Uniform convergence methods in Hilbert-Kunz theory

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A Dissertation presented to the Graduate Faculty
of the University of Virginia in Candidacy for the Degree of
Doctor of Philosophy

Department of Mathematics

University of Virginia
May, 2015

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Acknowledgements

Foremost, I want to express my sincere gratitude to my mentors. As an undergraduate, I was lucky to be guided and have an opportunity to learn from Dmitry Trushin and Evgeny Golod. For my first steps as a mathematician they provided encouragement, example, and valuable advice.

In graduate school, I worked with Craig Huneke and Hailong Dao. Craig Huneke, my advisor, has offered endless amount of support throughout my graduate days. His astonishing intuition has been always extremely mysterious and helpful. This work started with a question posed by him and grew under his influence.

I am fortunate to have Hailong Dao as a mentor and a friend. Himself being a younger mathematician than my advisor, Long has offered me a different perspective on our field and career.

I would like to thank the committee members Edith Clowes, Mikhail Ershov, and Andrew Obus for their service.

Abstract of the dissertation

Hilbert-Kunz multiplicity is an invariant of a local ring containing a field of positive characteristic. In this work, we study its continuity properties as a function on a variety.

First, we develop a theory of equimultiplicity for Hilbert-Kunz multiplicity. Remarkably, it is quite similar to the classical equimultiplicity. The theory is then applied to show that a stronger form of upper semi-continuity does not hold.

Later, using uniform convergence ideas we prove that a weaker form of upper semi-continuity holds. As an application, we obtain that the maximum value locus of Hilbert-Kunz multiplicity is closed.

Chapter 1

Introduction

1.1 On the title

Interest in asymptoptical behavior appears throughout all mathematics, and commutative algebra is not an exception. Asymptotical nature appears in different forms within commutative algebra and one of the most important sources is the collection of the powers of an ideal and the Rees algebra of an ideal as a way to impose additional structure on the powers. From this source stem many other instances of asymptotical nature; in this work we will encounter Hilbert functions, (Hilbert-Samuel) multiplicity, analytic spread, integral closure, and the Artin-Rees lemma.

These invariants have been known for many years, have been studied extensively, and have had strong impact. The present definition of multiplicity was introduced in 1951 by Samuel, as a generalization of an earlier work of Chevalley (1945). Multiplicity was introduced for geometrical purposes and served well these purposes. For example, it found a use in Hironaka's celebrated work on resolution of singularities. Rees algebras were introduced by Rees in late 50s and have been useful, in particular, for connections with integral closure. They also have a geometric side; the projective scheme of the Rees algebra of an ideal is the blow-up of the spectrum along the subvariety defined by the ideal.

Over a field of positive characteristic, there is a different driving force behind asymptotical invariants: the Frobenius endomorphism. Given by a simple map $x \mapsto x^p$, it can be repeated over and over, and these iterations can be used in different ways, leading to various invariants that express its asymptotical properties.

This thesis is devoted to one such invariant, Hilbert-Kunz multiplicity. The corresponding iterations were introduced by Kunz in 1969. His work was ahead of its time and eventually led to various characteristic p methods in study of singularities. Hilbert-Kunz multiplicity itself was defined by Monsky in 1983. However, the active development started only in 90s, after Hochster and Huneke introduced the theory of tight closure and gave a new momentum to the positive characteristic methods in commutative algebra.

At the present moment, Hilbert-Kunz multiplicity is still a very mysterious invariant and there is much to understand. In this work, our main interest is its global behavior: Hilbert-Kunz multiplicity is a local invariant defined at a point, and we want to study it as a function on a variety. Namely, we will investigate its continuity properties as a way to understand the geometrical behavior of Hilbert-Kunz multiplicity. Our source of motivation is the corresponding theory for Hilbert-Samuel multiplicity that was developed after Hironaka's work on the resolution of singularities.

While various forms of continuity represent a certain kind of uniform behavior, it is not the uniformity that is referred to in the title. Since Hilbert-Kunz multiplicity is defined as a limit, analytic ideas related to convergence appear naturally. One such idea is the idea of uniform convergence.

The theme of uniformity arises often in commutative algebra, and sometimes yields good fruit. In tight closure theory, test elements were developed by Hochster and Huneke and have led to spectacular developments. In this work, test elements are necessary in Chapter 4. Another notable example is Kollár's proof of an effective Nullstellensatz using uniform annihilation of local cohomology.

For Hilbert-Kunz multiplicity this side of the story starts in the work of Tucker where he established uniform convergence with respect to $\mathfrak{m}$-primary ideals. As a consequence, Tucker obtained a "volume=multiplicity" kind of formula that allowed him to prove the existence of F-signature, another new asymptotical invariant in positive characteristic. We give another instances of uniform convergence and use them to what we believe to be a great success.

Our main results stem from two very different instances of uniform convergence. In this first instance (Chapter 4), we use it "locally" to interchange the limits of a bisequence, thus giving a powerful formula that will be used as a first stone in our theory of equimultiplicity. The second use of uniform convergence in Chapter 5 is "global", as we use it to control convergence rate at different points. This gives a way to obtain a result about the limits from the information about the terms of the corresponding sequences.

Together with Tucker's original contribution, this demonstrates how powerful and different are the uniform convergence methods in Hilbert-Kunz theory.

1.2 Structure of the thesis

First, we review some prerequisites in Chapter 2. Chapter 3 can be seen as a somewhat more technical extension of the introduction. There we discuss the classical theory of equimultiplicity to motivate the main problem and discuss our approach.

The main goal of this work is to understand whether Hilbert-Kunz multiplicity is upper semi-continuous. The main results of this thesis are in Chapters 4 and 5. In Chapter 4 we build an equimultiplicity theory for Hilbert-Kunz multiplicity following an outline that we gave for Hilbert-Samuel multiplicity. In the foundation, we use our first uniform convergence result; and as a consequence of our theory, we show that Hilbert-Kunz multiplicity need not be locally constant, which is a strong form of semi-continuity.

In Chapter 5, we consider a weaker form of the question and show that Hilbert-Kunz multiplicity is indeed upper semi-continuous under very minor technical assumptions. Our main tool is a globalized version of uniform convergence for Hilbert-Kunz multiplicity. We finish with a list of open questions related to our research in Chapter 6.

Now, let us discuss the main chapters in more details.

1.2.1 Chapter 4

In Section 4.1, we establish uniform convergence of a certain bisequence (Theorem 4.1.6) and use it to interchange the limits and obtain a very useful formula (Corollary 4.1.9).

We immediately apply our findings for a simple case of an one-dimensional ideal in Section 4.2. Most of the results will be generalized later, but we establish the foundation and obtain applications. The most important result for the following is Proposition 4.2.7 that will be used to give our main counterexample and show that Hilbert-Kunz multiplicity need not to be locally constant.

In Section 4.3, we continue to develop a general theory of equimultiplicity. Namely, we define an ideal I to be equimultiple for Hilbert-Kunz multiplicity if there exists a sequence $x_1, \ldots, x_m$ in R such that its image in R/I is a system of parameters and

$$\mathrm{e}_{\mathrm{HK}}(I + (x_1, \ldots, x_m)) = \sum_{P \in \mathrm{Minh}(I)} \mathrm{e}_{\mathrm{HK}}((x_1, \ldots, x_m), R/P)\, \mathrm{e}_{\mathrm{HK}}(I, R_P),$$

where $\mathrm{Minh}(I)$ denotes the set of minimal prime ideals P of I such that $\dim R/P = \dim R/I$. In Theorem 4.3.5 and Theorem 4.3.7, we use tight closure to give a different characterization of equimultiplicity. Building on this, in Proposition 4.3.10 we show that our definition is independent of the sequence $x_1, \ldots, x_m$.

The obtained result compares very well to the classical theory that will be discussed in Chapter 3.

1.2.2 Chapter 5

In this chapter, we will prove that Hilbert-Kunz multiplicity is upper semi-continuous in a F-finite ring or an algebra of essentially finite type over an excellent local ring (Theorem 5.4.3). Though that restriction is of technical nature, any affine algebra or a complete ring satisfies it.

After the introduction, we establish two uniform convergence results that will help us to treat the two cases of the main theorem. In Section 5.2 we treat F-finite case and in Section 5.3 we establish results that will be used for algebras of essentially finite type over an excellent local ring. Then we employ these uniform convergence results to reduce the problem from Hilbert-Kunz multiplicity as a limit of a sequence to a fixed term of the sequence. And upper semi-continuity of a fixed term was established by Kunz.

Chapter 2

Preliminaries

In this chapter we discuss preliminaries needed later on. Most of the material here is not new, and an experienced reader may want to proceed directly to the next chapter.

We assume that all rings are commutative and Noetherian. For a finite length module M, we let $\lambda(M)$ denote its length, i.e. the common length of its composition series. We assume basic understanding of commutative algebra on the level of Matsumura's book ([22]), but will try to refresh crucial notions.

2.1 Hilbert-Samuel multiplicity

We start with the classical theory of Hilbert-Samuel multiplicity.

2.1.1 Basic properties

Definition 2.1.1. Let $(R, \mathfrak{m})$ be a local ring of dimension d and I be an $\mathfrak{m}$-primary ideal. Let M be a finitely generated R-module. The multiplicity of M with respect of I is defined to be

$$\mathrm{e}(I, M) = \lim_{n\to\infty} \frac{d!\,\lambda(M/I^nM)}{n^d}.$$

In fact, $\lambda(M/I^nM)$ is a polynomial in n (the Hilbert-Samuel polynomial) of degree $\dim M$ for all n sufficiently large. Thus $\mathrm{e}(I,M)=0$ if and only if $\dim M<d$. It is a custom to denote $\mathrm{e}(I,R)$ as $\mathrm{e}(I)$.

Proposition 2.1.2. *Let $(R,\mathfrak{m})$ be a local ring, and let I be an $\mathfrak{m}$-primary ideal. If $0\to K\to L\to M\to 0$ is a short exact sequence of finitely generated R-modules, then $\mathrm{e}(I,M)=\mathrm{e}(I,K)+\mathrm{e}(I,N)$.*

Proof. Tensoring the short exact sequence with R/I^n, we obtain an exact sequence

$$K/I^nK\to L/I^nL\to M/I^nM\to 0.$$

Taking lengths, we get an inequality $\lambda(L/I^nL)\le\lambda(K/I^nK)+\lambda(M/I^nM)$.

For the other direction, we note that the following sequence is exact

$$0\to K/(I^nL\cap K)\to L/I^nL\to M/I^nM\to 0.$$

Thus,

$$\lambda(L/I^nL)=\lambda(M/I^nM)+\lambda(K/(I^nL\cap K))=\lambda(M/I^nM)+\lambda(K/(I^{n-c}(I^cL\cap K))),$$

where the last equality holds since, by the Artin-Rees Lemma, there exists an integer c such that $I^nL\cap K=I^{n-c}(I^cL\cap K)$ for all $n\ge c$. Note that $I^{n-c}(I^cL\cap K)\subseteq I^{n-c}K$, so $\lambda(K/(I^nL\cap K))\le\lambda(K/I^{n-c}K)$ and, combining all estimates, we get

$$\lambda(M/I^nM)+\lambda(K/I^{n-c}K)\le\lambda(L/I^nL)\le\lambda(M/I^nM)+\lambda(K/I^nK).$$

Since $\lim\limits_{n\to\infty}\lambda(K/I^{n-c}K)/n^d=\lim\limits_{n\to\infty}\lambda(K/I^nK)/n^d$, the claim follows. $\square$

In the next result $\mathrm{Minh}(R)$ denotes the set of all minimal prime ideals P of R such that $\dim R/P=\dim R$.

Proposition 2.1.3 (Associativity formula)**.** *Let $(R, \mathfrak{m})$ be a local ring, I be an $\mathfrak{m}$-primary ideal, and M be a finitely generated R-module. Then*

$$\mathrm{e}(I, M) = \sum_{P \in \mathrm{Minh}(R)} \mathrm{e}(I, R/P)\, \lambda_{R_P}(M_P).$$

Proof. Let $0 = M_0 \subset M_1 \subset \ldots \subset M_N = M$ be a prime filtration of M, so there are exact sequences

$$0 \to M_{n-1} \to M_n \to R/Q_n \to 0$$

where Q_n is a prime ideal. Using Proposition 2.1.2, we derive

$$\mathrm{e}(I, M) = \mathrm{e}(I, M_{N-1}) + \mathrm{e}(I, R/Q_{N-1}) = \ldots = \sum_{n=0}^{N-1} \mathrm{e}(I, R/Q_n).$$

However, $\mathrm{e}(I, R/Q_n) = 0$ unless $\dim R/Q_n = \dim R$, i.e. $Q_n \in \mathrm{Minh}(R)$.

Let $P \in \mathrm{Minh}(R)$, we want to find how many times P appears among all Q_n. To do this, localize the prime filtration at P and note that $\lambda((R/Q_n)_P) = 1$ if $Q_n = P$ and is zero otherwise. Thus the factor R/P appears $\lambda_{R_P}(M_P)$ times in the filtration and the claim follows. □

The following simple observation is useful.

Lemma 2.1.4. *Let $(R, \mathfrak{m})$ be a local ring of dimension d, M be a finitely generated R-module, and I be an $\mathfrak{m}$-primary ideal. Then $\mathrm{e}(I^n, M) = n^d\, \mathrm{e}(I, M)$ for all n.*

Proof. By definition,

$$\mathrm{e}(I^n, M) = \lim_{m \to \infty} \frac{d!\, \lambda(M/I^{nm}M)}{m^d} = n^d \lim_{m \to \infty} \frac{d!\, \lambda(M/I^{nm}M)}{(nm)^d} = n^d\, \mathrm{e}(I, M).$$

□

2.1.2 Multiplicity of parameter ideals

First recall the following definition.

Definition 2.1.5. Let $(R, \mathfrak{m})$ be a local ring of dimension d. A sequence of elements $x_1, \ldots, x_d$ is called a system of parameters if $(x_1, \ldots, x_d)$ is an $\mathfrak{m}$-primary ideal.

An ideal I is called a parameter ideal if it can be generated by a system of parameters.

Parameter ideals have some special properties, in particular, Lech ([20]) established a special formula for the multiplicity of a parameter ideal. The following result is usully called Lech's lemma, nonetheless it appeared as Theorem 2 in [20].

Lemma 2.1.6. *Let $(R, \mathfrak{m})$ be a local ring and let $x_1, \ldots, x_d$ be a system of parameters. Then*

$$\mathrm{e}((x_1, \ldots, x_d)) = \lim_{\min(n_1, \ldots, n_d) \to \infty} \frac{\lambda(R/(x_1^{n_1}, \ldots, x_d^{n_d}))}{n_1 \cdots n_d}.$$

Similar to Lemma 2.1.4, we obtain the following consequence of Lech's lemma.

Corollary 2.1.7. *Let $(R, \mathfrak{m})$ be a local ring and let $x_1, \ldots, x_d$ be a system of parameters. Then for any vector $(n_1, \ldots, n_d) \in \mathbb{N}^d$*

$$\mathrm{e}((x_1^{n_1}, \ldots, x_d^{n_d})) = n_1 \cdots n_d \, \mathrm{e}((x_1, \ldots, x_d)).$$

Lech's lemma was used to obtain the following variation of the associativity formula for parameter ideals ([20, Theorem 1]). Recall that $\mathrm{Minh}(I)$ denotes the set of all minimal primes P of an ideal I such that $\dim R/P = \dim R/I$.

Proposition 2.1.8. *Let* $(R, \mathfrak{m})$ *be a local ring and let* $x_1, \ldots, x_d$ *be a system of parameters. Then for any* $0 \le i \le d$

$$\mathrm{e}((x_1, \ldots, x_d)) = \sum_{P \in \mathrm{Minh}((x_1, \ldots, x_i))} \mathrm{e}((x_{i+1}, \ldots, x_d), R/P)\, \mathrm{e}((x_1, \ldots, x_i)R_P).$$

Also, when dealing with parameter ideals, we will often use the following "filtration by the powers of x".

Lemma 2.1.9. *Let* R *be a ring of dimension* 1 *and* x *be a parameter. Then for any positive integer* n

$$\lambda(R/(x^n)) = n\,\lambda(R/(x)) - \sum_{k=1}^{n-1} \lambda\left(\frac{(0:x)}{x^k(0:x^{k+1})}\right).$$

In particular, $\lambda(R/(x^n)) \le n\,\lambda(R/(x))$ *and the equality holds for all* n *if and only if* x *is not a zero divisor.*

Proof. For any n we can surject $R/(x^n)$ onto $R/(x)$ and obtain the exact sequence

$$0 \to (x)/(x^n) \to R/(x^n) \to R/(x) \to 0.$$

Now note that we can surject R onto $(x)/(x^n)$ by mapping $1 \mapsto x + (x^n)$. The kernel of this map is the ideal $(x^n) : x = (x^{n-1}) + 0 : x$. These observations imply that

$$\lambda(R/(x^n)) = \lambda(R/(x)) + \lambda\left(R/((x^{n-1}) + 0 : x)\right),$$

so

$$\lambda(R/(x^n)) = \lambda(R/(x)) + \lambda(R/(x^{n-1})) - \lambda\left(\frac{(x^{n-1}) + 0 : x}{(x^{n-1})}\right).$$

Now,

$$\frac{(x^{n-1}) + 0 : x}{(x^{n-1})} \cong \frac{(0:x)}{(x^{n-1}) \cap (0:x)} = \frac{(0:x)}{x^{n-1}(0:x^n)}$$

and the claim follows by induction on n.

For the second part, we observe that $0 : x^n$ form an ascending chain of ideals as n increases, so it stabilizes and $x^k(0 : x^{k+1}) = x^k(0 : x^k) = 0$ for a large k. Therefore, the equality holds if and only if $0 : x = 0$. □

Corollary 2.1.10. *Let $(R, \mathfrak{m})$ be a local ring of dimension 1 and x be a parameter. Then $\mathrm{e}(x) \leq \lambda(R/(x))$.*

Moreover, the equality holds if and only if x is regular, i.e. R is Cohen-Macaulay.

Proof. By the definition and the lemma

$$\mathrm{e}(x) = \lim_{n\to\infty} \frac{\lambda(R/(x^n))}{n} \leq \lim_{n\to\infty} \frac{n\,\lambda(R/(x))}{n}$$

and this shows the first part.

For the second part we note that, for any ideal I and an element t, $I \cap (t) = t(I : t)$, thus $(x^k) \cap (0 : x) = x^k(0 : x^{k+1})$ for all k. Since $(0 : x^k)$ form an ascending chain of ideals, it stabilizes, so $x^k(0 : x^{k+1}) = x^k(0 : x^k) = 0$ for any k sufficiently large, say, for $k \geq N$.

Therefore, the lemma above gives us the following estimate

$$\lambda(R/(x^n)) = n\,\lambda(R/(x)) - \sum_{k=1}^{N-1} \lambda((0 : x)/(x^k) \cap (0 : x)) - (n - N + 1)\,\lambda(0 : x).$$

Hence, dividing by n and taking the limit, we obtain that

$$\mathrm{e}(x) = \lambda(R/(x)) - \lambda(0 : x),$$

and the claim follows. □

Using the associativity formula we can generalize this result for higher dimension, but, first, we need the following lemma.

Lemma 2.1.11. *Let $(R, \mathfrak{m})$ be a local ring and let x be a parameter. Suppose $R/(x)$ is Cohen-Macaulay and R_P is Cohen-Macaulay for all minimal primes P of (x). Then R is Cohen-Macaulay.*

Proof. If $\dim R = 1$, then $\mathfrak{m}$ is a minimal prime of (x), and the assertion is trivial, so we assume that $\dim R > 1$.

After localizing if necessary, we can assume that $R_\mathfrak{p}$ is Cohen-Macaulay for any $\mathfrak{p} \neq \mathfrak{m}$ and containing x. Then $0 : x^n$ has finite length for any n, because if $x \notin \mathfrak{p}$ then the image of x is invertible in $R_\mathfrak{p}$ and if $x \in \mathfrak{p} \neq \mathfrak{m}$ then x is regular in $R_\mathfrak{p}$.

Since $(0 : x^n)$ is a finite length submodule of R, for any element $y \in \mathfrak{m}$ there is a power N such that $y^N(0 : x^n) = 0$. But, since $R/(x)$ is a Cohen-Macaulay ring of dimension at least one, it has a regular element y, so the image of $(0 : x^n)$ in $R/(x)$ is zero, i.e. $(0 : x^n) \subseteq (x)$. Then one can easily check that $0 : x^n = x(0 : x^{n+1})$. Since $0 : x^n$ is an ascending chain of ideals, it stabilizes for some $n = N$. Then $0 : x^N = x(0 : x^N)$, so, by Nakayama's lemma, $0 : x = 0$. Thus x is regular and R Cohen-Macaulay. □

Proposition 2.1.12. *Let $(R, \mathfrak{m})$ be a local ring and $x_1, \ldots, x_d$ be a system of parameters. Then* $\mathrm{e}((x_1, \ldots, x_d)) \leq \lambda(R/(x_1, \ldots, x_d))$.

Moreover, the equality holds if and only if R is Cohen-Macaulay.

Proof. We use induction on d, the base case is Corollary 2.1.10.

Now, by the associativity formula (Proposition 2.1.8) and the induction base,

$$\begin{aligned}
\mathrm{e}((x_1,\dots,x_d)) &= \sum_{P\in \mathrm{Minh}(x_1)} \mathrm{e}((x_2,\dots,x_d),R/P)\,\mathrm{e}(x_1R_P)\\
&\le \sum_{P\in \mathrm{Minh}(x_1)} \mathrm{e}((x_2,\dots,x_d),R/P)\,\lambda(R_P/(x_1)R_P).
\end{aligned}$$

Moreover, by the usual associativity formula,

$$\sum_{P\in \mathrm{Minh}(x_1)} \mathrm{e}((x_2,\dots,x_d),R/P)\,\lambda(R_P/(x_1)R_P) = \mathrm{e}((x_2,\dots,x_d),R/(x_1)).$$

Thus, by the induction hypothesis for $d-1$ elements in $R/(x_1)$

$$\sum_{P\in \mathrm{Minh}(x_1)} \mathrm{e}((x_2,\dots,x_d),R/P)\,\lambda(R_P/(x_1)R_P) \le \lambda(R/(x_1,\dots,x_d)).$$

Now, suppose that $\mathrm{e}((x_1,\dots,x_d)) = \lambda(R/(x_1,\dots,x_d))$. By the first part of the proof, we get that $R/(x_1)$ is Cohen-Macaulay, so $\mathrm{Minh}(x_1) = \mathrm{Min}(x_1)$. Then the first formula yields that R_P is Cohen-Macaulay for all minimal primes P of x_1 and we are done by Lemma 2.1.11.

Last, suppose that R is Cohen-Macaulay. We use Lech's lemma and filter by the powers of x_i by repeatedly applying Lemma 2.1.9

$$\begin{aligned}
\mathrm{e}((x_1,\dots,x_d)) &= \lim_{n\to\infty}\frac{\lambda(R/(x_1^n,\dots,x_d^n))}{n^d} = \lim_{n\to\infty}\frac{\lambda(R/(x_1^n,\dots,x_{d-1}^n,x_d))}{n^{d-1}} = \dots\\
&= \lambda(R/(x_1,\dots,x_d)).
\end{aligned}$$

□

2.2 Rings of positive characteristic

For a prime number p, we say that a ring R has characteristic p if R is an algebra over the prime field $\mathbb{Z}/p\mathbb{Z}$. The key characteristic of such rings is the Frobenius endomorphism $\phi\colon R \to R$ defined by $a \mapsto a^p$. Since p is the characteristic of the ring, $(a+b)^p = a^p + b^p$, so ϕ is an endomorphism.

2.2.1 Frobenius endomorphism

Definition 2.2.1. Let R be a commutative ring of positive characteristic p and e be a positive integer. Then the eth iteration of the Frobenius endomorphism gives R an additional structure of an R-algebra, $\phi^e\colon R \to R$, $a \mapsto a^{p^e}$. This R-algebra will be denoted by $F^e_* R$.

More generally, let M be an R-module. Then we use $F^e_* M$ to denote the R-module obtained from M via restriction of scalars along the Frobenius iterated e times. Namely, $F^e_* M$ is isomorphic to M as an abelian group, but the action of R is given by p^e-powers, i.e. if $m \in M$ is considered as element of $F^e_* M$, then $a \cdot m = a^{p^e} m$.

Now, let us list some properties of F_*.

Proposition 2.2.2. *1. F_* is an exact functor,*

2. F_* commutes with localization, i.e. $F_*(S^{-1}M) \cong S^{-1}(F_* M)$ for any multiplicatively closed set S.

Proof. First, F_* is exact, since it is a restriction of scalars.

For the second property, we observe that there is no difference between inverting s^p or s. □

Remark 2.2.3. If R is a domain, we can define $R^{1/p}$ to be a subring of the algebraic closure of the fraction field of R that consists of all elements a such that $a^p \in R$. Then $R^{1/p}$ is isomorphic to $F_* R$ as an R-algebra.

For example, if $R = k[x_1, \ldots, x_n]$ then $F_*R \cong R^{1/p} = k^{1/p}[x_1^{1/p}, \ldots, x_n^{1/p}]$.

Definition 2.2.4. Let R be a commutative ring of characteristic $p > 0$ and I an ideal in R. Let $q = p^e$ for some positive integer e, then $I^{[q]}$ is the ideal generated by the image of I under the eth iterate of the Frobenius, i.e. $I^{[q]} = (\{a^q \mid a \in I\})$.

Proposition 2.2.5. *Let R be a commutative ring of characteristic $p > 0$, M an R-module, and I an ideal. Then $R/I \otimes_R F_*^e M \cong F_*^e(M/I^{[q]}M)$.*

Proof. Tensoring an exact sequence

$$0 \to I \to R \to R/I \to 0$$

with $F_*^e M$, we obtain

$$I \otimes_R F_*^e M \to F_*^e M \to R/I \otimes_R F_*^e M \to 0.$$

Note that the image of $I \otimes_R F_*^e M$ in $F_*^e M$ is $I(F_*^e M) = F_*^e(I^{[q]}M)$ by the definition of the R-action on F_*M. Moreover, since F_*^e is exact, $F_*^e M/F_*^e(I^{[q]}M) \cong F_*^e(M/I^{[q]}M)$, and the claim follows. □

For the next result, we need a bit of notation.

Definition 2.2.6. Let R be a ring of characteristic $p > 0$. For a prime ideal $\mathfrak{p}$ of R, we denote $\alpha(\mathfrak{p}) = \log_p[k(\mathfrak{p}) : k(\mathfrak{p})^p]$, where $k(\mathfrak{p}) = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$ is the residue field of $\mathfrak{p}$.

We understand how length changes under F_*^e.

Proposition 2.2.7. *Let $(R, \mathfrak{m}, k)$ be a local ring of characteristic $p > 0$ and let M be a finite length module. Then*

1. $\lambda_R(M) = \lambda_{F^e_* R}(F^e_* M)$.

2. $\lambda_R(F^e_* M) = [k : k^{p^e}]\, \lambda_R(M) = p^{e\alpha(R)}\, \lambda_R(M)$. *So, if* k *is perfect,* $\lambda_R(F^e_* M) = \lambda_R(M)$.

Proof. Consider a composition series of M

$$0 = M_0 \subset M_1 \subset \ldots \subset M_l = M,$$

where the factors $M_{n+1}/M_n \cong k$. Since F^e_* is an exact functor and $F^e_* k$ is the residue field of $F^e_* R$, applying F^e_* to the original series we obtain a composition series for $F^e_* M$ and the first claim follows.

The second claim also follows after taking the length over R of the new composition series, note that $F^e_* k \cong k^{1/p^e}$, so $\lambda_R(F^e_* k) = [k^{1/p^e} : k] = [k : k^{p^e}]$ since the Frobenius is a field isomorphism of k and k^p. □

2.2.2 F-finite rings

Definition 2.2.8. Let R be a ring of characteristic $p > 0$. We say that R is F-finite if $F_* R$ is a finitely generated R-module.

A quotient of an F-finite ring is F-finite, since $F_*(R/I)$ is a quotient of a finitely generated module $R/I \otimes F_* R \cong F_*(R/I^{[p]})$. A localization of a F-finite ring is *F*-finite since F_* respects localization. Thus, from Remark 2.2.3 we obtain that any affine algebra over an F-finite (e.g. perfect) field is F-finite.

We will need the following result of Kunz ([19, Proposition 2.3]).

Proposition 2.2.9. *Let R be F-finite and let $\mathfrak{p} \subseteq \mathfrak{q}$ be prime ideals. Then* $\alpha(\mathfrak{p}) = \alpha(\mathfrak{q}) + \dim R_{\mathfrak{q}}/\mathfrak{p}R_{\mathfrak{q}}$.

This result allows us to control the change in length between M and F_*M in localizations. For example, the next corollary shows that we can compute the rank of F_*M.

Corollary 2.2.10. *Let $(R, \mathfrak{m})$ be a reduced F-finite local ring of dimension d and M be a finitely generated R-module. Then for any minimal prime $\mathfrak{p} \in \operatorname{Minh}(R)$ of R the modules $M_{\mathfrak{p}}^{\oplus p^{\alpha(R)+d}}$ and $(F_*M)_{\mathfrak{p}}$ are isomorphic.*

In particular, if R is equidimensional and M has rank r, then the rank of F_*M is $rp^{\alpha(R)+d}$.

Proof. By Proposition 2.2.9, $\alpha(\mathfrak{p}) = \alpha(R) + d$ for any $\mathfrak{p} \in \operatorname{Minh}(R)$. Note that by Proposition 2.2.7

$$\lambda_{R_{\mathfrak{p}}}(F_*(M_{\mathfrak{p}})) = p^{\alpha(\mathfrak{p})}\, \lambda_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}).$$

Since R is reduced, $R_{\mathfrak{p}}$ is a field, hence the vector spaces $F_*(M_{\mathfrak{p}})$ and $\oplus_{p^{\alpha(P)}} M_{\mathfrak{p}}$ are isomorphic. Last, recall that $F_*(M_{\mathfrak{p}}) \cong (F_*M)_{\mathfrak{p}}$ since F_* commutes with localization. □

F-finite rings have a few nice properties. Most importantly, F-finite rings are excellent by a theorem of Kunz ([19, Theorem 2.5]). Let us recall the definition of an excellent ring. We refer the reader to Chapter 13 of Matsumura's book ([22]) for a detailed treatment.

Definition 2.2.11. Let R be a Noetherian ring. We say that R is excellent if the following conditions hold.

1. Any finitely generated R-algebra is catenary.

2. The formal fibers of R are geometrically regular, i.e. for any prime $\mathfrak{p}$ and for every finite field extension L of $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$ the ring $\widehat{R_{\mathfrak{p}}} \otimes_{R_{\mathfrak{p}}} L$ is a regular ring.

3. For every finitely generated R-algebra B, the set of all prime ideal P such that B_P is regular (the regular locus of B) is open.

The definition of an excellent was made to make it satisfy geometric properties of most "naturally appearing" rings. In particular, complete rings and affine algebras are excellent. The class of excellent rings is stable under localization, taking quotients, and finite extensions.

Remark 2.2.12. Besides the properties given in the definition, excellent rings also have well-behaved completions.

For example, if R is excellent and reduced, then $\widehat{R}$ is also reduced, i.e. R is analytically unramified. If R is excellent and equidimensional ($\operatorname{Minh}(R) = \operatorname{Min}(R)$), then $\widehat{R}$ is also equidimensional, i.e. R is formally equidimensional.

The regular locus of an excellent ring is open by the definition. However, it happens that many other loci are also open. We will only need the following result ([9, 7.8.3(iv)]).

Proposition 2.2.13. *Let R be an excellent ring. Then the Cohen-Macaulay locus of R*

$$\{\mathfrak{p} \in \operatorname{Spec} R \mid R_{\mathfrak{p}} \text{ is Cohen-Macaulay}\}$$

is an open subset of $\operatorname{Spec} R$.

Even if R is not F-finite we can often pass to an F-finite ring via the following recipe.

Remark 2.2.14. Let $(R, \mathfrak{m})$ be a local ring of characterstic $p > 0$.

By Cohen's structure theorem ([22, page 265]), $\widehat{R}$ has a coefficient field k. So, if $x_1, \ldots, x_n$ are the generators of $\mathfrak{m}\widehat{R}$, there is a surjection $k[[T_1, \ldots, T_n]] \to \widehat{R}$ where T_1 are indeterminates and the map acts by $T_i \mapsto x_i$.

Let $S = \widehat{R} \otimes_{k[[T_1,\ldots,T_n]]} k^\infty[[T_1, \ldots, T_n]]$, where k^∞ is the perfect closure of k. Then $R \to S$ is faithfully flat as a composition of faithfully flat maps $R \to \widehat{R} \to S$. Moreover, S is complete with a perfect residue field, so, since it is a homomorphic image of the power series ring over a perfect field, it is F-finite.

