Weak commensurability of Zariski-dense subgroups of algebraic groups defined over fields of positive characteristic

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A Dissertation presented to the Graduate Faculty
of the University of Virginia in Candidacy for the Degree of
Doctor of Philosophy

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August, 2015

Abstract

In [20], Prasad and Rapinchuk introduce and analyze a new relationship, called 'weak commensurability,' between (Zariski-dense) abstract subgroups of the groups of K-rational points of connected semisimple algebraic groups. Numerous results have been shown for algebraic groups defined over fields of characteristic zero, but not for fields of positive characteristic. The main purpose of this work is to extend the notion of weak commensurability to fields of positive characteristic, specifically to prove and analyze characteristic $p > 0$ analogs of the results from [20].

We develop several characteristic $p > 0$ results showing that weakly commensurable Zariski-dense subgroups must share structural properties. Specifically, we show that the Zariski-closure of two weakly commensurable Zariski-dense subgroups of absolutely almost simple groups must have the same Killing-Cartan type. The trace field of a subgroup is the field generated by 1 and the traces of elements in the Zariski-dense subgroup. We show that a pth power of the trace field of one Zariski-dense subgroup contained in the trace field of a weakly commensurable subgroup. We also prove a similar statement when we replace the trace field with the minimal Galois extension of the trace field such that the Zariski-closure of the subgroup is an inner form over this field. We also show that discreteness is a property that is shared by weakly commensurable subgroups.

Acknowledgements

I would first like to express my immense gratitude to my advisor Andrei Rapinchuk. The impact of his advice and guidance on both my appreciation for mathematics and my skills as a researcher. While I may not be pursuing a career in academia after graduation, his affect on my current level of knowledge and capabilities is significant.

Second, I would like to thank all of the other professors at UVa that have directed my education. Your training and advice has been invaluable.

Third, I would like to thank all of my friends for making my six years in Charlottesville enjoyable. A special thanks goes out to Sean Clark, Maria Iribarren, Zach Bradshaw, and Sean Droms for making the time I spend in the department both entertaining and exciting. Also, thank you Zack Svoboda for being an excellent friend over the years and making sure that I had fun outside of the department.

Finally, I would like to thank my first teachers: my parents, my sister Ariel, and my brothers Cody, Evan, and Jesse. Your support has both enabled me and motivated me to succeed. And Cody: I would be honored to be your groomsman.

Contents

Chapter 1

Introduction

1.1 Weak commensurability in positive characteristic

In [20], Prasad and Rapinchuk introduce and analyze a new relationship between (Zariski-dense) abstract subgroups of the groups of K-rational points of connected semisimple algebraic groups defined over a field K called 'weak commensurability.' This relationship is defined in term of the eigenvalues of the individual elements of the abstract subgroups and does not involve any structural connections between the subgroups. The results demonstrated in [20] show that for fields K of characteristic zero, the weak commensurability of (Zariski-dense) subgroups impose strong restrictions on the ambient semisimple algebraic groups over K, as well as the minimal fields of definition of the subgroups. The main goal of this paper is to prove analogous results for weakly commensurable (Zariski-dense) subgroups of the groups of K-rational points of connected absolutely almost simple groups defined over a field

K of positive characteristic.

We begin with the definition of weak commensurability. Let K be an infinite field of arbitrary characteristic, $\overline{K}$ be a fixed algebraic closure, and let K^{sep} be the separable closure of K in $\overline{K}$.

Definition. Let $\gamma_1 \in \text{GL}_{n_1}(K)$ and $\gamma_2 \in \text{GL}_{n_2}(K)$ be semisimple elements of infinite order (for some positive integers n_1 and n_2). Let $\lambda_1, ..., \lambda_{n_1}$ be the eigenvalues of γ_1 and let $\mu_1, ..., \mu_{n_2}$ be the eigenvalues of γ_2 (taken in a separable closure K^{sep} of K).

1. The elements γ_1 and γ_2 are **weakly commensurable elements** if there exist integers $a_1, ..., a_{n_1} \in \mathbb{Z}$ and $b_1, ..., b_{n_2} \in \mathbb{Z}$ such that

$$\lambda_1^{a_1} ... \lambda_{n_1}^{a_{n_1}} = \mu_1^{b_1} ... \mu_{n_2}^{b_{n_2}} \neq 1.$$

2. If $G_i \subseteq \text{GL}_{n_i}$ is a reductive algebraic group defined over K and $\Gamma_i \subseteq G_i(K)$ is a Zariski-dense subgroup of $G_i(K)$ for $i = 1, 2$, the subgroups Γ_1 and Γ_2 are **weakly commensurable subgroups** if every semisimple element of Γ_1 of infinite order is weakly commensurable to some semisimple element of Γ_2 of infinite order and vice-versa.

The study of weak commensurability is useful to the study of a novel form of rigidity called *eigenvalue rigidity*, discussed in [22]. Classically, the rigidity of lattices $\Gamma \subseteq G(K)$ of algebraic groups refers the property that under appropriate assumptions, a homomorphism of lattices $\Gamma_1 \to \Gamma_2$ (virtually) extends to a homomorphism of the semisimple Lie groups $G_1 \to G_2$ containing Γ_1 and Γ_2. As a consequence, sufficiently large (e.g. higher rank arithmetic) Zariski-dense subgroups $\Gamma \subset G(K)$ of a semisimple Lie group G determine the field of definition up to isomorphism and the ambient algebraic group G. The motivation behind eigenvalue rigidity is that rather than the structural information obtained from the Zariski-dense subgroup Γ, one should be able to determine the field of definition as a subfield of K and the ambient algebraic group G from Γ if one uses information about the eigenvalues of Γ. This allows one to examine rigidity in the context of any Zariski-dense subgroup of G, not just higher rank arithmetic groups. The study of weak commensurability is and important step towards making this relationship explicit.

The following theorems provide three basic results about weakly commensurable Zariski-dense subgroups of the groups of K-rational points of two connected absolutely almost simple groups defined over a field K of positive characteristic.

Theorem A. *Let K be an infinite finitely generated field of characteristic $p > 0$. Let G_i be a connected absolutely almost simple algebraic group defined over K for $i = 1, 2$. Let $\Gamma_i \subseteq G_i(K)$ be a finitely generated Zariski-dense subgroup for $i = 1, 2$, and suppose that Γ_1 and Γ_2 are weakly commensurable. Then G_1 and G_2 have the same Killing-Cartan type, or one is of type B_n and the other is of type C_n.*

To state the next result, we need to introduce the notion of the minimal field of definition of a Zariski-dense subgroup. Let K be as above, let G be a connected absolutely almost simple algebraic group defined over K, let $\Gamma \subseteq G(K)$ be a Zariski-dense subgroup, and let $\mathrm{Ad}\colon G \to \mathrm{GL}(\mathrm{Lie}(G))$ be the adjoint representation of G. The *minimal field of definition* of Γ is the smallest subfield of Γ such that there exists a basis of the Lie algebra $\mathrm{Lie}(G)$ such that $\mathrm{Ad}\,\Gamma$ may be conjugated to a subgroup of a linear group with matrix coefficients in K_Γ. See Ch. 2, Section 2.1 for more information on minimal fields of definition.

For fields K of characteristic zero, the minimal field of definition of Γ is the same as the field generated by 1 and the set of all traces, $\mathrm{tr}(\mathrm{Ad}(\gamma))$ for $\gamma \in \Gamma$ (due to [30, Cor. to Thm. 1]). Let Γ_1 and Γ_2 be two finitely generated Zariski-dense subgroups of $G_1(K)$ and $G_2(K)$ respectively (where G_i is a connected absolutely almost simple algebraic group defined over a field K of characteristic zero for $i = 1, 2$). If Γ_1 is weakly commensurable to Γ_2, then [20, Thm. 2] shows that their minimal fields of definition must be equal, i.e. $K_{\Gamma_1} = K_{\Gamma_2}$.

In the case where the field K has characteristic $p > 0$, we have to modify the definition of the trace field due to the fact that the adjoint representation is not necessarily irreducible over fields of characteristic p for some small primes p (see [11] for a precise exposition of this fact). Therefore, we have a more complicated definition of the trace field, explicitly given by Definition 2.2.8. Let G be a connected semisimple group defined over K of (absolute) rank n, and let $\Gamma \subseteq G(K)$ be a Zariski-dense subgroup. Except for the cases where $p = 2$, 3, or $p|(n + 1)$, our given definition is precisely the subfield of K generated by 1 and the traces of elements of $\mathrm{Ad}\,\Gamma$, as in the characteristic zero case. In the positive characteristic case, we continue to denote the trace field of a Zariski-dense subgroup $\Gamma \subseteq G(K)$ by K_Γ.

Frobenius isogenies have a significant effect on our results. For example, let G be an adjoint algebraic group defined over a field K and let $\Gamma \subseteq G(K)$ be a finitely generated Zariski-dense subgroup. Let $\mathrm{Fr} : G \to \mathrm{Fr}(G)$ be the Frobenius isogeny and consider the group $\mathrm{Fr}(\Gamma)$. We observe that the following are true.

- The group $\mathrm{Fr}(\Gamma)$ is a finitely generated Zariski-dense subgroup of $\mathrm{Fr}(G)(K)$.
- Let $\gamma \in \Gamma$ be a semisimple element. If the eigenvalues of γ are $\lambda_1, ..., \lambda_n$, then the eigenvalues of $\gamma^p = \mathrm{Fr}(\gamma)$ are $\lambda_1^p, ..., \lambda_n^p$, hence γ and $\mathrm{Fr}(\gamma)$ are weakly commensurable elements.
- Therefore, Γ is weakly commensurable to $\mathrm{Fr}(\Gamma)$. Since $\mathrm{tr}(\gamma^p) = \mathrm{tr}(\gamma)^p$, we know that $K_{\mathrm{Fr}(\Gamma)} = K_\Gamma^p$ (except possibly in the cases where the adjoint representation is not irreducible).

Therefore, Γ is always weakly commensurable to $\mathrm{Fr}(\Gamma)$ but the fields of definition for these two groups will typically not coincide. However, they will coincide "up to a power of Frobenius," which is made evident by the following results.

Theorem B. *Let K be an infinite finitely generated field of characteristic $p > 0$. Let G_i be a connected absolutely almost simple algebraic group defined over K for $i = 1, 2$. Let $\Gamma_i \subseteq G_i(K)$ be a finitely generated Zariski-dense subgroup for $i = 1, 2$,* *and suppose that* Γ_1 *and* Γ_2 *are weakly commensurable. Let* K_i *be the trace field of* Γ_i *for* $i = 1, 2$. *Then there exist integers* k_1, k_2 *such that*

$$(K_1)^{p^{k_1}} \subseteq K_2 \text{ and } (K_2)^{p^{k_2}} \subseteq K_1.$$

We also prove a similar result to Theorem B concerning the minimal Galois extensions of the minimal fields of definition of Γ_1 and Γ_2 over which the groups G_1 and G_2 become inner forms respectively. We refer to Chapter 2, Section 2.1 for the definition of an inner form.

Theorem C. *Let* K, G_i, *and* $\Gamma_i \subseteq G_i(K)$ *for* $i = 1, 2$ *be defined in the same way as in Theorem B above. For* $i = 1, 2$, *let* K_i *be the trace field of* Γ_i *and let* L_i *be the minimal Galois extension of* K_i *such that* G_i *becomes an inner form of a split group over* L_i. *Then there exist integers* k_1, k_2 *such that*

$$(L_1)^{p^{k_1}} \subseteq L_2 \text{ and } (L_2)^{p^{k_2}} \subseteq L_1.$$

After proving the above theorems, we examine the case where K is a local field of characteristic $p > 0$ in Chapter 3, Section 3.3. Note that for a connected absolutely simple adjoint algebraic group defined over K, the group of rational points $G(K)$ becomes a locally compact topological group, so we can examine the induced topology on any abstract subgroup $\Gamma \subseteq G(K)$. The following theorem tells us that for local fields K, weak commensurability imposes a strong restriction on the induced topology of the Zariski-dense subgroups of the group of K-points of a connected absolutely almost simple algebraic group defined over K.

Theorem D (Analog of [20, Prop. 5.6])**.** *Suppose that K is a local field of characteristic $p > 0$, and suppose that for $i = 1, 2$, G_i is a connected absolutely simple adjoint algebraic group defined over K. For $i = 1, 2$, suppose that $\Gamma_i \subseteq G_i(K)$ is a Zariski-dense finitely generated subgroup, and suppose that Γ_1 is weakly commensurable to Γ_2. Then Γ_2 is discrete if and only if Γ_1 is discrete.*

The proofs of these theorems use a variety of algebraic and number-theoretic techniques. First and foremost, we make extensive use of Pink's results found in [16] and [17] concerning strong approximation of Zariski-dense subgroups of connected semisimple algebraic groups defined over fields of arbitrary characteristic. This work serves are a generalization of the strong approximation property for Zariski-dense subgroups in the characteristic zero case, which is used extensively in the proofs of the characteristic zero analogs of our results. In characteristic zero, the strong approximation property of Zariski-dense subgroups can be traced back to Platonov's proof of the strong approximation property for connected semisimple algebraic groups defined over number fields [18]. Platonov's proof relies in an essential way on the following theorem originally due to Cartan: a closed subgroup of a Lie group is a Lie group [4, pg. 340]. Unfortunately, Cartan's theorem is no longer true in positive characteristic, so the work of Pink is essential to our work. We provide the precise statements of these strong approximation results in Chapter 2, Section 2.2.

1.2 Generic elements in positive characteristic

Furthermore, one of the key ingredients in the proofs of the above results is the existence of *generic elements* in Zariski-dense subgroups of the group of K-rational points of a connected absolutely almost simple algebraic group defined over a field of positive characteristic. We start with the definition of generic tori and generic elements.

Let K be an infinite field, fix an algebraic closure $\overline{K}$ of K, and let K^{sep} be the separable closure of K in $\overline{K}$. Let G be a connected absolutely almost simple algebraic group defined over K, and fix a maximal K-torus T of G. Let $\Phi := \Phi(G,T)$ and $\mathcal{W} := \mathcal{W}(G,T)$ be the root system and Weyl group of G respectively. Let K_T denote the (minimal) splitting field of T over K. Note that K_T/K is a Galois extension.

The Galois group $\text{Gal}(K_T/K)$ naturally acts on the character group of T, $X(T)$. This action takes $\Phi(G,T)$ to $\Phi(G,T)$, so it induces an injective homomorphism

$$\theta_T \colon \text{Gal}(K_T/K) \longrightarrow \text{Aut}\Phi(G,T). \tag{1.2.1}$$

Definition. We say that the torus T is a **generic over** K if

$$\theta_T(\mathrm{Gal}(K_T/K)) \supseteq \mathcal{W}(G,T).$$

A regular semisimple element $g \in G(K)$ is said to be **generic over** K if the torus $T = Z_G(g)^\circ$ is a generic torus over K.

In [19], Prasad and Rapinchuk prove that a Zariski-dense subgroup $\Gamma \subseteq G(K)$ of a connected absolutely almost simple algebraic group G defined over a finitely generated field K of characteristic zero must contain a K-generic element.

Generally, we can specify a finite number of local conditions and show that a maximal K-torus that satisfies these local conditions must be generic. Using this method, we prove the existence of these elements in the positive characteristic case in Chapter 3, Section 3.1.

Theorem E (Analog of [19, Theorem 3(i)]). *Let K be an infinite field of characteristic $p > 0$. Let r be the number of nontrivial conjugacy classes in $\mathcal{W}$ and let S be a set of nontrivial non-archimedean valuations on K such that K_v is locally compact and G is K_v-split. For each $v \in S$, we can choose a maximal K_v-torus T_v of G such that any K-torus T that is $G(K_v)$-conjugate to T_v for all $v \in S$ is a generic torus over K.*

The main result concerning generic elements shows the existence of generic elements in Zariski-dense subgroups.

Theorem F (Analog of [19, Theorem 1])**.** *Let G be a connected absolutely almost simple algebraic group defined over an infinite finitely generated field K of characteristic $p > 0$ and let $\Gamma \subseteq G(K)$ be a finitely generated Zariski-dense subgroup. Then Γ contains a generic K-element (of infinite order).*

The proof of this fact relies on both Theorem E and the strong approximation results introduced in Chapter 2, Section 2.2. The existence of generic elements is invaluable to our examination of weak commensurability and we use this concept in the proofs of many of the theorems in the subsequent sections.

1.3 Notation

G	An algebraic group
G_R	An algebraic group considered as a group over the algebra R
G^A	The algebraic group scheme with Hopf algebra A
A_{red}	The reduced ring of A
$G(R)$	The R-points of an algebraic group
$\overline{G}$	The adjoint group of the algebraic group G
$\widetilde{G}$	The simply connected cover of the algebraic group G
G°	The connected component of the identity of the algebraic group G
$Z(G)$	The center of G
$Z_G(H)$	The centralizer of the subgroup H in the algebraic group G
$Z_G(x)$	The centralizer of $x \in G(K)$ in the algebraic group G
$N_G(H)$	The normalizer of the subgroup H in the algebraic group G
$\mathcal{R}(G)$	The radical of the algebraic group G
$\mathcal{R}_u(G)$	The unipotent radical of the algebraic group G
G_{reg}	The open subvariety of regular elements of the algebraic group G
$K[G]$	The ring of regular functions of the group G
K^{sep}	The separable closure of the field K
$\overline{K}$	The algebraic closure of the field K
K_v	The completion of the field K with respect to the v-adic norm

$\mathcal{O}_v$	The local ring of integers in K_v
K_v^{ur}	The maximal unramified extension of K_v in K_v^{sep}
V^K	The set of all pairwise inequivalent absolute values on K
V_f^K	The subset of V^K consisting of non-archimedean absolute values
K_T	The splitting field of the torus T over the field K
$X(T)$	The set of (absolute) characters of the group T
Ad	The adjoint representation
K_Γ	The minimal field for a Zariski-dense subgroup $\Gamma \subseteq G(K)$
Fr	The Frobenius morphism
(F, G, Γ)	Standard triple consisting of the semisimple ring F, the group scheme G over F, and the Zariski-dense subgroup $\Gamma \subseteq G(F)$
$\text{Lie}(G)$	The Lie algebra of the algebraic group G
df	The derivative at the identity of the group scheme morphism $f\colon G \to H$
$\Phi(G,T)$, $\Phi(G)$	The root system of the reductive group G (with respect to the maximal torus T if specified)
$\Delta(G,T)$, $\Delta(G)$	A set of simple roots for the reductive group G (with respect to the maximal torus T if specified)
$\mathcal{W}(G,T)$, $\mathcal{W}(G)$	The Weyl group of the reductive group G (with respect to the maximal torus T if specified)
$\text{Dyn}(G)$	The Dynkin diagram of the reductive group G

Chapter 2

Background information

In this chapter we summarize a number of mostly well-known facts concerning linear algebraic groups over an arbitrary field K. For this summary, we assume a basic understanding of algebraic geometry and commutative algebra.

2.1 Algebraic groups

Group schemes

Let K be a field of arbitrary characteristic. Let A be a commutative (unital, associative) K-algebra with multiplication $m\colon A \otimes_K A \to A$ and unit $e\colon K \to A$. Assume that we have K-algebra homomorphisms

$$\begin{aligned} &\Delta\colon A \to A \otimes_K A && \text{(comultiplicaton)} \\ &\iota\colon A \to A && \text{(co-inverse)} \\ &\epsilon\colon A \to K && \text{(co-unit)} \end{aligned}$$

which satisfy that following conditions.

(a) The diagram

$$\begin{array}{ccc} A & \xrightarrow{\quad\Delta\quad} & A \otimes_K A \\ {\scriptstyle\Delta}\downarrow & & \downarrow{\scriptstyle 1_A \otimes \Delta} \\ A \otimes_K A & \xrightarrow{\Delta \otimes 1_A} & A \otimes_K A \otimes_K A \end{array}$$

commutes.

(b) The map

$$A \xrightarrow{\Delta} A \otimes_K A \xrightarrow{\epsilon \otimes 1_A} K \otimes_K A = A,$$

coincides with the identity map $1_A \colon A \longrightarrow A$.

(c) The composite maps

$$A \xrightarrow{\Delta} A \otimes_K A \xrightarrow{\iota \otimes 1_A} A \otimes_K A \xrightarrow{m} A$$

and

$$A \xrightarrow{\epsilon} K \xrightarrow{e} A$$

coincide.

An K-algebra A with the above maps is called a **(commutative) Hopf algebra over** K.

Let A be a commutative Hopf algebra over K. An ideal J of A such that

$$\Delta(J) \subseteq J \otimes_K A + A \otimes_K J, \quad \iota(J) \subseteq J, \text{ and } \epsilon(J) = 0$$

is called a **Hopf ideal** of A. If J is a Hopf ideal of A, then A/J has a natural Hopf algebra structure.

Let Alg_K denote the category of commutative (unital, associative) K-algebras with morphisms given by K-algebra homomorphisms. Let A be a Hopf algebra over K. For any object R in Alg_K, one defines a product on the set

$$G^A(R) = \mathrm{Hom}_{Alg_K}(A, R)$$

by the formula $fg = m_R(f \otimes g) \circ \Delta$ for $f, g \in G^A(R)$, where $m_R \colon R \otimes_K R \longrightarrow R$ is multiplication in R. The defining properties of a Hopf algebra imply that this product is associative and with a (left) identity given by the composition $A \xrightarrow{\epsilon} K \xrightarrow{e_R} R$ and (left) inverse given by $f^{-1} := f \circ \iota$. It is straightforward to verify that $G^A(R)$ is an abstract group with multiplication and inversion given by these operations.

For any K-algebra homomorphism $f \colon R \to S$, there is a group homomorphism

$$G^A(f) \colon G^A(S) \longrightarrow G^A(R), \ g \mapsto f \circ g,$$

which implies that we have a representable covariant functor

$$G^A\colon Alg_K \longrightarrow Grps.$$

An **affine group scheme** G over K is a functor $G\colon Alg_K \longrightarrow Grps$ that is isomorphic to G^A for some Hopf algebra A. By Yoneda's lemma, the Hopf algebra A is uniquely determined by G up to isomorphism, which we denote by $A = K[G]$. We call the algebra $K[G]$ the **ring of regular functions on** G. For any object R in Alg_K, the group $G(R)$ is called the **group of** R**-points of** G.

A **group scheme homomorphism** $f\colon G \longrightarrow H$ is a natural transformation of functors. By Yoneda's lemma, this is completely determined by a unique Hopf algebra homomorphism $f^*\colon K[H] \longrightarrow K[G]$.

Suppose that G is a group scheme represented by the Hopf algebra A, and H is a group scheme represented by the Hopf algebra A/J for some Hopf ideal J of A. Then consider the group scheme homomorphism $\rho\colon H \to G$ induced by $A \to A/J$. For each object R of Alg_K, $\rho_R\colon H(R) \to G(R)$ is injective, hence we can naturally identify $H(R)$ with a subgroup of $G(R)$. When this is the case, the group scheme H is called a **(closed) subgroup of** G. The subgroup H is said to be **normal** in G if $H(R)$ is normal in $G(R)$ for all objects R of Alg_K.

Let $f\colon G \longrightarrow H$ be a group scheme homomorphism. The subgroup $\ker(f)$ of G is defined to be the functor

$$\ker(f)(R) = \{g \in G(R) \mid f_R(g) = 1\}.$$

It is not hard to show that this is the subgroup of G associated to the Hopf algebra $K[G]/(f^*(I)K[G])$, where $I = \ker(\epsilon\colon K[H] \to K)$, i.e. the augmentation ideal of $K[H]$.

We say that an affine group scheme G over K is an **algebraic group scheme over** K if the K-algebra $K[G]$ is finitely generated. We will work exclusively with algebraic group schemes.

Example 2.1.1.

1. Let $A = K$ with a trivial Hopf structure. Then $G^A = 1$, the trivial group scheme, i.e. $G^A(R) = \{e\}$ for all objects R in Alg_K.

2. Consider the K-algebra, $A = K[X_{ij}\ (i,j = 1,...,n), \det(X_{11},...,X_{nn})^{-1}]$. We can endow A with the following Hopf algebra structure. We define the homomorphisms Δ, ϵ, and ι on generators as follows:

$$\Delta(X_{ij}) = \sum_{k=1}^{n} X_{ik} \otimes X_{kj}, \quad \epsilon(X_{ij}) = \delta_{ij},$$

$$\iota(X_{ij}) = (-1)^{i+j} \det(X_{11},...,X_{nn})^{-1} \det(M_{ij}),$$

where δ_{ij} is the Dirac delta function and M_{ij} is the $(n-1) \times (n-1)$ minor of the matrix (X_{ij}) formed by removing the ith row and jth column.

Note that the above values imply the following.

$$\Delta(\det(X_{11},...,X_{nn})) = \det(X_{11},...,X_{nn}) \otimes \det(X_{11},...,X_{nn}),$$

$$\epsilon(\det(X_{11},...,X_{nn})) = 1, \text{ and } \iota(\det(X_{11},...,X_{nn})) = \det(X_{11},...,X_{nn})^{-1}.$$

The definition of A as a Hopf algebra implies that $G^A(R) = \mathrm{GL}_n(R)$ for all commutative K-algebras R (see for example [13, §20.A, pg. 326]). The algebraic group scheme G^A is denoted $\mathrm{GL}_{n,K}$ or GL_n if the field K is clear from the context.

3. Consider $A = K[\mathrm{GL}_n]/(\det(X) - 1)$, where $X = (X_{ij})$. It is not difficult to show that $(\det(X) - 1)$ is a Hopf ideal, so the Hopf algebra structure on A comes from the Hopf algebra structure on $K[\mathrm{GL}_n]$. Then $G^A(R) = \mathrm{SL}_n(R)$ for all commutative K-algebras R, and we denote the algebraic group scheme G^A by $\mathrm{SL}_{n,K}$ or just SL_n if the field K is clear from the context.

4. Let $A = K[x]$ and $B = K[x, x^{-1}]$. The Hopf algebra structure on A is defined on the generator x by

$$\Delta_A(x) = x \otimes 1 + 1 \otimes x, \quad \epsilon_A(x) = 0, \quad \iota_A(x) = -x.$$

On the algebra B, it is defined on the generator x by

$$\Delta_B(x) = x \otimes x, \quad \epsilon_B(x) = 1, \quad \iota_B(x) = x^{-1}.$$

The algebraic group scheme G^A is denoted by $\mathbb{G}_a$ and called the additive group scheme. The algebraic group scheme G^B is denoted by $\mathbb{G}_m$ and called the multiplicative group scheme. As functors, $G^A(R) = R$ and $G^B(R) = R^\times$ for all objects R in Alg_K.

5. Let V be a finite dimensional K-vector space and let R be an object in Alg_K.

Define the group scheme $\mathbf{GL}(V)$ by $\mathbf{GL}(V)(R) = \mathrm{GL}(V \otimes_K R)$, where $\mathrm{GL}(V)$ is the classical group of invertible linear transformations of the vector space V. The group scheme $\mathbf{GL}(V)$ is represented by $A = \mathrm{End}_K(V)$.

Let A be a Hopf algebra over K and let $A_{red} = A/\mathrm{nil}(A)$, where $\mathrm{nil}(A)$ is the nilradical of A. We say that G^A is **connected** if and only if A_{red} is a domain. In other words, G^A is connected if and only if $\mathrm{Spec}(A)$ is connected as an affine scheme.

Recall that a finite dimensional K-algebra E is called *étale* if it is a product of separable field extensions of K. Let $\pi_0(A)$ denote the maximal étale K-subalgebra of A. It is also a Hopf algebra, and the inclusion $\pi_0(A) \subseteq A$ induces a surjective group scheme homomorphism $G^A \longrightarrow G^{\pi_0(A)}$. The kernel of this map, denoted G°, is called **the connected component of the identity of** G. The connected component G° is a connected group scheme [31, §6.7, pg. 51].

