Probing Strong-Field Gravity with Gravitational Waves

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Abstract

With the historic detection of gravitational waves (GWs) by the LIGO and Virgo collaborations (LVC) in 2015, an era of GW astronomy began. Almost fifty binary merger events have been detected from the first two and a half observing runs of LVC. Ripples in the curvature of spacetime created by coalescing compact binaries allowed us to perform tests of gravity in strong and dynamical field regimes that were previously unexplored. Testing gravity in such a field regime is of particular importance to probe modified theories of gravity, which are viable modifications to general relativity (GR), motivated from both theoretical and observational aspects. This thesis considers both theory-agnostic and theory-specific approaches for testing a number of modified theories of gravity with GWs.

Theory-agnostic or model-independent tests have the advantage over theory-specific tests, as one can map the results of one particular test to several theories. Adopting parametrized post-Einsteinian formalism, we introduce generic deviations to the amplitude and phase of gravitational waveforms from those of GR. We derive analytic expressions of such deviations in a number of theories. We further perform numerical analyses with GW events GW150914 and GW151226 to achieve bounds on such deviations. Finally, we map such bounds to some modified theories of gravity to achieve constraints on those theories. A critical feature of our work is that we keep non-GR deviations in both phase and amplitude of the waveform while performing numerical analyses, while most works done previously focused on corrections to the GW phase only.

For theory-specific cases, we first consider higher-dimensional scenarios. One of the many avenues of modifying the four-dimensional theory of GR is to introduce extra dimensions. Such modifications are motivated by string theories in order to achieve a quantum theory of gravity. We study theories that contain extra dimensions compactified on circles. In particular, we compute modifications induced by compact extra dimensions to the binding energy of binaries and the luminosity of GWs generated by them. We compute the GW phase using such quantities and compare it with GW and binary pulsar observations. Such comparisons show inconsistency between the prediction of our model and the observations, which rules out the class of simple compactified higher-dimensional models.

Finally, we study GW memory effects, a set of strong-field GW phenomena yet to be detected. Such effects manifest as permanent changes in the GW strain and its time integrals after the passage of GWs. They are closely related to asymptotic symmetries of the spacetime and corresponding conserved charges. GW memory effects are well studied in GR but need to be carefully investigated in theories beyond GR. To do so, we consider Brans-Dicke theory which contains a massless scalar field nonminimally coupled to gravity. GWs in Brans-Dicke theory can have three polarizations— two tensor modes (which are present in GR) and one scalar or breathing mode. We study Brans-Dicke theory in Bondi-Sachs framework and derive asymptotically flat solutions, asymptotic symmetries, and associated fluxes of conserved charges. We find that the connection between symmetries and memory effects in Brans-Dicke theory is different from that of GR. In particular, the symmetries are the same as those of GR, but there are two memory effects associated with the non-GR breathing polarization not related to spacetime symmetries.

We further implement the connection between memory effects and fluxes of conserved charges to compute GW memories associated with tensor polarizations in Brans-Dicke theory. We derive memory effects generated by quasi-circular nonprecessing binaries and find that tensor memories in Brans-Dicke theory have two unique features. First, they depend on the sky angles differently from those of GR, which can potentially constrain Brans-Dicke theory with future space-based GW detectors. Second, in terms of binary's relative velocity, they start at a lower order than those of GR.

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List of Abbreviations

GR	General relativity
GW	Gravitational wave
BH	Black hole
NS	Neutron star
BBH	Binary black hole
BNS	Binary Neutron Star
PN	post-Newtonian
PPE	parameterized post-Einsteinian
EDGB	Einstein-dilaton-Gauss-Bonnet
DCS	Dynamical Chern-Simons
BD	Brans-Dicke
DEF	Damour-Esposito-Fareśe
EMD	Einstein-Maxwell-dilaton
BHLMXB	Low-Mass X-ray Binary
IMR	Inspiral-merger-ringdown
SNR	Signal-to-noise-ratio
PSR	Pulsar
RS	Randall-Sandrum
KK	Kaluza-Klein
TT	Transverse-Traceless
NZ	Near-zone
FZ	Far-zone
LL	Landau-Lifshitz
DIRE	Direct integration of relaxed Einstein equations
BMS	Bondi-Metzner-Sachs
CM	Center-of-mass

List of Symbols

m_A	Mass of the A-th component of a binary
χ_A	Dimensionless spin of A-th component of a binary
χ	Effective spin parameter $(m_1\chi_1 + m_2\chi_2) / (m_1 + m_2)$
M_t	Total mass of a binary $(m_1 + m_2)$
δm	Mass difference of binary components $(m_1 - m_2)$
δ_m	Weighted mass difference $(m_1 - m_2)/M_t$
μ	Reduced mass of a binary $(m_1 m_2)/M_t$
η	Symmetric mass ratio μ/M_t
$\mathcal{M}_{ch}$	Chirp mass $\eta^{3/5} M_t$
r_{12}	Binary separation
$\tilde{h}(f)$	Gravitational waveform in the frequency domain [Eq. (2.1)]
ϕ_p	phase of GWs in the time domain
t_0	Time of binary coalescence
$\phi_{p,0}$	GW phase at the time of coalescence
Ψ_p	phase of GWs in the frequency domain
$(\beta_{\rm PPE}, b)$	Parametrized post-Einsteinian phase correction parameters
$(\alpha_{\rm PPE}, a)$	Parametrized post-Einsteinian amplitude correction parameters
Ω	Orbital angular frequency
f	Frequency of the quadrupolar mode of GWs Ω/π
$\underline{u}$	Effective relative velocity of binary constituents $(\pi\mathcal{M}_{ch} f)^{1/3}$
α_A	Scalar charge of the A-th component of a binary
s_A	Sensitivity of the A-th component of a binary
$\bar{\alpha}_{\rm EDGB}$	Einstein-dilaton-Gauss-Bonnet coupling parameter
$\zeta_{\rm EDGB}$	Einstein-dilaton-Gauss-Bonnet dimensionless coupling $16\pi\bar{\alpha}_{\rm EDGB}^2/M_t^4$
$\bar{\alpha}_{\rm DCS}$	Dynamical Chern-Simon coupling parameter
$\zeta_{\rm DCS}$	Dynamical Chern-Simon dimensionless coupling $16\pi\bar{\alpha}_{\rm DCS}^2/M_t^4$
G_C	Gravitational constant in conservative sector in a varying-G theory
G_D	Gravitational constant in dissipative sector in a varying-G theory

