Stability and Convergence of Approximate Solutions to the Moore-Gibson-Thompson Equation

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Chapter 1

Physical Origins

1.1 Infinite speed of propagation

Solutions to the familiar heat equation exhibit a property known as infinite speed of propagation. Informally, information from the initial data has an effect on the entire spatial domain instantaneously. This property can be seen through a variety of phenomena.

1.1.1 Maximum principle

Suppose that $u_0 \in L^2(\Omega)$, positive and nonzero but with compact support inside Ω. Then variants of the maximum principle will give that $u(x,t) > 0$ for all $t > 0$, everywhere in Ω!. See for example sources cited in Brezis [2]. (p345)

1.1.2 Fundamental solutions

Suppose that the domain $\Omega = \mathbb{R}^n$. Then it is known that if u_0 is sufficiently well behaved, say compactly supported, we can write the solution to the heat equation as a convolution:

$$u(x,t) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{\frac{-|x-y|^2}{4t}} u_0(y)\, dy$$

for $t > 0$. See for example Kesavan [11] and sources within.

This is more specialized than the example of the maximum principle, but it shows explicitly the dependance of u on u_0. One can see that even a point disturbance in the initial data u_0 has instant effects for all $t > 0$ at all points x.

1.1.3 Analyticity

Although not strictly the same as infinite speed of propagation, the property of analyticity of the semigroup corresponding to the heat equation shows another phenomenon occuring with "infinite speed". Abstractly, if a semigroup $T(t)$ is analytic on a space X, then for all $x \in X$, $T(t)x \in D(A)$ for all $t > 0$, A the generator of the semigroup.

For a simple heat equation $X = L^2(\Omega)$, $A = -\Delta$ with 0 Dirichlet boundary conditions so that $D(A) = H^2(\Omega) \cap H^1_0(\Omega)$. Therefore the analyticity of the semigroup means that the solution is instantly globally smoother than the initial data, which is only required to be in $L^2(\Omega)$.

In fact the smoothing effect is even stronger for analytic semigroups. For any $t > 0$ fix a time ϵ, $0 < \epsilon < t$ and then

$$AT(t)x = AT(\epsilon)T(t-\epsilon)x = T(\epsilon)AT(t-\epsilon)x$$

where the first equality is the semigroup property and the second is the fact that the generator commutes with the semigroup against elements in $D(A)$. But then we see that $T(t-\epsilon)x \in D(A)$ so that $T(\epsilon)A(t-\epsilon)x \in D(A)$, and so $T(t)x \in D(A^2)$. Inductively we find that $T(t)x \in D(A^k)$ for all $k \geq 0$. So the smoothing effect is very strong as well as global in the domain.

1.2 Maxwell-Cattaneo law

The source of the infinite speed of propgation is Fourier's law for heat flux

$$\vec{q} = -K\nabla\theta$$

Where θ is the temperature and $\vec{q}$ is the heat flux. Informally examining this equation suggests that heat conduction begins immediately in the presence of a nonzero temperature gradient. A proposed alternative law that does not lead to infinite speed of propagation is the Maxwell-Cattaneo law

$$\tau\dot{\vec{q}} + \vec{q} = -K\nabla\theta$$

Where the dot indicates a material derivative that has been taken to simply indicate a time derivative [9]. The constant τ is called the relaxation time, a material-specific quantity representing the lag time before heat conduction takes place across neighboring thermal elements. This quantity has been experimentally determined for a variety of materials [5]. See Christov and Jordan [5] for a discussion of this law.

Fourier's law is used in the field of nonlinear acoustics in the derivation of the classical Kuznetsov and Westervelt equation. Substitution of the Maxwell Cattaneo law in for Fourier's law leads to a new equation.

1.3 Physical derivation of the Moore-Gibson-Thompson equation

First we briefly recap the derivation of the Kuznetsov and Westervelt equation, as presented in Kaltenbacher and Lasiecka [1] and [9]. The physical quantities involved in thermo-viscous flow in a compressible fluid are

- The acoustic particle velocity $\vec{v}$ with potential ψ
- The acoustic pressure u
- The mass density ρ
- The temperature θ

- The heat flux q
- And the entropy η.

The relations used are conservation of mass, momentum, and energy:

$$\rho_t + \nabla(\rho\vec{v}) = 0$$

$$\rho(\vec{v}_t + (\vec{v}\cdot\nabla)\vec{v} = -\nabla p + (\frac{4\mu_V}{3} + \zeta_V)\Delta\vec{v}$$

$$\rho\theta(\eta_t + \vec{v}\cdot\nabla\eta) = -\nabla\cdot\vec{q} + \mathbb{T} : \mathbb{D}$$

Where $\mathbb{D} = \frac{1}{2}(\nabla\vec{v} + (\nabla\vec{v})^T)$, $\mathbb{T} = -p\mathbb{I} + 2\mu_V\mathbb{D} + \lambda(\nabla\cdot\vec{v})\mathbb{I}$, : denotes the Frobenius product, or elementwise multiplication. The variable μ_V is the shear viscosity and $\zeta_V = \lambda + \frac{2}{3}\mu_V$ is the bulk viscosity. Also we make use of the state equation: Write the pressure as $p = p_0 + p_\sim$ where $\nabla p_0 = 0$, and the density by $\rho = \rho_0 + \rho_\sim$ with $(\rho_0)_t = 0$ and then the relation is

$$p_\sim = \rho_0 c^2\left(\frac{\rho_\sim}{\rho_0} + \frac{B}{2A}\left(\frac{\rho_\sim}{\rho_0}\right)^2 + \frac{\gamma-1}{\chi c^2}(\eta-\eta_0)\right)$$

The new parameters are c for the speed of sound, B/A the parameter of nonlinearity, χ the coefficient of volume expansion. See [9] for more on the derivation. At this point the application of Fourier's law and neglecting third order and higher terms in the varying quantities of the pressure, velocity, and density would place you at Kuznetsov's equation

$$\psi_{tt} - c^2\Delta\psi - \delta\Delta\psi_t = \left(\frac{B}{2c^2A}(\psi_t)^2 + |\nabla\psi|^2\right)_t$$

The reduction of the Kuznetsov equation to the Westervelt is to neglect the local nonlinear effects of the gradient term, to obtain

$$\psi_{tt} - c^2\Delta\psi - \delta\Delta\psi_t = \left(\frac{1}{c^2}\left(1 + \frac{b}{2A}\right)(\psi_t)^2\right)$$

Where in either equation ψ is the acoustic velocity potential.

Note that the linear Kuznetsov or Westervelt equation has the algebraic structure of a strongly damped wave equation, which is known to correspond to an analytic semigroup on the appropriate space [9]. If we instead use the Maxwell-Cattaneo law and make the analogous reduction from the Kuznetsov to Westervelt equation, we arrive at the equation

$$\tau u_{ttt} + (1 - 2ku)u_{tt} - c^2\Delta u - b\Delta u_t = 2k(u_t)^2$$

Where u is the vairable denoting the acoustic pressure, $k = (1 + \frac{B}{2A})c^{-2}\rho^{-1}$. We can link the pressure and velocity potential via $u = \rho\psi_t$. This equation will be called the Jordan-Moore-Gibson-Thompson-(Westervelt) equation, while its linearization will be refered to as the Moore-Gibson-Thompson equation. As will be seen, this equation does not correspond to an analytic semigroup.

1.3.1 Interpretation of constants

The Jordan-Moore-Gibson-Thompson equation has the constants τ, c, b, k as part of the equation. These constants will have theoretical importance for the well posedness and stability of the PDE. Viewing the equation as a physical model rather than an abstract problem gives additonal meaning to these quantities. As mentioned above, τ represents the thermal relaxation time of the medium. The parameter c is the speed of sound. The constants corresponding to the nonlinearity are all encoded together as k, while $b = \delta + \tau c^2$ where δ is the diffusivity of the sound. In fact we can place these constants in relation to each other by use of the Mach number M via the relation $M^2 c^2 = \frac{\delta}{\tau}$. See [8] for more details.

Chapter 2

The abstract Moore-Gibson-Thompson PDE

2.1 Abstract model

Investigation of the full nonlinear problem will rest on having solid well posedness and stability results for the linearized model. As the linear model will be handled by functional analytic methods we can state the results in greater generality by translating the concrete linear MGT PDE to an abstract operator equation on real Hilbert space.

To that end let A be an operator on a real Hilbert space H satisfying

Assumption 2.1.1.

- *A is unbounded, selfadjoint, and positive.*
- *A is closed and densely defined.*
- *A is elliptic, which implies a Poincaré-type inequality, $\|u\| \leq C \left\|A^{1/2}u\right\|$ for $u \in D(A^{1/2})$.*
- *$0 \in \rho(A)$.*

- *A has compact resolvent.*

Then the abstract linear problem is to find a function $u(t)$, $u(t) \in H$ satisfying

$$\tau u_{ttt} + \alpha u_{tt} + c^2 Au + bAu_t = 0 \tag{2.1.1}$$

With initial conditions $u(0) = u_0, u_t(0) = u_1, u_{tt}(0) = u_2$, and the constants $\tau, \alpha, k, b, c > 0$.

The question of wellposedness of this problem depends critically on which spaces your initial data lie in. In [9, 8] two spaces are considered for wellposedness, while in [13] there are several more. We will make use of these spaces directly when posing our own results, as well as drawing on the wellposedness and stability of the PDE on these spaces. We refer to [13] for a full discussion, but catalog several useful spaces here and then follow with a discussion of wellposedness on them.

2.2 Possible state spaces

With respect to the variables (u, u_t, u_{tt}), [9] establishes the wellposedness of the MGT equation on the spaces $\mathcal{H} = D(A^{1/2}) \times D(A^{1/2}) \times H$ and $\mathcal{H}_1 = D(A) \times D(A^{1/2}) \times H$. To avoid ambiguity, I will refer to spaces from [13] with a hat, as both papers adopt the convention of using $\mathcal{H}$ with various subscripts to denote the state spaces.

Model 2 in [13] stems from a selection of variables (z, z_t, u), where $z = \frac{c^2}{b}u + u_t$. In these variables, the space $\hat{\mathcal{H}}_1 = D(A^{1/2}) \times H \times D(A^{1/2})$ is considered. However

$(z, z_t, u) \in \hat{\mathcal{H}}_1$ is equivalent to $(u, u_t, u_{tt}) \in D(A^{1/2}) \times D(A^{1/2}) \times H$: $\frac{c^2}{b}u + u_t \in D(A^{1/2})$ and $u \in D(A^{1/2})$ implies that $u_t \in D(A^{1/2})$, $\frac{c^2}{b}u_t + u_{tt} \in H$ and $u_t \in D(A^{1/2})$ implies that $u_{tt} \in H$, and vice versa.

There is a weaker space considered in [13], again with respect to (z, z_t, u), $\hat{\mathcal{H}}_0 = D(A^{1/2}) \times H \times H$. This space is not as natural when translated into the (u, u_t, u_{tt}) variables. We are given that $u \in H$, but then this only allows us to infer from $z \in D(A^{1/2})$ that $u_t \in H$ and therefore as $z_t \in H$ we also have $u_{tt} \in H$. But this process does not reverse; clearly u and u_t in H will not yield $z \in D(A^{1/2})$.

The space $\mathcal{H}_1$ used in [9] has a partner in [13], which is the space $\hat{\mathcal{H}}_3 = D(A^{1/2}) \times H \times D(A)$. We are given here $u \in D(A)$ and thus $u_t \in D(A^{1/2})$ from the first coordinate, and then $u_{tt} \in H$ from the second. This argument does reverse and these spaces are equivalent.

In addition there is a useful stronger space considered in [13], the space $\hat{\mathcal{H}}_2 = D(A) \times D(A^{1/2}) \times D(A)$ (again with respect to the variables z, z_t, u). This is equivalent to the original variables $(u, u_t, u_{tt}) \in D(A) \times D(A) \times D(A^{1/2})$. We will see that this is just the domain of the semigroup generator on the space $\mathcal{H}_1 = \hat{\mathcal{H}}_3$.

For reasons of spatial smoothness and time differentiability it is useful to know the domains of the semigroup generator on these spaces. However although these operators have smoothing effects on the z and z_t components, unfortunately they do not smooth the u component and therefore quickly saturate with respect to the smoothness of the original variables.

On the space $\mathcal{H} = \hat{\mathcal{H}}_1$, modulo bounded or compact perturbations, the generator $\mathcal{A}$ has the structure

$$\begin{pmatrix} 0 & I & 0 \\ -\frac{b}{\tau}A & -\gamma I & 0 \\ 0 & 0 & \frac{c^2}{b}I \end{pmatrix}$$

For $\gamma = \alpha - \frac{\tau c^2}{b}$.

The domains of the generator on these spaces will be briefly computed in the sections to follow, and then the results summarized afterwards.

2.2.1 Domain on $\mathcal{H} = \hat{\mathcal{H}}_1$

Thus on the space $\hat{\mathcal{H}}_1$ we have domain determined by the conditions

- $(z_1, z_2, u) \in \hat{\mathcal{H}}_1 = D(A^{1/2}) \times H \times D(A^{1/2})$.
- $z_2 \in D(A^{1/2})$.
- $-Az_1 - \gamma z_2 \in H$ hence since $z_2 \in H$, $z_1 \in D(A)$.

- $u \in D(A^{1/2})$.

Thus we see that the domain of the generator $\mathcal{A}$ on $\hat{\mathcal{H}}_1$ is exactly $D(A) \times D(A^{1/2}) \times D(A^{1/2})$. The domain of $\mathcal{A}^2$ is similarly determined.

- $(z_1, z_2, u) \in D(\mathcal{A}) = D(A) \times D(A^{1/2}) \times D(A^{1/2})$.
- $z_2 \in D(A)$.
- $-Az_1 - \gamma z_2 \in D(A^{1/2})$ hence since $z_2 \in D(A^{1/2})$ this is equivalent to $z_1 \in D(A^{3/2})$.
- $u \in D(A^{1/2})$.

And so on, so that $D(\mathcal{A}^k) = D(A^{\frac{1+k}{2}}) \times D(A^{k/2}) \times D(A^{1/2})$. Note that this gives limited smoothness in the original u variables. As u never goes beyond $D(A^{1/2})$ we find that no other component does either, although we will get u, u_t, u_{tt} all in $D(A^{1/2})$ for $k \geq 1$.

2.2.2 Domain on $\mathcal{H}_1 = \hat{\mathcal{H}}_3$

In this case we will do computations directly in the original variables. Recall the space $\hat{\mathcal{H}}_3$ is equivalent to the original variable space H_2. The generator here, unperturbed, is given by

$$\mathcal{A} = \begin{pmatrix} 0 & I & 0 \\ 0 & 0 & I \\ -\frac{c^2}{\tau}A & -\frac{b}{\tau}A & \frac{\alpha}{\tau}I \end{pmatrix}$$

With domain

- $(u_1, u_2, u_3) \in \mathcal{H}_3 = D(A) \times D(A^{1/2}) \times H$.
- $u_2 \in D(A)$.
- $u_3 \in D(A^{1/2})$.
- $-c^2Au_1 - bAu_2 - \alpha u_3 \in H$. But we already have $u_3 \in H$, and $u_2 \in D(A)$, and $u_1 \in D(A)$, so this condition does not provide or require any new membership.

Thus $D(\mathcal{A}) = D(A) \times D(A) \times D(A^{1/2})$. Consider now the domain of the square.

- $(u_1, u_2, u_3) \in D(\mathcal{A}) = D(A) \times D(A) \times D(A^{1/2})$.
- $u_2 \in D(A)$.
- $u_3 \in D(A)$.
- $-c^2Au_1 - bAu_2 - \alpha u_3 \in D(A^{1/2})$ but this time we know only $u_3 \in D(A^{1/2})$ so we can only draw out that $-A(c^2u_1 + bu_2) \in D(A^{1/2})$, so $z \in D(A^{3/2})$.

This gives $D(\mathcal{A}^2) = \{(u_1, u_2, u_3) \in D(A) \times D(A) \times D(A) \mid (c^2u_1 + bu_2) \in D(A^{3/2})\}$. At this point we see that taking a second power has improved our situation by smoothing

u_{tt} but taking further powers will provide no further smoothing in the u variables. Taking one more power for illustration,

- $(u_1, u_2, u_3) \in D(\mathcal{A}^2) = D(A) \times D(A) \times D(A)$ such that $z_1 = \frac{c^2}{b} u_1 + u_2 \in D(A^{3/2})$.
- $u_2 \in D(A)$
- $u_3 \in D(A)$
- $-\frac{c^2}{\tau} A u_1 - \frac{b}{\tau} A u_2 - \frac{\alpha}{\tau} u_3 \in D(A)$. This last condition is thus $z_1 \in D(A^2)$.
- $\frac{c_2}{\tau} u_2 + u_3 = z_2 \in D(A^{3/2})$.

So we see the condition on z_1 is elevated to a condition on z_2, which lags 1/2 power behind. In summary,

$$D(\mathcal{A}^k) = \{(u_1, u_2, u_3) \in D(A) \times D(A) \times D(A) \mid z_1 \in D(A^{\frac{k+1}{2}}), z_2 \in D(A^{\frac{k}{2}})\}$$

Observe that $\hat{\mathcal{H}}_3 = D(\mathcal{A})$.

2.2.3 Summary

Here we briefly collect the information above on the definition of three possible state spaces for the problem, the domain of the generator in each space in the appropriate variables, and the consequences for the original variables (u, u_t, u_{tt}).

Space	Definition	Domain of generator $\mathcal{A}^k$
$\mathcal{H}^1$	$(z, z_t, u) \in D(A^{1/2}) \times H \times D(A^{1/2})$ $(u, u_t, u_{tt}) \in D(A^{1/2}) \times D(A^{1/2}) \times H$	$(z, z_t, u) \in D(A^{\frac{1+k}{2}}) \times D(A^{k/2}) \times D(A^{1/2})$ $\implies (u, u_t, u_{tt}) \in D(A^{1/2}) \times D(A^{1/2}) \times D(A^{1/2})$
${\mathcal{H}_1}^2$	$(u, u_t, u_{tt}) \in D(A) \times D(A^{1/2}) \times H$	$k = 1: (u, u_t, u_{tt}) \in D(A) \times D(A) \times D(A^{1/2})$ $k \geq 2: D(A)^3 : z_1 \in D(A^{\frac{1+k}{2}}), z_2 \in D(A^{\frac{k}{2}})$

We also will frequently have occasion to refer to a lower and higher energy corresponding to solutions of (2.1.1), which are topologically equivalent to the norms on the state spaces $\mathcal{H}$ and $\mathcal{H}_1$:

$$
\begin{aligned}
E_u(T) &= \frac{\tau}{2}\left\|u_{tt}(T) + \frac{c^2}{b}u_t(T)\right\|^2 + \frac{c^4}{2b}\left\|A^{1/2}u(T) + \frac{b}{c^2}A^{1/2}u_t(T)\right\|^2 \\
&\quad + \left(\frac{c^2\alpha}{2b} - \frac{c^4\tau}{2b^2}\right)\|u_t(T)\|^2 + \left\|A^{1/2}u(T)\right\|^2 \\
E_{1,u}(T) &= E_u(T) + \|Au(T)\|^2
\end{aligned}
$$

2.3 Wellposedness and stability

Although we phrased the abstract problem (2.1.1) with all parameters stricly positive we will discuss for a moment the implications each parameter's existence has on wellposedness. As briefly mentioned before, if $\tau = 0$ then we are in the case of an

[1]This is $\mathcal{H}_1$ in [13]

[2]This is $\mathcal{H}_3$ in [13]

abstract strongly damped wave equation, however $\tau > 0$ is exactly what corresponds to making use of the Maxwell-Cattaneo law rather than Fourier's. If $c = 0$ then by change of variable to $w = u_t$ we are led to consider $\tau w_{tt} + \alpha w_t + bAw$, or simply a damped wave equation. The constant α is not critical for wellposedness and could potentially take on any value. However $b \neq 0$ is critical for the wellposedness of the abstract problem [9]. This makes a contrast with the $\tau = 0$ case, where b is responsible for exponential stablilty. Here $b > 0$ will grant wellposedness and stability will come from a balancing of the parameters of the equation, as will be seen.

We now state a collection of results on the linear model.

Theorem 2.3.1. *Suppose that $b, \tau, c > 0$ and $\alpha \in \mathbb{R}$. Then the abstract MGT equation (2.1.1) generates a strongly continuous semigroup (in fact a group) on any of the spaces $\mathcal{H}_i$, $i = 1, 2$.*

This is proved for $i = 1, 2$ in [9] and extended to other spaces as well, for reference, in [13]. By energy methods in [9] and by spectral computations in [13] this result is extended to exponential stability.

Theorem 2.3.2. *Suppose that $b, \tau, c > 0$ and also that $\gamma = \alpha - \frac{\tau c^2}{b} > 0$. Then the semigroup $T_i(t)$, $i = 1, 3$ on the respective spaces is exponentially stable, in other words there exists $C > 0$, $\omega > 0$ such that $\|T_i(t)\|^2_{\mathcal{L}(\hat{\mathcal{H}}_i)} \leq Ce^{-\omega t}$.*

We should comment specifically on the nature of the stability analysis in [9] and

[8] as it will be critical at several steps in this work. When we say vaguely that these results follow from energy methods what we specifically mean is that they follow from multiplying the PDE by suitable multipliers (u, u_t, etc) and performing calculus to extract information on the behavior of the norm of the solution in certain spaces. This process can be vindicated from the get-go by semigroup well-posedness, giving existence of the multipliers in an appropriate space to form inner products in, or by stipulating a weak solution - which then will later be evidently a nontrivial definition by virtue of semigroup solutions. However in either case the calculus is done with a function u that has three time derivatives. A mild solution (homogeneous or nonhomogeneous) coming from semigroup theory will not be that regular. This problem is easily overcome by performing the calculations on classical solutions, and then using a density argument. The details of this process are laid out for completeness in the next lemma.