Also, note that $\mathfrak{m}S$ is the maximal ideal of S. So by the lemma below, we obtain that S preserves length, and therefore $\mathrm{e_{HK}}(I, M) = \mathrm{e_{HK}}(I, S \otimes_R M)$ for any $\mathfrak{m}$-primary ideal I and finitely generated module M.

Lemma 2.2.15. *Let $(R, \mathfrak{m})$ be a local ring and let $(S, \mathfrak{n})$ be a flat local R-algebra such that $\dim S = \dim R$ (equivalently, $\mathfrak{m}S$ is $\mathfrak{n}$-primary). Then for any R-module M of finite length, $\lambda_S(M \otimes_R S) = \lambda_R(M)\, \lambda_S(S/\mathfrak{m}S)$.*

Proof. Consider a composition series of M

$$0 = M_0 \subset M_1 \ldots \subset M_L = M.$$

Since S is flat, tensoring the composition series above with S we obtain a filtration of $M \otimes_R S$:

$$0 = M_0 \otimes_R S \subset M_1 \otimes_R S \ldots \subset M_L \otimes_R S = M \otimes_R S.$$

Because the original filtration was a composition series, for each quotient

$$(M_{n+1} \otimes_R S)/(M_n \otimes_R S) \cong M_{n+1}/M_n \otimes_R S \cong R/\mathfrak{m} \otimes_R S = S/\mathfrak{m}S$$

and the claim follows. □

2.3 Hilbert-Kunz multiplicity

For convenience we will denote $q = p^e$ where e may vary.

Definition 2.3.1. Let $(R, \mathfrak{m})$ be a local ring of characteristic $p > 0$, I be an $\mathfrak{m}$-primary ideal, and M a finitely generated R-module. The (normalized) Hilbert-Kunz function of M with respect to I is the sequence

$$q \mapsto \lambda_{\mathrm{q}}(I, M) := \frac{\lambda(M/I^{[q]}M)}{q^{\dim R}}.$$

The Hilbert-Kunz multiplicity of M with respect to I is defined to be

$$\mathrm{e}_{\mathrm{HK}}(I, M) = \lim_{q \to \infty} \lambda_{\mathrm{q}}(I, M) = \lim_{q \to \infty} \frac{\lambda(M/I^{[q]}M)}{q^{\dim R}}.$$

If $M = R$, we will omit it, and will write $\mathrm{e}_{\mathrm{HK}}(I)$ for $\mathrm{e}_{\mathrm{HK}}(I, R)$. Also, $\mathrm{e}_{\mathrm{HK}}(\mathfrak{m})$ is often denoted by $\mathrm{e}_{\mathrm{HK}}(R)$ and called the Hilbert-Kunz multiplicity of R.

The study of Hilbert-Kunz function starts in the work of Kunz ([18, 19]), while Monsky showed that the limit exists in [24]. We will provide a generalization of Monsky's argument later in Section 4.1, see Remark 4.1.7.

In general, $\lambda(M/I^{[q]}M)$ is not a polynomial, so it is much trickier to prove the existence of the limit. In fact, the known proof only establishes that the sequence is Cauchy (see Remark 4.1.7) and does not give a way to compute it. Thus, computation of Hilbert-Kunz multiplicity is a challenge: there is no algorithm to compute it even for hypersurfaces. Even more, it is usually very hard to calculate it by hand, so only a handful of examples are known.

Example 2.3.2. First, the two theories agree if dimension is at most one. This follows from the inequalities $\mathrm{e}(R)/d! \le \mathrm{e}_{\mathrm{HK}}(R) \le \mathrm{e}(R)$ that will be established in Remark 3.1.14 and the proof of Lemma 2.3.12.

For an Artinian local ring $(R, \mathfrak{m})$, this can be observed very easily. Namely, $\mathfrak{m}^{[q]} = \mathfrak{m}^n = 0$ for all large n and q, so $\mathrm{e}_{\mathrm{HK}}(\mathfrak{m}) = \mathrm{e}(\mathfrak{m}) = \lambda(R)$.

However, in general, the multiplicities do not have to be equal. In the following example, $\mathrm{e}_{\mathrm{HK}}(R) = 168/61$ while $\mathrm{e}(R) = 4$.

Example 2.3.3. In [10] Han and Monsky have computed the Hilbert-Kunz function of a hypersurface

$$R = \mathbb{Z}/5\mathbb{Z}[[x_1, x_2, x_3, x_4]]/(x_1^4 + \ldots + x_4^4).$$

They obtained that

$$\lambda_{\mathrm{q}}(R) = \frac{168}{61} 5^{3e} - \frac{107}{61} 3^e$$

which is not a polynomial in $q = 5^e$.

Remark 2.3.4. This example shows another difference between two multiplicities: Hilbert-Kunz multiplicity need not be an integer. In fact, recently, Brenner ([1]) was able to show that Hilbert-Kunz multiplicity could be even irrational. However, his proof is not constructive and we do not have an explicit example. On the other hand, there is a conjectured example of transcendental Hilbert-Kunz multiplicity due to Monsky.

There is an another characterization of $\mathrm{e_{HK}}(\mathfrak{m})$. Suppose that k, the residue field of R, is perfect. Then, by Proposition 2.2.7,

$$\lambda_R(k \otimes_R F^e_* R) = \lambda_R(F^e_*(R/\mathfrak{m}^{[q]})) = \lambda_R(R/\mathfrak{m}^{[q]}),$$

so $\lambda_R(R/\mathfrak{m}^{[q]})$ is the minimal number of generators of $F^e_* R$ (as an R-module). Thus, Hilbert-Kunz multiplicity measures the asymptotic number of generators of $F^e_* R$.

Using this characterization, Kunz ([18, Proposition 3.2]) observed the following.

Lemma 2.3.5. *Let $(R, \mathfrak{m})$ be a local ring of dimension d. Then $\lambda_{\mathrm{q}}(R) \geq 1$ for all q. In particular, $\mathrm{e_{HK}}(R) \geq 1$.*

Proof. Using Remark 2.2.14, it is enough to assume that R is complete with a perfect residue field. Now, in $\mathfrak{p} \in \mathrm{Minh}(R)$, then $\lambda(R/\mathfrak{m}^{[q]}) \geq \lambda(R/(\mathfrak{m}^{[q]} + \mathfrak{p}))$, so $\lambda_{\mathrm{q}}(R) \geq \lambda_{\mathrm{q}}(R/\mathfrak{p})$. So we may assume that R is a complete local domain.

We claim that the minimal number of generators of M is greater or equal than the rank of M for any finitely generated module M. To see this we note that the surjection $R^m \to M \to 0$ induces a surjection $L^m \to M \otimes_R L \to 0$.

Using the claim for $M = F^e_* R$, we obtain that $\lambda(R/m^{[q]}) \geq q^d$ by Corollary 2.2.10. ∎

2.3.1 Properties

Now, we want to discuss some basic properties of Hilbert-Kunz multiplicity. First, as a corollary to Lech's lemma (2.1.6), we obtain that for parameter ideals the two multiplicity theories are same.

Corollary 2.3.6. *Let $(R, \mathfrak{m})$ be a local ring of characteristic $p > 0$ and let $x_1, \dots, x_d$ be a system of parameters. Then*

$$\mathrm{e_{HK}}((x_1, \dots, x_d)) = \mathrm{e}((x_1, \dots, x_d)).$$

Example 2.3.7. Let $(R, \mathfrak{m})$ be a regular local ring of characteristic $p > 0$. Then $\mathrm{e}(R) = \mathrm{e_{HK}}(R)$ since $\mathfrak{m}$ is generated by a system of parameters. Moreover, a repeated application of Lemma 2.1.9 shows that $\lambda_{\mathrm{q}}(\mathfrak{m}) = q^{\dim R}$, so, $\mathrm{e}(R) = \mathrm{e_{HK}}(R) = 1$.

We establish two easy lemmas on "iteration" of Frobenius powers.

Lemma 2.3.8. *Let $(R, \mathfrak{m})$ be a local ring of characteristic $p > 0$, J be an $\mathfrak{m}$-primary ideal, and M a finitely generated R-module. Then for any q, $\mathrm{e_{HK}}(J^{[q]}, M) = q^{\dim R}\,\mathrm{e_{HK}}(J, M)$.*

Proof. Observe that

$$\mathrm{e_{HK}}(J^{[q]}, M) = \lim_{q' \to \infty} \frac{\lambda(M/J^{[qq']}M)}{(q')^{\dim R}} = \lim_{(qq') \to \infty} q^{\dim R} \frac{\lambda(M/J^{[qq']}M)}{(qq')^{\dim R}} = q^{\dim R}\,\mathrm{e_{HK}}(J, M).$$

□

Lemma 2.3.9. *Let $(R, \mathfrak{m})$ be an F-finite local ring of characteristic $p > 0$, J be an $\mathfrak{m}$-primary ideal, and M a finitely generated R-module. Then for any e, $\mathrm{e_{HK}}(J, F^e_* M) = p^{e(\dim R + \alpha(R))}\,\mathrm{e_{HK}}(J, M)$.*

Proof. By definition and Proposition 2.2.5,

$$\mathrm{e_{HK}}(J, F^e_* M) = \lim_{e' \to \infty} \frac{\lambda(R/J^{[p^{e'}]} \otimes_R (F^e_* M))}{p^{e \dim R}} = \lim_{e' \to \infty} \frac{\lambda(F^e_*(M/J^{[p^{e+e'}]}M))}{p^{e \dim R}}.$$

Now, by Proposition 2.2.7,

$$\lambda(F^e_*(M/J^{[p^{e+e'}]}M)) = p^{e\alpha(R)}\,\lambda(M/J^{[p^{e+e'}]}M)$$

and the claim follows.

□

Hilbert-Kunz multiplicity satisfies many properties of the classical multiplicity. For example, it is additive.

Proposition 2.3.10. *Let* $(R, \mathfrak{m})$ *be a local ring of characteristic* $p > 0$*, and let* J *be an* $\mathfrak{m}$*-primary ideal. If* $0 \to K \to L \to M \to 0$ *is a short exact sequence of finitely generated* R*-modules, then* $\mathrm{e_{HK}}(J, L) = \mathrm{e_{HK}}(J, K) + \mathrm{e_{HK}}(J, M)$*.*

Proof. First, by Remark 2.2.14, we can assume that R is F-finite.

As it will be observed in Remark 4.1.5, $\mathrm{e_{HK}}(J, M) = \mathrm{e_{HK}}(J, N)$ if M and N are two modules such that $M_P \cong N_P$ for any minimal prime P such that $\dim R/P = \dim R$.

If R is reduced, R_P is a field for any minimal prime P of R. Thus, $L_P \cong K_P \oplus M_P$ as vector spaces of same dimension. Therefore, the observation helps us to show that $\mathrm{e_{HK}}(L) = \mathrm{e_{HK}}(K \oplus M) = \mathrm{e_{HK}}(K) + \mathrm{e_{HK}}(M)$, where the last equality readily follows from additivity of length and the definition of Hilbert-Kunz multiplicity.

If R is not reduced, its nilradical is nilpotent, so there exists an e such that the nilradical acts as zero on $F^e_* L$. Thus, we obtain an exact sequence of $R/\sqrt{0}$-modules,

$$0 \to F^e_* K \to F^e_* L \to F^e_* M \to 0$$

and the argument above showes that $\mathrm{e_{HK}}(J, F^e_* L) = \mathrm{e_{HK}}(J, F^e_* K) + \mathrm{e_{HK}}(J, F^e_* M)$. Moreover, by Lemma 2.3.9, $\mathrm{e_{HK}}(J, F^e_* L) = p^{e(\dim R + \alpha(R))} \mathrm{e_{HK}}(J, L)$ and the assertion follows. □

From additivity, the proof Proposition 2.1.3 derives the associativity formula.

Proposition 2.3.11. *Let $(R,\mathfrak{m})$ be a local ring of characteristic $p>0$, I be an $\mathfrak{m}$-primary ideal, and M be a finitely generated R-module. Then*

$$\mathrm{e_{HK}}(I,M)=\sum_{P\in \mathrm{Minh}(R)} \mathrm{e_{HK}}(I,R/P)\,\lambda_{R_P}(M_P).$$

Hilbert-Kunz multiplicity also detects dimension of a module.

Lemma 2.3.12. *Let $(R,\mathfrak{m})$ be a local ring, M a finitely generated R-module, and I be an $\mathfrak{m}$-primary ideal. Then $\mathrm{e_{HK}}(I,M)=0$ if and only if $\dim M<\dim R$.*

Proof. Recall that $\dim M=\dim S$ for $S=R/0:_R M$. So, if $\dim S<\dim R$, then we observe that

$$\mathrm{e_{HK}}(IS,M)=\lim_{q\to\infty}\frac{\lambda(M\otimes_S S/I^{[q]}S)}{q^{\dim S}}=\lim_{q\to\infty}\frac{\lambda(M\otimes_R R/I^{[q]})}{q^{\dim S}}$$

exists, so $\mathrm{e_{HK}}(I,M)=0$.

For the other direction, we use that $I^{[q]}\subseteq I^q$ for all q, so $\mathrm{e_{HK}}(I,M)\geq \mathrm{e}(I,M)/d!$. Now, the claim follows from the analogous property of Hilbert-Samuel multiplicity.

□

The following result is due to Kunz ([19, Proposition 3.2]). It also has a corresponding result for Hilbert-Samuel multiplicity, but we will not need it.

Proposition 2.3.13. *Let $(R,\mathfrak{m})$ be a local ring and let x be an element of a system of parameters. Then $\mathrm{e_{HK}}(R)\leq \mathrm{e_{HK}}(R/(x))$.*

Proof. Since $x^q\in\mathfrak{m}^{[q]}$, we use Lemma 2.1.9 to obtain that

$$\lambda(R/\mathfrak{m}^{[q]})\leq q\,\lambda(R/(\mathfrak{m}^{[q]}+(x))).$$

Now, since x is a part of a system of parameters, $\dim R/(x) = \dim R - 1$, and the claim follows, after dividing both sides with $q^{\dim R}$ and taking the limit. □

Hilbert-Kunz multiplicity behaves well with respect to completion.

Lemma 2.3.14. *Let $(R, \mathfrak{m})$ be a local ring, I be an $\mathfrak{m}$-primary ideal, and M a finitely generated R-module. Then $\mathrm{e}_{\mathrm{HK}R}(I, M) = \mathrm{e}_{\mathrm{HK}\widehat{R}}(I\widehat{R}, \widehat{M})$.*

Proof. This follows from Lemma 2.2.15. □

The following bound is quite useful.

Lemma 2.3.15. *Let $(R, \mathfrak{m})$ be a local ring of characteristic $p > 0$. Then for any $\mathfrak{m}$-primary ideal I and every q*

$$\lambda(R/I^{[q]}) \leq \lambda(R/I)\, \lambda(R/\mathfrak{m}^{[q]}).$$

In particular, $\mathrm{e}_{\mathrm{HK}}(I) \leq \lambda(R/I)\, \mathrm{e}_{\mathrm{HK}}(\mathfrak{m})$.

Proof. Let $l = \lambda(R/I)$ and consider the composition series of R/I:

$$I = I_0 \subset I_1 \subset \ldots \subset I_l = R,$$

so $I_{n+1}/I_n \cong R/\mathfrak{m}$ for all $l > n \geq 0$, i.e. we have $I_{n+1} = (I_n, x_n)$ and $I_n \colon x_n = \mathfrak{m}$.

Consider the filtration

$$I^{[q]} = {I_0}^{[q]} \subset {I_1}^{[q]} \subset \ldots \subset {I_l}^{[q]} = R.$$

This may not be a composition series anymore, but we can bound the length of each factor

$$\frac{{I_{n+1}}^{[q]}}{{I_n}^{[q]}} = \frac{({I_n}^{[q]}, x_n^q)}{{I_n}^{[q]}} \cong \frac{(x_n^q)}{(x_n^q) \cap {I_n}^{[q]}} \cong \frac{R}{{I_n}^{[q]} : x_n^q}.$$

Since $\mathfrak{m}x_n \subseteq I_n$, we get that $\mathfrak{m}^{[q]}x_n^q \subseteq {I_n}^{[q]}$, thus $\lambda(R/{I_n}^{[q]} \colon x_n^q) \leq \lambda(R/\mathfrak{m}^{[q]})$ and the claim follows. □

2.4 Tight closure

Tight closure was introduced by Hochster and Huneke and has a tremendous number of applications. For our goals, we do not need to dig deep, but we still need discuss some aspects of the theory. For a more detailed exposition we refer a curious reader to [12].

Definition 2.4.1. Let R be a ring and let I be an ideal of R. Let R° denote $R \setminus \cup_{P \in \mathrm{Min}(R)} P$. The tight closure I^* of I consists of all elements x of R such that there exists a fixed element $c \in R^\circ$ (i.e. not contained in any minimal prime of R), such that

$$cx^q \in I^{[q]}$$

for all sufficiently large q.

Now we explore some of the properties of tight closure. First, we will need the following form of prime avoidance. The proof is taken from Kaplansky ([17, Theorem 124]).

Lemma 2.4.2. *Let $P_1, \ldots, P_n$ be prime ideals in a commutative ring R, and let I be an ideal in R. Suppose $x \in R$ is such that (I, x) is not contained in $\cup P_i$, then there exists an element $a \in I$, such that $x + a \notin \cup P_i$.*

Proof. We can assume that no two of P_i are comparable, because if P_k is contained in some other prime, then the union will not change after ommiting it.

After relabeling, we assume that $x \in P_1, \dots, P_k$ but x is not contained in any of $P_{k+1}, \dots, P_n$. If $k = 0$, then $x + 0$ is the required element, so assume $k > 0$. By our assumption on (I, x), there exists an element $b \in I \setminus \{P_1 \cup \dots \cup P_k\}$. Also, by the classical prime avoidance, there exists an element $c \in (P_{k+1} \cap \dots \cap P_n) \setminus \{P_1 \cup \dots \cup P_k\}$.

Last, one can easily check that $x + bc$ satisfies the assertion. □

Proposition 2.4.3. *Let R be a ring and let I, J be ideals of R.*

(1) I^* is an ideal.

(2) $(I^*)^* = I^*$,

(3) If $I \subseteq J$ then $I^* \subseteq J^*$.

(4) For all q $(I^*)^{[q]} \subseteq (I^{[q]})^*$.

(5) $x \in I^*$ if and only if the image of x is in $(IR/P)^*$ for all minimal primes P.

(6) I^* is the preimage of $(IR_{red})^*$ in R.

Proof. The first four properties are straightforward.

For (5), if $x \in I^*$ then the tight closure equations still hold true modulo a minimal prime, so one direction follows. Let P_i be minimal primes, then there are $c_i \notin P_i$, such that for all $q >> 0$

$$c_i x^q \in I^{[q]} + P_i.$$

By Lemma 2.4.2, we can find representatives for c_i in R°. Now choose arbitrary t_i contained in all minimal primes except for P_i. Set $c = \sum_i t_i c_i$, then for all $q >> 0$

$$cx^q = \sum t_i c_i x^q \in I^{[q]} + \cap P_i.$$

Now, the nilradical is nilpotent, so $(\cap P_i)^{[q']} = 0$ for some q', and thus for all $q >> 0$

$$c^{q'} x^{qq'} \in I^{[qq']},$$

so $x \in I^*$.

For (6), its easy to see that every element in I^* is in the preimage. If $\bar{c} \in (R_{red})^\circ$ than its preimage is in R°, so if $\bar{x} \in (IR_{red})^*$, then we have an equation for its preimage

$$cx^q \in I^{[q]} + \sqrt{0}.$$

But as in the previous property, we can raise the whole equation to the power q' such that $(\sqrt{0})^{[q']} = 0$ and obtain that x is in the tight closure of I. □

2.4.1 Test Elements

A notable difference between theories of tight and integral closure is our ability to find a uniform element that tests tight closure for all ideals (and, even, modules).

Definition 2.4.4. Let R be a Noetherian ring of characteristic $p > 0$. We say that an element $c \in R^\circ$ is a test element, if, for every ideal I and element $x \in R$, $x \in I^*$ if and only if $cx^q \in I^{[q]}$ for all q.

Test elements make tight closure to be a very powerful tool and have had a great use. Even more, the test ideal (the ideal generated by test elements) became a subject of separate research due to its fantastic connections with birational geometry. After a tremendous amount of work in [13], Hochster and Huneke obtained the following existence theorem.

Theorem 2.4.5. *Let R be reduced and either F-finite or of essentially finite type over an excellent local ring. Then R has a test element.*

Now, let us show some of the remarkable properties that test elements give to tight closure.

Lemma 2.4.6. *Let $(R, \mathfrak{m})$ be a local ring of positive characteristic with a test element c. If I and J are two ideals then*

$$I^* = \cap_q (I, J^{[q]})^* = \cap_n (I, J^n)^*.$$

Proof. This easily follows from the definition: if x belongs to the intersection, then

$$cx^{q'} \in \cap_n (I, J^n)^{[q']} \text{ (or } \cap_q (I, J^{[q]})^{[q']})$$

and we are done by Krull's intersection theorem. □

As an easy consequence of this result, we obtain that tight closure of any ideal is the intersection of tightly closed $\mathfrak{m}$-primary ideals. However, we do not need R to be local for this.

Lemma 2.4.7. *Let R be a ring characteristic $p > 0$ with a test element c. Then for any ideal I*

$$I^* = \bigcap_{\mathfrak{m}} \bigcap_{n} (I + \mathfrak{m}^n)^*$$

where the first intersection is taken over all maximal ideals $\mathfrak{m}$.

In particular, I^* *is the intersection of tightly closed ideals primary to maximal ideals.*

Proof. Clearly, $I^* \subseteq \bigcap_{\mathfrak{m}} \bigcap_n (I + \mathfrak{m}^n)^*$. Let x be an element of the intersection. If $x \notin I^*$, then, since c is a test element, $cx^q \notin I^{[q]}$ for some q. Therefore, there exists a maximal ideal $\mathfrak{m}$ such that non-inclusion still holds in $R_{\mathfrak{m}}$. Then by Krull's intersection theorem applied in $R_{\mathfrak{m}}$, $cx^q \notin I^{[q]} + \mathfrak{m}^n$ for some n. In particular, $x \notin (I + \mathfrak{m}^n)^*$, a contradiction. □

2.4.2 Connection with Hilbert-Kunz multiplicity

Now we discuss a very useful connection between Hilbert-Kunz multiplicity and tight closure. For two ideals $I \subset J$, it is easy to see that $\mathrm{e_{HK}}(I) \geq \mathrm{e_{HK}}(J)$, but an equality may hold despite that the two ideals are distinct. The following theorem, due to Hochster and Huneke, describes when does it happen.

Recall that a local ring is formally unmixed if $\operatorname{Ass}(\widehat{R}) = \operatorname{Minh}(\widehat{R})$. For example, a complete domain is formally unmixed.

Theorem 2.4.8. *Let* $(R, \mathfrak{m})$ *be a formally unmixed local ring and* $I \subseteq J$ *are ideals. Then* $J \subseteq I^*$ *if and only if* $\mathrm{e_{HK}}(I) = \mathrm{e_{HK}}(J)$.

The proof of the "if" direction is quite involved, so we do not present it. The following lemma shows that the Hilbert-Kunz multiplicity can be computed using the filtration $(I^{[q]})^*$, it can be thought of as a generalization of the "only if" direction. We are interested in this filtration, since it is often more useful then the usual filtration $I^{[q]}$.

Remark 2.4.9. If R has a test element c, then, by definition, $c\sqrt{0} = c0^* = 0$. Since c does not belong to any minimal prime, it follows that R_P is reduced for any minimal prime P of R. Therefore the dimension of the nilradical of R is less than $\dim R$. Now, from the additivity of Hilbert-Kunz multiplicity on exact sequences and Lemma 2.3.12, we obtain that $\mathrm{e_{HK}}(I, M) = \mathrm{e_{HK}}(I, R_{red} \otimes_R M)$ for any $\mathfrak{m}$-primary ideal I and module M.

Lemma 2.4.10. *Let $(R, \mathfrak{m})$ be a local ring of characteristic $p > 0$, I be an $\mathfrak{m}$-primary ideal, and M a finitely generated R-module. If R has a test element c, then*

$$\lim_{q\to\infty} \frac{1}{q^d}\lambda(M/(I^{[q]})^*M) = \mathrm{e_{HK}}(I, R_{red} \otimes_R M) = \mathrm{e_{HK}}(I, M).$$

Proof. First, by definition and (6) of Proposition 2.4.3, c is still a test element in R_{red}. Also, Proposition 2.4.3 shows that $R/(I^{[q]})^* \cong R_{red}/(I^{[q]}R_{red})^*$, and, using the previous remark, we assume that R is reduced.

Now, consider an exact sequence

$$R \xrightarrow{c} R \to R/(c) \to 0.$$

Since $c(I^{[q]})^* \subseteq I^{[q]}$, we obtain that the sequence

$$R/(I^{[q]})^* \otimes_R M \xrightarrow{c} R/I^{[q]} \otimes_R M \to R/(c, I^{[q]}) \otimes_R M \to 0.$$

is still exact. Together with inclusion $I^{[q]} \subseteq (I^{[q]})^*$ this shows that

$$\lambda(M/(I^{[q]})^*M) \leq \lambda(M/I^{[q]}M) \leq \lambda(M/(I^{[q]})^*M) + \lambda(M/(c, I^{[q]})M).$$

Since c is not contained in any minimal prime, $\dim M/cM \leq \dim R/(c) < \dim R$. Therefore $\mathrm{e}_{\mathrm{HK}}(I, M) = \lim_{q\to\infty} \frac{1}{q^d}\,\lambda(M/(I^{[q]})^*M)$. □

We will need the following corollary to deal with localization of tight closure. Unfortunately, tight closure does not commute with localization, but there is still an inclusion $I^*R_{\mathfrak{p}} \subseteq (IR_{\mathfrak{p}})^*$, where the first closure is taken in R and the second is in $R_{\mathfrak{p}}$. Thus, the corollary allows us to compute the Hilbert-Kunz multiplicity of $R_{\mathfrak{p}}$ by taking the filtration $(\mathfrak{p}^{[q]})^*R_{\mathfrak{p}}$.

Corollary 2.4.11. *Let $(R, \mathfrak{m})$ be a local ring of characteristic $p > 0$ with a test element c. If I_q is a sequence of ideals such that $\mathfrak{m}^{[q]} \subseteq I_q \subseteq (\mathfrak{m}^{[q]})^*$ then*

$$\lim_{q\to\infty} \frac{1}{q^d}\,\lambda(R/I_q) = \mathrm{e}_{\mathrm{HK}}(R).$$

Proof. The claim follows from Lemma 2.4.10, since the inclusions $\mathfrak{m}^{[q]} \subseteq I_q \subseteq (\mathfrak{m}^{[q]})^*$ give that

$$\mathrm{e}_{\mathrm{HK}}(R) \geq \lim_{q\to\infty} \frac{1}{q^d}\,\lambda(R/I_q) \geq \lim_{q\to\infty} \frac{1}{q^d}\,\lambda(R/(\mathfrak{m}^{[q]})^*) = \mathrm{e}_{\mathrm{HK}}(R).$$

□

More importantly, there is a partial converse to this inequality. It will useful later, as it provides us a way to detect when a filtration is in tight closure.

Lemma 2.4.12. *Let $(R, \mathfrak{m})$ be an formally unmixed local ring of characteristic $p > 0$ and I be an $\mathfrak{m}$-primary ideal. If I_q is a sequence of ideals such that $I^{[q]} \subseteq I_q$, $I_q^{[q']} \subseteq I_{qq'}$ for all q, q', and*

$$\lim_{q\to\infty} \frac{\lambda\,(R/I_q)}{q^d} = \mathrm{e}_{\mathrm{HK}}(I),$$

then $I_q \subseteq (I^{[q]})^*$ for all q.

Proof. By the assumptions on I_q,

$$I^{[qq']} \subseteq I_q^{[q']} \subseteq I_{qq'}.$$

Therefore, by Lemma 2.3.8

$$q^d \, \mathrm{e_{HK}}(I) = \mathrm{e_{HK}}(I^{[q]}) \geq \mathrm{e_{HK}}(I_q) \geq \lim_{q \to \infty} \frac{\lambda\,(R/I_{qq'})}{(q')^d} = q^d \, \mathrm{e_{HK}}(I),$$

so $\mathrm{e_{HK}}(I^{[q]}) = \mathrm{e_{HK}}(I_q)$. Hence by Theorem 2.4.8, $I_q \subseteq (I^{[q]})^*$. □

The following lemma helps us to understand what it means for an element to be regular modulo tight closures of consecutive powers. Recall that for an ideal I and element $x \notin I$ we denote $I : x^\infty = \bigcup_n I : x^n$.

Lemma 2.4.13. *Let R be a local ring of characteristic $p > 0$, I be an ideal, and x an element. Suppose R has a test element c, then the following are equivalent:*

(a) x is not a zero divisor modulo $(I^{[q]})^*$ for any q,

(b) $I^{[q]} : x^\infty \subseteq (I^{[q]})^*$ for all q,

(c) for all q there are ideals I_q such that x is not a zero divisor modulo I_q and $I^{[q]} \subseteq I_q \subseteq (I^{[q]})^*$.

Proof. $(a) \Rightarrow (b)$ since $I^{[q]} : x^\infty \subseteq (I^{[q]})^* : x^\infty$. $(b) \Rightarrow (c)$ trivially. For the last implication, we observe that if $ax \in (I^{[q]})^*$ for some q, then $ca^{q'}x^{q'} \in I^{[qq']}$ for all $q' >> 0$. But, since $x^{q'}$ is not a zero divisor modulo $I_{qq'}$,

$$ca^{q'} \in I_{qq'} \subseteq (I^{[qq']})^*.$$

Since R has a test element, these equations imply that $a \in (I^{[q]})^*$. □

Corollary 2.4.14. *Let* $(R, \mathfrak{m})$ *be a local ring of positive characteristic and* $\mathfrak{p}$ *be a prime ideal. Let* $L_q = \mathfrak{p}^{[q]} R_{\mathfrak{p}} \cap R$ *be the* $\mathfrak{p}$*-primary component of* $\mathfrak{p}^{[q]}$*. If* R *has a test element* c*, the following are equivalent:*

(a) $(\mathfrak{p}^{[q]})^*$ *is* $\mathfrak{p}$*-primary for any* q*,*

(b) $L_q \subseteq (\mathfrak{p}^{[q]})^*$ *for all* q*,*

(c) for all q *there exist* $\mathfrak{p}$*-primary ideals* I_q *such that* $\mathfrak{p}^{[q]} \subseteq I_q \subseteq (\mathfrak{p}^{[q]})^*$*.*

Proof. Clearly, $I_q = L_q$ yields $(b) \Rightarrow (c)$.

For an ideal I such that $\sqrt{I} = \mathfrak{p}$, we can characterise its $\mathfrak{p}$-primary part as the smallest among the ideals containing I and such that any element $x \notin \mathfrak{p}$ is not a zero divisor modulo that ideal. Thus, for $(c) \Rightarrow (a)$, we note that $\mathfrak{p}^{[q]} : x^\infty \subseteq I_q$ and use the lemma above.

For the last implication, we just note that $\mathfrak{p}^{[q]} R_{\mathfrak{p}} \cap R \subseteq (\mathfrak{p}^{[q]})^* R_{\mathfrak{p}} \cap R$.

□

Chapter 3

Multiplicities and Singularities

3.1 Introduction

An important stimulus for the development of multiplicity theories is their applications in singularity theory. Hilbert-Samuel theory enjoyed a growth spurt in 60-70s due to its appearance in the celebrated work of Hironaka on resolution of singularities in characteristic 0. It is impossible to give even a modest overview in a few pages, so we restrict ourselves to earlier foundations.

The first step was made by Nagata ([26, Theorem 40.6]).

Theorem 3.1.1. *Let* $(R, \mathfrak{m})$ *be a formally unmixed local ring. Then* $\mathrm{e}(R) = 1$ *if and only if* R *is regular.*

We recall that a local ring is formally unmixed if $\mathrm{Ass}(\widehat{R}) = \mathrm{Minh}(\widehat{R})$, i.e. for every associated prime P of $\widehat{R}$, $\dim \widehat{R}/P = \dim R$. Since the completion of a regular ring is still regular and a regular local ring is a domain, a regular local ring is formally unmixed. This assumption is needed to guarantee that there are no low dimensional components, as they do not contribute to the limit.

Example 3.1.2. For example, let $R = k[[x, y]]/x(x, y)$. Then R is not regular, but

$e(R) = 1$. Note that R has an embedded prime ideal (x, y).

Nagata's result was extended to Hilbert-Kunz multiplicity by Watanabe and Yoshida ([33]). In fact, since $e_{HK}(R) \leq e(R)$ (see Remark 3.1.14), this result is stronger than Nagata's characterization.

