Affine Quantum Symmetric Pairs: Multiplication Formulas and Canonical Bases

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Abstract

One breakthrough in the theory of quantum groups is the construction of the canonical bases for quantum groups by Lusztig and Kashiwara. For type A, there is a geometric construction for (idempotented) quantum group together with a canonical basis due to Beilinson, Lusztig and MacPherson (BLM) using a stabilization procedure on a family of quantum Schur algebras of type A. Two essential ingredients in their work are a multiplication formula and a monomial basis.

In this dissertation, we provide a BLM-type construction for affine type C. We realize the affine q-Schur algebras of type C as an endomorphism algebra of a certain permutation module of affine Hecke algebras, and then establish a multiplication formula on the Schur algebra level. We provide a direct construction of monomial bases for Schur algebras, which is also adapted to produce monomial bases for affine type A. Via a BLM-type stabilization on the Schur algebras, we construct an algebra $\dot{\mathbf{K}}_n^{\mathfrak{c}}$ admitting canonical basis. We obtain that $(\mathbf{K}_n^{\mathfrak{a}}, \mathbf{K}_n^{\mathfrak{c}})$ forms a quantum symmetric pair in the spirit of Letzter and Kolb, where $\mathbf{K}_n^{\mathfrak{a}} \simeq \mathbf{U}(\widehat{\mathfrak{gl}}_n)$ is a quantum group of affine type A. The affine type C construction above is associated to an involution on Dynkin diagrams of affine type A. For other three types of involutions, we construct similar stabilization algebras admitting compatible canonical bases.

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I could not have gotten where I am without the love and support from Ching.

Contents

Chapter 1

Introduction

1.1 Background

Around 1985, the quantum groups were introduced by Drinfel'd and Jimbo [Dr86, Jim86], which have played important roles in representation theory, quantum topology, mathematical physics and many other areas. One of the most important developments in the theory of quantum groups is the construction of the canonical bases for quantum groups and their integrable modules by Lusztig and Kashiwara [Lu90, Ka91], which motivated further advances including categorification.

1.1.1 BLM construction for idempotented quantum $\mathfrak{gl}_n$

In [BLM90], Beilinson, Lusztig and MacPherson developed a geometric construction for the idempotented quantum group of type A together with its canonical basis. They started with a geometric realization of q-Schur algebras using partial flag varieties of type A. The standard basis of q-Schur algebra $\mathbf{S}_{n,d}$ can be parametrized by the set of $n \times n$ matrices over $\mathbb{N}$ whose entries add to d. Under this parametrization, a

(Rth divided power of) Chevalley generator of the Schur algebra is associated to a matrix $A = (a_{ij})$ having a unique non-zero off-diagonal entry, and the entry is either $a_{i,i+1} = R$ or $a_{i+1,i} = R$ for some $1 \leqslant i \leqslant n-1$. By deriving multiplication formulas with Chevalley generators on the Schur algebras, they provided a construction of a monomial basis satisfying the following properties:

(M1) A basis element is bar-invariant;

(M2) The transition matrix from this basis to the standard basis is unitriangular.

On the other hand, by showing that the structure constants in the multiplication formulas behave well, they developed a stabilization procedure that constructs a limit algebra, which can be identified with the modified quantum group of type A admitting both monomial and canonical bases.

$$\underset{q\text{-Schur algebra } (d\geqslant 1)}{\mathbf{S}_{n,d}} \quad \underset{d\to\infty}{\overset{\text{stabilization}}{\Longrightarrow}} \quad \underset{\text{stabilization algebra}}{\underleftarrow{\mathrm{Stab}}(\mathbf{S}_{n,d}) := \dot{\mathbf{K}}_n} \quad \simeq \quad \underset{\substack{\text{idempotented}\\ \text{quantum group}}}{\dot{\mathbf{U}}(\mathfrak{gl}_n)}$$

1.1.2 BLM-type constructions for affine type A

The BLM construction has been partially generalized to affine type A by Ginzburg-Vasserot [GV93] and by Lusztig [Lu99] via a geometric realization of affine q-Schur algebras using affine partial flags. The standard basis for affine Schur algebras of type A is parametrized by the set of periodic $\mathbb{Z} \times \mathbb{Z}$ $\mathbb{N}$-matrices with a fixed "size". Similar to finite type A, a (divided power of) Chevalley generator of the affine Schur algebra is associated to a matrix $A = (a_{ij})$ having a unique non-zero off-diagonal entry in each period, and that entry in each period is either $a_{i,i+1}$ or $a_{i,i-1}$ for some $i \in \mathbb{Z}$. A new phenomenon in affine types is that the Chevalley generators only generate a proper subalgebra. In order to generate the full Schur algebra, one needs a larger generating set associated to bidiagonal matrices.

Recently, Du and Fu provided another BLM-type construction [DF14, DF15] for affine type A. Instead of the geometric realization of Schur algebras, they use an algebraic realization of Schur algebras as endomorphism algebras of certain permutation modules of extended affine Hecke algebras. They further proved remarkable multiplication formulas with bidiagonal generators. They provided a family of monomial bases for affine Schur algebras by adapting monomial bases for the Ringel-Hall algebras of cyclic quivers due to Deng-Du-Xiao (cf. [DDX07]). The construction therein is quite involved and it is not clear how to understand their bases along the line of [BLM90].

1.1.3 BLM-type constructions and quantum symmetric pairs

For type B/C, Bao, Kujawa, Li and Wang [BKLW14] provided BLM-type constructions using a realization of the q-Schur algebra $\mathbf{S}^{\mathrm{B}}_{n,d}$ as a convolution algebra via partial flag varieties of type B/C. In this case, the standard basis of $\mathbf{S}^{\mathrm{B}}_{n,d}$ is parametrized by the set of centro-symmetric $n \times n$ $\mathbb{N}$-matrices. By developing multiplication formulas with Chevalley generators, they constructed stabilization algebras $\dot{\mathbf{U}}^{\imath}_n$ and $\dot{\mathbf{U}}^{\jmath}_n$ (depending on the parity of n) admitting canonical bases. The non-idempotented quantum algebras $\mathbf{U}^\jmath_n$ and $\mathbf{U}^\imath_n$ are not the Drinfel'd-Jimbo type quantum groups of type B/C, they are coideal subalgebras of the type A quantum group $\mathbf{U}(\mathfrak{gl}_n)$ in the sense that the comultiplication Δ of $\mathbf{U}(\mathfrak{gl}_n)$ sends $\mathbf{U}^\imath$ to $\mathbf{U}^\imath \otimes \mathbf{U}(\mathfrak{gl}_n)$, and sends $\mathbf{U}^\jmath$ to $\mathbf{U}^\jmath \otimes \mathbf{U}(\mathfrak{gl}_n)$. Moreover, $(\mathbf{U}(\mathfrak{gl}_n), \mathbf{U}^\imath_n)$ and $(\mathbf{U}(\mathfrak{gl}_n), \mathbf{U}^\imath_n)$ form quantum symmetric pairs, whose theory is developed and studied in [Le02, Ko14]. Recall that a symmetric pair $(\mathfrak{g}, \mathfrak{g}^\theta)$ consists of a Lie algebra $\mathfrak{g}$ and its fixed point subalgebra $\mathfrak{g}^\theta$ for some involution $\theta : \mathfrak{g} \to \mathfrak{g}$. A quantum symmetric pair $(\mathbf{U}, \mathbf{B})$ is a quantum analog of a symmetric pair, in the sense that $\mathbf{B}$ is a special coideal subalgebra as in [Ko14, Definition 5.1]. (See also [FL14] for a BLM-type stabilization for finite type D.)

1.1.4 The q-Schur algebras

For type A, the q-Schur algebra $\mathbf{S}^{\text{alg}}_{n,d}$ was introduced in the work of Dipper-James [DJ89] as an endomorphism algebra of certain module over Hecke algebra. As a consequence of the Schur-Jimbo duality [Jim86] between the quantum group $\mathbf{U}(\mathfrak{gl}_n)$ and the Hecke algebra of type A, it follows that $\mathbf{S}^{\text{alg}}_{n,d}$ is a quotient of $\mathbf{U}(\mathfrak{gl}_n)$. In this context, $\mathbf{S}^{\text{alg}}_{n,d}$ can be identified with the aforementioned convolution algebra $\mathbf{S}_{n,d}$ of pairs of partial flags. Moreover, the canonical bases of Schur algebras can be obtained either by a geometric approach [BLM90] using intersection cohomology, or by an algebraic approach [Du92] using canonical bases for Hecke algebras.

Beyond type A, there are different notions of "q-Schur algebras" arising from modular representations of algebraic groups or quantum groups at roots of unity (cf. [DDPW08] and the reference therein). We are interested in the Hecke-algebraic approach of q-Schur algebras along the line of Dipper-James. In finite types B/C, the convolution algebra $\mathbf{S}^{\mathrm{B}}_{n,d}$ constructed in [BKLW14] can also be realized algebraically as an endomorphism algebra of certain module over Hecke algebra of type B/C. Such an algebra is introduced by Green for even n (referred as the hyperoctahedral Schur algebra in [Gr97]). By a Schur-type duality in [BW13], the Schur algebra $\mathbf{S}^{\mathrm{B}}_{n,d}$ can be identified with a quotient of the coideal subalgebra $\mathbf{U}^{\imath}_n$ of $\mathbf{U}(\mathfrak{gl}_n)$.

1.2 Main results

It is natural to ask for an affinization of the previous results on BLM-type constructions and q-Schur algebras. We will concentrate on affine type C, for which a geometric approach has been developed in a joint work [FLLLW1] with Z. Fan, Y. Li, L. Luo and W. Wang. A comprehensive treatment for the Hecke-algebraic approach will appear in [FLLLW2]. In this dissertation we provide part of the Hecke-algebraic approach.

Let us start with affine type A, note that the monomial bases for the affine Schur algebra $\mathbf{S}^{\mathfrak{a}}_{n,d}$ used in [DF14] trace back to Hall algebras of cyclic quivers [DDX07], while for classical types, the monomial basis elements are constructed directly by multiplying Chevalley generators in a suitable order. We provide a direct construction [LL15] in the same spirit for monomial bases of $\mathbf{S}^{\mathfrak{a}}_{n,d}$ by multiplying bidiagonal generators in a suitable order.

Theorem A (Theorem 2.4.2, Corollary 2.4.3)**.** Algorithm 2.4.1 produces a monomial basis (and hence a canonical basis) for $\mathbf{S}^{\mathfrak{a}}_{n,d}$.

From now on we switch to affine type C. We start with studying the affine Schur algebra $\mathbf{S}^{\mathfrak{c}}_{n,d}$ as an endomorphism algebra of certain permutation modules over affine Hecke algebras of type C (cf. (3.2.1)). In order to develop a BLM-type construction, one needs to derive a multiplication formula with the generating elements for $\mathbf{S}^{\mathfrak{c}}_{n,d}$, and to construct a monomial basis of $\mathbf{S}^{\mathfrak{c}}_{n,d}$. There are two crucial differences comparing to the previous work:

1. The affine type C analogue of Chevalley generators or bidiagonal generators do not form a generating set for $\mathbf{S}^{\mathfrak{c}}_{n,d}$;

2. The constructions of monomial bases in previous work do not generalize naively to a construction of a monomial basis for $\mathbf{S}^{\mathfrak{c}}_{n,d}$.

Precisely speaking, the characteristic basis $\{e_A\}_{A\in\Xi_{n,d}}$ of $\mathbf{S}^{\mathfrak{c}}_{n,d}$ is parametrized by the set $\Xi_{n,d}$ (cf. Section 3.2) of n-periodic centro-symmetric $\mathbb{Z}\times\mathbb{Z}$ $\mathbb{N}$-matrices with size d. In light of the centro-symmetry condition, it is reasonable to hope that $\mathbf{S}^{\mathfrak{c}}_{n,d}$ is generated by the elements parametrized by the tridiagonal matrices $A = (a_{ij})$ in the sense that $a_{ij} = 0$ unless $|i - j| \leqslant 1$. Indeed, this fact follows once we obtain the corresponding multiplication formula. One of the difficulties in deriving the multiplication formula is the appearance of certain nontrivial structure constants for affine Hecke algebras (cf. Remark 4.1.2). We now paraphrase the multiplication formula (cf. Theorem 4.4.7 for details) and the upshots below.

Theorem B. For $A, B \in \Xi_{n,d}$ with B being tridiagonal, we establish a multiplication formula for $e_B * e_A \in \mathbf{S}^{\mathfrak{c}}_{n,d}$ with explicit coefficients. Moreover, the set $\{e_A \mid A \in \Xi_{n,d}$ is tridiagonal$\}$ is a generating set for the Schur algebra $\mathbf{S}^{\mathfrak{c}}_{n,d}$.

We then define the standard basis element $[A]$ by normalizing e_A so that $\overline{[A]} = [A]+$ lower terms, with respect to a partial order $\leqslant_{\mathrm{alg}}$ on $\Xi_{n,d}$. A key ingredient in our construction of monomial bases is the admissible pairs. We show that (cf. Lemma 5.2.4) if (B, A) is an admissible pair, then the leading coefficient for the highest term in $[B] * [A]$ is one. Using this lemma, we first construct a semi-monomial basis $\{m'_A\}_{A \in \Xi_{n,d}}$ by multiplying the tridiagonal generators in a suitable order. Another new phenomenon for affine type C is that the generating element $[B]$ with B being tridiagonal is not necessarily bar-invariant. Nevertheless, the semi-monomial basis can be adapted to a monomial basis $\{m_A\}_{A \in \Xi_{n,d}}$.

Theorem C (Theorem 5.2.8, Proposition 5.2.11)**.** The Schur algebra $\mathbf{S}^{\mathfrak{c}}_{n,d}$ admits both monomial and canonical bases.

With the results on the Schur algebra level, we can now construct the stabilization algebra $\dot{\mathbf{K}}^{\mathfrak{c}}_n$ as outlined below:

$$\underset{\text{Affine } q\text{-Schur algebra } (d \geqslant 1)}{\mathbf{S}^{\mathfrak{c}}_{n,d}} \quad \overset{\text{stabilization}}{\underset{d \to \infty}{\Longrightarrow}} \quad \underset{\text{stabilization algebra}}{\underleftarrow{\mathrm{Stab}}(\mathbf{S}^{\mathfrak{c}}_{n,d}) := \dot{\mathbf{K}}^{\mathfrak{c}}_n}$$

Let $\dot{\mathbf{K}}^{\mathfrak{c}}_n$ be the free $\mathbb{Z}[v, v^{-1}]$-module generated by $\{[A]\}_{A \in \tilde{\Xi}_n}$, where $\tilde{\Xi}_n$ is adapted from $\bigcup_{d \in \mathbb{N}} \Xi_{n,d}$ by allowing diagonal entries to be negative integers. Therefore, for any

$A \in \tilde{\Xi}_n$, the matrix $A + pI$ lies in $\Xi_{n,d+pn/2}$ for any large enough even integer p, where $I = (\delta_{ij})$ is the identity matrix. We show that $\dot{\mathbf{K}}^{\mathfrak{c}}_n$ has a unique associative algebra structure in the sense that for any $B, A \in \tilde{\Xi}_n$, the structure constants for $[B]*[A] \in \dot{\mathbf{K}}^{\mathfrak{c}}_n$ are compatible with the structure constants for $[B+pI]*[A+pI] \in \mathbf{S}^{\mathfrak{c}}_{n,d+pn/2}$ for all even p that is large enough. In other words, the multiplication formula with tridiagonal generators has an analogue on the stabilization algebra level. Therefore, we can "lift" the monomial basis for $\mathbf{S}^{\mathfrak{c}}_{n,d}$ to the stabilization algebra level to construct both monomial and canonical bases for $\dot{\mathbf{K}}^{\mathfrak{c}}_n$.

Theorem D (Corollary 6.1.4, Theorem 6.1.5)**.** We have an algebra $\dot{\mathbf{K}}^{\mathfrak{c}}_n$ arising from stabilization on the Schur algebras $\mathbf{S}^{\mathfrak{c}}_{n,d}$. Moreover, $\dot{\mathbf{K}}^{\mathfrak{c}}_n$ admits both monomial and canonical bases.

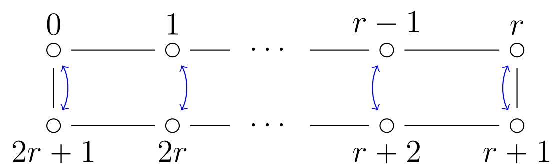
Moreover, by identifying $\dot{\mathbf{K}}^{\mathfrak{c}}_n$ with a similar stabilization algebra $\dot{\mathbf{K}}^{\mathfrak{c},\mathrm{geo}}_n$ in [FLLLW1], we can relate the algebras $\dot{\mathbf{K}}^{\mathfrak{c}}_n$ and $\mathbf{S}^{\mathfrak{c}}_{n,d}$ by a natural surjective map $\Phi_{n,d} : \dot{\mathbf{K}}^{\mathfrak{c}}_n \to \mathbf{S}^{\mathfrak{c}}_{n,d}$ given by

$$[A] \mapsto \begin{cases} [A] & \text{if } A \in \Xi_{n,d}, \\ 0 & \text{otherwise,} \end{cases}$$

In [FLLLW1, 9.7] we show that $\Phi_{n,d}$ is a homomorphism that also preserves canonical basis. It is standard that one can construct the non-idempotented stabilization algebras $\mathbf{K}^{\mathfrak{c}}_n$ (resp., $\mathbf{K}^{\mathfrak{a}}_n$) from $\dot{\mathbf{K}}^{\mathfrak{c}}_n$ (resp., $\dot{\mathbf{K}}^{\mathfrak{a}}_n$) by taking certain infinite sum. We obtain that $(\mathbf{K}^{\mathfrak{a}}_n, \mathbf{K}^{\mathfrak{c}}_n)$ forms a quantum symmetric pair, for which the detail will appear in [FLLLW2].

Note that the above construction for affine type C is associated to an involution (of type $\jmath\jmath$) on the Dynkin diagram of affine type A depicted in Figure 1.1 below. For

Figure 1.1: Dynkin diagram of type $A^{(1)}_{2r+1}$ with involution of type $\jmath\jmath \equiv \mathfrak{c}$.



other types of involutions $\jmath\imath, \imath\jmath$ and $\imath\imath$ (cf. Figures 7.1, 7.2, 7.3, respectively.), we also construct stabilization algebras $\dot{\mathbf{K}}^{\jmath\imath}_n, \dot{\mathbf{K}}^{\imath\jmath}_n, \dot{\mathbf{K}}^{\imath\imath}_n$ together with their canonical bases. There are subquotient relations among the four stabilization algebras which preserve their canonical bases.

1.3 Organization

In Chapter 2, we provide a direct construction producing a monomial basis (and hence a canonical basis) for affine Schur algebras of type A (cf. Theorem 2.4.2 and Corollary 2.4.3).

In Chapter 3, we study the affine q-Schur algebras of type C as endomorphism algebras. We also identify this algebraic realization with the geometric realization via affine flag varieties as in [FLLLW1].

Chapter 4 is devoted to the proof of the multiplication formula (Theorem 4.4.7) for tridiagonal generators.

In Chapter 5 we adapt the construction in Chapter 2 to produce a monomial basis (and hence a canonical basis) for affine Schur algebra of type C (cf. Theorem 5.2.8 and Proposition 5.2.11).

In Chapter 6 we construct a stabilization algebra $\dot{\mathbf{K}}^{\mathfrak{c}}_n$ from the affine Schur algebras of type C. We then show that $\dot{\mathbf{K}}^{\mathfrak{c}}_n$ can be identified with a similar stabilization algebra defined in a geometric framework (cf. [FLLLW1]), and it follows that $\mathbf{K}^{\mathfrak{c}}_n$ (resp., $\dot{\mathbf{K}}^{\mathfrak{c}}_n$) is a coideal subalgebra of $\mathbf{K}^{\mathfrak{a}}_n$ (resp., $\dot{\mathbf{U}}(\widehat{\mathfrak{gl}}_n)$).

Chapter 7 provides a formulation of three more variants of the stabilization algebras for different types of involutions on the Dynkin diagram of affine type A.

Chapter 2

Affine Schur algebras of type A

In this chapter we recall first some standard facts about the extended affine Weyl groups of type A, the corresponding Hecke algebras, and the affine q-Schur algebras of type A as an endomorphism algebra of certain q-permutation modules. We provide a direct construction producing a monomial basis (and hence a canonical basis) for the affine Schur algebra (cf. Theorem 2.4.2 and Corollary 2.4.3).

Throughout the dissertation, let $\mathbb{N} = \{0, 1, 2, \ldots\}$ be the set of natural numbers. Denote by $[a..b], [a..b), (a..b]$ and $(a..b)$ the integer intervals for $a, b \in \mathbb{Z}$. Let v be an indeterminate over $\mathbb{Q}$, and let $[a] = \frac{v^{2a}-1}{v^2-1}$ for $a \in \mathbb{Z}$. In this chapter, let n, d be fixed positive integers.

2.1 Affine Hecke algebras

Let W be the Weyl group of type $\widetilde{A}_{d-1}$ generated by $S = \{s_1, s_2, \ldots, s_d = s_0\}$. The extended Weyl group $\widetilde{W}$ is generated by W and π satisfying $\pi s_i \pi^{-1} = s_{i-1}$ for $i = 1, \ldots, d$. It is well-known that $\widetilde{W}$ can be identified as a permutation subgroup of

$\mathbb{Z}$ satisfying $g(i+d) = g(i) + d$ for all $i \in \mathbb{Z}, g \in \widetilde{W}$. In this identification each s_i is mapped to the permutation $\prod_{k\in\mathbb{Z}} (kd+i, kd+i+1)$ and π is mapped to the permutation $t \mapsto t+1$ for $t \in \mathbb{Z}$. Denote the length function on W by ℓ. Notice that each $g \in \widetilde{W}$ can be uniquely expressed as $g = \pi^z w$ for some $z \in \mathbb{Z}$ and $w \in W$, so the notion of length on W can be extended to $\widetilde{W}$ by requiring $\ell(\pi) = 0$, or equivalently, $\ell(g) = \ell(w)$.

Lemma 2.1.1. *The length of $g \in \widetilde{W}$ is given by*

$$\ell(g) = |\{(i,j) \in [1..d] \times \mathbb{Z} \mid i < j, g(i) > g(j)\}|.$$

Proof. Let $g = \pi^z w$ for some $z \in \mathbb{Z}$ and $w \in W$. Notice that $g(i) > g(j)$ is equivalent to $w(i) > w(j)$ and hence the lemma reduces to the case $z = 0$, which follows from [BB05, (8.30)]. ☐

Denote the set of (weak) compositions of d into n parts by

$$\Lambda^{\mathfrak{a}}_{n,d} = \{\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{N}^n \mid \sum_{i=1}^{n} \lambda_i = d\}. \tag{2.1.1}$$

Throughout this chapter we write $\Lambda = \Lambda^{\mathfrak{a}}_{n,d}$ for short. For each $\lambda \in \Lambda$, denote by W_λ the parabolic subgroup of W with respect to λ generated by $S\backslash\{s_{\lambda_1}, s_{\lambda_1+\lambda_2}, \ldots, s_{\lambda_1+\ldots+\lambda_{n-1}}\}$. For each $z \in \mathbb{Z}$, let $\lambda + z$ be the composition in Λ such that $W_{\lambda+z} = \pi^{-z} W_\lambda \pi^z$.

Example 2.1.2. Let $n = 3, d = 6, z = 4$ and $\lambda = (1,2,3) \in \Lambda$. We have $W_\lambda = \langle s_2, s_4, s_5, s_6\rangle$, $W_{\lambda+4} = \langle s_6, s_2, s_3, s_4\rangle$ and hence $\lambda + 4 = (1,4,1) \in \Lambda$.

Let $\mathcal{D}_\lambda = \{w \in \widetilde{W} \mid \ell(wg) = \ell(w) + \ell(g) \text{ for } g \in W_\lambda\}$. Then $\mathcal{D}_\lambda$ (resp., $\mathcal{D}_\lambda^{-1}$) is the set of distinguished right (resp. left) coset representatives of W_λ in $\widetilde{W}$. Denote by $\mathcal{D}_{\lambda\mu} = \mathcal{D}_\lambda \cap \mathcal{D}_\mu^{-1}$ the set of distinguished double coset representatives.

Lemma 2.1.3 (Howlett). *Let $\lambda, \mu \in \Lambda$, and let $g \in \mathcal{D}_{\lambda\mu}$. Then*

(a) *There is a unique $\delta = \delta(\lambda, g, \mu) \in \Lambda^{\mathfrak{a}}_{n',d}$ for some n' such that*

$$W_\delta = g^{-1} W_\lambda g \cap W_\mu.$$

(b) *The map $W_\lambda \times (\mathcal{D}_\delta \cap W_\mu) \to W_\lambda g W_\mu$ sending (x, y) to xgy is a bijection satisfying $\ell(xgy) = \ell(x) + \ell(g) + \ell(y)$.*

Proof. Part (a) follows from [Gr99, Lemma 2.2.2] and Part (b) is known (cf. [DDPW08, Theorem 4.18]). □

Let $\leqslant$ be the (strong) Bruhat order on W. Extend it to $\widetilde{W}$ by

$$\pi^{z_1} w_1 \leqslant \pi^{z_2} w_2 \text{ if and only if } z_1 = z_2, w_1 \leqslant w_2. \tag{2.1.2}$$

The extended affine Hecke algebra $\mathcal{H} = \mathcal{H}(\widetilde{W})$ associated to $\widetilde{W}$ is a $\mathbb{Z}[v, v^{-1}]$-algebra with a basis $\{T_g \mid g \in \widetilde{W}\}$ (cf. e.g., [Gr99, Proposition 1.2.3]) satisfying $T_w T_{w'} = T_{ww'}$ if $\ell(w) + \ell(w') = \ell(ww')$ and $(T_s + 1)(T_s - v^2) = 0$ for $s \in S$. For a finite subset $X \subset \widetilde{W}$ and for each $\lambda \in \Lambda$, let

$$T_X = \sum_{w \in X} T_w \quad \text{and} \quad x_\lambda = T_{W_\lambda}. \tag{2.1.3}$$

Following [KL79], denote by $\{C'_w \mid w \in W\}$ the Kazhdan-Lusztig basis of the Hecke algebra $\mathcal{H}(W)$ associated to W. For each $w \in W$, we have $C'_w = v^{-\ell(w)} \sum_{y \leqslant w} P_{y,w} T_y$, where $P_{y,w} \in \mathbb{Z}[v^2]$ is the Kazhdan-Lusztig polynomial. Note that $\mathcal{H} = \mathcal{H}(\widetilde{W})$ contains $\mathcal{H}(W)$ as a subalgebra, we define $C'_g = T_\pi^z C'_w \in \mathcal{H}$ for each $g = \pi^z w \in \widetilde{W}$ with $w \in W, z \in \mathbb{Z}$.

Statements in Lemma 2.1.4 below are known for non-extended Weyl groups and Hecke algebras (cf. [Cur85, Theorem 1.2(i)], [DDPW08, Corollary 4.19]). It seems that the extended version is taken for granted for the experts (cf. [DF14, Lemma 7.1]), and we provide a proof here for completeness.

Lemma 2.1.4. *Let $\lambda, \mu \in \Lambda, g = \pi^z w \in \mathcal{D}_{\lambda\mu}$ for some $w \in W$ and $z \in \mathbb{Z}$. Denote by $w_\circ^\nu$ the longest element in W_ν for any composition ν. Then:*

(a) *The longest element $g^+_{\lambda\mu}$ in $W_\lambda g W_\mu$ is given by $g^+_{\lambda\mu} = w_\circ^\lambda g w_\circ^{\delta(\lambda,g,\mu)} w_\circ^\mu$. In particular,*

$$\ell(g^+_{\lambda\mu}) = \ell(w_\circ^\lambda) + \ell(g) - \ell(w_\circ^{\delta(\lambda,g,\mu)}) + \ell(w_\circ^\mu).$$

(b) $W_\lambda g W_\mu = \{x \in \widetilde{W} \mid g \leqslant x \leqslant g^+_{\lambda\mu}\}$.

(c) *There exists $c^{(\lambda,\mu)}_{x,g} \in \mathbb{Z}[v, v^{-1}]$ such that*

$$T_{W_\lambda g W_\mu} = v^{\ell(g^+_{\lambda\mu})} C'_{g^+_{\lambda\mu}} + \sum_{\substack{x \in \mathscr{D}_{\lambda\mu} \\ x < g}} c^{(\lambda,\mu)}_{x,g} C'_{x^+_{\lambda\mu}}.$$

In particular, $x_\mu = v^{\ell(w_\circ^\mu)} C'_{w_\circ^\mu}$.

Proof. By [DDPW08, Corollary 4.19], we have $w^+_{\lambda+z,\mu} = w_\circ^{\lambda+z} w w_\circ^{\delta(\lambda+z,w,\mu)} w_\circ^\mu$. Note that $W_\lambda g W_\mu = W_\lambda \pi^z w W_\mu = \pi^z W_{\lambda+z} w W_\mu$. Hence

$$g^+_{\lambda\mu} = \pi^z w^+_{\lambda+z,\mu} = w_\circ^\lambda g w_\circ^{\delta(\lambda+z,w,\mu)} w_\circ^\mu.$$

By Lemma 2.1.3(a), we have

$$W_{\delta(\lambda+z,w,\mu)} = w^{-1} W_{\lambda+z} w \cap W_\mu = w^{-1} \pi^{-z} W_\lambda \pi^z w \cap W_\mu = W_{\delta(\lambda,g,\mu)},$$

which implies (a). In particular,

$$\ell(w^+_{\lambda+z,\mu}) = \ell(g^+_{\lambda\mu}).$$

Again by [DDPW08, Corollary 4.19], we have $W_{\lambda+z}wW_\mu = \{y \in W \mid w \leqslant y \leqslant w^+_{\lambda+z,\mu}\}$ and hence

$$W_\lambda g W_\mu = \{x = \pi^z y \mid g = \pi^z w \leqslant x \leqslant g^+_{\lambda\mu}\}.$$

Therefore,

$$\begin{aligned} T_{W_\lambda g W_\mu} &= T^z_\pi T_{W_{\lambda+z}wW_\mu} = T^z_\pi \Big(v^{\ell(w^+_{\lambda+z,\mu})} C'_{w^+_{\lambda+z,\mu}} + \sum_{y<w} c^{(\lambda+z,\mu)}_{y,w} C'_{y^+_{\lambda+z,\mu}} \Big) \\ &= v^{\ell(g^+_{\lambda\mu})} C'_{g^+_{\lambda\mu}} + \sum_{\pi^z y < g} c^{(\lambda+z,\mu)}_{y,w} T^z_\pi C'_{y^+_{\lambda+z\mu}}. \end{aligned}$$

We are done by recognizing $x = \pi^z y$ and $c^{(\lambda,\mu)}_{x,g} = c^{(\lambda+z,\mu)}_{y,w}$. □

2.2 Affine Schur algebras

For $\lambda, \mu \in \Lambda$ and $g \in \mathcal{D}_{\lambda\mu}$, denote by $\phi^g_{\lambda\mu} \in \mathrm{Hom}_{\mathcal{H}}(x_\mu\mathcal{H}, \mathcal{H})$ the right $\mathcal{H}$-linear map sending x_μ to $T_{W_\lambda g W_\mu}$. Thanks to Lemma 2.1.3(b), we have $T_{W_\lambda g W_\mu} = x_\lambda T_g T_{\mathscr{D}_\delta \cap W_\mu}$ for some $\delta \in \Lambda^{\mathfrak{a}}_{n',d}$ and hence $\phi^g_{\lambda\mu} \in \mathrm{Hom}_{\mathcal{H}}(x_\mu\mathcal{H}, x_\lambda\mathcal{H})$. The affine q-Schur algebra is defined by

$$\mathbf{S}^{\mathfrak{a}}_{n,d} = \mathbf{S}^{\mathfrak{a}}_{n,d}(n,d) = \mathrm{End}_{\mathcal{H}}\Big(\bigoplus_{\lambda\in\Lambda} x_\lambda\mathcal{H} \Big) = \bigoplus_{\lambda,\mu\in\Lambda} \mathrm{Hom}_{\mathcal{H}}(x_\mu\mathcal{H}, x_\lambda\mathcal{H}).$$

There is also a geometric definition for $\mathbf{S}^{\mathfrak{a}}_{n,d}$ as given in [Lu99]. It is known (cf. [Gr99, Theorem 2.2.4]) that $\{\phi^g_{\lambda\mu} \mid \lambda, \mu \in \Lambda, g \in \mathcal{D}_{\lambda\mu}\}$ forms a basis of $\mathbf{S}^{\mathfrak{a}}_{n,d}$.

Let $\Theta_n = \bigcup_{d\in\mathbb{N}} \Theta_{n,d}$, where $\Theta_{n,d}$ is the set of $\mathbb{Z}\times\mathbb{Z}$ matrices over $\mathbb{N}$ in which each element $A = (a_{ij})_{ij}$ satisfies the following conditions:

(T1) $a_{ij} = a_{i+n,j+n}$ for all $i, j \in \mathbb{Z}$;

(T2) $\sum_{i=1}^{n} \sum_{j\in\mathbb{Z}} a_{ij} = d$.

