrix}_{q^2} \begin{bmatrix} k+\lambda-m \\ a \end{bmatrix}_{q^2} B_{\bar{0}}^{(2m-2c-2a)}\eta$$

$$= \sum_{k=0}^{m} \sum_{a+c=k} (-1)^k q^k$$

$$\times (q^2)^{a(m-\lambda-1)-c(k+\lambda-m)} \begin{bmatrix} m-\lambda-1 \\ c \end{bmatrix}_{q^2} \begin{bmatrix} k+\lambda-m \\ a \end{bmatrix}_{q^2} B_{\bar{0}}^{(2m-2k)}\eta$$

$$\overset{(**)}{=} \sum_{k=0}^{m} (-1)^k q^k \begin{bmatrix} k-1 \\ k \end{bmatrix}_{q^2} B_{\bar{0}}^{(2m-2k)}\eta$$

$$= B_{\bar{0}}^{(2m)}\eta = LHS(16.4),$$

where we used $\begin{bmatrix} c+\lambda-m \\ c \end{bmatrix}_{q^2} {}_{=(-1)^c} \begin{bmatrix} m-\lambda-1 \\ c \end{bmatrix}_{q^2}$ in (*), and (**) follows by (16.10).

Therefore, (16.6) indeed provides the inverse transition matrix from $\imath\mathcal{CB}$ to $\mathcal{CB}$, and hence the identity (16.6) holds. □

*Remark* 16.5. The formulas in Proposition 16.4 can be reformulated uniformly, regardless of the parity of $n \in \mathbb{N}$: for $\ell \in \mathbb{N}$,

$$F^{(n-2\ell)}\eta = \sum_{c=0}^{\lfloor \frac{n}{2} \rfloor - \ell} (-1)^c q^{-(2\ell+1)c} \begin{bmatrix} \ell+c \\ c \end{bmatrix}_{q^2} B_{\overline{n}}^{(n-2\ell-2c)}\eta,$$

$$F^{(n-1-2\ell)}\eta = \sum_{c=0}^{\lfloor \frac{n-1}{2} \rfloor - \ell} (-1)^c q^{-(2\ell+3)c} \begin{bmatrix} \ell+c \\ c \end{bmatrix}_{q^2} B_{\overline{n}}^{(n-1-2\ell-2c)}\eta.$$

Similarly, the formulas in Lemma 16.3 can be reformulated uniformly: for $\ell \in \mathbb{N}$,

$$B_{\overline{n}}^{(n-2\ell)}\eta = \sum_{c=0}^{\lfloor \frac{n}{2} \rfloor - \ell} q^{-2c^2-(2\ell-1)c} \begin{bmatrix} \ell+c \\ c \end{bmatrix}_{q^2} F^{(n-2\ell-2c)}\eta,$$

$$B_{\overline{n}}^{(n-1-2\ell)}\eta = \sum_{c=0}^{\lfloor \frac{n-1}{2} \rfloor - \ell} q^{-2c^2-(2\ell+1)c} \begin{bmatrix} \ell+c \\ c \end{bmatrix}_{q^2} F^{(n-1-2\ell-2c)}\eta.$$

## 16.3 Rank one formulas

Set $\mathbf{U}_i$ to be the subalgebra of $\mathbf{U}$ generated by $E_i, F_i, K_i^{\pm 1}$. We consider an irreducible $\mathbf{U}_i$-modules $L(n)$ for fixed $n > 0, i \in \mathbb{I}$ in this subsection, and the goal is to find the formula of the braid group operator $\mathbf{T}'_{i,-1}$ on $L(n)$, in terms of $\imath$-divided powers $B_{i,\overline{p}}^{(k)}, \overline{p} \in \mathbb{Z}/2$.

### Special parameter case

We set the parameter $\varsigma_i = q_i^{-1}$ for $i \in \mathbb{I}$ in this subsection, unless otherwise specified. Write $b_k$ for $B_{i,\overline{n}}^{(k)}\eta$ and $v_k$ for $F_i^{(k)}\eta$. Explicitly, the action of $\mathcal{T}'_{i,-1}$ is given by

$$\mathcal{T}'_{i,-1}(v_k) = (-1)^k(-q_i)^{(2k-n)/2} q_i^{-k(n-k+1)} v_{n-k},$$

$$\mathcal{T}''_{i,+1}(v_k) = (-1)^{n-k}(-q_i)^{(2k-n)/2} q_i^{(n-k)(k+1)} v_{n-k}.$$

Define for $\overline{p} \in \mathbb{Z}/2\mathbb{Z}$

$$f_{\overline{p}}(B_i) := \sum_{\substack{k\geq 0 \\ \overline{k}=\overline{p}}} (-q_i)^{-k/2} B_{i,\overline{k}}^{(k)}, \qquad \overline{f}_{\overline{p}}(B_i) := \sum_{\substack{k\geq 0 \\ \overline{k}=\overline{p}}} (-q_i)^{k/2} B_{i,\overline{k}}^{(k)}. \tag{16.11}$$

**Proposition 16.6.** *For the parameter $\varsigma_i = q_i^{-1}$, the actions of operators $\mathbf{T}'_{i,-1}, \mathbf{T}''_{i,+1}$ on $L(n)$ are given by*

$$\begin{aligned} \mathbf{T}'_{i,-1} &= f_{\overline{n}}(B_i), \qquad & \mathbf{T}''_{i,-1} &= (-1)^n f_{\overline{n}}(B_i), \\ \mathbf{T}''_{i,+1} &= \overline{f}_{\overline{n}}(B_i), \qquad & \mathbf{T}'_{i,+1} &= (-1)^n \overline{f}_{\overline{n}}(B_i), \end{aligned} \tag{16.12}$$

*Proof.* The formulas of $\mathbf{T}'_{i,+1}, \mathbf{T}''_{i,-1}$ are obtained by applying the bar involution $\psi^\imath$ to formulas of $\mathbf{T}'_{i,-1}, \mathbf{T}''_{i,+1}$ respectively. Hence, we only need to show (16.12) for $\mathbf{T}'_{i,-1}, \mathbf{T}''_{i,+1}$.

We claim that it suffices to show (16.12) on the vector $\eta$. Suppose that $\mathbf{T}'_{i,-1}\eta = f_{\overline{n}}(B_i)\eta$. Since $\mathbf{T}'_{i,-1}(B^{(k)}_{i,\overline{n}}) = B^{(k)}_{i,\overline{n}}$, we have for any $0 \leq k \leq n$

$$\mathbf{T}'_{i,-1} b_k = \mathbf{T}'_{i,-1}(B^{(k)}_{i,\overline{n}}\eta) = \mathbf{T}'_{i,-1}(B^{(k)}_{i,\overline{n}})\mathbf{T}'_{i,-1}\eta = B^{(k)}_{i,\overline{n}} f_{\overline{n}}(B_i)\eta = f_{\overline{n}}(B_i) b_k.$$

This implies that $\mathbf{T}'_{i,-1} = f_{\overline{n}}(B_i)$ on $L(n)$ since $\{b_k | 0 \leq k \leq n\}$ is a basis for $L(n)$. Similar arguments work for the symmetry $\mathbf{T}''_{i,+1}$.

Let $A = (a_{k\ell})$ be the transition matrix of the canonical and $\imath$-canonical bases and denote its inverse by $A^{-1} = (a'_{k\ell})$, i.e., for $0 \leq \ell, k \leq n$, we have

$$b_k = \sum_{\ell \leq k} a_{k\ell} v_\ell, \qquad v_k = \sum_{\ell \leq k} a'_{k\ell} b_\ell. \tag{16.13}$$

We shall often write $\overline{v}$ for $\psi(v), v \in L(n)$. Since $b_k$ are fixed by $\psi^\imath$, we have

$$\Upsilon_{i,\varsigma} \overline{b_k} = b_k, \qquad (0 \leq k \leq n). \tag{16.14}$$

We first formulate the action of $\mathbf{T}'_{i,-1}$ on $\eta$. A primary computation of the $t$-action on $b_0 = v_0 = \eta$ gives the following equality,

$$
\begin{aligned}
(-q_i)^{n/2}\mathbf{T}'_{i,-1}\eta =&(-q_i)^{n/2}\Upsilon_{i,\varsigma}\cdot \mathcal{T}'_{i,-1}(v_0) \\
=&\Upsilon_{i,\varsigma}v_n = \Upsilon_{i,\varsigma}\sum_{\ell}\overline{a'_{n\ell}b_\ell} = \sum_{\ell}\overline{a'_{n\ell}}b_\ell = \sum_{\ell}\overline{a'_{n\ell}}B^{(\ell)}_{i,\overline{n}}\eta.
\end{aligned}
\tag{16.15}
$$

Note the formulas (16.6) and (16.9) for $m=\lambda$ give the following uniform formula:

$$
F_i^{(n)}\eta = \sum_{c=0}^{\lfloor \frac{n}{2}\rfloor}(-1)^c q_i^{-c}B^{(n-2c)}_{i,\overline{n}}\eta.
$$

i.e., $a'_{n\ell}$ equals $(-1)^c q_i^{-c}$ if $\ell = n-2c$, and equals 0 otherwise. Therefore, by (16.15), the action of $\mathbf{T}'_{i,-1}$ on $\eta$ is given by the formula below,

$$
\mathbf{T}'_{i,-1}\eta = (-q_i)^{-n/2}\sum_{c=0}^{\lfloor \frac{n}{2}\rfloor}(-1)^c q_i^{c}B^{(n-2c)}_{i,\overline{n}}\eta = \sum_{\overline{k}=\overline{n},0\le k\le n}(-q_i)^{-\frac{k}{2}}B^{(k)}_{i,\overline{k}}\eta. \tag{16.16}
$$

Since $B^{(k)}_{i,\overline{n}}\eta = 0$ for $k>n$, we obtain that $\mathbf{T}'_{i,-1}\eta = f_{\overline{n}}(B_i)\eta$ as desired.

We next formulate the action of $\mathbf{T}''_{i,+1}$ on $\eta$. By (16.14), $\mathbf{T}''_{i,+1}$ acts on the vector $b_0 = v_0$ by

$$
\begin{aligned}
\mathbf{T}''_{i,+1}(v_0) =&\mathcal{T}''_{i,+1}(\Upsilon^{-1}_{i,\varsigma}v_0) = \mathcal{T}''_{i,+1}(v_0) = (-q_i)^{n/2}v_n \\
=&(-q_i)^{n/2}\sum_{\ell}a'_{nl}b_\ell = (-q_i)^{n/2}\sum_{\ell}a'_{nl}B^{(l)}_{i,\overline{n}}v_0.
\end{aligned}
$$

Recall that $a'_{n\ell}$ equals $(-1)^c q_i^{-c}$ if $\ell = n-2c$. Hence, the action of $\mathbf{T}''_{i,+1}$ on $L(n)$

is given by

$$\mathbf{T}''_{i,+1} = (-q_i)^{n/2} \sum_{c=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^c q_i^{-c} B_{i,\overline{n}}^{(n-2c)} = \sum_{\overline{k}=\overline{n}, 0 \le k \le n} (-q_i)^{\frac{k}{2}} B_{i,\overline{k}}^{(k)} \eta. \tag{16.17}$$

Since $B_{i,\overline{n}}^{(k)} \eta = 0$ for $k > n$, we obtain that $\mathbf{T}''_{i,+1}\eta = \overline{f}_{\overline{n}}(B_i)\eta$ as desired.

□

*Remark* 16.7. The relation between formulas of $\mathbf{T}''_{i,+1}, \mathbf{T}'_{i,-1}$ on $L(n)$ is given by

$$\mathbf{T}''_{i,+1} = (-1)^n \psi^\imath(\mathbf{T}'_{i,-1}).$$

**General parameter case**

For $(\mathbf{U}, \mathbf{U}^\imath_{\boldsymbol{\varsigma}})$ with a general parameter $\boldsymbol{\varsigma} = (\varsigma_i)_{i \in \mathbb{I}}$, define

$$f_{\overline{p},\boldsymbol{\varsigma}}(B_i) := \sum_{\substack{k \ge 0 \\ \overline{k} = \overline{p}}} (-q_i^2 \varsigma_i)^{-k/2} B_{i,\overline{k},\boldsymbol{\varsigma}}^{(k)}. \tag{16.18}$$

$$\overline{f}_{\overline{p},\boldsymbol{\varsigma}}(B_i) := \sum_{\substack{k \ge 0 \\ \overline{k} = \overline{p}}} (-1)^{k/2} \varsigma_i^{-k/2} B_{i,\overline{k},\boldsymbol{\varsigma}}^{(k)}. \tag{16.19}$$

**Proposition 16.8.** *The actions of operators $\mathbf{T}'_{i,e}, \mathbf{T}''_{i,e}$ on $L(n)$ are given by*

$$\mathbf{T}'_{i,-1} = f_{\overline{n},\boldsymbol{\varsigma}}(B_i), \qquad \mathbf{T}''_{i,-1} = (-1)^n f_{\overline{n},\boldsymbol{\varsigma}}(B_i), \tag{16.20}$$

$$\mathbf{T}''_{i,+1} = \overline{f}_{\overline{n},\boldsymbol{\varsigma}}(B_i). \qquad \mathbf{T}'_{i,+1} = (-1)^n \overline{f}_{\overline{n},\boldsymbol{\varsigma}}(B_i), \tag{16.21}$$

*Proof.* These formulas are proved by applying scaling automorphisms to (16.12). □

Note that, for the distinguished parameter $\boldsymbol{\varsigma}_\circ = (-q_i^{-2})_{i \in \mathbb{I}}$, the formula (16.18) takes the simplest form

$$\mathbf{T}'_{i,-1} = \begin{cases} \sum_{p=0}^{\infty} B^{(2p)}_{i,\bar{0},\varsigma_\circ}, & \text{if } n = 2l, \\ \sum_{p=0}^{\infty} B^{(2p+1)}_{i,\bar{1},\varsigma_\circ}, & \text{if } n = 2l+1. \end{cases} \tag{16.22}$$

## 16.4 Compatible actions of $\mathbf{T}'_{i,e}, \mathbf{T}''_{i,e}$ on $\mathbf{U}^\imath$-modules

Consider an quantum symmetric pair $(\mathbf{U}, \mathbf{U}^\imath)$ of arbitrary split type. We show that rank one formulas defined in Proposition 16.8 give rise to compatible relative braid symmetries on integrable $\mathbf{U}^\imath$-modules.

Let $M$ be an integrable $\mathbf{U}^\imath$-module and $v \in M$ be an $\imath$weight vector of $\imath$weight $\overline{p} \in \mathbb{Z}/2\mathbb{Z}$ with respect to $B_i, i \in \mathbb{I}$. Define linear operators $\mathbf{T}'_{i,e}, \mathbf{T}''_{i,e}$ on $M$ by

$$\begin{aligned} \mathbf{T}'_{i,-1} v &= f_{\overline{p},\boldsymbol{\varsigma}}(B_i) v, \qquad & \mathbf{T}''_{i,-1} v &= (-1)^{\delta_{\overline{p}=\overline{1}}} f_{\overline{p},\boldsymbol{\varsigma}}(B_i) v, \\ \mathbf{T}''_{i,+1} v &= \overline{f}_{\overline{p},\boldsymbol{\varsigma}}(B_i) v, \qquad & \mathbf{T}'_{i,+1} v &= (-1)^{\delta_{\overline{p}=\overline{1}}} \overline{f}_{\overline{p},\boldsymbol{\varsigma}}(B_i) v, \end{aligned} \tag{16.23}$$

where $f_{\overline{p},\boldsymbol{\varsigma}}(B_i), \overline{f}_{\overline{p},\boldsymbol{\varsigma}}(B_i)$ defined in (16.18)-(16.19) are summations of $\imath$divided powers. Since $B_i$ acts locally nilpotently on $M$, these linear operators are well-defined.

As shown in Propositions 16.6,16.8, these new operators $\mathbf{T}'_{i,e}, \mathbf{T}''_{i,e}$ coincide with operators introduced in Definition 15.3, when acting on finite dimensional $\mathbf{U}$-modules.

**Theorem 16.9.** *Fix $i \in \mathbb{I}, e = \pm 1$. Let $M$ be an integrable $\mathbf{U}^\imath$-module. For any $x \in \mathbf{U}^\imath, v \in M$, we have*

$$\mathbf{T}'_{i,e}(xv) = \mathbf{T}'_{i,e}(x) \cdot \mathbf{T}'_{i,e}(v), \qquad \mathbf{T}''_{i,e}(xv) = \mathbf{T}''_{i,e}(x) \cdot \mathbf{T}''_{i,e}(v). \tag{16.24}$$

*Proof.* We prove Theorem 16.9 in the remaining part of this subsection. We shall prove (16.24) for the operator $\mathbf{T}'_{i,-1}$ and the proof for other operators can be obtained similarly.

An integrable module $M$ is spanned by $\imath$weight vectors and hence we assume $v$ to be a $\imath$weight vector.

If $x, y \in \mathbf{U}^\imath$ both satisfy (16.24), then $xy$ also satisfy (16.24). Hence, it suffices to check (16.24) when $x$ are generators of $\mathbf{U}^\imath$. For $x = B_i$, it is clear from the definition that the actions of $B_i, \mathbf{T}'_{i,-1}$ on $M$ commutes with each other, and then (16.24) follows since $\mathbf{T}'_{i,-1}(B_i) = B_i$.

It remains to check (16.24) for $x = B_j, j \neq i$. Recall that $b_{i,j;0} = B_j$ and $b_{i,j;\alpha} = \mathbf{T}'_{i,-1}(B_j)$. Then the desired relation (16.24) in this case is proved in Proposition 16.11 below, and the proof uses Proposition 16.10. $\square$

Write $\alpha$ for $-c_{ij}$ in the rest of this section.

**Proposition 16.10.** *For arbitrary $c_{ij}$ and the parameter $\varsigma_i = -q_i^{-2}$, we have the following formulas*

$$b_{i,j;\alpha} B_{i,\overline{k}}^{(k)} = \sum_{x=0}^{\alpha} q_i^{(k-x)(\alpha-x)} \left( \sum_{y=0}^{\lceil \frac{\alpha-x}{2} \rceil} (-1)^y q_i^{2y(\lceil \frac{\alpha-x}{2} \rceil - 1 - \alpha + x)} \begin{bmatrix} \lceil \frac{\alpha-x}{2} \rceil \\ y \end{bmatrix}_{q_i^2} B_{i,\overline{k+c_{ij}}}^{(k-x-2y)} \right) b_{i,j;\alpha-x}, \tag{16.25}$$

$$b_{i,j;\alpha} B_{i,\overline{k+1}}^{(k)} = \sum_{x=0}^{\alpha} q_i^{(k-x)(\alpha-x)} \left( \sum_{y=0}^{\lfloor \frac{\alpha-x}{2} \rfloor} (-1)^y q_i^{2y(\lfloor \frac{\alpha-x}{2} \rfloor - \alpha + x)} \begin{bmatrix} \lfloor \frac{\alpha-x}{2} \rfloor \\ y \end{bmatrix}_{q_i^2} B_{i,\overline{k+1+c_{ij}}}^{(k-x-2y)} \right) b_{i,j;\alpha-x}, \tag{16.26}$$

*where $\lceil a \rceil$ means the smallest integer bigger or equal than $a$ and $\lfloor a \rfloor$ means the biggest integer smaller or equal than $a$.*

The proof for Proposition 16.10 will be given in Appendix C.

**Proposition 16.11.** *We have the following identity for* $j \neq i, \overline{p} \in \mathbb{Z}/2\mathbb{Z}$

$$b_{i,j;-c_{ij}} f_{\overline{p},\varsigma}(B_i) = f_{\overline{p}+\overline{c_{ij}},\varsigma}(B_i) b_{i,j;0}. \tag{16.27}$$

*Proof.* We shall prove (16.27) for the parameter $\varsigma_i = -q_i^{-2}$. The general parameter case follows by applying the rescaling automorphism.

Recall from Definition 8.1 that the element $b_{i,j;m} \in \mathbf{U}^\imath$ for that special parameter is defined by the following recursive relation

$$\begin{aligned} &- q_i^{-(c_{ij}+2m)} b_{i,j;m} B_i + B_i b_{i,j;m} \\ =&[m+1]_i b_{i,j;m+1} + [c_{ij}+m-1]_i q_i^{-2m-c_{ij}} b_{i,j;m-1}. \end{aligned} \tag{16.28}$$

It is known that $\mathbf{T}'_{i,-1} = b_{i,j;-c_{ij}}$ and $b_{i,j;m} = 0$ if $m > -c_{ij}$ or $m < 0$.

Recall from (16.22) that, for the parameter $\varsigma_i = -q_i^{-2}$, we have

$$f_{\bar{0}}(B_i) = \sum_{k=0}^{\infty} B_{i,\bar{0}}^{(2k)}, \qquad f_{\bar{1}}(B_i) = \sum_{k=0}^{\infty} B_{i,\bar{1}}^{(2k+1)}.$$

By Proposition 16.10, we have

$$b_{i,j;\alpha} f_{\overline{p}}(B_i) = \sum_{x \geq 0} \xi_{\alpha,x,\overline{p}} B_{i,j,\alpha-x}, \tag{16.29}$$

where

$$\xi_{\alpha,x,\overline{p}} = \sum_{k:\overline{k}=\overline{p}} \sum_{y=0}^{\lceil \frac{\alpha-x}{2} \rceil} (-1)^y q_i^{(k-x)(\alpha-x)+2y(\lceil \frac{\alpha-x}{2} \rceil - 1-\alpha+x)} \begin{bmatrix} \lceil \frac{\alpha-x}{2} \rceil \\ y \end{bmatrix}_{q_i^2} B_{i,\overline{p+c_{ij}}}^{(k-x-2y)}. \tag{16.30}$$

Setting $d = k - x - 2y$ above, we can rewrite the $q_i$-power in (16.30) as

$$(k-x)(\alpha-x) + 2y(\lceil \frac{\alpha-x}{2} \rceil - 1 - \alpha + x) = d(\alpha - x) + 2y\big(\lceil \frac{\alpha-x}{2} \rceil - 1\big).$$

Hence, the coefficient of $B^{(d)}_{i,\overline{p+c_{ij}}}$ in $\xi_{\alpha,x,\overline{p}}$ for any integer $d \geq 0$, can be computed as follows:

$$\begin{aligned}
[B^{(d)}_{i,\overline{p+c_{ij}}}]\xi_{\alpha,x,\overline{p}} &= \delta_{\overline{d}=\overline{p-x}} \sum_{y=0}^{\lceil \frac{\alpha-x}{2} \rceil} (-1)^y q_i^{d(\alpha-x)+2y\left(\lceil \frac{\alpha-x}{2} \rceil -1\right)} \begin{bmatrix} \lceil \frac{\alpha-x}{2} \rceil \\ y \end{bmatrix}_{q_i^2} \\
&= \delta_{\overline{d}=\overline{p-x}} q_i^{d(\alpha-x)} \sum_{y=0}^{\lceil \frac{\alpha-x}{2} \rceil} (-1)^y (q_i^{-2})^{y\left(1-\lceil \frac{\alpha-x}{2} \rceil\right)} \begin{bmatrix} \lceil \frac{\alpha-x}{2} \rceil \\ y \end{bmatrix}_{q_i^{-2}} \\
&= \begin{cases} 0 & \text{if } x < \alpha, \\ \delta_{\overline{d}=\overline{p-\alpha}} & \text{if } x = \alpha, \end{cases}
\end{aligned}$$

where the last equality follows by a standard $v$-binomial identity (with $v = q_i^{-2}$); cf. [Lus93, 1.3.4].

Summarizing, we can now rewrite (16.30) as

$$\xi_{\alpha,x,\overline{p}} = \begin{cases} 0 & \text{if } x < \alpha, \\ \sum_{d:\overline{d}=\overline{p-\alpha}} B^{(d)}_{i,\overline{1}} & \text{if } x = \alpha. \end{cases}$$

Therefore, (16.29) becomes

$$b_{i,j;\alpha} f_{\overline{p}}(B_i) = \sum_{d:\overline{d}=\overline{p+c_{ij}}} B^{(d)}_{i,\overline{p+c_{ij}}} \cdot B_{i,j,0}. \tag{16.31}$$

Hence, we have proved (16.27) as desired. □

# Appendix A

# Proofs of Proposition 5.11 and Table 3

In this Appendix, we shall provide constructive proofs for Proposition 5.11 and verify the rank 2 formulas for $\widetilde{\mathbf{T}}'_{i,-1}(B_j)$ in Table 3. The proofs are based on type-by-type computations in $\widetilde{\mathbf{U}}$ for each rank two Satake diagram. Along the way, we will also specify a reduced expression for $\mathbf{r}_i$ in $W$.

## 1 Some preparatory lemmas

Denote the $t$-commutator

$$[C, D]_t = CD - tDC,$$

for various $q$-powers $t$. Let $(\mathbb{I} = \mathbb{I}_\bullet \cup \mathbb{I}_\circ, \tau)$ be an arbitrary Satake diagram. Recall that $B_i = F_i + \widetilde{\mathcal{T}}_{w_\bullet}(E_{\tau i})K'_i$ and $B^\sigma_i = F_i + K_i \widetilde{\mathcal{T}}^{-1}_{w_\bullet}(E_{\tau i})$.

**Lemma 1.1.** *Suppose that $i, j \in \mathbb{I}_\circ$ such that $j \notin \{i, \tau i\}$. Then we have*

$$[B_i^\sigma, F_j]_{q^{-(\alpha_i,\alpha_j)}} = [F_i, F_j]_{q^{-(\alpha_i,\alpha_j)}}, \tag{1.1}$$

$$[B_i, \widetilde{\mathcal{T}}_{w_\bullet}(\widetilde{E}_{\tau j})K'_j]_{q^{-(\alpha_i,\alpha_j)}} = q^{(\alpha_i, w_\bullet(\alpha_{\tau j}))}\widetilde{\mathcal{T}}_{w_\bullet}\big([E_{\tau i}, E_{\tau j}]_{q^{-(\alpha_i,\alpha_j)}}\big)K'_i K'_j. \tag{1.2}$$

*Proof.* Follows by a simple computation and using the identity $[E_k, F_j] = 0$, for $k \neq j$. $\square$

Introduce the following operator (see Lemma 4.4 for some of the notations)

$$\mathcal{D} := \widetilde{\mathcal{T}}_{w_0}\widetilde{\mathcal{T}}_{w_\bullet}\widehat{\tau}_0\widehat{\tau}. \tag{1.3}$$

We shall formulate several basic properties for $\mathcal{D}$ below. A systematic use of $\mathcal{D}$ throughout Appendices A and B will allow us to reduce the proofs of many challenging identities to easier ones.

