Homological methods, singularities, and numerical invariants

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Abstract of the dissertation

The contents of this dissertation are various in nature and background. A connection between them is a common effort to measure the singularities of a commutative Noetherian ring using homological tools, and associated numerical invariants.

First, we study the index of a Gorenstein local ring, a numerical invariant that is defined in terms of Auslander's delta invariant. In particular, we find a counterexample to a conjecture of Songqing Ding relating the index and the minimal Löewy length of an Artinian reduction of the ring. Successively, we focus on two numerical invariants for rings of prime characteristic, the F-pure threshold and the diagonal F-threshold. We relate them with a third number, the a-invariant, proving most of a conjecture made by Hirose, Watanabe, and Yoshida. Finally, we study Golod rings. We present an example of a quotient of a polynomial ring over a field by a product of monomial ideals that is not Golod. This answers, in negative, a question of Welker.

Chapter 1

Introduction

The study of the singularities of a ring is a central topic in commutative algebra. A celebrated Theorem of Auslander, Buchsbaum, and Serre from 1956 established a first clear connection between the theory of singularities and homological algebra: a local ring is regular if and only if the residue field has finite projective dimension. This is also equivalent to any module having finite projective dimension.

Since a great deal of research in commutative algebra is directed at rings that are not regular, considerable effort has been devoted to develop tools to measure how singular a ring could be. If a local ring is not regular, then Auslander-Buchsbaum-Serre's Theorem implies that the maximal ideal has infinite projective dimension. However, one can get information about singularities by looking, for instance, at the projective dimension of other type of ideals. For example, a local ring is Cohen-Macaulay if and only if the projective dimension of any ideal generated by a system of parameters is finite. Cohen-Macaulayness is a weaker condition than regularity, but such rings are still, generally speaking, very well behaved.

This is just one instance of many results of this kind, that typically come from ways of generalizing conditions equivalent to regularity. These characterizations sometimes make use of numerical invariants, rather than conditions on actual modules or ideals. Usually, the advantage of this approach is that numbers may be easier to deal with; however, they may not remember all the information that the actual modules encode. One very classical example is the Hilbert-Samuel multiplicity: roughly speaking, this is a measure of "how fast" the powers of the maximal ideal in a local ring grow. It is a well known result of Nagata that, as long as the completion at the maximal ideal is unmixed, a local ring is regular if and only if the multiplicity equals one. However, the restriction on the completion cannot be removed, as the easy example $k[\![x,y,z]\!]/(xy,xz)$ shows.

We now focus more specifically on the topics that we study in this dissertation.

The first invariant that we investigate is the index of a Gorenstein local ring. It is defined in terms of Auslander's delta invariant, and it is directed at the study of the Maximal Cohen-Macaulay modules of a ring. Maximal Cohen-Macaulay modules over a local ring provide significant information about singularities: a local ring $(R,\mathfrak{m})$ is regular if and only if every Maximal Cohen-Macaulay module over R is free. In terms of the index, this is equivalent to saying that $\operatorname{index}(R) = 1$. When R is not regular, the index provides a measure of how singular the ring is: for instance, if a Gorenstein local ring with infinite residue field is not regular, and has minimal multiplicity, then $\operatorname{index}(R) = 2$. In 1993, Songqing Ding conjectured that the index of a Gorenstein local ring is equal to the minimal Löewy length of an Artinian reduction, an invariant that is called the generalized Löewy length. A regular local ring $(R, \mathfrak{m})$ has generalized Löewy length one, since $R/\mathfrak{m}$ is an Artinian reduction of R. Roughly speaking, the generalized Löewy length measures "how short" an Artinian reduction can be. Equality between these two invariants implies good and, somehow, expected properties about ideals of finite projective dimension, as explained in Section 2.2. The main goal of Chapter 2 is to exhibit several counterexamples to Ding's conjecture.

In 1969, an amazing result of Kunz produced another homological device to detect the singularities of a ring. This tool is only directly applicable to rings containing a field of prime characteristic p, but it can be extended to a greater generality by using methods of reduction to positive characteristic. The reason is that such rings come equipped with a surprisingly useful map, the Frobenius endomorphism. Kunz's Theorem says that a local ring is regular if and only if the Frobenius endomorphism is flat. The Frobenius is a very simple map: it raises every element of the ring to its p-th power, but it turns out to be surprisingly powerful. One of the reasons is that one can apply this map over and over again, obtaining significant asymptotic information. It also allows to make sense out of taking p-th roots of elements of the ring, and this somehow compensates for the lack of analytic methods for rings in positive characteristic, as wonderfully explained in [9]. To be a bit more specific, given either a local or a standard graded ring $(R, \mathfrak{m})$ of prime characteristic p on which the Frobenius map acts injectively, let $R^{1/p}$ denote the rings of p-th roots of R. The fact that an element $f^{c/p}$, where $c \in \mathbb{N}$ and $f \in R$, is not inside the module $\mathfrak{m}R^{1/p}$

is somehow saying that the function $1/f^{c/p}$ does not blow-up at the point $\mathfrak{m}$. In fact, $f^{c/p}$ is an element of $R^{1/p}$, and being not inside $\mathfrak{m}R^{1/p}$ is saying that its "value" at the point $\mathfrak{m}$ is non-zero. For any $f \in \mathfrak{m}$, for small non-negative values of c we have that $f^{c/p} \notin \mathfrak{m}R^{1/p}$, while $f^{c/p} \in \mathfrak{m}R^{1/p}$ for $c \gg 0$. Moreover, one can iterate the process of taking p-th roots, studying whether for some $f \in R$ it is the case or not that f^{c/p^e} is inside $\mathfrak{m}R^{1/p^e}$ for a certain value $c \in \mathbb{N}$. This produces the notion of F-threshold of an element $f \in R$, which is the supremum of all values $c/p^e \in \mathbb{Z}[1/p]$ for which f^{c/p^e} is not inside $\mathfrak{m}R^{1/p^e}$. This number is denoted by $c^{\mathfrak{m}}(f)$, for a given $f \in R$, and it can be generalized to $c^I(\mathfrak{a})$ for any two ideals $\mathfrak{a}, I \subseteq R$ for which $\mathfrak{a} \subseteq \sqrt{I}$. When R is regular, the so-called diagonal F-threshold $c^{\mathfrak{m}}(\mathfrak{m})$ measures the maximal order of a splitting $R^{1/p^e} \to R$, in a sense that we make more precise in Section 4.1. Researchers usually refer to this number as the F-pure threshold, which is a characteristic p invariant that has close connections with the log-canonical thresholds in equal characteristic 0. The log-canonical threshold is an invariant that can be defined starting from a resolution of singularities, and the relation with the F-pure threshold is via reduction to positive characteristic methods. If R is not regular, the two notions of diagonal F-threshold and F-pure threshold may differ. In general, the F-pure threshold seems to have a better control of the singularities of the ring; for example, it is equal to the Krull dimension if and only if the ring is regular. On the other hand, relations between the diagonal F-threshold $c^{\mathfrak{m}}(\mathfrak{m})$ and singularities, or connections with other invariants in characteristic zero, are still quite obscure. In Chapters 3 and 4 we study F-pure thresholds and diagonal F-thresholds of standard graded algebras over a field. We relate them with a third number, called the a-invariant, proving most of a conjecture of Hirose, Watanabe and Yoshida. We also introduce the notion of F-pure regular sequence, that is, a regular sequence that preserves some good splitting properties of the ring. The maximal length of an F-pure regular sequence in the Gorenstein case is surprisingly controlled by the F-pure threshold, and the existence of such sequences is guaranteed, at least when the base field is infinite, by certain "Bertini-type" theorems.

The last topic that we investigate in this dissertation is the class of Golod rings. A well established way to study singularities of a local ring $(R, \mathfrak{m}, k)$, or a standard graded k-algebra R, is to study the k-vector space dimensions $\beta_i := \dim_k \operatorname{Tor}_i^R(k, k)$, which are called the Betti numbers of k. More globally, one can focus on the shape of the generating series of the Betti numbers, $P_R(t) = \sum_{i \geqslant 0} \beta_i t^i$, which is called the Poincaré series of R. A restatement of Auslander-Buchsbaum-Serre's Theorem is that R is regular if and only if $P_R(t)$ is a polynomial. The study of the Poincaré series is a classical problem in Commutative Algebra, and one of the fundamental questions about $P_R(t)$, attributed by Serre and Kaplansky, was whether $P_R(t)$ is a rational function. This means that it can be expressed as a ratio of two polynomials with integer coefficients. The problem remained unsolved for several decades, until Anick showed that there exist Artinian rings whose Poincaré series is irrational. However, Serre showed that the Poincaré series of a ring is always bounded above, coefficient-wise, by a rational series. The upper bound is sharp, and rings for which equality holds are called Golod. The class of Golod ring is a very interesting and, in some sense, a very mysterious one. For instance, the only rings that are simultaneously Golod and Gorenstein are hypersurfaces. In addition, there is no relation between the Cohen-Macaulay property and the Golod property, and Golod rings do not behave well with respect to standard operations like localization or, in general, going modulo regular elements. The main purpose of Chapter 5 is to give an example of a product of two monomial ideals that does not define a Golod ring. This is quite unexpected, given that several partial results seemed to indicate that products of ideals may always have been Golod. For example, powers of homogeneous ideals in a polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$ are always Golod by [34].

1.1 Structure of the dissertation

The main achievement in Chapters 2 is a counterexample to a conjecture of Ding. Chapters 3 and 4 are devoted to the study of certain numerical invariants in prime characteristic, and to the proof of a conjecture made by Hirose, Watanabe, and Yoshida. In Chapter 4 we also introduce and study some related notions, such as the concept of F-pure regular sequence. Chapter 5 is focused on the study of Golod rings, and the main achievement is an example that answers, in negative, a question of Welker. The contents of the main Chapters will be briefly described in the following subsections.

1.1.1 Chapter 2

In Section 2.3 we establish some conditions that are equivalent to Auslander's delta invariant $\delta(R/\mathfrak{m}^n)$ being equal to one, for one-dimensional Gorenstein local rings $(R, \mathfrak{m}, k)$. This easily allows to compute the index of such rings. One of these formulations has been crucial in order to understand what can go wrong in a potential counterexample to Ding's conjecture. Namely, Proposition 2.3.2 (iii) suggested that the equality $\mathfrak{m}^{n+1} \cap (x) = x\mathfrak{m}^n$, where x is a non zero-divisor in R, was somehow required in order for the conjecture to hold in general. This equality is true when the associated graded ring is Cohen-Macaulay, by Valabrega-Valla's Theorem, but can easily fail in general. This led to a series of counterexamples, exhibited in Section 2.4.

1.1.2 Chapter 3

In Section 3.1 we collect some facts about graded modules and local cohomology. In the successive sections, we mainly focus on the prime characteristic case, looking at interactions of local cohomology modules $H^i_{\mathfrak{m}}(R)$ and F-purity, that is, the splitting of the natural inclusion $R \subseteq R^{1/p}$. In Section 3.4 we recall some homogeneous Fedder-type criteria to describe explicitly the R^{1/p^e}-module structure of the module of R-homomorphisms from R^{1/p^e} to R. The results in Chapter 3 are essentially well known, but sometimes hard to find in the literature.

1.1.3 Chapter 4

In Section 4.1 we exhibit a formulation of the F-pure threshold of a homogeneous ideal in a standard graded k-algebra that was probably known to experts, but never recorded in this generality. This is crucial in order to attack a conjecture of Hirose, Watanabe and Yoshida. In Section 4.2 we prove most of the conjecture, and find a counterexample to the remaining part, at least in the generality in which the theorem is stated. In Section 4.3 we focus on F-pure thresholds of Gorenstein standard graded k-algebras, for which $\operatorname{fpt}(R) = -a(R)$ by our results in Section 4.2. In general, it is quite hard to keep control of the F-pure threshold of a ring when going modulo a homogeneous regular element. On the other hand, it is clear what happens to the a-invariant in this instance. The fact that $\operatorname{fpt}(R) = -a(R)$ is crucial to set up an induction, since it allows to control each step when going modulo a homogeneous regular element. This led to the notion of F-pure regular sequence, that is, a homogeneous regular sequence that preserves F-purity when going modulo any of its elements. In Section 4.4 we establish an upper bound for the length of an F-pure regular sequence, that we prove to be sharp in case the base field k is infinite. In fact, we obtain an existence result for F-pure regular sequences in the spirit of the Bertini Theorems for smooth hyperplane cuts on a variety. More specifically, we show that a generic choice of linear forms in a Gorenstein F-pure ring is an F-pure regular sequence, as long as the length of the sequence is at most $\operatorname{fpt}(R)$.

1.1.4 Chapter 5

In Section 5.2 we answer in negative a question of Volkmar Welker: we give examples of standard graded k-algebras, over any field k, defined as quotients of polynomial rings by products of homogeneous ideals, that are not Golod. Even more interestingly, since the homogeneous ideals in question are monomial, these examples contradict a previous Theorem of Sayed-Fakhari and Welker, which was stating that the question had positive answer in the monomial case. In Section 5.3, we study the strongly Golod property of rational powers of monomial ideals, proving that $I^{p/q}$ is strongly Golod whenever I is strongly Golod and $p \geqslant q$. As a consequence, we show that $I^{p/q}$ is strongly Golod if $p \geqslant 2q$. This is somehow consistent with a result of Herzog and Huneke, which states that regular powers I^d of homogeneous ideals are strongly Golod. Note, however, that rational powers $I^{p/q}$ do not need to be regular powers, even when q divides p, since they are defined using integral closure. In Section 5.4 we introduce the notion of lcm-strongly Golod, that generalizes the concept of squarefree-strongly Golod, introduced by Herzog and Huneke. We show that lcm-strongly Golod ideals are weakly Golod, that is, they have trivial multiplication on Koszul homology in positive degrees. Finally, in Section 5.5, we study some sufficient conditions to conclude that a product of ideals is Golod, and ask several general questions on Golod rings.

Chapter 2

Ding's Conjecture

The central purpose of this chapter is to exhibit several counterexamples to a conjecture made by Songqing Ding in 1993, regarding the index of a local ring. The main results that we present appear in [15].

2.1 Preliminaries

Throughout, $(R, \mathfrak{m}, k)$ will denote a commutative Noetherian local ring with identity, with unique maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$. In addition, M will denote a finitely generated R-module. We will use $\lambda(M)$ to denote the length of a module, that is, the length of any composition series of M. Furthermore, $\mu(M)$ will denote the minimal number of generators of a module, that is, the length of $M/\mathfrak{m}M \cong k \otimes_R M$. We will assume a basic knowledge of commutative algebra, on the level of Atiyah-Macdonald [4] and Matsumura [52].

2.1.1 The index of a local ring and Ding's conjecture

Let $(R, \mathfrak{m}, k)$ be a local ring. We recall here some known results about Auslander's delta invariant and the index of a local ring. For more details we refer the reader to [18] and [48, Chapter 11]. We will say that R is Cohen-Macaulay if $\operatorname{depth}(R) = \dim(R)$. For a non zero finitely generated R-module X, we say that X is maximal Cohen-Macaulay (MCM for short) if $\operatorname{depth}(X) = \dim(R)$. For a finitely generated R-module X, the free rank of X, denoted $\text{f-rank}(X)$ is the maximal number of copies of R splitting out of X. In other words, X can be written as $X \cong R^{\text{f-rank}(X)} \oplus N$, where N has no free summands. This number is well defined over a local ring. In fact, having a free splitting of X is equivalent to having an R-linear map $\varphi : X \to R$ such that $\varphi(x) \notin \mathfrak{m}$ for some $x \in X$. Therefore, setting $I_X := \{x \in X \mid \varphi(x) \in \mathfrak{m} \text{ for all } \varphi \in \operatorname{Hom}_R(X, R)\}$, we have that $I_X \subseteq X$ is an R-submodule, and $\text{f-rank}(X) = \lambda(X/I_X)$ is independent of the direct sum decomposition.

Definition 2.1.1. *Let $(R, \mathfrak{m}, k)$ be a Cohen-Macaulay local ring and let M be a finitely generated R-module. The Auslander delta invariant of M is defined as*

$$\delta(M) = \min\{\text{f-rank}(X) \mid X \textit{ is MCM and there exists a surjection } X \to M \to 0\}.$$

Remark 2.1.2. There is always a surjection $R^{\mu(M)} \to M \to 0$, and since R is MCM, and M is finitely generated, we obtain that $\delta(M) \leqslant \mu(M) < \infty$. In addition, given a surjection $M \to N \to 0$ of finitely generated R-modules, for any surjective map $X \to M \to 0$ one obtains a surjection $X \to N \to 0$. Therefore $\delta(M) \geqslant \delta(N)$.

Using the notion of minimal MCM approximation, originally due to Auslander and Buchweitz [5], one can compute $\delta(M)$ by looking at a specific choice of a MCM module mapping onto M.

Definition 2.1.3. *A MCM approximation of a finitely generated R-module M is an exact sequence*

$$0 \longrightarrow Y_M \xrightarrow{\ i\ } X_M \longrightarrow M \longrightarrow 0,$$

where X_M is a finitely generated MCM R-module, and Y_M is a finitely generated R-module of finite injective dimension. The approximation is called minimal if Y_M and X_M have no common direct summand via i.

When R is Cohen Macaulay, minimal MCM approximations of a finitely generated R-module M always exist, and they are unique up to isomorphism of short exact sequences inducing the identity on M [48, Theorem 11.17 and Proposition 11.13].

We describe how to construct a minimal MCM approximation of a Cohen-Macauly module M such that $\operatorname{depth}(R) - \operatorname{depth}(M) = 1$, in case R is Gorenstein. This argument can be used, for instance, when R is a one-dimensional Gorenstein ring, and M has finite length. Consider the module $M^\vee = \operatorname{Ext}^1_R(M, R)$, and say that $\mu(M^\vee) = t$. Then, there exists a short exact sequence $0 \to \Omega \to R^t \to M^\vee \to 0$, with Ω a MCM R-module. This follows from the depth-lemma [12], and the fact that $\operatorname{depth}(M) = \operatorname{depth}(R) - 1$. Furthermore, Ω and R^t have no common free summand via the map in the short exact sequence. Now apply $\operatorname{Hom}_R(-, R)$ to such a sequence, to obtain a long exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(M^\vee, R) \longrightarrow \mathrm{Hom}_R(R^t, R) \longrightarrow \mathrm{Hom}_R(\Omega, R) \longrightarrow \mathrm{Ext}^1_R(M^\vee, R) \longrightarrow 0$$

where the last zero follows from the fact that $\mathrm{Ext}^1_R(R^t, R) = 0$, since R^t is free. Note that $\mathrm{Hom}_R(M^\vee, R) = 0$, and $\mathrm{Ext}^1_R(M^\vee, R) \cong M$, by duality for Gorenstein rings [12]. A quick way to see the vanishing of $\mathrm{Hom}_R(M^\vee, R)$ is to notice that, since $\mathrm{depth}(M) < \mathrm{depth}(R)$, there is a non zero-divisor $x \in R$ that kills the module $M^\vee$, and hence the module $\mathrm{Hom}_R(M^\vee, R)$. But this is a submodule of $\mathrm{Hom}_R(R^t, R) \cong R^t$, which is torsion free. Thus $\mathrm{Hom}_R(M^\vee, R) = 0$. Putting these facts together, we get a short exact sequence

$$0 \longrightarrow R^t \longrightarrow \mathrm{Hom}_R(\Omega, R) \longrightarrow M \longrightarrow 0.$$

Note that R^t has finite injective dimension, because R is Gorenstein [12]. Furthermore, the module $\mathrm{Hom}_R(\Omega, R)$ is MCM, because Ω is MCM. Finally, R^t and $\mathrm{Hom}_R(\Omega, R)$ are finitely generated, and they have no common summands via the map in the sequence, because otherwise the modules Ω and R^t in the first sequence would have a common free summand. Therefore the one above is a minimal MCM approximation of M.

Remark 2.1.4. Let $0 \to Y_M \to X_M \to M \to 0$ be a MCM approximation of an R-module M. Then, given any surjection $X \to M \to 0$ from a MCM R-module X, there exists a map $X \to X_M$ such that the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & X & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & Y_M & \longrightarrow & X_M & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

In fact, since Y_M has finite injective dimension, and X is MCM, we obtain that $\mathrm{Ext}^1_R(X, Y_M) = 0$ [48, Theorem 11.2]. This implies that the map $\mathrm{Hom}_R(X, X_M) \to \mathrm{Hom}_R(X, M)$ is surjective, proving the claim. In addition, every two lifts differ by an element of $\mathrm{Hom}_R(X, Y_M)$.

In our context, minimal MCM approximation are very useful, because one can read Auslander's delta invariant from them.

Proposition 2.1.5 ([48], Proposition 11.27)**.** *If $0 \to Y_M \to X_M \to M \to 0$ is a minimal MCM approximation of M, then $\delta(M) = \text{f-rank}(X_M)$.*

Proof. Let $\delta' = \text{f-rank}(X_M)$ and set $\delta = \delta(M)$. Clearly $\delta \leqslant \delta'$, since X_M is MCM, and we are given a surjection $X_M \to M \to 0$. Conversely, consider any surjection $X = \underline{X} \bigoplus R^{\oplus \delta} \to M \to 0$, with $\underline{X}$ a MCM module with no free summands. Also, write $X_M = \underline{X_M} \bigoplus R^{\oplus \delta'}$. By Remark 2.1.4, we can find a lift $\alpha : \underline{X} \to X_M$ such that the following diagram commutes:

$$\begin{array}{ccccccccc} & & & & \underline{X} & \xrightarrow{\pi} & M & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle \alpha} & & \| & & \\ 0 & \longrightarrow & Y_M & \longrightarrow & \underline{X_M} \bigoplus R^{\oplus \delta'} & \xrightarrow{p} & M & \longrightarrow & 0 \end{array}$$

Since $\underline{X}$ has no free summands, the image of the composition

$$\underline{X} \xrightarrow{\alpha} \underline{X_M} \bigoplus R^{\oplus\delta'} \twoheadrightarrow R^{\oplus\delta'}$$

is contained in $\mathfrak{m}R^{\oplus\delta'}$. Therefore, $\alpha(\underline{X})$ contains no minimal generators of $M/p(\underline{X_M})$ and, thus, $\mu(M/p(\underline{X_M})) \leqslant \mu(M/p(\alpha(\underline{X})))$.

Claim. Let $0 \to Z \to W \xrightarrow{p} M \to 0$ be a short exact sequence. Assume that $W = \underline{W} \bigoplus R^{\oplus n}$, with $\underline{W}$ a MCM module with no non-zero free summands. Then $\mu(M/p(\underline{W})) \leqslant n$, with equality if the one above is a minimal MCM approximation of M

Proof of the Claim. To prove the claim, consider the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \ker(p|_{\underline{W}}) & \to & Z & \to & \ker(\overline{p}) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \underline{W} & \to & W & \to & R^{\oplus n} & \to & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \overline{p}} & & \\
0 & \to & p(\underline{W}) & \to & M & \to & M/p(\underline{W}) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

The rightmost column shows that $\mu(M/p(\underline{W})) \leqslant \mu(R^{\oplus n}) = n$. Now assume that the middle column is a minimal MCM approximation of M. If $\mu(M/p(\underline{W})) < n$, then $\ker(\overline{p})$ has a non-zero free summand. By exactness of the first row, and commutativity of the upper-right square, this must be a non-zero free summand that Z and W have in common, contradicting minimality. $\square$

Now the proposition immediately follows. In fact, by the Claim, we have

$$\delta = \text{f-rank}(X) \geqslant \mu(M/\pi(\underline{X})) = \mu(M/p(\alpha(\underline{X}))) \geqslant \mu(M/p(\underline{X_M})) = \delta'.$$

$\square$

We now focus on $\delta(M)$ for some special choices of M: for any integer $n \geqslant 1$ consider $\delta(R/\mathfrak{m}^n)$. It follows from Remark 2.1.2 that

$$0 \leqslant \delta(R/\mathfrak{m}) \leqslant \delta(R/\mathfrak{m}^2) \leqslant \ldots\ldots \leqslant \delta(R/\mathfrak{m}^n) \leqslant \delta(R/\mathfrak{m}^{n+1}) \leqslant \ldots\ldots \leqslant 1,$$

Definition 2.1.6. *Let* $(R, \mathfrak{m}, k)$ *be a Gorenstein local ring. The index of* R *is*

$$\text{index}(R) := \inf\{n \mid \delta(R/\mathfrak{m}^n) = 1\}.$$

Note that, potentially, $\delta(R/\mathfrak{m}^n)$ could be zero for all $n \in \mathbb{N}$, so that $\text{index}(R) = \infty$. However, the index is finite when R is Gorenstein [18, Theorem 1.1]. In fact, if R is Gorenstein and $x_1, \ldots, x_d$ is a maximal regular sequence in R, the beginning of a minimal free resolution of $R/(x_1, \ldots, x_d)$ is

$$0 \longrightarrow \Omega \longrightarrow R \longrightarrow R/(x_1, \ldots, x_d) \longrightarrow 0,$$

and Ω has finite injective dimension since R and $\overline{R}$ do. Therefore the one above is a MCM approximation of $R/(x_1, \ldots, x_d)$, and it is easily seen to be minimal. For example, it is minimal because R is local, hence it is an indecomposable R-module.

Then, we get that $\delta(R/(x_1,\ldots,x_d)) = 1$ for any choice of a maximal regular sequence $x_1,\ldots,x_d \in R$. In particular, if one chooses $n \gg 0$, so that $\mathfrak{m}^n \subseteq (x_1,\ldots,x_d)$, which is possible because $\sqrt{(x_1,\ldots,x_d)} = \mathfrak{m}$, then there is a surjection $R/\mathfrak{m}^n \to R/(x_1,\ldots,x_d) \to 0$, and this gives $\delta(R/\mathfrak{m}^n) = 1$. Therefore $\operatorname{index}(R) \leqslant n < \infty$.

Definition 2.1.7. *Let $(R,\mathfrak{m},k)$ be a local ring of dimension d. For an R-module M of finite length, we define the Löewy length of M as $\ell\ell(M) := \min\{n \in \mathbb{N} \mid \mathfrak{m}^n M = 0\}$. If M is a finitely generated R-module of dimension $c > 0$, we define the generalized Löewy length of M to be*

$$\mathrm{g}\ell\ell(M) = \min\left\{\ell\ell\left(\frac{M}{(x_1,\ldots,x_c)M}\right) \,\middle|\, x_1,\ldots,x_c \text{ a system of parameters for } M\right\}.$$

The argument before the definition shows that, when R is Gorenstein, we have that $\operatorname{index}(R) \leqslant \ell\ell(R/(x_1,\ldots,x_d))$ for any system of parameters $x_1,\ldots,x_d$ for R. Therefore the inequality $\operatorname{index}(R) \leqslant \mathrm{g}\ell\ell(R)$ always holds.

We are finally in a position to state Ding's conjecture.

Conjecture 2.1.8. *Let $(R,\mathfrak{m},k)$ be a Gorenstein local ring of dimension d. Then*

$$\operatorname{index}(R) = \mathrm{g}\ell\ell(R).$$

Herzog showed that the conjecture is true for homogeneous Gorenstein algebras over an infinite field [32] (extending the concepts in an obvious way to the graded case). Later, Ding generalized this result proving that it holds true if the associated graded ring $\operatorname{gr}_{\mathfrak{m}}(R)$ is Cohen-Macaulay [19, Theorem 2.1]. Hashimoto and Shida pointed out in [29] that Ding's result needs the residue field k to be infinite, and the conjecture may fail for rings with finite residue field, even if $\mathrm{gr}_{\mathfrak{m}}(R)$ is Cohen-Macaulay [29, Example 3.2].

2.1.2 Hilbert functions and superficial elements

We now review some basic facts about associated graded rings, Hilbert functions, and superficial elements. The literature regarding these topics is extremely rich; however, we will restrict ourselves to the aspects that will be relevant for this dissertation. More complete references for the contents that follow are [43, 59].

Definition 2.1.9. *Let $(R, \mathfrak{m}, k)$ be a local ring. The associated graded ring of R is defined as*

$$\mathrm{gr}_{\mathfrak{m}}(R) = \bigoplus_{n \geqslant 0} \mathfrak{m}^n/\mathfrak{m}^{n+1} = R/\mathfrak{m} \oplus \mathfrak{m}/\mathfrak{m}^2 \oplus \mathfrak{m}^2/\mathfrak{m}^3 \oplus \dots$$

This object is a graded k-algebra, generated by its degree one part $\mathfrak{m}/\mathfrak{m}^2$. Such algebras are called standard graded. Given an element $x \in R$ of order d, we will denote by $x^* \in \mathfrak{m}^d/\mathfrak{m}^{d+1}$ its image in the associated graded ring. The multiplication in $\mathrm{gr}_{\mathfrak{m}}(R)$ is defined, for two homogeneous elements $x^* \in \mathfrak{m}^d/\mathfrak{m}^{d+1}$ and $y^* \in \mathfrak{m}^n/\mathfrak{m}^{n+1}$ coming from $x, y \in R$, as the image $(xy)^* \in \mathfrak{m}^{d+n}/\mathfrak{m}^{d+n+1}$ of the element xy. We then extend by linearity to sum of homogeneous elements.

Proposition 2.1.10. *Let $P = k[x_1, \dots, x_n]$ be a polynomial ring over a field, and let $\mathfrak{n} = (x_1, \dots, x_n)$. Assume that Q is either the completion $\widehat{P}$ with respect to $\mathfrak{n}$, or the* *localization $P_{\mathfrak{n}}$. Let $R := Q/I$ for some ideal $I \subseteq Q$, and let $\mathfrak{m}$ be the maximal ideal of R. Then the associated graded ring $\operatorname{gr}_{\mathfrak{m}}(R)$ is isomorphic, as a graded k-algebra, to P/I^*, where $I^* = (f^* \mid f \in I)$ is the initial ideal of I, that is, the homogeneous ideal of P generated by the initial forms f^* of elements $f \in I$.*

Proof. Let $\mathcal{M}$ be the maximal ideal of Q, so that $\mathfrak{m} = \mathcal{M}/I$. For an integer $n \geqslant 0$, we have that

$$\frac{\mathfrak{m}^n}{\mathfrak{m}^{n+1}} = \frac{\mathcal{M}^n + I}{\mathcal{M}^{n+1} + I} \cong \frac{\mathcal{M}^n}{\mathcal{M}^n \cap I + \mathcal{M}^{n+1}}$$

from the second isomorphism theorem. Note that $\mathcal{M}^n/\mathcal{M}^{n+1} \cong \mathfrak{n}^n/\mathfrak{n}^{n+1}$ as k-vector spaces, via $k \cong P/\mathfrak{n} \cong R/\mathfrak{m}$. Furthermore

$$I^* := \bigoplus_{n \geqslant 0} \frac{\mathcal{M}^n \cap I + \mathcal{M}^{n+1}}{\mathcal{M}^{n+1}} \subseteq \bigoplus_{n \geqslant 0} \frac{\mathcal{M}^n}{\mathcal{M}^{n+1}} \cong \bigoplus_{n \geqslant 0} \frac{\mathfrak{n}^n}{\mathfrak{n}^{n+1}} \cong P$$

is easily seen to be a homogeneous ideal of P, which is the ideal I^* generated be the initial forms f^* of elements $f \in I$. In fact, note that the initial form of an element $f \in \mathcal{M}^n \smallsetminus \mathcal{M}^{n+1}$ is precisely the image of $f \in \mathcal{M}^n \cap I$ modulo $\mathcal{M}^{n+1}$. Finally, we have that the ring operations are preserved by these isomorphisms, therefore

$$\operatorname{gr}_{\mathfrak{m}}(R) \cong \bigoplus_{n \geqslant 0} \frac{\mathcal{M}^n}{\mathcal{M}^n \cap I + \mathcal{M}^{n+1}} \cong \bigoplus_{n \geqslant 0} \frac{\mathcal{M}^n/\mathcal{M}^{n+1}}{(\mathcal{M}^n \cap I + \mathcal{M}^{n+1})/\mathcal{M}^{n+1}} \cong P/I^*$$

as graded k-algebras. □

Definition 2.1.11. *Let $(R, \mathfrak{m}, k)$ be a local ring. The Hilbert function $HF_R : \mathbb{N} \to \mathbb{N}$ of R is defined as*

$$HF_R(n) := \lambda(\mathfrak{m}^n/\mathfrak{m}^{n+1}) \qquad \text{for all } n \geqslant 0.$$

Note that, by definition, the Hilbert function of R coincides with the Hilbert function of its associated graded ring.

Remark 2.1.12. For $n \gg 0$, the Hilbert function is a polynomial of degree $\dim(R)-1$. In particular, when $\dim(R) = 1$, the Hilbert function $HF_R(n) = \lambda(\mathfrak{m}^n/\mathfrak{m}^{n+1})$ is eventually constant.

Remark 2.1.13. For given integers d, n there is a unique expression

$$d = \binom{k_n}{n} + \binom{k_{n-1}}{n-1} + \ldots + \binom{k_1}{1}$$

with $k_n > k_{n-1} > \ldots > k_1 \geqslant 0$. See, for instance, [12, Lemma 4.2.6] for more details. We define

$$d^{\langle n \rangle} := \binom{k_n+1}{n+1} + \binom{k_{n-1}+1}{n} + \ldots + \binom{k_1+1}{2}.$$

We now recall, without proof, a well-known theorem of Macaulay on Hilbert functions of standard graded algebras over a field.