For $i, j \in \mathbb{Z}$, define a matrix $E^{ij} = (E^{ij}_{xy})_{xy} \in \Theta_{n,1}$ by

$$E^{ij}_{xy} = \begin{cases} 1 & \text{if } (x, y) = (i + rn, j + rn) \text{ for the same } r \in \mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases} \tag{2.2.1}$$

For each matrix $T = (t_{ij})_{ij} \in \Theta_n$, define its row sum vector $\mathrm{ro}_{\mathfrak{a}}(T) = (\mathrm{ro}_{\mathfrak{a}}(T)_1, \ldots, \mathrm{ro}_{\mathfrak{a}}(T)_n)$ and its column sum vector $\mathrm{co}_{\mathfrak{a}}(T) = (\mathrm{co}_{\mathfrak{a}}(T)_1, \ldots \mathrm{co}_{\mathfrak{a}}(T)_n)$ by

$$\mathrm{ro}_{\mathfrak{a}}(T)_k = \sum_{j\in\mathbb{Z}} t_{kj}, \quad \mathrm{co}_{\mathfrak{a}}(T)_k = \sum_{i\in\mathbb{Z}} t_{ik}, \quad k = 1, \ldots, n. \tag{2.2.2}$$

For each $\lambda \in \Lambda$ and $i = 1, \ldots, n$, we define integral intervals with respect to λ by

$$\mathcal{R}^{\lambda}_i = (\sum_{k=1}^{i-1} \lambda_k \,..\, \sum_{k=1}^{i} \lambda_k]. \tag{2.2.3}$$

It is known that $\Theta_{n,d}$ parametrizes a basis of $\mathbf{S}^{\mathfrak{a}}_{n,d}$ in [VV99, §7.4] and [DF15].

Lemma 2.2.1. *The map*

$$\kappa : \{(\lambda, g, \mu) \mid \lambda, \mu \in \Lambda, g \in \mathcal{D}_{\lambda\mu}\} \longrightarrow \Theta_{n,d}$$

is a bijection sending (λ, g, μ) *to* $A = (a_{ij})_{ij}$ *where* $a_{ij} = |\mathcal{R}^{\lambda}_i \cap g\mathcal{R}^{\mu}_j|$.

For $A = (a_{ij})_{ij} = \kappa(\lambda, g, \mu) \in \Theta_{n,d}$, set $e_A = \phi^g_{\lambda\mu}$. Hence $\{e_A \mid A \in \Theta_d\}$ forms a basis of $\mathbf{S}^{\mathfrak{a}}_{n,d}$. For each $j = 1, \ldots, n$, let $(\delta^{(j)}_1, \ldots, \delta^{(j)}_{k_j}) \in \Lambda^{\mathfrak{a}}_{k_j,\lambda_j}$ for some $k_j \in \mathbb{N}$ be the composition obtained from $(\ldots, a_{-1,j}, a_{0j}, a_{1j}, \ldots)$ by deleting all zero entries. Define $\delta(A) \in \Lambda^{\mathfrak{a}}_{n',d}$ by

$$\delta(A) = (\delta^{(1)}_1, \ldots, \delta^{(1)}_{k_1}, \delta^{(2)}_1, \ldots, \delta^{(n)}_1, \ldots, \delta^{(n)}_{k_n}). \tag{2.2.4}$$

Lemma 2.2.2. *Let $A = \kappa(\lambda, g, \mu)$. Then $W_{\delta(A)} = g^{-1}W_\lambda g \cap W_\mu$. In particular, $\delta(A)$ is equal to $\delta(\lambda, g, \mu)$ described in Lemma 2.1.3.*

Proof. By [BB05, Proposition 8.3.4], for each composition $\lambda \in \Lambda^{\mathfrak{a}}_{n,d}$ we have $W_{S\backslash\{s_{\lambda_{0,i}}\}} = \mathrm{Stab}[\lambda_{0,i} + 1..d + \lambda_{0,i}]$ (here are below Stab stands for stabilizer), and hence

$$W_\lambda = \bigcap_{i=1}^{n} \mathrm{Stab}[\lambda_{0,i} + 1..d + \lambda_{0,i}] = \bigcap_{i=1}^{n} \mathcal{R}^\lambda_i = \bigcap_{i\in\mathbb{Z}} \mathcal{R}^\lambda_i.$$

Therefore,

$$g^{-1}W_\lambda g \cap W_\mu = \Big(\bigcap_{i\in\mathbb{Z}} \mathrm{Stab}(g^{-1}\mathcal{R}^\lambda_i)\Big) \cap \Big(\bigcap_{j\in\mathbb{Z}} \mathrm{Stab}(\mathcal{R}^\mu_j)\Big) = \bigcap_{(i,j)\in\mathbb{Z}^2} \mathrm{Stab}(g^{-1}\mathcal{R}^\lambda_i \cap \mathcal{R}^\mu_j).$$

It then follows from definition that $W_{\delta(A)} = g^{-1}W_\lambda g \cap W_\mu$. □

Set $\ell(A) = \ell(g)$ for $A = \kappa(\lambda, g, \mu)$. The following lemma and a two-page proof can be found in [DF15, Lemma 3.2(2)]. Here we provide a much shorter proof by combining Lemma 2.1.1 and Lemma 2.2.1.

Lemma 2.2.3. *Assume that $A \in \Theta_{n,d}$. Then*

$$\ell(A) = \sum_{\substack{i\in\mathbb{Z}\\ 1\leqslant j\leqslant n}} \sum_{\substack{x<i\\ y>j}} a_{ij}a_{xy} = \sum_{\substack{1\leqslant i\leqslant n\\ j\in\mathbb{Z}}} \sum_{\substack{x>i\\ y<j}} a_{ij}a_{xy}.$$

Proof. Let $A = \kappa(\lambda, g, \mu)$ for some $\lambda, \mu \in \Lambda$ and $g \in \mathcal{D}_{\lambda\mu}$. By Lemma 2.2.1, for all $i, j \in \mathbb{Z}$ there is a natural bijection $\mathcal{R}_i^\lambda \cap g\mathcal{R}_j^\mu \leftrightarrow \{(g(s), s) \in \mathcal{R}_i^\lambda \times \mathcal{R}_j^\mu\}$ between sets of size a_{ij}. Note that for $(g(s), s), (g(t), t) \in \mathcal{R}_i^\lambda \times \mathcal{R}_j^\mu$, the condition $s < t$ is equivalent to the condition $g(s) < g(t)$ since $g \in \mathcal{D}_{\lambda\mu}$. Hence if $(s, t) \in \mathcal{R}_j^\mu \times \mathcal{R}_y^\mu$ satisfies both $s < t$ and $g(s) > g(t)$, then j must be smaller than y.

Under these bijections, the set of pairs $(s, t) \in \mathbb{Z}^2$ satisfying "$s < t, g(s) > g(t)$ and $s \in [1..d]$" becomes the set of quadruples $(g(s), s, g(t), t) \in \mathcal{R}_i^\lambda \times \mathcal{R}_j^\mu \times \mathcal{R}_x^\lambda \times \mathcal{R}_y^\mu$ satisfying "$j < y, i > x$ and $1 \leqslant j \leqslant n$". The first assertion follows. The second assertion follows from that $\ell(g) = \ell(g^{-1})$ and $\kappa(\mu, g^{-1}, \lambda) = {}^{\mathrm{t}}A$. ∎

Example 2.2.4. Let $n = 2, d = 10$ and

$$A = 3E_{10} + 4E_{12} + E_{23} + 2E_{24} = \left[\begin{array}{cc:cc:cc} \ddots & 4 & & & & \\ & & 1 & 2 & & \\ \hdashline & 3 & 0 & 4 & & \\ & & 0 & 0 & 1 & 2 \\ \hdashline & & & 3 & & 4 \\ & & & & & \ddots \end{array}\right] \in \Theta_{2,10}.$$

We have $\delta(A) = (a_{01}, a_{02}, a_{12}, a_{32}) = (1, 2, 4, 3) \in \Lambda_{4,10}^{\mathfrak{a}}$ and $\ell(A) = 3(1 + 2) = 9$.

For $A \in \Theta_{n,d}$, let

$$d_A^{\mathfrak{a}} = \sum_{\substack{1 \leqslant i \leqslant n \\ j \in \mathbb{Z}}} \sum_{\substack{x \leqslant i \\ y > j}} a_{ij} a_{xy}, \qquad [A] = v^{-d_A^{\mathfrak{a}}} e_A. \tag{2.2.5}$$

It is clear that $\{[A] \mid A \in \Theta_{n,d}\}$ is a basis of $\mathbf{S}_{n,d}^{\mathfrak{a}}$, which is called the *standard basis* (cf. [Lu99]). The following is due to Du-Fu [DF14, Lemma 7.1], and we offer a slightly different argument.

Lemma 2.2.5. *For* $A = \kappa(\lambda, g, \mu) \in \Theta_{n,d}$*, we have* $d_A^{\mathfrak{a}} = \ell(g_{\lambda\mu}^{+}) - \ell(w_\circ^\mu)$.

Proof. Let $\delta = \delta({}^{\mathrm{t}}A) = (\delta_1^{(1)}, \dots, \delta_{k_1}^{(1)}, \delta_1^{(2)}, \dots, \delta_1^{(n)}, \dots, \delta_{k_n}^{(n)})$ as in (2.2.4). So $\lambda_i = \sum_{j=1}^{k_i} \delta_j^{(i)}$, $W_\delta \simeq W_{\delta(A)}$ and hence $\ell(w_\circ^{\delta(A)}) = \ell(w_\circ^\delta)$. We have $\ell(g_{\lambda\mu}^{+}) - \ell(w_\circ^\mu) = \ell(g) + \ell(w_\circ^\lambda) - \ell(w_\circ^\delta)$ by Lemma 2.1.4(a), where

$$\begin{aligned} \ell(g) &= \sum_{\substack{1\leqslant i\leqslant n\\ j\in\mathbb{Z}}} \sum_{\substack{x<i\\ y>j}} a_{ij}a_{xy}, \\ \ell(w_\circ^\lambda) - \ell(w_\circ^\delta) &= \sum_{i=1}^{n} \binom{\lambda_i}{2} - \sum_{i=1}^{n'} \binom{\delta_i}{2} = \sum_{i=1}^{n} \left(\binom{\sum_{j=1}^{k_i} \delta_j^{(i)}}{2} - \sum_{j=1}^{k_i} \binom{\delta_j^{(i)}}{2} \right) = \sum_{i=1}^{n} \sum_{y>j} \delta_j^{(i)} \delta_y^{(i)} \\ &= \sum_{\substack{1\leqslant i\leqslant n\\ j\in\mathbb{Z}}} \sum_{\substack{x=i\\ y>j}} a_{ij}a_{xy}. \end{aligned}$$

Therefore, $\ell(g_{\lambda\mu}^{+}) - \ell(w_\circ^\mu) = \sum_{\substack{1\leqslant i\leqslant n\\ j\in\mathbb{Z}}} \sum_{\substack{x\leqslant i\\ y>j}} a_{ij}a_{xy} = d_A^{\mathfrak{a}}$. □

Denote the bar involution on $\mathcal{H}$ by $^{-} : \mathcal{H} \to \mathcal{H}, \quad v \mapsto v^{-1}, \quad T_g \mapsto T_{g^{-1}}^{-1}$. Following [Du92, Proposition 3.2], the bar involution on $\mathbf{S}_{n,d}^{\mathfrak{a}}$ can be described as follows: for each $f \in \mathrm{Hom}_{\mathcal{H}}(x_\mu\mathcal{H}, x_\lambda\mathcal{H})$, let $\overline{f} \in \mathrm{Hom}_{\mathcal{H}}(x_\mu\mathcal{H}, x_\lambda\mathcal{H})$ be the map sending v to v^{-1} and $C'_{w_\circ^\mu}$ to $\overline{f(C'_{w_\circ^\mu})}$. Equivalently,

$$\overline{f}(x_\mu H) = v^{2\ell(w_\circ^\mu)}\overline{f(x_\mu)}H \quad \text{for all} \quad H \in \mathcal{H}.$$

In particular, for $A = \kappa(\lambda, g, \mu) \in \Theta_{n,d}$, by Lemma 2.1.4 we have

$$e_A(C'_{w_\circ^\mu}) = v^{\ell(g_{\lambda\mu}^{+}) - \ell(w_\circ^\mu)} C'_{g_{\lambda\mu}^{+}} + \sum_{\substack{x\in\mathcal{D}_{\lambda\mu}\\ x<g}} v^{-\ell(w_\circ^\mu)} c_{x,g}^{(\lambda,\mu)} C'_{x_{\lambda\mu}^{+}}, \tag{2.2.6}$$

$$\overline{e_A}(C'_{w_\circ^\mu}) = v^{\ell(w_\circ^\mu) - \ell(g_{\lambda\mu}^{+})} C'_{g_{\lambda\mu}^{+}} + \sum_{\substack{x\in\mathcal{D}_{\lambda\mu}\\ x<g}} v^{\ell(w_\circ^\mu)} \overline{c_{x,g}^{(\lambda,\mu)}} C'_{x_{\lambda\mu}^{+}}. \tag{2.2.7}$$

Proposition 2.2.6. *Assume that* $A = \kappa(\lambda, g, \mu) \in \Theta_{n,d}$. *There exists* $\gamma_{x,g}^{(\lambda,\mu)} \in \mathbb{Z}[v, v^{-1}]$ *for each* $x \in \mathcal{D}_{\lambda\mu}$ *such that*

$$\overline{[A]} = [A] + \sum_{\substack{x \in \mathcal{D}_{\lambda\mu} \\ x < g}} \gamma_{x,g}^{(\lambda,\mu)} [\kappa(\lambda, x, \mu)].$$

Proof. By Lemma 2.2.5, Equations (2.2.6) and (2.2.7) can be rewritten as

$$\begin{aligned} [A](C'_{w_\circ^\mu}) &= C'_{g^+_{\lambda\mu}} + \sum_{\substack{x \in \mathcal{D}_{\lambda\mu} \\ x < g}} v^{-\ell(g^+_{\lambda\mu})} c_{x,g}^{(\lambda,\mu)} C'_{x^+_{\lambda\mu}}, \\ \overline{[A]}(C'_{w_\circ^\mu}) &= C'_{g^+_{\lambda\mu}} + \sum_{\substack{x \in \mathcal{D}_{\lambda\mu} \\ x < g}} v^{\ell(g^+_{\lambda\mu})} \overline{c_{x,g}^{(\lambda,\mu)}} C'_{x^+_{\lambda\mu}}. \end{aligned}$$

If $\ell(g) = 0$ (i.e. $g = \pi^z$ for some z) then $\overline{[A]} = [A]$ and we are done. For arbitrary g, it follows from an easy induction on $\ell(g)$. □

A matrix $A = (a_{ij})$ is called *bidiagonal* if either $a_{ij} = 0$ for all $j \neq i, i+1$ or $a_{ij} = 0$ for all $j \neq i, i-1$.

Corollary 2.2.7. (*[DF14, Lemma 7.2]*) *If* $A \in \Theta_{n,d}$ *is bidiagonal then* $[A]$ *is bar-invariant.*

Proof. By Lemma 2.2.3, $\ell(A) = 0$ for any bidiagonal matrix A and we are done. □

We define a partial order $\leqslant_a$ on Θ by $A \leqslant_a B$ if and only if $\mathrm{ro}_{\mathfrak{a}}(A) = \mathrm{ro}_{\mathfrak{a}}(B)$, $\mathrm{co}_{\mathfrak{a}}(A) = \mathrm{co}_{\mathfrak{a}}(B)$ and $\sigma_{i,j}(A) \leqslant \sigma_{i,j}(B)$ for all $i \neq j$ where

$$\sigma_{i,j}(A) = \begin{cases} \sum\limits_{x \leqslant i, y \geqslant j} a_{xy} & \text{if } i > j, \\ \sum\limits_{x \geqslant i, y \leqslant j} a_{xy} & \text{if } i < j. \end{cases}$$

In the following the expression “lower terms” represents a linear combination of smaller elements with respect to the partial order $\leqslant_a$. Here we provide an algebraic proof of [BLM90, Lemma 3.6].

Lemma 2.2.8. *Assume that $A = \kappa(\lambda, g, \mu)$ and $B = \kappa(\lambda, h, \mu)$. If $h \leqslant g$ then $B \leqslant_a A$.*

Proof. By [BB05, Proposition 8.3.7], th condition $h \leqslant g$ is equivalent to that $h[i,j] \leqslant g[i,j]$ for all $i,j \in \mathbb{Z}$, where $g[i,j] = |\{(a, g(a)) \in \mathbb{Z}_{\leqslant i} \times \mathbb{Z}_{\geqslant j}\}|$. The bijections $\mathcal{R}_x^\lambda \cap g\mathcal{R}_y^\mu \leftrightarrow \{(g(i), i) \in \mathcal{R}_x^\lambda \times \mathcal{R}_y^\mu\}$ for $x, y \in \mathbb{Z}$ give that, for $i < j$,

$$g[i,j] = \sum_{\substack{x \geqslant i \\ y \leqslant j}} |\mathcal{R}_x^\lambda \cap g\mathcal{R}_y^\mu| = \sum_{\substack{x \geqslant i \\ y \leqslant j}} a_{xy} = \sigma_{ij}(A). \tag{2.2.8}$$

Applying (2.2.8) to ${}^{\mathrm{t}}A = \kappa(\mu, g^{-1}, \lambda)$, we have $g^{-1}[j,i] = \sigma_{ij}(A)$ for $i > j$. Therefore, the condition $h \leqslant g$ implies that $B \leqslant_a A$. □

Corollary 2.2.9. *For $A \in \Theta_{n,d}$, we have*

$$\overline{[A]} = [A] + \text{lower terms}.$$

Proof. It follows by combining Proposition 2.2.6 and Lemma 2.2.8. □

2.3 Multiplication formulas with bidiagonal generators

For each $A \in \Theta_n$, let $\mathrm{diag}(A) = (\delta_{ij}a_{ij})_{ij} \in \Theta_n$ and let $A^{\pm} \in \Theta_n$ be such that

$$A = A^{\pm} + \mathrm{diag}(A). \tag{2.3.1}$$

For any matrix $T = (t_{ij})_{ij} \in \Theta_n$, denote the matrix obtained by shifting every entry of T up by one row as

$$\widehat{T} = (\widehat{t}_{ij})_{ij}, \quad \widehat{t}_{ij} = t_{i+1,j}. \tag{2.3.2}$$

On the other hand, denote the matrix obtained by shifting every entry of T down by one row as

$$\check{T} = (\check{t}_{ij})_{ij}, \quad \check{t}_{ij} = t_{i-1,j}. \tag{2.3.3}$$

For $A = (a_{ij})_{ij}, B = (b_{ij})_{ij} \in \Theta_n$, define

$$\begin{bmatrix} A+B \\ A \end{bmatrix} = \prod_{\substack{1 \leqslant i \leqslant n \\ j \in \mathbb{Z}}} \frac{[a_{ij}+b_{ij}][a_{ij}+b_{ij}-1]\dots[b_{ij}+1]}{[a_{ij}][a_{ij}-1]\dots[1]}.$$

The following remarkable multiplication formulas were due to [DF15, Proposition 3.6].

Lemma 2.3.1. *Assume that $A, B \in \Theta_{n,d}$, $\mathrm{ro}_{\mathfrak{a}}(A) = \mathrm{co}_{\mathfrak{a}}(B)$ and B is bidiagonal. Let $\Theta_\alpha = \{T \in \Theta_n \mid \mathrm{ro}_{\mathfrak{a}}(T) = \alpha\}$ for $\alpha \in \Lambda$.*

(a) *If B is upper triangular (i.e. $B^{\pm} = \sum \alpha_i E_{i-1,i}$), then*

$$[B] * [A] = \sum_{T \in \Theta_\alpha} v^{\beta(A,T)} \overline{\begin{bmatrix} A - T + \widehat{T} \\ A - T \end{bmatrix}} [A - T + \widehat{T}], \tag{2.3.4}$$

where

$$\beta(A,T) = \sum_{1\leqslant i\leqslant n}\sum_{j\leqslant y}\widehat{t}_{ij}(a_{iy}-t_{iy}) - \sum_{1\leqslant i\leqslant n}\sum_{j<y}t_{ij}(a_{iy}-t_{iy}).$$

(*b*) *If* B *is lower triangular (i.e.* $B^{\pm} = \sum \alpha_i E_{i+1,i}$*), then*

$$[B]*[A] = \sum_{T\in\Theta_\alpha} v^{\beta'(A,T)}\overline{\begin{bmatrix} A-T+\check{T} \\ A-T \end{bmatrix}}[A-T+\check{T}], \tag{2.3.5}$$

where

$$\beta'(A,T) = \sum_{1\leqslant i\leqslant n}\sum_{j\geqslant y}\check{t}_{ij}(a_{iy}-t_{iy}) - \sum_{1\leqslant i\leqslant n}\sum_{j>y}t_{ij}(a_{iy}-t_{iy}).$$

Algorithm 2.3.2. Assume that $A, B \in \Theta_{n,d}$, $\mathrm{ro}_{\mathfrak{a}}(A) = \mathrm{co}_{\mathfrak{a}}(B)$ and B is bidiagonal. We produce a matrix $M \in \Theta_{n,d}$ as follows.

(*a*) If B is upper triangular (i.e. $B^{\pm} = \sum \alpha_i E_{i-1,i}$), then:

(1) For each row i, find the unique j such that $\alpha_i \in \big(\sum_{y>j} a_{iy} .. \sum_{y\geqslant j} a_{iy}\big]$.

(2) Construct a matrix $T_+ = \sum_{i=1}^{n}\big((\alpha_i - \sum_{y>j} a_{iy})E_{ij} + \sum_{y>j} a_{iy}E_{iy}\big)$.

(3) Let $M = A - T_+ + \widehat{T}_+$.

(*b*) If B is lower triangular (i.e. $B^{\pm} = \sum \alpha_i E_{i+1,i}$), then:

(1) For each row i, find the unique j such that $\alpha_i \in \big(\sum_{y<j} a_{iy} .. \sum_{y\leqslant j} a_{iy}\big]$.

(2) Construct a matrix $T_+ = \sum_{i=1}^{n}\big((\alpha_i - \sum_{y<j} a_{iy})E_{ij} + \sum_{y<j} a_{iy}E_{iy}\big)$.

(3) Let $M = A - T_+ + \check{T}_+$.

That is, the matrix M is obtained from A by "shifting" up (or down) entries by one row starting from the rightmost (or leftmost) nonzero entries on each row.

Lemma 2.3.3. *The highest term (with respect to $\leqslant_a$) in* (2.3.4) *or in* (2.3.5) *exists and its corresponding matrix is the matrix M described in Algorithm 2.3.2.*

Proof. If B is upper triangular, then each term on the right-hand side of (2.3.4) must be of the form $[A-T+\widehat{T}]$ for some $T \in \Theta_\alpha$ such that $a_{ij}-t_{ij}+\widehat{t}_{ij} \geqslant 0$ for all $i,j \in \mathbb{Z}$. Note that

$$\sigma_{ij}(A - E_{xy} + \widehat{E}_{xy}) = \begin{cases} \sigma_{ij}(A)+1 & \text{if } j < i = x-1, j \leqslant y, \\ \sigma_{ij}(A)-1 & \text{if } j > i = x, j \geqslant y, \\ \sigma_{ij}(A) & \text{otherwise.} \end{cases}$$

It follows immediately that, for each i,

$$\ldots <_a (A - E_{i,-1} + \widehat{E}_{i,-1}) <_a (A - E_{i0} + \widehat{E}_{i0}) <_a (A - E_{i1} + \widehat{E}_{i1}) <_a \ldots$$

Therefore, for any $T \in \Theta_\alpha$ we have $A - T + \widehat{T} \leqslant_a A - T_+ + \widehat{T}_+ = M$.

The case that B is lower triangular is similar and skipped. ∎

Example 2.3.4. Let $n = 2$, $B^\pm = 2E_{12} + 1E_{23}$ and $A = 2E_{12} + 3E_{21} + E_{22} + E_{23}$, that is,

$$B = \left[\begin{array}{c:cc:c} \ddots & 1 & & \\ \hdashline & * & 2 & \\ & & * & 1 \\ \hdashline & & & \ddots \end{array}\right], \quad A = \left[\begin{array}{c:cc:c} \ddots & 1 & & \\ \hdashline & & 2 & \\ & 3 & 1 & 1 \\ \hdashline & & & \ddots \end{array}\right].$$

Then $\alpha_1 = 1 \in \Big(\sum_{y>2} a_{1y} .. \sum_{y\geqslant 2} a_{1y}\Big] = (0..2]$ and $\alpha_2 = 2 \in \Big(\sum_{y>2} a_{2y} .. \sum_{y\geqslant 2} a_{2y}\Big] = (1..2]$. Therefore

$$T_+ = \left[\begin{array}{c|cc|c} \ddots & 1 & & \\ \hline & & 1 & \\ & & 1 & 1 \\ \hline & & & \ddots \end{array}\right], \quad M = \left[\begin{array}{c|cc|c} \ddots & 0^{\circlearrowleft} & 1 & \\ \hline & & 2^{\circlearrowleft} & 1 \\ & 3 & 0^{\circlearrowleft} & 0^{\circlearrowleft} \\ \hline & & & \ddots \end{array}\right].$$

We call a pair (B, A) of matrices to be *admissible* if either of the following conditions (A1) or (A2) holds.

(A1) $B^{\pm} = \sum_{i=1}^{n} m_i E_{i,i+1}$ for some $m_i \in \mathbb{N}$, and

$A^{\pm} = \sum_{i=1}^{n} \sum_{j\leqslant k} a_{i,i+j} E_{i,i+j}$ for some $k \in \mathbb{Z}$, where $a_{i,i+k} \geqslant m_i$ for all i;

(A2) $((b_{-i,-j})_{ij}, (a_{-i,-j})_{ij})$ satisfies Condition (A1).

That is, if (B, A) satisfies Condition (A1), we have

$$B = \left[\begin{array}{c|ccccc} \ddots & & & & & \\ \ddots & m_n & & & & \\ \hline & * & m_1 & & & \\ & & * & m_2 & & \\ & & & * & \ddots & \\ & & & & & \ddots \end{array}\right], \; A = \left[\begin{array}{cccc} \ddots & & & \\ \hline \ddots & * + m_n & & \\ \ddots & * & * + m_1 & \\ \ddots & * & * & \ddots \\ \ddots & \ddots & \ddots & \ddots \end{array}\right], \; M = \left[\begin{array}{cccc} \ddots & & & \\ \ddots & m_n & & \\ \hline \ddots & *^{\circlearrowleft} & m_1 & \\ \ddots & * & *^{\circlearrowleft} & \ddots \\ \ddots & * & * & \ddots \\ \ddots & \ddots & \ddots & \ddots \end{array}\right].$$

Theorem 2.3.5. *If (B, A) is admissible then* $[B] * [A] = [M] +$ lower terms.

Proof. We only prove when B is upper triangular since the other case is similar. Due to Lemma 2.3.3, it remains to show that the coefficient for $[M]$ is one. If (B, A) is admissible, then $T_+ = \sum_{i=1}^{n} m_i E_{i,i+k}$ and hence

$$\begin{bmatrix} A - T_+ + \widehat{T}_+ \\ A - T_+ \end{bmatrix} = \prod_{1 \leqslant i \leqslant n} \left(\prod_{j < i+k} \frac{[(A-T_+)_{ij}] \ldots [1]}{[(A-T_+)_{ij}] \ldots [1]} \right) = 1.$$

Note that by definition of admissible pairs we have $\sum_{j \leqslant y} (a_{iy} - t_{iy}) = 0$ for each nonzero $\widehat{t}_{ij}$ and $\sum_{j<y} (a_{iy} - t_{iy}) = 0$ for each nonzero t_{ij}. Hence $\beta(A, T_+) = 0$. □

2.4 Constructing monomial bases

Below we provide an algorithm that generates a monomial basis in a diagonal-by-diagonal manner involving only admissible pairs (see also [FL14] for a diagonal-by-diagonal construction in a finite type setting).

Algorithm 2.4.1. For each $A = (a_{ij})_{ij} \in \Theta_{n,d}$, we construct upper bidiagonal matrices $B^{(1)}, \ldots, B^{(x)}$ and lower bidiagonal matrices $B_{(1)}, \ldots, B_{(y)}$ as follows:

1. Initialization: $t = 0$, $U^{(0)} = A$.

2. If $U^{(t)}$ is a lower triangular matrix, then go to Step (5) (denote this t by x). Otherwise, denote the outermost nonzero upper diagonal of the matrix $U^{(t)} = (u_{ij}^{(t)})_{ij}$ by $T_+^{(t)} = \sum_{i=1}^{n} u_{i,i+k}^{(t)} E_{i,i+k}$ for some $k > 0$.

3. Define matrices

$$B^{(t+1)} = \sum_{i=1}^{n} u_{i,i+k}^{(t)} E_{i,i+1} + \text{a diagonal determined by (2.4.1)},$$
$$U^{(t+1)} = U^{(t)} - T_+^{(t)} + \check{T}_+^{(t)}.$$

4. Increase t by one and then go to Step (2).

5. Set $L^{(0)} = U^{(x)}$ and set $s = 0$.

6. If $L^{(s)}$ is a lower bidiagonal matrix (denote this s by y), then set $B_{(y)} = L^{(y)}$ and end the algorithm. Otherwise, denote the outermost nonzero lower diagonal of the matrix $L^{(s)} = (l_{ij}^{(s)})_{ij}$ by $T_{+,(s)} = \sum_{i=1}^{n} l_{i+k,i}^{(s)} E_{i+k,i}$ for some $k > 0$.

7. Define matrices

$$\begin{aligned} B_{(s+1)} &= \sum_{i=1}^{n} l_{i+k,i}^{(s)} E_{i+1,i} + \text{a diagonal determined by (2.4.1)}, \\ L^{(s+1)} &= L^{(s)} - T_{+,(s)} + \widehat{T}_{+,(s)}. \end{aligned}$$

8. Increase s by one and then go back to Step (6).

Here the diagonal entries are uniquely determined by

$$\begin{aligned} &\mathrm{ro}_{\mathfrak{a}}(B^{(1)}) = \mathrm{ro}_{\mathfrak{a}}(A), && \mathrm{co}_{\mathfrak{a}}(B^{(i)}) = \mathrm{ro}_{\mathfrak{a}}(B^{(i+1)}) && \text{for } i = 1, \ldots, x-1, \\ &\mathrm{co}_{\mathfrak{a}}(B^{(x)}) = \mathrm{ro}_{\mathfrak{a}}(B_{(1)}), && \mathrm{co}_{\mathfrak{a}}(B_{(i)}) = \mathrm{ro}_{\mathfrak{a}}(B_{(i+1)}) && \text{for } i = 1, \ldots, y-1. \end{aligned} \tag{2.4.1}$$

Theorem 2.4.2. *For $A \in \Theta_{n,d}$, the matrices $B^{(1)}, \ldots, B^{(x)}, B_{(1)}, \ldots, B_{(y)} \in \Theta_{n,d}$ in Algorithm 2.4.1 satisfy that*

$$[B^{(1)}] * \cdots * [B^{(x)}] * [B_{(1)}] * \cdots * [B_{(y)}] = [A] + \text{lower terms}.$$

Proof. For each admissible pair (Y, X), let M be the matrix corresponding to the highest term in $[Y] * [X]$ (cf. Algorithm 2.3.2). For any matrix $X' <_a X$, let M' be the matrix that corresponds to the highest term in $[Y] * [X']$. By construction we have $M' <_a M$, and hence

$$[Y] * ([X] + \text{lower terms}) = [M] + \text{lower terms}.$$

Algorithm 2.4.1 guarantees that each pair $(B^{(j)}, U^{(j)})$ or $(B_{(j)}, L^{(j)})$ is admissible (here it is understood that $L^{(y-1)} = B_{(y)}$ and $U^{(x-1)} = L^{(0)}$). Hence by Theorem 2.3.5,

$$\begin{aligned}
&[B^{(1)}] * \cdots * [B^{(x-1)}] * [B_{(1)}] * [B_{(2)}] * \cdots * \big([B_{(y-1)}] * [B_{(y)}]\big) \\
&= [B^{(1)}] * \cdots * [B^{(x-1)}] * [B_{(1)}] * [B_{(2)}] * \cdots * [B_{(y-2)}] * \big([L^{(y-2)}] + \text{lower terms}\big) \\
&= \dots \\
&= [B^{(1)}] * \cdots * [B^{(x-1)}] * \big([L^{(0)}] + \text{lower terms}\big) \\
&= [B^{(1)}] * \cdots * [B^{(x-1)}] * \big([U^{(x-1)}] + \text{lower terms}\big) \\
&= \dots \\
&= [A] + \text{lower terms}.
\end{aligned}$$

□

For each $A \in \Theta_{n,d}$, we define

$$m_A = [B^{(1)}] * \cdots * [B^{(x)}] * [B_{(1)}] * \cdots * [B_{(y)}]. \tag{2.4.2}$$

Corollary 2.4.3. *The set* $\{m_A \mid A \in \Theta_{n,d}\}$ *forms a basis of the* $\mathbb{Z}[v, v^{-1}]$*-algebra* $\mathbf{S}^{\mathfrak{a}}_{n,d}$ *(called a monomial basis). Moreover,* m_A *is bar invariant for each* $A \in \Theta_{n,d}$*.*

Proof. The first assertion is clear from Theorem 2.4.2. The second assertion follows from Corollary 2.2.7. □

Remark 2.4.4. In [DDX07], Deng, Du and Xiao constructed a family of monomial bases for the Hall algebra of the cyclic quiver. Any such basis can be adapted to a monomial basis for $\mathbf{S}^{\mathfrak{a}}_{n,d}$ using surjections from the double Hall algebra of the cyclic quiver to $\mathbf{S}^{\mathfrak{a}}_{n,d}$ (cf. [DF14]). But their monomial bases are less explicit, and the relation to our monomial basis is unclear.

