C^*-algebras and their finite-dimensional representations

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A Dissertation presented to the Graduate Faculty
of the University of Virginia in Candidacy for the Degree of
Doctor of Philosophy

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May, 2018

Abstract

We investigate C^*-algebras whose structure can be recovered from their finite- dimensional representations, i.e. the class of so-called residually finite-dimensional (RFD) C^*-algebras. We show that these are exactly the C^*-algebras that contain a dense subset of elements that attain their norm under a finite-dimensional representation. Moreover, we prove that this subset is the whole space precisely when every irreducible representation of the C^*-algebra is finite-dimensional, which is equivalent to the C^*-algebra having no simple infinite-dimensional AF subquotient.

We then use the residual finite-dimensionality of certain universal C^*-algebras to formulate and sharpen a von Neumann-type inequality for noncommutative *-polynomials. Finally, we consider a noncommutative $*$-polynomial analogue to a two-variable von Neumann inequality, showing that it holds for all noncommutative $*$-polynomials in two variables if and only if Connes' Embedding Problem has a positive solution.

Acknowledgements

Many thanks to my advisor, David Sherman, for his patience, guidance, and ample questions. I am also grateful to Don Hadwin, Nate Brown, Chris Ramsey, and numerous other mathematicians, for their generosity and enlightening conversations.

I would like to thank my friends and family for their support. In particular, I want to thank my grandmother, Lois George, for instilling in me a love of learning, my Aunt, Joan Fulmer, for her unwavering love, Raphaël Clouâtre for his encouragement, and Michael Reeks and Kristen Bevalet for their camaraderie.

Finally, to my mother, Elizabeth Courtney, thank you for years of love and sacrifice. You are and always will be my hero.

In memory of L.H. George, my granddaddy.

Contents

Chapter 1

Introduction

A C^*-algebra is a complex Banach algebra satisfying the C^*-relation $\|x^*x\| = \|x\|^2$. More concretely, it can be thought of as a self-adjoint, norm-closed subalgebra of bounded operators $B(\mathcal{H})$ on some Hilbert space $\mathcal{H}$. Two significant classes of examples of C^*-algebras are the algebras of $n \times n$ complex matrices, $\mathbb{M}_n(\mathbb{C})$ (or just $\mathbb{M}_n$), and spaces of complex valued functions on (locally) compact Hausdorff spaces, $C(X, \mathbb{C})$ (or just $C(X)$). The first are notably finite-dimensional, and the second commutative. Consequently, the study of C^*-algebras is sometimes likened to infinite-dimensional linear algebra or non-commutative topology.

In fact, most C^*-algebras of interest, such as C^*-algebras arising from infinite groups or topological dynamical systems, are infinite-dimensional, which often makes them difficult to describe and study. For this reason, the study of finite-dimensional approximations of C^*-algebras have become indispensable to the theory. Actually, the influence of finite-dimensional approximation properties goes far beyond mere utility, and they have found themselves at the heart of many of the most important problems in the field (see [17], [51], [86], and [88] to name a few).

Of course, "finite-dimensional approximation properties" is highly ambiguous and can be unpacked in many ways. For the present work, we will consider how well norms of elements are approximated by the norms of their images under finite-dimensional representations. If the finite-dimensional representations of the C^*-algebra separate the points, we can recover significant information about the structure of the algebra from these representations. We say a C^*-algebra is **residually finite-dimensional** (RFD) if it has a separating family of finite-dimensional representations. In this case, the direct sum of the finite-dimensional representations yields a faithful (i.e., isometric) representation of the algebra into a direct product of matrix algebras: the algebra is "block-diagonalizable" with finite-dimensional blocks. One of the first results on RFD C^*-algebras, due to Choi ([16, Theorem 7]), is the fact that the full C^*-algebra $C^*(\mathbb{F}_n)$ of the free group on $n \in \overline{\mathbb{N}}$ generators is RFD ([16]). In the ensuing years, various characterizations of RFD C^*-algebras have been obtained (e.g. in [26, 6, 37]), and various classes of C^*-algebras were proved to be RFD. A notable class of RFD C^*-algebras are those whose irreducible representations are all finite-dimensional. We call such C^*-algebras **FDI** for "Finite-Dimensional Irreps". This class includes, in particular, n-subhomogeneous C^*-algebras, which have no irreducible representations of dimension more than n.

Examples of RFD C^*-algebras arising from groups include full group C^*-algebras of amenable maximally periodic groups [9], surface groups and fundamental groups of closed hyperbolic 3-manifolds that fiber over the circle [73], and many 1-relator groups with non-trivial center [39]. Other classes of RFD C^*-algebras include amalgamated products of commutative C^*-algebras [53], projective C^*-algebras [57], the soft torus C^*-algebra [22], and certain just-infinite C^*-algebras [30]. There also exist notable examples and non-examples among universal C^*-algebras. For instance, the universal C^*-algebra generated by a contraction, the universal C^*-algebra generated by a partial isometry [14], and universal C^*-algebras generated by algebraic contractions [59] are all RFD; whereas the universal C^*-algebra generated by an isometry (better known as the Toeplitz algebra) is not.(This list of examples is certainly incomplete.) The class of RFD C^*-algebras is also closed under free products [26] (see also [29]), minimal tensor products [15], extensions, and subalgebras.

In [27] Fritz, Netzer and Thom improve on Choi's proof of [16, Theorem 7] to prove that every element in the group algebra $\mathbb{C}\mathbb{F}_n$ actually *attains* its universal norm under some finite-dimensional unitary representation. Viewing $\mathbb{C}\mathbb{F}_n$ as a dense subalgebra of $C^*(\mathbb{F}_n)$, it is natural to ask whether there exists in other RFD C^*-algebras a dense subset of elements that attain their norm under a finite-dimensional representation. In section 3.1, we prove that this is indeed true. Moreover, this characterizes RFD C^*-algebras (Corollary 3.1.3).

Looking at the result of Fritz, Netzer and Thom, one can ask further questions. For instance, are there elements in $C^*(\mathbb{F}_n)$ other than the elements of $\mathbb{C}\mathbb{F}_n$ that attain their norm under a finite-dimensional representation? Could this be true for all elements?

In Section 3.2, we prove that all elements of a C^*-algebra attain their norm under a finite-dimensional representation if and only if the C^*-algebra has no infinite-dimensional irreducible representation, i.e the C^*-algebra is FDI (Theorem 3.2.4). In particular, this theorem implies the existence of elements in $C^*(\mathbb{F}_n)$ that do not attain their norm under a finite-dimensional representation. Moreover, we show that A is FDI if and only if A has no C^*-subalgebra which surjects onto some simple, infinite-dimensional AF-algebra.

We then turn our attention to the study of the set of elements in an RFD C^*-algebra that attain their norms under some finite-dimensional representation. A first natural question is whether or not this set has any algebraic structure. In Theorem

3.3.2 we prove that this subset is additively closed if and only if it is multiplicatively closed if and only if the set is the entire C^*-algebra, i.e., the C^*-algebra is FDI. In particular, it implies that there exist elements in $C^*(\mathbb{F}_n)\backslash\mathbb{C}\mathbb{F}_n$ that attain their norm under a finite-dimensional representation.

Knowing that a C^*-algebra A is RFD tells us *that* the norms of the images of a given element approach the norm of the element as the dimension of the representations grows, i.e. for each $a \in A$,

$$\|a\| = \lim_{n\to\infty} \sup\{\|\pi(a)\| : \pi : A \to \mathbb{M}_n\}.$$

But what can be said about *how* the norms of the images grow as the dimension of the representations grows? Let's make this more precise. For a C^*-algebra A and $k \in \mathbb{N}$, we define a seminorm $\|\cdot\|_{\mathbb{M}_k}$ by

$$\|a\|_{\mathbb{M}_k} = \sup\{\|\pi(a)\| \mid \pi : A \to \mathbb{M}_k\},$$

for all $a \in A$. If we form a sequence of seminorms $(\|a\|_{\mathbb{M}_k})_{k\in\mathbb{N}}$ for each $a \in A$, what sort of sequences will arise?

Due to restrictions imposed by our proof techniques, we will want to instead index our sequence of seminorms by the dimensions of the finite-dimensional irreducible representations for A. This isn't such a bad restriction since any finite-dimensional representation of A can be written as a finite-direct sum of irreducible representations. To that end, assume A has irreducible representations of dimensions $n_1 < n_2 < ... < \infty$, and let

$$\Lambda(A) := \{(\|a\|_{\mathbb{M}_{n_k}})_{k\in\mathbb{N}} : a \in A\}$$

. With this terminology, we consider two questions,

1. Given a non-decreasing sequence $(\lambda_k) \in \ell^\infty_+$, can we find an $a \in A$ with $\|a\|_{\mathbb{M}_{n_k}} = \lambda_k$ for each k? In other words, what sequences are in $\Lambda(A)$?

2. Given an element $a \in A$, what is its sequence of seminorms $(\|a\|_{\mathbb{M}_{n_k}})$?

In Chapter 4, we investigate these two questions. Theorem 4.1.1 shows that $\Lambda(A)$ contains the set of all nondecreasing sequences of positive numbers which are eventually constant. We show that those two sets coincide exactly when A is FDI (Corollary 4.1.4). When A is RFD but not FDI, we describe the behavior of some sequences in $\Lambda(A)$ that are not eventually constant (Theorem 4.1.5). Our results, when relevant, also hold for C^*-algebras, which have irreducible representations in only finitely many dimensions.

For the second question, we consider a specific collection of elements of $\mathbb{C}\mathbb{F}_n$ that attain their norm under some finite dimensional representation and ask for an upper bound for the dimension required to witness this. In their proof that any element of $\mathbb{C}\mathbb{F}_n$ achieves its norm in a finite-dimensional representation, Fritz, Netzer, and Thom give an upper bound for the dimension of such a representation ([27, Lemma 2.7]): if ℓ is the length of the longest word in the support of the element, they show that such a representation can be chosen of dimension no more than $4n^\ell$. In Section 4.2, we find a better bound on the dimension for binomials in $\mathbb{C}\mathbb{F}_n$. In Theorem 4.2.2 we prove that for any nontrivial, reduced word $w \in \mathbb{F}_n$ of length ℓ and any $\lambda \in \mathbb{T}$, there exists a representation $\pi : C^*(\mathbb{F}_n) \to \mathbb{M}_{2\ell}$ such that the spectrum of $\pi(w)$ contains λ. From this theorem we deduce (Proposition 4.2.1) that any element of the form $\alpha w_1 + \beta w_2$, where $\alpha, \beta \in \mathbb{C}$ and $w_1, w_2 \in \mathbb{F}_n$, attains its norm under a 2ℓ-dimensional representation of $C^*(\mathbb{F}_2)$; here ℓ is the length of the reduced word $w_2^{-1}w_1$.

One of our main tools in Chapters 3 and 4 is AF-telescopes, which were proven to be projective by Loring and Pedersen in [58]. The proof of Theorem 3.2.4 develops a technique for building and lifting elements with desired properties from AF-telescopes. This technique is repeatedly applied in Chapters 3 and 4 and has since found applications in [75].

In Chapters 5 and 6, we switch from studying residual finite-dimensionality to using it. An inspiration for these chapters is a famous inequality due to von Neumann ([61]).

Theorem (von Neumann's Inequality). *Let T be an operator on a Hilbert space with $\|T\| \leq 1$. Then for any polynomial p in $\mathbb{C}[z]$,*

$$\|p(T)\| \leq \|p\|,$$

where $\|p\| = \sup_{z\in\overline{\mathbb{D}}} |p(z)|$ and $\mathbb{D}$ denotes the unit disk of $\mathbb{C}$.

Because of its elegance and utility, von Neumann's Inequality has become canon in operator theory, and its applications and extensions to various contexts are still the subject of a wide range of research. See [4], [84], [65], [21], [43], and [52], to name a few.

This inequality is intimately tied with power dilations of contractive Hilbert space operators. For a contractive operator T on some Hilbert space $\mathcal{H}$, a power dilation of T is an operator S on some Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ as a subspace such that $T^n = P|_{\mathcal{H}}S^n|_{\mathcal{H}}$ for all $n \geq 0$, where $P_{\mathcal{H}}$ denotes the orthogonal projection onto $\mathcal{H}$. Often S has nicer properties than T, e.g., S is a unitary. Two notable theorems in the theory of dilations are Sz.-Nagy's dilation theorem ([82]), which says that any contraction T on a Hilbert space admits a unitary power dilation, and Andô's dilation theorem ([4]), which says that any pair of commuting contractions has a simultaneous power dilation to a pair of commuting unitaries. The former yields an elegant proof of von Neumann's Inequality, and the latter yields an extension of the inequality to the case of polynomials in two commuting contractions (see [65, Chapters 4 and 5], respectively). On the other hand, the Kaijser-Varopoulos counterexample to von Neumann's Inequality for commuting triples of contractive operators ([84]) shows that Andô's dilation theorem fails to extend to commuting n-tuples of contractive operators. Since the discovery of this obstruction, many extensions have been given when the n-tuple of commuting contractions is assumed to have extra properties, e.g. [21] and [43].

In the present work, we consider analogous inequalities where the polynomial is replaced with a noncommutative $*$-polynomial. To see what would be an appropriate analogue to von Neumann's Inequality, we first rephrase it to say that, in order to find the maximal norm of $p(T)$ as T ranges over all contractive Hilbert space operators, it suffices to consider contractive Hilbert space operators on $\mathbb{C}$. This sort of statement can be more easily tracked by employing universal structures. Since we will soon be concerned with $*$-polynomials, we will employ universal C^*-algebras.

We denote the universal unital C^*-algebra generated by a contraction[1], or just the universal unital contraction algebra for short, by

$$C_u^*\langle x : \|x\| \leq 1\rangle.$$

It has the universal property that, given any contractive operator T on a Hilbert space $\mathcal{H}$, the assignment $x \mapsto T$ induces a surjective unital $*$-homomorphism $C_u^*\langle x : \|x\| \leq 1\rangle \to C^*(T, I_{\mathcal{H}})$. The distinguished generator is called a universal contraction, and this title is also bestowed on the image of x under any faithful nondegenerate

[1]This is isomorphic to the universal unital C^*-algebra of the operator space $\mathbb{C}$, $C_u^*\langle\mathbb{C}\rangle$. See [68, Chapter 8] for a description.

representation.

Example 1.0.1. For each $n \in \mathbb{N}$, let $\{D_{n,k}\}_{k\in\mathbb{N}}$ be a dense subset of the unit ball of $\mathbb{M}_n$. In his proof for [36, Theorem 5.1], Hadwin shows that $S := \oplus_n \oplus_k D_{n,k}$ is a universal contraction. A more direct proof was given later in [42, Proposition 4.5].

In terms of a universal contraction operator, von Neumann's Inequality can be rephrased as follows.

Theorem 1.0.2 (von Neumann's Inequality)**.** *Let x be a universal contraction operator on some Hilbert space, and let $p \in \mathbb{C}[z]$. Then,*

$$\begin{aligned}\|p(x)\| &= \sup_{|z|\le 1} |p(z)| \\ &= \sup\{\|\pi(p(x))\| \mid \pi : C^*(x) \to \mathbb{C}\} \\ &= \max\{\|\pi(p(x))\| \mid \pi : C^*(x) \to \mathbb{C}\}.\end{aligned}$$

This theorem fails when p is replaced with a noncommutative $*$-polynomial. For example, $q(z) = zz^* - z^*z$ will vanish on $\mathbb{C}$, whereas its maximum norm is witnessed by an operator on $\mathbb{C}^2$. Though one-dimensional contraction operators are insufficient to determine the maximal norm of $q(T)$ for any noncommutative $*$-polynomial q, one can hope it will suffice to use only finite-dimensional contractions. Standard power dilation arguments are not the way to go here because we would actually need $*$-monomial dilations. (Though we will see at the end of Section 5.3 that "asymptotic" dilations can still produce some results.) We circumvent the need for any dilation argument by using finite-dimensional approximations of universal C^*-algebras.

If x is a universal contraction and q is a noncommutative $*$-polynomial, then for any contractive Hilbert space operator T, $\|q(T)\| \le \|q(x)\|$. Moreover, from

[36, Theorem 5.1] (see Example 1.0.1) or more generally from [67, Theorem 4.1], we know that $C_u^*\langle x : \|x\| \leq 1\rangle$ is RFD, which means that, for any noncommutative $*$-polynomial q,

$$\begin{aligned}\|q(x)\| &= \sup\{\|\pi(q(x))\| \mid \pi : C_u^*\langle x : \|x\| \leq 1\rangle \to B(\mathcal{H}), \dim(\mathcal{H}) < \infty\} \\ &= \sup\{\|q(\pi(x))\| \mid \pi : C_u^*\langle x : \|x\| \leq 1\rangle \to B(\mathcal{H}), \dim(\mathcal{H}) < \infty\} \\ &= \sup\{\|q(a)\| \mid a \in \mathbb{M}_n(\mathbb{C}), n \geq 1, \|a\| \leq 1\}\end{aligned}$$

Thus, we arrive at the following noncommutative $*$-polynomial analogue for von Neumann's inequalituy.

Proposition 1.0.3. *Let T be contractive Hilbert space operator and q any noncommutative $*$-polynomial. Then,*

$$\|q(T)\| \leq \sup\{\|q(a)\| : a \in \mathbb{M}_n(\mathbb{C}), n \geq 1, \|a\| \leq 1\}.$$

We shall see in Section 5.2 that the supremum given in Proposition 1.0.3 is actually a maximum for any noncommutative $*$-polynomial q. In fact, if $\deg(q) \leq \ell$, then Theorem 5.2.1 says that, for any contractive Hilbert space operator T,

$$\|q(T)\| \leq \sup\{\|q(a)\| : a \in \mathbb{M}_{2^\ell}(\mathbb{C}), \|a\| \leq 1\}.$$

Even without placing restrictions on our $*$-polynomials, we can still sharpen Proposition 1.0.3. Using a result of Herrero ([44, Corollary 4.8]), one can show that the maximal norm of $q(T)$, where q is any $*$-polynomial and T any contractive Hilbert space operator, can be determined by fixing a $\lambda \in \mathbb{D}$ and considering only matrices in the family

$$\begin{aligned} V_\lambda &= \bigcup_n \{a \in \mathbb{M}_n : \ \|a\| \le 1, (a-\lambda)^n = 0\} \\ &= \bigcup_n \{a \in \mathbb{M}_n : \ \|a\| \le 1, \sigma(a) = \{\lambda\}\}. \end{aligned}$$

In the special case where $\lambda = 0$, this says that, to determine the maximum norm of $q(T)$ as T ranges over contractive Hilbert space operators, it suffices to consider only contractive *nilpotent* matrices.

In Section 5.3, we provide a new proof of this sharpening as a corollary to Theorem 5.3.6, which states that any faithful, nondegenerate representation of $C^*_u\langle x : \|x\| \le 1\rangle$ on a separable, infinite-dimensional Hilbert space asymptotically factors through a family of universal algebras, each generated by a contraction satisfying a certain polynomial relation, such as $z^n = 0$.

Two crucial components of the proof of Theorem 5.3.6 are [59, Theorem 10] and [5, Theorem 2.7]. In [59, Theorem 10] Loring and Shulman prove that, for $C > 0$ and $r \in \mathbb{C}[z]$ a polynomial, the universal C^*-algebra of the relation $r(x) = 0$, $\|x\| \le C$ is RFD. In [5, Theorem 2.7] Apostol, Foias, and Voiculescu solve Halmos's (reformulated) seventh problem by characterizing the norm closure of nilpotent operators on a separable Hilbert space.

In Chapter 6, we explore an analogue to a two-variable von Neumann inequality that arises as a corollary to Andô's dilation theorem.

Corollary 1.0.4 ([4])**.** *Let T_1 and T_2 be commuting contractions on a Hilbert space, and let $p \in \mathbb{C}[z_1, z_2]$ be a polynomial in two variables. Then*

$$\|p(T_1, T_2)\| \le \sup_{z_1, z_2 \in \overline{\mathbb{D}}} |p(z_1, z_2)|.$$

Given that we have such an extension of von Neumann's inequality to polynomials in two variables, could we likewise extend Proposition 1.0.3 or even Corollary 5.3.10 to noncommutative $*$-polynomials in two variables? Two potential analogies arise. The first involves $*$-polynomials in two variables and pairs of commuting contractions, and the other involves $*$-polynomials in two variables and pairs of **doubly commuting** contractions. We will call a pair of contractions y_1, y_2 in a self-adjoint algebra doubly commuting (d.c.) if

$$y_1 y_2 = y_2 y_1 \text{ and } y_1 y_2^* = y_2^* y_1.$$

Note that the later criteria also forces $y_1^* y_2 = y_2 y_1^*$. If y_1 and y_2 are operators on the same Hilbert space, then they are d.c. if $C^*(y_1) \subset C^*(y_2)'$, but neither C^*-algebra is assumed to be abelian.

Andô's extension of von Neumann's inequality famously fails for more than two commuting contractions ([84]), but it does hold for any finite number of d.c. contractions ([83]). Hence, we focus for now on the d.c. contraction analogy to Corollary 1.0.4. Question 6.2.1 asks, given a noncommutative $*$-polynomial q in two variables, can the maximal norm of the operator $q(T_1, T_2)$, as (T_1, T_2) ranges over pairs of d.c. contractive Hilbert space operators, be determined by considering only finite-dimensional d.c. contractions? Question 6.2.2 asks, if so, could we hope to mimic Corollary 5.3.10 and show that, to bound the norm of $\|q(T_1, T_2)\|$, it actually suffices to consider only finite-dimensional d.c. contractions that satisfy certain spectral criteria such as nilpotency? In Lemma 6.2.6, we show that these two questions are equivalent.