Lemma 2.3.3. *Suppose that for a semigroup $T(t)$ on a space H with generator A and (necessarily) dense domain $D(A)$, we have the bounds,*

$$\|T(t)x\| \leq C\,\|x\| \;\; \textit{for } x \in D(A)$$

Then in fact,

$$\|T(t)x\| \leq C\,\|x\| \;\; \textit{for } x \in H$$

Proof. Pick $x \in H$ and fix $\epsilon > 0$. Since $D(A)$ is dense in H pick a subsequence $x_n \in D(A)$ converging to x. The standard semigroup growth bound says that $T(t)x \leq$

$Me^{\omega t}\|x\|$, for some $M > 1, \omega > 0$. Therefore at the initial time we find $\|x_n\| \leq \|x\| + \|x_n - x\|$, while at the later time we find

$$\begin{aligned}\|T(t)x\| &\leq \|T(t)x_n\| + \|T(t)(x_n - x)\| \\ &\leq C\|x_n\| + Me^{\omega t}\|x_n - x\| \\ &\leq C(\|x\| + \|x_n - x\|) + Me^{\omega t}\|x_n - x\|\end{aligned}$$

Therefore if we pick n such that $\|x_n - x\| \leq \frac{\epsilon}{2\max\{Me^{\omega t}, C\}}$ we will find that $\|T(t)x\| \leq C\|x\| + \epsilon$. $\square$

2.3.1 Comparison to other results

We comment briefly on the treatment of this problem given in [4]. In Theorem 3.10 there, well-posedness of mild solutions to the nonhomogeneous problem is given by the following mapping: Initial data in $D(A) \times D(A^{1/2}) \times H$ and forcing term $f \in L^1([0,\infty); H)$ yield a mild solution $u \in C^1([0,\infty), D(A^{1/2})) \cap C^2([0,\infty), H)$, with energy boundedness measured

$$\|u_{tt}(t)\| + \left\|A^{1/2}u_t(t)\right\| + \left\|A^{1/2}u(t)\right\| \leq C(\|f\|_1 + \|Au_0\| + \left\|A^{1/2}u_1\right\| + \|u_2\|)$$

The semigroup approach in [13], [9], and [8] yields better well-posedness, with the same stability results - there will be no loss of regularity from $u_0 \in D(A)$ to $u(t)$ only in $D(A^{1/2})$. Under the assumptions given above we have the initial data on the space $\mathcal{H}_1$. The forcing term acts on the third component of the space, so to that end let

$F(t) = (0, 0, f(t)) \in L^1([0,\infty), \mathcal{H}_1)$. Denoting U for the vector (u, u_t, u_{tt}) and $T(t)$ for the semigroup generated by $\mathcal{A}$, the mild solution given by

$$U(t) = T(t)U_0 + \int_0^\infty T(t-s)F(s)\,ds$$

has regularity $U(T) \in \mathcal{H}_1$ (see e.g, Def. 4.2.3 in [12]), and thus $u(t) \in C([0,\infty); D(A)) \cap C^1([0,\infty); D(A^{1/2})) \cap C^2([0,\infty), H)$. The stability result follows immediately the in the same topology simply by boundedness of the energy, $\|T(t)x\| \leq C\,\|x\|$.

In the case of Theorem 3.11 from [4], the results from [13], [9], and [8] obtain the same regularity but with fewer assumptions. To summarize: [4] requires initial data in $D(A^{3/2}) \times D(A) \times D(A^{1/2})$, and $f \in L^1([0,\infty); D(A^{1/2})) \cap C([0,\infty); H)$, and gives a solution $u \in C^3([0,\infty), H) \cap C^2([0,\infty), D(A^{1/2})) \cap C^1([0,\infty)D(A))$. To contrast, assume in the formulation above only that the initial data are in $D(\mathcal{A})$ on $\mathcal{H}_1$, $D(A) \times D(A) \times D(A^{1/2})$, and use the same assumption on f acting in the third coordinate so that $F(t) = (0, 0, f(t)) \in D(\mathcal{A})$. We check the hypotheses of Cor. 4.2.6 from [12]: For any $T > 0$,

- $F \in L^1([0,T); \mathcal{H}_1)$ and is $C([0,T); H)$.
- $F(t) \in D(\mathcal{A})$ for each $0 < t < T$.
- $\mathcal{A}F(t) = (0, f(t), \frac{-\alpha}{\tau} f(t)) \in L^1([0,T); \mathcal{H}_1)$.

Therefore for $U_0 \in D(\mathcal{A})$ we have (in the language of [12]) a classical solution with regularity $U \in C^1((0,T);\mathcal{H}_1)$ and $U(t) \in D(\mathcal{A})$ for $0 < t < T$. This means $u \in C^3([0,T);H) \cap C^2([0,T);D(A^{1/2})) \cap C^1([0,T);D(A))$.

2.3.2 Influence of b

In the results above it was explicit that $b > 0$. Consider the equation when $b = 0$.

$$\tau u_{ttt} + \alpha u_{tt} + c^2 A u = 0$$

This equation is not well-posed. For an informal argument, consider the would-be semigroup generator,

$$\begin{pmatrix} 0 & I & 0 \\ 0 & 0 & I \\ -\frac{c^2}{\tau}A & 0 & \frac{-\alpha}{\tau}I \end{pmatrix}$$

And note that the $-\alpha I$ entry in the $(3,3)$ coordinate is bounded regardless of what state space is selected. Therefore modulo a bounded perturbation we would also have that

$$\begin{pmatrix} 0 & I & 0 \\ 0 & 0 & I \\ \frac{-c^2}{\tau}A & 0 & 0 \end{pmatrix}$$

is a semigroup generator. However the assocated differential equation would then be

$$\tau u_{ttt} + c^2 A u = 0$$

It is a result of Fattorini [7] that this third-order equation is not well-posed unless the operator A is bounded. Thus the perturbed matrix is not a generator, and so the original cannot be either. This argument is only an informal one because one would have to be more specific about the admissable state spaces that we assume generation on - of course with respect to absurd spaces the operator IS bounded! We mention a result from [9] on this issue.

Theorem 2.3.4. *Let $b = 0$. Then the equation is not well-posed on $\mathcal{H}$.*

Proof. Let A have eigenvectors ϕ_n and eigenvalues $\nu_n > 0$ such that $\nu_n \to \infty$. This is a consequence of our assumptions on A, namely that it is positive and unbounded. Consider the perturbed generator again,

$$\mathcal{B} = \begin{pmatrix} 0 & I & 0 \\ 0 & 0 & I \\ \frac{-c^2}{\tau}A & 0 & 0 \end{pmatrix}$$

Consider the eigenvalue equation for $\mathcal{B}$,

$$\begin{pmatrix} 0 & I & 0 \\ 0 & 0 & I \\ \frac{-c^2}{\tau}A & 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} \lambda v_1 \\ \lambda v_2 \\ \lambda v_3 \end{pmatrix}$$

So that

$$v_2 = \lambda v_1$$

$$v_3 = \lambda v_2$$

$$\frac{-c^2}{\tau} A v_1 = \lambda v_3$$

Hence $Av_1 = \lambda^3 \frac{-\tau}{c^2} v_1$ indicates that $\lambda^3 \frac{-\tau}{c^2}$ is an eigenvalue of A, so $\lambda^3 = \frac{-c^2}{\tau}\nu_n$. Since ν_n is real and c^2, τ are positive, this is a cube root of a negative collection of values. This means the roots will contain one branch on the negative real axis, and two conjugate branches with positive real part. As ν_n are unbounded, the real part of the cube roots will be unbounded as well. This violates a necessary conditon for generation of a semigroup [12]. □

2.4 Discussion of assumptions on A

The abstract assumptions we have made on the operator A are suitable generalizations of the physical model under consideration. The action of A under consideration originally was $Au = -\Delta u$ with the function u defined on a bounded domain Ω which we assume to have C^2-smooth boundary. If we impose 0-Dirichlet boundary conditions then such A would be self-adjoint and strictly positive, whereas with a Neumann boundary condition it would only be non-negative.

The assumption of compact resolvent is a reasonable assumption on a elliptic operator, essentially as a consequence of the Sobolev embedding theorem. To sketch the argument, let A to be a Dirichlet Laplacian as above. Consider a bounded set

S_M so that $f \in S_M \implies \|f\|^2_{L^2(\Omega)} \leq M$. For such an f we have

$$\begin{aligned} A^{-1}f = u \Leftrightarrow f = Au \\ &\implies \|u\|^2_{H^2(\Omega)} \leq \|f\|^2_{L^2(\Omega)} \\ &\implies \|u\|^2_{H^2(\Omega)} \leq M \end{aligned}$$

There in the second line we have used a regularity theorem for the Dirichlet Laplacian. Then this computation shows that the image $A^{-1}S_M$ is a bounded set in $H^2(\Omega)$. But then $H^2(\Omega)$ includes boundedly into $H^1(\Omega)$ and $H^1(\Omega)$ is compact in $L^2(\Omega)$ (regardless of the dimension of Ω) by the Rellich-Kondrachov theorem. Thus the set $A^{-1}S_M$ is compact in $L^2(\Omega)$, and so A^{-1} is a compact operator.

Further note that the language “compact resolvent” is not ambiguous with respect to which particular resolvent we choose. By the resolvent identity, if $\lambda, \omega \in \rho(A)$

$$R(\omega, A) = (\lambda - \omega)R(\lambda, A)R(\omega, A) + R(\lambda, A)$$

and so we see that if $R(\lambda, A)$ is compact then so is $R(\omega, A)$ for any $\omega \in \rho(A)$.

2.5 Nonlinear well posedness

We now summarize the results from [8] on the nonlinear concrete full nonlinear JMGT equation. The discussion to follow will depend on a suitable definition of a mild solution, as well as evolution operators. The essential method of argument is to consider the factor $(1-2ku)$ as an operator applied to u_{tt}, $\alpha(t)u_t t$. One can see that this leads to a slightly different creature from a semigroup, as time is not autonomous - the mapping $\alpha(t)$ cares specifically on what time it is, rather than simply how long the process has been in motion. A thorough study of mild solutions to the problem $\tau u_{ttt} + \alpha(t)u_{tt} + b\Delta u_t + c^2\Delta u = f$ will lead to construction of solutions to (2.5.1) by a fixed point procedure. Full detail on this method is contained in [8].

We are interested in the equation,

$$\tau u_{ttt} + (1-2ku)u_{tt} - c^2\Delta u - b\Delta u_t = 2k(u_t)^2 \text{ in } \Omega \times (0,T) \tag{2.5.1}$$

with Ω a C^2-smooth bounded domain in $\mathbb{R}^d$, $1 \leq d \leq 3$, homogeneous boundary conditions $u = 0$ on $\partial\Omega$, and initial conditions $u(0) = u_0, u_t(0) = u_1, u_{tt}(0) = u_2$. We can link this to the operator A considered in the abstract modelling of the linear problem by $A = -\Delta$ with 0 Dirichlet boundary conditions on $L^2(\Omega)$, so that

- $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$.
- $D(A^{1/2}) = H_0^1(\Omega)$.
- $\mathcal{H}_1 = D(A) \times D(A^{1/2}) \times H = (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega) \times L^2(\Omega)$
- Make use of an energy function $E_{1,u}(t)^2 \approx \|u\|^2_{H^2(\Omega)} + \|u_t\|^2_{H_0^1(\Omega)} + \|u_{tt}\|^2_{L^2(\Omega)}$.

A mild solution to this equation is a function $U(t) = [u(t), u_t(t), u_{tt}(t)]$ satisfying

- $U \in C([0,T]; H_1)$

- U satisfies the integral form of (2.5.1),

$$U(t) = U_1^{\alpha(u)}(t,0)U(0) + \int_0^t U_1^{\alpha(u)} D^U(s)\, ds$$

Where $D^U(s) = [0, 0, 2k(u_t(s))^2]$ and $U_1^{\alpha(u)}$ is the evolution operator corresponding to the linear homogenous evolution problem

$$\tau u_{ttt} + \alpha u_{tt} + b\Delta u_t + c^2 \Delta u = 0$$

where α is a *operator* (possibly, importantly, a multiplier) rather than a constant.

Theorem 2.5.1. *Let $\delta > 0$ and $\gamma_* = 1 - \frac{\tau c^2}{b}$.*

Regardless of γ_*, for any $T > 0$ there exist a ball such that if the initial data is inside this ball in the $\mathcal{H}_1$ norm then there is a unique mild solution on $[0, T]$, depending continuously on the initial data. Alternately, for any initial data in H_1 there exists a particular time interval on which the solution exists.

If in addition $\gamma_* > 0$ then the solutions exist globally in time: for any $C > 0$ there is a $\rho_C > 0$ such that if $E_{1,u}(0) \leq \rho_C$ then a mild solution U exists for all t and $E_{1,u}(t) \leq C\, \forall t > 0$.

Further global solutions are exponentially stable: If $E_{1,u}(0) \leq \rho_C$ then there exists $\omega > 0$, $C_1 > 0$ such that $E_{1,u}(t) \leq C_1 e^{-\omega t}$ for all $t > 0$.

If one further assumes that $U(0) \in \mathcal{D} = D(A) \times D(A) \times D(A^{1/2})$ *then the mild solutions are in fact classical - i.e. are* $C^1((0,T); H_1) \cap C([0,T]; \mathcal{D})$ *satisfying (2.5.1) for each* t.

We note that the condition $\delta > 0$ is automatically satisfied if $b, \tau > 0$ and therefore is analogous to the requirement that $b > 0$ for the linear problem.

2.6 Finite element results

In chapter 4 we provide a finite element formulation for the linear homogeneous Moore-Gibson-Thompson equation. We summarize briefly some results on time-stability and convergence of this method, see the chapter for full details and proofs.

Theorem 2.6.1. *Let* Ω *be a bounded domain with smooth boundary, and consider the operator* $-\Delta$ *with* 0*-Dirichlet boundary conditions. Let* S_h *be a chosen collection of finite element subspaces of* $H_0^1(\Omega)$*,* h *the discretization parameter going to* 0*. Let* u_n *be the finite element solution corresponding to* $n = n(h)$ *number of elements at this discretization step. Let* $v_{h,i}$*,* $i = 0, 1, 2$ *be the chosen representations of the initial data in* S_h*, and let* $U_0 = (u_0, u_1, u_2)$*. Suppose the physical constants* α, τ, b, c^2 *are all positive. Then,*

1. *The solution* $(u_n(t), u_{n,t}(t), u_{n,tt}(t))$ *is bounded by the projections of the initial*

data,

$$\|(u_n(t), u_{n,t}(t), u_{n,tt}(t))\|_{\mathcal{H}} \leq C \|(v_{h,0}, v_{h,1}, v_{h,2})\|_{\mathcal{H}}$$

2. *If $\gamma = \alpha - \frac{\tau c^2}{b} > 0$ they in fact exponentially decay, for some $C > 0$ $\omega > 0$,*

$$\|(u_n(t), u_{n,t}(t), u_{n,tt}(t))\|_{\mathcal{H}} \leq Ce^{-\omega t} \|(v_{h,0}, v_{h,1}, v_{h,2})\|_{\mathcal{H}}$$

3. *If the order of accuracy r of the finite element spaces S_h obeys $r \geq 2$, and if the initial data are approximated to order 2 by $v_{h,i}$ as well, and if the initial data for the continuous problem obeys $U_0 \in D(\mathcal{A}^2)$, then for each $T > 0$ there exists a $C_T > 0$ such that for $0 \leq s \leq T$*

$$\left\|A^{1/2}(u_n(s) - u(s))\right\| + \left\|A^{1/2}(u_{n,t}(s) - u_t(s))\right\| + \|u_{n,tt}(s) - u_{tt}(s)\| \leq C_T h$$

Where C_T includes the measurement of U_0 in $D(\mathcal{A}^2)$.

2.7 Spectral method results

In chapter 5 a method of computing approximate solutions by use of the eigenvectors of the differential operator A - for example a Laplacian - is put forward. The analysis there proves the stability of the finite-dimensional solutions presented, as well as then proving convergence to the continuous solution given sufficiently smooth initial data. The results are summarized coarsely here, see chapter 5 for full details and proofs.

Theorem 2.7.1. *Suppose A is an operator satisfying assumptions 2.1.1. Let u_n be the approximate solution to the abstract linear homogeneous Moore-Gibson-Thompson* *equation arising from the spectral method. Let* $U_0 = (u_0, u_1, u_2)$ *be the inital data for the continuous problem. Let* $P_n : H \to H$ *be the projection of an element in* H *down to the subspace spanned by the first* n *eigenvectors of* A*. Given the parameters* $b \neq 0$, $\tau, \alpha, c^2 > 0$, *then*

1. *The solution* $(u_n(t), u_{n,t}(t), u_{n,tt}(t))$ *is bounded by the projection of the initial data*

$$\|(u_n(t), u_{n,t}(t), u_{n,tt}(t))\|_{\mathcal{H}} \leq C \|(P_n u_0, P_n u_1, P_n u_2)\|_{\mathcal{H}}$$

$$\|(u_n(t), u_{n,t}(t), u_{n,tt}(t))\|_{\mathcal{H}_1} \leq C \|(P_n u_0, P_n u_1, P_n u_2)\|_{\mathcal{H}_1}$$

2. *If* $\gamma = \alpha - \frac{\tau c^2}{b} > 0$, $(u_n, u_{n,t}, u_{n,tt})$ *is exponentially stable on* $\mathcal{H}$ *or* $\mathcal{H}_1$, *provided the initial data* U_0 *is in* $\mathcal{H}$ *or* $\mathcal{H}_1$ *respectively.*

3. *If* $U_0 \in D(\mathcal{A})$ *on* $\mathcal{H}_1$ *then the approximate solution converges to the continuous solution in the sense that*

$$\left\|A^{1/2}(u(t) - u_n(t))\right\|^2 + \left\|A^{1/2}(u_t(t) - u_{n,t}(t))\right\|^2 + \|u_{tt}(t) - u_{n,tt}(t)\|^2 \leq C \frac{1}{\lambda_{n+1}} \left(\|Au_0\|^2 + \|Au_1\|^2 + \left\|A^{1/2}u_2\right\|^2\right)$$

These results are illustrated in chapter 6.

Chapter 3

Stability of weak solutions to the abstract PDE

3.1 Abstract stability inequalities

The computational methods, both finite element and spectral, considered in this work are based on a weak formulation of the concrete or abstract PDE in question. Although they differ substantially in details the basic methodology is to obtain a solution that satisfies the weak formulation when pulling test functions only from a finite dimensional subspace of $D(A^{1/2})$. Stability as computed in [13] is based on direct spectral and operator-theoretic calculations, whereas in [9] and [8] energy methods are employed. Energy methods are particularly well-suited for application to weak solutions, and therefore we can apply similar calculations, posed abstractly in terms of the operator A, in the context of a finite dimensional solution. This exact inequality will be used several times to come.

Theorem 3.1.1. *Let $V \subseteq D(A^{1/2})$, and let $u(t)$ be such that $(u, u_t, u_{tt}) \in V \times V \times H$ for all $t \in [0, T]$, and suppose also that u solves the weak form of the abstract Moore-*

Gibson-Thompson equation on V: For all $\phi \in V$, for all $t \in [0,T]$,

$$\tau(u_{ttt},\phi) + \alpha(u_{tt},\phi) + b(A^{1/2}u_t, A^{1/2}\phi) + c^2(A^{1/2}u, A^{1/2}\phi) = 0$$

Let

$$\hat{E}(T) = \frac{\tau}{2}\left\|u_{tt}(T) + \frac{c^2}{b}u_t(T)\right\|^2 + \frac{b}{2}\left\|\frac{c^2}{b}A^{1/2}u(T) + A^{1/2}u_t(T)\right\|^2 + \left(\frac{c^2\alpha}{2b} - \frac{c^4\tau}{2b^2}\right)\|u_t(T)\|^2$$

And let

$$\hat{E}_0(T) = \left\|A^{1/2}u(T)\right\|^2$$

For $E(T) = \hat{E}(T) + \hat{E}_0(T)$ we have that

$$E(T) \leq CE(0)$$

And in fact,

$$E(T) + C\int_0^T E(s)\ ds \leq C_1E(0)$$

Remark 3.1.1. *The quantities in the expression for the energy are somewhat obfuscated topologically but are algebraically convenient. Examination reveals that $\sqrt{E}$ is norm equivalent to $\sqrt{\|A^{1/2}u\|^2 + \|A^{1/2}u_t\|^2 + \|u_{tt}\|^2}$. The additon of $\hat{E}_0$ is necessary for the equivalance, otherwise there is a lack of control of u in $D(A^{1/2})$.*

Proof. We will begin with the boundedness of the energy. Multiplying by u_{tt} and integrating by parts we have

$$\tau(u_{ttt},u_{tt}) + \alpha(u_{tt},u_{tt}) + c^2(A^{1/2}u, A^{1/2}u_{tt}) + b(A^{1/2}u_t, A^{1/2}u_{tt}) = 0$$

Rewrite the first and last terms as a derivative to get

$$\frac{d}{dt}\left[\frac{\tau}{2}\left\|u_{tt}\right\|^2+\frac{b}{2}\left\|A^{1/2}u_t\right\|^2+c^2(A^{1/2}u,A^{1/2}u_t)\right]+\alpha\left\|u_{tt}\right\|^2-c^2\left\|A^{1/2}u_t\right\|^2=0 \quad (3.1.1)$$

Multiply by u_t to obtain

$$\frac{d}{dt}\left[\tau(u_{tt},u_t)+\frac{c^2}{2}\left\|A^{1/2}u\right\|^2+\frac{\alpha}{2}\left\|u_t\right\|^2\right]+b\left\|A^{1/2}u_t\right\|^2-\tau\left\|u_{tt}\right\|^2=0 \quad (3.1.2)$$

where we have made use of the fact that $\frac{d}{dt}(u_{tt},u_t)=(u_{ttt},u_t)+\left\|u_{tt}\right\|^2$.

