Annular Link Homology Theories and their Homotopical Refinements

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Abstract

Homology theories of links which categorify quantum link invariants have been developed over the past twenty years, starting with Khovanov's seminal categorification of the Jones polynomial. This thesis focuses on links in the thickened annulus and develops annular link homology in two main directions. First, we present joint work with Krushkal and Willis in which a stable homotopy refinement of Beliakova-Putyra-Wehrli's quantum annular homology is constructed. Second, we introduce equivariant annular link homology. This is comprised of $\mathfrak{sl}_2$ annular homology via filtrations, joint work with Khovanov in the $\mathfrak{sl}_2$ and $\mathfrak{sl}_3$ setting via foam evaluation and universal construction, and a treatment of the general $\mathfrak{gl}_N$ setting via foams.

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Contents

Chapter 1

Introduction

Knot theory, the study of knots in links in 3-space, constitutes a significant part of low-dimensional topology. Of central importance is the notion of *link invariants*, which are used to distinguish two links and shed light on intricate topological information. The celebrated Jones polynomial $V(L)$, discovered by Vaughan Jones in 1984 [31], was a revolutionary link invariant that not only distinguished many links in 3-space, but also led to resolutions of long-standing conjectures in knot theory [82] and ushered in the field of quantum topology. Generalizations of the Jones polynomial, such as the Reshetikhin-Turaev invariants [70] demonstrated a deep connection between low-dimensional topology and representation theory, while connections to mathematical physics were discovered by Witten [88].

In 1999, Mikhail Khovanov [33] introduced what is now called *Khovanov homology*, a bigraded cohomology theory of links $Kh(L)$ which *categorifies* $V(L)$, in the sense that the graded Euler characteristic of $Kh(L)$ is equal to $V(L)$. Khovanov homology is a strictly stronger invariant than the Jones polynomial [7]; it detects the unknot [47], which remains an open question for $V(L)$; and, perhaps most importantly, it enjoys *functoriality* properties which are invisible at the decategorified level. Functoriality was famously used by Rasmussen [69] to provide a combinatorial proof of Milnor's conjecture regarding slice genus of torus knots. Generalizations and extensions of Khovanov's original homology for links in 3-space are plentiful, employing methods from many areas of mathematics, namely (higher) representation theory, symplectic geometry, and algebraic geometry.

Extensions of Khovanov homology to links in 3-manifolds besides $\mathbb{R}^3$ remain, for the most part, undiscovered. The first such extension was introduced by Asaeda-Przytycki-Sikora [6] for interval bundles over a surface. The special case of the thickened annulus $\mathbb{A} \times I$, where $I := [0,1]$ and $\mathbb{A} := S^1 \times I$, is known as *annular Khovanov homology*. It can be defined by taking the associated graded of an appropriate filtration on the Khovanov chain complex. Annular homology is the most widely explored among the few known extensions of Khovanov homology to other 3-manifolds, and moreover it has a richer structure than homology of links in $\mathbb{R}^3$: it is a triply graded theory and carries representation-theoretic importance [25].

This thesis develops new features and generalizations of annular homology in two main directions:

(I) Constructing a stable homotopy refinement of *quantum* annular Khovanov homology. This is joint work with Krushkal and Willis [5] and is the content of Chapter 3.

(II) Introducing annular versions of *equivariant* Khovanov homology and of the foam-theoretic approach to Khovanov-Rozansky $\mathfrak{sl}_N$ homology [42, 50, 67, 73, 24]. This is partially based on joint work with Khovanov [3] and occupies Chapter 4.

Let us give context for the above two items.

Regarding item (I), Lipshitz-Sarkar [57] defined a *stable homotopy refinement* of Khovanov homology. To a link $L \subset \mathbb{R}^3$ they associate a *CW* spectrum $\mathcal{X}(L)$ whose stable homotopy type is an invariant of L and whose cohomology is isomorphic to $Kh(L)$. The salient feature of the homotopical refinement is that it admits stable cohomology operations. Lipshitz-Sarkar give combinatorial formulas for the Steenrod operations $\mathbf{Sq}^1$ and $\mathbf{Sq}^2$, which are used to show that $\mathcal{X}(L)$ is a non-trivial extension of $Kh(L)$, in the sense that $\mathcal{X}(L)$ is not simply a wedge of Moore spaces [59]. Seed [79] used Steenrod operations to show that $\mathcal{X}(L)$ is a strictly stronger link invariant than $Kh(L)$. As another application, Lipshitz-Sarkar [58] extracted a refinement of Rasmussen's s-invariant from the homotopy type.

Beliakova-Putyra-Wehrli [11] introduced a sophisticated deformation of annular Khovanov homology, called *quantum* annular Khovanov homology. For a link L in the thickened annulus, they construct a triply graded homology $Kh_{\mathbb{A}_\mathfrak{q}}(L)$ of $\Bbbk := \mathbb{Z}[\mathfrak{q}, \mathfrak{q}^{-1}]$-modules. A remarkable aspect of their theory is that $Kh_{\mathbb{A}_\mathfrak{q}}(L)$ admits an action of the quantum group $U_\mathfrak{q}(\mathfrak{sl}_2)$. We use a modification of Beliakova-Putyra-Wehrli's chain complex, given by setting

$\mathfrak{q}^r = 1$ before taking homology, resulting in a homology theory $Kh^r_{\mathbb{A}_\mathfrak{q}}(L)$ of $\Bbbk/(\mathfrak{q}^r - 1)$-modules. We construct a stable homotopy refinement of this theory.

Theorem A ([5, Theorem 1.1])**.** Let L be an oriented link in the thickened annulus $\mathbb{A} \times I$. Then for each $r \geq 2$, there exists a $\mathbb{Z}/r\mathbb{Z}$-equivariant spectrum $\mathcal{X}^r_{\mathbb{A}_\mathfrak{q}}(L)$ which is well-defined up to equivariant homotopy equivalence and whose cohomology is isomorphic to the quantum annular homology $Kh^r_{\mathbb{A}_\mathfrak{q}}(L)$, as modules over $\mathbb{Z}[\mathbb{Z}/r\mathbb{Z}] = \Bbbk/(\mathfrak{q}^r - 1)$.

We stress two key points in the construction of $\mathcal{X}^r_{\mathbb{A}_\mathfrak{q}}(L)$. First, it is crucial in Lipshitz-Sarkar's definition of $\mathcal{X}(L)$ to have a suitable basis for the Khovanov chain complex. The situation in quantum annular homology is more involved, because there is no such canonical choice of generators: there is a family of preferred choices of generators, which are related to each other by multiplication by a (non-uniform) power of $\mathfrak{q}$. Second, the element $\mathfrak{q}$ is interpreted as a generator of the cyclic group $\mathbb{Z}/r\mathbb{Z}$, which gives the cohomology of the $\mathbb{Z}/r\mathbb{Z}$-spectrum $\mathcal{X}^r_{\mathbb{A}_\mathfrak{q}}(L)$ the structure of a module over $\mathbb{Z}[\mathbb{Z}/r\mathbb{Z}] = \Bbbk/(\mathfrak{q}^r - 1)$, and thus allows for a comparison between $H^*(\mathcal{X}^r_{\mathbb{A}_\mathfrak{q}}(L); \mathbb{Z})$ and $Kh^r_{\mathbb{A}_\mathfrak{q}}(L)$.

Setting $\mathfrak{q} = 1$ (equivalently, taking $r = 1$) in the quantum annular complex recovers annular Khovanov homology and the $\mathfrak{sl}_2$ action discovered by Grigsby-Licata-Wehrli [25]. On the other hand, a homotopy refinement $\mathcal{X}_\mathbb{A}(L)$ of annular Khovanov homology is readily defined. The following result is a spectral analogue of setting $\mathfrak{q} = 1$.

Theorem B ([5, Theorem 1.2])**.** The quotient of $\mathcal{X}^r_{\mathbb{A}_\mathfrak{q}}(L)$ by the $\mathbb{Z}/r\mathbb{Z}$ action recovers $\mathcal{X}_\mathbb{A}(L)$.

We also associate maps of spectra to cobordisms, analogous to those in [58] for $\mathcal{X}(L)$.

Theorem C ([5, Theorem 6.1])**.** For any generically embedded cobordism $W \subset \mathbb{A} \times [0,1] \times [0,1]$ between two annular links L_0 and L_1, there exists a map of spectra $\varphi^r_W : \mathcal{X}^r_{\mathbb{A}_\mathfrak{q}}(L_1) \to \mathcal{X}^r_{\mathbb{A}_\mathfrak{q}}(L_0)$, whose induced map on cohomology $(\varphi^r_W)^*$ equals the map induced by W on quantum annular Khovanov homology.

Theorems A, B and C are restated and proven as Theorems 3.0.1, 3.0.2, and 3.7.2 in Chapter 3. The necessary background regarding quantum annular homology and (one approach to) Khovanov stable homotopy refinements is presented in Section 2.4.2 and Section 2.6, respectively.

We now discuss item (II). There exist versions of Khovanov homology which are, in an appropriate sense, universal. These link homology theories, systematically studied in [37, 40], are associated with equivariant cohomology of $\mathbb{CP}^1$, explaining the terminology *equivariant* Khovanov homology. By specializing parameters in a suitable way, equivariant Khovanov homology reduces to Khovanov homology and also to important deformations of Khovanov homology, namely the filtered Lee [55] and Bar-Natan [8] deformations, from which subtle topological information is extracted, most notably by Rasmussen [69].

Beyond the Jones polynomial, Reshetikhin-Turaev [70] defined a family of link polynomials which depend on a simple Lie algebra $\mathfrak{g}$ and a labeling of each link component by a representation of the corresponding quantum group $U_q(\mathfrak{g})$. The Jones polynomial is an instance of these invariants as the special case where $\mathfrak{g} = \mathfrak{sl}_2$ and all components are labeled by the fundamental 2-dimensional representation. Murakami-Ohtsuki-Yamada [65] discovered that, if $\mathfrak{g} = \mathfrak{sl}_N$ and components are labeled by quantum exterior powers of the fundamental representation, then the Reshetikhin-Turaev invariant can be computed combinatorially from the link diagram. Each crossing is replaced by a $\mathbb{Z}[q, q^{-1}]$-linear combination of *webs*, which are certain trivalent graphs giving a diagrammatic description of (a piece of) the representation category of $U_q(\mathfrak{sl}_N)$ (see [48] for the $\mathfrak{sl}_3$ setting and [18] for general N). Each web is then evaluated to a Laurent polynomial by recursively applying the *MOY relations*.

The program of categorifying higher rank invariants began in work of Khovanov [35] in the $\mathfrak{sl}_3$ setting and was continued in the $\mathfrak{sl}_N$ setting by Khovanov-Rozansky [42] and Wu [90] via matrix factorizations. Using higher representation theory, Webster [86] has categorified the Reshetikhin-Turaev invariant in full generality. By now there exists a wealth of approaches to categorifying Reshetikhin-Turaev invariants for links in $\mathbb{R}^3$. By studying deformations of higher rank link homology, Lewark-Lobb [56] introduced a generalization of Rasmussen's s-invariant, producing new bounds on slice genus of knots.

Of central importance to this thesis is the approach via *foams*, which are singular surfaces viewed as cobordisms between webs. Foams were introduced in Khovanov's categorification of the $\mathfrak{sl}_3$ polynomial, and developed further by many authors [12, 50, 67, 62].

Of particular focus will be the Robert-Wagner closed foam evaluation [73]. Robert-Wagner give a state-sum evaluation $\langle - \rangle_{\mathrm{RW}}$ of closed foams which is miraculously valued in the ring of symmetric polynomials $R_N = \mathbb{Z}[x_1, \ldots, x_N]^{S_N}$ [73, Proposition 2.19]. This allows them to define, for any web Γ, the *state space* $\langle\Gamma\rangle$ via universal construction: $\langle\Gamma\rangle$ is spanned by all foams from the empty web to Γ, with relations determined by the evaluation $\langle - \rangle_{\mathrm{RW}}$. They prove that $\langle\Gamma\rangle$ is a free graded R_N-module of graded rank equal to the MOY evaluation of Γ [73, Theorem 3.30]. Thus Robert-Wagner evaluation categorifies the MOY relations; the chain complex categorifying the $\mathfrak{sl}_N$ link invariant is assembled from these state spaces.

Key benefits of this approach include

- its combinatorial nature,
- that it naturally produces equivariant link homology, and
- the *strict* functoriality of the resulting link homology, proven in the $N = 2$ case by Blanchet [12] and in the colored, general N case by Ehrig-Tubbenhauer-Wedrich [24].

Webs, foams, and foam evaluation are discussed in Section 2.5.

Remark 1.0.1. As explained in the introduction of [73], Robert-Wagner, strictly speaking, work with $\mathfrak{gl}_N$ webs and foams. Both $\mathfrak{sl}_N$ and $\mathfrak{gl}_N$ terminology appears in the literature. In this document we will say $\mathfrak{gl}_N$ when referring to the webs and foams in [73, 24].

Item (II) addresses the problem of defining equivariant Khovanov homology and Khovanov-Rozansky homology for links in the thickened annulus. Non-equivariant annular Khovanov-Rozansky homology was introduced in [68] using categorical traces of categorified quantum groups [49, 39]. We thus give a foamy construction of annular Khovanov-Rozansky homology which enjoys the benefits of being purely combinatorial, having equivariance naturally built in, and, in the $\mathfrak{gl}_N$ setting, strict functoriality.

A unifying aspect of all the equivariant annular link homology theories discussed below is that the ground ring is a polynomial ring rather than the subring of symmetric polynomials, which is in contrast to existing equivariant homology for links in 3-space. Robert-Wagner foam evaluation takes values in symmetric polynomials, allowing state spaces to be constructed over the smaller ring. Passing to the larger ground ring introduces additional flexibility that is apparently needed for annular versions of these theories. This phenomenon appears already in the simplest instance of our constructions, the $\mathfrak{sl}_2$ setting via filtrations. See the introduction to Chapter 4 for a more thorough discussion.

One method to extend equivariant $\mathfrak{sl}_2$ homology to the annular setting is to take the associated graded of a suitably defined filtration, analogous to the definition of (non-equivariant) annular homology. This strategy is applied in Section 4.1 and has appeared in [2].

For the reasons stated above, it is desirable to have a foam-based approach to higher rank link homology. The main obstacle in the annular setting is assigning a suitable module to a web in the annulus $\mathbb{A}$. As discussed above, universal construction assigns to a planar object a certain quotient of the free module generated by all cobordisms bounding that object. However, a single non-contractible circle in the annulus does not bound any surface in the thickened annulus. To overcome this, the thickened annulus is modeled as the complement of a distinguished line $\mathcal{L} \subset \mathbb{R}^3$. Foams are allowed to generically intersect $\mathcal{L}$, and they must carry additional decorations at intersection points, which contribute to the foam evaluation. We call these *anchored foams*; this perspective is based on joint work with Khovanov [3].

In addition to quantum gradings, anchored foams also carry *annular* gradings, coming from intersections with $\mathcal{L}$. In the $\mathfrak{sl}_2$ and $\mathfrak{sl}_3$ case, annular degrees give $\mathbb{Z}$ and $\mathbb{Z}\oplus\mathbb{Z}$ gradings, respectively, and in the $\mathfrak{gl}_N$ setting they give a $\mathbb{Z}^N$ grading. This is expected from the point of view of representation theory. It was shown in [25] that annular Khovanov homology carries an $\mathfrak{sl}_2$ action, with the annular grading corresponding to the weight space decomposition. More generally, the annular $\mathfrak{sl}_N$ homology in [68] is shown to carry an action of $\mathfrak{sl}_N$. We note, however, that the equivariant annular homology theories presented in this thesis are not known to admit actions by the corresponding Lie algebras.

Section 4.2 gives a foam perspective on the equivariant annular $\mathfrak{sl}_2$ homology introduced in Section 4.1. We define an evaluation of anchored surfaces, valued in the polynomial ring $\mathbb{Z}[\alpha_1, \alpha_2]$. When the surface is disjoint from $\mathcal{L}$, the evaluation agrees with evaluation of closed surfaces in equivariant Khovanov homology. Applying the universal construction to anchored surface evaluation yields state spaces for collections of disjoint circles in the annulus. As in annular Khovanov homology, state spaces are bigraded, carrying quantum and annular gradings. This assembles into a functor $\langle - \rangle : \mathbf{ACob} \to \mathbb{Z}[\alpha_1, \alpha_2]-\mathrm{ggmod}$ from the category of cobordisms in the thickened annulus to the category of bigraded $\mathbb{Z}[\alpha_1, \alpha_2]$-modules.

Theorem D ([3])**.** Let $L \subset \mathbb{A} \times [0,1]$ be an annular link with diagram D. Applying the above functor $\langle - \rangle$ term-wise to the cube of resolutions of D yields a chain complex $C(D)$ of free bigraded $\mathbb{Z}[\alpha_1, \alpha_2]$-modules whose chain homotopy class is an invariant of L. Moreover, $C(D)$ is isomorphic to the chain complex defined in Section 4.1.

The construction of anchored surface evaluation occupies Section 4.2. Identification of state spaces (and their bigraded rank) is Theorem 4.2.10. That the resulting homology agrees with the filtration approach from Section 4.1 is stated as Theorem 4.2.19.

Section 4.3 addresses $\mathfrak{sl}_3$ homology in the annular setting.

Theorem E ([3])**.** There exists an evaluation of anchored $\mathfrak{sl}_3$ foams valued in the ring of polynomials $\mathbb{Z}[x_1, x_2, x_3]$. The resulting state spaces of $\mathfrak{sl}_3$ webs in the annulus are free triply graded $\mathbb{Z}[x_1, x_2, x_3]$-modules. Applying the state-space construction to the $\mathfrak{sl}_3$ cube of resolutions of an annular link diagram D yields a chain complex of free triply graded $\mathbb{Z}[x_1, x_2, x_3]$-modules whose chain homotopy class is an invariant of the annular link represented by D.

Anchored $\mathfrak{sl}_3$ foam evaluation is defined in Sections 4.3.1 and 4.3.2. Theorem 4.3.32 identifies state spaces of annular webs. The contents of Section 4.2 and Section 4.3 are joint work with Khovanov [3].

Section 4.4 addresses the general $\mathfrak{gl}_N$ setting. The main result is summarized as follows.

Theorem F. For each $N \geq 2$, there exists an evaluation of anchored $\mathfrak{gl}_N$ foams valued in the ring of polynomials $R'_N = \mathbb{Z}[x_1, \ldots, x_N]$. The resulting state spaces of $\mathfrak{gl}_N$ webs in the annulus are free $\mathbb{Z} \oplus \mathbb{Z}^N$-graded R'_N-modules.

Given an annular link $L \subset \mathbb{A} \times [0,1]$ with diagram $D \subset \mathbb{A}$, there exists a chain complex $C(D)$ of $\mathbb{Z} \oplus \mathbb{Z}^N$-graded R'_N-modules whose homology $H(D)$ is an invariant of the isotopy class of L. For a link cobordism $S \subset \mathbb{A} \times [0,1] \times [0,1]$ from L_0 to L_1, there exists a map on the homology $S_* : H(L_0) \to H(L_1)$, which is independent of the isotopy class of S and is functorial with respect to composition of cobordisms.

Anchored $\mathfrak{gl}_N$ foam evaluation is defined in Section 4.4.1. Local relations are established in Section 4.4.2, allowing us to identify state spaces assigned to annular $\mathfrak{gl}_N$ webs in Section 4.4.3. The resulting link homology is discussed in Section 4.4.4 .

Chapter 2

Background

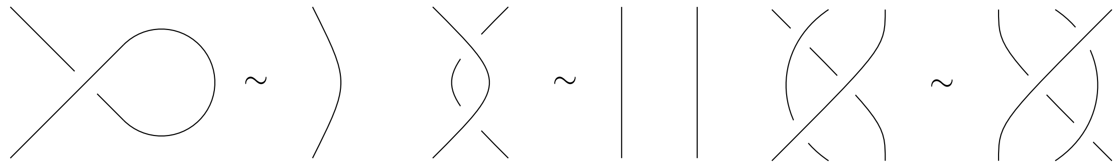
2.1 The Jones polynomial

Throughout this document, a *link* L will mean a smooth embedding of a disjoint union of finitely many circles, $\coprod_{i=1}^{k} S^1$, into the 3-sphere S^3. The number $k \geq 0$ is the *number of components of* L. An *ambient isotopy* is an orientation-preserving diffeomorphism $S^3 \to S^3$. Two links L_0, L_1 are *isotopic* if there is an ambient isotopy sending L_0 to L_1. An *oriented link* is a link $L \subset S^3$ with a choice of orientation on each component. Oriented links are isotopic if there is an ambient isotopy which sends one link to the other and preserves the orientation on each component.

Every link L can be represented as a *link diagram*, a generic projection of L onto a plane. If L is oriented then every diagram of L inherits an orientation, depicted by drawing arrows on the components. A classical result, proven in the 1920's by Reidemeister, states that two diagrams D and D' represent isotopic links if and only if they are related by a finite sequence of *Reidemeister moves*, shown in Figure 2-1. If L is oriented, then there are oriented analogues of the moves.

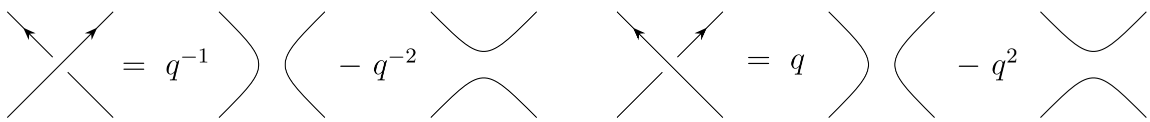
Determining whether two links are isotopic is, in general, a difficult question. One method to conclude that two links are *not* isotopic is to define a *link invariant*, which for our purposes is a function f from the set of all link diagrams into some set, such that if D and D' are related by a Reidemeister move, then $f(D) = f(D')$.

We now define the *Jones polynomial*, discovered in the seminal work of Jones [31]. The

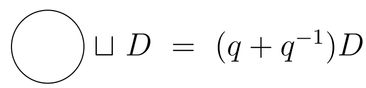


(a) Reidemeister I. (b) Reidemeister II. (c) Reidemeister III.

Figure 2-1: Reidemeister moves relating diagrams of isotopic links.



(a) Resolving a positive crossing. (b) Resolving a negative crossing.



(c) Circle removal.

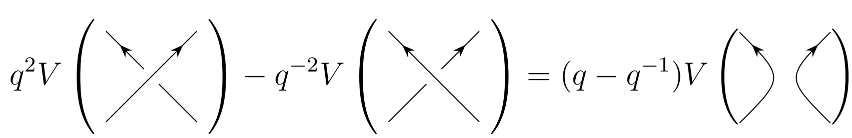
Figure 2-2: The algorithm to compute the Jones polynomial.

definition given below is a modification of Kauffman's bracket approach to the Jones polynomial [32].

Definition 2.1.1. Let D be a link diagram for an oriented link L. The Jones polynomial $V(D)$ is defined as follows. First, replace each crossing of D by the linear combination shown in Figure 2-2a or 2-2b, depending on the orientation of the strands involved in the crossing. If D has n crossings, this results in a 2^n-term sum of planar diagrams with coefficients given by monomials $\pm q^a$. Each planar diagram is a collection of some number of disjoint circles in the plane, which we evaluate using the rule in Figure 2-2c, with the understanding that $V(\varnothing) = 1$; that is, a planar diagram with k circles evaluates to $(q + q^{-1})^k$. By construction, $V(D) \in \mathbb{Z}[q, q^{-1}]$.

Proposition 2.1.2. *If D and D' are oriented link diagrams that are related by a Reidemeister move, then $V(D) = V(D')$. Consequently, $V(-)$ is an invariant of oriented links.*

Using the relations in Figure 2-2a and 2-2b, one obtains the following *skein relation*.



(2.1)

2.2 Graded rings and modules

This section records some algebraic notions regarding graded rings, modules, and chain complexes.

A *graded* ring is a commutative ring R together with a decomposition as a direct sum of (additive) abelian subgroups

$$R = \bigoplus_{i \in \mathbb{Z}} R_i,$$

such that the multiplication in R satisfies $R_i \cdot R_j \subset R_{i+j}$.

Let R be a graded ring. A *graded R-module* is a (left) R-module M, together with a direct sum decomposition $M = \bigoplus_{i \in \mathbb{Z}} M$, where each M_i is an additive subgroup of M, such that $R_i \cdot M_j \subset M_{i+j}$. Note that R is a graded R-module. An element $m \in M_d$ is called *homogeneous of degree* $\deg(m) = d$; the element 0 is homogeneous of any degree.

Often one focuses on the case when R is *trivially graded*, meaning that $R_0 = R$ and $R_i = \{0\}$ for $i \neq 0$. In this case, a graded R-module is an R-module with a distinguished $\mathbb{Z}$-indexed direct sum decomposition into R-submodules. In Section 2.3.4 we will consider rings with nontrivial gradings. The key graded rings in this thesis are $R = \mathbb{Z}$ with the trivial grading and polynomial rings $R = \mathbb{Z}[x_1, \ldots, x_N]$ where each x_i has degree 2.

Given graded R-modules M and N, an R-linear map $f : M \to N$, and $d \in \mathbb{Z}$, we say that f is *a graded map of degree d* if $f(M_i) \subset N_{i+d}$ for all $i \in \mathbb{Z}$. We will write $R-\mathrm{gmod}$ to denote the category of graded R-modules and graded maps (of any degree) between them.

Direct sum and tensor product extend naturally to the category $R-\mathrm{gmod}$. For graded R-modules M and N, define a grading on $M \oplus N$ by setting $(M \oplus N)_i = M_i \oplus N_i$, and on $M \otimes_R N$ by setting a simple tensor $m \otimes n$, for homogeneous $m \in M, n \in N$, to be homogeneous of degree $\deg(m) + \deg(n)$.

The category $R-\mathrm{gmod}$ comes equipped with a $\mathbb{Z}$-action called a *grading shift functor*, denoted $M \mapsto M\{n\}$. As an R-module, $M = M\{n\}$, but as a graded R-module we have

$$(M\{n\})_i = M_{i-n}. \tag{2.2}$$

A *chain*[1] *complex of graded R-modules* consists of

- A graded R-module C^i for each $i \in \mathbb{Z}$, such that $C^i = \{0\}$ for all but finitely many i.

- For each $i \in \mathbb{Z}$, a degree zero map $\partial_i : C^i \to C^{i+1}$ such that $\partial_{i+1}\partial_i = 0$.

We write such a chain complex as (C, ∂), sometimes omitting the differential from the notation.

Given two chain complexes C and D, a *chain map* $f : C \to D$ consists of a degree zero R-linear map $f_i : C^i \to D^i$ for each $i \in \mathbb{Z}$, such that $f_{i+1}\partial_i = \partial_i f_i$ for all i.

Write $C^{i,j}$ to denote the degree j part of the i-th chain group; note that, in general $C^{i,j}$ is only an abelian group, not necessarily an R-module. Each homology group $H^i(C)$ is graded, with its degree j part given by $H^{i,j}(C)$, the homology of $C^{i-1,j} \xrightarrow{\partial_{i-1}} C^{i,j} \xrightarrow{\partial_i} C^{(i+1),j}$.

Definition 2.2.1. Let C and D be chain complexes of graded R-modules. Two chain maps $f, g : C \to D$ are *chain homotopy equivalent* if there exist degree zero maps $F_i : C^i \to D^{i-1}$ such that $\partial_{i-1}F_i + F_{i+1}\partial_i = f - g$ for all $i \in \mathbb{Z}$.

A chain map $f : C \to D$ is a *chain homotopy equivalence* if there exists a chain map $g : D \to C$ such that fg and gf are chain homotopy equivalent to id_D and id_C, respectively. We say that C and D are homotopy equivalent, written $C \simeq D$, if there exists a chain homotopy equivalence $f : C \to D$.

Lemma 2.2.2. *If $f : C \to D$ is a chain homotopy equivalence, then it induces an isomorphism of graded R-modules $H^i(C) \cong H^i(D)$ for each i.*

Suppose that $R = \mathbb{Z}$ with the trivial grading and M is a finitely generated graded abelian group. Define the *graded rank* of M to be

$$\mathrm{rank}_q(M) = \sum_{i \in \mathbb{Z}} \mathrm{rank}(M_i) q^i \in \mathbb{Z}[q, q^{-1}].$$

[1] While the differential in Khovanov's chain complex increases homological degree, it is nevertheless standard to refer to it as a *homology* rather than *cohomology* theory. For this reason we write *chain* complex rather than *cochain* complex.

If C is a chain complex of finitely generated graded abelian groups, then the *graded Euler characteristic* of C is defined to be

$$\chi_q(C) := \sum_{i\in\mathbb{Z}}(-1)^i \operatorname{rank}_q(C^i).$$

Lemma 2.2.3. *For a chain complex C of graded abelian groups,*

$$\chi_q(C) = \sum_{i\in\mathbb{Z}}(-1)^i \operatorname{rank}_q(H^i).$$

It follows that if chain complexes C and D are chain homotopy equivalent, then $\chi_q(C) = \chi_q(D)$.

More generally, one can consider a ring graded over any abelian monoid $\mathcal{M}$. Precisely, given an abelian monoind $\mathcal{M}$, an $\mathcal{M}$-graded ring is a ring R with a decomposition into additive subgroups $R = \bigoplus_{m\in\mathcal{M}} R_m$ such that $R_m \cdot R_n \subset R_{m+n}$ for all $m, n \in \mathcal{M}$. The notions of graded R-modules and chain complexes extend naturally to this setting. When $\mathcal{M} = \mathbb{Z}$, we recover the definitions given above.

We will be interested later in $\mathbb{Z}^n$-graded rings and graded modules over them. If R is a $\mathbb{Z}^2$-graded ring, we write $R-$ ggmod for the category of graded R-modules, and we call such a module *bigraded*. The chain groups in annular Khovanov homology, recalled in Section 2.4.1, are bigraded.

2.3 Khovanov homology

In the seminal work [33], Khovanov introduced a *categorification* of the Jones polynomial. We summarize some of the main results here.

Theorem 2.3.1 ([33])**.** *Let D be a diagram for an oriented link L. There exists a chain complex $CKh(D)$ of graded abelian groups such that $\chi_q(CKh(D)) = V(L)$. Moreover, if D' is another diagram of L, then $CKh(D) \simeq CKh(D')$.*

Calculations demonstrate that Khovanov homology is a strictly stronger invariant than the Jones polynomial, in the sense that there exist links L_1 and L_2 such that $V(L_1) = V(L_2)$

but $Kh(L_1)$ and $Kh(L_2)$ are non-isomorphic, c.f. [7]. Kronheimer-Mrowka proved that Khovanov homology detects the unknot [47], while this question for the Jones polynomial remains open. In addition to being a powerful invariant, Khovanov homology enjoys numerous structural properties. We review the key property, called *functoriality*.

Let $L, L' \subset \mathbb{R}^3$ be oriented links. A *cobordism from L to L'* is a smoothly and properly embedded compact oriented surface $S \subset \mathbb{R}^3 \times [0,1]$ such that $\partial S = \overline{L} \sqcup L'$, where $\overline{L}$ denotes L with the reversed orientation. Two link cobordisms are isotopic if there exists an ambient isotopy of $\mathbb{R}^3 \times [0,1]$ sending one to the other while fixing the boundary of $\mathbb{R}^3 \times [0,1]$ pointwise.

Any link cobordism S is isotopic to one presented as a sequence of *elementary cobordisms*. This consists of a finite sequence of oriented link diagrams $D_1, \ldots, D_k$, where D_1 and D_k are diagrams of L and L' respectively, and each D_{i+1} is obtained from D_i by either planar isotopy, a Reidemeister move, or a handle attachment (cup, cap or saddle).

To each elementary cobordism $S_i : D_i \to D_{i+1}$, Khovanov defined a chain map $CKh(S_i) : CKh(D_i) \to CKh(D_{i+1})$ [33, Section 6.3]. Setting $CKh(S) = CKh(S_{k-1}) \circ \cdots \circ CKh(S_0)$, Khovanov conjectured that the induced map on homology is independent of the decomposition of S into elementary pieces, up to an overall multiplication by ± 1 [33, Conjecture 1]. The conjecture was first proven by Jacobsson [29] by a thorough case check; alternative, more conceptual arguments are given in [8] and [36]. Several approaches to fixing the sign ambiguity have been proposed [21, 17, 12, 84, 77].

Theorem 2.3.2 ([29, 8, 36])**.** *Let $L_0, L_1 \subset \mathbb{R}^3$ be oriented links, and let $S \subset \mathbb{R}^3 \times [0,1]$ be an oriented cobordism from $L_0 \subset \mathbb{R}^3 \times \{0\}$ to $L_1 \subset \mathbb{R}^3 \times \{1\}$. There exists a chain map $CKh(S) : CKh(L_0) \to CKh(L_1)$ of degree $-\chi(S)$. The chain homotopy class of $CKh(S)$ is an invariant of the isotopy class of S up to multiplication by ± 1.*

A key application of functoriality is Rasmussen's celebrated proof of Milnor's conjecture on the slice genus of positive knots [69], originally proven by Kronheimer and Mrowka [46] using gauge theory.

2.3.1 The Bar-Natan category and the formal complex

This section reviews the construction of Khovanov homology via the Bar-Natan category, introduced in [8]. Let us give an executive summary: Bar-Natan defines a graded, additive category $\widetilde{\mathcal{BN}}$ and associates to every oriented link diagram D a chain complex $[[D]]$ over $\widetilde{\mathcal{BN}}$. The chain homotopy class of $[[D]]$ is proven to invariant under Reidemeister moves. In order to obtain Khovanov homology, one applies a particular functor $\widetilde{\mathcal{BN}} \to \mathbb{Z}-\mathrm{gmod}$ term-wise to $[[D]]$. One benefit of this approach is that invariance is proven at an earlier level: any reasonable[2] functor from $\widetilde{\mathcal{BN}}$ into an abelian category yields an algebraic invariant of links. We now recall the Bar-Natan category $\widetilde{\mathcal{BN}}$ and the construction of the chain complex $[[D]]$. Key examples of such functors which yield link homology are discussed in Sections 2.3.3 and 2.3.4. Section 2.3.2 discusses the dotted Bar-Natan category, a slight but useful modification of $\widetilde{\mathcal{BN}}$.

Let R be a commutative ring. We begin by reviewing some categorical notions.

Definition 2.3.3. 1. A category $\mathcal{C}$ is called *R-linear* if each hom set $\mathrm{Hom}_{\mathcal{C}}(x, y)$ is an R-module and composition in $\mathcal{C}$ is R-bilinear.

2. An R-linear category is *additive* if it admits all finite products and coproducts.

3. Suppose that R is a $\mathbb{Z}$-graded ring. An R-linear category $\mathcal{C}$ is *graded* if

   - each morphism space is a graded R-module, such that composition $\mathrm{Hom}_{\mathcal{C}}(y, z) \otimes_R \mathrm{Hom}_{\mathcal{C}}(x, y) \to \mathrm{Hom}_{\mathcal{C}}(x, y)$ is a graded map of degree zero, and
   - $\mathcal{C}$ comes equipped with a $\mathbb{Z}$-action, denoted $x \mapsto x\{n\}$, called a *grading shift*. As R-modules, $\mathrm{Hom}_{\mathcal{C}}(x, y) = \mathrm{Hom}_{\mathcal{C}}(x\{n\}, y\{m\})$ for all $n, m \in \mathbb{Z}$, but as graded R-modules we have $\mathrm{Hom}_{\mathcal{C}}(x\{n\}, y\{m\})_i = \mathrm{Hom}_{\mathcal{C}}(x, y)_{i+m-n}$.

Remark 2.3.4. An R-linear category is also called a category *enriched in* $R-\mathrm{mod}$. Our notion of a $\mathbb{Z}$-linear category is called *pre-additive* in [8].

We now discuss some categorical constructions.

[2] The functor should, at the very least, be additive on objects and linear on morphism spaces.

- Given a category $\mathcal{C}$, we can form a R-linear category $R\mathcal{C}$ in a natural way: objects of $R\mathcal{C}$ are the same as objects of $\mathcal{C}$, and $\mathrm{Hom}_{R\mathcal{C}}(x, y)$ is defined to be the free R-module generated by $\mathrm{Hom}_{\mathcal{C}}(x, y)$. Composition is defined by extending the composition in $\mathcal{C}$ in an R-bilinear manner: if $f_1, \ldots, f_n \in \mathrm{Hom}_{\mathcal{C}}(x, y), g_1, \ldots, g_m \in \mathrm{Hom}_{\mathcal{C}}(y, z)$, and $r_1, \ldots, r_n, s_1, \ldots, s_m \in R$, then composition of $f = \sum_{i=1}^{n} r_i f_i$ and $g = \sum_{j=1}^{m} s_j g_j$ in $R\mathcal{C}$ is
$$g \circ f := \sum_{i,j} r_i s_j g_j f_i.$$
Note also that there is a faithful functor $\mathcal{C} \to R\mathcal{C}$, so we view $\mathcal{C}$ as a subcategory of $R\mathcal{C}$. The category $R\mathcal{C}$ is called the *R-linear closure* of $\mathcal{C}$.

- Suppose that R is graded and $\mathcal{C}$ is an R-linear category satisfying the first bullet point in item (3) of Definition 2.3.3. We can upgrade $\mathcal{C}$ to a graded category by declaring objects to be symbols $x\{n\}$ for $n \in \mathbb{Z}$, $x \in \mathrm{ob}(\mathcal{C})$, and defining a grading on the new morphism spaces in the unique way such that the definition of a graded category is satisfied: if $f \in \mathrm{Hom}_{\mathcal{C}}(x, y)$ is homogeneous of degree d then, viewed as a morphism from $x\{n\}$ to $y\{m\}$, it is homogeneous of degree $d + m - n$. We call this the *graded closure* of $\mathcal{C}$.

- Any R-linear category $\mathcal{C}$ can be upgraded to an additive category $\mathcal{C}^{\oplus}$ as follows. Objects of $\mathcal{C}^{\oplus}$ are formal direct sums $x_1 \oplus \cdots \oplus x_n$ of objects of $\mathcal{C}$. A morphism in $\mathcal{C}^{\oplus}$ from $x_1 \oplus \cdots \oplus x_n$ to $y_1 \oplus \cdots \oplus y_m$ is an $m \times n$ matrix whose (i, j)-th entry is a morphism $x_j \to y_i$ in $\mathcal{C}$. Composition of morphisms in $\mathcal{C}^{\oplus}$ is defined by usual matrix multiplication, where multiplication of entries is given by composition in $\mathcal{C}$. The category $\mathcal{C}^{\oplus}$ is called the *additive closure* of $\mathcal{C}$.

Definition 2.3.5. Let $\mathcal{C}$ be a R-linear category. Define the category of complexes over $\mathcal{C}$, denoted $\mathrm{Kom}(\mathcal{C})$, to have objects finite-length chains of composable morphisms in $\mathcal{C}$, $C = (\cdots \to C^i \xrightarrow{\partial_i} C^{i+1} \xrightarrow{\partial_{i+1}} \cdots)$, such that $\partial_{i+1} \circ \partial_i = 0$ for all j. A morphism f from C to D is a collection of morphisms $f_i : C^i \to D^i$ which satisfy $\partial_i f_i = f_{i+1} \partial_i$.

Two morphisms $f, g : C \to D$ are *chain homotopic* if there exists a sequence of morphisms $F_i : C^i \to D^{i+1}$ such that $\partial_{i+1} F_i + F_{i+1} \partial_i = f_i - g_i$ for all i. Two complexes C and D are *chain homotopy equivalent* if there exist morphisms $f : C \to D$, $g : D \to C$ such that gf and fg are chain homotopy equivalent to id_C and id_D, respectively.

Moreover, if R is a graded ring and $\mathcal{C}$ is a graded R-linear category, then we require all morphisms in $\mathcal{C}$ appearing above to be of degree zero.

For the time being, we will focus on the case $R = \mathbb{Z}$ with the trivial grading, in which case we simply say *linear* rather than $\mathbb{Z}$-linear. The more general situation, when R is a polynomial ring with nontrivial grading, will appear in Section 2.3.4. If R is a $\mathbb{Z}^N$-graded ring, then there is a natural analogue (which we do not spell out explicitly) of item (3) of Definition 2.3.3.

We will now define the graded category $\widetilde{\mathcal{BN}}$. The formal complex $[[D]]$ associated to an oriented link diagram D will be a chain complex over $\widetilde{\mathcal{BN}}$, or, in other words, an object in $\mathrm{Kom}(\widetilde{\mathcal{BN}})$.

Definition 2.3.6.

1. Let **Cob** denote the following category. An object of **Cob** is a (possibly empty) collection of disjoint simple closed curves in the plane $\mathbb{R}^2$. Define $\mathrm{Hom}_{\mathbf{Cob}}(Z_0, Z_1)$ to be the set of embedded cobordisms from Z_0 to Z_1 in $\mathbb{R}^2 \times I$, modulo ambient isotopy which fixes the boundary of $\mathbb{R}^2 \times I$ pointwise. The identity morphism of an object Z is the product cobordism $Z \times I$. Composition of morphisms is given by stacking one on top of the other, gluing along their common boundary, and rescaling the interval direction.

2. Consider now the linear closure $\mathbb{Z}\mathbf{Cob}$; a morphism in $\mathbb{Z}\mathbf{Cob}$ is illustrated in Figure 2-3. Note that $\mathbb{Z}\mathbf{Cob}$ satisfies the first bullet point in Definition 2.3.3, by defining the degree of a cobordism S to be $-\chi(S)$.

3. Let $\mathbf{Cob}_{/l}$ denote the quotient of $\mathbb{Z}\mathbf{Cob}$ by the local Bar-Natan relations shown in Figure 2-4. Here *local* means that the relations hold inside of a small ball, where the cobordisms differ as in the linear combination depicted in the relation, and outside of this small ball the cobordisms are identical. Note that the Bar-Natan relations are homogeneous, so that the grading on hom-spaces in $\mathbb{Z}\mathbf{Cob}$ descends to a grading on hom-spaces in $\mathbf{Cob}_{/l}$.

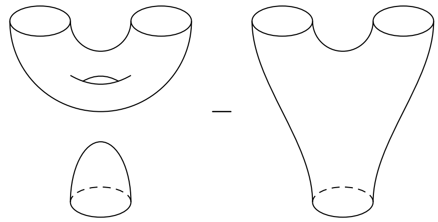
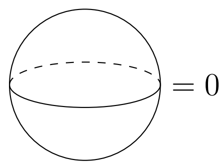
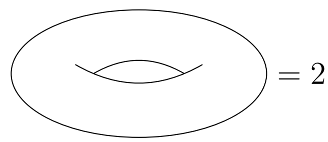


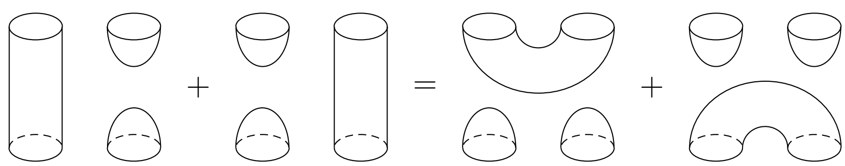
Figure 2-3: A morphism in $\mathbb{Z}\mathbf{Cob}$ from one circle in the plane to two un-nested circles in the plane. Cobordisms are read bottom to top.



(a) Sphere



(b) Torus



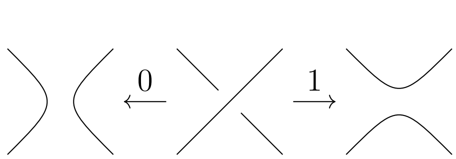
(c) Four tubes

Figure 2-4: Relations in $\widetilde{\mathcal{BN}}$.

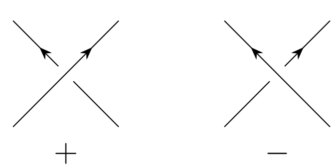
4. Finally, let $\widetilde{\mathcal{BN}}$ denote the additive closure of the graded closure of $\mathbf{Cob}_{/l}$.

Let D be a diagram for an oriented link L. We are now ready to define the Bar-Natan complex $[[D]]$. To begin, one first forms the *cube of resolutions* as follows.

Label the crossings of the D by $1, \ldots, n$. Every crossing may be resolved in two ways, called the *0-smoothing* and *1-smoothing*, as shown in Figure (2-5a). For each $u = (u_1, \ldots, u_n) \in \{0,1\}^n$, perform the u_i-smoothing at the i-th crossing. The resulting diagram is a collection of disjoint simple closed curves in the plane, which we view as an object in $\widetilde{\mathcal{BN}}$ and denote



(a) The two smoothings of a crossing.



(b) Positive and negative crossings.

Figure 2-5

by D_u. Viewing elements of $\{0,1\}^n$ as vertices of an n-dimensional cube $[0,1]^n$, decorate the vertex u by the smoothing D_u.

Let $v = (v_1, \ldots, v_n)$ and $u = (u_1, \ldots, u_n)$ be vertices which differ only in the i-th entry, where $v_i = 0$ and $u_i = 1$. Then the diagrams D_v and D_u are the same outside of a small disk around the i-th crossing. There is a natural cobordism from D_v to D_u, which is a saddle inside this disk around the i-th crossing and the identity (product cobordism) elsewhere. We will call this the *saddle cobordism* from D_v to D_u, and denote it by $d_{v,u}$. Decorate each edge of the n-dimensional cube by these saddle cobordisms. We now have a commutative cube in the category $\widetilde{\mathcal{BN}}$.

A *sign assignment* is a label $s_{v,u} \in \{0,1\}$ on each edge such that, for each square face of the cube, the sum of labels of its edges is equal to $1 \mod 2$. It follows that multiplying the edge map $d_{v,u}$ by $(-1)^{s_{v,u}}$ results in an anti-commutative cube. We refer the interested reader to [8, Section 2.7] and [57, Definition 4.5], for further discussion about sign assignments.

For $u = (u_1, \ldots, u_n) \in \{0,1\}^n$, let $|u| = \sum_i u_i$. Now, form the chain complex $[[D]]$ by setting

$$[[D]]^i = \bigoplus_{|u|=i+n_-} D_u\{n_- - n_+ - i\}$$

where n_-, n_+ are the number of negative and positive crossings in D (see Figure 2-5b), and the brackets $\{-\}$ denotes the formal grading shift in $\widetilde{\mathcal{BN}}$. The differential is given on each summand by the edge map $(-1)^{s_{v,u}} d_{v,u}$. Anti-commutativity of the cube ensures that the differential in $[[D]]$ squares to zero.

Note that each cobordism decorating an edge in the cube of resolutions has Euler characteristic -1. The grading shifts ensure that the differential in $[[D]]$ is of degree zero. Hence we view $[[D]]$ as an object in $\mathrm{Kom}(\widetilde{\mathcal{BN}})$.

Remark 2.3.7. There are two internal grading conventions in the literature. The internal grading shifts in the curly brackets given above agree with those in [34, 37] but are opposite those in [33, 8].

Theorem 2.3.8 ([8, Theorem 1, Theorem 3]). *If diagrams D and D' are related by a Reidemeister move, then $[[D]]$ and $[[D']]$ are chain homotopy equivalent.*

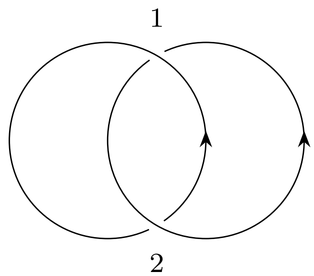


Figure 2-6: A diagram for the Hopf link, with crossings ordered.

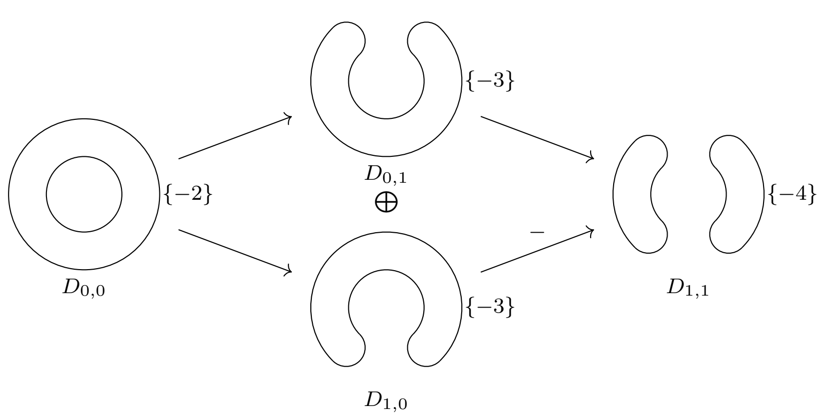
The complex $[[D]]$ can also be obtained by assigning to each crossing one of the two complexes

(2.3)

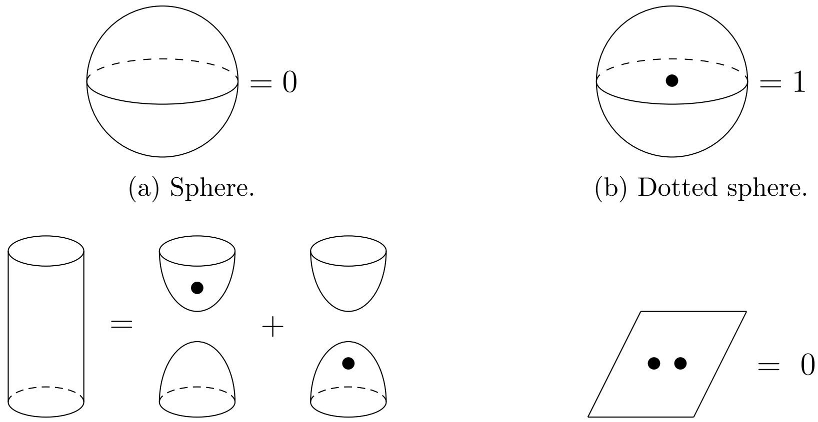
where the underlined terms are in homological degree zero. The complex $[[D]]$ is then assembled by tensoring these local pieces in a planar algebra manner.

Remark 2.3.9. A key insight in Bar-Natan's construction is that the complex $[[D]]$ can be defined for *tangles* rather than links. Invariance under Reidemeister moves is proven at this local level.

Example 2.3.10. Consider the diagram D shown in Figure 2-6, with the indicated ordering on its crossings. Its associated complex $[[D]]$ is shown below in (2.4). The term $D_{0,0}$ is in homological degree zero.



(2.4)



(a) Sphere.

(b) Dotted sphere.

(c) Neck-cutting.

(d) Two dots.

Figure 2-7: Relations in the dotted Bar-Natan category $\mathcal{BN}$.

Section 2.3.3 details how to obtain Khovanov homology from the formal complex $[[D]]$. A key feature of Bar-Natan's construction is that invariance with respect to Reidemeister moves is proven already at this "topological" stage, before passing to an algebraic category.

2.3.2 The dotted Bar-Natan category

We recall from [8, Section 11.2] a minor but useful modification of $\widetilde{\mathcal{BN}}$. Let $Z_0, Z_1 \subset \mathbb{R}^2$ be two objects in **Cob** (that is, each of Z_0 and Z_1 is a collection of disjoint simple closed curves in the plane). A *dotted* cobordism from Z_0 to Z_1 is an embedded cobordism S from Z_0 to Z_1 such that each component of S carries finitely many marked points called *dots*. A dot may move freely along the component it lies on, but it may not jump across components of S.

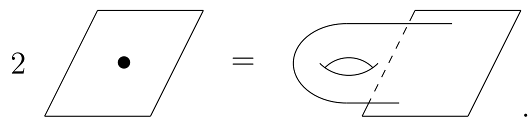
Let $\mathbf{Cob}_\bullet$ be the category of dotted cobordisms, with composition defined in the natural way. Morphism spaces in its linear closure $\mathbb{Z}\mathbf{Cob}_\bullet$ carry a grading, defined by

$$\deg(S) = -\chi(S) + 2d(S), \tag{2.5}$$

where $d(S)$ denotes the number of dots on S. Degree is additive with respect to composition of cobordisms, so $\mathbb{Z}\mathbf{Cob}_\bullet$ satisfies the first bullet point in Definition 2.3.3. Let $\mathbf{Cob}_{\bullet,/l}$ denote the quotient of $\mathbb{Z}\mathbf{Cob}_\bullet$ by the local relations shown in Figure 2-7; note that these relations are homogeneous with respect to the above grading, so morphism spaces in $\mathbf{Cob}_{\bullet,/l}$ inherit the grading.

Definition 2.3.11. Define $\mathcal{BN}$ to be the additive closure of the graded closure of $\mathbf{Cob}_{\bullet,/l}$.

Note that the neck-cutting relation, Figure 2-7c, implies that twice a dot is equal to a handle:

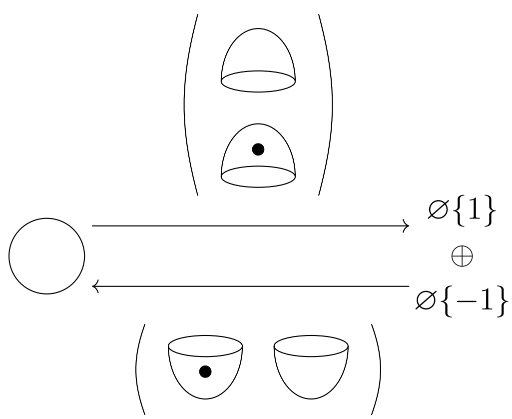


Therefore if we extend scalars from $\mathbb{Z}$ to $\mathbb{Z}[1/2]$, then there is no difference between the dotted and undotted Bar-Natan categories.

The following proposition, called *delooping*, implies that any object in $\mathcal{BN}$ is isomorphic to a direct sum of grading-shifted empty diagrams.

Proposition 2.3.12 ([9, Lemma 4.1])**.** *Let $Z \subset \mathbb{R}^2$ be collection of $k \geq 1$ disjoint embedded circles. Let Z' denote Z with one circle removed. As an object in $\mathcal{BN}$, Z is isomorphic to $Z'\{1\} \oplus Z'\{-1\}$.*

Proof. Using the relations in Figure 2-7, we see that the following maps are mutually inverse isomorphisms.



∎

Given an oriented link diagram D, one can construct the formal complex $[[D]]$ over the category $\mathcal{BN}$ by following exactly the procedure explained in Section 2.3.1. Note that the relations in Figure 2-7 imply the relations in Figure 2-4. It follows that $[[D]]$, as a chain complex over $\mathcal{BN}$, is an invariant up to chain homotopy equivalence of the oriented link represented by D.

2.3.3 Frobenius algebras and homological link invariants

We will now review, to the extent necessary, the notions of topological quantum field theories and Frobenius algebras.

Definition 2.3.13. A *Frobenius system* is a tuple $\mathcal{F} = (R, A, \eta, \varepsilon, \Delta)$ where R and A are commutative rings, $\eta : R \to A$ is a ring inclusion, $\varepsilon : A \to R$ is an R-module map, and $\Delta : A \to A \otimes_R A$ is an (A, A)-bimodule homomorphism. Moreover, the following diagrams must commute

$$\begin{tikzcd} A \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & A \otimes_R A \arrow[d, "\Delta \otimes \mathrm{id}_A"] \\ A \otimes_R A \arrow[r, "\mathrm{id}_A \otimes \Delta"] & A \otimes_R A \otimes_R A \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\Delta"] \arrow[dr, "\Delta"'] & A \otimes_R A \arrow[d, "\tau"] \\ & A \otimes_R A \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\Delta"] \arrow[dr, "\mathrm{id}_A"'] & A \otimes_R A \arrow[d, "\varepsilon \otimes \mathrm{id}_A"] \\ & A \end{tikzcd},$$

where the map τ in the second diagram is the involution $\tau(a \otimes b) = b \otimes a$.

The ring R is the *ground ring* and A is the *Frobenius algebra*. Multiplication in A will be denoted m. The maps η, ε, and Δ are called the *unit, counit* and *comultiplication*, respectively. We will often write $\mathcal{F} = (R, A)$ when the structure maps are clear from context, and moreover we will sometimes refer only to the Frobenius algebra A when both R and the structure maps are clear.

Let $\mathbf{Cob}'$ denote the category of abstract[3] 2-dimensional cobordisms. Objects of $\mathbf{Cob}'$ are oriented closed 1-manifolds (i.e., a finite disjoint union of oriented circles). A morphism from Z_0 to Z_1 is a 2-dimensional, oriented, compact manifold S, together with a homeomorphism

$$\partial S \cong \overline{Z_0} \sqcup Z_1$$

where $\overline{Z_0}$ denotes Z_0 with the reverse orientation. Cobordisms are considered up to orientation-preserving homeomorphism which commutes with the above homeomorphisms on the boundary. Composition of cobordisms $S_0 : Z_0 \to Z_1$ and $S_1 : S_1 \to S_2$ is defined in the natural way, by gluing along the common boundary Z_1. Note that disjoint union makes $\mathbf{Cob}'$ into a symmetric monoidal category. Likewise, the category of R-modules, denoted $R-$ mod, is symmetric monoidal via tensor product.

[3]Here *abstract* means not embedded in 3-space, to distinguish from morphims in **Cob**.

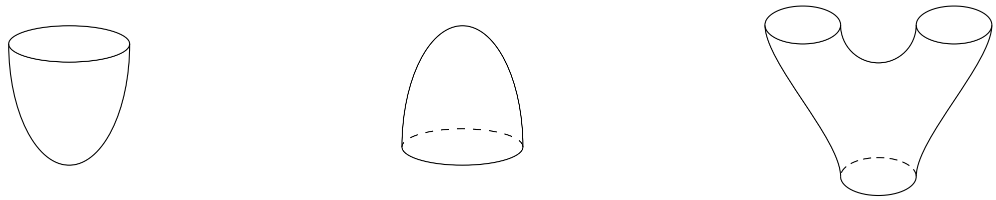


Figure 2-8: The elementary cup, cap, and saddle cobordisms. Every morphism in $\mathbf{Cob}'$ can be written as a composition of cobordisms where precisely one component is one of these three cobordisms, and the other components are the identity cobordism on the remaining circles.

Definition 2.3.14. Let R be a commutative ring. A $(1+1)$-*dimensional TQFT* is a symmetric monoidal functor $\mathbf{Cob}' \to R-\mathrm{mod}$.

A Frobenius system $\mathcal{F} = (R, A, \eta, \varepsilon, \Delta)$ defines a $(1+1)$-dimensional TQFT, also denoted $\mathcal{F}$, from $\mathbf{Cob}'$ to $R-\mathrm{mod}$. Given a closed 1-manifold $\mathbb{Z}$, set[4] $\mathcal{F}(Z) = A^{\otimes_R |Z|}$, where $|Z|$ is the number of components of Z. To define $\mathcal{F}$ on a cobordism S, pick a Morse decomposition of S to write S as a composition $S = S_1 \circ \cdots \circ S_k$ where each S_i is an *elementary cobordism*: a disjoint union of an identity cobordism and either a cup (0-handle), cap (2-handle), or saddle (1-handle); see Figure 2-8. To the cup and cap cobordisms assign the maps $\eta : R \to A$ and $\varepsilon : A \to R$, respectively. To a saddle cobordism, assign the multiplication in A, $m : A \otimes_R A \to A$ if the saddle goes from two circles to one circle, and assign $\Delta : A \to A \otimes_R A$ if the saddle goes from one circle to two circles. Define $\mathcal{F}(S)$ to be the composition of these maps, $\mathcal{F}(S) = \mathcal{F}(S_1) \circ \cdots \circ \mathcal{F}(S_k)$.

Remark 2.3.15. Note that objects and morphisms in $\mathbf{Cob}'$ are oriented manifolds, while those in $\mathbf{Cob}$ do not come with a prescribed orientation. Objects of $\mathbf{Cob}$ are, however, embedded in $\mathbb{R}^2$, which can be used to orient objects and morphisms, leading to a functor $\mathbf{Cob} \to \mathbf{Cob}'$ (see [34, Section 2.1]). Computing maps assigned by a Frobenius system does not involve considering orientations, so this minor adjustment is inconsequential.

Theorem 2.3.16 ([1, 23])**.** *The map $\mathcal{F}(S)$ is independent of the decomposition of S into elementary cobordisms.*

Remark 2.3.17. In fact, more can be said: monoidal functors $\mathbf{Cob}' \to R-\mathrm{mod}$ are in bijection with Frobenius systems with ground ring R, [23]. We refer the interested reader to

[4] Since every object in $\mathbf{Cob}'$ is a disjoint union circles, the assignment on objects is fixed by requiring $\mathcal{F}$ to be monoidal.

[44] for a comprehensive treatment of this rich theory.

We now describe the Frobenius system underlying Khovanov homology.

Definition 2.3.18. Let $\mathcal{F}_0 = (R_0, A_0, \eta_0, \varepsilon_0, \Delta_0)$, where $R_0 = \mathbb{Z}$, $A_0 = \mathbb{Z}[X]/(X^2)$, and $\eta_0 : Z \to Z[X]/(X^2)$ the natural inclusion. Note that A_0 is a free abelian group with basis $\{1, X\}$. The map $\varepsilon_0 \colon A_0 \to \mathbb{Z}$ is given by $1 \mapsto 0$, $X \mapsto 1$, and $\Delta_0 : A_0 \to A_0 \otimes A_0$ is given by $1 \mapsto X \otimes 1 + 1 \otimes X, X \mapsto X \otimes X$.

Let S be a dotted cobordism. Following the correspondence between (1+1)-dimensional TQFTs and Frobenius algebras outlined above, we write S as a composition of elementary dotted cobordisms, each of which is either an undotted cup, cap or saddle, or an identity (product) cobordism carrying precisely one dot. Associate to an undotted elementary cobordisms the linear maps as detailed above, and to dotted identity cobordism multiplication by $X \in A_0$ on the circle corresponding to the dotted component. Given a dotted cobordism S from Z_0 to Z_1, let $\mathcal{F}_0(S) : A_0^{|Z_0|} \to A_0^{|Z_1|}$ denote the induced map; following the discussion preceding Theorem 2.3.16, it is straightforward to deduce that $\mathcal{F}_0(S)$ is independent of the choice of decomposition of S into elementary cobordisms. We obtain a functor $\mathcal{F}_0 : \mathbb{Z}\mathbf{Cob}_{\bullet} \to \mathbb{Z}-\text{mod}$.

Lemma 2.3.19. *The functor $\mathcal{F}_0 : \mathbb{Z}\mathbf{Cob}_{\bullet} \to \mathbb{Z}-\text{mod}$ factors through the dotted Bar-Natan relations shown in Figure 2-7.*

Proof. The dotted Bar-Natan relations (Figure 2-7) correspond to the structure of A_0 in the following way. Then the sphere relation corresponds to $\varepsilon_0(\eta_0(1)) = 0$, while the dotted sphere comes from $\varepsilon_0(X) = 1$. The two dots relation corresponds to the relation $X^2 = 0$ in A_0. Neck-cutting is a topological incarnation of the algebraic relation

$$y = X\varepsilon_0(y) + \varepsilon_0(Xy),$$

which holds for every $y \in A_0$. □

Definition 2.3.20. If $Z \subset \mathbb{R}^2$ consists of n circles, then $\mathcal{F}_0(Z)$ is a free abelian group with basis $y_1 \otimes \cdots \otimes y_n$, where each $y_i \in \{1, X\}$. We will refer to these basis elements as *standard Khovanov generators.*

Define the *quantum grading*, denoted qdeg, on A_0 by setting

$$\mathrm{qdeg}(1) = -1 \qquad \mathrm{qdeg}(X) = 1. \tag{2.6}$$

Remark 2.3.21. Viewing A_0 as an algebra, it is more natural to set 1 and X in degrees 0 and 2, respectively, to make the multiplication grading-preserving. However, when viewing A_0 as an R_0-module, degrees are balanced around 0 as above. Moreover, we note that elsewhere in the literature the quantum gradings of 1 and X are opposite those given above. See also Remark 2.3.7.

Given a dotted cobordism S, it is straightforward to verify that $\mathcal{F}_0(S)$ is a graded map of degree $\deg(S)$, where $\deg(S)$ is as defined in Equation (2.5). Combined with Lemma 2.3.19, we obtain a functor

$$\mathcal{F}_0 : \mathcal{BN} \to \mathbb{Z}\text{-}\,\mathrm{gmod},$$

which is degree-preserving on each hom-space. To summarize, $\mathcal{F}_0$ sends an object $\bigoplus_{i=1}^{k} Z_i\{n_i\}$ to $\bigoplus_{i=1}^{k} A_0^{|Z_0|}\{n_i\}$ and a morphism to the corresponding matrix of linear maps.

Definition 2.3.22. Given an oriented link L with diagram D, let $CKh_0(D)$ denote the chain complex of graded R-modules obtained by applying $\mathcal{F}_0$ term-wise to $[[D]]$.

Grading shifts in the definition of $[[D]]$ ensure that the differential is degree preserving. Theorem 2.3.8 implies that the chain homotopy class of $CKh_0(D)$ is an invariant of L, moreover, $CKh_0(D)$ is precisely the complex constructed in [33, Section 7], after negating the quantum gradings (see Remark 2.3.21).

Remark 2.3.23. Note that, while we work with $\mathcal{BN}$ above, the discussion holds just as well if working over the undotted version $\widetilde{\mathcal{BN}}$.

We now expand on the relation between dots in $\mathcal{BN}$ and the structure of the Frobenius algebra A_0. Consider the representable functor $\mathrm{Hom}_{\mathcal{BN}}(\varnothing, -) : \mathcal{BN} \to \mathbb{Z}\text{-}\,\mathrm{gmod}$. The following result is essentially the content of [8, Exercise 9.3], and is also stated in [8, Section 11.2].

Proposition 2.3.24. *The functors* $\mathrm{Hom}_{\mathcal{BN}}(\varnothing, -)$ *and* $\mathcal{F}_0$ *and are naturally isomorphic.*

Figure 2-9: Distinguished basis elements for $\mathrm{Hom}_{\mathcal{BN}}(\varnothing, Z)$ when Z is a single circle. The undotted and dotted cup cobordisms correspond to 1 and X in A_0, respectively.

Proof. Let $Z \subset \mathbb{R}^2$ be a collection of n disjoint simple closed curves, ordered from $1, \ldots, n$. For $B \subset \{1, \ldots, n\}$, let Σ_B denote the cobordism from $\varnothing$ to Z whose underlying surface is a disjoint union of cup cobordisms, where the i-th cup cobordism has a dot if $i \in B$. See Figure 2-9 when $n = 1$. Note that each Σ_B is homogeneous of degree $2|B| - n$, where $|B|$ denotes the cardinality of B.

We first argue that $\mathrm{Hom}_{\mathcal{BN}}(\varnothing, Z)$ is free with basis $\{\Sigma_B \mid B \subset \{1, \ldots, n\}\}$. Using the neck-cutting relation, Figure 2-7c, we see that this set spans $\mathrm{Hom}_{\mathcal{BN}}(\varnothing, Z)$. To verify linear independence, let Σ_B^* denote the cobordism from Z to $\varnothing$ whose underlying surface is a disjoint union of disks, and where the disk bounding the i-th circle is dotted if $i \notin B$. The sphere relations, Figure 2-7a and Figure 2-7b, imply $\Sigma_B^* \Sigma_C = \delta_{B,C}$ for all $B, C \subset \{1, \ldots, n\}$. This proves linear independence of our claimed basis.

The natural isomorphism from $\mathrm{Hom}_{\mathcal{BN}}(\varnothing, -)$ to $\mathcal{F}_0$ has components $\phi_Z : \mathrm{Hom}_{\mathcal{BN}}(\varnothing, Z) \to F_0(Z)$, defined by sending Σ_B to the standard generator $y_1 \otimes \cdots y_n$ given by $y_i = 1$ if $i \notin B$ and $y_i = X$ if $i \in B$. In other words, an undotted cup corresponds to 1 and a dotted cup corresponds X. To show that the ϕ_Z assemble into a natural isomorphism, it suffices to show that they commute with elementary cobordisms, which is a straightforward case check. $\square$

2.3.4 Equivariant Khovanov homology

The Frobenius algebra A_0 from Section 2.3.3 is not the only one which yields link homology when applied to $[[D]]$. Alternative TQFTs were explored in [8, Section 9], which are defined by introducing further relations into $\widetilde{\mathcal{BN}}$ and applying a representable functor. A systematic treatment of Frobenius algebras in the context of link homology appeared in [37]; see also [40]. We review some of these theories here. In this section, all cobordisms are assumed to potentially carry dots. We also fix an oriented link L with diagram D.

Definition 2.3.25. 1. Let $\mathcal{F}_E$ be the Frobenius system with ground ring $R_E = \mathbb{Z}[E_1, E_2]$ and Frobenius algebra $A_E = \mathbb{Z}[X]/(X^2 - E_1X + E_2)$. Note that A_E is a free R_E-module with basis $\{1, X\}$. The counit ε_E is given by $\varepsilon_E(1) = 0$, $\varepsilon_E(X) = 1$, and comultiplication is $\Delta_E(1) = 1 \otimes X + X \otimes 1 - E_1 1 \otimes 1$, $\Delta_E(X) = X \otimes X - E_2 1 \otimes 1$.

2. Let $\mathcal{F}_\alpha$ be the Frobenius system with ground ring $R_\alpha = \mathbb{Z}[\alpha_1, \alpha_2]$ and Frobenius algebra $A_\alpha = \mathbb{Z}[X]/((X - \alpha_1)(X - \alpha_2))$. As above, A_α is a free R_α-module with basis $\{1, X\}$. The counit ε_α is given by $\varepsilon_\alpha(1) = 0$, $\varepsilon_\alpha(X) = 1$, and comultiplication is $\Delta_\alpha(1) = 1 \otimes X + X \otimes 1 - (\alpha_1 + \alpha_2)$, $\Delta_\alpha(X) = X \otimes X - \alpha_1\alpha_2 1 \otimes 1$.

Denote the $(1 + 1)$-dimensional TQFTs assigned to (R_E, A_E) and (R_α, A_α) by $\mathcal{F}_E$ and $\mathcal{F}_\alpha$, respectively. Define a grading on R_E and R_α by setting $\deg(E_1) = 2, \deg(E_2) = 4$ and $\deg(\alpha_1) = \deg(\alpha_2) = 2$. Likewise, make A_E (resp. A_α) into graded modules over R_E (resp. R_α) by setting $\mathrm{qdeg}(1) = -1$, $\mathrm{qdeg}(X) = 1$. Under the theories $\mathcal{F}_E$ and $\mathcal{F}_\alpha$, a cobordism S induces a graded map of degree $\deg(S) = -\chi(S) + 2d(S)$.

The following is a straightforward check.

Lemma 2.3.26. *The TQFTs $\mathcal{F}_E$ and $\mathcal{F}_\alpha$ factor through the Bar-Natan relations in Figure 2-4, descending to functors*

$$\mathcal{F}_E : \mathcal{BN} \to R_E\text{– gmod}, \quad \mathcal{F}_\alpha : \mathcal{BN} \to R_\alpha\text{– gmod}\,.$$

Consequently, applying either $\mathcal{F}_E$ or $\mathcal{F}_\alpha$ to the Bar-Natan complex $[[D]]$ yields a homological link invariant. We denote these complexes by $CKh_E(D)$ and $CKh_\alpha(D)$, and their homologies by $Kh_E(L)$ and $Kh_\alpha(L)$.

The theory $\mathcal{F}_E$ is denoted by $\mathcal{F}_5$ in [37]; the above E_1 and E_2 correspond to h and $-t$, respectively, in that reference. The link homology theory given by $\mathcal{F}_E$ is sometimes called *universal* Khovanov homology, due to [37, Proposition 5].

Note that the Frobenius algebra A_0 is the cohomology with $\mathbb{Z}$ coefficients of $\mathbb{CP}^1$. As noted in [37, Example 2], R_E and A_E are the the $U(2)$-equivariant cohomology of a point and $\mathbb{CP}^1$, respectively. Likewise, R_α and A_α are the $U(1)\times U(1)$-equivariant cohomology of a point and $\mathbb{CP}^1$, respectively. This explains the terminology *equivariant* Khovanov homology.

We note that $\mathcal{F}_\alpha$ is obtained from $\mathcal{F}_E$ via extension of scalars. Consider the inclusion $(R_E, A_E) \hookrightarrow (R_\alpha, A_\alpha)$ given by identifying E_1 and E_2 with the elementary symmetric polynomials in α_1, α_2:

$$E_1 \mapsto \alpha_1 + \alpha_2, \quad E_2 \mapsto \alpha_1\alpha_2.$$

Indeed, under the above inclusion, the defining relation $X^2 - E_1X + E_2$ of A_E factors as $(X - \alpha_1)(X - \alpha_2)$, the defining relation of A_α. Therefore $\mathcal{F}_\alpha$ is equal to the composition $\mathcal{F}_E(-) \otimes_{R_E} R_\alpha$. We will use this inclusion throughout.

It follows that the chain complex $CKh_\alpha(D)$ is simply obtained by extending scalars, $CKh_\alpha(D) \cong CKh_E(D) \otimes_{R_E} R_\alpha$. Moreover, as discussed in [40, Section 1.2], R_α is a flat (in fact, free) R_E-module, so there is also an isomorphism $Kh_\alpha(L) \cong Kh_E(L) \otimes_{R_E} R_\alpha$. Nevertheless, enlarging the ground ring yields additional flexibility that is crucial for defining equivariant and foam versions of annular link homology, as will be demonstrated in Chapter 4. In light of this, we will focus on the larger theory $\mathcal{F}_\alpha$.