Theorem 2.1.14. *([51], [12, Theorem 4.2.10]) Let k be a field and let A be a standard graded k-algebra. Then*

$$HF_A(n+1) \leqslant HF_A(n)^{\langle n \rangle} \quad \textit{for all } n \geqslant 1.$$

Proposition 2.1.15. *Let $(R, \mathfrak{m}, k)$ be a local ring, and let $x \in R$. Then, there is a surjective homomorphism $\pi : \mathrm{gr}_{\mathfrak{m}}(R)/(x^*) \to \mathrm{gr}_{\mathfrak{m}/(x)}(R/(x))$ of graded rings, which is of degree zero. In other words, for all $n \in \mathbb{N}$, the map π sends the degree n part of $\mathrm{gr}_{\mathfrak{m}}(R)/(x^*)$ to the degree n part of $\mathrm{gr}_{\mathfrak{m}/(x)}(R/(x))$.*

Proof. For $n \in \mathbb{N}$, the degree n part of $\mathrm{gr}_{\mathfrak{m}}(R)/(x^*)$ is

$$[\mathrm{gr}_{\mathfrak{m}}(R)/(x^*)]_n = \frac{\mathfrak{m}^n}{\mathfrak{m}^{n+1} + x\mathfrak{m}^n}.$$

On the other hand, the degree n part of $\mathrm{gr}_{\mathfrak{m}/(x)}(R/(x))$ is

$$\left[\mathrm{gr}_{\mathfrak{m}/(x)}(R/(x))\right]_n = \frac{\mathfrak{m}^n + (x)}{\mathfrak{m}^{n+1} + (x)} \cong \frac{\mathfrak{m}^n}{\mathfrak{m}^{n+1} + \mathfrak{m}^n \cap (x)}.$$

Since $x\mathfrak{m}^{n-1} \subseteq \mathfrak{m}^n \cap (x)$, for all $n \in \mathbb{N}$ there is a k-vector space surjective map $[\mathrm{gr}_{\mathfrak{m}}(R)/(x^*)]_n \to \left[\mathrm{gr}_{\mathfrak{m}/(x)}(R/(x))\right]_n$, which gives rise to a degree preserving surjection of graded rings $\pi : \mathrm{gr}_{\mathfrak{m}}(R)/(x^*) \to \mathrm{gr}_{\mathfrak{m}/(x)}(R/(x))$. □

We now turn our attention to superficial elements, connecting their properties with the ones of the associated graded ring.

Let $(R, \mathfrak{m}, k)$ be a local ring. For a non-zero element $x \in R$, we denote by $\mathrm{ord}(x)$ the order of x, that is the largest integer n such that $x \in \mathfrak{m}^n$. The order is well defined, because $\bigcap_n \mathfrak{m}^n = (0)$ by Krull's Intersection Theorem.

Definition 2.1.16. *Let $(R, \mathfrak{m}, k)$ be a local ring and let $x \in \mathfrak{m}$ with $\mathrm{ord}(x) = d \geqslant 1$. Then x is said to be a superficial element of order d if there exists an integer c such that $(\mathfrak{m}^{n+d} : x) \cap \mathfrak{m}^c = \mathfrak{m}^n$ for all $n \geqslant c$.*

Remark 2.1.17. If $\mathrm{depth}(R) > 0$, then any superficial element is a non zero-divisor of R. The converse is true for a general non zero-divisor of R, i.e., a non zero-divisor that avoids a certain closed set with respect to the Zariski topology. Superficial elements of any order exist if the residue field is infinite.

Proposition 2.1.18. *Let* $(R, \mathfrak{m}, k)$ *be a one-dimensional local ring. A non zero-divisor* $x \in R$ *with* $\operatorname{ord}(x) = d$ *is a superficial element if and only if the initial form* x^* *is a homogeneous parameter of degree* d *in the associated graded ring* $\operatorname{gr}_m(R)$.

Proof. Let $G := \operatorname{gr}_{\mathfrak{m}}(R)$, and let $\mathcal{M}$ be the irrelevant maximal ideal of G. Let $(0) = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_s \cap J$ be a primary decomposition of the ideal (0) in G, where $\mathfrak{p}_i = \sqrt{\mathfrak{q}_i}$ is a minimal prime of $\operatorname{gr}_{\mathfrak{m}}(R)$ for all i, and either $\sqrt{J} = \mathcal{M}$ or $J = G$. Assume that x is superficial of order d, so that there is $c \in \mathbb{N}$ such that $(\mathfrak{m}^{n+d} : x) \cap \mathfrak{m}^c = \mathfrak{m}^n$ for all $n \geqslant c$. To prove that x^* is a parameter, we have to show that $x^* \notin \mathfrak{p}_j$ for all $j = 1, \ldots, s$. For all i, we have that $\mathfrak{p}_i = (0 :_G y_i^*)$ for some homogeneous $y_i^* \in G$. In fact, since G is graded, all associated primes are homogeneous. Let $z_i \in R$ be an element of order c such that its image z_i^* in G is not contained in $\mathfrak{p}_i$. This means that $z_i^* y_i^* \neq 0$. Notice that $\mathfrak{p}_i = (0 :_G y_i^*) \subseteq (0 :_G z_i^* y_i^*)$ and, since associated primes are maximal among annihilators, we must have $\mathfrak{p}_i = (0 :_G z_i^* y_i^*)$. Let $w_i = z_i y_i$, and notice that $z_i^* y_i^*$ is the image in G of w_i; in addition, note that $\operatorname{ord}(w_i) \geqslant c$. Putting things together, if w_i^* denotes the image of w_i in G, for all $i = 1, \ldots, s$ we have that $\mathfrak{p}_i = (0 :_G w_i^*)$, with $\operatorname{ord}(w_i) = d_i \geqslant c$, for all $i = 1, \ldots, s$. Now, assume that $x^* \in \mathfrak{p}_i$ for some i. Then $x^* w_i^* = 0$, and this means that $x w_i \in \mathfrak{m}^{d+d_i+1}$. By choice of d_i, we then have $w_i \in (\mathfrak{m}^{d+d_i+1} : x) \cap \mathfrak{m}^{d_i} \subseteq \mathfrak{m}^{d_i+1}$, and hence $\operatorname{ord}(w_i) > d_i$, which is a contradiction.

Conversely, assume that x^* is a parameter of degree d, so that $x^* \notin \mathfrak{p}_i$ for all $i = 1, \ldots, s$. Here, we are using that $\dim(G) = \dim(R) = 1$, so that all primes $\mathfrak{p}_i$ are minimal in G. Since J is either $\mathcal{M}$-primary or the whole ring, we can choose $c \in \mathbb{N}$ such that $\mathfrak{m}^s/\mathfrak{m}^{s+1} \subseteq J$ for all $s \geqslant c-1$. Now, let $z \in (\mathfrak{m}^{n+d} : x) \cap \mathfrak{m}^c$, for some $n \geqslant c$ and, by way of contradiction, assume that $z \notin \mathfrak{m}^n$. Without loss of generality, we can assume that $z \in \mathfrak{m}^{n-1}$. Then, we have that $zx \in \mathfrak{m}^{n+d}$, so that $z^*x^* = 0$. Using the primary decomposition of (0) in G and the fact that $x^* \notin \mathfrak{p}_i$ for all i, this means that

$$z^* \in (0 :_G x^*) = (\mathfrak{q}_1 :_G x^*) \cap \ldots (\mathfrak{q}_s :_G x^*) \cap (J :_G x) \subseteq \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_s.$$

Finally, since $n-1 \geqslant c-1$, we have that $z^* \in \mathfrak{m}^{n-1}/\mathfrak{m}^n \subseteq J$. Thus, we obtain that $z^* \in \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_s \cap J = (0)$, and hence $z \in \mathfrak{m}^n$. This gives the desired contradiction. □

Lemma 2.1.19. *[43, Lemma 8.5.3] Let $(R, \mathfrak{m}, k)$ be a local ring, and let $x \in \mathfrak{m}$ be a non zero-divisor of order d. Then x is superficial if and only if $\mathfrak{m}^{n+d} : x = \mathfrak{m}^n$ for all $n \gg 0$.*

Proof. Assume that x is superficial. There exists $c \in \mathbb{N}$ such that $(\mathfrak{m}^{n+d} : x) \cap \mathfrak{m}^c = \mathfrak{m}^n$ for all $n \geqslant c$. Since $\operatorname{ord}(x) = d$, it is clear that $\mathfrak{m}^n \subseteq \mathfrak{m}^{n+d} : x$ for all $n \geqslant 1$. For the other containment, it is enough to show that $\mathfrak{m}^{n+d} : x \subseteq \mathfrak{m}^c$ for all $n \gg 0$. Let $y \in \mathfrak{m}^{n+d} : x$, so that $xy \in \mathfrak{m}^{n+d} \cap (x)$. By the Artin-Rees Lemma [52, Theorem 8.5] there exists $t \in \mathbb{N}$ such that $\mathfrak{m}^{n+d} \cap (x) \subseteq x\mathfrak{m}^{n+d-t}$ for all $n \geqslant t$. Choose $n \geqslant t-d+c$ so that $xy \in x\mathfrak{m}^{n+d-t} \subseteq x\mathfrak{m}^c$. Since x is a non zero-divisor for R, this implies that $y \in x\mathfrak{m}^c : x = \mathfrak{m}^c$, as required.

For the converse, assume that $\mathfrak{m}^{n+d} : x = \mathfrak{m}^n$ for all $n \geqslant c$, for some integer $c \in \mathbb{N}$. Then, for all $n \geqslant c$, we have $(\mathfrak{m}^{n+d} : x) \cap \mathfrak{m}^c \subseteq \mathfrak{m}^{n+d} : x = \mathfrak{m}^n \subseteq (\mathfrak{m}^{n+d} : x) \cap \mathfrak{m}^c$, forcing equality. Hence, x is a superficial element. □

We recall the definition of Hilbert-Samuel multiplicity of a ring. To be precise, the following is the Hilbert-Samuel multiplicity of the ring with respect to the maximal ideal, when the ring has positive dimension. The multiplicity can be also defined, more generally, for any finitely generated R-module with respect to any $\mathfrak{m}$-primary ideal, even for Artinian rings. However, we will stick to the simplified version, since we will not need this notion in its full generality.

Definition 2.1.20. *Let $(R, \mathfrak{m}, k)$ be a local ring of dimension $d > 0$. The Hilbert-Samuel multiplicity of R is*

$$e(R) := \lim_{n\to\infty} \frac{(d-1)!\ HF_R(n)}{n^{d-1}}.$$

Remark 2.1.21. From Remark 2.1.12 we deduce that, when R is one-dimensional, $HF_R(n) = \lambda(\mathfrak{m}^n/\mathfrak{m}^{n+1}) = e(R)$ for all $n \gg 0$.

Proposition 2.1.22. *Let $(R, \mathfrak{m}, k)$ be a one-dimensional Cohen-Macaulay local ring, and let $x \in \mathfrak{m}$ be a non zero-divisor. If $d = \operatorname{ord}(x)$, then*

$$\lambda(R/(x)) \geqslant d \cdot e(R).$$

Furthermore, equality holds if and only if x is a superficial element of order d.

Proof. For all $n \geqslant 1$ we have an exact sequence

$$0 \longrightarrow \frac{\mathfrak{m}^{n+d} : x}{\mathfrak{m}^n} \longrightarrow \frac{R}{\mathfrak{m}^n} \xrightarrow{\cdot x} \frac{R}{\mathfrak{m}^{n+d}} \longrightarrow \frac{R}{\mathfrak{m}^{n+d} + (x)} \longrightarrow 0.$$

By Remark 2.1.21, using that $\dim(R) = 1$, we conclude that there exists an integer N such that $\lambda\left(\mathfrak{m}^n/\mathfrak{m}^{n+1}\right) = e(R)$ and $\mathfrak{m}^{n+d} \subseteq (x)$ for all $n \geqslant N$. For $n \geqslant N$ we have

$$\begin{aligned}
\lambda(R/(x)) = & \\
& = \lambda\left(\frac{R}{\mathfrak{m}^{n+d} + (x)}\right) \\
& = \lambda\left(\frac{\mathfrak{m}^n}{\mathfrak{m}^{n+d}}\right) + \lambda\left(\frac{\mathfrak{m}^{n+d} : x}{\mathfrak{m}^n}\right) \\
& = d \cdot e(R) + \lambda\left(\frac{\mathfrak{m}^{n+d} : x}{\mathfrak{m}^n}\right) \geqslant \ d \cdot e(R).
\end{aligned}$$

Finally, equality holds if and only if $\mathfrak{m}^{n+d} : x = \mathfrak{m}^n$ for all $n \gg 0$ and, by Lemma 2.1.19, this happens if and only if x is a superficial element. □

2.1.3 Irreducibility of polynomials in two variables

We refer to [27, Section 19.5] for more details about some notions and results that we are about to use.

A simple cycle in $\mathbb{R}^2$ is a graph that has the same number of vertices and edges, and such that every vertex has degree exactly two. A polygon is the closure of the interior of a simple cycle in $\mathbb{R}^2$, with the Euclidean topology.

Definition 2.1.23. *We say that a polygon is convex if, given any two of its points, the line segment connecting them is entirely contained inside the polygon.*

By vertex or edge of a polygon we will mean a vertex or edge of the graph that forms its boundary.

Definition 2.1.24. *A polygon $N \subseteq \mathbb{R}^2$ is said to be a convex lattice polygon if there exists a finite subset A of the non-negative integer lattice $\mathbb{Z}^2_{\geqslant 0}$ such that N is the convex hull of A, that is, N is the smallest convex polygon that contains A.*

Given two polygons M and N, we define

$$M + N := \{(m_1 + n_1, m_2 + n_2) \mid (m_1, m_2) \in M, (n_1, n_2) \in N\}.$$

Note that, if M and N are two convex lattice polygons, then so is $M+N$. Futhermore, the vertices of $M + N$ come from the sum of a vertex in M with a vertex in N.

Definition 2.1.25. *We say that a convex lattice polygon is integer reducible if it can be written as the sum of two convex lattice polygons, each consisting of at least two points. Otherwise, it is said to be is integer irreducible.*

One way to obtain a convex lattice polygon is starting from a polynomial in two variables. Let k be a field. For a polynomial $f \in k[x, y]$, the Newton polygon N_f of f is the convex hull of all points $(i, j) \in \mathbb{Z}^2_{\geqslant 0}$, where ax^iy^j appears as a monomial in f with non-zero coefficient $a \in k$.

Proposition 2.1.26. *[27, Theorem 19.7] Let f be a polynomial that is not divisible by either x or y. If N_f is integer irreducible, then the polynomial f is irreducible.*

Proof. Assume that $f = gh$ for some non constant polynomials $g, h \in k[x, y]$. By assumption, neither g or h is divisible by x or y, therefore N_g and N_h consist of at least two points. Every exponent of a monomial appearing in f is of the form

$(i + a, j + b) = (i, j) + (a, b)$ for some $(i, j) \in N_g$ and $(a, b) \in N_h$. Hence, the exponents of monomials in f all lie inside $N_g + N_h$, and thus the convex hull N_f of such points is itself contained in $N_g + N_h$, since $N_g + N_h$ is convex. Conversely, every vertex of the polygon $N_g + N_h$ is the sum of a vertex in N_g and one in N_h. This means that any vertex $(i + a, j + b)$ of $N_g + N_h$ comes from a pair of exponents (i, j) of a monomial in g and a pair of exponents (a, b) of a monomial h. Thus, $(i+a, j+b)$ is a pair of exponents of a monomial in $f = gh$. Therefore $(i + a, j + b) \in N_f$ for all vertices of $N_g + N_h$, and hence $N_g + N_h \subseteq N_f$ by convexity of N_f. □

Lemma 2.1.27. *[27, Corollary 19.2] If a convex lattice polygon N has en edge with vertices $(0, m)$ and $(n, 0)$, with m and n relatively prime, and N is contained in the triangle with vertices $(0, m), (n, 0), (0, 0)$, then N is integer irreducible.*

Proof. Suppose that $N = N_1 + N_2$, for two convex lattice polygons N_1 and N_2 with at least two points. Then N_1 must contain two integer points of the form $(n_1, 0), (0, m_1)$, and N_2 must contain two integer points of the form $(n_2, 0), (0, m_2)$, and none of them is the point $(0, 0)$. In fact, since $N = N_1 + N_2$, there are two points $(i, j) \in N_1$ and $(a, b) \in N_2$ such that $(i + a, j + b) = (n, 0)$. Since all these real numbers are non-negative, we have that $i + a = n$ and $j = b = 0$. In addition, if $(i, 0) \in N_1$ and $(a, 0) \in N_2$ are such that $i + a = n$, then $0 < i, a < n$. In fact, suppose by way of contradiction that $i = n$, so that $(i, 0) = (n, 0) \in N_1$. Then, for any point $(0, 0) \neq (c, d) \in N_2$, we have $(n, 0) + (c, d) = (n + c, d) \in N$, but this is impossible since N is contained in the triangle with vertices $(0, m), (n, 0), (0, 0)$, and $(n + c, d)$ is outside of it. Furthermore, if we let n_1 and n_2 be the maximal values among the ones for which $(n_1, 0) \in N_1$ and $(n_2, 0) \in N_2$, then n_1 and n_2 are forced to be integers satisfying $0 < n_1, n_2 < n$, and such that $n_1+n_2 = n$. Similarly, one can show that $(0, m) = (0, m_1)+(0, m_2)$ for some integer points $(0, m_1) \in N_1$ and $(0, m_2) \in N_2$, both different from $(0, 0)$. Now, consider the points $(n_1, 0) + (0, m_2) \in N$ and $(n_2, 0) + (0, m_1) \in N$. Since N is contained in the triangle with vertices $(0, m), (n, 0), (0, 0)$, we must have $m_2 \leqslant m - \frac{m}{n}n_1$ and $m_1 \leqslant m - \frac{m}{n}n_2$. Note that, since $m_1, m_2 \in \mathbb{Z}$, and $\gcd(m, n) = 1$, equality cannot hold in any of the two inequalities. In fact, suppose for instance that $m_2 = m - \frac{m}{n}n_1$, then $nm_2 = nm - mn_1$. Since $m - m_2 = m_1$, we have that $nm_1 = mn_1$, and since $\gcd(m, n) = 1$, then m divides m_1. This implies that $m_1 = m$, and $m_2 = 0$, which is a contradiction. Thus, $m_2 < m - \frac{m}{n}n_1$ and $m_1 < m - \frac{m}{n}n_2$. Therefore, adding them, we obtain that

$$m = m_1 + m_2 < 2m - \frac{m}{n}(n_1 + n_2) = m,$$

which is a contradiction. Therefore N is integer irreducible. $\square$

Remark 2.1.28. We note that the methods of this subsection can be generalized to polynomials in more than two variables, modifying the definitions and the claims accordingly. We presented only the two variable version of them, since we will need these tools only in such generality.

2.2 A motivation behind Ding's conjecture

Ideals and modules of finite projective dimension often play a crucial role: they share good properties, and they usually imply nice conditions about the ring itself. For example, one form of Bass's Question states that if a ring has a module of finite length and finite projective dimension, then it is Cohen-Macaulay. Bass's Question is now a theorem, since it follows from the Intersection Theorem. The latter has been proved by Peskine-Szpiro for rings of positive characteristic, or essentially of finite type over a field [55, 56], extended by Hochster to the general equicharacteristic case [38], and finally proven by Paul Roberts in the mixed characteristic case [57]. Ideals generated by a system of parameters in a Cohen-Macaulay local ring are resolved by the Koszul complex, and, in a sense, they are special ideals of finite projective dimension. The following fits into a more general frame of questions about minimality for ideals of finite projective dimension.

Question 2.2.1. Let $(R, \mathfrak{m}, k)$ be a Cohen-Macaulay local ring. Does there exist an ideal I generated by a full system of parameters for R, such that $\ell\ell(R/I) \leqslant \ell\ell(R/J)$ for all $\mathfrak{m}$-primary ideals J of finite projective dimension?

If, for a Gorenstein ring R, Ding's conjecture holds true, then Question 2.2.1 has positive answer for such an R. In fact, if J is any ideal of finite projective dimension, then

$$0 \longrightarrow J \longrightarrow R \longrightarrow R/J \longrightarrow 0,$$

is a minimal MCM approximation of R/J. The only thing to show that is not entirely clear is that J has finite injective dimension. This is because we have a finite resolution $0 \to F_d \to F_{d-1} \to \ldots \to F_1 \to R \to R/J \to 0$, where the R-modules F_j are free for all j. From the exact sequence $0 \to F_d \to F_{d-1} \to \Omega_{d-1} \to 0$ we deduce that $\mathrm{id}_R(\Omega_{d-1}) < \infty$, since both F_d and F_{d-1} have finite injective dimension, being free. Repeating the argument with the short exact sequences $0 \to \Omega_{j+1} \to F_j \to \Omega_j \to 0$, we get down to the sequence $0 \to \Omega_2 \to F_1 \to J \to 0$, and we conclude that $\mathrm{id}_R(J) < \infty$, as claimed.

By Proposition 2.1.5, we then conclude that $\delta(R/J) = 1$. Now, let $n = \ell\ell(R/J)$, so that we have an inclusion $\mathfrak{m}^n \subseteq J$. This gives a surjection $R/\mathfrak{m}^n \to R/J \to 0$ and, by Remark 2.1.2, it follows that $\delta(R/\mathfrak{m}^n) = 1$. As a consequence, $\mathrm{index}(R) \leqslant n$. Finally, we are assuming that Ding's conjecture holds true for R, therefore $\mathrm{g}\ell\ell(R) = \mathrm{index}(R) \leqslant n$. This precisely means that there exists an ideal I, generated by a full system of parameters, such that $\ell\ell(R/I) \leqslant n = \ell\ell(R/J)$.

Remark 2.2.2. We note that, even if Ding's conjecture fails for a ring R, Question 2.2.1 may still have positive answer. For instance, if $\dim(R) = 1$, every non-zero ideal I of finite projective dimension must be generated by a parameter. In fact, given the short exact sequence $0 \to I \to R \to R/I \to 0$, it follows that I has to be free of rank one. Therefore $I = (x)$ for some non zero-divisor $x \in R$.

2.3 One-dimensional rings

We now turn back to the index and Ding's conjecture. In particular, we focus our attention on one-dimensional rings, for which we will present a result that allows us to get a manageable description of the index. We start with an explicit description for the dual of an ideal. See [43, Section 2.4] for more details.

Lemma 2.3.1. *Let* $(R, \mathfrak{m}, k)$ *be a local ring, and let* $I \subseteq R$ *be an ideal containing a non zero-divisor* x*. Then* $\operatorname{Hom}_R(I, R) \cong (x) : I$*.*

Proof. Consider the map $\Phi : \operatorname{Hom}_R(I, R) \to (x) : I$ that sends $\varphi \in \operatorname{Hom}_R(I, R)$ to $\Phi(\varphi) = \varphi(x)$. Fist of all, we want to show that it is well defined. Let $y \in I$, then $\varphi(x)y = \varphi(xy) = x\varphi(y) \in (x)$, therefore $y \in (x) : I$. It is easy to see that it is a R-module homomorphism. To show injectivity, assume that $\varphi(x) = \psi(x)$ for two homomorphisms $\varphi, \psi \in \operatorname{Hom}_R(I, R)$. Then, for all $y \in I$, we obtain

$$x\varphi(y) = \varphi(xy) = y\varphi(x) = y\psi(x) = \psi(xy) = x\psi(y).$$

Since x is a non zero-divisor, we conclude that $\varphi(y) = \psi(y)$ for all $y \in I$ and, thus $\varphi = \psi$. Hence, Φ is injective. To show surjectivity, let $z \in (x) : I$, and set $\varphi(y) = \frac{yz}{x}$ for all $y \in I$. We claim that $\varphi \in \operatorname{Hom}_R(I, R)$. It is clearly R-linear, so it is enough to show that $\varphi(I) \subseteq R$. Let $y \in I$, then $yz \in (x)$, say $yz = xw$, for some $w \in R$. Therefore $\varphi(y) = \frac{xw}{x} = w \in R$, as claimed. This shows that Φ is surjective and, hence, an isomorphism. $\square$

Proposition 2.3.2. *Let $(R, \mathfrak{m}, k)$ be a one-dimensional Gorenstein local ring and let $x \in \mathfrak{m}$ be a non zero-divisor. For $n \geqslant 1$, the following facts are equivalent*

(i) $\delta(R/\mathfrak{m}^n) = 1$.

(ii) $x^n \in \mathfrak{m}((x^n) : \mathfrak{m}^n)$.

(iii) $x\mathfrak{m}^n : \mathfrak{m} \subseteq (x)$.

(iv) $\Delta \notin x\mathfrak{m}^n : \mathfrak{m}$ *for any* $\Delta \in (x) : \mathfrak{m}$, $\Delta \notin (x)$.

Proof. It is shown in [20, Proposition 1.2] that under our assumptions

$$\delta(R/\mathfrak{m}^n) = 1 + \mu(\operatorname{Ext}^1_R(R/\mathfrak{m}^n, R)) - \mu(\operatorname{Hom}_R(\mathfrak{m}^n, R)).$$

Therefore $\delta(R/\mathfrak{m}^n) = 1$ if and only if $\mu(\operatorname{Ext}^1_R(R/\mathfrak{m}^n, R)) = \mu(\operatorname{Hom}_R(\mathfrak{m}^n, R))$. Notice that, by dimension shifting, we have

$$\operatorname{Ext}^1_R(R/\mathfrak{m}^n, R) \cong \operatorname{Hom}_R(R/\mathfrak{m}^n, R/(x^n)) \cong \frac{(x^n) : \mathfrak{m}^n}{(x^n)}.$$

On the other hand, by Lemma 2.3.1, we have that $\operatorname{Hom}_R(\mathfrak{m}^n, R) \cong (x^n) : \mathfrak{m}^n$, and putting these facts together we obtain that $\delta(R/\mathfrak{m}^n) = 1$ if and only if $\mu\left(\frac{(x^n):\mathfrak{m}^n}{(x^n)}\right) = \mu((x^n) : \mathfrak{m}^n)$, if and only if $x^n \in \mathfrak{m}((x^n) : \mathfrak{m}^n)$. This shows that *(i)* and *(ii)* are equivalent. Using that R is Gorenstein, by duality *(ii)* holds if and only if

$$(x) = (x^{n+1}) : (x^n) \supseteq (x^{n+1}) : (\mathfrak{m}((x^n) : \mathfrak{m}^n)) = \big((x^{n+1}) : ((x^n) : \mathfrak{m}^n)\big) : \mathfrak{m}.$$

Since x is a non zero-divisor, we have that $((x^n) : \mathfrak{m}^n) = ((x^{n+1}) : x\mathfrak{m}^n)$, and by duality we obtain that $(x^{n+1}) : ((x^{n+1}) : x\mathfrak{m}^n) = x\mathfrak{m}^n$. Therefore *(ii)* is equivalent to *(iii)*. Finally, *(iii)* clearly implies *(iv)*. For the converse, assume that *(iii)* does not hold, that is $x\mathfrak{m}^n : \mathfrak{m} \not\subseteq (x)$. Consider the following short exact sequence

$$0 \longrightarrow \frac{(x)}{x\mathfrak{m}^n} \longrightarrow \frac{R}{x\mathfrak{m}^n} \longrightarrow \frac{R}{(x)} \longrightarrow 0,$$

which, applying $\operatorname{Hom}_R(k,-)$, induces an exact sequence on socles

$$0 \longrightarrow \frac{(x)\cap(x\mathfrak{m}^n : \mathfrak{m})}{x\mathfrak{m}^n} \xrightarrow{\ \psi\ } \frac{x\mathfrak{m}^n : \mathfrak{m}}{x\mathfrak{m}^n} \xrightarrow{\ \varphi\ } \frac{(x) : \mathfrak{m}}{(x)}.$$

Note that the module $\operatorname{soc}(R/(x))$ is isomorphic to k, therefore φ is either surjective or it is zero. By assumption, we have that $x\mathfrak{m}^n : \mathfrak{m} \not\subseteq (x)$, therefore we conclude that $(x) \cap (x\mathfrak{m}^n : \mathfrak{m}) \subsetneq x\mathfrak{m}^n : \mathfrak{m}$. As a consequence, ψ is not an isomorphism, and thus φ is surjective. This means that there exists a choice of $\Delta \in (x) : \mathfrak{m}$, $\Delta \notin (x)$, such that $\Delta \in x\mathfrak{m}^n : \mathfrak{m}$, proving that *(iv)* does not hold. □

As a corollary, we easily recover Ding's conjecture for one-dimensional Gorenstein rings with infinite residue field, and whose associated graded ring is Cohen-Macaulay [19, Theorem 2.1].

Corollary 2.3.3. *Let $(R,\mathfrak{m},k)$ be a one-dimensional Gorenstein local ring, and assume that the associated graded ring $\operatorname{gr}_{\mathfrak{m}}(R)$ has a homogeneous non zero-divisor of degree one. This condition is satisfied, for example, if $\operatorname{gr}_{\mathfrak{m}}(R)$ is Cohen-Macaulay and k is infinite. Then $\operatorname{index}(R) = \operatorname{g}\ell\ell(R)$.*

Proof. We only have to show that $\operatorname{index}(R) \geqslant \operatorname{g}\ell\ell(R)$ as the other inequality always holds. Let $n = \operatorname{index}(R)$ and let $x \in \mathfrak{m}$ be a non zero-divisor such that x^* is a non zero-divisor in $\mathrm{gr}_{\mathfrak{m}}(R)$ of degree one. In particular, $\mathrm{ord}(x) = 1$. By Proposition 2.3.2 *(iii)* we have that $x\mathfrak{m}^n : \mathfrak{m} \subseteq (x)$. By way of contradiction suppose that $\mathfrak{m}^n \nsubseteq (x)$, so that we can choose $\Delta \in \mathfrak{m}^n$ which represents a non-zero socle element in $R/(x)$. Then $\Delta\mathfrak{m} \subseteq \mathfrak{m}^{n+1} \cap (x) = x\mathfrak{m}^n$ by Valabrega-Valla's Theorem [67, Theorem 2.3], because x^* is a non zero-divisor in $\mathrm{gr}_{\mathfrak{m}}(R)$ of degree one. Hence $\Delta \in x\mathfrak{m}^n : \mathfrak{m} \subseteq (x)$, and this contradicts the fact that Δ is chosen to be non-zero in $R/(x)$. Thus $\mathfrak{m}^n \subseteq (x)$ and $\mathrm{g}\ell\ell(R) \leqslant n = \mathrm{index}(R)$. □

Example 2.3.4. [29, Example 3.2] Consider the one-dimensional hypersurface $R = \mathbb{F}_2[\![x, y]\!]/(xy(x + y))$. In this case, $\mathrm{index}(R) = 3$, and $\mathrm{g}\ell\ell(R) = 4$, therefore Ding's conjecture fails. Note that for this ring there does not exist a homogeneous non zero-divisor of degree one in $\mathrm{gr}_{(x,y)R}(R)$, and thus Corollary 2.3.3 (or [19, Theorem 2.1]) cannot be applied.

2.4 The counterexample

If $(R, \mathfrak{m}, k)$ is a one-dimensional Gorenstein local ring, the index of R can be checked on any non zero-divisor $x \in R$, just by testing what is the minimal $n \in \mathbb{N}$ for which any of the equivalent conditions in Proposition 2.3.2 are satisfied. On the other hand, a generic choice of a non zero-divisor $x \in R$ will give a maximal value of $\ell\ell(R/(x))$, whereas $\mathrm{g}\ell\ell(R)$ is defined as the minimum of such Löewy lengths. Therefore, to verify that a potential candidate is a counterexample, one should take in account the

Löewy length of $R/(x)$ for every non zero-divisor $x \in R$. We now present our first counterexample to Ding's conjecture.

Let k be a field, and let $S = k[x, y, z]_{(x,y,z)}$, with maximal ideal $\mathfrak{n}$. Consider

$$I = (x^2 - y^5, xy^2 + yz^3 - z^5)S \subseteq S$$

and let $R := S/I$. We now use the methods developed in Subsection 2.1.3 to show that R is a domain.

Lemma 2.4.1. *For any field k, the ideal $J := (x^2-y^5, xy^2+yz^3-z^5) \subseteq k[x, y, z] =: T$ is prime.*

Proof. Since $y \in T$ is a non zero-divisor modulo J, it suffices to show that $JT[y^{-1}]$ is a prime in the localization $T[y^{-1}]$ of T at the element y. After some algebraic manipulations, we obtain

$$\begin{aligned} JT[y^{-1}] =& (x^2 - y^5, x + y^{-1}z^3 - y^{-2}z^5)T[y^{-1}] \\ &= ((-y^{-1}z^3 + y^{-2}z^5)^2 - y^5, x + y^{-1}z^3 - y^{-2}z^5)T[y^{-1}]. \end{aligned}$$

Set $x' := x + y^{-1}z^3 - y^{-2}z^5$, then we have

$$JT[y^{-1}] = (y^{-2}z^6 - 2y^{-3}z^8 + y^{-4}z^{10} - y^5, x')T[y^{-1}] = (z^{10} - 2z^8y + z^6y^2 - y^9, x')T[y^{-1}].$$

Note that $T[y^{-1}] = k[x, y, z, y^{-1}] = k[x', y, z, y^{-1}]$. Since $x' \in JT[y^{-1}]$, we have that $JT[y^{-1}]$ is prime if and only if $(z^{10} - 2z^8y + z^6y^2 - y^9)Q[y^{-1}]$ is prime, where $Q = k[y, z]$. Because $y \in Q$ is a non zero-divisor modulo $(z^{10} - 2z^8y + z^6y^2 - y^9)Q$, we finally reduce the claim to showing that the polynomial $f = z^{10} - 2z^8y + z^6y^2 - y^9$ is irreducible in Q. The exponents of the pure powers of z and y are 10 and 9, so they are relatively prime. Furthermore, the pairs of exponents $(8,1)$ and $(6,2)$ of the other monomials $-2z^8y$ and z^6y^2 appearing in f are contained in the triangle with vertices $(0,0)$, $(0,9)$, $(10,0)$. It follows from Proposition 2.1.26 and Lemma 2.1.27 that f is irreducible. $\square$

Lemma 2.4.1 shows that, for any field k, the ring $R = S/I$ defined above is a domain. Denote by $\mathfrak{m}$ the maximal ideal $\mathfrak{n}/I$ of R. The ring R is a one-dimensional complete intersection. We used the computer algebra programs CoCoA [1] and Macaulay2 [26] to obtain the following information: the associated graded ring of R with respect to $\mathfrak{m}$ is $G := \operatorname{gr}_{\mathfrak{m}}(R) \cong P/I^*$, where $P = k[X,Y,Z]$ and

$$I^* = (X^2, XY^2, XYZ^3, YZ^6) = (X^2, Y) \cap (X, Z^6) \cap (X^2, Y^2, Z^3) \subseteq P.$$

From now on, we will identify G with P/I^*. The Hilbert function of R is

n	0	1	2	3	4	5	6	7	8	...
$HF_R(n)$	1	3	5	6	7	7	8	8	8	...

with $HF_R(n) = 8 = e(R)$ for all $n \geqslant 6$. See the Ending Remarks on page 44 for comments about the use of computer algebra programs for this example.

