Infinite Controversy: Reactions to Cantor's Theory

A Research Paper submitted to the Department of Engineering and Society

Presented to the Faculty of the School of Engineering and Applied Science
University of Virginia • Charlottesville, Virginia

In Partial Fulfillment of the Requirements for the Degree
Bachelor of Science, School of Engineering

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Spring 2024

On my honor as a University Student, I have neither given nor received unauthorized aid on this assignment as defined by the Honor Guidelines for Thesis-Related Assignments

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Introduction

When considering a cosmological result concerning the age of the universe, one would be prudent to keep in mind the beliefs and convictions of the scientists involved. For example, one might consider whether their spiritual or secular predispositions biased their study towards finding results supporting their faith or lack thereof (Porta et al., 2016). It is precisely this view of science – that is, as a social activity – that brought about the sociology of scientific knowledge (SSK) framework, which emphasizes how scientific results are shaped by the preconceived beliefs and opinions of scientists. The study of the development of mathematics demands a similar approach. Unlike the age of the universe, which most people, whether spiritual or secular, will surely agree is a finite number, the mathematical community has historically been divided (and to a certain extent, still is) over the existence of the infinite.

As mathematics has evolved, different schools of mathematicians have evolved along with it. In the late nineteenth century, the predominant schools were formalism and pre-intuitionism, the latter of which influenced or was closely related to schools such as intuitionism, constructivism, and finitism.

In 1874, a mathematician named Georg Cantor published a proof suggesting that two sets, each containing infinitely many elements, can be of different "sizes." Today, this sentiment is often stated informally as "some infinities are bigger than others" and is generally accepted by the mathematical community.[1] However, this was not always the case. Though some contemporaries enthusiastically embraced Cantor's theory, such as Bertrand Russell, who described him as "one of the greatest intellects of the nineteenth century," others were not so eager. Leopold Kronecker, Cantor's former teacher, labeled him a "scientific charlatan" and

[1] The distinction between "countable" and "uncountable" infinity is now presented as fact in most undergraduate courses in real analysis, following the proof that will be outlined below.

“corrupter of youth” while Henri Poincaré referred to his work as “a grave mathematical malady, a perverse pathological illness that would one day be cured” (Dauben, 1990). In this paper, we will analyze the responses Cantor’s theory elicited from several opposing schools of mathematicians, from both critics and proponents, whose disagreement stemmed largely from fundamentally differing views on infinity, religion, and the nature of mathematics. Further, we will explore how these responses have shaped the current mathematical climate, study traces of dissidence lingering today, and hypothesize why it is that Cantor’s theory seems to have won out in the long run. Completely divorced from its mathematical content, Cantor’s story offers a compelling lesson for us today: an unconventional, possibly even heretical, idea may prove to be exactly correct; as prominent liberal thinker John Stuart Mill wrote, “We can never be sure that the opinion we are endeavoring to stifle is a false opinion; and if we were sure, stifling it would be an evil still” (Mill, 2002).

Cantor’s Theory

To understand the disagreement, it is necessary to first familiarize oneself with Cantor’s theory that “infinite sets can have different cardinalities.” A *set* is simply a collection of objects. For example, $A = \{a, b, c\}$ and $B = \{\alpha, \beta, \gamma, \delta\}$ are sets containing three Latin letters and four Greek letters respectively. Intuitively, the “size” of B is in some sense larger than that of A. Mathematicians formalize this notion of size as a set’s *cardinality*. For a finite set, its cardinality is simply the number of elements it contains. Many of the most important sets, however, are infinite, such as the set of non-negative integers and the set of real numbers, commonly denoted as $\mathbb{N}$ and $\mathbb{R}$ respectively. Therefore, we would like to generalize our definition of cardinality to any (possibly infinite) set, being careful to do so in such a way that ensures this generalization is consistent with our definition of cardinality for finite sets.