We remark that the latter question is formally analogous to [1, Theorem 3.1], which states that, for pairs of commuting matrices of norm strictly less than 1, Corollary 1.0.4 can be sharpened to looking at the maximum modulus on a (distinguished) variety, which is contained in the bidisk.[2] Indeed, if we think of each V_λ as a "matrix variety," then Corollary 5.3.10, says that for any noncommutative *-polynomial q and any contractive Hilbert space operator T

$$\|q(T)\| \leq \|q\|_{V_\lambda}$$

So, a positive answer to Question 6.2.2 would imply that the same holds if we replace q with a $*$-polynomial in two variables, T with a pair of d.c. contractive Hilbert space operators, and $(z - \lambda)$ with a pair of polynomials.

However, it turns out that Questions 6.2.1 and 6.2.2 are both equivalent to a famous open problem in operator algebras, which was first posed by Alain Connes in [17] and is often dubbed Connes' Embedding Problem/Conjecture/Question. In ensuing years, Connes' embedding conjecture has been proven equivalent to a variety of other important conjectures from many subfields of operator algebras, noncommutative geometry, and quantum information theory. In particular, it is equivalent to Kirchberg's QWEP conjecture [51] and essentially equivalent to Tsirelson's problem [47].

The equivalence of 6.2.1 and 6.2.2 with Connes' Embedding Problem can be proved as a corollary to a result of Pisier ([68, Proposition 16.13]), who has remarked that contents of this proposition were likely already known to Kirchberg. We provide a different proof of this fact in Lemma 6.2.7, which involves showing that $C^*(\mathbb{F}_2)$ embeds relatively weakly injectively into $C^*_u\langle x : \|x\| \leq 1\rangle$; that is, there are a unital embedding of $C^*(\mathbb{F}_2)$ into $C^*_u\langle x : \|x\| \leq 1\rangle$ and a ucp map (actually $*$-homomorphism) $C^*_u\langle x : \|x\| \leq 1\rangle \to C^*(\mathbb{F}_2)^{**}$ that restricts to the natural inclusion of $C^*(\mathbb{F}_2)$ into $C^*(\mathbb{F}_2)^{**}$.

[2] This has been extended in some cases to more general pairs – or tuples – of commuting contractive Hilbert space operators, e.g. [69], [70], and [1].

Chapter 2

Preliminaries

Here we formally introduce a few terms and concepts that will be used throughout. Those terms and concepts whose use and mention are confined to a chapter will be introduced in the relevant chapter.

2.1 C^*-tensor norms

Let A, B, and C be C^*-algebras and $\mathcal{H}$ a Hilbert space.

We denote an algebraic tensor product (with natural involution) by

$$A \odot B = \left\{ \sum_{i=1}^{n} a_i \otimes b_i : a_i \in A, b_i \in B, \text{ for } 1 \le i \le n, n \in \mathbb{N} \right\}.$$

To make this a C^*-algebra, we complete it with respect to a C^*-norm $\| \cdot \|_\alpha$ (which will actually be a cross norm[1]). We denote the C^*-algebra $\overline{A \odot B}^{\| \cdot \|_\alpha} = A \otimes_\alpha B$.

However, we can define multiple C^*-norms $\| \cdot \|_\alpha$ on $A \odot B$, which are often distinct. (We even have a name for C^*-algebras whose algebraic tensor product with any other algebra has a unique C^*-norm. Such algebras are called **nuclear**.) Two of these norms will be of primary interest to us.

[1] $\| \cdot \|_\alpha$ is a cross norm if $\|a \otimes b\|_\alpha = \|a\|_A \|b\|_B$ for any $a \in A$ and $b \in B$.

Definition 2.1.1 (Tensor Product Norms)**.** For $x = \sum_{i=1}^{n} a_i \otimes b_i \in A \odot B$ (the algebraic tensor product)

- $\|x\|_{max} = \sup\{\|\pi(x)\| : \pi : A \odot B \to B(\mathcal{H})$ is an irrep $\}$
- $\|x\|_{min} = \| \sum_{i=1}^{n} \pi_A(a_i) \otimes \pi_B(b_i)\|_{B(\mathcal{H}_A \otimes \mathcal{H}_B)}$

  where $(\pi_A, \mathcal{H}_A)$ and $(\pi_B, \mathcal{H}_B)$ are *any* faithful reps of A and B, resp.

The completion $A \otimes_{max} B = \overline{A \odot B}^{\|\cdot\|_{\max}}$ is often called the **projective C^*-tensor product**; the completion $A \otimes_{min} B = \overline{A \odot B}^{\|\cdot\|_{\min}}$ is often called the **injective C^*-tensor product**.

Theorem 2.1.2 (Universal Property of $\otimes_{max}$)**.** *For any C^*-algebra C, any *-homomorphism $A \odot B \to C$ extends uniquely to a *-homomorphism $A \otimes_{max} B \to C$. In particular, if $\pi_A : A \to C$ and $\pi_B : B \to C$ are *-homomorphisms with commuting ranges, then the map $\pi_A \times \pi_B$ extends uniquely to $A \otimes_{max} B$.*

See [15, Chapter 3] for a proof.

Thanks to the universality of $\|\cdot\|_{\max}$ and a theorem due to Takesaki (see [76] or [15]), we know these are truly norms and that for any $x \in A \odot B$ and any C^*-norm $\|\cdot\|_\alpha$ on $A \odot B$,

$$\|x\|_{\min} \leq \|x\|_\alpha \leq \|x\|_{\max}.$$

More formally, we have natural surjective *-homomorphisms

$$A \otimes_{max} B \to A \otimes_{\alpha} B \to A \otimes_{min} B.$$

Consequently, to prove that $A \odot B$ has a unique C^*-norm, it suffices to show that this surjection is injective. This is often indicated by saying $A \otimes_{max} B = A \otimes_{min} B$ without indicating the isomorphism.

2.2 Universal C^*-algebras

In this section, we will define a universal C^*-algebra $C^*\langle \mathfrak{G} : \mathcal{R} \rangle$ on a set of generators $\mathfrak{G}$ subject to relations $\mathcal{R}$, which are conditions on the generators, such as norm restrictions or algebraic relations. We will need a little more discussion before we can say formally what we will take as relations. Our definitions and constructions follow those from [57] and [12].

A **representation** of $(\mathfrak{G}, \mathcal{R})$ is a set of operators $\{y_n\}_{n \in I}$ on a Hilbert space $\mathcal{H}$ that satisfy $\mathcal{R}$. Let $Alg(\mathfrak{G})$ denote the free $*$-algebra on $\mathfrak{G}$. Any representation of $(\mathfrak{G}, \mathcal{R})$ on $\mathcal{H}$ extends uniquely to a $*$-homomorphism $Alg(\mathfrak{G}) \to B(\mathcal{H})$. As in [12] (and [57]), we will call a pair $(\mathfrak{G}, \mathcal{R})$ admissible if the following hold.[2]

1. There exists a representation of $(\mathfrak{G}, \mathcal{R})$;

2. $\mathcal{R}$ is preserved under $*$-homomorphisms; and

3. The relations $\mathcal{R}$ are preserved by direct sums, i.e. if $\{y_n^{(\alpha)}\}_{n \in I} \subseteq B(\mathcal{H}_\alpha)$ satisfy $\mathcal{R}$ for all $\alpha \in J$, then so do $\{\oplus_\alpha y_n^{(\alpha)}\}_{n \in I}$.

4. The value
$$\|a\|_u = \sup\{\phi(a) : \phi \text{ a rep of } (\mathfrak{G}, \mathcal{R})\}$$
is finite for any $a \in Alg(\mathfrak{G})$.

[2] This definition is an amalgamation of Blackadar's definition in [12] and Loring's [57, Theorem 3.1.1].

When $(\mathfrak{G}, \mathcal{R})$ is admissible, $\|\cdot\|_u$ gives us a C^*-norm on $Alg(\mathfrak{G})$. The completion of $Alg(\mathfrak{G})/\|\cdot\|_u$ under $\|\cdot\|_u$ is the universal C^*-algebra generated by $\mathfrak{G}$ subject to $\mathcal{R}$, which we will denote by

$$C^*\langle \mathfrak{G} : \mathcal{R}\rangle.$$

It has the universal property that, given any representation $\{y_n\}_{n\in I}$ of $(\mathfrak{G}, \mathcal{R})$ on some Hilbert space, there is a surjective $*$-homomorphism

$$C^*\langle \mathfrak{G} : \mathcal{R}\rangle \to C^*(\{y_n\}_{n\in I})$$

sending $x_n \mapsto y_n$ for each $n \in I$.

We will restrict ourselves henceforth to finite generating sets $\mathfrak{G} = \{x_n\}_{n\leq N}$ and to relations of the forms

- $\|x_n\| \leq c_n$ (for some $c_n > 0$) and
- $q(x_{i_1}, ..., x_{i_k}) = 0$ for some $*$-polynomial q and $x_{i_1}, ..., x_{i_k} \in \mathfrak{G}$.

. With these restrictions, we will be guaranteed that $(\mathfrak{G}, \mathcal{R})$ is admissible as long as $\mathcal{R}$ enforces a norm bound on each of the generators (for each $x_n \in \mathfrak{G}$, $\exists\ c_n > 0$ such that $\|x_n\| \leq c_n$) and as long as some representation of $(\mathfrak{G}, \mathcal{R})$ exists.

In general $C^*\langle \mathfrak{G}, \mathcal{R}\rangle$ is not assumed to be unital. To construct a universal unital C^*-algebra, one simply adds a unit variable to $\mathfrak{G}$ and the appropriate relations to $\mathcal{R}$ (and assume all representations are nondegenerate, i.e. the unit in $\mathfrak{G}$ maps to the unit in $B(\mathcal{H})$). The resulting algebra is universal in the category of unital C^*-algebras with unital $*$-homomorphisms and is denoted

$$C^*_u\langle \mathfrak{G} : \mathcal{R}\rangle.$$

Because our theorems often need units, we will often restrict ourselves to universal unital C^*-algebras.

Example 2.2.1. 1. In the introduction we saw the universal unital C^*-algebra generated by a contraction, i.e. $C_u^*\langle x : \|x\| \leq 1\rangle$. Note that $C^*\langle x : \|x\| \leq 1\rangle$ is not assumed to be unital (and, in fact, cannot be unital by Remark 2.3.2).

2. Let $r \in \mathbb{C}[z]$ be a polynomial with at least one root in $\overline{\mathbb{D}}$. We denote the universal unital C^*-algebra of the relations $\|x\| \leq 1$ and $r(x) = 0$ by

$$C_u^*\langle x : \|x\| \leq 1,\ r(x) = 0\rangle.$$

This C^*-algebra is identified by the universal property that, for any operator y on some Hilbert space $\mathcal{H}$ such that $\|y\| \leq 1$ and $r(y) = 0$, the map sending x to y induces a surjective unital $*$-homomorphism from $C_u^*\langle x : \|x\| \leq 1, r(x) = 0\rangle$ onto $C^*(y, 1_{\mathcal{H}})$. (For a polynomial $r \in \mathbb{C}[z]$, and an operator y on some Hilbert space $\mathcal{H}$, we interpret the constant term of $r(y)$ as the corresponding scalar multiples of the identity $1_{\mathcal{H}}$.)

Universal unital C^*-algebras are considered universal in the category of unital C^*-algebras with unital $*$-homomorphisms. In this setting, one should take care to guarantee that the algebra has any representations at all. (If the C^*-algebra is non-unital, there's always the zero representation.) Requiring that our polynomial r has at least one root in $\overline{\mathbb{D}}$ guarantees that this universal C^*-algebra has at least one representation.

3. We will also be interested in universal C^*-algebras with more than one generator. For instance, the universal C^*-algebra generated by a pair of doubly commuting

contractions is denoted

$$C_u^*\langle x_1, x_2 : \|x_i\| \leq 1, [x_1, x_2] = 0, [x_1, x_2^*] = 0\rangle.$$

By the universal property of the maximal C^*-tensor product, this is isomorphic to

$$C_u^*\langle x : \|x\| \leq 1\rangle \otimes_{max} C_u^*\langle x : \|x\| \leq 1\rangle$$

via the map that sends generators to generators.

4. Given a discrete group G, we define involution on the group algebra $\mathbb{C}G$ by defining $g^* = g^{-1}$ for all $g \in G$ and extending linearly. Any unitary representation of G in the unitary group on some Hilbert space $\mathcal{H}$ extends uniquely to a $*$-homomorphism from $\mathbb{C}G$ to $B(\mathcal{H})$. With this, we can define the universal C^*-norm on $\mathbb{C}G$ (sometimes called a universal unitary norm) by

$$\|a\|_u = \sup\{\|\phi(a)\| : \phi|_G \text{ a unitary representation of } G\}$$

for each $a \in \mathbb{C}G$. The closure of the involutive algebra $\mathbb{C}G$ with respect to this norm is called the **full group C^*-algebra of G.**[3]

If G has presentation $\langle \mathfrak{G}|\mathcal{R}\rangle$, then

$$C^*\langle \mathfrak{G}|\mathcal{R}\rangle = C^*(G).$$

In particular, the full group C^*-algebra for a free group $\mathbb{F}_n$ on n generators is

[3]The other commonly used group C^*-algebra is the reduced group C^*-algebra, $C_r^*(G)$ or $C_\lambda^*(G)$. Here norm is inherited from the left regular representation. The two C^*-algebras coincide if and only if the group is amenable.

the universal C^*-algebra generated by n unitaries.

All of these universal C^*-algebras exist by the preceding abstract nonsense.

Remark 2.2.2. The universal (nonunital) C^*-algebra generated by two doubly commuting contractions is not isomorphic to the maximum tensor product of two copies of the universal (nonunital) contraction algebra. It will (isomorphically) contain the maximum tensor product as an ideal. (See [12, Example 1.3 (g)].) To see this, let $C^*(y_1)$ and $C^*(y_2)$ be two copies of the universal contraction algebra, and let $\pi : C^*(y_1) \otimes_{max} C^*(y_2) \to B(\mathcal{H})$. Let $\pi_i : C^*(y_i) \to B(\mathcal{H})$ be nondegenerate faithful representations for $i = 1, 2$ with commuting ranges such that $\pi = \pi_1 \times \pi_2$.[4] This gives us

$$\overline{\operatorname{span}\{\pi(a_1)\pi(a_2) : a_i \in C^*(y_i)\}} = \pi(C^*(y_1) \otimes_{max} C^*(y_2)).$$

On the other hand, we have a canonical $*$-homomorphism

$$\sigma : C^*\langle x_1, x_2 : x_i \text{ doubly comm cont}\rangle \to B(\mathcal{H})$$

induced by $x_i \mapsto \pi_i(y_i)$. But

$$\overline{\operatorname{span}\{\pi(a_1)\pi(a_2) : a_i \in C^*(y_i)\}} \lhd C^*(\pi_1(y_1), \pi_2(y_2))$$

and is a proper ideal unless it contains $\pi_1(y_1)$ and $\pi_2(y_2)$, which is not necessarily the case.

[4]These are non-trivial to construct in the non-unital case. See [15, Theorem 3.2.6] for a proof.

2.3 Projective C^*-algebras

We say a C^*-algebra A is **projective** ([12], [57]) if it is a projective object in the category of C^*-algebras with $*$-homomorphisms. In other words, given C^*-algebras B and C with surjective $*$-homomorphism $q : B \to C$, any $*$-homomorphism $\phi : A \to C$ lifts to a $*$-homomorphism $\psi : A \to B$ such that $q \circ \psi = \phi$, and we have the following commutative diagram.

$$\begin{array}{ccc} & & B \\ & \overset{\psi}{\nearrow} & \downarrow q \\ A & \xrightarrow[\phi]{} & C \end{array}$$

By [12, Proposition 2.5], a C^*-algebra is projective iff its unitization is projective in the category of unital C^*-algebras with unital $*$-homomorphisms.

Proposition 2.3.1. *$C^*\langle x : \|x\| \leq 1\rangle$ is projective.*

This is a consequence of the fact that any contraction in a C^*-quotient lifts to a contraction, and this implies that $C^*_u\langle x : \|x\| \leq 1\rangle$ is projective as a unital C^*-algebra.

Remark 2.3.2. By [12, Proposition 2.3], a projective C^*-algebra is contractible, which means it cannot be unital. In particular, $C^*\langle x : \|x\| \leq 1\rangle$ is not unital.

Corollary 2.3.3. *[36, Theorem 5.1] $C^*_u\langle x : \|x\| \leq 1\rangle$ is RFD.*

This corollary has a multitude of proofs. Given the faithful representation $C^*(S)$ constructed in [36, Theorem 5.1] (see Example 1.0.1), one has the representations already laid out. Alternatively, identifying $C^*_u\langle x : \|x\| \leq 1\rangle$ with $C^*_u\langle \mathbb{C}\rangle$, it follows from [67, Theorem 4.1]. It also follows from the fact that any (separable) projective C^*-algebra is RFD (see [57, Theorem 11.2.1] or [37]) and the fact that any C^*-algebra is RFD iff its unitization is also RFD. Moreover, it can be proved by adapting arguments from the proof of [16, Theorem 7]. Surely this list is not exhaustive.

2.4 AF C^*-algebras and their mapping telescopes

In this section, we briefly introduce AF C^*-algebras and AF mapping telescopes (also called AF-telescopes), limiting ourselves to properties and descriptions relevant to the work at hand. For a more thorough introduction of AF C^*-algebras, the reader is referred to any number of standard texts in C^*-algebras such as [11], [10], [56]. For a more detailed introduction to AF mapping telescopes, see [57] or [58].

An **Approximately Finite-dimensional** C^*-algebra or **AF algebra** is a C^*-algebra which is (isomorphic to) the inductive limit of a sequence of finite-dimensional C^*-algebras. We shall take a slightly less standard (albeit equivalent) definition. For us, an AF algebra is a C^*-algebra A with a nested squence of finite-dimensional C^*-subalgebras

$$A_1 \subset A_2 \subset A_3 \subset ... \subset A$$

whose union is dense in A. This is the same as saying that A is the direct limit of these subalgebras where the connecting maps are simply embeddings.

Two important examples of AF algebras are $\mathbb{M}_{2^\infty}$, often called the **CAR algebra**, and $K(\mathcal{H})$, the compact operators on a separable Hilbert space (often assumed to be ℓ^2 for the sake of arrays). For $\mathbb{M}_{2^\infty}$, the associated sequence of finite-dimensional C^*-subalgebras is assumed to be

$$\mathbb{C} \subset \mathbb{M}_2 \subset \mathbb{M}_4 \subset ... \subset \mathbb{M}_{2^n} \subset ... \subset \mathbb{M}_{2^\infty},$$

where $\mathbb{M}_{2^n}$ is identified with a subalgebra of $\mathbb{M}_{2^{n+1}}$ by the map $a \mapsto a \oplus a$. The sequence of finite-dimensional C^*-subalgebras of $K(\ell^2)$ is assumed to be

$$\mathbb{C} \subset \mathbb{M}_2 \subset \mathbb{M}_3 \subset ... \subset \mathbb{M}_n \subset ... \subset K(\ell^2),$$

where $\mathbb{M}_n$ is identified with a subalgebra of $\mathbb{M}_{n+1}$ via $a \mapsto a \oplus 0$ and $K(\ell^2)$ is identified with the direct limit of this sequence.

Now, let $B = \overline{\bigcup B_n}$ be an inductive limit of an increasing sequence of C^*-algebras

$$B_1 \subset B_2 \subset \ldots \subset B$$

with injective connecting maps. The **mapping telescope** of (B_n) is the C^*-subalgebra of the mapping cone of B, $CB = C_0(0, \infty] \otimes B = C_0((0, \infty], B)$, defined by

$$T(B) = \{f \in C_0((0, \infty], B) | f(t) \in B_{\lceil t \rceil} \ \forall t \in (0, \infty)\}$$

where $\lceil t \rceil = \min\{n \in \mathbb{N} : n \geq t\}$. The following visual highlights the intuition behind the name.

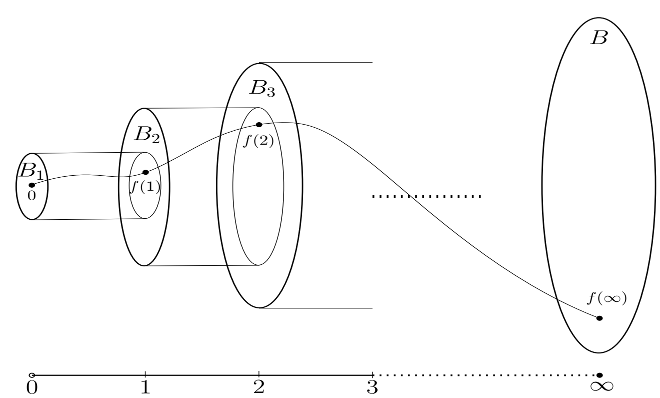


Figure 2.1: Mapping Telescope

When each B_n is finite-dimensional, we call $T(B)$ an **AF-telescope**. Obviously the mapping telescope depends on the sequence (B_n), but we will use the notation $T(B)$ as opposed to $T(B_1, B_2, ...)$ and specify the inductive sequence when necessary. In particular, we denote the mapping telescope corresponding to the inductive sequence $(\mathbb{M}_{2^n})_n$ by $T(\mathbb{M}_{2^\infty})$ and the mapping telescope corresponding to the sequence

$(\mathbb{M}_n)_n$ by $T(K(\mathcal{H}))$.

