# LABOR MARKET IMPLICATIONS OF TECHNOLOGICAL CHANGE

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## ABSTRACT

The dissertation explores the labor market implications of technological change, with a focus on income inequality and the future impacts of artificial intelligence (AI).

Chapter 1 develops a model of hierarchical production organizations to examine how technological change affects the entire income distribution, particularly at the top. The model shows that if machines can only perform simple tasks, workers' wages rise more than managers' wages. However, if machines can perform sufficiently complex tasks, managers' wages increase while workers' wages fall, leading to greater income concentration at the top. The chapter also explores the potential equalizing effects of AI systems capable of automating managerial functions.

Chapter 2, in joint work with Anton Korinek, introduces an economic framework to evaluate alternative scenarios of technological progress culminating in artificial general intelligence (AGI). The analysis shows that the effects of automation on wages depend on the distribution of task complexity and the race between automation and capital accumulation. Wage growth can be sustained if capital accumulation outpaces automation. However, if all tasks can be automated, wages eventually collapse regardless of the rate of capital accumulation. The chapter also examines several extensions, such as the role of fixed factors, automating technological progress, societal limits on automation, and heterogeneous worker skills.

Chapter 3, in joint work with Eric Young, studies the constrained inefficiency of capital accumulation in economies with incomplete markets. The chapter develops two-period models to characterize the forces determining the constrained-efficient capital stock under various assumptions on the production function.

Overall, the dissertation provides insights into the complex relationship between technological change, labor markets, and income inequality, while also exploring the potential economic impacts of future developments in AI.

*To Davin*

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# Chapter 1

# Machines and Superstars: Technological Change and Top Labor Incomes

## 1.1 Introduction

A notable change in the U.S. labor income distribution over the past four decades is that the top earners have experienced a faster income growth than the rest of the economy (Atkinson, Piketty, and Saez 2011), while those in the lower parts of the distribution have under-performed (Acemoglu and D. Autor 2011). As a result, the U.S. economy witnessed a divergence between top earners and the rest, indicating that the gains from economic growth have been concentrated. Evidence suggests that technology is a major driving force behind the trend.[1]

In this paper, I build a model of hierarchical production organizations to examine the effects of technological change on income distribution with a focus on top incomes. The model builds on Garicano and Rossi-Hansberg (2006) and Antràs, Garicano, and Rossi-Hansberg (2006), featuring workers and machines. Workers differ in skill levels. Production requires solving problems that differ in complexity, where problems can be interpreted as work tasks. Workers spend time on production and their skill levels determine the maximum complexity of problems they can solve. Similarly, machines also differ in the maximum complexity of

[1] Acemoglu and Restrepo (2022b) is a recent work on the stagnation of middle and low incomes, and rising wage premium for skilled workers. Acemoglu (2002) and Aghion and Howitt (2008) provide a review of the literature on rising wage inequality. For evidence on top earners' increased income, see Piketty, Saez, and Zucman (2018). Kaplan and Rauh (2010) examine specific high-income occupations including top executives, investors, and lawyers. They conclude that evidence supports theories of skill-biased technological change and superstars.

problems they can solve, which I refer to as the complexity of machines.

Workers and machines form hierarchical organizations with multiple layers to efficiently use skills. Workers solve relatively simple problems and pass complex problems to their managers. Those with relatively high skill levels become managers who supervise workers. Thus, organizations are hierarchical in the sense that they consist of layers and higher layers solve more complex problems. Machines also solve relatively simple problems and pass the unsolved problems to the upper layer. However, machines cannot supervise workers or other machines.

To examine the effects of technological change on income distribution, I consider an increase in the maximum complexity of machines. Specifically, I model technological change as the introduction of new machines that can solve more complex problems than the existing machines. First, I show that if machines only solve sufficiently simple problems, then organizations have three layers with machines in the bottom and workers in the middle. In this case, workers delegate simple problems that machines can solve and focus on more complex problems. Also, I find that technological change raises workers' wages more than managers' wages. The most skilled managers can even experience a decline in their wages. Intuitively, technological change allows workers to employ more advanced machines and solve more problems, which tends to raise workers' wages. On the other hand, managers gain less because they do not supervise machines directly and the skill levels of their subordinates do not increase.

On the contrary, if the maximum complexity of machines passes a certain threshold, then organizations have two layers with machines and workers in the bottom layer supervised by managers. In this case, machines compete with workers for managers. Technological change increases managers' wages and reduces workers' wages. Moreover, more skilled managers benefit more than less skilled managers. As a result, income distribution becomes more concentrated at the top. The mechanism behind this "cascading effect" of technology is that the most skilled managers employ the most complex machines. Therefore, these managers benefit directly from technological advances and experience the largest wage increases. Additionally, new machines exert downward pressure on the wages of workers and existing machines. The analysis highlights how the complexity of tasks automated by machines matters for the complementarity between machines and skills. The results also demonstrate how gains from technological change could be concentrated among the top earners.

These results offer insights into the decline of top income shares during the mid-20th century and their subsequent increase since the 1980s. One interpretation is that machines during the former period, exemplified by conveyor belts and type writers, were only capable of very simple tasks. Through the lens of the model, low-skilled workers could directly use these machines and were augmented, but high-skilled workers (i.e. managers) who supervise the low-skilled workers, did not benefit as much. With advances in technology in the following years, machines became capable of many tasks done by low-skilled workers. Examples of such machines are robot arms and personal computers. The introduction of more advanced machines increased the supply of skills that augment high-skilled workers, while substituting for low-skilled workers. My model generates patterns that are qualitatively consistent with the changes in top income shares.

After establishing the main results, I explore the implications of advanced AI systems for managerial functions. Due to the latest progress, AI systems have gained a wide range of new capabilities that allow the automation of cognitive tasks. While some express concerns about higher inequality due to automation (A. Agrawal, J. Gans, and Goldfarb (2019), Mindell, Reynolds, et al. (2022)), others emphasize the potential equalizing effects of AI (Brynjolfsson, Li, and Raymond 2023) as AI distributes the knowledge possessed by high-skilled workers. Motivated by these developments, I analyze a setup where machines can supervise workers.[2] To do so, I modify the model so that machines (or algorithms in this context) can supervise workers and substitute for managers.

Another modification of the model is that machines are more efficient in supervision than managers. Specifically, I assume that supervision incurs costs in terms of time for both managers and machines, but the cost is lower for machines. Lower supervision costs allow machines to supervise a larger number of workers. In an extreme case, the supervision cost of machines can be negligible, which captures the low inference costs, or fast computation,

[2]Here, I am interested in the effects of wide-spread algorithmic management on income distribution. Note that Lee et al. (2015) define algorithmic management as “software algorithms that assume managerial functions and surrounding institutional devices that support algorithms in practice” and examine ride sharing services such as Uber and Lyft.

of modern AI systems.

In the modified setup, I show that if machines have sufficiently small supervision costs, then the number of workers supervised by a machine is no longer limited by the time constraint, and thus time is no longer scarce for machines. As a result, machines supervise all workers below a certain threshold skill level. Moreover, the model yields three results on the effects of algorithmic management on income distribution. First, inequality between workers and managers decreases. Second, inequality among workers declines as the least skilled workers benefit the most from algorithmic management. Third, inequality among the remaining managers rises because the least skilled managers face direct competition with machines.

The main results shed light on the role of technological change in shaping the dynamics of top income shares. In particular, previous research takes two approaches. First, at least since Tinbergen (1956), economists have recognized that technological change may favor certain groups of workers over others. Specifically, technological change is skill-biased if it raises skilled wages, such as those for college graduates, more than unskilled wages.[3] Moreover, recent work shows that advances in automation technologies account for the decline in the wages of a wide range of workers in the US (Acemoglu and Restrepo 2022b).[4] The approach taken in this line of research divides workers into fixed groups, for example by education, and mainly aims to understand the changes in relative wages between these groups. While the approach provides insights on rising skill premium and stagnant wages of automated occupations, it does not speak to why gains from economic growth are increasing more for those at higher income levels. Put differently, how are the superstars growing even more successful?

That observation led to the second approach in the literature, which focuses on the superstar effect in the labor market (Rosen (1981), Garicano (2000), Garicano and Rossi-Hansberg (2006)). A main theme is that recent technology disproportionately complements the highest skills, giving rise to "superstars" who are significantly more successful than the rest.

[3]See Katz and Murphy (1992b) and Acemoglu (1998) for early contributions on skill-biased technological changes.

[4]Acemoglu and Restrepo (2020) estimates the negative effects of robots on employment and wages. There is also literature on routine-biased technological change that focuses on the polarization of the labor markets. See, for example, Goos, Manning, and Salomons (2014) and related work.

However, this approach does not speak to the divergence, since the 1980s, between top incomes and lower incomes, mentioned earlier in the introduction. Moreover, the superstars literature does not account for the potential role of technology during the mid-20th century when the growth of top income was outpaced by that for lower incomes. In this paper, I merge these two approaches and theoretically examine the relationship between technology and top incomes.

The analysis of the implications of algorithmic management highlights a channel through which future AI systems could reduce income inequality by replacing high-skilled workers. The results in this paper speak to the recent evidence that AI, or large language models specifically, benefits low-skilled workers more than high-skilled workers (Noy and Zhang (2023), Brynjolfsson, Li, and Raymond (2023), Peng et al. (2023)). Relatedly, there have been discussions on the possibility of high-skilled workers being replaced by AI (Webb (2020), Hui, Reshef, and Zhou (2023), A. Agrawal, J. S. Gans, and Goldfarb (2023), Edward W Felten, Raj, and Seamans (2023a)). This paper provides a framework to examine the mechanism that gives rise to an equalizing effect of AI. Moreover, the model shows that advanced AI can lead to much lower labor market inequality by reducing worker differences due to managerial quality or task-specific knowledge.

Additionally, low supervision costs represent a new type of costs that fall significantly in the digital economy (Goldfarb and Tucker 2019). Previous work has also examined the effects of falling cost of supervision as a result of advances in information and communication technology (ICT) (Garicano and Rossi-Hansberg 2006). Nonetheless, this paper extends the analysis by considering different supervision costs between managers and machines to study the extreme case where the supervision cost of machines is arbitrarily small. More specifically, even if the mass of machines is close to zero, so that there is a "single" machine, a significant fraction of workers may be supervised by machines if the supervision cost of machines is also close to zero.[5]

The paper proceeds as follows: Section 1.2 develops the model. Section 1.3 contrasts two distinct outcomes on the effects of technology and discusses how the insights apply to the trends in top income shares before and after the 1980s. Section 1.4 is more focused

[5] In contrast, high supervision costs of managers limit their scope of supervision.

towards the implications of future AI systems for managerial functions and how top inequality could decrease with the introduction of management by machines. Section 1.5 concludes by outlining the directions for future work.

## 1.2 Model

### 1.2.1 Environment and Strategies

**Agents and Endowment** The economy is populated by a unit mass of agents and lasts one period. There is one good in the economy. Agents differ in their skill levels that are exogenously given. Agents are uniformly distributed on an interval $[1-\Delta, 1]$. Here, $1-\Delta$ is the lowest skill level of agents. Agents are endowed with one unit of time. Agents have a linear utility function over the consumption of the good.

**Machines** There are machine owners who are endowed with one machine each. Machine owners differ in the complexity of machines, which parallels the skill level of agents. Machine owners are also uniformly distributed on an interval $[\theta, \theta+\phi] \subset [0,1]$ with a distribution function $G$.

The mass of machine owners is *not* normalized and the density function of the distribution for machines is

$$g(x) = \begin{cases} \mu, & \text{if } x \in [\theta, \theta+\phi] \\ 0, & \text{otherwise} \end{cases}$$

where $\mu \geq 0$. Thus, the total mass of machines existing in the economy is $\mu\phi$, which is not necessarily equal to one. In other words, $\mu\phi$ can be interpreted as the mass of machines relative to the agents. Note that if $\mu = 0$ then the model collapses to the standard setup of hierarchical organizations without machines.

The parameters $\theta$ and $\phi$ are not restricted as long as the interval of machines is a subset of the unit interval. The parameter $\theta$ is the skill level of the least skilled machines and $\theta+\phi$ is the skill level of the most skilled machines. In other words, $\theta+\phi$ is the maximum complexity of the problems that machines can solve.[6] Figure 1.1 illustrates these cases. The first highlighted interval captures "rudimentary" machines that can only solve very simple problems relative to workers. On the other hand, the second interval is "advanced" machines on which machines can solve more complex and a wider range of problems.

**Production and Organizations** To produce the good, agents must spend time and solve problems. Agents can spend their time either to generate problems or supervise others. It costs one unit of time to generate a unit mass of problems. Problems differ in their levels of complexity and are uniformly distributed from 0 to 1. Agents solve problems with complexity levels lower than their skill levels. The amount of output produced by an agent is determined by the mass of problems solved. Thus, in autarky, an agent with skill $x$ solves problems from 0 to $x$, and the amount of output is $x$, which is the mass of problems solved. In short, the skill level of an agent equals the amount of output produced by the agent. I assume machine owners cannot produce in autarky.

Alternatively, agents can form organizations, which can also include machines. Organizations are hierarchical and consist of one layer of managers and lower layers of workers and machines. I assume that an organization can have three layers at most. Also, the mass of managers in each organization is normalized to be one so that there is a "single" manager at the top of the hierarchy. Thus, each manager corresponds to an organization consisting of workers and machines in lower layers.

Similarly with workers, a machine with complexity $x$ generates a unit mass of problems and solves problems from 0 to $x$, producing output of amount $x$. However, machines cannot supervise workers or other machines. So machines and workers in the bottom layer generate problems and pass the unsolved ones to their managers. Note that if the intervals $[\theta, \theta + \phi]$ and $[1 - \Delta, 1]$ overlap, then workers and machines on the overlapping region solve exactly the same amount of problems.

As an example, suppose an organization has two layers. A worker with skill $x_1 \in [1-\Delta, 1]$ solves problems $[0, x_1]$, of which mass equals $x_1$. Then the worker passes on the unsolved

[6] For example, early computer vision techniques allowed computers to only categorize pictures of relatively simple objects but recent advances have led to decreasing error rates and applications to more complex problems in real-world situations such as inventory management and self-driving cars.

problems to the manager in the upper layer. The manager, whose skill is $x_2 > x_1$, analyzes the problems by spending time $h < 1$ per unit mass of problems and instantaneously solves those that are less difficult than $x_2$.[7] Here, $h$ is the supervision cost in terms of time spent per unit mass of problems by a manager. It is time required for managers to analyze the problems received from workers and communicate the results after analyzing them. Since there is no layer above the manager, the measure of problems solved by the worker together with the manager is $x_2$. Therefore, the final output is $x_2$.

It is worth noting the differences between generating problems and supervision. Generating problems can be interpreted as the actual production process or activities that require physical involvement. On the other hand, supervision can be considered purely cognitive in the sense that it is only about providing solutions to unsolved problems without engaging in the production activity itself (Garicano 2000). Thus, managers spend their time by analyzing and communicating the unsolved problems and not by actually solving the problems because they only provide the solutions to the workers.[8]

Relatedly, the restriction on machines can be interpreted as follows. Machines in this economy are robots that are capable of physical activities but require workers to operate them. Thus, higher values of the parameter $\phi$ can be interpreted as the improvements in industrial machinery during the previous century, which evolved from conveyor belts to robot arms that assemble complex objects.[9]

**Strategies** At the beginning of the period, agents decide whether to become workers or managers.[10] Those who become workers earn $w_1(x_1)$ and those who become managers hire workers to supervise and earn $w_2(x_2)$. Agents choose whichever yields higher income. Another way to interpret the environment is that agents trade tasks the market. Workers sort

---

[7] Like in the existing models of hierarchical organizations, agents specialize either in production (i.e. generating problems) or supervision in equilibrium. Moreover, Garicano (2000) shows that agents in the upper layer has higher skill levels than those in lower layers in equilibrium.

[8] Examples for the role of workers and managers include call center staff members and their superiors, and research assistants and professors. In the latter example, research assistants do the basic work (i.e. generate problems) and ask their professors about difficult issues they cannot resolve on their own.

[9] Of course, the interpretation of machines is not restricted to industrial robots and physical tasks. Other examples include type writers/word processors and grammar checkers/smart chatbots.

[10] Agents can also choose to produce in autarky. Throughout the paper, I restrict attention to the parameter space where all agents are in organizations so I focus on this case.

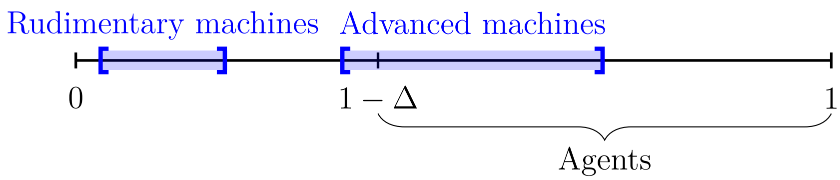



Figure 1.1: Machines and agents

tasks and send the difficult ones to managers. Managers pay $w_1(x_1)$ to workers in return and earn $w_2(x_2)$ themselves by completing the tasks.

In addition to the workers, managers can employ machines as well. A machine owner with machine complexity $x$ receives $w_m(x)$ as a compensation for passing on problems. Here, the subscript $m$ can be either 0 or 1, where 0 indicates the organization has three layers and machines are in the bottom and 1 indicates two layers. In three-layer organizations, all machines are in the bottom layer (layer 0) supervised by workers and are compensated according to a wage function $w_0(\cdot)$.

In two-layer organizations, workers and machines are in the same layer (layer 1) directly supervised by managers. Managers do not distinguish between workers and machines, as long as they solve the same amount of problems (or have the same "skill" levels). Thus, managers form organizations with workers and machines randomly. By the law of large numbers, the fraction of machines within an organization is the same as the economy-wide fraction of machines at the complexity level. Note that if there are two layers then machines and workers solving the same amount of problems face the same wage schedule $w_1(\cdot)$.

The problem of a manager is as follows. If there are two layers in the organization then the manager only chooses the skill level of direct subordinates and solves

$$w_2(x_2) = \max_{x_1, n_1} n_1 x_2 - n_1 w_1(x_1) \tag{1.1}$$

subject to the time constraints

$$h(1 - x_1)n_1 \leq 1$$

where $n_1$ is the number of workers hired.[11] Here, managers receive what is left of the total output ($n_1 x_2$) after compensating the workers ($n_1 w_1(x_1)$). Due to the time constraint, agents can spend only up to the time endowment. The amount of time spent can be broken down into three parts. First, a manager spends $h$ to observe each problem. Each worker passes on unsolved problems of mass $1 - x_1$ and there are $n_1$ workers under the supervision of the manager. Since it is optimal to spend all time endowment, the time constraint holds with equality, which pins down the number of workers

$$n_1 = \frac{1}{h(1 - x_1)}$$

Note that $n_1$ is increasing in $x_1$. Intuitively, more skilled workers allow the manager to hire more workers because they require less supervision time from their manager. With the saved time per worker, the manager can supervise a larger group of workers. By substituting the above expression into the objective, the manager's problem can be written as

$$w_2(x_2) = \max_{x_1} \frac{x_2 - w_1(x_1)}{h(1 - x_1)}$$

Panel (a) in Figure 1.2 illstrates this case where the top circle is the manager supervising workers and machines.

If there are three layers with machines in the bottom layer, then the manager chooses the skill level of workers as well as that of machines.

$$w_2(x_2) = \max_{x_0, x_1, n_0, n_1} n_1 n_0 x_2 - n_1 w_1(x_1) - n_1 n_0 w_0(x_0) \tag{1.2}$$

subject to the time constraints

$$h(1 - x_1) n_1 n_0 \leq 1$$
$$h(1 - x_0) n_0 \leq 1$$

[11] Here, I assume that managers choose workers of only one skill level. In the following subsection, I show that the assumption is without loss of generality. For a related discussion, see Antràs, Garicano, and Rossi-Hansberg (2006) and their working paper version.

where $n_1$ is the number of workers as above and $n_0$ is the number of machines per worker. Thus, the span of control of the manager, or the total mass of problems generated in the organization, is given by $n_1 n_0$. In the three-layer case, the total cost is the sum of compensations paid to workers and machines. As in the two-layer case, the manager's problem can be rewritten by substituting in the time constraints

$$w_2(x_2) = \max_{x_0, x_1, n_0, n_1} \frac{x_2}{h(1-x_1)} - \frac{1-x_0}{1-x_1} w_1(x_1) - \frac{w_0(x_0)}{h(1-x_1)}$$

Panel (b) of Figure 1.2 depicts this case where the manager directly supervises workers only (middle circles) and workers supervise machines (bottom squares).

It is worth noting that the production function of an organization is supermodular since the total output is given by $x_2/h(1-x_1)$ that take $x_1$ and $x_2$ as inputs. Taking the derivatives, it follows that

$$\frac{\partial^2}{\partial x_1 \partial x_2}\left(\frac{x_2}{h(1-x_1)}\right) > 0$$

Intuitively, all managers can produce more if they hire better workers but the increase is larger for better managers than worse managers. Similarly, all workers become more productive if better managers supervise them but the productivity gain is greater for better workers than worse workers.

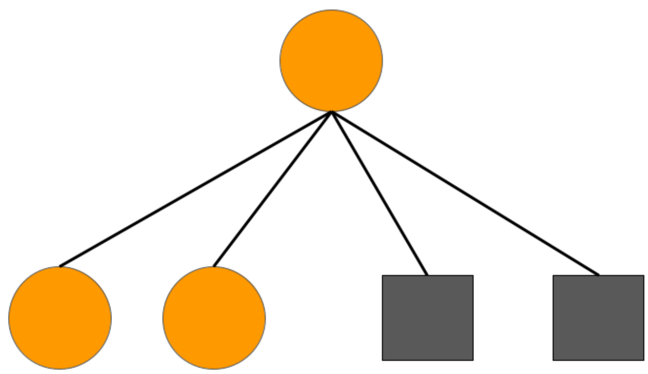
### 1.2.2 Equilibrium

#### Equilibrium Characterization

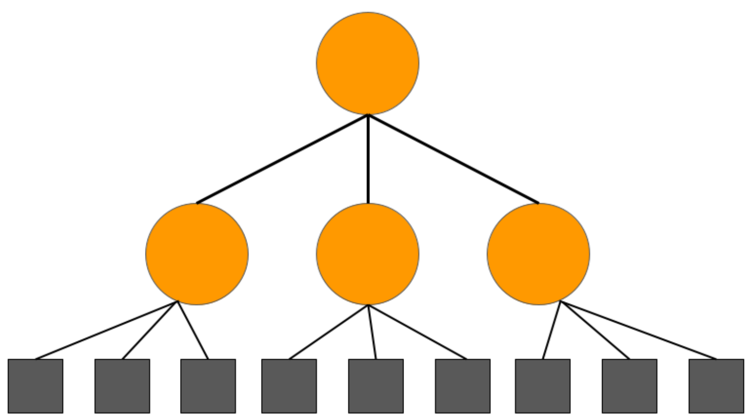
I start with the definition of a competitive equilibrium in this economy.

**Definition 1.1** (Equilibrium)**.** A competitive equilibrium consists of (i) an allocation of agents between workers and managers, (ii) a set of machines in the market, (iii) the number of layers $L \in \{2, 3\}$ in organizations, (iv) wage functions, (v) a mapping from managers to workers such that

1. agents maximize their utility given the wages, the assignment function, and threshold skill levels;



(a) When machines compete with workers



(b) When machines augment workers

Figure 1.2: Organizational structure

*Note:* Upper circles are managers, lower circles are workers, and squares are machines.

2. markets clear for all skill levels.

Since there are no market imperfections in the economy, a competitive equilibrium is Pareto optimal. Therefore, to obtain the decentralized allocation, it suffices to solve the planner's problem.

The planner matches workers and managers to maximize the total output of the economy. The following lemma shows that each manager hires workers of only one skill level.

**Lemma 1.2.** *Each manager supervises workers of only one skill level.*

*Proof.* See Appendix A.1.1. □

The intuition behind the proof is that the planner exploits supermodularity to efficiently allocate the time of agents. In a decentralized equilibrium, wages adjust to support the first-best allocation. Thus, I focus on the manager's problems stated in Section 1.2.1 throughout the paper.

To characterize an equilibrium of this economy, I begin with the labor market clearing conditions. In equilibrium, the labor markets must clear for all skill levels with the supply and demand for workers equalize.

Suppose there are two layers and consider the assignment of managers on an interval $[x_2, x_2 + dx_2]$ to workers on $[x_1, x_1 + dx_1]$. Then the labor market clearing condition requires

$$[f(x_1) + g(x_1)]dx_1 = n_1 f(x_2)dx_2$$

where the left-hand side is the supply of workers as a sum of agents and machines. The right-hand side is the demand for workers, which is the demand from each manager $n_1(x_2)$ multiplied by the number of managers $f(x_2)dx_2$. The demand of manager for workers is $n_1 = 1/h(1 - x_1)$, and thus the condition can be written as

$$f(x_1) + g(x_1) = \frac{1}{h(1 - x_1)} f(x_2) \frac{dx_2}{dx_1}$$

Denote the equilibrium relationship between $x_1$ and $x_2$ by the assignment function $x_2 =$

$a(x_1)$. Then the labor market clearing condition, or the assignment equation, is

$$f(x_1) + g(x_1) = \frac{1}{h(1-x_1)} f(a(x_1))a'(x_1) \tag{1.3}$$

Equation (1.3) is a differential equation that pins down the equilibrium assignment function $a(\cdot)$ together with a boundary condition. Notice that the slope of the assignment function is increasing in the supply of workers. Thus, the addition of machines, or a new pool of workers in general, makes the assignment steeper. Intuitively, a steeper assignment function implies that slightly better workers are supervised by much better managers than worse workers. The addition of machines, therefore, increases the difference of workers in their managers' skill levels.

A similar logic applies to the case with three layers. The difference is that the equilibrium assignment function is defined over two separate intervals. Suppose machines on $[x_0, x_0+dx_0]$ and workers on $[x_1, x_1 + dx_1]$ are matched with managers on $[x_2, x_2 + dx_2]$. Then

$$g(x_0)dx_0 = n_1 n_0 f(x_2)dx_2$$
$$f(x_1)dx_1 = n_1 f(x_2)dx_1$$

where the left-hand side is the supply of machines and workers, and the right-hand side is the demand of managers. Again, substituting the span of control into the equations I have

$$g(x_0) = \frac{1}{h(1-x_1)} f(x_2)\frac{dx_2}{dx_0}$$
$$f(x_1) = \frac{1-x_0}{1-x_1} f(x_2)\frac{dx_2}{dx_1}$$

In this case, the assignment function $a(\cdot)$ maps the set of machines into the set of workers and maps the set of workers into the set of managers. Then the market clearing conditions are

$$g(x_0) = \frac{1}{h(1-a(x_0))} f(a(a(x_0)))a'(a(x_0))a'(x_0)$$
$$f(a(x_0)) = \frac{1-x_0}{1-a(x_0)} f(a(a(x_0)))a'(a(x_0))$$

By dividing the first equation with the second and substituting $x_1 = a(x_0)$, the final expression for the labor market clearing conditions is

$$g(x_0) = \frac{1}{h(1-x_0)} f(a(x_0))a'(x_0)$$
$$f(x_1) = \frac{1-a^{-1}(x_1)}{1-x_1} f(a(x_1))a'(x_1)$$

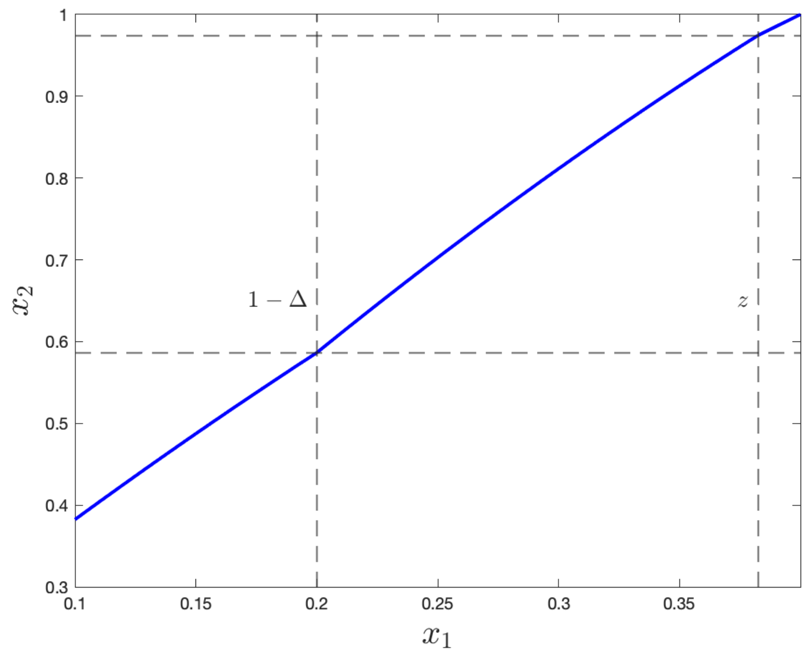
Note that the above conditions are assignment equations for adjacent layers. The same conditions can be derived with the matching between machines and workers, and workers and managers.

The equilibrium allocation exhibits positive sorting since the slope of the assignment function is always strictly positive. To gain intuition, consider the optimal allocation of this economy. Due to the supermodularity of the production function, the planner wants highly productive managers to be matched with highly productive workers. And in fact, this is also true in the decentralized equilibrium since it coincides with the first-best allocation.[12] Note also that the sorting is strictly positive and so the assignment function is one-to-one as well as onto. That is, it is not optimal for a manager to supervise an interval of workers because it is more efficient to spend all available time only on the most skilled workers.

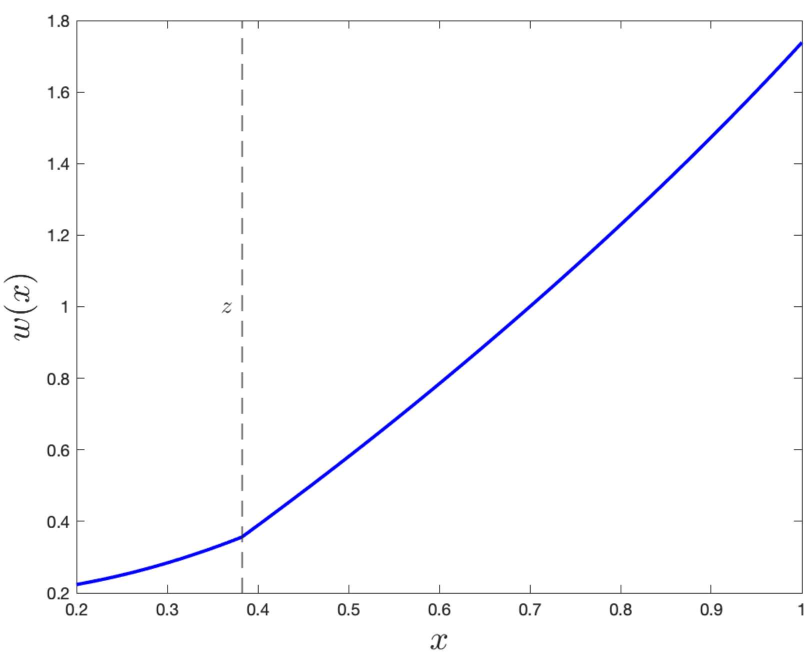
Also, note that the assignment function is concave. The intuition for concavity is that more skilled managers supervise an increasingly larger number of workers and thus more skilled workers are supervised by relatively similar managers. Put differently, the inverse assignment function, which maps managers into workers, is convex. Convexity implies that the difference in worker skill is larger for more skilled managers. The reason is that more skilled managers supervise a large number of workers and slightly less productive managers can only supervise much less productive workers that are left for them.

Wages are such that support the equilibrium assignment of agents. Thus, equilibrium wage functions are determined by the first-order conditions of the manager's problem. In

[12] As in Garicano and Rossi-Hansberg (2006), the complementarities are reflected in the wages and the markets are complete. Thus the decentralized equilibrium is efficient.




(a) Assignment function




(b) Wage function

Figure 1.3: Equilibrium assignment and wages

the case with two layers, the first-order condition to (1.1) is

$$w_1'(x_1) = \frac{x_2 - w_1(x_1)}{1 - x_1} \tag{1.4}$$

The left-hand side is the marginal cost of hiring slightly more skilled workers, which is higher wages paid to the workers. The right-hand side is the benefit, which is the gains in output as a result of expanding the span of control.

Likewise, the first-order conditions determine the equilibrium wage functions in the three layer case as well. By taking the derivatives with respect to $x_0$ and $x_1$, I obtain

$$\begin{aligned}
&[x_1] \quad \frac{x_2}{h(1-x_1)^2} = \frac{1-x_0}{1-x_1} w_1'(x_1) + \frac{1-x_0}{(1-x_1)^2} w_1(x_1) + \frac{w_0(x_0)}{h(1-x_1)^2} \\
&[x_0] \quad \frac{w_1(x_1)}{1-x_1} = \frac{w_0'(x_0)}{h(1-x_1)}
\end{aligned}$$

The left-hand side of the first condition tells that, by hiring slightly more skilled workers, the manager can increase output. As the right-hand side shows, costs increase as well because (i) more skilled workers require higher wages, (ii) the manager can hire more workers, and (iii) more skilled workers supervise more machines.

In the second condition, the benefit of hiring slightly more productive machines is the reduction in the compensation paid to the workers because more productive machines makes each worker more productive and thus the manager may reduce the total number of workers. The cost is again higher wages paid to machines.

Convexity of the wage function reflects the superstar effect in this economy. The income difference between adjacent managers is larger for more skilled managers. Thus, near the upper tail of the income distribution, slightly more skilled managers earn significantly more than less skilled managers. The focus of the comparative static analysis in Section 1.3 is how this convexity, and thus the superstar effect, depends on technological change captured by an increase in $\phi$.

As shown by existing work, such as Garicano and Rossi-Hansberg (2006), the set of agents in each layer is connected in equilibrium. Thus, the allocation of agents is summarized by a threshold skill level $z$. The following lemma shows that it is indeed the case in the current economy as well.

**Lemma 1.3** (Occupational Choice)**.** *For some $z$, agents with skill levels higher than $z$ become managers, and those below become workers. Agents exactly at $z$ are indifferent.*

*Proof.* See Appendix A.1.2. □

The threshold $z$ is an outcome of occupational choice, which divides workers and managers in equilibrium. With the addition of machines, there is another threshold that characterize the stock of machines available in equilibrium. The following lemma shows that only sufficiently productive machine owners enter the market for problems due to the presence of an entry cost.

**Lemma 1.4** (Entry Threshold of Machines)**.** *Suppose there is a sufficient amount of machines. Then, for some $\theta^* \geq \theta$, only machine owners above $\theta^*$ enter the market and those below the threshold do not enter. Machine owners exactly at $\theta^*$ are indifferent.*

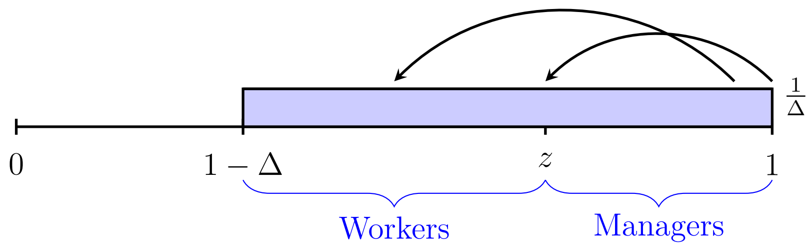
*Proof.* See Appendix A.1.3. □

Lemmas 1.3 and 1.4 imply that the sets of workers and machines that participate in the labor market are $[1 - \Delta, z]$ and $[\theta^*, \theta + \phi]$. Note that if $\theta^* > \theta$ then the entry condition is binding and thus there are fewer machines in the market than the total amount available.[13]

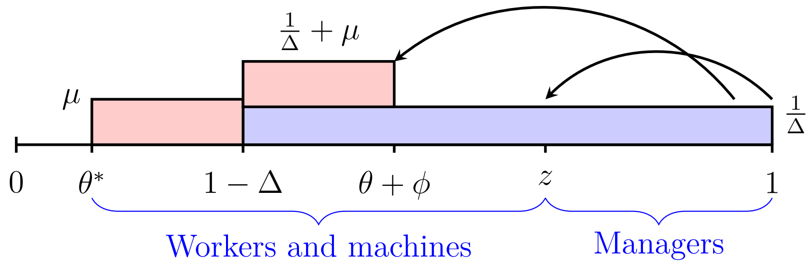
Figure 1.4 illustrates the assignment of agents and machines given the thresholds. Panel (a) is the assignment of workers and managers, which corresponds to the basic model of production hierarchies developed in, for example, Garicano and Rossi-Hansberg (2006). The arrow from 1 to $z$ is the matching between the most skilled managers and workers. The second arrow is the matching between less skilled managers and workers.

Panel (b) includes machines, which are indicated by the additional rectangles in red. The density of machines is $\mu$ and thus the density of all workers and machines on the overlapping region is $1/\Delta + \mu$. Therefore, workers on this region face a greater competition for managers. Note that in the figure, the most skilled managers still supervise the most skilled workers at $z$.

[13] The reason for an entry cost is technical. Since machines do not choose occupations, there needs to be a mechanism that ensures that the supply of and demand for machines are met in equilibrium, which is what $\theta^*$ does. If $\theta$ is sufficiently high then the entry cost becomes small relative to the wages that machines receive and so $\theta^* = \theta$.




(a) Agents only




(b) Agents and machines

Figure 1.4: Allocation of agents and machines

*Note:* Agents are blue and machines are red.

Figure 1.3 shows an equilibrium assignment and wage functions. Panel (a) is the assignment function for an economy where all organizations have two layers. The figure shows a mapping from the lower layer consisting of workers and machines to the upper layer of managers. Note that the mapping is defined piecewise over the intervals $[\theta, 1-\Delta]$, $[1-\Delta, z]$, and $[z, \theta+\phi]$, on which the function is concave.

Panel (b) of Figure 1.3 shows the wage function against skill levels. The wage function is continuous at $z$ because of the indifference condition for between workers and managers. Moreover, the wage function is convex for both workers and managers. The reason is the supermodularity of the production function. The marginal product of workers is increasing in skill level, which justifies the convexity of workers' wages. Also, managers' wages are convex because the output function is convex in the manager's skill level. Intuitively, more skilled managers are able to expand their span of control to a greater extent by hiring more skilled workers and saving their time spent on each worker. Since more skilled managers supervise a larger number of workers, even a small increase in workers' skill turns into a large gain in terms of the span of control. And the equilibrium wage function supports the allocation exhibiting positive sorting.

### Existence and Uniqueness

I focus on the cases where all organizations have the same structure, i.e. the same number of layers, in equilibrium. In particular, I restrict my attention to the cases where (i) all organizations have two layers with machines and workers both located in the bottom layer and (ii) all organizations have three layers with only machines in the bottom layer below workers. In each of these cases, machines either complement workers or compete with them for managers as different layers are complementary to each other. Thus, agents who are in separate layers from machines are complemented, linked by the assignment function. On the other hand, if machines are in the same layer as workers, then they raise the supply of workers that managers face and increase the competition for managers.

The following propositions show the existence and uniqueness of an equilibrium that supports two or three layers. Moreover, the results reveals how the supervision cost $h$ and the overall technology level $\theta$ determine the equilibrium organizational structure.

**Proposition 1.5** (Equilibrium with Two Layers)**.** *The economy has a unique equilibrium where all orgaziations have two layers with workers and machines in the bottom layer if all machines are sufficiently productive and the supervision cost $h$ falls into some interval $I_h^2 \subset [0, 1]$.*

*Proof.* See Appendix A.1.4. □

The proof of Proposition 1.5 shows that, in order for all organizations to have two layers in equilibrium, the level of technology must be sufficiently high so that managers directly supervise machines. Otherwise, managers may have an incentive to switch to an organization with three layers to delegate the supervision of less skilled workers/machines to more skilled workers.

On the contrary, if machines have sufficiently low skill levels then they are located in the bottom layer below workers, and thus organizations have three layers. The following proposition proves the existence and uniqueness of such an equilibrium.

**Proposition 1.6** (Equilibrium with Three Layers)**.** *The economy has a unique equilibrium*

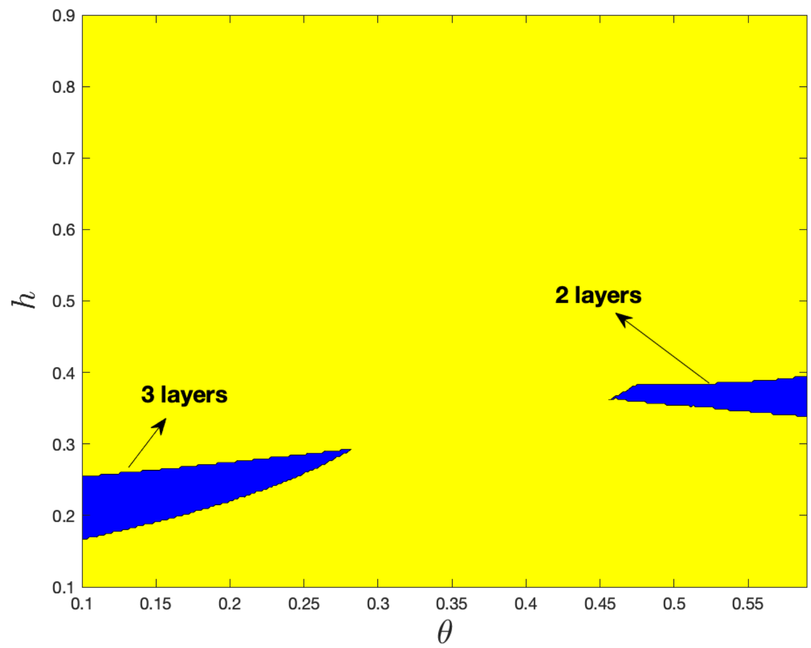



Figure 1.5: Equilibrium on the $(\theta, h)$ space

*where all orgaziations have three layers with only machines in the bottom layer supervised by workers if all machines have sufficiently low skill levels and the supervision cost $h$ falls into some interval $I_h^3 \subset [0, 1]$.*

*Proof.* See Appendix A.1.5. □

Unlike in the two-layer case, $\theta$ must be sufficiently low for organizations to have three layers. Otherwise, managers may have an incentive to hire machines directly or hire workers only because machines become expensive.