This is accomplished via surjections; a surjection from a set X to a set Y is a function f such that for every element y of Y, there exists an element x of X such that f(x) = y. Colloquially, f is a surjection if given an arbitrary element of Y, we can find an element of X that maps to it. If there exists a surjection from X to Y, we say that the cardinality of X is at least as large as the cardinality of Y, since the surjection suggests we can "cover" Y with elements of X. To illustrate this concept of surjection, recall the finite sets A and B discussed previously. We concluded that the cardinality of B is at least that of A, because B contains more elements. Hence, for this new definition of cardinality in terms of surjections to be consistent with our intuitive understanding of cardinality for finite sets, we should be able to demonstrate a surjection from B to A, and indeed,

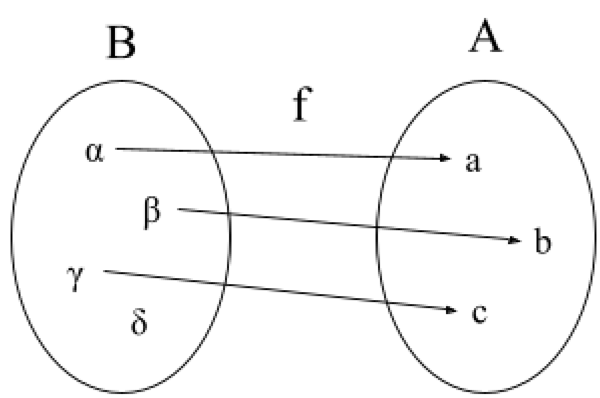

Figure 1: Surjection from B to A

Figure 1 is a surjection because for every element of A, there exists an element of B that maps to it. Thus, there exists a surjection from B to A, so we may conclude that the cardinality of B must be at least the cardinality of A.[2] For finite sets, this definition is exactly the same as a set having more elements than another (a surjection from B to A exists if and only if B has more

[2]A mathematical idea analogous to *Hume's principle* in philosophy.

elements of A), however, it also allows us to measure the cardinality of infinite sets, whose elements we cannot count directly.

Having introduced the fundamental concepts, we now proceed by outlining Cantor's main argument. The set of natural numbers, $\mathbb{N}$, is an infinite set; he then reasons that were there only one "size" of infinity, then for any other infinite set A, we should be able to find a surjection from $\mathbb{N}$ to A – that is, conclude that the cardinality of $\mathbb{N}$ is at least that of A. If we can demonstrate that this cannot happen for some A, then it must follow that the cardinality of $\mathbb{N}$ is *not* at least the cardinality of A, and thus the cardinality of A is a "larger" infinity. Cantor's proof that there exists no such surjection relies on a line of argumentation known as *proof by contradiction*; if one assumes the negation of a statement and is able to derive a contradiction, then it must be that the statement is true.

In 1891, Cantor published his *diagonal argument*, which is the version of the proof most commonly taught to mathematics students today. The argument proceeds by showing there exists no surjection from $\mathbb{N}$ to the closed interval [0, 1] (the set of all real numbers between 0 and 1, including both endpoints), and is constructive – by assuming such a surjection exists, one can construct an element of [0, 1] with no corresponding element of $\mathbb{N}$ that mapping to it, in which case the "surjection" is not a surjection at all – a contradiction. Hence, it must be that there exists *no* surjection from $\mathbb{N}$ to [0, 1], so the cardinality of $\mathbb{N}$ is *not* at least the cardinality of [0, 1], or equivalently, the cardinality of [0, 1] is a "larger" infinity than the cardinality of $\mathbb{N}$.[3]

Critics

The term *pre-intuitionism* was coined retroactively in 1951 by L. E. J. Brouwer, in order to distinguish its members from its philosophical successor, *intuitionism* (Brouwer, 1981). A

[3] Today, we refer to the cardinality of $\mathbb{N}$ as *countable infinity*, and use *uncountable infinity* for larger varieties.

primary tenet of pre-intuitionism and its derivatives is a dissatisfaction with proof by contradiction. To most of the mathematical community, it is acceptable to define a term, for example *group*, and prove by contradiction that it cannot be that no objects satisfying the definition of a group exist. Therefore, one may conclude that *group* is a useful definition – it refers to a non-empty set of mathematical objects. However, the pre-intuitionist eschews this logic: unless one can provide a concrete example satisfying the definition at hand, the definition itself is of little value.