Scale (3.1.2) by $\frac{c^2}{b}$ and add to (3.1.1) to obtain

$$\frac{d}{dt}\left[\frac{c^2\tau}{b}(u_{tt},u_t)+\frac{c^4}{2b}\left\|A^{1/2}u\right\|^2+\frac{c^2\alpha}{2b}\left\|u_t\right\|^2+\frac{\tau}{2}\left\|u_{tt}\right\|^2+\frac{b}{2}\left\|A^{1/2}u_t\right\|^2+c^2(A^{1/2}u,A^{1/2}u_t)\right]$$
$$+\left(\alpha-\frac{\tau c^2}{b}\right)\left\|u_{tt}\right\|^2=0$$

(3.1.3)

We have required that $\alpha-\frac{\tau c^2}{b}$ is positive, but there are two troublesome inner products inside the time derivative. With an eye towards doing away with them, note that

$$\frac{b}{2}\left\|\frac{c^2}{b}A^{1/2}u+A^{1/2}u_t\right\|^2=\frac{c^4}{2b}\left\|A^{1/2}u\right\|^2+\frac{b}{2}\left\|A^{1/2}u_t\right\|^2+c^2(A^{1/2}u,A^{1/2}u_t)$$

and

$$\frac{\tau}{2}\left\|u_{tt}+\frac{c^2}{b}u_t\right\|^2=\frac{\tau}{2}\left\|u_{tt}\right\|^2+\frac{c^2\tau}{b}(u_{tt},u_t)+\frac{c^4\tau}{2b^2}\left\|u_t\right\|^2$$

Making substitutions we arrive at

$$\frac{d}{dt}\left[\frac{\tau}{2}\left\|u_{tt}+\frac{c^2}{b}u_t\right\|^2+\frac{b}{2}\left\|\frac{c^2}{b}A^{1/2}A^{1/2}u+u_t\right\|^2+\left(\frac{c^2\alpha}{2b}-\frac{c^4\tau}{2b^2}\right)\left\|u_t\right\|^2\right]+\gamma\left\|u_{tt}\right\|^2=0$$

(3.1.4)

Therefore, integrating on $[0,t]$ we have

$$\left[\frac{\tau}{2}\left\|u_{tt}(t)+\frac{c^2}{b}u_t(t)\right\|^2+\frac{b}{2}\left\|\frac{c^2}{b}A^{1/2}u(t)+A^{1/2}u_t(t)\right\|^2+\frac{c^2}{2b}\gamma\left\|u_t(t)\right\|^2\right]$$
$$+\gamma\int_0^t\|u_{tt}(s)\|^2\,ds=\left[\frac{\tau}{2}\left\|u_{tt}(0)+\frac{c^2}{b}u_t(0)\right\|^2+\frac{b}{2}\left\|A^{1/2}u(0)+\frac{c^2}{b}A^{1/2}u_t(0)\right\|^2+\frac{c^2}{2b}\gamma\left\|u_t(0)\right\|^2\right]$$

Thus since all quantities are positive we have from the third term that $\|u_t\|^2 \leq C_1\hat{E}(0)$, where C depends only on $\{\alpha,\tau,b,c\}$. Then from the first term we have $\|u_{tt}\|^2 \leq C_2\hat{E}(0)$, and also $\int_0^t\|u_{tt}\|^2\,ds \leq C_3\hat{E}(0)$ again with the constants only depending on parameters of the equation.

Now consider (3.1.2). Rearrange and integrate to get

$$\frac{c^2}{2}\left\|A^{1/2}u(s)\right\|^2\Big|_0^t+b\int_0^t\left\|A^{1/2}u_t(s)\right\|^2\,ds=\tau\int_0^t\|u_{tt}(s)\|^2\,ds-\tau(u_{tt}(s),u_t(s))\Big|_0^t-\frac{\alpha}{2}\|u_t(s)\|^2\Big|_0^t$$

Thus in terms of magnitude

$$\frac{c^2}{2}\left\|A^{1/2}u(s)\right\|^2\Big|_0^t+b\int_0^t\left\|A^{1/2}u_t(s)\right\|^2\,ds\leq\tau\int_0^T\|u_{tt}(s)\|^2\,ds+\tau\|u_{tt}(t)\|\,\|u_t(t)\|$$
$$+\frac{\alpha}{2}\|u_t(t)\|^2+\tau\|u_{tt}(0)\|\,\|u_t(0)\|+\frac{\alpha}{2}\|u_t(0)\|^2$$

Therefore $\left\|A^{1/2}u(t)\right\|^2 \leq \left\|A^{1/2}u(0)\right\|^2+C_4\hat{E}(0) \leq C_5\left[\hat{E}_0(0)+\hat{E}(0)\right]$, and since from before $\left\|A^{1/2}u(t)+\frac{c^2}{b}A^{1/2}u_t(t)\right\|^2 \leq C_6\hat{E}(0)$ we can extract $\left\|A^{1/2}u_t(t)\right\|^2 \leq C_7\left[\hat{E}_0(0)+\hat{E}(0)\right]$.

Now we have that each of $\left\|A^{1/2}u\right\|^2$, $\left\|A^{1/2}u_t\right\|^2$, $\|u_{tt}\|^2$ are bounded by $C_i\left[\hat{E}_0(0)+\hat{E}(0)\right]$, where all C_i are functions only of the parameters of the equation. Using equivalence of norm squared and energy and Poincare's inequality we have our result.

As for the integrability, reinspection of the previous calculations extracts $\int_0^t \|u_{tt}(s)\|^2 \, ds$ and $\int_0^t \left\|A^{1/2}u_t(s)\right\|^2 \, ds$ are both bounded by $C\left[\hat{E}_0(0) + \hat{E}(0)\right]$ for arbitrary t. All that is left is to have an appropriate bound on $\int_0^\infty \left\|A^{1/2}u\right\|^2 dt$. This is done by the following calculation:

Multiply by u and integrate by parts to obtain

$$\tau(u_{ttt}, u) + \alpha(u_{tt}, u) + c^2(A^{1/2}u, A^{1/2}u) + b(A^{1/2}u_t, A^{1/2}u) = 0$$

Then use the following identities:

$$(u_{ttt}, u) = \frac{d}{dt}(u_{tt}, u) - (u_{tt}, u_t) = \frac{d}{dt}(u_{tt}, u) - \frac{1}{2}\frac{d}{dt}\|u_t\|^2 \tag{3.1.5}$$

$$(u_{tt}, u) = \frac{d}{dt}(u_t, u) - \|u_t\|^2 \tag{3.1.6}$$

$$(A^{1/2}u_t, A^{1/2}u) = \frac{1}{2}\frac{d}{dt}\left\|A^{1/2}u\right\|^2 \tag{3.1.7}$$

Combining these we obtain

$$\frac{d}{dt}\left[\tau(u_{tt}, u) + \alpha(u_t, u) + b\left\|A^{1/2}u\right\|^2\right] + \left\|A^{1/2}u\right\|^2 = \frac{d}{dt}\tau\|u_t\|^2 + \alpha\|u_t\|^2 \tag{3.1.8}$$

Thus integrating

$$\int_0^t \left\|A^{1/2}u(s)\right\|^2 ds = \tau\|u_t(s)\|^2\Big|_0^t + \alpha\int_0^t \|u_t(s)\|^2 \, ds + \tau(u_{tt}(s), u(s))\Big|_t^0$$
$$+\alpha(u_t(s), u(s))\Big|_t^0 + b\left\|A^{1/2}u(s)\right\|^2\Big|_t^0$$

Any individual of $\|u\|, \|u_t\|, \|u_{tt}\|, \|A^{1/2}u\|$ is bounded by $C\sqrt{\hat{E}_0(0) + \hat{E}(0)}$, and we also have $\int_0^t \|u_t(s)\|^2 \, ds \leq C\left[\hat{E}_0(0) + \hat{E}(0)\right]$. Therefore for any $t > 0$,

$$\int_0^t \left\|A^{1/2}u_h(s)\right\|^2 \, ds \leq C\left[\hat{E}_0(0) + \hat{E}(0)\right]$$

□

Exponential stability in the E norm follows from Pazy-Datko's theorem, since the norm is integrable.

Corollary 3.1.2. *There exist positive constants C, ω, such that*

$$E(T) \leq Ce^{-\omega T}E(0)$$

We can also make use of these computations to derive stability results for a non-homogeneous equation.

Theorem 3.1.3. *Let $V \subseteq D(A^{1/2})$, and let $u(t)$ be such that $(u, u_t, u_{tt}) \in V \times V \times H$ for all $t \in [0, T]$, and suppose also that u solves: For all $\phi \in V$, for all $t \in [0, T]$,*

$$\tau(u_{ttt}, \phi) + \alpha(u_{tt}, \phi) + b(A^{1/2}u_t, A^{1/2}\phi) + c^2(A^{1/2}u, A^{1/2}\phi) = (f, \phi)$$

for some function $f(t) \in L^2((0, T); H)$. Then there exists a positive constant C depending on the parameters of the equation and the Poincare-type constant for A such that

$$E(T) + \int_0^T E(s) \, ds \leq C\left(E(0) + \int_0^T \|f(s)\|^2 \, ds\right)$$

Proof. Throughout this proof C will stand for a generic constant depending on algebraic inequalities and the constants α, τ, c, b. Constants with dependence on other factors will be noted specifically. Apply the same multipliers u_{tt} and u_t, and form the same linear combination. This will arrive at

$$\frac{d}{dt}\hat{E}(t) + \gamma \|u_{tt}(t)\|^2 = (f(t), u_{tt}(t) + \frac{c^2}{b}u_t(t))$$

Integrate,

$$\hat{E}(t) + \gamma \int_0^t \|u_{tt}(s)\|^2 \, ds = \hat{E}(0) + \int_0^t (f(s), u_{tt}(s) + \frac{c^2}{b}u_t(s))\, ds$$

Since $\left\|u_{tt}(s) + \frac{c^2}{b}u_t(s)\right\|$ is part of $\sqrt{\hat{E}(s)}$,

$$\hat{E}(t) + \gamma \int_0^t \|u_{tt}(s)\|^2 \, ds = \hat{E}(0) + C\int_0^t \|f(s)\| \sqrt{\hat{E}(s)}\, ds$$

Use $ab \leq \frac{a^2}{2\epsilon} + \frac{\epsilon b^2}{2}$,

$$\hat{E(t)} + \gamma \int_0^t \|u_{tt}(s)\|^2 \, ds = \hat{E}(0) + C\int_0^t \frac{\epsilon}{2} \|f(s)\|^2 + \frac{1}{2\epsilon}\hat{E}(s)\, ds \tag{3.1.9}$$

Although $E(t)$ does not determine a norm for the variables u, u_t, u_{tt} without the addition of $\hat{E}_0$, it does give some information. Specifically, since it includes $\|u_t\|^2$ alone, if $\hat{E}(t) \leq M$ then $\|u_t\|^2 \leq CM$ and also $\left\|u_{tt} + \frac{c^2}{b}u_t\right\|^2 \leq CM$ therefore $\|u_{tt}\|^2 \leq CM$.

Now go back to multiplication by u_t:

$$\frac{d}{dt}\left[\tau(u_{tt}, u_t) + \frac{c^2}{2}\left\|A^{1/2}u\right\|^2 + \frac{\alpha}{2}\|u_t\|^2\right] + b\left\|A^{1/2}u_t\right\|^2 - \tau\|u_{tt}\|^2 = (f, u_t)$$

After integration and bounding this gives

$$\frac{c^2}{2}\left\|A^{1/2}u(t)\right\|^2 + b\int_0^t \left\|A^{1/2}u_t(s)\right\|^2\,ds \le \frac{c^2}{2}\left\|A^{1/2}u(0)\right\|^2 + \tau\int_0^t \|u_{tt}(s)\|^2\,ds +$$
$$\tau\,\|u_{tt}(s)\|\,\|u_t(s)\||_0^t + \frac{\alpha}{2}\|u_t(s)\|^2\Big|_0^t + \tau\int_0^t \|f(s)\|\,\|u_t(s)\|\,ds$$

For the term $\tau\int_0^t \|f(s)\|\,\|u_t(s)\|\,ds$, we wish to transfer the u_t quantity to the left side, absorbing into the $\left\|A^{1/2}u_t\right\|^2$ integral there. To do so we will use our Poincaré-type inequality, introducing a constant dependent on the domain Ω, $\|v\|^2 \le K_\Omega\left\|A^{1/2}v\right\|^2$. Then we will use again $ab \le \frac{a^2}{2\epsilon_1} + \frac{\epsilon_1 b^2}{2}$ with $\epsilon_1 = \frac{b}{K_\Omega\tau}$. The other terms have bounds coming from equation (3.1.9).

$$\frac{c^2}{2}\left\|A^{1/2}u(t)\right\|^2 + b\int_0^t \left\|A^{1/2}u_t(s)\right\|^2\,ds \le \frac{c^2}{2}\left\|A^{1/2}u(0)\right\|^2 + \hat{E}(0) + \frac{C\epsilon}{2}\int_0^t \|f(s)\|^2\,ds$$
$$+\frac{1}{2\epsilon}\int_0^t \hat{E}(s)\,ds + \frac{\tau^2 K_\Omega}{2b}\int_0^t \|f(s)\|^2\,ds + \frac{b}{2}\int_0^t \left\|A^{1/2}u_t(s)\right\|^2\,ds$$

So,

$$\frac{c^2}{2}\left\|A^{1/2}u(t)\right\|^2 + \frac{b}{2}\int_0^t \left\|A^{1/2}u_t(s)\right\|^2\,ds \le \frac{c^2}{2}\left\|A^{1/2}u(0)\right\|^2 + \hat{E}(0) + (\frac{C\epsilon}{2} + C_\Omega)\int_0^t \|f(s)\|^2\,ds$$
$$+\frac{1}{2\epsilon}\int_0^t \hat{E}(s)\,ds$$

Combining this inequality with equation (3.1.9) gives us

$$\hat{E}(t) + \frac{c^2}{2}\left\|A^{1/2}u_t(t)\right\|^2 + \gamma\int_0^t \|u_{tt}(s)\|^2\,ds + \frac{b}{2}\int_0^t \left\|A^{1/2}u_t(s)\right\|^2\,ds \le 2\hat{E}(0) + \frac{c^2}{2}\left\|A^{1/2}u_t(0)\right\|^2$$
$$+ (C\epsilon + C_\Omega)\int_0^t \|f(s)\|^2\,ds + \frac{1}{\epsilon}\int_0^t \hat{E}(s)\,ds \qquad (3.1.10)$$

Now multiply by u.

$$\tau(u_{ttt}, u) + \alpha(u_{tt}, u) + c^2(A^{1/2}u, A^{1/2}u) + b(A^{1/2}u_t, A^{1/2}u) = (f, u)$$

Use the identities from before

$$\begin{aligned}
(u_{ttt}, u) &= \frac{d}{dt}(u_{tt}, u) - (u_{tt}, u_t) = \frac{d}{dt}(u_{tt}, u) - \frac{1}{2}\frac{d}{dt}\|u_t\|^2 \\
(u_{tt}, u) &= \frac{d}{dt}(u_t, u) - \|u_t\|^2 \\
(A^{1/2}u_t, A^{1/2}u) &= \frac{1}{2}\frac{d}{dt}\left\|A^{1/2}u\right\|^2
\end{aligned}$$

So that

$$\tau\frac{d}{dt}(u_{tt}, u) - \frac{\tau}{2}\frac{d}{dt}\|u_t\|^2 + \alpha\left(\frac{d}{dt}(u_t, u) - \|u_t\|^2\right) + c^2\left\|A^{1/2}u\right\|^2 + \frac{b}{2}\frac{d}{dt}\left\|A^{1/2}u\right\|^2 = (f, u)$$

Integrate and rearrange

$$\begin{aligned}
\frac{\tau}{2}\|u_t(t)\|^2 + \frac{b}{2}\left\|A^{1/2}u(t)\right\|^2 + c^2\int_0^t \left\|A^{1/2}u(s)\right\|^2 ds &= \frac{\tau}{2}\|u_t(0)\|^2 + \frac{b}{2}\left\|A^{1/2}u(0)\right\|^2 \\
+ \alpha\int_0^t \|u_t(s)\|^2 ds + \tau\left(u_{tt}(s), u(s)\right)\Big|_t^0 &+ \alpha\left(u_t(s), u(s)\right)\Big|_t^0 + \int_0^t (f(s), u(s))\, ds
\end{aligned}$$

Bound (f, u) by $\|f\|\,\|u\|$, use our Poincaré-type inequality on u, then split the result as usual with $\epsilon = \frac{c^2}{C_\Omega}$ to eliminate the u-term's dependence on C_Ω, and then combine it with its partner on the left. Bound other terms by $E(t)$ or $E(0)$, as appropriate, again using a Poincare inequaltiy on the integral term for $\|u_t\|^2$. This gives,

$$\begin{aligned}
&\frac{\tau}{2}\|u_t(t)\|^2 + \frac{b}{2}\left\|A^{1/2}u(t)\right\|^2 + \frac{c^2}{2}\int_0^t \left\|A^{1/2}u(s)\right\|^2 \, ds \\
&\qquad \leq C_\Omega \left(E(0) + E(t) + \int_0^t \left\|A^{1/2}u_t(s)\right\|^2 \, ds + \frac{1}{2c^2}\int_0^t \|f(s)\|^2 \, ds \right) \\
&\qquad \leq C_\Omega \left(E(0) + C_{\epsilon,\Omega}\int_0^t \|f(s)\|^2 \, ds + \frac{1}{\epsilon}\int_0^t \hat{E}(s)\, ds \right)
\end{aligned}$$

Add together this inequality and equation (3.1.10), factor out algebraic and physical constants, and find that

$$E(t) + \int_0^t E(s)\, ds \leq C_\Omega \left(E(0) + C_{\epsilon,\Omega}\int_0^t \|f(s)\|^2 \, ds + \frac{1}{\epsilon}\int_0^t E(s)\, ds \right)$$

Select $\epsilon = \frac{1}{2C_\Omega}$ to find that

$$E(t) + \int_0^t E(s)\, ds \leq C_\Omega \left(E(0) + \int_0^t \|f(s)\|^2 \, ds \right)$$

□

Corollary 3.1.4. *With reference to the previous theorem, if $f \in L^2((0,\infty); H)$ then the solution is exponentially stable in the E norm.*

Chapter 4

Finite element solutions to the PDE

One of the approximation schemes for computation of an approximate solution is the finite element method. An advantage of finite elements is that you do not need any particularly special geometry of Ω, provided that you do not mind a little fiddling around near the boundary. A disadvantage of finite elements is that they do not play well with higher norms. As we will see, a typical finite element solution might consist of piecewise cubic function, and in such a case Au is no longer an element, and thus not an admissible multiplier to the weak formulation. However this multiplier is critical for energy calculations in some of the spaces of interest. We present a formulation of a finite element solution to the linear MGT equation, abstracting the boundary of the domain and the specific elements used behind the *order of accuracy* of the space of elements. Stablity and convergence results are included.

The method will be formulated in terms of the abstract model under abstract assumptions on the approximating subspaces. An application will be given to a concrete example with typical elements at the end.

4.1 Preliminaries; Stationary problem

We will build our formulation and analysis of approximate solutions to the MGT equation by starting with an approximation scheme for the stationary problem $Au = f$. Let A be an operator obeying assumptions (2.1.1). Let S_h be a family of subspaces of $D(A^{1/2})$ depending on a parameter h of discretization going to 0, with basis $\{\chi_i(x)\}_{i=0}^{n(h)}$. Further, assume the following:

Assumption 4.1.1. *The operator A is a differential operator of order s_1, so that $H = H^{s_0}(\Omega)$, $D(A) \subseteq H^{s_0+s_1}(\Omega)$, and $D(A^{1/2}) \subseteq H^{s_0+s_1/2}(\Omega)$, where notationally $H^0(\Omega) = L^2(\Omega)$.*

It obeys an elliptic regularity estimate,

$$\|u\|_{s_1} \leq C \|Au\|$$

And the spaces S_h have a close-approximation property: there exists an integer r such that

$$\inf_{\chi \in S_h} \left(\|v - \chi\| + h^{s/2} \left\|A^{1/2}(v - \chi)\right\| \right) \leq Ch^s \|v\|_s$$

For all $1 \leq s \leq r$, $v \in H^s(\Omega) \cap D(A^{1/2})$. Call r the order of accuracy of the family S_h.

Assumption 4.1.2. *Suppose that $u_0, u_1, u_2 \in D(A^{1/2})$ and that $u_0 \in H^{i_0}(\Omega)$, $u_1 \in H^{i_1}(\Omega)$, $u_2 \in H^{i_2}(\Omega)$ for some positive constants $i_j \leq r$. Suppose that $(u(t), u_t(t), u_{tt}(t))$*

are in $D(A^{1/2})$ and $H^{k_j}(\Omega)$, $k = 0, 1, 2$ respectively. Suppose also that $u_{ttt} \in D(A^{1/2}) \cap H^{k_3}(\Omega)$. The indices k_j are all less than r as well.

By way of explanation, the numbers i_j, k_j represent the highest index that we can apply through the estimate in 4.1.1.