**Lemma 1.2.** *We have*

$$\mathcal{D}(B_i^\sigma) = -q^{-(\alpha_i,\alpha_i)}B_i\widetilde{\mathcal{T}}_{w_\bullet}(\mathcal{K}_{\tau i}^{-1}), \tag{1.4}$$

$$\mathcal{D}(F_j) = -q_j^{-2}\widetilde{\mathcal{T}}_{w_\bullet}(E_{\tau j})K'_j\widetilde{\mathcal{T}}_{w_\bullet}(\mathcal{K}_{\tau j}^{-1}). \tag{1.5}$$

*Proof.* We rewrite the identity (4.16) as follows:

$$\begin{aligned} B_i\widetilde{\mathcal{T}}_{\mathbf{r}_i}(\mathcal{K}_{\tau_{\bullet,i}\tau i}) &= -q^{-(\alpha_i, w_\bullet\alpha_{\tau i})}\widetilde{\mathcal{T}}_{w_\bullet}\widetilde{\mathcal{T}}_{w_{\bullet,i}}(B^\sigma_{\tau_{\bullet,i}\tau i}) \\ &= -q^{-(\alpha_i, w_\bullet\alpha_{\tau i})}\widetilde{\mathcal{T}}_{w_\bullet}\widetilde{\mathcal{T}}_{w_0}(B^\sigma_{\tau_0\tau i}) = -q^{-(\alpha_i, w_\bullet\alpha_{\tau i})}\mathcal{D}(B_i^\sigma). \end{aligned} \tag{1.6}$$

Since $\widetilde{\mathcal{T}}_{\mathbf{r}_i}(\mathcal{K}_{\tau_{\bullet,i}\tau i}) = \widetilde{\mathcal{T}}_{w_\bullet}\widetilde{\mathcal{T}}_{w_{\bullet,i}}(\mathcal{K}_{\tau_{\bullet,i}\tau i}) = \varsigma_{i,\diamond}^2\widetilde{\mathcal{T}}_{w_\bullet}(\mathcal{K}_{\tau i}^{-1})$, the formula (1.4) follows from (1.6).

By Lemma 4.4, we have $\mathcal{D}(F_j) = -K^{-1}_{w_\bullet(\tau j)}\widetilde{\mathfrak{T}}_{w_\bullet}(E_{\tau j}) = -q_j^{-2}\widetilde{\mathfrak{T}}_{w_\bullet}(E_{\tau j})K'_j\widetilde{\mathfrak{T}}_{w_\bullet}(\mathcal{K}^{-1}_{\tau j})$. This proves (1.5). ☐

**Lemma 1.3.** *The operator $\mathcal{D}$ commutes with $\widetilde{\mathfrak{T}}_{\mathbf{r}_i}, \widetilde{\mathfrak{T}}_j$, for $i \in \mathbb{I}_\circ, j \in \mathbb{I}_\bullet$.*

*Proof.* Since $w_0 s_k = s_{\tau_0 k} w_0$, for $k \in \mathbb{I}$, we have $\widetilde{\mathfrak{T}}_{w_0}\widetilde{\mathfrak{T}}_k^{-1} = \widetilde{\mathfrak{T}}_{w_0 s_k} = \widetilde{\mathfrak{T}}_{s_{\tau_0 k} w_0} = \widetilde{\mathfrak{T}}^{-1}_{\tau_0 k}\widetilde{\mathfrak{T}}_{w_0}$. Hence, $\widetilde{\mathfrak{T}}_{w_0}\widetilde{\mathfrak{T}}_k = \widetilde{\mathfrak{T}}_{\tau_0 k}\widetilde{\mathfrak{T}}_{w_0}$ for any $k \in \mathbb{I}$. Therefore, $\widetilde{\mathfrak{T}}_{w_0}\widehat{\tau}_0$ commutes with $\widetilde{\mathfrak{T}}_k$ $(k \in \mathbb{I})$ and thus commutes with $\widetilde{\mathfrak{T}}_{\mathbf{r}_i}, \widetilde{\mathfrak{T}}_j$, for $i \in \mathbb{I}_\circ, j \in \mathbb{I}_\bullet$.

Similarly, one can show that $\widetilde{\mathfrak{T}}_{w_\bullet}\widehat{\tau}$ commutes with $\widetilde{\mathfrak{T}}_j$, for $j \in \mathbb{I}_\bullet$. Hence, by definition (1.3), the operator $\mathcal{D}$ commutes with $\widetilde{\mathfrak{T}}_j$ for $j \in \mathbb{I}_\bullet$.

On the other hand, by definition (2.21), $\widetilde{\mathfrak{T}}_{\mathbf{r}_i}$, for $i \in \mathbb{I}_\circ$, commutes with both $\widetilde{\mathfrak{T}}_{w_\bullet}$ and $\widehat{\tau}$. Hence, $\mathcal{D}$ also commutes with $\widetilde{\mathfrak{T}}_{\mathbf{r}_i}$. ☐

# 2 Split types of rank 2

Consider rank 2 split Satake diagrams $(\mathbb{I} = \mathbb{I}_\circ = \{i, j\}, \mathrm{Id})$. In this case, we have $\mathbf{r}_i = s_i,\ B_i^\sigma = F_i + K_i E_i$.

**The case $c_{ij} = -1$**

In this case, according to the first line of Table 3, Proposition 5.11 is reformulated and proved as follows.

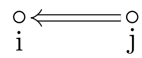
**Lemma 2.1.** *We have*

$$\widetilde{\mathfrak{T}}_i^{-1}(F_j) = [B_i^\sigma, F_j]_q, \qquad \widetilde{\mathfrak{T}}_i^{-1}(E_j K'_j) = [B_i, E_j K'_j]_q. \tag{2.1}$$

*Proof.* Follows immediately by Lemma 1.1 and the definition of $\widetilde{\mathfrak{T}}_i$. ☐

**The case** $c_{ij} = -2$

In this case, the rank 2 Satake diagram is given by




and according to Table 3, Proposition 5.11 can be reformulated and proved as follows.

**Lemma 2.2.** *We have*

$$\widetilde{\mathfrak{T}}_i^{-1}(F_j) = \frac{1}{[2]_i}\big[B_i^\sigma, [B_i^\sigma, F_j]_{q_i^2}\big] - q_i^2 F_j K_i K_i', \tag{2.2}$$

$$\widetilde{\mathfrak{T}}_i^{-1}(E_j K_j') = \frac{1}{[2]_i}\big[B_i, [B_i, E_j K_j']_{q_i^2}\big] - q_i^2 E_j K_j' K_i K_i'. \tag{2.3}$$

*Proof.* We prove the formula (2.2). By Lemma 1.1, we have $[B_i^\sigma, F_j]_{q_i^2} = [F_i, F_j]_{q_i^2}$. By Proposition 4.2, we have $\widetilde{\mathfrak{T}}_i^{-1}(F_j) = \frac{1}{[2]_i}\big[F_i, [F_i, F_j]_{q_i^2}\big]$. Now we compute the first term on RHS (2.2) using Lemma 1.1 as follows:

$$\begin{aligned}
\big[B_i^\sigma, [B_i^\sigma, F_j]_{q_i^2}\big] &= \big[B_i^\sigma, [F_i, F_j]_{q_i^2}\big] \\
&= \big[F_i, [F_i, F_j]_{q_i^2}\big] + \big[K_i E_i, [F_i, F_j]_{q_i^2}\big] \\
&= [2]_i \widetilde{\mathfrak{T}}_i^{-1}(F_j) + K_i[\frac{K_i - K_i'}{q_i - q_i^{-1}}, F_j]_{q_i^2} \\
&= [2]_i \widetilde{\mathfrak{T}}_i^{-1}(F_j) + q_i^2 [2]_i F_j K_i K_i'.
\end{aligned}$$

Hence the formula (2.2) holds.

We next prove the formula (2.3). In this case, we read (1.3) as $\mathcal{D} = \widetilde{\mathfrak{T}}_{w_0}$, and note that $\mathcal{K}_i = \widetilde{k}_i$. By Lemma 1.3, $\mathcal{D}$ commutes with $\widetilde{\mathfrak{T}}_i^{-1}$. Applying this operator $\mathcal{D}$ to the formula (2.2) and then using (1.4)–(1.5), we obtain

$$\widetilde{\mathfrak{T}}_i^{-1}(E_jK'_j)\widetilde{\mathfrak{T}}_i^{-1}(\widetilde{k}_j^{-1}) = \frac{q_i^{-4}}{[2]_i}\big[B_i\widetilde{k}_i^{-1},[B_i\widetilde{k}_i^{-1},E_jK'_j\widetilde{k}_j^{-1}]_{q_i^2}\big] - q_i^2E_jK'_j\widetilde{k}_j^{-1}\widetilde{\mathfrak{T}}_{w_0}(\widetilde{k}_i). \quad (2.4)$$

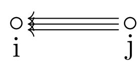
Recall our symmetries $\widetilde{\mathfrak{T}}_j$ are defined in §4.1 by normalizing a variant of Lusztig's symmetries $\widetilde{\mathfrak{T}}''_{j,+1}$. In this case, we have $\widetilde{\mathfrak{T}}_{w_0}(\widetilde{k}_i) = q_i^{-4}\widetilde{k}_i^{-1}$ and $\widetilde{\mathfrak{T}}_i^{-1}(\widetilde{k}_j^{-1}) = q_i^{-4}\widetilde{k}_j^{-1}\widetilde{k}_i^{-2}$. Hence, since $\widetilde{k}_i, \widetilde{k}_j$ are central, (2.4) is simplified as the following formula

$$\widetilde{\mathfrak{T}}_i^{-1}(E_jK'_j)\widetilde{k}_j^{-1}\widetilde{k}_i^{-2} = \Big(\frac{1}{[2]_i}\big[B_i,[B_i,E_jK'_j]_{q_i^2}\big] - q_i^2E_jK'_jK_iK'_i\Big)\widetilde{k}_j^{-1}\widetilde{k}_i^{-2}, \quad (2.5)$$

which clearly implies the formula (2.3). □

**The case $c_{ij} = -3$**

Consider the Satake diagram of split type $G_2$




In this case, we have $q_i = q$ and $q_j = q^3$.

**Lemma 2.3.** *We have*

$$\big[K_iE_i,[F_i,F_j]_{q^3}\big]_q = q^3[3]F_jK_iK'_i, \quad (2.6)$$

$$\Big[K_iE_i,\big[F_i,[F_i,F_j]_{q^3}\big]_q\Big]_{q^{-1}} = q(1+[3])[B_i^\sigma,F_j]_{q^3}K_iK'_i. \quad (2.7)$$

*Proof.* The first identity (2.6) is derived as follows:

$$\mathrm{LHS}(2.6) = K_i\big[E_i,[F_i,F_j]_{q^3}\big] = K_i\Big[\frac{K_i-K_i^{-1}}{q-q^{-1}},F_j\Big]_{q^3} = q^3[3]K_iK'_iF_j = \mathrm{RHS}(2.6).$$

We next compute

$$\begin{aligned}
\mathrm{LHS}(2.7) &= K_i\Big[E_i, \big[F_i, [F_i, F_j]_{q^3}\big]_q\Big] \\
&= K_i\big[\frac{K_i - K_i^{-1}}{q-q^{-1}}, [F_i, F_j]_{q^3}\big]_q + K_i\big[F_i, [\frac{K_i - K_i'}{q-q^{-1}}, F_j]_{q^3}\big]_q \\
&= qK_iK_i'[F_i, F_j]_{q^3} + q^3[3]K_i\big[F_i, K_i'F_j\big]_q \\
&= (q + q[3])[F_i, F_j]_{q^3}K_iK_i' \\
&= (q + q[3])[B_i^\sigma, F_j]_{q^3}K_iK_i',
\end{aligned}$$

where the last equality follows from Lemma 1.1. This proves (2.7). □

According to Table 3, Proposition 5.11 can be reformulated and proved as follows.

**Lemma 2.4.** *We have*

$$\begin{aligned}
\widetilde{\mathfrak{T}}_i^{-1}(F_j) = \frac{1}{[3]!}&\Big[B_i^\sigma, \big[B_i^\sigma, [B_i^\sigma, F_j]_{q^3}\big]_q\Big]_{q^{-1}} \\
&- \frac{1}{[3]!}\Big(q(1+[3])[B_i^\sigma, F_j]_{q^3} + q^3[3][B_i^\sigma, F_j]_{q^{-1}}\Big)\widetilde{k}_i.
\end{aligned} \tag{2.8}$$

*Proof.* By Proposition 4.2, we have $\widetilde{\mathfrak{T}}_i^{-1}(F_j) = \frac{1}{[3]!}\Big[F_i, \big[F_i, [F_i, F_j]_{q^3}\big]_q\Big]_{q^{-1}}$. By Lemma 1.1, we have $[B_i^\sigma, F_j]_{q^3} = [F_i, F_j]_{q^3}$. Then we have

$$\Big[B_i^\sigma, \big[B_i^\sigma, [B_i^\sigma, F_j]_{q^3}\big]_q\Big]_{q^{-1}} = \Big[B_i^\sigma, \big[F_i, [F_i, F_j]_{q^3}\big]_q\Big]_{q^{-1}} + \Big[B_i^\sigma, \big[K_iE_i, [F_i, F_j]_{q^3}\big]_q\Big]_{q^{-1}}. \tag{2.9}$$

Using Lemma 2.3, we rewrite RHS (2.9) as

$$\begin{aligned}
&\Big[F_i, \big[F_i, [F_i, F_j]_{q^3}\big]_q\Big]_{q^{-1}} + \Big[K_iE_i, \big[F_i, [F_i, F_j]_{q^3}\big]_q\Big]_{q^{-1}} + q^3[3][B_i^\sigma, F_j]_{q^{-1}}K_iK_i' \\
&= [3]!\widetilde{\mathfrak{T}}_i^{-1}(F_j) + q(1+[3])[B_i^\sigma, F_j]_{q^3}K_iK_i' + q^3[3][B_i^\sigma, F_j]_{q^{-1}}K_iK_i'.
\end{aligned} \tag{2.10}$$

Now the desired formula (2.8) follows from (2.9)-(2.10). $\square$

**Lemma 2.5.** *We have*

$$\widetilde{\mathcal{T}}_i^{-1}(E_jK_j') = \frac{1}{[3]!}\Big[B_i, \big[B_i, [B_i, E_jK_j']_{q^3}\big]_q\Big]_{q^{-1}} - \frac{1}{[3]!}\big(q(1+[3])[B_i, E_jK_j']_{q^3} - q^3[3][B_i, E_jK_j']_{q^{-1}}\big)\widetilde{k}_i. \tag{2.11}$$

*Proof.* In this case, $\mathcal{K}_i = \widetilde{k}_i$ and $\mathcal{K}_j = \widetilde{k}_j$ are central. By (1.4)–(1.5), we have

$$\mathcal{D}(F_j) = -q_j^{-2}E_jK_j'\widetilde{k}_j^{-1}, \qquad \mathcal{D}(B_i^\sigma) = -q^{-2}B_i\widetilde{k}_i^{-1}. \tag{2.12}$$

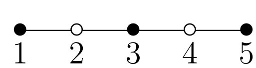
Recall from Lemma 1.3 that $\mathcal{D}$ commutes with $\widetilde{\mathcal{T}}_i$. Applying $\mathcal{D}$ to (2.8) and then using (2.12), we have

$$\begin{aligned}
&\widetilde{\mathcal{T}}_i^{-1}(E_jK_j')\widetilde{\mathcal{T}}_i^{-1}(\widetilde{k}_j^{-1}) \\
&= -q^{-6}\frac{1}{[3]!}\Big[B_i, \big[B_i, [B_i, E_jK_j']_{q^3}\big]_q\Big]_{q^{-1}}\widetilde{k}_j^{-1}\widetilde{k}_i^{-3} \\
&+ q^{-2}\frac{1}{[3]!}\big(q(1+[3])[B_i^\sigma, E_jK_j']_{q^3} - q^3[3][B_i^\sigma, E_jK_j']_{q^{-1}}\big)\mathcal{D}(\widetilde{k}_i)\widetilde{k}_j^{-1}\widetilde{k}_i^{-1}.
\end{aligned} \tag{2.13}$$

Since $s_i(\alpha_j) = \alpha_j + 3\alpha_i$, by Proposition 4.2, we have $\widetilde{\mathcal{T}}_i^{-1}(\widetilde{k}_j^{-1}) = -q^{-6}\widetilde{k}_j^{-1}\widetilde{k}_i^{-3}$. Note also that $\mathcal{D}(\widetilde{k}_i) = q^{-4}\widetilde{k}_i^{-1}$. Hence, (2.13) implies the desired formula (2.11). $\square$

# 3 Type AII

Consider the rank 2 Satake diagram of type $\mathrm{AII}_5$



$$\mathbf{r}_4 = s_4 s_3 s_5 s_4.$$

In this case, Proposition 5.11 is reformulated and proved as follows.

**Lemma 3.1.** *We have*

$$\widetilde{\mathcal{T}}_{\mathbf{r}_4}^{-1}(F_2) = [\widetilde{\mathcal{T}}_3(B_4^{\sigma}), F_2]_q,$$
$$\widetilde{\mathcal{T}}_{\mathbf{r}_4}^{-1}\big(\widetilde{\mathcal{T}}_{w_\bullet}(E_2)K_2'\big) = [\widetilde{\mathcal{T}}_3(B_4), \widetilde{\mathcal{T}}_{w_\bullet}(E_2)K_2']_q.$$

*Proof.* The first formula follows by $\widetilde{\mathcal{T}}_{\mathbf{r}_4}^{-1} = \widetilde{\mathcal{T}}_{4354}^{-1}$, Proposition 4.2, and the formula (1.1).

We prove the second formula. By (1.4)–(1.5), we have

$$\mathcal{D}(F_2) = -q^{-2}\widetilde{\mathcal{T}}_{w_\bullet}(E_2)K_2'\widetilde{\mathcal{T}}_{w_\bullet}(\mathcal{K}_2^{-1}), \qquad \mathcal{D}(B_4^{\sigma}) = -q^{-2}B_4\widetilde{\mathcal{T}}_{w_\bullet}(\mathcal{K}_4^{-1}). \tag{3.1}$$

Recall from Lemma 1.3 that the operator $\mathcal{D}$ in (1.3) commutes with $\widetilde{\mathcal{T}}_3, \widetilde{\mathcal{T}}_{\mathbf{r}_4}$. Applying the operator $\mathcal{D}$ to both sides of the first formula and then using (3.1), we have

$$\widetilde{\mathcal{T}}_{\mathbf{r}_4}^{-1}\big(\widetilde{\mathcal{T}}_{w_\bullet}(E_2)K_2'\big)\widetilde{\mathcal{T}}_{w_\bullet,4}(\mathcal{K}_2^{-1}) = -q^{-2}[\widetilde{\mathcal{T}}_3(B_4)\widetilde{\mathcal{T}}_5(\mathcal{K}_4^{-1}), \widetilde{\mathcal{T}}_{w_\bullet}(E_2)K_2'\widetilde{\mathcal{T}}_{w_\bullet}(\mathcal{K}_2^{-1})]_q. \tag{3.2}$$

For a weight reason, we have

$$\widetilde{\mathcal{T}}_5(\mathcal{K}_4^{-1})\widetilde{\mathcal{T}}_{w_\bullet}(E_2)K_2' = q\widetilde{\mathcal{T}}_{w_\bullet}(E_2)K_2'\widetilde{\mathcal{T}}_5(\mathcal{K}_4^{-1}),$$
$$\widetilde{\mathcal{T}}_{w_\bullet}(\mathcal{K}_2^{-1})\widetilde{\mathcal{T}}_3(B_4) = q\widetilde{\mathcal{T}}_3(B_4)\widetilde{\mathcal{T}}_{w_\bullet}(\mathcal{K}_2^{-1}).$$

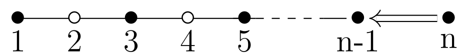
Using these two identities, we simplify (3.2) as

$$\widetilde{\mathcal{T}}_{\mathbf{r}_4}^{-1}\big(\widetilde{\mathcal{T}}_{w_\bullet}(E_2)K_2'\big)\widetilde{\mathcal{T}}_{w_\bullet,4}(\mathcal{K}_2^{-1}) = -q^{-1}[\widetilde{\mathcal{T}}_3(B_4), \widetilde{\mathcal{T}}_{w_\bullet}(E_2)K_2']_q\widetilde{\mathcal{T}}_5(\mathcal{K}_4^{-1})\widetilde{\mathcal{T}}_{w_\bullet}(\mathcal{K}_2^{-1}). \tag{3.3}$$

Finally, by Proposition 4.2, we have $\widetilde{\mathcal{T}}_{w_{\bullet,4}}(\mathcal{K}_2) = -q\widetilde{\mathcal{T}}_5(\mathcal{K}_4)\widetilde{\mathcal{T}}_{w_\bullet}(\mathcal{K}_2)$. Hence, (3.3) implies the second desired formula. ☐

# 4 Type CII$_n$, $n \geq 5$

Consider the rank 2 Satake diagram of type CII$_n$, for $n \geq 5$:




$$\varsigma_{2,\diamond} = -q_2^{-1}, \qquad \varsigma_{4,\diamond} = -q_4^{-1/2}$$

$$\mathbf{r}_4 = s_{4\cdots n\cdots 4}s_3s_{4\cdots n\cdots 4}.$$

Note that $q_2 = q_4 = q$. The notation $4\cdots n\cdots 4$ (with the local minima/maxima indicated) denotes a sequence $4\ 5\cdots n-1\ n\ n-1\cdots 5\ 4$, and we denote $s_{4\cdots n\cdots 4} = s_4\cdots s_n\cdots s_4$.

In this case, Proposition 5.11 is reformulated and proved as Lemmas 4.1–4.2 below.

**Lemma 4.1.** *We have*

$$\widetilde{\mathcal{T}}_{\mathbf{r}_4}^{-1}(F_2) = \big[[\widetilde{\mathcal{T}}_{5\cdots n\cdots 5}(B_4^\sigma), \widetilde{\mathcal{T}}_3(B_4^\sigma)]_q, F_2\big]_q - q\widetilde{\mathcal{T}}_3^{-2}(F_2)\widetilde{\mathcal{T}}_3(K_4')\widetilde{\mathcal{T}}_{5\cdots n\cdots 5}(K_4). \tag{4.1}$$

*Proof.* Since $s_{5\cdots n\cdots 5}s_4s_{5\cdots n\cdots 5}(\alpha_4) = \alpha_4$, we have $\widetilde{\mathcal{T}}_4^{-1}\widetilde{\mathcal{T}}_{5\cdots n\cdots 5}^{-1}(F_4) = \widetilde{\mathcal{T}}_{5\cdots n\cdots 5}(F_4)$. Then

$$\begin{aligned}\widetilde{\mathcal{T}}_{\mathbf{r}_4}^{-1}(F_2) = \widetilde{\mathcal{T}}_{4\cdots n\cdots 4}^{-1}[F_3, F_2]_q &= \big[\widetilde{\mathcal{T}}_4^{-1}\widetilde{\mathcal{T}}_{5\cdots n\cdots 5}^{-1}([F_4, F_3]_q), F_2\big]_q \\ &= \Big[\big[\widetilde{\mathcal{T}}_{5\cdots n\cdots 5}(F_4), [F_4, F_3]_q\big]_q, F_2\Big]_q.\end{aligned}$$

On the other hand, we compute RHS (4.1) as follows. First, note that

$$[K_4\widetilde{\mathcal{T}}_{w_\bullet}^{-1}(E_4), F_3]_q = q^{-1}\widetilde{\mathcal{T}}_{5\cdots n\cdots 5}^{-1}(E_4)K_3K_4,$$

and hence

$$\big[[K_4\widetilde{\mathcal{T}}_{w_\bullet}(E_4), F_3]_q, F_2\big]_q = [\widetilde{\mathcal{T}}_{5\cdots n\cdots 5}^{-1}(E_4), F_2]K_3K_4 = 0.$$

Thus, we have

$$\begin{aligned}
&\big[[\widetilde{\mathcal{T}}_{5\cdots n\cdots 5}(B_4^\sigma), \widetilde{\mathcal{T}}_3(B_4^\sigma)]_q, F_2\big]_q \\
&= \Big[\big[\widetilde{\mathcal{T}}_{5\cdots n\cdots 5}(B_4^\sigma), [B_4^\sigma, F_3]_q\big]_q, F_2\Big]_q \\
&= \Big[\big[\widetilde{\mathcal{T}}_{5\cdots n\cdots 5}(B_4^\sigma), [F_4, F_3]_q\big]_q, F_2\Big]_q \\
&= \Big[\big[\widetilde{\mathcal{T}}_{5\cdots n\cdots 5}(F_4), [F_4, F_3]_q\big]_q, F_2\Big]_q + \Big[\big[\widetilde{\mathcal{T}}_{5\cdots n\cdots 5}(K_4)\widetilde{\mathcal{T}}_3^{-1}(E_4), [F_4, F_3]_q\big]_q, F_2\Big]_q \\
&= \widetilde{\mathcal{T}}_{\mathbf{r}_4}^{-1}(F_2) + q\Big[\big[\widetilde{\mathcal{T}}_3^{-1}(E_4), [F_4, F_3]_q\big], F_2\Big]_q \widetilde{\mathcal{T}}_{5\cdots n\cdots 5}(K_4) \\
&= \widetilde{\mathcal{T}}_{\mathbf{r}_4}^{-1}(F_2) + q\Big[[\widetilde{\mathcal{T}}_3^{-1}(F_3), F_3], F_2\Big]_{q^2} \widetilde{\mathcal{T}}_3(K_4')\widetilde{\mathcal{T}}_{5\cdots n\cdots 5}(K_4) \\
&= \widetilde{\mathcal{T}}_{\mathbf{r}_4}^{-1}(F_2) + q\widetilde{\mathcal{T}}_3^{-2}(F_2)\widetilde{\mathcal{T}}_3(K_4')\widetilde{\mathcal{T}}_{5\cdots n\cdots 5}(K_4),
\end{aligned}$$

as desired. This proves the formula (4.1). □

**Lemma 4.2.** *We have*

$$\begin{aligned}
\widetilde{\mathcal{T}}_{\mathbf{r}_4}^{-1}(\widetilde{\mathcal{T}}_{w_\bullet}(E_2)K_2') = \big[[\widetilde{\mathcal{T}}_{5\cdots n\cdots 5}(B_4), \widetilde{\mathcal{T}}_3(B_4)]_q, \widetilde{\mathcal{T}}_{w_\bullet}(E_2)K_2'\big]_q \\
- q\widetilde{\mathcal{T}}_3^{-2}\big(\widetilde{\mathcal{T}}_{w_\bullet}(E_2)K_2'\big)\widetilde{\mathcal{T}}_3(K_4')\widetilde{\mathcal{T}}_{5\cdots n\cdots 5}(K_4). \qquad (4.2)
\end{aligned}$$