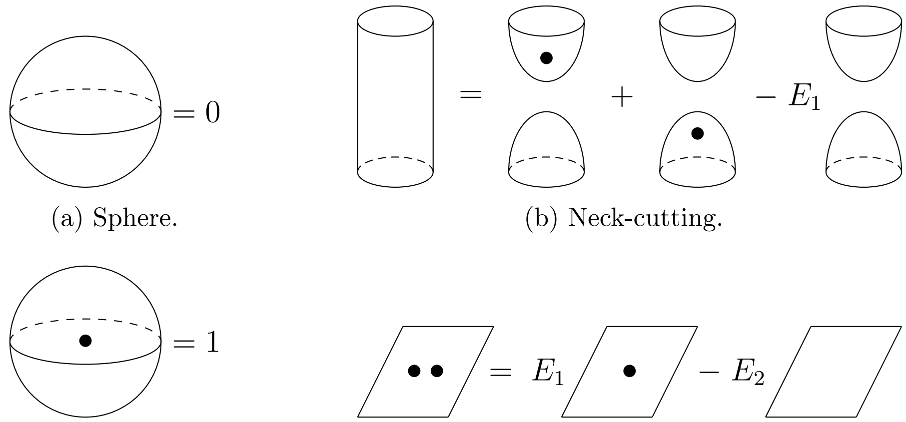
Neither $\mathcal{F}_E$ nor $\mathcal{F}_\alpha$ factors through the relations in 2-7 (as in Section 2.3.3, a dot is interpreted as multiplication by X). Consider the R_α-linear completion $R_\alpha\mathbf{Cob}_\bullet$ of $\mathbf{Cob}_\bullet$. For a dotted cobordism S, setting $\deg(S) = -\chi(S) + 2d(S)$ as in Equation (2.5), we see that the first bullet point of item (3) in Definition 2.3.3 is satisfied. Let $\mathbf{Cob}_{\alpha/l}$ denote the quotient of $R_\alpha\mathbf{Cob}_\bullet$ by the relations shown in Figure 2-10. Note that these relations are homogeneous, so that the grading on morphism spaces descends to $\mathbf{Cob}_{\alpha/l}$.

Definition 2.3.27. Let $\mathcal{BN}_\alpha$ be the additive closure of the graded closure of $\mathbf{Cob}_{\alpha/l}$.

Proposition 2.3.28. *The functor $\mathcal{F}_\alpha : R_\alpha\mathbf{Cob}_\bullet \to R_\alpha-\mathrm{gmod}$ factors through the relations in Figure 2-10 and hence descends to a functor $\mathcal{F}_\alpha : \mathcal{BN}_\alpha \to R_\alpha-\mathrm{gmod}$.*

Proof. The sphere relation follows from $\varepsilon_\alpha\eta_\alpha = 0$, and the dotted sphere relation follows from $\varepsilon_\alpha(X) = 1$. Neck cutting is a topological version of the equality $y = X\varepsilon_\alpha(y) + \varepsilon(Xy) - E_1\varepsilon(y)$, which holds for all $y \in A_\alpha$. Finally, the two dots relation is precisely the identity $0 = X^2 - E_1X + E_2 = (X - \alpha_1)(X - \alpha_2)$ in A_α. □

It is easy to see that the relations in Figure 2-10 imply the relations in Figure 2-4, so one may just as well form the complex $[[D]]$ over $\mathcal{BN}_\alpha$ and apply $\mathcal{F}_\alpha$ to obtain $CKh_\alpha(D)$.



(a) Sphere. (b) Neck-cutting. (c) Dotted sphere. (d) Two dots.

Figure 2-10: Relations in $\mathcal{BN}_\alpha$.

Remark 2.3.29. Note that setting $\alpha_1 = \alpha_2 = 0$ (i.e., applying $\otimes_{R_\alpha}\mathbb{Z}$, where R_α acts on $\mathbb{Z}$ by sending α_1, α_2 to zero, to the morphism spaces in $\mathcal{BN}_\alpha$) reduces the relations in Figure 2-10 to those of 2-7. From the algebraic viewpoint, this is saying that the relation $(X - \alpha_1)(X - \alpha_2) = 0$ in A_α becomes $X^2 = 0$, the defining relation in A_0, upon setting $\alpha_1 = \alpha_2 = 0$. Setting $\alpha_1 = 0$ and renaming $\alpha_2 = t$ yields the (graded) version of Bar-Natan homology, denoted $\mathcal{F}_3$ in [37], while setting $\alpha_1 = 0$, $\alpha_2 = 1$ collapses the grading into a filtration. Likewise, setting $\alpha_1 = \pm 1, \alpha_2 = \mp 1$ reduces the defining relation to $X^2 = 1$, recovering Lee's filtered deformation of Khovanov homology [55]. The filtered Lee and Bar-Natan homologies are crucial for extracting topological information, most famously in the work of Rasmussen [69].

We end this section by recalling from [40] a further extension of the Frobenius system $\mathcal{F}_\alpha$.

Definition 2.3.30. Let $\mathcal{D} := (\alpha_1 - \alpha_2)^2$ denote the discriminant of the quadratic polynomial $(X - \alpha_1)(X - \alpha_2) \in R_\alpha[X]$, let $R_{\alpha\mathcal{D}} := R_\alpha[\mathcal{D}^{-1}]$ denote the ring obtained by inverting $\mathcal{D}$ (equivalently, one may invert $\alpha_1 - \alpha_2$), and let $A_{\alpha\mathcal{D}} := A_\alpha \otimes_{R_\alpha} R_{\alpha\mathcal{D}}$ be the extension of A_α to an $R_{\alpha\mathcal{D}}$-algebra. Consider the functor $\mathcal{F}_{\alpha\mathcal{D}}$ given by the composition

$$\mathcal{BN}_\alpha \xrightarrow{\mathcal{F}_\alpha} R_\alpha\text{– gmod} \to R_{\alpha\mathcal{D}}\text{– gmod}$$

where the second functor is extension of scalars, $(-) \otimes_{R_\alpha} R_{\alpha\mathcal{D}}$. For a link $L \subset \mathbb{R}^3$ with diagram D, write

$$CKh_{\alpha\mathcal{D}}(D) := \mathcal{F}_{\alpha\mathcal{D}}([[D]])$$

to denote the resulting chain complex. The chain complex $CKh_{\alpha\mathcal{D}}(D)$ is an invariant of L up to chain homotopy equivalence, and we will denote its homology by $Kh_{\alpha\mathcal{D}}(L)$.

The elements

$$e_1 = \frac{X - \alpha_1}{\alpha_2 - \alpha_1}, \quad e_2 = \frac{X - \alpha_2}{\alpha_1 - \alpha_2} \in A_{\alpha\mathcal{D}}. \tag{2.7}$$

form a basis for $A_{\alpha\mathcal{D}}$ and satisfy $e_1 + e_2 = 1,\ e_1^2 = e_1,\ e_2^2 = e_2,\ e_1e_2 = 0$. It follows that the algebra $A_{\alpha\mathcal{D}}$ decomposes as a product, $A_{\alpha\mathcal{D}} = R_{\alpha\mathcal{D}}e_1 \times R_{\alpha\mathcal{D}}e_2$. With respect to the basis $\{e_1, e_2\}$, comultiplication in $A_{\alpha\mathcal{D}}$ is simply given by

$$\begin{aligned} \Delta(e_1) &= (\alpha_2 - \alpha_1)e_1 \otimes e_1, \\ \Delta(e_1) &= (\alpha_1 - \alpha_2)e_2 \otimes e_2. \end{aligned} \tag{2.8}$$

As noted in [40, Section 1.2], the TQFT $\mathcal{F}_{\alpha\mathcal{D}}$ is essentially the Lee deformation [55]. By [55, Theorem 4.2], the Lee homology of a k-component link is free (over $\mathbb{Q}$) of rank 2^k. A quick alternate proof can be found in the final remark in [87]; see also [10]. The statement of the following proposition is written in [40, Section 1.2], and the arguments in [87] apply without modification.

Proposition 2.3.31. *For an oriented link $L \subset \mathbb{R}^3$ with k components, the homology $Kh_{\alpha\mathcal{D}}(L)$ is a free $R_{\alpha\mathcal{D}}$-module of rank 2^k.*

2.4 Annular Khovanov homology

We give an overview of annular Khovanov homology, also known as annular Asaeda-Przytycki-Sikora (APS) homology. It was originally defined in [6] as part of a broader categorification of the Kauffman bracket skein module of I-bundles over surfaces. This theory is sometimes called *sutured* annular Khovanov homology. In Section 2.4.2 we will give an overview of the Beliakova-Putyra-Wehrli *quantum* annular homology [11], a sophisticated deformation of annular Khovanov homology.

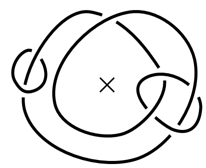
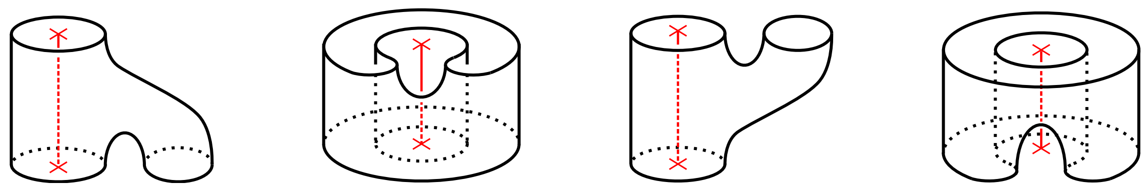


Figure 2-11: An annular link diagram.



(a) Type (I) (b) Type (II) (c) Type (III) (d) Type (IV)

Figure 2-12: Annular saddle cobordisms involving at least one essential circle.

2.4.1 The (classical) annular TQFT

Let $\mathbb{A} := S^1 \times I$ denote the annulus. An *annular link* is a link in the thickened annulus $\mathbb{A} \times I$, and a diagram for an annular link is a generic projection onto $\mathbb{A} \times \{0\}$. Fix an embedding of $\mathbb{A}$ into $\mathbb{R}^2$ in some standard way, for instance $\mathbb{A} = \{x \in \mathbb{R}^2 \mid 1 \leq |x| \leq 2\}$, so that an annular link diagram and all of its smoothings are drawn in the punctured plane

$$\mathcal{P} := \mathbb{R}^2 \setminus \{(0,0)\}.$$

Identifying the interior of $\mathbb{A}$ with $\mathcal{P}$, we represent the annulus in the plane by simply indicating the puncture using the symbol $\times$. Figure 2-11 illustrates an example of an annular link diagram. By a *circle* in $\mathbb{A}$ we mean a smoothly and properly embedded S^1 in $\mathbb{A}$. There are two kinds of circles in $\mathbb{A}$: *trivial* circles, which are contractible in $\mathbb{A}$, and *essential* ones, which are not contractible.

A (dotted) *annular cobordism* is a smoothly and properly embedded surface in $\mathbb{A} \times I$ decorated by dots. Cobordisms are considered up to ambient isotopy fixing the boundary of $\mathbb{A} \times I$ and are allowed to carry dots, as in Section 2.3.2. Annular cobordisms will be depicted in $\mathcal{P} \times I$, with the complement of $\mathcal{P} \times I$ in $\mathbb{R}^2 \times I$ drawn as a red vertical segment; see Figure 2-12 for examples.

Annular cobordisms carry two gradings, the quantum grading $\mathrm{qdeg}(S) = -\chi(S) + 2d(S)$

as in Equation (2.5), and the *annular grading* $\operatorname{adeg}(S) = 0$. Let $\mathbf{Cob}_{\bullet}(\mathbb{A})$ denote the category of dotted annular cobordisms, $\mathbf{Cob}_{\bullet/l}(\mathbb{A})$ the quotient of $\mathbb{Z}\mathbf{Cob}_{\bullet}(\mathbb{A})$ by the local relations in Figure 2-7, and $\mathcal{BN}(\mathbb{A})$ the additive closure of the graded closure of $\mathbf{Cob}_{\bullet/l}(\mathbb{A})$.

To summarize: an object of $\mathcal{BN}(\mathbb{A})$ is of the form $Z_1\{n_1, a_1\} \oplus \cdots \oplus Z_k\{n_k, a_k\}$, where each Z_i is a collection of disjoint simple closed curves in $\mathbb{A}$ and the integers n_i and a_i are formal shifts in qdeg and adeg, respectively. Morphisms are matrices whose entries are formal $\mathbb{Z}$-linear combinations of dotted annular cobordisms, modulo isotopy relative to the boundary, and subject to the local relations shown in Figure 2-4.

We now describe the *annular TQFT*

$$\mathcal{F}_{\mathbb{A}} \colon \mathcal{BN}(\mathbb{A}) \to \mathbb{Z}-\text{ggmod},$$

where $\mathbb{Z}-$ ggmod denotes the category of bigraded ($\mathbb{Z}\times\mathbb{Z}$-graded) abelian groups. Let $Z \subset \mathbb{A}$ be a collection of n trivial and m essential circles. Viewing $\mathbb{A} \times I$ as a subspace of $\mathbb{R}^2 \times I$, apply the TQFT $\mathcal{F}_0$ from Section 2.3.3,

$$\mathcal{F}_0(Z) = A_0^{\otimes n} \otimes A_0^{\otimes m}.$$

Recall that $\mathcal{F}_0(Z)$ carries a quantum grading, Equation (2.6). Define a second grading, called the *annular* grading and denoted adeg, on $\mathcal{F}_0(Z)$ as follows. A tensor factor A_0 corresponding to a trivial circle is concentrated in annular degree 0. For a factor A_0 corresponding to an essential circle, let

$$v_0 = 1, \qquad v_1 = X$$

denote a basis for this copy of A_0, and set

$$\operatorname{adeg}(v_0) = -1 \qquad \operatorname{adeg}(v_1) = 1. \tag{2.9}$$

Bigradings are summarized in Figure 2-13.

The underlying abelian group of $\mathcal{F}_{\mathbb{A}}(Z)$ is defined to be $\mathcal{F}_0(Z)$, with the bigrading given by $(\text{qdeg}, \text{adeg})$. For a cobordism $S \subset \mathbb{A} \times I$, first view S as a surface in $\mathbb{R}^2 \times I$ and consider

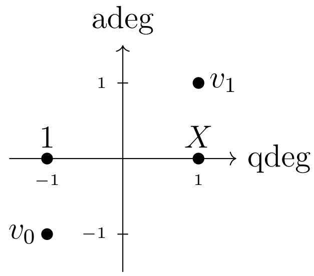


Figure 2-13: Bigradings, where $\{1, X\}$ is a homogeneous basis for trivial circles, and $\{v_0, v_1\}$ is a homogeneous basis for essential ones.

the map $\mathcal{F}_0(S)$. The following lemma says that $\mathcal{F}_0(S)$ is non-decreasing with respect to adeg.

Lemma 2.4.1 ([74, Section 2]). *The map $\mathcal{F}_0(S)$ splits as a sum*

$$\mathcal{F}_0(S) = \mathcal{F}_0(S)_0 + \mathcal{F}_0(S)_+ \tag{2.10}$$

where $\mathcal{F}_0(S)_0$ preserves adeg *and $\mathcal{F}_0(S)_+$ increases* adeg.

Definition 2.4.2. Given an annular cobordism S, define $\mathcal{F}_{\mathbb{A}}(S) := \mathcal{F}_0(S)_0$, the adeg-preserving part of $\mathcal{F}_0(S)$.

Lemma 2.4.3. *The above definition of $\mathcal{F}_{\mathbb{A}}$ assembles into a functor $\mathcal{F}_{\mathbb{A}} : \mathcal{BN}(\mathbb{A}) \to \mathbb{Z}-$* ggmod.

Proof. That $\mathcal{F}_{\mathbb{A}}$ factors through the relations in Figure 2-7 is clear from the definition. To show that $\mathcal{F}_{\mathbb{A}}$ is functorial, consider annular cobordisms $S_0 : Z_0 \to Z_1$ and $S_1 : Z_1 \to Z_2$. Using the notation in Equation (2.10), we have

$$\begin{aligned}
\mathcal{F}_0(S_1 S_0) &= \mathcal{F}_0(S_1)\mathcal{F}(S_0) \\
&= ((\mathcal{F}_0(S_1)_0 + \mathcal{F}_0(S_1)_+) (\mathcal{F}_0(S_0)_0 + \mathcal{F}_0(S_0)_+) \\
&= \mathcal{F}_0(S_1)_0\mathcal{F}_0(S_0)_0 + G_+
\end{aligned}$$

where G_+ strictly increases the annular degree. It follows that the adeg-preserving part of $\mathcal{F}_0(S_1S_0)$, which is precisely $\mathcal{F}_{\mathbb{A}}(S_1S_0)$, is equal to $\mathcal{F}_0(S_1)_0\mathcal{F}_0(S_0)_0 = \mathcal{F}_{\mathbb{A}}(S_1)\mathcal{F}_{\mathbb{A}}(S_0)$. $\square$

The functor $\mathcal{F}_{\mathbb{A}} : \mathcal{BN}(\mathbb{A}) \to \mathbb{Z}-$ ggmod is called the *annular TQFT*. By construction, an annular cobordism S is assigned a map of (qdeg, adeg)-bidegree $(-\chi(S) + 2d(S), 0)$.

To distinguish the bigraded modules assigned to trivial and essential circles, write

$$V_{\mathbb{A}} = \mathcal{F}_{\mathbb{A}}(C)$$

if $C \subset \mathbb{A}$ is an essential circle, with basis $\{v_0, v_1\}$, and keep the notation $A_0 = \mathcal{F}_{\mathbb{A}}(C)$ when C is trivial, with basis $\{1, X\}$. Then if $Z \subset \mathbb{A}$ consists of n trivial and m essential circles, the module assigned to Z is

$$\mathcal{F}_{\mathbb{A}}(Z) = A_0^{\otimes n} \otimes V_{\mathbb{A}}^{\otimes m}.$$

Note that the construction of the Bar-Natan complex from Section 2.3.1 is completely local. For an oriented annular link L with diagram D, its crossings are away from the puncture $\times$, and we may form the chain complex $[[D]]$ over the category $\mathcal{BN}(\mathbb{A})$ exactly as described in Section 2.3.1. The only modification is that qdeg shifts $\{-\}$ are rewritten as $(\mathrm{qdeg}, \mathrm{adeg})$ shifts $\{-, 0\}$ in $\mathcal{BN}(\mathbb{A})$.

Proposition 2.4.4. *Let L be an oriented annular link, and let D, D' be two annular link diagrams for L. Then $[[D]], [[D']] \in \mathrm{Kom}(\mathcal{BN}(\mathbb{A}))$ are chain homotopy equivalent via bidegree-preserving maps.*

Proof. Isotopies of annular links are described by Reidemeister moves away from the puncture. That $[[D]]$ and $[[D']]$ are chain homotopy equivalent follows from the local arguments in the proof of [8, Theorem 1]. An inspection of the explicit chain homotopy equivalences assigned to Reidemeister moves verifies that they preserve annular degree. □

Definition 2.4.5. Let D be a diagram for an oriented annular link L. Define the *annular Khovanov complex*, denoted $CKh_{\mathbb{A}}(D)$, to be the chain complex obtained by applying $\mathcal{F}_{\mathbb{A}}$ term-wise to $[[D]]$,

$$CKh_{\mathbb{A}}(D) := \mathcal{F}_{\mathbb{A}}([[D]])$$

Denote the homology of the above complex by $Kh_{\mathbb{A}}(D)$. We write $Kh_{\mathbb{A}}(L)$ to be $Kh_{\mathbb{A}}(D')$ for any diagram D' for L.

Proposition 2.4.4 implies that $CKh_{\mathbb{A}}(D)$ is an invariant of L up to chain homotopy equivalence. Annular Khovanov homology $Kh_{\mathbb{A}}(L)$ is triply graded: it carries homological, quantum, and annular gradings.

An *elementary annular cobordism* is one that has a single non-degenerate critical point with respect to the height function $\mathbb{A} \times I \to I$. It is a disjoint union of a product cobordism and a single cup, cap, or saddle. By Definition 2.4.2, an elementary cobordism S whose boundary consists of only trivial circles is assigned the same map by $\mathcal{F}_0$ and $\mathcal{F}_{\mathbb{A}}$. We record the maps assigned to the four elementary saddles involving at least one essential circle, Figure 2-12.

$$\begin{aligned} V_{\mathbb{A}} \otimes A_0 &\xrightarrow{\text{(I)}} V_{\mathbb{A}} \\ v_0 \otimes 1 &\mapsto v_0 \\ v_1 \otimes 1 &\mapsto v_0 \\ v_0 \otimes X &\mapsto 0 \\ v_1 \otimes X &\mapsto 0 \end{aligned} \tag{2.11}$$

$$\begin{aligned} V_{\mathbb{A}} \otimes V_{\mathbb{A}} &\xrightarrow{\text{(II)}} A_0 \\ v_0 \otimes v_0 &\mapsto 0 \\ v_1 \otimes v_0 &\mapsto X \\ v_0 \otimes v_1 &\mapsto X \\ v_1 \otimes v_1 &\mapsto 0 \end{aligned} \tag{2.12}$$

$$\begin{aligned} V_{\mathbb{A}} &\xrightarrow{\text{(III)}} V_{\mathbb{A}} \otimes A_0 \\ v_0 &\mapsto v_0 \otimes X \\ v_1 &\mapsto v_1 \otimes X \end{aligned} \tag{2.13}$$

$$\begin{aligned} A_0 &\xrightarrow{\text{(IV)}} V_{\mathbb{A}} \otimes V_{\mathbb{A}} \\ 1 &\mapsto v_0 \otimes v_1 + v_1 \otimes v_0 \\ X &\mapsto 0 \end{aligned} \tag{2.14}$$

Remark 2.4.6. There is an evident symmetry of the above maps given by interchanging v_0 and v_1. Grigsby-Licata-Wehrli [25] showed that the annular Khovanov complex carries an action of the Lie algebra $\mathfrak{sl}_2(\mathbb{Z})$. The module $V_{\mathbb{A}}$ assigned to an essential circle is the fundamental representation, with weights given by adeg, and the module A_0 assigned to a trivial circle is the trivial two-dimensional representation.

From (2.11), we see that X acts trivially on any essential circle. It follows that a cobordism with a component that carries a dot and a closed curve which is nonzero in $\pi_1(\mathbb{A} \times I)$ is assigned the zero map by $\mathcal{F}_{\mathbb{A}}$. Thus $\mathcal{F}_{\mathbb{A}}$ factors through the relation shown in Figure 2-14, called Boerner's relation [14]. Indeed, for an essential circle $C \subset \mathbb{A}$, there are no nonzero endomorphisms of $\mathcal{F}_{\mathbb{A}}(C)$ of bidegree $(2, 0)$.

The category $\mathcal{BN}(\mathbb{A})$ has a monoidal product given by taking two copies $\mathbb{A}_1, \mathbb{A}_2$ of $\mathbb{A}$ and gluing the boundary component $S^1 \times \{1\}$ of $\mathbb{A}_1$ to the boundary component $S^1 \times \{0\}$ of $\mathbb{A}_2$. The annular TQFT $\mathcal{F}_{\mathbb{A}}$ is evidently monoidal.

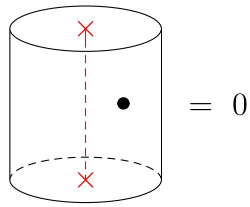


Figure 2-14: Boerner's relation.

2.4.2 Quantum annular homology

This section outlines the construction of the Beliakova-Putyra-Wehrli quantum annular link homology [11]. Their theory is built over a commutative ring $\Bbbk$ and a unit $\mathfrak{q} \in \Bbbk$. We set $\Bbbk := \mathbb{Z}[\mathfrak{q}, \mathfrak{q}^{-1}]$, the ring of Laurent polynomials in $\mathfrak{q}$ with integer coefficients, and the distinguished unit $\mathfrak{q} \in \Bbbk$ is the same $\mathfrak{q}$ appearing in $\mathbb{Z}[\mathfrak{q}, \mathfrak{q}^{-1}]$. The main object is the *quantum annular TQFT*

$$\mathcal{F}_{\mathbb{A}_\mathfrak{q}} : \mathcal{BN}_\mathfrak{q}(\mathbb{A}) \to \Bbbk\text{-}\,\mathrm{gmod},$$

where $\mathcal{BN}_\mathfrak{q}(\mathbb{A})$ is a certain deformation of $\mathcal{BN}(\mathbb{A})$. We will give an overview of the functor $\mathcal{F}_{\mathbb{A}_\mathfrak{q}}$ and state a main theorem [11, Theorem 6.3].

Remark 2.4.7. As mentioned above, we work over the Laurent polynomial ring $\Bbbk$ throughout this section. To construct the stable homotopy refinement in Chapter 3, we will tensor the resulting theory with $\Bbbk_r := \Bbbk/(\mathfrak{q}^r - 1)$.

Definition 2.4.8. Let $n, m \geq 0$. A *planar* (n, m)*-tangle* is a smooth and proper embedding of $n + m$ intervals and a finite number of circles into I^2, such that n boundary points map to points in $I \times \{0\}$ and m boundary points map to points in $I \times \{1\}$. See Figure 2-15 for an example.

Let T_0 and T_1 be planar (n, m)-tangles. A *cobordism* from T_0 to T_1 is a smoothly and properly embedded compact surface S in I^3, such that $S \cap I^2 \times \{i\} = T_i$, $S \cap (\{i\} \times I^2) = \varnothing$ for $i = 1, 2$, and both $S \cap (I \times \{0\} \times I)$ and $S \cap (I \times \{1\} \times I)$ are vertical intervals above the $n + m$ points comprising the boundary of T_0 and T_1. Tangle cobordisms are considered up to ambient isotopy of I^3 fixing ∂I^3 pointwise.

Let $\mathcal{BN}(n, m)$ denote the Bar-Natan category of the rectangle with n points on the bottom and m on top. Its objects are formal direct sums of formally graded planar (n, m)

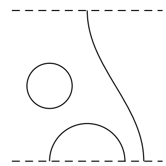


Figure 2-15: A planar (3,1)-tangle

tangles. Morphisms in $\mathcal{BN}(n,m)$ are matrices whose entries are formal $\Bbbk$-linear combinations of embedded dotted cobordisms in I^3 between planar (n,m)-tangles, subject to the relations in Figure 2-7.

A *seam* of $\mathbb{A} = S^1 \times I$, denoted μ, is an interval $\{*\} \times I$. In our representation of the interior of the annulus as the puncture plane $\mathcal{P}$, we will fix the seam as the positive x-axis, emanating from the puncture $\times$. See Figure 2-17 for an example.

We now recall the quantum Bar-Natan category of the annulus $\mathcal{BN}_{\mathfrak{q}}(\mathbb{A})$, a deformation of $\mathcal{BN}(\mathbb{A})$. The objects of $\mathcal{BN}_{\mathfrak{q}}(\mathbb{A})$ are nearly the same as those of $\mathcal{BN}(\mathbb{A})$, with the slight modification that curves in $\mathbb{A}$ must be transverse to μ. Morphisms in $\mathcal{BN}_{\mathfrak{q}}(\mathbb{A})$ are also similar to those in $\mathcal{BN}(\mathbb{A})$. In $\mathcal{BN}_{\mathfrak{q}}(\mathbb{A})$, isotopic cobordisms are equal if the isotopy fixes the *membrane* $\mu \times I \subset \mathbb{A} \times I$. Otherwise, the cobordisms are scaled by a power of $\mathfrak{q}$ according to the degree of the part of the cobordism that passes through the membrane during the isotopy, accounting also for the coorientation of the membrane induced by the standard orientation of the core circle of $\mathbb{A}$. These will be referred to as *trace moves.* The relations are depicted in Figure $2-16$; for details see [11, Section 6.2]. Moreover, the dotted Bar-Natan relations in Figure 2-7 are imposed, where the local pictures are understood to be disjoint from the membrane.

Remark 2.4.9. In [11, Section 6.2], $\mathcal{BN}_{\mathfrak{q}}(\mathbb{A})$ is obtained from a 2-categorical construction known as a (twisted) horizontal trace, see [11, Definition 3.3]. We will not need the details of this construction, so we do not review it.

General position implies that if two annular cobordisms are isotopic, then they are related by a sequence of trace moves and isotopies which fix the membrane. Therefore, if two cobordisms $S, S' \subset \mathbb{A} \times I$ are isotopic, then $S = \mathfrak{q}^k S'$ as morphisms in $\mathcal{BN}_{\mathfrak{q}}(\mathbb{A})$, for some $k \in \mathbb{Z}$; see also [11, Proposition 6.2].

A *configuration* $\mathscr{C}$ is a collection of disjoint simple closed curves in $\mathbb{A}$ which are transverse

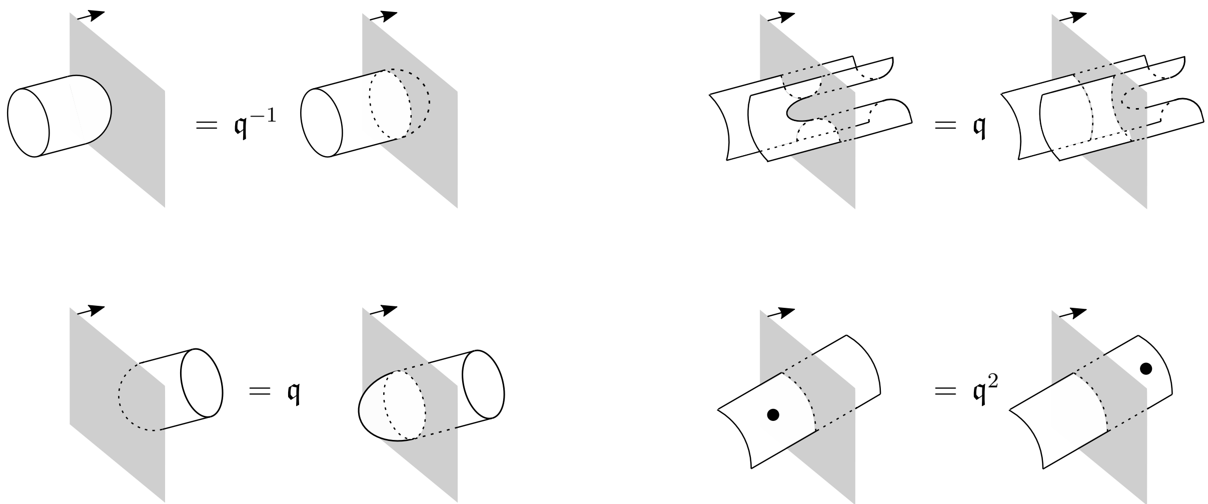


Figure 2-16: The trace relations in $\mathcal{BN}_{\mathfrak{q}}(\mathbb{A})$.

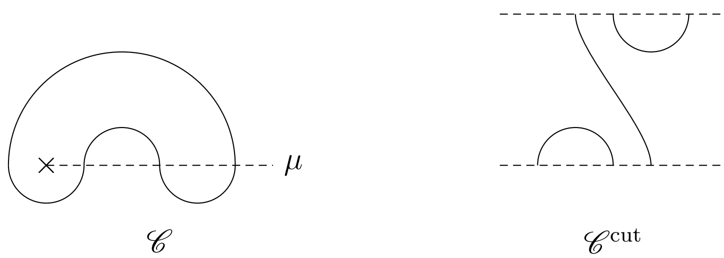


Figure 2-17: Cutting open a configuration $\mathscr{C}$ along μ to obtain the $(3,3)$-tangle $\mathscr{C}^{\mathrm{cut}}$

to μ. Note that an object of $\mathcal{BN}_{\mathfrak{q}}(\mathbb{A})$ is a formal direct sum of formally graded configurations. Given a configuration $\mathscr{C}$ which intersects μ in n points, we can cut along μ to obtain a planar (n,n)-tangle $\mathscr{C}^{\mathrm{cut}}$. Figure 2-17 shows an example.

A construction of Chen-Khovanov [19] (see also work of Stroppel [81] and Brundan-Stroppel [16]) yields graded $\Bbbk$-algebras A^k for each $k \geq 0$, and a functor

$$\mathcal{F}_{CK} : \mathcal{BN}(n,m) \to \mathrm{gBimod}(A^n, A^m)$$

where $\mathrm{gBimod}(A^n, A^m)$ is the category of graded (A^n, A^m)-bimodules. Let $\mathbb{I}^n$ denote the planar tangle consisting of n vertical strands. Then, by definition of $\mathcal{F}_{CK}$, we have $\mathcal{F}_{CK}(\mathbb{I}^n) = A^n$. A thorough account of Chen-Khovanov algebras and bimodules will not be needed, so we do not review the construction.

Remark 2.4.10. Strictly speaking, Chen-Khovanov define $\mathbb{Z}$-algebras. The $\Bbbk$-algebras above are obtained by simply extending scalars to $\Bbbk$.

The *quantum Hochschild homology*, denoted qHH and defined in [11, Section 3.8.5], is a deformation of the usual Hochschild homology of bimodules. It takes as input a graded $\Bbbk$-algebra B and a graded (B, B)-bimodule M. The output $qHH(B, M) = \bigoplus_{i \geq 0} qHH_i(B, M)$ is a $\Bbbk$-module. Due to [11, Proposition 6.6] (stating that $qHH_i(\mathcal{F}_{CK}(\mathscr{C}^{\text{cut}})) = 0$ for $i > 0$), we restrict our focus to qHH_0. It follows immediately from the definition of qHH that

$$qHH_0(B, M) = M/\text{span}_{\Bbbk}\{bm - \mathfrak{q}^{|b|}mb \mid b \in B, m \in M\}, \tag{2.15}$$

where $|b|$ denotes the degree of b.

We are now ready to define $\mathcal{F}_{\mathbb{A}_\mathfrak{q}}$ on objects. Let $\mathscr{C}$ be a configuration which intersects μ in n points. Using the Chen-Khovanov functor, form the (A^n, A^n)-bimodule $\mathcal{F}_{CK}(\mathscr{C}^{\text{cut}})$. The quantum annular TQFT $\mathcal{F}_{\mathbb{A}_\mathfrak{q}}$ is then defined on objects by

$$\mathcal{F}_{\mathbb{A}_\mathfrak{q}}(\mathscr{C}) := qHH(A^n, \mathcal{F}_{CK}(\mathscr{C}^{\text{cut}})).$$

By [11, Proposition 6.6], we have $qHH_i(\mathcal{F}_{CK}(\mathscr{C}^{\text{cut}})) = 0$ for $i > 0$. Suppose $\mathscr{C}$ consists of n essential curves each intersecting the seam once. Then $\mathscr{C}^{\text{cut}} = \mathbb{I}^n$, so

$$\mathcal{F}_{\mathbb{A}_\mathfrak{q}}(\mathscr{C}) = qHH_0(A^n, A^n).$$

Let $A_0^n \subset A^n$ denote the subalgebra consisting of elements of degree 0. By [11, Proposition 6.6], the inclusion $A_0^n \hookrightarrow A^n$ induces an isomorphism $qHH_0(A_0^n, A_0^n) \cong qHH_0(A^n, A^n)$. Moreover, A_0^n is freely generated over $\Bbbk$ by 2^n elements $x_1, \ldots, x_{2^n}$, which are the *primitive idempotents* of [11, Section 5.5]. They are in bijection with the *cup diagrams* and satisfy $x_i x_j = \delta_{ij} x_i$. It follows from (2.15) that

$$qHH_0(A_0^n, A_0^n) \cong \Bbbk^{2^n}.$$

Every configuration $\mathscr{C}$ is isomorphic in $\mathcal{BN}_\mathfrak{q}(\mathbb{A})$ to a configuration $\mathscr{C}^\circ$ in which every component intersects the seam at most once. If $\mathscr{C}$ has e essential and t trivial circles, then by delooping, one obtains

$$\mathcal{F}_{\mathbb{A}_{\mathfrak{q}}}(\mathscr{C}) \cong \mathcal{F}_{\mathbb{A}_{\mathfrak{q}}}(\mathscr{C}^{\circ}) \cong \Bbbk^{2^{e+t}}.$$

The above isomorphism is, however, not canonical, and a general configuration $\mathscr{C}$ does not have a canonical choice of basis. This is an obstacle for defining the stable homotopy refinement of quantum annular homology. An extensive discussion of choosing a basis is the content of Section 3.2.

We have so far only explained the definition of $\mathcal{F}_{\mathbb{A}_{\mathfrak{q}}}$ on objects. The full construction of $\mathcal{F}_{\mathbb{A}_{\mathfrak{q}}}$ in [11] follows from a general theory of (twisted) horizontal traces of bicategories, which we will not describe. The definition of $\mathcal{F}_{\mathbb{A}_{\mathfrak{q}}}$ on morphisms follows from this general theory as well. The rest of this subsection describes the set-up for [11, Theorem 6.3], which is stated as our Theorem 2.4.13, and which is the main computational tool.

Let $\mathcal{BBN}_{\mathfrak{q}}(\mathbb{A})$ denote the quotient of $\mathcal{BN}_{\mathfrak{q}}(\mathbb{A})$ by Boerner's relation, Figure 2-14. The functor $\mathcal{F}_{\mathbb{A}_{\mathfrak{q}}}$ factors through $\mathcal{BBN}_{\mathfrak{q}}(\mathbb{A})$. Given a diagram D for an annular link L such that D is transverse to μ and the crossings are disjoint from μ, we form the cube of resolutions $[[D]]$ in the usual manner and view the result as a chain complex over the category $\mathcal{BBN}_{\mathfrak{q}}(\mathbb{A})$.

Definition 2.4.11. If D is a diagram for an annular link L which is transverse to the seam, define the *quantum annular Khovanov complex of D* to be

$$CKh_{\mathbb{A}_{\mathfrak{q}}}(D) := \mathcal{F}_{\mathbb{A}_{\mathfrak{q}}}([[D]]).$$

The chain complex $CKh_{\mathbb{A}_{\mathfrak{q}}}(D)$ is an invariant of L up to chain homotopy equivalence by [11, Proposition 6.8].

Definition 2.4.12 ([11, Appendix A.1])**.** Let **TL** denote the additive closure of the formally graded Temperley-Lieb category. Its objects are formal direct sums of formally graded finite collections of points on a line, and morphisms are $\Bbbk$-linear combinations of planar tangles between the points, modulo planar isotopy and the local relation that a circle is set to $\mathfrak{q}+\mathfrak{q}^{-1}$. Composition is given by stacking planar tangles, see Figure 2-18 for an example.

There is a functor $S^1 \times (-) : \mathbf{TL} \to \mathcal{BN}_{\mathfrak{q}}(\mathbb{A})$, which sends a collection of n points to n essential circles in $\mathbb{A}$, each intersecting μ once, and sends a planar tangle T to the cobordism

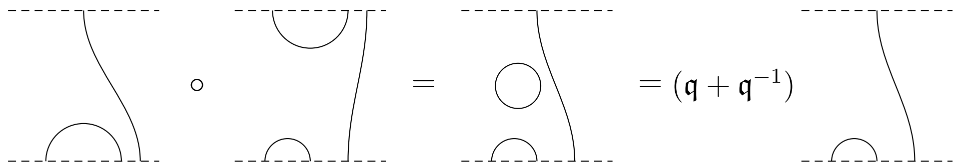


Figure 2-18: Composition and relations in **TL**.

$S^1 \times T$. A circle evaluates to $\mathfrak{q}+\mathfrak{q}^{-1}$ in **TL**. On the other hand, the relations in $\mathcal{BN}_{\mathfrak{q}}(\mathbb{A})$ imply that a torus wrapping once around the annulus evaluates to $\mathfrak{q}+\mathfrak{q}^{-1}$, ensuring that $S^1 \times (-)$ is well-defined.

Let $\mathrm{gRep}(U_{\mathfrak{q}}(\mathfrak{sl}_2))$ denote the category of graded representations of $U_{\mathfrak{q}}(\mathfrak{sl}_2)$. We follow the conventions established in [11, Appendix A.1] concerning $U_{\mathfrak{q}}(\mathfrak{sl}_2)$; also see Section 3.9 below for further discussion. There is another functor $\mathcal{F}_{\mathbf{TL}} : \mathbf{TL} \to \mathrm{gRep}(U_{\mathfrak{q}}(\mathfrak{sl}_2))$, defined as follows. Let $V_1 = \langle v_{-1}, v_1\rangle$ be the fundamental representation of $U_{\mathfrak{q}}(\mathfrak{sl}_2)$ and $V_1^* = \langle v_{-1}^*, v_1^*\rangle$ its dual. Let $V_{\mathfrak{q}}$ be a free $\Bbbk$-module with basis $\{v_+, v_-\}$. Consider two $\Bbbk$-linear isomorphisms $\alpha : V_1 \to V_{\mathfrak{q}}$ and $\beta : V_1^* \to V_{\mathfrak{q}}$ defined by

$$\alpha : v_1 \mapsto v_+ \qquad\qquad \beta : v_1^* \mapsto v_-$$

$$\alpha : v_{-1} \mapsto v_- \qquad\qquad \beta : v_{-1}^* \mapsto \mathfrak{q}^{-1} v_+$$

These equip $V_{\mathfrak{q}}$ with two actions of $U_{\mathfrak{q}}(\mathfrak{sl}_2)$, which are detailed in [11, Appendix A.1]. Note that, while V_1 and V_1^* are isomorphic as $U_{\mathfrak{q}}(\mathfrak{sl}_2)$-modules, the composition $\beta^{-1}\circ\alpha : V_1 \to V_1^*$ is not $U_{\mathfrak{q}}(\mathfrak{sl}_2)$-linear.

The functor $\mathcal{F}_{\mathbf{TL}}$ assigns $V_{\mathfrak{q}}^{\otimes n}$ to a collection of n points. Since $\beta^{-1}\alpha$ is not $U_{\mathfrak{q}}(\mathfrak{sl}_2)$-linear, there is an ambiguity in specifying the $U_{\mathfrak{q}}(\mathfrak{sl}_2)$-module structure on $V_{\mathfrak{q}}^{\otimes n}$. The convention is that the m-th point is assigned V_1 if m is odd, and V_1^* if m is even, so that the module assigned to n points is given the $U_{\mathfrak{q}}(\mathfrak{sl}_2)$-action according to the identification

$$V_{\mathfrak{q}}^{\otimes n} \cong V_1 \otimes V_1^* \otimes V_1 \otimes \cdots$$

To define the value of $\mathcal{F}_{\mathbf{TL}}$ on any planar tangle, it suffices to specify its value on caps and cups. For a cap $\cap$, $\mathcal{F}_{\mathbf{TL}}$ assigns the evaluation map $ev : V_{\mathfrak{q}} \otimes V_{\mathfrak{q}} \to \Bbbk$, defined by

$$v_+ \otimes v_+ \mapsto 0 \qquad\qquad v_+ \otimes v_- \mapsto \mathfrak{q}$$
$$v_- \otimes v_- \mapsto 0 \qquad\qquad v_- \otimes v_+ \mapsto 1$$

On a cup $\cup$, $\mathcal{F}_{\mathbf{TL}}$ assigns the coevaluation $coev : \Bbbk \to V_{\mathfrak{q}} \otimes V_{\mathfrak{q}}$, defined by

$$1 \mapsto v_+ \otimes v_- + \mathfrak{q}^{-1} v_- \otimes v_+$$

The above assignments extend to define $\mathcal{F}_{\mathbf{TL}}(T)$ for any planar tangle T, by writing T as a composition of tangles consisting of either a cup or cap together with the identity tangle on the remaining points. It is well-known that the map $\mathcal{F}_{\mathbf{TL}}(T)$ is independent of the decomposition of T into such pieces. Finally, the relation $coev \circ ev = \mathfrak{q} + \mathfrak{q}^{-1}$ implies that the above assignments assemble into a functor $\mathcal{F}_{\mathbf{TL}} : \mathbf{TL} \to \mathrm{gRep}(U_{\mathfrak{q}}(\mathfrak{sl}_2))$.

The evaluation map is always identified with either $V_1 \otimes V_1^* \to \Bbbk$ or $V_1^* \otimes V_1 \to \Bbbk$, and the coevaluation is identified with either $\Bbbk \to V_1 \otimes V_1^*$ or $\Bbbk \to V_1^* \otimes V_1$. With these identifications, the cap and cup are assigned $U_{\mathfrak{q}}(\mathfrak{sl}_2)$-linear maps by $\mathcal{F}_{\mathbf{TL}}$.

We have now explained the functors $\mathcal{F}_{\mathbf{TL}} : \mathbf{TL} \to \mathrm{gRep}(U_{\mathfrak{q}}(\mathfrak{sl}_2))$ and $S^1 \times (-) : \mathbf{TL} \to \mathcal{BN}_{\mathfrak{q}}(\mathbb{A})$. To compare $\mathcal{F}_{\mathbf{TL}}$ with the composition $\mathcal{F}_{\mathbb{A}_{\mathfrak{q}}} \circ (S^1 \times (-))$ in the statement of Theorem 2.4.13, the value of $\mathcal{F}_{\mathbb{A}_{\mathfrak{q}}}$ on n essential circles intersecting the seam once needs to be given a $U_{\mathfrak{q}}(\mathfrak{sl}_2)$-module structure. Recall that the Chen-Khovanov functor assigns the $\Bbbk$-algebra A^n to the planar tangle $\mathbb{I}^n$ consisting of n vertical strands. There is distinguished $\Bbbk$-linear isomorphism

$$qHH_0(A^n, A^n) \cong V_{\mathfrak{q}}^{\otimes n} \tag{2.16}$$

which we now describe. Recall that the inclusion $A_0^n \hookrightarrow A_n$ induces an isomorphism on qHH_0, and that $qHH_0(A_0^n, A_0^n)$ has a distinguished $\Bbbk$-basis $\{x_1, \ldots, x_{2^n}\}$ corresponding to cup diagrams. Chen-Khovanov in [19, Section 6] assign to each x_i an element $p_i \in V_{\mathfrak{q}}^{\otimes n}$ such that the collection $\{p_i\}$ forms a basis of $V_{\mathfrak{q}}^{\otimes n}$. The isomorphism (2.16) is obtained by composing $qHH_0(A^n, A^n) \cong qHH_0(A_0^n, A_0^n)$ with the assignment $x_i \mapsto p_i$.

Theorem 2.4.13. ([11, Theorem 6.3]) *There is a commuting diagram*

$$\begin{tikzcd} \mathbf{TL} \arrow[rr, "S^1\times(-)"] \arrow[dr, "\mathcal{F}_{\mathbf{TL}}"'] & & \mathcal{BBN}_{\mathfrak{q}}(\mathbb{A}) \arrow[dl, "\mathcal{F}_{\mathbb{A}_{\mathfrak{q}}}"] \\ & \mathrm{gRep}(U_{\mathfrak{q}}(\mathfrak{sl}_2)) & \end{tikzcd}$$

with the horizontal functor an equivalence of categories.

The above theorem is a key tool for carrying out calculations in quantum annular homology, allowing one to bypass unraveling the categorical machinery of twisted horizontal traces used to define $\mathcal{F}_{\mathbb{A}_{\mathfrak{q}}}$. In particular, it will play an important role in determining the values of the differential on generators in Section 3.2.1.

2.5 Categorification of the Reshetikhin-Turaev invariants

Reshetikhin-Turaev [70] introduced a wealth of quantum invariants of oriented links. [5] As a rough summary, consider an oriented link diagram D where each component is labeled by some representation of the quantum group $U_q(\mathfrak{g})$, for some fixed Lie algebra $\mathfrak{g}$. The representations labeling components are often called *colors*. Given such input, Reshetikhin-Turaev define a Laurent polynomial invariant of L.

This thesis considers the case $\mathfrak{g} = \mathfrak{sl}_N$ or $\mathfrak{g} = \mathfrak{gl}_N$, and where components are colored by (quantum) exterior powers of the fundamental representation V. In this case, the colors consist of labels on each component by an integer $i \in \{0, \ldots, N\}$, where i corresponds to the i-th exterior power $\Lambda_q^i V$. In this situation, Murakami-Ohtsuki-Yamada [65] showed that the Reshetikhin-Turaev invariant can be combinatorially computed from a link diagram by resolving each crossing as a particular linear combination of webs, and then evaluating each resulting closed web using the MOY relations (Figure 2-24). The Jones polynomial corresponds to $\mathfrak{g} = \mathfrak{sl}_2$ and where all components are colored are labeled 1 (i.e., all components are colored by the fundamental representation).

A categorification of the invariant for $\mathfrak{sl}_3$ and all components colored 1 was introduced

[5] In fact, the Reshetkhin-Turaev define an invariant of *tangles*, but we consider only the restriction to links

Figure 2-19: The orientations of edges at each trivalent vertex of an $\mathfrak{sl}_3$ web must be either all outgoing or all incoming.

by Khovanov in [35]; we give an overview of this construction in Section 2.5.1. Khovanov-Rozansky [42] categorified the invariant for all $\mathfrak{sl}_N$, again in the case where strands are colored by 1. An extension to all colors was given by Wu [90]. An equivariant version of $\mathfrak{sl}_3$ homology was constructed in [63], and a generalization to $\mathfrak{sl}_N$ homology was introduced in [45] in the uncolored case and in [89] for colored links.

The approach to categorified $\mathfrak{sl}_N$ invariants considered in this thesis is via Robert-Wagner closed foam evaluation [73], which we review in Section 2.5.3. Key benefits of this construction include its combinatorial nature, that it produces naturally equivariant link homology, and the *strict* functoriality of the resulting link homology, proven by Ehrig-Tubbenhauer-Wedrich [24].

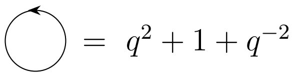
2.5.1 Categorification of the $\mathfrak{sl}_3$ polynomial

This section gives an overview of the $\mathfrak{sl}_3$ link polynomial via the Kuperberg bracket [48] and Khovanov's categorification of this invariant [35]. We will follow the normalization and conventions in [35], except the one in the following remark.

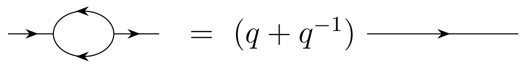
Remark 2.5.1. Note that [35] uses the opposite conventions for positive and negative crossings, see [35, Figure 8].

Definition 2.5.2. An $\mathfrak{sl}_3$ *web* is a planar trivalent graph $\Gamma \subset \mathbb{R}^2$ embedded in the plane, which may have closed loops with no vertices. Moreover, edges and loops of Γ carry orientations such that each vertex is either a source or a sink, as shown in Figure 2-19. In this section we will simply write *web* rather than $\mathfrak{sl}_3$ web.

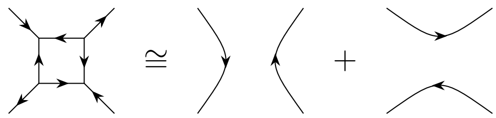
Given an $\mathfrak{sl}_3$ web Γ, the *Kuperberg bracket* of Γ, denoted $\langle\Gamma\rangle_{\mathrm{Kup}}$ is a Laurent polynomial invariant of Γ, computed as follows. A straightforward Euler characteristic argument shows that any web Γ contains either an innermost circle, a bigon, or a square. Repeatedly applying



(a) An innermost circle.



(b) A bigon face.



(c) A square face.

Figure 2-20: The local relations used to recursively compute the Kuperberg bracket $\langle \Gamma \rangle_{\text{Kup}}$.

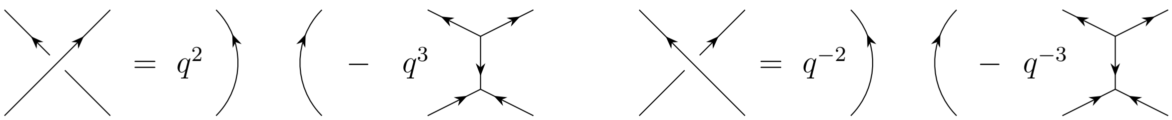
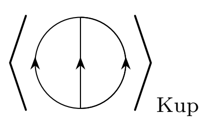


Figure 2-21: Crossing resolution used to compute the $\mathfrak{sl}_3$ polynomial.

the relations in Figure 2-20 simplifies any web into a $\mathbb{Z}_{\geq 0}[q, q^{-1}]$-linear combinations of the empty web, which evaluates to 1. That $\langle \Gamma \rangle_{\text{Kup}}$ is well-defined is proven in [48].

Example 2.5.3. By first applying the bigon relation and then the circle relation, we have



$$= (q + q^{-1})(q^2 + 1 + q^{-2}).$$

We now define the $\mathfrak{sl}_3$ link polynomial. Let D be a diagram of an oriented link L. Resolve each crossing according to the rule in Figure 2-21, resulting in a linear combination of webs, and evaluate each web using the Kuperberg bracket to obtain a Laurent polynomial $\langle D \rangle_{\text{Kup}} \in \mathbb{Z}[q, q^{-1}]$.

Proposition 2.5.4 ([35, Proposition 2])**.** *The polynomial $\langle D \rangle_{\text{Kup}}$ is invariant under Reidemeister moves, and hence is an invariant of the oriented link L.*

We now give a brief overview Khovanov's chain complex $CKh_3(D)$ categorifying the $\mathfrak{sl}_3$ polynomial, following [35, Section 4]. To each web Γ, Khovanov assigns a free, graded abelian group $\mathcal{F}_3(\Gamma)$, and shows that the graded rank of $\mathcal{F}_3(\Gamma)$ is equal to the Kuperberg bracket of Γ [35, Corollary 2]. This is proven by categorifying the relations in Figure 2-20. We do not discuss the definition of $\mathcal{F}_3$; an equivariant version of this construction using foam evaluation

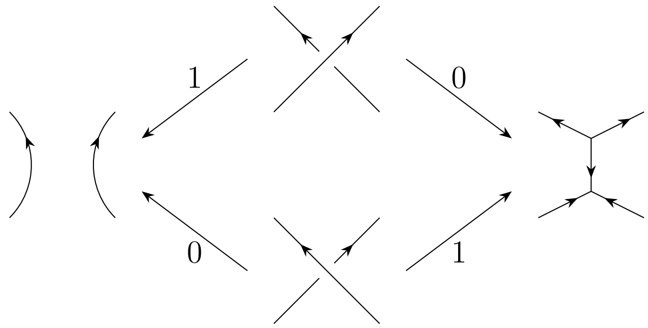


Figure 2-22: The 0- and 1-smoothings used to define the $\mathfrak{sl}_3$ chain complex.

in the style of Robert-Wagner is given in Section 4.3.

Let D be an n-crossing diagram of an oriented link L. Similar to the discussion in Section 2.3.1 , first form the cube of resolutions as follows. Using the rules in Figure 2-22, each vertex $u \in \{0,1\}^n$ is assigned a web D_u. If vertices u and v differ in exactly one entry, where $u_i = 0, v_i = 1$, then there is an associated map $\mathcal{F}_3(D_u) \to \mathcal{F}_3(D_v)$, and decorating edges by these maps results in a commutative cube. Add minus signs to edges to make the cube anti-commutative. The chain complex $CKh_3(D)$ is given by

$$CKh_3^i(D) = \bigoplus_{|u|=i+n_+} \mathcal{F}_3(D_u)\{2(n_+ - n_-) - i\}, \tag{2.17}$$

and the differential is given on each summand by the sum of all (signed) edge maps which start at the corresponding vertex.

Theorem 2.5.5 ([35, Theorem 1, Proposition 1])**.** *The chain homotopy class of $CKh_3(D)$ is invariant under Reidemeister moves, and the graded Euler characteristic of $CKh_3(D)$ equals the $\mathfrak{sl}_3$ polynomial $\langle D\rangle_{\mathrm{Kup}}$.*

2.5.2 MOY webs

In this section we fix an integer $N \geq 1$.

Definition 2.5.6. A $\mathfrak{gl}_N$ *web* (also called a *MOY web*) is a embedded trivalent graph $\Gamma \subset \mathbb{R}^2$, which may also contain closed loops with no vertices. Moreover, edges and loops of Γ are oriented and carry weights in $\{0, \dots, N\}$, called the *thickness* of the edge, such that the

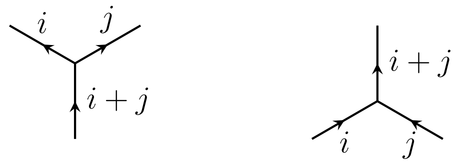


Figure 2-23: The flow condition near each trivalent vertex in a $\mathfrak{gl}_N$ web.

flow condition shown in Figure 2-23 is satisfied at each vertex. Let Web_N denote the set of planar isotopy classes of $\mathfrak{gl}_N$ webs. We will often simply write *web* instead of $\mathfrak{gl}_N$ web in this section.

Definition 2.5.7. For $n \in \mathbb{Z}$, The *quantum integer* $[n]$ is defined to be

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}} = q^{n-1} + q^{n-3} + \cdots + q^{3-n} + q^{1-n} \in \mathbb{Z}_{\geq 0}[q, q^{-1}].$$

If $k > 0$, set

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n][n-1]\cdots[n-k+1]}{[k][k-1]\cdots[2][1]}.$$

Also define $\begin{bmatrix} n \\ 0 \end{bmatrix} = 1$.

Remark 2.5.8. It is straighforward to verify the identity

$$\begin{bmatrix} n+1 \\ k \end{bmatrix} = q^{-k}\begin{bmatrix} n \\ k \end{bmatrix} + q^{n-k+1}\begin{bmatrix} n \\ n-1 \end{bmatrix},$$

which holds for all n and k with $k > 0$. It follows that $\begin{bmatrix} n \\ k \end{bmatrix} \in \mathbb{Z}_{\geq 0}[q, q^{-1}]$.

Theorem 2.5.9 ([65], see also [90, Theorem 2.4])**.** *There is a unique function* $\langle - \rangle_N$: $\mathrm{Web}_N \to \mathbb{Z}[q, q^{-1}]$ *which factors through the local relations in Figure 2-24.*

Indeed, the evaluation $\langle \Gamma \rangle_N$ for any web Γ can be recursively computed using the relations in Figure 2-24; this is part of the proof of uniqueness of the evaluation in [90, Theorem 2.4]. That applying the MOY relations to simplify a web results in a well-defined polynomial follows from the state-sum formulation of $\langle - \rangle_N$ defined in [65].

The evaluation $\langle - \rangle_N$ can be used to define the Reshetikhin-Turaev $\mathfrak{sl}_N$ invariant of a colored link L. This invariant is obtained by resolving crossings in a particular way, resulting

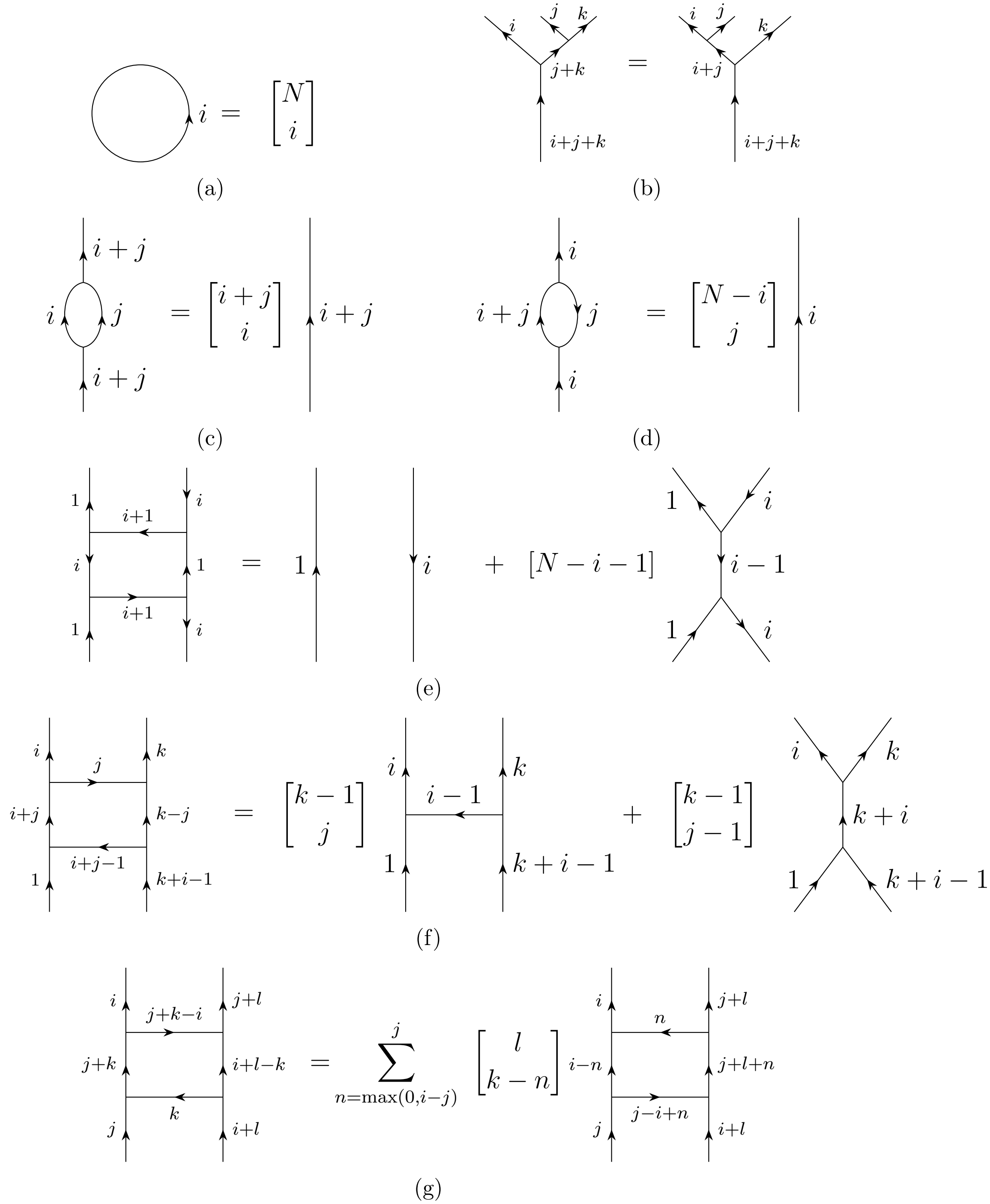


Figure 2-24: The MOY relations.

in a sophisticated $\mathbb{Z}[q, q^{-1}]$-linear combination of webs, and then evaluating each web to obtain a Laurent polynomial. See [65, Section 5] and [90, Section 2.5] for details.

2.5.3 Robert-Wagner foam evaluation

This section gives an overview of Robert-Wagner foam evaluation, which is used to categorify the MOY calculus, as we will detail in Section 2.5.4. Later in Section 4.4 we extend Robert-Wagner foam evaluation to the annular setting.

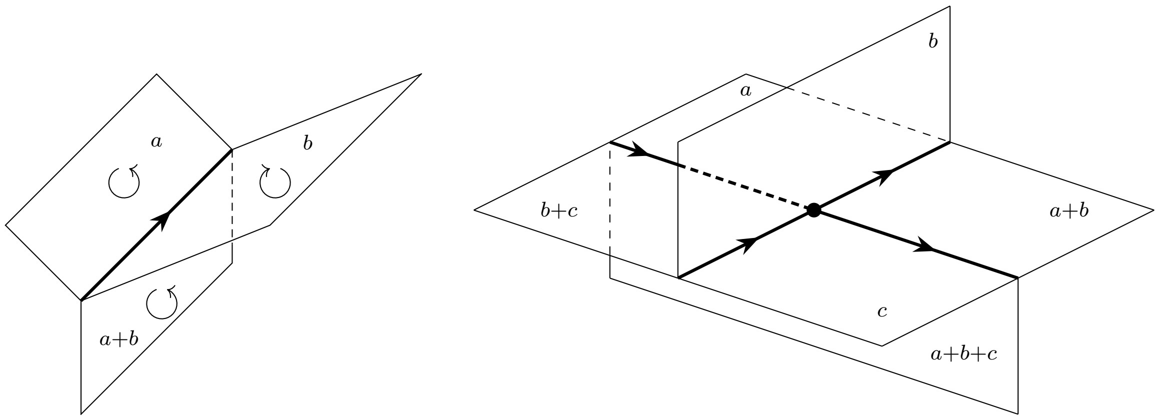
Definition 2.5.10. A (closed) $\mathfrak{gl}_N$ *foam* F is a compact, PL, 2-dimensional CW complex with the following properties and data.

- Every point of F has neighborhood homeomorphic to either a disk, the letter Y times an interval, or the cone on the 1-skeleton of a tetrahedron; the latter two types of points, illustrated in Figure 2-25, are called *singular* points. The *singular graph of* F, denoted $s(F)$, consists of all singular points. Note that $s(F)$ may contain closed loops with no vertices. Edges and loops in $s(F)$ are called *bindings*; moreover, the edges are called *interval bindings*. A *singular vertex* of F is a vertex of $s(F)$, shown in Figure 2-25b.

- A *facet* of F is a connected component of $F \setminus s(F)$; the set of facets is denoted $f(F)$. Each facet $f \in f(F)$ of F carries a label in $\{0, \dots, N\}$ called its *thickness* and denoted $\mathrm{th}(f)$.

- The singular graph and facets of F must be oriented, and moreover these orientations must be compatible with thicknesses in the following way. Three facets meeting along a binding must have thickness a, b, and $a+b$. The orientation of the binding must agree with the orientations of the thickness a and b facets and disagree with the orientation of thickness $a+b$ facet. See Figure 2-25 for a summary.

- Foams moreover carry *decorations*, consisting of a homogeneous symmetric polynomial P_f in $\mathrm{th}(f)$ variables for each facet f. The variables are all of degree two. We will indicate these polynomial decorations by an arrow pointing from a polynomial to the



(a) An interval binding, where three facets meet. (b) A singular vertex, where six facets meet.

Figure 2-25: A summary of the thickness and orientation conventions at singular points.

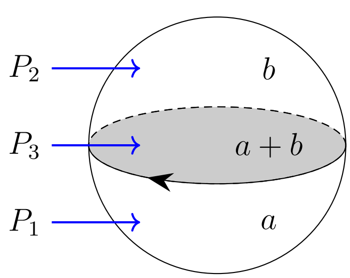
facet it decorates, see (2.18). When depicting foams, a facet with no such decoration appearing indicates that it is decorated by the constant polynomial 1.

- Finally, foams considered in the present paper will always be PL embedded in $\mathbb{R}^3$. As for webs, we will often write *foam* instead of $\mathfrak{gl}_N$ foam.

Remark 2.5.11. One may consider abstract foams (not embedded in $\mathbb{R}^3$). In this case a cyclic order of the three facets at each interval binding must be fixed. In our situation, when all foams are embedded, the cyclic order is determined by the left-hand rule.

We will generally omit orientations of facets from the pictures; they are determined by the orientation of the binding and the thickness of facets.

Example 2.5.12. Consider the foam F shown in Equation 2.18. It consists of three disks with thicknesses a, b, and $a+b$ glued along their common boundary, with symmetric polynomial labels P_1, P_2, and P_3, respectively. The orientations on facets can be determined from the orientation of the singular circle.



(2.18)

Definition 2.5.13. Let F be a $\mathfrak{gl}_N$ foam. Consider the following quantities.

- For a facet f of thickness a, set $d(f) = a(N-a)\chi(f)$.
- For an interval binding i which meets facets of thicknesses a, b, and $a+b$, set $d(i) = ab + (a+b)(N-a-b)$.
- For a singular vertex v which meets facets of thicknesses $a, b, c, a+b, a+c, b+c$, and $a+b+c$, set $d(v) = ab + ac + bc + (N-a-b-c)(a+b+c)$.

Define the degree of a foam F to be

$$\deg(F) = -\sum_{f \in f(F)} d(f) + \sum_{\substack{i \text{ an} \\ \text{interval binding}}} d(i) - \sum_{\substack{v \text{ a} \\ \text{singular vertex}}} d(i) + \sum_{f \in f(F)} \deg(P_f)$$

We introduce the following notation.

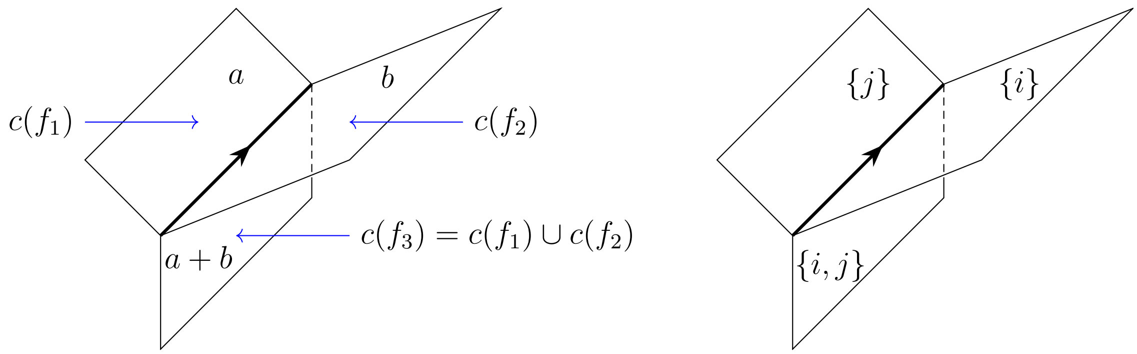
- Let $[N] = \{1, 2, \ldots, N\}$.
- For a set A, we let 2^A denote its powerset.
- We will write $\#A$ to denote the cardinality of a finite set A.

Definition 2.5.14. A *coloring* of a foam F is a function $c : f(F) \to 2^{[N]}$ such that $\#c(f) = \mathrm{th}(f)$ for all $f \in f(F)$. Moreover, if three facets f_1, f_2, f_3, of thicknesses a, b and $a+b$ respectively, meet at a singular edge, then we must have $c(f_1) \sqcup c(f_2) = c(f_3)$. This condition is illustrated in Figure 2-26a. Let $\mathrm{adm}(F)$ denote the set of colorings of F.

For $i \in [N]$, we say a facet $f \in f(F)$ is *colored* i (according to c) if $i \in c(f)$; more generally, for $I \subset [N]$, we say f is colored I if $I \subset c(f)$.

Now fix a foam F and a coloring c.

- For $1 \leq i \leq N$, let $F_i(c)$ denote the union of all i-colored facets of F.
- For $1 \leq i \neq j \leq N$, let $F_{ij}(c)$ denote the union of facets colored either i or j but not both.



(a) The condition on a foam coloring c around each singular edge.

(b) A positive (i,j) binding, where $1 \leq i < j \leq N$.

Figure 2-26

- Let $1 \leq i < j \leq N$ and let β be a binding joining facets f_1, f_2, f_3 with f_1 colored i, f_2 colored j, and f_3 colored $\{i,j\}$. We say β is *positive* with respect to (i,j) if, according to the left-hand rule, the cyclic ordering is (f_1, f_2, f_3); otherwise we say β is negative. This is summarized in Figure 2-26b. Let

$$\theta_{ij}^{+}(c)$$

denote the number of positive $(i.j)$ bindings.

- Recall that facets of foams are decorated by symmetric polynomials. If $A \subset [N]$ and $P \in \mathbb{Z}[y_1, \ldots, y_m]$ is a symmetric polynomial in $m = \#A$ variables, let $P(A) := P((x_a)_{a \in A}) \in \mathbb{Z}[x_1, \ldots, x_N]$.

Definition 2.5.15. Define the following rings.

- $R'_N = \mathbb{Z}[x_1, \ldots, x_N]$.
- $R_N \subset R'_N$ the subring of symmetric polynomials.
- $R''_N = R'_N[(x_i - x_j)^{-1} \mid 1 \leq i < j \leq N]$ the extension of R'_N obtained by inverting $x_i - x_j$ for all $i < j$. We have an inclusion of rings $R_N \subset R'_N \subset R''_N$. These rings are all graded by setting $\deg(x_i) = 2$.

We are now ready to introduce Robert-Wagner closed foam evaluation [73, Definition 2.12].

Definition 2.5.16. Let F be a foam and $c \in \mathrm{adm}(F)$ a coloring. Define

$$\begin{aligned} s(F,c) &= \sum_{i=1}^{N} i\chi(F_i(c))/2 + \sum_{1\le i<j\le N} \theta^+_{ij}(c), \\ P(F,c) &= \prod_{f\in f(F)} P_f(c(f)), \\ Q(F,c) &= \prod_{1\le i<j\le N} (x_i - x_j)^{\chi(F_{ij}(c))/2}, \\ \langle F, c\rangle_{\mathrm{RW}} &= (-1)^{s(F,c)} \frac{P(F,c)}{Q(F,c)}. \end{aligned}$$

Finally, the evaluation of F is given by $\langle F\rangle_{\mathrm{RW}} = \sum_{c\in \mathrm{adm}(F)} \langle F, c\rangle_{\mathrm{RW}}$

Note that Definition 2.5.14 implies that $F_i(c)$ and $F_{ij}(c)$ are closed orientable surfaces and hence have even Euler characteristic. A priori, $\langle F\rangle_{\mathrm{RW}}$ is valued in R''_N. However, Robert Wagner prove the following.