Theorem 2.4.2. *Let $R = S/I$ be as above. Then*

$$\operatorname{index}(R) = 5 < 6 = \operatorname{g}\ell\ell(R).$$

Proof. Using CoCoA [1] and Macaulay2 [26] we obtain that

$$y\mathfrak{m}^5 : \mathfrak{m} = (y^5, xy^3, yz^4, xyz^3, y^3z^2, xy^2z^2, y^4z)R \subseteq yR$$

and

$$y\mathfrak{m}^4 : \mathfrak{m} = (y^4, yz^3, y^2z^2, xyz^2, y^3z, xy^2z, xz^4)R \not\subseteq yR.$$

Therefore $\operatorname{index}(R) = 5$ by Proposition 2.3.2. On the other hand, a direct computation shows that $\mathfrak{m}^6 \subseteq yR$, therefore $\mathrm{g}\ell\ell(R) \leqslant 6$, and we want to show that equality holds. To do this, we need to prove that for any $f \in \mathfrak{m} \setminus \{0\}$ we have $\ell\ell(R/fR) \geqslant 6$. Assume the contrary, i.e., that there exists $f \in \mathfrak{m}$ such that $\mathfrak{m}^5 \subseteq fR$. Lifting to S we get that $\mathfrak{n}^5 \subseteq I + (f) =: J$, that is $J = J + \mathfrak{n}^5 = (x^2, xy^2 + yz^3, f) + \mathfrak{n}^5$.

First, assume that $\operatorname{ord}(f) = 1$, and let f^* be the initial form of f. Since $\operatorname{ord}(f) = 1$, f can be made a part of a minimal set of generators of $\mathfrak{n}$: say that $f, a, b \in \mathfrak{n}$ are linearly independent modulo $\mathfrak{n}^2$. Furthermore, we have the equality $J \cap \mathfrak{n}^2 + \mathfrak{n}^3 = (x^2, f^2, af, bf) + \mathfrak{n}^3$, so that

$$3 \leqslant \lambda((J^*)_2) = \lambda\left(\frac{J \cap \mathfrak{n}^2 + \mathfrak{n}^3}{\mathfrak{n}^3}\right) \leqslant 4,$$

and such length is equal to three if and only if $x^2 \in (f^2, af, bf) + \mathfrak{n}^3$. But, by choice of f, a, b, this happens if and only if $f = ux + g$, for some unit $u \in S$ and some some $g \in S$ of order at least two. Then, for $h = u^{-1}g$, we have

$$\mathfrak{n}^5 + (f) \subseteq J = (x^2, xy^2 + yz^3, ux + g) + \mathfrak{n}^5 = (h^2, -hy^2 + yz^3, ux + g) + \mathfrak{n}^5 \subseteq \mathfrak{n}^4 + (f).$$

Notice that $\lambda\left(\dfrac{\mathfrak{n}^4+(f)}{\mathfrak{n}^5+(f)}\right)=5$, and

$$\lambda\left(\frac{J}{\mathfrak{n}^5+(f)}\right)=\lambda\left(\frac{(h^2,-hy^2+yz^3)+(\mathfrak{n}^5+(f))}{\mathfrak{n}^5+(f)}\right)\leqslant 2,$$

therefore $\lambda\left(\dfrac{\mathfrak{n}^4+(f)}{J}\right)\geqslant 3$. But this is a contradiction, because

$$\frac{\mathfrak{n}^4+(f)}{J}\subseteq \operatorname{soc}(S/J)=\operatorname{soc}(R/fR),$$

which is simple because R is Gorenstein. We ruled out the case $\lambda((J^*)_2)=3$, so we are left with the case $\lambda((J^*)_2)=4$. Under such condition, the Hilbert function of R/fR, which is the Hilbert function of P/J^*, is

n	0	1	2	3	4	5	6	7	...
$HF_{R/fR}(n)$	1	2	2	h	k	0	0	0	...

where $k\leqslant 1$ because R is Gorenstein and because $\mathfrak{m}^5\subseteq fR$ by assumption, and $h\leqslant 2$ by Macaulay's Theorem 2.1.14. On the other hand, $\lambda(R/fR)\geqslant e(R)=8$ by Proposition 2.1.19, therefore we necessarily have $k=1$ and $h=HF_{R/fR}(3)=HF_{P/J^*}(3)=2$. In addition, again by Proposition 2.1.19, f must be a superficial element. Since

$$I^*=(X^2,Y)\cap(X,Z^6)\cap(X^2,Y^2,Z^3)$$

and f is superficial if and only if $f^*\notin\bigcup_{\mathfrak{p}\in\min(G)}\mathfrak{p}=(X,Y)G\cup(X,Z)G$ by Proposition 2.1.18, we conclude that $f^*\notin(X,Y)$. Let $\mathcal{M}=(X,Y,Z)$ be the irrelevant maximal ideal of P. Then, since $f^*\in\mathcal{M}\smallsetminus(\mathcal{M}^2\cup(X,Y))$, we have that $(f^*,X,Y)=\mathcal{M}$. Let $K:=I^*+(f^*)\subseteq P$, then we have

$$\lambda(K_3)\geqslant\lambda\left(\frac{(X^3,X^2Y,XY^2)+f^*\mathcal{M}^2}{\mathcal{M}^4}\right)=9.$$

But $HF_P(3) = 10$, therefore $HF_{P/K}(3) \leqslant 1$. In particular, we see that $HF_{P/K}(3) < HF_{P/J^*}(3)$. On the other hand, by Proposition 2.1.15, there is always a surjective homomorphism of graded rings

$$P/K \to P/J^* \to 0, \tag{2.4.1}$$

which is homogeneous of degree zero. This gives the desired contradiction. We analyzed all possible cases when $\operatorname{ord}(f) = 1$, so let us assume now that $\operatorname{ord}(f) = 2$. Again, let $J := I + (f)$ and $K := I^* + (f^*)$. In this case $J \cap \mathfrak{n}^2 + \mathfrak{n}^3 = (x^2, f) + \mathfrak{n}^3$, so that

$$1 \leqslant \lambda((J^*)_2) = \lambda\left(\frac{(x^2, f) + \mathfrak{n}^3}{\mathfrak{n}^3}\right) \leqslant 2,$$

and such length is equal to one if and only if $f = ux^2 + g$ for some unit $u \in S$ and g of order at least three. Assume that we are in the latter case, then the Hilbert function of R/fR, which is the Hilbert function of P/J^*, is

n	0	1	2	3	4	5	6	7	...
$HF_{R/fR}(n)$	1	3	5	h	k	0	0	0	...

with $k = HF_{R/fR}(4) \leqslant 1$, because R/fR is a zero-dimensional Gorenstein local ring and $\mathfrak{m}^5 \subseteq fR$. By Macaulay's Theorem 2.1.14 we must have $h \leqslant 7$. Also $\lambda(R/fR) > 16$ by Proposition 2.1.19, because $(f^*) = (X^2)$ is contained in the minimal prime (X, Y) of G, so that f cannot be superficial. Therefore $h = HF_{R/fR}(3) = HF_{P/J^*}(3) = 7$ and $k = 1$ are forced. On the other hand, $(f^*) = (X^2) \subseteq I^*$, therefore $K = I^*$ and $HF_{P/K}(3) = HF_G(3) = HF_R(3) = 6$. This is again a contradiction because of the surjection (2.4.1). Thus we can assume that $\lambda((J^*)_2) = 2$. In this case the Hilbert function of R/fR is

n	0	1	2	3	4	5	6	7	...
$HF_{R/fR}(n)$	1	3	4	h	k	0	0	0	...

with $k \leqslant 1$. In addition, Macaulay's Theorem 2.1.14 implies that $h \leqslant 5$. Thus $\lambda(R/fR) \leqslant 14 < 16 = 2e(R)$, contradicting Proposition 2.1.19. Finally, let us assume that $\text{ord}(f) \geqslant 3$, so that $\lambda(R/fR) \geqslant 24$ by Proposition 2.1.19. Since $HF_{R/fR}(4) \leqslant 1$ because R is Gorenstein and we are assuming that $\mathfrak{m}^5 \subseteq fR$, the maximal possible Hilbert function for R/fR is

n	0	1	2	3	4	5	6	7	...
$HF_{max}(n)$	1	3	6	10	1	0	0	0	...

But then $\lambda(R/fR) \leqslant 21 < 24$, a contradiction. This shows that for any $f \in \mathfrak{m} \smallsetminus \{0\}$ we have $\mathfrak{m}^5 \not\subseteq fR$, and completes the proof that $\text{g}\ell\ell(R) = 6 > 5 = \text{index}(R)$. □

2.5 Further examples and remarks

Given that the conjecture fails in general, even for complete intersection domains of dimension one, one may wonder if the conjecture is true for some smaller classes of rings. A ring R is called quasi-homogeneous if it isomorphic to the completion of a positively graded k-algebra at the irrelevant maximal ideal. In the counterexample of Section 3, it is not possible to give weights to the variables x, y and z that make the completion $\widehat{R}$ of R quasi-homogeneous. This does not say that R is not quasi-homogeneous, since such weights could exist for a different choice of minimal generators of the maximal ideal $\mathfrak{m}$, but we were not able to find it. On the other hand, it is easy to find a quasi-homogeneous counterexample if one is not looking for a domain:

Example 2.5.1. Let $S = k[\![x, y, z]\!]$, where k is any field, and let $\mathfrak{n} = (x, y, z)$ be the maximal ideal of S. Consider the one-dimensional complete intersection $R := S/I$, where

$$I = (x^2 - y^5, xy^2 + yz^3)S.$$

The ring R is quasi-homogeneous, since it is the completion of the positively graded ring $k[x, y, z]/(x^2 - y^5, xy^2 + yz^3)$ with weights $w(x) = 15$, $w(y) = 6$ and $w(z) = 7$ at the maximal ideal (x, y, z). Using CoCoA [1] and Macaulay2 [26], we checked that

$$z\mathfrak{m}^5 : \mathfrak{m} = (xy^2z, z^5, xz^4, y^3z^2, y^4z, y^2z^3, xyz^3)R \subseteq zR,$$

and

$$z\mathfrak{m}^4 : \mathfrak{m} = (xy^2, x^2y, z^4, xz^3, y^2z^2, xyz^2, y^3z)R \nsubseteq zR.$$

Therefore $\operatorname{index}(R) = 5$, and because $\mathfrak{m}^6 \subseteq (y - z)$ we also have that $\mathrm{g}\ell\ell(R) \leqslant 6$. On the other hand, note that $I + \mathfrak{n}^5 = (x^2, xy^2 + yz^3) + \mathfrak{n}^5$, $I^* = (X^2, XY^2, XYZ^3, YZ^6) \subseteq k[X, Y, Z]$ and $e(R) = 8$ are the same as in the counterexample of Theorem 2.4.2. Therefore, using the same proof, we get that $\mathrm{g}\ell\ell(R) = 6$.

Remark 2.5.2. In [20, Corollary 3.3], Ding claims that the conjecture is true for what he calls gradable rings such that $\operatorname{depth}(\operatorname{gr}_{\mathfrak{m}}(R)) \geqslant \dim(R) - 1$. In our notation, gradable rings correspond to quasi-homogeneous rings. His argument is not correct, since he uses results that need a standard grading, i.e. the weights of all the minimal generators of $\mathfrak{m}$ must be one. Example 2.5.1 gives a counterexample to his statement.

Another direction of investigation is to consider Gorenstein analytically irreducible rings, i.e. local rings such that the completion at the maximal ideal is a domain.

Remark 2.5.3. Note that the example in Theorem 2.4.2 is not analytically irreducible. We thank William Heinzer for suggesting the following argument:

$$\widehat{R} \cong \frac{k[\![x,y,z]\!]}{(x^2-y^5, xy^2+yz^3-z^5)} \cong \frac{k[\![t^2,t^5,z]\!]}{(z^5-t^2z^3-t^9)} \subseteq \frac{k[\![t,z]\!]}{(z^5-t^2z^3-t^9)} := T,$$

is an integral extension. The inclusion follows from the fact that t^4 is in the conductor $\widehat{R} :_{\widehat{R}} T$, and it is a non zero-divisor in $\widehat{R}$. The initial form $z^5 - t^2z^3$ has two relatively prime non-constant factors z^3 and $z^2 - t^2$ in $\mathbb{Q}[\![t,z]\!]$, therefore T is not a domain [47, Theorem 16.6]. Since $\widehat{R}$ and T have the same total ring of fractions, $\widehat{R}$ is also not a domain.

However, there is an analytically irreducible counterexample:

Example 2.5.4. Let $S = \mathbb{Q}[x,y,z]_{(x,y,z)}$ and let $\mathfrak{n} = (x,y,z)S$ be the maximal ideal of S. Consider the one-dimensional domain

$$R := \mathbb{Q}[t^8+t^{10}, t^9, t^{20}+t^{36}]_{(t^8+t^{10},t^9,t^{20}+t^{36})}$$

and let $\mathfrak{m} = (t^8 + t^{10}, t^9, t^{20} + t^{36})R$ be its maximal ideal. Using CoCoA or Macaulay2 one sees that $R \cong S/I$, where

$$I = (z^2 + f_1, y^4 - x^2z + 2y^2z + z^2 + f_2) \subseteq S$$

for some $f_1, f_2 \in \mathfrak{n}^5$. In particular, R is a complete intersection. We checked with Macaulay2 and CoCoA that $\mathfrak{m}^6 \subseteq xR$, $x\mathfrak{m}^5 : \mathfrak{m} \subseteq xR$ and $x\mathfrak{m}^4 : \mathfrak{m} \not\subseteq xR$, hence $\operatorname{index}(R) = 5$ and $\mathrm{g}\ell\ell(R) \leqslant 6$. On the other hand, there does not exist $f \in S$ such that $\mathfrak{n}^5 \subseteq I + (f)$, otherwise $I + (f) = (z^2, y^4 - x^2z + 2y^2z, f) + \mathfrak{n}^5$ and since $e(R) = 8$ one can use arguments which are analogous to the ones used in the proof of Theorem 2.4.2 to show that this cannot happen. Therefore $\mathrm{g}\ell\ell(R) = 6$. Finally, to show that R is analytically irreducible, notice that $R \subseteq \mathbb{Q}[t]_{\mathfrak{m}} := V$ is an integral birational extension, and since V is normal we have in fact that V is the integral closure of R in its field of fractions. The ring V is semi-local, with maximal ideals $N_1, \ldots, N_s$. Since $N_i \cap R = \mathfrak{m}$, each N_i contains t^9, and hence it must contain t. So V has only one maximal ideal, namely $(t)V$, and thus it is local. There is a well-known one to one correspondence between maximal ideals in the integral closure of R and minimal primes in $\widehat{R}$ [54, Exercise 1 p. 122], therefore $\widehat{R}$ is a domain.

Remark 2.5.5. If R is an equicharacteristic one-dimensional complete local domain, the integral closure $\overline{R}$ of R in its quotient field is isomorphic to a power series ring $k[\![t]\!]$. Thus, R is of the form $k[\![f_1, \ldots, f_n]\!]$ for some $f_1, \ldots, f_n \in k[\![t]\!]$. Ding proved that the conjecture is true when the f_i's are monomials in t [18, Proposition 2.6].

This is no longer true if the f_i's are not monomials, as Example 2.5.4 shows. In fact, with the notation introduced above, we have that $\widehat{R}$ is a one-dimensional complete equicharacteristic local domain, and by [29, Lemma 3.3 and Corollary 5.2] we have that

$$\operatorname{index}(\widehat{R}) = \operatorname{index}(R) = 5 < 6 = \mathrm{g}\ell\ell(R) = \mathrm{g}\ell\ell(\widehat{R}).$$

Ending Remarks. We conclude by making some comments about the role of computer algebra programs in the proofs contained in this chapter. In Theorem 2.4.2, we justify certain arguments, such as the inclusion $y\mathfrak{m}^5 : \mathfrak{m} \subseteq yR$, by saying that we checked them with CoCoA and Macaulay2. We also verified the validity of these statements by hand. These are just tedious computations, adding no real content to the argument, and we decided not include them in this dissertation. Analogous considerations apply to Example 2.5.1. On the other hand, in Example 2.5.4, both the equations in S defining R and the inclusion $x\mathfrak{m}^5 : \mathfrak{m} \subseteq xR$ are rather hard computationally, hence the correctness of Example 2.5.4 relies heavily on a careful analysis of the answers given by CoCoA and Macaulay2.

Chapter 3

Graded rings in positive characteristic

In this chapter, we introduce and develop some of the concepts that we will need in Chapter 4. The results in this chapter are essentially well-known to experts. However, in some cases they are hard to pinpoint in the literature. In fact, they are often stated only for local rings, while we need their analogous restatement in the standard graded setting. We will only treat the notions that will be relevant for us, trying to keep the exposition as self-contained as possible.

3.1 Local cohomology and grading

We start by defining local cohomology modules. Let R be a commutative Noetherian ring with 1. Let $I \subseteq R$ be an ideal, and let $f_1, \ldots, f_t$ be elements in R that generate I. Consider the Čech complex, $\check{\mathrm{C}}^\bullet(\underline{f}; R)$:

$$0 \longrightarrow R \longrightarrow \bigoplus_{i=1}^{t} R_{f_i} \longrightarrow \bigoplus_{i,j=1}^{t} R_{f_i f_j} \longrightarrow \ldots\ldots \longrightarrow R_{f_1 \cdots f_t} \longrightarrow 0,$$

where every homomorphism is a localization map with an appropriate sign. More specifically, for $(i_1, \ldots, i_s) \in \mathbb{N}^s$ and $(j_1, \ldots, j_{s+1}) \in \mathbb{N}^{s+1}$ we consider the sets $A := \{i_1, \ldots, i_s\}$ and $B := \{j_1, \ldots, j_{s+1}\}$. We want to define a map $\delta_{A,B} : R_{f_{i_1} \cdots f_{i_s}} \to R_{f_{j_1} \cdots f_{j_{s+1}}}$. If $A \not\subseteq B$, then set $\delta_{A,B} = 0$. Otherwise, we have that $B = A \cup \{j_m\}$, for some $1 \leqslant m \leqslant s+1$. We then define $\delta_{A,B} : R_{f_{i_1} \cdots f_{i_s}} \to R_{f_{i_1} \cdots f_{i_s} f_{j_m}}$ to be the natural localization map, multiplied by $(-1)^{m-1}$. Since every module in the Čech complex is a direct sum of modules of this form, this defines the differential on $\check{C}^\bullet(\underline{f}; R)$. It is a routine computation to check that this yields a complex of R-modules.

Definition 3.1.1. *Let $I = (f_1, \ldots, f_t)$ be an ideal in R, and let M be an R-module. We define the i-th local cohomology of M with support in I to be*

$$H^i_I(M) := H^i(\check{\mathrm{C}}^\bullet(\underline{f}; R) \otimes_R M).$$

We record some well-known facts about local cohomology. A standard, but exhaustive reference for them is [11].

Properties. (i) We have that $H^i_I(-)$ is a functor from the category of R-modules to itself. These functors are, in general, not exact. However, for any $I \subseteq R$, $H^0_I(-)$ is left-exact. Moreover, if $(R, \mathfrak{m}, k)$ is either local or standard graded, and if $d = \dim(R)$, the functor $H^d_{\mathfrak{m}}(-)$ is right exact.

(ii) The local cohomology modules $H^i_I(M)$ do not depend on the choice of generators, $f_1, \ldots, f_t$, of I. In fact, they only depend on the radical of I.

(iii) The smallest $i \in \mathbb{Z}$ such that $H^i_I(M) \neq 0$ is equal to $\operatorname{grade}(I, M)$, that is, it equals the maximal length of a regular sequence on M contained in I.

(iv) As a consequence of Property (iii), if $(R, \mathfrak{m}, k)$ is either local or standard graded, the smallest $i \in \mathbb{Z}$ such that $H^i_{\mathfrak{m}}(R) \neq 0$ is the maximal length of a regular sequence (homogeneous in the second case) inside $\mathfrak{m}$, i.e., the depth of the ring.

(v) If $(R, \mathfrak{m}, k)$ is either local or graded, and M is a finitely generated (graded in the second case) R-module of Krull dimension c, then $H^c_{\mathfrak{m}}(M) \neq 0$.

(vi) If $(R, \mathfrak{m}, k)$ is either local or graded, and M is a finitely generated (graded) R-module, then the modules $H^i_{\mathfrak{m}}(M)$ are Artinian, for all $i \in \mathbb{Z}$.

We now review some results about grading and graded modules, and relate them with local cohomology. Assume that $(R, \mathfrak{m}, k)$ is standard graded, and that $I \subseteq R$ is a homogeneous ideal, so that we can choose homogeneous generators $f_1, \ldots, f_t$ for I. Note that, since R is $\mathbb{N}$-graded, and the modules in the Čech complex $\check{C}^\bullet(\underline{f}; R)$ are localization at the homogeneous elements $f_1, \ldots, f_t$, the modules in the Čech complex are $\mathbb{Z}$-graded. The maps are graded as well and, as a consequence, the modules $H^i_I(R)$ are $\mathbb{Z}$-graded for all $i \in \mathbb{Z}$.

The a-invariant of a graded ring was introduced by Goto and Watanabe in [25]. It is defined in terms of degrees of the top local cohomology module $H^d_{\mathfrak{m}}(R)$. More generally, we consider a-invariants that can be defined starting from all the modules $H^i_{\mathfrak{m}}(R)$, for $i \in \mathbb{Z}$.

Definition 3.1.2. *For all $i \in \mathbb{Z}$ such that $H^i_{\mathfrak{m}}(R) \neq 0$, we define the i-th a-invariant of R to be $a_i(R) := \max\left\{t \in \mathbb{Z} \mid H^i_{\mathfrak{m}}(R)_t \neq 0\right\}$. If $H^i_{\mathfrak{m}}(R) = 0$, we set $a_i(R) = -\infty$.*

The definition makes sense since, for all $i \in \mathbb{Z}$, we have that $[H^i_{\mathfrak{m}}(R)]_t = 0$ for all $t \gg 0$, because such a module is Artinian. These numbers can be related to several other important invariants in Commutative Algebra. Just to name a few, if $R = S/I$ is a graded quotient of a polynomial ring S over a field, then the Castelnuovo-Mumford regularity can be obtained as $\operatorname{reg}_S(R) = \max\{a_i(R) + i \mid i \in \mathbb{Z}\}$ (see [11, Theorem 16.3.7]). In addition, when R is Cohen-Macaulay, $a_d(R)$ is the initial degree of a graded canonical module of R (see [11, Remarks 14.5.21]).

3.2 Rings of positive characteristic

Assume that R is a Noetherian ring containing a field of positive characteristic p. Then R comes equipped with a very powerful tool, namely the Frobenius endomorphism $F : R \to R$. The map F raises every element $r \in R$ to its p-th power r^p and, since $\operatorname{char}(R) = p > 0$, this is a ring homomorphism. One of the reasons why this turns out to be an extremely useful is that we can apply F over and over again. In other words, for $e \in \mathbb{N}$, we get the e-th iterated Frobenius endomorphism $F^e : R \to R$ that raises every $r \in R$ to its p^e-th power r^{p^e}.

If R is reduced, we denote by R^{1/p^e} the ring of p^e-th roots of R, that is, the ring that we obtain by adjoining all the p^e-th roots of elements in R, inside some integral extension. We have that R and R^{1/p^e} are abstractly isomorphic as rings, just by associating $r \in R$ with $r^{1/p^e} \in R^{1/p^e}$. Under this isomorphism, ideals $I \subseteq R$ correspond to $I^{1/p^e} = \{r \in R^{1/p^e} \mid r^{p^e} \in I\}$, which are ideals in R^{1/p^e}, and conversely. For an ideal $I \subseteq R$, we have that $IR^{1/p^e} \subseteq R^{1/p^e}$ is an ideal. Under the isomorphism above, this corresponds to $I^{[p^e]} := (i^{p^e} \mid i \in I) \subseteq R$, the ideal of p^e-th powers of elements in I. We will study this correspondence more formally in Proposition 3.2.4.

The reason why it is useful to consider the rings R^{1/p^e} is that, for all $e \geqslant 1$ we can identify $F^e : R \to R$ with the inclusion $R \subseteq R^{1/p^e}$. This is better explained by the following commutative diagram:

$$\begin{array}{ccc} R & \xrightarrow{\;F\;} & R \\ \Big\| & & \Big\downarrow \cong \\ R & \hookrightarrow & R^{1/p} \end{array}$$

We clearly have that $R \subseteq R^{1/p}$ is an integral extension, but it does not have to be finite. An example where it is not finite is $R = \mathbb{F}_p(x_1, x_2, \ldots)$, a field generated over $\mathbb{F}_p$ by infinitely many transcendental elements.

Definition 3.2.1. *Let R be a reduced Noetherian ring of positive characteristic p. We say that R is F-finite if $R^{1/p}$ is a finitely generated R-module.*

Note that R is F-finite if and only if R^{1/p^e} is finitely generated as an R-module for any (equivalently, for all) integer $e \geqslant 1$. This follows from the fact that for any $e \geqslant 1$ the inclusion $R^{1/p^{e-1}} \subseteq R^{1/p^e}$ is isomorphic to $R \subseteq R^{1/p}$, and $R \subseteq R^{1/p^e}$ can be thought of as the chain of inclusions $R \subseteq R^{1/p} \subseteq R^{1/p^2} \subseteq \ldots \subseteq R^{1/p^{e-1}} \subseteq R^{1/p^e}$.

It turns out that F-finiteness is not a very restrictive condition to require. The following proposition shows that, for standard algebras over a field, the example of a non-F-finite field that we provided before is essentially the only relevant case.

Proposition 3.2.2. *[21, Lemma 1.5] If $(R, \mathfrak{m}, k)$ is a standard graded reduced k-algebra, then R is F-finite if and only if k is F-finite.*

Proof. Since R is a quotient of a polynomial ring $S = k[x_1, \ldots, x_n]$, it is enough to show that S is F-finite if and only if k is F-finite, i.e., $[k^{1/p} : k] < \infty$. In fact, if $R = S/I$ and $S^{1/p}$ is a finitely generated S-module, then by base change we obtain that $S^{1/p} \otimes_S S/I \cong S^{1/p}/IS^{1/p}$ is a finitely generated $S/I = R$-module. But then, because $IS^{1/p} \subseteq I^{1/p}$, we obtain a surjection $S^{1/p}/IS^{1/p} \to S^{1/p}/I^{1/p} \cong R^{1/p} \to 0$ of R-modules, which shows that $R^{1/p}$ is a finitely generated R-module as well.

The proof that if S if F-finite then k is F-finite is along the lines of the argument that we just used, this time using $\mathfrak{n} = (x_1, \ldots, x_n)$ instead of I. Conversely, if $k^{1/p}$ is finitely generated over k, we get that $k^{1/p}[x_1, \ldots, x_d]$ is a finitely generated $S = k[x_1, \ldots, x_n]$-module. Furthermore, $S^{1/p} \cong k^{1/p}[x_1^{1/p}, \ldots, x_d^{1/p}]$ is generated by $\{x_1^{i_1/p} \cdots x_d^{i_d/p} \mid 0 \leqslant i_1, \ldots, i_d \leqslant p-1\}$ as a $k^{1/p}[x_1, \ldots, x_n]$-module. Therefore, $S^{1/p}$ is a finitely generated S-module, i.e., S is F-finite. $\square$

Remark 3.2.3. We will show in Proposition 3.3.9 that, in case S is a polynomial ring, the set of generators for $S^{1/p}$ as an S-module displayed in the proof is in fact a basis.

In the rest of this section, we relate the R-modules R^{1/p^e} with grading and local cohomology. In what follows, assume that $(R, \mathfrak{m}, k)$ is standard graded and reduced. We have noticed before that $R \cong R^{1/p^e}$ as rings. Note that, since R is $\mathbb{N}$-graded, the module R^{1/p^e} has a a natural $\dfrac{\mathbb{N}}{p^e}$-grading. In fact, given an element $x \in R^{1/p^e}$ we have that $x^{p^e} \in R$. Then, we can write it as $x^{p^e} = r_{d_1} + \ldots + r_{d_t}$ for some $r_{d_j} \in R$ that are homogeneous of degree $d_j \in \mathbb{N}$. Thus, we obtain that $x = r_{d_1}^{1/p^e} + \ldots + r_{d_t}^{1/p^e}$, and each element $r_{d_j}^{1/p^e}$ is homogeneous of degree $\dfrac{d_j}{p^e} \in \dfrac{\mathbb{N}}{p^e}$. This shows that $R^{1/p^e} = \sum_{i \geqslant 0} R_{i/p^e}^{1/p^e}$, where $R_{i/p^e}^{1/p^e} := (R_i)^{1/p^e}$. In addition, the sum is direct, since if $x \in R_{i/p^e}^{1/p^e} \cap R_{j/p^e}^{1/p^e}$ for some $i \neq j$, then $x^{p^e} \in R_i \cap R_j = \{0\}$, so that $x = 0$ because R is reduced. Finally, since R_1 is finite dimensional over k and $R = k[R_1]$, we obtain that $R_{1/p^e}^{1/p^e}$ is finite dimensional over k^{1/p^e}, and $R^{1/p^e} = k^{1/p^e}[R_{1/p^e}^{1/p^e}]$.

We now focus on graded modules over these two rings. Let $\mathcal{C} = \text{mod}_{\mathbb{Z}}(R)$ be the category of $\mathbb{Z}$-graded R-modules, with $\mathbb{Z}$-graded R-linear homomorphisms. Let $\mathcal{D} = \text{mod}_{\mathbb{Z}/p^e}(R^{1/p^e})$ be the category of $\dfrac{\mathbb{Z}}{p^e}$-graded R^{1/p^e}-modules, with R^{1/p^e}-linear homomorphisms that are $\dfrac{\mathbb{Z}}{p^e}$-graded.

Proposition 3.2.4. *The categories $\mathcal{C}$ and $\mathcal{D}$ are isomorphic.*

Proof. To show that $\mathcal{C} \cong \mathcal{D}$, we need to define two functors $\Phi_e : \mathcal{C} \to \mathcal{D}$ and $\Psi_e : \mathcal{D} \to \mathcal{C}$ such that $\Psi_e \circ \Phi_e = 1_{\mathcal{C}}$ and $\Phi_e \circ \Psi_e = 1_{\mathcal{D}}$.

If M is any R-module, then define $\Phi_e(M)$ to be the the same as M as an abelian group, with R^{1/p^e}-module structure given by $x \cdot m = x^{p^e} m \in M$ for all $x \in R^{1/p^e}$ and all $m \in M$. If $M = \bigoplus_{i \in \mathbb{Z}} M_i$ is $\mathbb{Z}$-graded, for all $\dfrac{i}{p^e} \in \dfrac{\mathbb{Z}}{p^e}$ we set $\Phi_e(M)_{i/p^e} = M_i$, so that $\Phi_e(M) = \bigoplus_{i/p^e \in \mathbb{Z}/p^e} \Phi_e(M)_{i/p^e}$ is $\dfrac{\mathbb{Z}}{p^e}$-graded. This module is usually denoted by

M^{1/p^e}.

Conversely, if N is an R^{1/p^e}-module, then $\Psi_e(N)$ is defined to be the same as N as an abelian group, with R-module action given by $r \cdot n = r^{1/p^e} n$ for all $r \in R$ and $n \in N$. If $N = \bigoplus_{i/p^e \in \mathbb{Z}/p^e} N_{i/p^e}$ is $\dfrac{\mathbb{Z}}{p^e}$-graded, then set $\Psi_e(N)_i = N_{i/p^e}$ for all $i \in \mathbb{Z}$, and this makes $\Psi_e(N) = \bigoplus_{i \in \mathbb{Z}} \Psi_e(N)_i$ into a $\mathbb{Z}$-graded R-module.

Finally, given $f : M \to N$ an R-module homomorphism that is graded of degree $t \in \mathbb{Z}$, we define $\Phi_e(f) : \Phi_e(M) \to \Phi_e(N)$ to be $\Phi_e(f)(\Phi_e(m)) = \Phi_e(f(m))$. This is a R^{1/p^e}-homomorphism, and $\Phi_e(f)(\Phi_e(M_i)) \subseteq \Phi_e(N_{d+i}) = N_{(d+i)/p^e}$, for all $i \in \mathbb{Z}$, so that $\Phi_e(f)$ is graded of degree d/p^e. Analogous arguments applies to graded R^{1/p^e}-homomorphisms. It is clear from the definitions that Φ_e and Ψ_e preserve compositions and identities. It also follows immediately that the compositions of these functors give identities, showing that the categories $\mathcal{C}$ and $\mathcal{D}$ are isomorphic. $\square$

Remark 3.2.5. Given the definitions of Φ_e and Ψ_e, it is clear that these functors are exact.