*Proof.* By Lemma 1.3, the operator $\mathcal{D}$ in (1.3) commutes with $\widetilde{\mathcal{T}}_3, \widetilde{\mathcal{T}}_{5\cdots n\cdots 5}, \widetilde{\mathcal{T}}_{\mathbf{r}_4}$. Applying $\mathcal{D}$ to (4.1) and then using (1.4)-(1.5), we obtain

$$\begin{aligned}
&\widetilde{\mathfrak{T}}_{\mathbf{r}_4}^{-1}(\widetilde{\mathfrak{T}}_{w_\bullet}(E_2)K_2'\widetilde{\mathfrak{T}}_{w_{\bullet,4}}(\mathcal{K}_2^{-1})\\
&= q^{-4}\big[[\widetilde{\mathfrak{T}}_{5\cdots n\cdots 5}(B_4)\widetilde{\mathfrak{T}}_3(\mathcal{K}_4^{-1}),\widetilde{\mathfrak{T}}_3(B_4)\widetilde{\mathfrak{T}}_{5\cdots n\cdots 5}(\mathcal{K}_4^{-1})]_q,\widetilde{\mathfrak{T}}_{w_\bullet}(E_2)K_2'\widetilde{\mathfrak{T}}_{w_\bullet}(\mathcal{K}_2^{-1})\big]_q\\
&- q\widetilde{\mathfrak{T}}_3^{-2}\big(\widetilde{\mathfrak{T}}_{w_\bullet}(E_2)K_2'\big)\widetilde{\mathfrak{T}}_{w_\bullet}(\mathcal{K}_2^{-1})\mathcal{D}(\widetilde{\mathfrak{T}}_3(K_4')\widetilde{\mathfrak{T}}_{5\cdots n\cdots 5}(K_4)).
\end{aligned} \tag{4.3}$$

Recalling $ck_i$ from (3.23), we have

$$\begin{aligned}
\mathcal{K}_4B_4 &= q^{-3}B_4\mathcal{K}_4,\\
\widetilde{\mathfrak{T}}_{w_\bullet}(\mathcal{K}_2)\widetilde{\mathfrak{T}}_{5\cdots n\cdots 5}(B_4)\widetilde{\mathfrak{T}}_3(B_4) &= \widetilde{\mathfrak{T}}_{5\cdots n\cdots 5}(B_4)\widetilde{\mathfrak{T}}_3(B_4)\widetilde{\mathfrak{T}}_{w_\bullet}(\mathcal{K}_2),\\
\mathcal{D}(\widetilde{\mathfrak{T}}_3(K_4')\widetilde{\mathfrak{T}}_{5\cdots n\cdots 5}(K_4)) &= q^{-1}\widetilde{\mathfrak{T}}_{5\cdots n\cdots 5}(\mathcal{K}_4^{-1})\widetilde{\mathfrak{T}}_3(\mathcal{K}_4^{-1})\widetilde{\mathfrak{T}}_3(K_4')\widetilde{\mathfrak{T}}_{5\cdots n\cdots 5}(K_4).
\end{aligned}$$

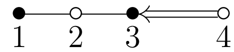
Using these formulas, we simplify (4.3) as

$$\begin{aligned}
&\widetilde{\mathfrak{T}}_{\mathbf{r}_4}^{-1}(\widetilde{\mathfrak{T}}_{w_\bullet}(E_2)K_2'\widetilde{\mathfrak{T}}_{w_{\bullet,4}}(\mathcal{K}_2^{-1})\\
&= q^{-1}\big[[\widetilde{\mathfrak{T}}_{5\cdots n\cdots 5}(B_4),\widetilde{\mathfrak{T}}_3(B_4)]_q,\widetilde{\mathfrak{T}}_{w_\bullet}(E_2)K_2'\big]_q\widetilde{\mathfrak{T}}_3(\mathcal{K}_4^{-1})\widetilde{\mathfrak{T}}_{5\cdots n\cdots 5}(\mathcal{K}_4^{-1})\widetilde{\mathfrak{T}}_{w_\bullet}(\mathcal{K}_2^{-1})\\
&- \widetilde{\mathfrak{T}}_3^{-2}\big(\widetilde{\mathfrak{T}}_{w_\bullet}(E_2)K_2'\big)\widetilde{\mathfrak{T}}_3(K_4')\widetilde{\mathfrak{T}}_{5\cdots n\cdots 5}(K_4)\widetilde{\mathfrak{T}}_{5\cdots n\cdots 5}(\mathcal{K}_4^{-1})\widetilde{\mathfrak{T}}_3(\mathcal{K}_4^{-1})\widetilde{\mathfrak{T}}_{w_\bullet}(\mathcal{K}_2^{-1}).
\end{aligned} \tag{4.4}$$

Finally, by (3.23) we have $\widetilde{\mathfrak{T}}_{w_{\bullet,4}}(\mathcal{K}_2) = q\widetilde{\mathfrak{T}}_3(\mathcal{K}_4)\widetilde{\mathfrak{T}}_{5\cdots n\cdots 5}(\mathcal{K}_4)\widetilde{\mathfrak{T}}_{w_\bullet}(\mathcal{K}_2)$. Therefore, the formula (4.2) follows from (4.4). □

# 5 Type CII$_4$

Consider the rank 2 Satake diagram of type CII$_4$:




$\mathbf{r}_4 = s_4 s_3 s_4, \qquad \mathbf{r}_2 = s_2 s_1 s_3 s_2.$

In this case, Proposition 5.11 is reformulated and proved as Lemmas 5.1–5.2 below.

**Lemma 5.1.** *We have*

$$\widetilde{\mathfrak{T}}_{\mathbf{r}_4}^{-1}(F_2) = \big[[B_4^\sigma, F_3]_{q_4}, F_2\big]_{q_3}, \tag{5.1}$$

$$\widetilde{\mathfrak{T}}_{\mathbf{r}_2}^{-1}(F_4) = \big[\widetilde{\mathfrak{T}}_3(B_2^\sigma), [\widetilde{\mathfrak{T}}_3(B_2^\sigma), F_4]_{q_3^2}\big] - (q_3 - q_3^{-1})[F_3, F_4]_{q_3^2} E_1 K_2 K_2' K_3. \tag{5.2}$$

*Proof.* The first formula (5.1) follows by a direct computation.

We prove (5.2). We have

$$\widetilde{\mathfrak{T}}_{\mathbf{r}_2}^{-1}(F_4) = \big[\widetilde{\mathfrak{T}}_2^{-1}(F_3), [\widetilde{\mathfrak{T}}_2^{-1}(F_3), F_4]_{q_3^2}\big] = \big[\widetilde{\mathfrak{T}}_3(F_2), [\widetilde{\mathfrak{T}}_3(F_2), F_4]_{q_3^2}\big].$$

Hence, recalling that $B_2^\sigma = F_2 + K_2 \widetilde{\mathfrak{T}}_{13}^{-1}(E_3)$, we have

$$\begin{aligned}
\big[\widetilde{\mathfrak{T}}_3(B_2^\sigma), [\widetilde{\mathfrak{T}}_3(B_2^\sigma), F_4]_{q_3^2}\big] &= \big[\widetilde{\mathfrak{T}}_3(B_2^\sigma), [\widetilde{\mathfrak{T}}_3(F_2), F_4]_{q_3^2}\big] \\
&= \big[\widetilde{\mathfrak{T}}_3(F_2), [\widetilde{\mathfrak{T}}_3(F_2), F_4]_{q_3^2}\big] + \big[K_2 K_3 \widetilde{\mathfrak{T}}_1^{-1}(E_2), [\widetilde{\mathfrak{T}}_3(F_2), F_4]_{q_3^2}\big] \\
&= \widetilde{\mathfrak{T}}_{\mathbf{r}_2}^{-1}(F_4) + \big[[\widetilde{\mathfrak{T}}_1^{-1}(E_2), \widetilde{\mathfrak{T}}_3(F_2)], F_4\big]_{q_3^2} K_2 K_3 \\
&= \widetilde{\mathfrak{T}}_{\mathbf{r}_2}^{-1}(F_4) + (q_3 - q_3^{-1})[F_3, F_4]_{q_3^2} E_1 K_2 K_2' K_3.
\end{aligned}$$

Thus, (5.2) is proved. $\square$

**Lemma 5.2.** *We have*

$$\widetilde{\mathfrak{T}}_{\mathbf{r}_4}^{-1}\big(\widetilde{\mathfrak{T}}_{w_\bullet}(E_2) K_2'\big) = \big[[B_4, F_3]_{q_4}, \widetilde{\mathfrak{T}}_{w_\bullet}(E_2) K_2'\big]_{q_3}, \tag{5.3}$$

$$\widetilde{\mathfrak{T}}_{\mathbf{r}_2}^{-1}\big(\widetilde{\mathfrak{T}}_{w_\bullet}(E_4) K_4'\big) = \big[\widetilde{\mathfrak{T}}_3(B_2), [\widetilde{\mathfrak{T}}_3(B_2), \widetilde{\mathfrak{T}}_{w_\bullet}(E_4) K_4']_{q_3^2}\big]$$

$$-(q_3-q_3^{-1})[F_3,\widetilde{\mathfrak{T}}_{w_\bullet}(E_4)K_4']_{q_3^2}E_1K_2K_2'K_3. \tag{5.4}$$

*Proof.* We shall prove the formula (5.4) only, and skip a similar proof for (5.3).

By Lemma 1.3, the operator $\mathcal{D}$ defined in (1.3) commutes with $\widetilde{\mathfrak{T}}_3, \widetilde{\mathfrak{T}}_{\mathbf{r}_2}$. Applying $\mathcal{D}$ to the identity (5.2) and then using (1.4)-(1.5), we have

$$\begin{aligned}
&\widetilde{\mathfrak{T}}_{\mathbf{r}_2}^{-1}\big(\widetilde{\mathfrak{T}}_{w_\bullet}(E_4)K_4'\big)\widetilde{\mathfrak{T}}_{w_{\bullet,2}}(\mathcal{K}_4^{-1})\\
&=q_2^{-4}\big[\widetilde{\mathfrak{T}}_3(B_2)\widetilde{\mathfrak{T}}_1(\mathcal{K}_2^{-1}),[\widetilde{\mathfrak{T}}_3(B_2)\widetilde{\mathfrak{T}}_1(\mathcal{K}_2^{-1}),\widetilde{\mathfrak{T}}_{w_\bullet}(E_4)K_4'\widetilde{\mathfrak{T}}_{w_\bullet}(\mathcal{K}_4^{-1})]_{q_3^2}\big]\\
&-(q_3-q_3^{-1})q_3^{-4}[F_3K_3K_3'^{-1},\widetilde{\mathfrak{T}}_{w_\bullet}(E_4)K_4'\widetilde{\mathfrak{T}}_{w_\bullet}(\mathcal{K}_4^{-1})]_{q_3^2}E_1K_1^{-1}K_1'\mathcal{D}(K_2K_2'K_3).
\end{aligned}\tag{5.5}$$

For a weight reason, we have

$$\begin{aligned}
\widetilde{\mathfrak{T}}_1(\mathcal{K}_2^{-1})\widetilde{\mathfrak{T}}_{w_\bullet}(E_4)&=q_3^2\widetilde{\mathfrak{T}}_{w_\bullet}(E_4)\widetilde{\mathfrak{T}}_1(\mathcal{K}_2^{-1}),\\
\widetilde{\mathfrak{T}}_{w_\bullet}(\mathcal{K}_4^{-1})\widetilde{\mathfrak{T}}_3(B_2)&=q_3^2\widetilde{\mathfrak{T}}_3(B_2)\widetilde{\mathfrak{T}}_{w_\bullet}(\mathcal{K}_4^{-1}),\\
\widetilde{\mathfrak{T}}_1(\mathcal{K}_2^{-1})\widetilde{\mathfrak{T}}_3(B_2)\widetilde{\mathfrak{T}}_{w_\bullet}(E_4)&=\widetilde{\mathfrak{T}}_3(B_2)\widetilde{\mathfrak{T}}_{w_\bullet}(E_4)\widetilde{\mathfrak{T}}_1(\mathcal{K}_2^{-1}),\\
\widetilde{\mathfrak{T}}_1(\mathcal{K}_2^{-1})\widetilde{\mathfrak{T}}_{w_\bullet}(\mathcal{K}_4^{-1})\widetilde{\mathfrak{T}}_3(B_2)&=\widetilde{\mathfrak{T}}_3(B_2)\widetilde{\mathfrak{T}}_1(\mathcal{K}_2^{-1})\widetilde{\mathfrak{T}}_{w_\bullet}(\mathcal{K}_4^{-1}).
\end{aligned}$$

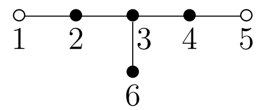
We also have $\mathcal{D}(K_2K_2'K_3)=q_2^{-2}\widetilde{\mathfrak{T}}_{w_\bullet}(K_2K_2')^{-1}K_3$. Hence, (5.5) is simplified as

$$\begin{aligned}
&\widetilde{\mathfrak{T}}_{\mathbf{r}_2}^{-1}\big(\widetilde{\mathfrak{T}}_{w_\bullet}(E_4)K_4'\big)\widetilde{\mathfrak{T}}_{w_{\bullet,2}}(\mathcal{K}_4^{-1})\\
&\quad=q_2^{-2}\big[\widetilde{\mathfrak{T}}_3(B_2),[\widetilde{\mathfrak{T}}_3(B_2),\widetilde{\mathfrak{T}}_{w_\bullet}(E_4)K_4']_{q_3^2}\big]\widetilde{\mathfrak{T}}_{w_\bullet}(\mathcal{K}_4^{-1})\widetilde{\mathfrak{T}}_1(\mathcal{K}_2^{-1})^2\\
&\quad-(q_3-q_3^{-1})q_2^{-2}[F_3,\widetilde{\mathfrak{T}}_{w_\bullet}(E_4)K_4']_{q_3^2}E_1K_2K_2'K_3\widetilde{\mathfrak{T}}_{w_\bullet}(\mathcal{K}_4^{-1})\widetilde{\mathfrak{T}}_1(\mathcal{K}_2^{-1})^2.
\end{aligned}\tag{5.6}$$

By the definition of $\mathcal{K}_i$ in (3.23), we have $\widetilde{\mathfrak{T}}_{w_{\bullet,2}}(\mathcal{K}_4^{-1})=q_2^{-2}\widetilde{\mathfrak{T}}_{w_\bullet}(\mathcal{K}_4^{-1})\widetilde{\mathfrak{T}}_1(\mathcal{K}_2^{-1})^2$. Thus, (5.6) implies the desired formula (5.4). □

# 6 Type EIV

Consider the rank 2 Satake diagram of type EIV:




$$\mathbf{r}_1 = s_1 s_2 s_3 s_4 s_6 s_3 s_2 s_1.$$

In this case, Proposition 5.11 is reformulated and proved as Lemma 6.1 below.

**Lemma 6.1.**

$$\widetilde{\mathcal{T}}_{\mathbf{r}_1}^{-1}(F_5) = \big[\widetilde{\mathcal{T}}_4\widetilde{\mathcal{T}}_3\widetilde{\mathcal{T}}_2(B_1^\sigma), F_5\big]_q, \tag{6.1}$$

$$\widetilde{\mathcal{T}}_{\mathbf{r}_1}^{-1}\big(\widetilde{\mathcal{T}}_{w_\bullet}(E_5)K_5'\big) = \big[\widetilde{\mathcal{T}}_4\widetilde{\mathcal{T}}_3\widetilde{\mathcal{T}}_2(B_1), \widetilde{\mathcal{T}}_{w_\bullet}(E_5)K_5'\big]_q. \tag{6.2}$$

*Proof.* We prove the formula (6.1). Indeed, we have

$$\begin{aligned}\widetilde{\mathcal{T}}_{\mathbf{r}_1}^{-1}(F_5) = \widetilde{\mathcal{T}}_1^{-1}\widetilde{\mathcal{T}}_2^{-1}\widetilde{\mathcal{T}}_3^{-1}[F_4, F_5]_q &= [\widetilde{\mathcal{T}}_1^{-1}\widetilde{\mathcal{T}}_2^{-1}\widetilde{\mathcal{T}}_3^{-1}(F_4), F_5]_q \\ &= [\widetilde{\mathcal{T}}_4\widetilde{\mathcal{T}}_3\widetilde{\mathcal{T}}_2(F_1), F_5]_q = [\widetilde{\mathcal{T}}_4\widetilde{\mathcal{T}}_3\widetilde{\mathcal{T}}_2(B_1^\sigma), F_5]_q.\end{aligned}$$

We next prove the formula (6.2). Recall from Lemma 1.3 that $\widetilde{\mathcal{T}}_j$, for $j \in \mathbb{I}_\bullet$, commutes with $\mathcal{D}$ in (1.3). Applying $\mathcal{D}$ to the formula (6.1) and then using (1.4)-(1.5), we have

$$\begin{aligned}&\widetilde{\mathcal{T}}_{\mathbf{r}_1}^{-1}\big(\widetilde{\mathcal{T}}_{w_\bullet}(E_5)K_5'\big)\widetilde{\mathcal{T}}_{w_{\bullet,1}}(\mathcal{K}_5^{-1}) \\ &= -q^{-2}\big[\widetilde{\mathcal{T}}_{432}(B_1)\widetilde{\mathcal{T}}_{632}(\mathcal{K}_1^{-1}), \widetilde{\mathcal{T}}_{w_\bullet}(E_5)K_5'\widetilde{\mathcal{T}}_{w_\bullet}(\mathcal{K}_5^{-1})\big]_q.\end{aligned} \tag{6.3}$$

By a weight consideration, we have

$$\widetilde{\mathcal{T}}_{w_\bullet}(\mathcal{K}_5^{-1})\widetilde{\mathcal{T}}_{432}(B_1) = q\widetilde{\mathcal{T}}_{432}(B_1)\widetilde{\mathcal{T}}_{w_\bullet}(\mathcal{K}_5^{-1}),$$
$$\widetilde{\mathcal{T}}_{632}(\mathcal{K}_1^{-1})\widetilde{\mathcal{T}}_{w_\bullet}(E_5) = q\widetilde{\mathcal{T}}_{w_\bullet}(E_5)\widetilde{\mathcal{T}}_{632}(\mathcal{K}_1^{-1}).$$

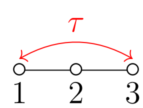
Hence, using these two identities, (6.3) is simplified as

$$\widetilde{\mathcal{T}}_{\mathbf{r}_1}^{-1}\big(\widetilde{\mathcal{T}}_{w_\bullet}(E_5)K_5'\big)\widetilde{\mathcal{T}}_{w_{\bullet,1}}(\mathcal{K}_5^{-1})$$
$$= -q^{-1}\big[\widetilde{\mathcal{T}}_{432}(B_1), \widetilde{\mathcal{T}}_{w_\bullet}(E_5)K_5'\big]_q\widetilde{\mathcal{T}}_{632}(\mathcal{K}_1^{-1})\widetilde{\mathcal{T}}_{w_\bullet}(\mathcal{K}_5^{-1}). \quad (6.4)$$

Finally, by the definition (3.23) of $\mathcal{K}_i$, $\widetilde{\mathcal{T}}_{w_{\bullet,1}}(\mathcal{K}_5^{-1}) = -q^{-1}\widetilde{\mathcal{T}}_{632}(\mathcal{K}_1^{-1})\widetilde{\mathcal{T}}_{w_\bullet}(\mathcal{K}_5^{-1})$. Then (6.4) implies the desired formula (6.2). □

# 7 Type AIII$_3$

Consider the rank 2 Satake diagram of type AIII$_3$:




$$\varsigma_{1,\diamond} = \varsigma_{3,\diamond} = -q^{-1}, \qquad \varsigma_{2,\diamond} = -q^{-2}$$

$$\mathbf{r}_1 = s_1s_3, \qquad \mathbf{r}_2 = s_2.$$

In this case, Proposition 5.11 is reformulated and proved as the following lemma.

**Lemma 7.1.** *We have*

$$\widetilde{\mathcal{T}}_{\mathbf{r}_1}^{-1}(F_2) = \big[B_3^\sigma, [B_1^\sigma, F_2]_q\big]_q - qF_2K_3K_1', \quad (7.1)$$
$$\widetilde{\mathcal{T}}_{\mathbf{r}_1}^{-1}(E_2K_2') = \big[B_3, [B_1, E_2K_2']_q\big]_q - qE_2K_2'K_3K_1'. \quad (7.2)$$

*Proof.* By Lemma 1.1, we have $[B_1^\sigma, F_2]_q = [F_1, F_2]_q$. Then the first term on the RHS of (7.1) is computed as follows:

$$\begin{aligned}\big[B_3^\sigma, [B_1^\sigma, F_2]_q\big]_q &= \big[K_3 E_1, [F_1, F_2]_q\big]_q + \big[F_3, [F_1, F_2]_q\big]_q \\ &= q\big[[E_1, F_1], F_2\big]_q K_3 + \big[F_3, [F_1, F_2]_q\big]_q \\ &= q[\frac{K_1 - K_1'}{q - q^{-1}}, F_2]_q K_3 + \big[F_3, [F_1, F_2]_q\big]_q \\ &= q F_2 K_3 K_1' + \big[F_3, [F_1, F_2]_q\big]_q \\ &= \widetilde{\mathcal{T}}_{13}^{-1}(F_2) + q F_2 K_3 K_1'.\end{aligned}$$

This proves the formula (7.1).

We next prove (7.2). In this case, $\tau_0 = \tau \neq \mathrm{Id}$, $\tau_{\bullet,1} = \mathrm{Id}$, and we simplify $\mathcal{D}$ in (1.3) as $\mathcal{D} = \widetilde{\mathcal{T}}_{w_0}$. We also have $\mathcal{K}_i = \widetilde{k}_i$ for $i = 1, 2, 3$. Applying the operator $\mathcal{D} = \widetilde{\mathcal{T}}_{w_0}$ to the identity (7.1) and then using (1.4)-(1.5), we have

$$\widetilde{\mathcal{T}}_{\mathbf{r}_1}^{-1}(E_2 K_2')\widetilde{\mathcal{T}}_{\mathbf{r}_1}(\widetilde{k}_2^{-1}) = q^{-4}\big[B_3 \widetilde{k}_1^{-1}, [B_1 \widetilde{k}_3^{-1}, E_2 K_2' \widetilde{k}_2^{-1}]_q\big]_q - q E_2 K_2' \widetilde{k}_2^{-1} \mathcal{D}(K_3 K_1'). \quad (7.3)$$

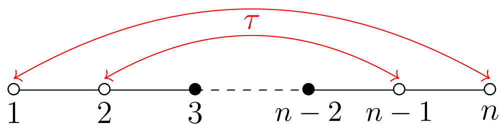
We have $\mathcal{D}(K_3 K_1') = q^{-2} \widetilde{k}_1^{-1} \widetilde{k}_3^{-1} K_3 K_1'$. Note also that $\widetilde{k}_2$ is central and $\widetilde{k}_3, \widetilde{k}_1$ commute with $E_2$. Hence, (7.3) can be rewritten as

$$\begin{aligned}\widetilde{\mathcal{T}}_{\mathbf{r}_1}^{-1}(E_2 K_2')\widetilde{\mathcal{T}}_{\mathbf{r}_1}(\widetilde{k}_2^{-1}) &= q^{-2}\big[B_3, [B_1, E_2 K_2']_q\big]_q \widetilde{k}_1^{-1} \widetilde{k}_3^{-1} \widetilde{k}_2^{-1} \\ &\quad - q^{-1} E_2 K_2' \widetilde{k}_2^{-1} \widetilde{k}_1^{-1} \widetilde{k}_3^{-1} K_3 K_1'. \quad (7.4)\end{aligned}$$

Finally, since $\mathbf{r}_1(\alpha_2) = \alpha_2 + \alpha_1 + \alpha_3$, we have $\widetilde{\mathcal{T}}_{\mathbf{r}_1}(\widetilde{k}_2^{-1}) = q^{-2} \widetilde{k}_2^{-1} \widetilde{k}_1^{-1} \widetilde{k}_3^{-1}$. Therefore the desired formula (7.2) follows from (7.4). □

# 8 Type AIII$_n, n \geq 4$

Consider the rank 2 Satake diagram of type AIII$_n, n \geq 4$:




$$\varsigma_{1,\diamond} = \varsigma_{n,\diamond} = -q^{-1}, \qquad \varsigma_{2,\diamond} = \varsigma_{n-1,\diamond} = -q^{-1/2},$$

$$\mathbf{r}_1 = s_1 s_n, \qquad \mathbf{r}_2 = s_2 \cdots s_{n-1} \cdots s_2.$$

We first have a simple observation.