Since we have assumed that $0 \in \rho(A)$, we know that the problem $Au = f$ has a solution $u \in D(A)$ for any $f \in H$. To arrive at a finite-dimensional solution, consider the weak formulation: find a function $u \in D(A^{1/2})$ satisfying

$$(A^{1/2}u, A^{1/2}\phi) = (f, \phi)\, \forall \phi \in D(A^{1/2})$$

Restrict this weak formulation to only include test functions on S_h, and seeking a solution in S_h, By linearity we thus only need consider basis functions, yielding: find a function $u_h \in S_h$ solving

$$(A^{1/2}u_h, A^{1/2}\chi_j) = (u_h, \chi_j)\, \forall \chi_j, 1 \leq j \leq n(h)$$

Using the fact that $u_h = \sum \alpha_i \chi_i$ we can write this system as a matrix equation

$$[(A^{1/2}\chi_i, A^{1/2}\chi_j)]_{i,j=1}^{n(h)} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} (f, \chi_1) \\ \vdots \\ (f, \chi_n) \end{pmatrix}$$

We see that this matrix, called the stiffness matrix, is positive definite, for if (α_i) is an n-vector,

$$
\begin{aligned}
([(A^{1/2}\chi_i, A^{1/2}\chi_j)]_{i,j=1}^{n(h)}(\alpha_i),(\alpha_j)) &= \sum_{i,j=1} n\alpha_i\alpha_j(A^{1/2}\chi_i, A^{1/2}\chi_j) \\
&= (\sum \alpha_i A^{1/2}\chi_i, \sum \alpha_j A^{1/2}\chi_j) \\
&= \left\| A^{1/2}\sum \alpha_i\chi_i \right\|^2 \\
&\geq 0
\end{aligned}
$$

And then if $\left\| A^{1/2}\sum \alpha_i\chi_i \right\| = 0$, by the coercivity of the bilinear form assoicated with A we can apply a Poincare inequality to determine that $\|\sum \alpha_i\chi_i\| \leq \left\| A^{1/2}\sum \alpha_i\chi_i \right\| = 0$ and so $\sum \alpha_i\chi_i = 0$ as an element of H and thus $\alpha_i = 0$ for all i.

Therefore we can invert the stiffness matrix and determine the finite element solution u_h from the coefficients. We can prove a convergence result for this u_h as follows. This proof and the preceeding discussion is a generalization of material from [14] and [10].

Theorem 4.1.3. *For u and u_h as above, we have*

$$
\|u - u_h\| + h^{s_1/2}\left\| A^{1/2}(u - u_h) \right\| \leq Ch^s \|u\|_s
$$

Where $1 \leq s \leq \min\{r, s_1\}$.

Proof. We make use of the fact that $u - u_h$ is orthogonal to S_h with respect to the inner product $(A^{1/2}u, A^{1/2}v)$. Let χ be arbitrary in S_h. Then $u_h - \chi$ is in S_h and so,

$$\begin{aligned}\left\|A^{1/2}(u-u_h)\right\|^2 &= (A^{1/2}(u-u_h), A^{1/2}(u-u_h)) + (A^{1/2}(u-u_h), A^{1/2}(u_h-\chi)) \\ &= (A^{1/2}(u-u_h), A^{1/2}(u-\chi)) \\ &\le \left\|A^{1/2}(u-u_h)\right\| \left\|A^{1/2}(u-\chi)\right\|\end{aligned}$$

And so for all $\chi \in S_h$ we have the inequality $\left\|A^{1/2}(u-u_h)\right\| \le \left\|A^{1/2}(u-\chi)\right\|$, whence by assumption 4.1.1

$$\left\|A^{1/2}(u-u_h)\right\| \le \inf_{\chi \in S_h} \left\|A^{1/2}(u-\chi)\right\| \le Ch^{s-s_1/2} \left\|u\right\|_s$$

The other term in the estimate is a little more tricky. Consider $\phi \in H$ arbitrary, $\psi \in D(A)$ the solution to $A\psi = \phi$. Pick $\chi \in S_h$ and then

$$\begin{aligned}(u_h - u, \phi) &= (u_h - u, A\psi) \\ &= (A^{1/2}(u_h-u), A^{1/2}\psi) \\ &= (A^{1/2}(u_h-u), A^{1/2}(\psi-\chi)) \\ &\le \left\|A^{1/2}(u_h-u)\right\| \left\|A^{1/2}(\psi-\chi)\right\|\end{aligned}$$

So that for all $\chi \in S_h$, $(u_h - u, \phi)/\left\|A^{1/2}(u_h-u)\right\| \le \left\|A^{1/2}(\psi-\chi)\right\|$ which proves

$$(u_h - u, \phi) \le \left\|A^{1/2}(u_h-u)\right\| \inf_{\chi \in S_h} \left\|A^{1/2}(\psi-\chi)\right\|$$

But from our previous estimates we already know

$$\left\|A^{1/2}(u_h-u)\right\| \le Ch^{s-s_1/2} \left\|u\right\|_s$$

And from assumption 4.1.1

$$\inf_{\chi \in S_h} \left\| A^{1/2}(\psi - \chi) \right\| \leq Ch^{s_1/2} \left\| \psi \right\|_{s_1}$$

But because of the regularity estimate $\|\psi\|_{s_1} \leq C \|\phi\|$. Then finally by picking $\phi = u - u_h$ we find that

$$\|u - u_h\| \leq Ch^s \|u\|_s$$

□

4.2 Equation and semidiscrete problem

Consider the abstract MGT equation ,

$$\tau u_{ttt} + \alpha u_{tt} - c^2 Au - bAu_t = 0$$

with initial conditions

$$u(t = 0) = u_0, u_t(t = 0) = u_1, u_{tt}(t = 0) = u_2$$

We wish to study stability and convergence of finite element solutions to this problem. To give a finite element formulation we need a variational form and a family of finite dimensional spaces in which we we seek our solutions.

Consider the variational form of the original equation.

$$\tau \left(u_{ttt}, \phi\right) + \alpha \left(u_{tt}, \phi\right) + c^2 \left(A^{1/2}u, A^{1/2}\phi\right) + b \left(A^{1/2}u_t, A^{1/2}\phi\right) = 0 \ \forall \phi \in D(A^{1/2})$$

By restricting this equation to one on S_h rather than all of $D(A^{1/2})$ we arrive at the semidiscrete problem: find a function $u_h(x,t) \in S_h$ for each t such that

$$\tau\left(u_{h,ttt},\phi\right)+\alpha\left(u_{h,tt},\phi\right)+c^2\left(A^{1/2}u_h,A^{1/2}\phi\right)+b\left(A^{1/2}u_{h,t},A^{1/2}\phi\right)=0\ \forall\phi\in S_h$$

with associated initial conditions $u_h(0)=v_{h,0}$, $u_{h,t}=v_{h,1}$, $u_{h,tt}=v_{h,2}$, where $v_{h,i}$ is some interoplant of the initial data u_i in S_h.

4.3 Existence of finite element solutions

We seek a $u_h(x,t)=\sum_{i=1}^{n}\gamma_i(t)\chi_i(x)$ solving the semidiscrete problem. That is, for $\phi=\sum_{j=1}^{n}\beta_j\chi_j\in S_h$ we wish to have

$$\tau\sum_{i=1}^{n}\left(\gamma_i^{'''}(t)\chi_i(x),\phi\right)+\alpha\sum_{i=1}^{n}\left(\gamma_i^{''}(t)\chi_i(x),\phi\right)+c^2\sum_{i=1}^{n}\left(\gamma_i(t)A^{1/2}\chi_i(x),A^{1/2}\phi\right)$$
$$+b\sum_{i=1}^{n}\left(\gamma_i^{'}(t)A^{1/2}\chi_i(x),A^{1/2}\phi\right)=0$$

In the case of $\phi=\chi_j$ we rewrite

$$\sum_{i=1}^{n}\gamma_i^{'''}\left(\chi_i,\chi_j\right)=-\frac{\alpha}{\tau}\sum_{i=1}^{n}\gamma_i^{''}(t)\left(\chi_i,\chi_j\right)-\frac{c^2}{\tau}\sum_{i=1}^{n}\gamma_i\left(A^{1/2}\chi_i,A^{1/2}\chi_j\right)-\frac{b}{\tau}\sum_{i=1}^{n}\gamma_i^{'}\left(A^{1/2}\chi_i,A^{1/2}\chi_j\right)$$

Rewrite in terms of the following matrices. The matrix M_h has entries (χ_i,χ_j). The matrix A_h has entries $(A^{1/2}\chi_i,A^{1/2}\chi_j)$. Then using the notation $\vec{\gamma}$ for the n-vector with components γ_i we find it suffices if

$$\vec{\gamma}^{'''}=-\frac{\alpha}{\tau}\vec{\gamma}^{''}-\frac{c^2}{\tau}M_h^{-1}A_h\vec{\gamma}-\frac{b}{\tau}M_h^{-1}A_h\vec{\gamma}^{'}$$

provided that M_h is invertible.

Lemma 4.3.1. *Given a collection $\{\phi_i\}_{i=1}^n$ of linearly independent vectors in a Hilbert space, the matrix $M = \{(\phi_i, \phi_j)\}$ is invertible.*

Proof. Consider a linear combination of the rows,

$$\sum_{i=1}^{n} \alpha_i \begin{bmatrix} (\phi_1, \phi_i) \\ \dots \\ (\phi_n, \phi_i) \end{bmatrix} = \sum_{i=1}^{n} \begin{bmatrix} (\phi_1, \alpha_i \phi_i) \\ \dots \\ (\phi_n, \alpha_i \phi_i) \end{bmatrix}$$

$$= \begin{bmatrix} (\phi_1, \sum_{i=1}^n \alpha_i \phi_i) \\ \dots \\ (\phi_n, \sum_{i=1}^n \alpha_i \phi_i) \end{bmatrix}$$

This linear combination is $\vec{0}$ only if $\sum_{i=1}^n \alpha_i \phi_i \perp \phi_j \ \forall j$, but this is only possible if the linear combination is trivial since $\sum_{i=1}^n \alpha_i \phi_i$ is in the span of $\{\phi_j\}$. □

Thus we write the matrix ODE

$$\frac{d}{dt} \begin{pmatrix} \vec{\gamma} \\ \vec{\gamma}' \\ \vec{\gamma}'' \end{pmatrix} = \begin{pmatrix} 0 & I & 0 \\ 0 & 0 & I \\ -\frac{c^2}{\tau} M_h^{-1} A_h & -\frac{b}{\tau} M_h^{-1} A_h & -\frac{\alpha}{\tau} I \end{pmatrix} \begin{pmatrix} \vec{\gamma} \\ \vec{\gamma}' \\ \vec{\gamma}'' \end{pmatrix}$$

With initial conditions $\vec{\gamma}(0) = \vec{\gamma}_0$, $\vec{\gamma}'(0) = \vec{\gamma}_1$, $\vec{\gamma}''(0) = \vec{\gamma}_2$, where

$$\vec{\gamma}_i = M_h^{-1} \begin{bmatrix} (v_{h,i}, \phi_1) \\ \dots \\ (v_{h,i}, \phi_n) \end{bmatrix}$$

This ODE yields the finite element solution $u_h(x, t)$.

4.4 Stability of finite element solutions

We previously derived an abstract stability result that applies to our solutions, and thus the result stated there about exponential stability in the $\mathcal{H}$ norm applies. In fact we can do better than simply applying the Pazy-Datko theorem, which derives bounds in terms of constants from the closed graph and uniform boundedness theorem, and thus may vary based on h. By examination and modification of the proof given in [12] we can avoid the reliance on these theorems for this particular problem. I will essentially repeat and elaborate on the proof given there for completeness, with the first two steps replaced by the estimates found before.

Theorem 4.4.1. $(E+E_0)(t) \leq Me^{-\mu t}$ *where* M *and* μ *can be selected to not depend on* h*.*

Proof. Write our solution $(u_h(t), u_{h,t}(t), u_{h,tt}(t))$ as an element of the Banach space $X = S_h \times S_h \times S_h$ topologized by $|||x|||^2 = \|x_1\|^2_{H_0^1} + \|x_2\|^2_{H_0^1} + \|x_3\|^2_{L^2}$, with x the inital data $(u_h(0), u_{h,t}(0), u_{h,tt}(0))$. Then we can write the solution as a semigroup applied to the initial data, $T(t)x$. Reading our previous inequalities in terms of this notation we have

$$\int_0^\infty |||T(s)x|||^2 \, ds \leq C_1 \, |||x|||^2$$

and

$$|||T(t)x||| \leq C_2 \, |||x|||$$

For $x \in X$ define the function

$$t_x(\rho) = \sup\{t \,:\, |||T(s)x||| \geq \rho\,|||x||| \text{ for } 0 \leq s \leq t\}$$

Then $t_x(\rho) < \infty$ for $\rho > 0$, since otherwise $|||T(s)x|||^2$ would not be integrable. Also for $\rho < 1$ we have that $|||T(0)x||| = |||x||| \geq \rho\,|||x|||$ and, since $t \to |||T(t)x|||$ is continuous, $t_x(\rho) > 0$. Thus we can select $\rho < \min\{1, C_2^{-1}\}$ and we find

$$t_x(\rho)\rho^2\,|||x|||^2 \leq \int_0^{t_x(\rho)} |||T(s)x|||^2\,ds \leq \int_0^{\infty} |||T(s)x|||\,ds \leq C_1\,|||x|||^2$$

So therefore $t_x(\rho) \leq \frac{C_1}{\rho^2} = t_0$. In other words, by time $t > t_0$ $|||T(t)x|||$ has certainly decreased below a factor of ρ. We can use this fact and semigroup algebra to infer exponential decay.

More specifically, when $t > t_0$ we have that

$$|||T(t)x||| \leq \|T(t - t_x(\rho))\|\,|||T(t_x(\rho))x||| \leq C_2\rho\,|||x|||$$

Now choose any time step $\epsilon > 0$, let $t_1 = t_0 + \epsilon$, consider any time $t = nt_1 + s$, $0 \leq s < t_1$. We obtain from this

$$\|T(t)\| \leq \|T(s)\|\,\|T(nt_1)\| \leq C_2(\rho C_2)^n = \frac{C_2}{\rho C_2}(\rho C_2)^{n+1}$$
$$\leq M(\rho C_2)^{\frac{nt_1+s}{t_1}} \leq Me^{-(-(t/t_1)\ln(\rho C_2))} = Me^{-\mu t}$$

Where $M = \frac{C_2}{\rho C_2}$, $\mu = -\frac{1}{t_1}\ln(\rho C_2)$. Note since $\rho C_2 < 1$ we have $\mu > 0$ as desired. The constants μ and M depend explicitly on C_2 and ρ. In turn, ρ depends itself on C_2, and C_2 is given to us independent of h, as desired. The constant t_1 depends on ϵ

which is arbitrary and t_0, which depends on C_1, which is given independent of h as well.

□

4.5 Projections and Operators

We will make use of two projections and an operator:

$$P_h : D(A^{1/2}) \to S_h \text{ defined by } (P_h f, \phi) = (f, \phi)\, \forall \phi \in S_h$$

$$R_h : D(A^{1/2}) \to S_h \text{ defined by } \left(A^{1/2} R_h f, A^{1/2}\phi\right) = \left(A^{1/2} f, A^{1/2}\phi\right) \forall \phi \in S_h$$

$$A_h : S_h \to S_h \text{ defined by} \,(A_h\phi, \psi) = \left(A^{1/2}\phi, A^{1/2}\psi\right) \forall \phi, \psi \in S_h$$

4.6 Convergence

We wish to estimate the error $u_h - u$ in various spaces. We do so by use of an intermediate projection, splitting $u_h - u$ into $\theta = u_h - R_h u$ and $\rho = R_h u - u$. The advantage of this approach is that ρ has good bounds because of theorem 4.1.3, while θ will be seen to solve a weak form of the original PDE - but with nonzero forcing term. Similar bounds as in the stability calculation will extract control over the norm of $\theta(t)$ in terms of the initial error $\theta(0)$ and higher time derivatives $\rho_{tt}(s), \rho_{ttt}(s)$.

These will in turn allow for extraction of powers of h, in line with our assumptions above, provided that the continuous solution u is sufficiently smooth. Details follow.

4.6.1 Analysis of θ

By writing out terms we find that $\theta(t)$ solves the equation on S_h

$$\tau\theta_{ttt} + \alpha\theta_{tt} - b\Delta_h\theta_t - c^2\Delta_h\theta = -P_h(\tau\rho_{ttt} + \alpha\rho_{tt})$$

in a weak sense, provided that $\theta, \theta_t, \theta_{tt}, \theta_{ttt}$ all exist in S_h - which is ensured if u, u_t, u_{tt}, u_{ttt} all exist in $D(A^{1/2})$.

This allows us to employ the same multipliers used in the stability calculations to derive bounds on the error term $\theta(t)$.

Theorem 4.6.1. *The follow bounds hold, where C and K denote generic constants that do not depend on T or h.*

$$\|\theta(t)\|^2 \leq C\left(\sum_{k=0}^{4} E_\theta(0)^{\frac{k}{4}}\left(\int_0^t \|P_h(\tau\rho_{ttt}(s) + \alpha\rho_{tt}(s))\|\ ds\right)^{2-\frac{k}{2}}\right)$$
$$\left\|A^{1/2}\theta(t)\right\|^2 \leq C\left(\sum_{k=0}^{4} E_\theta(0)^{\frac{k}{4}}\left(\int_0^t \|P_h(\tau\rho_{ttt}(s) + \alpha\rho_{tt}(s))\|\ ds\right)^{2-\frac{k}{2}}\right)$$
$$\left\|A^{1/2}\theta_t(t)\right\|^2 \leq C\left(\sum_{k=0}^{4} E_\theta(0)^{\frac{k}{4}}\left(\int_0^t \|P_h(\tau\rho_{ttt}(s) + \alpha\rho_{tt}(s))\|\ ds\right)^{2-\frac{k}{2}}\right)$$
$$\|\theta_{tt}(t)\| \leq C\left(\sqrt{E_\theta(0)} + \left(\sqrt{E_\theta(0)}\int_0^t \|P_h(\tau\rho_{ttt}(s) + \alpha\rho_{tt}(s))\|\ ds\right)^{1/2} + \int_0^t \|P_h(\tau\rho_{ttt}(s) + \alpha\rho_{tt}(s))\|\ ds\right)$$

Remark 4.6.1. *The significance of these inequalities is that the error term θ has bounds in terms of the error of the initial data and the companion error term ρ. Both of these terms will be shown to be small, provided the initial data is smooth enough and the initial approximation scheme has a high enough degree of approximation.*

Proof. By multiplying by θ_t and θ_{tt} we find the following energy relation:

$$E_\theta(T) + \gamma \int_0^T \|\theta_{tt}\|^2 \, ds = E_\theta(0) - \int_0^T (P_h(\tau\rho_{ttt} + \alpha\rho_{tt}), \theta_{tt} + \frac{c^2}{b}\theta_t) \, ds \qquad (4.6.1)$$

In addition we have

$$\begin{aligned} \frac{c^2}{2} \left\|A^{1/2}\theta\right\|^2\Big|_0^T + b\int_0^T \left\|A^{1/2}\theta_t\right\|^2 ds \leq & \tau \int_0^T \|\theta_{tt}\|^2 \, ds + \|\theta_{tt}\| \, \|\theta_t\||_0^T + \frac{\alpha}{2} \|\theta_t\|^2\Big|_0^T \\ & + \int_0^T (P_h(\tau\rho_{ttt} + \alpha\rho_{tt}), \theta_t) \, ds \end{aligned} \qquad (4.6.2)$$

From (4.6.1) we have that

$$\left\|\theta_{tt}(t) + \frac{c^2}{b}\theta_t(t)\right\|^2 \leq E_\theta(0) + \int_0^t \|P_h(\tau\rho_{ttt}(s) + \alpha\rho_{tt}(s))\| \left\|\theta_{tt}(s) + \frac{c^2}{b}\theta_t(s)\right\| ds$$

Using a variant of Gronwall's inequality found in section 7.1 we obtain from this

$$\left\|\theta_{tt} + \frac{c^2}{b}\theta_t\right\| \leq \sqrt{E_\theta(0)} + \frac{1}{2}\int_0^t \|P_h(\tau\rho_{ttt}(s) + \alpha\rho_{tt}(s))\| \, ds$$

Thus likewise we find

$$\begin{aligned} \|\theta_t(t)\|^2 &\leq E_\theta(0) + \int_0^t \|P_h(\tau\rho_{ttt}(s) + \alpha\rho_{tt}(s))\| \left\|\theta_{tt}(s) + \frac{c^2}{b}\theta_t(s)\right\| ds \\ &\leq E_\theta(0) + \sqrt{E_\theta(0)} \int_0^t \|P_h(\tau\rho_{ttt}(s) + \alpha\rho_{tt}(s))\| \, ds + \frac{1}{2}\left(\int_0^t \|P_h(\tau\rho_{ttt}(s) + \alpha\rho_{tt}(s))\| \, ds\right)^2 \end{aligned}$$

So since both $\|\theta_t(t)\|$ and $\left\|\theta_{tt}(t) + \frac{c^2}{b}\theta_t(t)\right\|$ are bounded by

$$\sqrt{E_\theta(0)}+\left(\sqrt{E_\theta(0)}\int_0^t \|P_h(\tau\rho_{ttt}(s) + \alpha\rho_{tt}(s))\|\ ds\right)^{\frac{1}{2}}+\int_0^t \|P_h(\tau\rho_{ttt}(s) + \alpha\rho_{tt}(s))\|\ ds$$

we have

$$\begin{aligned}\|\theta_{tt}(t)\| \leq & C\left(\sqrt{E_\theta(0)} + \left(\sqrt{E_\theta(0)}\int_0^t \|P_h(\tau\rho_{ttt}(s) + \alpha\rho_{tt}(s))\|\ ds\right)^{\frac{1}{2}}\right. \\ & \left. + \int_0^t \|P_h(\tau\rho_{ttt}(s) + \alpha\rho_{tt}(s))\|\ ds\right)\end{aligned}$$

Similarly we have

$$\begin{aligned}\int_0^T \|\theta_{tt}\|^2\ ds \leq & E_\theta(0) + \sqrt{E_\theta(0)}\int_0^t \|P_h(\tau\rho_{ttt}(s) + \alpha\rho_{tt}(s))\|\ ds \\ & + \frac{1}{2}\left(\int_0^t \|P_h(\tau\rho_{ttt}(s) + \alpha\rho_{tt}(s))\|\ ds\right)^2\end{aligned}$$

Then putting this all together into (4.6.2) and applying the previous bounds we have

$$\frac{c^2}{2}\left\|A^{1/2}\theta(t)\right\|^2 \leq K\left(\sum_{k=0}^{4} E_\theta(0)^{\frac{k}{4}}\left(\int_0^t \|P_h(\tau\rho_{ttt}(s) + \alpha\rho_{tt}(s))\|\ ds\right)^{2-\frac{k}{2}}\right)$$

Then thanks to Poincare's inequality we have

$$\|\theta(t)\| \leq C\left(\sum_{k=0}^{4} E_\theta(0)^{\frac{k}{4}}\left(\int_0^t \|P_h(\tau\rho_{ttt}(s) + \alpha\rho_{tt}(s))\|\ ds\right)^{2-\frac{k}{2}}\right)^{\frac{1}{2}}$$

Now applying the same analysis this time to the $\left\|A^{1/2}\theta + \frac{c^2}{b}A^{1/2}\theta_t\right\|^2$ term yields

$$\left\|A^{1/2}\theta_t(t)\right\| \leq C\left(\sum_{k=0}^{4} E_\theta(0)^{\frac{k}{4}}\left(\int_0^t \|P_h(\tau\rho_{ttt}(s) + \alpha\rho_{tt}(s))\|\ ds\right)^{2-\frac{k}{2}}\right)^{\frac{1}{2}}$$

□

This theorem transfers the analysis of $\theta(t)$ to that of $\theta(0)$ and $\rho_{tt}(s), \rho_{ttt}(s)$. We will next address these quantities.