Theorem 3.1.3. *Let* $(R, \mathfrak{m})$ *be a formally unmixed local ring of characteristic* $p > 0$*. Then* $e_{HK}(R) = 1$ *if and only if* R *is regular.*

Of course, it is nice to have an invariant that detects regularity, but more is needed if we hope to use it in an algorithm for a resolution of singularities.

Let us give a naive approach to a resolution of singularities. First, we look for the set of the "worst" singularity measured by the maximal value of an invariant on the variety. Since we would like to blow-up along this set, the set had better be closed. During this procedure, we want the maximal value to decrease considerably, so after finitely many steps we will get a regular scheme.

So, very roughly, we want an invariant that detects regularity, whose maximal value locus is closed, and that behaves quite well under a blow-up (or, perhaps, under a "good" blow-up).

This strategy can be made to work in characteristic 0 with multiplicity used as a part of the controlling invariant. However, the proof fails badly in characteristic p and the problem is wide open.

The following property guarantees us that the maximal value set is closed.

Definition 3.1.4. Let X be a topological space and $f\colon X \to \mathbb{R}$ be a function. We say that f is locally constant if for every $a \in \mathbb{R}$, the set

$$X_{\leq a} = \{x \in X \mid f(x) \leq a\}$$

is open.

Remark 3.1.5. Let us briefly review Zariski's topology on the spectrum and set up some notation. We refer to Chapter 1 of [22] for more details.

For a ring R, we can make $\operatorname{Spec} R$ into a topological space by setting the closed sets to be of the form

$$V(I) = \{\mathfrak{p} \in \operatorname{Spec} R \mid I \subseteq \mathfrak{p}\}$$

for an arbitrary set (or, equivalently, ideal) I. Naturally, $V(I)$ can be identified with the spectrum of R/I.

The open sets are defined as complements, but any open set is a (possibly, infinite) union of principal open sets

$$D_s = \{\mathfrak{p} \in \operatorname{Spec} R \mid s \notin \mathfrak{p}\}$$

for an element $s \in R$. A principal open set D_s can be identified with $\operatorname{Spec} R_s$, so if we restrict our attention to D_s, we may aswell consider prime ideals in R_s.

Since the spectrum $X = \operatorname{Spec} R$ of a Noetherian ring is Noetherian, the ascending chain (as a increases) of open sets $X_{\leq a}$ stabilizes, so a locally constant function f attains its maximum. Furthermore, this property provides a stratification of $\operatorname{Spec} R$ by locally closed sets $X_{=a} = X_{\leq a} \cap X_{\geq a}$. We note that the set $X_{\geq a}$ is closed as the intersection of closed sets $X_{>b}$ for all $b > a$.

We can consider any kind of multiplicity as a function on the spectrum, by setting $\mathrm{e}(\mathfrak{p}) = \mathrm{e}(R_{\mathfrak{p}})$ for any prime $\mathfrak{p}$. In [26, Theorem 40.3] Nagata showed that Hilbert-Samuel multiplicity satisfies the desired property.

Theorem 3.1.6. *Let R be an excellent locally equidimensional ring. Then the Hilbert-Samuel multiplicity is locally constant on* $\operatorname{Spec} R$*.*

We are going to put some effort into proving this theorem, as it will allow us to develop results that will further guide us in our quest in Hilbert-Kunz theory. First of all, we will need Nagata's criterion ([22, 22.B]), a common tool used to prove that something is open.

Proposition 3.1.7. *Let R be a ring. A subset U of* $\operatorname{Spec} R$ *is open if and only if the following two conditions are satisfied:*

1. *U is stable under generalization, i.e. if $\mathfrak{q} \in U$ and $\mathfrak{p} \subseteq \mathfrak{q}$, then $\mathfrak{p} \in U$,*
2. *U contains a nonempty open subset of $V(\mathfrak{p})$ for any $\mathfrak{p} \in U$.*

The first condition is verified via the following result ([26, Theorem 40.1]).

Proposition 3.1.8. *Let $(R, \mathfrak{m})$ be a formally equidimensional local ring and $\mathfrak{p}$ be a prime ideal.*

$$\mathrm{e}(\mathfrak{p}) \leq \mathrm{e}(\mathfrak{m}).$$

Now, we are left to check the second condition. First of all, it is enough to consider only principal open sets. So, we need to show that there exists $s \notin \mathfrak{p}$ such that $D_s \cap V(\mathfrak{p}) \subseteq X_{\leq a}$ whenever $\mathrm{e}(\mathfrak{p}) \leq a$. Second, if $\mathrm{e}(\mathfrak{p}) = a$, the previous proposition shows that the only way to enforce the inclusion above is to force multiplicity to be constant on $D_s \cap V(\mathfrak{p})$.

Thus we just showed that the following assertion is equivalent to Theorem 3.1.6.

Theorem 3.1.9. *Let R be an excellent locally equidimensional ring. Then for any prime ideal $\mathfrak{p}$, there is an element $s \notin \mathfrak{p}$, such that multiplicity is constant on $V(\mathfrak{p}) \cap D_s$, i.e. for every prime ideal $\mathfrak{q} \supseteq \mathfrak{p}$, $\mathrm{e}(\mathfrak{q}) = \mathrm{e}(\mathfrak{p})$ provided that $s \notin \mathfrak{q}$.*

This naturally leads to study prime ideals $\mathfrak{p}$ such that $\mathrm{e}(\mathfrak{p}) = \mathrm{e}(\mathfrak{q})$ for every prime $\mathfrak{q}$ containing $\mathfrak{p}$. We will call this property "equimultiplicity".

3.1.1 Analytic Spread and Equimultiplicity

We will need some results of the theory of integral closure and analytic spread. For an in-depth treatment, a good reference is the monograph devoted to the subject by Swanson and Huneke ([30]).

Definition 3.1.10. Let I be an ideal in ring R. An element $x \in R$ is integral over I if it satisfies an equation of the form

$$x^n + a_1 x^{n-1} + \ldots + a_n = 0$$

where for all i the coefficient a_i belongs to I^i.

The set of all elements of R that are integral over I is called the integral closure of I and is denoted $\overline{I}$. It can be verified that $\overline{I}$ is an ideal that coincides with its integral closure.

Definition 3.1.11. Let $(R, \mathfrak{m})$ be a local ring and let I be an ideal. The analytic spread of I, $\ell(I)$, is the Krull dimension of the ring

$$R[It]/\mathfrak{m}R[It] \cong R/\mathfrak{m} \oplus I/\mathfrak{m}I \oplus I^2/\mathfrak{m}I^2 \oplus \dots.$$

The following result ([30, Proposition 8.3.7] gives us a relation between integral closure and analytic spread.

Proposition 3.1.12. *Let $(R, \mathfrak{m})$ be a local ring with an infinite residue field and let I be an ideal. Then $\ell(I)$ is the minimal number of elements needed to generate I up to integral closure, i.e. the minimum among numbers of generators of ideals J such that $\overline{J} = \overline{I}$.*

If the residue field is finite, then assertion holds for some power of I.

The analytic spread is naturally bounded ([30, Corollary 8.3.9]).

Lemma 3.1.13. *Let R be a ring and I an ideal in R. Then $\operatorname{ht} I \leq \ell(I) \leq \dim R$.*

Remark 3.1.14. There is a powerful connection between Hilbert-Samuel multiplicity and integral closure. In fact, Theorem 2.4.8 was modeled after the corresponding results for integral closure.

Let us sketch an easy application of the theory. Let $(R, \mathfrak{m})$ be a local ring of positive characteristic with an infinite residue field. Then $\ell(\mathfrak{m}) = \dim R$ by the preceeding lemma, so Proposition 3.1.12 asserts that there is a parameter ideal J such that $\overline{J} = \overline{\mathfrak{m}}$. Also, $\overline{\mathfrak{m}} = \mathfrak{m}$ since $\mathfrak{m}$ is maximal. This forces $\mathrm{e}(J) = \mathrm{e}(\mathfrak{m})$, since multiplicity does not change within the integral closure, similarly to Theorem 2.4.8.

On the other hand, since $J \subseteq \mathfrak{m}$, $\mathrm{e_{HK}}(\mathfrak{m}) \leq \mathrm{e_{HK}}(J)$. But since J is a parameter ideal, $\mathrm{e_{HK}}(J) = \mathrm{e}(J)$ by Corollary 2.3.6, so $\mathrm{e_{HK}}(\mathfrak{m}) \leq \mathrm{e}(\mathfrak{m})$.

In fact, we could extend the residue field using a suitable flat extension, so the result holds in full generality.

Definition 3.1.15. Let R be a ring, I an ideal, and $x_1, \ldots, x_r$ elements of R. We say that $x_1, \ldots, x_r$ are a system of parameter modulo I, if the images of $x_1, \ldots, x_r$ in R/I are a system of parameters.

We will need the following result of Rees ([28]) that characterizes analytic spread using integral closures of the consecutive powers.

Lemma 3.1.16. *Let $(R, \mathfrak{m})$ be a formally equidimensional local ring and I be an ideal. Then $\ell(I) \leq \dim R - r$ if and only if there exists a system of parameters $x_1, \ldots, x_r$ modulo I such that for all $0 \leq i < r$ and for all n*

$$x_{i+1} \text{ is not a zero divisor modulo } \overline{(I, x_1, \ldots, x_i)^n} = \overline{(I^n, x_1^n, \ldots, x_i^n)}.$$

Proof. We will use the result of Burch ([4]): if x is regular modulo $\overline{I^n}$ for all n, then $\ell((I, x)) = \ell(I) + 1$.

Suppose $\ell(I) \leq \dim R - r$. We use induction on r. The base case of $r = 0$ is trivial. Now, if $\ell(I) < \dim R$, a result of McAdam and Ratliff ([23], [27], [30, Theorem 5.4.6]) states that $\mathfrak{m}$ is not an associated prime of $\overline{I^n}$ for all n. Since $\bigcup_n \mathrm{Ass}(\overline{I^n})$ is finite (Brodmann, [3]), by prime avoidance, there is an element x regular modulo $\overline{I^n}$ for all n. Now, the assertion follows by induction applied to (I, x).

For the other direction, we use the result of Burch to obtain that $\ell(I) + r = \ell((I, x_1, \ldots, x_r)) \leq \dim R$. ∎

This can be considered as an Auslander-Buchsbaum type formula, where analytic spread plays the role of projective dimension and a regular sequence is considered "up to integral closure". Such a sequence was called an asymptotic prime sequence by Rees.

Theorem 3.1.17. *Let $(R, \mathfrak{m}, k)$ be a formally equidimensional local ring. For an ideal I in R the following conditions are equivalent:*

(a) $\ell(I) = \mathrm{ht}(I)$,

(b) for every (equivalently, some) parameter ideal J modulo I

$$\mathrm{e}(I + J) = \sum_{P \in \mathrm{Minh}(I)} \mathrm{e}(J, R/P)\,\mathrm{e}(I, R_P),$$

(c) if k is infinite, there is a system of parameter $J = (x_1, \ldots, x_r)$ modulo I which is a part of a system of parameter is R, and such that

- *if* $r < \dim R$ *then* $\sum_{P \in \mathrm{Minh}(I)} \mathrm{e}(J, R/P)\,\mathrm{e}(IR_P) = \mathrm{e}(IR/J)$
- *if* $r = \dim R$ *then* $\sum_{P \in \mathrm{Minh}(I)} \mathrm{e}(J, R/P)\,\mathrm{e}(IR_P) = \mathrm{e}(J)$,

(d) for every (equivalently, some) system of parameters $(x_1, \ldots, x_r)$ modulo I, all $0 \le i < r$, and all n

$$x_{i+1} \text{ is regular modulo } \overline{(I, x_1, \ldots, x_i)^n}.$$

Proof. For the equivalence of the first three conditions we refer the reader to Lipman's survey ([21, Theorem 4]).

Now, (d) implies (a) by the lemma above. The converse follows from the proof of Lemma 3.1.16. Namely, observe that $\ell(IR_P) = \mathrm{ht}(IR_P)$ for any prime ideal P

containing I. It follows that $\overline{I^n}$ has no embedded associated primes (this was noted by Ratliff in [27, Corollary 11]), thus we can take any parameter in the proof of Lemma 3.1.16.

□

Remark 3.1.18. It is important to note that the equivalent characterizations represent extremal conditions. For example, (a) asserts that the analytic spread attains its minimal possible value (Lemma 3.1.13). Similarly, via Lemma 3.1.16, r is the maximal length that we could hope for in (d). With a bit more work one can show that, in general, $\mathrm{e}(I+J) \geq \sum_{P\in \mathrm{Minh}(I)} \mathrm{e}(J, R/P)\, \mathrm{e}(I, R_P)$, so (b) is also an extremal condition.

An ideal I satisfying the equivalent conditions of the theorem is called *equimultiple*. This is motivated by the following special case ([21, Corollary, p. 121]).

Corollary 3.1.19. *Let $(R, \mathfrak{m})$ be a formally equidimensional local ring and let $\mathfrak{p}$ be a prime ideal such that $R/\mathfrak{p}$ is a regular. The following are equivalent:*

1. $\mathrm{e}(\mathfrak{p}) = \mathrm{e}(\mathfrak{m})$,

2. $\ell(\mathfrak{p}) = \mathrm{ht}(\mathfrak{p})$,

3. *for every (equivalently, some) parameter ideal J modulo $\mathfrak{p}$*

$$\mathrm{e}(\mathfrak{p} + J) = \mathrm{e}(J, R/\mathfrak{p})\, \mathrm{e}(\mathfrak{p}),$$

4. *for every (equivalently, some) system of parameters $(x_1, \ldots, x_r)$ modulo $\mathfrak{p}$, for all $0 \leq i < r$, and all n*

$$x_{i+1} \textit{ is regular modulo } \overline{(\mathfrak{p}, x_1, \ldots, x_i)^n}.$$

Now, we can sketch the proof of Theorem 3.1.9.

Proof. If $\mathfrak{p}$ is a maximal ideal, there is nothing to prove, so assume that it is proper.

Since $R/\mathfrak{p}$ is an excellent domain, its regular locus is open and non empty, so there is an element $f \notin \mathfrak{p}$ such that $R_f/\mathfrak{p}R_f$ is regular.

Now, by Lemma 3.1.13, $\ell(\mathfrak{p}R_\mathfrak{p}) = \dim R_\mathfrak{p} = \operatorname{ht} \mathfrak{p} := h$. Since $\mathfrak{p}$ is a proper prime ideal, the residue field $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$ is infinite. Thus by Proposition 3.1.12 there are elements $x_1, \ldots, x_h \in R_\mathfrak{p}$ such that $\overline{(x_1, \ldots, x_h)} = \mathfrak{p}R_\mathfrak{p}$. Now, we can collect denominators and invert an element s to guarantee that R_s contains all x_i and all coefficients of some integral dependence equations of a system of generators of $\mathfrak{p}$. Therefore, $\overline{(x_1, \ldots, x_h)} = \mathfrak{p}R_s$, and the claim follows from the characterization in Corollary 3.1.19. □

3.2 Hilbert-Kunz functions

Inspired by the classical results of Hilbert-Samuel multiplicity, we want to understand whether the theory translates to positive characteristic. We hope that the new multiplicity could be useful for resolution of singularities as it captures more information. Let us illustrate it with the following example.

Example 3.2.1. Let us list Hilbert-Kunz multiplicities of the well-known class of Du Val (ADE) singularities. Du Val surfaces have many amazingly different characterizations, for example, they classify rational double points in characteristic zero or $p > 5$. Thus all of them have multiplicity two. However, Watanabe and Yoshida ([33]) obtained the following Hilbert-Kunz multiplicities:

(A_n) $\mathrm{e_{HK}}\,(k[[x,y,z]]/(z^{n+1}+xy)) = 2 - \frac{1}{n+1}$,

(D_n) $\mathrm{e_{HK}}\,(k[[x,y,z]]/(x^2+zy^2+z^{n-1})) = 2 - \frac{1}{4n-8}$ for $p > 2$,

(E_6) $\mathrm{e_{HK}}\,(k[[x,y,z]]/(x^2+y^3+z^4)) = 2 - \frac{1}{24}$ for $p > 3$,

(E_7) $\mathrm{e_{HK}}\,(k[[x,y,z]]/(x^2+y^3+xz^3)) = 2 - \frac{1}{48}$ for $p > 3$,

(E_8) $\mathrm{e_{HK}}\,(k[[x,y,z]]/(x^2+y^3+z^5)) = 2 - \frac{1}{120}$ for $p > 5$.

Another characterization of Du Val singularities comes from invariant theory. Over $k = \mathbb{C}$, the hypersurfaces listed above arise as the completions of invariant rings of $\mathbb{C}[x,y]$ by the action of a finite subgroup $G \subset SL_2(\mathbb{C})$. Then the Hilbert-Kunz multiplicities above are computed by the formula $2 - 1/|G|$, thus providing us with more information than the usual multiplicity.

The first step was made by Kunz, who started to study behavior of a fixed Hilbert-Kunz function $\lambda_q(\mathfrak{p}) = \lambda(R_\mathfrak{p}/\mathfrak{p}^{[q]}R_\mathfrak{p})/q^{\operatorname{ht}\mathfrak{p}}$. In [19, Proposition 3.3] Kunz obtained the following result that verifies the first condition of Nagata's criterion (Proposition 3.1.7). The proof below is due to Huneke and Yao ([16]).

Theorem 3.2.2. *Let $(R,\mathfrak{m})$ be a local ring and $\mathfrak{p}$ be a prime ideal such that $\operatorname{ht}\mathfrak{p} + \dim R/\mathfrak{p} = \dim R$. Then for all q*

$$\lambda_q(\mathfrak{p}) \le \lambda_q(\mathfrak{m}).$$

Proof. By induction, it is enough to consider the case when $\dim R/\mathfrak{p} = 1$.

Let $f \in \mathfrak{m} \setminus \mathfrak{p}$ be arbitrary. First, by Corollary 2.1.10 and the associativity formula (Proposition 2.1.3), for all q we have

$$\lambda(R/((\mathfrak{p},f)^{[q]})) = \lambda(R/(\mathfrak{p}^{[q]},f^q)) \ge \mathrm{e}(f^q, R/\mathfrak{p}^{[q]}) = \mathrm{e}(f^q, R/\mathfrak{p})\,\lambda_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{p}^{[q]}R_\mathfrak{p}).$$

By Lemma 2.1.4 and Corollary 2.1.10,

$$\mathrm{e}(f^q, R/\mathfrak{p}) = q\,\mathrm{e}(f, R/\mathfrak{p}) = q\,\lambda(R/(\mathfrak{p}, f)).$$

So, combining this with the previous formula, we see that

$$\lambda(R/((\mathfrak{p}, f)^{[q]})) \geq q\,\lambda(R/(\mathfrak{p}, f))\,\lambda_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{p}^{[q]}R_\mathfrak{p}).$$

Moreover, by Lemma 2.3.15, $\lambda(R/\mathfrak{m}^{[q]})\,\lambda(R/(\mathfrak{p}, f)) \geq \lambda(R/(\mathfrak{p}, f)^{[q]})$, so combining our findings we obtain that $\lambda(R/\mathfrak{m}^{[q]}) \geq q\,\lambda(R_\mathfrak{p}/\mathfrak{p}^{[q]}R_\mathfrak{p})$. Now the result follows after dividing by $q^{\dim R}$ and noting that $\dim R = \dim R_\mathfrak{p} + 1$ by the assumption. □

By taking limits, we obtain the following corollary.

Corollary 3.2.3. *Let R be an equidimensional catenary ring. Then for all q and for all prime ideals $\mathfrak{p} \subseteq \mathfrak{q}$*

$$\mathrm{e}_{\mathrm{HK}}(\mathfrak{p}) \leq \mathrm{e}_{\mathrm{HK}}(\mathfrak{q}).$$

And the second part of Nagata's criterion can be verified fairly easily. Kunz did it in [19, Corollary 3.4] and Shepherd-Barron ([29]) obtained a different proof that we present below.

Theorem 3.2.4. *If R is an excellent locally equidimensional ring, then for any fixed q the qth Hilbert-Kunz function λ_q is locally constant on* $\operatorname{Spec} R$.

Proof. We use Nagata's criterion. Theorem 3.2.2 verifies the first condition. So, we want to show that for an arbitrary prime ideal $\mathfrak{p}$ there is an element $s \notin \mathfrak{p}$ such that λ_q is constant on $V(\mathfrak{p}) \cap D_s$.

Take an arbitrary prime filtration of $R/\mathfrak{p}^{[q]}$:

$$0 \subset M_1 \subset \ldots \subset M_i \subset \ldots M_N = R/\mathfrak{p}^{[q]},$$

where $M_{i+1}/M_i \cong R/P_i$. By prime avoidance there is an element $s \in \bigcap_{P_i \neq \mathfrak{p}} P_i \setminus \mathfrak{p}$, so the original prime filtration of $R/\mathfrak{p}^{[q]}$ induces a prime filtration of $R_s/\mathfrak{p}^{[q]}R_s$ where $R_s/\mathfrak{p}R_s$ is the only prime factor. Thus the length of this filtration is $\lambda(R_\mathfrak{p}/\mathfrak{p}^{[q]}R_\mathfrak{p})$.

Since $R/\mathfrak{p}$ is an excellent domain, its regular locus is open and non-empty, so there exists an element $f \notin \mathfrak{p}$ such that $R_f/\mathfrak{p}R_f$ is regular. Let $\mathfrak{m} \in V(\mathfrak{p}) \cap D_{sf}$. Then $\mathfrak{m}R_\mathfrak{m} = (\mathfrak{p}, x_1, \ldots, x_r)R_\mathfrak{m}$ for a regular sequence $x_1, \ldots, x_r$ modulo $\mathfrak{p}$. Note that

$$R_\mathfrak{m}/\mathfrak{m}^{[q]}R_\mathfrak{m} = R_\mathfrak{m}/(\mathfrak{p}^{[q]}, x_1^q, \ldots, x_r^q)R_\mathfrak{m} \cong R_\mathfrak{m}/\mathfrak{p}^{[q]}R_\mathfrak{m} \otimes_{R_\mathfrak{m}} R_\mathfrak{m}/(x_1, \ldots, x_r)^{[q]}R_\mathfrak{m},$$

so, tensoring the prime filtration above with $R_\mathfrak{m}/(x_1, \ldots, x_r)^{[q]}$, we obtain

$$\lambda_{R_\mathfrak{m}}(R_\mathfrak{m}/\mathfrak{m}^{[q]}R_\mathfrak{m}) \leq \lambda(R_\mathfrak{p}/\mathfrak{p}^{[q]}R_\mathfrak{p})\, \lambda_{R_\mathfrak{m}}\left(R_\mathfrak{m}/\mathfrak{p}R_\mathfrak{m} \otimes_{R_\mathfrak{m}} R_\mathfrak{m}/(x_1, \ldots, x_r)^{[q]}R_\mathfrak{m}\right).$$

Because $x_1, \ldots, x_r$ is a regular sequence in $R_\mathfrak{m}/\mathfrak{p}R_\mathfrak{m}$, it follows from Lemma 2.1.9 that

$$\lambda_{R_\mathfrak{m}}\left(R_\mathfrak{m}/(\mathfrak{p}, (x_1, \ldots, x_r)^{[q]})\right) = q^{\dim R_\mathfrak{m}/\mathfrak{p}R_\mathfrak{m}}\, \lambda_{R_\mathfrak{m}}\left(R_\mathfrak{m}/(\mathfrak{p}, x_1, \ldots, x_r)\right) = q^{\dim R_\mathfrak{m}/\mathfrak{p}R_\mathfrak{m}}.$$

Thus, the previous estimate shows that $\lambda_\mathrm{q}(\mathfrak{m}) \leq \lambda_\mathrm{q}(\mathfrak{p})$, so, by Theorem 3.2.2, $\lambda_\mathrm{q}(\mathfrak{m}) = \lambda_\mathrm{q}(\mathfrak{p})$. Note that $\operatorname{ht} \mathfrak{p} + \dim R_\mathfrak{m}/\mathfrak{p} = \dim R_\mathfrak{m}$ since R is catenary and locally equidimensional. □

3.3 Equimultiplicity theory for Hilbert-Kunz multiplicity

Motivated by Theorem 3.1.9 we may ask whether Theorem 3.2.4 extends for the limit.

Question 3.3.1. Let R be an excellent locally equidimensional ring. Is the Hilbert-Kunz multiplicity locally constant on $\operatorname{Spec} R$?

As a first step, if we try to follow the treatment of Hilbert-Samuel multiplicity from Section 3.1, we see that the first condition of Nagata's criterion is still satisfied by Corollary 3.2.3. So, we need to translate Theorem 3.1.9 and, again, we need to study equimultiplicity, this time in Hilbert-Kunz theory.

A curious reader may wonder if the preceeding theorem can be used to answer this question. A natural approach would be to hope that we can ever make the total Hilbert-Kunz functions to be equal, but this is hardly possible. The devil hides in the need to invert an element to force constancy of λ_{q} for a fixed q. So we have infinitely many open sets D_{s_q} whose intersection might not be open. This suggest that we need to study the behavior of Hilbert-Kunz functions more deeply.

In fact, it is quite easy to precisely characterize the constancy of a fixed Hilbert-Kunz function. This characterization can be also used to give a different proof of Theorem 3.2.4; this time we use that the Cohen-Macaulay locus is open in excellent rings (Proposition 2.2.13).

Proposition 3.3.2. *Let $(R, \mathfrak{m})$ be a local ring of characteristic $p > 0$ and $\mathfrak{p}$ be a prime ideal of R such that $R/\mathfrak{p}$ is regular and $\operatorname{ht} \mathfrak{p} + \dim R/\mathfrak{p} = \dim R$. Then, for a fixed q, $\lambda_{\mathrm{q}}(\mathfrak{m}) = \lambda_{\mathrm{q}}(\mathfrak{p})$ if and only if $R/\mathfrak{p}^{[q]}$ is Cohen-Macaulay.*

Therefore the (normalized) Hilbert-Kunz functions of $\mathfrak{m}$ and $\mathfrak{p}$ coincide for all q if and only if $R/\mathfrak{p}^{[q]}$ are Cohen-Macaulay for all q.

Proof. Let $x_1, \dots, x_m$ be a minimal system of generators of $\mathfrak{m}$ modulo $\mathfrak{p}$. By the associativity formula, Corollary 2.3.6, and Lemma 2.3.8

$$\mathrm{e}((x_1,\dots,x_m)^{[q]}, R/\mathfrak{p}^{[q]}) = \mathrm{e}((x_1,\dots,x_m)^{[q]}, R/\mathfrak{p})\,\lambda_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{p}^{[q]}R_\mathfrak{p}) = q^m\,\lambda_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{p}^{[q]}R_\mathfrak{p}).$$

Since $q^m\,\lambda_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{p}^{[q]}R_\mathfrak{p}) = q^{\dim R}\,\lambda_\mathrm{q}(\mathfrak{p})$, we obtain by Proposition 2.1.12 that

$$q^{\dim R}\,\lambda_\mathrm{q}(\mathfrak{m}) = \lambda(R/(\mathfrak{p}^{[q]}, x_1^q,\dots,x_m^q)) \geq \mathrm{e}((x_1,\dots,x_m)^{[q]}, R/\mathfrak{p}^{[q]}) = q^{\dim R}\,\lambda_\mathrm{q}(\mathfrak{p}).$$

Thus, $\lambda_\mathrm{q}(\mathfrak{m}) = \lambda_\mathrm{q}(\mathfrak{p})$ if and only if $\lambda(R/(\mathfrak{p}^{[q]}, x_1^q,\dots,x_m^q)) = \mathrm{e}((x_1,\dots,x_m)^{[q]}, R/\mathfrak{p}^{[q]})$. However, by Proposition 2.1.12, the latter holds if and only if $R/\mathfrak{p}^{[q]}$ is Cohen-Macaulay. □

While the characterization is simple and natural, it is not clear whether it could be used for our modest needs, i.e. a generalization of the proof of Theorem 3.1.9. We do not see a way to force this condition by inverting an element, since, again, we need to intersect infinitely many open sets.

This is why we need to go further and try to characterize equality of Hilbert-Kunz multiplicities. It will be much harder to achieve, and the next chapter is devoted to this problem. This characterization will be of similar spirit to Proposition 3.3.2, and, having a better control over Hilbert-Kunz multiplicity, we will learn that, in fact, the conditions of Proposition 3.3.2 cannot be forced for all q by inverting a single element.

Chapter 4

Equimultiplicity for Hilbert-Kunz multiplicity

In this chapter we will derive a Hilbert-Kunz analogue of Theorem 3.1.17. As a consequence, we are able to characterize a prime ideal $\mathfrak{p}$ in a local ring $(R, \mathfrak{m})$ such that $R/\mathfrak{p}$ is regular and $\mathrm{e}_{\mathrm{HK}}(\mathfrak{m}) = \mathrm{e}_{\mathrm{HK}}(\mathfrak{p})$ similarly to Corollary 3.1.19. Later, we will use our findings to obtain a negative answer to Question 3.3.1.

First, we need a uniform convergence result to derive a critical formula. We closely follow the uniform convergence method of Kevin Tucker ([31]), tayloring the estimates for our purpose.

4.1 A uniform convergence result

4.1.1 Overview of the proof

Before proceeding to technicalities, let us sketch the ideas of the proof. Over a local ring $(R, \mathfrak{m})$ we are going to prove uniform convergence (with respect to q) of the bisequence

$$\frac{\lambda(M/(I^{[q]} + J^{[qq']})M)}{q^{\dim R} q'^{\dim R/I}}$$

where I is an ideal, M is a finitely generated module, and J is an $\mathfrak{m}$-primary ideal. Observe that for $I = 0$ and $q' = 1$ we recover the usual Hilbert-Kunz function. After establishing uniform convergence, we use it to interchange the limits (with respect to q and q') of the bisequence.

Uniform convergence will be established by showing that the sequence is Cauchy with an appropriate estimate; to do the bookkeeping of estimates we follow Tucker's treatment in [31]. Tucker's proof can be viewed as a careful adaptation of the original proof of existence of Hilbert-Kunz multiplicity by Monsky ([24]).

With the exception of minor tweaks and variations, Monsky's approach is the only known proof of the existence of Hilbert-Kunz multiplicity. In the proof he uses the idea of Proposition 2.2.5 to compare

$$\lambda(M/(I^{[q]} + J^{[qq']})M) \text{ and } \lambda(R/(I^{[q]} + J^{[qq']}) \otimes_R F_*M)$$

instead of directly comparing two consecutive terms of the bisequence. This requires R to be F-finite, but we can reduce to this situation via Remark 2.2.14.

4.1.2 A variation of Tucker's result

We start with an upper bound for a function that we study.

Lemma 4.1.1. *Let $(R, \mathfrak{m})$ be a local ring and I be an ideal. Then there exists a constant C such that*

$$\lambda(R/(I^{[q]} + \mathfrak{m}^{[qq']})) \leq C{q'}^{\dim R/I} q^{\dim R}$$

for all q, q'.

Proof. Let $x_1, \dots, x_h$ be elements of R such that their images form a system of parameters in R/I. Then

$$\lambda(R/(I^{[q]} + \mathfrak{m}^{[qq']})) \leq \lambda(R/(I^{[q]} + (x_1, \dots, x_h)^{[qq']})) = \lambda(R/(I^{[q]} + (x_1^{qq'}, \dots, x_h^{qq'}))).$$

Filtering by the powers of x_i^q (i.e. repeatedly applying Lemma 2.1.9), we get that

$$\lambda(R/(I^{[q]} + (x_1^{qq'}, \dots, x_h^{qq'}))) \leq (q')^h \, \lambda(R/(I, x_1, \dots, x_h)^{[q]}).$$

Since $(I, x_1, \dots, x_h)$ is an $\mathfrak{m}$-primary ideal, it contains a system of parameters, say, $y_1, \dots, y_d$. Then, filtering by the powers of y_i, we obtain that

$$\lambda(R/(I, x_1, \dots, x_h)^{[q]}) \leq \lambda(R/(y_1, \dots, y_d)^{[q]}) \leq q^d \, \lambda(R/(y_1, \dots, y_d)).$$

Last, let $C = \lambda(R/(y_1, \dots, y_d))$ and observe that $d = \dim R$ and $h = \dim R/I$. □

Corollary 4.1.2. *Let $(R, \mathfrak{m})$ be a local ring, let J be an $\mathfrak{m}$-primary ideal, and I be an arbitrary ideal. If M is a finitely generated R-module, then there exists a constant D (independent of q') such that for all q, q'*

$$\lambda(M/(I^{[q]} + J^{[qq']})M) \leq Dq'^{\dim R/I} q^{\dim M}.$$

Proof. Since J is $\mathfrak{m}$-primary, $\mathfrak{m}^{[q_0]} \subseteq \mathfrak{m}$ for some q_0, thus if the result holds for $J = \mathfrak{m}$

$$\lambda(M/(I^{[q]} + J^{[qq']})M) \leq \lambda(M/(I^{[q]} + \mathfrak{m}^{[qq'q_0]})M) \leq\leq (Dq_0^{\dim M})q'^{\dim R/I} q^{\dim M}.$$

Therefore we assume that $J = \mathfrak{m}$.