The Lie algebra of a group scheme

Let G be an algebraic group scheme over K and let $A = K[G]$. A derivation $D \in \mathrm{Der}(A, A)$ is said to be left-invariant if $\Delta \circ D = (1_A \otimes D) \circ \Delta$. The set of left-invariant derivations forms an K-vector subspace of $\mathrm{Der}(A, A)$, and it is denoted $\mathrm{Lie}(G)$. It takes some work to show that $\mathrm{Lie}(G)$ is a Lie subalgebra of $\mathrm{Der}(A, A)$ with the bracket operation given by $[D_1, D_2] = D_1 \circ D_2 - D_2 \circ D_1$ for all left-invariant derivations $D_1, D_2 \in \mathrm{Lie}(G)$. We call the Lie algebra $\mathrm{Lie}(G)$ the **Lie algebra of** G [31, §12.1, pg. 93].

Remark 2.1.2. If G is an algebraic group scheme, then $\dim_K(\mathrm{Lie}(G)) < \infty$. See [13, Cor. 21.2, pg. 334] for details.

Example 2.1.3.

- If $G = \mathrm{GL}_{n,K}$, then $\mathrm{Lie}(G) = \mathrm{M}_n(K)$ with the Lie bracket given by $[A, B] = AB - BA$. (See [31, §12.3(a), pg. 95].)
- If $G = \mathrm{SL}_{n,K}$, then $\mathrm{Lie}(G) = \{X \in \mathrm{M}_n(K) \mid \mathrm{tr}(X) = 0\}$, with the same Lie bracket as $\mathrm{M}_n(K)$, namely $[A, B] = AB - BA$. (See [31, §12.3(b), pg. 95].)

Note that this construction is functorial. Given an K-morphism of algebraic groups $f\colon G \longrightarrow H$, it induces a homomorphism of Lie algebras over K, denoted $df\colon \mathrm{Lie}(G) \longrightarrow \mathrm{Lie}(H)$. The homomorphism df is called the **derivative of f (at the identity)** [31, §12.2, pg. 94].

Let R be an object in Alg_K. Let $g \in G(R)$, and let $\psi_g\colon G(R) \longrightarrow G(R)$ be the automorphism of G defined by conjugation by g, i.e. $\psi_g(h) = g^{-1}hg$ for $h \in G(R)$. Note that this defines an automorphism of the algebraic group G as a group scheme over R. Thus, we may take the derivative of ψ_g to get the R-linear isomorphism

$$d\psi_g\colon \mathrm{Lie}(G) \otimes_K R \longrightarrow \mathrm{Lie}(G) \otimes_K R,$$

taking for granted that the Lie algebra of G as a group scheme over R is the same as the tensor product of the Lie algebra of the group scheme G over K and the algebra R. For any $g \in G(R)$, define $\mathrm{Ad}_R(g)\colon = d\psi_g \in \mathrm{GL}(\mathrm{Lie}(G) \otimes_K R)$.

Definition 2.1.4. The group scheme homomorphism $\mathrm{Ad}\colon G \longrightarrow \mathbf{GL}(\mathrm{Lie}(G))$ that we have defined above is called the **adjoint representation of** G.

Smoothness

Let G be a connected algebraic group scheme over K and let $A = K[G]$. Then A_{red} is a domain. The **dimension of** G, denoted $\dim(G)$, is the transcendence degree of the field of fractions of A_{red} over K. If G is not connected, then define $\dim(G) = \dim(G^\circ)$.

Example 2.1.5.

- $\dim(\mathrm{GL}_{n,K}) = n$.
- $\dim(\mathrm{SL}_{n,K}) = n - 1$.
- $\dim(\mathbb{G}_a) = \dim(\mathbb{G}_m) = 1$.

Let $I = \ker(A \xrightarrow{\epsilon} K)$. As stated above, this is a maximal Hopf ideal called the **augmentation ideal** of A. If we localize A with respect to the ideal I, denoted A_I, the resulting ring A_I is a local ring with maximal ideal $M = IA_I$ and residue field $K = A_I/M$. Then M/M^2 has a natural K-vector space structure. We say that local ring A_I is **regular** if $\dim_K(M/M^2) = \dim_{\mathrm{Krull}}(A)$.

Lemma 2.1.6 ([13, Lemma 21.8, pg. 337])**.** *If G is an algebraic group scheme over the field K, then* $\dim_K(\mathrm{Lie}(G)) \geq \dim(G)$. *Equality holds if and only if the local ring A_I is regular.*

Definition 2.1.7. If an algebraic group G satisfies $\dim_K(\mathrm{Lie}(G)) = \dim(G)$, we say that G is **smooth**. A smooth algebraic group scheme is called an **algebraic group**.

It is not difficult to show that G is smooth if and only if $A \otimes_K L$ is reduced for any field extension L/K. When G is smooth and connected, then $K[G]$ is a domain and we define $K(G) := \mathrm{Frac}(K[G])$, called **the field of rational functions on** G.

Example 2.1.8.

1. The group schemes $\mathbb{G}_a$ and $\mathbb{G}_m$ are smooth over any field K since $\dim(\mathbb{G}_a) = \dim(\mathbb{G}_m) = 1$ and $\mathrm{Lie}(\mathbb{G}_a) = K$ and $\mathrm{Lie}(\mathbb{G}_m) = K$.

2. Let K be a field of characteristic $p > 0$. Let $A = K[x]/(x^n - 1)$ with the Hopf structure on A given by $\Delta(x) = x \otimes x$, $\iota(x) = x^{-1}$, and $\epsilon(x) = 1$ and consider the group scheme G^A. When $(n, p) = 1$, this is a smooth group scheme since the polynomial $x^n - 1$ is separable over K. When $n|p$, then this is a non-smooth group scheme since A is not reduced.

Remark 2.1.9. Classically, affine algebraic groups over an algebraically closed field can be realized as an affine variety $\mathrm{Spec}(A)$ endowed with a group structure. These conditions imply that A must be a reduced finitely generated Hopf algebras A, which corresponds to the notion of a smooth algebraic group scheme that we have defined above. This is the reason why we call smooth algebraic group schemes algebraic groups.

Rational points of algebraic groups

Definition 2.1.10. Let $g \in G(K)$. Let $P_g(t)$ be the characteristic polynomial of $\mathrm{Ad}_K(g) \in \mathrm{GL}(\mathrm{Lie}(G))(K)$. We say that g is **K-regular** if the multiplicity of $(t-1)$ in $P_g(t)$ is minimal.

Proposition 2.1.11. *The set of regular elements is Zariski-open in $G(K)$.*

Proof. See [3, Thm. 12.3, pg. 161] □

Let V be a finite dimensional K-vector space. Recall that an element $a \in \mathrm{End}(V)$ is **semisimple** if there is an $\overline{K}$-basis of $V \otimes_K \overline{K}$ consisting of eigenvectors of a, so a is a diagonal matrix with respect to this basis. We say $a \in \mathrm{End}(V)$ is **nilpotent** if $a^n = 0$ for some integer $n \geq 0$ and we say that a is **unipotent** if $a - 1$ is nilpotent. Notice that if $\mathrm{char}(K) = p > 0$, then a is unipotent if and only if $a^{p^s} = 1$ for some $s \geq 1$.

Definition 2.1.12. Let G be a linear algebraic group. We say that an element $x \in G(K)$ is **semisimple** (resp. **unipotent**) if the image of x via an embedding of $G(K)$ into $\mathrm{GL}_n(K)$ is a semisimple (resp. unipotent) operator in $\mathrm{GL}_n(K)$.

Unipotent radicals and radicals

For this section, we assume that F is an algebraically closed field. Let G be a connected algebraic group defined over F.

Definition 2.1.13. A **unipotent linear algebraic group** is an algebraic group U defined over F such that the elements of $U(F)$ are all unipotent. A **solvable linear algebraic group** is an algebraic group W defined over F such that $W(F)$ is a solvable abstract group.

Definition 2.1.14.

(a) Let $\mathcal{R}_u(G)$ be the unique maximal connected unipotent normal F-subgroup of G, called the **geometric unipotent radical of** G.

(b) Let $\mathcal{R}(G)$ be the unique maximal connected solvable normal F-subgroup of G, called the **geometric radical of** G.

(c) The group G is **reductive** if $\mathcal{R}_u(G)(F) = 1$.

(d) The group G is **semisimple** if $\mathcal{R}(G)(F) = 1$.

Since $\mathcal{R}_u(G) \subseteq \mathcal{R}(G)$, all semisimple groups are reductive.

Example 2.1.15.

- The algebraic group of upper triangular matrices $\mathbb{T}_n$ is not reductive because the subgroup of upper triangular unipotent matrices is normal in $\mathbb{T}_n$.

- A torus is reductive but not semisimple.

- The algebraic group $\mathrm{GL}_{n,F}$ is reductive, but not semisimple. The algebraic group $\mathrm{SL}_{n,F}$ is semisimple.

Proposition 2.1.16. *The algebraic F-group $G/\mathcal{R}_u(G)$ is reductive and the algebraic F-group $G/\mathcal{R}(G)$ is semisimple.*

Proof. See [3, §11.21, pg. 157]. □

Note that for imperfect subfields $F' \subset F$, the geometric unipotent radical $\mathcal{R}_u(G)$ may fail to descend to an algebraic F'-group. If G' is a connected algebraic group over F', then it is clear that $\mathcal{R}_u(G')_{\overline{F'}} \subseteq \mathcal{R}_u(G'_{\overline{F'}})$ and $\mathcal{R}(G')_{\overline{F'}} \subseteq \mathcal{R}(G'_{\overline{F'}})$, but they need not be equal (see [7] for numerous examples).

However, when $G'_{\overline{F'}}$ is reductive (resp. semisimple) then $\mathcal{R}_u(G')$ is trivial (resp. $\mathcal{R}(G')$ is trivial). In this case, we say that the F'-group G' is reductive (resp. semisimple).

Algebraic tori

Definition 2.1.17. An **n-dimensional K-torus** is an algebraic K-group that is $\overline{K}$-isomorphic to $(\mathbb{G}_m)^n$ for some $n > 0$.

Let T be an n-dimensional torus defined over a field K. A **character of** T, $\chi \in K^{\text{sep}}[T]$, is a regular function with the property that for all $t_1, t_2 \in T(K^{\text{sep}})$,

$$\chi(t_1 t_2) = \chi(t_1)\chi(t_2).$$

The set of all characters is denoted by $X(T)$. We can also identify $X(T)$ with the set of group scheme morphisms $\text{Hom}(T, \text{GL}_1)$.

Let $\mathscr{G} = \text{Gal}(K^{\text{sep}}/K)$. Note that $\mathscr{G}$ naturally acts on $X(T)$ as follows. For all $\sigma \in \mathscr{G}$, $\chi \in X(T)$, and $t \in T(K^{\text{sep}})$, let

$$\sigma \cdot \chi(t) = \sigma(\chi(\sigma^{-1}(t))).$$

We define the K-characters of T, $X_K(T)$ to be the characters fixed by this action, $X_K(T) := X(T)^{\mathscr{G}}$. In other words, $X_K(T)$ is the set of characters of T that are defined over the field K. It is clear that $X_{K^{\text{sep}}}(T) = X(T)$. Note that we often write the group of characters $X(T)$ with additive notation.

Note that every character $X(\text{GL}_1)$ has the form $\chi(x) = x^m$ for $x \in (K^{\text{sep}})^\times$ and some $m \in \mathbb{Z}$. Therefore, $X(\text{GL}_1) \cong \mathbb{Z}$. Furthermore, for a rank n torus T, $X(T) = (X(\mathbb{G}_m))^n = \mathbb{Z}^n$ (see [3, §8.5, pg. 114]).

Definition 2.1.18. We say that the K-torus T is **split over** K if $X_K(T) = X(T)$. We say that T is **anisotropic over** K if $X_K(T) = \{1\}$.

Note that saying an n-dimensional torus is split over K is equivalent to saying that the isomorphism $T \to (\mathbb{G}_m)^n$ is defined over K.

Proposition 2.1.19. *Every K-torus is split over some finite separable extension of K.*

Proof. See [3, Prop. 8.11, pg. 117] □

Consider a rational representation $T \longrightarrow \mathbf{GL}(V)$. If $\alpha \in X(T)$, define

$$V_\alpha = \{v \in V \otimes_K \overline{K} \mid t \cdot v = \alpha(t)v \text{ for all } t \in T(\overline{K})\}.$$

Since T is split over $\overline{K}$, we have the decomposition $V \otimes_K \overline{K} = \bigoplus_{\alpha \in X(T)} V_\alpha$. We say that α is a **weight of T in V** if $V_\alpha \neq 0$.

Remark 2.1.20. Let T be an n-dimensional torus and let $t \in T(\overline{K})$. Let V be the representation space given the representation coming from the diagonal embedding, $T(\overline{K}) \cong (\overline{K}^\times)^n \hookrightarrow \mathrm{GL}_n(\overline{K})$, and let $\alpha \in X(T)$ be a weight of this representation. Note that in this case, $\alpha(t) \in \overline{K}^\times$ is an eigenvalue of t with eigenvector $v \in V_\alpha$. Since the product of characters is also a character, $\chi(t)$ is contained in the multiplicative subgroup of $\overline{K}^\times$ generated by the eigenvalues of t for all $\chi \in X(T)$.

Example 2.1.21. Consider the algebraic torus T over $\mathbb{R}$,

$$T(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in \mathrm{M}_2(\mathbb{R}) \;\middle|\; a^2 + b^2 = 1 \right\}.$$

This is a 1-dimensional torus over $\mathbb{R}$. The eigenvalues of every element in $T(\mathbb{R})$ are of the form $a + ib$ and $a - ib$, so by Remark 2.1.20, the only possible character is the trivial character. Thus, $X_{\mathbb{R}}(T) = \{1\}$, and T is $\mathbb{R}$-anisotropic.

If we consider the complex points of T, $T(\mathbb{C})$ is isomorphic to the group $T' = \{\mathrm{diag}(z, z^{-1}) \mid x \in \mathbb{C}\}$, hence T is $\mathbb{C}$-split.

Let G be a semisimple group defined over the field K.

Definition 2.1.22. A **maximal K-torus** of G is an K-subgroup of G that is a torus and not strictly contained in any other K-torus.

Proposition 2.1.23. *Let $T \subseteq G$ be a maximal K-torus of the semisimple K-group G.*

(i) *If T' is another maximal K-torus of G, then T and T' are conjugate in $G(K^{\mathrm{sep}})$.*

(ii) *When K is not an algebraic extension of a finite field, there exists an element $t \in T(\overline{K})$ that generates a Zariski-dense subgroup of T.*

(iii) *Assume that K is infinite. Then every semisimple element of $G(K)$ lies in the K-points of some maximal torus.*

(iv) *A semisimple element $x \in G(K)$ is regular if and only if $Z_G(x)^\circ$ is a maximal torus.*

(v) *The torus T is maximal if and only if $T = Z_G(T)^\circ$.*

Proof. For (i), see [7, Prop. A.2.10, pg. 401].

For (ii), see [3, §8, Prop. 8.8].

For (iii), see [27, Cor. 13.3.8, pg. 231].

For (iv), see [3, Prop. 12.1 pg. 160].

For (v), if $C := Z_G(T)^\circ = T$, then T is maximal since every torus containing T is contained in C.

If T is maximal, then since C is reductive its radical $\mathcal{R}(C)$ is a torus. Clearly, $T \subseteq \mathcal{R}(C)$, and so $T = \mathcal{R}(C)$ since T is maximal. Thus C/T is semisimple and $\operatorname{rank}(C/T) = 0$ since a nontrivial torus of C/T would correspond to a torus in C properly containing T. But C is smooth and connected, hence C/T is trivial and $C = T$. □

Definition 2.1.24. We say that the group G is **K-split** if it contains an K-split maximal torus, and that the group is **K-anisotropic** if every maximal torus in G is K-anisotropic. We say that G is **K-isotropic** if it is not K-anisotropic.

Semisimple algebraic groups

We would like to first classify all split semisimple groups. We first recall the basic theory of root systems.

Let E be a Euclidean space with positive definite symmetric bilinear form $(\cdot, \cdot)$. A **root system** in E is a finite set $\Phi \subseteq E$ of non-zero vectors that satisfy the following conditions:

(RS1) The roots span E.

(RS2) The only scalar multiples of $\alpha \in \Phi$ are α and $-\alpha$.

(RS3) For any two roots $\alpha, \beta \in \Phi$,

$$s_\alpha(\beta) := \alpha - 2\frac{(\alpha, \beta)}{(\beta, \beta)}\beta \in \Phi.$$

(RS4) For any two roots $\alpha, \beta \in \Phi$,

$$2\frac{(\alpha, \beta)}{(\beta, \beta)} \in \mathbb{Z}.$$

We call $s_\alpha \colon \Phi \to \Phi$ **the reflection associated to the root** α.

Sometimes (RS4) may be omitted, and such root systems satisfying it are called **crystallographic**. Similarly, sometimes (RS2) may be omitted, and such root systems satisfying it are called **reduced**. All of the root systems arising in this paper are crystallographic and reduced. We note that non-reduced root systems do arise naturally in this context when one considers relative root systems (see [27, 15.3.9, pg. 260]), but this will not be necessary for this paper.

An **isomorphism of root systems** (E, Φ) and (E', Φ') is an E-linear isomorphism $f\colon E \to E'$ such that $F(\Phi) = \Phi'$. The automorphism group $\mathrm{Aut}(\Phi)$ is a finite group and the subgroup $\mathcal{W}(\Phi)$ of $\mathrm{Aut}(\Phi)$ generated by all the reflections s_α, $\alpha \in \Phi$, is called the **Weyl group of** Φ.

Let Φ_1 and Φ_2 be root systems in V_1 and V_2 respectively, and let $\Phi = \Phi_1 \cup \Phi_2$ and $V = V_1 \oplus V_2$. Then Φ is a root system in V, called the sum of Φ_1 and Φ_2 and denoted $\Phi = \Phi_1 + \Phi_2$. We say that Φ is **irreducible** if it cannot be written as a sum of two (or more) root systems. Any root system decomposes uniquely into a sum of irreducible root systems (see [13, §24, pg. 353]).

Definition 2.1.25. Let Φ be a root system in the Euclidean space E. A subset Φ^+ of Φ is called **a system of positive roots** if there exists an $x \in E$ with $(\alpha, x) \neq 0$ for all $\alpha \in R$ such that

$$\Phi^+ = \{\alpha \in \Phi \mid (\alpha, x) > 0\}.$$

Since $(\cdot, \cdot)$ is a $\mathcal{W}(\Phi)$-invariant symmetric bilinear form, $\mathcal{W}(\Phi)$ acts on the set of positive root systems, i.e. $w\Phi^+$ is a system of positive roots for all $w \in \mathcal{W}(\Phi)$, and it is not difficult to show that this action is simply transitive.

Definition 2.1.26. Two systems of positive roots $\Phi+$ and $\widetilde{\Phi}^+$ are called **adjacent** if $|\Phi^+ \cap \widetilde{\Phi}^+| = |\Phi^+| - 1$.

Definition 2.1.27. Let Φ be a root system and let Φ^+ be a system of positive roots. Define the following subset of Φ^+:

$$\Delta := \{\alpha \in \Phi^+ \mid s_\alpha \Phi^+ \text{ is adjacent to } \Phi^+\}.$$

The set Δ is called the **set of simple roots** of Φ^+.

It is clear for all $w \in \mathcal{W}(\Phi)$ and a set of simple roots Δ, $w\Delta$ is also a set of simple roots and the action of $\mathcal{W}(\Phi)$ on the set of sets of simple roots is simply transitive.

A subset $\Pi \subset \Phi$ is **basis** of the root system Φ if for any $\alpha \in \Phi$,

$$\alpha = \sum_{\beta \in \Pi} n_\beta \beta$$

for some uniquely determined n_β such that either $n_\beta \leq 0$ for all $\beta \in \Pi$ or $n_\beta \geq 0$ for all $\beta \in \Pi$.

Proposition 2.1.28. *The set of simple roots Δ is a basis for Φ.*

Proof. See [3, Cor. 1 to Thm. 14.8, pg. 189] □

We define a graph, called the **Dynkin diagram** of Φ, which has Δ as its set of vertices. The vertices α and β are connected by $m_{\alpha\beta}$ edges, where

$$m_{\alpha\beta} = \left(\frac{2(\alpha, \beta)}{(\beta, \beta)}\right)\left(\frac{2(\beta, \alpha)}{(\alpha, \alpha)}\right).$$

If α and β are two incident vertices and $(\alpha, \alpha) > (\beta, \beta)$, then we label the edges connecting α and β with a "$>$" pointing toward β to indicate that the root α is longer than β.

Let G be a semisimple algebraic group defined over the field K and suppose that G is split with maximal split torus T. Let $\mathfrak{g} := \mathrm{Lie}(G)$, and consider the weight space decomposition with respect to the adjoint representation restricted to T, $\mathrm{Ad}|_T : T \to \mathbf{GL}(\mathfrak{g})$,

$$\mathfrak{g} = \bigoplus_{\alpha \in X(T)} \mathfrak{g}_\alpha.$$

Definition 2.1.29. The nonzero weights of the adjoint representation for a maximal torus $T \subseteq G$ are called the **roots of G with respect to T** and the set of all roots is denoted $\Phi(G, T)$.

Definition 2.1.30. The **Weyl group of G with respect to T** is defined to be $\mathcal{W}(G, T) = N_G(T)(\overline{K})/T(\overline{K})$

Proposition 2.1.31. *Let $\Phi := \Phi(G, T)$ be the set of roots of the semisimple group G. Let Q be the* **root lattice***, the subgroup of $X(T)$ generated by Φ, and let $E = \mathbb{R} \otimes_{\mathbb{Z}} Q$.*

(a) There exists a $\mathcal{W}(G, T)$-invariant positive definite symmetric bilinear form on E, denoted $(\cdot, \cdot)$.

(b) The pair (E, Φ) is a root system with respect to the above $\mathcal{W}(G, T)$-invariant bilinear form.

Proof. See [3, Thm. 14.8, pg. 189]. □

Proposition 2.1.32.

(a) *There exists an isomorphism* $\mathcal{W}(G,T) \cong \langle s_\alpha \,|\, \alpha \in \Phi(G,T)\rangle$.

(b) *The conjugation of maximal tori* $T \to T' = gTg^{-1}$ *induces a natural bijection of root systems* $\Phi(G,T) \to \Phi(G,T')$ *and a natural isomorphism of Weyl groups* $\mathcal{W}(G,T) \cong \mathcal{W}(G,T')$. *In particular,* $\Phi(G,T)$ *and* $\mathcal{W}(G,t)$ *are invariant up to isomorphism with respect to the choice of maximal torus.*

Proof. Part (a) is a consequence of the proof of [3, Thm. 14.8, pg. 189].

Part (b) follows from Proposition 2.1.23(i). If T and T' are two maximal tori, then there exists a $g \in G(\overline{K})$ such that $T'(\overline{K}) = g^{-1}T(\overline{K})g$. The conjugation isomorphism $\psi_g \colon T(\overline{K}) \to T'(\overline{K})$ induces a natural bijection $\psi_g^* \colon X(T') \to X(T)$ which induces a natural bijection between the root system $\Phi(G,T')$ and the root system $\Phi(G,T)$. This bijection induces a natural isomorphism between the Weyl groups $\mathcal{W}(G,T)$ and $\mathcal{W}(G,T')$. □

Isogenies

Let N be an algebraic group scheme over the field K, and let $C = K[N]$. We say that N is a **finite group scheme** if C is a finite dimensional K-vector space.

Example 2.1.33.

1. Let $p = \text{char}(K)$ and consider $N = K[x]/(x^p)$. Then the group scheme $\alpha_p := G^N$ is a finite group scheme.

2. Similarly, let $p = \text{char}(K)$ and consider $M = K[x]/(x^p - 1)$. Then the group scheme $\mu_p := G^M$ is a finite group scheme.

Definition 2.1.34. Let $f\colon G \longrightarrow H$ be a surjective homomorphism of group schemes (i.e. $f^*\colon K[H] \longrightarrow K[G]$ is injective), and let $N = \ker(f)$. The homomorphism f is called an **isogeny** if N is a finite group scheme, and it is called a **central isogeny** if $N(R)$ is contained in $Z(G)(R)$ for every object R in Alg_K and $\ker(df)$ is contained in the center of $\text{Lie}(G)$.

Example 2.1.35. Let $n > 0$ be natural number. Define $f\colon \mathbb{G}_m \to \mathbb{G}_m$ by $f_R(x) = x^n$ for all commutative K-algebras R. Then f is a central isogeny of group schemes.

Definition 2.1.36. Let G and H be connected semisimple linear algebraic K-groups and $\pi\colon G \longrightarrow H$ be an isogeny. We say that π is **purely inseparable** (resp. **separable**) if the induced inclusion of function fields $K(H) \hookrightarrow K(G)$ is a purely inseparable (resp. separable) extension.

Proposition 2.1.37. *Suppose that $\pi\colon G \longrightarrow H$ is a separable isogeny of connected semisimple K-groups. Then π is a central isogeny.*

Proof. See [3, §22.3, pg. 247]. □

Definition 2.1.38.

1. Two algebraic groups H and H' are called **(strictly) isogenous** if there exists an algebraic K-group G and two (central) isogenies $G \to H$ and $G \to H'$.

2. A connected semisimple algebraic group H is called **simply connected** if every central isogeny $H' \to H$, for H' connected, is an isomorphism.

3. A connected semisimple algebraic group H is called **adjoint** if every central isogeny $H \to H'$, for H' connected, is an isomorphism.

Proposition 2.1.39. *Let G be a connected semisimple algebraic K-group. Then there exists a sequence*

$$\widetilde{G} \xrightarrow{\widetilde{\pi}} G \xrightarrow{\overline{\pi}} \overline{G},$$

such that $\widetilde{G}$ is a simply connected K-group and $\overline{G}$ is an adjoint K-group, $\widetilde{\pi}$ and $\overline{\pi}$ are central K-isogenies, and $\widetilde{G}$, $\overline{G}$, $\widetilde{\pi}$, and $\overline{\pi}$ are unique up to K-isomorphism.

Proof. See [29, Prop. 2, pg. 42]. □

The groups $\widetilde{G}$ and $\overline{G}$ are called the **simply connected cover** and **adjoint group** of G respectively.

Definition 2.1.40. A connected algebraic group G is called **K-simple** if G is non-commutative and $G(K)$ has no normal algebraic subgroup except $G(K)$ and $\{e\}$. A connected algebraic group G is called **K-almost simple** if it is isogenous to an K-simple group.

For example, the group SL_n is K-almost simple and the group PSL_n is K-simple for $n > 1$ and any infinite field K.

Theorem 2.1.41. *Let K be a field and let G be a semisimple simply connected K-group. Then G is a direct product of K-almost simple simply connected groups.*

Proof. See [29, §3.1.1 pg. 46]. □

Proposition 2.1.42. *A K-split semisimple group G is K-simple if and only if $\Phi(G)$ is an irreducible root system. A simply connected (resp. adjoint) K-split semisimple group G is the direct product of uniquely determined simple subgroups G_i and $\Phi(G) = \sum_i \Phi(G_i)$.*

Proof. See [13, Prop. 25.8, pg. 357]. □

Definition 2.1.43. We say that the algebraic K-group G is **absolutely almost simple** if it is almost simple over the algebraic closure $\overline{K}$.

The group G is absolutely almost simple if and only if the root system $\Phi(G)$ is an irreducible root system.

Let K'/K be a finite separable extension and suppose that H is a semisimple

K'-group. Then we can define an algebraic K-group $R_{K'/K}(H)$ as a functor by

$$R_{K'/K}(H)(R) = H(R \otimes_K K')$$

for all objects R in Alg_K. The algebraic K-group $R_{K'/K}(H)$ is called called the **restriction of scalars** of the group H from K' to K (also known as **Weil restriction**).

Theorem 2.1.44. *Let G be a K-almost simple simply connected K-group. Then there exists a finite separable extension K' over K and an absolutely almost simple simply connected group H defined over K' such that $G \cong R_{K'/K}(H)$.*

Proof. See [29, §3.1.1 pg. 46]. □

The isogenies described in Theorems 2.1.41 and 2.1.44 are called **almost direct products**. Since semisimple groups essentially break up as almost direct products of connected absolutely almost simple groups, we will restrict ourselves to the study of connected absolutely almost simple groups. We hope to address the more general cases in the future.

One of the benefits to restricting our study to connected absolutely almost simple groups is because they are classified by their root systems.