Proposition 2.5.17 ([73, Proposition 2.19])**.** *For any foam F, its evaluation $\langle F\rangle_{\mathrm{RW}}$ lies in R_N, the ring of symmetric polynomials.*

2.5.4 Universal construction and categorification of the MOY calculus

Robert-Wagner use universal construction applied to $\langle - \rangle_{\mathrm{RW}}$ to build state spaces for $\mathfrak{gl}_N$ webs, and they show that applying the universal construction to the evaluation $\langle - \rangle_{\mathrm{RW}}$ categorifies the MOY relations [73, Theorem 3.30]. Precisely, to any $\mathfrak{gl}_N$ web Γ, Rober-Wagner define a R_N-module $\langle \Gamma\rangle_{\mathrm{RW}}$. They show that $\langle \Gamma\rangle_{\mathrm{RW}}$ is a free R_N-module with graded rank equal to $\langle \Gamma\rangle_N$. We recall the construction here. A positive integer N is fixed throughout.

Definition 2.5.18. Let Γ_0, Γ_1 be $\mathfrak{gl}_N$ webs. A $\mathfrak{gl}_N$ *foam with boundary* (Γ_0, Γ_1) consists of an intersection of a $\mathfrak{gl}_N$ foam F with $\mathbb{R}^2 \times [0,1]$ such that, for some $\varepsilon > 0$, $F \cap (\mathbb{R}^2 \times [0,\varepsilon]) = (-\Gamma_0) \times [0,\varepsilon]$, $F \cap (\mathbb{R}^2 \times [1-\varepsilon, 1]) = \Gamma_1 \times [1-\varepsilon, \varepsilon]$. Foams with boundary are considered up to ambient isotopy of $\mathbb{R}^2 \times [0,1]$ which is the identity near the boundary of $\mathbb{R}^2 \times [0,1]$. The degree of a foam with boundary is defined as in Definition 2.5.13.

A $\mathfrak{gl}_N$ foam with boundary (Γ_0, Γ_1) is viewed as a cobordism from Γ_0 to Γ_1. Let $\mathbf{Foam}_N$ denote the category with objects $\mathfrak{gl}_N$ webs and morphisms foams with boundary. Composition of foams with boundary is given, as usual, by stacking one on top of the other and re-scaling.

Let Γ be a $\mathfrak{gl}_N$ web. Define $\mathrm{Fr}(\Gamma)$ to be the free R_N-module generated by all foams from the empty web $\varnothing$ to Γ. Next, define a bilinear pairing

$$\langle -, - \rangle : \mathrm{Fr}(\Gamma) \times \mathrm{Fr}(\Gamma) \to R_N$$

by setting $\langle F, G \rangle = \left\langle \overline{F} G \right\rangle_{\mathrm{RW}}$, where $\overline{F}$ denotes the foam $\Gamma \to \varnothing$ obtained by reflecting F through a horizontal plane. Note that the pairing $\langle -, - \rangle$ is symmetric.

Definition 2.5.19. The *state space of* Γ, denoted $\langle \Gamma \rangle_{\mathrm{RW}}$, is defined to be the quotient of $\mathrm{Fr}(\Gamma)$ by the kernel of $\langle -, - \rangle$:

$$\begin{aligned} \ker(\langle -, - \rangle) &:= \{x \in \mathrm{Fr}(\Gamma) \mid \langle x, y \rangle = 0 \text{ for all } y \in \mathrm{Fr}(\Gamma)\}, \\ \langle \Gamma \rangle_{\mathrm{RW}} &:= \mathrm{Fr}(\Gamma) / \ker(\langle -, - \rangle) \end{aligned}$$

Since the bilinear pairing respects degrees, the degree of foams descends to a grading on $\langle \Gamma \rangle_{\mathrm{RW}}$. Given a foam cobordism $F : \Gamma_0 \to \Gamma_1$, there is an induced map $\mathrm{Fr}(\Gamma_0) \to \mathrm{Fr}(\Gamma_1)$ defined by sending a basis element $G : \varnothing \to \Gamma_0$ to the composition $FG : \varnothing \to \Gamma_1$. The definition of the state space immediately implies that we obtain an induced map $\langle F \rangle_{\mathrm{RW}} : \langle \Gamma_0 \rangle_{\mathrm{RW}} \to \langle \Gamma_1 \rangle_{\mathrm{RW}}$, which is graded of degree $\deg(F)$. Moreover, this assignment is functorial: if $F' : \Gamma_1 \to \Gamma_2$ is a foam cobordism, then $\langle F'F \rangle_{\mathrm{RW}} = \langle F' \rangle_{\mathrm{RW}} \langle F \rangle_{\mathrm{RW}}$. Thus the state space construction yields a functor $\mathbf{Foam}_N \to R_N-\mathrm{gmod}$.

The above state space assignment is an instance of *universal construction*, due to [13]. Note that functoriality follows immediately, but this is at the cost of defining state spaces to be quotients of infinite-dimensional modules. Identifying state spaces obtained from a general evaluation involves establishing "local relations" satisfied by the evaluation. In the case of $\langle - \rangle_{\mathrm{RW}}$, the relevant relations are established in [73, Section 3.2].

The following theorem is the main result of [73]. It says that universal construction applied to the evaluation $\langle - \rangle_{\mathrm{RW}}$ categorifies the MOY relations.

Theorem 2.5.20 ([73, Theorem 3.30]). *For any $\mathfrak{gl}_N$ web, the state space $\langle \Gamma \rangle_{\mathrm{RW}}$ is a free R_N-module of graded rank equal to $\langle \Gamma \rangle_N$, the MOY evaluation of Γ.*

The above theorem is proven by lifting, in an appropriate sense, the MOY relations (Figure 2-24) to isomorphisms of state spaces.

2.6 Khovanov spectra

This section reviews the spectral refinement of Khovanov homology, originally defined by Lipshitz and Sarkar in [57] using framed flow categories and (a modification of) the Cohen-Jones-Segal [22] construction. We follow the approach in [52] via Burnside functors, which is more categorical but bypasses the many technicalities of framed flow categories. Another construction of a Khovanov homotopy type was given in [26]. By [52, Theorem 3], all three constructions are equivalent.

Before delving into the details, let us give an overview of the main results. Let L be an oriented link and D a diagram for L containing n_- negative crossings.

Theorem 2.6.1 ([57]). *There exists a CW suspension spectrum $\mathcal{X}_{Kh}(D)$ such that its cellular cochain complex is isomorphic to the Khovanov complex $CKh_0(D)$,*

$$C^*_{\mathrm{cell}}(\mathcal{X}_{Kh}(D)) \cong CKh^*_0(D).$$

Moreover, $\mathcal{X}_{Kh}(D)$ splits as a wedge sum of subcomplexes, $\mathcal{X}_{Kh}(D) = \bigvee_{j \in \mathbb{Z}} \mathcal{X}^j_{Kh}(D)$, such that $C^*_{\mathrm{cell}}(\mathcal{X}^j_{Kh}(D)) \cong CKh^{*,j}_0(D)$. If D and D' are related by a Reidemeister move, then $\mathcal{X}_{Kh}(D)$ and $\mathcal{X}_{Kh}(D')$ are stably homotopy equivalent.

The spectrum $\mathcal{X}_{Kh}(D)$ is defined by first building a based CW complex $\mathcal{X}_N(D)$ associated to a particular functor (a *Burnside functor*) and a choice of integer $N >> 0$. For $M \geq N$, the CW complexes $\mathcal{X}_N(D)$ and $\mathcal{X}_M(D)$ are related via $\mathcal{X}_M(D) \simeq \Sigma^{M-N}\mathcal{X}_N(D)$, where Σ^* denoted reduced suspension and $\simeq$ denotes homotopy equivalence of based topological spaces. Cells of $\mathcal{X}_N(D)$ except the basepoint are in bijection with Khovanov generators of $CKh_0(D)$.

The CW complex $\mathcal{X}_N(D)$ splits as a wedge sum over the quantum gradings, and there is an isomorphism of chain complexes

$$C^*_{\text{cell}}(\mathcal{X}^j_N(D))[-N-n_-] \cong CKh_0^{*,j}(D),$$

obtained by mapping cells to the corresponding generators (the brackets $[n]$ denote an upwards homological degree shift by n).

The spectrum $\mathcal{X}_{Kh}(D)$ is then obtained by taking a suitable formal desuspension of the suspension spectrum of $\mathcal{X}_N(D)$,

$$\mathcal{X}_{Kh}(D) := \Sigma^{-N-n_-}\Sigma^\infty \mathcal{X}_N(D).$$

The terminology *stable homotopy equivalence* means, in this context, that if D' is obtained from D by a Reidemeister move, then the CW complexes $\mathcal{X}_N(D)$ and $\mathcal{X}_N(D')$ become homotopy equivalent after suspending each space some number of times. The spectrum $\mathcal{X}_{Kh}(D)$ is an invariant of L, allowing the notation $\mathcal{X}_{Kh}(L)$.

It is natural to ask if one can find homotopical refinements of other versions of other link homology theories. If one intends to use the Burnside functor approach, then a negative answer to this question for the Lee deformation was given in [52] (see in particular [52, Figure 4.1]). A homotopy type for Bar-Natan homology was introduced in [76]. Some proposals for defining a stable homotopy refinement of Khovanov-Rozansky $\mathfrak{sl}_N$ homology for $N \geq 3$ have been outlined in [27, 30, 43].

2.6.1 Burnside functors

Following the general strategy of [52], the first step towards lifting Khovanov homology to a spectrum is to build a Burnside functor from the cube category $\underline{2}^n$ [52, Section 2.1] to the Burnside category $\mathscr{B}$ [52, Section 4.1] which encodes the information underlying the chain complex $CKh_0(D)$ in a higher categorical manner. This encoding is, however, not determined by the Khovanov chain complex. In this section we review the general framework of such categories and functors.

We begin by recalling the *cube category* $\underline{2}^n$ from [52, Section 2.1].

Definition 2.6.2. The objects of $\underline{2}^n$ are the elements of $\{0,1\}^n$, thought of as vertices of the n-dimensional cube I^n. There is a natural partial order on $\{0,1\}^n$: for vertices $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$, write $u \geq v$ if each $u_i \geq v_i$. The set of morphisms $\mathrm{Hom}_{\underline{2}^n}(u, v)$ is defined to be empty unless $u \geq v$, in which case $\mathrm{Hom}_{\underline{2}^n}(u, v)$ consists of a single element, denoted $\varphi_{u,v}$. In particular, given $u, w \in \{0,1\}^n$ with $u \geq w$, in $\underline{2}^n$ we have $\varphi_{u,w} = \varphi_{v,w} \circ \varphi_{u,v}$ for any v such that $u \geq v \geq w$.

Remark 2.6.3. Note that the edges in $\underline{2}^n$ point in the opposite direction of those in the cube of resolutions of a link diagram.

We establish some useful notation and definitions regarding $\underline{2}^n$.

Definition 2.6.4. Let $u, v \in \{0,1\}^n$ be vertices.

- If $u = (u_1, \ldots, u_n)$, define $|u| := \sum_i u_i$.
- Write $u \geq_k v$ if $u \geq v$ and $|u| - |v| = k$. In particular, $u \geq_1 v$ means there is a saddle cobordism from D_v to D_u in the cube of resolutions of D.
- If $u \geq_k v$, then u and v specify a *k-dimensional sub-cube*, which is the full subcategory of $\underline{2}^n$ with objects consisting of all vertices w satisfying $u \geq w \geq v$.

We now recall the notion of *correspondences*, following the exposition in [52, Section 4.1]. Correspondences are sometimes called *spans* elsewhere in the literature.

Definition 2.6.5.

- For sets X and Y, a *correspondence* from X to Y is a triple (A, s, t) where A is a set and $s : A \to X$, $t : A \to Y$ are functions, called the *source map* and *target map*, respectively. We will often denote a correspondence by $X \xleftarrow{s} A \xrightarrow{t} Y$ or simply $X \leftarrow A \rightarrow Y$.
- Given correspondences $X \xleftarrow{s_A} A \xrightarrow{t_A} Y$ and $Y \xleftarrow{s_B} B \xrightarrow{t_B} Z$, their composition $(B, s_B, t_B) \circ (A, s_A, t_A)$ is the correspondence (C, s, t) from X to Z obtained as the fiber product
$$C = B \times_Y A = \{(b, a) \in B \times A \mid s_B(b) = t_A(a)\}$$
with the source and target maps $s(b, a) = s_A(a), t(b, a) = t_B(b)$.

- A *morphism* from a correspondence $X \xleftarrow{s_A} A \xrightarrow{t_A} Y$ to a correspondence $X \xleftarrow{s_B} B \xrightarrow{t_B} Y$ is a bijection $f : A \to B$ which commutes with the source and target maps. That is, f is a bijection fitting into the commutative diagram

$$\begin{tikzcd} & A \arrow[dl, "s_A"'] \arrow[dr, "t_A"] \arrow[dd, "f"] & \\ X & & Y. \\ & B \arrow[ul, "s_B"] \arrow[ur, "t_B"'] & \end{tikzcd}$$

Composition of correspondences be summarized by the fiber product diagram

$$\begin{tikzcd} & & C \arrow[dl] \arrow[dr] \arrow[ddll, bend right, "s"'] \arrow[ddrr, bend left, "t"] & & \\ & A \arrow[dl, "s_A"'] \arrow[dr, "t_A"] & & B \arrow[dl, "s_B"'] \arrow[dr, "t_B"] & \\ X & & Y & & Z. \end{tikzcd}$$

The collection of sets, correspondences between them, and morphisms of correspondences forms a *bicategory* in the language of [11] or, equivalently, a *weak 2-category* in the language of [52]. The objects are sets, 1-morphisms are correspondences, and 2-morphisms are morphisms of correspondences. We will use the terms bicategory and weak 2-category interchangebly. A quick reference for the notion of bicategories is [11, Appendix A.4].

The identity 1-morphism of a set X is the identity correspondence

$$X \xleftarrow{\mathrm{id}_X} X \xrightarrow{\mathrm{id}_X} X.$$

Given correspondences $X \xleftarrow{s_A} A \xrightarrow{t_A} Y$ and $Z \xleftarrow{s_B} B \xrightarrow{t_B} X$, the compositions $(A, s_A, t_A) \circ (X, \mathrm{id}_X, \mathrm{id}_X)$ and $(X, \mathrm{id}_X, \mathrm{id}_X) \circ (B, s_B, t_B)$ are not equal to (A, s_A, t_A) and (B, s_B, t_B), but there are natural 2-morphisms

$$(A, s_A, t_A) \circ (X, \mathrm{id}_X, \mathrm{id}_X) \cong (A, s_A, t_A)$$
$$(X, \mathrm{id}_X, \mathrm{id}_X) \circ (B, s_B, t_B) \cong (B, s_B, t_B)$$

Similarly, composition of correspondences is not strictly associative, but is associative up to natural isomorphism. This is the sense in correspondences form a weak 2-category, as opposed to a strict 2-category.

We are now ready to define the Burnside category $\mathscr{B}$ and Burnside functors $\underline{2}^N \to \mathscr{B}$, as in [52, Section 4.1].

Definition 2.6.6. The *Burnside category*, denoted $\mathscr{B}$, is the bicategory of finite sets and finite correspondences.

Remark 2.6.7. Even though $\mathscr{B}$ is a bicategory, it will always be referred to as a category.

The construction of Khovanov homotopy types in [52], [78] utilizes functors $F : \underline{2}^n \to \mathscr{B}$, which we explain here. First, make $\underline{2}^n$ into a (strict) 2-category by introducing only identity 2-morphisms. There is a notion of a *lax 2-functor* between 2-categories, and also of a *strictly unitary lax 2-functor*. The complete definitions, consisting of a slew of data and naturality conditions, can be found in [52, Definition 4.2] and [52, Defintion 4.3].

Definition 2.6.8. A *Burnside functor* is a strictly unitary lax 2-functor $F : \underline{2}^n \to \mathscr{B}$.

We will only be interested in the above type of functor between 2-categories. Lemma 2.6.9 below specifies the data needed to define a Burnside functor uniquely up to natural isomorphism (see Definition 2.6.11 for the definition of natural isomorphism of Burnside functors). In light of this, we do not recall definitions of the various types of functors between 2-categories, and instead refer the interested reader to the above references.

Lemma 2.6.9 (([52, Lemma 4.5], [51, Proposition 4.3]))**.** *Consider the following data:*

- *A finite set $F(u)$ for each vertex $u \in \underline{2}^n$.*
- *A finite correspondence $F(\varphi_{u,v})$ from $F(u)$ to $F(v)$ for each pair of vertices $u, v \in \underline{2}^n$ with $u \geq_1 v$.*
- *A 2-morphism*

$$F_{u,v,v',w} : F(\varphi_{v,w}) \circ F(\varphi_{u,v}) \to F(\varphi_{v',w}) \circ F(\varphi_{u,v'})$$

for each 2-dimensional face of $\underline{2}^n$ with vertices u, v, v', w satisfying $u \geq_1 v, v' \geq_1 w$.

Suppose also that the above data satisfies the following conditions:

(1) $F^{-1}_{u,v,v',w} = F_{u,v',v,w}$

(2) *For every* 3*-dimensional sub-cube of* $\underline{2}^n$ *as in Figure 2-27a, the hexagon of Figure 2-27b commutes.*

$$
\begin{tikzcd}
 & v' \arrow[rr] \arrow[dd] & & w \arrow[dd] \\
u \arrow[ur] \arrow[rr] \arrow[dd] & & v \arrow[ur] \arrow[dd] & \\
 & w' \arrow[rr] & & z \\
v'' \arrow[ur] \arrow[rr] & & w'' \arrow[ur] &
\end{tikzcd}
$$

(a) 3D cube

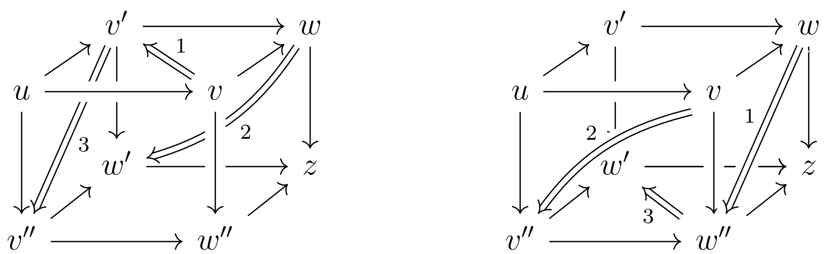
$$
\begin{tikzcd}
 & \circ \arrow[r, "F_{v',w,w',z}\times\mathrm{id}"] & \circ \arrow[dr, "\mathrm{id}\times F_{u,v',v'',w'}"] & \\
\circ \arrow[ur, "\mathrm{id}\times F_{u,v,v',w}"] \arrow[dr, "F_{v,w,w'',z}\times\mathrm{id}"'] & & & \circ \\
 & \circ \arrow[r, "\mathrm{id}\times F_{u,v,v'',w''}"'] & \circ \arrow[ur, "F_{v'',w'',w',z}\times\mathrm{id}"'] &
\end{tikzcd}
$$

(b) The hexagon relation

Figure 2-27

Then the data can be extended to a strictly unitary lax 2*-functor* $F : \underline{2}^n \to \mathscr{B}$*, which is unique up to natural isomorphism.*

The hexagon of Figure 2-27b comes from two ways traversing the faces of the 3-dimensional cube, starting from the correspondence $F(\varphi_{w,z}) \circ F(\varphi_{v,w}) \circ F(\varphi_{u,v})$ and ending at $F(\varphi_{w',z}) \circ F(\varphi_{v'',w'}) \circ F(\varphi_{u,v''})$. The top half of the hexagon comes from traversing the faces as shown in the left side of (2.19), and the bottom half comes from traversing the faces as shown in the right side of (2.19). Lemma 2.6.9 states that verifying the hexagon relation suffices guarantee that the functor is coherent on all higher dimensional cubes as well.



(2.19)

Remark 2.6.10. We make some comments about verifying the hexagon relation which will be useful in proving Theorem 3.5.2. Suppose we have 2-morphisms $F_{u,v,v',w}$ for each square face $u \geq_1 v, v' \geq_1 w$ of $\underline{2}^n$, which satisfy $F^{-1}_{u,v,v',w} = F_{u,v',v,w}$ as in condition (1) of Lemma 2.6.9. Then verifying the hexagon relation is equivalent to the following. Start at the correspondence $F(\varphi_{w,z}) \circ F(\varphi_{v,w}) \circ F(\varphi_{u,v})$ and traverse the six faces of the cube using the 2-morphisms; i.e., first move across the three faces in the left part of (2.19), and then move across the remaining three faces as in the right part of (2.19), except in the reverse order. Composing these six 2-morphisms yields a 2-morphism

$$\Phi_{u,v,w,z} : F(\varphi_{w,z}) \circ F(\varphi_{v,w}) \circ F(\varphi_{u,v}) \to F(\varphi_{w,z}) \circ F(\varphi_{v,w}) \circ F(\varphi_{u,v}).$$

Verifying commutativity of the hexagon of Lemma 2.6.9 is equivalent to verifying that $\Phi_{u,v,w,z}$ is the identity. Moreover, for each 3-dimensional sub-cube of $\underline{2}^n$, it suffices to verify $\Phi_{u,v,w,z} = \mathrm{id}$ for just one tuple of vertices $u \geq_1 v \geq_1 w \geq_1 z$ within the sub-cube.

Furthermore, such verifications are immediate under certain circumstances. Let A denote the correspondence $F(\varphi_{w,z}) \circ F(\varphi_{v,w}) \circ F(\varphi_{u,v})$, with source and target maps $s : A \to F(u)$, $t : A \to F(z)$. Suppose that for every $x \in F(u)$ and $y \in F(z)$, $s^{-1}(x) \cap t^{-1}(y)$ is either empty or has one element. Let $a \in A$ and let $a' = \Phi_{u,v,w,z}(a)$. Since $\Phi_{u,v,w,z}(a)$ is a 2-morphism, we have $s(a') = s(a)$ and $t(a') = t(a)$. Then $a' = a$, so the hexagon relation is satisfied for this 3-dimensional sub-cube. In this situation, we will say that this 3-dimensional cube is *simple*

.

We end this section with a discussion of natural transformations (including natural isomorphisms) of Burnside functors.

To start, note that there is a canonical identification $\underline{2}^{n+1} = \underline{2} \times \underline{2}^n$. In the context of natural transformations, we will think of $\underline{2}^{n+1} = \underline{2} \times \underline{2}^n$ as two "horizontal" copies of $\underline{2}^n$ with vertical edges connecting them, pointing downwards. The top copy of $\underline{2}^n$ corresponds to $\{1\} \times \underline{2}^n \subset \underline{2} \times \underline{2}^n$, and likewise the bottom copy corresponds to $\{0\} \times \underline{2}^n \subset \underline{2} \times \underline{2}^n$. Recall that for $u \geq v$, $\varphi_{u,v}$ denotes the unique element in $\mathrm{Hom}_{\underline{2}^n}(u, v)$. We distinguish two types of morphisms in $\underline{2} \times \underline{2}^n$. First, for each $u \in \underline{2}^n$, there is a morphism

$$(\varphi_{1,0}, \mathrm{id}_u) : (1, u) \to (0, u).$$

We denote this edge by e_u, and think of it a vertical arrow

$$\begin{tikzcd} (1,u) \arrow[d, "e_u"'] \\ (0,u) \end{tikzcd}$$

The second type consists of morphisms in $\underline{2} \times \underline{2}^n$ of the form

$$(\mathrm{id}_i, \varphi_{u,v}) : (i,u) \to (i,v)$$

where $i \in \{0,1\}$, and $u,v \in \underline{2}^n$ with $u \geq v$. These will be denoted $\varphi^i_{u,v}$ and viewed as living in the "horizontal" cube $\{i\} \times \underline{2}^n$.

With these conventions, the diagram

$$\begin{tikzcd} & (i,v') \arrow[rr, "\varphi^i_{v',w}"] & & (i,w) \\ (i,u) \arrow[ur, "\varphi^i_{u,v'}"] \arrow[rr, "\varphi^i_{u,v}"'] & & (i,v) \arrow[ur, "\varphi^i_{v,w}"'] & \end{tikzcd} \tag{2.20}$$

lives in the horizontal cube $\{i\} \times \underline{2}^n$, and the diagram

$$\begin{tikzcd} (1,u) \arrow[r, "\varphi^1_{u,v}"] \arrow[d, "e_u"'] & (1,v) \arrow[d, "e_v"] \\ (0,u) \arrow[r, "\varphi^0_{u,v}"'] & (0,v) \end{tikzcd} \tag{2.21}$$

is between the horizontal cubes. We will often not label some or all of the edges, with the understanding that the above conventions (2.20) and (2.21) are followed.

Definition 2.6.11. A *natural transformation* $\eta : F_1 \to F_0$ of Burnside functors $F_1, F_0 : \underline{2}^n \to \mathscr{B}$ is a functor $\eta : \underline{2}^{n+1} \to \mathscr{B}$ such that the restriction of η to $\{i\} \times \underline{2}^n$ is equal to F_i. We say η is a *natural isomorphism* if $\eta(e_u) : F_1(u) \to F_0(u)$ is an isomorphism in $\mathscr{B}$ for each vertex u.

Due to Lemma 2.6.9, in order to define a natural transformation $\eta : F_1 \to F_0$, one needs to specify a correspondence $\eta(e_u)$ for each vertical edge e_u, a 2-morphism $\eta_{u,v} : \eta(e_v) \circ \eta(\varphi^1_{u,v}) \to$

$\eta(\varphi^0_{u,v}) \circ \eta(e_u)$ for each vertical face as in (2.21), and verify the hexagon relation of Figure 2-27.

2.6.2 Totalizations of Burnside functors

Associated to any Burnside functor $\underline{2}^n \to \mathscr{B}$ is a chain complex, called the *totalization*.

Definition 2.6.12 ([51, Definition 5.1], [78, Section 3.6])**.** For a set X, let $\mathcal{A}(X)$ denote the free abelian group generated by X. Given a correspondence $X \xleftarrow{s} A \xrightarrow{t} Y$, define a map $\mathcal{A}(A) : \mathcal{A}(X) \to \mathcal{A}(Y)$ by

$$\mathcal{A}(A)(x) = \sum_{a \in s^{-1}(x)} t(a). \tag{2.22}$$

Let $F : \underline{2}^n \to \mathscr{B}$ be a Burnside functor. For $u \geq v$, let $A_{u,v}$ denote the correspondence assigned by F to the morphism $\varphi_{u,v} : u \to v$. The complex $\mathrm{Tot}(F)$ is defined by

$$\mathrm{Tot}(F) := \bigoplus_{u \in \underline{2}^n} \mathcal{A}(F(u))$$

with the term $\mathcal{A}(F(u))$ in homological degree $|u|$. The differential

$$\partial : \bigoplus_{|u|=i+1} \mathcal{A}(F(u)) \to \bigoplus_{|v|=i} \mathcal{A}(F(v))$$

is given on summands by maps $\partial_{u,v} : \mathcal{A}(F(u)) \to \mathcal{A}(F(v))$, for $|u| = i+1$, $|v| = i$, defined as

$$\partial_{u,v}(x) = (-1)^{s_{u,v}} \mathcal{A}(A_{u,v}).$$

In the above, $s_{u,v} \in \{0,1\}$ is a sign assignment on edges, ensuring that $\partial^2 = 0$ (see [8, Section 2.7], also [57, Definition 4.5] for a discussion of $s_{u,v}$).

Remark 2.6.13. Note that the above differential decreases homological grading.

Given a natural transformation $\eta : F_1 \to F_0$, there is an induced chain map $\mathrm{Tot}(\eta) : \mathrm{Tot}(F_1) \to \mathrm{Tot}(F_0)$, defined on each summand by $\mathcal{A}(\eta(e_u)) : \mathcal{A}(F_1(u)) \to \mathcal{A}(F_0(u))$.

2.6.3 From Burnside functors to spaces

In [52, Section 4], Lawson-Lipshitz-Sarkar explain how to build a CW complex $\|F\|_k$ associated to a Burnside functor F and an integer $k >> 0$. In this section we record the key properties of this construction; a thorough treatment is given in Sections 3.6.3 and 3.6.4, where we will need an equivariant analogue.

Proposition 2.6.14 ([52, Proposition 6.1], [78, Section 4.4])**.** *Let $F : \underline{2}^n \to \mathscr{B}$ be a Burnside functor. For each $k \geq n$, there exists a based CW complex $\|F\|_k$ satisfying the following properties.*

1. *If $k' > k$, then $|F|_{k'}$ is homotopy equivalent to $\Sigma^{k'-k}\|F\|_k$.*

2. *The cells of $\|F\|_k$, except the basepoint, are in bijection with $\coprod_{u \in \underline{2}^n} F(u)$.*

3. *Its reduced shifted cellular chain complex $\widetilde{C}^{cell}_*(\|F\|_k)[-k]$ is isomorphic to the totalization $\mathrm{Tot}(F)$, with the cells mapping to the corresponding generators.*

4. *If $\eta : F_1 \to F_0$ is a natural transformation of Burnside functors, then there is a cellular map $\|F_1\|_k \to \|F_0\|_k$. Under the identification $\widetilde{C}^{cell}_*(\|F\|_k)[-k] \cong \mathrm{Tot}(F)$, the induced map on cellular chain complexes agrees with $\mathrm{Tot}(\eta)$.*

5. *If $\eta : F_1 \to F_0$ is a natural transformation such that the induced map $\mathrm{Tot}(\eta) : \mathrm{Tot}(F_1) \to \mathrm{Tot}(F_0)$ is a chain homotopy equivalence, then the induced map $\|F_1\|_k \to \|F_0\|_k$ is a homotopy equivalence.*

2.6.4 The Khovanov stable homotopy type

In this section we define the Khovanov stable homotopy type. We begin by describing the Burnside functor F_{Kh} associated to a link diagram; its construction is sketched in [52, Example 4.21]. Let D be a link diagram with n crossings. Recall from Definition 2.3.18 and Definition 2.3.20 the Frobenius system $\mathcal{F}_0$ and that, for each $u \in \underline{2}^n$, the module $\mathcal{F}_0(D_u)$ assigned to the smoothing D_u has a standard basis consisting of *Khovanov generators*, each of which is a choice of label of either 1 or X on every circle in D_u.

The Burnside functor F_{Kh} is defined so that its totalization $\mathrm{Tot}(F_{Kh})$ (Definition 2.6.12) is isomorphic (up to a homological grading shift) to the Khovanov complex $CKh_0(D)$, with the distinguished basis of the totalization mapping to the Khovanov generators. However, constructing the Burnside functor requires strictly more data than is contained in the Khovanov chain complex, namely the specification of 2-morphisms across square faces.

Definition 2.6.15 ([52, Example 4.21])**.** Fix an n-crossing link diagram D. Define the Burnside functor $F_{Kh} : \underline{2}^n \to \mathscr{B}$ as follows (dependence on D is omitted from the notation).

- For a vertex $u \in \underline{2}^n$, $F_{Kh}(u)$ is the set of Khovanov generators of $\mathcal{F}_0(D_u)$.
- Consider an edge $u \geq_1 v$ in $\underline{2}^n$. Let $d_{v,u} : \mathcal{F}_0(D_v) \to \mathcal{F}_0(D_u)$ denote the corresponding map induced by the saddle cobordism from D_v to D_u. For a Khovanov generator $x \in \mathcal{F}_0(D_v)$, an examination of multiplication and comultipilication in A_0 (Definition 2.3.18) shows that, when writing $d_{v,u}(x)$ as a linear combination of Khovanov generators of $\mathcal{F}_0(D_u)$, all coefficients are in $\{0,1\}$. We say $y \in \mathcal{F}_0(D_u)$ *appears in* $d_{v,u}(x)$ if its coefficient is 1. Define the correspondence $A_{u,v}$ from $F_{Kh}(u)$ to $F_{Kh}(v)$ to be

$$A_{u,v} = \{(y,x) \in F_{Kh}(u) \times F_{Kh}(v) \mid y \text{ appears in } d_{v,u}(x)\},$$

  with source and target maps given by projection.
- Consider a square face $u \geq_1 v, v' \geq_1 w$ in $\underline{2}^n$. It corresponds to performing two saddle cobordisms on D_w. Consider the two composition correspondences $A_{v,w} \times_{F_{Kh}(v)} A_{u,v}$ and $A_{v',w} \times_{F_{Kh}(v')} A_{u,v'}$, with source and target maps denoted s, t and s', t', respectively. In [57, Lemma 5.7], Lipshitz-Sarkar show, besides a special case discussed below, that for any $z \in F_{Kh}(u)$, $x \in F_{Kh}(w)$, the two sets $s^{-1}(z) \cap t^{-1}(x)$ and $(s')^{-1}(z) \cap (t')^{-1}(x)$ are either both empty or both consist of one element. It follows that, outside of this special case, the 2-morphism $\phi_{u,v,v',w} : A_{v,w} \times_{F_{Kh}(v)} A_{u,v} \to A_{v',w} \times_{F_{Kh}(v')} A_{u,v'}$ is uniquely determined.

  The remaining case is called the *ladybug configuration*. It occurs when one circle in D_w splits into two circles in each of $D_v, D_{v'}$, and then the two circles in each of $D_v, D_{v'}$ merge into one circle in D_w; see Figure 2-28. In this case, one can check that

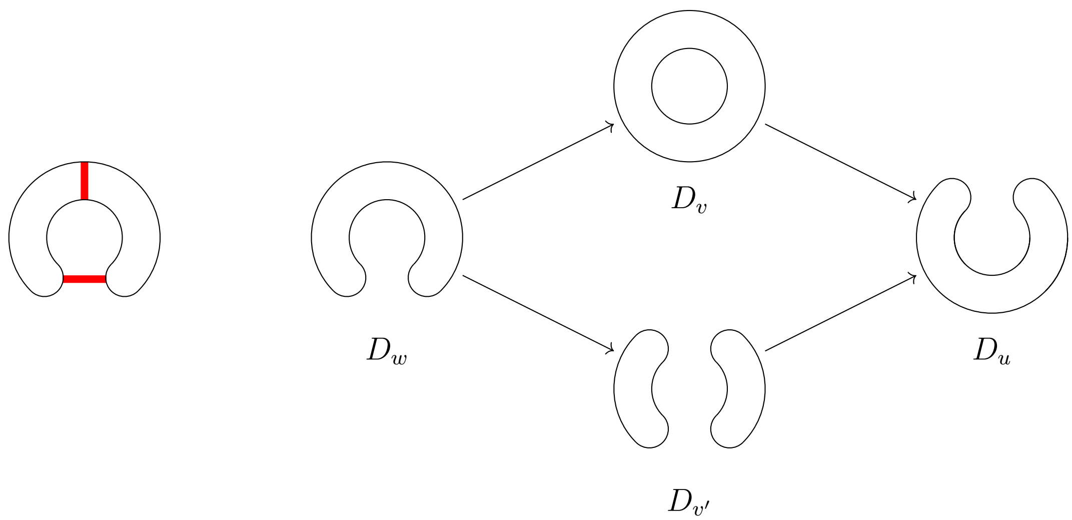


Figure 2-28: The ladybug configuration and its cube of resolutions. On the left, thick red arcs indicate where 1-handles will be attached.

$s^{-1}(z) \cap t^{-1}(x)$ and $(s')^{-1}(z) \cap (t')^{-1}(x)$ are either both empty or both consist of two elements, and the 2-morphism is not uniquely determined in the latter case. In [57, Section 5.4] (see in particular [57, Figure 5.1]), Lipshitz-Sarkar define a bijection, called the *ladybug matching* by using one of two global choices, which we now describe.

Let Z denote the circle in D_w which is split into two circles Z_v, Z'_v in D_v and $Z_{v'}, Z'_{v'}$ in $D_{v'}$. To define the 2-morphism, one needs a bijection between $\{Z_v, Z'_v\}$ and $\{Z_{v'}, Z'_{v'}\}$.

Represent the two saddle cobordisms as surgery arcs on D_w (Figure 2-28). The two surgery arcs split Z into four arcs, and the following procedure is used to specify two of them: traveling along the two surgery arcs (in either direction) and turning *right* specifies two arcs in Z. See Figure 2-29, where these two distinguished arcs are drawn bold. The two arcs, which we label 1 and 2 (the exact labeling is irrelevant), appear in both intermediate diagrams D_v (on the circles Z_v, Z'_v) and $D_{v'}$ (on the circles $Z_{v'}, Z'_{v'}$). The desired bijection between these two pairs of circles is then given by matching the circle labeled 1 (resp. 2) in D_v with the circle labeled 1 (resp. 2) in $D_{v'}$.

Remark 2.6.16. Note that the ladybug matching described above is made by turning *right* at the surgery arcs. One could just as well have turned *left*. It is shown in [57, Proposition 6.5] that the resulting spectrum is independent of this choice.

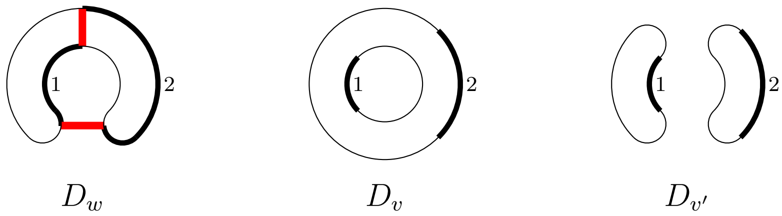


Figure 2-29: The ladybug matching made with the *right pair*. Traveling along the two surgery arcs and turning right specifies two arcs (drawn bold and labeled $1, 2$) in D_w. These two distinguished arcs appear in the diagrams $D_v, D_{v'}$ and determine the bijection by matching the 1-labeled circle with the 1-labeled circle and the 2-labeled circle with the 2-labeled circle

The above definition specifies the data needed for Lemma 2.6.9. The following proposition is proven (using different language) in [57, Section 5.5]; it involves reducing the problem to a small list of 3-dimensional cubes and then verifying that the relation does hold in each case.

Proposition 2.6.17 ([57, Section 5.5])**.** *The data in Definition 2.6.15 satisfies the hexagon relation of Lemma 2.6.9, and hence defines a Burnside functor F_{Kh}.*

Let D be an n-crossing diagram for an oriented link L, with n_- negative crossings and n_+ positive crossings. Consider the corresponding Burnside functor $F_{Kh} : \underline{2}^n \to \mathscr{B}$. For $k >> 0$, let $\|F_{Kh}\|_k$ denote the CW complex of Proposition 2.6.14. By dualizing the isomorphism in item (3) of Proposition 2.6.14, we see that the reduced cellular *co*chain complex of $\|F_{Kh}\|_k$ is isomorphic (up to a homological shift) to the Khovanov chain complex $CKh_0(D)$,

$$C^*_{\text{cell}}(\|F_{Kh}\|_k)[-k - n_-] \cong CKh_0(D).$$

We are now ready to define the Khovanov spectrum $\mathcal{X}_{Kh}(D)$ from Theorem 2.6.1.

Definition 2.6.18. The *Khovanov spectrum* of D is defined to be

$$\mathcal{X}_{Kh}(D) = \Sigma^{-k-n_-}\Sigma^\infty\|F_{Kh}\|_k,$$

the $(k + n_-)$-fold desuspension of the suspension spectrum of $\|F_{Kh}\|_k$.

Chapter 3

Stable homotopy refinement of quantum annular homology

The main result of this chapter is a construction of a stable homotopy refinement of quantum annular homology, which was reviewed in Section 2.4.2. All the results presented in this chapter are from joint work with Krushkal and Willis [5].

We work over the Laurent polynomial ring $\Bbbk := \mathbb{Z}[\mathfrak{q}, \mathfrak{q}^{-1}]$, and tensor the quantum annular chain complex with $\Bbbk_r := \mathbb{Z}[\mathfrak{q}, \mathfrak{q}^{-1}]/(\mathfrak{q}^r - 1)$, where $r \geq 2$; see the beginning of Section 3.5 for a detailed discussion of this modification. Denote the resulting chain complex by $Kh^r_{\mathbb{A}_\mathfrak{q}}(L)$.

Theorem 3.0.1 ([5, Theorem 1.1])**.** *Let L be an oriented link in the thickened annulus $\mathbb{A} \times I$. Then for each $r \geq 2$, there exists a naive*[1] *$\mathbb{Z}/r\mathbb{Z}$-equivariant spectrum $\mathcal{X}^r_{\mathbb{A}_\mathfrak{q}}(L)$ which is well-defined up to equivariant homotopy equivalence and whose cohomology is isomorphic to the quantum annular homology $Kh^r_{\mathbb{A}_\mathfrak{q}}(L)$, as modules over $\mathbb{Z}[\mathbb{Z}/r\mathbb{Z}] = \mathbb{Z}[\mathfrak{q}, \mathfrak{q}^{-1}]/(\mathfrak{q}^r - 1)$.*

The definition of $\mathcal{X}^r_{\mathbb{A}_\mathfrak{q}}(D)$ is given for link diagrams D in Definition 3.6.12 in Section 3.6.5; the proof of invariance with respect to all choices involved (including choice of diagram) is presented there via Theorems 3.6.13 and 3.6.14. We also study structural properties of the spectra, including maps induced by cobordisms, which is the content of Theorem 3.7.2

The construction begins by giving a concrete description of generators and of the differential, starting from the quantum Hochschild homology definition [11] of the quantum

[1]The terminology *naive* is to distinguish from the *genuine* G-spectra studied in equivariant stable homotopy theory. See the end of Section 3.6.5 for further discussion.

annular TQFT $\mathcal{F}_{\mathbb{A}_\mathfrak{q}}(D)$ (see Section 2.4.2 for an overview of $\mathcal{F}_{\mathbb{A}_\mathfrak{q}}$). In fact, there is an important distinction between (annular) Khovanov homology and the quantum annular homology $Kh_{\mathbb{A}_\mathfrak{q}}(L)$. In Khovanov homology, the module assigned to each resolution of the link diagram has a preferred collection of generators, and this is a crucial feature used in constructions of stable homotopy refinements. On the other hand, in the context of quantum annular homology, generators are well-defined only up to a multiple of a power of $\mathfrak{q}$. The proof of Theorem 3.0.1 involves a careful analysis of this indeterminacy and its relation to the group action on the spectrum.

Moreover, the differential depends in a non-trivial way on the combinatorics of a given curve configuration in the annulus. A detailed analysis of the saddle maps defining the differential, using the definition in terms of the quantum annular TQFT $\mathcal{F}_{\mathbb{A}_\mathfrak{q}}$, is given in Section 3.2.1. For $r > 2$, the powers of $\mathfrak{q}$ appearing in the differential affect the construction of the Burnside functor, similar to how the signs appearing in odd Khovanov homology affect the analysis in [78].

A stable homotopy refinement $\mathcal{X}_{\mathbb{A}}(L)$ of classical annular Khovanov homology may be defined along the lines of [57, 52] and is discussed in Section 3.5.3. The following result is spectral analogue to setting $\mathfrak{q} = 1$ in the quantum annular chain complex; its proof is presented in Section 3.8.

Theorem 3.0.2 ([5, Theorem 1.2]). *The quotient of $\mathcal{X}^r_{\mathbb{A}_\mathfrak{q}}(L)$ by the action of $\mathbb{Z}/r\mathbb{Z}$ recovers $\mathcal{X}_{\mathbb{A}}(L)$.*

Note that no group action is present on the link $L \subset \mathbb{A} \times I$, so the context for our work is different from that in [15, 66, 80]. Therefore $\mathcal{X}^r_{\mathbb{A}_\mathfrak{q}}(L)$ may be thought of as an "equivariant refinement" of $\mathcal{X}_{\mathbb{A}}(L)$, a structure that is not apparent in other constructions of the annular spectrum $\mathcal{X}_{\mathbb{A}}(L)$.

3.1 Notational and diagrammatic conventions

This section establishes conventions that will be used throughout this chapter. In order to adhere to notation in [11], we make some minor adjustments to the conventions established in Section 2.3.3 and Section 2.4.1 regarding (annular) Khovanov homology.

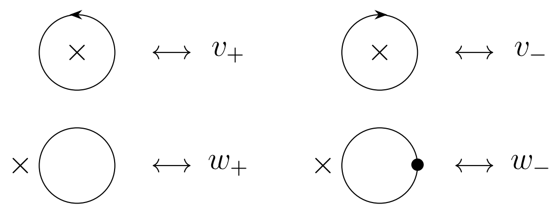


Figure 3-1: Diagrammatic representation of generators.

The main change in this chapter is that all quantum gradings are negated. See Remark 2.3.7 and Remark 2.3.21 for a discussion of the two, opposite grading conventions in the literature. In particular, a cobordism S will induces a map of degree $\chi(S) - 2d(S)$; compare with Equation (2.5). We write the modules assigned to a trivial and essential circle by the classical annular TQFT as

$$W = \mathbb{Z}w_- \oplus \mathbb{Z}w_+ \quad \text{and} \quad V_{\mathbb{A}} = \mathbb{Z}v_- \oplus \mathbb{Z}v_+,$$

respectively. For a trivial circle C, the basis element w_+ (resp. w_-) in $\mathcal{F}_{\mathbb{A}}(C)$ is the image of $1 \in \mathbb{Z}$ under the cup cobordism (resp. dotted cup cobordism). Because of the grading negation, quantum gradings are given by $\text{qdeg}(w_-) = -1$, $\text{qdeg}(w_+) = 1$. Beliakova-Putyra-Wehrli use a modified quantum grading, given as the difference between the usual quantum degree and the annular degree, which is more natural in the setting of annular link homology (see also Remark 4.2.18). Bigradings are recorded in Equation (3.1) below.

$$\begin{aligned} \text{qdeg}(v_\pm) &= 0 & \text{qdeg}(w_\pm) &= \pm 1 \\ \text{adeg}(v_\pm) &= \pm 1 & \text{adeg}(w_\pm) &= 0 \end{aligned} \tag{3.1}$$

Then for $Z \subset \mathbb{A}$ a collection of disjoint simple closed curves with e essential and t trivial circles, the free abelian group $\mathcal{F}_{\mathbb{A}}(Z)$ has a standard basis consisting of a label of v_- or v_+ on each essential circle, and w_- or w_+ on each trivial one. Following the conventions in [11], a generator of $\mathcal{F}_{\mathbb{A}}(Z)$ will be represented as a choice of counterclockwise or clockwise orientations on each essential circle, corresponding to v_+ and v_- respectively, and either a dot or no dot on each trivial circle, corresponding to w_- and w_+. This is shown in Figure 3-1. We will often switch between the diagrammatic and algebraic representations of generators.

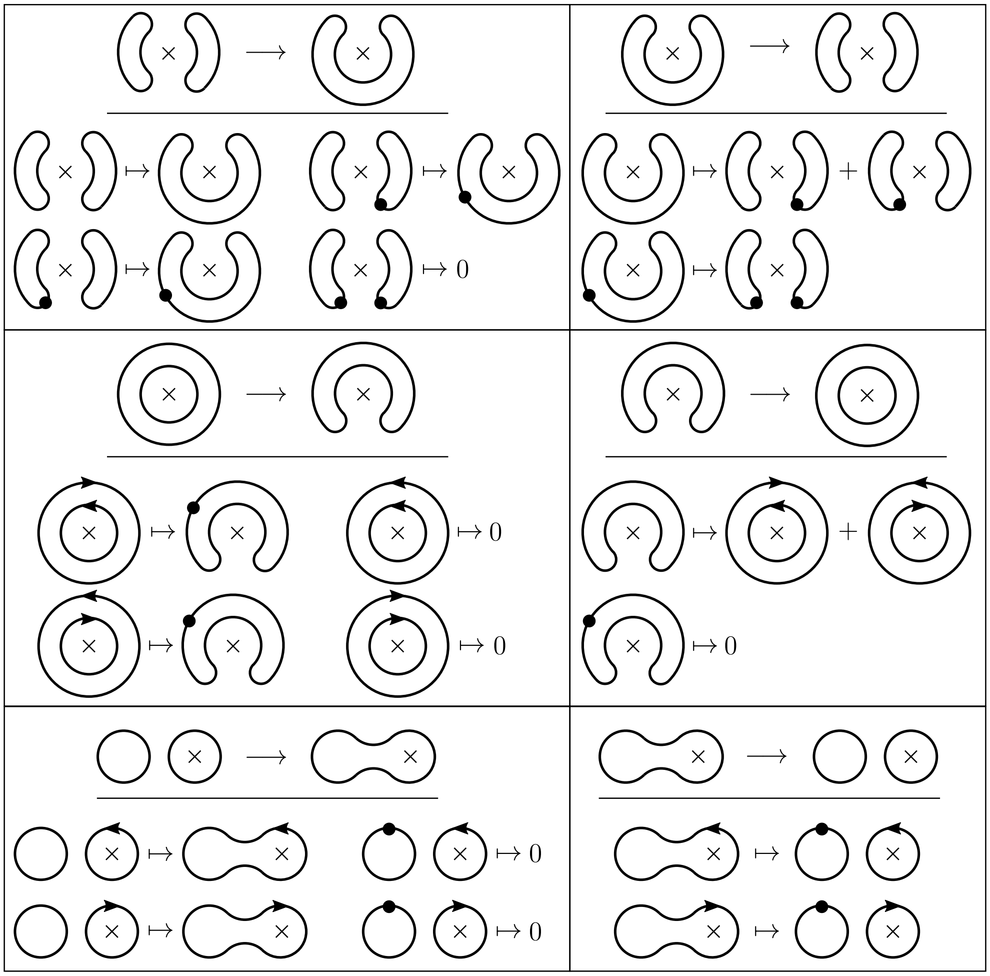


Figure 3-2: Surgery formulas in classical annular Khovanov homology. The table lists topological types of surgeries and their corresponding maps on generators.

The maps assigned to cobordisms by $\mathcal{F}_{\mathbb{A}}$ are defined by taking the adeg-preserving part, as in Section 2.4.1. Figure 3-2 records, in the diagrammatic notation, maps assigned to each of the six types of annular saddle cobordisms.

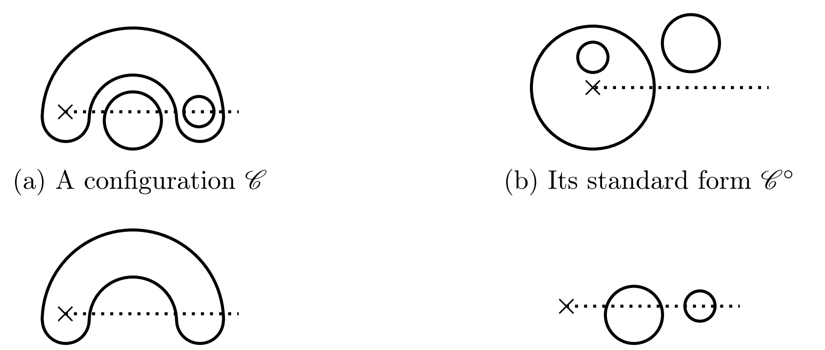
3.2 Fixing generators

Recall from Definition 2.6.15 that a key ingredient in constructing Khovanov homotopy refinements is having a fixed set of generators for the module assigned to each collection of circles. Due to the definition of the quantum annular TQFT $\mathcal{F}_{\mathbb{A}_q}$, discussed in Section 2.4.2, the situation is more complicated in quantum annular homology. Indeed, consider a trivial circle $C \subset \mathbb{A}$. In classical annular homology, the module assigned to C has a distinguished basis w_+ and w_-, given as the image of $1 \in \mathbb{Z}$ under the undotted and dotted cup cobordism from the empty diagram to C. However, in quantum annular homology, there is not necessarily a canonical choice for the dotted cup cobordism. If, for example, C intersects the seam μ twice, then there are two dot placements, related to each other by sliding the dot through the membrane, which introduces a power of $\mathfrak{q}$ (Figure 2-16). This subsection explains how to fix generators for a general configuration.

Definition 3.2.1. A configuration $\mathscr{C} \subset \mathbb{A}$ is *standard* if every component intersects the seam in at most one point. Every configuration $\mathscr{C}$ is isotopic in $\mathbb{A}$ to a standard configuration, denoted $\mathscr{C}^\circ$, which is unique up to planar isotopy of the cut-open planar tangle.

Next we explain how Theorem 2.4.13 gives a canonical choice of generators for $\mathcal{F}_{\mathbb{A}_\mathfrak{q}}(\mathscr{C})$ when $\mathscr{C}$ is standard.

For a configuration $\mathscr{C}$, we will write $\mathscr{C}_E$ to denote the essential circles in $\mathscr{C}$, and $\mathscr{C}_T$ to denote the trivial circles. Figure 3-3 illustrates these conventions.



(a) A configuration $\mathscr{C}$

(b) Its standard form $\mathscr{C}^\circ$

(c) The essential circles $\mathscr{C}_E$ in $\mathscr{C}$.

(d) The trivial circles $\mathscr{C}_T$ in $\mathscr{C}$.

Figure 3-3: A depiction of various configurations associated to a given configuration.

We will often suppress the notation $\mathcal{F}_{\mathbb{A}_\mathfrak{q}}(\mathscr{C})$ when it is clear from context; that is, $x \in \mathscr{C}$ means $x \in \mathcal{F}_{\mathbb{A}_\mathfrak{q}}(\mathscr{C})$. Likewise, for a cobordism $S : \mathscr{C} \to \mathscr{C}'$, we will often write $S(x)$ to mean $\mathcal{F}_{\mathbb{A}_\mathfrak{q}}(S)(x)$, where $\mathcal{F}_{\mathbb{A}_\mathfrak{q}}(S) : \mathcal{F}_{\mathbb{A}_\mathfrak{q}}(\mathscr{C}) \to \mathcal{F}_{\mathbb{A}_\mathfrak{q}}(\mathscr{C}')$ is the induced map.

Given a cobordism $S \subset \mathbb{A} \times I$, denote by $\overline{S}$ its reflection in the I-coordinate. For cobordisms $\mathscr{C} \xrightarrow{S_1} \mathscr{C}'$ and $\mathscr{C}' \xrightarrow{S_2} \mathscr{C}''$, we will write S_2S_1 to denote their composition.

Suppose $C \subset \mathbb{A}$ is a trivial circle intersecting μ transversely, and let $n = |C \cap \mu|$ be the number of intersection points. The circle C bounds an embedded disk $D \subset \mathbb{A}$, and we may push the interior of D down into the I coordinate to obtain a cobordism $\Sigma \subset \mathbb{A} \times I$ from the empty set to C which intersects the membrane in exactly $n/2$ arcs. We refer to Σ as the *cup cobordism on* C. Similarly, we may pull D up into the I coordinate to obtain the *cap cobordism on* C, which is simply the reflection $\overline{\Sigma}$ of Σ.

Let W denote the $\Bbbk$-module assigned by $\mathcal{F}_{\mathbb{A}_\mathsf{q}}$ to a trivial circle C which is disjoint from the seam. The module W is free of rank 2. with standard generators w_+, w_-. The standard generator w_+ (resp. w_- is the image of $1 \in \Bbbk$ under the undotted (resp. dotted) cup cobordism on C. Therefore, we will often identify $w_\pm$ with these cup cobordisms. Diagrammatically, we will signify that a trivial circle C in $\mathscr{C}$ is labelled by w_- by drawing a dot on C, as in Figure 3-1.

Suppose $\mathscr{C}$ is a standard configuration with e essential circles and t trivial circles. Order the trivial circles in some way, and order the essential circles from the innermost (closest to the puncture) to the outermost. The exact ordering of the trivial circles is irrelevant, but it is important to order the essential circles in this way in light of Theorem 2.4.13 and the asymmetry of the evaluation and coevaluation maps.

We have that

$$\mathcal{F}_{\mathbb{A}_\mathsf{q}}(\mathscr{C}) = V_\mathsf{q}^{\otimes e} \otimes W^{\otimes t}, \tag{3.2}$$

where the tensor products above are understood to be over $\Bbbk$, and the identification of the value of $\mathcal{F}_{\mathbb{A}_\mathsf{q}}$ on e standard essential circles with $V_\mathsf{q}^{\otimes e}$ is the isomorphism from (2.16). The modules V_q and W are each bigraded, carrying a quantum grading qdeg and an annular grading adeg. The degrees of generators are as in (3.1). A standard generator will be written as

$$v_{a_1} \otimes \cdots \otimes v_{a_e} \otimes w_{b_1} \otimes \cdots \otimes w_{b_t}$$

where each $a_i, b_j \in \{-, +\}$, the v_{a_i} label the essential circles, and the w_{b_j} label the trivial circles. We will often shorten the notation to $v_\mathcal{I} \otimes w_\mathcal{J}$, where $\mathcal{I}$ is a sequence of $\pm$ labelling the essential circles and $\mathcal{J}$ is a sequence of $\pm$ labelling the trivial ones. Note also that each standard generator $x = v_\mathcal{I} \otimes w_\mathcal{J} \in \mathscr{C}$ of a standard configuration $\mathscr{C}$ is the image of $v_\mathcal{I}$ under the cobordism

$$\Sigma_\mathcal{J} : \mathscr{C}_E \to \mathscr{C}$$

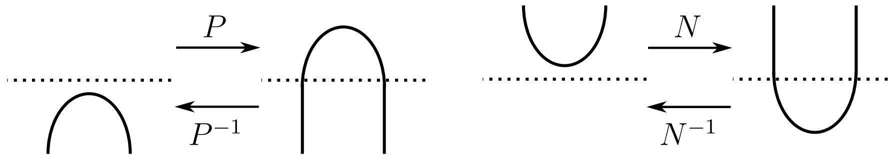


Figure 3-4: The isotopies P, P^{-1}, N, and N^{-1} in the proof of Lemma 3.2.2.

which is the identity on $\mathscr{C}_E$ and a cup cobordism on each trivial circle, with some cups possibly carrying dots as specified by the labels $\mathcal{J}$. Diagrammatically, we will use the same convention as in Figure 3-1.

The following lemma concerns general (not necessarily standard) configurations; the argument is similar to the proof of [11, Lemma 6.4].

Lemma 3.2.2. *Let $\mathscr{C}, \mathscr{C}' \subset \mathbb{A}$ be two isotopic configurations. Let ϕ be an isotopy from $\mathscr{C}$ to $\mathscr{C}'$, and denote by $S : \mathscr{C} \to \mathscr{C}'$ the cylindrical cobordism in $\mathbb{A} \times I$ formed by ϕ. Then S is an isomorphism in $\mathcal{BN}_{\mathfrak{q}}(\mathbb{A})$, with $S^{-1} = \mathfrak{q}^k \overline{S}$ for some $k \in \mathbb{Z}$.*

Proof. Isotopic cobordisms are equal in $\mathcal{BN}_{\mathfrak{q}}(\mathbb{A})$ if the isotopy between them fixes the membrane. We may therefore assume that the isotopy ϕ is a sequence of the local moves in Figure $3-4$, denoted P, P^{-1}, N, and N^{-1}.

Let p denote the number of moves of type P or P^{-1}, let n denote the number of moves of type N or N^{-1}, and set $k = n - p$. It follows from the relations in Figure 2-16 that

$$\mathfrak{q}^k \overline{S} S = \mathrm{id}_{\mathscr{C}}, \ \mathfrak{q}^k S \overline{S} = \mathrm{id}_{\mathscr{C}'} .$$

□

Lemma 3.2.3. *Let $\mathscr{C}$ be a standard configuration. Let ϕ be a component-preserving isotopy from $\mathscr{C}$ to itself, with corresponding cobordism $S : \mathscr{C} \to \mathscr{C}$. For any standard generator $x \in \mathscr{C}$, $\mathfrak{q}^k S(x) = x$ for some $k \in \mathbb{Z}$.*

Proof. As discussed above, each standard generator $x = v_{\mathcal{I}} \otimes w_{\mathcal{J}} \in \mathscr{C}$ is the image of $v_{\mathcal{I}}$ under a cobordism

$$\Sigma_{\mathcal{J}} : \mathscr{C}_E \to \mathscr{C}$$

which is the identity on $\mathscr{C}_E$ and a cup cobordism on all circles in $\mathscr{C}_T$, with dot placement on cups specified by $\mathcal{J}$. Every component of the cobordism $S\Sigma_{\mathcal{J}}$ is either an undotted annulus between essential circles or a possibly dotted disk with trivial boundary. Each disk can be isotoped to a cup cobordism on its trivial boundary circle at the cost of multiplying by a power of $\mathfrak{q}$. Then, at the cost of introducing further powers of $\mathfrak{q}$, the remaining annuli may be isotoped to the identity cobordism on $\mathscr{C}_E$ while fixing each cup cobordism. This yields $\mathfrak{q}^k S\Sigma_{\mathcal{J}} = \Sigma_{\mathcal{J}}$ for some $k \in \mathbb{Z}$, so $\mathfrak{q}^k S(x) = \mathfrak{q}^k S\Sigma_{\mathcal{J}}(v_{\mathcal{I}}) = \Sigma_{\mathcal{J}}(v_{\mathcal{I}}) = x$. □

Lemma 3.2.4. *Let $\mathscr{C}$ be a configuration and let ϕ_1, ϕ_2 two isotopies from $\mathscr{C}^\circ$ to $\mathscr{C}$ that induce the same correspondence between components. Denote the corresponding cobordisms by $S_1, S_2 : \mathscr{C}^\circ \to \mathscr{C}$. If $x \in \mathscr{C}^\circ$ is a standard generator, then $S_1(x) = \mathfrak{q}^k S_2(x)$ for some $k \in \mathbb{Z}$.*

Proof. Consider the component-preserving cobordism $\overline{S_1} S_2 : \mathscr{C}^\circ \to \mathscr{C}^\circ$, which is formed by the isotopy $\phi_1^{-1}\phi$. By Lemma 3.2.3, we have $\mathfrak{q}^m \overline{S_1} S_2(x) = x$ for some $m \in \mathbb{Z}$. By Lemma 3.2.2, we know $\mathfrak{q}^\ell \overline{S_1} S_1 = \mathrm{id}_{\mathscr{C}}$ for some $\ell \in \mathbb{Z}$. Then

$$\mathfrak{q}^\ell \overline{S_1} S_1(x) = x = \mathfrak{q}^m \overline{S_1} S_2(x),$$

and we obtain

$$S_1(x) = \mathfrak{q}^{m-\ell} S_2(x).$$

□

Remark 3.2.5. Given S_1, S_2 as in Lemma 3.2.4, in general the power k of $\mathfrak{q}$ depends on the generator $x \in \mathscr{C}^\circ$.

Lemma 3.2.4 may be interpreted as follows. Suppose $\mathscr{C}$ has t trivial and e essential circles. The cobordisms $S_1, S_2 : \mathscr{C}^\circ \to \mathscr{C}$ induce isomorphisms

$$V_q^{\otimes e} \otimes W^{\otimes t} = \mathcal{F}_{\mathbb{A}_\mathfrak{q}}(\mathscr{C}^\circ) \xrightarrow[\sim]{S_1} \mathcal{F}_{\mathbb{A}_\mathfrak{q}}(\mathscr{C}) \xleftarrow[\sim]{S_2} \mathcal{F}_{\mathbb{A}_\mathfrak{q}}(\mathscr{C}^\circ) = V_q^{\otimes e} \otimes W^{\otimes t},$$

and Lemma 3.2.4 says that the matrix of the composite automorphism $S_2^{-1} S_1$ of $V_q^{\otimes e} \otimes W^{\otimes t}$ is diagonal with entries powers of $\mathfrak{q}$.

So far the discussion concerned only generators of standard configurations. Next we consider generators for arbitrary configurations.

Definition 3.2.6. Fix a configuration $\mathscr{C}$ and an isotopy from $\mathscr{C}^\circ$ to $\mathscr{C}$. Let S denote the resulting cobordism $\mathscr{C}^\circ \to \mathscr{C}$. The *generators of* $\mathcal{F}_{\mathbb{A}_\mathfrak{q}}(\mathscr{C})$ *corresponding to the cobordism* S, are the images of the standard generators of $\mathscr{C}^\circ$ under S. We will also write generators of $\mathscr{C}$ as $v_\mathcal{I} \otimes w_\mathcal{J}$, which is to be understood as the image of the corresponding standard generator of $\mathscr{C}^\circ$. Note that this image $v_\mathcal{I} \otimes w_\mathcal{J}$ depends on the choice of isotopy $\mathscr{C}^\circ \to \mathscr{C}$, which we suppress from the notation.

By Lemma 3.2.4, these generators of $\mathscr{C}$ are well-defined up to multiplication by a (non-uniform, according to Remark 3.2.5) power of $\mathfrak{q}$, and also a possible re-ordering of the trivial circles in $\mathscr{C}^\circ$ which corresponds to a permutation of the indexing set $\mathcal{J}$. We assume throughout that there is a fixed isotopy $\mathscr{C}^\circ \to \mathscr{C}$, which will often not be named. Likewise, an unnamed cobordism $\mathscr{C} \to \mathscr{C}^\circ$ denotes the inverse of $\mathscr{C}^\circ \to \mathscr{C}$.

As discussed earlier, a standard generator $x \in \mathscr{C}^\circ$ is the image of the corresponding standard generator of $\mathscr{C}_E^\circ$ under a cobordism $\Sigma_\mathcal{J} : \mathscr{C}_E^\circ \to \mathscr{C}^\circ$, where $\Sigma_\mathcal{J}$ is the identity on $\mathscr{C}_E^\circ$ and a cup cobordism on each circle in $\mathscr{C}_T^\circ$, with some cups possibly carrying dots. Up to a power of $\mathfrak{q}$, the cobordism $S\Sigma_\mathcal{J}$ represents a cobordism which traces out an isotopy $\mathscr{C}_E^\circ \to \mathscr{C}_E$ and is a cup cobordism on each circle in $\mathscr{C}_T$. Then each standard generator of $\mathscr{C}$ can also be realized as the image of a cup cobordism on each trivial circle and an isotopy on the essential circles.

Note also that we do not make any assumptions about how these isotopies are picked for different configurations within a single cube of resolutions.

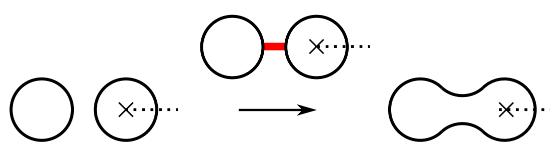
3.2.1 Computations of saddle maps

In this subsection we compute saddle maps in quantum annular homology using the relations in $\mathcal{BBN}_\mathfrak{q}(\mathbb{A})$ and Theorem 2.4.13. These results will be used in the formulation of the quantum annular Burnside functor in Section 3.5.

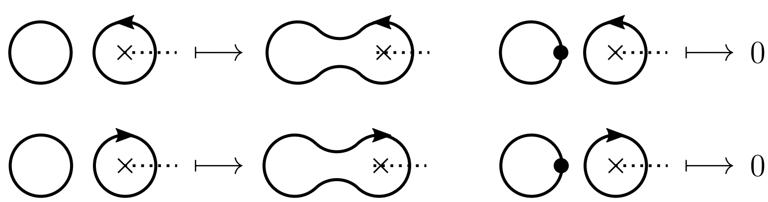
We start with several examples; the general case is treated in Proposition 3.2.13. Saddle maps for various types of configurations (where intersections with the seam are minimal) are summarized in Figure 3-5. In the first two examples the calculation relies on the Boerner relation, Figure 2-14, and relations satisfied by cobordisms in the Bar-Natan category, Figure 2-7. Specifically, one uses delooping (see Proposition 2.3.12) and the neck-cutting relation. See also the proof of [11, Proposition 5.3]. Note that delooping makes sense only for trivial circles in the annulus.

To analyze our saddle maps, we will use the language of *surgery arcs*, as in [57, Section 2]. For a configuration $\mathscr{C}$, a surgery arc is an interval embedded in $\mathbb{A}$ whose endpoints lie on $\mathscr{C}$ and whose interior is disjoint from $\mathscr{C}$. In the construction of quantum annular homology, link diagrams are assumed transverse to μ, and all crossings are away from μ. We may then assume that surgery arcs are disjoint from the seam. For a configuration $\mathscr{C}$ with a surgery arc, let $s(\mathscr{C})$ denote the configuration obtained by surgery on the arc. There is a saddle cobordism $\mathscr{C} \to s(\mathscr{C})$, which is well-defined in $\mathcal{BN}_{\mathsf{q}}(\mathbb{A})$. In terms of the cube of resolutions of a link diagram, a surgery arc may be placed at a 0-smoothing to indicate that there will be a saddle cobordism at that smoothing.

Example 3.2.7. For the saddle



between standard configurations, we have the following formulas



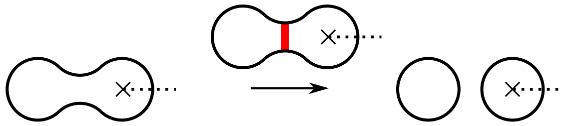
Algebraically, this is written as

$$v_+ \otimes w_+ \mapsto v_+ \qquad\qquad v_+ \otimes w_- \mapsto 0$$

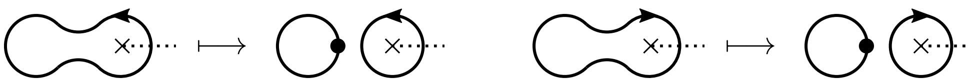
$$v_- \otimes w_+ \mapsto v_- \qquad\qquad v_- \otimes w_- \mapsto 0$$

These can be deduced from Boerner's relation (Figure 2-14) and the fact that the two standard generators of a trivial circle are picked out by an undotted cup and a once dotted cup.

Example 3.2.8. For the saddle



we have the formulas



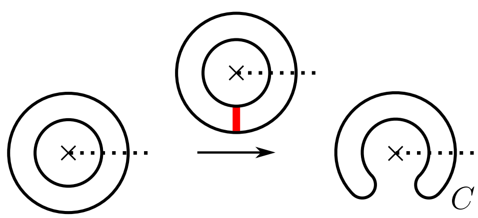
which, algebraically, can be written as

$$v_+ \mapsto v_+ \otimes w_- \qquad\qquad v_- \mapsto v_- \otimes w_-$$

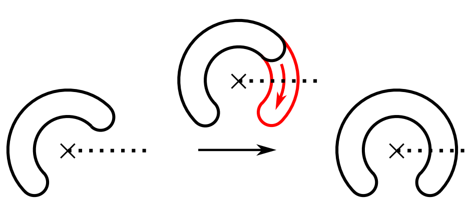
This can be deduced by cutting the neck along the trivial circle which splits off as a result of the saddle, and then applying Boerner's relation.

The next two examples are also discussed in [11, Section 6.4].

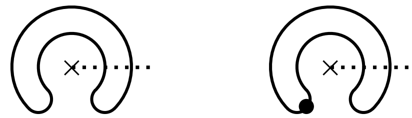
Example 3.2.9. Let S denote the following saddle



Let C denote the trivial circle on the right-hand side above. Since C is not standard, we need to pick an isotopy to specify its generators. For the sake of calculation, we pick the following isotopy



which specifies generators, represented diagrammatically, as



w_+ $\qquad$ w_-

These generators are the images of $1 \in \Bbbk$ under the undotted and dotted cup cobordisms on C, respectively. Since the cup cobordism on C intersects the membrane, the placement of the dot is relevant, and the diagram shows where the dot is placed.

Now, let Σ denote the undotted cap cobordism on C, and let Σ' denote the dotted cap cobordism on C, with the dot placed as in the generator w_-. Using the relations in $\mathcal{BN}_{\mathfrak{q}}(\mathbb{A})$, we obtain

$$\Sigma(w_+) = 0 \qquad\qquad \Sigma(w_-) = \mathfrak{q}^{-1}$$

$$\Sigma'(w_+) = \mathfrak{q}^{-1} \qquad\qquad \Sigma'(w_-) = 0$$

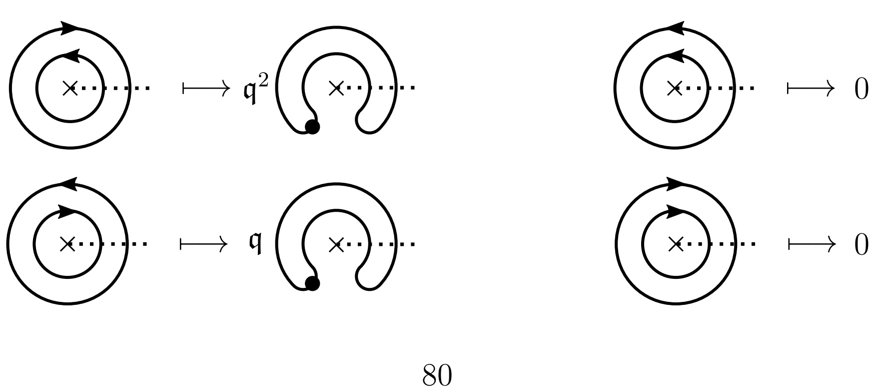
Composing with the saddle S, observe that $\Sigma S = S^1 \times \cap$ in $\mathcal{BBN}_{\mathfrak{q}}(\mathbb{A})$, and that $\Sigma' S = 0$ by Boerner's relation. We can now write down formulas for S. For example, we may write

$$S(v_+ \otimes v_-) = \alpha w_+ + \beta w_- \tag{3.3}$$

for some $\alpha, \beta \in \Bbbk$. Applying Σ to the above equality, we obtain

$$\Sigma S(v_+ \otimes v_-) = \mathfrak{q}^{-1}\beta.$$

Theorem 2.4.13 implies that $\Sigma S(v_+ \otimes v_-) = ev(v_+ \otimes v_-) = \mathfrak{q}$, so $\beta = \mathfrak{q}^2$. By applying Σ' to both sides of (3.3), we obtain $\alpha = 0$. A similar argument for the other generators yields the full table of formulas for S:

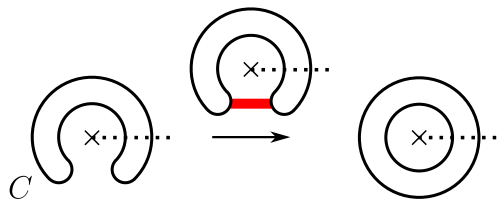


Equivalently,

$$S(v_+ \otimes v_-) = \mathfrak{q}^2 w_- \qquad S(v_+ \otimes v_+) = 0$$

$$S(v_- \otimes v_+) = \mathfrak{q} w_- \qquad S(v_- \otimes v_-) = 0$$

Example 3.2.10. Let S denote the following saddle.



