# Products of Closed $k$-Schur Katalan Functions


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## Abstract

Following applications of Catalan functions to resolve conjectures about $k$-Schur functions, which are Schubert homology representatives of the affine Grassmannian, the $K$-theoretic Catalan functions (or Katalan functions) have offered a rich combinatorial structure with connections to the K-homology $K_*(\mathrm{Gr})$, Hopf isomorphic to $\Lambda_{(k)}$. We prove a multiplication rule for the closed $k$-Schur Katalan functions conjectured to be equivalent to a cancellation-free product that matches Lenart and Maeno's Monk rule for quantum Grothendieck polynomials under a $K$-theoretic analogue of the Peterson isomorphism. Our methods include root expansions of the $k$-Schur root ideal, the combinatorics of covers, and the relationship between the two. We conclude with an involution that proves the equivalence of certain Katalan functions; we offer progress alongside this result which could establish the cancellation-free conjecture.

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CHAPTER 1

# Introduction

During the late 19th century, Hermann Schubert began the study of enumerative geometry to count the solutions of generic line intersections. For example, we may ask how many lines in the projective space $\mathbb{P}^3$ over the complex numbers intersect four given lines. Schubert used enumerative methods to answer questions like this, invoking what he called the "Principle of Conservation of Number," but unfortunately the methods initially led to computational errors for harder enumerative geometry questions due to degenerate cases involving counting with multiplicity. Hilbert's 15th problem asks to put Schubert's enumerative methods on a rigorous foundation, leading to the modern-day theory of Schubert calculus.

Schubert's questions have since been reformulated in terms of structure constants in the cup product of certain classes in the cohomology ring of the Grassmannian. Combinatorially, the study of symmetric functions has proven to be a useful tool; a surjective ring homomorphism from the the ring of symmetric functions to the cohomology of the Grassmanian that maps Schur functions to Schubert classes ensures that every relation among symmetric functions manifests as a relation on $H^*(\mathrm{Gr}^n(\mathbb{C}^m))$. Schubert calculus has since broadened in scope and now encompasses the study of generalized cohomology theories. Ongoing in these efforts is the search for symmetric function representatives whose structure constants match those of the cohomology theory. We continue that practice in this work with a focus on the interplay between symmetric functions and quantum $K$-theory, focusing on combinatorial rules for the $K$-theoretic Catalan functions introduced as a $K$-theoretic counterpart to the study of the affine Grassmanian.

## 1.1. Symmetric function theory

Viewing the ring of symmetric functions, $\Lambda$, as a $\mathbb{Z}$-algebra, its homogeneous components $\Lambda^n$ have bases parameterized by partitions of $n$. Therefore, all bases of $\Lambda$ can be indexed by the set of all partitions. A central question given such a basis for $\Lambda$ is how to express basis elements in terms of other known bases. We may also ask how to explicitly describe the multiplication structure constants of basis elements.

Though many unique bases exist for symmetric functions, we often focus our study on families that capture an underlying geometric picture. One of the most omnipresent bases that does just this is the basis of Schur functions, $s_\lambda$. The Schur functions have been studied widely, and there are numerous expansions of these functions that translate back and forth between other known bases. Moreover, the celebrated Littlewood-Richardson tells us precisely what the multiplication structure constants are for any two Schur functions.

The $k$-Schur symmetric functions, $s_\lambda^{(k)}$, are a basis for $\Lambda_{(k)} = \mathbb{Z}[s_1, s_2, \ldots, s_k]$ that were motivated by the study of the Macdonald positivity conjecture [**LLM**]. While constructed in [**LLM**] from sums of tableaux using the charge statistic, another formulation of $s_\lambda^{(k)}$ was conjectured by [**LM03**], and a specialization of this formulation was shown in [**Lam08**] to have geometric significance for the affine Grassmannian. A central question in the study of the $k$-Schur symmetric functions is determining the structure coefficients of $s_\lambda^{(k)} s_\mu^{(k)} = \sum_\nu c_{\lambda\mu}^\nu s_\nu^{(k)}$. After several years of stilted progress, [**BMPS19**] made strides by reconciling the many definitions for the $k$-Schur functions. Central to this work was identifying the $k$-Schur functions with a subset of the Catalan functions, which generalize the Hall-Littlewood polynomials. Catalan functions were initially introduced in the study of Euler characteristics of vector bundles on the flag variety [**Bro9420, SW, Che10, Pan10**]. These functions are a large family of symmetric functions indexed by both a weight and a *root ideal*; root ideals are upper order ideals of the poset $\Delta^+$, the set of labels for the positive roots of the root system of type $A_{\ell-1}$. [**BMPS19**] advanced the study of Catalan functions in light of the $k$-Schur functions by giving the expansion of $k$-Schur functions (and all Catalan functions with partition weight) into Schur functions and proving that $s_\lambda^{(k)}$ are a Schur positive basis of $\Lambda_{(k)}$, satisfy a dual Pieri rule, and have a special "shift invariance" property.

Geometrically, the $k$-Schur functions are Schubert representatives for the homology of the affine Grassmannian $\mathrm{Gr} = G(\mathbb{C}((t)))/G(\mathbb{C}[[t]])$ of $G = \mathrm{SL}_{k+1}$ [Lam08]. A $K$-theoretic counterpart to this study has emerged. The $K$-homology of Gr can be viewed as a subalgebra of the affine $K$-NilHecke algebra of Kostant and Kumar [KK] and there is a Hopf isomorphism $K_*(\mathrm{Gr}) \cong \Lambda_{(k)}$ [LSS]. Schubert representatives are given by a basis of inhomogeneous symmetric functions called $K-k$-Schur functions, $g_\lambda^{(k)} \in \Lambda_{(k)}$. To prove positivity results and a branching property for these functions, [BMS] extended the Catalan functions to an inhomogenous family of symmetric functions called the Katalan functions and reformulated $g_\lambda^{(k)}$ using this definition. The Katalan functions are indexed by a root ideal, weight, and multiset, and they contain numerous specializations to well known symmetric functions such as Catalan functions, Schur functions, dual Grothendieck functions, and $k$-Schur functions. Moreover, the Katalan functions were used to define the closed $k$-Schur Katalan functions, $\tilde{\mathfrak{g}}_\lambda^{(k)}$, a basis for $\Lambda_{(k)}$ which were introduced as conjecturally the images of the Lenart-Maeno quantum Grothendieck polynomials under a $K$-theoretic analog of the Peterson isomorphism. Closed $k$-Schur Katalan functions are indexed by partitions; their definition invokes the $k$-Schur root ideal, which is a distinguished root ideal constructed from a partition weight. [BMS] showed that the closed $k$-Schur Katalan functions are unitriangularly related to $K$-$k$-Schur functions and satisfy shift invariance, and they conjectured a $k$-rectangle property equivalent to Takigiku's $k$-rectangle property. This $k$-rectangle property was established in [See21], but more sophisticated multiplication results relevant to Lenart and Maeno's rule were initially intractable due to the complexity of the ensuing combinatorics.

## 1.2. Survey of Results

Givental and Lee [GL] studied the quantum $K$-theory ring $\mathrm{QK}(\mathrm{Fl}_{k+1})$, which is a ring defined as a deformation of $K(\mathrm{Fl}_{k+1})$, the Grothendieck ring of coherent sheaves on $\mathrm{Fl}_{k+1}$, the variety of complete flags in $\mathbb{C}^{k+1}$. In [KM], Kirillov and Maeno provided a conjectural presentation of $\mathrm{QK}(\mathrm{Fl}_{k+1})$. Ikeda-Iwao-Maeno [IIM] defined an explicit ring isomorphism $\Phi$ between localizations of $K_*(\mathrm{Gr}_{SL_{k+1}})$ and $\mathrm{QK}(\mathrm{Fl}_{k+1})$ and studied the images of *quantum Grothendieck polynomials*

$\{\mathfrak{G}^Q_w\}_{w\in S_{k+1}}$ [**LM**], recently proven to represent quantum Schubert classes in $\mathrm{QK}(\mathrm{Fl}_{k+1})$ [**Kat18, ACT, LNS**].

We focus here on a multiplication rule whose image under the Peterson isomorphism is conjectured to be the Monk rule for quantum Grothendieck polynomials, introduced by Lenart and Maeno in [**LM**]. Monk rules are explicit formulas giving structure constants of a product of a special class and a generic one, and such a rule is enough to determine the underlying ring structure. [**IIN**] proved [**BMS**]'s conjecture that the closed k-Schur Katalan functions are identified with the Schubert structure sheaves in the $K$-homology of the affine Grassmannian, also studying a $K$-theoretic Peterson isomorphism that Ikeda, Iwao, and Maeno [**IIM**] constructed based the unipotent solution of the relativistic Toda lattice of Ruijsenaars. In the case of the image we focus on, the product of closed $k$-Schur Katalan functions has one function indexed by a generic partition and another weighted by a $k$-rectangle minus a box (Definition 3.2.2). This work is a combinatorial investigation of properties of root ideals. We leverage bounce paths, or certain series of roots which can be removed from a root ideal while maintaining the root ideal's defining properties, to prove the equivalence of various sums of Katalan functions using properties of root ideals called mirrors, ceilings, and walls. We construct the set $I^{d,k}_\mu$, a set of integers defined via bounce paths in the $k$-Schur root ideal, which gives a combinatorial structure for the product of a generic analogue to a closed $k$-Schur Katalan function and a closed $k$-Schur Katalan function weighted by a rectangle minus a box partition.

Theorem 3.3.16. For $\mu \in \mathrm{Par}^\ell_k$ reduced and $R^*_d$ a $k$-rectangle minus a box (Definition 3.2.2),

$$\tilde{\mathfrak{g}}^{(k)}_\mu \tilde{\mathfrak{g}}^{(k)}_{R^*_d} = \sum_{D\subset I^{d,k}_\mu D\neq\emptyset} (-1)^{|D|+1} \tilde{\mathfrak{g}}^{(k)}_{\lambda-\epsilon_D},$$

where, if $r$ is the smallest index of $\lambda := \mu \cup R_d$ such that $\lambda_r > \lambda_d$, $I^{d,k}_\mu = \cup_{i\in[r+d,r+2d-1]}\mathrm{downpath}_{\Delta^k(\lambda)}(i)$ (Definition 3.3.12).

A major challenge in making Theorem 3.3.16 cancellation-free is that while the combinatorics of the Monk-type multiplication formula for quantum Grothendieck polynomials in [**LM**] involves nonempty paths in a subgraph of the quantum Bruhat graph, a weighted directed graph defined on $S_n$, the combinatorics for the closed $k$-Schur Katalan rules involves the affine symmetric group $\hat{\mathcal{S}}_n$. $\hat{\mathcal{S}}_n$ underscores the combinatorics of both the $k$-Schur functions and the $k$-Schur Katalan functions. From [LM], it is known that $k$-bounded partitions, the affine Grassmannian words of $\hat{\mathcal{S}}_{k+1}$, and $(k+1)$-cores are in bijection, where an $n$-core is a partition such that none of the cells have hook length $n$. To prove $k$-Schur straightening in [BMPS19], the authors established a dictionary between constructions on cores to constructions on bounce paths. To make progress here, we leverage bounce path properties to prove that certain Katalan functions are equivalent, invoking the properties of covers (Definition 4.1.2).

Theorem 5.0.11. For $\mu \in \mathrm{Par}_k^\ell$ reduced and $R_d^*$ a $k$-rectangle minus a box (Definition 3.2.2),

$$\tilde{\mathfrak{g}}_\mu^{(k)}\tilde{\mathfrak{g}}_{R_d^*}^{(k)} = \sum_{D\subset I_\mu^{d,k}, D\notin\mathcal{D}_1, D\neq\emptyset} (-1)^{|D|+1}\tilde{\mathfrak{g}}_{\lambda-\epsilon_D}^{(k)},$$

where $\mathcal{D}_1$ is the set of all $D \subset I_\mu^{d,k}$ such that there exists some $a_x \in D$ satisfying three conditions:

$$\lambda_{a_x+h_x^D} = \lambda_{a_x+h_x^D+1}, \tag{1.2.1}$$

$$[a_x, a_x + h_x^D + 1] \subset I_\mu^{d,k}, \tag{1.2.2}$$

$$a_x + h_x^D + 1 \in D \Rightarrow |\mathrm{uppath}_{\Psi^{D,x}}(a_x + h_x^D + 1)| < |\mathrm{uppath}_{\Psi^{D,x}}(a_x + h_x^D)|\ (Definition\ 3.3.12). \tag{1.2.3}$$

Based on this result, we propose a cancellation-free conjecture,

Conjecture 5.0.3. For $\mu \in \mathrm{Par}_k^\ell$ reduced and $R_d^*$ a $k$-rectangle minus a box (Definition 3.2.2),

$$\tilde{\mathfrak{g}}_\mu^{(k)}\tilde{\mathfrak{g}}_{R_d^*}^{(k)} = \sum_{D\in\mathcal{D}}(-1)^{|D|+1}\tilde{\mathfrak{g}}_{\lambda-\epsilon_D}^{(k)},$$

where

$$\mathcal{D} = \{D \subset I_\mu^{d,k} | [a_x, a_x + h_x^D] \subset I_\mu^{d,k} \text{ for every } a_x \in D, \text{ and if } \lambda_{a_x+h_x^D} = \lambda_{a_x+h_x^D+1},$$

then $a_x + h_x^D + 1 \in D$ and $|\mathrm{uppath}_{\Psi^{D,x}}(a_x + h_x^D + 1)| \geq |\mathrm{uppath}_{\Psi^{D,x}}(a_x)|\}$ (*Definition* 3.3.12).

We conclude with progress towards resolving this conjecture and integrating the theory of covers into the result.

Proposition 6.2.2. For all $D = \{a_1, \ldots, a_t\} \in \mathcal{D}$,

$$\tilde{\mathfrak{g}}^{(k)}_{\lambda-\epsilon_D} = \tilde{\mathfrak{g}}^{(k)}_{\mathrm{cover}_{a_t}(\ldots(\mathrm{cover}_{a_1}(\lambda))\ldots)} \text{ (\textit{Definition} 4.1.2).}$$

In Chapter 2, we reference the necessary groundwork from the theory of symmetric functions, especially that of Catalan and K-theoretic Catalan functions. We build on this theory in Chapter 3, proving a litany of combinatorial tools for manipulating $K$-theoretic Catalan functions and their products. Chapter 3 illustrates an expansion for $\tilde{g}^{(k)}_{R^*_d}\tilde{g}^{(k)}_{\lambda}$ in terms of closed $k$-Schur Katalan functions. In Chapter 4, we offer results that address combinatorial questions motivated by the rule offered in Chapter 3, leveraging covers to re-express certain families of Katalan functions. Chapter 5 offers a refinement of the formula offered in Chapter 3 using an involution to establish partial cancellation. We conclude in Chapter 6 with a description of future work and progress towards the cancellation-free conjecture.

CHAPTER 2

# Background

## 2.1. Symmetric functions

The ring of symmetric functions provides a beautiful illustration of the interplay between combinatorial and algebraic structures. We will make liberal use of this structure to establish our results. The following subsection summarizes fundamental results in the theory, see [**Sta99**] for proofs.

**2.1.1. Symmetric polynomial basics.** For $m \in \mathbb{Z}_{>0}$, we say $f \in \mathbb{Z}[x_1, x_2, \dots x_m]$ has *homogeneous degree d* if $f = \sum_{|\alpha|=d} c_\alpha x^\alpha$, where $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_m^{\alpha_m}$ for any $\alpha = (\alpha_1, \dots, \alpha_m)$. There is a degree-preserving $S_m$-action on $\mathbb{Z}[x_1, x_2, \dots x_m]$ given by $\sigma.f(x_1, \dots, x_m) = f(x_{\sigma(1)}, \dots, x_{\sigma(m)})$ for $\sigma \in S_m$. The *ring of symmetric polynomials* on $m$ indeterminates is

$$\Lambda_m := \{f \in \mathbb{Z}[x_1, x_2, \dots x_m] | \sigma.f = f \ \forall \sigma \in S_n\}.$$

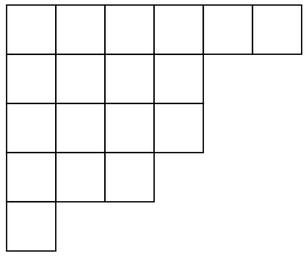
We note that $\Lambda_m$ is a subring of $\mathbb{Z}[x_1, x_2, \dots x_m]$ and a graded $\mathbb{Z}$-algebra. We index bases of symmetric polynomials by *partitions*.

Definition 2.1.1. For any $\ell \in \mathbb{Z}_{>0}$, we say $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell \in \mathbb{Z}_{\geq 0}^\ell$ is a *partition* if $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell \geq 0$. We denote set of all such partitions as $\mathrm{Par}_\ell$ and say the *length* of $\lambda$, denoted $\ell(\lambda)$, is its number of non-zero parts. More generally, we let Par denote the set of all partitions. For any $\mu \in \mathrm{Par}$, we call the *size* of $\mu$ the sum of its parts, denoting $|\mu| = \mu_1 + \cdots + \mu_{\ell(\mu)}$.

We often identify partitions with *Young diagrams*.

Definition 2.1.2. A *Young diagram* is a finite collection of left-justified boxes with rows of non-increasing length when viewed from north to south (top to bottom).

EXAMPLE 2.1.3. We identify the partition $(6, 4, 4, 3, 1)$ with a Young diagram:

$(6, 4, 4, 3, 1) \leftrightarrow$ 

For any partition $\lambda$, we define the *monomial symmetric polynomials* as

$$m_\lambda(x_1, \ldots, x_m) := \sum_\alpha x^\alpha,$$

where $\alpha$ ranges over all distinct permutations of $\lambda$. By construction, $\{m_\lambda\}_{\lambda\in\mathrm{Par}_m}$ is a natural candidate for a $\mathbb{Z}$-basis of $\Lambda_m$. For $r \in \mathbb{Z}_{>0}$, we also define the *complete symmetric polynomial*, $h_r(x_1, \ldots, x_m) := \sum_{1\leq i_1\leq i_2\leq\dot{\leq} i_r\leq m} x_{i_1} \ldots x_{i_r}$, and for $\alpha = \mathbb{Z}^\ell$, the *homogeneous symmetric polynomials* are $h_\alpha := h_{\alpha_1} \ldots h_{\alpha_\ell}$, where $h_r = 0$ for $r < 0$ and $h_0 = 1$.

PROPOSITION 2.1.4. *For $m \in \mathbb{Z}_{>0}$, $\Lambda_m = \mathbb{Z}[h_1, h_2, \ldots h_m]$. Moreover, $\{h_\lambda\}_{\lambda\in Par_m}$ forms a basis for $\Lambda_m$.*

Equipped with the definition of the homogeneous symmetric polynomials, we can define the Schur polynomials using the Jacobi-Trudi identity.

DEFINITION 2.1.5. For $\alpha \in \mathbb{Z}^\ell$, we define the *Schur polynomial*

$$s_\alpha = \det(h_{\alpha_i+j-i}).$$

As the most omnipresent family of symmetric polynomials, the Schur polynomials have been defined with many other equivalent formulations, such as via Young tableaux and raising operators, and each definition offers different benefits. For example, Proposition 2.2.1 offers a formulation in terms of raising operators.

PROPOSITION 2.1.6. *We have the following facts about Schur polynomials.*

*(1) For all $r \in \mathbb{Z}$, $s_r = h_r$.*

*(2)* $\{s_\lambda\}_{\lambda \in Par_m}$ *forms a basis for* $\Lambda_m$*.*

### 2.1.2. Symmetric functions.

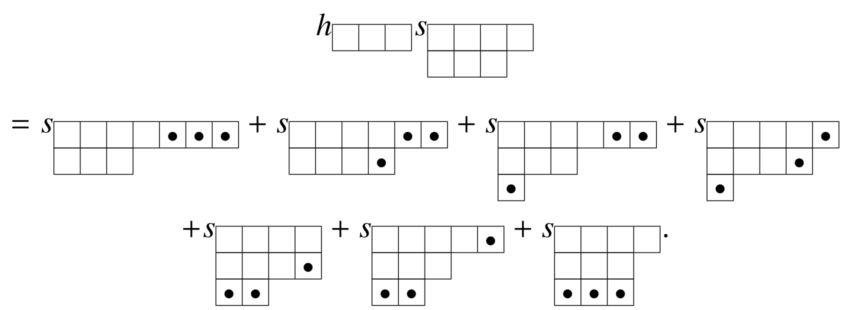
DEFINITION 2.1.7. A *symmetric function* is a symmetric infinite series with bounded degree, where we say $f \in \mathbb{Z}[[x_1, x_2, \dots]]$ is a *symmetric infinite series* if the coefficient of $x_\alpha$ is equal to the coefficient of $x_\beta$ whenever the underlying multisets of entries of $\alpha$ and $\beta$ are equal (not counting 0).

We denote the ring of symmetric functions by $\Lambda$. For $X = x_1, x_2, \dots$ an infinite alphabet of variables and $\alpha \in \mathbb{Z}^\ell$, we define the the complete homogeneous symmetric functions $h_\alpha(X) = h_{\alpha_1}(X) \dots h_{\alpha_\ell}(X)$ and the Schur functions $s_\alpha(X)$.

PROPOSITION 2.1.8. *(Pieri rule). For* $\lambda$ *any partition and* $r \in \mathbb{Z}_{\geq 0}$*,*

$$h_r s_\lambda = \sum_{\substack{\mu \in Par, |\mu| = |\lambda| + r \\ \mu_1 \geq \lambda_1 \geq \mu_2 \geq \lambda_2 \geq \dots \lambda_\ell \geq \mu_{\ell+1}}} s_\mu$$

EXAMPLE 2.1.9. Setting $r = 3$ and $\lambda = (4, 3)$ in the Pieri rule yields




## 2.2. Catalan functions

In this section, we introduce the Catalan functions, a family of symmetric functions which vastly generalizes the Schur functions. The Catalan symmetric functions were motivated by a conjecture of Chen and Haiman [Che10] that the $k$-Schur functions (Section 1.1) are a subclass of a family of symmetric functions indexed by pairs $(\Psi, \gamma)$ consisting of an upper order ideal $\Psi$

of positive roots (of which there are Catalan many) and a weight $\gamma \in \mathbb{Z}^\ell$. In [Che10], Chen-Haiman investigated their Schur expansions and conjectured a positive combinatorial formula when $\gamma_1 \geq \gamma_2 \geq \dots$, and Panyushev [Pan10] proved a cohomological vanishing theorem to establish Schur positivity of a large subclass of Catalan functions. Interpolating between symmetric functions bases is of central interest to algebraic combinatorialists, and in this sense, the study of the Catalan functions in [BMPS19, BMPS20] is particularly notable: these functions allow a method to interpolate between the complete homogeneous symmetric functions and the Schur functions.

**2.2.1. Raising operators.** Raising operators were introduced by Young [You32] and formalized rigorously by Garsia-Remmel [GR79, GR81]. Given $i < j$, the *raising operator* $R_{ij}$ acts on any integer sequence $\alpha \in \mathbb{Z}^\ell$ via $R_{ij}\alpha = \alpha + \epsilon_i - \epsilon_j$, where for any integer $m$, $\epsilon_m$ is the vector of zeroes in all coordinates besides the $m$th, which is a 1. We define raising operators on symmetric functions by defining their action on the basis of homogeneous symmetric functions: $R_{ij}h_\alpha := h_{\alpha+\epsilon_i-\epsilon_j}$. We also use the notation $\epsilon_S := \sum_{s\in S} \epsilon_s$ and $\varepsilon_{(a,b)} := \epsilon_a - \epsilon_b$.

Proposition 2.2.1. *For $\gamma \in \mathbb{Z}^\ell$,*

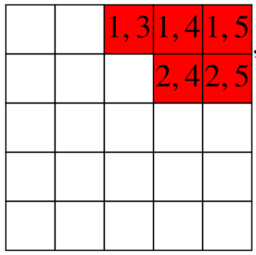
$$s_\gamma = \prod_{1\leq i<j\leq \ell} (1 - R_{ij}) h_\gamma$$

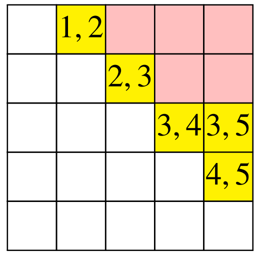
**2.2.2. Root ideals and generalizations.** In what follows, we summarize some relevant combinatorial tools for defining the Catalan functions and their $K$-theoretic analog; see [BMPS19, BMPS20, BMS] for further reference. We fix a positive integer $\ell$ and use the notation $\Delta^+_\ell = \Delta^+ := \{(i, j) | 1 \leq i < j \leq \ell\}$. We use the notation $[a, b]$ for $\{i \in \mathbb{Z} | a \leq i \leq b\}$ and $[n] = [1, n]$. For a set $S \subset [\ell]$, denote $\epsilon_S = \sum_{i\in S} \epsilon_i$, and for $\alpha = (i, j) \in \Delta^+_\ell$, denote by $\varepsilon_\alpha = \epsilon_i - \epsilon_j$ the corresponding positive root (not to be confused with $\epsilon_{\{i,j\}} = \epsilon_i + \epsilon_j$).

Definition 2.2.2. A *root ideal* $\Psi$ is an upper order ideal of the poset $\Delta^+_\ell$ with partial order given by $(a, b) \leq (c, d)$ when $a \geq c + \ell$ and $b \leq d$.

We will represent root ideals via grids, and we will often also invoke the lower order ideal $\Delta^+ \setminus \Psi$.

Example 2.2.3. Let $\Psi = \{(1,3),(1,4),(1,5),(2,4),(2,5)\} \subseteq \Delta_5^+$. The root ideal $\Psi$ and $\Delta^+ \setminus \Psi = \{(1,2),(2,3),(3,4),(3,5),(4,5)\}$ are depicted by

$\Psi =$ 

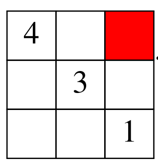
, $\Delta^+ \setminus \Psi =$ 


**2.2.3. Catalan functions.** In 2010, Chen [Che10] and Panyushev [Pan10] introduced a family of symmetric functions known as Catalan functions that were further studied in [BMPS19], [BMPS20]. In full generality, Catalan functions involve a parameter $t$, but we will only work with $t = 1$, as this specialization is necessary for applications to Schubert calculus.

Definition 2.2.4. A Catalan function, indexed by a pair $(\Psi, \gamma)$ consisting of a root ideal $\Psi$ and a weight $\gamma \in \mathbb{Z}^\ell$, is defined by

$$H(\Psi;\gamma) = \prod_{(i,j)\in\Delta^+\setminus\Psi} (1 - R_{ij}) h_\gamma.$$

Example 2.2.5. One convenient way to represent a Catalan function is via a grid representation. With $\gamma = (4,3,1)$ and $\Psi = \{(1,3)\}$,

$H(\Psi;\gamma) =$ 

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In this case,

$$H(\Psi;\gamma) = (1 - R_{1,2})(1 - R_{2,3}) h_{(4,3,1)} = h_{(4,3,1)} - h_{(4,4,0)} - h_{(5,2,1)} + h_{(5,3,0)}$$

The Catalan functions specialize to several well-known families of symmetric functions. For example, it is immediate that $H(\Delta^+;\gamma) = h_\gamma$. We also have $H(\emptyset;\gamma) = s_\gamma$ when using the raising operator definition of the Schur functions presented in 2.2.1.

**2.2.4. $k$-Schur Functions.** The Macdonald polynomials form a basis for the ring of symmetric functions over the field $\mathbb{Q}(q,t)$, and an impressive body of research has focused on the Macdonald positivity conjecture: the Schur expansion coefficients of the (Garsia) modified Macdonald polynomials $H_\mu(\mathbf{x};q,t)$ lie in $\mathbb{N}[q,t]$. Lapointe, Lascoux, and Morse [LLM] considerably strengthened this conjecture, constructing a family of functions and conjecturing (i) they form a basis for the space $\Lambda^k = \mathrm{span}_{\mathbb{Q}(q,t)}\{H_\mu(\mathbf{x};q,t)\}_{\mu_1\leq k}$, (ii) they are Schur positive, and (iii) the expansion of $H_\mu(\mathbf{x};q,t) \in \Lambda^k$ in this basis has coefficients in $\mathbb{N}[q,t]$. Due to challenges in further progress using these intricately constructed functions, many conjecturally equivalent candidates were proposed. Informally, all these candidates are now called $k$-Schur functions. Here we invoke the Catalan-theoretic definition with $t = 1$. This definition invokes a generalization of weights; let

$$\widetilde{\mathrm{Par}}^k_\ell = \{\mu \in \mathbb{Z}_{\leq k} | \mu_1 + \ell - 1 \geq \mu_2 + \ell - 2 \geq \cdots \geq \mu_\ell\}.$$

Note that $\mathrm{Par}^k_\ell \subseteq \widetilde{\mathrm{Par}}^k_\ell$, but $\widetilde{\mathrm{Par}}^k_\ell$ in general contains non-partitions.

Definition 2.2.6. For $\lambda \in \widetilde{\mathrm{Par}}^k_\ell$, the associated *$k$-Schur function* is $s^{(k)}_\lambda := H(\Delta^k(\lambda);\lambda)$.

It was proven in [LM08] that the $k$-Schur functions $\{s^{(k)}_\mu(\mathbf{x})\}_{\mu\in\mathrm{Par}^k_\ell}$ form a basis for $\Lambda^k$. In [BMPS19], it was shown that the $k$-Schur functions satisfy a straightening rule quite similar to that of ordinary Schur functions. This rule shows that analogs of the $k$-Schur functions weighted by nonpartitions are either equal to 0 or equal to a power of $t$ multiplied by a $k$-Schur function with partition weight. The combinatorics of this rule led to the necessity of covers, which play an important role in this work, and are defined in Chapter 4.

## 2.3. K-theoretic Catalan functions

We now extend the Catalan functions to an inhomogenous family of symmetric functions using additional information from a multiset $M$ supported on $\{1, \ldots, \ell\} = [\ell]$. We use the notation that for such a multiset $M$ on $[\ell]$, its multiplicity function is denoted $m_M : [\ell] \to \mathbb{Z}_{\geq 0}$.

These $K$-theoretic Catalan functions, or *Katalan functions* were introduced in [**BMS**] in order to reformulate the *K-k-Schur functions*, $g_\lambda^{(k)} \in \Lambda_{(k)}$, which are a basis of inhomogeneous symmetric functions that give Schubert representatives for then $K$-homology $K_*(\mathrm{Gr})$, Hopf isomorphic to $\Lambda_{(k)}$ [**LSS**].