Let K be the annihilator of M and let n be the minimal number of generators of M. Then there exists a surjection $(R/K)^n \to M \to 0$, so, after tensoring with

$R/(I^{[q]}+\mathfrak{m}^{[qq']})$, we obtain from Lemma 4.1.1 the estimate

$$\lambda(M/(I^{[q]}+\mathfrak{m}^{[qq']})M) \le n\,\lambda(R/(K+I^{[q]}+\mathfrak{m}^{[qq']})) \le nC{q'}^{\dim R/(I+K)}q^{\dim(R/K)}$$
$$\le nC{q'}^{\dim R/I}q^{\dim M}.$$

□

Remark 4.1.3. Suppose for two R-modules, M and N, $M_P \cong N_P$ for any minimal prime $P \in \mathrm{Minh}(R)$. If R is reduced, we want to observe that $S^{-1}M \cong S^{-1}N$ where S is the complement to the union of all primes in $\mathrm{Minh}(R)$. This follows from the isomorphism $S^{-1}R \cong \prod_{P\in \mathrm{Minh}(R)} R_P$.

Lemma 4.1.4. *Let $(R,\mathfrak{m})$ be a local ring, let J be an $\mathfrak{m}$-primary ideal, and I be an arbitrary ideal. Let M,N be finitely generated R-modules. Moreover, suppose $M_P \cong N_P$ for any minimal prime P such that $\dim R/P = \dim R$. Then there exists a constant C independent of q' and such that for all q,q'*

$$|\,\lambda(M/(I^{[q]}+J^{[qq']})M) - \lambda(N/(I^{[q]}+J^{[qq']})N)| < C{q'}^{\dim R/I}q^{\dim R-1}.$$

Proof. Let $S = R \setminus \cup_{\mathrm{Minh}(R)}P$. By the previous remark, $S^{-1}M = S^{-1}N$, so, since $S^{-1}\operatorname{Hom}_R(M,N) = \operatorname{Hom}_{S^{-1}R}(S^{-1}M, S^{-1}N)$, there exist homomorphisms $M \to N$ and $N \to M$ that become isomorphisms after localization by S. Thus we have exact sequences

$$M \to N \to K_1 \to 0,$$

$$N \to M \to K_2 \to 0$$

where $\dim K_1, \dim K_2 < \dim R$, since $S^{-1}K_1 = S^{-1}K_2 = 0$.

Tensoring the first exact sequence with $R/(I^{[q]}+J^{[qq']})$ and taking length, we obtain that

$$\lambda(N/(I^{[q]}+J^{[qq']})N) \le \lambda(M/(I^{[q]}+J^{[qq']})M) + \lambda(K_1/(I^{[q]}+J^{[qq']})K_1),$$

while the second sequence yields

$$\lambda(M/(I^{[q]}+J^{[qq']})M) \le \lambda(N/(I^{[q]}+J^{[qq']})N) + \lambda(K_2/(I^{[q]}+J^{[qq']})K_2).$$

By Corollary 4.1.2, there are constants C_1 and C_2 such that $\lambda(K_1/(I^{[q]}+J^{[qq']})K_1) \le C_1 {q'}^{\dim R/I} q^{\dim R-1}$ and $\lambda(K_2/(I^{[q]}+J^{[qq']})K_2) \le C_2 {q'}^{\dim R/I} q^{\dim R-1}$. Combining the estimates together, we derive that

$$|\,\lambda(N/(I^{[q]}+J^{[qq']})N) - \lambda(M/(I^{[q]}+J^{[qq']})M)| \le \max(C_1, C_2) {q'}^{\dim R/I} q^{\dim R-1}.$$

□

Remark 4.1.5. Provided that Hilbert-Kunz multiplicity exists, Lemma 4.1.4 shows that if M and N are two modules and $M_P \cong N_P$ for any minimal prime P such that $\dim R/P = \dim R$, then $\mathrm{e}_{\mathrm{HK}}(J, M) = \mathrm{e}_{\mathrm{HK}}(J, N)$ for any $\mathfrak{m}$-primary ideal J.

Theorem 4.1.6. *Let $(R, \mathfrak{m})$ be a reduced F-finite local ring of dimension d, J an $\mathfrak{m}$-primary ideal, and I be an ideal. Then for any finitely generated R-module M there exists a constant C such that*

$$|\,\lambda(M/(I^{[q]}+J^{[qq']})M) - q^d\, \mathrm{e}_{\mathrm{HK}}(I+J^{[q']}, M)| < Cq^{d-1}{q'}^{\dim R/I}$$

for all q, q'. In particular, the bisequence

$$\frac{\lambda(M/(I^{[q]}+J^{[qq']})M)}{q^{\dim R}{q'}^{\dim R/I}}$$

converges uniformly with respect to q.

Proof. By Corollary 2.2.10, $(F_*M)_P$ and $M_P^{\oplus p^{\alpha(R)+d}}$ are isomorphic for any minimal prime $P \in \mathrm{Minh}(R)$. Thus, we can apply Lemma 4.1.4 to $M^{\oplus p^{\alpha(R)+d}}$ and F_*M and get that

$$|p^{\alpha(R)+d}\,\lambda(M/(I^{[q]}+J^{[qq']})M)-\lambda(F_*M/(I^{[q]}+J^{[qq']})F_*M)|<Cq'^{\dim R/I}q^{d-1}$$

for any q, q' and a constant C depending only on M and I. By Proposition 2.2.7,

$$\lambda(F_*M/(I^{[q]}+J^{[qq']})F_*M)=p^{\alpha(R)}\,\lambda(M/(I^{[qp]}+J^{[qpq']})M).$$

Therefore, using that $p^{-\alpha(R)}C \leq C$,

$$|p^{d}\,\lambda(M/(I^{[q]}+J^{[qq']})M)-\lambda(M/(I^{[qp]}+J^{[qpq']})M)|<Cq'^{\dim R/I}q^{d-1}. \tag{4.1.1}$$

Now, we prove by induction on q_1 that for all q, q'

$$|(q_1)^{d}\,\lambda(M/(I^{[q]}+J^{[qq']})M)-\lambda(M/(I^{[qq_1]}+J^{[qq_1q']})M)|<Cq'^{\dim R/I}(qq_1/p)^{d-1}\frac{q_1-1}{p-1}. \tag{4.1.2}$$

The induction base of $q_1 = p$ is (4.1.1). Now, assume that the claim holds for q_1 and we want to prove it for q_1p.

First, (4.1.1) applied to qq_1 gives

$$|p^{d}\,\lambda(M/(I^{[qq_1]}+J^{[qq_1q']})M)-\lambda(M/(I^{[qq_1p]}+J^{[qq_1pq']})M)|<Cq'^{\dim R/I}(qq_1)^{d-1}, \tag{4.1.3}$$

and, multiplying the induction hypothesis by p^d, we get

$$|(q_1p)^{d}\,\lambda(M/(I^{[q]}+J^{[qq']})M)-p^{d}\,\lambda(M/(I^{[qq_1]}+J^{[qq_1q']})M)|<Cq'^{\dim R/I}(qq_1)^{d-1}\frac{q_1p-p}{p-1}. \tag{4.1.4}$$

Using that $p^d\,\lambda(M/(I^{[qq_1]}+J^{[qq_1q']})M)$ appears in both (4.1.3) and (4.1.4), one obtains

$$|(q_1p)^d\,\lambda(M/(I^{[q]}+J^{[qq']})M)-\lambda(M/(I^{[qq_1p]}+J^{[qq_1pq']})M)|<$$
$$<Cq'^{\dim R/I}(qq_1)^{d-1}\left(\frac{q_1p-p}{p-1}+1\right)=Cq'^{\dim R/I}(qq_1)^{d-1}\left(\frac{q_1p-1}{p-1}\right),$$

and the induction step follows.

Now, dividing (4.1.2) by q_1^d, we obtain

$$|\,\lambda(M/(I^{[q]}+J^{[qq']})M)-\frac{1}{q_1^d}\,\lambda(M/(I^{[qq_1]}+J^{[qq_1q']})M)|<Cq^{d-1}\cdot\frac{q_1-1}{p-1}\cdot\frac{1}{q_1p^{d-1}}\leq Cq^{d-1}.$$

Thus, if we let $q_1\to\infty$ and note that $\mathrm{e}_{\mathrm{HK}}(I^{[q]}+J^{[qq']},M)=q^d\,\mathrm{e}_{\mathrm{HK}}(I+J^{[q']})$, we get that

$$|\,\lambda(M/(I^{[q]}+J^{[qq']})M)-q^d\,\mathrm{e}_{\mathrm{HK}}(I+J^{[q']},M)|<Cq^{d-1},$$

and the claim follows. □

Remark 4.1.7. The equation

$$|\,\lambda(M/(I^{[q]}+J^{[qq']})M)-\frac{1}{q_1^d}\,\lambda(M/(I^{[qq_1]}+J^{[qq_1q']})M)|<Cq^{d-1}$$

yields (for $I=0$ and $q'=1$) that the sequence $\lambda(M/J^{[q]})M)/q^d$ is Cauchy. This shows the limit in Definition 2.3.1 exists if R is F-finite and the next corollary establishes it in full generality.

Corollary 4.1.8. *Let $(R,\mathfrak{m})$ be a local ring of dimension d, J an $\mathfrak{m}$-primary ideal, and I be an aribtrary ideal. Then there is a q_0 such that for any finitely generated R-module M there exists a constant C such that*

$$|\,\lambda(M/(I^{[q]}+J^{[qq']})M)-q^d\,\mathrm{e}_{\mathrm{HK}}(I+J^{[q']},M)|<Cq^{d-1}q'^{\dim R/I}$$

for all q' and all $q \geq q_0$. In particular, the bisequence

$$\frac{\lambda(M/(I^{[q]} + J^{[qq']})M)}{q^{\dim R} q'^{\dim R/I}}$$

converges uniformly with respect to q.

Proof. First, we reduce to the case where R is F-finite. Using the recipe in Remark 2.2.14, we can find a faithfully flat F-finite extension S of R such that $\mathfrak{m}S$ is the maximal ideal of S. Hence, for any Artinian R-module A, $\lambda_S(A \otimes_R S) = \lambda_R(A)$.

Now, there is $q_0 = p^{e_0}$ such that $(\sqrt{0S})^{q_0} = 0$. Naturally, $N = F_*^{e_0}(S \otimes_R M)$ is a S_{red}-module, where $S_{red} = S/\sqrt{0S}$. Since S_{red} is reduced and F-finite, we can apply Theorem 4.1.6 and find a constant C such that

$$|\, \lambda_S(S_{red}/(I^{[q]} + J^{[qq']})S_{red} \otimes_{S_{red}} N) - q^d \operatorname{e_{HK}}((I + J^{[q']})S_{red}, N)| < Cq^{d-1} \qquad (4.1.5)$$

for all q, q'.

Now, observe that

$$\frac{S_{red}}{(I^{[q]} + J^{[qq']})S_{red}} \otimes_{S_{red}} N \cong \frac{S}{(I^{[q]} + J^{[qq']})S} \otimes_S N \cong F_*^{e_0}\left(\frac{S \otimes_R M}{(I^{[qq_0]} + J^{[qq_0q']})(S \otimes_R M)}\right)$$
$$\cong F_*^{e_0}\left(\frac{M}{(I^{[qq_0]} + J^{[qq'q_0]})M} \otimes_R S\right).$$

So,

$$\lambda_S\left(\frac{N}{(I^{[q]} + J^{[qq']})N}\right) = \lambda_S\left(\frac{M}{(I^{[qq_0]} + J^{[qq'q_0]})M} \otimes_R S\right) = \lambda_R(M/(I^{[qq_0]} + J^{[qq'q_0]})M).$$

Therefore, by definition,

$$\operatorname{e_{HK}}((I + J^{[q']})S_{red}, N) = \operatorname{e_{HK}}((I + J^{[q']})^{[q_0]}, M) = q_0^d \operatorname{e_{HK}}(I + J^{[q']}, M)$$

and we can rewrite (4.1.5) as

$$|\, \lambda_R(M/(I^{[qq_0]} + J^{[qq_0q']})M) - (qq_0)^d \operatorname{e_{HK}}(I + J^{[q']}, M)| < Cq^{d-1} \leq C(qq_0)^{d-1}.$$

By setting $q = qq_0$, we get that for all $q \geq q_0$ and all q'

$$| \lambda_R(M/(I^{[q]} + J^{[qq']})M) - q^d \operatorname{e}_{\mathrm{HK}}(I + J^{[q']}, M)| < Cq^{d-1}.$$

□

Now, we can establish the main result of this section, which will be the basic tool of our theory of equimultiplicity.

Corollary 4.1.9. *Let* $(R, \mathfrak{m})$ *be a local ring, and* J *be an* $\mathfrak{m}$*-primary ideal. If* I *is an ideal such that* $\dim R/I + \operatorname{ht} I = \dim R$*, then*

$$\lim_{q' \to \infty} \operatorname{e}_{\mathrm{HK}}(I + J^{[q']}, M) = \sum_{P \in \operatorname{Minh}(I)} \operatorname{e}_{\mathrm{HK}}(JR/P, R/P) \operatorname{e}_{\mathrm{HK}}(IR_P, M_P).$$

Proof. We have proved that the double sequence

$$\frac{\lambda(M/(I^{[q]} + J^{[qq']})M)}{q^{\dim R} q'^{\dim R/I}}$$

converges uniformly with respect to q. Moreover, the limit with respect to q' exists for any q since

$$\lim_{q' \to \infty} \frac{\lambda(M/(I^{[q]} + J^{[qq']})M)}{q^{\dim M} q'^{\dim R/I}} = \frac{\operatorname{e}_{\mathrm{HK}}(J^{[q]}R/I^{[q]}, M/I^{[q]}M)}{q^{\dim R}},$$

where $\operatorname{e}_{\mathrm{HK}}(J^{[q]}R/I^{[q]}, M/I^{[q]}M)$ is taken over the ring $R/I^{[q]}$. Thus, we the iterated limits of the double sequence are equal, i.e.

$$\begin{aligned} \lim_{q' \to \infty} \frac{\operatorname{e}_{\mathrm{HK}}(I + J^{[q']}, M)}{q'^{\dim R/I}} &= \lim_{q' \to \infty} \lim_{q \to \infty} \frac{\lambda(M/(I^{[q]} + J^{[qq']})M)}{q^{\dim R} q'^{\dim R/I}} \\ &= \lim_{q \to \infty} \lim_{q' \to \infty} \frac{\lambda(M/(I^{[q]} + J^{[qq']})M)}{q^{\dim R} q'^{\dim R/I}} = \lim_{q \to \infty} \frac{\operatorname{e}_{\mathrm{HK}}(J^{[q]}R/I^{[q]}, M/I^{[q]}M)}{q^{\dim R}}. \end{aligned}$$

By Lemma 2.3.8, $\operatorname{e}_{\mathrm{HK}}(J^{[q]}R/I^{[q]}, M/I^{[q]}M) = q^{\dim R/I} \operatorname{e}_{\mathrm{HK}}(JR/I^{[q]}, M/I^{[q]}M)$. Note that $\sqrt{I} = \sqrt{I^{[q]}}$, so $\dim R/I = \dim R/I^{[q]}$ and $\operatorname{Minh}(I) = \operatorname{Minh}(I^{[q]})$. Moreover, by the associativity formula,

$$\mathrm{e}_{\mathrm{HK}}(JR/I^{[q]}, M/I^{[q]}M) = \sum_{P\in \mathrm{Minh}(I)} \mathrm{e}_{\mathrm{HK}}(JR/P, R/P)\, \lambda_{R_P}(M_P/I^{[q]}M_P).$$

Hence

$$\begin{aligned}\lim_{q\to\infty} \frac{\mathrm{e}_{\mathrm{HK}}(J^{[q]}R/I^{[q]}, M/I^{[q]}M)}{q^{\dim R}} &= \lim_{q\to\infty} \frac{\mathrm{e}_{\mathrm{HK}}(JR/I^{[q]}, M/I^{[q]}M)}{q^{\operatorname{ht} I}} \\ &= \sum_{P\in \mathrm{Minh}(I)} \mathrm{e}_{\mathrm{HK}}(JR/P, R/P) \lim_{q\to\infty} \frac{\lambda_{R_P}(M_P/I^{[q]}M_P)}{q^{\operatorname{ht} I}}\end{aligned}$$

and the claim follows, since $\operatorname{ht} I = \operatorname{ht} P$.

□

Corollary 4.1.10. *Let $(R, \mathfrak{m})$ be a local ring, J an $\mathfrak{m}$-primary ideal, and $\mathfrak{p}$ be a prime ideal such that* $\dim R/\mathfrak{p} + \operatorname{ht}\mathfrak{p} = \dim R$. *Then*

$$\lim_{q\to\infty} \frac{\mathrm{e}_{\mathrm{HK}}(\mathfrak{p} + J^{[q]})}{q^{\dim R/\mathfrak{p}}} = \mathrm{e}_{\mathrm{HK}}(JR/\mathfrak{p}, R/\mathfrak{p})\, \mathrm{e}_{\mathrm{HK}}(R_{\mathfrak{p}}).$$

When $R/\mathfrak{p}$ is regular, this corollary will help us to connect $\mathrm{e}_{\mathrm{HK}}(\mathfrak{m})$ to $\mathrm{e}_{\mathrm{HK}}(\mathfrak{p})$.

4.2 Equimultiplicity for ideals of dimension one

We will start developing the theory in the easiest case. Some of the results obtain here will be used later; moreover, it will help us to highlight some connections.

Lemma 4.2.1. *Let $(R, \mathfrak{m})$ be a local ring, I be an ideal, and x be a parameter modulo I. Suppose* $\dim R/I = \dim R - 1$. *Then*

$$\frac{1}{n}\, \lambda(R/(I, x^n)) \geq \frac{1}{n+1}\, \lambda(R/(I, x^{n+1})).$$

Proof. Observe that

$$\frac{I+(x^k)}{I+(x^{k+1})} \cong \frac{(x^k)}{(x^k)\cap I+(x^{k+1})} \cong \frac{R}{I:x^k+(x)}.$$

Thus we get the formula

$$\lambda\left(\frac{R}{(I,x^{k+1})}\right) = \lambda\left(\frac{R}{(I,x^k)}\right) + \lambda\left(\frac{R}{I:x^k+(x)}\right).$$

First of all, setting $n = k$ in the formula, we see that it is enough to show that $\lambda(R/(I,x^n)) \geq n\,\lambda\left(R/(I:x^n+(x))\right)$. Second, using the formula above for consecutive values of k, we obtain that

$$\lambda\left(R/(I,x^n)\right) = \lambda\left(R/(I,x)\right) + \sum_{k=1}^{n-1} \lambda\left(R/(I:x^k+(x))\right) \geq n\,\lambda\left(R/(I:x^n+(x))\right),$$

where the last inequality holds since $I : x^k \subseteq I : x^n$ for all $0 \leq k \leq n$. □

Corollary 4.2.2. *In the setting of the lemma, we have*

$$\frac{1}{n}\,\mathrm{e}_{\mathrm{HK}}(I+(x^n)) \geq \frac{1}{n+1}\,\mathrm{e}_{\mathrm{HK}}(I+(x^{n+1})).$$

Proof. Apply the lemma to $I^{[q]}$ and x^q and take the limit as $q \to \infty$. □

Recall that a discrete valuation ring (DVR) is a regular local ring of dimension one. A DVR is a principal ideal domain.

Corollary 4.2.3. *Let $(R,\mathfrak{m})$ to be a local ring of characteristic $p > 0$. Let $\mathfrak{p}$ be a prime ideal in R such that $R/\mathfrak{p}$ is a DVR and* $\operatorname{ht}\mathfrak{p} = \dim R - 1$. *If x is a parameter modulo $\mathfrak{p}$ then the sequence*

$$\frac{1}{n}\,\mathrm{e}_{\mathrm{HK}}(\mathfrak{p}+(x^n))$$

monotonically decreases to its limit $\lambda(R/(\mathfrak{p},x))\,\mathrm{e}_{\mathrm{HK}}(\mathfrak{p})$.

Proof. By Corollary 4.2.2, the sequence of positive reals $\frac{1}{n}\operatorname{e_{HK}}(\mathfrak{p}+(x^n))$ is decreasing, hence converges. Moreover, by Corollary 4.1.10,

$$\lim_{q'\to\infty}\frac{\operatorname{e_{HK}}(\mathfrak{p}+(x^{q'}))}{q'} = \operatorname{e_{HK}}(x, R/\mathfrak{p})\operatorname{e_{HK}}(\mathfrak{p}) = \lambda(R/(\mathfrak{p}, x))\operatorname{e_{HK}}(\mathfrak{p}),$$

where the last equality holds since $R/\mathfrak{p}$ is regular. □

Corollary 4.2.4. *Let $(R, \mathfrak{m})$ to be a local ring of characteristic $p > 0$. Let $\mathfrak{p}$ be a prime ideal in R such that $R/\mathfrak{p}$ is a DVR and $\operatorname{ht}\mathfrak{p} = \dim R - 1$. Then the following are equivalent:*

1. $\operatorname{e_{HK}}(\mathfrak{m}) = \operatorname{e_{HK}}(\mathfrak{p})$,
2. $\operatorname{e_{HK}}(\mathfrak{p} + (x^n)) = n\operatorname{e_{HK}}(\mathfrak{m})$ *for all n and for all (equivalently, some) minimal generators x of $\mathfrak{m}/\mathfrak{p}$,*
3. $\operatorname{e_{HK}}(\mathfrak{p} + (y)) = \lambda(R/(\mathfrak{p}, y))\operatorname{e_{HK}}(\mathfrak{m})$ *for any (equivalently, some) element $y \notin \mathfrak{p}$.*

Proof. First, since $R/\mathfrak{p}$ is a DVR, any parameter is a product of an invertible element and a power of a minimal generator, so the last two claims are equivalent.

The previous corollary shows that the sequence $\operatorname{e_{HK}}(\mathfrak{p} + (x^n))/n$ is monotonically decreasing to its limit $\operatorname{e_{HK}}(\mathfrak{p})$. Since $\mathfrak{m} = \mathfrak{p} + (x)$, the equality between the first term and the limit holds if and only if the sequence is constant. □

Now, we summarize our results in a criterion for equimultiplicity similar to Corollary 3.1.19. Recall that a formally unmixed ($\operatorname{Ass}(\widehat{R}) = \operatorname{Minh}(\widehat{R})$) local ring is formally equidimensional ($\operatorname{Min}(\widehat{R}) = \operatorname{Minh}(\widehat{R})$), thus for any prime ideal $\mathfrak{p}$ in R we must have $\operatorname{ht}\mathfrak{p} + \dim R/\mathfrak{p} = \dim R$ (for example, see [30, Lemma B.4.2]).

Theorem 4.2.5. *Let $(R, \mathfrak{m})$ be a formally unmixed local ring of characteristic $p > 0$ and $\mathfrak{p}$ be a prime ideal such that $R/\mathfrak{p}$ is a DVR. Furhermore, suppose that R has a test element. Let $L_q = \mathfrak{p}^{[q]} : \mathfrak{m}^\infty$ to denote the saturation of $\mathfrak{p}^{[q]}$.*

Then the following are equivalent:

(a) $\mathrm{e}_{\mathrm{HK}}(\mathfrak{m}) = \mathrm{e}_{\mathrm{HK}}(\mathfrak{p})$,

(b) for any (equivalently, some) element $x \notin \mathfrak{p}$

$$\mathrm{e}_{\mathrm{HK}}(\mathfrak{p} + (x)) = \lambda(R/(\mathfrak{p}, x))\, \mathrm{e}_{\mathrm{HK}}(\mathfrak{m}),$$

(c) For all n and some (all) $x \notin \mathfrak{p}$,

$$\lim_{q \to \infty} \frac{1}{q^d} \lambda(R/(\mathfrak{p}^{[q]} : x^{nq} + (x^q))) = \lambda(R/(\mathfrak{p}, x))\, \mathrm{e}_{\mathrm{HK}}(\mathfrak{m}),$$

(d) For some (all) $x \notin \mathfrak{p}$, $\lim_{q \to \infty} \frac{1}{q^d} \lambda(R/(L_q + (x^q))) = \lambda(R/(\mathfrak{p}, x))\, \mathrm{e}_{\mathrm{HK}}(\mathfrak{m})$,

(e) $(\mathfrak{p}^{[q]})^*$ is $\mathfrak{p}$-primary for any q.

Proof. The first two conditions are equivalent by Corollary 4.2.4. Since

$$\lambda(R/(\mathfrak{p}^{[q]}, x^{(n+1)q})) = \lambda(R/(\mathfrak{p}^{[q]}, x^{nq})) + \lambda(R/(\mathfrak{p}^{[q]} : x^{nq} + (x^q)))$$

we get that (b) and (c) are equivalent.

On the other hand, since x^q is not a zero-divisor modulo L_q, Corollary 2.1.10 gives that

$$\lambda(R/(L_q, x^q)) = \mathrm{e}(x^q, R/L_q) = q\, \mathrm{e}(x, R/\mathfrak{p})\, \lambda_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{p}^{[q]}).$$

So, the limit exists and is equal $\lambda(R/(\mathfrak{p}, x))\, \mathrm{e}_{\mathrm{HK}}(\mathfrak{p})$. Therefore, (d) and (a) are equivalent too.

By Lemma 2.4.12, (d) implies that $L_q \subseteq (\mathfrak{p}^{[q]}, x)^*$ for any parameter x, hence $L_q \subseteq (\mathfrak{p}^{[q]})^*$ by Lemma 2.4.6. Moreover, since $\dim R/\mathfrak{p} = 1$, L_q is also the $\mathfrak{p}$-primary part of $\mathfrak{p}^{[q]}$, so Corollary 2.4.14 shows that (d) implies the last condition. And the converse holds by Lemma 2.4.10. □

Remark 4.2.6. We want to point out a direct analogy between Proposition 3.3.2 and equivalence of (a) and (e) in the theorem.

Now, let us recover condition (c) of Theorem 3.1.17.

Proposition 4.2.7. *Let $(R, \mathfrak{m})$ be a formally unmixed local ring with a test element c. Let $\mathfrak{p}$ be a prime ideal of R such that $R/\mathfrak{p}$ is a regular ring. Then the following are equivalent*

1. $\mathrm{e}_{\mathrm{HK}}(\mathfrak{p}) = \mathrm{e}_{\mathrm{HK}}(R/(x))$ *for some minimal generator x of $\mathfrak{m}$ modulo $\mathfrak{p}$,*
2. $\mathrm{e}_{\mathrm{HK}}(\mathfrak{p}) = \mathrm{e}_{\mathrm{HK}}(R/(y))$ *for some parameter y of $\mathfrak{m}$ modulo $\mathfrak{p}$,*
3. $\mathrm{e}_{\mathrm{HK}}(\mathfrak{p}) = \mathrm{e}_{\mathrm{HK}}(\mathfrak{m})$.

Proof. (1) ⇒ (2) is obvious. By Proposition 2.3.13, $\mathrm{e}_{\mathrm{HK}}(R) \leq \mathrm{e}_{\mathrm{HK}}(R/(y))$ for any parameter y in R. Thus, we always have inequalities $\mathrm{e}_{\mathrm{HK}}(\mathfrak{p}) \leq \mathrm{e}_{\mathrm{HK}}(\mathfrak{m}) \leq \mathrm{e}_{\mathrm{HK}}(R/(y))$, and (2) ⇒ (3) follows.

Now, we proceed to the last implication. First, suppose that $\dim R/\mathfrak{p} = 1$. If $\dim R = 1$, $\mathfrak{p}$ is a minimal prime. By the associativity formula, using Lemma 2.3.5 we observe that

$$\mathrm{e}_{\mathrm{HK}}(\mathfrak{m}) = \sum_{P \in \mathrm{Min}(R)} \mathrm{e}_{\mathrm{HK}}(\mathfrak{m}, R/P)\, \lambda(R_P) \geq \lambda(R_{\mathfrak{p}}) = \mathrm{e}_{\mathrm{HK}}(\mathfrak{p}).$$

But $e_{HK}(\mathfrak{p}) = e_{HK}(\mathfrak{m})$, so $\mathfrak{p}$ is the unique minimal prime. Take a minimal generator x of $\mathfrak{m}$ modulo $\mathfrak{p}$. Since R is unmixed, x is not a zero divisor. Then, by the associativity formula and Corollary 2.1.10, we derive that

$$e_{HK}(R/(x)) = \lambda(R/(x)) = e(x, R) = e(x, R/\mathfrak{p})\, \lambda(R_{\mathfrak{p}}) = e_{HK}(\mathfrak{p}).$$

Now, suppose that $\dim R \geq 2$. Then, by prime avoidance, there exists a minimal generator x of $\mathfrak{m}$ modulo $\mathfrak{p}$ such that x does not belong to any minimal prime of (c). We claim that

$$\lim_{q\to\infty} \frac{1}{q^{d-1}}\, \lambda(R/((\mathfrak{p}^{[q]})^*, x)) \geq e_{HK}(R/(x)).$$

To see this, we tensor the exact sequence

$$R \xrightarrow{c} R \to R/(c) \to 0$$

with $R/(\mathfrak{p}^{[q]}, x)$ and observe that $c((\mathfrak{p}^{[q]})^*, x) \subseteq (\mathfrak{p}^{[q]}, x)$. Hence the sequence

$$R/((\mathfrak{p}^{[q]})^*, x) \xrightarrow{c} R/(\mathfrak{p}^{[q]}, x) \to R/(c, x, \mathfrak{p}^{[q]}) \to 0$$

is also exact. Note that $\dim R/(x, c) = \dim R/(x) - 1$. Therefore, taking the limit, we get that $\lim_{q\to\infty} \frac{1}{q^{d-1}}\, \lambda(R/((\mathfrak{p}^{[q]})^*, x)) \geq e_{HK}(R/(x))$.

Since $e_{HK}(\mathfrak{p}) = e_{HK}(\mathfrak{m})$, $(\mathfrak{p}^{[q]})^*$ are $\mathfrak{p}$-primary for all q. Hence the minimal generator x is not a zero divisor modulo $(\mathfrak{p}^{[q]})^*$ and, by Corollary 2.1.10,

$$\lambda(R/\mathfrak{m}^{[q]}) \geq \lambda(R/((\mathfrak{p}^{[q]})^*, x^q)) = q\, \lambda(R/((\mathfrak{p}^{[q]})^*, x)).$$

It follows that $e_{HK}(\mathfrak{p}) = e_{HK}(\mathfrak{m}) \geq e_{HK}(R/(x))$, but $e_{HK}(\mathfrak{m}) \leq e_{HK}(R/(x))$ by Proposition 2.3.13.

For the general case, we induct on $\dim R/\mathfrak{p}$. Let y be a minimal generator of $\mathfrak{m}$ and let $\mathfrak{q} = (\mathfrak{p}, y)$, then $R/\mathfrak{q}$ is a regular ring, so $\mathfrak{q}$ is prime. Note that $\mathrm{e}_{\mathrm{HK}}(\mathfrak{q}) = \mathrm{e}_{\mathrm{HK}}(\mathfrak{m})$ and $\dim R/\mathfrak{q} < \dim R/\mathfrak{p}$, so, by the induction hypothesis, $\mathrm{e}_{\mathrm{HK}}(\mathfrak{p}) = \mathrm{e}_{\mathrm{HK}}(\mathfrak{q}) = \mathrm{e}_{\mathrm{HK}}(\mathfrak{m}R/(x))$ for some minimal generator x of $\mathfrak{m}$ modulo $\mathfrak{p}$. □

Remark 4.2.8. In many cases, we should be able to choose any minimal generator of $\mathfrak{m}$ modulo $\mathfrak{p}$ in the lemma above. Namely, this will hold if the ideal generated by test elements has height at least two; for example, if R is a an excellent normal domain. In this case we will be able to choose a test element c such that $\dim R/(c, x) \leq \dim R - 2$.

We point an easy consequence of the previous proof.

Corollary 4.2.9. *Let $(R, \mathfrak{m})$ be a formally unmixed local ring and $\mathfrak{p}$ be a prime ideal of R such that $R/\mathfrak{p}$ is a regular ring. Suppose that x is an element of R such that $R/(x)$ has a test element. If $\mathrm{e}_{\mathrm{HK}}(\mathfrak{m}) = \mathrm{e}_{\mathrm{HK}}(\mathfrak{p})$ then $\mathrm{e}_{\mathrm{HK}}(R/(x)) = \mathrm{e}_{\mathrm{HK}}(\mathfrak{p})$.*

Proof. The proof is essentially same as in the proposition, except that we derive

$$\lim_{q\to\infty} \frac{1}{q^{d-1}} \lambda(R/((\mathfrak{p}^{[q]})^*, x)) = \mathrm{e}_{\mathrm{HK}}(R/(x))$$

by directly applying Lemma 2.4.10. □

4.2.1 On the difference between consecutive terms

First we would like to recall the following definition.