$\omega_{\rm BD}$	Brans-Dicke coupling parameter
$s(t)$	Detector output $n(t)+h(t)$
$n(t)$	Noise in the detector output
$h(t)$	GW signal in the detector output
$S_n(f)$	Noise spectral density
$(A\|B)$	Inner product of A and B weighted by detector sensitivity [Eq. (3.2)]
ρ	Signal-to-noise ratio $\sqrt{(h\|h)}$
Γ_{ij}	Fisher information matrix $(\partial_i h\|\partial_j h)$ [Eq. (3.6)]
α_r	Right ascension angle
δ	Declination angle
$\psi_{\rm polar}$	Polarization angle
ι	Inclination angle
z	Redshift
D_L	Luminosity distance
H_0	Hubble constant 70 km $\mathrm{s}^{-1}\mathrm{Mpc}^{-1}$ [1]
Ω_Λ	Relative universal vacuum energy density 0.6889 [1]
Ω_Λ	Relative universal matter density 0.3111 [1]
$\mathcal{M}_z$	Red-shifted chirp mass $\mathcal{M}_{ch}(1+z)$
ϕ	Dilaton field in Einstein-Maxwell-dilaton theory [Eqs. (4.32)-(4.33)]
$\mathfrak{R}$	Radius of compactification of an extra dimension
ℓ	Length of a compact extra dimension
$G^{(5)}$	Bare gravitational constant in $5d$ Einstein-Hilbert action
$G_{\rm N}$	Newton's constant in a Kaluza-Klein reduced theory [Eq. (4.27)]
$\mathcal{G}_{MN}$	Metric tensor in a $5d$ Kaluza-Klein theory [Eq. (4.30)]
$\mathcal{G}^{(5,c)}$	Retarded Green's function in a $5d$ Kaluza-Klein theory
$\mathfrak{z}$	Radial distance in a $5d$ Kaluza-Klein theory
$\mathcal{R}[\mathcal{G}_{MN}]$	Ricci scalar of metric $\mathcal{G}_{MN}$
t	Time coordinate in harmonic gauge
R	Radial coordinate in harmonic gauge
y^A	Angular coordinates in harmonic gauge
u	Retarded time in Bondi gauge
r	Radial coordinate in Bondi gauge
x^A	Angular coordinates in Bondi gauge
$g_{\mu\nu}$	Metric of a $4d$ spacetime

$R^{\mu}{}_{\nu\alpha\beta}$	Riemann tensor of metric $g_{\mu\nu}$
$R_{\mu\nu}$	Ricci tensor $R^{\mu}{}_{\alpha\mu\beta}$
$\mathcal{R}$	Ricci scalar R^{μ}_{μ} [Eq. (5.12)]
$T_{\mu\nu}$	Stress-energy tensor
$G_{\mu\nu}$	Einstein tensor $R_{\mu\nu} - \frac{1}{2}\mathcal{R}\, g_{\mu\nu}$
h_{AB}	Conformal 2-metric
$\mathscr{R}$	Ricci scalar of metric h_{AB}
q_{AB}	unit-sphere metric
$\eth_A$	covariant derivative compatible with metric q_{AB}
$Ð^2$	Laplacian of metric q_{AB}
$\mathcal{M}(u, x^A)$	Bondi mass aspect [Eq. (5.32b)]
Φ	Scalar field in a scalar-tensor theory in Einstein frame
λ	Scalar field in a scalar-tensor theory in Jordan frame
c_{AB}	Shear tensor [Eq. (5.22a)]
N_{AB}	News tensor $\partial_u c_{AB}$
$L_A(u, x^A)$	Bondi Angular momentum aspect [Eq. (5.24)]
$\alpha(x^A)$	Supertranslation [Eq. (5.39)]
Y^A	Conformal Killing vector on the 2-sphere [Eq. (5.38)]

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Chapter 1

Introduction

The most successful theory of gravitation so far is general relativity (GR), put forth by Albert Einstein in 1915, which describes gravity as a curvature of spacetime. The theory is appreciated for its elegant mathematical structure and solid conceptual foundation and has passed all the tests with high precision [2]. Nevertheless, some theoretical and observational motivations lead to the demand for a modification to GR. Regarding theoretical reasons, GR is a purely classical theory and incompatible with quantum mechanics, which describes microscopic phenomena. Quantum effects can be important in the case of strong gravitational fields at Planck scale [3,4], such as in the vicinity of black holes (BHs) and the very early universe. A complete description of such scenarios requires a consistent theory of quantum gravity. On the observational side, the accelerated expansion of the universe [5–12] and anomalous galactic rotation curves [13–19] suggest that one may need to go beyond GR to explain such cosmological phenomena [1]. Many theories of gravity have been proposed

[1] Alternatively, one can introduce dark energy or dark matter and work in the framework of GR. However, the states of dark matter and dark energy are currently unknown.

which modify the known theory of GR, for example, by introducing additional fields, interactions, or extra dimensions. Collectively such theories are dubbed as modified theories of gravity [20–22].

One of the many remarkable predictions of GR which is also shared by modified theories of gravity is the existence of gravitational waves (GWs): ripples in spacetime that are produced by explosive events such as coalescing binary black holes (BBHs) or neutron stars. In 2015, the Laser Interferometer Gravitational-Wave Observatory (LIGO) Scientific Collaborations and the Virgo Collaboration (LVC) detected GWs from a pair of black holes (BHs) for the first time [23, 24]. This historic event opened new possibilities for testing gravity in the strong-field and dynamical regime [25–27]. During the first three observing runs LVC detected fifty BBH merger events and two binary neutron star (BNS) events [28,29], furthering the advance of the relatively new field of GW astronomy [30–35].

Before the detection of GWs, tests of gravity were performed predominantly by solar system experiments, observations of radio pulsars, cosmological observations, and table-top experiments. Each such observation explores a particular regime of field strength and length scale. For example, solar system experiments constrain gravity in the weak-field and slow-motion environment, allowing one to probe only first order relativistic corrections to Newtonian dynamics [2,36]. As with pulsar timing observations of neutron stars (NSs), we can achieve both strong-field and weak-field tests of gravity to some extent [37–46]. This is because NSs are compact objects that are sources of strong-field gravity; however, the separation between the NSs in a binary pulsar is large, and thus the binary components are slowly moving. Cosmological observations can test gravity at length scales that are many orders of magnitude larger compared to other tests, although in weak-field regime [21, 36, 47–49]. Such tests of gravity include observations of cosmic microwave background (CMB) radiation [50,51, 51–54], studies of Big Bang Nucleosynthesis [55–61], weak gravitational lensing [62–66] and observations of galaxies [36]. Table-top experiments take place in a weak-field regime and have been able to probe deviation to gravitational inverse-square law to a micrometer scale [67, 68]. Probing gravity with GW observations is unique compared to all other tests, as GWs originate from strong and highly dynamical field regimes of colliding compact objects (see figure 1.1). In particular, accessing such a regime is vital to constrain modified theories of gravity. Many viable modified theories of gravity coincide with GR in the weak-field and slow-motion regime, making strong-field tests the most promising sector for exploring their predictions.

With GW events detected so far, several tests of GR in strong-field regimes have been performed [25,27,30–34,69,70]. For example, model-independent tests of gravity with BBH mergers have been carried out by evaluating the amount of residuals in the detected signals from best-fit waveforms [25, 32]. Consistency tests of GR between inspiral and post-inspiral phases were also performed, by measuring the remnant BH's mass and spin independently from the two phases [25, 35]. Adding the Virgo detector to GW observations let us search for non-tensorial polarizations [32, 70].