**2.3.1. Katalan functions.** We require the following inhomogeneous versions of the complete symmetric polynomials. For $m, r \in \mathbb{Z}$, define

$$k_m^{(r)} = \sum_{i=0}^{m} \binom{r+i-1}{i} h_{m-i}.$$

Then for $\gamma \in \mathbb{Z}^\ell$, let $g_\gamma = \det(k_{\gamma_i+j-i}^{(i-1)})_{1\leq i,j\leq \ell}$. When $\gamma$ is a partition, these are the *dual stable Grothendieck polynomials*, first studied implicitly in [**Len00**] and determinantally formulated in [**LN**]. We use an alternative characterization as in [**BMS**]:

$$g_\gamma = \prod_{1\leq i<j\leq \ell} (1-R_{ij}) k_\gamma\,, \quad \text{where } k_\gamma := k_{\gamma_1}^{(0)} k_{\gamma_2}^{(1)} \cdots k_{\gamma_\ell}^{(\ell-1)}\,. \tag{2.3.1}$$

Definition 2.3.1. For a root ideal $\Psi \subset \Delta_\ell^+$, a multiset $M$ with $\mathrm{supp}(M) \subset \{1, \ldots, \ell\}$, and $\gamma \in \mathbb{Z}^\ell$, we define the *Katalan function*

$$K(\Psi; M; \gamma) := \prod_{j\in M}(1-L_j) \prod_{(i,j)\in\Psi} (1-R_{ij})^{-1} g_\gamma\,, \tag{2.3.2}$$

where the *lowering operator* $L_j$ acts on the subscripts of $g_\gamma \in \Lambda$ by $L_j g_\gamma = g_{\gamma-\epsilon_j}$.

We will often appeal to the following alternative formulation of the Katalan functions.

Proposition 2.3.2. [**BMS**] *For a root ideal $\Psi \subset \Delta_\ell^+$, a multiset $M$ with supp$(M) \subset \{1, \ldots, \ell\}$, and $\gamma \in \mathbb{Z}^\ell$,*

$$K(\Psi; M; \gamma) = \prod_{j \in M} (1 - L_j) \prod_{(i,j) \in \Delta^+ \setminus \Psi} (1 - R_{ij})\, k_\gamma\,.$$

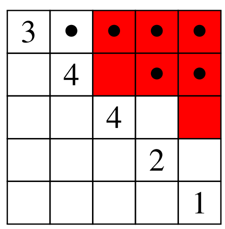
Although Katalan functions are defined for arbitrary multisets, we mainly work with those where the associated multiset comes from a root ideal $\mathcal{L} \subset \Delta_\ell^+$ via the function

$$L(\mathcal{L}) = \bigsqcup_{(i,j) \in \mathcal{L}} \{j\}\,. \tag{2.3.3}$$

In this scenario, we will often abuse notation to write $K(\Psi; \mathcal{L}; \gamma) = K(\Psi; L(\mathcal{L}); \gamma)$.

Given a root ideal $\Psi \subset \Delta_\ell^+$, a multiset $M$ on $[\ell]$, and $\gamma \in \mathbb{Z}^\ell$, we represent the Katalan function $K(\Psi; M; \gamma)$ by the $\ell \times \ell$ grid of boxes (labelled by matrix-style coordinates) with the boxes of $\Psi$ shaded, $m_M(a)$ $\bullet$'s in column $a$ (assuming $m_M(a) < a$), and the entries of $\gamma$ written along the diagonal.

Example 2.3.3. Let $\Psi = \{(1,3), (1,4), (1,5), (2,3), (2,4), (2,5), (3,5)\} \subset \Delta_5^+$, $M = \{2,3,4,4,5,5\}$, and $\gamma = (3,4,4,2,1)$. $K(\Psi, M, \gamma)$ is depicted by:

$K(\Psi; M; \gamma) =$ 

.

The family of Katalan functions contains several well-studied symmetric function bases.

Proposition 2.3.4. [**BMS**] *Let $\gamma \in \mathbb{Z}^\ell$.*

*(1) The Katalan functions contain the family of Catalan functions: $K(\Psi; \Delta_\ell^+; \gamma) = H(\Psi; \gamma)$ for any root ideal $\Psi \subset \Delta_\ell^+$. In particular, $K(\emptyset; \Delta_\ell^+; \gamma) = s_\gamma$ and $K(\Delta_\ell^+; \Delta_\ell^+; \gamma) = h_\gamma$.*

*(2) $K(\emptyset; \emptyset; \gamma) = g_\gamma$.*

*(3) $K(\Delta_\ell^+; \emptyset; \gamma) = k_\gamma$*

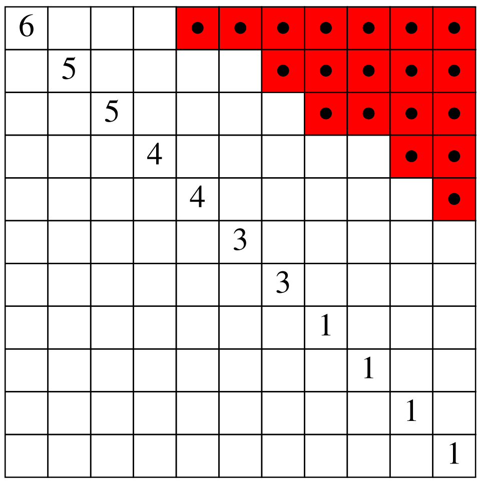
*(4) $K(\Delta^k(\mu); \Delta_\ell^+; \mu) = s_\mu^{(k)}$,*

*where $\Delta^k(\mu)$ is the k-Schur root ideal.*

An important class of Katalan functions are defined using the $k$-Schur root ideal, defined as follows.

DEFINITION 2.3.5. For fixed $k \in \mathbb{Z}_{>0}$, let $\mathrm{Par}_\ell^k = \{(\lambda_1, \ldots, \lambda_\ell \in \mathbb{Z}^\ell : k \geq \lambda_1 \geq \ldots \lambda_\ell \geq 0\}$. Then for $\lambda \in \mathrm{Par}_\ell^k$, we define the *k-Schur root ideal* $\Delta^k(\lambda) = \{(i, j) \in \Delta_\ell^+ | k - \lambda_i + i < j\}$.

EXAMPLE 2.3.6. Let $\lambda = (6, 5, 5, 4, 4, 3, 3, 1, 1, 1, 1) \in \mathrm{Par}_{11}^9$. Then $K(\Delta^9(\lambda), \Delta^9(\lambda), \lambda)$ is depicted by:



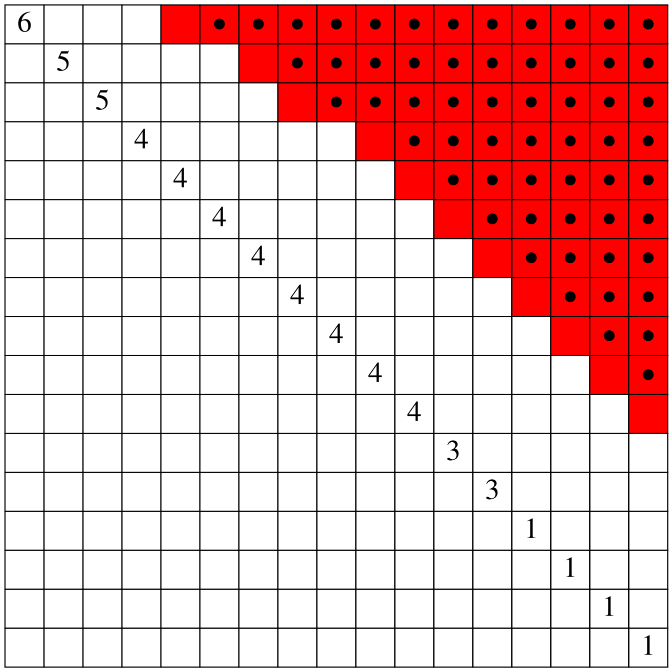
REMARK 2.3.7. [**BMS**] Let $\lambda \in \mathrm{Par}_\ell^k$, $\Psi = \Delta^k(\lambda)$, and $\mathcal{L} = \Delta^{k+1}(\lambda)$. Let $z$ be the lowest nonempty row of $\Psi$. For $x \in [z]$, $\Psi$ does not have a wall in rows $x, x+1$. Hence, for all $x \in [\ell - 1]$, either $\Psi$ has a ceiling in columns $x, x+1$ or has removable roots $(y, x)$ and $(y+1, x+1)$. In the latter case, if $y \neq x - 1$, then $\Psi$ has a mirror in rows $y, y+1$.

DEFINITION 2.3.8. For $\lambda \in \mathrm{Par}_\ell^k$, define the *k-Schur Katalan function* by

$$\mathfrak{g}_\lambda^{(k)} = K(\Delta^k(\lambda); \Delta^{k+1}(\lambda); \lambda)\,.$$

Example 2.3.9. For $\lambda = (6,5,5,4,4,4,4,4,4,4,3,3,1,1,1,1) \in \mathrm{Par}^9_{17}$, we have

$\mathfrak{g}^{(9)}_\lambda =$



The $k$-Schur Katalan functions have been proven to coincide exactly with the $K$-$k$-Schur functions $g^{(k)}_\lambda$ ( [**BMS**]). This Katalan function formula is considerably more direct and explicit than any previously known description of the $K$-$k$-Schur functions and readily resolves several outstanding conjectures, including positive branching:

Proposition 2.3.10. [**BMS**] *For any* $\lambda \in Par^k$,

$$g^{(k)}_\lambda = \sum_{\mu \in Par^{k+1}} a_{\lambda\mu} g^{(k+1)}_\mu,$$

*where* $(-1)^{|\lambda|-|\mu|} a_{\lambda\mu} \in \mathbb{Z}_{\geq 0}$.

**2.3.2. Closed $k$-Schur Katalan functions.** As mentioned in Chapter 1, the Katalan functions have a relationship with quantum Schubert polynomials. The quantum $K$-theory ring $Q\mathcal{K}(\mathrm{Fl}_{k+1})$ can be identified with a quotient of $\mathbb{C}[z_1, \ldots, z_{k+1}, Q_1, \ldots, Q_k]$ [**KM, ACT17**].

Definition 2.3.11. A *$k$-rectangle* is a partition of the form $R_i := (k+1-i)^i$ for $i \in [k]$. We call a partition *reduced* if it contains no $k$-rectangles.

Example 2.3.12. If $k = 5$, $R_2 = (4,4)$ and $R_4 = (2,2,2,2)$ are both examples of $k$-rectangles.

We define $\sigma_i = \sum_{\mu\subseteq R_i} g_\mu$ for $i \in [k]$, and we set $\sigma_0 = \sigma_{k+1} = g_{R_0} = g_{R_{k+1}} = 1$. In [IIM], Ikeda, Iwao, and Maeno gave the following description of a $K$-theoretic version of the Peterson isomorphism:

$$\Phi : Q\mathcal{K}(\mathrm{Fl}_{k+1})[Q_1^{-1},\dots,Q_k^{-1}] \xrightarrow{\sim} \Lambda_{(k)}[g_{R_1}^{-1},\dots,g_{R_k}^{-1},\sigma_1^{-1},\dots,\sigma_k^{-1}]$$

$$z_i \mapsto \frac{g_{R_i}\sigma_{i-1}}{g_{R_{i-1}}\sigma_i},\ Q_i \mapsto \frac{g_{R_{i-1}}g_{R_{i+1}}}{g_{R_i^2}}.$$

[LM] defined the *quantum Grothendieck polynomials* $\{\mathfrak{G}_w^Q\}_{w\in S_{k+1}} \subseteq Q\mathcal{K}(\mathrm{Fl}_{k+1})$ as the image of the ordinary Grothendieck polynomials $\{\mathfrak{G}_w\}_{w\in S_{k+1}}$ under a quantization map $\hat{Q} : K(\mathrm{Fl}_{k+1}) \to Q\mathcal{K}(\mathrm{Fl}_{k+1})$. In [IIM], the authors described the image the quantum Grothendieck polynomials $\mathfrak{G}_{w_{\lambda,d}}^Q$ under the Peterson isomorphism, where $\lambda \cup R_d$ and $w_{\lambda,d}$ is the corresponding *d-Grassmanian permutation*, defined in [IIM] Remark 6.7. For a general permutation $w \in S_n$, [IIM] conjectured the existence of some polynomial $\tilde{g}_w \in \Lambda_{(n)}$ such that

$$\Phi(\mathfrak{G}_w^Q) = \frac{\tilde{g}_w}{\prod_{d\in D(w)} g_{R_d}},$$

where the *descent set* of $w$ is $D(w) := \{i : w_i > w_{i+1}\}$. In [BMS], the authors conjectured that Ikeda's functions have the following description in terms of Katalan functions.

Definition 2.3.13. For $\lambda \in \mathrm{Par}_\ell^k$, the *closed k-Schur Katalan function* is

$$\tilde{\mathfrak{g}}_\lambda^{(k)} = K(\Delta^k(\lambda); \Delta^k(\lambda); \lambda)\,.$$

The conjecture in [BMS] is defined via a map $\theta : S_{k+1} \to \mathrm{Par}^k$ which we now construct. For $w = w_1 \dots w_{k+1} \in S_{k+1}$ in one-line notation, the *inversion sequence* of $w$ is $\mathrm{Inv}(w) \in \mathbb{Z}_{\geq 0}^k$ given by $\mathrm{Inv}_i(w) = |\{j > i : w_i > w_j\}|$. Define an injection $\zeta : S_{k+1} \to \mathrm{Par}^k$ by letting column $i$ of $\zeta(w)$ be

$$\binom{k+1-i}{2} + \mathrm{Inv}_i(w_0 w),$$

for all $i \in [k]$, where $w_0$ is the longest element of $S_{k+1}$. We call an element of $\mathrm{Par}^k$ *irreducible* if it has at most $k-i$ parts of size $i$, or equivalently, it contains no $k$-rectangle as a subsequence. For any $\mu \in \mathrm{Par}^k$, define the unique irreducible partition $\mu_\downarrow$ by deleting from $\mu$ all of the $k$-rectangles it contains as a subsequence. Set $\theta(w) = \zeta(w)_\downarrow$. It is shown in [**BMPS20**] that $\theta$ is the same as the map $\lambda$ from [**LS12, IIM**].

Theorem 2.3.14. [**IIN**] *For any* $w \in S_{k+1}$,

$$\Phi(\mathfrak{G}^Q_w) = \frac{\tilde{\mathfrak{g}}^{(k)}_\lambda}{\prod_{d \in D(w)} g_{R_d}},$$

*where* $\lambda = \theta(w)^{\omega_k}$ *and* $\omega_k$ *denotes the k-conjugate, an involution on* $Par^k$ *introduced in* [**LM05**].

Proposition 2.3.15. [**BMS**, Proposition 2.15] *The closed k-Schur Katalan functions* $\{\tilde{\mathfrak{g}}^{(k)}_\lambda\}_{\lambda \in Par^k}$ *form a basis for* $\Lambda_{(k)}$.

CHAPTER 3

# Manipulating Katalan functions

In this chapter, we develop a combinatorial framework that leads us to a novel multiplication rule satisfied by the closed $k$-Schur Katalan functions.

## 3.1. Katalan basics

We establish some notation convenient to the combinatorics of positive roots, and as a consequence, to root ideals. For the most part, these definitions coincide with those of [**See21**].

DEFINITION 3.1.1. For a subset of positive roots $S \subseteq \Delta_\ell^+$, we say there is

*a wall in rows* $r, r+1, \dots, r+d$ if every root in row $r+d$ of $S$ $(r+d, y) \in S$ implies $(x, y) \in S$ for all $x \in [r, r+d]$ and $(r+d, z) \in S$ for all $z \geq y$

*a ceiling in columns* $c, c+1, \dots, c+d$ if every root in column $s$ of $S$ satisfies $(x, c) \in S$ implies $(x, y) \in S$ for all $y \in [c, c+d]$ and $(z, c) \in S$ for all $z \leq x$.

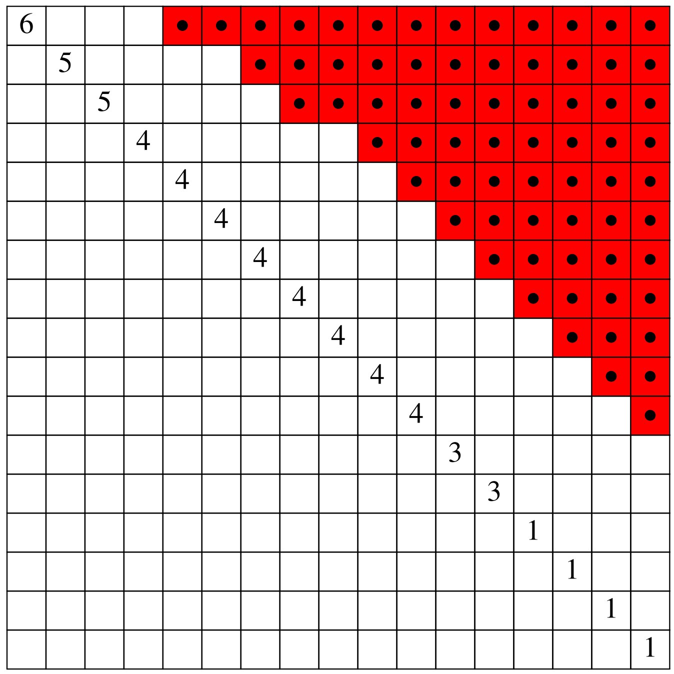
DEFINITION 3.1.2. For $x \in [\ell]$ and a subset of positive roots $S \subseteq \Delta_\ell^+$,

- Let $j = \min\{c | (x, c) \in S\}$. If $j$ is defined and $(r, j) \notin S$ for all $r > x$, we say $\mathrm{down}_S(x) = j$; otherwise $\mathrm{down}_S(x)$ is undefined.
- Let $i = \max\{r | (r, x) \in S\}$. If $i$ is defined and $(i, c) \notin S$ for all $c < x$, we say $\mathrm{up}_S(x) = i$; otherwise $\mathrm{up}_S(x)$ is undefined.

We call a root $(x, y) \in S$ *removable* if $S \setminus (x, y)$ has a ceiling in columns $y-1, y$ or a wall in column $x-1, x$. The *bounce graph* of $S$ is the graph on the vertex set $[\ell]$ with edges $(r, \mathrm{down}_S(r))$ for each $r \in [\ell]$ such that $\mathrm{down}_S(r)$ is defined. The bounce graph of $S$ is a disjoint union of paths called *bounce paths* of $S$.

EXAMPLE 3.1.3. For $\lambda = (6, 5, 5, 4, 4, 4, 4, 4, 4, 4, 3, 3, 1, 1, 1, 1) \in \mathrm{Par}^9_{17}$, we have

$\tilde{\mathfrak{g}}^{(9)}_\lambda =$

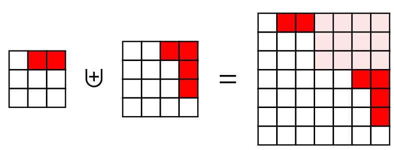



As evidenced by this, the root ideal $\Delta^9(\lambda)$ has a wall in rows $[12, 17]$, a ceiling in columns $[1, 4]$, a ceiling in columns $[5, 6]$, and a ceiling in columns $[8, 9]$. An example of a bounce path in $\Delta^9(\lambda)$ is $\{1, 5, 11, 17\}$.

DEFINITION 3.1.4. Given root ideals $\Psi \subset \Delta^+_\ell$ and $\Psi' \subset \Delta^+_{\ell'}$, we define the root ideal $\Psi \uplus \Psi' \subset \Delta^+_{\ell+\ell'}$ to be the result of placing $\Psi$ and $\Psi'$ catty-corner and including the full $\ell \times \ell'$ rectangle of roots in between. Equivalently, $\Psi \uplus \Psi'$ is determined by

$$\Delta^+_{\ell+\ell'} \setminus (\Psi \uplus \Psi') = (\Delta^+_\ell \setminus \Psi) \;\sqcup\; \{(i + \ell, j + \ell) | (i, j) \in \Delta^+_{\ell'} \setminus \Psi'\} .$$

For example, using lighter shading to emphasize the $\ell \times \ell'$ rectangle,



LEMMA 3.1.5. [**BMS**] *Given $\lambda \in \mathbb{Z}^\ell, \mu \in \mathbb{Z}^{\ell'}$, root ideals $\Psi, \mathcal{L} \subset \Delta^+_\ell$, and root ideals $\Psi', \mathcal{L}' \subset \Delta^+_{\ell'}$, we have*

$$K(\Psi; \mathcal{L}; \lambda) K(\Psi'; \mathcal{L}'; \mu) = K(\Psi \uplus \Psi'; \mathcal{L} \uplus \mathcal{L}'; (\lambda, \mu)) ,$$

*where* $(\lambda, \mu) := (\lambda_1, \ldots, \lambda_\ell, \mu_1, \ldots, \mu_{\ell'})$.

Note that as alternative notation, we may write $\lambda\mu := (\lambda, \mu)$ when $\lambda$ and $\mu$ are both lists of integers (not necessarily partitions, nor is $\lambda\mu$ itself necessarily a partition).

### 3.1.1. Root expansions.

PROPOSITION 3.1.6. [See21, Lemma 4.5.3] *Let* $S \subseteq \Delta^+$*,* $M$ *on* $[\ell]$ *be a multiset, and* $\mu \in \mathbb{Z}^\ell$*. Then,*

*(1) for any root* $\beta \notin S$*,*

$$K(S; M; \mu) = K(S \cup \beta; M; \mu) - K(S \cup \beta; M; \mu + \varepsilon_\beta)\,;$$

*(2) for any root* $\alpha \in S$*,*

$$K(S; M; \mu) = K(S \setminus \alpha; M; \mu) + K(S; M; \mu + \varepsilon_\alpha)\,;$$

*(3) for any* $y \in M$*,*

$$K(S; M; \mu) = K(S; M \setminus y; \mu) - K(S; M \setminus y; \mu - \epsilon_y)\,;$$

*(4) for any* $y \in [\ell]$*,*

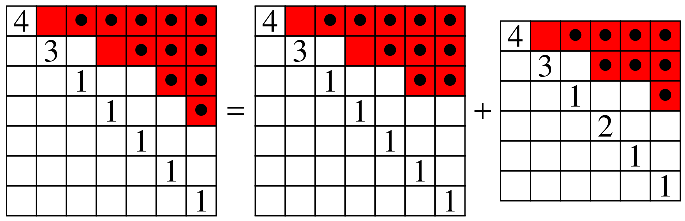
$$K(S; M; \mu) = K(S; M \sqcup y; \mu) + K(S; M; \mu - \epsilon_y)\,.$$

LEMMA 3.1.7. [BMS] *Let* $\Psi \subset \Delta^+_\ell$*,* $M$ *be a multiset on* $[\ell]$*, and* $\mu \in \mathbb{Z}^\ell$ *with* $\mu_\ell = 1$*. If* $\ell \in M$ *and* $\Psi$ *has a removable root* $\alpha = (x, \ell)$ *for some* $x$*, then*

$$K(\Psi; M; \mu) = K(\Psi \setminus \alpha; M \setminus \ell; \mu) + K(\hat{\Psi}; \hat{M} \sqcup x; (\mu_1, \ldots, \mu_{\ell-1}) + \epsilon_x)\,,$$

*where* $\hat{\Psi} = \{(i, j) \in \Psi | j < \ell\}$ *and* $\hat{M} = \{j \in M | j < \ell\}$*.*

Example 3.1.8. We apply Lemma 3.1.7 to the following scenario, with $\ell = 7$ and root $\alpha = (4, 7)$:




By applying root expansions, we obtain useful mirror lemmas that will be relevant to manipulating Katalan functions.

Lemma 3.1.9. [See21] *Suppose an arbitrary subset* $\Psi \subset \Delta_\ell^+$, *a multiset* $M$ *on* $[\ell]$, $\mu \in \mathbb{Z}^\ell$, *and* $z \in [\ell - 1]$ *satisfy*

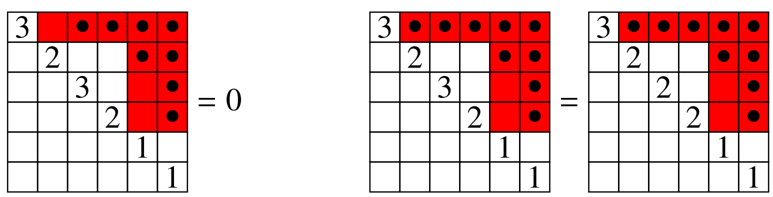
*(1)* $\Psi$ *has a ceiling in columns* $z, z + 1$*;*

*(2)* $\Psi$ *has a wall in rows* $z, z + 1$*;*

*(3)* $\mu_z = \mu_{z+1} - 1$.

*If* $m_M(z + 1) = m_M(z) + 1$, *then* $K(\Psi; M; \mu) = 0$. *If* $m_M(z) = m_M(z + 1)$, *then* $K(\Psi; M; \mu) = K(\Psi; M; \mu - \epsilon_{z+1})$*,*

Example 3.1.10. For $z = 2$, Lemma 3.1.9 applies in the following two situations:




Corollary 3.1.11. *Suppose a subset* $\Psi \subset \Delta_\ell^+$, *a multiset* $M$ *on* $[\ell]$, $\mu \in \mathbb{Z}^\ell$, *and* $z \in [\ell - 1]$ *satisfy*

*(1)* $\Psi$ *has a removable root in column* $z + 1$*;*

*(2)* $\Psi$ *has a wall in rows* $z, z + 1$*;*

*(3)* $\mu_z = \mu_{z+1} - 1$.

*If* $m_M(z + 1) = m_M(z) + 1$, *then* $K(\Psi; M; \mu) = K(\Psi; M; \mu + \varepsilon_{(up_\Psi(z+1), z+1)})$.

LEMMA 3.1.12. [See21, Lemma 4.5.3] *Suppose* $S \subseteq \Delta_\ell^+$, *multiset* $M$ *on* $[\ell]$, $\gamma \in \mathbb{Z}^\ell$, *and* $j \in [\ell]$ *satisfy*

*(1)* $S$ *has a removable root* $(i, j)$ *in column* $j$*;*

*(2)* $S$ *has a ceiling in columns* $j, j+1$ *and a wall in rows* $j, j+1$*;*

*(3)* $m_M(j+1) = m_M(j) + 1$*;*

*(4)* $\gamma_j = \gamma_{j+1}$.

*Then,* $K(S; M; \gamma) = K(S; M \setminus j; \gamma) = K(S \setminus (i, j); M; \gamma)$.

LEMMA 3.1.13. [See21, Lemma 4.5.3] *Suppose* $S \subseteq \Delta_\ell^+$, *multiset* $M$ *on* $[\ell]$, $\gamma \in \mathbb{Z}^\ell$, *and* $j \in [\ell]$ *satisfy*

*(1)* $j \in M$*;*

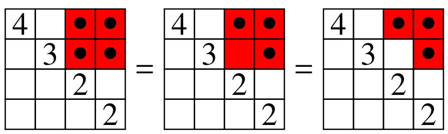
*(2)* $S$ *has a ceiling in columns* $j, j+1$ *and a wall in rows* $j, j+1$*;*

*(3)* $m_M(j+1) = m_M(j)$*;*

*(4)* $\gamma_j = \gamma_{j+1}$.

*Then,* $K(S; M; \gamma) = K(S; M \setminus j; \gamma)$. *If, in addition,* $S$ *has a removable root* $(i, j)$ *in column* $j$*, then* $K(S; M; \gamma) = K(S \setminus (i, j); M \setminus j; \gamma)$.

EXAMPLE 3.1.14. By Lemma 3.1.13,




## 3.2. $k$-Rectangle and minus box machinery

The closed $k$-Schur Katalan functions satisfy a beautiful multiplication rule that Seelinger proved in his thesis.

PROPOSITION 3.2.1. [See21, Theorem 4.4.5] *For* $d \in [k]$ *and* $\mu \in Par_\ell^k$, $g_{R_d}\, \tilde{\mathfrak{g}}_\mu^{(k)} = \tilde{\mathfrak{g}}_{\mu \cup R_d}^{(k)}$, *where* $\mu \cup R_d$ *is the partition made by combining the parts of* $\mu$ *and those of* $R_d$ *and then sorting.*

We may also consider products of closed $k$-Schur Katalan functions wherein we index with a *$k$-rectangle-minus a box*.

$\tilde{\mathfrak{g}}^{(9)}_{\mu \cup R_6} =$

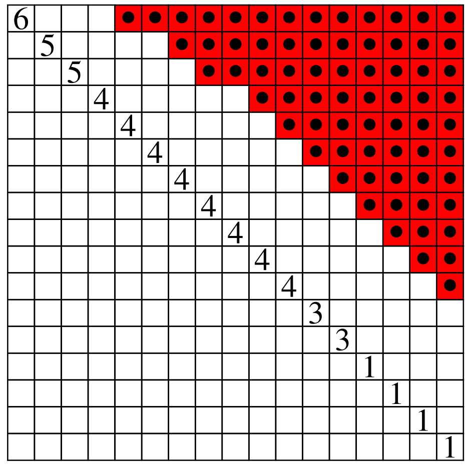



FIGURE 1. The $k$-rectangle property with $\mu = 6544331111$

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DEFINITION 3.2.2. A *k-rectangle minus a box* is a partition of length $i$ of the form $R_i^* := (k+1-i, k+1-i, \ldots, k+1-i, k+1-i-1)$ for $i \in [k]$.

The rule's complexity grows substantially upon considering such products. Via computer experimentation, one can verify that the product need not produce a single closed $k$-Schur Katalan function in general.