Example 2.4.5. Let $n = 2, d = 21$, and let

$$A = E_{01} + 2E_{02} + 3E_{03} + 4E_{12} + 5E_{21} + 6E_{20} = \left[\begin{array}{c|cc|c} \ddots & & & \\ 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 4 & 0 \\ 6 & 5 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 \\ & & & \ddots \end{array}\right] \in \Theta_{2,21}.$$

We have $[A] = [B^{(1)}] * [U^{(1)}] +$ lower terms, where

$$B^{(1)} = \left[\begin{array}{cc|cc|cc} \ddots & \ddots & & & & \\ & * & 3 & & & \\ \hline & & * & 0 & & \\ & & & * & 3 & \\ \hline & & & & * & \ddots \\ & & & & & \ddots \end{array}\right], \quad U^{(1)} = \left[\begin{array}{c|cc|c} \ddots & & & \\ 0 & 1 & 2 & \\ \hline 0 & 0 & 4 & 3 \\ 6 & 5 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 \\ & & & \ddots \end{array}\right].$$

$[U^{(1)}] = [B^{(2)}] * [U^{(2)}] +$ lower terms, where

$$B^{(2)} = \left[\begin{array}{cc:cc:cc} \ddots & \ddots & & & & \\ & * & 2 & & & \\ \hdashline & & * & 3 & & \\ & & & * & 2 & \\ \hdashline & & & & * & \ddots \\ & & & & & \ddots \end{array}\right], \quad U^{(2)} = \left[\begin{array}{cc:cc:cc} \ddots & & & & & \\ & 0 & 4\curvearrowright & & & \\ \hdashline & 0 & 0 & 6\curvearrowright & & \\ & 6 & 5 & 0 & 4\curvearrowright & \\ \hdashline & 0 & 0 & 0 & 0 & \\ & & & & & \ddots \end{array}\right].$$

$[U^{(2)}] = [B^{(3)}] * [U^{(3)}] +$ lower terms, where

$$B^{(3)} = \left[\begin{array}{cc:cc:cc} \ddots & \ddots & & & & \\ & * & 4 & & & \\ \hdashline & & * & 6 & & \\ & & & * & 4 & \\ \hdashline & & & & * & \ddots \\ & & & & & \ddots \end{array}\right], \quad U^{(3)} = \left[\begin{array}{cc:cc:cc} \ddots & & & & & \\ & 6 & & & & \\ \hdashline & 0 & 4\curvearrowright & & & \\ & 6 & 5 & 6\curvearrowright & & \\ \hdashline & 0 & 0 & 0 & 4\curvearrowright & \\ & & & & & \ddots \end{array}\right].$$

$[U^{(3)}] = [B_{(1)}] * [L^{(1)}] +$ lower terms, where

$$B_{(1)} = \left[\begin{array}{cc:cc:cc} \ddots & & & & & \\ \ddots & * & & & & \\ \hdashline & 6 & * & & & \\ & & 0 & * & & \\ \hdashline & & & 6 & * & \\ & & & & \ddots & \ddots \end{array}\right], \quad L^{(1)} = \left[\begin{array}{cc:cc:cc} \ddots & & & & & \\ & 6 & & & & \\ \hdashline & 6 & 4 & & & \\ & \curvearrowleft & 5 & 6 & & \\ \hdashline & & & 6 & 4 & \\ & & & \curvearrowleft & & \ddots \end{array}\right].$$

Hence $L^{(1)} = B_{(2)}$ and

$$[A] = [B^{(1)}] * [B^{(2)}] * [B^{(3)}] * [B_{(1)}] * [B_{(2)}] + \text{ lower terms.}$$

Chapter 3

Affine Schur algebras of type C

In this chapter we recall some standard facts about the affine Weyl groups of type C and the corresponding Hecke algebras. We come up with a new formulation of length formula (cf. Lemma 3.1.1). We then study the affine q-Schur algebras of type C as endomorphism algebras of certain q-permutation modules for Hecke algebras. In particular, we show that the bases for Schur algebras can be parametrized by a set of $\mathbb{Z} \times \mathbb{Z}$ periodic centro-symmetric $\mathbb{N}$-matrices.

From now on, let $n = 2r + 2, D = 2d + 2$ be fixed positive integers.

3.1 Affine Hecke algebras

Let W be the Weyl group of type $\widetilde{C}_d$ (or $C_d^{(1)}$) generated by $S = \{s_0, s_1, \ldots, s_d\}$ with Dynkin diagram

$$\begin{array}{ccccccccc} \circ & \Longrightarrow & \circ & — & \cdots & — & \circ & \Longleftarrow & \circ \\ 0 & & 1 & & & & d-1 & & d \end{array}$$

It is known ([BB05], [EE98]) that (W, S) is a Coxeter group and W can be identified as a permutation subgroup of $\mathbb{Z}$ satisfying

$$g(i+D) = g(i) + D, g(-i) = -g(i) \quad \text{for} \quad i \in \mathbb{Z}, g \in W$$

Note that we always have, for $g \in W$,

$$g(0) = 0 \quad \text{and} \quad g(d+1) = d+1.$$

Also, w is uniquely determined by its value on $\{1, 2, \ldots, d\}$, and we write

$$w = \begin{pmatrix} 1 & 2 & \ldots & d \\ a_1 & a_2 & \ldots & a_d \end{pmatrix}_{\mathfrak{c}} = [a_1, a_2, \ldots, a_d]_{\mathfrak{c}}. \tag{3.1.1}$$

to mean that $w(i) = a_i$ for $1 \leqslant i \leqslant d$. Here we adapt a slightly different notation than the Weyl group $\widetilde{S}_d^C$ in [BB05] by inserting fixed points $d + 1 + D\mathbb{Z}$. Precisely speaking, there is an identification $\widetilde{S}_d^C \to W$ given by

$$g' \mapsto [\iota(g'(1)), \ldots, \iota(g'(d))]_{\mathfrak{c}}, \tag{3.1.2}$$

where $\iota : \mathbb{Z} \to \mathbb{Z}, i \mapsto i + \left\lceil \frac{i-d}{D-1} \right\rceil$ is the bijection induced by inserting $d + 1 + D\mathbb{Z}$. In particular, we have

$$\iota(g'(i + k(D-1))) = g(i + kD), \quad -d \leqslant i \leqslant d, k \in \mathbb{Z}. \tag{3.1.3}$$

Denote the length function on W by ℓ. Now we give an interpretation of length under the identification above.

Lemma 3.1.1. *The length of $g \in W$ is given by*

$$\ell(g) = \frac{1}{2}|\{(i,j) \in [1..d] \times \mathbb{Z} \mid {}^{i>j}_{g(i)<g(j)} \text{ or } {}^{i<j}_{g(i)>g(j)}\}|. \tag{3.1.4}$$

Proof. Let $g' \in \widetilde{S}_d^C$ be the element identified with g. It is known [BB05, (8.44), (8.45)] that

$$\ell(g') = \mathrm{inv}_B(g'(1), \ldots, g'(d)) + \sum_{1 \leqslant i \leqslant j \leqslant d} \left(\left\lfloor \frac{|g'(i) - g'(j)|}{D-1} \right\rfloor + \left\lfloor \frac{|g'(i) + g'(j)|}{D-1} \right\rfloor \right), \tag{3.1.5}$$

where by [BB05, (8.2)] we have

$$\mathrm{inv}_B(g'(1), \ldots, g'(d)) = |\{(i,j) \in [1..d]^2 \mid {}_{g'(i) > g'(j)}^{i<j}\}| + |\{(i,j) \in [1..d]^2 \mid {}_{g'(-i) > g'(j)}^{i \leqslant j}\}|. \tag{3.1.6}$$

Since that ι is order preserving, the statement $g'(i) < g'(j)$ is equivalent to $g(i) < g(j)$ for all $i, j \in \mathbb{Z}$. Hence we have

$$\mathrm{inv}_B(g'(1), \ldots, g'(d)) = |\{(i,j) \in [1..d]^2 \mid {}_{g(i) > g(j)}^{i<j}\}| + |\{(i,j) \in [1..d]^2 \mid {}_{g(-i) > g(j)}^{i \leqslant j}\}|. \tag{3.1.7}$$

A detailed calculation shows that

$$\mathrm{inv}_B(g'(1), \ldots, g'(d)) = \frac{1}{2}|\{(i,j) \in [1..d] \times [-d..d] \mid {}_{g(i) < g(j)}^{i>j} \text{ or } {}_{g(i) > g(j)}^{i<j}\}|. \tag{3.1.8}$$

Let $\widetilde{S}_N$ be the Weyl group of affine type A as in [BB05, Section 8.3]. It is also known [BB05, (8.31)] that for $g' \in \widetilde{S}_d^C \subset \widetilde{S}_{D-1}$, $i, j \in [1..d]$, we have

$$\begin{aligned} \left\lfloor \frac{|g'(i) - g'(j)|}{D-1} \right\rfloor &= |\{k \in \mathbb{Z} \mid g'(j) > g'(i + k(D-1))\}| \\ &+ |\{k \in \mathbb{Z} \mid g'(i) > g'(j + k(D-1))\}|. \end{aligned} \tag{3.1.9}$$

Similarly, for $g \in W \subset \widetilde{S}_D$, we have

$$\begin{aligned} \left\lfloor \frac{|g(i) - g(j)|}{D} \right\rfloor &= |\{k \in \mathbb{Z} \mid g(j) > g(i + k(D))\}| \\ &+ |\{k \in \mathbb{Z} \mid g(i) > g(j + k(D))\}|. \end{aligned} \tag{3.1.10}$$

By (3.1.3), we obtain that

$$\left\lfloor \frac{|g'(i) - g'(j)|}{D-1} \right\rfloor = \left\lfloor \frac{|g(i) - g(j)|}{D} \right\rfloor.$$

Another detailed calculation shows that

$$\begin{aligned}
\sum_{1 \leqslant i \leqslant j \leqslant d} \left\lfloor \tfrac{|g(i)+g(j)|}{D} \right\rfloor &= \sum_{1 \leqslant i < j \leqslant d} \big(|\{k \geqslant 1 \mid g(j) > g(-i+kD)\}| + |\{k \geqslant 1 | g(i) < g(-j-kD)\}|\big) \\
&+ \sum_{1 \leqslant i \leqslant d} \big(|\{k \geqslant 1 \mid g(i) > g(-i+kD)\}| + |\{k \geqslant 1 | g(i) < g(-i-kD)\}|\big) \\
&= \frac{1}{2} \sum_{k=1}^{\infty} \begin{pmatrix} & |\{(j, -i+kD) \mid 1 \leqslant i < j \leqslant d, g(j) > g(-i+kD)\}| \\ + & |\{(i, -j+kD) \mid 1 \leqslant i < j \leqslant d, g(i) > g(-j+kD)\}| \\ + & |\{(i, -j-kD) \mid 1 \leqslant i < j \leqslant d, g(i) < g(-j-kD)\}| \\ + & |\{(j, -i-kD) \mid 1 \leqslant i < j \leqslant d, g(j) < g(-i-kD)\}| \\ + & |\{(i, -i+kD) \mid 1 \leqslant i \leqslant d, g(i) > g(-i+kD)\}| \\ + & |\{(i, \frac{kD}{2}) \mid 1 \leqslant i \leqslant d, g(i) > g(\frac{kD}{2})\}| \\ + & |\{(i, -i-kD) \mid 1 \leqslant i \leqslant d, g(i) < g(-i-kD)\}| \\ + & |\{(i, -\frac{kD}{2}) \mid 1 \leqslant i \leqslant d, g(i) < g(-\frac{kD}{2})|\} \end{pmatrix},
\end{aligned}$$

and

$$\begin{aligned}
\sum_{1 \leqslant i \leqslant j \leqslant d} \left\lfloor \tfrac{|g(i)-g(j)|}{D} \right\rfloor &= \sum_{1 \leqslant i < j \leqslant d} \big(|\{k \geqslant 1 \mid g(i) > g(j+kD)\}| + |\{k \geqslant 1 \mid g(j) > g(i+kD)\}|\big) \\
&= \frac{1}{2} \sum_{k=1}^{\infty} \begin{pmatrix} & |\{(i, j+kD) \mid 1 \leqslant i < j \leqslant d, g(i) > g(j+kD)\}| \\ + & |\{(j, i-kD) \mid 1 \leqslant i < j \leqslant d, g(i-kD) > g(j)\})| \\ + & |\{(j, i+kD) \mid 1 \leqslant i < j \leqslant d, g(j) > g(i+kD)\}| \\ + & |\{(i, j-kD) \mid 1 \leqslant i < j \leqslant d, g(j-kD) > g(i)\}|) \end{pmatrix}.
\end{aligned}$$

The lemma then follows by summing them up. □

By using the convention in (3.1.1), the generators of W can be denoted by

$$
\begin{aligned}
s_0 &= [-1, 2, 3, \ldots, d-1, d]_{\mathfrak{c}}, \\
s_d &= [1, 2, 3, \ldots, d-1, d+2]_{\mathfrak{c}}, \\
s_i &= [1, \ldots, i-1, i+1, i, i+2, \ldots, d]_{\mathfrak{c}} \quad \text{for} \quad i = 1, \ldots d-1.
\end{aligned} \tag{3.1.11}
$$

Denote the set of (weak) compositions of d into $r+2$ parts (where "weak" means a possible zero part is allowed) by

$$\Lambda^{\mathfrak{c}}_{r,d} = \left\{ \lambda = (\lambda_0, \ldots, \lambda_{r+1}) \in \mathbb{N}^{r+2} \;\middle|\; \sum_{i=0}^{r+1} \lambda_i = d \right\}. \tag{3.1.12}$$

From now on, write $\Lambda = \Lambda^{\mathfrak{c}}_{r,d}$. For each $\lambda \in \Lambda$, denote by W_λ the parabolic (finite) subgroup with respect to λ generated by $S \backslash \{s_{\lambda_0}, s_{\lambda_{0,1}}, \ldots, s_{\lambda_{0,r}}\}$ where $\lambda_{0,i} = \lambda_0 + \lambda_1 + \ldots + \lambda_i$ for $0 \leqslant i \leqslant r$ and $\lambda_{0,0} = \lambda_0$. We define integral intervals with respect to λ by

$$R^\lambda_i = \begin{cases} [-\lambda_0..\lambda_0] & \text{if } i = 0, \\ (\lambda_{0,i-1}..\lambda_{0,i}] & \text{if } i \in [1..r], \\ [d+1-\lambda_{r+1}..d+1+\lambda_{r+1}] & \text{if } i = r+1, \end{cases} \tag{3.1.13}$$

and we extend the definition R^λ_i for all $i \in \mathbb{Z}$ recursively by letting

$$R^\lambda_{-i} = \{-x \mid x \in R^\lambda_i\}, \quad R^\lambda_{i+n} = \{x + D \mid x \in R^\lambda_i\}. \tag{3.1.14}$$

Lemma 3.1.2.

$$W_\lambda = \bigcap_{i=0}^{r+1} \mathrm{Stab} R^\lambda_i.$$

Proof. By [BB05, Proposition 8.4.4] we have that for each i,

$$W_{S\setminus\{s_{\lambda_{0,i}}\}} = \mathrm{Stab}([-\lambda_{0,i}..\lambda_{0,i}]) \cap \mathrm{Stab}([\lambda_{0,i}+1..D-\lambda_{0,i}-1]).$$

The lemma follows by taking the intersection $W_\lambda = \bigcap_{i=0}^{r} W_{S\setminus\{s_{\lambda_{0,i}}\}}$. □

Let $\mathscr{D}_\lambda = \{w \in W \mid \ell(wg) = \ell(w) + \ell(g) \text{ for } g \in W_\lambda\}$. Then $\mathscr{D}_\lambda$ (resp., $\mathscr{D}_\lambda^{-1}$) is the set of distinguished right (resp. left) coset representatives of W_λ in W. Denote by $\mathscr{D}_{\lambda\mu} = \mathscr{D}_\lambda \cap \mathscr{D}_\mu^{-1}$ the set of distinguished double coset representatives.

Lemma 3.1.3. *Let $g \in W$ and let $\lambda \in \Lambda$. Then the following are equivalent:*

(a) $g \in \mathscr{D}_\lambda$;

(b) g^{-1} *is order-preserving on* $R_i^\lambda, i \in [0 \,..\, r+1]$;

(c) g^{-1} *is order-preserving on* $R_i^\lambda, i \in \mathbb{Z}$.

Proof. By the argument following [BB05, Proposition 8.4.4], we have

$$\mathscr{D}_\lambda = \left\{ g \in W \;\middle|\; \begin{array}{l} g^{-1}(0) < \ldots < g^{-1}(\lambda_0), \\ g^{-1}(1+\lambda_{0,i}) < \ldots < g^{-1}(\lambda_{0,i+1}), \forall i \in [1..r-1], \\ g^{-1}(1+\lambda_{0,r}) < \ldots < g^{-1}(d+1) \end{array} \right\}.$$

Note that $g(-i) = -g(i)$ and $g(0) = 0$, so the condition "$g^{-1}(0) < \ldots < g^{-1}(\lambda_0)$" is equivalent to $g^{-1}(-\lambda_0) < \ldots < g^{-1}(0) = 0 < \ldots < g^{-1}(\lambda_0)$. Similarly, we have $g^{-1}(d+1-\lambda_{r+1}) < \ldots < g^{-1}(d+1) = d+1 < \ldots < g^{-1}(d+1+\lambda_{r+1})$. The equivalence of the latter two conditions follows from the periodic condition $g(i+D) = g(i)+D$ for all i, g. □

Proposition 3.1.4 (Howlett)**.** *Let $\lambda, \mu \in \Lambda$, and let $g \in \mathscr{D}_{\lambda\mu}$. Then*

(a) *There is a weak composition $\delta = \delta(\lambda, g, \mu) \in \Lambda_{r',d}$ for some r' such that*

$$W_\delta = g^{-1} W_\lambda g \cap W_\mu.$$

(b) *The map $W_\lambda \times (\mathscr{D}_\delta \cap W_\mu) \to W_\lambda g W_\mu$ sending (x, y) to xgy is a bijection satisfying $\ell(xgy) = \ell(x) + \ell(g) + \ell(y)$.*

(c) *The map $(\mathscr{D}_\delta \cap W_\mu) \times W_\delta \to W_\mu$ sending (x, y) to xy is a bijection satisfying $\ell(x) + \ell(y) = \ell(xy)$.*

Proof. See [DDPW08, Proposition 4.16, Lemma 4.17 and Theorem 4.18]. □

The Hecke algebra $\mathbf{H} = \mathbf{H}(W)$ of type $\widetilde{C}_d$ is a $\mathbb{Z}[v, v^{-1}]$-algebra with a basis $\{T_g \mid g \in W\}$ satisfying

$$\begin{aligned} &T_w T_{w'} = T_{ww'} && \text{if } \ell(ww') = \ell(w) + \ell(w'), \\ &(T_s + 1)(T_s - v^2) = 0 && \text{for } s \in S. \end{aligned}$$

For a finite subset $X \subset W$ and for $\lambda \in \Lambda$, set

$$T_X = \sum_{w \in X} T_w \quad \text{and} \quad x_\lambda = T_{W_\lambda}. \tag{3.1.15}$$

3.2 Affine Schur algebras

For $\lambda, \mu \in \Lambda$ and $g \in \mathscr{D}_{\lambda\mu}$, denote by $\phi^g_{\lambda\mu} \in \mathrm{Hom}_{\mathbf{H}}(x_\mu \mathbf{H}, \mathbf{H})$ the right $\mathbf{H}$-linear map sending x_μ to $T_{W_\lambda g W_\mu}$. Thanks to Proposition 3.1.4 (b), we have $T_{W_\lambda g W_\mu} =$

$x_\lambda T_g T_{\mathscr{D}_\delta \cap W_\mu}$ for some $\delta \in \Lambda_{r',d}$ and hence $\phi^g_{\lambda\mu} \in \mathrm{Hom}_{\mathbf{H}}(x_\mu \mathbf{H}, x_\lambda \mathbf{H})$. The affine Schur algebra is defined by

$$\mathbf{S}^{\mathfrak{c}}_{n,d} = \mathrm{End}_{\mathbf{H}}\Big(\bigoplus_{\lambda \in \Lambda} x_\lambda \mathbf{H}\Big) = \bigoplus_{\lambda,\mu \in \Lambda} \mathrm{Hom}_{\mathbf{H}}(x_\mu \mathbf{H}, x_\lambda \mathbf{H}). \tag{3.2.1}$$

It is known that $\{\phi^g_{\lambda\mu} \mid \lambda, \mu \in \Lambda, g \in \mathscr{D}_{\lambda\mu}\}$ is a $\mathbb{Z}[v, v^{-1}]$-basis of $\mathbf{S}^{\mathfrak{c}}_{n,d}$.

Recall that we assume throughout the dissertation that $n = 2r+2$ and $D = 2d+2$. Set $\Xi_{n,d}$ to be the subset of $\Theta_{n,d}$ (cf. Section 2.2) in which each element $A = (a_{ij})$ satisfies additionally that

- $a_{-i,-j} = a_{ij}$ for all $i, j \in \mathbb{Z}$;
- a_{00} and $a_{r+1,r+1}$ are odd;
- $\sum_{1 \leqslant i \leqslant n} \sum_{j \in \mathbb{Z}} a_{ij} = D$.

For any $T = (t_{ij}) \in \Theta_n$, set

$$T_\theta = (t_{\theta,ij}), \quad t_{\theta,ij} = t_{ij} + t_{-i,-j}.$$

Let $\Xi_n = \bigcup_{d \in \mathbb{N}} \Xi_{n,d}$. For $A = (a_{ij}) \in \Xi_n$, we set

$$a'_{ij} = \begin{cases} \frac{1}{2}(a_{ij} - 1) & \text{if } i = j \in \mathbb{Z}(r+1), \\ a_{ij} & \text{otherwise.} \end{cases} \tag{3.2.2}$$

For any $A \in \Xi_n$, we define its type C row sum vector $\mathrm{ro}_{\mathfrak{c}}(A) = (\mathrm{ro}_{\mathfrak{c}}(A)_0, \ldots, \mathrm{ro}_{\mathfrak{c}}(A)_{r+1})$

and type C column sum vector $\mathrm{co}_{\mathfrak{c}}(A) = (\mathrm{co}_{\mathfrak{c}}(A)_0, \ldots, \mathrm{co}_{\mathfrak{c}}(A)_{r+1})$ by

$$\mathrm{ro}_{\mathfrak{c}}(A)_k = \begin{cases} \sum_{j\geqslant 0} a'_{0j} & \text{if } k = 0, \\ \sum_{j\leqslant r+1} a'_{r+1,j} & \text{if } k = r+1, \\ \mathrm{ro}_{\mathfrak{a}}(T)_k & \text{otherwise.} \end{cases}$$
$$\mathrm{co}_{\mathfrak{c}}(A)_k = \begin{cases} \sum_{i\geqslant 0} a'_{i0} & \text{if } k = 0, \\ \sum_{i\leqslant r+1} a'_{i,r+1} & \text{if } k = r+1, \\ \mathrm{co}_{\mathfrak{a}}(A)_k & \text{otherwise.} \end{cases} \tag{3.2.3}$$

Each $A \in \Xi_n$ is uniquely determined by $\{a_{ij} \mid (i,j) \in I^+\}$, where

$$I^+ = (\{0\} \times \mathbb{N}) \sqcup ([1..r] \times \mathbb{Z}) \sqcup (\{r+1\} \times \mathbb{Z}_{\leqslant r+1}) \tag{3.2.4}$$

is the index set corresponding to the "first half-period". On I^+, let $\leqslant$ be the lexicographical order such that $(i,j) \leqslant (x,y)$ if and only if $i < x$ or ($i = x$ and $j \leqslant y$). With these notation, Ξ_n can be expressed as follows:

$$\Xi_n = \{A \mid A \in E^{00} + E^{r+1,r+1} + \sum_{(i,j)\in I^+} \mathbb{N} E_\theta^{ij}\}. \tag{3.2.5}$$

We also introduce a partial order "$\leqslant$" on Ξ_n (and on Θ_n) by

$$(a_{ij}) \leqslant (b_{ij}) \Leftrightarrow a_{ij} \leqslant b_{ij} (\forall i,j). \tag{3.2.6}$$

Next, we introduce a "higher-level" structure of Ξ_n, which is used in the proof of the multiplication formula. Let $\Xi^{\mathcal{P}}_{n,d}$ be the set of $\mathbb{Z} \times \mathbb{Z}$ matrices with entries being subsets of $\mathbb{Z}$ in which each element $\mathcal{A} = (\mathcal{A}_{ij})$ satisfies that

(P0) For $z \in \mathbb{Z}$, there exists a unique (i,j) such that $z \in \mathcal{A}_{ij}$;

(P1) $\mathcal{A}_{i+n,j+n} = \{x + D \mid x \in \mathcal{A}_{ij}\}$ for all $i,j \in \mathbb{Z}$;

(P2) $\mathcal{A}_{-i,-j} = \{-x \mid x \in \mathcal{A}_{ij}\}$ for all $i,j \in \mathbb{Z}$;

(P3) $0 \in \mathcal{A}_{00}$ and $d+1 \in \mathcal{A}_{r+1,r+1}$;

(P4) $\sum_{1 \leqslant i \leqslant n} \sum_{j \in \mathbb{Z}} |\mathcal{A}_{ij}| = D$.

Set $\Xi_n^{\mathcal{P}} = \bigcup_{d \in \mathbb{N}} \Xi_{n,d}^{\mathcal{P}}$. Again, each $\mathcal{A}$ is uniquely determined by $\{\mathcal{A}_{ij} \mid (i,j) \in I^+\}$. Now we define a map κ' sending each triple $(\lambda, g, \mu) \in \Lambda \times W \times \Lambda$ to a $\mathbb{Z} \times \mathbb{Z}$ matrix $(|R_i^\lambda \cap gR_j^\mu|)$. It is clear that the image of κ' lies in $\Xi_{n,d}$. We further define

$$\kappa : \{(\lambda, g, \mu) \mid \lambda, \mu \in \Lambda, g \in \mathscr{D}_{\lambda\mu}\} \longrightarrow \Xi_{n,d} \tag{3.2.7}$$

by $\kappa(\lambda, g, \mu) = \kappa'(\lambda, g, \mu)$.

Algorithm 3.2.1. For each $A = (a_{ij}) \in \Xi_{n,d}$, we define a matrix $A_{\mathrm{std}}^{\mathcal{P}} \in \Xi_{n,d}^{\mathcal{P}}$ by "row-reading" as follows (see (3.2.2) for a'_{ij}):

1. Set $(A_{\mathrm{std}}^{\mathcal{P}})_{00} = \left[-a'_{00} \,..\, a'_{00}\right]$ and $(A_{\mathrm{std}}^{\mathcal{P}})_{r+1,r+1} = \left[d{+}1{-}a'_{r+1,r+1} \,..\, d{+}1{+}a'_{r+1,r+1}\right]$.

2. For $(i,j) \in I_{\mathfrak{a}}^+$, where

$$I_{\mathfrak{a}}^+ = I^+ \backslash \{(0,0), (r+1,r+1)\}, \tag{3.2.8}$$

   set

$$(A_{\mathrm{std}}^{\mathcal{P}})_{ij} = \Big(\sum_{l=0}^{i-1} \mathrm{ro}_{\mathfrak{c}}(A)_l + \sum_{k<j} a_{ik} \,..\, \sum_{l=0}^{i-1} \mathrm{ro}_{\mathfrak{c}}(A)_l + \sum_{k \leqslant j} a_{ik}\Big].$$

3. For $(i,j) \notin I^+$, $(A^{\mathcal{P}}_{\mathrm{std}})_{ij}$ is determined by Conditions (P1) and (P2).

For any $\mathcal{A} \in \Xi^{\mathcal{P}}_{n,d}$, it is obvious that $|\mathcal{A}| := (|\mathcal{A}_{ij}|) \in \Xi_{n,d}$. Moreover, $|A^{\mathcal{P}}_{\mathrm{std}}| = A$.

Algorithm 3.2.2. For any $\mathcal{A} = (\mathcal{A}_{ij}) \in \Xi^{\mathcal{P}}_{n,d}$, set $A = (a_{ij}) = |\mathcal{A}|$. We define a Weyl group element $g^{\mathrm{std}} = g^{\mathrm{std}}(\mathcal{A}) \in W$, which sends $k \in \mathbb{Z}$ to $g^{\mathrm{std}}(k) \in \mathbb{Z}$, using "column-reading" as follows:

1. For $(i,j) \in I^+$ and $a_{ij} > 0$, we set

$$I^{(i,j)} = \begin{cases} [-a'_{ij}..a'_{ij}], & \text{if } (i,j) = (0,0) \text{ or } (r+1, r+1), \\ [1..a_{ij}], & \text{otherwise.} \end{cases}$$

   Then set $\mathcal{A}_{ij} = \{a^{(i,j)}_l \mid l \in I^{(i,j)}\}$ such that $a^{(i,j)}_l < a^{(i,j)}_{l+1}$ for admissible l.

2. For $k = 1, \ldots, d$, find the unique $(i,j) \in I^+$ and $m \in [1..a'_{ij}]$ such that

$$k = \sum_{(x,y) \in I^+, (x,y) < (i,j)} a'_{yx} + m,$$

   and then set $g^{\mathrm{std}}(k) = a^{(j,i)}_m$.

3. For $k \notin [1..d]$, $g^{\mathrm{std}}(k)$ is determined recursively by

$$g^{\mathrm{std}}(k + D) = g^{\mathrm{std}}(k) + D = -g^{\mathrm{std}}(-k).$$

Example 3.2.3. Let $d = 7, D = 16, r = 0, n = 2$, and let

$$A = \left[\begin{array}{ccc:c:ccc}
\ddots & & & \ddots & & & \\
 & \ddots & & & 2 & & \\
 & & 5 & & & 2 & \\
\hdashline
\ddots & & & 3 & & & 2 \\
\hdashline
 & 2 & & & 5 & & \ddots \\
 & & 2 & & & 3 & \\
 & & & \ddots & & & \ddots
\end{array}\right] = 3E^{00} + 2E^{03}_{\theta} + 2E^{1,-2}_{\theta} + 5E^{11}.$$

We have $\mathrm{ro}_{\mathfrak{c}}(A) = \mathrm{co}_{\mathfrak{c}}(A) = (3,4)$, and

$$A^{\mathcal{P}}_{\mathrm{std}} = \left[\begin{array}{cc:c:ccc}
 & & \ddots & & & \\
\ddots & & & \{-14,-13\} & & \\
 & [-10..-6] & & & \{-5,-4\} & \\
\hdashline
 & & [-1..1] & & & \{2,3\} \\
\hdashline
\{4,5\} & & & [6..10] & & \ddots \\
 & \{13,14\} & & & [15..17] & \\
 & & \ddots & & & \ddots
\end{array}\right].$$

On the other hand, we have $g^{\mathrm{std}}(A^{\mathcal{P}}_{\mathrm{std}}) = [1, 20, 21, -14, -13, 6, 7]_{\mathfrak{c}}$ (see (3.1.1)).

Lemma 3.2.4. (*a*) *The map* $\kappa : \{(\lambda, g, \mu) \mid \lambda, \mu \in \Lambda, g \in \mathscr{D}_{\lambda\mu}\} \to \Xi_{n,d}$ *is a bijection.*

(*b*) *Let* $A = \kappa(\lambda, g, \mu)$ *for some* $\lambda, \mu \in \Lambda, g \in \mathscr{D}_{\lambda\mu}$. *Then* $\lambda = \mathrm{ro}_{\mathfrak{c}}(A)$, $\mu = \mathrm{co}_{\mathfrak{c}}(A)$, *and* $g = g^{\mathrm{std}}(A^{\mathcal{P}}_{\mathrm{std}})$.