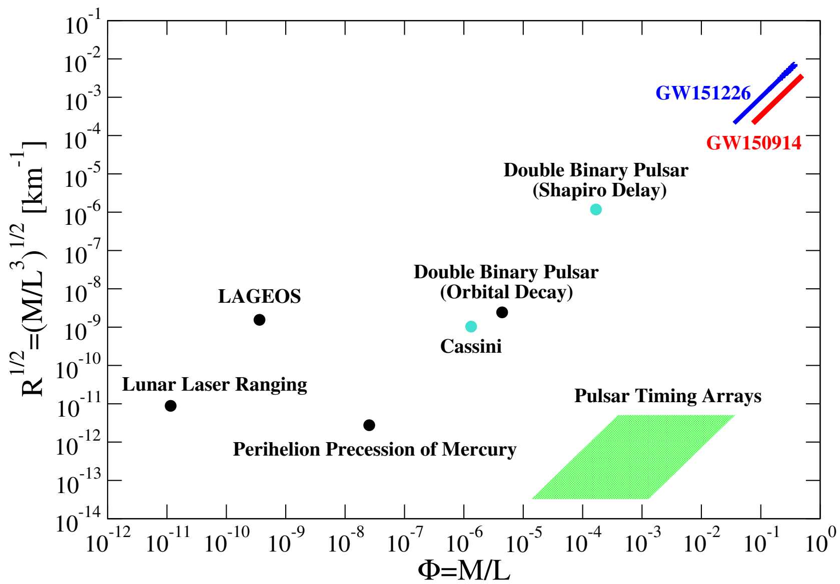

Figure 1.1: Curvature-potential phase space probed by various experiments that test gravity. The vertical axis shows the inverse of characteristic curvature length, and the horizontal axis shows the characteristic gravitational potential. M and L denote typical mass and size of the system being probed. GW150914 [23] and GW151226 [24] sample a curvature regime and gravitational potential that is strong (and dynamical, illustrated by the finite range the curves take over). The finite region of the pulsar timing array is because of the range in GW frequency and total mass of supermassive BBHs such arrays possibly detect in the future. (Image taken from Ref. [27])

With the BNS merger event GW170817, the arrival time difference between GWs and electromagnetic (EM) waves has been utilized to constrain the violation of Lorentz invariance and to execute a new test of the equivalence principle through the Shapiro time delay [71]. A constraint on the propagation speed of GWs with the same event has been used to rule out several modified theories of gravity that aim to explain the accelerating expansion of the universe [72–79]. An analysis of 31 BBH merger events constrained modified dispersion relation and graviton mass [69]. GW observations have also been used to constrain extra dimensional models and time-varying Planck mass [69,80]. This thesis will present several works that contributed to probing strong-field gravity with GWs, including both theory-specific tests and theory-agnostic or model-independent tests.

The central part of this thesis consists of four main projects: (1) probing modified theories of gravity in a model-independent way with GWs [26,81], (2) probing compact extra dimensions with GWs [80], (3) asymptotically flat solutions, asymptotic symmetries, and GW memory effects in Brans-Dicke theory [82], (4) gravitational memory waveforms generated by quasi-circular nonprecessing binaries in Brans-Dicke theory. We now provide an introduction to the topics covered by the above projects before we summarize important results in section 1.1.

Let us first discuss theory-agnostic or model-independent approaches of probing gravity with GWs. A model-independent approach of probing modified theories of gravity parametrizes the deviation of the gravitational waveform from that of GR in a theory-agnostic way. It is possible to derive gravitational waveform in each theory and compare it with GW observations; nevertheless, a more effective way is to conduct theory-agnostic tests at first and subsequently map the information to specific non-GR theory parameters. Some early works in this direction were done in Refs. [83–85] where the authors considered post-Newtonian [2] (PN) terms in the GR waveform as independent of each other and proposed to study the consistency among them. However, one shortcoming of such an approach is that it cannot accommodate the non-GR effects entering at PN orders absent in GR, such as -1PN order which occurs in, e.g., scalar-tensor theories. A new framework called *parametrized post-Einsteinian* (PPE) formalism was proposed by Yunes and Pretorius, which overcome such shortcomings by introducing generic corrections at any PN order to both the phase and the amplitude [86,87] (see figure. 1.2). Such a formalism was later extended to include time domain waveforms [88], eccentric binaries [89], and a sudden turn on of non-GR effects [90, 91]. A model-independent data analysis pipeline named TIGER was also developed in Refs. [92, 93]. The LVC implemented the generalized IMRPhenom (gIMR) waveform model, which has a one-to-one correspondence with the PPE framework in the inspiral part of the waveform phase, to carry out tests of gravity with the GW phase [25, 27, 30, 32, 71, 94].

In our work [26, 81], we aim to carry out model-independent tests of gravity with both amplitude and phase corrections, and as such, we implement the PPE formal-

[2] A post-Newtonian (PN) expansion of the gravitational waveform is an expansion in the small parameter v^2/c^2, where c is the speed of light, and v is the typical velocity of binary components.

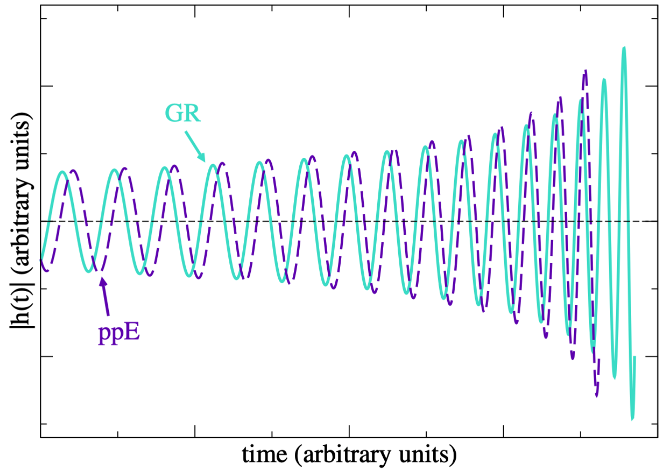

Figure 1.2: Gravitational waveform in GR and a PPE waveform with non-GR corrections injected at -1PN order. PPE formalism can capture non-GR effects that enter in the waveform at a PN order absent in GR. (Image taken from Ref. [95])

ism [86, 87]. Most works done previously in testing gravity with GWs focused on corrections to the GW phase only, given that the matched filtering technique is more sensitive to phase corrections than amplitude ones. However, there are scenarios where probing the amplitude correction is beneficial, such as the parity-violating theories, which predict amplitude birefringence [3] [96–99]. Probing amplitude corrections is also important in constraining theories that predict stochastic GW backgrounds [100, 101]. Moreover, theories with flat extra dimensions [102], Horndeski gravity [103], and $f(R)$ gravity [104] predict possible amplitude damping that scales with the cosmological distance. Finally, if the waveform template does not accommodate a non-GR effect in amplitude, it may cause systematic errors in other parameters like the luminosity

[3] In some parity-violating theories, one of the circularly-polarized modes is amplified while the other one is suppressed. Such an effect is called amplitude birefringence and enters only in circularly-polarized modes.

distance.