We now define a version of the a-invariants for the modules R^{1/p^e}.

Definition 3.2.6. *Let $i \in \mathbb{Z}$. We define the i-th a-invariant of R^{1/p^e} to be*

$$a_i(R^{1/p^e}) = \max\left\{ t \in \frac{\mathbb{Z}}{p^e} \mid H^i_{\mathfrak{m}}(R^{1/p^e})_t \neq 0 \right\}$$

if $H^i_{\mathfrak{m}}(R^{1/p^e}) \neq 0$. Otherwise, we set $a_i(R^{1/p^e}) = -\infty$.

Using the isomorphisms Φ_e and Ψ_e described above, it turns out that the a-invariants of the modules R^{1/p^e} are directly related to the a-invariants of R.

Lemma 3.2.7. *Let $(R, \mathfrak{m}, k)$ be a standard graded reduced F-finite algebra over a field k of positive characteristic p. Then, for all $i \in \mathbb{Z}$ and all integers $e \geqslant 1$, we have* $a_i(R^{1/p^e}) = \dfrac{a_i(R)}{p^e}$.

Proof. Let $x_1, \ldots, x_d$ be a full homogeneous system of parameters in R, so that $\sqrt{(x_1, \ldots, x_d)} = \mathfrak{m}$. Consider the Čech complex $\check{C}^\bullet(x_1, \ldots, x_d; R)$ that gives the modules $H^i_\mathfrak{m}(R)$ as cohomology. After tensoring with R^{1/p^e}, we obtain a new complex, whose homology are the modules $H^i_\mathfrak{m}(R^{1/p^e})$, by definition. Note that, via the functor Ψ_e, the complex $\check{C}^\bullet(x_1, \ldots, x_d; R) \otimes_R R^{1/p^e}$ of $\dfrac{\mathbb{Z}}{p^e}$-graded R^{1/p^e}-modules corresponds to the complex $\check{C}(x_1^{p^e}, \ldots, x_d^{p^e}; R)$ of graded $\mathbb{Z}$-modules.. Since $\sqrt{(x_1^{p^e}, \ldots, x_d^{p^e})} = \mathfrak{m}$, i.e., $x_1^{p^e}, \ldots, x_d^{p^e}$ is still a full homogeneous system of parameters of R, and local cohomology only depends on the radical of the ideal that such elements generate, we obtain that the cohomology of $\Psi_e(\check{C}^\bullet(x_1, \ldots, x_d; R) \otimes_R R^{1/p^e}) = \check{C}^\bullet(x_1^{p^e}, \ldots, x_d^{p^e}; R)$ is still $H^i_\mathfrak{m}(R)$. Using that $\Phi_e \circ \Psi_e = 1_\mathcal{D}$ and that Ψ_e is an exact functor, we obtain

$$\begin{aligned} H^i_\mathfrak{m}(R^{1/p^e}) &= \Phi_e(\Psi_e(H^i(\check{C}^\bullet(x_1, \ldots, x_d; R) \otimes_R R^{1/p^e}))) \\ &\cong \Phi_e(H^i(\check{C}^\bullet(x_1^{p^e}, \ldots, x_d^{p^e}; R)) = \Phi_e(H^i_\mathfrak{m}(R)) \end{aligned}$$

as graded $\dfrac{\mathbb{Z}}{p^e}$-modules. Finally, since $\Phi_e(H^i_\mathfrak{m}(R))_{s/p^e} = \Phi_e(H^i_\mathfrak{m}(R)_s)$ for all $s \in \mathbb{Z}$ by definition of the functor Φ_e, we obtain that

$$\begin{aligned} a_i(R^{1/p^e}) = &\max\left\{t \in \frac{\mathbb{Z}}{p^e} \mid \Phi_e(H^i_\mathfrak{m}(R))_t \neq 0\right\} = \\ &\frac{\max\{s \in \mathbb{Z} \mid \Phi_e(H^i_\mathfrak{m}(R)_s) \neq 0\}}{p^e} = \frac{a_i(R)}{p^e}. \end{aligned}$$

□

3.3 F-purity and local cohomology

We start by introducing the central notion of this section, that is, F-purity. F-pure rings have been first investigated by Hochster and Roberts in [40], and by many other authors since then (for instance see [2, 65]).

Definition 3.3.1. *A Noetherian ring R of positive prime characteristic p is called F-pure if the Frobenius endomorphism $F : R \to R$ is a pure homomorphism, that is, $F \otimes 1 : R \otimes_R M \to R \otimes_R M$ is injective for all R-modules M.*

Another notion, that is closely related, is the one of F-split ring.

Definition 3.3.2. *A Noetherian ring R of positive prime characteristic p is called F-split if $F : R \to R$ is a split monomorphism.*

F-split rings are clearly F-pure, and if R is an F-pure ring, F itself is injective and R must be a reduced ring. Given the identification between the Frobenius map and the natural inclusion $R \subseteq R^{1/p}$, we have that R is F-split if and only if $R \subseteq R^{1/p}$ is a split inclusion. If R is an F-finite ring, being F-pure is equivalent to being F-split (see [40, Corollary 5.3]). Since, in what follows, we will always assume that R is F-finite, we use the word F-pure to refer to both these notions.

Example 3.3.3. If $R = \mathbb{F}_2[x,y]/(x^3+y^3)$, then R is not F-pure. In fact, if so, there would be a graded map $\varphi : R^{1/2} \to R$, such that $\varphi(1) = 1$. Note that $\varphi(x^2) = x^2\varphi(1) = x^2$. In addition $x^2 + y(xy)^{1/2} = 0$ in $R^{1/2}$, because $(x^2 + y(xy)^{1/2})^2 =$

$x(x^3+y^3)=0$ in R. Therefore, we have that

$$0=\varphi(x^2-y(xy)^{1/2})=x^2-y\varphi((xy)^{1/2}),$$

which implies that $x^2 \in (y)$ in R. This is a contradiction, therefore the inclusion $R \subseteq R^{1/2}$ does not split, and R is not F-pure.

Example 3.3.4. If $R=\mathbb{F}_2[x,y]/(xy)$, then R is F-pure. In fact, by Proposition 3.3.9, we have that $\{1,x^{1/2},y^{1/2},(xy)^{1/2}\}$ is a basis of $(\mathbb{F}_2[x,y])^{1/2}=\mathbb{F}_2[x^{1/2},y^{1/2}]$ as an $\mathbb{F}_2[x,y]$-module. As a consequence, we have that $\{1,x^{1/2},y^{1/2}\}$ generates $R^{1/2}$ as an R-module, where we disregard $(xy)^{1/2}$ since it is equal to zero in $R^{1/2}$. Define $\varphi: R^{1/2}\to R$ by $\varphi(1)=1$, and $\varphi(x^{1/2})=\varphi(y^{1/2})=0$. Then φ is a splitting of the natural inclusion $R\subseteq R^{1/2}$. The only thing to show here is that φ is well defined. It suffices to show that if $f+gx^{1/2}+hy^{1/2}=0$ in $R^{1/2}$, for some $f,g,h\in R$, then $f=0$. Lifting such a relation to $\mathbb{F}_2[x,y]$, this means that $f^2+g^2x+h^2y\in(xy)$. Write $f=A+\sum_{i=1}^b B_ix^i+\sum_{j=1}^c C_jy^j$ modulo (xy), for some $A,B_i,C_j\in\mathbb{F}_2$, so that $f^2=A+\sum_{i=1}^b B_ix^{2i}+\sum_{j=1}^c C_jy^{2j}$ modulo (xy). Note that no term in g^2x or h^2y will have a pure power in x or y of even degree. Therefore, by degree considerations, we conclude that $B_i=0=C_j$ for all $i=1,\ldots,b$ and $j=1,\ldots,c$. Finally, again by degree considerations, we also conclude that $A=0$, so that $f\in(xy)$. But this precisely means that $f=0$ in R. Thus the splitting φ is well defined, and R is F-pure.

Let R be a Noetherian ring containing a field of positive characteristic p. Then, for any multiplicatively closed set $W\subseteq R$, we obtain a map $F_W: R_W\to R_{F(W)}$, induced by the Frobenius endomorphism F on R. We claim that $R_{F(W)} = R_W$. Clearly $R_{F(W)} \subseteq R_W$, since $F(W) \subseteq W$. Conversely, let $x \in R_W$. Then we can assume that $x = \dfrac{r}{w}$ for some $r \in R$ and $w \in W$. We can rewrite x as $x = \dfrac{w^{p-1}r}{w^p} \in R_{F(W)}$, and this shows that the other inclusion is also true. In conclusion, we obtain an induced endomorphism $F : R_W \to R_W$, which is precisely the Frobenius endomorphism on R_W. Now let $W \subseteq R$ be a multiplicatively closed set, and consider the natural localization map $\varphi : R \to R_W$. Let $K = \ker(\varphi)$ and $I = \operatorname{Im}(\varphi)$. Note that $F(K) \subseteq K$, since $\varphi(r^p) = r^{p-1}\varphi(r) = 0$ for all $r \in R$. In addition, $F(I) \subseteq I$, because if $s = \varphi(r)$, then $ws = wr$ for some $w \in W$. But then, $w^p s^p = w^p r^p$, which means that $\varphi(r^p) = s^p$, and hence $s^p = F(s) \in I$. Therefore we get commutative squares

$$\begin{array}{ccc} K & \hookrightarrow & R \\ \downarrow F & & \downarrow F \\ K & \hookrightarrow & R \end{array} \qquad\qquad \begin{array}{ccc} R_W & \xrightarrow{\varphi} & I \\ \downarrow F & & \downarrow F \\ R_W & \xrightarrow{\varphi} & I \end{array}$$

We now specialize to local cohomology modules. Let $I = (f_1, \ldots, f_t) \subseteq R$ be an ideal. Since the modules $H^i_I(R)$ are obtained from the Čech complex, which is essentially constructed from localization maps, we obtain a commutative diagram

$$\begin{array}{ccccccccccccc} 0 & \to & R & \to & \bigoplus_{i=1}^t R_{f_i} & \to & \bigoplus_{i,j=1}^t R_{f_if_j} & \to & \cdots\cdots & \to & R_{f_1\cdots f_t} & \to & 0 \\ & & \downarrow F & & \downarrow F & & \downarrow F & & & & \downarrow F & & \\ 0 & \to & R & \to & \bigoplus_{i=1}^t R_{f_i} & \to & \bigoplus_{i,j=1}^t R_{f_if_j} & \to & \cdots\cdots & \to & R_{f_1\cdots f_t} & \to & 0 \end{array}$$

As explained in the previous discussion, F preserves kernels and images.Therefore we obtain induced maps in cohomology $H^i_I(F) : H^i_I(R) \to H^i_I(R)$ for all $i \in \mathbb{Z}$, that we denote by F_i. Note that F_i is not necessarily an inclusion. However, this is the case for F-pure rings.

Proposition 3.3.5. *Let R be an F-finite and F-pure ring. Then, for all $i \in \mathbb{Z}$ and all ideal $I \subseteq R$, the map $F_i : H^i_I(R) \to H^i_I(R)$ is a split inclusion.*

Proof. By assumption, we have that the Frobenius map $F : R \hookrightarrow R$ is a split inclusion. That is, there exists a ring homomorphism $\varphi : R \to R$ such that $\varphi \circ F = 1_R$. Then, by functoriality of local cohomology $H^i_{\mathfrak{m}}(-)$, we obtain induced maps $F_i = H^i_I(F) : H^i_I(R) \to H^i_I(R)$ and $\varphi_i := H^i_I(\varphi) : H^i_I(R) \to H^i_I(R)$. In addition, we have that $\varphi_i \circ F_i = H^i_I(\varphi \circ F) = H^i_I(1_R) = 1_{H^i_I(R)}$. Therefore, the map F_i is a split inclusion. □

Proposition 3.3.6. *[50, 65] Let $(R, \mathfrak{m}, k)$ be an F-finite and F-pure ring. For an integer $e \geqslant 1$, let $R \subseteq R^{1/p^e}$ be the natural inclusion. For an integer $i \in \mathbb{N}$, let $\psi_i : H^i_{\mathfrak{m}}(R) \to H^i_{\mathfrak{m}}(R^{1/p^e})$ be the induced map on local cohomology. Then, the map $H^i_{\mathfrak{m}}(R) \otimes R^{1/p^e} \to H^i_{\mathfrak{m}}(R^{1/p^e})$, induced by $v \otimes r^{1/p^e} \mapsto r^{1/p^e}\psi_i(v)$ is surjective.*

Proof. The statement is equivalent to showing that the R^{1/p^e}-span of $H^i_{\mathfrak{m}}(R)$ generates $H^i_{\mathfrak{m}}(R^{1/p^e})$. Equivalently, we are going to show that the image of $H^i_{\mathfrak{m}}(R)$ under the map $H^i_{\mathfrak{m}}(F^e) : H^i_{\mathfrak{m}}(R) \to H^i_{\mathfrak{m}}(R)$ generates $H^i_{\mathfrak{m}}(R)$ as an R-module. In addition, it is enough to show the claim for $e = 1$, since we can then reiterate the argument. As in Proposition 3.3.5, we denote by $\varphi : R \to R$ a splitting of the Frobenius map, we denote the map $H^i_{\mathfrak{m}}(F) : R \to R$ by F_i, and its splitting by φ_i. For all $r \in R$ and all $v \in H^i_{\mathfrak{m}}(R)$, we have $\varphi_i(r \cdot v) = \varphi(r) \cdot \varphi_i(v)$. Let $v \in H^i_{\mathfrak{m}}(R)$ be an arbitrary element.

For all integers $n \geqslant 0$ consider the following R-submodules of $H^i_{\mathfrak{m}}(R)$:

$$M_n = R\text{-span}\{F_i^n(v), F_i^{n+1}(v), \ldots\} \subseteq H^i_{\mathfrak{m}}(R).$$

We have a descending chain $M_0 \supseteq M_1 \supseteq \ldots \supseteq M_n \supseteq M_{n+1} \supseteq \ldots$ which stabilizes, since $H^i_{\mathfrak{m}}(R)$ is Artinian. Let n be the minimum integer for which $M_n = M_{n+1}$. If $n = 0$, we obtain that $v \in M_0 = M_1$. This means that there exist $r_1, \ldots, r_t$ such that $v = \sum_{j=1}^t r_j F^j(v)$, concluding the proof in this case. Assume $n > 0$. Since $F^n(v) \in M_n = M_{n+1}$, there exist $r_1, \ldots, r_t \in R$ such that $F_i^n(v) = \sum_{j=1}^t r_j F_i^{n+j}(v)$. Apply φ_i:

$$F^{n-1}(v) = \varphi_i(F^n(v)) = \sum_{j=1}^t \varphi_i(r_j F_i^{n+j}(v)) = \sum_{j=1}^t \varphi(r_j) F^{n+j-1}(v).$$

Since $\varphi(r_j) \in R$, we obtain that $F^{n-1}(v) \in M_n$. Then $M_{n-1} = M_n$, contradicting the minimality of n. This concludes the proof.

□

Proposition 3.3.7. *[40, Proposition 2.4] Let $(R, \mathfrak{m}, k)$ be a standard graded F-pure k-algebra. Then $a_i(R) \leqslant 0$ for all $i \in \mathbb{Z}$.*

Proof. If $H^i_{\mathfrak{m}}(R) = 0$, then $a_i(R) = -\infty$ by definition. Suppose that $H^i_{\mathfrak{m}}(R) \neq 0$, and assume that $a_i(R) > 0$. Let $v \in H^i_{\mathfrak{m}}(R)_{a_i(R)}$ be a non-zero element of degree $a_i(R)$. Since R is F-pure, we have that the map $F_i : H^i_{\mathfrak{m}}(R) \to H^i_{\mathfrak{m}}(R)$, induced by the Frobenius endomorphism on R, is injective by Proposition 3.3.5. But $F_i(v) \in H^i_{\mathfrak{m}}(R)$ is then a non-zero element of degree $pa_i(R) > a_i(R)$, which is a contradiction. □

We end this section by introducing another notion of F-singularity, namely, strong F-regularity. It has been defined for F-finite rings by Hochster and Huneke in [39], in relation to the notion of tight closure and test ideals.

Definition 3.3.8. *An F-finite reduced standard graded k-algebra R is called strongly F-regular if for any element $f \in R \setminus \{0\}$, there exists $e \in \mathbb{N}$ such that the inclusion $f^{1/p^e}R \subseteq R^{1/p^e}$ splits.*

Examples of strongly F-regular rings include F-finite polynomial rings.

Proposition 3.3.9. *Let $S = k[x_1, \ldots, x_n]$, where k is an F-finite field of positive characteristic p. Then S^{1/p^e} is a finitely generated free S-module for all $e \geqslant 1$. In particular, S is strongly F-regular.*

Proof. Since k is a field, we have that k^{1/p^e} is a finitely generated free k-module by Proposition 3.2.2. Therefore, we have that $k^{1/p^e}[x_1, \ldots, x_n]$ is a finitely generated free S-module. In addition, $S^{1/p^e} \cong k^{1/p^e}[x_1^{1/p^e}, \ldots, x_n^{1/p^e}]$ is a free $k^{1/p^e}[x_1, \ldots, x_n]$-module, with basis $\{(x_1^{i_1} \cdots x_n^{i_n})^{1/p^e} \mid 0 \leqslant i_1, \ldots, i_n \leqslant p^e - 1\}$. In fact, using Proposition 3.2.4 and the functors Φ_e and Ψ_e, this is equivalent to the fact that the elements $\{x_1^{i_1} \cdots x_n^{i_n} \mid 0 \leqslant i_1, \ldots, i_n \leqslant p^e - 1\}$ form a $k \cong S/\mathfrak{n}$-basis of $S/\mathfrak{n}^{[p^e]}$.

For strong F-regularity, if $f \in S$ is a non-zero element, choose $e \gg 0$ such that $f \notin \mathfrak{n}^{[p^e]}$. This integer exists, since otherwise $f \in \bigcap_e \mathfrak{n}^{[p^e]} = (0)$. Then $f^{1/p^e} \notin \mathfrak{n}S^{1/p^e}$, and hence it is not a minimal generator of S^{1/p^e} as an S-module. By freeness, this means that it can be made part of a basis. In particular, we can define a splitting of the map $f^{1/p^e}S \subseteq S^{1/p^e}$, as desired. □

Remark 3.3.10. Strongly F-regular rings are F-pure, since one can choose $f = 1$ in the definition, and get that the inclusion $R \subseteq R^{1/p^e}$ splits for some (equivalently, all) integers $e \geqslant 1$. Strongly F-regular rings are normal and Cohen-Macaulay domains. Note that F-pure rings need not be either normal or Cohen-Macaulay.

Example 3.3.11. Let $R = k[x, y, z]/(xy, xz)$. Then R is a two-dimensional F-pure ring, but it is not Cohen-Macaulay, since $\operatorname{depth}(R) = 1$. In particular, R is not S_2, hence not normal.

3.4 Fedder-type results

We now present an extremely useful explicit description of $\operatorname{Hom}_R(R^{1/p^e}, R)$, proved by Richard Fedder in [22], and some of its consequences.

Let $(S, \mathfrak{n}, k)$ be an F-finite standard graded Gorenstein reduced ring, and let $e \geqslant 1$ be an integer. Then $\operatorname{Hom}_S(S^{1/p^e}, S)$ is a $\dfrac{\mathbb{N}}{p^e}$-graded S^{1/p^e}-module, that is a canonical module of S^{1/p^e} [12, Theorem 3.3.7 (b)]. Since S and S^{1/p^e} are isomorphic as rings, we have that S^{1/p^e} is also Gorenstein. Therefore, there exists an S-homomorphism Tr_e that generates $\operatorname{Hom}_S(S^{1/p^e}, S)$ as a S^{1/p^e}-module. Furthermore, it is unique up to multiplication by a unit in S^{1/p^e}.

When $S = k[x_1, \ldots, x_n]$ is a standard graded polynomial ring over a field k of positive characteristic p, we can describe Tr_e more explicitly.

Definition 3.4.1. *Let* $S = k[x_1, \ldots, x_n]$ *be a polynomial ring over an* F*-finite field. The* e*-trace map* $\mathrm{Tr}_e \in \mathrm{Hom}_S(S^{1/p^e}, S)$ *is defined by*

$$\mathrm{Tr}_e\left(\lambda x_1^{\alpha_1/p^e} \cdots x_n^{\alpha_n/p^e}\right) = \begin{cases} \lambda x_1^{\frac{\alpha_1 - p^e + 1}{p^e}} \cdots x_n^{\frac{\alpha_n - p^e + 1}{p^e}} & \lambda \in k,\ \alpha_i \equiv p^e - 1 \ mod\ p^e\ \forall i \\ 0 & otherwise \end{cases}$$

on monomials, and extended by linearity to S^{1/p^e}*.*

Remark 3.4.2. Note that, for all integers $e, e' \geqslant 1$, we have $\mathrm{Tr}_{e'} \circ \mathrm{Tr}_e^{1/p^{e'}} = \mathrm{Tr}_{e'+e}$, where $\mathrm{Tr}_e^{1/p^{e'}} : S^{1/p^{e+e'}} \to S^{1/p^{e'}}$ is defined as $\mathrm{Tr}_e^{1/p^{e'}}(r^{1/p^{e+e'}}) = \left(\mathrm{Tr}_e(r^{1/p^e})\right)^{1/p^{e'}}$.

Lemma 3.4.3. *[21, Lemma 1.6] Let* $S = k[x_1, \ldots, x_n]$ *be a polynomial ring over an* F*-finite field of positive characteristic* p*. Let* H *be a (possibly improper) ideal of* S^{1/p^e}*, and let* $J \subseteq R$ *be an ideal in* R*. Then* $\mathrm{Tr}_e(f^{1/p^e} H) \subseteq J$ *if and only if* $f^{1/p^e} \in (JS^{1/p^e} :_{S^{1/p^e}} H)$*.*

Proof. We note that, if $f^{1/p^e} H \subseteq JS^{1/p^e}$, then $\mathrm{Tr}_e(f^{1/p^e} H) \subseteq \mathrm{Tr}_e(JS^{1/p^e}) \subseteq J$, because Tr_e is S-linear. We now prove the converse implication, that is, we assume that $\mathrm{Tr}_e(f^{1/p^e} H) \subseteq J$. Since $H \subseteq S^{1/p^e}$ is an ideal, we have that this happens if and only if $\mathrm{Tr}_e(f^{1/p^e} h S^{1/p^e}) \subseteq J$ for all $h \in H$. Recall that S^{1/p^e} is a free S-module by Proposition 3.3.9, and let $\{\mu_i^{1/p^e}\}$ be a basis. In addition, let $\{\eta_j\}$ be its dual basis. Since $\eta_j \in \mathrm{Hom}_S(S^{1/p^e}, S)$, which is a cyclic S^{1/p^e}-module, we have that $\eta_j = \mathrm{Tr}_e(\theta_j^{1/p^e} \cdot -)$, for some $\theta_j^{1/p^e} \in S^{1/p^e}$. Let $h \in H$ be any element. By assumption, for all i, we have that $\mathrm{Tr}_e(f^{1/p^e} h \mu_i^{1/p^e}) = \nu_i$, for some $\nu_i \in J$. On the other hand, using that $\{\mu_i^{1/p^e}\}$ and $\{\eta_j = \mathrm{Tr}_e(\theta_j^{1/p^e} \cdot -)\}$ are dual bases, we obtain that

$$\mathrm{Tr}_e(\sum_j \nu_j \theta_j^{1/p^e} \mu_i^{1/p^e}) = \sum_j \nu_j \eta_j(\mu_i^{1/p^e}) = \nu_i$$

for all i. Therefore $\mathrm{Tr}_e(f^{1/p^e} h \cdot -) = \nu_i = \mathrm{Tr}_e(\sum_j \nu_j \theta_j^{1/p^e} \cdot -)$, that is,

$$\mathrm{Tr}_e\left((f^{1/p^e} h - \sum_j \nu_j \theta_j^{1/p^e}) \cdot -\right) = 0.$$

Hence $f^{1/p^e} h = \sum_j \nu_j \theta_j^{1/p^e} \in JS^{1/p^e}$, as desired. □

Theorem 3.4.4. *[21, Corollary to Lemma 1.6] Let $S = k[x_1, \ldots, x_n]$ be a polynomial ring over an F-finite field k of positive characteristic p. Let $I \subseteq S$ be a homogeneous ideal, and let $R = S/I$. We have an isomorphism of R-modules*

$$\Gamma_e : \frac{(IS^{1/p^e} :_{S^{1/p^e}} I^{1/p^e})}{IS^{1/p^e}} \xrightarrow{\cong} \mathrm{Hom}_R(R^{1/p^e}, R)$$

which is graded of degree $-\frac{n(p^e-1)}{p^e}$, and such that $\Gamma_e(f^{1/p^e}) = \varphi_{f,e}$ is the image of $\mathrm{Tr}_e(f^{1/p^e} \cdot -)$ inside $\mathrm{Hom}_R(R^{1/p^e}, R)$.

Proof. Given $f^{1/p^e} \in (IS^{1/p^e} :_{S^{1/p^e}} I^{1/p^e})$ it is clear that this defines an R-module map $\varphi_{f,e} \in \mathrm{Hom}_R(R^{1/p^e}, R)$ by setting $\varphi_{f,e}(r^{1/p^e}) = \mathrm{Tr}_e(f^{1/p^e} r^{1/p^e})$ for all $r \in R$. Here we are identifying elements in S with their images in $R = S/I$.

Since S^{1/p^e} is a free S-module, every homomorphism $\phi \in \mathrm{Hom}_R(R^{1/p^e}, R) = \mathrm{Hom}_S(S^{1/p^e}/I^{1/p^e}, S/I)$ can be lifted to a homomorphism $\phi_0 \in \mathrm{Hom}_S(S^{1/p^e}, S)$

$$\begin{array}{ccccc}
(\bigoplus S) \cong S^{1/p^e} & \longrightarrow & S^{1/p^e}/I^{1/p^e} & \longrightarrow & 0 \\
\downarrow{\scriptstyle \phi_0} & & \downarrow{\scriptstyle \phi} & & \\
S & \longrightarrow & S/I & \longrightarrow & 0
\end{array}$$

Since $\operatorname{Hom}_S(S^{1/p^e}, S)$ is a cyclic S^{1/p^e}-module, we have that $\phi_0 = \operatorname{Tr}_e(f^{1/p^e} \cdot -)$ for some $f^{1/p^e} \in S^{1/p^e}$. In addition, $f^{1/p^e} \in (IS^{1/p^e} :_{S^{1/p^e}} I^{1/p^e})$ by Lemma 3.4.3 applied to $H = I^{1/p^e}$ and $J = I$. This shows surjectivity.

Clearly $\Gamma_e(IS^{1/p^e}) = 0$, since $\operatorname{Tr}_e(IS^{1/p^e}) \subseteq I$. If $\Gamma_e(f^{1/p^e}) = \varphi_{f,e} = 0$, then we have that $\operatorname{Tr}_e(f^{1/p^e}S^{1/p^e}) \subseteq I$. Using again Lemma 3.4.3 with $H = S^{1/p^e}$ and $J = I$, we conclude that $f^{1/p^e} \in IS^{1/p^e}$, showing injectivity. It is easy to check that the map in question is R-linear and, hence, an isomorphism. Finally, the isomorphism is graded, since it sends an element $f^{1/p^e} \in (IS^{1/p^e} :_{S^{1/p^e}} I^{1/p^e})$ of degree $\frac{i}{p^e} \in \frac{\mathbb{N}}{p^e}$ to the graded homomorphism $\varphi_{f,e} = \operatorname{Tr}_e(f^{1/p^e} \cdot -)$. For any homogeneous element $r^{1/p^e} \in R^{1/p^e}$ of degree $\frac{j}{p^e} \in \frac{\mathbb{N}}{p^e}$, the map $\varphi_{f,e}$ sends r^{1/p^e} either to zero, or to a non-zero element of degree $\frac{i+j-n(p^e-1)}{p^e}$, by the definition of the trace map Tr_e. Therefore, $\varphi_{f,e}$ has degree $\frac{i-n(p^e-1)}{p^e} = \deg(f) - \frac{n(p^e-1)}{p^e}$, which means that the isomorphism is graded of degree $-\frac{n(p^e-1)}{p^e}$, as desired. □

Lemma 3.4.5. *Let $S = k[x_1, \dots, x_n]$ be a polynomial ring over an F-finite field k of positive characteristic p. Let $\mathfrak{n} = (x_1, \dots, x_n)$, let $I \subseteq S$ be a homogeneous ideal, and set $R = S/I$. Let $e \geqslant 1$ be an integer and let $g^{1/p^e} \in R^{1/p^e}$ be a homogeneous element of degree $\frac{i}{p^e}$, for some $i \in \mathbb{N}$. Then, the R-module inclusion $g^{1/p^e}R \subseteq R^{1/p^e}$ splits if and only if there exists a homogeneous element $f^{1/p^e} \in (IS^{1/p^e} :_{S^{1/p^e}} I^{1/p^e})$ of degree $\frac{n(p^e-1)-i}{p^e}$ such that $f^{1/p^e}g^{1/p^e} \equiv (x_1 \cdots x_n)^{(p^e-1)/p^e}$ modulo $\mathfrak{n}S^{1/p^e}$.*

Proof. Assume that $g^{1/p^e}R \subseteq R^{1/p^e}$ splits. Then, there exists an R-module map $\varphi : R^{1/p^e} \to R$ such that $\varphi(g^{1/p^e}) = 1$. In addition, the splitting φ can be chosen to be graded of degree $-\dfrac{i}{p^e}$. In fact, this is clear if φ is graded, since $\deg(g^{1/p^e}) = \dfrac{i}{p^e}$ and $\varphi(g^{1/p^e}) = 1$. If not, since R is F-finite, φ can be written as a sum of graded R-linear homomorphisms $\varphi = \varphi_1 + \ldots + \varphi_t$ [23, Lemma 4.2]. Since g^{1/p^e} is homogeneous, it follows that $\varphi_j(g^{1/p^e}) = 1$ for some j, so we can replace φ by φ_j and assume that φ is graded of degree $-\dfrac{i}{p^e}$. In this proof, we will not distinguish between elements of S or S^{1/p^e} and their images in R and R^{1/p^e}. By Theorem 3.4.4, there exists a homogeneous $f^{1/p^e} \in (IS^{1/p^e} :_{S^{1/p^e}} I^{1/p^e})$ such that $\varphi = \varphi_{f,e}$. Furthermore, since the isomorphism Γ_e described in Theorem 3.4.4 is the inverse of the one we need here, f^{1/p^e} must have degree $\dfrac{n(p^e-1)-i}{p^e}$. Finally, notice that if $f^{1/p^e}g^{1/p^e} \in \mathfrak{n}S^{1/p^e}$, then $\mathrm{Tr}_e(f^{1/p^e}g^{1/p^e}) \subseteq \mathfrak{n}$, so that $\varphi_{f,e}(g^{1/p^e}) \subseteq \mathfrak{m}$, by S-linearity of the trace map Tr_e and by definition of $\varphi_{f,e}$. But this is not possible, since $\varphi_{f,e}(g^{1/p^e}) = 1$. Therefore, $f^{1/p^e}g^{1/p^e}$ is a homogeneous element of degree $\dfrac{n(p^e-1)}{p^e}$ that is not contained in $\mathfrak{n}S^{1/p^e}$. This implies that $f^{1/p^e}g^{1/p^e} \equiv \lambda(x_1\cdots x_n)^{(p^e-1)/p^e}$ modulo $\mathfrak{n}S^{1/p^e}$ for some constant $\lambda \in k^{1/p^e} \smallsetminus \{0\}$. If $\lambda \in k^{1/p^e} \smallsetminus k$, then we have that $\mathrm{Tr}_e(f^{1/p^e}g^{1/p^e}) = 0$, by the definition of the trace map, contradicting again that $\varphi_{f,e}(g^{1/p^e}) = 1$. Therefore, we get that $\lambda \in k \smallsetminus \{0\}$. Finally, we have that $1 = \varphi_{f,e}(g^{1/p^e}) = \lambda$, as desired.

Conversely, assume that $f^{1/p^e} \in (IS^{1/p^e} :_{S^{1/p^e}} I^{1/p^e})$ is such that $f^{1/p^e}g^{1/p^e} \equiv (x_1\cdots x_n)^{(p^e-1)/p^e}$ modulo $\mathfrak{n}S^{1/p^e}$. Consider the R-module map $\varphi_{f,e} : R^{1/p^e} \to R$. Since the degree of the homogeneous element $f^{1/p^e}g^{1/p^e}$ is $\dfrac{n(p^e-1)}{p^e}$, we conclude that $\mathrm{Tr}_e(f^{1/p^e}g^{1/p^e}) = \mathrm{Tr}_e((x_1\cdots x_n)^{(p^e-1)/p^e}) = 1$, by the definition of the trace map. Therefore, we obtain that $\varphi_{f,e}(g^{1/p^e}) = 1$, that is, the inclusion $g^{1/p^e}R \subseteq R^{1/p^e}$

splits. □

As an immediate corollary, after identifying the colon ideal $(IS^{1/p} :_{S^{1/p}} I^{1/p})$ with $(I^{[p]} : SI)$, we recover the well know criterion of Fedder for F-purity of quotients of polynomial rings.