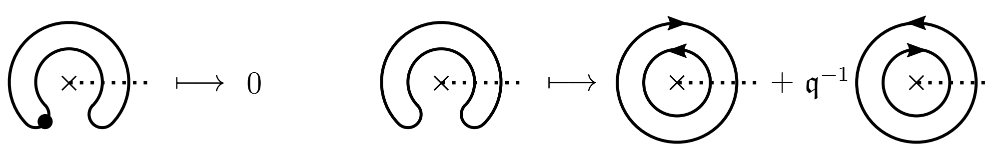
Pick generators w_+ and w_- for the left-hand trivial circle C as in Example 3.2.9. Let Σ and Σ' be the undotted and dotted cups on C, so that $w_+ = \Sigma(1)$ and $w_- = \Sigma'(1)$. Observe that $S\Sigma' = 0$ by Boerner's relation, so

$$S(w_-) = 0$$

Finally, note that $S\Sigma = S^1 \times \cup$. By Theorem 2.4.13, we obtain

$$S(w_+) = v_+ \otimes v_- + \mathfrak{q}^{-1} v_- \otimes v_+.$$

Diagrammatically, the formulas for S are



Saddle maps for various topological types of configurations, including the result of calculations in examples 3.2.7 - 3.2.10, are summarized in Figure 3-5. The next example details the calculation of a saddle map in which the configurations intersect the seam in four points.

Example 3.2.11. Here is a slightly more involved version of Example 3.2.9. Consider the configurations $\mathscr{C}_1$ and $\mathscr{C}_2$, and the saddle $S : \mathscr{C}_1 \to \mathscr{C}_2$ as shown in Figure 3-6.

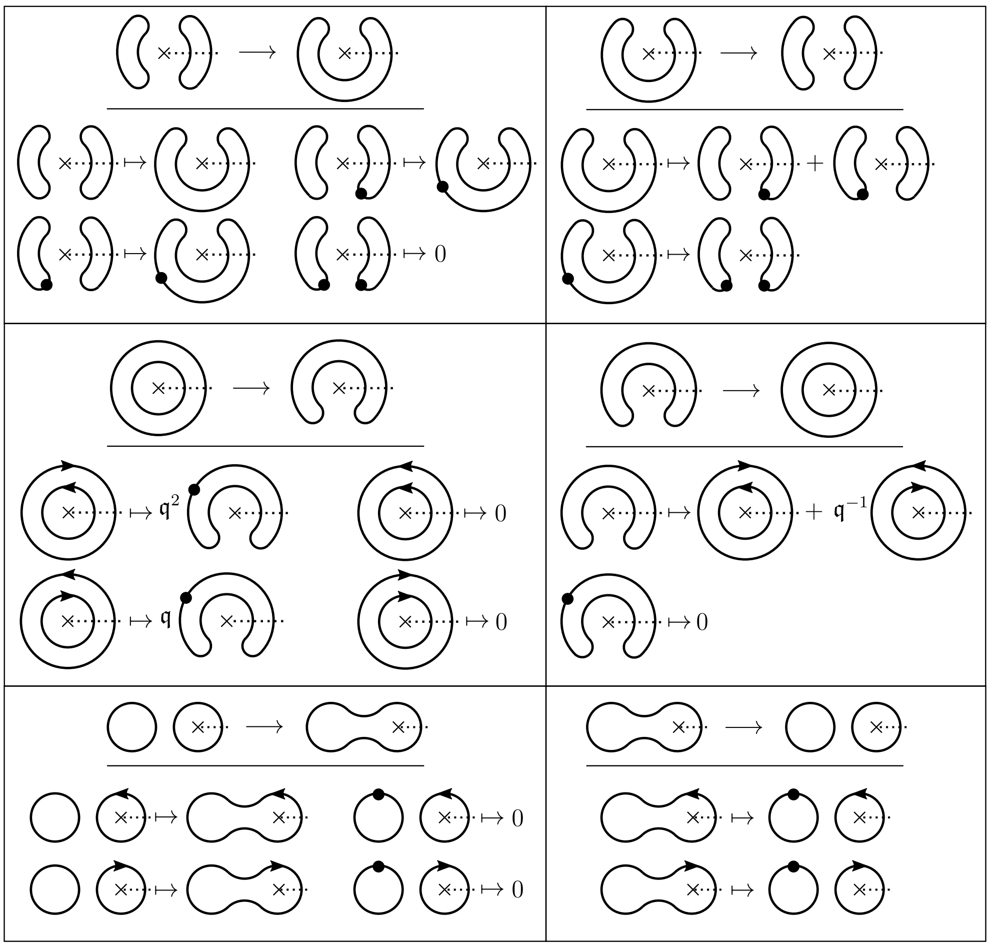


Figure 3-5: Surgery formulas in quantum annular Khovanov homology for curves with minimal intersections with the seam.

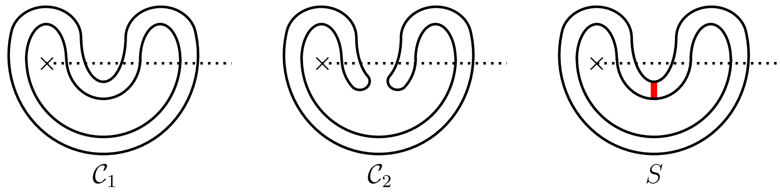


Figure 3-6

Fix generators for $\mathscr{C}_1$ and $\mathscr{C}_2$ using the isotopies S_1 and S_2 depicted in Figure 3-7 and Figure 3-8.

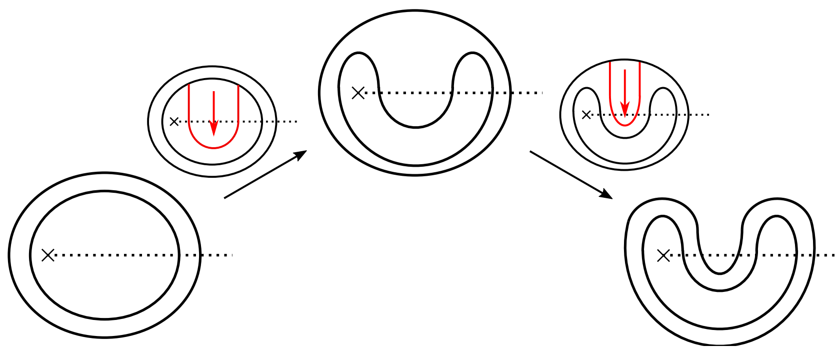


Figure 3-7: An isotopy $S_1 : \mathscr{C}_1^\circ \to \mathscr{C}_1$.

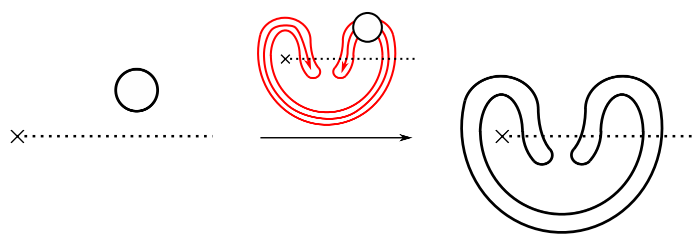


Figure 3-8: An isotopy $S_2 : \mathscr{C}_2^\circ \to \mathscr{C}_2$.

Since generators of $\mathscr{C}_2$ are the images of the standard generators of $\mathscr{C}_2^\circ$ under the isomorphism $S_2 : \mathscr{C}_2^\circ \to \mathscr{C}_2$, it suffices to write down formulas for the composition

$$\mathscr{C}_1^\circ \xrightarrow{S_1} \mathscr{C}_1 \xrightarrow{S} \mathscr{C}_2 \xrightarrow{S_2^{-1}} \mathscr{C}_2^\circ \tag{3.4}$$

Note that $S_2^{-1} = \mathfrak{q}^3\overline{S_2}$ (see Lemma 3.2.2). Let Φ denote the composition (3.4).

Let Σ and Σ' be the undotted and dotted cap cobordisms, respectively, on the trivial circle $\mathscr{C}_2^\circ$. Note that $\Sigma'\Phi = 0$ by Boerner's relation. Applying a trace move from Figure 2-16 to the part of the cobordism depicted in (3.5), we see that

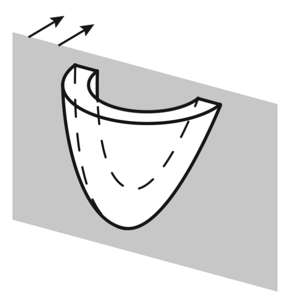
$$\Sigma\overline{S_2}SS_1$$

is equal to $\mathfrak{q}^{-1}(S^1 \times \cap)$, so that

$$\Sigma\Phi = \mathfrak{q}^2(S^1 \times \cap)$$

Arguing as in Example 3.2.9, we obtain

$$S(v_+ \otimes v_-) = \mathfrak{q}^3 w_- \qquad S(v_+ \otimes v_+) = 0$$
$$S(v_- \otimes v_+) = \mathfrak{q}^2 w_- \qquad S(v_- \otimes v_-) = 0$$



(3.5)

Remark 3.2.12. Here is a slightly different way to finish the computation in Example 3.2.11, which will be used in Proposition 3.2.13. In the morphism $\Phi : \mathscr{C}_1^\circ \to \mathscr{C}_2^\circ$, we may cut the neck along a small push-off of the trivial circle $\mathscr{C}_2^\circ$ to write Φ as a sum of two dotted cobordisms. One of the summands is 0 by Boerner's relation, and the other is isotopic to a disjoint union of

$$S^1 \times \cap$$

and a dotted cup cobordism on $\mathscr{C}_2^\circ$, which is denoted $\overline{\Sigma'}$ using the notation of Example 3.2.11. Then, using the trace relations in $\mathcal{BN}_\mathfrak{q}(\mathbb{A})$, we see that

$$\Phi = \mathfrak{q}^2(S^1 \times \cap) \sqcup \overline{\Sigma'},$$

and the formulas in Example 3.2.11 follow.

Example 3.2.11 shows that there is considerable complexity in computing the saddle map when curves have multiple intersections with the seam. The next proposition extends Examples 3.2.9 - 3.2.11 to the case of arbitrary configurations. It will be important for the analysis in Section 3.2.2. Recall the numbering of circles discussed in the paragraph preceding (3.2).

Proposition 3.2.13. *Let $\mathscr{C}$ be a configuration with a surgery arc A. Let $S : \mathscr{C} \to s(\mathscr{C})$ denote the saddle.*

(1) *Suppose both endpoints of A are on a trivial circle C, and that surgery along A splits C into two essential circles. Assume C is first in the ordering on trivial circles of $\mathscr{C}$, and it splits into the i-th and $(i+1)$-th essential circles in $s(\mathscr{C})$. Let $x = v_{\mathcal{I}'} \otimes v_{\mathcal{I}''} \otimes w_+ \otimes w_{\mathcal{J}}$ be a generator of $\mathscr{C}$ in which C is undotted, where $\mathcal{I}'$ labels the first $i-1$ essential circles. Then*

$$S(x) = \mathfrak{q}^a v_{\mathcal{I}'} \otimes v_+ \otimes v_- \otimes v_{\mathcal{I}''} \otimes w_{\mathcal{J}} + \mathfrak{q}^{a-1} v_{\mathcal{I}'} \otimes v_- \otimes v_+ \otimes v_{\mathcal{I}''} \otimes w_{\mathcal{J}}$$

for some $a \in \mathbb{Z}$.

(2) *Suppose the endpoints of A are on the i-th and $(i+1)$-th essential circles of $\mathscr{C}$. Consider the generators*

$$y_1 = v_{\mathcal{I}'} \otimes v_+ \otimes v_- \otimes v_{\mathcal{I}''} \otimes w_{\mathcal{J}}$$
$$y_2 = v_{\mathcal{I}'} \otimes v_- \otimes v_+ \otimes v_{\mathcal{I}''} \otimes w_{\mathcal{J}}$$

of $\mathscr{C}$, where $\mathcal{I}'$ labels the first $i-1$ essential circles. Then

$$S(y_1) = \mathfrak{q}^{b+1} v_{\mathcal{I}'} \otimes v_{\mathcal{I}''} \otimes w_- \otimes w_{\mathcal{J}}$$
$$S(y_2) = \mathfrak{q}^{b} v_{\mathcal{I}'} \otimes v_{\mathcal{I}''} \otimes w_- \otimes w_{\mathcal{J}}$$

for some $b \in \mathbb{Z}$.

Proof. For both (1) and (2), it is enough to show that the result holds after applying $s(\mathscr{C}) \to s(\mathscr{C})^\circ$.

(1). The generator x is the image of $v_{\mathcal{I}'} \otimes v_{\mathcal{I}''}$ under the composition $\mathscr{C}_E^\circ \xrightarrow{\Sigma_{\mathcal{J}}} \mathscr{C}^\circ \to \mathscr{C}$, where $\Sigma_{\mathcal{J}}$ is a cup cobordism on trivial circles in $\mathscr{C}^\circ$, with the cups dotted according to $\mathcal{J}$. Note that the cup on C is undotted. Let Ψ denote the composition

$$\mathscr{C}_E^\circ \xrightarrow{\Sigma_{\mathcal{J}}} \mathscr{C}^\circ \to \mathscr{C} \xrightarrow{S} s(\mathscr{C}) \to s(\mathscr{C})^\circ$$

The cobordism Ψ is isotopic to a disjoint union of

$$S^1 \times \big|\cdots\big| \bigcup \big|\cdots\big| \tag{3.6}$$

and cup cobordisms on each trivial circle in $s(\mathscr{C})^\circ$, with dots placed according to $\mathcal{J}$. By Theorem 2.4.13, the cobordism (3.6) induces the map

$$v_{\mathcal{I}'} \otimes v_{\mathcal{I}''} \mapsto v_{\mathcal{I}'} \otimes v_+ \otimes v_- \otimes v_{\mathcal{I}''} + \mathfrak{q}^{-1} v_{\mathcal{I}'} \otimes v_- \otimes v_+ \otimes v_{\mathcal{I}''}$$

and the result follows.

(2). The generators y_1 and y_2 are the images of $v_{\mathcal{I}'} \otimes v_+ \otimes v_- \otimes v_{\mathcal{I}''}$ and $v_{\mathcal{I}'} \otimes v_- \otimes v_+ \otimes v_{\mathcal{I}''}$, respectively, under the cobordism

$$\mathscr{C}_E^\circ \xrightarrow{\Sigma_{\mathcal{J}}} \mathscr{C}^\circ \to \mathscr{C}$$

where $\Sigma_{\mathcal{J}}$ is a cup cobordism on trivial circles with dots placed according to $\mathcal{J}$. Let Φ denote the composition

$$\mathscr{C}_E^\circ \xrightarrow{\Sigma_{\mathcal{J}}} \mathscr{C}^\circ \to \mathscr{C} \xrightarrow{S} s(\mathscr{C}) \to s(\mathscr{C})^\circ$$

Let $C \subset s(\mathscr{C})$ denote the trivial circle obtained by surgery along A, and let $C' \subset s(\mathscr{C})^\circ$ denote the corresponding circle. In the cobordism Φ, we may cut the neck along C' to write Φ as a sum of two cobordisms. One of the summands is 0 by Boerner's relation, and the other is isotopic to the disjoint union of

$$S^1 \times \big|\cdots\big| \bigcap \big|\cdots\big| \tag{3.7}$$

and cup cobordisms on each trivial circle. Note also that the the cup cobordism on C', resulting from neck-cutting, is dotted. Finally, Theorem 2.4.13 implies that the cobordism (3.7) induces the map

$$v_{\mathcal{I}'} \otimes v_+ \otimes v_- \otimes v_{\mathcal{I}''} \mapsto \mathfrak{q} v_{\mathcal{I}'} \otimes v_{\mathcal{I}''}$$
$$v_{\mathcal{I}'} \otimes v_- \otimes v_+ \otimes v_{\mathcal{I}''} \mapsto v_{\mathcal{I}'} \otimes v_{\mathcal{I}''},$$

and the desired result follows. □

We end this subsection with a discussion about recovering classical annular homology. Consider the map $\Bbbk \to \mathbb{Z}$ which is the identity on $\mathbb{Z} \subset \Bbbk$ and sends $\mathfrak{q}$ to 1. It induces a functor $(-) \otimes_{\Bbbk} \mathbb{Z} : \Bbbk-\mathrm{gmod} \to \mathbb{Z}-\mathrm{gmod}$. Thus one can consider the composition

$$\mathcal{BBN}_{\mathfrak{q}}(\mathbb{A}) \xrightarrow{\mathcal{F}_{\mathbb{A}_{\mathfrak{q}}}} \Bbbk-\mathrm{gmod} \to \mathbb{Z}-\mathrm{gmod}$$

which we denote $\mathcal{F}_{\mathbb{A}_{\mathfrak{q}}} \otimes_{\Bbbk} \mathbb{Z}$. Tensoring with $\mathbb{Z}$ forgets the action of $\mathfrak{q}$, in the sense that isotopic cobordisms induce equal maps under $\mathcal{F}_{\mathbb{A}_{\mathfrak{q}}} \otimes_{\Bbbk} \mathbb{Z}$ even when the isotopy does not fix the membrane. Let $\mathscr{C}$ be a configuration with e essential and t trivial circles. By Lemma 3.2.4, there is a canonical isomorphism

$$\mathcal{F}_{\mathbb{A}_{\mathfrak{q}}}(\mathscr{C}) \otimes_{\Bbbk} \mathbb{Z} \cong V_{\mathbb{A}}^{\otimes e} \otimes_{\mathbb{Z}} (\mathbb{Z}w_- \oplus \mathbb{Z}w_+)^{\otimes t} = \mathcal{F}_{\mathbb{A}}(\mathscr{C})$$

obtained by picking any isotopy $\mathscr{C}^\circ \to \mathscr{C}$. It is implicit in [11] that $\mathcal{F}_{\mathbb{A}_{\mathfrak{q}}} \otimes_{\Bbbk} \mathbb{Z}$ is the classical annular functor $\mathcal{F}_{\mathbb{A}}$; indeed this is straightforward to verify by using the relations in $\mathcal{BBN}_{\mathfrak{q}}(\mathbb{A})$ and Theorem 2.4.13, as in the proof of Proposition 3.2.13.

Lemma 3.2.14. *Let $\mathscr{C}$ be a configuration with a single surgery arc, and let $S : \mathscr{C} \to s(\mathscr{C})$ denote the saddle. Let $x \in \mathscr{C}$ be a generator. Then*

$$S(x) = \sum \varepsilon_y y$$

where the sum is over generators of $s(\mathscr{C})$ and each ε_y is either 0 or a power of $\mathfrak{q}$. Moreover, $\varepsilon_y \neq 0$ if and only if y appears in $\mathcal{F}_{\mathbb{A}}(S)(x)$, where $\mathcal{F}_{\mathbb{A}}$ is the classical (unquantized) annular

Figure 3-9: The three ladybug configurations in the annulus.

TQFT.

Proof. There are six types of saddles to check, corresponding to merges and splits between various combinations of essential and trivial circles as in Figure 3-2. The first part of the lemma was verified for two of these types of saddles in Proposition 3.2.13. It is straightforward to verify the lemma for the other four using similar arguments. The second statement follows from the discussion preceding the lemma. □

3.2.2 Ladybug configurations

In this subsection, we analyze the *ladybug configuration*, which was recalled in Definition 2.6.15 (see in particular Figure 2-29). Examining ladybug configurations is crucial in the construction of Khovanov homotopy types in various contexts. We start with a discussion of ladybug configurations in classical annular homology. We will then examine a particular type of ladybug configuration in quantum annular homology, and we will indicate how the analysis differs from that in classical annular homology.

Let us briefly recall the notion of a ladybug configuration. A circle $C \subset \mathbb{A}$ with two surgery arcs forms a ladybug configuration if the endpoints of the two arcs alternate around C. We will say a configuration $\mathscr{C}$ with surgery arcs has a ladybug configuration if a circle C in $\mathscr{C}$ and two of the surgery arcs forms a ladybug configuration.

First, consider ladybug configurations in classical annular homology. Let $C \subset \mathbb{A}$ be a circle carrying two surgery arcs A_1 and A_2 which form a ladybug configuration. Figure 3-9 illustrates the three possibilities in the annulus.

For $i = 1, 2$, denote by $\mathscr{C}_i$ the configuration obtained by performing surgery on C along A_i, and let $d_i : \mathcal{F}_{\mathbb{A}}(C) \to \mathcal{F}_{\mathbb{A}}(\mathscr{C}_i)$ denote the maps assigned to the saddles in classical annular Khovanov homology. Let $\mathscr{C}'$ denote the final configuration, obtained by performing surgery on C along both A_1 and A_2. When C is essential, as in Figure 3-9a, the formulas in Figure 3-2 show that composing two saddle maps yields 0. Now consider the cases where C is trivial, as in Figure 3-9b and Figure 3-9c. The dotted generator w_- is sent to 0 by the composition of two saddle maps. On the other hand, the two summands appearing in each of $d_1(w_+)$ and $d_2(w_+)$ are mapped to the same element in the final configuration $\mathscr{C}'$. The case of Figure 3-9c is illustrated in Figure 3-10.

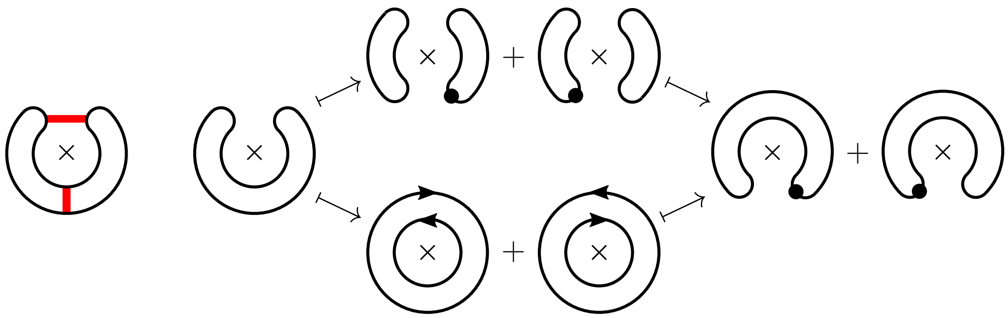


Figure 3-10

When constructing stable homotopy refinements framework, it is crucial to have a bijection between these intermediate generators which appear in $d_1(w_+)$ and $d_2(w_+)$, such that these bijections are coherent, in the sense of Lemma 2.6.9. The bijections are the *ladybug matchings* defined in [57, Section 5.4] and recalled in Definition 2.6.15 in the non-annular setting.

In the case of Figure 3-9b, the annulus and seam play no role, and the usual ladybug matching can be used without significant alteration for both classical and quantum annular homology. The remainder of this subsection examines the ladybug configuration of the type in Figure 3-9c in quantum annular homology.

Let $\mathscr{C}$ be a configuration with two surgery arcs A_T and A_E, both having endpoints on a trivial circle C in $\mathscr{C}$, such that surgery along A_T splits C into two trivial circles and surgery along A_E splits C into two essential circles, cf. Figure 3-11. Let $s_T(\mathscr{C})$ and $s_E(\mathscr{C})$ denote the configurations obtained by surgery along A_T and A_E respectively. Let $\mathscr{C}'$ denote the final configuration, obtained by surgery along both arcs. Let $C' \in \mathscr{C}'$ denote the circle obtained

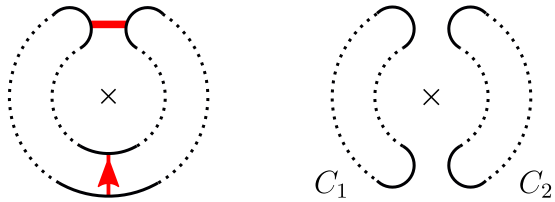


Figure 3-12: The ordering convention on C_1 and C_2.

by both surgeries. We have the commutative square.

$$\begin{tikzcd}
 & s_T(\mathscr{C}) \arrow[dr, "m_T"] & \\
\mathscr{C} \arrow[ur, "\Delta_T"] \arrow[dr, "\Delta_E"'] & & \mathscr{C}' \\
 & s_E(\mathscr{C}) \arrow[ur, "m_E"'] &
\end{tikzcd} \tag{3.8}$$

Figure 3 − 11 exhibits the specific instance of this set-up corresponding directly to the configuration in Figure 3-9c, but there are many such cases depending on how C intersects the seam. As usual, we will not distinguish between cobordisms and their induced maps.

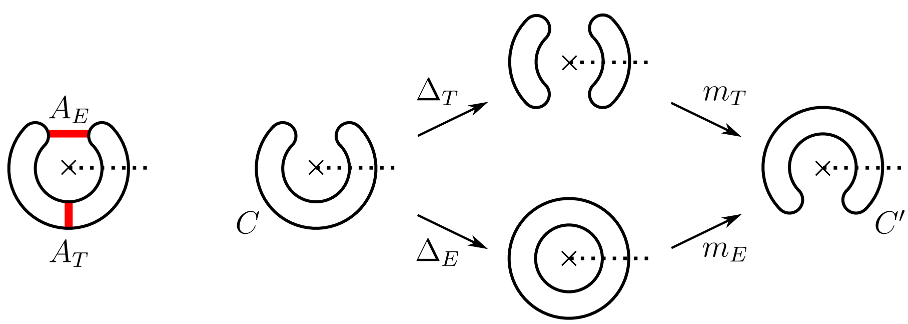


Figure 3-11

Recall that, for computational purposes, trivial circles in configurations are ordered in some way. We assume that C occurs first in the ordering on trivial circles in $\mathscr{C}$. Surgery along A_T splits C into two trivial circles in $s_T(\mathscr{C})$, which we assume are the first two trivial circles in $s_T(\mathscr{C})$. Finally, we order these first two circles as follows. Orient the arc A_T such that it points from the outer essential circle in $s_E(\mathscr{C})$ to the inner one. Declare that the first circle C_1 is to the left of A_T and the second C_2 is to the right of A_T. This ordering convention is illustrated in Figure 3-12.

Let $x = v_{\mathcal{I}} \otimes w_+ \otimes w_{\mathcal{J}} \in \mathscr{C}$ be a generator in which C is undotted. By Lemma 3.2.14, we obtain

$$\Delta_T(x) = \mathfrak{q}^k v_{\mathcal{I}} \otimes w_- \otimes w_+ \otimes w_{\mathcal{J}} + \mathfrak{q}^\ell v_{\mathcal{I}} \otimes w_+ \otimes w_- \otimes w_{\mathcal{J}}$$

for some $k, \ell \in \mathbb{Z}$.

The following corollary implies that, in the quantum annular setting, there is no need for a ladybug matching for this ladybug configuration because intermediate generators are mapped to different elements in the final configuration.

Corollary 3.2.15. *With the notation established above,*

$$m_E(\Delta_E(x)) = \mathfrak{q}^m v_{\mathcal{I}} \otimes w_- \otimes w_{\mathcal{J}} + \mathfrak{q}^{m+2} v_{\mathcal{I}} \otimes w_- \otimes w_{\mathcal{J}},$$

for some $m \in \mathbb{Z}$. Moreover, one of $m_T(\mathfrak{q}^k v_{\mathcal{I}} \otimes w_- \otimes w_+ \otimes w_{\mathcal{J}})$ or $m_T(\mathfrak{q}^\ell v_{\mathcal{I}} \otimes w_+ \otimes w_- \otimes w_{\mathcal{J}})$ is equal to $\mathfrak{q}^m v_{\mathcal{I}} \otimes w_- \otimes w_{\mathcal{J}}$, and the other is equal to $\mathfrak{q}^{m+2} v_{\mathcal{I}} \otimes w_- \otimes w_{\mathcal{J}}$.

Proof. If whole configuration $\mathcal{C}$ consists of just the circle C intersecting the seam in two points as in Figure 3-11, then the first statement can be readily checked using the formulas in Figure 3-5. (Also see figure 3-15 below.) In full generality the first statement follows from Proposition 3.2.13. The second statement is a direct consequence of the commutativity of the square (3.8). □

Remark 3.2.16. Note that generators for each configuration depend, up to a power of $\mathfrak{q}$, on a choice of a cobordism: see Definition 3.2.6 and discussion following it. The exponents k, ℓ, and m of $\mathfrak{q}$ above are determined by the cobordisms chosen for the different configurations.

It will be crucial for Section 3.8 to know which of $m_T(\mathfrak{q}^k v_{\mathcal{I}} \otimes w_- \otimes w_+ \otimes w_{\mathcal{J}})$ or $m_T(\mathfrak{q}^\ell v_{\mathcal{I}} \otimes w_+ \otimes w_- \otimes w_{\mathcal{J}})$ is equal to $\mathfrak{q}^m v_{\mathcal{I}} \otimes w_- \otimes w_{\mathcal{J}}$. This is addressed in the following proposition.

Proposition 3.2.17. *With the above notation,*

$$m_T(\mathfrak{q}^k v_{\mathcal{I}} \otimes w_- \otimes w_+ \otimes w_{\mathcal{J}}) = \mathfrak{q}^m v_{\mathcal{I}} \otimes w_- \otimes w_{\mathcal{J}}$$
$$m_T(\mathfrak{q}^\ell v_{\mathcal{I}} \otimes w_+ \otimes w_- \otimes w_{\mathcal{J}}) = \mathfrak{q}^{m+2} v_{\mathcal{I}} \otimes w_- \otimes w_{\mathcal{J}}$$

Proof. The generator $x = v_{\mathcal{I}} \otimes w_+ \otimes w_{\mathcal{J}}$ is the image of $v_{\mathcal{I}}$ under $\mathscr{C}_E^\circ \xrightarrow{\Sigma_{\mathcal{J}}} \mathscr{C}^\circ \xrightarrow{S} \mathscr{C}$. Here $\Sigma_{\mathcal{J}}$ is the usual disjoint union of the identity cobordism on the essential part together with an undotted cup corresponding to w_+ on C and various other cups dotted according to $\mathcal{J}$, while S is some chosen cobordism from the standard $\mathscr{C}^\circ$ to $\mathscr{C}$, used to fix generators. The resulting disk in $S\Sigma_{\mathcal{J}}$ bounding C may be isotoped to a cup cobordism on C, yielding a new cobordism $\Sigma' : \mathscr{C}_E^\circ \to \mathscr{C}$, so that $S\Sigma_{\mathcal{J}} = \mathfrak{q}^c \Sigma'$ for some $c \in \mathbb{Z}$.

The trivial circle C bounds a disk D in the annulus. Note that A_T lies inside D, so we may push it into the cup cobordism on C in Σ' to obtain an arc A. We may also pull A_T onto the saddle Δ_T to obtain another arc A'. Performing neck-cutting on the circle $A \cup A'$ on $S\Sigma'$ yields

$$\Delta_T \Sigma' = S_1 + S_2$$

where S_1 and S_2 are labelled such that S_1 is dotted on C_1 and S_2 is dotted on C_2. Figure 3-13 shows the local picture near A_T; the surgery arc A_T is decorated by an arrowhead.

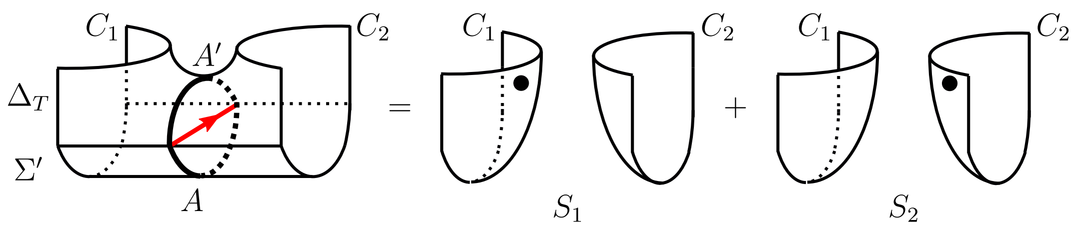


Figure 3-13

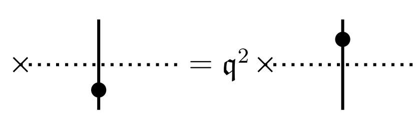
This yields

$$\Delta_T(x) = \Delta_T S\Sigma_{\mathcal{J}}(v_{\mathcal{I}}) = \mathfrak{q}^c \Delta_T \Sigma'(v_{\mathcal{I}}) = \mathfrak{q}^c S_1(v_{\mathcal{I}}) + \mathfrak{q}^c S_2(v_{\mathcal{I}}),$$

and it follows that

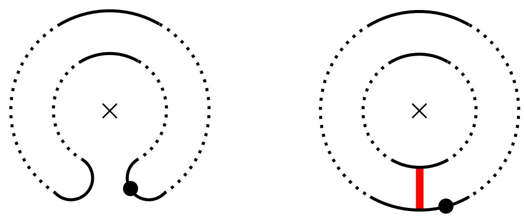
$$\mathfrak{q}^c S_1(v_{\mathcal{I}}) = \mathfrak{q}^k v_{\mathcal{I}} \otimes w_- \otimes w_+ \otimes w_{\mathcal{J}}$$
$$\mathfrak{q}^c S_2(v_{\mathcal{I}}) = \mathfrak{q}^\ell v_{\mathcal{I}} \otimes w_+ \otimes w_- \otimes w_{\mathcal{J}}.$$

The relation[2]



[2] This is a planar depiction of the dot-sliding relation in Figure 2-16.

implies that $\mathfrak{q}^d m_T S_1 = m_T S_2$ for some $d \in \mathbb{Z}$. To compute $\mathfrak{q}^d$, we need to move the dot on S_2 along the circle C' until it is in the same position as the dot on S_1, and count (with sign) the number of times the dot intersects the membrane during this process. This signed count is the same as the signed intersection between the seam and one of the essential circles in $s_E(\mathscr{C})$ obtained by surgery on C. The situation is depicted below in (3.9); the dot on the right diagram needs to be moved along the circle to the other side of the surgery arc, without intersecting the surgery arc in the process.



(3.9)

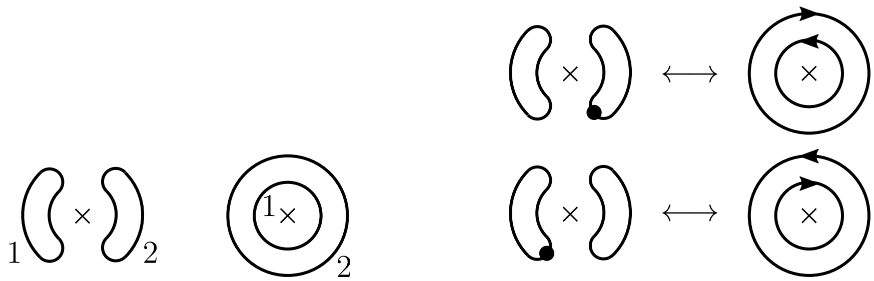
Our convention of ordering C_1 and C_2 (see Figure 3-12) guarantees that the dot is moved counter-clockwise along an essential circle in $s_E(\mathscr{C})$, so that $m_T S_2 = \mathfrak{q}^2 m_T S_1$. Therefore

$$\mathfrak{q}^2 m_T(\mathfrak{q}^k v_{\mathcal{I}} \otimes w_- \otimes w_+ \otimes w_{\mathcal{J}}) = m_T(\mathfrak{q}^\ell v_{\mathcal{I}} \otimes w_+ \otimes w_- \otimes w_{\mathcal{J}}),$$

and the statement of the proposition follows. $\square$

Recall from Definition 2.6.15 that there is always a global "right" or "left" choice which is made at the very beginning of defining the ladybug matching (see Figure 2-29). Consider the classical annular differential for the ladybug configuration of Figure 3-10. The ladybug matching made with the left choice identifies the circles in the middle smoothings as shown in Figure 3-14a. Then the ladybug matching pairs up the intermediate generators appearing in $\Delta_T(w_+)$ and $\Delta_E(w_+)$ as shown in Figure 3-14b.

Now consider the quantum annular surgery formulas for the same configuration (Figure 3-11) which are detailed in Figure 3-15.



(a)

(b)

Figure 3-14: The ladybug matching in classical annular homology for the configuration in Figure 3-10, made with the *left* pair.

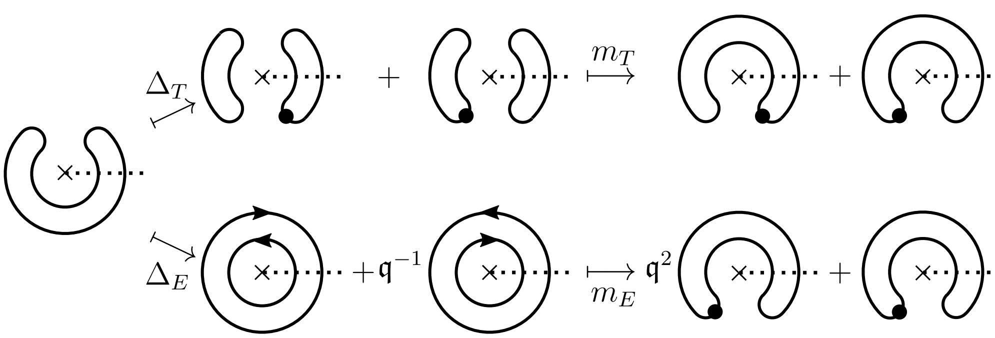
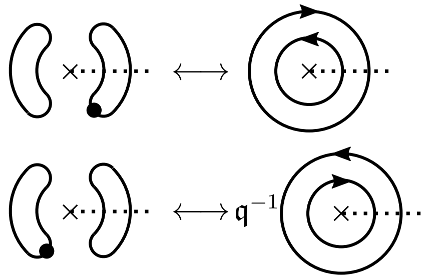


Figure 3-15: Note that the term with coefficient $\mathfrak{q}^2$ on bottom right matches the term directly above it, because dragging the dot across the seam amounts to multiplication by $\mathfrak{q}^2$.

We see that the intermediate generators are paired up as in (3.10).



(3.10)

Algebraically, the matching is

$$w_+ \otimes w_- \longleftrightarrow v_+ \otimes v_-$$
$$w_- \otimes w_+ \longleftrightarrow \mathfrak{q}^{-1} v_- \otimes v_+$$

where the ordering on trivial circles follows the convention illustrated in Figure 3-12. After forgetting powers of $\mathfrak{q}$, the matching in (3.10) is consistent with the matching in Figure 3-14. Our remaining goal in this section is to show that the matching forced by powers of $\mathfrak{q}$ (Corollary 3.2.15 and Proposition 3.2.17) agrees with the ladybug matching made with the left pair, as stated in Proposition 3.2.18.

We will use the notation and conventions established in this section. By Proposition 3.2.13 (1), we can write

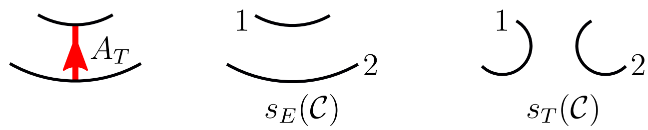
$$\Delta_E(x) = \mathfrak{q}^a v_{\mathcal{I}'} \otimes v_+ \otimes v_- \otimes v_{\mathcal{I}''} \otimes w_{\mathcal{J}} + \mathfrak{q}^{a-1} v_{\mathcal{I}'} \otimes v_- \otimes v_+ \otimes v_{\mathcal{I}''} \otimes w_{\mathcal{J}}$$

for some $a \in \mathbb{Z}$. Proposition 3.2.13 (2), Corollary 3.2.15 , and Proposition 3.2.17 imply that in the quantum setting, the pairing on intermediate generators is forced to be

$$\begin{aligned} \mathfrak{q}^\ell v_{\mathcal{I}} \otimes w_+ \otimes w_- \otimes w_{\mathcal{J}} &\longleftrightarrow \mathfrak{q}^a v_{\mathcal{I}'} \otimes v_+ \otimes v_- \otimes v_{\mathcal{I}''} \otimes w_{\mathcal{J}} \\ \mathfrak{q}^k v_{\mathcal{I}} \otimes w_- \otimes w_+ \otimes w_{\mathcal{J}} &\longleftrightarrow \mathfrak{q}^{a-1} v_{\mathcal{I}'} \otimes v_- \otimes v_+ \otimes v_{\mathcal{I}''} \otimes w_{\mathcal{J}}. \end{aligned} \tag{3.11}$$

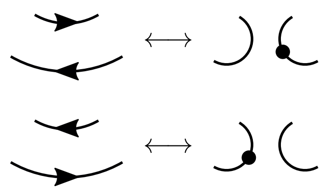
Proposition 3.2.18. *Given a ladybug configuration (C, A_T, A_E) of the type described in the paragraph preceding Equation (3.8), the matching in Equation (3.11) is the same as the ladybug matching made with the left pair.*

Proof. A look at the surgery arc A_T shows that the left choice makes the following identification on circles in $s_E(\mathscr{C})$ and $s_T(\mathscr{C})$.



(3.12)

Therefore the ladybug matching makes the following identification on generators



(3.13)

Comparing with our ordering convention on the circles in $s_T(\mathscr{C})$ in Figure 3-12, we see that this is consistent with the matching in (3.11). $\square$

3.3 The equivariant Burnside category

Let G be a finite group. The *G-equivariant Burnside category*, denoted $\mathscr{B}_G$, is an equivariant analogue of $\mathscr{B}$. Objects of $\mathscr{B}_G$ are finite free G-sets. A 1-morphism from X to Y is a triple (A, s, t) where A is another finite free G-set, and $s : A \to X$, $t : A \to Y$ are G-equivariant maps. We will call such a triple (A, s, t) an *equivariant correspondence*. Given equivariant correspondences $X \xleftarrow{s_A} A \xrightarrow{t_A} Y$ and $Y \xleftarrow{s_B} B \xrightarrow{t_B} Z$, the composition $(B, s_B, t_B) \circ (A, s_A, t_A)$ is the same as in $\mathscr{B}$. The G-action on $B \times_Y A$ is inherited from the diagonal G-action on $B \times A$; that is, $g(b, a) = (gb, ga)$. The additional requirement on 2-morphisms between correspondences is that they be G-equivariant.

The equivariant Burnside category is discussed in [78, Section 3.3] in the case $G = \mathbb{Z}_2$. Lemma 3.2 in [78] (our Lemma 2.6.9) gives sufficient conditions for defining a functor $F : \underline{2}^n \to \mathscr{B}_{\mathbb{Z}_2}$, and the same conditions clearly work for general G. The modification to the data of Lemma 2.6.9 is that all sets should be finite free G-sets and all set maps should be equivariant. Note that if $G = \{1\}$, then $\mathscr{B}_{\{1\}} = \mathscr{B}$, so everything stated about $\mathscr{B}_G$ in the following sections holds just as well for $\mathscr{B}$.

We will later be interested in the *quotient functor* $(-)/G : \mathscr{B}_G \to \mathscr{B}$, which simply takes the quotient of all sets and set maps. Explicitly, the quotient functor sends a G-set X to the set of orbits $X/G = X/(x \sim gx)$. For G-sets X and Y and an equivariant map $f : X \to Y$, there is an induced map $f/G : X/G \to Y/G$, given by $(f/G)([x]) = [f(x)]$. The quotient functor sends an equivariant correspondence $X \xleftarrow{s} A \xrightarrow{t} Y$ to the correspondence $X/G \xleftarrow{s/G} A/G \xrightarrow{t/G} Y/G$. Likewise, a 2-morphism $f : A \to B$ is assigned $f/G : A/G \to B/G$.

Recall totalizations of Burnside functors from Section 2.6.2. If F is an equivariant Burnside functor taking values in $\mathscr{B}_G$, we may consider it as taking values in $\mathscr{B}$ by forgetting the group action, and construct the chain complex $\mathrm{Tot}(F)$. Note that, if X is a G-set, then $\mathcal{A}(X)$ is naturally a $\mathbb{Z}[G]$-module. Moreover, If $X \xleftarrow{s} A \xrightarrow{t} Y$ is an equivariant correspondence, then the map $\mathcal{A}(A)$ is $\mathbb{Z}[G]$-linear. Thus if $F : \underline{2}^n \to \mathscr{B}$ is an equivariant Burnside functor, then $\mathrm{Tot}(F)$ is a complex of $\mathbb{Z}[G]$-modules.

3.4 A Strategy for constructing natural isomorphisms

As in Definition 2.6.11, given equivariant Burnside functors $F_1, F_0 : \underline{2}^n \to \mathscr{B}_G$, a *natural transformation* from F_1 to F_0 is an equivariant Burnside functor $\eta : \underline{2}^{n+1}$ such that the restriction of η to $\{i\} \times \underline{2}^n$ is equal to F_i. We say η is a *natural isomorphism* if $\eta(e_u) : F_1(u) \to F_0(u)$ is an isomorphism in $\mathscr{B}_G$ for each vertex u.

We will have several occasions to show that two equivariant Burnside functors are isomorphic (Propositions 3.5.3, 3.5.4, and 3.8.1). The general strategy is the same in all these cases, so we outline it here.

Note that a correspondence $X \xleftarrow{s} A \xrightarrow{t} Y$, thought of as a morphism in $\mathscr{B}_G$, is an isomorphism if and only if s and t are bijective. In particular, given an equivariant bijection $t : X \to Y$, the correspondence $X \xleftarrow{\mathrm{id}} X \xrightarrow{t} Y$ is an isomorphism in $\mathscr{B}_G$, with inverse $Y \xleftarrow{t} X \xrightarrow{\mathrm{id}} X$ (up to a canonical identification of a set with the diagonal in its cartesian square).

Suppose we are given two functors $F_1, F_0 : \underline{2}^n \to \mathscr{B}_G$ and equivariant bijections ψ_u :

$F_1(u) \to F_0(u)$ for each vertex $u \in \underline{2}^n$. For $u \geq_1 v$, let

$$F_i(u) \xleftarrow{s^i_{u,v}} A^i_{u,v} \xrightarrow{t^i_{u,v}} F_i(v)$$

be the correspondence assigned to the edge $\varphi_{u,v} : u \to v$ by F_i, for $i = 0, 1$. Suppose also that for each $u \geq_1 v$, the following conditions hold.

(NI 1) $A^i_{u,v} \subset F_i(u) \times F_i(v)$, and the G-action on $A^i_{u,v}$ is inherited from the diagonal G-action on $F_i(u) \times F_i(v)$ (i.e., $g(x, y) = (gx, gy)$ for $g \in G$, $(x, y) \in F_i(u) \times F_i(v)$).

(NI 2) The map $F_1(u) \times F_1(v) \xrightarrow{\psi_u \times \psi_v} F_0(u) \times F_0(v)$ restricts to an bijection $A^1_{u,v} \to A^0_{u,v}$, denoted $\psi_{u,v}$.

(NI 3) The source and target maps, $s^i_{u,v}$ and $t^i_{u,v}$, are restrictions of the projections $F_i(u) \twoheadleftarrow F_i(u) \times F_i(v) \twoheadrightarrow F_i(v)$.

In this situation, we have a systematic method for building a natural isomorphism $\eta : F_1 \to F_0$ using Lemma 2.6.9 as follows. Define η on objects by

$$\eta(i, u) := F_i(u)$$

for $i \in \{0, 1\}$ and $u \in \{0, 1\}^n$. We then define η on each vertical edge $e_u : (1, u) \to (0, u)$ by

$$\eta(e_u) = \left(F_1(u) \xleftarrow{\mathrm{id}} F_1(u) \xrightarrow{\psi_u} F_0(u)\right),$$

That is, the underlying set of the correspondence $\eta(e_u)$ is simply $F_1(u)$, the source map is the identity, and the target map is the given equivariant bijection ψ_u.

We have now specified η on objects and edges. It remains to define the 2-morphisms for each square face of $\underline{2}^{n+1}$. Since η must restrict to F_i on $\{i\} \times \underline{2}^n$, we need only to specify a 2-morphism

$$\eta_{u,v} : F_1(v) \times_{F_1(v)} A^1_{u,v} \to A^0_{u,v} \times_{F_0(u)} F_1(u).$$

corresponding to the vertical square faces (2.21) of $\underline{2} \times \underline{2}^n$.

The situation is illustrated in Figure 3-16. Note that every element of $F_1(v) \times_{F_1(v)} A^1_{u,v}$ is of the form (y, x, y), where $(x, y) \in A^1_{u,v} \subset F_1(u) \times F_1(v)$. Likewise, an element of

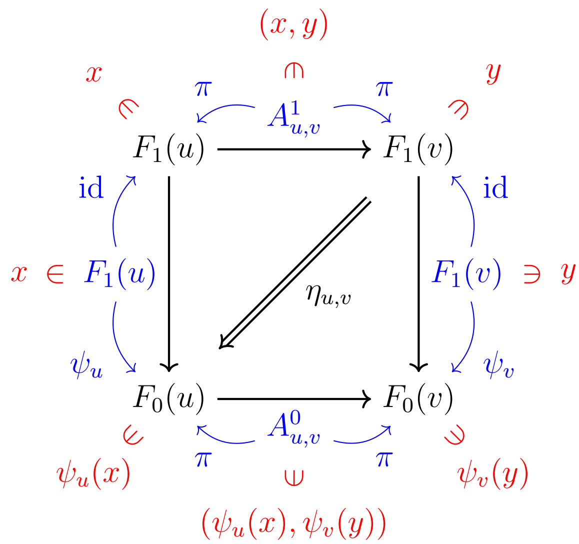


Figure 3-16: We draw the square diagram required for building our natural isomorphism $\eta : F_1 \to F_0$. The correspondences are indicated along each edge together with their source and target maps, drawn as curved arrows. Specific elements are also indicated, showing how the 2-morphism $\eta_{u,v}$ (indicated by the double arrow) should be defined such that the entire diagram is consistent.

$A^0_{u,v} \times_{F_0(u)} F_1(u)$ is of the form $(a, b, \psi_u^{-1}(a))$, where $(a, b) \in A^0_{u,v} \subset F_0(u) \times F_0(v)$. Condition (NI 1) ensures that the bijections

$$\begin{aligned} F_1(v) \times_{F_1(v)} A^1_{u,v} &\to A^1_{u,v} \\ (y, x, y) &\mapsto (x, y) \end{aligned} \qquad\qquad \begin{aligned} A^0_{u,v} &\to A^0_{u,v} \times_{F_0(u)} F_1(u) \\ (a, b) &\mapsto (a, b, \psi_u^{-1}(a)) \end{aligned}$$

are equivariant. Then the composition

$$F_1(v) \times_{F_1(v)} A^1_{u,v} \to A^1_{u,v} \xrightarrow{\psi_{u,v}} A^0_{u,v} \to A^0_{u,v} \times_{F_0(u)} F_1(u) \tag{3.14}$$

is given by $(y, x, y) \mapsto (\psi_u(x), \psi_v(y), x)$, and condition (NI 2) guarantees that it is also an equivariant bijection. Moreover, condition (NI 3) ensures that this composition commutes with the source and target maps. Therefore, we may define the 2-morphism $\eta_{u,v}$ to be the composition (3.14).

To extend η to a natural transformation, one still needs to check the hexagon relation of Lemma 2.6.9. We need only to verify commutativity of the hexagon coming from a three dimensional cube of the form

$$\begin{tikzcd}
 & (1,v') \arrow[rr] \arrow[dd] & & (1,w) \arrow[dd] \\
(1,u) \arrow[ur] \arrow[rr] \arrow[dd] & & (1,v) \arrow[ur] \arrow[dd] & \\
 & (0,v') \arrow[rr] & & (0,w) \\
(0,u) \arrow[ur] \arrow[rr] & & (0,v) \arrow[ur] &
\end{tikzcd}$$

Let

$$\phi^i_{u,v,v',w} : A^i_{v,w} \times_{F_i(v)} A^i_{u,v} \to A^i_{v',w} \times_{F_i(v')} A^i_{u,v'}$$

be the 2-morphism assigned by the functor F_i corresponding to the horizontal square face

$$\begin{tikzcd}
 & (i,v') \arrow[rr, "\varphi^i_{v',w}"] & & (i,w) \\
(i,u) \arrow[ur, "\varphi^i_{u,v'}"] \arrow[rr, "\varphi^i_{u,v}"'] & & (i,v) \arrow[ur, "\varphi^i_{v,w}"'] &
\end{tikzcd}$$

of $\underline{2} \times \underline{2}^n$. In this situation, checking that the hexagon of Lemma 2.6.9 commutes comes down to verifying commutativity of the diagram in (3.15) below.

$$\begin{tikzcd}
A^1_{v,w} \times_{F_1(v)} A^1_{u,v} \arrow[r, "\phi^1_{u,v,v',w}"] \arrow[d, "\psi_{v,w} \times \psi_{u,v}"'] & A^1_{v',w} \times_{F_1(v')} A^1_{u,v'} \arrow[d, "\psi_{v',w} \times \psi_{u,v'}"] \\
A^0_{v,w} \times_{F_0(v)} A^0_{u,v} \arrow[r, "\phi^0_{u,v,v',w}"] & A^0_{v',w} \times_{F_0(v')} A^0_{u,v'}
\end{tikzcd} \tag{3.15}$$

If the diagram (3.15) commutes, then η extends to a natural transformation $\eta : F_1 \to F_0$. Moreover, since each $\eta(e_u) : F_1(u) \to F_0(u)$ is an isomorphism in $\mathscr{B}_G$, the natural transformation η is a natural isomorphism of Burnside functors.

3.5 The quantum annular Burnside functor

In this section, we construct the *quantum annular Burnside functor* corresponding to an annular link diagram D. Before giving the outline of the section, we emphasize one small but important caveat. In the quantum annular theory over the base ring $\Bbbk = \mathbb{Z}[\mathfrak{q}, \mathfrak{q}^{-1}]$, every configuration is assigned a module which has infinite rank over $\mathbb{Z}$, with generators of the form $\mathfrak{q}^k x$ for $k \in \mathbb{Z}$. In our set-up, this would correspond to assigning an infinite set to each vertex in the cube of resolutions. This would require considering spaces of infinitely many boxes in Section 3.6, and also of CW-complexes with a $\mathbb{Z}$-action. Although we believe that such a version of the theory could be worked out, in the present paper we stay in the context of finite cyclic group actions. This is motivated in part by the fact that a substantial part of equivariant homotopy theory is formulated for compact group actions. To this end, we make the following modification to the quantum annular complex.

For $r > 0$, set $\Bbbk_r := \Bbbk/(\mathfrak{q}^r - 1)$. Let $\mathcal{F}^r_{\mathbb{A}_\mathfrak{q}}$ denote the composition

$$\mathcal{BBN}_\mathfrak{q}(\mathbb{A}) \xrightarrow{\mathcal{F}_{\mathbb{A}_\mathfrak{q}}} \Bbbk\text{– gmod} \xrightarrow{(-)\otimes_\Bbbk \Bbbk_r} \Bbbk_r\text{– gmod}\,. \tag{3.16}$$

We can define a modified quantum annular homology by applying $\mathcal{F}^r_{\mathbb{A}_\mathfrak{q}}$ to each vertex in the cube of resolutions of an annular link diagram D. The result is the same as applying $(-) \otimes_\Bbbk \Bbbk_r$ to the quantum annular chain complex $CKh_{\mathbb{A}_\mathfrak{q}}(D)$. Every vertex is assigned a free $\Bbbk_r$-module, and the formulas in Section 2.4.2 remain true, modulo the additional relation $\mathfrak{q}^r = 1$. At $r = 1$ we obtain the classical annular TQFT $\mathcal{F}_\mathbb{A}$.

With this modification in place, we proceed as follows. Given an annular link diagram D with n crossings, we will define the quantum annular Burnside functor $F_\mathfrak{q} : \underline{2}^n \to \mathscr{B}_G$, where

$$G = \langle \mathfrak{q} \mid \mathfrak{q}^r = 1 \rangle$$

is the finite cyclic group of order r with distinguished generator $\mathfrak{q}$. Note that there is a natural ring isomorphism $\mathbb{Z}[G] \cong \Bbbk_r$, so that it is possible to compare the cellular cohomology of the stable homotopy type, which is a $\mathbb{Z}[G]$-module, with the modified quantum annular homology, which is a $\Bbbk_r$-module. The dependence on r will be omitted from the group G and the Burnside functor $F_\mathfrak{q}$ in order to simplify the notation.

The totalization $\mathrm{Tot}(F_\mathfrak{q})$ should recover the quantum annular complex

$$CKh_{\mathbb{A}_\mathfrak{q}}(D) \otimes_\Bbbk \Bbbk_r,$$

so we already know what $F_\mathfrak{q}$ should assign to vertices and edges of $\underline{2}^n$. The subtlety is that generators are defined only up to a power of $\mathfrak{q}$, and the formulas for the differential depend non-trivially on the configuration, so the full extent of the analysis in Sections 3.2, 3.2.1 is used here. Once $F_{\mathfrak{q}}$ is determined on vertices and edges, it remains to assign specific bijections to the (identity) 2-morphisms in $\underline{2}^n$ and check the hexagon relation of Lemma 2.6.9. This will be done in Section 3.5.1 for the case $r > 2$; this restriction is to guarantee that $\mathfrak{q}^2 \neq 1$, allowing the use of Corollary 3.2.15 to simplify the analysis. We will show in Proposition 3.5.3 that the Burnside functor $F_{\mathfrak{q}}$ is independent of the choice of generators for each vertex, and in Section 3.5.2 we will show that planar isotopies of the link diagram induce natural isomorphisms of the corresponding Burnside functors. Finally in Section 3.5.3 we will address the cases $r = 1, 2$.

3.5.1 The quantum annular Burnside functor for a link diagram

Let D be a diagram for an annular link with n crossings, which are assumed to be disjoint from the seam. For each $u \in \{0,1\}^n$, let D_u denote the smoothing of D corresponding to u. Fix $r > 2$, and set $G = \langle \mathfrak{q} \mid \mathfrak{q}^r = 1 \rangle$. We will specify the data of the *quantum annular Burnside functor* $F_{\mathfrak{q}} : \underline{2}^n \to \mathscr{B}_G$.

For each vertex $u \in \{0,1\}^n$, pick a set of generators $\Gamma(u)$ of D_u, following Section 3.2. Define $F_{\mathfrak{q}}$ on vertices by

$$F_{\mathfrak{q}}(u) = G \times \Gamma(u). \tag{3.17}$$

The G-action on $F_{\mathfrak{q}}(u)$ is on the first factor: $\mathfrak{q}^k \cdot (\mathfrak{q}^\ell, x) = (\mathfrak{q}^{k+\ell}, x)$. We will write elements of $F_{\mathfrak{q}}(u)$ as $\mathfrak{q}^k x$ instead of $(\mathfrak{q}^k, x)$.

For $u \geq_1 v$, let $d_{v,u}$ denote the map assigned to the edge $v \to u$ by the modified quantum annular functor $\mathcal{F}^r_{\mathbb{A}_{\mathfrak{q}}}$. Recall from Lemma 3.2.14 that for each $x \in \Gamma(v)$,

$$d_{v,u}(x) = \sum_{y \in \Gamma(u)} \varepsilon_y y$$

where each coefficient ε_y is either 0 or $\mathfrak{q}^k$ for some $k \in \mathbb{Z}$. We will say that $\mathfrak{q}^k y$ *appears in* $d_{v,u}(\mathfrak{q}^\ell x)$ if in the equation

$$d_{v,u}(\mathfrak{q}^\ell x) = \sum_{y \in \Gamma(u)} \varepsilon_y y,$$

the coefficient ε_y is equal to $\mathfrak{q}^k$. For $u \geq_1 v$, define the correspondence $A_{u,v} \subset F_{\mathfrak{q}}(u) \times F_{\mathfrak{q}}(v)$ from $F_{\mathfrak{q}}(u)$ to $F_{\mathfrak{q}}(v)$ by

$$A_{u,v} = \{(\mathfrak{q}^k y, \mathfrak{q}^\ell x) \in F_{\mathfrak{q}}(u) \times F_{\mathfrak{q}}(v) \mid \mathfrak{q}^k y \text{ appears in } d_{v,u}(\mathfrak{q}^\ell x)\} \tag{3.18}$$

The source and target maps of $A_{u,v}$ are the projections to $F_{\mathfrak{q}}(u)$ and $F_{\mathfrak{q}}(v)$, respectively. Note that $A_{u,v}$ is a sub G-set of $F_{\mathfrak{q}}(u) \times F_{\mathfrak{q}}(v)$ since $d_{v,u}(\mathfrak{q}x) = \mathfrak{q}d_{v,u}(x)$.

We will show that the above data extends to a Burnside functor $F_q : \underline{2}^n \to \mathscr{B}_G$. The following lemma will be useful in our analysis of the hexagon relation.

Lemma 3.5.1. *Let $\mathscr{C}$ be a configuration with three surgery arcs A_1, A_2, and A_3. Let $\mathscr{C}'$ denote the circles in $\mathscr{C}$ containing the endpoints of the surgery arcs. Assume there is a circle C in $\mathscr{C}'$ such that $C \cup A_1 \cup A_2$ forms a ladybug configuration. Then one of the following holds.*

(1) *The diagram $\mathscr{C}' \cup A_1 \cup A_2 \cup A_3$ is trivial in the annulus; i.e. $\mathscr{C}'$ and the three surgery arcs lie in a disk in $\mathbb{A}$.*

(2) *The composition of three edge maps is* 0.

(3) *The* 3*-dimensional cube is simple (see the discussion in Remark 2.6.10).*

(4) *$C \cup A_1 \cup A_2$ is trivial in the annulus and disjoint from A_3.*

Proof. If C is essential in the annulus, then (2) follows from the neck-cutting and Boerner's relations (see Figures 2-4, 2-14). We may therefore assume that C is trivial. Let C' denote the (necessarily trivial) circle obtained by performing surgery along both A_1 and A_2. Note that the result of composing the two saddle maps corresponding to A_1 and A_2 will send any generator of $\mathscr{C}$ in which C is undotted to a sum of elements in which C' is dotted, and will send any generator which is dotted on C to 0. It therefore suffices to consider the effect of surgery along A_3 on a dotted C'.

First, assume that $C \cup A_1 \cup A_2$ is trivial but $\mathscr{C}' \cup A_1 \cup A_2 \cup A_3$ is not. There are several cases to consider. If neither endpoint of A_3 is on C, then (4) holds. If precisely one endpoint of A_3 is on C, then the other endpoint must be on another circle $\overline{C}$, as in Figure 3-17a. In this situation, Boerner's relation implies that (2) holds. Finally, suppose both endpoints of A_3 are on C, as in Figure 3-17b. Then surgery along A_3 must split C' into two essential circles. In this situation, (2) holds again, since a dotted trivial circle splitting into two essential circles is sent to 0.

(a) One endpoint of A_3 is on C

(b) Both endpoints of A_3 are on C

Figure 3-17: The two cases where $C \cup A_1 \cup A_2$ is trivial. Note that these are only schematic depictions, since the interaction of the curves with the seam could be complicated.

Now suppose that the diagram $C \cup A_1 \cup A_2$ is non-trivial. Then we are in the situation of Section 3.2.2 (see Figure 3-11 for example) where Corollary 3.2.15 shows we have a sum of terms $y + \mathfrak{q}^2 y$ in which C' is dotted. If surgery along A_3 either splits off a trivial circle from C' or merges C' with another trivial circle, then (3) holds since the two terms retain distinct powers of $\mathfrak{q}$ (see Figure 3-5). If surgery along A_3 splits C' into two essential circles or merges C' with an essential circle, then (2) holds as above.

$\square$

Theorem 3.5.2. *There is a functor $F_{\mathfrak{q}} : \underline{2}^n \to \mathscr{B}_G$ which extends the data* (3.17) *and* (3.18)*.*

Proof. Following Lemma 2.6.9, it remains to define the 2-morphisms

$$\phi_{u,v,v',w} : A_{v,w} \times_{F_{\mathfrak{q}}(v)} A_{u,v} \to A_{v',w} \times_{F_{\mathfrak{q}}(v')} A_{u,v'}$$

for each square face of $\underline{2}^n$ with vertices $u \geq_1 v, v' \geq_1 w$, and to verify the hexagon relation.

For all cases except the ladybug configuration, the 2-morphism $\phi_{u,v,v',w}$ is uniquely determined by the property that it commutes with the source and target maps. Therefore we need to consider only the ladybug configurations. Assume a circle C in D_w carries two surgery arcs as in the ladybug configuration. We distinguish three cases, as in Figure 3-9.

(a) C is essential.

(b) C is trivial, and surgery along both arcs results in trivial circles.

(c) C is trivial, surgery along one arc produces two trivial circles, and surgery along the other arc produces two essential circles.

For (a), the composition of two edge maps is 0. Therefore

$$A_{v,w} \times_{F_{\mathsf{q}}(v)} A_{u,v} = \varnothing = A_{v',w} \times_{F_{\mathsf{q}}(v')} A_{u,v'},$$

and there is no 2-morphism to specify. For (b), we rely on the ladybug matching made with the *left pair* (see [57, Section 5.4]). Finally, for (c), note that generators dotted on C are sent to 0 by the composition of two edges. For generators undotted on C, Corollary 3.2.15 implies that $\phi_{u,v,v',w}$ is uniquely determined by the property that it commutes with the source and target maps.

It remains to verify the hexagon relation. Let $\mathscr{C}$ denote a configuration with three surgery arcs. We may assume that two of the three surgery arcs form a ladybug configuration, since otherwise the 3-dimensional cube is simple. Then the analysis consists of the four cases in Lemma 3.5.1. In case (1), the verification reduces to classical Khovanov homology (see, for example, [51, Proposition 6.1]). For case (2), the composition of three correspondences coming from any three edge maps is empty, so there is nothing to check. Similarly, In case (3), the hexagon relation follows from the discussion in Remark 2.6.10. Finally, case (4) is straightforward to check by hand since the disjoint arc A_3 cannot interfere with the classical Khovanov ladybug matching used on $C \cup A_1 \cup A_2$. □

Proposition 3.5.3. *Up to natural isomorphism, F_{q} is independent of the choices of generators $\Gamma(u)$.*

Proof. For each $u \in \{0,1\}^n$, let $\Gamma(u)$, $\Gamma'(u)$ be two sets of generators of D_u, obtained by picking different isotopies S, S' from standard configurations $D_u^\circ, D_u^{\circ\prime}$ to D_u. The natural isomorphism is straightforward to construct when $S' = SS''$ where $S'' : D_u^{\circ\prime} \to D_u^\circ$ is a planar isotopy of standard configurations, similar to the usual reindexing of circles in Khovanov homology under planar isotopy of a link diagram. Thus it is enough to consider the case when $D_u^{\circ\prime} = D_u^\circ$ and $S, S' : D_u^\circ \to D_u$ induce the same correspondence on components.

Let $F_{\mathsf{q}}, F'_{\mathsf{q}} : \underline{2}^n \to \mathscr{B}_G$ denote the corresponding functors, and let $A_{u,v}$, $A'_{u,v}$ denote the correspondences assigned to edges by $F_{\mathfrak{q}}$ and $F'_{\mathfrak{q}}$ respectively. We will use the strategy of Section 3.4 to build a natural isomorphism $F_{\mathfrak{q}} \to F'_{\mathfrak{q}}$.

There is a clear bijection $\Gamma(u) \to \Gamma'(u)$, denoted $x \mapsto x'$. Lemma 3.2.4 says that for any $x \in \Gamma(u)$, there exists $m_x \in \mathbb{Z}$ such that $x = \mathfrak{q}^{m_x} x'$. Consider the equivariant bijection $\psi_u : F_{\mathfrak{q}}(u) \to F'_{\mathfrak{q}}(u)$ defined by $\psi_u(\mathfrak{q}^k x) = \mathfrak{q}^{k+m_x} x'$. Observe that conditions (NI 1) and (NI 3) in Section 3.4 hold by definition of $F_{\mathfrak{q}}$ and $F'_{\mathfrak{q}}$. Condition (NI 2) holds since $x = \mathfrak{q}^{m_x} x'$. To complete the proof, observe that the diagram (3.15) commutes, since the vertical maps act on generators by multiplication by powers of $\mathfrak{q}$, and thus do not interfere with the ladybug matchings. □

3.5.2 Isotoping the link diagram

In this section we show that a planar isotopy of link diagrams induces a natural isomorphism between the corresponding quantum annular Burnside functors. Elementary isotopies away from the seam are readily dealt with, but isotopies which involve intersections with the seam need more care. These results will also be used to show Reidemeister invariance for the stable homotopy type in Section 3.6.5.

Proposition 3.5.4. *Let D be a link diagram with n crossings, and let D' be a link diagram obtained from D by one of the following moves.*

(1) *Moving an arc (as in the $P^{\pm 1}$ and $N^{\pm 1}$ moves of Figure 3 – 4) across the seam.*

(2) *Moving a crossing across the seam (see Figure (3.19))*

Let $F_{\mathfrak{q}}$ and $F'_{\mathfrak{q}}$ be Burnside functors for D and D' respectively. Then $F_{\mathfrak{q}}$ is naturally isomorphic to $F'_{\mathfrak{q}}$.

Proof. We will again follow the strategy of Section 3.4. By Proposition 3.5.3, we are free to choose generators for each configuration D_u using any isotopy $D_u^\circ \to D_u$, and likewise for D'_u. For each $u \in \underline{2}^n$, let $S_u : D_u^\circ \to D_u$ be the cobordism formed by a fixed choice of isotopy from D_u° to D_u. There is also an obvious isotopy $R_u : D_u \to D'_u$, corresponding to the moves in the statement of the proposition. Since $D_u^\circ = D_u'^\circ$, we can choose generators for D'_u using the cobordism $R_u S_u$. For $u, v \in \underline{2}^n$ with $u \geq_1 v$, let $d_{v,u}$ and $d'_{v,u}$ denote the maps assigned to the edge $v \to u$ in $[[D]]$ and $[[D']]$, respectively.

Suppose we are in the situation (1). We have equivariant bijections $\psi_u : F_{\mathfrak{q}}(u) \to F'_{\mathfrak{q}}(u)$ for each u, given by $\psi_v(\mathfrak{q}^k x) = \mathfrak{q}^k R_u(x)$ (as usual, we do not distinguish between a cobordism and its induced map). Conditions (NI 1) and (NI 3) are satisfied by definition. For each $u, v \in \underline{2}^n$ with $u \geq_1 v$, we have

$$R_u d_{v,u} = d'_{v,u} R_v.$$

It follows that, for $\mathfrak{q}^k y \in F_{\mathfrak{q}}(u)$ and $\mathfrak{q}^\ell x \in F_{\mathfrak{q}}(v)$, $\mathfrak{q}^k y$ appears in $d_{v,u}(\mathfrak{q}^\ell x)$ if and only if $\mathfrak{q}^k R_u(y)$ appears in $d'_{v,u}(\mathfrak{q}^k R_v(x))$. Therefore condition (NI 2) is satisfied as well. Observe that the diagram (3.15) commutes since we do not interfere with any potential ladybug matchings.

For case (2), the maps ψ_u need to be modified slightly in order to satisfy (NI 2). We illustrate one case in detail. Suppose D and D' are as shown in (3.19). Assume also that the crossing shown is first in the ordering of crossings.

(3.19)

Let $u \in \underline{2}^n$. If $u_1 = 0$, define $\psi_u : F_{\mathfrak{q}}(u) \to F'_{\mathfrak{q}}(u)$ as in case (1). If $u_1 = 1$, then define ψ_u by

$$\psi_u(\mathfrak{q}^k x) = \mathfrak{q}^{k+1} R_u(x)$$

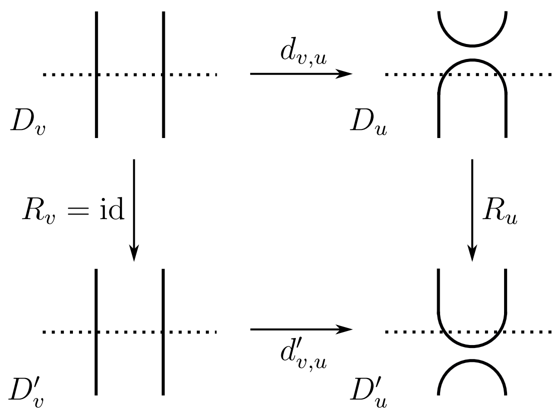
We will now verify that condition (NI 2) holds for this choice of equivariant bijections $\{\psi_u\}_{u \in \underline{2}^n}$. Let $u, v \in \underline{2}^n$ with $u \geq_1 v$. If $u_1 = v_1$, then the edge maps $d_{v,u}$ and $d'_{v,u}$ are induced by changing the smoothing at a crossing away from the one shown in (3.19). As in case (1) above, we have

$$R_u d_{v,u} = d'_{v,u} R_v,$$

so condition (NI 2) holds. Suppose now that $u_1 = 1$ and $v_1 = 0$. Then

$$d'_{v,u} R_v = \mathfrak{q} R_u d_{v,u}, \tag{3.20}$$

where the factor of $\mathfrak{q}$ comes from moving a saddle across the membrane (see Figure 2-16). The situation is depicted in the (noncommutative!) diagram (3.21).