We next consider probing extra dimensions with GWs. Extra dimensions arise in string theories that aim to achieve a quantum theory of gravity, and such dimensions can leave an imprint on GWs. For instance, in flat (non-compact) D-dimensional spacetime, GWs decay as $1/R^{(D-2)/2}$ [102], where R is the distance travelled by GWs. Constraints on such models have also been computed with GW observations [31,105]. In our work, we consider GWs in a $5d$ spacetime, with the fifth dimension being compactified on a circle (see figure 1.3) and with the matter constrained on the $4d$ spacetime called "brane". Such a model of gravity is called the Kaluza-Klein (KK) theory [106]. In a KK theory, one considers Einstein-Hilbert action in $5d$. One can further adopt a metric that is periodic in the extra dimension since the extra dimension is compactified on a circle. Plugging the metric in Einstein-Hilbert action, it is possible to integrate out the extra dimension to obtain an effective $4d$ action. Such a procedure is called KK reduction, and the effective $4d$ action represents a theory containing tensor, vector, and scalar fields, known as Einstein-Maxwell-dilaton (EMD) theory [107–110].

In warped or compactified extra dimensional models, KK compactification leads to scalar polarizations (due to massless scalar fields), as well as massive KK modes with frequencies much higher than the range of ground-based detectors [111,112]. Such KK modes can produce a stochastic GW background [113] and modify quasinormal modes after binary mergers [114, 115]. Furthermore, in the RS-II braneworld model [116],

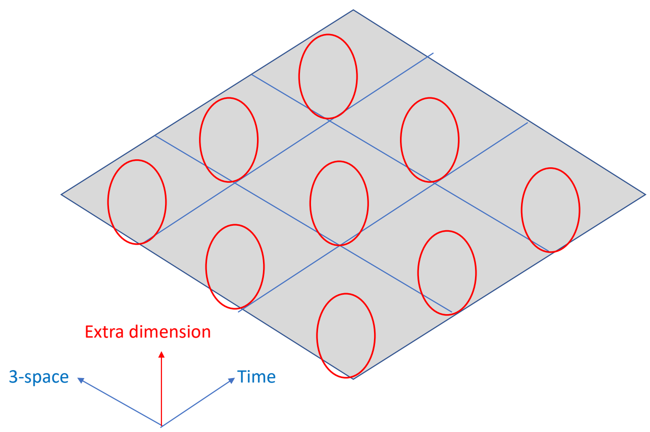

Figure 1.3: An extra dimension compactified on a circle. The flat surface represents $4d$ spacetime or "brane" that we live in. The fifth dimension is small and curled up, and can be visualized as a circle at each point of the flat spacetime.

BHs can evaporate classically [117, 118], modifying the orbital evolution of BHs and changing the waveform from that of $4d$ GR. Such cases have been considered to place bounds on the size of the extra dimension with GW150914 [27], although the bounds are much weaker than the current most stringent bound obtained from table-top experiments [67, 68]. Tidal deformabilities of BHs and NSs have been computed in braneworld models and have been used to constrain the brane tension with GW170817 [119, 120]. Finally, one can consider yet another fact that GWs can propagate through the higher dimensional bulk, while EM waves are constrained to travel on a brane. The difference in the propagation of such two waves can probe the extra dimensions [121–125], and such a case has been applied to GW170817 [126–129], although the bounds are much weaker than those of table-top experiments.

We next explore GW memory effects in a modified theory of gravity. To under-

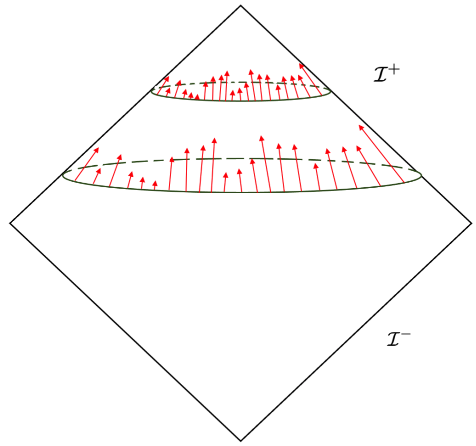

Figure 1.4: Supertranslations represented in a conformally compactified spacetime. $\mathcal{I}^+$ and $\mathcal{I}^-$ denote future and past null infinity, respectively. A green circle denotes a 2-sphere at future null infinity. Under a supertranslation, each point on the 2-sphere is shifted independently along the direction of retarded time, as indicated by red arrows. On the other hand, under an ordinary (Poincaré) translation, each point would be shifted by the same amount.

stand our work, let us first introduce memory effects and related formulations in GR. GW memory effects are enduring changes in the GW strain and its time integrals after bursts of GWs [130–132]. Such effects are too small to be detected from an individual event with current ground-based GW detectors; nevertheless, they could be detected from a population of BBH mergers observed by the Advanced LIGO and Virgo detectors for several years [133–135]. On the other hand, future space-based detector LISA holds promise to detect such effects from a single event [136, 137]. Third-generation ground-based detectors such as Einstein Telescope or Cosmic Explorer [138] can potentially detect such effects as well.

Memory effects in GR are closely linked to symmetries of asymptotically flat spacetime. The asymptotic symmetry group of such a spacetime consists of a semidirect product of Lorentz group and an infinite dimensional subgroup of translations called the supertranslations (see figure 1.4), forming the Bondi-Metzner-Sachs (BMS) group [139–141]. The Lorentz part of the BMS group can also be extended to include all conformal Killing vectors on the 2-sphere called superrotations (or superboosts) [142–145] or all smooth vector fields on the 2-sphere [146, 147], and the resultant group is the extended BMS group. There are conserved charges associated with BMS symmetries [4] [148–151]. Charges conjugate to superstranslation symmetries are termed as supermomenta. The ones corresponding to Lorentz symmetries are six components of the relativistic angular momentum, which can be divided into center-of-mass (CM) and spin parts. In the case of the extended BMS group, charges corresponding to the Lorentz part are super CM and super spin charges (or super-angular momenta) [152, 153].

As mentioned, there are memory effects corresponding to asymptotic symmetries, and they can be computed from associated fluxes of conserved charges or BMS flux balance laws [152,154]. Supertranslation symmetries are associated with displacement memory effects predicting that after a burst of GWs, the separation between two distant initially comoving observers will have a permanent shift [153,155] (demonstrated in figure 1.5). On the other hand, Super CM and super spin charges are related to center-of-mass and spin memory effects, respectively. Such effects predict changes in the separation that grows linearly with time for two observers with an initial relative

[4] These charges are conserved in the sense that the difference in the charges between two times is equal to the flux of the quantity between these two times.

velocity [152, 154, 156]. In the case of GWs generated by compact binaries, the displacement memory effect enters as a shift in GW strain, while CM and spin memory effects enter in the time integral of the strain. BMS flux balance laws provide an efficient way of computing GW memory effects from waveforms without memories. Such a technique makes use of both Bondi gauge and harmonic gauge quantities and has been implemented to derive displacement, spin, and CM memory waveforms from compact binaries in GR [154, 157–162]. On the other hand, it is possible to capture memory effects working in harmonic gauge alone, but it requires one to go to a sufficiently high PN order in the waveform (2.5PN order in case of displacement memory waveforms in GR [163, 164]).