Corollary 3.4.6. *[21, Proposition 1.7] Let $S = k[x_1, \ldots, x_n]$ be an F-finite polynomial ring over a field k of positive characteristic p. Let $\mathfrak{n} = (x_1, \ldots, x_n)$, and let $I \subseteq S$ be a homogeneous ideal. Then the ring $R = S/I$ is F-pure if and only if $(I^{[p^e]} :_S I) \not\subseteq \mathfrak{n}^{[p^e]}$ for some (equivalently, for all) integer $e \geqslant 1$.*

Remark 3.4.7. Let $S = k[x_1, \ldots, x_n]$, and let $\mathfrak{n} = (x_1, \ldots, x_n)$. Let $R = S/(f)$ be a hypersurface, for some homogeneous $0 \neq f \in S$. We have that R is F-pure if and only if $((f^p) :_S f) \not\subseteq \mathfrak{n}^{[p]}$. Since f is a regular element in S, we obtain that R is F-pure if and only if $f^{p-1} \notin \mathfrak{n}^{[p]}$. In particular, the ring in Example 3.3.3 is not F-pure, since $(x^3 + y^3) \in (x^2, y^2)$, while the ring in Example 3.3.4 is F-pure, because $xy \notin (x^2, y^2)$.

We now turn our attention to compatible ideals. They have been introduced first by Schwede in [61].

Definition 3.4.8. *[61] Suppose that R is an F-finite ring. An ideal $J \subseteq R$ is said to be compatible if $\phi(J^{1/p^e}) \subseteq J$ for all integers $e \geqslant 1$ and all R-linear maps $\phi : R^{1/p^e} \to R$.*

Note that, if R is an F-pure ring, then so is R/I for any compatible ideal I. In fact, if $\varphi : R^{1/p} \to R$ is such that $\varphi(1) = 1$, giving a splitting of the inclusion

$R \subseteq R^{1/p}$, we can defined $\overline{\varphi} : R^{1/p}/I^{1/p} \to R/I$ as the map induced by φ on the residue class rings. This is possible precisely because $\varphi(I^{1/p}) \subseteq I$. Then $\overline{\varphi}(1) = 1$, so that it gives a splitting of the inclusion $R/I \subseteq (R/I)^{1/p}$, showing that R/I is F-pure.

Using Theorem 3.4.4, we can give a precise description of what compatible ideals of graded quotients of polynomial rings look like when lifted to the ambient space.

Proposition 3.4.9. *Let S be a polynomial ring over an F-finite field. Let $I \subseteq S$ be a homogeneous ideal, and let $R = S/I$. Let $J \subseteq R$ be a homogeneous ideal, and let $\widetilde{J}$ denote its pullback in S. We have that J is compatible if and only if $(IS^{1p^e} :_{S^{1/p^e}} I^{1/p^e}) \subseteq (\widetilde{J}S^{1/p^e} :_{S^{1/p^e}} \widetilde{J}^{1/p^e})$ for all $e \geqslant 1$.*

Proof. Let J be a compatible ideal in R, and let $e \geqslant 1$ be an integer. For any $f^{1/p^e} \in \left(IS^{1/p^e} :_{S^{1/p^e}} I^{1/p^e}\right)$, the map $\varphi_{f,e} = \mathrm{Tr}_e(f^{1/p^e} \cdot -)$ as in Theorem 3.4.4 is an element of $\mathrm{Hom}_R(R^{1/p^e}, R)$. By compatibility of J, we have that $\varphi_{f,e}(J^{1/p^e}) \subseteq J$, which means that $\mathrm{Tr}_e(f^{1/p^e}\widetilde{J}^{1/p^e}) \subseteq \widetilde{J}$, after lifting to S. By Lemma 3.4.3, we obtain that $f^{1/p^e} \in \left(\widetilde{J}S^{1/p^e} :_{S^{1/p^e}} \widetilde{J}^{1/p^e}\right)$, as desired.

We now prove the converse statement. Let $e \geqslant 1$ be an integer, and let $\phi \in \mathrm{Hom}_R(R^{1/p^e}, R)$. By Theorem 3.4.4 there exists $f^{1/p^e} \in (IS^{1/p^e} :_{S^{1/p^e}} I^{1/p^e})$ such that $\phi = \mathrm{Tr}_e(f^{1/p^e} \cdot -)$. By assumption, $f^{1/p^e}\widetilde{J}^{1/p^e} \subseteq JS^{1/p^e}$, therefore $\mathrm{Tr}_e(f^{1/p^e}\widetilde{J}^{1/p^e}) \subseteq \mathrm{Tr}_e(\widetilde{J}S^{1/p^e}) = \widetilde{J}\,\mathrm{Tr}_e(S^{1/p^e}) \subseteq \widetilde{J}$. This implies that $\phi(J^{1/p^e}) \subseteq J$, hence J is a compatible ideal. $\square$

Chapter 4

F-thresholds of graded rings

The results contained in this Chapter have been obtained in joint work with Luis Núñez-Betancourt, and they appear in [17]. One of the main goals is to study the following conjecture, made by Hirose, Watanabe and Yoshida in [37]:

Conjecture 4.0.10. *Let $(R, \mathfrak{m}, k)$ be a standard graded strongly F-regular ring. Let* $\operatorname{fpt}(R)$ *be the F-pure threshold, let $c(R)$ be the diagonal F-threshold and let $a(R)$ be the a-invariant of R. Then*

(i) $\operatorname{fpt}(R) \leqslant -a(R) \leqslant c(R)$.

(ii) R is Gorenstein if and only if $\operatorname{fpt}(R) = -a(R)$.

We will show that a more general version of *(i)* holds even for rings that are just F-pure. In addition, if R is Gorenstein F-pure, then $\operatorname{fpt}(R) = -a(R)$, but the converse is not true. However, at the best of our knowledge, the "if" direction of *(ii)* remains open for strongly F-regular rings.

4.1 F-thresholds: definitions and basic properties

Definition 4.1.1. *[66] Let R be a Noetherian ring, containing $\mathbb{F}_p$, which is F-finite and F-pure, and let $I \subseteq R$ be an ideal. For a real number $\lambda \geqslant 0$, we say that (R, I^λ) is F-pure if for every $e \gg 0$, there exists an element $f_e \in I^{\lfloor (p^e-1)\lambda \rfloor}$ such that the inclusion of R-modules $f_e^{1/p^e} R \subseteq R^{1/p^e}$ splits. Here, $\lfloor \alpha \rfloor$ denotes the largest integer that is less than or equal to α.*

Remark 4.1.2. Note that $(R, I^0) = (R, R)$ being F-pure simply means that R is F-pure, according to Definition 3.2.1. Therefore, if R is F-pure, then the pair (R, I^λ) is F-pure at least for $\lambda = 0$.

Definition 4.1.3. *[66] Let $(R, \mathfrak{m}, k)$ be Noetherian ring, containing $\mathbb{F}_p$, which is F-finite and F-pure, and let $I \subseteq R$ be a homogeneous ideal. The F-pure threshold of I is*

$$\operatorname{fpt}(I) = \sup\{\lambda \in \mathbb{R}_{\geqslant 0} \mid (R, I^\lambda) \textit{ is } F\textit{-pure}\}.$$

When $(R, \mathfrak{m}, k)$ is either local or standard graded, and $I = \mathfrak{m}$, we denote the F-pure threshold by $\operatorname{fpt}(R)$.

We are now aiming at a different way of characterizing the F-pure threshold of an ideal. We start with some auxiliary notions.

Definition 4.1.4. *[2] Let $(R, \mathfrak{m}, k)$ be a standard graded k-algebra which is F-finite and F-pure. For all integers $e \geqslant 1$ consider the ideals*

$$I_e(R) := \{r \in R \mid \varphi(r^{1/\mathfrak{p}^e}) \in \mathfrak{m} \textit{ for every } \varphi \in \operatorname{Hom}_R(R^{1/p^e}, R)\}.$$

In addition, we define $\mathcal{P}(R) := \bigcap_e I_e(R)$. *We will show that* $\mathcal{P}(R)$ *is a homogeneous prime ideal, called the splitting prime of* R. *We also define the splitting dimension of* R *to be* $\operatorname{sdim}(R) := \dim(R/\mathcal{P}(R))$.

Remark 4.1.5. For an integer $e \geqslant 1$ let

$$J_e := \{r \in R \mid \varphi(r^{1/\mathfrak{p}^e}) \in \mathfrak{m} \text{ for every homogeneous } \varphi \in \operatorname{Hom}_R(R^{1/p^e}, R)\}.$$

Then clearly $I_e \subseteq J_e$. On the other hand, for $r \in J_e$ and any $\varphi \in \operatorname{Hom}_R(R^{1/p^e}, R)$, we have that $\varphi = \varphi_1 + \ldots + \varphi_t$ is a sum of graded maps in $\operatorname{Hom}_R(R^{1/p^e}, R)$ by [23, Lemma 4.2], since R is F-finite. Then $\varphi_j(r^{1/p^e}) \in \mathfrak{m}$ for all $j = 1, \ldots, t$, therefore $\varphi(r^{1/p^e}) \in \mathfrak{m}$, that is, $r \in I_e$.

Remark 4.1.6. Let $r \in R$ be a homogeneous element such that $r \notin I_e(R)$. Then there is a map $\varphi \in \operatorname{Hom}_R(R^{1/p^e}, R)$ such that $\varphi(r^{1/p^e}) = 1$. In fact, by Remark 4.1.5, that there exists $e \geqslant 1$, and a graded homomorphism $\psi \in \operatorname{Hom}_R(R^{1/p^e}, R)$, such that $\psi(r^{1/p^e}) \notin \mathfrak{m}$. Since both r^{1/p^e} and ψ are homogeneous, this implies that $\psi(r^{1/p^e}) = \lambda \in k \smallsetminus \{0\}$. We can then define a splitting as $\varphi := \lambda^{-1}\psi$.

The following proposition gives basic properties of the ideals $I_e(R)$ and of the splitting prime $\mathcal{P}(R)$ for standard graded k-algebras. We refer to [2, Theorem 3.3, and Theorem 4.7] and [61, Remark 4.4] for the analogous statements for local rings.

Proposition 4.1.7. *Let* $(R, \mathfrak{m}, k)$ *be an* F*-finite* F*-pure standard graded* k*-algebra. Then*

(i) $I_e(R)$ *and* $\mathcal{P}(R)$ *are homogeneous ideals.*

(ii) $\mathcal{P}(R)$ *is a prime ideal.*

(iii) $\mathcal{P}(R)$ *is the largest compatible ideal of* R.

(iv) $R/\mathcal{P}(R)$ *is strongly* F*-regular.*

Proof. (i) Let $e \geqslant 1$. If $r \in I_e(R)$, then $\varphi(r^{1/p^e}) \in \mathfrak{m}$ for all homogeneous $\varphi \in \operatorname{Hom}_R(R^{1/p^e}, R)$. Let $r = r_0 + r_1 + \ldots + r_t \in I_e(R)$, with $r_i \in R$ of degree d_i. We want to show that, for all $\varphi \in \operatorname{Hom}_R(R^{1/p^e}, R)$, we have $\varphi(r_i^{1/p^e}) \in \mathfrak{m}$ for all i. Assume that $\varphi \in \operatorname{Hom}_R(R^{1/p^e}, R)$ is homogeneous of degree k. Then

$$\varphi(r^{1/p^e}) = \varphi(r_0^{1/p^e}) + \ldots + \varphi(r_t^{1/p^e}) \in \mathfrak{m},$$

and each $\varphi(r_i^{1/p^e})$ now has degree $d_i + k$. Since $\mathfrak{m}$ is homogeneous, we get $\varphi(r_i^{1/p^e}) \in \mathfrak{m}$, showing that $r_i \in I_e(R)$ for all $i = 1, \ldots, t$. In addition, $\mathcal{P}(R)$ is homogeneous since it is an intersection of homogeneous ideals.

(ii) Let $x, y \in R$ be elements not inside $\mathcal{P}(R)$. Since $\mathcal{P}(R)$ is homogeneous, we can assume that x and y are homogeneous elements. Then, there exist $e, e' \geqslant 1$ and maps $\varphi : R^{1/p^e} \to R$ and $\psi : R^{1/p^{e'}} \to R$ such that $\varphi(x^{1/p^e}) = 1$ and $\psi(y^{1/p^{e'}}) = 1$. Define $\psi^{1/p^e} : R^{1/p^{e+e'}} \to R^{1/p^e}$ as

$$\psi^{1/p^e}(r^{1/p^{e+e'}}) := \left(\psi(r^{1/p^{e'}})\right)^{1/p^e} \in R^{1/p^e}$$

for all $r^{1/p^{e+e'}} \in R^{1/p^{e+e'}}$. Note that ψ^{1/p^e} is an R^{1/p^e}-linear map, because ψ is R-linear. Consider the R-linear map $\varphi \circ \psi^{1/p^e} : R^{1/p^{e+e'}} \to R$. Note that

$$\varphi \circ \psi^{1/p^e}\left(\left(x^{p^{e'}} y\right)^{1/p^{e+e'}}\right) = \varphi\left(x^{1/p^e}\left(\psi(y^{1/p^{e'}})\right)^{1/p^e}\right) = \varphi(x^{1/p^e}) = 1.$$

Therefore, $x^{p^{e'}}y \notin I_{e+e'}(R)$, and in particular $xy \notin I_{e+e'}(R)$. Finally, this implies that $xy \notin \mathcal{P}(R) = \bigcap_{e \geqslant 1} I_e(R)$.

(iii) First, we show that $\mathcal{P}(R)$ is compatible. Let $x \in \mathcal{P}(R)$, so that for all $e \geqslant 1$ and all homogeneous $\varphi \in \operatorname{Hom}_R(R^{1/p^e}, R)$ we have $\varphi(x^{1/p^e}) =: y \in \mathfrak{m}$. Suppose that $y \notin \mathcal{P}(R)$. There exists $e' \geqslant 1$ and $\psi \in \operatorname{Hom}_R(R^{1/p^{e'}}, R)$ such that $\psi(y^{1/p^{e'}}) = 1$. Consider $\psi \circ \varphi^{1/p^{e'}}$. Then we have $\psi \circ \varphi^{1/p^{e'}}(x^{1/p^{e+e'}}) = \psi(y^{1/p^{e'}}) = 1$, which implies that $x \notin I_{e+e'}$. Therefore, $x \notin \mathcal{P}(R)$, which is a contradiction. We then conclude that $y \in \mathcal{P}(R)$, and thus $\mathcal{P}(R)$ is compatible. Now suppose that $J \subseteq R$ is a homogeneous compatible ideal, and let $z \in J$. Then, for all $e \geqslant 1$ and all homogeneous $\varphi \in \operatorname{Hom}_R(R^{1/p^e}, R)$, we get $\varphi(z^{1/p^e}) \in J \subseteq \mathfrak{m}$, so that $z \in \mathcal{P}(R)$.

(iv) Let $T := R_{\mathfrak{m}}$. By [49, Lemma 4.2], we have that $R/\mathcal{P}(R)$ is strongly F-regular if and only $(R/\mathcal{P}(R))_{\mathfrak{m}} \cong T/\mathcal{P}(R)_{\mathfrak{m}}$ is strongly F-regular. Let $x \in T/\mathcal{P}(R)_{\mathfrak{m}}$ be a non-zero element, and let x denote also a lift of such an element to T. Then $wx \in R$ for some integer $w \in R \smallsetminus \mathfrak{m}$, and $wx \notin \mathcal{P}(R)$, otherwise $x \in \mathcal{P}(R)_{\mathfrak{m}}$. Then, there exists $e \geqslant 1$ and $\varphi \in \operatorname{Hom}_R(R^{1/p^e}, R)$ such that $\varphi((wx)^{1/p^e}) \notin \mathfrak{m}$. Since $\mathcal{P}(R)$ is compatible, this gives a map $\psi \in \operatorname{Hom}_{R/\mathcal{P}(R)}((R/\mathcal{P}(R))^{1/p^e}, R/\mathcal{P}(R))$, such that $\psi((wx)^{1/p^e}) \notin \mathfrak{m}/\mathcal{P}(R)$. After localizing at $\mathfrak{m}$, we then obtain a map $\psi_{\mathfrak{m}} : (T/\mathcal{P}(R)_{\mathfrak{m}})^{1/p^e} \to T/\mathcal{P}(R)_{\mathfrak{m}}$ such that $\psi_{\mathfrak{m}}((wx)^{1/p^e})$ is a unit in $T/\mathcal{P}(R)_{\mathfrak{m}}$. It follows that $(wx)^{1/p^e}T/\mathcal{P}(R)_{\mathfrak{m}} \subseteq (T/\mathcal{P}(R)_{\mathfrak{m}})^{1/p^e}$ splits, hence so does $x^{1/p^e}T/cP(R) \subseteq (T/\mathcal{P}(R)_{\mathfrak{m}})^{1/p^e}$, since $w \notin \mathfrak{m}$. Then $T/\mathcal{P}(R)_{\mathfrak{m}}$ is strongly F-regular, and so is $R/\mathcal{P}(R)$.

□

We now make a crucial definition. The purpose is to measure how deep one can choose an element f inside an ideal, giving a splitting $f^{1/p^e}R \subseteq R^{1/p^e}$ for some e.

Definition 4.1.8. *Let $(R, \mathfrak{m}, k)$ be an F-finite F-pure standard graded k-algebra. Let $J \subseteq R$ be a homogeneous ideal. Then, we define*

$$b_J(p^e) = \max\{r \in \mathbb{N} \mid J^r \not\subseteq I_e(R)\}.$$

Remark 4.1.9. Note that, by definition, for all $e \geqslant 1$ there exists an element $g \in J^{b_J(p^e)} \smallsetminus I_e$, that we can choose to be homogeneous. Then, by Remark 4.1.6, the inclusion $g^{1/p^e}R \subseteq R^{1/p^e}$ splits. As a consequence, taking $J = \mathfrak{m}$, we have that for all $e \geqslant 1$ we can find a homogeneous element g^{1/p^e} of degree $\dfrac{b_{\mathfrak{m}}(p^e)}{p^e}$ such that the inclusion $g^{1/p^e}R \subseteq R^{1/p^e}$ splits.

Lemma 4.1.10. *Let $(R, \mathfrak{m}, k)$ be a standard graded k-algebra which is F-finite and F-pure. Let $J \subseteq R$ be a homogeneous ideal. Then, $p \cdot b_J(p^e) \leqslant b_J(p^{e+1})$.*

Proof. Let $f \in J^{b_J(p^e)} \smallsetminus I_e(R)$ be a homogeneous element. Then, $f^{1/p^e}R \subseteq R^{1/p^e}$ splits as map of R-modules. In particular, there exists $\varphi : R^{1/p^e} \to R$ such that $\varphi(f^{1/p^e}) = 1$. Since R is F-pure, the inclusion $R^{1/p^e} \subseteq R^{1/p^{e+1}}$ splits. Therefore, there exists $\psi : R^{1/p^{e+1}} \to R^{1/p^e}$ such that $\psi(1) = 1$. Consider the map $\theta := \varphi \circ \psi : R^{1/p^{e+1}} \to R$. Then note that $\theta(f^{p/p^{e+1}}) = \varphi(\psi(f^{1/p^e})) = \varphi(f^{1/p^e}) = 1$. Therefore, we have that $f^p \notin I_{e+1}(R)$. On the other hand, $f^p \in J^{p \cdot b_J(p^e)}$, therefore $p \cdot b_J(p^e) \leqslant b_J(p^{e+1})$. □

We now present a characterization of the F-pure threshold that may be known to experts (see [30, Key Lemma] for principal ideals). However, we were not able to find it in the literature in the generality we need. This characterization will be crucial in order to attack Conjecture 4.0.10.

Proposition 4.1.11. *Let $(R, \mathfrak{m}, k)$ be a standard graded k-algebra which is F-finite and F-pure. Let $J \subseteq R$ be a homogeneous ideal. Then*

$$\operatorname{fpt}(J) = \lim_{e\to\infty} \frac{b_J(p^e)}{p^e}.$$

Proof. For all integers $e \geqslant 1$ there exists a homogeneous $f \in J^{b_J(p^e)} \setminus I_e(R)$, by definition of $b_J(p^e)$. Then, the inclusion $f^{1/p^e}R \to R^{1/p^e}$ splits by Remark 4.1.6. Therefore, we have $\frac{b_J(p^e)}{p^e} \in \{\lambda \in \mathbb{R}_{\geqslant 0} \mid (R, J^\lambda) \text{ is } F\text{-pure}\}$. Hence, for all $e \geqslant 1$, we have $\frac{b_J(p^e)}{p^e} \leqslant \operatorname{fpt}(J)$. In particular, $\left\{\frac{b_J(p^e)}{p^e}\right\}$ is a bounded sequence and thus, by Lemma 4.1.10 it converges to a limit, because it is non-decreasing. In addition, we conclude that $\lim\limits_{e\to\infty} \frac{b_J(p^e)}{p^e} \leqslant \operatorname{fpt}(J)$.

For the converse inequality, let $\sigma \in \{\lambda \in \mathbb{R}_{\geqslant 0} \mid (R, J^\lambda) \text{ is } F\text{-pure}\}$. For $e \gg 0$, so that $J^{\lfloor (p^e-1)\sigma \rfloor} \not\subseteq I_e(R)$, by definition of $b_J(p^e)$. Then, we obtain that $\frac{\lfloor (p^e-1)\sigma \rfloor}{p^e} \leqslant \frac{b_J(p^e)}{p^e}$, and therefore, taking limits as $e \to \infty$:

$$\sigma = \lim_{e\to\infty} \frac{\lfloor (p^e-1)\sigma \rfloor}{p^e} \leqslant \lim_{e\to\infty} \frac{b_J(p^e)}{p^e}.$$

Since this holds for all σ such that (R, J^σ) is F-pure, we obtain that $\operatorname{fpt}(J) \leqslant \lim\limits_{e\to\infty} \frac{b_J(p^e)}{p^e}$, as desired. □

Remark 4.1.12. One can make similar definitions for Noetherian local rings $(R, \mathfrak{m}, k)$. Then an analogous restatement of Proposition 4.1.11 for F-finite F-pure local rings is also true, and the proof is essentially the same, mutatis mutandis.

We end this section by defining the F-threshold of an ideal I with respect to another ideal J. It has first been introduced by Huneke, Mustaţă, Takagi and Watanabe in [42]. F-thresholds are closely related to the theories of tight closure and integral closure.

Definition 4.1.13. *Let R be Noetherian ring containing $\mathbb{F}_p$. Assume that R is F-finite, and let I, J be two ideals such that $I \subseteq \sqrt{J}$. Let $\nu_I^J(p^e) = \max\{r \in \mathbb{N} \mid I^r \not\subseteq J^{[p^e]}\}$, and define*

$$c_-^J(I) = \liminf_{e\to\infty} \frac{\nu_I^J(p^e)}{p^e} \quad \text{and} \quad c_+^J(I) = \limsup_{e\to\infty} \frac{\nu_I^J(p^e)}{p^e}.$$

If the two limits coincide, we denote the common value by $c^J(I)$ and call it the F-threshold of I with respect to J.

The previous limit exists for F-pure rings [42, Lemma 2.3]. When $(R, \mathfrak{m}, k)$ is either local or standard graded, and $I = J = \mathfrak{m}$, we denote $c^{\mathfrak{m}}(\mathfrak{m})$ by $c(R)$, and call it the diagonal F-threshold of R.

4.2 F-thresholds and a-invariants

In this section, we start the proof of Conjecture 4.0.10, in the more general setting of F-pure standard graded k-algebras. We start with the relation between the F-pure threshold and the a-invariants, as conjectured in the first inequality of 4.0.10 *(i)*.

Theorem 4.2.1. *Let $(R, \mathfrak{m}, k)$ be a standard graded k-algebra which is F-finite and F-pure. Then $\operatorname{fpt}(R) \leqslant -a_i(R)$ for every $i \in \mathbb{N}$.*

Proof. If $H^i_{\mathfrak{m}}(R) = 0$ there is nothing to prove, since $a_i(R) = -\infty$. Let $i \in \mathbb{N}$ be such that $H^i_{\mathfrak{m}}(R) \neq 0$. For all integers $e \geqslant 1$, pick a homogeneous element $f_e \in \mathfrak{m}^{b_{\mathfrak{m}}(p^e)} \smallsetminus I_e(R)$. The map

$$\begin{array}{rcl} R & \hookrightarrow & R^{1/p^e} \\ 1 & \longmapsto & f_e^{1/p^e} \end{array}$$

splits, and the inclusion is homogeneous of degree $\dfrac{b_{\mathfrak{m}}(p^e)}{p^e}$. Applying the i-th local cohomology functor, we get a homogeneous split inclusion $H^i_{\mathfrak{m}}(R) \hookrightarrow H^i_{\mathfrak{m}}(R^{1/p^e})$, which is still of degree $\dfrac{b_{\mathfrak{m}}(p^e)}{p^e}$. Let $v \in H^i_{\mathfrak{m}}(R)_{a_i(R)}$ be an element in the top graded part of $H^i_{\mathfrak{m}}(R)$, which has degree $a_i(R)$. Under the inclusion above, this maps to a nonzero element of degree $a_i(R)+\dfrac{b_{\mathfrak{m}}(p^e)}{p^e}$ in $H^i_{\mathfrak{m}}(R^{1/p^e})$. Therefore, by definition of the i-th a-invariant of R^{1/p^e}, we have that $a_i(R) + \dfrac{b_{\mathfrak{m}}(p^e)}{p^e} \leqslant a_i(R^{1/p^e}) = \dfrac{a_i(R)}{p^e}$. Taking limits as $e \to \infty$ we obtain the desired claim:

$$a_i(R) + \operatorname{fpt}(R) = a_i(R) + \lim_{e\to\infty} \frac{b_{\mathfrak{m}}(p^e)}{p^e} \leqslant \lim_{e\to\infty} \frac{a_i(R)}{p^e} = 0.$$

□

Before stating a consequence of Theorem 4.2.1, we recall that we defined the splitting dimension of R as $\operatorname{sdim}(R) = \dim(R/\mathcal{P}(R))$.

Corollary 4.2.2. *Let $(R, \mathfrak{m}, k)$ be a standard graded k-algebra which is F-finite and F-pure. If $a_i(R) = 0$ for some i, then* $\operatorname{sdim}(R) = 0$.

Proof. If $a_i(R) = 0$ for some i, we have that $\operatorname{fpt}(R) = 0$ by Theorem 4.2.1. Then, we have that $b_e = 0$ for every $e \in \mathbb{N}$ by Lemma 4.1.10 and Proposition 4.1.11. As a consequence, $\mathfrak{m} \subseteq I_e$ for every $e \in \mathbb{N}$. Since $I_e(R) \subseteq \mathfrak{m}$ holds true because R is F-pure, we have that $\mathfrak{m} = I_e(R)$ for every $e \in \mathbb{N}$. Hence, $\mathcal{P}(R) = \mathfrak{m}$, and $\operatorname{sdim}(R) = 0$. □

We now want to study how the F-pure threshold varies going modulo compatible ideals. The following result allows us to keep track the degrees of the splittings via some kind of duality.

Lemma 4.2.3. *Let $S = k[x_1, \ldots, x_n]$ be a polynomial ring over an F-finite field k. Let $\mathfrak{n} = (x_1, \ldots, x_n)$ denote the maximal homogeneous ideal. Let $I \subseteq S$ be a homogeneous ideal such that $R := S/I$ is an F-pure ring, and let $\mathfrak{m} = \mathfrak{n}R$. Then,*

$$\min\left\{ t \in \frac{\mathbb{N}}{p^e} \ \middle| \ \left[\frac{(IS^{1/p^e} :_{S^{1/p^e}} I^{1/p^e}) + \mathfrak{n}S^{1/p^e}}{\mathfrak{n}S^{1/p^e}}\right]_t \neq 0 \right\} = \frac{n(p^e-1) - b_{\mathfrak{m}}(p^e)}{p^e}$$

Proof. Let $e \geqslant 1$ be an integer. By Remark 4.1.9 there exists a homogeneous element $g^{1/p^e} \in R^{1/p^e}$ of degree $b_{\mathfrak{m}}(p^e)/p^e$ such that $g^{1/p^e}R \subseteq R^{1/p^e}$ splits. By Lemma 3.4.5, there exists a homogeneous element $f^{1/p^e} \in (IS^{1/p^e} :_{S^{1/p^e}} I^{1/p^e})$ of degree $\dfrac{n(p^e-1) - b_{\mathfrak{m}}(p^e)}{p^e}$ such that $f^{1/p^e}g^{1/p^e} \notin \mathfrak{n}S^{1/p^e}$. A fortiori, we have that $f^{1/p^e} \notin \mathfrak{n}S^{1/p^e}$. Therefore, the right-hand-side is greater than or equal to the left-hand-side of the equation. If there were a homogeneous element $f^{1/p^e} \in (IS^{1/p^e} :_{S^{1/p^e}} I^{1/p^e})$ of degree $s = \dfrac{j}{p^e} < \dfrac{n(p^e-1) - b_{\mathfrak{m}}(p^e)}{p^e}$ that is not inside $\mathfrak{n}S^{1/p^e}$, then by Lemma 3.4.5 we would be able to find a homogeneous element $h^{1/p^e} \in R^{1/p^e}$ of degree $\dfrac{n(p^e-1) - j}{p^e} > \dfrac{b_{\mathfrak{m}}(p^e)}{p^e}$ such that the inclusion $h^{1/p^e}R \subseteq R^{1/p^e}$ splits. But this contradicts the definition of $b_{\mathfrak{m}}(p^e)$, given Remark 4.1.9. □

Theorem 4.2.4. *Let $(R, \mathfrak{m}, k)$ be a standard graded k-algebra which is F-finite and F-pure, and let $J \subseteq R$ be a homogeneous compatible ideal. Then, we have* $\operatorname{fpt}(R) \leqslant \operatorname{fpt}(R/J)$. *In particular,* $\operatorname{fpt}(R) \leqslant \operatorname{fpt}(R/\mathcal{P}(R)) \leqslant \operatorname{sdim}(R)$.

Proof. Let $S = k[x_1, \ldots, x_n]$ be a polynomial ring such that there exists a surjection $S \to R$, and let $\mathfrak{n} = (x_1, \ldots, x_n)$, so that $\mathfrak{m} = \mathfrak{n}R$. Let I denote the kernel of the surjection. Let $\widetilde{J} \subseteq S$ be the pullback of J. We have that $(IS^{1/p^e} :_{S^{1/p^e}} I^{1/p^e}) \subseteq (\widetilde{J}S^{1/p^e} :_{S^{1/p^e}} \widetilde{J}^{1/p^e})$ for every $e \in \mathbb{N}$ by Proposition 3.4.9. Then,

$$\alpha := \min\left\{ t \in \frac{\mathbb{N}}{p^e} \;\middle|\; \left[\frac{(\widetilde{J}S^{1/p^e} :_{S^{1/p^e}} \widetilde{J}^{1/p^e}) + \mathfrak{n}S^{1/p^e}}{\mathfrak{n}S^{1/p^e}}\right]_t \neq 0 \right\} \leqslant$$
$$\leqslant \min\left\{ t \in \frac{\mathbb{N}}{p^e} \;\middle|\; \left[\frac{(IS^{1/p^e} :_{S^{1/p^e}} I^{1/p^e}) + \mathfrak{n}S^{1/p^e}}{\mathfrak{n}S^{1/p^e}}\right]_t \neq 0 \right\} =: \beta.$$

As a consequence, we get

$$b_{\mathfrak{m}}(p^e) = \frac{n(p^e-1)}{p^e} - \beta \leqslant \frac{n(p^e-1)}{p^e} - \alpha = b_{\mathfrak{m}/J}(p^e)$$

by Lemma 4.2.3. Then, $\operatorname{fpt}(R) \leqslant \operatorname{fpt}(R/J)$. The last claim follows from the fact that the splitting prime is compatible by Proposition 4.1.7 *(iii)*, and $\operatorname{fpt}(R/\mathcal{P}(R)) \leqslant \operatorname{sdim}(R)$ by [66, Proposition 2.6(1)]. □

We finally focus on the diagonal F-threshold $c(R)$. Note that

$$\max\left\{ t \in \frac{1}{p^e} \cdot \mathbb{Z} \;\middle|\; \left[R^{1/p^e}/\mathfrak{m}R^{1/p^e}\right]_t \neq 0 \right\} = \frac{\nu_{\mathfrak{m}}^{\mathfrak{m}}(p^e)}{p^e}.$$

Before proving the second inequality in Conjecture 4.0.10 *(i)*, we set some more notation. Observe that the inclusion $R \subseteq R^{1/p^e}$ endows R with a $\dfrac{\mathbb{N}}{p^e}$-grading, that is compatible with the original $\mathbb{N}$-grading on R. In fact, an element $r \in R$ of degree $n \in \mathbb{N}$ can be viewed as an element of degree $n = \dfrac{p^e n}{p^e} \in \dfrac{\mathbb{N}}{p^e}$. Then, for $\dfrac{i}{p^e} \in \dfrac{\mathbb{N}}{p^e}$, we set $R(i/p^e)$ to the same as R, but with $\dfrac{\mathbb{Z}}{p^e}$-grading given by $R(i/p^e)_{j/p^e} = R_{i/p^e+j/p^e}$. Note that, since R is $\mathbb{N}$-graded, if $i+j$ is not a positive multiple of p^e, then $R(i/p^e)_{j/p^e} = 0$.