Figure 1.5 illustrates how changes in parameters $\theta$ and $h$ determine whether or not the allocation of interest is an actual equilibrium output. The blue region in the figure is where one of the two allocations described in Section 1.2.2 is an equilibrium: (i) two layers with machines and workers in the lower layer, and (ii) three layers with machines in the bottom and workers in the middle layer. The yellow region is where neither of the allocations is an equilibrium because there are some agents who can profitably deviate from the allocations.[14]

[14] Also, it is possible that other types of organizations arise in equilibrium. Although it may be interesting to explore various possibilities for the relationship between the parameters and organizations, in this paper I focus on the blue region and the corresponding allocations.

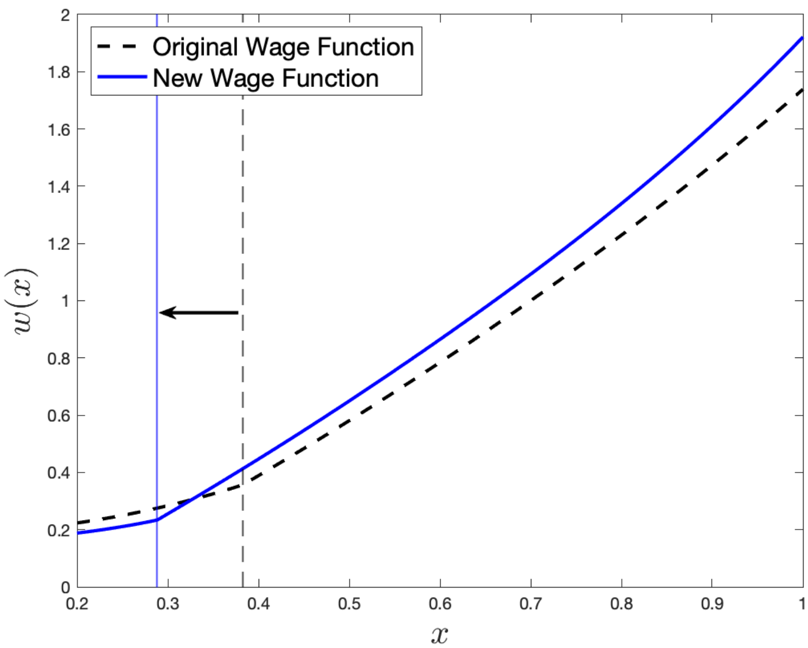
The blue region is separated into two parts. The right part indicates where the allocation with two layers is an equilibrium. To gain intuition, it is worth noting that the right part of the blue region is the intersection of the region where agents do not have the incentive to switch to three layers and the region where there is no self-employment. If the supervision cost $h$ is too low then supervision becomes less costly and thus managers have the incentive to form organizations with three layers. On the other hand, if $h$ is too high then forming organizations becomes too costly and it is more profitable to be self-employed instead of spending time on communication in organizations. Thus, the allocation in Proposition 1.5 requires $h$ to be in the "goldilocks" zone that rules out both incentives to deviate. Moreover, the equilibrium requires technology level $\theta$ to be sufficiently high given the heterogeneity in machines $\phi$. Otherwise, the most skilled machines are not matched with the most skilled managers, which is needed for the results in Section 1.3. Lastly, the right part of the blue region ends at lower values of $\theta$ (around 0.45 in the figure). This is because managers can reduce compensation for workers by delegating the easiest problems to machines with low skill levels.

The left part is where three layers are an equilibrium outcome. Again, the equilibrium is in the goldilocks zone where $h$ is neither too high or too low. If $h$ is too high, then it is not efficient to maintain three layers because communication becomes costly. Instead, managers may profitably deviate by switching to two layers. On the other hand, if $h$ is too low, then workers become more efficient in supervising machines and so the demand for machines is too high compared to the existing stock.

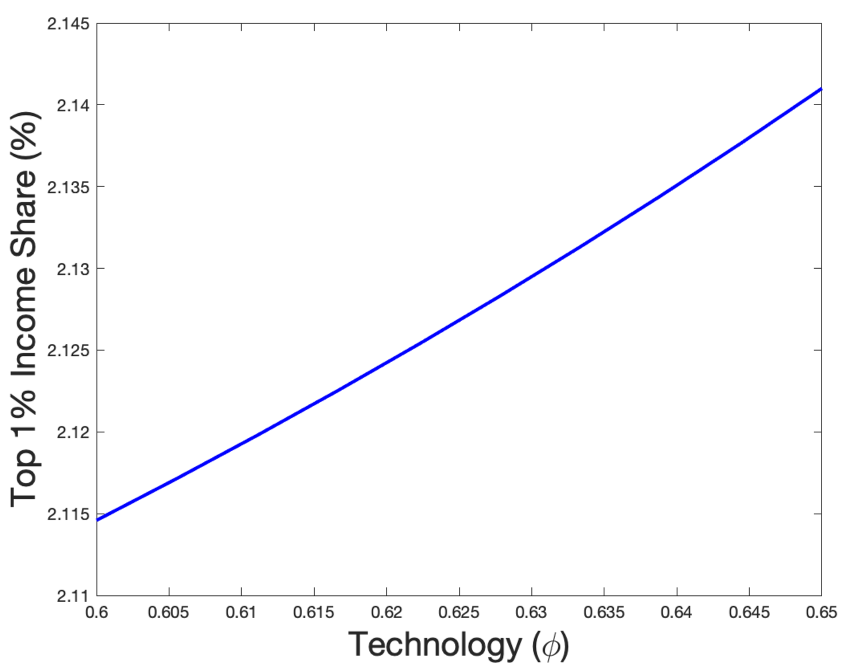
## 1.3 Technological Change and Top Incomes

A key takeaway from the previous section is that the complementarity among managers, workers, and machines depends on the organizational structure determined by the level of technology. This subsection examines how technological change may have distinct effects on top incomes depending on whether machines complement workers or managers.

I define technological change as an increase in $\phi$, which expands the interval of machines $[\theta, \theta + \phi]$. Technological change introduces new machines that can solve more difficult prob-




(a) Changes in wages




(b) Top 1% income share and advances in machines

Figure 1.6: When machines compete with workers

lems than the existing ones. If workers are in a different layer than machines, workers may benefit from the complementarity in the production technology. On the other hand, if workers are in the same layer as machines, then it may be managers who benefit from technological change while workers experience falling wages due to the increased competition for managers. The next two subsections show that this is indeed the case.

### 1.3.1 When Machines Compete with Workers

For the comparative static analysis, I focus on the parameter space to which Proposition 1.5 applies. Furthermore, I am interested in the allocation where the most advanced machines are matched with the most skilled managers. This is true if $\theta + \phi > z$. That is, if the most advanced machines can solve more difficult problems than any worker, then the most skilled managers employ the most advanced machines due to positive sorting.

In the allocation with two layers, $z$ admits a closed-form expression. By observing the equilibrium threshold $z$ and the inequality, it follows that $\theta + \phi$ must be sufficiently high given the other parameters. Thus, I impose the following assumption on the parameters $\theta$ and $\phi$. The maximum complexity of machines is sufficiently high.

Since machines cannot supervise others, all machines including those on $[z, \theta + \phi]$ are supervised by managers. In particular, the most advanced machines are supervised by the most skilled managers as a result of positive sorting.

Now suppose technological change leads to a rise in $\phi$ by $d\phi$. A rise in $\phi$ introduces new machines that have higher skill levels than the existing ones by extension of the interval $[\theta + \phi, \theta + \phi + d\phi]$. As a result of the introduction of new machines $[\theta + \phi, \theta + \phi + d\phi]$, managers may now hire better subordinates while workers are matched with worse managers than before due to greater competition for managers. Thus, technological change is skill-biased in the sense that managers' wages rise but workers' wages generally fall, possibly except for those near the threshold $z$.[15] The following proposition shows the distributional effects of technology between managers and workers.

**Proposition 1.7** (Skill-biased Technological Change)**.** *Suppose organizations have two layers with managers supervising workers and machines directly. Also, machines are sufficiently*

[15] Allowing for $z$ to adjust in equilibrium, the marginal workers may earn more by switching into managers.

*complex, satisfying Assumption 1.3.1. Then, greater complexity of machines increases managers' wages but reduces workers' wages.*

*Proof.* See Appendix A.1.6. □

Proposition 1.7 follows because machines complement managers but compete with workers. Thus, new machines provide a larger and improved stock of workers from which managers can choose from and increase competition among workers at the same time.

In addition to the distributional effect between skill groups, the gains are heterogeneous among managers. Specifically, the setup, and models of hierarchies more broadly, produces a superstar effect that is reflected in the convex wage functions. The magnitude of this superstar effect depends on the technology, creating heterogeneous effects even among managers.

**Proposition 1.8** (Cascading Effects of Technological Change)**.** *As machines become more complex, wages rise more for more skilled managers.*

*Proof.* See Appendix A.1.7. □

The intuition is that the introduction of more productive machines increases the superstar effect through the supermodularity in the production function. Supermodularity matters for positive sorting between managers and workers. Since the most skilled managers hire the best machines, they experience the largest gain in their span of control through technological change, which is reflected in their wage increases.

A corollary of Proposition 1.8 is that technological change increases the income share of top earners.

**Corollary 1.9** (Rising Top Income Shares)**.** *Let* $p \in [0, 1]$ *indicate the top earners on the interval* $[1 - p, 1]$*. Then top income shares increase with* $\phi$ *for sufficiently small* $p$*.*

*Proof.* See Appendix A.1.8. □

Corollary 1.9 shows that the model can potentially account for the rise in top labor income shares in the US during the past four decades. This is the period when new technologies, such as personal computers and industrial robots spread across the economy. The result suggests that more advanced technologies favor superstars and lead to greater concentration of income.

### 1.3.2 When Machines Augment Workers

Propositions 1.7 and 1.8 extend the results in the literature on skill-biased technological change and automation, which show that skilled (or non-routine) workers benefit more from technological change than unskilled (or routine) workers. Specifically, prop:sbtc establishes a link between gains from technological change and skill levels. As in the existing work, the high-skilled group (that is, managers) gains from technological change while the low-skilled group (that is, workers) loses. Proposition 1.8 goes beyond the existing results and shows that the gains from technological change are increasing in skill level and thus concentrated at the most skilled agents within the high-skilled group.

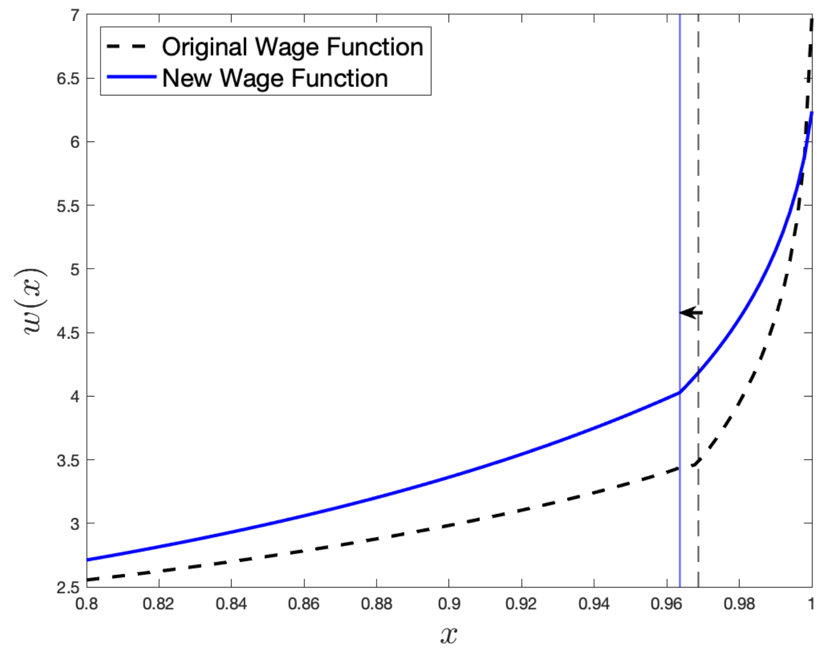
Another aspect of the current model that extends the previous work is the role of technology level. As the beginning of this section alludes to, the model may have different implications for top income if machines are not advanced enough to replace workers. As Proposition 1.6 states, machines complement workers if the level of technology is low, and it turns out that technological change reduces income concentration at the top.

**Proposition 1.10** (Effects on Top Income)**.** *Technological change reduces top income shares.*

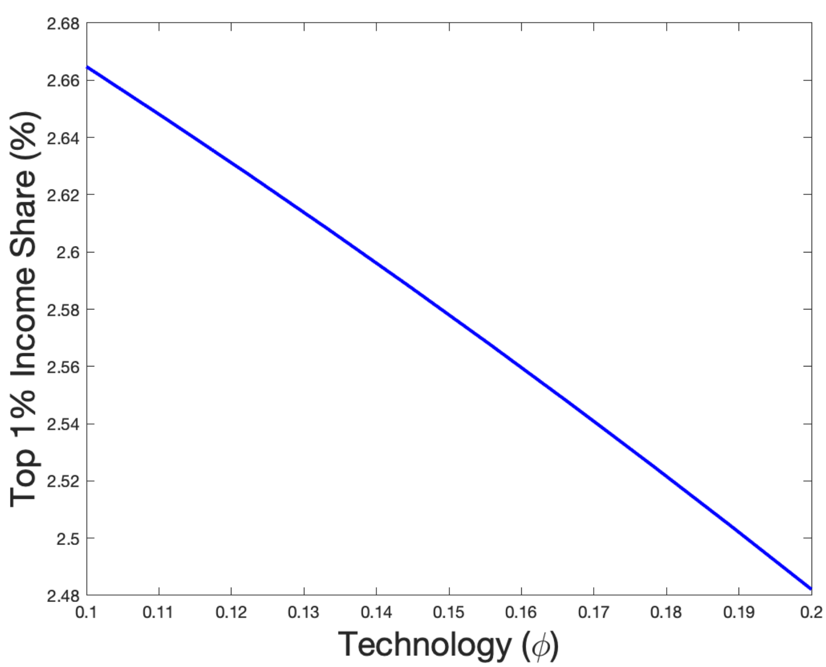
*Proof.* See Appendix A.1.9. □

The proof of the result shows that technological change increases workers' wages and thus distributes the output of an organization in favor of the workers. As technological change introduces more productive machines, workers supervise more advanced machines and solve more problems. On the contrary, managers do not directly benefit from technological change because their subordinates remain the same with the occupational threshold fixed. Since the span of control, and thus the skill level of subordinates, is what matters for the equilibrium wages, managers have only limited gains. Moreover, in this setup, technological change is an equalizing force among managers. That is, less skilled managers gain more from technological change than more skilled managers.

Along with Proposition 1.7, the above result shows that technological change can have opposite effects depending on the level of technology. If machines can only solve the easiest problems and thus are able to assist workers but not managers, workers gain the most from technological change. However, the effect can reverse when technology is sufficiently




(a) Changes in wages




(b) Top 1% income share and advances in machines

Figure 1.7: When machines compete with workers

advanced so that machines become capable of most or all of the tasks assigned to workers.

Figure 1.7 illustrates the effects of technological change with three layers. Panel (a) shows the changes in the wage function. As the solid curve indicates the most skilled workers gain the most. On the other hand, the most skilled managers experience a fall in their wages.

To understand the mechanism, observe that technological advances allow workers to supervise more complex machines than before and process more problems. In other words, workers increase their span of control by supervising a larger mass of machines that are also more complex. Therefore, each worker now passes on more unsolved problems to their managers, which increases the demand for managers. In the new equilibrium, this requires an increase in the mass of managers through a lower value of the threshold $z$ indicated by the left arrow in panel (a).

The positive sorting between machines and workers explains the result that the most skilled workers gain the most from technological advances. On the other hand, changes in managers' wages, especially the most skilled managers, are explained by workers' wage increases that distributes output of the organization in favor of the workers, output losses through the equilibrium effect of lower $z$ leading to the supervision of less skilled workers, and the changes in machines' compensation.

In particular, the equilibrium effect through the threshold $z$ mainly affects the most skilled managers because they supervise less skilled workers in the new equilibrium. This can significantly reduce the wages of the most skilled managers since the amount of output tends to be more sensitive to workers' productivity run by more skilled managers. Indeed, Figure 1.8 shows that the output losses due to the matching with less skilled workers drive the decline in the wages of the most skilled managers. The blue solid curve is the total wage changes, which consists of three terms reflecting the changes in the output of the organization (dotted), total compensation to workers (dashed), and total compensation to machines (dash-dotted). Interestingly, the effect of the increases in workers' wages is offset by the equilibrium effect through the threshold as managers supervise less skilled workers who are paid less.

Overall, technological change acts as an equalizing force that reduces income concentration. Panel (b) in Figure 1.7 shows that top 1% income share is a decreasing function of $\phi$,

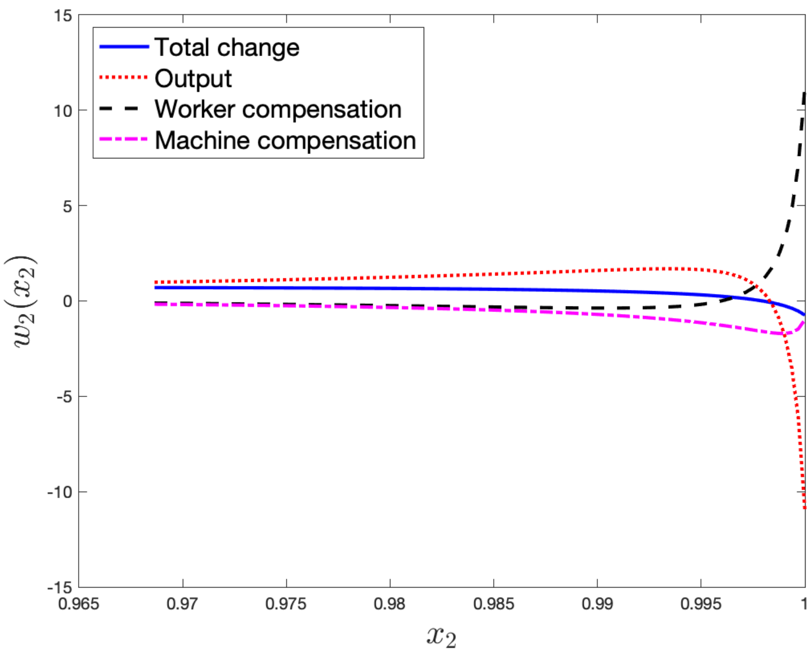



Figure 1.8: Decomposition of changes in managers' wages

unlike with complex machines in Section 1.3.1.

### 1.3.3 Discussion on the Trend in Top Income Inequality and Technology

The model provides a potential explanation for the diverging trend in top income shares around the early 1980s in the U.S. First, note that top income shares were declining during the mid-20th century before increasing abruptly since the 1980s. This was when machines were relatively rudimentary in terms of the tasks they could perform. For example, until the late 1970s industrial robots had limited applications to a relatively narrow range of tasks (Gasparetto, Scalera, et al. 2019).

But since the 1980s industrial robots gained more flexibility and became capable of performing significantly more advanced computations than before.[16] The 1980s is also when robots spread across various sectors outside of the automotive industry. In the language of the model of this paper, machines have become comparable to workers in terms of the

[16] As Gasparetto, Scalera, et al. (2019) write, the 1980s was "the time when the robots became even more versatile, by exploiting important improvements both with respect to the hardware and the software."

complexity of problems they can solve as in Section 1.3.1. One of the main contributions of the model is to generate this non-monotonic relationship between technological change and income concentration.

Relatedly, evidence suggests that technological advances have contributed to the rise of high-income professional workers such as lawyers and investors, allowing them to operate at a greater scale (Kaplan and Rauh 2010). This is consistent with the results in this section since more complex machines increase the size of the organizations that the most skilled managers supervise.

## 1.4 Algorithmic Management and Income Distribution

The analysis so far assumes that machines are only capable of production tasks, that is, producing problems. Specifically, I have intentionally restricted the attention to whether machines replace workers or not. So machines in the previous setup are "narrow" in the sense that they are only suitable for a small set of relatively simple tasks.

However, recent advances in AI have allowed machines to be capable of a broader range of tasks, including those related to management. For example, the latest vision processing technology allows automation of inventory management. The application of AI in recruitment is a relevant example for human resources.

By extrapolating from these latest advances, one can imagine a scenario where machines become fully capable of running organizations. In this section, I ask: What are the implications of algorithmic management, or "machine managers," for income distribution, especially top incomes? Will future AI systems have qualitatively different effects on income inequality compared to previous automation technologies? I answer these questions by modifying the model so that machines can substitute for managers and have a lower supervision cost than managers.

### 1.4.1 Modifications

Denote the supervision cost of machine managers by $h_m$, which is potentially different from $h$. In particular, I am interested in the case where $h_m < h$. This implies that machines are more efficient in supervising than managers and thus are capable of forming larger organizations. Nonetheless, machines are still equally efficient in production as workers. In other words, machines have comparative advantage in supervision.

By considering a lower supervision cost of machines, I can examine the effects of technological change on knowledge workers. As in Garicano (2000), the model can be interpreted as follows. Workers specialize in production that requires physical activities. Unlike managers, workers provide mainly the physical resources that are less affected by technologies for cognitive tasks.[17] On the other hand, managers specialize in providing knowledge that complements these physical activities. Thus, technological change in this section can be interpreted as advances in cognitive automation.

### 1.4.2 Equilibrium with Algorithmic Management

I proceed as in Section 1.3.1 and focus on the equilibrium in which organizations have two layers. Agents solve the same problems as before. Machine managers solve

$$w_2(x_2; h_m) = \max_{x_1} \frac{x_2 - w_1(x_1)}{h_m(1 - x_1)}.$$

Note that the wage function $w_2(\cdot)$ of machines is different from that of human managers because of $h_m$. However, the equilibrium wage function $w_1(\cdot)$ of workers does not directly depend on $h_m$ because the first-order condition is the same as (1.4). Thus, given a wage schedule, both managers and machines with the same skill level hire the same workers because they have the same first-order conditions.

Note that any dependence of $w_1$ on $h$ and $h_m$ is through the assignment function. For any value $x_1$ on $[1 - \Delta, z]$, the assignment equation becomes

$$f(x_1) + g(x_1) = \left[\frac{1}{h(1 - x_1)} \cdot f\left(a(x_1)\right) + \frac{1}{h_m(1 - x_1)} \cdot g(a(x_1))\right] a'(x_1) \tag{1.5}$$

As before, the left-hand side is the demand for workers by managers. The right-hand side is the supply of workers. Generally, both the supply and demand are a sum of agents and

[17]For example, even though GPS can perfectly find routes, Uber drivers still need to drive cars to the destination.

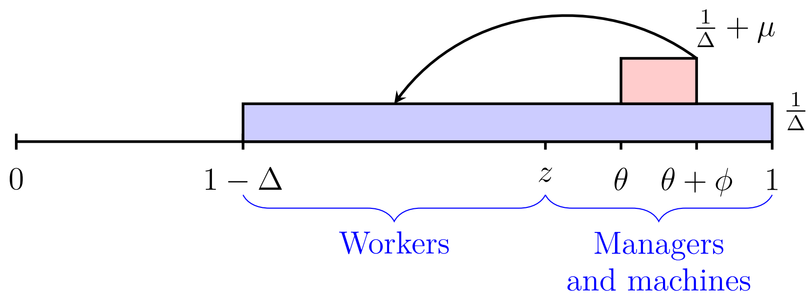



Figure 1.9: Assignment of machine managers

machines.

If $h_m = h$ then the assignment function takes the standard form. Also, there exists an occupational threshold $z$ that divides workers and managers as before. Machines are equivalent to workers or managers with the same skill level depending on whether they are below or above the threshold.

Figure 1.9 illustrates how machine managers affect the allocation of workers and managers. Unlike in Figure 1.4, machines now *reduce* the competition among workers for managers as the density on the interval $[\theta, \theta + \phi]$ increases by the density of machines $\mu$. In other words, machines increase the supply of managers.

To isolate the effects of technological change through algorithmic management, I restrict the technology parameters so that all machines become managers in equilibrium.

**Lemma 1.11** (Machine Managers)**.** *Given the supervision cost $h_m$ of machines, if machines are sufficiently productive then all machines become managers in equilibrium.*

*Proof.* See Appendix A.1.10. □

Note that the set of workers is divided into three intervals if $z < \theta$: $[1 - \Delta, \underline{y}]$, $[\underline{y}, \bar{y}]$, and $[\bar{y}, z]$ where $1 - \Delta < \underline{y} < \bar{y} < z$. The first interval is the set of workers who are matched with the mangers below $\theta$. The second interval is the set of workers who are matched with machines and managers with equivalent skill levels. Lastly, the third interval is the set of workers who are matched with the managers above $\theta + \phi$. Note that the thresholds $\underline{y}$ and $\bar{y}$ depend on parameters $\theta$, $\phi$, and $h_m$.[18] The following lemma summarizes the segregation of workers.

[18]See Appendix B.1.4 for the assignment functions.

**Lemma 1.12** (Segregation of Workers)**.** *If machines are sufficiently productive then workers are segregated depending on their managers. That is, there exist threshold values* $\underline{y}$ *and* $\bar{y}$*, with* $\underline{y} < \bar{y}$*, such that (i) workers on* $[1-\Delta, \underline{y}]$ *are supervised by managers less productive than machines; (ii) workers on* $[\underline{y}, \bar{y}]$ *are supervised by machines; (iii) workers on* $[\bar{y}, z]$ *are supervised by managers more productive than machines.*

*Proof.* See Appendix A.1.11. □

Using the assignment function, I solve for the wage function using the first-order condition of managers. As in Section 1.2, the equilibrium wage function in this economy is continuous, monotonically increasing, and convex. Notably, machines earn more than managers because $h_m < h$ and they are matched with the same workers paying them the same wages if they have the same skill levels.

### 1.4.3 Distributional Effects of Machine Managers

To see the distributional effects of advances in machine managers, consider an increase in $\phi$ as before. First, advances in machines increase the mass of agents that become workers. To see this, note that new machines allow more workers to be supervised by machines both because there are now more machines in the economy and new machines supervise more workers than the existing ones. The demand for workers rises, leading to increases in workers' wages. As a result, the threshold skill level $z$ rises and the mass of managers decreases.

**Lemma 1.13.** *Technological change increases the occupational threshold* $z$*, and thus reduces the number of managers.*

*Proof.* See Appendix A.1.13. □

An increase in $\phi$ has two offsetting effects on managers' wages. The first effect is that more advanced machines increase the demand for workers and their wages. Thus, higher wages of workers reduce managers' wages as the total compensation for workers within organizations increases. On the other hand, the second effect is that, as $z$ increases, managers are matched with more skilled workers, and thus they can hire more skilled workers.

Figure 1.10 shows that the first effect generally dominates the second, and thus most

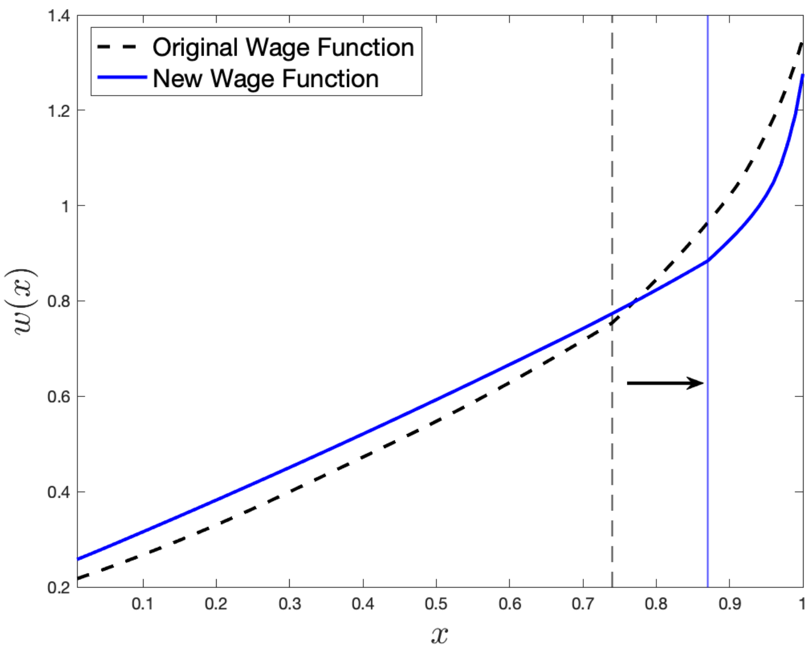
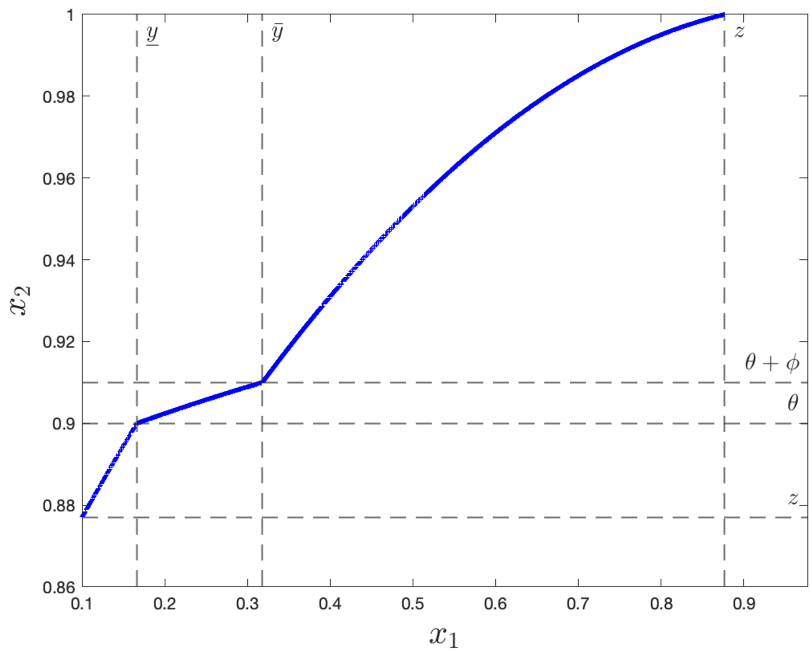



Figure 1.10: Wage functions when $h_m = h$

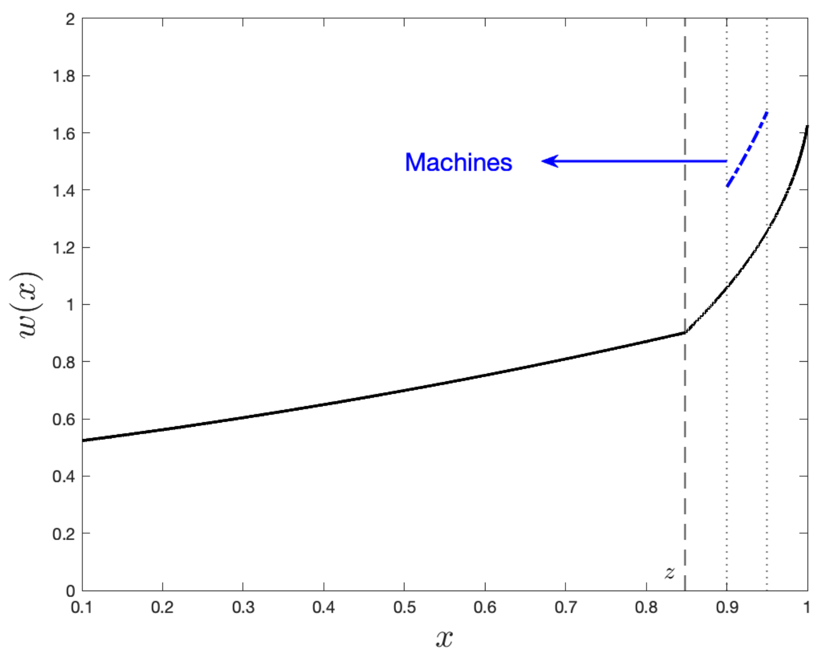
agents who were previously managers (on the right of the dashed vertical line) experience falling wages. The result is somewhat similar to that in Section 1.3.2. The main difference is how machines augment workers. In Section 1.3.2, machines augment workers "from below" because they solve easier problems and allow workers to focus on more complex ones. On the other hand, machines in this section augment workers "from above" because they solve more difficult problems than workers. Technological change makes workers more productive because more larger fractions of the problems they generate are solved.

### 1.4.4 Supervision Costs and the Nonrivalry of Machines

Machines in the current setup best represent softwares that automate complex managerial tasks. Unlike individual physical machines, AI systems can be deployed to multiple instances simultaneously. In particular, if computations are cheap, then it is possible to deploy AI systems at a large scale without affecting its performance for each instance (e.g. algorithmic management at ride-sharing companies). In an extreme case where computation costs are negligible, the AI systems become almost "nonrivalrous" as the inference costs do not limit




(a) Assignment




(b) Wages

Figure 1.11: Assignment and wages when $h_m < h$

the scale at which they are deployed.[19]

In the current model, $h_m$ represents the cost of computation for such tasks. As a starting point, I consider the case where $h_m$ is slightly lower than $h$, so that machines are more efficient in supervision than managers but the gap is not too large. In this case, the assignment function has kinks at the ends of the interval of machines as Figure 1.11a shows.

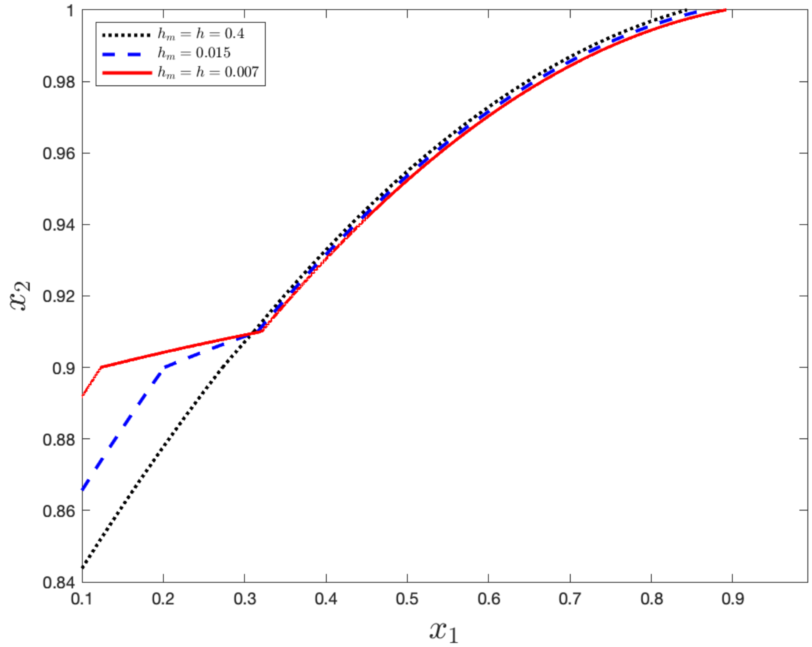
Figure 1.11a illustrates the assignment function in the case where $h_m < h$. The $x$-axis is the skill level of workers on $[1-\Delta, z]$. The $y$-axis is the skill level of managers on $[z, 1]$. The dashed vertical lines labeled $\underline{y}$ and $\bar{y}$ divide the workers into three groups as explained above. Notice that the assignment function is flatter on the middle region. Intuitively, workers with different skill levels are supervised by relatively similar managers (or machines). Because there are more managers on $[\theta, \theta + \phi]$ than other points on $[z, 1]$, there is less competition for managers among workers. The flat part of the figure implies that machines with $h_m < h$ allow a larger group of workers to be supervised by the managers on $[\theta, \theta + \phi]$. As $h_m$ falls the middle part of the assignment function becomes even flatter.

Figure 1.11b depicts the wage functions. The solid curve is the wages of workers and managers separated by the vertical dahsed line at $z$ on the $x$-axis. Machines earn wages according to the dash-dotted curve that is located strictly above the solid curve. Intuitively, machines have a lower cost of supervision which allows them to supervise a larger mass of workers than managers with the same skill levels. Therefore, machines earn a higher level of income.

What happens as $h_m$ declines further? Changes to the assignment function are illustrated in Figure 1.12a. The dotted curve is the assignment function with $h_m = h$, which is the case where machines and managers are perfectly substitutable. As $h_m$ falls, the curve on $[\underline{y}, \bar{y}]$ becomes flatter. Moreover, $\underline{y}$ falls as well, meaning that the set of workers supervised by the least skilled managers shrinks.

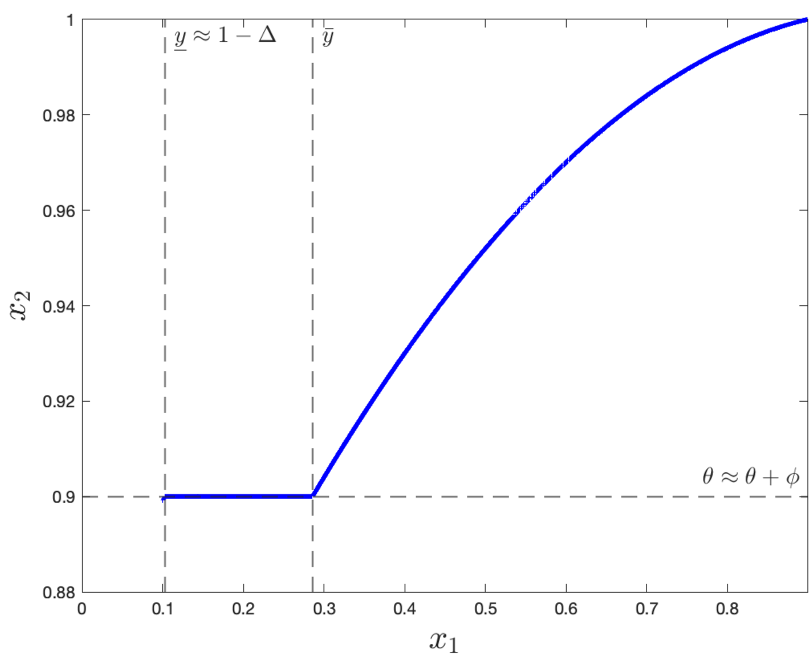
In the limit where $h_m$ approaches some very low value, workers are segregated into two groups: those supervised by machines (and managers on the overlapping region) and managers who solve more complex problems than machines. In this limit case, all workers below

[19] In addition to ride-sharing companies, another example is language models such as ChatGPT and Claude, and their limits on usage due to computational constraints.




(a) Falling $h_m$ and flattening assignment function

*Note:* The dotted curve is the case where $h_m = h$. As $h_m$ falls, the assignment function changes to the dashed curve and then the solid curve.




(b) Assignment function with nonrivalrous machines

Figure 1.12: Assignment function and the supervision cost $h_m$ of machines

$\bar{y}$ are supervised by machines. Moreover, the mass of workers supervised by machines is not constrained by the time endowment.

**Proposition 1.14** (Nonrivalry of Machines)**.** *Suppose workers are segregated as in Lemma 1.12. Then the occupational threshold $z$ increases as the supervision cost $h_m$ of machines declines. Moreover, for given values of $\theta$, $\phi$, and $\mu$, there exists $\underline{h}$ such that as $h_m$ approaches $\underline{h}$ from above, (i) $z \to \theta$ and (ii) $\underline{y} \to 1-\Delta$. In other words, for any positive mass of machines, all workers below some threshold are supervised by machines if the supervision cost $h_m$ is sufficiently low.*

*Proof.* See Appendix A.1.12. □

A key takeaway from Proposition 1.14 is that machines become nonrivalrous in the limit case as $h_m \to \underline{h}$. In other words, time is not scarce for machines any more. As a result, workers are segregated into two groups: the more skilled group, who are supervised by managers, and the less skilled group, who are supervised by machines.

Also, the result holds for arbitrarily small values of $\mu$ and $\phi$. As the proof of Proposition 1.14 suggests, even if there is a very small mass of machines, because of small values of $\mu$, the segregation result holds for sufficiently small $h_m$. Thus, in the limit case where $\phi \to 0$, $\mu \to 0$, and $h_m \to 0$, machines are nonrivalrous in the sense that the cost of the supervision of an additional problem is negligible. I interpret this case as a "single" software supervising a large number of workers.

Note that the segregation result applies as long as $\theta$ is sufficiently high. Thus, as $\theta$ approaches one, it is straightforward to imagine that the productive group shrinks because machines supervise more and more workers. The following result is a corollary of Proposition 1.14 that verifies this intuition.

**Corollary 1.15** (Nonrivalry and the Automation of Management)**.** *Suppose machines are nonrivalrous in the sense of Proposition 1.14. Then as $\theta$ approaches one all agents become workers.*

Corollary 1.15 allows for a speculation about the implications of advanced AI systems on organizational structure, occupational choice, and income distribution. As $\theta$ approaches one, machines become comparable to the most skilled managers in their ability to solve problems.