Theorem 2.1.45 (Classification theorem)**.** *Let G be a connected absolutely almost simple group. Then the root system of G, $\Phi(G)$, is exactly one of type A_n $(n \geq 1)$, B_n $(n \geq 2)$, C_n $(n \geq 3)$, D_n $(n \geq 4)$, E_6, E_7, E_8, F_4, or G_2, called the* **(Killing-**

Cartan) type of G. *If* G *and* G' *are strictly isogenous, then* $\Phi(G)$ *and* $\Phi(G')$ *have the same type.*

Proof. See [29, Thm. 1, pg. 34]. □

Inseparable isogenies

Let G be an algebraic group defined over K, where $\mathrm{char}(K) = p > 0$. Let $A := K[G]$.

Let $f \in \mathrm{Hom}_K(A, R)$. For $a \in A$, consider the map $a \mapsto f(a)^{p^n}$. This is a ring homomorphism from A to R^{p^n}, but it takes the action of K on A to an action of K^{p^n} on R^{p^n}. We would like to 'fix' this map so that it is an algebra homomorphism.

Define a K-module structure on R^p by $\alpha \cdot r = \alpha^{p^n} r$ for $\alpha \in K$, $r \in R^p$. Under this action, $f^{p^n} : A \longrightarrow R^p$ is a K-algebra homomorphism. The group valued functor defined by

$$R \mapsto \mathrm{Hom}_K(A, R^{p^n})$$

for all objects R in Alg_K is an algebraic group. It is represented by the Hopf-algebra $A \otimes_\sigma K$, where the tensor product is twisted by the p^nth power, i.e. $a \otimes \alpha = (\alpha)^{p^n} a \otimes 1$ for all $a \in A$ and $\alpha \in K$ (see [16, §1, pg. 448]). The homomorphism of algebraic groups defined on R-points by the homomorphism

$$\mathrm{Hom}_K(A, R) \longrightarrow \mathrm{Hom}_K(A, R^{p^n})$$

is actually an isogeny called the n**th Frobenius isogeny**, denoted by $\mathrm{Fr}^n : G \longrightarrow$

$\mathrm{Fr}^n(G)$.

The isogeny Fr^{n} is purely-inseparable. Any composition of Frobenius isogenies is also a Frobenius isogeny. This isogeny is also purely inseparable and not central when G is connected and non-commutative.

Let G be a linear algebraic group over K. We would like to determine how the Frobenius isogeny acts on the R-rational points of G. Fix an embedding $G \subset \mathrm{GL}_N$. It suffices to examine Fr^n for $\mathrm{GL}_N(R)$.

Let $f \in \mathrm{GL}_N(R)$. This is given by a K-algebra homomorphism $f\colon K[\mathrm{GL}_N] \to R$, and the images of the generators X_{ij} of $K[\mathrm{GL}_N]$ determine the matrix entries of a matrix realization of f. In other words, $[f(X_{ij})] = [a_{ij}]$ is a matrix in $\mathrm{GL}_N(R)$ in the classical sense. Let us compute the image of X_{ij} under $\mathrm{Fr}^n(f)$:

$$\mathrm{Fr}^{\mathrm{n}}(f)(X_{ij}) = f(X_{ij})^{p^n} = a_{ij}^{p^n}.$$

Therefore we have shown that on a matrix group, the Frobenius isogeny is the map

$$[a_{ij}]_{i,j} \mapsto \left[a_{ij}^{p^n}\right]_{i,j}.$$

Non-standard isogenies

For a few special cases, there exist purely inseparable isogenies between connected semisimple groups which cannot be obtained from central isogenies and Frobenius isogenies. In particular, there are a few cases where the Frobenius isogeny can be factored in a nontrivial way. We call such isogenies **non-standard isogenies**.

Remark 2.1.46. We note that other sources refer to non-standard isogenies as non-central special isogenies or very special isogenies, but we follow the language used in Pink [16].

Proposition 2.1.47. *Let G be a connected absolutely almost simple group defined over K, a field of characteristic $p > 0$. Let Φ be the root system of G and suppose that Φ contains roots of different lengths so that the squared length ratio is equal to p. Then $\mathrm{Fr}\colon G \longrightarrow \mathrm{Fr}(G)$ factors through totally separable isogenies*

$$G \xrightarrow{\phi} G^\sharp \xrightarrow{\phi^\sharp} \mathrm{Fr}(G),$$

such that neither ϕ nor $\phi^\sharp$ is an isomorphism. The algebraic group $G^\sharp$ is another absolutely almost simple group over K. Let $\phi^\sharp$ be the root system of $G^\sharp$. The possibilities for $(p, \Phi, \Phi^\sharp)$ are listed in the following table.

p	*Type of* Φ	*Type of* $\Phi^\sharp$
2	B_n $(n \geq 2)$	C_n
2	C_n $(n \geq 2)$	B_n
2	F_4	F_4
3	G_2	G_2

Proof. See [16, Prop. 1.6]. □

Example 2.1.48. Let $K = \overline{K}$ be an algebraically closed field of characteristic 2 and let G be the simply connected almost simple group of type B_l for some $l > 2$. Let the set of long roots be denoted by Φ_l and short roots be denoted by Φ_s. In this case, the subalgebra of $\mathfrak{g} := \mathrm{Lie}(G)$ spanned by the short roots is an $\mathrm{Ad}(G)$-invariant subspace of $\mathfrak{g}$ (see [11, Table 1]). In this case, $\psi\colon G \to G^\sharp$ induces a map, called an isogeny of root data, between the root systems. The isogeny theorem and isogenies of root data are discussed at length in [7, Thm. A.4.10, pg. 418]. The isogeny of root data corresponding to the isogeny ψ is given by

$$f\colon \Phi \to \Phi^\sharp,$$

where $f(\alpha) = \alpha$ if α is a long root and $f(\alpha) = 2\alpha$ if α is a short root. Showing that this is the correct isogeny of root data (see [7, Prop. 7.1.5, pg. 223] for details). Note that this isogeny maps short roots in Φ to long roots in $\Phi^\sharp$ and long roots in Φ to short roots in $\Phi^\sharp$.

The above isogeny of root data maps Φ onto a root system of type C_l, so $G^\sharp$ is an almost simple group of type C_l.

The corresponding map $\psi\colon G^\sharp \to \mathrm{Fr}(G)$ induces an isogeny of root data:

$$f^\sharp\colon \Phi^\sharp \to 2\Phi,$$

where $f(\alpha^\sharp) = \alpha^\sharp$ if $\alpha^\sharp$ is a long root in $\Phi^\sharp$ and $f(\alpha^\sharp) = 2\alpha^\sharp$ if $\alpha^\sharp$ is a short root in $\Phi^\sharp$.

The composition $f^\sharp \circ f \colon \Phi \to 2\Phi$ is multiplication by 2, which corresponds to the Frobenius isogeny. Furthermore, when G is simply connected, $G^\sharp$ is also simply connected.

Example 2.1.49. For a more concrete example, we note that the isogeny $\psi : G \to G^\sharp$ can also be realized in the following way. Let K be an algebraically closed field of characteristic 2, let $V = K^{2l+1}$ be a vector space, and let q be the quadratic form on V defined by

$$q(x_0, x_1, ..., x_{2l}) = x_0^2 + \sum_{i=1}^{l} x_i x_{l+i}.$$

Note that q is a non-degenerate quadratic form, and the spin group $\mathrm{Spin}(q)$ of this quadratic form is a group of type B_l. The associated bilinear form $B_q(x, y) = q(x + y) - q(x) - q(y)$ has the one dimensional radical $V^\perp = Ke_0$, which induces a non-singular alternating form $\overline{B_q}$ on the $2l$-dimensional vector space $\overline{V} = V/V^\perp$. The symplectic group of $\overline{B_q}$, $\mathrm{Sp}(\overline{B_q})$, is a simply connected group of type C_l. It is clear that any linear transformation that preserves the quadratic form q preserves the form $\overline{B_q}$ on $\overline{V}$. Therefore, we have a induced homomorphism $\mathrm{SO}(q) \to \mathrm{Sp}(\overline{B_q})$ (and it takes a little work to show that this is actually surjective). The isogeny ψ can be realized as a composition of the maps

$$\mathrm{Spin}(q) \to \mathrm{SO}(q) \to \mathrm{Sp}(\overline{B_q}).$$

Classification of isogenies

All isogenies between connected (split) semisimple groups can be obtained from central isogenies, Frobenius isogenies, and the non-standard isogenies. The following theorem proven in [16] is invaluable to our work.

Theorem 2.1.50. *Let $f\colon G \longrightarrow H$ be an isogeny between two connected absolutely simple adjoint groups defined over a field K of characteristic p.*

(a) If $p = 0$, then f is an isomorphism.

(b) If $p > 0$, and G possesses no non-standard isogenies, then there exists an integer $n \geq 0$ and an isomorphism $\psi\colon \mathrm{Fr}^n(G) \longrightarrow H$ such that $f = \psi \circ \mathrm{Fr}^n$.

(c) If $p > 0$ and G possesses non-standard isogenies, then there exists an integer $n \geq 0$ and an isogeny $\psi\colon \mathrm{Fr}^n(G) \longrightarrow H$ with non-vanishing derivative such that $f = \psi \circ \mathrm{Fr}^n$. Furthermore, either ψ is an isomorphism or there exists an isomorphism $\chi\colon \mathrm{Fr}^n(G)^\sharp \longrightarrow H$ such that $\psi = \chi \circ \phi$ where ϕ is the non-standard isogeny $\mathrm{Fr}^n(G) \xrightarrow{\phi} \mathrm{Fr}^n(G)^\sharp$.

Proof. See [16, Thm. 1.7]. □

Note that for a K-defined isogeny $f\colon G \to H$, this does not imply that the above morphisms $\psi\colon \mathrm{Fr}^n(G) \longrightarrow H$ and $\chi\colon \mathrm{Fr}^n(G)^\sharp \longrightarrow H$ are defined over K. Only when G (and therefore H) is split over K can we determine that these are defined over K.

Inner and outer forms

In this section, we will assume that the reader has knowledge about the basic results from Galois cohomology. For more details, we refer the reader to [26].

Let G be a semisimple group defined over K.

The following definition is due to Tits in [29, pg. 39]. For this section, let $\mathscr{G} := \mathrm{Gal}(K^{\mathrm{sep}}/K)$ be the absolute Galois group, $S \subseteq G$ be a maximal K-split torus in G, and let T be a maximal K-torus of G containing S.

If $S = T$, then it is clear that G is split over K. If $Z_G(S) = T$, then we say that G is **quasi-split over** K.

Remark 2.1.51. The semisimple group G is quasi-split over K if and only if there exists a Borel subgroup of G defined over K. This is the traditional definition of quasi-split and it is equivalent to our definition [27, 16.2, pg. 271].

Let $\Phi(G,T)$ be a the root system of G, let $\Delta := \Delta(G,T)$ be a system of simple roots in $\Phi(G,T)$, and let $\mathcal{W} = N_G(T)(\overline{K})/T(\overline{K})$ be the Weyl group of G. Let Δ_0 be the subset of Δ consisting of those simple roots that vanish on S.

We first define an action, called the **$*$-action**, of $\mathscr{G}$ on Δ. Recall that $\mathscr{G}$ naturally acts on $X(T)$. Note that for the set of simple roots Δ, $\sigma(\Delta)$ is again a system of simple roots. Since $\mathcal{W}$ acts simply transitively on the systems of simple roots, there exists a well-defined element $w \in \mathcal{W}$ such that $w(\sigma(\Delta)) = \Delta$. We define the $*$-action of $\mathscr{G}$ on Δ by $\sigma * \Delta = w \circ \sigma(\Delta)$ for all $\sigma \in \mathscr{G}$. Note that the $*$-action of $\mathscr{G}$ fixes both

Δ_0, and $\Delta - \Delta_0$.

Definition 2.1.52. If the $*$-action of $\mathscr{G}$ on Δ is trivial (resp. nontrivial), we say that the group G is an **inner form over** K (resp. **outer form over** K).

Note that the $*$-action of $\mathscr{G}$ induces a diagram automorphism $\tau \in \mathrm{Aut}(\mathrm{Dyn}(G))$. We call the data consisting the triple (Δ, Δ_0, τ) the **Tits index** of the group G.

To represent the Tits index, the Dynkin diagram of G is drawn so that vertices belonging to the same $\mathscr{G}$ orbit are close to each other, and the orbits whose elements belong to $\Delta - \Delta_0$, called **distinguished orbits**, are circled.

Inner forms also have a Galois cohomological description, as demonstrated by the following lemma proven in [20]. Let G be a connected absolutely almost simple group defined over K. Given a (continuous) Galois 1-cocycle $z \in Z^1(K, \mathrm{Aut}(G))$, we can form the group ${}_zG$, the algebraic group twisted by the 1-cocycle z as follows.

Let $K^{\mathrm{sep}}[G]$ be the Hopf algebra of regular functions of G over K^{sep}. This is naturally a $\mathscr{G}$-module. Given a 1-cocycle $z \in Z^1(K, \mathrm{Aut}G)$, we define a new action of $\mathscr{G}$ on $K^{\mathrm{sep}}[G]$ by the following rule. Let $f \in K^{\mathrm{sep}}[G]$ and $\sigma \in \mathscr{G}$ and note that $z_\sigma \in \mathrm{Aut}_{K^{\mathrm{sep}}}(G)$. Therefore, $\sigma \circ z_\sigma \in \mathrm{Aut}_{K^{\mathrm{sep}}}(G)$. Let $(\sigma \circ z_\sigma)^* \in \mathrm{Aut}_{K^{\mathrm{sep}}-alg}(K^{\mathrm{sep}}[G])$ be the automorphism of Hopf algebras corresponding to $\sigma \circ z_\sigma$. Define the twisted action of $\mathscr{G}$ on $K^{\mathrm{sep}}[G]$ by

$$\sigma \cdot f = (\sigma \circ z_\sigma)^*(f),$$

for all $f \in K^{\mathrm{sep}}[G]$, $\sigma \in \mathscr{G}$. Let $K[{}_zG]$ be the fixed points of $K^{\mathrm{sep}}[G]$ under this new action of $\mathscr{G}$. The twisted group ${}_zG$ is algebraic group scheme corresponding to $K[{}_zG]$.

Lemma 2.1.53. *Let G be a connected absolutely almost simple group defined over a field K. Then $G(K)$ is an inner form over K (i.e. the $*$-action is trivial) if and only if $G(K)$ is isomorphic to $({}_zG_0)(K)$, where G_0 is a connected absolutely almost simple K-split group and z is a 1-cocycle with values in $\mathrm{Int}(G_0)$, i.e. $z \in Z^1(K, \mathrm{Int}(G_0))$. Furthermore, there exists a finite Galois extension L/K such that G is an inner form over L.*

Proof. See [20, Lemma 4.1]. □

Fields of definition of a Zariski-dense subgroup

Let K be a field and let V be an K-vector space of finite dimension. Let $A \subseteq K$ be a subring of K.

Definition 2.1.54. A set $L \subseteq V$ is called an A**-lattice** if it is a finitely generated A-module such that $K \otimes_A L \cong V$.

Remark 2.1.55. Note that if A is a principal ideal domain, then every A-lattice has a basis that is also a basis for V over K. Thus, if A is a principal ideal domain, then L is free.

Definition 2.1.56. Let $\Delta \subseteq \mathrm{End}(V)$. Let $A \subseteq K$ be a subring of K such that V contains a Δ-invariant A-lattice. Then A is called a **ring of definition** for Δ. We also say that Δ is **definable** over A.

Lemma 2.1.57. *Let A be an integrally closed Noetherian ring. If Δ is definable over A, then* $\mathrm{tr}(\Delta) \subseteq A$.

If we apply Lemma 2.1.57 to the setting of connected semisimple algebraic groups, then we see that given the adjoint representation of a connected semisimple algebraic group G defined over a field K, $\mathrm{Ad}_K\colon G(K) \to \mathrm{GL}(\mathrm{Lie}(G))$, if $\Gamma \subseteq G(A)$ then $\mathrm{tr}(\mathrm{Ad}_K(\gamma)) \in A$. With some additional conditions, we are able to find a characterization of the ring A such that the converse is true. By a slight abuse of notation, we use Ad when referring to Ad_K for the rest of this section.

Theorem 2.1.58. *Let K be a field of characteristic* 0*, and let $A \subseteq K$ be an integrally closed Noetherian ring. Let $\Gamma \subseteq G(K)$ be a Zariski-dense subgroup.*

(a) *Then A is a ring of definition for* $\mathrm{Ad}(\Gamma) \subseteq \mathrm{GL}(\mathrm{Lie}(G))$ *if and only if* $\mathrm{tr}(\mathrm{Ad}(\gamma)) \in A$ *for all $\gamma \in \Gamma$.*

(b) *There exists a minimal field of definition for the group* $\mathrm{Ad}(\Gamma)$*, and it is the subfield of K generated by* 1 *and the elements* $\mathrm{tr}(\mathrm{Ad}(\gamma))$ *for all $\gamma \in \Gamma$.*

Sketch of the proof. See [30, Theorem 1] for the complete argument.

The proof of (a) consists of a close examination of the regular function $f_0 := \mathrm{tr}\mathrm{Ad} \in K[G]$. Suppose that $\mathrm{tr}(\mathrm{Ad}(\gamma)) \in A$ for all $\gamma \in \Gamma$. Let $V(f_0)$ be the vector subspace of $K[G]$ generated by left translations of f_0 by elements of $G(K)$. The resulting representation, denoted $\rho\colon G(K) \to \mathrm{GL}(V(f_0))$ is a multiple of the adjoint representation, so the kernel of the action of $G(K)$ on $V(f_0)$ is finite.

Consider the set

$$L = \{f \in V(f_0) \mid f(\gamma) \in A \text{ for all } \gamma \in \Gamma\}.$$

This is an $\rho(\Gamma)$-invariant A-submodule of $V(f_0)$. Since Γ is Zariski-dense in $G(K)$, this implies that $L \otimes_A K$ is $\rho(G)$-invariant.

Choose elements $\gamma_1, ..., \gamma_n \in \Gamma$ such that the map $\sigma\colon V(f_0) \to K^n$ defined by

$$\sigma(f) = (f(\gamma_1), ..., f(\gamma_n))$$

is an isomorphism.

Linear dependence of elements of L over $\mathrm{Frac}(A)$ implies linear dependence of elements of L over K. Therefore, $\sigma(L) \otimes K \cong K^n \cong V(f_0)$, which implies that L is an A-lattice of $V(f_0)$ and that $\rho(\Gamma)$ (hence $\mathrm{Ad}(\Gamma)$) is definable over A.

The proof of (b) is an immediate consequence of (a). □

We call the field generated by the traces Γ the **trace field of** Γ, and since for fields of characteristic zero, the trace field and the minimal field of definition coincide, we denote the trace field by K_Γ as well.

We will have to modify the notion of the field of definition in order for these results to hold over fields of arbitrary characteristic. These nuances are addressed by Pink in [16] and [17] and summarized in Section 2.2.

Local fields

For now, let K be any field.

Definition 2.1.59. We say that $|\cdot|\colon K \to \mathbb{R}_{\geq 0}$ is an **absolute value** on K if for all $x, y \in K$, it satisfies the following:

(a) $|x| = 0$ if and only if $x = 0$,

(b) $|xy| = |x||y|$,

(c) and $|x + y| \leq |x| + |y|$.

A trivial absolute value is one for which $|x| = 1$ for all $x \neq 0$. We assume that all absolute values are nontrivial unless otherwise stated.

Definition 2.1.60. A **local field** is a topological field that is locally compact with respect to a non-discrete topology.

Let K be a local field, let μ be the Haar measure of the additive group of K, and let $A \subset K$ be any measurable set. Consider the function $|\cdot|\colon K \to \mathbb{R}_{\geq 0}$ defined by

$$|x| = \frac{\mu(xA)}{\mu(A)}.$$

Fact 2.1.61.

(a) The function $|\cdot|$ defined above is an absolute value on K.

(b) The metric topology induced by the absolute value is the same as the original topology.

Proof. Part (a) is a consequence of [32, Prop. 1, pg. 4]. Part (b) is a consequence of [32, Cor. 1, pg. 5]. □

Definition 2.1.62. We say that the absolute value $|\cdot|\colon K \to \mathbb{R}_{\geq 0}$ is **archimedean** if $|n \cdot 1| > |1|$ for all integers $n > 1$, and **non-archimedean** otherwise.

Since $p \cdot 1 = 0$, it is clear that all absolute values are non-archimedean when $\mathrm{char}(K) = p > 0$. When the absolute value on K is archimedean (resp. non-archimedean) we typically say that the field K is archimedean (resp. non-archimedean) as well.

Fact 2.1.63. If K is a complete local field with archimedean absolute value, then $K = \mathbb{R}$ or $K = \mathbb{C}$.

Proof. See [12, Ch. II, §4, pg. 85-88]. □

Lemma 2.1.64. *Let K be a local field. Then K is complete.*

Proof. Let $\tilde{K}$ be the completion of K, and let $x \in \tilde{K}$. Let $S_x = \{y \in K; |y| = |x|\}$.

This is a compact set since the absolute value is continuous. Let $\{x_n\}$ be a sequence in K that converges to x. Since K is locally compact, there exists a large enough $N \in \mathbb{Z}$ so that x_n is contained in a compact set $U \subset K$ for all $n > N$. Since U contains all of its limit points, $x \in U$. □

Fact 2.1.65. A non-archimedean absolute value satisfies the ultrametric inequality. Specifically, $|x + y| \leq \max\{|x|, |y|\}$ for all $x, y \in K$. Furthermore, $|x + y| = \max\{|x|, |y|\}$ if $|x| \neq |y|$.

Suppose now that K is a non-archimedean local field.

Definition 2.1.66. We define the following:

(a) **the ring of integers of** K, $\mathcal{O}_K = \{a \in K; |a| \leq 1\}$,

(b) **the maximal ideal of** $\mathcal{O}$, $\mathfrak{m}_K = \{a \in K; |a| < 1\}$,

(c) **the group of units** $\mathcal{O}_K^\times = \mathcal{O}_K \setminus \mathfrak{m}_K$, and

(d) **the residue field** $k = \mathcal{O}_K/\mathfrak{m}_K$.

Lemma 2.1.67.

(a) The ring $\mathcal{O}_K$ is a local ring.

(b) The ideal $\mathfrak{m}_K$ is closed.

(c) The ring $\mathcal{O}_K$ is a principal ideal domain.

Proof.

Part (a): Let $\mathfrak{n} \subseteq \mathcal{O}_K$ be a maximal ideal. Then $\mathcal{O}_K/\mathfrak{n}$ is a field. Let $x \in \mathfrak{m}_K$ and suppose that $x \notin \mathfrak{n}$.

If $x \in \mathcal{O}_K \setminus \mathfrak{n}$, then there exists a $y \in \mathcal{O}_K$ such that $xy - 1 \in \mathfrak{n}$. Since $|x| < 1$ and $|y| \leq 1$, we have that $|xy| \neq 1$. Since $|xy - 1| = \max\{|xy|, 1\}$ when $|xy| \neq 1$, we have that $|xy - 1| = 1$ and so $xy - 1$ is a unit. This is a contradiction, so $\mathfrak{m}_K = \mathfrak{n}$.

Part (b): Let $\{r_n\}$ be a sequence in $\mathfrak{m}_K$ that converges to some r in K. Since $|r_n| = |r| < 1$ for large enough n, we have that $r \in \mathfrak{m}_K$. Thus, $\mathfrak{m}_K$ is closed.

Part (c): The ideal $\mathfrak{m}_K$ is closed and uniformly bounded, which implies that $\mathfrak{m}_K$ is compact. Since the norm is continuous, we have that there exists some $\pi \in \mathfrak{m}_K$ such that $|\pi| \geq |a|$ for all $a \in \mathfrak{m}_K$.

Let $x \in K$. Then there exists some $n \in \mathbb{Z}$ such that

$$|\pi|^{n+1} < |x| \leq |\pi|^n,$$

so $1 < |x\pi^{-(n+1)}|$ and $|x\pi^{-n}| \leq 1$, so $x\pi^{-n} \in \mathcal{O}_K$. If $|x\pi^{-n}| \neq 1$, we have that $|x\pi^{-n}| \leq |\pi|$, so $|x\pi^{-(n+1)}| \leq 1$. This is a contradiction, so $x\pi^{-n} \in \mathcal{O}_K^\times$.

Thus, every element $x \in K$ has a unique presentation $x = \pi^n u$ for some $n \in \mathbb{Z}$ and $u \in \mathcal{O}_K^\times$. Therefore, $\mathfrak{m}_K = (\pi)$. □

A generator of $\mathfrak{m}_K$ is called a **uniformizer** of the field K. Given a uniformizer π, every element $x \in K$ has a unique decomposition $x = \pi^n u$ for $n \in \mathbb{Z}$ and $u \in \mathcal{O}_K^\times$.

Definition 2.1.68. A **discrete valuation** on K is a surjective function $v\colon K \longrightarrow \mathbb{Z} \cup \{\infty\}$ satisfying

1. $v(0) = \infty$,
2. $v(xy) = v(x) + v(y)$ for all $x, y \in K$, and
3. $v(x + y) \geq \min\{v(x), v(y)\}$ for all $x, y \in K$.

A ring (resp. field) with a discrete valuation is called a **discrete valuation ring (resp. field)**.

Fact 2.1.69. Any local principal ideal domain that is not a field is a discrete valuation ring.

Proof. See [2, Prop. 9.2, pg. 94]. □

In particular, $\mathcal{O}_K$ is a discrete valuation ring. In the proof of Lemma 2.1.67(c), we showed that every element x of $\mathcal{O}_K$ has the form $x = \pi^n u$ for some $n \geq 0$ and $u \in \mathcal{O}_K^\times$. An example of a discrete valuation on $\mathcal{O}_K$ is given by the function $v(x) = v(\pi^n u) = n$.

Note that we can extend any discrete valuation v on $\mathcal{O}_K$ to K by setting

$$v\left(\frac{a}{b}\right) = v(a) - v(b).$$

Given a discrete valuation v on K, we can define an absolute value on K, called the **v-adic norm**, given by the expression

$$|x|_v = d^{v(x)}$$

for any real number $0 < d < 1$.

Thus, K is a complete discrete valuation field with valuation $v\colon K \longrightarrow \mathbb{Z} \cup \{\infty\}$. Since the v-adic norm and the absolute value induce the same topology on K, we may assume that we have renormalized the absolute value so that $|\cdot| = d^{v(\cdot)}$.

Theorem 2.1.70. *Every local field of characteristic zero with a non-archimedean absolute value is equal to a finite extension of the p-adic numbers $\mathbb{Q}_p$ for some p. Every local field of characteristic $p > 0$ is isomorphic to a local field of the form $\mathbb{F}_q((T))$ for some indeterminant T and $q = p^r$.*

Proof. For the characteristic $p > 0$ part, see [25, II.4, Thm. 2, pg. 33]. For the characteristic zero case, see [25, II.5, Thm. 3, pg. 36]. □

Global fields

Definition 2.1.71. We say that K is a **global field** if K is a finite field extension of either the field of rational numbers $\mathbb{Q}$ or a field of rational functions in one variable over a finite field.

If K is a finite extension of $\mathbb{Q}$, let $R = \mathbb{Z}$. If K is a finite extension of $\mathbb{F}_q(t)$, let $R = \mathbb{F}_q[t] \subset K$. In both cases, let $\mathcal{O}_K$ be the integral closure of R in K. It is clear that $\mathrm{Frac}(\mathcal{O}_K) = K$.

Note that for the characteristic $p > 0$ case, the choice of $R = \mathbb{F}_q[t]$ is not unique. However, many of the number-theoretic constructions are invariant with respect to the choice of R.

Recall that a domain R is a *Dedekind domain* if R is Noetherian, integrally closed in its field of fractions, and every prime ideal of R is maximal.

Proposition 2.1.72.