Although Memory effects and their correspondence with symmetries and conserved charges are well studied in GR, there remain open questions regarding memories in a non-GR theory. Are symmetries of asymptotically flat spacetimes in modified theories of gravity the same as those of GR? As there are additional polarizations in modified theories of gravity [2, 87, 165], one expects additional memory effects. Do the additional memories have a similar relationship with asymptotic symmetries as those of GR? To answer such questions, we pick one of the simplest possible modified theory of gravity: Brans-Dicke (BD) theory [166]. BD theory is a scalar-tensor theory with a single massless scalar field nonminimally coupled to gravity [36]. Scalar-tensor theories emerge in the contexts of string theories, inflation [21, 167], and the cosmic acceleration of the Universe [168–170]. The massless scalar field in BD theory leads to

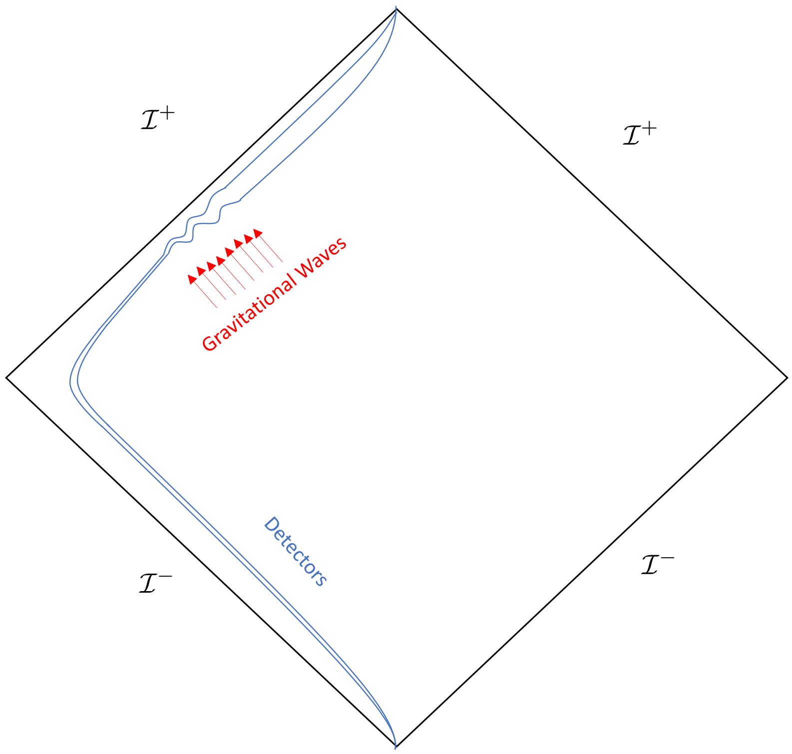

Figure 1.5: Displacement memory effect presented in a conformally compactified spacetime. A pair of initially comoving detectors near future null infinity undergo a temporary oscillation when GWs pass by them. After the passage of GWs, the separation between the detectors is shifted permanently due to the displacement memory effect.

an additional breathing type polarization of GWs that produces a relative contraction and expansion of a freely falling ring of particles in the plane transverse to the direction of GW propagation. We will explore asymptotically flat solutions, asymptotic symmetries, and GW memory effects in BD theory, as well as gravitational memory waveforms generated by compact binaries in such a theory.

1.1 Executive Summary

We now move on to explaining the primary outcomes of this thesis briefly. A more in-depth summary of each project can be found in the concluding sections of chapters covered by the project. Theory-agnostic tests of gravity with GWs are considered in chapters 2 and 3. Chapter 4 explores GWs in higher dimensional gravity with compact extra dimensions. Chapter 5 studies BD theory in Bondi-Sachs form, while chapter 6 presents displacement and spin memory waveforms generated by compact binaries in BD theory.

1.1.1 Model-independent tests of modified theories of gravity

Let us summarize the model-independent works considered in this thesis. First, we compute analytic expressions of PPE waveforms (the deviation from GR phase and amplitude in a PPE foramlism are called the PPE phase and amplitude parameters, respectively, and the waveform is called the PPE waveform) in terms of theory parameters in a collection of modified theories of gravity [81]. The theories we consider include scalar-tensor theories [171,172], Einstein-dilaton-Gauss-Bonnet (EDGB) gravity [173], dynamical Chern-Simon (DCS) gravity [27, 174], Einstein-Æther theory [175], Khronometric gravity [175], noncommutative gravity [176], and varying-G theories [81, 177]. Each such theory violates one or more fundamental pillars of GR (parity invariance, strong equivalence principle, Lorentz invariance, absence of extra dimensions etc.) predicting a small non-GR deviation in GWs. The analytical expressions of PPE parameters are summarized in table 2.1 and table 2.2 in chapter 2.

Theories	Repr. Parameter	Constraints					
		GW150914			GW151226		
		Phase	Amplitude	Combined	Phase	Amplitude	Combined
EDGB [173]	$\sqrt{\lvert\bar{\alpha}_{\text{EDGB}}\rvert}$ [km]	(50.5)	(76.3)	(51.5)	4.32	10.5	4.32
	ζ_{EDGB}	3.62	32.4	3.91	0.0207	0.709	0.0207
Scalar-Tensor [171, 172]	$\lvert\dot{\Phi}\rvert$ [10^4/sec]	(3.64)	(7.30)	(3.77)	1.09	(5.60)	1.09
	$\lvert m_1\dot{\Phi}\rvert$	6.87	16.4	7.15	0.688	3.66	0.688
Varying-G [81, 177]	$\lvert\dot{G}_0/G_0\rvert$ [10^6/yr]	7.30	137	7.18	0.0224	0.382	0.0220

Table 1.1: 90% credible constraints on representative parameters of various modified theories of gravity from GW150914 and GW151226. For each of the GW events, "phase" and "amplitude" correspond to the cases where we include non-GR corrections only to the GW phase and amplitude respectively, while "combined" is the case where we include both corrections in the waveform and reduce the two constraints to a single one according to Sec. 3.2. $\bar{\alpha}_{\text{EDGB}}$ is the EDGB coupling parameter which is related to the dimensionless coupling by $\zeta_{\text{EDGB}} \equiv 16\pi\bar{\alpha}_{\text{EDGB}}^2/M_t^4$ with M_t being the total mass of the binary. $m_1\dot{\Phi}$ corresponds to a dimensionless parameter in scalar-tensor theories where m_1 is the mass of the primary BH while Φ is the scalar field. The bounds are derived by assuming subdominant non-GR corrections, which is realized whenever $\zeta_{\text{EDGB}} \ll 1$ ($m_1\dot{\Phi} \ll 1$) in EDGB (scalar-tensor) gravity. Numbers inside brackets mean such criterion is violated and the constraints are unreliable. G is the gravitational constant with the subscript 0 representing the time of coalescence. An overhead dot denotes a derivative with respect to time.