(3.21)

Let $\mathfrak{q}^k y \in F_{\mathfrak{q}}(u)$ and $\mathfrak{q}^\ell x \in F_{\mathfrak{q}}(v)$. It follows from (3.20) that $\mathfrak{q}^k y$ appears in $d_{v,u}(\mathfrak{q}^\ell x)$ if and only if $\mathfrak{q}^{k+1} R_u(y)$ appears in $d'_{v,u}(\mathfrak{q}^\ell R_v(x))$. Therefore condition (NI 2) is satisfied. Again, the hexagon relation is satisfied because the 2-morphisms do not interfere with the ladybug matching.

□

3.5.3 The cases $r = 1, 2$ and the classical annular homotopy type

Our proof of Theorem 3.5.2 relies on $r > 2$. In this section we address the cases $r = 1, 2$ in that order.

Let D be a diagram for an annular link with n crossings. When $r = 1$, the modified quantum annular chain complex

$$CKh_{\mathbb{A}_{\mathfrak{q}}}(D) \otimes_{\Bbbk} \Bbbk/(\mathfrak{q} - 1)$$

is just the classical annular chain complex $CKh_{\mathbb{A}}(D)$ (see the discussion preceding Lemma 3.2.14). We sketch how to define the annular Burnside functor, denoted F_1, for the classical annular Khovanov chain complex below. An alternative construction, using topological Hochschild homology of Chen-Khovanov spectra for tangles, was introduced in [53].

Let $F_{Kh} : \underline{2}^n \to \mathscr{B}$ denote the usual Khovanov Burnside functor (see definition 2.6.15) where the ladybug matching is made with the *left pair*. For $u \geq_1 v$, let

$$d^{\mathbb{A}}_{v,u} : \mathcal{F}_{\mathbb{A}}(D_v) \to \mathcal{F}_{\mathbb{A}}(D_u)$$

denote the classical annular differential, and let $d^{Kh}_{v,u}$ denote the usual Khovanov differential. We have that $d^{Kh}_{v,u} = d^{\mathbb{A}}_{v,u} + d'_{v,u}$ (see Lemma 2.4.1 and Definition 2.4.2).

Recall that annular Khovanov generators may be taken to be the usual Khovanov generators (where the annular link diagram is considered as a planar diagram under the inclusion $\mathbb{A} \subset \mathbb{R}^2$), so set $F_1(u) = F_{Kh}(u)$ for each $u \in \underline{2}^n$. For $u \geq_1 v$, let $A^{Kh}_{u,v}$ denote the correspondence assigned by F_{Kh} to the edge $\varphi_{u,v} : u \to v$. Define the correspondence $A^{\mathbb{A}}_{u,v}$ from $F_1(u)$ to $F_1(v)$ by

$$A^{\mathbb{A}}_{u,v} := \{(y, x) \in F_1(u) \times F_1(v) \mid y \text{ appears in } d^{\mathbb{A}}_{v,u}(x)\},$$

with the obvious source and target maps, and set $F_1(\varphi_{u,v}) = A^{\mathbb{A}}_{u,v}$. Note that $A^{\mathbb{A}}_{u,v} \subset A^{Kh}_{u,v}$.

Let $\phi^{Kh}_{u,v,v',w}$ be the 2-morphism assigned by F_{Kh} to the square face with vertices $u \geq_1 v, v' \geq_1 w$. One can check that $\phi^{Kh}_{u,v,v',w}$ restricts to

$$\phi^{\mathbb{A}}_{u,v,v',w} : A^{\mathbb{A}}_{v,w} \times_{F_1(v)} A^{\mathbb{A}}_{u,v} \to A^{\mathbb{A}}_{v',w} \times_{F_1(v')} A^{\mathbb{A}}_{u,v'}.$$

Taking $\phi^{\mathbb{A}}_{u,v,v',w}$ to be the 2-morphisms assigned to square faces by F_1, the conditions of Lemma 2.6.9 are satisfied as a consequence of the construction of F_{Kh}.

When $r = 2$, Lemma 3.5.1 and the ensuing analysis in case (c) of the proof of Theorem 3.5.2 do not hold, since $\mathfrak{q}^2 = 1$. Instead, we rely on the ladybug matching made with the left pair to define the 2-morphism in case (c) of Theorem 3.5.2. Let us verify the hexagon relation, using the formulation in Remark 2.6.10. Start with an element $x = (\mathfrak{q}^i a, \mathfrak{q}^j b, \mathfrak{q}^k c, \mathfrak{q}^\ell d)$ in the correspondence obtained as the composition of correspondences for three consecutive edge maps. Going around the six faces of the cube, the 2-morphisms send x to an element $x' = (\mathfrak{q}^i a, \mathfrak{q}^{j'} b', \mathfrak{q}^{k'} c', \mathfrak{q}^\ell d)$ in the same correspondence. It follows from classical annular case, where powers of $\mathfrak{q}$ are disregarded, that labels on the circles in the generators b and c match the labels on the circles in the generators b' and c', respectively. Then $b = b'$, and since both $\mathfrak{q}^j b$ and $\mathfrak{q}^{j'} b' = \mathfrak{q}^{j'} b$ appear in the image of $\mathfrak{q}^i a$ under a saddle map, Lemma 3.2.14 implies that $\mathfrak{q}^j = \mathfrak{q}^{j'}$. Likewise, $\mathfrak{q}^k c = \mathfrak{q}^{k'} c'$, and we conclude that the hexagon relation is satisfied.

3.6 From Burnside functors to stable homotopy types

This section describes a general framework for obtaining a spectrum from a Burnside functor. This general construction is then applied to the case of the quantum annular Burnside functor, establishing the main result of the paper, Theorem 3.0.1. In more detail in Section 3.6.1 we recall box maps and their required properties for the non-equivariant case as in [52, Section 5]. Then in Section 3.6.2 we discuss G-equivariant box maps via a slight generalization of the ideas established in [78], ensuring that the required properties are still satisfied. Sections 3.6.3 and 3.6.4 describe how to use box maps to pass from an equivariant Burnside functor F to a G-CW complex realizing F, following [78, Section 4], by taking the homotopy colimit (see [85]) of an appropriate diagram. Finally in Section 3.6.5 we apply this theory to the quantum annular Burnside functor of Section 3.5.1 to define the equivariant spectrum $\mathcal{X}^r_{\mathbb{A}_\mathfrak{q}}(L)$ and check that it is well-defined, proving Theorem 3.0.1.

We emphasize some differences and similarities between the constructions presented in this chapter and those appearing elsewhere in the literature. Note that there is no group action on the we consider, so our box maps and homotopy coherent refinements are different from those in [15, 66, 80]. Functors $\underline{2}^n \to \mathscr{B}_{\mathbb{Z}/2\mathbb{Z}}$ are considered in [78], and there the authors introduce actions of $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, which involve an internal action on boxes. We are however interested in an external G-action, given by permuting boxes.

3.6.1 Box maps

We begin by reviewing a key part of the non-equivariant case allowing us to set some notation following [52, Section 5.1].

A k-dimensional box is $\prod_{i=1}^k [a_i, b_i] \subset \mathbb{R}^k$. For two k-dimensional boxes B and B', there is a canonical homeomorphism $B \xrightarrow{\sim} B'$, obtained by scaling and translating the ambient space $\mathbb{R}^k$. Fix an identification $S^k = [0,1]^k/\partial([0,1]^k)$, so that for any k-dimensional box B, $B/\partial B$ is canonically identified with S^k.

Suppose we have a correspondence $X \xleftarrow{s} A \xrightarrow{t} Y$. Pick disjoint k-dimensional boxes $\{B_x\}_{x\in X}$. Following [52], let

$$E(\{B_x\}, s)$$

denote the space of all collections $\{B_a\}_{a\in A}$ of disjoint k-dimensional boxes such that $B_a \subset B_{s(a)}$. A point $e = \{B_a\} \in E(\{B_x\}, s)$ determines a map

$$\Phi(e, A) : \bigvee_{x\in X} S^k_x \to \bigvee_{y\in Y} S^k_y \tag{3.22}$$

defined as follows. On each wedge summand, $\Phi(e, A)$ is the composition

$$S^k_x = B_x/\partial B_x \to B_x/(B_x \setminus \bigcup_{a\in s^{-1}(x)} B_a^{\mathrm{int}}) = \bigvee_{a\in s^{-1}(x)} B_a/\partial B_a \xrightarrow{t} \bigvee_{y\in Y} S^k_y \tag{3.23}$$

where the first map is a quotient and the last map sends the sphere $B_a/\partial B_a$ to $B_{t(a)}/\partial B_{t(a)} = S^k_{t(a)}$ via the canonical homeomorphism $B_a \cong B_{t(a)}$. A map of this form is said to *refine the correspondence* $X \xleftarrow{s} A \xrightarrow{t} Y$.

Suppose we have correspondences $X \xleftarrow{s_A} A \xrightarrow{t_A} Y$ and $Y \xleftarrow{s_B} B \xrightarrow{t_B} Z$ with boxes $\{B_x\}_{x\in X}$ and $\{B_y\}_{y\in Y}$. Given $e \in E(\{B_x\}_{x\in X}, s_A)$ and $e' \in E(\{B_y\}_{y\in Y}, s_B)$, we can consider the preimage of the boxes in e' under the map $\Phi(e, A)$. An important point in the proof of existence and uniqueness of spatial refinements is that $\Phi(e, A)^{-1}(e')$ is a collection of little boxes in $\{B_x\}_{x\in X}$ labelled by the composition $C := B \times_Y A$; that is,

$$\Phi(e, A)^{-1}(e') \in E(\{B_x\}_{x\in X}, s_C)$$

where $s_C : C \to X$ is the source map of the composition. This is explained in Figure 3-18.

3.6.2 Equivariant box maps

Throughout this section, we will continue to use G to denote a finite cyclic group, but all of the statements below generalize to more general finite groups acting freely on finite sets.

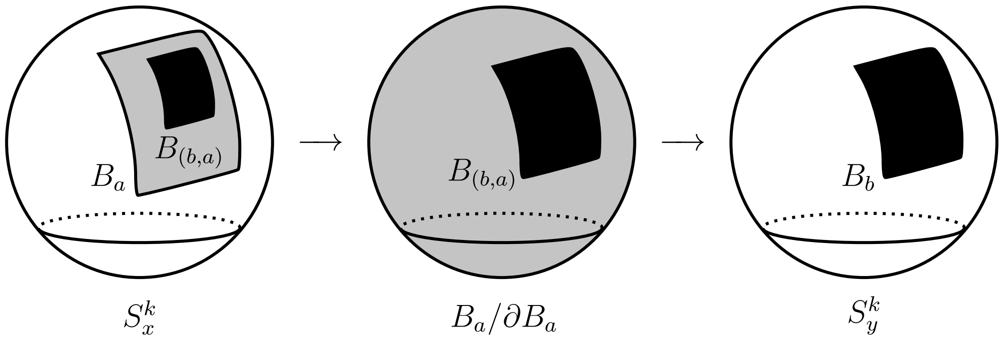


Figure 3-18: Fix two collections of subboxes $e \in E(\{B_x\}_{x\in X}, s_A)$ $e' \in E(\{B_y\}_{y\in Y}, s_B)$ and an element (b,a) in the composition correspondence $C = B \times_Y A$. Then by definition e' gives us a box $B_b \subset S^k_y$ (in black on the right) where $y = s_B(b) = t_A(a)$ and e gives us a box $B_a \subset S^k_x$ where $x = s_A(a)$ (in gray on the left). The maps in the figure are those appearing in (3.23), composing to give $\Phi(e, A)$. If we pull back the box B_b under this map (restricted to S^k_x), we obtain a smaller subbox $B_{(b,a)} \subset B_a \subset S^k_x$ for any such pair $(b,a) \in C$. Taking the collection of such boxes and preimages together, we see that $\Phi(e,A)^{-1}(e') \in E(\{B_x\}_{x\in X}, s_C)$.

Consider a G-equivariant correspondence $X \xleftarrow{s} A \xrightarrow{t} Y$. Then the spaces $\bigvee_{x\in X} S^k_x$ and $\bigvee_{y\in Y} S^k_y$ inherit a G-action via the canonical homeomorphism $B_x \cong B_{gx}$. In terms of the wedge summands, this G-action permutes the copies of S^k. However, to obtain a G-equivariant refinement $\bigvee_{x\in X} S^k_x \to \bigvee_{y\in Y} S^k_y$, a second condition needs to be imposed on the sub-boxes $\{B_a\} \in E(\{B_x\}, s)$.

Definition 3.6.1. Let $E_G(\{B_x\}, s)$ denote the subset of $E(\{B_x\}, s)$ consisting of the little boxes $\{B_a\}$ satisfying the following property: for each $g \in G$, $x \in X$, and $a \in s^{-1}(x)$, the canonical homeomorphism $B_x \to B_{gx}$ restricts to a homeomorphism $B_a \to B_{ga}$.

Note that with the above condition in place, the restricted homeomorphism $B_a \to B_{ga}$ is also the canonical one. In particular, if we identify all boxes B_{gx} canonically with $[0,1]^k$, then the subboxes $B_{ga} \subset B_{gx}$ for various g all have the same image in $[0,1]^k$.

Lemma 3.6.2. *Let $X \xleftarrow{s} A \xrightarrow{t} B$ be an equivariant correspondence. If $e \in E_G(\{B_x\}, s)$, then the induced box map $\Phi(e, A)$ is G-equivariant.*

Proof. Let $g \in G$ and $x \in X$. We need to verify commutativity of the following diagram

$$\begin{tikzcd}
S^k_x \arrow[r] \arrow[d, "g"'] & \bigvee_{a \in s^{-1}(x)} B_a/\partial B_a \arrow[r] \arrow[d, "\cong"] & \bigvee_{y \in Y} S^k_y \arrow[d, "g"] \\
S^k_{gx} \arrow[r] & \bigvee_{a \in s^{-1}(x)} B_{ga}/\partial B_{ga} \arrow[r] & \bigvee_{y \in Y} S^k_y
\end{tikzcd}$$

where the horizontal maps are those of (3.23) and the vertical maps are induced by canonical homeomorphisms between boxes. The left square commutes since $e \in E_G(\{B_x\}, s)$, and the right square commutes because all the maps there are induced by canonical homeomorphisms of boxes. $\square$

In the construction of stable homotopy refinements, it is crucial that the space of little boxes is highly connected and that boxes pull back to boxes. The remainder of this subsection is devoted to verifying these properties for $E_G(\{B_x\}, s)$. For the following statement, recall the notation for the quotient functor, introduced in Section 3.3.

Lemma 3.6.3. *Let $X \xleftarrow{s} A \xrightarrow{t} Y$ be a correspondence of G-sets and equivariant maps between them. Then we have the following homeomorphism of spaces of sub-boxes:*

$$E_G(\{B_x\}_{x \in X}, s) \cong E(\{B_{[x]}\}_{[x] \in X/G}, s/G).$$

Proof. For a fixed $x \in X$, a sub-collection of little boxes $\{B_a\}_{a \in s^{-1}(x)}$ in B_x determines the little boxes $\{B_c\}_{c \in s^{-1}(gx)}$ in B_{gx} for all $g \in G$. Explicitly, for each $a \in s^{-1}(x)$ and $g \in G$, $B_{ga} \subset B_{gx}$ is the image of B_a under the canonical homeomorphism $B_x \to B_{gx}$, and since s is a G-equivariant map, $c \in s^{-1}(gx)$ if and only if $c = ga$ for some $a \in s^{-1}(x)$.

Thus a collection of equivariant boxes in $E_G(\{B_x\}, s)$ is equivalent to a (non-equivariant) choice of boxes in each G-orbit of X for each G-orbit of A, which is the meaning of $E(\{B_{[x]}\}_{[x] \in X/G}, s/G)$. $\square$

Corollary 3.6.4. *$E_G(\{B_x\}, s)$ is $(k-2)$-connected.*

Proof. This follows from Lemma 3.6.3 together with the connectivity for spaces of (non-equivariant) sub-boxes shown in [52, Lemma 5.18]. $\square$

Lemma 3.6.5. *Let $X \xleftarrow{s_A} A \xrightarrow{t_A} Y$ and $Y \xleftarrow{s_B} B \xrightarrow{t_B} Z$ be equivariant correspondences. Let $\{B_x\}_{x\in X}$ and $\{B_y\}_{y\in Y}$ be collections of k-dimensional boxes, and let $e \in E_G(\{B_x\}, s_A)$, $e' \in E_G(\{B_y\}, s_B)$. Then $\Phi(e, A)^{-1}(e') \in E_G(\{B_x\}_{x\in X}, s_C)$.*

Proof. Fix $g \in G$. Let $B_{(b,a)} \in \Phi(e, A)^{-1}(e')$ and set $y = s_B(b)$ and $x = s_A(a)$ as in Figure 3-18. We need to show that the canonical homeomorphism $B_x \to B_{gx}$ sends $B_{(b,a)}$ to $B_{(gb,ga)}$. By construction, $\Phi(e, A)(B_{(b,a)}) = B_b$. Since $e \in E_G(\{B_x\}_{x\in X}, s_A)$, we also know that $gB_b = B_{gb}$. Recall from Lemma 3.6.2 that $\Phi(e, A)$ is G-equivariant. It follows that

$$B_{gb} = g\left(\Phi(e, A)(B_{(b,a)})\right) = \Phi(e, A)\left(gB_{(b,a)}\right).$$

Since $e' \in E_G(\{B_y\}_{y\in Y}, s_B)$ and $B_{(b,a)} \subset B_a$, we have that $gB_{(b,a)} \subset gB_a = B_{ga}$. Then $gB_{(b,a)}$ is a box contained in B_{ga}, which is sent to B_{gb} by $\Phi(e, A)$. It follows that $gB_{(b,a)} = B_{(gb,ga)}$, which completes the proof.

□

3.6.3 From Burnside functors to spatial refinements

In this section we describe how to use box maps to transform a Burnside functor $F : \underline{2}^n \to \mathscr{B}_G$ into a certain homotopy coherent diagram of spaces, called a *spatial refinement* of F, in an equivariant manner.

Let Top_*^G denote the category of based G-spaces. We refer the reader to [78, Section 4.2] for the definition and discussion of homotopy coherent diagrams and homotopy colimits in the equivariant setting, parallel to the non-equivariant treatment in [52, Section 4.2]. The following is an equivariant analogue of [52, Definition 5.21] and [52, Proposition 5.22] (and extends [78, Definition 4.11] and [78, Proposition 4.12] from $\mathbb{Z}/2\mathbb{Z}$ to general finite groups G).

Definition 3.6.6. Fix a small category $\mathscr{C}$ and a strictly unitary lax 2-functor $F : \mathscr{C} \to \mathscr{B}_G$. A *$k$-dimensional spatial refinement* of F is a homotopy coherent diagram $\widetilde{F}_k : \mathscr{C} \to \mathrm{Top}_*^G$ satisfying

(1) For any $u \in \mathscr{C}$, $\widetilde{F}_k(u) = \bigvee_{x \in F(u)} S^k$.

(2) For any sequence $u_0 \xrightarrow{f_1} \cdots \xrightarrow{f_m} u_m$ of morphisms in $\mathscr{C}$ and $t \in I^{m-1}$, the map

$$\widetilde{F}_k(f_m, \dots, f_1)(t) : \bigvee_{x \in F(u_0)} S^k \to \bigvee_{y \in F(u_m)} S^k$$

is a box map which refines the correspondence $F(f_m \circ \cdots \circ f_1)$. Note that, by assumption, the diagram lands in Top_*^G, so that the map in (2) is also G-equivariant.

Proposition 3.6.7. *Let $\mathscr{C}$ be a small category in which every sequence of composable non-identity morphisms has length at most n, and let $F : \mathscr{C} \to \mathscr{B}_G$ be a strictly unitary lax 2-functor.*

(1) *If $k \geq n$, there is a k-dimensional spatial refinement of F.*

(2) *If $k \geq n+1$, then any two k-dimensional spatial refinements of F are equivariantly weakly equivalent (see [78, Section 4.2] for a detailed discussion of weak equivalences between spatial refinements).*

(3) *If $\widetilde{F}_k$ is a k-dimensional spatial refinement of F, then the (reduced) suspension of each $\widetilde{F}_k(u)$ and of each $\widetilde{F}_k(f_m, \dots, f_1)(t)$ gives a $(k+1)$-dimensional spatial refinement of F.*

Proof. This is completely parallel to the proofs of [52, Proposition 5.22] and [78, Proposition 4.12]. The modification is that the maps

$$e_{f_m, \dots, f_1} : I^{m-1} \to E(\{B_x\}, s_{f_m \circ \cdots \circ f_1})$$

should land in $E_G(\{B_x\}, s_{f_m \circ \cdots \circ f_1})$, which is still highly connected by Corollary 3.6.4. We have also verified that equivariant boxes pull back to equivariant boxes in Lemma 3.6.5. Note that suspension respects the group action, which permutes spheres. $\square$

3.6.4 From spatial refinements to realizations

In this section we discuss how to pass from a G-equivariant homotopy coherent spatial refinement $\widetilde{F}_k$ to a realization $\|F\|_k \in \mathrm{Top}_*^G$. We also recall some of the cellular properties of such $\|F\|_k$ and the induced maps between them.

Following [52, Definition 5.1], we begin by defining a slight enlargement of the cube category $\underline{2}^n$, denoted $\underline{2}^n_+$. The objects of $\underline{2}^n_+$ are $\mathrm{ob}(\underline{2}^n) \cup \{*\}$; that is, $\underline{2}^n_+$ has an extra object added. Set $\mathrm{Hom}_{\underline{2}^n_+}(u,v) = \mathrm{Hom}_{\underline{2}^n}(u,v)$ if $u, v \in \underline{2}^n$. Otherwise, for $u \in \underline{2}^n \setminus \{0\}$, set $\mathrm{Hom}_{\underline{2}^n_+}(u,*)$ to consist of a single morphism. Finally, set $\mathrm{Hom}_{\underline{2}^n_+}(0,*) = \mathrm{Hom}_{\underline{2}^n_+}(*,0) = \mathrm{Hom}_{\underline{2}^n_+}(*,u) = \varnothing$.

Let $F : \underline{2}^n \to \mathscr{B}_G$ be a Burnside functor, and let $\widetilde{F}_k : \underline{2}^n \to \mathrm{Top}_*^G$ be a k-dimensional spatial refinement of F. Extend $\widetilde{F}_k$ to a homotopy coherent diagram $\widetilde{F}_k^+ : \underline{2}^n_+ \to \mathrm{Top}_*^G$ by setting $\widetilde{F}_k^+(*)$ to be a single point. Following [78, Definition 4.9], define the space

$$\|F\|_k := \operatorname{hocolim} \widetilde{F}_k^+, \tag{3.24}$$

called a *realization* of F. Since the homotopy coherent diagram $\widetilde{F}_k$ takes values in Top_*^G, the space $\|F\|_k$ is again a based G-space.

There is a cell structure on $\|F\|_k$, called the *coarse cell structure* in Section 4.4 of [78], with the cells of $\|F\|_k$ in bijection with $\coprod_{u \in \underline{2}^n} F(u)$. This cell structure is described in [52, Section 6].

Lemma 3.6.8. *With the above notation,*

(1) *The G-action on $\|F\|_k$ is cellular, and the bijection*

$$\{\text{Cells of } \|F\|_k\} \longleftrightarrow \coprod_{u \in \underline{2}^n} F(u)$$

is G-equivariant.

(2) *The G-action on $\|F\|_k$ is free away from the basepoint.*

(3) *The weak equivalences of Proposition 3.6.7 induce equivariant homotopy equivalences on realizations. Thus $\|F_k\|$ is well-defined and $\Sigma\|F\|_k \simeq \|F\|_{k+1}$.*

Proof. Statement (1) follows from inspecting the cell decomposition in [52, Proposition 6.1] (see also the discussion in [78, Section 4.4]). Statement (2) follows from (1) and the fact that G acts freely on the set $\coprod_{u \in \underline{2}^n} F(u)$.

For (3), we first note that (1) and (2) imply our realizations are G-CW complexes (the action of G is cellular and its fixed set is the basepoint, which is trivially a subcomplex). Then as described in [78, Section 4.5], equivariant weak equivalences of homotopy coherent diagrams induce equivariant weak equivalences on their homotopy colimits (which can be taken to be cellular), which induce equivariant homotopy equivalences for large enough k by the equivariant Whitehead theorem. $\square$

Now recall from Section 3.3 that the totalization $\mathrm{Tot}(F)$ of an equivariant Burnside functor $F : \underline{2}^n \to \mathscr{B}_G$ is a complex of $\mathbb{Z}[G]$-modules, and that any natural transformation $\eta : F_1 \to F_0$ between two such functors induces a $\mathbb{Z}[G]$-linear chain map $\mathrm{Tot}(\eta) : \mathrm{Tot}(F_1) \to \mathrm{Tot}(F_0)$. Also recall [78, Lemma 4.15], which in particular says that if $F_1, F_0 : \underline{2}^n \to \mathscr{B}$ are Burnside functors and $\eta : F_1 \to F_0$ is a natural transformation, then there is an induced map $\eta : \|F_1\|_k \to \|F_0\|_k$ for any realization. We record [78, Proposition 4.16] below relating these notions, where the notation $[k]$ denotes an upwards homological shift by k (that is, $C^i[k] := C^{i-k}$).

Proposition 3.6.9 ([78, Proposition 4.16]). *Given $F : \underline{2}^n \to \mathscr{B}$, then its reduced shifted cellular chain complex $\widetilde{C}_*^{cell}(\|F\|_k)[-k]$ is isomorphic to* $\mathrm{Tot}(F)$*, with the cells mapping to the corresponding generators. If $\eta : F_1 \to F_0$ is a natural transformation of Burnside functors, then the map $\|F_1\|_k \to \|F_0\|_k$ is cellular, and the induced cellular chain map agrees with* $\mathrm{Tot}(\eta)$.

When F takes values in $\mathscr{B}_G$, the above discussion shows that $\widetilde{C}_*^{cell}(\|F\|_k)[-k]$ is a $\mathbb{Z}[G]$-module, and the isomorphism

$$\widetilde{C}_*^{cell}(\|F\|_k)[-k] \cong \mathrm{Tot}(F)$$

of Proposition 3.6.9 is an isomorphism of complexes of $\mathbb{Z}[G]$-modules (see [78, Proposition 4.23]). Likewise, a natural transformation η between two equivariant Burnside functors induces an equivariant cellular map on their realizations that recovers $\mathrm{Tot}(\eta)$ on the chain complex level.

Lemma 3.6.10. ([78, Lemma 4.17]) *Let $F_1, F_0 : \underline{2}^n \to \mathscr{B}_G$ be equivariant Burnside functors, and let $\|F_1\|_k$, $\|F_0\|_k$ be k-dimensional spatial refinements. If $\eta : F_1 \to F_0$ is a natural transformation such that the induced map $\mathrm{Tot}(\eta) : \mathrm{Tot}(F_1) \to \mathrm{Tot}(F_0)$ is a chain homotopy equivalence, then the induced map $\eta : \|F_1\|_k \to \|F_0\|_k$ is an equivariant homotopy equivalence.*

Proof. The argument is the same as in [78, Lemma 4.17]. The previous discussion shows that the induced map on spaces $\eta : \|F_1\|_k \to \|F_0\|_k$ is G-equivariant and cellular. We may take k big enough, so that the realizations are simply connected. Note that the G-action on the realization is free away from the basepoint. Since $\mathrm{Tot}(\eta)$ is a $\mathbb{Z}[G]$-linear isomorphism on homology, the equivariant Whitehead theorem implies that $\eta : \|F_1\|_k \to \|F_0\|_k$ is an equivariant homotopy equivalence. □

Remark 3.6.11. Although the discussion in [78, Section 4.4] actually makes use of spatial refinements out of the category $(\underline{2}_+)^n$ rather than $\underline{2}^n_+$, the cellular structures described on the realizations there induce equivalent cellular structures on our realizations via a simple quotient within each cell. This is implicit in their reference to [52, Section 6] which builds the coarse structure for realizations using $\underline{2}^n_+$ as we are here; the alternative use of $(\underline{2}_+)^n$ in [52] is denoted by $\underline{2}^n_{\dagger}$ there.

3.6.5 The quantum annular spectrum $\mathcal{X}^r_{\mathbb{A}_\mathfrak{q}}(L)$

Finally, we turn to defining our quantum annular spectrum and proving Theorem 3.0.1. Let D be a diagram for an annular link with n_- negative crossings, and fix $G = \mathbb{Z}/r\mathbb{Z}$ for some $r \geq 2$. Let $F_\mathfrak{q} : \underline{2}^n \to \mathscr{B}_G$ be a quantum annular Burnside functor for D as provided by Theorem 3.5.2. Let $\widetilde{F}^+_{\mathfrak{q},k} : \underline{2}^n_+ \to \mathrm{Top}^G_*$ be a k-dimensional spatial refinement extended to the enlarged cube category $\underline{2}^n_+$, as described in Section 3.6.3 and the beginning of Section 3.6.4. Let $\|F_\mathfrak{q}\|_k$ be the realization, as defined in (3.24).

Definition 3.6.12. Define the *quantum annular Khovanov spectrum* $\mathcal{X}^r_{\mathbb{A}_\mathfrak{q}}(D)$ to be the suspension spectrum of $\|F_\mathfrak{q}\|_k$, desuspended $k + n_-$ times; that is,

$$\mathcal{X}^r_{\mathbb{A}_\mathfrak{q}}(D) := \Sigma^{-k-n_-}\left(\Sigma^\infty \|F_\mathfrak{q}\|_k\right).$$

By dualizing the isomorphism in Proposition 3.6.9, we obtain the following isomorphism of $\mathbb{Z}[G]$-modules:

$$C^*(\mathcal{X}^r_{\mathbb{A}_\mathfrak{q}}(D)) \cong CKh^*_{\mathbb{A}_\mathfrak{q}}(D) \otimes_{\Bbbk} \Bbbk_r. \tag{3.25}$$

Theorem 3.6.13. *For a fixed annular link diagram D and $r \geq 2$, the naive G-spectrum $\mathcal{X}^r_{\mathbb{A}_\mathfrak{q}}(D)$ is well-defined; that is, different choices during the construction yield equivariantly stably homotopy equivalent spectra.*

Proof. The construction of $\mathcal{X}^r_{\mathbb{A}_\mathfrak{q}}(D)$ requires a choice of generators at each vertex of the cube to build $F_\mathfrak{q}$, together with a choice of spatial refinement of $F_\mathfrak{q}$. Proposition 3.5.3 says that any two choices of generators give naturally isomorphic Burnside functors, which in turn yield equivariantly stably homotopy equivalent spectra by Lemma 3.6.10. Meanwhile, as long as k is large enough, any two spatial refinements yield homotopy equivalent realizations by Lemma 3.6.8, and thus equivariantly stably homotopy equivalent spectra. □

Finally, we address the independence of choice of diagram with the following theorem.

Theorem 3.6.14. *Let D and D' be two annular link diagrams for the same annular link $L \subset \mathbb{A} \times I$. Then $\mathcal{X}^r_{\mathbb{A}_\mathfrak{q}}(D)$ is equivariantly stably homotopy equivalent to $\mathcal{X}^r_{\mathbb{A}_\mathfrak{q}}(D')$, and as such we may use the notation $\mathcal{X}^r_{\mathbb{A}_\mathfrak{q}}(L)$ to denote the quantum annular G-equivariant stable homotopy type of L.*

Proof. The diagrams D and D' are connected by a series of moves corresponding either to the annular isotopies of Section 3.5.2 or Reidemeister moves. Isotopies were shown to induce natural isomorphisms of Burnside functors (Proposition 3.5.4), which therefore induce equivariant stable equivalences by Lemma 3.6.10.

With such planar equivalences at hand, we can assume that any Reidemeister move takes place in a disk disjoint from the seam μ. Such moves then induce homotopy equivalences in precisely the same fashion as they do for the classical Khovanov homotopy type [57, Section 6]. That is, any Reidemeister move corresponds to finding subfaces of the relevant cube corresponding to acyclic subcomplexes (or quotient complexes) in the totalization. (These subfaces are referred to as upwards- and downwards-closed subcategories in the original treatment of [57].) The complements of these acyclic faces can then be included into the large cube; the inclusion induces a map on stable homotopy types that gives an isomorphism on homology, and therefore is a stable equivalence by Whitehead's theorem. All of this continues to hold in the equivariant setting so long as the group G is finite - face inclusions induce equivariant maps by definition. □

As in [52, Section 4.7] and [78, Section 3.9], we also have a splitting of the functor $F_{\mathfrak{q}}$ into a coproduct over the two gradings qdeg and adeg (see (3.1)). Thus the spectrum also splits as a wedge sum

$$\mathcal{X}^r_{\mathbb{A}_{\mathfrak{q}}}(D) = \bigvee_{j,k} \mathcal{X}^{r;j,k}_{\mathbb{A}_{\mathfrak{q}}}(D)$$

where j corresponds to qdeg and k corresponds to adeg. As in [57, Theorem 1.1] (see also [51, Theorem 1]), Theorems 3.6.13 and 3.6.14 respect this splitting, as does Equation (3.25) as indicated below:

$$C^*(\mathcal{X}^{r;j,k}_{\mathbb{A}_{\mathfrak{q}}}(D)) \cong CKh^{*,j,k}_{\mathbb{A}_{\mathfrak{q}}}(D) \otimes_{\Bbbk} \Bbbk_r.$$

We end this section with some remarks about the spectrum $\mathcal{X}^r_{\mathbb{A}_{\mathfrak{q}}}(L)$. Although the construction above was aimed at building a naive G-spectrum, one could also construct a genuine G-spectrum in a similar manner by applying the functor Σ^∞_G, rather than Σ^∞, to the realization $\|F_{\mathfrak{q}}\|_k$. This functor produces genuine G-spectra using smash products with all G-representation spheres, rather than using only spheres with trivial G-action as Σ^∞ does.

We also note that, in general, G acts on $\mathcal{X}^r_{\mathbb{A}_{\mathfrak{q}}}(L)$ in a nontrivial way. Precisely, $\mathcal{X}^r_{\mathbb{A}_{\mathfrak{q}}}(L)$ does not decompose into a wedge product which is simply permuted by G. This is already evident on homology due to the calculation in [11, Proposition 6.9] for the annular closure of $(2,n)$ torus links. For an appropriate n, homological degree i, and q-degree j, the quantum annular homology is of the form $Kh^{i,j}_{\mathbb{A}_{\mathfrak{q}}}(T_{2,n}) = \Bbbk_r/(\mathfrak{q}^2+1)$.

3.7 Maps on spectra induced by annular link cobordisms

In [58, Section 3] the authors assign to a link cobordism $W \subset \mathbb{R}^3 \times [0,1]$ between two links $L_0 \subset \mathbb{R}^3 \times \{0\}$ and $L_1 \subset \mathbb{R}^3 \times \{1\}$ a map of spectra

$$\varphi_W : \mathcal{X}(L_1) \to \mathcal{X}(L_0),$$

such that the induced map on cohomology

$$\varphi_W^* : H^*(\mathcal{X}(L_0)) \to H^*(\mathcal{X}(L_1))$$

recovers the corresponding link cobordism maps W_* in Khovanov homology as studied in [33, 29, 8]. The map φ_W is constructed by first decomposing W into elementary cobordisms whose planar projections correspond to either Reidemeister moves or Morse moves (births/cups, saddles, or deaths/caps) and assigning maps to each elementary cobordism. A generically embedded W determines such a decomposition. It was conjectured in [58] that isotopic cobordisms induce stably homotopic maps, and this was recently proven in [54].

Now consider a cobordism $W \subset (\mathbb{A}\times I)\times[0,1]$ between two annular links $L_0 \subset \mathbb{A}\times I\times\{0\}$ and $L_1 \subset \mathbb{A} \times I \times \{1\}$ that is transverse to the 3-dimensional membrane $\mu \times I \times [0,1]$. In [11] the authors show that there is an induced map

$$W_* : Kh_{\mathbb{A}_\mathfrak{q}}(L_0) \to Kh_{\mathbb{A}_\mathfrak{q}}(L_1)$$

on the quantum annular homology, defined using the general theory of twisted horizontal traces and shadows established in [11, Section 3], as well as the functoriality of Chen-Khovanov bimodules under tangle cobordisms ([19, Proposition 6]). The map induced by W on the chain complex level can be determined by the sequence of maps given in [11, Equation (7.2)].

By [11, Theorem B], an isotopy of W can alter the map W_* by a sign change and a power of $\mathfrak{q}$; the sign ambiguity is inherited from the similar statement in the usual Khovanov homology, while the power of $\mathfrak{q}$ comes from the ability to isotope parts of W through the membrane. If one instead demands that isotopies fix the membrane, then W_* is well-defined up to a sign.

The main goal in this section is to prove Theorem 3.7.2. We first establish some preliminary results in the next section, where we compute W_* explicitly for certain elementary cobordisms; this computation is used in the proof of Theorem 3.7.2.

3.7.1 Elementary cobordisms

This section establishes a technical lemma needed for the proof of Theorem 3.7.2. We compare two ways of constructing a map on quantum annular chain complexes for certain elementary annular link cobordisms W. On the one hand, W induces a chain map W_* as defined in [11, Equation (7.2)]. On the other hand, for each type of elementary annular link cobordism W, we can define a second map $W_\bullet$ tailored towards the maps on spectra corresponding to the constructions in this chapter (mainly those in Sections 3.2.1 and 3.5.2). Our goal will be to show that these two maps $W_*, W_\bullet$ differ at most by some power of $\mathfrak{q}$ in all cases.

We begin by describing the general construction of the map W_*. Let $W \subset \mathbb{A} \times I \times [0,1]$ be a corbordism between annular links L and L' which intersect the membrane in k and ℓ points respectively, and let T, T' denote the tangles obtained by cutting L and L' along the membrane. Then W intersects the 3-dimensional membrane $\mu \times I \times [0,1]$ in a (k,ℓ)-tangle P. As in [11, Section 7.1], we represent W by a tangle cobordism $\widetilde{W} : PT \to T'P$:

$$\begin{tikzcd} k \arrow[r, "T"] \arrow[d, "P"'] & k \arrow[d, "P"] \arrow[dl, Rightarrow, "\widetilde{W}"'] \\ \ell \arrow[r, "T'"'] & \ell \end{tikzcd} \tag{3.26}$$

The chain map $W_* : CKh_{\mathbb{A}_\mathfrak{q}}(L) \to CKh_{\mathbb{A}_\mathfrak{q}}(L')$ is then given in each homological grading and quantum grading by the formula (7.2) in [11].

To describe the map $W_\bullet$, we distinguish four types of elementary cobordisms.

I. Reidemeister moves away from the seam.

II. Morse moves away from the seam.

Figure 3-19

III. Moving an arc across the seam as in the $P^{\pm 1}$ and $N^{\pm 1}$ moves of Figure 3-4.

IV. Moving a crossing through the seam as in Figure 3-19.

For Type I moves, we define $W_\bullet$ as in [33] and [8]; that is, a Reidemeister move is assigned its chain homotopy equivalence.

For the remaining types of moves, let D and D' denote the diagrams for L and L' differing locally as indicated by the elementary cobordism W, and let n be the number of crossings. For each $u \in \{0,1\}^n$, W induces an annular cobordism $R_u : D_u \to D'_u$ in $\mathbb{A} \times [0,1]$. The quantum annular TQFT $\mathcal{F}_{\mathbb{A}_\mathfrak{q}}$ assigns a map to this cobordism; after tensoring with $\Bbbk_r$, we write this as

$$\mathcal{F}^r_{\mathbb{A}_\mathfrak{q}}(R_u) : \mathcal{F}^r_{\mathbb{A}_\mathfrak{q}}(D_u) \to \mathcal{F}^r_{\mathbb{A}_\mathfrak{q}}(D'_u).$$

If we pick out generators for $\mathcal{F}^r_{\mathbb{A}_\mathfrak{q}}(D_u)$ and $\mathcal{F}^r_{\mathbb{A}_\mathfrak{q}}(D'_u)$ via cobordisms from standard configurations as in Section 3.2, then the image of these generators under this map can be computed by composing cobordisms. We will use shifted copies of this map to define the components $W_{\bullet u} : \mathcal{F}^r_{\mathbb{A}_\mathfrak{q}}(D_u) \to \mathcal{F}^r_{\mathbb{A}_\mathfrak{q}}(D'_u)$ of $W_\bullet$ on each smoothing individually. Note that this is precisely how the natural isomorphisms of the quantum annular Burnside functors are determined in Section 3.5.2 (although there we omitted the functor $\mathcal{F}^r_{\mathbb{A}_\mathfrak{q}}$ from the notation).

If W is of Type II or Type III, we define $W_{\bullet u}$ on each smoothing to be $\mathcal{F}^r_{\mathbb{A}_\mathfrak{q}}(R_u)$. Notice that for Type II moves, this is equivalent to defining $W_\bullet$ as in [33] and [8] where Morse moves are assigned either the unit, saddle map, or counit on each smoothing, corresponding to 0-handle, 1-handle, and 2-handle attachments respectively.

Finally, if W is of Type IV, then there are four cases to consider depending on the type of crossing and the direction of movement across the seam. In all of these cases, we will define

$$W_{\bullet u} := \mathfrak{q}^a \mathcal{F}^r_{\mathbb{A}_\mathfrak{q}}(R_u), \tag{3.27}$$

for some power $a \in \mathbb{Z}$ which is determined by the resolution of the crossing near the seam. If the smoothing corresponding to u resolves this crossing into two parallel lines each intersecting μ once, we set $a := 0$ in the formula (3.27) for $W_{\bullet u}$, so that $W_{\bullet u} = \mathcal{F}^r_{\mathbb{A}_\mathfrak{q}}(R_u)$ once again. Note that R_u is just the identity cobordism in this case, so $W_{\bullet u}$ is the identity map. Otherwise, we set $a := 1$ for the moves (a) and (d), and $a := -1$ for moves (b) and (c).

Note that in all cases, $W_\bullet$ is a chain map; for Type I and II moves this follows from the definitions, while for Type III and IV moves this follows from the trace relations in Figure 2-16. Note also that, for Type III and IV moves, the maps $W_\bullet$ defined here are precisely the totalizations of the natural isomorphisms built in Proposition 3.5.4.

Lemma 3.7.1. *Let L and L' be annular links and let $W : L \to L'$ be an elementary cobordism.*

(1) *If W is of Type I, II, or III, then $W_* = W_\bullet$.*

(2) *If W is Type IV, then $W_* = \mathfrak{q}^m W_\bullet$ for some m which depends only on the sign of the crossing involved in the move.*

Proof. Throughout the proof, we will use $C(-)$ to denote the Chen-Khovanov complex of bimodules, which is denoted by $C_{CK}(-)$ in [11, Section 5.5].

The first thing to notice is that, in any case where the intersection tangle $P = W \cap \mu \times I \times [0, 1]$ has no crossings, the formula [11, (7.2)] for W_* simplifies drastically. There is no summation over indices i' since $C(P)$ has only one term. In all such cases (which include Types I, II, and III here), one shows that W_* is equal to $W_\bullet$ by direct comparison.

Finally, for elementary cobordisms W of Type IV, we focus on the case (a) from Figure 3-19. Observe that the tangle P in this case has a single crossing, that T and T' can be written as $T = T''P$ and $T' = PT''$, and that $\widetilde{W} : PT \to T'P$ is the identity cobordism. All of this is illustrated in Figure 3-20.

Using $\otimes$ to denote the tensor product over the relevant Chen-Khovanov arc algebra, the map $\widetilde{W}_* : C^i(T) \otimes C^{i'}(P) \to C^{i'}(P) \otimes C^i(T')$ is the identity on the summand $C^{i'}(P) \otimes C^{i-i'}(T'') \otimes C^{i'}(P)$ which appears in both $C^i(T) \otimes C^{i'}(P)$ and $C^{i'}(P) \otimes C^i(T')$, and $\widetilde{W}_*$ is 0 on the other summands. In particular, there is no need for summing over various i' in the

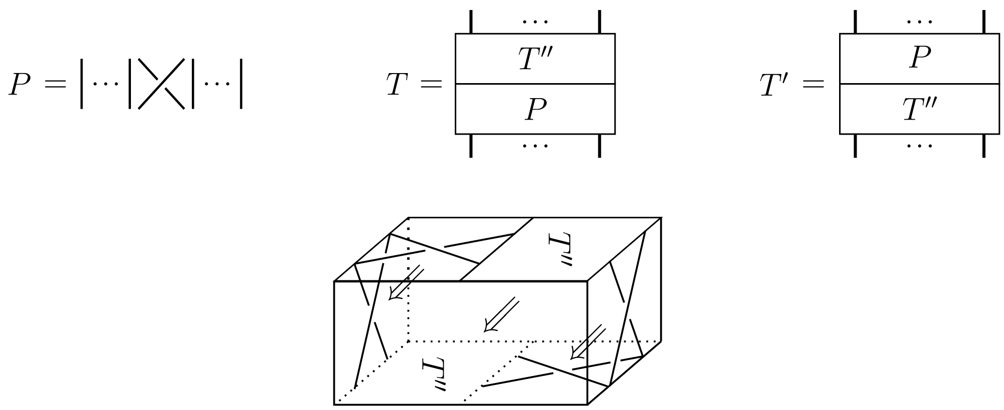


Figure 3-20

formula [11, (7.2)] for W_*, and the only possible difference between W_* and $W_\bullet$ acting on any given generator is in the use of the third map θ in [11, (7.2)] which permutes the tensor factors and multiplies generators $x \otimes y \otimes \alpha$ by a power of $\mathfrak{q}$ according to the grading of x as defined in [11, Section 5.5].

To analyze this potential difference, let P_0, P_1 denote the 0- and 1-resolutions of P. When constructing $W_\bullet$ on each resolution, we view x as living in either $\mathcal{F}_{CK}(P_0)$ or $\mathcal{F}_{CK}(P_1)$. If $x \in \mathcal{F}_{CK}(P_1)$, there is an extra factor of $\mathfrak{q}^1$ in our map (recall that we are considering case (a) amongst the Type IV moves; see the paragraph following (3.27)). However, in the definition of W_*, we view x as living in

$$\mathcal{F}_{CK}(P) = \big[\mathcal{F}_{CK}(P_0)\{m\} \to \mathcal{F}_{CK}(P_1)\{m+1\}\big],$$

where the grading shift m depends on the sign of the crossing. And so the map θ multiplies generators by an extra overall factor of $\mathfrak{q}^m$ when defining W_* as compared to how it would act when defining $W_\bullet$, as desired.

The proof for case (c) of Figure 3-19 is similar. For the cases (b) and (d), note that the map $W_\bullet$ is precisely the inverse of the map defined in (a) and (c), respectively. Moreover, the cobordisms W of moves (b) and (d) are inverses (in the category $Links_{\mathfrak{q}}(\mathbb{A})$, see [11, Proposition 6.8]) to the cobordisms of (a) and (c), respectively. This completes the proof.

$\square$

3.7.2 Cobordism maps of quantum annular spectra

We are now ready to state and prove the main result in this section.

Theorem 3.7.2 ([5, Theorem 6.1])**.** *Fix $r \in \mathbb{N}$. A generically embedded cobordism $W \subset \mathbb{A} \times I \times [0,1]$ between two annular links L_0 and L_1 induces a map*

$$\varphi_W^r : \mathcal{X}^r_{\mathbb{A}_\mathfrak{q}}(L_1) \to \mathcal{X}^r_{\mathbb{A}_\mathfrak{q}}(L_0)$$

whose induced map on cohomology

$$(\varphi_W^r)^* : H^*(\mathcal{X}^r_{\mathbb{A}_\mathfrak{q}}(L_0)) \to H^*(\mathcal{X}^r_{\mathbb{A}_\mathfrak{q}}(L_1))$$

equals the map W_* on quantum annular Khovanov homology over the ring $\Bbbk_r$.

Remark 3.7.3. We stress that Theorem 3.7.2 assigns a map to cobordisms W that come with a particular decomposition into a sequence of elementary cobordisms in the thickened annulus with membrane. The map φ_W^r is not known to be an invariant of W, even up to a factor of $\pm\mathfrak{q}^k$. But see Remark 3.7.4 below.

Proof of Theorem 3.7.2. A generic annular cobordism determines a sequence of elementary cobordisms (called *elementary string interactions* in [8, 25]), which are either Reidemeister moves or Morse moves. When accounting for the presence of the membrane $\mu \times I \times [0,1]$, there are certain additional elementary isotopies of a link through the seam which must be considered: we have the P and N moves of Figure 3-4, as well as pushing a crossing through the seam, as in (3.19). Meanwhile, that W is generic implies that all Reidemeister moves and Morse moves occur away from the seam.

For all elementary isotopies of the link, we already have stable homotopy equivalences via Theorem 3.6.14. As in the case for S^3, we wish to use the inverses of these maps. It is clear that such inverses induce the maps $W_\bullet$ described in the Appendix, and so according to Lemma 3.7.1, any such map will recover its corresponding W_* up to some power of $\mathfrak{q}$. We may thus compose any such stable homotopy equivalence with some iterate of the group action on $\mathcal{X}^r_{\mathbb{A}_\mathfrak{q}}(L_0)$ to define φ_W^r which induces precisely W_*.

Meanwhile, Morse moves induce natural transformations of Burnside functors in the same manner as they do in S^3: births induce correspondences which involve a w_+ label on the new (trivial, disjoint from the seam) circle; deaths induce correspondences which place a w_- label on the dying (trivial, disjoint from the seam) circle; and saddles utilize the higher dimensional cube which would be built if the diagram had a crossing placed at the point of the saddle. Carrying out these constructions equivariantly does not present any new issues, leading to constructions of maps φ^r_W via Proposition 3.6.9 which again induce the maps $W_\bullet$ of the Appendix. For these moves (Type II in Lemma 3.7.1) we have $W_* = W_\bullet$, concluding the proof of the Theorem. $\square$

Remark 3.7.4. Let W, W' be two isotopic annular link cobordisms with corresponding maps on spectra $\varphi^r_W, \varphi^r_{W'}$ via Theorem 3.7.2. Let $\tau_{\mathfrak{q}}$ denote the map on spectra determined by the action of the distinguished generator of the group $G = \mathbb{Z}/r\mathbb{Z}$. Then it is reasonable to conjecture that there exists some $m \in \mathbb{Z}$ such that the maps φ^r_W and $\tau^m_{\mathfrak{q}} \circ \varphi^r_{W'}$ are stably homotopic.

This is based on the similar conjecture regarding homotopy functoriality in [58] for cobordisms in $\mathbb{R}^3 \times [0,1]$, proven in [54]. Notice that the composition with the map $\tau^m_{\mathfrak{q}}$ recovers the ambiguity in the power of $\mathfrak{q}$ which is known to exist for the corresponding maps on the quantum annular homology.

Let $\mathbb{S}$ denote the sphere spectrum and define

$$\mathcal{X}^r_{\mathbb{A}_{\mathfrak{q}}}(\varnothing) := \bigvee_G \mathbb{S} \tag{3.28}$$

where G acts by permuting the wedge summands as usual. Then we have the following corollary for closed surfaces in $\mathbb{A} \times D^2$ formed by sweeping out a link in the S^1 direction. It is a spectral analogue of [11, Theorem D].

Corollary 3.7.5. *Let L be a link in the 3-ball B^3, and consider the surface $\widehat{W} = S^1 \times L$ in $\mathbb{A} \times D^2 \cong S^1 \times B^3$. Let W denote a copy of $\widehat{W}$ perturbed to be generic, viewed as a cobordism from $\varnothing$ to itself. Then the map*

$$\varphi^r_W : \mathcal{X}_{\mathbb{A}_{\mathfrak{q}}}(\varnothing) \longrightarrow \mathcal{X}_{\mathbb{A}_{\mathfrak{q}}}(\varnothing)$$

induces the map on quantum annular homology

$$(\varphi_W^r)^* : Kh_{\mathbb{A}_\mathfrak{q}}(\varnothing) = \Bbbk_r \longrightarrow \Bbbk_r = Kh_{\mathbb{A}_\mathfrak{q}}(\varnothing)$$

which is given by multiplication by the Jones polynomial of L, considered as an element of $\Bbbk_r$, up to a sign and a power of $\mathfrak{q}$ (where the standard basis of the groups $\Bbbk_r \cong \bigoplus_G \mathbb{Z}$ is written as $\{1, \mathfrak{q}, \dots \mathfrak{q}^{r-1}\}$).

Remark 3.7.6. In order to make sense of assigning a wedge of sphere spectra to the empty diagram $\varnothing$ in terms of Burnside functors, we assign to $\varnothing$ the functor $F_\mathfrak{q} : \underline{2}^0 = \{0\} \to \mathscr{B}_G$ defined by setting $F_\mathfrak{q}(0) := G \times \{1\}$, where $1 \in \mathcal{F}_{\mathbb{A}_\mathfrak{q}}(\varnothing) \otimes_\Bbbk \Bbbk_r = \Bbbk_r$ is the chosen generator. The spatial refinement is then a wedge of r spheres with no box maps, and in the homotopy colimit there is nothing to identify except the basepoint of $\bigvee_G S^k$ with the new basepoint in $\underline{2}^0_+$. Thus the final space is just a wedge of r spheres with the natural action, desuspended k times, and its reduced cohomology is isomorphic to $\Bbbk_r$ as a $\Bbbk_r$-module.

Proof of Corollary 3.7.5. Theorem 3.7.2 implies that $(\varphi_W^r)^* = W_*$, and by [11, Proposition 6.8], we also have $W_* = \pm\mathfrak{q}^k \widehat{W}_*$ for some $k \in \mathbb{Z}$. Finally, [11, Theorem D] states that $\widehat{W}_*$ is multiplication by the Jones polynomial of L. In fact, the more general statement about Lefschetz traces in [11, Theorem D] also applies here. □

3.8 Taking the quotient

Our goal in this section is to prove Theorem 3.0.2, stating that the quotient $\mathcal{X}^r_{\mathbb{A}_\mathfrak{q}}(D)/G$ of the quantum annular homotopy type is stably homotopy equivalent to the classical annular homotopy type $\mathcal{X}_\mathbb{A}(D)$. This is accomplished in two stages.

First we show that the quotient of $F_\mathfrak{q}$ is naturally isomorphic to the classical annular Burnside functor F_1 defined in Section 3.5.3. This will follow from Proposition 3.2.18 which establishes that the matching forced by powers of $\mathfrak{q}$ in the quantum theory agrees with the ladybug matching made with the left pair in the classical theory.

Next we show that taking the quotient of a spatial refinement for F_q yields a spatial refinement for F_1. The result will then follow from the property that homotopy colimits commute with taking quotients.

Let D be a diagram for an annular link with n crossings. Let $F_1 : \underline{2}^n \to \mathscr{B}$ denote the classical annular Khovanov Burnside functor, where the ladybug matching is made with the left choice. Recall the quotient functor $(-)/G : \mathscr{B}_G \to \mathscr{B}$ from Section 3.3. We can compose $F_\mathfrak{q} : \underline{2}^n \to \mathscr{B}_G$ with the quotient functor to obtain a Burnside functor $F_\mathfrak{q}/G : \underline{2}^n \to \mathscr{B}$. We will also use $(-)/G : \mathrm{Top}_*^G \to \mathrm{Top}_*$ to denote the quotient functor on G-spaces. It will be clear from context which functor is used.

Proposition 3.8.1. *The functors $F_\mathfrak{q}/G : \underline{2}^n \to \mathscr{B}$ and $F_1 : \underline{2}^n \to \mathscr{B}$ are naturally isomorphic.*

Proof. We will use the strategy of Section 3.4 to build a natural isomorphism $\eta : F_\mathfrak{q}/G \to F_1$. For $u \in \underline{2}^n$, there is a natural identification

$$F_\mathfrak{q}(u)/G = (G \times \Gamma(u))/G \cong \Gamma(u) = F_1(u).$$

Let $\psi_u : F_\mathfrak{q}(u)/G \to F_1(u)$ be the above bijection. For $u \geq_1 v$, let $A_{u,v}$ denote the correspondence assigned by F_q to the edge $u \to v$, and let $A'_{u,v}$ denote the correspondence assigned by F_1. There is an injection

$$A_{u,v}/G \hookrightarrow F_\mathfrak{q}(u)/G \times F_\mathfrak{q}(v)/G$$

given by $[\mathfrak{q}^k y, \mathfrak{q}^\ell x] \mapsto ([x],[y])$. We will identify $A_{u,v}/G$ with its image in $F_\mathfrak{q}(u)/G \times F_\mathfrak{q}(v)/G$. By Lemma 3.2.14, the map

$$\psi_u \times \psi_v : F_\mathfrak{q}(u)/G \times F_\mathfrak{q}(v)/G \to F_1(u) \times F_1(v)$$

restricts to a bijection

$$\psi_u \times \psi_v : A_{u,v}/G \to A'_{u,v}.$$

Thus conditions (NI 1), (NI 2), and (NI 3) of Section 3.4 are satisfied. It remains to verify that the diagram (3.15) commutes. Recall that we have used the ladybug matching made with the left pair for both $F_\mathfrak{q}$ and F_1. Then commutativity of (3.15) follows from Proposition 3.2.18. $\square$

Note that any homotopy coherent diagram in Top_*^G can be composed with the quotient functor $(-)/G : \mathrm{Top}_*^G \to \mathrm{Top}_*$ to give a homotopy coherent diagram in Top_* as in [78, Section 4.2].

Proposition 3.8.2. *Let $\widetilde{F_\mathfrak{q}} : \underline{2}^n \to \mathrm{Top}_*^G$ be a d-dimensional spatial refinement of $F_\mathfrak{q}$. Then the homotopy coherent diagram $\widetilde{F_\mathfrak{q}}/G$, obtained by applying $(-)/G$ to each $\widetilde{F_\mathfrak{q}}(v)$ and each $\widetilde{F_\mathfrak{q}}(f_m, \dots, f_1)$, is a d-dimensional spatial refinement of $F_\mathfrak{q}/G$.*

Proof. On a vertex $u \in \underline{2}^n$, since $F_\mathfrak{q}(u) = G \times F_1(u)$, it is again clear that the quotient

$$(\widetilde{F_\mathfrak{q}}/G)(u) = \widetilde{F_\mathfrak{q}}(u)/G = \Big(\bigvee_{\mathfrak{q}^k x \in F_\mathfrak{q}(u)} S^d\Big)/G$$

is canonically identified with

$$\widetilde{F_\mathfrak{q}/G}(u) = \bigvee_{x \in F_\mathfrak{q}(u)/G} S^d.$$

The key point is to recognize that, for any correspondence $A = F_\mathfrak{q}(f)$ assigned to some morphism $f : u \to v$ in $\underline{2}^n$ (with source and target maps s and t, respectively), the quotient of a box map refining A is itself a box map which refines the quotient of A. That is to say, given a choice of equivariant little boxes

$$e = \{B_a\}_{a \in A} \in E_G(\{B_{\mathfrak{q}^k x}\}_{\mathfrak{q}^k x \in F_\mathfrak{q}(u)}, s)$$

which induces a map

$$\Big(\bigvee_{\mathfrak{q}^k x \in F_\mathfrak{q}(u)} S^d\Big) \xrightarrow{\Phi(e,A)} \Big(\bigvee_{\mathfrak{q}^\ell y \in F_\mathfrak{q}(v)} S^d\Big),$$

the image of the boxes e in the quotient gives a new collection of little boxes

$$e/G := \{B_a/G\}_{a \in A} \cong \{B_{[a]}\}_{[a] \in A/G} \in E(\{B_x\}_{x \in F_\mathfrak{q}(u)/G}, s/G)$$

such that the following diagram commutes:

$$\begin{tikzcd}
\left(\bigvee_{\mathfrak{q}^k x\in F_{\mathfrak{q}}(u)} S^d\right)/G \arrow[r, "\Phi(e,A)/G"] \arrow[d, "\cong"'] & \left(\bigvee_{\mathfrak{q}^\ell y\in F_{\mathfrak{q}}(v)} S^d\right)/G \arrow[d, "\cong"] \\
\bigvee_{x\in F_{\mathfrak{q}}(u)/G} S^d \arrow[r, "\Phi(e/G,A/G)"] & \bigvee_{y\in F_{\mathfrak{q}}(v)/G} S^d
\end{tikzcd}.$$

Note that the group is simply permuting equivalent boxes, and recall that all correspondences coming from edges of the cube are subsets of the products of their source and target. Furthermore, all boxes remain distinct after taking the quotient. Thus the quotient boxes e/G fit into the commuting diagram as required, and the homotopy coherent diagram $\widetilde{F_{\mathfrak{q}}}/G$ can be identified with a spatial refinement of $F_{\mathfrak{q}}/G$. $\square$

Recall the enlarged cube category $\underline{2}^n_+$ from Section 3.6.4. Any homotopy coherent diagram $D : \underline{2}^n \to \mathrm{Top}_*$ (resp. $D : \underline{2}^n \to \mathrm{Top}_*^G$) can be extended to $D^+ : \underline{2}^n_+ \to \mathrm{Top}_*$ (resp. $D^+ : \underline{2}^n_+ \to \mathrm{Top}_*^G$) by setting $D^+(*)$ to be a single point space. Take k-dimensional spatial refinements of $F_{\mathfrak{q}}$, F_1, and $F_{\mathfrak{q}}/G$, denoted $\widetilde{F_{\mathfrak{q}}}$, $\widetilde{F_1}$, and $\widetilde{F_{\mathfrak{q}}/G}$ respectively (suppressing the subscript k). Extend each of them to diagrams $\widetilde{F_{\mathfrak{q}}}^+$, $\widetilde{F_1}^+$, and $\widetilde{F_{\mathfrak{q}}/G}^+$ out of $\underline{2}^n_+$, and take the corresponding homotopy colimits $\|F_{\mathfrak{q}}\|_k$, $\|F_1\|_k$, and $\|F_{\mathfrak{q}}/G\|_k$. We also have the homotopy coherent diagram $\widetilde{F_{\mathfrak{q}}}/G$; its two extensions $\widetilde{F_{\mathfrak{q}}}^+/G$ and $(\widetilde{F_{\mathfrak{q}}}/G)^+$ are equal.

Corollary 3.8.3. $\|F_1\|_k \simeq (\|F_{\mathfrak{q}}\|_k)/G$.

Proof. By Proposition 3.8.1, Proposition 3.6.7, and Lemma 3.6.10, there is a homotopy equivalence $\|F_1\|_k \simeq \|F_{\mathfrak{q}}/G\|_k$. By Proposition 3.8.2 and Proposition 3.6.7, there is also a homotopy equivalence $\|F_{\mathfrak{q}}/G\|_k \simeq \mathrm{hocolim}(\widetilde{F_{\mathfrak{q}}}/G)^+$. Since $(\widetilde{F_{\mathfrak{q}}}/G)^+ = \widetilde{F_{\mathfrak{q}}}^+/G$, we obtain

$$\mathrm{hocolim}(\widetilde{F_{\mathfrak{q}}}/G)^+ = \mathrm{hocolim}\left(\widetilde{F_{\mathfrak{q}}}^+/G\right).$$

Finally, homotopy colimits commute with the quotient functor $(-)/G$. This is clear from the definition of homotopy colimit, but is also stated explicitly as property (ho-4) in [78, Section 4.2]. Therefore

$$\operatorname{hocolim}\left(\widetilde{F_{\mathfrak{q}}}^{+}/G\right) \cong \operatorname{hocolim}(\widetilde{F_{\mathfrak{q}}})/G = (\|F_{\mathfrak{q}}\|_k)/G.$$

$\square$

Remark 3.8.4. The techniques in this section can also be used to show that, for any subgroup $\mathbb{Z}/s\mathbb{Z}$ of $\mathbb{Z}/r\mathbb{Z}$, the quotient $\mathcal{X}^r_{\mathbb{A}_{\mathfrak{q}}}(D)/(\mathbb{Z}/s\mathbb{Z})$ recovers the naive $(\mathbb{Z}/r\mathbb{Z})/(\mathbb{Z}/s\mathbb{Z})$-spectrum $\mathcal{X}^{r/s}_{\mathbb{A}_{\mathfrak{q}}}(D)$.

3.9 Towards lifting the $U_{\mathfrak{q}}(\mathfrak{sl}_2)$ action

This section concerns lifting the $U_{\mathfrak{q}}(\mathfrak{sl}_2)$ action on quantum annular homology, constructed in [11, Theorem D], to the level of spectra. We start by briefly summarizing the relevant background; see [11, Appendix A.1] for more details. Let $U_{\mathfrak{q}}(\mathfrak{sl}_2)$ be the $\Bbbk$-algebra generated by E, F, K and K^{-1} subject to the relations

$$\begin{aligned} KE &= \mathfrak{q}^2 EK & KK^{-1} &= 1 = K^{-1}K \\ KF &= \mathfrak{q}^{-2} FK & K - K^{-1} &= (\mathfrak{q} - \mathfrak{q}^{-1})(EF - FE) \end{aligned} \tag{3.29}$$

Let $\mathscr{C}$ be a configuration consisting of e essential circles and t trivial circles, with corresponding standard configuration $\mathscr{C}^\circ$. Recall from Section 2.4.2 that $\mathcal{F}_{\mathbb{A}_{\mathfrak{q}}}(\mathscr{C}^\circ) \cong V_{\mathfrak{q}}^{\otimes e} \otimes W^{\otimes t}$ carries an action of $U_{\mathfrak{q}}(\mathfrak{sl}_2)$ via an identification

$$V_{\mathfrak{q}}^{\otimes e} \cong V_1 \otimes V_1^* \otimes V_1 \otimes \cdots$$

where V_1 is the fundamental representation of $U_{\mathfrak{q}}(\mathfrak{sl}_2)$, and W is the trivial 2-dimensional representation. Fix an isotopy from $\mathscr{C}^\circ$ to $\mathscr{C}$. Then $\mathcal{F}_{\mathbb{A}_{\mathfrak{q}}}(\mathscr{C})$ inherits a $U_{\mathfrak{q}}(\mathfrak{sl}_2)$-action via the isomorphism $\mathcal{F}_{\mathbb{A}_{\mathfrak{q}}}(\mathscr{C}^\circ) \cong \mathcal{F}_{\mathbb{A}_{\mathfrak{q}}}(\mathscr{C})$. This action descends to a $U_{\mathfrak{q}}(\mathfrak{sl}_2)$ action on the homology $Kh_{\mathbb{A}_{\mathfrak{q}}}(L)$ for any annular link L.

The stable homotopy type $\mathcal{X}^r_{\mathbb{A}_{\mathfrak{q}}}(L)$ is constructed for the modified quantum annular functor $\mathcal{F}^r_{\mathbb{A}_{\mathfrak{q}}}$; see the discussion in the beginning of Section 3.5. In what follows we will also denote by $U_\mathfrak{q}(\mathfrak{sl}_2)$ the result of applying $(-) \otimes_\Bbbk \Bbbk_r$ to the algebra defined above. It has the same generators and relations, with the additional relation that $\mathfrak{q}^r = 1$. It is natural to ask if the actions of $E, F, K^{\pm 1}$ on $Kh_{\mathbb{A}_\mathfrak{q}}(L)$ can be lifted to the homotopy type. For each $J \in \{E, F, K, K^{-1}\}$, this would mean constructing an endomorphism $\mathcal{J} : \mathcal{X}^r_{\mathbb{A}_\mathfrak{q}}(L) \to \mathcal{X}^r_{\mathbb{A}_\mathfrak{q}}(L)$ such that the induced map on cohomology $\mathcal{J}^*$ is equal to the action of J.

Conjecture 3.9.1 ([5, Conjecture 1.4])**.** *The action of $U_\mathfrak{q}(\mathfrak{sl}_2)$ on quantum annular homology $Kh_{\mathbb{A}_\mathfrak{q}}(L)$ can be lifted to an action on $\mathcal{X}^r_{\mathbb{A}_\mathfrak{q}}(L)$.*

Aside from lifting the action of generators, the above conjecture also concerns lifting, in an appropriate sense, the defining $U_\mathfrak{q}(\mathfrak{sl}_2)$ relations. See Remark 3.9.3 for further discussion.

Let $F_1, F_0 : \underline{2}^n \to \mathscr{B}_G$ be Burnside functors. Recall from Proposition 3.6.9 (and the discussion following it) that a natural transformation $\eta : F_1 \to F_0$ induces a cellular map $\|F_1\|_k \to \|F_0\|_k$ which agrees with the map $\mathrm{Tot}(\eta) : \mathrm{Tot}(F_1) \to \mathrm{Tot}(F_0)$.

Each of E, F, K and K^{-1} can be viewed as $\Bbbk$-linear endomorphisms of $\mathcal{F}^r_{\mathbb{A}_\mathfrak{q}}(\mathscr{C})$. To address the above conjecture, it is natural to first ask whether the generators $E, F, K^{\pm 1}$ lift to natural endomorphisms of the quantum Burnside functor $F_\mathfrak{q}$ constructed in Section 3.5. Proposition 3.9.2 below answers this affirmatively for $K^{\pm 1}$; as we shall see, the situation for E and F is more complicated.

Let J denote one of E, F, K or K^{-1}. For a generator $x \in F_{\mathbb{A}_\mathfrak{q}}(\mathscr{C})$,

$$Jx = \sum_y \varepsilon_y y$$

where the sum is over generators and each ε_y is either 0 or of the form $\pm \mathfrak{q}^k$. Note that the appearance of negatively signed coefficients in odd Khovanov homology was dealt with by using signed correspondences ([78, Section 3.2]) and signed box maps ([78, Section 4.1]).

Let D be a diagram for an annular link with n crossings, and fix a corresponding Burnside functor $F_\mathfrak{q} : \underline{2}^n \to \mathscr{B}_G$ for D. For $u \in \underline{2}^n$, one can define the signed correspondence

$$J_u := \{(\mathfrak{q}^k y, \mathfrak{q}^\ell x) \in F_q(u) \times F_\mathfrak{q}(v) \mid \pm \mathfrak{q}^k y \text{ appears in } J(\mathfrak{q}^\ell x)\} \tag{3.30}$$

from $F_\mathfrak{q}(u)$ to $F_\mathfrak{q}(u)$, with the obvious source and target maps. The sign map $\sigma : J_u \to \mathbb{Z}_2 =$

$\{-1,1\}$ returns the sign of $\mathfrak{q}^k y$. Such a correspondence J_u is equivariant since J is $\Bbbk$-linear. In the case when $J = K^{\pm 1}$, the signs are not needed and we have the following lifts.

Proposition 3.9.2. *Let D be a diagram for an annular link with n crossings, and let $F_{\mathfrak{q}} : \underline{2}^n \to \mathscr{B}_G$ be a Burnside functor for D. Then there is a natural isomorphism $\mathcal{K}^{\pm 1} : F_{\mathfrak{q}} \to F_{\mathfrak{q}}$ which extends the correspondences $K_u^{\pm 1}$ of (3.30).*

Proof. We will use the strategy of Section 3.4. For each $u \in \underline{2}^n$ we define the required equivariant bijection $\psi_u^{\pm} : F_{\mathfrak{q}}(u) \to F_{\mathfrak{q}}(u)$ by

$$\psi_u^{\pm}(\mathfrak{q}^k x) = \mathfrak{q}^{k \mp \mathrm{adeg}(x)} x$$

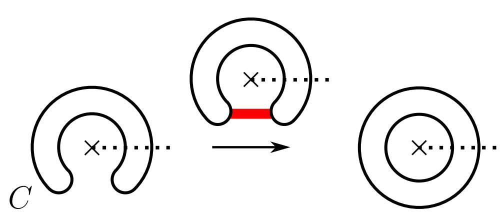
for all generators $x \in D_u$. Now let $u \geq_1 v$. The conditions (NI 1) and (NI 3) of Section 3.4 have already been checked on correspondences assigned to edges $\varphi_{u,v} : u \to v$ by $F_{\mathfrak{q}}$. In order to check condition (NI 2), we let $\mathfrak{q}^k y \in F_{\mathfrak{q}}(u)$, $\mathfrak{q}^\ell x \in F_{\mathfrak{q}}(v)$ be elements such that $\mathfrak{q}^k y$ appears in $d_{v,u}(q^\ell x)$. Since $d_{v,u}$ preserves annular degree, we have $\mathrm{adeg}(x) = \mathrm{adeg}(y)$, so $\mathfrak{q}^{k \mp \mathrm{adeg}(y)} y$ appears in $d_{v,u}(\mathfrak{q}^{\ell - \mp \mathrm{adeg}(x)} x)$. This implies condition (NI 2).