(a) *The ring $\mathcal{O}_K$ is a Dedekind domain.*

(b) *For all $x \in \mathcal{O}_K \setminus \{0\}$, the principal ideal (x) decomposes as a unique product of prime ideals $\mathfrak{p} \subseteq \mathcal{O}_K$,*

$$(x) = \prod_{\mathfrak{p} \text{ prime}} \mathfrak{p}^{v_{\mathfrak{p}}(x)}.$$

(c) *For each prime ideal $\mathfrak{p}$, the function $x \mapsto v_{\mathfrak{p}}(x)$ is a discrete valuation on $\mathcal{O}_K$, assuming that we set $v_{\mathfrak{p}}(0) = \infty$.*

Note that part (b) is true for any ideal of $\mathcal{O}_K$, not just principal ideals. We only need the statement for principal ideals in order to define the valuation associated to a prime ideal $\mathfrak{p}$.

Proof. Part (a) follows from the Krull-Akizuki theorem (see [15, Cor. to Thm. 11.7, pg. 85]).

For part (b), see [10, Prop. 1.12, pg. 40].

Part (c) can be verified by examining the properties of prime ideals. For example, suppose $x \in \mathfrak{p}^n$ and $y \in \mathfrak{p}^m$, and m and n are the minimal such integers.

Then $xy \in \mathfrak{p}^{m+n}$ and $m+n$ is the smallest such integer such that this is true, so $v_\mathfrak{p}(xy) = v_\mathfrak{p}(x) + v_\mathfrak{p}(y)$. For $r = \min\{n, m\}$, $x + y \in \mathfrak{p}^r$ so $v_\mathfrak{p}(x+y) \leq \min\{m, n\}$. □

Extend $v_\mathfrak{p}$ to all of K by setting $v_\mathfrak{p}(x/y) = v_\mathfrak{p}(x) - v_\mathfrak{p}(y)$ for $x, y \in \mathcal{O}_K$.

Each valuation induces a metric on K, making K a topological field. We say that two discrete valuations are **equivalent** if they induce the same topology on K. An equivalence class of a discrete valuation is called a **place**. If $v_\mathfrak{p}$ is a representative of a place for some prime ideal $\mathfrak{p} \subseteq \mathcal{O}_K$, we say that $\mathfrak{p} \subseteq K$ is a **prime of** K.

Define V_f^K to be the set of places on K. We let V^K denote the set of all pairwise inequivalent absolute values on K, and we identify V_f^K with the subset of V^K consisting of those absolute values coming from places $v \in V_f^K$.

Given a place v on K, K becomes a metric space with respect to the metric coming from the v-adic norm. We can complete K with respect to this absolute value and we denote this completion by K_v. This is a local field, and let $\mathcal{O}_v$ be the ring of integers in K_v. If $\mathfrak{p} \subseteq K$ is a prime, we write $\mathcal{O}_\mathfrak{p}$ and $K_\mathfrak{p}$ to be the completions with respect to the $v_\mathfrak{p}$-adic norm of $\mathcal{O}_K$ and K respectively.

Let L/K be a finite Galois extension and let $\mathfrak{p}$ be a prime in K. We say that a prime $\mathfrak{b} \subset L$ **lies above** $\mathfrak{p}$ if $\mathfrak{b} \cap K = \mathfrak{p}$, and denote this by $\mathfrak{b}|\mathfrak{p}$. Every prime $\mathfrak{p}$ has a unique decomposition in $\mathcal{O}_L$ as

$$\mathfrak{p}\mathcal{O}_L = \mathfrak{b}_1^{e_1}...\mathfrak{b}_r^{e_r},$$

for primes $\mathfrak{b}_i|\mathfrak{p}$ and integers $e_i \geq 1$. The integer e_i is called the **ramification index of $\mathfrak{b}_i$ over $\mathfrak{p}$** and is also denoted $e(\mathfrak{b}_i|\mathfrak{p})$. If we define $f_i = [L_{\mathfrak{b}_i} : K_{\mathfrak{p}}]$ for $i = 1...r$, then

$$[L : K] = \sum_{i=1}^{r} e_i f_i.$$

The integer f_i is called the **relative degree of $\mathfrak{b}_i$ over $\mathfrak{p}$** and is also denoted as $f(\mathfrak{b}_i|\mathfrak{p})$. We say that $\mathfrak{b}_i$ is **unramified** over $\mathfrak{p}$ if $e_i = 1$.

Let $\mathscr{G} := \mathrm{Gal}(L/K)$. We now define two important subgroups of $\mathscr{G}$:

$$D(\mathfrak{b}|\mathfrak{p}) = \{\sigma \in \mathscr{G} \mid \sigma\mathfrak{b} = \mathfrak{b}\},$$

$$I(\mathfrak{b}|\mathfrak{p}) = \{\sigma \in \mathscr{G} \mid \sigma w \equiv w \pmod{\mathfrak{b}}, \text{ for all } w \in \mathcal{O}_{\mathfrak{b}}\}.$$

The first group $D(\mathfrak{b}|\mathfrak{p})$ is called the **decomposition group of $\mathfrak{b}$ over $\mathfrak{p}$** and the second group $I(\mathfrak{b}|\mathfrak{p})$ is called the **inertia group of $\mathfrak{b}$ over $\mathfrak{p}$**.

Proposition 2.1.73. *The orders* $|D(\mathfrak{b}|\mathfrak{p})| = e(\mathfrak{b}|\mathfrak{p})f(\mathfrak{b}|\mathfrak{p})$ *and* $|I(\mathfrak{b}|\mathfrak{p})| = e(\mathfrak{b}|\mathfrak{p})$.

Proof. See [23, Lemma 9.4, Thm. 9.6, pg. 118]. □

Let $l_{\mathfrak{b}}$ and $k_{\mathfrak{p}}$ be the residue fields of $L_{\mathfrak{b}}$ and $K_{\mathfrak{p}}$ respectively. Suppose that $\mathfrak{b}|\mathfrak{p}$ is unramified. Then there exist isomorphisms

$$D(\mathfrak{b}|\mathfrak{p}) \cong \mathrm{Gal}(L_{\mathfrak{b}}/K_{\mathfrak{p}}) \cong \mathrm{Gal}(l_{\mathfrak{b}}/k_{\mathfrak{p}}).$$

Note that $\mathrm{Gal}(l_{\mathfrak{b}}/k_{\mathfrak{p}})$ is a cyclic group of order $f(\mathfrak{b}|\mathfrak{p})$ and the automorphism $\phi\colon l_{\mathfrak{b}} \to l_{\mathfrak{b}}$ defined by

$$\phi(x) = x^{|k_{\mathfrak{p}}|}, \text{ for all } x \in l_{\mathfrak{b}},$$

generates $\mathrm{Gal}(l_{\mathfrak{b}}/k_{\mathfrak{p}})$. We can find an element $\sigma \in D(\mathfrak{b}|\mathfrak{p})$ such that σ is mapped to ϕ under the above isomorphism. The automorphism ϕ is called the **Frobenius automorphism** and is denoted by $[L/K, \mathfrak{b}]$.

Let S_K be the set of primes of K, and let $M \subseteq S_K$ be a subset. The **Dirichlet density of** M, denoted $\delta(M)$, is given by the following limit, provided that the limit exists. If it doe not exist, we say that M does not have Dirichlet density.

$$\delta(M) = \lim_{s \to 1^+} \frac{\sum_{\mathfrak{p} \in M} |k_{\mathfrak{p}}|^{-s}}{\sum_{\mathfrak{p} \in S_K} |k_{\mathfrak{p}}|^{-s}}$$

It is clear from the definition that when $\delta(M)$ exists $0 \leq \delta(M) \leq \delta(S_K) = 1$.

Theorem 2.1.74 (Tchebotarev's density theorem)**.** *Let L/K be a Galois extension of global fields with Galois group $\mathscr{G}$. For all conjugacy classes $C \subseteq \mathscr{G}$, let*

$$S_C = \{\mathfrak{p} \subseteq K \textit{ prime} \mid [L/K, \mathfrak{b}] \in C, \textit{ for all } \mathfrak{b} \subseteq L \textit{ such that } \mathfrak{b}|\mathfrak{p} \textit{ is unramified}\},$$

where $[L/K, \mathfrak{b}]$ is the Frobenius automorphism. The set S_C has Dirichlet density in the primes of K equal to $|C|/|\mathscr{G}|$.

Proof. See [23, Theorem 9.13A]. □

Remark 2.1.75. A consequence of Theorem 2.1.74 is that given an irreducible polynomial $f(x)$ over a global field K, we can find infinitely many primes $\mathfrak{p} \subseteq K$ such that $f(x)$ splits completely over the completion $K_\mathfrak{p}$ (since we can think of $[L/K, \mathfrak{b}]$ as the generator of the decomposition group $D(\mathfrak{b}|\mathfrak{p}) \cong \mathrm{Gal}(L_\mathfrak{b}/K_\mathfrak{p})$ whenever $\mathfrak{b}$ is unramified over $\mathfrak{p}$).

In fact, if $f(x)$ splits as $f(x) = f_1(x)...f_k(x)$ where $\deg(f_i) = n_i$ over $K_\mathfrak{p}$ for a given prime $\mathfrak{p} \subseteq K$, there are infinitely many other primes $\mathfrak{q} \subseteq K$ such that $f(x)$ splits as $f(x) = g_1(x)...g_k(x)$ where $\deg(g_i) = n_i$ over $K_\mathfrak{q}$.

The implicit function theorem

In this section, we review the classical theory of K-analytic manifolds and the corresponding version of the implicit function theorem. We begin by introducing the definition of a K-analytic manifold.

Throughout this section, let K be a non-archimedean local field complete with respect to the absolute value $|\cdot|$. For all $n > 0$ and $x \in K^n$, let $B_r(x)$ denote the open ball about $x \in K^n$ of radius $r > 0$.

Definition 2.1.76.

1. Let $V \subset K^n$ be open in the induced topology on K^n and let $\phi\colon U \to K$ be a function. The function ϕ is said to be **analytic** in V if for each $x \in V$ there is a formal power series f_x and a radius $r := r_x > 0$ such that
   (a) the open ball $B_r(x) \subset V$,
   (b) the power series f_x converges in $B_r(0)$, and
   (c) for all $h \in B_r(0)$, $\phi(x+h) = f_x(h)$.
2. Let $V \subset K^n$ be open in the induced topology on K^n and let $\phi = (\phi_1, ..., \phi_m)\colon U \to K^m$ be a function. Then ϕ is said to be analytic if ϕ_i is analytic for $i = 1, ..., m$.

Definition 2.1.77. Let X be a topological space.

1. A **chart** on X is a triple $C = (U_C, \phi_C, n_C)$ such that
   (a) the set $U_C \subset X$ is open,
   (b) the integer $n_C \in \mathbb{Z}$ is greater than 0, and
   (c) the continuous map $\phi_C\colon U_C \to K^{n_C}$ has open image $\phi_C(U_C) \subset K^{n_C}$ and ϕ_C induces a homeomorphism between U_C and $\phi_C(U_C) \subset K^{N_C}$.

2. Two charts, C and C', on X are said to be **compatible** if, setting $V = U_C \cap U_{C'}$, the maps $\phi_{C'} \circ \phi_C^{-1}|_{\phi_C(V)}$ and $\phi_C \circ \phi_{C'}^{-1}|_{\phi_{C'}(V)}$ are analytic.

3. A family $\{C_i\}_{i\in I}$ of charts on X is said to **cover** X is $\bigcup_{i\in I} U_{C_i} = X$.

4. An **atlas** A on X is a family of charts on X which cover X such that all the charts in the family are mutually compatible.

5. Two atlases A and A' of X are **compatible** if $A \cup A'$ is an atlas of X.

Note that compatibility of atlases is an equivalence relation. See [24, LG 3.2] for a proof of this fact.

Definition 2.1.78.

1. Let X be a topological space. A K**-analytic manifold** structure on X is an equivalence class of compatible atlases. If X has a K-analytic manifold structure, then we call X a K-analytic manifold (or just analytic manifold if K is clear from context).

2. Let X and Y be K-analytic manifolds. A function $f\colon X \to Y$ is said to be an **analytic function** or a **morphism** if f is continuous and f is locally given by analytic functions. That is, there exist atlases A of X and B of Y such that if

$C \in A$ and $D \in B$ are charts, then, setting $W = U_C \cap f^{-1}(U_D)$, the composite

$$\phi_C(W) \xrightarrow{\phi_C^{-1}} W \xrightarrow{f} U_D \xrightarrow{\phi_D} \phi_D(U_D)$$

is analytic.

Example 2.1.79.

- Let $X = K^n$. Let A be the collection of charts (X, ϕ, n) where $\phi\colon X \to K^n$ is a linear isomorphism. Each pair of charts is compatible, so A is an atlas and X is a K-analytic manifold.

- The group $\mathrm{GL}_n(K)$ has a K-analytic structure such that multiplication and inversion are morphisms. See [24, LG 4.4] for a proof.

- Let $G \subseteq \mathrm{GL}_n$ be a K-defined algebraic subgroup. Then the group of rational points $G(K)$ has a K-analytic structure such that multiplication and inversion are morphisms. See [24, LG 4.5] for a proof.

Let X be an K-analytic manifold and let $x \in X$. Let F_x be the set of all pairs (U, ϕ), where U is an open neighborhood of x and $\phi\colon U \to K$ is an analytic function. We say two elements (U, ϕ), (V, ψ) of F_x are equivalent if there is an open neighborhood W of x such that $W \subset U \cap V$ and $\phi|_W = \psi|_W$. The set of equivalence classes of F_x is denoted $\mathcal{O}(x)$ and is called the **germs of analytic functions at** x. One can show that $\mathcal{O}(x)$ is a local ring [24, LG 3.8-9]. The canonical map $F_x \longrightarrow K$ that sends $(U,\phi) \in F_x$ to $\phi(x) \in K$ induces a canonical homomorphism $\theta\colon \mathcal{O}(x) \longrightarrow K$. The kernel of θ is a maximal ideal $\mathfrak{m}_x$.

Definition 2.1.80.

1. Let X be a K-analytic manifold and let $x \in X$. The **tangent space of** X **at** x is defined to be

$$\mathcal{T}_x(X) = \mathrm{Hom}_K(\mathfrak{m}_x/\mathfrak{m}_x^2, K).$$

2. Let $f \in \mathcal{O}(x)$. Since $f - f(x) \in \mathfrak{m}_x$, we can consider the image of this function modulo $\mathfrak{m}_x^2$. Define $df_x := f - f(x) \mod \mathfrak{m}_x^2 \in \mathfrak{m}_x/\mathfrak{m}_x^2$ and call this the **differential of** f **at** x.

3. Let Y be a second K-analytic manifold, let $y \in Y$, and let $\phi\colon X \longrightarrow Y$ be a morphism such that $\phi(x) = y$. Define the **derivative of** ϕ **at** x to be the map $\mathcal{T}_x(\phi)\colon \mathcal{T}_x(X) \longrightarrow \mathcal{T}_y(Y)$ by the formula

$$\mathcal{T}_x(\phi)(v)(df_y) = v(d(f \circ \phi)_x)$$

for all $v \in \mathcal{T}_x(X)$ and $f \in \mathcal{O}(x)$.

Theorem 2.1.81. *Let G be an algebraic group over K and let $e \in G(K)$ be the identity element. Then there is a canonical bijection between* $\mathrm{Lie}(G)$ *and* $\mathcal{T}_e(G(K))$.

Proof. See [24, LG 3.12] and [31, §12.2, pg. 93]. □

Definition 2.1.82. Let X be a K-analytic manifold, let $x \in X$, and let $f_1, ..., f_m$ be analytic functions on a neighborhood U of x. Let $F(y) = (f_1(y), ..., f_m(y))$ for $y \in U$. We say that $\{f_i\}_{i=1,...,m}$ defines a **coordinate system at** x if there exists an open neighborhood U' of x, contained in U, such that $(U', F|_{U'}, m)$ is a chart on X.

In this context, we get a version of the implicit function theorem that is valid over a non-archimedean local field of any characteristic.

Theorem 2.1.83 (The implicit function theorem)**.** *Let X be a K-analytic manifold, let $x \in X$, and let $f_1, ..., f_m$ be analytic functions on a neighborhood U of x. Let $F(y) = (f_1(y), ..., f_m(y))$ for $y \in U$. Then $\{f_i\}_{i=1,...,m}$ defines a coordinate system at x if and only if $\{d(f_i)_x\}_{i=1,...,m}$ forms a K-basis of $\mathfrak{m}_x/\mathfrak{m}_x^2$.*

Proof. See [24, LG 3.13]. □

Corollary 2.1.84. *Let X and Y be K-analytic manifolds, let $x \in X$, $y \in Y$, and let $\phi\colon X \to Y$ be a morphism such that $\phi(x) = y$. Then the following are equivalent:*

1. *The linear map $\mathcal{T}_x(\phi)\colon \mathcal{T}_x(X) \to \mathcal{T}_y(Y)$ is surjective.*

2. *There exist open neighborhoods $U \subset X$ of x and $V \subset Y$ of y and a morphism $\sigma\colon V \to U$ such that $\phi(U) \subset V$ and $\phi \circ \sigma = \mathrm{Id}_V$.*

Proof. See the discussion in [24, LG 3.16]. □

Note that the existence of $\sigma\colon V \to U$ in Corollary 2.1.84(2) implies that in fact $\phi(U) = V$. Therefore, we have the following immediate corollary.

Corollary 2.1.85. *Let X and Y be K-analytic manifolds and let $\phi\colon X \to Y$ be a morphism. Suppose that the linear map $\mathcal{T}_x(\phi)\colon \mathcal{T}_x(X) \to \mathcal{T}_{\phi(x)}(Y)$ is surjective for all $x \in X$. Then ϕ is an open map.*

Now, let G be a connected algebraic group over K, let $g \in G(K)$ and let λ_g : $G(K) \to G(K)$ be the K-morphism defined by $h \mapsto gh$. Note that the composition $\lambda_{g^{-1}} \circ \lambda_g = \mathrm{Id}_{G(K)}$, so λ_g is a K-isomorphism of K-analytic manifolds. Therefore, the K-linear map $\mathcal{T}_e(\lambda_g)$ is a K-isomorphism between $\mathcal{T}_e(G(K))$ and $\mathcal{T}_g(G(K))$.

Corollary 2.1.86. *Let G and H be connected algebraic groups defined over K and let $\phi\colon G \to H$ be a homomorphism (of algebraic groups). Let $x \in G(K)$ and let $\phi_K(x) = y \in H(K)$. Suppose that the linear map $\mathcal{T}_x(\phi_K)\colon \mathcal{T}_x(G(K)) \to \mathcal{T}_y(H(K))$ is surjective. Then ϕ_K is an open map.*

Proof. It suffices to prove that for $x' \in G(K)$ and $y' = \phi_K(x')$, the linear map $\mathcal{T}_{x'}(\phi_K)\colon \mathcal{T}_{x'}(G(K)) \to \mathcal{T}_{y'}(H(K))$ is surjective. This follows from the fact that the composition

$$\mathcal{T}_{x'}(G(K)) \xrightarrow{\mathcal{T}_{x'}\left(\lambda_{x(x')^{-1}}\right)} \mathcal{T}_x(G(K)) \xrightarrow{\mathcal{T}_x(\phi_K)} \mathcal{T}_y(H(K)) \xrightarrow{\mathcal{T}_y\left(\lambda_{y'(y)^{-1}}\right)} \mathcal{T}_{y'}(H(K))$$

is surjective and equal to $\mathcal{T}_{x'}(\phi_K)$. □

As above, let K be a non-archimedean local field complete with respect to the absolute value $|\cdot|$.

Lemma 2.1.87. *Let G be a connected absolutely almost simple group over K and let X be an irreducible K-variety. Let $G \times X \to X$ be a K-regular action of G on X. Then for $x \in X(K)$, the $G(K)$ orbit of x, $G(K)x \subset X(K)$ is open in $X(K)$.*

Proof. Let $x \in X(K)$ and let Y be the Zariski closure of the orbit $G(K)x$ in X. Then

$$\phi : G \longrightarrow Y$$

defined on K points by $g \mapsto gx$ is a dominant K-morphism by construction. This implies that the induced morphism on the tangent spaces, $\mathcal{T}_g(\phi)$, is surjective. By Corollary 2.1.85, the map ϕ_K is open at the point $g \in G(K)$. Since $\phi_K(hg) = h\phi_K(g)$ for all $h \in G(K)$, then ϕ_K is open at all points $h \in G(K)$. In particular, $G(K)x$ is open in $X(K)$. □

Weak approximation and the variety of maximal tori

Now, let K be a field (not necessarily local).

Definition 2.1.88. Let X and Y be irreducible varieties over K and let $\phi\colon X \to Y$ be a rational morphism.

1. The morphism ϕ is called a **birational morphism** if there exists and inverse rational map $\phi^{-1}\colon Y \to X$. If both ϕ and ϕ^{-1} are defined over K, then ϕ is called an **K-birational morphism**.

2. Varieties that are (K-)birationally isomorphic to an affine space are called **(K-)rational**.

Definition 2.1.89. Let K be a field, let X be an algebraic variety defined over K, and let S be a finite set of inequivalent valuations on K. We say that X **satisfies the weak approximation property** with respect to S if the diagonal embedding $X(K) \longrightarrow \prod_{v\in S} X(K_v)$ is dense, where the topology on the product is the product topology.

Proposition 2.1.90. *Let K be a field and let S be a finite set of inequivalent valuations on K.*

1. *If X and Y are biregularly isomorphic varieties over K, then both have weak approximation with respect to S, or neither have.*

2. *Let X be an irreducible, smooth K-rational variety. Then X satisfies the weak approximation property with respect to S.*

Proof.

1. The biregular isomorphism induces a natural homeomorphism of topological

spaces.

2. The K-rationality of X means there exists a biregular K-isomorphism $\phi\colon U \longrightarrow W$ between open subsets $U \subseteq \mathbb{A}^l$ ($l = \dim X$) and $W \subseteq X$. The variety $\mathbb{A}^l$ has weak approximation (see [9, Prop. 3.7, pg. 6]). This implies that W has weak approximation, i.e., $W(K)$ is dense in $\prod_{v\in S} W(K_v)$. Note that $W(K_v)$ is dense in $X(K_v)$ for $v \in S$ by [3, AG 13.7, pg. 29]. It follows that $W(K)$, and thus certainly $X(K)$, are dense in $\prod_{v\in S} X(K_v)$.

□

Let G be an absolutely almost simple group defined over the field K and let $T \subset G$ be a maximal K-torus and let $N := N_G(T)$ be the normalizer of T in G. It follows from Proposition 2.1.23(i) that for $g \in G(K)$, the map $T_g = gTg^{-1} \mapsto gN$ gives a bijection between K-maximal tori of G and the K-points of G/N.

Definition 2.1.91. The variety $\mathscr{T} := G/N$ is called **the variety of maximal tori of** G.

Note that $\mathscr{T}(K)$ corresponds to the maximal K-tori of G and that up to K-isomorphism, $\mathscr{T}$ does not depend on the choice of maximal torus T.

Theorem 2.1.92. *The variety of maximal tori $\mathscr{T}$ is K-rational.*

Proof. See [1, XIV, Thm. 6.1]. □

Corollary 2.1.93. *The variety of maximal tori* $\mathscr{T}$ *has the weak approximation property with respect to a finite set of inequivalent valuations* S*.*

Proof. Follows immediately from Proposition 2.1.90 and Theorem 2.1.92. □

Proposition 2.1.94. *Let* K *be a field and let* S *be a finite set of inequivalent valuations on* K*. Let* G *be a connected absolutely almost simple group over* K *and let* $T_v \subset G$ *be a maximal* K_v*-torus for all* $v \in S$*. Then the* $G(K_v)$*-conjugacy class of* T_v *is open in* $\mathscr{T}(K_v)$ *for all* $v \in S$*, and in particular the exists a maximal* K*-torus* T *that is* $G(K_v)$*-conjugate to each* T_v*.*

Proof. Pick $v \in S$. Note that $\mathscr{T}$ is irreducible and $G(K_v)$ acts on $\mathscr{T}(K_v)$ in the following way. Let $g \in G(K_v)$ and $xN \in \mathscr{T}(K_v)$.

$$(g, xN) \mapsto gxN,$$

If we consider the identification of points of $\mathscr{T}(K_v)$ with maximal K_v-tori of G, $xN \mapsto xT_vx^{-1}$, we see that $gxN \mapsto g(xT_vx^{-1})g^{-1}$. In particular, the orbit of $N \in \mathscr{T}(K_v)$ is open in $\mathscr{T}(K_v)$ by Lemma 2.1.87 and corresponds to the $G(K_v)$-conjugacy class of T_v. Since the $G(K_v)$-orbit of T_v is open for all $v \in S$, there exists a maximal K-torus T conjugate to each T_v over $G(K_v)$ by Corollary 2.1.93. □

2.2 Strong approximation in positive characteristic

In order to prove the main results of our paper, it is necessary to use a form of strong approximation for Zariski-dense subgroups of connected absolutely almost simple groups. In characteristic zero, the strong approximation of Zariski-dense subgroups can be translated to a problem concerning Lie algebras (see [19, Lemma 2]). As stated previously, this is not possible in characteristic $p > 0$. Instead, we use the results due to Pink, proved in [16] and [17]. To show this property holds, Pink uses so-called *(weak) quasi-models*. To state the approximation results that we use, we begin with a definition of a (weak) quasi-model.

Weak quasi-models

To define these (weak) quasi-models, we need to introduce the following notation.

Definition 2.2.1. For $i = 1, ..., l$, let F_i be a field (of arbitrary characteristic). Furthermore, we make the additional restriction that the fields F_i must all be local or they must all be global. Define

$$F = \bigoplus_{i=1}^{l} F_i.$$

For each $i = 1, ...l$, let G_i be a connected absolutely simple adjoint group defined over the field F_i. Define the group scheme G over the ring F as the product

$$G = \prod_{i=1}^{l} G_i.$$

Let

$$G(F) = \prod_{i=1}^{l} G_i(F_i),$$

and call $G(F)$ the group of F-rational points of G. Let $\Gamma_i \subseteq G_i(F_i)$ be a Zariski-dense subgroup of $G_i(F_i)$ and let

$$\Gamma = \prod_{i=1}^{l} \Gamma_i.$$

We call triples (F, G, Γ) with the above structure **standard triples**.

Let (F, G, Γ) and (F, H, Δ) be two standard triples. We say that $\phi\colon G \to H$ is an **F-isogeny** if ϕ is surjective and the restriction of ϕ to each direct factor G_i, denoted ϕ_i, maps onto a simple factor H_i of H and $\phi_i\colon G_i \to H_i$ is an F_i-isogeny for all $i = 1, ..., l$. In particular, ϕ is an isomorphism, central, purely inseparable, or non-standard if each ϕ_i is an isomorphism, central, purely inseparable, or non-standard respectively.

For a standard triple (F, G, Γ), define its Lie algebra (over F) to be the formal product of the the Lie algebras (over F_i) of the simple factors, namely

$$\mathrm{Lie}(G) = \prod_{i=1}^{l} \mathrm{Lie}(G_i).$$

In this way, we can define the derivative at the identity of an F-isogeny ϕ, denoted $d\phi$, to be the map

$$d\phi\colon \mathrm{Lie}(G) \longrightarrow \mathrm{Lie}(H),$$

where $d\phi_i = d\phi|_{\mathrm{Lie}(G_i)}\colon \mathrm{Lie}(G_i) \to \mathrm{Lie}(H_i)$ is a homomorphism of Lie algebras $\mathrm{Lie}(G_i)$ and $\mathrm{Lie}(H_i)$ over F_i found by taking the derivative at the identity of the F_i-isogeny $\phi_i\colon G_i \to F_i$. We say that $d\phi$ is **nowhere vanishing** if each $d\phi_i$ is an isomorphism of F_i-Lie algebras.

Let $E = \bigoplus_{i=1}^{l} E_i$ be a semisimple ring with the same structure as those in the definition of a standard triple. Suppose that F is a semisimple ring that is also an E-algebra of finite type. Then

$$F = \bigoplus_{i=1}^{l} V_i,$$

where V_i is a finite dimensional E_i-vector space. Since F is semisimple, each V_i is the direct product of fields, denoted

$$V_i = \bigoplus_{j=1}^{n_i} F_{i,j},$$

where each $F_{i,j}$ is a finite field extension of E_i.