Definition 4.2.10. Let $(R, \mathfrak{m})$ be a local ring and M an R-module. The 0th local cohomology of M with the support in $\mathfrak{m}$ is defined to be

$$\mathrm{H}^0_{\mathfrak{m}}(M) = \{x \in M \mid \mathfrak{m}^n x = 0 \text{ for some } n\} = \cup_n (0 :_M \mathfrak{m}^n).$$

If M is finitely generated, $\mathrm{H}^0_{\mathfrak{m}}(M)$ is the maximal finite length submodule of M. In this case, $\mathfrak{m}^N \mathrm{H}^0_{\mathfrak{m}}(M) = 0$ for all large N.

If $M = R/I$ for some ideal I, then one can easily verify that

$$\mathrm{H}^0_{\mathfrak{m}}(R/I) \cong \frac{I : \mathfrak{m}^\infty}{I},$$

where $I : \mathfrak{m}^\infty$ is the largest term of the ascending chain of ideals $I : \mathfrak{m}^n$. Moreover, if $\dim R/I = 1$, then $I : \mathfrak{m}^\infty = I : x^\infty$ for any parameter x modulo I. This follows from $\mathfrak{m}$-primarity of (I, x), so $\mathfrak{m}$ and (x) are cofinal modulo I.

We can estimate the difference between the consecutive terms in the following way.

Proposition 4.2.11. *Let $(R, \mathfrak{m})$ to be a local ring of characteristic $p > 0$ and $\mathfrak{p}$ be a prime ideal of height $\dim R - 1$ and such that $R/\mathfrak{p}$ is a DVR. Let x be a minimal generator of $\mathfrak{m}$ modulo $\mathfrak{p}$. Then for all n*

$$\mathrm{e}_{\mathrm{HK}}(\mathfrak{p} + (x^{n+1})) \geq \mathrm{e}_{\mathrm{HK}}(\mathfrak{p} + (x^n)) + \mathrm{e}_{\mathrm{HK}}(\mathfrak{p}).$$

Moreover, if there exists a constant c such that $\mathfrak{m}^{cq} \mathrm{H}^0_{\mathfrak{m}}(R/\mathfrak{p}^{[q]}) = 0$ for all q, then

$$\mathrm{e}_{\mathrm{HK}}(\mathfrak{p} + (x^{n+1})) = \mathrm{e}_{\mathrm{HK}}(\mathfrak{p} + (x^n)) + \mathrm{e}_{\mathrm{HK}}(\mathfrak{p}) \text{ for all } n \geq c.$$

Proof. From the isomorphism $R/(\mathfrak{p}^{[q]} : x^{nq} + (x^q)) \cong \frac{\mathfrak{p}^{[q]} + (x^{nq})}{\mathfrak{p}^{[q]} + (x^{(n+1)q})}$, we get

$$\lambda\left(R/(\mathfrak{p}^{[q]}, x^{(n+1)q})\right) = \lambda\left(R/(\mathfrak{p}^{[q]}, x^{nq})\right) + \lambda\left(R/(\mathfrak{p}^{[q]} : x^{nq} + (x^q))\right).$$

Therefore,

$$\mathrm{e}_{\mathrm{HK}}(\mathfrak{p} + (x^{n+1})) - \mathrm{e}_{\mathrm{HK}}(\mathfrak{p} + (x^n)) = \lim_{q \to \infty} \frac{1}{q^d} \lambda\left(R/(\mathfrak{p}^{[q]} : x^{nq} + (x^q))\right).$$

Furthermore, by Corollary 2.1.10 and the associativity formula,

$$\lambda\left(R/(\mathfrak{p}^{[q]}:x^{nq}+(x^q))\right)\geq \mathrm{e}\left(x^q, R/(\mathfrak{p}^{[q]}:x^{nq})\right) = q\,\lambda_{R_\mathfrak{p}}\left(R_\mathfrak{p}/(\mathfrak{p}^{[q]}:x^{nq})R_\mathfrak{p}\right)\mathrm{e}(x, R/\mathfrak{p}).$$

Since $(\mathfrak{p}^{[q]}:x^{nq})R_\mathfrak{p} = \mathfrak{p}^{[q]}R_\mathfrak{p}$, by taking the limit in the inequality above, we derive that

$$\lim_{q\to\infty}\frac{1}{q^d}\lambda\left(R/(\mathfrak{p}^{[q]}:x^{nq}+(x^q))\right)\geq \mathrm{e}_{\mathrm{HK}}(\mathfrak{p}).$$

Hence $\mathrm{e}_{\mathrm{HK}}(\mathfrak{p}+(x^{n+1}))\geq \mathrm{e}_{\mathrm{HK}}(\mathfrak{p}+(x^n))+\mathrm{e}_{\mathrm{HK}}(\mathfrak{p})$.

For the second assertion, we note that our assumption gives that $\mathfrak{p}^{[q]}:x^{nq}=\mathfrak{p}^{[q]}:x^\infty$ for $n\geq c$. Since x is not a zero divizor modulo $\mathfrak{p}^{[q]}:x^\infty$, by Corollary 2.1.10 and the associativity formula,

$$\lambda\left(R/(\mathfrak{p}^{[q]}:x^{nq}+(x^q))\right)= \mathrm{e}\left(x^q, R/(\mathfrak{p}^{[q]}:x^{nq})\right) = \mathrm{e}(x^q, R/\mathfrak{p})\,\lambda_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{p}^{[q]}R_\mathfrak{p}).$$

Moreover, using that $R/\mathfrak{p}$ is regular, we observe that

$$\lambda\left(R/(\mathfrak{p}^{[q]}:x^{nq}+(x^q))\right)= \mathrm{e}(x^q, R/\mathfrak{p})\,\lambda_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{p}^{[q]}R_\mathfrak{p}) = q\,\lambda_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{p}^{[q]}R_\mathfrak{p}),$$

and the assertion follows after taking the limit. □

Remark 4.2.12. We want to remark that it is believed that for any ideal I there exists c such that

$$\mathfrak{m}^{cq}\,\mathrm{H}^0_\mathfrak{m}(R/I^{[q]})=0.$$

However, the only known case is the case of a homogeneous ideal I in a graded ring R (with the maximal homogenenous ideal $\mathfrak{m}$) such that $\dim R/I=1$ (Huneke, [15]).

Corollary 4.2.13. *Let $(R,\mathfrak{m})$ be a local ring of characteristic $p>0$ and let $\mathfrak{p}$ be a prime ideal in R such that $R/\mathfrak{p}$ is a DVR and* $\operatorname{ht}\mathfrak{p}=\dim R-1$. *If x is a minimal* *generator of* $\mathfrak{m}$ *modulo* $\mathfrak{p}$*, then* $\mathrm{e_{HK}}(\mathfrak{m}) = \mathrm{e_{HK}}(\mathfrak{p})$ *if and only if for some (equivalently, all)* n $\mathrm{e_{HK}}(\mathfrak{p} + (x^n)) = n\,\mathrm{e_{HK}}(\mathfrak{p})$.

In other words, $\mathrm{e_{HK}}(\mathfrak{m}) = \mathrm{e_{HK}}(\mathfrak{p})$ *if and only if the sequence* $\frac{1}{n}\mathrm{e_{HK}}(\mathfrak{p} + (x^n))$ *stabilizes.*

Proof. By Corollary 4.2.3, $n\,\mathrm{e_{HK}}(\mathfrak{m}) \geq \mathrm{e_{HK}}(\mathfrak{p} + (x^n)) \geq n\,\mathrm{e_{HK}}(\mathfrak{p})$, so one direction follows.

We prove the other direction by reverse induction on n, and the base case of $n = 1$ is trivial. By Proposition 4.2.11 and our assumption, we have

$$n\,\mathrm{e_{HK}}(\mathfrak{p}) = \mathrm{e_{HK}}(\mathfrak{p} + (x^n)) \geq \mathrm{e_{HK}}(\mathfrak{p} + (x^{n-1})) + \mathrm{e_{HK}}(\mathfrak{p}) \geq (n-1)\,\mathrm{e_{HK}}(\mathfrak{p}) + \mathrm{e_{HK}}(\mathfrak{p}).$$

Thus, $\mathrm{e_{HK}}(\mathfrak{p} + (x^{n-1})) = (n-1)\,\mathrm{e_{HK}}(\mathfrak{p})$. $\square$

Corollary 4.2.14. *Let* $(R, \mathfrak{m})$ *be a local ring and* $\mathfrak{p}$ *be a prime ideal of height* $\dim R - 1$ *such that* $R/\mathfrak{p}$ *is a DVR. Suppose there exists a constant* c *such that* $\mathfrak{m}^{qc}\,\mathrm{H}^0_{\mathfrak{m}}(R/\mathfrak{p}^{[q]}) = 0$*. Then* $\mathrm{e_{HK}}(\mathfrak{m}) = \mathrm{e_{HK}}(\mathfrak{p})$ *if and only if* $\lim\limits_{q\to\infty} \frac{\lambda(\mathrm{H}^0_{\mathfrak{m}}(R/\mathfrak{p}^{[q]}))}{q^d} = 0$.

Proof. Let x be a minimal generator of $\mathfrak{m}$ modulo $\mathfrak{p}$. By our assumption, $\mathfrak{p}^{[q]} : x^\infty = \mathfrak{p}^{[q]} : x^{cq}$, so

$$(x^{cq}) \cap (\mathfrak{p}^{[q]} : x^{cq}) = (x^{cq})\left((\mathfrak{p}^{[q]} : x^{cq}) : x^{cq}\right) = (x^{cq})\left(\mathfrak{p}^{[q]} : x^{2cq}\right) = (x^{cq})(\mathfrak{p}^{[q]} : x^{cq}) \subseteq \mathfrak{p}^{[q]}.$$

Thus

$$\frac{\mathfrak{p}^{[q]} : x^{cq} + (x^{cq})}{\mathfrak{p}^{[q]} + (x^{cq})} \cong \frac{\mathfrak{p}^{[q]} : x^{cq}}{\mathfrak{p}^{[q]} + (x^{cq}) \cap (\mathfrak{p}^{[q]} : x^{cq})} \cong \frac{\mathfrak{p}^{[q]} : x^{cq}}{\mathfrak{p}^{[q]}} = \frac{\mathfrak{p}^{[q]} : x^{\infty}}{\mathfrak{p}^{[q]}} \cong \mathrm{H}^0_{\mathfrak{m}}(R/\mathfrak{p}^{[q]}),$$

and we obtain that

$$\lambda(R/(\mathfrak{p}^{[q]} + (x^{cq}))) = \lambda(R/(\mathfrak{p}^{[q]} : x^{cq} + (x^{cq}))) + \lambda(\mathrm{H}^0_{\mathfrak{m}}(R/\mathfrak{p}^{[q]})).$$

Again, since x is regular on $\mathfrak{p}^{[q]} : x^{\infty} = \mathfrak{p}^{[q]} : x^{cq}$, Corollary 2.1.10 implies that

$$\lambda(R/(\mathfrak{p}^{[q]} : x^{cq} + (x^{cq}))) = \mathrm{e}(x^{cq}, R/\mathfrak{p}^{[q]}),$$

and we can rewrite the formula above as

$$\lambda(R/(\mathfrak{p}^{[q]} + (x^{cq}))) = \mathrm{e}(x^{cq}, R/\mathfrak{p}^{[q]}) + \lambda(\mathrm{H}^0_{\mathfrak{m}}(R/\mathfrak{p}^{[q]})). \tag{4.2.1}$$

By the associativity formula, $\mathrm{e}(x^{cq}, R/\mathfrak{p}^{[q]}) = cq\,\mathrm{e}(x, R/\mathfrak{p}^{[q]}) = cq\,\lambda_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}/\mathfrak{p}^{[q]}R_{\mathfrak{p}})$, so dividing (4.2.1) by q^d and taking the limit we obtain

$$\mathrm{e}_{\mathrm{HK}}(\mathfrak{p} + (x^c)) = c\,\mathrm{e}_{\mathrm{HK}}(\mathfrak{p}) + \lim_{q\to\infty} \frac{\lambda(\mathrm{H}^0_{\mathfrak{m}}(R/\mathfrak{p}^{[q]}))}{q^d}.$$

Now the assertion follows from the previous corollary. □

4.3 The general case

In this section, we study equimultiple ideals for Hilbert-Kunz multiplicity. We will find that these should be ideals I such that for any (or, as we will show, some) system of parameters $J = (x_1, \ldots, x_m)$ modulo I

$$\mathrm{e}_{\mathrm{HK}}(I + J) = \sum_{P \in \mathrm{Minh}(I)} \mathrm{e}_{\mathrm{HK}}(J, R/P)\,\mathrm{e}_{\mathrm{HK}}(I, R_P).$$

This could be seen as a direct analogue of condition (b) of Thereom 3.1.17.

4.3.1 Preliminaries

First, we observe that this condition is extremal.

Lemma 4.3.1. *Let $(R, \mathfrak{m})$ to be a local ring of characteristic $p > 0$ and let I be an ideal such that $\operatorname{ht} I + \dim R/I = \dim R$. Then for any parameter ideal J modulo I*

$$\mathrm{e}_{\mathrm{HK}}(I + J) \geq \sum_{P \in \mathrm{Minh}(I)} \mathrm{e}_{\mathrm{HK}}(J, R/P)\, \mathrm{e}_{\mathrm{HK}}(IR_P).$$

Proof. First, by Proposition 2.1.12 and Corollary 2.1.7,

$$\lambda(R/(I + J)^{[q]}) \geq \mathrm{e}(J^{[q]}, R/I^{[q]}) = q^{\dim R/I}\, \mathrm{e}(J, R/I^{[q]}).$$

So, by the associativity formula,

$$\mathrm{e}_{\mathrm{HK}}(I + J) \geq \lim_{q \to \infty} \sum_{P \in \mathrm{Minh}(I)} \frac{1}{q^{\operatorname{ht} I}}\, \mathrm{e}(J, R/P)\, \lambda(R_P/I^{[q]}R_P)$$

and the claim follows. □

To make our notation less cumbersome, in the following we are going to write $\mathrm{e}_{\mathrm{HK}}(I, x_1, \ldots, x_m)$ instead of $\mathrm{e}_{\mathrm{HK}}((I, x_1, \ldots, x_m))$.

Proposition 4.3.2. *Let $(R, \mathfrak{m})$ to be a local ring of characteristic $p > 0$ and let I be an ideal in R such that $\operatorname{ht} I = \dim R - \dim R/I$. If $x_1, \ldots, x_m$ are a system of parameters modulo I then*

$$\lim_{\min(n_i) \to \infty} \frac{1}{n_1 \cdots n_m} \mathrm{e}_{\mathrm{HK}}(I, x_1^{n_1}, \ldots, x_m^{n_m}) = \sum_{P \in \mathrm{Minh}(I)} \mathrm{e}_{\mathrm{HK}}((x_1, \ldots, x_m), R/P)\, \mathrm{e}_{\mathrm{HK}}(IR_P).$$

Proof. Let $(n_1, \ldots, n_m) \in \mathbb{N}^m$ be an arbitrary vector and let $n = \min(n_1, \ldots, n_m)$ and $N = \max(n_1, \ldots, n_m)$. Then Corollary 4.2.2 shows that

$$\frac{1}{N^m} \mathrm{e}_{\mathrm{HK}}(I, x_1^N, \ldots, x_m^N) \leq \frac{1}{n_1 \cdots n_m} \mathrm{e}_{\mathrm{HK}}(I, x_1^{n_1}, \ldots, x_m^{n_m}) \leq \frac{1}{n^m} \mathrm{e}_{\mathrm{HK}}(I, x_1^n, \ldots, x_m^n).$$

Therefore,

$$\lim_{\min(n_i) \to \infty} \frac{1}{n_1 \cdots n_m} \mathrm{e}_{\mathrm{HK}}(I, x_1^{n_1}, \ldots, x_m^{n_m}) = \lim_{n \to \infty} \frac{1}{n^m} \mathrm{e}_{\mathrm{HK}}(I, x_1^n, \ldots, x_m^n).$$

Moreover, by Corollary 4.2.2, the sequence $\frac{1}{n^m}\,\mathrm{e}_{\mathrm{HK}}(I, x_1^n, \ldots, x_m^n)$ is monotonically decreasing, so its limit exists and computed by looking at a subsequence. But, by Corollary 4.1.9,

$$\lim_{q'\to\infty} \frac{1}{q'^m}\,\mathrm{e}_{\mathrm{HK}}(I, x_1^{q'}, \ldots, x_m^{q'}) = \sum_{P\in\mathrm{Minh}(I)} \mathrm{e}_{\mathrm{HK}}((x_1, \ldots, x_m), R/P)\,\mathrm{e}_{\mathrm{HK}}(IR_P).$$

□

Corollary 4.3.3. *In the assumptions of Proposition 4.3.2, if*

$$\mathrm{e}_{\mathrm{HK}}(I, x_1, \ldots, x_m) = \sum_{P\in\mathrm{Minh}(I)} \mathrm{e}_{\mathrm{HK}}((x_1, \ldots, x_m), R/P)\,\mathrm{e}_{\mathrm{HK}}(IR_P)$$

then for any vector $(n_1, \ldots, n_m) \in \mathbb{N}^m$

$$\mathrm{e}_{\mathrm{HK}}(I, x_1^{n_1}, \ldots, x_m^{n_m}) = \sum_{P\in\mathrm{Minh}(I)} \mathrm{e}_{\mathrm{HK}}((x_1^{n_1}, \ldots, x_m^{n_m}), R/P)\,\mathrm{e}_{\mathrm{HK}}(IR_P).$$

Proof. By Corollary 4.2.2,

$$\mathrm{e}_{\mathrm{HK}}(I, x_1, \ldots, x_m) \geq \frac{\mathrm{e}_{\mathrm{HK}}(I, x_1^{n_1}, \ldots, x_m^{n_m})}{n_1 \cdots n_m}.$$

Moreover, by Lemma 4.3.1 and Corollary 2.1.7,

$$\frac{\mathrm{e}_{\mathrm{HK}}(I, x_1^{n_1}, \ldots, x_m^{n_m})}{n_1 \cdots n_m} \geq \sum_{P\in\mathrm{Minh}(I)} \frac{\mathrm{e}_{\mathrm{HK}}((x_1^{n_1}, \ldots, x_m^{n_m}), R/P)}{n_1 \cdots n_m}\,\mathrm{e}_{\mathrm{HK}}(IR_P)$$

$$= \sum_{P\in\mathrm{Minh}(I)} \mathrm{e}_{\mathrm{HK}}((x_1, \ldots, x_m), R/P)\,\mathrm{e}_{\mathrm{HK}}(IR_P) = \mathrm{e}_{\mathrm{HK}}(I, x_1, \ldots, x_m).$$

□

Corollary 4.3.4. *Let* $(R, \mathfrak{m})$ *to be a local ring of characteristic* $p > 0$ *and* $\mathfrak{p}$ *be a prime ideal in* R *such that* $R/\mathfrak{p}$ *is a regular local ring and* $\operatorname{ht}\mathfrak{p} = \dim R - \dim R/\mathfrak{p}$. *Then the following are equivalent:*

(i) $e_{HK}(\mathfrak{m}) = e_{HK}(\mathfrak{p})$,

(ii) $e_{HK}(\mathfrak{p}, x_1^{n_1}, \ldots, x_m^{n_m}) = n_1 \cdots n_m \, e_{HK}(\mathfrak{m})$ *for all vectors* $(n_1, \ldots, n_m)$ *and any system of minimal generators* $x_1, \ldots, x_n$ *of* $\mathfrak{m}$ *modulo* $\mathfrak{p}$,

(iii) $e_{HK}(\mathfrak{p} + I) = \lambda(R/(\mathfrak{p} + I)) \, e_{HK}(\mathfrak{m})$ *for any system of parameters* I *modulo* $\mathfrak{p}$.

Proof. Clearly, $(iii) \Rightarrow (ii) \Rightarrow (i)$. For $(i) \Rightarrow (iii)$, first, by Lemma 4.3.1, $e_{HK}(\mathfrak{p}+I) \geq \lambda(R/(\mathfrak{p} + I)) \, e_{HK}(\mathfrak{p})$. On the other hand, Lemma 2.3.15 gives that

$$e_{HK}(\mathfrak{p} + I) \leq \lambda(R/(\mathfrak{p} + I)) \, e_{HK}(\mathfrak{m}) = \lambda(R/(\mathfrak{p} + I)) \, e_{HK}(\mathfrak{p}).$$

□

After some preliminary results, we are going to strengthen the proposition and show that the equality $e_{HK}(\mathfrak{p}+I) = \lambda(R/(\mathfrak{p}+I)) \, e_{HK}(\mathfrak{p})$ for some system of parameters I forces $e_{HK}(\mathfrak{m}) = e_{HK}(\mathfrak{p})$.

4.3.2 Main results

The next fundamental theorem can be seen as an analogue of implication $(b) \Rightarrow (e)$ of Theorem 3.1.17.

Theorem 4.3.5. *Let* $(R, \mathfrak{m})$ *be a formally unmixed local ring of characteristic* $p > 0$ *with a test element* c*. Let* I *be an ideal and suppose for some system of parameters* $J = (x_1, \ldots, x_m)$ *modulo* I,

$$e_{HK}(I + J) = \sum_{P \in \operatorname{Minh}(I)} e_{HK}(J, R/P) \, e_{HK}(I, R_P).$$

Then $(I, x_1, \ldots, x_{i-1})^{[q]} : x_i^{\infty} \subseteq (I^{[q]}, x_1, \ldots, x_{i-1})^*$ *for all* q *and* $1 \leq i \leq m$.

In particular, x_i is not a zero divisor modulo $((I, x_1, \ldots, x_{i-1})^{[q]})^*$ for all i and q.

Proof. First, observe that the second claim follow from the first via Lemma 2.4.13.

Let d be the dimension of R. For a fixed k, let $L = (x_1, x_2, \ldots, x_{i-1}, x_{i+1}^k, \ldots, x_m^k)$. For any n, q, q' we have inclusions

$$\left(I, L, x_i^k\right)^{[qq']} \subseteq \left((I, x_1, \ldots, x_{i-1})^{[q]} : x_i^{nq}\right)^{[q']} + \left(L, x_i^k\right)^{[qq']} \subseteq (I, L)^{[qq']} : x_i^{nqq'} + \left(x_i^{kqq'}\right). \tag{4.3.1}$$

Hence, after dividing by q'^d and taking the limit, we obtain that

$$\begin{aligned} \mathrm{e_{HK}}\left((I, L, x_i^k)^{[q]}\right) &\geq \mathrm{e_{HK}}\left((I, x_1, \ldots, x_{i-1})^{[q]} : x_i^{nq} + (L, x_i^k)^{[q]}\right) \\ &\geq \lim_{q' \to \infty} \frac{1}{(q')^d} \lambda\left(\frac{R}{(I, L)^{[qq']} : x_i^{nqq'} + (x_i^{kqq'})}\right). \end{aligned} \tag{4.3.2}$$

By Corollary 4.3.3 and Corollary 2.1.7, for all n

$$\mathrm{e_{HK}}(I, L, x_i^n) = n \sum_{P \in \mathrm{Minh}(I)} \mathrm{e}((x_i, L), R/P)\, \mathrm{e_{HK}}(I, R_P).$$

In particular, using Lemma 2.3.8 we obtain that

$$\mathrm{e_{HK}}\left((I, L, x_i^k)^{[q]}\right) = q^d\, \mathrm{e_{HK}}\left(I, L, x_i^k\right) = kq^d \sum_{P \in \mathrm{Minh}(I)} \mathrm{e}((x_i, L), R/P)\, \mathrm{e_{HK}}(I, R_P).$$

Moreover, from the isomorphism $R/((I, L)^{[q]} : x_i^{nq}, x_i^{kq}) \cong (I, L, x_i^n)^{[q]}/(I, L, x_i^{n+k})^{[q]}$, we get the exact sequence

$$0 \to R/((I, L)^{[q]} : x_i^{nq}, x_i^{kq}) \to R/(I, L, x_i^{n+k})^{[q]} \to R/(I, L, x_i^n)^{[q]} \to 0.$$

Together with the previous computation, the sequence gives that for all n and k

$$\lim_{q \to \infty} \frac{1}{q^d} \lambda\left(\frac{R}{(I, L)^{[q]} : x_i^{nq} + (x_i^{kq})}\right) = k \sum_{P \in \mathrm{Minh}(I)} \mathrm{e}((x_i, L), R/P)\, \mathrm{e_{HK}}(I, R_P),$$

so, we compute

$$\lim_{(qq')\to\infty} \frac{q^d}{(qq')^d}\,\lambda\left(\frac{R}{(I,L)^{[qq']} : x_i^{nqq'} + (x_i^{kqq'})}\right) = kq^d \sum_{P\in \mathrm{Minh}(I)} \mathrm{e}((x_i, L), R/P)\, \mathrm{e}_{\mathrm{HK}}(I, R_P).$$

Thus, by (4.3.1) and (4.3.2)

$$\mathrm{e}_{\mathrm{HK}}\left((I, x_1, \ldots, x_{i-1})^{[q]} : x_i^{nq}, (L, x_i^k)^{[q]}\right) = \mathrm{e}_{\mathrm{HK}}\left((I, L, x_i^k)^{[q]}\right).$$

Therefore, by Theorem 2.4.8, $(I, x_1, \ldots, x_{i-1})^{[q]} : x_i^{nq} \subseteq \left((I, L, x_i^k)^{[q]}\right)^*$. Now, since n is arbitrary, we have

$$(I, x_1, \ldots, x_{i-1})^{[q]} : x_i^{\infty} \subseteq \bigcap_k \left((I, x_1, \ldots, x_{i-1}, x_i^k, \ldots, x_m^k)^{[q]}\right)^*$$

and the assertion follows from Lemma 2.4.6. □

Now, we can establish the converse to Theorem 4.3.5. But first we will need the following definition.

Definition 4.3.6. Let R be a ring and $c \in R^{\circ}$. We say that c is a locally stable test element if the image of c in R_P is a test element for any prime P.

While this condition is stronger than that of a test element, in fact, the known results on existence of tests elements provide us locally stable test elements. In particular, Theorem 2.4.5 asserts that locally stable test elements exists for F-finite domains and algebras of essentially finite type over an excellent local domain.

Theorem 4.3.7. *Let $(R, \mathfrak{m})$ be a formally unmixed local ring of characteristic $p > 0$ with a locally stable test element c. Let I be an ideal and $J = (x_1, \ldots, x_m)$ be a system of parameters modulo I. Then*

$$\mathrm{e}_{\mathrm{HK}}(I + J) = \sum_{P\in \mathrm{Minh}(I)} \mathrm{e}_{\mathrm{HK}}(J, R/P)\, \mathrm{e}_{\mathrm{HK}}(I, R_P)$$

if and only if x_i is not a zero divisor modulo $((I, x_1, \ldots, x_{i-1})^{[q]})^*$ for all i and q.

Proof. One direction follows from Theorem 4.3.5.

For the converse, we use induction on m. If $m = 1$, then by Proposition 2.1.12 and by the associativity formula (Proposition 2.1.3)

$$\lambda\left(R/((I^{[q]})^*, x_1^q)\right) = q\,\mathrm{e}(x_1, R/(I^{[q]})^*) = q \sum_{P\in\mathrm{Minh}(I)} \mathrm{e}(x_1, R/P)\,\lambda(R_P/(I^{[q]})^* R_P).$$

So, by Lemma 2.4.10, $\mathrm{e_{HK}}(I, x_1) \le \sum_{P\in\mathrm{Minh}(I)} \mathrm{e}(x_1, R/P)\,\mathrm{e}(I, R_P)$ and the converse holds by Lemma 4.3.1.

Now, by the induction hypothesis,

$$\mathrm{e_{HK}}(I{+}J) = \mathrm{e_{HK}}((I, x_1){+}(x_2, \ldots, x_m)) = \sum_{Q\in\mathrm{Minh}((I,x_1))} \mathrm{e_{HK}}((x_2, \ldots, x_m), R/Q)\,\mathrm{e_{HK}}((I, x_1), R_Q).$$

Since c is locally stable, by Corollary 2.4.11 $((I^{[q]})^*, x_1^q)R_Q$ still can be used to compute $\mathrm{e_{HK}}((I, x_1), R_Q)$. Thus, same way as in the first step, we obtain

$$\mathrm{e_{HK}}((I, x_1), R_Q) = \sum_{P\in\mathrm{Minh}(IR_Q)} \mathrm{e}(x_1, R_Q/PR_Q)\,\mathrm{e_{HK}}(I, R_P).$$

Combining these results, we get

$$\mathrm{e_{HK}}(I{+}J) = \sum_{Q\in\mathrm{Minh}((I,x_1))} \mathrm{e}((x_2, \ldots, x_m), R/Q) \sum_{P\in\mathrm{Minh}(IR_Q)} \mathrm{e}(x_1, R_Q/PR_Q)\,\mathrm{e_{HK}}(I, R_P).$$

Observe that $\mathrm{Minh}(IR_Q) = \mathrm{Min}(IR_Q)$, since x_1 is a parameter modulo I and R_Q/IR_Q has dimension 1. Hence, any prime $P \in \mathrm{Minh}(IR_Q)$ is just a minimal prime of I contained in Q.

Therefore, we can change the order of summations to get

$$\mathrm{e_{HK}}(I + J) = \sum_{P\in\mathrm{Min}(I)} \mathrm{e_{HK}}(I, R_P) \sum_{P\subset Q\in\mathrm{Minh}((I,x_1))} \mathrm{e}(x_1, R_Q/PR_Q)\,\mathrm{e}((x_2, \ldots, x_m), R/Q),$$

where the second sum is taken over all primes $Q \in \operatorname{Minh}((I, x_1))$ that contain P. For such Q we must have $\dim R/P \geq \dim R/Q + 1 = \dim R/I$, because x_1 is a parameter modulo I. So, in fact, the first sum could be taken over $\operatorname{Minh}(I)$. Furthermore, since x_1 is a parameter modulo I and $P \in \operatorname{Minh}(I)$, x_1 is a parameter modulo P and $\dim R/(I, x_1) = \dim R/(P, x_1)$. Hence, the second sum is taken over $Q \in \operatorname{Minh}((P, x_1))$, and we rewrite the formula as

$$\mathrm{e}_{\mathrm{HK}}(I+J) = \sum_{P \in \operatorname{Minh}(I)} \mathrm{e}_{\mathrm{HK}}(I, R_P) \sum_{Q \in \operatorname{Minh}((P, x_1))} \mathrm{e}(x_1, R_Q/PR_Q)\, \mathrm{e}((x_2, \ldots, x_m), R/Q).$$

Last, by the associativity formula for parameter ideals (Proposition 2.1.8), for any P

$$\mathrm{e}_{\mathrm{HK}}(J, R/P) = \sum_{Q' \in \operatorname{Minh}((P, x_1))} \mathrm{e}((x_2, \ldots, x_m), R/Q')\, \mathrm{e}(x_1, R'_Q/PR'_Q),$$

and the claim follows. □

Corollary 4.3.8. *Let $(R, \mathfrak{m})$ be a formally unmixed local ring of positive characteristic with a locally stable test element c. Let I be an ideal and $J = (x_1, \ldots, x_m)$ be a system of parameters modulo I. If*

$$\mathrm{e}_{\mathrm{HK}}(I+J) = \sum_{P \in \operatorname{Minh}(I)} \mathrm{e}_{\mathrm{HK}}(J, R/P)\, \mathrm{e}_{\mathrm{HK}}(I, R_P)$$

then for any $0 \leq k \leq m$

$$\mathrm{e}_{\mathrm{HK}}(I+J) = \sum_{Q \in \operatorname{Minh}((I, x_1, \ldots, x_k))} \mathrm{e}_{\mathrm{HK}}((x_{k+1}, \ldots, x_m), R/P)\, \mathrm{e}_{\mathrm{HK}}((I, x_1, \ldots, x_k), R_Q).$$

Proof. First, by Theorem 4.3.5, x_i is not a zero divisor modulo $((I, x_1, \ldots, x_{i-1})^{[q]})^*$ for all i and q. Now, since this holds for all $i \geq k$, Theorem 4.3.7 shows the assertion. □

For the next result, we record the following consequence of Lemma 2.4.2.

Corollary 4.3.9. *Let $(R, \mathfrak{m})$ be a local ring and let $x_1, \ldots, x_d$ and $y_1, \ldots, y_d$ be two systems of parameters. Then there exists a linear combination $x' = x_d + a_1x_1 + \ldots + a_{d-1}x_{d-1}$ with coefficients in R such that $x_1, \ldots, x_{d-1}, x'$ and $y_1, \ldots, y_{d-1}, x'$ are systems of parameters.*

Proof. First, it is easy to see that $x_1, \ldots, x_{d-1}, x'$ is still system of parameters for any choice of the coefficients a_i.