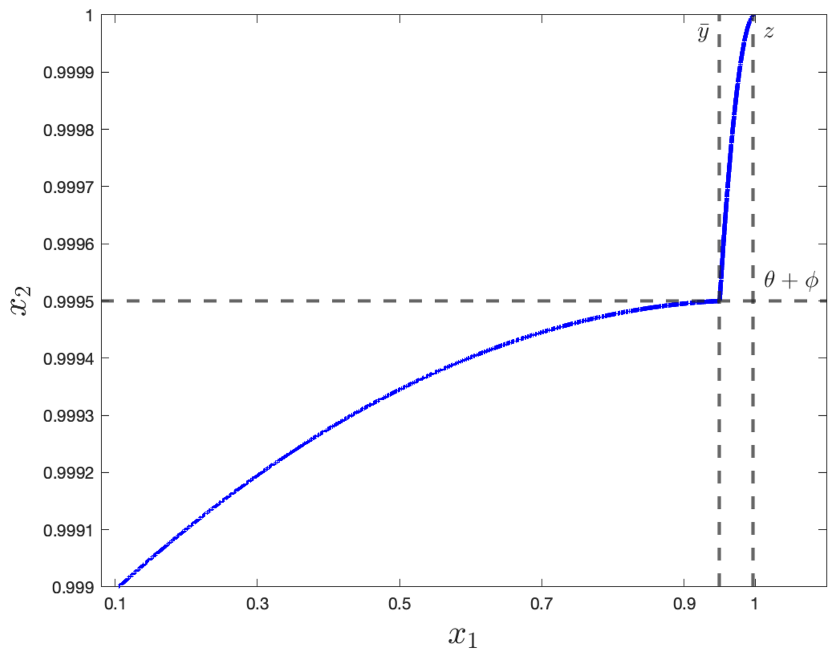



Figure 1.13: Automation of management

As a result, all workers benefit from switching their managers to the machines because of the complementarity between workers and managers. And the nonrivalry of machines implies that all workers are supervised by machines at the limit $h_m \to \underline{h}$.

Figure 1.13 illustrates how the automation of management affects the assignment function. Notice that, compared with Figure 1.12b, technology level $\theta$ is closer to one (at 0.999) and the heterogeneity among machines is very small ($\phi \approx 0$). Moreover, machines supervise all workers below $\bar{y}$, which is almost all workers in the economy. The remaining managers are those who are more skilled than machines (above $\theta + \phi = 0.9995$) and supervise workers on $[\bar{y}, z]$, which is much smaller than before. According to Corollary 1.15, as $\theta$ approaches one, the steep part of the assignment function on $[\bar{y}, z]$ collapses and the flatter part dominates.

The limit case illustrated in Figure 1.13 captures a world where all workers are supervised by an algorithm that outperforms any humans.[20] The model predicts that such algorithms have equalizing effects by erasing productivity differences between workers that arise from managers' skills. In other words, advanced algorithms spread the knowledge to less skilled workers, which only the most skilled workers and managers had. Therefore, the limit case

[20]Such an algorithm would be a superintelligence (Bostrom 2014b).

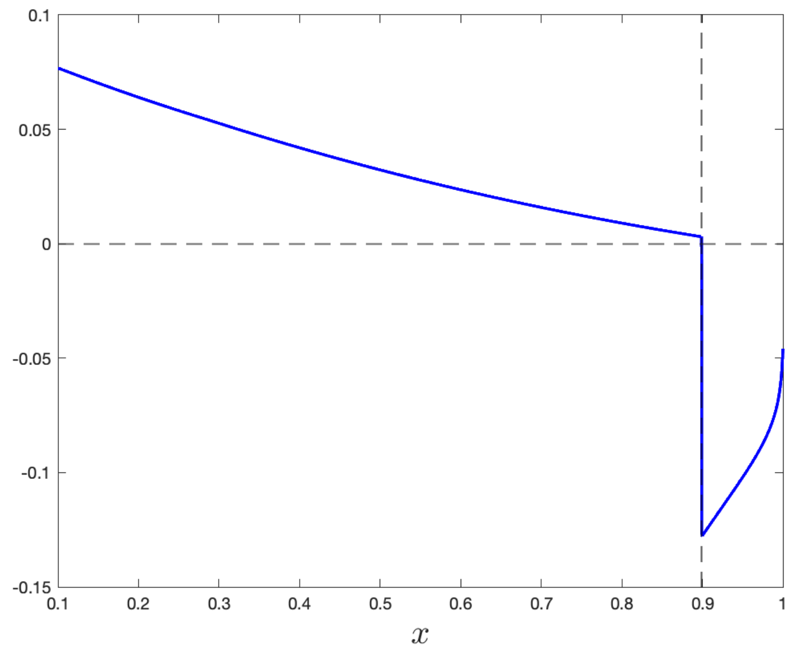



Figure 1.14: Normalized changes in wages ($\frac{\partial w}{\partial \phi}\frac{\phi}{w}$) as $\phi$ increases

suggests a possibility that a superintelligence significantly reduces labor market inequality by eliminating worker differences due to managerial quality.[21]

**Nonrivalrous Machines and Income Inequality** Nonrivalrous machines have similar effects on income distribution as in the case with $h_m = h$. Suppose $h_m \approx 0$ so that $\underline{y} \approx 1 - \Delta$ and $z \approx \theta$ as discussed in Proposition 1.14. Moreover, assume $\phi$ is small so that machines are on a very narrow interval.

Figure 1.14 illustrates how wages change at each skill level as machines advance. The vertical dashed line is the threshold $z$ and the blue solid curve is the changes in wages as the parameter $\phi$ increases by a small amount. As the figure shows, workers' wages rise due to the advances in machines, while managers experience falling wages. Note that it is the rise in the demand for the relatively less productive workers on $[\underline{y}, \bar{y}]$ that drives the overall upward shift in workers' wage function. Intuitively, increases in wages on $[\underline{y}, \bar{y}]$ lead to increases in wages on the other parts on $[1 - \Delta, z]$ due to the monotonicity of the wage function. Thus,

[21]However, this result may be relevant only in the short run as a software superintelligence is developed but constrained by physical actuators.

the gains from technological change trickles up from less skilled to more skilled workers.

In addition to falling income inequality between workers and managers, it turns out that technological change has different distributional implications among managers compared to workers. In particular, wages decline more for less skilled managers than more skilled managers. As a result, income inequality *among* managers rises. Intuitively, technological change has a first-order negative effect on managers' wages by increasing the compensation for workers. At the same time, managers face a larger supply of workers through the increase in $z$, which tends to offset the first effect. However, the most skilled managers are those who benefit from the second effect. More skilled managers gain increasingly more from this second effect. Thus, the equilibrium wage function $w_2(\cdot)$ becomes more convex.

It is worth noting that the results on falling income inequality among workers is consistent with early evidence on the effects of AI. Brynjolfsson, Li, and Raymond (2023) find that low-skilled workers gain more from AI in the context of call centers. Their interpretation is that AI spreads knowledge to low-skilled workers who have less experiences than high-skill workers and lack such knowledge. Noy and Zhang (2023) and Peng et al. (2023) find similar results in other settings and report equalizing effects of AI. Through the lens of the model, AI managers help workers by sending them the knowledge required for solving problems. More skilled workers do not gain as much because they need supervision for problems that AI managers cannot solve either.

## 1.5 Conclusions

The paper was partly motivated by the discussion on who will be augmented by AI. While AI could be mainly a continuation of previous automation technologies examined in the literature, early evidence suggests the opposite may happen. The results of the paper show that a crucial factor is the maximum complexity of automated tasks. In particular, the model highlights the vertical structure of production processes and how AI would fit into it.

On the concerns related to AI automation and income inequality, the model predicts the divergence in income distribution observed over the past four decades may persist. As AI workers grow increasingly useful, human workers whose skills have become more abundant will face a greater competition. On the other hand, those who can leverage their skills over other workers, either human or AI, are more likely to succeed.[22] The difference from the previous waves of automation is that the degree of income concentration can be greater.

The model also suggests a possibility that future AI systems reduce income concentration as they substitute for high-skilled workers. Thus, compared to AI workers, AI managers can have a positive effect on worker demand. It is possible that both AI workers and AI managers affect the labor market as technology advances. The net effect on income inequality depends on which of these forces dominates.

There are several avenues for future work. First, an unexplored channel in this paper is the implications of technology ownership for income and wealth inequality. As the results in Section 1.4 suggest, owners of technology can earn significantly higher shares of income as technology advances. As workers earn smaller shares of total income, the ownership of technology may become a major determinant of overall income inequality. Second, recent progress in AI presents potential for artificial general intelligence (AGI), or complete automation of tasks done by humans, which can have profound implications for economic growth and overall labor demand. **korinek2023scenarios** is one attempt to examine various possibilities brought by AGI. Lastly, it is important to understand the welfare effects of technological change and policy implications, which can be challenging because of the presence of market imperfections and limited policy tools.

[22]In a related context, "agency" may be particularly important as AI advances (Seetharaman and Wells 2023).

# Chapter 2

# Scenarios for the Transition to AGI

## 2.1 Introduction

Recent advances in AI promise significant productivity gains, but have also renewed fears about the displacement of labor. A growing number of both AI researchers and industry leaders suggest that it is time for humanity to prepare for the possibility that we may soon reach Artificial General Intelligence (AGI) – AI that can perform all cognitive tasks at human levels and thus automate them.[1] This raises a number of fundamental economic questions. What would the transition to AGI look like? What would AGI imply for output, wages, and ultimately human welfare? Would wages rise or collapse?

Our paper introduces an economic framework to think about these questions and evaluate alternative scenarios of technological progress that may culminate in AGI. Our starting assumption is that human work can be decomposed into unchanging atomistic tasks that differ in how complex they are. Advances in technology make ever more complex tasks amenable to automation. We capture this by assuming that there is a threshold of task complexity that can be automated at a given time, captured by an *automation index*. This index grows exogenously over time, in line with regularities such as Moore's Law. Although our results hold more broadly, we suggest that in the Age of AI, a natural measure of task complexity is the amount of compute (shorthand for computational resources) required for the execution of a task by machines. Some tasks, such as adding up numbers in a spreadsheet, can be performed with minimal computation. In contrast, others require a substantial amount of computation for machines, despite seeming natural and effortless

[1] This includes, for example, the CEOs of the three leading AGI labs, OpenAI's Sam Altman, Google Deepmind's Demis Hassabis, and Anthropic's Dario Amodei (Kruppa 2023; Time 2024). It also includes the world's most renowned AI researchers, for example two of the godfathers of deep learning, Geoffrey Hinton and Yoshua Bengio.

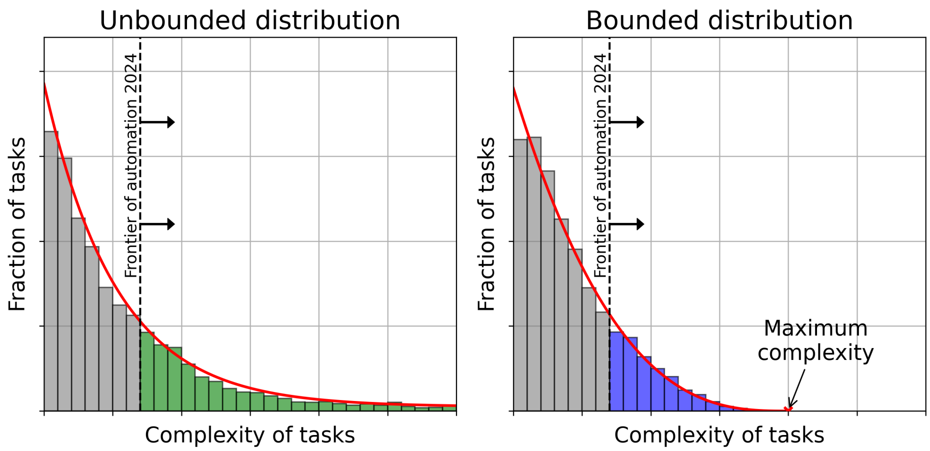



Figure 2.1: Unbounded and bounded distributions of tasks in complexity space

for humans, such as navigating a bipedal body over an uneven surface. We describe how tasks differ in computational complexity using a distribution function that captures tasks in complexity space or, referring to our preferred interpretation, *tasks in compute space*.[2]

Throughout the paper, we analyze two opposing cases for the distribution of tasks in complexity space, which result in sharply different economic outcomes. First, we consider the possibility that human tasks are of unbounded complexity, illustrated in the left-hand panel of Figure 2.1. In this case, advances in the automation index, illustrated by the rightward movement of the vertical "frontier of automation," imply that more and more tasks are automated over time, but that there always remain tasks and by extension jobs that cannot be automated. Second, we consider a bounded distribution of task complexity, which reflects that the computational capabilities of the human brain are finite, as discussed, e.g., in Carlsmith (2020). Bounded distributions result in full automation within finite time when the frontier of automation crosses the maximum complexity of tasks performed by humans. An alternative interpretation for tasks being to complex to automate is that society may choose not to automate certain tasks even when it is feasible to do so. This may apply, for example, to some of the tasks performed by priests, judges, or lawmakers.

We lay out an economic model in which atomistic tasks are gross complements that are

[2]Although the computational complexity of tasks is most evident for cognitive tasks, the automation of physical tasks is also greatly constrained by the computational complexity involved, as captured, e.g., by Moravec's paradox (Moravec 1988). We analyze an extension that explicitly accounts for cognitive and physical tasks in Section 4.

combined to produce final goods. In the spirit of Zeira (1998) and Acemoglu and Restrepo (2018) and Acemoglu and Restrepo (2022a), all tasks can be performed by labor, and automated tasks can be performed by either labor or capital. However, unlike in the described works, our main focus is on the edge cases that arise as we come close to full automation.

Our analysis begins by examining the equilibrium under fixed supplies of capital and labor. We show that automation can have dramatic impacts on wages and output even before it reaches all tasks. There exists a threshold level of the automation index that separates two distinct regions. As long as the index remains below the threshold, labor remains scarce relative to capital, and wages remain high. However, once the automation index surpasses the threshold, the economy enters a second region, where the scarcity of labor is alleviated, despite the presence of some tasks that still need to be performed by humans. In this region, labor and capital become perfect substitutes at the margin so wages decline starkly to equal the marginal product of capital. The economy exhibits behavior akin to an $AK$ model.

Next, we characterize the effects of automation on the economy's factor price frontier (FPF), which reflects all possible combinations of factor prices that may result from a given level of technology under different capital/labor ratios. The FPF provides general insights into the effects of automation that do not depend on specific assumptions on capital accumulation. We find that for a given level of automation, wages lie within a bounded interval that expands as the automation index rises – but only as long as automation is incomplete. Once all tasks are automated, the factor price frontier discontinuously collapses to a single point at which the effective returns to labor and capital are equalized. For given factor endowments, the effects of automation on wages are hump-shaped: for low levels of automation, advances in automation increase wages as the economy becomes more productive, but for higher levels of automation, wages decline due to the displacement of labor.

We analyze dynamic settings and show that the effects on wages are determined by a *race between automation and capital accumulation*. In addition to the previous two opposing effects on wages from rising productivity and labor displacement, automation also triggers capital accumulation that moves the economy up on the factor price frontier, increasing wages. We characterize an upper bound on output and wages that is reached in the limit case that the capital stock can instantaneously adjust to its optimal level whenever automation advances. We show a powerful analytic result: For any optimizing representative agent with linearly separable intertemporal preferences, the effects of automation on output and wages will lie between a lower bound captured by the constant-capital case and the described upper bound.

When the complexity distribution of tasks is bounded, full automation is reached in finite time and leads to a collapse in wages, no matter what savings behavior the representative agent pursues. For unbounded complexity distributions of tasks, we show that if the tail of remaining tasks is sufficiently thick, wages will rise forever. By contrast, if the tail of unautomated tasks is too thin, wages will eventually collapse.

Next, we simulate a range of scenarios to illustrate our findings numerically. (Figure 2.8 shows the main results.) We start with a "business-as-usual scenario," which captures the traditional notion that a constant fraction of tasks is automated each period, similar to Aghion, B. Jones, and C. Jones (2019). This corresponds to a Pareto distribution for task complexity together with exponential growth in the automation index. Since the maximum complexity of tasks in this scenario is unbounded, true AGI will not be reached in finite time. In our calibration, both output and wages rise forever in this scenario, at a pace similar to what advanced countries have experienced over the past century.

Next we consider two AGI scenarios that span the range of estimates provided by Geoffrey Hinton, one of the godfathers of deep learning, who estimated in May 2023 that AGI may be reached within 5 to 20 years—after declaring that he had "suddenly switched [his] views on whether these things are going to be more intelligent than us." In our "baseline AGI scenario" we assume a bounded task distribution such that full automation is obtained within 20 year.[3] In an "aggressive AGI scenario" we assume a shorter-tailed distribution that implies full automation within five years. Our simulation results imply ten times faster growth than in the business-as-usual scenario, especially in the aggressive AGI scenario. However, wages collapse as the economy approaches full automation.

In a fourth scenario, we consider the possibility that there is a large bout of automation in the near term—for example because AI rapidly automates cognitive jobs—but that there

[3]As economists, we hope that the computational complexity of at least some atomistic tasks that go into writing economics papers is far into that right tail. Alas, this might be wishful thinking.

remains a long tail of tasks that are harder to automate. As a result of the initial bout of automation, the economy enters the region in which labor loses its relative scarcity value, and wages in our simulation collapse. However, after capital accumulation has caught up sufficiently, labor becomes sufficiently scarce again so that the economy returns to region 1 and wages rise in line with output growth.

We extend our baseline model to analyze several additional important considerations. First, we consider the role of fixed factors (such as minerals or matter) and show that they may pose a bottleneck that holds back economic growth and worsens the outlook for wages, ultimately leading to stagnation accompanied by a wage collapse. Next, we add an innovation sector to analyze the potential for automating technological progress and show that this lifts the returns of all factors including wages. We illustrate that sufficient automation may give rise to a growth singularity whereby output takes off.

Furthermore, we analyze societal choices to retain certain jobs as exclusively human even when they can be automated (e.g., priests and judges), and show that a sufficient volume of such "nostalgic jobs" may help to keep labor sufficiently scarce so that wages continue to grow even when full automation is technically possible. We analyze the wage-maximizing rate of automation and show that slowing down automation in an AGI scenario may deliver significant gains to workers albeit at the cost of forgoing a growing fraction of output.

Next, we evaluate the impact of automation on workers with heterogeneous skills and susceptibility to being automated. We find that automation in such a scenario may give rise to an ever-declining fraction of superstar workers earnings ever-growing wages, whereas the majority of the labor force is starkly devalued by automation. Finally, we explore the role of compute as an example of specific capital that is tailored to automating specific tasks. We observe that in the short term, compute may earn very high returns, but after an adjustment period during which sufficient compute has been accumulated (and which may last long), compute may become just another form of capital that earns the same return as all other types of capital.

**Related Literature** The foundational work of Aghion, B. Jones, and C. Jones (2019) explores the impact of artificial intelligence on economic growth, offering valuable insights into how technological advances in AI, including AGI, might influence future economic trajectories. C. I. Jones (2023) underscores the risk of technological progress, emphasizing existential risk—a concept crucial in AGI discussions. Davidson (2023) analyzes a model of the factors that may lead to a take-off in economic growth if technology advances near AGI but does not focus on the wage implications. Besiroglu, Emery-Xu, and Thompson (2022) show how advances in AI may accelerate growth by speeding up R&D, and Erdil and Besiroglu (2023) review the factors by which AGI may give rise to exponential growth. Trammell and Korinek (2023) provide a useful survey on the broader implications of advanced artificial intelligence on economic growth.

A critical body of literature explores the dynamics between labor and automation. Seminal works by Acemoglu and Restrepo (2018) and Acemoglu and Restrepo (2022a) and D. Autor (2019) provide insights into how automation reshapes labor markets, focusing on technology as a substitute for individual worker tasks or how workers and tasks can complement or substitute for technology. Eloundou et al. (2023) and Edward W. Felten, Raj, and Seamans (2023b) provide excellent empirical analyses of which tasks are amenable to automation by the current wave of foundation models. These studies offer a useful lens for understanding the economic implications of AI before AGI is reached. Our contribution to these strands of literature is to look at the limit case of what happens if either all work tasks are automated or we asymptote towards a world in which all tasks are automated.

Our paper is also related to a broader literature on AGI and superintelligence literature. Good (1965) was the first to articulate the potential of an intelligence explosion if AGI is reached. Bostrom (2014a) provides a comprehensive exploration of superintelligence, highlighting the potential capabilities of AGI and the profound implications these might have for society. Yudkowsky (2013) discusses several of the economic implications of the transition to AGI.

## 2.2 A Compute-Centric Model of Automation

### 2.2.1 Tasks in Compute Space

**compute** /km-pyüt/

**verb:** to determine by calculation

*The system computed the length of the shortest path.*

**noun:** the combined computational resources available for information processing tasks

*Modern AI relies on vast amounts of compute.*

**Etymology:** Derived from the Latin verb "computare," meaning "to count, sum up, or reckon together," the word "compute" entered the English language as a verb in the 16th century. The noun form "compute" gained prominence more recently with the advent of high-performance digital computers and the increasing need to describe the resources required for computation.

**Atomistic Job Tasks** A central assumption of our analysis is that the work performed by humans is composed of tasks and sub-tasks – or unchanging atomistic task – that differ in how easily they are automated. In our baseline model, we focus on cognitive tasks and their potential for automation. In this setting, an atomistic task is a well-defined computational assignment that contributes to the accomplishment of a larger job task.

These atomistic tasks are fundamental and are significantly smaller than the tasks that are listed in O*Net. Table 2.1 lists, for example, the top-5 O*Net tasks of economists: to study data; conduct and disseminate research; compile, analyze and report data; supervise research; and teach. Each of these O*Net "job tasks" involves a wide variety of different atomistic tasks. For example, the O*Net task "teach theories of economics" may require first planning the overall task, recalling different economic theories, synthesizing a structure, preparing slides, formulating lectures, synthesizing speech and affect, decoding and responding to student questions, preparing problem sets, distributing problem sets, grading problem sets, and so on—all while keeping track of the plan. It may also require tasks such

- Study economic and statistical data in area of specialization, such as finance, labor, or agriculture.
- Conduct research on economic issues, and disseminate research findings through technical reports or scientific articles in journals.
- Compile, analyze, and report data to explain economic phenomena and forecast market trends, applying mathematical models and statistical techniques.
- Supervise research projects and students' study projects.
- Teach theories, principles, and methods of economics.

Table 2.1: Top-5 Tasks performed by economists (O*Net database)

as recognizing emotional expressions on students' faces, using theory of mind to evaluate student progress and dynamically adjust the structure, etc.

All of these tasks involve a set of basic human brain functions, which constitute a form of computation. Some of these functions are easily performed by machines and therefore highly susceptible to cognitive automation (Korinek 2023), whereas others are more difficult. What matters for our purposes here is how computation-intensive they are using machines.

Recent literature on technology on labor markets observes that innovation typically gives rise to new job tasks (e.g., Acemoglu and Restrepo 2018; D. Autor 2019). This holds true when viewed from the perspective of high-level job tasks such as those captured by O*Net. However, when viewed from an atomistic level that reflects basic brain functions, innovation merely recombines atomistic tasks in novel ways to produce novel high-level tasks and jobs. For example, the novel task of "prompt engineering" may require atomistic tasks such as defining a desired output, crafting an initial prompt, entering it, reading the output, evaluating it, deciding whether to iterate, and finally sharing the output—all functions that existed long before the invention of generative AI systems that triggered prompt engineering.

**Task Complexity and Compute Intensity** Our baseline model emphasizes differences in complexity as a key dimension when studying the automation potential of tasks. Our preferred interpretation for what makes tasks difficult to automate is their compute intensity, which refers to the amount of computational resources required to perform a specific task. Compute intensity can easily be measured by the amount of floating point operations (FLOP) that need to be executed to perform a given task. The computational complexity for machines to execute a task often differs starkly from how easy or difficult it is for humans.[4] Still, it is the computational complexity for machines that determines whether a task can be automated.

**Advances in Computing** One of the main drivers of recent advances in AI has been the increased availability of computing power. Moore's Law, first described by Gordon Moore (1965), describes that the performance of cutting-edge computer chips doubles approximately every two years. The regularity has held for the past sixty years. Additionally, the amount of compute deployed in cutting-edge AI systems has grown even faster over the past decade, doubling roughly every six months, as shown in Sevilla et al. (2022) and depicted in Figure 2.2. Improvements in algorithms have further accelerated the growth in capabilities of cutting-edge AI systems (Besiroglu, Emery-Xu, and Thompson 2022).

For our analysis below, we assume that there is an automation index that captures the maximum complexity of tasks that can be automated. This index grows exponentially at an exogenous rate, mirroring the type of advances captured by Moore's Law and Figure 2.2. As the automation index increases, a growing mass of tasks can be automated.

### 2.2.2 Baseline Model

Consider a representative household in a static economy who is endowed with $L = 1$ units of labor and $K > 0$ units of capital. There is a continuum of tasks that differ in their computational complexity $i$. The distribution function $\Phi(i)$ reflects the cumulative mass of tasks with complexity $\leq i$ and satisfies $\Phi(0) = 0$ and $\lim_{i\to\infty} \Phi(i) = 1$. If the distribution function is differentiable, we call its derivative $\phi(i)$ the density of tasks of complexity $i$. Examples are shown in Figure 2.1.

To produce aggregate output $Y$, we combine all the tasks of different complexity using a

[4]For example, Moravec (1988) observed that some tasks that *feel* easy to execute for humans, such as vision processing, employ a large amount of dedicated grey matter and are, in fact, computationally quite intensive. There are also certain tasks that require very little compute in dedicated machines but that are difficult for the human brain since it has not evolved for them: for example, arithmetic operations take just one FLOP on a basic computer but require significant amounts of grey matter for human brains to perform, likely involving the equivalent of billions of FLOP.

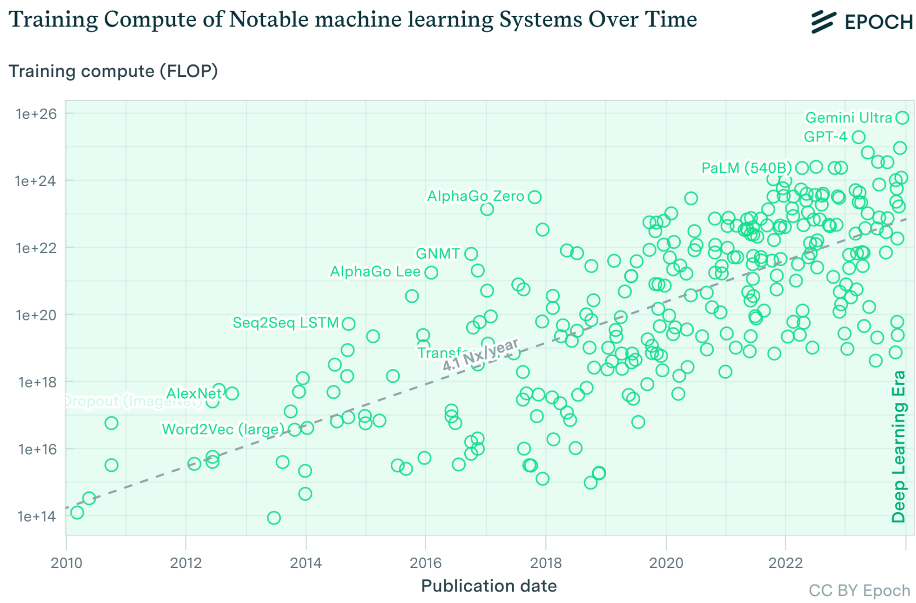



Figure 2.2: Training compute of frontier AI systems over time
(Copyright © 2024 by Epoch under a CC-BY-4.0 license; Sevilla et al. (2022).)

CES aggregator with elasticity of substitution $\sigma$.

$$Y = A\left[\int_i y(i)^{\frac{\sigma-1}{\sigma}} d\Phi(i)\right]^{\frac{\sigma}{\sigma-1}} \tag{2.1}$$

where $y(i)$ is the amount of type $i$ tasks employed in the production of output. We generally assume $\sigma < 1$, reflecting that the atomistic tasks are gross complements.

Each task is performed using capital $k(i)$ and labor $\ell(i)$ according to the production function

$$y(i) = a_K(i)k(i) + a_L(i)\ell(i) \tag{2.2}$$

where the coefficients $a_K(i)$ and $a_L(i)$ reflect the efficiency of capital and labor. We assume that the exogenous index $I$ reflects the state of automation and defines a complexity threshold such that all tasks below the threshold can be performed with either capital or labor but all the tasks above the threshold require labor. We normalize the technological parameters

$a_K(i) = a_L(i) = 1$ except that $a_K(i) = 0$ if $i \geq I$. In other words,

$$y(i) = \begin{cases} k(i) + \ell(i) & \text{for } i < I \\ \ell(i) & \text{for } i \geq I \end{cases}$$

**Strategies** The representative agent supplies her endowments of labor and capital every period at the prevailing factor prices $w$ and $R$ and makes no interesting economic decisions. The representative firm in the economy maximizes profits by hiring capital $k(i)$ and labor $\ell(i)$ for each task at the prevailing factor prices $w$ and $R$ to produce $y(i)$, which is then combined to produce final output. The firm's maximization problem is

$$\max_{k(i),\ell(i)} Y - R\int_i k(i)d\Phi(i) - w\int_i \ell(i)d\Phi(i) \quad \text{s.t.} \quad (1),(2)$$

**Equilibrium** An equilibrium in the baseline model consists of a set of $\{k(i), \ell(i), y(i)\}_{i\geq 0}$ and factor prices $w$ and $R$ such that the representative firm solves its maximization problem and markets for capital and labor clear, i.e.,

$$\int_i k(i)d\Phi(i) = K \qquad \int_i \ell(i)d\Phi(i) = 1$$

Since there are no market imperfections, the described equilibrium also constitutes the first-best of the economy.

### 2.2.3 Equilibrium: Characterizing Two Regions

**Scarcity of Labor** For given factor endowments $(K, L)$, there are two possible regimes for the scarcity of labor, depending on the level of the automation index $I$: If the index is low enough so that labor is relatively scarce, then the return on labor is greater than the return on capital, $w > R$, and the scarce labor is employed solely in those tasks that cannot be automated.

Conversely, if the state of automation is sufficiently advanced that only a small fraction of tasks are exclusive to human labor, then $w = R$ holds, and labor is employed not only in the remaining unautomated tasks but also in some of the automated tasks. At the margin, capital and labor are perfect substitutes.

**Lemma 2.1** (Scarcity of labor)**.** *For given $(K, L)$, there is a threshold value for the state of automation $\hat{I}$ that is defined by*

$$\Phi\left(\hat{I}\right) = \frac{K/L}{1 + K/L} \tag{2.3}$$

*and increasing in the $K/L$-ratio such that there are two regions:*

***Region 1:*** *If $I < \hat{I}$, then labor is scarce compared to capital. In this regime, labor is employed only for tasks with $i > I$. Output is*

$$Y = F\left(K, L; I\right) = A\left[K^{\frac{\sigma-1}{\sigma}}\Phi(I)^{\frac{1}{\sigma}} + L^{\frac{\sigma-1}{\sigma}}(1 - \Phi(I))^{\frac{1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}} \tag{2.4}$$

*and wages satisfy*

$$w = A^{\frac{\sigma-1}{\sigma}}\left(Y/L\right)^{\frac{1}{\sigma}} \cdot (1 - \Phi(I))^{\frac{1}{\sigma}} > R$$

***Region 2:*** *If $I \geq \hat{I}$, then the relative scarcity of labor is relieved, and labor earns the same return as capital $w = R = A$; if the inequality is strict, some labor is deployed alongside capital for tasks with $i < I$, and labor and capital are perfect substitutes for the marginal task. Output is given by the linear function*

$$Y = F\left(K, L\right) = A\left(K + L\right) \tag{2.5}$$

*Conversely, for given $I$, there is a threshold $\kappa\left(I\right) = \Phi(I)/\left[1 - \Phi\left(I\right)\right]$ such that the economy is in region 1 if $K/L > \kappa\left(I\right)$ and in region 2 if $K/L \leq \kappa(I)$. The threshold $\kappa(I)$ is increasing in $I$, i.e., if $K/L$ is marginally above the threshold, further automation pushes the economy from region 1 into region 2 where the scarcity of labor is relieved.*

*Proof.* Assume first that all labor is employed in tasks with $i \geq I$ and observe that the symmetry of the production function across all tasks implies that an identical amount of capital $k = K/\Phi(I)$ will be employed in each task below the threshold and an identical amount of labor $\ell = L/(1 - \Phi(I))$ for each task above the threshold for given aggregate $K$

and $L$. The production function can then be written as

$$\begin{aligned} Y = F(K, L; I) &= A\left[k^{\frac{\sigma-1}{\sigma}}\Phi(I) + \ell^{\frac{\sigma-1}{\sigma}}(1-\Phi(I))\right]^{\frac{\sigma}{\sigma-1}} \\ &= A\left[K^{\frac{\sigma-1}{\sigma}}\Phi(I)^{\frac{1}{\sigma}} + L^{\frac{\sigma-1}{\sigma}}(1-\Phi(I))^{\frac{1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}} \end{aligned}$$

proving equation (2.4).

The firm's optimization problem implies the first-order conditions

$$F_K = A^{\frac{\sigma-1}{\sigma}}Y^{\frac{1}{\sigma}} \cdot K^{-\frac{1}{\sigma}}\Phi(I)^{\frac{1}{\sigma}} = R \tag{2.6}$$

$$F_L = A^{\frac{\sigma-1}{\sigma}}Y^{\frac{1}{\sigma}} \cdot L^{-\frac{1}{\sigma}}(1-\Phi(I))^{\frac{1}{\sigma}} = w \tag{2.7}$$

By comparing these two expressions, we can see that the return on capital $R$ is less than the return on labor $w$ as long as $K^{-\frac{1}{\sigma}}\Phi(I)^{\frac{1}{\sigma}} < L^{-\frac{1}{\sigma}}(1-\Phi(I))^{\frac{1}{\sigma}}$ or, equivalently, $k > \ell$, i.e., the capital assigned to each automated task is greater than the labor assigned to unautomated tasks. Expressing this in terms of aggregate supplies of factors, the condition is

$$\frac{K}{L} > \kappa(I) := \frac{\Phi(I)}{1-\Phi(I)} \tag{2.8}$$

The right-hand side is an increasing function of $I$ that goes from 0 to $\infty$. By implication, there is a value of $I$ such that the inequality is violated for all $I > \hat{I}$.

When that threshold is crossed, it is more efficient to allocate some labor to tasks with $i < I$, and the marginal unit of labor is perfectly substituable with capital. By implication, $k(i) = \ell(i) = K + L$, and the CES aggregator simplifies to equation (2.5). Alternatively, the threshold for $\Phi(I)$ can be expressed explicitly as an increasing function of the $K/L$-ratio by solving for

$$\Phi\left(\hat{I}\right) = \frac{K/L}{1+K/L}$$

The remaining results stated in the lemma follow immediately. □

Intuitively, region 1 reflects the world as we have experienced it over the past 200 years, in which capital and labor are complementary in production, and labor is comparatively scarce. Figure 2.3 shows that for given factor supplies, a higher automation index $I$ increases the

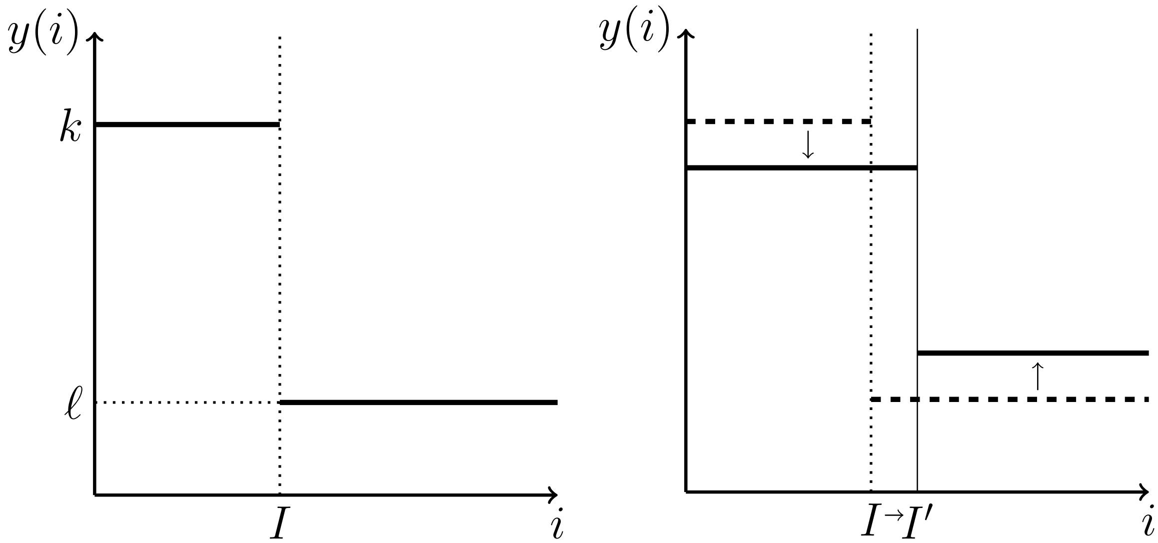



Figure 2.3: Automation and the scarcity of labor

mass of tasks that can be accomplished with capital, implying that the available capital is spread over a greater number of tasks and becomes scarcer. Conversely, automation reduces the mass of tasks that is exclusive to labor, implying that the available labor can be concentrated on fewer tasks and becomes less scarce. As the automation index reaches the threshold $\hat{I}$, there are so few tasks left that are exclusive to labor that labor no longer enjoys a scarcity advantage over capital, and the returns on the two factors are equated.

Note that the threshold $\hat{I}$ depends solely on relative factor supplies, not on the elasticity of substitution $\sigma$ between capital and labor. As soon as labor is no longer scarce, it will be used interchangably with capital in the marginal task, and this holds even when individual tasks are highly complementary as reflected by low values of the elasticity of substitution (as long as $\sigma > 0$).

### 2.2.4 Factor Price Frontier (FPF)

For an analysis of the effects of advances in automation $I$ on factor returns, let us characterize the factor price frontier associated with the firm's technology. The factor price frontier depicts all possible combinations of factor prices $R$ and $w$ that will result from different proportions of factor supplies $K$ and $L$ in a competitive economy with profit-maximizing firms under a given technology.

**Lemma 2.2** (Factor Price Frontier (FPF))**.** *For a given automation index $I$, the factor price frontier slopes downwards, starting from a limiting point $w^*(I) = A(1-\Phi(I))^{\frac{1}{\sigma-1}}$ and $R = 0$ as $K/L \to \infty$ to the point $w = R = A$ when $K/L \leq \kappa(I)$. Increases in $A$ move the FPF proportionately outwards. Increases in $I$ raise $w^*(I)$ and swivel the factor price frontier clock-wise.*

*Proof.* We obtain the factor price frontier from the aggregate cost function, which is the dual of the aggregate production function. The associated unit cost function represents the minimum cost at which a competitive optimizing firm can produce one unit of final output, given factor prices $w$ and $R$. In the region of $I < \hat{I}$, the unit cost function associated with equation (2.4) is

$$C(w, R; I) = \frac{1}{A}\left(R^{1-\sigma}\Phi(I) + w^{1-\sigma}(1-\Phi(I))\right)^{\frac{1}{1-\sigma}}$$

Since we employed the final good as the numeraire good, this cost function needs to equal 1 in a competitive economy. The factor price frontier when $I < \hat{I}$ is thus given by all pairs of $(w, R)$ that satisfy the equation $C\,(w, R; I) = 1$, or equivalently,

$$w = \left(\frac{A^{1-\sigma} - R^{1-\sigma}\Phi(I)}{1-\Phi(I)}\right)^{\frac{1}{1-\sigma}} \tag{2.9}$$

Asymptotically, as $K/L$ goes to infinity, we can see from equations (2.6) and (2.7) that the return to capital $R$ goes to zero, whereas the wage converges to

$$w^*(I) = \lim_{K/L\to\infty} w = A\left[0\cdot\Phi(I)^{\frac{1}{\sigma}} + (1-\Phi(I))^{\frac{1}{\sigma}}\right]^{\frac{1}{\sigma-1}} (1-\Phi(I))^{\frac{1}{\sigma}} = A(1-\Phi(I))^{\frac{1}{\sigma-1}} \tag{2.10}$$

Conversely, when $I \geq \hat{I}$, the cost function is simply $C\,(w, R; I) = \min\{w, R\}/A$, and the factor price frontier is degenerate and consists of a single point $w = R = A$. $\square$

The factor price frontier is illustrated in Figure 2.4. The area above the 45 degree line corresponds to Region 1 of Lemma 1, reflecting a high capital-labor ratio $K/L > \kappa(I)$ and $w > R$. Higher capital intensity $K/L$ moves factor returns up and to the left along the

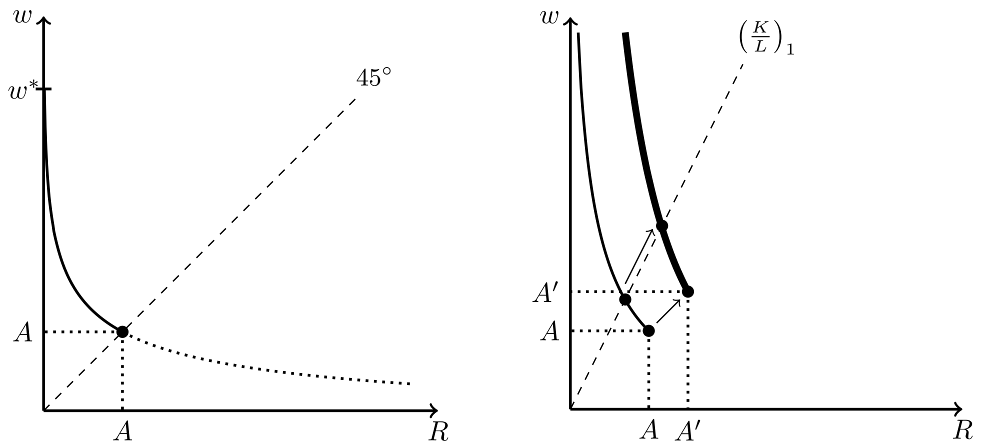



Figure 2.4: Factor price frontier and its dependence on $A$

frontier, i.e., it increases $w$ and reduce $R$. Conversely, when $K/L \leq \kappa(I)$, we enter Region 2 of the lemma, and the factor price frontier corresponds to a single dot on the 45 degree line at which $w = R = A$.