Next, we perform numerical analyses (Fisher analyses with Monte-Carlo simulations) implementing PPE waveforms and utilizing LIGO/Virgo posterior samples of the first two GW events: GW150914 and GW151226. Through Fisher analyses with Monte-Carlo simulations, we obtain bounds on PPE parameters at various PN orders [26]. Mapping such bounds to specific theories, we achieve constraints on EDGB gravity, scalar-tensor theories, and varying-G theories from GW observations, which we present in table 1.1. In particular, we obtain a reliable bound on the time-evolution of a scalar field in a scalar-tensor theory from GW observations for the first time. We derive constraints from the GW phase, GW amplitude and combining those from phase and amplitude for each theory. By comparing such constraints, we conclude that bounds coming from amplitude corrections can be comparable to those from the phase corrections for massive binaries like GW150914. Furthermore, a combined constraint differs from that coming from phase only at most by 4%, which validates many previous studies that focused on GW phase only.

1.1.2 Probing compact extra dimensions with GWs

To probe compact extra dimensions with GWs, we work in the formalism of GR in $5d$ instead of considering the effective $4d$ theory. This is because it is much easier to work with one tensor field (metric) than a collection of tensor, scalar, and vector fields. Besides, we wish to extend our work to an arbitrary number of compact extra dimensions, which is possible if we work in the higher dimensional framework. We assume that binaries are point-like sources located on a $4d$ brane, and the extra dimension is much smaller compared to binary separation. Through direct integration of relaxed Einstein equations (DIRE) [5] [178] procedure, we compute GW perturbations, modified Kepler's law, and GW luminosity. Finally, with the help of modified Kepler's law and GW luminosity, we derive the GW phase in the frequency domain and compare it with observations [80].

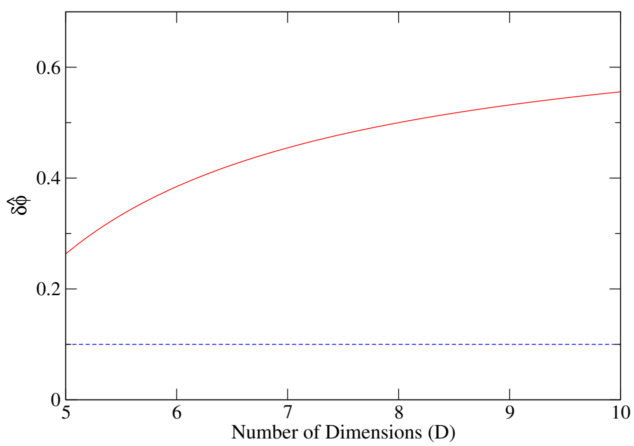

Figure 1.6: The fractional difference ($\delta\hat{\phi}$) of the GW phase with respect to that of the $4d$ GR as a function of the number of dimensions (D), represented by the solid red curve [80]. The blue dashed line shows the upper bound placed on $\delta\hat{\phi}$ from the combined events of the first and second observing runs of the LIGO/Virgo [32].

Our work shows that in the $5d$ case, leading order GW luminosity and waveform

[5]DIRE approach is a framework for computing GW emission from isolated gravitating sources in PN approximation [2, 178]. In such an approach, the Einstein equations are cast into flat spacetime wave equations along with a harmonic gauge condition, and the source term contains both matter and GW stress-energy pseudotensor. The equations are then integrated over the past light cone to obtain GW perturbations by splitting the integration regime into the near zone and far zone. We implement the DIRE approach because the conventional quadrupole formula of gravitational radiations does not hold in the presence of compact extra dimensions [80].

phase differs from that of a $4d$ binary of same masses and separation distance considered in GR by 20.8% and 26% respectively. Because there is a degeneracy of the leading order non-GR phase correction with the chirp mass, it is difficult to disprove the theory with GW observations alone. However, one can consider binary pulsar observations where additional measurements of post-Keplerian parameters break this degeneracy, and one can rule out this theory by comparing it with such observations. Furthermore, gravitational waveforms in the corresponding effective $4d$ theory (EMD theory) have been derived previously [108–110]. We compare our results with that of the EMD theory and find that the results are consistent with each other. We further generalize our work to an arbitrary number of extra compact dimensions, which shows that the discrepancy of the calculated GW phase with the GW and pulsar observations increases with the number of dimensions (shown in figure 1.6), which effectively *rule out* this class of theories that contain extra dimensions compactified on circles.

1.1.3 Asymptotically flat solutions, asymptotic symmetries, and GW memory effects in Brans-Dicke theory

We study BD theory in the Bondi-Sachs framework [139,140] [6] in chapter 5 and derive the asymptotically flat solutions, asymptotic symmetries and associated conserved charges, and GW memory effects [82]. We find that in such a theory, the asymptotic

[6] Bondi-Sachs framework is a metric-based formulation of gravity where coordinates are well suited to the null hypersurfaces and the null geodesics of the spacetime.

symmetry group is interestingly the same as GR, i.e., the BMS group. We further study the effect of GWs and scalar waves on freely falling observers and find two new memories in addition to memories present in GR [82]. These two memories are related to the additional breathing polarization and are called scalar memories (to distinguish them from tensor memories of GR). One of the scalar memories is a displacement memory effect explained in figure 1.7, and the other one depends on the initial relative velocity of test masses and grows linearly with time. Since the number of spacetime symmetries remains the same as that of GR, new scalar memories cannot be computed from fluxes of conserved charges (or BMS flux balance laws), which constrain only tensor memories. Nevertheless, the tensor memories themselves receive non-GR contributions from the scalar energy flux, and we can compute those contributions, which we discuss next.

1.1.4 GW memory effects generated by compact binaries in Brans-Dicke theory

Implementing the results of chapter 5, we study memory effects in gravational waveforms in Brans-Dicke theory in chapter 6. Scalar and tensor wave solutions for compact binaries in BD theory have already been computed in harmonic gauge coordinates using the DIRE approach previously [179, 180]. However, similar to the case of GR, one needs to go to sufficiently high PN order in the waveforms to capture memory effects in such an approach. Ref. [179] found displacement memory effect