For the sake of consistency, we define the AF-telescope for inductive sequences of the form $\mathbb{M}_{n_1} \subset \mathbb{M}_{n_2} \subset \ldots \subset \mathbb{M}_{n_N}$, as

$$T(\mathbb{M}_{n_N}) := \{f \in C_0((0,\infty], \mathbb{M}_{n_N}) | f(t) \in \mathbb{M}_{n_{\lceil t \rceil}} \ \forall t \in (0, n_N]\}.$$

In [58], Loring and Pedersen proved that all AF-telescopes are projective. This fact will be used repeatedly throughout Chapters 3 and 4.

2.5 Type I and GCR C^*-algebras

In Chapters 3 and 4, we rely on a result (Theorem 2.5.1) of Glimm and Sakai. The particular formulation we would like to cite is not so readily found in the literature, so we briefly describe it here.

Let $\mathcal{H}$ be a Hilbert space. A C^*-algebra A is called **GCR** if $K(\mathcal{H}) \subseteq \pi(A)$ for any irreducible representation $(\pi, \mathcal{H})$ of A. In particular, all FDI C^*-algebras are GCR. It is due to a deep theorem of Glimm and Sakai that a C^*-algebra is GCR iff it is type I, i.e. the double commutant of any nondegenerate representation is a type I von Neumann algebra (see [28] for the classic theorem and [72] for the nonseparable case). We will call all such algebras GCR.

A C^*-algebra is NGCR (antiliminal) if it contains no nonzero abelian elements, i.e. there is no nonzero $x \in A$ so that $\overline{x^*Ax}$ is commutative. Glimm ([28]) and Sakai ([71]) have shown that an NGCR C^*-algebra must have a subquotient isomorphic to the CAR algebra, i.e. it has a subalgebra that surjects onto the CAR algebra. Since a GCR C^*-algebra is characterized as having no NGCR quotients (see [10, Section IV.1.3]), we arrive at the following formulation of the result.

Theorem 2.5.1 ([28], [71]). *Let A be a C^*-algebra that is not GCR. Then A has a subquotient isomorphic to the CAR algebra.*

2.6 The Chinese Remainder Theorem

In Sections 3.3 and 4.1 we will use the Chinese Remainder Theorem for C^*-algebras:

Theorem 2.6.1 (Chinese Remainder Theorem). *Let A be a C^*-algebra and $I_1, \ldots, I_k$ be closed two-sided ideals in A such that $I_i + I_j = A$, when $i \neq j$. Then the map*

$$\phi : a + \bigcap_{i=1}^{k} I_i \mapsto (a + I_1, \ldots, a + I_k)$$

gives a $*$-isomorphism from $A/(\bigcap_{i=1}^{k} I_i)$ to $A/I_1 \oplus \ldots \oplus A/I_k$.

In particular, if A is a C^*-algebra, and $q_1, ..., q_n$ are $*$-homomorphisms mapping A onto simple C^*-algebras $C_1, ..., C_n$, then $q = \oplus_{i=1}^{n} q_i : A \to \bigoplus_{i=1}^{n} C_i$ is a surjective $*$-homomorphism. Indeed, since each C_i is simple, each $I_i := \ker(q_i)$ is a maximal ideal, and so $I_i + I_j = A$, for each $i \neq j$. Since $\ker(\oplus_{i=1}^{n} q_i) = \bigcap_{i=1}^{N} I_i$, we have $\left(\oplus_{i=1}^{N} q_i\right)(A) \simeq A/(\bigcap_{i=1}^{n} I_i)$ and hence, by the Chinese Reminder Theorem,

$$\left(\oplus_{i=1}^{N} q_i\right)(A) = \bigoplus_{i=1}^{n} q_i(A) = \bigoplus_{i=1}^{n} C_i.$$

Chapter 3

Elements of C^*-algebras attaining their norm in a finite-dimensional representation

Loosely speaking, a C^*-algebra is RFD if its finite-dimensional irreducible representations are dense among its irreducible representations. In this chapter, we show that a C^*-algebra is RFD if and only if the subset of elements that attain their norm under a finite-dimensional representation is dense in the algebra. Moreover, we shall see that every irreducible representation is finite-dimensional if and only if every element attains its norm under a finite-dimensional representation. However, we do not prove this equivalence directly. Our proof goes through another equivalent characterization in terms of AF subquotients (Theorem 3.2.4). To prove this last characterization, we develop a technique using AF mapping telescopes that we later use to describe the algebraic properties of the aforementioned set of elements in a C^*-algebra that attain their norm under a finite-dimensional representation (Theorem 3.3.2).

This chapter is taken from joint work with Tatiana Shulman in [18].

3.1 A characterization of RFD C^*-algebras

In this section, we characterize RFD C^*-algebras as being exactly those which have a dense subset of elements that attain their norm under a finite-dimensional representation. In fact, we prove that, for any residually class $\mathcal{C}$ C^*-algebra (i.e. an algebra with a separating family of representations which are class $\mathcal{C}$) the set of elements that attain their norm under a class $\mathcal{C}$ representations is dense.

First, we give a well-known characterization for a family of representations to be separating.

Lemma 3.1.1. *Let A be a C^*-algebra and $\mathcal{F}$ be a separating family of its representations. Then for each $a \in A$,*

$$\|a\| = \sup_{\pi \in \mathcal{F}} \|\pi(a)\|.$$

Proof. Since $\mathcal{F}$ is separating, the representation $a \mapsto \oplus_{\pi \in \mathcal{F}} \pi(a)$ is injective. It follows from spectral theory for normal operators on C^*-algebras that any injective $*$-homomorphism on a C^*-algebra is isometric. □

Theorem 3.1.2. *Let A be a C^*-algebra, $\mathcal{F}$ a family of representations of A, and define*

$$A_{\mathcal{F}} := \{a \in A \mid \|a\| = \max_{\pi \in \mathcal{F}} \|\pi(a)\|\}.$$

Then the following are equivalent:

(i) $A_{\mathcal{F}}$ is dense in A.

(ii) $\mathcal{F}$ is a separating family of representations of A.

In the proof, we use a trick with polar decomposition, which is folklore nowadays, but was first done in [2].

Proof. If we assume (i), then for any $a \in A\backslash\{0\}$, we can choose $b \in A_{\mathcal{F}}\backslash\{0\}$ such that $\|a-b\| < \frac{1}{4}\|a\|$ and $\pi \in \mathcal{F}$ so that $\|\pi(b)\| = \|b\|$. Then,

$$\|a\| - \|\pi(a)\| = |\|a\| - \|b\| + \|\pi(b)\| - \|\pi(a)\|| \leq \|a-b\| + \|\pi(b-a)\| < \frac{1}{2}\|a\|.$$

Hence, $0 < \frac{1}{2}\|a\| < \|\pi(a)\|$, i.e. $\mathcal{F}$ is a separating family of representations.

Now, assume (ii), and let $a \in A\backslash\{0\}$ and $\epsilon > 0$. By Lemma 3.1.1 there exists $\pi \in \mathcal{F}$ such that $\|a\| \leq \|\pi(a)\| + \epsilon$. Embed $\tilde{A}$ into $B(\mathcal{H})$ for some $\mathcal{H}$, where $\tilde{A}$ is the unitization of A, and let

$$a = u|a|$$

be the polar decomposition of a in $B(\mathcal{H})$. Define a function $f : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$f(t) = \begin{cases} t & ; \ t \in [0, \|\pi(a)\|] \\ \\ \|\pi(a)\| & ; \ t \in (\|\pi(a)\|, \infty) \end{cases}$$

Let $b = uf(|a|)$. We claim that $b \in A_{\mathcal{F}}$ and $\|b-a\| < \epsilon$. First, note that $b \in A$. Indeed, $f(t) = tg(t)$ where

$$g(t) = \begin{cases} 1 & ; \ t \in [0, \|\pi(a)\|] \\ \\ \dfrac{\|\pi(a)\|}{t} & ; \ t \in (\|\pi(a)\|, \infty) \end{cases}$$

Then g is continuous on $[0, \infty)$, and $g(|a|) \in \tilde{A}$. Hence $b = uf(|a|) = ag(|a|) \in A$ since A is an ideal in $\tilde{A}$.

To show that $b \in A_{\mathcal{F}}$, it will suffice to show that $\|b\| \leq \|\pi(b)\|$. If A is non-unital, let π' denote the unique unital extension of π to $\tilde{A}$, and if A is unital, let $\pi' = \pi$. Then, since $g(t) = 1$ when $t \in [0, \|\pi(a)\|]$, we have that $\pi'(g(|a|)) = g(\pi(|a|)) = 1$ in $\pi'(\tilde{A})$ and hence

$$\pi(b) = \pi(ag(|a|)) = \pi(a)g(\pi(|a|)) = \pi(a).$$

This gives us that

$$\|b\| \leq \|f(|a|)\| = \sup_{t \in \sigma(|a|)} |f(t)| \leq \|\pi(a)\| = \|\pi(b)\|,$$

Finally,

$$\begin{aligned}
\|a - b\| &= \|u|a| - uf(|a|)\| \\
&\leq \||a| - f(|a|)\| \\
&= \sup_{t \in \sigma(|a|)} |t - f(t)| \\
&\leq \|a\| - \|\pi(a)\| \leq \epsilon.
\end{aligned}$$

Hence, $A = \overline{A_{\mathcal{F}}}$. □

Corollary 3.1.3. *The following are equivalent for a C^*-algebra A:*

(i) The set $\{a \in A : \|a\| = \max_{\substack{\pi \in Irr_n(A) \\ n < \infty}} \|\pi(a)\|\}$ is dense in A.

(ii) A is RFD.

A natural question now is how to characterize the class of C^*-algebras for which every element attains its norm under some finite-dimensional representation. For example, is this true for $C^*(\mathbb{F}_n)$?

It turns out that the answer is "no" for any C^*-algebra that has an infinite-dimensional irreducible representation, including $C^*(\mathbb{F}_n)$. We will address this in the next section.

3.2 A characterization of FDI C^*-algebras

We begin with a key lemma that is intuitively clear and must be known to specialists.

Lemma 3.2.1. *Let $T(B)$ be an AF-telescope with associated inductive sequence (B_n). Then any irreducible representation $(\pi, \mathcal{H})$ of $T(B)$ factorizes through a point evaluation ev_t, for some $t \in (0, \infty]$. Moreover, when B_n are all simple and distinct, if $\dim \mathcal{H} \leq rank(B_n)$ for some n, then $t \leq n$.*

Proof. Let π be an irreducible representation of $T(B)$. Put

$$I = \{f \in T(B) | f(\infty) = 0\}. \tag{3.2.1}$$

Note that I is a closed ideal in $T(B)$ and so $\pi|_I$ is either irreducible or zero. If it is zero, then π factorizes through $T(B)/I \simeq B$ and hence through ev_∞. So we assume now that $\pi|_I$ is non-zero and irreducible. For each $n \geq 1$, define the closed ideal $I_n \triangleleft I$ by

$$I_n := \{f \in I | f(t) = 0 \ \forall \ t \geq n\}$$

Then (I_n) is a nested sequence of closed, two-sided ideals with $I = \overline{\bigcup_n I_n}$. Thus there must exist n such that $\pi|_{I_n}$ is non-zero and therefore irreducible. Let

$$\tilde{I}_n := \{f \in T(B)|\ f(t) = f(n)\ \forall\ t \geq n\}.$$

Then I_n is an ideal in $\tilde{I}_n$, and so $\pi|_{I_n}$ extends uniquely to an irreducible representation (in fact $\pi|_{\tilde{I}_n}$) of $\tilde{I}_n$. So, it will be sufficient to prove that any irreducible representation, say ρ, of $\tilde{I}_n$ factorizes through a point evaluation.

We will prove it by induction. Since B_1 is isomorphic to a direct sum of matrix algebras, the claim for $\tilde{I}_1 \simeq C_0(0,1] \otimes B_1$ reduces to the claim for $C_0(0,1] \otimes \mathbb{M}_n$ for some n. But all irreducible representations of $C_0(0,1] \otimes \mathbb{M}_n$ are of the form $\eta \otimes id$ where η is an irrep of $C_0(0,1]$, i.e. a point evaluation. Now, assume the claim holds for $(n-1)$. Let J_n be the closed ideal in $\tilde{I}_n$ defined by

$$J_n = \{f \in \tilde{I}_n \mid f|_{(0,n-1]} = 0\} \simeq C_0((n-1,n]) \otimes B_n.$$

If ρ does not vanish on J_n, then it is irreducible on J_n and hence factorizes through a point evaluation. So we can assume ρ vanishes on J_n. Then ρ factorizes through the map $\phi : \tilde{I}_n \to \tilde{I}_n/J_n \cong \tilde{I}_{n-1}$ given by

$$\phi(f)(t) = \begin{cases} f(t) & ; \quad t \in (0, n-1] \\ \\ f(n-1) & ; \quad t \in (n-1, \infty]. \end{cases}$$

Hence ρ factorizes through a point evaluation by the induction hypothesis.

Thus, $\pi|_{I_n}$ factorizes through a point evaluation. Since an irreducible representation of an ideal extends uniquely to a representation of the whole C^*-algebra, we conclude that π factorizes through a point evaluation.

Moreover, if each B_n is simple, then any irreducible representation of $T(B)$ is equivalent to a point evaluation ev_t for some $t \in (0, \infty]$, in which case the image of the representation is isomorphic to $B_{\lceil t \rceil}$. □

Remark 3.2.2. Recall that a C^*-algebra is **n-subhomogeneous** if all of its irreducible representations are of dimension no more than $n \in \mathbb{N}$. Clearly any subhomogenous C^*-algebra is FDI, but there exist many FDI C^*-algebras that are not subhomogeneous. For instance, if B is a UHF algebra or $K(\ell^2)$, then I in (3.2.1) is not subhomogeneous.

More such examples come from group theory. In [60], Moore proves that a locally compact group has a finite bound for the dimensions of its irreducible unitary representations if and only if it has an open abelian subgroup of finite index. On the other hand, he also shows in [60] that a locally compact group has all of its irreducible unitary representations of finite dimension if and only if it is a projective limit of Lie groups with the same property; and a Lie group has this property if and only if it has an open subgroup of finite index that is compact modulo its center. Consequently, examples of FDI but non-subhomogeneous C^*-algebras include, for instance, the full group C^*-algebra of a locally compact Lie group whose irreducible representations are all finite-dimensional but which has no open abelian subgroups of finite index.

On the other hand, if G is a discrete group, Thoma shows in [79] and [80] that all irreducible unitary representations of G are finite-dimensional iff they are all of bounded finite dimension iff the group is type I iff the group is virtually abelian. In other words, for a discrete group G, the following are equivalent:

1. $C^*(G)$ is subhomogeneous

2. $C^*(G)$ is FDI

3. $C^*(G)$ is GCR

4. G is virtually abelian.

Lemma 3.2.3. *For any simple, infinite-dimensional AF-algebra B with inductive sequence (B_n), there is an element $f \in T(B)$ such that $\|\pi(f)\| < \|f\| = \|f(\infty)\|$ for any finite-dimensional representation π of $T(B)$.*

Proof. Recall that any finite-dimensional representation π of $T(B)$ is a finite direct sum of irreducible representations, $\pi_1, ...\pi_n$. Since B has no finite-dimensional representations, it follows from Lemma 3.2.1 that, for each $1 \leq k \leq n$, there exists $t_k \in (0, \infty)$ such that π_k factorizes through ev_{t_k}. Hence, for each $1 \leq k \leq n$ and each $g \in T(B)$, $\|\pi_k(g)\| \leq \|g(t_k)\|$, which implies that $\|\pi(g)\| \leq \max_{1\leq k\leq n} \|g(t_k)\|$.

Now, let $0 \neq x \in B_1 \subset B$ and define $f \in T(B)$ by $f(t) = (1 - e^{-t})x$, and let π be a finite-dimensional representation of $T(B)$. Since the $\|f(t)\|$ is a strictly increasing function, by the above argument, there exists a finite set $F \subset (0, \infty)$ such that $\|\pi(f)\| \leq \max_{t\in F} \|f(t)\| < \|f(\infty)\| = \|f\|$. □

Now we are ready to give the main theorem of this section.

Theorem 3.2.4. *The following are equivalent for any C^*-algebra A:*

(i) A is FDI.

(ii) For each $a \in A$ there exists a representation $(\pi, \mathcal{H})$ of A with dim$(\mathcal{H}) < \infty$ such that $\|a\| = \|\pi(a)\|$.

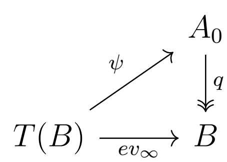
(iii) A does not have an infinite-dimensional simple AF-algebra as a subquotient.

Proof. To see that (i) implies (ii), recall that for any $a \in A$, there exists a pure state φ on A such that $|\varphi(a^*a)| = \|a^*a\|$. Applying the GNS construction to φ gives an irreducible representation π_φ and unit vector ξ_φ such that

$$\|\pi_\varphi(a)\xi_\varphi\| = \|a\|.$$

Since A is FDI, we know π_φ is finite-dimensional.

To show that (ii) implies (iii), suppose $A_0 \subseteq A$ is a C^*-subalgebra, B is a simple, infinite-dimensional AF-algebra with inductive sequence (B_n), and $q : A_0 \to B$ a surjective $*$-homomorphism. Let $T(B)$ be the mapping telescope for (B_n). Since AF-telescopes are projective ([58]), there is a $*$-homomorphism $\psi : T(B) \to A_0$ such that $q \circ \psi = ev_\infty$, i.e. the following diagram commutes.



Let $f \in T(B)$ be the element guaranteed by Lemma 3.2.3, and let $a := \psi(f)$. Then $\|a\| = \|f\|$ since

$$\|a\| = \|\psi(f)\| \leq \|f\| = \|f(\infty)\| = \|ev_\infty(f)\| = \|q(a)\| \leq \|a\|.$$

If $\|a\| = \|\pi(a)\|$, for some finite-dimensional representation π of A, then f attains its norm under the finite-dimensional representation $\pi \circ \psi$ of $T(B)$ which, by Lemma 3.2.3, is not true. Thus $\|a\| > \|\pi(a)\|$ for any finite-dimensional representation π of A.

To show that (iii) implies (i), we notice first that (iii) implies that A is GCR. Indeed otherwise A would have a subquotient isomorphic to the CAR algebra $\mathbb{M}_{2^\infty}$ by Theorem 2.5.1. Assume now that A does have an infinite-dimensional irreducible representation $(\pi, \mathcal{H})$. Since A is GCR, $K(\mathcal{H}) \subseteq \pi(A)$. Let $\mathcal{H}' \subseteq \mathcal{H}$ be an infinite-dimensional separable subspace, and let $P_{\mathcal{H}'}$ denote the projection of $\mathcal{H}$ onto $\mathcal{H}'$. Since $K(\mathcal{H}') \oplus 0|_{\mathcal{H}'^{\perp}}$ is singly generated, we can choose $x \in A$ such that $\pi(C^*(x)) = K(\mathcal{H}') \oplus 0|_{\mathcal{H}'^{\perp}}$. Then $C^*(x)$ is a subalgebra of A, and $P_{\mathcal{H}'}\pi P_{\mathcal{H}'} : C^*(x) \to K(\mathcal{H}')$ is a surjective $*$-homomorphism . □

Remark 3.2.5. Rephrasing the theorem, we can say that a C^*-algebra A contains an element a with $\|a\| > \|\pi(a)\|$ for any finite-dimensional representation $(\pi, \mathcal{H})$ of A if and only if A has an infinite-dimensional irreducible representation. Since $C^*(\mathbb{F}_n)$ has a faithful irreducible representation (necessarily infinite-dimensional) ([16]), we conclude that there are elements which do not attain their norm under a finite-dimensional representation. Recall that in [27], the authors show that no such element lies in $\mathbb{C}\mathbb{F}_n$.

If the C^*-algebra is subhomogeneous, we can say a little more.

Proposition 3.2.6. *For any C^*-algebra A and any $n < \infty$, the following are equivalent.*

(i) A is n-subhomogeneous (i.e. every irreducible representation is of dimension no more than n).

(ii) For each $a \in A$ there exists a representation $(\pi, \mathcal{H})$ of A of dimension no more than n such that $\|a\| = \|\pi(a)\|$.

(iii) A has a separating family of finite-dimensional representations of dimension no more than n.

(iv) The set $\{a \in A : \|a\| = \max_{\substack{\pi \in Irr_k(A) \\ k \leq n}} \|\pi(a)\|\}$ is dense in A.