EXAMPLE 3.2.3. Let $k = 5$ and consider $\mu = (5,4,3,3,1)$ and $R_3^* = (3,3,2)$. Then we have the cancellation-free expression:

$$\tilde{\mathfrak{g}}^{(k)}_{\mu}\tilde{\mathfrak{g}}^{(k)}_{R_3^*} =$$

$$\tilde{\mathfrak{g}}^{(5)}_{(5,4,4,4,2,2,2,1)} + \tilde{\mathfrak{g}}^{(5)}_{(5,4,3,3,3,3,3)} - \tilde{\mathfrak{g}}^{(5)}_{(5,4,4,4,2,2,2)} + \tilde{\mathfrak{g}}^{(5)}_{(5,5,3,3,3,2,2,1)} - \tilde{\mathfrak{g}}^{(5)}_{(5,4,3,3,3,2,2,1)}$$

$$+\tilde{\mathfrak{g}}^{(5)}_{(5,4,3,3,3,2,2)} - \tilde{\mathfrak{g}}^{(5)}_{(5,4,3,3,2,2,2)} + \tilde{\mathfrak{g}}^{(5)}_{(5,4,3,3,2,2,2,1)} - \tilde{\mathfrak{g}}^{(5)}_{(5,5,3,3,2,2,2,1)} - \tilde{\mathfrak{g}}^{(5)}_{(5,5,3,3,3,2,2)}$$

$$+\tilde{\mathfrak{g}}^{(5)}_{(5,5,3,3,2,2,2)} + \tilde{\mathfrak{g}}^{(5)}_{(5,4,3,3,3,3,2,1)} - \tilde{\mathfrak{g}}^{(5)}_{(5,4,4,3,2,2,2,1)} - \tilde{\mathfrak{g}}^{(5)}_{(5,4,3,3,3,3,2)} + \tilde{\mathfrak{g}}^{(5)}_{(5,4,4,3,2,2,2)}.$$

EXAMPLE 3.2.4. Let $k = 5$ and consider $\mu = (4,4,2,2)$ and $(3,3,2) = R_3^*$. We have:

$$\tilde{\mathfrak{g}}^{(k)}_{\mu}\tilde{\mathfrak{g}}^{(k)}_{R_3^*} =$$

$$\tilde{\mathfrak{g}}^{(5)}_{(4,4,3,3,3,1,1)} + \tilde{\mathfrak{g}}^{(5)}_{(4,4,3,3,3,2,1)} + \tilde{\mathfrak{g}}^{(5)}_{(4,4,3,3,2,2,2)} - \tilde{\mathfrak{g}}^{(5)}_{(4,4,3,3,2,2,1)} - \tilde{\mathfrak{g}}^{(5)}_{(4,4,3,3,3,1,1)} - \tilde{\mathfrak{g}}^{(5)}_{(4,4,3,3,2,1,1)} + \tilde{\mathfrak{g}}^{(5)}_{(4,4,3,3,2,1,1)}$$

$$= \tilde{\mathfrak{g}}^{(5)}_{(4,4,3,3,2,2,2)}.$$

In [See21], a host of root ideal machinery is introduced to facilitate the proof of the $k$-rectangle property for the closed $k$-Schur functions. Much of this machinery is indeed applicable to the manipulation of these functions upon replacing the indexing $k$-rectangle with a $k$-rectangle minus a box. The following section summarizes the appropriate tools that carry over. We note several consequences of these tools to the rectangle minus a box product.

**3.2.1. Root subsets.** While rules like Lemma 3.1.5 make writing products of Katalan functions as one Katalan function straightforward, the result of this technique is insufficient to make progress on the $k$-rectangle minus a box rule. To that end, our first goal is to write the product as one advantageously designed Katalan function, which we accomplish in Lemma 3.2.14 using root subsets (as opposed to ideals).

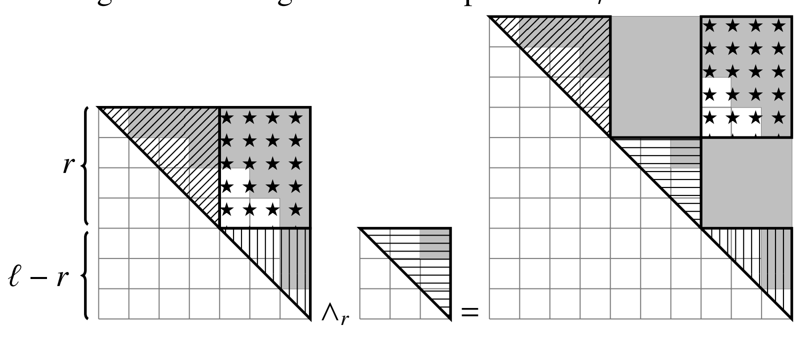
Definition 3.2.5. (1) Given a subset $S \subset \mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$ and $a \leq b$, let $S|_{[a,b]} \subset \Delta^+_{b-a+1}$ be the root ideal given by the roots of $S$ in $[a,b] \times [a,b]$. More precisely,

$$S|_{[a,b]} = \{(i-a+1, j-a+1) \in \Delta^+_{b-a+1} \mid (i,j) \in S \cap ([a,b] \times [a,b])\}.$$

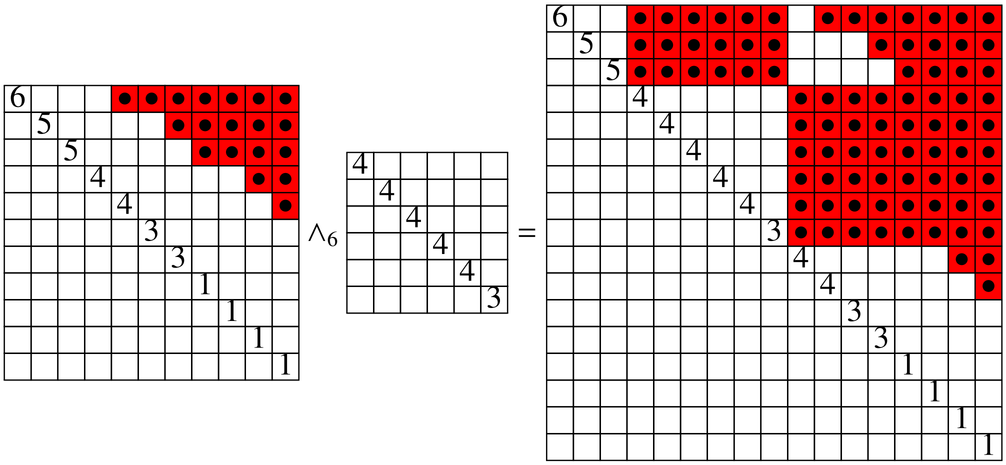
(2) Consider two subsets of roots $S \subset \Delta^+_\ell$ and $T \subset \Delta^+_m$. For any $r \in [\ell]$, we define $S \wedge_r T \subset \Delta^+_{\ell+m}$ to be a generalization of $\uplus$ where we slice $S$ into 3 regions and position them around $T$ to form a new root subset. Precisely,

$$S \wedge_r T = (S|_{[1,r]} \uplus T \uplus S|_{[r+1,\ell]}) \setminus \{(i, m+j) \in \Delta^+_{\ell+m} \mid (i,j) \in \Delta^+_\ell \setminus S, i \leq r, \text{ and } j > r\}.$$

The schematic below gives a visual guide for the operation $\wedge_r$.



EXAMPLE 3.2.6. Consider the following example with $\ell = 11$ and $r = 6$.



LEMMA 3.2.7. [See21, Lemma 4.5.6] *Consider root subsets (not necessarily ideals)* $\Psi, \mathcal{L} \subset \Delta^+_\ell$, $\Psi', \mathcal{L}' \subset \Delta^+_{\ell'}$, *as well as* $\lambda \in \mathbb{Z}^\ell$ *and* $\mu \in \mathbb{Z}^{\ell'}$. *Then, for any* $r \in [\ell]$, *we have*

$$K(\Psi;\mathcal{L};\lambda)K(\Psi';\mathcal{L}';\mu) = K(\Psi \wedge_r \Psi';\mathcal{L} \wedge_r \mathcal{L}';(\lambda_1,\dots,\lambda_r,\mu_1,\dots,\mu_{\ell'},\lambda_{r+1},\dots,\lambda_\ell)).$$

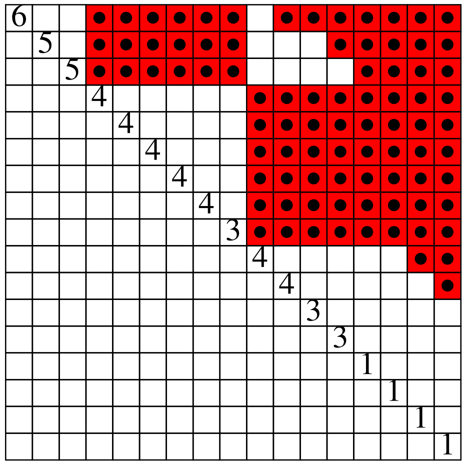
COROLLARY 3.2.8. *Let* $k, \ell \geq 1$, $\mu \in Par^k_\ell$ *and* $d \in [k]$. *Furthermore, let* $r$ *be the number such that* $\mu_r > k+1-d$ *but* $\mu_{r+1} \leq k+1-d$, *taking* $\mu_0 = \infty$ *and* $\mu_{\ell+1} = 0$. *Then,*

$$\tilde{\mathfrak{g}}^{(k)}_\mu \tilde{\mathfrak{g}}^{(k)}_{R^*_d} = K(\Delta^k(\mu) \wedge_r \emptyset_d;\Delta^k(\mu) \wedge_r \emptyset_d;(\mu_1,\dots,\mu_r,k+1-d,\dots,k+1-d,k-d,\mu_{r+1},\dots,\mu_\ell))$$

PROOF. This result follows from Lemma 3.2.7 by taking $\Psi = \mathcal{L} = \Delta^k(\mu)$ and $\Psi' = \mathcal{L}' = \Delta^k(R^*_d)$, noting that $\Delta^k(R^*_d) = \emptyset_d$ by construction. □

EXAMPLE 3.2.9. For $k = 9$, $\mu = 65544331111$, and $d = 6$, we get $r = 3$ and Corollary 3.2.8 gives the following.

$\tilde{\mathfrak{g}}^{(9)}_{65544331111}\tilde{\mathfrak{g}}^{(9)}_{R_6^*} =$



$\tilde{\mathfrak{g}}^{(9)}_{\mu}\tilde{\mathfrak{g}}^{(9)}_{R_6^*} =$

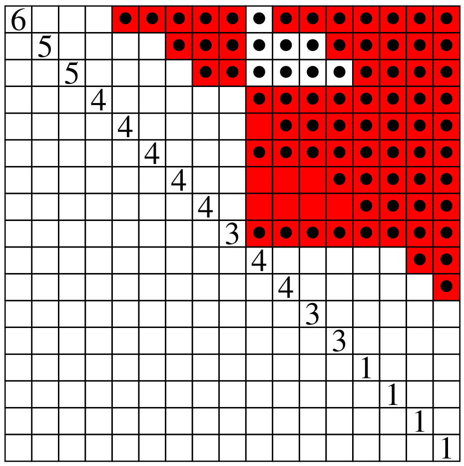


FIGURE 2. An example of Lemma 3.2.14 with $\mu = 6544331111$.

LEMMA 3.2.10. [See21, Lemma 4.5.9] *For $\mu \in Par^k_\ell$, $d \in [k]$, and $r$ such that $\mu_r > k+1-d$ but $\mu_{r+1} \leq k+1-d$, taking $\mu_0 = \infty$ and $\mu_{\ell+1} = 0$, let $\nu = (\mu_1, \ldots, \mu_r)$ and $\eta = (\mu_{r+1}, \ldots, \mu_\ell)$. Then,*

$$\tilde{\mathfrak{g}}^{(k)}_{R_d}\tilde{\mathfrak{g}}^{(k)}_{\mu} = K(\Psi; \Psi; \nu R_d \eta)\,,$$

*where $\Psi = (\Delta^k(\nu R_d) \uplus \Delta^k(\eta)) \setminus \Theta$ for $\Theta = \{(i, d+j) \mid (i,j) \in \Delta^+_\ell \setminus \Delta^k(\mu), i \leq r, \text{ and } j > r\}$.*

Combining the proof of 3.2.10 in [See21] with the $k$-rectangle property provides a stronger result which will be useful in the following section.

Corollary 3.2.11. *With the same notation as 3.2.10, let $D$ be a set such that $x \in D$ implies $x \in [r+1, \ell]$. Then*

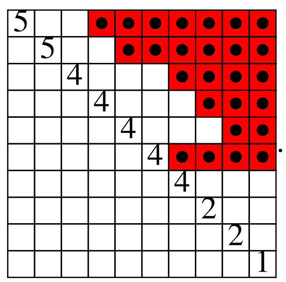
$$\tilde{\mathfrak{g}}^{(k)}_{R_d} K(\Delta^k(\mu - \epsilon_D); \Delta^k(\mu - \epsilon_D); \mu - \epsilon_D) = K(\Delta^k(\mu \cup R_d - \epsilon_D); \Delta^k(\mu \cup R_d - \epsilon_D); \mu \cup R_d - \epsilon_D)$$

$$= K(\Psi; \Psi; \mu \cup R_d - \epsilon_D)\,.$$

Definition 3.2.12. We call a list of roots $B^c_{(m,n)} := \{(c, i) : i \in [m, n]\}$ a *root-bar*.

Example 3.2.13. Let $\mu = (5, 5, 4, 2, 2, 1)$, $k = 6$, and $R_d = (4, 4, 4, 4)$. We have

$$K(\Delta^k(\mu \cup R_d) \cup B^6_{(7,9)}; \Delta^k(\mu \cup R_d) \cup B^6_{(7,9)}; \mu \cup R_d) =$$




Lemma 3.2.14. *For $\mu \in Par^k_\ell$, $d \in [k]$, and $r$ such that $\mu_r > k+1-d$ but $\mu_{r+1} \le k+1-d$, taking $\mu_0 = \infty$ and $\mu_{\ell+1} = 0$, let $\nu = (\mu_1, \ldots, \mu_r)$ and $\eta = (\mu_{r+1}, \ldots, \mu_\ell)$. Then,*

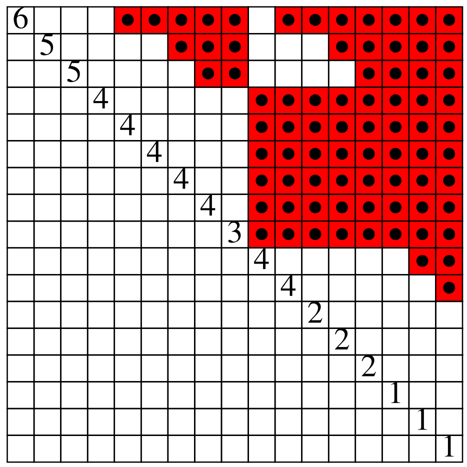
$$\tilde{\mathfrak{g}}^{(k)}_{R^*_d} \tilde{\mathfrak{g}}^{(k)}_{\mu} = K(\Psi; \Psi; \nu R^*_d \eta)\,,$$

*where $\Psi = (\Delta^k(\nu R^*_d) \uplus \Delta^k(\eta)) \setminus \Theta$ for $\Theta = \{(i, d+j) \mid (i, j) \in \Delta^+_\ell \setminus \Delta^k(\mu), i \le r,\ and\ j > r\}$.*

Proof. We start with $\tilde{\mathfrak{g}}^{(k)}_{R^*_d} \tilde{\mathfrak{g}}^{(k)}_{\mu} = K(\Psi'; \Psi'; \nu R^*_d \eta)$ for $\Psi' = \Delta^k(\mu) \wedge_r \emptyset_d$ as in Corollary 3.2.8. We then iteratively remove roots in columns $r+1, \ldots, r+(k-\nu_x)$ for $x = r, r-1, \ldots, 1$ of $\Psi$ by repeated applications of Lemma 3.1.13. We start by removing $k - \nu_r$ roots from row $r$ in order from left to right, removing all of $B^r_{(r+1, r+(k-\nu_r))}$. Each time, the conditions of Lemma 3.1.13 are met. Note $r + k - \nu_r = r + k - k - 1 + d = r + d - 1$; by our choice of $r$, it must be that $(r, r+d-1) \in \Psi$ and $(r, r+d) \in \Psi$, so that we need not apply Lemma 3.1.13 to the root $(r, r+d-1)$ (the conditions would fail). We then continue the process by removing $k - \nu_{r-1}$ roots from row $r-1$, $B^{r-1}_{(r+1, r+(k-\nu_{r-1}))}$, in order from left to right; by the same logic as Remark 2.3.7, $\Delta^k(\nu \uplus R_d^*) \setminus B^r_{(r+1,r+(k-\nu_r))} \setminus B^{r-1}_{(r+1,r+(k-\nu_{r-1}))}$ is "wall-free;" therefore, at each root removal, the conditions of Lemma 3.1.13 are met again. The process continues in this manner, moving on to row $r-2$, then $r-3$, and so on; the method continues to apply for higher rows by similar reasoning until we arrive at $\Psi$. □

EXAMPLE 3.2.15. If $\mu = (6, 5, 5, 4, 4, 2, 2, 2, 1, 1, 1)$, $k = 9$, and $R_d = (4, 4, 4, 4, 4, 4)$, Lemma 3.2.14 implies

$\tilde{\mathfrak{g}}^{(k)}_{R_d^*} \tilde{\mathfrak{g}}^{(k)}_{\mu} =$




**3.2.2. Root removals.** Before proceeding to next steps that expand the $k$-rectangle minus a box product, we reference several properties of root ideals which will serve us later. These laws will guide us into removing roots east of $R_d^*$ from the result of Lemma 3.2.14.

LEMMA 3.2.16. [See21, Lemma 4.5.14] *Suppose a subset of roots $S \subset \Delta^+_\ell$, a multiset $M$ on $[\ell]$, $\gamma \in \mathbb{Z}^\ell$, and $y \in [\ell]$ satisfy*

*(1) $S$ has a root in row $y$ and $z = \mathrm{down}_S(y)$ is defined;*

*(2) $z \in M$;*

*(3) $S$ has a ceiling in columns $y-1, y$ and a wall in rows $y-1, y$;*

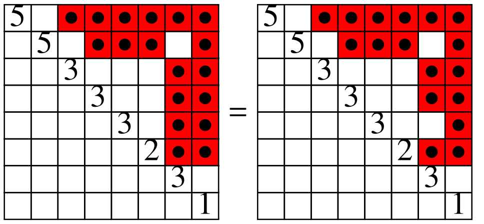
*(4) $m_M(y-1) = m_M(y)$;*

*(5) $\gamma_y = \gamma_{y-1}$.*

*Then, we have*

$$K(S; M; \gamma) = K(S \setminus (y, z); M \setminus \{z\}; \gamma) .$$

EXAMPLE 3.2.17. Lemma 3.2.16 gives the following equality.



LEMMA 3.2.18. [See21, Lemma 4.5.16] *Suppose a subset of roots* $S \subset \Delta_\ell^+$, *a multiset* $M$ *on* $[\ell]$, $\gamma \in \mathbb{Z}^\ell$, *and* $1 \leq x < y < z \leq \ell$ *satisfy*

*(1)* $y = \mathrm{down}_S(x)$;

*(2)* $(y, z) \in S$ *but* $(y, m) \notin S$ *if* $m < z$;

*(3)* $(x, z) \notin S$;

*(4)* $(x, y-1) \notin S$ *and* $S \cup (x, y-1)$ *has a ceiling in columns* $y-1, y$;

*(5)* $S$ *has a wall in rows* $y-1, y$;

*(6)* $m_M(y-1) = m_M(y) - 1$;

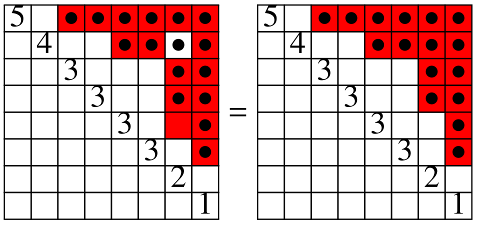
*(7)* $\gamma_{y-1} = \gamma_y$.

*Then,*

$$K(S; M; \gamma) = K((S \cup (x, z)) \setminus (y, z); M; \gamma)\,.$$

Note that in [See21], Lemma 3.2.18 states instead that $z = \mathrm{down}_S(y)$ as opposed to the condition that $(y, z) \in S$ but $(y, m) \notin S$ if $m < z$, but in practice, the result applies to this more general wording.

EXAMPLE 3.2.19. Lemma 3.2.18 gives the following equality with $y = 5$, $x = 2$, and $z = 7$.



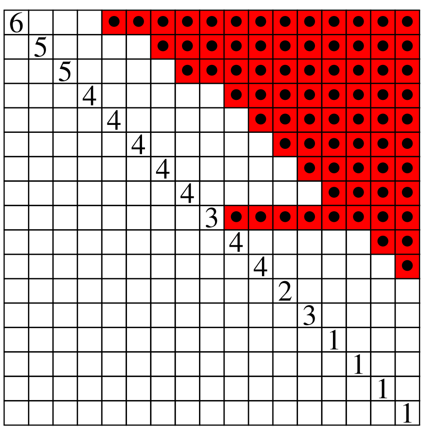
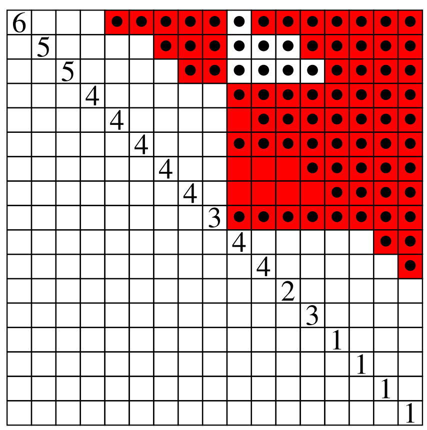
## 3.3. The root bar approach

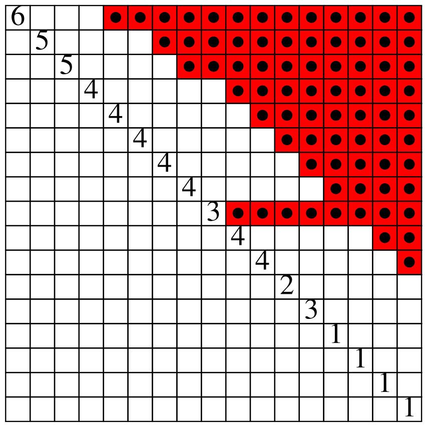
We now leverage root removals to make the next meaningful step in rewriting the $k$-rectangle minus a box product. Our approach is more generic than a direct rewriting of Lemma 3.2.14; we begin by developing some useful root subset machinery.

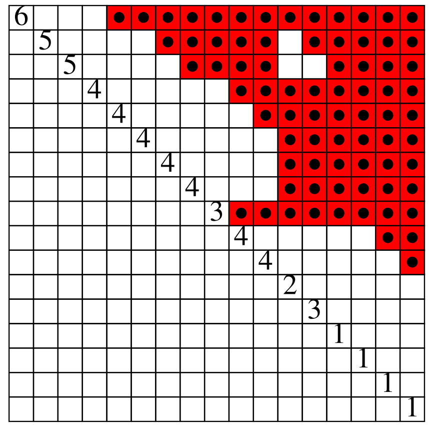
DEFINITION 3.3.1. Fix integers $d, r$ and $\ell \geq 2d + r$ and consider $b \in [r+1, r+d]$. A set $\Psi \subset \Delta_\ell^+$ which is a wall-free root ideal in rows $\leq r+d$ and consists of $\{(i, j) : j \geq d+i\}$ in rows $i \in [r+1, b]$ is called a $(d, r, b)$-staircase. When $\Psi$ is a $(d, r, b)$-staircase, $\Theta = \{(i, j+d) : (i, j) \in \Delta_\ell^+ \setminus \Psi, i \leq r, j \in [a, b]\} \subset \Psi$ for any $a \in [r+1, r+d]$, and we can define the set

$$\Psi(d, r, a, b) = \Psi \cup \{(i, j+d) : i \in [a+1, b], j \in [a, i-1]\} \setminus \Theta\,.$$

EXAMPLE 3.3.2. The set $\Psi = \Delta^k(\mu \cup R_d^*) \cup B^{r+d}_{[r+d+1, r+2d]}$ is a $(d, r, r+d-1)$-staircase and $\Psi(d, r, r+1, r+d-1) = (\Delta^k(\nu R_d^*) \uplus \Delta^k(\eta)) \setminus \Theta$ where $\Theta = \{(i, d+j) \mid (i, j) \in \Delta_\ell^+ \setminus \Delta^k(\mu), i \leq r < j\}$.

$\Psi =$  $\Psi(6, 3, 4, 8) =$ 

$\Psi =$ 


$\Psi(6,3,6,8) =$ 


Remark 3.3.3. For any $(d,r,b)$-staircase $\Psi$, if $(i,j)$ where $i \le r$ and $down_\Psi(r) \le j \le r+d$, then $(i,j) \notin \Delta^+ \backslash \Psi$. Hence, $\Psi(d,r,a,b) = \Psi$ when $a \ge down_\Psi(r)$.

Lemma 3.3.4. *Fix integers $d, r$ and $\ell \ge 2d + r$. Let $\Psi \subset \Delta^+_\ell$ be a $(d,r,b)$-staircase for some $down_\Psi(r) \le b \le r+d$. If $\gamma \in \mathbb{Z}^\ell$ satisfies $\gamma_{a'} = \cdots = \gamma_b$ for some $r < a' \le r+d$, then for all $a' \le a \le r+d$, $K(S,S;\gamma) = K(\tilde{S},\tilde{S};\gamma)$ where $S = \Psi(d,r,a,b)$ and $\tilde{S} = \Psi(d,r,a',b)$.*

Proof. For all $a \ge down_\Psi(r)$, $\Psi = \Psi(d,r,a,b)$ by Remark 3.3.3. Hence we consider $a' < a \le b$, and it suffices to prove the result for $a' = a-1$; here $S$ and $\tilde{S}$ differ only in column $d+a'$. We do this by iteration with an identity on Katalan functions indexed by sets of roots which differ by at most two roots in column $d+a'$. Let $S^0 = \tilde{S}$ and for $t = 1,2,\dots,b-a'$, iteratively define

$$S^t = \begin{cases} S^{t-1} \backslash \{(y, d+a')\} \cup \{(up_{S^{t-1}}(y), d+a') & \text{if } up_{S^{t-1}}(y) \text{ exists} \\ S^{t-1} \backslash \{(y, d+a')\} & \text{otherwise,} \end{cases} \tag{3.3.1}$$

where $y = b+1-t$. By construction, $S^{t-1}$ matches $\Psi$ in columns $\le d+r$ and has a wall in rows $y-1, y$. Hence, if $x = up_{S^{t-1}}(y)$ exists, the conditions of Lemma 3.2.18 hold for $M = S^{t-1}$, implying that $K(S^{t-1}, S^{t-1};\gamma) = K(S^t, S^t;\gamma)$. If $up_{S^{t-1}}(y)$ does not exist, $y, y-1$ has a ceiling since $y \le b \le down_{S^{t-1}}(r)$ and $S^{t-1}$ is wall-free in rows $\le r$. The conditions of Lemma 3.2.16 are met and again $K(S^{t-1}, S^{t-1};\gamma) = K(S^t, S^t;\gamma)$. Therefore, $K(S^0, S^0;\gamma) = K(S^{b-a'}, S^{b-a'};\gamma)$.

It remains to check that $S^{b-a'} = S$. For this, we need $\cup_t \{(up_{S^{t-1}}(y = b+1-t), d+a') : up_{S^{t-1}}(y) \text{ exist } \}$ to be $\{(i, d+a') : (i,a') \in \Delta^+_\ell \backslash \Psi, i \le r\}$; indeed, the number of roots in column $a'$

of $\Delta_\ell^+ \backslash \Psi$ is the number of removable corners between $a'$ and $down_\Psi(r) \le b$ in $\Psi$. Since $\Psi$ and $S^{t-1}$ match in these columns, these corners are in the columns where $up_{S^{t-1}}(y)$ exists. □

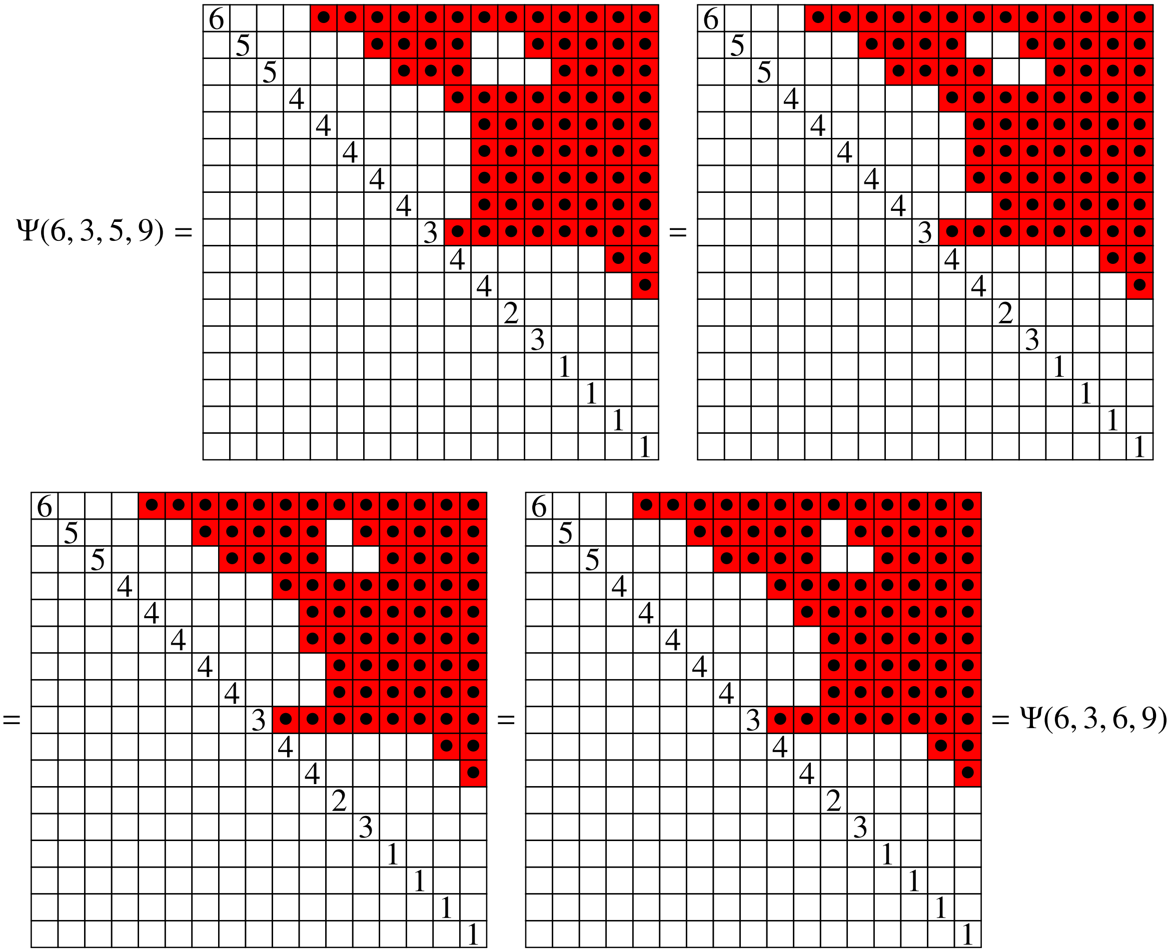



FIGURE 3. The algorithmic proof of Lemma 3.3.4 applied to an example.

Applications of this lemma are important to our study of $\tilde{\mathfrak{g}}^{(k)}_{R^*_d}\tilde{\mathfrak{g}}^{(k)}_\mu$. For the remainder of the section, fix $\mu \in \mathrm{Par}^k_\ell$, $d \in [k]$, and $r$ such that $\mu_r > k+1-d$ but $\mu_{r+1} \le k+1-d$, taking $\mu_0 = \infty$ and $\mu_{\ell+1} = 0$. Let $\nu = (\mu_1, \ldots, \mu_r)$ and $\eta = (\mu_{r+1}, \ldots, \mu_\ell)$.