4.6.2 Analysis of $E_\theta(0)$

Reverse our initial separation and write $\theta(0) = (u_h(0) - u(0)) + (u(0) - R_h u(0)) = (v_{h,0} - u_0) + \rho(0)$. Thus simplifying by equivalent norms,

$$E_\theta(0) \leq C \left(\left\| A^{1/2}(v_{h,0} - u_0) \right\|^2 + \left\| A^{1/2}(v_{h,1} - u_1) \right\|^2 + \| v_{h,2} - u_2 \|^2 \right.$$
$$\left. + \left\| A^{1/2}\rho(0) \right\|^2 + \left\| A^{1/2}\rho_t(0) \right\|^2 + \| \rho_{tt}(0) \|^2 \right)$$

Because of assumption 4.1.2, the first three terms can extract some factor of h. Because of theorem 4.1.3, the latter three terms also have good bounds. The exact degree that can be employed depends on the order of accuracy and the smoothness of the original solution u. Exact calculations will follow.

4.6.3 Analysis of ρ

Theorem 4.6.2. *For all $0 < t \leq T$, if $k_0, k_1, k_2 \geq \frac{s1}{2}$ (this implies $r \geq \frac{s1}{2}$) and u, u_t, u_{tt} are all in $D(A^{1/2})$,*

$$\|\rho(t)\| \leq Ch^{k_0} \|u(t)\|_{k_0}$$

$$\left\| A^{1/2}\rho(t) \right\| \leq Ch^{k_0 - \frac{s_1}{2}} \|u(t)\|_{k_0}$$

$$\|\rho_t(t)\| \leq Ch^{k_1} \|u_t(t)\|_{k_1}$$

$$\left\| A^{1/2}\rho_t(t) \right\| \leq Ch^{k_1 - \frac{s_1}{2}} \|u_t(t)\|_{k_1}$$

$$\|\rho_{tt}(t)\| \leq Ch^{k_2} \|u_{tt}(t)\|_{k_2}$$

and also, if $i_0, i_1 \geq \frac{s1}{2}$

$$\left\|A^{1/2}\rho(0)\right\| \leq Ch^{i_0-\frac{s_1}{2}} \left\|u_0\right\|_{i_0}$$

$$\left\|A^{1/2}\rho_t(0)\right\| \leq Ch^{i_1-\frac{s_1}{2}} \left\|u_1\right\|_{i_1}$$

$$\left\|\rho_{tt}(0)\right\| \leq Ch^{i_2} \left\|u_0\right\|_{i_2}$$

Proof. Supose that $v \in D(A^{1/2})$. Then $Av \in D(A^{1/2})'$. On the one hand, R_h works by the definition

$$(A^{1/2}R_hv, A^{1/2}\chi) = (A^{1/2}v, A^{1/2}\chi)$$

for all $\chi \in S_h$. On the other hand, in our study of the stationary problem at the start of this chapter we solved the problem

$$(A^{1/2}v_h, A^{1/2}\chi) = (f, \chi)$$

for all $\chi \in S_h$, and any $f \in D(A^{1/2})'$. By moving a half power across the inner product on the right hand side of the definition of R_h we find that these two problems coincide. Therefore we can apply theorem (4.1.3) to analyze quantities involving $R_hu - u$, provided that the u in question is at least in $D(A^{1/2})$. Bearing in mind that $\rho = R_hu - u$ we find exactly the result.

□

Corollary 4.6.3. *Suppose that* $k_2, k_3 > 0$ *and* u_{tt}, u_{ttt} *are in* $D(A^{1/2})$. *Then,*

$$\int_0^T \left\|\alpha\rho_{tt}(s) + \tau\rho_{ttt}(s)\right\| \, ds \leq h^{k_2}\alpha \int_0^T \left\|u_{tt}(s)\right\|_{k_2} \, ds + h^{k_3}\tau \int_0^T \left\|u_{ttt}(s)\right\|_{k_3} \, ds$$

Proof. This follows just from the same considerations applied to u_{tt} and u_{ttt}. Notice that we need both k_2 and k_3 to be positive - so that the factors of h are meaningful, and also that they live in $D(A^{1/2})$ - so that the estimates apply. □

4.6.4 Application

The calculations above give the tools to infer an explicit convergence rate for approximate solutions. To demonstrate we will specialize to a particular PDE. Suppose that $A = -\Delta$ with 0-Dirichlet boundary conditions on the space $H = L^2(\Omega)$. In this case, $D(A) = H^2 \cap H_0^1(\Omega)$, $s_1 = 2$, and $D(A^{1/2}) = H_0^1(\Omega)$. From examining the preceeding theorems we find that we need

- We need for sure $u, u_t, u_{tt} \in D(A^{1/2})$ will mean that $k_0, k_1, k_2 \geq 1$. However this will not suffice, since if $k_0, k_1 = 1$ then we will extract h^0 from the estimates in 4.6.2.
- Therefore we need $r > 1$ and $k_0, k_1 > 1$,
- We likewise need $i_0, i_1 > 1$, $i_2 > 0$.
- We will also need $k_3 > 0$.

The space $\mathcal{H}$ is not suitable for this analysis, because there is no smoothing in the (u, u_t, u_{tt}) variables past $D(A^{1/2})$ and therefore we cannot attain $k_0, k_1 > 1$.

The space $\mathcal{H}_1$ will suffice. Suppose that $(u_0, u_1, u_2) \in D(\mathcal{A})^2$. Then by semigroup wellposedness $u_{ttt}(s) \in D(A^{1/2})$, so that $k_3 = 1$. Then also we will have $(u(s), u_t(s), u_{tt}(s)) \in D(\mathcal{A})^2$, which will mean $k_0 = k_1 = k_2 = i_0 = i_1 = i_2 = 2$, $k_3 = 1$. This means that

$$\int_0^T \|\alpha\rho_{tt}(s) + \tau\rho_{ttt}(s)\| \, ds \leq \alpha h^2 \int_0^T \|u_{tt}(s)\|_2 \, ds + \tau h \int_0^T \|u_{ttt}(s)\|_1 \, ds$$

Or more succinctly, $C_T(h^2 + h)$. We also then see that

$$\begin{aligned} E_\theta(0) &\leq C\left(I(h) + \left\|A^{1/2}\rho(0)\right\|^2 + \left\|A^{1/2}\rho_t(0)\right\|^2 + \|\rho_{tt}(0)\|^2\right) \\ &\leq C\left(I(h) + h^2\|u_0\|_2^2 + h^2\|u_1\|_2^2 + h^4\|u_2\|_2^2\right) \\ &\leq C\left(I(h) + h^2 + h^2 + h^4\right) \end{aligned}$$

Returning to 4.6.1 we see that the right hand bound for many of the quantities is

$$C\left(\sum_{k=0}^{4} E_\theta(0)^{\frac{k}{4}}\left(\int_0^t \|P_h(\tau\rho_{ttt}(s) + \alpha\rho_{tt}(s))\| \, ds\right)^{2-\frac{k}{2}}\right)$$

For the sake of legibility set $T_1 = E_\theta(0)$ and $T_2 = \int_0^t \|P_h(\tau\rho_{ttt}(s) + \alpha\rho_{tt}(s))\| \, ds$. Then this sum is

$$T_2^2 + T_2^{3/4}T_1^{1/4} + T_2T_1^{1/2} + T_2^{1/2}T_1^{3/4} + T_1$$

Then from the above we find that $T_1 \leq Ch^2$, supposing that the initial interpolation error $I(h)$ is at least as good as h^2, and $T_2 \leq C_T h$. Thus this quantitiy is bounded by

$$C_T\left(h^2 + h^{3/2}h^{1/2} + hh + h^{1/2}h^{3/2} + h^2\right) = C_T h^2$$

Therefore all of $\|\theta(s)\|, \left\|A^{1/2}\theta(s)\right\|, \left\|A^{1/2}\theta_t(s)\right\|$ are bounded by $C_T h$ (note the difference between $\|\theta\|^2$ and $\|\theta\|$). The bounds are different for $\|\theta_{tt}\|$, which is bounded by $C\left(T_1^{1/2} + T_1^{1/4}T_2^{1/2} + T_2\right)$. But again these terms match to a power of h and we get $\|\theta_{tt}(s)\| \leq C_T h$. We have derived,

Corollary 4.6.4. *Suppose that A is the Laplacian described at the start of the section, that the initial approximation scheme has error $I(h)$ converging at least as fast as h^2, that the order of accuracy r is at least 2, that the initial data u_0, u_1, u_2 are in the domain of the generator $\mathcal{A}^2$ on the space $\mathcal{H}_1$ as described in section 2.1, and the time interval $[0,T]$, $0 < s \leq T$. Then using the separation $u_h(s) - u(s) = \theta(s) + \rho(s)$,*

$$\begin{aligned}
\left\|A^{1/2}(u_h(s) - u(s))\right\| &\leq \left\|A^{1/2}\theta(s)\right\| + \left\|A^{1/2}\rho(s)\right\| \\
&\leq C_T h + Ch\,\|u(s)\|_2 \\
&\leq C_T h \\
\left\|A^{1/2}(u_{h,t}(s) - u_t(s))\right\| &\leq \left\|A^{1/2}\theta_t(s)\right\| + \left\|A^{1/2}\rho_t(s)\right\| \\
&\leq C_T h + Ch\,\|u_t(s)\|_2 \\
&\leq C_T h \\
\|(u_{h,tt}(s) - u_{tt}(s))\| &\leq \|\theta_{tt}(s)\| + \|\rho_{tt}(s)\| \\
&\leq C_T h + Ch^2\,\|u_{tt}(s)\|_2 \\
&\leq C_T h
\end{aligned}$$

And thus we get exactly rate of convergence h with the error measured on the space

$D(A^{1/2}) \times D(A^{1/2}) \times H$.

We note that we could improve the smoothness on $\mathcal{H}_1$ by taking the initial data in $D(\mathcal{A}^3)$, which would place $u_{ttt}(s) \in D(A)$ and so $k_3 = 2$ as well. However this would not produce a better convergence estimate. As we saw in the calculations above there was an exact matching so that the combinations of T_1 and T_2 were uniformly h^2. As k_3 improves T_2 but not T_1 we find that the overall rate is still h^2 at that stage in the calculation.

We remark that we could not have gotten by with less: on the space $\mathcal{H}_1$, $(u_0, u_1, u_2) \in D(\mathcal{A})$ is not good enough, because this yields only $u_{ttt} \in H$.

Chapter 5

Spectral solutions to the abstract PDE

In addition to the finite element method, we construct a scheme for computing an approximate solution to the abstract equation consisting of a linear combination of the eigenvectors of the operator A. This has the advantage that the test functions and constructed solutions are very smooth and thus we can apply higher energy estimates via multipliers involving full powers of A. When A is a particular operator and Ω has good geometry these eigenfunctions are known explicitly, although their existence in general will follow from our assumptions.

5.1 Discussion of spectral properties of A

We recall quickly our assumptions 2.1.1 on A. We can draw a lot of information about A from the fact that it has compact resolvent. Since A is selfadjoint, so is A^{-1} and so the spectrums of A and A^{-1} are both real. If ϕ is an eigenvector of A^{-1} with eigenvalue μ, then since $A^{-1}\phi = \mu\phi$ we can conclude that $\phi \in D(A)$ since A^{-1}

maps to the domain of A. Therefore we can apply A across that equation to find that $\frac{1}{\mu}\phi = A\phi$ and thus ϕ is also an eigenvector of A with eigenvalue $\lambda = \frac{1}{\mu}$, and conversely. Thus we can pass directly from information about the eigenvalues of A^{-1} to the eigenvalues of A.

Operating only on the assumption that A is closed with compact resolvent, we can apply the spectral theorem for compact symmetric operators on a Hilbert space to A^{-1}. This will give us that A^{-1} has a possibly terminating orthonormal sequence ϕ_i of eigenvectors with associated eigenvalues μ_i, and that for any $x \in H$, $A^{-1}x = \sum \mu_i (x, \phi_i)\phi_i$. One can see at once that this sum cannot be finite. Each of the ϕ_i are members of the domain of A and so a finite sum of them is as well. Then by applying A to this finite sum we would find that $x = \sum (x, \phi_i)\phi_i$, so that $x \in D(A)$. However x was arbitrary in H and then we would have $D(A) = H$. By closure of A this would imply that A is bounded. Therefore there are infinitely many μ_i and therefore they accumulate at 0. Thus the eigenvalues λ_i of A tend to ∞.

In fact one can conclude more. The operator A^{-1} certainly does not have 0 for an eigenvalue: $A^{-1}x = 0$ means $x = 0$ by application of A. The spectral theorem further gives us that if 0 is not an eigenvalue then the collection of eigenvectors is not only infinite but also a complete set in H. Furthermore, in this case the range of A^{-1} is exactly those elements $x \in H$ such that the sum $\sum \frac{1}{\mu_i}(x, \phi_i)\phi_i$ is convergent.

But the range of A^{-1} is $D(A)$! Thus we have justified the correctness of the formally appealing representations

$$A^{-1}x = \sum \mu_i(x,\phi_i)\phi_i \text{ for } x \in H$$

$$x = \sum (x,\phi_i)\phi_i \text{ for } x \in H$$

$$Ax = \sum \frac{1}{\mu_i}(x,\phi_i)\phi_i = \sum \lambda_i(x,\phi_i)\phi_i \text{ for } x \in D(A)$$

This also gives us a way to define the domain and action of fractional powers of A.

Let S_n denote the subspace of H spanned by the first n eigenvectors. For any element $u \in H$ there is associated its projection down to S_n. Write this projection as $P_n u = \sum_{i=1}^{n}(u,\phi_i)\phi_i$.

Lemma 5.1.1. *For $u \in D(A^{1/2})$, $\|u - P_n u\|^2 \leq \frac{1}{\lambda_{n+1}} \left\|A^{1/2}u\right\|^2$.*

Proof.

$$\begin{aligned}
\|u - P_n u\|^2 &= \left\| \sum_{i=n+1}^{\infty} (u,\phi_i)\phi_i \right\|^2 \\
&= \frac{1}{\lambda_{n+1}} \left\| \sum_{i=n+1}^{\infty} \lambda_{n+1}^{1/2}(u,\phi_i)\phi_i \right\|^2 \\
&\leq \frac{1}{\lambda_{n+1}} \left\| \sum_{i=n+1}^{\infty} \lambda_i^{1/2}(u,\phi_i)\phi_i \right\|^2 \\
&\leq \frac{1}{\lambda_{n+1}} \left\|A^{1/2}u\right\|^2
\end{aligned}$$

□

Corollary 5.1.2. *For $u \in D(A)$,* $\left\| A^{1/2}(u - P_n u) \right\|^2 \leq \frac{1}{\lambda_{n+1}} \left\| Au \right\|^2$.

Proof. First note that $A^{1/2}$ commutes with P_n, and then if $u \in D(A)$ we have that $A^{1/2}u \in D(A^{1/2})$ and so the previous result applies to $A^{1/2}u$. □

5.2 Formulation of spectral method

We restate the abstract Moore-Gibson-Thompson PDE for visual reference,

$$\tau u_{ttt} + \alpha u_{tt} - c^2 Au - bAu_t = 0$$

with initial conditions

$$u(t = 0) = u_0, u_t(t = 0) = u_1, u_{tt}(t = 0) = u_2$$

Consider a solution u on any of the spaces considered previously. By using the structure of the PDE we can determine the otherwise unknown coefficients (u, ϕ_i). Write $(u, \phi_i) = \gamma_i(t)$, so that $u = \sum \gamma_i(t)\phi_i(x)$. Multiply the equation by any of the eigenvectors ϕ_j. By orthogonality this reduces the equation to done only featuring the jth coefficients,

$$\tau\gamma_{j,ttt}(t) + \alpha\gamma_{j,tt}(t) + b\lambda_j\gamma_{j,t}(t) + c^2\lambda_j\gamma_j(t) = 0$$

We can write this as a matrix ODE

$$\frac{d}{dt}\begin{pmatrix} \gamma_j \\ \gamma_{j,t} \\ \gamma_{j,tt} \end{pmatrix} = \begin{pmatrix} 0 & I & 0 \\ 0 & 0 & I \\ -\frac{\lambda_j c^2}{\tau} & -\frac{\lambda_j b}{\tau} & -\frac{\alpha}{\tau} \end{pmatrix} \begin{pmatrix} \gamma_j \\ \gamma_{j,t} \\ \gamma_{j,tt} \end{pmatrix}$$

With $\gamma_j(0) = (u_0, \phi_j)$ and likewise for the other initial conditions. Then this method allows one to explicitly determine the the coefficients (u, ϕ_i), and therefore the projection $P_n u$ can be computed for any n. Note that $u_n = P_n u$ solves the weak form of the PDE

$$\tau(u_{n,ttt}, \psi) + \alpha(u_{n,tt}, \psi) + b(A^{1/2}u_{n,t}, A^{1/2}\psi) + c^2(A^{1/2}u_n, A^{1/2}\psi) = 0 \ \forall \psi \in S_n$$

$$u_n(0) = P_n u_0, \ u_{n,t}(0) = P_n u_1, \ u_{n,tt}(0) = P_n u_2$$

This method also easily adapts to a nonhomogeneous equation, with nonzero forcing term. Suppose that $f \in H$, so that it has an eigenvector decomposition. Then the equation

$$\tau u_{ttt} + \alpha u_{tt} - c^2 Au - bAu_t = f$$

after being hit with ϕ_j will once again reduce to one in only the jth coordinate,

$$\tau\gamma_{j,ttt}(t) + \alpha\gamma_{j,tt}(t) + b\lambda_j\gamma_{j,t}(t) + c^2\lambda_j\gamma_j(t) = f_j = (f, \phi_j)$$

This gives us the matrix equation for this component,

$$\frac{d}{dt}\begin{pmatrix} \gamma_j \\ \gamma_{j,t} \\ \gamma_{j,tt} \end{pmatrix} = \begin{pmatrix} 0 & I & 0 \\ 0 & 0 & I \\ -\frac{\lambda_j c^2}{\tau} & -\frac{\lambda_j b}{\tau} & -\frac{\alpha}{\tau} \end{pmatrix} \begin{pmatrix} \gamma_j \\ \gamma_{j,t} \\ \gamma_{j,tt} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \frac{1}{\tau} f_j \end{pmatrix}$$

With initial conditions as before. This can be solved, for example by the variation of parameters formula for matrix exponentials.

We now consider the stability of the finite dimensional solution $u_n = P_n u$.

5.3 Stability of spectral solutions

Note that the multipliers $u_n, u_{n,t}$, and $u_{n,tt}$ are all in S_n and therefore are admissible multipliers in the weak formulation that u_n itself solves. The abstract stability results 3.1.1 apply to these solutions. However, unlike the finite element method, these solutions are very smooth and therefore we can apply higher multipliers such as Au_n. This will lead to a stability result in terms of a higher norm.

Theorem 5.3.1. *With reference to the spectral approximations $u_n = P_n u$ to the homogeneous MGT equation, there exists a constant $C > 0$ such that,*

$$E_1(t) + \int_0^T E_1(s)\ ds \leq CE_1(0)$$

Recall that $E_1(t)$ is equivalent to $\|Au(t)\|^2 + \left\|A^{1/2}u_t(t)\right\|^2 + \|u_{tt}(t)\|^2 \sim E(t) + \|Au(t)\|^2$.