Thus we can build a natural transformation $\eta^{\pm}$ as in Section 3.4; the diagram 3.15 commutes, since ψ_u simply multiplies generators by powers of $\mathfrak{q}$. Finally, note that $K^{\pm 1} x = \mathfrak{q}^{\pm \mathrm{adeg}(x)} x$, so

$$K_u^{\pm} = \{(\mathfrak{q}^k x, \mathfrak{q}^{k \mp \mathrm{adeg}(x)} x) \mid \mathfrak{q}^k x \in F_{\mathfrak{q}}(u)\}$$

is naturally identified with $\eta^{\pm}(e_u)$ via $(\mathfrak{q}^k x, \mathfrak{q}^{k \mp \mathrm{adeg}(x)} x) \mapsto \mathfrak{q}^k x$. $\square$

We note that when $J = E$ or $J = F$, this overall strategy does not produce such a lift. Consider the saddle S from Example 3.2.10, thought of as the cube of resolutions for a link diagram with one crossing. Let $u = 1$ and $v = 0$ denote the vertices of the cube $\underline{2}$, and let $d : \mathcal{F}^r_{\mathbb{A}_{\mathfrak{q}}}(D_v) \to \mathcal{F}^r_{\mathbb{A}_{\mathfrak{q}}}(D_u)$ denote the differential. Let A denote the correspondence $F_{\mathfrak{q}}(\varphi_{u,v})$ from $F_{\mathfrak{q}}(u)$ to $F_{\mathfrak{q}}(v)$ assigned by the Burnside functor $F_{\mathfrak{q}}$.



The surgery formulas are

$$d(w_-) = 0 \qquad\qquad d(w_+) = v_+ \otimes v_- + \mathfrak{q}^{-1} v_- \otimes v_+,$$

and actions of E and F are given by

$$\begin{aligned} Ew_+ &= 0 \qquad & E(v_+ \otimes v_-) &= -v_+ \otimes v_+ \qquad & E(v_- \otimes v_+) &= \mathfrak{q} v_+ \otimes v_+ \\ Fw_+ &= 0 \qquad & F(v_+ \otimes v_-) &= v_- \otimes v_+ \qquad & F(v_- \otimes v_+) &= -\mathfrak{q} v_- \otimes v_- \end{aligned}$$

The correspondence $J_v \times_{F_{\mathfrak{q}}(v)} A$ is empty, whereas the correspondence $A \times_{F_{\mathfrak{q}}(u)} J_u$ is non-empty, containing two oppositely signed elements.

Remark 3.9.3. The above cancellations are already present at the classical ($\mathfrak{q} = 1$) level. In joint work with Krushkal and Willis [4], a lift of the action of the standard generators E, F, H of $\mathfrak{sl}_2$ to the annular Khovanov spectrum was defined. The construction uses the flow category formulation of Khovanov spectra in [57], and the methods do not extend immediately to the setting of $\mathcal{X}^r_{\mathbb{A}_\mathfrak{q}}(L)$.

One may also ask for some notion of a lift of the relations (3.29), or, in the $\mathfrak{q} = 1$ setting, of the $\mathfrak{sl}_2$ relations. The morphism space $[X, Y]$ between spectra X and Y forms an abelian group; moreover the endomorphisms $[X, X]$ are a ring under composition. The usual commutator bracket of $f, g \in [X, X]$, given by $fg - gf$, makes $[X, X]$ into a Lie algebra. A lift of the relations means verifying whether the map $\mathfrak{sl}_2 \to [\mathcal{X}_\mathbb{A}(L), \mathcal{X}_\mathbb{A}(L)]$ constructed in [4] is a Lie algebra homomorphism. In the quantum annular setting, if actions of $E, F \in U_\mathfrak{q}(\mathfrak{sl}_2)$ are lifted to maps of spectra, one may also ask if the resulting map from the free $\Bbbk_r$ algebra on generators E, F, K, K^{-1} into $[\mathcal{X}^r_{\mathbb{A}_\mathfrak{q}}(L), \mathcal{X}^r_{\mathbb{A}_\mathfrak{q}}(L)]$ factors through the relations in (3.29). This would produce a homotopical representation of the quantum group $U_\mathfrak{q}(\mathfrak{sl}_2)$.

Chapter 4

Equivariant annular homology

This chapter discusses constructions of equivariant Khovanov and Khovanov-Rozansky homology for links in the thickened annulus. A unifying feature of these theories is that the ground ring is the ring of polynomials rather than the subring of symmetric polynomials, contrary to other constructions of equivariant link homology which can be built over the smaller ground ring. Let us demonstrate this phenomenon in the simplest setting.

Recall the Frobenius systems $\mathcal{F}_E = (R_E, A_E)$ and $\mathcal{F}_\alpha = (R_\alpha, A_\alpha)$ from Definition 2.3.25. As explained in Section 2.3.4, $(R_\alpha A_\alpha)$ is an extension of (R_E, A_E) by identifying $E_1, E_2 \in R_E$ with elementary symmetric polynomials in variables α_1, α_2. We note an incompatibility of annular Khovanov homology with the theory $\mathcal{F}_E$. Let M be a free $\mathbb{Z}\times\mathbb{Z}$-graded R_E-module with basis m_-, m_+ in bidegrees $(-1,-1)$ and $(1,1)$, respectively, thought of as the module assigned to a single essential circle by an annular version of $\mathcal{F}_E$. Suppose $g\colon M \to M$ is an R_E-linear map of bidegree $(2,0)$. Then necessarily

$$g(m_-) = nE_1 m_- \tag{4.1}$$

for some $n \in \mathbb{Z}$. In particular, if g is the map assigned to the cobordism in Figure 4-1, then the defining relation $X^2 - E_1X + E_2 = 0$ of A_E implies

$$g^2(m_-) - E_1 g(m_-) + E_2 m_- = 0. \tag{4.2}$$

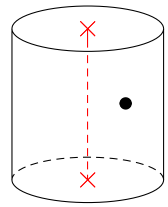


Figure 4-1: Dotted product cobordism on an essential circle.

However, Equations (4.1) and (4.2) are incompatible.

In Section 4.1 we define an equivariant lift of annular Khovanov homology by defining a suitable filtration on the larger theory $\mathcal{F}_\alpha$. Some structural properties of the resulting link homology are also discussed. The results of Section 4.1 have appeared in [2].

The remainder of this chapter develops a foam evaluation approach, in the spirit of Robert-Wagner [73], to annular link homology. Constructions in the $\mathfrak{sl}_2, \mathfrak{sl}_3$, and the general $\mathfrak{gl}_N$ setting are presented in Sections 4.2, 4.3, and 4.4, respectively. The results of Section 4.2 and Section 4.3 have appeared in joint work with Khovanov [3].

As discussed in Chapter 1, the main obstacle is determining how to assign an appropriate module to a web in the annulus (equivalently, in the punctured plane). Recall from the discussion of universal construction in Section 2.5.4 that a planar object is assigned a quotient of the free module generated by all cobordisms bounding that object. However, a single non-contractible circle in the annulus does not bound any surface in the thickened annulus. The approach is to model $\mathbb{A} \times I$ as the complement of a distinguished line $\mathcal{L} \subset \mathbb{R}^3$. Surfaces and foams are allowed to generically intersect $\mathcal{L}$, and they must carry additional decorations at these intersection points, which contribute factors to the foam evaluation. Similar to the earlier discussion, evaluation of these foams takes values in the ring of polynomials rather than in its subring ring of symmetric polynomials, contrary to Robert-Wagner foam evaluation.

4.1 Equivariant annular $\mathfrak{sl}_2$ link homology via filtrations

We are interested in an annular version of the equivariant link homology theory $\mathcal{F}_\alpha$ described in Section 2.3.4. Precisely, the goal is to fill in the dashed arrow in the diagram

$$\begin{tikzcd} \mathcal{BN}_\alpha(\mathbb{A}) \arrow[r, dashed, "\mathcal{G}_\alpha"] \arrow[d] & R_\alpha-\text{ggmod} \arrow[d] \\ \mathcal{BN}(\mathbb{A}) \arrow[r, "\mathcal{F}_{\mathbb{A}}"] & \mathbb{Z}-\text{ggmod} \end{tikzcd}$$

where the vertical arrows are obtained by setting $\alpha_1 = \alpha_2 = 0$. The earlier discussion justifies working with R_α rather than the subring of symmetric polynomials R_E. The desired functor $\mathcal{G}_\alpha$ is defined in Section 4.1.1 by taking the associated graded of a suitably defined annular degree filtration, in the spirit of the definition of $\mathcal{F}_{\mathbb{A}}$ given in Section 2.4.1.

We note that Boerner's relation, Figure 2-14, does not hold in the theory $\mathcal{G}_\alpha$; the map assigned to a dotted identity cobordism on a collection of essential circles is given in Equation (4.8). Maps assigned to saddle cobordisms are recorded in Equations (4.4) – (4.7). In Section 4.1.2 we invert $\mathcal{D}$ in the annular theory and show that the rank of the resulting homology depends only on the number of components.

4.1.1 The equivariant annular TQFT $\mathcal{G}_\alpha$

Let $Z \subset \mathbb{A}$ be a collection of disjoint simple closed curves, and view Z as embedded in $\mathbb{R}^2$. Consider $\mathcal{F}_\alpha(Z)$ with the following additional annular grading, denoted adeg as in Equation (2.9). Define elements of A_α,

$$v_0 = 1, \qquad v_1 = X - \alpha_1,$$
$$v_0' = 1, \qquad v_1' = X - \alpha_2,$$

with the annular gradings

$$\operatorname{adeg}(v_0) = \operatorname{adeg}(v_0') = -1, \qquad \operatorname{adeg}(v_1) = \operatorname{adeg}(v_1') = 1. \tag{4.3}$$

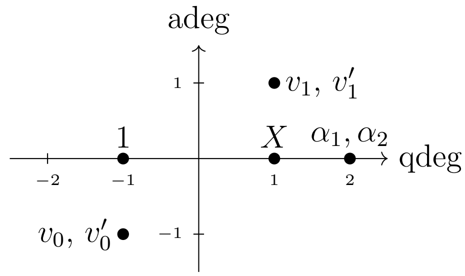


Figure 4-2: Bigradings, where $\{1, X\}$ corresponds to trivial circles, and $\{v_0, v_1\}$, $\{v_0', v_1'\}$ correspond to essential ones.

Remark 4.1.1. We note that the notation v_0, v_1 was used in Section 2.4.1 in the saddle formulas for non-equivariant annular homology.

Both $\{v_0, v_1\} = \{1, X - \alpha_1\}$ and $\{v_0', v_1'\} = \{1, X - \alpha_2\}$ is an R_α-basis for A_α. Together with the quantum grading qdeg, these bases equip A_α with two (isomorphic) structures of a bigraded R_α-module, with the bigrading given by $(\text{qdeg}, \text{adeg})$. The ground ring R_α lies in annular degree 0.

Let $Z \subset \mathbb{A}$ consist of n trivial and m essential circles, with the essential circles ordered from innermost (closest to the puncture $\times$) to outermost. Define the annular grading on

$$\mathcal{F}_\alpha(Z) = A_\alpha^{\otimes n} \otimes A_\alpha^{\otimes m}$$

by declaring that every copy of A_α corresponding to a trivial circle is concentrated in annular degree 0 and that the copy of A_α corresponding to the i-th essential circle $(1 \le i \le m)$ is given the homogeneous basis

$$\{v_0, v_1\} = \{1, X - \alpha_1\}$$

if i is odd and

$$\{v_0', v_1'\} = \{1, X - \alpha_2\}$$

if i is even. In other words, the essential circles are assigned one of two adeg-homogeneous bases, $\{1, X - \alpha_1\}$ or $\{1, X - \alpha_2\}$, in an alternating manner, with the innermost circle assigned $\{1, X - \alpha_1\}$. Bigradings are summarized in Figure 4-2

As in Section 2.4.1, it is convenient to distinguish the modules assigned to essential and trivial circles. Let V_α and V_α' denote A_α with homogeneous bases $\{v_0, v_1\}$ and $\{v_0', v_1'\}$, respectively. Then, for a collection of circles $Z \subset \mathbb{A}$, the i-th essential circle in Z is assigned V_α if i is odd and V'_α if i is even. We reserve the notation A_α for the module assigned to a trivial circle. Note that interchanging $\alpha_1 \leftrightarrow \alpha_2$ also interchanges $v_0 \leftrightarrow v'_0$ and $v_1 \leftrightarrow v'_1$.

With the above gradings at hand, we now prove the analogue of Lemma 2.4.1 in the equivariant setting.

Lemma 4.1.2. *Let $S \subset \mathbb{A} \times I$ be an elementary cobordism. Viewing S as a cobordism in $\mathbb{R}^2 \times I$, the map $\mathcal{F}\alpha(S)$ splits as a sum*

$$\mathcal{F}_\alpha(S) = \mathcal{F}_\alpha(S)_0 + \mathcal{F}_\alpha(S)_2$$

where $\mathcal{F}_\alpha(S)_0$ preserves adeg *and $\mathcal{F}_\alpha(S)_2$ increases* adeg *by* 2*.*

Proof. If the saddle component of S involves only trivial circles then the claim is immediate, since $\mathcal{F}_\alpha(S) = \mathcal{F}_\alpha(S)_0$ in this case. We verify the claim for the four elementary cobordisms in Figure 2-12 by rewriting $\mathcal{F}_\alpha(S)$ in terms of the bases for the circles involved. Terms where adeg is increased by 2 are boxed.

$$V_\alpha \otimes A_\alpha \xrightarrow{\text{(I)}} V_\alpha$$
$$v_0 \otimes 1 \mapsto v_0$$
$$v_1 \otimes 1 \mapsto v_1$$
$$v_0 \otimes X \mapsto \alpha_1 v_0 + \boxed{v_1}$$
$$v_1 \otimes X \mapsto \alpha_2 v_1$$

$$V_\alpha \otimes V'_\alpha \xrightarrow{\text{(II)}} A_\alpha$$
$$v_0 \otimes v'_0 \mapsto \boxed{1}$$
$$v_1 \otimes v'_0 \mapsto X - \alpha_1$$
$$v_0 \otimes v'_1 \mapsto X - \alpha_2$$
$$v_1 \otimes v'_1 \mapsto 0$$

$$V_\alpha \xrightarrow{\text{(III)}} V_\alpha \otimes A_\alpha$$
$$v_0 \mapsto v_0 \otimes X - \alpha_2 v_0 \otimes 1 + \boxed{v_1 \otimes 1}$$
$$v_1 \mapsto v_1 \otimes X - \alpha_1 v_1 \otimes 1$$

$$A_\alpha \xrightarrow{\text{(IV)}} V_\alpha \otimes V'_\alpha$$
$$1 \mapsto v_0 \otimes v'_1 + v_1 \otimes v'_0$$
$$X \mapsto \alpha_1 v_0 \otimes v'_1 + \alpha_2 v_1 \otimes v'_0 + \boxed{v_1 \otimes v'_1}$$

Our assignment for essential circles depends on nesting, so strictly speaking the above calculations do not handle all cases. However, note that for types (I) and (II), the position of the essential circle does not change, and for types (III) and (IV), the two essential circles involved in the saddle must be consecutive in the ordering. Thus a full verification amounts to interchanging $v_0 \leftrightarrow v_0'$, $v_1 \leftrightarrow v_1'$ in the input of above maps. One may check that this amounts to interchanging $v_0 \leftrightarrow v_0'$, $v_1 \leftrightarrow v_1'$, and $\alpha_1 \leftrightarrow \alpha_2$ in the output. $\square$

Corollary 4.1.3. (1) *Let $S \subset \mathbb{A} \times I$ be a cobordism. Viewing S as a cobordism in $\mathbb{R}^2 \times I$, the map $\mathcal{F}_{\mathbb{A}}(S)$ splits as a sum*

$$\mathcal{F}_\alpha(S) = \mathcal{F}_\alpha(S)_0 + \mathcal{F}_\alpha(S)_+$$

where $\mathcal{F}_\alpha(S)_0$ preserves adeg *and $\mathcal{F}_\alpha(S)_+$ strictly increases* adeg.

(2) *Let $S_1, S_2 \subset \mathbb{A} \times I$ be composable cobordisms. Then*

$$\mathcal{F}_\alpha(S_2 S_1)_0 = \mathcal{F}_\alpha(S_2)_0 \mathcal{F}_\alpha(S_1)_0.$$

Proof. For (1), write S as a composition $S = S_n \cdots S_1$ where each S_i is an elementary cobordism. Functoriality of $\mathcal{F}_\alpha$ and Lemma 4.1.2 yield

$$\begin{aligned}
\mathcal{F}_\alpha(S) &= \mathcal{F}_\alpha(S_n) \cdots \mathcal{F}_\alpha(S_1) \\
&= (\mathcal{F}_\alpha(S_n)_0 + \mathcal{F}_\alpha(S_n)_2) \cdots (\mathcal{F}_\alpha(S_1)_0 + \mathcal{F}_\alpha(S_1)_2) \\
&= \mathcal{F}_\alpha(S_n)_0 \cdots \mathcal{F}_\alpha(S_1)_0 + \text{terms that increase adeg}\,.
\end{aligned}$$

Therefore

$$\mathcal{F}_\alpha(S)_0 = \mathcal{F}_\alpha(S_n)_0 \cdots \mathcal{F}_\alpha(S_1)_0$$

is the desired adeg-preserving part, and the remaining terms constitute $\mathcal{F}_\alpha(S)_+$. Statement (2) follows from (1) in a similar fashion. $\square$

We are now ready for the main theorem.

Theorem 4.1.4. *There exists a functor $\mathcal{G}_\alpha \colon \mathcal{BN}_\alpha(\mathbb{A}) \to R_\alpha-$*ggmod *such that the following* *diagram commutes*

$$\begin{array}{ccc} \mathcal{BN}_\alpha(\mathbb{A}) & \xrightarrow{\mathcal{G}_\alpha} & R_\alpha\text{– ggmod} \\ \downarrow & & \downarrow \\ \mathcal{BN}(\mathbb{A}) & \xrightarrow{\mathcal{F}_\mathbb{A}} & \mathbb{Z}\text{– ggmod} \end{array}$$

where the vertical arrows are obtained by setting $\alpha_1 = \alpha_2 = 0$.

Proof. For a collection of circles $Z \subset \mathbb{A}$, set

$$\mathcal{G}_\alpha(S) := \mathcal{F}_\alpha(Z),$$

with the bigrading (qdeg, adeg) as defined earlier in this section. For a cobordism $S \subset \mathbb{A} \times I$, set

$$\mathcal{G}_\alpha(S) := \mathcal{F}_\alpha(S)_0$$

as in Corollary 4.1.3 (1). That $\mathcal{G}_\alpha$ is well-defined on cobordisms and factors through the relations in $\mathcal{BN}_\alpha(\mathbb{A})$ follows from the analogous statements for $\mathcal{F}_\mathbb{A}$. Corollary 4.1.3 (2) implies functoriality of $\mathcal{G}_\alpha$. Finally, commutativity of the diagram follows from deleting the boxed terms and setting $\alpha_1 = \alpha_2 = 0$ in the maps appearing in the proof of Lemma 4.1.2, and comparing the result with the maps (2.11)–(2.14). □

Maps assigned to the four elementary saddles in Figure 2-12 are recorded below. The full set of maps – that is, if other essential circles are present – can be obtained by interchanging $\alpha_1 \leftrightarrow \alpha_2$.

$$\begin{aligned} V_\alpha \otimes A_\alpha &\xrightarrow{\text{(I)}} V_\alpha \\ v_0 \otimes 1 &\mapsto v_0 \\ v_1 \otimes 1 &\mapsto v_1 \\ v_0 \otimes X &\mapsto \alpha_1 v_0 \\ v_1 \otimes X &\mapsto \alpha_2 v_1 \end{aligned} \tag{4.4}$$

$$\begin{aligned} V_\alpha \otimes V'_\alpha &\xrightarrow{\text{(II)}} A_\alpha \\ v_0 \otimes v'_0 &\mapsto 0 \\ v_1 \otimes v'_0 &\mapsto X - \alpha_1 \\ v_0 \otimes v'_1 &\mapsto X - \alpha_2 \\ v_1 \otimes v'_1 &\mapsto 0 \end{aligned} \tag{4.5}$$

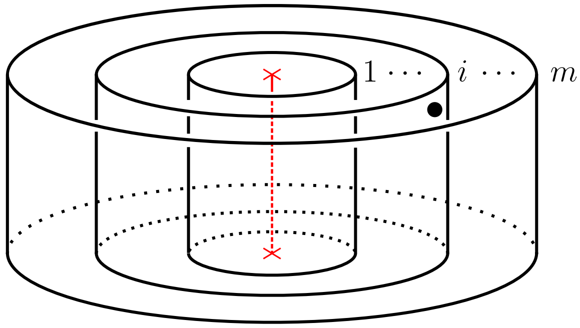


Figure 4-3: Product cobordism on $m > 0$ essential circles, with the i-th component dotted

$$\begin{aligned} V_\alpha &\xrightarrow{\text{(III)}} V_\alpha \otimes A_\alpha \\ v_0 &\mapsto v_0 \otimes X - \alpha_2 v_0 \otimes 1 \\ v_1 &\mapsto v_1 \otimes X - \alpha_1 v_1 \otimes 1 \end{aligned} \tag{4.6}$$

$$\begin{aligned} A_\alpha &\xrightarrow{\text{(IV)}} V_\alpha \otimes V'_\alpha \\ 1 &\mapsto v_0 \otimes v'_1 + v_1 \otimes v'_0 \\ X &\mapsto \alpha_1 v_0 \otimes v'_1 + \alpha_2 v_1 \otimes v'_0 \end{aligned} \tag{4.7}$$

Let $Z \subset \mathbb{A}$ consist of $m > 0$ essential circles, and let C be the i-th essential circle in Z. Consider the cobordism S whose underlying surface is the identity cobordism $Z \times I$, with a single dot on the component $C \times I$, as shown in Figure 4-3. Then $\mathcal{G}_\alpha(S)$ is the identity on all tensor factors except the one corresponding to C, and on C it is given by the left-hand side of (4.8) if i is odd, and the right-hand side if i is even.

$$\begin{aligned} V_\alpha &\to V_\alpha \\ v_0 &\mapsto \alpha_1 v_0 \\ v_1 &\mapsto \alpha_2 v_1 \end{aligned} \qquad \begin{aligned} V'_\alpha &\to V'_\alpha \\ v'_0 &\mapsto \alpha_2 v'_0 \\ v'_1 &\mapsto \alpha_1 v'_1 \end{aligned} \tag{4.8}$$

Observe that the functor $\mathcal{G}_\alpha$ is not monoidal, since the action of X on an essential circle depends on its nestedness.

Let $L \subset \mathbb{A} \times I$ be an oriented link with diagram D. Let

$$CKh^{\mathbb{A}}_\alpha(D) := \mathcal{G}_\alpha([[D]])$$

denote the chain complex obtained by applying $\mathcal{G}_\alpha$ to the formal complex $[[D]]$. The differential preserves bidegree, and the complex is an invariant of L up to bidegree-preserving chain homotopy equivalence.

The remainder of this section discusses variants of $\mathcal{G}_\alpha$. Instead of setting both $\alpha_1 = \alpha_2 = 0$, it is possible to set only $\alpha_1 = 0$ and rename the remaining parameter α_2 to $\alpha_2 = h$. Denote the resulting Frobenius pair by (R_h, A_h). Explicitly,

$$R_h = \mathbb{Z}[h],\ A_h = R_h[X]/(X^2 - hX).$$

It may also be obtained from (R_E, A_E) by setting $E_1 = h$, $E_2 = 0$; note that the obstruction to working with the Frobenius pair (R_E, A_E) (see Equations (4.1) and (4.2)) disappears when $E_2 = 0$. Collapsing (R_h, A_h) further to characteristic 2 (that is, applying $(-) \otimes_{R_h} \mathbb{Z}_2[h]$) recovers Bar-Natan's theory [8, Section 9.3]. We expect that the resulting annular homology is related to [83].

Let $L \subset \mathbb{A} \times I$ be an oriented link with diagram D. Viewing D as a diagram in $\mathbb{R}^2$ and applying $\mathcal{F}_{\mathbb{A}}$ to $[[D]]$ yields a chain complex $CKh_\alpha(D)$ of bigraded R_α-modules. Letting ∂ denote the differential, Lemma 4.1.2 implies that ∂ splits as a sum $\partial = \partial_0 + \partial_2$, where ∂_0 is of bidegree $(0, 0)$ and ∂_2 is of bidegree $(0, 2)$. As in [28], we can introduce an extra parameter β to account for ∂_2. Let $R_{\alpha\beta} = R_\alpha[\beta]$ with β in bidegree $(0, -2)$, and let $CKh^{\mathbb{A}}_{\alpha\beta}(D)$ be the chain complex over $R_{\alpha\beta}$ obtained by extending scalars,

$$CKh^{\mathbb{A}}_{\alpha\beta}(D) := CKh_\alpha(D) \otimes_{R_\alpha} R_{\alpha\beta}$$

in homological degree i and differential ∂_β given by

$$\partial_\beta := \partial_0 + \beta\partial_2.$$

Note that ∂_β preserves bidegree. Maps assigned to the four elementary saddles in Figure 2-12 are given below.

$$\begin{aligned} V_\alpha \otimes A_\alpha &\xrightarrow{\text{(I)}} V_\alpha \\ v_0 \otimes 1 &\mapsto v_0 \\ v_1 \otimes 1 &\mapsto v_1 \\ v_0 \otimes X &\mapsto \alpha_1 v_0 + \ \beta v_1 \\ v_1 \otimes X &\mapsto \alpha_2 v_1 \end{aligned} \qquad \begin{aligned} V_\alpha \otimes V'_\alpha &\xrightarrow{\text{(II)}} A_\alpha \\ v_0 \otimes v'_0 &\mapsto \beta \\ v_1 \otimes v'_0 &\mapsto X - \alpha_1 \\ v_0 \otimes v'_1 &\mapsto X - \alpha_2 \\ v_1 \otimes v'_1 &\mapsto 0 \end{aligned}$$

$$\begin{aligned} V_\alpha &\xrightarrow{\text{(III)}} V_\alpha \otimes A_\alpha \\ v_0 &\mapsto v_0 \otimes X - \alpha_2 v_0 \otimes 1 + \beta v_1 \otimes 1 \\ v_1 &\mapsto v_1 \otimes X - \alpha_1 v_1 \otimes 1 \end{aligned} \qquad \begin{aligned} A_\alpha &\xrightarrow{\text{(IV)}} V_\alpha \otimes V'_\alpha \\ 1 &\mapsto v_0 \otimes v'_1 + v_1 \otimes v'_0 \\ X &\mapsto \alpha_1 v_0 \otimes v'_1 + \alpha_2 v_1 \otimes v'_0 + \beta v_1 \otimes v'_1 \end{aligned}$$

4.1.2 Inverting $\mathcal{D}$ in equivariant annular homology

Recall the Frobenius pair $(R_{\alpha\mathcal{D}}, A_{\alpha\mathcal{D}})$ from [40], which was reviewed in Definition 2.3.30. Let $\mathcal{G}_{\alpha\mathcal{D}}$ denote the composition

$$\mathcal{BN}_\alpha(\mathbb{A}) \xrightarrow{\mathcal{G}_\alpha} R_\alpha\text{– ggmod} \to R_{\alpha\mathcal{D}}\text{– ggmod}$$

where the second functor is extension of scalars. Consider the following elements of $A_{\alpha\mathcal{D}}$,

$$\begin{aligned} \overline{v}_0 &:= v_0 = 1, & \overline{v}_1 &:= \frac{v_1}{\alpha_2 - \alpha_1} = \frac{X - \alpha_1}{\alpha_2 - \alpha_1}, \\ \overline{v}'_0 &:= v'_0 = 1, & \overline{v}'_1 &:= \frac{v'_1}{\alpha_1 - \alpha_2} = \frac{X - \alpha_2}{\alpha_1 - \alpha_2}. \end{aligned}$$

As in Section 4.1.1, let $V_{\alpha\mathcal{D}}$ and $V'_{\alpha\mathcal{D}}$ denote the module $A_{\alpha\mathcal{D}}$ with distinguished homogeneous bases $\{\overline{v}_0, \overline{v}_1\}$ and $\{\overline{v}'_0, \overline{v}'_1\}$, respectively. For a collection of circles $Z \subset \mathbb{A}$, the i-th essential circle in Z is assigned $V_{\alpha\mathcal{D}}$ if i is odd and $V'_{\alpha\mathcal{D}}$ if i is even. The notation $A_{\alpha\mathcal{D}}$ is reserved for trivial circles, with distinguished basis $\{e_1, e_2\}$, see Equation (2.7). Bigradings are summarized in Figure 4-4.

With respect to these bases, the maps assigned to the four elementary saddles in Figure

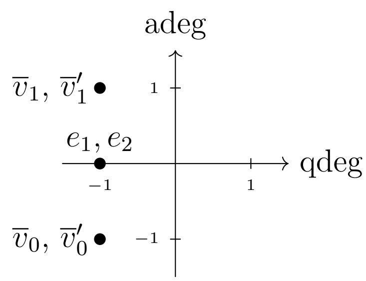


Figure 4-4: Bigradings

2-12 are recorded below.

$$\begin{aligned} V_{\alpha\mathcal{D}} \otimes A_{\alpha\mathcal{D}} &\xrightarrow{\text{(I)}} V_{\alpha\mathcal{D}} \\ \overline{v}_0 \otimes e_1 &\mapsto 0 \\ \overline{v}_1 \otimes e_1 &\mapsto \overline{v}_1 \\ \overline{v}_0 \otimes e_2 &\mapsto \overline{v}_0 \\ \overline{v}_1 \otimes e_2 &\mapsto 0 \end{aligned} \tag{4.9}$$

$$\begin{aligned} V_{\alpha\mathcal{D}} \otimes V'_{\alpha\mathcal{D}} &\xrightarrow{\text{(II)}} A_{\alpha\mathcal{D}} \\ \overline{v}_0 \otimes \overline{v}_0' &\mapsto 0 \\ \overline{v}_1 \otimes \overline{v}_0' &\mapsto e_1 \\ \overline{v}_0 \otimes \overline{v}_1' &\mapsto e_2 \\ \overline{v}_1 \otimes \overline{v}_1' &\mapsto 0 \end{aligned} \tag{4.10}$$

$$\begin{aligned} V_{\alpha\mathcal{D}} &\xrightarrow{\text{(III)}} V_{\alpha\mathcal{D}} \otimes A_{\alpha\mathcal{D}} \\ \overline{v}_0 &\mapsto (\alpha_1 - \alpha_2)\overline{v}_0 \otimes e_2 \\ \overline{v}_1 &\mapsto (\alpha_2 - \alpha_1)\overline{v}_1 \otimes e_1 \end{aligned} \tag{4.11}$$

$$\begin{aligned} A_{\alpha\mathcal{D}} &\xrightarrow{\text{(IV)}} V_{\alpha\mathcal{D}} \otimes V'_{\alpha\mathcal{D}} \\ e_1 &\mapsto (\alpha_2 - \alpha_1)\overline{v}_1 \otimes \overline{v}_0' \\ e_2 &\mapsto (\alpha_1 - \alpha_2)\overline{v}_0 \otimes \overline{v}_1' \end{aligned} \tag{4.12}$$

To obtain the full set of maps – that is, if other essential circles are present – one interchanges $\alpha_1 \leftrightarrow \alpha_2$, which has the effect of interchanging $\overline{v}_0 \leftrightarrow \overline{v}_0'$, $\overline{v}_1 \leftrightarrow \overline{v}_1'$, and $e_1 \leftrightarrow e_2$. They are recorded below for convenience.

$$\begin{aligned} V'_{\alpha\mathcal{D}} \otimes A_{\alpha\mathcal{D}} &\to V'_{\alpha\mathcal{D}} \\ \overline{v}_0' \otimes e_1 &\mapsto \overline{v}_0' \\ \overline{v}_1' \otimes e_1 &\mapsto 0 \\ \overline{v}_0' \otimes e_2 &\mapsto 0 \\ \overline{v}_1' \otimes e_2 &\mapsto \overline{v}_1' \end{aligned} \tag{4.13}$$

$$\begin{aligned} V'_{\alpha\mathcal{D}} \otimes V_{\alpha\mathcal{D}} &\to A_{\alpha\mathcal{D}} \\ \overline{v}_0' \otimes \overline{v}_0 &\mapsto 0 \\ \overline{v}_1' \otimes \overline{v}_0 &\mapsto e_2 \\ \overline{v}_0' \otimes \overline{v}_1 &\mapsto e_1 \\ \overline{v}_1' \otimes \overline{v}_1 &\mapsto 0 \end{aligned} \tag{4.14}$$

$$\begin{aligned} V'_{\alpha\mathcal{D}} &\to V'_{\alpha\mathcal{D}} \otimes A_{\alpha\mathcal{D}} \\ \overline{v}'_0 &\mapsto (\alpha_2 - \alpha_1)\overline{v}'_0 \otimes e_1 \\ \overline{v}'_1 &\mapsto (\alpha_1 - \alpha_2)\overline{v}'_1 \otimes e_2 \end{aligned} \tag{4.15}$$

$$\begin{aligned} A_{\alpha\mathcal{D}} &\to V'_{\alpha\mathcal{D}} \otimes V_{\alpha\mathcal{D}} \\ e_1 &\mapsto (\alpha_2 - \alpha_1)\overline{v}'_0 \otimes \overline{v}_1 \\ e_2 &\mapsto (\alpha_1 - \alpha_2)\overline{v}'_1 \otimes \overline{v}_0 \end{aligned} \tag{4.16}$$

These maps may be written uniformly in the following way. Let $Z \subset \mathbb{A}$ be a collection of circles, and label each circle in Z by one of the letters $\mathbf{a}$ or $\mathbf{b}$. From such a labeling we obtain a distinguished basis element of $\mathcal{G}_{\alpha\mathcal{D}}(Z)$ by using the correspondence

$$\mathbf{a} \leftrightarrow e_1, \ \mathbf{b} \leftrightarrow e_2 \tag{4.17}$$

for a trivial circle, and

$$\mathbf{a} \leftrightarrow \begin{cases} \overline{v}_1 & i \text{ is odd} \\ \overline{v}'_0 & i \text{ is even} \end{cases}, \ \mathbf{b} \leftrightarrow \begin{cases} \overline{v}_0 & i \text{ is odd} \\ \overline{v}'_1 & i \text{ is even} \end{cases} \tag{4.18}$$

on the i-th essential circle. Then the saddle maps are

$$\begin{aligned} V_{\alpha\mathcal{D}} \otimes A_{\alpha\mathcal{D}} &\xrightarrow{\text{(I)}} V_{\alpha\mathcal{D}} \\ \mathbf{b} \otimes \mathbf{a} &\mapsto 0 \\ \mathbf{a} \otimes \mathbf{a} &\mapsto \mathbf{a} \\ \mathbf{b} \otimes \mathbf{b} &\mapsto \mathbf{b} \\ \mathbf{a} \otimes \mathbf{b} &\mapsto 0 \end{aligned} \tag{4.19}$$

$$\begin{aligned} V_{\alpha\mathcal{D}} \otimes V'_{\alpha\mathcal{D}} &\xrightarrow{\text{(II)}} A_{\alpha\mathcal{D}} \\ \mathbf{b} \otimes \mathbf{a} &\mapsto 0 \\ \mathbf{a} \otimes \mathbf{a} &\mapsto \mathbf{a} \\ \mathbf{b} \otimes \mathbf{b} &\mapsto \mathbf{b} \\ \mathbf{a} \otimes \mathbf{b} &\mapsto 0 \end{aligned} \tag{4.20}$$

$$\begin{aligned} V_{\alpha\mathcal{D}} &\xrightarrow{\text{(III)}} V_{\alpha\mathcal{D}} \otimes A_{\alpha\mathcal{D}} \\ \mathbf{b} &\mapsto (\alpha_1 - \alpha_2)\mathbf{b} \otimes \mathbf{b} \\ \mathbf{a} &\mapsto (\alpha_2 - \alpha_1)\mathbf{a} \otimes \mathbf{a} \end{aligned} \tag{4.21}$$

$$\begin{aligned} A_{\alpha\mathcal{D}} &\xrightarrow{\text{(IV)}} V_{\alpha\mathcal{D}} \otimes V'_{\alpha\mathcal{D}} \\ \mathbf{a} &\mapsto (\alpha_2 - \alpha_1)\mathbf{a} \otimes \mathbf{a} \\ \mathbf{b} &\mapsto (\alpha_1 - \alpha_2)\mathbf{b} \otimes \mathbf{b} \end{aligned} \tag{4.22}$$

Moreover, the same formulas hold with $V_{\alpha\mathcal{D}}$ and $V'_{\alpha\mathcal{D}}$ interchanged.

For an annular link L with diagram D, let

$$CKh^{\mathbb{A}}_{\alpha\mathcal{D}}(D) := \mathcal{G}_{\alpha\mathcal{D}}([[D]])$$

denote the chain complex obtained by applying $\mathcal{G}_{\alpha\mathcal{D}}$ to $[[D]]$. It is an invariant of L up to chain homotopy equivalence, so we may write $Kh^{\mathbb{A}}_{\alpha\mathcal{D}}(L)$ to denote the homology of $CKh^{\mathbb{A}}_{\alpha\mathcal{D}}(D)$, for any diagram D of L.

Theorem 4.1.5. *Let $L \subset \mathbb{A} \times I$ be a link with diagram D. Viewing L as a link in $\mathbb{R}^3$, there is a* qdeg*-preserving isomorphism $\varphi\colon CKh^{\mathbb{A}}_{\alpha\mathcal{D}}(D) \xrightarrow{\sim} CKh_{\alpha\mathcal{D}}(D)$.*

Proof. For a smoothing D_u, the inclusion $\mathbb{A} \hookrightarrow \mathbb{R}^2$ induces an isomorphism

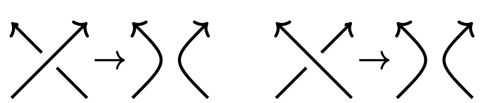
$$\varphi_u\colon \mathcal{G}_{\alpha\mathcal{D}}(D_u) \to \mathcal{F}_{\alpha\mathcal{D}}(D_u),$$

defined in terms of the basis elements labeled by $\mathbf{a}$ and $\mathbf{b}$ by $\mathbf{a} \mapsto e_1$, $\mathbf{b} \mapsto e_2$. Comparing the formulas (4.19)–(4.22) with multiplication and comultiplication in $A_{\alpha\mathcal{D}}$, we see that each of the maps φ_u commute with cobordism maps and thus assemble into an isomorphism $\varphi\colon CKh^{\mathbb{A}}_{\alpha\mathcal{D}}(D) \to CKh_{\alpha\mathcal{D}}(D)$. It is evident from Figure 4-4 that each φ_u preserves qdeg. Quantum grading shifts in both chain complexes are the same, so the isomorphism φ preserves qdeg as well. $\square$

The following is immediate from Proposition 2.3.31.

Corollary 4.1.6. *For a link $L \subset \mathbb{A} \times I$ with k components, the homology $Kh^{\mathbb{A}}_{\alpha\mathcal{D}}(L)$ is a free $R_{\alpha\mathcal{D}}$-module of rank 2^k.*

We recall the *canonical generators* for Lee homology, following [55] and [69]. Let $L \subset \mathbb{A} \times I$ be a link with diagram D. Given an orientation o on L, let $D_o \subset \mathbb{A}$ denote the result of performing the oriented resolution at each crossing,



Each of the resulting circles is naturally oriented. Assign a mod 2 invariant to each circle C as follows. First, consider the number of circles in D_o separating C from infinity, mod 2. Add 1 if C is counterclockwise oriented, and add 0 otherwise. Now that each circle in D_o is labeled by 0 or 1, use the correspondence $0 \leftrightarrow \mathbf{a}$, $1 \leftrightarrow \mathbf{b}$ to label each circle by $\mathbf{a}$ or $\mathbf{b}$, and finally use (4.17) and (4.18) to obtain a generator $\mathfrak{s}_o$ in $CKh^{\mathbb{A}}_{\alpha\mathcal{D}}(D)$.

For a collection of oriented circles $Z \subset \mathbb{A}$, let $w(Z)$ denote the winding number of Z. That is, $w(Z)$ equals the number of counterclockwise essential circles minus the number of clockwise ones. If $C_1, \ldots, C_m$ are the essential circles in Z, then

$$w(Z) = \sum_{i=1}^{m} w(C_i).$$

Proposition 4.1.7. *Let $L \subset \mathbb{A} \times I$ be a link with diagram D, and let o be an orientation of L. Let m be the number of essential circles in the oriented resolution D_o. Then*

$$\mathrm{adeg}(\mathfrak{s}_o) = (-1)^m w(L, o)$$

where $w(L, o)$ is the winding number of L with respect to the orientation o.

Proof. First note that $w(L, o) = w(D_o)$. It is straightforward to verify that each essential circle C in D_o contributes $(-1)^m w(C)$ to the annular degree of $\mathfrak{s}_o$. The claim follows, since trivial circles do not contribute to the annular degree or the winding number. $\square$

4.2 Anchored $\mathfrak{sl}_2$ link homology

4.2.1 Anchored surfaces and their evaluations

Consider the integral polynomial ring $R_\alpha = \mathbb{Z}[\alpha_1, \alpha_2]$ in two variables α_1, α_2. Define a grading on R_α by setting

$$\deg(\alpha_1) = \deg(\alpha_2) = 2. \tag{4.23}$$

Denote by τ the nontrivial involution of $\{1, 2\}$, given by $\tau(i) = 3 - i$ for $i \in \{1, 2\}$. Also denote by τ the induced involution of R_α which permutes α_1, α_2, so that $\tau(\alpha_i) = \alpha_{\tau(i)} = \alpha_{3-i}$.

Let R_E be the τ-invariant subring of R_α, which consists of symmetric polynomials in α_1, α_2. The subring R_E is itself a polynomial ring, $R_E = \mathbb{Z}[E_1, E_2]$, where E_1, E_2 are elementary symmetric polynomials in α_1, α_2,

$$E_1 = \alpha_1 + \alpha_2, \quad E_2 = \alpha_1\alpha_2.$$

Degrees of E_1 and E_2 are 2 and 4, respectively.

Let $\mathcal{L} \subset \mathbb{R}^3$ denote the z-axis, $\mathcal{L} = \{(0,0)\} \times \mathbb{R}$. Let $S \subset \mathbb{R}^3$ be a closed, smoothly embedded surface which intersects $\mathcal{L}$ transversely. The surface S may be decorated by dots, disjoint from $\mathcal{L}$, that can otherwise float freely on components of S. The intersection points $S \cap \mathcal{L}$ are called *anchor points*. Fix a labeling ℓ, which is a map from the set of anchor points to $\{1, 2\}$,

$$\ell \colon S \cap \mathcal{L} \to \{1, 2\}.$$

Order the anchor points by $1, \dots, 2k$, read from bottom to top, so that the labeling ℓ consists of a choice $\ell(j) \in \{1, 2\}$ for each $1 \leq j \leq 2k$. We will define an evaluation

$$\langle S \rangle \in R_\alpha.$$

for S with the fixed labeling ℓ, which is omitted from the notation.

Let $\mathrm{Comp}(S)$ denote the set of connected components of S. A *coloring* of S is a function $c : \mathrm{Comp}(S) \to \{1, 2\}$, and we denote by $\mathrm{adm}(S)$ the set of colorings of S. A surface S has $2^{|\mathrm{Comp}(S)|}$ colorings. For a coloring c and $i = 1, 2$, let $d_i(c)$ denote the number of dots on components colored i. Let S_2 denote the union of the 2-colored components. For $1 \leq j \leq 2k$, let $c(j)$ denote the color of the j-th anchor point, induced by c, which may in general be different from the fixed label $\ell(j)$. Define

$$\langle S, c \rangle = (-1)^{\chi(S_2)/2} \frac{\alpha_1^{d_1(c)} \alpha_2^{d_2(c)} \left(\prod_{j=1}^{2k} (\alpha_{c(j)} - \alpha_{\ell(j)}) \right)^{1/2}}{(\alpha_1 - \alpha_2)^{\chi(S)/2}}. \tag{4.24}$$

Note that $\chi(S_2)$ is even since S_2 is a closed surface in $\mathbb{R}^3$. Let us explain the square root in the above equation.

Each component S' of S can be made disjoint from $\mathcal{L}$ via a homotopy. Since the mod 2 intersection number is preserved under homotopy, it follows that S' intersects $\mathcal{L}$ at an even number of points $p_1, \ldots, p_{2m}$, which can be ordered as encountered along $\mathcal{L}$, from bottom to top. Suppose S' is colored by $c(S') = j$, and moreover S' contains an anchor point labeled j. Then the product $\prod_{j=1}^{2m}(\alpha_{c(j)} - \alpha_{\ell(j)}) = 0$, since it contains a term $\alpha_j - \alpha_j = 0$, and the entire evaluation $\langle S, c\rangle$ is then zero. Thus, the evaluation (4.24) is only nonzero when the anchor points on a component S' colored j are all labeled by the complementary color $\tau(j)$. In this case, each component contributes an even number of factors of either $\alpha_1 - \alpha_2$ or $\alpha_2 - \alpha_1$ to the product $\prod_{j=1}^{2m}(\alpha_{c(j)} - \alpha_{\ell(j)})$, and we define the square root to be $(\alpha_1 - \alpha_2)^m$ or $(\alpha_2 - \alpha_1)^m$, respectively. If S' has no anchor points, this term is 1 and can be removed from the product.

Note that the evaluation is the product of evaluations of individual components,

$$\langle S, c\rangle \;=\; \prod_{S' \in \mathrm{Comp}\,(S)} \langle S', c(S')\rangle\,. \tag{4.25}$$

Thus, if a connected component S' is colored 1 by $c' = c(S')$, has $2k$ anchor points all labeled 2, and carries d dots, then

$$\langle S', c'\rangle = \alpha_1^d(\alpha_1 - \alpha_2)^{k - \chi(S')/2}. \tag{4.26}$$

If S' is colored 2 by $c' = c(S')$, has $2k$ anchor points all labeled 1 and carries d dots, then

$$\langle S', c'\rangle = (-1)^{\chi(S')/2+k}\alpha_2^d(\alpha_1 - \alpha_2)^{k-\chi(S')/2} = \alpha_2^d(\alpha_2 - \alpha_1)^{k-\chi(S')/2}. \tag{4.27}$$

Otherwise, if one of the anchor points has the same label as the color of S', the evaluation $\langle S', c'\rangle = 0$ and $\langle S, c\rangle = 0$.

Define the evaluation of S by

$$\langle S\rangle = \sum_c \langle S, c\rangle\,, \tag{4.28}$$

where the sum is over all colorings of S. Note that if $S \cap \mathcal{L} = \varnothing$, then $\langle S\rangle$ agrees with the evaluation of closed surfaces in equivariant Khovanov homology [73, 40]. Also note that $\langle S\rangle = 0$ if a component of S has two anchor points with different labels 1, 2.

We have

$$\langle S\rangle \;=\; \prod_{S'\in \mathrm{Comp}\, S} \langle S'\rangle\,, \tag{4.29}$$

that is, evaluation of S is the product of evaluations over connected components of S.

We can rewrite $\langle S\rangle$ as follows. First, suppose S is connected, carrying d dots, with $2k \geq 0$ anchor points. For $i = 1, 2$, let c_i denote the coloring of S by i. Define

$$\langle S, c_1\rangle = \frac{\alpha_1^d((\alpha_1-\alpha_{\ell(1)})\cdots(\alpha_1-\alpha_{\ell(2k)}))^{1/2}}{(\alpha_1-\alpha_2)^{\chi(S)/2}}, \tag{4.30}$$

$$\langle S, c_2\rangle = (-1)^{\chi(S)/2}\frac{\alpha_2^d((\alpha_2-\alpha_{\ell(1)})\cdots(\alpha_2-\alpha_{\ell(2k)}))^{1/2}}{(\alpha_1-\alpha_2)^{\chi(S)/2}}. \tag{4.31}$$

Again, square roots in the above equations are taken in the natural way. If S has oppositely labeled anchor points then both (4.30) and (4.31) are zero. If all anchor points are labeled 1, then (4.30) is zero, whereas (4.31) is equal to

$$\langle S, c_2\rangle = (-1)^{\chi(S)/2}\frac{\alpha_2^d(\alpha_2-\alpha_1)^k}{(\alpha_1-\alpha_2)^{\chi(S)/2}}.$$

On the other hand, if all anchor points are labeled by 2 then (4.31) is zero and (4.30) equals

$$\frac{\alpha_1^d(\alpha_1-\alpha_2)^k}{(\alpha_1-\alpha_2)^{\chi(S)/2}}.$$

Then for connected S with anchor points, we have

$$\langle S\rangle = \langle S, c_1\rangle + \langle S, c_2\rangle\,,$$

where at most one of the summands $\langle S, c_i\rangle$ is nonzero.

Clearly the evaluation is multiplicative under disjoint union. That is, if $S = S_1 \sqcup \cdots \sqcup S_n$, then

$$\langle S\rangle = \langle S_1\rangle \cdots \langle S_n\rangle\,.$$

Remark 4.2.1. Unlike closed foam evaluations appearing elsewhere [73, 41, 38, 40, 75], our evaluation does not in general produce a symmetric polynomial. The following examples illustrate this.

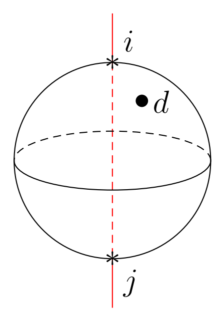
Example 4.2.2. Let S be a sphere intersecting $\mathcal{L}$ in two points with labels i and j and carrying d dots. If $i \neq j$, then each coloring c yields $\langle S, c\rangle = 0$. If both anchor points are labeled 1, then only coloring S by 2 contributes to the sum, and we have

$$\langle S\rangle = \langle S, c_2\rangle = -\frac{\alpha_2^d(\alpha_2 - \alpha_1)}{\alpha_1 - \alpha_2} = \alpha_2^d.$$

On the other hand, if both anchor points are labeled 2, then

$$\langle S\rangle = \langle S, c_1\rangle = \alpha_1^d.$$

This is summarized pictorially in (4.32). Both signs are positive since $k+\chi(S^2)/2 = 1+1 = 2$ is even.



$$= \delta_{ij}\tau(\alpha_i)^d \tag{4.32}$$

Note that these evaluations are not symmetric in α_1, α_2.

Example 4.2.3. More generally, let S be a genus g surface with d dots and $2k$ anchor points. If $k = 0$ (that is, if S is disjoint from $\mathcal{L}$) then the evaluation is

$$\langle S\rangle = \frac{\alpha_1^d + (-1)^{g-1}\alpha_2^d}{(\alpha_1 - \alpha_2)^{1-g}}.$$

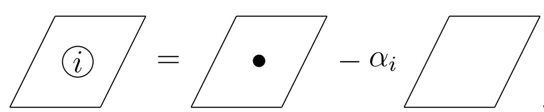
On the other hand, if $k > 0$, then

$$\langle S \rangle = \begin{cases} \alpha_2^d(\alpha_2 - \alpha_1)^{k+g-1} & \text{if } \ell(1) = \cdots = \ell(2k) = 1, \\ \alpha_1^d(\alpha_1 - \alpha_2)^{k+g-1} & \text{if } \ell(1) = \cdots = \ell(2k) = 2, \\ 0 & \text{otherwise.} \end{cases} \tag{4.33}$$

Proposition 4.2.4. *For any anchored surface $S \subset \mathbb{R}^3$ with d dots and $2k$ anchor points, its evaluation $\langle S \rangle$ is a homogeneous polynomial in α_1 and α_2 of degree $-\chi(S) + 2d + 2k$.*

Proof. If S does not intersect $\mathcal{L}$, then this follows from Example 4.2.3. Suppose that S intersects $\mathcal{L}$. It suffices to verify the statement for connected surfaces. If S intersects $\mathcal{L}$, then the statement follows from (4.33), since $k > 0$. $\square$

We recall the following notation from [40]. For $i = 1, 2$, we allow surfaces to carry decorations (i) consisting of i inscribed in a small circle. They must be disjoint from $\mathcal{L}$ and are allowed to float freely along the connected component on which they appear. We call these *shifted* dots. Diagrammatically, a shifted dot (i) is the difference between a dot and α_i,



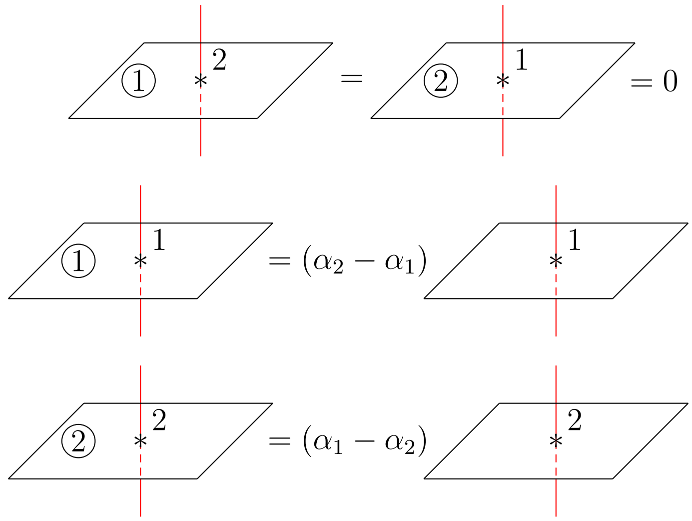
(4.34)

Lemma 4.2.5. *Let S be an anchored foam and let $S \cup (i)$ denote the anchored foam obtained by placing a shifted dot (i) on some component S' of S. Then*

$$\langle S \cup (i) \rangle = \begin{cases} 0 & \textit{if } S' \textit{ has an anchor point labeled } \tau(i) \\ (-1)^i(\alpha_1 - \alpha_2)\,\langle S \rangle & \textit{if all anchor points on } S' \textit{ are labeled } i. \end{cases}$$

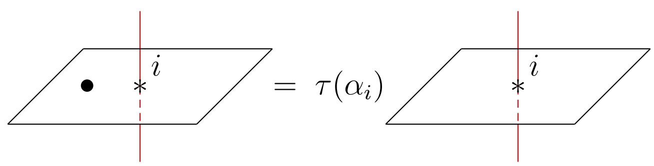
Proof. This is clear from the definitions. $\square$

Lemma 4.2.5 is summarized diagrammatically in the relations (4.35).



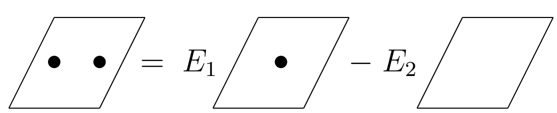
(4.35)

Alternatively, the skein relations (4.35) may written compactly as in (4.36).

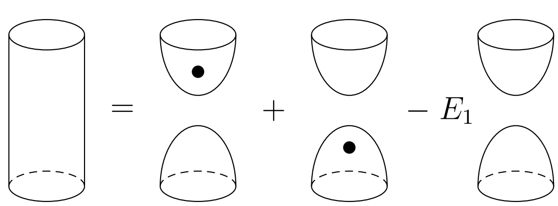


(4.36)

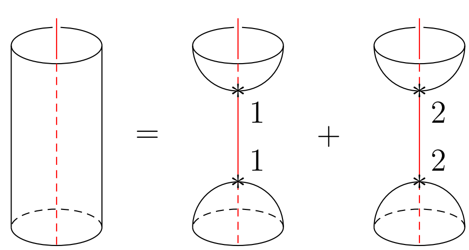
Lemma 4.2.6. *The local relations* (4.37), (4.38), (4.39), *and* (4.40) *hold.*



(4.37)



(4.38)



(4.39)

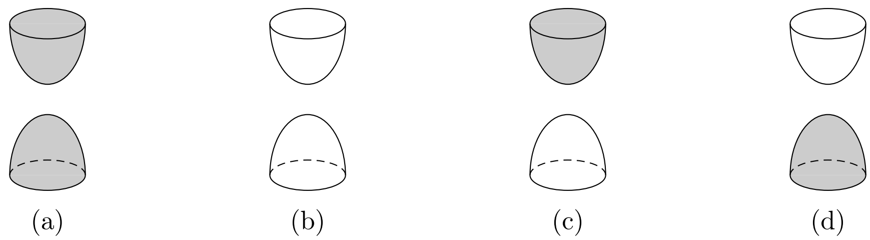
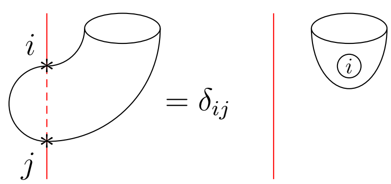


Figure 4-5: Local models for colorings of F. Shaded indicates color 1 and solid white indicates color 2.



(4.40)

Proof. The relations (4.37) and (4.40) are straightforward. Let us now verify equation (4.38), which is proved in the same way as for non-anchored foams, see [40, Lemma 3.5]. Let S denote the surface on the left, and let F denote the surface obtained by surgering S as shown on the right. Denote by F^t (resp. F^b) the surface obtained from F by placing an additional dot on the top (resp. bottom) depicted disk. Note that anchor points, as well as their labels, are the same for F^t, F^b, and F. Colorings of F, F^t, and F^b are in a canonical bijection. There are four local models for a coloring of F, illustrated in Figure 4-5.

Let c be a coloring of F of the type shown in Figure 4-5c, with the corresponding coloring of F^t and F^b still denoted by c. We have

$$\left\langle F^t, c\right\rangle = \alpha_1 \langle F, c\rangle \qquad \left\langle F^b, c\right\rangle = \alpha_2 \langle F, c\rangle \,,$$

hence $\langle F^t, c\rangle + \left\langle F^b, c\right\rangle - E_1 \langle F, c\rangle = 0$. A similar calculation holds for a coloring c of Figure 4-5d type.

There is a natural bijection between colorings of S and colorings of F of Figures 4-5a and 4-5b types. Let c be a coloring of F of Figure 4-5a type, and continue to denote by c

the corresponding coloring of S. Then

$$\chi(F) = \chi(S) + 2 \qquad\qquad \chi(F_2(c)) = \chi(S_2(c)),$$
$$\left\langle F^t, c\right\rangle = \alpha_1 \left\langle F, c\right\rangle \qquad\qquad \left\langle F^b, c\right\rangle = \alpha_1 \left\langle F, c\right\rangle,$$

so we have

$$\left\langle F^t, c\right\rangle + \left\langle F^b, c\right\rangle - E_1 \left\langle F, c\right\rangle = (\alpha_1 - \alpha_2) \left\langle F, c\right\rangle = \left\langle S, c\right\rangle.$$

Finally, if c is a coloring of F of the Figure 4-5b type, then

$$\chi(F) = \chi(S) + 2 \qquad\qquad \left\langle F^t, c\right\rangle = \alpha_2 \left\langle F, c\right\rangle,$$
$$\chi(F_2(c)) = \chi(S_2(c)) + 2 \qquad\qquad \left\langle F^b, c\right\rangle = \alpha_2 \left\langle F, c\right\rangle,$$

which yields

$$\left\langle F^t, c\right\rangle + \left\langle F^b, c\right\rangle - E_1 \left\langle F, c\right\rangle = (\alpha_2 - \alpha_1) \left\langle F, c\right\rangle = -(\alpha_2 - \alpha_1)\frac{\left\langle S, c\right\rangle}{\alpha_1 - \alpha_2} = \left\langle S, c\right\rangle.$$

We now address equation (4.39), where anchor points are present. Let S denote the surface on the left-hand side of the equality. Let F^1, F^2 denote the two anchored foams obtained by surgery on S in which the new anchor points are both labeled 1 or 2, respectively, so that (4.39) reads $\langle S\rangle = \langle F^1\rangle + \langle F^2\rangle$. For each $i = 1, 2$ there are four local models for a coloring of F^i, shown in Figure 4-6. Colorings c in Figure 4-6c and Figure 4-6d evaluate to zero for both $i = 1, 2$,

$$\left\langle F^1, c\right\rangle = \left\langle F^2, c\right\rangle = 0$$

and they don't correspond to any colorings of S. There is a natural bijection between colorings of S and colorings of F^i of the types in Figures 4-6a and 4-6b.

Let c be a coloring of S in which the depicted region of S in (4.39) is colored 1, with the corresponding colorings of F^1 and F^2 still denoted c. We have immediately that $\langle F^1, c\rangle = 0$. On the other hand,

$$\chi(F^2) = \chi(S) + 2, \qquad \chi(F_2^2(c)) = \chi(S_2(c)),$$

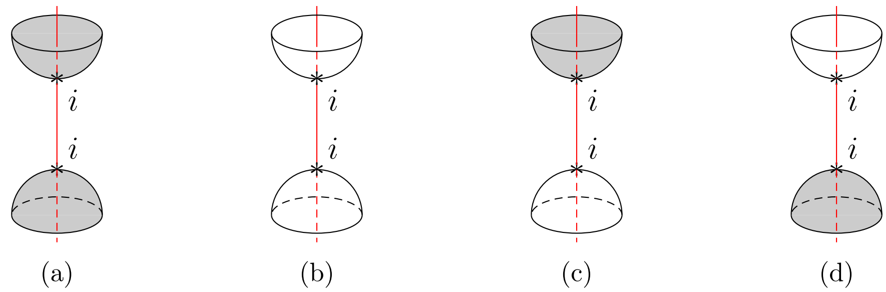


Figure 4-6: Local models for colorings of F^i. Shaded indicates color 1 and unshaded indicates color 2.

and F^2 has two additional anchor points compared to S, both labeled 2 and their regions colored 1. Therefore

$$\left\langle F^1, c\right\rangle + \left\langle F^2, c\right\rangle = \left\langle F^2, c\right\rangle = (\alpha_1 - \alpha_2)\frac{\langle S, c\rangle}{\alpha_1 - \alpha_2} = \langle S, c\rangle\,.$$

Now let c be a coloring of S in which the depicted region of (4.39) is colored 2, and continue to denote by c the corresponding colorings of F^1 and F^2. Then $\langle F^2, c\rangle = 0$. Since

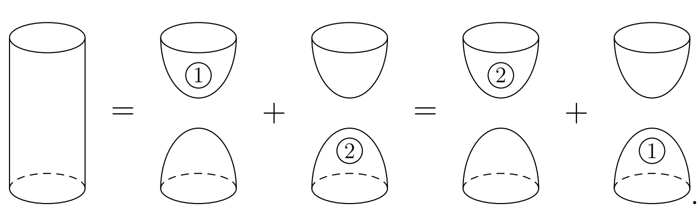
$$\chi(F^1) = \chi(S) + 2, \qquad \chi(F^1_2(c)) = \chi(S_2(c)) + 2,$$

and F^1 contains two more anchor points labeled 1 and colored 2 than S does, we obtain

$$\left\langle F^1, c\right\rangle + \left\langle F^2, c\right\rangle = \left\langle F^1, c\right\rangle = -(\alpha_2 - \alpha_1)\frac{\langle S, c\rangle}{\alpha_1 - \alpha_2} = \langle S, c\rangle\,.$$

Relation $\langle S\rangle = \langle F^1\rangle + \langle F^2\rangle$ in Figure (4.39) follows. $\square$

Equation (4.38) can also be written using shifted dots,



(4.41)

4.2.2 State spaces

Following [13, 40], we can apply the universal construction to the evaluation described above. Let $\mathcal{P} = \mathbb{R}^2 \backslash (0,0)$ denote the punctured plane. Given a collection C of disjoint simple closed curves in $\mathcal{P}$, let $\mathrm{Fr}(C)$ denote the free R_α-module with a basis consisting of properly embedded compact surfaces $S \subset \mathbb{R}^2 \times (-\infty, 0]$ with $\partial S = C$ and which are transverse to the ray $\mathcal{L}_- := (0,0) \times (-\infty, 0]$. The intersection $S \cap \mathcal{L}_-$ is a 0-submanifold of $\mathcal{L}_-$ and consists of finitely many points. Moreover, each such surface S must carry a labeling, a map

$$\ell = \ell_S : S \cap \mathcal{L}_- \to \{1, 2\}$$

from the set of its intersection points with the ray $\mathcal{L}_-$ (its *anchor points*) to $\{1, 2\}$. For a basis element $S \in \mathrm{Fr}(C)$, let $\overline{S} \subset \mathbb{R}^2 \times [0, \infty)$ denote its reflection through the plane $\mathbb{R}^2$. Labels of anchor points do not change upon reflection. For two basis elements $S, S' \in \mathrm{Fr}(C)$ denote by $\overline{S}S'$ the closed anchored surface obtained by gluing $\overline{S}$ to S' along their common boundary C.

Define a bilinear form

$$(-,-)\colon\ \mathrm{Fr}(C) \times \mathrm{Fr}(C) \to R_\alpha \tag{4.42}$$

by setting $(S, S') = \left\langle \overline{S}S' \right\rangle$. A direct computation shows that the form is symmetric, since for a closed surface T the evaluation satisfies $\left\langle \overline{T} \right\rangle = \langle T \rangle$.

Define the *state space of* C, denoted $\langle C \rangle$, to be the quotient of $\mathrm{Fr}(C)$ by the kernel

$$\{x \in \mathrm{Fr}(C) \mid (x, y) = 0 \text{ for all } y \in \mathrm{Fr}(C)\}$$

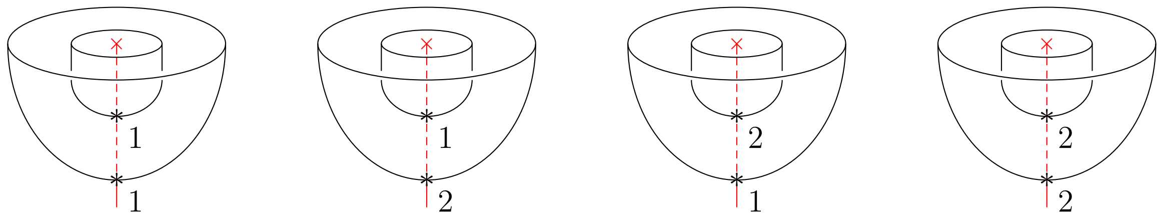
of this bilinear form. For a basis element $S \in \mathrm{Fr}(C)$, we will write $[S]$ to denote its equivalence class in $\langle C \rangle$.

Equip the ground ring R_α with a bigrading by placing α_1, α_2 in bidegree $(2, 0)$. We extend this bigrading $(\mathrm{qdeg}, \mathrm{adeg})$ to $\mathrm{Fr}(C)$ as follows. For a basis element $S \in \mathrm{Fr}(C)$ with d dots and m anchor points, set the quantum grading $\mathrm{qdeg}(S) \in \mathbb{Z}$ to be

$$\mathrm{qdeg}(S) = -\chi(S) + 2d + m. \tag{4.43}$$

	label 1	label 2
i odd	1	-1
i even	-1	1

Figure 4-7: The contribution of the i-th anchor point on S to $\mathrm{adeg}(S)$.



(a) (0,0) (b) (0,-2) (c) (0,2) (d) (0,0)

Figure 4-8: The (qdeg, adeg) bidegrees of some anchored surfaces whose boundary consists of two non-contractible circles.

Note that if S is a closed surface, then $\langle S\rangle \in R_\alpha$ is a homogeneous polynomial of degree $\mathrm{qdeg}(S)$, following the degree convention (4.23).

Next, let $\ell(1), \ldots, \ell(m)$ denote the labels of the anchor points of S, ordered from bottom to top, and define the annular grading $\mathrm{adeg}(S) \in \mathbb{Z}$ by setting

$$\mathrm{adeg}(S) = \sum_{i=1}^{m} (-1)^{i+\ell(i)}. \tag{4.44}$$

In other words, if the i-th anchor point p_i is labeled 1, then it contributes 1 to adeg if i is odd and -1 if i is even. Likewise, if p_i has label 2 then it contributes -1 if i is odd and 1 if i is even, see also Figure 4-7. Multiplication by α_1, α_2 increases $(\mathrm{qdeg}, \mathrm{adeg})$-bidegree by $(2, 0)$.

Example 4.2.7. Let C consist of two non-contractible circles. The bidegree (qdeg, adeg) of the four anchored surfaces in $\mathrm{Fr}(C)$ whose underlying surface consists of two disks each intersecting $\mathcal{L}_-$ once are recorded in Figure 4-8.

Lemma 4.2.8. *Let S be an anchored surface. Then $\langle S\rangle = 0$ or $\mathrm{adeg}(S) = 0$.*

Proof. If some component of S has anchor points with different labels then $\langle S\rangle = 0$. Assume that all anchor points on any component of S are labeled identically. We also assume that S intersects $\mathcal{L}$, otherwise $\mathrm{adeg}(S) = 0$ is immediate. As usual, order the anchor points

$p_1, \ldots, p_m$ from bottom to top.

Take a generic half-plane P in $\mathbb{R}^3$ containing the anchor line $\mathcal{L}$, so that $P \cap S$ consists of finitely many arcs (with boundary on $\mathcal{L}$) and circles (disjoint from $\mathcal{L}$). For any arc a in $P \cap S$ with boundary $\partial a = \{p_i, p_j\}$, necessarily i and j have opposite parities, and moreover $\ell(p_i) = \ell(p_j)$ by assumption. Therefore the total contribution of the anchor points p_i and p_j to $\operatorname{adeg}(S)$ is zero. Summing over all arcs in $P \cap S$ yields the statement of the lemma.

□

The subspace $\ker((,)) \subset \operatorname{Fr}(C)$ respects this bigrading on $\operatorname{Fr}(C)$. Consequently, the bigrading descends to the state space $\langle C \rangle$.

Note that the relations (4.38) and (4.39) are bi-homogeneous. Let $S \in \operatorname{Fr}(C)$ be a basis element of the form $S = S_1 \sqcup S_2$ where $S_1, S_2 \in \operatorname{Fr}(C)$ are anchored surfaces with S_2 closed. Then in $\langle C \rangle$ we have

$$[S] = \langle S_2 \rangle \, [S_1], \quad \langle S_2 \rangle \in R_\alpha. \tag{4.45}$$

Moreover, the relation (4.45) is bi-homogeneous. That it is homogeneous with respect to qdeg follows from the fact that $\langle S_2 \rangle \in R_\alpha$ is a polynomial of degree $\operatorname{qdeg}(S_2)$. Lemma 4.2.8 ensures that $\operatorname{adeg}(S_2) = \operatorname{adeg}(\langle S_2 \rangle) = 0$, so $\operatorname{adeg}(S) = \operatorname{adeg}(S_1)$.

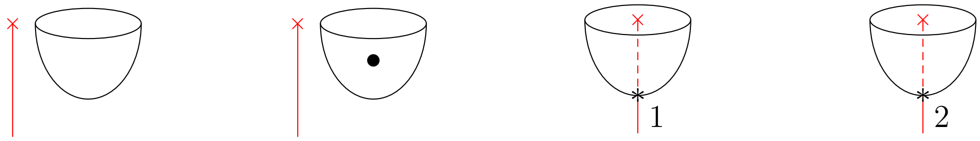
Given a bigraded module $M = \bigoplus_{(i,j) \in \mathbb{Z}^2} M_{i,j}$ over a commutative domain such that each $M_{i,j}$ has finite rank, define its *graded rank* to be

$$\operatorname{rank}_q(M) = \sum_{i,j} \operatorname{rank}(M_{i,j}) q^i a^j.$$

Lemma 4.2.9. *Let $C \subset \mathcal{P}$ be a single circle. Then the state space $\langle C \rangle$ is a free R_α-module of rank 2. Moreover, we have*

$$\operatorname{rank}_q(\langle C \rangle) = \begin{cases} q + q^{-1} & \textit{if } C \textit{ is contractible} \\ a + a^{-1} & \textit{if } C \textit{ is non-contractible.} \end{cases}$$

Proof. If C is contractible, then by applying the neck-cutting relation (4.38) near C and evaluating closed anchored surfaces as in equation (4.45), we see that $\langle C \rangle$ is spanned by the two elements S and $S_\bullet$ shown in Figures 4-9a and 4-9b. Bidegrees of S and $S_\bullet$ are $(-1, 0)$



(a) S (b) $S_\bullet$ (c) S_1 (d) S_2

Figure 4-9: Basis elements for the state space of a single circle C. The first two surfaces form a basis if C is contractible, and the last two form a basis if C is non-contractible.

and $(1,0)$, respectively. Computing the matrix of the bilinear form (4.42) for these elements yields

$$\begin{pmatrix} \overline{S}S & \overline{S}S_\bullet \\ \overline{S_\bullet}S & \overline{S_\bullet}S_\bullet \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & E_1 \end{pmatrix},$$

which is invertible, thus $S, S_\bullet$ constitute a basis for $\langle C \rangle$.

Now suppose C is non-contractible. Applying the neck-cutting relation (4.39) near C and evaluating closed anchored surfaces shows that the two elements S_1, S_2 depicted in Figures 4-9c and 4-9d span $\langle C \rangle$. Bidegrees of S_1 and S_2 are $(0,1)$ and $(0,-1)$, respectively. The matrix of the bilinear form is

$$\begin{pmatrix} \overline{S_1}S_1 & \overline{S_1}S_2 \\ \overline{S_2}S_1 & \overline{S_2}S_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

hence S_1, S_2 are linearly independent and constitute a basis of $\langle C \rangle$. $\square$

Theorem 4.2.10. *Let $C \subset \mathcal{P}$ consist of n contractible circles and m non-contractible circles. Then the state space $\langle C \rangle$ is a free R_α-module of rank 2^{n+m}. Moreover, we have*

$$\operatorname{rank}_q(\langle C \rangle) = (q+q^{-1})^n (a+a^{-1})^m.$$

Proof. Consider a 2^{n+m} element set $B(C)$ of basis vectors of $\operatorname{Fr}(C)$ consisting of surfaces S satisfying

- Each component of S is a disk.
- Each disk in S with contractible boundary is disjoint from $\mathcal{L}_-$ and carries either zero or one dot.