**Lemma 8.1.** *For any $3 \leq s \leq n-2$, $\widetilde{\mathcal{T}}_{2\cdots n-2}(F_{n-1})$ is fixed by $\widetilde{\mathcal{T}}_s$.*

*Proof.* Recall from Proposition 4.2 that $\widetilde{\mathcal{T}}_s$ satisfies the braid relation. Then we have

$$\begin{aligned}
\widetilde{\mathcal{T}}_s \widetilde{\mathcal{T}}_{2\cdots n-2}(F_{n-1}) &= \widetilde{\mathcal{T}}_{2\cdots s-2} \widetilde{\mathcal{T}}_s \widetilde{\mathcal{T}}_{s-1} \widetilde{\mathcal{T}}_s \widetilde{\mathcal{T}}_{s+1\cdots n-2}(F_{n-1}) \\
&= \widetilde{\mathcal{T}}_{2\cdots s-2} \widetilde{\mathcal{T}}_{s-1} \widetilde{\mathcal{T}}_s \widetilde{\mathcal{T}}_{s-1} \widetilde{\mathcal{T}}_{s+1\cdots n-2}(F_{n-1}) \\
&= \widetilde{\mathcal{T}}_{2\cdots s-2} \widetilde{\mathcal{T}}_{s-1} \widetilde{\mathcal{T}}_s \widetilde{\mathcal{T}}_{s+1\cdots n-2} \widetilde{\mathcal{T}}_{s-1}(F_{n-1}) \\
&= \widetilde{\mathcal{T}}_{2\cdots s-2} \widetilde{\mathcal{T}}_{s-1} \widetilde{\mathcal{T}}_s \widetilde{\mathcal{T}}_{s+1\cdots n-2}(F_{n-1}) = \widetilde{\mathcal{T}}_{2\cdots n-2}(F_{n-1}).
\end{aligned}$$

Hence, $\widetilde{\mathcal{T}}_{2\cdots n-2}(F_{n-1})$ is fixed by $\widetilde{\mathcal{T}}_s$ for $3 \leq s \leq n-2$. □

In this case, Proposition 5.11 is reformulated and proved as Lemmas 8.2–8.3 below.

**Lemma 8.2.** *We have*

$$\widetilde{\mathcal{T}}_{\mathbf{r}_1}^{-1}(F_2) = [B_1^\sigma, F_2]_q, \tag{8.1}$$

$$\widetilde{\mathcal{T}}_{\mathbf{r}_2}^{-1}(F_1) = \big[\widetilde{\mathcal{T}}_{w_\bullet}(B_{n-1}^\sigma), [B_2^\sigma, F_1]_q\big]_q - F_1 K_2' K_{w_\bullet(\alpha_{n-1})}. \tag{8.2}$$

*Proof.* The formula (8.1) follows from Lemma 1.1.

We prove (8.2). By a direct computation, we have

$$\begin{aligned}\widetilde{\mathcal{T}}_{\mathbf{r}_2}^{-1}(F_1) &= \big[\widetilde{\mathcal{T}}_{2\cdots n-1\cdots 3}^{-1}(F_2), [F_2, F_1]_q\big]_q \\ &= \big[\widetilde{\mathcal{T}}_{2\cdots n-2}^{-1}\widetilde{\mathcal{T}}_{2\cdots n-2}(F_{n-1}), [F_2, F_1]_q\big]_q \\ &= \big[\widetilde{\mathcal{T}}_{3\cdots n-2}(F_{n-1}), [F_2, F_1]_q\big]_q \\ &= \big[\widetilde{\mathcal{T}}_{w_\bullet}(F_{n-1}), [F_2, F_1]_q\big]_q,\end{aligned}$$

where the last equality follows by applying Lemma 8.1 and noting that $w_\bullet(\alpha_{n-1}) = s_{3\cdots n-2}(\alpha_{n-1})$. Recalling that $B_{n-1}^\sigma = F_{n-1} + K_{n-1}\widetilde{\mathcal{T}}_{w_\bullet}^{-1}(E_2)$, we compute the RHS of (8.2) as follows:

$$\begin{aligned}\big[\widetilde{\mathcal{T}}_{w_\bullet}(B_{n-1}^\sigma), [B_2^\sigma, F_1]_q\big]_q &= \big[\widetilde{\mathcal{T}}_{w_\bullet}(B_{n-1}^\sigma), [F_2, F_1]_q\big]_q \\ &= \big[\widetilde{\mathcal{T}}_{w_\bullet}(F_{n-1}), [F_2, F_1]_q\big]_q + \big[\widetilde{\mathcal{T}}_{w_\bullet}(K_{n-1})E_2, [F_2, F_1]_q\big]_q \\ &= \widetilde{\mathcal{T}}_{\mathbf{r}_2}^{-1}(F_1) + \big[E_2, [F_2, F_1]_q\big]\widetilde{\mathcal{T}}_{w_\bullet}(K_{n-1}) \\ &= \widetilde{\mathcal{T}}_{\mathbf{r}_2}^{-1}(F_1) + F_1K_2'K_{w_\bullet(\alpha_{n-1})}.\end{aligned}$$

This proves the formula (8.2). □

**Lemma 8.3.** *We have*

$$\widetilde{\mathcal{T}}_{\mathbf{r}_1}^{-1}\big(\widetilde{\mathcal{T}}_{w_\bullet}(E_{n-1})K_2'\big) = [B_1, \widetilde{\mathcal{T}}_{w_\bullet}(E_{n-1})K_2']_q, \tag{8.3}$$

$$\widetilde{\mathcal{T}}_{\mathbf{r}_2}^{-1}(E_nK_1') = \big[\widetilde{\mathcal{T}}_{w_\bullet}(B_{n-1}), [B_2, E_nK_1']_q\big]_q - E_nK_1'K_2'K_{w_\bullet(\alpha_{n-1})}. \tag{8.4}$$

*Proof.* Note that $\widetilde{\mathfrak{T}}_{\mathbf{r}_1} = \widetilde{\mathfrak{T}}_1 \widetilde{\mathfrak{T}}_n$ commutes with $\widetilde{\mathfrak{T}}_{w_\bullet}$. Hence, we have

$$\begin{aligned}
\widetilde{\mathfrak{T}}_{\mathbf{r}_1}^{-1}\big(\widetilde{\mathfrak{T}}_{w_\bullet}(E_{n-1})K_2'\big) &= \varsigma_{1,\diamond}^{-1/2}\widetilde{\mathfrak{T}}_{w_\bullet}\widetilde{\mathfrak{T}}_{\mathbf{r}_1}^{-1}(E_{n-1})K_2'K_1' \\
&= \varsigma_{1,\diamond}^{-1}\widetilde{\mathfrak{T}}_{w_\bullet}\big([E_{n-1}E_n]_{q^{-1}}\big)K_2'K_1' \\
&= \widetilde{\mathfrak{T}}_{w_\bullet}[E_n, E_{n-1}]_q K_2'K_1' \\
&= [B_1, \widetilde{\mathfrak{T}}_{w_\bullet}(E_{n-1})K_2']_q
\end{aligned}$$

where the last step follows from Lemma 1.1. Hence, we have proved (8.3).

We next prove (8.4). In this case, $\tau_0 = \tau$, $\widetilde{\mathfrak{T}}_{w_\bullet}(\mathcal{K}_n) = \mathcal{K}_n = \widetilde{k}_n$, and we simplify $\mathcal{D}$ in (1.3) as $\mathcal{D} = \widetilde{\mathfrak{T}}_{w_\bullet}\widetilde{\mathfrak{T}}_{w_0}$. Applying $\mathcal{D}$ to (8.2) and then using (1.4)-(1.5), we have

$$\begin{aligned}
&\widetilde{\mathfrak{T}}_{\mathbf{r}_2}^{-1}(E_nK_1')\widetilde{\mathfrak{T}}_{w_{\bullet,2}}(\widetilde{k}_n^{-1}) \\
&\quad = q^{-4}\Big[\widetilde{\mathfrak{T}}_{w_\bullet}(B_{n-1})\mathcal{K}_2^{-1}, [B_2\widetilde{\mathfrak{T}}_{w_\bullet}(\mathcal{K}_{n-1}^{-1}), E_nK_1'\widetilde{k}_n^{-1}]_q\Big]_q \\
&\qquad - E_nK_1'\widetilde{k}_n^{-1}\mathcal{D}(K_2'K_{w_\bullet(\alpha_{n-1})}).
\end{aligned} \tag{8.5}$$

For a weight reason, we have

$$\begin{aligned}
\widetilde{\mathfrak{T}}_{w_\bullet}(\mathcal{K}_{n-1}^{-1})E_n &= qE_n\widetilde{\mathfrak{T}}_{w_\bullet}(\widetilde{k}_{n-1}^{-1}), \\
\widetilde{k}_n^{-1}B_2 &= qB_2\widetilde{k}_n^{-1}, \\
\mathcal{K}_2^{-1}B_2E_n &= q^2B_2E_n\mathcal{K}_2^{-1}, \\
\widetilde{k}_n^{-1}\widetilde{\mathfrak{T}}_{w_\bullet}(\mathcal{K}_{n-1}^{-1})\widetilde{\mathfrak{T}}_{w_\bullet}(B_{n-1}) &= q^2\widetilde{\mathfrak{T}}_{w_\bullet}(B_{n-1})\widetilde{k}_n^{-1}\widetilde{\mathfrak{T}}_{w_\bullet}(\mathcal{K}_{n-1}^{-1}).
\end{aligned}$$

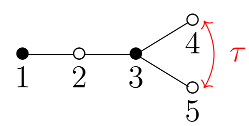
In addition, by (3.23), we have $\mathcal{D}(K_2'K_{w_\bullet\alpha_{n-1}}) = q^{-1}\widetilde{\mathfrak{T}}_{w_\bullet}(\mathcal{K}_{n-1}^{-1})\mathcal{K}_2^{-1}K_2'K_{w_\bullet\alpha_{n-1}}$. Using these formulas, we rewrite (8.5) as

$$\widetilde{\mathcal{T}}_{\mathbf{r}_2}^{-1}(E_nK_1')\widetilde{\mathcal{T}}_{w_\bullet,2}(\widetilde{k}_n^{-1}) = q^{-1}\Big[\widetilde{\mathcal{T}}_{w_\bullet}(B_{n-1}), [B_2, E_nK_1']_q\Big]_q \widetilde{k}_n^{-1}\widetilde{\mathcal{T}}_{w_\bullet}(\mathcal{K}_{n-1}^{-1})\mathcal{K}_2^{-1}$$
$$- q^{-1}E_nK_1'\widetilde{k}_n^{-1}\widetilde{\mathcal{T}}_{w_\bullet}(\mathcal{K}_{n-1}^{-1})\mathcal{K}_2^{-1}K_2'K_{w_\bullet\alpha_{n-1}}. \quad (8.6)$$

Finally, we have $\widetilde{\mathcal{T}}_{w_\bullet,2}(\widetilde{k}_n^{-1}) = q^{-1}\widetilde{k}_n^{-1}\widetilde{\mathcal{T}}_{w_\bullet}(\mathcal{K}_{n-1}^{-1})\mathcal{K}_2^{-1}$. Then the formula (8.4) follows from (8.6). □

# 9 Type DIII$_5$

Consider the rank 2 Satake diagram of type DIII$_5$:




$$\varsigma_{2,\diamond} = -q^{-1}, \qquad \varsigma_{4,\diamond} = \varsigma_{5,\diamond} = -q^{-1/2},$$

$$\mathbf{r}_2 = s_2s_1s_3s_2, \qquad \mathbf{r}_4 = s_4s_5s_3s_4s_5.$$

In this case, Proposition 5.11 is reformulated and proved as Lemmas 9.1–9.2 below.

**Lemma 9.1.** *We have*

$$\widetilde{\mathcal{T}}_{\mathbf{r}_2}^{-1}(F_4) = [\widetilde{\mathcal{T}}_3(B_2^\sigma), F_4]_q, \quad (9.1)$$

$$\widetilde{\mathcal{T}}_{\mathbf{r}_4}^{-1}(F_2) = \big[B_4^\sigma, [\widetilde{\mathcal{T}}_3(B_5^\sigma), F_2]_q\big]_q - \widetilde{\mathcal{T}}_3^{-2}(F_2)K_4K_5'K_3'. \quad (9.2)$$

*Proof.* The proof for (9.1) is similar to that of Lemma 3.1, and thus omitted.

We prove (9.2). By a direct computation, we have

$$\widetilde{\mathcal{T}}_{\mathbf{r}_4}^{-1}(F_2) = \Big[\big[F_4, [F_5, F_3]_q\big]_q, F_2\Big]_q = \big[F_4, [\widetilde{\mathcal{T}}_3(F_5), F_2]_q\big]_q.$$

Note that $B_5^\sigma = F_5 + K_5\widetilde{\mathfrak{T}}_3^{-1}(E_4)$. Since $[\widetilde{\mathfrak{T}}_3(K_5)E_4, F_2]_q = q[E_4, F_2]K_3K_5 = 0$, we have $[\widetilde{\mathfrak{T}}_3(B_5^\sigma), F_2]_q = [\widetilde{\mathfrak{T}}_3(F_5), F_2]_q$. We now compute the first term of RHS (9.2) as

$$\begin{aligned}
\big[B_4^\sigma, [\widetilde{\mathfrak{T}}_3(B_5^\sigma), F_2]_q\big]_q &= \big[B_4^\sigma, [\widetilde{\mathfrak{T}}_3(F_5), F_2]_q\big]_q \\
&= \big[F_4, [\widetilde{\mathfrak{T}}_3(F_5), F_2]_q\big]_q + \big[K_4\widetilde{\mathfrak{T}}_3^{-1}(E_4), [\widetilde{\mathfrak{T}}_3(F_5), F_2]_q\big]_q \\
&= \widetilde{\mathfrak{T}}_{\mathbf{r}_4}^{-1}(F_2) + K_4\big[\widetilde{\mathfrak{T}}_3^{-1}(E_4), [\widetilde{\mathfrak{T}}_3(F_5), F_2]_q\big] \\
&= \widetilde{\mathfrak{T}}_{\mathbf{r}_4}^{-1}(F_2) - q^{-1}\big[[E_3, F_3]_{q^2}, F_2\big]_q K_4K_5' \\
&= \widetilde{\mathfrak{T}}_{\mathbf{r}_4}^{-1}(F_2) + \widetilde{\mathfrak{T}}_3^{-2}(F_2)K_4K_5'K_3'.
\end{aligned}$$

This proves (9.2). ☐

**Lemma 9.2.** *We have*

$$\widetilde{\mathfrak{T}}_{\mathbf{r}_2}^{-1}(\widetilde{\mathfrak{T}}_{w_\bullet}(E_5)K_4') = [\widetilde{\mathfrak{T}}_3(B_2), \widetilde{\mathfrak{T}}_{w_\bullet}(E_5)K_4']_q, \tag{9.3}$$

$$\widetilde{\mathfrak{T}}_{\mathbf{r}_4}^{-1}(\widetilde{\mathfrak{T}}_{w_\bullet}(E_2)K_2') = \big[B_4, [\widetilde{\mathfrak{T}}_3(B_5), \widetilde{\mathfrak{T}}_{w_\bullet}(E_2)K_2']_q\big]_q - \widetilde{\mathfrak{T}}_3^{-2}(\widetilde{\mathfrak{T}}_{w_\bullet}(E_2)K_2')K_4K_5'K_3'. \tag{9.4}$$

*Proof.* We prove (9.4). The proof for (9.3) is easier and hence omitted.

By Lemma 1.3, the operator $\mathcal{D}$ defined in (1.3) commutes with $\widetilde{\mathfrak{T}}_3, \widetilde{\mathfrak{T}}_{\mathbf{r}_4}$. Applying $\mathcal{D}$ to (9.2) and using (1.4)-(1.5), we have

$$\begin{aligned}
&\widetilde{\mathfrak{T}}_{\mathbf{r}_4}^{-1}(\widetilde{\mathfrak{T}}_{w_\bullet}(E_2)K_2')\widetilde{\mathfrak{T}}_{w_{\bullet,4}}(\mathcal{K}_2^{-1}) \\
&\quad = q^{-4}\big[B_4\widetilde{\mathfrak{T}}_3(\mathcal{K}_5^{-1}), [\widetilde{\mathfrak{T}}_3(B_5)\mathcal{K}_4^{-1}, \widetilde{\mathfrak{T}}_{w_\bullet}(E_2)K_2'\widetilde{\mathfrak{T}}_{w_\bullet}(\mathcal{K}_2^{-1})]_q\big]_q \\
&\qquad - \widetilde{\mathfrak{T}}_3^{-2}(\widetilde{\mathfrak{T}}_{w_\bullet}(E_2)K_2')\widetilde{\mathfrak{T}}_{w_\bullet}(\mathcal{K}_2^{-1})\mathcal{D}(K_4K_5'K_3').
\end{aligned} \tag{9.5}$$

For a weight reason, we have

$$\mathcal{K}_4^{-1}\widetilde{\mathfrak{T}}_{w_\bullet}(E_2) = q\widetilde{\mathfrak{T}}_{w_\bullet}(E_2)\mathcal{K}_4^{-1},$$
$$\widetilde{\mathfrak{T}}_{w_\bullet}(\mathcal{K}_2^{-1})\widetilde{\mathfrak{T}}_3(B_5) = q\widetilde{\mathfrak{T}}_3(B_5)\widetilde{\mathfrak{T}}_{w_\bullet}(\mathcal{K}_2^{-1}),$$
$$\widetilde{\mathfrak{T}}_3(\mathcal{K}_5^{-1})\widetilde{\mathfrak{T}}_3(B_5)\widetilde{\mathfrak{T}}_{w_\bullet}(E_2) = q^2\widetilde{\mathfrak{T}}_3(B_5)\widetilde{\mathfrak{T}}_{w_\bullet}(E_2)\widetilde{\mathfrak{T}}_3(\mathcal{K}_5^{-1}),$$
$$\mathcal{K}_4^{-1}\widetilde{\mathfrak{T}}_{w_\bullet}(\mathcal{K}_2^{-1})B_4 = q^2 B_4\mathcal{K}_4^{-1}\widetilde{\mathfrak{T}}_{w_\bullet}(\mathcal{K}_2^{-1}).$$

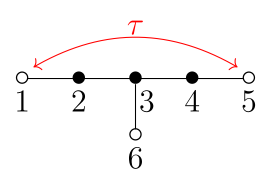
We also have $\mathcal{D}(K_4K_5'K_3') = q^{-1}\widetilde{\mathfrak{T}}_3(\mathcal{K}_5^{-1})\mathcal{K}_4^{-1}K_4K_5'K_3'$. Hence (9.5) is written as

$$\begin{aligned}
&\widetilde{\mathfrak{T}}_{\mathbf{r}_4}^{-1}(\widetilde{\mathfrak{T}}_{w_\bullet}(E_2)K_2')\widetilde{\mathfrak{T}}_{w_{\bullet,4}}(\mathcal{K}_2^{-1})\\
&\quad = q^{-1}\big[B_4, [\widetilde{\mathfrak{T}}_3(B_5), \widetilde{\mathfrak{T}}_{w_\bullet}(E_2)K_2']_q\big]_q\widetilde{\mathfrak{T}}_{w_\bullet}(\mathcal{K}_2^{-1})\widetilde{\mathfrak{T}}_3(\mathcal{K}_5^{-1})\mathcal{K}_4^{-1}\\
&\quad - q^{-1}\widetilde{\mathfrak{T}}_3^{-2}(\widetilde{\mathfrak{T}}_{w_\bullet}(E_2)K_2')K_4K_5'K_3'\widetilde{\mathfrak{T}}_{w_\bullet}(\mathcal{K}_2^{-1})\widetilde{\mathfrak{T}}_3(\mathcal{K}_5^{-1})\mathcal{K}_4^{-1}.
\end{aligned} \tag{9.6}$$

Finally, by definition of $\mathcal{K}_i$ (3.23), we have $\widetilde{\mathfrak{T}}_{w_{\bullet,4}}(\mathcal{K}_2^{-1}) = q^{-1}\widetilde{\mathfrak{T}}_{w_\bullet}(\mathcal{K}_2^{-1})\widetilde{\mathfrak{T}}_3(\mathcal{K}_5^{-1})\mathcal{K}_4^{-1}$. Thus, (9.6) implies (9.4). □

# 10 Type EIII

Consider the rank 2 Satake diagram of type EIII:




$$\varsigma_{1,\diamond} = \varsigma_{5,\diamond} = -q^{-1/2}, \qquad \varsigma_{6,\diamond} = -q^{-1}$$
$$\mathbf{r}_1 = s_1\cdots s_5\cdots s_1, \qquad \mathbf{r}_6 = s_6s_3s_2s_4s_3s_6$$
$$w_\bullet = s_3s_2s_4s_3s_2s_4 = s_2s_4s_3s_2s_4s_3.$$

In this case, Proposition 5.11 is reformulated and proved as Lemmas 10.1–10.2 below.

**Lemma 10.1.** *We have*

$$\widetilde{\mathfrak{T}}_{\mathbf{r}_6}^{-1}(F_1) = [\widetilde{\mathfrak{T}}_{23}(B_6^\sigma), F_1]_q, \tag{10.1}$$

$$\widetilde{\mathfrak{T}}_{\mathbf{r}_1}^{-1}(F_6) = \big[\widetilde{\mathfrak{T}}_4(B_5^\sigma), [\widetilde{\mathfrak{T}}_{32}(B_1^\sigma), F_6]_q\big]_q - \widetilde{\mathfrak{T}}_{32323}^{-1}(F_6) K_1' K_2' K_3' K_4 K_5 \tag{10.2}$$

*Proof.* We have

$$\widetilde{\mathfrak{T}}_{\mathbf{r}_6}^{-1}(F_1) = \widetilde{\mathfrak{T}}_{632}^{-1}(F_1) = [\widetilde{\mathfrak{T}}_{63}^{-1}(F_2), F_1]_q = [\widetilde{\mathfrak{T}}_{23}(F_6), F_1]_q = [\widetilde{\mathfrak{T}}_{23}(B_6^\sigma), F_1]_q.$$

Hence, (10.1) follows.

We next prove (10.2). We have

$$\begin{aligned}\widetilde{\mathfrak{T}}_{\mathbf{r}_1}^{-1}(F_6) &= \widetilde{\mathfrak{T}}_{1\cdots5\cdots3}^{-1}(F_6) = \widetilde{\mathfrak{T}}_{123}^{-1}[\widetilde{\mathfrak{T}}_{454}^{-1}(F_3), F_6]_q = \widetilde{\mathfrak{T}}_{123}^{-1}[\widetilde{\mathfrak{T}}_{34}(F_5), F_6]_q \\ &= \big[\widetilde{\mathfrak{T}}_4(F_5), [\widetilde{\mathfrak{T}}_{12}^{-1}(F_3), F_6]_q\big]_q = \big[\widetilde{\mathfrak{T}}_4(F_5), [\widetilde{\mathfrak{T}}_{32}(F_1), F_6]_q\big]_q.\end{aligned} \tag{10.3}$$

Recall that $B_1^\sigma = F_1 + K_1 \widetilde{\mathfrak{T}}_{w_\bullet}^{-1}(E_5)$. Hence,

$$\begin{aligned}[\widetilde{\mathfrak{T}}_{32}(B_1^\sigma), F_6]_q &= [\widetilde{\mathfrak{T}}_{32}(F_1), F_6]_q + [K_{123}\widetilde{\mathfrak{T}}_3\widetilde{\mathfrak{T}}_{434}^{-1}(E_5), F_6]_q \\ &= [\widetilde{\mathfrak{T}}_{32}(F_1), F_6]_q + K_{123}[\widetilde{\mathfrak{T}}_4^{-1}(E_5), F_6] = [\widetilde{\mathfrak{T}}_{32}(F_1), F_6]_q.\end{aligned} \tag{10.4}$$

On the other hand, we have

$$\begin{aligned}\widetilde{\mathfrak{T}}_{32323}^{-1}(F_6) &= \widetilde{\mathfrak{T}}_{323}^{-1}[\widetilde{\mathfrak{T}}_2^{-1}(F_3), F_6]_q = \widetilde{\mathfrak{T}}_{323}^{-1}[\widetilde{\mathfrak{T}}_3(F_2), F_6]_q \\ &= -[\widetilde{\mathfrak{T}}_3^{-1}(E_2 K_2'^{-1}), \widetilde{\mathfrak{T}}_{232}^{-1}(F_6)]_q \\ &= -q^{-1}[\widetilde{\mathfrak{T}}_3^{-1}(E_2), \widetilde{\mathfrak{T}}_{23}^{-1}(F_6)]_{q^2} K_2'^{-1} K_3'^{-1}\end{aligned}$$

$$= -q^{-1}\big[\widetilde{\mathcal{T}}_3^{-1}(E_2), [\widetilde{\mathcal{T}}_2^{-1}(F_3), F_6]_q\big]_{q^2} K_2'^{-1} K_3'^{-1}$$
$$= -q^{-1}\big[[\widetilde{\mathcal{T}}_3^{-1}(E_2), \widetilde{\mathcal{T}}_2^{-1}(F_3)]_{q^2}, F_6\big]_q K_2'^{-1} K_3'^{-1}, \tag{10.5}$$

We now rewrite RHS (10.2) as follows:

$$\begin{aligned}
&\big[\widetilde{\mathcal{T}}_4(B_5^\sigma), [\widetilde{\mathcal{T}}_{32}(B_1^\sigma), F_6]_q\big]_q \\
&\overset{(10.4)}{=} \big[\widetilde{\mathcal{T}}_4(B_5^\sigma), [\widetilde{\mathcal{T}}_{32}(F_1), F_6]_q\big]_q \\
&= \big[\widetilde{\mathcal{T}}_4(F_5), [\widetilde{\mathcal{T}}_{32}(F_1), F_6]_q\big]_q + \big[K_4K_5\widetilde{\mathcal{T}}_{232}^{-1}(E_1), [\widetilde{\mathcal{T}}_{32}(F_1), F_6]_q\big]_q \\
&\overset{(10.3)}{=} \widetilde{\mathcal{T}}_{\mathbf{r}_1}^{-1}(F_6) + K_4K_5\big[\widetilde{\mathcal{T}}_{32}^{-1}(E_1), [\widetilde{\mathcal{T}}_{32}(F_1), F_6]_q\big] \\
&= \widetilde{\mathcal{T}}_{\mathbf{r}_1}^{-1}(F_6) - q^{-1}\big[[\widetilde{\mathcal{T}}_3^{-1}(E_2), \widetilde{\mathcal{T}}_3(F_2)]_{q^2}, F_6\big]_q K_1' K_4 K_5 \\
&\overset{(10.5)}{=} \widetilde{\mathcal{T}}_{\mathbf{r}_1}^{-1}(F_6) + \widetilde{\mathcal{T}}_{32323}^{-1}(F_6) K_1' K_2' K_3' K_4 K_5.
\end{aligned}$$

Therefore, the formula (10.2) follows. □

**Lemma 10.2.** *We have*

$$\widetilde{\mathcal{T}}_{\mathbf{r}_6}^{-1}\big(\widetilde{\mathcal{T}}_{w_\bullet}(E_5)K_1'\big) = [\widetilde{\mathcal{T}}_{23}(B_6), \widetilde{\mathcal{T}}_{w_\bullet}(E_5)K_1']_q, \tag{10.6}$$
$$\begin{aligned}\widetilde{\mathcal{T}}_{\mathbf{r}_1}^{-1}\big(\widetilde{\mathcal{T}}_{w_\bullet}(E_6)K_6'\big) &= \big[\widetilde{\mathcal{T}}_4(B_5), [\widetilde{\mathcal{T}}_{32}(B_1), \widetilde{\mathcal{T}}_{w_\bullet}(E_6)K_6']_q\big]_q \\ &\quad - \widetilde{\mathcal{T}}_{32323}^{-1}\big(\widetilde{\mathcal{T}}_{w_\bullet}(E_6)K_6'\big)K_1'K_2'K_3'K_4K_5. \end{aligned}\tag{10.7}$$

*Proof.* Recall from Lemma 1.3 that the operator $\mathcal{D}$ defined in (1.3) commutes with each of the automorphisms $\widetilde{\mathcal{T}}_4, \widetilde{\mathcal{T}}_{32}, \widetilde{\mathcal{T}}_{23}, \widetilde{\mathcal{T}}_{\mathbf{r}_1}, \widetilde{\mathcal{T}}_{\mathbf{r}_6}$.