Proof. The implications $(i) \Rightarrow (ii) \Rightarrow (iii)$ are immediate, and $(iii) \Leftrightarrow (iv)$ follows immediately from Theorem 3.1.2. Now, assume (iii). We denote the primitive ideal space of A (the space of kernels of all irreducible representations of A with the hull-kernel topology) by $\mathrm{Prim}(A)$, and we denote by $\mathrm{Prim}_n(A)$ the subspace of $\mathrm{Prim}(A)$ consisting of kernels of n-dimensional irreducible representations of A. Assume (iii). Then the irreducible representations of dimensional at most n form a separating family, which implies that $\bigcup_{k\leq n} \mathrm{Prim}_k(A)$ is dense in $\mathrm{Prim}(A)$. Indeed, for any C^*-algebra B and any set $X \subseteq \mathrm{Prim}(B)$ associated to a separating family of irreducible representations of B, the intersection of all ideals in X is the 0 ideal, which is contained in every element in $\mathrm{Prim}(B)$. Hence every element in $\mathrm{Prim}(B)$ is in the closure of X. Furthermore, $\bigcup_{k\leq n} \mathrm{Prim}_k(A)$ is closed in $\mathrm{Prim}(A)$ for any $n < \infty$ by [20, Proposition 3.6.3 (i)]. Hence, $\mathrm{Prim}(A) = \bigcup_{k\leq n} \mathrm{Prim}_k(A)$. But this implies that every irreducible representation of A must have dimension at most n. □

3.3 Algebraic structure of the set of finite-dimensional norm-attaining elements

In this section, we want to study the algebraic structure of the set

$$A_{00} = \{a \in A : \|a\| = \max_{\substack{\pi\in \mathrm{Irr}_n(A)\\ n<\infty}} \|\pi(a)\|\},$$

for any C^*-algebra A with sufficiently many representations. Is this set an algebra? Is it additively or multiplicatively closed?

First, we will need a technical fact that will be useful in this section and the next chapter.

Proposition 3.3.1. *Suppose A is RFD and $A_0 \subseteq A$ is a non-subhomogeneous subalgebra. Then there exists an unbounded sequence $(n_k)_{k\in\mathbb{N}}$ in $\mathbb{N}$ and irreducible representations $\pi_k : A_0 \to \mathbb{M}_{n_k}$ such that each π_k is a subrepresentation of $\pi'_k|_{A_0}$, denoted $\pi_k \leq \pi'_k|_{A_0}$, for some finite-dimensional representation π'_k of A.*

Proof. Let $\mathcal{F}$ be a separating family of finite-dimensional irreducible representations of A. Then the collection $\{\pi|_{A_0} : \pi \in \mathcal{F}\}$ is a separating family of representations of A_0. Let

$$\mathcal{F}_0 = \{\sigma \in \mathrm{Irr}(A_0) : \sigma \leq \pi|_{A_0} \text{ for some } \pi \in \mathcal{F}\}.$$

Then $\mathcal{F}_0$ separates the points of A_0. If the set $\{\dim(\sigma) | \ \sigma \in \mathcal{F}_0\}$ is bounded, then A_0 is subhomogeneous by Proposition 3.2.6. □

Theorem 3.3.2. *Suppose A is an RFD C^*-algebra. Then the following are equivalent:*

1. *A is FDI.*
2. *A_{00} is closed under addition.*
3. *A_{00} is closed under multiplication.*

Proof. Clearly (1) ⇒ (2) and (1) ⇒ (3).

To show that (2) ⇒ (1), we assume A is not FDI. We demonstrate the existence of $a_1, a_2 \in A$ such that a_1 and a_2 achieve their norm under a finite-dimensional representation, but $a_1 + a_2$ does not. By Theorem 3.2.4, there exists a subalgebra $A_0 \subseteq A$ and a simple infinite-dimensional AF-algebra B with inductive sequence (B'_n) such that A_0 surjects onto B. Moreover, we can take B to be either $\mathbb{M}_{2^\infty}$ with inductive sequence $(\mathbb{M}_{2^n})$ or $\mathbb{K}(\ell^2)$ with inductive sequence $(\mathbb{M}_n)$. By Proposition 3.3.1, there is a finite-dimensional nonzero irreducible representation π_0 of A_0 and a finite-dimensional representation π_0' of A such that π_0 is a subrepresentation of $\pi_0'|_{A_0}$. Then for some n,

$$B_{n-1}' \hookrightarrow \pi_0(A_0) \hookrightarrow B_n'.$$

So, we can find a new inductive sequence (B_k) where $B_k = B_k'$ for $k < n$, $B_n \simeq \pi_0(A_0)$, and $B_k = B_{k-1}'$ for $k > n$. We still call the inductive limit B, and let $\pi : A_0 \to B$ be a surjection. Since B_n and B are simple, it follows from the Chinese Remainder Theorem (and the discussion in Section 2.6) that

$$q := \pi \oplus \pi_0 : A_0 \to B \oplus B_n$$

is a surjection. Let

$$\phi := ev_\infty \oplus ev_n : T(B) \to B \oplus B_n.$$

The projectivity of $T(B)$ again yields the following commutative diagram.

$$\begin{array}{ccc} & \overset{\psi}{\nearrow} & A_0 \\ & & \Big\downarrow q \\ T(B) & \xrightarrow[\phi]{} & B \oplus B_n \end{array}$$

Let $b \in B_1$ with $\|b\| = 1$. Define $f_1 \in T(B)$ by

$$f_1(t) = \begin{cases} \frac{2}{n} t b & ; \quad t \in (0, n] \\ \frac{2}{n}(2n - t) b & ; \quad t \in (n, 2n] \\ (1 - e^{2n-t}) b & ; \quad t \in (2n, \infty]. \end{cases}$$

Define $f_2 \in T(B)$ by

$$f_2(t) = \begin{cases} -f_1(t) & ; \quad t \in (0, 2n] \\ \\ 0 & ; \quad t \in (2n, \infty]. \end{cases}$$

Then for $i = 1, 2$,

$$\|f_i\| = \|f_i(n)\|,$$

but for any finite-dimensional representation ρ of $T(B)$,

$$\|f_1 + f_2\| = \|(f_1 + f_2)(\infty)\| = 1 > \|\rho(f_1 + f_2)\|.$$

Let $a_i = \psi(f_i)$ for $i = 1, 2$. By the same argument as in Theorem 3.2.4, $\|a_i\| = \|f_i\|$ for $i = 1, 2$, and $\|a_1 + a_2\| = \|f_1 + f_2\|$. Also as in the proof of Theorem 3.2.4, $\|a_1 + a_2\| > \|\rho(a_1 + a_2)\|$ for any finite-dimensional representation ρ of A. Since $\|a_i\| = \|f_i\|$ for $i = 1, 2$, we know that each a_i attains its norm under some finite-dimensional representation of A_0. To see that they will attain their norms under some finite-dimensional representation of A, note that since

$$q(a_i) = f_i(\infty) \oplus f_i(n)$$

for $i = 1, 2$, we have that

$$\|\pi_0(a_i)\| = \|f_i(n)\| = \|a_i\|$$

for $i = 1, 2$. Because $\pi_0 \leq \pi_0'|_{A_0}$, it follows that $\|\pi_0(a_i)\| = \|a_i\|$ for $i = 1, 2$.

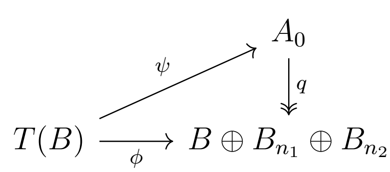
To show that (3) $\Rightarrow$ (1), assume that A is not FDI. Again, by Theorem 3.2.4, there exists a subalgebra $A_0 \subseteq A$ and a simple infinite-dimensional AF-algebra B with inductive sequence (B'_n) such that A_0 surjects onto B, and again, we take B to be either $\mathbb{M}_{2^\infty}$ or $\mathbb{K}(\ell^2)$. By the Proposition 3.3.1, there exist irreducible representations π_1 and π_2 of A_0 and π'_1 and π'_2 of A such that $\pi_i : A_0 \to \mathbb{M}_{k_i}$ with $k_1 < k_2$ and $\pi_i \leq \pi'_i|_{A_0}$ for $i = 1, 2$. As before, we intertwine $\mathbb{M}_{k_1}$ and $\mathbb{M}_{k_2}$ into the inductive sequence (B_n) of B and let $n_1 < n_2$ so that now $B_{n_i} = \mathbb{M}_{k_i}$ for $i = 1, 2$. Let $\pi : A_0 \to B$ be a surjection. As before, we use the Chinese remainder theorem to get a surjection

$$q := \pi \oplus \pi_1 \oplus \pi_2 : A_0 \to B \oplus B_{n_1} \oplus B_{n_2}.$$

Letting

$$\phi := ev_\infty \oplus ev_{n_1} \oplus ev_{n_2} : T(B) \to B \oplus B_{n_1} \oplus B_{n_2},$$

the projectivity of $T(B)$ yields the following commutative diagram.



Let $b \in B_1$ self-adjoint with $\|b\| = 1$. Define $f_1 \in T(B)$ by

$$f_1(t) = \begin{cases} \frac{t}{n_1} b & ; \quad t \in (0, n_1] \\ \\ \frac{-1}{n_2 - n_1}(t - n_2) b & ; \quad t \in (n_1, n_2] \\ \\ (1 - e^{n_2 - t}) b & ; \quad t \in (n_2, \infty]. \end{cases}$$

Define $f_2 \in T(B)$ by

$$f_2(t) = \begin{cases} \frac{t}{n_2} b & ; \quad t \in (0, n_2] \\ \\ b & ; \quad t \in (n_2, \infty]. \end{cases}$$

Then, $\|f_i\| = \|f_i(n_i)\|$ for $i = 1, 2$, and

$$\|f_1 f_2\| = \|f_1 f_2(\infty)\| = \|b^2\| = 1 > \|f_1 f_2(t)\|$$

for all $t < \infty$. Let $a_1 = \psi(f_1)$ and $a_2 = \psi(f_2)$. As before, $\|a_i\| = \|f_i\|$ and $\|a_1 a_2\| = \|f_1 f_2\|$. Moreover, $\|a_1 a_2\| > \|\rho(a_1 a_2)\|$ for all finite-dimensional representations ρ of A. Also as before, we have for $i = 1, 2$,

$$\|a_i\| \geq \|\pi_i'(a_i)\| \geq \|\pi_i(a_i)\| = \|f_i(n_i)\| = \|a_i\|. \qquad \square$$

Remark 3.3.3. Our proof needed the C^*-algebra to be RFD in order to guarantee that there exists one or two finite-dimensional representations that do not vanish on A_0. If such representations are known to exist, the the RFD assumption can be dropped.

As a consequence, if a C^*-algebra is RFD but not FDI, then the set of elements in a C^*-algebra that achieve their norm under a finite-dimensional representation does not form an algebra. In particular, we have the following corollary.

Corollary 3.3.4. *For $n < \infty$, there exists an element in $C^*(\mathbb{F}_n) \backslash \mathbb{C}\mathbb{F}_n$ that achieves its norm under some finite-dimensional representation.*

Chapter 4

Behavior of elements of C^*-algebras under finite-dimensional representations

If a C^*-algebra has sufficiently many finite-dimensional representations, the norm of any element can be approximated by the norms of its images under these representations. If we allow degenerate representations (i.e. we don't require that $\pi(A)$ have trivial null space), then the norms of these images will grow as the dimension of the representations increase. In this chapter, we investigate what this growth looks like. One of our primary tools is the technique using AF mapping telescopes that we developed in the proof of Theorem 3.2.4.

This is taken from joint work with Tatiana Shulman in [18].

4.1 Growth of finite-dimensional norms

Let $n \in \mathbb{N}$. If a C^*-algebra has a representation of dimension no more than n, we define a seminorm $\|\cdot\|_{\mathbb{M}_n}$ on A by

$$\|a\|_{\mathbb{M}_n} = \sup\{\|\pi(a)\| \mid \pi : A \to \mathbb{M}_n\},$$

for all $a \in A$. We do not require representations to be non-degenerate and so by $\pi : A \to \mathbb{M}_n$ we mean a representation of dimension not larger than n. Equivalently we can say that

$$\|a\|_{\mathbb{M}_n} = \sup\{\|\pi(a)\|\}$$

where supremum is taken over all irreducible representations of dimension not larger than n.

Now, suppose that $\{n_1, n_2, \ldots\}$ is the nonempty set of dimensions of all *irreducible* finite-dimensional representations of a C^*-algebra A, arranged in increasing order, with $\kappa = |\{n_1, n_2, ...\}|$. Then for each $a \in A$ we get a sequence

$$(\|a\|_{\mathbb{M}_{n_k}})_\kappa \in \mathbb{R}_+^\kappa.$$

In general we would like to know what sequences of numbers can be obtained in this way. Namely, define the set $\Lambda(A)$ by

$$\Lambda(A) = \{(\|a\|_{\mathbb{M}_{n_k}})_\kappa \mid a \in A\}.$$

Since we allow degenerate representations, all such sequences will be nondecreasing. In this section we prove that, for any C^*-algebra A with at least one finite-dimensional representation, $\Lambda(A)$ contains the set of all nondecreasing sequences of κ positive numbers which are eventually constant (Theorem 4.1.1). Moreover, we show that A is an FDI-algebra if and only if the two sets coincide. (Corollary 4.1.4).

Theorem 4.1.1. *Let $N \in \mathbb{N}$, and $0 \leq \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_N$ be a sequence of non-negative numbers. Suppose that a C^*-algebra A has irreducible representations of* *dimensions* $n_1 < n_2 < \ldots < n_N$ *(and possibly of some other dimensions too). Then there exists* $a \in A$ *such that* $\|a\|_{\mathbb{M}_{n_k}} = \lambda_k$, *for* $1 \leq k \leq N$. *In addition* a *can be chosen such that* $\|a\| = \|a\|_{\mathbb{M}_{n_N}}$.

Proof. For each $i \leq N$, let $\pi_i : A \to \mathbb{M}_{n_i}$ be an irreducible representation with kernel I_i, i.e. $A/I_i \simeq \pi_i(A) = \mathbb{M}_{n_i}$. Since each $\mathbb{M}_{n_i}$ is simple, we have by the Chinese Remainder Theorem (Section 2.6) that $q = \oplus_{i=1}^N \pi_i : A \to \mathbb{M}_{n_1} \oplus \ldots \oplus \mathbb{M}_{n_N}$ is a surjective $*$-homomorphism. Now, consider standard embeddings

$$\mathbb{M}_{n_1} \subset \mathbb{M}_{n_2} \subset \ldots \subset \mathbb{M}_{n_N},$$

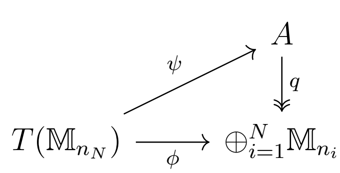
and let $T(\mathbb{M}_{n_N})$ denote the corresponding AF-telescope. For $i \leq N$, let $ev_i : T(\mathbb{M}_{n_N}) \to \mathbb{M}_{n_i}$ denote the evaluation map, and let

$$\phi = \oplus_{i=1}^N ev_i : T(\mathbb{M}_{n_N}) \to \mathbb{M}_{n_1} \oplus \ldots \oplus \mathbb{M}_{n_N}.$$

Since $T(\mathbb{M}_{n_N})$ is projective, ϕ lifts to some $*$-homomorphism $\psi : T(\mathbb{M}_{n_N}) \to A$ so that

$$q \circ \psi = \phi,$$

giving us the following commutative diagram.



Let $f \in T(\mathbb{M}_{n_N})$ be any element such that $\|f(t)\|$ is a nondecreasing function on

$(0, \infty]$ with $\|f(i)\| = \lambda_i$ for each $i \leq N$, and let $a = \psi(f)$. Then

$$q(a) = \oplus_{i=1}^{N} \pi_i(a) = \oplus_{i=1}^{N} \pi_i(\psi(f)) = \phi(f) = \oplus_{i=1}^{N} f(i).$$

Hence for any $k \leq N$,

$$\|\pi_k(a)\| = \|f(k)\| = \lambda_k,$$

which implies

$$\|a\|_{\mathbb{M}_{n_k}} \geq \lambda_k.$$

On the other hand, for any representation π of A of dimension not larger than n_k, $\pi \circ \psi$ is a representation of $T(\mathbb{M}_{n_N})$ of dimension not larger than n_k and hence factors through a finite direct sum of evaluations at some points in $(0, k]$ by Lemma 3.2.1. Since $\|f(t)\|$ is a nondecreasing function and $\|f(k)\| = \lambda_k$, it follows that

$$\|\pi(a)\| = \|\pi(\psi(f))\| \leq \lambda_k.$$

Thus for each $k \leq N$,

$$\|a\|_{\mathbb{M}_{n_k}} = \lambda_k.$$

If we additionally chose f to attain its norm at the point n_N, then we would have

$$\lambda_N = \|f\| \geq \|\psi(f)\| = \|a\| \geq \|q(a)\| = \| \oplus_{k=1}^{N} f(k)\| = \lambda_N$$

whence $\|a\| = \lambda_N$. □

Corollary 4.1.2. *Let G be a discrete group with representations of dimensions $n_1 < n_2 < \infty$, and let $\epsilon > 0$. Then there is an $a \in \mathbb{C}G$ such that $\|a\|_{\mathbb{M}_{n_1}} \leq \epsilon$ and $\|a\|_{\mathbb{M}_{n_2}} > \|a\| - \epsilon$.*

Proof. Since $\mathbb{C}G$ is dense in $C^*(G)$, the statement follows from Theorem 4.1.1. $\square$

Remark 4.1.3. In case $G = \mathbb{F}_n$ with $n < \infty$, Thom proves in [77] that for any $d > 0$ and $\epsilon > 0$, there exists a nontrivial $w \in \mathbb{F}_n$ such that

$$\sup\{\|\pi(1-w)\| : \pi : C^*(\mathbb{F}_n) \to \mathbb{M}_d\} < \epsilon.$$

However, he does not state (in [77]) for which $m > d$ we know that we have $\|1 - w\|_{\mathbb{M}_m} > \|1-w\| - \epsilon$. Corollary 4.1.2 guarantees that for any $m > d$, we can find some element with this behavior, but, in contrast to Thom, we do not know what this element looks like.

On the other hand, it follows from [27, Lemma 2.7] that, to have $\|1 - w\|_{\mathbb{M}_m} > \|1 - w\| - \epsilon$, it suffices to take $m \geq 4n^\ell$, where ℓ is the length of w. We show in Section 4.2 that it actually suffices to take $m \geq 2\ell$.

Corollary 4.1.4. *Assume A has irreducible representations of $\kappa > 0$ many distinct finite dimensions. Then*

$$\Lambda(A) \supseteq \{(\lambda_n)_{n \leq \kappa} \mid 0 \leq \lambda_n \leq \lambda_{n+1} \ \forall\ n \leq \kappa \textit{ and } (\lambda_n) \textit{ is eventually constant}\}$$

Moreover, the two sets are equal if and only if A is FDI.

Proof. It follows from Theorem 4.1.1 and Theorem 3.2.4. $\square$

Theorem 4.1.1 says we can find an element that attains the prescribed norms λ_k for $k \leq N$. Theorem 3.2.4 says that if a C^*-algebra is not FDI, then we can find an element that does not achieve its norm under any finite-dimensional representation. The following theorem says that if we assume the C^*-algebra is RFD and not FDI, then we can find an element that does both.

Theorem 4.1.5. *Suppose A is RFD and has an infinite-dimensional irreducible representation. Then there exist $M_1 < M_2 < \ldots < \infty$ such that for any finite sequence $0 \le \lambda_1 \le \lambda_2 \le ... \le \lambda_N \le \lambda$ there is $a \in A$ such that $\|a\|_{\mathbb{M}_{M_k}} = \lambda_k$ for $1 \le k \le N$, $\|a\| = \lambda$, and, in the case $\lambda > \lambda_N$, $\|a\| \ne \|a\|_{\mathbb{M}_{M_k}}$ for any $k < \infty$.*

Proof. Since A has an infinite-dimensional irreducible representation, by Theorem 3.2.4 there is a C^*-subalgebra $A_0 \subseteq A$, a simple infinite-dimensional AF-algbera B, and a surjective $*$-homomorphism $\pi : A_0 \to B$. Again, we can and do take B to be either $\mathbb{M}_{2^\infty}$ or $\mathbb{K}(\ell^2)$.

Since A_0 is a C^*-subalgebra of an RFD C^*-algebra, it is also RFD. Since B is a quotient of A_0, A_0 is not subhomogeneous. It then follows from Proposition 3.3.1 that we can build sequences $(j_i), (m_i)$, and (M_i) of positive integers such that

$$j_1 < 2^{j_1} < m_1 \le M_1 < j_2 < 2^{j_2} < m_2 \le M_2 < \ldots,$$

and such that for each i, there exists an irreducible representation π_i of A_0 and a representation π_i' of A such that $\dim(\pi_i) = m_i$, $\dim(\pi_i') = M_i$, and $\pi_i \le \pi_i'|_{A_0}$.

Since B is simple, again using the Chinese Reminder Theorem (Section 2.6), we conclude that the $*$-homomorphism

$$q = (\oplus_{i=1}^N \pi_i) \oplus \pi : A_0 \to \mathbb{M}_{m_1} \oplus \ldots \oplus \mathbb{M}_{m_N} \oplus B$$

is surjective. If $B = \mathbb{K}(\ell^2)$ let

$$\phi = (\oplus_{i=1}^N (ev_{m_i})) \oplus ev_\infty : T(B) \to \mathbb{M}_{m_1} \oplus \ldots \oplus \mathbb{M}_{m_N} \oplus B.$$

If $B = \mathbb{M}_{2^\infty}$ let

$$\phi = (\oplus_{i=1}^{N}(ev_{j_i})) \oplus ev_\infty : T(B) \to \mathbb{M}_{m_1} \oplus \ldots \oplus \mathbb{M}_{m_N} \oplus B$$

where we view $\mathbb{M}_{2^{j_i}}$ as a subalgebra of $\mathbb{M}_{m_i}$ via the standard inclusion. Since $T(B)$ is projective, ϕ lifts to some $*$-homomorphism $\psi : T(B) \to A_0$. Thus

$$q \circ \psi = \phi.$$

Choose $f \in C_0(0, \infty]$ so that f is nonnegative, nondecreasing, $f(\infty) = \lambda$, and for each $i \leq N$,

$$f|_{[j_i, M_i]} \equiv \lambda_i;$$

in case $\lambda > \lambda_N$, we also require $f(t) < \lambda$ for each $t < \infty$. We can view f as an element of $T(B)$ by the standard embdedding of $\mathbb{C}$ to B. Let $a = \psi(f)$.