Second, we let $P_1, \ldots, P_n$ be the minimal primes of $(y_1, \ldots, y_{d-1})$ and use the avoidance lemma above for $x = x_d$ and $I = (x_1, \ldots, x_{d-1})$. □

After all the preliminary work, we can establish that our definition of an equimultiple ideal is independent on the choice of a parameter ideal.

Proposition 4.3.10. *Let $(R, \mathfrak{m})$ be a formally unmixed local ring of characteristic $p > 0$ with a locally stable test element c and let I be an ideal. If for some system of parameters $J = (x_1, \ldots, x_m)$ modulo I*

$$\mathrm{e}_{\mathrm{HK}}(I + J) = \sum_{P \in \mathrm{Minh}(I)} \mathrm{e}_{\mathrm{HK}}(J, R/P)\, \mathrm{e}_{\mathrm{HK}}(I, R_P),$$

then same is true for all systems of parameters.

Proof. We use induction on m and Theorem 4.3.7. If $\dim R/I = 1$, then, by Theorem 4.3.5, our assumption shows that $R/(I^{[q]})^*$ is Cohen-Macaulay for any q, so any parameter is regular.

Let $(y_1, \ldots, y_m)$ be an arbitrary system of parameters modulo I. Then using Corollary 4.3.9, we can find an element of the form $x' = x_d + a_1x_1 + \ldots + a_{m-1}x_{m-1}$ such that

$y_1, \ldots, y_{m-1}, x'$ is still a system of parameters modulo I. Note that $(x_1, \ldots, x_{m-1}, x') = (x_1, \ldots, x_m) = J$, so the original formula still holds. By Corollary 4.3.8, we get that

$$\mathrm{e}_{\mathrm{HK}}(I+J) = \mathrm{e}_{\mathrm{HK}}(I, x', x_1, \ldots, x_{m-1}) = \sum_{Q \in \mathrm{Minh}((I,x'))} \mathrm{e}_{\mathrm{HK}}((x_1, \ldots, x_{m-1}), R/Q)\, \mathrm{e}_{\mathrm{HK}}((I, x'), R_Q),$$

so, by the induction hypothesis aplied to (I, x'),

$$\mathrm{e}_{\mathrm{HK}}(I, x', y_1, \ldots, y_{m-1}) = \sum_{Q \in \mathrm{Minh}((I,x'))} \mathrm{e}_{\mathrm{HK}}((y_1, \ldots, y_{m-1}), R/Q)\, \mathrm{e}_{\mathrm{HK}}((I, x'), R_Q).$$

Using Theorem 4.3.5 on (I, x'), we get that y_i is regular modulo $((I, x', y_1, \ldots, y_{i-1})^{[q]})^*$ for any i and q. But since x' is also regular modulo $(I^{[q]})^*$ for all q, Theorem 4.3.7 implies that

$$\mathrm{e}_{\mathrm{HK}}(I, x', y_1, \ldots, y_{m-1}) = \sum_{P \in \mathrm{Minh}(I)} \mathrm{e}_{\mathrm{HK}}((x', y_1, \ldots, y_{m-1}), R/P)\, \mathrm{e}_{\mathrm{HK}}(I, R_P).$$

After permuting the sequence, and using Theorem 4.3.5 we see that x' is not a zero divisor modulo $((I, y_2, \ldots, y_n)^{[q]})^*$ for all q. Now, again, both x' and y_m are parameters modulo $((I, y_1, \ldots, y_{m-1})^{[q]})^*$, so y_m is regular too. □

Motivated by Proposition 4.3.10 and Theorem 4.3.7, we introduce the following definition.

Definition 4.3.11. Let $(R, \mathfrak{m})$ be a local ring and let I be an ideal. We say that I satisfies colon capturing, if for every system of parameters $x_1, \ldots, x_m$ in R/I, for every $0 \le i < m$, and every q

$$((I, x_1, \ldots, x_i)^{[q]})^* : x_{i+1} \subseteq ((I, x_1, \ldots, x_i)^{[q]})^*.$$

The well-known result of tight closure theory asserts that unders mild conditions, 0 satisfies colon capturing. We note that the tight closure is taken in R, so this property is different from colon capturing in R/I.

Remark 4.3.12. The colon capturing property asserts that any system of parameters in R/I is "regular up to tight closure" modulo I. So it not very surprising that it could be checked for a single system of parameters.

With this definition, we can summarize our findings in an analogue of equivalence (b) and (d) of Theorem 3.1.17.

Theorem 4.3.13. *Let $(R, \mathfrak{m})$ be a formally unmixed local ring of characteristic $p > 0$ with a locally stable test element c and let I be an ideal. Then the following are equivalent:*

1. *I satisfies colon capturing,*

2. *for some (equivalently, every) ideal J which is a system of parameters modulo I,*

$$\mathrm{e}_{\mathrm{HK}}(I+J) = \sum_{P \in \mathrm{Minh}(I)} \mathrm{e}_{\mathrm{HK}}(J, R/P)\, \mathrm{e}_{\mathrm{HK}}(I, R_P).$$

Proof. This was proved in Theorem 4.3.7 and Proposition 4.3.10. □

In the special case of prime ideals with regular quotients we obtain the following characterization.

Corollary 4.3.14. *Let $(R, \mathfrak{m})$ be an formally unmixed local ring of characteristic $p > 0$ with a locally stable test element c and let $\mathfrak{p}$ be a prime ideal such that $R/\mathfrak{p}$ is a regular local ring. Then the following are equivalent:*

(a) $\mathrm{e}_{\mathrm{HK}}(\mathfrak{m}) = \mathrm{e}_{\mathrm{HK}}(\mathfrak{p})$,

(b) For any (equivalently, some) system of parameters J modulo $\mathfrak{p}$

$$\mathrm{e}_{\mathrm{HK}}(\mathfrak{p}, J) = \lambda(R/(\mathfrak{p}, J))\, \mathrm{e}_{\mathrm{HK}}(\mathfrak{m}),$$

(c) For any (equivalently, some) system of parameters J modulo $\mathfrak{p}$

$$\mathrm{e}_{\mathrm{HK}}(\mathfrak{p}, J) = \lambda(R/(\mathfrak{p}, J))\, \mathrm{e}_{\mathrm{HK}}(\mathfrak{p}),$$

(d) $\mathfrak{p}$ satisfies colon capturing.

Proof. The first two conditions are equivalent by Corollary 4.3.4, (a), (c), (e) are equivalent by the previous theorem. □

This theorem has a notable corollary. First, recall that a ring R of characteristic $p > 0$ is called weakly F-regular if $I^* = I$ for every ideal I in R. For example, any regular ring is weakly F-regular and direct summands of weakly F-regular rings are weakly F-regular.

Corollary 4.3.15. *Let $(R, \mathfrak{m})$ be a weakly F-regular excellent local domain and let $\mathfrak{p}$ be a prime ideal such that $R/\mathfrak{p}$ is regular. Then $\mathrm{e}_{\mathrm{HK}}(\mathfrak{m}) = \mathrm{e}_{\mathrm{HK}}(\mathfrak{p})$ if and only if the Hilbert-Kunz functions of R and $R_{\mathfrak{p}}$ coincide.*

Proof. Since all ideals in R are tightly closed, from the preceeding theorem we obtain that $R/\mathfrak{p}^{[q]}$ is Cohen-Macaulay for all q. Hence the assertion follows from Proposition 3.3.2. □

4.3.3 Further generalizations

We will develop some general reductions for the equimultiplicity condition and use them to generalize the obtained results.

First, we show that equimultiplicity can be checked modulo minimal primes.

Lemma 4.3.16. *Let* $(R, \mathfrak{m})$ *be a local ring of characteristic* $p > 0$ *and* $\mathfrak{p}$ *be a prime ideal such that* $\operatorname{ht}\mathfrak{p} + \dim R/\mathfrak{p} = \dim R$. *Then* $\mathrm{e_{HK}}(\mathfrak{m}) = \mathrm{e_{HK}}(\mathfrak{p})$ *if and only if* $\operatorname{Minh}(R) = \operatorname{Minh}(R_\mathfrak{p})$ *and* $\mathrm{e_{HK}}(\mathfrak{m}R/P) = \mathrm{e_{HK}}(\mathfrak{p}R/P)$ *for any* $P \in \operatorname{Minh}(R)$.

In particular, if R *is catenary, then* $\mathrm{e_{HK}}(\mathfrak{m}) = \mathrm{e_{HK}}(\mathfrak{p})$ *if and only if* $P \subseteq \mathfrak{p}$ *and* $\mathrm{e_{HK}}(\mathfrak{m}R/P) = \mathrm{e_{HK}}(\mathfrak{p}R/P)$ *for all* $P \in \operatorname{Minh}(R)$.

Proof. If $Q \in \operatorname{Minh}(R_\mathfrak{p})$, by definition, $\dim R_\mathfrak{p}/QR_\mathfrak{p} = \operatorname{ht}\mathfrak{p}$, so $Q \in \operatorname{Minh}(R)$ by the assumption on $\mathfrak{p}$. Moreover, if R is catenary, it is easy to check that, in fact, $\operatorname{Minh}(R_\mathfrak{p}) = \{P \in \operatorname{Minh}(R) \mid P \subseteq \mathfrak{p}\}$.

By the associativity formula we have:

$$\mathrm{e_{HK}}(\mathfrak{m}) = \sum_{P \in \operatorname{Minh}(R)} \mathrm{e_{HK}}(\mathfrak{m}, R/P)\,\lambda(R_P),$$

and, also by Corollary 3.2.3,

$$\mathrm{e_{HK}}(\mathfrak{p}) = \sum_{Q \in \operatorname{Minh}(R_\mathfrak{p})} \mathrm{e_{HK}}(\mathfrak{p}, R_\mathfrak{p}/QR_\mathfrak{p})\,\lambda(R_Q) \leq \sum_{Q \in \operatorname{Minh}(R_\mathfrak{p})} \mathrm{e_{HK}}(\mathfrak{m}, R/Q)\,\lambda(R_Q).$$

Since the second sum is contained in the sum appearing in the expression for $\mathrm{e_{HK}}(\mathfrak{m})$, the claim follows. □

The lemma can be easily generalized, but we decided to leave the special case for clarity. A more general lemma can be found right after the following easy corollary.

Corollary 4.3.17. *Let* $(R, \mathfrak{m})$ *be a local ring of characteristic* $p > 0$ *and* $\mathfrak{p}$ *be a prime ideal such that* $\operatorname{ht}\mathfrak{p} + \dim R/\mathfrak{p} = \dim R$. *Then* $\mathrm{e_{HK}}(\mathfrak{m}) = \mathrm{e_{HK}}(\mathfrak{p})$ *if and only if* $\mathrm{e_{HK}}(\mathfrak{m}R_{red}) = \mathrm{e_{HK}}(\mathfrak{p}R_{red})$.

Proof. Since $\operatorname{Minh}(R) = \operatorname{Minh}(R_{red})$, this immediately follows from the previous lemma applied to R and R_{red}. $\square$

Lemma 4.3.18. *Let $(R, \mathfrak{m})$ be a local ring of characteristic $p > 0$ and I be an ideal such that $\operatorname{ht} I + \dim R/I = \dim R$. Let J be a system of parameters modulo I. Then*

$$\mathrm{e}_{\mathrm{HK}}(I + J) = \sum_{Q \in \operatorname{Minh}(I)} \mathrm{e}_{\mathrm{HK}}(J, R/Q)\, \mathrm{e}_{\mathrm{HK}}(IR_Q)$$

if and only if the following two conditions hold:

(a) $\operatorname{Minh}(I + P) \subseteq \operatorname{Minh}(I)$ *for all* $P \in \operatorname{Minh}(R)$*,*

(b) $\mathrm{e}_{\mathrm{HK}}(I + J, R/P) = \sum_{Q \in \operatorname{Minh}(IR/P)} \mathrm{e}_{\mathrm{HK}}(J, R/Q)\, \mathrm{e}_{\mathrm{HK}}(IR_Q/PR_Q)$ *for all* $P \in \operatorname{Minh}(R)$*.*

Proof. First, observe that if $P \in \operatorname{Minh}(R)$ and $Q \in \operatorname{Minh}(I)$ such that $P \subseteq Q$, then $\dim R/I \geq \dim R/(I+P) \geq \dim R/Q = \dim R/I$, so $Q \in \operatorname{Minh}(I+P)$ and the image of Q in R/P is in $\operatorname{Minh}(IR/P)$. Moreover, in this case, $\dim R/(I+P) = \dim R/I$, so $\operatorname{Minh}(I + P) \subseteq \operatorname{Minh}(I)$. And the converse is also true: if $\operatorname{Minh}(I + P) \subseteq \operatorname{Minh}(I)$ then P is contained in some $Q \in \operatorname{Minh}(I)$.

By the associativity formula for $\mathrm{e}_{\mathrm{HK}}(IR_Q)$

$$\sum_{Q \in \operatorname{Minh}(I)} \mathrm{e}_{\mathrm{HK}}(J, R/Q)\, \mathrm{e}_{\mathrm{HK}}(IR_Q) = \sum_{Q \in \operatorname{Minh}(I)} \mathrm{e}_{\mathrm{HK}}(J, R/Q) \sum_{P \in \operatorname{Minh}(R_Q)} \mathrm{e}_{\mathrm{HK}}(IR_Q/PR_Q)\, \lambda(R_P).$$

If $P \in \operatorname{Minh}(R_Q)$, by definition, $\dim R_Q/PR_Q = \operatorname{ht} Q$. So, since $Q \in \operatorname{Minh}(I)$ and $\dim R/I + \operatorname{ht} I = \dim R$, $P \in \operatorname{Minh}(R)$.

Let $\Lambda = \cup \operatorname{Minh}(R_Q) \subseteq \operatorname{Minh}(R)$ where the union is taken over all $Q \in \operatorname{Minh}(I)$.

In the formula above, we change the order of summations to obtain

$$\sum_{Q\in \mathrm{Minh}(I)} \mathrm{e_{HK}}(J, R/Q)\,\mathrm{e_{HK}}(IR_Q) = \sum_{P\in\Lambda} \lambda(R_P) \sum_{\substack{Q\in \mathrm{Minh}(I)\\ P\in \mathrm{Minh}(R_Q)}} \mathrm{e_{HK}}(J, R/Q)\,\mathrm{e_{HK}}(IR_Q/PR_Q).$$

By the observation in the beginning of the proof,

$$\sum_{\substack{Q\in \mathrm{Minh}(I)\\ P\in \mathrm{Minh}(R_Q)}} \mathrm{e_{HK}}(J, R/Q)\,\mathrm{e_{HK}}(IR_Q/PR_Q) = \sum_{Q'\in \mathrm{Minh}(IR/P)} \mathrm{e_{HK}}(J, R/Q')\,\mathrm{e_{HK}}(IR'_Q/PR'_Q).$$

If the first sum is not empty (i.e. $P \subseteq Q$ for some $Q \in \mathrm{Minh}(I)$), then J is still a system of parameters modulo $I + P$ because it is a system of parameters modulo Q. Thus, in this case, by Lemma 4.3.1,

$$\sum_{\substack{Q\in \mathrm{Minh}(I)\\ P\in \mathrm{Minh}(R_Q)}} \mathrm{e_{HK}}(J, R/Q)\,\mathrm{e_{HK}}(IR_Q/PR_Q) \le \mathrm{e_{HK}}(I + J, R/P).$$

But now, we can use the associativity formula for $I + J$, so

$$\begin{aligned}\sum_{Q\in \mathrm{Minh}(I)} \mathrm{e_{HK}}(J, R/Q)\,\mathrm{e_{HK}}(IR_Q) &= \sum_{P\in\Lambda} \lambda(R_P) \sum_{\substack{Q\in \mathrm{Minh}(I)\\ P\in \mathrm{Minh}(R_Q)}} \mathrm{e_{HK}}(J, R/Q)\,\mathrm{e_{HK}}(IR_Q/PR_Q)\\ &\le \sum_{P\in \mathrm{Minh}(R)} \lambda(R_P)\,\mathrm{e_{HK}}(I + J, R/P) = \mathrm{e_{HK}}(I + J),\end{aligned}$$

which finishes the proof. □

Corollary 4.3.19. *Let $(R, \mathfrak{m})$ be a local ring of characteristic $p > 0$ and I be an ideal such that $\operatorname{ht} I + \dim R/I = \dim R$. Let J be a system of parameters modulo I. Then*

$$\mathrm{e_{HK}}(I + J) = \sum_{Q\in \mathrm{Minh}(I)} \mathrm{e_{HK}}(J, R/Q)\,\mathrm{e_{HK}}(IR_Q)$$

if and only if

$$\mathrm{e_{HK}}(I + J, R_{red}) = \sum_{Q\in \mathrm{Minh}(I)} \mathrm{e_{HK}}(J, R/Q)\,\mathrm{e_{HK}}(I(R_{red})_Q).$$

Proof. This follows from Lemma 4.3.18, since both conditions are equivalent for R and R_{red}. $\square$

Equimultiplicity is stable under completion.

Lemma 4.3.20. *Let $(R, \mathfrak{m})$ be a local ring of positive characteristic $p > 0$ and I be an ideal such that $\operatorname{ht} I + \dim R/I = \dim R$. Let J be a system of parameters modulo I. Then*

$$\mathrm{e}_{\mathrm{HK}}(I + J) = \sum_{Q \in \mathrm{Minh}(I)} \mathrm{e}_{\mathrm{HK}}(J, R/Q)\, \mathrm{e}_{\mathrm{HK}}(IR_Q)$$

if and only if

$$\mathrm{e}_{\mathrm{HK}}((I + J)\widehat{R}) = \sum_{P \in \mathrm{Minh}(I\widehat{R})} \mathrm{e}_{\mathrm{HK}}(J\widehat{R}/P, \widehat{R}/P)\, \mathrm{e}_{\mathrm{HK}}(I\widehat{R}_P).$$

Proof. Let $Q \in \mathrm{Minh}(I)$. Since $\widehat{R/Q} = \widehat{R}/Q\widehat{R}$, $\mathrm{e}_{\mathrm{HK}}(J, R/Q) = \mathrm{e}_{\mathrm{HK}}(J\widehat{R}/Q, \widehat{R}/Q)$ by Lemma 2.3.14. So, using the associativity formula for $\mathrm{e}_{\mathrm{HK}}(J\widehat{R}/Q, \widehat{R}/Q)$,

$$\mathrm{e}_{\mathrm{HK}}(J, R/Q) = \mathrm{e}_{\mathrm{HK}}(J\widehat{R}/Q, \widehat{R}/Q) = \sum_{P \in \mathrm{Minh}(Q\widehat{R})} \mathrm{e}_{\mathrm{HK}}(J\widehat{R}/P, \widehat{R}/P)\, \lambda(\widehat{R}_P/Q\widehat{R}_P).$$

Since there is a flat map $R_Q \to \widehat{R}_Q \to \widehat{R}_P$, it follows from Lemma 2.2.15 that $\mathrm{e}_{\mathrm{HK}}(IR_Q)\, \lambda(\widehat{R}_P/Q\widehat{R}_P) = \mathrm{e}_{\mathrm{HK}}(I\widehat{R}_P)$. Therefore

$$\sum_{Q \in \mathrm{Minh}(I)} \mathrm{e}_{\mathrm{HK}}(J, R/Q)\, \mathrm{e}_{\mathrm{HK}}(IR_Q) = \sum_{Q \in \mathrm{Minh}(I)} \sum_{P \in \mathrm{Minh}(Q\widehat{R})} \mathrm{e}_{\mathrm{HK}}(J\widehat{R}/P, \widehat{R}/P)\, \mathrm{e}_{\mathrm{HK}}(I\widehat{R}_P).$$

Moreover, $\cup_Q \mathrm{Minh}(Q\widehat{R}) = \mathrm{Minh}(I\widehat{R})$, because $\dim R/I = \dim \widehat{R}/I\widehat{R} = \dim \widehat{R}/Q\widehat{R}$. Thus we obtain that

$$\sum_{Q \in \mathrm{Minh}(I)} \mathrm{e}_{\mathrm{HK}}(J, R/Q)\, \mathrm{e}_{\mathrm{HK}}(IR_Q) = \sum_{P \in \mathrm{Minh}(I\widehat{R})} \mathrm{e}_{\mathrm{HK}}(J\widehat{R}/P, \widehat{R}/P)\, \mathrm{e}_{\mathrm{HK}}(I\widehat{R}_P).$$

Last, by Lemma 2.3.14, $\mathrm{e}_{\mathrm{HK}}(I + J) = \mathrm{e}_{\mathrm{HK}}((I + J)\widehat{R})$ and the claim follows. $\square$

Corollary 4.3.21. *Let $(R, \mathfrak{m})$ be an excellent equidimensional local ring of characteristic $p > 0$ and let I be an ideal. Then the following are equivalent:*

1. *I satisfies colon capturing (as in Definition 4.3.11),*

2. *for some (equivalently, every) ideal J which is a system of parameters modulo I,*

$$\mathrm{e_{HK}}(I + J) = \sum_{P \in \mathrm{Minh}(I)} \mathrm{e_{HK}}(J, R/P)\, \mathrm{e_{HK}}(I, R_P).$$

Proof. By Proposition 2.4.3 and Corollary 4.3.19, both conditions are independent of the nilradical. Thus we can assume that R is reduced, so since R is excellent, by Theorem 2.4.5, it has a locally stable test element. Last, since R is an excellent equidimensional reduced ring, it is formally unmixed and we can apply Theorem 4.3.13. □

Corollary 4.3.22. *Let $(R, \mathfrak{m})$ be a local ring of positive characteristic $p > 0$ and I be an ideal such that $\operatorname{ht} I + \dim R/I = \dim R$. If for some system of parameters $J = (x_1, \ldots, x_m)$ modulo I,*

$$\mathrm{e_{HK}}(I + J) = \sum_{P \in \mathrm{Minh}(I)} \mathrm{e_{HK}}(J, R/P)\, \mathrm{e_{HK}}(I, R_P),$$

then same is true for all systems of parameters.

Proof. First, we use Lemma 4.3.20 to reduce the question to the completion of R, note that $\operatorname{ht} I\widehat{R} + \dim \widehat{R}/I\widehat{R} = \dim \widehat{R}$. Thus we assume that R is complete.

Now, condition (a) of Lemma 4.3.18 is independent of J. So, it is enough to show that the claim holds in a complete domain. But a complete domain has a locally stable test element by Theorem 2.4.5 and the claim follows from Proposition 4.3.10. □

4.4 Applications

First, we note the following consequence of our machinery.

Proposition 4.4.1. *Let* $(R, \mathfrak{m})$ *be a formally unmixed local ring of characteristic* $p > 0$ *with a locally stable test element* c*. Moreover, let* $\mathfrak{p}$ *be a prime ideal of* R *such that* $R/\mathfrak{p}$ *is regular and* $\mathrm{e}_{\mathrm{HK}}(\mathfrak{m}) = \mathrm{e}_{\mathrm{HK}}(\mathfrak{p})$*. Then* $(\mathfrak{p}^{[q]})^*$ *are* $\mathfrak{p}$*-primary for all* q*.*

Proof. Suppose $\mathfrak{q}$ is an embedded prime of $(\mathfrak{p}^{[q]})^*$ for some q. By definition, there exists $u \notin (\mathfrak{p}^{[q]})^*$ such that $u\mathfrak{q} \in (\mathfrak{p}^{[q]})^*$, i.e.

$$cu^{q'}\mathfrak{q}^{[qq']} \subseteq \mathfrak{p}^{[qq']}.$$

Let $x \in \mathfrak{q}$ be a parameter modulo $\mathfrak{p}$. Then, by Theorem 4.3.5, $C_q = \mathfrak{p}^{[q]} : x^\infty \subseteq (\mathfrak{p}^{[q]})^*$. Therefore, since $\mathfrak{p}^{[qq']} \subseteq C_{qq'}$ and x is a nonzerodivizor on $C_{qq'}$, it follows from the tight closure equation above that

$$cu^{q'} \in C_{qq'} \subseteq (\mathfrak{p}^{[qq']})^*.$$

Now, multiplying by c again,

$$c^2u^{q'} \in c(\mathfrak{p}^{[qq']})^* \subseteq \mathfrak{p}^{[qq']},$$

hence $u \in (\mathfrak{p}^{[q]})^*$, a contradiction. □

In the excellent case, we do not need to assume existence of a test element.

Corollary 4.4.2. *Let* $(R, \mathfrak{m})$ *be an excellent equidimensional local ring of characteristic* $p > 0$ *and* $\mathfrak{p}$ *be a prime ideal such that* $R/\mathfrak{p}$ *is a regular ring. If* $\mathrm{e}_{\mathrm{HK}}(\mathfrak{m}) = \mathrm{e}_{\mathrm{HK}}(\mathfrak{p})$*, then* $(\mathfrak{p}^{[q]})^*$ *are* $\mathfrak{p}$*-primary for all* q*.*

Proof. Suppose there exists an embedded prime $\mathfrak{q}$ of $(\mathfrak{p}^{[q]})^*$. Then we have an inclusion $\mathfrak{q}u \subseteq (\mathfrak{p}^{[q]})^*$ for some $u \notin (\mathfrak{p}^{[q]})^*$. We know that an element is in tight closure if and only if it is in tight closure modulo minimal primes (Proposition 2.4.3). Therefore for some minimal prime $\mathfrak{p}_i$, $\overline{u} \notin (\mathfrak{p}^{[q]}R/\mathfrak{p}_i)^*$ but $\mathfrak{q}u \subseteq (\mathfrak{p}^{[q]}R/\mathfrak{p}_i)^*$. Now, by Lemma 4.3.16, $\mathrm{e}_{\mathrm{HK}}(R_{\mathfrak{m}}/\mathfrak{p}_i) = \mathrm{e}_{\mathrm{HK}}(R_{\mathfrak{p}}/\mathfrak{p}_i)$ and Proposition 4.4.1 finishes the proof.

□

4.4.1 Equimultiplicity and localization of tight closure

We start with well-known lemmas.

Lemma 4.4.3. *Let R be a ring of positive characteristic and S a multiplicatively closed subset of R. Then every element of $(S^{-1}R)^\circ$ is the product of a unit in $S^{-1}R$ and element in the image of R°.*

Proof. Let $P_1, \dots P_n$ be the minimal primes of R and assume $P_1, \dots, P_h$ are the only minimal primes that meet S. If $h = n$, then $S^{-1}R = 0$, and the claim is not interesting. Let $c \in (S^{-1}R)^\circ$. After multiplication by a unit $S^{-1}R$, we assume that $c \in R$.

Let $I = P_{h+1} \cap \ldots \cap P_n$. Note that the image of I in $S^{-1}R$ is the nilradical of $S^{-1}R$. Let N be such that $I^N S^{-1}R = 0$. Using that c is not contained in $P_{h+1}, \dots, P_n$ and prime avoidance, one can check that (c, I^N) is not contained in the union of the minimal primes of R. Thus by Lemma 2.4.2, there exists $v \in I^N$, such that $c + v \in R^\circ$. Moreover, the image of v in $S^{-1}R$ is zero, so $c + v$ and c have same image and the claim follows. □

It is known that tight closure does not commute with localization even for hypersurfaces. The localization problem has attained much attention from the early days of the theory, and a positive answer was attained in special cases, e.g. in the next lemma. However, Brenner and Monsky have recently obtained a counterexample in [2], we will discuss it in more details shortly.

Lemma 4.4.4. *Let R be a ring of characteristic $p > 0$ with a test element c and I be an ideal of R. Then $I^*S^{-1}R = (IS^{-1}R)^*$ for any multiplicatively closed subset S disjoint from $\bigcup_q \operatorname{Ass}(R/(I^{[q]})^*)$.*

Proof. Since $I^*S^{-1}R \subseteq (IS^{-1}R)^*$ always, we need to show the opposite inclusion. If $x/s \in (IS^{-1}R)^*$ for $x \in R$, by definition, for all sufficient large q

$$d\frac{x^q}{s^q} \in I^{[q]}S^{-1}R,$$

where we can choose $d \in R^\circ$ by the previous lemma. After collecting the denominators we observe that there are elements $s_q \in S$ such that

$$ds_q x^q \in I^{[q]} \subseteq (I^{[q]})^*,$$

for all sufficiently large q.

By the assumption on S, s_q is not a zero divisor modulo $(I^{[q]})^*$ for all q, so $dx^q \in (I^{[q]})^*$ for all sufficiently large q. Therefore, $cdx^q \in I^{[q]}$ and the claim follows. ∎

Lemma 4.4.5. *Let R be a ring of characteristic $p > 0$. Suppose R has a test element c. The following are equivalent:*

(a) $(\mathfrak{p}^{[q]})^*$ *are* $\mathfrak{p}$*-primary for all* q*,*

(b) $(\mathfrak{p}^{[q]}R_{\mathfrak{q}})^*$ *are* $\mathfrak{p}$*-primary for all* q *and all prime ideals* $\mathfrak{q} \supseteq \mathfrak{p}$*,*

(c) $(\mathfrak{p}^{[q]}R_{\mathfrak{m}})^*$ *are* $\mathfrak{p}$*-primary for all* q *and all maximal ideals* $\mathfrak{m}$ *that contain* $\mathfrak{p}$*.*

Proof. The previous lemma shows $(a) \Rightarrow (b)$ and (b) clearly implies (c). So, we need to show $(c) \Rightarrow (a)$.

Suppose there exists an embedded prime ideal $\mathfrak{q}$. Then there is u such that $\mathfrak{q}u \subseteq (\mathfrak{p}^{[q]})^*$, but $u \notin (\mathfrak{p}^{[q]})^*$. By Lemma 2.4.7, $\mathfrak{p}^{[q]}$ is contained in an ideal J primary to some maximal ideal $\mathfrak{m}$ and such that $u \notin J^*$. Now the previous lemma shows that $(J)^*R_{\mathfrak{m}} = (JR_{\mathfrak{m}})^*$, so

$$u\mathfrak{p}'R_{\mathfrak{m}} \subseteq (\mathfrak{p}^{[q]})^*R_{\mathfrak{m}} \subseteq (\mathfrak{p}^{[q]}R_{\mathfrak{m}})^* \subseteq (JR_{\mathfrak{m}})^* = J^*R_{\mathfrak{m}}.$$

On the other hand, $u \notin J^*R_{\mathfrak{m}}$ since J is $\mathfrak{m}$-primary. Thus $u \notin (\mathfrak{p}^{[q]}R_{\mathfrak{m}})^*$ and hence $\mathfrak{p}'$ consists of zero divisors on $(\mathfrak{p}^{[q]}R_{\mathfrak{m}})^*$. □

Now, we can globalize Corollary 4.4.2.

Corollary 4.4.6. *Let* R *be a locally equidimensional ring and let* $\mathfrak{p}$ *be a prime ideal such that* $R/\mathfrak{p}$ *is regular. If* $\mathrm{e}_{\mathrm{HK}}(\mathfrak{m}) = \mathrm{e}_{\mathrm{HK}}(\mathfrak{p})$ *for all maximal (equivalently, all prime) ideals* $\mathfrak{m}$ *containing* $\mathfrak{p}$*, then* $(\mathfrak{p}^{[q]})^*$ *are* $\mathfrak{p}$*-primary for all* q*.*

The following result is a global version of Theorem 4.2.5.

Corollary 4.4.7. *Let* R *be an excellent domain and let* $\mathfrak{p}$ *be a prime ideal such that* $R/\mathfrak{p}$ *is a regular ring of dimension one. Then* $\mathrm{e}_{\mathrm{HK}}(\mathfrak{m}) = \mathrm{e}_{\mathrm{HK}}(\mathfrak{p})$ *for all maximal ideals* $\mathfrak{m}$ *containing* $\mathfrak{p}$ *if and only if* $(\mathfrak{p}^{[q]})^*$ *is* $\mathfrak{p}$*-primary for all* q*.*

Remark 4.4.8. It seems that equimultiplicity is a very strong condition.

For simplicity, suppose R is an excellent domain and $\mathfrak{p}$ is a prime ideal of dimension 1. Then by the previous corollary, if there exists an open equimultiple subset of Max(R/$\mathfrak{p}$), then we can find an element $f \notin \mathfrak{p}$ such that $(\mathfrak{p}^{[q]}R_f)^*$ is $\mathfrak{p}$-primary for all q. In view of Lemma 4.4.4, this forces tight closure of all $\mathfrak{p}^{[q]}R_f$ to commute with localization at any multiplicatively closed set.

Since tight closure does not commute with localization in general, we would like to check what happens in the known counterexample.

4.4.2 The Brenner-Monsky example

Now it is time to apply our results to get a negative answer to Question 3.3.1.