LEMMA 3.3.5. *Let $\Psi^x = \Delta^k(\mu \cup R^*_d) \cup B^{r+d}_{[x, r+2d]}$ for $x \in [r+d+1, r+2d]$. For any $\gamma \in \mathbb{Z}^n$ such that $\gamma_{r+1} = \cdots = \gamma_{r+d-1}$,*

$$K(\Psi^{r+d+1}; \Psi^{r+d+1}; \gamma) = K(\tilde{S}; \tilde{S}; \gamma), \tag{3.3.2}$$

*for* $\tilde{S} = (\Delta^k(\nu R_d^*) \uplus \Delta^k(\eta)) \setminus \Theta$ *where* $\Theta = \{(i, d+j) \mid (i,j) \in \Delta_\ell^+ \setminus \Delta^k(\mu), i \le r < j\}$. *If* $\gamma_{r+d-1} = \gamma_{r+d}$ *holds as well,*

$$K(\Psi^x; \Psi^x; \gamma) = K(\Delta^k(\mu \cup R_d); \Delta^k(\mu \cup R_d); \gamma)\,. \tag{3.3.3}$$

Proof. Set $b = r + d$ and fix $x \in [b+1, b+d]$. Consider the $(d, r, b-1)$-staircase $\Psi^x$. Since $r$ was chosen so that $\nu_r > k + 1 - d$, we have $down_{\Psi^x}(r) < r + d$ and hence $down_{\Psi^x}(r) \le b - 1$. We note that $\Psi^x(d, r, b, b-1) = \Psi^x$ and so, because $x - d \in [r+1, b]$, Lemma 3.3.4 gives

$$K(\Psi^x(d, r, x-d, b-1); \Psi^x(d, r, x-d, b-1); \beta) = K(\Psi^x; \Psi^x; \beta) \tag{3.3.4}$$

for any $\beta \in \mathbb{Z}^\ell$ where $\beta_{r+1} = \cdots = \beta_{b-1}$. In the case when $x = b+1$, this proves (3.3.2) since its left hand side is $\tilde{S}$ (see Example 3.3.2).

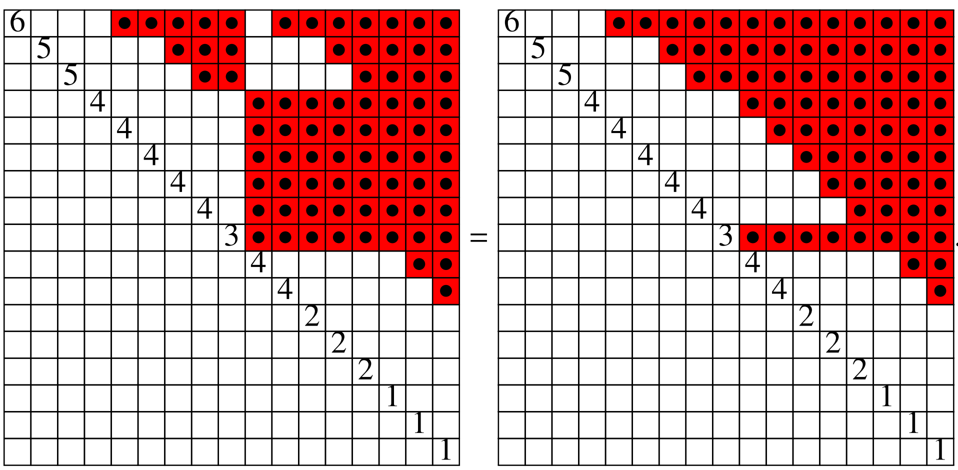
Let $\gamma \in \mathbb{Z}^\ell$ where $\gamma_{r+1} = \cdots = \gamma_b$, so for $\beta = \gamma$, (3.3.4) holds. Now consider the $(d, r, b)$-staircase $\hat{\Psi} = \Delta^k(\mu \cup R_d)$. Since $b \ge down_{\hat{\Psi}}(r)$, $\hat{\Psi}(d, r, b, b) = \hat{\Psi}$ by Remark 3.3.3 and thus

$$K(\hat{\Psi}(d, r, a', b); \hat{\Psi}(d, r, a', b); \gamma) = K(\hat{\Psi}; \hat{\Psi}, \gamma) \tag{3.3.5}$$

for any $a' \in [r+1, b]$ by Lemma 3.3.4. Identity (3.3.3) follows from the observation that $\hat{\Psi}(d, r, x-d, b) = \Psi^x(d, r, x-d, b-1)$ and equating (3.3.5) and (3.3.4). □

Example 3.3.6. If $\mu = (6, 5, 5, 4, 4, 2, 2, 2, 1, 1, 1)$, $k = 9$, and $R_d = (4, 4, 4, 4, 4, 4)$, Lemma 3.3.5 with $x = r + d - 1$ applied to the result of Lemma 3.2.14 implies

$\tilde{\mathfrak{g}}^{(k)}_{R_d^*}\tilde{\mathfrak{g}}^{(k)}_{\mu} =$



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**3.3.1. Root bar expansions.** To rewrite the result of Lemma 3.3.5 as a sum of closed $k$-Schur Katalan functions, we appeal to root expansions on the root bar. This process will advance in stages, each a refinement of the previous. Our first observation is a corollary to Lemma 3.3.5.

COROLLARY 3.3.7. *Fix $x \in [r+d+1, r+2d]$ and let $\Psi^x = \Delta^k(\mu \cup R_d^*) \cup B^{r+d}_{[x,r+2d]}$. For any $\gamma \in \mathbb{Z}^\ell$ where $\gamma_{r+1} = \cdots = \gamma_{r+d-1}$ and $\gamma_{r+d} = \gamma_{r+d-1} - 1$,*

$$K(\Psi^x;\Psi^x;\gamma) = K(\Psi^{x+1};\Psi^{x+1};\gamma) - K(\Psi^{x+1};\Psi^{x+1};\gamma-\epsilon_x) + K(\Delta^k(\mu\cup R_d);\Delta^k(\mu\cup R_d);\gamma+\epsilon_{r+d}-\epsilon_x)\,.$$

PROOF. Expand on the root $(r+d, x) \in \Psi^x$ with Lemma 3.1.6 to obtain

$$K(\Psi^x;\Psi^x;\gamma) = K(\Psi^{x+1};\Psi^x;\gamma) \;+\; K(\Psi^x;\Psi^x;\gamma+\epsilon_{r+d}-\epsilon_x)\,. \tag{3.3.6}$$

Replace the first term by $K(\Psi^{x+1};\Psi^{x+1};\gamma) - K(\Psi^{x+1};\Psi^{x+1};\gamma-\epsilon_x)$ using Lemma 3.1.6 to delete the lowering operator $x$ and apply (3.3.3) of Lemma 3.3.5 to the rightmost term. □

LEMMA 3.3.8. *For $\mu \in Par^k_\ell$, $d \in [k]$, and $r$ such that $\mu_r > k+1-d$ but $\mu_{r+1} \le k+1-d$, take $\mu_0 = \infty$ and $\mu_{\ell+1} = 0$. Then,*

$$\tilde{\mathfrak{g}}^{(k)}_{R_d^*}\tilde{\mathfrak{g}}^{(k)}_\mu = \sum_{D\subseteq[r+d,r+2d-1]} (-1)^{|D|+1} K(\Delta^k(\mu\cup R_d);\Delta^k(\mu\cup R_d);\mu\cup R_d-\epsilon_D).$$

PROOF. Let $\gamma = \nu R_d^* \eta$. Lemma 3.2.14 gives that $\tilde{\mathfrak{g}}^{(k)}_{R_d^*}\tilde{\mathfrak{g}}^{(k)}_\mu = K(\tilde{S};\tilde{S};\gamma)$ for $\tilde{S} = (\Delta^k(\nu R_d^*) \uplus \Delta^k(\eta)) \setminus \Theta$ where $\Theta = \{(i, d+j) \mid (i,j) \in \Delta^+_\ell \setminus \Delta^k(\mu), i \le r < j\}$. We then apply (3.3.2) in Lemma 3.3.5 to obtain

$$\tilde{\mathfrak{g}}^{(k)}_{R_d^*}\tilde{\mathfrak{g}}^{(k)}_\mu = K(\Psi^{r+d+1};\Psi^{r+d+1};\gamma)\,,$$

where $\Psi^x = \Delta^k(\mu \cup R_d^*) \cup B^{r+d}_{[x,r+2d]}$ is defined for any $x \in [r+d+1, r+2d]$. Let $s = r+d+1$ and apply the three term recurrence from Corollary 3.3.7 to the right hand side:

$$\tilde{\mathfrak{g}}^{(k)}_{R_d^*}\tilde{\mathfrak{g}}^{(k)}_\mu = K(\Psi^{s+1};\Psi^{s+1};\gamma) - K(\Psi^{s+1};\Psi^{s+1};\gamma-\epsilon_s) + K(\Delta^k(\mu\cup R_d);\Delta^k(\mu\cup R_d);\mu\cup R_d-\epsilon_s)\,.$$

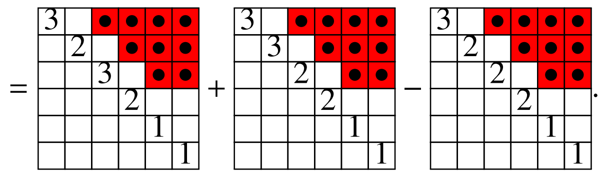
The three term recurrence can again be applied, now to the first two terms. By iteration,

$$\tilde{\mathfrak{g}}^{(k)}_{R^*_d}\tilde{\mathfrak{g}}^{(k)}_{\mu} = \sum_{\substack{|D|\geq 0\\ D\subset[s,r+2d-1]}} (-1)^{|D|}K(\Psi^{r+2d-1};\Psi^{r+2d-1};\gamma-\epsilon_D)+ \sum_{\substack{|D|>0\\ D\subset[s,r+2d-1]}} (-1)^{|D|-1}K(\Delta^k(\mu\cup R_d);\Delta^k(\mu\cup R_d);\mu\cup R_d-\epsilon_D)\,.$$

The sums can be combined by noting that $\Psi^{r+2d-1} = \Delta^k(\mu \cup R_d)$ and, for each $D \subset [s, r + 2d - 1]$, $\gamma - \epsilon_D = \mu \cup R_d - \epsilon_{D'}$ for the set $D' = D \cup \{r + d\}$. □

Example 3.3.9. Let $k = 4$, $R^*_2 = (3, 2)$, $\mu = (3, 2, 1, 1)$. Lemma 3.3.8 implies

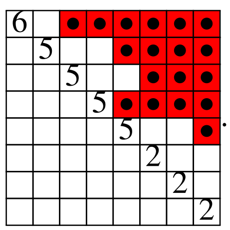
$$\tilde{\mathfrak{g}}^{(4)}_{R^*_2}\tilde{\mathfrak{g}}^{(4)}_{(3,2,1,1)}$$




**3.3.2. Downpath expansions.** To further refine the result of Lemma 3.3.8, we extend our root expansions further within the resulting root ideals. While the results of this lemma no longer feature root bars, the resulting root ideals may not coincide with the $k$-Schur root ideal for their indexing weight. We will address these issues by leveraging *downpaths* (Definition 3.3.12), which are at the heart of Catalan function combinatorics. Not only do the combinatorics of downpaths play a role in rewriting Lemma 3.3.8, but they are also essential to re-indexing generically weighted (non-partition) Katalan functions in all following chapters.

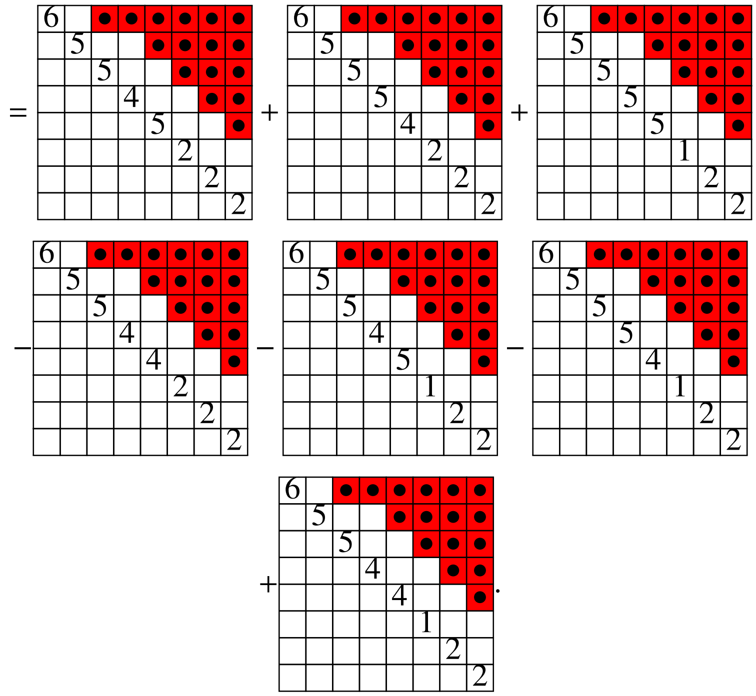
Example 3.3.10. Given $k = 7$, $\mu = (6, 5, 2, 2, 2)$, and $d = 3$, Lemma 3.2.14 and Lemma 3.3.5 combine to imply that

$$\tilde{\mathfrak{g}}^{(k)}_{R^*_d}\tilde{\mathfrak{g}}^{(k)}_{\mu} =$$

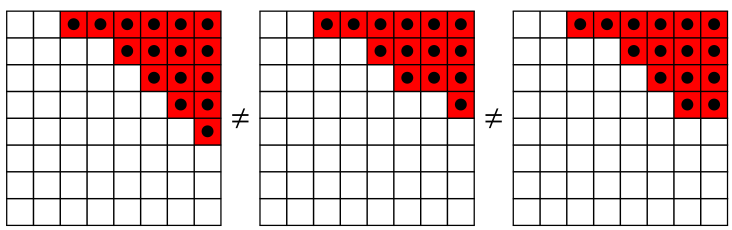


Further, Lemma 3.3.8 implies

$$\tilde{\mathfrak{g}}^{(k)}_{R_d^*}\tilde{\mathfrak{g}}^{(k)}_{\mu} = \sum_{D\subseteq[4,6]} (-1)^{|D|+1} K(\Delta^k(\mu\cup R_d); \Delta^k(\mu\cup R_d); \mu\cup R_d - \epsilon_D$$




While many of these resulting functions are not indexed by partition weights, we note that even those which *are* as such are not closed $k$-Schur Katalan functions; consider the root ideal of each Katalan function written in this result,




These latter root ideals correspond to the $\Delta^k(6,5,5,4,4,2,2,2) = \Delta^k(6,5,5,4,4,1,2,2)$ and $\Delta^k(6,5,5,5,4,2,2,2) = \Delta^k(6,5,5,5,4,1,2,2)$, respectively.

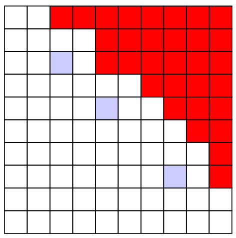
Proposition 3.3.11. [**BMPS19**] *For $\mu \in \widetilde{Par}^k_\ell$, the western border of the root ideal $\Delta^k(\mu)$ consists of the roots $(i, k+1-\mu_i+i)$ for $i$ such that $k+1-\mu_i+i \le \ell$. Moreover, $(i, k+1-\mu_i+i)$ is a removable root of $\Delta^k(\mu)$ if and only if $\mu_i \ge \mu_{i+1}$ and $k+1-\mu_i+i \le \ell$.*

Definition 3.3.12. Let $\Psi \subseteq \Delta_\ell^+$ be an arbitrary subset of positive roots. For each vertex $r \in [\ell]$, distinguish $\mathrm{top}_\Psi(r)$ to be the minimum element of the bounce path of $\Psi$ containing $r$. For $a, b \in [\ell]$ in the same bounce path of $\Psi$ with $a \leq b$, we define

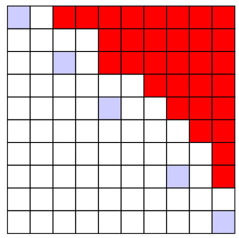
$$\mathrm{downpath}_\Psi(a, b) = \{a, \mathrm{down}_\Psi(a), \mathrm{down}_\Psi^2(a), \ldots, b\},$$

i.e., the set of indices in this path lying between $a$ and $b$. We also set $\mathrm{uppath}_\Psi(r)$ to be $\mathrm{downpath}_\Psi(\mathrm{top}_\Psi(r), r)$ for any $r \in [\ell]$.

Example 3.3.13. A downpath and uppath for the root ideal $\Psi$ are given below:



$\mathrm{downpath}_\Psi(3, 8) = \{3, 5, 8\}$



$\mathrm{uppath}_\Psi(10) = \{10, 8, 5, 3, 1\}$

Remark 3.3.14. For $\mu \in \widetilde{\mathrm{Par}}_\ell^k$ and fixed $R_d$, let $r$ be such that $\mu_r > k + 1 - d$ and $\mu_{r+1} \leq k + 1 - d$. Then for $D \subseteq [r + d, r + 2d - 1]$, $x, y \in D$ implies that $x$ and $y$ have disjoint downpaths in $\Delta^k(\mu \cup R_d)$. To see why, first assume without loss of generality that $x < y$.

But then if $\mathrm{down}^c_{\Delta^k(\mu\cup R_d)}(y) \in \mathrm{downpath}_{\Delta^k(\mu\cup R_d)}(x)$, for some $c$, we would have that $y \in \mathrm{downpath}_{\Delta^k(\mu\cup R_d)}(x)$ or $x \in \mathrm{downpath}_{\Delta^k(\mu\cup R_d)}(y)$, as $\mathrm{down}_{\Delta^k(\mu\cup R_d)}(i)$ is uniquely defined for any $i \in \ell + k + 1 - d$. In particular, $\mathrm{downpath}_{\Delta^k(\mu\cup R_d)}(x)$ and $\mathrm{downpath}_{\Delta^k(\mu\cup R_d)}(y)$ are the same as the corresponding downpaths for $x$ and $y$ had $\mu$ been a partition. Given that $\mu \in \widetilde{\mathrm{Par}}_\ell^k$, we only possibly have that $\mathrm{downpath}_{\Delta^k(\mu\cup R_d)}(x)$ and $\mathrm{downpath}_{\Delta^k(\mu\cup R_d)}(y)$ are shorter downpaths than those of $\Delta^k(\nu \cup R_d)$ for $\nu \in \mathrm{Par}$, but have no further differences (Proposition 3.3.11). Both with this in mind, we note that both $y \in \mathrm{downpath}_{\Delta^k(\mu\cup R_d)}(x)$ and $x \in \mathrm{downpath}_{\Delta^k(\mu\cup R_d)}(y)$ are impossible: the former is because $\mathrm{down}_{\Delta^k(\mu\cup R_d)}(x) > y$ as $k - (\mu \cup R_d)_x + x + 1 \geq r + 2d$ and the latter is because $x < y$.

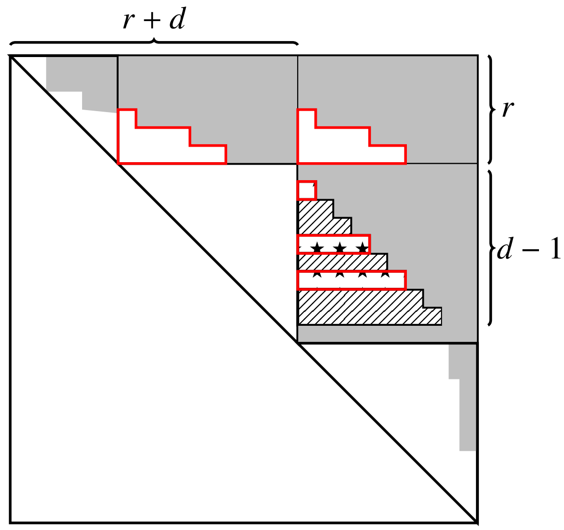



FIGURE 4. The sections outlined in red and starred are roots which correspond to the sections with just red outlining. Together with the section of roots containing north east lines, these roots demonstrate the difference between $\Delta^k(\lambda_{\leq r}R_d) \uplus \Delta^k(\lambda_{>r})$ and $\Delta^k(\mu \cup R_d^*) \cup B^{r+d}_{(r+d+1,r+d+1+d)}$. In the proof of Lemma 3.3.8, we obtain this Katalan function after applying Lemma 3.2.14, and applying Lemma 3.3.5 removes the red/starred and north-east lined sections.

REMARK 3.3.15. $\mathrm{downpath}_{\Delta^k(\mu)}(x) = \mathrm{downpath}_{\Delta^k(\mu-\epsilon_Q)}(x)$ where $Q \cap \mathrm{downpath}_{\Delta^k(\mu)}(x) = \emptyset$. The roots of $\Delta^k(\mu)$ and $\Delta^k(\mu - \epsilon_Q)$ are the same in all rows $j \notin Q$. On the other hand, if $j \in Q$, the roots only differ by deletion of a single root in $\Delta^k(\mu - \epsilon_Q)$. Therefore, for all $y \notin Q$, we have $\mathrm{down}_{\Delta^k(\mu)}(y) = \mathrm{down}_{\Delta^k(\mu-\epsilon_Q)}(y)$.

THEOREM 3.3.16. *Let $\lambda = \mu \cup R_d$ and define $I^{d,k}_\mu := \cup_{i\in[r+d,2d+r-1]} \mathrm{downpath}_{\Delta^k(\mu\cup R_d)}(i)$. Then*

$$\tilde{\mathfrak{g}}^{(k)}_{R_d^*}\tilde{\mathfrak{g}}^{(k)}_\mu = \sum_{D\subset I^{d,k}_\mu, D\neq\emptyset} (-1)^{|D|+1} K(\Delta^k(\mu \cup R_d - \epsilon_D); \Delta^k(\mu \cup R_d - \epsilon_D); \mu \cup R_d - \epsilon_D)$$

$$= \sum_{D\subset I^{d,k}_\mu, D\neq\emptyset} (-1)^{|D|+1} \tilde{\mathfrak{g}}^{(k)}_{\lambda-\epsilon_D}. \tag{3.3.7}$$

PROOF. Apply Lemma 3.3.8. Suppose $D = \{a_1, a_2, \ldots, a_m\}$ with $a_1 < a_2 < \cdots < a_m$ where each $a_i \in [r + d, r + 2d - 1]$. If $\mathrm{down}_{\Delta^k(\mu\cup R_d)}(a_1)$ exists, then by applying Proposition 3.1.6,

$$K(\Delta^k(\mu \cup R_d); \Delta^k(\mu \cup R_d); \mu \cup R_d - \epsilon_D) =$$

$$K(\Delta^k(\mu \cup R_d - \epsilon_{a_1}); \Delta^k(\mu \cup R_d - \epsilon_{a_1}); \mu \cup R_d - \epsilon_D)$$
$$-K(\Delta^k(\mu \cup R_d - \epsilon_{a_1}); \Delta^k(\mu \cup R_d - \epsilon_{a_1}); \mu \cup R_d - \epsilon_{D \cup \mathrm{down}_{\Delta^k(\mu \cup R_d)}(a_1)})$$
$$+K(\Delta^k(\mu \cup R_d); \Delta^k(\mu \cup R_d); \mu \cup R_d - \epsilon_{D \setminus a_1 \cup \mathrm{down}_{\Delta^k(\mu \cup R_d)}(a_1)}).$$

Note that on the other hand, if $\mathrm{down}_{\Delta^k(\mu \cup R_d)}(a_1)$ does not exist, we must have that $\Delta^k(\mu \cup R_d)$ contains no root of the form $(a_1, j)$ for any $j$, so $\Delta^k(\mu \cup R_d) = \Delta^k(\mu \cup R_d - \epsilon_{a_1})$. Therefore, in this case,

$$K(\Delta^k(\mu \cup R_d); \Delta^k(\mu \cup R_d); \mu \cup R_d - \epsilon_D) = K(\Delta^k(\mu \cup R_d - \epsilon_{a_1}); \Delta^k(\mu \cup R_d - \epsilon_{a_1}); \mu \cup R_d - \epsilon_D).$$

In either case, expanding $K(\Delta^k(\mu \cup R_d - \epsilon_{a_1}); \Delta^k(\mu \cup R_d - \epsilon_{a_1}); \mu \cup R_d - \epsilon_D)$ on $a_2$ through Proposition 3.1.6 implies

$$K(\Delta^k(\mu \cup R_d); \Delta^k(\mu \cup R_d); \mu \cup R_d - \epsilon_D) =$$
$$= K(\Delta^k(\mu \cup R_d - \epsilon_{\{a_1,a_2\}}); \Delta^k(\mu \cup R_d - \epsilon_{\{a_1,a_2\}}); \mu \cup R_d - \epsilon_D)$$
$$-K(\Delta^k(\mu \cup R_d - \epsilon_{\{a_1,a_2\}}); \Delta^k(\mu \cup R_d - \epsilon_{\{a_1,a_2\}}); \mu \cup R_d - \epsilon_{D \cup \mathrm{down}_{\Delta^k(\mu \cup R_d)}(a_2)})$$
$$+K(\Delta^k(\mu \cup R_d - \epsilon_{a_1}); \Delta^k(\mu \cup R_d - \epsilon_{a_1}); \mu \cup R_d - \epsilon_{D \setminus a_2 \cup \mathrm{down}_{\Delta^k(\mu \cup R_d)}(a_2)})$$
$$-K(\Delta^k(\mu \cup R_d - \epsilon_{a_1}); \Delta^k(\mu \cup R_d - \epsilon_{a_1}); \mu \cup R_d - \epsilon_{D \cup \mathrm{down}_{\Delta^k(\mu \cup R_d)}(a_1)})$$
$$+K(\Delta^k(\mu \cup R_d); \Delta^k(\mu \cup R_d); \mu \cup R_d - \epsilon_{D \setminus a_1 \cup \mathrm{down}_{\Delta^k(\mu \cup R_d)}(a_1)})$$

if $\mathrm{down}_{\Delta^k(\mu \cup R_d)}(a_1)$ exists. Note that here, we make use of the fact that $\mathrm{down}_{\Delta^k(\mu \cup R_d)}(a_2) = \mathrm{down}_{\Delta^k(\mu \cup R_d - \epsilon_{a_1})}(a_2)$. Moreover, by applying Remark 3.3.15 followed by Remark 3.3.14, we see that any $a_i, a_j \in D$ have disjoint downpaths in $\Delta^k(\mu \cup R_d)$; therefore, no Katalan function in the expansion above is indexed by weight decremented by 2 in any coordinate. As before, if $\mathrm{down}_{\Delta^k(\mu \cup R_d)}(a_2)$ does not exist, we must have that $\Delta^k(\mu \cup R_d)$ contains no root of the form $(a_2, j)$ for any $j$, so $\Delta^k(\mu \cup R_d - \epsilon_{a_1}) = \Delta^k(\mu \cup R_d - \epsilon_{\{a_1,a_2\}})$. Therefore, in this case,

$$K(\Delta^k(\mu \cup R_d - \epsilon_{a_1}); \Delta^k(\mu \cup R_d - \epsilon_{a_1}); \mu \cup R_d - \epsilon_D) = K(\Delta^k(\mu \cup R_d - \epsilon_{\{a_1,a_2\}}); \Delta^k(\mu \cup R_d - \epsilon_{\{a_1,a_2\}}); \mu \cup R_d - \epsilon_D).$$

We continue with $a_3, a_4$, and so on until we obtain the top-level term $K(\Delta^k(\mu \cup R_d - \epsilon_D); \Delta^k(\mu \cup R_d - \epsilon_D); \mu \cup R_d - \epsilon_D)$. In particular, we obtain the expansion:

$$K(\Delta^k(\mu \cup R_d); \Delta^k(\mu \cup R_d); \mu \cup R_d - \epsilon_D)$$

$$= K(\Delta^k(\mu \cup R_d - \epsilon_D); \Delta^k(\mu \cup R_d - \epsilon_D); \mu \cup R_d - \epsilon_D)$$

$$- \sum_{i=1}^{B} K(\Delta^k(\mu \cup R_d - \epsilon_{\{a_1,\dots,a_i\}}); \Delta^k(\mu \cup R_d - \epsilon_{\{a_1,\dots,a_i\}}); \mu \cup R_d - \epsilon_{D \cup \mathrm{down}_{\Delta^k(\mu \cup R_d)}(a_i)})$$

$$+ \sum_{i=1}^{B} K(\Delta^k(\mu \cup R_d - \epsilon_{\{a_1,\dots,a_{i-1}\}}); \Delta^k(\mu \cup R_d - \epsilon_{\{a_1,\dots,a_{i-1}\}}); \mu \cup R_d - \epsilon_{D \setminus a_i \cup \mathrm{down}_{\Delta^k(\mu \cup R_d)}(a_i)}),$$

where $B$ is defined as the maximal index of the $a_i$ such that $\mathrm{down}_{\Delta^k(\mu \cup R_d)}(a_i)$ exists. Then for $i \in [B]$, we rewrite the terms in the resulting summations by expanding on $a_{i+1}, a_{i+2}$, and so on, noting that the downpath of $\mathrm{down}_{\Delta^k(\mu \cup R_d)}(a_i)$ is disjoint from the downpaths of each of $a_{i+1}, a_{i+2}, \dots, a_m$. In particular, replacing $D$ with $\{a_{i+1}, a_{i+2}, \dots, a_m\} \cup \mathrm{down}_{\Delta^k(\mu \cup R_d)}(a_i)$ and $\mu \cup R_d$ with $\mu \cup R_d - \epsilon_{\{a_1,\dots,a_i\}}$, we may iterate the initial method of expansion for $-K(\Delta^k(\mu \cup R_d - \epsilon_{\{a_1,\dots,a_i\}}); \Delta^k(\mu \cup R_d - \epsilon_{\{a_1,\dots,a_i\}}); \mu \cup R_d - \epsilon_{D \cup \mathrm{down}_{\Delta^k(\mu \cup R_d)}(a_i)})$ if $\mathrm{down}^2_{\Delta^k(\mu \cup R_d)}(a_i)$ exists. Here, we make use of the fact that $\mathrm{down}_{\Delta^k(\mu \cup R_d)}(a_i) > a_m$ (Remark 3.3.14) to be sure that we do not decrement by two.

Similarly, replacing $D$ with $\{a_{i+1}, a_{i+2}, \dots, a_m\} \cup \mathrm{down}_{\Delta^k(\mu \cup R_d}(a_i)$ and $\mu \cup R_d$ with $\mu \cup R_d - \epsilon_{\{a_1,\dots,a_{i-1}\}}$, we can use the expansion method for $K(\Delta^k(\mu \cup R_d - \epsilon_{\{a_1,\dots,a_{i-1}\}}); \Delta^k(\mu \cup R_d - \epsilon_{\{a_1,\dots,a_{i-1}\}}); \mu \cup R_d - \epsilon_{D \setminus a_i \cup \mathrm{down}_{\Delta^k(\mu \cup R_d)}(a_i)})$. As before, we use Remark 3.3.14 to be sure that we do not decrement by two.