The right panel of Figure 2.4 shows how an increase in the level of technology $A$ pushes out the factor price frontier – for any ratio of $K/L$, it scales the returns of all factors proportionately. This exemplifies how the factor price frontier serves as a convenient tool to describe how factor returns are impacted by technological changes across any levels of factors supplies.

### The Automation Path on the Factor Price Frontier

We next turn to the effects of automation for a given capital stock $K$ or equivalently, capital intensity $k = K/L$. Then it is easy to see that:

**Lemma 2.3** (Automation and Output)**.** *An increase in automation $d\Phi(I)$ raises output as long as $I < \hat{I}$, and leaves output unaffected otherwise.*

*Proof.* For $I < \hat{I}$, the result follows by differentiating expression (2.4),

$$\frac{dY}{d\Phi(I)} = \frac{1}{\sigma - 1} A^{\frac{\sigma-1}{\sigma}} Y^{\frac{1}{\sigma}} \cdot \left( k^{\frac{\sigma-1}{\sigma}} - \ell^{\frac{\sigma-1}{\sigma}} \right)$$

Given $\sigma < 1$, the derivative is positive as long as $k > \ell$, which is the condition for being in region 1 in Lemma 2.1 in which the production function is relevant. For $I \geq \hat{I}$, the relevant production function is (2.5), which is independent of $I$. $\square$

Intuitively, for output to rise, capital must be sufficiently abundant, delivering a productivity gain from deploying the amply available capital to a greater number of tasks. This is frequently termed the *productivity effect* of automation.

Let us look at factor returns next.

**Lemma 2.4** (Automation and Factor Returns)**.** *(i) An increase in automation $d\Phi(I)$ always raises $R$ as long as $I < \hat{I}$. The effect on $w$ is hump-shaped: there is a threshold $I^*(K/L)$ with $\Phi(I^*(\cdot)) \in (0,1)$ such that wages $w$ rise in $\Phi(I)$ as long as $I < I^*(K/L)$ or, equivalently, as long as $K/L > \kappa^*(I)$, but decline in $\Phi(I)$ for $I > I^*(K/L)$ or, equivalently, $K/L < \kappa^*(I)$.*

*(ii) For $\Phi(I) = 0$, the return on capital is $R = 0$, and wages equal $w = A$. For $\Phi(I) \geq \kappa/(1+\kappa)$, both equal $R = w = A$. The latter condition always holds if $\Phi(I) = 1$.*

*Proof.* The limit results follow readily from equations (2.6) and (2.7) and from the second part of Lemma 1. By differentiating equation (2.6), with respect to $\Phi(I)$, we can see that automation always raises the return on capital.

To see how automation affects wages, consider the derivative of $\log w$ with respect to $\Phi$ from the firm's optimality condition (2.7):

$$\frac{d\log w}{d\Phi(I)} = \frac{1}{\sigma-1}\frac{1}{\sigma}\left(k^{\frac{\sigma-1}{\sigma}} - \ell^{\frac{\sigma-1}{\sigma}}\right)(Y/A)^{\frac{1-\sigma}{\sigma}} - \frac{1}{\sigma}\frac{1}{1-\Phi(I)}. \tag{2.11}$$

The first term reflects the productivity effect of automation, which is positive under condition (2.8), reflecting that producing the marginal task using a relatively more abundant $k$ units of capital rather than a scarce $\ell$ units of labor increases output. The second term captures the *displacement effect* of automation and reduces labor income. It reflects that the labor used in each unautomated task $\ell = L/(1-\Phi)$ increases, as captured by the term in the denominator, thereby pulling down the marginal product of labor.

As $I$ rises, wages rise at first—at $\Phi(I) = 0$ we find $\frac{d\log w}{d\Phi(I)} = \frac{1}{1-\sigma} > 0$. As $I$ becomes larger, the first term in (2.4) declines, reaching zero for $I = \hat{I}$, and the absolute value of the second term grows and eventually dominates the first term—in the limit of $\Phi \to 1$, the second term becomes infinitely large. Thus, there exists an intermediate value $I^*(K/L)$ after which

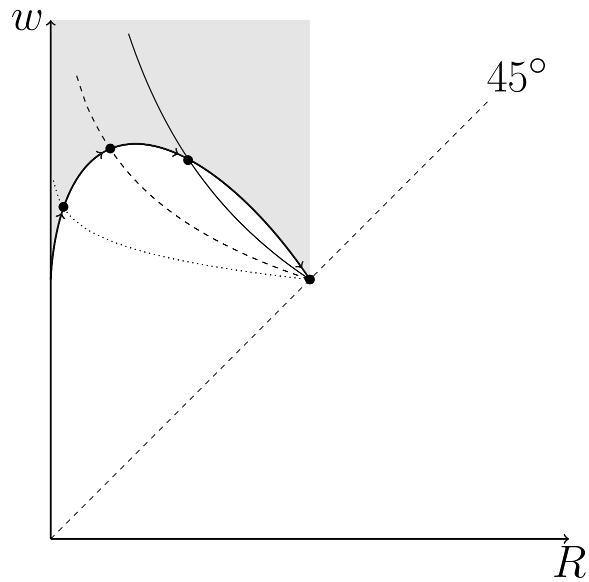



Figure 2.5: Factor price frontier and automation

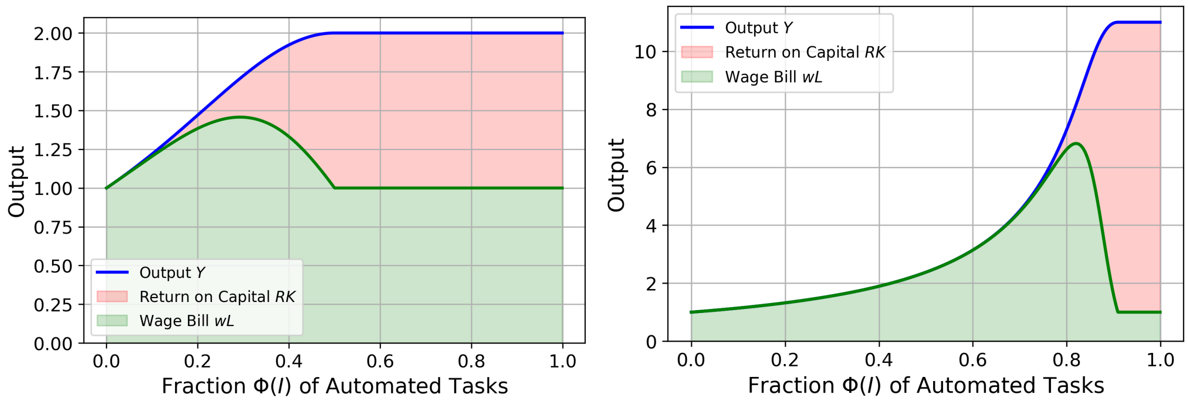
further automation reduces wages. Notice that the first term is increasing in $K/L$ whereas the second term is independent of $K/L$. The threshold can alternatively be expressed as $K/L < \kappa^*(I)$.

$\square$

Figure 2.5 illustrates that an increase in automation $I$ "rotates" the factor price frontier clockwise, for example, from the dotted to the dashed and solid lines. If the economy is in the labor-scarce region 1 (above the 45 degree line), automation raises wages for a given return of capital and also the maximum wage level $w^*(I)$. For given $K/L$, the path of factor prices that results from rising automation $I$ is illustrated by the hump-shaped bold line with arrows in the figure. Along the path, $R$ rises continually whereas $w$ at first rises but eventually falls. When automation reaches $\hat{I}(K/L)$, the economy ends up in the degenerate equilibrium with $w = R = A$ on the 45-degree line.

### 2.2.5 Automation and Factor Earnings

Figure 2.6 shows the effects of automation on total output for given factor supplies as well as its split into the wage bill and the total returns to capital. The horizontal axis depicts the fraction $\Phi(I)$ of automated tasks, which goes from zero to one. The left-hand panel illustrates




(a) Equal factor endowments (b) Abundant effective capital

Figure 2.6: Static equilibria under rising automation

the case of equal capital and labor endowments, $K = L = 1$, and modest complementarity with an elasticity of substitution $\sigma = 0.5$ between the two. As long as the economy is in the scarce-labor region (Region 1), output is a strictly monotonic function of automation. At first, automation almost exclusively benefits labor, and the returns to capital are minuscule. But as automation increases and we come closer to Region 2, the wage bill reaches a ceiling and starts to decline. Further automation still raises output, but the returns to capital grow faster than output, at the expense of the wage bill. When Region 2 is reached at $\Phi = 0.5$, both factors earn equal returns. Given equal endowments, this translates into capital and labor shares of one-half each.

The right panel of the figure shows an alternative scenario in which the effective supply of capital is ten times higher than labor, i.e., $L = 1$ and $K = 10$, and in which the two are strong complements with $\sigma = 0.2$. The abundance of capital and the strong complementarity imply that the region in which most of the benefits go to labor is much larger, but so is the drop in the wage bill once a critical threshold is surpassed: whereas wages *seem* to be growing exponentially in $\Phi(I)$ up until $\Phi \approx 0.80$, they experience a precipitous decline by about 85% starting around $\Phi \approx 0.83$, accompanied by a meteoric rise in the returns to capital. When Region 2 is reached at $\Phi = 10/11$, factor returns are equalized, and given the relative factor endowments, the capital share of the economy is ten times the labor share. This example highlights that the fate of labor can change rapidly when certain thresholds are crossed.

Crucially, the effect of automation on output—and per-capita income—depends on the capital available. In the illustration in the left panel, full automation merely doubles output; in the right panel, output grows eleven-fold. This observation naturally leads us to the next step of our analysis—to analyze how automation interacts with capital accumulation in a dynamic setting.

## 2.3 Dynamics: The Race between Automation and Capital Accumulation

The dynamics of output and wages depend not only on technological advances—captured by the automation index $I$—but also on capital accumulation and by extension on the savings behavior of the agents in the economy. This section analyzes these forces in a dynamic setting.

### 2.3.1 Automation Scenarios

**Progress in automation** We assume that the automation index $I$ grows exponentially over time at an exogenous rate of $g$, reflecting Moore's law and similar regularities. For an initial $I_0$, the time path of $I$ (omitting the time index $t$ for conciseness) is given by

$$I = I_0 e^{gt}$$

We can equivalently write that $\log I = \log I_0 + gt$ grows linearly at the rate $g$.

We consider different distributions $\Phi(i)$ of task complexity or tasks in compute space to capture alternative scenarios for the advent of AGI:

**Business-As-Usual Scenario (Unbounded Distribution)** We model unbounded complexity distributions of tasks $\Phi(i)$ as Pareto, implying that $\log i$ is described by an exponential distribution, $\log i \sim Exp(\lambda)$ with decay parameter $\lambda$. The resulting cumulative distribution function is $\Phi(i) = 1 - e^{-\lambda \log i}$. If the automation index $I$ grows exponentially at rate $g$, the fraction of non-automated tasks declines at rate $\lambda \cdot g$. This distribution has an infinite right tail, meaning that there will always be tasks that cannot be automated.

**Baseline and Aggressive AGI Scenarios (Bounded Distributions)** For our AGI scenarios, we assume a bounded complexity distribution of tasks to capture the scenario that the tasks that can be performed by human brains is limited by an upper bound so automation crosses the threshold $\hat{I}$ within finite time. We assume that $\Phi(i)$ follows a power function $\Phi(i) = 1-(1-\log i/\log I^{\max})^{\beta}$ with $\beta = 1$ and with normalization $I^{\max} = I_0 e^{gT}$ such that all tasks are automated after $T$ years.[5] Following Hinton's predictions, we set $T = 20$ in the baseline AGI scenario and $T = 5$ in the aggressive AGI scenario. For $I > I^{\max}$, we keep $\Phi(i) = 1$ capturing full automation.

**Bout of Automation (Mixed Distribution)** We consider a fourth scenario in which rapid advances in AI automate a large fraction of tasks within a short time span, but in which we assume that there remains an unbounded tail of tasks that cannot be automated, for example, because of legal or cultural reasons. Analytically, we assume a mixture of the two scenarios above. Specifically, $\Phi(i)$ is defined as $\Phi(i) = \omega\left[1-(1-\log i/\log I^{\max})^{\beta}\right] + (1-\omega)\left[1-e^{-\lambda\log i}\right]$ where $\omega \in [0,1]$ is a weight parameter. We assume the same values for the parameters of the Pareto and power function distributions as in the previous two cases.

### 2.3.2 Consumer Problem

The representative household seeks to maximize its lifetime utility by choosing consumption $C_t$ over time:

$$\max_{\{C_t\}} U = \int_0^{\infty} e^{-\rho t} u(C_t) dt \tag{2.12}$$

subject to the law of motion for capital:

$$\dot{K}_t = F(K_t, L_t; I_t) - \delta K_t - C_t \tag{2.13}$$

for given $K_0$. The current-value Hamiltonian for this problem is:

$$H_c = u(C_t) + \mu_t \left[F(K_t, L_t) - \delta K_t - C_t\right]$$

[5]Power function distributions are a special case of beta distributions with a beta parameter $\alpha = 1$.

The first-order conditions with respect to consumption and capital are:

$$\frac{\partial H_c}{\partial C_t} = u'(C_t) - \mu_t = 0$$
$$\frac{\partial H_c}{\partial K_t} = \mu_t \left[F_K - \delta\right] = -\dot{\mu}_t + \rho\mu_t$$

Differentiating the first optimality condition with respect to time yields $u''(C_t)\dot{C}_t = \dot{\mu}_t$, and substituting into the second optimality condition gives

$$\frac{\dot{C}_t}{C_t} = \frac{1}{\eta(C_t)}\left[F_K(K_t, L_t) - \rho - \delta\right] \tag{2.14}$$

where $\eta(C_t) = -\frac{u''(C_t)C_t}{u'(C_t)}$ is the elasticity of intertemporal substitution.

**Limit Behavior in Region 2** When the economy is in region 2, then $F_K = A$. If the agent's utility function exhibits constant elasticity of substitution $\eta$, then the Euler equation implies a constant growth rate of consumption

$$g_C = \frac{\dot{C}_t}{C_t} = \frac{A - \rho - \delta}{\eta} \tag{2.15}$$

Let us assume that $A > \rho + \delta$ so consumption growth is positive and consider the case that the economy remains in region 2 forever—for example, because full automation $\Phi(I) = 1$ has been reached. Then the economy will converge towards a balanced growth path in which $g_Y = g_K = g_C$ as in (2.15) and the savings rate $s^\infty = 1 - C/Y$ is constant. From (2.13), we obtain that

$$g_K = \frac{\dot{K}_t}{K_t} = \frac{sA(K_t + L)}{K_t} - \delta$$

As $\lim_{t\to\infty} L/K_t = 0$, we can equate $g_C = g_K$ and solve for the long-run savings rate

$$s^\infty = \frac{A - \rho - \delta + \eta\delta}{A\eta} = \frac{1}{\eta} - \frac{\rho + (1-\eta)\delta}{A\eta}$$

**Bounds** Assume an initial $I_0$ and $K_0$ that satisfy $F_K\left(K_0, L; I_0\right) \geq \rho + \delta$, i.e., there was no excessive capital accumulation in the past. Then the following proposition holds for any intertemporal utility function that is linearly separable as specified in (2.12) with a twice continuously differentiable, increasing, and strictly concave period utility function $u(C)$:

**Proposition 2.5** (Bounds for Output and Wages)**.** *For any distribution $\Phi(i)$ of tasks in compute space and exogenous growth in the automation index $I_t$, the paths of capital, output, and wages lie between lower and upper bounds $K^- \leq K_t \leq K_t^+$, $Y_t^- \leq Y_t \leq Y_t^+$ and $w_t^- \leq w_t \leq w_t^+$.*

*The lower bounds are defined by the fixed-capital case with $K^- = K_0$ $\forall t$ and $Y_t^- = F(K^-, L, I_t)$, $w_t^- = F_L(K^-, L, I_t)$. The lower bound on wages first rises in $I_t$ and then declines in $I_t$. It declines to $A$ in finite time if full automation is reached asymptotically, i.e., if $\lim_{I\to\infty}\Phi(I) = 1$.*

*If $\Phi(I_t) < 1$, an upper bound $K_t^+$ for capital is defined by $F_K\left(K_t^+, L, I_t\right) = R = \rho + \delta$ $\forall t$ as long as a solution exists; otherwise we set $K_t^+ = \infty$. The upper bounds for output and wages are $Y_t^+ = F(K_t^+, L, I_t)$ and $w_t^+ = F_L(K_t^+, L, I_t)$. All three upper bounds are increasing in the automation index $I_t$. If automation is full, $\Phi(I_t) = 1$, the upper bounds are $K_t^+ = \infty$ and $Y_t^+ = \infty$, and the upper bound on wages discontinuously collapses to $w_t^+ = A$.*

*Proof.* Observe that for any twice continuously differentiable period utility function that is increasing and strictly concave, the elasticity in the Euler equation (2.14) satisfies $\eta(C_t) \in (0, \infty)$. Consumption on the optimal path is increasing as long as $F_K > \rho + \delta$ and constant when $F_K = \rho + \delta$. Our characterization of the factor price frontier delivers most of the remaining results.

For the lower bound, observe that increases in $I$ and $\Phi(I)$ raise the marginal product $F_K$ for given $K$, triggering additional capital accumulation, which raises output and wages above the lower bound. For the upper bound, observe that by the Euler equation, capital accumulation will never exceed the upper threshold $K_t^+$, which is given by

$$K_t^+ = \frac{A^\sigma L(1-\Phi(I_t))^{\frac{1}{\sigma-1}}\Phi(I_t)}{(R^{\sigma-1} - A^{\sigma-1}\Phi(I_t))^{\frac{\sigma}{\sigma-1}}}. \tag{2.16}$$

For given $I_t$, output and wages are increasing in $K_t$, implying that they must lie between the lower and upper bounds defined by $K^-$ and $K_t^+$. If $\Phi(I_t) = 1$, the production function is $AK$-style, and Lemma 2.4 implies that $w_t = A$. ∎

On the factor price frontier, the lower bound on wages $w^-$ is pinned down by the automation path in Figure 2.5; it collapses to $A$ in finite time if the economy asymptotically converges to full automation. As long as $\Phi(I) < 1$, the upper bound on wages $w_t^+$ is pinned down by the intersection of the corresponding factor price frontier with a vertical line at $R = \rho + \delta$ and rises without bounds in $I$. However, when full automation $\Phi(I) = 1$ is reached, the upper bound on wages $w_t^+$ discontinuously collapses to $A$, which equals the lower bound and must therefore equal the equilibrium wage. This result is independent of intertemporal preferences and savings behavior and occurs in finite time if the distribution of task complexity $\Phi(I)$ is bounded, as in our two AGI scenarios.

**The Balancing Savings Rate** To further investigate the race between automation and capital accumulation, we analyze the threshold at which the wage effects of automation and capital accumulation precisely offset each other. For this, we take the total differential of the equilibrium wage, $w_t = F_L(K_t, L; I_t)$, and set $dw_t = 0$ to find

$$F_{KL}(K_t, L; I_t)\frac{dK_t}{dt} + F_{LI}(K_t, L; I_t)\frac{dI_t}{dt} = 0 \tag{2.17}$$

Suppose, for simplicity, that $\delta = 0$ so we can denote the savings rate at $t$ by $s_t = \dot{K}_t/Y_t$. Also, note that $F_{LI}(K_t, L; I_t)\frac{dI_t}{dt} = F_{L\Phi}\dot{\Phi}_t$. Then

$$s_t Y_t \cdot F_{KL} = -F_{L\Phi}\dot{\Phi}_t$$

The left-hand side is the increase in wages due to capital accumulation. The right-hand side is the change in wages due to automation. As we observed above in Lemma 2.4, the term $F_{L\Phi}$ encompasses the productivity effect and the displacement effect of automation on wages. Dividing by the cross-derivative $F_{KL}$, the fraction $\frac{F_{L\Phi}}{F_{KL}}$ captures the wage effects of automation relative to capital accumulation. After some algebra, we obtain

$$\frac{F_{L\Phi}}{F_{KL}} = \left[\frac{\sigma}{1-\sigma}(k/\ell)^{\frac{1-\sigma}{\sigma}} - \left(\kappa + \frac{1}{1-\sigma}\right)\right]k$$

The first term in the brackets is the productivity effect that is increasing in the relative abundance of capital $k/\ell$ – if capital is very abundant compared to labor, then using capital for newly automated tasks significantly raises output. The second term is the displacement effect that is increasing in $\kappa$ and thus in the automation index $I$. (Recall that for the given level of automation $I$, $\kappa(I) = \Phi(I)/[1 - \Phi(I)]$ reflects the threshold of the capital/labor ratio below which the economy is in region 2 such that the scarcity of labor is lifted.) Intuitively, if a large fraction of tasks has already been automated, then further automation of marginal tasks will result in a large fall in labor demand since the automated labor has to be reallocated to an ever smaller set of human-only tasks.

By rearranging terms, we obtain the following expression for the savings rate that, given $\Phi$ and $g$, perfectly offsets the effect of automation on wages

$$\tilde{s}_t = \left[\left(\kappa + \frac{1}{1-\sigma}\right) - \frac{\sigma}{1-\sigma}(k/\ell)^{\frac{1-\sigma}{\sigma}}\right] \cdot \frac{K_t}{Y_t} \cdot \frac{\phi_t I_t}{\Phi_t} \cdot g \tag{2.18}$$

The condition tells us that, to offset the effect of automation on wages, the savings rate must be increasing with (i) the displacement effect net of the productivity effect, (ii) the capital-output ratio, (iii) the relative mass of automated tasks at the current compute threshold for task automation, and (iv) the growth of compute $g$. Intuitively, a large fraction of output must be invested if (i) the displacement effect reduces wages significantly, or (ii) there is already a large amount of capital stock in the economy, or (iii) a large amount of tasks are being automated, or (iv) automation is fast.

The expression in (2.18) tells us about the threshold level for the savings rate at time $t$ above which wages rise and below which wages fall, given the extent of automation occurring at time $t$. In other words, it characterizes the short-run behavior of wages as an outcome of the race between automation and capital accumulation.

**Long-Run Dynamics for Unbounded Task Distributions** To further illuminate the trade-off in (2.18), we turn to the long-run dynamics of wages. To do so, we start by characterizing the conditions for the existence of a balanced growth path (BGP). We define a BPG as an equilibrium path on which output and capital stock grow at a constant rate and factor shares remain constant.

**Lemma 2.6.** *Suppose that as $t$ increases, $\Phi(I_t) \to 1$, and focus on the limit case. Then the return to capital converges to $A$. Moreover, output and capital stock grow at the rate $(A-\rho-\delta)/\eta$ and the savings rate converges to $(A-\rho-\delta+\eta\delta)/A\eta$.*

*Proof.* If the economy is in region 1 in the limit, then the production function converges to

$$\lim_{\Phi\to 1} A\left[K^{\frac{\sigma-1}{\sigma}}\Phi^{\frac{1}{\sigma}} + L^{\frac{\sigma-1}{\sigma}}(1-\Phi)^{\frac{1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}} = AK$$

If the economy is in region 2 in the limit, then $F(K,L) = A(K+L)$. In both cases,

$$\lim_{\Phi\to 1} F_K = A$$

As a result, the Euler equation implies

$$\frac{\dot{C}_t}{C_t} = \frac{1}{\eta}[F_K - \rho - \delta] \to \frac{1}{\eta}[A-\rho-\delta]$$

as $\Phi \to 1$. Output and capital must grow at the same rate, which implies that the savings rate must satisfy

$$\frac{\dot{C}_t}{C_t} = \frac{\dot{K}_t}{K_t} = \frac{s^\infty Y_t}{K_t} - \delta$$

Since $\lim_{\Phi\to 1}\frac{Y_t}{K_t} = A$, this requires that $\dot{K}_t/K_t \to s^\infty A - \delta$. Therefore, we have

$$s^\infty A - \delta = \frac{1}{\eta}[A-\rho-\delta]$$
$$s^\infty = \frac{A-\rho-\delta+\eta\delta}{A\eta}$$

□

Since we are interested in the long-run dynamics of wages, we make a simplifying assumption that the savings rate is given exogenously at a constant value $s^\infty$, which can be interpreted as the long-run savings rate. Under the assumption, $s$ is a key parameter determining the rate of capital accumulation. Depending on the value of $s^\infty$ relative to the rate of automation $g$, the race between automation and capital accumlulation can result in three

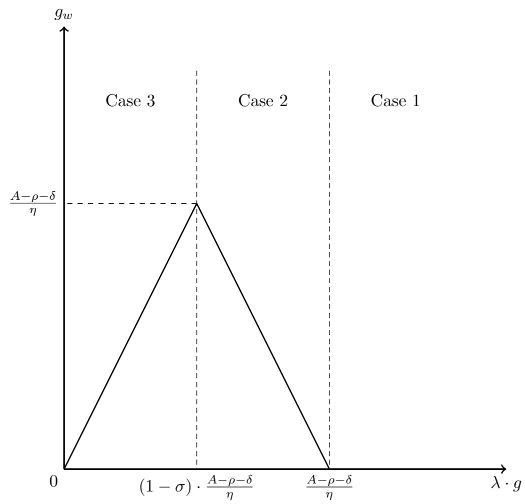



Figure 2.7: Wage growth rate ($g_w$) as a function of the rate of automation ($\lambda g$)

possible outcomes. The following proposition summarizes the results.

**Proposition 2.7** (Race between automation and capital accumulation)**.** *Suppose the complexity distribution of tasks is Pareto and that the economy starts in region 1, i.e., $I_0 < \hat{I}_0$. Then the growth of wages and long-run labor shares are characterized by two thresholds on the rate of automation $\lambda g$:*

1. If $\lambda g > \frac{A-\rho-\delta}{\eta}$ then $\lim_{t\to\infty} w_t = A$ and the labor share converges to zero.
2. If $\frac{A-\rho-\delta}{\eta} \cdot (1-\sigma) < \lambda g \leq \frac{A-\rho-\delta}{\eta}$ then wages grow exponentially at an asymptotic rate $\frac{1}{\sigma}\left(\frac{A-\rho-\delta}{\eta} - \lambda g\right)$ and the labor share converges to one.
3. Lastly, if $\lambda g \leq \frac{A-\rho-\delta}{\eta} \cdot (1-\sigma)$ then wages grow exponentially at an asymptotic rate $\frac{\lambda g}{1-\sigma}$ and the labor share converges to $1 - \left[\frac{(A-\rho-\delta+\eta\delta)/\eta}{\frac{\lambda g}{1-\sigma}+\delta}\right]^{\frac{\sigma-1}{\sigma}}$.

*Proof.* See Appendix A.2.1. □

Intuitively, the proposition illustrates how wages evolve as the result of a race between automation and capital accumulation. As observed above, the fraction $\frac{A-\rho-\delta}{\eta}$ is proportional to the long-run savings rate of the economy. In the first case, if the rate of task automation $\lambda g$ is too high compared the savings rate, then the automation index $I$ crosses the threshold $\hat{I}$ in finite time and the economy transitions into region 2, where wages collapse to $A$ and remain stagnant. If the rate of task automation $\lambda g$ is at an intermediate value, then wage growth is constrained by capital accumulation. Wages grow perpetually at rate $\frac{1}{\sigma}(\frac{A-\rho-\delta}{\eta} - \lambda g)$, which

| Parameter | Value | Description |
|---|---|---|
| $\rho$ | 0.04 | Discount rate |
| $\eta$ | 2 | Risk aversion parameter |
| $\delta$ | 0.1 | Depreciation rate |
| $\sigma$ | 0.5 | Elasticity of substitution |
| $A$ | 0.5 | Total factor productivity |
| $L$ | 1 | Labor endowment |
| $\Phi_0$ | 0.608 | Initial fraction of automated tasks |
| $K_0$ | 4.6 | Initial capital stock |

Table 2.2: Parameter values for the numerical illustration

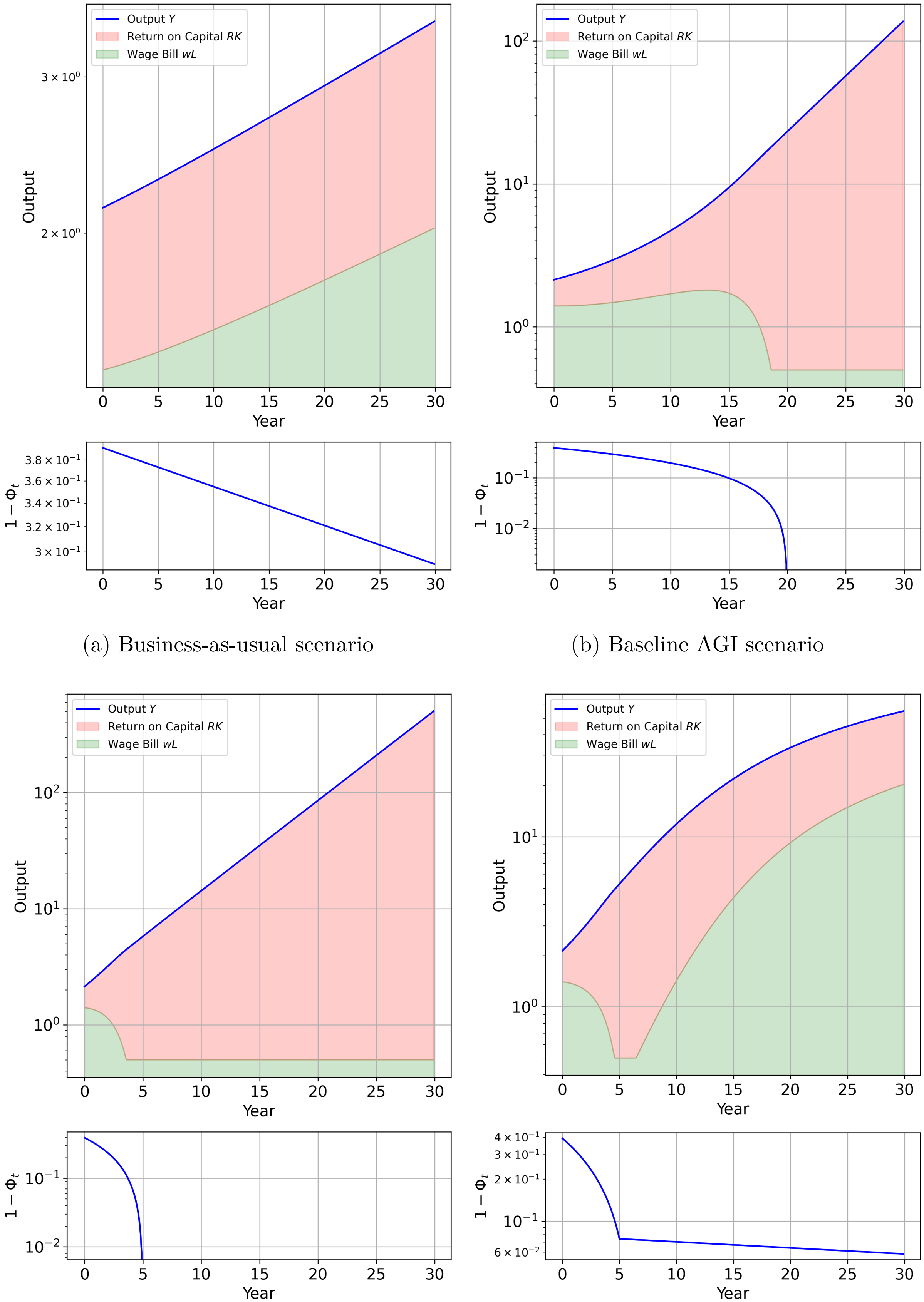
is proportional to the savings rate minus the rate of automation. Finally, if $\lambda g$ is low enough, then the rate of automation rather than capital accumulation constrains wage growth. In other words, wage growth depends on how fast automation increases the efficiency of factor allocation and allows the utilization of abundant capital. Indeed, the growth rate of wages (and of the entire economy) in this regime is increasing in the rate of automation.

Figure 2.7 illustrates the three cases in Proposition 2.7. The figure plots the long-run growth rate of wages as a function of the rate of automation. If $\lambda g$ is sufficiently low as in case 1, then the wage growth rate is increasing in the rate of automation as the upward-sloping part of the curve indicates. Once $\lambda g$ surpasses the first threshold value, the growth rate of wages starts to decline as $\lambda g$ increases further. Lastly, if $\lambda g$ surpasses the second threshold value, then wages do not grow in the long run and stay at $A$.

### 2.3.3 Numerical Illustration

To provide an illustration of the theoretical results, we present simulations of the four automation scenarios described in Section 3.1. Table 2.2 summarizes the parameter values that were common to all the simulations. The first five parameters are standard in the literature, and $L = 1$ is a normalization. We chose $\Phi_0$ and $K_0$ to match a 66% initial labor share with capital at its steady state for that level of technology.

Figure 2.8 presents the results. Panel (a) shows the traditional automation scenario




(a) Business-as-usual scenario

(b) Baseline AGI scenario

(c) Aggressive AGI scenario

(d) Mixed scenario

Figure 2.8: Simulations of the four scenarios

"business-as-usual," in which $\Phi(i)$ reflects a rate of task automation of $\lambda g = 0.01$ per year. The upper part of the panel shows the output, split into the returns to capital (red, upper area) and the wage bill (green, lower area), on a logarithmic scale. The lower part of the panel shows the fraction of unautomated tasks $1 - \Phi$ on a logarithmic scale—for panel (a), this is a straight line, capturing exponential decay. We observe that in the "business-as-usual" scenario, output grows at approximately 2% per year, and both the returns to capital and the wage bill rise approximately in tandem (with a small decline in the labor share due to the effects of automation). Note that this scenario corresponds to case 3 in Proposition 2.7, i.e., capital accumulation is sufficiently fast so that growth is constrained by the speed of automation.

Panels (b) and (c) show the AGI scenarios, in which the fraction of unautomated tasks collapses to zero in 20 or 5 years, respectively. In the baseline AGI scenario, wages grow slightly during the initial periods but then collapse before full automation is reached. After the collapse, wages are equal to the returns to capital, and the economy remains in region 2 where labor and capital are perfectly substitutable, with steady-state growth of 18% per year. In the aggressive AGI scenario, the wage collapse happens after about 3 years. Since the scarcity of labor is relieved earlier than in the baseline AGI scenario, the growth take-off occurs earlier.

Panel (d) shows the "bout-of-automation" scenario. During the initial periods, a large fraction of tasks are automated, leading to wage collapse similar to the aggressive AGI scenario as the economy enters region 2—labor is abundant because of the rapid automation and comparatively low capital stock. However, over time, the economy accumulates more capital, making labor scarcer again. Around year 9, the economy has accumulated sufficient capital so that it returns to region 1. Wages rise above $A$ and start growing again in line with further (slower) advances in automation and further capital accumulation. This scenario illustrates the possibility that labor demand may collapse due to rapid automation but recover later because of a long tail of tasks that cannot be automated.

## 2.4 Extensions

### 2.4.1 Fixed Factors and the Return of Scarcity

If labor is dethroned as the most important factor of production, it becomes useful to disentangle the remaining factors, which have traditionally been lumped together into "capital" in the economic models of the Industrial Age. Let us distinguish between factors that are in fixed supply and factors that are reproducible and can therefore be accumulated. We continue to call all reproducible factors "capital," including compute, robots, power plants, and factories. By contrast, factors in fixed supply include land, space, minerals, or solar radiation.[6] It is difficult to predict which scarce factors will matter the most in an AGI-powered future – in the short term, it is likely that microchips and the semiconductor fabrication equipment ("fabs") used for producing these chips will be bottlenecks, but these are clearly reproducible. By contrast, the raw materials going into the production of chips, for example certain rare earth minerals, are irreproducible. In the longer-term, matter or, equivalently, energy ($E = mc^2$) may be the ultimately source of scarcity.

For the purposes of our analysis, we incorporate a fixed factor in our analysis that we label $M$ for minerals or matter. We assume that the aggregate production function is a Cobb-Douglas aggregator of the task composite and $M$,

$$Y = A \left[ \int_i y(i)^{\frac{\sigma-1}{\sigma}} d\Phi(i) \right]^{\frac{\sigma}{\sigma-1} \cdot \alpha} M^{1-\alpha} \tag{2.19}$$

where $\alpha \in [0, 1]$ is the share of the composite among total output. Then, a version of Lemma 2.1 applies, separating two regimes:

**Lemma 2.8.** *For given $(K, L)$, the automation threshold $\hat{I}$ is defined by (2.3) as in the original lemma and is independent of $M$. It defines two regions:*

***Region 1:*** *If $I < \hat{I}$, then labor is scarce compared to capital and employed only for unautomated tasks. Output is given by*

[6] During the Industrial Age, labor was considered in fixed supply at the relevant times scale – raising humans took so long that their supply could be approximated as exogenous – whereas human capital was a reproducible factor.

$$Y = F\left(K, L, M; I\right) = A\left[K^{\frac{\sigma-1}{\sigma}}\Phi(I)^{\frac{1}{\sigma}} + L^{\frac{\sigma-1}{\sigma}}(1-\Phi(I))^{\frac{1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}\cdot\alpha} M^{1-\alpha} \tag{2.20}$$

*Wages and the returns to $M$ satisfy*

$$w = \alpha A\left[K^{\frac{\sigma-1}{\sigma}}\Phi(I)^{\frac{1}{\sigma}} + L^{\frac{\sigma-1}{\sigma}}(1-\Phi(I))^{\frac{1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}\cdot\alpha-1} L^{-\frac{1}{\sigma}}(1-\Phi(I))^{\frac{1}{\sigma}} M^{1-\alpha} > R$$
$$Q = (1-\alpha)Y/M$$

***Region 2:*** *If $I \geq \hat{I}$, then the relative scarcity of labor is relieved; if the inequality is strict, labor and capital are perfect substitutes for the marginal task. Output is given by*

$$Y = F\left(K, L, M\right) = A(K+L)^{\alpha} M^{1-\alpha} \tag{2.21}$$

*Wages and the return to $M$ satisfy*

$$w = R = \alpha A(K+L)^{\alpha-1} M^{1-\alpha} \tag{2.22}$$
$$Q = (1-\alpha)A(K+L)^{\alpha} M^{-\alpha}$$

*Proof.* The proof follows along the same lines as the proof of Lemma 2.1. □

The presence of the fixed factor $M$ does not affect the key characteristics of the production function described in Lemma 2.1 such as the threshold for the automation index beyond which labor is no longer scarce compared to capital. Similar results apply for the effects of automation on wages:

**Lemma 2.9** (Automation and Wages with $M$)**.** *For given capital intensity $K/L$, an increase in automation $d\Phi(I)$ always raises $R$ for $I < \hat{I}$. The effects on $w$ is hump-shaped: there is a threshold $I^*(K/L)$ with $\Phi(I^*(\cdot)) \in (0,1)$ such that wages $w$ rise in $\Phi(I)$ as long as $I < I^*(K/L)$ but decline in $\Phi(I)$ for $I > I^*(K/L)$. The threshold $I^*$ with $M$ is lower than in Lemma 2.4. In the limit cases of $\Phi(I) = 0$ and $\Phi(I) \geq \kappa/(1+\kappa)$, wages are given by (2.22). The limit is reached for any $K/L$ ratio if $\Phi(I) = 1$.*

*Proof.* The effect of automation on wages for a given $K/L$-ratio is similar to Lemma 2.4:

$$\frac{d\log w}{d\Phi(I)} = \left(\frac{\sigma}{\sigma-1}\alpha - 1\right)\frac{1}{\sigma}\left(k^{\frac{\sigma-1}{\sigma}} - \ell^{\frac{\sigma-1}{\sigma}}\right)(Y/A)^{\frac{1-\sigma}{\sigma}} - \frac{1}{\sigma}\frac{1}{1-\Phi(I)} \tag{2.23}$$

The only difference is the multiplicative term $\frac{\sigma}{\sigma-1}\alpha - 1$, which is smaller than $\frac{1}{\sigma-1}$ for $\alpha < 1$. Thus, the productivity effect is smaller with the fixed factor $M$, meaning that wages start to decline at lower levels of $I$. □

Although the presence of a fixed factor preserves the key results on the automation threshold and the wage effects of automation, the long-run dynamics of the economy change—for the worse. In particular, we find that wages will always decline to the return on capital as the economy will always enter region 2 in finite time.

**Proposition 2.10.** *If* $\lim \Phi(I) = 1$*, then the economy enters region 2 in finite time, and wages equal the returns on capital* $w = R = \rho + \delta$*. The labor share equals* $\alpha L/(K^* + L)$*, where* $K^*$ *is defined by*

$$w = R = \alpha A(K^* + L)^{\alpha-1}M^{1-\alpha} = \rho + \delta$$

*Proof.* If the economy is in region 2 after some finite time $T$, it will converge towards a steady state in which $(K^* + L)$ are pinned down by the Euler equation (2.14), resulting in the expression above. We observe that $K^*$ is the maximum capital level that an optimizing agent will accumulate in this economy since $F_K < \rho + \delta$ for any region 1 allocation, as can be seen from the economy's factor price frontier. This implies that the economy will enter region 2 no later than when the automation threshold reaches the scarcity of labor threshold $\hat{I}$ s.t. $\Phi(\hat{I}) = K^*/L/(1 + K^*/L)$, as defined in Lemma 1.