Proof. We shall show that for fixed $\lambda, \mu \in \Lambda$, the restriction of κ to $\mathscr{D}_{\lambda\mu}$

$$\kappa_{\lambda\mu} : \mathscr{D}_{\lambda\mu} \longrightarrow \Xi_{n,d}(\lambda, \mu) = \{A \in \Xi_{n,d} \mid \mathrm{ro}_{\mathfrak{c}}(A) = \lambda, \mathrm{co}_{\mathfrak{c}}(A) = \mu\} \tag{3.2.9}$$

is a bijection. We start with the restriction of κ' to W, denoted by $\kappa'_{\lambda\mu} : W \to \Xi_{n,d}(\lambda, \mu)$. Let $\Xi^{\mathcal{P}}_{n,d}(\lambda, \mu) = \{\mathcal{A} \in \Xi^{\mathcal{P}}_{n,d} \mid \mathrm{ro}_{\mathfrak{c}}(|\mathcal{A}|) = \lambda, \mathrm{co}_{\mathfrak{c}}(|\mathcal{A}|) = \mu\}$. Note that $\kappa'_{\lambda\mu}$ is the composition of two maps

$$\begin{array}{rcl} \kappa^{\mathcal{P}}_{\lambda\mu} : W & \to & \Xi^{\mathcal{P}}_{n,d}(\lambda, \mu) \\ g & \mapsto & (R^{\lambda}_i \cap gR^{\mu}_j) \end{array} \quad \text{and} \quad \begin{array}{rcl} |\cdot|_{\lambda\mu} : \Xi^{\mathcal{P}}_{n,d}(\lambda, \mu) & \to & \Xi_{n,d}(\lambda, \mu). \\ \mathcal{A} & \mapsto & |\mathcal{A}| \end{array}$$

It is easy to check that for each $\mathcal{A} \in \Xi^{\mathcal{P}}_{n,d}(\lambda, \mu)$, $\kappa^{\mathcal{P}}_{\lambda\mu}(g^{\mathrm{std}}(\mathcal{A})) = \mathcal{A}$ and hence $\kappa^{\mathcal{P}}_{\lambda\mu}$ is a surjection. Furthermore, given $g \in (\kappa^{\mathcal{P}}_{\lambda\mu})^{-1}(\mathcal{A})$, by Lemma 3.1.3 we have that

$$g = g^{\mathrm{std}}(\mathcal{A}) \text{ if and only if } g \in \mathscr{D}_{\lambda}.$$

Thus the restriction $\kappa^{\mathcal{P}}_{\lambda\mu}|_{\mathscr{D}_{\lambda}}$ is a bijection. On the other hand, for each $A \in \Xi_{n,d}(\lambda, \mu)$ we have $|A^{\mathcal{P}}_{\mathrm{std}}| = A$, and hence $|\cdot|_{\lambda\mu}$ is a surjection. Moreover, given $\mathcal{A} \in |\cdot|^{-1}(A)$, by Lemma 3.1.3 again, we have that

$$\mathcal{A} = A^{\mathcal{P}}_{\mathrm{std}} \text{ if and only if } g^{\mathrm{std}}(\mathcal{A}) \in \mathscr{D}^{-1}_{\mu}.$$

Therefore $(|\cdot|_{\lambda\mu} \circ \kappa^{\mathcal{P}}_{\lambda\mu})|_{\mathscr{D}_{\lambda\mu}} = \kappa_{\lambda\mu}$ is a bijection, whence (a). Part (b) can be clearly read off from the proceeding argument. $\square$

For each $A = \kappa(\lambda, g, \mu) \in \Xi_{n,d}$, set

$$e_A = \phi^{g}_{\lambda\mu}. \tag{3.2.10}$$

Hence $\{e_A \mid A \in \Xi_{n,d}\}$ forms a $\mathbb{Z}[v, v^{-1}]$-basis of $\mathbf{S}^{\mathfrak{c}}_{n,d}$. For $A \in \Xi_{n,d}$, we define $\delta(A) = (\delta(A)_0, \ldots, \delta(A)_{r'+1}) \in \Lambda_{r',d}$ for some r' by the following procedure:

1. Set $\delta_0 = \frac{1}{2}(a_{00} - 1)$ (possibly zero); set $(\delta^{(0)}_1, \ldots, \delta^{(0)}_{k_0})$ for some $k_0 \in \mathbb{N}$ to be the composition of $\mathrm{co}_{\mathfrak{c}}(A)_0 - \delta_0$ obtained from $(a_{10}, a_{20}, \ldots)$ by deleting all zero entries.

2. For each $j = 1, \ldots, r$, set $(\delta^{(j)}_1, \ldots, \delta^{(j)}_{k_j}) \in \Lambda_{k_j,\lambda_j}$ for some $k_j \in \mathbb{N}$ to be the composition of $\mathrm{co}_{\mathfrak{c}}(A)_j$ obtained from $(\ldots,\ a_{-1,j}, a_{0j}, a_{1j}, \ldots)$ by deleting all zero entries.

3. Set $(\delta^{(r+1)}_1, \ldots, \delta^{(r+1)}_{k_{r+1}})$ for some $k_{r+1} \in \mathbb{N}$ to be the composition of $\mathrm{co}_{\mathfrak{c}}(A)_{r+1} - \delta_{r'+1}$ obtained by deleting all zero entries from $(\ldots, a_{r-1,r+1}, a_{r,r+1})$, where $\delta_{r'+1} = \frac{1}{2}(a_{r+1,r+1} - 1)$ (possibly zero) and $r' = k_0 + \ldots + k_{r+1}$.

4. Finally, set
$$\delta(A) = (\delta_0, \delta^{(0)}_1, \ldots, \delta^{(0)}_{k_0}, \delta^{(1)}_1, \ldots, \delta^{(1)}_{k_1}, \ldots, \delta^{(r+1)}_1, \ldots, \delta^{(r+1)}_{k_{r+1}}, \delta_{r'+1}). \qquad (3.2.11)$$

Proposition 3.2.5. *Let $A = \kappa(\lambda, g, \mu)$ for some $\lambda, \mu \in \Lambda, g \in \mathscr{D}_{\lambda\mu}$. Then $W_{\delta(A)} = g^{-1}W_\lambda g \cap W_\mu$. Namely, $\delta(A)$ is one possible weak composition δ described in Proposition 3.1.4.*

Proof. By Lemma 3.1.2, we have
$$\begin{aligned} g^{-1}W_\lambda g \cap W_\mu &= \Big(\bigcap_{i=0}^{r+1} \mathrm{Stab}(g^{-1}R^\lambda_i)\Big) \cap \Big(\bigcap_{j=0}^{r+1} \mathrm{Stab}(R^\mu_j)\Big) \\ &= \bigcap_{(i,j)\in I^+} \mathrm{Stab}(g^{-1}R^\lambda_i \cap R^\mu_j) = \bigcap_{(i,j)\in I^+} \mathrm{Stab}(g^{-1}(A^{\mathcal{P}}_{\mathrm{std}})_{ij}) = W_{\delta(A)}. \end{aligned}$$

□

Remark 3.2.6. We can remove any zeroes that are not in the first or the last place in a weak composition in Λ without changing the parabolic subgroup, i.e. $W_{(2,0,2)} = W_{(2,2)}$. On the other hand, removing zeroes in the first or the last place changes the parabolic subgroup. For example, $W_{(0,1,1,0)}, W_{(0,1,1)}, W_{(1,1,0)}$ and $W_{(1,1)}$ are four distinct parabolic subgroups.

For $T = (t_{ij}) \in \Theta_n$, define

$$[T]^!_{\mathfrak{a}} = \prod_{i=1}^{n} \prod_{j\in\mathbb{Z}} [t_{ij}]^!_{\mathfrak{a}}, \quad \text{where} \quad [t_{ij}]^!_{\mathfrak{a}} = \prod_{k=1}^{t_{ij}} [k]. \tag{3.2.12}$$

For $A = (a_{ij}) \in \Xi_n$, define (see (3.2.2) for a'_{ij})

$$[A]^!_{\mathfrak{c}} = \prod_{(i,j)\in I^+} [a_{ij}]^!_{\mathfrak{c}}, \quad \text{where} \quad [a_{ij}]^!_{\mathfrak{c}} = \begin{cases} \prod_{k=1}^{a'_{ij}} [2k] & \text{if } i = j \in \mathbb{Z}(r+1, r+1), \\ [a_{ij}]^!_{\mathfrak{a}} & \text{otherwise.} \end{cases} \tag{3.2.13}$$

Alternatively, we have (see (3.2.8) for $I^+_{\mathfrak{a}}$):

$$[A]^!_{\mathfrak{c}} = [a_{00}]^!_{\mathfrak{c}} [a_{r+1,r+1}]^!_{\mathfrak{c}} \prod_{(i,j)\in I^+_{\mathfrak{a}}} [a_{ij}]^!_{\mathfrak{a}}. \tag{3.2.14}$$

Lemma 3.2.7. *For any $A \in \Xi_n$, we have $[A]^!_{\mathfrak{c}} = \sum_{w\in W_{\delta(A)}} q^{\ell(w)}$.*

Proof. Denote the Weyl group of type A_{m-1} (resp. C_m) by S_m (resp. W_{C_m}). It is well-known that the Poincare polynomial for S_m and W_{C_m} are, respectively,

$$\sum_{w\in S_m} q^{\ell(w)} = \prod_{k=1}^{m} [k] = [m]^!_{\mathfrak{a}} \quad \text{and} \quad \sum_{w\in W_{C_m}} q^{\ell(w)} = \prod_{k=1}^{m} [2k] = [m]^!_{\mathfrak{c}}.$$

Since $W_{\delta(A)} \simeq W_{C_{\delta_0}} \times S_{\delta_1} \times S_{\delta_2} \times \cdots \times S_{\delta_{r'}} \times W_{C_{\delta_{r'+1}}}$, we obtain

$$\sum_{w \in W_{\delta(A)}} q^{\ell(w)} = \prod_{i \in \{0, r'+1\}} \Big(\sum_{w \in W_{C_{\delta_i}}} q^{\ell(w)} \Big) \prod_{i=1}^{r'} \Big(\sum_{w \in S_{\delta_i}} q^{\ell(w)} \Big) = [A]^!_{\mathfrak{c}}.$$

□

Lemma 3.2.8. *Let $A = \kappa(\lambda, g, \mu)$ for some $\lambda, \mu \in \Lambda, g \in \mathscr{D}_{\lambda\mu}$. Then*

$$x_\lambda T_g x_\mu = [A]^!_{\mathfrak{c}} e_A(x_\mu).$$

Proof. Let $\delta = \delta(A)$. By Proposition 3.1.4 (c), we have

$$x_\mu = \sum_{x \in W_\mu} T_x = \sum_{\substack{w \in \mathscr{D}_\delta \cap W_\mu \\ y \in W_\delta}} T_{wy} = \sum_{w \in \mathscr{D}_\delta \cap W_\mu} T_w \sum_{y \in W_\delta} T_y = T_{\mathscr{D}_\delta \cap W_\mu} x_\delta.$$

Note that $x_\mu T_w = q^{\ell(w)} x_\mu$ for any $w \in W_\mu$ and thus $x_\mu x_\delta = \sum_{w \in W_\delta} q^{\ell(w)} x_\mu = [A]^!_{\mathfrak{c}} x_\mu$ since $W_\delta \subset W_\mu$. Therefore we have

$$x_\lambda T_g x_\mu = x_\lambda T_g T_{\mathscr{D}_\delta \cap W_\mu} x_\delta = e_A(x_\mu) x_\delta = e_A(x_\mu x_\delta) = [A]^!_{\mathfrak{c}} e_A(x_\mu).$$

□

For each $A = \kappa(\lambda, g, \mu) \in \Xi_{n,d}$, the length $\ell(A)$ is defined to be the length $\ell(g)$ of the corresponding Weyl group element. By rephrasing Lemma 3.1.1, we are able to express $\ell(A)$ as a polynomial in the matrix elements as follows:

Proposition 3.2.9. *Let $A = (a_{ij}) \in \Xi_n$ and a'_{ij} be the one in* (3.2.2). *Then the length of A is given by*

$$\ell(A) = \frac{1}{2} \Big(\sum_{(i,j) \in I^+} \Big(\sum_{\substack{x<i \\ y>j}} + \sum_{\substack{x>i \\ y<j}} \Big) a'_{ij} a_{xy} \Big). \tag{3.2.15}$$

3.3 A comparison with geometric realization

We now show that the Schur algebra $\mathbf{S}^{\mathfrak{c}}_{n,d}$ defined in Section 3.2 can be identified with the Schur algebra in [FLLLW1] as a convolution algebra. In the following we only provide a minimum setup. The readers may refer to [FLLLW1, Chapter 3 and 4] for more details.

Let $F = k((\epsilon))$ be the formal Laurent series over a finite field k of $q = v^2$ elements, and let $\mathrm{Sp}_F(2d)$ be the symplectic group with coefficients in F. Set

$$\mathbf{S}^{\mathfrak{c},\mathrm{geo}}_{n,d} = \mathcal{A}_{\mathrm{Sp}_F(2d)}(\mathcal{X}^{\mathfrak{c}}_{n,d} \times \mathcal{X}^{\mathfrak{c}}_{n,d}) \tag{3.3.1}$$

to be the convolution algebra, where $\mathcal{X}^{\mathfrak{c}}_{n,d}$ is the variety of n-step flags of affine type C (cf. [FLLLW1, Section 3.2]) in an F-vector space V of rank $2d$, and $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$. Denote by e^{geo}_A the characteristic function on the orbit $\mathcal{O}_A$. It is known that the Hecke algebra $\mathbf{H}$ can also be identified as a convolution algebra

$$\mathbf{H} = \mathcal{A}_{\mathrm{Sp}_F(2d)}(\mathcal{Y}^{\mathfrak{c}}_d \times \mathcal{Y}^{\mathfrak{c}}_d), \tag{3.3.2}$$

where $\mathcal{Y}^{\mathfrak{c}}_d$ (cf. $\mathcal{Y}^{\mathfrak{c}}$ in [FLLLW1, Section 3.1]) is the variety of complete flags of affine type C in V.

Lemma 3.3.1. *There is an algebra isomorphism* $\mathbf{S}^{\mathfrak{c}}_{n,d} \simeq \mathbf{S}^{\mathfrak{c},\mathrm{geo}}_{n,d}$.

Proof. Let $\psi : \mathbf{S}^{\mathfrak{c}}_{n,d} \to \mathbf{S}^{\mathfrak{c},\mathrm{geo}}_{n,d}$ be the linear map sending e_A to e^{geo}_A for all $A \in \Xi_{n,d}$. Since $\Xi_{n,d}$ parameterizes the basis of both algebras, ψ is a bijection. We now show that ψ is an algebra homomorphism. Fix $A, B, C \in \Xi_{n,d}$, and let $\lambda, \mu, \nu \in \Lambda$ and

$g_1, g_2, g_3 \in W$ be such that

$$A = \kappa(\lambda, g_1, \mu), \quad B = \kappa(\mu, g_2, \nu), \quad C = \kappa(\lambda, g_3, \nu).$$

Set $g_{AB}^C(v) \in \mathcal{A}$ to be such that

$$g_{AB}^C(v) = \# \left\{ \widetilde{L} \in \mathcal{X}_{n,d}^{\mathfrak{c}} \;\middle|\; (L, \widetilde{L}) \in \mathcal{O}_A, (\widetilde{L}, L') \in \mathcal{O}_B, (L, L') \in \mathcal{O}_C \right\}, \tag{3.3.3}$$

for some fixed $L, L' \in \mathcal{X}_{n,d}^{\mathfrak{c}}$. Therefore,

$$e_A^{\mathrm{geo}} * e_B^{\mathrm{geo}} = \sum_C g_{AB}^C(v)\; e_C^{\mathrm{geo}}. \tag{3.3.4}$$

For $x, y, z \in W$, set $B_{xy}^z(v) \in \mathcal{A}$ to be such that

$$T_x T_y = \sum_z B_{xy}^z(v) T_z. \tag{3.3.5}$$

For $g \in W$, let $\mathcal{O}_g$ be the orbit $\mathcal{O}_{\kappa(\omega,g,\omega)}$ where $\omega = (0, 1, 1, \ldots, 1, 0) \in \Lambda_{d,d}$. It is known that

$$B_{xy}^z(v) = \# \left\{ \widetilde{L} \in \mathcal{Y}_d^{\mathfrak{c}} \;\middle|\; (L, \widetilde{L}) \in \mathcal{O}_x, (\widetilde{L}, L') \in \mathcal{O}_y, (L, L') \in \mathcal{O}_z \right\}, \tag{3.3.6}$$

for some fixed $L, L' \in \mathcal{Y}_d^{\mathfrak{c}}$. Then for all $z \in W_\lambda g_3 W_\nu$ we have

$$g_{AB}^C(v) = \pi_\mu(v)^{-1} \sum_{\substack{x \in W_\lambda g_1 W_\mu \\ y \in W_\mu g_2 W_\nu}} B_{xy}^z(v), \tag{3.3.7}$$

where $\pi_\mu(v)$ is the cardinality of the fiber of the projection $\mathcal{Y}_d^{\mathfrak{c}} \to \mathrm{Sp}_F(2d)(L_\mu)$, which is given by

$$\pi_\mu(v) = \sum_{x \in W_\mu} v^{2\ell(x)}. \tag{3.3.8}$$

Also, by direct computation, we have

$$x_\mu^2 = \pi_\mu x_\mu. \tag{3.3.9}$$

Therefore,

$$e_A * e_B(x_\nu) = e_A(x_\mu)T_{g_2}T_{\mathscr{D}_{\delta(B)}\cap W_\nu} = \pi_\mu^{-1}e_A(x_\mu^2)T_{g_2}T_{\mathscr{D}_{\delta(B)}\cap W_\nu} = \pi_\mu^{-1}T_{W_\lambda g_1 W_\mu}T_{W_\mu g_2 W_\nu}, \tag{3.3.10}$$

and hence

$$e_A * e_B(x_\nu) = \pi_\mu^{-1} \sum_{z\in W_\lambda g_3 W_\nu} \sum_{\substack{x\in W_\lambda g_1 W_\mu \\ y\in W_\mu g_2 W_\nu}} B_{xy}^z T_z = \sum_{z\in W_\lambda g_3 W_\nu} g_{AB}^C T_z. \tag{3.3.11}$$

Finally, we have $e_A * e_B = \sum_C g_{AB}^C(v)\ e_C$. □

Chapter 4

Multiplication formulas

This chapter is devoted to the multiplication formula with tridiagonal generators. The proof is quite involved. An essential idea here is to identify the standard basis element e_A with its corresponding "higher-level" matrix with entries being subsets of $\mathbb{Z}$ in light of Algorithm 3.2.1. We also provide two special cases of the multiplication formula that are analogous to the multiplication formulas with semisimple generators in affine type A, and with Chevalley generators as in finite type B/C.

4.1 Structure constants

From now on, fix $B = \kappa(\lambda, g_1, \mu)$ and $A = \kappa(\mu, g_2, \nu)$ for some $\lambda, \mu, \nu \in \Lambda$, $g_1 \in \mathscr{D}_{\lambda\mu}$, and $g_2 \in \mathscr{D}_{\mu\nu}$. Recall e_A from (3.2.10).

Lemma 4.1.1. *Let $\delta = \delta(B)$ (see (3.2.11)). We have*

$$e_B * e_A(x_\nu) = \frac{1}{[A]^!_{\mathfrak{c}}} x_\lambda T_{g_1} T_{(\mathscr{D}_\delta \cap W_\mu) g_2} x_\nu.$$

Proof. From Lemma 3.2.8 we have

$$e_B e_A(x_\nu) = \frac{1}{[A]^!_{\mathfrak{c}}} e_B(x_\mu T_{g_2} x_\nu) = \frac{1}{[A]^!_{\mathfrak{c}}} e_B(x_\mu) T_{g_2} x_\nu = \frac{1}{[A]^!_{\mathfrak{c}}} x_\lambda T_{g_1} T_{\mathscr{D}_\delta \cap W_\mu} T_{g_2} x_\nu.$$

Since $g_2 \in \mathscr{D}_\mu^{-1}$, so $T_w T_{g_2} = T_{wg_2}$ for all $w \in \mathscr{D}_\delta \cap W_\mu$. Therefore $T_{\mathscr{D}_\delta \cap W_\mu} T_{g_2} = T_{(\mathscr{D}_\delta \cap W_\mu) g_2}$ and we are done. □

Remark 4.1.2. For $w \in W_\mu$, although $T_{g_1} T_w = T_{g_1 w}$ and $T_w T_{g_2} = T_{wg_2}$, it is not true that $T_{g_1} T_w T_{g_2} = T_{g_1 w g_2}$ in general. Therefore we need to write out $T_{g_1} T_{wg_2}$ in order to have a useful multiplication formula.

For $w \in \mathscr{D}_\delta \cap W_\mu$, let $\Delta(w) \subset W$ be the finite set such that

$$T_{g_1} T_{wg_2} = \sum_{\sigma \in \Delta(w)} c^{(w,\sigma)} T_{g_1 \sigma w g_2}, \quad c^{(w,\sigma)} \in \mathbb{Z}[q]. \tag{4.1.1}$$

For $\sigma \in \Delta(w)$, denote the shortest representative in the double coset $W_\lambda(g_1 \sigma w g_2) W_\nu$ by $y^{(w,\sigma)} \in \mathscr{D}_{\lambda\nu}$. Namely, $g_1 \sigma w g_2 = w_\lambda^{(\sigma)} y^{(w,\sigma)} w_\nu^{(\sigma)}$ for some $w_\lambda^{(\sigma)} \in W_\lambda, w_\nu^{(\sigma)} \in W_\nu$. In particular, $T_{g_1 \sigma w g_2} = T_{w_\lambda^{(\sigma)}} T_{y^{(w,\sigma)}} T_{w_\nu^{(\sigma)}}$. We further let $A^{(w,\sigma)} = (a_{ij}^{(w,\sigma)}) = \kappa(\lambda, y^{(w,\sigma)}, \nu)$.

Proposition 4.1.3. *Let $\delta = \delta(B)$, and let $c^{(w,\sigma)}, \Delta(w)$ be defined as in* (4.1.1). *Then*

$$e_B * e_A = \sum_{\substack{w \in \mathscr{D}_\delta \cap W_\mu \\ \sigma \in \Delta(w)}} c^{(w,\sigma)} q^{\ell(g_1 \sigma w g_2) - \ell(y^{(w,\sigma)})} \frac{[A^{(w,\sigma)}]^!_{\mathfrak{c}}}{[A]^!_{\mathfrak{c}}} e_{A^{(w,\sigma)}}. \tag{4.1.2}$$

Proof. Combining Lemma 4.1.1 and (4.1.1), we have

$$e_B * e_A(x_\nu) = \frac{1}{[A]^!_{\mathfrak{c}}} \sum_{\substack{w \in \mathscr{D}_\delta \cap W_\mu \\ \sigma \in \Delta(w)}} c^{(w,\sigma)} x_\lambda T_{g_1 \sigma w g_2} x_\nu.$$

For $\sigma \in \Delta(w)$, we have $x_\lambda T_{g_1\sigma w g_2} x_\nu = q^{\ell(w_\lambda^{(\sigma)})+\ell(w_\nu^{(\sigma)})} x_\lambda T_{y^{(w,\sigma)}} x_\nu$, and hence it follows by applying Lemma 3.2.8. □

For general g_1, it is unlikely to obtain an explicit description for $c^{(w,\sigma)}$ since it is equivalent to obtaining explicitly the structure constants for Hecke algebras. Also, it is not clear what are the pairs (w,σ) such that $A^{(w,\sigma)}$ represent the same matrix. As a consequence, the multiplication formula above (i.e., Proposition 4.1.3) does not afford the stabilization. In the following we discuss the special case when B is tridiagonal, whose multiplication formula affords a stabilization that generates the desired affine coideal subalgebra.

4.2 Shortest representatives

From now on, we assume that $B = (b_{ij}) = \kappa(\lambda, g_1, \mu)$ is tridiagonal. By slightly abuse of notations, set

$$\delta = (b'_{00}, b_{10}, b_{01}, b_{11}, \ldots, b_{i,i-1}, b_{i-1,i}, b_{ii}, \ldots, b_{n,r+1}, b'_{r+1,r+1}).$$

That is, unlike $\delta(B)$ defined in (3.2.11), here the intermediate terms δ_1, …, δ_{3r+2}, δ'_1, …, δ'_{3r+2} can be zero. Note that the two conventions coincide (i.e., $\delta = \delta(B)$) when the intermediate terms are all nonzero.

Lemma 4.2.1. *For all i, $R_i^\mu = R_{3i-1}^\delta \cup R_{3i}^\delta \cup R_{3i+1}^\delta$ and $g_1^{-1} R_i^\lambda = R_{3i-2}^\delta \cup R_{3i}^\delta \cup R_{3i+2}^\delta$*

.

Proof. By construction. □

Recall the partial order $\leqslant$ on Ξ_n by (3.2.6). For $A, B \in \Xi_n$, we set

$$\Theta_{B,A} = \{T \in \Theta_n \mid T_\theta \leqslant A, \mathrm{ro}_{\mathfrak{a}}(T)_i = b_{i-1,i} \text{ for all } i\}. \tag{4.2.1}$$

We define a map $\varphi : \mathscr{D}_\delta \cap W_\mu \to \Theta_{B,A}$ by

$$\varphi(w)_{ij} = |R^\delta_{3i-1} \cap w g_2 R^\nu_j|. \tag{4.2.2}$$

Lemma 4.2.2. *The map φ is well-defined and surjective.*

Proof. For any $w \in \mathscr{D}_\delta \cap W_\mu$, we have

$$\mathrm{ro}_{\mathfrak{a}}(\varphi(w))_i = |R^\delta_{3i-1}| = b_{i-1,i}.$$

Moreover, by Lemma 4.2.1 and that $w^{-1} \in W_\mu$, we have

$$\begin{aligned}\varphi(w)_{\theta,ij} &= |R^\delta_{3i-1} \cap w g_2 R^\nu_j| + |R^\delta_{3(-i)-1} \cap w g_2 R^\nu_{-j}| \\ &= |(R^\delta_{3i-1} \cup R^\delta_{3i+1}) \cap w g_2 R^\nu_j| \\ &\leqslant |R^\mu_i \cap w g_2 R^\nu_j| \\ &= |w^{-1} R^\mu_i \cap g_2 R^\nu_j| = |R^\mu_i \cap g_2 R^\nu_j| = a_{ij}.\end{aligned}$$

To show that φ is surjective, for any $T \in \Theta_{B,A}$ we construct an element $w_{A,T} \in \varphi^{-1}(T)$ as follows. For all i, j, we set

$$\mathcal{T}^-_{ij} = \text{subset of } (A^{\mathcal{P}}_{\mathrm{std}})_{ij} \text{ consisting of the smallest } t_{ij} \text{ elements.} \tag{4.2.3}$$

$$\mathcal{T}^+_{ij} = \text{subset of } (A^{\mathcal{P}}_{\mathrm{std}})_{ij} \text{ consisting of the largest } t_{-i,-j} \text{ elements.} \tag{4.2.4}$$

$$\mathcal{T}^0_{ij} = (A^{\mathcal{P}}_{\mathrm{std}})_{ij} - \mathcal{T}^+_{ij} - \mathcal{T}^-_{ij}. \tag{4.2.5}$$

Note that $\sum_{j\in\mathbb{Z}} |\mathcal{T}_{ij}^-| = \mathrm{ro}_{\mathfrak{a}}(T)_i = |R^\delta_{3i-1}|$, $\sum_{j\in\mathbb{Z}} |\mathcal{T}_{ij}^+| = \mathrm{ro}_{\mathfrak{a}}(T)_{-i} = |R^\delta_{3i+1}|$. There is a unique $w_{A,T} = \prod_{i=0}^{r+1} w^{(i)}_{A,T} \in \mathscr{D}_\delta \cap W_\mu$ such that $w^{(i)}_{A,T} \in \mathrm{Perm}(R^\mu_i)$ is determined by

$$w^{(i)}_{A,T}(x) \in \begin{cases} R^\delta_{3i-1}, & \text{if } x \in \bigcup_j \mathcal{T}_{ij}^- \\ R^\delta_{3i+1}, & \text{if } x \in \bigcup_j \mathcal{T}_{ij}^+ \\ R^\delta_{3i}, & \text{otherwise.} \end{cases} \tag{4.2.6}$$

The uniqueness follows from that $w^{-1}_{A,T}$ is order-preserving on each R^δ_i (c.f. Lemma 3.1.3).

□

Lemma 4.2.3. *For $T \in \Theta_{B,A}$, the element $w_{A,T}$ determined by (4.2.6) is the minimal length element in $\varphi^{-1}(T)$. Moreover, its length is given by*

$$\begin{aligned}\ell(w_{A,T}) = &\sum_{\substack{1\leqslant i\leqslant r\\ j\in\mathbb{Z}}} \Big(t_{ij}\sum_{k<j}(A-T)_{ik} + t_{-i,-j}\sum_{k>j}(A-T_\theta)_{ik}\Big)\\ &+ \sum_{\substack{j\leqslant 0\\ k<j}} t_{0j}(A-T)_{0k} + \sum_{j>0} t_{0j}\Big(\sum_{k\leqslant -j}(A-T)_{0k} + \sum_{|k|<j}(A-T_\theta)_{0k}\Big) - \sum_{j>0}\binom{1+t_{0j}}{2}\\ &+ \sum_{\substack{j\leqslant r+1\\ k<j}} t_{r+1,j}(A-T)_{r+1,k} + \sum_{j>r+1} t_{r+1,j}\Big(\sum_{k\leqslant n-j}(A-T)_{r+1,k} + \sum_{|k-r-1|<j}(A-T_\theta)_{r+1,k}\Big)\\ &- \sum_{j>r+1}\binom{1+t_{r+1,j}}{2}.\end{aligned} \tag{4.2.7}$$

Proof. It is due to the construction of $w_{A,T}$. □

Lemma 4.2.4. *Let $T \in \Theta_{B,A}$, we have*

$$\sum_{w\in\varphi^{-1}(T)} q^{\ell(w)} = q^{\ell(w_{A,T})}\frac{[A]^!_{\mathfrak{c}}}{[A-T_\theta]^!_{\mathfrak{c}}[T]^!_{\mathfrak{a}}}. \tag{4.2.8}$$

Proof. For each $(i,j) \in I_{\mathfrak{a}}^{+}$, its contribution to $\sum_{w\in\varphi^{-1}(T)} q^{\ell(w)}$ is $\frac{[a_{ij}]^!_{\mathfrak{a}}}{[t_{ij}]^!_{\mathfrak{a}}[t_{-i,-j}]^!_{\mathfrak{a}}[a_{ij}-(t_\theta)_{ij}]^!_{\mathfrak{a}}}$.

For the (k,k)-th entry where $k \in \{0, r+1\}$, we need the well-known q-binomial theorem

$$\sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix} q^{\frac{r(r-1)}{2}} x^r = \prod_{k=1}^{n} (1+q^{k-1}x). \tag{4.2.9}$$

Recall the notation a'_{ij} from (3.2.2). The contribution of (k,k)-th entry to $\sum_{w\in\varphi^{-1}(T)} q^{\ell(w)}$ is given by

$$\begin{aligned}
\sum_{x+y=t_{kk}} \begin{bmatrix} a'_{kk} \\ x \end{bmatrix} \begin{bmatrix} a'_{kk}-x \\ y \end{bmatrix} q^{\frac{x(x+1)}{2}+x(a'_{kk}-t_{kk})} &= \begin{bmatrix} a'_{kk} \\ t_{kk} \end{bmatrix} \sum_{x=0}^{t_{kk}} \begin{bmatrix} t_{kk} \\ x \end{bmatrix} q^{\frac{x(x-1)}{2}} (q^{a'_{kk}+1-t_{kk}})^x \\
&= \begin{bmatrix} a'_{kk} \\ t_{kk} \end{bmatrix} \prod_{i=1}^{t_{kk}} (1+q^{i-1}q^{a'_{kk}-t_{kk}+1}) \\
&= \begin{bmatrix} a'_{kk} \\ t_{kk} \end{bmatrix} \prod_{i=1}^{t_{kk}} \frac{[a_{kk}+1-2i]}{[a'_{kk}+1-i]} \\
&= \prod_{i=1}^{t_{kk}} \frac{[a_{kk}+1-2i]}{[t_{kk}+1-i]} \\
&= \frac{[a_{kk}]^!_{\mathfrak{c}}}{[a_{kk}-2t_{kk}]^!_{\mathfrak{c}}[t_{kk}]^!_{\mathfrak{a}}}.
\end{aligned}$$

Thus

$$\begin{aligned}
\sum_{w\in\varphi^{-1}(T)} q^{\ell(w)} &= q^{\ell(w_{A,T})} \prod_{(i,j)\in I_{\mathfrak{a}}^{+}} \frac{[a_{ij}]^!_{\mathfrak{a}}}{[t_{ij}]^!_{\mathfrak{a}}[t_{-i,-j}]^!_{\mathfrak{a}}[a_{ij}-(t_\theta)_{ij}]^!_{\mathfrak{a}}} \prod_{k=0,r+1} \frac{[a_{kk}]^!_{\mathfrak{c}}}{[a_{kk}-2t_{kk}]^!_{\mathfrak{c}}[t_{kk}]^!_{\mathfrak{a}}} \\
&= q^{\ell(w_{A,T})} \frac{[A]^!_{\mathfrak{c}}}{[A-T_\theta]^!_{\mathfrak{c}}[T]^!_{\mathfrak{a}}}.
\end{aligned}$$

□

Example 4.2.5. Let $r = 2, n = 6, d = 8$ and $D = 18$. Let $B = E^{00} + 2\sum_{1\leqslant i,j\leqslant 2} E_\theta^{ij} + E^{33}$

and $A = E^{00} + \sum_{i=1}^{2}\sum_{j=1}^{4} E_\theta^{ij} + E^{33}$, Namely,

$$B = \left[\begin{array}{c|c|cc:c:c} \ddots & & & & & \\ \hline & 1 & & & & \\ \hline & & 2 & 2 & & \\ & & 2 & 2 & & \\ \hdashline & & & & 1 & \\ \hdashline & & & & & \ddots \end{array}\right] \quad \text{and} \quad A = \left[\begin{array}{c|c|cc:c:c} \ddots & & & & & \\ \hline & 1 & & & & \\ \hline & & 1 & 1 & 1 & 1 \\ & & 1 & 1 & 1 & 1 \\ \hdashline & & & & 1 & \\ \hdashline & & & 1 & 1 & \ddots \end{array}\right],$$

where the column/row surrounded by solid lines is the 0th column/row. Therefore, $\lambda = \mu = (0,4,4,0), \nu = (0,2,4,2)$. Hence

$$\mathscr{D}_\delta \cap W_\mu = \{xy \mid x \in S_1, y \in S_2\},$$

where $S_1 = \left\{\begin{array}{cc} [1,2,3,4]_\mathfrak{c}, & [3,1,2,4]_\mathfrak{c}, \\ [1,3,2,4]_\mathfrak{c}, & [3,1,4,2]_\mathfrak{c}, \\ [1,3,4,2]_\mathfrak{c}, & [3,4,1,2]_\mathfrak{c} \end{array}\right\}$ and $S_2 = \left\{\begin{array}{cc} [5,6,7,8]_\mathfrak{c}, & [7,5,6,8]_\mathfrak{c}, \\ [5,7,6,8]_\mathfrak{c}, & [7,5,8,6]_\mathfrak{c}, \\ [5,7,8,6]_\mathfrak{c}, & [7,8,5,6]_\mathfrak{c} \end{array}\right\}$.