- Each disk in S with non-contractible boundary intersects $\mathcal{L}_-$ exactly once, and its intersection point may be labeled by either 1 or 2.

That $B(C)$ spans $\langle S \rangle$ follows from applying the two neck-cutting relations (4.38) and (4.39) near the circles in C and evaluating closed anchored surfaces. Linear independence of $B(C)$ and the statement regarding graded rank follow from the computations in Lemma 4.2.9. $\square$

Elements of the basis $B(C)$ constructed above are *standard generators*. For such a $\Sigma \in B(C)$ with d dots and anchor points labeled $\ell_1, \ldots, \ell_m$, we have

$$\operatorname{qdeg}(\Sigma) = -n + 2d, \quad \operatorname{adeg}(\Sigma) = \sum_{i=1}^{m} (-1)^{i+\ell(i)}. \tag{4.46}$$

Let $C_0, C_1 \subset \mathcal{P}$ be two collections of disjoint circles in the punctured plane. An *anchored cobordism from C_0 to C_1* is a smoothly and properly embedded compact surface $S \subset \mathbb{R}^2 \times [0,1]$ with boundary $\partial S = C_0 \sqcup C_1$, such that $C_i \subset \mathbb{R}^2 \times \{i\}$, $i = 0, 1$. Moreover, S is required to intersect the arc $\mathcal{L}_{[0,1]} := (0,0) \times [0,1]$ transversely and come equipped with a labeling of these intersection points (called *anchor points*), which is a map

$$\ell = \ell_S \colon S \cap \mathcal{L}_{[0,1]} \to \{1, 2\}$$

from the set of its anchor points to $\{1, 2\}$. Anchored cobordisms are allowed to carry dots which can float on components but cannot jump to a different component.

We compose anchored cobordisms in the usual manner. For anchored cobordisms $S_1 \colon C_0 \to C_1$, $S_2 \colon C_1 \to C_2$, let $S_2 S_1 \colon C_0 \to C_2$ denote the anchored cobordism obtained by gluing along the common boundary C_1 and re-scaling. Labels of anchor points of $S_2 S_1$ are inherited from labels of S_1 and S_2.

As above, if an anchored cobordism S from C_0 to C_1 has m anchor points and carries d dots, define

$$\operatorname{qdeg}(S) = -\chi(S) + 2d + m.$$

Let $\ell(1), \ldots, \ell(m)$ denote the labels of anchor points of S, ordered from bottom to top, and let n be the number of non-contractible circles in C_0. Set

$$\operatorname{adeg}(S) = (-1)^n \sum_{i=1}^{m} (-1)^{i+\ell(i)}.$$

Remark 4.2.11. If $C_0 = \varnothing$, then S is a basis element of $\operatorname{Fr}(C_1)$, and moreover the two degrees $\operatorname{qdeg}(S)$, $\operatorname{adeg}(S)$ defined above for anchored cobordisms agree with the definitions in (4.43) and (4.44) for elements of $\operatorname{Fr}(C_1)$.

An anchored cobordism S from C_0 to C_1 induces an R_α-linear map

$$S\colon \operatorname{Fr}(C_0) \to \operatorname{Fr}(C_1)$$

defined on the basis by gluing along the common boundary C_0. The definition of state spaces via universal construction immediately implies that we have an induced map

$$\langle S \rangle : \langle C_0 \rangle \to \langle C_1 \rangle . \tag{4.47}$$

Lemma 4.2.12. *Let $S_1\colon C_0 \to C_1$ and $S_2\colon C_1 \to C_2$ be anchored cobordisms. Then*

$$\operatorname{qdeg}(S_2S_1) = \operatorname{qdeg}(S_2) + \operatorname{qdeg}(S_1), \quad \operatorname{adeg}(S_2S_1) = \operatorname{adeg}(S_2) + \operatorname{adeg}(S_1).$$

In particular, $\langle S_1 \rangle : \langle C_0 \rangle \to \langle C_1 \rangle$ is a map of bidegree $(\operatorname{qdeg}(S_1), \operatorname{adeg}(S_1))$.

Proof. The first equality involving qdeg is straightforward. Let n and m denote the number of non-contractible circles in C_0 and C_1 respectively, and let k denote the number of anchor points of S_1. We have

$$\operatorname{adeg}(S_2S_1) = \operatorname{adeg}(S_1) + (-1)^{n+m+k} \operatorname{adeg}(S_2).$$

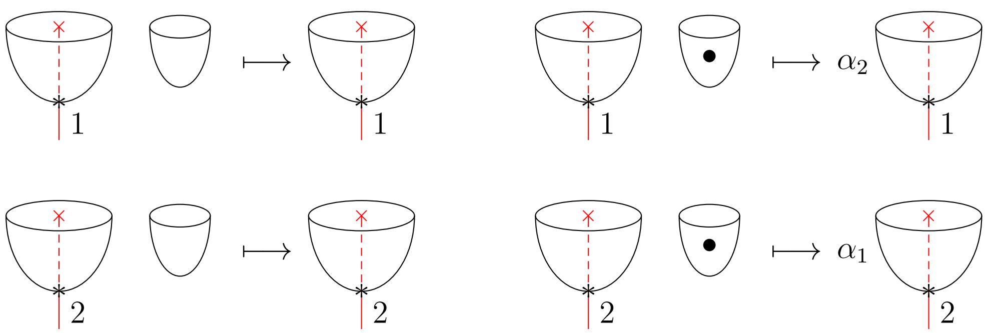
Note $n+m+k$ is even, since it is equal to the number of anchor points of the closed surface obtained by gluing disks to all boundary circles of S_1.

The final statement concerning the bidegree of $\langle S_1 \rangle$ follows from interpreting generators of $\langle C_0 \rangle$ as anchored cobordisms $\varnothing \to C_0$, as in Remark 4.2.11. $\square$

Definition 4.2.13. An *annular cobordism* is an anchored cobordism $S \subset \mathbb{R}^2 \times [0,1]$ which is disjoint from the arc $\mathcal{L}_{[0,1]} = \{(0,0)\} \times [0,1]$. An *elementary* annular cobordism is one with a single non-degenerate critical point with respect to the height function $\mathbb{R}^2 \times [0,1] \to [0,1]$.

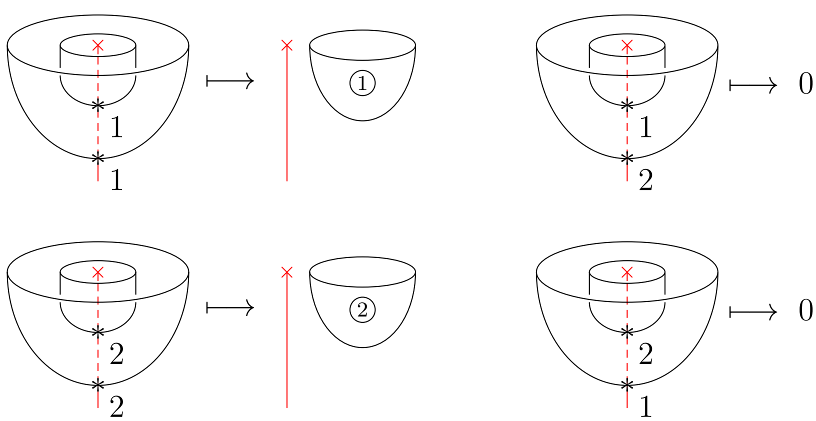
Elementary annular cobordisms consist of a union of a product cobordism with a cup, cap, or saddle. Every annular cobordism may be obtained by composing finitely many elementary ones. Cup and cap annular cobordisms always have contractible boundary. Recall that there are four types of elementary annular saddles involving at least one non-contractible circle, shown in Figure 2-12. In the next four examples we write down the maps assigned to these four cobordisms in the standard bases of state spaces, as defined in the proof of Theorem 4.2.10. We also use the notation of shifted dots from (4.34).

Example 4.2.14. Figure 2-12a map. The calculation for this map follows at once from the skein relation (4.36).



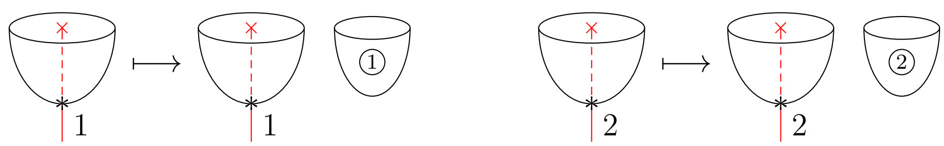
(4.48)

Example 4.2.15. Figure 2-12b map. This calculation follows easily from the skein relation (4.40).



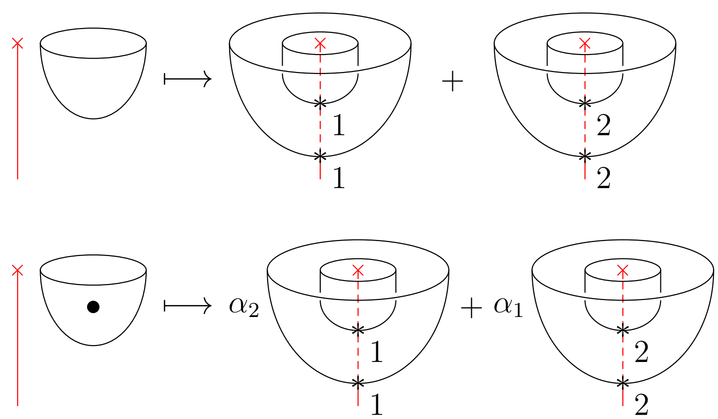
(4.49)

Example 4.2.16. Figure 2-12c map. A convenient way to perform this calculation is to use neck-cutting with shifted dots (4.41) near the contractible circle and then simplify using the relations (4.35).



(4.50)

Example 4.2.17. Figure 2-12d map. The neck-cutting relation (4.39) is helpful here. For the dotted cup we also use (4.36) to simplify further.



(4.51)

Recall the involution τ of R_α that transposes α_1, α_2 and extend it to an antilinear involution, also denoted τ, of the free R_α-module $\mathrm{Fr}(C)$ as follows. Involution τ on $\mathrm{Fr}(C)$ sends a surface S to the same surface with the labeling ℓ of anchor points reversed and acts on linear combinations by

$$\tau\left(\sum_i \lambda_i S_i\right) = \sum_i \tau(\lambda_i)\tau(S_i).$$

For a closed surface S we have, by direct computation, $\langle \tau(S) \rangle = \tau(\langle S \rangle)$, showing compatibility of the two involutions. If S, in addition, carries shifted dots, involution τ reverses their labels, so that $\tau(①) = ②$ and $\tau(②) = ①$. Involution τ descends to an involution, also denoted τ, on $\langle C \rangle$. Annular degree is negated under τ, $\mathrm{adeg}(\tau(S)) = -\mathrm{adeg}(S)$, for an anchored cobordism S.

	1	X	v_0	v_1	v_0'	v_1'
qdeg$'$	-1	1	0	0	0	0
adeg	0	0	-1	1	-1	1

Figure 4-10: The $(\text{qdeg}', \text{adeg})$-bidegrees of relevant elements, where $\{1, X\}$ is a basis for a contractible circle and $\{v_0, v_1\}, \{v_0', v_1'\}$ are bases for non-contractible circles.

4.2.3 Equivariant annular $\mathfrak{sl}_2$ homology via foam evaluation

Let **ACob** denote the category whose objects consist of collections of finitely many disjoint simple closed curves in the punctured plane $\mathcal{P}$. A morphism from C_0 to C_1 in **ACob** is an anchored cobordism from C_0 to C_1, up to ambient isotopy fixing the boundary pointwise and mapping $\mathcal{L}_{[0,1]}$ to itself. Let **ACob$'$** denote the subcategory of **ACob** with the same objects as **ACob** but whose morphisms are isotopy classes of annular cobordisms, disjoint from the anchor line $\mathcal{L}$. The composition of annular cobordisms is again annular.

Let $R_\alpha-$ggmod denote the category of bigraded R_α-modules and homogeneous maps (of any bidegree) between them. We have a functor

$$\langle - \rangle : \mathbf{ACob} \to R_\alpha - \text{ggmod}$$

which sends a collection of circles $C \subset \mathcal{P}$ to the state space $\langle C \rangle$ and sends an anchored cobordism S from C_0 to C_1 to the map $\langle S \rangle : \langle C_0 \rangle \to \langle C_1 \rangle$ as in (4.47). By Lemma 4.2.12, $\langle S \rangle$ is a map of bidegree $(\text{qdeg}(S), \text{adeg}(S))$. We can restrict to the category of annular cobordisms to get a functor

$$\langle - \rangle' : \mathbf{ACob}' \to R_\alpha - \text{ggmod},$$

which assigns to an annular cobordism S a map $\langle S \rangle' = \langle S \rangle$ of bidegree $(\text{qdeg}(S), 0)$. The restriction $\langle - \rangle'$ does not change the state space assigned to a collection of circles $C \subset \mathcal{P}$.

On the other hand, a functor $\mathcal{G}_\alpha : \mathbf{ACob}' \to R_\alpha-$ggmod was constructed in Theorem 4.1.4. To compare $\mathcal{G}_\alpha$ with $\langle - \rangle'$, we will use a modified quantum grading qdeg$'$ on $\mathcal{G}_\alpha$, given by $\text{qdeg}' = \text{qdeg} - \text{adeg}$ and summarized in Figure 4-10.

Remark 4.2.18. The modified quantum grading qdeg$'$ appears elsewhere in the literature and is apparently more natural for annular link homology. In [25] this grading was denoted j'. Similarly, quantum annular link homology carries the modified quantum grading, see Section 3.1.

Theorem 4.2.19. *The functors* $\langle - \rangle' : \mathbf{ACob}' \to R_\alpha-\,\mathrm{ggmod}$ *and* $\mathcal{G}_\alpha \colon \mathbf{ACob}' \to R_\alpha-\,\mathrm{ggmod}$ *are naturally isomorphic via bidegree-preserving maps.*

Proof. Let $C \subset \mathcal{P}$ be a collection of circles. We will define an R_α-linear, bidegree preserving isomorphism $\Phi_C \colon \langle C \rangle \to \mathcal{G}_\alpha(C)$ and show that it is natural with respect to annular cobordisms.

Let n and m denote the number of contractible and non-contractible circles in C, respectively. Fix an ordering $1, \ldots, n$ on the contractible circles in C. The R_α-module $\mathcal{G}_\alpha(C)$ is free with basis given by elements of the form

$$y_1 \otimes \cdots y_n \otimes z_1 \otimes \cdots \otimes z_m,$$

where each y_i is in $\{1, X\}$, specifying a basis element of the i-th contractible circle, and each z_j is in either $\{v_0, v_1\}$ or $\{v_0', v_1'\}$, depending on nesting, specifying basis elements of the non-contractible circles. The ordering of factors $z_1 \otimes \cdots \otimes z_m$ corresponding to non-contractible circles is from outermost to innermost as usual, so that the first factor z_1 labels the outermost non-contractible circle.

We now define the isomorphism $\Phi_C \colon \langle C \rangle \to \mathcal{G}_\alpha(C)$. Recall the standard basis $B = B(C)$ for $\langle C \rangle$ defined in the proof of Theorem 4.2.10. For $\Sigma \in B$ with anchor points labeled $\ell_1, \ldots, \ell_m$, read from bottom to top, set

$$\Phi_C(\Sigma) = y_1 \otimes \cdots y_n \otimes z_1 \otimes \cdots \otimes z_m,$$

where $y_i = 1$ if the corresponding cup in Σ is undotted and $y_i = X$ if the corresponding cup

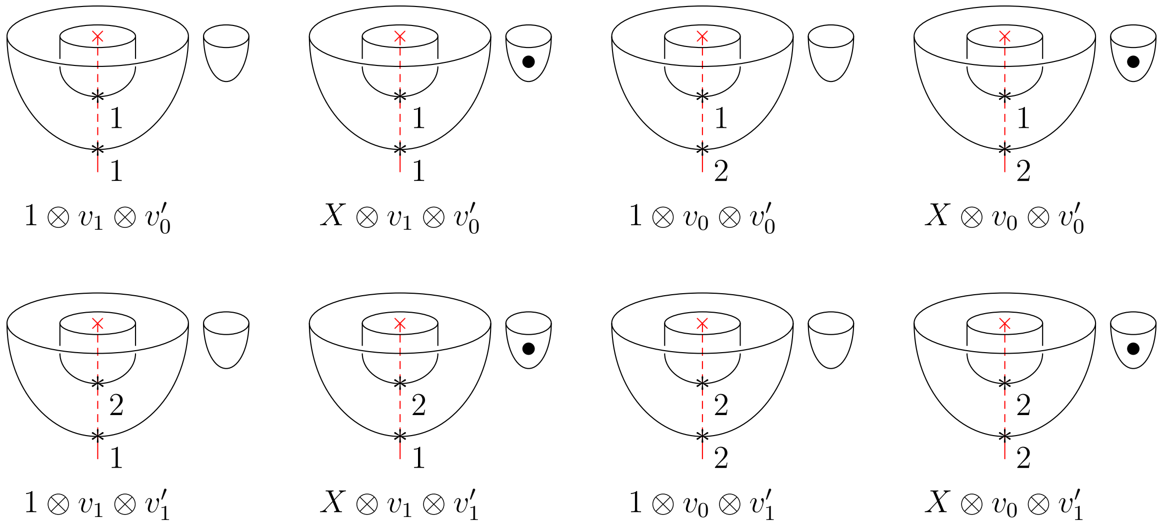


Figure 4-11: An example of the isomorphism Φ_C when C consists of one contractible circle and two non-contractible circles. Basis elements Σ of $\langle C \rangle$ are drawn with the corresponding basis element $\Phi_C(\Sigma) \in \mathcal{G}_\alpha(C)$ written underneath.

in Σ is dotted. The generators z_j of non-contractible circles are determined using the rule

$$z_j = \begin{cases} v_1 & \text{if } j \text{ is odd and } \ell_j = 1 \\ v_0 & \text{if } j \text{ is odd and } \ell_j = 2 \\ v_0' & \text{if } j \text{ is even and } \ell_j = 1 \\ v_1' & \text{if } j \text{ is even and } \ell_j = 2 \end{cases}$$

See Figure 4-11 for an example of the assignment Φ_C when $n = 1$, $m = 2$. By comparing the bidegree formula (4.46) for Σ with the bidegree of $\Phi_C(\Sigma)$ (see Figure 4-10), we see that Φ_C is a bidegree-preserving isomorphism. Recall that we use the modified quantum grading qdeg' for $\mathcal{G}_\alpha(C)$, see Figure 4-10.

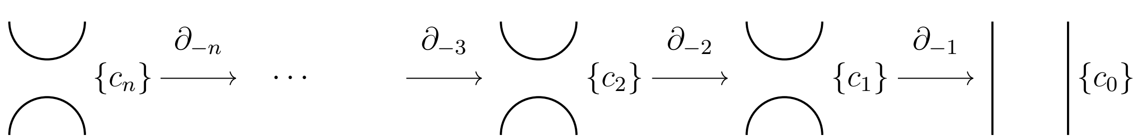
Now let $S\colon C_1 \to C_2$ be an annular cobordism. To complete the proof, we check that the square

$$\begin{tikzcd} \langle C_1 \rangle \arrow[r, "\Phi_{C_1}"] \arrow[d, "\langle S \rangle"'] & \mathcal{G}_\alpha(C_1) \arrow[d, "\mathcal{G}_\alpha(S)"] \\ \langle C_2 \rangle \arrow[r, "\Phi_{C_2}"] & \mathcal{G}_\alpha(C_2) \end{tikzcd}$$

commutes. If all the boundary circles of S are contractible, then commutativity of the square is straightforward. Otherwise, if S has at least one non-contractible boundary circle, it suffices to consider the case where S is one of the elementary annular cobordisms depicted in Figure 2-12. Formulas for these maps were recorded in Examples 4.2.14 – 4.2.17. Comparing with the formulas (4.4) – (4.7) completes the proof. $\square$

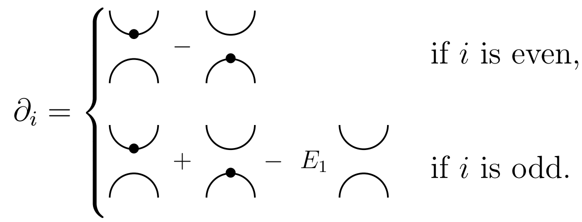
For an oriented link $L \subset \mathbb{A} \times [0,1]$ in the thickened annulus, a generic projection of L onto $\mathbb{A} \times \{0\}$ yields a link diagram D in the interior of $\mathbb{A}$. Identifying the interior of $\mathbb{A}$ with the punctured plane $\mathcal{P}$, we may form the cube of resolutions of D in the usual way, with all smoothings drawn in $\mathcal{P}$. The result is a commutative cube in the category $\mathbf{ACob}'$. Introducing signs to make the cube anti-commutative, taking direct sums along diagonals, adding homological and quantum grading shifts, and applying the functor $\langle - \rangle' : \mathbf{ACob}' \to R_\alpha-$ ggmod, one obtains a chain complex $C(D)$ of bigraded R_α-modules. Diagrams representing isotopic annular links are related by Reidemeister moves away from the puncture. It follows that the chain homotopy class of $C(D)$ is an invariant of the annular link L. We write $H(L)$ to denote the homology of $C(L)$. Theorem 4.2.19 implies that there is an isomorphism of chain complexes $C(D) \cong CKh^{\mathbb{A}}_\alpha(D)$.

Example 4.2.20. As an explicit example, let σ denote the positive crossing generator of the 2-strand braid group, and let L_n denote the annular link obtained as the annular closure of σ^{-n}. Consider the complex $C(n)$ shown in (4.52).



(4.52)

The right-most term is in homological degree zero and the quantum grading shifts c_i are given by $c_0 = n$ and $c_i = n + 2i - 1$ for $1 \leq i \leq n$. The right-most differential ∂_{-1} is the saddle cobordism, and for $-n \leq i \leq -2$ the differentials are



The above schematic depiction of ∂_i is interpreted as follows: each ∂_i is an R_α-linear combination of surfaces, each of which is given by the product cobordism on the depicted planar tangle, with a dot on a component of the surface if the corresponding tangle component is dotted. Finally, we apply the functor $\langle - \rangle' : \mathbf{ACob}' \to R_\alpha-\text{ggmod}$ to obtain a complex of R_α-modules.

By induction on n and using abstract Gaussian elimination [9, Lemma 4.2], one can show that he chain complex $C(L_n)$ is chain homotopy equivalent to the annular closure of $C(n)$.

Note that the annular closure of chain groups in negative homological degree are each a contractible circle, contributing a free module with basis 1 and X (represented by the surfaces S and $S_\bullet$ in Figure 4-9). In homological degree zero the result is two essential circles. We also see that, upon taking the annular closure, that $\partial_i = 0$ for i even, and that ∂_i for $i > 1$ odd is given by $\partial_i(1) = 2X - E_1$, $\;\partial_i(X) = E_1 X - 2E_2$, which is injective. The differential ∂_{-1} is the map in Example 4.2.17, which is also injective. Therefore, in homological degree $i \leq 0$, the homology is

$$H^i(L_n) = \begin{cases} 0 & \text{if } i \text{ is odd ,} \\ \dfrac{R_\alpha\{n-2i-2,0\} \oplus R_\alpha\{n-2i,0\}}{\langle(-E_1,2),(-2E_2,E_1)\rangle} & \text{if } i < 0, i \text{ even,} \\ R_\alpha\{n,-2\} \oplus R_\alpha\{n,2\} \oplus (R_\alpha\{n,0\}/\langle \alpha_2 - \alpha_1\rangle) & \text{if } i = 0, \end{cases}$$

where $\{j,k\}$ denotes an upwards (qdeg, adeg) shift of (j,k), and the angled brackets denote the R_α-submodule generated by the enclosed elements.

4.3 Anchored $\mathfrak{sl}_3$ link homology

In this section we recall (oriented) $\mathfrak{sl}_3$ foams, which were introduced in [35] in the context of $\mathfrak{sl}_3$ link homology. An equivariant analogue was defined in [63], see also [64, 20, 61, 60, 71] for various aspects of $\mathfrak{sl}_3$ foams and link homology. In Section 4.3.1 we define an evaluation of oriented $\mathfrak{sl}_3$ foams via colorings in the style of Robert-Wagner and show in Theorem 4.3.34 that our evaluation agrees with that of [63]. In Section 4.3.2 we deform the evaluation in the presence of the anchor line $\mathcal{L}$; the main result of that section, Theorem 4.3.21, shows that our evaluation is always a polynomial rather than a rational function. Section 4.3.3 establishes key local relations, which are used in Section 4.3.4 to identify state spaces of $\mathfrak{sl}_3$ webs in the punctured plane. Finally, in Section 4.3.5 we describe the resulting annular link homology theory.

We establish the following notation for rings that will be used throughout this section; they are the $N = 3$ specializations of rings in Definition 2.5.15, as well as new rings which will be needed for the anchored foam evaluation.

- $R_3' = \mathbb{Z}[x_1, x_2, x_3]$ is the ring of polynomials in three variables.
- $R_3 = \mathbb{Z}[E_1, E_2, E_3]$ the subring of R_3' that consists of symmetric polynomials in x_1, x_2, x_3, with generators E_i being elementary symmetric polynomials:

$$E_1 = x_1 + x_2 + x_3,$$
$$E_2 = x_1x_2 + x_1x_3 + x_2x_3,$$
$$E_3 = x_1x_2x_3.$$

- $R_3'' = R_3'[(x_1 - x_2)^{-1}, (x_2 - x_3)^{-1}, (x_1 - x_3)^{-1}]$ is a localization of R_3' given by inverting $x_i - x_j$, for $1 \le i < j \le 3$.
- $\widetilde{R}_3' = R_3'[\sqrt{x_1 - x_3}, \sqrt{x_2 - x_3}, \sqrt{x_1 - x_3}]$ is the extension of R_3' obtained by introducing square roots of $\sqrt{x_i - x_j}$, for $1 \le i < j \le 3$.
- $\widetilde{R}_3'' = \widetilde{R}_3'[(x_1 - x_2)^{-1}, (x_2 - x_3)^{-1}, (x_1 - x_3)^{-1}]$ is a suitable localization of the ring $\widetilde{R}_3'$.

All five of these rings are graded by setting $\deg(x_1) = \deg(x_2) = \deg(x_3) = 2$. Inclusions of the above rings are summarized in the following diagram.

$$\begin{array}{ccccc} & & \widetilde{R}_3' & \subset & \widetilde{R}_3'' \\ & & \cup & & \cup \\ R_3 & \subset & R_3' & \subset & R_3'' \end{array} \tag{4.53}$$

4.3.1 Oriented $\mathfrak{sl}_3$ foams and their evaluations

We begin by recalling the definition of (oriented) $\mathfrak{sl}_3$ foams from [35, Section 3.2]. The terminology *oriented* is to distinguish these from the foams appearing in [41], but we will often drop this descriptor.

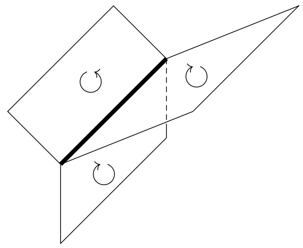
Definition 4.3.1. A (closed) $\mathfrak{sl}_3$ *pre-foam* F consists of the following data.

- An orientable surface F' with connected components $F_1, \ldots, F_k$ and a partition of the boundary components of F' into triples. The underlying CW structure of F is obtained by identifying the three circles in each triple. The image of the three circles in each triple becomes a single circle in F, called a *singular circle*. The image of the surfaces F_i are called *facets*. Three facets meet at each singular circle. Let $f(F)$ denote the set of facets of F.

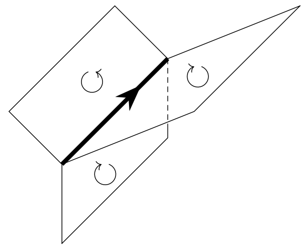
- For each singular circle Z, we fix a cyclic ordering of the three facets meeting at Z. There are two possible choices of cyclic ordering for each Z.

- Each facet may carry some number of dots, which are allowed to float freely along the facet but cannot cross singular circles.

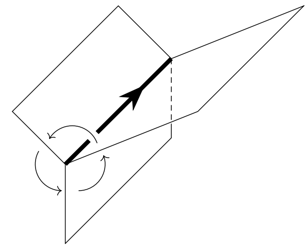
An $\mathfrak{sl}_3$ *foam* is a pre-foam as above equipped with an embedding into $\mathbb{R}^3$, along with an orientation on each facet such that any two of the three facets meeting at each singular circle are incompatibly oriented, as shown in Figure 4-12a. Each singular circle Z acquires an induced orientation, see Figure 4-12b. This induced orientation on Z specifies a cyclic ordering of the three facets meeting at Z by following the left-hand rule, Figure 4-12c, and we require this to match the cyclic ordering specified by the pre-foam F.



(a) Orientations of three facets meeting at a singular circle.



(b) The induced orientation of a singular circle.



(c) The induced cyclic ordering.

Figure 4-12

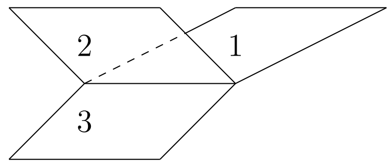


Figure 4-13: The local model for a pre-admissible coloring near a singular circle.

For a pre-foam F, let $\Theta(F)$ denote the set of its singular circles and $\theta(F) = |\Theta(F)|$ the number of singular circles. Each $Z \in \Theta(F)$ has a neighborhood homeomorphic to the product of a circle S^1 and the letter Y.

Definition 4.3.2. Let F be an $\mathfrak{sl}_3$ pre-foam. A *coloring* of F is a function $c : f(F) \to \{1, 2, 3\}$ such that the three circles meeting at each singular circles have distinct colors, as shown in Figure 4-13.

For a coloring c and $1 \leq i \neq j \leq 3$, let $F_{ij}(c)$ denote the union of facets colored i or j. Note that $F_{ij}(c)$ is a closed surface. We say that c is *admissible* if each $F_{ij}(c)$ is orientable. Let $\mathrm{adm}(F)$ denote the set of admissible colorings.

Remark 4.3.3. Note that if F is a foam (meaning, a pre-foam embedded in $\mathbb{R}^3$), then each $F_{ij}(c)$ is a closed surface in $\mathbb{R}^3$, so all colorings of F are admissible.

Remark 4.3.4. In order to keep the notation reasonable, in this section we will often repurpose the notation from Section 2.5.3 (for example, the bi-colored surface $F_{ij}(c)$ described in the above definition). We warn the reader to keep in mind the distinction between $\mathfrak{sl}_3$ foams considered here and the $\mathfrak{gl}_N$ foams from Section 2.5.3.

For $1 \leq i \leq 3$, let $F_i(c)$ be the surface consisting of all facets of F which are colored i by c; the surface $F_i(c)$ is orientable and has $\theta(F)$ boundary components. Denote by $\overline{F}_i(c)$ the closed surface obtained by gluing disks along boundary components of $F_i(c)$. We have

$$\begin{aligned}\chi(\overline{F}_i(c)) &= \chi(F_i(c)) + \theta(F),\ 1 \le i \le 3 \\ \chi(F_{ij}(c)) &= \chi(F_i(c)) + \chi(F_j(c)),\ 1 \le i < j \le 3.\end{aligned} \tag{4.54}$$

The three facets meeting at each singular circle are colored by i, j, k, we use i, j, k to denote the three elements of $\{1, 2, 3\}$. We now define quantities $\theta^{\pm}(c)$, $\theta^{\pm}_{ij}(c)$ associated with the set of singular circles $\Theta(F)$ and the admissible coloring c.

Definition 4.3.5. Let F be a pre-foam with admissible coloring c, and let $1 \le i < j \le 3$. A singular circle $Z \in \Theta(F)$ is *positive* with respect to (i, j) if the cyclic ordering of the colors of the three facets meeting at Z is $(i\ k\ j)$. If F is a foam, then an equivalent formulation is as follows: when looking along the orientation of Z with the facet colored k drawn below, the i-colored facet is to the left of the j-colored facet. Otherwise, we say Z is negative with respect to (i, j). See Figure 4-14a for a pictorial definition. Let $\theta^{+}_{ij}(c)$ (resp. $\theta^{-}_{ij}(c)$) denote the number of positive (resp. negative) circles with respect to (i, j). We have

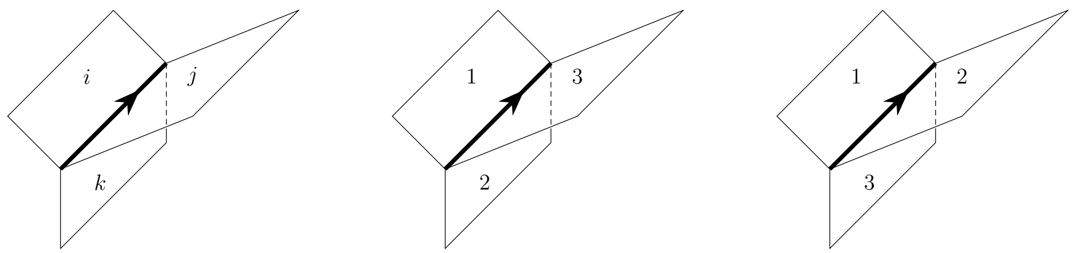
$$\theta^{+}_{ij}(F, c) + \theta^{-}_{ij}(F, c) = \theta(F).$$

We say that a singular circle Z is *positive* with respect to c if the colors of the three facets meeting at Z are $(1\ 2\ 3)$ in the cyclic ordering, and otherwise Z is negative, see Figures 4-14b and 4-14c. Let $\theta^{+}(F, c)$ (resp. $\theta^{-}(F, c)$) denote the number of positive (resp. negative) circles in F with respect to c. We have

$$\theta^{+}(F, c) + \theta^{-}(F, c) = \theta(F). \tag{4.55}$$

We will often omit F from the notation and simply write θ, $\theta^{\pm}_{ij}(c)$, and $\theta^{\pm}(c)$.

We now define the evaluations $\langle F, c\rangle$ and $\langle F\rangle$. For a pre-foam F, $c \in \mathrm{adm}(F)$, and



(a) A positive (i,j)-circle, where $i<j$. (b) A positive singular circle. (c) A negative singular circle.

Figure 4-14

$1 \leq i \leq 3$, let $d_i(c)$ denote the number of dots on facets colored i. Define

$$P(F,c) = \prod_{i=1}^{3} x_i^{d_i(c)} \tag{4.56}$$

$$Q(F,c) = \prod_{1\leq i<j\leq 3} (x_i - x_j)^{\chi(F_{ij}(c))/2} \tag{4.57}$$

$$s(F,c) = \sum_{i=1}^{3} i\chi(\overline{F}_i(c))/2 + \sum_{1\leq i<j\leq 3} \theta_{ij}^{+}(c). \tag{4.58}$$

Set

$$\langle F,c\rangle = (-1)^{s(F,c)} \frac{P(F,c)}{Q(F,c)}, \tag{4.59}$$

$$\langle F\rangle = \sum_{c\in \mathrm{adm}(F)} \langle F,c\rangle\,. \tag{4.60}$$

A priori, the evaluations $\langle F,c\rangle$ and $\langle F\rangle$ lie in the ring R_3'' (see diagram (4.53)).

In what follows, we use the symbol $\equiv$ to mean equality modulo 2. Note that

$$\sum_{i=1}^{3} i\chi(\overline{F}_i(c))/2 \equiv \frac{\chi(\overline{F}_1(c)) + \chi(\overline{F}_3(c))}{2}, \tag{4.61}$$

since $\chi(\overline{F}_2(c))$ is even. Moreover, from (4.54) we obtain

$$\sum_{i=1}^{3} i\chi(\overline{F}_i(c))/2 \equiv \theta + \sum_{i=1}^{3} i\chi(F_i(c))/2. \tag{4.62}$$

Lemma 4.3.6. *For a pre-foam F and $c \in \mathrm{adm}(F)$, we have*

$$\sum_{1 \leq i < j \leq 3} \theta^+_{ij}(c) \equiv \theta^+(c).$$

It follows that

$$s(F, c) \equiv \sum_{i=1}^{3} i\chi(F_i(c))/2 + \theta^-(c). \tag{4.63}$$

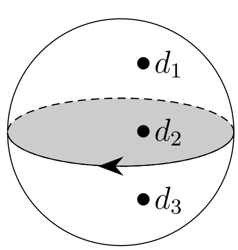
Proof. Let $Z \in \Theta(F)$. Observe that if Z is positive with respect to c, then it contributes only to $\theta^+_{13}(c)$. Likewise, if Z is negative then it contributes to both $\theta^+_{12}(c)$ and $\theta^+_{23}(c)$ but not to $\theta^+_{13}(c)$, which verifies the first equality. The second equality follows from equations (4.62) and (4.55). □

Example 4.3.7. Let F be a 2-sphere S^2 with d dots. For $1 \leq i \leq 3$, let $c_i \in \mathrm{adm}(F)$ color F by i. We have

$$\begin{aligned}
\langle F \rangle &= \langle F, c_1 \rangle + \langle F, c_2 \rangle + \langle F, c_3 \rangle \\
&= -\frac{x_1^d}{(x_1 - x_2)(x_1 - x_3)} + \frac{x_2^d}{(x_1 - x_2)(x_2 - x_3)} - \frac{x_3^d}{(x_1 - x_3)(x_2 - x_3)} \\
&= \frac{-x_1^d(x_2 - x_3) + x_2^d(x_1 - x_3) - x_3^d(x_1 - x_2)}{(x_1 - x_2)(x_2 - x_3)(x_1 - x_3)} \\
&= -s_{(d-2,0,0)}(x_1, x_2, x_3) = -h_{d-2}(x_1, x_2, x_3) = -\sum_{i+j+k=d-2} x_1^i x_2^j x_3^k,
\end{aligned}$$

where $s_{(d-2,0,0)}(x_1, x_2, x_3)$ is the Schur function of the partition $(d-2, 0, 0)$, and $h_{d-2}(x_1, x_2, x_3)$ is the complete symmetric function of degree $d - 2$. In particular $\langle F \rangle = 0$ if $d = 0$ or $d = 1$, and $\langle F \rangle = -1$ if $d = 2$.

Example 4.3.8. Let F be the theta foam shown in (4.64).



(4.64)

Given any $c \in \operatorname{adm}(F)$, each capped-off surface $\overline{F}_i(c)$ and each bicolored surface $F_{ij}(c)$ is a 2-sphere. In particular,

$$s(F,c) \equiv \theta^+(c).$$

For $\sigma \in S_3$, let $c(\sigma) \in \operatorname{adm}(F)$ denote the coloring which colors the top facet by $\sigma(1)$, the middle facet by $\sigma(2)$, and the bottom facet by $\sigma(3)$. We have

$$\langle F \rangle = \sum_{\sigma \in S_3} \langle F, c(\sigma) \rangle = \frac{\sum_{\sigma \in S_3} (-1)^{\theta^+(c(\sigma))} x_{\sigma(1)}^{d_1} x_{\sigma(2)}^{d_2} x_{\sigma(3)}^{d_3}}{(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)},$$

and moreover

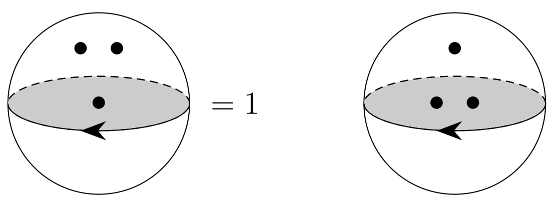
$$\theta^+(c(\sigma)) \equiv |\sigma|,$$

where $|\sigma|$ is the length of σ.

Therefore if $d_1 \geq d_2 \geq d_3$, we have

$$\langle F \rangle = s_{(d_1-2, d_2-1, d_3)}(x_1, x_2, x_3),$$

the Schur function with partition (d_1-2, d_2-1, d_3). In particular, $\langle F \rangle = 0$ if d_1, d_2, d_3 are not distinct. If d_1, d_2, d_3 are distinct and $d_1 + d_2 + d_3 \leq 3$, then up to cyclic permutation there are two choices, for which the evaluation is recorded in (4.65).



$= 1 \qquad\qquad = -1 \qquad (4.65)$

The symmetric group S_3 naturally acts on $\operatorname{adm}(F)$ and on the five rings in the diagram (4.53). The following lemma is analogous to [73, Lemma 2.17].

Lemma 4.3.9. *Let F be a pre-foam, $c \in \operatorname{adm}(F)$, and $\sigma \in S_3$. Then*

$$\sigma(\langle F, c \rangle) = \langle F, \sigma(c) \rangle .$$

Proof. We may assume that σ is a transposition $(i\ i+1)$ for $i = 1, 2$. We have

$$\sigma(P(F,c)) = P(F,\sigma(c)), \qquad \sigma(Q(F,c)) = (-1)^{\chi(F_{i(i+1)}(c))/2} Q(F,\sigma(c)).$$

Let $k \in \{1,2,3\} \setminus \{i, i+1\}$. Note that a singular circle Z is positive with respect to c if and only if Z is negative with respect to $\sigma(c)$, so

$$\theta^+(c) + \theta^+(\sigma(c)) = \theta = \theta^-(c) + \theta^-(\sigma(c)).$$

Moreover, we have

$$F_i(c) = F_{i+1}(\sigma(c)),\ F_{i+1}(c) = F_i(\sigma(c)),\ F_k(c) = F_k(\sigma(c)).$$

Therefore

$$\begin{aligned} s(F,c) - s(F,\sigma(c)) &= \frac{\chi(F_{i+1}(c)) - \chi(F_i(c))}{2} + \theta^-(c) - \theta^-(\sigma(c)) \\ &\equiv \frac{\chi(F_{i+1}(c)) - \chi(F_i(c))}{2} + \theta \\ &\equiv \frac{\chi(F_{i+1}(c)) + \chi(F_i(c))}{2} \\ &\equiv \frac{\chi(F_{i(i+1)}(c))}{2}, \end{aligned}$$

which completes the proof. □

Corollary 4.3.10. *The evaluation $\langle F \rangle$ is a symmetric rational function.*

Later we will prove that $\langle F \rangle$ is in fact a polynomial, see Corollary 4.3.22.

Lemma 4.3.11. *Let $i \in \{1,2\}$, let F be a pre-foam, and let $c \in \operatorname{adm}(F)$ be an admissible coloring. Suppose $c' \in \operatorname{adm}(F)$ is obtained from c by an $(1,2)$-Kempe move along a surface $\Sigma \subset F_{12}(c)$. Then*

$$s(F,c) \equiv s(F,c') + \frac{\chi(\Sigma)}{2}.$$

Proof. Note that this is analogous to [73, Lemma 2.20]. Letting $\theta(\Sigma)$ denote the number of seam circles on Σ, we have

$$\theta^-(c) + \theta^-(c') \equiv \theta(\Sigma) \equiv \chi(F_1(c) \cap \Sigma).$$

Note also that

$$\chi(F_1(c)) - \chi(F_1(c')) = \chi(F_1(c) \cap \Sigma) - \chi(F_2(c) \cap \Sigma),$$
$$\chi(F_2(c)) - \chi(F_2(c')) = \chi(F_2(c) \cap \Sigma) - \chi(F_1(c) \cap \Sigma).$$

We compute:

$$\begin{aligned}
s(F, c) - s(F, c') &\equiv \frac{\chi(F_1(c)) - \chi(F_1(c'))}{2} + \frac{2(\chi(F_2(c)) - \chi(F_2(c')))}{2} + \theta(\Sigma) \\
&\equiv \frac{\chi(F_2(c) \cap \Sigma) - \chi(F_1(c) \cap \Sigma)}{2} + \chi(F_1(c) \cap \Sigma) \\
&\equiv \frac{\chi(\Sigma)}{2}.
\end{aligned}$$

□

4.3.2 Anchored $\mathfrak{sl}_3$ foams and their evaluations

In this section we introduce (oriented) anchored $\mathfrak{sl}_3$ foams and their evaluations. Recall the anchor line $\mathcal{L}$, which is the z-axis in $\mathbb{R}^3$.

Definition 4.3.12. An *anchored $\mathfrak{sl}_3$ foam* F consists of an $\mathfrak{sl}_3$ foam $F' \subset \mathbb{R}^3$ that may intersect the anchor line $\mathcal{L}$ at finitely many points away from the singular circles of F', so that each intersection point belongs to some facet of F', and moreover these intersections are required to be transverse. Denote by $\mathrm{an}(F) = F' \cap \mathcal{L}$ the set of intersection points (anchor points) of F. The anchor points carry labels in $\{1, 2, 3\}$; that is, F comes equipped with a fixed map

$$\ell \colon \mathrm{an}(F) \to \{1, 2, 3\}.$$

Fix an anchored foam F and an admissible coloring c of the underlying foam F'. Each anchor point $p \in \mathrm{an}(F)$ lying on a facet f inherits a color $c(p) := c(f)$.

For $i \in \{1,2,3\}$, let i', i'' denote the complementary elements, so that $\{i, i', i''\} = \{1,2,3\}$. Define the evaluation

$$\langle F, c\rangle = (-1)^{s(F,c)} \frac{P(F,c)}{Q(F,c)} \left(\prod_{p \in \mathrm{an}(F)} (-1)^{c(p)-1} (x_{c(p)} - x_{\ell(p)'})(x_{c(p)} - x_{\ell(p)''}) \right)^{1/2}, \tag{4.66}$$

where $P(F,c)$, $Q(F,c)$ and $s(F,c)$ are as defined in equations (4.56), (4.57), and (4.58), respectively.

Let us explain the square root in equation (4.66). If $c(p) \neq \ell(p)$ for some $p \in \mathrm{an}(F)$, then one of $x_{c(p)} - x_{\ell(p)'}$ or $x_{c(p)} - x_{\ell(p)''}$ is zero, so $\langle F, c\rangle = 0$. Assume then that $c(p) = \ell(p)$ for every anchor point $p \in \mathrm{an}(F)$. If p is labeled i, then it contributes

$$(-1)^{i-1}(x_i - x_j)(x_i - x_k)$$

to the product under the square root. More concretely, the product of the two terms under the square root, for a fixed anchor point p, is equal to

$$\begin{aligned} &(x_1 - x_2)(x_1 - x_3) \quad \text{if} \quad c(p) = 1, \\ &(x_1 - x_2)(x_2 - x_3) \quad \text{if} \quad c(p) = 2, \\ &(x_1 - x_3)(x_2 - x_3) \quad \text{if} \quad c(p) = 3. \end{aligned}$$

So, a priori, the evaluation $\langle F, c\rangle$ lies in $\widetilde{R}_3''$ (see diagram (4.53) and the surrounding discussion for notations of various rings). In light of the above discussion, we make the following definition.

Definition 4.3.13. Given an anchored foam F with underlying foam F', we say that $c \in \mathrm{adm}(F')$ is an admissible coloring of the anchored foam F if for each $p \in \mathrm{an}(F)$, the color of p equals the label of p; that is, $c(p) = \ell(p)$. Denote by $\mathrm{adm}(F)$ the set of admissible colorings of the anchored foam F.

The following proposition says that no square roots appear in $\langle F, c\rangle$.

Proposition 4.3.14. *The evaluation $\langle F, c\rangle$ is an element of R_3''.*

Proof. Let $\mathrm{an}_i(F)$ be the number of anchor points p with $c(p) = i$. Then for $1 \le i < j \le 3$, $\mathrm{an}_i(F) + \mathrm{an}_j(F)$ is equal to the number of intersection points of $F_{ij}(c)$ with $\mathcal{L}$, which is even since $F_{ij}(c)$ is a closed surface. On the other hand, by the above discussion of contributions of anchor points, the exponent of $x_i - x_j$ under the square root in equation (4.66) is precisely $\mathrm{an}_i(F) + \mathrm{an}_j(F)$. $\square$

If $c \in \mathrm{adm}(F)$, we can rewrite the square root in equation (4.66) as

$$\widetilde{Q}(F,c) \;:=\; \prod_{1\le i<j\le 3} (x_i - x_j)^{(\mathrm{an}(i)+\mathrm{an}(j))/2} \tag{4.67}$$

and rewrite the evaluation $\langle F, c\rangle$ as

$$\begin{aligned} \langle F, c\rangle \;&:=\; (-1)^{s(F,c)} \frac{P(F,c)\widetilde{Q}(F,c)}{Q(F,c)} \\ &= (-1)^{s(F,c)} P(F,c) \prod_{1\le i<j\le 3} (x_i - x_j)^{(\mathrm{an}(i)+\mathrm{an}(j)-\chi(F_{ij}(c)))/2}. \end{aligned} \tag{4.68}$$

Definition 4.3.15. For an anchored foam F, define its evaluation to be

$$\langle F\rangle = \sum_{c\in\mathrm{adm}(F)} \langle F, c\rangle\,. \tag{4.69}$$

By Proposition 4.3.14, $\langle F\rangle \in R_3''$. We will see in Theorem 4.3.21 that in fact no denominators appear, and that $\langle F\rangle \in R_3' = \mathbb{Z}[x_1, x_2, x_3]$.

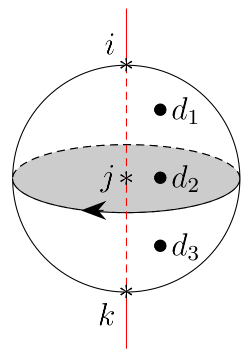
Remark 4.3.16. As in the discussion following equation (4.66), if c is an admissible coloring of the underlying foam F' but not of the anchored foam F, then the evaluation (4.66) is still well-defined and equal to zero. Even if we do not restrict the notion of admissible colorings of an anchored foam to those which color anchor points according to their labels, additional summands in the evaluation will each be 0, and thus not contribute nothing.

Example 4.3.17. Let F be a 2-sphere S^2 carrying d dots and intersecting $\mathcal{L}$ twice. Then $\langle F\rangle = 0$ unless both anchor points are labeled by $i \in \{1, 2, 3\}$. In this case, there is one admissible coloring c which colors F by i. We see that $s(F,c) \equiv i$, and the evaluation is

$$\langle F \rangle = (-1)^i x_i^d.$$

Example 4.3.18. Consider the theta foam F whose facets each intersect $\mathcal{L}$ exactly once, shown in (4.70). There is one admissible coloring c, and we have

$$\langle F \rangle = \langle F, c \rangle = \begin{cases} x_i^{d_1} x_j^{d_2} x_k^{d_3} & \text{if } (i,j,k) = (1,3,2) \text{ or a cyclic permutation,} \\ -x_i^{d_1} x_j^{d_2} x_k^{d_3} & \text{if } (i,j,k) = (1,2,3) \text{ or a cyclic permutation.} \end{cases}$$



(4.70)

The symmetric group S_3 acts on all five of the rings in diagram (4.53). Recall also that S_3 acts on the set of admissible colorings of an non-anchored foam (i.e., those considered in Section 4.3.1). However, for an anchored foam F, $c \in \mathrm{adm}(F)$, and $\sigma \in S_3$, the coloring $\sigma(c)$ is in general not admissible for F.

Consider instead the anchored foam $\sigma(F)$ defined as follows. The underlying foam of $\sigma(F)$ agrees with the underlying foam of F. If anchor points of F are labeled by $\ell\colon \mathrm{an}(F) \to \{1,2,3\}$, then the anchor points of $\sigma(F)$ are labeled by $\sigma(l)\colon p \mapsto \sigma(\ell(p))$. Note that σ provides a bijection $\mathrm{adm}(F) \cong \mathrm{adm}(\sigma(F))$ via $c \mapsto \sigma(c)$. The following lemma says that the evaluations $\langle F \rangle$ and $\langle \sigma(F) \rangle$ differ by a sign, and moreover the sign depends only on σ and on labels of anchor points of F.

Lemma 4.3.19. *For an anchored foam F, $c \in \mathrm{adm}(F)$, and $\sigma \in S_3$, we have*

$$\sigma\left(\langle F, c \rangle\right) = (-1)^{\varepsilon(F,\sigma)} \left\langle \sigma(F), \sigma(c) \right\rangle,$$

where

$$\varepsilon(F,\sigma) = \sum_{\substack{1\le i<j\le 3,\\ \sigma(i)>\sigma(j)}} \frac{\mathrm{an}(i)+\mathrm{an}(j)}{2}. \tag{4.71}$$

It follows that

$$\sigma\left(\langle F\rangle\right) = (-1)^{\varepsilon(F,\sigma)}\left\langle \sigma(F)\right\rangle.$$

Proof. By Lemma 4.3.9, we have

$$\sigma\left((-1)^{s(F,c)}\frac{P(F,c)}{Q(F,c)}\right) = (-1)^{s(\sigma(F),\sigma(c))}\frac{P(\sigma(F),\sigma(c))}{Q(\sigma(F),\sigma(c))}.$$

It is clear that

$$\sigma(\widetilde{Q}(F)) = (-1)^{\varepsilon(F,\sigma)}\widetilde{Q}(\sigma(F)),$$

and the first equality follows. For the second equality, we have

$$\begin{aligned}\sigma\left(\langle F\rangle\right) &= \sum_{c\in\mathrm{adm}(F)}\sigma\left(\langle F,c\rangle\right)\\ &= (-1)^{\varepsilon(F,\sigma)}\sum_{c\in\mathrm{adm}(F)}\left\langle\sigma(F),\sigma(c)\right\rangle\\ &= (-1)^{\varepsilon(F,\sigma)}\left\langle\sigma(F)\right\rangle.\end{aligned}$$

□

For $1\le i\ne j\le 3$, consider the ring

$$R''_{ij} := R'_3[(x_i-x_k)^{-1},(x_j-x_k)^{-1}].$$

Each R''_{ij} is a subring of R''_3. A permutation $\sigma\in S_3$ sends R''_{ij} isomorphically onto $R''_{\sigma(i)\sigma(j)}$.

We recall the following local operation on foam colorings from [73].

Definition 4.3.20. Let F be an anchored foam, $c\in\mathrm{adm}(F)$, $1\le i<j\le 3$, and $\Sigma\subset F_{ij}(c)$ a closed sub-surface which is disjoint from $\mathcal{L}$. Let $c'\in\mathrm{adm}(F)$ denote the coloring which swaps i and j colors on the facets of Σ, and leaves all other facets colored according to c. The coloring c' is said to obtained from c by an (i,j) *Kempe move.*

We are now ready for the main result of this section.

Theorem 4.3.21. *The evaluation $\langle F\rangle$ of an anchored foam is an element of R_3', the polynomial ring in variables x_1, x_2, x_3.*

Proof. The proof is similar to that of [41, Theorem 2.17] and [73, Proposition 2.19]. By Lemma 4.3.19, it suffices to show that $\langle F\rangle \in R_{12}''$ for any anchored foam F. This is because we may take a permutation $\sigma \in S_3$ sending 1 to i and 2 to j, and consider the anchored foam $\sigma^{-1}(F)$. Then $\langle \sigma^{-1}(F)\rangle \in R_{12}''$ implies that

$$\pm\langle F\rangle = \pm\left\langle \sigma(\sigma^{-1}(F))\right\rangle = \pm\sigma\left(\left\langle \sigma^{-1}(F)\right\rangle\right) \in R_{ij}'',$$

where the first equality comes from Lemma 4.3.19. It will then follow that

$$\langle F\rangle \in R_{12}'' \cap R_{23}'' \cap R_{13}'' = R_3'.$$

Let us show that $\langle F\rangle \in R_{12}''$. Partition $\mathrm{adm}(F)$ into equivalence classes as follows. For $c \in \mathrm{adm}(F)$, the class C_c containing c consists of colorings obtained from c by performing a sequence of $(1,2)$ Kempe moves along surfaces in $F_{12}(c)$ which are disjoint from $\mathcal{L}$. If $F_{12}(c)$ has n connected components, $k \geq 0$ of which are disjoint from $\mathcal{L}$, then C_c consists of 2^k elements. We will show that

$$\sum_{c' \in C_c} \langle F, c'\rangle \in R_{12}'',$$

which will conclude the proof.

Write $\Sigma := F_{12}(c)$ as a disjoint union

$$\Sigma = \Sigma' \cup \Sigma_1 \cup \cdots \cup \Sigma_k,$$

where each Σ_a, $a = 1, \ldots, k$ is connected and disjoint from $\mathcal{L}$, and where each component of Σ' intersects $\mathcal{L}$. For $i = 1, 2$ and $a = 1, \ldots, k$, let $t_i(a)$ denote the number of dots on i-colored facets (according to c) of Σ_a, and let t_3 denote the number of dots on 3-colored facets (according to c) of F. We claim that

$$\sum_{c' \in C_c} \langle F, c' \rangle = \frac{x_3^{t_3} \cdot \prod_{a=1}^{k} \left(x_1^{t_1(a)} x_2^{t_2(a)} + (-1)^{\chi(\Sigma_a)/2} x_2^{t_1(a)} x_1^{t_2(a)} \left(\frac{(x_1 - x_3)}{(x_2 - x_3)}^{\ell_{\Sigma_a}(c)/2} \right) \right) \cdot \widetilde{Q}(F)}{(x_1 - x_2)^{\chi(\Sigma)/2} (x_1 - x_3)^{\chi(F_{13}(c))/2} (x_2 - x_3)^{\chi(F_{23}(c))/2}} \tag{4.72}$$

where

- $\ell_{\Sigma_a}(c) \in 2\mathbb{Z}$ is an even integer such that

$$\chi(F_{13}(c')) = \chi(F_{13}(c)) - \ell_{\Sigma_a}(c) \ \text{ and } \ \chi(F_{23}(c')) = \chi(F_{23}(c)) + \ell_{\Sigma_a}(c)$$

  for the coloring $c' \in C_c$ which is obtained from c by a $(1, 2)$ Kempe move along Σ_a. See [41, Lemma 2.12 (3)] for details regarding this integer.

- $\widetilde{Q}(F)$ is the contribution from the anchor points of F, equation (4.67).

To verify the claimed equality, expand the product to obtain 2^k terms, each of which corresponds to one of the 2^k colorings in C_c. That the sign is correct follows from Lemma 4.3.11.

Finally, we argue that $(x_1 - x_2)^{\chi(\Sigma)/2}$ divides the numerator of (4.72). Positive contributions to $\chi(\Sigma)$ come from 2-sphere components of Σ. Each Σ_a which is a 2-sphere contributes one to the exponent $\chi(\Sigma)/2$. On the other hand, the corresponding factor in the product in the numerator of (4.72) is divisible by $x_1 - x_2$. The remaining positive contributions to $\chi(\Sigma)/2$ come from 2-sphere components of Σ'. Such a component Σ_0 contains at least two anchor points, each labeled 1 or 2, so the contribution from Σ_0 can be cancelled with terms in $\widetilde{Q}(F)$. □

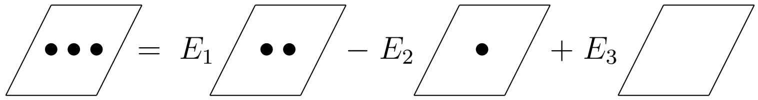
Corollary 4.3.22. *If F is a pre-foam or a foam which is disjoint from $\mathcal{L}$, then $\langle F \rangle \in R_3$, the ring of symmetric polynomials in x_1, x_2, x_3.*

Proof. This follows from Lemma 4.3.19 and Theorem 4.3.21. □

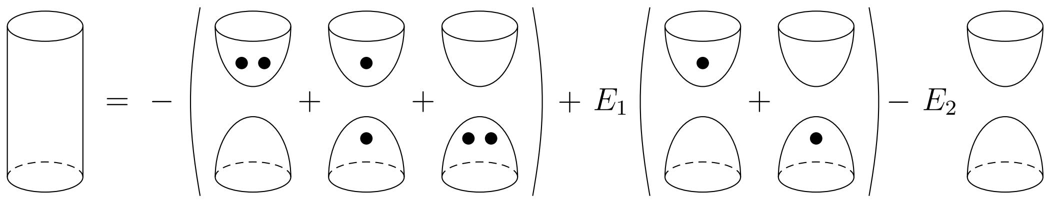
4.3.3 Skein relations

In this section we record several local relations involving oriented anchored $\mathfrak{sl}_3$ foams.

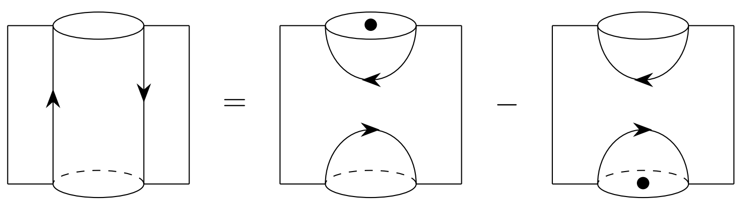
Lemma 4.3.23. *The local relations* (4.73)*,* (4.74)*,* (4.75)*, and* (4.76) *hold for anchored foams. Seam lines are drawn in bold in relation* (4.76) *to clarify the picture.*



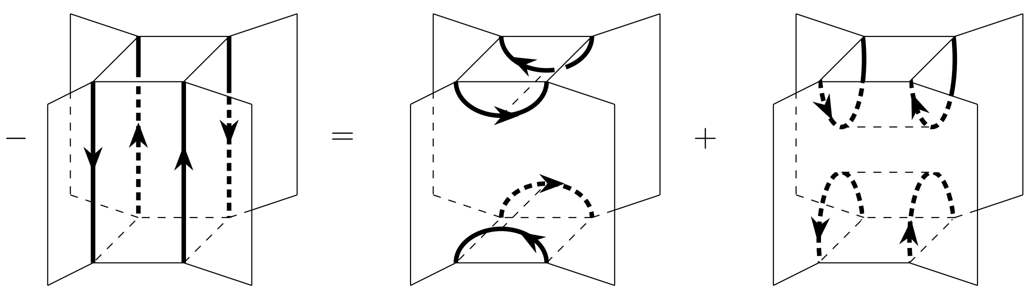
(4.73)



(4.74)



(4.75)



(4.76)

Proof. Proofs of these four relations are similar to Propositions 2.33, 2.22, 2.23, and 2.24 in [41], respectively, with the caveat that we must carefully keep track of the sign (4.58). Moreover, S_3 symmetry is used in [41] to simplify the calculations. For any one of the above four relations, anchor points and their labels are the same for all the foams depicted, so Lemma 4.3.19 implies that we may use S_3 symmetry in a similar manner.

We verify relations (4.74) and (4.75), and leave the remaining two relations to the reader. Let F denote the foam appearing on the left-hand side of the equality. The six foams on the right-hand side are identical except for placement of dots. We denote them by $G^1, \dots, G^6$, so that the relation reads

$$\langle F\rangle = -\left(\left\langle G^1\right\rangle + \left\langle G^2\right\rangle + \left\langle G^3\right\rangle\right) + E_1\left(\left\langle G^4\right\rangle + \left\langle G^5\right\rangle\right) - E_2\left\langle G^6\right\rangle.$$

Admissible colorings of $G^1, \dots, G^6$ are in canonical bijection. For $c \in \operatorname{adm}(G^1)$, let

$$\langle G, c\rangle := -\left(\left\langle G^1, c\right\rangle + \left\langle G^2, c\right\rangle + \left\langle G^3, c\right\rangle\right) + E_1\left(\left\langle G^4, c\right\rangle + \left\langle G^5, c\right\rangle\right) - E_2\left\langle G^6, c\right\rangle.$$

There are two types of colorings of G^1: those which color the two depicted disks the same, and those which color them differently. Those of the first type are in canonical bijection with colorings of F.

Suppose $c \in \operatorname{adm}(G^1)$ colors both disks the same color, say i, and denote by $c \in \operatorname{adm}(G^2) \cong \cdots \cong \operatorname{adm}(G^6)$ and $c' \in \operatorname{adm}(F)$ the corresponding colorings. We will show that $\langle F, c'\rangle = \langle G, c\rangle$. We may assume $i = 1$. Then

$$\left\langle G^1, c\right\rangle = \left\langle G^2, c\right\rangle = \left\langle G^3, c\right\rangle = x_1^2\left\langle G^6, c\right\rangle, \quad \left\langle G^4, c\right\rangle = \left\langle G^5, c\right\rangle = x_1\left\langle G^6, c\right\rangle,$$

which yields

$$\langle G, c\rangle = -3x_1^2\left\langle G^6, c\right\rangle + 2E_1x_1\left\langle G^6, c\right\rangle - E_2\left\langle G^6, c\right\rangle = -(x_1 - x_2)(x_1 - x_3)\left\langle G^6, c\right\rangle.$$

To compare this with $\langle F, c'\rangle$, observe that

$$\chi(F_1(c')) + 2 = \chi(G_1^6(c)), \quad \chi(F_2(c')) = \chi(G_2^6(c)), \quad \chi(F_3(c')) = \chi(G_3^6(c)),$$

which implies $s(F, c') \equiv s(G, c) + 1$. Moreover, we have

$$\chi(F_{12}(c')) + 2 = \chi(G_{12}^6(c)), \quad \chi(F_{13}(c')) + 2 = \chi(G_{13}^6(c)), \quad \chi(F_{23}(c')) = \chi(G_{23}^6(c)).$$

Therefore

$$\left\langle G^6, c\right\rangle = -\frac{\langle F, c'\rangle}{(x_1 - x_2)(x_1 - x_3)},$$

which verifies $\langle F, c'\rangle = \langle G, c\rangle$.

To complete the proof, suppose that c colors the top depicted disk by i and the bottom disk by j, with $i \neq j$. We have

$$\left\langle G^1, c\right\rangle = x_i^2 \left\langle G^6, c\right\rangle, \quad \left\langle G^2, c\right\rangle = x_i x_j \left\langle G^6, c\right\rangle, \quad \left\langle G^3, c\right\rangle = x_j^2 \left\langle G^3, c\right\rangle,$$

$$\left\langle G^4, c\right\rangle = x_i \left\langle G^6, c\right\rangle, \quad \left\langle G^5, c\right\rangle = x_j \left\langle G^6, c\right\rangle.$$

Therefore $\langle G, c\rangle = 0$, which concludes the proof of relation (4.74).

Let us now demonstrate relation (4.75). Let F denote the foam on the left-hand side, and G^t, G^b the two foams on the right-hand side, so that the relation reads $\langle F\rangle = \langle G^t\rangle - \left\langle G^b\right\rangle$. There are two types of colorings of G^t (and of G^b). Let $\mathrm{adm}_0(G^t)$ denote the colorings where the two front half-cups are colored the same, and so are the two back half-cups. Let $\mathrm{adm}_1(G^t)$ denote the colorings where the two front half-cups are colored differently (and, consequently, so are the two back half-cups). Likewise, decompose $\mathrm{adm}(G^b)$ as a disjoint union of $\mathrm{adm}_0(G^t)$ and $\mathrm{adm}_1(G^t)$. There is a natural bijection

$$\mathrm{adm}(F) \cong \mathrm{adm}_0(G^t) \cong \mathrm{adm}_0(G^b).$$

For $c \in \mathrm{adm}(F)$, let $c^t \in \mathrm{adm}_0(G^t)$ and $c^b \in \mathrm{adm}_0(G^b)$ denote the corresponding colorings. There is also a natural bijection $\mathrm{adm}_1(G^t) \cong \mathrm{adm}_1(G^b)$; given $d \in \mathrm{adm}_1(G^t)$, let $d' \in \mathrm{adm}_1(G^b)$ denot the corresponding coloring. We will show that

$$\langle F, c\rangle = \left\langle G^t, c^t\right\rangle - \left\langle G^b, c^b\right\rangle, \tag{4.77}$$

$$0 = \left\langle G^t, d\right\rangle - \left\langle G^b, d'\right\rangle, \tag{4.78}$$

for all $c \in \mathrm{adm}(F)$ and all $d \in \mathrm{adm}_1(G^t)$, from which the desired relation follows.

By Lemma 4.3.19, it suffices to verify (4.77) when $c \in \mathrm{adm}(F)$ colors the two side facets by 1, the back facet 2, and the front facet 3. We have

$$\begin{aligned}
\chi(F_{12}(c)) &= \chi(G^t_{12}(c^t)) = \chi(G^b_{12}(c^b)),\\
\chi(F_{13}(c)) &= \chi(G^t_{13}(c^t)) = \chi(G^b_{13}(c^b)),\\
\chi(F_{23}(c)) + 2 &= \chi(G^t_{23}(c^t)) = \chi(G^b_{23}(c^b)).
\end{aligned}$$

On the other hand, $P(G^t, c^t) = x_2 P(F, c)$, $P(G^b, c^b) = x_3 P(F, c)$. Labels of anchor points for all three foams are the same, so equation (4.77) follows if we show that signs for all three foams are equal. We have

$$\begin{aligned}
\chi(F_1(c)) &= \chi(G^t_1(c^t)) + 1 = \chi(G^b_1(c^b)) + 1,\\
\chi(F_2(c)) &= \chi(G^t_2(c^t)) - 1 = \chi(G^b_2 c^b)) - 1,\\
\chi(F_3(c)) &= \chi(G^t_3(c^t)) - 1 = \chi(G^b_3 c^b)) - 1.
\end{aligned}$$

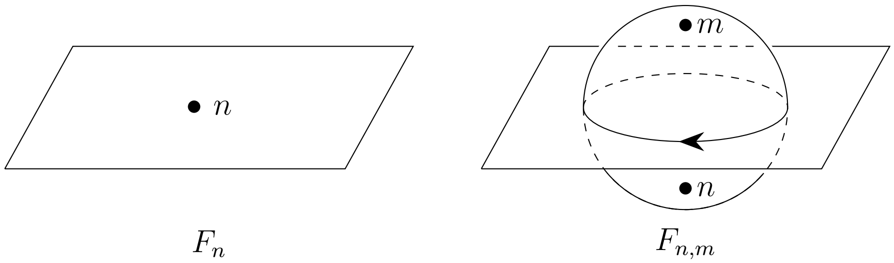
The singular intervals depicted in the relation are all part of positive circles, so $\theta^-(F, c) = \theta^-(G^t, c^t) = \theta^-(G^b, c^b)$. Therefore $s(F, c) = s(G^t, c^t) = s(G^b, c^b)$, verifying equation (4.77).

Let us now prove equation (4.78). For any $d \in \mathrm{adm}_1(G^t)$, it is straightforward to verify the equalities $P(G^t, d) = P(G^b, d')$, $Q(G^t, d) = Q(G^b, d')$, $\widetilde{Q}(G^t, d) = \widetilde{Q}(G^b, d')$, and $s(G^t, d) = s(G^b, d')$, which finishes the proof.

□

Lemma 4.3.24. *Let F be an anchored foam. Denote by $F_{n,m}$ the anchored foam obtained from F by adding a bubble (disjoint from $\mathcal{L}$) to some facet in F, with the two new facets carrying n and m dots respectively, such that the facet with n dots directly precedes the facet with m dots in the cyclic ordering. Let F_n denote the foam obtained from F by adding n dots to the same facet. This is shown in (4.79). Then*

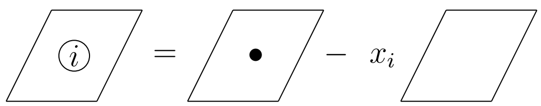
$$\begin{aligned}
&\langle F_{n,n}\rangle = 0,\\
&\langle F_{1,0}\rangle = -\langle F_{0,1}\rangle = \langle F\rangle,\\
&\langle F_{2,0}\rangle = -\langle F_{0,2}\rangle = E_1\langle F\rangle - \langle F_1\rangle.
\end{aligned}$$



(4.79)

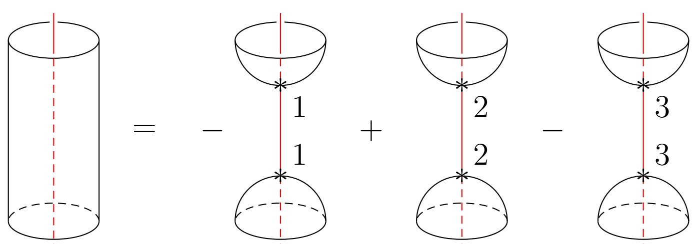
Remark 4.3.25. The relations in Lemmas 4.3.23 and 4.3.24 also hold for pre-foams.

Similar to the $\mathfrak{sl}_2$ setting, for anchored $\mathfrak{sl}_3$ foams we allow shifted dots $ⓘ = \bullet - x_i$ $(1 \leq i \leq 3)$ on a facet.

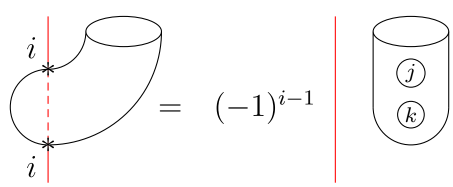


They must be disjoint from $\mathcal{L}$ and are allowed to float freely on their facets but cannot cross seam lines.