We first prove the formula (10.6). Applying $\mathcal{D}$ to (10.1) and then using (1.4)-(1.5), we obtain

$$\widetilde{\mathcal{T}}_{\mathbf{r}_6}^{-1}\big(\widetilde{\mathcal{T}}_{w_\bullet}(E_5)K_1'\big)\widetilde{\mathcal{T}}_{w_\bullet,6}(\mathcal{K}_5^{-1})$$

$$= -q^{-2}[\widetilde{\mathfrak{T}}_{23}(B_6)\widetilde{\mathfrak{T}}_{432}(\mathcal{K}_6^{-1}), \widetilde{\mathfrak{T}}_{w_\bullet}(E_5)K_1'\widetilde{\mathfrak{T}}_{w_\bullet}(\mathcal{K}_5^{-1})]_q$$

$$= -q^{-1}[\widetilde{\mathfrak{T}}_{23}(B_6), \widetilde{\mathfrak{T}}_{w_\bullet}(E_5)K_1']_q\widetilde{\mathfrak{T}}_{432}(\mathcal{K}_6^{-1})\widetilde{\mathfrak{T}}_{w_\bullet}(\mathcal{K}_5^{-1}), \tag{10.8}$$

where the last equality follows by a weight consideration. On the other hand, we have $\widetilde{\mathfrak{T}}_{w_{\bullet,6}}(\mathcal{K}_5^{-1}) = -q^{-1}\widetilde{\mathfrak{T}}_{432}(\mathcal{K}_6^{-1})\widetilde{\mathfrak{T}}_{w_\bullet}(\mathcal{K}_5^{-1})$. Thus the formula (10.6) follows from (10.8).

We next prove the formula (10.7). Applying $\mathcal{D}$ in the identity (1.3) to (10.2) and using (1.4)-(1.5), we obtain

$$\begin{aligned}
&\widetilde{\mathfrak{T}}_{\mathbf{r}_1}^{-1}\big(\widetilde{\mathfrak{T}}_{w_\bullet}(E_6)K_6'\big)\widetilde{\mathfrak{T}}_{w_{\bullet,1}}(\mathcal{K}_6^{-1})\\
&= q^{-4}\big[\widetilde{\mathfrak{T}}_4(B_5)\widetilde{\mathfrak{T}}_4\widetilde{\mathfrak{T}}_{w_\bullet}(\mathcal{K}_1^{-1}), [\widetilde{\mathfrak{T}}_{32}(B_1)\widetilde{\mathfrak{T}}_{32}\widetilde{\mathfrak{T}}_{w_\bullet}(\mathcal{K}_5^{-1}), \widetilde{\mathfrak{T}}_{w_\bullet}(E_6)K_6'\widetilde{\mathfrak{T}}_{w_\bullet}(\mathcal{K}_6^{-1})]_q\big]_q\\
&- \widetilde{\mathfrak{T}}_{32323}^{-1}\big(\widetilde{\mathfrak{T}}_{w_\bullet}(E_6)K_6'\big)\widetilde{\mathfrak{T}}_{w_\bullet}(\mathcal{K}_6^{-1})\mathcal{D}(K_1'K_2'K_3'K_4K_5).
\end{aligned} \tag{10.9}$$

Note that $\widetilde{\mathfrak{T}}_4\widetilde{\mathfrak{T}}_{w_\bullet}(\mathcal{K}_1^{-1}) = \widetilde{\mathfrak{T}}_{32}(K_1^{-1})\widetilde{\mathfrak{T}}_4(K_5')^{-1}$ and $\widetilde{\mathfrak{T}}_{32}\widetilde{\mathfrak{T}}_{w_\bullet}(\mathcal{K}_5^{-1}) = \widetilde{\mathfrak{T}}_4(K_5^{-1})\widetilde{\mathfrak{T}}_{32}(K_1')^{-1}$. We also note that $K_1'K_2'K_3'K_4K_5 = \widetilde{\mathfrak{T}}_{32}(K_1')\widetilde{\mathfrak{T}}_4(K_5)$ and then $\mathcal{D}(K_1'K_2'K_3'K_4K_5) = q^{-1}\widetilde{\mathfrak{T}}_4(K_5')^{-1}\widetilde{\mathfrak{T}}_{32}(K_1^{-1})$. Hence, (10.9) can be rewritten as

$$\begin{aligned}
&\widetilde{\mathfrak{T}}_{\mathbf{r}_1}^{-1}\big(\widetilde{\mathfrak{T}}_{w_\bullet}(E_6)K_6'\big)\widetilde{\mathfrak{T}}_{w_{\bullet,1}}(\mathcal{K}_6^{-1})\\
&= q^{-4}\big[\widetilde{\mathfrak{T}}_4(B_5)\widetilde{\mathfrak{T}}_{32}(K_1^{-1})\widetilde{\mathfrak{T}}_4(K_5')^{-1}, [\widetilde{\mathfrak{T}}_{32}(B_1)\widetilde{\mathfrak{T}}_4(K_5^{-1})\widetilde{\mathfrak{T}}_{32}(K_1')^{-1}, \widetilde{\mathfrak{T}}_{w_\bullet}(E_6)K_6'\widetilde{\mathfrak{T}}_{w_\bullet}(\mathcal{K}_6^{-1})]_q\big]_q\\
&- q^{-1}\widetilde{\mathfrak{T}}_{32323}^{-1}\big(\widetilde{\mathfrak{T}}_{w_\bullet}(E_6)K_6'\big)\widetilde{\mathfrak{T}}_{w_\bullet}(\mathcal{K}_6^{-1})\widetilde{\mathfrak{T}}_4(K_5')^{-1}\widetilde{\mathfrak{T}}_{32}(K_1^{-1}).
\end{aligned} \tag{10.10}$$

For a weight reason, we have

$$\widetilde{\mathfrak{T}}_4(K_5^{-1})\widetilde{\mathfrak{T}}_{32}(K_1')^{-1}\widetilde{\mathfrak{T}}_{w_\bullet}(E_6) = q\widetilde{\mathfrak{T}}_{w_\bullet}(E_6)\widetilde{\mathfrak{T}}_4(K_5^{-1})\widetilde{\mathfrak{T}}_{32}(K_1')^{-1}$$

$$\widetilde{\mathfrak{T}}_{w_\bullet}(\mathcal{K}_6^{-1})\widetilde{\mathfrak{T}}_{32}(B_1) = q\widetilde{\mathfrak{T}}_{32}(B_1)\widetilde{\mathfrak{T}}_{w_\bullet}(\mathcal{K}_6^{-1})$$

$$\widetilde{\mathfrak{T}}_{32}(K_1)\widetilde{\mathfrak{T}}_4(K_5')[\widetilde{\mathfrak{T}}_{32}(B_1), \widetilde{\mathfrak{T}}_{w_\bullet}(E_6)K_6']_q = q^{-2}[\widetilde{\mathfrak{T}}_{32}(B_1), \widetilde{\mathfrak{T}}_{w_\bullet}(E_6)K_6']_q\widetilde{\mathfrak{T}}_{32}(K_1)\widetilde{\mathfrak{T}}_4(K_5'),$$

$$\widetilde{\mathcal{T}}_4(K_5^{-1})\widetilde{\mathcal{T}}_{32}(K_1')^{-1}\widetilde{\mathcal{T}}_{w_\bullet}(\mathcal{K}_6^{-1})\widetilde{\mathcal{T}}_4(B_5) = q^2\widetilde{\mathcal{T}}_4(B_5)\widetilde{\mathcal{T}}_4(K_5^{-1})\widetilde{\mathcal{T}}_{32}(K_1')^{-1}\widetilde{\mathcal{T}}_{w_\bullet}(\mathcal{K}_6^{-1}).$$

Using the above four identities, we rewrite (10.10) as

$$\begin{aligned}
&\widetilde{\mathcal{T}}_{\mathbf{r}_1}^{-1}\big(\widetilde{\mathcal{T}}_{w_\bullet}(E_6)K_6'\big)\widetilde{\mathcal{T}}_{w_{\bullet,1}}(\mathcal{K}_6^{-1})\\
&= q^{-1}\big[\widetilde{\mathcal{T}}_4(B_5),[\widetilde{\mathcal{T}}_{32}(B_1),\widetilde{\mathcal{T}}_{w_\bullet}(E_6)K_6']_q\big]_q\widetilde{\mathcal{T}}_{w_\bullet}(\mathcal{K}_6^{-1})\widetilde{\mathcal{T}}_{32}(K_1K_1')^{-1}\widetilde{\mathcal{T}}_4(K_5K_5')^{-1}\\
&- q^{-1}\widetilde{\mathcal{T}}_{32323}^{-1}\big(\widetilde{\mathcal{T}}_{w_\bullet}(E_6)K_6'\big)K_1'K_2'K_3'K_4K_5\widetilde{\mathcal{T}}_{w_\bullet}(\mathcal{K}_6^{-1})\widetilde{\mathcal{T}}_{32}(K_1K_1')^{-1}\widetilde{\mathcal{T}}_4(K_5K_5')^{-1}. \quad (10.11)
\end{aligned}$$

Moreover, we have $\widetilde{\mathcal{T}}_{w_{\bullet,1}}(\mathcal{K}_6^{-1}) = q^{-1}\widetilde{\mathcal{T}}_{w_\bullet}(\mathcal{K}_6^{-1})\widetilde{\mathcal{T}}_{32}(K_1K_1')^{-1}\widetilde{\mathcal{T}}_4(K_5K_5')^{-1}$. Thus, (10.11) implies the desired formula (10.7). □

# Appendix B

# Proofs of Proposition 6.5 and Table 4

In this appendix, we shall provide constructive proofs for Proposition 6.5 and verify the formulas for $\widetilde{\mathbf{T}}''_{i,+1}(B_j)$ in Table 4. The proofs in various types bear much similarity, and they are made a little easier by taking advantage of the results in Appendix A.

## 1 Preparatory identities

We prepare some identities which are valid in all types. Recall that $w_0$ denote the longest element of the Weyl group $W$ and $\tau_0$ is the diagram automorphism associated to $w_0$. Recall the operator $\mathcal{D} = \widetilde{\mathfrak{T}}_{w_0}\widetilde{\mathfrak{T}}_{w_\bullet}\widehat{\tau}_0\widehat{\tau}$ from definition (1.3).

**Lemma 1.1.** *For $i \neq j \in \mathbb{I}_{\circ,\tau}$, we have*

$$\mathcal{D}(F_j) = -q_j^{-2}\widetilde{\mathfrak{T}}_{w_\bullet}(E_{\tau j})K'_j\widetilde{\mathfrak{T}}_{w_\bullet}(\mathcal{K}_{\tau j}^{-1}), \tag{1.1}$$

$$\mathcal{D}(B_i) = -q_i^{-2}\widehat{B}_i\widetilde{\mathcal{T}}_{w_\bullet}(\mathcal{K}_{\tau i}^{-1}). \tag{1.2}$$

*Proof.* The formula (1.1) is copied from the identity (1.5), for convenience of citation in this appendix. By the formula (4.17) and the definition of $\widehat{B}_i$ (6.7), we have

$$\begin{aligned}\widehat{B}_i = \widetilde{\mathcal{T}}_{\mathbf{r}_i}\big(\widetilde{\mathbf{T}}'_{i,-1}(B_i)^\imath\big) &= -q^{-(\alpha_i, w_\bullet\alpha_{\tau i})}\widetilde{\mathcal{T}}_{w_\bullet}\widetilde{\mathcal{T}}_{w_{\bullet,i}}(B_{\tau_{\bullet,i}\tau i})\widetilde{\mathcal{T}}_{w_\bullet}\widetilde{\mathcal{T}}_{w_{\bullet,i}}(\mathcal{K}^{-1}_{\tau_{\bullet,i}\tau i})\\ &= -q^{(\alpha_i,\alpha_i)}\widetilde{\mathcal{T}}_{w_\bullet}\widetilde{\mathcal{T}}_{w_{\bullet,i}}(B_{\tau_{\bullet,i}\tau i})\widetilde{\mathcal{T}}_{w_\bullet}(\mathcal{K}_{\tau i}). \end{aligned}\tag{1.3}$$

By Lemma 4.4, $\widetilde{\mathcal{T}}_{w_{\bullet,i}}\widehat{\tau}_{\bullet,i}(B_i) = \widetilde{\mathcal{T}}_{w_0}\widehat{\tau}_0(B_i)$. Hence, the identity (1.2) follows by a reformulation of (1.3). □

# 2 Split types of rank 2

Consider a rank two Satake diagram ($\mathbb{I} = \mathbb{I}_\circ, \mathrm{Id}$) of split type. In this case, we have $B_i = F_i + E_iK_i'$ and $\mathcal{K}_i = \widetilde{k}_i$.

**The case $c_{ij} = -1$**

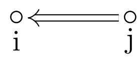
According to Table 4, Proposition 6.5 can be reformulated as the following lemma, which can be proved by the definition of $\widetilde{\mathcal{T}}_j$.

**Lemma 2.1.** *We have*

$$\widetilde{\mathcal{T}}_i(F_j) = [F_j, B_i]_q, \qquad \widetilde{\mathcal{T}}_i(E_jK_j') = [E_jK_j', B_i]_q. \tag{2.1}$$

**The case $c_{ij} = -2$**

In this case, the rank two Satake diagram is given by




According to Table 4, Proposition 6.5 can be reformulated and proved as follows.

**Lemma 2.2.** *We have*

$$\widetilde{\mathcal{T}}_i(F_j) = \frac{1}{[2]_i}\big[[F_j, B_i]_{q_i^2}, B_i\big] - q_i^2 F_j K_i K_i', \tag{2.2}$$

$$\widetilde{\mathcal{T}}_i(E_j K_j') = \frac{1}{[2]_i}\big[[E_j K_j', \widehat{B}_i]_{q_i^2}, \widehat{B}_i\big] - q_i^2 E_j K_j' K_i K_i'. \tag{2.3}$$

*Proof.* The firs formula (2.2) is obtained by applying $\sigma$ to the formula of $\widetilde{\mathcal{T}}_i^{-1}(F_j)$ in Lemma 2.2.

We prove the second formula (2.3) next. Recall from Lemma 1.3 that $\mathcal{D}$ commutes with $\widetilde{\mathcal{T}}_i$. By (1.1)–(1.2), we have

$$\mathcal{D}(F_j) = -q_j^{-2} E_j K_j' \widetilde{k}_j^{-1}, \qquad \mathcal{D}(B_i) = -q^{-2}\widehat{B_i}\widetilde{k}_i^{-1}. \tag{2.4}$$

Applying $\mathcal{D}$ to (2.2) and then using (2.4), we have

$$\widetilde{\mathcal{T}}_i(E_j K_j' \widetilde{k}_j^{-1}) = \frac{1}{[2]_i} q_i^{-4}\big[[E_j K_j' \widetilde{k}_j^{-1}, \widehat{B}_i \widetilde{k}_i^{-1}]_{q_i^2}, \widehat{B}_i \widetilde{k}_i^{-1}\big] - q_i^2 E_j K_j' \widetilde{k}_j^{-1} \mathcal{D}(K_i K_i'). \tag{2.5}$$

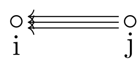
In this case $\mathcal{D}(K_i K_i') = q^{-4} K_i K_i' \widetilde{k}_i^{-2}$. Since $\widetilde{k}_i$ are central, (2.5) is simplified as

$$\widetilde{\mathcal{T}}_i(E_j K_j')\widetilde{\mathcal{T}}_i(\widetilde{k}_j^{-1}) = q_i^{-4}\Big(\frac{1}{[2]_i}\big[[E_j K_j', \widehat{B}_i]_{q_i^2}, \widehat{B}_i\big] - q_i^2 E_j K_j'\Big)\widetilde{k}_j^{-1}\widetilde{k}_i^{-2}. \tag{2.6}$$

Finally, by Proposition 4.2, we have $\widetilde{\mathcal{T}}_i(\widetilde{k}_j^{-1}) = q_i^{-4}\widetilde{k}_j^{-1}\widetilde{k}_i^{-2}$. Hence, (2.6) implies the desired identity (2.3). $\square$

**The case $c_{ij} = -3$**

Consider the Satake diagram of split type $G_2$



In this case, we have $q_i = q$ and $q_j = q^3$. According to Table 4, Proposition 6.5 can be reformulated and proved as follows.

**Lemma 2.3.** *We have*

$$\begin{aligned}
\widetilde{\mathcal{T}}_i(F_j) &= \frac{1}{[3]!}\Big[\big[[F_j, B_i]_{q^3}, B_i\big]_q, B_i\Big]_{q^{-1}} \\
&\quad - \frac{1}{[3]!}\Big(q(1+[3])[F_j, B_i]_{q^3} + q^3[3][F_j, B_i]_{q^{-1}}\Big)\widetilde{k}_i. \qquad (2.7)\\
\widetilde{\mathcal{T}}_i(E_jK_j') &= \frac{1}{[3]!}\Big[\big[[E_jK_j', \widehat{B}_i]_{q^3}, \widehat{B}_i\big]_q, \widehat{B}_i\Big]_{q^{-1}} \\
&\quad - \frac{1}{[3]!}\Big(q(1+[3])[F_j, \widehat{B}_i]_{q^3} + q^3[3][F_j, \widehat{B}_i]_{q^{-1}}\Big)\widetilde{k}_i. \qquad (2.8)
\end{aligned}$$

*Proof.* The first formula (2.7) is obtained by applying $\sigma$ to (2.8). We prove the second formula (2.8).

By (1.1)-(1.2), we have

$$\mathcal{D}(F_j) = -q_j^{-2}E_jK_j'\widetilde{k}_j^{-1}, \qquad \mathcal{D}(B_i) = -q^{-2}\widehat{B_i}\widetilde{k}_i^{-1}. \qquad (2.9)$$

Note that $\widetilde{k}_i, \widetilde{k}_j$ are central. Recall from Lemma 1.3 that $\mathcal{D}$ commutes with $\widetilde{\mathcal{T}}_i$.

Applying $\mathcal{D}$ to (2.7) and then using (2.9), we have

$$\widetilde{\mathcal{T}}_i(E_jK'_j)\widetilde{\mathcal{T}}_i(\widetilde{k}_j^{-1}) = -q^{-6}\frac{1}{[3]!}\Big[\big[[E_jK'_j,\widehat{B}_i]_{q^3},\widehat{B}_i\big]_q,\widehat{B}_i\Big]_{q^{-1}}\widetilde{k}_j^{-1}\widetilde{k}_i^{-3} \\ + q^{-2}\frac{1}{[3]!}\Big(q(1+[3])[F_j,\widehat{B}_i]_{q^3} - q^3[3][F_j,\widehat{B}_i]_{q^{-1}}\Big)\widetilde{k}_j^{-1}\widetilde{k}_i^{-1}\mathcal{D}(\widetilde{k}_i). \tag{2.10}$$

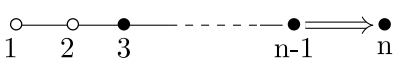
In this case $\mathcal{D} = \widetilde{\mathcal{T}}_{w_0}$, and then we have $\mathcal{D}(\widetilde{k}_i) = q^{-4}\widetilde{k}_i^{-1}$. Hence, we simplify (2.10) as

$$\widetilde{\mathcal{T}}_i(E_jK'_j)\widetilde{\mathcal{T}}_i(\widetilde{k}_j^{-1}) = -q^{-6}\frac{1}{[3]!}\Big[\big[[E_jK'_j,\widehat{B}_i]_{q^3},\widehat{B}_i\big]_q,\widehat{B}_i\Big]_{q^{-1}}\widetilde{k}_j^{-1}\widetilde{k}_i^{-3} \\ + q^{-6}\frac{1}{[3]!}\Big(q(1+[3])[F_j,\widehat{B}_i]_{q^3} - q^3[3][F_j,\widehat{B}_i]_{q^{-1}}\Big)\widetilde{k}_j^{-1}\widetilde{k}_i^{-3}\widetilde{k}_i. \tag{2.11}$$

Finally, since $s_i(\alpha_j) = \alpha_j + 3\alpha_i$, we have $\widetilde{\mathcal{T}}_i(\widetilde{k}_j^{-1}) = -q^{-6}\widetilde{k}_j^{-1}\widetilde{k}_i^{-3}$. Thus, (2.11) implies the desired formula (2.8). $\square$

# 3 Type BI, DI, DIII$_4$

Consider the rank two Satake diagrams of type BI$_n$, for $n \geq 3$:




$$t_a := s_a \cdots s_n \cdots s_a, \qquad (3 \leq a \leq n).$$

According to Table 4, Proposition 6.5 is reformulated and proved as Lemma 3.1.

**Lemma 3.1.** *We have*

$$\widetilde{\mathcal{T}}_{\mathbf{r}_2}(F_1) = \big[[F_1,B_2]_{q_2},\widetilde{\mathcal{T}}_{w_\bullet}^{-1}(B_2)\big]_{q_2} - q_2F_1\widetilde{\mathcal{T}}_{w_\bullet}(K'_2)K_2. \tag{3.1}$$

$$\widetilde{\mathcal{T}}_{\mathbf{r}_2}(E_1K'_1) = \big[[E_1K'_1,\widehat{B}_2]_{q_2},\widetilde{\mathcal{T}}_{w_\bullet}^{-1}(\widehat{B}_2)\big]_{q_2} - q_2E_1K'_1\widetilde{\mathcal{T}}_{w_\bullet}(K'_2)K_2. \tag{3.2}$$

*Proof.* The first formula (3.1) is obtained from Lemma 5.2 by applying $\sigma$. We shall derive the second formula (3.2) from (3.1) as follows.

By (1.1)–(1.2), we have

$$\mathcal{D}(F_1) = -q_1^{-2} E_1 K_1' \mathcal{K}_1^{-1}, \qquad \mathcal{D}(B_2) = -q_2^{-2} \widehat{B}_2 \widetilde{\mathcal{T}}_{w_\bullet}(\mathcal{K}_2^{-1}). \tag{3.3}$$

Recall from Lemma 1.3 that $\mathcal{D}$ commutes with $\widetilde{\mathcal{T}}_{w_\bullet}, \widetilde{\mathcal{T}}_{\mathbf{r}_2}$. Applying $\mathcal{D}$ to (3.1) and then using (3.3), we obtain

$$\begin{aligned}\widetilde{\mathcal{T}}_{\mathbf{r}_2}(E_1 K_1') \widetilde{\mathcal{T}}_{\mathbf{r}_2}(\mathcal{K}_1^{-1}) &= q_2^{-4} \big[ [E_1 K_1' \mathcal{K}_1^{-1}, \widehat{B}_2 \widetilde{\mathcal{T}}_{w_\bullet}(\mathcal{K}_2^{-1})]_{q_2}, \widetilde{\mathcal{T}}_{w_\bullet}^{-1}(\widehat{B}_2) \mathcal{K}_2^{-1} \big]_{q_2} \\ &\quad - q_2 E_1 K_1' \mathcal{K}_1^{-1} \mathcal{D}(\widetilde{\mathcal{T}}_{w_\bullet}(K_2') K_2). \end{aligned} \tag{3.4}$$

In this case, $\mathcal{K}_1 = \widetilde{k}_1$ is central and both $\mathcal{K}_2, \widetilde{\mathcal{T}}_{w_\bullet}(\mathcal{K}_2)$ commute with $E_1$. For the weight reason, we have $\mathcal{K}_2 \widehat{B}_2 = q^{-2} \widehat{B}_2 \mathcal{K}_2$. Note also that

$$\mathcal{D}(\widetilde{\mathcal{T}}_{w_\bullet}(K_2') K_2) = q_2^{-2} \widetilde{\mathcal{T}}_{w_\bullet}(K_2') K_2 \mathcal{K}_2^{-1} \widetilde{\mathcal{T}}_{w_\bullet}(\mathcal{K}_2^{-1}).$$

Then, using these formulas, we rewrite (3.4) as

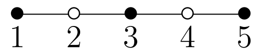
$$\begin{aligned}\widetilde{\mathcal{T}}_{\mathbf{r}_2}(E_1 K_1') \widetilde{\mathcal{T}}_{\mathbf{r}_2}(\mathcal{K}_1^{-1}) &= q_2^{-2} \big[ [E_1 K_1', \widehat{B}_2]_{q_2}, \widetilde{\mathcal{T}}_{w_\bullet}^{-1}(\widehat{B}_2) \big]_{q_2} \mathcal{K}_1^{-1} \mathcal{K}_2^{-1} \widetilde{\mathcal{T}}_{w_\bullet}(\mathcal{K}_2^{-1}) \\ &\quad - q_2^{-1} E_1 K_1' \widetilde{\mathcal{T}}_{w_\bullet}(K_2') K_2 \mathcal{K}_1^{-1} \mathcal{K}_2^{-1} \widetilde{\mathcal{T}}_{w_\bullet}(\mathcal{K}_2^{-1}). \end{aligned} \tag{3.5}$$

Since $\mathbf{r}_2(\alpha_1) = \alpha_1 + \alpha_2 + w_\bullet(\alpha_2)$, by Proposition 4.2 we have $\widetilde{\mathcal{T}}_{\mathbf{r}_2}(\mathcal{K}_1) = q_2^2 \mathcal{K}_1 \mathcal{K}_2 \widetilde{\mathcal{T}}_{w_\bullet}(\mathcal{K}_2)$. Hence, (3.5) implies the desired formula (3.2). ☐

*Remark* 3.2. Similar formulas can be derived for types $\mathrm{DI}_n$ ($n \geq 5$) and $\mathrm{DIII}_4$, since the formulas of $\widetilde{\mathbf{T}}'_{i,-1}(B_j)$ for these two types and type BI are unified; compare §5.2.