We also have

i	-2	-1	0	1	2	3
R^δ_{3i-1}	$\varnothing$	$[-4..-3]$	$\varnothing$	$\varnothing$	$[5..6]$	$\varnothing$

and

j	0	1	2	3	4
$g_2R^\nu_j$	$\{0\}$	$\{1,5\}$	$\{2,6,10,14\}$	$\{3,7,9,11,15\}$	$\{4,8,12,16\}$

There are nine distinct matrices $T = \varphi(w) \in \mathrm{Im}(\varphi)$ with the -1st and 2nd rows given by

$T_{-1,*}$	$\{x \in S_1 \mid \varphi(xy) = T\}$	$T_{2,*}$	$\{y \in S_2 \mid \varphi(xy) = T\}$
$\left[\begin{array}{c:c:cc\|c\|c} \ddots &&&&& \\ \hdashline & 0 &&&& \\ \hdashline 1 & 1 & 0 & 0 && \\ \hline &&&& 0 & \\ \hline &&&&& \ddots \end{array}\right]$	$\{[1,2,3,4]_{\mathfrak{c}}\}$	$\left[\begin{array}{c\|c\|cc:c:c} \ddots &&&&& \\ \hline & 0 &&&& \\ \hline && 1 & 1 & 0 & 0 \\ \hdashline &&&& 0 & \\ \hdashline &&&&& \ddots \end{array}\right]$	$\{[5,6,7,8]_{\mathfrak{c}}\}$
$\left[\begin{array}{c:c:cc\|c\|c} \ddots &&&&& \\ \hdashline & 0 &&&& \\ \hdashline 1 & 0 & 1 & 0 && \\ \hline &&&& 0 & \\ \hline &&&&& \ddots \end{array}\right]$	$\left\{\begin{array}{c} [1,3,2,4]_{\mathfrak{c}}, \\ [1,3,4,2]_{\mathfrak{c}}, \\ [3,1,2,4]_{\mathfrak{c}}, \\ [3,1,4,2]_{\mathfrak{c}} \end{array}\right\}$	$\left[\begin{array}{c\|c\|cc:c:c} \ddots &&&&& \\ \hline & 0 &&&& \\ \hline && 1 & 0 & 1 & 0 \\ \hdashline &&&& 0 & \\ \hdashline &&&&& \ddots \end{array}\right]$	$\left\{\begin{array}{c} [5,7,6,8]_{\mathfrak{c}}, \\ [5,7,8,6]_{\mathfrak{c}}, \\ [7,5,6,8]_{\mathfrak{c}}, \\ [7,5,8,6]_{\mathfrak{c}} \end{array}\right\}$
$\left[\begin{array}{c:c:cc\|c\|c} \ddots &&&&& \\ \hdashline & 0 &&&& \\ \hdashline 0 & 0 & 1 & 1 && \\ \hline &&&& 0 & \\ \hline &&&&& \ddots \end{array}\right]$	$\{[3,4,1,2]_{\mathfrak{c}}\}$	$\left[\begin{array}{c\|c\|cc:c:c} \ddots &&&&& \\ \hline & 0 &&&& \\ \hline && 0 & 0 & 1 & 1 \\ \hdashline &&&& 0 & \\ \hdashline &&&&& \ddots \end{array}\right]$	$\{[7,8,5,6]_{\mathfrak{c}}\}$

4.3 Multiplication formulas for Hecke algebras

In this section, we deal with (4.1.1) for $B = \kappa(\lambda, g_1, \mu)$ is tridiagonal. In this special case, g_1 is the permutation "swapping" R^δ_{3i-2} and R^δ_{3i-1} for any i, and hence it can be written as $g_1 = \prod_{i=1}^{r+1} g_1^{(i)}$, where $g_1^{(i)} \in W$ is determined by

$$g_1^{(i)}(x) = \begin{cases} x + b_{i-1,i} & \text{if } x \in R^\delta_{3i-2} \subset R^\mu_{i-1}, \\ x - b_{i,i-1} & \text{if } x \in R^\delta_{3i-1} \subset R^\mu_{i}, \\ x & \text{if } x \in [1..d] \backslash (R^\delta_{3i-2} \cup R^\delta_{3i-1}). \end{cases} \tag{4.3.1}$$

Lemma 4.3.1. *For $i = 1, \ldots, r+1$, assume that $R^\delta_{3i-2} = [m+1..m+\alpha]$ and $R^\delta_{3i-1} = [m+\alpha+1..m+\alpha+\beta]$ for some $m, \alpha, \beta \in \mathbb{N}$. Then $g_1^{(i)}$ has a reduced expression*

$$g_1^{(i)} = (s_{m+\beta} \cdots s_{m+2} s_{m+1})(s_{m+\beta+1} \cdots s_{m+2}) \cdots (s_{m+\beta+\alpha-1} \cdots s_{m+\alpha}). \tag{4.3.2}$$

Proof. By direct computation, $s_{m+\beta+t-1} \cdots s_{m+t}$ is the permutation on $[m+t..m+t+\beta]$ sending

$$m+\beta+t \mapsto m+t, \quad m+\beta+t-1 \mapsto m+\beta+t, \quad \ldots, \quad m+t \mapsto m+t+1.$$

Lemma then follows from (4.3.1). □

For any $i = 1, 2, \ldots, r+1$ and $w \in \mathscr{D}_\delta \cap W_\mu$, define $K^{(i)}_w$ to be the set in which element is a product of disjoint transpositions such that each transposition $(j,k)_\mathfrak{c}$ satisfies

$$j \in R^\delta_{3i-2}, \quad k \in R^\delta_{3i-1}, \quad (wg_2)^{-1}(k) < (wg_2)^{-1}(j). \tag{4.3.3}$$

We also define

$$K_w := \left\{ \prod_{i=1}^{r+1} \sigma^{(i)} \;\middle|\; \sigma^{(i)} \in K_w^{(i)} \right\}. \tag{4.3.4}$$

Since each elements of $K_w \subset W$ is a product of disjoint transpositions, We note here that $\sigma^{-1} = \sigma$ for any $\sigma \in K_w$. For $w \in \mathscr{D}_\delta \cap W_\mu$, denote the number of disjoint transpositions in $\sigma \in K_w$ by

$$n(\sigma) = \sum_{i=1}^{r+1} s_i, \quad \text{if} \quad \sigma = \prod_{i=1}^{r+1} \prod_{l=1}^{s_i} (j_l^{(i)} k_l^{(i)})_{\mathfrak{c}}. \tag{4.3.5}$$

We also set $h(w,\sigma) = |H(w,\sigma)|$, where

$$H(w,\sigma) = \bigcup_{i=1}^{r+1} \left\{ (j,k) \in R^\delta_{3i-2} \times R^\delta_{3i-1} \;\middle|\; \begin{array}{c} (wg_2)^{-1}\sigma(j) > (wg_2)^{-1}(k), \\ (wg_2)^{-1}(j) > (wg_2)^{-1}\sigma(k) \end{array} \right\} \tag{4.3.6}$$

for $w \in \mathscr{D}_\delta \cap W_\mu$, $\sigma \in K_w$.

Lemma 4.3.2. *Assume that $w_1, w_2 \in \mathscr{D}_\delta \cap W_\mu$. If $\varphi(w_1) = \varphi(w_2)$, then $K_{w_1} = K_{w_2}$ and $H(w_1,\sigma) = H(w_2,\sigma)$ for any $\sigma \in K_{w_1} = K_{w_2}$.*

Proof. Since $\varphi(w_1) = \varphi(w_2)$, we have that for any $x \in R^\delta_{3i-2} \cup R^\delta_{3i-1}$, $w_1^{-1}(x)$ and $w_2^{-1}(x)$ lie in the same entry of $A^{\mathcal{P}}_{\text{std}}$. Thus for any $j \in R^\delta_{3i-2}$ and $k \in R^\delta_{3i-1}$, $g_2^{-1}w_1^{-1}(k) < g_2^{-1}w_1^{-1}(j)$ if and only if $g_2^{-1}w_2^{-1}(k) < g_2^{-1}w_2^{-1}(j)$. So $K_{w_1} = K_{w_2}$ and $H(w_1,\sigma) = H(w_2,\sigma)$ by the definition (4.3.3) and (4.3.6). □

As a consequence, for $T \in \Theta_{B,A}$, the set below is well-defined:

$$K(T) = K_w \quad \text{for some} \quad w \in \varphi^{-1}(T), \tag{4.3.7}$$

so is

$$h(T,\sigma) = h(w,\sigma) \quad \text{for some} \quad w \in \varphi^{-1}(T), \sigma \in K(T). \tag{4.3.8}$$

In below we compute the non-trivial structure constants for Hecke algebras which reflect the fact that the tridiagonal generator is not necessary bar-invariant (cf. Remark 5.2.10).

Theorem 4.3.3. *For $w \in \mathscr{D}_\delta \cap W_\mu$, we have*

$$T_{g_1} T_{wg_2} = \sum_{\sigma \in K_w} (q-1)^{n(\sigma)} q^{h(w,\sigma)} T_{g_1 \sigma w g_2}. \tag{4.3.9}$$

See (4.3.5) *and* (4.3.6) *for $n(\sigma)$ and $h(w,\sigma)$, respectively.*

Proof. It suffices to show that for all $1 \leqslant i \leqslant r+1$, we have

$$T_{g_1^{(i)}} T_{wg_2} = \sum_{\sigma \in K_w^{(i)}} (q-1)^{n(\sigma)} q^{h(w,\sigma)} T_{g_1^{(i)} \sigma w g_2} \tag{4.3.10}$$

By Lemma 4.3.1, we have

$$T_{g_1^{(i)}} = (T_{m+\beta} \cdots T_{m+1})(T_{m+\beta+1} \cdots T_{m+2}) \cdots (T_{m+\beta+\alpha-1} \cdots T_{m+\alpha}). \tag{4.3.11}$$

Write $g = s_{m+\beta+\alpha-1} \cdots s_{m+\alpha}$ and $x = wg_2$ for short. We start with showing

$$T_g T_{wg_2} = q^{Q(x,j,\mathbb{1})} T_{gx} + (q-1) \sum_{\substack{k \in R^\delta_{3i-1} \\ x^{-1}(j) > x^{-1}(k)}} q^{Q(x,j,(j,k))} T_{g(j,k)x}, \tag{4.3.12}$$

where

$$\begin{aligned} Q(x,j,\sigma) &= |\{k \in R^\delta_{3i-1} \mid x^{-1}(k) < (\sigma x)^{-1}(j)\}| \\ &= |\{k \in R^\delta_{3i-1} \mid (\sigma x)^{-1}(k) < (\sigma x)^{-1}(j)\}|, \end{aligned} \tag{4.3.13}$$

which counts the number of elements in R^{δ}_{3i-1} that are to the left of j on the one-line notation for σx. Recall first that for $s_i \in S, y \in W$, that $\ell(s_i y) = \ell(y) - 1$ is equivalent to $y^{-1}(i+1) < y^{-1}(i)$. So we have

$$T_i T_y = \begin{cases} T_{s_i y} & \text{if } y^{-1}(i+1) > y^{-1}(i), \\ qT_{s_i y} + (q-1)T_y & \text{if } y^{-1}(i+1) < y^{-1}(i). \end{cases}$$

Let k_m be the mth smallest number in the set $\{k \in R^{\delta}_{3i-1} \mid x^{-1}(j) > x^{-1}(k)\}$. If this set is empty, then $T_g T_x = T_{gx}$ and we are done. Now we assume that this set is non-empty. Let $y = s_{k_1-2}s_{k_1-1}\ldots s_j x$, we have

$$y^{-1}(k_1) = x^{-1}(k_1) < x^{-1}(j) = y^{-1}(k_1 - 1),$$

and thus

$$T_{k_1-1}\ldots T_{j+1}T_jT_x = T_{k_1-1}T_{s_{k_1-2}\ldots s_j x} = qT_{s_{k_1-1}s_{k_1-2}\ldots s_j x} + (q-1)T_{s_{k_1-2}\ldots s_j x}.$$

On the other hand, by assumption x^{-1} is order-preserving on R^{δ}_{3i-2} and thus each k_m must be in R^{δ}_{3i-1}. Now we show that the RHS is $(q-1)^2$-free: we will show that if the coefficient of T_y in $T_aT_{a-1}\ldots T_jT_x$ is a nonzero multiple of $q-1$, then y must be of the form

$$y = s_a s_{a-1}\ldots \widehat{s}_b \ldots s_j x, \quad a \geqslant b \geqslant k_1 - 1, b = k_i - 1 \text{ for some } i.$$

The initial case (i.e., $(q-1)T_{s_{k_1-1}\ldots s_j x}$) is indeed of the form (with $a = b = k_1 - 1$). It suffices to show that if y is of the form, then $\ell(s_{j+a+1}y) > \ell(y)$ and hence further multiplication does not produce $(q-1)$'s anymore.

Note that $y^{-1}(a+2) = x^{-1}(a+2)$ and

$$y^{-1}(a+1) = \begin{cases} x^{-1}(a+1) & \text{if } a = b, \\ \\ x^{-1}(b+1) & \text{if } a > b. \end{cases}$$

Since $a+2 > a+1 \geqslant b+1 \geqslant k_1$, so these numbers all lie in R^{δ}_{3i-1}. Again by the assumption that x^{-1} is order-preserving on R^{δ}_{3i-1}, we have $x^{-1}(a+2) > x^{-1}(a+1) \geqslant x^{-1}(b+1)$ and hence $y^{-1}(a+2) > y^{-1}(a+1)$. In other words, we have

$$T_g T_x = q^{Q(x,j,\mathbb{1})} T_{gx} + (q-1) \sum_{k=k_i} q^{i-1} T_{s_{j+\beta-1} \cdots \widehat{s}_{k-1} \cdots s_j x},$$

where $s_{j+\beta-1} \ldots \widehat{s}_{k-1} \ldots s_j x = g s_j \ldots s_{k-2} s_{k-1} s_{k-2} \ldots s_j x = g(j,k)x$. Finally,

$$i - 1 = \#\{k \in R^{\delta}_{3i-1} \mid k < k_i, x^{-1}(k) < x^{-1}(j)\} = Q(x, j, (j, k_i)).$$

We repeat the procedure. Let $\eta = (m+\alpha, m+\alpha+x)_{\mathfrak{c}}$, we have

$$\begin{aligned} & \{k \in R^{\delta}_{3i-1} | (\eta w g_2)^{-1}(k) < (\eta w g_2)^{-1}(m+\alpha-1)\} \\ = \; & \{k \in R^{\delta}_{3i-1} | (w g_2)^{-1}(k) < (w g_2)^{-1}(m+\alpha-1)\} \backslash \{m+\alpha+x\} \\ = \; & \{k \in R^{\delta}_{3i-1} | (w g_2)^{-1}(k) < (w g_2)^{-1}\eta(m+\alpha-1), (w g_2)^{-1}\eta(k) < (w g_2)^{-1}(m+\alpha-1)\}. \end{aligned}$$

Hence

$$\begin{aligned} & (T_{s_{m+\beta+\alpha-2}} \cdots T_{s_{m+\alpha-1}})(T_{s_{m+\beta+\alpha-1}} \cdots T_{s_{m+\alpha}}) T_{w g_2} \\ = \; & \sum_{\zeta} (q-1)^{n(\zeta)} q^{h'(\zeta)} T_{s_{m+\beta+\alpha-2} \cdots s_{m+\alpha-1} s_{m+\beta+\alpha-1} \cdots s_{m+\alpha} \zeta w g_2}. \end{aligned} \tag{4.3.14}$$

where ζ runs over $\mathbb{1}, (m+\alpha-1, k_1)_{\mathfrak{c}}, (m+\alpha, k_2)_{\mathfrak{c}}, (m+\alpha-1, k_1)_{\mathfrak{c}}(m+\alpha, k_2)_{\mathfrak{c}}, (k_1 \neq k_2)$

with $wg_2{}^{-1}(m+\alpha-1) > wg_2{}^{-1}(k_1)$ and $wg_2{}^{-1}(m+\alpha) > wg_2{}^{-1}(k_2)$, and

$$h'(\zeta) = \left| \left\{ (j,k) \in R^{\delta}_{3i-2} \times R^{\delta}_{3i-1} \;\middle|\; \begin{array}{c} j = m+\alpha \text{ or } m+\alpha-1, \\ (wg_2)^{-1}\zeta(j) > (wg_2)^{-1}(k), (wg_2)^{-1}(j) > (wg_2)^{-1}\zeta(k) \end{array} \right\} \right|.$$

Equation (4.3.10) follows by repeating similar arguments several times.

□

With the assumption that B is tridiagonal, (4.1.1) can be written as

$$T_{g_1} T_{wg_2} = \sum_{\sigma \in K_w} (q-1)^{n(\sigma)} q^{h(w,\sigma)} T_{g_1 \sigma w g_2}. \tag{4.3.15}$$

Example 4.3.4. Retain the notation as in Example 4.2.5. In this section, we use a two-by-four submatrix for short when there is no ambiguity. That is,

$$B = \begin{array}{|c|c|c|c|} \hline 2 & 2 & 0 & 0 \\ \hline 2 & 2 & 0 & 0 \\ \hline \end{array}, \quad A = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 1 \\ \hline 1 & 1 & 1 & 1 \\ \hline \end{array}.$$

Thus, $g_1 = g^{\text{std}}\left(\begin{array}{|c|c|c|c|} \hline 12 & 34 & & \\ \hline 56 & 78 & & \\ \hline \end{array}\right) = [1,2,5,6,3,4,7,8]_{\mathfrak{c}}$ is obtained by column-reading (see (3.1.1)). Also, we have

i	0	1	2	3	4	5	6	7	8	9
R_i^δ	$\{0\}$			$[1..2]$	$[3..4]$	$[5..6]$	$[7..8]$			$\{9\}$

That is, $g_1 = g_1^{(2)} = (s_4 s_3)(s_5 s_4)$ with $m = \alpha = \beta = 2$. Also, $g_2 = g^{\text{std}}\left(\begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 & 7 & 8 \\ \hline \end{array}\right) = [1,5,2,6,3,7,4,8]_{\mathfrak{c}}$.

Now write $T_{\begin{array}{|c|c|c|c|}\hline a & b & c & d \\ \hline e & f & g & h \\ \hline\end{array}} = T_x$ where $x = g^{\mathrm{std}}\left(\begin{array}{|c|c|c|c|}\hline a & b & c & d \\ \hline e & f & g & h \\ \hline\end{array}\right)$ for short. We have

$$T_4 T_{\begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 & 7 & 8 \\ \hline\end{array}} = qT_{\begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 5 \\ \hline 4 & 6 & 7 & 8 \\ \hline\end{array}} + (q-1)T_{\begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 & 7 & 8 \\ \hline\end{array}},$$

$$T_5 T_4 T_{\begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 & 7 & 8 \\ \hline\end{array}} = q^2 T_{\begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 6 \\ \hline 4 & 5 & 7 & 8 \\ \hline\end{array}} + (q-1)\Big(qT_{\begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 5 \\ \hline 4 & 6 & 7 & 8 \\ \hline\end{array}} + T_{\begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 6 & 5 & 7 & 8 \\ \hline\end{array}}\Big).$$

Let us deal with the simplest case $w = \mathbb{1}$ here. We have

$$K_w^{(1)} = K_w^{(3)} = \{\mathbb{1}\}, \quad K_w^{(2)} = K_w = \{\mathbb{1}, (3,5), (3,6), (4,5), (4,6), (3,5)(4,6), (3,6)(4,5)\},$$

and $T_{g_1} T_{wg_2} = \sum (q-1)^{n(\sigma)} q^{h(\sigma)} T_{g_1 \sigma w g_2}$ with

$T_{g_1\sigma w g_2}$	σ	$n(\sigma)$	$h(w,\sigma)$
$T_{\begin{array}{\|c\|c\|c\|c\|}\hline \ & \ & 5 & 6 \\ \hline 3 & 4 & \ & \ \\ \hline\end{array}}$	$\mathbb{1}$	0	4
$T_{\begin{array}{\|c\|c\|c\|c\|}\hline \ & \ & 4 & 6 \\ \hline 3 & 5 & \ & \ \\ \hline\end{array}}$	$(3,6)$	1	3
$T_{\begin{array}{\|c\|c\|c\|c\|}\hline \ & \ & 3 & 6 \\ \hline 5 & 4 & \ & \ \\ \hline\end{array}}$	$(3,5)$	1	2
$T_{\begin{array}{\|c\|c\|c\|c\|}\hline \ & \ & 5 & 4 \\ \hline 3 & 6 & \ & \ \\ \hline\end{array}}$	$(4,6)$	1	2
$T_{\begin{array}{\|c\|c\|c\|c\|}\hline \ & \ & 3 & 4 \\ \hline 5 & 6 & \ & \ \\ \hline\end{array}}$	$(3,5)(4,6)$	2	1
$T_{\begin{array}{\|c\|c\|c\|c\|}\hline \ & \ & 5 & 3 \\ \hline 6 & 4 & \ & \ \\ \hline\end{array}}$	$(4,5)$	1	1
$T_{\begin{array}{\|c\|c\|c\|c\|}\hline \ & \ & 4 & 3 \\ \hline 6 & 5 & \ & \ \\ \hline\end{array}}$	$(3,6)(4,5)$	2	0

Lemma 4.3.5. *If $w \in \mathscr{D}_\delta \cap W_\mu$ and $\sigma \in K_w$, then*

$$\ell(g_1) + \ell(w) + \ell(g_2) = \ell(g_1\sigma w g_2) + n(\sigma) + 2h(w,\sigma). \tag{4.3.16}$$

Proof. It follows from (4.3.15). □

4.4 Multiplication formulas with tridiagonal generators

For any $w \in \mathscr{D}_\delta \cap W_\mu$ and $\sigma \in K_w$, denote the shortest representative in the double coset $W_\lambda g_1 \sigma w g_2 W_\nu$ by $y^{(w,\sigma)} \in \mathscr{D}_{\lambda\nu}$. We further set $A^{(w,\sigma)} = (a_{ij}^{(w,\sigma)}) = \kappa(\lambda, y^{(w,\sigma)}, \nu)$. Now, for each element $\sigma = \prod_{i=1}^{r+1} \sigma^{(i)} \in K_w$ such that $\sigma^{(i)} \in K_w^{(i)}$, we fix the unique expression $\sigma^{(i)} = \prod_{l=1}^{s_i} (j_l^{(i)}, k_l^{(i)})_{\mathfrak{c}}$ satisfying

$$j_1^{(i)} < j_2^{(i)} < \cdots < j_{s_i}^{(i)}. \tag{4.4.1}$$

We further set $s_{-i} = s_{i+1}$ for $0 \leqslant i \leqslant r$ and

$$j_l^{(-i)} = k_{s_{i+1}-l+1}^{(i+1)}, \quad k_l^{(-i)} = j_{s_{i+1}-l+1}^{(i+1)} \quad \text{for} \quad 0 \leqslant i \leqslant r, 1 \leqslant l \leqslant s_i. \tag{4.4.2}$$

Hence the permutations $\sigma^{(-i)} = \prod_{l=1}^{s_{-i}} (j_l^{(-i)}, k_l^{(-i)})_{\mathfrak{c}}$ for $0 \leqslant i \leqslant r$ satisfy (4.4.1) as well. For $w \in \mathscr{D}_\delta \cap W_\mu$,we define a map $\psi_w : K_w \to \Theta_n$ by

$$\begin{aligned} \psi_w(\sigma)_{ij} &= |R_{3i-1}^\delta \cap \sigma(R_{3i-2}^\delta) \cap w g_2 R_j^\nu| \\ &= |\{k_l^{(i)}\}_{l=1}^{s_i} \cap w g_2 R_j^\nu|. \end{aligned} \tag{4.4.3}$$

For any matrix T, recall $\widehat{T}$ from (2.3.2). Assume that $S = \psi_w(\sigma)$ for some $w \in \mathscr{D}_\delta \cap W_\mu$ and $\sigma \in K_w$. By (4.4.2) we have

$$\begin{aligned} \widehat{s}_{ij} &= |\{k_l^{(i+1)}\}_l \cap wg_2R_j^\nu|, \\ s_{-i,-j} &= |\{j_l^{(i+1)}\}_l \cap wg_2R_j^\nu|, \\ (\widehat{s})_{-i,-j} &= |\{j_l^{(i)}\}_l \cap wg_2R_j^\nu|. \end{aligned} \tag{4.4.4}$$

For any matrix $S = (s_{ij})$, denote by $S^\dagger = (s_{ij}^\dagger)$ the matrix obtained by rotating the matrix $\widehat{S}$ by 180 degrees and then shifting up entries by one row, namely,

$$s_{ij}^\dagger = s_{1-i,-j} = (\widehat{s})_{-i,-j}. \tag{4.4.5}$$

For $T \in \Theta_{B,A}$ (cf. (4.2.1)), we set

$$\Gamma_T = \{S \in \Theta \mid S \leqslant T, \mathrm{ro}_{\mathfrak{a}}(S) = \mathrm{ro}_{\mathfrak{a}}(S^\dagger)\}. \tag{4.4.6}$$

Lemma 4.4.1. *For $w \in \mathscr{D}_\delta \cap W_\mu$, we have*

$$\psi_w(K_w) \subset \Gamma_{\varphi(w)}.$$

Proof. For each $\sigma \in K_w$, it follows from (4.4.2) that $\mathrm{ro}_{\mathfrak{a}}(\psi_w(\sigma)) = \mathrm{ro}_{\mathfrak{a}}(\psi_w(\sigma)^\dagger)$. Also, by Lemma 4.2.2) we have

$$|\{k_1^{(i)}, k_2^{(i)}, \ldots, k_{s_i}^{(i)}\} \cap wg_2R_j^\nu| \leqslant |R_{3i-1}^\delta \cap wg_2R_j^\nu| = \varphi(w)_{ij},$$

and hence $\psi_w(\sigma) \leqslant T$. □

For $T \in \Theta_{B,A}, S \in \Gamma_T$, set

$$A^{(T,S)} = A - (T-S)_\theta + (\widehat{T-S})_\theta, \tag{4.4.7}$$

Lemma 4.4.2. *For $w \in \mathscr{D}_\delta \cap W_\mu$ and $\sigma \in K_w$, we have*

$$A^{(w,\sigma)} = A^{(T,S)}, \tag{4.4.8}$$

where $T = \varphi(w)$ (cf. (4.2.2)) and $S = \psi_w(\sigma)$ (cf. (4.4.3)).

Proof. By the definitions (4.2.2) and (4.4.3), we have

$$\begin{aligned} (T-S)_{ij} &= |(R^\delta_{3i-1} - \sigma(R^\delta_{3i-2})) \cap wg_2R^\nu_j|, \\ (T-S)_{-i,-j} &= |(R^\delta_{3i+1} - \sigma(R^\delta_{3i+2})) \cap wg_2R^\nu_j|, \\ (\widehat{T-S}))_{ij} &= |(R^\delta_{3i+2} - \sigma(R^\delta_{3i+1})) \cap wg_2R^\nu_j|, \\ (\widehat{T-S})_{-i,-j} &= |(R^\delta_{3i-2} - \sigma(R^\delta_{3i-1})) \cap wg_2R^\nu_j|. \end{aligned} \tag{4.4.9}$$

Recall from Lemma 4.2.1 that $g_1^{-1}R^\lambda_i = R^\delta_{3i-2} \cup R^\delta_{3i} \cup R^\delta_{3i+2}$ We have

$$\begin{aligned} a^{(w,\sigma)}_{ij} &= |R^\lambda_i \cap g_1\sigma wg_2R^\nu_j| = |\sigma g_1^{-1}R^\lambda_i \cap wg_2R^\nu_j| \\ &= |\sigma(R^\delta_{3i-2}) \cap wg_2R^\nu_j| + |R^\delta_{3i} \cap wg_2R^\nu_j| + |\sigma(R^\delta_{3i+2}) \cap wg_2R^\nu_j|. \end{aligned} \tag{4.4.10}$$

Again by Lemma 4.2.1, we have $R^\mu_i = R^\delta_{3i-1} \cup R^\delta_{3i} \cup R^\delta_{3i+1}$, and hence

$$\begin{aligned} a_{ij} &= |R^\delta_{3i-1} \cup R^\delta_{3i} \cup R^\delta_{3i+1} \cap wg_2R^\nu_j| \\ &= a^{(w,\sigma)}_{ij} - ((T-S)_\theta)_{ij} + ((\widehat{T-S})_\theta)_{ij}. \end{aligned}$$

□

Example 4.4.3. Following Example 4.2.5, we choose T to be the matrix

$$T = \left[\begin{array}{c:c:cc|c|cc:c:c}
\ddots & & & & & & & & \\ \hdashline
 & 0 & & & & & & & \\ \hdashline
 & & & & & & & & \\
1 & 1 & 0 & 0 & & & & & \\ \hline
 & & & & 0 & & & & \\ \hline
 & & & & & & & & \\
 & & & & & 1 & 1 & 0 & 0 \\ \hdashline
 & & & & & & & 0 & \\ \hdashline
 & & & & & & & & \ddots
\end{array}\right] = E^{21} + E^{22} + E^{-1,-4} + E^{-1,-3}.$$

For any matrix $M = (m_{ij})$, we use another short-hand notation by writing

$$M_{\mathfrak{a}} = \sum_{(i,j)\in I^+} m_{\theta,ij} E^{ij}. \tag{4.4.11}$$

Therefore we have

$$T_{\mathfrak{a}} = \begin{array}{|c|c|c|c|} \hline 0 & 0 & 1 & 1 \\ \hline 1 & 1 & 0 & 0 \\ \hline \end{array}.$$

In this case we have $\varphi^{-1}(T) = \{\mathbb{1}\}$. Moreover, by Example 4.3.4 we have

$$K(T) = \{\mathbb{1}, (3,5)_{\mathfrak{c}}, (3,6)_{\mathfrak{c}}, (4,5)_{\mathfrak{c}}, (4,6)_{\mathfrak{c}}, (3,5)_{\mathfrak{c}}(4,6)_{\mathfrak{c}}, (3,6)_{\mathfrak{c}}(4,5)_{\mathfrak{c}}\}.$$

The complete list of $S \in \Gamma_T$ is given by

S	$S_{\mathfrak{a}}$	$\psi_w^{-1}(S)$
0	$\begin{array}{\|c\|c\|c\|c\|}\hline 0&0&0&0\\\hline 0&0&0&0\\\hline\end{array}$	$\{\mathbb{1}\}$
$E^{-1,-3}+E^{21}$	$\begin{array}{\|c\|c\|c\|c\|}\hline 0&0&1&0\\\hline 1&0&0&0\\\hline\end{array}$	$\{(3,5)_{\mathfrak{c}}\}$
$E^{-1,-3}+E^{22}$	$\begin{array}{\|c\|c\|c\|c\|}\hline 0&0&1&0\\\hline 0&1&0&0\\\hline\end{array}$	$\{(3,6)_{\mathfrak{c}}\}$
$E^{-1,-4}+E^{21}$	$\begin{array}{\|c\|c\|c\|c\|}\hline 0&0&0&1\\\hline 1&0&0&0\\\hline\end{array}$	$\{(4,5)_{\mathfrak{c}}\}$
$E^{-1,-4}+E^{22}$	$\begin{array}{\|c\|c\|c\|c\|}\hline 0&0&0&1\\\hline 0&1&0&0\\\hline\end{array}$	$\{(4,6)_{\mathfrak{c}}\}$
T	$\begin{array}{\|c\|c\|c\|c\|}\hline 0&0&1&1\\\hline 1&1&0&0\\\hline\end{array}$	$\{(3,5)_{\mathfrak{c}}(4,6)_{\mathfrak{c}},(3,6)_{\mathfrak{c}}(4,5)_{\mathfrak{c}}\}$

We define an element $\sigma_{w,S} = \prod_{i=1}^{r+1}\prod_{l=1}^{s_i}(j_l^{(i)},k_l^{(i)})_{\mathfrak{c}} \in \psi_w^{-1}(S)$ satisfying the following conditions.

(S1) $k_1^{(i)} < k_2^{(i)} < \cdots < k_{s_i}^{(i)}$ for all i.

(S2) $w^{-1}(\{k_l^{(i)}\}_l) \cap g_2R_j^{\nu}$ consists of the largest s_{ij} elements in $w^{-1}R_{3i-1}^{\delta} \cap g_2R_j^{\nu}$ for all i;

It follows from (4.4.1) that Conditions (S1) and (S2) together imply the condition below:

(S3) $w^{-1}(\{j_l^{(i)}\}_l) \cap g_2R_j^{\nu}$ consists of the smallest s_{ij} elements in $w^{-1}R_{3i-2}^{\delta} \cap g_2R_j^{\nu}$ for all i.

For $S \in \Gamma_T$, recall $S^\dagger$ from (4.4.5) and set

$$[\![S]\!] = \prod_{i=1}^{r+1} [\![S]\!]_i\,, \tag{4.4.12}$$

where

$$[\![S]\!]_i = \prod_{j\in\mathbb{Z}} \begin{bmatrix} \sum_{k\leqslant j}(S-S^\dagger)_{ik} \\ s^\dagger_{i,j+1} \end{bmatrix}_{\mathfrak{a}} [s^\dagger_{i,j+1}]^!_{\mathfrak{a}}. \tag{4.4.13}$$

Each $[\![S]\!]_i$ counts the "quantum number" of pairs (x, y) in the following sense:

1. The element x contributes to the ith row of S. That is, $x \in \{k_l^{(i)}\}_{l=1}^{s_i}$;
2. The element y contributes to the ith row of $S^\dagger$. That is, $y \in \{j_l^{(i)}\}_{l=1}^{s_i}$;
3. The element x is "to the left" of y as elements in $A^{\mathcal{P}}_{\text{std}}$.