Theorem 4.2.5. *Let R be an F-finite standard graded k-algebra, and let $d = \dim(R)$. Then, $-a_d(R) \leqslant c^{\mathfrak{m}}_{-}(\mathfrak{m})$. Furthermore, if R is F-pure, then $-a_i(R) \leqslant c(R)$ for every i such that $H^i_{\mathfrak{m}}(R) \neq 0$.*

Proof. We fix $i \in \mathbb{N}$ such that $H^i_{\mathfrak{m}}(R) \neq 0$. Let $v_1, \ldots, v_r$ be a minimal system of homogeneous generators of R^{1/p^e} as an R-module, with degrees $\gamma_1, \ldots, \gamma_r \in \dfrac{\mathbb{N}}{p^e}$. As noted before, we can view R as a $\dfrac{\mathbb{N}}{p^e}$-graded module. We then have a degree zero surjective map

$$\bigoplus_{j=0}^{r} R(-\gamma_j) \xrightarrow{\pi} R^{1/p^e} \longrightarrow 0$$

where $\pi_j : R(-\gamma_j) \to R^{1/p^e}$ maps 1 to v_j. This induces a degree zero homomorphism

$$\bigoplus_{i=0}^{j} H^i_{\mathfrak{m}}(R(-\gamma_j)) \xrightarrow{\varphi} H^i_{\mathfrak{m}}(R^{1/p^e}).$$

If $i = d$, then φ is surjective by right exactness of $H^d_{\mathfrak{m}}(-)$. We now prove that φ is also surjective for $i \neq d$, when R is F-pure. In this case, the natural inclusion $R \subseteq R^{1/p^e}$ induces an inclusion $H^i_{\mathfrak{m}}(R) \subseteq H^i_{\mathfrak{m}}(R^{1/p^e})$. We have that the map ψ_i : $H^i_{\mathfrak{m}}(R) \otimes_R R^{1/p^e} \to H^i_{\mathfrak{m}}(R^{1/p^e})$ is surjective by Proposition 3.3.6. By right exactness of tensor products, we also have that

$$H^i_{\mathfrak{m}}(R) \otimes_R \left(\bigoplus_{j=0}^{r} R(-\gamma_j) \right) \xrightarrow{1 \otimes \pi} H^i_{\mathfrak{m}}(R) \otimes_R R^{1/p^e}$$

is surjective. Thus, φ is surjective, because $\varphi = \theta \circ (1 \otimes \pi)$.

We have now that φ is surjective under the given assumptions, for all $i \in \mathbb{Z}$ such that $H^i_{\mathfrak{m}}(R) \neq 0$. Since $\dfrac{\nu^{\mathfrak{m}}_{\mathfrak{m}}(p^e)}{p^e} = \max\{\gamma_1, \dots, \gamma_j\}$, since φ is surjective we have that

$$a_i(R^{1/p^e}) = \frac{a_i(R)}{p^e} \leqslant \max\{a_d(R(-\gamma_i)) \mid i = 0, \dots, j\} = a_i(R) + \frac{\nu_{\mathfrak{m}}(p^e)}{p^e}.$$

Taking limits as $e \to \infty$, we obtain that

$$0 = \liminf_{e\to\infty} \frac{a_i(R)}{p^e} \leqslant \liminf_{e\to\infty} \left(a_i(R) + \frac{\nu_{\mathfrak{m}}(p^e)}{p^e} \right) = a_i(R) + c^{\mathfrak{m}}_{-}(\mathfrak{m}).$$

Finally, we note that if R is F-pure $c^{\mathfrak{m}}_{-}(\mathfrak{m}) = c(R)$, and this concludes the proof. □

4.3 F-thresholds of Gorenstein rings

Suppose that $(R, \mathfrak{m}, k)$ is an F-finite standard graded Gorenstein k-algebra. Let $S = k[x_1, \dots, x_n]$, and let $I \subseteq S$ be a homogeneous ideal such that $R \cong S/I$ as graded rings. Since $\operatorname{Hom}_R(R^{1/p}, R)$ is a cyclic $R^{1/p}$-module, we have that for all integers $e \geqslant 1$ there exist homogeneous polynomials $f_e^{1/p^e} \in S^{1/p^e}$ such that $(IS^{1/p^e} :_{S^{1/p^e}} I^{1/p^e}) = f_e^{1/p^e} S^{1/p^e} + IS^{1/p^e}$, by Theorem 3.4.4. In fact, even more is true: if $(IS^{1/p} :_{S^{1/p}} I^{1/p}) = f^{1/p}S^{1/p} + IS^{1/p}$ for some $f^{1/p} \in S^{1/p}$, then we can choose $f_e := f^{1+p+\dots+p^{e-1}}$, i.e, for all $e \geqslant 1$ we have $(IS^{1/p^e} :_{S^{1/p^e}} I^{1/p^e}) = f^{(1+p+\dots+p^{e-1})/p^e} S^{1/p^e} + IS^{1/p^e}$.

Remark 4.3.1. When $R = S/I$ is F-pure, we have $(IS^{1/p^e} :_{S^{1/p^e}} I^{1/p^e}) \not\subseteq \mathfrak{n}S^{1/p^e}$ by Fedder's criterion (Corollary 3.4.6). If R is Gorenstein, we have $(IS^{1/p^e} :_{S^{1/p^e}} I^{1/p^e}) =$

$f^{(1+p+\cdots+p^{e-1})/p^e}S^{1/p^e} + IS^{1/p^e}$ for some homogeneous element $f \in S$. In particular,

$$\min\left\{t \in \frac{\mathbb{N}}{p^e} \;\middle|\; \left[\frac{(IS^{1/p^e} :_{S^{1/p^e}} I^{1/p^e}) + \mathfrak{n}S^{1/p^e}}{\mathfrak{n}S^{1/p^e}}\right]_t \neq 0\right\} = \frac{\deg(f)(1+p+\ldots+p^{e-1})}{p^e}$$

We now prove the "only if" direction of Conjecture 4.0.10 *(ii)*, only assuming that the ring is F-pure.

Theorem 4.3.2. *Let $(R, \mathfrak{m}, k)$ be a Gorenstein standard graded k-algebra which is F-finite and F-pure, and let $d = \dim(R)$. Then we have $\operatorname{fpt}(R) = -a_d(R)$.*

Proof. Let $S = k[x_1, \ldots, x_n]$ be a polynomial ring, and let $I \subseteq S$ be a homogeneous ideal such that $R \cong S/I$ as graded rings. Let $\mathfrak{n} = (x_1, \ldots, x_n)$, so that $\mathfrak{m} = \mathfrak{n}R$. Let $a = a_d(R)$. Consider the natural map $S^{1/p}/IS^{1/p} \to S^{1/p}/I^{1/p}$ induced by the inclusion $IS^{1/p} \subseteq I^{1/p}$. Then such a map extends to a map of complexes $\psi_\bullet$ from a minimal free resolution of $S^{1/p}/IS^{1/p}$ to a minimal free resolution of $S^{1/p}/I^{1/p}$ as $S^{1/p}$-modules. Furthermore, such a map $\psi_\bullet$ can be chosen graded of degree zero. Since $a_d(R^{1/p}) = \dfrac{a}{p}$, we have that the last homomorphism in the map of complexes is $S(-n-a) \to S((-n-a)/p)$, and it is then given by multiplication by a homogeneous element $f^{1/p} \in S^{1/p}$. Furthermore, we have that $IS^{1/p} :_{S^{1/p}} I^{1/p} = f^{1/p}S^{1/p} + IS^{1/p}$ [68, Lemma 1]. Since $\psi_\bullet$ is homogeneous of degree zero, we have that $\deg(f^{1/p}) = \dfrac{(p-1)(n+a)}{p}$. In addition, recall that, for all $e \geqslant 1$, we have $(IS^{1/p^e} :_{S^{1/p^e}} I^{1/p^e}) =$

$f^{(1+p+\ldots+p^{e-1})/p^e}S^{1/p^e} + IS^{1/p^e}$. By Remark 4.3.1 and Lemma 4.2.3, we obtain that

$$\begin{aligned}
\operatorname{fpt}(R) &= \lim_{e\to\infty} \frac{n(p^e-1) - (\deg(f)\cdot(1+p+\ldots+p^{e-1}))}{p^e} \\
&= \lim_{e\to\infty} \frac{n(p^e-1)}{p^e} - \lim_{e\to\infty} \frac{\deg(f)\cdot(1+p+\ldots+p^{e-1})}{p^e} \\
&= n - \frac{\deg(f)}{p-1} \\
&= n - \frac{(p-1)(n+a)}{p-1} \\
&= -a.
\end{aligned}$$

□

We now give an example to show that an F-finite and F-pure standard graded k-algebra such that $\operatorname{fpt}(R) = -a_d(R)$ is not necessarily Gorenstein.

Example 4.3.3. Let $S = k[x, y, z]$ with k a perfect field of characteristic $p > 0$, and let $\mathfrak{n} = (x, y, z)$ be its homogeneous maximal ideal.

$$I = (xy, xz, yz) = (x, y) \cap (x, z) \cap (y, z) \subseteq S.$$

Let $R = S/I$, with maximal ideal $\mathfrak{m} = \mathfrak{n}/I$. Note that R is a one-dimensional Cohen-Macaulay F-pure ring. Consider the short exact sequence

$$0 \longrightarrow R \longrightarrow \frac{S}{(x,y)} \oplus \frac{S}{(x,z)\cap(y,z)} \longrightarrow \frac{S}{(x,y)+(x,z)\cap(y,z)} \cong k \longrightarrow 0.$$

Then, we get a long exact sequence of local cohomology modules

$$0 \longrightarrow k \longrightarrow H^1_{\mathfrak{m}}(R) \longrightarrow H^1_{\mathfrak{n}}(S/(x,y)) \oplus H^1_{\mathfrak{n}}(S/(xy,z)) \longrightarrow \ldots$$

The maps in this sequence are homogeneous of degree zero. Thus, $a_1(R) \geqslant 0$, because k injects into $H^1_{\mathfrak{m}}(R)$. On the other hand, since R is F-pure, we have that $a_1(R) \leqslant 0$; therefore, $\operatorname{fpt}(R) = a_1(R) = 0$. However, R is not Gorenstein, since the canonical module $\omega_R \cong (x, y)/(xy, xz + yz)$ has two generators.

Remark 4.3.4. The ring in Example 4.3.3 is not strongly F-regular. In fact, strongly F-regular rings are normal, and normal local rings of dimension one are regular. However, the ring above, when localized at $\mathfrak{m}$, is not even Gorenstein since the canonical module $\omega_{R_\mathfrak{m}} \cong ((x, y)/(xy, xz + yz))_\mathfrak{m}$ is again not cyclic. Therefore, Example 4.3.3 is not a counterexample to Conjecture 4.0.10 *(ii)*.

4.4 F-pure regular sequences

We now aim at an interpretation of the F-pure threshold of a standard graded Gorenstein k-algebra as the maximal length of a regular sequence that preserves F-purity. We start with an auxiliary lemma that will allow us to start the process of finding F-pure regular elements inside a ring.

Lemma 4.4.1. *Let $S = k[x_1, \ldots, x_n]$ be a polynomial ring over and F-finite field, with the standard grading, and let $\mathfrak{n} = (x_1, \ldots, x_n)$ be its irrelevant maximal ideal. Let $I \subseteq S$ be an ideal such that $R = S/I$ is an F-pure ring, and let $\mathfrak{m} = \mathfrak{n}R$. Then, $\mathcal{P}(R) = \mathfrak{m}$ if and only if $(I^{[p^e]} :_S I) \subseteq (\mathfrak{n}^{[p^e]} :_S \mathfrak{n})$ for all integers $e \geqslant 1$.*

Proof. By Proposition 4.1.7 *(iii)*, we have that $\mathcal{P}(R)$ is a compatible ideal. If $\mathcal{P}(R) =$

$\mathfrak{m}$, by Proposition 3.4.9 we obtain that $(I^{[p^e]} :_S I) \subseteq (\mathfrak{n}^{[p^e]} :_S \mathfrak{n})$, as desired. Conversely, if $(I^{[p^e]} :_S I) \subseteq (\mathfrak{n}^{[p^e]} :_S \mathfrak{n})$, then we have that $\mathfrak{m}$ is a compatible ideal, again by Proposition 3.4.9. Since R is F-pure, we have that $1 \notin I_e$ for all integers $e \geqslant 1$. In particular, $\mathcal{P}(R) = \bigcap_{e \geqslant 1} I_e \subseteq \mathfrak{m}$, because the ideals I_e are homogeneous. Since $\mathcal{P}(R)$ is the largest compatible ideal of R by Proposition 4.1.7 *(iii)*, and $\mathfrak{m}$ is maximal, we get that $\mathcal{P}(R) = \mathfrak{m}$. □

Proposition 4.4.2. *Let $S = k[x_1, \ldots, x_n]$ be a polynomial ring over an F-finite infinite field k. Let $\mathfrak{n} = (x_1, \ldots, x_n)$ denote the maximal homogeneous ideal. Let $I \subseteq S$ be a homogeneous ideal such that $R = S/I$ is an F-pure ring, and let $\mathfrak{m} = \mathfrak{n}R$. Let $f \in (IS^{1/p} :_{S^{1/p}} I^{1/p}) \smallsetminus \mathfrak{n}S^{1/p}$ be a homogeneous element. If $\deg(f^{1/p}) \leqslant \dfrac{(p-1)(n-1)}{p}$, then there exists a linear form $\ell \in S$ such that:*

1. *$\ell^{p-1} f \notin \mathfrak{n}^{[p]}$.*
2. *the class of ℓ in R does not belong to $\mathcal{P}(R)$.*
3. *ℓ is a non zero-divisor in R.*

Proof. Write $f = \sum_{|\alpha| = \deg(f)} c_\alpha x^\alpha$, with $c_\alpha \in k$. Let $\ell(y) = y_1 x_1 + \ldots + y_n x_n \in S[y_1, \ldots, y_n]$ be a generic linear form. We note that

$$\ell(y)^{p-1} = \sum_{|\theta| = p-1} g_\theta(y) x^\theta,$$

where $g_\theta(y) = \frac{(p-1)!}{\theta_1! \cdots \theta_n!} y^\theta \in k[y_1, \ldots, y_n]$. Since $f \notin \mathfrak{n}^{[p]}$, there exists $x^\beta \in \operatorname{supp}\{f\}$ such that $x^\beta \notin \mathfrak{n}^{[p]}$. Since $|\beta| \leq (p-1)(n-1)$ and $x^\beta \notin \mathfrak{n}^{[p]}$, there exists $x^\gamma \in \mathfrak{n}^{p-1}$ such that $x^\gamma x^\beta \notin \mathfrak{n}^{[p]}$ by the pigeonhole principle. Let

$$h := \sum_{\beta+\gamma=\theta+\alpha} c_\alpha g_\theta(y) \in k[y_1,\dots,y_n]$$

We note that $h \neq 0$ because $c_\beta g_\theta \neq 0$. In addition, h is the coefficient of $x^{\theta+\gamma}$ in $\ell(y)^{p-1} f$. We note that $\mathcal{P}(R) \neq \mathfrak{m}$ by Lemma 4.4.1, since $\deg(f) \leqslant (p-1)(n-1)$. Therefore, we have that $\mathcal{P}(R) \cap \mathfrak{m} \neq \mathfrak{m}$. Since k is an infinite field, we can pick a point $v \in k^n$ such that $h(v) \neq 0$ and the class of $\ell(v)$ does not belong to $\mathcal{P}(R)$. We set $\ell := \ell(v)$. By our construction, $x^{\beta+\gamma} \in \operatorname{supp}\{\ell^{p-1} f\}$ and $x^{\beta+\gamma} \notin \mathfrak{n}^{[p]}$. In addition, $\ell \notin \mathcal{P}(R)$. Since the pullback of $\mathcal{P}(R)$ to S contains every associated prime of R (see [2], for example), we have that ℓ is a non zero-divisor in R. □

Note that, for ℓ as in Proposition 4.4.2, if we set $I' := I + (\ell)$ we have that the ring S/I' is again F-pure. In fact, for f as above, we have that $\ell^{p-1} f \in (I'^{[p]} : I') \smallsetminus \mathfrak{n}^{[p]}$, and F-purity follows by Fedder's criterion 3.4.6.

As a consequence of these results, and of Theorem 4.3.2, we give an interpretation of the F-pure threshold of a standard graded Gorenstein F-pure algebra in terms of the maximal length of a regular sequence that preserves F-purity. We start by introducing the notion of F-pure regular sequence.

Definition 4.4.3. *Let $(R, \mathfrak{m}, k)$ be a standard graded k algebra which is F-finite and F-pure. We say that a homogeneous regular sequence $f_1, \dots, f_r$ is an F-pure regular sequence if $R/(f_1, \dots, f_i)$ is an F-pure ring for all $i = 1, \dots, r$.*

Lemma 4.4.4. *Let $(R, \mathfrak{m}, k)$ be a standard graded k-algebra. If f is a regular element of degree $d > 0$, then $d + a_i(R) \leqslant a_{i-1}(R/(f))$ for all $i \in \mathbb{N}$ such that $H^i_{\mathfrak{m}}(R) \neq 0$.*

Proof. Suppose that $H^i_{\mathfrak{m}}(R) \neq 0$. Consider the homogeneous short exact sequence

$$0 \longrightarrow R(-d) \xrightarrow{\ f\ } R \longrightarrow R/(f) \longrightarrow 0.$$

For all $j \in \mathbb{Z}$, this gives rise to an exact sequence of k-vector spaces

$$\ldots \longrightarrow H^{i-1}_{\mathfrak{m}}(R/(f))_j \longrightarrow H^i_{\mathfrak{m}}(R)_{j-d} \longrightarrow H^i_{\mathfrak{m}}(R)_j \longrightarrow \ldots$$

Since $d > 0$, for $j = a_i(R) + d$ we have that $H^i_{\mathfrak{m}}(R)_j = 0$. Then,

$$H^{i-1}_{\mathfrak{m}}(R/(f))_{a_i(R)+d} \longrightarrow H^i_{\mathfrak{m}}(R)_{a_i(R)} \longrightarrow 0$$

is a surjection. We note that $H^i_{\mathfrak{m}}(R)_{a_i(R)} \neq 0$, which yields $H^{i-1}_{\mathfrak{m}}(R/(f))_{a_i(R)+d} \neq 0$, and hence $a_{i-1}(R/(f)) \geqslant a_i(R) + d$. □

Corollary 4.4.5. *Let $(R, \mathfrak{m}, k)$ be a standard graded k-algebra which is F-finite and F-pure. If $f_1, \ldots, f_r$ is a homogeneous F-pure regular sequence of degrees $d_1, \ldots, d_r$, then $\sum_{j=1}^{r} d_j \leqslant \min\{-a_i(R) \mid i \in \mathbb{N}\}$.*

Proof. We proceed by induction on $r \geqslant 1$. Assume that $r = 1$. If $H^i_{\mathfrak{m}}(R) = 0$, we have that $d_1 \leqslant -a_i(R) = \infty$, therefore there is nothing to prove in this case. If $H^i_{\mathfrak{m}}(R) \neq 0$, by Lemma 4.4.4 we have that $a_i(R) + d_1 \leqslant a_{i-1}(R/(f_1))$. Since $R/(f_1)$ is F-pure, it follows from Proposition 3.3.7 that $a_{i-1}(R/(f)) \leqslant 0$, and hence $d_1 \leqslant -a_i(R)$. Thus, $d_1 \leqslant -a_i(R)$ for all $i \in \mathbb{N}$, that is, $d_1 \leqslant \min\{-a_i(R) \mid i \in \mathbb{N}\}$.

This concludes the proof of the base case. For $r > 1$, if $H^i_{\mathfrak{m}}(R) = 0$ we have that $\sum_{i=1}^r d_i \leqslant -a_i(R) = \infty$ and, again, there is nothing to prove in this case. Assume that $H^i_{\mathfrak{m}}(R) \neq 0$. By induction, we get that $\sum_{j=2}^r d_j \leqslant -a_s(R/(f_1))$ for all $s \in \mathbb{N}$. In particular, we have that $\sum_{j=2}^r d_j \leqslant -a_{i-1}(R/(f_1))$. By Lemma 4.4.4, we have that $-a_{i-1}(R/(f_1)) \geqslant -a_i(R) - d_1$. Combining the two inequalities, and rearranging the terms in the sum, we obtain $\sum_{j=1}^r d_i \leqslant -a_i(R)$. Therefore, we obtain $\sum_{j=1}^r d_j \leqslant \min\{-a_i(R) \mid i \in \mathbb{N}\}$ □

We are finally in a position to state and prove the main result of this section.

Theorem 4.4.6. *Let $(R, \mathfrak{m}, k)$ be a Gorenstein standard graded k-algebra which is F-finite and F-pure, and let $d = \dim(R)$. If $f_1, \ldots, f_r$ is an F-pure regular sequence, then $r \leqslant \operatorname{fpt}(R)$. Furthermore, if k is infinite, then there exists an F-pure regular sequence consisting of $\operatorname{fpt}(R)$ linear forms.*

Proof. By Theorem 4.3.2, we have that $\operatorname{fpt}(R) = -a_d(R)$. The first claim follows from Corollary 4.4.5. For the second claim, let $S = k[x_1, \ldots, x_n]$ be a polynomial ring and let $I \subseteq S$ be a homogeneous ideal such that $R \cong S/I$ as graded rings. We proceed by induction on $\operatorname{fpt}(R)$. The case $\operatorname{fpt}(R) = 0$ is trivial. We now assume $\operatorname{fpt}(R) > 0$. From the proof of Theorem 4.3.2, we have that $(IS^{1/p} :_{S^{1/p}} I^{1/p}) = f^{1/p}S^{1/p} + IS^{1/p}$ for a homogeneous element $f^{1/p} \in S^{1/p} \smallsetminus \mathfrak{n}S^{1/p}$, of degree $\deg(f^{1/p}) \leqslant \dfrac{(p-1)(n + a_d(R))}{p}$. Since $a_d(R) = -\operatorname{fpt}(R) < 0$ by assumption, there exists a linear non zero-divisor

$\ell_1 \in R$ such that $R/(\ell_1)$ is F-pure by Proposition 4.4.2. Consider

$$0 \longrightarrow H^{d-1}_{\mathfrak{m}}(R/(\ell_1)) \longrightarrow H^d_{\mathfrak{m}}(R)(-1) \xrightarrow{\ell_1} H^d_{\mathfrak{m}}(R) \longrightarrow 0,$$

that is a homogeneous short exact sequence of degree zero, obtained from the natural short exact sequence by applying the local cohomology functors. It follows that $a_{d-1}(R/(\ell_1)) = a_d(R) + 1$. Since $R/(\ell_1)$ is Gorenstein, we have that $\operatorname{fpt}(R/(\ell_1)) = -a_{d-1}(R/(\ell_1)) = -a_d(R) - 1 = \operatorname{fpt}(R) - 1$ by Theorem 4.3.2. The claim now follows by induction. $\square$

Chapter 5

Golod rings

The results contained in this chapter appear in [16]. We specialize to the case when R is a Noetherian positively graded algebra over a field k, but most of the results, appropriately restated, hold also in the case of Noetherian local rings, as well as in some reasonable generalizations of these two concepts. The main problem that we study is a question of Volkmar Welker:

Question 5.0.7. [53, Problem 6.18] Let k be a field, and let R be a Noetherian positively graded k-algebra, and let I, J be two proper homogeneous ideals in R. Is the ring R/IJ always Golod?

We will provide a negative answer to this question, exhibiting examples of products of monomial ideals IJ inside a polynomial ring S such that S/IJ is not Golod. We will also study some positive cases, where we will be able to conclude that the ring is Golod or, at least, weakly Golod.

5.1 Preliminaries on Golod rings

Let k be a field, and let $R = \bigoplus_{i\geqslant 0} R_i$ denote a Noetherian positively graded algebra over $R_0 = k$. In these assumptions, the modules $\operatorname{Tor}_i^R(k,k)$ are finitely generated k-vector spaces, whose dimensions are $\beta_i := \dim_k \operatorname{Tor}_i^R(k,k)$, the Betti numbers of k as an R-module.

Definition 5.1.1. *The Poincaré series of R is defined as the generating series of the integers $\{\beta_i\}_{i\geqslant 0}$, that is*

$$P_R(t) = \sum_{i\geqslant 0} \beta_i t^i \in \mathbb{Z}[\![t]\!].$$

A natural question, attributed to Serre and Kaplansky, is whether this series is always rational. This means that $P_R(t)$ is the ratio of two polynomials with integer coefficients. This is not the case in general, as shown by Anick in 1982 [3].

Example 5.1.2. The following ring has irrational Poincaré series:

$$R := \frac{\mathbb{Q}[x_1, x_2, x_3, x_4, x_5]}{((x_1^2, x_2^2, x_4^2, x_5^2, x_1x_2, x_4x_5, x_1x_3 + x_3x_4 + x_2x_5) + (x_1, \ldots, x_5)^3)}.$$

Classes of rings for which the Poincaré series are rational include Koszul algebras, algebras defined by monomial ideals [8], Artinian Gorenstein compressed algebras with socle degree $s \neq 3$ [58], and some special rings, for which the resolution of k has a particular structure, such as complete intersections.

Even though $P_R(t)$ may not be rational, it is always bounded above, coefficient-wise, by a rational series. We now describe this upper bound, proved by Serre.

Since R is Noetherian and graded, it is a finitely generated as an algebra over k. We can then find a polynomial ring $S = k[x_1, \dots, x_n]$ that maps onto R. We can also assume that this is done minimally: if $\varphi : S \to R$ is the surjection, we can assume that $\ker(\varphi) \subseteq (x_1, \dots, x_n)^2$. In this case n, i.e, the number of variables of S, is called the embedding dimension of R. Serre showed that $P_R(t)$ is bounded above by the following rational series:

$$P_R(t) \preceq \frac{(1+t)^n}{1 - \sum_{i=1}^{n} \dim_k \left(\operatorname{Tor}_i^S(k, R)\right) t^{i+1}}. \tag{5.1.1}$$

The symbol $\preceq$ denotes inequality term by term. The bound is sharp, and the case when the Poincaré series is maximal produces the class of rings at the center of investigation in this chapter.

Definition 5.1.3. *The ring R is called Golod if equality holds in (5.1.1).*

Remark 5.1.4. As an evident consequence of the definition, Golod rings constitute another class of algebras that have rational Poincaré series.

There are several other ways to characterize Golod rings. In fact, Golod rings were named after Evgenii S. Golod, who proved that the upper bound in (5.1.1) is achieved if and only if the Eagon resolution is minimal [24]. This happens if and only if all the Massey operations of the ring vanish. We now describe the Massey operations in more details. We essentially follow the treatment of [6], and we refer to Section 5.2 in that book and to [28, Chapter 4] for more details.

As noted before, since R is a Noetherian graded k-algebra, we can write $R \cong S/I$, where $S = k[x_1, \ldots, x_n]$ is a polynomial ring, and $I \subseteq S$ is a homogeneous ideal. We can always assume that $I \subseteq (x_1, \ldots, x_n)^2$. Let $K_\bullet$ be the Koszul complex on the elements $x_1, \ldots, x_n$ of S, which is a minimal free resolution of k over S. We have that K_1 is a free S-module of rank n, and we denote by $\{e_1, \ldots, e_n\}$ a basis. In addition, we have that $K_i \cong \bigwedge_i K_1$ for all $i = 1, \ldots, n$. On a basis element $e_{t_1} \wedge \ldots \wedge e_{t_i}$ of K_i, where $t_1, \ldots, t_i \in \{1, \ldots, n\}$ are i distinct integers, the differential $\delta_i : K_i \to K_{i-1}$ is given by the following formula

$$\delta_i(e_{t_1} \wedge \ldots \wedge e_{t_i}) = \sum_{j=1}^{i} (-1)^{j-1} \, x_{t_j} \; e_{t_1} \wedge \ldots \wedge e_{t_{j-1}} \wedge e_{t_{j+1}} \wedge \ldots \wedge e_{t_i},$$

and extended by linearity to K_i. Let $K_\bullet(R) = K_\bullet \otimes_S R$ be the Koszul complex on R. We denote by $Z_\bullet(R)$ the Koszul cycles, and by $H_\bullet(R)$ the Koszul homology on R. We will use δ also to denote the differential on the Koszul complex $K_\bullet(R)$ on R, and we denote by $[-]$ the equivalence class of a Koszul cycle in the homology $H_\bullet(R)$.

The following formulation of the Massey operations is taken from [34].

Definition 5.1.5. *Let R be a positively graded k-algebra, let $\mathfrak{m}$ be the irrelevant maximal ideal of R, and let $K_\bullet(R)$ be the Koszul complex on R. We say that all the Massey operations on R vanish if for each subset $\mathcal{S}$ of homogeneous elements of $\bigoplus_{i=1}^{n} H_i(R)$ there exists a function θ, defined on the set of finite sequences of elements from $\mathcal{S}$, with values in $\mathfrak{m} \oplus (\bigoplus_{i=1}^{n} K_i(R))$, such that*

(1) For each $h \in \mathcal{S}$, we have that $\theta(h)$ is a Koszul cycle, and $h = [\theta(h)]$.

(2) If $h_1, \dots, h_m$ is a sequence in $\mathcal{S}$, with $m > 1$, then

$$\delta(\theta(h_1, \dots, h_m)) = \sum_{j=1}^{m-1} \overline{\theta(h_1, \dots, h_j)}\theta(h_{j+1}, \dots, h_m),$$

where $\overline{a} = (-1)^{i+1}a$ if $a \in K_i(R)$.

Theorem 5.1.6 (Golod)**.** *[24] A ring R is Golod if and only if all the Massey operations on R vanish.*

The proof of Golod's Theorem is quite involved, and would take us far from the purposes of this chapter. We refer to [28, Theorem 4.2.2] for a proof that the Massey operations are trivial if and only if the Eagon resolution of k is minimal. It is then a matter of counting ranks to show that the Eagon resolution is minimal if and only if Serre's upper bound is achieved.

From the equivalent definition of Golod rings in terms of Massey operations, we readily obtain the following corollary.

Corollary 5.1.7. *If a ring R is Golod, then the multiplication on the positive degree Koszul homology $H_{\geqslant 1}(R)$ is identically zero.*

Proof. Note that Definition 5.1.5 (2) and Theorem 5.1.6 imply that, for any two $h_1, h_2 \in \mathcal{S}$, we have that $\theta(h_1)\theta(h_2) = \delta(\pm\theta(h_1, h_2))$ is a boundary. Therefore, choosing $\mathcal{S}$ to be a k-basis for $\bigoplus_{i=1}^{n} H_i(R)$, we obtain that $h_1 \cdot h_2 = [\theta(h_1)][\theta(h_2)] = [\delta(\pm(\theta(h_1, h_2)))] = 0$ in $H_\bullet(R)$ for all $h_1, h_2 \in \mathcal{S}$. In particular, the multiplication on $H_{\geqslant 1}(R)$ is trivial. □

Since we will repeatedly use this property in the following sections, we give a name to rings that satisfy this condition.

Definition 5.1.8. *A ring R is called weakly Golod if the second Massey operation is zero, that is, if the multiplication on the positive homological degree part of the Koszul homology is trivial.*

5.2 Examples of products that are not Golod

In this Section we give a negative answer to the following question of Welker:

Question 5.2.1. [53, Problem 6.18] Let k be a field, and let R be positively graded k-algebra. Let I, J be two proper homogeneous ideals in R. Is the ring R/IJ always Golod?

The general belief, supported by strong computational evidence, was that this question had positive answer. The first result in this direction is a theorem of Herzog and Steurich [35]: let S be a polynomial ring over a field, and let I,J be two proper homogeneous ideals of S. If $I \cap J = IJ$, then S/IJ is Golod. Another reason to believe that Question 5.2.1 had positive answer comes from a result of Avramov and Golod [7], which says that Golod rings are never Gorenstein, unless they are hypersurfaces. This is consistent with a result of Huneke [41], according to which S/IJ is never Gorenstein, unless I and J are principal. More recently, Herzog and Huneke show that, if I is a homogeneous ideal in a polynomial ring S over a field of characteristic zero, then, for all $d \geqslant 2$, the ring S/I^d is Golod [34, Theorem 2.3 (d)]. Another similar result of Herzog, Welker and Yassemi, states that S/I^d is Golod for all $d \gg 0$, with no assumption on the characteristic of k [36].

We are now ready for the first example.

Example 5.2.2. *Let k be a field, and let $S = k[x, y, z, w]$, with the standard grading. Let $\mathfrak{n} = (x, y, z, w)$ be the irrelevant maximal ideal, consider the monomial ideal $J = (x^2, y^2, z^2, w^2)$ and let*

$$I := \mathfrak{n}J = (x^3, x^2y, x^2z, x^2w, xy^2, y^3, y^2z, y^2w, xz^2, yz^2, z^3, z^2w, xw^2, yw^2, zw^2, w^3).$$

Then, the ring $R = S/I$ is not Golod.