# 4 Type AII

Consider the Satake diagram of type AII




$$\mathbf{r}_4 = s_4 s_3 s_5 s_4.$$

In this case, according to Table 4, Proposition 6.5 is reformulated and proved as Lemma 4.1 below.

**Lemma 4.1.** *We have*

$$\widetilde{\mathcal{T}}_{\mathbf{r}_4}(F_2) = \big[F_2, \widetilde{\mathcal{T}}_3^{-1}(B_4)\big]_q, \tag{4.1}$$

$$\widetilde{\mathcal{T}}_{\mathbf{r}_4}\big(\widetilde{\mathcal{T}}_{w_\bullet}(E_2)K_2'\big) = \big[\widetilde{\mathcal{T}}_{w_\bullet}(E_2)K_2', \widetilde{\mathcal{T}}_3^{-1}(\widehat{B}_4)\big]_q. \tag{4.2}$$

*Proof.* The first formula (4.1) is obtained by applying $\sigma$ to Lemma 3.1. We shall derive the second formula (4.2) from (4.1).

By (1.1)–(1.2), we have

$$\mathcal{D}(F_2) = -q^{-2}\widetilde{\mathcal{T}}_{w_\bullet}(E_2)K_2'\widetilde{\mathcal{T}}_{w_\bullet}(\mathcal{K}_2^{-1}), \qquad \mathcal{D}(B_4) = -q^{-2}\widehat{B}_4\widetilde{\mathcal{T}}_{w_\bullet}(\mathcal{K}_4^{-1}). \tag{4.3}$$

Recall from Lemma 1.3 that $\mathcal{D}$ commutes with $\widetilde{\mathcal{T}}_{\mathbf{r}_4}, \widetilde{\mathcal{T}}_3$. Applying $\mathcal{D}$ to (4.1) and then using (4.3), we have

$$\widetilde{\mathcal{T}}_{\mathbf{r}_4}\big(\widetilde{\mathcal{T}}_{w_\bullet}(E_2)K_2'\big)\widetilde{\mathcal{T}}_{w_{\bullet,4}}(\mathcal{K}_2^{-1}) = -q^{-2}\big[\widetilde{\mathcal{T}}_{w_\bullet}(E_2)K_2'\widetilde{\mathcal{T}}_{w_\bullet}(\mathcal{K}_2^{-1}), \widetilde{\mathcal{T}}_3^{-1}(\widehat{B}_4)\widetilde{\mathcal{T}}_5(\mathcal{K}_4^{-1})\big]_q. \tag{4.4}$$

For the weight consideration, we have the following two commutator relations

$$\widetilde{\mathscr{T}}_{w_\bullet}(\mathcal{K}_2^{-1})\widetilde{\mathscr{T}}_3^{-1}(\widehat{B}_4) = q\widetilde{\mathscr{T}}_3^{-1}(\widehat{B}_4)\widetilde{\mathscr{T}}_{w_\bullet}(\mathcal{K}_2^{-1}),$$
$$\widetilde{\mathscr{T}}_5(\mathcal{K}_4^{-1})\widetilde{\mathscr{T}}_{w_\bullet}(E_2) = q\widetilde{\mathscr{T}}_{w_\bullet}(E_2)\widetilde{\mathscr{T}}_5(\mathcal{K}_4^{-1}).$$

Using these two formulas, we rewrite (4.4) as

$$\widetilde{\mathscr{T}}_{\mathbf{r}_4}\big(\widetilde{\mathscr{T}}_{w_\bullet}(E_2)K_2'\big)\widetilde{\mathscr{T}}_{w_{\bullet,4}}(\mathcal{K}_2^{-1}) = -q^{-1}\big[\widetilde{\mathscr{T}}_{w_\bullet}(E_2)K_2', \widetilde{\mathscr{T}}_3^{-1}(\widehat{B}_4)\big]_q\widetilde{\mathscr{T}}_{w_\bullet}(\mathcal{K}_2^{-1})\widetilde{\mathscr{T}}_5(\mathcal{K}_4^{-1}). \quad (4.5)$$

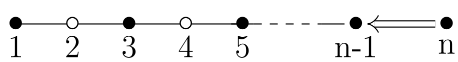
Finally, since $w_{\bullet,4}(\alpha_2) = w_\bullet(\alpha_2) + s_5(\alpha_4)$, by Proposition 4.2, we have

$$\widetilde{\mathscr{T}}_{w_{\bullet,4}}(\mathcal{K}_2) = -q\widetilde{\mathscr{T}}_{w_\bullet}(\mathcal{K}_2)\widetilde{\mathscr{T}}_5(\mathcal{K}_4).$$

Hence, (4.5) implies the desired formula (4.2). □

# 5 Type CII$_n$, $n \geq 5$

Consider the Satake diagram of type CII$_n$, for $n \geq 5$:




$$\mathbf{r}_4 = s_{4\cdots n\cdots 4}s_3s_{4\cdots n\cdots 4}.$$

Note that $q_2 = q_4 = q$ in this case. According to Table 4, Proposition 6.5 is reformulated and proved as Lemma 5.1 below.

**Lemma 5.1.** *We have*

$$\widetilde{\mathscr{T}}_{\mathbf{r}_4}(F_2) = \big[F_2, [\widetilde{\mathscr{T}}_3^{-1}(B_4), \widetilde{\mathscr{T}}_{5\cdots n\cdots 5}^{-1}(B_4)]_q\big]_q$$

$$- q^2\widetilde{\mathcal{T}}_3^2(F_2)\widetilde{\mathcal{T}}_3(\mathcal{K}_4), \tag{5.1}$$

$$\widetilde{\mathcal{T}}_{\mathbf{r}_4}\big(\widetilde{\mathcal{T}}_{w_\bullet}(E_2)K_2'\big) = \big[\widetilde{\mathcal{T}}_{w_\bullet}(E_2)K_2', [\widetilde{\mathcal{T}}_3^{-1}(\widehat{B}_4), \widetilde{\mathcal{T}}_{5\cdots n\cdots 5}^{-1}(\widehat{B}_4)]_q\big]_q$$
$$- q^2\widetilde{\mathcal{T}}_3^2\widetilde{\mathcal{T}}_{w_\bullet}(E_2)K_2'\widetilde{\mathcal{T}}_3(\mathcal{K}_4). \tag{5.2}$$

*Proof.* The first formula (5.1) is obtained by applying $\sigma$ to Lemma 4.1. We shall derive the second formula (5.2) from (5.1) as follows.

By (1.1)–(1.2), we have

$$\mathcal{D}(B_4) = -q^{-2}\widehat{B}_4\widetilde{\mathcal{T}}_{w_\bullet}(\mathcal{K}_4^{-1}), \qquad \mathcal{D}(F_2) = -q^{-2}\widetilde{\mathcal{T}}_{w_\bullet}(E_2)K_2'\widetilde{\mathcal{T}}_{w_\bullet}(\mathcal{K}_2^{-1}). \tag{5.3}$$

Recall from Lemma 1.3 that $\mathcal{D}$ commutes with $\widetilde{\mathcal{T}}_3, \widetilde{\mathcal{T}}_{5\cdots n\cdots 5}$ and $\widetilde{\mathcal{T}}_{\mathbf{r}_4}$. Note that $\mathcal{D}\big(\widetilde{\mathcal{T}}_3(\mathcal{K}_4)\big) = q^{-1}\widetilde{\mathcal{T}}_{5\cdots n\cdots 5}(\mathcal{K}_4^{-1})$. Applying $\mathcal{D}$ to (5.1), by (5.3), we have

$$\widetilde{\mathcal{T}}_{\mathbf{r}_4}\big(\widetilde{\mathcal{T}}_{w_\bullet}(E_2)K_2'\big)\widetilde{\mathcal{T}}_{w_\bullet,4}(\mathcal{K}_2^{-1})$$
$$= q^{-4}\big[\widetilde{\mathcal{T}}_{w_\bullet}(E_2)K_2'\widetilde{\mathcal{T}}_{w_\bullet}(\mathcal{K}_2^{-1}), [\widetilde{\mathcal{T}}_3^{-1}(\widehat{B}_4)\widetilde{\mathcal{T}}_{5\cdots n\cdots 5}(\mathcal{K}_4^{-1}), \widetilde{\mathcal{T}}_{5\cdots n\cdots 5}^{-1}(\widehat{B}_4)\widetilde{\mathcal{T}}_3(\mathcal{K}_4^{-1})]_q\big]_q$$
$$+ q\widetilde{\mathcal{T}}_3^2\widetilde{\mathcal{T}}_{w_\bullet}(E_2)K_2'\widetilde{\mathcal{T}}_{w_\bullet}(\mathcal{K}_2^{-1})\widetilde{\mathcal{T}}_{5\cdots n\cdots 5}(\mathcal{K}_4^{-1}). \tag{5.4}$$

For weight reason, we have the following commutator relations:

$$\mathcal{K}_4^{-1}\widehat{B}_4 = q^3\widehat{B}_4\mathcal{K}_4^{-1},$$
$$\widetilde{\mathcal{T}}_{w_\bullet}(\mathcal{K}_2^{-1})\widetilde{\mathcal{T}}_3^{-1}(\widehat{B}_4) = q\widetilde{\mathcal{T}}_3^{-1}(\widehat{B}_4)\widetilde{\mathcal{T}}_{w_\bullet}(\mathcal{K}_2^{-1})$$
$$\widetilde{\mathcal{T}}_{w_\bullet}(\mathcal{K}_2^{-1})\widetilde{\mathcal{T}}_{5\cdots n\cdots 5}^{-1}(\widehat{B}_4) = q^{-1}\widetilde{\mathcal{T}}_{5\cdots n\cdots 5}^{-1}(\widehat{B}_4)\widetilde{\mathcal{T}}_{w_\bullet}(\mathcal{K}_2^{-1})$$
$$\big[\widetilde{\mathcal{T}}_{5\cdots n\cdots 5}(\mathcal{K}_4^{-1})\widetilde{\mathcal{T}}_3(\mathcal{K}_4^{-1}), \widetilde{\mathcal{T}}_{w_\bullet}(E_2)\big] = 0.$$

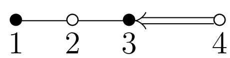
Using these formulas, we rewrite (5.4) as

$$
\begin{aligned}
&\widetilde{\mathcal{T}}_{\mathbf{r}_4}\big(\widetilde{\mathcal{T}}_{w_\bullet}(E_2)K_2'\big)\widetilde{\mathcal{T}}_{w_{\bullet,4}}(\mathcal{K}_2^{-1})\\
&= q^{-1}\big[\widetilde{\mathcal{T}}_{w_\bullet}(E_2)K_2', [\widetilde{\mathcal{T}}_3^{-1}(\widehat{B}_4), \widetilde{\mathcal{T}}_{5\cdots n\cdots 5}^{-1}(\widehat{B}_4)]_q\big]_q \widetilde{\mathcal{T}}_{w_\bullet}(\mathcal{K}_2^{-1})\widetilde{\mathcal{T}}_{5\cdots n\cdots 5}(\mathcal{K}_4^{-1})\widetilde{\mathcal{T}}_3(\mathcal{K}_4^{-1})\\
&+ q\widetilde{\mathcal{T}}_3^2\widetilde{\mathcal{T}}_{w_\bullet}(E_2)K_2'\widetilde{\mathcal{T}}_3(\mathcal{K}_4)\widetilde{\mathcal{T}}_{w_\bullet}(\mathcal{K}_2^{-1})\widetilde{\mathcal{T}}_{5\cdots n\cdots 5}(\mathcal{K}_4^{-1})\widetilde{\mathcal{T}}_3(\mathcal{K}_4^{-1}). \qquad (5.5)
\end{aligned}
$$

In addition, by Proposition 4.2, we have $\widetilde{\mathcal{T}}_{w_{\bullet,4}}(\mathcal{K}_2) = q\widetilde{\mathcal{T}}_{w_\bullet}(\mathcal{K}_2)\widetilde{\mathcal{T}}_{5\cdots n\cdots 5}(\mathcal{K}_4)\widetilde{\mathcal{T}}_3(\mathcal{K}_4)$. Hence (5.5) implies the desired formula (5.2). □

# 6 Type CII$_4$

Consider the Satake diagram of type CII$_4$




$$\mathbf{r}_4 = s_4 s_3 s_4, \qquad \mathbf{r}_2 = s_2 s_1 s_3 s_2.$$

In this case, Proposition 6.5 is reformulated and proved as Lemma 6.1 below.

**Lemma 6.1.** *We have*

$$
\begin{aligned}
\widetilde{\mathcal{T}}_{\mathbf{r}_2}(F_4) &= \big[[F_4, \widetilde{\mathcal{T}}_3^{-1}(B_2)]_{q_3^2}, \widetilde{\mathcal{T}}_3^{-1}(B_2)\big]\\
&\quad - (q_3 - q_3^{-1})[F_4, F_3]_{q_3^2} E_1 K_2 K_2' K_3', \qquad (6.1)\\
\widetilde{\mathcal{T}}_{\mathbf{r}_2}\big(\widetilde{\mathcal{T}}_{w_\bullet}(E_4)K_4'\big) &= \big[[\widetilde{\mathcal{T}}_{w_\bullet}(E_4)K_4', \widetilde{\mathcal{T}}_3^{-1}(\widehat{B}_2)]_{q_3^2}, \widetilde{\mathcal{T}}_3^{-1}(\widehat{B}_2)\big]\\
&\quad - (q_3 - q_3^{-1})[\widetilde{\mathcal{T}}_{w_\bullet}(E_4)K_4', F_3]_{q_3^2} E_1 K_2 K_2' K_3'. \qquad (6.2)
\end{aligned}
$$

*Proof.* The first formula (6.1) is obtained by applying $\sigma$ to Lemma 5.1. We derive the second formula (6.2) from (6.1) as follows. By (1.1)–(1.2), we have

$$\mathcal{D}(F_4) = -q_4^{-2}\widetilde{\mathcal{T}}_{w_\bullet}(E_4)K_4'\widetilde{\mathcal{T}}_3(\mathcal{K}_4^{-1}), \qquad \mathcal{D}(B_2) = -q_2^{-2}\widehat{B_2}\widetilde{\mathcal{T}}_{w_\bullet}(\mathcal{K}_2^{-1}). \tag{6.3}$$

Recall from Lemma 1.3 that $\mathcal{D}$ commutes with $\widetilde{\mathcal{T}}_{\mathbf{r}_2}, \widetilde{\mathcal{T}}_3$. Applying $\mathcal{D}$ to (6.1) and then using (6.3), we have

$$\begin{aligned}
&\widetilde{\mathcal{T}}_{\mathbf{r}_2}\big(\widetilde{\mathcal{T}}_{w_\bullet}(E_4)K_4'\big)\widetilde{\mathcal{T}}_{\mathbf{r}_2}\widetilde{\mathcal{T}}_3(\mathcal{K}_4^{-1})\\
&= q_2^{-4}\big[[\widetilde{\mathcal{T}}_{w_\bullet}(E_4)K_4'\widetilde{\mathcal{T}}_3(\mathcal{K}_4^{-1}), \widetilde{\mathcal{T}}_3^{-1}(\widehat{B}_2)\widetilde{\mathcal{T}}_1(\mathcal{K}_2^{-1})]_{q_3^2}, \widetilde{\mathcal{T}}_3^{-1}(\widehat{B}_2)\widetilde{\mathcal{T}}_1(\mathcal{K}_2^{-1})\big]\\
&+ (q_3 - q_3^{-1})[\widetilde{\mathcal{T}}_{w_\bullet}(E_4)K_4'\widetilde{\mathcal{T}}_3(\mathcal{K}_4^{-1}), \mathcal{D}(F_3)]_{q_3^2}\mathcal{D}(E_1K_2K_2'K_3').
\end{aligned} \tag{6.4}$$

By Lemma 4.4, we have $\mathcal{D}(F_3) = F_3K_3K_3'^{-1}$ and $\mathcal{D}(E_1) = E_1K_1^{-1}K_1'$. We then have

$$\mathcal{D}(E_1K_2K_2'K_3') = q_2^{-2}E_1K_2K_2'K_3'\widetilde{\mathcal{T}}_1(\mathcal{K}_2^{-1})\widetilde{\mathcal{T}}_{w_\bullet}(\mathcal{K}_2^{-1}).$$

Hence, we rewrite (6.4) as

$$\begin{aligned}
&\widetilde{\mathcal{T}}_{\mathbf{r}_2}\big(\widetilde{\mathcal{T}}_{w_\bullet}(E_4)K_4'\big)\widetilde{\mathcal{T}}_{\mathbf{r}_2}\widetilde{\mathcal{T}}_3(\mathcal{K}_4^{-1})\\
&= q_2^{-4}\big[[\widetilde{\mathcal{T}}_{w_\bullet}(E_4)K_4'\widetilde{\mathcal{T}}_3(\mathcal{K}_4^{-1}), \widetilde{\mathcal{T}}_3^{-1}(\widehat{B}_2)\widetilde{\mathcal{T}}_1(\mathcal{K}_2^{-1})]_{q_3^2}, \widetilde{\mathcal{T}}_3^{-1}(\widehat{B}_2)\widetilde{\mathcal{T}}_1(\mathcal{K}_2^{-1})\big] \qquad (6.5)\\
&+ q_2^{-2}(q_3 - q_3^{-1})[\widetilde{\mathcal{T}}_{w_\bullet}(E_4)K_4'\widetilde{\mathcal{T}}_3(\mathcal{K}_4^{-1}), F_3K_3K_3'^{-1}]_{q_3^2}E_1K_2K_2'K_3'\widetilde{\mathcal{T}}_1(\mathcal{K}_2^{-1})\widetilde{\mathcal{T}}_{w_\bullet}(\mathcal{K}_2^{-1}).
\end{aligned}$$

For a weight reason, we have the following identities:

$$\begin{aligned}
\widetilde{\mathcal{T}}_1(\mathcal{K}_2^{-1})\widetilde{\mathcal{T}}_{w_\bullet}(E_4) &= q_2^2\widetilde{\mathcal{T}}_{w_\bullet}(E_4)\widetilde{\mathcal{T}}_1(\mathcal{K}_2^{-1}),\\
\widetilde{\mathcal{T}}_3(\mathcal{K}_4^{-1})\widetilde{\mathcal{T}}_3^{-1}(\widehat{B}_2) &= q_2^2\widetilde{\mathcal{T}}_3^{-1}(\widehat{B}_2)\widetilde{\mathcal{T}}_3(\mathcal{K}_4^{-1}),\\
\widetilde{\mathcal{T}}_1(\mathcal{K}_2^{-1})\widetilde{\mathcal{T}}_3(\widehat{B}_2) &= q_2^{-2}\widetilde{\mathcal{T}}_3(\widehat{B}_2)\widetilde{\mathcal{T}}_1(\mathcal{K}_2^{1}),
\end{aligned}$$

$$[\widetilde{\mathcal{T}}_3(\mathcal{K}_4^{-1}), F_3] = 0 = [K_3 K_3'^{-1}, \widetilde{\mathcal{T}}_{w_\bullet}(E_4)].$$

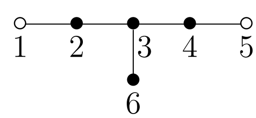
Using the above identities, we rewrite (6.5) as

$$\begin{aligned}
&\widetilde{\mathcal{T}}_{\mathbf{r}_2}\big(\widetilde{\mathcal{T}}_{w_\bullet}(E_4)K_4'\big)\widetilde{\mathcal{T}}_{\mathbf{r}_2}\widetilde{\mathcal{T}}_3(\mathcal{K}_4^{-1}) \\
&= q_2^{-2}\big[[\widetilde{\mathcal{T}}_{w_\bullet}(E_4)K_4', \widetilde{\mathcal{T}}_3^{-1}(\widehat{B}_2)]_{q_3^2}, \widetilde{\mathcal{T}}_3^{-1}(\widehat{B}_2)\big]\widetilde{\mathcal{T}}_3(\mathcal{K}_4^{-1})\widetilde{\mathcal{T}}_1(\mathcal{K}_2^{-2}) \\
&+ q_2^{-2}(q_3 - q_3^{-1})[\widetilde{\mathcal{T}}_{w_\bullet}(E_4)K_4', F_3]_{q_3^2} E_1 K_2 K_2' K_3' \widetilde{\mathcal{T}}_3(\mathcal{K}_4^{-1})\widetilde{\mathcal{T}}_1(\mathcal{K}_2^{-2}). \qquad (6.6)
\end{aligned}$$

Finally, note that $\widetilde{\mathcal{T}}_{\mathbf{r}_2}\widetilde{\mathcal{T}}_3(\mathcal{K}_4^\imath) = q_2^2\widetilde{\mathcal{T}}_3(\mathcal{K}_4)\widetilde{\mathcal{T}}_1(\mathcal{K}_2^2)$, and thus (6.6) implies the desired identity (6.2). □

# 7 Type EIV

Consider the Satake diagram of type EIV




$$\mathbf{r}_1 = s_1 s_2 s_3 s_4 s_6 s_3 s_2 s_1.$$

Denote $\widetilde{\mathcal{T}}_4^{-1}\widetilde{\mathcal{T}}_3^{-1}\widetilde{\mathcal{T}}_2^{-1}$ by $\widetilde{\mathcal{T}}_{432}^{-1}$. According to Table 4, Proposition 6.5 is reformulated and proved as Lemma 7.1 below.

**Lemma 7.1.** *We have*

$$\widetilde{\mathcal{T}}_{\mathbf{r}_1}(F_5) = [F_5, \widetilde{\mathcal{T}}_{432}^{-1}(B_1)]_q, \qquad (7.1)$$

$$\widetilde{\mathcal{T}}_{\mathbf{r}_1}\big(\widetilde{\mathcal{T}}_{w_\bullet}(E_5)K_5'\big) = [\widetilde{\mathcal{T}}_{w_\bullet}(E_5)K_5', \widetilde{\mathcal{T}}_{432}^{-1}(\widehat{B}_1)]_q. \qquad (7.2)$$

*Proof.* The first formula (7.1) is obtained by applying $\sigma$ to $\widetilde{\mathcal{T}}_{\mathbf{r}_1}^{-1}(F_5)$, which is calculated in §6. We shall derive the second formula (7.2) from (7.1) as follows.

Recall from Lemma 1.3 that $\mathcal{D}$ commutes with each of $\widetilde{\mathcal{T}}_{\mathbf{r}_1}, \widetilde{\mathcal{T}}_{432}^{-1}$. By (1.1), we have $\mathcal{D}(F_5) = -q^{-2}\widetilde{\mathcal{T}}_{w_\bullet}(E_5)K_5'\widetilde{\mathcal{T}}_{w_\bullet}(\mathcal{K}_5^{-1})$. Applying $\mathcal{D}$ to (7.1) and using (1.2), we have

$$\begin{aligned}&\widetilde{\mathcal{T}}_{\mathbf{r}_1}\big(\widetilde{\mathcal{T}}_{w_\bullet}(E_5)K_5'\big)\widetilde{\mathcal{T}}_{w_\bullet,1}(\mathcal{K}_5^{-1})\\ &= -q^{-2}[\widetilde{\mathcal{T}}_{w_\bullet}(E_5)K_5'\widetilde{\mathcal{T}}_{w_\bullet}(\mathcal{K}_5^{-1}), \widetilde{\mathcal{T}}_{432}^{-1}(\widehat{B}_1)\widetilde{\mathcal{T}}_{632}(\mathcal{K}_1^{\imath,-1})]_q.\end{aligned} \tag{7.3}$$

Moreover, we have

$$\begin{aligned}\widetilde{\mathcal{T}}_{w_\bullet}(\mathcal{K}_5^{-1})\widetilde{\mathcal{T}}_{432}^{-1}(\widehat{B}_1) &= q\widetilde{\mathcal{T}}_{432}^{-1}(\widehat{B}_1)\widetilde{\mathcal{T}}_{w_\bullet}(\mathcal{K}_5^{-1}),\\ \widetilde{\mathcal{T}}_{632}(\mathcal{K}_1^{-1})\widetilde{\mathcal{T}}_{w_\bullet}(E_5)K_5' &= q\widetilde{\mathcal{T}}_{w_\bullet}(E_5)K_5'\widetilde{\mathcal{T}}_{632}(\mathcal{K}_1^{-1}).\end{aligned}$$

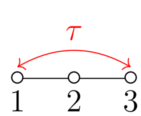
Hence, using these two identities, we rewrite (7.3) as

$$\begin{aligned}&\widetilde{\mathcal{T}}_{\mathbf{r}_1}\big(\widetilde{\mathcal{T}}_{w_\bullet}(E_5)K_5'\big)\widetilde{\mathcal{T}}_{w_\bullet,1}(\mathcal{K}_5^{-1})\\ &= -q^{-1}[\widetilde{\mathcal{T}}_{w_\bullet}(E_5)K_5', \widetilde{\mathcal{T}}_{432}^{-1}(\widehat{B}_1)]_q\widetilde{\mathcal{T}}_{w_\bullet}(\mathcal{K}_5^{\imath,-1})\widetilde{\mathcal{T}}_{632}(\mathcal{K}_1^{\imath,-1}).\end{aligned} \tag{7.4}$$

Finally, note that $\widetilde{\mathcal{T}}_{w_\bullet,1}(\mathcal{K}_5^{-1}) = -q^{-1}\widetilde{\mathcal{T}}_{w_\bullet}(\mathcal{K}_5^{-1})\widetilde{\mathcal{T}}_{632}(\mathcal{K}_1^{\imath,-1})$, and hence (7.4) implies (7.2). □

# 8 Type $\mathrm{AIII}_3$

Consider the following Satake diagram of type $\mathrm{AIII}_3$



$$\varsigma_{1,\diamond} = \varsigma_{3,\diamond} = -q^{-1}, \qquad \varsigma_{2,\diamond} = -q^{-2}$$

$$\mathbf{r}_1 = s_1 s_3, \qquad \mathbf{r}_2 = s_2.$$

In this case, Proposition 6.5 is reformulated and proved as Lemma 8.1 below.