First, let us introduce the Brenner-Monsky hypersurface

$$R = F[x, y, z, t]/(z^4 + xyz^2 + (x^3 + y^3)z + tx^2y^2),$$

where F is an algebraic closure of $\mathbb{Z}/2\mathbb{Z}$. Since R is a quotient of a polynomial ring over an algebraically closed field, it is F-finite. Also, R is a domain, so, in particular, any localization of R has a test element.

Let $P = (x, y, z)$ then $R/P \cong F[t]$ is a regular ring and P is prime. In [2], Brenner and Monsky showed that tight closure does not commute with localization at P. Namely, they showed that $y^3z^3 \notin (P^{[4]})^*$, but the image of y^3z^3 is contained in $(P^{[4]}S^{-1}R)^*$ for $S = F[t] \setminus \{0\}$.

We want to understand the values of Hilbert-Kunz multiplicity on the maximal ideals containing P. First, we will need the following result of Monsky.

Theorem 4.4.9. *Let K be an algebraically closed field of characteristic 2. For $\alpha \in K$ let $R_\alpha = K[[x,y,z]]/(z^4 + xyz^2 + (x^3+y^3)z + \alpha x^2y^2)$. Then*

1. $\mathrm{e}_{\mathrm{HK}}(R_\alpha) = 3 + \frac{1}{2}$, *if $\alpha = 0$,*

2. $\mathrm{e}_{\mathrm{HK}}(R_\alpha) = 3 + 4^{-m}$, *if $\alpha \neq 0$ is algebraic over $\mathbb{Z}/2\mathbb{Z}$, where $m = [\mathbb{Z}/2\mathbb{Z}(\lambda) : \mathbb{Z}/2\mathbb{Z}]$ for λ such that $\alpha = \lambda^2 + \lambda$*

3. $\mathrm{e}_{\mathrm{HK}}(R_\alpha) = 3$ *if α is transcendental over $\mathbb{Z}/2\mathbb{Z}$.*

Proof. The last two cases are computed by Monsky in [25]. For the first case, we note that in characterstic 2 we can factor

$$z^4 + xyz^2 + (x^3+y^3)z = z(x+y+z)((x+y+z)^2 + zy).$$

Thus by the associativity formula,

$$\mathrm{e}_{\mathrm{HK}}(R_0) = \mathrm{e}_{\mathrm{HK}}(K[x,y]) + \mathrm{e}_{\mathrm{HK}}(K[x,y,z]/(x+y+z)) + \mathrm{e}_{\mathrm{HK}}(K[x,y,z]/((x+y+z)^2+zy))$$

and the claim follows. □

Using the developed machinery we derive the following result from Monsky's computations.

Proposition 4.4.10. *Let $R = F[x,y,z,t]/(z^4 + xyz^2 + (x^3+y^3)z + tx^2y^2)$, where F is the algebraic closure of $\mathbb{Z}/2\mathbb{Z}$. Then $\mathrm{e}_{\mathrm{HK}}(P) = 3$ for a prime ideal $P = (x,y,z)$ in R, but $\mathrm{e}_{\mathrm{HK}}(\mathfrak{m}) > 3$ for any maximal ideal $\mathfrak{m}$ containing P.*

Proof. First of all, in the notation of the preceeding theorem, Cohen's structure theorem ([22, p.211]) shows that $\widehat{R_P} \cong R_t$ for $K = F(t)$, so, by Lemma 2.3.14, $\mathrm{e}_{\mathrm{HK}}(R_P) = 3$.

Second, since F is algebraically closed, all maximal ideals containing P are of the form $(P, t-\alpha)$ for $\alpha \in F$. By Monsky's result, $\mathrm{e}_{\mathrm{HK}}(R_{\mathfrak{m}}/(t-\alpha)) > 3 = \mathrm{e}_{\mathrm{HK}}(P)$, since α is algebraic. So, since $R/(t-\alpha)$ is reduced, $\mathrm{e}_{\mathrm{HK}}(\mathfrak{m}) > \mathrm{e}_{\mathrm{HK}}(P)$ by Corollary 4.2.9. □

Thus, we obtain that Hilbert-Kunz multiplicity is not locally constant.

Corollary 4.4.11. *The set* $\{\mathfrak{q} \mid \mathrm{e}_{\mathrm{HK}}(\mathfrak{q}) \leq 3\}$ *is not open.*

Proof. If the set was open, its intersection with $V(P)$ should be open and non-empty. In particular, all but finitely many maximal ideals $\mathfrak{m}$ containing P should belong to the open set. □

Another application of our methods is a quick calculation of the associated primes of $P^{[q]}$. Using the calculations that Monsky made to obtain Theorem 4.4.9, Dinh ([5]) proved that $\bigcup_q \mathrm{Ass}(P^{[q]})$ is infinite. However, he was only able to show that the maximal ideals corresponding to the irreducible factors of $1 + t + t^2 + \ldots + t^q$ appear as associated primes, while our methods give all associated primes of the Frobenius powers and their tight closures.

Proposition 4.4.12. *In the Brenner-Monsky example,*

$$\bigcup_q \mathrm{Ass}(P^{[q]})^* = \bigcup_q \mathrm{Ass}(P^{[q]}) = \mathrm{Spec}\, R/P.$$

In particular, it is infinite.

Proof. Clearly, P is an associated prime, so we need to check the maximal ideals.

First, we prove that any prime $\mathfrak{m}$ that contains P is associated to some $(P^{[q]})^*$. If not, then $(P^{[q]})^* R_{\mathfrak{m}}$ are P-primary for all q. Note that $(P^{[q]})^* R_{\mathfrak{m}} \subseteq (P^{[q]} R_{\mathfrak{m}})^*$, thus by Corollary 2.4.14 $(P^{[q]}R_{\mathfrak{m}})^*$ is P-primary for any q. Therefore, by Theorem 4.2.5, $\mathrm{e}_{\mathrm{HK}}(\mathfrak{m}) = \mathrm{e}_{\mathrm{HK}}(P)$, a contradiction.

For the second claim, let $\mathfrak{m}$ be any maximal ideal containing P. Since $\mathrm{e}_{\mathrm{HK}}(P) < \mathrm{e}_{\mathrm{HK}}(\mathfrak{m})$ and R/P is regular, $\mathfrak{m}$ is an associated prime of $(P^{[q]}R_{\mathfrak{m}})^*$ for some q. Thus, there exists $u \notin (P^{[q]}R_{\mathfrak{m}})^*$ such that

$$cu^{q'}\mathfrak{m}^{[q']} \subseteq P^{[qq']}R_{\mathfrak{m}}.$$

Now, if $\mathfrak{m}$ is not an associated prime of any $P^{[qq']}$, then we would have $u \in (P^{[q]}R_{\mathfrak{m}})^*$, a contradiction. □

Remark 4.4.13. The presented example shows that Hilbert-Kunz multiplicity need not to be locally constant if tight closure does not commute with localization. However, it is not clear whether it should be locally constant if we assume that tight closure commutes with localization. Even in this case, $\bigcup_q \mathrm{Ass}(\mathfrak{p}^{[q]})^*$ might be infinite, and it is not clear why the intersection of the embedded primes must be greater than $\mathfrak{p}$.

Chapter 5

Upper semi-continuity of Hilbert-Kunz multiplicity

5.1 Introduction

In the previous chapter (Corollary 4.4.11) we have shown that Hilbert-Kunz multiplicity is not locally constant. However, we may ask whether there is a weaker notion of continuity that still holds.

Definition 5.1.1. Let X be a topological space. A real-valued function f is upper semi-continuous if for any $a \in \mathbb{R}$ the set $\{x \in X \mid f(x) < a\}$ is open in X.

Remark 5.1.2. While this condition is weaker and does not provide a nice stratification, its maximum value locus is still closed. Namely, when a is increasing the open sets $X_{<a} = \{x \in X \mid f(x) < a\}$ form an ascending chain. Since R is Noetherian, the chain stabilizes, thus giving that $\sup_X f(x) = M$ is finite. Moreover, as n increases $X_{<M-1/n}$ is also an ascending chain of open sets, so the supremum must be attained. Then $X_{=M} = X \setminus X_{<M}$ is closed.

It should be noted that the two continuity notions agree for "discrete" functions.

For example, since Hilbert-Samuel multiplicity is integrally valued, we have equality

$$\{\mathfrak{p} \in \operatorname{Spec} R \mid \mathrm{e}(\mathfrak{p}) \leq n\} = X_{\leq n} = X_{<n+1} = \{\mathfrak{p} \in \operatorname{Spec} R \mid \mathrm{e}(\mathfrak{p}) < n+1\}$$

which shows that the two notions coincide in this case. Thus, in the literature, upper semi-continuity often denotes the stronger propery of locally constancy.

Motivated by Theorem 3.1.6 and this observation, we now ask if Hilbert-Kunz multiplicity is upper semi-continuous. And since we are able to prove this in almost full generality, we boldly call it a conjecture.

Conjecture 5.1.3. Let R be an excellent locally equidimensional ring of characteristic $p > 0$. Then the Hilbert-Kunz multiplicity is upper semi-continuous on $\operatorname{Spec} R$.

This question was asked by Enescu and Shimomoto in [6], although it seems that the authors meant locally constancy in their question.

As in Chapter 3, we need to show openness of certain sets, so Nagata's criterion (Proposition 3.1.7) will be helpful. By Corollary 3.2.3, the first condition of Nagata's criterion is satisfied. So, following the treatment of Section 3.1, we can restate the condition of upper semi-continuity in the following form.

Proposition 5.1.4. *Let R be a locally equidimensional ring. Then the Hilbert-Kunz multiplicity is upper semi-continuous on* $\operatorname{Spec} R$ *if and only if for any prime ideal* $\mathfrak{p}$ *and any* $\varepsilon > 0$ *there exists* $s \notin \mathfrak{p}$ *such that for all prime ideals* $\mathfrak{q} \in D_s \cap V(\mathfrak{p})$

$$\mathrm{e}_{\mathrm{HK}}(\mathfrak{q}) < \mathrm{e}_{\mathrm{HK}}(\mathfrak{p}) + \varepsilon.$$

It is also easy to show that we can restrict ourselves to domains.

Proposition 5.1.5. *Let R be a locally equidimensional ring. If the Hilbert-Kunz multiplicity is upper semi-continuous in $R/\mathfrak{p}$ for all minimal primes $\mathfrak{p}$ of R, then the Hilbert-Kunz multiplicity is upper semi-continuous in R.*

Proof. Given ε, we want to find an element $s \notin \mathfrak{p}$, such that for any ideal $\mathfrak{q}$ containing $\mathfrak{p}R_s$ of R_s, $\mathrm{e_{HK}}(\mathfrak{q}) < \mathrm{e_{HK}}(\mathfrak{p}) + \varepsilon$.

For $i = 1 \dots n$ let $\mathfrak{p}_i$ be the minimal primes of R. Inverting an element, we may assume that all $\mathfrak{p}_i$ are contained in $\mathfrak{p}$. By the assumption, there exist elements $s_i \notin \mathfrak{p}$, such that in the corresponding subsets of $\operatorname{Spec} R/\mathfrak{p}_i$,

$$\mathrm{e_{HK}}(\mathfrak{q}R/\mathfrak{p}_i) < \mathrm{e_{HK}}(\mathfrak{p}R/\mathfrak{p}_i) + \varepsilon/(n\,\lambda_{R_{\mathfrak{p}_i}}(R_{\mathfrak{p}_i}))\,.$$

Now, if we invert the product s of s_i, we obtain that for any ideal $\mathfrak{q}$ of R_s that contains $\mathfrak{p}$, by the associativity formula for Hilbert-Kunz multiplicity (Proposition 2.3.11),

$$\begin{aligned} \mathrm{e_{HK}}(\mathfrak{q}) = \sum_{i=1}^{n} \mathrm{e_{HK}}(\mathfrak{q}R/\mathfrak{p}_i)\,\lambda_{R_{\mathfrak{p}_i}}(R_{\mathfrak{p}_i}) < \\ < \sum_{i=1}^{n} \left(\mathrm{e_{HK}}(\mathfrak{p}R/\mathfrak{p}_i) + \frac{\varepsilon}{n\,\lambda_{R_{\mathfrak{p}_i}}(R_{\mathfrak{p}_i})}\right)\lambda_{R_{\mathfrak{p}_i}}(R_{\mathfrak{p}_i}) = \mathrm{e_{HK}}(\mathfrak{p}) + \varepsilon. \end{aligned}$$

□

Corollary 5.1.6. *Hilbert-Kunz multiplicity is upper semi-continuous in locally equidimensional excellent rings if and only if for any excellent domain R, prime ideal $\mathfrak{p}$ of R, and $\varepsilon > 0$, there exists $s \notin \mathfrak{p}$ such that for all prime ideals $\mathfrak{q} \in V(\mathfrak{p}) \cap D_s$*

$$\mathrm{e_{HK}}(\mathfrak{q}) < \mathrm{e_{HK}}(\mathfrak{p}) + \varepsilon.$$

Proof. We just note that a quotient of an excellent ring is excellent. □

Now, let us give a quick overview of the proof of the main theorem. We will verify the condition of Proposition 5.1.4, by reducing the problem to a fixed Hilbert-Kunz function λ_{q} and using Theorem 3.2.4. To do the reduction, we will establish a uniform convergence result: we will show that one can choose $s \notin \mathfrak{p}$ and control the convergence rate on $D_s \cap V(\mathfrak{p})$. Thus, there is a such q that λ_{q} is sufficiently close to e_{HK} on $D_s \cap V(\mathfrak{p})$.

In the next two sections we will prove the uniform convergence results needed for the proof. Section 5.2 will deal with F-finite rings and more technical Section 5.3 will be needed to work with algebras of essentially finite type over an excellent local ring.

5.2 Globally uniform Hilbert-Kunz estimates for F-finite rings

In this section, we once again rebuild Tucker's uniform Hilbert-Kunz estimates from [31] in order to control the rate of convergence of the Hilbert-Kunz function globally.

To make the proof less cumbersome, we use notation $\operatorname{ht} \mathfrak{p}/I = \dim R_{\mathfrak{p}}/IR_{\mathfrak{p}}$ for a prime ideal $\mathfrak{p}$ and an ideal I.

Lemma 5.2.1 (Key lemma)**.** *Let R be an excellent ring of characteristic $p > 0$ and $\mathfrak{p}$ a prime ideal of R. Let M be a finitely generated R-module. There exists a constant C (depending only on M and $\mathfrak{p}$) and an element $s \notin \mathfrak{p}$, such that for any prime ideal $\mathfrak{q} \in D_s \cap V(\mathfrak{p})$ and for all q, we have*

$$\lambda_{R_{\mathfrak{q}}} \left(M_{\mathfrak{q}}/\mathfrak{q}^{[q]} M_{\mathfrak{q}}\right) \leq C q^{\dim M_{\mathfrak{q}}}.$$

Proof. Assume that $M = R/P$ is a cyclic module for a prime ideal P. If $\mathfrak{p}$ does not contain P, we can invert any $s \in P \setminus \mathfrak{p}$, so $M_s = 0$ and the assertion is trivially true. Hence, assume $P \subseteq \mathfrak{p}$.

First, invert an element to make $R/\mathfrak{p}$ regular; this is possible since $R/\mathfrak{p}$ is an excellent domain. Let $S = R/P$, then $S/\mathfrak{p}S \cong R/\mathfrak{p}$ is regular too. The assertion is also trivial if $\mathfrak{p}$ is a maximal ideal, so, by the proof of Theorem 3.1.9, we can invert an element outside of $\mathfrak{p}S$ and assume that $\mathfrak{p}S$ contains a parameter ideal $\underline{y} = (y_1, \ldots, y_h)$ where $h = \operatorname{ht} \mathfrak{p}S$. Since $S/\underline{y}$ is excellent, we can make it Cohen-Macaulay after inverting another element. And since $(S/\underline{y})_{\mathfrak{p}}$ is Artinian, so Cohen-Macaulay, the element could be chosen outside of $\mathfrak{p}$.

Let $\mathfrak{q}$ be an arbitrary prime ideal in the obtained localization that contains $\mathfrak{p}$. Since $R_{\mathfrak{q}}/\mathfrak{p}R_{\mathfrak{q}}$ is a regular local ring, there exists a system of parameters $\underline{x} = (x_1, \ldots, x_m)$ that generates $\mathfrak{q}R_{\mathfrak{q}}$ modulo $\mathfrak{p}R_{\mathfrak{q}}$. Since $(\underline{y}, \underline{x})^{[q]} S_{\mathfrak{q}} \subseteq \mathfrak{q}^{[q]} S_{\mathfrak{q}}$, Lemma 2.1.9 shows that

$$\lambda_{S_{\mathfrak{q}}} \left(S_{\mathfrak{q}} / \mathfrak{q}^{[q]} S_{\mathfrak{q}} \right) \leq \lambda_{S_{\mathfrak{q}}} \left(S_{\mathfrak{q}} / (\underline{y}, \underline{x})^{[q]} S_{\mathfrak{q}} \right) \leq q^{\operatorname{ht} \mathfrak{q}/P} \lambda_{S_{\mathfrak{q}}} \left(S_{\mathfrak{q}} / (\underline{y}, \underline{x}) S_{\mathfrak{q}} \right).$$

Moreover, using the associativity formula (Proposition 2.1.3), we get

$$\lambda_{S_{\mathfrak{q}}} \left(S_{\mathfrak{q}} / (\underline{y}, \underline{x}) S_{\mathfrak{q}} \right) = \mathrm{e} \left(\underline{x}, S_{\mathfrak{q}} / \mathfrak{p} S_{\mathfrak{q}} \right) \lambda_{S_{\mathfrak{p}}} \left(S_{\mathfrak{p}} / \underline{y} S_{\mathfrak{p}} \right) = \lambda_{S_{\mathfrak{p}}} \left(S_{\mathfrak{p}} / \underline{y} S_{\mathfrak{p}} \right).$$

Note that $\mathrm{e}(\underline{x}, S_{\mathfrak{q}}/\mathfrak{p}S_{\mathfrak{q}}) = 1$, since $\underline{x}$ generates the maximal ideal of a regular local ring $S_{\mathfrak{q}}/\mathfrak{p}S_{\mathfrak{q}}$. Therefore, we obtain that

$$\lambda_{S_{\mathfrak{q}}} \left(S_{\mathfrak{q}} / \mathfrak{q}^{[q]} S_{\mathfrak{q}} \right) \leq q^{\operatorname{ht} \mathfrak{q}/P} \lambda_{S_{\mathfrak{p}}} \left(S_{\mathfrak{p}} / (\underline{y}) S_{\mathfrak{p}} \right) = \lambda_{S_{\mathfrak{p}}} \left(S_{\mathfrak{p}} / (\underline{y}) S_{\mathfrak{p}} \right) q^{\dim M_q}$$

and we proved the assertion for the constant $C = \lambda_{S_{\mathfrak{p}}} \left(S_{\mathfrak{p}} / (\underline{y}) S_{\mathfrak{p}} \right)$ independent of $\mathfrak{q}$.

By choosing a prime filtration of M over R, we can reduce the general case to $M = R/P$. Namely, if P_i are prime ideals appearing in the prime filtration, then

$$\lambda_{R_\mathfrak{q}}\left(M_\mathfrak{q}/\mathfrak{q}^{[q]}M_\mathfrak{q}\right) \leq \sum_i \lambda_{R_\mathfrak{q}}\left((R/P_i)_\mathfrak{q}/\mathfrak{q}^{[q]}(R/P_i)_\mathfrak{q}\right).$$

Since there are finitely many primes P_i, we can invert finitely many elements in order to force the claim for all R/P_i. Also, note that $\dim M_\mathfrak{q}$ is the maximum of $\dim R_\mathfrak{q}/P_iR_\mathfrak{q}$ over the primes in a prime filtration. So,

$$\lambda_{R_\mathfrak{q}}\left(M_\mathfrak{q}/\mathfrak{q}^{[q]}M_\mathfrak{q}\right) \leq \sum_i \lambda_{R_\mathfrak{q}}\left(R_\mathfrak{q}/(P_i + \mathfrak{q}^{[q]})R_\mathfrak{q}\right) \leq \sum_i C_i q^{\operatorname{ht} \mathfrak{q}/P_i} \leq \left(\sum_i C_i\right) q^{\dim M_\mathfrak{q}}.$$

□

Now, we derive from the Key lemma the following result.

Corollary 5.2.2. *Let R be an excellent ring of characteristic $p > 0$ and $\mathfrak{p}$ be a prime ideal of R. Suppose M and N are finite R-modules such that their localizations at every minimal prime are isomorphic. Then there exists a constant C and an element $s \notin \mathfrak{p}$, such that for any prime ideal $\mathfrak{q} \in D_s \cap V(\mathfrak{p})$ and for all q, we have*

$$|\,\lambda_{R_\mathfrak{q}}\left(M_\mathfrak{q}/\mathfrak{q}^{[q]}M_\mathfrak{q}\right) - \lambda_{R_\mathfrak{q}}\left(N_\mathfrak{q}/\mathfrak{q}^{[q]}N_\mathfrak{q}\right)| \leq Cq^{\operatorname{ht} \mathfrak{q} - 1}.$$

Proof. By the assumptions, we have an exact sequence

$$N \to M \to K \to 0,$$

where $K_P = 0$ for every minimal prime P. By Lemma 5.2.1, we can find an element s_1 such that for some constant C_1 and all $\mathfrak{q} \in D_{s_1} \cap V(\mathfrak{p})$

$$\lambda_{R_\mathfrak{q}}\left(M_\mathfrak{q}/\mathfrak{q}^{[q]}M_\mathfrak{q}\right) - \lambda_{R_\mathfrak{q}}\left(N_\mathfrak{q}/\mathfrak{q}^{[q]}N_\mathfrak{q}\right) \leq \lambda_{R_\mathfrak{q}}\left(K_\mathfrak{q}/\mathfrak{q}^{[q]}K_\mathfrak{q}\right) \leq C_1 q^{\dim K_\mathfrak{q}}.$$

Since $K_P = 0$ for any minimal prime P, $\dim K_{\mathfrak{q}} \leq \operatorname{ht} \mathfrak{q} - 1$.

To finish the proof, we switch M and N in the first part of the argument, i.e. apply it to the sequence

$$M \to N \to L \to 0.$$

Hence, by inverting an element s_2, we will get

$$\lambda_{R_{\mathfrak{q}}}\left(N_{\mathfrak{q}}/\mathfrak{q}^{[q]}M_{\mathfrak{q}}\right) - \lambda_{R_{\mathfrak{q}}}\left(M_{\mathfrak{q}}/\mathfrak{q}^{[q]}M_{\mathfrak{q}}\right) \leq \lambda_{R_{\mathfrak{q}}}\left(L_{\mathfrak{q}}/\mathfrak{q}^{[q]}L_{\mathfrak{q}}\right) \leq C_2 q^{\dim L_{\mathfrak{q}}} \leq C_2 q^{\operatorname{ht} \mathfrak{q} - 1},$$

and the claim follows for $C = \max(C_1, C_2)$ and $s = s_1 s_2$. □

Theorem 5.2.3. *Let R be an F-finite domain and let $\mathfrak{p}$ be an arbitrary prime ideal. Then there exists an element $s \notin \mathfrak{p}$ such that for any $\varepsilon > 0$ there is q_0 such that for all $q > q_0$*

$$\left|\lambda_{R_{\mathfrak{q}}}\left(R_{\mathfrak{q}}/\mathfrak{q}^{[q]}R_{\mathfrak{q}}\right)/q^{\operatorname{ht}\mathfrak{q}} - \mathrm{e}_{\mathrm{HK}}(\mathfrak{q})\right| < \varepsilon$$

for all prime ideals $\mathfrak{q} \in D_s \cap V(\mathfrak{p})$.

Proof. By Corollary 2.2.10, $R^{\oplus p^{\alpha(0)}}$ and $R^{1/p}$ are isomorphic localized at the minimal prime 0. So, by Corollary 5.2.2, we can invert an element and obtain a global bound

$$\left|\lambda_{R_{\mathfrak{q}}}\left(R_{\mathfrak{q}}^{\oplus p^{\alpha(0)}}/\mathfrak{q}^{[q]}R_{\mathfrak{q}}^{\oplus p^{\alpha(0)}}\right) - \lambda_{R_{\mathfrak{q}}}\left(R_{\mathfrak{q}}^{1/p}/\mathfrak{q}^{[q]}R_{\mathfrak{q}}^{1/p}\right)\right| < Cq^{\operatorname{ht}\mathfrak{q}-1},$$

for an arbitrary prime ideal $\mathfrak{q}$ containing $\mathfrak{p}$.

As in the proof of Theorem 4.1.6, by Proposition 2.2.9 and Proposition 2.2.7, we obtain from the formula above the estimate

$$\left|p^{\operatorname{ht}\mathfrak{q}+\alpha(\mathfrak{q})}\lambda_{R_{\mathfrak{q}}}\left(R_{\mathfrak{q}}/\mathfrak{q}^{[q]}R_{\mathfrak{q}}\right) - p^{\alpha(\mathfrak{q})}\lambda_{R_{\mathfrak{q}}}\left(R_{\mathfrak{q}}/\mathfrak{q}^{[qp]}R_{\mathfrak{q}}\right)\right| < Cq^{\operatorname{ht}\mathfrak{q}-1}, \text{ so}$$

$$\left|p^{\operatorname{ht}\mathfrak{q}}\lambda_{R_{\mathfrak{q}}}\left(R_{\mathfrak{q}}/\mathfrak{q}^{[q]}R_{\mathfrak{q}}\right) - \lambda_{R_{\mathfrak{q}}}\left(R_{\mathfrak{q}}/\mathfrak{q}^{[qp]}R_{\mathfrak{q}}\right)\right| < p^{-\alpha(\mathfrak{q})}Cq^{\operatorname{ht}\mathfrak{q}-1} \leq Cq^{\operatorname{ht}\mathfrak{q}-1}. \quad (5.2.1)$$

Now, we prove by induction on q' that

$$\left|(q')^{\operatorname{ht}\mathfrak{q}}\,\lambda_{R_\mathfrak{q}}\left(R_\mathfrak{q}/\mathfrak{q}^{[q]}R_\mathfrak{q}\right)-\lambda_{R_\mathfrak{q}}\left(R_\mathfrak{q}/\mathfrak{q}^{[qq']}R_\mathfrak{q}\right)\right|<C(qq'/p)^{\operatorname{ht}\mathfrak{q}-1}\frac{q'-1}{p-1}. \tag{5.2.2}$$

The induction base of $q'=p$ is (5.2.1). Now, assume that the claim holds for q' and we want to prove it for $q'p$.

First, (5.2.1) applied to qq' gives

$$\left|p^{\operatorname{ht}\mathfrak{q}}\,\lambda_{R_\mathfrak{q}}\left(R_\mathfrak{q}/\mathfrak{q}^{[qq']}R_\mathfrak{q}\right)-\lambda_{R_\mathfrak{q}}\left(R_\mathfrak{q}/\mathfrak{q}^{[qq'p]}R_\mathfrak{q}\right)\right|<C(qq')^{\operatorname{ht}\mathfrak{q}-1}, \tag{5.2.3}$$

and, multiplying the induction hypothesis by $p^{\operatorname{ht}\mathfrak{q}}$, we get

$$\left|(q'p)^{\operatorname{ht}\mathfrak{q}}\,\lambda_{R_\mathfrak{q}}\left(R_\mathfrak{q}/\mathfrak{q}^{[q]}R_\mathfrak{q}\right)-p^{\operatorname{ht}\mathfrak{q}}\,\lambda_{R_\mathfrak{q}}\left(R_\mathfrak{q}/\mathfrak{q}^{[qq']}R_\mathfrak{q}\right)\right|<C(qq')^{\operatorname{ht}\mathfrak{q}-1}\frac{pq'-p}{p-1}. \tag{5.2.4}$$

Combining (5.2.3) and (5.2.4) results in

$$\left|(q'p)^{\operatorname{ht}\mathfrak{q}}\,\lambda_{R_\mathfrak{q}}\left(R_\mathfrak{q}/\mathfrak{q}^{[q]}R_\mathfrak{q}\right)-\lambda_{R_\mathfrak{q}}\left(R_\mathfrak{q}/\mathfrak{q}^{[qq'p]}R_\mathfrak{q}\right)\right|<C(qq')^{\operatorname{ht}\mathfrak{q}-1}\left(\frac{q'p-p}{p-1}+1\right),$$

and the induction step follows.

Now, dividing (5.2.2) by $q'^{\operatorname{ht}\mathfrak{q}}$, we obtain

$$\left|\lambda_{R_\mathfrak{q}}\left(R_\mathfrak{q}/\mathfrak{q}^{[q]}R_\mathfrak{q}\right)-\frac{1}{q'^{\operatorname{ht}\mathfrak{q}}}\,\lambda_{R_\mathfrak{q}}\left(R_\mathfrak{q}/\mathfrak{q}^{[qq']}R_\mathfrak{q}\right)\right|<Cq^{\operatorname{ht}\mathfrak{q}-1}\cdot\frac{q'-1}{p-1}\cdot\frac{1}{q'p^{\operatorname{ht}\mathfrak{q}-1}}\leq Cq^{\operatorname{ht}\mathfrak{q}-1}.$$

Thus, if we let $q'\to\infty$, we get that

$$\left|\lambda_{R_\mathfrak{q}}\left(R_\mathfrak{q}/\mathfrak{q}^{[q]}R_\mathfrak{q}\right)-q^{\operatorname{ht}\mathfrak{q}}\,\mathrm{e}_{\mathrm{HK}}(\mathfrak{q})\right|<Cq^{\operatorname{ht}\mathfrak{q}-1},$$

and the claim follows. □

5.3 Uniform estimates for a flat extension

In this section we prove convergence estimates of Theorem 5.2.3 for algebras of essentially finite type over a complete domain. To do so, we use existence of a faithfully flat F-finite extension, and we relativize the estimates of the previous section to use in the extension.

Lemma 5.3.1. *Let R be a locally equidimensional excellent ring and S be an R-algebra. Let I be an ideal in R, M be an S-module such that $\operatorname{Supp} M \subseteq V(IS)$, and $\mathfrak{p}$ be a prime ideal of R. Then there exists an element $s \notin \mathfrak{p}$ and a constant C such that for any prime ideal $\mathfrak{q} \in V(\mathfrak{p}) \cap D(s)$ and for any prime ideal Q in S minimal over $\mathfrak{q}S$*

$$\lambda_{S_Q}\left(M_Q/\mathfrak{q}^{[q]}M_Q\right) \leq Cq^{\operatorname{ht}\mathfrak{q}/I}\,\lambda_{S_Q}\left(S_Q/\mathfrak{q}S_Q\right).$$

Proof. If I is not contained in $\mathfrak{p}$ we can invert an element and make M to be zero. So assume $I \subseteq \mathfrak{p}$.

Since R is excellent, by the proof of Lemma 5.2.1, we can invert an element $s \notin \mathfrak{p}$ to assume that $R/\mathfrak{p}$ regular and there is a parameter ideal $\underline{y}$ in R/I such that $\mathfrak{p}R/I$ is the only minimal prime of $\underline{y}$ and $R/(\underline{y}+I)$ is Cohen-Macaulay. We claim that the required bound holds for this s.

Similarly to the proof of Lemma 5.2.1, by taking a prime filtration of M we reduce the statement to $M = S/J$, where J is a prime ideal in S that contains IS. So

$$\lambda_{S_Q}\left(S_Q/(\mathfrak{q}^{[q]}S+J)S_Q\right) \leq \lambda_{S_Q}\left(S_Q/(\mathfrak{q}^{[q]}+I)S_Q\right).$$

By tensoring a composition series of $R_{\mathfrak{q}}/(\mathfrak{q}^{[q]}+I)R_{\mathfrak{q}}$ with S_Q, we get

$$\lambda_{S_Q}(S_Q/\left(\mathfrak{q}^{[q]}+I)S_Q\right) \leq \lambda_{R_{\mathfrak{q}}}\left(R_{\mathfrak{q}}/(\mathfrak{q}^{[q]}+I)R_{\mathfrak{q}}\right)\lambda_{S_Q}\left(S_Q/\mathfrak{q}S_Q\right).$$

Since $R/\mathfrak{p}$ is regular, we can write $\mathfrak{q}R_{\mathfrak{q}} = (\mathfrak{p}+(\underline{x}))R_{\mathfrak{q}}$, where $\underline{x}$ is a system of parameters of $\mathfrak{q}/\mathfrak{p}$. Because $(\underline{y},\underline{x})^{[q]} \subseteq \mathfrak{q}^{[q]}$, Lemma 2.1.9 gives

$$\begin{aligned}\lambda_{R_{\mathfrak{q}}}\left(R_{\mathfrak{q}}/\left(\mathfrak{q}^{[q]}+I\right)R_{\mathfrak{q}}\right) &\leq \lambda_{R_{\mathfrak{q}}}\left(R_{\mathfrak{q}}/\left((\underline{y},\underline{x})^{[q]}+I\right)R_{\mathfrak{q}}\right)\\ &\leq q^{\operatorname{ht}\mathfrak{p}/I+\operatorname{ht}\mathfrak{q}/\mathfrak{p}}\lambda_{R_{\mathfrak{q}}}\left(R_{\mathfrak{q}}/\left(\underline{y},\underline{x},I\right)R_{\mathfrak{q}}\right) \leq q^{\operatorname{ht}\mathfrak{q}/I}\lambda_{R_{\mathfrak{q}}}\left(R_{\mathfrak{q}}/\left(\underline{y},\underline{x},I\right)R_{\mathfrak{q}}\right).\end{aligned}$$

Now, since $R/(\underline{y}+I)$ is Cohen-Macaulay,

$$\lambda_{R_{\mathfrak{q}}}\left(R_{\mathfrak{q}}/\left(\underline{y},\underline{x},I\right)R_{\mathfrak{q}}\right) = \mathrm{e}\left(\underline{x}, R_{\mathfrak{q}}/(\underline{y},I)R_{\mathfrak{q}}\right)$$

and, by the associativity formula,

$$\mathrm{e}\left(\underline{x}, R_{\mathfrak{q}}/(\underline{y},I)R_{\mathfrak{q}}\right) = \mathrm{e}\left(\underline{x}, R_{\mathfrak{q}}/\mathfrak{p}R_{\mathfrak{q}}\right)\lambda_{R_{\mathfrak{p}}}\left(R_{\mathfrak{p}}/(\underline{y},I)R_{\mathfrak{p}}\right) = \lambda_{R_{\mathfrak{p}}}\left(R_{\mathfrak{p}}/(\underline{y},I)R_{\mathfrak{p}}\right),$$

where $\mathrm{e}\left(\underline{x}, R_{\mathfrak{q}}/\mathfrak{p}R_{\mathfrak{q}}\right) = 1$ due to the choice of $\underline{x}$.