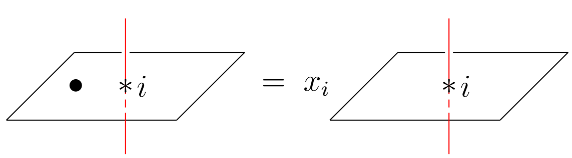
Lemma 4.3.26. *The local relations shown in* (4.80), (4.81), (4.82) , *and* (4.83) *hold.*



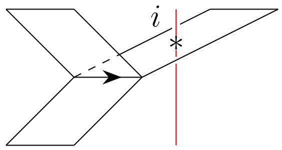
(4.80)

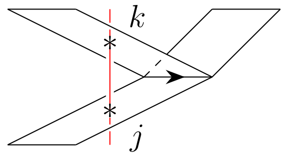


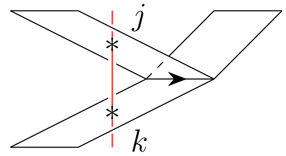
(4.81)



(4.82)

$(x_j - x_k)$ 

$=$ 

$+$ 

(4.83)

In the last equation we assume $j < k$.

Proof. We verify equation (4.80); the other relations are easier and left to the reader. Let F denote the foam on the left-hand side, and let G^1, G^2, G^3 denote the three foams on the right-hand side, with superscript corresponding to labels of the anchor points. For $1 \le i \le 3$, let $\mathrm{adm}_i(F)$ consist of all admissible colorings of F which color the depicted tube by i. There is a natural bijection $\mathrm{adm}_i(F) \cong \mathrm{adm}(G^i)$.

Given $c \in \mathrm{adm}_i(F)$, let $c' \in \mathrm{adm}(G^i)$ denote the corresponding coloring. Clearly $P(G^i, c') = P(F, c)$. Let j, k denote the two elements in $\{1, 2, 3\} \setminus \{i\}$. To compare the denominators $Q(G^i, c')$ and $Q(F, c)$, we have

$$\chi(G^i_{ij}(c')) = \chi(F_{ij}(c)) + 2, \quad \chi(G^i_{ik}(c')) = \chi(F_{ik}(c)) + 2, \quad \chi(G^i_{jk}(c')) = \chi(F_{jk}(c)),$$

so $Q(G^i, c') = \pm(x_i - x_j)(x_i - x_k)Q(F, c)$ (the sign is there only to account for the ordering of subscripts). Note that the additional factor $\pm(x_i - x_j)(x_i - x_k)$ precisely cancels with contributions from the two depicted anchor points of G^i. Therefore

$$\langle F, c\rangle = \pm \left\langle G^i, c' \right\rangle .$$

It remains to show that the above sign is equal to $(-1)^i$. We have

$$\chi(F_j(c)) = \chi(G^i_j(c')),\ \chi(F_k(c)) = \chi(G^j_k(c')),\ \chi(F_i(c)) = \chi(G^i_i(c')) - 2,\ \theta^{\pm}(F, c) = \theta^{\pm}(G^i, c'),$$

so $s(F, c) \equiv s(G^i, c') + i$ as needed.

$\square$

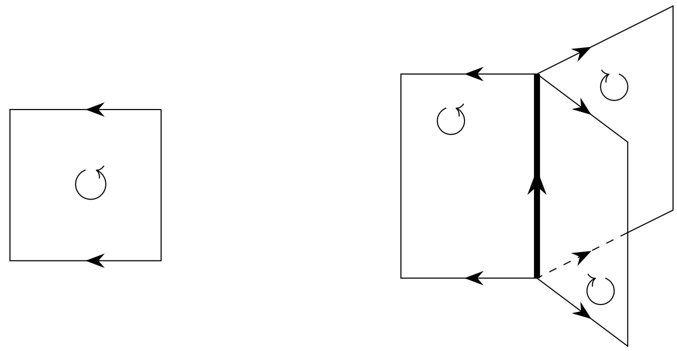


Figure 4-15: Our convention for the induced orientation on the webs $\partial_0 V$ (bottom) and $\partial_1 V$ (top).

4.3.4 State spaces

In this section we define state spaces via the usual recipe applied to anchored foam evaluation. Recall from Definition 2.5.2 the notion of an $\mathfrak{sl}_3$ web.

Definition 4.3.27. An *annular* $\mathfrak{sl}_3$ web is an $\mathfrak{sl}_3$ web embedded in the punctured plane $\mathcal{P}$.

A *anchored* $\mathfrak{sl}_3$ *foam with boundary* V is the intersection of a closed anchored $\mathfrak{sl}_3$ foam $F \subset \mathbb{R}^3$ with a $\mathbb{R}^2 \times [0,1]$ such that $F \cap (\mathcal{P} \times \{i\})$, $i = 0,1$ is a web (in particular, F is disjoint from the two points $(0,0,0)$ and $(0,0,1)$), and dots of F are disjoint from $\mathbb{R}^2 \times \{i\}$ and from $\mathcal{L}$. Foams with boundary are considered up to ambient isotopy of $\mathbb{R}^2 \times [0,1]$ which fixes the boundary of $\mathbb{R}^2 \times [0,1]$ point-wise and maps the line segment $\mathcal{L}_{[0,1]} := \{(0,0)\} \times [0,1]$ to itself.

For a foam with boundary V, let $\mathrm{an}(V) = V \cap \mathcal{L}_{[0,1]}$ denote its intersection points with the anchor line, called *anchor points*. Each anchor point p is required to carry a label $\ell(p) \in \{1,2,3\}$.

The orientation of facets of V induces an orientation on the boundary webs $\partial_0 V := V \cap (\mathbb{R}^2 \times \{0\})$ and $\partial_1 V := V \cap (\mathbb{R}^2 \times \{1\})$ via the convention in Figure 4-15. We view V as a cobordism from the web $\partial_0 V$ to the web $\partial_1 V$. A closed anchored foam is then a cobordism from the empty web to itself. In this section, we will often refer to anchored $\mathfrak{sl}_3$ foams with boundary simply as *foams* when the meaning is clear from context. Composition WV of foams V, W with $\partial_1 V = \partial_0 W$ is defined in the natural way. We obtain a category **AFoam** of $\mathfrak{sl}_3$ webs and anchored $\mathfrak{sl}_3$ foams.

Given a foam cobordism V, let $|d(V)|$ the number of dots on V. The quantum grading qdeg(V) of V is defined to be

$$\operatorname{qdeg}(V) = 2\left(|d(V)| + |\operatorname{an}(V)| - \chi(V)\right) + \chi(\partial V). \tag{4.84}$$

Lemma 4.3.28. *If V and W are composable foam cobordisms, then* $\operatorname{qdeg}(WV) = \operatorname{qdeg}(W) + \operatorname{qdeg}(V)$. *If F is a closed anchored foam, then* $\operatorname{qdeg}(F) = \deg(\langle F \rangle)$.

Proof. This is clear from the definitions. □

As in Definition 4.2.13, by an *annular foam* we mean a foam (with boundary) which is disjoint from $\mathcal{L}$. The composition of two annular foams is again annular.

There is an involution ω defined by reflecting a foam with boundary through $\mathbb{R}^2 \times \{1/2\}$. We have $\partial_1 V = \partial_0(\omega(V))$ and $\partial_0 V = \partial_1(\omega(V))$ for any foam with boundary V. Given a web $\Gamma \subset \mathcal{P}$, let $\operatorname{Fr}(\Gamma)$ denote the free R_3'-module generated by foams with boundary V from the empty web to Γ (that is, $\partial_0 V = \varnothing$, $\partial_1 V = \Gamma$). Define a bilinear form

$$(-,-)\colon\ \operatorname{Fr}(\Gamma) \times \operatorname{Fr}(\Gamma) \to R_3'$$

by $(V, W) = \omega(V)W$. This bilinear form is symmetric since $\langle F \rangle = \langle \omega(F) \rangle$ for any closed foam F. The state space $\langle \Gamma \rangle$ is the quotient of $\operatorname{Fr}(\Gamma)$ by the kernel

$$\ker((-,-)) = \{x \in \operatorname{Fr}(\Gamma) \mid (x, y) = 0 \text{ for all } y \in \operatorname{Fr}(\Gamma)\}$$

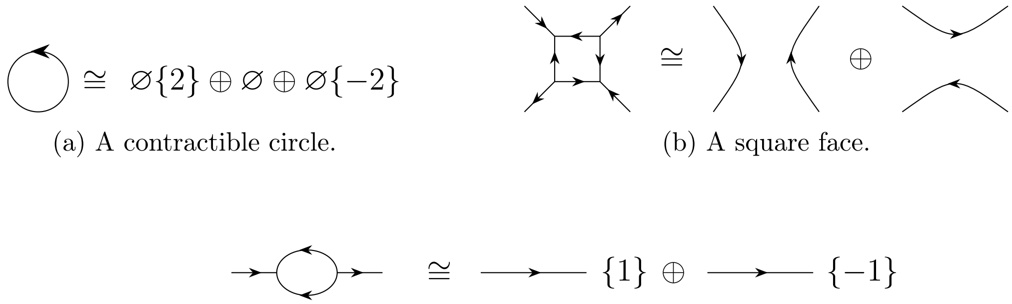
of the bilinear form,

$$\langle \Gamma \rangle := \operatorname{Fr}(\Gamma)/\ker((-,-)).$$

The state space $\langle \Gamma \rangle$ inherits the grading from $\operatorname{Fr}(\Gamma)$ since $(-,-)$ is degree-preserving. A foam with boundary V from Γ_0 to Γ_1 naturally induces a map

$$\langle V \rangle :\ \langle \Gamma_0 \rangle \to \langle \Gamma_1 \rangle$$

of degree qdeg(V), defined by sending the equivalence class of a basis element $U \in \operatorname{Fr}(\Gamma_0)$ to the equivalence class of $VU \in \operatorname{Fr}(\Gamma_1)$. This assignment is functorial with respect to



(a) A contractible circle.

(b) A square face.

(c) A bigon face.

Figure 4-16: Local relations for state spaces of oriented $\mathfrak{sl}_3$ webs, where the depicted regions do not contain the puncture.

composition of foams, $\langle WV \rangle = \langle W \rangle \langle V \rangle$ for composable V, W.

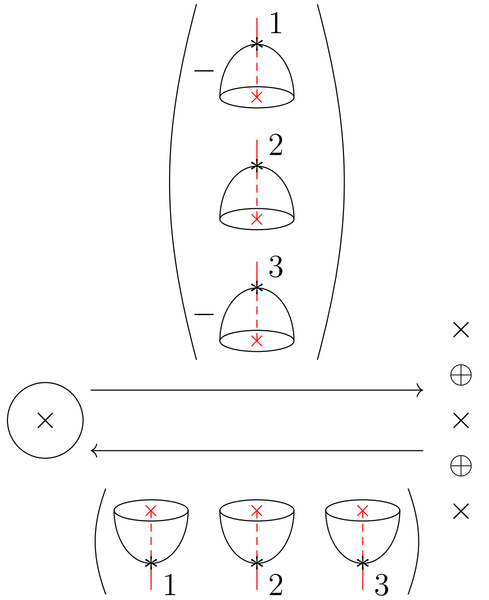
Lemma 4.3.29. *The three local isomorphisms shown in Figure 4-16 hold.*

Proof. The arguments for relations (a), (b), and (c) of the figure are analogous to Propositions 7, 9, and 8, respectively, of [35]. The relevant relations are given in Lemma 4.3.23 and Lemma 4.3.24. $\square$

Proposition 4.3.30. *Let $\Gamma \subset \mathcal{P}$ be a web with a non-contractible circle C which bounds a disk in $\mathbb{R}^2 \setminus \Gamma$, and let $\Gamma' = \Gamma \setminus C$ be the web obtained by removing C. Then there is an isomorphism*

$$\langle \Gamma \rangle \cong \langle \Gamma' \rangle \oplus \langle \Gamma' \rangle \oplus \langle \Gamma' \rangle$$

given by the maps shown in (4.85) (orientation of the circle is omitted).



(4.85)

Proof. This follows from Example 4.3.17 and the neck-cutting relation (4.80). ☐

Lemma 4.3.31. *Any $\mathfrak{sl}_3$ web $\Gamma \subset \mathbb{R}^2$ has at least two bounded faces with at most four edges each.*

Proof. We may assume that Γ is connected. Let v, e, f denote the number of vertices, edges, and faces (including the unbounded face) of Γ, respectively. Label the faces $1, \dots, f$, and for $1 \leq i \leq f$, let r_i denote the number of edges comprising the boundary of the i-th face. We have

$$\sum_{i=1}^{f} r_i = 2e = 3v, \tag{4.86}$$

where the second equality holds since Γ is trivalent. The underlying graph of Γ is bipartite, so each r_i is even. Suppose for the sake of contradiction that at most one bounded face of Γ has four or fewer edges. Then equation (4.86) implies

$$\sum_{i=1}^{f} r_i > 6(f-2),$$

so $12 > 6f - 3v$. On the other hand, an Euler characteristic computation gives

$$12 = 6(f - e + v) = 6f - 3v,$$

which is a contradiction.

□

For a (non-annular) $\mathfrak{sl}_3$ web, recall the Kuperberg polynomial from Section 2.5.1. Recall also that a *Tait* coloring of a trivalent graph is an edge coloring by three colors such that the three edges meeting at every trivalent vertex have distinct colors.

Theorem 4.3.32. *For any web $\Gamma \subset \mathcal{P}$, the state space $\langle \Gamma \rangle$ is a free graded R_3'-module of rank equal to the number of Tait colorings of Γ. Moreover, if Γ is contractible, then the graded rank of $\langle \Gamma \rangle$ equals the Kuperberg polynomial of Γ, viewed as a web in $\mathbb{R}^2$.*

Proof. Lemma 4.3.31 (1) guarantees that we can reduce $\langle \Gamma \rangle$ to a direct sum of empty webs by recursively applying the local isomorphisms in Lemma 4.3.29 and Proposition 4.3.30. It is then clear that the rank equals the number of Tait colorings, since each relation respects the number of Tait colorings upon decategorifying.

If Γ is contractible then $\langle \Gamma \rangle$ can be simplified using only the isomorphisms in Lemma 4.3.29. Upon taking graded ranks, these isomorphisms recover the recursive relations for computing the Kuperberg polynomial. □

Theorem 4.3.32 does not address the graded rank of state spaces of non-contractible webs. These may be computed recursively. As a special case, if Γ consists of n contractible and m non-contractible circles, then $\langle \Gamma \rangle$ is free of graded rank $3^m(q^2 + 1 + q^{-2})^n$.

Given a web $\Gamma \subset \mathcal{P}$, we can forget the puncture and the anchor line $\mathcal{L}$ and apply the universal construction to the evaluation (4.60). Precisely, let $\mathrm{Fr}(\Gamma)_{\mathrm{forget}}$ denote the free R_3-module generated by all foams with boundary Γ (forgetting the anchor line). By Corollary 4.3.22, we can define the bilinear form $(-,-)\colon \mathrm{Fr}(\Gamma)_{\mathrm{forget}} \times \mathrm{Fr}(\Gamma)_{\mathrm{forget}} \to R_3$ and the corresponding state space $\langle \Gamma \rangle_{\mathrm{forget}}$ in the usual way. Thus we obtain state spaces for webs in $\mathbb{R}^2$, functorial with respect to foams in $\mathbb{R}^2 \times [0,1]$. These state spaces and maps induced by foams are graded via equation (4.84), where $|\mathrm{an}(V)| = 0$.

Proposition 4.3.33. *For a contractible web $\Gamma \subset \mathcal{P}$, there is a degree-preserving isomorphism*

$$\langle \Gamma \rangle \cong \langle \Gamma \rangle_{\mathrm{forget}},$$

natural with respect to foams with contractible boundary and which are disjoint from $\mathcal{L}$.

Proof. This follows from Theorem 4.3.32. $\square$

On the other hand, Mackaay-Vaz [63] define an evaluation $\langle - \rangle_{\mathrm{MV}}$ for oriented $\mathfrak{sl}_3$ pre-foams and use it to define an equivariant (also called *universal*) version of the $\mathfrak{sl}_3$ link homology introduced in [35]. They work over the ground ring $\mathbb{Z}[a,b,c]$ and associate a state space $\langle \Gamma \rangle_{\mathrm{MV}}$ to each web $\Gamma \subset \mathbb{R}^2$ via the universal construction applied to their pre-foam evaluation $\langle - \rangle_{\mathrm{MV}}$. To compare with our situation, identify $\mathbb{Z}[a,b,c]$ with the ring $R_3 = \mathbb{Z}[E_1,E_2,E_3]$ of symmetric functions in x_1,x_2,x_3 via a ring isomorphism φ defined by $\varphi(a) = E_1$, $\varphi(b) = -E_2$, $\varphi(c) = E_3$.

Theorem 4.3.34. *For any closed pre-foam F, we have*

$$\langle F \rangle = \varphi\left(\langle F \rangle_{\mathrm{MV}}\right).$$

It follows that there are isomorphisms $\langle \Gamma \rangle_{\mathrm{forget}} \cong \langle \Gamma \rangle_{\mathrm{MV}} \otimes_{\mathbb{Z}[a,b,c]} R_3$ for any web $\Gamma \subset \mathbb{R}^2$, natural with respect to maps induced by foams with boundary.

Proof. The evaluation $\langle - \rangle_{\mathrm{MV}}$ is defined by applying the local relations (3D), (CN), (S), and (Θ) in [63, Section 2.1] to reduce any foam to an element of $\mathbb{Z}[a,b,c]$. Under the change of variables $a \mapsto E_1$, $b \mapsto -E_2$, $c \mapsto E_3$, these four relations hold for our evaluation $\langle - \rangle$ by relation (4.73), relation (4.74), Example 4.3.7, and Example 4.3.8. The statement follows. $\square$

As in the $\mathfrak{sl}_2$ setting considered in Section 4.2, we can define an additional grading on oriented $\mathfrak{sl}_3$ foams and state spaces. Define the abelian group

$$\Lambda = \mathbb{Z}w_1 \oplus \mathbb{Z}w_2 \oplus \mathbb{Z}w_3/(w_1+w_2+w_3), \tag{4.87}$$

on three generators and one relation. Λ is a free abelian group of rank two.

Orient the anchor line $\mathcal{L}$ from bottom to top. For an anchored foam V with boundary and $p \in \mathrm{an}(V)$ an anchor point lying on some facet f, let $s(p) \in \{\pm 1\}$ denote the oriented intersection number between f and $\mathcal{L}$ ($s(p)$ does not depend on the label of p), see Figure

(a) $s(p) = 1$ (b) $s(p) = -1$

Figure 4-17: The oriented intersection number between a facet and $\mathcal{L}$.

4-17 for the convention. Define the *annular degree of* V to be

$$\operatorname{adeg}(V) = \sum_{p \in \operatorname{an}(V)} s(p) w_{\ell(p)} \in \Lambda. \tag{4.88}$$

Proposition 4.3.35. *If F is a closed anchored foam with an admissible coloring c, then* $\operatorname{adeg}(F) = 0$.

Proof. The intersection of F with a generic half-plane that bounds $\mathcal{L}$ is an oriented web Γ with boundary points on $\mathcal{L}$. An admissible coloring c of F induces a Tait coloring of Γ. The boundary points (one-valent vertices) of Γ are colored according to their label. The sum in (4.88) may be rewritten as the sum of terms $\pm(w_1 + w_2 + w_3) = 0$ over all trivalent vertices of Γ, where the sign is $+1$ if all edges are incoming and -1 if all edges are outgoing. Each i-colored inner edge e of Γ bounds two trivalent vertices and contributes $\pm(w_i - w_i) = 0$ since e is oriented towards one of its boundary vertices and away from the other. The remaining edges, with one or both endpoints on $\mathcal{L}$, contribute precisely $\operatorname{adeg}(F)$. $\square$

Let $\Gamma \subset \mathcal{P}$ be an (annular oriented) $\mathfrak{sl}_3$ web. An anchored foam $F \subset \mathbb{R}^2 \times (-\infty, 0]$ with $\partial F = \Gamma$ has a well-defined degree $\operatorname{adeg}(F) \in \Lambda$ via (4.88). Furthermore, we equip the coefficient ring R_3' with a Λ-grading by setting all elements to be degree 0. This makes the free R_3'-module $\operatorname{Fr}(\Gamma)$ into an Λ-graded module, and Proposition 4.3.35 implies that the kernel of the bilinear form on $\operatorname{Fr}(\Gamma)$ is Λ-graded as well. Consequently, the grading descends to a Λ-grading on the state space $\langle \Gamma \rangle$. A foam V with boundary induces a map $\langle V \rangle : \langle -\partial_0 \Gamma \rangle \to \langle \partial_1 \Gamma \rangle$ which changes adeg by $\operatorname{adeg}(V)$. If V has no anchor points, it induces an annular degree zero map between the state spaces of its boundaries. The state space of a contractible web is concentrated in annular degree zero.

Remark 4.3.36. The Λ-grading on $\langle\Gamma\rangle$ is the analogue of grading on finite-dimensional $\mathfrak{sl}_3$ representations by the weight lattice. In fact, in the non-equivariant version of our construction, where all x_i's are set to 0 upon closed foam evaluation (and state spaces are defined accordingly, over a ground field rather than the ring R_3'), the state space $\langle\Gamma\rangle$ is naturally an $\mathfrak{sl}_3$-representation. We also refer the reader to Queffelec-Rose [68] for the construction of sutured annular $\mathfrak{sl}_n$-homology, with state spaces of annular webs carrying an $\mathfrak{sl}_n$-action. In the equivariant case, it is not clear how to define an $\mathfrak{sl}_3$-action or what is the substitute for it.

Recall that **AFoam** denotes the category of $\mathfrak{sl}_3$ webs in $\mathcal{P}$ and anchored cobordisms between them. Morphism spaces in this category are triply graded via $(\mathrm{qdeg}, \mathrm{adeg})$. The state space construction assembles into a functor

$$\langle - \rangle : \mathbf{AFoam} \to R_3' - \mathrm{g}_3 \,\mathrm{mod}$$

landing in the category of triply-graded R_3'-modules.

This functor respects the trigradings on the hom spaces in the two categories. Restricting to the subcategory of annular cobordisms and their linear combinations, the induced maps have annular degree 0.

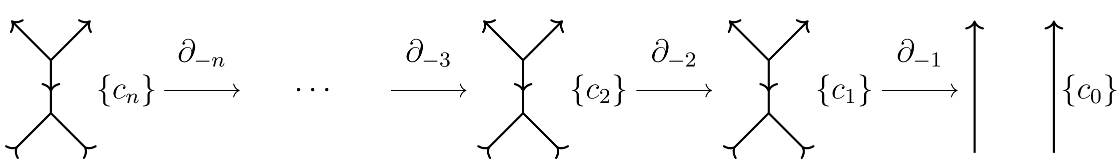
4.3.5 Annular $\mathfrak{sl}_3$ link homology

We now explain how to obtain annular link homology. Let $L \subset \mathbb{A} \times [0,1]$ be an oriented annular link. Projecting onto $\mathbb{A} \times \{0\} = \mathbb{A}$ and identifying the interior of $\mathbb{A}$ with the punctured plane $\mathcal{P}$, we obtain a link diagram $D \subset \mathcal{P}$. Following the construction of Khovanov's $\mathfrak{sl}_3$ complex [35, Section 4], which was recalled in Section 2.5.1, form the $\mathfrak{sl}_3$ cube of resolutions of D, with each resolution web D_u drawn in the punctured plane $\mathcal{P}$. Edges in the cube are decorated by either the zip or unzip foam shown in Figure 4-18. Applying the functor $\langle - \rangle : \mathbf{AFoam} \to R_3' - \mathrm{g}_3 \,\mathrm{mod}$ yields a commutative cube of free $\mathbb{Z} \oplus \Lambda$-graded R_3'-modules. Collapse the cube into a chain complex and introduce homological and quantum grading shifts exactly as in Equation (2.17), where the degree shifts $\{-\}$ are replaced by bidegree shifts $\{-, 0\}$. Denote the resulting chain complex by $C(D)$.

Figure 4-18: The zip (left) and unzip (right) foams decorating the edges in the $\mathfrak{sl}_3$ cube of resolutions.

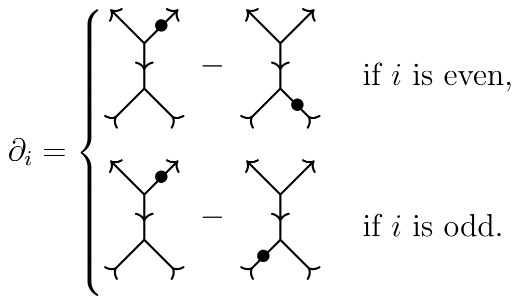
Diagrams in $\mathcal{P}$ representing isotopic annular links are related by Reidemeister moves away from the puncture. Proofs of Reidemeister invariance in [63] are local, and all local relations (away from $\mathcal{L}$) on foams in [63] also hold for our evaluation $\langle - \rangle$ by (4.74), Example 4.3.7, and Example 4.3.8. It follows that the chain homotopy class of $C(D)$ is an invariant of the annular link L. We define *equivariant annular* $\mathfrak{sl}_3$ *homology* as cohomology groups $H(C(D))$. Foams between webs appearing in the cube of resolutions are disjoint from $\mathcal{L}$, so the differential preserves annular degree throughout the complex. Consequently, equivariant annular $\mathfrak{sl}_3$ link homology carries a homological grading as well as an internal $\mathbb{Z} \oplus \Lambda$-grading (deg, adeg). Cohomology groups $H(C(D))$ are triply graded R'_x-modules.

We conclude this section with an explicit calculation. Let σ denote the positive crossing generator of the 2-strand braid group, let L_n denote the annular link diagram obtained as the annular closure of σ^n, and let $C(L_n)$ denote the corresponding chain complex. Consider the complex $C(n)$ shown in (4.89).



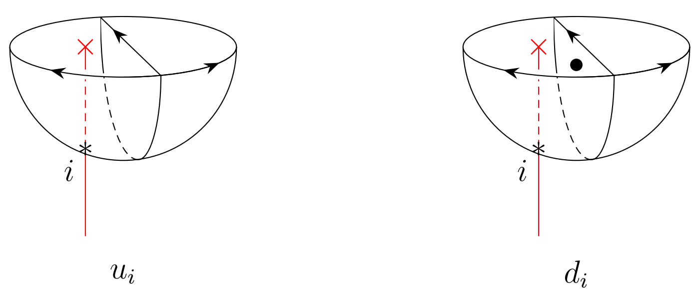
(4.89)

The right-most term is in homological degree zero and the quantum grading shifts c_i are $c_0 = 2n$ and $c_i = 2n + 2i - 1$ for $1 \leq i \leq n$. The right-most differential ∂_{-1} is the unzip cobordism, and for $-n \leq i \leq -2$ the differentials are



As in Example 4.2.20, in the above schematic depiction of the differential, each web corresponds to an identity foam in which a facet carries a dot if the corresponding edge of the web is dotted. By induction and Gaussian elimination [9, Lemma 4.2], one can show that the chain complex $C(L_n)$ is chain homotopy equivalent to the annular closure of $C(n)$.

Upon taking annular closures, the differential ∂_i for even i is zero. Consider the annular closure Γ of the web appearing in negative homological degree. By Theorem 4.3.32, the state space of Γ is a free R'_N-module of rank six, and we choose a basis $\{u_1, d_1, u_2, d_2, u_3, d_3\}$ shown in (4.90).



(4.90)

Quantum and annular bidegrees of u_i and d_i are $(-1, -w_i)$ and $(1, -w_i)$, respectively (not accounting for grading shifts).

After taking the annular closure, the differential ∂_i, for $i \leq -3$ odd, is given as the difference of foams $F - G$, where F puts a dot on the right-most facet and G puts a dot on the middle facet. We have

$$F(u_i) = (x_j + x_k)u_i - d_i \qquad F(d_i) = x_j x_k u_i$$

$$G(u_i) = d_i \qquad G(d_i) = (x_j + x_k)d_i - x_j x_k u_i.$$

In particular, ∂_i for $i \leq -3$ odd is injective.

Let us now compute the right-most differential ∂_{-1}. Let Γ_0 denote the web consisting of two essential counterclockwise oriented circles, which is the annular closure of the term in homological degree zero in $C(n)$. For $1 \leq i, j \leq 3$, let $g_{ij} : \varnothing \to \Gamma_0$ be the foam consisting of two cups, each intersecting $\mathcal{L}$ once, with the anchor point of the inner cup labeled i and the anchor point of the outer cup labeled j. By Proposition 4.3.30, $\{g_{ij}\}_{1\leq i,j\leq 3}$ is a basis for $\langle \Gamma_0 \rangle$. The generator g_{ij} is in quantum degree $2n$ and in annular degree $w_i + w_j = -w_k$. Let $Z : \Gamma \to \Gamma_0$ denote the unzip cobordism. By applying the neck-cutting relation, Equation (4.80), near the two circles comprising Γ_0, we write Zu_i as a sum

$$Zu_i = \sum_{1\leq s,t\leq 3} (-1)^{s+t} g_{st} \sqcup \tau_{st},$$

where τ_{st} is a theta foam as in Example 4.3.18, with no dots, and anchor points labeled i, s, t read from bottom to top. These theta foams evaluate to zero unless $\{i, s, t\} = \{1, 2, 3\}$, and otherwise they evaluate to ± 1. Moreover, $\langle \tau_{st} \rangle = - \langle \tau_{ts} \rangle$. Therefore we have

$$Zu_i = \pm(g_{jk} - g_{kj})$$

A similar procedure yields $Zd_i = \pm(x_j g_{jk} - x_k g_{kj})$.

Therefore, in homological degree $s \leq 0$ and annular degree $-w_i$, the homology of L_n is given by

$$H^{s,-w_i}(L_n) = \begin{cases} 0 & \text{if } s \text{ is odd,} \\ \dfrac{R'_N\{2n-2s-2\} \oplus R'_N\{2n-2s\}}{\langle (x_j + x_k, -2), (2x_j x_k, x_j + x_k) \rangle} & \text{if } s < 0,\ s \text{ is even,} \\ \left(R'_N/(x_j - x_k)\right)\{2n\} & \text{if } s = 0. \end{cases}$$

4.4 Anchored $\mathfrak{gl}_N$ link homology

Throughout this section, a positive integer N will be fixed, and all link components are colored by an integer in $\{0, \dots, N\}$. Recall the rings R_N, R'_N, R''_N from Definition 2.5.15. We introduce the following additional notation.

- $\widetilde{R_N}' = R'_N[\sqrt{x_i - x_j} \mid 1 \le i < j \le N]$ the extension of R'_N obtained by introducing square roots of $x_i - x_j$ for $i < j$.
- $\widetilde{R_N}'' = \widetilde{R_N}'[(x_i - x_j)^{-1} \mid 1 \le i < j \le N]$ the ring obtained by inverting $x_i - x_j$, $i < j$, in R''_x.

The rings $\widetilde{R_N}', \widetilde{R_N}''$ are graded by setting variables x_i to be degree two. Inclusions between these rings is summarized in the diagram (4.91).

$$\begin{array}{ccccc} & & \widetilde{R_N}' & \subset & \widetilde{R_N}'' \\ & & \cup & & \cup \\ R_N & \subset & R'_N & \subset & R''_N \end{array} \tag{4.91}$$

4.4.1 Anchored $\mathfrak{gl}_N$ foams and their evaluations

In this section we extend the Robert-Wagner foam evaluation from Section 2.5.3 to the annular setting.

Definition 4.4.1. An *anchored* $\mathfrak{gl}_N$ *foam* is a $\mathfrak{gl}_N$ foam F such that intersections of F with $\mathcal{L}$ occur transversely in the interior of facets of F. An intersection point of F with $\mathcal{L}$ is called an *anchor point*, and we let $\mathrm{an}(F) = F \cap \mathcal{L}$ denote the set of anchor points. For $p \in \mathrm{an}(F)$ lying on a facet f, we define its *thickness* $\mathrm{th}(p)$ to be the thickness of f.

Anchored foams must also come equipped with a *label* of each anchor point p, which consists of a subset $\ell(p) \subset [N] = \{1, \dots, N\}$ of cardinality equal to the thickness of the label on which p lies.

The *underlying foam* of F is the $\mathfrak{gl}_N$ foam obtained by forgetting anchor points and their labels.

A *coloring* of an anchored foam means a coloring, in the sense of Definition 2.5.14, of the underlying foam. Let F be an anchored foam and c a coloring of F. For an anchor point $p \in \mathrm{an}(F)$ lying on a facet $f \in f(F)$, its color $c(p)$ is defined to be the color of the facet, $c(p) := c(f)$.

We establish some notation before introducing anchored foam evaluation.

- For $A \subset [N]$, let $\#A$ denote the cardinality of A.
- For $A \subset [N]$, let $\overline{A} = [N] \setminus A$ denote its complement.
- For $A, B \subset [N]$, let $[A \leq B]$ denote the subset of $A \times B$ consisting of pairs (i, j) with $i \leq j$. Likewise, let $[A < B]$ denote the subset of $A \times B$ consisting of pairs (i, j) with $i < j$.
- For $A, B \subset [N]$, set

$$\Pi(A, B) := \prod_{(i,j)\in[A\leq B]} (x_i - x_j) \prod_{(i,j)\in[B\leq A]} (x_i - x_j).$$

Lemma 4.4.2. *For $A, B, C \subset [N]$, we have*

1. $\Pi(A, B) = 0$ *if and only if* $A \cap B \neq \varnothing$.
2. $\Pi(A, B) = (-1)^{\#[B\leq A]} \prod_{(i,j)\in A\times B}(x_i - x_j)$.
3. $\Pi(A, B) = \Pi(B, A)$.
4. *If* $B \cap C = \varnothing$, *then* $\Pi(A, B \cup C) = \Pi(A, B)\Pi(A, C)$.

Proof. All statements are immediate from the definition. $\square$

Definition 4.4.3. Let F be an anchored foam and c a coloring of F. For $p \in \mathrm{an}(F)$, define $\widetilde{Q}(F, c, p) = \Pi\left(c(p), \overline{\ell(p)}\right)$, and set

$$\widetilde{Q}(F, c) = \left(\prod_{p\in\mathrm{an}(F)} \widetilde{Q}(F, c, p)\right)^{1/2}, \tag{4.92}$$

$$\langle F, c\rangle = (-1)^{s(F,c)} \frac{P(F, c)}{Q(F, c)} \cdot \widetilde{Q}(F, c), \tag{4.93}$$

where $s(F, c), P(F, c)$, and $Q(F, c)$ are as in Definition 2.5.16.

Let us pause to comment on the above definition. As in Definition 2.5.16, the evaluation $\langle F\rangle$ of an anchored foam F will be the sum of $\langle F, c\rangle$ over all colorings. Note that if $c(p) \neq \ell(p)$

for some $p \in \mathrm{an}(F)$ then $\widetilde{Q}(F,c,p) = \widetilde{Q}(F,c) = \langle F,c \rangle = 0$. In light of this, we restrict the set of admissible colorings as follows.

Definition 4.4.4. For an anchored foam F, let $\mathrm{adm}(F)$ denote the set of colorings c of F such that $c(p) = \ell(p)$ for each $p \in \mathrm{an}(F)$. Define

$$\langle F \rangle = \sum_{c \in \mathrm{adm}(F)} \langle F, c \rangle .$$

Due to the presence of the square root in $\widetilde{Q}(F,c)$, a priori we have $\langle F,c \rangle \in \widetilde{R_N}''$ (see the diagram (4.91) and the discussion above it for definitions of various rings). The following lemma is analogous to Proposition 4.3.14 and shows that no square roots appear.

Lemma 4.4.5. *For an anchored foam F and $c \in \mathrm{adm}(F)$, we have $\langle F,c \rangle \in R_N''$.*

Proof. For $1 \leq i < j \leq N$, a factor of $x_i - x_j$ appears under the square root in the definition of $\widetilde{Q}(F,c)$ when either $i \in \ell(p) = c(p), j \notin \ell(p) = c(p)$ or $j \in \ell(p) = c(p), i \notin \ell(p) = c(p)$. Thus the power of $x_i - x_j$ appearing under the square root is equal to the number of intersection points between $F_{ij}(c)$ and $\mathcal{L}$, which is even since $F_{ij}(c)$ is a closed surface in $\mathbb{R}^3$. □

Remark 4.4.6. In view of the proof of Lemma 4.4.5, we may write

$$\widetilde{Q}(F,c) = \prod_{1 \leq i < j \leq N} (x_i - x_j)^{\#(F_{ij}(c) \cap \mathcal{L})/2}.$$

Letting $\mathrm{an}(i,j)$ be the number of anchor points of F which contain exactly one of i or j in their labels, we have $\#(F_{ij}(c) \cap \mathcal{L}) = \mathrm{an}(i,j)$, so that the above expression of $\widetilde{Q}(F,c)$ is independent of the admissible coloring c. Compare with Equation (4.67).

The remainder of this subsection is devoted to proving Proposition 4.4.8, which states that no denominators appear in $\langle F \rangle$. Fix an anchored foam F and a permutation $\sigma \in S_N$. Let $\sigma(F)$ be the anchored foam whose underlying foam is the same as F but the label of each anchor point p is permuted by σ: if $\ell_F(p)$ denotes the label according to F, then the label $\ell_{\sigma(F)}(p)$ according to $\sigma(F)$ is given by $\ell_{\sigma(F)}(p) = \sigma(\ell(p))$. Likewise, if $c \in \mathrm{adm}(F)$ is admissible, then $\sigma(c) \in \mathrm{adm}(\sigma(F))$ denotes the coloring which colors a facet f by $\sigma(c(f))$. Note that S_N acts on the rings in diagram (4.91) by permuting the indices of variables.

Define

$$\varepsilon(F,\sigma) = \sum_{\substack{1\le i<j\le N,\\ \sigma(i)>\sigma(j)}} \mathrm{an}(i,j)/2.$$

The following lemma is a $\mathfrak{gl}_N$ analogue of Lemma 4.3.19.

Lemma 4.4.7. *With the notation established above, we have*

$$\sigma\left(\langle F,c\rangle\right) = (-1)^{\varepsilon(F,\sigma)} \langle \sigma(F),\sigma(c)\rangle .$$

In particular, $\langle\sigma(F)\rangle = \pm\langle F\rangle$, where the sign depends only on labels of anchor points of F and on σ.

Proof. By [73, Lemma 2.17], we have

$$\sigma\left((-1)^{s(F,c)}\frac{P(F,c)}{Q(F,c))}\right) = (-1)^{s(\sigma(F),\sigma(c))}\frac{P(\sigma(F),\sigma(c))}{Q(\sigma(F),\sigma(c))}.$$

Using the reformulation of $\widetilde{Q}(F,c)$ in Remark 4.4.6, we see that $\sigma(\widetilde{Q}(F,c)) = (-1)^{\varepsilon(F,\sigma)}\widetilde{Q}(\sigma(F),\sigma(c))$, which completes the proof. ☐

For $1\le i<j\le N$, consider the subring

$$R'_{ij} = R'_N[(x_k-x_\ell)^{-1} \mid 1\le k<\ell\le N, (k,\ell)\neq(i,j)]$$

of R''_N. We have $R'_{ij} = R'_{ji}$,

$$\bigcap_{1\le i<j\le N} R'_{ij} = R'_N,$$

and $\sigma\in S_N$ sends R'_{ij} isomorphically onto $R'_{\sigma(i),\sigma(j)}$.

We recall the following local operation on colors from [73], called a *Kempe move*. Let F be an anchored foam, $c\in\mathrm{adm}(F)$ a coloring, and $\Sigma\subset F_{ij}(c)$ a (not necessarily connected) sub-surface which is disjoint from $\mathcal{L}$. Let $c'\in\mathrm{adm}(F)$ which agrees with c on facets not in Σ, and in which i and j have been exchanged for facets in Σ.

Proposition 4.4.8. *For any anchored foam F, we have $\langle F\rangle\in R'_N$.*

Proof. The argument is similar to that of [73, Proposition 2.19]. First note that it suffices to show $\langle F\rangle \in R'_{12}$. To see this, let $i<j$ and take $\sigma \in S_N$ with $\sigma(1)=i$, $\sigma(2)=j$. If $\langle F\rangle \in R'_{12}$ for any anchored foam F, then by Lemma 4.4.7 we have

$$\sigma\left(\left\langle \sigma^{-1}(F)\right\rangle\right) = \pm\langle F\rangle \in R'_{ij},$$

which implies $\langle F\rangle \in R'_N$.

Let us now show that $\langle F\rangle \in R'_{12}$. Partition $\operatorname{adm}(F)$ as follows: the equivalence class C_c of $c \in \operatorname{adm}(F)$ is the set of colorings obtained by $(1,2)$-Kempe moves on components of $F_{12}(c)$ which are disjoint from $\mathcal{L}$. We will show that

$$\sum_{c'\in C_c} \langle F, c\rangle = \frac{A}{B}$$

where $A, B \in R'_N$ are polynomials and B is not divisible by $x_1 - x_2$.

Write $F_{12}(c) = \Sigma_{\text{an}} \sqcup \Sigma_1 \sqcup \cdots \sqcup \Sigma_r$, where $\Sigma_1,\ldots,\Sigma_r$ are connected and disjoint from $\mathcal{L}$, and each component of Σ_{an} intersects $\mathcal{L}$. Let $\sigma \in S_N$ denote the transposition $(1\ 2)$. Introduce the following notation

$$\begin{aligned}
P_s(F,c) &= \prod_{f \text{ a facet in } \Sigma_s} P(c(f))\\
P'(F,c) &= \prod_{f \text{ not a facet in } \cup_{s=1}^r \Sigma_s} P(c(f))\\
\widehat{Q}(F,c) &= \frac{Q(F,c)\prod_{k\geq 3,s}(x_1-x_k)^{\ell_{\Sigma_s}(c,k)/2}}{(x_1-x_2)^{\chi(F_{12}(c))/2}}\\
\widehat{P}_s(F,c) &= P_s(F,c)\prod_{k\geq 3}(x_1-x_k)^{\ell_{\Sigma_s}(c,k)/2}\\
T_s(F,c) &= \widehat{P}_s(F,c) + (-1)^{\chi(\Sigma_s)/2}\sigma\left(\widehat{P}_s(F,c)\right),
\end{aligned}$$

where $\ell_{\Sigma_s}(c,k)$ is the (even) integer defined in [73, Lemma 2.10].

Arguing as in the proof of [73, Proposition 2.19], we have

$$\sum_{c'\in C_c} \langle F, c\rangle = (-1)^{s(F,c)} \frac{P'(F,c)}{\widehat{Q}(F,c)} \left(\prod_{s=1}^{r} (x_1 - x_2)^{-\chi(\Sigma_s)/2} T_s(F,c) \right) \frac{\widetilde{Q}(F,c)}{(x_1 - x_2)^{\chi(\Sigma_{\mathrm{an}})/2}}.$$

The term $\widehat{Q}(F,c)$ is not divisible by $(x_1 - x_2)$. Therefore denominators of the form $x_1 - x_2$ can appear in one of the two following ways. The first is when some Σ_s is a 2-sphere. In this case $T_s(F,c)$ is antisymmetric in x_1, x_2 and hence divisible by $x_1 - x_2$, allowing to cancel this factor of $x_1 - x_2$ in the denominator. The second is when some component $\Sigma' \subset \Sigma_{\mathrm{an}}$ is a 2-sphere. In this case Σ' contains at least two anchor points p_1, p_2, each containing either 1 or 2 in their labels. Thus their contribution

$$\sqrt{\widetilde{Q}(F,c,p_1)\widetilde{Q}(F,c,p_2)}$$

to $\widetilde{Q}(F,c)$ is divisible by $x_1 - x_2$, allowing to cancel with this contribution of $x_1 - x_2$ to the denominator. □

Closed anchored foams carry two types of gradings, which will induce gradings on state spaces.

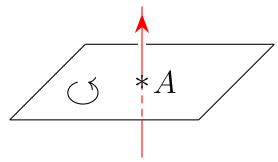
Definition 4.4.9. The *quantum grading*, denoted qdeg, of an anchored foam F is defined as

$$\mathrm{qdeg}(F) = \deg(F^{\mathrm{un}}) + \sum_{p\in \mathrm{an}(F)} \mathrm{th}(p)(N - \mathrm{th}(p)),$$

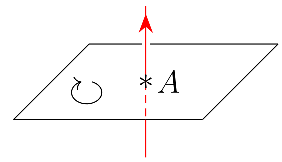
where F^{un} is the underlying foam and $\deg(F^{\mathrm{un}})$ is as in Definition 2.5.13.

Anchored foams carry an additional $\mathbb{Z}^N$-grading, called the *annular degree*. Let $w_1, \ldots, w_N$ denote the standard basis of $\mathbb{Z}^N$. Given $A \subset [N]$, let $w_A = \sum_{i\in A} w_i$. For an anchored foam F, we define $\mathrm{adeg}(F) \in \mathbb{Z}^N$ as follows. Orient the anchor line $\mathcal{L}$ from bottom to top. For an anchor point $p \in \mathrm{an}(F)$ lying on a facet f, denote by $s(p) \in \{\pm 1\}$ the oriented intersection number of f and $\mathcal{L}$ at p (see Figure 4-19). Set

$$\mathrm{adeg}(F) = \sum_{p\in \mathrm{an}(F)} s(p) w_{\ell(p)}.$$



(a) $s(p) = 1$



(b) $s(p) = -1$

Figure 4-19: The oriented intersection number between a facet and $\mathcal{L}$.

The ground ring R'_N is $\mathbb{Z} \oplus \mathbb{Z}^N$-graded by setting $\mathrm{qdeg}(x_i) = 2$ and $\mathrm{adeg}(x_i) = 0$ for $1 \leq i \leq N$. The following lemma says that anchored $\mathfrak{gl}_N$ foam evaluation respects degrees.

Lemma 4.4.10. *If F is a closed anchored foam with an admissible coloring, then* $\mathrm{qdeg}(F) = \mathrm{qdeg}(\langle F \rangle)$ *and* $\mathrm{adeg}(F) = \mathrm{adeg}(\langle F \rangle) = 0$.

Proof. The first statement is clear from the formula for $\langle F, c \rangle$, equation (4.93). The argument for the second statement is similar to the proof of Proposition 4.3.35. Consider the intersection of F with a generic half-plane containing $\mathcal{L}$, resulting in a web Γ with boundary on $\mathcal{L}$. Each edge e in Γ is the intersection of the half-plane with a facet f of F, and we set $c(e) := c(f)$. For each internal vertex v of Γ, with edges e_1, e_2, e_3 incident to v, let

$$w(v) := \varepsilon_1 \sum_{i \in c(e_1)} w_i + \varepsilon_2 \sum_{i \in c(e_2)} w_i + \varepsilon_3 \sum_{i \in c(e_3)} w_i,$$

where $\varepsilon_j = 1$ if e_j is oriented towards v and $\varepsilon_j = -1$ if e_j is oriented away from v. Each $w(v) = 0$ since c is a coloring of F, so

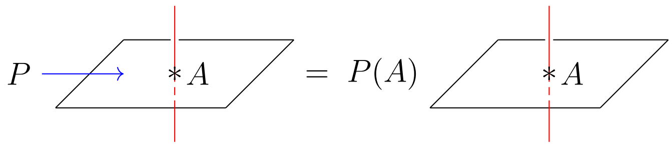
$$\sum_{v \text{ a vertex of } \Gamma} w(v) = 0. \tag{4.94}$$

On the other hand, $\mathrm{adeg}(F)$ is equal to the above sum. Indeed, each internal edge, with both endpoints disjoint from $\mathcal{L}$, contributes the same quantity with opposite signs. Thus the above sum can be written as a contribution over edges with one or both endpoints on $\mathcal{L}$, which is precisely $\mathrm{adeg}(F)$. □

4.4.2 Local relations

This section establishes local relations satisfied by anchored $\mathfrak{gl}_N$ foam evaluation.

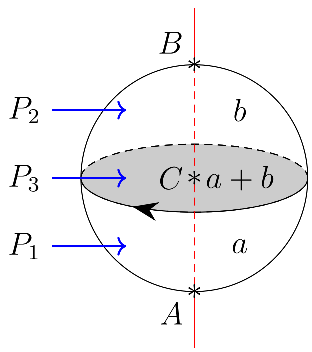
Lemma 4.4.11. *The local relation* (4.95) *holds.*



(4.95)

Proof. This is immediate from the definition. □

Lemma 4.4.12. *Consider the foam* F *shown in* (4.96)*.*



(4.96)

Then

$$\langle F\rangle = \begin{cases} (-1)^{\#[B<A]+\sum_{i\in C} i}\, P_1(A)P_2(B)P_3(C) & \textit{if } A\cup B = C, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. If $C \neq A \cup B$ then F has no admissible colorings. Suppose then that $C = A \cup B$, in which case A and B are disjoint, and there is a single admissible coloring c. We have $P(F, c) = P_1(A)P_2(B)P_3(A \cup B)$. For $i < j$, the bicolored surface $F_{ij}(c)$ is either empty or a 2-sphere. The latter occurs in two cases: first, when (i, j) or (j, i) is in $A \times B$, where the 2-sphere is the union of the thickness a and b facets. The second case when (i, j) or (j, i) is in $C \times \overline{C}$, where the 2-sphere is the union of the thickness $a+b$ facet and either the thickness a or the thickness b facet. Thus

$$Q(F, c) = \Pi(A, B)\Pi(C, \overline{C}).$$

Contributions from anchor points also equal the above term:

$$\begin{aligned}\widetilde{Q}(F,c) &= \left(\Pi(A,\overline{A})\cdot\Pi(B,\overline{B})\cdot\Pi(C,\overline{C})\right)^{1/2}\\ &= \left(\Pi(A,\overline{C})\cdot\Pi(A,B)\cdot\Pi(B,\overline{C})\cdot\Pi(B,A)\cdot\Pi(C,\overline{C})\right)^{1/2}\\ &= \Pi(A,B)\Pi(C,\overline{C}).\end{aligned}$$

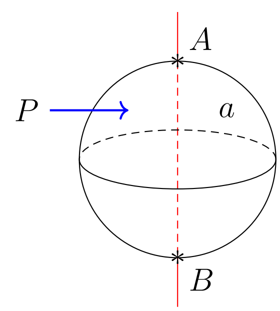
It remains to compute the sign. For $i\in[N]$, the monochrome surface $F_i(c)$ is empty whenever $i\notin K$, and otherwise for $i\in C$, $F_i(c)$ is a 2-sphere. Letting Z denote the depicted singular circle and fixing $i<j$, we see that Z is positive with respect to (i,j) if and only if $i\in B$, $j\in A$. So we obtain

$$s(F,c)=\sum_{i\in C} i+\#[B<A].$$

□

The following lemmas will be crucial for identifying state spaces assigned to annular webs.

Lemma 4.4.13. *Consider the foam F shown in (4.97), consisting of a thickness a sphere decorated by the symmetric polynomial P and which intersects $\mathcal{L}$ twice, with anchor points labeled A and B.*



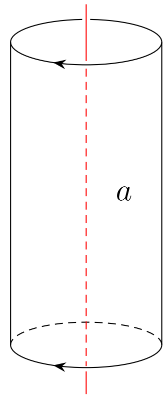
(4.97)

Then

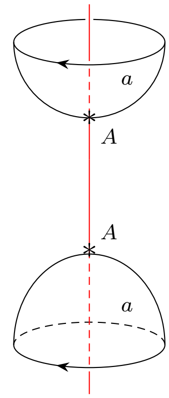
$$\langle F\rangle=\begin{cases}(-1)^{\sum_{i\in A} i}P(A) & \text{if } A=B,\\ 0 & \text{otherwise.}\end{cases}$$

Proof. There is one admissible coloring c which colors F by A. We have $s(F,c)=\sum_{i\in A} i$, $P(F,c)=P(A)$, and $Q(F,c)=\Pi(A,\overline{A})=\widetilde{Q}(F,c)$. □

Lemma 4.4.14. *The local relation* (4.98) *holds.*



$$= \sum_{\substack{A\subset[N]\\ \#A=a}} (-1)^{\sum_{i\in A} i} \tag{4.98}$$



Proof. Denote by F the foam on the left-hand side on the equality. For each $A \subset [N]$ of order a, denote by $\mathrm{adm}_A(F)$ the admissible colorings of F in which the depicted annulus is colored by A, and let G^A be the foam summand on the right-hand side of the equality whose top anchor point is labeled A. There is a natural bijection $\mathrm{adm}_A(F) \cong \mathrm{adm}(G^A)$. For $c \in \mathrm{adm}_A(F)$ with corresponding coloring $c' \in \mathrm{adm}(G^A)$, we will show that

$$\langle F, c\rangle = (-1)^{\sum_{i\in A} i} \left\langle G^A, c'\right\rangle,$$

from which relation (4.98) follows.

For $1 \leq i \leq N$, we have

$$\chi(G_i^A(c')) = \begin{cases} \chi(F_i(c)) + 2 & \text{if } i \in A, \\ \chi(F_i(c)) & \text{if } i \notin A. \end{cases}$$

It follows that $s(F, c) = s(G^A, c') + \sum_{i\in A} i$.

Next, we have $P(G^A, c') = P(F, c)$. For $i < j$, we have

$$\chi(G_{ij}^A(c')) = \begin{cases} \chi(F_{ij}(c)) & \text{if } i, j \in A \text{ or } i \notin A, j \notin A \\ \chi(F_{ij}(c)) + 2 & \text{if exactly one of } i, j \text{ is in } A \end{cases}$$

so

$$Q(G^A, c') = Q(F, c) \cdot \Pi(A, \overline{A}).$$

On the other hand, we have

$$\widetilde{Q}(G^A, c') = \widetilde{Q}(F, c) \cdot \Pi(A, \overline{A}),$$

which verifies $\langle F, c\rangle = \left\langle G^A, c'\right\rangle$ and completes the proof of the lemma.

□

4.4.3 State spaces of annular $\mathfrak{gl}_N$ webs

In this section we define state spaces for annular $\mathfrak{gl}_N$ webs using universal construction applied to anchored foam evaluation. The main result is Theorem 4.4.21, which identifies state spaces.

Definition 4.4.15. An *annular* $\mathfrak{gl}_N$ *web* is a $\mathfrak{gl}_N$ web embedded in the punctured plane $\mathcal{P}$. Annular webs are considered up to ambient isotopy of $\mathcal{P}$. In this section, we will say annular web rather than annular $\mathfrak{gl}_N$ web.

Let Γ_0, Γ_1 be annular webs. A *anchored cobordism* from Γ_0 to Γ_1 is a $\mathfrak{gl}_N$ foam with boundary $F \subset \mathbb{R}^2 \times I$ (see Definition 2.5.18) from Γ_0 to Γ_1 (viewing $\Gamma_0, \Gamma_1 \subset \mathbb{R}^2$) such that F intersects the segment $\mathcal{L}_{[0,1]} = \{(0,0)\} \times I$ in the interior of its facets, these intersection points are transverse, and each intersection point p is labeled by a subset $\ell(p) \subset [N]$ of cardinality equal to the thickness of the facet on which p lies. As for closed anchored foams, we let $\mathrm{an}(F) = F \cap \mathcal{L}$ denote the *anchor points.*

Anchored cobordisms are considered up to ambient isotopy of $\mathbb{R}^2 \times I$ which is the identity near $\partial(\mathbb{R}^2 \times I)$ and maps $\mathcal{L}_{[0,1]}$ to itself. Define $\mathrm{qdeg}(F)$ and $\mathrm{adeg}(F)$ for an anchored cobordism F as in Definition 4.4.9.

Let $\mathbf{AFoam}_N$ denote the category of anchored cobordisms. It is straightforward to see that bidegrees qdeg and adeg are additive under composition of anchored cobordisms.

We define the state space of an annular web Γ in the usual way. Let $\mathrm{Fr}(\Gamma)$ denote the free R'_N-module generated by all anchored cobordisms from the empty web to Γ, and let $\langle\Gamma\rangle$ denote the kernel of $\mathrm{Fr}(\Gamma)$ by the kernel of the symmetric bilinear form

$$\langle -, - \rangle : \mathrm{Fr}(\Gamma) \times \mathrm{Fr}(\Gamma) \to R'_N, \quad \langle F, G\rangle = \left\langle \overline{F} G \right\rangle.$$

Lemma 4.4.10 implies that qdeg and adeg descend to gradings on $\langle\Gamma\rangle$, so $\langle\Gamma\rangle$ is $\mathbb{Z}\oplus\mathbb{Z}^N$-graded. It follows immediately from definition that an anchored cobordism $F:\Gamma_0\to\Gamma_1$ induces a map $\langle F\rangle:\langle\Gamma_0\rangle\to\langle\Gamma_1\rangle$ of degree $(\mathrm{qdeg}(F),\mathrm{adeg}(F))$. Thus we obtain a functor

$$\langle - \rangle : \mathbf{AFoam}_N \to R'_N\!-\mathrm{g}_{N+1}\,\mathrm{mod},$$

where $R'_N\!-\mathrm{g}_{N+1}\,\mathrm{mod}$ is the category of $\mathbb{Z}\oplus\mathbb{Z}^N$-graded R'_N-modules.

For a $\mathbb{Z}\oplus\mathbb{Z}^N$-graded R'_N-module M, we will write grading shifts as $\{(i,j)\}$ where $i\in\mathbb{Z}$ and $j=(j_1,\dots,j_N)\in\mathbb{Z}^N$. Given $f\in\mathbb{Z}_{\geq 0}[q^{\pm 1},a_1^{\pm 1},\dots,a_N^{\pm 1}]$, write $f=\sum_{I=(i,j_1,\dots,j_N)\in\mathbb{Z}^{N+1}} m_I q^i a_1^{j_1}\cdots a_N^{j_N}$, and set

$$M\{f\}:=\bigoplus_{I=(i,j_1,\dots,j_N)\in\mathbb{Z}^{N+1}} M^{\oplus m_I}\{(i,j_1,\dots,j_N)\}.$$

A free graded R_N-module is of the form $R_N\{f\}$, and we define its graded rank to be f. By definition the graded rank of $M\{f\}$ is equal to f times the graded rank of M.

We establish the following sequence of lemmas before proving Theorem 4.4.21.

Lemma 4.4.16. *Let Γ be an annular web, and let Γ' be the annular web obtained from Γ by deleting all the edges of thickness zero. Then there is an isomorphism of state spaces $\langle\Gamma\rangle\cong\langle\Gamma'\rangle$.*

Proof. The argument is the same as in [73, Claim 3.33]. □

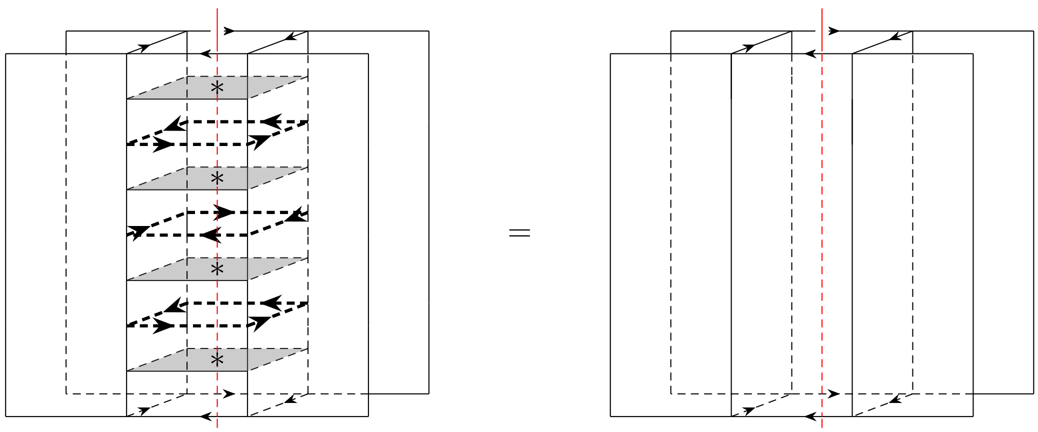
Definition 4.4.17 ([73, Notation 3.6, Definition 3.7])**.** Let C be an oriented cycle in an annular web Γ, and let Γ' be the annular web obtained from Γ by reversing the orientation of each edge in C and replacing its thickness i by $N-i$. The web Γ' is said to be obtained by a *cycle move along* C. The cycle C is *face-like* if it bounds a disk in $\mathbb{R}^2\setminus\Gamma$.

Note that face-like cycles are allowed to bound a disk which contains the puncture.

Lemma 4.4.18. *If Γ' is obtained from Γ by a cycle move, then $\langle\Gamma'\rangle\cong\langle\Gamma\rangle$.*

Proof. By [73, Lemma 3.9], it suffices to prove the claim when C is face-like. If C bounds a disk which does not contain the puncture, then the isomorphism can be taken to be the same one as in [73, Claim 3.34]. Suppose that C bounds a disk which contains the puncture. We claim that the relation (4.99) holds, where:

- the shaded facets have thickness N,
- the bold dashed arrows are not seam lines indicating a meeting of facets, but rather the orientation on the edges of the web that would be created if the foam were sliced horizontally,
- the relation holds not only for squares but for any face-like cycle, which is why most orientations and thicknesses are omitted.



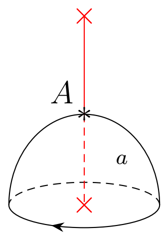
(4.99)

Since anchor points on thickness N facets do not contribute to the evaluation, the identity 4.99 follows from the proof of [73, Lemma 3.21]. Therefore a modification of the foams witnessing the isomorphism in [73, Claim 3.34], given by passing the anchor line through the thicknesses N facets, yields the desired isomorphism in the case where C bounds a disk containing the puncture.

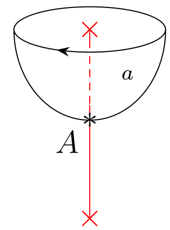
□

Recall that $w_1, \ldots, w_N$ denote the standard basis vectors of $\mathbb{Z}^N$, and that $w_A = \sum_{i \in A} w_i$, for $A \subset [N]$.

Theorem 4.4.19. *Let Γ be an annular web containing a non-contractible clockwise oriented circle Z of thickness a which bounds a disk in $\mathbb{R}^2 \setminus \Gamma$. Let $\Gamma' = \Gamma \setminus Z$ denote the annular web obtained by removing Z from Γ. There is an isomorphism $\langle \Gamma \rangle \cong \bigoplus_{\substack{A \subset [N] \\ \#A = a}} \langle \Gamma' \rangle \{(0, -w_A)\}$. This*



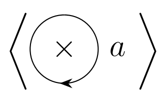
(a) The anchored cobordism $F^A : \Gamma \to \Gamma'$.



(b) The anchored cobordism $G^A : \Gamma' \to \Gamma$.

Figure 4-20: The foams F^A and G^A in the proof of Theorem 4.4.19

is depicted in (4.100).



$$\cong \bigoplus_{\substack{A\subset\{1,\ldots,N\}\\ \#A=a}} \langle\varnothing\rangle\,\{(0,-w_A)\} \tag{4.100}$$

Moreover, reversing the orientation negates the degree shifts.

Proof. For $A \subset [N]$ with $\#A = a$, let $F^A : \Gamma \to \Gamma'$ denote the anchored cobordism depicted in Figure 4-20a, and let $G^A : \Gamma' \to \Gamma$ denote the anchored cobordism depicted in Figure 4-20b (only the relevant part of the foams are shown - outside of the depicted region they are the identity on Γ'). It is straightforward to verify that $\mathrm{qdeg}(F^A) = \mathrm{qdeg}(G^A) = 0$, $\mathrm{adeg}(F^A) = w_A$, and $\mathrm{adeg}(G^A) = -w_A$.

Define $\Phi : \langle\Gamma\rangle \to \langle\Gamma'\rangle$ to be the $\binom{N}{a} \times 1$ matrix of foams with entries

$$(-1)^{\sum_{i\in A} i}\left\langle F^A\right\rangle,$$

and $\Psi : \langle\Gamma'\rangle \to \langle\Gamma\rangle$ to be the $1 \times \binom{N}{a}$ matrix of foams with entries $\left\langle G^A\right\rangle$. Lemma 4.4.14 and Lemma 4.4.13 imply, respectively, that $\Psi\Phi = \mathrm{id}$ and $\Phi\Psi = \mathrm{id}$. $\square$

Consider any one of MOY relations in Figure 2-24, where the depicted relations are contained in disks disjoint from the puncture, and write it in the form $\Gamma = \sum_i q^{n_i}\Gamma_i$. By a *categorified* MOY relation we mean an isomorphism of state spaces $\langle\Gamma\rangle = \bigoplus_i \langle\Gamma_i\rangle\,\{(n_i, 0)\}$.

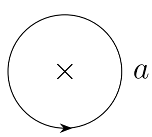
Lemma 4.4.20. *All the categorified MOY relations hold for anchored $\mathfrak{gl}_N$ state spaces.*

Proof. Robert-Wagner show that there is an isomorphism of state spaces

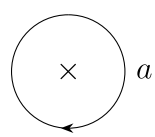
$$\langle \Gamma \rangle_{\mathrm{RW}} = \bigoplus_i \langle \Gamma_i \rangle_{\mathrm{RW}} \{n_i\}.$$

For most of the relations, these isomorphisms are realized explicitly via maps induced by foams: categorified versions of Figures 2-24a, 2-24b, 2-24c, 2-24d, and 2-24g follow from the relations [73, Equation (10), Claim 3.35, Equation (12), Equation (13), Claim 3.36], respectively. The remaining relations are Figures 2-24e and 2-24f. These too can be realized via foams. As noted in [72, Lemma 4.2], the relation Figure 2-24e is a special case of Figure 2-24g. Figure 2-24f is a combination of specializing Figure 2-24g and applying a cycle move, and cycle moves can be realized by anchored foams which induce isomorphism of state spaces by Lemma 4.4.18. The arguments in [73] establishing foam relations are completely local and apply unchanged in the annular setting when all the local foam relations are disjoint from $\mathcal{L}$. □

Theorem 4.4.21. *Let Γ be an annular web. Then $\langle \Gamma \rangle$ is a free R'_N-module. Its graded rank can be computed by applying the MOY relations in Figure 2-24 (where the local pictures are all disjoint from the puncture) and the additional relations shown in* (4.101)*.*



$$= \sum_{\substack{A \subset [N] \\ \#A = a}} \prod_{i \in A} a_i$$

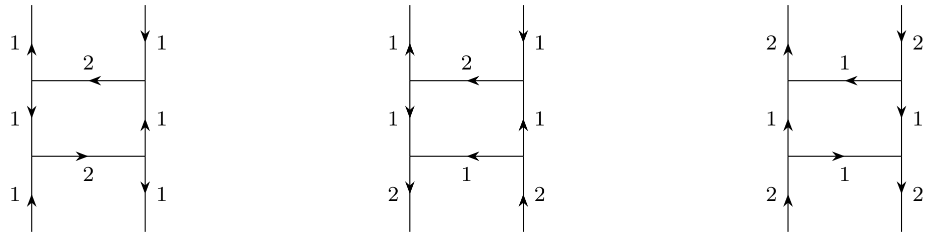


$$= \sum_{\substack{A \subset [N] \\ \#A = a}} \prod_{i \in A} a_i^{-1} \qquad (4.101)$$

Proof. By Lemma 4.4.16, we may assume that Γ has no edges of thickness zero. Suppose that all edges of Γ are of thickness 1 or 2. In this case the underling graph of Γ, not including closed loops, is bipartite. By Lemma 4.3.31, Γ contains either a closed loop, bigon, or square region. Every closed innermost[1] loop can be removed by using either the categorified MOY relation Figure 2-24a or Theorem 4.4.19. There are two types of bigon faces in this case, which can be removed using the categorified MOY relations Figure 2-24c and Figure 2-24d. Finally, for this step, consider a square face of Γ. Up to rotation and reflection, there are

[1]Innermost means that the loop bounds a disk in $\mathbb{R}^2 \setminus \Gamma$ (the disk may contain the puncture).

three types of square faces:



$$(4.102)$$

The left square can be simplified using the relation Figure 2-24e. The other two are obtained from this one by a cycle move, so by Lemma 4.4.18 all square faces can be simplified. Applying these reductions and removing closed innermost loops as necessary, we reduce $\langle\Gamma\rangle$ to a direct sum of empty webs.

The rest of the argument follows as in the proof of [90, Theorem 2.4], keeping in mind that all closed innermost loops can be removed.

□

4.4.4 Equivariant annular Khovanov-Rozansky homology

In this section we define the chain complex associated to a colored, oriented annular link L with diagram D, following [24, Definition 3.3].

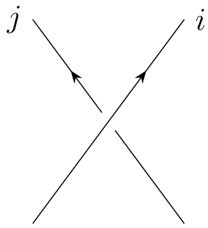
Definition 4.4.22. To a positive crossing with overstrand colored i and understrand colored j, with $i \geq j$ (Figure 4-21a) assign the complex

$$\langle\Gamma\rangle_0 \{(c_0, 0)\} \to \langle\Gamma_1\rangle \{(c_1, 0)\} \to \cdots \to \langle\Gamma_j\rangle \{(c_j, 0)\},$$

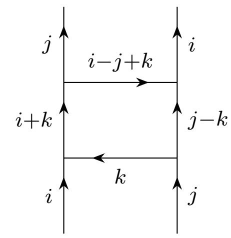
where Γ_0 is in homological degree zero, the grading shifts are given by $c_k = -k - j(N-j)$ (applied only to the quantum grading), the webs Γ_k are shown in Figure 4-21b, and the maps are induced by a single foam, shown in Figure 4-21c.

If $i < j$ or the crossing is negative, then the complex assigned to the crossing is obtained from the one above in the manner described in [24, Definition 3.3].

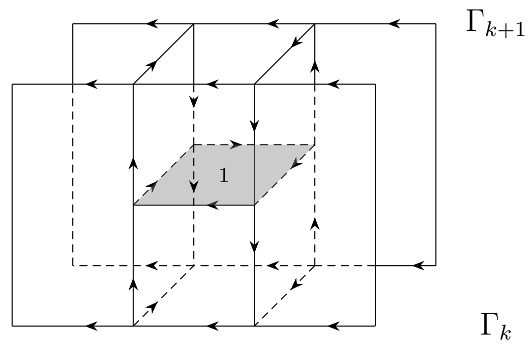
Finally, the chain complex $C_N(D)$ of an annular link diagram D is obtained by replacing each crossing with the corresponding complex and tensoring them together in a planar



(a) A positive crossing, with $i \geq j$.



(b) The web Γ_k, $0 \leq k \leq j$, appearing in homological degree k.



(c) The foam cobordism $\Gamma_k \to \Gamma_{k+1}$ inducing the differential. The shaded facet has thickness 1. Thicknesses and orientations of facets are determined by Γ_k and Γ_{k+1}.

Figure 4-21

algebra fashion.

The differential is induced by foams which are disjoint from $\mathcal{L}$ and hence preserves annular degree. Chain groups in $C_N(D)$ and homology groups $H_N(D)$ are $\mathbb{Z} \oplus \mathbb{Z}^N$-graded R'_N-modules.

Proposition 4.4.23. *If D' is an annular link diagram obtained from D by a Reidemeister move, then $C_N(D)$ and $C_N(D')$ are chain homotopy equivalent. Consequently, the $\mathbb{Z} \oplus \mathbb{Z}^N$-graded homology $H_N(D)$ is an invariant of L up to isomorphism.*

Proof. Ehrig-Tubbenhauer-Wedrich establish *local* invariance of colored $\mathfrak{gl}_N$ homology [24, Theorem 3.5]. Since Reidemeister moves occur away from the puncture, it follows that $C_N(D)$ and $C_N(D')$ are chain homotopy equivalent. $\square$

Consider colored annular link diagrams D_0 and D_1 where D_1 is obtained from D_0 by a Reidemeister move or a Morse move (cup, cap, or saddle); saddles must involve strands of the same color. There is an induced chain map $C_N(D_0) \to C_N(D_1)$, given by the natural foam map for handle attachments and otherwise given by chain homotopy equivalences realizing Reidemeister moves. This map has bidegree $(d, 0)$, where $d = 0$ for Reidemeister moves, $d = -i(N-i)$ for a cup or a cap involving a circle of color i, and $d = i(N-i)$ for a saddle involving strands of color i.

A *colored* cobordism is a link cobordism $S \subset \mathbb{R}^3 \times I$ in which each component is labeled by an element of $\{0, \ldots, N\}$. The boundary links $S \cap \mathbb{R}^3 \times \{0\}$ and $S \cap \mathbb{R}^3 \times \{1\}$ are then naturally colored. Any colored link cobordism can be represented as a sequence of the elementary Reidemeister or Morse cobordisms described above. A *cobordism* between colored links L, L' is a colored cobordism S such that the induced coloring on the boundary of S agrees with the coloring of L and L'. These notions extend in a straightforward manner to cobordisms in $\mathbb{A} \times I \times I$ between annular links.

Theorem 4.4.24. *Let $S \subset \mathbb{A} \times I \times I$ be a colored cobordism between annular links L and L'. Write S as a composition of elementary cobordisms $D_0 \xrightarrow{S_0} D_1 \xrightarrow{S_1} \cdots \xrightarrow{S_{k-1}} D_k$, where D_0 and D_k are diagrams for L and L'. Up to chain homotopy equivalence, the induced chain map $S_{k-1} \circ \cdots \circ S_0 : C_N(D_0) \to C_N(D_k)$ is independent of the choice of decomposition of S into elementary pieces. Denote the map on homology by S_*. Then the assignment $S \mapsto S_*$ is functorial with respect to composition of cobordisms: $S'_* S_* \simeq (S'S)_*$ for any cobordism $S' : L' \to L''$.*

Proof. This follows from [24, Theorem 4.5]. $\square$

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