Proof. To cut down on subscripts, for this proof u will stand for the approximation $P_n u$. Multiply by Au:

$$\frac{b}{2}\frac{d}{dt}\|Au\|^2 + c^2\|Au\|^2 = -\tau(u_{ttt}, Au) - \alpha(u_{tt}, Au)$$

On the right use the identities

$$\frac{d}{dt}(u_t, Au) = (u_{tt}, Au) + \left\|A^{1/2}u_t\right\|^2$$

$$\frac{d}{dt}(u_{tt}, Au) = (u_{ttt}, Au) + \frac{1}{2}\frac{d}{dt}\left\|A^{1/2}u_t\right\|^2$$

So that

$$\frac{b}{2}\frac{d}{dt}\left\|Au\right\|^2 + c^2\left\|Au\right\|^2 = \frac{\tau}{2}\frac{d}{dt}\left\|A^{1/2}u_t\right\|^2 - \tau\frac{d}{dt}(u_{tt}, Au) + \alpha\left\|A^{1/2}u_t\right\|^2 - \alpha\frac{d}{dt}(u_t, Au)$$

After integration this becomes

$$\begin{aligned}\frac{b}{2}\left\|Au(t)\right\|^2 + c^2\int_0^t \left\|Au(s)\right\|^2\,ds &= \frac{b}{2}\left\|Au(0)\right\|^2 + \frac{\tau}{2}\left.\left\|A^{1/2}u_t\right\|^2\right|_0^t \\ &- \tau\left.(u_{tt}, Au)\right|_0^t + \alpha\int_0^t \left\|A^{1/2}u_t(s)\right\|^2\,ds - \alpha\left.(u_t, Au)\right|_0^t\end{aligned}$$

We will now use the inequality $ab \le \frac{a^2}{2\epsilon} + \frac{\epsilon b^2}{2}$, first with $\epsilon = \frac{b}{4\tau}$ and then with $\epsilon = \frac{4\alpha}{b}$. This will produce

$$\begin{aligned}\le \frac{b}{2}\left\|Au(0)\right\|^2 + \frac{\tau}{2}\left.\left\|A^{1/2}u_t\right\|^2\right|_0^t + \tau\left.\left(\frac{b}{8\tau}\left\|Au\right\|^2 + \frac{2\tau}{b}\left\|u_{tt}\right\|^2\right)\right|_0^t + \alpha\int_0^t \left\|A^{1/2}u_t(s)\right\|^2\,ds \\ + \alpha\left.\left(\frac{2\alpha}{b}\left\|u_t\right\|^2 + \frac{b}{8\alpha}\left\|Au\right\|^2\right)\right|_0^t\end{aligned}$$

We now can collect terms involving $\left\|Au(t)\right\|^2$ and $\left\|Au(0)\right\|^2$ to the left and right side respectively, arriving at

$$\begin{aligned}\frac{b}{4}\left\|Au(t)\right\|^2 + c^2\int_0^t \left\|Au(s)\right\|^2\,ds &\le \frac{b}{4}\left\|Au(0)\right\|^2 + \frac{\tau}{2}\left.\left\|A^{1/2}u_t\right\|^2\right|_0^t + \left.\frac{2\tau^2}{b}\left\|u_{tt}\right\|^2\right|_0^t \\ &+ \alpha\int_0^t \left\|A^{1/2}u_t(s)\right\|^2\,ds + \left.\frac{2\alpha^2}{b}\left\|u_t\right\|^2\right|_0^t\end{aligned}$$

All the right hand terms other than $\left\|Au(0)\right\|^2$ are bounded by the initial energy, by the lower stability result. Therefore,

$$\frac{b}{4}\left\|Au(t)\right\|^2 + c^2\int_0^t \left\|Au(s)\right\|^2\,ds \le \frac{b}{4}\left\|Au(0)\right\|^2 + CE(0)$$

Since $E_1(t) = E(t) + \|Au(t)\|^2$, we combine this with the lower stability inequality to find

$$E_1(t) + \int_0^t E_1(s)\,ds \le CE_1(0)$$

□

Exponential stability in E_1 norm follows from Pazy-Datko's theorem, and we already had exponential stability in the E norm via the abstact inequalities.

Corollary 5.3.2. *There exist positive constants C, C_1, ω, ω_1 such that, with respect to the approximations $u_n = P_n u$,*

$$E(T) \le Ce^{-\omega T} E(0)$$

$$E_1(T) \le C_1 e^{-\omega_1 T} E_1(0)$$

We can also build upon these calculations to produce stability statements for the nonhomogeneous equation.

Corollary 5.3.3. *Suppose that $u_n = P_n u$ is the approximation to a solution u to the nongomogeneous MGT equation, with nonzero right hand side $f \in L^2((0,T); H)$. Then,*

$$E_1(t) + \int_0^t E_1(s)\,ds \le C\left(E_1(0) + \int_0^t \|f(s)\|^2\,ds\right)$$

Proof. The presence of a nonzero f term would contribute (f, Au) after multiplication by Au. Following the calculations from before, this leads to

$$\frac{b}{4}\|Au(t)\|^2 + c^2 \int_0^t \|Au(s)\|^2\,ds \le \frac{b}{4}\|Au(0)\|^2 + CE(0) + \int_0^t (f(s), Au(s))\,ds$$

Use the inequality $(f(s), Au(s)) \leq \|f(s)\| \|Au(s)\| \leq \frac{c^2}{2} \|Au(s)\|^2 + \frac{1}{2c^2} \|f(s)\|^2$, so that

$$\frac{b}{4} \|Au(t)\|^2 + \frac{c^2}{2} \int_0^t \|Au(s)\|^2 \, ds \leq \frac{b}{4} \|Au(0)\|^2 + CE(0) + \int_0^t \|f(s)\|^2 \, ds$$

Adding in the lower stability as before we have the result. □

5.4 Convergence of spectral solutions

As indicated in lemma 5.1.1 and corollary 5.1.2, the convergence of the approximations u_n have good error bounds when compared to the solutions u provided that u has good membership in the domain of A. This reduces the question of convergence of the solutions to the spectral method scheme to one of a semigroup calculation, selecting smooth enough initial data to ensure the desired memerships hold, paired with use of stability inequalities to bound the right hand side of lemma 5.1.1 and cor. 5.1.2.

Theorem 5.4.1. *Suppose that the initial data* $\overline{U}_0$ *are in* $\mathcal{H}$. *Then there exist constants* $\omega_0, \omega_1, C_0, C_1 > 0$ *such that*

$$\|u(t) - P_n u(t)\|^2 \leq C_0 \frac{1}{\lambda_{n+1}} e^{-\omega_0 t} \left(\left\|A^{1/2} u_0\right\|^2 + \left\|A^{1/2} u_1\right\|^2 + \|u_2\|^2 \right)$$

And

$$\|u_t(t) - P_n u_t(t)\|^2 \leq C_1 \frac{1}{\lambda_{n+1}} e^{-\omega_1 t} \left(\left\|A^{1/2} u_0\right\|^2 + \left\|A^{1/2} u_1\right\|^2 + \|u_2\|^2 \right)$$

Suppose further that the intitial data are in $D(\mathcal{A})$ *on* $\mathcal{H}$. *Then there exist constants* $\omega_2, C_2 > 0$ *such that*

$$\|u_{tt}(t) - P_n u_{tt}(t)\|^2 \leq C_2 \frac{1}{\lambda_{n+1}} e^{-\omega_2 t} \left(\|Az_0\|^2 + \left\|A^{1/2}u_1\right\|^2 + \left\|A^{1/2}u_2\right\|^2 \right)$$

Finally, if in additon we have $\overline{U}_0 \in D(\mathcal{A}^2)$ *there exist constants* $\omega_3, C_3 > 0$ *such that*

$$\|u_{ttt}(t) - P_n u_{ttt}(t)\|^2 \leq C_3 \frac{1}{\lambda_{n+1}} e^{-\omega_3 t} \left(\left\|A^{3/2}z_0\right\|^2 + \|Az_1\|^2 + \left\|A^{1/2}u_2\right\|^2 \right)$$

Proof. For the first case, well-posedness on $\mathcal{H}$ will give that $u(t)$ and $u_t(t)$ exist in $D(A^{1/2})$. Therefore we are justified in the estimates

$$\|u(t) - P_n u(t)\|^2 \leq \frac{1}{\lambda_{n+1}} \left\|A^{1/2}u(t)\right\|^2$$

$$\|u_t(t) - P_n u_t(t)\|^2 \leq \frac{1}{\lambda_{n+1}} \left\|A^{1/2}u_t(t)\right\|^2$$

The norms on the right hand side of these inequalities can be bounded via the (exponential) stability on the space $\hat{\mathcal{H}}_1$, see theorem 2.3.2

$$\left\|A^{1/2}u(t)\right\|^2 \leq C_0 e^{-\omega_0 t} \left(\left\|A^{1/2}u_0\right\|^2 + \left\|A^{1/2}u_1\right\|^2 + \|u_2\|^2 \right)$$

$$\left\|A^{1/2}u_t(t)\right\|^2 \leq C_1 e^{-\omega_1 t} \left(\left\|A^{1/2}u_0\right\|^2 + \left\|A^{1/2}u_1\right\|^2 + \|u_2\|^2 \right)$$

This covers the first case.

Moving on to the next case, if the inital data are in $D(\mathcal{A})$ then $u_{tt}(t) \in D(A^{1/2})$ and we are justified in applying

$$\|u_{tt}(t) - P_n u_{tt}(t)\|^2 \leq \frac{1}{\lambda_{n+1}} \left\|A^{1/2}u_{tt}(t)\right\|^2$$

However the stability estimates will not immediately provide us decay or even boundedness of $u_{tt} \in D(A^{1/2})$. To overcome this problem we will morally differentiate the MGT equation with respect to time and then make a change of variables $w = u_t$, and then apply stability estimates to w. More precisely however, we will give appropriate initial data that will produce the $w = u_t$ sought, and also ensure well-posedness and stability for this new variable. This strategy will be reused repeatedly for the remainder of the convergence results.

Consider new initial conditions

$$w_0 = u_1$$

$$w_1 = u_2$$

$$w_2 = -\frac{\alpha}{\tau}u_2 - \frac{b}{\tau}Az_0$$

Note the use of z_0 in w_2. Formally if we simply attempted to write $\overline{W}_0 = \mathcal{A}\overline{U}_0$ to match up a time derivative we would have our third row read $w_2 = -\frac{\alpha}{\tau}u_2 - \frac{b}{\tau}Au_1 - \frac{c^2}{\tau}Au_0$. However the smoothness of our u-variables saturates at $D(A^{1/2})$ and we are not permitted to apply A to u_0 and u_1 directly. Membership in the domain of $\mathcal{A}$ instead gives that $z_0 \in D(A)$, and therefore we must take u_0 and u_1 in combination. Under this formulation these initial states provide a solution w on $\mathcal{H}$ that corresponds to u_t. The stability estimate for this variable includes $\left\|A^{1/2}w_t(t)\right\|^2 = \left\|A^{1/2}u_{tt}(t)\right\|^2$, bounded in terms of the w initial states:

$$\left\|A^{1/2}u_{tt}(t)\right\|^2 \leq C_2 e^{-\omega_2 t}\left(\left\|A^{1/2}u_1\right\|^2 + \left\|A^{1/2}u_2\right\|^2 + \|Az_0\|^2\right)$$

Note that the lower $\|u_2\|^2$ is absorbed into $\left\|A^{1/2}u_2\right\|^2$. This will provide the desired estimate.

For the final case the situation is the same. The assumption that the initial data is in $D(\mathcal{A}^2)$ yields the infomation that $u_{ttt}(t) \in D(A^{1/2})$ so that we can apply

$$\|u_{ttt}(t) - P_n u_{ttt}(t)\|^2 \leq \frac{1}{\lambda_{n+1}}\left\|A^{1/2}u_{ttt}(t)\right\|^2$$

But as before the naive stablity estimate will not provide for $u_{ttt}(t)$ at all, let alone in $D(A^{1/2})$. Therefore we turn to essentially $w = u_{tt}$ with the initial data

$$w_0 = u_2$$

$$w_1 = -\frac{\alpha}{\tau}u_2 - \frac{b}{\tau}Az_0$$
$$w_2 = \frac{\alpha^2}{\tau^2}u_2 + \frac{\alpha b}{\tau^2}Az_0 - \frac{b}{\tau}Az_1$$

The assumption that the initial data are in $D(\mathcal{A}^2)$ give that $z_0 \in D(A^{3/2})$ and $z_1 \in D(A)$ so that the initial w variables live in $\mathcal{H}_1$, and then the energy estimates on the $w_t = u_{ttt}$ level read

$$\left\|A^{1/2}u_{ttt}(t)\right\|^2 \leq C_3 e^{-\omega_3 t}\left(\left\|A^{1/2}u_2\right\|^2 + \left\|A^{3/2}z_0\right\|^2 + \|Az_1\|^2\right)$$

Where again the lower terms have been absorbed into the highest powers of A. $\square$

Theorem 5.4.2. *Suppose that $\overline{U}_0 \in \mathcal{H}_1$. Then there exist constants $\omega_0, C_0 > 0$ such that*

$$\left\|A^{1/2}(u(t) - P_n u(t))\right\|^2 \leq C_0 \frac{1}{\lambda_{n+1}} e^{-\omega_0 t} \left(\|Au_0\|^2 + \left\|A^{1/2}u_1\right\|^2 + \|u_2\|^2\right)$$

Suppose next that $\overline{U}_0 \in D(\mathcal{A})$ on $\mathcal{H}_1$. Then there exist $\omega_1, C_1 > 0$ such that

$$\left\|A^{1/2}(u_t(t) - P_n u(t))\right\|^2 \leq C_1 \frac{1}{\lambda_{n+1}} e^{-\omega_1 t} \left(\|Au_0\|^2 + \|Au_1\|^2 + \left\|A^{1/2}u_2\right\|^2\right)$$

Suppose next that $\overline{U}_0 \in D(\mathcal{A}^2)$ on $\mathcal{H}_1$. Then there exist $\omega_2, C_2 > 0$ such that

$$\left\|A^{1/2}(u_{tt}(t) - P_n u_{tt}(t))\right\|^2 \leq C_2 \frac{1}{\lambda_{n+1}} e^{-\omega_2 t} \left(\left\|A^{3/2}z_0\right\|^2 + \|Au_1\|^2 + \|Au_2\|^2\right)$$

Finally, suppose that $\overline{U}_0 \in D(\mathcal{A}^3)$ on $\mathcal{H}_1$. Then there exist $\omega_3, C_3 > 0$ such that

$$\left\|A^{1/2}(u_{ttt}(t) - P_n u_{ttt}(t))\right\|^2 \leq C_3 \frac{1}{\lambda_{n+1}} e^{-\omega_3 t} \left(\left\|A^2 z_0\right\|^2 + \left\|A^{3/2}z_1\right\|^2 + \|Au_2\|^2\right)$$

Proof. For the first case, observe that this will correspond to $u(t)$ existing in $D(A)$. Therefore we can apply the estimate

$$\left\|A^{1/2}(u(t) - P_n u(t))\right\|^2 \leq \frac{1}{\lambda_{n+1}} \|Au(t)\|^2$$

We refer to Theorem 2.3.2 for the stability of our continuous solution u in the E_1 norm - note that our assumption is compatable with the requirement of well-posedness on the higher energy space in that paper. Therefore we can bound $\|Au(t)\|^2$ by the initial data in this topology.

For the second case, on $\mathcal{H}_1$ our assumption will give us that $u_t(t) \in D(A)$. This permits the application of

$$\left\|A^{1/2}(u_t(t) - P_n u_t(t))\right\|^2 \leq \frac{1}{\lambda_{n+1}} \left\|Au_t(t)\right\|^2$$

Now we wish to use stability results to bound the right-hand side. We wish to consider $w = u_t$. We use the initial data $w_0 = u_1, w_1 = u_2, w_2 = -\frac{\alpha}{\tau}u_2 - \frac{b}{\tau}Au_1 - \frac{c^2}{\tau}Au_0$. Note that we do have sufficient smoothness from u_0 and u_1 for A to be applied to them directly, and likewise $w_0 \in D(A)$, $w_1 \in D(A^{1/2})$, and $w_2 \in H$, which is sufficient for the E_1 well-posedness and stability. So then we obtain positive constants C, ω such that

$$\left\|Au_t(t)\right\|^2 \leq Ce^{-\omega t}\left(\left\|Au_1\right\|^2 + \left\|A^{1/2}u_2\right\|^2 + \left\|-\frac{\alpha}{\tau}u_2 - \frac{b}{\tau}Au_1 - \frac{c^2}{\tau}Au_0\right\|^2\right)$$

Bounding to keep only the dominant norms, we have our result.

Finally, with $\overline{U}_0 \in D(\mathcal{A}^2)$ we have that $u_{tt} \in D(A)$, thus

$$\left\|A^{1/2}(u_{tt}(t) - P_n u_{tt}(t))\right\|^2 \leq \frac{1}{\lambda_{n+1}} \left\|Au_{tt}(t)\right\|^2$$

To appeal to stability we must consider the variable $w = u_{tt}$. The appropriate initial data will be as before

$$w_0 = u_2$$

$$w_1 = -\frac{b}{\tau}Az_0 - \frac{\alpha}{\tau}u_2$$

$$w_2 = -\frac{b}{\tau}Az_1 + \frac{\alpha b}{\tau^2}Az_0 + \frac{\alpha^2}{\tau^2}u_2$$

Which for stability we will require to be in $\mathcal{H}_1$. The use of z_0 in w_1 is critical, because our assumptions on $\overline{U}_0$ do not provide that Au_0 and Au_1 are in $D(A^{1/2})$. However they do place $z_0 \in D(A^{3/2})$. Therefore the well-posedness and stability give us (after bounding)

$$\|Au_{tt}(t)\|^2 \leq C_2 e^{-\omega_2 t}\left(\|Au_2\|^2 + \left\|A^{3/2}z_0\right\|^2 + \|Az_1\|^2\right)$$

To remain in u variables to the greatest extent possible, note that Au_2 and Az_1 are controlled by Au_2 and Au_1.

For our final case, since the initial data are in $D(\mathcal{A}^3)$ we have ensured that $u_{ttt}(t) \in D(A)$, allowing applicability of the estimate

$$\left\|A^{1/2}(u_{ttt}(t) - P_n u_{ttt}(t))\right\|^2 \leq \frac{1}{\lambda_{n+1}} \|Au_{ttt}(t)\|^2$$

We are thus led to consider the variable $w = u_{ttt}$, which will have the following initial conditions:

$$w_0 = -\frac{b}{\tau}Az_0 - \frac{\alpha}{\tau}u_2$$

$$w_1 = -\frac{b}{\tau}Az_1 + \frac{\alpha b}{\tau^2}Az_0 + \frac{\alpha^2}{\tau^2}u_2$$

$$w_2 = \frac{b^2}{\tau^2}A^2 z_0 - \frac{\alpha^2 b}{\tau^2}Az_0 + \frac{\alpha b}{\tau^2}Az_1 + \frac{b}{\tau^2}\gamma Au_2 - \frac{\alpha^3}{\tau^3}u_2$$

Recall that $\gamma = \alpha - \frac{\tau c^2}{b}$. Our assumption that the initial data is in $D(\mathcal{A}^3)$ also gives $z_0 \in D(A^2)$ and $z_1 \in D(A^{\frac{3}{2}})$, along with $u_0, u_1, u_2 \in D(A)$. Therefore the initial data for w lie in the appropriate space for well-posedness and decay of the higher energy. $\square$

We wish now to repeat these calculations for the nonhomogeneous problem. The time differentiability is more sensitive in this case so we will begin with an overview.

Lemma 5.4.3. *Suppose that $u_0 \in H$, $F(s) \in L^1(0,t;H)$. Then there exists an explicit mild solution to the problem*

$$u_t = Au + F$$

$$u(0) = u_0$$

Given by the formula

$$u(t) = T(t)u_0 + \int_0^t T(t-s)F(s)\,ds$$

If furthermore $u_0 \in D(A)$ and $F(s)$ is either $C^1(0,t;H)$ or is integrable in the domain of A on $(0,t)$, then this is in fact a solution, i.e. $u(s) \in D(A)$ and satisfies the differential equation on H.

If we assume that $u_0 \in D(A)$ and that $F(s)$ is in $C^1(0,t;H)$, then the solution to the equation

$$w_t = Aw + F_t$$

$$w(0) = Au(0) + F(0)$$

In fact satisfies $w = u_t$.

Proof. The first two claims are simply theorems from Pazy [12], section 4.2. For the final claim, which will be of use to us shortly, note that under our assumptions F_t is in $L^1(0,t;H)$ and $Au(0) + F(0) \in H$, so therefore this problem has at least the mild solution. We wish to show that $w = u_t$. Therefore, we will differentiate u. To make the identity clear we will make a change of variable: We can rewrite $\int_0^t T(t-s)F(s)$ as $\int_0^t T(s)F(t-s)\,ds$. Therefore,

$$\frac{d}{dt}u(t) = T(t)Au(0) + \int_0^t T(s)F'(t-s)\,ds + T(t)F(0)$$

Where we have used the fact that $u_0 \in D(A)$ in the first term, and the second two terms come from differentiation of the integral. We can now undo the change of variables in the integral, which would leave us with

$$u_t(t) = T(t)Au_0 + T(t)F(0) + \int_0^t T(t-s)F'(s)\,ds$$

But the variation of parameters solution for w reads

$$w(t) = T(t)w(0) + \int_0^t T(t-s)F'(s)\,ds$$

And as $w(0) = Au(0) + F(0)$, this proves our claim. □

Theorem 5.4.4. *Suppose with respect to the nonhomogeneous problem the intial data* $\overline{U}_0 \in \mathcal{H}$, *and* $f \in L^2(0,t;H)$. *Then*

$$\|P_n u(t) - u(t)\|^2 \leq C\frac{1}{\lambda_{n+1}}\left(\left\|A^{1/2}u_0\right\|^2 + \left\|A^{1/2}u_1\right\|^2 + \|u_2\|^2 + \int_0^t \|f(s)\|^2\,ds\right)$$

And

$$\|P_n u_t(t) - u_t(t)\|^2 \leq C\frac{1}{\lambda_{n+1}}\left(\left\|A^{1/2}u_0\right\|^2 + \left\|A^{1/2}u_1\right\|^2 + \|u_2\|^2 + \int_0^t \|f(s)\|^2\,ds\right)$$

If $\overline{U}_0 \in D(\mathcal{A})$ *on* $\mathcal{H}$, $f \in C^1(0,t;H)$, *then*

$$\|P_n u_{tt}(t) - u_{tt}(t)\|^2 \leq C\frac{1}{\lambda_{n+1}}\left(\|Az_0\|^2 + \left\|A^{1/2}u_1\right\|^2 + \left\|A^{1/2}u_2\right\|^2 + \|f(0)\|^2 + \int_0^t \|f'(s)\|^2\,ds\right)$$

If $\overline{U}_0 \in D(\mathcal{A})^2,\ f(0) \in D(A^{1/2}),\ f \in C^2(0,t;H),$

$$\|P_n u_{ttt}(t) - u_{ttt}(t)\|^2 \leq C\frac{1}{\lambda_{n+1}}\left(\left\|A^{3/2}z_0\right\|^2 + \|Az_1\|^2 + \left\|A^{1/2}u_2\right\|^2 + \left\|A^{1/2}f(0)\right\|^2 + \|f'(0)\|^2 + \int_0^t \|f''(s)\|^2\, ds\right)$$

Proof. The first two claims are proven identically to the first two claims in theorem 5.4.1, with the mofidication that we use the stability estimates for the nonhomogeneous problem, which contribute the integral of f term.