□

Intuitively, Proposition 2.10 tells us that if there is a fixed factor then automation eventually outpaces capital accumulation regardless of the distribution of tasks. This contrasts with Proposition 2.7, which shows that wages may grow indefinitely if a sufficient amount of tasks is always left to labor.

Figure 2.9 illustrates the implications under the assumption that the Cobb-Douglas for

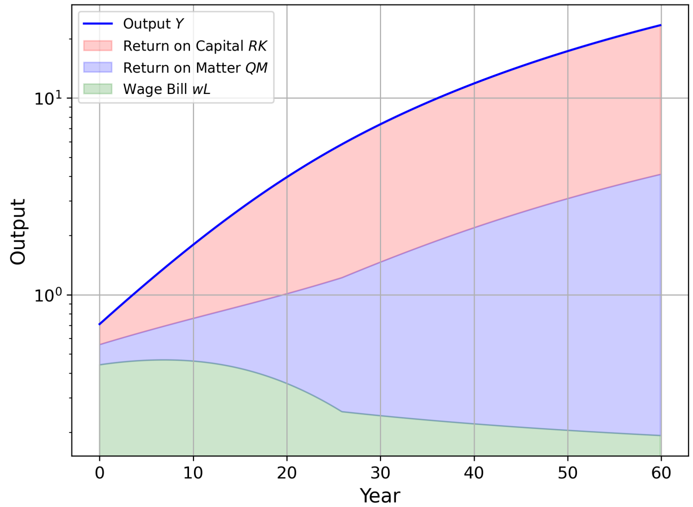



Figure 2.9: Factor shares with fixed factor $M$ in traditional scenario

$M$ is $(1-\alpha) = .10$ in the "traditional scenario," in which a constant fraction of tasks is automated every period. As can be seen, wages peak after about 10 years, and the economy enters region 2 after 25 years, slowly converging to the steady-state level of capital $K^*$. In stark contrast to our simulation results in Section 3, this illustrates that even though there is an infinite tail of unautomated tasks, the presence of a fixed factor bottlenecks capital accumulation and implies that labor loses its scarcity status in finite time.

### 2.4.2 Automating Technological Progress

Our analysis so far has focused on automation as the only form of technological advancement and has taken as given the technology parameter $A$, which is considered as the main driver of productivity gains in the neoclassical growth model. This has allowed us to derive a number of powerful results on the effects of automation in goods production on output and wages. However, there are widespread predictions that advances in AI not only will make output production more efficient but also will speed up technological progress (Aghion, B. Jones, and C. Jones 2019; A. K. Agrawal, McHale, and Oettl 2023; Davidson 2023).

At the most basic level, the production of R&D that drives technological progress consists of atomistic computational tasks—like any other production process described earlier in the paper. For example, A. K. Agrawal, McHale, and Oettl (2023) suggest that scientific hypothesis generation can be viewed as the making of predictions over a vast combinatorial space. We denote the complexity distribution of tasks involved in R&D by the distribution function "Gamma" $\Gamma(i)$, which may differ from the complexity distribution of tasks $\Phi(i)$ involved in producing output—perhaps R&D involves on average more complex computational tasks. W.l.o.g., we assume that our ability to automate both R&D and production are captured by the same automation index $I$.

Building on our earlier task production function and on the endogenous growth setup of C. I. Jones (1995), we assume that advances in the technology parameter $A$ are driven by an ideas production function combining atomistic computational tasks $\{x(i)\}$ that involve computational complexity as reflected in the distribution function $\Gamma(i)$,

$$\log \dot{A} = \log A^{\theta} + \int \log x(i) d\Gamma(i),$$

where the parameter $\theta$ captures the potential for knowledge spillovers or for decreasing returns to knowledge accumulation. Similarly with the production of final goods, we assume that automated tasks can be performed by either capital or labor whereas unautomated tasks require labor,

$$x\left(i\right) = \begin{cases} k_A\left(i\right) + \ell_A\left(i\right) & \text{for } i < I, \\ \ell_A\left(i\right) & \text{for } i \geq I. \end{cases}$$

To keep our analysis tractable, we assume that there is an exogenous supply of knowledge workers $L_A = 1$ that can only work in ideas production in addition to the unit supply of workers $L_Y = 1$ who are solely engaged in final output production. Moreover, we assume unitary elasticity of substitution between tasks in the output production function so $\sigma = 1$. Analogs of lemma 1 hold for both production functions. As long as labor is scarce (region 1), we observe that the production functions of final output and knowledge satisfy $F(K, L) \simeq AK_Y^{\Phi(I)} L_Y^{1-\Phi(I)}$ and $\dot{A} \simeq A^{\theta} K_A^{\Gamma(I)} L_A^{1-\Gamma(I)}$ where $K_Y$ and $K_A$ are the aggregate amounts of capital employed in final output or knowledge production.

Following the hypothesis of Aghion, B. Jones, and C. Jones (2019) and the proof of Trammell and Korinek (2023), it can then be shown that once automation in the two production functions has proceeded sufficiently, growth in the described economy will experience what

Aghion, B. Jones, and C. Jones (2019) term a "type II singularity." The intuition is that a rapidly growing capital stock generates an explosion in R&D output and technological progress that feeds on itself, resulting infinite output in finite time. The following proposition states this result formally under the assumptions of a constant savings rate $s \in (0, 1)$, a constant allocation of capital across final output and ideas production, and no depreciation for tractability.

**Proposition 2.11.** *The economy enters a path of super-exponential growth in technology $A$ and output $Y$ that diverges to infinity in finite time once the automation index $I$ reaches the level $I^{\heartsuit}$such that*

$$\frac{\Gamma\left(I^{\heartsuit}\right)}{\left(1-\Phi\left(I^{\heartsuit}\right)\right)(1-\theta)} > 1$$

*The marginal product of labor in the production of final output diverges to infinity alongside output.*

*Proof.* Observe that once the described threshold has been passed, the production functions $F(K, L) \simeq AK_Y^{\Phi(I^{\heartsuit})} L_Y^{1-\Phi(I^{\heartsuit})}$ and $\dot{A} \simeq A^{\theta} K_A^{\Gamma(I^{\heartsuit})} L_A^{1-\Gamma(I^{\heartsuit})}$ for constant fractions of capital $K_Y = cK$ and $K_A = (1-c)K$ are lower bounds for the actual production functions using the (still increasing) automation level $\Phi(I)$ and $\Gamma(I)$. The result is then a direct application of the proof in Trammell and Korinek (2023) (Appendix B.2). Note that $w_t \geq A_t$ no matter if output production is in Region 1 or Region 2 as defined in Lemma 1. Therefore the marginal product of labor in the production of final output also diverges to infinity. $\square$

The condition in the proposition depends on three parameters – sufficient automation in the production of ideas $\Gamma(\cdot)$, sufficient automation in the production of final output $\Phi(\cdot)$, and sufficient returns to the accumulation of knowledge. Remarkably, as long as the production of final output has been sufficiently automated (sufficiently high $\Phi$), the remaining two parameters can take on any finite levels. This highlights that sufficient automation in the production of final output and thus capital accumulation will always lead to an explosion in growth.

Figure 2.10 illustrates the path of wages under a CES production function for output and optimal savings, numerically illustrating the explosive path of output growth and type-II singularity that we identified in the proposition. The convexity of the curve on a log-

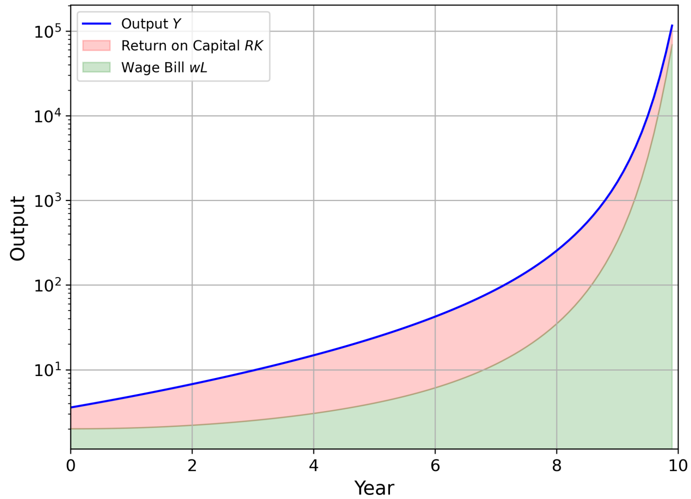



Figure 2.10: Output and wage growth under technological progress

scale indicates that the growth rate of wages is ever-increasing due to the acceleration of technological progress. (Note that the log scale hides that the labor share of output is declining.) In the figure, we have cut off the simulation at $t = 10$. The singularity occurs shortly thereafter.

In summary, even if automation induces wages to collapse to the returns to capital at $A$, rapid technological progress from the automation of R&D allows workers to benefit from the advancement of AI once sufficient automation has taken place.

More generally, the force described in this subsection is plausible, and sufficient progress in AI will likely indeed lead to rapid technological advances and increases in living standards. At the same time, it is also likely that both the production of output and of ideas will eventually be bottlenecked by fixed factors, as we emphasized in Section 2.4.1. A model that comprehensively incorporates both effects is beyond the scope of this paper.

### 2.4.3 Nostalgic Jobs or Limits on Automation

Our baseline model assumed that the automation of work was driven solely by technological factors, occurring as soon as the compute requirements of performing specific tasks were reached. However, even if it is technologically feasible to perform certain tasks, our society may decide that it is preferably for those tasks to remain exclusively human. For example, Korinek and Juelfs (2023) observe that jobs such as priests, judges, or lawmakers may remain exclusively human long after the time when they can be performed at equal or superior levels by machines, labelling such jobs "nostalgic jobs."

For the purposes of our analysis, we assume that there is a separate distribution function $\Psi(I)$ that captures how far the automation index $I$ must advance for society to choose to automate task $I$ – in addition to the distribution $\Phi\,(I)$ capturing the technological possibility of automation. The inequality $\Psi\,(I) \leq \Phi(I)$ reflects that society can only choose to automate tasks that are feasible to automate. The inequality is strict if there are tasks that could be automated from a technical perspective but aren't for societal reasons. If $\lim_{I\to\infty}\Psi(I) < \Phi(I) \leq 1$, then this captures that there are tasks that humans choose to never automate even though they could be.

The described setup can also capture situations in which tasks are delegated to machines with a delay, i.e., for higher levels of the automation index $I$ than what is technologically feasible. Korinek and Juelfs (2023) describe two reasons for why this may occur: First, as the capabilities of machines to perform certain tasks become better and better than human abilities, it may become increasingly untenable for the tasks to be left to humans. For example, if AI systems demonstrably become much fairer judges with fewer biases and noise than human judges, it may become untenable to leave many judicial deicisons to error-prone humans. Second, with sufficient advances in robotics, it may become more and more difficult to distinguish humans and AI-powered robots performing human services. They observe that a robot priest with greater emotional intelligence than humans and a more comprehensive theory of human minds than a human priest may be able to perform the tasks typically performed by human priests quite perfectly, or intentionally somewhat imperfectly so as to not give away that it is a robot. Both of these categories require that the performance of AI systems is sufficiently above human levels, corresponding to a sufficiently high level of the automation index $I$.

**Maximizing Wage Growth** Consider the problem of a government with the objective to maximize wage growth by imposing limits on automation and choosing an optimal path $\Psi(I) \leq \Psi(I)$. The following result characterizes the optimal $\Psi(I)$ among all Pareto distributions—given exponential advances in the automation index $I$, this amounts to the government choosing an optimal constant rate of automation per time period.

**Proposition 2.12** (Maximizing Wage Growth)**.** *Suppose $\Psi$ is a Pareto distribution defined as $\Psi(I_t) = 1 - I_0^{-\lambda} e^{-\lambda g t}$ where $I_0$ is the initial automation index and $\lambda g$ is the rate of task automation. Then the long-run growth rate of wages is maximized for $\lambda g = (1-\sigma) \cdot \frac{A-\rho-\delta}{\eta}$ , assuming that $\Psi(I) \leq \Psi(I) \forall I$ for this distribution. As a result, wages grow at rate $\frac{A-\rho-\delta}{\eta}$.*

*Proof.* The proof follows from Proposition 2.7. The rate of automation is lowest in case 3. And the wage growth rate is increasing in $\lambda g$. Thus, the wage growth rate increases until $\lambda g = (1-\sigma) \cdot \frac{A-\rho-\delta}{\eta}$. Once $\lambda g$ surpasses $(1-\sigma) \cdot \frac{A-\rho-\delta}{\eta}$, the growth rate of wages decreases in $\lambda g$ until $\lambda g = \frac{A-\rho-\delta}{\eta}$ at which the growth rate equals zero. Therefore, the maxmum growth rate of wages is $\frac{A-\rho-\delta}{\eta}$ at $\lambda g = (1-\sigma) \cdot \frac{A-\rho-\delta}{\eta}$. Figure 2.7 provides a graphical illustration of this finding—the peak of the wage growth rate as a function of $\lambda g$ is $\frac{A-\rho-\delta}{\eta}$. $\square$

Figure 2.11 shows what happens if we slow down progress in the "baseline AGI scenario" from Section 3 so that wages growth is maximized. Up until period 14, a wage-maximizing planner is constrained by the natural pace of automation and sets $\Psi(I) = \Phi(I)$ over that stretch. After that point, the baseline AGI scenario implies rapid declines in the labor share, but the planner sets $\Psi(I) < \Phi(I)$ to slow down effective automation. The left panel of the figure shows the paths of output and wages, and the right panel depicts the two variables in relative terms for the two scenarios. Up until period 14, the paths in the two scenarios roughly coincide (with a minor gap opening since the AGI scenario triggers rapid capital accumulation in advance of the economy achieving full automation). Thereafter, the wage-maximizing planner obtains a path of exponentially growing wages, as predicted by the proposition, whereas wages in the AGI scenario collapse. Notably, the right-hand panel also illustrates the output cost of foregoing the possibility of full automation. As can be seen, the output cost of holding back automation is low at first, but eventually, almost 100% of the output potential of the economy is lost by holding back automation.

Our finding illustrates that slowing down automation may be a powerful tool to increase wages, albeit it comes at the cost of reducing output growth. The described policy is feasible under both of the AGI scenarios simulated in the previous section and always results in

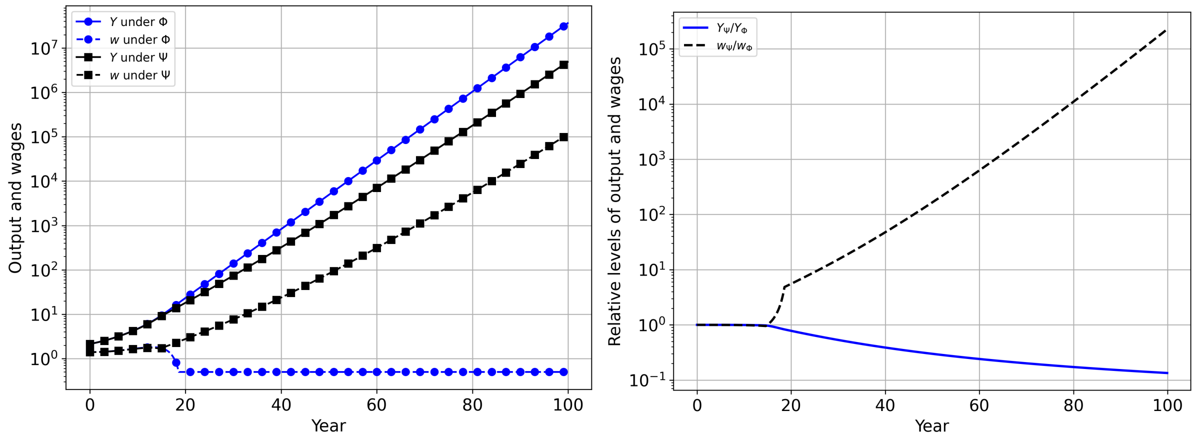



Figure 2.11: Comparison of output and wages under $\Phi$ and $\Psi$

exponentially growing wages instead of the collapse wihin a matter of years that would otherwise occur when AGI automates human tasks too quickly.

### 2.4.4 Heterogeneous Worker Skills

When labor is heterogeneous, individuals are hit by the effects of automation at different times, depending on the extent to which their skills are automated. In practice, workers differ along many different dimensions, and each worker's labor may be complemented or substituted for in different ways by technological advances. One of the classical ways of accounting for heterogeneity in the labor market, going back to Katz and Murphy (1992a), is to split workers into skilled and unskilled based on a threshold level of educational attainment. An additional distinction, introduced by D. H. Autor, Levy, and Murnane (2003), was to categorize workers according to whether they hold cognitive or manual jobs performing routine or non-routine activities. Under the described paradigm, we could capture the distribution of tasks in compute space separately for each of the resulting buckets (e.g., routine cognitive workers), and analyze how advances in computing capabilities will affect that type of workers. Recent advances in AI have raised the possibility that many cognitive tasks, including non-routing tasks, may be automated relatively soon (e.g. Korinek 2023). However, ongoing advances in robotics make it likely that non-routine manual jobs will be similarly affected to cognitive tasks by the recent wave of progress in foundation models (Ahn et al. 2022).

For our purposes here, we found it useful to consider labor that differs in uni-dimensional but continuous manner. We assume that workers differ in an exogenous parameter that we label skill $J$, which reflects the maximum level of task complexity that the worker can perform. Workers' skill levels are described by the distribution function $\Upsilon(J)$. For analytical simplicity, we assume that $\Phi(I) \geq \Upsilon(I)$.[7]

For a level of the automation index $I$, a fraction $\Upsilon(I)$ of workers are perfectly substitutable by machines and earn wage $w_j = A$. A fraction $1 - \Upsilon(I)$ is not substitutable, but given that the remaining workers are sufficiently skilled, they are all effective substitutes for each other and earn wage $w_j = F_L(K + \Upsilon(I), 1 - \Upsilon(I))$. In contrast to our baseline model, this captures the concern that automation may make workers on the lower rungs of the skill distribution redundant, whereas workers who are able to perform at higher levels of skill may benefit from automation.

In the long run, assuming less than full automation ($\Phi(I) < 1$ for any finite $I$), the share of workers who are gainfully employed will decline over time and will asymptote to $1 - \lim_{J\to\infty} \Upsilon(J)$, i.e., only workers who can perform unautomated tasks with arbitrary computational complexity will earn higher returns than capital. If $\Upsilon(J) = 1$ for finite $J$, then the role of all human labor will lose its scarcity value in the same manner as in the AGI scenarios in our baseline model. Conversely, if $\Upsilon(J)$ asymptotes to 1, then there may be ever-growing inequality among workers: an ever-declining fraction of workers at the top may see incomes rise without bounds, whereas a fraction of the population that asymptotes towards one will see wages collapse to the level that equates the return on capital $A$.

**Heterogeneity in both skill and productivity** The described setup could easily be extended to include heterogeneity in individual worker productivity in addition to heterogeneity in skill. Assume that workers not only have different skill levels $J_j$ but are also endowed with different efficiency units of labor $L_j$ per time period. This may capture, for example, that there may be two economists who can both write papers up to complexity $J$, but one of them is twice as fast at it than the other. This could explain the empirial observation that

[7] This assumption implies that for any level of automation $I$, there are sufficiently many skilled workers at all unautomated complexity levels left so that we can treat unautomated workers as perfect substitutes for each other.

workers in the same occupation sometimes earn significantly different wages.

**Complementary human capital** An alternative lens that may be relevant in the current era of cognitive automation is that workers possess different levels of human capital that is affected by automation. To keep our discussion simple, assume again that each worker $j$ is characterized by a skill level $J_j$ as well as an exogenous amount of human capital $H_j > 0$, which enables them to supply $L_j = H_j$ efficiency units of labor per time period. As the automation index $I$ surpasses a given worker with skill level $J_j$, the human capital that they possessed is fully devalued. The loss is greater and more painful for workers with more human capital.

### 2.4.5 Compute as Specific Capital

An important feature of the ongoing AI take-off is the scarcity of compute. In our baseline model, we followed the standard neoclassical practice of modeling capital as uniform, capable of being deployed in the production of any task. As illustrated in Figure 2.3, automation of new tasks then implies that the existing capital stock can be smoothly allocated to a larger number of tasks, unlocking immediate productivity gains.

In practice, however, many types of capital are specific to the task for which they were created and difficult or impossible to reallocate, corresponding to what the literature has traditionally called putty-clay capital.[8] In the current context, the most salient type of specific capital on which AI systems rely is compute, which is in very limited supply, slowing down the deployment of AI systems for new tasks. Another example of specific capital is organizational capital, including the capital derived from investments into developing new processes for deploying new technologies in firms.

We expand our framework by assuming that each unit of capital investment is specific to a task $i$ and can only be invested once the task is automated, i.e., once $I \geq i$. This leaves the task production function (2.2) unaffected but modifies the capital accumulation constraint: instead of a single law of motion for capital (2.13), the consumer needs to separately

[8] The notion was first introduced by Leif Johansen (1959) – once putty has been turned into clay, it cannot be turned into another shape – and expanded by Solow (1962) and others.

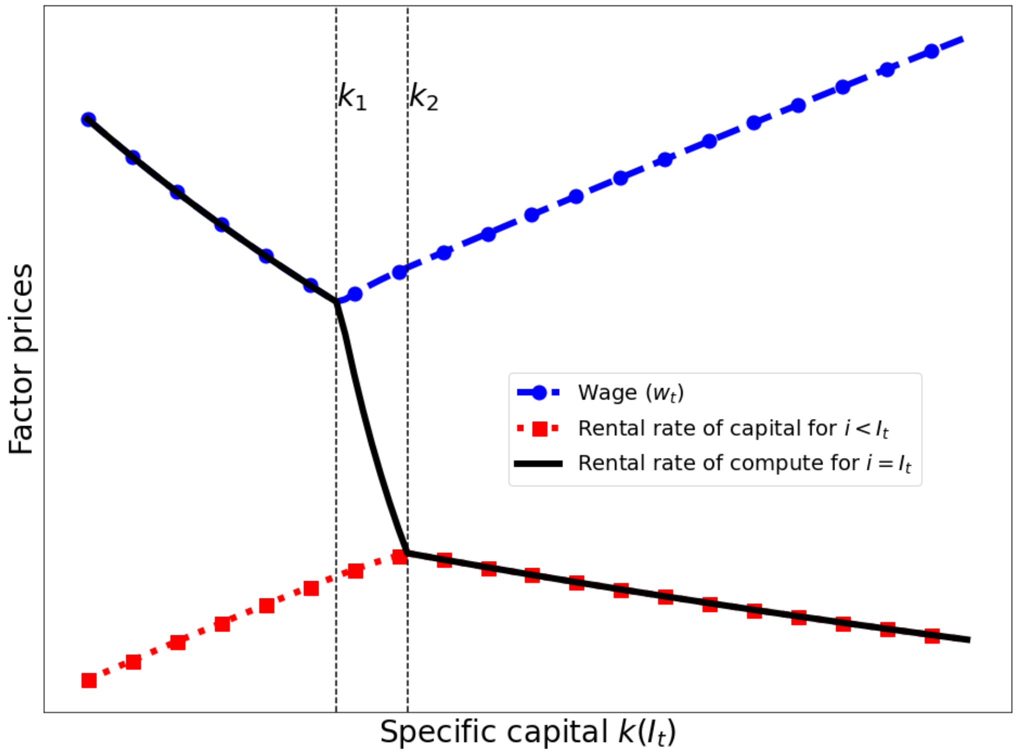



Figure 2.12: Specific capital and factor prices

keep track of each type of capital $k(i)$ since capital that is deployed for one task cannot be redeployed later. In an economy in which automation is proceeding slowly and steadily, the consumer problem is unchanged as the resulting constraints on capital redeployment are slack—every instant of time, a density $\phi(I_t)$ of new tasks is automated, and sufficient capital for those tasks is instantaneously accumulated. By contrast, if the economy experiences a bout of progress that leads to a discrete mass of tasks suddenly being amenable to automation, the accumulation of the relevant specific capital may lag behind. The rapid rise of LLMs at the time of writing may be an example of such a bout.

To illustrate this analytically, assume that a discrete mass of tasks $\Delta_t = \Phi(I_t) - \Phi(I_{t^-}) > 0$ is automated at time $t$, and let us interpret the specific capital $k(I_t)$ required for these tasks as compute. At time $t$, no compute has been accumulated yet, $k(I_t) = 0$, so all type-$I_t$ tasks are performed by humans at wage $w$, even though they could technically be automated.

Figure 2.12 illustrates the factor returns as a function of the accumulation of compute $k(I_t)$ while holding the inputs of labor and other capital constant: at first, $k(I_t) = 0$, and the economy starts out at the left side of the figure where labor and compute are, at the margin, perfect substitutes so the returns on the two are equated. The rental rate on traditional capital is comparatively low.

Over time, compute capital $k(I_t)$ is accumulated and progressively substitutes for labor. As long as compute remains below the first threshold $k(I_t) < k_1$, illustrated by the first vertical line in the figure, labor and compute remain perfect substitutes at the margin, but wages $w$ decline with the addition of more compute, whereas the return on traditional capital rises. Within this region, all capital investment goes into compute. Once sufficient compute ($k_1$) is accumulated so that all humans are replaced from type-$I_t$ tasks, all labor is allocated to the remaining unautomated tasks with $i > I_t$, and the marginal product of compute decouples from wages. All capital investment continues to be devoted to compute; wages $w_t$ and the returns to traditional capital rise whereas the return on compute $k(I_t)$ declines sharply until it reaches the marginal product of all other types of capital. This is the middle region between $k_1$ and $k_2$ where only the return on compute, captured by the dotted curve, is decreasing. Once the second threshold is passed, the marginal product of compute and other capital is equated. Any additional capital investment is spread proportionately across all types of specific capital $k(i), i \leq I$, and leads to a decline in the return on capital. In summary, the race between automation and capital accumulation leads to a non-monotonic response of wages, depicted by the blue curve marked with dots, and the returns to traditional capital, depicted by the red curve marked with squares.

The following proposition characterizes the thresholds for the amount of specific capital and summarizes the non-monotonic response of factor prices to the accumulation of this specific capital analytically.

**Proposition 2.13** (Specific capital and factor returns). *Suppose that the current amount of the specific capital is given by $k(I_t)$. There are threshold values $k_1$ and $k_2 > k_1$ such that (i) if $k(I_t) < k_1$ then the wage decreases and the rental rate of the traditional capital increases with $k(I_t)$, (ii) if $k_1 \leq k(I_t) < k_2$ then both the wage and the rental rate of traditional capital increase with $k(I_t)$, and (iii) if $k(I_t) \geq k_2$ then specific capital $k(I_t)$ is only accumulated* *alongside traditional capital, and the wage increases with capital accumulation.*

*Proof.* See appendix. □

In summary, rapid advances in automation may lead to episodes in which certain types of specific capital (like compute) may exhibit very high returns, but since capital is reproducible, the resulting accumulation of specific capital will ultimately dissipate the excess returns. The implication is that after an adjustment period, specific capital for newly automated processes will be just another form of capital earning the market rate of return.

## 2.5 Conclusions

This paper models the economic impact of the transition towards artificial general intelligence on output and wages. We develop a compute-centric framework that represents work as consisting of tasks that vary in their computational complexity and study how exponential growth in computing power will affect automation and the advent of artificial general intelligence (AGI).

The paper illuminates how different plausible assumptions about the complexity distribution of tasks across "compute space" translate into dramatically different scenarios for economic outcomes. If the task distribution has an infinite Pareto tail, reflecting unlimited complexity of human work, then the we show that wages can rise indefinitely if the tail is sufficiently thick, as capital accumulation automates ever more complex tasks but there always remains enough for human labor. However, if the Pareto tail is too thin, then automation ultimately outpaces capital accumulation and causes a collapse in wages.

Moreover, if the complexity of tasks humans can perform is bounded, mirroring computational limits on human cognition, then we demonstrate that wages would at first surge as machines displace more and more human labor, but would eventually collapse, even before full AGI is reached.

Beyond these scenarios, the paper provides several powerful general insights. Using the economy's factor price frontier, we show that the effects of automation follow an inverse U-shape, first increasing wages by utilizing abundant capital but eventually decreasing wages due to labor displacement. We show that sufficient capital accumulation is essential to prevent automation from depressing wages. Adding fixed factors like land causes wages to eventually decline. Yet automating innovation itself can restart wage growth after an initial automation-driven collapse.

The novel compute-centric approach opens up a new perspective for analyzing the economic impact of artificial intelligence. Interesting next steps include incorporating labor and capital adjustment costs, modeling endogenous innovation, analyzing distributional impacts more fully, studying macroeconomic dynamics and policies, and evaluating the possibility of an intelligence explosion with AGI.

By presenting several rigorous scenarios for how the transition to AGI may unfold, we hope that this paper will make an important contribution to enabling economists, policymakers and the public to examine alternative futures and to prepare for the technological transformations on the horizon.

# Chapter 3

# Constrained Efficiency of Capital under Capital-Skill Complementarity

## 3.1 Introduction

In economies with incomplete insurance, agents respond by accumulating a buffer stock of savings (Huggett 1993; Aiyagari 1994). This extra demand for assets drives down the return, creating a pecuniary externality (Davila et al. 2012; Park 2018). Davila et al. (2012) show that the nature of this pecuniary externality – that is, whether a planner would desire more or less capital – depends on the nature of the households with high marginal utility of consumption. If these "consumption-poor" agents have primarily asset income (think of an economy in which agents become unemployed), then the planner wants to reduce capital in order to raise the return; if instead the poor have primarily labor income (think of an economy in which agents draw low productivity), the planner wants to increase the capital stock to raise wages.

In this paper, we study how this mechanism operates in the presence of capital-skill complementarity (Krusell et al. 2000). In this model, there are two types of households, high-skilled and low-skilled. Capital complements high-skilled labor but substitutes low-skilled labor. There are two possible nestings of this production technology. In the first nesting, from Krusell et al. (2000), capital and high-skilled labor form a composite good that is then a substitute for low-skilled labor; in the second nesting, capital and low-skilled labor form a composite good that is a complement to high-skilled labor. The distinctions are important – in the first case, an increase in the capital stock raises both wages, but increases inequality because the high-skilled wage rises more, but in the second case an increase in the capital stock reduces low-skilled wages.

The second case poses a problem for the planner, and is therefore our main focus. Consumption-poor agents in this model have primarily labor income, so the planner wants to raise their wages. If labor supply is inelastic, the planner cannot raise the wages of both high-skilled and low-skilled workers, because the capital stock affects them in different directions. In a two-period model, we characterize the forces that determine the optimal capital stock in the second period. To do so, we isolate the pure effect of the skill premium; all households enter the second period with identical capital stocks, but some become high-skilled and some low-skilled. As a result, the effect of the return to capital is identical, and the key forces act through the relative wage.

We extend the baseline setup of inelastic labor supply in two ways. First, we consider elastic labor supply. Including elastic labor supply frees the planner to adjust wages in opposite directions, by adjusting the relative labor inputs of the types of workers. We show that capital accumulation benefits only those who are high-skilled and supply a sufficiently large amount of labor. The rest are hurt because capital accumulation reduces their capital income as well as labor income if they are low-skilled.

Second, we extend the model to distinguish structure and equipment capital. Structure capital complements both high-skilled and low-skilled labor. On the other hand, equipment capital complements high-skilled labor while subsituting for low-skilled labor. We find that, in addition to the forces described in the baseline setup, there is an additional downward pressure on the accumulation of equipment capital. The reason is that equipment capital crowds out structure capital, which complements all labor types.

The analysis of the paper is relevant for understanding the trade-offs that arise as policy-makers address inefficiencies associated with technological advances. In particular, governments in reality are often subject to the same market structure as the private agents. For example, a source of market incompleteness associated with technological advances is that workers cannot perfectly insure against all possible risks of future automation. The paper lays out the key considerations in such circumstances.[1]

[1]For recent work on the policy implications of automation, see Beraja and Zorzi (2024), Guerreiro, Rebelo, and Teles (2022), and Korinek and Stiglitz (2018). Unlike these papers, the focus of our paper is the role of pecuniary externalities and the problem of a constrained planner.

## 3.2 Baseline Model

To analyze the determinants of the constrained-efficient level of aggregate capital, we examine two-period models and characterize conditions under which there is over- or under-accumulation of capital. We begin with the baseline model.

### 3.2.1 Model Setup and Optimality

The economy lasts for two periods and is populated by a unit mass of individuals. Individuals are ex ante heterogeneous due to their initial endowment $y_1$, which is distributed according to the distribution $\Gamma(\cdot)$. The borrowing of the individuals is only restricted by their (lowest possible) future income. Future income $y_2 \in \{w_L, w_H\}$ is random and determined by the realization of the type of each individual in period 2. An individual becomes type $i \in \{L, H\}$ with probability $\pi_i$ where $\sum_i \pi_i = 1$ and inelastically supplies one unit of labor to earn income $w_i$. The expectation operator $\mathbb{E}_i$ indicates that the source of randomness is the future type. Each individual earns the lifetime value $V(\cdot)$.

$$V(y_1; K) = \max_{c_1, c_2, a \in [0, y_1]} u(c_1) + \beta \mathbb{E}_i [u(c_2)]$$

subject to the budget constraints $c_1 + a \leq y_1$ and $c_2 \leq ra + y_2$. The representative firm combines high-skilled labor ($N_H$), low-skilled labor ($N_L$), and capital ($K$) using a constant returns to scale production technology $F(\cdot)$ that satisfies $F_{KL} < 0 < F_{KH}$ where $F_{Ki} = \frac{\partial^2 F}{\partial K \partial N_i}$. Intuitively, capital complements high-skilled labor but replaces low-skilled labor.

### 3.2.2 Constrained Efficiency of Aggregate Capital Stock

Consider how the value of an individual depends on the level of aggregate capital by taking the derivative of $V(\cdot)$ with respect to $K$:

$$\frac{dV}{dK} = \beta \mathbb{E}_i \left[ \frac{d}{dK} u(F_K a + F_i) \right] = \beta \sum_i \pi_i u'(F_K a + F_i)(F_{KK} a + F_{Ki}).$$

The term $F_{KK}a + F_{Ki}$ captures the pecuniary externalities. To see how the welfare of individuals with different asset holdings and labor types are affected, we rewrite the above expression as

$$\begin{aligned}\frac{dV}{dK} =&\beta u'\left(F_K a + F_H\right) F_{KH}\pi_H \\ &\times \left[\left(1-\frac{a}{K}\right)(B+1)\left((\chi-1)\pi_L+1\right)+(\chi-1)\left(B\left(1-\pi_L\right)-\pi_L\right)\right]\end{aligned} \tag{3.1}$$

where

$$B := \frac{F_{KL}\pi_L}{F_{KH}\pi_H} \quad \text{and} \quad \chi := \frac{u'\left(F_K a + F_L\right)}{u'\left(F_K a + F_H\right)}.$$

Note that $B$ is the changes in low-skill wages relative to high-skill wages and captures the strength of the distributional effects due to capital-skill complementarity. Thus, in addition to the relative amount of savings $(a/K)$, the sign of the derivative in (3.1) depends on the distributional effects captured by $B$.

To see this dependence more clearly, suppose the agents receive different amounts of endowment but there is no income risk. That is, we shut down the heterogeneity in labor types. Then $\chi = B = 1$, so that

$$\frac{dV}{dK} =\beta u'\left(F_K a + F_H\right) F_{KH}\pi_H \cdot 2\left(1-\frac{a}{K}\right). \tag{3.2}$$

It follows that $\frac{dV}{dK} < 0$ if $1 - \frac{a}{K} < 0$. In other words, the average savings $K$ is the threshold savings level above which agents are hurt from an increase in aggregate capital.

Now suppose the agents are homogeneous in their endowments but there is income risk so that wealth heterogeneity is shut down. Then all individuals hold the same amount of asset $a = K$, so that

$$\frac{dV}{dK} =\beta u'\left(F_K K + F_H\right) F_{KH}\pi_H\left(\chi-1\right)\left(B\left(1-\pi_L\right)-\pi_L\right) < 0. \tag{3.3}$$

Thus, at the decentralized equilibrium there is an overaccumulation of capital. Because of market incompleteness, agents hold excess capital for self-insurance, pushing returns down too far compared to what the planner would want. The following proposition summarizes how capital-skill complementarity affects the threshold savings relative to aggregate capital.

**Proposition 3.1.** *Suppose that agents' skill types are random in the second period and also that the production function $F(\cdot)$ is such that $F_{KL} < 0 < F_{KH}$. Then the threshold savings relative to aggregate capital is lower than the no-risk case in (3.2).*

*Proof.* Continuing from (3.1), set $\frac{dV}{dK} = 0$ so that

$$\left(1 - \frac{a}{K}\right)(B+1)\left((\chi-1)\pi_L + 1\right) + (\chi - 1)\left(B(1-\pi_L) - \pi_L\right) = 0$$

By rearranging the terms, we obtain the threshold relative savings

$$\left(\frac{a}{K}\right)^* = \frac{B\chi + 1}{(B\chi+1) + (\chi-1)(\pi_L - B\pi_H)}$$

Note that $0 < \left(\frac{a}{K}\right)^* < 1$, and so the threshold is lower than the threshold level in the case without heterogeneous labor types in (3.2). □

Therefore, the presence of capital-skill complementarity puts a downward pressure on the constrained-efficient aggregate capital stock due to the distributional effects across labor types. A decrease in capital stock can be beneficial because it reduces income risks and supports low-skill wages.

## 3.3 Extensions

### 3.3.1 Elastic Labor Supply

If labor supply is elastic then the planner has an additional means to redistribute from rich to poor agents. More specifically, the planner can dictate agents with high productivity shocks within each skill group to work more than they would in a decentralized equilibrium and those with low productivity shocks to work less. As a result, the planner can redistribute towards the consumption-poor.

The setup is the same as before with the only difference being the presence of disutility from labor supply in the second period. Let $n$ be the work hours and $v(n)$ be the disutility from work so that the utility function in period 2 is

$$U(c,n) = u(c) - v(n) = \frac{c^{1-\gamma}}{1-\gamma} - \kappa \frac{n^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}}$$

where $\eta$ is the Frisch elasticity of labor supply. Then the problem of agents becomes

$$V = \max u(c_1) + \beta \mathbb{E} U(c_2, n)$$

subject to

$$\begin{aligned} c_1 + a &\leq y_1 \\ c_2 &\leq w_i z_i n_i + ra \\ a &\in [-w_L z_{min}/r, y_1] \\ n &\in [0, 1] \end{aligned}$$

where $z_{min}$ is the minimum level of the productivity shock.

We are interested in signing the pecuniary externality associated with the labor supply decision. As before, the constrained planner chooses the aggregates taking into account the influence on prices. That is, the planner chooses $K$ and $\{N_i\}_{i \in \{L,H\}}$ that maximize the weighted sum of individual values. Consider the value of an agent with initial endowment $y_1$.

$$V = \max u(y_1 - a) + \beta \mathbb{E}\left[u(F_i z_i n_i + F_K a) + v(n_i)\right]$$

Since the type and the labor productivity in the second period are random the expectation operator is over the type and $z$. Next, we sign the pecuniary externalities on individuals' values as $K$ and $N_i$ change.

Suppose the planner raises $K$ slightly. Then the change in an individual's value is:

$$\begin{aligned}\frac{\partial V}{\partial K} =& \beta \mathbb{E}_i \left[ u'(F_i z_i n_i + F_K a)(F_{Ki} z_i n_i + F_{KK} a) \right] \\ =& \beta u'(H) F_{KH} N_H \left[ \pi_L \chi \left( B \frac{z_L n_L}{N_L} - (B+1) \frac{a}{K} \right) + \pi_H \left( \frac{z_H n_H}{N_H} - (B+1) \frac{a}{K} \right) \right] \end{aligned} \tag{3.4}$$

where $u'(H)$ is the marginal utility of high-skilled workers. Whether an agent benefit from an increase in aggregate capital is determined by his relative savings and relative labor supply in each realization of the labor type. Suppose an agent is a net saver. Then an increase in capital reduces capital income regardless of labor type. Also, if he becomes low-skilled then labor income falls as well. However, if he is high-skilled then his labor income rises. These effects are weighted by the relative sensitivity of low-skill wage $B$, the probability for each type $\pi_i$, and the marginal rate of substitution $\chi$. For example, if the marginal rate of substitution is very high then an increase in aggregate capital is welfare-reducing in expectation because it reduces the labor and capital income of low-skill agents.