This could explain why some pre-intuitionist mathematicians opposed Cantor's theory, for his argument relied fundamentally on deriving a contradiction from the assumption that a surjection exists. While his 1891 proof is constructive, in that he identifies a specific element of [0, 1] that is not mapped to by any element of $\mathbb{N}$, his earlier proofs were not, and were therefore unsurprisingly rejected by this segment of the mathematical community.

The finitists were a related school, whose central tenet was that they accepted only finite mathematical objects to exist. Their issue with Cantor's theory is clear: not only did they reject the assertion that the cardinalities of $\mathbb{N}$ and [0, 1] were not equal, but rejected the existence of $\mathbb{N}$ and [0,1] entirely. For many, the motivation for finitist beliefs was religious – infinitude was an inherently godly quality; it was perceived as sacrilege to ascribe it to anything other than the divine. Cantor himself was a Lutheran and believed his mathematical work to have revolutionary theological value, saying "From me, Christian philosophy will be offered for the first time the true theory of the infinite" (Dauben, 1990). However, not all theists were as eager to explore the religious consequences of Cantor's theory. Such was the case for prominent finitist and former teacher of Cantor's, Leopold Kronecker, who famously said, "God created the integers; all else is the work of man." Moreover, on his student, he wrote, "I don't know what predominates in

Cantor's theory – philosophy or theology, but I am sure that there is not any mathematics here" (Zenkin, 2004).

Secular mathematicians had reason to be finitist as well. Since the time of Aristotle, mathematicians drew a distinction between *potential infinity* and *actual infinity*; potential infinity refers to an interminable process, for example, one can never finish enumerating the natural numbers because each one has a successor. *Actual infinity*, in contrast, concerns treating an interminable process as if it had been completed, like working with the set of natural numbers, for example. In rejecting the concept of actual infinity, Aristotle resolved Zeno's dichotomy paradox[4], and thereby set a precedent for others doing so, which persisted until relatively recently. In this way, finitists acknowledged the infinitude of natural numbers, yet simultaneously rejected the existence of $\mathbb{N}$, and therefore certainly Cantor's theory.

Proponents

Among Cantor's proponents was the formalist school of mathematicians, who were guided by the idea that mathematics need not be representative of reality, but instead more akin to a kind of abstract game. As Weir writes, "This idea has some intuitive plausibility: consider the tyro toiling at multiplication tables or the student using a standard algorithm for differentiating or integrating a function." These processes are not unlike a chess player searching for the optimal move. From the formalist perspective, mathematics is about successively inventing definitions, applying rules, and exploring the consequences. For example, the conclusion that every square is a rectangle is a simple consequence of the standard definitions of squares and rectangles. It follows directly from our definitions, even if no squares or rectangles

[4] The paradox states that to walk 1 meter, one must first walk ½ meter, which in turn necessitates one to walk ¼ meter, and so on. Repeating this division *ad infinitum*, Zeno concludes that to walk 1 meter requires one to complete infinitely many tasks, and therefore, motion is impossible, contradicting common sensibility.

exist in reality (Weir, 2019). In this way, the formalist approach to mathematics is disconnected from the semantic meaning of the statements being studied and is more concerned with their syntax; A formalist is content with the deduction "if all zlorps are zlop, and all zlops are zleep, then all zlorps are zleep," irrespective of what zlorps, zlops, and zleep actually are. A structurally-identical statement, such as "if all squares are rectangles, and all rectangles are quadrilaterals, then all squares are quadrilaterals" is an equally valid deduction. Although we can *interpret* the argument using our familiar geometric notions of squares, rectangles, and quadrilaterals, the formalist chooses to be divorced from any such interpretation. Formalists, then, generally embraced Cantor's theory. Prominent formalist David Hilbert, for instance, wrote, "No one shall expel us from the paradise which Cantor has created for us!" (Zenkin, 2004). To most formalists, Cantor's conclusions were likely regarded more as curious consequences of the definitions of surjection and cardinality, rather than philosophically revolutionary (or even heretical) results.