Suppose that (E, H, Δ) is a standard triple and let F be an E-algebra of finite type. Then define the extension of scalars, denoted $H \times_E F$, to be the product

$$H \times_E F = \prod_{i=1}^{l} \prod_{j=1}^{n_i} H_i \times_{E_i} F_{i,j}.$$

The F-rational points of $H \times_E F$ are given by the rule

$$(H \times_E F)(F) = \prod_{i=1}^{l} \prod_{j=1}^{n_i} H_i(F_{i,j}).$$

This makes sense since each H_i is defined over E_i for each i and for a given i, each $F_{i,j}$ is a finite extension of E_i for all j, so H_i is defined over $F_{i,j}$ as well.

Definition 2.2.2. A **weak quasi-model** of the standard triple (F, G, Γ) is a triple (E, H, ϕ) such that

(a) F is a semisimple E-algebra of finite type,

(b) $\phi\colon H \times_E F \to G$ is an F-isogeny such that

(c) $\Gamma \subseteq \phi(H(E)) \subseteq G(F)$.

When ϕ has nowhere vanishing derivative $d\phi$, we call (E, H, ϕ) a **quasi-model**.

Note that if (E, H, ϕ) is a (weak) quasi-model of the standard triple (F, G, Γ), then $(E, H, \phi^{-1}(\Gamma))$ is a standard triple as well.

Proposition 2.2.3. *For any weak quasi-model* (E, H, ϕ) *of* (F, G, Γ)*, there exists* *a ring endomorphism* $\tau\colon F \to F$*, which on each simple summand* F_i *is either the identity or a power of the Frobenius morphism, and a quasi-model* (E_1, H_1, ϕ_1) *of* (F, G, Γ)*, such that* $E_1 = \tau(E)$*. Clearly,* (E, H, ϕ) *is already a quasi-model if* τ *is an isomorphism.*

Proof. See [16, Prop. 3.3]. □

Definition 2.2.4. We call the standard triple (F, G, Γ) **minimal** if and only if for every (weak) quasi-model (E, H, ϕ), $E = F$ and ϕ is an F-isomorphism. The (weak) quasi-model (E, H, ϕ) is called **minimal** if $(E, H, \phi^{-1}(\Gamma))$ is a minimal standard triple.

Theorem 2.2.5. *For every standard triple* (F, G, Γ)*, there exists a minimal quasi-model* (E, H, ϕ)*. If* (E', H', ϕ') *is another minimal quasi-model of* (F, G, Γ)*, then* $E = E'$ *and there exists an* E*-isomorphism between* H *and* H'*.*

Proof. See [16, Thm. 3.6] □

Theorem 2.2.5 combined with Proposition 2.2.3 implies that minimal weak quasi-models are in fact quasi-models. Therefore, we are able to drop the "weak" modifier in the subsequent results.

Weak quasi-models over fields

In order to make the definitions in the previous section more clear, we write down the definition of (weak) quasi-model for group schemes over fields F. In other words,

$l = 1$ in the above definition.

Let F be a field, G is an absolutely simple adjoint group defined over F, and $\Gamma \subset G(F)$ is a Zariski-dense subgroup. Then (F, G, Γ) is a standard triple according to our definition.

In this case, the definitions of F-isogeny and the Lie algebra of G over F coincide with the definitions given in Subsection 2.1 and Subsection 2.1 respectively.

Let E be a field and (E, H, Δ) is a standard triple. Let F be an E-algebra of finite type. This implies that F is a direct product of field extensions of E, denoted F_i. Explicitly,

$$F = \bigoplus_{i=1}^{n} F_i.$$

The extension of scalars $H \times_E F$ is then just

$$H \times_E F = \prod_{i=1}^{n} H_i \times_E F_i.$$

If F is a field extension of E, then $H \times_E F = H_F$, the group H considered as an F-group.

Let (F, G, Γ) be a standard triple with F a field. A *weak quasi-model* of (F, G, Γ) is then a triple (E, H, ϕ) such that E is a finite subextension of F, $\phi\colon H \times_E F \to G$ is an F-isogeny such that $\Gamma \subset \phi(H(E))$.

Strong approximation results

Let (F, G, Γ) be a standard triple. Let $\widetilde{G}$ be the simply connected cover of G, which we define to be the direct product of the simply connected covers of each of the G_i, and let $\pi\colon \widetilde{G} \longrightarrow G$ be the corresponding central isogeny. Let Γ' be the subgroup of $\widetilde{G}(F)$ generated by elements of the form $\pi^{-1}([\gamma_1, \gamma_2])$ for all $\gamma_1, \gamma_2 \in \Gamma$.

Theorem 2.2.6. *Let (E, G, Γ) be a minimal standard triple. Then when E is a direct product of local fields and Γ' is contained in a compact subgroup of $\widetilde{G}(E)$, the closure of Γ' is open in $\widetilde{G}(E)$.*

Proof. See [16, Thm. 0.2] and [17, Thm. 0.2]. □

Furthermore, the minimal semisimple ring E of a minimal triple (E, G, Γ) can be described explicitly. To do this, we first need to examine the adjoint representation of a connected absolutely almost simple algebraic group over fields of positive characteristic.

For now, we examine the case where $l = 1$ in the definition of the standard triple. In other words, suppose that E is an infinite field of characteristic $p > 0$. Let G be a connected absolutely almost simple adjoint algebraic group over E, and let $\widetilde{G}$ be its simply connected cover with central isogeny $\pi\colon \widetilde{G} \to G$. Let $\mathfrak{g} := \mathrm{Lie}(G)$ and $\widetilde{\mathfrak{g}} := \mathrm{Lie}(\widetilde{G})$.

The kernel of the induced map $d\pi\colon \widetilde{\mathfrak{g}} \to \mathfrak{g}$ is the center of the Lie algebra, denoted $\mathfrak{z}$. Define $\overline{\mathfrak{g}}$ to be the image of $\widetilde{\mathfrak{g}}$ in $\mathfrak{g}$ via $d\pi$, and let $\mathfrak{z}^*$ be the quotient $\mathfrak{g}/\overline{\mathfrak{g}}$.

Let R be an object in Alg_E. Since $\pi\colon \widetilde{G} \to G$ is central, any elements $\tilde{z}, \tilde{z}' \in \ker(\pi)(R)$ are in the R-points of the center of $\widetilde{G}$, so they have the property that $[\tilde{a}\tilde{z}, \tilde{b}\tilde{z}'] = [\tilde{a}, \tilde{b}]$ for $\tilde{a}, \tilde{b} \in \widetilde{G}(R)$. Since $\widetilde{G}/\ker(\pi) \cong G$, this implies that the commutator on $\widetilde{G}$, $[-,-]\colon \widetilde{G} \times \widetilde{G} \to \widetilde{G}$, descends to an E-morphism

$$[-,-]^{\sim}\colon G \times G \to \widetilde{G}.$$

If we take the derivative of this morphism (at the identity) with respect to the second argument, we get a morphism

$$\widetilde{\mathrm{Ad}}_G\colon G \longrightarrow \underline{\mathrm{Hom}}(\mathfrak{g}, \widetilde{\mathfrak{g}}),$$

where $\underline{\mathrm{Hom}}(\mathfrak{g}, \widetilde{\mathfrak{g}})$ is the group scheme over E defined as follows. For any object R in the category Alg_E, we have $\underline{\mathrm{Hom}}(\mathfrak{g}, \widetilde{\mathfrak{g}})(R) = \mathrm{Hom}_{E-lin}(\mathfrak{g} \otimes_E R, \widetilde{\mathfrak{g}} \otimes_E R)$.

Let $\hat{\mathfrak{g}} = \widetilde{\mathfrak{g}} \oplus \mathfrak{z}^*$. Let $\iota\colon \widetilde{\mathfrak{g}} \to \hat{\mathfrak{g}}$ be the inclusion of $\widetilde{\mathfrak{g}}$ in the first summand. Then $\mathfrak{g} \cong \overline{\mathfrak{g}} \oplus \mathfrak{z}^*$ for some arbitrary but fixed embedding of E-vector spaces $\mathfrak{z}^* \hookrightarrow \mathfrak{g}$. Let $\omega\colon \hat{\mathfrak{g}} \to \mathfrak{g}$ be the composite map of $d\pi$ on the first summand and the above isomorphism.

Define the morphism

$$\hat{\rho}\colon G \longrightarrow \mathrm{GL}(\hat{\mathfrak{g}})$$

by the following rule.

Let R be an object in Alg_E, and let $g \in G_i(R)$. Define $\hat{\rho}$ to be the group scheme morphism defined on R-points by

$$\hat{\rho}_R(g) = 1_{\hat{\mathfrak{g}}_R} + \iota_R \circ \widetilde{\mathrm{Ad}}_R(g) \circ \omega_R.$$

It is straightforward to check that this morphism defines a representation which makes $\hat{\mathfrak{g}}$ into a G-module. It is clear that when $\ker(d\pi) = \{0\}$, this is the normal adjoint representation of G.

Recall that V is said to be a **constant representation** of G if the representation $\rho_V \colon G \to \mathrm{GL}(V)$ has vanishing derivative, i.e. $d\rho(x) = 0 \in \mathfrak{gl}(V)$ for all $x \in \mathrm{Lie}(G)$, and V is a **non-constant representation** otherwise.

Proposition 2.2.7 ([16, Prop. 1.11]). *Let G, $\hat{\mathfrak{g}}$, $\widetilde{\mathfrak{g}}$, $\overline{\mathfrak{g}}$, $\mathfrak{g}$, $\mathfrak{z}$, $\mathfrak{z}^*$, and $\hat{\rho}$ be as defined above.*

(a) The representations $\mathfrak{z}$ and $\mathfrak{z}^*$ are constant representations with the same dimension. The dimension is greater than 0 only when p divides the index of the root lattice in the weight lattice. It is greater than 1 if and only if $p = 2$ and G is of type D_n for some even n. Then the dimension is 2.

(b) When G does not possess non-standard isogenies, then $\overline{\mathfrak{g}}$ is an absolutely irreducible non-constant representation of G. Furthermore, it is the unique simple quotient G-module of $\widetilde{\mathfrak{g}}$ and the unique simple G-submodule of $\mathfrak{g}$, which implies

that it is the unique simple non-constant subquotient of $\hat{\mathfrak{g}}$.

(c) *When* G *has non-standard isogenies, then* $\mathfrak{g}$ *has a unique simple non-constant* G*-submodule* $\overline{\mathfrak{g}}_S$ *and* $\widetilde{\mathfrak{g}}$ *has a unique simple non-constant quotient* G*-module* $\overline{\mathfrak{g}}_L$*. These are pairwise inequivalent absolutely irreducible representations of* G*, and they are the two unique non-constant simple subquotients of* $\hat{\mathfrak{g}}$.

Sketch of the Proof. All of the above statements can be verified by explicitly looking at the sub-representations of $\hat{\mathfrak{g}}$. These correspond to ideals of the Lie algebra $\widetilde{\mathfrak{g}}$. By looking at [11, Table 1], we can determine all ideals of the Lie algebra coming from a group of a given type over a field of a given characteristic. The only remaining difficulty occurs in part (c) of the proposition, but this can may be proved by examining the derivative of the non-standard isogeny $\phi\colon G \to G^{\#}$, denoted $d\phi\colon \mathfrak{g} \to \mathfrak{g}^{\#}$. Since $d\phi$ is trivial on a root space if and only if that root is long, we can find $\overline{\mathfrak{g}}_S$ and $\overline{\mathfrak{g}}_L$ in the kernel and image of $d\phi$ respectively using [11, Table 1]. This leads to a very complete description of any Jordan-Hölder series for $\hat{\mathfrak{g}}$, summarized by the diagrams in [16, Prop. 1.11]. □

Denote the G-representation corresponding to the module $\overline{\mathfrak{g}}$ (resp. $\overline{\mathfrak{g}}_S$, $\overline{\mathfrak{g}}_L$) by ρ (resp. ρ_S, ρ_L).

Let (F, G, Γ) be a standard triple such that

$$F = \bigoplus_{i=1}^{l} F_i, \quad G = \prod_{i=1}^{l} G_i, \quad \Gamma = \prod_{i=1}^{l} \Gamma_i.$$

Definition 2.2.8. When G_i does not admit any non-standard isogenies, let ρ_i be the absolutely irreducible non-constant representation of G_i described by Proposition 2.2.7. Define E_{ρ_i} to be the subfield of F_i generated by 1 and $\mathrm{tr}(\rho_i(\Gamma_i))$.

When G_i admits non-standard isogenies, then let $\rho_{i,S}$ and $\rho_{i,L}$ be the unique absolutely irreducible non-constant representations of G_i described by Proposition 2.2.7. Define E_{ρ_i} to be the subfield of F_i generated by 1, $\mathrm{tr}(\rho_{i,S}(\Gamma_i))$, and $\mathrm{tr}(\rho_{i,L}(\Gamma_i))$.

In each case, we call E_{ρ_i} the **trace field** of Γ_i.

Proposition 2.2.9. *Let (F, G, Γ) be a standard triple and let (E, H, ϕ) be a minimal quasi-model. Then E has the form*

$$E = \bigoplus_{i=1}^{l} E_i.$$

For each $i = 1, ...l$, let E_{ρ_i} be the field defined above. Let $\mathrm{char}(E_i) = p_i$. *If $p_i \neq 2$ or 3, then the factor E_i is the closure of E_{ρ_i} in the local case and $E_i = E_{\rho_i}$ in the global case. If $p_i = 2$ or 3, then the closure of E_{ρ_i} is either E_i or $E_i^{p_i}$ in the local case and E_{ρ_i} is either E_i or $E_i^{p_i}$ in the global case.*

Proof. See [16, Prop. 3.3] □

In the case where E is a field, we have the following corollary.

Corollary 2.2.10. *Let (E, G, Γ) be a minimal triple such that $l = 1$ in the definition. Specifically, E is a local or global field, G is a connected absolutely almost simple* *adjoint algebraic group over E, and $\Gamma \subseteq G(E)$ is a Zarisi-dense subgroup.*

(a) Suppose that E is a local field. Then one of the following is true.

(i) The characteristic of E is $p \neq 2, 3$ and the closure of $E_\Gamma := E_\rho$ is E.

(ii) The characteristic of E is $p = 2$ or 3 and the closure of $E_\Gamma := E_\rho$ is E.

(iii) The characteristic of E is $p = 2$ or 3, the closure of E_ρ is E^p, and there exists some purely inseparable element α in E such that the closure of $E_\Gamma := E_\rho(\alpha)$ is E.

(b) Suppose that E is a global field. Then $E_\Gamma := E_\rho$ or $E_\Gamma := E_\rho^{1/p}$, and $E_\Gamma = E$ by Proposition 2.2.9.

In each of these cases, call E_Γ the **minimal field of** Γ*. In other words, it is the smallest field extension of the trace field such that (the closure of) E_Γ is the field E such that (E, G, Γ) is a minimal standard triple.*

Note that different choices of α in Definition 2.2.8(iii) may yield different minimal fields, but this will not affect our results. In the local case, we only need the trace field to be a dense subfield of the minimal local field E described by Theorem 2.2.5.

Chapter 3

Proofs of main results

3.1 Generic elements in Zariski-dense subgroups

Results from field theory

Suppose that E is a field with discrete valuation v. Throughout this section, we denote the completion of E with respect to v by E_v and the ring of integers of E_v by $\mathcal{O}(E_v)$.

Proposition 3.1.1. *Let E' be a finitely generated infinite field of characteristic $p > 0$, let E/E' be a purely transcendental extension with transcendence basis X, and let F/E be a finite separable extension. Let R be a finitely generated infinite integral domain such that E is its field of fractions. Then there exist infinitely many non-archimedean, pairwise inequivalent valuations v on E such that E_v is locally compact, $R \subseteq \mathcal{O}(E_v)$, $F_w = E_v$ for any extension $w|v$ of v, and $v(x) = 0$ for all $x \in X$.*

Proof. The field $\mathbb{F}_p$ is the prime subfield of E'. Since $\mathbb{F}_p$ is prefect, there exists a transcendence basis $\{s_1, ..., s_l\}$ of E' such that E' is separable over $P = \mathbb{F}_p(s_1, ..., s_l)$

(see for example [14, Ch. VIII, §, Cor. 4.4, pg. 365]). Furthermore, E is separable over $P' = \mathbb{F}_p(s_1, ..., s_l, X)$. The extension F/P' is also separable, so let α be a primitive element of F/P'. Let $A = \mathbb{F}_p[s_1, ..., s_l, X]$ and $B = A[\alpha]$. Since R is finitely generated, there exists some $a \in A$ such that $R \subseteq B[1/a]$.

We can also choose a so that the minimal monic polynomial $f(z)$ of α over P has coefficients in $A[1/a]$. Since $f(z)$ is separable, it must be prime to its derivative. Hence, there exist $q(z), r(z) \in A[z]$ so that

$$q(z)f(z) + r(z)f'(z) = b$$

for some non-zero $b \in A$. Set $c = ab\left(\prod_{x \in X} x\right)$.

Let $C = \mathbb{F}_p[t]$. Define $\epsilon \colon A \longrightarrow C$ be a homomorphism that takes ab to $t \in C$ and $x \mapsto 1$ for all $x \in X$. This implies that ϵ is surjective and $\epsilon(c) \neq 0$.

Let $L := \mathbb{F}_p(t)$, the field of fractions of $\epsilon(A)$. We can invoke Tchebotarev's density theorem (see Theorem 2.1.74) to say that for any separable extension $\mathcal{L}/L$, there exist infinitely many discrete valuations u of L such that for any extension w of u to $\mathcal{L}$, the completions $\mathcal{L}_w$ and L_u coincide.

Let $g(z) := f^{\epsilon}(z) \in C[z]$, the polynomial obtained by applying ϵ to the coefficients of $f(z)$. Since $f(z)$ is prime to its derivative, then $g(z)$ is prime to its derivative as well. Hence, $g(z)$ is separable. For a given $r \in L^{\times}$, we know that $u(r) \geq 0$ for all but finitely many non-archimedean valuations u of L. This implies that we can find infinitely many valuations u such that $\epsilon(c_1), ..., \epsilon(c_l)$ lie in the valuation ring of u, $u(\epsilon(c)) = 0$, and $g(z)$ splits completely over the completion L_u. Let u be such a valuation.

Let $\mathcal{O}(L_u)$ be the ring of integers in L_u, $\mathfrak{m}_u$ be its maximal ideal, and let k_u be the residue field. Since $g(z)$ is monic with coefficients in $\mathcal{O}(L_u)$ and $\mathcal{O}(L_u)$ is integrally closed, the roots of $g(z)$ are in $\mathcal{O}(L_u)$. Thus, the residue polynomial $\overline{g}(z)$ is a product of linear factors over k_u.

Since $\mathcal{O}(L_u)$ is uncountable, it is possible to find elements $t_1, ..., t_l \in \mathcal{O}(L_u)$ such that $t_1, ..., t_l$ are algebraically independent over $\mathbb{F}_p(t)$ and satisfy the conditions $t_i \equiv c_i \pmod{\mathfrak{m}_u}$ for all $i = 1, ..., l$. Let $\sigma \colon P \longrightarrow L_u$ be the embedding sending s_i to t_i. Let $h(z) := f^\sigma(z)$, the polynomial formed by applying σ to the coefficients of $f(z)$.

Since $\overline{h}(z) = \overline{g}(z)$, $\overline{h}(z)$ splits into linear factors. Furthermore,

$$\overline{q^\sigma}(z)\overline{h}(x) + \overline{r^\sigma}(z)\overline{h'}(z) = \overline{b} \neq 0,$$

which implies that $\overline{h}(z)$ is prime to $\overline{h'}(z)$. By Hensel's lemma, $h(z)$ splits in $\mathcal{O}(E_u)$ into linear factors.

Since $h(z)$ splits, for any embedding $\overline{\sigma} \colon F \hookrightarrow \overline{L_u}$ extending σ (where $\overline{L_u}$ denotes the algebraic closure of L_u), we have that $\overline{\sigma}(F) \subseteq L_u$ by our choice of u. If v is a valuation of E obtained by pulling back u under one of these embeddings, then $E_v = L_u$ and $\mathcal{O}(E_v) = \mathcal{O}(L_u)$ by construction. Moreover, $F_w = E_v = L_u$ for any extension $w|v$. Since $\overline{\sigma}(\alpha)$ is a root of $h(x)$, we know that $\overline{\sigma}(\alpha) \in \mathcal{O}(E_v)$. Note that all factors of c in A are units in $\mathcal{O}(E_v)$. Since $v(a) = 0$, we conclude that $R \subseteq \mathcal{O}(E_v)$. Furthermore, $v(x) = 0$ for all $x \in X$. □

Remark 3.1.2. Proposition 3.1.1 implies that there exist infinitely many nontrivial non-archimedean places v' on E' with an embedding

$$\epsilon_{v'} \colon F \hookrightarrow E'_{v'},$$

such that the field F is dense in the locally compact completion E'_v. Furthermore, all such valuations v on E lying above such a v', i.e. $v' = v|_{E'}$, have the property that $E'_{v'} = E_v$.

Proof of Theorem E

For the rest of the section, let G be a connected absolutely almost simple algebraic group over a finitely generated infinite field K of characteristic $p > 0$. Let $\Gamma \subseteq G(K)$ be a finitely generated Zariski-dense subgroup, and let $E' \subseteq K$ be the minimal field of Γ as defined by Definition 2.2.8. Furthermore, let $R \subseteq K$ be a finitely generated ring such that $\Gamma \subseteq G(R)$.

Let X be a transcendence basis of K/E' and define $E := E'(X)$. Let $F \subseteq K$ be the separable closure of E in K.

Lemma 3.1.3. *There exist infinitely many valuations discrete v on F such that* *(a)* F_v *is locally compact,*

(b) for $w = v|_{E'}$, $E'_w = F_v$,

(c) $R \subseteq \mathcal{O}(E'_w)$, *and*

(d) G *splits over* $E'_w = F_v$.

Proof. Assume the same set-up as the proof of Proposition 3.1.1. This proposition already guarantees the existence of an infinite set with properties (a)-(c). We claim that (d) follows automatically from the proof of Proposition 3.1.1. Note that G becomes split over some finite separable extension and we have chosen valuations u on L so that for a given finite separable extension $\mathcal{L}/L$ and for any valuation $w|u$ on $\mathcal{L}$, we have guaranteed that $\mathcal{L}_w = L_u$. We are able choose $\mathcal{L}$ to be large enough so that G splits over $\mathcal{L}$. Thus, G splits over $L_u = F_v$. □

Remark 3.1.4. Since K/F is a purely inseparable extension, any valuation on F extends uniquely to a valuation on K. Therefore, we will abuse notation slightly and use the same symbols to denote valuations on F and K.

Let T be any maximal torus of G and let $\mathcal{W} := \mathcal{W}(G, T)$. Let r be the number of nontrivial conjugacy classes of $\mathcal{W}$ (which does not depend on T by Proposition 2.1.32). Invoke Lemma 3.1.3 to find a set S of r nontrivial pairwise inequivalent non-archimedean valuations of K such that for every $v \in S$, the completion K_v is locally compact and G splits over K_v. To prove Theorem E, we first construct the appropriate T_v for each $v \in S$.

First notice that for each $v \in S$, the simply connected cover $\widetilde{G}$ splits over K_v. Therefore, we can consider a K_v-split maximal torus $\widetilde{T}_{0,v}$ in $\widetilde{G}$. Let $\widetilde{\mathscr{T}}$ be the K-variety of maximal tori of $\widetilde{G}$. By Proposition 2.1.94, the $\widetilde{G}(K_v)$-conjugacy class of $\widetilde{T}_{0,v}$ is open in $\widetilde{\mathcal{T}}(K_v)$ for all $v \in S$, the weak approximation property implies that there exists a maximal K-torus $\widetilde{T}_0$ of $\widetilde{G}$ that is $\widetilde{G}(K_v)$-conjugate to $\widetilde{T}_{0,v}$ for all $v \in S$. Set $T_0 = \pi(\widetilde{T}_0)$ (where $\pi\colon \widetilde{G} \to G$ is the canonical central isogeny).

Fix any bijection between the r nontrivial conjugacy classes of $\mathcal{W}$ and the set S. Let c_v be the corresponding conjugacy class of $\mathcal{W}$. For $w \in \mathcal{W}$, let $[w]$ denote the conjugacy class of w in $\mathcal{W}$, and for $U \subseteq \mathcal{W}$, let $[U]$ be the set of conjugacy classes of $\mathcal{W}$ that intersect U. Let T, T' be two K_v-split maximal tori in G, and let $g \in G(K_v)$ such that $T'(K_v) = gT(K_v)g^{-1}$.

Let $\phi \in \mathrm{Aut}\Phi(G,T)$, and let $\beta \in \Phi(G,T')$. Then $\beta \circ \mathrm{Int}(g) \in \Phi(G,T)$, so $\phi(\beta \circ \mathrm{Int}(g)) \in \Phi(G,T)$. Define

$$\iota_g(\phi)(t') = \phi(\beta \circ \mathrm{Int}(g))(gt'g^{-1}),$$

for all $\beta \in \Phi(G,T')$ and $t' \in T'(K_v)$. Then it is clear that the map

$$\iota_g\colon \mathrm{Aut}\Phi(G,T) \to \mathrm{Aut}\Phi(G,T'),$$

defined above is an isomorphism.

If we are given another $g' \in G(K_v)$ such that $T'(K_v) = g'T(K_v)(g')^{-1}$, then note that $g'g^{-1} \in N_G(T')(K_v)$. Since $\mathcal{W}(G,T') \cong N_G(T')(\overline{K_v})/T'(\overline{K_v})$, note that $\iota_{g'} = \mathrm{Int}(w') \circ \iota_g$, where w' is the class of $g'g^{-1}$ in $\mathcal{W}(G,T')$.

In other words, $[\iota_g(w)] = [\iota_{g'}(w)]$ in $[\mathcal{W}(G,T')]$ for all $w \in \mathcal{W}(G,T)$. Let $[w] \in [\mathcal{W}(G,T)]$. Fix some $g \in G(K_v)$ such that $T'(K_v) = gT(K_v)g^{-1}$ and define the map

$$\iota_{T,T'} \colon [\mathcal{W}(G,T)] \to [\mathcal{W}(G,T')]$$

by

$$\iota_{T,T'}([w]) = [\iota_g(w)].$$

The work above shows that this map is well-defined and bijective. Furthermore, it is clear that $\iota_{T,T} = \mathrm{id}$, $\iota_{T,T'} = \iota_{T',T}^{-1}$, and $\iota_{T_1,T_3} = \iota_{T_2,T_3} \circ \iota_{T_1,T_2}$.

Lemma 3.1.5. *For each $v \in S$, there exists a maximal K_v-torus T_v of G such that*

$$c_v \in \iota_{T_v,T_0}([\theta_{T_v}(\mathrm{Gal}((K_v)_{T_v}/K_v)) \cap \mathcal{W}(G,T_v)]).$$

Proof. Since T_0 and $\widetilde{T}_0$ are split over K_v, we know that $\mathcal{W}(G,T_0) \cong N_G(T_0)(K_v)/T_0(K_v)$ and $\mathcal{W}(\widetilde{G},\widetilde{T}_0) \cong N_{\widetilde{G}}(\widetilde{T}_0)(K_v)/\widetilde{T}_0(K_v)$.

The idea behind the proof is the following observation. Let $N = N_{\widetilde{G}}(\widetilde{T}_0)$ and consider the variety of maximal tori $\mathscr{T} = \widetilde{G}/N$. Furthermore, $\widetilde{G}$ acts by left multiplication on $\mathscr{T}$ (which corresponds to conjugation by $\widetilde{G}$ on the set of maximal tori) and the elements of the orbit set $\widetilde{G}(K_v) \setminus \mathscr{T}(K_v)$ are in bijective correspondence with the $\widetilde{G}(K_v)$-conjugacy classes of maximal tori.

Let

$$\mathscr{C} := \ker(H^1(K_v, N) \longrightarrow H^1(K_v, \widetilde{G})).$$

It is well-known that there is a natural bijection

$$\delta \colon \mathscr{C} \longrightarrow \widetilde{G}(K_v) \setminus \mathscr{T}(K_v).$$

See for example [20, Lemma 9.1].

By [5, Section 4.7], we know that $H^1(K_v, \widetilde{G})$ is trivial. Therefore, $\mathscr{C} = H^1(K_v, N)$ and conjugacy classes of maximal K_v-tori may be completely described by 1-cocycles with values in $N(K_v^{\mathrm{sep}})$. As a result, we first construct such a cocycle.