By Lemma 3.2.1, for any $i \in \mathbb{N}$ and any representation ρ of A of dimension not larger than M_i we have

$$\|\rho(a)\| = \|\rho(\psi(f))\| \leq \|f(M_i)\|.$$

Hence for any $i \leq N$

$$\|a\|_{\mathbb{M}_{M_i}} \leq \lambda_i, \tag{4.1.1}$$

and for all $i \in \mathbb{N}$,

$$\|a\|_{\mathbb{M}_{M_i}} < \lambda. \tag{4.1.2}$$

On the other hand, when $B = \mathbb{K}(\ell^2)$,

$$(\pi_1(a), \ldots, \pi_N(a), \pi(a)) = q(a) = q(\psi(f)) = \phi(f) = (f(m_1), \ldots, f(m_N), f(\infty)),$$

and we conclude that $\pi_i(a) = f(m_i) = \lambda_i$ for $i \leq N$; when $B = \mathbb{M}_{2^\infty}$,

$$(\pi_1(a), \ldots, \pi_N(a), \pi(a)) = q(a) = q(\psi(f)) = \phi(f) = (f(j_1), \ldots, f(j_N), f(\infty)),$$

and we conclude that $\pi_i(a) = f(j_i) = \lambda_i$ for $i \leq N$. In either case,

$$\|a\|_{\mathbb{M}_{M_i}} \geq \|\pi_i'(a)\| \geq \|\pi_i(a)\| = \lambda_i, \tag{4.1.3}$$

for $i \leq N$. By (4.1.1) and (4.1.3) we have for each $i \leq N$,

$$\|a\|_{\mathbb{M}_{M_i}} = \lambda_i. \tag{4.1.4}$$

We also have

$$\lambda = \|\phi(f)\| = \|q(\psi(f))\| = \|q(a)\| \leq \|a\| = \|\psi(f)\| \leq \|f\| = \lambda,$$

and thus

$$\|a\| = \lambda. \tag{4.1.5}$$

By (4.1.2), (4.1.4) and (4.1.5) we are done. □

4.2 A bound on dimension

Fritz, Netzer, and Thom show in [27, Lemma 2.7] that, for any $n < \infty$, any element in $\mathbb{CF}_n$ will achieve its norm in a representation of dimension no more than $4n^\ell$, where ℓ is the length of the longest word in the support of the element. In this section, we improve the bound on the dimension for binomials in $\mathbb{CF}_n$. Namely, we show the following proposition.

Proposition 4.2.1. *Let $\alpha, \beta \in \mathbb{C}$. Let $w_1, w_2 \in \mathbb{F}_n$ be distinct reduced words for some $n < \infty$, and let ℓ be the length of the reduced word $w_2^{-1}w_1$. Then, there exists a representation $\pi : C^*(\mathbb{F}_n) \to \mathbb{M}_{2\ell}$ such that*

$$\|\pi(\alpha w_1 + \beta w_2)\| = |\alpha| + |\beta|.$$

We say a word w in $\mathbb{F}_n$ is **balanced** if it lies in the commutator subgroup $[\mathbb{F}_n, \mathbb{F}_n]$ of $\mathbb{F}_n$. In other words, any representation $C^*(\mathbb{F}_n) \to \mathbb{C}$ maps $w \mapsto 1$, e.g. $w = x_1x_2x_1^{-1}x_2^{-1}$. Notice that if the reduced word $w_2^{-1}w_1$ is not balanced, then this norm will be achieved by a representation $C^*(\mathbb{F}_n) \to \mathbb{C}$ sending all but one generator to 1. For example, the word $w = x_1x_2x_1^{-2}$ is not balanced, and the map $C^*(\mathbb{F}_2) \to \mathbb{C}$ sending $x_1 \to 1$ and $x_2 \to -1$ will send $1 - w$ to an element in $\mathbb{C}$ of norm 2. However, for a balanced word there is no such map. Since balanced words constitute the class of nontrivial examples, we will assume $w := w_2^{-1}w_1$ is balanced.

Notice that for any representation $\pi : C^*(\mathbb{F}_n) \to B(\mathcal{H})$,

$$\begin{aligned}\|\pi(\alpha w_1 + \beta w_2)\| &= \|\alpha\pi(w_2^{-1}w_1) + \beta I_{\mathcal{H}}\| \\ &= \|\alpha\pi(w) + \beta I_{\mathcal{H}}\| \\ &= \max_{\lambda\in\sigma(\pi(w))} |\alpha\lambda + \beta|\end{aligned}$$

where $w := w_2^{-1}w_1$ and $\sigma(\pi(w))$ denotes the spectrum of an operator $\pi(w) \in B(\mathcal{H})$. This maximum equals $|\alpha| + |\beta|$ if and only if $\frac{\mathrm{sgn}(\beta)}{\mathrm{sgn}(\alpha)} \in \sigma(\pi(w))$, and so Proposition 4.2.1 will follow from the following theorem.

Theorem 4.2.2. *For any nontrivial, balanced, reduced word $w \in \mathbb{F}_n$ of length ℓ and any $\lambda \in \mathbb{T}$, there exists a representation $\pi : C^*(\mathbb{F}_n) \to \mathbb{M}_{2\ell}$ such that $\lambda \in \sigma(\pi(w))$.*

For the proof we will first need two lemmas.

Lemma 4.2.3. *Let $w \in \mathbb{F}_n$ be a nontrivial, balanced, reduced word. Define for each $1 \leq d < \infty$*

$$\Sigma_d := \bigcup\{\sigma(\pi(w))|\ \pi : C^*(\mathbb{F}_n) \to \mathbb{M}_d\}.$$

Then for each $1 \leq d < \infty$, there is a $\theta_d \in [0, \pi]$ such that

$$\Sigma_d = \{e^{i\theta}|\ \theta \in [-\theta_d, \theta_d]\}$$

with $\theta_d \leq \theta_{d+1}$ for each $1 \leq d < \infty$.

Proof. Since (Σ_d) is clearly a nested sequence, we need only to show that for each

$1 \leq d < \infty$,

$$\Sigma_d = \{e^{i\theta} |\ \theta \in [-\theta_d, \theta_d]\}$$

for some $\theta_d \in [0, \pi]$.

Fix $1 \leq d < \infty$. We first, observe that Σ_d is symmetric about $\mathbb{R}$ and centered at 1. Indeed, since $\lambda \in \sigma(w(u_1, ..., u_n))$ for some $u_1, ..., u_n \in \mathcal{U}(d)$ implies that $\overline{\lambda} \in \sigma(w(\overline{u_1}, ..., \overline{u_n}))$, where $\overline{u}$ denotes the complex conjugate of the matrix u. Clearly $1 \in \Sigma_d$.

By these observations, it will suffice to show that Σ_d is the union of two continuous images of the compact, path-connected space $\prod_{k=1}^n \mathcal{U}(d)$, which have nontrivial intersection.

To that end, let $w : \prod_{k=1}^n \mathcal{U}(d) \to \mathcal{U}(d)$ denote the word map; let $\psi : \mathcal{U}(d) \to [-1, 1]$ be given by $u \mapsto \min_{\lambda \in \sigma(u)} \mathrm{Re}(\lambda)$, and let $\gamma : [-1, 1] \to \{e^{i\theta} |\ \theta \in [0, \pi]\}$ be given by $r \mapsto r + i\sqrt{1 - r^2}$. Then, $\gamma \circ \psi \circ w$ is a continuous map on $\prod_{k=1}^n \mathcal{U}(d)$ whose image is the compact, path-connected arc $\{e^{i\theta} |\ \theta \in [0, \theta_d]\}$ where
$\theta_d = \max\{\theta \in [0, \pi] |\ e^{i\theta} \in \Sigma_d\}$. Likewise, the image of $\bar{\gamma} \circ \psi \circ w$ is the path-connected arc $\{e^{i\theta} |\ \theta \in [-\theta_d, 0]\}$ for the same $\theta_d \in [0, \pi]$. Hence, the union of the two is the compact, path connected arc $\{e^{i\theta} |\ \theta \in [-\theta_d, \theta_d]\} = \Sigma_d$. □

By Lemma 4.2.3, we can conclude that, if for some $1 \leq d < \infty$ there exists a representation $\pi : C^*(\mathbb{F}_n) \to \mathbb{M}_d$ such that $-1 \in \sigma(\pi(1 - w))$, then $\Sigma_d = \mathbb{T}$. The following lemma tells us for which $1 \leq d < \infty$ we can expect such a representation.

Lemma 4.2.4. *For any nontrivial, balanced, reduced word $w \in \mathbb{F}_n$ of length ℓ, there are (permutation) matrices $u_1, ..., u_n \in \mathcal{U}(2\ell)$ such that $-1 \in \sigma(w(u_1, ..., u_n))$.*

Our proof is inspired by Schreier's proof of residual finiteness of $\mathbb{F}_n$.

Proof. The goal will be to construct a map from $\mathbb{F}_n$ into $\mathcal{S}_{2\ell}$, which maps w to a permutation in $\mathcal{S}_{2\ell}$ containing the two-cycle $(1, \ell+1)$. Composing this map with the natural inclusion of $\mathcal{S}_{2\ell}$ into $\mathcal{U}(2\ell)$ will yield a map from $\mathbb{F}_n$ to $\mathcal{U}(2\ell)$ mapping w to a permutation matrix with -1 as an eigenvalue.

Write $w = x_{i_\ell}^{\epsilon_\ell} \cdots x_{i_1}^{\epsilon_1}$ where $\epsilon_i \in \{\pm 1\}$ and $i_k \in \{1, ..., \ell\}$.

First, we claim that we may assume $i_\ell \neq i_1$. Indeed, suppose $i_\ell = i_1$. Let $d \geq 0$ and $u_1, ..., u_n \in \mathcal{U}(d)$, and denote $w(u_1, ..., u_n) := u_{i_\ell}^{\epsilon_\ell} \cdots u_{i_1}^{\epsilon_1}$. If $\epsilon_1 = -\epsilon_\ell$, then

$$\sigma(w(u_1, ..., u_n)) = \sigma(u_{i_1}^{-\epsilon_1} w(u_1, ..., u_n) u_{i_1}^{-\epsilon_1}).$$

This reduction will not change the length of the word and must terminate since w is balanced, reduced, and nontrivial. Hence, we may assume $\epsilon_1 = \epsilon_\ell$. Moreover, since $-1 \in \sigma(w(u_1, ..., u_n))$ iff $-1 \in \sigma((w(u_1, ..., u_n))^*)$, we may assume $\epsilon_1 = 1$. Let j and k denote the multiplicity of x_{i_1} at the end and beginning of w, respectively, and assume $j \leq k$. Now, write $w = x_{i_1}^j x_{i_{\ell-j}}^{\epsilon_{\ell-j}} \cdots x_{i_{k+1}}^{\epsilon_{k+1}} x_{i_1}^k$ where $x_{i_{\ell-j}} \neq x_{i_1} \neq x_{i_{k+1}}$. Then,

$$\sigma(w(u_1, ..., u_n)) = \sigma(u_{i_1}^{-j} w(u_1, ..., u_n) u_{i_1}^{j}).$$

Again, the length of the word is unchanged. Hence, we assume $i_1 \neq i_\ell$.

Now, define a map $\phi : \mathbb{F}_n \to \mathcal{S}_{2\ell}$ by mapping the generators x_i to permutations σ_i such that for each $1 \leq k \leq \ell$ we require that

$$\begin{gathered} \sigma_{i_k}(k) = k+1, \text{ if } \epsilon_k = 1 \\ \text{and} \\ \sigma_{i_k}^{-1}(k) = k+1, \text{ if } \epsilon_k = -1. \end{gathered} \tag{4.2.1}$$

Note that, since $\ell \neq 1$ and $i_1 \neq i_\ell$, the values for $\sigma_{i_1}^{\epsilon_1}(\ell+1)$ and $\sigma_{i_\ell}^{-\epsilon_\ell}(1)$ are not determined by the conditions in (4.2.1), which means $\phi(w)^{-1}(1)$ and $\phi(w)(\ell+1)$ are not determined by the conditions in (4.2.1). Hence, we are free to require for $1 \leq k \leq \ell - 1$, that

$$\begin{gathered} \sigma_{i_k}(\ell+k) = l+k+1, \text{ if } \epsilon_k = 1 \\ \text{and} \\ \sigma_{i_k}^{-1}(\ell+k) = l+k+1, \text{ if } \epsilon_k = -1; \end{gathered} \tag{4.2.2}$$

and for σ_{i_ℓ} that

$$\begin{gathered} \sigma_{i_\ell}(2\ell) = 1, \text{ if } \epsilon_\ell = 1 \\ \text{and} \\ \sigma_{i_\ell}(1) = 2\ell, \text{ if } \epsilon_\ell = -1. \end{gathered} \tag{4.2.3}$$

Aside from these restrictions, the $\sigma_{i_k}^{\epsilon_k}$ are free to take any values. The following table provides an illustration of the enforced mappings.

	1		2		...		ℓ		$\ell+1$		$\ell+2$		...		2ℓ		1
$\sigma_{i_1}^{\epsilon_1}$		$\searrow$								$\searrow$							
	1		2		...		ℓ		$\ell+1$		$\ell+2$		...		2ℓ		1
$\sigma_{i_2}^{\epsilon_2}$				$\searrow$								$\searrow$					
$\vdots$																	
	1		2		...		ℓ		$\ell+1$		$\ell+2$		...		2ℓ		1
$\sigma_{i_\ell}^{\epsilon_\ell}$								$\searrow$								$\searrow$	
	1		2		...		ℓ		$\ell+1$		$\ell+2$		...		2ℓ		1

The conditions in (4.2.1) guarantee that $\phi(w) = \sigma_{i_\ell}^{\epsilon_\ell} \ldots \sigma_{i_1}^{\epsilon_1}$ will map $1 \mapsto \ell + 1$. The conditions in (4.2.2) and (4.2.3) guarantee that $\phi(w)$ maps $\ell + 1 \mapsto 1$. Hence, $\phi(w)$ has the two-cycle $(1, \ell + 1)$ as desired. □

Remark 4.2.5. This proof likely uses too much "space," and the argument may still work with $\mathcal{U}(l + 1)$ instead. One would have to be very careful with choosing permutations with respect to the given word.

We are now ready to prove Theorem 4.2.2.

Proof of Theorem 4.2.2. Let $w \in \mathbb{F}_n$ be a nontrivial, balanced, reduced word of length ℓ, and let $\lambda \in \mathbb{T}$. By Lemma 4.2.4, there exists a representation $\pi : C^*(\mathbb{F}_n) \to \mathbb{M}_{2\ell}$ for which $-1 \in \sigma(\pi(w))$. By Lemma 4.2.3, there then exists a representation $\pi' : C^*(\mathbb{F}_n) \to \mathbb{M}_{2\ell}$ with $\lambda \in \sigma(\pi'(w))$. □

Chapter 5

A von Neumann-type inequality for noncommutative $*$-polynomials

In this chapter, we offer two sharpenings of the noncommutative $*$-polynomial analogue to von Neumann's inequality given in Proposition 1.0.3. The first gives an upper bound for the dimension required to witness the maximal norm of a $*$-polynomial whose entries range over all Hilbert space contractions. The second, which follows from a result of Herrero, says that, for any noncommutative $*$-polynomial q, the norm of $q(T)$ as T ranges over all Hilbert space contractions is actually bounded by the norm of $q(a)$ as a ranges over all matrices with spectrum $\{\lambda\}$ for fixed $\lambda \in \mathbb{D}$. In terms of C^*-algebras, this says that, not only is $C_u^*\langle x : \|x\| \leq 1\rangle$ RFD, but its separating family of finite-dimensional representations can be chosen so that the generator is sent to such matrices. In the case where $\lambda = 0$, this can be rephrased to say that there exists a countable family of nilpotent matrices whose direct sum is a universal contraction operator. We provide a new proof of this sharpening using Theorem 5.3.6, which rests heavily on a classical theorem of Apostol, Foias, and Voiculescu ([5]) and a new result of Loring and Shulman ([59]).

5.1 Quasidiagonal representations

Before we proceed, we must first establish a few preliminary definitions and results pertaining to quasidiagonal (QD) operators, C^*-algebras, and representations.

Definition 5.1.1. A C^*-algebra A is **quasidiagonal** (QD) if there exists a net of completely positive contractive maps $\phi_n : A \to \mathbb{M}_{k_n}$ that are asymptotically multiplicative and asymptotically isometric, i.e. for any $a, b \in A$,

- $\|\phi_n(ab) - \phi_n(a)\phi_n(b)\| \to 0$ and
- $|\|a\| - \|\phi_n(a)\|| \to 0$.

It is easy to see that a separable RFD C^*-algebra is QD since it embeds faithfully into some product $\prod_{n\geq 1} \mathbb{M}_{k_n}(\mathbb{C})$.

Definition 5.1.2. A set of operators $\Omega \subseteq B(\mathcal{H})$ is called **quasidiagonal** (QD) if there exists an increasing sequence of finite-rank projections $P_1 \leq P_2 \leq ...$ converging strongly to I such that $\|P_nT - TP_n\| \to 0$ for every $T \in \Omega$.

A representation π of a C^*-algebra A on $\mathcal{H}$ is called **quasidiagonal** (QD) if $\pi(A)$ is a QD set of operators on $\mathcal{H}$.

Note that π being a QD representation of A is not equivalent to $\pi(A)$ being a QD C^*-algebra (see [15, Remark 7.5.3]). The issue arises from a Fredholm index obstruction, which can be avoided by taking essential representations, i.e. the image contains no nonzero compact operators. In fact, Voiculescu has shown in [85] that a C^*-algebra A is QD iff every faithful essential representation of A is quasidiagonal.

Definition 5.1.3. An operator T on $\mathcal{H}$ is called **quasitriangular** if there exists an increasing sequence of finite-rank projections $P_1 \leq P_2 \leq ...$ converging strongly to

I such that $\|TP_n - P_nTP_n\| \to 0$; it is called **bi-quasitriangular** if T and T^* are quasitriangular.

A QD operator is automatically bi-quasitriangular, and so if π is a QD representation of a C^*-algebra A, then every $\pi(a) \in \pi(A)$ is bi-quasitriangular. In particular, if π is an essential representation of a separable RFD C^*-algebra, then every $\pi(a) \in \pi(A)$ is bi-quasitriangular.

5.2 A maximum in finite dimensions

For an analytic polynomial p and universal contraction x, von Neumann's Inequality (Theorem 1.0.2) actually gives us a maximum,

$$\|p(x)\| = \max\{|p(a)| : a \in \overline{\mathbb{D}}\}.$$

It turns out that we also have a maximum for $*$-polynomials. Moreover, the dimension required to achieve this maximum value depends only on the degree of q.

Theorem 5.2.1. *Let q be a noncommutative $*$-polynomial of degree d. Then*

$$\|q(x)\| = \sup\{\|q(a)\| : a \in \mathbb{M}_n, n \leq 2^d, \|a\| \leq 1\}.$$

The proof is a simplified version of Fritz, Netzer, and Thom's proof of [27, Lemma 2.7], which itself is an adaptation of Choi's argument in [16, Theorem 7] that $C^*(\mathbb{F}_n)$ is RFD.

Proof. Let q be a noncommutative $*$-polynomial of degree d. From the GNS construction, we know that there exists some representation $\pi_0 : C^*_u\langle x : \|x\| \leq 1\rangle \to B(\mathcal{H}_0)$

and a unit vector $\xi \in \mathcal{H}_0$ so that $\|q(x)\| = \|\pi_0(q(x))\xi\|$. Define

$$\mathcal{H} = \mathrm{span}\{\pi_0(g(x)) : g \text{ is a } *\text{-monomial of degree } \ell(g) \leq d\}.$$

Note that $\dim(\mathcal{H}) \leq 2^d$. Let $P \in B(\mathcal{H}_0)$ be the orthogonal projection onto $\mathcal{H}$. Then $P\pi_0(x)P$ is a contraction, and so the assignment $x \mapsto P\pi_0(x)P$ induces a unital $*$-homomorphism $\pi : C_u^*\langle x : \|x\| \leq 1\rangle \to C^*(P\pi_0(x)P, I_{\mathcal{H}}) \subseteq B(\mathcal{H})$. Moreover,

$$\begin{aligned}\|\pi(a)\xi\| &= \|\pi(q(x))\xi\| = \|q(P\pi_0(x)P)\xi\| \\ &= \|q(\pi_0(x))\xi\| = \|\pi_0(q(x))\xi\| = \|\xi\|.\square\end{aligned}$$

Let A_{00} denote the set of elements in $C_u^*\langle x : \|x\| \leq 1\rangle$ that achieve their norm under some finite-dimensional representation as in Section 3.3. Then Theorem 5.2.1 says that the set $\mathcal{P}_*$ of noncommutative $*$-polynomials in x is contained in A_{00}. As the next theorem shows, not every element of $C_u^*\langle x : \|x\| \leq 1\rangle$ achieves its norm under a finite-dimensional representation, but there are some elements in $C_u^*\langle x : \|x\| \leq 1\rangle \backslash \mathcal{P}_*$ that do.