But how did the formalist approach gain traction at all, considering its evident failure to consider practical significance? To develop one potential answer, we will briefly discuss the notion of a *metric space*, which generalizes the idea of distance between points in space. A ubiquitous component of K-12 math education is the distance formula

$$d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

which enables one to compute the Euclidean distance between two points in a plane; if you are walking from one corner of a park to the opposite corner, the distance formula provides a method to compute the minimal distance you must traverse. But now, suppose you are instead walking in a big city where the horizontal and vertical lines at the integers (such as $x = -3$, $y = 7$, etc.) correspond to roads. That is, to get from one point to another, you are physically unable to walk diagonally through a city block. To account for this restriction, we may adopt a new distance formula, commonly referred to as *Manhattan distance*:

$$d((x_1, y_1), (x_2, y_2)) = |x_2 - x_1| + |y_2 - y_1|.$$

In many cases, Manhattan distance may prove to be a more useful notion of distance than the classical Euclidean interpretation. For example, suppose you are working on an algorithm for Google Maps that estimates the time required to walk from one point to another. In a city such as New York, where one is likely confined to walking along a grid of parallels, using Manhattan distance will likely yield the more accurate model.

For another example, we now know that the Earth is (roughly) a sphere, however the Euclidean distance formula ignores the curvature of the Earth, modeling it as a plane. We may be comfortable with this assumption for points that lie relatively close together, but for points that lie farther apart, the Euclidean distance formula may yield a number that differs considerably from the true shortest path between the points.[5] Therefore, an algorithm estimating flight times would require yet another distance formula to account for this discrepancy. All of the above are examples of *metrics*, which generalize the concept of distance. Essentially, mathematicians observed the need for alternative distance formulae, but that it would be burdensome to prove the same facts over and over for each of them. Instead, they distilled the key properties that any notion of distance should obey (i.e. that the distance between any point and itself is zero, etc.) into the definition of an object called a metric. We can then prove facts about metrics in general without relying on any particular notion of distance. For example, we can prove many results from calculus for arbitrary metric spaces. If tomorrow the need arises for a new distance function, we need only prove that it satisfies the definition of a metric, and these results of

[5] The shortest path between two points on a sphere is known as a *geodesic*.

calculus would be inherited for free. Such a departure from any particular interpretation of distance would appeal to the formalist and makes mathematical theories extremely generalizable.

Another appealing feature of formalism stems from the observation that theory has historically often preceded application. Complex numbers, for instance, were first introduced in 1545 by Gerolamo Cardano purely as a theoretical oddity when he observed that defining a quantity i such that $i^2 = -1$ could yield solutions to equations previously thought to have none, such as $x^2 + x + 1 = 0$. Today, complex numbers are widely used in electrical engineering and play a crucial role in quantum physics, such as in the Schrödinger equation. Similarly, the quaternions were famously first described by Hamilton in 1843 when he carved their governing equation

$$i^2 = j^2 = k^2 = ijk = -1$$

into the Brougham Bridge in Dublin (Baez, 2004). Again, though at the time of their introduction they were another purely theoretical curiosity, today they are used extensively in robotics and computer graphics to represent rotations in three-dimensional space. Finally, modern modular arithmetic dates as far back as Gauss, but it was not until 1977 that it was used as the basis for the RSA encryption system (Koblitz, 2007). In each of these cases, applications for these abstract mathematical concepts were discovered long after the concepts were introduced in a purely theoretical capacity. It is thus important not to disregard novel results or ideas solely because they do not produce immediate practical significance or align with our current understanding of physics.

To Infinity and Beyond!

We now arrive at the big question: why is that Cantor's theory has earned relatively ubiquitous acceptance today? As we have discussed above, the formalist view is convenient in that it allows for generalization, such as proving classical calculus results over arbitrary metric spaces. We have seen that there is potentially great value in studying math which may initially appear not representative of reality or with no apparent applications. Further, we posit that the success of Cantor's theory also stemmed from a desire for unification between the fields of mathematics. Indeed, geometry was done long before even Euclid, and algebra as well, but these subjects were treated very separately. It was not until Descartes introduced coordinate systems that we became able to translate between geometric objects, such as circles, lines, and parabolas, and algebraic equations which they obey.