Since $\widetilde{G}$ is the same Killing-Cartan type as G, the central isogeny $\pi \colon \widetilde{G} \to G$ induces an isomorphism

$$\tilde{\pi} \colon \mathcal{W}(\widetilde{G}, \widetilde{T}_0) \longrightarrow \mathcal{W}(G, T_0).$$

Let $\tilde{c}_v$ be the conjugacy class in $\mathcal{W}(\widetilde{G}, \widetilde{T}_0)$ corresponding to c_v, and let $x \in \tilde{c}_v$ be a representative of $\tilde{c}_v$. Since $\widetilde{G}$ splits over K_v, it is K_v-isomorphic to the simply connected Chevalley group over K_v. Define

$$X_\alpha(K_v) = \{x_\alpha(t) \mid t \in K_v\}.$$

We call this subgroup the elementary root subgroup corresponding to the root $\alpha \in \Phi(\widetilde{G}, \widetilde{T}_0)$. See Section 3 of [28] for more details as to how such x_α are defined. Define $\widetilde{N}_0 := N_{\widetilde{G}}(\widetilde{T}_0)$ and consider the elements

$$w_\alpha(t) = x_\alpha(t)x_{-\alpha}(-t^{-1})x_\alpha(t) \in \widetilde{N}_0(K_v),$$

for $\alpha \in \Phi(\widetilde{G}, \widetilde{T}_0)$ and $t \in K_v$. Note that [28, Lemma 22, Section 3] implies that $\widetilde{N}_0(K_v)$ contains $w_\alpha(1)$ for all $\alpha \in \Phi(\widetilde{G}, \widetilde{T}_0)$ up to K_v-isomorphism.

In particular, there exists a Chevalley $\mathbb{Z}$-subscheme $\mathcal{N}$ such that when we extend scalars, $\mathcal{N} \times_{\mathbb{Z}} K_v = (\widetilde{N}_0)_{K_v}$, and $w_\alpha(1) \in \mathcal{N}(\mathbb{Z})$, as well as split-torus $\mathbb{Z}$-subscheme $\mathcal{T}$ such that $\mathcal{T} \times_{\mathbb{Z}} K_v = (\widetilde{T}_0)_{K_v}$. The group $\mathcal{N}(\mathbb{Z})/\mathcal{T} \cong \mathcal{W}(\widetilde{G}, \widetilde{T}_0)$ and since $|\mathcal{T}(\mathbb{Z})| < \infty$, elements of $\mathcal{N}(\mathbb{Z})$ have finite order and $\mathcal{N}(\mathbb{Z})$ contains representatives for all elements of $\mathcal{W}(\widetilde{G}, \widetilde{T}_0)$. After extending scalars of $\mathcal{N}$ to K_v, we can conclude that $\widetilde{N}_0(K_v)$ contains a set of all conjugacy class representatives of $\mathcal{W}(\widetilde{G}, \widetilde{T}_0)$, and each representation of finite order. Let $y \in \widetilde{N}_0(K_v)$ be an element congruent to x modulo $\widetilde{T}_0(K_v)$ (of finite order).

Fix a separably closed extension K_v^{sep} as well as a maximal unramified extension

K_v^{ur} in K_v^{sep}. Consider the map

$$\zeta\colon \hat{\mathbb{Z}} \longrightarrow \widetilde{N}_0(K_v)$$

defined by $\zeta(1) = y$. K_v is a locally compact field, so $\text{Gal}(K_v^{ur}/K_v) \cong \hat{\mathbb{Z}}$ (see [6, Ch. 1, Section 7]). Since y has finite order, we can think of ζ as a continuous 1-cocycle on $\text{Gal}(K_v^{ur}/K_v)$ with values in $\widetilde{N}_0(K_v^{ur})$.

We can extend ζ to the absolute Galois group $\mathcal{G}_v = \text{Gal}(K_v^{\text{sep}}/K_v)$ with values in $\widetilde{N}_0(K_v^{\text{sep}})$. We know that $H^1(K_v, \widetilde{G}) = \{1\}$ (see [5, Section 4.7]).

This implies that there exists some $g \in \widetilde{G}(K_v^{\text{sep}})$ such that $\zeta(\gamma) = g^{-1}\gamma(g)$ for all $\gamma \in \mathcal{G}_v$. Let $\widetilde{T}_v$ be the K_v^{sep}-torus $\widetilde{T}_v(K_v^{\text{sep}}) = g\widetilde{T}_0(K_v^{\text{sep}})g^{-1}$ and $T_v = \pi(\widetilde{T}_v)$.

To show $\widetilde{T}_v$ is defined over K_v, we need to show that it is $\mathcal{G}_v$-stable. Let $\gamma \in \mathcal{G}_v$. Then

$$\begin{aligned}
\gamma(\widetilde{T}_v) &= \gamma(g)\widetilde{T}_0\gamma(g)^{-1} \\
&= g(g^{-1}\gamma(g))\widetilde{T}_0(g^{-1}\gamma(g))^{-1}g^{-1} \\
&= g\widetilde{T}_0 g^{-1} \\
&= \widetilde{T}_v,
\end{aligned}$$

since $g^{-1}\gamma(g) \in \widetilde{N}_0(K_v^{\text{sep}})$. Thus, $\widetilde{T}_v$ and T_v are defined over K_v.

We now need to show that $\theta_{T_0}(\text{Gal}((K_v)_{T_v}/K_v))$ intersects $\tilde{c}_v$. Let $\alpha_0 \in \Phi(\widetilde{G}, \widetilde{T}_0)$.

Define $\alpha \in \Phi(\widetilde{G}, \widetilde{T}_v)$ by

$$\alpha(t) = \alpha_0(g^{-1}tg)$$

for all $t \in \widetilde{T}_v(K_v^{\text{sep}})$. Let

$$\iota := \iota_g \colon \operatorname{Aut}\Phi(\widetilde{G}, \widetilde{T}_0) \longrightarrow \operatorname{Aut}\Phi(\widetilde{G}, \widetilde{T}_v)$$

be the map induced by conjugation by g. Since $\widetilde{T}_0$ is K_v-split, α_0 is defined over K_v. Let $t \in \widetilde{T}_0(K_v^{\text{sep}})$ and $\gamma \in \operatorname{Gal}(K_v^{\text{sep}}/K_v)$. Then

$$\begin{aligned}
\iota \circ \theta_{\widetilde{T}_0}(\gamma)(\alpha_0(t)) &= \gamma(\alpha)(gtg^{-1}) \\
&= \gamma(\alpha(\gamma^{-1}(g)\gamma^{-1}(t)\gamma^{-1}(g)^{-1})) \\
&= \gamma(\alpha_0(g^{-1}\gamma^{-1}(g)\gamma^{-1}(t)(\gamma^{-1}(g)^{-1}g))) \\
&= \alpha_0((g^{-1}\gamma(g))^{-1}tg^{-1}\gamma(g)) \\
&= \zeta(\gamma)(\alpha_0)(t),
\end{aligned}$$

since $\zeta(\gamma) = g^{-1}\gamma(g)$.

If we let $\overline{\zeta}$ be the image of ζ in $\mathcal{W}(\widetilde{G}, \widetilde{T}_0)$, then

$$\iota\left(\theta_{\widetilde{T}_0}\left(\operatorname{Gal}\left(\frac{(K_v)_{\widetilde{T}_v}}{K_v}\right)\right)\right) = \operatorname{Im}(\overline{\zeta}) \subseteq \mathcal{W}(\widetilde{G}, \widetilde{T}_0).$$

Since $\overline{\zeta}(\phi) = x \in \tilde{c}_v$, where ϕ is the Frobenius element in $\operatorname{Gal}(K_v^{ur}/K_v)$, we know that $\theta_{\tilde{T}_0}(\mathrm{Gal}((K_v)_{T_v}/K_v))$ intersects $\tilde{c}_v$ nontrivially, hence it intersects c_v nontrivially when we apply $\tilde{\pi}$. □

To complete the proof of Theorem E, we need the following general lemma.

Lemma 3.1.6. *Let $\mathscr{G}$ be a finite group and suppose that $\mathscr{H} \leq \mathscr{G}$ is a subgroup that intersects all the conjugacy classes of $\mathscr{G}$. Then $\mathscr{G} = \mathscr{H}$.*

Proof. Since $\mathscr{H}$ intersects all the conjugacy classes of $\mathscr{G}$, we have the decomposition

$$\mathscr{G} = \bigcup_{g \in \mathscr{G}} g\mathscr{H}g^{-1}.$$

If there exist $g_1, g_2 \in \mathscr{G}$ such that $g_1\mathscr{H} = g_2\mathscr{H}$, then $\mathscr{H}g_1^{-1} = \mathscr{H}g_2^{-1}$. Therefore, $g_1\mathscr{H}g_1^{-1} = g_2\mathscr{H}g_2^{-1}$. If $\{g_1, ..., g_r\}$ is a set of distinct coset representatives for $\mathscr{G}/\mathscr{H}$, then

$$\mathscr{G} = \bigcup_{i=1}^{r} g_i\mathscr{H}g_i^{-1}.$$

The number of elements on the left is $|\mathscr{G}|$, and the number of elements on the right is $[\mathscr{G} : \mathscr{H}]|\mathscr{H}| = |\mathscr{G}|$, which implies that the union of $g_i\mathscr{H}g_i^{-1}$ is disjoint. Since each $g_i\mathscr{H}g_i^{-1}$ contains the identity, this is a contradiction unless $\mathscr{G} = \mathscr{H}$. □

Proof of Theorem E. For each $v \in S$, invoke Lemma 3.1.5 to construct the maximal K_v-torus T_v. We know that there exists a maximal K-torus T that is $G(K_v)$-conjugate to each of the T_v for $v \in S$ by Proposition 2.1.94.

Let T be such a maximal K-torus. Since T is $G(K_v)$-conjugate to T_v, we see that the splitting fields of T and T_v over K_v must be the same. We need to check that the following diagram commutes.

$$\begin{array}{ccc} \mathrm{Gal}((K_v)_T/K_v) & \xrightarrow{\mathrm{id}} & \mathrm{Gal}((K_v)_{T_v}/K_v) \\ \downarrow{\scriptstyle \theta_T} & & \downarrow{\scriptstyle \theta_{T_v}} \\ \mathrm{Aut}\Phi(G,T) & \xrightarrow{\iota_g} & \mathrm{Aut}\Phi(G,T_v) \end{array}$$

Let $\sigma \in \mathrm{Gal}((K_v)_T/K_v)$, $t' \in T_v(K_v)$, and $\beta \in \Phi(G,T_v)$. Since $g \in G(K_v)$, it is σ-invariant, so we see that

$$\begin{aligned} \iota_g(\theta_T(\sigma))(\beta)(t') &= \theta_T(\sigma)(\beta \circ \mathrm{Int}(g))(g^{-1}t'g), \\ &= \sigma(\beta \circ \mathrm{Int}(g)(\sigma^{-1}(gt'g^{-1}))), \\ &= \sigma(\beta(\sigma^{-1}(t'))), \\ &= \theta_{T_v}(\sigma)(\beta)(t'). \end{aligned}$$

Therefore, the above diagram commutes.

The Galois group $\mathrm{Gal}((K_v)_T/K_v)$ naturally embeds into $\mathrm{Gal}(K_T/K)$. Therefore, know that $\theta_T(\mathrm{Gal}(K_T/K))$ has nontrivial intersection with every conjugacy class of $\mathcal{W}(G,T)$. By Lemma 3.1.6, we know that $\mathcal{W}(G,T) \subseteq \theta_T(\mathrm{Gal}(K_T/K))$, i.e. the torus T is generic over K. □

Notice that the proof of Theorem E prescribes a the set of places S and tori T_v for

$v \in S$ and shows that any T that is $G(K_v)$-conjugate to each T_v will be K-generic. In the way, we can construct a K-generic torus T with any desired local behavior by simply enlarging the set of places S.

Corollary 3.1.7 (Analog of [20, Cor. 3.2]). *Let G and K be as above. Let S' be a finite set of nontrivial non-archimedean valuations on K such that K_v is locally compact for all $v \in S'$. Let T_v be a maximal K_v-torus for all $v \in S'$. Then there exists a K-generic torus T that is $G(K_v)$-conjugate to T_v for all $v \in S'$.*

Proof. Let r be the number of nontrivial conjugacy classes of $\mathcal{W}$. Suppose L is a finite extension of K such that G splits over L. By Lemma 3.1.3, we know that there exists an infinite number of places w on L such that G splits over K_w. Let $S = \{v_1, ..., v_r\}$ be a set of r places on K that are pullbacks of the w-adic valuations where L embeds into K_w. Then, G is split over K_{v_i} for all $i = 1, ..., r$. Since there are infinitely many choices for these places, we may choose S so that $S \cap S' = \emptyset$.

Let T_{v_i} be the K_{v_i}-torus constructed by Lemma 3.1.5. Since K_v is locally compact for all $v \in S \cup S'$, we know that the tori in the $G(K_v)$-conjugacy class of T_v are points in an open subset of $\mathcal{T}(K_v)$ by the implicit function theorem. As above, $\mathcal{T}$ has the weak approximation property, so there exists some maximal K-torus T that is $G(K_v)$-conjugate to T_v for all $v \in S \cup S'$. By Theorem E, this implies that T is K-generic. $\square$

Proof of Theorem F

Let $\omega\colon G \to \overline{G}$ be the central isogeny from G to its adjoint group $\overline{G}$. Let $T_0 \subseteq G$ be a maximal torus and let $\overline{T}_0 = \omega(T_0)$. The isogeny induces an isomorphism

$$\tilde{\omega}\colon \mathcal{W}(G, T_0) \to \mathcal{W}(\overline{G}, \overline{T}_0).$$

It is clear that $K_{T_0} \supseteq K_{\overline{T}_0}$. Therefore, T_0 is generic over K if $\overline{T}_0$ is generic over K, so we may assume that $G = \overline{G}$ is adjoint without loss of generality.

As before, we invoke Lemma 3.1.3 to find a set S of non-archimedean pairwise inequivalent valuations on K such that $|S| = r$.

Define $G_S := \prod_{v \in S} G(K_v)$ and let

$$\delta_S\colon G(K) \longrightarrow G_S \tag{3.1.1}$$

be the diagonal embedding.

Lemma 3.1.8. *Let G and S be as above and let $r = |S|$. Consider the topology on $G(K_v)$ induced by the v-adic topology on K_v. There exists a subset U of $G_S = \prod_{v \in S} G(K_v)$ that is open in the v-adic topology with the following properties:*

(a) U intersects every open subgroup of G_S, and

(b) $\delta_S^{-1}(\delta_S(G(K) \cap U))$ consists of K-generic elements.

Proof. Let $v \in S$, and let T_v be a maximal K_v-torus of G satisfying the conclusion of Lemma 3.1.5. Consider the map

$$\phi \colon G(K_v) \times T_v(K_v) \longrightarrow G(K_v)$$

given by $\phi(g, t) = gtg^{-1}$.

Let $(T_v)_{\text{reg}}(K_v)$ be the set of K_v-regular elements in $T_v(K_v)$, which we know is dense by Proposition 2.1.11. We claim that the set $U(T_v, v) = \phi(G(K_v), (T_v)_{\text{reg}}(K_v))$ is a open subset of $G(K_v)$ that intersects every open subgroup of $G(K_v)$.

Let $(g, t) \in G(K_v) \times (T_v)_{\text{reg}}(K_v)$. The homomorphism ϕ factors as

$$G(K_v) \times T_v(K_v) \xrightarrow{\phi_1} G(K_v) \times T_v(K_v) \xrightarrow{\phi_2} G(K_v) \xrightarrow{\phi_3} G(K_v),$$

where $\phi_1(x, y) = (g^{-1}x, t^{-1}y)$, $\phi_2(x, y) = t^{-1}xtyx^{-1}$, and $\phi_3(z) = gtzg^{-1}$. The ϕ_1 and ϕ_3 are isomorphisms of K_v-analytic manifolds. Compute the K_v-tangent spaces

$$\mathcal{T}_{(g,t)}(G \times T_v) \xrightarrow{\mathcal{T}_{(g,t)}(\phi_1)} \mathcal{T}_{(e,e)}(G \times T_v) \xrightarrow{\mathcal{T}_{(e,e)}(\phi_2)} \mathcal{T}_e(G) \xrightarrow{\mathcal{T}_e(\phi_3)} \mathcal{T}_g(G).$$

Since $\mathcal{T}_{(g,t)}(\phi_1)$ and $\mathcal{T}_e(\phi_3)$ are K_v-vector space isomorphisms, the differential $\mathcal{T}_{(g,t)}\phi$ is surjective if and only if $\mathcal{T}_{(e,e)}(\phi_2)$ is surjective. The map $\mathcal{T}_{(e,e)}(\phi_2)$ us an isomorphism of Lie algebras, so we may compute $\mathcal{T}_{(e,e)}(\phi_2)$ using the discussion in [31, §12.2, pg. 93].

Let $K_v[\epsilon] = K_v[x]/(x^2)$. We may compute $\mathcal{T}_{(e,e)}(\phi_2)$ by computing the kernel of the $\epsilon \to 0$ map

$$G(K_v[\epsilon]) \times T_v(K_v[\epsilon]) \xrightarrow{\phi_2} G(K_v[\epsilon])/$$

For $(1+\epsilon X, 1+\epsilon Y) \in G(K_v[\epsilon]) \times T_v(K_v[\epsilon])$,

$$\begin{aligned}(1+\epsilon X, 1+\epsilon Y) &\mapsto t^{-1}(1+\epsilon X)t(1+\epsilon Y)(1-\epsilon X) \\ &= (1+\epsilon(t^{-1}Xt))(1+\epsilon Y)(1-\epsilon X) \\ &= 1+\epsilon(t^{-1}Xt + Y - X).\end{aligned}$$

For $(X,Y) \in \mathcal{T}_{(e,e)}(G \times T)$, the value $\mathcal{T}_{(e,e)}(\phi_2)(X,Y) = \mathrm{Ad}(t)X - X + Y$. Since t is regular, the 1-eigenspace of $\mathrm{Ad}(t)$ is $\mathcal{T}_e(T)$. Therefore, the differential $\mathcal{T}_{(e,e)}(\phi_2)$ is surjective, and therefore $\mathcal{T}_{(g,t)}\phi$ is surjective.

Since $\mathcal{T}_{(g,t)}\phi$ is surjective, ϕ is an open map by Corollary 2.1.86. Thus, $U(T,v)$ is open. For any open subgroup $\Omega \subseteq G(K_v)$, $T_v(K_v) \cap \Omega$ is Zariski-dense in $T_v(K_v)$, so it contains an element of $(T_v)_{\mathrm{reg}}(K_v)$. Thus, $U(T_v,v) \cap \Omega \neq \emptyset$.

Let $U = \prod_{v \in S} U(T_v, v)$. If $\delta_S(x) \in \delta_S(G(K)) \cap U$, then x is a K-element that is K_v-regular semisimple for all $v \in S$, hence it is K-regular semisimple. In other words, and K-torus that becomes a maximal K_v-torus after extending scalars is a priori a maximal K-torus. Let $T = Z_G(x)^\circ$ and note that $\theta_T(\mathrm{Gal}(K_T/K)) \supseteq \mathcal{W}(G,T)$ by construction. Thus, x is K-generic. □

As above, let E' be the minimal field of Γ, let X be a transcendence basis of K/E', and let $E = E'(X)$. Let F be the separable closure of E in K.

If T is a torus defined over F, then note that since $F_T K = K_T$ and $F_T \cap K = F$, a classic result from Galois theory states that

$$\mathrm{Gal}(K_T/K) \cong \mathrm{Gal}(F_T/F).$$

By the above isomorphism, any F-generic torus is automatically a K-generic torus. Therefore, we will find an F-generic element in Γ, which will automatically be K-generic.

Suppose now that G is adjoint. Fix an embedding $G \subseteq \mathrm{GL}_N$ and consider the matrix realization of $\Gamma \subseteq \mathrm{GL}_N(K)$. Let $\Gamma_F = \Gamma \cap G(F)$. Since G is semisimple, $G(F)$ is Zariski-dense in G and Γ_F is not empty, but not necessarily dense. Let H be the Zariski-closure of Γ_F in G. Let $T \subset G$ be a maximal torus and let $\gamma \in T_{\mathrm{sep}}(K) \cap \Gamma$. The element γ is regular semisimple. If $[K : F] = p^s$, consider γ^{p^s}. This is a regular element and the eigenvalues of γ^{p^s} lie in a Galois extension of F. Therefore, there exists $g \in G(K)$ such that $g\gamma^{p^s}g^{-1} \in G(F)$, and in particular gHg^{-1}. Therefore, H contains a maximal torus of the same rank as T. Therefore $\mathrm{rank}(H) = \mathrm{rank}(G)$ and Γ_F must be Zariski-dense. Therefore, it suffices to prove Theorem F under the assumption that $\Gamma \subseteq G(F)$.

Proof of Theorem F. Let Γ' be the subgroup of $\widetilde{G}(F)$ generated by elements of the form $\pi^{-1}([\gamma_1, \gamma_2])$ for all $\gamma_1, \gamma_2 \in \Gamma$. Since Γ is finitely generated, there exists a finitely generated subring $R \subseteq F$ such that $\Gamma \subseteq G(R)$ and $\Gamma' \subseteq \widetilde{G}(R)$.

By Lemma 3.1.3, we know that there exists a set S of r distinct nontrivial non-archimedean discrete valuations on E' such that for every $v \in S$, there exits an embedding

$$\epsilon_v \colon F \to E'_v,$$

and $\widetilde{G}$ splits over E'_v. Thus, we can apply Theorem E to construct maximal E'_v-tori $T_v \subseteq G$ such that any E'-torus that is $G(E'_v)$-conjugate to each T_v is E'-generic. Since $E'_v = F_w$ for any valuation $w|v$, any such F-torus is F-generic as well.

Let δ_S be the diagonal embedding of $G(F)$ into G_S.

Let

$$\tilde{\delta}_S \colon \widetilde{G}(F) \longrightarrow \widetilde{G}_S := \prod_{v \in S} \widetilde{G}(E'_v)$$

be the diagonal embedding of corresponding simply connected covers.

From this set of tori T_v for $v \in S$, we also construct the open subset $U \subseteq \widetilde{G}_S$ specified by Lemma 3.1.8.

Let H be the closure of $\tilde{\delta}_S(\Gamma')$ in $\widetilde{G}_S$. By Proposition 2.2.9, we see that since E'_v is the closure of the minimal field, so $\bigoplus_{v \in S} E'_v$ is the minimal semisimple ring for G_S. Because $\Gamma' \subseteq \widetilde{G}(R) \subseteq \widetilde{G}(\mathcal{O}_v)$ for each $v \in S$, we see that $\delta_S(\Gamma)$ is contained in a compact subgroup of G_S. Thus, $(\oplus_S E'_v, G_S, \delta_S(\Gamma))$ is a minimal triple. By Theorem 2.2.6, we know that H is an open subgroup in $\widetilde{G}_S$, so $Y = H \cap U \neq \emptyset$. Furthermore,

$\tilde{\delta}_S(\Gamma') \cap Y$ is dense in Y, so $\tilde{\delta}_S(\Gamma') \cap Y \neq \emptyset$. Let $\tilde{x} \in \Gamma'$ such that $\tilde{\delta}_S(\tilde{x}) \in \tilde{\delta}_S(\Gamma') \cap Y$.

By construction, $\tilde{x}$ is a regular semisimple E-element of infinite order. Let $\widetilde{T} = Z_{\widetilde{G}}(\tilde{x})^\circ$, then $\pi(\widetilde{T}) = T = Z_G(x)^\circ$ is a maximal F-torus in G as well and $\pi(\tilde{x}) = x \in \Gamma$. Furthermore, $\theta_T(\mathrm{Gal}(F_T/F)) \supseteq \mathcal{W}(G,T)$. Thus, x is generic over F.

□

3.2 Weak commensurability

We will now prove a few elementary results on weak commensurability. It is useful to start by defining the notion of a 'neat' subgroup with the properties described in [21] and show that these subgroups exist in the positive characteristic case.

Definition 3.2.1. Let $H \subseteq \mathrm{GL}_n(F)$. We say that the element $x \in H$ is **neat** if the subgroup of $(\overline{F})^\times$ generated by the eigenvalues of x contains no nontrivial root of unity. We say that H is a **neat subgroup** if all elements $x \in H$ are neat.

Proposition 3.2.2. *Let G be a connected absolutely almost simple connected algebraic group defined over an infinite finitely generated field F of characteristic $p > 0$ and let $\Gamma \subseteq G(F)$ be a finitely generated Zariski-dense subgroup. Then Γ contains a neat subgroup of finite index.*

For the proof of the analogous proposition in the characteristic zero, see [21, Theorem 6.11]. The proof is somewhat simpler in the positive characteristic case.

Proof. Since Γ is finitely generated, there exists a finitely generated subring $R \subseteq F$ such that $\Gamma \subseteq G(R)$. By Proposition 3.1.1, we can say that there exists some non-archimedean discrete valuation v on F with locally compact completion F_v such that $R \subseteq \mathcal{O}_v$. Let $\mathfrak{m} = R \cap \mathfrak{p}_v$, where $\mathfrak{p}_v$ is the maximal ideal of $\mathcal{O}_v$. Then $\mathfrak{m}$ is a proper maximal ideal of R and $R/\mathfrak{m}$ embeds into $k_v := \mathcal{O}_v/\mathfrak{p}_v$, the residue field associated to v. Hence, $R/\mathfrak{m}$ is a finite field.

Choose an embedding $G(F) \hookrightarrow \mathrm{GL}_n(F)$. Let $\eta\colon R \to R/\mathfrak{m}$ be the natural projection, and let $\widetilde{\eta}\colon G(R) \to G(R/\mathfrak{m})$ be the induced surjective homomorphism. Let $H = \ker(\widetilde{\eta})$. We claim that H is neat.

Let $x \in H$. Then $x \in \mathrm{GL}_n(\mathcal{O}_v)$ and $x \equiv 1 \mod \mathfrak{p}_v$. The eigenvalues of x all lie in some finite integral extension $\mathcal{O}'$ of $\mathcal{O}_v$, and let $\mathfrak{p}'$ be a maximal ideal of $\mathcal{O}'$ such that $\mathcal{O}_v \cap \mathfrak{p}' = \mathfrak{p}_v$. Take $\det(x - \lambda I) = 0 \mod \mathfrak{p}'$ to see that $(1-\lambda)^n \equiv 0 \mod \mathfrak{p}'$. In particular, each eigenvalue is a unit in $\mathcal{O}'$. The lemma will be proved if we can show that $1 + \mathfrak{p}'$ contains no nontrivial roots of unity.

Choose a uniformizer π of $\mathfrak{p}'$. Note that

$$\frac{1+\mathfrak{p}'}{1+(\mathfrak{p}')^2} \cong k_v,$$

via the isomorphism (to the additive group k_v) that sends $[1+\pi u] \mod 1+(\mathfrak{p}')^2$ to $u \in k_v$.

Suppose that $\mu \in 1 + \mathfrak{p}'$ is a nontrivial root of unity and suppose $r > 1$ is the smallest such integer such that $\mu^r = 1$. Then $\mu = 1 + \pi u$ for $u \neq 0$. Then μ is sent to $u \in k_v^\times$ under this isomorphism, and we see that $ru = 0$ in k_v. Therefore, $p|r$, so let r' be an integer such that $r = pr'$. Therefore,

$$\mu^{pr'} = 1 \quad \text{implies that} \quad (\mu^{r'})^p - 1 = (\mu^{r'} - 1)^p = 0.$$

Therefore, $\mu^{r'} = 1$. Since r' is strictly smaller that r and r is minimal, we have reached a contradiction.

Thus, $\Gamma' = \Gamma \cap H$ is neat and of finite index in Γ. □

For each $i = 1, 2$, let G_i be connected absolutely almost simple algebraic groups defined over an infinite finitely generated field F of characteristic $p > 0$ and let $\Gamma_i \subseteq G_i(F)$ be a finitely generated Zariski-dense subgroup.

Using the existence of neat subgroups, we are able to prove the following elementary lemmas that describe the behavior of weakly commensurable subgroups. These results are analogous to Lemma 2.3 and Lemma 2.4 in [20].