For the third inequality, since $\overline{U}_0 \in D(\mathcal{A})$ and $f \in C^1(0,t;H)$, $w = u_t$ solving

$$\overline{W}_t = \mathcal{A}\overline{W} + F_t$$

$$F = (0,0,f)^\intercal, \overline{W}(0) = \mathcal{A}\overline{U}(0) + F(0)$$

exists, per lemma 5.4.3. Thus it will have structurally the same stability as u, and the initial energy will be that of $\mathcal{A}\overline{U}_0$ (as before in theorem 5.4.1), with the additon of $F(0)$.

For the final inequality, $\overline{U}_0 \in D(\mathcal{A}^2)$, $F(0) \in D(\mathcal{A})$, so $\mathcal{A}(\mathcal{A}\overline{U}_0 + F(0)) \in \mathcal{H}$. Also f is in $C^2(0,t;H)$, so $F_t \in C^1(0,t;H)$. Therefore $v = w_t = u_{tt}$ exists solving

$$\overline{V}_t = \mathcal{A}\overline{V} + F_{tt}$$

$$\overline{V}(0) = \mathcal{A}(\mathcal{A}\overline{U}_0 + F(0)) + F_t(0)$$

Since we have assumed that $f(0)$ is in $D(A^{1/2})$ we have that $F(0) \in D(\mathcal{A})$ and we assumed that $\overline{U}_0 \in D(\mathcal{A}^2)$ so we can measure the initial energy for V in $\mathcal{H}$ as

$$\underbrace{\left\|A^{3/2}z_0\right\|^2 + \|Az_1\|^2 + \left\|A^{1/2}u_2\right\|^2}_{\text{initial homogeneous energy}} + \underbrace{\|f(0)\|^2 + \left\|A^{1/2}f(0)\right\|^2}_{\text{contribution of } \mathcal{A}F(0)} + \|f'(0)\|^2$$

□

Theorem 5.4.5. *Suppose* $\overline{U}_0 \in \mathcal{H}_1$, $f \in L^2(0,t;H)$. *Then,*

$$\left\|A^{1/2}(u(t) - P_n u(t))\right\|^2 \leq C\frac{1}{\lambda_{n+1}}\left(\|Au_0\|^2 + \left\|A^{1/2}u_1\right\|^2 + \|u_2\|^2 + \int_0^t \|f(s)\|^2\, ds\right)$$

Suppose that $\overline{U}_0 \in D(\mathcal{A})$ *on* $\mathcal{H}_1$, $f \in C^1(0,t;H)$. *Then,*

$$\left\|A^{1/2}(u_t(t) - P_n u_t(t))\right\|^2 \leq C\frac{1}{\lambda_{n+1}}\left(\|Au_0\|^2 + \|Au_1\|^2 + \left\|A^{1/2}u_2\right\|^2 + \|f(0)\|^2 + \int_0^t \|f'(s)\|^2\, ds\right)$$

Suppose that $\overline{U}_0 \in D(\mathcal{A}^2)$ *on* $\mathcal{H}_1$, $f \in C^2(0,t;H)$, $f(0) \in D(A^{1/2})$. *Then,*

$$\left\|A^{1/2}(u_{tt}(t) - P_n u_{tt}(t))\right\|^2 \leq C\frac{1}{\lambda_{n+1}}\left(\left\|A^{3/2}z_0\right\|^2 + \|Au_1\|^2 + \|Au_2\|^2 + \left\|A^{1/2}f(0)\right\|^2 + \|f'(0)\|^2 + \int_0^t \|f''(s)\|^2\, ds\right)$$

Suppose finally that $\overline{U}_0 \in D(\mathcal{A}^3)$ *on* $\mathcal{H}_1$, $f \in C^3(0,t;H)$, $f(0) \in D(A)$, $f'(0) \in D(A^{1/2})$. *Then,*

$$\left\|A^{1/2}(u_{ttt}(t) - P_n u_{ttt}(t))\right\|^2 \leq C\frac{1}{\lambda_{n+1}}\left(\left\|A^2 z_0\right\|^2 + \left\|A^{3/2}z_1\right\|^2 + \|Au_2\|^2 + \|Af(0)\|^2 + \left\|A^{1/2}f'(0)\right\|^2 + \|f_{tt}(0)\|^2 + \int_0^t \|f'''(s)\|^2\, ds\right)$$

Proof. The first claim follows just as in theorem 5.4.2, with the addition of the integral owing to the presence of f in the stability estimates for $A^{1/2}u$.

For the second claim, if $\overline{U}_0 \in D(\mathcal{A})$ and $f \in C^1(0,t;H)$ then $w = u_t$ exists with stability estimates following from

$$\overline{W}_t = \mathcal{A}\overline{W} + F'$$

$$F = (0,0,f)^\intercal, \overline{W}_0 = \mathcal{A}\overline{U}_0 + F(0)$$

Then the stability of Au_t will come from the w-energy, where as in 5.4.2 $\mathcal{A}\overline{U}_0$ will contribute $\|Au_0\|^2, \|Au_1\|^2$, and $\left\|A^{1/2}u_2\right\|$ and $F(0)$ contributes $\|f(0)\|^2$.

Under the assumptions of the third claim, $v = u_{tt}$ exists with

$$\overline{V}_t = \mathcal{A}\overline{V} + F''$$

$$\overline{V}_0 = \mathcal{A}\overline{W}_0 + F'(0) = \mathcal{A}^2\overline{U}_0 + \mathcal{A}F(0) + F'(0)$$

$$F = (0,0,f)^\intercal$$

and therefore v has stability estimates following the nonhomoegenous problem. As before in theorems 5.4.1 and 5.4.2 at this point we no longer have enough individual smoothness from u_0, u_1, u_2 to make the transit from the z variables back to u and therefore our final stability statement properly involves z.

$$\|Au_{tt}(t)\|^2 \leq \underbrace{\left\|A^{3/2}z_0\right\|^2 + \|Au_1\|^2 + \|Au_2\|^2}_{\textit{homogeneous energy for } \mathcal{A}^2\overline{U}_0} + \underbrace{\left\|A^{1/2}f(0)\right\|^2}_{\mathcal{A}F(0)} + \underbrace{\|f'(0)\|^2}_{F'(0)} + \int_0^t \|f''(s)\|^2\,ds$$

Note as well that we absorbed lower powers of A to keep only the dominant norms.

For the final claim, $y = u_{ttt}$ exists by our assumptions with

$$\overline{Y}_t = \mathcal{A}\overline{Y} + F'''$$

$$\overline{Y}_0 = \mathcal{A}\overline{V}_0 + F''(0) = \mathcal{A}^3\overline{U}_0 + \mathcal{A}^2 F(0) + \mathcal{A}F'(0) + F''(0)$$

$$F = (0, 0, f)^\intercal$$

Then the stability calculation for $\overline{Y}$ in $\mathcal{H}_1$ says,

$$\begin{aligned}\|Au_{ttt}\|^2 \leq{}& \underbrace{\left\|A^2 z_0\right\|^2 + \left\|A^{3/2} z_1\right\|^2 + \|Au_2\|^2}_{\textit{homogeneous energy of } \mathcal{A}^3\overline{U}_0} + \underbrace{\|Af(0)\|^2}_{\mathcal{A}^2 F(0)} + \underbrace{\left\|A^{1/2} f'(0)\right\|^2}_{\mathcal{A}F'(0)} \\ &+ \underbrace{\|f_{tt}(0)\|^2}_{F''(0)} + \int_0^t \|f'''(s)\|^2 \, ds\end{aligned}$$

$\square$

Chapter 6

Numerical Illustrations

6.1 Stability analysis for the spectral method

Recall that the solutions obtained by the spectral method have the structure $P_n u(x,t) = \sum_{i=1}^{n} \gamma_i(t)\phi_i(x)$, where the ϕ_i are eigenvectors of a differential operator A and the $\gamma_i(t)$ are determined by a matrix ODE

$$\frac{d}{dt}\begin{pmatrix} \gamma_j \\ \gamma_{j,t} \\ \gamma_{j,tt} \end{pmatrix} = \begin{pmatrix} 0 & I & 0 \\ 0 & 0 & I \\ -\frac{\lambda_j c^2}{\tau} & -\frac{\lambda_j b}{\tau} & -\frac{\alpha}{\tau} \end{pmatrix} \begin{pmatrix} \gamma_j \\ \gamma_{j,t} \\ \gamma_{j,tt} \end{pmatrix}$$

Let M_j denote the matrix on the right-hand side of this equation. Writing $\bar{\gamma}_j = \begin{pmatrix} \gamma_j \\ \gamma_{j,t} \\ \gamma_{j,tt} \end{pmatrix}$, M for the $3n \times 3n$ matrix direct sum of the M_j, we can simultaneously write all n of the above matrix ODEs by

$$\frac{d}{dt}\begin{pmatrix}\bar{\gamma}_1\\ \vdots\\ \bar{\gamma}_n\end{pmatrix} = \begin{pmatrix}M_1 & \cdots & 0\\ 0 & \ddots & 0\\ 0 & \cdots & M_n\end{pmatrix}\begin{pmatrix}\bar{\gamma}_1\\ \vdots\\ \bar{\gamma}_n\end{pmatrix}$$

Or,

$$\frac{d}{dt}\begin{pmatrix}\bar{\gamma}_1\\ \vdots\\ \bar{\gamma}_n\end{pmatrix} = M\begin{pmatrix}\bar{\gamma}_1\\ \vdots\\ \bar{\gamma}_n\end{pmatrix}$$

The eigenvalues of M are just the collection of the eigenvalues of the individual M_j, and the matrix exponential of M is just a direct sum of the matrix exponentials of the M_j. As a finite-dimensional system, the stability of the solution - meaning here the vectors $\bar{\gamma}_j$ - is determined exactly by the placement of the spectrum. We will demonstrate the effects of the constants of the problem α, b, τ, c^2 on the location of the spectrum, and show plots of the higher and lower energy for computed solutions.

6.1.1 Description of procedures

All the figures to follow were produced using Wolfram Mathematica 8.

For the illustrations of the spectrum our procedures are as follows. For figures 6.1 through 6.3 we treat the eigenvalues λ_j as a continuous parameter ν in the matrices M_j and then plot the spectrum of the resulting matrices. The values for the physical parameters are given in each figure, and ν varies over a range with upper bound 1000 and lower bound as indicated in each figure.

For figures 6.4 through 6.8 we examine the sensitivity of low eigenvalues, represented by $\nu = 2$, to the physical constants of the equation. In each figure one constant varies over a given range and again the spectra of the matrices M_j are plotted over the variation in each variable. The square circle and diamond are placed arbitrarily to coarsely illustrate the rate of change.

Figure 6.9 is produced in the same manner, but this time comparing the α-variation for $\nu = 2$ with $\nu = 3$. The spectrum has conjugate complex branches and also values along the real axis. This figure is interested in studying only the complex branch. The optimal values can be verified with an optimization method on the real part of the spectrum in each case.

Moving on the the plots of the energy in figures 6.10 through 6.16, we compute the solutions to the spectral method in the concrete case $\Omega = [0,1]$, $A = \Delta = \frac{d^2}{dx^2}$, with 0-Dirichlet boundary conditions, $H = L^2(\Omega)$, $D(A^{1/2}) = H_0^1(\Omega)$, $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$. Then the associated eigenvectors of A are $\phi_i = \sqrt{2}\sin(i\pi x)$ with eigenvalues $\lambda_i = i^2\pi^2$. In these figures we take $n = 50$. The initial data used are $u_0 = 100x(x-1)$, $u_1 = u_2 = 0$. Under those selections we then form the matrices M_j as discussed in the start of this chapter, and then solve the matrix ODE for M via matrix exponentials for M_j. This allows us to construct the time components $\bar{\gamma}_j$. We then use this to construct the solutions $P_n u = \sum_{j=1}^n \gamma_j \phi_j$, $P_n u = \sum_{j=1}^n \gamma_{j,t} \phi_j$, $P_n u = \sum_{j=1}^n \gamma_{j,tt} \phi_j$. For computational convenience the time derivatives are taken from the vectors $\bar{\gamma}_j$ rather than differentiating $\gamma(t)$. Powers of A applied to these elements will be essential for the energy plots, and they are constructed formally rather than via differentiation - $A^\theta P_n u = \sum_{j=1}^n \lambda^\theta \gamma_j \phi_j$, and likewise for u_t and u_{tt} as needed. Finally, the norms are computed formally using the eigenvector structure, rather than by direct integration, so that e.g. $\left\|A^{1/2} P_n u(t)\right\|^2 = \left(A^{1/2} P_n u(t), A^{1/2} P_n u(t)\right) = \sum_{j=1}^n |\lambda_j| \gamma_j(t)^2$.

6.1.2 Location of the spectrum

First, we treat the operator A simply as an abstract unbounded operator with an unbounded collection of positive eigenvalues. We model this case by treating λ_j as a continous parameter ν, $0 < \nu \to \infty$, and compute the eigenvalues of M as a function of ν. A concrete operator A with particular eigenvalues will have a spectrum belonging to this set. This will allow us to demonstrate the overall effect of the parameters on the location of the spectrum.

As discussed in [8] and [13], there are two key values determining the spectral behavior for the continuous problem. First the value $p_1 = \frac{\gamma}{2\tau}$, which determines a vertical asymptote at $Re\lambda = -p_1$, second the value $p_2 = \frac{c^2}{b}$ which determines an interval $[-\frac{\gamma}{\tau} - p_2, -p_2]$ containing real spectral values. The spectrum for the continuous linear problem in fact has continuous spectrum in this interval.

In figure 6.1 we show a computation for $\alpha = 1.5, b = 2, c = 1, \tau = 1$, so that $\gamma = 1 > 0$, $p_1 = 0.5$, $p_2 = 0.5$. The parameter ν representing the eigenvalues runs from .005 to 1000 for this figure. Note the matching of p_1 and p_2 creates a matching of the asymptote in the spectrum with the band along the real axis.

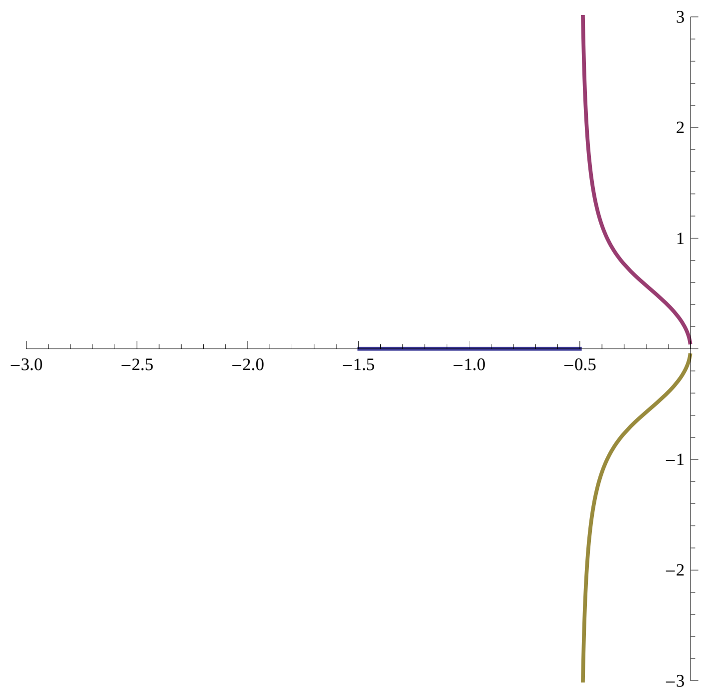

Figure 6.1: $\alpha = 1.5, b = 2, c = 1, \tau = 1, \nu > .005$

We see that the stability can be quite poor - although we have placed the asymptote and the band at $Re\lambda = -0.5$, we have parts of the spectrum reaching out to the imaginary axis. This instability is partly caused by the low range of eigenvalues

- increasing the minimal value for ν to .5 and then 1 creates the spectra as shown in figures 6.2 and 6.3.

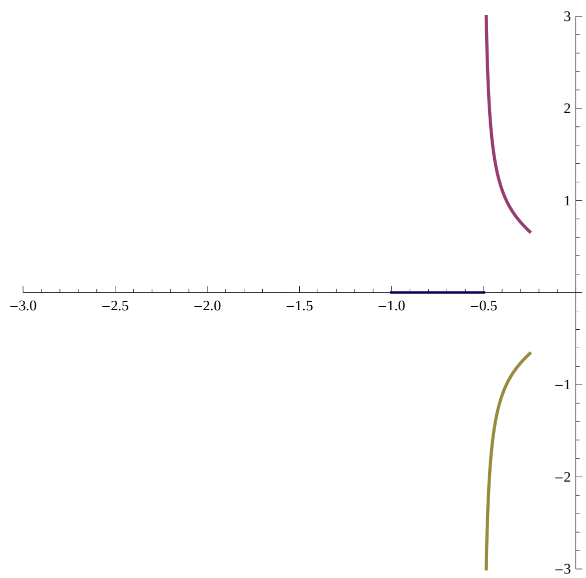

Figure 6.2: $\alpha = 1.5, b = 2, c = 1, \tau = 1, \nu > .5$

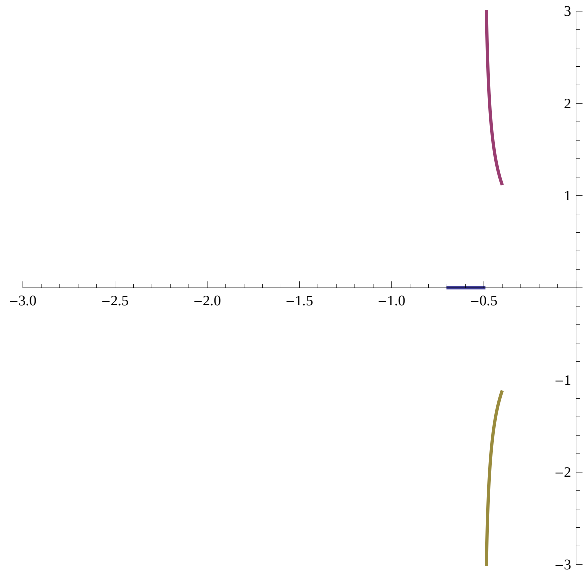

Figure 6.3: $\alpha = 1.5, b = 2, c = 1, \tau = 1, \nu > 1$

This situation creates a paried question - although the asymptote and interval control the large-mode behavior of the spectrum, the placement of the finitely many low eigenvalues tending toward the axis is unpleasant. How do the parameters of the problem impact the placement of the low eigenvalues? We chart in figure 6.4 the impact on the $\nu = 2$ eigenvalue as b varies. We repeat the procedure, holding b fixed and varying α, in figure 6.5. The impact of c is in 6.6, and τ is tracked in 6.7.

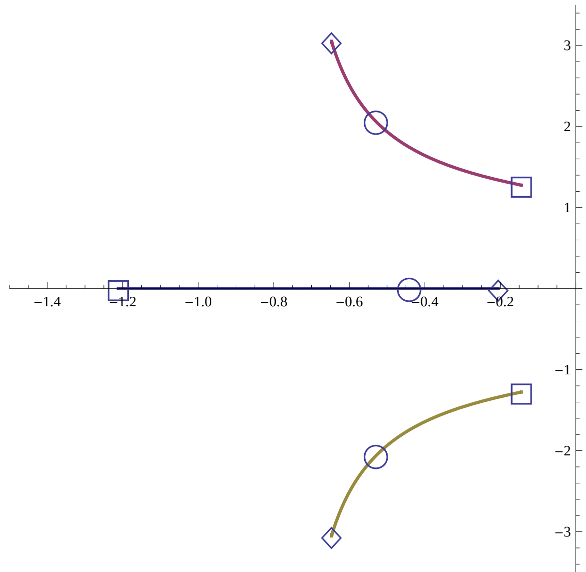

Figure 6.4: Variation of the $\nu = 2$ eigenvalue with $\alpha = 1.5, c = \tau = 1$, b varying from 1 to 5. Square, circle, and diamond are $b = 1, 2.5, 5$ respectively.

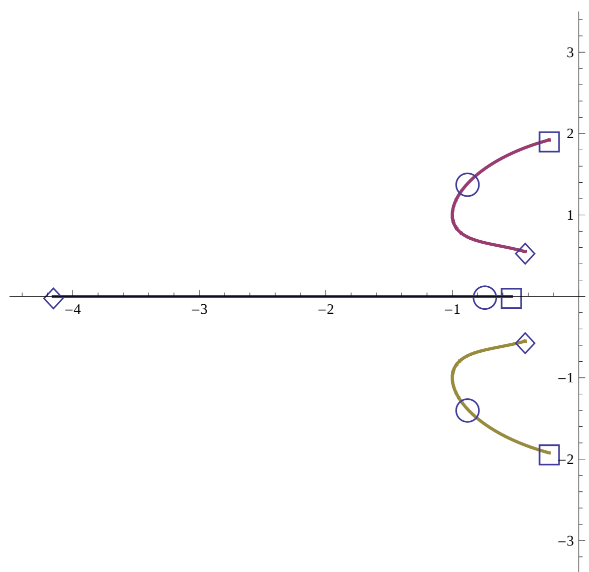

Figure 6.5: Variation of the $\nu = 2$ eigenvalue with $b = 2, c = \tau = 1$, α varying from 1 to 5. Square, circle, and diamond are $\alpha = 1, 2.5, 5$ respectively.