Lemma 4.4.4. *Let $T \in \Theta_{B,A}$ and $S \in \Gamma_T$. For any $w \in \varphi^{-1}(T)$, we have*

$$\sum_{\sigma\in\psi_w^{-1}(S)} q^{-h(w,\sigma)} = \begin{bmatrix} T \\ S \end{bmatrix}_{\mathfrak{a}} [\![S]\!]\, q^{-h(w,\sigma_{w,S})} \tag{4.4.14}$$

Proof. Let $s_i = \mathrm{ro}_{\mathfrak{a}}(S)_i = \mathrm{ro}_{\mathfrak{a}}(S^\dagger)_i$ for all i. By definition, each $\sigma \in \psi_w^{-1}(S)$ can be reconstructed by the following steps:

1. For $1 \leqslant i \leqslant r+1, j \in \mathbb{Z}$, choose $s^\dagger_{i,j}$ elements from the set $R^\delta_{3i-2} \cap wg_2R^\nu_j$.
2. Let $j_\sigma = \{j_1^{(i)}, j_2^{(i)}, \dots, j_{s_i}^{(i)}\}$ be the set of elements chosen from $\bigcup_{j\in\mathbb{Z}} R^\delta_{3i-2} \cap wg_2R^\nu_j$ such that
$$j_1^{(i)} < j_2^{(i)} < \cdots < j_{s_i}^{(i)}.$$
3. For $1 \leqslant i \leqslant r+1, j \in \mathbb{Z}$, choose $s_{i,j}$ elements from the set $R^\delta_{3i-1} \cap wg_2R^\nu_j$.

4. Let $k_\sigma = \{k_1^{(i)}, k_2^{(i)}, \ldots, k_{s_i}^{(i)}\}$ be the set of elements chosen from $\bigcup_{j\in\mathbb{Z}} R^\delta_{3i-1} \cap wg_2 R^\nu_j$ such that

$$(wg_2)^{-1}(j_s^{(i)}) > (wg_2)^{-1}(k_s^{(i)}), \quad \text{for} \quad s = 1, 2, \ldots, s_i.$$

   Note that it is not necessary that $k_1^{(i)} < k_2^{(i)} < \cdots < k_{s_i}^{(i)}$.

5. Set $\sigma = \prod_{i=1}^{r+1} (j_1^{(i)}, k_1^{(i)})_{\mathfrak{c}} \cdots (j_{s_i}^{(i)}, k_{s_i}^{(i)})_{\mathfrak{c}}$.

For those $\sigma \in \psi_w^{-1}(S)$ having the same k_σ (say $k_\sigma = K$), we pick a representative

$$\sigma^{\lhd K} = \prod_{i=1}^{r+1} (j_1^{(i)}, k_1^{(i)})_{\mathfrak{c}} \cdots (j_{s_i}^{(i)}, k_{s_i}^{(i)})_{\mathfrak{c}}$$

such that $k_1^{(i)} < k_2^{(i)} < \cdots < k_{s_i}^{(i)}$. Hence the sum over such σ is then

$$\sum_{\sigma\in K_w, k_\sigma = K} q^{-h(w,\sigma)} = [\![S]\!]\, q^{-h(w,\sigma_{\lhd K})}, \tag{4.4.15}$$

In other words, any σ showed up in (4.4.15) must be of the form

$$\sigma = \prod_{i=1}^{r+1} (j_1^{(i)}, k_{\epsilon(1)}^{(i)})_{\mathfrak{c}} \cdots (j_{s_i}^{(i)}, k_{\epsilon(s_i)}^{(i)})_{\mathfrak{c}}, \quad \epsilon \in S_{s_i}, \quad (wg_2)^{-1}(j_s^{(i)}) > (wg_2)^{-1}(k_{\epsilon(s)}^{(i)}),$$

where S_{s_i} is the symmetric group.on s_i letters. By a detailed calculation, we have

$$\sum_\epsilon q^{\ell(\epsilon)} = \prod_{j\in\mathbb{Z}} \begin{bmatrix} \sum_{l\leqslant j} (S - S^\dagger)_{il} \\ s^\dagger_{i,j+1} \end{bmatrix}_{\mathfrak{a}} [s^\dagger_{i,j+1}]^!_{\mathfrak{a}}.$$

Also, we have

$$\sum_{\sigma\in K_w, k_\sigma = K} q^{-h(w,\sigma)} = \prod_{i=1}^{r+1} \sum_\epsilon q^{\ell(\epsilon)} q^{-h(w,\sigma^{\lhd K})},$$

where ϵ runs over all elements in S_{s_i} such that

$$(wg_2)^{-1}(j_s^{(i)}) > (wg_2)^{-1}(k_{\epsilon(s)}^{(i)}), \quad \text{for all} \quad s.$$

Therefore (4.4.15) holds. By the construction of $\sigma_{w,S}$ we have

$$\sum_K q^{-h(w,\sigma^{\lhd K})} = \begin{bmatrix} T \\ S \end{bmatrix}_{\mathfrak{a}} q^{-h(w,\sigma_{w,S})}. \tag{4.4.16}$$

The lemma folloes by combining (4.4.15) and (4.4.16). □

Example 4.4.5. Following Example 4.4.3, we pick the element $S = T \in \Gamma_T$. Thus

$$S_{\mathfrak{a}} = \begin{array}{|c|c|c|c|}\hline 0&0&1&1\\\hline 1&1&0&0\\\hline\end{array}, \quad S_{\mathfrak{a}}^{\dagger} = \begin{array}{|c|c|c|c|}\hline 1&1&0&0\\\hline 0&0&1&1\\\hline\end{array}, \quad (\sum_{k\leqslant j}(S-S^{\dagger})_{ik})_{ik} = \begin{array}{|c|c|c|c|}\hline 0&1&2&1\\\hline 1&2&1&0\\\hline\end{array}.$$

Therefore

$$\begin{bmatrix} T \\ S \end{bmatrix}_{\mathfrak{a}} = 1, \quad [\![S]\!] = \left[\begin{array}{c} \begin{array}{|c|c|c|c|}\hline &&&\\\hline 1&2&1&0\\\hline\end{array} \\ \\ \begin{array}{|c|c|c|c|}\hline &&&\\\hline 0&1&1&0\\\hline\end{array} \end{array}\right] = [2], \quad h(w,\sigma_{w,S}) = 1.$$

On the other hand, we have $\psi_w^{-1}(S) = \{(3,5)_{\mathfrak{c}}(4,6)_{\mathfrak{c}}, (3,6)_{\mathfrak{c}}(4,5)_{\mathfrak{c}}\}$, and hence

$$LHS = q^{-0} + q^{-1} = q^{-1}[2] = RHS.$$

For $T \in \Theta_{B,A}$ and $S \in \Gamma_T$ we set

$$n(S) = \sum_{i=1}^{r+1} \mathrm{ro}_{\mathfrak{a}}(S)_i, \tag{4.4.17}$$

and

$$\begin{aligned} h(T,S) = &\sum_{i=1}^{r+1}\sum_{j=-\infty}^{\infty} s_{ij}\Big(\sum_{k=-\infty}^{j} t_{ik} - \frac{s_{ij}+1}{2}\Big) \\ &+ \sum_{i=1}^{r+1}\sum_{j=-\infty}^{\infty} (t_{1-i,-j} - s_{1-i,-j})\Big(\sum_{k=-\infty}^{j-1} t_{ik} + \sum_{k=j}^{\infty} s_{ik} - \sum_{k=j+1}^{\infty} s_{1-i,-k}\Big). \end{aligned} \tag{4.4.18}$$

Lemma 4.4.6. *For $T \in \Theta_{B,A}$ and $S \in \Gamma_T$, we have $n(\sigma_{w,S}) = n(S)$ and $h(T,\sigma_{w,S}) = h(T,S)$.*

Proof. The first statement is obvious since $n(\sigma_{w,S})$ is the number of disjoint transpositions for $\sigma_{w,S}$. To compute $h(T,\sigma_{w,S})$, we count the elements in $H(T,\sigma_{w,S})$. There are $\sum_{i=1}^{r+1}\sum_{j=-\infty}^{\infty}(t_{1-i,-j}-s_{1-i,-j})(\sum_{k=-\infty}^{j-1}t_{ik}+\sum_{k=j}^{\infty}s_{ik}-\sum_{k=j+1}^{\infty}s_{1-i,-k})$ elements $(u,v)\in H(T,\sigma_{w,S})$ such that $\sigma_{w,S}(u)=u$ while there are $\sum_{i=1}^{r+1}\sum_{j=-\infty}^{\infty}s_{ij}(\sum_{k=-\infty}^{j}t_{ik}-\frac{s_{ij}+1}{2})$ elements $(u,v)\in H(T,\sigma_{w,S})$ such that u appears in the disjoint transpositions of $\sigma_{w,S}$. □

Finally, for $A,B\in\Xi_n, T\in\Theta_{B,A}$ and $S\in\Gamma_T$, we set

$$\ell(A,B,S,T)=\ell(A)+\ell(B)-\ell(A^{(T,S)})+\ell(w_{A,T}). \tag{4.4.19}$$

We are now in the position of proving the multiplication formula.

Theorem 4.4.7. *Let $A,B\in\Xi_{n,d}$ with B being tridiagonal and $\mathrm{ro}_{\mathfrak{c}}(A)=\mathrm{co}_{\mathfrak{c}}(B)$. Let $[\![S]\!], n(S), \ell(A,B,S,T), h(S,T), A^{(T,S)}$ be defined as in* (4.4.12), (4.4.17), (4.4.19), (4.4.18), (4.4.7) *respectively. We have*

$$e_B*e_A=\sum_{\substack{T\in\Theta_{B,A}\\ S\in\Gamma_T}}(q-1)^{n(S)}q^{\ell(A,B,S,T)-n(S)-h(S,T)}[A;S;T]\ e_{A^{(T,S)}}, \tag{4.4.20}$$

where

$$[A;S;T]_{\mathfrak{c}}=\frac{[A^{(T,S)}]^{!}_{\mathfrak{c}}}{[T-S]^{!}_{\mathfrak{a}}[S]^{!}_{\mathfrak{a}}[A-T_\theta]^{!}_{\mathfrak{c}}}\,[\![S]\!]\,, \tag{4.4.21}$$

Precisely, we have

$$\begin{aligned}[A;S;T]=&\prod_{(i,j)\in I_{\mathfrak{a}}^{+}}\begin{bmatrix}(A-T_\theta)+s_{ij}+s_{-i,-j}+(\widehat{T-S})_{ij}+(\widehat{T-S})_{-i,-j}\\ (A-T_\theta);s_{ij};s_{-i,-j};(\widehat{T-S})_{ij};(\widehat{T-S})_{-i,-j}\end{bmatrix}\\ &\cdot\prod_{k\in\{0,r+1\}}\left(\frac{\prod_{i=1}^{s_{kk}+(\widehat{T-S})_{kk}}[a_{kk}-2t_{kk}-1+2i]}{[s_{kk}]^{!}_{\mathfrak{a}}[(\widehat{T-S})_{kk}]^{!}_{\mathfrak{a}}}\right)\cdot[\![S]\!]\,.\end{aligned} \tag{4.4.22}$$

Proof.

$$
\begin{aligned}
& e_B * e_A \\
&= \sum_{\substack{w\in\mathscr{D}_\delta\cap W_\mu\\ \sigma\in K_w}} (q-1)^{n(\sigma)} q^{h(w,\sigma)+\ell(g_1\sigma w g_2)-\ell(y^{(w,\sigma)})} \frac{[A^{(w,\sigma)}]^!_{\mathfrak{c}}}{[A]^!_{\mathfrak{c}}} e_{A^{(w,\sigma)}} && \text{by (4.1.2), (4.3.15)} \\
&= \sum_{\substack{w\in\mathscr{D}_\delta\cap W_\mu\\ \sigma\in K_w}} (q-1)^{n(\sigma)} q^{\ell(g_1)+\ell(w)+\ell(g_2)-\ell(y^{(w,\sigma)})-n(\sigma)-h(w,\sigma)} \frac{[A^{(w,\sigma)}]^!_{\mathfrak{c}}}{[A]^!_{\mathfrak{c}}} e_{A^{(w,\sigma)}} && \text{by (4.3.16)} \\
&= \sum_{\substack{T\in\Theta_{B,A}\\ S\in\Gamma_T}} (q-1)^{n(S)} q^{\ell(A,B,S,T)-n(S)-h(S,T)} [A;S;T] e_{A^{(T,S)}}. && \text{by (4.2.8), (4.4.14)}
\end{aligned}
$$

Finally, $A^{(T,S)} = A - (T-S)_\theta + (\widehat{T-S})_\theta$ because of Lemma 4.4.2. □

4.5 Multiplication formulas with quasi-bidiagonal generators

For any matrix $T \in \Theta_n$, recall $\mathrm{diag}(T)$ and $T^\pm$ from (2.3.1). In this sectoin we discuss the special case when $B^\pm = \sum_{i=0}^{r} b_{i,i+1} E_\theta^{i,i+1}$ or $B^\pm = \sum_{i=0}^{r} b_{i+1,i} E_\theta^{i+1,i}$. Note that $g_1 = \mathbb{1}$

here. Namely,

$$
B = \left[\begin{array}{ccc|c|ccc}
\ddots & & & & & & \\
\ddots & * & & & & & \\
& b_{12} & * & & & & \\
\hline
& & b_{01} & * & b_{01} & & \\
\hline
& & & & * & b_{12} & \\
& & & & & * & \ddots \\
& & & & & & \ddots
\end{array}\right]
\quad \text{or} \quad
\left[\begin{array}{ccc|c|ccc}
\ddots & \ddots & & & & & \\
& * & b_{21} & & & & \\
& & * & b_{10} & & & \\
\hline
& & & * & & & \\
\hline
& & & b_{10} & * & & \\
& & & & b_{21} & * & \\
& & & & & \ddots & \ddots
\end{array}\right].
$$

Below is a special case of multiplication formula (see Theorem 4.4.7), which is analogous to the multiplication formulas in affine type A.

Theorem 4.5.1. *If* $B^{\pm} = \sum_{i=0}^{r} b_{i,i+1} E_{\theta}^{i,i+1}$ *or* $\sum_{i=0}^{r} b_{i+1,i} E_{\theta}^{i+1,i}$ *and* $\mathrm{ro}_{\mathfrak{c}}(A) = \mathrm{co}_{\mathfrak{c}}(B)$*. Then*

$$e_B * e_A = \sum_{T \in \Theta_{B,A}} q^{\ell(w_{A,T}) + \ell(A) - \ell(A^{(T,0)})} \frac{[A^{(T,0)}]_{\mathfrak{c}}!}{[A - T^{\theta}]_{\mathfrak{c}}! [T]_{\mathfrak{a}}!} e_{A^{(T,0)}}.$$

Proof. This is due to (4.4.20), where S is always the zero matrix therein. □

Let ϵ_{ij}^{θ} be the (i,j)-th entry of $E_{\theta}^{i,j}$. That is,

$$\epsilon_{ij}^{\theta} = \begin{cases} 2 & \text{if } (i,j) \in \mathbb{Z}(r+1, r+1); \\ 1 & \text{otherwise.} \end{cases} \tag{4.5.1}$$

Below is a another special case with Chevalley generators, whose coefficients are compatible with the multiplication formulas for finite type B/C.

Corollary 4.5.2. *Let* $0 \leqslant h \leqslant r$.

(a) If $B^{\pm} = E_{\theta}^{h,h+1}$ *and* $\mathrm{ro}_{\mathfrak{c}}(A) = \mathrm{co}_{\mathfrak{c}}(B)$, *then*

$$e_B * e_A = \sum_{\substack{p\in\mathbb{Z} \\ a_{h+1,p}\geqslant\epsilon^{\theta}_{h+1,p}}} q^{\sum\limits_{j>p} a_{hj}}[a_{hp}+1]e_{A+E_{\theta}^{hp}-E_{\theta}^{h+1,p}}.$$

(b) If $B^{\pm} = E_{\theta}^{h+1,h}$ *and* $\mathrm{ro}_{\mathfrak{c}}(A) = \mathrm{co}_{\mathfrak{c}}(B)$, *then*

$$e_B * e_A = \sum_{\substack{p\in\mathbb{Z} \\ a_{hp}\geqslant\epsilon^{\theta}_{hp}}} q^{\sum\limits_{j<p} a_{h+1,j}}[a_{h+1,p}+1]e_{A-E_{\theta}^{hp}+E_{\theta}^{h+1,p}}.$$

Chapter 5

Monomial bases for affine Schur algebras of type C

In this chapter, we first define a partial order on the index set of affine q-Schur algebra that refines the Bruhat order. Then we show that applying the bar involution on any standard basis element $[A]$ leads to itself plus a combination of lower terms with respect to this partial order. We provide an elementary construction (Algorithm 5.2.7) of a semi-monomial basis using the multiplication formula on admissible pairs (cf. Section 5.2). We then obtain a monomial basis (Proposition 5.2.11) and a canonical basis.

5.1 Bar involutions and standard bases

By slightly abuse of notation, let $\leqslant$ be the (strong) Bruhat order on W. Following [KL79], denote by $\{C'_w\}$ the Kazhdar-Lusztig basis of the Hecke algebra $\mathbf{H}$ characterized by the following condition.

$$C'_w = v^{-\ell(w)} \sum_{y \leqslant w} P_{yw} T_y \quad (w \in W),$$

where $P_{yw} \in \mathbb{Z}[v^2]$ is the Kazhdar-Lusztig polynomial.

For $\lambda, \mu \in \Lambda$, set $g^+_{\lambda\mu}$ to be the longest element in $W_\lambda g W_\mu$ for $g \in \mathscr{D}_{\lambda\mu}$, and set $w^\mu_\circ = \mathbb{1}^+_{\mu\mu}$ to be the longest element in the (finite) parabolic subgroup $W_\mu = W_\mu \mathbb{1} W_\mu$.

Lemma 5.1.1. *Let $\lambda, \mu \in \Lambda$, $g \in \mathscr{D}_{\lambda\mu}$, and let $\delta = \delta(\lambda, g, \mu)$ (see Proposition 3.1.4). Then:*

(a) $g^+_{\lambda\mu} = w^\lambda_\circ g w^\delta_\circ w^\mu_\circ$. *In particular,*

$$\ell(g^+_{\lambda\mu}) = \ell(w^\lambda_\circ) + \ell(g) - \ell(w^\delta_\circ) + \ell(w^\mu_\circ).$$

(b) $W_\lambda g W_\mu = \{w \in W \mid g \leqslant w \leqslant g^+_{\lambda\mu}\}$.

(c) *There exists* $c^{(\lambda,\mu)}_{w,g} \in \mathbb{Z}[v, v^{-1}]$ *such that*

$$T_{W_\lambda g W_\mu} = v^{\ell(g^+_{\lambda\mu})} C'_{g^+_{\lambda\mu}} + \sum_{\substack{w \in \mathscr{D}_{\lambda\mu} \\ w < g}} c^{(\lambda,\mu)}_{w,g} C'_{w^+_{\lambda\mu}}.$$

In particular, $x_\mu = v^{\ell(w^\mu_\circ)} C'_{w^\mu_\circ}$.

Proof. See [Cur85, Theorem 1.2 (ii), (1.11)] and [DDPW08, Corollary 4.19]. □

Denote the bar involution on $\mathbf{H}$ by $^- : \mathbf{H} \to \mathbf{H}$, $v \mapsto v^{-1}, T_w \mapsto T^{-1}_{w^{-1}}$. By [KL79, Theorem 1.1], C'_w is bar-invariant for $w \in W$. Following [Du92, Proposition 3.2], we define the bar involution on $\mathbf{S}^{\mathfrak{c}}_{n,d}$ as follows: for each $f \in \mathrm{Hom}_{\mathbf{H}}(x_\mu \mathbf{H}, x_\lambda \mathbf{H})$, let $\overline{f} \in \mathrm{Hom}_{\mathbf{H}}(x_\mu \mathbf{H}, x_\lambda \mathbf{H})$ be the map sending $C'_{w^\mu_\circ}$ to $\overline{f(C'_{w^\mu_\circ})}$. Equivalently,

$$\overline{f}(x_\mu H) = v^{2\ell(w^\mu_\circ)} \overline{f(x_\mu)} H \quad \text{for} \quad H \in \mathbf{H}.$$

In particular, for $A = \kappa(\lambda, g, \mu) \in \Xi_n$, by Lemma 5.1.1 we have

$$e_A(C'_{w_\circ^\mu}) = v^{\ell(g^+_{\lambda\mu}) - \ell(w_\circ^\mu)} C'_{g^+_{\lambda\mu}} + \sum_{\substack{w \in \mathscr{D}_{\lambda\mu} \\ w < g}} v^{-\ell(w_\circ^\mu)} c^{(\lambda,\mu)}_{w,g} C'_{w^+_{\lambda\mu}}, \tag{5.1.1}$$

$$\overline{e_A}(C'_{w_\circ^\mu}) = v^{\ell(w_\circ^\mu) - \ell(g^+_{\lambda\mu})} C'_{g^+_{\lambda\mu}} + \sum_{\substack{w \in \mathscr{D}_{\lambda\mu} \\ w < g}} v^{\ell(w_\circ^\mu)} \overline{c^{(\lambda,\mu)}_{w,g}} C'_{w^+_{\lambda\mu}}. \tag{5.1.2}$$

For $A \in \Xi_n$, we define a number

$$d_A = \frac{1}{2}\Big(\sum_{(i,j) \in I^+} \Big(\sum_{x \leqslant i, y > j} + \sum_{x \geqslant i, y < j} \Big) a'_{ij} a_{xy} \Big). \tag{5.1.3}$$

It can be checked that $d_A \in \mathbb{Z}$. Set

$$[A] = v^{-d_A} e_A.$$

Then $\{[A] \mid A \in \Xi_{n,d}\}$ is a $\mathbb{Z}[v, v^{-1}]$-basis of $\mathbf{S}^{\mathfrak{c}}_{n,d}$, which we call the *standard basis*.

Proposition 5.1.2. *Assume that $A = \kappa(\lambda, g, \mu) \in \Xi_n$. There exists $\gamma^{(\lambda,\mu)}_{w,g} \in \mathbb{Z}[v, v^{-1}]$ for each $w \in \mathscr{D}_{\lambda\mu}$ such that*

$$\overline{[A]} = [A] + \sum_{\substack{w \in \mathscr{D}_{\lambda\mu} \\ w < g}} \gamma^{(\lambda,\mu)}_{w,g} [\kappa(\lambda, w, \mu)].$$

Proof. Set $\delta = \delta(A)$, by Lemma 5.1.1(b) we have

$$\ell(g^+_{\lambda\mu}) - \ell(w_\circ^\mu) = \ell(g) + \ell(w_\circ^\lambda) - \ell(w_\circ^\delta).$$

Here

$$\begin{aligned}
\ell(w_\circ^\lambda) - \ell(w_\circ^\delta) &= \lambda_0^2 + \sum_{i=1}^{r} \binom{\lambda_i}{2} + \lambda_{r+1}^2 - \Big((\delta_0)^2 + \sum_{i=1}^{r'} \binom{\delta_i}{2} + (\delta_{r'+1})^2 \Big) \\
&= 2 \sum_{0 \leqslant j < x} a'_{0j} a_{0x} + \sum_{j>0} \binom{a_{0j}+1}{2} + \sum_{\substack{j<y \\ 1 \leqslant i \leqslant r}} a_{ij} a_{iy} \\
&+ 2 \sum_{j<x \leqslant r+1} a_{r+1,j} a'_{r+1,x} + \sum_{j<r+1} \binom{a_{r+1,j}+1}{2} \\
&= \frac{1}{2} \Big(\sum_{(i,j) \in I^+} \big(\sum_{\substack{x=i \\ y>j}} + \sum_{\substack{x=i \\ y<j}} \big) a'_{ij} a_{xy} \Big) \\
&= d_A - \ell(A).
\end{aligned}$$

Hence

$$d_A = \ell(g^+_{\lambda\mu}) - \ell(w_\circ^\mu). \tag{5.1.4}$$

Therefore, Equations (5.1.1) and (5.1.2) can be rewritten as

$$[A](C'_{w_\circ^\mu}) = C'_{g^+_{\lambda\mu}} + \sum_{\substack{w \in \mathscr{D}_{\lambda\mu} \\ w<g}} v^{-\ell(g^+_{\lambda\mu})} c^{(\lambda,\mu)}_{w,g} C'_{w^+_{\lambda\mu}}, \tag{5.1.5}$$

$$\overline{[A]}(C'_{w_\circ^\mu}) = C'_{g^+_{\lambda\mu}} + \sum_{\substack{w \in \mathscr{D}_{\lambda\mu} \\ w<g}} v^{\ell(g^+_{\lambda\mu})} \overline{c^{(\lambda,\mu)}_{w,g}} C'_{w^+_{\lambda\mu}}. \tag{5.1.6}$$

If $\ell(g) = 0$, then $\overline{[A]} = [A]$ and we are done. For arbitrary g, the proposition follows from an easy induction on $\ell(g)$. □

Now we define a partial order $\leqslant_{\mathrm{alg}}$ on Ξ_n by $A \leqslant_{\mathrm{alg}} B$ if and only if $\mathrm{ro}_{\mathfrak{c}}(A) = \mathrm{ro}_{\mathfrak{c}}(B)$, $\mathrm{co}_{\mathfrak{c}}(A) = \mathrm{co}_{\mathfrak{c}}(B)$ and $\sigma_{i,j}(A) \leqslant \sigma_{i,j}(B)$ for all $i < j$, where

$$\sigma_{i,j}(A) = \sum_{x \leqslant i, y \geqslant j} a_{xy}. \tag{5.1.7}$$

Here the subscript "alg" stands for algebraic. In the following the expression "lower terms" represents a linear combination of smaller elements with respect to $\leqslant_{\mathrm{alg}}$.

Lemma 5.1.3. *Assume that* $A = \kappa(\lambda, g, \mu)$ *and* $B = \kappa(\lambda, h, \mu)$. *If* $h \leqslant g$ *then* $B \leqslant_{\mathrm{alg}} A$.

Proof. By [BB05, Theorem 8.4.8], the condition $h \leqslant g$ is equivalent to that $h[s,t] \leqslant g[s,t]$ for all $s, t \in \mathbb{Z}$, where $g[s,t] = |\{(g(a), a) \in \mathbb{Z}_{\geqslant t} \times \mathbb{Z}_{\leqslant s}\}|$. The bijections $R_x^\lambda \cap gR_y^\mu \leftrightarrow \{(g(a), a) \in R_x^\lambda \times R_y^\mu\}$ for $x, y \in \mathbb{Z}$ give that, for $i < j$,

$$\sigma_{ij}(A) = \sum_{\substack{x \leqslant i \\ y \geqslant j}} a_{xy} = \sum_{\substack{x \geqslant -i \\ y \leqslant -j}} |R_x^\lambda \cap gR_y^\mu| = g[s,t],$$

where s is the largest element in R_{-i}^λ and t is the smallest element in R_{-j}^μ. Therefore, $\sigma_{ij}(B) = h[s,t] \leqslant g[s,t] = \sigma_{ij}(A)$. □

Corollary 5.1.4. *If* $A \in \Xi_n$, *then* $\overline{[A]} = [A] + \text{lower terms}$.

Proof. It follows by combining Proposition 5.1.2 and Lemma 5.1.3. □

In order to construct a limit algebra via the BLM stabilization procedure, one needs to show that the coefficients in the multiplication formula "behave well". It is standard to put the multiplication formula in the standard basis (see [BLM90, Lemma 3.4(a2)]). In our case, Theorem 4.4.7 can be written as

$$[B] * [A] = \sum_{\substack{T \in \Theta_{B,A} \\ S \in \Gamma_T}} v^{\beta(A,S,T)} (v^2 - 1)^{n(S)} \overline{[A; S; T]}_{\mathfrak{c}} \, [A^{(T,S)}], \tag{5.1.8}$$

for some $\beta(A,S,T)$. What we need is an explicit formula for $\beta(A,S,T)$, which can be derived as follows. Let $\gamma(A,S,T)$ to be the integer such that

$$\overline{[A;S;T]} = v^{\gamma(A,S,T)}[A;S;T]. \tag{5.1.9}$$

Lemma 5.1.5. *Let $A \in \Xi_n, T \in \Theta_{B,A}$ for some tridiagonal matrix B, and $S \in \Gamma_T$. Then*

$$\begin{aligned}\gamma(A,S,T) = &- \sum_{(i,j)\in I_{\mathfrak{a}}^+} (s_{\theta,ij} + (\widehat{T-S})_{\theta,ij})(s_{\theta,ij} + (\widehat{T-S})_{\theta,ij} + 2a_{ij} - 2t_{\theta,ij} - 1)\\ &- \sum_{k\in\{0,r+1\}} (a_{kk} - 1 - (T-S)_{\theta,kk} + (\widehat{T-S})_{\theta,kk})(s_{\theta,kk} + (\widehat{T-S})_{\theta,kk})\\ &+ 2\sum_{i=1}^{n}\sum_{j\in\mathbb{Z}}\left(\binom{(T-S)}{2} + \binom{s_{ij}}{2}\right)\\ &+ 2\sum_{i=1}^{r+1}\sum_{j\in\mathbb{Z}} s^\dagger_{i,j+1}(s^\dagger_{i,j+1} - \sum_{k\leqslant j}(S-S^\dagger)_{ik}) - \binom{s^\dagger_{i,j+1}}{2}.\end{aligned} \tag{5.1.10}$$

In particular, by setting $d'(A) = 2\ell(A) - d_A$, we have

$$\beta(A,S,T) = d'_B + d'_A - d'_{A^{(T,S)}} + \ell(w_{A,T}) + \gamma(A,S,T). \tag{5.1.11}$$

Proof. By direct computation, we have

$$\begin{aligned}\overline{[A]^!_{\mathfrak{a}}} &= q^{-\sum_{i=1}^{n}\sum_{j\in\mathbb{Z}}\binom{a_{ij}}{2}}[A]^!_{\mathfrak{a}},\\ \overline{[A]^!_{\mathfrak{c}}} &= q^{-(a'_{00})^2-(a'_{r+1,r+1})^2-\sum_{(i,j)\in I_{\mathfrak{a}}^+}\binom{a_{ij}}{2}}[A]^!_{\mathfrak{c}},\\ \overline{[\![S]\!]} &= q^{\sum_{i=1}^{r+1}\sum_{j\in\mathbb{Z}} s^\dagger_{i,j+1}(s^\dagger_{i,j+1}-\sum_{k\leqslant j}(S-S^\dagger)_{ik})-\binom{s^\dagger_{i,j+1}}{2}}[\![S]\!].\end{aligned}$$

The lemma follows from putting them together. ∎

Summarizing (5.1.8) and (5.1.11), we obtain the multiplication formula with tridiagonal generators for standard bases of $\mathbf{S}^{\mathfrak{c}}_{n,d}$.

Theorem 5.1.6. *Retain the assumptions as in Theorem 4.4.7. We have*

$$[B] * [A] = \sum_{\substack{T \in \Theta_{B,A} \\ S \in \Gamma_T}} v^{\beta(A,S,T)} (v^2 - 1)^{n(S)} \overline{[A; S; T]}_{\mathfrak{c}} \, [A^{(T,S)}].$$

5.2 Constructing monomial bases

A pair (B, A) of matrices in Ξ_n^2 is called *admissible* if the following conditions hold (see (2.3.1) for the notation $\pm$):

1. $B^{\pm} = \sum\limits_{i=1}^{n} b_{i,i+1} E_{\theta}^{i,i+1}$;
2. $A^{\pm} = \sum\limits_{j=1}^{k} \sum\limits_{i=1}^{n} a_{i,i+j} E_{\theta}^{i,i+j}$ for some $k \in \mathbb{N}$, where $a_{i,i+k} \geqslant b_{i,i+1}$ for all i.

Algorithm 5.2.1. Assume that $A, B \in \Xi_{n,d}$, $\mathrm{ro}(A) = \mathrm{co}(B)$ and B is tridiagonal. We produce a matrix $M = M(B, A) \in \Xi_{n,d}$ as follows.

(1) For each row i, find the unique j such that $b_{i,i+1} \in \big(\sum\limits_{y>j} a_{iy} .. \sum\limits_{y \geqslant j} a_{iy} \big]$.

(2) Construct a matrix $T^+ \in \Theta_n$ by

$$T^+ = \sum_{i=1}^{n} \Big((b_{i,i+1} - \sum_{y>j} a_{iy}) E_{\theta}^{ij} + \sum_{y>j} a_{iy} E_{\theta}^{iy} \Big).$$

(3) Set $M = A^{(T^+)}$.