Lemma 3.2.3. *For $i = 1, 2$, let Γ_i and $G_i(F)$ be as above. Suppose that Γ_1 and Γ_2 are weakly commensurable. For $i = 1, 2$, suppose that Δ_i is a subgroup of $G_i(F)$ that is commensurable with Γ_i. Then Δ_1 and Δ_2 are weakly commensurable.*

Proof. By Proposition 3.2.2, we can find a neat subgroup $\Theta \subseteq \Gamma_1 \cap \Delta_1$ of finite index. Let $\delta_1 \in \Delta_1$ be a semisimple element of infinite order. Pick $n_1 \geq 1$ such that

$\gamma_1 = \delta_1^{n_1} \in \Theta$. Since Γ_1 and Γ_2 are weakly commensurable, there exist tori T_i such that $\gamma_i \in T_i(F)$ and there exist $\chi_i \in X(T_i)$ for $i = 1, 2$ such that

$$\chi_1(\gamma_1) = \chi_2(\gamma_2) \neq 1.$$

Pick $n_2 \geq 1$ such that $\delta_2 = \gamma_2^{n_2} \in \Gamma_2 \cap \Delta_2$. Then

$$\chi_1(\gamma_1)^{n_2} = \chi_1(\gamma_1^{n_2}) = \chi_1(\delta_1^{n_1 n_2}) = \chi_2(\delta_2).$$

Since $\gamma_1 \in \Theta$, $\chi_1(\gamma_1)^{n_2} \neq 1$. Therefore δ_1 and δ_2 are weakly commensurable elements, which implies that Δ_1 and Δ_2 are weakly commensurable. $\square$

Lemma 3.2.4. *For $i = 1, 2$, let $\pi_i \colon G_i \to G_i'$ be an F-isogeny of connected absolutely almost simple algebraic groups, and let Γ_i be a finitely generated Zariski-dense subgroup of $G_i(F)$. Then Γ_1 and Γ_2 are weakly commensurable if and only if $\Gamma_1' = \pi_1(\Gamma_1)$ and $\Gamma_2' = \pi_2(\Gamma_2)$ are weakly commensurable.*

Proof. Suppose that Γ_1' and Γ_2' are weakly commensurable. Let $\gamma_1 \in \Gamma_1$ be a semisimple element of infinite order. Then there exists a semisimple $\gamma_2 \in \Gamma_2$ of infinite order such that for $i = 1, 2$, there exists a maximal F-torus T_i' of G_i' and a character χ_i' of T_i' such that $\pi_i(\gamma_i) \in T_i'(F)$ and

$$\chi_1'(\pi_1(\gamma_1)) = \chi_2'(\pi_2(\gamma_2)) \neq 1.$$

Then for $i = 1, 2$, let $T_i = \pi_i^{-1}(T_i')$ be the maximal F-torus of G_i and note that $\gamma_i \in T_i(F)$. Let $\pi_i^*\colon X(T_i') \longrightarrow X(T_i)$ be the induced map on character groups. Let $\chi_i = \pi_i^*(\chi_i') \in X(T_i)$. Then

$$\chi_1(\gamma_1) = \chi_2(\gamma_2) \neq 1,$$

and Γ_1 and Γ_2 are weakly commensurable.

Suppose now that Γ_1 and Γ_2 are weakly commensurable. Use Proposition 3.2.2, pick neat subgroups of finite index Δ_i of Γ_i for $i = 1, 2$. By Lemma 3.2.3, it suffices to show that $\pi_1(\Delta_1)$ and $\pi_2(\Delta_2)$ are weakly commensurable. Let δ_1 be a nontrivial semisimple element of infinite order Δ_1. There exists a $\delta_2 \in \Delta_2$ such that for $i = 1, 2$, there exists a maximal F-torus T_i of G_i with $\delta_i \in T_i(F)$ and a character $\chi_i \in X(T_i)$ so that

$$\chi_1(\delta_1) = \chi_2(\delta_2) \neq 1.$$

Set $T_i' = \pi_i(T_i)$. Then $\pi_i(\delta_i) \in T_i'(F)$. Let $m = |(\ker \pi_1)(F)||(\ker \pi_2)(F)|$.

There exist characters $\chi_i' \in X(T_i')$ such that $\chi_i(t)^m = \chi_i' \circ \pi_i(t)$ for $t \in T_i(F)$. Since Δ_1 is neat, $\chi_1(\delta_1)$ is not an mth root of unity. Thus

$$\chi_1'(\pi_1(\delta_1)) = \chi_2'(\pi_2(\delta_2)) = \chi_1(\delta_1)^m \neq 1.$$

Thus, $\pi_1(\Delta_1)$ and $\pi_2(\Delta_2)$ are weakly commensurable. □

Lemma 3.2.4 is useful in that we are able to make the assumption that each G_i is simply connected or adjoint for $i = 1, 2$ without loss of generality. The lemma shows that the existence of weakly commensurable subgroups in the adjoint case implies their existence in the simply connected case and vice versa, so we may reduce or problems to the cases where the groups adjoint or simply connected with relative impunity.

Lemma 3.2.5. *Let T be a K-generic torus, and K_T is its splitting field over K. Let $t \in T(K)$ be an element of infinite order, let $\chi \in X(T)$ be a nontrivial character and let $\lambda = \chi(t)$. Then the Galois conjugates $\sigma(\lambda)$ with $\sigma \in \mathrm{Gal}(K_T/K)$ generate K_T over K.*

Proof. The set $\Phi(G, T)$ forms a generating set for the vector space $V = X(T) \otimes_{\mathbb{Z}} \mathbb{Q}$. The Weyl group $\mathcal{W}(G, T)$ acts irreducibly on V, and because $\mathrm{Gal}(K_T/K)$ contains the Weyl group, the vector space V has no $\mathrm{Gal}(K_T/K)$-invariant subspaces.

It suffices to show that if $\tau \in \mathrm{Gal}(K_T/K)$ with the property

$$\tau(\sigma(\lambda)) = \sigma(\lambda) \text{ for all } \sigma \in \mathrm{Gal}(K_T/K),$$

then $\tau = 1$. Let $\tau \in \mathrm{Gal}(K_T/K)$. Then

$$(\sigma^{-1}\tau\sigma)(\lambda) = (\sigma^{-1}\tau\sigma(\chi))(t) = \lambda.$$

Note that $(-\chi + \sigma^{-1}\tau\sigma(\chi)$ is a character that takes the value 1 at t. The element $t \in$

$T(K)$ generates a Zariski-dense subgroup of T since T is K-generic, hence irreducible. This implies that the character must be trivial on all of $T(\overline{K})$. The elements $\sigma(\chi)$ for $\sigma \in \mathrm{Gal}(K_T/K)$ span the $\mathbb{Q}$-vector space $V = X(T) \otimes_{\mathbb{Z}} \mathbb{Q}$. Since τ acts trivially on V, it must be the case that $\tau = 1$.

□

We also make use of the following theorem.

Theorem 3.2.6. *Let K be a local field of positive characteristic and let G be a semisimple group over K. Then there exists an maximal K-anisotropic torus $T \subset G$.*

Proof. See the discussion in [8, §2.4]. Specifically for semisimple G over K, a K-minisotropic torus is K-anisotropic. □

Proof of Theorem A

Proof of Theorem A. Without loss of generality, we can assume that K is large enough so that G_i is defined and split over K for both $i = 1, 2$. Let E_i be the minimal field of Γ_i for $i = 1, 2$, and let F_i be the separable closure of E_i in K. Since K/F_i is purely inseparable, we know that $[K_T : K] = [(F_i)_T : F_i]$ for $i = 1, 2$ and any maximal F_i-torus $T \subseteq G_i$. By applying Theorem F, we know that there exists a generic element $\gamma_1 \in \Gamma_1$ such that $T_1 = Z_G(\gamma_1)^{\circ}$ and

$$\theta_{T_1}(\mathrm{Gal}(K_{T_1}/K)) \supseteq \mathcal{W}(G_1, T_1).$$

Since Γ_1 and Γ_2 are weakly commensurable, we have that there exist $\gamma_2 \in G_2(K)$, a maximal torus $T_2 \subseteq G_2$, and characters $\chi_i \in X(T_i)$ such that

$$\chi_1(\gamma_1) = \chi_2(\gamma_2) = \lambda \neq 1.$$

Note that K_{T_1} is generated by all the Galois conjugates $\sigma(\lambda)$ for $\sigma \in \mathrm{Gal}(K^{\mathrm{sep}}/K)$. Since all of these conjugates belong to K_{T_2} as well, we have that $K_{T_1} \subseteq K_{T_2}$. Since G_2 splits over K, it must be an inner form over K, so by Lemma 2.1.53, we have that

$$\theta_{T_2}(\mathrm{Gal}(K_{T_2}/K)) \subseteq \mathcal{W}(G_2, T_2).$$

This implies that $|\mathcal{W}(G_1, T_1)|$ divides $|\mathcal{W}(G_2, T_2)|$. By symmetry, $|\mathcal{W}(G_1, T_1)| = |\mathcal{W}(G_2, T_2)|$.

□

Products of minimal triples

The proof of Theorem B requires a few technical lemmas and some results from the theory of quasi-models. Namely, we would like to explicitly describe the behavior of quasi-models formed by taking the product of two minimal standard triples. First, we state the following result due Pink [16].

Lemma 3.2.7. *Suppose that $F = F_1 \oplus F_2$ for local fields F_1 and F_2, $G = G_1 \times G_2$, and*

$\Gamma = \Gamma_1 \times \Gamma_2$ such that G_i is a connected absolutely simple adjoint group over F_i, and Γ_i is compact Zariski-dense subgroup of $G_i(F_i)$ for each $i = 1, 2$. Suppose that for $i = 1, 2$ each triple $(F_i, G_i, \pi_i(\Gamma))$ is minimal, and ρ_i be the non-constant representation of G_i that occurs as the subquotient of the adjoint representation. Define $\rho = \rho_1 \oplus \rho_2$ to be the representation of G over F. Then exactly one of the following is true:

(a) $E_\rho = E_{\rho_1} \oplus E_{\rho_2}$, *or*

(b) there is a quasi-model (E, H, ϕ) of (F, H, Γ) such that E is a field, ϕ is an isomorphism, and $\rho = \rho_0 \circ \phi$, where ρ_0 is a representation of H.

Proof. See [16, Prop. 3.13] □

Proposition 3.2.8. *Let G be a connected absolutely simple adjoint algebraic group defined over a local field L with Zariski-dense compact subgroup $\Gamma \subseteq G(L)$. Let $\iota\colon L \to F$ be an embedding of L into some field F. Let G' be the algebraic F-group obtained by extension of scalars given by ι and let $\psi\colon G(L) \to G'(F)$ be the induced homomorphism. If (K, G, Γ) is a minimal standard triple for some $K \subseteq L$, then $(\iota(K), G', \psi(\Gamma))$ is also minimal.*

Proof. Let (E', H', ϕ) be a minimal quasi-model of $(\iota(K), G', \psi(\Gamma))$. Let $F' = \iota(K)$, so there is an isomorphism $\iota^{-1}\colon F' \to K$, and let $E = \iota^{-1}(E') \subseteq K$. The embedding

$\iota|_{E'} \colon E' \to K$ induces a E'-morphism

$$\zeta \colon H' \to H,$$

where H is the group obtained from H' by extension of scalars via $\iota|_{E'}$. By construction, $E \subseteq K$, $H(K) \cong G(K)$ and $\zeta \circ \psi(\Gamma) \cong \Gamma$ over K. This implies that $(E, H, \zeta \circ \psi)$ is a quasi-model for (K, G, Γ). Hence, $E = K$ since (K, G, Γ) is minimal. Therefore, $E' = F'$ as well. The F'-isogeny

$$\phi \colon H' \to G',$$

is totally inseparable and induces a K-isomorphism

$$\zeta \circ \phi \colon H \to G.$$

Note that $\zeta \circ \phi$ has non-zero derivative and does not have a non-standard isogeny as a factor. Thus ϕ has non-zero derivative and has no non-standard isogeny as a factor, so ϕ is an isomorphism as well. □

The following Proposition reinterprets the result of Lemma 3.2.7 in a way that will be useful to the proof of Theorem B.

Proposition 3.2.9. *Let G be a connected absolutely simple adjoint group defined* *over a local field L of characteristic $p > 0$. Suppose that Γ is a compact Zariski-dense subgroup of $G(L)$ and that (L, G, Γ) is a minimal standard triple. Let $K := K_\Gamma \subseteq L$ be the minimal field of Γ and let v be the discrete valuation on K obtained by restricting the discrete valuation on L. Suppose that for $i = 1, 2$, we can construct embeddings $\iota^{(i)}\colon L \to K_v$. Let $G^{(i)}$ be the algebraic K_v-group obtained from G by extension of scalars via $\iota^{(i)}$, and let $\psi^{(i)}\colon G(L) \to G^{(i)}(K_v)$ be the induced homomorphism. Then exactly one of the following is true:*

(a) $G^{(1)} \cong G^{(2)}$ *and* $\iota^{(1)}(K) = \iota^{(2)}(K)$.

(b) The triple $(\iota^{(1)}(L) \oplus \iota^{(2)}(L), G^{(1)} \times G^{(2)}, \psi^{(1)}(\Gamma) \times \psi^{(2)}(\Gamma))$ *is minimal.*

Proof. Note that since (L, G, Γ) is minimal, the triple $(\iota^{(1)}(L) \oplus \iota^{(2)}(L), G^{(1)} \times G^{(2)}, \psi^{(1)}(\Gamma) \times \psi^{(2)}(\Gamma))$ is fibre-wise minimal by Proposition 3.2.8. This implies that we can invoke Lemma 3.2.7 to say that either $E_\rho = E_{\rho_1} \oplus E_{\rho_2}$ or there is a quasi-model (E, H, ϕ) of (F, H, Γ) such that E is a field, ϕ is an isomorphism, and $\rho = \rho_0 \circ \phi$, where ρ_0 is a representation of H. Let (F, H, ϕ) be a minimal quasi-model of the triple $(\iota^{(1)}(L) \oplus \iota^{(2)}(L), G^{(1)} \times G^{(2)}, \psi^{(1)}(\Gamma) \times \psi^{(2)}(\Gamma))$.

Suppose that $E_\rho = E_{\rho_1} \oplus E_{\rho_2}$. It is clear that $E_\rho \cap \iota^{(i)}(L) \subseteq E_{\rho_i}$. Since we either have $E_{\rho_i} = \iota^{(i)}(L)$ or $E_{\rho_i} = \iota^{(i)}(L)^p$ for $i = 1, 2$, we can conclude that $F = \iota^{(1)}(L) \oplus \iota^{(2)}(L)$.

Consider the isomorphism

$$\phi\colon H(F) \to G^{(1)}(\iota^{(1)}(L)) \times G^{(2)}(\iota^{(2)}(L)).$$

Denote the projection onto the ith compontent of $G^{(1)} \times G^{(2)}$ by π_i for $i = 1, 2$. For each $i = 1, 2$, define the $\iota^{(i)}(L)$-groups $H_i := (\pi_i \circ \phi)^{-1}(G^{(i)})$. It is then clear that $H = H_1 \times H_2$ and the isomorphism

$$\phi\colon H(F) \to G^{(1)}(\iota^{(1)}(L)) \times G^{(2)}(\iota^{(2)}(L))$$

factors into isomorphisms $H_1 \cong G^{(1)}$ and $H_2 \cong G^{(2)}$.

This implies that the triple $(\iota^{(1)}(L) \oplus \iota^{(2)}(L), G^{(1)} \times G^{(2)}, \psi^{(1)}(\Gamma) \times \psi^{(2)}(\Gamma))$ is minimal.

Suppose that E_ρ is a field. Again, this implies that F is a field. We know that $\iota^{(1)}(L) \oplus \iota^{(2)}(L)$ is of finite type over F, which implies that $F \subseteq \iota^{(1)}(L)$ is a finite extension. We have that $H \times_F \iota^{(1)}(L) \cong G^{(1)}(\iota^{(1)}(L))$, which is only possible if $F = \iota^{(1)}(L)$ by the minimality of the fibres and the uniqueness of the closure of the minimal field. Thus, $G^{(1)} \cong G^{(2)}$ and $\iota^{(1)}(L) = \iota^{(2)}(L)$. Since K is dense in L, $\iota^{(i)}(L)$ is entirely determined by the image of K. Therefore, $\iota^{(1)}(K) = \iota^{(2)}(K)$. □

Corollary 3.2.10. *With the set-up of Proposition 3.2.9, if* $\iota^{(1)}(K) \neq \iota^{(2)}(K)$, *then* *the closure of the image of the homomorphism*

$$\delta_v\colon \Gamma' \longrightarrow \widetilde{G}^{(1)}(K_v) \times \widetilde{G}^{(2)}(K_v)$$

is open in $\widetilde{G}^{(1)}(K_v) \times \widetilde{G}^{(2)}(K_v)$.

Proof. If $\iota^{(1)}(K) \neq \iota^{(2)}(K)$, then $(\iota^{(1)}(L) \oplus \iota^{(2)}(L), G^{(1)} \times G^{(2)}, \psi^{(1)}(\Gamma) \times \psi^{(2)}(\Gamma))$ is a minimal standard triple. The Corollary follows immediately from Theorem 2.2.6. □

Proof of Theorem B

Before proving the main result of this section, we prove the following general field theoretic result.

Lemma 3.2.11. *Let* $F \subseteq F_1 \subsetneq F_2 \subseteq K$ *be a tower of finitely generated infinite fields of characteristic* $p > 0$ *such that* K *is separable over* F_1, F_1 *is a purely transcendental extension of* F, *and let* $R \subseteq K$ *be a finitely generated subring. Then there exists an infinite set of discrete valuations* v *over* F *such that for each* v, *there are embeddings* $\iota_1, \iota_2\colon K \to F_v$ *such that*

(1) both $\iota_1(R)$ *and* $\iota_2(R)$ *are contained in* $\mathcal{O}_v$,

(2) the restriction $\iota_1|_{F_1} = \iota_2|_{F_1}$, *but* $\iota_1|_{F_2} \neq \iota_2|_{F_2}$.

Proof. Since K is a finite separable extension of the field F_1, let M be a the Galois closure of K over F_1. Then there exists a $\sigma \in \mathrm{Gal}(M/F_1)$ which acts nontrivially on F_2. Let R_0 be the subring of K generated by R and $\sigma(R)$. Since M is a finitely generated field and R_0 is a finitely generated ring, by Proposition 3.1.1, there exist infinitely many discrete valuations v on F such that for each valuation v, we have the embedding

$$\iota_v \colon M \to F_v$$

such that $\iota_v(R_0) \subseteq \mathcal{O}_v$. Then the embeddings

$$\iota_1 = \iota_v|_K \text{ and } \iota_2 = (\iota_v \circ \sigma)|_K$$

satisfy the requirements of the Lemma by construction. □

Proof of Theorem B. Let G_1 and G_2 be two connected absolutely simple adjoint algebraic groups defined over a finitely generated field F of characteristic $p > 0$. For $i = 1, 2$, let $\Gamma_i \subseteq G_i(F)$ be a finitely generated Zariski-dense subgroup and let $K_i := K_{\Gamma_i}$ be the minimal field of Γ_i defined by Corollary 2.2.10. Suppose that Γ_1 and Γ_2 are weakly commensurable.

By Corollary 2.2.10, the trace field of Γ_i is either equal to the minimal field K_{Γ_i} or it is a purely inseparable subextension, it suffices to prove the conclusion of the theorem for the minimal fields. Specifically, we would like to show that there exists some integer $k \geq 0$ such that $(K_1)^{p^k} \subseteq K_2$. To show this, we shall assume that for all integers $k \geq 0$, $(K_1)^{p^k} \not\subseteq K_2$ and reach a contradiction.

The compositum of the fields K_1K_2 might have nontrivial transcendence degree over K_2. Let $X \subset K_1$ be a finite transcendence basis such that K_1K_2 is algebraic over $K_2(X)$.

The theorem is proved in two steps. First, we prove the case where the extension K_1K_2 over K_2 is separable. Then we show that the general case can be reduced to this case.

Case 1: K_1K_2 is separable over $K_2(X)$:

Define $K := K_1K_2$, and let L be a Galois extension of K such that G_1 and G_2 are split over L. We can replace $(K_1)^{p^k} \nsubseteq K_2$ for any k with the weaker assumption $K_1 \nsubseteq K_2$. Therefore, we now assume that $K_1 \nsubseteq K_2$ and that $K/K_2(X)$ is separable.

Let r be the number of nontrivial conjugacy classes of the Weyl group of G_1. By Lemma 3.1.3, there exists an (arbitrarily large) finite set of places $S_L \subseteq V^K$ such that $K_v = L_w$ for all places w on L over $v \in S_L$. For a given $v \in S_L$, define $v_i := v|_{K_i}$. By construction, $K_v = (K_1)_{v_1} = (K_2)_{v_2}$. Define $S := \{v|_{K_2} \mid v \in S_L\} \subseteq V^{K_2}$. Furthermore, find a subset $S' \subseteq \{v|_{K_1} \mid v \in S_L \text{ and } v|_{K_2} \in S\}$. Choose S_L large enough so that $|S| = |S'| = r$, and identify the sets S and S' via that bijection that takes $v_1 \in S'$ to the unique place $v_2 \in S$ such that $(K_1)_{v_1} = (K_2)_{v_2}$.

Therefore, the set S contains r inequivalent discrete valuations on K_2, and for each $v \in S$, there exists a unique $v' \in S'$ such that $(K_2)_v = (K_1)_{v'}$ and an embedding $\iota_v \colon L \longrightarrow (K_2)_v$ such that $\iota_v(R) \subseteq \mathcal{O}_v$ (where $\mathcal{O}_v \subseteq (K_2)_v$ is the ring of integers). Note that G_1 is a $(K_2)_v$-group and $((K_2)_v, G_1, \Gamma_1)$ is a minimal standard triple by construction. Let $\psi_v\colon \Gamma_1 \longrightarrow G_1(\mathcal{O}_v)$ be the homomorphism induced by the embedding ι_v. Let Γ_1' be the group generated by $[\Gamma_1, \Gamma_1]^{\sim} \subseteq \widetilde{G_1}(\mathcal{O}_v)$. By Theorem 2.2.6, we know that the closure of the image of the homomorphism

$$\delta_S\colon \Gamma_1' \longrightarrow \prod_{v\in S} \widetilde{G_1}(\mathcal{O}_v) := \widetilde{G}_S$$

is open. By Lemma 3.1.8, there exists an open subset $U \subseteq \widetilde{G}_S$ such that for any $\tilde{\gamma} \in \Gamma_1'$ with $\delta_S(\tilde{\gamma}) \in U$, $\tilde{\gamma}$ is L-generic. Let $\pi\colon \widetilde{G_1} \to G_1$ is the canonical central isogeny, and let $\gamma = \pi(\tilde{\gamma})$. For the L-torus $T = Z_{G_1}(\gamma)^{\circ}$, we have that

$$\theta_T(\mathrm{Gal}(L_T/L)) \supseteq \mathcal{W}(G_1, T).$$

Since L is separable over $K_2(X)$, we can apply Lemma 3.2.11 to the tower of fields $K_2 \subseteq K_2(X) \subsetneq K \subseteq L$. Therefore, there exists a discrete valuation $w \in V^{K_2} \setminus S$ such that $(K_2)_w = (K_1)_{w'}$ for some $w' \in V^{K_2} \setminus S'$, and embeddings $\iota^{(1)}, \iota^{(2)}\colon L \longrightarrow (K_2)_w$ such that $\iota^{(1)}(K_2) = \iota^{(2)}(K_2)$, but $\iota^{(1)}(K) \neq \iota^{(2)}(K)$. We also have that $\iota^{(i)}(R) \subseteq \mathcal{O}_w$ for $i = 1, 2$. Let $G_1^{(i)}$ be the algebraic $(K_2)_w$-group obtained from G_1 by extension of scalars via $\iota^{(i)}$ for $i = 1, 2$.

Let $\psi^{(i)}\colon \Gamma_1 \longrightarrow {G_1}^{(i)}(\mathcal{O}_v)$ be the induced homomorphism. The fact that $\iota^{(1)}(K_2) = \iota^{(2)}(K_2)$ and $\iota^{(1)}(K) \neq \iota^{(2)}(K)$ implies that $\iota^{(1)}(K_1) \neq \iota^{(2)}(K_1)$. By Corollary 3.2.10, we have that the closure of the image of the homomorphism

$$\delta_w \colon \Gamma_1' \longrightarrow \widetilde{G_1}^{(1)}(\mathcal{O}_w) \times \widetilde{G_1}^{(2)}(\mathcal{O}_w)$$

is open. Since $w \notin S$, it follows that the closure of the image of the homomorphism

$$\delta \colon \Gamma_1' \longrightarrow \widetilde{G}_S \times \widetilde{G_1}^{(1)}(\mathcal{O}_w) \times \widetilde{G_1}^{(2)}(\mathcal{O}_w) := \widetilde{G}_{S,w}$$

is open in $\widetilde{G}_{S,w}$.

Since $L \subseteq (K_2)_w$, $G_1^{(1)}$ splits over $(K_2)_w$. Let $T^{(1)}$ be a $(K_2)_w$-split maximal torus of $G_1^{(1)}$. By applying Theorem 3.2.6, we can find a $(K_2)_w$-anisotropic maximal torus $T^{(2)} \subseteq G_1^{(2)}$.

For $i = 1, 2$ define the function

$$\phi_i \colon \widetilde{G}_1^{(i)}((K_2)_w) \times \widetilde{T}^{(i)}((K_2)_w) \to \widetilde{G}_1^{(i)}((K_2)_w)$$

in the same way as ϕ from Lemma 3.1.8. Namely, let $\phi_i(g, t) = gtg^{-1}$ for all $g \in \widetilde{G}_1^{(i)}((K_2)_w)$ and $t \in \widetilde{T}^{(i)}((K_2)_w)$ for $i = 1, 2$.

Define $U^{(i)} := \phi_i(\widetilde{G_1}^{(i)}((K_2)_w), \widetilde{T}^{(i)}_{\text{reg}}((K_2)_w))$. Note that $U^{(i)}$ intersects every open subgroup of $\widetilde{G_1}^{(i)}((K_2)_w)$ for $i = 1, 2$.

Thus, there exists some $\tilde{\gamma}_1 \in \Gamma_1'$ such that

$$\delta(\tilde{\gamma}_1) \in U \times U^{(1)} \times U^{(2)}.$$

Then for $\gamma_1 = \pi(\tilde{\gamma}_1)$, the torus $T_1 = Z_{G_1}(\gamma_1)^\circ$ is a maximal K_1-torus of G_1 since $\gamma_1 \in \Gamma_1 \subseteq G_1(K_1)$.

Since Γ_1 and Γ_2 are weakly commensurable, there exists a maximal K_2-torus T_2 of G_2 and $\gamma_2 \in \Gamma_2 \cap T_2(K_2)$ such that

$$\chi_1(\gamma_1) = \chi_2(\gamma_2) = \lambda \neq 1,$$

for some characters $\chi_i \in X(T_i)$ for $i = 1, 2$. Note that this implies that λ is algebraic over both K_1 and K_2.

Let $\mathscr{K}_i$ be the field over K_i generated by $\sigma(\lambda)$, where $\sigma \in \mathrm{Gal}(K_i^{\mathrm{sep}}/K_i)$ and let $\mathscr{L}$ be the field over L generated by $\sigma(\lambda)$, where $\sigma \in \mathrm{Gal}(L^{\mathrm{sep}}/L)$.

Note that $\mathrm{Gal}(L^{\mathrm{sep}}/L)$ naturally embeds into $\mathrm{Gal}(K_i^{\mathrm{sep}}/K_i)$ for $i = 1, 2$. This implies that $\mathscr{L} \subseteq \mathscr{K}_i L$ for $i = 1, 2$.