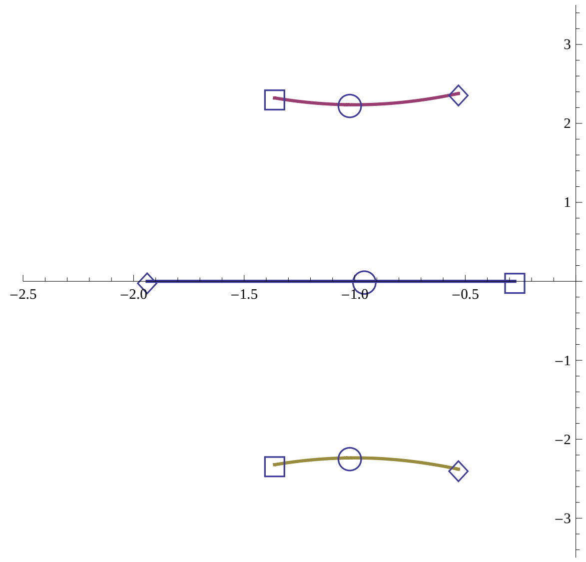

Figure 6.6: Variation of the $\nu = 2$ eigenvalue with $\alpha = 3, b = 4, \tau = 1$, c varying from 1 to 2.4. Square, circle, and diamond are $c = 1, 1.7, 2.4$ respectively.

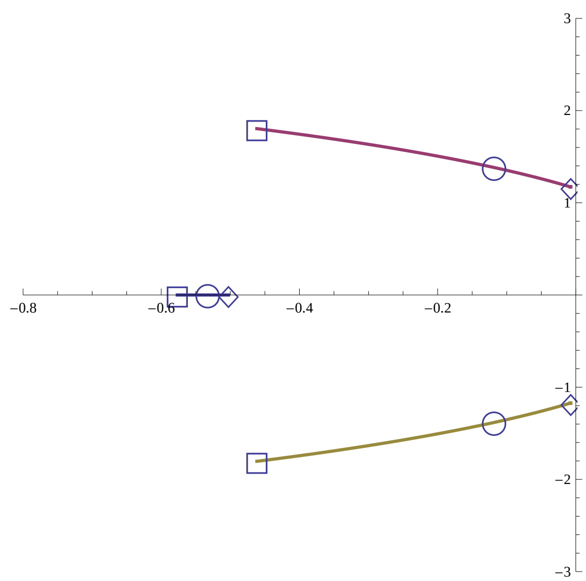

Figure 6.7: Variation of the $\nu = 2$ eigenvalue with $\alpha = 1.5, b = 2, c = 1$, τ varying from 1 to 2.9. Square, circle, and diamond are $\tau = 1, 1.95, 2.9$ respectively.

Observe in 6.5 that there appears to be an optimal selection of α with respect to the location of the two complex sections. This optimal placement changes for differing eigenvalues as we see below in figures 6.8 and 6.9.

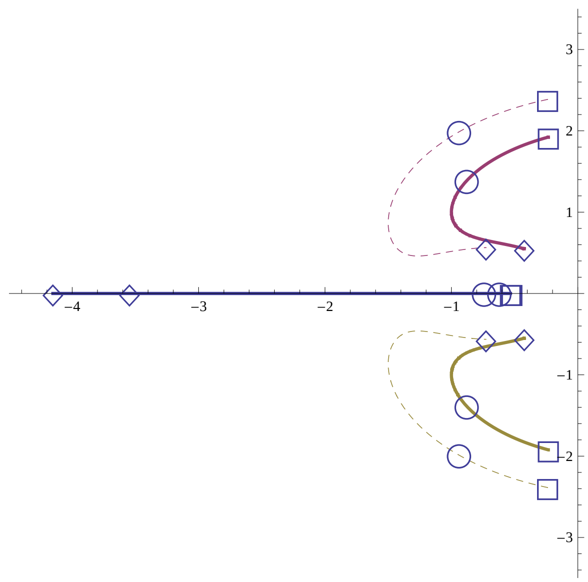

Figure 6.8: Comparison of $\nu = 2$ and $\nu = 3$ with variation in α. Solid and dashed line respectively correspond to $\nu = 2$, 3.

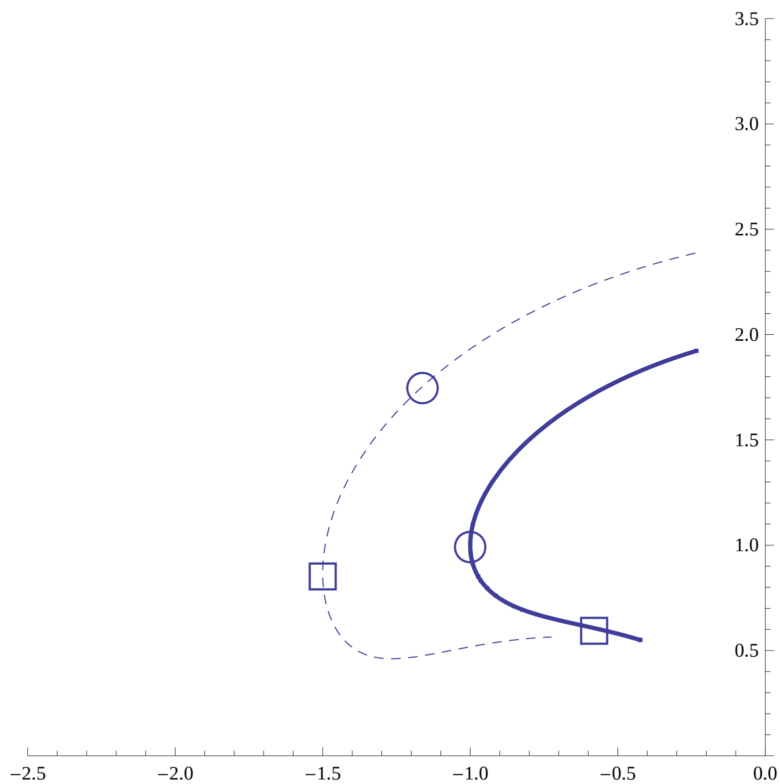

Figure 6.9: Approximately optimal pairs are $\nu = 2$, $\alpha = 3$ and $\nu = 3$, $\alpha = 4$. Solid and dashed lines indicate $\nu = 2, 3$, circle indicates $\alpha = 3$ and square marks $\alpha = 4$.

Thus as we see the placement of the low modes is a sensitive issue.

6.1.3 Illustration of stability results

We include some energy plots for solutions obtained from the spectral method, in the case $\Omega = [0,1]$, $A = \Delta = \frac{d^2}{dx^2}$, with 0-Dirichlet boundary conditions, $H = L^2(\Omega)$, $D(A^{1/2}) = H_0^1(\Omega)$, $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$. Then the associated eigenvectors of A are $\phi_i = \sqrt{2}\sin(i\pi x)$ with eigenvalues $\lambda_i = i^2\pi^2$. The initial data used are $u_0 = 100x(x-1)$, $u_1 = u_2 = 0$.

First we show instability in the case $\gamma < 0$

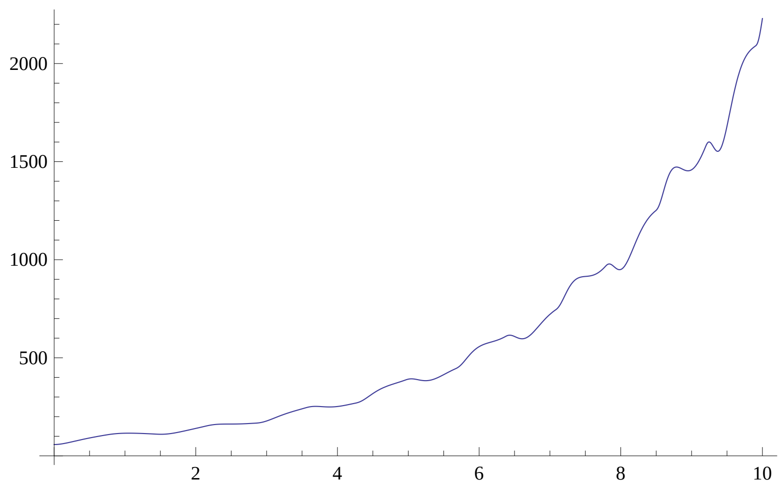

Figure 6.10: Instability in the $D(A^{1/2}) \times D(A^{1/2}) \times H$ norm when $\gamma = -1 < 0$, $\alpha = 1, b = .5, c = 1, \tau = 1, n = 50$. The x-axis plots time, y-axis plots energy in the indicated norm.

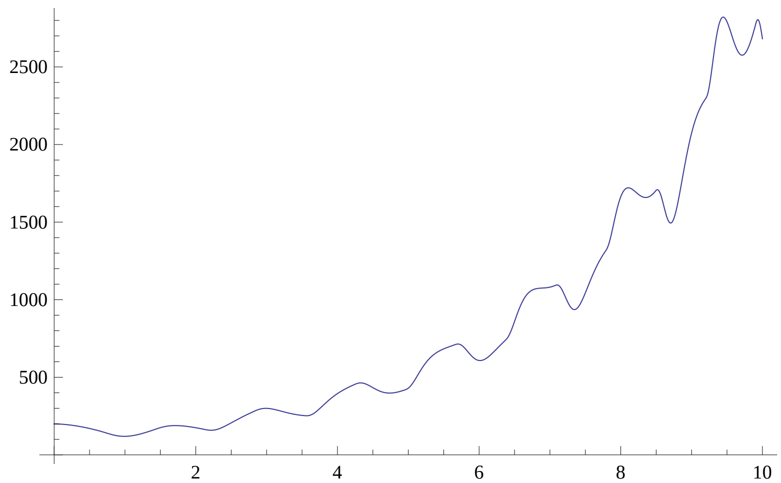

Figure 6.11: Instability in the $D(A) \times D(A^{1/2}) \times H$ norm when $\gamma = -1 < 0$, $\alpha = 1, b = .5, c = 1, \tau = 1, n = 50$. The x-axis plots time, y-axis plots energy in the indicated norm.

Now we show exponential stability for $\gamma > 0$

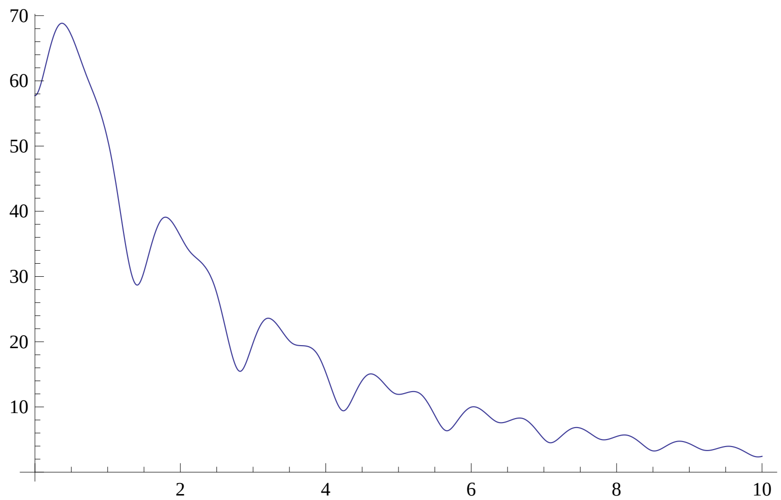

Figure 6.12: Stability in the $D(A^{1/2}) \times D(A^{1/2}) \times H$ norm when $\gamma = .5 > 0$, $\alpha = 1, b = 2, c = 1, \tau = 1, n = 50$. The x-axis plots time, y-axis plots energy in the indicated norm.

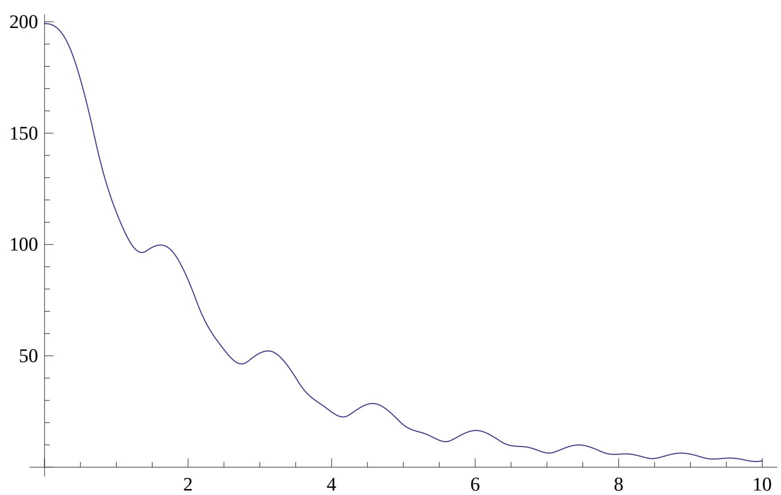

Figure 6.13: Stability in the $D(A) \times D(A^{1/2}) \times H$ norm when $\gamma = .5 > 0$, $\alpha = 1, b = 2, c = 1, \tau = 1, n = 50$. The x-axis plots time, y-axis plots energy in the indicated norm.

Finally we see a sort of conservation in the case $\gamma = 0$,

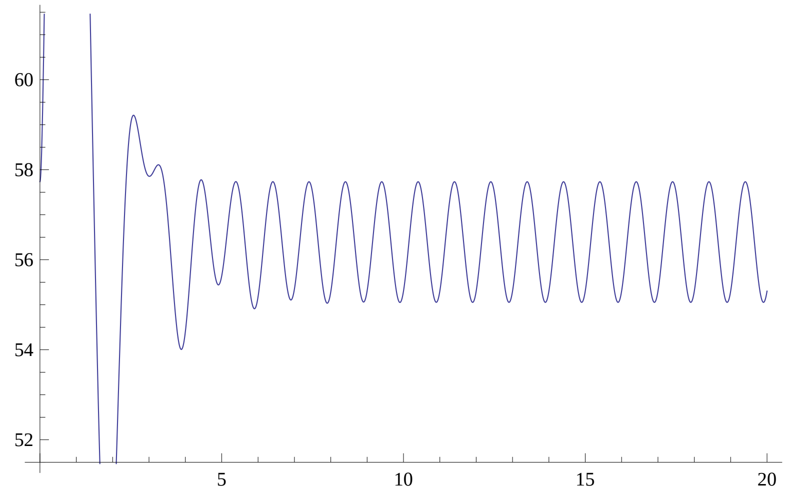

Figure 6.14: Boundedness in the $D(A^{1/2}) \times D(A^{1/2}) \times H$ norm when $\gamma = 0$, $\alpha = 1, b = 1, c = 1, \tau = 1, n = 50$. The x-axis plots time, y-axis plots energy in the indicated norm.

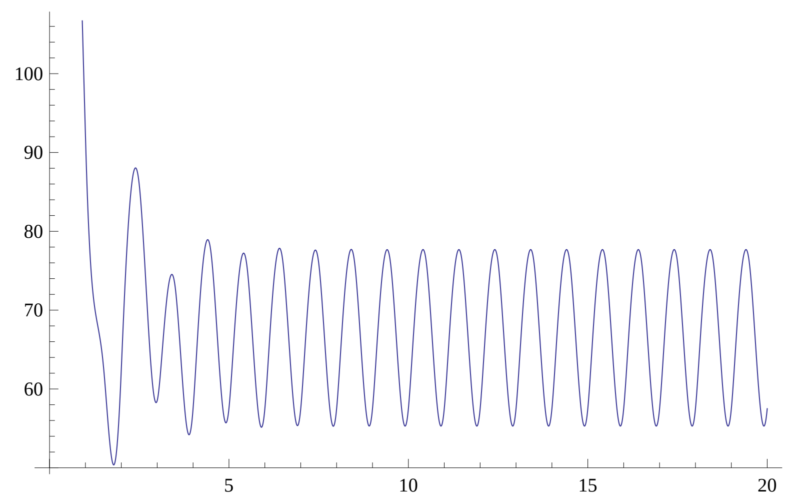

Figure 6.15: Boundedness in the $D(A) \times D(A^{1/2}) \times H$ norm when $\gamma = 0$, $\alpha = 1, b = 1, c = 1, \tau = 1, n = 50$. The x-axis plots time, y-axis plots energy in the indicated norm.

We should comment on the oscillation in these graphs, especially in those demonstrating the stability. One might expect that the result from the energy calculations was that the derivative of the energy was strictly negative when $\gamma > 0$ - see for example equation (3.1.4) - and therefore these graphs should not admit any increase at any point. However closer inspection will show that the quantities in the energy calculations which have negative derivative are not the full norm in either the higher or lower energy. Rather it is only part of the norm, and also is partially in terms of the other variables z and z_t. We include here a graph of the term guarenteed to strictly decay from the calculus:

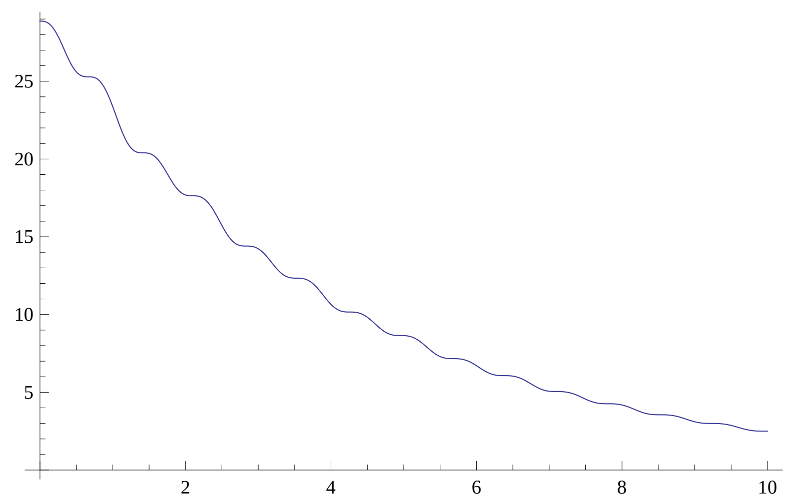

Figure 6.16: Monotone decrease of $\frac{b}{2}\left\|A^{1/2}(u_t + \frac{c^2}{b}u)\right\|^2 + \frac{\tau}{2}\left\|u_{tt} + \frac{c^2}{b}u_t\right\|^2 + \frac{c^2}{2b}\gamma\left\|u_t\right\|^2$ when $\gamma = .5$, $\alpha = 1, b = 2, c = 1, \tau = 1, n = 50$. The x-axis plots time, y-axis plots the indicated quantity.

A final figure is a brief attempt at a demonstration of the convergence properties of this method. Of course as the continuous solution is inaccessible we cannot directly show the error. However, we can demonstrate the accumulation of the energy plots with variation in n. It turns out to be very difficult to producde an interesting plot of this phenomenon because the convergence is quite fast. Also as can be seen from the convergence theory for the spectral method, there is an exponential time weight in the error that strengthens convergence at larger times.

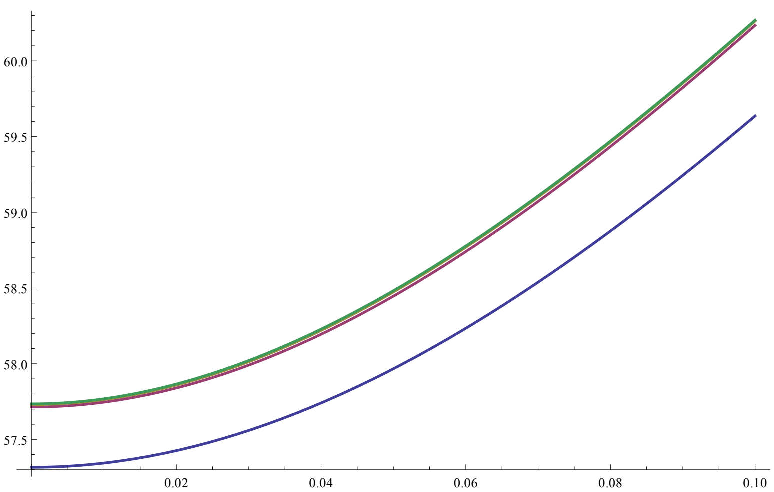

Figure 6.17: Accumulation of the energy in the $D(A^{1/2}) \times D(A^{1/2}) \times H$ norm for $\gamma = .5 > 0$, $\alpha = 1, b = 2, c = 1, \tau = 1$, lines in increasing order are $n = 1, 5, 10, 25$. The x-axis is time and the y-axis is the norm of the solution.

Chapter 7

Appendices

7.1 A Gronwall Inequality

The following lemma appears in [3]:

Lemma 7.1.1. *Let $m(t) \in L^1(0, T; \mathbf{R})$ be a positive function, let $a \geq 0$ be a constant. Suppose $\phi(t) \in C[0, T]$ is a real-valued function satisfying*

$$\frac{1}{2}\phi^2(t) \leq \frac{1}{2}a^2 + \int_0^t m(s)\phi(s)\, ds$$

for each $t \in [0, T]$. Then,

$$|\phi(t)| \leq a + \int_0^t m(s)\, ds$$

for all t in$[0, T]$.

It also appears with a slightly different proof in [6] The proof is repeated here for convenience.

Proof. Fix an $\epsilon > 0$ and define $\psi_\epsilon(t) = \frac{1}{2}(a + \epsilon)^2 + \int_0^t m(s)\phi(s)\, ds$. Then note that since ϕ is continuous and m is L^1, $\frac{d\psi_\epsilon}{dt} = m(t)\phi(t)$, and $\frac{1}{2}\phi^2(t) = \psi_0(t) \le \psi_\epsilon(t)$. Thus,

$$\frac{d\psi_\epsilon(t)}{dt} \le m(t)\sqrt{2}\sqrt{\psi_\epsilon(t)}$$

Now, $\phi_\epsilon(t) \ge \frac{1}{2}\epsilon^2$, so $\sqrt{\psi_\epsilon(t)}$ is differentiable for each t and

$$\frac{d}{dt}\sqrt{\psi_\epsilon(t)} = \frac{1}{2\sqrt{\psi_\epsilon(t)}}\frac{d\psi_\epsilon}{dt} \le \frac{1}{\sqrt{2}}m(t)$$

Thus

$$\sqrt{\psi_\epsilon(t)} \le \sqrt{\psi_\epsilon(0)} + \frac{1}{\sqrt{2}}\int_0^t m(s)\,ds$$

And so finally

$$|\phi(t)| \le \sqrt{2}\sqrt{\psi_\epsilon(t)} \le a + \epsilon + \int_0^t m(s)\,ds$$

Let ϵ go to zero and we have the claimed inequality. $\square$

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