**Proposition 3.2.** *In an economy with elastic labor supply decisions, the threshold relative savings level is higher for (i) low-skilled agents with below-average work hours and (ii) high-skilled agents with above-average work hours.*

*Proof.* Set (3.4) to zero to derive the threshold level.

$$\left( \frac{a}{K} \right)^* = \frac{(\pi_L \chi + \pi_H) + \pi_L \chi \left[ B \left( \frac{z_L n_L}{N_L} - 1 \right) - 1 \right) + \pi_H \left( -B + \frac{z_H n_H}{N_H} - 1 \right]}{\pi_L \chi + \pi_H}$$

Observe that if $\frac{z_L n_L}{N_L} < 1$ then the first term in the square brackets contributes positively to the threshold level. On the other hand, if $\frac{z_H n_H}{N_H} > 1$ then the second term contributes positively. □

To understand the pecuniary externalities associated with labor supply decisions, suppose the planner chooses the aggregate supply of labor before skill types realize. That is, the planner chooses labor supply in the veil of ignorance. Then we have:

$$\frac{\partial V}{\partial N_L} = \beta \mathbb{E}_{i,z} \left[ u'(F_i z_i n_i + F_K a)(F_{Li} z_i n_i + F_{LK} a) \right].$$

Notice that we can also interpret the above expression as the economy-wide pecuniary externalities in the homogeneous wealth case. Abusing the notation $\Gamma(\cdot)$, we can write the expectation as follows:

$$\begin{aligned}\frac{\partial V}{\partial N_L} =&\beta\sum_i \pi_i \int u'(F_i z_i n_i + F_K a)(F_{Li} z_i n_i + F_{LK} a)\Gamma(dz_i)\\ =&\beta\sum_i \pi_i \int u'(F_i z_i n_i + F_K a)(F_{Li} z_i n_i + F_{LK} a)\Gamma(dz_i)\\ =&\beta\pi_L \int u'(F_L z_L n_L + F_K a)(F_{LL} z_L n_L + F_{LK} a)\Gamma(dz_L)\\ &+\beta\pi_H \int u'(F_H z_H n_H + F_K a)(F_{LH} z_H n_H + F_{LK} a)\Gamma(dz_H)\end{aligned}$$

If $z_i$ follows a single distribution and agents draw only one productivity regardless of the type then we can combine the two integrals.

$$\frac{\partial V}{\partial N_L} =\beta \int u'(H) F_{LH} N_H \left[\pi_L \chi \left(D_L \frac{z n_L}{N_L} - (D_L+1)\frac{a}{K}\right) + \pi_H \left(\frac{z n_H}{N_H} - (D_L+1)\frac{a}{K}\right)\right]\Gamma(dz)$$

where $D_L := \frac{F_{LL} N_L}{F_{LH} N_H}$, which is negative since $F_{LL} < 0$ due to concavity of $F(\cdot)$. The last expression is similar to what we derive with Euler equation. Note that $D_L + 1 = \frac{-F_{LK} K}{F_{LH} N_H} > 0$ if $F_{LK} < 0$. Thus, the interpretation is similar as well. Low-skilled agents are worse off due to an increase in aggregate supply of low-skilled labor because both their labor income and capital income fall given that they are net savers. If their borrowing is sufficiently large then even low-skilled agents can be better off because greater low-skilled labor in the aggregate reduces the marginal product of capital and thus debt burden. High-skilled agents can be better off if their labor income is sufficiently higher than the savings. If the amount of savings is large enough then high-skilled workers can be worse off because of smaller capital income. They are unambiguously better off if they are net borrowers.

Now suppose that the planner chooses the aggregate supply of high-skilled labor. Then we can obtain a similar expression as before.

$$\frac{\partial V}{\partial N_H} =\beta \int u'(H) F_{HL} N_L \left[\pi_L \chi \left(\frac{z n_L}{N_L} - (D_H+1)\frac{a}{K}\right) + \pi_H \left(D_H \frac{z n_H}{N_H} - (D_H+1)\frac{a}{K}\right)\right]\Gamma(dz)$$

where $D_H := \frac{F_{HH}N_H}{F_{HL}N_L}$, which is negative due to concavity of $F(\cdot)$ and complementarity between $N_L$ and $N_H$. Note that $D_H+1 = \frac{-F_{HK}K}{F_{HL}N_L} < 0$ since high-skilled labor is complementary to both capital and low-skilled labor. Therefore, all agents with positive savings gain from an increase in $N_H$ as their capital income rises. Low-skilled agents unambiguously benefit overall because their labor is complementary to high-skilled labor. High-skilled workers can lose because of the decline in their labor income. High-skilled agents with borrowing lose unambiguously because their labor income falls and debt burden rises. The effect is unambiguous for low-skilled agents with borrowing.

### 3.3.2 Capital Equipment and Structure

The assumption so far has been that all types of capital complement high-skilled labor and substitutes for low-skilled labor. Now we consider two different types of capital: equipment and structure. Capital equipment is broadly defined as the type of capital that complements high-skilled labor but substitutes for low-skilled labor. Examples include industrial robots and computers. On the other hand, capital structure complements both types of labor as well as equipment capital. Examples are buildings and infrastructure.

Again, we modify the production function and leave the other parts of the model. Households supply labor inelastically and solve the following problem:

$$V(y_1; A, K) = \max_{c_1, a \in [-w_L/r, y_1]} u(c_1) + \beta \mathbb{E}_i u(c_2)$$

subject to period budget constraints $c_1 + a \leq y_1$ and $c_2 \leq ra + y_2$. Note that the aggregate state variables are aggregate asset $A$ in the economy and capital structure $K$. $A$ is the sum of structure $S$ and equipment $K$. The labor income in the second period $y_2$ depends on the realization of type so $y_2 = w_i$.

We denote the aggregate production function by $J(\cdot)$ is a Cobb-Douglas aggregator of capital structure and the composite output $F(\cdot)$ with the factor share parameter $\alpha$.

$$Y = J(S, K, N_L, N_H) \tag{3.5}$$

$$= S^{1-\alpha} F(K, N_L, N_H)^{\alpha} \tag{3.6}$$

To examine the pecuniary externalities associated with different types of capital, we derive the derivatives of $V$ with respect to $A$ and $K$.

$$\begin{aligned}\frac{\partial V}{\partial A} =&\beta\mathbb{E}\left[u'(J_i + J_A a)(J_{Ai} + J_{AA}a)\right] \\ =&\beta\mathbb{E}\left[u'(J_i + J_S a)(J_{Si} + J_{SS}a)\right] \qquad (3.7)\\ \frac{\partial V}{\partial K} =&\beta\mathbb{E}\left[u'(J_i + J_A a)(J_{Ki} + J_{KA}a)\right] \\ =&\beta\mathbb{E}\left[u'(J_i + J_S a)(J_{Ki} + J_{KS}a)\right] \qquad (3.8)\end{aligned}$$

In (3.7) and (3.8), the second derivatives capture the pecuniary externalities. We can rewrite the pecuniary externality terms in (3.7) more succinctly as follows:

$$J_{Si} + J_{SS}a = J_{SS}S\left(\frac{B_i}{\pi_i} + \frac{a}{S}\right)$$

where $B_i := \frac{J_{Si}\pi_i}{J_{SS}S}$ is the (negative) fraction of income accruing to labor $i \in \{H, L\}$ as regular capital rises. The first-order condition now becomes

$$\frac{\partial V}{\partial A} = \beta u'(J_H + J_S a)J_{SS}S\left[B_H + B_L\chi + \frac{a}{S}(\pi_H + \pi_L\chi)\right] \qquad (3.9)$$

Note that $B_H, B_L < 0$ and $J_{SS} < 0$. If an agent has a sufficiently large amount of savings (large $a$) then the welfare of the agent can rise by reducing aggregate wealth. Thus, $\frac{\partial V}{\partial A}$ captures the mechanism that is the focus of Davila et al. (2012).

In the presence of capital structure, an increase in capital equipment has an additional effect on wages compared to the baseline setup, which operates through the adjustment in the composition of total wealth.

**Proposition 3.3.** *Suppose there are two types of capital as specified in (3.6). Whether an agent worse off by an increase in capital equipment depends on (i) the amount of savings, (ii) the strength of capital-skill complementarity, and (iii) the crowding-out effect.*

*Proof.* Notice that we can rearrange the terms as follows.

$$\begin{aligned}\frac{\partial V}{\partial K} =& \beta \mathbb{E}_i \left[ u'(J_i + J_A a) \left( J_{Ki} - J_{KS} a \right) \right] \\ &- \beta \mathbb{E}_i \left[ u'(J_i + J_K a) \left( J_{Si} - J_{SS} a \right) \right]\end{aligned}$$

Again, using Euler's theorem we can write the pecuniary externality term as follows.

$$\begin{aligned}J_{Ki} - J_{KS} a =& J_{KK} K \left[ D_i \left( \frac{1}{\pi_i} + \frac{a}{S} \right) + (D_j + 1) \frac{a}{S} \right] \\ J_{Si} - J_{SS} a =& J_{SS} S \left( \frac{B_i}{\pi_i} - \frac{a}{S} \right)\end{aligned}$$

where $D_i := \frac{J_{Ki}\pi_i}{J_{KK}K}$ is the (negative) fraction of income accruing to labor $i$ as capital equipment rises and $B_i := \frac{J_{Si}\pi_i}{J_{SS}S}$ as before. Note that some of the terms have different signs than before because the pecuniary externality in the capital income has now a negative sign. Thus, the derivative becomes

$$\begin{aligned}\frac{\partial V}{\partial K} =& \beta u'(J_H + J_S a) J_{KK} K \left[ D_H + \chi D_L + (D_H + D_L + 1) \frac{a}{S} (\pi_H + \chi \pi_L) \right] \\ &- \beta u'(J_H + J_S a) J_{SS} S \left[ B_H + B_L \chi - \frac{a}{S} (\pi_H + \pi_L \chi) \right] \end{aligned} \tag{3.10}$$

Note that we are interested in the case where $\frac{\partial J}{\partial K \partial N_L} < 0$. In this case, it follows that $D_L > 0$ and $D_H + D_L + 1 > 0$. □

By observing (3.10), we can understand who is worse off by an increase in capital equipment. First, if an agents has a sufficiently large amount of savings or the decline in routine wages is sufficiently large then the terms in the square brackets can sum up a positive value, which implies that the first term is negative. Thus, there is downward pressure on robot capital, which is already discussed in the baseline setup.

More importantly, the second term in (3.10) captures the crowding-out effect of increasing capital equipment. It is negative if $a > 0$ because $B_H, B_L < 0$. Since capital structure complements all other input factors, there is an increase in capital structure is beneficial or, equivalently, there is a downward pressure on capital equipment. In other words, an increase in capital equipment reduces the incomes of all workers as it crowds out capital structure.

## 3.4 Conclusions

This paper explores the mechanisms through which capital-skill complementarity affects the pecuniary externalities of capital accumulation. At the center of the analysis is the distributional effects across different types of labor. In particular, the specific form of capital-skill complementarity assumed in the paper puts a downward pressure on the optimal capital stock to support low-skill wages and reduce income risks.

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# Appendices

# Appendix A

# Proofs

## A.1 Proofs for Chapter 1

### A.1.1 Proof of Lemma 1.2

Suppose a manager $x_2$ supervises workers $x_1$ and $x_1' < x_1$. There can be workers on $(x_1', x_1)$ who are supervised by managers less skilled than $x_2$. The planner can reallocate the time of the manager $x_2$ away from workers $x_1'$ to more skilled workers on $(x_1', x_1)$ by replacing the less skilled managers. This is because of the supermodularity of the production function. Likewise, there can also be workers on $(x_1', x_1)$ who are supervised by managers more skilled than $x_2$. Then the planner can increase output by reallocating the time of the manager $x_2$ away from workers $x_1$ to the less skilled workers on $(x_1', x_1)$.

Also, note that managers of a single skill level cannot supervise a continuum of workers with a positive measure because agents at each skill level has zero measure. Otherwise, the labor market clearing condition is violated, which requires that the supply of workers of a positive measure is met with the same measure of demand from managers. See Antras, Garicano, and Rossi-Hansberg (2005) for related discussions.

### A.1.2 Proof of Lemma 1.3

Suppose that the set of human workers is $[1-\Delta, y_1] \cup [y_2, y_3]$ and the set of managers is $[y_1, y_2] \cup [y_3, 1]$ with $1-\Delta < y_1 < y_2 < y_3 < 1$ so the set of workers is not connected. For this allocation to be an equilibrium, the wage functions must be continuous at the thresholds $y_1$, $y_2$, and $y_3$. Otherwise, marginal agents have the incentive to switch into occupations that yield higher income. Moreover, agents should not have an incentive to changes matches.

Let $\hat{w}_1(\cdot)$ and $\hat{w}_2(\cdot)$ denote the wage functions of the workers and managers on $[1-\Delta, y_1]$

and $[y_1, y_2]$. Also, let $\tilde{w}_1(\cdot)$ and $\tilde{w}_2(\cdot)$ denote the wage functions of workers and managers on $[y_2, y_3]$ and $[y_3, 1]$. Now consider the incentive of manager $x_2 = y_3$ who hires workers $x_1 = y_2 - \epsilon$

$$\begin{aligned}\pi(y_3, y_2 - \epsilon) &= \frac{y_3 - \hat{w}_2(y_2 - \epsilon)}{h(1 - y_2 + \epsilon)} \\ \frac{\partial \pi(y_3, y_2 - \epsilon)}{\partial \epsilon} &= -\frac{\hat{w}_2'(y_2 - \epsilon)}{h(1 - y_2 + \epsilon)} + \frac{y_3 - \hat{w}_2(y_2 - \epsilon)}{h(1 - y_2 + \epsilon)^2}\end{aligned}$$

Then the manager does not have an incentive to hire workers $y_2 - \epsilon$ if

$$\left.\frac{\partial \pi(y_3, y_2 - \epsilon)}{\partial \epsilon}\right|_{\epsilon \to 0} = -\frac{\hat{w}_2'(y_2)}{h(1 - y_2)} + \frac{y_3 - \hat{w}_2(y_2)}{h(1 - y_2)^2} < 0$$

However, the derivative is in fact positive since

$$-\frac{\hat{w}_2'(y_2)}{h(1 - y_2)} + \frac{y_3 - \hat{w}_2(y_2)}{h(1 - y_2)^2} = -\frac{\hat{w}_2'(y_2) - \tilde{w}_1'(y_2)}{h(1 - y_2)}$$

and

$$\hat{w}_2'(y_2) = \frac{1}{h(1 - y_1)} > 1 > \frac{y_3 - \hat{w}_2(y_2)}{1 - y_2} = \tilde{w}_1'(y_2)$$

The last inequality follows from $y_3 < 1$ and $\hat{w}_2(y_2) \geq y_2$, which is necessary for all agents to be matched. This shows that an allocation where the set of workers is disconnected cannot be an equilibrium.

### A.1.3 Proof of Lemma 1.4

Wages increase in skill level. Given a wage schedule, since machine owners maximize income, they enter only if their wages are greater than the entry cost. Higher $\mu$ makes the assignment function steeper and thus the least skilled machines earn less. If $\mu$ is sufficiently high then wages of the least skilled machines may be lower than $\epsilon$, which implies $\theta^* \geq \theta$.

### A.1.4 Proof of Proposition 1.5

For each statement in the proposition, I start by showing that there exists a unique allocation satisfying the optimality conditions of managers and the labor market clearing conditions. Then I argue that such an allocation is an actual equilibrium by showing that agents have no incentive to deviate from the given allocation.

**Existence and uniqueness of a solution**: An equilibrium is characterized by the two differential equations for the optimality of the manager's problem and the market clearing condition. Since the coefficient functions are continuous, these differential equations have a unique solution (Boyce and DiPrima 2020).

**Equilibrium**: The next step is to show that the allocation is actually an equilibrium. To begin with, note that $h$ must not be too low for the economy to have an equilibrium. Given $\mu$, $h$ must be above some threshold to ensure $z < 1$. Intuitively, if there is a large amount of machines and managers are very efficient (low $h$) then the supply of problems (or workers and machines) is not met by the demand from managers.

In the following, I show that agents do not have the incentive to deviate when fixing the number of layers at two. As Lemma 1.3 shows, the set of workers is connected and thus the allocation is characterized by the threshold $z$. First, agents slightly below $z$ do not become managers. Suppose workers slightly below $z$ by $\delta > 0$ deviate by hiring the least skilled workers ($x_1 = \theta$). Then they earn less wages. To see this, consider the following derivative

$$\frac{\partial w_1(z-\delta)}{\partial \delta} = -w_1(z-\delta)$$

By setting $\delta = 0$ I get

$$\left.\frac{\partial w_1(z-\delta)}{\partial \delta}\right|_{\delta=0} = -w_1(z) < 0$$

If these workers hire the least skilled workers $\theta$ as managers then the payoff would be

$$\frac{z-\delta-w_1(\theta)}{h(1-\theta)} = w_2(z) - \frac{\delta}{h(1-\theta)}$$

Taking the derivative of the deviation payoff, I get

$$\frac{\partial}{\partial \delta}\frac{z-\delta-w_1(\theta)}{h(1-\theta)} = -\frac{1}{h(1-\theta)} < 0$$

Note that $w_1'(z) = \frac{\hat{x}_2 - w_1(z)}{1-z}$. Also, $\hat{x}_2 < 1$ under Assumption 1.3.1 and $w_1(z) > z$ since I am interested in an equilibrium where all agents are matched. Thus, $-w_1'(z) > -1/h(1-\theta)$, which implies workers slightly below $z$ are worse off if they become managers.

Agents slightly above $z$ do not have an incentive to deviate by switching occupations. Because there are machines and managers' wages increase more steeply at $z$, the most skilled managers do not hire agents slightly above $z$ as workers.

Now I show that the managers do not have the incentive to add additional layers by examining the incentive of the most skilled managers $x_2 = 1$. Suppose that a manager with skill one deviates and switches to an organization with three layers. Since machines cannot supervise, the manager hires workers $x_1 \in [1-\Delta, z]$ as his direct subordinates. Also, workers $x_0 < x_1$ are hired in the bottom layer. The manager pays the prevailing market wages $w_1(x_1)$ and $w_1(x_0)$ to the wprkers. The manager, thus, solves

$$\hat{w}_2(x_2) = \max_{x_0,x_1} \hat{n}_1\hat{n}_0 x_2 - \hat{n}_1\hat{w}_1(x_1) - \hat{n}_1\hat{n}_0\hat{w}_1(x_0)$$

subject to

$$h(1-x_1)\hat{n}_1\hat{n}_0 \le 1$$
$$h(1-x_0)\hat{n}_0 \le 1$$

where the hat indicates the variables associated with the deviating manager. Note that $\hat{w}_1(\cdot)$ is the workers' wages which equal $w_2(x)$ if $x \ge z$ and $w_1(x)$ if $x < z$. The manager has no incentive to pay strictly above the equilibrium wages because workers will accept any offer greater than the market wage. Also, the manager cannot pay below because the workers will not accept the offer.

Substituting the time constraints into the objective function and taking the derivatives, I obtain

$$
\begin{aligned}
&[x_0] \quad \hat{w}_1(x_1) = \frac{1}{h}\hat{w}_1'(x_0) \\
&[x_1] \quad \frac{x_2 - \hat{w}_1(x_0)}{h(1-x_1)^2} = \frac{1-x_0}{(1-x_1)^2}\hat{w}_1'(x_1) + \frac{1-x_0}{1-x_1}\hat{w}_1(x_1)
\end{aligned}
$$

Note that the optimality of the deviating manager requires $x_1 \geq x_0$, otherwise the manager may benefit by directly supervising the workers in the bottom layer. Then there are three cases to consider: (i) $x_0 \leq \theta + \phi$, $x_1 \geq z$, (ii) $x_0 < x_1 \leq z$, and (iii) $\theta + \phi \leq x_0 < x_1$.

Consider the first case. The FOCs are

$$
\begin{aligned}
&[x_0] \quad w_2(x_1) = \frac{1}{h}w_1'(x_0) \\
&[x_1] \quad \frac{x_2 - w_1(x_0)}{h(1-x_1)^2} = \frac{1-x_0}{(1-x_1)^2}w_2(x_1) + \frac{1-x_0}{1-x_1}w_2'(x_1)
\end{aligned}
$$

The FOC with respect to $x_0$ can be rewritten as

$$
\frac{x_1 - w_1(a^{-1}(x_1))}{h(1-a^{-1}(x_1))} = \frac{1}{h}\frac{a(x_0) - w_1(x_0)}{1-x_0}
$$

where the left-hand side uses the equilibrium wage function of manager and the assignment function. The right-hand side uses the first-order condition of managers in the original allocation and the assignment function. Note that the equation is $w_2(x_1) = w_2(a(x_0))$. Since the equilibrium wage function $w_2(\cdot)$ is monotonically increasing in $x_1$, it is one-to-one. Thus, the pair $(x_0, x_1)$ must satisfy

$$
x_1 = a(x_0)
$$

With three layers, the manager may reduce the amount of compensation paid to workers by hiring more productive machines that allow the manager to hire fewer workers. Together with the above relationship, the FOC with respect to $x_1$ pins down the optimal choice of $x_0$

and $x_1$. Rearrange the terms to obtain

$$\frac{x_2 - w_1(x_0)}{h(1-x_0)} = w_2(x_1) + (1-x_1)w_2'(x_1)$$

Using the conditions that characterize the original allocation, the right-hand side can be written as

$$\begin{aligned} w_2(x_1) + (1-x_1)w_2'(x_1) =& \frac{x_1 - w_1(a^{-1}(x_1))}{h(1-a^{-1}(x_1))} + (1-x_1)\frac{1}{h(1-a^{-1}(x_1))} \\ =& \frac{1 - w_1(a^{-1}(x_1))}{h(1-a^{-1}(x_1))} \end{aligned}$$

Together with $x_1 = a(x_0)$, I have

$$\begin{aligned} \frac{x_2 - w_1(x_0)}{h(1-x_0)} &= \frac{1 - w_1(x_0)}{h(1-x_0)} \\ &\implies x_2 = 1 \end{aligned}$$

which is true only if $x_2 = 1$ in the first place. In other words, if $x_2 = 1$ any choice of subordinates satisfying $x_1 = a(x_0)$ yields the same payoff. Note that if $x_2 < 1$ then the deviating manager chooses $x_0 = \theta$ and $x_1 = z$ to avoid paying high labor costs from hiring more productive subordinates. The payoff of manager $x_2$ is then

$$\hat{w}_2(x_2) = \frac{x_2}{h(1-a(x_0))} - \frac{1-x_0}{1-a(x_0)}w_2(a(x_0)) - \frac{w_1(x_0)}{h(1-a(x_0))}$$

Substituting in $w_2(a(x_0)) = \frac{a(x_0)-w_1(x_0)}{h(1-x_0)}$, I obtain

$$\begin{aligned} \hat{w}_2(x_2) &= \frac{x_2 - w_1(x_0) - h(1-x_0)w_2(a(x_0))}{h(1-a(x_0))} \\ &= \frac{x_2 - w_1(x_0) - a(x_0) + w_1(x_0)}{h(1-a(x_0))} \\ &= \frac{x_2 - a(x_0)}{h(1-a(x_0))} \end{aligned}$$

For $x_2 = 1$, it follows that $\hat{w}_2(x_2) = 1/h$. Note that $w_2(1) = \frac{1-w_1(\theta+\phi)}{h(1-(\theta+\phi))}$. Thus, the manager does not have an incentive to deviate if $\frac{1-w_1(\theta+\phi)}{h(1-(\theta+\phi))} > \frac{1}{h}$, which is equivalent to

$$w_1(\theta+\phi) < \theta+\phi \tag{A.1}$$

Condition (A.1) is true if the supply of machines $\mu$ at each point is sufficiently large so that workers' wages are low.

Now I turn to the second case where $x_1 \leq z$. In this case, the first-order conditions are

$$\begin{aligned}
&[x_0] \quad w_1(x_1) = \frac{1}{h} w_1'(x_0) \\
&[x_1] \quad \frac{x_2 - w_1(x_0)}{h(1-x_1)^2} = \frac{1-x_0}{(1-x_1)^2} w_1'(x_1) + \frac{1-x_0}{1-x_1} w_1(x_1)
\end{aligned}$$

Rearrange the FOC with respect to $x_1$ to obtain

$$\frac{x_2 - w_1(x_0)}{h(1-x_0)} = a(x_1)$$

And combined with the optimality condition in the original allocation, the FOC with respect to $x_0$ is

$$w_2(a(x_0)) = w_1(x_1)$$

Since $x_1 \leq z$, the only values of $x_0$ and $x_1$ satisfying the above condition are $x_0 = \theta < 1 - \Delta$ and $x_1 = z$. However, it does not generally satisfy the first-order condition with respect to $x_1$. Thus, there is no solution corresponding to the second case.

Lastly, consider the third case where $x_0 \geq \theta + \phi$. The first-order conditions are

$$\begin{aligned}
&[x_0] \quad w_2(x_1) = \frac{1}{h} w_2'(x_0) \\
&[x_1] \quad \frac{x_2 - w_2(x_0)}{h(1-x_1)^2} = \frac{1-x_0}{(1-x_1)^2} w_2'(x_1) + \frac{1-x_0}{1-x_1} w_2(x_1)
\end{aligned}$$

First, note that any choice with $x_0 > \theta + \phi$ is not profitable because there is a discontinuous increase from $w_1(\theta+\phi)$ to $w_2(\theta+\phi+\delta)$ for any positive $\delta$. Instead, consider the case where

$x_0 = \theta + \phi$ and $x_1 > x_0$. Then the manager $x_2 = 1$ does not have an incentive to deviate if

$$w_2(x_2) = \frac{x_2 - w_1(\theta+\phi)}{h(1-(\theta+\phi))} > \frac{x_2 - w_1(\theta+\phi) - h(1-(\theta+\phi))w_2(x_1)}{h(1-x_1)} = \hat{w}_2(x_2)$$

Note that the deviating manager pays direct subordinates $w_2(x_1)$ since $x_1 > \theta+\phi$. Rearrange the terms on the right-hand side so that

$$w_2(x_1)\frac{1-(\theta+\phi)}{x_1-(\theta+\phi)} > w_2(1)$$

Note that the left-hand side diverges to infinity as $x_1 \to \theta+\phi$ and approaches $w_2(1)$ as $x_1 \to 1$. A sufficient condition for the inequality to hold is that the left-hand side term is monotonically decreasing in $x_1$. Take the derivative

$$\frac{dLHS}{dx_1} = \left(1 - \frac{x_1 - w_1(a^{-1}(x_1))}{x_1-(\theta+\phi)}\right)\frac{1-(\theta+\phi)}{x_1-(\theta+\phi)}\frac{1}{h(1-a^{-1}(x_1))}$$

The derivative is negative if $w_1(a^{-1}(x_1)) < \theta+\phi$, which is true if $\mu$ is sufficiently large and thus (A.1) holds. Thus, managers do not have an incentive to deviate and add another layer in the third case either.

### A.1.5 Proof of Proposition 1.6

As in Proposition 1.5, I start by showing that there exists a unique allocation satisfying the optimality conditions of managers and the labor market clearing conditions. Then I argue that such an allocation is an actual equilibrium by showing that agents have no incentive to deviate from the given allocation.

**Existence and uniqueness of a solution**: Given the assignment function, the equilibrium wage functions $w_0$ and $w_1$ solve the system of equations given by the first-order conditions. Rearrange the terms so that

$$[x_0] \quad w_1(a(x_0)) = \frac{1}{h}w_0'(x_0), \quad x_0 \in [\theta, \theta+\phi]$$

$$[x_1] \quad \frac{a(x_1)}{h(1-x_1)(1-a^{-1}(x_1))} = w_1'(x_1) + \frac{w_1(x_1)}{1-x_1} + \frac{w_0(a^{-1}(x_1))}{h(1-x_1)(1-a^{-1}(x_1))}, \quad x_1 \in [1-\Delta, z]$$

Again, the above system of equations has at most a unique solution (Boyce and DiPrima 2020). Thus, there exists a unique allocation that satisfies the first-order conditions of the managers, the boundary conditions for the assignment function, and the continuity of the wage functions.

**Equilibrium**: The allocation described above requires $\theta^*$ to adjust and clear the labor market. Since the supply of machines on the market cannot be greater than the endowment of machines in the economy, it must be that $\theta^* \geq \theta$ in equilibrium. The demand for machines increases as $h$ falls and thus $h$ must not be too low.

To prove that the allocation described above is an actual equilibrium, I consider whether agents have the incentive to deviate from the allocation described above when (i) the number of layers is fixed and (ii) managers are allowed to choose two layers. Fixing the number of layers at three, I show that machines and workers are in separate layers, and the occupational threshold $z$ separates workers and managers. Since machines cannot supervise others, including other machines, it suffices to show that all workers are in the middle layer.

Consider the incentive of the most skilled manager $x_2 = 1$. Suppose the manager deviates and hires the least skilled workers $x_1 = 1 - \Delta$ in the bottom layer, instead of machines $x_0 = \theta + \phi$. The deviation is not profitable if

$$\begin{aligned} w_2(1) &> \hat{w}_2(1) \\ \frac{w_1(1-\Delta) - w_0(\theta+\phi)}{(1-\Delta) - (\theta+\phi)} &> w_1(z) \end{aligned}$$

Note that the left-hand side is increasing in $\theta$ because $w_0(\cdot)$ is convex. On the other hand, the right-hand side is falling in $\theta$ because $w_1(z) = w_0'(\theta+\phi)/h$ from the FOC with respect to $x_0$. Thus, $\theta$ must be sufficiently small.

To show that $z$ separates managers and workers, I show that $w_1'(z) < w_2'(z)$. If it were true then the most skilled manager cannot profitably deviate by hiring agents above $z$. To see this, consider the incentive of the best manager hiring agents slightly above $z$

$$\pi(1, z+\delta) = \frac{1 - w_0(\theta+\phi)}{h(1-z-\delta)} - \frac{1-\theta-\phi}{1-z-\epsilon} w_2(z+\delta)$$

For the allocation to be an equilibrium, it must be that

$$\lim_{\delta\to 0}\frac{\partial\pi}{\partial\delta}=\frac{1-\theta-\phi}{1-z}\left(w_1'(z)-w_2'(z)\right)<0$$

which holds if $w_1'(z)-w_2'(z)<0$.

To find the conditions for the inequality to hold, consider the FOC with respect to $x_1$ at $x_1=z$:

$$w_1'(z)=\frac{1}{1-\theta-\phi}\frac{1}{h(1-z)}\left(1-w_1(z)h(1-\theta-\phi)-w_0(\theta+\phi)\right)$$

Since $w_2'(z)=1/h\Delta$, I need to show

$$w_1'(z)=\frac{1}{1-\theta-\phi}\frac{1}{h(1-z)}\left(1-w_1(z)h(1-\theta-\phi)-w_0(\theta+\phi)\right)<\frac{1}{h\Delta}$$

Rearrange the terms to obtain

$$\frac{1}{1-z}\left(\frac{\Delta(1-w_0(\theta+\phi))}{1-\theta-\phi}-w_1(z)h\Delta\right)<1$$

Since the indifference condition implies $w_1(z)=w_2(z)$, I have

$$w_1(z)=w_2(z)=\frac{z}{h\Delta}-\frac{1-\theta^*}{\Delta}w_1(1-\Delta)-\frac{w_0(\theta^*)}{h\Delta}$$

and substituting the above into the brackets it follows that

$$\begin{aligned}&\frac{\Delta(1-w_0(\theta+\phi))}{1-\theta-\phi}-\left(\frac{z}{h\Delta}-\frac{1-\theta^*}{\Delta}w_1(1-\Delta)-\frac{w_0(\theta^*)}{h\Delta}\right)h\Delta\\=&\frac{\Delta(1-w_0(\theta+\phi))}{1-\theta-\phi}-z+h(1-\theta^*)w_1(1-\Delta)+\epsilon\end{aligned}$$

I need to show that the last expression is strictly less than $1-z$

$$\frac{\Delta(1-w_0(\theta+\phi))}{1-\theta-\phi}+h(1-\theta^*)w_1(1-\Delta)+\epsilon<1$$

Rearrange the first term so that

$$\frac{\Delta}{1-\theta-\phi}(1-w_0(\theta+\phi)-h(1-\theta-\phi)w_1(z))+h\Delta w_1(z)+h(1-\theta^*)w_1(1-\Delta)+\epsilon<1$$

Note that $h\Delta w_1(z)=h\Delta w_2(z)=z-w_0(\theta^*)-h(1-\theta^*)w_1(1-\Delta)$ is the wage of manager $x_2=z$ per machine and thus

$$\frac{1-w_0(\theta+\phi)}{h(1-\theta-\phi)}<\frac{1-w_0(\theta^*)-h(1-\theta^*)w_1(1-\Delta)}{h\Delta} \tag{A.2}$$

The left-hand side is the payoff of manager $x_2=1$ if he deviates and hires only one layer of machines $x_0=\theta+\phi$. The right-hand side is the payoff of the same manager if he deviates and hires workers $x_1=1-\Delta$ and machines $x_0=\theta$. The condition requires that the first deviation strategy is dominated by the second. This is true for values of $\theta$ that are sufficiently small. Suppose $\theta$ is large so that $\theta^*$ is close to $1-\Delta$ and $\phi$ is small so that $\theta+\phi\approx\theta^*$. Then the difference between the left-hand side and the right-hand side is

$$\begin{aligned}&\frac{1-w_0(\theta+\phi)}{h(1-\theta-\phi)}-\frac{1-w_0(\theta^*)-h(1-\theta^*)w_1(1-\Delta)}{h\Delta}\\&\approx w_1(1-\Delta)>0\end{aligned}$$

which violates (A.2). If $\theta^*$ is sufficiently low then $z$ characterizes the equilibrium allocation of agents.

The next step is to prove that managers do not have an incentive to switch to two layers. I show this in the case of the marginal managers at $x_2=z$. This is sufficient because more skilled managers only run at least as many layers as less productive managers.[1]

Suppose a manager $x_2=z$ deviates from the allocation described above and switches to two layers. The deviating manager faces a trade-off between a fall in total labor cost due to fewer layers and a rise in total labor cost due to more direct subordinates. Note that it is more profitable to choose $x_1\geq 1-\Delta$ than $x_1\in[\theta,\theta+\phi]$ if $\theta+\phi$ is sufficiently small.[2] Since

[1]See the discussion on the equilibrium number of layers in Garicano and Rossi-Hansberg (2006)

[2]On the other hand, if $\theta+\phi$ were close to $1-\Delta$ then the manager hires machines as direct subordinates to exploit the fact that $w_0(\theta+\phi)$ is strictly less than $w_1(1-\Delta)$, which is true for sufficiently small $\epsilon$.

optimality requires $x_1 < z$, consider the following problem

$$\hat{w}_2(z) = \max_{x_1 \in [1-\Delta, z)} \pi(z, x_1)$$

where $\pi(z, x_1) \equiv \frac{z - w_1(x_1)}{h(1-x_1)}$. First, I show that $x_1 = 1 - \Delta$ is dominated by the original choice of organization structure that yields $w_2(z)$ if $(1-h)(1-\Delta)$ is sufficiently large. To see this, consider the difference in payoffs

$$w_2(z) - \pi(z, 1-\Delta) = \frac{(1 - h(1-\theta^*))w_1(1-\Delta) - w_0(\theta^*)}{h\Delta} \tag{A.3}$$

which is nonnegative (and so there is no incentive to deviate) if

$$(1 - h(1-\theta^*))w_1(1-\Delta) \geq \epsilon$$

The left-hand side is bounded below by

$$(1 - h(1-\theta^*))w_1(1-\Delta) \geq (1 - h + h\theta) \cdot \frac{1 - \Delta - w_0(\theta^*)}{h(1-\theta^*)}$$

where, in the last inequality, $\frac{1-\Delta-w_0(\theta^*)}{h(1-\theta^*)}$ is the outside option of the least skilled workers as a manager supervising machines. In equilibrium, it must be that $w_1(1-\Delta) \geq \frac{1-\Delta-w_0(\theta^*)}{h(1-\theta^*)}$ to rule out deviation of workers $x_1 = 1 - \Delta$. The right-hand side is bounded further below by

$$(1 - h + h\theta) \cdot \frac{1 - \Delta - w_0(\theta^*)}{h(1-\theta^*)} \geq \left(\frac{1}{h(1-\theta)} - 1\right)(1 - \Delta - \epsilon)$$

A sufficient condition to (A.3) is

$$\theta \geq \frac{\epsilon - (1-h)(1-\Delta)}{h(1-\Delta)}$$

The last inequality requires that, given $\Delta$ and $\theta$, $h$ must be sufficiently small. Also, the condition trivially holds if $\epsilon - (1-h)(1-\Delta) < 0$ because $\theta \geq 0$.[3]

[3] And I assume that the entry cost $\epsilon$ is small.

Now I show that any $x_1 > 1-\Delta$ yields a lower payoff than $x_1 = 1-\Delta$. That is,

$$\frac{z - w_1(1-\Delta)}{h\Delta} \geq \frac{z - w_1(x_1^*)}{h(1-x_1^*)}$$

for $x_1 \in (1-\Delta, z)$. Consider the change in payoff as $x_1$ rises from $1-\Delta$ to $1-\Delta+\delta$

$$\pi(z, 1-\Delta+\delta) = \frac{z - w_1(1-\Delta+\delta)}{h(1-(1-\Delta+\delta))} = \frac{z - w_1(1-\Delta+\delta)}{h(\Delta-\delta)}$$
$$\frac{\partial\pi}{\partial\delta} = -\frac{w_1'(1-\Delta+\delta)}{h(\Delta-\delta)} + \frac{z - w_1(1-\Delta+\delta)}{h(\Delta-\delta)^2}$$

Letting $\delta \to 0$, I have

$$\left.\frac{\partial\pi}{\partial\delta}\right|_{\delta\to 0} = -\frac{1}{h\Delta}\left(w_1'(1-\Delta) + \frac{w_1(1-\Delta)}{\Delta}\right) + \frac{z}{h\Delta^2}$$

I want to show that $\left.\frac{\partial\pi}{\partial\delta}\right|_{\delta\to 0} < 0$. Note that the FOC with respect to $x_1$ implies

$$w_1'(1-\Delta) + \frac{w_1(1-\Delta)}{\Delta} = \frac{1}{1-\theta^*}\frac{z-\epsilon}{h\Delta}$$

Substituting the above expression into the derivative, I can write

$$\left.\frac{\partial\pi}{\partial\delta}\right|_{\delta\to 0} = \left(1 - \frac{1}{h(1-\theta^*)}\right)\frac{z-\epsilon}{h\Delta^2} + \frac{\epsilon}{h\Delta^2}$$

The goal is to show that the derivative is negative. Rearrange the terms to obtain

$$\epsilon < \left(\frac{1}{h(1-\theta^*)} - 1\right)(z-\epsilon)$$

Note that the right-hand side is bounded below by

$$\left(\frac{1}{h(1-\theta^*)} - 1\right)(z-\epsilon) > \left(\frac{1}{h(1-\theta)} - 1\right)(1-\Delta-\epsilon)$$

Thus, a sufficient condition for the inequality to hold is

$$\left(\frac{1}{h(1-\theta)}-1\right)(1-\Delta-\epsilon)>\epsilon$$

Again rearrange the terms so that

$$1-\frac{1}{h}\left(1-\frac{\epsilon}{1-\Delta}\right)<\theta$$

Again, given $\epsilon$, $\Delta$, and $\theta$, the inequality holds if the value of $h$ is sufficiently small. This shows that if $\theta$ and $h$ are sufficiently small then the allocation where all organizations have three layers is indeed an equilibrium.

### A.1.6 Proof of Proposition 1.7

I show that workers' wages fall as $\phi$ rises. Consider the following derivative of the equilibrium wage function

$$\frac{\partial w_1(x_1;\phi)}{\partial \phi}=\frac{\partial a(x_1;\phi)}{\partial \phi}+\frac{\partial C_k}{\partial \phi}(1-x_1) \tag{A.4}$$

where $x_1 \in [\theta, \theta+\phi]$ and $C_k$, $k=1,2,3$ is a function of parameters defined in Appendix B.1.1. The first term in (A.4) is the change in the assignment function as an increase in $\phi$ raises the competition among workers and machines. The second term ensures that the wage function is continuous after an increase in $\phi$. Note that

$$\begin{aligned}\frac{\partial a(x_1;\phi)}{\partial \phi}&<0\\ \frac{\partial C_k}{\partial \phi}&>0\end{aligned}$$

I show that the first term dominates the second term if $\theta$ is sufficiently high. To see this, consider the case with $k=1$. Then

$$\frac{\partial w_1(x_1;\phi)}{\partial \phi}=\frac{\partial z}{\partial \phi}\left[1-\frac{(1-z+1/h)+hz(1-z+1/h)+\mu\Delta z+\hat{x}_2-(1-\Delta)-\mu\Delta\theta}{(1-z+1/h)^2}\cdot(1-x_1)\right]$$

The goal is to show that the second term in the square brackets is smaller than one. Note that the denominator is increasing in $\theta$ and the numerator is decreasing in $\theta$. Also, $1 - x_1 < 1 - \theta$ and so there are values of $\theta$ sufficiently large that deliver the desired inequality. This proves the proposition since a fall in workers' wages directly raises managers' wages.

### A.1.7 Proof of Proposition 1.8

To show that more skilled managers gain more from technological change, note that

$$\frac{\partial w_2(x_2;\phi)}{\partial x_2} = \frac{1}{h(1 - a^{-1}(x_2))}$$

by the envelope theorem. It suffices to show that the above derivative itself is increasing in $\phi$ since

$$\begin{aligned}\frac{\partial^2 w_2(x_2;\phi)}{\partial x_2 \partial \phi} &= \frac{\partial^2 w_2(x_2;\phi)}{\partial \phi \partial x_2} \\ &= \frac{1}{h(1 - x_1)^2} \cdot \frac{\partial x_1}{\partial \phi}\end{aligned}$$

Note that since $\frac{\partial z}{\partial \phi} < 0$, it follows that $\frac{\partial a(x_1;\phi)}{\partial \phi} < 0$. By the implicit function theorem,

$$\frac{\partial x_1}{\partial \phi} > 0$$

which implies $\frac{\partial^2 w_2(x_2;\phi)}{\partial x_2 \partial \phi} > 0$ and proves the last statement of the proposition.