We obtain

$$K(\Delta^k(\mu \cup R_d); \Delta^k(\mu \cup R_d); \mu \cup R_d - \epsilon_D)$$

$$= \sum (-1)^{|J|+1} K(\Delta^k(\mu \cup R_d - \epsilon_J); \Delta^k(\mu \cup R_d - \epsilon_J); \mu \cup R_d - \epsilon_J)$$

where the sum ranges over all nonempty sets $J$ such that $j \in J$ implies $j \in \mathrm{downpath}_{\Delta^k(\mu \cup R_d)}(a_i)$ for some $a_i \in D$ and $j \geq r + d$.

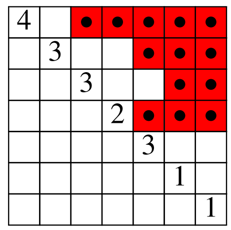
□

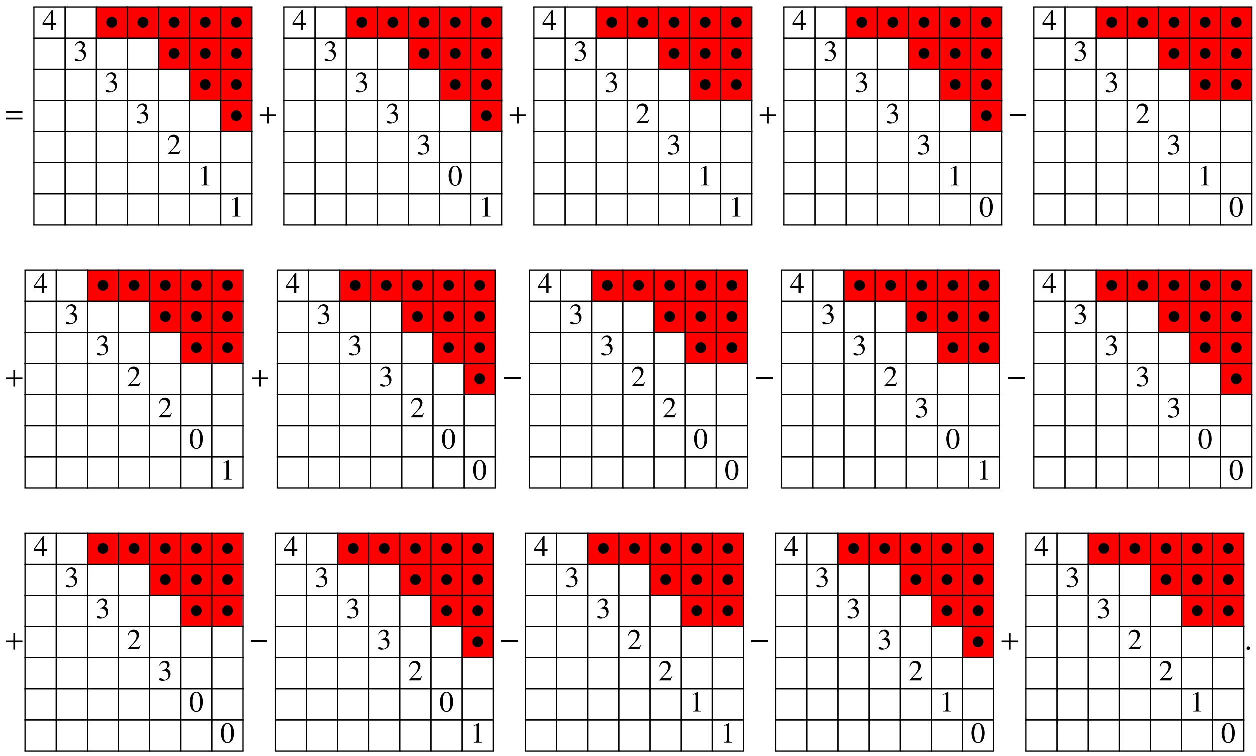
Remark 3.3.17. The summand of Proposition 3.3.16 ranges over all nonempty sets consisting of $x$ such that $x \geq r + d$ and $x \in \mathrm{downpath}_{\Delta^k(\mu \cup R_d)}(z)$ for some $z \in [r + 1, r + d]$ (note this implies

$(\mu \cup R_d)_z = k+1-d)$. This is because for $i \in [d]$, we have $\mathrm{down}_{\Delta^k(\mu\cup R_d)}(r+i) = k-(k+1-d)+r+i = r+i+d$.

EXAMPLE 3.3.18. Theorem 3.3.16 applied to $k = 5$, $\mu = (4,3,1,1)$, $R_3 = (3,3,3)$, and $\lambda = (4,3,3,3,3,1,1)$ implies

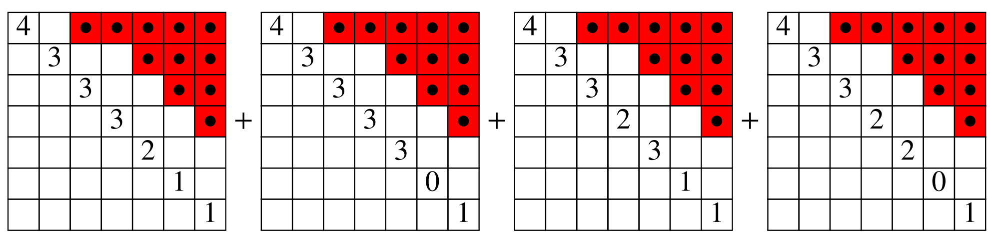
$$\tilde{\mathfrak{g}}^{(5)}_{R_3^*}\tilde{\mathfrak{g}}^{(5)}_{(4,3,1,1)} =$$

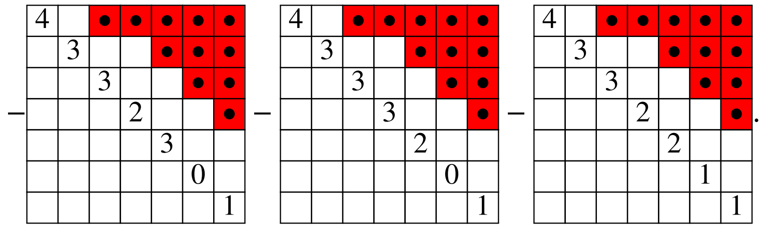




Note that on the other hand, Lemma 3.3.8 provides a coarser sum of Katalan functions:

$$\tilde{\mathfrak{g}}^{(5)}_{R_3^*}\tilde{\mathfrak{g}}^{(5)}_{(4,3,1,1)} =$$





CHAPTER 4

# Covers and equivalent functions

In their generic definition, Katalan functions are not unique. Because we know from Proposition 2.3.15 that closed $k$-Schur Katalan functions $\{\tilde{\mathfrak{g}}_\lambda^{(k)}\}_{\lambda \in \mathrm{Par}^k}$ form a basis for $\Lambda_{(k)}$, there must exist a cancellation-free sum of closed $k$-Schur Katalan functions equivalent to 3.3.7 weighted by partitions. There are two central questions that remain despite the result of Theorem 3.3.16:

- Which functions in 3.3.7 are equivalent to one another? In other words, which cancellations can we realize within this sum?
- How can we index the functions in 3.3.7 by partitions, as opposed to simply by weights?

Example 4.0.1. The product $\tilde{\mathfrak{g}}^{(5)}_{R_3^*}\tilde{\mathfrak{g}}^{(5)}_{(4,3,1,1)}$ is written in Example 3.3.18 as a sum closed $k$-Schur Katalan functions; the terms the terms appearing in the sum include closed $k$-Schur Katalan functions indexed by (but not limited to) the weights $\{4,3,3,3,3,0,1\}$, $\{4,3,3,2,2,0,1\}$, $\{4,3,3,3,2,0,0\}$, $\{4,3,3,2,2,0,0\}$, $\{4,3,3,2,3,0,1\}$, $\{4,3,3,3,3,0,0\}$, $\{4,3,3,2,3,0,0\}$, and $\{4,3,3,3,2,0,1\}$.

However, we have that

$$\tilde{\mathfrak{g}}^{(5)}_{\{4,3,3,3,3,0,1\}} + \tilde{\mathfrak{g}}^{(5)}_{\{4,3,3,3,2,0,0\}} - \tilde{\mathfrak{g}}^{(5)}_{\{4,3,3,2,3,0,1\}} - \tilde{\mathfrak{g}}^{(5)}_{\{4,3,3,3,3,0,0\}} + \tilde{\mathfrak{g}}^{(5)}_{\{4,3,3,2,3,0,0\}} - \tilde{\mathfrak{g}}^{(5)}_{\{4,3,3,3,2,0,1\}} = 0,$$

where the coefficients (signs) for each function match the result of Example 3.3.18. Moreover, the remaining functions produced in Example 3.3.18 can all be rewritten with partition weights:

$$\tilde{\mathfrak{g}}^{(5)}_{\{4,3,3,3,2,1,1\}} + \tilde{\mathfrak{g}}^{(5)}_{\{4,3,3,2,2,0,1\}} + \tilde{\mathfrak{g}}^{(5)}_{\{4,3,3,2,3,1,1\}} + \tilde{\mathfrak{g}}^{(5)}_{\{4,3,3,3,3,1,0\}}$$

$$-\tilde{\mathfrak{g}}^{(5)}_{\{4,3,3,2,3,1,0\}} - \tilde{\mathfrak{g}}^{(5)}_{\{4,3,3,2,2,1,1\}} - \tilde{\mathfrak{g}}^{(5)}_{\{4,3,3,3,2,1,0\}} + \tilde{\mathfrak{g}}^{(5)}_{\{4,3,3,2,2,1,0\}} - \tilde{\mathfrak{g}}^{(5)}_{\{4,3,3,2,2,0,0\}}$$

$$= \tilde{\mathfrak{g}}^{(5)}_{\{4,3,3,3,2,1,1\}} + \tilde{\mathfrak{g}}^{(5)}_{\{4,3,3,3,2\}} + \tilde{\mathfrak{g}}^{(5)}_{\{4,4,3,2,2,1,1\}} + \tilde{\mathfrak{g}}^{(5)}_{\{4,3,3,3,3,1\}}$$

$$-\tilde{\mathfrak{g}}^{(5)}_{\{4,4,3,2,2,1\}} - \tilde{\mathfrak{g}}^{(5)}_{\{4,3,3,2,2,1,1\}} - \tilde{\mathfrak{g}}^{(5)}_{\{4,3,3,3,2,1\}} + \tilde{\mathfrak{g}}^{(5)}_{\{4,3,3,2,2,1\}} - \tilde{\mathfrak{g}}^{(5)}_{\{4,3,3,2,2\}}$$

$$= \tilde{\mathfrak{g}}^{(5)}_{R_3^*}\tilde{\mathfrak{g}}^{(5)}_{(4,3,1,1)}.$$

## 4.1. Covers and $k$-Schur straightening

Our primary goal is to rewrite each closed $k$-Schur Katalan function in the sum produced by Theorem 3.3.16 either with a partition weight or as 0. Rules of this form that apply for any $\gamma \in \mathbb{Z}^\ell$ are called straightening laws.

Proposition 4.1.1. *(Schur function straightening).* [**BMPS19**, Proposition 4.1] *For any $\gamma \in \mathbb{Z}^\ell$,*

$$s_\gamma(\boldsymbol{x}) = \begin{cases} sgn(\gamma+\rho)s_{sort(\gamma+\rho)-\rho}(\boldsymbol{x}) & \text{if } \gamma+\rho \text{ has distinct nonnegative parts,} \\ 0 & \text{otherwise,} \end{cases} \tag{4.1.1}$$

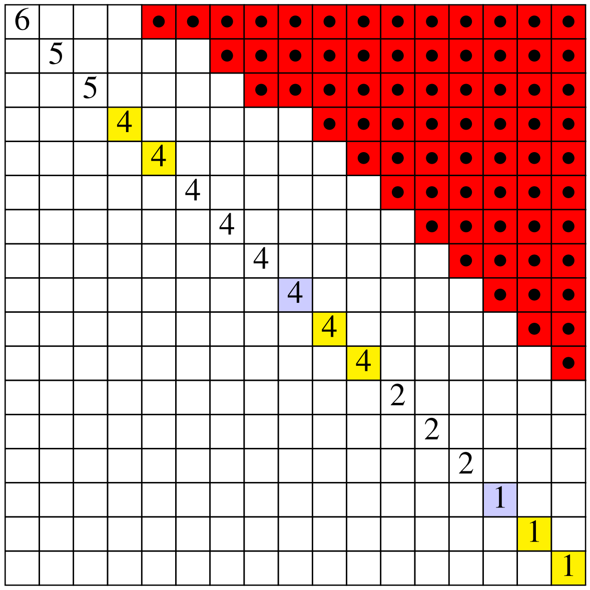
*where $\rho = (\ell-1, \ell-2, \ldots, 0)$, sort($\beta$) denotes the weakly decreasing sequence obtained by sorting $\beta$, and sgn($\beta$) denotes sign of the shortest permutation taking $\beta$ to sort($\beta$).*

In a similar vein, any $k$-Schur Catalan can be written as a single $k$-Schur Catalan function with partition weight or zero [**BMPS19**]. This $k$-Schur Catalan straightening rule from [**BMPS19**] relies on *covers*, which are combinatorial tools that we will define and generalize in this section to apply to closed $k$-Schur Katalan functions.

Definition 4.1.2. Let $\lambda \in \widetilde{Par}^k_\ell$ and $z \in [\ell]$. Set $\mu = \lambda - \epsilon_z$ and $\Psi = \Delta^k(\mu)$. Let $c = |uppath_\Psi(z)|$. If $z = \ell$ or $\lambda_z > \lambda_{z+1}$ or $up^c_\Psi(z+1)$ is undefined, then set $h = 0$; otherwise, set $y + 1 = up^c_\Psi(z+1)$ and let $h \in [\ell - z]$ be as large as possible such that $\mu$ is constant on each of the intervals $[z+1, z+h]$, $[up_\Psi(z), up_\Psi(z)+h]$, $[up^2_\Psi(z), up^2_\Psi(z)+h], \ldots, [top_\Psi(z), top_\Psi(z)+h]$, and $[y+1, y+h]$.

Define $cover_z(\lambda) = \lambda + \epsilon_{[y+1,y+h]} - \epsilon_{[z,z+h]}$. If $y$ is undefined or, equivalently, $h = 0$, then $cover_z(\lambda) = \mu$.

Example 4.1.3. If $\lambda = (6,5,5,4,4,4,4,4,4,4,4,2,2,2,1,1,1)$ and $k = 9$, we have $\text{cover}_{15}(\lambda) = (6,5,5,5,5,4,4,4,4,4,4,2,2,2)$. In this case, $h = 2$ and $y + 1 = 4$. This computation is evident when viewing the $k$-Schur root ideal with the relevant uppaths colored:




$$\text{uppath}(15)_{\Delta^k(\lambda)} = \{15, 9\}$$

$$\text{uppath}(16)_{\Delta^k(\lambda)} = \{16, 10, 4\}$$

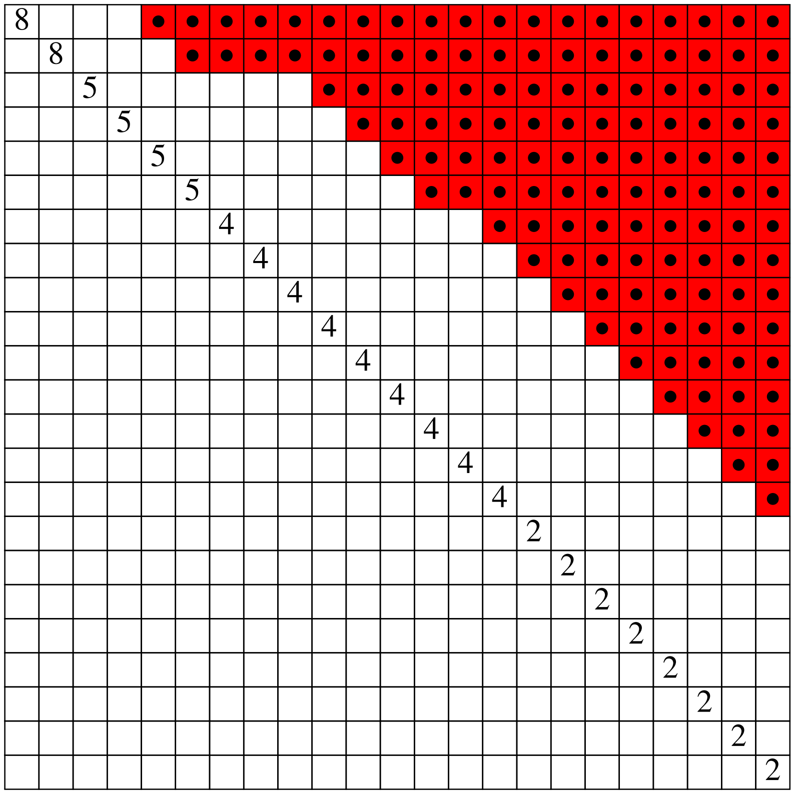
$$\text{uppath}(17)_{\Delta^k(\lambda)} = \{17, 11, 5, 1\}$$

While the uppath of 17 is longer than that of both 15 and 16, we only consider up to and including $\text{up}^{2+1}(17)$, as $|\text{uppath}(15)_{\Delta^k(\lambda)}| = 2$.

Lemma 4.1.4. [**BMPS19**] *For $\lambda \in Par$, the intervals $[z+1, z+h], \dots, [top_\Psi(z), top_\Psi(z) + h]$, and $[y+1, y+h]$ in Definition 4.1.2 are pairwise disjoint.*

Remark 4.1.5. Suppose $\lambda \in \text{Par}_\ell^k$ and $z \in [\ell]$. Let $D = \{a_1, \dots, a_m\} \subseteq [\ell]$ be such that $a_i \notin \text{uppath}_{\Delta^k(\lambda)}(z + i)$ for all $i \in [h]$, where $h$ is as in the definition of cover for $\text{cover}_z(\lambda)$. Then $\lambda - \epsilon_D \in \widetilde{Par}_\ell^k$ and $\text{cover}_z(\lambda) - \epsilon_D = \text{cover}_z(\lambda - \epsilon_D)$. In particular, the calculation of $y$ and $h$ for both $\text{cover}_z(\lambda)$ and $\text{cover}_z(\lambda - \epsilon_D)$ will be the same, as $\Psi$ will be unchanged on the intervals listed in the definition of cover.

Example 4.1.6. While some covers are partitions, they need not be. Let $k = 11$, and consider the $k$-Schur root ideal corresponding to $\lambda = (8, 8, 5, 5, 5, 5, 4, 2, 2, 2, 2, 2, 2, 2, 2) \cup R_4$,



We have then that $\text{cover}_{22}(\lambda) = \lambda + \epsilon_7 - \epsilon_{[22,23]} \in \text{Par}$ and $\text{cover}_{17}(\lambda) = \lambda + \epsilon_{[3,6]} - \epsilon_{[17,21]} \notin \text{Par}$, as $|\text{uppath}_{\Delta^k(\lambda)}(22)| = 2$, while $|\text{uppath}_{\Delta^k(\lambda)}(x)| \geq 3$ for all $x \in [18, 21]$.

Example 4.1.7. Cover computations need not commute. For example, with $k = 16$, we have that

$$\begin{aligned}
&\text{cover}_{13}(\text{cover}_6(11, 11, 11, 11, 11, 10, 10, 10, 10, 8, 8, 6, 5, 5, 5, 5)) \\
&= (12, 12, 12, 11, 11, 10, 10, 10, 9, 8, 8, 6, 4, 4, 4, 4), \\
&\neq \text{cover}_6(\text{cover}_{13}(11, 11, 11, 11, 11, 10, 10, 10, 10, 8, 8, 6, 5, 5, 5, 5)) \\
&= (12, 12, 12, 11, 11, 9, 10, 10, 10, 8, 8, 6, 4, 4, 4, 4).
\end{aligned}$$

THEOREM 4.1.8. [**BMPS19**]*(k-Schur function straightening) Maintaining the notation of Definition 4.1.2 for cover, h, and c, we have*

$$s_\mu^{(k)} = t^{h*c} s^{(k)}_{cover_z(\lambda)}$$

*and this is equal to 0 if* $cover_z(\lambda) \notin Par^k_\ell$.

## 4.2. Covers and Katalan functions

We now leverage the properties of root ideals and covers to straighten certain families of Katalan functions. While these results are in a more limited scope than the Schur and $k$-Schur straightening laws, their implications for the closed $k$-Schur Katalan functions assist us in answering both questions raised at the beginning of this chapter.

LEMMA 4.2.1. *For* $\lambda \in \widetilde{Par}^k_\ell$ *and* $cover_z(\lambda) = \lambda + \epsilon_{[y+1,y+h]} - \epsilon_{[z,z+h]}$ *(or* $cover_z(\lambda) = \lambda - \epsilon_z$ *with* $h = 0$*), let* $S = \{s_1, \ldots, s_m\} \subseteq [\ell]$ *be such that for all* $i \in [m]$*, either* $s_i > z + h$ *or* $s_i < z$ *but* $s_i \notin \{up_{\Delta^k(\lambda)}(z+j), up^2_{\Delta^k(\lambda)}(z+j), \ldots, up^c_{\Delta^k(\lambda)}(z+j)\}$ *for all* $j \in [h]$*, where* $c = |uppath_{\Delta^k(\lambda)}(z+1)|$*. Let* $\Psi_1, \Psi_2 \subseteq \Delta^+$ *be such that for all* $(a,b) \in \Psi_1$ *or* $(a,b) \in \Psi_2$*, we have* $a > z+h$ *and* $b > k-\lambda_z+z+h+1$*.*

*Then*

$$K(\Delta^k(\lambda - \epsilon_z - \epsilon_S) \cup \Psi_1; \Delta^k(\lambda - \epsilon_z - \epsilon_S) \cup \Psi_2; \lambda - \epsilon_z - \epsilon_S)$$

$$= K(\Delta^k(cover_z(\lambda) - \epsilon_S) \cup \Psi_1; \Delta^k(cover_z(\lambda) - \epsilon_S) \cup \Psi_2; cover_z(\lambda) - \epsilon_S).$$

PROOF. If $h = 0$, then $\lambda - \epsilon_z - \epsilon_S = \mathrm{cover}_z(\lambda) - \epsilon_S$ immediately. Otherwise, there is a removable root $(\mathrm{up}_{\Delta^k(\lambda)}(z+1), z+1)$ in $\Delta^k(\lambda - \epsilon_z - \epsilon_S)$. We apply Corollary 3.1.11 to this removable root:

$$K(\Delta^k(\lambda - \epsilon_z - \epsilon_S) \cup \Psi_1; \Delta^k(\lambda - \epsilon_z - \epsilon_S) \cup \Psi_2; \lambda - \epsilon_z - \epsilon_S)$$

$$= K(\Delta^k(\lambda - \epsilon_z - \epsilon_S) \cup \Psi_1; \Delta^k(\lambda - \epsilon_z - \epsilon_S) \cup \Psi_2; \lambda - \epsilon_{\{z,z+1\}} + \epsilon_{\mathrm{up}_{\Delta^k(\lambda)}(z+1)} - \epsilon_S).$$

Noting then that by definition of cover, $\Delta^k(\mathrm{cover}_z(\lambda))$ has a mirror in columns $\{z, z+1\}$ (in fact, more generally, in rows $\{\mathrm{up}_{\Delta^k(\lambda)}(z+1), \mathrm{up}_{\Delta^k(\lambda)}(z+2), \ldots, \mathrm{up}_{\Delta^k(\lambda)}(z+h)\}$ as well), we may then employ Lemma 3.1.13 followed by Lemma 3.2.16:

$$K(\Delta^k(\lambda - \epsilon_z - \epsilon_S) \cup \Psi_1; \Delta^k(\lambda - \epsilon_z - \epsilon_S) \cup \Psi_2; \lambda - \epsilon_{\{z,z+1\}} + \epsilon_{\mathrm{up}_{\Delta^k(\lambda)}(z+1)} - \epsilon_S)$$

$$= K(\Delta^k(\lambda - \epsilon_z + \epsilon_{\mathrm{up}_{\Delta^k(\lambda)}(z+1)} - \epsilon_S) \cup \Psi_1; \Delta^k(\lambda - \epsilon_z + \epsilon_{\mathrm{up}_{\Delta^k(\lambda)}(z+1)} - \epsilon_S) \cup \Psi_2; \lambda - \epsilon_{\{z,z+1\}} + \epsilon_{\mathrm{up}_{\Delta^k(\lambda)}(z+1)} - \epsilon_S)$$

$$= K(\Delta^k(\lambda - \epsilon_{\{z,z+1\}} + \epsilon_{\mathrm{up}_{\Delta^k(\lambda)}(z+1)} - \epsilon_S) \cup \Psi_1; \Delta^k(\lambda - \epsilon_{\{z,z+1\}} + \epsilon_{\mathrm{up}_{\Delta^k(\lambda)}(z+1)} - \epsilon_S) \cup \Psi_2; \lambda - \epsilon_{\{z,z+1\}} + \epsilon_{\mathrm{up}_{\Delta^k(\lambda)}(z+1)} - \epsilon_S).$$

By making use of the observation about mirrors in additional rows, and the fact that $\Delta^k(\mathrm{cover}_z(\lambda))$ also has a mirror in rows $\{z, z+1, \ldots, z+h\}$, we can iterate this process with $z+2$ and $\mathrm{up}_{\Delta^k(\lambda)}(z+2)$, then $z+3$ and $\mathrm{up}_{\Delta^k(\lambda)}(z+3)$, and so on until $z+h$ and $\mathrm{up}_{\Delta^k(\lambda)}(z+h)$:

$$K(\Delta^k(\lambda - \epsilon_z) \cup \Psi_1; \Delta^k(\lambda - \epsilon_z) \cup \Psi_2; \lambda - \epsilon_z - \epsilon_S)$$

$$= K(\Delta^k(\lambda - \epsilon_{[z,z+h]} + \epsilon_{[\mathrm{up}_{\Delta^k(\lambda)}(z+1), \mathrm{up}_{\Delta^k(\lambda)}(z+h)]}) \cup \Psi_1; \Delta^k(\lambda - \epsilon_{[z,z+h]} + \epsilon_{[\mathrm{up}_{\Delta^k(\lambda)}(z+1), \mathrm{up}_{\Delta^k(\lambda)}(z+h)]}) \cup \Psi_2;$$

$$\lambda - \epsilon_{[z,z+h]} + \epsilon_{[\mathrm{up}_{\Delta^k(\lambda)}(z+1), \mathrm{up}_{\Delta^k(\lambda)}(z+h)]} - \epsilon_S).$$

If $\mathrm{up}_{\Delta^k(\lambda)}(z+h) = y+h$, then the proof is complete. Otherwise, we iterate the procedure, starting with $\mathrm{up}_{\Delta^k(\lambda)}(z+1)$ and $\mathrm{up}^2_{\Delta^k(\lambda)}(z+1)$, and continuing until $\mathrm{up}_{\Delta^k(\lambda)}(z+h)$ and $\mathrm{up}^2_{\Delta^k(\lambda)}(z+h)$. By definition of $h$ in covers (which implies stability of $\lambda$ on not just one interval but a series of intervals), we guarantee the necessary mirrors that make the procedure possible at each iteration. We continue in this fashion until establishing the result. □

Example 4.2.2. Because $\mathrm{downpath}_{\Delta^4(3,3,3,2,2,1)}(2) = \{2,4\}$, we may apply Lemma 4.2.1 with $z = 2$ to obtain

$$K(\Delta^4((3,3,3,2,2,1) - \epsilon_{\{2,4\}}); \Delta^4((3,3,3,2,2,1) - \epsilon_{\{2,4\}}); (3,3,3,2,2,1) - \epsilon_{\{2,4\}})$$

$$= K(\Delta^4(\mathrm{cover}_2((3,3,3,2,2,1)) - \epsilon_{\{4\}}); \Delta^4(\mathrm{cover}_2((3,3,3,2,2,1)) - \epsilon_{\{4\}}); \mathrm{cover}_2((3,3,3,2,2,1)) - \epsilon_{\{4\}})$$

$$= K(\Delta^4((4,2,2,2,2,1) - \epsilon_{\{4\}}); \Delta^4((4,2,2,2,2,1) - \epsilon_{\{4\}}); (4,2,2,2,2,1) - \epsilon_{\{4\}}).$$

Note that, in this case, we could not have applied the lemma with $z = 4$.

## 4.3. Equivalent functions

We now leverage the combinatorics of root ideals and covers to show that certain closed $k$-Schur Katalan functions indexed by different weights in fact are equivalent.

LEMMA 4.3.1 (Mirror Lemma). [BMS] *Let* $\Psi \subset \Delta_\ell^+$ *be a root ideal,* $M$ *a multiset on* $[\ell]$, $\mu \in \mathbb{Z}^\ell$, *and* $1 \le y \le z < \ell$ *be indices in the same bounce path of* $\Psi$ *satisfying*

*(1)* $\Psi$ *has a ceiling in columns* $y, y+1$*;*

*(2)* $\Psi$ *has a mirror in rows* $x, x+1$ *for all* $x \in downpath_\Psi(y, up_\Psi(z))$*;*

*(3)* $\Psi$ *has a wall in rows* $z, z+1$*;*

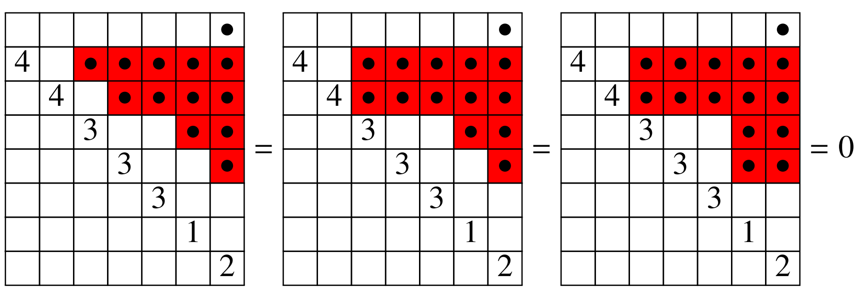
*(4)* $m_M(x+1) = m_M(x) + 1$ *for all* $x \in downpath_\Psi(down_\Psi(y), z)$*;*

*(5)* $\mu_x = \mu_{x+1}$ *for all* $x \in downpath_\Psi(y, up_\Psi(z))$*;*

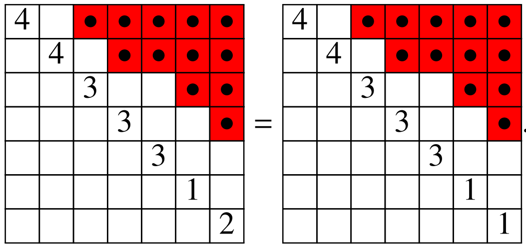
*(6)* $\mu_z = \mu_{z+1} - 1$.

*If* $m_M(y+1) = m_M(y) + 1$*, then* $K(\Psi; M; \mu) = 0$*. If* $m_M(y+1) = m_M(y)$*, then* $K(\Psi; M; \mu) = K(\Psi; M; \mu - \epsilon_{z+1})$.