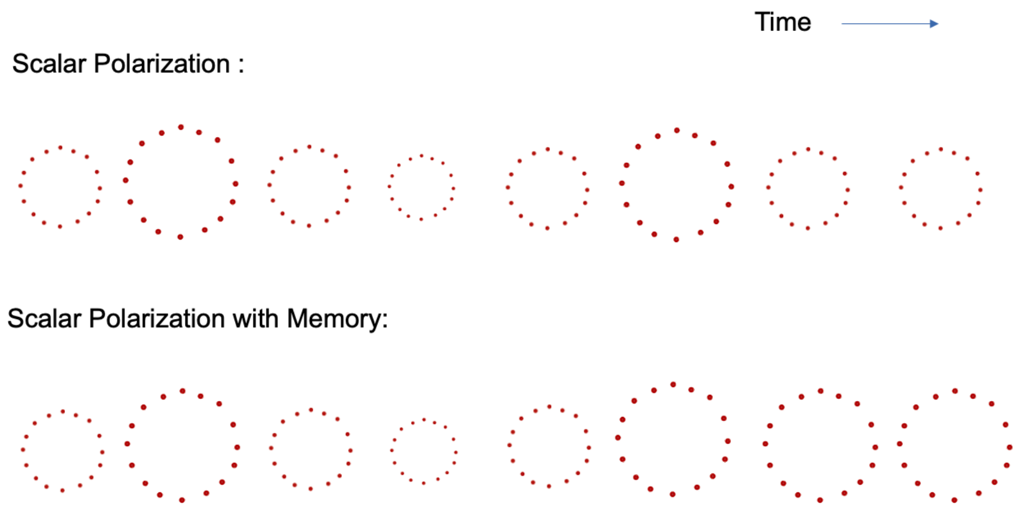

Figure 1.7: Displacement memory effect associated with the scalar or breathing polarization. The figures on the top show the effect of scalar polarization on a ring of particles without taking into account memory effects. GWs and scalar waves propagate into the page, and the evolution of the ring of particles with time is shown from left to right. The ring of particles expands and contracts sequentially and is left in the initial circular shape once the waves have passed. The figures on the bottom show the same scenario but with a scalar displacement memory effect. A lasting uniform expansion (or contraction) occurs, and the ring of particles retains an expanded (or contracted) shape after the waves have left.

in the tensor waveform in BD theory using such an approach at 1.5PN, which after integrating over binary's history becomes effectively -1PN order. We adopt the alternative approach of using relatively low PN order waveforms found in harmonic gauge to BMS flux balance laws in Bondi gauge that we derived in chapter 5. To do so, we perform coordinate transformation from Bondi gauge to harmonic gauge. Such a transformation allows us express Bondi gauge quantities in BMS balance laws in terms of the harmonic gauge quantities already available in Refs. [179, 180]. Next, we evaluate the BMS balance laws to obtain displacement and spin memory effects in BD theory for a quasi-circular nonspinning binary. The waveforms we compute in this approach are complete to Newtonian order (0PN), valid under the assumption that non-GR BD corrections to waveforms are small compared to GR contributions. There are two features of tensor memory waveforms in BD theory that are different from their GR counterparts. First, the leading PN order of such waveforms is at -1PN due to the scalar dipole radiation. Second, the 0PN part of such waveforms has a dependence on the inclination angle that is different from those of GR. Our waveforms can potentially be used to constrain BD theory with memory effects, e.g., by considering projected events with the space-based detector LISA [181], or by stacking multiple signals from projected events with LIGO/Virgo or third-generation detectors [182, 183]. Especially, the difference in the dependence on inclination angle between BD theory and GR can be implemented to constrain BD theory through a hypothesis test described in Ref. [184].

1.2 Organization and conventions

Let us present a brief outline of this thesis. Chapters 2 and 3 focus on model-independent tests of gravity. In particular, in chapter 2 we present the analytical results of PPE waveforms in various modified theories of gravity. In chapter 3, we perform data analyses with events GW150914 and GW151226 in a theory-agnostic approach and then map the bounds to specific theories. Chapter 4 discusses GWs in theories with compact extra dimensions, focusing on modified Kepler's law, GW perturbations, GW luminosity, and GW phase in such theories. In chapter 5, we study BD theory in the Bondi-Sachs framework and present asymptotically flat solutions, asymptotic symmetries, and associated conserved charges, and GW memory effects. Using the results of chapter 5, in chapter 6. we compute gravitational memory waveforms in BD theory generated by quasi-circular nonspinning binaries in PN approximations. At the end of each chapter, we present a concluding section highlighting the important results of that chapter. We relegate some of the more technical details to Appendices.

Throughout this thesis, we use units in which the speed of light $c = 1$, and we use the conventions for the metric and curvature tensors given in [185]. Greek indices $(\mu, \nu, \alpha, \dots)$ represent four-dimensional spacetime indices, while indices with circumflex diacritic (e.g., $\hat{\alpha}$) represent those of an orthonormal tetrad. Lower case Latin indices $(i, j, k, \dots)$ denote spatial indices in quasi-Cartesian harmonic coordinates in four-dimensional spacetime. Uppercase Latin indices $A, B, C, \dots, H$ represent indices on the 2-sphere. On the other hand, uppercase Latin indices $M, N, \ldots, S$ denote spacetime indices in a higher-dimensional scenario, while I, J, K, L denote spatial indices in the same setup.

Chapter 2

PPE Waveforms in Various Modified Theories of Gravity

2.1 Introduction

In this chapter, we derive PPE waveforms in various modified theories of gravity [1]. Many of previous literature focused on deriving phase corrections since matched filtering is more sensitive to such phase corrections than to amplitude corrections. Having said this, there are situations where amplitude corrections are more useful to probe, such as amplitude birefringence in parity-violating theories of gravity [96–99] and testing GR with astrophysical stochastic GW backgrounds [100]. We first derive PPE amplitude and phase corrections in terms of generic modifications to the frequency evolution and Kepler's third law that determine the waveform in Fourier domain. For our purpose, this formalism is more useful than that in [87], which derives the amplitude and phase corrections in terms of generic modifications to the binding energy of a binary and the GW luminosity. We follow the original PPE framework and focus

[1]This chapter is based on the following paper: *Parameterized Post-Einsteinian Gravitational Waveforms in Various Modified Theories of Gravity*; Tahura, Sharaban; Yagi, Kent; Phys. Rev. D 98, 084042 (2018)

on deriving leading PN corrections in tensorial modes only [86, 186]. Non-tensorial GW modes also typically exist in theories beyond GR, though at least in scalar-tensor theories, the amplitude of a scalar polarization is of higher PN order than amplitude corrections to tensor modes [87, 187].

Non-GR corrections can enter in the gravitational waveform through activation of different theoretical mechanisms, which can be classified as generation mechanisms and propagation mechanisms [27]. Generation mechanisms take place close to the source (binary), while propagation mechanisms occur in the far-zone and accumulate over distance as the waves propagate. In this chapter, we focus on the former[2]. The PPE parameters in various modified theories of gravity are summarized in Tables 2.1 (phase corrections) and 2.2 (amplitude corrections). Some of the amplitude corrections were derived here for the first time. We also correct some errors in previous literature.

The rest of the chapter is organized as follows: In Sec. 2.2, we revisit the standard PPE formalism. In Sec. 2.3, we derive the PPE parameters in some example theories following the formalism in Sec. 2.2. In Sec. 2.4, we derive the PPE parameters in varying-G theories. We summarize our work and discuss possible future prospects in Sec. 2.5. Appendix A discusses the original PPE formalism. In App. B, we derive the frequency evolution in varying-G theories from the energy-balance law. We use the geometric units $G = c = 1$ throughout this chapter except for varying-G theories.

[2] PPE waveforms due to modifications in the propagation sector can be found in [27, 188, 189], which have been used for GW150914, GW151226 [27] and GW170104 [94] to constrain the mass of the graviton and Lorentz violation.