### A.1.8 Proof of Corollary 1.9

Denote the total wages of top $p$ earners by $T_p$. Then

$$T_p \equiv \int_{1-p}^{p} w_2(x_2) dx_2$$

For the income share of top $p$ earners to rise with $\phi$, the growth rate of $T_p$ must be higher than the growth rate of total income in the economy. A sufficient condition for a higher growth rate of $T_p$ is

$$\frac{\partial^2 w_2(x_2;\phi)}{\partial x_2 \partial \phi} > \frac{\partial w_2(x_2;\phi)}{\partial x_2}$$

That is, if the increase in $w_2(\cdot)$ due to an increase in $\phi$ grows faster with $x_2$ than $w_2(\cdot)$ itself, then the growth rate is increasing in $x_2$. Note that, from the proof of Proposition 1.8,

$$\frac{\partial^2 w_2(x_2;\phi)}{\partial x_2 \partial \phi} = \frac{\partial w_2(x_2;\phi)}{\partial x_2} \cdot \frac{1}{1-x_1} \cdot \frac{1-(\theta+\phi)}{1-x_1}$$

Since $1/(1-x_1) > 1$, I have the desired inequality for $x_1 = \theta + \phi$. Moreover, $\frac{1-(\theta+\phi)}{1-x_1}$ is continuous in $x_1$ so there is a left neighborhood of $\theta+\phi$ such that the inequality holds. This proves that the income share of top $p$ agents is increasing in $\phi$ for small $p > 0$.

### A.1.9 Proof of Proposition 1.10

Note that all existing machines' wages must fall because the supply of machines increases with an increase in the parameter $\phi$. In an equilibrium where all machines are supervised by workers, higher $\phi$ allows workers to supervise more advanced machines and thus increase the mass of problems that are drawn and passed to their managers. As a result, there need to be more managers in the new equilibrium to solve more problems, which implies lower $z$.

I show that more skilled managers gain less from an increase in $\phi$ than less skilled managers. To do so, consider the cross derivative $\frac{\partial w_2(x_2;\phi)}{\partial \phi \partial x_2}$. By the envelop theorem, I have

$$\frac{\partial w_2(x_2;\phi)}{\partial x_2} = \frac{1}{h(1-x_1)}$$

Then, it follows that

$$\frac{\partial^2 w_2(x_2;\phi)}{\partial \phi \partial x_2} = \frac{1}{h(1-x_1)^2} \frac{\partial e(x_2;\phi)}{\partial \phi}$$

where $e(x_2;\phi)$ is the employment function, or an inverse of the assignment function. Since $z$ decreases in the new equilibrium, there are fewer workers as the most skilled workers become managers. Then it must be that managers supervise less skilled workers, which implies

$\frac{\partial e(x_2;\phi)}{\partial \phi} < 0$. This shows that $\frac{\partial^2 w_2(x_2;\phi)}{\partial \phi \partial x_2} < 0$ and the proposition.

### A.1.10 Proof of Lemma 1.11

Assume $z < \theta$ is true and solve for an equilibrium. Then verify that $z < \theta$ is indeed true for sufficiently high values of $\theta$. See Appendix B.1.4 for details.

### A.1.11 Proof of Lemma 1.12

Solve for the assignment function assuming $z < \theta$ taking $z$, $\underline{y}$, and $\bar{y}$ as given. Then find the thresholds using the continuity conditions on the assignment function. See Appendix B.1.4 for details.

### A.1.12 Proof of Proposition 1.14

For the first statement, I show

$$\frac{\partial z}{\partial h_m} < 0$$

The occupational threshold $z$ is pinned down by the continuity conditions of the assignment function

$$\begin{aligned} a(1-\Delta) =& z \\ a(\underline{y}) =& \theta \\ a(\bar{y}) =& \theta + \phi \end{aligned}$$

As I show in Appendix B.1.4, $z$ is the solution to the following equation that combines the conditions

$$F \equiv -\frac{h}{2}\left[z\frac{2}{h} + \Delta^2 - \frac{2}{h}\theta - \phi\frac{2}{\Phi}\right] + \frac{h}{2}(1-z)^2 + 1 - \theta - \phi = 0 \tag{A.5}$$

Then

$$\frac{\partial F}{\partial z} = -1 - h(1-z) < 0$$

$$\frac{\partial F}{\partial h_m} = -\frac{h}{2} \cdot \left(\phi \frac{2}{\Phi^2}\right) \frac{\partial \Phi}{\partial h_m} = -\frac{\phi h}{\Phi^2} \frac{\partial \Phi}{\partial h_m} < 0$$

Recall that $\Phi \equiv \frac{1/\Delta}{1/h\Delta + \mu/h_m}$ and so

$$\begin{aligned}
\Phi =& \frac{\frac{1}{\Delta}}{\frac{1}{h\Delta} + \frac{\mu}{h_m}} = \frac{\frac{h_m}{\Delta}}{\frac{h_m}{h\Delta} + \mu} \\
\frac{\partial \Phi}{\partial h_m} =& \frac{\frac{1}{\Delta}}{\frac{h_m}{h\Delta} + \mu} - \frac{\frac{h_m}{\Delta}}{\left(\frac{h_m}{h\Delta} + \mu\right)^2} \frac{1}{h\Delta} = \frac{\frac{1}{\Delta}}{\frac{h_m}{h\Delta} + \mu} \left(1 - \frac{\frac{h_m}{h\Delta}}{\left(\frac{h_m}{h\Delta} + \mu\right)^2}\right) > 0
\end{aligned}$$

Thus, by the implicit function theorem I have

$$\frac{\partial z}{\partial h_m} = -\frac{\partial F/\partial h_m}{\partial F/\partial z} < 0$$

To see that $z$ converges to $\theta$ as $h_m$ falls, suppose $z = \theta$. Then the corresponding value of $h_m$ is

$$h_m = \underline{h} \equiv \frac{h\phi\mu\Delta}{\frac{h}{2}\Delta^2 - \frac{h}{2}(1-\theta)^2 + \theta - 1} \tag{A.6}$$

which requires $\Delta$ and $\theta$ to be sufficiently large to be positive. Denote the above expression by $\underline{h}$. It follows that as $h_m \to \underline{h}$ $z \to \theta$. Moreover, from the continuity of the assignment function $\underline{y}$ is

$$\underline{y} = 1 - \sqrt{z\frac{2}{h} + \Delta^2 - \frac{2}{h}\theta}$$

which implies that $\underline{y} \to 1 - \Delta$ as $h_m \to \underline{y}$.

### A.1.13 Proof of Lemma 1.13

Using (A.5), derive

$$\frac{\partial F}{\partial \phi} = \frac{\partial}{\partial \phi}\left(\frac{h\phi}{\Phi}\right) - 1 = \frac{h}{\Phi} - 1 = 1 + \frac{\mu\Delta}{h_m} - 1 = \frac{\mu\Delta}{h_m} > 0$$

Thus, by the implicit function theorem

$$\frac{\partial z}{\partial \phi} = -\frac{\partial F/\partial \phi}{\partial F/\partial z} > 0$$

which proves the proposition.

## A.2 Proofs for Chapter 1

### A.2.1 Proof of Proposition 2.7

To begin with, we show that the growth of capital stock is approximately exponential at some constant rate. If the economy is in region 1, the production function is CES. After some algebra, the growth rate of capital is

$$\begin{aligned}\frac{\dot{K}_t}{K_t} &= \frac{s_t A\left(K_t^{\frac{\sigma-1}{\sigma}}\Phi_t^{\frac{1}{\sigma}} + L^{\frac{\sigma-1}{\sigma}}(1-\Phi_t)^{\frac{1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}}}{K_t} - \delta \\ &= s_t A\left(\Phi_t^{\frac{1}{\sigma}} + K_t^{-\frac{\sigma-1}{\sigma}}L^{\frac{\sigma-1}{\sigma}}(1-\Phi_t)^{\frac{1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}} - \delta \\ &= s_t A\left(1 + L^{\frac{\sigma-1}{\sigma}}\Big/ K_t^{\frac{\sigma-1}{\sigma}}\left(\frac{\Phi_t}{1-\Phi_t}\right)^{\frac{1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}} \cdot \Phi_t^{\frac{1}{\sigma-1}} - \delta\end{aligned}$$

Note that the growth rate of capital stock depends on the behavior of $\Omega_t \equiv K_t^{\frac{\sigma-1}{\sigma}}\left(\frac{\Phi_t}{1-\Phi_t}\right)^{\frac{1}{\sigma}}$. In particular, $\frac{\Phi_t}{1-\Phi_t}$ grows approximately at an exponential rate under the Pareto assumption because

$$
\begin{aligned}
\frac{\Phi_t}{1-\Phi_t} &= \frac{1-I_t^{-\lambda}}{I_t^{-\lambda}} \\
&I_t^{\lambda} - 1 \\
&= I_0^{\lambda} e^{\lambda g t} - 1 \\
&\approx I_0^{\lambda} e^{\lambda g t}
\end{aligned}
$$

We consider three cases and see whether each case is consistent with derivation above. First, suppose $\Omega_t \to \infty$. Then capital stock must grow at a sufficiently low rate at least for large $t$. That is, the long-run growth rate of capital $g_K$ must satisfy

$$
g_K < \frac{\lambda g}{1-\sigma}
$$

In this case, the growth rate of capital is

$$
\begin{aligned}
\frac{\dot{K}_t}{K_t} &\to sA\Big(1+0\Big)^{\frac{\sigma}{\sigma-1}} \cdot 1 - \delta \\
&= s^{\infty} A - \delta
\end{aligned}
$$

where $s^{\infty}$ is the long-run savings rate defined in Lemma 2.6. Then we have $g_K = s^{\infty} A - \delta < \frac{\lambda g}{1-\sigma}$ and thus the following upper bound on the long-run savings rate

$$
s^{\infty} < \frac{1}{A}\left(\frac{\lambda g}{1-\sigma} + \delta\right)
$$

Secondly, consider the case where $\Omega_t \to 0$. Then capital stock must grow at a rate such that

$$
g_K > \frac{\lambda g}{1-\sigma}
$$

In this case, the growth rate of capital converges to a negative value

$$\begin{aligned} \frac{\dot{K}_t}{K_t} &\to sA \cdot 0 \cdot 1 - \delta \\ &= -\delta \end{aligned}$$

But this contradicts the lower bound on $g_K$ and puts an upper bound on the growth rate of capital stock.

Lastly, suppose $\Omega_t$ converges to a nonzero constant. Then it must be the case that

$$g_K = \frac{\lambda g}{1-\sigma}$$

which requires $\Omega_t \to \Omega$ where $\Omega$ is some constant satisfying $\frac{\dot{K}_t}{K_t} \to s^{\infty} A \left(1 + L^{\frac{\sigma-1}{\sigma}} \Big/ \Omega\right)^{\frac{\sigma}{\sigma-1}} - \delta = \frac{\lambda g}{1-\sigma}$. In the three cases, capital stock grows asymptotically at either $s^{\infty}A - \delta$ or $\lambda g/(1-\sigma)$.

To characterize the long-run labor income share, note that the labor share is

$$\begin{aligned} LS_t &= \frac{w_t L}{Y_t} \\ &= \frac{A^{\frac{\sigma-1}{\sigma}} (Y_t/L)^{\frac{1}{\sigma}} (1-\Phi_t)^{\frac{1}{\sigma}} L}{Y_t} \\ &= A^{\frac{\sigma-1}{\sigma}} Y_t^{\frac{1}{\sigma}-1} L^{1-\frac{1}{\sigma}} (1-\Phi_t)^{\frac{1}{\sigma}} \\ &= A^{\frac{\sigma-1}{\sigma}} Y_t^{-\frac{\sigma-1}{\sigma}} L^{\frac{\sigma-1}{\sigma}} (1-\Phi_t)^{\frac{1}{\sigma}} \\ &= A^{\frac{\sigma-1}{\sigma}} A^{-\frac{\sigma-1}{\sigma}} \left( K_t^{\frac{\sigma-1}{\sigma}} \Phi_t^{\frac{1}{\sigma}} + L^{\frac{\sigma-1}{\sigma}} (1-\Phi_t)^{\frac{1}{\sigma}} \right)^{-1} L^{\frac{\sigma-1}{\sigma}} (1-\Phi_t)^{\frac{1}{\sigma}} \\ &= \left( K_t^{\frac{\sigma-1}{\sigma}} \Phi_t^{\frac{1}{\sigma}} + L^{\frac{\sigma-1}{\sigma}} (1-\Phi_t)^{\frac{1}{\sigma}} \right)^{-1} L^{\frac{\sigma-1}{\sigma}} (1-\Phi_t)^{\frac{1}{\sigma}} \\ &= \frac{L^{\frac{\sigma-1}{\sigma}} (1-\Phi_t)^{\frac{1}{\sigma}}}{K_t^{\frac{\sigma-1}{\sigma}} \Phi_t^{\frac{1}{\sigma}} + L^{\frac{\sigma-1}{\sigma}} (1-\Phi_t)^{\frac{1}{\sigma}}} \\ &= \frac{L^{\frac{\sigma-1}{\sigma}}}{K_t^{\frac{\sigma-1}{\sigma}} \left(\frac{\Phi_t}{1-\Phi_t}\right)^{\frac{1}{\sigma}} + L^{\frac{\sigma-1}{\sigma}}} \\ &= \frac{L^{\frac{\sigma-1}{\sigma}}}{\Omega_t + L^{\frac{\sigma-1}{\sigma}}} \end{aligned}$$

In the first case, $\Omega_t \to 1$ and so $LS_t \to 0$. In the second case, $\Omega_t \to 0$ and so $LS_t \to 1$. Lastly, in the third case, $\Omega_t \to \bar{K}^{\frac{\sigma-1}{\sigma}} \bar{I}^{\frac{1}{\sigma}}$. Since $s^\infty A\left(1+L^{\frac{\sigma-1}{\sigma}} \Big/ \Omega\right)^{\frac{\sigma}{\sigma-1}} - \delta = \frac{\lambda g}{1-\sigma}$, it follows that

$$\begin{aligned}
s^\infty A\left(1+L^{\frac{\sigma-1}{\sigma}} \Big/ \Omega\right)^{\frac{\sigma}{\sigma-1}} - \delta &= \frac{\lambda g}{1-\sigma} \\
\left(1+L^{\frac{\sigma-1}{\sigma}} \Big/ \Omega\right)^{\frac{\sigma}{\sigma-1}} &= \frac{1}{s^\infty A}\left(\frac{\lambda g}{1-\sigma}+\delta\right) \\
\left(\frac{\Omega+L^{\frac{\sigma-1}{\sigma}}}{\Omega}\right)^{\frac{\sigma}{\sigma-1}} &= \frac{1}{s^\infty A}\left(\frac{\lambda g}{1-\sigma}+\delta\right) \\
\left(\frac{\Omega}{\Omega+L^{\frac{\sigma-1}{\sigma}}}\right)^{\frac{\sigma}{\sigma-1}} &= \frac{s^\infty A}{\frac{\lambda g}{1-\sigma}+\delta} \\
\left(1-\frac{L^{\frac{\sigma-1}{\sigma}}}{\Omega+L^{\frac{\sigma-1}{\sigma}}}\right)^{\frac{\sigma}{\sigma-1}} &= \frac{s^\infty A}{\frac{\lambda g}{1-\sigma}+\delta} \\
1-\frac{L^{\frac{\sigma-1}{\sigma}}}{\Omega+L^{\frac{\sigma-1}{\sigma}}} &= \left[\frac{s^\infty A}{\frac{\lambda g}{1-\sigma}+\delta}\right]^{\frac{\sigma-1}{\sigma}} \\
\frac{L^{\frac{\sigma-1}{\sigma}}}{\Omega+L^{\frac{\sigma-1}{\sigma}}} &= 1-\left[\frac{s^\infty A}{\frac{\lambda g}{1-\sigma}+\delta}\right]^{\frac{\sigma-1}{\sigma}}
\end{aligned}$$

Therefore, $LS_t \to 1-\left[\frac{s^\infty A}{\frac{\lambda g}{1-\sigma}+\delta}\right]^{\frac{\sigma-1}{\sigma}} = 1-\left[\frac{(A-\rho-\delta+\eta\delta)/\eta}{\frac{\lambda g}{1-\sigma}+\delta}\right]^{\frac{\sigma-1}{\sigma}}$.

If the economy starts in region 1 then the economy stays in region 1 as long as

$$\frac{\dot{\hat{I}}_t}{\hat{I}_t} \geq \frac{\dot{I}_t}{I_t}$$

That is, the threshold grows faster than the automation index. Under the Pareto assumption, the inequality is equivalent to

$$\frac{\dot{K}_t}{K_t} \geq \frac{1+K_t/L}{K_t/L} \cdot \lambda g$$

where $\frac{1+K_t/L}{K_t/L}$ converges to one from above. Thus, the above inequality delivers a lower bound on the savings rate for the economy to asymptotically stay in region 1:

$$s^\infty A - \delta > \lambda g \tag{A.7}$$

The inequality ensures that capital accumulation is sufficiently fast compared to automation. If it is violated then $I_t$ crosses $\hat{I}_t$ eventually and wages collapse to $A$.

To further examine how capital accumulation and automation shape the asymptotic behavior of wages, consider the growth rate of wages

$$\frac{\dot{w}_t}{w_t} = \frac{1}{\sigma}\frac{\dot{Y}_t}{Y_t} - \frac{1}{\sigma}\frac{\dot{\Phi}_t}{1-\Phi_t}$$

derived from $w_t = F_L$ and

$$\begin{aligned}\log F_L &= \log A + \frac{1}{\sigma}\log(Y/A) - \frac{1}{\sigma}\log L + \frac{1}{\sigma}\log(1-\Phi(I)) \\ \frac{d\log F_L}{dt} &= \frac{1}{\sigma}\frac{d\log Y}{dt} + \frac{1}{\sigma}\frac{d\log(1-\Phi(I))}{dt} \\ &= \frac{1}{\sigma}\frac{d\log Y}{dt} + \frac{1}{\sigma}\frac{1}{1-\Phi}(-\dot{\Phi})\end{aligned}$$

. The above equation shows that the growth rate consists of output growth and the displacement effect of automation. Note that the growth rate of output is

$$\frac{\dot{Y}_t}{Y_t} = S_K\frac{\dot{K}_t}{K_t} + \frac{1}{1-\sigma}\frac{\dot{\Phi}_t}{1-\Phi_t}\left(S_L - S_K\frac{1-\Phi_t}{\Phi_t}\right)$$

where $S_K \equiv \frac{K_t^{\frac{\sigma-1}{\sigma}}\Phi_t^{\frac{1}{\sigma}}}{K_t^{\frac{\sigma-1}{\sigma}}\Phi_t^{\frac{1}{\sigma}}+L^{\frac{\sigma-1}{\sigma}}(1-\Phi_t)^{\frac{1}{\sigma}}}$ and $S_L \equiv \frac{L^{\frac{\sigma-1}{\sigma}}(1-\Phi_t)^{\frac{1}{\sigma}}}{K_t^{\frac{\sigma-1}{\sigma}}\Phi_t^{\frac{1}{\sigma}}+L^{\frac{\sigma-1}{\sigma}}(1-\Phi_t)^{\frac{1}{\sigma}}}$, omitting time subscripts for notational simplicity. The first term is growth due to capital accumulation and the second term is growth due to the productivity effect of automation. The wage growth rate is then

$$\frac{\dot{w}_t}{w_t} = \frac{1}{\sigma}\left[S_K\frac{\dot{K}_t}{K_t} + \frac{1}{1-\sigma}\frac{\dot{\Phi}_t}{1-\Phi_t}\left(S_L - S_K\frac{1-\Phi_t}{\Phi_t}\right) - \frac{\dot{\Phi}_t}{1-\Phi_t}\right] \tag{A.8}$$

That is, wages rise as long as capital accumulation and the productivity effect dominate the displacement effect. In fact, this is another version of (2.18), which can be seen by setting $\dot{w}_t = 0$, and tells us what determines the growth rate of wages. Under the Pareto assumption, we have

$$\frac{\dot{w}_t}{w_t} = \frac{1}{\sigma}\left[S_K\frac{\dot{K}_t}{K_t} + \frac{1}{1-\sigma}\cdot\lambda g\cdot\left(S_L - S_K\frac{1-\Phi_t}{\Phi_t}\right) - \lambda g\right]$$

where $\lambda g$ is the rate of automation adjusted by the decay rate of the fraction of tasks for labor.

Notice that

$$S_K = \frac{(*)}{(*) + L^{\frac{\sigma-1}{\sigma}}}$$

If $\lambda g > (1-\sigma)(s^{\infty}A - \delta)$ (i.e. the first case in the beginning of the proof) then $S_K \to 1$ since $(*) \to \infty$. Note that

$$\frac{\dot{w}_t}{w_t} = \frac{1}{\sigma}\left[S_K \frac{\dot{K}_t}{K_t} + \frac{1}{1-\sigma} \cdot \lambda g \cdot \left(S_L - S_K \frac{1-\Phi_t}{\Phi_t}\right) - \lambda g\right]$$

As $t \to \infty$, the growth rate of wages converges as follows

$$\begin{aligned}
\frac{\dot{w}_t}{w_t} &\to \frac{1}{\sigma}\left[1 \cdot (s^{\infty}A - \delta) + \frac{1}{1-\sigma} \cdot \lambda g \cdot \left(0 - 1 \cdot 0\right) - \lambda g\right] \\
&= \frac{1}{\sigma}\left[s^{\infty}A - \delta - \lambda g\right]
\end{aligned}$$

Thus, if $s^{\infty}A - \delta > \lambda g$ (but $s^{\infty}A - \delta < \lambda g/(1-\sigma)$) then wages grow exponentially at an asymptotic rate $\frac{1}{\sigma}\left[s^{\infty}A - \delta - \lambda g\right] = \frac{1}{\sigma}\left[\frac{A-\rho-\delta}{\eta} - \lambda g\right]$.

In the case where capital stock asymptotically grows at $\lambda g/(1-\sigma)$, $S_K$ converges to one. As a result, the growth rate of wages converges as follows

$$\begin{aligned}
\frac{\dot{w}_t}{w_t} &\to \frac{1}{\sigma}\left[S_K \frac{\lambda g}{1-\sigma} + \frac{1}{1-\sigma} \cdot \lambda g \cdot \left(1 - S_K - S_K \cdot 0\right) - \lambda g\right] \\
&= \frac{1}{\sigma}\left[S_K \frac{\lambda g}{1-\sigma} + \frac{\lambda g}{1-\sigma} \cdot (1 - S_K) - \lambda g\right] \\
&= \frac{1}{\sigma}\left[\frac{\lambda g}{1-\sigma} - \lambda g\right] \\
&= \frac{\lambda g}{1-\sigma}
\end{aligned}$$

$$\therefore \frac{\dot{w}_t}{w_t} \to \frac{\lambda g}{1-\sigma}$$

If $s^{\infty}A - \delta \leq \lambda g$ then wages decline until the automation index crosses the threshold and collapse to $A$, as (A.7) indicates.

### A.2.2 Proof of Proposition 2.13

*Proof of Proposition 2.13.* The production function can be written as

$$Y_t = A\left(K_t^{\frac{\sigma-1}{\sigma}}\Phi(I_{t-})^{\frac{1}{\sigma}} + K(I)_t^{\frac{\sigma-1}{\sigma}}\Delta_t^{\frac{1}{\sigma}} + L^{\frac{\sigma-1}{\sigma}}(1-\Phi(I_t))^{\frac{1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}}$$
$$F_K = A\left(K_t^{\frac{\sigma-1}{\sigma}}\Phi(I_{t-})^{\frac{1}{\sigma}} + K(I)_t^{\frac{\sigma-1}{\sigma}}\Delta_t^{\frac{1}{\sigma}} + L^{\frac{\sigma-1}{\sigma}}(1-\Phi(I_t))^{\frac{1}{\sigma}}\right)^{\frac{1}{\sigma-1}} K_t^{-\frac{1}{\sigma}}\Phi(I_{t-})^{\frac{1}{\sigma}}$$
$$F_{K(I_t)} = A\left(K_t^{\frac{\sigma-1}{\sigma}}\Phi(I_{t-})^{\frac{1}{\sigma}} + K(I)_t^{\frac{\sigma-1}{\sigma}}\Delta_t^{\frac{1}{\sigma}} + L^{\frac{\sigma-1}{\sigma}}(1-\Phi(I_t))^{\frac{1}{\sigma}}\right)^{\frac{1}{\sigma-1}} K(I)_t^{-\frac{1}{\sigma}}\Delta_t^{\frac{1}{\sigma}}$$
$$F_L = A\left(K_t^{\frac{\sigma-1}{\sigma}}\Phi(I_{t-})^{\frac{1}{\sigma}} + K(I)_t^{\frac{\sigma-1}{\sigma}}\Delta_t^{\frac{1}{\sigma}} + L^{\frac{\sigma-1}{\sigma}}(1-\Phi(I_t))^{\frac{1}{\sigma}}\right)^{\frac{1}{\sigma-1}} L^{-\frac{1}{\sigma}}(1-\Phi(I_t))^{\frac{1}{\sigma}}$$

If $F_L < F_{K(I_t)}$, then the specific capital and labor are perfectly substitutable. That is,

$$\frac{K(I_t)}{L} < \frac{\Delta_t}{1-\Phi(I_t)}$$
$$k(I_t) = \frac{K(I_t)}{\Delta_t} < \frac{L}{1-\Phi(I_t)}$$

in which case, the production function can be written as

$$Y_t = A\left(K_t^{\frac{\sigma-1}{\sigma}}\Phi(I_{t-})^{\frac{1}{\sigma}} + (K(I_t)+L)^{\frac{\sigma-1}{\sigma}}(1-\Phi(I_{t-}))^{\frac{1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}}$$

If $F_K > F_{K(I_t)}$, then the specific capital and the traditional capital are perfectly substitutable. That is,

$$\frac{K(I_t)}{K_t} > \frac{\Delta_t}{\Phi(I_{t-})}$$
$$k(I_t) = \frac{K(I_t)}{\Delta_t} > \frac{K_t}{\Phi(I_{t-})}$$

in which case, the production function can be written as

$$Y_t = A\left((K_t + K(I_t))^{\frac{\sigma-1}{\sigma}}\Phi(I_t)^{\frac{1}{\sigma}} + L^{\frac{\sigma-1}{\sigma}}(1-\Phi(I_t))^{\frac{1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}}$$

□

# Appendix B

# Additional Results

## B.1 Additional Results for Chapter 1

### B.1.1 Two Layers

In the following discussion, I lay out the steps to solve for an equilibrium.

1. Given the thresholds, solve for the assignment functions each of which satisfies the labor market clearing equation on each interval.
2. Given the thresholds and the assignment functions, solve for workers' and machines' wages using the first-order conditions.
3. Pin down the thresholds using the assignment functions and the wage functions.

**Assignment** Assuming $\theta < 1-\Delta$, there are three intervals on which assignment functions and workers' wage functions are defined. First, $\mathcal{I}_1 = [\theta, 1-\Delta]$ is the machine-only region. Second, $\mathcal{I}_2 = [1-\Delta, z]$ is where workers and machines co-exist. Third, $\mathcal{I}_3 = [z, \theta+\phi]$ is populated only by machines that are more productive than any worker.

On $\mathcal{I}_1$, the labor market clearing equation is given by

$$g(x_1) = \frac{1}{h(1-x_1)} f(a(x_1))a'(x_1), \;\; x_1 \in \mathcal{I}_1$$

On $\mathcal{I}_2$, however, managers face a larger pool of workers to choose from since there are now machines as well. In this case, the labor market clearing equation takes the following form:

$$(f(x_1) + g(x_1)) = \frac{1}{h(1-x_1)} f(a(x_1))a'(x_1), \;\; x_1 \in \mathcal{I}_2$$

As is clear from the above equation, if there were no machines ($\mu = 0$) then the labor market clearing equation would be the same as the first expression. A positive value of $\mu$ implies that there is a greater supply of workers that pass on unsolved problems, and thus greater competition for managers. Lastly, the interval $\mathcal{I}_3$ is where advanced machines are. On this interval, the assignment function connects machines with top managers since these machines are the most skilled workers available to managers.

$$g(x_1) = \frac{1}{h(1-x_1)} f(a(x_1))a'(x_1), \;\; x_1 \in \mathcal{I}_3$$

where, again, the right-hand side is the demand of managers for workers.

Since $f(x) = 1/\Delta$ and $g(x) = \mu$ over the relevant supports, the labor market clearing conditions become:

$$\begin{aligned} a'(x_1) &= \mu\Delta h(1-x_1), \;\; x_1 \in \mathcal{I}_1 \\ a'(x_1) &= (1+\mu\Delta)h(1-x_1), \;\; x_1 \in \mathcal{I}_2 \\ a'(x_1) &= \mu\Delta h(1-x_1), \;\; x_1 \in \mathcal{I}_3 \end{aligned}$$

These are separate differential equations that can be solved independently given boundary conditions. To guarantee the continuity of the entire assignment function and market clearing over the entire interval $\mathcal{I}$, I impose the following boundary conditions:

$$\begin{aligned} a_1(\theta) &= z \\ a_2(z) &= \hat{x}_2 \\ a_3(\theta+\phi) &= 1 \end{aligned}$$

Given the thresholds, these conditions guarantee that the assignment function is one-to-one and onto as a mapping between $\mathcal{I}$ and $[z, 1]$. The equilibrium assignment function given the thresholds is then

$$a(x_1) = \begin{cases} a_1(x_1) = -\frac{\mu\Delta h}{2}(1-x_1)^2 + \frac{\mu\Delta h}{2}(1-\theta)^2 + z, \ x_1 \in \mathcal{I}_1 \\ a_2(x_1) = -\frac{(1+\mu\Delta)h}{2}(1-x_1)^2 + \frac{(1+\mu\Delta)h}{2}(1-z)^2 + \hat{x}_2, \ x_1 \in \mathcal{I}_2 \\ a_3(x_1) = -\frac{\mu\Delta h}{2}(1-x_1)^2 + \frac{\mu\Delta h}{2}(1-(\theta+\phi))^2 + 1, \ x_1 \in \mathcal{I}_3 \end{cases}$$

Additional conditions for continuity pin down the equilibrium thresholds

$$\begin{aligned} a_1(1-\Delta) = \tilde{x}_2 &\implies \tilde{x}_2(z) \\ a_2(\theta) = \tilde{x}_2 &\implies \hat{x}_2(z) \\ a_3(z) = \hat{x}_2 &\implies z \end{aligned}$$

which give

$$\begin{aligned} z =& \frac{h + 1 - \sqrt{1 + h^2\Delta^2 + (1+\mu\Delta)h^2((1-\theta)^2 - (1-(\theta+\phi))^2)}}{h} \\ \tilde{x}_2 =& -\frac{\mu\Delta h}{2}\Delta^2 + \frac{\mu\Delta h}{2}(1-\theta)^2 + z \\ \hat{x}_2 =& \tilde{x}_2 + \frac{(1+\mu\Delta)h}{2}(1-\theta)^2 - \frac{(1+\mu\Delta)h}{2}(1-z)^2 \end{aligned}$$

Consider the following comparative statics result

$$\frac{\partial a(x_1;\phi)}{\partial \phi} = \begin{cases} \frac{\partial z}{\partial \phi} < 0, \ x_1 \in \mathcal{I}_1 \\ -(1+\mu\Delta)h(1-z)\frac{\partial z}{\partial \phi} + \frac{\partial \hat{x}_2}{\partial \phi} = \frac{\partial z}{\partial \phi} < 0, \ x_1 \in \mathcal{I}_2 \\ -\mu\Delta h(1-(\theta+\phi)) < 0, \ x_1 \in \mathcal{I}_3 \end{cases}$$

**Wages** The equilibrium wage function is determined by the first-order condition

$$w_1'(x_1) = \frac{x_2 - w_1(x_1)}{1 - x_1}$$

The general solution to the above differential equation is

$$w_1(x_1) = \begin{cases} a(x_1) - \mu\Delta h x_1(1-x_1) + C_1(1-x_1), \;\; x_1 \in \mathcal{I}_1 \\ a(x_1) - (1+\mu\Delta) h x_1(1-x_1) + C_2(1-x_1), \;\; x_1 \in \mathcal{I}_2 \\ a(x_1) - \mu\Delta h x_1(1-x_1) + C_3(1-x_1), \;\; x_1 \in \mathcal{I}_3 \end{cases}$$

where $C_k$, $k = 1, 2, 3$, is some constant. Workers' wages must be continuous at $1-\Delta$ and $z$. Moreover, the marginal managers must earn the same amount as the marginal workers. Thus, the constants of integration must satisfy

$$\begin{aligned} a(1-\Delta) - \mu\Delta h(1-\Delta)\Delta + C_1\Delta =& a(1-\Delta) - (1+\mu\Delta)h(1-\Delta)\Delta + C_2\Delta \\ a(z) - (1+\mu\Delta)hz(1-z) + C_2(1-z) =& a(x_1) - \mu\Delta hz(1-z) + C_3(1-z) \\ a(z) - \mu\Delta hz(1-z) + C_3(1-z) =& w_2(z) \end{aligned}$$

The three conditions pin down the constants:

$$\begin{aligned} C_1 =& \frac{\mu\Delta\theta - \hat{x}_2 + (1+\mu\Delta)hz(1-z) - h(1-\Delta)(1-z)}{1 - z + 1/h} \\ C_2 =& C_1 + h(1-\Delta) \\ C_3 =& C_2 - hz \end{aligned}$$

To derive $\frac{\partial w_1(a^{-1}(x_2);\phi)}{\partial \phi}$, take the derivative of $C_1$ with respect to $\phi$

$$\begin{aligned} \frac{dC_1}{d\phi} =& \frac{1}{1-z+1/h}\left(-\frac{\partial \hat{x}_2}{\partial \phi} + (1+\mu\Delta)h(1-2z)\frac{\partial z}{\partial \phi} + h(1-\Delta)\frac{\partial z}{\partial \phi}\right) \\ &+ \frac{\mu\Delta\theta - \hat{x}_2 + \mu\Delta hz(1-z) - h(1-\Delta)(1-z)}{(1-z+1/h)^2}\frac{\partial z}{\partial \phi} \end{aligned}$$

After rearranging the terms, it follows that

$$\begin{aligned} \frac{dC_1}{d\phi} =& \frac{\partial z}{\partial \phi}\frac{1}{(1-z+1/h)^2} \\ & \times \left[-(1-z+1/h) - \mu\Delta z - hz(1-z+1/h) + (1-\Delta) + \mu\Delta\theta - \hat{x}_2\right] > 0 \end{aligned}$$

### B.1.2 Three Layers

**Assignment** To begin with, the equilibrium assignment functions solve the following differential equations

$$
\begin{aligned}
g(x_0) &= \frac{1}{h(1-x_0)} f(\tilde{a}(x_0))\tilde{a}'(x_0), \;\; x_0 \in [\theta^*, \theta+\phi] \\
f(x_1) &= \frac{1-x_0}{1-x_1} f(a(x_1))a'(x_1), \;\; x_1 \in [1-\Delta, z]
\end{aligned}
$$

where $\tilde{a}$ is the assignment function that connects machines and workers, and $a$ is the assignment function that connects workers and managers. Assuming $g(x) = \mu$ and $f(x) = 1/\Delta$ as before, and imposing the terminal condition $\tilde{a}(\theta+\phi) = z$, I have

$$
\tilde{a}(x_0) = -\frac{\mu\Delta h}{2}(1-x_0)^2 + \frac{\mu\Delta h}{2}(1-\theta-\phi)^2 + z, \;\; x_0 \in [\theta^*, \theta+\phi]
$$

Note that the employment function is

$$
1-\tilde{e}(x_1) = 1-\tilde{a}^{-1}(x_1) = \sqrt{(1-\theta-\phi)^2 + (z-x_1)/\Phi}, \;\; x_1 \in [1-\Delta, z]
$$

where $\Phi \equiv \mu\Delta h/2$. The equilibrium assignment function $a(\cdot)$ from workers to managers is:

$$
\begin{aligned}
a(x_1) =& 2\Phi\sqrt{(1-\theta-\phi)^2 + (t-z)/\Phi} \\
& \times \left[\left(\frac{2}{3}\frac{1}{\Phi} - 1\right)t + 1 + \frac{2}{3}(1-\theta-\phi)^2 - \frac{2}{3}\frac{1}{\Phi}z\right]\Bigg|_{1-\Delta}^{x_1} + z
\end{aligned}
$$

where $t$ is a dummy variable.

The market for machines clears through adjustments in the entry threshold $\theta^*$ that is determined by the free entry condition.

**Lemma B.1** (Entry threshold)**.** *There exists a cutoff skill level $\theta^* \in [\theta, \theta+\phi]$ such that $w_0(\theta^*) = \epsilon$. Thus, marginal machine producer earns zero profits in equilibrium.*

*Proof.* If $w_0(\theta^*) > \epsilon$ in equilibrium then machine producers with lower skills can enter and $\theta^*$ must fall. If $w_0(\theta^*) < \epsilon$ then the opposite holds. $\square$

Furthermore, the occupational threshold $z$ is pinned down by the condition $a(z) = 1$.

**Wages** Given the assignment functions and the thresholds, I can find the wage functions by solving the system of differential equations. The boundary conditions are

$$w_0(\theta^*) = \epsilon$$
$$w_1(z) = w_2(z)$$

The first condition states that the least skilled machines earn zero net income. This is because managers can lower the machine rent down to the entry cost and still hire them. The second condition ensures that marginal managers are indifferent to becoming workers.

$$w_2(x_2) = \max_{x_1 \in [1-\Delta, z]} n_1 n_0 \left(x_2 - x_1 + p(x_1)\right)$$
$$w_1(x_1) = \max_{x_0 \in [\theta, \theta+\phi]} n_0 \left(x_1 - x_0 + p(x_0) - p(x_1)\right)$$

where

$$p(x_1) = x_1 - x_0 - \frac{1}{n_0} w_1(x_1) + p(x_0)$$
$$p(x_0) = x_0 - w_0(x_0)$$

Substitute $n_0 = \frac{1}{h(1-x_0)}$ and $n_1 = \frac{1}{h(1-x_1)}$ into the problem.

$$w_2(x_2) = \max_{x_1 \in [1-\Delta, z]} \frac{x_2 - x_1 + p(x_1)}{h(1-x_1)}$$
$$w_1(x_1) = \max_{x_0 \in [\theta, \theta+\phi]} \frac{x_1 - x_0 + p(x_0) - p(x_1)}{h(1-x_0)}$$

where

$$p(x_1) = x_1 - x_0 - h(1-x_0) w_1(x_1) + p(x_0)$$
$$p(x_0) = x_0 - w_0(x_0)$$

### B.1.3 Solving the Model under Alternative Parameterizations

To verify the uniqueness of an equilibrium at each point on the parameter space, I calculate the allocation and see whether agents have an incentive to deviate from the allocation. To do so, I solve the allocation by imposing different boundary conditions and checking which boundary conditions are appropriate.

There are several cases to consider for both the two-layer and three-layer cases. The procedure follows the steps below:

1. Calculate thresholds for all possible cases.
2. Determine which case is valid.
3. Solve for the assignment function and the wage function.
4. Check the incentive to deviate.