Lemma 5.2.2. *The highest term (with respect to* $\leqslant_{\mathrm{alg}}$*) in* (5.1.8) *exists and its corresponding matrix is the matrix* M *described in Algorithm 5.2.1.*

Proof. Note that

$$\sigma_{ij}(A^{(E^{xy},0)}) = \begin{cases} \sigma_{ij}(A)+1 & \text{if } j < i = x-1, j \leqslant y, \\ \sigma_{ij}(A)-1 & \text{if } j > i = x, j \geqslant y, \\ \sigma_{ij}(A) & \text{otherwise.} \end{cases} \tag{5.2.1}$$

It follows that $A^{(E^{ij},0)} <_{\text{alg}} A^{(E^{i,j+1},0)}$ for all $i, j \in \mathbb{Z}$. Therefore, for any $T \in \Theta_{B,A}, S \in \Gamma_T$ we have $A^{(T-S)} \leqslant_{\text{alg}} A^{(T)} \leqslant_{\text{alg}} A^{(T^+)} = M$. □

The corollary below is a direct consequence of (5.2.1)

Corollary 5.2.3. *If $A' <_{\text{alg}} A$, then $M(B, A') <_{\text{alg}} M(B, A)$.*

Lemma 5.2.4. *If (B, A) is admissible, then $[A; 0; T^+]_{\mathfrak{c}} = 1$.*

Proof. Now $T^+ = \sum_{i=1}^{n} b_{i,i+1} E_\theta^{i,i+k}$, and hence

$$[A; 0; T^+]_{\mathfrak{c}} = \frac{1}{[T^+]^!_{\mathfrak{a}}} \frac{[A + \widehat{T}^+_\theta - T^+_\theta]^!_{\mathfrak{c}}}{[A - T^+_\theta]^!_{\mathfrak{c}}} = \frac{1}{\prod_{i=1}^{n} [m_i]} \cdot \prod_{i=1}^{n} [m_i] = 1.$$

□

Following Lemma 5.2.4, (5.1.8) can be rewritten as

$$[B] * [A] = v^{\beta(A,0,T^+)}[M] + \text{lower terms.} \tag{5.2.2}$$

One can show that $\beta(A, 0, T^+) = 0$ by a direct but lengthy computation. Here we present a more elegant proof via the bar involution.

Lemma 5.2.5. *If B is tridiagonal, and $B' <_{\text{alg}} B$, then B' is also tridiagonal. Moreover, $(B')^\pm < B^\pm$.*

Proof. Since $B' \leqslant_{\text{alg}} B$, we have

$$\sigma_{i,i+2}(B') \leqslant \sigma_{i,i+2}(B) = 0 \quad \text{for} \quad i = 1, \ldots, n.$$

Therefore $\sigma_{i,i+2}(B') = 0$ for all i and hence B' is tridiagonal. Also, we have

$$\sigma_{i,i+1}(B') = b'_{i,i+1} \leqslant \sigma_{i,i+1}(B) = b_{i,i+1} \quad \text{for} \quad i = 1, \ldots, n.$$

Since $B' \neq B$, hence $(B')^{\pm} < B^{\pm}$. □

Lemma 5.2.6. *If (B, A) is admissible, then $\beta(A, 0, T^+) = 0$. In other words,*

$$[B] * [A] = [M(B, A)] + \text{ lower terms.}$$

Proof. Write $M = M(B, A)$. By taking bar on (5.2.2), we get

$$\overline{[B]} * \overline{[A]} = v^{-\beta(A,0,T^+)}\overline{[M]} + \text{lower terms.}$$

By Proposition 5.1.2, we have

$$\Big([B] + \sum_{B' <_{\text{alg}} B} \gamma_{B,B'}[B']\Big) * \Big([A] + \sum_{A' <_{\text{alg}} A} \gamma_{A,A'}[A']\Big) = v^{-\beta(A,0,T^+)}[M] + \text{lower terms.}$$

For any $B' <_{\text{alg}} B$, by Lemma 5.2.5 we know that $(B')^{\pm} < B^{\pm}$, and hence $M(B', A) <_{\text{alg}} M$, by construction. Also, by Corollary 5.2.3 we have

$$M(B', A') <_{\text{alg}} M(B', A) <_{\text{alg}} M.$$

Therefore,

$$\begin{aligned}\Big([B] + \sum_{B' <_{\text{alg}} B} \gamma_{B,B'}[B']\Big) * \Big([A] + \sum_{A' <_{\text{alg}} A} \gamma_{A,A'}[A']\Big) &= [B] * [A] + \text{lower terms} \\ &= v^{\beta(A,0,T^+)}[M] + \text{lower terms.}\end{aligned}$$

By comparing the leading coefficient, we have $\beta(A, 0, T^+) = 0$. □

Below we provide an algorithm that constructs a monomial basis in a diagonal-by-diagonal manner involving only admissible pairs. See [LL15] for a similar algorithm.

Algorithm 5.2.7. For each $A = (a_{ij}) \in \Xi_{n,d}$, we construct tridiagonal matrices $B^{(1)}, \ldots, B^{(x)}$ as follows:

1. Initialization: set $t = 0$, and set $A^{(0)} = A$.

2. If $A^{(t)}$ is a tridiagonal matrix, then end the algorithm. Otherwise, denote the outermost nonzero diagonal of the matrix $A^{(t)} = (a_{ij}^{(t)})$ by $(T^+)^{(t)} = \sum\limits_{i=1}^{n} a_{i,i+k}^{(t)} E_\theta^{i,i+k}$ for some $k > 0$.

3. Define matrices
$$B^{(t+1)} = \sum_{i=1}^{n} a_{i,i+k}^{(t)} E_\theta^{i,i+1} + \text{a diagonal determined by } \ (5.2.3),$$
$$A^{(t+1)} = A^{(t)} - (T^+)^{(t)} + (\widehat{T}^+)^{(t)}.$$

4. Increase t by one and then go to Step (2).

Here the diagonal entries are uniquely determined by

$$\mathrm{co}_{\mathfrak{c}}(B^{(t-1)}) = \mathrm{ro}_{\mathfrak{c}}(B^{(t)}), \quad t = 1, \ldots, x-1. \tag{5.2.3}$$

Theorem 5.2.8. *For each $A \in \Xi_{n,d}$, the matrices $B^{(1)}, \ldots, B^{(x)} \in \Xi_{n,d}$ in Algorithm 5.2.7 satisfy that*

$$[B^{(1)}] * [B^{(2)}] * \cdots * [B^{(x)}] = [A] + \text{lower terms}.$$

Proof. Algorithm 5.2.7 guarantees that each pair $(B^{(j)}, A^{(j)})$ is admissible and $M(B^{(j)}, A^{(j)}) = A^{(j-1)}$ for $j = 1, \ldots, x-1$. Hence by Corollary 5.2.3 and by Lemma 5.2.6, we have

$$\begin{aligned}
&[B^{(1)}] * [B^{(2)}] * \cdots * \big([B^{(x-1)}] * [B^{(x)}]\big) \\
&= [B^{(1)}] * [B^{(2)}] * \cdots * \big([B^{(x-1)}] * [A^{(x-1)}]\big) \\
&= [B^{(1)}] * [B^{(2)}] * \cdots * [B^{(x-2)}] * \big([A^{(x-2)}] + \text{lower terms}\big) \\
&= [A] + \text{lower terms.}
\end{aligned}$$

□

Example 5.2.9. In the followings we give an example to show how the algorithm works. Note that by Lemma 5.2.6 the leading coefficient in each intermediate step is indeed one. Let $r = 2$, $n = 6$, and

$$A = \left[\begin{array}{c|cc|c|cc|c}
 & 3 & & & & & \\ \hline
3 & 2 & 2 & & & & \\ \hline
0 & 0 & 5 & 4 & & & \\
3 & 4 & 1 & 1 & 3 & & \\ \hline
 & 3 & 2 & 1 & 2 & 3 & \\ \hline
 & & 3 & 1 & 1 & 4 & 3 \\
 & & & 4 & 5 & 0 & 0 \\ \hline
 & & & & 2 & 2 & 3 \\ \hline
 & & & & & 3 &
\end{array}\right].$$

We have

$$[A] = [B^{(1)}] * [A^{(1)}] + \text{lower terms, where}$$

$$B^{(1)} = \begin{bmatrix} 3 & & & & & & \\ * & 2 & & & & & \\ 3 & * & 4 & & & & \\ & 3 & * & 3 & & & \\ & & 3 & * & 3 & & \\ & & & 3 & * & 3 & \\ & & & & 4 & * & 3 \\ & & & & & 2 & * \\ & & & & & & 3 \end{bmatrix}, A^{(1)} = \begin{bmatrix} & 0 & & & & & \\ 3 & 5^{\uparrow 3} & 0 & & & & \\ 3 & 0 & 7^{\uparrow 2} & 0 & & & \\ 0^{\downarrow 3} & 7 & 1 & 5^{\uparrow 4} & 0 & & \\ & 0^{\downarrow 3} & 5 & 1 & 5^{\uparrow 3} & 0 & \\ & & 0^{\downarrow 3} & 5 & 1 & 7^{\uparrow 3} & 0 \\ & & & 0^{\downarrow 4} & 7 & 0 & 3^{\uparrow 3} \\ & & & & 0^{\downarrow 2} & 5 & 3 \\ & & & & & 0^{\downarrow 3} & \end{bmatrix}.$$

$$[A^{(1)}] = [B^{(2)}] * [A^{(2)}] + \text{lower terms, where}$$

$$B^{(2)} = \begin{bmatrix} 3 & & & & & & \\ * & 5 & & & & & \\ 3 & * & 7 & & & & \\ & 7 & * & 5 & & & \\ & & 5 & * & 5 & & \\ & & & 5 & * & 7 & \\ & & & & 7 & * & 3 \\ & & & & & 5 & * \\ & & & & & & 3 \end{bmatrix}, A^{(2)} = \begin{bmatrix} & & & & & & \\ 9^{\uparrow 3} & 0 & & & & & \\ 0^{\downarrow 3} & 12^{\uparrow 5} & 0 & & & & \\ & 0^{\downarrow 7} & 13^{\uparrow 7} & 0 & & & \\ & & 0^{\downarrow 5} & 11^{\uparrow 5} & 0 & & \\ & & & 0^{\downarrow 5} & 13^{\uparrow 5} & 0 & \\ & & & & 0^{\downarrow 7} & 12^{\uparrow 7} & 0 \\ & & & & & 0^{\downarrow 5} & 9^{\uparrow 3} \\ & & & & & & \end{bmatrix}.$$

Finally, we have

$$[A] = [B^{(1)}] * [B^{(2)}] + \text{ lower terms.}$$

For $A \in \Xi_n$, we define

$$m'_A = [B^{(1)}] * [B^{(2)}] * \cdots * [B^{(x)}]. \tag{5.2.4}$$

It is clear that $\{m'_A \mid A \in \Xi_{n,d}\}$ is also a $\mathbb{Z}[v, v^{-1}]$-basis of $\mathbf{S}^{\mathfrak{c}}_{n,d}$, which we call a semi-monomial basis.

Remark 5.2.10. In general, the element m'_A is not bar-invariant since for an arbitrary tridiagonal matrix B, $[B]$ is not necessarily bar-invariant. For example, take $n = 2r + 2 = 4$. Let

$$B = E^{00} + 2E^{01}_{\theta} + E^{10}_{\theta} + E^{22} = \kappa((2, 1, 0), s_2 s_1, (1, 2, 0)).$$

The matrix B is not minimal with respect to the Bruhat order nor the algebraic partial order $\leqslant_{\text{alg}}$, since we have

$$B' = \kappa((2, 1, 0), \mathbb{1}, (1, 2, 0)) = 3E^{00} + E^{01}_{\theta} + E^{11}_{\theta} + E^{22},$$

with $\mathbb{1} < s_2 s_1$.

By a standard argument one construct a monomial basis $\{m_A \mid A \in \Xi_{n,d}\}$ via the semi-monomial basis $\{m'_A \mid A \in \Xi_{n,d}\}$.

Proposition 5.2.11. *There exists a $\mathbb{Z}[v, v^{-1}]$-basis $\{m_A \mid A \in \Xi_{n,d}\}$ of $\mathbf{S}^{\mathfrak{c}}_{n,d}$ satisfying the following properties.*

1. *m_A is bar-invariant;*

2. *$m_A = [A]$+lower terms.*

5.3 A comparison of canonical bases

By a standard argument [Lu93, 24.2], one can construct a canonical basis $\{\{A\} \mid A \in \Xi_{n,d}\}$ from the monomial basis $\{m_A \mid A \in \Xi_{n,d}\}$ as below.

Corollary 5.3.1. *There exists a unique $\mathbb{Z}[v, v^{-1}]$-basis $\{\{A\} \mid A \in \Xi_{n,d}\}$ of $\mathbf{S}^{\mathfrak{c}}_{n,d}$ satisfying the following properties:*

1. *$\{A\}$ is bar-invariant;*

2. *$\{A\} = [A] + \sum_{B <_{\rm alg} A} \pi_{B,A}[B]$ for $\pi_{B,A} \in v^{-1}\mathbb{Z}[v^{-1}]$.*

By a similar construction to [Du92], we can define another canonical basis $\{\{A\}' \mid A \in \Xi_{n,d}\}$ as follows. For $A = \kappa(\lambda, g, \mu)$ for some $\lambda, \mu \in \Lambda, g \in \mathscr{D}_{\lambda\mu}$, let $d^{(\lambda,\mu)}_{w,g}$ be the coefficients such that

$$C'_{g^+_{\lambda\mu}} = \sum_{w \in \mathscr{D}_{\lambda\mu}} d^{(\lambda,\mu)}_{w,g} T_{W_\lambda w W_\mu}. \tag{5.3.1}$$

In other words, $(d^{(\lambda,\mu)}_{w,g})_{w,g}$ is the inverse matrix of $(c^{(\lambda,\mu)}_{w,g})_{w,g}$. Define

$$\{A\}' = v^{\ell(w^\mu_\circ)} \sum_{w \in \mathscr{D}_{\lambda\mu}} d^{(\lambda,\mu)}_{w,g} e_{\kappa(\lambda,w,\mu)}. \tag{5.3.2}$$

The set $\{\{A\}' \mid A \in \Xi_{n,d}\}$ is a basis by construction. Precisely, $\{A\}' \in \mathrm{Hom}_{\mathbf{H}}(x_\mu, x_\lambda) \subset \mathbf{S}^{\mathfrak{c}}_{n,d}$ is the map sending x_μ to

$$v^{\ell(w^\mu_\circ)} \sum_{w \in \mathscr{D}_{\lambda\mu}} d^{(\lambda,\mu)}_{w,g} T_{W_\lambda w W_\mu} = v^{\ell(w^\mu_\circ)} C'_{w^+_{\lambda\mu}}. \tag{5.3.3}$$

Equivalently, we have

$$\{A\}'(C'_{w^\mu_\circ}) = \{A\}'(v^{-\ell(w^\mu_\circ)} x_\mu) = C'_{w^+_{\lambda\mu}}. \tag{5.3.4}$$

Hence, $\{A\}'$ is bar-invariant. By a detailed calculation we can show that the $\{\{A\}' \mid A \in \Xi_{n,d}\}$ satisfies the second property in Corollary 5.3.1, and hence the two canonical bases coincide by the uniqueness.

On the other hand, there is a canonical basis $\{\{A\}^{\text{geo}} \mid A \in \Xi_{n,d}\}$ for $\mathbf{S}^{\mathfrak{c},\text{geo}}_{n,d}$ (cf. [FLLLW1, (4.2.12)]) arising from intersection cohomology. Using the identification of Schur algebras in Section 3.3, we can also show that all canonical bases coincide. We summarize the above as a proposition.

Proposition 5.3.2. *The three canonical bases* $\{\{A\} \mid A \in \Xi_{n,d}\}, \{\{A\}' \mid A \in \Xi_{n,d}\}$ *and* $\{\{A\}^{\text{geo}} \mid A \in \Xi_{n,d}\}$ *match under the identification in Lemma 3.3.1.*

Chapter 6

Stabilization algebras of affine type C

In this chapter we construct a stabilization algebra $\dot{\mathbf{K}}_n^{\mathfrak{c}}$ from a family of affine Schur algebras of type C. The idea mostly follows [BLM90]. However, there are still technical details to be clarified for affine type C (see Propositions 6.1.1 and 6.1.2). We then show that $\dot{\mathbf{K}}_n^{\mathfrak{c}}$ can be identified with a similar stabilization algebra defined in a geometric framework (cf. [FLLLW1]), and it follows that $\dot{\mathbf{K}}_n^{\mathfrak{c}}$ is an affine coideal subalgebra of $\dot{\mathbf{U}}(\widehat{\mathfrak{gl}}_n)$.

6.1 A BLM-type stabilization

Let $\tilde{\Xi}_n$ be the set of $\mathbb{Z} \times \mathbb{Z}$ matrices over $\mathbb{Z}$ in which each element $A = (a_{ij})$ satisfies the following conditions:

(T1) $a_{ij} = a_{n+i,n+j}$ for all i, j;

(X1) $a_{-i,-j} = a_{ij}$ for all i, j;

(X2) a_{00} and $a_{r+1,r+1}$ are odd;

(X4) $a_{ij} \in \mathbb{N}$ for all $i \neq j$.

For each $A \in \tilde{\tilde{\Xi}}_n$ and $p \in \mathbb{N}$, let ${}_pA = A + pI$ where I is the identity matrix. Thus there exists large enough even number p such that ${}_pA \in \Xi_n$. Let v' be an indeterminate (independent of v), and let $\mathcal{R}_1$ be the subring of $\mathbb{Q}(v)[v']$ generated by

$$\prod_{i=1}^{t} \frac{v^{2(a+i)}v'^2 - 1}{v^{2i} - 1}, \quad \prod_{i=1}^{t} \frac{v^{4(a+i)}v'^2 - 1}{v^{2i} - 1}, \text{ and } v^a \quad a \in \mathbb{Z}, t \in \mathbb{Z}_{>0}. \tag{6.1.1}$$

Let $\mathcal{R}_2$ be the subring of $\mathbb{Q}(v)[v', v'^{-1}]$ generated by

$$\begin{array}{ll} \prod_{i=1}^{t} \frac{v^{2(a+i)}v'^2 - 1}{v^{2i} - 1}, & \prod_{i=1}^{t} \frac{v^{4(a+i)}v'^2 - 1}{v^{2i} - 1}, \\ \prod_{i=1}^{t} \frac{v^{-2(a+i)}v'^{-2} - 1}{v^{-2i} - 1}, & \prod_{i=1}^{t} \frac{v^{-4(a+i)}v'^{-2} - 1}{v^{-2i} - 1}, \text{ and } v^a, \quad a \in \mathbb{Z}, t \in \mathbb{Z}_{>0}. \end{array} \tag{6.1.2}$$

Proposition 6.1.1. *Let $A_1, \dots, A_f \in \tilde{\tilde{\Xi}}_n$ be such that $\mathrm{co}_{\mathfrak{c}}(A_i) = \mathrm{ro}_{\mathfrak{c}}(A_{i+1})$ for all i. Then there exists matrices $Z_1, \dots, Z_m \in \tilde{\tilde{\Xi}}_n$ and $G_i(v, v') \in \mathcal{R}_1$ such that for even integer $p \gg 0$,*

$$[{}_pA_1] * [{}_pA_2] * \dots * [{}_pA_f] = \sum_{i=1}^{m} G_i(v, v^{-p})[{}_pZ_i]. \tag{6.1.3}$$

Proof. The proof follows exactly the idea as in [BLM90]. However, since the multiplication formula here is much more complicated, it is not obvious whether the coefficients are good enough to afford a stabilization. Below we give some explicit formulas to convince the readers that we can indeed derive a stabilization procedure. By Theorem 5.2.8, we may assume that $f = 2$, $A = A_2$ and $B = A_1 = \sum \alpha_i E_\theta^{i,i+1}$ is tridiagonal. For any even p making all entries in A_i positive, we can apply (5.1.8) and obtain

$$[{}_pB] * [{}_pA] = \sum_{\substack{T\in\Theta_{B,{}_pA}\\ S\in\Gamma_T}} v^{\beta({}_pA,S,T)}(v^2-1)^{n(S)}\overline{[{}_pA;S;T]}\ [{}_pA^{(T)}]. \tag{6.1.4}$$

We shall prove that for some polynomial $G^{(1)}$,

$$\beta({}_pA,S,T) = \beta(A,S,T) + pG^{(1)}(b_{i,i\pm1}, a_{ij}, t_{ij}).$$

Recall that $\beta({}_pA,S,T) = d'({}_pB) + d'({}_pA) - d'({}_pA^{(T_\theta - S_\theta)}) + \ell(w_{{}_pA,T}) + \gamma({}_pA,S,T)$. The difference can be obtained by computing the difference for each term. We have:

$$\begin{aligned}
d'({}_pB) - d'(B) &= -p\sum_{i=0}^{2r+1} b_{i,i+1},\\
d'({}_pA) - d'(A) &= \frac{p}{2}\sum_{i=0}^{2r+1}\Big(\sum_{x<i,y<i} a_{xy} - \sum_{y>i} a_{iy}\Big)\\
&\quad + \frac{p}{2}\sum_{(i,j)\in I^+}\Big(\sum_{x=1}^{r}\Big(\sum_{i<x<j} + \sum_{i>x>j}\Big)a_{ij} - \sum_{i\neq j} a_{ij}\Big),\\
\ell(w_{{}_pA,T}) - \ell(w_{A,T}) &= p\sum_{i=0}^{2r+1}\sum_{j>i} t_{ij},\\
\gamma({}_pA,S,T) - \gamma(A,S,T) &= -p\Big(\sum_{k\in\{0,r+1\}} (S+\widehat{T}-\widehat{S})_{\theta,kk} - 2\sum_{(i,j)\in I_{\mathfrak{a}}^+} (S+\widehat{T}-\widehat{S})_{\theta,ij}\Big).
\end{aligned} \tag{6.1.5}$$

Combining these yields to the desirable polynomials $G^{(1)}$. On the other hand, set

$$\begin{aligned}
a_{ij}^{(1)} &= p\delta_{ij} + (A - T_\theta)_{ij} + s_{-i,-j} + (\widehat{T-S})_{ij} + (\widehat{T-S})_{-i,-j},\\
a_{ij}^{(2)} &= p\delta_{ij} + (A - T_\theta)_{ij} + (\widehat{T-S})_{ij} + (\widehat{T-S})_{-i,-j},\\
a_{ij}^{(3)} &= p\delta_{ij} + (A - T_\theta)_{ij} + (\widehat{T-S})_{-i,-j},\\
a_{ij}^{(4)} &= p\delta_{ij} + (A - T_\theta)_{ij},\\
a_{kk}^{(5)} &= p + a_{kk} - 2t_{kk} - 1 + 2s_{kk} \in 2\mathbb{Z},\\
a_{kk}^{(6)} &= p + a_{kk} - 2t_{kk} - 1 \in 2\mathbb{Z}.
\end{aligned} \tag{6.1.6}$$

We have

$$\begin{aligned}
[{}_pA; S; T] = &\prod_{(i,j)\in I_{\mathfrak{a}}^+} \Big(\prod_{l=1}^{s_{ij}} \frac{[a_{ij}^{(1)} + l]}{[l]} \prod_{l=1}^{s_{-i,-j}} \frac{[a_{ij}^{(2)} + l]}{[l]} \prod_{l=1}^{(\widehat{T-S})_{ij}} \frac{[a_{ij}^{(3)} + l]}{[l]} \prod_{l=1}^{(\widehat{T-S})_{-i,-j}} \frac{[a_{ij}^{(4)} + l]}{[l]} \Big)\\
&\cdot \prod_{k\in\{0,r+1\}} \left(\prod_{l=1}^{(\widehat{T-S})_{kk}} \frac{[a_{kk}^{(5)} + 2l]}{[l]} \prod_{l=1}^{s_{kk}} \frac{[a_{kk}^{(6)} + 2l]}{[l]} \right) \cdot [\![S]\!]
\end{aligned} \tag{6.1.7}$$

The "type-A quantum binomials" are of the form

$$\prod_{i=1}^{t} \frac{[a+i+p]}{[i]} = \prod_{i=1}^{t} \frac{v^{2(a+i)}v^{2p} - 1}{v^{2i} - 1}, \quad a \in \mathbb{Z}, t \in \mathbb{Z}_{>0},$$

while the "type-C quantum binomials" are of the form

$$\prod_{i=1}^{t} \frac{[2(a+i)+p)]}{[i]} = \prod_{i=1}^{t} \frac{v^{4(a+i)}v^{2p} - 1}{v^{2i} - 1}, \quad a \in \mathbb{Z}, t \in \mathbb{Z}_{>0}.$$

They are indeed of the form $G(v, v^{-p})$ for some $G(v, v') \in \mathcal{R}_1$. □

Proposition 6.1.2. *Let $A \in \tilde{\Xi}_n$. Then there exists matrices $T_1, \ldots, T_m \in \tilde{\Xi}_n$ and $H_i(v, v') \in \mathcal{R}_2$ such that for even integer $p \gg 0$,*

$$\overline{[{}_pA]} = \sum_{i=1}^{m} H_i(v, v^{-p})[{}_pT_i]. \tag{6.1.8}$$

Proof. By Proposition 5.1.2, taking bar on $[{}_pB] * [{}_pA] = [{}_pM] + \sum_i G_i(v, v^{-p})[{}_pZ_i]$ leads to

$$[{}_pB] * [{}_pA] + \text{lower terms} = \overline{[{}_pM]} + \sum_i \overline{G_i(v, v^{-p})} * \big([{}_pZ_i] + \text{lower terms}\big)\,, \quad (6.1.9)$$

Following the idea of [BLM90, Proposition 4.3], one can show by induction that the coefficients showing up are indeed of the form $H(v, v^{-p})$ for some $H(v, v') \in \mathcal{R}_2$ as long as the initial case holds. Hence it suffices to prove the case when $A = (a_{ij})$ is tridiagonal. We shall prove this by another induction on $\ell(g)$, for which the initial case is trivial since $[A]$ is bar-invariant. Assume now that $\ell(g) > 0$.

Assume that $A = \kappa(\lambda, g, \mu)$ for some $\lambda, \mu \in \Lambda_{r,d}$, $g \in \mathscr{D}_{\lambda\mu}$. Then ${}_pA = \kappa({}_p\lambda, {}_pg, {}_p\mu)$ where

$${}_p\lambda = \Big(\lambda_0 + \frac{p}{2}, \lambda_1 + p, \ldots, \lambda_r + p, \lambda_{r+1} + \frac{p}{2}\Big), \quad {}_p\mu = \Big(\mu_0 + \frac{p}{2}, \mu_1 + p, \ldots, \mu_r + p, \mu_{r+1} + \frac{p}{2}\Big), \quad (6.1.10)$$

and ${}_pg = \prod_{i=0}^{r} {}_pg_i$ such that each ${}_pg_i$ swaps $p(i - \frac{1}{2}) + R^{\delta}_{3i+1}$ and $p(i - \frac{1}{2}) + R^{\delta}_{3i+2}$ for any i. That is, for $1 \leqslant x \leqslant d + rp$,

$${}_pg_i(x) = \begin{cases} x + a_{i-1,i} & \text{if } x - p(i - \frac{1}{2}) \in R^{\delta}_{3i+1}, \\ x - a_{i,i-1} & \text{if } x - p(i - \frac{1}{2}) \in R^{\delta}_{3i+2}, \\ x & \text{otherwise.} \end{cases} \quad (6.1.11)$$

Denote by ${}_pg^{+}_{\lambda\mu}$ the longest element in $W_{{}_p\lambda} \cdot {}_pg \cdot W_{{}_p\mu}$. By Lemma 5.1.1, we have

$$T_{W_{{}_p\lambda} \cdot {}_pg \cdot W_{{}_p\mu}} = v^{\ell({}_pg^{+}_{\lambda\mu})} C'_{{}_pg^{+}_{\lambda\mu}} + \sum_{\substack{w \in \mathscr{D}_{{}_p\lambda, {}_p\mu} \\ w < {}_pg}} c^{({}_p\lambda, {}_p\mu)}_{w, {}_pg} C'_{w^{+}_{{}_p\lambda, {}_p\mu}}. \quad (6.1.12)$$

For any $w \in \mathscr{D}_{{}_p\lambda,{}_p\mu}$, we set $A_w = \kappa({}_p\lambda, w, {}_p\mu)$. By Lemma 5.2.5 we know that $A_w \leqslant_{\text{alg}} {}_pA$, and hence $(A_w)^{\pm} \leqslant_{\text{alg}} {}_pA^{\pm} = A^{\pm}$. There is a unique tridiagonal matrix $\kappa(\lambda, x, \mu)$ such that ${}_p\kappa(\lambda, x, \mu) = A_w$. That is, $w = {}_px$ for some $x \in \mathscr{D}_{\lambda\mu}$ such that $x < g$. Therefore, (6.1.12) can be written as

$$T_{W_{{}_p\lambda} \cdot {}_pg \cdot W_{{}_p\mu}} = v^{\ell({}_pg^+_{\lambda\mu})} C'_{{}_pg^+_{\lambda\mu}} + \sum_{\substack{x \in \mathscr{D}_{\lambda\mu} \\ x < g}} c^{({}_p\lambda, {}_p\mu)}_{{}_px, {}_pg} C'_{{}_px^+_{\lambda\mu}}. \tag{6.1.13}$$

In particular, $T_{W_{{}_p\mu} \mathbb{1} W_{{}_p\mu}} = x_{{}_p\mu} = v^{\ell(w_\circ^{{}_p\mu})} C'_{w_\circ^{{}_p\mu}}$, where

$$\begin{aligned}
\ell(w_\circ^{{}_p\mu}) &= \ell(w_\circ^{\mu}) + \Big(\mu_0 + \frac{p}{2}\Big)^2 - \mu_0^2 + \sum_{i=1}^{r} \Big(\binom{\mu_i + p}{2} - \binom{\mu_i}{2} \Big) + \Big(\mu_{r+1} + \frac{p}{2}\Big)^2 - \mu_{r+1}^2 \\
&= \ell(w_\circ^{\mu}) + p\mu_0 + \frac{p^2}{4} + \sum_{i=1}^{r} p\mu_i + r\binom{p}{2} + p\mu_{r+1} + \frac{p^2}{4} \\
&= \ell(w_\circ^{\mu}) + \frac{p}{2}\Big(2d - r + p(r+1)\Big).
\end{aligned} \tag{6.1.14}$$

Therefore, by Lemma 5.1.1 again, for any $x \in \mathscr{D}_{\lambda\mu}$ such that $x \leqslant g$ with $A_x = \kappa(\lambda, x, \mu)$, we have

$$\begin{aligned}
\ell({}_px^+_{\lambda\mu}) - \ell(x^+_{\lambda\mu}) &= \Big(\ell(w_\circ^{{}_p\lambda}) - \ell(w_\circ^{\lambda})\Big) + \ell({}_px) - \ell(g) \\
&\quad - \Big(\ell(w_\circ^{\delta({}_pA_x)}) - \ell(w_\circ^{\delta(A)})\Big) + \Big(\ell(w_\circ^{{}_p\mu}) - \ell(w_\circ^{\mu})\Big) \\
&= \frac{p}{2}\Big(2(d + |A_x^{\pm}|) - r + p(r+1)\Big),
\end{aligned} \tag{6.1.15}$$

Here $|A_x^{\pm}|$ is the sum of off-diagonal entries of A_x over $I_{\mathfrak{a}}^+$. In particular, the leading coefficient ($=v^{\ell(g^+_{\lambda\mu})}$) in $T_{W_\lambda g W_\mu}$ is good enough to afford stablization. Moreover, by construction we know that the Kazhdan-Lusztig polynomial $P_{{}_px, {}_pg}$ is equal to $P_{x,g}$ for any even integer p. A similar argument shows that $c^{({}_p\lambda, {}_p\mu)}_{{}_px, {}_pg}$ is a product of $c^{(\lambda,\mu)}_{x,g}$ and a

v-power in terms of $\ell({}_py^+_{\lambda\mu}) - \ell(y^+_{\lambda\mu})$ for $x \leqslant y \leqslant g$. Hence, by (6.1.15), all coefficients in $T_{W_\lambda g W_\mu}$ are good enough to afford stablization.

On the other hand, we can derive from (6.1.13) that

$$\Big(\overline{[{}_pA]} - [{}_pA]\Big)(C'_{w_\circ^{p\mu}}) = \sum_{\substack{x\in\mathscr{D}_{\lambda\mu}\\ x<g}} \Big(v^{\ell({}_pg^+_{\lambda\mu})}\bar{c}^{({}_p\lambda,{}_p\mu)}_{{}_px,{}_pg} - v^{-\ell({}_pg^+_{\lambda\mu})}c^{({}_p\lambda,{}_p\mu)}_{{}_px,{}_pg}\Big)C'_{{}_px^+_{\lambda\mu}} \tag{6.1.16}$$

Combining (6.1.14), (6.1.15) and the inductive hypothesis, we have shown the existence of $H_i(v,v') \in \mathcal{R}_2$ for tridiagonal A. □

Proposition 6.1.3. *Retain the same notation for Proposition 6.1.1. We have:*

(*a*) *Let $\mathcal{K}_1$ be the free $\mathcal{R}_1$-module with basis $\{A \mid A \in \widetilde{\Xi}_n\}$. Then $\mathcal{K}_1$ has a unique associative $\mathcal{R}_1$-algebra structure in which the multiplication $*$ is given by*

$$A_1 * \cdots * A_f = \begin{cases} \sum\limits_{i=1}^m G_i(v,v')Z_i & \text{if } \mathrm{co}_{\mathfrak{c}}(A_i) = \mathrm{ro}_{\mathfrak{c}}(A_{i+1}) \text{ for all } i, \\ 0 & \text{otherwise.} \end{cases}$$

(*b*) *Let $\mathcal{K}_2$ be the same algebra with scalar extended to $\mathcal{R}_2$. Then the map $^- : \mathcal{K}_2 \to \mathcal{K}_2$ given by*

$$\overline{r(v,v')A} = r(v^{-1},v'^{-1})\sum_{i=1}^m H_i(v,v')T_i, \quad r(v,v') \in \mathcal{R}_2,$$

is an involution.

Let $\dot{\mathbf{K}}^{\mathfrak{c}}_n$ be the free $\mathbb{Z}[v,v^{-1}]$-module with basis $\{[A] \mid A \in \widetilde{\Xi}_n\}$.