Proof. Golod rings are weakly Golod, by Corollary 5.1.7. Therefore, to show that R is not Golod, it is enough to find two elements $\alpha, \beta \in H_{\geqslant 1}(R)$ such that $\alpha\beta \neq 0$. Consider the element $u = (e_x \wedge e_y) \otimes xy \in K_2(R) = K_2 \otimes_S R$. It is a Koszul cycle:

$$\delta_2(u) = e_y \otimes x^2y - e_x \otimes xy^2 = 0 \text{ in } K_1(R),$$

because $x^2y \in I$ and $xy^2 \in I$. Then, let $\alpha := [u] \in H_2(R)$ be its residue class in homology. Similarly, let $v = (e_z \wedge e_w) \otimes zw \in Z_2(R)$, and let $\beta := [v] \in H_2(R)$. We want to show that $uv = (e_x \wedge e_y \wedge e_z \wedge e_w) \otimes xyzw \in Z_4(R)$ is not a boundary, so that $\alpha\beta = [uv] \neq 0$ in $H_4(R)$. Note that $K_5(R) = 0$, hence such a product is zero in homology if and only if $uv \in K_4(R)$ is zero as a cycle. Since $K_4(R)$ is free over R, this happens if and only if $xyzw \in I$. But $xyzw \notin I$, as every monomial generator of I contains the square of a variable. $\square$

Important Remark. In [62, Theorem 1.1] Seyed Fakhari and Welker write that any product of proper monomial ideals in a polynomial ring over a field is Golod. The key step in their proof is to show that products of monomial ideals always satisfy the strong-GCD condition, that we now recall:

Definition 5.2.3. *[45, Definition 3.8] A monomial ideal $I \subseteq S$ satisfies the strong-GCD condition if there exists a linear order $\prec$ on the set* $\mathrm{MinGen}(I)$ *of minimal monomial generators of I such that, for any two monomials $u \prec v$ in* $\mathrm{MinGen}(I)$*, with* $\gcd(u, v) = 1$*, there exists a monomial $w \in \mathrm{MinGen}(I)$, $v \neq w$, with $u \prec w$ and such that w divides uv.*

The ring in Example 5.2.2 satisfies the strong-GCD condition, for example choosing the order on the monomial generators induced by the Lex order on the variables. In fact, the result of Seyed Fakhari and Welker is correct, but it only shows that products of monomial ideals satisfy the strong-GCD condition. The fact that monomial ideals that satisfy the strong-GCD condition are Golod is first stated by Jöllenbeck in [45, Theorem 7.5], provided an extra assumption, called Property (P), is satisfied, and then by Berglund and Jöllenbeck in [10, Theorem 5.5], where the extra assumption is removed. The mistake seems to be contained inside [45], and the validity of the claims made subsequently in [10] is then affected by this [46].

Another way to show that the ring in Example 5.2.2 is not Golod, is to compare the Poincaré series of R with the one expected for Golod rings. Using Macaulay2

[26], one can compute the first Betti numbers of k over R:

$$\dots \longrightarrow R^{11283} \longrightarrow R^{2312} \longrightarrow R^{493} \longrightarrow R^{98} \longrightarrow R^{22} \longrightarrow R^{4} \longrightarrow R \longrightarrow k \longrightarrow 0.$$

Therefore the Poincaré series of R is

$$P_R(t) = 1 + 4t + 422t^2 + 98t^3 + 493t^4 + 2312t^5 + 11283t^6 + \dots$$

On the other hand, the upper bound given by Serre's inequality is

$$\frac{(1+t)^4}{1 - 16t^2 - 30t^3 - 20t^4 - 5t^5} = 1 + 4t + 22t^2 + 98t^3 + 493t^4 + 2313t^5 + 11288t^6 + \dots$$

Since the two series are not coefficientwise equal, R is not Golod. We also checked that R is not Golod using the Macaulay2 command `isGolod(S/I)` which computes the generators of all the Koszul homology modules, and determines whether their products are zero.

5.2.1 Another example

The ring of Example 5.2.2 is not the first example of a non-Golod product of ideals that we discovered. In fact, the ring of Example 5.2.2 was suggested to us by Srikanth Iyengar, after some discussions about Example 5.2.4, that we describe in this Subsection.

The proof that the ring in Example 5.2.4 is not Golod relies on lifting Koszul cycles. More specifically, we use the double-complex proof of the fact that $\mathrm{Tor}^S_\bullet(k, S/I)$ can be computed in two ways, to lift a Koszul cycle to a specific element of a finitely generated k-vector space. The results that we use are very well-known, so we will not explain all the steps. We refer the reader to [69] or [60] for more details.

Let $S = k[x_1, \dots, x_n]$ be a polynomial ring over a field k, not necessarily standard graded, and let $\mathfrak{n}$ be the irrelevant maximal ideal of S. Let $I \subseteq \mathfrak{n}^2$ be a homogeneous ideal in S, and consider the residue class ring $R = S/I$. Since $K_\bullet$ is a free resolution of k over S, we have that $H_i(R) := H_i(K_\bullet \otimes_S R) \cong \operatorname{Tor}_i^S(k, R)$, and its dimension as a k-vector space is the i-th Betti number, β_i, of R as an S-module. On the other hand, if $F_\bullet \to R \to 0$ is a minimal free resolution of R over S, then $H_i(k \otimes_S F_\bullet) \cong k \otimes F_i$ is also isomorphic to $\operatorname{Tor}_i^S(k, R)$. There is map $\psi : Z_i(R) \to k \otimes F_i$, which is constructed by "lifting cycles". Since the boundaries map to zero via ψ, this induces a map $\overline{\psi} : H_i(R) \to k \otimes_S F_i$, which is an isomorphism. See [31] for a canonical way to construct Koszul cycles from elements in $k \otimes F_i$ (that is, a canonical choice of an inverse for ψ). We are now ready to illustrate the example.

Let k be a field, and let $S = k[a, b, c, d, x, y, z, w]$. Consider the monomial ideals $I_1 = (ax, by, cz, dw)$ and $I_2 = (a, b, c, d)$ inside S. Let $I := I_1 I_2$ be their product, and set $R = S/I$. Let $T = \mathbb{Z}[a, b, c, d, x, y, z, w]$, and let J be the ideal I inside T. Then, using the Macaulay2 command `res J`, we get a resolution of J over T

$$F_\bullet : 0 \longrightarrow T^5 \xrightarrow{\varphi_4} T^{20} \xrightarrow{\varphi_3} T^{30} \xrightarrow{\varphi_2} T^{16} \xrightarrow{\varphi_1} T \xrightarrow{\varphi_0} T/I \longrightarrow 0.$$

Assume that $\operatorname{char}(k) = p > 0$. We checked with Macaulay2 [26] that $(a^2x) \subseteq I(\varphi_1)$, where $I(\varphi_1)$ is the Fitting ideal of the map φ_1. This is still a regular element after tensoring with $- \otimes_{\mathbb{Z}} \mathbb{Z}/(p)$, so that $\operatorname{grade}(I(\varphi_1 \otimes 1_{\mathbb{Z}/(p)})) \geqslant 1$. Similarly, one can see that

$$(a^{12}x^3, b^{12}y^3) \subseteq I(\varphi_2) \qquad (a^{15}x^3, b^{15}y^3, c^{15}z^3) \subseteq I(\varphi_3) \qquad (a^5x, b^5y, c^5z, d^5w) \subseteq I(\varphi_4).$$

Since they stay regular sequences after tensoring with $-\otimes_{\mathbb{Z}} \mathbb{Z}/(p)$ we obtain that

$$\text{grade}(I(\varphi_i \otimes 1_{\mathbb{Z}/(p)})) \geqslant i$$

for all $i = 1, \dots, 4$. In addition, the ranks of the maps add up to the correct numbers after tensoring. By Buchsbaum-Eisenbud's criterion for exactness of complexes [14], $F_\bullet \otimes_{\mathbb{Z}} \mathbb{Z}/(p)$ is a minimal free resolution of $J \otimes_{\mathbb{Z}} \mathbb{Z}/(p)$ as an ideal of $T \otimes_Z \mathbb{Z}/(p) \cong \mathbb{Z}/(p)[a, b, c, d, x, y, z, w]$. Finally, since the map $T \otimes_{\mathbb{Z}} \mathbb{Z}/(p) \to S$ is faithfully flat, tensoring with $(F_\bullet \otimes_{\mathbb{Z}} \mathbb{Z}/(p)) \otimes_{\mathbb{Z}/(p)} S$ gives a minimal free resolution of I over S, using Buchsbaum-Eisenbud's criterion for exactness of complexes [14] once again. When $\text{char}(k) = 0$, one can use $\mathbb{Q}$ instead of $\mathbb{Z}/(p)$ and the same arguments can be applied.

Therefore we get a resolution

$$F_\bullet \otimes_{\mathbb{Z}} S : 0 \longrightarrow S^5 \xrightarrow{\varphi_4} S^{20} \xrightarrow{\varphi_3} S^{30} \xrightarrow{\varphi_2} S^{16} \xrightarrow{\varphi_1} S \xrightarrow{\varphi_0} R \longrightarrow 0.$$

Letting $E_j^{(i)}$ be the canonical bases of the modules $F_i \cong \bigoplus_{j=1}^{\beta_i} T$, for $i = 0, \dots, 4$, the matrices representing the differentials of the minimal free resolution of J over T are the same as the ones of a minimal free resolution of I over S. Here follows a description of such matrices. All the missing entries must be regarded as zeros:

φ_1	$E_1^{(1)}$	$E_2^{(1)}$	$E_3^{(1)}$	$E_4^{(1)}$	$E_5^{(1)}$	$E_6^{(1)}$	$E_7^{(1)}$	$E_8^{(1)}$	$E_9^{(1)}$	$E_{10}^{(1)}$	$E_{11}^{(1)}$	$E_{12}^{(1)}$	$E_{13}^{(1)}$	$E_{14}^{(1)}$	$E_{15}^{(1)}$	$E_{16}^{(1)}$
$E_1^{(0)}$	a^2x	abx	acx	adx	aby	b^2y	bcy	bdy	acz	bcz	c^2z	cdz	adw	bdw	cdw	d^2w

φ_2	$E_1^{(2)}$	$E_2^{(2)}$	$E_3^{(2)}$	$E_4^{(2)}$	$E_5^{(2)}$	$E_6^{(2)}$	$E_7^{(2)}$	$E_8^{(2)}$	$E_9^{(2)}$	$E_{10}^{(2)}$	$E_{11}^{(2)}$	$E_{12}^{(2)}$	$E_{13}^{(2)}$	$E_{14}^{(2)}$	$E_{15}^{(2)}$	$E_{16}^{(2)}$	$E_{17}^{(2)}$	$E_{18}^{(2)}$	$E_{19}^{(2)}$	$E_{20}^{(2)}$	$E_{21}^{(2)}$	$E_{22}^{(2)}$	$E_{23}^{(2)}$	$E_{24}^{(2)}$	$E_{25}^{(2)}$	$E_{26}^{(2)}$	$E_{27}^{(2)}$	$E_{28}^{(2)}$	$E_{29}^{(2)}$	$E_{30}^{(2)}$
$E_1^{(1)}$	$-b$	$-c$		$-d$																										
$E_2^{(1)}$	a		$-c$		$-d$								$-y$																	
$E_3^{(1)}$		a	b			$-d$														$-z$										
$E_4^{(1)}$				a	b	c																						$-w$		
$E_5^{(1)}$							$-b$	$-c$		$-d$			x																	
$E_6^{(1)}$							a		$-c$		$-d$																			
$E_7^{(1)}$								a	b			$-d$									$-z$									
$E_8^{(1)}$										a	b	c																	$-w$	
$E_9^{(1)}$														$-b$	$-c$		$-d$			x										
$E_{10}^{(1)}$														a		$-c$		$-d$			y									
$E_{11}^{(1)}$															a	b			$-d$											
$E_{12}^{(1)}$																	a	b	c											$-w$
$E_{13}^{(1)}$																						$-b$	$-c$		$-d$			x		
$E_{14}^{(1)}$																						a		$-c$		$-d$			y	
$E_{15}^{(1)}$																							a	b			$-d$			z
$E_{16}^{(1)}$																									a	b	c			

φ_3	$E_1^{(3)}$	$E_2^{(3)}$	$E_3^{(3)}$	$E_4^{(3)}$	$E_5^{(3)}$	$E_6^{(3)}$	$E_7^{(3)}$	$E_8^{(3)}$	$E_9^{(3)}$	$E_{10}^{(3)}$	$E_{11}^{(3)}$	$E_{12}^{(3)}$	$E_{13}^{(3)}$	$E_{14}^{(3)}$	$E_{15}^{(3)}$	$E_{16}^{(3)}$	$E_{17}^{(3)}$	$E_{18}^{(3)}$	$E_{19}^{(3)}$	$E_{20}^{(3)}$
$E_1^{(2)}$	c	d																		
$E_2^{(2)}$	$-b$		d																	
$E_3^{(2)}$	a			d													$-yz$			
$E_4^{(2)}$		$-b$	$-c$																	
$E_5^{(2)}$		a		$-c$														$-yw$		
$E_6^{(2)}$			a	b															$-zw$	
$E_7^{(2)}$					c	d														
$E_8^{(2)}$					$-b$		d										xz			
$E_9^{(2)}$					a			d												
$E_{10}^{(2)}$						$-b$	$-c$											xw		
$E_{11}^{(2)}$						a		$-c$												
$E_{12}^{(2)}$							a	b												$-zw$
$E_{13}^{(2)}$																	cz	dw		
$E_{14}^{(2)}$									c	d							$-xy$			
$E_{15}^{(2)}$									$-b$		d									
$E_{16}^{(2)}$									a			d								
$E_{17}^{(2)}$										$-b$	$-c$								xw	
$E_{18}^{(2)}$										a		$-c$								yw
$E_{19}^{(2)}$											a	b								
$E_{20}^{(2)}$																	$-by$		dw	
$E_{21}^{(2)}$																	ax			dw
$E_{22}^{(2)}$													c	d				$-xy$		
$E_{23}^{(2)}$													$-b$		d				$-xz$	
$E_{24}^{(2)}$													a			d				$-yz$
$E_{25}^{(2)}$														$-b$	$-c$					
$E_{26}^{(2)}$														a		$-c$				
$E_{27}^{(2)}$															a	b				
$E_{28}^{(2)}$																		$-by$	$-cz$	
$E_{29}^{(2)}$																		ax		$-cz$
$E_{30}^{(2)}$																			ax	by

φ_4	$E_1^{(4)}$	$E_2^{(4)}$	$E_3^{(4)}$	$E_4^{(4)}$	$E_5^{(4)}$
$E_1^{(3)}$	$-d$				
$E_2^{(3)}$	c				
$E_3^{(3)}$	$-b$				
$E_4^{(3)}$	a				$-yzw$
$E_5^{(3)}$		$-d$			
$E_6^{(3)}$		c			
$E_7^{(3)}$		$-b$			xzw
$E_8^{(3)}$		a			
$E_9^{(3)}$			$-d$		
$E_{10}^{(3)}$			c		$-xyw$
$E_{11}^{(3)}$			$-b$		
$E_{12}^{(3)}$			a		
$E_{13}^{(3)}$				$-d$	xyz
$E_{14}^{(3)}$				c	
$E_{15}^{(3)}$				$-b$	
$E_{16}^{(3)}$				a	
$E_{17}^{(3)}$					$-dw$
$E_{18}^{(3)}$					cz
$E_{19}^{(3)}$					$-by$
$E_{20}^{(3)}$					ax

Using this explicit expression for the differentials, we prove that R is not Golod.

Example 5.2.4. *Let k be a field, and let $S = k[a, b, c, d, x, y, z, w]$. Consider the monomial ideals $I_1 = (ax, by, cz, dw)$ and $I_2 = (a, b, c, d)$ inside S. Let*

$$I := I_1 I_2 = (a^2x, abx, acx, adx, aby, b^2y, bcy, bdy, acz, bcz, c^2z, cdz, adw, bdw, cdw, d^2w)$$

be their product, and set $R = S/I$. Then, the ring R is not Golod.

Proof. Let $0 \to F_4 \to F_3 \to F_2 \to F_1 \to F_0 \to R \to 0$ be a minimal free resolution of R over S, with maps $\varphi_j : F_j \to F_{j-1}, j = 1, \ldots, 4$, and $\varphi_0 : F_0 = S \to R$ being the natural projection. For each $i = 0, \ldots, 4$ and each free module $F_i = S^{\beta_i}$ fix standard bases $E_j^{(i)}$, $j = 1, \ldots, \beta_i$. In this way, the differentials can be represented by matrices, as shown before. We have the following staircase:

$$\begin{array}{ccccccccc}
 & & & & & & & & S \otimes_S S^5 \xrightarrow{\delta_0 \otimes 1_{S^5}} k \otimes_S S^5 \\
 & & & & & & & & \big\downarrow{\scriptstyle 1_S \otimes \varphi_4} \\
 & & & & & & K_1 \otimes_S S^{20} & \xrightarrow{\delta_1 \otimes 1_{S^{20}}} & S \otimes_S S^{20} \\
 & & & & & & \big\downarrow{\scriptstyle 1_{K_1} \otimes \varphi_3} & & \\
 & & & & K_2 \otimes_S S^{30} & \xrightarrow{\delta_2 \otimes 1_{S^{30}}} & K_1 \otimes_S S^{30} & & \\
 & & & & \big\downarrow{\scriptstyle 1_{K_2} \otimes \varphi_2} & & & & \\
 & & K_3 \otimes_S S^{16} & \xrightarrow{\delta_3 \otimes 1_{S^{16}}} & K_2 \otimes_S S^{16} & & & & \\
 & & \big\downarrow{\scriptstyle 1_{K_3} \otimes \varphi_1} & & & & & & \\
K_4 \otimes_S S & \xrightarrow{\delta_4 \otimes 1_S} & K_3 \otimes_S S & & & & & & \\
\big\downarrow{\scriptstyle 1_{K_4} \otimes \varphi_0} & & & & & & & & \\
K_4 \otimes_S R & & & & & & & &
\end{array}$$

Let $u = (e_x \wedge e_y) \otimes (ab)$, and $v = (e_z \wedge e_w) \otimes (cd)$, inside $K_2(R) = K_2 \otimes_S R$. As they are cycles, we can consider their classes $\alpha = [u]$ and $\beta = [v]$ in homology. We want to construct a lifting $\psi(uv)$ of the Koszul cycle $uv = (e_x \wedge e_y \wedge e_z \wedge e_w) \otimes (abcd) \in K_4(R)$. Given $(e_x \wedge e_y \wedge e_z \wedge e_w) \otimes (abcd) \in K_4 \otimes_S R$ we consider the lift $(e_x \wedge e_y \wedge e_z \wedge e_w) \otimes (abcd\ E_1^{(0)}) \in K_4 \otimes_S S$, and then apply the differential $\delta_4 \otimes 1_S$:

$$(\delta_4 \otimes 1_S)((e_x \wedge e_y \wedge e_z \wedge e_w) \otimes (abcd\ E_1^{(0)})) = \begin{array}{l} +(e_y \wedge e_z \wedge e_w) \otimes (abcdx E_1^{(0)}) \\ -(e_x \wedge e_z \wedge e_w) \otimes (abcdy\ E_1^{(0)}) \\ +(e_x \wedge e_y \wedge e_w) \otimes (abcdz\ E_1^{(0)}) \\ -(e_x \wedge e_y \wedge e_z) \otimes (abcdw\ E_1^{(0)}) \end{array}$$

This is now a boundary, and, in fact, it is equal to

$$(1_{K_3} \otimes \varphi_1) \begin{pmatrix} +(e_y \wedge e_z \wedge e_w) \otimes (cd\ E_2^{(1)}) \\ -(e_x \wedge e_z \wedge e_w) \otimes (cd\ E_5^{(1)}) \\ +(e_x \wedge e_y \wedge e_w) \otimes (ab\ E_{12}^{(1)}) \\ -(e_x \wedge e_y \wedge e_z) \otimes (ab\ E_{15}^{(1)}) \end{pmatrix}$$

Now we apply $\delta_3 \otimes 1_{S^{16}}$ to this element:

$$(\delta_3 \otimes 1_{S^{16}}) \begin{pmatrix} +(e_y \wedge e_z \wedge e_w) \otimes (cd\ E_2^{(1)}) \\ -(e_x \wedge e_z \wedge e_w) \otimes (cd\ E_5^{(1)}) \\ +(e_x \wedge e_y \wedge e_w) \otimes (ab\ E_{12}^{(1)}) \\ -(e_x \wedge e_y \wedge e_z) \otimes (ab\ E_{15}^{(1)}) \end{pmatrix} = \begin{array}{l} +(e_z \wedge e_w) \otimes (cdy\ E_2^{(1)} - cdx\ E_5^{(1)}) \\ -(e_y \wedge e_w) \otimes (cdz\ E_2^{(1)} - abx\ E_{12}^{(1)}) \\ +(e_y \wedge e_z) \otimes (cdw\ E_2^{(1)} - abx\ E_{15}^{(1)}) \\ +(e_x \wedge e_w) \otimes (cdz\ E_5^{(1)} - aby\ E_{12}^{(1)}) \\ -(e_x \wedge e_z) \otimes (cdw\ E_5^{(1)} - aby\ E_{15}^{(1)}) \\ +(e_x \wedge e_y) \otimes (abw\ E_{12}^{(1)} - abz\ E_{15}^{(1)}) \end{array}$$

This is a boundary. Namely, it is equal to

$$(1_{K_2}\otimes\varphi_2)\begin{pmatrix} -(e_z\wedge e_w)\otimes(cd\ E_{13}^{(2)}) \\ +(e_y\wedge e_w)\otimes(dz\ E_3^{(2)}+bx\ E_{17}^{(2)}+bd\ E_{20}^{(2)}) \\ -(e_y\wedge e_z)\otimes(cw\ E_5^{(2)}+bx\ E_{23}^{(2)}+bc\ E_{28}^{(2)}) \\ -(e_x\wedge e_w)\otimes(dz\ E_8^{(2)}+ay\ E_{18}^{(2)}+ad\ E_{21}^{(2)}) \\ +(e_x\wedge e_z)\otimes(cw\ E_{10}^{(2)}+ay\ E_{24}^{(2)}+ac\ E_{29}^{(2)}) \\ -(e_x\wedge e_y)\otimes(ab\ E_{30}^{(2)}) \end{pmatrix}.$$

We now apply the map $\delta_2\otimes 1_{S^{30}}$ to such a lift:

$$(\delta_2\otimes 1_{S^{30}})\begin{pmatrix} -(e_z\wedge e_w)\otimes(cd\ E_{13}^{(2)}) \\ +(e_y\wedge e_w)\otimes(dz\ E_3^{(2)}+bx\ E_{17}^{(2)}+bd\ E_{20}^{(2)}) \\ -(e_y\wedge e_z)\otimes(cw\ E_5^{(2)}+bx\ E_{23}^{(2)}+bc\ E_{28}^{(2)}) \\ -(e_x\wedge e_w)\otimes(dz\ E_8^{(2)}+ay\ E_{18}^{(2)}+ad\ E_{21}^{(2)}) \\ +(e_x\wedge e_z)\otimes(cw\ E_{10}^{(2)}+ay\ E_{24}^{(2)}+ac\ E_{29}^{(2)}) \\ -(e_x\wedge e_y)\otimes(ab\ E_{30}^{(2)}) \end{pmatrix}=$$

$$=\begin{matrix} +e_x\otimes(dzw\ E_8^2-czw\ E_{10}^{(2)}+ayw\ E_{18}^{(2)}+adw\ E_{21}^{(2)}-ayz\ E_{24}^{(2)}-acz\ E_{29}^{(2)}+aby\ E_{30}^{(2)}) \\ -e_y\otimes(dzw\ E_3^{(2)}-czw\ E_5^{(2)}+bxw\ E_{17}^{(2)}+bdw\ E_{20}^{(2)}-bxz\ E_{23}^{(2)}-bcz\ E_{28}^{(2)}+abx\ E_{30}^{(2)}) \\ +e_z\otimes(-cyw\ E_5^{(2)}+cxw\ E_{10}^{(2)}+cdw\ E_{13}^{(2)}-bxy\ E_{23}^{(2)}+axy\ E_{24}^{(2)}-bcy\ E_{28}^{(2)}+acx\ E_{29}^{(2)}) \\ -e_w\wedge(-dyz\ E_3^{(2)}+dxz\ E_8^{(2)}+cdz\ E_{13}^{(2)}-bxy\ E_{17}^{(2)}+axy\ E_{18}^{(2)}-bdy\ E_{20}^{(2)}+adx\ E_{21}^{(2)}). \end{matrix}$$

Again, this element is a boundary. In fact, it is equal to

$$(1_{K_1} \otimes \varphi_3) \begin{pmatrix} +e_x \otimes (zw\ E_7^{(3)} + a\ E_{20}^{(3)}) \\ -e_y \otimes (zw\ E_4^{(3)} + b\ E_{19}^{(3)}) \\ +e_z \otimes (xy\ E_{13}^{(3)} + c\ E_{18}^{(3)}) \\ -e_w \otimes (xy\ E_{10}^{(3)} + d\ E_{17}^{(3)}) \end{pmatrix}.$$

One more time, we apply $\delta_1 \otimes 1_{S^{20}}$, to get

$$(\delta_1 \otimes 1_{S^{20}}) \begin{pmatrix} +e_x \otimes (zw\ E_7^{(3)} + a\ E_{20}^{(3)}) \\ -e_y \otimes (zw\ E_4^{(3)} + b\ E_{19}^{(3)}) \\ +e_z \otimes (xy\ E_{13}^{(3)} + c\ E_{18}^{(3)}) \\ -e_w \otimes (xy\ E_{10}^{(3)} + d\ E_{17}^{(3)}) \end{pmatrix} =$$

$$= 1 \otimes (-yzw\ E_4^{(3)} + xzw\ E_7^{(3)} - xyw\ E_{10}^{(3)} + xyz\ E_{13}^{(3)} - dw\ E_{17}^{(3)} + cz\ E_{18}^{(3)} - by\ E_{19}^{(3)} + ax\ E_{20}^{(3)}).$$

This is a boundary: it is equal to $(1_S \otimes \varphi_4)(1 \otimes E_5^{(4)})$. When applying $\delta_0 \otimes 1_{S^5}$ to the lift, we finally get the image of uv under the map $\psi : Z_4(R) \to k \otimes_S S^5$. Namely:

$$\psi(uv) = (\delta_0 \otimes 1_{S^5})(1 \otimes E_5^{(4)}) = \overline{1} \otimes E_5^{(4)} \in k \otimes_S S^5,$$

and since the latter is non-zero (because it is part of a k-basis of $k \otimes_S S^5$) we obtain that uv is not a boundary of the Koszul complex. Thus, $\alpha\beta$ is non-zero in $H_4(R)$, and R is not Golod. □

Remark 5.2.5. With the same notation as in Example 5.2.4, we have that $x - a, y - b, z - c, w - d$ is a regular sequence modulo I. Modulo these linear forms, one recovers the ring of Example 5.2.2. In fact, by [6, Proposition 5.2.4] adapted to the standard graded case, we see that the ring in Example 5.2.4 is Golod if and only if the ring in Example 5.2.2 is Golod.

We end this section presenting another ideal that satisfies the strong-GCD condition (see Definition 5.2.3), and that is not Golod. Although it is not a product, it has the advantage of having fewer generators than our previous examples.

Example 5.2.6. Let $S = k[x, y, z]$, and let $I = (x^2y, xy^2, x^2z, y^2z, z^2)$. Set $R = S/I$. The ideal I satisfies the strong-GCD condition, for example choosing $x^2y \prec xy^2 \prec x^2z \prec y^2z \prec z^2$. Using Macaulay2 [26], we checked that the Poincaré series of R starts as

$$P_R(t) = 1 + 3t + 8t^2 + 21t^3 + 55t^4 + 144t^5 + 377t^6 + \dots$$

and that the right-hand side of Serre's inequality is

$$\frac{(1+t)^3}{1 - 5t^2 - 5t^3 - t^4} = 1 + 3t + 8t^2 + 21t^3 + 56t^4 + 148t^5 + 393t^6 + \dots$$

Therefore, R is not Golod. Alternatively, one can use the Macaulay2 command `isGolod(S/I)`, or one can show, with arguments similar to the ones used for the previous examples, that the product of Koszul cycles

$$((e_x \wedge e_y) \otimes xy) \cdot (e_z \otimes z) \in K_3(R)$$

is not zero in homology. Looking for a squarefree example, using polarization, one obtains that $I' = (axy, bxy, axz, byz, cz) \subseteq k[a, b, c, x, y, z]$ satisfies the strong GCD condition, and is not Golod.

5.3 Strongly Golod property for rational powers of monomial ideals

In this section, we present some positive results: we are able that quotients by some rational powers of monomial ideals are Golod. The techniques are, for most part, similar to the ones used in [34] for the integral closure.

Let k be a field, and let $S = k[x_1, \ldots, x_n]$, with $\deg(x_i) = d_i > 0$. We recall the definition of rational powers of an ideal.

Definition 5.3.1. *For an ideal $I \subseteq S$ and positive integers p, q define the ideal*

$$I^{p/q} := \{f \in R \mid f^q \in \overline{I^p}\}.$$

The integral closure of I^p inside the definition is needed in order to make the set into an ideal, and to make it independent of the choice of the representation of p/q as a rational number.

Remark 5.3.2. We would like to warn the reader about a potential source of confusion. When $p = q$, the ideal $I^{p/q} = I^{1/1}$ is the integral closure $\overline{I}$ of I, and should not be regarded as the ideal $I^1 = I$, even though the exponents $1/1$ and 1 are equal.

Remark 5.3.3. If $I \subseteq S$ is a monomial ideal, then so is $I^{p/q}$.

Proof. Let $f = \sum_{i=1}^{d} \lambda_i u_i \in I^{p/q}$, where $0 \neq \lambda_i \in k$ and u_i are monomials. Since $\overline{I^p}$ is monomial, we have that $f^{qr} \in I^{pr}$ for all integers $r \gg 0$ [33, Theorem 1.4.2]. In addition, I^{pr} is monomial, therefore every monomial appearing in f^{qr} belongs to I^{pr}, and in particular for any $i = 1, \dots, d$ we have that $u_i^{qr} \in I^{pr}$ for all $r \gg 0$. This shows that $u_i^q \in \overline{I^p}$, that is $u_i \in I^{p/q}$ for all $i = 1, \dots, d$, and hence $I^{p/q}$ is monomial. $\square$

In the rest of this section, we assume that the characteristic of k is zero.

Definition 5.3.4 ([34]). *A proper homogeneous ideal $I \subseteq S$ is called strongly Golod if $\partial(I)^2 \subseteq I$.*

Here, $\partial(I)$ denotes the ideal of S generated by the partial derivatives of elements in I. By [34, Theorem 1.1], if I is strongly Golod, then S/I is Golod. This condition, however, is only sufficient. For example, the ideal $I = (xy, xz) \subseteq k[x, y, z]$ is Golod [64], or [6, Proposition 5.2.5]. However, it is not strongly Golod. This example is not even squarefree strongly Golod (see Section 5.4 for the definition). In case I is monomial, being strongly Golod is equivalent to the requirement that, for all minimal monomial generators $u, v \in I$, and all integers i, j such that x_i divides u and x_j divides v, one has $uv/x_ix_j \in I$.

The following argument is a modification of [34, Proposition 3.1], and shows that strong Golodness is preserved if one takes "at least" the integral closure of an ideal.

Theorem 5.3.5. *Let $I \subseteq S$ be a strongly Golod monomial ideal. If $p \geqslant q$, then $I^{p/q}$ is strongly Golod.*

Proof. Let $u \in I^{p/q}$ be a monomial generator, then $u^{qr} \in I^{pr}$ for all $r \gg 0$. Let j be an index such that $x_j \mid u$, we claim that $(u/x_j)^{qr} \in I^{pr/2}$ for all even $r \gg 0$. Notice that if $x_j^2 \mid u$ then, for any even $r \gg 0$, we have

$$\left(\frac{u}{x_j}\right)^{qr} = u^{q(r/2)} \left(\frac{u}{x_j^2}\right)^{qr/2} \in I^{p(r/2)},$$

as desired. Now suppose that x_j divides u, but x_j^2 does not. Since for any $r \gg 0$ we have that $u^{qr} \in I^{pr}$, we can write

$$u^{qr} = m_1 m_2 \cdots m_{pr},$$

where $m_i \in I$ for all i. Again, we can assume that r is even. For $i = 1, \ldots, pr$ let d_i be the maximum non-negative integer such that $x_j^{d_i}$ divides m_i. Then we can rewrite

$$u^{qr} = m_1 \cdots m_a m_{a+1} \cdots m_{a+b} m_{a+b+1} \cdots m_{pr},$$

where $d_i = 0$ for $1 \leqslant i \leqslant a$, $d_i = 1$ for $a+1 \leqslant i \leqslant a+b$ and $d_i \geqslant 2$ for $a+b+1 \leqslant i \leqslant pr$. Because of the assumption $x_j^2 \nmid u$ we have that

$$qr = \sum_{i=1}^{pr} d_i = b + \sum_{i=a+b+1}^{pr} d_i \geqslant b + 2(pr - b - a).$$

But we assumed that $p \geqslant q$, therefore $pr \geqslant b+2(pr-b-a)$, which gives $a+b/2 \geqslant pr/2$ and also $a + \lfloor b/2 \rfloor \geqslant pr/2$ because $pr/2$ is an integer. Write

$$\left(\frac{u}{x_j}\right)^{qr} = \frac{u^{qr}}{x_j^{qr}} = m_1 \cdots m_a \frac{m_{a+1}}{x_j} \cdots \frac{m_{a+b}}{x_j} \frac{m_{a+b+1} \cdots m_{pr}}{x_j^{qr-b}},$$

then $m_{a+1} \ldots m_{a+b}/x_j^b \in I^{\lfloor b/2 \rfloor}$ because I is strongly Golod, so that $\partial(I)^b \subseteq I^{\lfloor \frac{b}{2} \rfloor}$. Furthermore, $m_1 \cdots m_a \in I^a$. Therefore

$$\left(\frac{u}{x_j}\right)^{qr} \in I^{a+\lfloor b/2 \rfloor} \subseteq I^{pr/2}.$$

Now let $v \in I^{p/q}$ be another monomial generator, and assume that $x_i | v$. Then, for all even $r \gg 0$ we have

$$\left(\frac{uv}{x_j x_i}\right)^{qr} \in I^{pr}$$

which implies that $uv/x_j x_i \in I^{p/q}$. Since u and v were arbitrary monomial generators, $I^{p/q}$ is strongly Golod. ☐

Corollary 5.3.6. *[34, Proposition 3.1] Let $I \subseteq S$ be a monomial strongly Golod ideal, then $\overline{I}$ is strongly Golod.*

Proof. Choose $p = q$ in Theorem 5.3.5. ☐

Proposition 5.3.7. *Let $I \subseteq S$ be a monomial ideal. If $p \geqslant 2q$, then $I^{p/q}$ is strongly Golod.*

Proof. Note that, for integers $a, b, c \in \mathbb{N}$, and an ideal J, we have $J^{ab/c} = (J^a)^{b/c}$. In fact, this follows from the fact that $\overline{(J^a)^b} = \overline{J^{ab}}$. To prove the proposition, note that I^2 is strongly Golod by [34, Theorem 2.3 (d)]. Therefore, by Theorem 5.3.5, we have that $I^{p/q} = (I^2)^{p/2q}$ is strongly Golod, since $p \geqslant 2q$ by assumption. ☐

Remark 5.3.8. Herzog and Huneke show that powers of ideals are strongly Golod. Proposition 5.3.7 reflects this behavior. However, even when $p \geqslant 2q$, rational powers $I^{p/q}$ need not be actual regular powers.