EXAMPLE 4.3.2. We can apply twice Lemma 3.2.16 to the Katalan function below (first with $(y, z) = (2, 3)$, then $(y, z) = (4, 6)$); from there, we apply the Mirror Lemma 4.3.1 with $y = z = 6$ to see that the function vanishes:



On the other hand, the Mirror Lemma 4.3.1 implies that




LEMMA 4.3.3. *Suppose a subset of roots* $\Psi \subset \Delta_\ell^+$*, a multiset* $M$ *on* $[\ell]$*,* $\gamma \in \mathbb{Z}^\ell$*, and* $1 \le y < z < \ell$ *indices in the same bounce path of* $\Psi$ *satisfy*

- $\Psi$ *has a ceiling in columns* $y, y+1$*;*
- $\Psi$ *has a mirror in rows* $x, x+1$ *for all* $x \in$ *downpath*$_\Psi(y, \mathrm{up}_\Psi(z))$*;*
- $\Psi$ *has a wall in rows* $z, z+1$*;*
- $z' = \mathrm{down}_\Psi(z+1)$ *exists and* $z' \in M$*;*
- $m_M(y+1) = m_M(y)$ *and* $m_M(x+1) = m_M(x) + 1$ *for all* $x \in \mathrm{downpath}_\Psi(\mathrm{down}_\Psi(y), z)$*;*
- $\gamma_x = \gamma_{x+1}$ *for all* $x \in \mathrm{downpath}_\Psi(y, \mathrm{up}_\Psi(z))$*;*
- $\gamma_z = \gamma_{z+1}$.

*Then* $K(\Psi; M; \gamma) = K(\Psi \setminus (z+1, z'); M \setminus \{z'\}; \gamma)$.

PROOF. Let $P = |\mathrm{downpath}_\Psi(y, \mathrm{up}_\Psi(z))|$. First we expand on $z' \in M$ to get

$$K(\Psi; M; \gamma) = K(\Psi; M \setminus \{z'\}; \gamma) - K(\Psi; M \setminus \{z'\}; \gamma - \epsilon_{z'}). \tag{4.3.1}$$

We expand on $(z+1, z') \in \Psi$ in the first term of the resulting difference to get

$$K(\Psi; M \setminus \{z'\}; \gamma) = K(\Psi \setminus (z+1, z'); M \setminus \{z'\}; \gamma) + K(\Psi; M \setminus \{z'\}; \gamma + \epsilon_{z+1} - \epsilon_{z'}).$$

However, by expanding on $(\mathrm{up}_\Psi(z+1), z+1)$ and subsequently applying Lemma 3.1.9, we have that $K(\Psi; M \setminus \{z'\}; \gamma + \epsilon_{z+1} - \epsilon_{z'}) = K(\Psi; M \setminus \{z'\}; \gamma + \epsilon_{\mathrm{up}_\Psi(z+1)} - \epsilon_{z'})$. Note that by Lemma 3.1.13, $K(\Psi; M \setminus \{z'\}; \gamma + \epsilon_{\mathrm{up}_\Psi(z+1)} - \epsilon_{z'}) = K(\Psi \cup (\mathrm{up}_\Psi(z+1), z+1); M \setminus \{z'\} \cup \{z+1\}; \gamma + \epsilon_{\mathrm{up}_\Psi(z+1)} - \epsilon_{z'})$. In fact, we may repeat a variation of this process: expand on $(\mathrm{up}_\Psi^2(z+1), \mathrm{up}_\Psi(z+1))$, then apply Lemma 3.1.9 to one of the resulting terms, then apply Lemma 3.1.13 with $j = \mathrm{up}_\Psi(z+1)$. If we further apply Lemma 3.2.16 with $y = \mathrm{up}_\Psi(z+1)$, we obtain

$$K(\Psi; M \setminus \{z'\}; \gamma + \epsilon_{z+1} - \epsilon_{z'}) = K(\Psi \cup (\mathrm{up}_\Psi^2(z+1), \mathrm{up}_\Psi(z+1)); M \setminus \{z'\} \cup \{\mathrm{up}_\Psi(z+1)\}; \gamma + \epsilon_{\mathrm{up}_\Psi^2(z+1)} - \epsilon_{z'}).$$

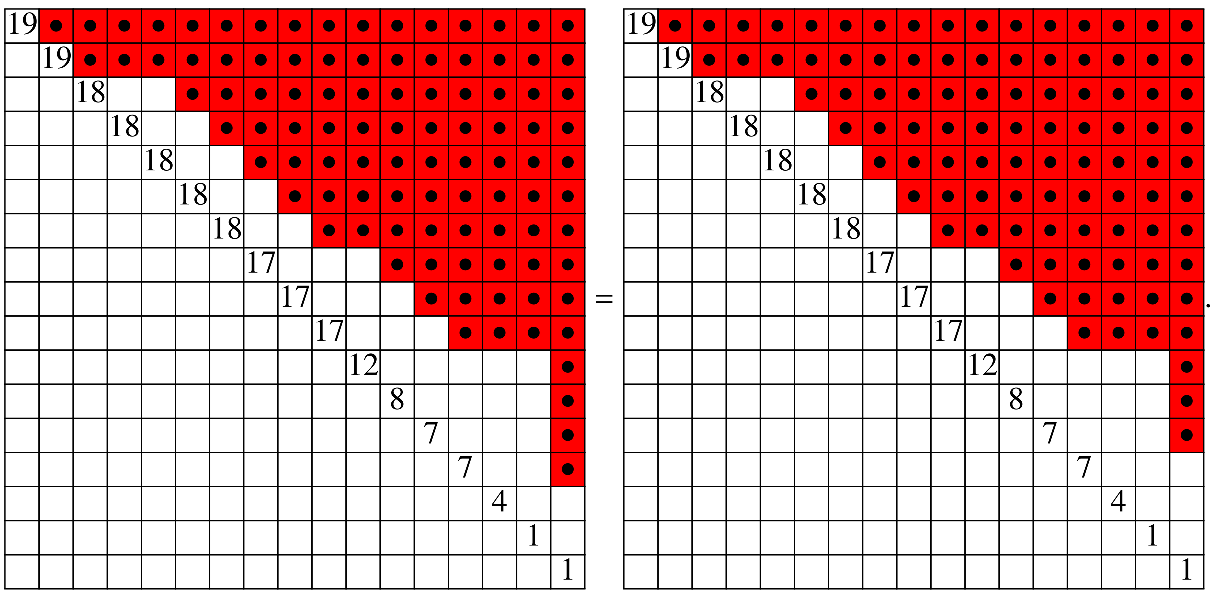
In total we may apply these steps $P$ times, expanding on $(\mathrm{up}_\Psi^i(z+1), \mathrm{up}_\Psi^{i-1}(z+1))$, then applying Lemma 3.1.9, then applying Lemma 3.1.13 with $j = \mathrm{up}_\Psi^{i-1}(z+1))$, and finally Lemma 3.2.16 with $y = \mathrm{up}_\Psi^{i-1}(z+1)$, $i = [1, P]$ to show that

$$K(\Psi; M \setminus \{z'\}; \gamma + \epsilon_{z+1} - \epsilon_{z'}) = K(\Psi; M \setminus \{z'\}; \gamma + \epsilon_{y+1} - \epsilon_{z'}).$$

But $K(\Psi; M \setminus \{z'\}; \gamma + \epsilon_{y+1} - \epsilon_{z'}) = K(\Psi; M \setminus \{z'\}; \gamma - \epsilon_{z'})$ due to Lemma 3.1.9. Substituting this into 4.3.1 establishes the result.

□

Example 4.3.4. Lemma 4.3.3 with $y = 3$ and $z = 13$ gives




Lemma 4.3.5. *For $\lambda \in \widetilde{Par}_\ell^k$ and $b \in [\ell - 1]$ such that $\lambda_{b+h} = \lambda_{b+h+1}$,*

$$\tilde{\mathfrak{g}}^{(k)}_{cover_b(\lambda)-\epsilon_{b+h+1}} = \tilde{\mathfrak{g}}^{(k)}_{\lambda-\epsilon_b},$$

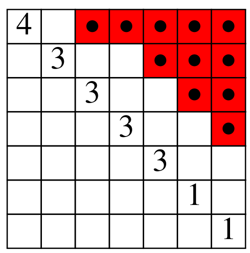
*where $h$ is computed as in Definition 4.1.2.*

Proof. For $\Psi = \Delta^k(\lambda - \epsilon_b)$, let $a^j = \text{up}^j_\Psi(b)$ for all $j \le c = |\text{uppath}_\Psi(b)|$. Let $d$ be maximal such that $\lambda_{a^j+h} = \lambda_{a^j+h+1}$ for all $j \le d$. Note that $0 \le d \le c$ by definition of $h$ in Definition 4.1.2 and the fact that $\lambda_{b+h} = \lambda_{b+h+1}$. We apply Lemma 4.2.1 and satisfy the conditions of Lemma 4.3.1 with $z = b + h$, which implies that

$$\tilde{\mathfrak{g}}^{(k)}_{\lambda-\epsilon_b} = \tilde{\mathfrak{g}}^{(k)}_{\text{cover}_b(\lambda)} = K(\Delta^k(\text{cover}_b(\lambda)); \Delta^k(\text{cover}_b(\lambda)); \text{cover}_b(\lambda) - \epsilon_{b+h+1}).$$

We have the conditions of Lemma 4.3.3 with $\gamma = \text{cover}_b(\lambda) - \epsilon_{b+h+1}$ and $z = b + h$, implying that the righthand side is $\tilde{\mathfrak{g}}^{(k)}_{\text{cover}_b(\lambda)-\epsilon_{b+h+1}}$. □

Example 4.3.6. Recall the $k$-Schur root ideal root ideal corresponding to $\lambda = (4,3,3,3,3,1,1) = (4,3,1,1) \cup R_3$, $k = 5$, introduced in Example 3.3.18,




With this root ideal and the descent sets produced in Example 3.3.18 in mind, we note that

$$\tilde{\mathfrak{g}}^{(k)}_{\{4,3,3,3,3,0,1\}} = \tilde{\mathfrak{g}}^{(k)}_{\text{cover}_6(\lambda)}$$

as the corresponding $h = 0$ for this cover. Therefore, due to Lemma 4.3.5, we have

$$\tilde{\mathfrak{g}}^{(k)}_{\{4,3,3,3,3,0,1\}} = \tilde{\mathfrak{g}}^{(k)}_{\{4,3,3,3,3,0,0\}},$$

which was another term produced by Example 3.3.18. Moreover,

$$\tilde{\mathfrak{g}}^{(k)}_{\{4,3,3,3,2,0,1\}} = \tilde{\mathfrak{g}}^{(k)}_{\text{cover}_5(\lambda-\epsilon_6)} = \tilde{\mathfrak{g}}^{(k)}_{\text{cover}_6(\text{cover}_5(\lambda))}$$

wherein both covers have $h = 0$. Therefore, due to Lemma 4.3.5, we have

$$\tilde{\mathfrak{g}}^{(k)}_{\{4,3,3,3,2,0,1\}} = \tilde{\mathfrak{g}}^{(k)}_{\{4,3,3,3,2,0,0\}},$$

which was another term produced by Example 3.3.18.

Finally, we have

$$\tilde{\mathfrak{g}}^{(k)}_{\{4,3,3,2,3,0,1\}} = \tilde{\mathfrak{g}}^{(k)}_{\mathrm{cover}_4(\lambda-\epsilon_6)} = \tilde{\mathfrak{g}}^{(k)}_{\{4,4,3,2,2,0,1\}} = \tilde{\mathfrak{g}}^{(k)}_{\mathrm{cover}_6(\mathrm{cover}_4(\lambda))},$$

where the first equality is due to Lemma 4.2.1, and the final equality holds as $h = 0$ for the second cover. Noting further that Lemma 4.2.1 also implies $\tilde{\mathfrak{g}}^{(k)}_{\{4,3,3,2,3,0,0\}} = \tilde{\mathfrak{g}}^{(k)}_{\mathrm{cover}_4(\lambda)}$, we have that due to Lemma 4.3.5,

$$\tilde{\mathfrak{g}}^{(k)}_{\{4,3,3,2,3,0,1\}} = \tilde{\mathfrak{g}}^{(k)}_{\{4,3,3,2,3,0,0\}},$$

justifying the observation made in Example 4.0.1 that a certain set of functions indexed by weights from Example 3.3.18 combines to vanish.

Lemma 4.3.7. *For $\lambda \in Par^k_\ell$ and $b \in [\ell-1]$ such that $cover_b(\lambda) \notin Par$, let $\Psi = \Delta^k(\lambda - \epsilon_b)$. For all $j \leq c = |uppath_\Psi(b)|$, denote $a^j = up^j_\Psi(b)$. Suppose there is some $j < c$ such that $\lambda_{a^j+h} > \lambda_{a^j+h+1}$ where $h$ corresponds to $cover_b(\lambda)$; let $a = a^J$ where this holds and $J$ is minimal. Then*

$$\tilde{\mathfrak{g}}^{(k)}_{\lambda-\epsilon_b} = \tilde{\mathfrak{g}}^{(k)}_{\lambda-\epsilon_{\{a,b\}}}.$$

Proof. First, we claim that $\tilde{\mathfrak{g}}^{(k)}_{\lambda-\epsilon_{\{a,b\}}} = \tilde{\mathfrak{g}}^{(k)}_{\mathrm{cover}_b(\lambda)-\epsilon_{b+h+1}}$. An application of Lemma 4.2.1 with $z = a$ implies $\tilde{\mathfrak{g}}^{(k)}_{\lambda-\epsilon_{\{a,b\}}} = \tilde{\mathfrak{g}}^{(k)}_{\mathrm{cover}_a(\lambda)-\epsilon_b}$. For each $j \in [1, h+1]$, $\mathrm{up}_{\Delta^k(\mathrm{cover}_a(\lambda))}(b+j)$ exists. Therefore, applying Lemma 4.2.1 to $z = b$ at this juncture yields $\tilde{\mathfrak{g}}^{(k)}_{\mathrm{cover}_a(\lambda)-\epsilon_b} = \tilde{\mathfrak{g}}^{(k)}_{\mathrm{cover}_b(\lambda)-\epsilon_{b+h+1}}$.

We next claim that $\tilde{\mathfrak{g}}^{(k)}_{\mathrm{cover}_b(\lambda)-\epsilon_{b+h+1}} - \tilde{\mathfrak{g}}^{(k)}_{\lambda-\epsilon_b} = 0$. We apply Lemma 4.2.1 to $\tilde{\mathfrak{g}}^{(k)}_{\lambda-\epsilon_b}$:

$$\tilde{\mathfrak{g}}^{(k)}_{\lambda-\epsilon_{\{a,b\}}} - \tilde{\mathfrak{g}}^{(k)}_{\lambda-\epsilon_b} = \tilde{\mathfrak{g}}^{(k)}_{\mathrm{cover}_b(\lambda)-\epsilon_{b+h+1}} - \tilde{\mathfrak{g}}^{(k)}_{\mathrm{cover}_b(\lambda)}.$$

We note that if $\Psi$ has no roots in the row $b+h+1$, then at this juncture, we can apply Proposition 3.1.6 to rewrite this difference as

$$K(\Delta^k(\mathrm{cover}_b(\lambda)); \Delta^k(\mathrm{cover}_b(\lambda)) \sqcup b+h+1; \mathrm{cover}_b(\lambda)),$$

to which we apply Lemma 3.1.13, then Lemma 3.1.9 to complete the claim.

Suppose on the other hand that $\Psi$ did have roots in the row $b+h+1$. There must be a ceiling in columns $\text{down}_\Psi(a+h), \text{down}_\Psi(a+h)+1$. Moreover, because $J$ was minimal, $\lambda$ is constant on the interval $[a^{J-1}, a^{J-1}+h+1]$. With this in mind, we can repeatedly apply Lemma 3.2.16 to add roots until arriving at row $b+h+1$, after which point we apply the lemma in reverse:

$$\tilde{\mathfrak{g}}^{(k)}_{\text{cover}_b(\lambda)-\epsilon_{b+h+1}}$$

$$= K(\Delta^k(\text{cover}_b(\lambda)-\epsilon_{b+h+1}) \sqcup (a^{J-1}+h); \Delta^k(\text{cover}_b(\lambda)-\epsilon_{b+h+1}) \sqcup (a^{J-1}+h); \text{cover}_b(\lambda)-\epsilon_{b+h+1})$$

$$= \dots$$

$$= K(\Delta^k(\text{cover}_b(\lambda)-\epsilon_{b+h+1}) \sqcup (a^{J-1}+h) \sqcup \cdots \sqcup (a^1+h);$$

$$\Delta^k(\text{cover}_b(\lambda)-\epsilon_{b+h+1}) \sqcup (a^{J-1}+h) \sqcup \cdots \sqcup (a^1+h); \text{cover}_b(\lambda)-\epsilon_{b+h+1})$$

$$= K(\Delta^k(\text{cover}_b(\lambda)-\epsilon_{b+h+1}) \sqcup (a^{J-1}+h) \sqcup \cdots \sqcup (a^1+h);$$

$$\Delta^k(\text{cover}_b(\lambda)-\epsilon_{b+h+1}) \sqcup (a^{J-1}+h) \sqcup \cdots \sqcup (a^1+h); \text{cover}_b(\lambda)-\epsilon_{b+h+1})$$

$$= K(\Delta^k(\text{cover}_b(\lambda)) \sqcup (a^{J-1}+h) \sqcup \cdots \sqcup (a^1+h);$$

$$\Delta^k(\text{cover}_b(\lambda)) \sqcup (a^{J-1}+h) \sqcup \cdots \sqcup (a^1+h); \text{cover}_b(\lambda)-\epsilon_{b+h+1})$$

$$= K(\Delta^k(\text{cover}_b(\lambda)) \sqcup (a^{J-1}+h) \sqcup \cdots \sqcup (a^2+h);$$

$$\Delta^k(\text{cover}_b(\lambda)) \sqcup (a^{J-1}+h) \sqcup \cdots \sqcup (a^2+h); \text{cover}_b(\lambda)-\epsilon_{b+h+1})$$

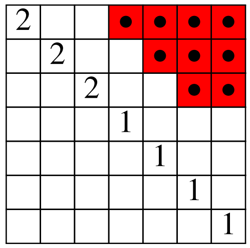
$$= \dots$$

$$= K(\Delta^k(\text{cover}_b(\lambda)); \Delta^k(\text{cover}_b(\lambda)); \text{cover}_b(\lambda)-\epsilon_{b+h+1}).$$

From this stage, the previous argument applies to prove the result.

□

EXAMPLE 4.3.8. Consider $k=4$, $\mu=(1,1,1,1)$, and the $k$-rectangle $(2,2,2)$; let $\lambda=\mu\cup(2,2,2)$,




By inspecting the $k$-Schur root ideal, it is evident that $\{6\} \subset I_{\mu}^{d,k}$. We note that $\text{cover}_6(\lambda) = (2, 2, 2, 1, 1, 0, 1)$, so the corresponding $h$-value is 0 and $\text{up}_{\Delta^k(\lambda)}(6) = 3$. Therefore, Lemma 4.3.7 implies that

$$\tilde{\mathfrak{g}}_{\lambda-\epsilon_{\{6\}}}^{(k)} = \tilde{\mathfrak{g}}_{\lambda-\epsilon_{\{3,6\}}}^{(k)}.$$

Note that in this case, $\{3, 6\} \subset I_{\mu}^{d,k}$. On the other hand, while $h = 0$ corresponds to $\text{cover}_6(\lambda)$, so $7 = 6 + h + 1$, we have that $\{7\} \not\subset I_{\mu}^{d,k}$ (and hence $\{6, 7\} \not\subset I_{\mu}^{d,k}$).

CHAPTER 5

# Cancellation

The straightening and cancellation laws introduced in Chapter 4 apply to individual closed $k$-Schur Katalan functions, so to answer the central questions raised at the start of the preceding chapter, we need a framework for grouping together the functions indexed by the set

$$I_{\mu}^{d,k} = \cup_{i \in [r+d, r+2d-1]} \text{downpath}_{\Delta^k(\lambda)}(i),$$

which weights the functions of the sum 3.3.7. In this chapter, we offer a result, Theorem 5.0.11, that leverages the structure of $I_{\mu}^{d,k}$ as well as our cancellation laws to prove that an important family of closed $k$-Schur Katalan functions in the sum 3.3.7 adds to 0.

Throughout this section, we use the notation that for $D \subset I_{\mu}^{d,k}$ with elements $a_1 < \cdots < a_t$, we let $\lambda^{D,0} = \lambda - \epsilon_D$; for $i > 0$, we have $\lambda^{D,i} = \text{cover}_{a_{i-1}}(\lambda^{D,i-1}) + \epsilon_{a_i}$, $\Psi^{D,i} = \Delta^k(\lambda^{D,i} - \epsilon_{a_i})$, and $h_i^D$ is the $h$-value of $\text{cover}_{a_i}(\lambda^{D,i})$. We will also often use the notation that $\lambda = \mu \cup R_d$ when a certain $\mu \in \text{Par}_k^{\ell}$ and $R_d$ a $k$-rectangle are specified.

Remark 5.0.1. If $D = \{a_1, \ldots, a_t\}$, then by definition, $\lambda_x^{D,1} = \lambda_x$ for $x < a_2$ and $\lambda_{a_2}^{D,1} = \lambda_{a_2} - 1$. Hence $\lambda_{a_2-1}^{D,1} > \lambda_{a_2}^{D,1}$, so $a_1 + h_1^D < a_2$. Therefore, $\lambda_x^{D,2} = \lambda_x$ for $a_2 \leq x < a_3$ and $\lambda_{a_3}^{D,2} = \lambda_{a_3} - 1$. Hence $\lambda_{a_3-1}^{D,2} > \lambda_{a_3}^{D,2}$. By iteration, we have that $a_x + h_x^D < a_{x+1}$ for all $1 \leq x \leq t-1$ and

$$\lambda_v^{D,x} = \begin{cases} \lambda_v & \text{for } v = a_x \text{ or for } v > a_x \text{ where } v \notin D \\ \lambda_v - 1 & \text{for all } v \in D \text{ where } v > a_x \end{cases} \tag{5.0.1}$$

### 5.0.1. Rewriting $I_{\mu}^{d,k}$.

Definition 5.0.2.

$$\mathcal{D} = \{D \subset I_{\mu}^{d,k} | [a_x, a_x + h_x^D] \subset I_{\mu}^{d,k} \text{ for every } a_x \in D, \text{ and if } \lambda_{a_x+h_x^D} = \lambda_{a_x+h_x^D+1},$$

then $a_x + h_x^D + 1 \in D$ and $|\mathrm{uppath}_{\Psi^{D,x}}(a_x + h_x^D + 1)| \geq |\mathrm{uppath}_{\Psi^{D,x}}(a_x)|\}$.

Conjecture 5.0.3. *If $\lambda = \mu \cup R_d$, then*

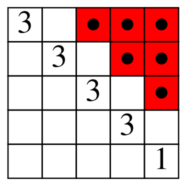
$$\tilde{\mathfrak{g}}^{(k)}_{R_d^*}\tilde{\mathfrak{g}}^{(k)}_{\mu} = \sum_{D \in \mathcal{D}, D \neq \emptyset} (-1)^{|D|+1} \tilde{\mathfrak{g}}^{(k)}_{\lambda - \epsilon_D}$$

$$= \sum_{D \in \mathcal{D}, D \neq \emptyset} (-1)^{|D|+1} \tilde{\mathfrak{g}}^{(k)}_{\lambda^D},$$

*where $\lambda^D = cover_{c_t}(\ldots(cover_{c_1}(\lambda)\ldots)$ is the chain of covers indexed by $D = \{c_1, \ldots, c_t\}$ ordered so that for all $1 \leq i \leq t$, we have $\lambda_{c_i} > \lambda_{c_{i+1}}$, or, if $\lambda_{c_i} = \lambda_{c_{i+1}}$ then $c_i > c_{i+1}$.*

Example 5.0.4. Consider $k = 4$, $\mu = (3, 3, 1)$, and $R_3 = (3, 3)$,

$$\tilde{\mathfrak{g}}^{(4)}_{\mu \cup R_3} =$$




In this case, $I_\mu^{d,k} = \{2, 3, 4, 5\}$ and $\mathcal{D}$ consists of the sets $\{4\}$, $\{5\}$, and $\{4, 5\}$:

- $\mathrm{cover}_4(\mu \cup R_3) = (3, 3, 3, 2, 1)$
- $\mathrm{cover}_5(\mu \cup R_3) = (3, 3, 3, 3)$
- $\mathrm{cover}_5(\mathrm{cover}_4(\mu \cup R_3)) = (3, 3, 3, 2)$

$$\tilde{\mathfrak{g}}^{(4)}_{(3,3,1)}\tilde{\mathfrak{g}}^{(5)}_{(3,2)} =$$

$$\tilde{\mathfrak{g}}^{(4)}_{\mathrm{cover}_4(\mu \cup R_3)} + \tilde{\mathfrak{g}}^{(4)}_{\mathrm{cover}_5(\mu \cup R_3)}$$

$$-\tilde{\mathfrak{g}}^{(4)}_{\mathrm{cover}_4(\mathrm{cover}_5(\mu \cup R_3))}.$$

Note that in this case, $\tilde{\mathfrak{g}}^{(4)}_{\mathrm{cover}_5(\mathrm{cover}_4(\mu \cup R_3))} = \tilde{\mathfrak{g}}^{(4)}_{\mathrm{cover}_4(\mathrm{cover}_5(\mu \cup R_3))}$.

We make progress towards Conjecture 5.0.3 by breaking the complement of $\mathcal{D}$ in $I_\mu^{d,k}$ into two disjoint sets; an involution on one of these sets is a promising approach to a proof of the conjecture.

Definition 5.0.5. We define $\mathcal{D}_1$ as the set of all $D \subset I_\mu^{d,k}$ such that there exists some $a_x \in D$ satisfying three conditions:

$$\lambda_{a_x+h_x^D} = \lambda_{a_x+h_x^D+1}, \tag{5.0.2}$$

$$[a_x, a_x + h_x^D + 1] \subset I_\mu^{d,k}, \tag{5.0.3}$$

$$a_x + h_x^D + 1 \in D \Rightarrow |\mathrm{uppath}_{\Psi^{D,x}}(a_x + h_x^D + 1)| < |\mathrm{uppath}_{\Psi^{D,x}}(a_x + h_x^D)|. \tag{5.0.4}$$

When it is understood that $D \in \mathcal{D}_1$, we let $j(D)$ denote the minimum of all $x \in [t]$ such that $a_x \in D$ satisfies 5.0.2, 5.0.3, and 5.0.4.

Our goal in this chapter is to show that closed $k$-Schur Katalan functions with weights indexed by members of $\mathcal{D}_1$ vanish from the result of Theoream 3.3.16.

**5.0.2. An involution on $\mathcal{D}_1$.** To rewrite Theorem 3.3.16 using $\mathcal{D}_1$, we first establish the existence of a useful involution on elements of $\mathcal{D}_1$.

Lemma 5.0.6. *An involution on $\mathcal{D}_1$ is given by the map defined by taking the image of $D \in \mathcal{D}_1$ with elements $a_1 < \cdots < a_t$ and $j = j(D)$ to be $\phi(D) = D \cup \{a_j + h_j^D + 1\}$ if $a_j + h_j^D + 1 \notin D$ and $\phi(D) = D \setminus \{a_j + h_j^D + 1\}$ otherwise.*

Proof. Assume $a_j + h_j^D + 1 \notin D$. Since $a_j + h_j^D < a_{j+1}$ by Remark 5.0.1, $D' := \{a_, \dots, a_j, a_j + h_j^D + 1, a_{j+1}, \dots, a_t\}$ is in ascending order. In particular, $\lambda^{D,j} - \epsilon_{a_j+h_j^D+1} = \lambda^{D',j}$. Therefore, $h_j^D = h_j^{D'}$ and $j(D') = j$. Because $\lambda^{D,j}_{a_j+h_j^D} = \lambda^D_{a_j+h_j^D+1}$ by Definition 5.0.5 and ( 5.0.1), the maximality of $h_j^D$ for $\mathrm{cover}_{a_j}(\lambda^{D,j})$ implies that $|\mathrm{uppath}_{\Psi^{D,j}}(a_j)| > |\mathrm{uppath}_{\Psi^{D,j}}(a_j + h_j^D + 1)|$. Because these uppaths are the same taken with respect to $\Psi^{D',j}$, we have the necessary conditions to ensure $D' \in \mathcal{D}_1$. Since $j(D') = j$, we also have that $\phi(D') = D$.

If $a_j + h_j^D + 1 \in D$, then $a_{j+1} = a_j + h_j^D + 1$ by Remark 5.0.1; let $D' = \{a_1, \dots, a_j, a_{j+2}, \dots, a_t\}$. Hence, $\lambda^{D',j}_{a_j+h_j^D} = \lambda_{a_j+h_j^D} = \lambda_{a_j+h_j^D+1} = \lambda^{D',j}_{a_j+h_j^D+1}$ by ( 5.0.1). Since $\lambda - \epsilon_D - \epsilon_{a_j+h_j^D+1} = \lambda - \epsilon_{D'}$

implies $\lambda^{D,j} + \epsilon_{a_j+h_j^D+1} = \lambda^{D',j}$, we have that $\mathrm{uppath}_{\Psi^{D,j}}(a_j + x) = \mathrm{uppath}_{\Psi^{D',j}}(a_j + x)$ for $0 \leq x \leq a_j + h_j^D + 1$. Therefore, $|\mathrm{uppath}_{\Psi^{D,j}}(a_j + h_j^D + 1)| < |\mathrm{uppath}_{\Psi^{D,j}}(a_j)|$ implies $|\mathrm{uppath}_{\Psi^{D',j}}(a_j + h_j^D + 1)| < |\mathrm{uppath}_{\Psi^{D',j}}(a_j)|$, ensuring that $h_j^D = h_j^{D'}$. We have then that $D' \in \mathcal{D}_1$, $j(D') = j(D)$, and $\phi(D') = D$. □

Example 5.0.7. Example 3.3.18 expresses the product $\tilde{\mathfrak{g}}^{(5)}_{R_3^*}\tilde{\mathfrak{g}}^{(5)}_{(4,3,1,1)}$ ($\lambda = (4, 3, 3, 3, 3, 1, 1)$) as a sum of closed $k$-Schur Katalan functions indexed by the following weights:

- $\{4, 3, 3, 3, 2, 1, 1\}, \{4, 3, 3, 2, 3, 1, 1\}, \{4, 3, 3, 3, 3, 1, 0\}, \{4, 3, 3, 2, 3, 1, 0\}, \{4, 3, 3, 2, 2, 1, 1\}, \{4, 3, 3, 3, 2, 1, 0\}, \{4, 3, 3, 2, 2, 1, 0\}, \{4, 3, 3, 2, 2, 0, 1\}, \{4, 3, 3, 2, 2, 0, 0\}$, which each correspond to a set $D \notin \mathcal{D}_1$ and,
- $\{4, 3, 3, 3, 3, 0, 1\}, \{4, 3, 3, 3, 2, 0, 0\}, \{4, 3, 3, 2, 3, 0, 1\}, \{4, 3, 3, 3, 3, 0, 0\}, \{4, 3, 3, 2, 3, 0, 0\}, \{4, 3, 3, 3, 2, 0, 1\}$, which each correspond to a set $D \in \mathcal{D}_1$.