**Two Layers**

**Case 1:** $\theta < \theta + \phi < 1 - \Delta < z$

$$\mathcal{I}_1 = [\theta, \theta + \phi]$$
$$\mathcal{I}_2 = [1 - \Delta, z]$$

$$g(x_1) = \frac{1}{h(1 - x_1)} f(a(x_1))a'(x_1), \;\; x_1 \in \mathcal{I}_1$$
$$f(x_1) = \frac{1}{h(1 - x_1)} f(a(x_1))a'(x_1), \;\; x_1 \in \mathcal{I}_2$$

Since $f(x) = 1/\Delta$ and $g(x) = \mu$ over the relevant supports, the labor market clearing conditions become:

$$a'(x_1) = \mu \Delta h(1 - x_1), \;\; x_1 \in \mathcal{I}_1$$
$$a'(x_1) = h(1 - x_1), \;\; x_1 \in \mathcal{I}_2$$

To guarantee the continuity of the entire assignment function and market clearing over the entire interval $\mathcal{I}$, I impose the following boundary conditions:

$$a_1(\theta) = z$$
$$a_2(z) = 1$$

Given the thresholds, these conditions guarantee that the assignment function is one-to-one and onto as a mapping between $\mathcal{I}$ and $[z, 1]$. The equilibrium assignment function given the thresholds is then

$$a(x_1) = \begin{cases} a_1(x_1) = -\frac{\mu\Delta h}{2}(1-x_1)^2 + \frac{\mu\Delta h}{2}(1-\theta)^2 + z, \;\; x_1 \in \mathcal{I}_1 \\ a_2(x_1) = -\frac{h}{2}(1-x_1)^2 + \frac{h}{2}(1-z)^2 + 1, \;\; x_1 \in \mathcal{I}_2 \end{cases}$$

Additional conditions for continuity pin down the equilibrium thresholds

$$a_1(\theta+\phi) = \hat{x}_2 \implies \hat{x}_2(z)$$
$$a_2(1-\Delta) = \hat{x}_2 \implies z$$

which give

$$z = 1 + 1/h - \sqrt{\Delta^2 + 1/h^2 + \mu\Delta\left[(1-\theta)^2 - (1-\theta-\phi)^2\right]}$$
$$\hat{x}_2 = -\frac{\mu\Delta h}{2}(1-\theta-\phi)^2 + \frac{\mu\Delta h}{2}(1-\theta)^2 + z$$

$$w_1(x_1) = \begin{cases} w_{11}(x_1) = a_1(x_1) - \mu\Delta h x_1(1-x_1) + C_1(1-x_1), \;\; x_1 \in \mathcal{I}_1 \\ w_{12}(x_1) = a_2(x_1) - \mu\Delta h x_1(1-x_1) + C_2(1-x_1), \;\; x_1 \in \mathcal{I}_2 \end{cases}$$

Using the conditions $w_{21}(\hat{x}_2) = w_{22}(\hat{x}_2)$ and $w_{12}(z) = w_2(z)$, the constants are

$$C_2 = \frac{\mu\Delta(1-\Delta-\phi) - 1 + \mu\Delta h z(1-z)}{1 - z + 1/h}$$
$$C_1 = C_2 + \mu\Delta h((\theta+\phi) - (1-\Delta))$$

**Case 2:** $\theta < 1 - \Delta < \theta + \phi < z$

$$\begin{aligned} \mathcal{I}_1 &= [\theta, 1-\Delta] \\ \mathcal{I}_2 &= [1-\Delta, \theta+\phi] \\ \mathcal{I}_3 &= [\theta+\phi, z] \end{aligned}$$

$$\begin{aligned} g(x_1) &= \frac{1}{h(1-x_1)} f(a(x_1))a'(x_1), \;\; x_1 \in \mathcal{I}_1 \\ f(x_1) + g(x_1) &= \frac{1}{h(1-x_1)} f(a(x_1))a'(x_1), \;\; x_1 \in \mathcal{I}_2 \\ f(x_1) &= \frac{1}{h(1-x_1)} f(a(x_1))a'(x_1), \;\; x_1 \in \mathcal{I}_2 \end{aligned}$$

Since $f(x) = 1/\Delta$ and $g(x) = \mu$ over the relevant supports, the labor market clearing conditions become:

$$\begin{aligned} a'(x_1) &= \mu\Delta h(1-x_1), \;\; x_1 \in \mathcal{I}_1 \\ a'(x_1) &= (1+\mu\Delta)h(1-x_1), \;\; x_1 \in \mathcal{I}_2 \\ a'(x_1) &= h(1-x_1), \;\; x_1 \in \mathcal{I}_3 \end{aligned}$$

To guarantee the continuity of the entire assignment function and market clearing over the entire interval $\mathcal{I}$, I impose the following boundary conditions:

$$\begin{aligned} a_1(\theta) &= z \\ a_2(\theta+\phi) &= \hat{x}_2 \\ a_3(z) &= 1 \end{aligned}$$

Given the thresholds, these conditions guarantee that the assignment function is one-to-one and onto as a mapping between $\mathcal{I}$ and $[z, 1]$. The equilibrium assignment function given the thresholds is then

$$a(x_1) = \begin{cases} a_1(x_1) \equiv -\frac{\mu\Delta h}{2}(1-x_1)^2 + \frac{\mu\Delta h}{2}(1-\theta)^2 + z, \;\; x_1 \in \mathcal{I}_1 \\ a_2(x_1) \equiv -\frac{(1+\mu\Delta)h}{2}(1-x_1)^2 + \frac{(1+\mu\Delta)h}{2}(1-\theta-\phi)^2 + \hat{x}_2, \;\; x_1 \in \mathcal{I}_2 \\ a_3(x_1) \equiv -\frac{h}{2}(1-x_1)^2 + \frac{h}{2}(1-z)^2 + 1, \;\; x_1 \in \mathcal{I}_3 \end{cases}$$

Additional conditions for continuity pin down the equilibrium thresholds

$$a_1(1-\Delta) = a_2(1-\Delta) \implies \hat{x}_2(z)$$
$$a_2(\theta+\phi) = a_3(\theta+\phi) \implies z$$

which give

$$\begin{aligned} z =& 1 + 1/h - \sqrt{1/h^2 + \Delta^2 + \mu\Delta[(1-\theta)^2 - (1-\theta-\phi)^2]} \\ \hat{x}_2 =& -\frac{h}{2}(1-\theta-\phi)^2 + \frac{h}{2}(1-z)^2 + 1 \end{aligned}$$

$$w_1(x_1) = \begin{cases} w_{11}(x_1) = a_1(x_1) - \mu\Delta h x_1(1-x_1) + C_1(1-x_1), \;\; x_1 \in \mathcal{I}_1 \\ w_{12}(x_1) = a_2(x_1) - (1+\mu\Delta)h x_1(1-x_1) + C_2(1-x_1), \;\; x_1 \in \mathcal{I}_2 \\ w_{13}(x_1) = a_3(x_1) - h x_1(1-x_1) + C_3(1-x_1), \;\; x_1 \in \mathcal{I}_3 \end{cases}$$

Using the continuity conditions $w_{11}(1-\Delta) = w_{12}(1-\Delta)$, $w_{12}(\theta+\phi) = w_{13}(\theta+\phi)$, and $w_{13}(z) = w_2(z)$, the constants are

$$\begin{aligned} C_1 =& C_2 - h(1-\Delta) \\ C_2 =& C_3 + \mu\Delta h(\theta+\phi) \\ C_3 =& \frac{\mu\Delta h - \mu\Delta(\theta+\phi) - \Delta + hz(1-z)}{1 - z + 1/h} \end{aligned}$$

**Case 3:** $\theta < 1-\Delta < z < \theta+\phi$ This is the original case.

$$\begin{aligned}
z =& \frac{h+1-\sqrt{1+h^2\Delta^2+(1+\mu\Delta)h^2((1-\theta)^2-(1-(\theta+\phi))^2)}}{h} \\
=& 1+1/h-\sqrt{1/h^2+\Delta^2+(1+\mu\Delta)[(1-\theta)^2-(1-(\theta+\phi))^2]} \\
\tilde{x}_2 =& -\frac{\mu\Delta h}{2}\Delta^2+\frac{\mu\Delta h}{2}(1-\theta)^2+z \\
\hat{x}_2 =& \tilde{x}_2+\frac{(1+\mu\Delta)h}{2}(1-\theta)^2-\frac{(1+\mu\Delta)h}{2}(1-z)^2
\end{aligned}$$

$$w_1(x_1) = \begin{cases} a(x_1)-\mu\Delta h x_1(1-x_1)+C_1(1-x_1), & x_1\in\mathcal{I}_1 \\ a(x_1)-(1+\mu\Delta)hx_1(1-x_1)+C_2(1-x_1), & x_1\in\mathcal{I}_2 \\ a(x_1)-\mu\Delta h x_1(1-x_1)+C_3(1-x_1), & x_1\in\mathcal{I}_3 \end{cases}$$

$$\begin{aligned}
C_1 =& \frac{\mu\Delta\theta-\hat{x}_2+(1+\mu\Delta)hz(1-z)-h(1-\Delta)(1-z)}{1-z+1/h} \\
C_2 =& C_1+h(1-\Delta) \\
C_3 =& C_2-hz
\end{aligned}$$

**Case 4:** $1-\Delta < \theta < \theta+\phi < z$

$$\mathcal{I}_1 = [1-\Delta,\theta]$$
$$\mathcal{I}_2 = [\theta,\theta+\phi]$$
$$\mathcal{I}_3 = [\theta+\phi,z]$$

$$\begin{aligned}
f(x_1) &= \frac{1}{h(1-x_1)}f(a(x_1))a'(x_1), \ \ x_1\in\mathcal{I}_1 \\
f(x_1)+g(x_1) &= \frac{1}{h(1-x_1)}f(a(x_1))a'(x_1), \ \ x_1\in\mathcal{I}_2 \\
f(x_1) &= \frac{1}{h(1-x_1)}f(a(x_1))a'(x_1), \ \ x_1\in\mathcal{I}_2
\end{aligned}$$

Since $f(x) = 1/\Delta$ and $g(x) = \mu$ over the relevant supports, the labor market clearing conditions become:

$$a'(x_1) = h(1 - x_1), \;\; x_1 \in \mathcal{I}_1$$
$$a'(x_1) = (1 + \mu\Delta)h(1 - x_1), \;\; x_1 \in \mathcal{I}_2$$
$$a'(x_1) = h(1 - x_1), \;\; x_1 \in \mathcal{I}_3$$

To guarantee the continuity of the entire assignment function and market clearing over the entire interval $\mathcal{I}$, I impose the following boundary conditions:

$$a(1 - \Delta) = z$$
$$a(\theta + \phi) = \hat{x}_2$$
$$a(z) = 1$$

Given the thresholds, these conditions guarantee that the assignment function is one-to-one and onto as a mapping between $\mathcal{I}$ and $[z, 1]$. The equilibrium assignment function given the thresholds is then

$$a(x_1) = \begin{cases} a_1(x_1) \equiv -\frac{h}{2}(1 - x_1)^2 + \frac{h}{2}(1 - \theta)^2 + z, \;\; x_1 \in \mathcal{I}_1 \\ a_2(x_1) \equiv -\frac{(1+\mu\Delta)h}{2}(1 - x_1)^2 + \frac{(1+\mu\Delta)h}{2}(1 - \theta - \phi)^2 + \hat{x}_2, \;\; x_1 \in \mathcal{I}_2 \\ a_3(x_1) \equiv -\frac{h}{2}(1 - x_1)^2 + \frac{h}{2}(1 - z)^2 + 1, \;\; x_1 \in \mathcal{I}_3 \end{cases}$$

Additional conditions for continuity pin down the equilibrium thresholds

$$a_1(\theta) = a_2(\theta) \implies \hat{x}_2(z)$$
$$a_2(\theta + \phi) = a_3(\theta + \phi) \implies z$$

which give

$$
\begin{aligned}
z =& \frac{1 + h - \sqrt{1 + 2\Delta h^2 \mu \phi - \Delta h^2 \mu \phi^2 - 2\Delta h^2 \mu \phi \theta}}{h} \\
\hat{x}_2 =& z + \frac{(1+\mu\Delta)h}{2}(1-\theta)^2 - \frac{(1+\mu\Delta)h}{2}(1-\theta-\phi)^2
\end{aligned}
$$

$$
w_1(x_1) = \begin{cases} w_{11}(x_1) = a_1(x_1) - hx_1(1-x_1) + C_1(1-x_1), \;\; x_1 \in \mathcal{I}_1 \\ w_{12}(x_1) = a_2(x_1) - (1+\mu\Delta)hx_1(1-x_1) + C_2(1-x_1), \;\; x_1 \in \mathcal{I}_2 \\ w_{13}(x_1) = a_3(x_1) - hx_1(1-x_1) + C_3(1-x_1), \;\; x_1 \in \mathcal{I}_3 \end{cases}
$$

Using the continuity conditions $w_{11}(\theta) = w_{12}(\theta)$, $w_{12}(\theta+\phi) = w_{13}(\theta+\phi)$, and $w_{13}(z) = w_2(z)$, the constants are

$$
\begin{aligned}
C_1 =& C_2 - \mu\Delta h\theta \\
C_2 =& C_3 + \mu\Delta h(\theta+\phi) \\
C_3 =& \frac{hz(1-z) - \Delta - \mu\Delta\phi}{1 - z + 1/h}
\end{aligned}
$$

**Case 5:** $1 - \Delta < \theta < z < \theta + \phi$

$$
\begin{aligned}
\mathcal{I}_1 &= [1-\Delta, \theta] \\
\mathcal{I}_2 &= [\theta, z] \\
\mathcal{I}_3 &= [z, \theta+\phi]
\end{aligned}
$$

$$
\begin{aligned}
f(x_1) &= \frac{1}{h(1-x_1)} f(a(x_1))a'(x_1), \;\; x_1 \in \mathcal{I}_1 \\
f(x_1) + g(x_1) &= \frac{1}{h(1-x_1)} f(a(x_1))a'(x_1), \;\; x_1 \in \mathcal{I}_2 \\
g(x_1) &= \frac{1}{h(1-x_1)} f(a(x_1))a'(x_1), \;\; x_1 \in \mathcal{I}_2
\end{aligned}
$$

Since $f(x) = 1/\Delta$ and $g(x) = \mu$ over the relevant supports, the labor market clearing conditions become:

$$a'(x_1) = h(1 - x_1), \;\; x_1 \in \mathcal{I}_1$$
$$a'(x_1) = (1 + \mu\Delta)h(1 - x_1), \;\; x_1 \in \mathcal{I}_2$$
$$a'(x_1) = \mu\Delta h(1 - x_1), \;\; x_1 \in \mathcal{I}_3$$

To guarantee the continuity of the entire assignment function and market clearing over the entire interval $\mathcal{I}$, I impose the following boundary conditions:

$$a_1(1 - \Delta) = z$$
$$a_2(z) = \hat{x}_2$$
$$a_3(\theta + \phi) = 1$$

Given the thresholds, these conditions guarantee that the assignment function is one-to-one and onto as a mapping between $\mathcal{I}$ and $[z, 1]$. The equilibrium assignment function given the thresholds is then

$$a(x_1) = \begin{cases} a_1(x_1) \equiv -\frac{h}{2}(1 - x_1)^2 + \frac{h}{2}\Delta^2 + z, \;\; x_1 \in \mathcal{I}_1 \\ a_2(x_1) \equiv -\frac{(1+\mu\Delta)h}{2}(1 - x_1)^2 + \frac{(1+\mu\Delta)h}{2}(1 - z)^2 + \hat{x}_2, \;\; x_1 \in \mathcal{I}_2 \\ a_3(x_1) \equiv -\frac{\mu\Delta h}{2}(1 - x_1)^2 + \frac{\mu\Delta h}{2}(1 - \theta - \phi)^2 + 1, \;\; x_1 \in \mathcal{I}_3 \end{cases}$$

Additional conditions for continuity pin down the equilibrium thresholds

$$a_1(\theta) = a_2(\theta) \implies \hat{x}_2(z)$$
$$a_2(z) = a_3(z) \implies z$$

which give

$$z = 1 + 1/h - \sqrt{\Delta^2 + 1/h^2 + \mu\Delta[(1-\theta)^2 - (1-\theta-\phi)^2]}$$
$$\hat{x}_2 = z + \frac{\mu\Delta h}{2}(1-\theta)^2 + \frac{h}{2}\Delta^2 - \frac{(1+\mu\Delta)h}{2}(1-z)^2$$

$$w_1(x_1) = \begin{cases} w_{11}(x_1) = a_1(x_1) - hx_1(1-x_1) + C_1(1-x_1), \;\; x_1 \in \mathcal{I}_1 \\ w_{12}(x_1) = a_2(x_1) - (1+\mu\Delta)hx_1(1-x_1) + C_2(1-x_1), \;\; x_1 \in \mathcal{I}_2 \\ w_{13}(x_1) = a_3(x_1) - \mu\Delta hx_1(1-x_1) + C_3(1-x_1), \;\; x_1 \in \mathcal{I}_3 \end{cases}$$

Using the continuity conditions $w_{11}(\theta) = w_{12}(\theta)$, $w_{12}(z) = w_{13}(z)$, and $w_{13}(z) = w_2(z)$, the constants are

$$C_1 = C_2 - \mu\Delta h\theta$$
$$C_2 = C_3 + hz$$
$$C_3 = \frac{1 - \Delta - z + \mu\Delta\theta - \hat{x}_2 + \mu\Delta hz(1-z)}{1 - z + 1/h}$$

**Three Layers**

**Case 1:** $\theta + \phi < 1 - \Delta$ To begin with, the equilibrium assignment functions solve the following differential equations

$$g(x_0) = \frac{1}{h(1-x_0)} f(\tilde{a}(x_0))\tilde{a}'(x_0), \;\; x_0 \in [\theta^*, \theta+\phi]$$
$$f(x_1) = \frac{1-x_0}{1-x_1} f(a(x_1))a'(x_1), \;\; x_1 \in [1-\Delta, z]$$

where $\tilde{a}$ is the assignment function that connects machines and workers, and $a$ is the assignment function that connects workers and managers. Assuming $g(x) = \mu$ and $f(x) = 1/\Delta$ as before, and imposing the terminal condition $\tilde{a}(\theta+\phi) = z$, I have

$$\tilde{a}(x_0) = -\frac{\mu\Delta h}{2}(1-x_0)^2 + \frac{\mu\Delta h}{2}(1-\theta-\phi)^2 + z, \;\; x_0 \in [\theta^*, \theta+\phi]$$

Note that the employment function is

$$1-\tilde{e}(x_1)=1-\tilde{a}^{-1}(x_1)=\sqrt{(1-\theta-\phi)^2+(z-x_1)/\Phi},\ \ x_1\in[1-\Delta,z]$$

where $\Phi\equiv\mu\Delta h/2$. With the initial condition $a(1-\Delta)=z$, the assignment function from workers to managers is

$$\begin{aligned}a(x_1)=&2\Phi\sqrt{(1-\theta-\phi)^2+(t-z)/\Phi}\\&\times\left[\left(\frac{2}{3}\frac{1}{\Phi}-1\right)t+1+\frac{2}{3}(1-\theta-\phi)^2-\frac{2}{3}\frac{1}{\Phi}z\right]\Bigg|_{1-\Delta}^{x_1}+z\end{aligned}$$

where $t$ is a dummy variable.

**Case 2:** $1-\Delta<\theta+\phi<z_1<z$

$$\begin{aligned}g(x_0)&=\frac{1}{h(1-x_0)}f(\tilde{a}(x_0))\tilde{a}'(x_0),\ \ x_0\in[\theta^*,1-\Delta]\\f(x_1)&=\frac{1-x_0}{1-x_1}f(a(x_1))a'(x_1),\ \ x_1\in[z_1,\tilde{x}_1]\\\tilde{a}(1-\Delta)&=\tilde{x}_1\end{aligned}$$

$$\begin{aligned}g(x_0)+f(x_0)&=\frac{1}{h(1-x_0)}f(\tilde{a}(x_0))\tilde{a}'(x_0),\ \ x_0\in[1-\Delta,\theta+\phi]\\f(x_1)&=\frac{1-x_0}{1-x_1}f(a(x_1))a'(x_1),\ \ x_1\in[\tilde{x}_1,\hat{x}_1]\\\tilde{a}(\theta+\phi)&=\hat{x}_1\end{aligned}$$

$$\begin{aligned}f(x_0)&=\frac{1}{h(1-x_0)}f(\tilde{a}(x_0))\tilde{a}'(x_0),\ \ x_0\in[\theta+\phi,z_1]\\f(x_1)&=\frac{1-x_0}{1-x_1}f(a(x_1))a'(x_1),\ \ x_1\in[\hat{x}_1,z]\\\tilde{a}(z_1)&=z\end{aligned}$$

$$\tilde{a}(x_0) = \begin{cases} \tilde{a}_1(x_0) \equiv -\frac{\mu\Delta h}{2}(1-x_0)^2 + \frac{\mu\Delta h}{2}\Delta^2 + \tilde{x}_1, \;\; x_0 \in [\theta^*, 1-\Delta] \\ \tilde{a}_2(x_0) \equiv -\frac{(1+\mu\Delta)h}{2}(1-x_0)^2 + \frac{(1+\mu\Delta)h}{2}(1-\theta-\phi)^2 + \hat{x}_1, \;\; x_0 \in [1-\Delta, \theta+\phi] \\ \tilde{a}_3(x_0) \equiv -\frac{h}{2}(1-x_0)^2 + \frac{h}{2}(1-z_1)^2 + z, \;\; x_0 \in [\theta+\phi, z_1] \end{cases}$$

$$a'(x_1) = \begin{cases} \frac{1-x_1}{\sqrt{\Delta^2 + \frac{2}{\mu\Delta h}(\tilde{x}_1 - x_1)}}, \;\; x_1 \in [z_1, \tilde{x}_1], \;\; a(z_1) = z \\ \frac{1-x_1}{\sqrt{(1-\theta-\phi)^2 + \frac{2}{(1+\mu\Delta)h}(\hat{x}_1 - x_1)}}, \;\; x_1 \in [\tilde{x}_1, \hat{x}_1], \;\; a(\tilde{x}_1) = \tilde{x}_2 \\ \frac{1-x_1}{\sqrt{(1-z_1)^2 + \frac{2}{h}(z - x_1)}}, \;\; x_1 \in [\hat{x}_1, z], \;\; a(\hat{x}_1) = \hat{x}_2 \end{cases}$$

**Assignment function 1:** The solution to the given differential equation

$$a'(x_1) = \frac{1-x_1}{\sqrt{\Delta^2 + \frac{2}{\mu\Delta h}(\tilde{x}_1 - x_1)}}$$

with the boundary condition $a(z_1) = z$ and $x_1 \in [z_1, \tilde{x}_1]$ is:

$$a(x_1) = \frac{1}{3}\left(-3AB + A^2B + 2A\tilde{x}_1B + Ax_1B + 3AC - A^2C - 2A\tilde{x}_1C - Az_1C + 3z\right)$$

where

$$A = \Delta h\mu,$$
$$B = \sqrt{\frac{\Delta^3 h\mu + 2\tilde{x}_1 - 2x_1}{\Delta h\mu}},$$
$$C = \sqrt{\frac{\Delta^3 h\mu + 2\tilde{x}_1 - 2z_1}{\Delta h\mu}}.$$

**Assignment function 2:** The general solution to the given differential equation

$$a'(x_1) = \frac{1-x_1}{\sqrt{(1-\theta-\phi)^2 + \frac{2}{(1+\mu\Delta)h}(\hat{x}_1 - x_1)}}$$

is:

$$
\begin{aligned}
a(x_1) &= \frac{DEF}{3} + C_1 \\
\text{where} \quad D &= h(1 + \Delta\mu), \\
E &= \sqrt{\frac{D(-1+\phi+\theta)^2 + 2(\hat{x}_1 - x_1)}{h + \Delta h\mu}}, \\
F &= -3 + 2\hat{x}_1 + D(-1+\phi+\theta)^2 + x_1.
\end{aligned}
$$

Here, $C_1$ is the constant of integration, which can be determined using the boundary condition $a(\tilde{x}_1) = \tilde{x}_2$. The constant $C_1$ can be determined using the boundary condition $a(\tilde{x}_1) = \tilde{x}_2$ as follows:

$$
\begin{aligned}
C_1 = -\frac{1}{3}\Bigg( & h(1+\Delta\mu)\sqrt{\frac{h(1+\Delta\mu)(-1+\phi+\theta)^2 + 2(\hat{x}_1 - \tilde{x}_1)}{h + \Delta h\mu}} \\
& \times (-3 + 2\hat{x}_1 + h(1+\Delta\mu)(-1+\phi+\theta)^2 + \tilde{x}_1)\Bigg) \\
& + \tilde{x}_2
\end{aligned}
$$

With this constant, you can fully specify the function $a(x_1)$ given the boundary condition. **Assignment function 3:** The solution to the given differential equation

$$
a'(x_1) = \frac{1 - x_1}{\sqrt{(1 - z_1)^2 + \frac{2}{h}(z - x_1)}}
$$

with the boundary condition $a(\hat{x}_1) = \hat{x}_2$ and $x_1 \in [\hat{x}_1, z]$ is:

$$a(x_1) = \frac{1}{3}\Big(3\hat{x}_2 + 3hG - h^2G - h\hat{x}_1G + 2h^2z_1G - h^2z_1^2G$$
$$- 2hzG - 3hH + h^2H + hx_1H - 2h^2z_1H + h^2z_1^2H + 2hzH\Big)$$

$$\text{where} \quad G = \sqrt{\frac{h - 2\hat{x}_1 - 2hz_1 + hz_1^2 + 2z}{h}},$$
$$H = \sqrt{\frac{h - 2x_1 - 2hz_1 + hz_1^2 + 2z}{h}}.$$

**Case 3:** $1 - \Delta < z_1 < \theta + \phi < z$

$$g(x_0) = \frac{1}{h(1-x_0)} f(\tilde{a}(x_0))\tilde{a}'(x_0), \;\; x_0 \in [\theta, 1-\Delta]$$
$$f(x_1) = \frac{1-x_0}{1-x_1} f(a(x_1))a'(x_1), \;\; x_1 \in [z_1, \tilde{x}_1]$$
$$\tilde{a}(1-\Delta) = \tilde{x}_1$$

$$g(x_0) + f(x_0) = \frac{1}{h(1-x_0)} f(\tilde{a}(x_0))\tilde{a}'(x_0), \;\; x_0 \in [1-\Delta, z_1]$$
$$f(x_1) = \frac{1-x_0}{1-x_1} f(a(x_1))a'(x_1), \;\; x_1 \in [\tilde{x}_1, \hat{x}_1]$$
$$\tilde{a}(z_1) = \hat{x}_1$$

$$f(x_0) = \frac{1}{h(1-x_0)} f(\tilde{a}(x_0))\tilde{a}'(x_0), \;\; x_0 \in [z_1, \theta+\phi]$$
$$f(x_1) = \frac{1-x_0}{1-x_1} f(a(x_1))a'(x_1), \;\; x_1 \in [\hat{x}_1, z]$$
$$\tilde{a}(\theta+\phi) = z$$

$$\tilde{a}(x_0) = \begin{cases} \tilde{a}_1(x_0) \equiv -\frac{\mu\Delta h}{2}(1-x_0)^2 + \frac{\mu\Delta h}{2}\Delta^2 + \tilde{x}_1, \ \ x_0 \in [\theta^*, 1-\Delta] \\ \tilde{a}_2(x_0) \equiv -\frac{(1+\mu\Delta)h}{2}(1-x_0)^2 + \frac{(1+\mu\Delta)h}{2}(1-z_1)^2 + \hat{x}_1, \ \ x_0 \in [1-\Delta, \theta+\phi] \\ \tilde{a}_3(x_0) \equiv -\frac{h}{2}(1-x_0)^2 + \frac{h}{2}(1-\theta-\phi)^2 + z, \ \ x_0 \in [\theta+\phi, z_1] \end{cases}$$

$$a'(x_1) = \begin{cases} \frac{1-x_1}{\sqrt{\Delta^2 + \frac{2}{\mu\Delta h}(\tilde{x}_1 - x_1)}}, \ \ x_1 \in [z_1, \tilde{x}_1], \ \ a(z_1) = z \\ \frac{1-x_1}{\sqrt{(1-\theta-\phi)^2 + \frac{2}{(1+\mu\Delta)h}(\hat{x}_1 - x_1)}}, \ \ x_1 \in [\tilde{x}_1, \hat{x}_1], \ \ a(\tilde{x}_1) = \tilde{x}_2 \\ \frac{1-x_1}{\sqrt{(1-z_1)^2 + \frac{2}{h}(z-x_1)}}, \ \ x_1 \in [\hat{x}_1, z], \ \ a(\hat{x}_1) = \hat{x}_2 \end{cases}$$

**Assignment function 1:** The solution to the given differential equation

$$a'(x_1) = \frac{1-x_1}{\sqrt{\Delta^2 + \frac{2}{\mu\Delta h}(\tilde{x}_1 - x_1)}}$$

with the boundary condition $a(z_1) = z$ and $x_1 \in [z_1, \tilde{x}_1]$ is:

$$\begin{aligned} a(x_1) = \frac{1}{3}\Bigg( & -3\Delta h\mu\sqrt{\frac{\Delta^3 h\mu + 2\tilde{x}_1 - 2x_1}{\Delta h\mu}} + \Delta^4 h^2\mu^2\sqrt{\frac{\Delta^3 h\mu + 2\tilde{x}_1 - 2x_1}{\Delta h\mu}} \\ & + 2\Delta h\mu\tilde{x}_1\sqrt{\frac{\Delta^3 h\mu + 2\tilde{x}_1 - 2x_1}{\Delta h\mu}} + \Delta h\mu x_1\sqrt{\frac{\Delta^3 h\mu + 2\tilde{x}_1 - 2x_1}{\Delta h\mu}} \\ & + 3\Delta h\mu\sqrt{\frac{\Delta^3 h\mu + 2\tilde{x}_1 - 2z_1}{\Delta h\mu}} - \Delta^4 h^2\mu^2\sqrt{\frac{\Delta^3 h\mu + 2\tilde{x}_1 - 2z_1}{\Delta h\mu}} \\ & - 2\Delta h\mu\tilde{x}_1\sqrt{\frac{\Delta^3 h\mu + 2\tilde{x}_1 - 2z_1}{\Delta h\mu}} - \Delta h\mu z_1\sqrt{\frac{\Delta^3 h\mu + 2\tilde{x}_1 - 2z_1}{\Delta h\mu}} \\ & + 3z\Bigg) \end{aligned}$$

**Assignment function 2:** The general solution to the given differential equation

$$a'(x_1) = \frac{1-x_1}{\sqrt{(1-\theta-\phi)^2 + \frac{2}{(1+\mu\Delta)h}(\hat{x}_1 - x_1)}}$$

is:

$$a(x_1) = \frac{h(1+\Delta\mu)\sqrt{\frac{h(1+\Delta\mu)(-1+\phi+\theta)^2+2(\hat{x}_1-x_1)}{h+\Delta h\mu}}(-3+2\hat{x}_1+h(1+\Delta\mu)(-1+\phi+\theta)^2+x_1)}{3}+C_1$$

Here, $C_1$ is the constant of integration, which can be determined using the boundary condition $a(\tilde{x}_1) = \tilde{x}_2$. The constant $C_1$ can be determined using the boundary condition $a(\tilde{x}_1) = \tilde{x}_2$ as follows:

$$\begin{aligned} C_1 = -\frac{1}{3}\Bigg( & h(1+\Delta\mu)\sqrt{\frac{h(1+\Delta\mu)(-1+\phi+\theta)^2+2(\hat{x}_1-\tilde{x}_1)}{h+\Delta h\mu}} \\ & \times(-3+2\hat{x}_1+h(1+\Delta\mu)(-1+\phi+\theta)^2+\tilde{x}_1)\Bigg) \\ & +\tilde{x}_2 \end{aligned}$$

With this constant, you can fully specify the function $a(x_1)$ given the boundary condition. **Assignment function 3:** The solution to the given differential equation

$$a'(x_1) = \frac{1-x_1}{\sqrt{(1-z_1)^2+\frac{2}{h}(z-x_1)}}$$

with the boundary condition $a(\hat{x}_1) = \hat{x}_2$ and $x_1 \in [\hat{x}_1, z]$ is:

$$
\begin{aligned}
a(x_1) = \frac{1}{3}\Bigg(& 3\hat{x}_2 + 3h\sqrt{\frac{h - 2\hat{x}_1 - 2hz_1 + hz_1^2 + 2z}{h}} - h^2\sqrt{\frac{h - 2\hat{x}_1 - 2hz_1 + hz_1^2 + 2z}{h}} \\
& - h\hat{x}_1\sqrt{\frac{h - 2\hat{x}_1 - 2hz_1 + hz_1^2 + 2z}{h}} + 2h^2 z_1\sqrt{\frac{h - 2\hat{x}_1 - 2hz_1 + hz_1^2 + 2z}{h}} \\
& - h^2 z_1^2\sqrt{\frac{h - 2\hat{x}_1 - 2hz_1 + hz_1^2 + 2z}{h}} - 2hz\sqrt{\frac{h - 2\hat{x}_1 - 2hz_1 + hz_1^2 + 2z}{h}} \\
& - 3h\sqrt{\frac{h - 2x_1 - 2hz_1 + hz_1^2 + 2z}{h}} + h^2\sqrt{\frac{h - 2x_1 - 2hz_1 + hz_1^2 + 2z}{h}} \\
& + hx_1\sqrt{\frac{h - 2x_1 - 2hz_1 + hz_1^2 + 2z}{h}} - 2h^2 z_1\sqrt{\frac{h - 2x_1 - 2hz_1 + hz_1^2 + 2z}{h}} \\
& + h^2 z_1^2\sqrt{\frac{h - 2x_1 - 2hz_1 + hz_1^2 + 2z}{h}} + 2hz\sqrt{\frac{h - 2x_1 - 2hz_1 + hz_1^2 + 2z}{h}}\Bigg)
\end{aligned}
$$

### B.1.4 Machine Management

**Assignment function** The equilibrium assignment function is the solution to the following equations

$$a'(x_1) = h(1 - x_1),\ \ a(\underline{y}) = \theta,\ \ x_1 \in [1 - \Delta, \underline{y}] \tag{B.1}$$

$$a'(x_1) = \frac{1/\Delta}{1/h\Delta + \mu/h_m}(1 - x_1),\ \ a(\bar{y}) = \theta + \phi,\ \ x_1 \in [\underline{y}, \bar{y}] \tag{B.2}$$

$$a'(x_1) = h(1 - x_1),\ \ a(z) = 1,\ \ x_1 \in [\bar{y}, z] \tag{B.3}$$

where the thresholds $z$, $\underline{y}$, and $\bar{y}$ satisfy the continuity conditions for each interval

$$
\begin{aligned}
a(1 - \Delta) =& z \\
a(\underline{y}) =& \theta \\
a(\bar{y}) =& \theta + \phi
\end{aligned}
$$

The equations characterize the assignment function on each of the three intervals. Furthermore, the continuity conditions ensure that the assignment function is continuous on $[1 - \Delta, z]$, which is necessary for the resulting assignment function to represent an equilibrium allocation.

$$a_1(x_1) = -\frac{h}{2}(1-x_1)^2 + \frac{h}{2}(1-\underline{y})^2 + \theta,\;\; x_1 \in [1-\Delta, \underline{y}] \tag{B.4}$$

$$a_2(x_1) = -\frac{\Phi}{2}(1-x_1)^2 + \frac{\Phi}{2}(1-\bar{y})^2 + \theta + \phi,\;\; x_1 \in [\underline{y}, \bar{y}] \tag{B.5}$$

$$a_3(x_1) = -\frac{h}{2}(1-x_1)^2 + \frac{h}{2}(1-z)^2 + 1,\;\; x_1 \in [\bar{y}, z] \tag{B.6}$$

where $\Phi \equiv \frac{1/\Delta}{1/h\Delta + \mu/h_m}$. The corresponding employment functions are

$$\begin{aligned}
e(x_2) =& 1 - \sqrt{\frac{\theta - x_2 + \frac{h}{2}(1-\underline{y})^2}{h/2}},\;\; x_2 \in [z, \theta] \\
e(x_2) =& 1 - \sqrt{\frac{\theta + \phi - x_2 + \frac{\Phi}{2}(1-\bar{y})^2}{\Phi/2}},\;\; x_2 \in [\theta, \theta+\phi] \\
e(x_2) =& 1 - \sqrt{\frac{1 - x_2 + \frac{h}{2}(1-z)^2}{h/2}},\;\; x_2 \in [\theta+\phi, 1]
\end{aligned}$$

$$\begin{aligned}
e(x_2) =& 1 - \sqrt{\frac{\theta - x_2}{h/2} + (1-\underline{y})^2},\;\; x_2 \in [z, \theta] \\
e(x_2) =& 1 - \sqrt{\frac{\theta + \phi - x_2}{\Phi/2} + (1-\bar{y})^2},\;\; x_2 \in [\theta, \theta+\phi] \\
e(x_2) =& 1 - \sqrt{\frac{1 - x_2}{h/2} + (1-z)^2},\;\; x_2 \in [\theta+\phi, 1]
\end{aligned}$$

**Thresholds** Suppose $\theta > z$. Given the assignment functions solved above, the thresholds $z$, $\underline{y}$, and $\bar{y}$ satisfy

$$\begin{aligned}
a(1-\Delta) =& z \\
a(\underline{y}) =& \theta \\
a(\bar{y}) =& \theta + \phi
\end{aligned}$$

More explicitly,

$$\begin{aligned}-\frac{h}{2}\Delta^2+\frac{h}{2}(1-\underline{y})^2+\theta&=z\\-\frac{\Phi}{2}(1-\underline{y})^2+\frac{\Phi}{2}(1-\bar{y})^2+\theta+\phi&=\theta\\-\frac{h}{2}(1-\bar{y})^2+\frac{h}{2}(1-z)^2+1&=\theta+\phi\end{aligned}$$

where $\Phi\equiv\frac{1/\Delta}{1/h\Delta+\mu/h_m}$. The second equation is

$$\begin{aligned}-\frac{\Phi}{2}(1-\underline{y})^2+\phi&=-\frac{\Phi}{2}(1-\bar{y})^2\\(1-\underline{y})^2-\phi\frac{2}{\Phi}&=(1-\bar{y})^2\end{aligned}$$

Using the first equation,

$$(1-\underline{y})^2=z\frac{2}{h}+\Delta^2-\frac{2}{h}\theta$$

$$\begin{aligned}(1-\bar{y})^2=&(1-\underline{y})^2-\phi\frac{2}{\Phi}\\=&z\frac{2}{h}+\Delta^2-\frac{2}{h}\theta-\phi\frac{2}{\Phi}\end{aligned}$$

The third equation is then

$$-\frac{h}{2}\left[z\frac{2}{h}+\Delta^2-\frac{2}{h}\theta-\phi\frac{2}{\Phi}\right]+\frac{h}{2}(1-z)^2+1-\theta-\phi=0$$

Since $z<1$, it follows that

$$z=1+1/h-\sqrt{1/h^2+\Delta^2+2\phi(1/h-1/\Phi)}$$

Given the value of $z$, $\underline{y}$ and $\bar{y}$ are

$$\begin{aligned}1-\underline{y}=&\sqrt{z\frac{2}{h}+\Delta^2-\frac{2}{h}\theta}\\1-\bar{y}=&\sqrt{z\frac{2}{h}+\Delta^2-\frac{2}{h}\theta-\phi\frac{2}{\Phi}}\end{aligned}$$

$$\begin{aligned}\underline{y}=&1-\sqrt{z\frac{2}{h}+\Delta^2-\frac{2}{h}\theta}\\\bar{y}=&1-\sqrt{z\frac{2}{h}+\Delta^2-\frac{2}{h}\theta-\phi\frac{2}{\Phi}}\end{aligned}$$

**Wages** The equilibrium wage function is determined by the first-order condition

$$w_1'(x_1)=\frac{x_2-w_1(x_1)}{1-x_1}$$

The general solution to the above differential equation is

$$w_1(x_1)=\begin{cases}w_{11}(x_1)=a_1(x_1)-hx_1(1-x_1)+C_1(1-x_1), & x_1\in[1-\Delta,\underline{y}]\\ w_{12}(x_1)=a_2(x_1)-\Phi x_1(1-x_1)+C_2(1-x_1), & x_1\in[\underline{y},\bar{y}]\\ w_{13}(x_1)=a_3(x_1)-hx_1(1-x_1)+C_3(1-x_1), & x_1\in[\bar{y},z]\end{cases}$$

where $\Phi\equiv\frac{1/\Delta}{1/h\Delta+\mu/h_m}$ and $C_k$, $k=1,2,3$, is some constant. Again, Workers' wages must be continuous at $1-\Delta$ and $z$. Moreover, the marginal managers must earn the same amount as the marginal workers. Thus, the constants of integration must satisfy

$$\begin{aligned}a(\underline{y})-h\underline{y}(1-\underline{y})+C_1(1-\underline{y})=&a(\underline{y})-\Phi\underline{y}(1-\underline{y})+C_2(1-\underline{y})\\a(\bar{y})-\Phi\bar{y}(1-\bar{y})+C_2(1-\bar{y})=&a(\bar{y})-h\bar{y}(1-\bar{y})+C_3(1-\bar{y})\\a(z)-hz(1-z)+C_3(1-z)=&w_2(z)=\frac{z-w_1(1-\Delta)}{h\Delta}\end{aligned}$$

Rearrange the terms to obtain

$$\begin{aligned}
-h\underline{y}(1-\underline{y}) + C_1(1-\underline{y}) =& -\Phi\underline{y}(1-\underline{y}) + C_2(1-\underline{y}) \\
-\Phi\bar{y}(1-\bar{y}) + C_2(1-\bar{y}) =& -h\bar{y}(1-\bar{y}) + C_3(1-\bar{y}) \\
1 - hz(1-z) + C_3(1-z) =& \frac{h(1-\Delta)\Delta - C_1\Delta}{h\Delta}
\end{aligned}$$

$$\begin{aligned}
C_1 =& h\underline{y} - \Phi\underline{y} + C_2 \\
C_2 =& \Phi\bar{y} - h\bar{y} + C_3 \\
1 - hz(1-z) + C_3(1-z) =& 1 - \Delta - \frac{C_1}{h}
\end{aligned}$$

Combine the first and second equations

$$\begin{aligned}
C_1 =& h\underline{y} - \Phi\underline{y} + \Phi\bar{y} - h\bar{y} + C_3 \\
=& C_3 + (h-\Phi)\underline{y} - (h-\Phi)\bar{y} \\
=& C_3 - (h-\Phi)(\bar{y} - \underline{y})
\end{aligned}$$

Substituting the above into the third equation, I obtain

$$\begin{aligned}
1 - hz(1-z) + C_3(1-z) =& 1 - \Delta - \frac{1}{h}C_3 + \frac{1}{h}(h-\Phi)(\bar{y} - \underline{y}) \\
C_3 =& \frac{hz(1-z) - \Delta + (h-\Phi)(\bar{y} - \underline{y})/h}{1 - z + 1/h}
\end{aligned}$$

### B.1.5 Machines as Middle Managers

The analysis so far is based on the organizational structure where machines are at the top of the hierarchy. This setup captures the long-term case where machines have the ability to run organizations on par with human managers.

However, even without the ability to run an entire organization, machines today have already automated various managerial tasks, as mentioned above. Thus, machines today are competing with middle managers who supervise workers but do not run entire organizations.

In order to examine this case, I modify the setup and analyze an equilibrium with machines in the middle layer. Consider the following assignment equations

$$f(x_1) = \left[\frac{1}{h(1-x_1)} \cdot f\left(a(x_1)\right) + \frac{1}{h_m(1-x_1)} \cdot g(a(x_1))\right] a'(x_1)$$
$$f(x_2) + g(x_2) = \frac{1}{h(1-x_2)} \cdot f\left(a(x_2)\right) a'(x_2)$$

The first equation links workers and middle managers, and is analogous to (1.5). The second equation links the middle managers and top managers. The equations assume that machines are only in the middle layer. As before, I solve this setup and verify that there is indeed an equilibrium that justifies the allocation.

With machines as middle managers, technological change may have the opposite effect on top income. Since it is the middle managers who are competing with machines, only the least skilled managers who are in the second layer experience a decline in their wages. On the other hand, top managers who supervise the middle managers are complemented by machines because they are in the upper layer.