Theorem 5.2.2. *Let A_{00} be defined as above. Then*

$$\mathcal{P}_* \subsetneq A_{00} \subsetneq C_u^*\langle x : \|x\| \leq 1\rangle.$$

Proof. By Theorem 3.2.4, we know that $A_{00} \subsetneq C_u^*\langle x : \|x\| \leq 1\rangle$ iff $C_u^*\langle x : \|x\| \leq 1\rangle$ has a simple, infinite-dimensional AF subquotient. The CAR algebra $\mathbb{M}_{2^\infty}$ is a simple, infinite-dimensional AF C^*-algebra, which is singly generated by [81]. Hence, it is isomorphic to a quotient of $C_u^*\langle x : \|x\| \leq 1\rangle$, and so

$$A_{00} \subsetneq C_u^*\langle x : \|x\| \leq 1\rangle.$$

It also follows from Theorem 3.3.2 that A_{00} is not closed under addition (or multiplication) and hence cannot equal $\mathcal{P}_*$, i.e.

$$\mathcal{P}_* \subsetneq A_{00}. \qquad \square$$

5.3 Noncommutative $*$-polynomials and nilpotent approximations

In this section, we study the relationship between representations of the universal unital contraction algebra, $C_u^*\langle x : \|x\| \leq 1\rangle$, and families of universal C^*-algebras, which satisfy certain algebraic relations.

Fix a separable Hilbert space $\mathcal{H}$. We begin with an observation which combines results of Choi for the free group $\mathbb{F}_\infty$ on countably many generators ([16, Theorem 1 & Corollary 2]) with Apostol, Foias, and Voiculescu's characterization of the norm closure of nilpotents in $B(\mathcal{H})$ ([5, Theorem 2.7]).

Proposition 5.3.1. *Any separable projective C^*-algebra A has no nontrivial projections, and if $(\pi, \mathcal{H})$ is a faithful representation of A, then π is essential.*

Moreover, if π is a faithful representation, then for any $a \in A$, $\pi(a)$ is the norm limit of nilpotent operators in $B(\mathcal{H})$ iff its spectrum is connected and contains 0.

Recall from Section 2.3 that a projective C^*-algebra is necessarily non-unital, but a unital C^*-algebra can be considered projective if we restrict the definition to unital

$*$-homomorphisms between unital C^*-algebras. Moreover, a non-unital C^*-algebra is projective iff its unitization is projective as a unital C^*-algebra.

Proof. Note that it will suffice to prove the unital case.

Since it is separable, A is isomorphic to a quotient of $C^*(\mathbb{F}_\infty)$. By projectivity, this isomorphism lifts to an embedding of A into $C^*(\mathbb{F}_\infty)$. Since $C^*(\mathbb{F}_\infty)$ has no nontrivial projections ([16, Theorem 1]), neither does A.[1] It then follows, by the same argument as for [16, Corollary 2], that any faithful representation of A trivially intersects $K(\mathcal{H})$.

For the second claim, let π be a faithful – and hence essential – representation of A. Recall from [5, Theorem 2.7] that an operator $a \in B(\mathcal{H})$ is the norm limit of nilpotent operators iff a is bi-quasitriangular, $\sigma(a)$ and $\sigma_e(a)$ are connected, and $0 \in \sigma_e(a)$. Since π is essential, $\sigma(\pi(a)) = \sigma_e(\pi(a))$ for any $\pi(a) \in \pi(A)$. Because A is projective, it is RFD by [57, Theorem 11.2.1], and hence it is also QD. Again, since π is essential, it follows from [85] that π is QD, and so every element $\pi(a) \in \pi(A)$ is automatically bi-quasitriangular. Hence we have that, $\pi(a)$ is the norm limit of nilpotent operators in $B(\mathcal{H})$ iff $\sigma(\pi(a))$ is connected and contains zero. □

Lemma 5.3.2. *Let $x \in B(\mathcal{H})$ be a universal contraction operator and $\lambda \in \mathbb{D}$. Then there exists a sequence of contractive operators $(y_n) \subseteq B(\mathcal{H})$ that converges in norm to x and satisfies $(y_n - \lambda I)^n = 0$ for each $n \geq 1$.*

Note that, since x is a universal contraction and since homomorphisms preserve invertibility, the spectrum of x must be as large as possible, i.e. the closed unit disk $\overline{\mathbb{D}}$. So, when $\lambda = 0$, this lemma quickly follows from Propositions 2.3.1 and 5.3.1. When $\lambda \neq 0$, we have to take a little care to ensure the y_n's are contractions.

Proof. Since $\sigma(x - \lambda I) = \mathbb{D} - \lambda$ is connected and contains 0, by Proposition 5.3.1, there is a sequence $(z_n) \subseteq B(\mathcal{H})$ of nilpotents so that $z_n \to x - \lambda I$ in norm. Then

[1] This fact also follows from [39, Proposition 3].

$z_n + \lambda I \to x$ in norm, and, by possibly repeating a few terms and adding a few zeros at the beginning, we can assume $((z_n + \lambda I) - \lambda I)^n = z_n^n = 0$ for all $n \geq 1$. However, each $z_n + \lambda I$ may not be a contraction. To remedy this, we define $y_n := c_n z_n + \lambda I$ for each $n \geq 1$ where

$$c_n = \begin{cases} 1 & : \|z_n + \lambda I\| \leq 1 \\ \frac{1-|\lambda|}{\|z_n+\lambda\|-|\lambda|} & : \|z_n + \lambda I\| > 1 \end{cases}$$

Then $(y_n - \lambda I)^n = (c_n z_n)^n = 0$ for each $n \geq 1$. Since $\|z_n + \lambda I\| \to \|x\| = 1$, it follows that $c_n \to 1$ and hence that $y_n \to x$ in norm. Moreover, if $\|z_n + \lambda I\| \leq 1$, then $y_n = c_n z_n + \lambda I = z_n + \lambda I$ is a contraction. If $\|z_n + \lambda I\| > 1$, then $1 > c_n$, and we have that

$$\begin{aligned} \|y_n\| = \|c_n z_n + \lambda I\| &= \|c_n(z_n + \lambda I) + (1 - c_n)\lambda I\| \leq c_n\|z_n + \lambda I\| + |\lambda||1 - c_n| \\ &= c_n\|z_n + \lambda I\| + |\lambda| - |\lambda|c_n = c_n(\|z_n + \lambda I\| - |\lambda|) + |\lambda| = (1 - |\lambda|) + |\lambda| = 1. \end{aligned}$$

Thus (y_n) is the desired sequence. □

Remark 5.3.3. Let S be Hadwin's universal contraction operator from Example 1.0.1. Herrero shows in [44, Corollary 4.8] (along with the remark that follows) that for $x = S$, the sequence (y_n) from Lemma 5.3.2 can moreover be chosen to be block-diagonal with finite-dimensional blocks.

Remark 5.3.4. The universal contraction $x \in B(\mathcal{H})$ from Lemma 5.3.2 cannot be the norm limit of nilpotent contractions that lie *in* $C^*(x, I)$. Indeed, if $a \in C^*(x, I)$ is nilpotent and $\pi : C^*(x, I) \to \mathbb{C}$ maps $x \mapsto 1$, then

$$\|x - a\| \geq \|\pi(x) - \pi(a)\| = \|1 - 0\| = 1.$$

Thanks to Don Hadwin for originally pointing out this fact.

More generally, we can say the following.

Corollary 5.3.5. *Let $x \in B(\mathcal{H})$ be a universal contraction operator. Suppose r is a polynomial with at least one root in $\mathbb{D}$. Then there exists a sequence of contractive operators $(y_n) \subseteq B(\mathcal{H})$ that converges in norm to x and satisfies $(r(y_n))^n = 0$ for each $n \geq 1$.*

Moreover, the same holds for any holomorphic function f that vanishes at some $\lambda \in \mathbb{D}$.

Proof. Let $r(z)$ be any polynomial with at least one root $\lambda \in \mathbb{D}$. Then $r(z) = (z - \lambda)s(z)$ for some polynomial s. By Lemma 5.3.2, there exists a sequence (y_n) of contractions in $B(\mathcal{H})$ so that $y_n \to x$ in norm and $(y_n - \lambda I)^n = 0$ for all $n \geq 0$. Then,

$$r(y_n)^n = (y_n - \lambda I)^n s(y_n)^n = 0.$$

This argument can be generalized to any holomorphic function $f(z)$ that decomposes as $f(z) = (z - \lambda)h(z)$ where $h(z)$ is holomorphic. □

Theorem 5.3.6. *Let $\pi : C_u^*\langle x : \|x\| \leq 1\rangle \to B(\mathcal{H})$ be a faithful, nondegenerate representation, $\lambda \in \mathbb{D}$, and $r(z) = z - \lambda$. For each $n \geq 1$, let*

$$\phi_n : C_u^*\langle x : \|x\| \leq 1\rangle \to C_u^*\langle x_n : \|x_n\| \leq 1,\ r(x_n)^n = 0\rangle$$

be the $*$-homomorphisms induced by mapping $x \mapsto x_n$. Then there exists a sequence of $*$-homomorphisms

$$\psi_n : C_u^*\langle x_n : \|x_n\| \leq 1,\ r(x_n)^n = 0\rangle \to B(\mathcal{H})$$

such that $\psi_n \circ \phi_n$ converges to π pointwise in norm.

Proof. By Lemma 5.3.2, there exists a sequence $(y_n) \subseteq B(\mathcal{H})$ that converges in norm to $\pi(x)$ and satisfies $\|y_n\| \leq 1$ and $(r(y_n))^n = 0$ for each $n \geq 1$. For each $n \geq 1$, let

$$\psi_n : C_u^*\langle x_n : \|x_n\| \leq 1,\ r(x_n)^n = 0\rangle \to C^*(y_n, I) \subseteq B(\mathcal{H})$$

be the canonical surjective unital $*$-homomorphsim induced by mapping $x_n \mapsto y_n$. Then, for any $a \in C_u^*\langle x : \|x\| \leq 1\rangle$,

$$\|\psi_n \circ \phi_n(a) - a\| \to 0.$$

□

From [44, Corollary 4.8] or Theorem 5.3.6, we can immediately conclude the following corollary.

Corollary 5.3.7. *The $*$-homomorphisms $\{\phi_n\}$ from Theorem 5.3.6 separate the elements of $C_u^*\langle x : \|x\| \leq 1\rangle$.*

Since the direct sum of a separating family of $*$-homomorphisms is injective, we also have the following formulation of Corollary 5.3.7.

Corollary 5.3.8. *Let $\lambda \in \mathbb{D}$ and $r(z) = z - \lambda$. If x denotes the generator of $C_u^*\langle x : \|x\| \leq 1\rangle$ and x_n denotes the generator of $C_u^*\langle x_n : \|x_n\| \leq 1,\ r(x_n)^n = 0\rangle$, then the unital $*$-homomorphism*

$$\oplus\phi_n : C_u^*\langle x : \|x\| \leq 1\rangle \to C^*(\oplus x_n, I) \subseteq \prod_n C_u^*\langle x_n : \|x_n\| \leq 1,\ r(x_n)^n = 0\rangle$$

is a $*$-isomorphism, where $\{\phi_n\}$ is the family from Theorem 5.3.6.

Proof. From Corollary 5.3.7, we know $\{\phi_n\}$ forms a separating family of unital *-homomorphisms on $C^*_u\langle x : \|x\| \leq 1\rangle$, and therefore, the map

$$\oplus\phi_n : C^*_u\langle x : \|x\| \leq 1\rangle \to C^*(\oplus x_n, I)$$

is injective. Since $\| \oplus x_n\| = \sup_n \|\phi_n(x)\| = 1$, the surjectivity of $\oplus\phi_n$ follows from universality of x. □

Remark 5.3.9. In the special case where $r(z) = z$, this corollary can be phrased more palatably to say that the direct sum of the family of universal nilpotent contractions of all finite orders is a universal contraction.

The next corollary says in essence that, not only is $C^*_u\langle x : \|x\| \leq 1\rangle$ RFD, but its separating family of finite-dimensional representations can be chosen so that the generator maps to a matrix with spectrum $\{\lambda\}$ for some fixed $\lambda \in \mathbb{D}$. Since Herrero showed that, for any $\lambda \in \mathbb{D}$, the universal contraction S from Example 1.0.1 is the norm limit of block-diagonal operators with spectrum $\{\lambda\}$, the blocks of this sequence can be used to generate such a family of finite-dimensional representations. We give a different proof using Theorem 5.3.6.

Corollary 5.3.10. *Let q be any noncommutative $*$-polynomial q in one variable, T any contractive Hilbert space operator T, and $\lambda \in \mathbb{D}$. Then,*

$$\begin{aligned}\|q(T)\| &\leq \sup\{\|q(a)\| : a \in \mathbb{M}_n, n \geq 1,\ \|a\| \leq 1,\ \textit{and}\ (a - \lambda I)^n = 0\} \\ &= \sup\{\|q(a)\| : a \in \mathbb{M}_n, n \geq 1, \|a\| \leq 1, \sigma(a) = \{\lambda\}\} \\ &= \|q\|_{V_\lambda},\end{aligned}$$

where V_λ is the noncommutative variety defined in Section 1.

Proof. Let $r(z) = z - \lambda$, and let $\{\phi_n\}$ be as in Theorem 5.3.6. By [59, Theorem 10], for each $n \geq 1$, $C^*_u\langle x_n : \|x_n\| \leq 1,\ r(x_n)^n = 0\rangle$ is RFD, which means it has separating family of finite-dimensional representations $\{\rho_{n,k}\}_{k\in\mathbb{N}}$. Then, for each $a \in C^*_u\langle x : \|x\| \leq 1\rangle$,

$$\|a\| = \sup_{n,k\in\mathbb{N}} \|(\rho_{n,k} \circ \phi_n)(a)\|.$$

Note that for each $n, k \in \mathbb{N}$, $(\rho_{n,k} \circ \phi_n)(x)$ is a finite-dimensional operator with $\|(\rho_{n,k} \circ \phi_n)(x)\| \leq 1$ which satisfies $r((\rho_{n,k} \circ \phi_n)(x))^n = 0$. Hence, for any noncommutative $*$-polynomial q,

$$\begin{aligned} \|q(x)\| &= \sup_{n,k\in\mathbb{N}} \|(\rho_{n,k} \circ \phi_n)(q(x))\| = \sup_{n,k\in\mathbb{N}} \|q((\rho_{n,k} \circ \phi_n)(x))\| \\ &\leq \sup\{\|q(a)\| : a \in \mathbb{M}_n(\mathbb{C}), n \geq 1,\ \|a\| \leq 1,\ \text{and } r(a)^n = 0\} \\ &= \|q\|_{V_\lambda} \leq \|q(x)\|. \end{aligned}$$

Because x is a universal contraction, the inequality is established for any contractive Hilbert space operator. □

Remark 5.3.11. In particular, we can think of a universal contraction operator as a countable direct sum of contractive nilpotent matrices.

However, it will not suffice to take the direct sum of $n \times n$ nilpotent Jordan blocks, or any weighted shift for that matter. Indeed, any weighted shift will satisfy the equation $(T^*T)(TT^*) - (TT^*)(T^*T) = 0$, and so will its image under a $*$-homomorphism.

Using Corollary 5.3.5, we can replace $r(z)$ in Theorem 5.3.6 and Corollaries 5.3.7, 5.3.8, and 5.3.10 with any polynomial with at least one root in $\mathbb{D}$. In fact, in all but Corollary 5.3.10, we can replace $r(z)$ with a holomorphic function that vanishes at some $\lambda \in \mathbb{D}$. We may also replace q in Corollary 5.3.10 with any noncommutative continuous function in the style of Hadwin, Kaonga, and Mathes in [38].

However, we cannot replace $\lambda \in \mathbb{D}$ with $\lambda \in \mathbb{C}\backslash\mathbb{D}$ in any of the preceding results. Of course, if $|\lambda| > 1$, then no contraction $y \in B(\mathcal{H})$ can satisfy $(y-\lambda I)^n = 0$ (because the spectral radius cannot exceed the norm). On the other hand, if $y \in B(\mathcal{H})$ is any contractive operator with $(y - \lambda I)^n = 0$ for some $n \geq 1$ and some $\lambda \in \partial\mathbb{D}$, then $y = \lambda I$. Indeed, by [59, Lemma 4], we know that y is unitarily equivalent to an upper triangular array $(y_{ij})_{i,j\geq 1}$ with $y_{ii} = \lambda$ for $i \geq 1$. But, since $\|y\| \leq 1$, it follows that $y_{ij} = 0$ for $i \neq j$, and $y = \lambda I$. In other words, for any $\lambda \in \partial\mathbb{D}$ and $n \geq 0$,

$$C_u^*\langle x : \|x\| \leq 1, (x-\lambda)^n = 0\rangle \simeq \mathbb{C}$$

via the map $x \mapsto \lambda$.

Remark 5.3.12. The previous argument shows that $\|N + \lambda I\| > 1$ for any $\lambda \in \partial\mathbb{D}$ and any nonzero nilpotent operator N. (Just take $y = N + \lambda I$, and run the previous argument as a contradiction.) However, this fails if we assume only that N is quasi-nilpotent, i.e. $\sigma(N) = \{0\}$. For example, let V be the Volterra integration operator, and $N = (1+V)^{-1} - I$. We know from [40, Problem 190] that $\|N+I\| = \|(1+V)^{-1}\| = 1$, and $\sigma(N) = \{0\}$. However, $N \neq 0$.

Since the representations $\{\rho_{n,k} \circ \phi_n\}$ from the proof of Corollary 5.3.10 are separating, by Theorem 3.1.2, $C_u^*\langle x : \|x\| \leq 1\rangle$ has a dense subset of elements that attain their norm under one of these representations. However, this dense subset need not contain $\mathcal{P}_*$, as the next example shows.

Example 5.3.13. Let x be a universal contraction. Then $x + x^*$ cannot attain its norm under a finite-dimensional representation mapping x to a nilpotent operator.

First note that $\|x+x^*\| = 2$, which can be seen by mapping x to 1. Now, suppose $a \in \mathbb{M}_n$ is nilpotent with $\|a\| = 1$. Since $a + a^*$ is self-adjoint, if $\|a + a^*\| = 2$, then there exists a unit vector v such that either

$$(a + a^*)v = 2v \text{ or } (a + a^*)v = -2v.$$

Note that av and a^*v are in the unit ball, and v and $-v$ are in the unit sphere. So, if $(a + a^*)v = 2v$, then $\frac{1}{2}(av + a^*v) = v$, which means $av = a^*v = v$ and so $1 \in \sigma(a)$. Similarly, if $(a + a^*)v = -2v$, then $-v$ is a convex combination of av and a^*v, which implies that $av = -v$ and $-1 \in \sigma(a)$. Either way, we contradict a being nilpotent.

We mentioned in the introduction that, although we can use universal C^*-algebras to circumvent dilation arguments, we can still modify some classical techniques to recover some of our results of this section. We conclude this section by proving a special case of Corollary 5.3.10 using an "asymptotic *-monomial dilation,"[2] as in Choi's proof that $C^*(\mathbb{F}_2)$ is RFD ([16, Theorem 7]).

Proposition 5.3.14. *Let T be a contractive Hilbert space operator. Then for any noncommutative $*$-polynomial q*

$$\|q(T)\| \leq \sup\{\|q(a)\| : a \in V_0\}.$$

Proof. For each $n > 1$, let $J_n \in \mathbb{M}_n$ denote the $n \times n$ Jordan block with eigenvalue 0. By [32, Proposition], there is a unit vector $\xi_n \in \mathbb{C}^n$ so that $w(J_n) = \langle J_n \xi_n, \xi_n \rangle = \cos \frac{\pi}{n+1}$, where $w(J_n)$ is the numerical radius of J_n. Let $U_n \in \mathbb{M}_n$ be a unitary so that

[2] Arveson has characterized in [8, Theorem 1.3.1] the contractive Hilbert space operators that can be power dilated to contractive nilpotent operators– for powers up to the order of the nilpotent. However, we would need $*$-monomial dilations to recover Proposition 5.3.14.

$U_n e_1 = \xi_n$ (where e_1 is from the standard basis of $\mathbb{C}^n$). For each $n > 1$, define

$$A_n := U_n^* J_n U_n = (a_{ij}^{(n)})_{1 \leq i,j \leq n}.$$

Then A_n is a nilpotent contraction with $a_{11}^{(n)} = \cos \frac{\pi}{n+1}$.

Let $\mathcal{H} = \ell^2$ with the standard basis $\{e_1, e_2, ...\}$; let P_n be the orthogonal projection of $\mathcal{H}$ onto span$\{e_1, ..., e_n\}$; let x be a universal contraction operator on $\mathcal{H}$; and let $x_n := P_n x P_n$ for each $n > 1$. Then $x_n \otimes A_n \in P_n B(\mathcal{H}) P_n \otimes \mathbb{M}_n$ is a nilpotent contraction, and, if we view $P_n B(\mathcal{H}) P_n$ as $\mathbb{M}_n$, then the assignments $x \mapsto x_n \otimes A_n$ induce a family of finite-dimensional representations $\pi_n : C^*(x, I) \to \mathbb{M}_{n^2}$ where $\pi_n(x)$ is nilpotent. To see that this family is separating, it will suffice to show that $\oplus_n \pi_n$ is isometric, which will follow from showing that $\|q(x)\| = \sup_n \|q(x_n \otimes A_n)\|$ for any noncommutative $*$-polynomial q. To that end, let q be any nonzero noncommutative $*$-polynomial, and let $\varepsilon > 0$.