We may inspect the relationship of each $D \in \mathcal{D}_1$ to Lemma 5.0.6:

- $\{4, 3, 3, 3, 3, 0, 1\}$ corresponds to $D = \{6\}$, $\phi(D) = \{6, 7\}$, and we have $\lambda - \epsilon_{\{6,7\}} = \{4, 3, 3, 3, 3, 0, 0\} \in \mathcal{D}_1$;
- $\{4, 3, 3, 3, 2, 0, 0\}$ corresponds to $D = \{5, 6, 7\}$, $\phi(D) = \{5, 6\}$, and we have $\lambda - \epsilon_{\{5,6\}} = \{4, 3, 3, 3, 2, 0, 1\} \in \mathcal{D}_1$;
- $\{4, 3, 3, 2, 3, 0, 1\}$ corresponds to $D = \{4, 6\}$, $\phi(D) = \{4, 6, 7\}$, and we have $\lambda - \epsilon_{\{4,6,7\}} = \{4, 3, 3, 2, 3, 0, 0\} \in \mathcal{D}_1$.

We note that these pairings were leveraged in Example 4.3.6.

While in light of Lemma 4.3.5, Lemma 5.0.6 is a compelling way to cancel elements of $\mathcal{D}_1$ from the sum produced in Theorem 3.3.16, it is important to recognize that $D \in \mathcal{D}_1$ may be such that $|D| > 1$. With this in mind, we note a corollary to Lemma 4.2.1 which suggests we can compute covers in ascending order:

Corollary 5.0.8. *For $\lambda = \mu \cup R_d$, let $D = \{a_1, \ldots, a_t\} \subset I_\mu^{d,k}$ where $a_1 < \cdots < a_t$. For any fixed $i \in [t]$, let $\Psi_1, \Psi_2 \subseteq \Delta^+$ be such that for all $(a, b) \in \Psi_1$ or $(a, b) \in \Psi_2$, we have $a > a_i + h_i^D$ and*

*$b > k - \lambda_{a_i} + a_i + h_i^D + 1$. Then*

$$K(\Delta^k(\lambda - \epsilon_D) \cup \Psi_1; \Delta^k(\lambda - \epsilon_D) \cup \Psi_2; \lambda - \epsilon_D)$$

$$= K(\Delta^k(\lambda^{D,i} - \epsilon_{a_i}) \cup \Psi_1; \Delta^k(\lambda^{D,i} - \epsilon_{a_i}) \cup \Psi_2; \lambda^{D,i} - \epsilon_{a_i}).$$

$$= K(\Delta^k(cover_{a_t}(\lambda^{D,t})) \cup \Psi_1; \Delta^k(cover_{a_t}(\lambda^{D,t})) \cup \Psi_2; cover_{a_t}(\lambda^{D,t})).$$

LEMMA 5.0.9. *Suppose that $D = \{a_1, \ldots, a_t\} \in \mathcal{D}_1$ (ascending order). Then*

$$(-1)^{|D|+1} K(\Delta^k(\mu \cup R_d - \epsilon_D); \Delta^k(\mu \cup R_d - \epsilon_D); \mu \cup R_d - \epsilon_D)$$

$$+(-1)^{|\phi(D)|+1} K(\Delta^k(\mu \cup R_d - \epsilon_{\phi(D)}); \Delta^k(\mu \cup R_d - \epsilon_{\phi(D)}); \mu \cup R_d - \epsilon_{\phi(D)}) = 0.$$

PROOF. Suppose without loss of generality that $|\phi(D)| = |D| + 1$. By applying Corollary 5.0.8, we have

$$(-1)^{|D|+1} K(\Delta^k(\mu \cup R_d - \epsilon_D); \Delta^k(\mu \cup R_d - \epsilon_D); \mu \cup R_d - \epsilon_D)$$

$$= (-1)^{|D|+1} K(\Delta^k(\lambda^{D,j+1} - \epsilon_{a_{j+1}}); \Delta^k(\lambda^{D,j+1} - \epsilon_{a_{j+1}}); \lambda^{D,j+1} - \epsilon_{a_{j+1}}),$$

where $j(D) = j$. Similarly,

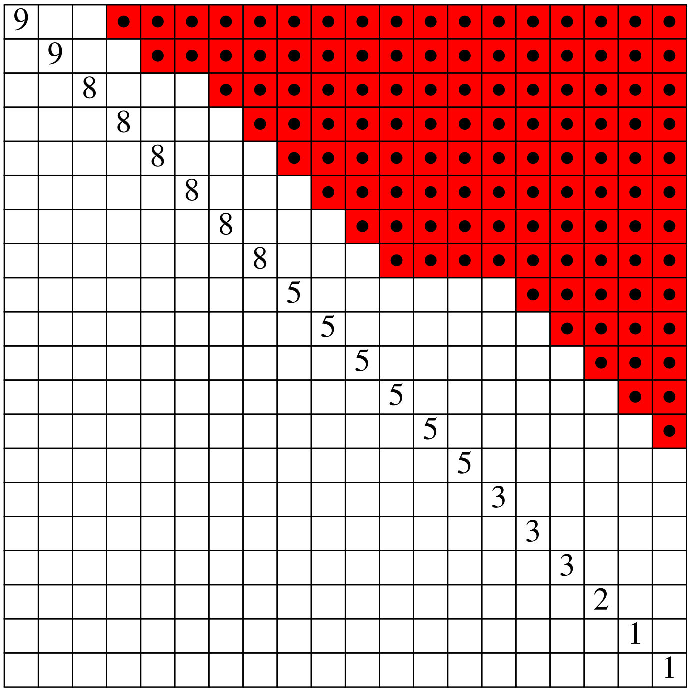
$$(-1)^{|\phi(D)|+1} K(\Delta^k(\mu \cup R_d - \epsilon_{\phi(D)}); \Delta^k(\mu \cup R_d - \epsilon_{\phi(D)}); \mu \cup R_d - \epsilon_{\phi(D)})$$

$$= (-1)^{|\phi(D)|+1} K(\Delta^k(\lambda^{D,j+1} - \epsilon_{a_{j+1}} - \epsilon_{a_j+h_j^D+1}); \Delta^k(\lambda^{D,j+1} - \epsilon_{a_{j+1}} - \epsilon_{a_j+h_j^D+1}); \lambda^{D,j+1} - \epsilon_{a_{j+1}} - \epsilon_{a_j+h_j^D+1}),$$

as $\text{cover}_{a_j}(\lambda^{\phi(D),j}) = \text{cover}_{a_j}(\lambda^{D,j}) - \epsilon_{a_j+h_j^D+1}$ due to our assumption that $D \in \mathcal{D}_1$ with $j(D) = j$. Lemma 4.3.5 completes the claim.

□

EXAMPLE 5.0.10. Consider $k = 11$, $\mu = (9, 9, 8, 8, 5, 5, 5, 5, 5, 5, 3, 3, 3, 2, 1, 1)$, $R_8 = (8, 8, 8, 8)$, and the $k$-Schur root ideal for $\lambda = \mu \cup R_8$:




If $D = \{6,9\} \subset I_\mu^{d,k}$, we have that as evidenced by the root ideal,

$$\lambda^{D,1} = \lambda - \epsilon_9$$

$$\lambda^{D,2} = (9,9,9,9,8,7,7,7,5,5,5,5,5,5,3,3,3,2,1,1) - \epsilon_9$$

$$\lambda^{D,3} = (9,9,9,9,8,7,7,7,4,5,5,5,5,5,3,3,3,2,1,1)$$

Because $\{10\} \subset I_\mu^{d,k}$, we have $D \in \mathcal{D}_1$ and $\phi(D) = \{6,9,10\}$, so by Lemma 5.0.9,

$$-\tilde{\mathfrak{g}}^{(k)}_{\lambda-\{6,9\}} + \tilde{\mathfrak{g}}^{(k)}_{\lambda-\{6,9,10\}} = 0.$$

THEOREM 5.0.11. *For $\mu \in Par_k^\ell$ reduced and $R_d^*$ a k-rectangle,*

$$\tilde{\mathfrak{g}}^{(k)}_\mu \tilde{\mathfrak{g}}^{(k)}_{R_d^*} = \sum_{D \subset I_\mu^{d,k}, D \notin \mathcal{D}_1, D \neq \emptyset} (-1)^{|D|+1} \tilde{\mathfrak{g}}^{(k)}_{\lambda-\epsilon_D}.$$

Proof. Lemma 5.0.6 implies that $\mathcal{D}_1 = S_1 \sqcup S_2$, where $S_1 = \{D \in \mathcal{D}_1 : a_j + h_j^D + 1 \in D\}$ and $S_2 = \{\phi(D) : D \in S_1\}$. We have

$$\sum_{D \in \mathcal{D}_1, D \neq \emptyset} (-1)^{|D|+1} \tilde{\mathfrak{g}}^{(k)}_{\lambda - \epsilon_D} = \sum_{D \in S_1} \left( (-1)^{|D|+1} \tilde{\mathfrak{g}}^{(k)}_{\lambda - \epsilon_D} + (-1)^{|\phi(D)+1|} \tilde{\mathfrak{g}}^{(k)}_{\lambda - \epsilon_{\phi(D)}} \right) = 0,$$

where the final equality is due to Lemma 5.0.9. Therefore, Theorem 3.3.16 implies that

$$\tilde{\mathfrak{g}}^{(k)}_{\mu} \tilde{\mathfrak{g}}^{(k)}_{R_d^*} = \sum_{D \subset I_\mu^{d,k}, D \neq \emptyset} (-1)^{|D|+1} \tilde{\mathfrak{g}}^{(k)}_{\lambda - \epsilon_D} = \sum_{D \subset I_\mu^{d,k}, D \notin \mathcal{D}_1, D \neq \emptyset} (-1)^{|D|+1} \tilde{\mathfrak{g}}^{(k)}_{\lambda - \epsilon_D}.$$

□

CHAPTER 6

# Future work

In the preceding chapters, we proved a multiplication rule for the product of closed $k$-Schur Katalan functions, one of which is weighted generically and another of which is indexed by the rectangle-minus-a-box partition $R_d^*$, by showing that several families of Katalan functions are in fact equivalent. The approach in Chapter 3 took several stages, leveraging the combinatorics of root ideals and root expansions, and it culminated in the construction of the set $I_\mu^{d,k}$. $I_\mu^{d,k}$ indexes the weights in the sum 3.3.7 using downpaths, and Chapters 4 and 5 laid out necessary groundwork to show that one subset of it, $\mathcal{D}_1$, in fact indexes a sum of closed $k$-Schur Katalan functions equivalent to 0. Equipped with these results, we conclude our work by exploring progress regarding $\mathcal{D}_1^\vee = \{D : D \subset I_\mu^{d,k}\} \setminus \mathcal{D}_1$. We offer future work building on 5.0.11 and conjecture that another subset of $I_\mu^{d,k}$, $\mathcal{D}_2$, has an analogous result to that of Theorem 5.0.11. We conjecture further that this would lead to a cancellation-free version of Theorem 5.0.11.

Definition 6.0.1.

$$\mathcal{D}_2 = \{D \in \mathcal{D}_1^\vee \mid \text{there exists } a_x \in D \text{ where } a_x + i \notin I_\mu^{d,k} \text{ for some } i \leq h_x^D + 1 \text{ if } \lambda_{a_x+h_x^D} = \lambda_{a_x+h_x^D+1}$$

$$\text{and } i \leq h_x^D \text{ if } \lambda_{a_x+h_x^D} > \lambda_{a_x+h_x^D+1}\}.$$

When it is understood that $D \in \mathcal{D}_2$, we let $j(D)$ denote the minimum of all $x \in [t]$ such that $a_x \in D$ satisfies the criterion characterizing $\mathcal{D}_2$. Further, we let $i(D) = \min\{i : a_{j(D)} + i \notin I_\mu^{d,k}\}$ and $u = \mathrm{up}_\Psi^f(a_{j(D)})$ where $\Psi = \Delta^k(\lambda)$ and $f = |\mathrm{uppath}_\Psi(a_{j(D)} + i(D))|$.

Remark 6.0.2. By definition, $\mathcal{D}_1$ and $\mathcal{D}_2$ are disjoint sets, and $\mathcal{D} = I_\mu^{d,k} \setminus (\mathcal{D}_1 \cup \mathcal{D}_2)$.

Conjecture 6.0.3.

$$\sum_{D\in\mathcal{D}_2}(-1)^{|D|+1}K(\Delta^k(\mu\cup R_d-\epsilon_D);\Delta^k(\mu\cup R_d-\epsilon_D);\mu\cup R_d-\epsilon_D)\ =0.$$

Example 6.0.4. Consider $k=4$, $\mu=(1,1,1,1)$, and the $k$-rectangle $(2,2,2)$; let $\lambda=\mu\cup(2,2,2)$.

In this case, we may inspect members of $I_\mu^{d,k}$ to determine whether or not they are elements of $\mathcal{D}_1$ or $\mathcal{D}_2$ (the following is not an exhaustive list of elements of $I_\mu^{d,k}$):

- $\text{cover}_3(\lambda)=(2,2,1,1,1,1,1)$, so $\{3\}\in\mathcal{D}$
- $\text{cover}_4(\lambda)=(2,2,2,0,1,1,1)$, so $\{4\}\in\mathcal{D}_1$
- $\text{cover}_5(\lambda)=(2,2,2,1,0,1,1)$, so $\{5\}\in\mathcal{D}_1$
- $\text{cover}_6(\lambda)=(2,2,2,1,1,0,1)$, so $\{6\}\in\mathcal{D}_2$
- $\text{cover}_6(\text{cover}_3(\lambda))=(2,2,2,1,1)$, so $\{3,6\}\in\mathcal{D}_2$.

As this example demonstrates, the chain of covers indexed by elements of $\mathcal{D}_2$ need not produce a partition (though they may).

Note that in this case, we have

$$\tilde{\mathfrak{g}}^{(4)}_{(2,1,1,1)}\tilde{\mathfrak{g}}^{(4)}_{(2,2,1)}=\tilde{\mathfrak{g}}^{(4)}_{\text{cover}_3((2,2,2)\cup(2,1,1,1))}.$$

It can be verified that here, every subset of $\{3,4,5,6\}$ besides $\{3\}$ is an element of either $\mathcal{D}_1$ or $\mathcal{D}_2$. In particular, in addition to the listed sets thusfar, we have:

- $\phi(\{4\})=\{4,5\}\in\mathcal{D}_1$,
- $\phi(\{5\})=\{5,6\}\in\mathcal{D}_1$,
- $\{3,4\}\in\mathcal{D}_1$,
- $\phi(\{3,4\})=\{3,4,5\}\in\mathcal{D}_1$,
- $\{3,5\}\in\mathcal{D}_1$,
- $\phi(\{3,5\})=\{3,5,6\}\in\mathcal{D}_1$,
- $\{4,6\}\in\mathcal{D}_1$,
- $\phi(\{4,6\})=\{4,5,6\}\in\mathcal{D}_1$,
- $\{3,4,6\}\in\mathcal{D}_1$,

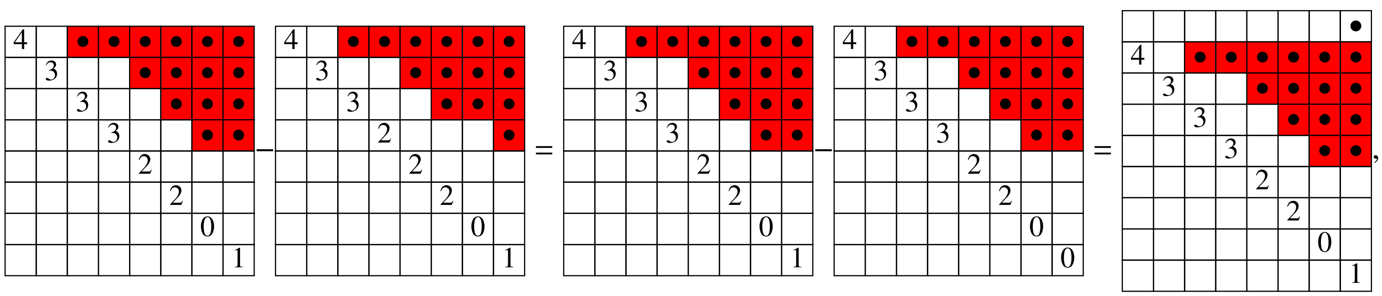
- $\phi(\{3,4,6\}) = \{3,4,5,6\} \in \mathcal{D}_1$.

## 6.1. Progress on $\mathcal{D}_2$

Because our conjecture is that the closed $k$-Schur Katalan functions with weights indexed by $\mathcal{D}_2$ in the sum from Theorem 5.0.11 vanish, a natural suggestion is to consider an involution on elements of $\mathcal{D}_2$ in the spirit of Lemma 5.0.6. However, attempts thus far to define a similar map on elements of $\mathcal{D}_2$ have failed. Of particular challenge is determining the appropriate function in the sum of Theorem 5.0.11 to "pair" with a given $\tilde{\mathfrak{g}}^{(k)}_\lambda$, $\lambda \in \mathcal{D}_2$, to establish an involution (and ultimately, an analog to Lemma 5.0.9).

Example 6.1.1. The sum produced by Theorem 3.3.16 applied to $\tilde{\mathfrak{g}}^{(5)}_{(4,2,2,1,1)}\tilde{\mathfrak{g}}^{(5)}_{R_3^*}$ includes a closed $k$-Schur Katalan function with weight $(4,3,3,3,2,2,0,1) = \lambda - \epsilon_D$, where $\lambda = (4,2,2,1,1) \cup (3,3,3)$ and $D = \{7\}$. Here, $D \in \mathcal{D}_2$: we have $\lambda^{D,2} = (4,3,3,3,2,2,0,1)$, $h_1^D = 0$, and $a_1 + 1 \notin I_\lambda^{3,5}$. Theorem 3.3.16 also produces a closed $k$-Schur Katalan function indexed by $(4,3,3,2,2,2,0,1) = \lambda - \epsilon_{D\cup\{4\}}$, and $D \cup \{4\} \in \mathcal{D}_2$ as well: we have $\lambda^{D,3} = (4,3,3,2,2,2)$, $h_2^D = 1$, and $a_2 + 1 \notin I_\lambda^{3,5}$. Note that

$$\tilde{\mathfrak{g}}^{(5)}_{\lambda-\epsilon_D} - \tilde{\mathfrak{g}}^{(5)}_{\lambda-\epsilon_{D\cup\{4\}}} =$$



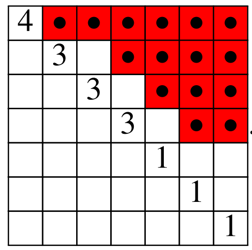
where the final equality is due to Proposition 3.1.6. However, due to Lemma 4.3.1, this final function vanishes.

Promising examples like this suggest a misleading candidate for an involution on $\mathcal{D}_2$. This failed candidate for the involution is taking the image of $D \in D_2$ with elements $a_1 < \cdots < a_t$ and $j = j(D)$, $i = i(D)$ to be $\tilde{\phi}(D) = D \cup \{u\}$ if $u \notin D$ and $\phi(D) = D \setminus \{u\}$ otherwise, where $u = \mathrm{up}^f_{\Psi_{D,j}}(a_j)$,

$f = |\mathrm{uppath}_{\Delta^k(\lambda)}(a_j + i)|$ (notation from Definition 6.0.1). Our next example illustrates evidence that such a definition fails.

Example 6.1.2. Let $k = 4$, $\mu = (4, 3, 1, 1, 1)$, and consider $R_3 = (3, 3)$.

$$\tilde{\mathfrak{g}}^{(4)}_{\mu\cup R_3} =$$




.

In this case, $I^{d,k}_{\mu} = \{3, 4, 5, 6\}$, and $\mathcal{D}$ consists of the sets $\{3\}, \{4\}$, and $\{3, 4\}$, and we have that:

$$\tilde{\mathfrak{g}}^{(4)}_{\mu} * \tilde{\mathfrak{g}}^{(4)}_{R_3^*} = \tilde{\mathfrak{g}}^{(4)}_{\mathrm{cover}_3(\mu\cup R_3)} + \tilde{\mathfrak{g}}^{(4)}_{\mathrm{cover}_4(\mu\cup R_3)} - \tilde{\mathfrak{g}}^{(4)}_{\mathrm{cover}_3(\mathrm{cover}_4(\mu\cup R_3))}.$$

We have $\{6\}, \{4, 6\}, \{3, 6\}, \{3, 4, 6\}, \{5, 6\}, \{3, 5, 6\}, \{3, 4, 5, 6\}, \{5\}, \{3, 5\}, \{3, 4, 5\} \in \mathcal{D}_2$, and $\{4, 5\}$, $\{4, 5, 6\} \in \mathcal{D}_1$. In light of Theorem 3.3.16, Theorem 5.0.11 implies that

$$\tilde{\mathfrak{g}}^{(4)}_{\mu\cup R_3 - \epsilon_{\{4,5\}}} - \tilde{\mathfrak{g}}^{(4)}_{\mu\cup R_3 - \epsilon_{\{4,5,6\}}} = 0.$$

One example of evidence that the proposed definition of $\tilde{\phi}$ fails in general is to consider $k = 4$, $\mu = (4, 3, 1, 1, 1)$, and $R_3 = (3, 3)$ as in Example 6.1.2. We have $\{5, 6\} \in \mathcal{D}_2$, as $h^{\{5,6\}}_2 = 0$ and $6 + 1 = 7 \notin I^{d,k}_{\mu}$; we also have $\{4, 5\} \in \mathcal{D}_1$, as $h^{\{4,5\}}_2 = 0$, so we meet the conditions of Definition 5.0.5 with $a_x = 5$, so $x = 2$ (as $6 = a_x + 1 \in I^{d,k}_{\mu}$). In this example, the involution $\phi$ on $\mathcal{D}_1$ defined in Lemma 5.0.6 is such that $\phi(\{4, 5\}) = \{4, 5, 6\}$, as while $6 \in \{4, 5, 6\}$, we have $|\mathrm{uppath}_{\Psi_{\{4,5\},x}}(6)| < |\mathrm{uppath}_{\Psi_{\{4,5\},x}}(a_x + h^{\{4,5\}}_x)|$. However, we have $4 = u = \mathrm{up}_{\Psi_{\{5,6\},2}}(6)$, and as discussed, $\{4, 5, 6\} \notin \mathcal{D}_2$. Through such exploration, it has become evident that exploring both $\mathcal{D}_2$ in general and the uppaths of elements in a given $D \in \mathcal{D}_2$ are essential to making progress. One possibility is that is that an involution can be defined by revising $\tilde{\phi}$ to map only certain $D \in \mathcal{D}_2$ such that $u \notin D$ to $D \cup \{u\}$. To that end, we offer a series of lemmas which could assist in finding a successful involution on $\mathcal{D}_2$.

LEMMA 6.1.3. *If $D \in \mathcal{D}_2$ with $u \notin D$ (notation from Definition 6.0.1), then*

$$\tilde{\mathfrak{g}}^{(k)}_{\lambda-\epsilon_D} = \tilde{\mathfrak{g}}^{(k)}_{\lambda-\epsilon_{D\cup\{u\}}}.$$

PROOF. Throughout this proof, let $D' = D \cup \{u\} = \{a_1, \ldots, a_m, u, a_{m+1}, \ldots, a_t\}$ (ascending order) and $j = j(D)$. We also use the notation that for any set $S = \{z_1, \ldots, z_q\}$, $y^S_x$ is the $y$-value corresponding to $\mathrm{cover}_{z_x}(\lambda^{S,x})$. We claim that $\lambda^{D',j+2} = \lambda^{D,j+1} - \epsilon_{a_j+h^D_j+1}$. To that end, we split into two cases depending on $h^D_m$. First assume $a_m + h^D_m < u$. In this case, $\lambda^{D,m+1} - \epsilon_u = \lambda^{D',m+1}$. Note also that in this case, $\lambda_{a_m+h^D_m} > \lambda_u$, as $D \notin \mathcal{D}_1$ and $j(D) = j > m$. We claim now that $h^D_j = h^{D'}_{m+1}$: due to Lemma 6.1.6, $\mathrm{uppath}_{\Psi^{D,j}}(a_j + v)$ and $\mathrm{uppath}_{\Psi^{D',m+1}}(u + v)$ agree on all elements less or equal to $u+v$, $0 \le v \le h^D_j + 1$, and $(\lambda^{D',m})_{u+i} = (\lambda^{D,j})_{u+i} > (\lambda^{D',m})_{u+i+1} = (\lambda^{D,j})_{u+i+1}$ by construction. We also note Lemma 6.1.6 implies that for all $m+1 \le v < j$, $y^D_v = y^{D'}_{v+1}$ and $h^D_v = h^{D'}_{v+1}$. Finally, we consider $\lambda^{D',j+2}$ and $\lambda^{D,j+1}$: we have $\mathrm{top}_{\Psi^{D',j+1}}(a_j) = \mathrm{down}_{\Psi^{D,j}}(u)$, so $y^{D'}_{j+2} + 1 = u \in \mathrm{uppath}_{\Psi^{D',j+1}}(a_j + 1)$ and $h^{D'}_{j+1} = h^D_j + 1$.

Suppose now that $a_m + h^D_m \ge u$. We first observe that this implies $a_m + h^D_m = u + i$, as $D \notin \mathcal{D}_1$ and $j(D) > m$ (so each $u + v \in I^{d,k}_\mu$ for $0 \le v \le i$). Note further that in this case, $u - 1 = a_m + h^{D'}_m \in \mathrm{uppath}_{\Psi^{D,j}}(a_j)$, $u \in \mathrm{uppath}_{\Psi^{D,j}}(a_j+1)$. We have $y^D_j + 1 = \mathrm{up}^q_{\Psi^{D,j}}(u)$, where $q = |\mathrm{uppath}_{\Psi^{D,j}}(u-1)|$. On the other hand, $[y^{D'}_m + 1, y^{D'}_m + h^{D'}_m] = [\mathrm{up}^c_{\Psi^{D,m}}(a_m + 1), \mathrm{up}^c_{\Psi^{D,m}}(u-1)]$, where $c = |\mathrm{uppath}_{\Psi^{D,m}}(a_m)|$. We have $\lambda_{a_m+h^{D'}_m} = \lambda_{a_m+h^{D'}_m+1}$, but because $D \in \mathcal{D}_2$, we have $|\mathrm{uppath}_{\Psi^{D',m}}(a_m+h^{D'}_m+1)| \ge |\mathrm{uppath}_{\Psi^{D',m}}(a_m+h^{D'}_m)|$, as $u-1 = a_m + h^{D'}_m \in \mathrm{uppath}_{\Psi^{D,j}}(a_j)$, $u \in \mathrm{uppath}_{\Psi^{D,j}}(a_j+1)$. We also note that because $u-1 \in \mathrm{uppath}_{\Psi^{D,j}}(a_j)$, $\mathrm{uppath}_{\Psi^{D,j}}(a_j)$ contains some $x \in \mathrm{uppath}_{\Psi^{D',m+1}}(u)$ where $x \in [y^D_m + 1, y^D_m + h^D_m]$. Similarly,

$(u+v) \in \mathrm{uppath}_{\Psi^{D,j}}(a_j + v + 1)$ for each $v \in [h^D_j - 1]$; these observations imply that $\mathrm{top}_{\Psi^{D',m+1}}(u) = \mathrm{top}_{\Psi^{D,j}}(a_j)$, $h^{D'}_{m+1} = h^D_j$, and $y^{D'}_{m+1} = y^D_j$. Finally, we claim that $h^{D'}_{j+1} = h^D_j + 1$ and $y^{D'}_{j+1} + 1 = \mathrm{up}^c_{\Psi^{D,m}}(u)$. This is because there is a ceiling in columns $\mathrm{down}_{\Psi^{D',j+1}}(\mathrm{up}^c_{\Psi^{D',m}}(u-1))$, $\mathrm{down}_{\Psi^{D',j+1}}(\mathrm{up}^c_{\Psi^{D',m}}(u-1))+1$ of $\Psi^{D',j+1}$, while for $1 \le v \le h^D_j$, we have $u+v-1 \in \mathrm{uppath}_{\Psi^{D',j+1}}(a_j+v)$ and $\mathrm{up}^{|\mathrm{uppath}_{\Psi^{D',j+1}}(a_j)|}_{\Psi^{D',j+1}}(a_j+v) = \mathrm{up}^c_{\Psi^{D,m}}(u+v-1)$. Granted then that $\lambda^{D',j+2} = \lambda^{D,j+1} - \epsilon_{a_j+h^D_j+1}$, we can apply Lemma 4.3.5. ∎

Lemma 6.1.4. *Suppose $D = \{a_1, \dots, a_t\} \in \mathcal{D}_2$, and let $u$, $\Psi$, and $j(D) = j$ be as the notation in Definition 6.0.1. If $u \notin D$, then for all $s \in uppath_\Psi(a_j)$ with $u \le s$ and $0 \le p \le h_j^D$, we have $s + p \notin D$.*

Proof. We first show that for all $s \in \text{uppath}_\Psi(a_j)$ with $u < s$, we have $s \notin D$. Suppose for the sake of contradiction that there exists an element in $D$ meeting this condition; fix $a_w$ as the largest such element in $D$. We claim this assumption implies there exists $a_z \in D$ such that for some $0 \le v \le h_z^D$:

- $a_z + v \in \text{uppath}_\Psi(a_j)$
- $a_z + v \in \text{uppath}_{\Psi^{D,j}}(a_j + 1)$
- $a_z > u$.

To justify this claim, it suffices to consider the case wherein $a_w \notin \text{uppath}_{\Psi^{D,j}}(a_j + 1)$. Then we have one of three possibilities: $h_w^D = 0$, $\lambda_{a_w} = \lambda_{a_w+1}$, and $a_w + 1 \notin D$ (so there is a wall in rows $a_w, a_w + 1$ of $\Psi^{D,j}$); $h_j^D = 0$ and there is a ceiling that blocks $\text{uppath}_{\Psi^{D,j}}(a_j + 1)$ from going up to $a_w$; or $a_w \in [y + 1, y + h_q^D]$ for at least one $a_q \in D$. In fact, the first situation is impossible: if this held and $a_w + 1 \notin I$, then we would meet the conditions of Definition 6.0.1 with $x = w$ (which is a contradiction as $j$ is minimal but $w < j$), but on the other hand, if $a_w + 1 \in I$, we meet the conditions of Definition 5.0.5, which contradicts the fact that because $D \in \mathcal{D}_2$, we have $D \notin \mathcal{D}_1$. The next possibility, that $h_j^D = 0$ and there is a ceiling that blocks $\text{uppath}_{\Psi^{D,j}}(a_j + 1)$ from going up to $a_w$, can also be eliminated: this assumption would imply $i(D) = 1$ and $u \ge a_w$ (a contradiction even if $u = a_w$ because we have assumed $u \notin D$).