**Lemma 8.1.** *We have*

$$\widetilde{\mathfrak{T}}_{\mathbf{r}_1}(F_2) = \big[[F_2, B_1]_q, B_3\big]_q - qF_2 K_3' K_1, \tag{8.1}$$

$$\widetilde{\mathfrak{T}}_{\mathbf{r}_1}(E_2 K_2') = \big[[E_2 K_2', \widehat{B}_1]_q, \widehat{B}_3\big]_q - qE_2 K_2' K_3' K_1. \tag{8.2}$$

*Proof.* The first identity (8.1) is obtained from (7.1) by applying $\sigma$.

We prove the second identity next. By (1.1), we have $\mathcal{D}(F_2) = -q^{-2} E_2 K_2' \mathcal{K}_2^{-1}$. Applying $\mathcal{D}$ to (8.1) and then using (1.2), we have

$$\widetilde{\mathfrak{T}}_{\mathbf{r}_1}(E_2 K_2' \mathcal{K}_2^{-1}) = q^{-4}\big[[E_2 K_2' \mathcal{K}_2^{-1}, \widehat{B}_1 \mathcal{K}_3^{-1}]_q, \widehat{B_3} \mathcal{K}_1^{-1}\big]_q - qE_2 K_2' \mathcal{K}_2^{-1} \mathcal{D}(K_3' K_1). \tag{8.3}$$

In this case, $\mathcal{K}_2$ is central and $[\mathcal{K}_i, E_2] = 0$ for $i = 1, 3$. Hence, (8.3) is rewritten as

$$\widetilde{\mathfrak{T}}_{\mathbf{r}_1}(E_2 K_2' \mathcal{K}_2^{-1}) = q^{-4}\big[[E_2 K_2', \widehat{B}_1]_q \mathcal{K}_2^{-1} \mathcal{K}_3^{-1}, \widehat{B_3} \mathcal{K}_1^{-1}\big]_q - qE_2 K_2' \mathcal{K}_2^{-1} \mathcal{D}(K_3' K_1). \tag{8.4}$$

For the weight reason, we have commutator relations

$$\mathcal{K}_2^{-1} \mathcal{K}_3^{-1} \widehat{B}_3 = q^2 \widehat{B}_3 \mathcal{K}_2^{-1} \mathcal{K}_3^{-1},$$

$$\mathcal{K}_1^{-1}[E_2 K_2', \widehat{B}_1]_q = q^2 [E_2 K_2', \widehat{B}_1]_q \mathcal{K}_1^{-1}.$$

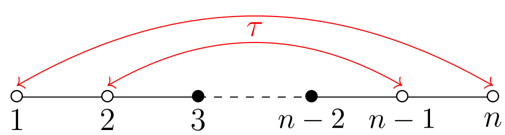
Note also that $\mathcal{D}(K_3' K_1) = q^{-2} K_3' K_1 \mathcal{K}_3^{-1} \mathcal{K}_1^{-1}$. Thus, using the above two commutator relations, we rewrite (8.4) as

$$\widetilde{\mathcal{T}}_{\mathbf{r}_1}(E_2K_2')\widetilde{\mathcal{T}}_{\mathbf{r}_1}(\mathcal{K}_2^{-1}) = q^{-2}\Big(\big[[E_2K_2', \widehat{B}_1]_q, \widehat{B}_3\big]_q - qE_2K_2'K_3'K_1\Big)\mathcal{K}_2^{-1}\mathcal{K}_3^{-1}\mathcal{K}_1^{-1}. \quad (8.5)$$

Finally, since $\widetilde{\mathcal{T}}_{\mathbf{r}_1}(\mathcal{K}_2) = q^2\mathcal{K}_1\mathcal{K}_2\mathcal{K}_3$, clearly (8.5) implies (8.2). □

# 9 Type AIII$_n, n \geq 4$

Consider the Satake diagram of type AIII$_n, n \geq 4$:




$$\varsigma_{1,\diamond} = \varsigma_{n,\diamond} = -q^{-1}, \qquad \varsigma_{2,\diamond} = \varsigma_{n-1,\diamond} = -q^{-1/2}$$

$$\mathbf{r}_1 = s_1 s_n, \qquad \mathbf{r}_2 = s_2 \cdots s_{n-1} \cdots s_2.$$

In this case, Proposition 6.5 is reformulated and proved as Lemma 9.1-9.2 below.

**Lemma 9.1.** *We have*

$$\widetilde{\mathcal{T}}_{\mathbf{r}_1}(F_2) = [F_2, B_1]_q, \quad (9.1)$$

$$\widetilde{\mathcal{T}}_{\mathbf{r}_2}(F_1) = \big[[F_1, B_2]_q, \widetilde{\mathcal{T}}_{w_\bullet}^{-1}(B_{n-1})\big]_q - qF_1K_2K'_{w_\bullet(\alpha_{n-1})}. \quad (9.2)$$

*Proof.* This two formulas are obtained from Lemma 8.2 by applying $\sigma$. □

**Lemma 9.2.** *We have*

$$\widetilde{\mathcal{T}}_{\mathbf{r}_1}\big(\widetilde{\mathcal{T}}_{w_\bullet}(E_{n-1})K_2'\big) = [F_2, \widehat{B}_1]_q, \quad (9.3)$$

$$\widetilde{\mathcal{T}}_{\mathbf{r}_2}(E_nK_1') = \big[[E_nK_1', \widehat{B}_2]_q, \widetilde{\mathcal{T}}_{w_\bullet}^{-1}(\widehat{B}_{n-1})\big]_q - qE_nK_1'K_2K'_{w_\bullet(\alpha_{n-1})}. \quad (9.4)$$

*Proof.* We prove the second formula (9.4). The first one can be obtained relatively easily by a similar strategy.

By (1.1), we have $\mathcal{D}(F_1) = -q^{-2}E_nK_1'\mathcal{K}_n^{-1}$. Recall from Lemma1.3 that $\mathcal{D}$ commutes with $\widetilde{\mathfrak{T}}_{\mathbf{r}_4}, \widetilde{\mathfrak{T}}_{w_\bullet}$. Applying the operator $\mathcal{D}$ to (9.2) and using (1.2), we have

$$\widetilde{\mathfrak{T}}_{\mathbf{r}_2}(E_nK_1'\mathcal{K}_n^{-1}) = q^{-4}\big[[E_nK_1'\mathcal{K}_n^{-1}, \widehat{B}_2\widetilde{\mathfrak{T}}_{w_\bullet}(\mathcal{K}_{n-1}^{-1})]_q, \widetilde{\mathfrak{T}}_{w_\bullet}^{-1}(\widehat{B}_{n-1})\mathcal{K}_2^{-1}\big]_q \tag{9.5}$$
$$- qE_nK_1'\mathcal{K}_n^{-1}\mathcal{D}(K_2K'_{w_\bullet(\alpha_{n-1})}).$$

For the weight reason, we have

$$\mathcal{K}_n^{-1}\widehat{B}_2 = q\widehat{B}_2\mathcal{K}_n^{-1},$$
$$\widetilde{\mathfrak{T}}_{w_\bullet}(\mathcal{K}_{n-1}^{-1})E_nK_1' = qE_nK_1'\widetilde{\mathfrak{T}}_{w_\bullet}(\mathcal{K}_{n-1}^{-1}),$$
$$\mathcal{K}_2^{-1}[E_nK_1', \widehat{B}_2]_q = q^2[E_nK_1', \widehat{B}_2]_q\mathcal{K}_2^{-1},$$
$$\mathcal{K}_n\widetilde{\mathfrak{T}}_{w_\bullet}(\mathcal{K}_{n-1}^{-1})\widetilde{\mathfrak{T}}_{w_\bullet}^{-1}(\widehat{B}_{n-1}) = q^2\widetilde{\mathfrak{T}}_{w_\bullet}^{-1}(\widehat{B}_{n-1})\mathcal{K}_n\widetilde{\mathfrak{T}}_{w_\bullet}(\mathcal{K}_{n-1}^{-1}).$$

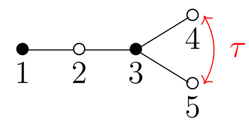
Note also that $\mathcal{D}(K_2K'_{w_\bullet(\alpha_{n-1})}) = q^{-1}K_2K'_{w_\bullet(\alpha_{n-1})}\mathcal{K}_2^{-1}\widetilde{\mathfrak{T}}_{w_\bullet}(\mathcal{K}_{n-1}^{-1})$. Hence, using the above relations, we can rewrite (9.5) as

$$\widetilde{\mathfrak{T}}_{\mathbf{r}_2}(E_nK_1')\widetilde{\mathfrak{T}}_{\mathbf{r}_2}(\mathcal{K}_n^{-1}) = q^{-1}\big[[E_nK_1', \widehat{B}_2]_q, \widetilde{\mathfrak{T}}_{w_\bullet}^{-1}(\widehat{B}_{n-1})\big]_q\mathcal{K}_n^{-1}\mathcal{K}_2^{-1}\widetilde{\mathfrak{T}}_{w_\bullet}(\mathcal{K}_{n-1}^{-1}) \tag{9.6}$$
$$- E_nK_1'K_2K'_{w_\bullet(\alpha_{n-1})}\mathcal{K}_n^{-1}\mathcal{K}_2^{-1}\widetilde{\mathfrak{T}}_{w_\bullet}(\mathcal{K}_{n-1}^{-1}).$$

Finally, by Proposition 4.2, we have $\widetilde{\mathfrak{T}}_{\mathbf{r}_2}(\mathcal{K}_n) = q\mathcal{K}_n\mathcal{K}_2\widetilde{\mathfrak{T}}_{w_\bullet}(\mathcal{K}_{n-1})$. Then clearly (9.6) implies the desired formula (9.4). ☐

# 10 Type DIII$_5$

Consider the Satake diagram of type DIII$_5$




$$\varsigma_{2,\diamond} = -q^{-1}, \qquad \varsigma_{4,\diamond} = \varsigma_{5,\diamond} = -q^{-1/2}$$

$$\mathbf{r}_2 = s_2 s_1 s_3 s_2, \qquad \mathbf{r}_4 = s_4 s_5 s_3 s_4 s_5.$$

In this case, according to Table 4, Proposition 6.5 is reformulated and proved as Lemma 10.1-10.2 below.

**Lemma 10.1.** *We have*

$$\widetilde{\mathcal{T}}_{\mathbf{r}_2}(F_4) = [F_4, \widetilde{\mathcal{T}}_3^{-1}(B_2)]_q, \tag{10.1}$$

$$\widetilde{\mathcal{T}}_{\mathbf{r}_4}(F_2) = \big[[F_2, \widetilde{\mathcal{T}}_3^{-1}(B_5)]_q, B_4\big]_q - q\widetilde{\mathcal{T}}_3^2(F_2)K_4'K_5K_3. \tag{10.2}$$

*Proof.* These two identities follow by applying $\sigma$ to Lemma 9.1. □

**Lemma 10.2.** *We have*

$$\widetilde{\mathcal{T}}_{\mathbf{r}_2}(\widetilde{\mathcal{T}}_{w_\bullet}(E_5)K_4') = [\widetilde{\mathcal{T}}_{w_\bullet}(E_5)K_4', \widetilde{\mathcal{T}}_3^{-1}(\widehat{B}_2)]_q, \tag{10.3}$$

$$\widetilde{\mathcal{T}}_{\mathbf{r}_4}(\widetilde{\mathcal{T}}_{w_\bullet}(E_2)K_2') = \big[[F_2, \widetilde{\mathcal{T}}_3^{-1}(\widehat{B}_5)]_q, \widehat{B}_4\big]_q - q\widetilde{\mathcal{T}}_3^2(F_2)K_4'K_5K_3. \tag{10.4}$$

*Proof.* We prove the second identity (10.4), while omitting a similar (and easier) proof for the first identity. By (1.1) and (1.2), we have

$$\mathcal{D}(F_2) = -q^{-2}\widetilde{\mathcal{T}}_{w_\bullet}(E_2)K_2'\widetilde{\mathcal{T}}_{w_\bullet}(\mathcal{K}_2^{-1}), \qquad \mathcal{D}(B_4) = -q^{-2}\widehat{B_4}\widetilde{\mathcal{T}}_{w_\bullet}(\mathcal{K}_5^{-1}). \tag{10.5}$$

Recall from Lemma 1.3 that $\mathcal{D}$ commutes with $\widetilde{\mathcal{T}}_{\mathbf{r}_4}, \widetilde{\mathcal{T}}_3$. Applying the operator $\mathcal{D}$ to (10.2) and then using (10.5), we have

$$\widetilde{\mathcal{T}}_{\mathbf{r}_4}\big(\widetilde{\mathcal{T}}_{w_\bullet}(E_2)K_2'\big)\widetilde{\mathcal{T}}_{w_\bullet,4}(\mathcal{K}_2^{-1}) = q^{-4}\big[[\widetilde{\mathcal{T}}_{w_\bullet}(E_2)K_2'\widetilde{\mathcal{T}}_{w_\bullet}(\mathcal{K}_2^{-1}), \widetilde{\mathcal{T}}_3^{-1}(\widehat{B}_5)\mathcal{K}_4^{-1}]_q, \widehat{B}_4\widetilde{\mathcal{T}}_3(\mathcal{K}_5^{-1})\big]_q \\ - q\widetilde{\mathcal{T}}_3^2\widetilde{\mathcal{T}}_{w_\bullet}(E_2)K_2'\widetilde{\mathcal{T}}_{w_\bullet}(\mathcal{K}_2^{-1})\mathcal{D}(K_4'K_5K_3). \tag{10.6}$$

For the weight reason, we have

$$\widetilde{\mathcal{T}}_{w_\bullet}(\mathcal{K}_2^{-1})\widetilde{\mathcal{T}}_3^{-1}(\widehat{B}_5) = q\widetilde{\mathcal{T}}_3^{-1}(\widehat{B}_5)\widetilde{\mathcal{T}}_{w_\bullet}(\mathcal{K}_2^{-1}),$$
$$\mathcal{K}_4^{-1}\widetilde{\mathcal{T}}_{w_\bullet}(E_2)K_2' = q\widetilde{\mathcal{T}}_{w_\bullet}(E_2)K_2'\mathcal{K}_4^{-1},$$
$$\widetilde{\mathcal{T}}_3(\mathcal{K}_5^{-1})[\widetilde{\mathcal{T}}_{w_\bullet}(E_2)K_2', \widetilde{\mathcal{T}}_3^{-1}(\widehat{B}_5)]_q = q^2[\widetilde{\mathcal{T}}_{w_\bullet}(E_2)K_2', \widetilde{\mathcal{T}}_3^{-1}(\widehat{B}_5)]_q\widetilde{\mathcal{T}}_3(\mathcal{K}_5^{-1}),$$
$$\widetilde{\mathcal{T}}_{w_\bullet}(\mathcal{K}_2^{-1})\mathcal{K}_4^{-1}\widehat{B}_4 = q^2\widehat{B}_4\widetilde{\mathcal{T}}_{w_\bullet}(\mathcal{K}_2^{-1})\mathcal{K}_4^{-1}.$$

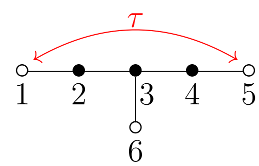
Note also that $\mathcal{D}(K_4'K_5K_3) = q^{-1}K_4'K_5K_3\mathcal{K}_4^{-1}\widetilde{\mathcal{T}}_3(\mathcal{K}_5^{-1})$. Hence, using the above formulas, we rewrite (10.6) as

$$\widetilde{\mathcal{T}}_{\mathbf{r}_4}\big(\widetilde{\mathcal{T}}_{w_\bullet}(E_2)K_2'\big)\widetilde{\mathcal{T}}_{w_\bullet,4}(\mathcal{K}_2^{-1}) = q^{-1}\big[[\widetilde{\mathcal{T}}_{w_\bullet}(E_2)K_2', \widetilde{\mathcal{T}}_3^{-1}(\widehat{B}_5)]_q, \widehat{B}_4\big]_q\widetilde{\mathcal{T}}_{w_\bullet}(\mathcal{K}_2^{-1})\mathcal{K}_4^{-1}\widetilde{\mathcal{T}}_3(\mathcal{K}_5^{-1}) \\ - \widetilde{\mathcal{T}}_3^2\widetilde{\mathcal{T}}_{w_\bullet}(E_2)K_2'K_4'K_5K_3\widetilde{\mathcal{T}}_{w_\bullet}(\mathcal{K}_2^{-1})\mathcal{K}_4^{-1}\widetilde{\mathcal{T}}_3(\mathcal{K}_5^{-1}). \tag{10.7}$$

Finally, by Proposition 4.2, the common factor $\widetilde{\mathcal{T}}_{w_\bullet}(\mathcal{K}_2^{-1})\mathcal{K}_4^{-1}\widetilde{\mathcal{T}}_3(\mathcal{K}_5^{-1})$ on the RHS equals $q\widetilde{\mathcal{T}}_{w_\bullet,4}(\mathcal{K}_2^{-1})$. Hence, the identity (10.7) implies (10.4) as desired. □

# 11 Type EIII

Consider the Satake diagram of type EIII




$$\varsigma_{1,\diamond} = \varsigma_{5,\diamond} = -q^{-1/2}, \qquad \varsigma_{6,\diamond} = -q^{-1}$$

$$\mathbf{r}_1 = s_1 \cdots s_5 \cdots s_1, \qquad \mathbf{r}_6 = s_6 s_3 s_2 s_4 s_3 s_6$$

$$w_\bullet = s_3 s_2 s_4 s_3 s_2 s_4 = s_2 s_4 s_3 s_2 s_4 s_3.$$

According to Table 4, Proposition 6.5 is reformulated and proved as Lemma 11.1-11.2 below.

**Lemma 11.1.** *We have*

$$\widetilde{\mathcal{T}}_{\mathbf{r}_6}(F_1) = [F_1, \widetilde{\mathcal{T}}_{23}^{-1}(B_6)]_q, \tag{11.1}$$

$$\widetilde{\mathcal{T}}_{\mathbf{r}_1}(F_6) = \big[[F_6, \widetilde{\mathcal{T}}_{32}^{-1}(B_1)]_q, \widetilde{\mathcal{T}}_4^{-1}(B_5)\big]_q - q\widetilde{\mathcal{T}}_{32323}(F_6)K_1K_2K_3K_4'K_5' \tag{11.2}$$

*Proof.* These two identities follow by applying $\sigma$ to the identities in Lemma 10.1. □

**Lemma 11.2.** *We have*

$$\widetilde{\mathcal{T}}_{\mathbf{r}_6}(\widetilde{\mathcal{T}}_{w_\bullet}(E_5)K_1') = [\widetilde{\mathcal{T}}_{w_\bullet}(E_5)K_1', \widetilde{\mathcal{T}}_{23}^{-1}(\widehat{B}_6)]_q, \tag{11.3}$$

$$\begin{aligned}\widetilde{\mathcal{T}}_{\mathbf{r}_1}(\widetilde{\mathcal{T}}_{w_\bullet}(E_6)K_6') = &\big[[\widetilde{\mathcal{T}}_{w_\bullet}(E_6)K_6', \widetilde{\mathcal{T}}_{32}^{-1}(\widehat{B}_1)]_q, \widetilde{\mathcal{T}}_4^{-1}(\widehat{B}_5)\big]_q \\ &- q\widetilde{\mathcal{T}}_{32323}\big(\widetilde{\mathcal{T}}_{w_\bullet}(E_6)K_6'\big)K_1K_2K_3K_4'K_5' \end{aligned}\tag{11.4}$$

*Proof.* We prove (11.4) here, while skipping the similar (and easier) proof for the other formula (11.3). By (1.1), we have $\mathcal{D}(F_6) = -q^{-2}\widetilde{\mathcal{T}}_{w_\bullet}(E_6)K_6'\widetilde{\mathcal{T}}_{w_\bullet}(\mathcal{K}_6^{-1})$. Recall from Lemma 1.3 that $\mathcal{D}$ commutes with $\widetilde{\mathcal{T}}_{\mathbf{r}_1}$ and $\widetilde{\mathcal{T}}_{32323}$. Applying $\mathcal{D}$ to (11.2) and then using (1.2), we have

$$\widetilde{\mathcal{T}}_{\mathbf{r}_1}(\widetilde{\mathcal{T}}_{w_\bullet}(E_6)K_6')\widetilde{\mathcal{T}}_{w_\bullet,1}(\mathcal{K}_6^{-1})$$

$$= q^{-4}\big[[\widetilde{\mathcal{T}}_{w_\bullet}(E_6)K_6'\widetilde{\mathcal{T}}_{w_\bullet}(\mathcal{K}_6^{-1}), \widetilde{\mathcal{T}}_{32}^{-1}(\widehat{B}_1)\widetilde{\mathcal{T}}_{4323}(\mathcal{K}_5^{-1})]_q, \widetilde{\mathcal{T}}_4^{-1}(\widehat{B}_5)\widetilde{\mathcal{T}}_{23243}(\mathcal{K}_1^{-1})\big]_q$$
$$- q\widetilde{\mathcal{T}}_{32323}\big(\widetilde{\mathcal{T}}_{w_\bullet}(E_6)K_6'\big)\widetilde{\mathcal{T}}_{w_\bullet}(\mathcal{K}_6^{-1})\mathcal{D}(K_1K_2K_3K_4'K_5') \tag{11.5}$$

Note that $\widetilde{\mathcal{T}}_{4323}(\mathcal{K}_5^{-1}) = \widetilde{\mathcal{T}}_4(K_5^{-1})\widetilde{\mathcal{T}}_{32}(K_1'^{-1})$ and $\widetilde{\mathcal{T}}_{23243}(\mathcal{K}_1^{-1}) = \widetilde{\mathcal{T}}_{32}(K_1^{-1})\widetilde{\mathcal{T}}_4(K_5'^{-1})$. For the weight reason, we have

$$\widetilde{\mathcal{T}}_{4323}(\mathcal{K}_5^{-1})\widetilde{\mathcal{T}}_{w_\bullet}(E_6) = q\widetilde{\mathcal{T}}_{w_\bullet}(E_6)\widetilde{\mathcal{T}}_{4323}(\mathcal{K}_5^{-1}),$$
$$\widetilde{\mathcal{T}}_{w_\bullet}(\mathcal{K}_6^{-1})\widetilde{\mathcal{T}}_{32}(B_1) = q\widetilde{\mathcal{T}}_{32}(B_1)\widetilde{\mathcal{T}}_{w_\bullet}(\mathcal{K}_6^{-1}),$$
$$\widetilde{\mathcal{T}}_{23243}(\mathcal{K}_1^{-1})[\widetilde{\mathcal{T}}_{w_\bullet}(E_6)K_6', \widetilde{\mathcal{T}}_{32}^{-1}(\widehat{B}_1)]_q = q^2[\widetilde{\mathcal{T}}_{w_\bullet}(E_6)K_6', \widetilde{\mathcal{T}}_{32}^{-1}(\widehat{B}_1)]_q\widetilde{\mathcal{T}}_{23243}(\mathcal{K}_1^{-1}),$$
$$\widetilde{\mathcal{T}}_{4323}(\mathcal{K}_5^{-1})\widetilde{\mathcal{T}}_{w_\bullet}(\mathcal{K}_6^{-1})\widetilde{\mathcal{T}}_4^{-1}(\widehat{B}_5) = q^2\widetilde{\mathcal{T}}_4^{-1}(\widehat{B}_5)\widetilde{\mathcal{T}}_{4323}(\mathcal{K}_5^{-1})\widetilde{\mathcal{T}}_{w_\bullet}(\mathcal{K}_6^{-1}).$$

Note that

$$\mathcal{D}(K_1K_2K_3K_4'K_5') = q^{-1}\widetilde{\mathcal{T}}_4(K_5'^{-1})\widetilde{\mathcal{T}}_{32}(K_1^{-1})$$
$$= K_1K_2K_3K_4'K_5'\widetilde{\mathcal{T}}_4(K_5K_5')^{-1}\widetilde{\mathcal{T}}_{32}(K_1K_1')^{-1}. \tag{11.6}$$

Hence, using (11.6) and the previous four identities, we rewrite (11.5) as

$$\widetilde{\mathcal{T}}_{\mathbf{r}_1}(\widetilde{\mathcal{T}}_{w_\bullet}(E_6)K_6')\widetilde{\mathcal{T}}_{w_{\bullet,1}}(\mathcal{K}_6^{-1})$$
$$= q^{-1}\big[[\widetilde{\mathcal{T}}_{w_\bullet}(E_6)K_6', \widetilde{\mathcal{T}}_{32}^{-1}(\widehat{B}_1)]_q, \widetilde{\mathcal{T}}_4^{-1}(\widehat{B}_5)\big]_q\widetilde{\mathcal{T}}_{w_\bullet}(\mathcal{K}_6^{-1})\widetilde{\mathcal{T}}_4(K_5K_5')^{-1}\widetilde{\mathcal{T}}_{32}(K_1K_1')^{-1}$$
$$- \widetilde{\mathcal{T}}_{32323}\big(\widetilde{\mathcal{T}}_{w_\bullet}(E_6)K_6'\big)K_1K_2K_3K_4'K_5'\widetilde{\mathcal{T}}_{w_\bullet}(\mathcal{K}_6^{-1})\widetilde{\mathcal{T}}_4(K_5K_5')^{-1}\widetilde{\mathcal{T}}_{32}(K_1K_1')^{-1} \tag{11.7}$$

Finally, by Proposition 4.2, we have $\widetilde{\mathcal{T}}_{w_{\bullet,1}}(\mathcal{K}_6) = q\widetilde{\mathcal{T}}_{w_\bullet}(\mathcal{K}_6)\widetilde{\mathcal{T}}_4(K_5K_5')\widetilde{\mathcal{T}}_{32}(K_1K_1')$. Then (11.7) implies the desired formula (11.4). □

# Appendix C

# Proof of Proposition 16.10

## 1 $(16.26)_k \Rightarrow (16.25)_{k+1}$

Recall that $\alpha$ denotes $-c_{ij}$.

**Lemma 1.1.** *We have, for any $a, b \in \mathbb{N}$,*

$$[2b] \begin{bmatrix} a \\ b \end{bmatrix}_{q_i^2} = [2a] \begin{bmatrix} a-1 \\ b-1 \end{bmatrix}_{q_i^2}.$$

**Lemma 1.2.** *We have for any $m \geq 0$,*

$$b_{i,j;m} B_i = q_i^{2m+c_{ij}} B_i b_{i,j;m} - [m+1] q_i^{2m+c_{ij}} b_{i,j;m+1} - [c_{ij} + m - 1] b_{i,j;m-1}.$$

We show that $(16.26)_k \Rightarrow (16.25)_{k+1}$ for fixed $\alpha$. We often omit the index $i$ for a quantum integer $[a]_i$.