Combining the inequalities, we obtain

$$\begin{aligned}\lambda_{S_Q}(S_Q/\left(\mathfrak{q}^{[q]}+I)S_Q\right) &\leq \lambda_{R_{\mathfrak{q}}}\left(R_{\mathfrak{q}}/(\mathfrak{q}^{[q]}+I)R_{\mathfrak{q}}\right)\lambda_{S_Q}\left(S_Q/\mathfrak{q}S_Q\right)\\ &\leq q^{\operatorname{ht}\mathfrak{q}/I}\lambda_{R_{\mathfrak{p}}}\left(R_{\mathfrak{p}}/(\underline{y},I)R_{\mathfrak{p}}\right)\lambda_{S_Q}\left(S_Q/\mathfrak{q}S_Q\right)\end{aligned}$$

and the claim holds for a constant $C = \lambda_{R_{\mathfrak{p}}}\left(R_{\mathfrak{p}}/(\underline{y},I)R_{\mathfrak{p}}\right)$ independent of $\mathfrak{q}$ and q. □

Remark 5.3.2. In Remark 2.2.14, we found that every local ring has a faithfully flat F-finite extension. However, due to its inseparability, certain properties of this extension could be very difficult to control, in particular, the extension is almost never reduced. In the proof of Theorem 2.4.5 Hochster and Huneke had to overcome these difficulties in order to translate their results from F-finite rings to algebras of finite type over a local ring. The following method is usually called the Gamma construction.

Let B be a complete local ring of positive characteristic $p > 0$ with the residue field K. By Cohen's structure theorem ([22, page 265]), there is a surjection from a power series ring $A = K[[x_1, \dots, x_n]]$ onto B.

The key idea is to modify the recipe of Remark 2.2.14 and use an intermediate field extension instead of the whole perfect closure of K. Namely, let Λ be a p-basis ([22, p. 269]) of K, and Γ be a subset of Λ such that $\Lambda \setminus \Gamma$ is finite. For a fixed integer e consider $K_e^\Gamma = K[\lambda^{1/p^e} \mid \lambda \in \Gamma]$ and define

$$A^\Gamma = \bigcup_e K_e^\Gamma[[x_1, \dots, x_n]].$$

One can show that A^Γ is a Noetherian complete local ring. Moreover, $K_e^\Gamma[[x_1, \dots, x_n]]$ is faithfully flat and purely inseparable over A for any e, so the direct limit A^Γ is faithfully flat and purely inseparable. With a bit more work, one can show that A^Γ is F-finite and for this we need $\Lambda \setminus \Gamma$ to be finite.

Now, we can define $B^\Gamma = B \otimes_A A^\Gamma$. Then, by the base change, B^Γ is an F-finite purely inseparable faithfully flat B-algebra. More importantly, certain properties of B or a fixed ideal of B can be preserved by avoiding finitely many elements of Λ, i.e. by a suitable choice of Γ. The next lemma ([13, Lemma 6.13]) is an application of this strategy.

Lemma 5.3.3. *Let B be a complete local domain and S be a B-algebra of essentially finite type. If S is a domain then there exists a purely inseparable faithfully flat F-finite B-algebra B^Γ such that $S \otimes_B B^\Gamma$ is a domain.*

Theorem 5.3.4. *Let B be a complete local domain. Let R be a domain that is a B-algebra of essentially finite type and $\mathfrak{p}$ be an arbitrary prime ideal in R. Then there exists an element $s \notin \mathfrak{p}$ such that for any $\varepsilon > 0$ there is q_0 such that for all $q > q_0$*

$$\left|\lambda_{R_\mathfrak{q}}\left(R_\mathfrak{q}/\mathfrak{q}^{[q]}R_\mathfrak{q}\right)/q^{\operatorname{ht}\mathfrak{q}} - \mathrm{e}_{\mathrm{HK}}(\mathfrak{q})\right| < \varepsilon$$

for all prime ideals $\mathfrak{q} \in D_s \cap V(\mathfrak{p})$.

Proof. We apply Lemma 5.3.3 to the quotient field L of R and obtain a B-algebra B^Γ. Note that $S = R \otimes_B B^\Gamma$ is F-finite, so $S^{1/p}$ is a finitely generated S-module.

By Lemma 5.3.3, $S \otimes_R L \cong B^\Gamma \otimes_B R \otimes_R L \cong B^\Gamma \otimes_B L$ is a domain. Since B^Γ is purely inseparable over B, $B^\Gamma \otimes_B L$ is integral over a field L, so it is a field. Since R is a subring of S, L the quotient field of R, and $S \otimes_R L$ is a field, then $S \otimes_R L$ must be the quotient field of S. Thus, by definition, the rank of the $S \otimes_R L$-vector space $(S \otimes_R L)^{1/p}$ is $p^{\alpha(0)}$.

Moreover, since taking p-roots commutes with localization, $(S)^{1/p} \otimes_R L \cong (S \otimes_R L)^{1/p}$, so it is a free module over the field $S \otimes_R L \cong B^\Gamma \otimes_B L$. Hence, we can invert an element f of R to make $S_f^{1/p}$ be a free module over S_f. Lifting this isomorphism, we obtain maps

$$0 \to S^{1/p} \to S^{\oplus p^{\alpha(0)}} \to M \to 0$$

and

$$0 \to S^{\oplus p^{\alpha(0)}} \to S^{1/p} \to N \to 0$$

such that $\operatorname{Supp} M$, $\operatorname{Supp} N \subseteq V(fS)$.

Using Lemma 5.3.1 to M and N, we can invert an element s and obtain that, for any prime $\mathfrak{q}$ containing $\mathfrak{p}$ and for any minimal prime Q of $\mathfrak{q}S$, $\lambda_{S_Q}\left(M_Q/\mathfrak{q}^{[q]}M_Q\right) \leq$

$C_1 q^{\operatorname{ht}\mathfrak{q}/(f)}\, \lambda_{S_Q}(S_Q/\mathfrak{q}S_Q)$ and $\lambda_{S_Q}\left(N_Q/\mathfrak{q}^{[q]}N_Q\right) \leq C_2 q^{\operatorname{ht}\mathfrak{q}/(f)}\, \lambda_{S_Q}(S_Q/\mathfrak{q}S_Q)$. Now we use the exact sequences above to estimate

$$\lambda_{S_Q}\left(S_Q^{\oplus p^{\alpha(0)}}/\mathfrak{q}^{[q]}S_Q^{\oplus p^{\alpha(0)}}\right) - \lambda_{S_Q}\left(S_Q^{1/p}/\mathfrak{q}^{[q]}S_Q^{1/p}\right) \leq \lambda_{S_Q}\left(M_Q/\mathfrak{q}^{[q]}M_Q\right),$$

$$\lambda_{S_Q}\left(S_Q^{1/p}/\mathfrak{q}^{[q]}S_Q^{1/p}\right) - \lambda_{S_Q}\left(S_Q^{\oplus p^{\alpha(0)}}/\mathfrak{q}^{[q]}S_Q^{\oplus p^{\alpha(0)}}\right) \leq \lambda_{S_Q}\left(N_Q/\mathfrak{q}^{[q]}N_Q\right).$$

Thus, by taking $C = \max(C_1, C_2)$ and noting that $\operatorname{ht}(f) = 1$, we obtain

$$\left|\lambda_{S_Q}\left(S_Q^{\oplus p^{\alpha(0)}}/\mathfrak{q}^{[q]}S_Q^{\oplus p^{\alpha(0)}}\right) - \lambda_{S_Q}\left(S_Q^{1/p}/\mathfrak{q}^{[q]}S_Q^{1/p}\right)\right| < Cq^{\operatorname{ht}\mathfrak{q}-1}\, \lambda_{S_Q}(S_Q/\mathfrak{q}S_Q).$$

So, since $\alpha(0) = \operatorname{ht}Q + \alpha(Q)$ by Proposition 2.2.9, Proposition 2.2.7 gives

$$x` \left|p^{\operatorname{ht}Q+\alpha(Q)}\, \lambda_{S_Q}\left(S_Q/\mathfrak{q}^{[q]}S_Q\right) - p^{\alpha(Q)}\, \lambda_{S_Q}\left(S_Q/\mathfrak{q}^{[qp]}S_Q\right)\right| < Cq^{\operatorname{ht}\mathfrak{q}-1}\, \lambda_{S_Q}(S_Q/\mathfrak{q}S_Q).$$

Note that S_Q is flat over $R_\mathfrak{q}$ and $\mathfrak{q}S_Q$ is Q-primary. Hence for any artinian $R_\mathfrak{q}$-module M,

$$\lambda_{S_Q}\left(M \otimes_{R_\mathfrak{q}} S_Q\right) = \lambda_{R_\mathfrak{q}}(M)\, \lambda_{S_Q}(S_Q/\mathfrak{q}S_Q).$$

Therefore, the estimate above can be rewritten as

$$\left|p^{\operatorname{ht}Q+\alpha(Q)}\, \lambda_{R_\mathfrak{q}}\left(R_\mathfrak{q}/\mathfrak{q}^{[q]}R_\mathfrak{q}\right) - p^{\alpha(Q)}\, \lambda_{R_\mathfrak{q}}\left(R_\mathfrak{q}/\mathfrak{q}^{[qp]}R_\mathfrak{q}\right)\right| < Cq^{\operatorname{ht}\mathfrak{q}-1}.$$

Since S is flat $\operatorname{ht}Q = \operatorname{ht}\mathfrak{q}$, so we obtain Equation 5.2.1 from Theorem 5.2.3:

$$\left|p^{\operatorname{ht}\mathfrak{q}}\, \lambda_{R_\mathfrak{q}}\left(R_\mathfrak{q}/\mathfrak{q}^{[q]}R_\mathfrak{q}\right) - \lambda_{R_\mathfrak{q}}\left(R_\mathfrak{q}/\mathfrak{q}^{[qp]}R_\mathfrak{q}\right)\right| < Cp^{-\alpha(Q)}q^{\operatorname{ht}\mathfrak{q}-1} \leq Cq^{\operatorname{ht}\mathfrak{q}-1};$$

and the proof follows the argument in Theorem 5.2.3.

$\square$

5.4 Proof of the main result and concluding remarks

Now, we want to finish the proof of upper semi-continuity of the Hilbert-Kunz multiplicity for F-finite rings and algebras of essentially finite type over an excellent local ring. To do this, we verify the second statement of Proposition 5.1.4.

First, we need two auxillary results that will help us apply Section 5.3. The upper bound of the following lemma is a slight improvement of a result of Kunz ([19, Proposition 3.9]) and the lower bound is due to Hanes ([11, Corollary IV.1]).

Lemma 5.4.1. *Let $(R, \mathfrak{m})$ be a local ring and let $(S, \mathfrak{n})$ be a faithfully flat extension. Then* $\mathrm{e}_{\mathrm{HK}}(R) \leq \mathrm{e}_{\mathrm{HK}}(S) \leq \mathrm{e}_{\mathrm{HK}}(S/\mathfrak{m}S)\,\mathrm{e}_{\mathrm{HK}}(R)$.

Proof. We prove the upper bound first. Consider a composition series of $R/\mathfrak{m}^{[q]}$

$$0 = M_0 \subset \ldots \subset M_{l_q} = R/\mathfrak{m}^{[q]}$$

for $l_q = \lambda_R(R/\mathfrak{m}^{[q]})$. After breaking the composition series, we get exact sequences

$$0 \to M_n \to M_{n+1} \to R/\mathfrak{m} \to 0.$$

Now, applying $\otimes_R S/\mathfrak{n}^{[q]}$, we obtain the exact sequences

$$M_n \otimes_R S/\mathfrak{n}^{[q]} \to M_{n+1} \otimes_R S/\mathfrak{n}^{[q]} \to S/(\mathfrak{m}S + \mathfrak{n}^{[q]}) \to 0.$$

Thus, it follows after taking lengths that

$$\lambda_S(S/\mathfrak{n}^{[q]}) = \lambda_S(R/\mathfrak{m}^{[q]} \otimes_R S/\mathfrak{n}^{[q]}) \leq l_q\, \lambda_S(S/(\mathfrak{m}+\mathfrak{n}^{[q]})) = \lambda_R(R/\mathfrak{m}^{[q]})\, \lambda_S(S/(\mathfrak{m}+\mathfrak{n}^{[q]})).$$

Since S is faithfully flat, $\dim S = \dim R + \dim S/\mathfrak{m}S$, so

$$\mathrm{e_{HK}}(S) \leq \lim_{q\to\infty} \frac{\lambda_R(R/\mathfrak{m}^{[q]})}{q^{\dim R}} \lim_{q\to\infty} \frac{\lambda_S(S/(\mathfrak{m}+\mathfrak{n}^{[q]}))}{q^{\dim S/\mathfrak{m}S}} = \mathrm{e_{HK}}(R)\,\mathrm{e_{HK}}(S/\mathfrak{m}S).$$

For the lower bound, first suppose that $\dim R = \dim S$. Then, we can compute $\lambda_S(S/\mathfrak{m}^{[q]}S) = \lambda_R(R/\mathfrak{m}^{[q]})\,\lambda_S(S/\mathfrak{m}S)$ by Lemma 2.2.15, and, therefore, $\mathrm{e_{HK}}(\mathfrak{m}S, S) = \mathrm{e_{HK}}(R)\,\lambda_S(S/\mathfrak{m}S)$. Moreover, by Lemma 2.3.15, $\mathrm{e_{HK}}(\mathfrak{m}S, S) \leq \mathrm{e_{HK}}(S)\,\lambda_S(S/\mathfrak{m}S)$ and the claim follows.

For the general case, by flatness of S, $\dim S = \operatorname{ht}\mathfrak{m}S + \dim S/\mathfrak{m}S$. So, there exists a minimal prime Q of $\mathfrak{m}S$ such that $\dim S/Q = \dim S/\mathfrak{m}S$. Thus, we can use Theorem 3.2.2 and get that $\mathrm{e_{HK}}(S) \geq \mathrm{e_{HK}}(S_Q)$. Now, $\dim S_Q = \dim R$ and we are done by the first case. □

The second lemma allows us to descend semi-continuity from a flat extension in a special case.

Lemma 5.4.2. *Let R be a ring and $f\colon R \to S$ be a faithfully flat R-algebra. Moreover, suppose f has regular fibers. Then the Hilbert-Kunz multiplicity is upper semi-continuous in S if and only if it is upper semi-continuous in R.*

Proof. Let Q be any prime in S and let $\mathfrak{p} = Q \cap R$. Note that $R_\mathfrak{p} \to S_Q$ is faithfully flat with regular fibers, so, by Lemma 5.4.1, $\mathrm{e_{HK}}(R_\mathfrak{p}) = \mathrm{e_{HK}}(S_Q)$. Thus, under our assumption, the Hilbert-Kunz multiplicity is constant in fibers, i.e. $\mathrm{e_{HK}}(R_\mathfrak{p}) = \mathrm{e_{HK}}(S_Q)$ for any prime ideal $\mathfrak{p}$ of R and any prime ideal Q of S such that $Q \cap R = \mathfrak{p}$.

Suppose upper semi-continuity holds in S. Let a be any real number and consider the closed set $V(I) = \{Q \mid Q \in \operatorname{Spec} S,\ \mathrm{e_{HK}}(Q) \geq a\}$. The argument above tells us that for any $Q \in V(I)$ any minimal prime of $(Q \cap R)S$ is also in $V(I)$. Hence we get that $V(I) = V(JS)$ where $J = I \cap R$.

We claim that $V(J) = \{\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Spec} R,\ \mathrm{e_{HK}}(\mathfrak{p}) \geq a\}$. Note that $J \in \mathfrak{p}$ if and only if $JS \subseteq Q$ for any prime Q in S that contracts to $\mathfrak{p}$, i.e. $\mathrm{e_{HK}}(\mathfrak{p}) = \mathrm{e_{HK}}(Q) \geq a$.

For the other direction, note that $f^*\colon \operatorname{Spec} S \to \operatorname{Spec} R$ is surjective, so, since $\mathrm{e_{HK}}$ is constant in fibers, we obtain that

$$\{Q \mid Q \in \operatorname{Spec} S, \mathrm{e_{HK}}(Q) < a\} = (f^*)^{-1}\{\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Spec} R, \mathrm{e_{HK}}(\mathfrak{p}) < a\}.$$

Hence it is open.

□

Theorem 5.4.3. *Let R be a locally equidimensional ring. Suppose that R is either F-finite or is an algebra of essentially finite type over an excellent local ring B. If $\mathfrak{p}$ be a prime ideal of R, then for any $\varepsilon > 0$ there exists $s \notin \mathfrak{p}$, such that for all prime ideals $\mathfrak{q} \in D_s \cap V(\mathfrak{p})$*

$$\mathrm{e_{HK}}(\mathfrak{q}) < \mathrm{e_{HK}}(\mathfrak{p}) + \varepsilon.$$

Proof. If R is not F-finite, first, consider the extension $R \to R \otimes_B \widehat{B}$. Since B is excellent, the natural map $B \to \widehat{B}$ is regular. So, by [22, Lemma 4, p. 253], $R \to R \otimes_B \widehat{B}$ satisfies the conditions of Lemma 5.4.2. Hence, by Proposition 5.1.4 and Lemma 5.4.2, we may assume that B is complete.

Note that the classes of rings that we consider are stable under taking quotients. So, by Proposition 5.1.5, we can assume that R is a domain.

By Theorem 5.2.3 and Theorem 5.3.4, there exists an element $s \notin p$ and a fixed power $q_0 = p^e$, such that for all $\mathfrak{q} \in D_s \cap V(\mathfrak{p})$

$$\left|\lambda_{R_\mathfrak{q}}\left(R_\mathfrak{q}/\mathfrak{q}^{[q_0]}R_\mathfrak{q}\right)/q_0^{\operatorname{ht}\mathfrak{q}} - \mathrm{e}_{\mathrm{HK}}(\mathfrak{q})\right| < \varepsilon/2.$$

In particular,

$$\left|\lambda_{R_\mathfrak{p}}\left(R_\mathfrak{p}/\mathfrak{p}^{[q_0]}R_\mathfrak{p}\right)/q_0^{\operatorname{ht}\mathfrak{p}} - \mathrm{e}_{\mathrm{HK}}(\mathfrak{p})\right| < \varepsilon/2.$$

Now, we can use Theorem 3.2.4, and obtain a non-empty subset $\mathfrak{p} \in U \subseteq V(\mathfrak{p})$ open in $V(\mathfrak{p})$ such that for any $\mathfrak{q} \in U$,

$$\lambda_{R_\mathfrak{q}}\left(R_\mathfrak{q}/\mathfrak{q}^{[q_0]}R_\mathfrak{q}\right)/q_0^{\operatorname{ht}\mathfrak{q}} = f_{q_0}(\mathfrak{q}) = f_{q_0}(\mathfrak{p}) = \lambda_{R_\mathfrak{p}}\left(R_\mathfrak{p}/\mathfrak{p}^{[q_0]}R_\mathfrak{p}\right)/q_0^{\operatorname{ht}\mathfrak{p}}.$$

Thus, we obtain that on $U \cap D_s$, $\lambda_{R_\mathfrak{p}}\left(R_\mathfrak{p}/\mathfrak{p}^{[q_0]}R_\mathfrak{p}\right)/q_0^{\operatorname{ht}\mathfrak{p}}$ is within $\varepsilon/2$ from both $\mathrm{e}_{\mathrm{HK}}(\mathfrak{p})$ and $\mathrm{e}_{\mathrm{HK}}(\mathfrak{q})$ and the statement follows. □

Corollary 5.4.4. *Let R be a locally equidimensional ring. Moreover, suppose that either R is F-finite or is an algebra of essentially finite type over an excellent local ring B. Then the Hilbert-Kunz multiplicity is upper semi-continuous on* $\operatorname{Spec} R$.

We would like to remark again that the assumptions are not very restrictive and any affine or complete domain satisfies them.

We note the following corollary of semi-continuity.

Corollary 5.4.5. *Let R be a Noetherian ring and suppose the Hilbert-Kunz multiplicity is upper semi-continuous on* $\operatorname{Spec} R$. *Then the Hilbert-Kunz multiplicity satisfies the ascending chain condition on* $\operatorname{Spec} R$, *i.e. any increasing sequence $e_1 = \mathrm{e}_{\mathrm{HK}}(\mathfrak{p}_1) \leq e_2 = \mathrm{e}_{\mathrm{HK}}(\mathfrak{p}_2) \leq \ldots$ stabilizes.*

Proof. Since e_{HK} is upper semi-continuous $U_i = \{\mathfrak{p} \mid \mathrm{e}_{\mathrm{HK}}(\mathfrak{p}) < e_i\}$ form an increasing sequence of open sets, so it stabilizes. □

Let us state the corollary discussed in Remark 5.1.2.

Corollary 5.4.6. *Let R be a locally equidimensional ring. Suppose that R is either F-finite or is an algebra of essentially finite type over an excellent local ring B. Then the maximum value locus of Hilbert-Kunz multiplicity is closed.*

Since Hilbert-Kunz multiplicity is upper semi-continuous but need not to be locally constant, it may attain infinitely many values. We will use the example of Proposition 4.4.10.

Corollary 5.4.7. *Let $R = F[x, y, z, t]/(z^4 + xyz^2 + (x^3 + y^3)z + tx^2y^2)$, where F is the algebraic closure of $\mathbb{Z}/2\mathbb{Z}$. Then e_{HK} attains infinitely many values on $\operatorname{Spec} R$.*

Proof. Let $P = (x, y, z)$. As we seen in Proposition 4.4.10, $\mathrm{e}_{\mathrm{HK}}(P) = 3$ and $\mathrm{e}_{\mathrm{HK}}(Q) > 3$ for any maximal ideal Q containing P. On the other hand, by upper semi-continuity, the set $X_n = \{Q \in \operatorname{Spec} R \mid \mathrm{e}_{\mathrm{HK}}(Q) < 3 + 1/n\}$ is open. In particular, it contains infinitely many maximal ideals containing P. Thus it easily follows that Hilbert-Kunz multiplicity attains infinitely many values on $V(P)$. □

Chapter 6

Questions

6.0.1 Further directions in Equimultiplicity

We would like to compare better the two equimultiplicity theories. There are many possible questions, but let us focus on the most interesting direction. Namely, we would like to know how does equality $e_{HK}(\mathfrak{p}) = e_{HK}(\mathfrak{m})$ affect the Hilbert-Samuel multiplicity.

First, one could hope that, it should imply the classical equimultiplicity, $e(\mathfrak{p}) = e(\mathfrak{m})$. Another direction to explore, is reduction to positive characteristic: what is a relation between the classical equimultiplicity in characteristic zero and Hilbert-Kunz equimultiplicity in reductions?

6.0.2 Equimultiplicity and Cohen-Macaulayness of closures

In view of Corollary 4.3.14, it is natural to ask whether equimultiplicity is in fact equivalent to $R/(\mathfrak{p}^{[q]})^*$ be Cohen-Macaulay, perhaps, for large q. Even more, by Proposition 3.3.2, we ask if $R/(\mathfrak{p}^{[q]})^*$ is forced to be Cohen-Macaulay if $R/\mathfrak{p}^{[q]}$ is Cohen-Macaulay for all q. This seems to be unlikely, but the author does not have a counterexample.

Also, motivated by Corollary 3.1.19, we could ask same questions about the integral closure of the powers of an ideal.

6.0.3 Analytic spread in tight closure theory

In view of Theorems 4.3.13 and 3.1.17, we would like to ask whether there is a correspondening notion of analytic spread for tight closure.

Epstein and Vraciu started to develop a theory of *-spread as an analogue of the size of a minimal reduction ([7, 32, 8]). However, it seems that this notion cannot provide a characterization in the spirit of laconic $\ell(I) = \operatorname{ht}(I)$, as it seems to be too weak.

Namely, Epstein and Vraciu in [8, Lemma 1] provided us with the following observation.

Lemma 6.0.8. *Let R be a Noetherian local ring of characteristic $p > 0$, let $f_1, \ldots, f_n$ be $*$-independent elements generating an ideal K, and let x be a parameter modulo K. Assume that R has a weak test element. Then there is some positive integer t such that $f_1, \ldots, f_n, x^t$ are $*$-independent.*

Together with [7, Proposition 2.3] and [32, Proposition 3.3], this gives us that, quite generally, for an arbitrary ideal I and a parameter x modulo I, $\ell^*((I, x^t)) = \ell^*(I) + 1$ for some t. This is quite pathological compared to Lemma 3.1.16, as x^t need not to be regular modulo I.

One may speculate that the size of a minimal reduction is not the "right" definition of analytic spread, since they are not equal if the residue field is finite. So, the author hope that there is a definition of analytic spread for tight closure that will generalize Lemma 3.1.16.

6.0.4 Hilbert-Kunz multiplicity and blowing-up

After we have showed semicontinuity of Hilbert-Kunz multiplicity, the next natural direction is to study its behavior under blow-ups. Of course, this is likely a hard question, but there is still a lot to explore.

First, as usually in Hilbert-Kunz theory, we need more examples. At the present moment, we do not have a single non-trivial computation of the maximum value locus of Hilbert-Kunz multiplicity. The structure of the locus will be certainly important for the main question.

6.0.5 Uniform annihilation of local cohomology

We want to finish with the most important and, possibly, the most difficult of our questions. As we discussed in Remark 4.2.12, it is believed that the following conjecture is true.

Conjecture 6.0.9. Let $(R, \mathfrak{m})$ be a local ring of characteristic $p > 0$. Then for any ideal I there exists a constant C such that for all q

$$\mathfrak{m}^{Cq} \operatorname{H}^0_{\mathfrak{m}}(R/I^{[q]}) = 0.$$

This conjecture is quite strong and, for example, implies that a localization of a weakly F-regular ring is still weakly F-regular. In fact, to show this it is enough to establish the conjecture only for ideals of dimension one (e.g. discussion after

Corollary 3.2 in [15]). Also, via the work of Hochster and Huneke ([14]), it will tell us more explicitly why tight closure does not localize.

Bibliography

[1] Holger Brenner. Irrational Hilbert-Kunz multiplicities. Preprint, available at http://arxiv.org/abs/1305.5873.

[2] Holger Brenner and Paul Monsky. Tight closure does not commute with localization. *Ann. of Math. (2)*, 171(1):571–588, 2010.

[3] Marcus Brodmann. Asymptotic stability of Ass(M/I^nM). *Proc. Amer. Math. Soc.*, 74(1):16–18, 1979.

[4] Lindsay Burch. Codimension and analytic spread. *Proc. Cambridge Philos. Soc.*, 72:369–373, 1972.

[5] Trung T. Dinh. Associated primes of the example of Brenner and Monsky. *Comm. Algebra*, 41(1):109–123, 2013.

[6] Florian Enescu and Kazuma Shimomoto. On the upper semi-continuity of the Hilbert-Kunz multiplicity. *J. Algebra*, 285(1):222–237, 2005.

[7] Neil Epstein. A tight closure analogue of analytic spread. *Math. Proc. Cambridge Philos. Soc.*, 139(2):371–383, 2005.

[8] Neil Epstein and Adela Vraciu. A length characterization of $*$-spread. *Osaka J. Math.*, 45(2):445–456, 2008.

[9] Alexander Grothendieck. Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. II. *Inst. Hautes Études Sci. Publ. Math.*, (24):231, 1965.

[10] Chungsim Han and Paul Monsky. Some surprising Hilbert-Kunz functions. *Math. Z.*, 214(1):119–135, 1993.

[11] Douglas Hanes. Bounds on multiplicities of local rings. *Comm. Algebra*, 30(8):3789–3812, 2002.

[12] Melvin Hochster and Craig Huneke. Tight closure, invariant theory, and the Briançon-Skoda theorem. *J. Amer. Math. Soc.*, 3(1):31–116, 1990.

[13] Melvin Hochster and Craig Huneke. F-regularity, test elements, and smooth base change. *Trans. Amer. Math. Soc.*, 346(1):1–62, 1994.

[14] Melvin Hochster and Craig Huneke. Localization and test exponents for tight closure. *Michigan Math. J.*, 48:305–329, 2000. Dedicated to William Fulton on the occasion of his 60th birthday.

[15] Craig Huneke. The saturation of Frobenius powers of ideals. *Comm. Algebra*, 28(12):5563–5572, 2000. Special issue in honor of Robin Hartshorne.

[16] Craig Huneke and Yongwei Yao. Unmixed local rings with minimal Hilbert-Kunz multiplicity are regular. *Proc. Amer. Math. Soc.*, 130(3):661–665, 2002.

[17] Irving Kaplansky. *Commutative rings*. The University of Chicago Press, Chicago, Ill.-London, revised edition, 1974.

[18] Ernst Kunz. Characterizations of regular local rings of characteristic p. *Amer. J. Math.*, 91:772–784, 1969.

[19] Ernst Kunz. On Noetherian rings of characteristic p. *Amer. J. Math.*, 98(4):999–1013, 1976.

[20] Christer Lech. On the associativity formula for multiplicities. *Ark. Mat.*, 3:301–314, 1957.

[21] Joseph Lipman. Equimultiplicity, reduction, and blowing up. In *Commutative algebra (Fairfax, Va., 1979)*, volume 68 of *Lecture Notes in Pure and Appl. Math.*, pages 111–147. Dekker, New York, 1982.

[22] Hideyuki Matsumura. *Commutative algebra*, volume 56 of *Mathematics Lecture Note Series*. Benjamin Cummings Publishing Co., Inc., Reading, Mass., second edition, 1980.

[23] Stephen McAdam. Asymptotic prime divisors and analytic spreads. *Proc. Amer. Math. Soc.*, 80(4):555–559, 1980.

[24] Paul Monsky. The Hilbert-Kunz function. *Math. Ann.*, 263(1):43–49, 1983.

[25] Paul Monsky. Hilbert-Kunz functions in a family: point-S_4 quartics. *J. Algebra*, 208(1):343–358, 1998.

[26] Masayoshi Nagata. *Local rings.* Interscience Tracts in Pure and Applied Mathematics, No. 13. Interscience Publishers a division of John Wiley & Sons New York-London, 1962.

[27] L. J. Ratliff, Jr. Note on asymptotic prime divisors, analytic spreads and the altitude formula. *Proc. Amer. Math. Soc.*, 82(1):1–6, 1981.

[28] D. Rees. Rings associated with ideals and analytic spread. *Math. Proc. Cambridge Philos. Soc.*, 89(3):423–432, 1981.

[29] N. I. Shepherd-Barron. On a problem of Ernst Kunz concerning certain characteristic functions of local rings. *Arch. Math. (Basel)*, 31(6):562–564, 1978/79.

[30] Irena Swanson and Craig Huneke. *Integral closure of ideals, rings, and modules*, volume 336 of *London Mathematical Society Lecture Note Series.* Cambridge University Press, Cambridge, 2006.

[31] Kevin Tucker. F-signature exists. *Invent. Math.*, 190(3):743–765, 2012.

[32] Adela Vraciu. $*$-independence and special tight closure. *J. Algebra*, 249(2):544–565, 2002.

[33] Kei-ichi Watanabe and Ken-ichi Yoshida. Hilbert-Kunz multiplicity and an inequality between multiplicity and colength. *J. Algebra*, 230(1):295–317, 2000.