For each $n > 1$, define $C_n = (c_{ij}^{(n)}) \in \mathbb{M}_{n-1}$ by $c_{ij}^{(n)} := a_{i+1,j+1}^{(n)}$ for each $1 \leq i, j \leq n-1$. Since $a_{11}^{(n)} = \cos \frac{\pi}{n+1} \to 1$ as $n \to \infty$, we have

$$\|[a_{1j}^{(n)}]_{2 \leq j \leq n}\|_{\ell_n^2} \to 0 \text{ and } \|[a_{i1}^{(n)}]_{2 \leq i \leq n}\|_{\ell_n^2} \to 0$$

as $n \to \infty$, and so

$$\begin{aligned} \|A_n - \cos(\tfrac{\pi}{n+1}) \oplus C_n\| &= \left\| \begin{pmatrix} 0 & a_{12}^{(n)} & ... & a_{1n}^{(n)} \\ a_{21}^{(n)} & 0 & ... & 0 \\ \vdots & \vdots & ... & \vdots \\ a_{1n}^{(n)} & 0 & ... & 0 \end{pmatrix} \right\| \\ &\leq \|[a_{1j}^{(n)}]_{2 \leq j \leq n}\|_{\ell_n^2} + \|[a_{i1}^{(n)}]_{2 \leq i \leq n}\|_{\ell_n^2} \to 0 \end{aligned}$$

as $n \to \infty$, which implies that

$$\|x_n \otimes A_n - x_n \otimes (\cos(\tfrac{\pi}{n+1}) \oplus C_n)\| \to 0$$

as $n \to \infty$. Now, q is Lipschitz on the unit ball of $\mathbb{M}_{n^2} \simeq P_n B(\mathcal{H}) P_n \otimes \mathbb{M}_n$ with Lipschitz constant independent of n. From this we see that

$$\|q(x_n \otimes A_n) - q((x_n \otimes (\cos(\tfrac{\pi}{n+1}) \oplus C_n))\| \xrightarrow[n\to\infty]{} 0. \tag{5.3.1}$$

Assume for simplicity that $\|q(x)\| = 1$. Since x_n converges strong-$*$ to x, so does $\cos(\frac{\pi}{n+1})x_n$, and we have

$$\|q(\cos(\tfrac{\pi}{n+1})x_n)\| > 1 - \frac{\varepsilon}{2} \tag{5.3.2}$$

for n sufficiently large. Combining 5.3.1 and 5.3.2, we have, for n sufficiently large,

$$\begin{aligned}\|q(x_n \otimes A_n)\| &> \|q(x_n \otimes (\cos(\tfrac{\pi}{n+1}) \oplus C_n))\| - \frac{\varepsilon}{2} \\ &= \|q((x_n \otimes \cos(\tfrac{\pi}{n+1})) \oplus (x_n \otimes C_n))\| - \frac{\varepsilon}{2} \\ &= \|q((x_n \otimes \cos(\tfrac{\pi}{n+1})) \oplus q(x_n \otimes C_n)\| - \frac{\varepsilon}{2} \\ &\geq \|q(\cos(\tfrac{\pi}{n+1})x_n)\| - \frac{\varepsilon}{2} \\ &> 1 - \frac{\varepsilon}{2} - \frac{\varepsilon}{2} = 1 - \varepsilon.\end{aligned}$$

Since ε was arbitrary, this shows that $\oplus_n \pi_n$ is an isomorphism. $\square$

Chapter 6

Noncommutative $*$-polynomials in two variables and Connes' Embedding Problem

With a noncommutative $*$-polynomial analogue of von Neumann's inequality in hand, it is natural to then ask whether such an analogue can be obtained for Andô's extension of von Neumann's inequality to polynomials in two variables whose entries are pairs of commuting contractions (Corollary 1.0.4). This extension famously fails in the case of polynomials in more than two variables unless the contractions are doubly commuting. So, if one wants to develop an analogue to Andô's inequality, why not start with the "nicer" case of $*$-polynomials whose entries are pairs of doubly commuting contractions? This leads us to the following questions.

Given any two-variable $*$-polynomial q, can we show that the norm of $q(T_1, T_2)$, as (T_1, T_2) ranges over pairs of doubly commuting Hilbert space contractions, is determined by considering only finite-dimensional contractions? If so, could it actually be determined by pairs of doubly commuting finite-dimensional *nilpotent* contractions? As we shall see, these two questions turn out to be equivalent, and both equivalent to Connes' Embedding Problem.

Connes' Embedding Problem(/Question/Conjecture) merits a brief introduction.

As was alluded in Chapter 1, finite-dimensional approximation properties prove to be, not just useful tools in studying C^*- and other operator algebras, but key components of many of the most important problems in the field. A quintessential example of this is a question of Alain Connes' ([17]), which asks whether any finite von Neumann algebra is "approximable" by matrix algebras.

Question 6.0.1 (Connes' Embedding Problem/Question/Conjecture)**.** Every finite von Neumann algebra with separable pre-dual is embeddable into an ultrapower $\mathcal{R}^\omega$ of the hyperfinite II_1-factor $\mathcal{R}$.

Instead of this formulation, we will consider another conjecture which was proven to be equivalent to Connes' Embedding Problem by Kirchberg in his ground breaking work [51]. We will state it in Section 6.1.3 after establishing a few preliminary definitions and results.

6.1 Preliminaries

6.1.1 Weak Injectivity and Weak Conditional Expectations

Though it is readily verifiable that, for any C^*-algebras A, B and C with $A \subseteq B$,

$$A \otimes_{min} C \subseteq B \otimes_{min} C,$$

the corresponding containment can easily fail for the maximal tensor product because subalgebras tend to have more representations.

Unfortunately, it is often difficult to verify directly whether or not the maximal tensor product with a certain algebra respects a given embedding. However, there is an extremely useful characterization for when this always occurs for a given embedding. The result is attributed to Lance ([54] see [15, Proposition 3.6.6] for a proof). We will give it as a proposition/definition hybrid.

Proposition 6.1.1. *Let A and B be unital C^*-algebras and $1_B \in A \subset B$ an inclusion. We say the inclusion is* **relatively weakly injective** *if any of the following equivalent conditions hold.*

1. *There exists a completely positive contractive (cpc) map $\phi : B \to A^{**}$ such that $\phi|_A$ agrees with the canonical embedding $A \hookrightarrow A^{**}$. Such a map is called a* **weak conditional expectation**.

2. *Given any representation $\pi : A \to B(\mathcal{H})$, there exists a ccp map $\rho : B \to \pi(A)''$ such that $\rho(a) = \pi(a)$ for all $a \in A$.*

3. *There exists a ucp map $\phi : B \to \pi_u(A)''$ so that $\phi(a) = \pi_u(a)$ for all $a \in A$ where $(\pi_u, \mathcal{H}_u)$ is the universal representation of A from the Gelfand-Naimark Theorem.*

4. *For any C^*-algebra C, there is a natural inclusion*

$$A \otimes_{max} C \subseteq B \otimes_{max} C.$$

If a unital C^*-algebra A embeds relatively weakly injectively into $B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$, then it is said to have Lance's **Weak Expectation Property** (WEP). As with the previous proposition, we have the following characterizations for the WEP, also due to Lance ([54]).

Proposition 6.1.2. *Let A be a unital C^*-algebra. The following are equivalent to A having the WEP.*

1. *There is a weak conditional expectation from* $B(\mathcal{H}_u)$ *onto* $\pi_u(A)''$ *that respects the image of* A *under its universal Gelfand-Naimark representation* π_u.

2. *For any* C^**-algebra* B *such that* A *embeds into* B*, there is a weak conditional expectation* $B \to A^{**}$.

3. *For any* C^**-algebra* B *such that* A *embeds into* B *and any representation* $\pi : A \to B(\mathcal{H})$ *there exists a cpc map* $\rho : B \to \pi(A)''$ *such that* $\rho(a) = \pi(a)$ *for all* $a \in A$.

Example 6.1.3. 1. Injective von Neumann algebras have the WEP. By Arveson's extension theorem, this includes $B(\mathcal{H})$.

2. Nuclear C^*-algebras have the WEP.

6.1.2 Local Lifting Property

Let A and B be C^*-algebras and J a closed two-sided ideal in B with quotient map $\pi : B \to B/J$. A cpc map $\phi : A \to B/J$ is **liftable** if there is a cpc map $\psi : A \to B$ such that $\pi \circ \psi = \phi$. We say a cpc map $\phi : A \to B/J$ is **locally liftable** if for any finite dimensional operator system $S \subseteq A$ there is a cpc map $\psi : S \to B$ such that $\pi \circ \psi = \phi|_S$.

A unital C^* algebra A has the **(local) lifting property** or (L)LP if any ucp[1] map from A into a quotient C^*-algebra is locally liftable. A non-unital C^*-algebra is liftable iff its unitization is.

Nuclear C^*-algebras have the LP by the Choi-Effros Lifting Theorem. Another key class of examples comes from Kirchberg ([50, Lemma 3.3]).

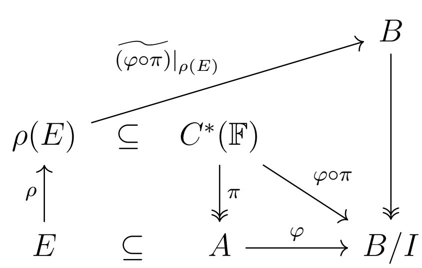
[1] Yes, we did unceremoniously shift from cpc maps to ucp maps, but it turns out to be sufficient to restrict ourselves to ucp maps. See [15] 13.1.2 for an argument.

Lemma 6.1.4. *For any free group* $\mathbb{F}$, $C^*(\mathbb{F})$ *has the LLP. Moreover,* $C^*(\mathbb{F})$ *has the LP if* $\mathbb{F}$ *has countably many generators.*

This is a crucial class because it allows us to greatly simplify checking whether or not a given C^*-algebra has the LLP.

Proposition 6.1.5. *Let* A *be a* C^**-algebra, and fix an identification of* A *with a quotient* $C^*(\mathbb{F})/J$ *of* $C^*(\mathbb{F})$ *for some free group* $\mathbb{F}$*. Then* A *has the LLP iff the identity on* $C^*(\mathbb{F})/J$ *is locally liftable.*

Proof. If A is non-unital, replace A with its unitization $\tilde{A}$. For simplicity, identify $A = C^*(\mathbb{F})/J$, and let $\pi : C^*(\mathbb{F}) \to C^*(\mathbb{F})/J$ be the quotient map. Let $E \subseteq A$ be a finite dimensional operator system and $\rho : E \to C^*(\mathbb{F})$ the lift of $id_A|_E$ guaranteed by assumption. Let B be a C^*-algebra with closed two-sided ideal I, and $\varphi : A \to B/I$ a ucp map. Then, $\varphi \circ \pi : C^*(\mathbb{F}) \to B/I$ is a ucp map and $\rho(E) \subseteq C^*(\mathbb{F})$ is a finite dimensional operator system. Then, since $C^*(\mathbb{F})$ has the LLP, there is a lift $\widetilde{(\varphi \circ \pi)}|_{\rho(E)}$ of $(\varphi \circ \pi)|_{\rho(E)}$ to B.



□

Since the $*$-isomorphism identifying a separable projective C^*-algebra with a quotient of $C^*(\mathbb{F}_\infty)$ lifts to a $*$-homomorphism, we immediately conclude the following.

Proposition 6.1.6. *Any projective C^*-algebra has the LLP (i.e. its unitization has the LLP).*

In particular, $C^*_u\langle x : \|x\| \leq 1\rangle$ has the LLP.

6.1.3 A few results of Kirchberg

In [51], Kirchberg establishes deep and elegant characterizations of the WEP and LLP as well as a "tensorial duality" between the two.

Proposition 6.1.7. *[51, Proposition 1.1] Let A and B be C^*-algebras, $\mathbb{F}$ a non-abelian free group, and $\mathcal{H}$ an infinite-dimensional Hilbert space.*

1. *If A has the LLP and B has the WEP, then $A \otimes_{min} B \simeq A \otimes_{max} B$.*
2. *$B(\mathcal{H}) \otimes_{min} B \simeq B(\mathcal{H}) \otimes_{max} B$ iff B has the LLP.*
3. *$A \otimes_{min} C^*(\mathbb{F}) \simeq A \otimes_{max} C^*(\mathbb{F})$ iff A has the WEP.*

Now we are also ready to state Kirchberg's characterization of Connes' Embedding Problem that we alluded to in the introduction of this chapter. This is an augmented and abridged version of [51, Proposition 8.1] (omitting some of the equivalent conditions and adding some that quickly follow).

Theorem 6.1.8 (Kirchberg)**.** *The following conjectures are equivalent.*

1. *Every finite von Neumann algebra with separable pre-dual is embeddable into an ultrapower $\mathcal{R}^\omega$ of the hyperfinite II_1-factor $\mathcal{R}$.*[2]

[2] For those familiar with the terminology, Kirchberg also includes in [51, Proposition 8.1] that this is equivalent to the existence of a $*$-homomorphism $h : M \to \mathcal{R}^\omega$ such that $\tau = \tau_\omega h$ where M is any separable von Neumann algebra M with faithful normal tracial state τ.

2. *The canonical surjection* $C^*(\mathbb{F}) \otimes_{max} C^*(\mathbb{F}) \to C^*(\mathbb{F}) \otimes_{min} C^*(\mathbb{F})$ *is injective (where* $\mathbb{F}$ *is any free group).*

3. $C^*(\mathbb{F}) \otimes_{max} C^*(\mathbb{F})$ *is RFD.*[3]

4. *The LLP implies the WEP.*

5. $C^*(\mathbb{F})$ *has the WEP.*

6.2 Andô's inequality for $*$-polynomials and Connes' Embedding Problem

We begin with explicit formulations of the two questions from the introduction pertaining to noncommutative $*$-polynomial analogues to Andô's von Neumann Inequality (Corollary 1.0.4). Recall that a pair of operators T_1, T_2 are called doubly commuting (d.c.) if $T_1T_2 = T_2T_1$ and $T_1T_2^* = T_2^*T_1$.

Question 6.2.1. Given any pair T_1 and T_2 of d.c. contractions on a Hilbert space and q a noncommutative $*$-polynomial in two variables, must the following inequality hold?

$$\|q(T_1, T_2)\| \leq \sup\{\|q(a_1, a_2)\| : a_i \in \mathbb{M}_n, n \geq 1, \|a_i\| \leq 1, a_i \text{ d.c.}\}$$

If Question 6.2.1 has an affirmative answer, might we be able to sharpen it, as we did in the one variable case in Corollary 5.3.10?

Question 6.2.2. Given any pair T_1 and T_2 of d.c. contractions on a Hilbert space, q a noncommutative $*$-polynomial in two variables, and $\lambda_1, \lambda_2 \in \mathbb{D}$, must the following

[3]This is not included in the proposition in [51]; however, it immediately follows from the proposition and Proposition A.0.1.

inequality hold?

$$\|q(T_1,T_2)\| \le \sup\{\|q(a_1,a_2)\| : a_i \in \mathbb{M}_n, n \ge 1, \|a_i\| \le 1, a_i \text{ d.c.}, (a_i - \lambda_i I)^n = 0\}$$
$$= \sup\{\|q(a_1,a_2)\| : a_i \in \mathbb{M}_n, n \ge 1, \|a_i\| \le 1, a_i \text{ d.c.}, \sigma(a_i) = \{\lambda_i\}\}$$

Of course an affirmative answer to Question 6.2.2 would imply an affirmative answer to Question 6.2.1. Though it is certainly unclear at this point, we shall see in Theorem 6.2.5 that the converse holds as well.

Remark 6.2.3. For matrices T_1 and T_2 with norms strictly less than 1, [1, Theorem 3.1] states that Corollary 1.0.4 can be sharpened to

$$\|p(T_1,T_2)\| \le \|p\|_V,$$

where V is some (distinguished) variety depending on T_1 and T_2. If Question 6.2.2 has an affirmative answer, then we could conclude that for any $*$-polynomial q in two variables and any d.c. contractions T_1 and T_2,

$$\|q(T_1,T_2)\| \le \|q\|_{V_{\lambda_1,\lambda_2}},$$

where λ_1, λ_2 are any elements of $\mathbb{D}$ and

$$V_{\lambda_1,\lambda_2} = \bigcup_n \{(a_1,a_2) \in \mathbb{M}_n : [a_1,a_2] = [a_1,a_2^*] = 0, \|a_i\| \le 1, (a_i - \lambda_i I)^n = 0, i = 1,2\}.$$

Remark 6.2.4. Notice that Question 6.2.1 is equivalent to asking whether the universal unital C^*-algebra generated by two d.c. contractions is RFD. Indeed, the universal unital C^*-algebra generated by two d.c. contractions is isomorphic to

$$C_u^*\langle x_1 : \|x_1\| \leq 1\rangle \otimes_{max} C_u^*\langle x_2 : \|x_2\| \leq 1\rangle,$$

via the map sending the pair of universal d.c. contractions to $x_1 \otimes 1$ and $1 \otimes x_2$.[4] In particular, this means we can think of $(x_1 \otimes 1, 1 \otimes x_2)$ as a universal pair of d.c. contractions.

From this perspective, we see that Question 6.2.1 is asking if $C_u^*\langle x_1 : \|x_1\| \leq 1\rangle \otimes_{max} C_u^*\langle x_2 : \|x_2\| \leq 1\rangle$ is RFD, and Question 6.2.2 is asking if the separating family of finite-dimensional representations can be chosen so that $x_1 \otimes 1$ and $1 \otimes x_2$ map to two d.c. contractions with spectra $\{\lambda_1\}$ and $\{\lambda_2\}$, respectively, where λ_1 and λ_1 are fixed elements in $\mathbb{D}$.

With this remark, a reader familiar with relevant literature may not be too surprised at our "answers" to Questions 6.2.1 and 6.2.2.

Theorem 6.2.5. *The following are equivalent.*

1. *Question 6.2.1 has an affirmative answer.*

2. *Question 6.2.2 has an affirmative answer.*

3. *Connes' Embedding Problem has an affirmative answer.*

The equivalence of (1) and (3) can be proved as a corollary to a result of Pisier ([68, Proposition 16.13]), who has remarked that contents of this proposition was likely already known to Kirchberg. We provide an alternative proof.

[4]For two C^*-algebras A and B, $A \otimes_{max} B$ and $A \otimes_{min} B$ refer to the completion of the algebraic tensor product $A \odot B$ of A and B with respect to the maximal and minimal C^*-norms, respectively. See [15, Chapter 3] for definitions and relevant properties, and see [12, Section 2] for a short discussion on universal C^*-algebras and tensor products.

Instead of proving this outright, it will be more illuminating to break the theorem (and its proof) into two pieces. Lemma 6.2.6 will establish the equivalence of (1) and (2), and Lemma 6.2.7 will establish the equivalence of (2) and (3).

From Remark 6.2.4, we see that (1) and (2) are equivalent iff the following lemma holds.

Lemma 6.2.6. *The following are equivalent.*

1. $C_u^*\langle x_1 : \|x_1\| \leq 1\rangle \otimes_{max} C_u^*\langle x_2 : \|x_2\| \leq 1\rangle$ *is RFD.*

2. *For any* $\lambda_1, \lambda_2 \in \mathbb{D}$, $C_u^*\langle x_1 : \|x_1\| \leq 1\rangle \otimes_{max} C_u^*\langle x_2 : \|x_2\| \leq 1\rangle$ *has a separating family of finite-dimensional representations* $\{\pi_k\}$ *where* $\sigma(\pi_k(x_1 \otimes 1)) = \{\lambda_1\}$ *and* $\sigma((\pi_k(1 \otimes x_2))) = \{\lambda_2\}$.

Proof. One implication is immediate; so, we only need to show (1) $\Rightarrow$ (2). With Corollary 5.3.10, this essentially amounts to the argument that the minimal tensor product of two RFD C^*-algebras is RFD.

Let $\lambda_1, \lambda_2 \in \mathbb{D}$. By Proposition A.0.1, since $C_u^*\langle x_1 : \|x_1\| \leq 1\rangle \otimes_{max} C_u^*\langle x_2 : \|x_2\| \leq 1\rangle$ is RFD, it has a unique C^*-tensor norm, which means it suffices to prove the claim for $C_u^*\langle x_1 : \|x_1\| \leq 1\rangle \otimes_{min} C_u^*\langle x_2 : \|x_2\| \leq 1\rangle$.