Corollary 6.1.4. $\dot{\mathbf{K}}^{\mathfrak{c}}_n$ *has a unique associative* $\mathbb{Z}[v,v^{-1}]$*-algebra structure in which the multiplication* $*$ *is given by*

$$[A_1] * \cdots * [A_f] = \begin{cases} \sum_{i=1}^{m} G_i(v,1)[Z_i] & \text{if } \mathrm{co}_{\mathfrak{c}}(A_i) = \mathrm{ro}_{\mathfrak{c}}(A_{i+1}) \text{ for all } i, \\ 0 & \text{otherwise.} \end{cases} \tag{6.1.17}$$

Also, the map $^- : \dot{\mathbf{K}}^{\mathfrak{c}}_n \to \dot{\mathbf{K}}^{\mathfrak{c}}_n$ *given by* $\overline{[A]} = \sum_{i=1}^{m} H_i(v,1)[T_i]$ *is an involution.*

Following a standard argument (cf. [Lu93, 24.2.1]), we have the following:

Theorem 6.1.5. *Let* $A \in \widetilde{\Xi}_n$*. There is a unique bar-invariant element*

$$\{A\} = [A] + \sum_{A' <_{\mathrm{alg}} A} \pi_{A',A}[A'] \in \dot{\mathbf{K}}^{\mathfrak{c}}_n$$

such that $\pi_{A',A} \in v^{-1}\mathbb{Z}[v^{-1}]$*.*

The elements $\{\{A\} \mid A \in \widetilde{\Xi}_n\}$ form an $\mathbb{Z}[v,v^{-1}]$-basis of $\dot{\mathbf{K}}^{\mathfrak{c}}_n$, which we call the *canonical basis.*

Remark 6.1.6. By the affine type A counterpart of Proposition 6.1.1, one can construct a stabilization algebra $\dot{\mathbf{K}}^{\mathfrak{a}}_n$ in a similar way thanks to the multiplication formulas (cf. Lemma 2.3.1) and monomial bases (cf. Theorem 2.4.2). The algebra $\dot{\mathbf{K}}^{\mathfrak{a}}_n$ is first introduced in [DF14] and it is shown that $\dot{\mathbf{K}}^{\mathfrak{a}}_n$ is isomorphic to $\dot{\mathbf{U}}(\widehat{\mathfrak{gl}}_n)$, the idempotented quantum affine $\mathfrak{gl}_n$.

6.2 Quantum symmetric pairs

We now show that the stabilization algebra $\dot{\mathbf{K}}^{\mathfrak{c}}_n$ defined in Section 6.1 is indeed a coideal subalgebra of $\dot{\mathbf{U}}(\widehat{\mathfrak{gl}}_n)$.

Let $\dot{\mathbf{K}}^{\mathfrak{c},\mathrm{geo}}_n$ be the stabilization algebra in [FLLLW1, Section 9.4], which has the same underlying vector space as $\dot{\mathbf{K}}^{\mathfrak{c}}_n$, while the multiplication structure constants are defined in a seemingly different way than (6.1.17) as below.

$$[A_1] * \cdots * [A_f] = \begin{cases} \sum_{i=1}^{m} G'_i(v,1)[Z_i] & \text{if } \mathrm{co}_{\mathfrak{c}}(A_i) = \mathrm{ro}_{\mathfrak{c}}(A_{i+1}) \text{ for all } i, \\ 0 & \text{otherwise}, \end{cases} \tag{6.2.1}$$

where G'_i are two-parameter polynomials (cf. [FLLLW1, (9.4.1)]) arising from structure constants for the Schur algebras $\mathbf{S}^{\mathfrak{c},\mathrm{geo}}_{n,d}$ as a convolution algebra for affine flag varieties.

Lemma 6.2.1. *There is an identification* $\dot{\mathbf{K}}^{\mathfrak{c}}_n = \dot{\mathbf{K}}^{\mathfrak{c},\mathrm{geo}}_n$ *as associate algebras.*

Proof. It suffices to show that the two algebras have the same structure constants, which can be reduced to showing that

$$G_i(v, v^p) = G'_i(v, v^p), \quad p \gg 0. \tag{6.2.2}$$

For large enough p, $G_i(v, v^p)$ and $G'_i(v, v^p)$ are the structure constants for the Schur algebras $\mathbf{S}^{\mathfrak{c}}_{n,d+pn}$ and $\mathbf{S}^{\mathfrak{c},\mathrm{geo}}_{n,d+pn}$, respectively. We are done due to the identification (cf. Lemma 3.3.1) on the Schur algebra level. □

As a consequence, the stabilization algebra $\dot{\mathbf{K}}^{\mathfrak{c}}_n$ admits a coassociative comultiplication inherited from the comultiplication on $\dot{\mathbf{K}}^{\mathfrak{c},\mathrm{geo}}_n$.

Corollary 6.2.2. *The stablization algebras for an (idempotented) quantum symmetric pair* $(\dot{\mathbf{K}}^{\mathfrak{a}}_n, \dot{\mathbf{K}}^{\mathfrak{c}}_n)$.

Proof. This is due to [FLLLW1, Proposition 9.5.3]. □

Now we set

$$\mathbb{Z}^{\mathfrak{c}}_n = \{\lambda = (\lambda_i)_i \in \mathbb{Z}^{\mathbb{Z}} \mid \lambda_0, \lambda_{r+1} \in (2\mathbb{Z}+1), \lambda_i = \lambda_{n+i} = \lambda_{-i}\}. \tag{6.2.3}$$

We define a completion $\widehat{\mathbf{K}}^{\mathfrak{c}}_n$ of $\dot{\mathbf{K}}^{\mathfrak{c}}_n$ to be the $\mathbb{Q}(v)$-vector space of all formal linear combinations $\sum_{A\in\widetilde{\Xi}_n} \xi_A[A] \quad (\xi_A \in \mathbb{Q}(v), [A] \in \dot{\mathbf{K}}^{\mathfrak{c}}_n)$ such that for any $\lambda \in \mathbb{Z}^{\mathfrak{c}}_n$, the sets $\{A \in \widetilde{\Theta}_n \mid \xi_A \neq 0, \mathrm{ro}_{\mathfrak{a}}(A) = \lambda\}$ and $\{A \in \widetilde{\Theta}_n \mid \xi_A \neq 0, \mathrm{co}_{\mathfrak{a}}(A) = \lambda\}$ are finite. The multiplication on $\widehat{\mathbf{K}}^{\mathfrak{c}}_n$ given by

$$\Big(\sum_A \xi_A[A]\Big) \cdot \Big(\sum_B \eta_B[B]\Big) = \sum_{A,B} \xi_A\eta_B([A] * [B]),$$

defines an algebra structure on $\widehat{\mathbf{K}}^{\mathfrak{c}}_n$, where $[A] * [B]$ is the product in $\dot{\mathbf{K}}^{\mathfrak{c}}_n$. Let Ξ^0_n be the subset of Ξ_n in which the diagonal entries are all zero for each element. For $\alpha \in \mathbb{Z}^{\mathfrak{c}}_n$ we associate a diagonal matrix

$$D_\alpha = (\delta_{ij}\alpha_i)_{ij}. \tag{6.2.4}$$

For each $\mathbf{j} = (j_0, \ldots, j_{r+1}) \in \mathbb{N}^{r+2}, A \in \Xi^0_n$, we define

$$A(\mathbf{j}) = \sum_{\alpha\in\mathbb{Z}^{\mathfrak{c}}_n} v^{\mathbf{j}\cdot\alpha}[A + D_\alpha] \in \widehat{\mathbf{K}}^{\mathfrak{c}}_n. \tag{6.2.5}$$

Let $\mathbf{K}_n^{\mathfrak{c}}$ be the $\mathbb{Q}(v)$-subspace of $\widehat{\mathbf{K}}_n^{\mathfrak{c}}$ spanned by $\{A(\mathbf{j}) \mid A \in \Xi_n^0, \mathbf{j} \in \mathbb{N}^{r+2}\}$. The detailed proof for the proposition below will appear in [FLLLW2].

Proposition 6.2.3. $\mathbf{K}_n^{\mathfrak{c}}$ *is a subalgebra of* $\widehat{\mathbf{K}}_n^{\mathfrak{c}}$ *generated by*

$$A(\mathbf{j}), 0(\mathbf{j}) \quad (A \in \Xi_n^0 \text{ is tridiagonal}, \mathbf{j} \in \mathbb{N}^{r+2}).$$

Moreover, $\mathbf{K}_n^{\mathfrak{c}}$ *is a coideal subalgebra of* $\mathbf{K}_n^{\mathfrak{a}}$.

Chapter 7

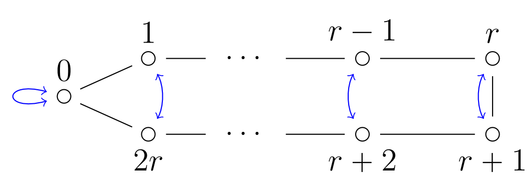
Stabilization algebras arising from different involutions

In this chapter we provide a formulation of three more variants of the stabilization algebras for different types of involutions on the Dynkin diagram of affine type A. We will present more details for the type $\imath\jmath$. We will merely formulating the main statements for types $\jmath\imath$ and $\imath\imath$.

7.1 Affine Schur algebras of type $\imath\jmath$

In the following we deal with the variant of affine q-Schur algebra of type $\imath\jmath$ corresponding to the involution as depicted below. Let $\Xi^{\imath\jmath}_{n,d}$ be the subset of $\Xi_{n,d}$ in which

Figure 7.1: Dynkin diagram of type $A^{(1)}_{2r}$ with involution of type $\imath\jmath$.



each element A satisfies additionally that

(X4) $\mathrm{ro}_{\mathfrak{c}}(A)_0 = 0 = \mathrm{ro}_{\mathfrak{c}}(A)_0$.

Let $\Lambda^{\imath\jmath}$ be the subset of $\Lambda = \Lambda_{r,d}$ in which each element $\lambda = (\lambda_i)$ satisfies additionally that $\lambda_0 = 0$. Recall from Lemma 3.2.4 that $\kappa : \{(\lambda, g, \mu) \mid \lambda, \mu \in \Lambda, g \in \mathscr{D}_{\lambda\mu}\} \to \Xi_{n,d}$ is a bijection. Similar argument leads to that

Lemma 7.1.1. *The restriction of κ^{-1} on $\Xi^{\imath\jmath}_{n,d}$ is a bijection. In particular, the map*

$$\kappa^{\imath\jmath} : \{(\lambda, g, \mu) \mid \lambda, \mu \in \Lambda^{\imath\jmath}, g \in \mathscr{D}_{\lambda\mu}\} \to \Xi^{\imath\jmath}_{n,d}$$

give by sending (λ, g, μ) to $(|R^{\lambda}_i \cap gR^{\mu}_j|)$ is a bijection.

Now we denote the affine q-Schur algebra of type $\imath\jmath$ by

$$\mathbf{S}^{\imath\jmath}_{n,d} = \mathrm{End}_{\mathbf{H}}\Big(\bigoplus_{\lambda\in\Lambda^{\imath\jmath}} x_\lambda \mathbf{H}\Big) \tag{7.1.1}$$

It is clear that $\mathbf{S}^{\jmath\imath}_{n,d}$ is naturally a subalgebra of $\mathbf{S}^{\mathfrak{c}}_{n,d}$. Moreover, both $\{e_A \mid A \in \Xi^{\imath\jmath}_{n,d}\}$ and $\{[A] \mid A \in \Xi^{\imath\jmath}_{n,d}\}$ are bases of $\mathbf{S}^{\jmath\imath}_{n,d}$ as a free $\mathbb{Z}[v, v^{-1}]$-module. Note that although Algorithm 5.2.7 applies to arbitrary $A \in \Xi^{\imath\jmath}_{n,d}$, the matrices produced does not lie in $\Xi^{\imath\jmath}_{n,d}$ in general. In order to define a monomial basis for $\mathbf{S}^{\jmath\imath}_{n,d}$, we need a modified matrix interpretation by collapsing those dummy rows and columns. Let $\mathbb{Z}^{\imath\jmath} = \mathbb{Z}\backslash n\mathbb{Z}$, and let $\check{\Xi}^{\imath\jmath}_{n,d}$ be the set of $\mathbb{Z}^{\imath\jmath} \times \mathbb{Z}^{\imath\jmath}$ matrices with entries in $\mathbb{N}$ in which each element $A = (a_{ij})$satisfies that:

(T1) $a_{ij} = a_{i+n,j+n}$ for all $i, j \in \mathbb{Z}^{\imath\jmath}$.

(X1) $a_{-i,-j} = a_{ij}$ for all $i, j \in \mathbb{Z}^{\imath\jmath}$.

(X2') $a_{r+1,r+1}$ is odd.

(X3') $\sum_{i=1}^{n-1} \sum_{j \in \mathbb{Z}^{\imath\jmath}} a_{ij} = D - 1$.

Let $f_c^{\imath\jmath} : \Xi_{n,d}^{\imath\jmath} \to \breve{\Xi}_{n,d}^{\imath\jmath}$ be the obvious collapsing map, and let $f_e^{\imath\jmath} : \breve{\Xi}_{n,d}^{\imath\jmath} \to \Xi_{n,d}^{\imath\jmath}$ be the expanding map that inserts suitable columns/rows making its image in $\Xi_{n,d}^{\imath\jmath}$. It is clear that $f_c^{\imath\jmath}$ and $f_e^{\imath\jmath}$ are bijections and they are inverse maps to each other.

Example 7.1.2. Let $r = 2$, $n = 6$, and

$$A = \left[\begin{array}{c:cc:c:cc:c}
3 & & & & & & \\ \hdashline
* & 4 & & & & & \\ \hdashline
3 & * & 5 & & & & \\
 & 2 & * & 0 & 1 & & \\ \hdashline
 & & 0 & 1 & 0 & & \\ \hdashline
 & & 1 & 0 & * & 2 & \\
 & & & & 5 & * & 3 \\ \hdashline
 & & & & & 4 & * \\ \hdashline
 & & & & & & 3
\end{array}\right].$$

That is, $A^{\pm} = E_{\theta}^{1,-1} + 2E_{\theta}^{12} + 3E_{\theta}^{23} + 4E_{\theta}^{32} + 5E_{\theta}^{21}$. Applying Algorithm 5.2.7, we get

$$[B^{(1)}] * [B^{(2)}] * [\mathrm{diag}] = [A] + \text{lower terms} \in \mathbf{S}_{n,d}^{\mathfrak{c}},$$

where

$$B^{(1)} = \left[\begin{array}{c:cc:c:cc:c}
 & & & & & & \\ \hdashline
* & & & & & & \\ \hdashline
 & * & & & & & \\
 & & * & 1 & & & \\ \hdashline
 & & & 1 & & & \\ \hdashline
 & & & 1 & * & & \\
 & & & & & * & \\ \hdashline
 & & & & & & * \\ \hdashline
 & & & & & &
\end{array}\right], B^{(2)} = \left[\begin{array}{c:cc:c:cc:c}
3 & & & & & & \\ \hdashline
* & 4 & & & & & \\ \hdashline
3 & * & 5 & & & & \\
 & 2 & * & 0 & & & \\ \hdashline
 & & 1 & 1 & 1 & & \\ \hdashline
 & & & 0 & * & 2 & \\
 & & & & 5 & * & 3 \\ \hdashline
 & & & & & 4 & * \\ \hdashline
 & & & & & & 3
\end{array}\right].$$

Note that here $B^{(2)} \notin \mathbf{S}^{\imath\jmath}_{n,d}$. On the other hand, we have

$$f_c^{\imath\jmath}(A) = \left[\begin{array}{c:cc|cc:c}
3 & & & & & \\ \hdashline
* & 4 & & & & \\ \hdashline
3 & * & 5 & & & \\
 & 2 & * & 1 & & \\ \hline
 & & 1 & * & 2 & \\
 & & & 5 & * & 3 \\ \hdashline
 & & & & 4 & * \\ \hdashline
 & & & & & 3
\end{array}\right].$$

Here the central solid lines represents the result of collapsing the stripes bounded by the central dashed lines. We have $f_c^{\imath\jmath}(A)^{\pm} = E_\theta^{1,-1} + 2E_\theta^{12} + 3E_\theta^{23} + 4E_\theta^{32} + 5E_\theta^{21}$.

We claim that in $\mathbf{S}^{\imath\jmath}_{n,d}$, matrices of this form behave like the tridiagonal ones in $\mathbf{S}^{\mathfrak{c}}_{n,d}$. In the followings we demonstrate another algorithm that generates $[A]$ for arbitrary $A \in \Xi^{\imath\jmath}_{n,d}$ bypassing elements in $\breve{\Xi}^{\imath\jmath}_{n,d}$:

Algorithm 7.1.3. For each element $A \in \Xi^{\imath\jmath}_{n,d}$, we define matrices ${}^{\imath}B^{(j)}$ as follows:

1. Apply a variant of Algorithm 5.2.7 on $f^{\imath\jmath}_c(A)$ and obtain matrices $B^{(j)} \in \Xi^{\imath\jmath}_{n,d}$.

2. Let ${}^{\imath}B^{(j)} = f^{\imath\jmath}_e(B^{(j)})$ for all j.

Theorem 7.1.4. *For each $A \in \Xi^{\imath\jmath}_{n,d}$, the matrices ${}^{\imath}B^{(j)}$, $j = 1, \dots, x$, in $\Xi^{\imath\jmath}_{n,d}$ produced by Algorithm 7.1.3 satisfy that*

$$[{}^{\imath}B^{(1)}] * [{}^{\imath}B^{(2)}] * \cdots * [{}^{\imath}B^{(x)}] = [A] + \text{lower terms} \in \mathbf{S}^{\imath\jmath}_{n,d}.$$

Proof. It remains to show that each lower term $[C]$ occurred lies in $\mathbf{S}^{\imath\jmath}_{n,d}$. By definition of multiplication on $\mathbf{S}^{\imath\jmath}_{n,d}$ we have $\mathrm{ro}_{\mathfrak{c}}(C) = \mathrm{ro}_{\mathfrak{c}}(A)$ and $\mathrm{co}_{\mathfrak{c}}(C) = \mathrm{co}_{\mathfrak{c}}(A)$. In particular, $\mathrm{ro}_{\mathfrak{c}}(C)_0 = \mathrm{ro}_{\mathfrak{c}}(A)_0 = 0$ and $\mathrm{co}_{\mathfrak{c}}(C)_0 = \mathrm{co}_{\mathfrak{c}}(A)_0 = 0$ since $A \in \Xi^{\imath\jmath}_{n,d}$. Therefore $C \in \Xi^{\imath\jmath}_{n,d}$ and $[C] \in \mathbf{S}^{\imath\jmath}_{n,d}$. □

7.2 Stabilization algebras of type $\imath\jmath$

Recall $\tilde{\Xi}_n$ from Section 6.1. We define

$$\tilde{\Xi}^{<}_n = \{A \in \tilde{\Xi}_n \mid a_{00} < 0\}, \quad \tilde{\Xi}^{>}_n = \{A \in \tilde{\Xi}_n \mid a_{00} > 0\}. \tag{7.2.1}$$

For any matrix $A \in \tilde{\Xi}_n$ and $p \in \mathbb{Z}$, we set

$${}_{\check{p}}A = A + p(\mathrm{diag}(0, 1, 1, \dots, 1, 0)). \tag{7.2.2}$$

Lemma 7.2.1. *For $A_1, A_2, \dots, A_f \in \widetilde{\Xi}_n^{>}$, there exists $Z_i \in \widetilde{\Xi}_n^{>}$ and $G_i(v, v') \in \mathbb{Q}(v)[v', v'^{-1}]$ ($i = 1, \dots, m$ for some m) such that*

$$[_{\breve{p}}A_1] * [_{\breve{p}}A_2] * \dots * [_{\breve{p}}A_f] = \sum_{i=1}^{m} G_i(v, v^{-p})[_{\breve{p}}Z_i], \quad \text{for all even integers } p \in \mathbb{Z}.$$

Proof. It is similar to the proof of Proposition 6.1.1. See [BKLW14, Lemma A.1] for some details that replacing p by $\breve{p}$ does not cause problems. □

As a corollary, the $\mathbb{Z}[v, v^{-1}]$-subspace $\dot{\mathbf{K}}_n^{\mathfrak{c}\,>}$ of $\dot{\mathbf{K}}_n^{\mathfrak{c}}$ spanned by $[A]$ for $A \in \widetilde{\Xi}_n^{>}$ is a stabilization algebra whose multiplicative structure is given by

$$[A_1] \cdot [A_2] \cdot \dots \cdot [A_f] = \begin{cases} \sum_{i=1}^{m} G_i(v, 1)[Z_i] & \text{if } \mathrm{co}_{\mathfrak{c}}(A_i) = \mathrm{ro}_{\mathfrak{c}}(A_{i+1}) \text{ for all } i, \\ 0 & \text{otherwise.} \end{cases} \tag{7.2.3}$$

It is routine to show that $\dot{\mathbf{K}}_n^{\mathfrak{c}\,>}$ has a monomial basis $\{m_A \mid A \in \widetilde{\Xi}_n^{>}\}$. By a similar argument to Proposition 6.1.2, it can be shown that $\dot{\mathbf{K}}_n^{\mathfrak{c}\,>}$ admits a compatible bar-involution. A standard argument then shows that $\dot{\mathbf{K}}_n^{\mathfrak{c}\,>}$ has a canonical basis (cf. [LW15]).

Let $\dot{\mathbf{K}}_n^{\imath\jmath}$ be the $\mathbb{Z}[v, v^{-1}]$-submodule of $\dot{\mathbf{K}}_n^{\mathfrak{c}\,>}$ generated by $[A]$ for $A \in \widetilde{\Xi}_n^{\imath\jmath}$, where

$$\begin{aligned} \widetilde{\Xi}_n^{\imath\jmath} &= \{A = (a_{ij}) \in \widetilde{\Xi}_n \mid a_{0i} = a_{i0} = \delta_{0i}\} \\ &= \{A \in \widetilde{\Xi}_n^{>} \mid \mathrm{co}_{\mathfrak{c}}(A)_0 = \mathrm{ro}_{\mathfrak{c}}(A)_0 = 0\}. \end{aligned} \tag{7.2.4}$$

Since that the bar-involution on $\dot{\mathbf{K}}_n^{\mathfrak{c}\,>}$ restricts to an involution on $\dot{\mathbf{K}}_n^{\imath\jmath}$, $\dot{\mathbf{K}}_n^{\mathfrak{c}\,>}$ and $\dot{\mathbf{K}}_n^{\imath\jmath}$ have compatible canonical bases.

Remark 7.2.2. The submodule of $\dot{\mathbf{K}}_n^{\mathfrak{c}}$ spanned by $[A]$ for $A \in \widetilde{\Xi}_n^{\imath\jmath}$ is not a subalgebra.

Now we realize $\dot{\mathbf{K}}_n^{\imath\jmath}$ as a subquotient of $\dot{\mathbf{K}}_n^{\mathfrak{c}}$. Define $\mathbf{J}_<$ to be the $\mathbb{Z}[v,v^{-1}]$-submodule of $\dot{\mathbf{K}}_n^{\mathfrak{c}}$ spanned by $[A]$ for all $A \in \tilde{\Xi}_n^<$.

Lemma 7.2.3. *The submodule $\mathbf{J}_<$ is a two-sided ideal of $\dot{\mathbf{K}}_n^{\mathfrak{c}}$*

Proof. It suffices to show that $[B]\cdot[A] \in \mathbf{J}_<$ for arbitrary $A \in \tilde{\Xi}_n^<$ and tridiagonal $B \in \tilde{\Xi}_n$. By the multiplication formula, the matrices corresponding to the terms showing up in $[B]\cdot[A]$ must be of the form

$$A^{(T-S)} = A - (T-S)_\theta + (\widehat{T-S})_\theta, \quad T \in \Theta_{B,A}, S \in \Gamma_T.$$

Suppose that the $(0,0)$-entry $a_{00} - 2(t_{00} - s_{00}) + 2(\widehat{T-S})_{00}$ is positive. Note that we have

$$[A;S;T] = \prod_{(i,j)\in I_{\mathfrak{a}}^+} \begin{bmatrix} (A-T_\theta) + s_{ij} + s_{-i,-j} + (\widehat{T-S}) + (\widehat{T-S})_{-i,-j} \\ (A-T_\theta); s_{ij}; s_{-i,-j}; (\widehat{T-S}); (\widehat{T-S})_{-i,-j} \end{bmatrix}$$
$$\cdot \prod_{k\in\{0,r+1\}} \left(\frac{\prod_{i=1}^{s_{kk}+(\widehat{T-S})_{kk}} [a_{kk} - 2t_{kk} - 1 + 2i]}{[s_{kk}]_{\mathfrak{a}}^![(\widehat{T-S})_{kk}]_{\mathfrak{a}}^!} \right) \cdot [\![S]\!] .$$

Therefore $[A;S;T] = 0$ and hence $[A^{(T-S)}] \in \mathbf{J}_<$. □

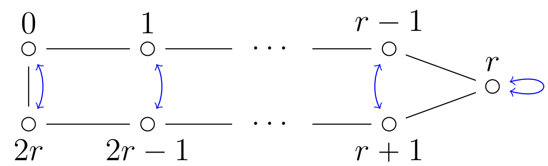
Finally, we realize $\dot{\mathbf{K}}_n^{\imath\jmath}$ as a subquotient (details omitted) by following [BKLW14] (see also [FL14]), where an algebra $\mathbf{U}^\imath$ is realized as a subquotient of an algebra $\mathbf{U}^\jmath$ with compatible canonical bases.

Proposition 7.2.4. *As an $\mathbb{Z}[v,v^{-1}]$-algebra, $\dot{\mathbf{K}}_n^{\imath\jmath}$ is naturally isomorphic to a subquotient of $\dot{\mathbf{K}}_n^{\mathfrak{c}}$, with compatible standard, monomial, and canonical bases.*

7.3 Stabilization algebras of type $\jmath\imath$

In the following we deal with the variant of affine q-Schur algebra of type $\jmath\imath$ corresponding to the involution as depicted below. Let $\Xi^{\jmath\imath}_{n,d}$ be the subset of $\Xi_{n,d}$ in which

Figure 7.2: Dynkin diagram of type $A^{(1)}_{2r}$ with involution of type $\jmath\imath$.



each element A satisfies additionally that

(X5) $\mathrm{ro}_{\mathfrak{c}}(A)_{r+1} = 0 = \mathrm{ro}_{\mathfrak{c}}(A)_{r+1}$.

Let $\Lambda^{\jmath\imath}$ be the subset of $\Lambda = \Lambda_{r,d}$ in which each element $\lambda = (\lambda_i)$ satisfies additionally that $\lambda_{r+1} = 0$.

Lemma 7.3.1. *The restriction of κ^{-1} on $\Xi^{\jmath\imath}_{n,d}$ is a bijection. In particular, the map*

$$\kappa^{\jmath\imath} : \{(\lambda, g, \mu) \mid \lambda, \mu \in \Lambda^{\jmath\imath}, g \in \mathscr{D}_{\lambda\mu}\} \to \Xi^{\jmath\imath}_{n,d}$$

give by sending (λ, g, μ) to $(|R^{\lambda}_i \cap gR^{\mu}_j|)$ is a bijection.

Now we denote the affine q-Schur algebra of type $\jmath\imath$ by

$$\mathbf{S}^{\jmath\imath}_{n,d} = \mathrm{End}_{\mathbf{H}}\Big(\bigoplus_{\lambda\in\Lambda^{\imath\jmath}} x_\lambda \mathbf{H}\Big) \tag{7.3.1}$$

$\mathbf{S}^{\jmath\imath}_{n,d}$ is also a subalgebra of $\mathbf{S}^{\mathfrak{c}}_{n,d}$, admitting compatible standard, monomial and canonical bases.

By repeating the process of the $\imath\jmath$ version. We construct an associative algebra $\dot{\mathbf{K}}_n^{\jmath\imath}$ with a basis $[A]$ parametrized by

$$\begin{aligned} \widetilde{\Xi}_n^{\jmath\imath} &= \{A = (a_{ij}) \in \widetilde{\Xi}_n \mid a_{r+1,i} = a_{i,r+1} = \delta_{r+1,i}\} \\ &= \{A \in \widetilde{\Xi}_n^{>} \mid \mathrm{co}_{\mathfrak{c}}(A)_{r+1} = \mathrm{ro}_{\mathfrak{c}}(A)_{r+1} = 0\}. \end{aligned} \tag{7.3.2}$$

All results for $\dot{\mathbf{K}}_n^{\imath\jmath}$ admit counterparts for $\dot{\mathbf{K}}_n^{\jmath\imath}$.

Proposition 7.3.2.

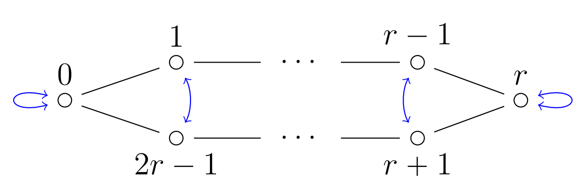
(a) The algebra $\dot{\mathbf{K}}_n^{\jmath\imath}$ admits a standard basis, a monomial basis, and a canonical basis.

(b) $\dot{\mathbf{K}}_n^{\jmath\imath}$ is a subquotient of $\dot{\mathbf{K}}_n^{\mathfrak{c}}$ with compatible canonical bases.

7.4 Stabilization algebras of type $\imath\imath$

In the following we deal with the variant of affine q-Schur algebra of type $\imath\imath$ corresponding to the involution as depicted below. Let $\Xi_{n,d}^{\imath\imath} = \Xi_{n,d}^{\imath\jmath} \cap \Xi_{n,d}^{\jmath\imath}$, $\Lambda^{\imath\imath} = \Lambda^{\jmath\imath} \cap \Lambda^{\imath\jmath}$.

Figure 7.3: Dynkin diagram of type $A_{2r-1}^{(1)}$ with involution of type $\imath\imath$.



Lemma 7.4.1. *The restriction of κ^{-1} on $\Xi^{\imath\imath}_{n,d}$ is a bijection. In particular, the map*

$$\kappa^{\imath\imath} : \{(\lambda, g, \mu) \mid \lambda, \mu \in \Lambda^{\imath\imath}, g \in \mathscr{D}_{\lambda\mu}\} \to \Xi^{\imath\imath}_{n,d}$$

give by sending (λ, g, μ) to $(|R^\lambda_i \cap gR^\mu_j|)$ is a bijection.

Now we denote the affine q-Schur algebra of type $\imath\imath$ by

$$\mathbf{S}^{\imath\imath}_{n,d} = \mathrm{End}_{\mathbf{H}}\Big(\bigoplus_{\lambda \in \Lambda^{\imath\imath}} x_\lambda \mathbf{H}\Big) \tag{7.4.1}$$

$\mathbf{S}^{\imath\imath}_{n,d}$ is naturally a subalgebra of $\mathbf{S}^{\imath\jmath}_{n,d}, \mathbf{S}^{\jmath\imath}_{n,d}$ and $\mathbf{S}^{\mathfrak{c}}_{n,d}$, admitting compatible standard, monomial and canonical bases. By a similar process, we construct an associative algebra $\dot{\mathbf{K}}^{\jmath\imath}_n$ with a basis $[A]$ parametrized by

$$\widetilde{\Xi}^{\imath\imath}_n = \widetilde{\Xi}^{\imath\jmath}_n \cap \widetilde{\Xi}^{\jmath\imath}_n. \tag{7.4.2}$$

We collect the main results for in the following. The proofs are very similar to the previous cases, and so we shall skip them to avoid redundancy.

Proposition 7.4.2.

(a) The algebra $\dot{\mathbf{K}}^{\imath\imath}_n$ admits a standard basis, a monomial basis, and a canonical basis.

(b) $\dot{\mathbf{K}}^{\imath\imath}_n$ is a subquotient of $\dot{\mathbf{K}}^{\jmath\imath}_n$ and $\dot{\mathbf{K}}^{\imath\jmath}_n$, with compatible canonical bases.

The interrelation among the four types can be summarized below. On the Schur algebra level, we have the following commuting diagram for inclusions of Schur algebras:

$$
\begin{array}{ccccc}
 & & \mathbf{S}^{\mathfrak{c}}_{n,d} & & \\
 & \nearrow\!\!\!\hookrightarrow & & \hookleftarrow\!\!\!\nwarrow & \\
\mathbf{S}^{\imath\jmath}_{n,d} & & & & \mathbf{S}^{\jmath\imath}_{n,d} \\
 & \nwarrow\!\!\!\hookleftarrow & & \hookrightarrow\!\!\!\nearrow & \\
 & & \mathbf{S}^{\imath\imath}_{n,d} & &
\end{array}
\tag{7.4.3}
$$

On the stabilization algebra level, we have the following diagram of subquotients:

$$
\begin{array}{ccccc}
 & & \dot{\mathbf{K}}^{\mathfrak{c}}_{n} & & \\
 & \overset{\mathfrak{sq}}{\nearrow} & & \overset{\mathfrak{sq}}{\nwarrow} & \\
\dot{\mathbf{K}}^{\imath\jmath}_{n} & & & & \dot{\mathbf{K}}^{\jmath\imath}_{n} \\
 & \overset{\mathfrak{sq}}{\nwarrow} & & \overset{\mathfrak{sq}}{\nearrow} & \\
 & & \dot{\mathbf{K}}^{\imath\imath}_{n} & &
\end{array}
\tag{7.4.4}
$$

where the notation $\mathbf{K}_1 \overset{\mathfrak{sq}}{\twoheadrightarrow} \mathbf{K}_2$ stands for the statement that $\mathbf{K}_2$ is a subquotient of $\mathbf{K}_1$. All the subquotients between various pairs of algebras preserve the canonical bases.

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