To show that opposite direction, notice that since $\delta(\gamma_1) \in U$, we see that T_1 is L-generic. By Lemma 3.2.5, $\mathscr{L} = L_{T_1}$, the splitting field of the torus T_1. We can therefore conclude that

$$|\mathrm{Gal}(\mathscr{L}/L)| \geq |\mathcal{W}(G_1, T_1)|.$$

Observe that $\mathscr{K}_i L$ is contained in the splitting field L_{T_i} (with equality for $i = 1$ since T_1 is L-irreducible). G_i splits over L, which implies that G_i is an inner form over L. By Lemma 2.1.53,

$$\theta_{T_i}(\mathrm{Gal}(L_{T_i}/L)) \subseteq \mathcal{W}(G_1, T_1).$$

Therefore,

$$|\mathrm{Gal}(\mathscr{K}_i L/L)| \leq |\mathcal{W}(G_1, T_1)| = |\mathcal{W}(G_2, T_2)| \leq |\mathrm{Gal}(\mathscr{L}/L)|,$$

and it follows that $\mathscr{L} = \mathscr{K}_1 L = \mathscr{K}_2 L$.

Note that the conclusion of Lemma 3.1.3 implies that $(K_2)_w = L_{w'}$ for all $w'|w$. Let $v_i = (\iota^{(i)})^{-1}(w|_{\iota^{(i)}(K_1)})$ for $i = 1, 2$. Thus v_i is a valuation on K_1 for $i = 1, 2$. By our choice of w, we can identify $(K_1)_{v_i}$ with $(K_2)_w$ for $i = 1, 2$.

By the construction of the set U, we know that the torus T_1 is $G_1^{(1)}((K_2)_w)$-conjugate to the $(K_2)_w$-split torus $T^{(1)}$. By the above identification, T_1 is isomorphic to a $(K_1)_{v_1}$-split torus.

Furthermore, T_1 is $G_1^{(2)}((K_2)_w)$-conjugate to the $(K_2)_w$-anisotropic torus $T^{(2)}$, hence T_1 is isomorphic to an anisotropic torus over $(K_1)_{v_2}$.

Since T_1 is isomorphic to a $(K_1)_{v_2}$-anisotropic torus, we have that for every nontrivial $\chi \in X(T_1)$, there exists some $\sigma \in \mathrm{Gal}((K_1)_{v_2}^{\mathrm{sep}}/(K_1)_{v_2})$ such that $\sigma \circ \chi \neq \chi$.

Since γ_1 generates a Zariski-dense subgroup, we have that

$$\sigma(\chi)(\gamma_1) \neq \chi(\gamma_1),$$

which implies that

$$\chi(\gamma_1) \notin (K_1)_{v_2} \text{ for all nontrivial } \chi \in X(T_1).$$

Since T_1 is isomorphic to a split torus over $(K_1)_{v_1}$, this implies that

$$\sigma(\lambda) = \sigma(\chi)(\gamma_1) \in (K_1)_{v_1}$$

for all $\sigma \in \mathrm{Gal}(K_1^{\mathrm{sep}}/K_1)$.

For $i = 1, 2$, extend the embedding

$$\iota^{(i)} \colon L \longrightarrow (K_2)_w$$

to an embedding

$$\widetilde{\iota^{(i)}} \colon \mathscr{L} \longrightarrow (K_2)_w^{\mathrm{sep}}.$$

From above, we see that $\widetilde{\iota^{(1)}}(\mathscr{K}_1) \subseteq (K_2)_w$. Since $\mathscr{K}_i L = \mathscr{L}$ for $i = 1, 2$, this implies that $\widetilde{\iota^{(i)}}(\mathscr{K}_2) \subseteq (K_2)_w$ as well.

On the other hand, we see that $\widetilde{\iota^{(2)}}(\mathscr{K}_1) \not\subseteq (K_2)_w$, so $\widetilde{\iota^{(2)}}(\mathscr{K}_2) \not\subseteq (K_2)_w$ as well.

Since $\widetilde{\iota^{(1)}}$ and $\widetilde{\iota^{(2)}}$ have the same restriction to K_2 and $\mathscr{K}_2/K_2$ is Galois, this implies that $\widetilde{\iota^{(1)}}\Big|_{\mathscr{K}_2}$ and $\widetilde{\iota^{(2)}}\Big|_{\mathscr{K}_2}$ differ by an element of $\mathrm{Gal}(\mathscr{K}_2/K_2)$. These elements fix $(K_2)_w$, so

$$\widetilde{\iota^{(2)}}(\mathscr{K}_2) \nsubseteq (K_2)_w \text{ and } \widetilde{\iota^{(2)}}(\mathscr{K}_2), \subseteq (K_2)_w$$

which is a contradiction. Therefore, $K_1 \subseteq K_2$ and the theorem is proved for the separable case.

Case 2: K_1K_2 is not separable over $K_2(X)$:

Let K be the separable closure of $K_2(X)$ in K_1K_2. Let L a separable extension of K_1K_2 such that G_1 and G_2 both split over L. Let L_2 be the separable closure of K in L. This implies that $K_2(X) \subseteq K \subseteq L_2$ is a separable tower of fields, and the extensions L/L_2 and K_1K_2/K are purely inseparable. Note that all the fields described above are finitely generated extensions of K_1 or K_2.

Since $(K_1)^{p^k} \nsubseteq K_2$ for any k, we know that $K_2 \neq K$. Let $l > 0$ be an integer such that $[K_1K_2 : K] = p^l$.

Consider the lth Frobenius isogeny,

$$\mathrm{Fr}^l \colon G_1 \longrightarrow \mathrm{Fr}^l(G_1),$$

which gives the following homomorphism on K_1K_2 points,

$$\hat{e} := \mathrm{Fr}^l_{K_1K_2} \colon G_1(K_1K_2) \longrightarrow G_1((K_1K_2)^{p^l}),$$

Let $G := \mathrm{Fr}^l(G_1)$. Note that G_1 and G have the same Killing-Cartan type and the Weyl groups of G_1 and G are the same. Furthermore, G is a K-group since $K \subset (K_1K_2)^{p^l}$. Define $\Gamma := \hat{e}(\Gamma_1)$, and let K_1' be the minimal field of Γ. By Lemma 3.2.4, Γ is weakly commensurable to Γ_2. We know that Γ is finitely generated as well, so we can find a finitely generated subring $R \subseteq L_2$ such that $\Gamma \subseteq G(R)$.

Let K_1' be the minimal field of Γ. Since Γ is defined over $K_1 \cap K$ by construction, we know that $K_1' \subseteq K_1 \cap K$.

We now compute the trace field of Γ. Fix an embedding $\Gamma_1 \subset \mathrm{GL}_N(K_1K_2)$, and suppose that $\gamma \in \Gamma_1$ has eigenvalues $\lambda_1, ..., \lambda_N$. Let ρ be the irreducible subquotient of the adjoint representation of G_1 (or the direct product of the two irreducible representations ρ_S and ρ_L) defined by Proposition 2.2.7. By choosing a basis of the Lie algebra $\mathfrak{gl}_N(K)$, we can realize $\rho(\gamma) \in \mathrm{GL}_{N^2}(K_1K_2)$. The trace of this element is given by

$$\mathrm{tr}(\rho(\gamma)) = \sum_{i,j} \lambda_i \lambda_j^{-1}.$$

Finding a common denominator, we see that

$$\begin{aligned}
\operatorname{tr}(\rho(\gamma)) &= \sum_{i,j} \lambda_i \lambda_j^{-1} \\
&= \sum_{i,j} \lambda_i \frac{\lambda_1 ... \hat{\lambda_j} ... \lambda_N}{\lambda_1 ... \lambda_N} \\
&= \sum_i \lambda_i \left(\sum_j \frac{\lambda_1 ... \hat{\lambda_j} ... \lambda_N}{\lambda_1 ... \lambda_N} \right) \\
&= \frac{s_1 s_{N-1}}{s_N},
\end{aligned}$$

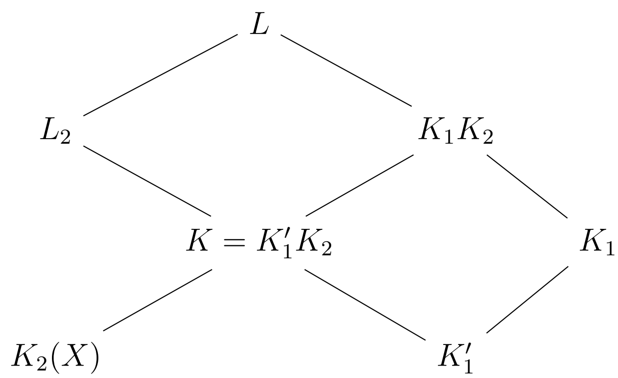
where

$$s_1 = \sum_i \lambda_i, \quad s_{N-1} = \sum_i \lambda_1 ... \hat{\lambda_i} ... \lambda_N, \quad s_N = \lambda_1 ... \lambda_N.$$

Note that s_1, s_{N-1}, and s_N are coefficients of the characteristic polynomial of γ. In particular, $\operatorname{tr}(\rho(\gamma)) \in K_1 K_2$. If we consider the image $\hat{e}(\gamma) \in \Gamma$, we see that $\hat{e}(\gamma)$ has eigenvalues $\lambda_1^{p^l}, ..., \lambda_N^{p^l}$. By the above computation,

$$\operatorname{tr}(\rho(\hat{e}(\gamma))) = \operatorname{tr}(\rho(\gamma))^{p^l}.$$

The minimal field K_1' is a purely inseparable extension of the trace field (the field generated by 1 and $\operatorname{tr}(\rho(\Gamma))$). Therefore, the extension $K_1 \cap K / K_1'$ is purely inseparable. Therefore, the extension $K/(K_1' K_2)$ is purely inseparable and separable, hence $K = K_1' K_2$. The relations between the defined fields are seen in the following diagram.



Note that the statement $(K_1)^{p^k} \not\subseteq K_2$ for all k implies that $K_1' \not\subseteq K_2$. But $K/K_2(X)$ is a separable extension and Γ is weakly commensurable to Γ_2. By Case 1, we know that $K_1' \subseteq K_2$, so we have a contradiction.

□

Corollary 3.2.12. *Suppose the same set-up as Theorem B and that F is a global field. Then $(K_1)^{p^{k_1}} = (K_2)^{p^{k_2}}$ for some $k_1, k_2 \in \mathbb{Z}$.*

Proof. If G_1 and G_2 are defined over a global field, then K_1 and K_2 are global fields as well (since they are infinite fields contained in a field F that has transcendence degree 1 over a finite field). The corollary will follow from the following general fact.

Let L be a global field. There exists a separating transcendence basis $x \in L$ such that $L/\mathbb{F}_p(x)$ is separable. Let α be a primitive element such that $L = \mathbb{F}_p(x, \alpha)$. If $f(t)$ is a minimal polynomial in $\mathbb{F}_p(x)[t]$ for α, then apply the pth power morphism to the coefficients of $f(t)$ to get the polynomial $f^p(t) \in \mathbb{F}_p(x^p)[t]$. Then α^p is a root $f^p(T)$, so $L^p = \mathbb{F}_p(x^p, \alpha^p)$ is separable over $\mathbb{F}_p(x^p)$ and $[L : \mathbb{F}_p(x)] = [L^p : \mathbb{F}_p(x^p)]$.

This implies that $[\mathbb{F}_p(x) : \mathbb{F}_p(x^p)] = [L : L^p] = p$.

Without loss of generality, suppose that there exists some k such that $(K_1)^{p^{k+1}} \subseteq K_2 \subseteq (K_1)^{p^k}$. Since $[(K_1)^{p^k} : (K_1)^{p^{k+1}}] = p$, then $K_2 = (K_1)^{p^k}$ or $K_2 = (K_1)^{p^{k+1}}$. □

Proof of Theorem C

Before proving Theorem C, we state the following field-theoretic lemma.

Lemma 3.2.13. *Let K and L be imperfect fields of characteristic $p > 0$ and suppose that $K \subseteq L$ is a finite purely inseparable extension. Then there is a natural continuous isomorphism*

$$\iota \colon \mathrm{Gal}\left(L^{\mathrm{sep}}/L\right) \longrightarrow \mathrm{Gal}\left(K^{\mathrm{sep}}/K\right).$$

Proof. Use [14, Ch. VI, Theorem 1.2, pg. 266] □

Proof of Theorem C. Again, let G_1 and G_2 be two connected absolutely almost simple adjoint algebraic groups defined over a finitely generated field F of characteristic $p > 0$. For $i = 1, 2$, let $\Gamma_i \subseteq G_i$ be a finitely generated Zariski-dense subgroup and let $K_i := K_{\Gamma_i}$ be the minimal field of Γ_i. Suppose that Γ_1 and Γ_2 are weakly commensurable.

For each $i = 1, 2$, let L_i be the minimal Galois extension of K_i such that G_i becomes an inner form over L_i. We would like to show that there exists some integer $k \geq 0$ such that $(L_1)^{p^k} \subseteq L_2$. Without loss of generality, suppose that $(K_1)^{p^k} \subseteq K_2$ by Theorem B.

Consider the kth Frobenius isogeny,

$$\mathrm{Fr}^k\colon : G_1 \longrightarrow \mathrm{Fr}^k(G_1),$$

and set $G_0 := \mathrm{Fr}^k(G_1)$, $\Gamma_0 := \mathrm{Fr}^k_{K_1}(\Gamma_1)$, and let $K_0 := K_{\Gamma_0}$ be the minimal field. Lemma 3.2.4 implies that Γ_0 and Γ_2 are weakly commensurable since Fr^k is an isogeny.

Let L_0 be the minimal Galois extension of K_0 such that G_0 becomes an inner form over L_0. Note that L_0 must be a finite purely inseparable subextension of L_1. To prove the theorem, it suffices to show that $L_0 \subseteq L_2$.

Suppose that $L_0 \not\subseteq L_2$. Let $L = L_0 L_2$. By Theorem F, there exists a semisimple element $\gamma_0 \in \Gamma_0$ such that $T_0 := Z_{G_0}(\gamma_0)^\circ$ is an L-generic torus. Let $\gamma_2 \in \Gamma_2$ be an element such that γ_0 is weakly commensurable to γ_2. By [20, Theorem 4.2], there exists a K_0K_2-isogeny $f\colon T_0 \longrightarrow T_2$. This induces a homomorphism between the character groups of T_2 and T_0:

$$f^*\colon X(T_2) \longrightarrow X(T_0).$$

Extending scalars by $\mathbb{Q}$, we get a $\mathbb{Q}$-linear isomorphism of vector spaces

$$f^*_{\mathbb{Q}}\colon X(T_2) \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow X(T_0) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

This induces a homomorphism

$$\overline{f}\colon \mathrm{GL}(X(T_0)\otimes_{\mathbb{Z}}\mathbb{Q}) \longrightarrow \mathrm{GL}(X(T_2)\otimes_{\mathbb{Z}}\mathbb{Q}).$$

Let L_0' be the separable closure of L_0 in L and let L_2' be the separable closure of L_2 in L. Then the following diagram commutes.

$$\begin{array}{ccc} \mathrm{Gal}(K_0^{\mathrm{sep}}/L_0') & \xrightarrow{\theta_{T_0}} & \mathrm{GL}(X(T_0)\otimes_{\mathbb{Z}}\mathbb{Q}) \\ \iota\downarrow & & \downarrow\overline{f} \\ \mathrm{Gal}(K_2^{\mathrm{sep}}/L_2') & \xrightarrow{\theta_{T_2}} & \mathrm{GL}(X(T_2)\otimes_{\mathbb{Z}}\mathbb{Q}) \end{array}$$

For any field extension F of K_0 in K_0^{sep} and a purely inseparable extension F' of F in K_2^{sep}, the map $\overline{f}$ induces an isomorphism between the images of $\mathrm{Gal}(K_0^{\mathrm{sep}}/F)$ under θ_{T_0} and $\theta_{T_2}(\mathrm{Gal}(K_2^{\mathrm{sep}}/F')$ under θ_{T_2}. Therefore,

$$|\theta_{T_0}(\mathrm{Gal}(K_0^{\mathrm{sep}}/F))| = |\theta_{T_2}(\mathrm{Gal}(K_2^{\mathrm{sep}}/F'))| \tag{3.2.1}$$

The assumption that $L_0 \not\subseteq L_2$ implies that $L_2 \subsetneq L$. Since G_2 is inner over L_2, we have that

$$\theta_{T_2}(\mathrm{Gal}(K_2^{\mathrm{sep}}/L_2)) = \mathcal{W}(G_2, T_2).$$

Since G_0 is not inner over L_2, by [20, Lemma 4.1] we know that

$$|\theta_{T_0}(\mathrm{Gal}(K_0^{\mathrm{sep}}/(L_2\cap K_0^{\mathrm{sep}}))| > |\mathcal{W}(G_0,T_0)| = |\mathcal{W}(G_2,T_2)| = |\theta_{T_2}(\mathrm{Gal}(K_2^{\mathrm{sep}}/L_2))|.$$

This contradicts (3.2.1), so $L_0 \subset L_2$.

□

3.3 Discrete subgroups

In this section, let F be a non-archimedean local field, let G be a connected absolutely almost simple group defined over F, and let Γ be a finitely generated Zariski-dense subgroup of $G(F)$. Since $G(F)$ has a locally compact topology, we can consider the induced topology on the subgroup Γ. The following lemmas will be used to prove Theorem D. We should note that many of these results are similar to those found in [16] and [17] with slight changes.

Lemma 3.3.1 (Analog of [16, Cor. 3.8])**.** *Let G be a connected, absolutely simple, adjoint algebraic group over a local (or global) field F. Suppose that Γ is a Zariski-dense subgroup of $G(F)$, and Δ is a Zariski-dense subgroup of $G(F)$ that is normal in Γ. If (F, G, Γ) is a minimal triple, then (F, G, Δ) is a minimal triple.*

Proof. Let (E, H, ϕ) be a minimal quasi-model of (F, G, Δ). Let $\gamma \in \Gamma$. Define $\mathrm{int}(\gamma)$ to be the inner automorphism corresponding to γ. Then $(E, H, \mathrm{int}(\gamma) \circ \phi)$ is a minimal quasi-model of (F, G, Δ). Therefore, both (E, H, ϕ) and $(E, H, \mathrm{int} \circ \phi)$ are minimal quasi-models of (F, G, Δ). By the uniqueness of minimal quasi-models, there exists some $\alpha \in \mathrm{Aut}(H)$ such that $\phi \circ \alpha = \mathrm{int}(\gamma) \circ \phi$.

Note that since ϕ is an isogeny with nowhere vanishing derivative, it factors the central isogeny $\pi\colon \widetilde{G} \to G$ (a consequence of Proposition 2.1.39 and Theorem 2.1.50). Since π induces an isomorphism $\mathrm{Out}(\widetilde{G}) \cong \mathrm{Out}(G)$, we get an isomorphism of $\mathrm{Out}(H \times_E F) \cong \mathrm{Out}(G)$ induced by ϕ. This implies that α must be an inner automorphism of H, so $\alpha = \mathrm{int}(\delta)$ for some $\delta \in H(E)$ since H is adjoint.

Then $\gamma = \phi(\delta) \in \phi(H(E))$, which implies that (E, H, ϕ) is also a minimal quasi-model for (F, G, Γ). Thus, ϕ is an isomorphism and $E = F$, so (F, G, Δ) is also minimal.

□

Corollary 3.3.2 (Analog of [17, Prop. 3.9])**.** *Let G be a connected, absolutely simple, adjoint algebraic group over a local (or global) field F. Suppose that Γ is a Zariski-dense subgroup of $G(F)$, and Δ is a Zariski-dense subgroup of $G(F)$ that is closed and of finite index in Γ. If (F, G, Γ) is a minimal triple, then (F, G, Δ) is a minimal triple.*

Proof. Let

$$\Delta' = \bigcap_{\gamma \in \Gamma} \gamma \Delta \gamma^{-1}.$$

Then Δ' is normal in Γ, so (F, G, Δ') is minimal by Lemma 3.3.1. Thus (F, G, Δ) must be minimal. □

Recall that a subgroup $\mathscr{H} \subseteq \mathscr{G}$ is **commensurated by** $\mathscr{G}$ if for all $g \in \mathscr{G}$, $g\mathscr{H}g^{-1}$ is commensurable to $\mathscr{H}$.

Lemma 3.3.3 (Analog of [17, Prop. 3.10])**.** *Let $G(F)$ be a connected, absolutely simple, adjoint algebraic group over a local (or global) field F. Suppose that Γ is a Zariski-dense subgroup of $G(F)$, and Δ is a Zariski-dense subgroup of $G(F)$ commensurated by Γ. Additionally, we assume that Δ is either compact in the local case or finitely generated in the global case. If (F, G, Γ) is a minimal triple, then (F, G, Δ) is a minimal triple.*

Proof. Let (E, H, ϕ) be a minimal quasi-model of (F, G, Δ). Let $\gamma \in \Gamma$. Define $\mathrm{int}(\gamma)$ to be the inner automorphism of G corresponding to γ. Then $(E, H, \mathrm{int}(\gamma) \circ \phi)$ is a minimal model of $(F, G, \gamma\Delta\gamma^{-1})$. Since Δ is commensurated by Γ, we know that $\Delta \cap \gamma\Delta\gamma^{-1}$ is finite index in each of Δ and $\gamma\Delta\gamma^{-1}$. By Corollary 3.3.2, both (E, H, ϕ) and $(E, H, \mathrm{int} \circ \phi)$ are minimal quasi-models of $(F, G, \Delta \cap \gamma\Delta\gamma^{-1})$.

By the exact same argument as the one in the proof of Lemma 3.3.1, ϕ is an isomorphism and $E = F$, so (F, G, Δ) is minimal. □

Lemma 3.3.4. *Suppose the G is a connected, absolutely simple, adjoint algebraic group defined over a local field F. Let $\Gamma \subseteq G(F)$ be a Zariski-dense subgroup. Suppose that the triple (F, G, Γ) is minimal. Then exactly one of the following is true:*

(a) $\overline{\Gamma}' \subseteq \widetilde{G}(F)$ *is open,*

(b) $\Gamma \subseteq G(F)$ *is discrete.*

Proof of Lemma 3.3.4. Let $\mathcal{K} \subseteq G(F)$ be a compact open subgroup. Let $\Delta = \Gamma \cap \mathcal{K}$.

Note that Δ is then an open compact subgroup of Γ in the subspace topology.

Let Θ be any open subgroup of Δ. Since $\{g\Theta\}_{g\in\Gamma}$ is an open cover of Δ and since Δ is relatively compact, it must have a finite subcover. Hence, $[\Delta : \Theta] < \infty$. Let $\gamma \in \Gamma$. Since the intersection of $\Delta \cap (\gamma\Delta\gamma^{-1})$ must have finite index in each of Δ and $\gamma\Delta\gamma^{-1}$, we see that Δ is commensurated by Γ.

Let H be the Zariski-closure of Δ in G. Note that Δ and $\gamma\Delta\gamma^{-1}$ are contained in $H(F)$. Since Γ is Zariski-dense and $H(F)$ is normalized by Γ, this implies that H is normal in G. Since G is simple, H is trivial or $H(F) = G(F)$.

If H is trivial, then $\Delta = \{1\}$ and is open compact. Hence, $\gamma\Delta = \{\gamma\}$ is open for all $\gamma \in \Gamma$. Thus, Γ is discrete.

If $H(F) = G(F)$, then Δ is Zariski-dense. By Lemma 3.3.3 and Theorem 2.2.6, we see that $\overline{\Delta}'$ is open in $\widetilde{G}(F)$.

Thus $\gamma\overline{\Delta}'$ is open for all $\gamma \in \overline{\Gamma}'$, so

$$\overline{\Gamma}' = \bigcup_{\gamma\in\overline{\Gamma}'} \gamma\overline{\Delta}'$$

is open in $\widetilde{G}(F)$. □

Proof of Theorem D

Proof of Theorem D. Suppose that Γ_1 is discrete and Γ_2 is not discrete. Let K_1 and K_2 be the minimal fields of Γ_1 and Γ_2 respectively. Let v be the discrete valuation on F.

By Theorem B, we know that $K_1^{p^l} \subseteq K_2$ for some $l \geq 0$. We first show that if Γ_1 is discrete, then $\mathrm{Fr}^l(\Gamma_1)$ is discrete.

Choose an embedding $\Gamma_1 \subset \mathrm{GL}_N(F)$. If $\Gamma := (\mathrm{Fr})^l(\Gamma_1)$ is not discrete, then there exists a convergent sequence $\mathrm{Fr}^l(\gamma_r)$ that converges to $1 \in \Gamma$ as $r \to \infty$. Via the embedding into $\mathrm{GL}_N(F)$, this becomes a convergent sequence of matrices

$$\mathrm{Fr}^l(\gamma_r) = [(a_{ij}^{(r)})^{p^l}]_{i,j} \longrightarrow 1.$$

Therefore, the (i,j)th matrix entry of the sequence of matrices is a convergent sequence in F that converges to $\delta_{i,j}$ (the Kronecker delta function). Since the pth power map is an automorphism of F, the sequence

$$(a_{ij}^{(r)})^{p^l} \longrightarrow \delta_{i,j}$$

induces a sequence

$$a_{ij}^{(r)} \longrightarrow \delta_{i,j}.$$

This implies that γ_r converges to 1 in Γ_1. This is impossible since Γ_1 is discrete, hence Γ is discrete.

Therefore, Γ is discrete and weakly commensurable to Γ_2. By replacing Γ_1 with Γ, we may assume without loss of generality that F is equal to the completion of K_2

with respect to the restriction of the v-adic norm on F. Furthermore, we may replace G_2 by its image under some purely inseparable isogeny so that the triple (F, G_2, Γ_2) is minimal.

By Lemma 3.3.4, we know that $\overline{\Gamma}_2'$ is open in $\widetilde{G}_2(F)$. Since Γ_2 is finitely generated, there exists some subring $R \subseteq K_2$ such that $\Gamma_2 \subseteq G_2(R)$. By Theorem 3.2.6, there exists a maximal F-anisotropic torus T_0 in G_2. Let r be the number of nontrivial conjugacy classes of the Weyl group $\mathcal{W}(G_2, T_0)$.

Let $U_0 = U(T_0, v)$ be the open subset of $\widetilde{G}_2(F)$ constructed in Lemma 3.1.8 associated to the torus T_0 and the valuation v on F.

By Proposition 3.1.1, we can pick r inequivalent valuations $v_1, ..., v_r$, such that v is not equivalent to v_j for $1 = 1, ..., r$, such that the completion $(K_2)_{v_j}$ is locally compact, and the embeddings $\iota_j \colon K_2 \longrightarrow (K_2)_{v_j}$ have the property that $\iota_j(R) \subseteq \mathcal{O}_{v_j}$ for all $j = 1, ..., r$.

Define the embedding

$$\delta_S \colon \widetilde{G}_2(K_2) \longrightarrow \widetilde{G}_S := \prod_{j=1}^{r} \widetilde{G_2}((K_2)_{v_j}).$$

as above. Since Γ_2' is mapped into a compact subgroup, the closure of the image $\delta_S(\Gamma_2')$ is open in $\widetilde{G}_S$. Invoke Lemma 3.1.8 to construct an open subset U of $\widetilde{G}_S$ such that $\delta_S^{-1}(\delta_S(G_2(K_2)) \cap U)$ consists of K_2-generic elements. Then $\mathcal{U} = U_0 \times U$ intersects every open subgroup of $\widetilde{G}_2(F) \times \widetilde{G}_S$. Let $\delta_{S,v}$ be the embedding of $\widetilde{G}_2(K_2)$

into $\widetilde{G}_2(F) \times \widetilde{G}_S$. Pick $\gamma_2 \in \pi(\overline{\delta_{S,v}(\Gamma_2')} \cap \mathcal{U})$. Lemma 3.1.8 implies that γ_2 is a K_2-generic element that generates a F-anisotropic torus in $G_2(F)$.

Since Γ_2 and Γ_1 are weakly commensurable, γ_2 is weakly commensurable to a semisimple $\gamma_1 \in \Gamma_1$ of infinite order. Let $T_1 = Z_{G_1}(\gamma_1)^\circ$. By the isogeny theorem (see [20, Theorem 4.2]), we know that T_2 is F-isogenous to T_1 which implies that T_1 is F-anisotropic. Note that $T_1(F)$ is compact and Γ_1 is discrete; thus $T_1(F) \cap \Gamma_1$ must be finite. However, it must contain the element γ_1, which has infinite order. Thus, we have a contradiction.

$\square$

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