On one hand, we have

$$b_{i,j;\alpha}B^{(k)}_{i,\overline{k+1}}B_i = [k+1]b_{i,j;\alpha}B^{(k+1)}_{i,\overline{k+1}}. \tag{1.1}$$

On the other hand, using Lemma 1.2 and $(16.26)_k$, we compute LHS as follows

$$\begin{aligned} b_{i,j;\alpha}B^{(k)}_{i,\overline{k+1}}B_i =& \sum_{x=0}^{\alpha} q_i^{(k-x)(\alpha-x)} \left( \sum_{y=0}^{\lfloor\frac{\alpha-x}{2}\rfloor} (-1)^y q_i^{2y(\lfloor\frac{\alpha-x}{2}\rfloor-\alpha+x)} \begin{bmatrix} \lfloor\frac{\alpha-x}{2}\rfloor \\ y \end{bmatrix}_{q_i^2} B^{(k-x-2y)}_{i,\overline{k+1+c_{ij}}} \right) \times \\ & \times \left( q_i^{\alpha-2x}B_i b_{i,j;\alpha-x} - q_i^{\alpha-2x}[\alpha-x+1]b_{i,j;\alpha-x+1} + [x+1]b_{i,j;\alpha-x-1} \right) \\ =& \sum_{x=0}^{\alpha} c_x b_{i,j;\alpha-x}, \end{aligned} \tag{1.2}$$

where the coefficient $c_x$ of $b_{i,j;\alpha-x}$ is given by

$$\begin{aligned} c_x =& q_i^{\alpha-2x} q_i^{(k-x)(\alpha-x)} \sum_{y=0}^{\lfloor\frac{\alpha-x}{2}\rfloor} (-1)^y q_i^{2y(\lfloor\frac{\alpha-x}{2}\rfloor-\alpha+x)} \begin{bmatrix} \lfloor\frac{\alpha-x}{2}\rfloor \\ y \end{bmatrix}_{q_i^2} B^{(k-x-2y)}_{i,\overline{k+1+c_{ij}}} B_i \\ & - q_i^{-k-1}[\alpha-x] q_i^{(k-x)(\alpha-x)} \sum_{y=0}^{\lfloor\frac{\alpha-x-1}{2}\rfloor} (-1)^y q_i^{2y(\lfloor\frac{\alpha-x-1}{2}\rfloor-\alpha+x+1)} \begin{bmatrix} \lfloor\frac{\alpha-x-1}{2}\rfloor \\ y \end{bmatrix}_{q_i^2} B^{(k-x-2y-1)}_{i,\overline{k+1+c_{ij}}} \\ & + [x] q_i^{(k-x+1)(\alpha-x+1)} \sum_{y=0}^{\lfloor\frac{\alpha-x+1}{2}\rfloor} (-1)^y q_i^{2y(\lfloor\frac{\alpha-x+1}{2}\rfloor-\alpha+x-1)} \begin{bmatrix} \lfloor\frac{\alpha-x+1}{2}\rfloor \\ y \end{bmatrix}_{q_i^2} B^{(k-x-2y+1)}_{i,\overline{k+1+c_{ij}}}. \end{aligned}$$

We simplify this formula in two cases:

(1) $\alpha - x$ is even. Then $k-x-2y$ and $k+1+\alpha$ have different parities, which

implies $B^{(k-x-2y)}_{i,\overline{k+1+c_{ij}}} B_i = [k-x-2y+1] B^{(k-x-2y+1)}_{i,\overline{k+1+c_{ij}}}$. By Lemma 1.1, we have

$$[\alpha-x] \begin{bmatrix} \lfloor \frac{\alpha-x-1}{2} \rfloor \\ y-1 \end{bmatrix}_{q_i^2} = [\alpha-x] \begin{bmatrix} \frac{\alpha-x}{2}-1 \\ y-1 \end{bmatrix}_{q_i^2} = [2y] \begin{bmatrix} \frac{\alpha-x}{2} \\ y \end{bmatrix}_{q_i^2}.$$

Hence, we have

$$\begin{aligned} c_x &= q_i^{\alpha-2x} q_i^{(k-x)(\alpha-x)} \sum_{y=0}^{\frac{\alpha-x}{2}} (-1)^y q_i^{y(-\alpha+x)} [k-x-2y+1] \begin{bmatrix} \frac{\alpha-x}{2} \\ y \end{bmatrix}_{q_i^2} B^{(k-x-2y+1)}_{i,\overline{k+1+c_{ij}}} \\ &\quad + q_i^{-k-1} q_i^{(k-x)(\alpha-x)} \sum_{y=1}^{\frac{\alpha-x}{2}} (-1)^y q_i^{(y-1)(-\alpha+x)} [2y] \begin{bmatrix} \frac{\alpha-x}{2} \\ y \end{bmatrix}_{q_i^2} B^{(k-x-2y+1)}_{i,\overline{k+1+c_{ij}}} \\ &\quad + [x] q_i^{(k-x+1)(\alpha-x+1)} \sum_{y=0}^{\frac{\alpha-x}{2}} (-1)^y q_i^{2y(\frac{\alpha-x}{2}-1-\alpha+x)} \begin{bmatrix} \frac{\alpha-x}{2} \\ y \end{bmatrix}_{q_i^2} B^{(k-x-2y+1)}_{i,\overline{k+1+c_{ij}}} \\ &= q_i^{(k+1-x)(\alpha-x)} [k+1] \sum_{y=0}^{\frac{\alpha-x}{2}} (-1)^y q_i^{y(-\alpha+x-2)} \begin{bmatrix} \frac{\alpha-x}{2} \\ y \end{bmatrix}_{q_i^2} B^{(k-x-2y+1)}_{i,\overline{k+1+c_{ij}}}. \end{aligned}$$

This formula, together with (1.1)-(1.2), verified the coefficient of $b_{i,j;\alpha-x}$ in $(16.25)_{k+1}$ for any $\alpha-x$ even.

(2) $\alpha-x$ is odd. Then $k-x-2y$ and $k+1+\alpha$ have the same parity, which implies

$$B^{(k-x-2y)}_{i,\overline{k+1+c_{ij}}} B_i = [k-x-2y+1] B^{(k-x-2y+1)}_{i,\overline{k+1+c_{ij}}} - q_i^{-1} [k-x-2y] B^{(k-x-2y-1)}_{i,\overline{k+1+c_{ij}}}.$$

Hence, we have

$$
\begin{aligned}
c_x = {} & q_i^{(k+1-x)(\alpha-x)-x} \sum_{y=0}^{\frac{\alpha-x-1}{2}} (-1)^y q_i^{y(-\alpha-1+x)} [k-x-2y+1] \begin{bmatrix} \frac{\alpha-x-1}{2} \\ y \end{bmatrix}_{q_i^2} B_{i,\overline{k+1+c_{ij}}}^{(k-x-2y+1)} \\
& + q_i^{(k+1-x)(\alpha-x)-x-1} \sum_{y=1}^{\frac{\alpha-x+1}{2}} (-1)^y q_i^{(y-1)(-\alpha-1+x)} [k-x-2y+2] \begin{bmatrix} \frac{\alpha-x-1}{2} \\ y-1 \end{bmatrix}_{q_i^2} B_{i,\overline{k+1+c_{ij}}}^{(k-x-2y+1)} \\
& + q_i^{-k-1} q_i^{(k-x)(\alpha-x)} [\alpha-x] \sum_{y=1}^{\frac{\alpha-x+1}{2}} (-1)^y q_i^{(y-1)(-\alpha+1+x)} \begin{bmatrix} \frac{\alpha-x-1}{2} \\ y-1 \end{bmatrix}_{q_i^2} B_{i,\overline{k+1+c_{ij}}}^{(k-x-2y+1)} \\
& + [x] q_i^{(k-x+1)(\alpha-x+1)} \sum_{y=0}^{\frac{\alpha-x+1}{2}} (-1)^y q_i^{y(-\alpha-1+x)} \begin{bmatrix} \frac{\alpha-x+1}{2} \\ y \end{bmatrix}_{q_i^2} B_{i,\overline{k+1+c_{ij}}}^{(k-x-2y+1)}
\end{aligned}
$$

Applying the following $q$-binomial identity to the first line of the above formula,

$$
\begin{bmatrix} \frac{\alpha-x-1}{2} \\ y \end{bmatrix}_{q_i^2} = q_i^{2y} \begin{bmatrix} \frac{\alpha-x+1}{2} \\ y \end{bmatrix}_{q_i^2} - q_i^{\alpha-x+1} \begin{bmatrix} \frac{\alpha-x-1}{2} \\ y-1 \end{bmatrix}_{q_i^2},
$$

we obtain

$$
\begin{aligned}
c_x = {} & q_i^{\alpha-x-k} q_i^{(k-x)(\alpha-x)} \sum_{y=1}^{\frac{\alpha-x+1}{2}} (-1)^y q_i^{(y-1)(-\alpha+1+x)} \begin{bmatrix} \frac{\alpha-x-1}{2} \\ y-1 \end{bmatrix}_{q_i^2} B_{i,\overline{k+1+c_{ij}}}^{(k-x-2y+1)} \\
& + q_i^{-k-1} q_i^{(k-x)(\alpha-x)} [\alpha-x] \sum_{y=1}^{\frac{\alpha-x+1}{2}} (-1)^y q_i^{(y-1)(-\alpha+1+x)} \begin{bmatrix} \frac{\alpha-x-1}{2} \\ y-1 \end{bmatrix}_{q_i^2} B_{i,\overline{k+1+c_{ij}}}^{(k-x-2y+1)}
\end{aligned}
$$

$$+ q_i^{(k-x+1)(\alpha-x)} \sum_{y=0}^{\frac{\alpha-x+1}{2}} (-1)^y q_i^{y(-\alpha+1+x)} [k-2y+1] \begin{bmatrix} \frac{\alpha-x+1}{2} \\ y \end{bmatrix}_{q_i^2} B_{i,\overline{k+1+c_{ij}}}^{(k-x-2y+1)}$$

$$= q_i^{(k-x)(\alpha-x)-k} [\alpha-x+1] \sum_{y=1}^{\frac{\alpha-x+1}{2}} (-1)^y q_i^{(y-1)(-\alpha+1+x)} \begin{bmatrix} \frac{\alpha-x-1}{2} \\ y-1 \end{bmatrix}_{q_i^2} B_{i,\overline{k+1+c_{ij}}}^{(k-x-2y+1)}$$

$$+ q_i^{(k-x+1)(\alpha-x)} \sum_{y=0}^{\frac{\alpha-x+1}{2}} (-1)^y q_i^{y(-\alpha+1+x)} [k-2y+1] \begin{bmatrix} \frac{\alpha-x+1}{2} \\ y \end{bmatrix}_{q_i^2} B_{i,\overline{k+1+c_{ij}}}^{(k-x-2y+1)}$$

Using Lemma 1.1, we further simplify the formula for $c_x$ as follows

$$c_x = q_i^{(k-x+1)(\alpha-x)-k-1} \left( \sum_{y=0}^{\frac{\alpha-x+1}{2}} (-1)^y q_i^{y(-\alpha+1+x)} [2y] \begin{bmatrix} \frac{\alpha-x+1}{2} \\ y \end{bmatrix}_{q_i^2} B_{i,\overline{k+1+c_{ij}}}^{(k-x-2y+1)} \right)$$

$$+ q_i^{(k-x+1)(\alpha-x)} \left( \sum_{y=0}^{\frac{\alpha-x+1}{2}} (-1)^y q_i^{y(-\alpha+1+x)} [k-2y+1] \begin{bmatrix} \frac{\alpha-x+1}{2} \\ y \end{bmatrix}_{q_i^2} B_{i,\overline{k+1+c_{ij}}}^{(k-x-2y+1)} \right)$$

$$= q_i^{(k-x+1)(\alpha-x)} [k+1] \left( \sum_{y=0}^{\frac{\alpha-x+1}{2}} (-1)^y q_i^{y(-\alpha-1+x)} \begin{bmatrix} \frac{\alpha-x+1}{2} \\ y \end{bmatrix}_{q_i^2} B_{i,\overline{k+1+c_{ij}}}^{(k-x-2y+1)} \right)$$

This formula, together with (1.1)-(1.2), verified the coefficient of $b_{i,j;\alpha-x}$ in $(16.25)_{k+1}$ for any $\alpha - x$ odd.

Therefore, combining above two cases, we have proved $(16.26)_k \Rightarrow (16.25)_{k+1}$.

## 2 $(16.25)_k + (16.26)_{k-1} \Rightarrow (16.26)_{k+1}$

In this subsection, we show that $(16.25)_k + (16.26)_{k-1} \Rightarrow (16.26)_{k+1}$ for fixed $\alpha$.

On one hand, we have

$$b_{i,j;\alpha}B^{(k)}_{i,\bar{k}}B_i + q_i^{-1}[k]b_{i,j;\alpha}B^{(k-1)}_{i,\bar{k}} = [k+1]b_{i,j;\alpha}B^{(k+1)}_{i,\bar{k}}. \tag{2.1}$$

On the other hand, using Lemma 1.2 and $(16.25)_k + (16.26)_{k-1}$, we compute LHS as follows

$$\begin{aligned}
& b_{i,j;\alpha}B^{(k)}_{i,\bar{k}}B_i + q_i^{-1}[k]b_{i,j;\alpha}B^{(k-1)}_{i,\bar{k}} \\
=& \sum_{x=0}^{\alpha} q_i^{(k-x)(\alpha-x)} \left( \sum_{y=0}^{\lceil\frac{\alpha-x}{2}\rceil} (-1)^y q_i^{2y(\lceil\frac{\alpha-x}{2}\rceil-1-\alpha+x)} \begin{bmatrix} \lceil\frac{\alpha-x}{2}\rceil \\ y \end{bmatrix}_{q_i^2} B^{(k-x-2y)}_{i,\overline{k+c_{ij}}} \right) \times \\
& \times \left( q_i^{\alpha-2x}B_i b_{i,j;\alpha-x} - q_i^{\alpha-2x}[\alpha-x+1]b_{i,j;\alpha-x+1} + [x+1]b_{i,j;\alpha-x-1} \right) \\
& + [k]\sum_{x=0}^{\alpha} q_i^{(k-1-x)(\alpha-x)-1} \left( \sum_{y=0}^{\lfloor\frac{\alpha-x}{2}\rfloor} (-1)^y q_i^{2y(\lfloor\frac{\alpha-x}{2}\rfloor-\alpha+x)} \begin{bmatrix} \lfloor\frac{\alpha-x}{2}\rfloor \\ y \end{bmatrix}_{q_i^2} B^{(k-1-x-2y)}_{i,\overline{k+c_{ij}}} \right) b_{i,j;\alpha-x} \\
=& \sum_{x=0}^{\alpha} d_x b_{i,j;\alpha-x},
\end{aligned} \tag{2.2}$$

where the coefficient $d_x$ of $b_{i,j;\alpha-x}$ is given by

$$\begin{aligned}
d_x =& q_i^{\alpha-2x}q_i^{(k-x)(\alpha-x)} \left( \sum_{y=0}^{\lceil\frac{\alpha-x}{2}\rceil} (-1)^y q_i^{2y(\lceil\frac{\alpha-x}{2}\rceil-1-\alpha+x)} \begin{bmatrix} \lceil\frac{\alpha-x}{2}\rceil \\ y \end{bmatrix}_{q_i^2} B^{(k-x-2y)}_{i,\overline{k+c_{ij}}} B_i \right) \\
& - q_i^{-k-1}[\alpha-x]q_i^{(k-x)(\alpha-x)} \left( \sum_{y=0}^{\lceil\frac{\alpha-x-1}{2}\rceil} (-1)^y q_i^{2y(\lceil\frac{\alpha-x-1}{2}\rceil-\alpha+x)} \begin{bmatrix} \lceil\frac{\alpha-x-1}{2}\rceil \\ y \end{bmatrix}_{q_i^2} B^{(k-x-2y-1)}_{i,\overline{k+c_{ij}}} \right) \\
& + [x]q_i^{(k-x+1)(\alpha-x+1)} \left( \sum_{y=0}^{\lceil\frac{\alpha-x+1}{2}\rceil} (-1)^y q_i^{2y(\lceil\frac{\alpha-x+1}{2}\rceil-\alpha+x-2)} \begin{bmatrix} \lceil\frac{\alpha-x+1}{2}\rceil \\ y \end{bmatrix}_{q_i^2} B^{(k-x-2y+1)}_{i,\overline{k+c_{ij}}} \right)
\end{aligned}$$

$$+ [k]q_i^{(k-1-x)(\alpha-x)-1} \left( \sum_{y=0}^{\lfloor\frac{\alpha-x}{2}\rfloor} (-1)^y q_i^{2y(\lfloor\frac{\alpha-x}{2}\rfloor-\alpha+x)} \begin{bmatrix} \lfloor\frac{\alpha-x}{2}\rfloor \\ y \end{bmatrix}_{q_i^2} B_{i,\overline{k+c_{ij}}}^{(k-1-x-2y)} \right).$$

We simplify this formula in two cases:

(1) $\alpha - x$ is odd. In this case, $k-x-2y$ and $k+\alpha$ has different parities, which implies $B_{i,\overline{k+c_{ij}}}^{(k-x-2y)} B_i = [k-x-2y+1] B_{i,\overline{k+c_{ij}}}^{(k-x-2y+1)}$.

We simplify $d_x$ as

$$d_x = q_i^{(k-x+1)(\alpha-x)-x} \sum_{y=0}^{\frac{\alpha-x+1}{2}} (-1)^y q_i^{y(-1-\alpha+x)} [k-x-2y+1] \begin{bmatrix} \frac{\alpha-x+1}{2} \\ y \end{bmatrix}_{q_i^2} B_{i,\overline{k+c_{ij}}}^{(k-x-2y+1)}$$

$$+ q_i^{-k-1}[\alpha-x] q_i^{(k-x)(\alpha-x)} \sum_{y=1}^{\frac{\alpha-x+1}{2}} (-1)^y q_i^{(y-1)(-1-\alpha+x)} \begin{bmatrix} \frac{\alpha-x-1}{2} \\ y-1 \end{bmatrix}_{q_i^2} B_{i,\overline{k+c_{ij}}}^{(k-x-2y+1)}$$

$$+ [x] q_i^{(k-x+1)(\alpha-x+1)} \sum_{y=0}^{\frac{\alpha-x+1}{2}} (-1)^y q_i^{y(-\alpha+x-3)} \begin{bmatrix} \frac{\alpha-x+1}{2} \\ y \end{bmatrix}_{q_i^2} B_{i,\overline{k+c_{ij}}}^{(k-x-2y+1)}$$

$$- [k] q_i^{(k-1-x)(\alpha-x)-1} \sum_{y=1}^{\frac{\alpha-x+1}{2}} (-1)^y q_i^{(y-1)(-1-\alpha+x)} \begin{bmatrix} \frac{\alpha-x-1}{2} \\ y-1 \end{bmatrix}_{q_i^2} B_{i,\overline{k+c_{ij}}}^{(k-x-2y+1)}$$

$$= q_i^{(k-x+1)(\alpha-x)} \sum_{y=0}^{\frac{\alpha-x+1}{2}} (-1)^y q_i^{y(-1-\alpha+x)} [k-2y+1] \begin{bmatrix} \frac{\alpha-x+1}{2} \\ y \end{bmatrix}_{q_i^2} B_{i,\overline{k+c_{ij}}}^{(k-x-2y+1)}$$

$$+ q_i^{-k-1}[\alpha-x] q_i^{(k-x+1)(\alpha-x)+1} \sum_{y=1}^{\frac{\alpha-x+1}{2}} (-1)^y q_i^{y(-1-\alpha+x)} \begin{bmatrix} \frac{\alpha-x-1}{2} \\ y-1 \end{bmatrix}_{q_i^2} B_{i,\overline{k+c_{ij}}}^{(k-x-2y+1)}$$

$$- [k] q_i^{(k-1-x)(\alpha-x)-1} \sum_{y=1}^{\frac{\alpha-x+1}{2}} (-1)^y q_i^{(y-1)(-1-\alpha+x)} \begin{bmatrix} \frac{\alpha-x-1}{2} \\ y-1 \end{bmatrix}_{q_i^2} B_{i,\overline{k+c_{ij}}}^{(k-x-2y+1)}.$$

Rewrite the first line using $[k-2y+1] = q_i^{-2y}[k+1] - q_i^{-k-1}[2y]$ and apply Lemma 1.1. Then we obtain

$$d_x = q_i^{(k-x+1)(\alpha-x)}[k+1] \sum_{y=0}^{\frac{\alpha-x+1}{2}} (-1)^y q_i^{y(-3-\alpha+x)} \begin{bmatrix} \frac{\alpha-x+1}{2} \\ y \end{bmatrix}_{q_i^2} B_{i,\overline{k+c_{ij}}}^{(k-x-2y+1)}$$
$$- q_i^{(k-1-x)(\alpha-x)+x-\alpha-1}[k+1] \sum_{y=1}^{\frac{\alpha-x+1}{2}} (-1)^y q_i^{y(-1-\alpha+x)} \begin{bmatrix} \frac{\alpha-x-1}{2} \\ y-1 \end{bmatrix}_{q_i^2} B_{i,\overline{k+c_{ij}}}^{(k-x-2y+1)}.$$

Finally, applying the following $q$-binomial identity to this formula of $d_x$

$$\begin{bmatrix} \frac{\alpha-x+1}{2} \\ y \end{bmatrix}_{q_i^2} = q_i^{2y} \begin{bmatrix} \frac{\alpha-x-1}{2} \\ y \end{bmatrix}_{q_i^2} + q_i^{x-\alpha+2y-1} \begin{bmatrix} \frac{\alpha-x-1}{2} \\ y-1 \end{bmatrix}_{q_i^2},$$

it is clear that $d_x/[k+1]$ equals the coefficient of $b_{i,j;\alpha-x}$ in $(16.26)_{k+1}$ for any $\alpha - x$ odd.

(2) $\alpha - x$ is even. In this case, $k - x - 2y$ and $k + \alpha$ has the same parity, which implies

$$B_{i,\overline{k+c_{ij}}}^{(k-x-2y)} B_i = [k-x-2y+1] B_{i,\overline{k+c_{ij}}}^{(k-x-2y+1)} - q_i^{-1}[k-x-2y] B_{i,\overline{k+c_{ij}}}^{(k-x-2y-1)}.$$

We simplify $d_x$ as

$$d_x = q_i^{\alpha-2x} q_i^{(k-x)(\alpha-x)} \sum_{y=0}^{\frac{\alpha-x}{2}} (-1)^y q_i^{y(-2-\alpha+x)} [k-x-2y+1] \begin{bmatrix} \frac{\alpha-x}{2} \\ y \end{bmatrix}_{q_i^2} B_{i,\overline{k+c_{ij}}}^{(k-x-2y+1)}$$
$$+ q_i^{\alpha-2x-1} q_i^{(k-x)(\alpha-x)} \sum_{y=1}^{\frac{\alpha-x}{2}+1} (-1)^y q_i^{(y-1)(-2-\alpha+x)} [k-x-2y+2] \begin{bmatrix} \frac{\alpha-x}{2} \\ y-1 \end{bmatrix}_{q_i^2} B_{i,\overline{k+c_{ij}}}^{(k-x-2y+1)}$$

$$+ q_i^{-k-1}[\alpha - x] q_i^{(k-x)(\alpha-x)} \sum_{y=1}^{\frac{\alpha-x}{2}+1} (-1)^y q_i^{(y-1)(-\alpha+x)} \begin{bmatrix} \frac{\alpha-x}{2} \\ y-1 \end{bmatrix}_{q_i^2} B_{i,\overline{k+c_{ij}}}^{(k-x-2y+1)}$$

$$+ [x] q_i^{(k-x+1)(\alpha-x+1)} \sum_{y=0}^{\frac{\alpha-x}{2}+1} (-1)^y q_i^{y(-2-\alpha+x)} \begin{bmatrix} \frac{\alpha-x}{2}+1 \\ y \end{bmatrix}_{q_i^2} B_{i,\overline{k+c_{ij}}}^{(k-x-2y+1)}$$

$$- [k] q_i^{(k-1-x)(\alpha-x)-1} \sum_{y=1}^{\frac{\alpha-x}{2}+1} (-1)^y q_i^{(y-1)(-\alpha+x)} \begin{bmatrix} \frac{\alpha-x}{2} \\ y-1 \end{bmatrix}_{q_i^2} B_{i,\overline{k+c_{ij}}}^{(k-x-2y+1)}.$$

Applying the following $q$-binomial identity to the above formula of $d_x$

$$\begin{bmatrix} \frac{\alpha-x}{2}+1 \\ y \end{bmatrix}_{q_i^2} = q_i^{2y} \begin{bmatrix} \frac{\alpha-x}{2} \\ y \end{bmatrix}_{q_i^2} + q_i^{x-\alpha+2y-2} \begin{bmatrix} \frac{\alpha-x}{2} \\ y-1 \end{bmatrix}_{q_i^2},$$

we obtain

$$d_x = q_i^{(k+1-x)(\alpha-x)} \sum_{y=0}^{\frac{\alpha-x}{2}} (-1)^y q_i^{y(-\alpha+x)} d'_{x,y} \begin{bmatrix} \frac{\alpha-x}{2} \\ y \end{bmatrix}_{q_i^2} B_{i,\overline{k+c_{ij}}}^{(k-x-2y+1)}$$

where

$$\begin{aligned} d'_{x,y} &= q_i^{-x-2y}[k-x-2y+1] + q_i^{k-x+1}[x] + \frac{[2y]}{[\alpha-x-2y+2]} \times \\ &\quad \times \left( q_i^{\alpha-2x-2y+1}[k-x-2y+2] + q_i^{-k-1}[\alpha-x] + q_i^{k-\alpha-1}[x] - q_i^{x-\alpha-1}[k] \right) \\ &= [k+1]. \end{aligned}$$

This formula shows that $d_x$ equals the coefficient of $b_{i,j;\alpha-x}$ in $(16.25)_{k+1}$ for any $\alpha - x$ even.

Therefore, combining above two cases, we have proved $(16.25)_k + (16.26)_{k-1} \Rightarrow$

$(16.26)_{k+1}$.

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