We know from the proof of Corollary 5.3.10 that there exist separating families of $*$-homomorphisms $\{\tau_n^{(i)}\}$ of $C_u^*\langle x_i : \|x_i\| \leq 1\rangle$ for $i = 1, 2$ so that $\sigma(\tau_n^{(i)}(x_i)) = \{\lambda_i\}$ for each $n \geq 1$. For $n, m \geq 1$, let $\pi_{n,m} = \tau_n^{(1)} \otimes \tau_m^{(2)}$ denote the $*$-homomorphism on $C_u^*\langle x_1 : \|x_1\| \leq 1\rangle \otimes_{min} C_u^*\langle x_2 : \|x_2\| \leq 1\rangle$ that maps $a \otimes b \mapsto \tau_n^{(1)}(a) \otimes \tau_m^{(2)}(b)$. Then $\{\pi_{n,m}\}_{m,n \in \mathbb{N}}$ form a separating family of finite-dimensional representations of $C_u^*\langle x_1 : \|x_1\| \leq 1\rangle \otimes_{min} C_u^*\langle x_2 : \|x_2\| \leq 1\rangle$. $\square$

By Theorem 6.1.8 and Remark 6.2.4, to show that (2) and (3) from Theorem 6.2.5 are equivalent, it suffices to prove the following lemma.

Lemma 6.2.7. *The following are equivalent.*

1. $C^*(\mathbb{F}_2) \otimes_{max} C^*(\mathbb{F}_2)$ *is RFD.*

2. $C^*_u\langle x_1 : \|x_1\| \leq 1\rangle \otimes_{max} C^*_u\langle x_2 : \|x_2\| \leq 1\rangle$ *is RFD.*

This follows immediately from a result of Pisier ([68, Proposition 16.13]), and the contents of this proposition were likely already known to Kirchberg. We provide an alternative proof.

Proof. To prove (1) $\Rightarrow$ (2), we first recall that $C^*_u\langle x : \|x\| \leq 1\rangle$ has the LLP (by Propositions 2.3.1 and 6.1.6). Then, by assumption and Theorem 6.1.8, it also has the WEP. Hence part 1 of Proposition 6.1.7 tells us that $C^*_u\langle x_1 : \|x_1\| \leq 1\rangle \odot C^*_u\langle x_2 : \|x_2\| \leq 1\rangle$ has a unique C^*-tensor norm, which implies (2) by Proposition A.0.1.

To prove (2) $\Rightarrow$ (1), it will suffice to show that $C^*(\mathbb{F}_2)$ embeds relatively weakly injectively into $C^*_u\langle x : \|x\| \leq 1\rangle$. Indeed, if this is true, then a few applications of Proposition 6.1.1 show that $C^*(\mathbb{F}_2) \otimes_{max} C^*(\mathbb{F}_2)$ embeds into $C^*_u\langle x_1 : \|x_1\| \leq 1\rangle \otimes_{max} C^*_u\langle x_2 : \|x_2\| \leq 1\rangle$. The claim then follows from the fact that residual finite-dimensionality passes to subalgebras.

To that end, let $\pi_u : C^*(\mathbb{F}_2) \to B(\mathcal{H}_u)$ be the universal representation of $C^*(\mathbb{F}_2)$, and let $u_1, u_2 \in B(\mathcal{H}_u)$ be the images of the generators under π_u. Let $a_1, a_2 \in B(\mathcal{H}_u)$ be self-adjoint elements such that $u_j = e^{ia_j}$ and

$$C^*(u_1, u_2) \subseteq C^*(a_1, a_2) = C^*(\tfrac{a_1}{\alpha} + i\tfrac{a_2}{\alpha}) \subseteq C^*(u_1, u_2)'',$$

where $\alpha = \|a_1 + ia_2\|$. Let $s_1 = \frac{1}{2}(x + x^*)$ and $s_2 = \frac{1}{2i}(x - x^*)$ in $C^*_u\langle x : \|x\| \leq 1\rangle$. Then, $e^{i\alpha s_j}$ are unitaries in $C^*_u\langle x : \|x\| \leq 1\rangle$ and

$$C^*(e^{i\alpha s_1}, e^{i\alpha s_2}) \subseteq C^*(s_1, s_2) = C^*_u\langle x : \|x\| \leq 1\rangle.$$

By universality, there exist surjective unital $*$-homomorphisms,

$$\phi : C^*(u_1, u_2) \to C^*(e^{i\alpha s_1}, e^{i\alpha s_2}) \text{ and } \psi : C_u^*\langle x : \|x\| \leq 1\rangle \to C^*(\tfrac{a_1}{\alpha} + i\tfrac{a_2}{\alpha})$$

with $\phi(u_j) = e^{i\alpha s_j}$ and $\psi(s_j) = \frac{a_j}{\alpha}$. Then,

$$\psi \circ \phi(u_j) = \psi(e^{i\alpha s_j}) = e^{i\alpha(\frac{1}{\alpha}a_j)} = e^{ia_j} = u_j.$$

Since this holds for the generators, it holds for $C^*(u_1, u_2)$, and hence $\psi \circ \phi|_{C^*(u_1,u_2)} = id_{C^*(u_1,u_2)}$, and this completes the proof.

□

Remark 6.2.8. We have not addressed the $*$-polynomial analogue for Corollary 1.0.4 for pairs of commuting contractions. A few partial results can be found in [70].

Remark 6.2.9. The universal (unital) C^*-algebra generated by a pair of (unrelated) contractions is RFD. This follows from [26, Theorem 3.2] since

$$C_u^*\langle x_1, x_2 : \|x_i\| \leq 1\rangle \simeq C_u^*\langle x : \|x\| \leq 1\rangle *_{\mathbb{C}} C_u^*\langle x : \|x\| \leq 1\rangle.$$

Actually, in this case, residual finite-dimensionality can be shown directly by adapting the proof of [16, Theorem 7].

Though it is unknown whether or not $C_u^*\langle x_1 : \|x_1\| \leq 1\rangle \otimes_{max} C_u^*\langle x_2 : \|x_2\| \leq 1\rangle$ is RFD, the proof of [15, Proposition 7.4.5] can be adapted to show that it is quasidiagonal. The proof relies heavily on Voiculescu's work on homotopy invariance ([85]) (specifically what appears in [15] as Proposition 7.3.5).

Proposition 6.2.10. *$C_u^*\langle x_1 : \|x_1\| \leq 1\rangle \otimes_{max} C_u^*\langle x_2 : \|x_2\| \leq 1\rangle$ is QD.*

Proof. Let $C^*(x_1, x_2, 1_{\mathcal{H}}) \subseteq B(\mathcal{H})$ be a faithful representation of $C_u^*\langle x_1 : \|x_1\| \leq 1\rangle \otimes_{max} C_u^*\langle x_2 : \|x_2\| \leq 1\rangle$. From [85], we know it suffices to show that $C^*(x_1, x_2, 1_{\mathcal{H}})$

is QD. For each $i = 1, 2$, define $\gamma_i : [0, 1] \to C^*(x_i, 1_{\mathcal{H}})$ to be the norm-continuous segment in the unit ball of $C^*(x_i, 1_{\mathcal{H}})$ such that $\gamma_i(0) = 1_{\mathcal{H}}$ and $\gamma_i(1) = x_i$. Then $\gamma_1(t)$ and $\gamma_2(t)$ are doubly commuting contractions for all $t \in [0, 1]$. So, for each $t \in [0, 1]$, we get $*$-homomorphisms

$$\sigma_t : C^*(x_1, x_2, 1_{\mathcal{H}}) \to C^*(\gamma_1(t), \gamma_2(t), 1_{\mathcal{H}})$$

induced by $x_1 \mapsto \gamma_1(t)$ and $x_2 \mapsto \gamma_2(t)$. Then, for each $a \in C^*(x_1, x_2, 1_{\mathcal{H}})$, the map $t \mapsto \sigma_t(a)$ is norm continuous, and so σ_1 and σ_0 are homotopic $*$-homomorphisms.

The map $\sigma_1 : C^*(x_1, x_2, 1_{\mathcal{H}}) \to B(H)$ is the identity representation, which is injective, and $\sigma_0 : C^*(x_1, x_2, 1_{\mathcal{H}}) \to \mathbb{C}1_{\mathcal{H}}$ is the trivial representation whose image is QD. By [15, Proposition 7.3.5], $C^*(x_1, x_2, 1_{\mathcal{H}})$ is QD. $\square$

Chapter 7

Future Work

In this final section, we will touch on some open questions related to the topics in this thesis.

7.1 Density and residual finite-dimensionality

Can Theorem 3.1.2 be adapted to other settings, such as non-self-adjoint operator algebras? As a potential first step, is there a proof of this theorem that does not use the functional calculus?

7.2 "Strongly RFD" C^*-algebras

I would like to characterize the class of C^*-algebras for which every quotient is RFD or "strongly RFD" C^*-algebras. (This is inspired by Don Hadwin's **strongly quasidiagonal** property studied in [35].) Strong residual finite-dimensionality clearly holds for FDI C^*-algebras, but it also holds for non-FDI C^*-algebras, e.g. RFD Just-Infinite C^*-algebras (see [30]).

Perhaps this question should first be restricted to group C^*-algebras or RFD AF C^*-algebras.

7.3 More questions on the set of finite-dimensional norm attaining elements

For an RFD C^*-algebra A, let

$$A_{00} = \{a \in A : \|a\| = \max_{\substack{\pi \in \mathrm{Irr}_n(A) \\ n<\infty}} \|\pi(a)\|\}$$

denote the set of elements of A that attain their norm under some finite-dimensional representation of A (as in Section 3.3).

Recall that [27, Lemma 2.7] shows that $\mathbb{C}\mathbb{F}_n \subseteq C^*(\mathbb{F}_n)_{00}$. Discussions with A. Thom indicate that this does not hold for all RFD groups. In general, it is unknown for which groups this does hold.

In general, when A is not FDI, Theorem 3.2.4 gives a nonconstructive proof that $A \backslash A_{00}$ is nonempty. Is there a constructive proof of this fact? In particular, can we construct an element in $C^*(\mathbb{F}_2) \backslash C^*(\mathbb{F}_2)_{00}$? One potential candidate involves an element built of words like those in [77, Corollary 1.2].

If A is RFD, Theorem 3.1.2 shows that A_{00} is dense in A, but that still does not paint a complete picture of how "big" A_{00} is in A. Is it meager? Is it co-meager? I would like to explore these questions. (The answer to the first should be no.)

If A is RFD but not FDI, then we know by Theorem 3.3.2 that A_{00} is not an algebra. However, that does not mean that it cannot contain a dense $*$-subalgebra, just as $C^*(\mathbb{F}_n)_{00}$ does. I would like to explore whether or not this always happens.

7.4 More questions on $\| \cdot \|_{M_n}$

Section 4 just scratched the surface of the behavior of the the norms of images of elements of an algebra under finite-dimensional representations.

For instance, even in the case of $C^*(\mathbb{F}_2)$, we can explicitly give $\|a\|_{M_n}$ for very few $a \in \mathbb{C}\mathbb{F}_2$. As a first step, it would help to determine the norm of a in $C^*(\mathbb{F}_2)$. Can we do this for trinomials?

What can be said about the elements of the set $\Lambda(A)$ (for say an RFD C^*-algebra A) if we define $\| \cdot \|_{M_n}$ instead to be the supremum over nondegenerate or even irreducible representations?

Finally, we remark that if A is RFD but not FDI, then $\Lambda(A)$ is conjecturally all nondecreasing sequences in ℓ^∞_+. However, since the Chinese Remainder Theorem holds for only finitely many ideals, our techniques from Theorem 4.1.5 can control only finitely many terms of a sequence $(\|a\|_{M_n})$.

7.5 A $*$-polynomial Andô's Theorem for commuting contractions

As we saw in Chapter 6, Andô's Theorem for $*$-polynomials in two doubly commuting variables (Question 6.2.1) is equivalent to Connes Embedding Problem. However, could we possibly show that the analogue holds for $*$-polynomials in commuting contractions?

7.6 Characterizing the WEP

We say a unital C^*-algebra A **characterizes the WEP** if, for any unital C^*-algebra B, $A \odot B$ has a unique C^*-tensor norm iff B has the WEP. If a C^*-algebra A characterizes the WEP, then $C^*(\mathbb{F}_2)$ has the WEP iff A has the WEP. In other words, Connes' Embedding Problem has a positive solution iff there exists a C^*-algebra that has the WEP and characterizes the WEP. To see this, suppose A characterizes the WEP. Since $B(\mathcal{H})$ has the WEP, $A \odot B(\mathcal{H})$ has a unique tensor norm. By part (2) of Proposition 6.1.7, this means A has the LLP. Now, if $C^*(\mathbb{F}_2)$ has the WEP, then $A \odot C^*(\mathbb{F}_2)$ has a unique C^*-tensor norm by part (1) of Proposition 6.1.7. On the other hand, if A has the WEP, then part (3) of Proposition 6.1.7, $A \odot C^*(\mathbb{F}_2)$ has a unique norm. By assumption, this means $C^*(\mathbb{F}_2)$ has the WEP.

Using the arguments from Lemma 6.2.7 along with Proposition 6.1.7, one can show that if a C^*-algebra A has the LLP and $C^*(\mathbb{F}_2)$ embeds relatively weakly injectively into A, then A characterizes the WEP. (In particular, $C^*_u\langle x : \|x\| \leq 1\rangle$ characterizes the WEP by the arguments from Lemma 6.2.7.)

Does the converse hold? That is, if A characterizes the WEP, does A have the LLP and does $C^*(\mathbb{F}_2)$ embed relatively weakly injectively into A? Note that we can conclude that a C^*-algebra that characterizes the WEP must have the LLP. Indeed, if A characterizes the WEP, then $A \odot B(\mathcal{H})$ must have a unique C^*-tensor norm, since $B(\mathcal{H})$ has the WEP. Then A has the LLP by Proposition 6.1.7. So, really the question is whether $C^*(\mathbb{F}_2)$ embeds relatively weakly injectively into any C^*-algebra that characterizes the WEP. These questions have notable resemblance to the von Neumann-Day conjecture, which asks whether any nonamenable group contains a copy of $\mathbb{F}_2$.

In [46, Section 3], the authors consider related questions in terms of group C^*-

algebras, though not with the emphasis on relatively weakly injective embeddings. In particular, in [46, Proposition 3.5] they show that, if $\mathbb{F}_2$ embeds into a discrete group G and $C^*(G)$ has the LLP, then $C^*(G)$ characterizes the WEP. They ask in the following remark whether these sufficient conditions are necessary. The preceding discussion shows that $C^*(G)$ having the LLP is indeed necessary.

Appendices

Appendix A

Residual finite-dimensionality

In this appendix we state and prove several relevant facts about RFD C^*-algebras that are often alluded to in the literature, but rarely (if ever) proved.

Proposition A.0.1. *If A and B are RFD C^*-algebras, then $A \otimes B$ has a unique C^*-norm (i.e. $A \otimes_{max} B = A \otimes_{min} B$ canonically) iff $A \otimes_{max} B$ is RFD.*

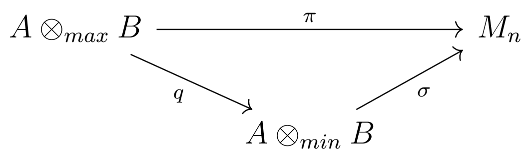
The argument is essentially that, for (unital) RFD C^*-algebras A and B, a finite-dimensional representation of $A \otimes_{max} B$ factors through $A \otimes_{min} B$.

Proof. Sufficiency follows immediately from the fact that $A \otimes_{min} B$ is RFD when A and B are RFD. For the necessity claim, first note that, since both the injective and projective tensor products are RFD, both norms are well-approximated by finite dimensional representations. So, to show that the two norms agree, it suffices to show that for any finite dimensional representation $\pi : A \otimes_{max} B \to M_n$, we can find a representation $\pi' : A \otimes_{min} B \to M_n$ so that

$$\left\| \sum_{i=1}^{n} \pi'(a_i) \otimes \pi'(b_i) \right\| \geq \left\| \sum_{i=1}^{n} \pi(a_i \otimes b_i) \right\|.$$

In fact, given any finite-dimensional representation $\pi : A \otimes_{max} B \to M_n$, we claim that $\pi|_{A \odot B}$ extends to a representation of σ $A \otimes_{min} B$ into M_n, i.e. π factors through

$A \otimes_{min} B$ via the quotient map and a representation σ of $A \otimes_{min} B$:



To that end, let $\pi : A \otimes_{max} B \to M_n$ be a finite dimensional representation and let $\pi_A : A \to M_n$ and $\pi_B : B \to M_n$ representations with commuting ranges so that

$$\pi(a \otimes b) = \pi_A(a)\pi_B(b)$$

for all $a \in A$ and $b \in B$. Then, the natural inclusions $\iota_A : \pi_A(A) \to M_n$ and $\iota_B : \pi_B(B) \to M_n$ are representations with commuting ranges, and so, by the universality of $\otimes_{max}$, there is a unique representation

$$(\iota_A \times \iota_B) : \pi_A(A) \otimes_{max} \pi_B(B) \to M_n$$

with $\pi_A(a) \otimes \pi_B(b) \mapsto \pi_A(a)\pi_B(b)$ for all $a \in A$ and $b \in B$.

On the other hand, let

$$\pi_A \otimes \pi_B : A \otimes_{min} B \to \pi_A(A) \otimes_{min} \pi_B(B)$$

be the representation induced by π_A and π_B, i.e. for all $a \in A$ and $b \in B$

$$(\pi_A \otimes \pi_B)(a \otimes b) = \pi_A(a) \otimes \pi_B(b).$$

Now, since $\pi_A(A)$ and $\pi_B(B)$ are finite-dimensional C^*-algebras, they are nuclear, i.e. $\pi_A(A) \otimes_{max} \pi_B(B) = \pi_A(A) \otimes_{min} \pi_B(B)$. Thus, we may define the representation

$$\sigma := (\iota_A \times \iota_B) \circ (\pi_A \otimes \pi_B) : A \otimes_{min} B \to M_n$$

of $A \otimes_{min} B$ where for $a \otimes b \in A \odot B$,

$$\sigma(a \otimes b) = (\iota_A \times \iota_B)(\pi_A(a) \otimes \pi_B(b)) = \pi_A(a)\pi_B(b) = \pi(a \otimes b).$$

Then, σ is an extension of $\pi|_{A\odot B}$. □

We switch between a C^*-algebra and its unitization often. Here we take care to ensure that residual finite-dimensionality is unaffected.

Proposition A.0.2. *Suppose*

$$0 \to I \to A \to A/I \to 0$$

is a short exact sequence of C^*-algebras. If I and A/I are RFD, then so is A.

If A is RFD, then so is I. However, we have many examples of quotients of RFD C^*-algebras which are not RFD. Indeed, any separable unital C^*-algebra is a quotient of $C^*(\mathbb{F}_\infty)$.

Proof. Let $q : A \to A/I$ be the quotient map. Let $\mathcal{F}$ and $\mathcal{E}$ be separating families of finite-dimensional representations for I and A/I respectively. Let $\mathcal{F}'$ be the family of extensions of the representations $\pi \in \mathcal{F}$ to A, and let $\mathcal{E}'$ be the family of representations $\tilde{\sigma} = \sigma \circ q$ for each $\sigma \in \mathcal{E}$. Now, suppose $a \in A\backslash\{0\}$ and $\tilde{\pi}(a) = 0$ for all $\tilde{\pi} \in \mathcal{F}'$. Then $a \notin I$, and so $q(a) \neq 0$. Hence, $\sigma(q(a)) = \tilde{\sigma}(a) \neq 0$ for some $\tilde{\sigma} \in \mathcal{E}'$. Hence $\mathcal{F}' \cup \mathcal{E}'$ forms a separating family of finite-dimensional representations of A. □

By taking the short exact sequence

$$0 \to A \to \tilde{A} \to \mathbb{C} \to 0,$$

we get the following corollary.

Corollary A.0.3. *A C^*-algebra is RFD iff its unitization is RFD.*

In what follows, for a C^*-algebra A, if A is nonunital, let $\tilde{A}$ denote its unitization, and if A is nonunital, take $\tilde{A} = A$.

We will want to translate this to the setting of tensor products. Considering the algebraic tensors, we see that $\widetilde{A \odot B} \subset \tilde{A} \odot \tilde{B}$, and the inclusion is proper unless $A = \tilde{A}$ and $B = \tilde{B}$. However, residual finite-dimensionality is inherited from $A \otimes_{max} B$ to even $\tilde{A} \otimes_{max} \tilde{B}$.

Proposition A.0.4. *Assume A and B are RFD. Then $A \otimes_{max} B$ is RFD iff $\tilde{A} \otimes_{max} \tilde{B}$ is RFD.*

Proof. Since $A \otimes_{max} B \lhd \tilde{A} \otimes_{max} B \lhd \tilde{A} \otimes_{max} \tilde{B}$, the $(\Leftarrow)$ claim follows. Now, assume that $A \otimes_{max} B$ is RFD, and consider the short exact sequence

$$0 \to A \otimes_{max} B \to \tilde{A} \otimes_{max} B \to \mathbb{C} \otimes_{max} B \to 0.$$

By Proposition A.0.1, $A \otimes_{max} B$ is RFD, and so is $\mathbb{C} \otimes_{max} B$. Then $\tilde{A} \otimes_{max} B$ is RFD by Proposition A.0.2. Repeating the argument also shows that $\tilde{A} \otimes_{max} \tilde{B}$ is RFD. □

In particular, we have that the universal C^*-algebra generated by two doubly commuting contractions is RFD iff the universal unital C^*-algebra generated by two doubly commuting contractions is RFD. This latter C^*-algebra is isomorphic to the maximum tensor product of two copies of the universal unital contraction algebra, which is RFD iff the maximum tensor product of two copies of the universal (nonunital) contraction algebra is RFD.

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