Assume instead then that $a_w \in [y + 1, y + h_q^D]$ for at least one $a_q \in D$; fix $q$ minimal such that this is so. Then we claim this implies $\text{cover}_{a_q}(\lambda^{D,q})$ necessarily decrements $\text{cover}_{a_{q-1}}(\lambda^{D,q-1})$ in a row $a_q + c \in \text{uppath}_\Psi(a_j)$, $0 \le c \le h_q^D$: even if $c = 0$ and $y + 1 = a_w$, the cover decrements in this row. From here, we may iterate our argument by replacing $a_w$ with $a_q + c$, beginning by noting that it suffices to consider the case that $a_q + c \notin \text{uppath}_{\Psi^{D,j}}(a_j + 1)$. We continue until we have some pair $z$, $v$, which meets the conditions of the claim; the process necessarily terminates when we obtain maximal $a_z + v$, $a_z \in D$ and $0 \le v \le h_z^D$, with $a_z + v \in \text{uppath}_\Psi(a_j)$.

We now note that with $i = i(D)$ as in Definition 6.0.1, we have $a_z + v + i \notin I_\mu^{d,k}$, so because $j$ is minimal and $j > z$, we must have one of two possible cases: if $\lambda_{a_z+h_z^D} = \lambda_{a_z+h_z^D+1}$, then we must have $v+i > h_z^D+1$, and if $\lambda_{a_z+h_z^D} > \lambda_{a_w+h_z^D+1}$, then we must have $v+i > h_z^D$. If we suppose $\lambda_{a_z+h_z^D} = \lambda_{a_z+h_z^D+1}$, then we either have a wall in rows $a_z + h_z^D, a_z + h_z^D + 1$ of $\Psi^{D,j}$ (so $a_z + h_z^D + 1 \notin D$) or we have $a_z + h_z^D + 1 \in D$. But a wall in rows $a_z + h_z^D, a_z + h_z^D + 1$ of $\Psi^{D,j}$ would reach a contradiction: were $[a_z, a_z + h_z^D + 1] \not\subseteq I_\mu^{d,k}$, we meet Definition 6.0.1 with $x = z < j$, and $[a_z, a_z + h_z^D + 1] \subseteq I_\mu^{d,k}$ would mean $D \in \mathcal{D}_1$. On the other hand, if $a_z + h_z^D + 1 \in D$, we may relabel $a_z + h_z^D + 1 = a_{z+1}$. We may then iterate this argument for $a_{z+1}$, and again as needed, getting some maximal $m \geq 1$ such that $\lambda$ is constant on $[a_z, a_{z+m}]$. We focus then on this $a_{z+m} \in D$, which is such that $a_{z+m} \in \text{uppath}_\Psi(a_j + s)$, $a_{z+m} \in \text{uppath}_{\Psi^{D,j}}(a_j + s + 1)$ for some $s \leq h_j^D$. Supposing then that $\lambda_{a_{z+m}+h_{z+m}^D} = \lambda_{a_{z+m}+h_{z+m}^D+1}$, we reach a contradiction analogous to our initial observations prohibiting a wall in rows $a_z + h_z^D, a_z + h_z^D + 1$ in $\Psi^{D,j}$, and because $m$ is maximal, this was the only possibility to eliminate when $\lambda_{a_{z+m}+h_{z+m}^D} = \lambda_{a_{z+m}+h_{z+m}^D+1}$. Otherwise, if $\lambda_{a_{z+m}+h_{z+m}^D} > \lambda_{a_{z+m}+h_{z+m}^D+1}$, we reach a contradiction as well: by construction of $i$, we must have that $\lambda$ is constant on $[a_z+v, a_z+v+i]$, but $j > z+m$ implies $a_z+v+i > a_{z+m}+h_{z+m}^D$. If we had instead began by assuming that $\lambda_{a_z+h_z^D} > \lambda_{a_z+h_z^D+1}$ (so $v + i > h_z^D$), we reach a similar contradiction: this assumption would imply $a_j + h_z^D + 1 \notin I_\mu^{d,k}$, so we would need $i = h_z^D + 1$, but by construction of $u$ this would mean $u = a_z$.

To see the generalization, we note that our preceding argument gives that $\text{up}_\Psi^e(a_j) = \text{up}_{\Psi^{D,j}}^e(a_j)$ for all $\text{up}_\Psi^e(a_j) > u$, so if $s + p \in D$ for some $s = \text{up}_\Psi^e(a_j) > u$, we may fix $s$ minimal, then $p$ minimal such that this is so. Then we may conclude that either $p > h_j^D$ or $s + p = y + 1$ for $y + 1$ corresponding to some $a_q \in D$. However, if $s + p = y + 1$, then $s + p = \text{up}_\Psi^m(a_q)$, i.e., $a_q = s' + p$ for $s' \in \text{uppath}_\Psi(a_j)$. If we repeat this argument for $s' + p$, then iterate it again as needed, we find some maximal $\tilde{s} + p \in D$ such that $\tilde{s} \in \text{uppath}_\Psi(a_j)$ and can conclude that $p > h_j^D$. □

The argument housed within Lemma 6.1.4 proves the following corollary.

Corollary 6.1.5. *If $D \in \mathcal{D}_2$ with $u \notin D$ (notation from Definition 6.0.1), suppose $D \cup \{u\} = \{a_1, \ldots, a_m, u, \ldots, a_t\}$ in ascending order. Then there exists no pair $m + 1 \leq z \leq t$, $0 \leq v \leq h_z^D$ such that:*

- $a_z + v \in uppath_{\Psi}(a_j)$
- $a_z + v \in uppath_{\Psi^{D,j}}(a_j + 1)$.

Lemma 6.1.6. *If $D \in \mathcal{D}_2$ with $u \notin D$ (notation from Definition 6.0.1), suppose $D \cup \{u\} = \{a_1, \ldots, a_m, u, \ldots, a_t\}$ in ascending order. Then for all $m < e < j = j(D)$ and all $0 < p \leq h_e^D$, we have*

$$\begin{aligned}&\{up_{\Psi^{D,e}}(a_e + p), up^2_{\Psi^{D,e}}(a_e + p), \ldots, up^M_{\Psi^{D,e}}(a_e + p)\}\\ &= \{up_{\Psi^{D\cup\{u\},e+1}}(a_e + p), up^2_{\Psi^{D\cup\{u\},e+1}}(a_e + p), \ldots, up^M_{\Psi^{D\cup\{u\},e+1}}(a_e + p)\},\end{aligned} \tag{6.1.1}$$

*where $M = |uppath_{\Psi^{D,e}}(a_e)|$.*

Proof. Suppose for the sake of contradiction that there exist $e$ and $p$ such that 6.1.1 does not hold. Then we may fix $m < e < j$ minimal such that this is so, then fix corresponding $p$ minimal. With these set, let $1 \leq c' \leq M$ be minimal such that $\text{up}^{c'}_{\Psi^{D,e}}(a_e + p) \neq \text{up}^{c'}_{\Psi^{D\cup\{u\},e+1}}(a_e + p)$. We have $u \in \text{uppath}_{\Psi^{D,e}}(a_e + p)$, as we have assumed $e$ and $p$ are minimal such that 6.1.1 does not hold. This implies one of two possibilities: either there exists some $\tilde{v}, \tilde{c}$ such that $a_{\tilde{v}} + \tilde{c} \in \text{downpath}_{\Psi}(u)$, where $\tilde{v} \leq e$ and $\tilde{c} \leq h^D_{\tilde{v}}$, or there exists $w \in \text{downpath}_{\Psi}(u)$ such that $w \in [y + 1, y + h^D_q]$ for some $q < e$. We claim that either case implies there exists a pair $z, v$ prohibited by Corollary 6.1.5.

To justify the existence of $z$ and $v$, first assume the existence of $a_{\tilde{v}} + \tilde{c}$ as described; set $a_{\tilde{v}} + \tilde{c}$ maximal such that the condition is met. It suffices to assume then that $a_{\tilde{v}} + \tilde{c} \notin \text{uppath}_{\Psi^{D,j}}(a_j + 1)$ (note: we may have that no $a_{\tilde{v}} + \tilde{c}$ existed at all, i.e. there exists $w \in \text{downpath}_{\Psi}(u)$ such that $w \in [y + 1, y + h^D_q]$ for some $q < v$, and we consider this separately). Then we have one of three possibilities: $h^D_{\tilde{v}} = \tilde{c}$, $\lambda_{a_{\tilde{v}}+\tilde{c}} = \lambda_{a_{\tilde{v}}+\tilde{c}+1}$, and $a_{\tilde{v}} + \tilde{c} + 1 \notin D$ (so there is a wall in rows $a_{\tilde{v}} + \tilde{c}, a_{\tilde{v}} + \tilde{c} + 1$ of $\Psi^{D,j}$); $h^D_j = 0$ and there is a ceiling that blocks $\text{uppath}_{\Psi^{D,j}}(a_j + 1)$ from going up to $a_{\tilde{v}} + \tilde{c}$; or $a_{\tilde{v}} + \tilde{c} \in [y + 1, y + h^D_q]$ for at least one $a_q \in D$. However, the first listed situation is impossible: if this held and $a_{\tilde{v}} + \tilde{c} + 1 \notin I^{d,k}_{\mu}$, then we would meet the conditions of Definition 6.0.1 with $x = \tilde{v}$ (which is a contradiction as $j$ is minimal but $\tilde{v} < j$), but on the other hand, if $a_{\tilde{v}} + \tilde{c} + 1 \in I^{d,k}_{\mu}$, we meet the conditions of Definition 5.0.5, which contradicts the fact that because $D \in \mathcal{D}_2$, we have $D \notin \mathcal{D}_1$. The next possibility, that $h^D_j = 0$ and there is a ceiling that blocks $\text{uppath}_{\Psi^{D,j}}(a_j + 1)$ from going up to $a_{\tilde{v}} + \tilde{c}$, can also be eliminated: this assumption would imply $i(D) = 1$ and $u \geq a_{\tilde{v}} + \tilde{c}$ (a contradiction even if $u = a_{\tilde{v}} + \tilde{c}$ because we have assumed $u \notin D$).

Assume instead then that $a_{\tilde{v}} + \tilde{c} \in [y + 1, y + h_q^D]$ for at least one $a_q \in D$; fix $q$ minimal such that this is so. Then we claim this implies $\text{cover}_{a_q}(\lambda^{D,q})$ necessarily decrements $\text{cover}_{a_{q-1}}(\lambda^{D,q-1})$ in a row $a_q + e' \in \text{uppath}_\Psi(a_j)$, $0 \leq e' \leq h_q^D$: even if $e' = 0$ and $y + 1 = a_{\tilde{v}} + \tilde{c}$, the cover decrements in this row. From here, we may iterate our argument by replacing $a_{\tilde{v}} + \tilde{c}$ with $a_q + e'$, beginning by noting that it suffices to consider the case that $a_q + e' \notin \text{uppath}_{\Psi^{D,j}}(a_j + 1)$. We continue until we have some pair $z$, $v$, which meets the conditions of Corollary 6.1.5; the process necessarily terminates when we obtain maximal $a_z + v$, $a_z \in D$ and $0 \leq v \leq h_z^D$, with $a_z + v \in \text{uppath}_\Psi(a_j)$. This final argument may be reproduced for the assumption that $w \in [y + 1, y + h_q^D]$ (this implies a decrement in a row $a_q + e'$ as before), completing our proof that there exists the element to which Corollary 6.1.5 shows is a contradiction.

□

Lemma 6.1.7. *If $D \in \mathcal{D}_2$ with $u \notin D$ (notation from Definition 6.0.1), suppose $D \cup \{u\} = \{a_1, \ldots, a_m, u, \ldots, a_t\}$ in ascending order. Then for all $m < e < j = j(D)$, we have that $y + 1$ corresponding to $\lambda^{D,e}$ is the same as that of $\lambda^{D\cup\{u\},e+1}$.*

Proof. Due to Lemma 6.1.6, $\text{uppath}_{\Psi^{D,e}}(a_e)$ necessarily coincides with $\text{uppath}_{\Psi^{D\cup\{u\},e+1}}(a_e)$ in all possible cases excepting that $u + c' \in \text{uppath}_{\Psi^{D,e}}(a_e)$ for some $c' \leq h_{m+1}^{D\cup\{u\}}$. We can conclude that $c' \neq 0$ because of Lemma 6.1.4. But if $u + c' \in \text{uppath}_{\Psi^{D,e}}(a_e)$ and $c' > 0$, then $y' + c' \in \text{uppath}_{\Psi^{D\cup\{u\},e+1}}(a_e)$, where $y'$ is the $y$-value corresponding to $\lambda^{D\cup\{u\},m+1}$. Because $y' + c' \in \text{uppath}_{\Psi^{D,e}}(a_e)$, we verify the claim in this case.

□

## 6.2. Progress on $\mathcal{D}$

We conclude with results about the subset of $I_\mu^{d,k}$ which conjecturally indexes a cancellation-free revision of Theorem 5.0.11.

Conjecture 6.2.1. *If $\lambda = \mu \cup R_d$, then*

$$\tilde{\mathfrak{g}}^{(k)}_{R^*_d}\tilde{\mathfrak{g}}^{(k)}_{\mu} = \sum_{D\in\mathcal{D}, D\neq\emptyset} (-1)^{|D|+1}\tilde{\mathfrak{g}}^{(k)}_{\lambda-\epsilon_D}.$$

An analog to Lemma 5.0.6 for $\mathcal{D}_2$ would establish this conjecture.

Proposition 6.2.2. *For $D = \{a_1, \ldots, a_t\} \in \mathcal{D}$, define cover ordering as*

$$(a_v, \lambda_{a_v}) < (a_w, \lambda_{a_w})$$

*if $\lambda_{a_v} > \lambda_{a_w}$ or $\lambda_{a_v} = \lambda_{a_w}$ and $a_v > a_w$. Then*

$$\tilde{\mathfrak{g}}^{(k)}_{\lambda-\epsilon_D} = \tilde{\mathfrak{g}}^{(k)}_{cover_{b_t}(\ldots(cover_{b_1}(\lambda)\ldots)},$$

*where $\{b_1, \ldots, b_t\} = D$ is the rearrangement of $\{a_1, \ldots, a_t\}$ dictated by cover ordering, i.e. $(b_x, \lambda_{b_x}) < (b_{x+1}, \lambda_{b_{x+1}})$ for all $x \in [t-1]$. Cover ordering for $\mathcal{D}$ can equivalently be defined as*

$$(a_v, \lambda_{a_v}) < (a_w, \lambda_{a_w})$$

*if $\lambda_{a_v} > \lambda_{a_w}$ or $\lambda_{a_v} = \lambda_{a_w}$ and $\lambda_{top_{\Psi D,v}(a_v)} > \lambda_{top_{\Psi D,v}(a_w)}$.*

Proof. First, we note Corollary 5.0.8 implies for any fixed $v \in [t]$,

$$\tilde{\mathfrak{g}}^{(k)}_{\lambda-\epsilon_D} = \tilde{\mathfrak{g}}^{(k)}_{\lambda^{D,v}-\epsilon_{a_v}}.$$

We next observe a consequence of the definition of $\mathcal{D}$: if $D \in \mathcal{D}$ contains a "run," i.e. a series of elements $\{a_x, a_{x+1}, \ldots, a_{x+x'}\}$ such that $\lambda_{a_x} = \cdots = \lambda_{a_{x+x'}}$, we claim we must have that

- $a_v + h^D_v + 1 = a_{v+1}$ for each $v \in [x, x+x'-1]$, and
- $|\mathrm{uppath}_{\Psi D,v}(a_{v+1})| > |\mathrm{uppath}_{\Psi D,v}(a_v)|$.

The first claim is due to the fact that $a_v + h^D_v + 1 < a_{v+1}$ would imply that $D \in \mathcal{D}_1$ if $[a_v, a_v + h^D_v + 1] \subset I^{d,k}_{\mu}$, and if $[a_v, a_v + h^D_v + 1] \not\subset I^{d,k}_{\mu}$, that $D \in \mathcal{D}_2$, which both contradict that $D \in \mathcal{D}$. Given then that $a_v + h^D_v + 1 = a_{v+1}$, the second claim holds to ensure that $D \notin \mathcal{D}_1$. Noting then that

$\text{uppath}_{\Psi^{D,v}}(a_v + c) = \text{uppath}_{\Delta^k(\text{cover}_{a_{v+1}}(\lambda^{D,v}+\epsilon_{a_{v+1}}))}(a_v + c)$ for all $c \leq h^D_v$ and $\text{uppath}_{\Psi^{D,v}}(a_{v+1} + c') = \text{uppath}_{\Psi^{D,v+1}}(a_{v+1} + c')$ for all $c' \leq h^D_{v+1}$, we have $\lambda^{D,x+2} = \text{cover}_{a_x}(\text{cover}_{a_{x+1}}(\lambda^{D,x} + \epsilon_{a_{x+1}}))$. Iterating this process, we have $\lambda^D_{x+x'} = \text{cover}_{a_x}(\text{cover}_{a_{x+1}}(\ldots(\text{cover}_{a_{x+x'}}(\lambda^D_x))\ldots))$.

Applying this logic to each series of runs as needed satisfies cover ordering as needed, and the fact that $|\text{uppath}_{\Psi^{D,v}}(a_{v+1})| > |\text{uppath}_{\Psi^{D,v}}(a_v)|$ gives us the second formulation of cover.

□

Remark 6.2.3. If weakened to remove the second formulation of cover ordering wherein $\lambda_{\text{top}_{\Psi^{D,v}}(a_w)} > \lambda_{\text{top}_{\Psi^{D,v}}(a_w)}$, Proposition 6.2.2 holds for $D \notin \mathcal{D}$.

**6.2.1. Chains in $\mathcal{D}$.** We seek a characterization using successively computed covers, or chains, for the surviving sets $D \subset I^{d,k}_\mu$ within the conjecturally cancellation-free formula proposed by Conjecture 5.0.3. Because this conjecture relies on $\mathcal{D}$, we conclude by observing a series of facts which assist us in computing covers indexed by a given $D \in \mathcal{D}$.

Remark 6.2.4. If within $D = \{z_1, \ldots, z_t\}$, there exists $x \in [t]$ such that $\text{uppath}_{\Delta^k(\lambda^{D,x+1})}(z_{x+1} + v)$ is disjoint from $\text{uppath}_{\Delta^k(\lambda^{D,x})}(z_x + v')$ for all $v \in [h^D_{x+1}]$ and $v' \in [h^D_x]$, then

$$\text{cover}_{z_x}(\text{cover}_{z_{x+1}}(\lambda^{D,x} + \epsilon_{z_{x+1}})) = \text{cover}_{z_{x+1}}(\text{cover}_{z_x}(\lambda^{D,x}) + \epsilon_{z_{x+1}}) = \lambda^{D,x+2} - \epsilon_{x+2}.$$

Lemma 6.2.5. *Suppose that for $D = \{z_1, \ldots, z_t\} \in \mathcal{D}$, there exists $x \in [t]$ such that $z_x + c \in$ uppath$_{\lambda^{D,x}}(z_{x+1})$ for some $0 < c \leq h^D_x$. Then up$^q_{\lambda^{D,x}}(z_{x+1}+v) =$ up$^q_{\lambda^{D,x+1}}(z_{x+1}+v)$ for all up$^q_{\lambda^{D,x}}(z_{x+1}+v) \leq y_x + c + v$, where $y_x$ is the y-value corresponding to cover$_{z_x}(\lambda^{D,x} + \epsilon_{z_x})$ and $v \leq \min(h^D_x - c, h^D_{x+1})$.*

Proof. Given that $z_x + c \in \text{uppath}_{\lambda^{D,x}}(z_{x+1})$, we have $y_x + c + v \in \text{uppath}_{\lambda^{D,x}}(z_{x+1} + v)$ for all $v \leq \min(h^D_x - c, h^D_{x+1})$ by definition of covers. However, while $z_x + c + v - 1 \in \text{uppath}_{\lambda^{D,x+1}}(z_{x+1} + v)$, we have $y_x + c + v \in \text{uppath}_{\lambda^{D,x+1}}(z_{x+1} + v)$ for two reasons: $\text{cover}_x(\lambda^{D,x} + \epsilon_{z_x})$ differs from $\lambda^{D,x} + \epsilon_{z_x}$ precisely in the intervals $[z_x, z_x + h^D_c]$ (a decrement) and $[y+1, y+h^D_c]$ (an increment), and moreover, the definition of covers implies that $\text{up}^w_{\lambda^{D,x}}(z_x+c+v) = \text{up}^w_{\lambda^{D,x}}(z_x+c+v-1)+1$ for all $\text{up}^w_{\lambda^{D,x}}(z_x+c+v) \leq y_x + c + v$. Therefore, the uppath of $z_{x+1} + v$ is the same in $\lambda^{D,x}$ and $\lambda^{D,x+1}$ (weakly) above $y_x + c + v$.

□

Remark 6.2.6. The assumption of Lemma 6.2.5 that $z_x + c \in \text{uppath}_{\lambda^{D,x}}(z_{x+1})$ is equivalent to assuming that $y_x + c \in \text{uppath}_{\lambda^{D,x}}(z_{x+1})$; the lemma could be restated to begin with this assumption.

Lemma 6.2.7. *If $D = \{z_1, \dots, z_t\} \in \mathcal{D}$, then for any $x \in [t]$, $\lambda^{D,x}$ may only possibly contain a wall in rows $z_{x-1} + h^D_{x-1}, z_x$ or $z_w, z_w + 1$, where $w \geq x$.*

Proof. Suppose for the sake of contradiction that $\lambda^{D,x}$ contains a wall in rows $z_v + h^D_v, z_v + h^D_v + 1$, $v < x - 1$. If $z_v + h^D_v + 1 \in I^{d,k}_\mu$, then $D \in \mathcal{D}_1$, a contradiction, but if $z_v + h^D_v + 1 \notin I^{d,k}_\mu$, then $D \in \mathcal{D}_2$, a contradiction as well. □

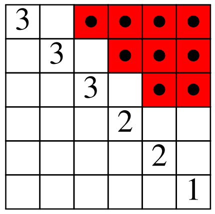
Lemma 6.2.8. *If $D = \{z_1, \dots, z_t\} \in \mathcal{D}$ and for some $i_1, i_2 \in [t]$, $z_{i_2} \in downpath_{\lambda^{D,i_1}}(z_{i_1})$, then $\lambda_{top_{\lambda^{D,i_1}}(z_{i_1})} > \lambda_{top_{\lambda^{D,i_2}}(z_{i_2})}$.*

Proof. We proceed by induction on $m$ where $z_{i_2} = \text{down}^m_{\lambda^{D,i_1}}(z_{i_1})$. As a base case, assume $m = 1$: $\lambda^{D,i_2}$ has a ceiling in columns $\text{down}_{\lambda^{D,i_1}}(z_{i_1}) - 1, \text{down}_{\lambda^{D,i_1}}(z_{i_1})$, so $\text{top}_{\lambda^{D,i_2}}(z_{i_2}) = z_{i_2}$.

Assume then that the inductive hypothesis holds for all positive integers until some fixed $m$, and suppose that $z_{i_2} = \text{down}^{m+1}_{\lambda^{D,i_1}}(z_{i_1})$. If $\text{up}^x_{\lambda^{D,i_2}}(z_{i_2}) = \text{up}^x_{\lambda^{D,i_1}}(z_{i_2})$ for all $\text{up}^x_{\lambda^{D,i_2}}(z_{i_2}) < z_{i_1}$, then $\text{top}_{\lambda^{D,i_2}}(z_{i_2}) = \text{down}_{\lambda^{D,i_1}}(z_{i_1}) < \text{top}_{\lambda^{D,i_1}}(z_{i_1})$. If instead there exists $x$ such that $\text{up}^x_{\lambda^{D,i_2}}(z_{i_2}) \neq \text{up}^x_{\lambda^{D,i_1}}(z_{i_2})$, then we may produce an exhaustive list $z_{v_1} + c_1 > z_{v_2} + c_2 > \cdots > z_{v_q} + c_q$ such that each $z_{v_e} + c_e \in \text{uppath}_{\lambda^{D,i_1}}(z_{i_2})$ and $c_e \in [h^D_{v_e}]$, and $z_{i_1} < z_{v_q} + c_q$. We now use as notation $y^D_p$ to indicate the $y$-value corresponding to a given $\text{cover}_p(\lambda^{D,p} + \epsilon_{z_p})$. We claim then that $y^D_{i_1} < y^D_{v_q}$. If we suppose first that $z_{i_1-1} + h^D_{i_1-1} + 1 \neq z_{i_1}$, then the claim holds due to the ceiling in columns $\text{down}_{\Psi^{D,i_1}}(z_{i_1}) - 1, \text{down}_{\Psi^{D,i_1}}(z_{i_1})$ of $\Psi^{D,v_q}$. If instead $z_{i_1-1} + h^D_{i_1-1} + 1 = z_{i_1}$, then because $D \in \mathcal{D}$, $y^D_{i_1} < y^D_{i_1-1}$. We have $z_{i_1} - 1 \in \text{uppath}_{\lambda^{D,v_q}}(z_{v_q} + c_q)$, and there is a ceiling in columns $\text{down}_{\lambda^{D,v_q}}(y^D_{i_1-1} + h^D_{i_1-1}), \text{down}_{\lambda^{D,v_q}}(y^D_{i_1-1} + h^D_{i_1-1}) + 1$ of $\lambda^{D,v_q}$, so that $\text{top}_{\lambda^{D,v_q}}(z_{i_1} - 1) > y^D_{i_1-1} + h^D_{i_1-1}$. By a similar token, we leverage Lemma 6.2.5 to claim that for all $e = q-1, q-2, \dots, 1$, we also have $y^D_{i_1} < y^D_{v_e}$. Therefore, if $N = \min\{y^D_{v_q}, \dots, y^D_{v_1}\}$, we have that the uppath of $z_{i_2}$ in $\Psi^{D,i_2}$ weakly above $N$ coincides with that of $z_{i_2}$ in $\Psi^{D,i_1+1}$. In other words, for all $\text{up}^x_{\Psi^{D,i_2}}(z_{i_2}) \leq N$, we have that $\text{up}^x_{\Psi^{D,i_2}}(z_{i_2}) = \text{up}^x_{\Psi^{D,i_1+1}}(z_{i_2})$. Therefore, the argument wherein $\text{up}^x_{\lambda^{D,i_2}}(z_{i_2}) = \text{up}^x_{\lambda^{D,i_1}}(z_{i_2})$ for all $\text{up}^x_{\lambda^{D,i_2}}(z_{i_2}) < z_{i_1}$ applies again.

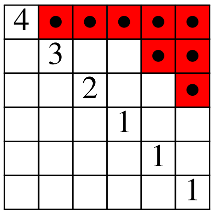
□

EXAMPLE 6.2.9. Let $k = 4$, $\mu = (3, 2, 2, 1)$, and $R_3 = (3, 3)$.




As evidenced by the $k$-Schur root ideal, we have that $4 = \mathrm{down}_{\Delta^k(\mu\cup R_3)}(2)$, $\mathrm{cover}_2(\mu \cup R_3) = (4, 2, 2, 2, 2, 1)$, and $\mathrm{cover}_4(\mathrm{cover}_2(\mu \cup R_3)) = (4, 3, 2, 1, 1, 1)$.

Lemma 6.2.8 guarantees that $3 > 2$, and if $\lambda = \mu \cup R_3$, we have




$$= \tilde{\mathfrak{g}}^{(4)}_{\mathrm{cover}_4(\mathrm{cover}_2(\lambda))} = \tilde{\mathfrak{g}}^{(4)}_{\lambda-\epsilon_{\{2,4\}}}.$$

LEMMA 6.2.10. *If $D = \{z_1, \dots, z_t\} \in \mathcal{D}$ and for some $x \in [t]$, $\lambda_{top_{\Psi^{D,x}}(z_x)} = \lambda_{top_{\Psi^{D,x}}(z_{x+1})}$ and $z_{x+1} \neq down_{\Psi}(z_x)$, then*

$$cover_{z_x}(cover_{z_{x+1}}(\lambda^{D,x} + \epsilon_{z_{x+1}})) = cover_{z_{x+1}}(cover_{z_x}(\lambda^{D,x}) + \epsilon_{z_{x+1}}).$$

PROOF. The statement is clear so long as we have that $\mathrm{uppath}_{\Psi^{D,x}}(z_{x+1})$ excludes every $z_x + c$, $c \in [h^D_x]$ (equivalently, $y_{x+1} \notin \mathrm{uppath}_{\Psi^{D,x}}(z_x + c)$, where $y_{x+1}$ is the $y$-value corresponding to $\mathrm{cover}_{z_{x+1}}(\lambda^{D,x} + \epsilon_{z_{x+1}})$), and $y_{x+1} + v \notin \mathrm{uppath}_{\Psi^{D,x}}(z_x)$ for any $v$ bounded by the $h$-value corresponding to $\mathrm{cover}_{z_{x+1}}(\lambda^{D,x} + \epsilon_{z_{x+1}})$. If indeed there exists $z_x + c \in \mathrm{uppath}_{\Psi^{D,x}}(z_{x+1})$, $c \in [h^D_x]$, then by Lemma 6.2.5, the uppath of each $z_{x+1} + w$, $w \in [h^D_{x+1}]$, is the same within $\Psi^{D,x}$ and $\Psi^{D,x+1}$ weakly above $y_x + h^D_x$, where $y_x$ is the $y$-value corresponding to $\mathrm{cover}_{z_x}(\lambda^{D,x})$. Moreover, this would imply that the $y$-value corresponding to $\mathrm{cover}_{z_{x+1}}(\lambda^{D,x})$ is strictly less than $y_x$, as the uppath of any $z_x + c$ in $\Psi^{D,x}$ must be strictly longer than that of $z_x$ by definition of cover. Therefore, we have that the $y$-value for $\mathrm{cover}_{z_x}(\lambda^{D,x})$ and $\mathrm{cover}_{z_x}(\lambda^{D,x+1})$ also remains the same. On the other hand, we eliminate the possibility that $y_{x+1} + v \notin \mathrm{uppath}_{\Psi^{D,x}}(z_x)$: this would imply that $z_x \in \mathrm{uppath}_{\Psi^{D,x}}(z_{x+1} + v)$ , which is impossible because there is a wall in rows $z_x, z_x + 1$ of $\Psi^{D,x}$ so long as $h_x^D > 0$, and if $h_x^D = 0$, the claim is immediately true.

□

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