Over time there has been a proliferation of "hybrid" fields of mathematics, including algebraic geometry, algebraic topology, differential geometry, and analytic number theory to name but a few. Today there is considerable emphasis placed on applying tools from one field to problems in another, and in order to do this, it is paramount to have a common language. Fortunately, Cantor's theory of sets provided mathematicians with such a common language, contributing to the unification of the field. For example, in algebraic topology one studies the fundamental groups of topological spaces. Groups are a basic object of study in algebra, and topological spaces in topology, but Cantor's theory allows us to think of both as sets with different structure imposed on them. In this way, set theory is a bridge between algebra and topology that makes studying algebraic topology possible, or at the very least, easier.

The axioms of set theory we use today, called ZFC[6], have evolved since Cantor[7], but still retain the key notions of surjections, cardinality, countable and uncountable infinity, and the axiom of infinity, which establishes the existence of a set containing infinitely many elements. While ZFC is the *de facto* mathematical foundation of today, there are some who do not accept it, objecting particularly to the axiom of infinity. One name which is often associated with finitism is Norman Wildberger, a mathematician specializing in geometry at the University of New South Wales. Wildberger staunchly believes that mathematics should remain as it began – a tool to model physical real-world phenomena with clear practical significance – and denounces belief in the existence of infinite objects as impractical and illusory. He writes, "the axiom 'an infinite set exists' is of equal value to an axiom that states 'the god of thunder exists'. We can claim it is consistent and cannot be disproved, but both these axioms are equally worthless and irrelevant in the real-world, just as are any deductions derived using these axioms" (Wildberger, 2016).

In 2005, Wildberger published *Divine Proportions: Rational Trigonometry to Universal Geometry*, in which he develops trigonometry and Euclidean geometry without appealing to real numbers, π, trigonometric functions, or the infinite. It is indeed impressive that this is possible, however, for other branches of mathematics, the situation is not as favorable for finitists; consider real analysis, which is from its foundation focused on limits of sequences, the idea of "completing" an infinite number of steps, which Wildberger so fervently rejects. Since real analysis has myriad applications both in other branches of mathematics, as well as adjacent fields like economics and physics, it is understandable that many do not agree with Wildberger.

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[6] Zermelo–Fraenkel set theory with the axiom of choice.
[7] Primarily in efforts to circumvent issues like Russell's paradox.

On the other hand, there are some even more extreme than Wildberger, who rejects only actual infinity; Doron Zeilberger of Rutgers University rejects potential infinity as well; that is, Zeilberger believes there is a largest finite number[8], to which adding 1 would result in something akin to an overflow error on a computer, perhaps looping back to 0 (Gefter, 2013). This idea is familiar in the context of binary representations of integers, as well as finite fields studied in abstract algebra, but with the caveat that most who study finite fields also believe in infinite fields, such as the real numbers.

Conclusion

Over the past century and a half, Cantor's theory has developed from a novel and incendiary set of ideas to a set of routine facts commonly emphasized in undergraduate coursework and only questioned by select niche circles. Some possible explanations for this shift include the gradual secularism, or a popularization of the view that mathematics need not be representative of reality; or perhaps it is simply that after enduring 150 years of criticism, Cantor's ideas no longer appear as radical as they might have once; lastly, perhaps it is due to set theory, the natural setting for cardinality and surjections, becoming the standard foundation of mathematics today. This shift is no doubt due to at least several of the above factors, in addition to many other developments not explicitly discussed in this paper.

Besides making for an entertaining story about the development of mathematical thought, we believe there is a deeper lesson to be learned from the history of Cantor's theory. Despite widespread initial criticism, the work of a supposed "scientific charlatan" now serves as a foundational pillar of modern mathematics. From this, we must acknowledge that no matter how radical one's idea may initially appear may sound, no matter how unconventional their belief,

[8] This radical viewpoint is sometimes given the name *ultrafinitism*.

and no matter how strongly they conflict with the majority, there is at least some chance that they are correct; so even if it is an impossible question to answer, it seems worth pondering: what controversial idea of today will be tomorrow's set theory?

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