If I is not strongly Golod and $2q > p \geqslant q$, it is not true in general that $I^{p/q}$ is strongly Golod, as the following family of examples shows.

Example 5.3.9. Let $2q > p \geqslant q$ be two positive integers and consider the ideal $I = (xy, z^q)$ inside the polynomial ring $S = k[x, y, z]$, where k is a field of characteristic zero. Then

$$(xy)^q (z^q)^{p-q} = (xyz^{p-q})^q \in I^p,$$

that is $xyz^{p-q} \in I^{p/q}$. Thus, $u := y^2 z^{2p-2q} \in \partial(I^{p/q})^2$. On the other hand, $u^q = y^{2q} z^{2pq-2q^2} \notin \overline{I^p}$ because the only monomial generator of I^p that can appear in an integral relation for u is $z^p q$. But

$$y^{2qn} z^{2pqn-2q^2 n} \notin (z^{pqn})$$

for any n because $pqn > 2pqn - 2q^2 n \iff p < 2q$, and we have the latter by assumption. As a consequence, $u \notin I^{p/q}$, and thus $I^{p/q}$ is not strongly Golod.

Remark 5.3.10. If we choose $p = q = 2$ in Example 5.3.9, we have in addition that $I^{p/q} = \overline{I} = I = (xy, z^2)$ is not even Golod, because it is a complete intersection of height two.

As a consequence, not all integrally closed ideals, even if assumed monomial, are Golod. A more trivial example is the irrelevant maximal ideal $\mathfrak{n}$ of S. However, as noted above in Corollary 5.3.6, if I is a strongly Golod monomial ideal, then $\overline{I}$ is strongly Golod. More generally, if $I \subseteq S = k[x_1, \ldots, x_n]$ is homogeneous, then $\overline{I^j}$ is strongly Golod for all $j \geqslant n + 1$ [34, Theorem 2.11]. It is still an open question whether $\overline{I^j}$ is strongly Golod, or, at least, Golod, for any ideal I and $j \geqslant 2$. Since for $j \geqslant 2$, the ideal I^j is strongly Golod, one can ask the following more general question, which has already been raised by Craig Huneke:

Question 5.3.11. [53, Problem 6.19] Let $I \subseteq S$ be a homogeneous strongly Golod ideal. Is $\overline{I}$ [strongly] Golod?

Remark 5.3.12. We checked with Macaulay2 [26] that the ideal I of Example 5.2.4 is integrally closed. Therefore, the integral closure of a product of ideals, even monomial ideals in a polynomial ring, may not be Golod.

We end the section with a more generic question about Golodness of the ideal $I^{3/2}$. Note that for each ideal $I = (xy, z^q)$ of the family considered in Example 5.3.9, the rational power $I^{3/2}$ is not strongly Golod. However, it is Golod. In fact, it is not hard to see that $\overline{I^3} = I^3 = (x^3y^3, x^2y^2z^q, xyz^{2q}, z^{3q})$. As a consequence, we have $I^{3/2} = (x^2y^2, xyz^{\lceil \frac{q}{2} \rceil}, z^{\lceil \frac{3q}{2} \rceil})$. Consider the linear form $x-y$, which is a non zero-divisor modulo $I^{3/2}$. The image of $I^{3/2}$ in the polynomial ring $S' = S/(x-y) \cong k[x,z]$ is $(x^4, x^2z^{\lceil \frac{q}{2} \rceil}, z^{\lceil \frac{3q}{2} \rceil})$. Such an ideal is easily seen to be strongly Golod, hence Golod. By [6, Proposition 5.2.4 (2)], the ideal $I^{3/2}$ is then Golod.

Question 5.3.13. Let $I \subseteq S$ be a proper homogeneous ideal. Is $I^{3/2}$ always Golod? Is it true if I is monomial?

5.4 lcm-strongly Golod monomial ideals

In this section we introduce the notion of lcm-strongly Golod monomial ideal, that generalizes the one of square-free strongly Golod, as defined in [34]. We then study the Golod property for this new class of ideals.

Let k be a field, and let $S = k[x_1, \dots, x_n]$, with $\deg(x_i) = d_i > 0$.

Definition 5.4.1. *Let $m \in S$ be a monomial, and let $I \subseteq S$ be a monomial ideal. Define $I_m \subseteq I$ to be the ideal of S generated by the monomials of I which divide m. We say that I is m-divisible if $I = I_m$.*

Remark 5.4.2. Note that, choosing $m = x_1 \cdots x_n$, then m-divisible simply means squarefree.

We now recall the Taylor resolution of a monomial ideal. Let $I \subseteq S$ be a monomial ideal, with minimal monomial generating set $\{m_1, \dots, m_t\}$. For each subset $\Lambda \subseteq [t] := \{1, \dots, t\}$ let $L_\Lambda := \operatorname{lcm}(m_i \mid i \in \Lambda)$. Let $a_\Lambda \in \mathbb{N}^n$ be the exponent vector of the monomial L_Λ, and let $S(-a_\Lambda)$ be the free module, with generator in multi-degree a_Λ. Consider the free modules $T_i := \bigoplus_{|\Lambda|=i} S(-a_\Lambda)$, with basis $\{e_\Lambda\}_{|\Lambda|=i}$. Also, set $F_0 := S$. The differential $\tau_i : T_i \to T_{i-1}$ acts on an element of the basis e_Λ, for $\Lambda \subseteq [t]$, $|\Lambda| = i$, as follows:

$$\tau_i(e_\Lambda) = \sum_{j \in \Lambda} \operatorname{sign}(j, \Lambda) \cdot \frac{L_\Lambda}{L_{\Lambda \smallsetminus \{j\}}} \cdot e_{\Lambda \smallsetminus \{j\}}$$

Here $\operatorname{sign}(j, \Lambda)$ is $(-1)^{s+1}$ if j is the s-th element in the ordering of $\Lambda \subseteq [t]$. The resulting complex is a free resolution of S/I over S, called the Taylor resolution. The following was already noted in [13, Corollary 3.2], and [44, Corollary to Theorem 1].

Remark 5.4.3. [13, Corollary 3.2] Let I be an m-divisible monomial ideal. Then, the Koszul homology $H_\bullet(S/I)$ is $\mathbb{Z}^n$-multigraded, and it is concentrated in multidegrees

$a_{\Lambda'} \in \mathbb{N}^n$ such that the monomial $x_1^{(a_{\Lambda'})_1} \cdots x_n^{(a'_\Lambda)_n} = L_{\Lambda'}$ divides m.

In [34], given a squarefree monomial ideal, Herzog and Huneke introduce the notion of squarefree strongly Golod monomial ideal. Given Remark 5.4.2, we generalize it to the notion of lcm-strongly Golod. Let I be a monomial ideal, and let $m := \operatorname{lcm}(I)$ be the least common multiple of the monomials appearing in the minimal monomial generating set of I. By definition, I is always m-divisible. Also, if I is m'-divisible for some other monomial m', then m divides m'.

In what follows, we assume that the characteristic of k is zero.

Definition 5.4.4. *Let $I \subseteq S$ be a monomial ideal, and let $m := \operatorname{lcm}(I)$ be as defined above. Let $\partial(I)^{[2]}$ denote the ideal $(\partial(I)^2)_m$. We say that I is* lcm-*strongly Golod if $\partial(I)^{[2]} \subseteq I$.*

The following is the main result of the section. It is a generalization of [34, Theorem 3.5].

Theorem 5.4.5. *Let $I \subseteq S$ be an* lcm-*strongly Golod monomial ideal. Then, S/I is weakly Golod.*

Proof. Let $m := \operatorname{lcm}(I)$, so that I is m-divisible and $\partial(I)^{[2]} \subseteq I$. By Remark 5.4.3, we can choose a k-basis of $H_\bullet(S/I)$ consisting of elements of multidegrees α_Λ, where $\underline{x}^{\alpha_\Lambda}$ divides m. Let a, b be two such elements. If ab has multidegree $\alpha \in \mathbb{N}^n$, such that $\underline{x}^\alpha$ does not divide m, then necessarily $ab = 0$ because of the multigrading on $H_\bullet(S/I)$. So assume that the multidegree α of ab is such that $\underline{x}^\alpha$ divides m. By [31],

a and b can be represented by cycles whose coefficients are k-linear combinations of elements in $\partial(I)$. Since I is monomial, so is $\partial(I)$. Because of the multidegree of ab, we then have that a and b can be represented by cycles whose coefficients are k-linear combinations of monomials $u, v \in \partial(I)$, such that the products uv divide m. Then $uv \in \partial(I)^{[2]} \subseteq I$ for each product uv appearing in these sums, and, as a consequence, $ab = 0$ in $H_\bullet(S/I)$. □

Remark 5.4.6. Given Theorem 5.4.5, it seems natural to ask whether the condition of being Golod and weakly Golod are equivalent, at least for some classes of rings. A recent example of Lukas Katthän shows that weakly Golod rings, even defined by monomial ideals, do not need to be Golod [46]. This is in contrast with the claim made in [10, Theorem 5.1], which, as noted in Section 5.2, relies on some erroneous statements made in [45].

Discussion 5.4.7. It is easy to see that being lcm-strongly Golod is only sufficient to be weakly Golod. For example, the ideal $(xy, xz) \subseteq k[x, y, z]$ is even Golod [64], but not lcm-strongly Golod. The proof of Theorem 5.4.5, as well as the proofs of [34, Theorem 1.1] and [34, Theorem 3.5], are based on a canonical description of Koszul cycles whose residue classes form a k-basis for the Koszul homology $H_\bullet(S/I)$ [31]. We want to suggest a slightly different definition of strong Golodness:

Potentially, one has to check that $\partial_i(f)\partial_j(g) \in I$ for any $f, g \in I$, and any $i, j = 1, \ldots, n$, where $\partial_i = \partial/\partial x_i$ and $\partial_j = \partial/\partial x_j$. However, by [31], each $\partial_i(f)$ appears as a factor in some coefficient of a Koszul cycle, which has the form $(e_i \wedge \ldots) \otimes \partial_i(f) \in$

$K_\bullet \otimes S/I = K_\bullet(S/I)$. Therefore, the corresponding product $\partial_i(f)\partial_j(g)$ will appear inside some coefficient of the form

$$(e_i \wedge e_j \wedge \ldots) \otimes \partial_i(f)\partial_j(g).$$

For $i = j$, we have that $e_i \wedge e_j = 0$. Hence we may consider only products $\partial_i(f)\partial_j(g)$, for $i \neq j$, in the definition of strongly Golod and lcm-strongly Golod. With this modification, the ideal (xy, xz) becomes lcm-strongly Golod. The ideal (x^2, xy) in the polynomial ring $k[x, y]$, which is lcm-strongly Golod, with this modification becomes strongly Golod. In fact, $\frac{xy}{x} \cdot \frac{xy}{x} = y^2 \notin (x^2, xy)$ is the product that is preventing it from being strongly Golod. However, the partial derivatives, in this case, are both with respect to x, so we can disregard such a product.

Here follows an example of a non-squarefree ideal which is lcm-strongly Golod, but not strongly Golod, even with the modified definition.

Example 5.4.8. Let k be a field of characteristic zero, and let $I = (x^2y^2, x^2z, y^2z) \subseteq k[x, y, z]$. Then I is not strongly Golod, even in the definition suggested above. In fact, $xz, yz \in \partial(I)$ come from taking derivative with respect to x and y, respectively, but their product is $xz \cdot yz \notin I$. However, such an element does not divide $\mathrm{lcm}(I) = x^2y^2z$, therefore it can be disregarded when looking at the lcm-strongly Golod condition. In fact, one can check that $\partial(I)^{[2]} \subseteq I$, that is, I is lcm-strongly Golod in this case.

As shown in [34, Proposition 3.7] for the squarefree part, if m is a monomial in S, then the m-divisible part of a strongly Golod monomial ideal is lcm-strongly Golod.

We record it in the next proposition.

Proposition 5.4.9. *Let $I \subseteq S$ be a strongly Golod monomial ideal, and let m be a monomial. Then I_m is* lcm-*strongly Golod. In particular, I is* lcm-*strongly Golod.*

Proof. We have that

$$\partial(I_m)^{[2]} = (\partial(I_m)^2)_m \subseteq (\partial(I)^2)_m \subseteq I_m.$$

□

As mentioned in Section 5.3, if I is a strongly Golod monomial ideal, then $\overline{I}$ is strongly Golod. It is natural to ask the following question:

Question 5.4.10. If $I \subseteq S$ is an lcm-strongly Golod monomial ideal, is $\overline{I}$ (lcm-strongly) Golod? For integers $p \geqslant q$, is the ideal $I^{p/q}$ (lcm-strongly) Golod?

The inequality $p \geqslant q$ seems reasonable to require, given previous results.

5.5 Golodness of products and further questions

In this final section, we give some sufficient conditions for a product of ideals to be strongly Golod, hence Golod. We end by asking some general questions about the Golod and the strongly Golod properties.

Let k be a field of characteristic zero, and let $S = k[x_1, \ldots, x_n]$ be a polynomial ring over k, with $\deg(x_i) = d_i > 0$. It is easy to see that arbitrary intersections of strongly Golod ideals are strongly Golod [34, Theorem 2.3 (a)]. Given a proper homogeneous ideal $I \subseteq S$, one may ask what is the intersection of all the strongly Golod ideals containing I. In other words, what is the smallest ideal that contains I and that is strongly Golod. Clearly, such an ideal must contain $I + \partial(I)^2$. On the other hand, note that $\partial(\partial(I)^2)) \subseteq \partial(I)$, therefore

$$\partial(I + \partial(I)^2)^2 \subseteq I + \partial(I)^2.$$

Thus, $I + \partial(I)^2$ is strongly Golod, and it is indeed the smallest strongly Golod ideal containing I.

We now introduce a sufficient condition, which is far from being necessary, for the product of two ideals to be strongly Golod.

Definition 5.5.1. *Let $S = k[x_1, \ldots, x_n]$ and let $I, J \subseteq S$ be two ideals. (I, J) is called a strongly Golod pair if $\partial(I)^2 \subseteq I : J$ and $\partial(J)^2 \subseteq J : I$.*

Note that, for examples of small size, the conditions from Definition 5.5.1 can easily be checked with the aid of a computer. The following proposition is the main motivation behind the definition.

Proposition 5.5.2. *If (I, J) is a strongly Golod pair, then IJ is strongly Golod.*

Proof. We noted above that the smallest strongly Golod ideal containing IJ is $IJ + \partial(IJ)^2$. In our assumptions, we have

$$\partial(IJ)^2 \subseteq (\partial(I)J + I\partial(J))^2 \subseteq \partial(I)^2J^2 + I^2\partial(J)^2 + IJ \subseteq IJ.$$

Therefore, $IJ + \partial(IJ)^2 = IJ$, which is then strongly Golod. □

Note that, looking at the proof of Proposition 5.5.2, one may notice that the conditions $\partial(I)^2 \subseteq IJ : J^2$ and $\partial(J)^2 \subseteq IJ : I^2$ are sufficient in order for the product IJ to be strongly Golod. However, when studying properties of the product IJ, one can replace the ideal I with $IJ : J$ without affecting the product. In fact:

$$IJ \subseteq (IJ : J)J \subseteq IJ,$$

forcing equality. Repeating the process, one gets an ascending chain of ideals containing I, that eventually stabilizes. Therefore one can assume that $IJ : J = I$. Similarly, one can assume that $IJ : I = J$. Therefore the conditions above become

$$\partial(I)^2 \subseteq IJ : J^2 = (IJ : J) : J = I : J,$$

which is precisely the requirement in the definition of strongly Golod pair. Of course, as long as one can write an ideal in terms of a Golod pair, one gets that the ideal is strongly Golod. Therefore, one may keep in mind the weaker colon conditions that come from the proof of Proposition 5.5.2. Examples of strongly Golod pairs include:

(1) (I^r, I^s), for any proper ideal $I \subseteq S$ and any integers $r, s \geqslant 1$.

(2) If I and J are strongly Golod, then (I, J) is a strongly Golod pair.

(3) If $I \subseteq J$ and I is strongly Golod, then (I, J) is a strongly Golod pair.

(4) $(I, I : \partial(I)^2)$ is a strongly Golod pair for any proper ideal $I \subseteq S$.

Remark 5.5.3. Let $I_1, \ldots, I_n$ be proper ideals in S. Assume that, for all $i = 1, \ldots, n$ there exists $j \neq i$ such that (I_i, I_j) is a strongly Golod pair, then the product $I := I_1 \cdots I_n$ is strongly Golod. In fact

$$\partial(I_1 I_2 \ldots I_n) \subseteq \partial(I_1) I_2 \ldots I_n + I_1 \partial(I_2) \ldots I_n + \ldots + I_1 I_2 \ldots \partial(I_n).$$

Thus

$$\partial(I)^2 \subseteq \partial(I_1)^2 I_2^2 \cdots I_n^2 + I_1^2 \partial(I_2)^2 \cdots I_n^2 + \ldots + I_1^2 I_2^2 \cdots \partial(I_n)^2 + I.$$

By assumption, for each i there exists $j \neq i$ such that $\partial(I_i)^2 I_j \subseteq I_i$, and the claim follows. More generally, one could define $(I_1, \ldots, I_n)$ to be a strongly Golod n-uple provided

$$\partial(I_i)^2 \subseteq I : (I_1 \cdots I_{i-1} \cdot I_{i+1} \cdots I_n)$$

for all $i = 1, \ldots, n$. Then, the above argument shows that if $(I_1, \ldots, I_n)$ is a strongly Golod n-uple, the product $I_1 \cdots I_n$ is strongly Golod.

All the conditions discussed above are sufficient, but evidently not necessary, for a product of two ideals to be Golod. We raise the following general question:

Question 5.5.4. Is there some relevant class of [pairs of] ideals for which products are [strongly] Golod?

In particular, note that in all the examples of Section 5.2, the ideals appearing in the product are not Golod. It is then natural to ask:

Question 5.5.5. If one of the two ideals I_1, I_2 [or both] is Golod, is then S/I_1I_2 Golod?

Another problem relating Golod rings to products is the following. Let I, J be two proper homogeneous ideals in a polynomial rings $S = k[x_1, \dots, x_n]$, with $\mathfrak{n} = (x_1, \dots, x_n)$. Suppose that S/IJ is Cohen-Macaulay. In [41], Huneke asks whether the Cohen-Macaulay type, that is, $t(S/IJ) = \dim_k \operatorname{Ext}^{\operatorname{depth}(S/IJ)}(k, S/IJ)$, is always at least the height of IJ. This was motivated by the fact that Gorenstein rings are never products, unless they are hypersurfaces. Thus, when S/IJ is Cohen-Macaulay and not a hypersurface, the type is always at least two. As noted in [41], the case when $I = \mathfrak{n}$ and J is $\mathfrak{n}$-primary, follows by Krull's height theorem. In our context, it seems natural to ask the following question:

Question 5.5.6. Let $I \subseteq S$ be a homogeneous ideal such that S/I is Cohen-Macaulay and Golod. Is it true that the Cohen-Macaulay type $t(S/I)$ is always at least $\operatorname{ht}(I)$? Is the Cohen-Macaulay assumption needed?

In [34, Proposition 2.12], Herzog and Huneke show that the Ratliff Rush filtration of a strongly Golod ideal is strongly Golod. We obtain a similar statement.

Proposition 5.5.7. *If (I, J) is a strongly Golod pair, then the ideal*

$$\bigcup_{n \geqslant 0} \left(I^{n+1}J : I^n\right)$$

is strongly Golod.

Proof. Let $f \in S$ be such that $fI^{n-1} \subseteq I^n J$ for some n. Then $fI^n \subseteq I^{n+1}J$. Let ∂ denote a partial derivative with respect to any variable. Taking partial derivatives, from the containment above we obtain that

$$\partial(f)I^n \subseteq fI^{n-1}\partial(I) + I^n\partial(I)J + I^{n+1}\partial(J) \subseteq I^n\partial(I)J + I^{n+1}\partial(J).$$

Let $f, g \in \bigcup_{n\geqslant 0} I^{n+1}J : I^n$ and choose $n \gg 0$ such that $fI^{n-1} \subseteq I^n J$ and $gI^{n-1} \subseteq I^n J$. Then

$$\partial(f)\partial(g)I^{2n} \subseteq I^{2n}\partial(I)^2J^2 + I^{2n+2}\partial(J)^2 + I^{2n+1}J \subseteq I^{2n+1}J$$

because $\partial(I)^2J \subseteq I$ and $I\partial(J)^2 \subseteq J$. □

In particular, Proposition 5.5.7 shows that the Ratliff-Rush closure of any power I^d, $d \geqslant 2$, is strongly Golod. In fact, it is enough to apply Proposition 5.5.7 to the strongly Golod pair (I^{d-1}, I). This already follows from [34, Proposition 2.12], since I^d is strongly Golod for any $d \geqslant 2$.

Given that the Ratliff-Rush closure of a strongly Golod ideal is strongly Golod we ask:

Question 5.5.8. Given a strongly Golod ideal $I \subseteq S$, is every coefficient ideal of I [strongly] Golod?

Question 5.5.8 is a more general version of Question 5.3.11. In fact, both the integral closure and the Ratliff-Rush closure are coefficient ideals. See [63] for details about coefficient ideals.

We conclude the section with two questions regarding the notion of strongly Golod ideal. The definition of strongly Golod ideals is restricted to homogeneous ideals in a polynomial ring $S = k[x_1, \ldots, x_n]$, with k a field of characteristic zero. This is because Herzog's canonical lift of Koszul cycles [31] can be applied only under these assumptions.

Question 5.5.9. Is there a suitable definition of strongly Golod for local rings, at least when the ring contains a field?

Question 5.5.10. Is there a notion of strongly Golod that does not require the characteristic of k to be zero?

Bibliography

[1] Abbott, J., Bigatti, A., and Lagorio, G. CoCoA-5: a system for doing Computations in Commutative Algebra. Available at http://cocoa.dima.unige.it.

[2] Aberbach, I. M., and Enescu, F. The structure of F-pure rings. *Math. Z. 250*, 4 (2005), 791–806.

[3] Anick, D. J. A counterexample to a conjecture of Serre. *Ann. of Math. (2) 115*, 1 (1982), 1–33.

[4] Atiyah, M. F., and Macdonald, I. G. *Introduction to commutative algebra.* Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1969.

[5] Auslander, M., and Buchweitz, R.-O. The homological theory of maximal Cohen-Macaulay approximations. *Mém. Soc. Math. France (N.S.)*, 38 (1989), 5–37. Colloque en l'honneur de Pierre Samuel (Orsay, 1987).

[6] Avramov, L. L. Infinite free resolutions. In *Six lectures on commutative algebra (Bellaterra, 1996)*, vol. 166 of *Progr. Math.* Birkhäuser, Basel, 1998, pp. 1–118.

[7] Avramov, L. L., and Golod, E. S. The homology of algebra of the Koszul complex of a local Gorenstein ring. *Mat. Zametki 9* (1971), 53–58.

[8] Backelin, J. Les anneaux locaux à relations monomiales ont des séries de Poincaré-Betti rationnelles. *C. R. Acad. Sci. Paris Sér. I Math. 295*, 11 (1982), 607–610.

[9] Benito, A., Faber, E., and Smith, K. E. Measuring singularities with Frobenius: the basics. In *Commutative algebra.* Springer, New York, 2013, pp. 57–97.

[10] Berglund, A., and Jöllenbeck, M. On the Golod property of Stanley-Reisner rings. *J. Algebra 315*, 1 (2007), 249–273.

[11] Brodmann, M. P., and Sharp, R. Y. *Local cohomology: an algebraic introduction with geometric applications*, vol. 60 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, 1998.

[12] Bruns, W., and Herzog, J. *Cohen-Macaulay rings*, vol. 39 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1993.

[13] Bruns, W., and Herzog, J. On multigraded resolutions. *Math. Proc. Cambridge Philos. Soc. 118*, 2 (1995), 245–257.

[14] Buchsbaum, D. A., and Eisenbud, D. What makes a complex exact? *J. Algebra 25* (1973), 259–268.

[15] De Stefani, A. A counterexample to a conjecture of Ding. *J. Algebra 452* (2016), 324–337.

[16] De Stefani, A. Products of ideals may not be Golod. *J. Pure Appl. Algebra 220*, 6 (2016), 2289–2306.

[17] De Stefani, A., and Núñez-Betancourt, L. F-thresholds of graded rings. *arxiv:1507.05459* (2015).

[18] Ding, S. A note on the index of Cohen-Macaulay local rings. *Comm. Algebra 21*, 1 (1993), 53–71.

[19] Ding, S. The associated graded ring and the index of a Gorenstein local ring. *Proc. Amer. Math. Soc. 120*, 4 (1994), 1029–1033.

[20] Ding, S. Auslander's δ-invariants of Gorenstein local rings. *Proc. Amer. Math. Soc. 122*, 3 (1994), 649–656.

[21] Fedder, R. F-purity and rational singularity. *Trans. Amer. Math. Soc. 278*, 2 (1983), 461–480.

[22] Fedder, R. F-purity and rational singularity in graded complete intersection rings. *Trans. Amer. Math. Soc. 301*, 1 (1987), 47–62.

[23] Fossum, R., and Foxby, H.-B. The category of graded modules. *Math. Scand. 35* (1974), 288–300.

[24] Golod, E. S. Homologies of some local rings. *Dokl. Akad. Nauk SSSR 144* (1962), 479–482.

[25] Goto, S., and Watanabe, K. On graded rings. I. *J. Math. Soc. Japan 30*, 2 (1978), 179–213.

[26] Grayson, D. R., and Stillman, M. E. Macaulay2, a software system for research in algebraic geometry. Available at http://www.math.uiuc.edu/Macaulay2/.

[27] Gruber, P. M. *Convex and discrete geometry*, vol. 336 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, Berlin, 2007.

[28] Gulliksen, T. H., and Levin, G. *Homology of local rings.* Queen's Paper in Pure and Applied Mathematics, No. 20. Queen's University, Kingston, Ont., 1969.

[29] Hashimoto, M., and Shida, A. Some remarks on index and generalized Loewy length of a Gorenstein local ring. *J. Algebra 187*, 1 (1997), 150–162.

[30] Hernández, D. J. F-purity of hypersurfaces. *Math. Res. Lett. 19*, 2 (2012), 389–401.

[31] Herzog, J. Canonical Koszul cycles. In *International Seminar on Algebra and its Applications (Spanish) (México City, 1991)*, vol. 6 of *Aportaciones Mat. Notas Investigación*. Soc. Mat. Mexicana, México, 1992, pp. 33–41.

[32] Herzog, J. On the index of a homogeneous Gorenstein ring. In *Commutative algebra: syzygies, multiplicities, and birational algebra (South Hadley, MA, 1992)*, vol. 159 of *Contemp. Math.* Amer. Math. Soc., Providence, RI, 1994, pp. 95–102.

[33] Herzog, J., and Hibi, T. *Monomial ideals*, vol. 260 of *Graduate Texts in Mathematics*. Springer-Verlag London, Ltd., London, 2011.

[34] Herzog, J., and Huneke, C. Ordinary and symbolic powers are Golod. *Adv. Math. 246* (2013), 89–99.

[35] Herzog, J., and Steurich, M. Golodideale der Gestalt $\mathfrak{a} \cap \mathfrak{b}$. *J. Algebra 58*, 1 (1979), 31–36.

[36] Herzog, J., Welker, V., and Yassemi, S. Homology of powers of ideals: Artin-Rees numbers of syzygies and the Golod property. *arXiv:1108.5862v1* (2011).

[37] Hirose, D., Watanabe, K.-i., and Yoshida, K.-i. F-thresholds versus a-invariants for standard graded toric rings. *Comm. Algebra 42*, 6 (2014), 2704–2720.

[38] Hochster, M. The equicharacteristic case of some homological conjectures on local rings. *Bull. Amer. Math. Soc. 80* (1974), 683–686.

[39] Hochster, M., and Huneke, C. Tight closure and strong F-regularity. *Mém. Soc. Math. France (N.S.)*, 38 (1989), 119–133. Colloque en l'honneur de Pierre Samuel (Orsay, 1987).

[40] HOCHSTER, M., AND ROBERTS, J. L. Rings of invariants of reductive groups acting on regular rings are Cohen-Macaulay. *Advances in Math. 13* (1974), 115–175.

[41] HUNEKE, C. Ideals defining Gorenstein rings are (almost) never products. *Proc. Amer. Math. Soc. 135*, 7 (2007), 2003–2005 (electronic).

[42] HUNEKE, C., MUSTAŢĂ, M., TAKAGI, S., AND WATANABE, K.-I. F-thresholds, tight closure, integral closure, and multiplicity bounds. *Michigan Math. J. 57* (2008), 463–483. Special volume in honor of Melvin Hochster.

[43] HUNEKE, C., AND SWANSON, I. *Integral closure of ideals, rings, and modules*, vol. 336 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2006.

[44] IYENGAR, S. Shifts in resolutions of multigraded modules. *Math. Proc. Cambridge Philos. Soc. 121*, 3 (1997), 437–441.

[45] JÖLLENBECK, M. On the multigraded Hilbert and Poincaré-Betti series and the Golod property of monomial rings. *J. Pure Appl. Algebra 207*, 2 (2006), 261–298.

[46] KATTHÄN, L. A non-Golod ring with a trivial product on its Koszul homology. *arxiv:1511.04883* (2015).

[47] KUNZ, E. *Introduction to plane algebraic curves*. Birkhäuser Boston, Inc., Boston, MA, 2005. Translated from the 1991 German edition by Richard G. Belshoff.

[48] LEUSCHKE, G. J., AND WIEGAND, R. *Cohen-Macaulay representations*, vol. 181 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2012.

[49] LYUBEZNIK, G., AND SMITH, K. E. Strong and weak F-regularity are equivalent for graded rings. *Amer. J. Math. 121*, 6 (1999), 1279–1290.

[50] MA, L. Finiteness properties of local cohomology for F-pure local rings. *Preprint* (2012).

[51] MACAULAY, F. S. Some Properties of Enumeration in the Theory of Modular Systems. *Proc. London Math. Soc. S2-26*, 1 (1927), 531.

[52] MATSUMURA, H. *Commutative ring theory*, second ed., vol. 8 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1989. Translated from the Japanese by M. Reid.

[53] McCullough, J., and Peeva, I. Infinite graded free resolutions. Available online at http://www.math.cornell.edu/ irena/papers/infiniteRes.pdf, 2013.

[54] Nagata, M. *Local rings*. Interscience Tracts in Pure and Applied Mathematics, No. 13. Interscience Publishers a division of John Wiley & Sons New York-London, 1962.

[55] Peskine, C., and Szpiro, L. Sur la topologie des sous-schémas fermés d'un shéma localement noethérien, définis comme support d'un faisceau cohérent localement de dimension projective finie. *C. R. Acad. Sci. Paris Sér. A-B 269* (1969), A49–A51.

[56] Peskine, C., and Szpiro, L. Dimension projective finie et cohomologie locale. Applications à la démonstration de conjectures de M. Auslander, H. Bass et A. Grothendieck. *Inst. Hautes Études Sci. Publ. Math.*, 42 (1973), 47–119.

[57] Roberts, P. Le théorème d'intersection. *C. R. Acad. Sci. Paris Sér. I Math. 304*, 7 (1987), 177–180.

[58] Rossi, M. E., and Şega, L. M. Poincaré series of modules over compressed Gorenstein local rings. *Adv. Math. 259* (2014), 421–447.

[59] Rossi, M. E., and Valla, G. *Hilbert functions of filtered modules*, vol. 9 of *Lecture Notes of the Unione Matematica Italiana*. Springer-Verlag, Berlin; UMI, Bologna, 2010.

[60] Rotman, J. J. *An introduction to homological algebra*, second ed. Universitext. Springer, New York, 2009.

[61] Schwede, K. Centers of F-purity. *Math. Z. 265*, 3 (2010), 687–714.

[62] Seyed Fakhari, S. A., and Welker, V. The Golod property for products and high symbolic powers of monomial ideals. *J. Algebra 400* (2014), 290–298.

[63] Shah, K. Coefficient ideals. *Trans. Amer. Math. Soc. 327*, 1 (1991), 373–384.

[64] Shamash, J. The Poincaré series of a local ring. *J. Algebra 12* (1969), 453–470.

[65] Singh, A. K., and Walther, U. Local cohomology and pure morphisms. *Illinois J. Math. 51*, 1 (2007), 287–298 (electronic).

[66] Takagi, S., and Watanabe, K.-i. On F-pure thresholds. *J. Algebra 282*, 1 (2004), 278–297.

[67] Valabrega, P., and Valla, G. Form rings and regular sequences. *Nagoya Math. J. 72* (1978), 93–101.

[68] Vraciu, A. Tight closure and linkage classes in Gorenstein rings. *Math. Z. 244*, 4 (2003), 873–885.

[69] Weibel, C. A. *An introduction to homological algebra*, vol. 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1994.