Theories	PPE Phase Parameters: Magnitude ($\beta_{\rm PPE}$)	PPE Phase Parameters: Exp. (b)	Binary Type
Scalar-Tensor [171, 172]	$-\frac{5}{7168}\eta^{2/5}(\alpha_1-\alpha_2)^2$	-7	Any
EDGB [173]	$-\frac{5}{7168}\zeta_{\rm EDGB}\frac{(m_1^2\tilde{s}_2^{\rm EDGB}-m_2^2\tilde{s}_1^{\rm EDGB})^2}{M_t^4\eta^{18/5}}$	-7	Any
DCS [27, 174]	$\frac{481525}{3670016}\eta^{-14/5}\zeta_{\rm DCS}\left[-2\delta_m\chi_a\chi_s+\left(1-\frac{4992\eta}{19261}\right)\chi_a^2 +\left(1-\frac{72052\eta}{19261}\right)\chi_s^2\right]$	-1	BH/BH
Einstein-Æther [175]	$-\frac{5}{3584}\eta^{2/5}\frac{(s_1^{\rm EA}-s_2^{\rm EA})^2}{[(1-s_1^{\rm EA})(1-s_2^{\rm EA})]^{4/3}}\left[\frac{(c_{14}-2)w_0^3-w_1^3}{c_{14}w_0^3w_1^3}\right]$	-7	Any
Khronometric [175]	$-\frac{5}{3584}\eta^{2/5}\frac{(s_1^{\rm kh}-s_2^{\rm kh})^2}{[(1-s_1^{\rm kh})(1-s_2^{\rm kh})]^{4/3}}\sqrt{\bar{\alpha}_{\rm kh}} \times\left[\frac{(\bar{\beta}_{\rm kh}-1)(2+\bar{\beta}_{\rm kh}+3\bar{\lambda}_{\rm kh})}{(\bar{\alpha}_{\rm kh}-2)(\bar{\beta}_{\rm kh}+\bar{\lambda}_{\rm kh})}\right]^{3/2}$	-7	Any
Noncommutative [176]	$-\frac{75}{256}\eta^{-4/5}(2\eta-1)\Lambda_{\rm NC}^2$	-1	BH/BH
Varying-G [177]	$\boldsymbol{-\frac{25}{851968}\eta_0^{3/5}\dot{G}_{C,0}[11M_{t,0}+3(s_{1,0}+s_{2,0} -\delta_{\dot{G}})M_{t,0}-41(m_{1,0}s_{1,0}+m_{2,0}s_{2,0})]}$	$\boldsymbol{-13}$	Any

Table 2.1: PPE corrections to the GW phase $\delta\Psi_p \equiv \beta_{\rm PPE}\underline{u}^b$ in Fourier space in various modified theories of gravity. $\underline{u} \equiv (\pi G_C \mathcal{M}_{ch} f)^{1/3}$, where $\mathcal{M}_{ch}$ and η are the chirp mass and the symmetric mass ratio of the binary respectively, and G_C is the conservative gravitational constant appearing in Kepler's third law. We adopt the unit $G_C \equiv 1$ in all theories except for the varying-G ones. The mass, sensitivity, and scalar charge of the Ath binary component are represented by m_A, s_A, and α_A respectively. $\zeta_{\rm EDGB}$ and $\zeta_{\rm DCS}$ are the dimensionless coupling constants in EDGB and DCS gravity respectively. $\tilde{s}_A^{\rm EDGB}$ are the spin-dependent factors of the scalar charges in EDGB gravity, given below Eq. (2.29) for BHs while 0 for ordinary stars. $\chi_{s,a}$ are the symmetric and antisymmetric combinations of dimensionless spin parameters and δ_m is the fractional difference in masses relative to the total mass M_t. The amount of Lorentz violation in Einstein-Æther theory and khronometric gravity is controlled by (c_1, c_2, c_3, c_4) and $(\bar{\alpha}_{\rm kh}, \bar{\beta}_{\rm kh}, \bar{\lambda}_{\rm kh})$ respectively. w_s is the propagation speed of the spin-s modes in Einstein-Æther theory given by Eqs. (2.36)-(2.38), and $c_{14} \equiv c_1 + c_4$. The representative parameter in noncommutative gravity is $\Lambda_{\rm NC}$. The subscript 0 in varying-G theories denotes that the quantity is measured at the time of coalescence t_0, while a dot refers to a time derivative. $\delta_{\dot{G}}$ is the fractional difference between the rates at which conservative and dissipative gravitational constants change in time. The former is G_C as already explained while the dissipative gravitational constant is defined as the one that enters in the GW luminosity through Eq. (2.8). The boldface expression indicates that it has been derived here for the first time.

Theories	PPE Amplitude Parameters	
	Magnitude (α_{PPE})	Exponent (a)
Scalar-Tensor [87, 187, 190]	$-\frac{5}{192}\eta^{2/5}(\alpha_1-\alpha_2)^2$	-2
EDGB	$-\frac{\mathbf{5}}{\mathbf{192}}\zeta_{\text{EDGB}}\frac{(\mathbf{m_1^2}\tilde{\mathbf{s}}_\mathbf{2}^{\text{EDGB}}-\mathbf{m_2^2}\tilde{\mathbf{s}}_\mathbf{1}^{\text{EDGB}})^\mathbf{2}}{\mathbf{M_t^4}\eta^{\mathbf{18/5}}}$	$\mathbf{-2}$
DCS	$\frac{\mathbf{57713}}{\mathbf{344064}}\eta^{\mathbf{-14/5}}\zeta_{\text{DCS}}\left[\mathbf{-2}\delta_\mathbf{m}\chi_\mathbf{a}\chi_\mathbf{s}+\left(\mathbf{1}-\frac{\mathbf{14976}\eta}{\mathbf{57713}}\right)\chi_\mathbf{a}^\mathbf{2}+\left(\mathbf{1}-\frac{\mathbf{215876}\eta}{\mathbf{57713}}\right)\chi_\mathbf{s}^\mathbf{2}\right]$	$\mathbf{+4}$
Einstein-Æther [175]	$-\frac{\mathbf{5}}{\mathbf{96}}\eta^{\mathbf{2/5}}\frac{(\mathbf{s_1^{\text{EA}}}-\mathbf{s_2^{\text{EA}}})^\mathbf{2}}{[(\mathbf{1}-\mathbf{s_1^{\text{EA}}})(\mathbf{1}-\mathbf{s_2^{\text{EA}}})]^{\mathbf{4/3}}}\left[\frac{(\mathbf{c_{14}}-\mathbf{2})\mathbf{w_0^3}-\mathbf{w_1^3}}{\mathbf{c_{14}w_0^3w_1^3}}\right]$	$\mathbf{-2}$
Khronometric [175]	$-\frac{\mathbf{5}}{\mathbf{96}}\eta^{\mathbf{2/5}}\frac{(\mathbf{s_1^{\text{kh}}}-\mathbf{s_2^{\text{kh}}})^\mathbf{2}}{[(\mathbf{1}-\mathbf{s_1^{\text{kh}}})(\mathbf{1}-\mathbf{s_2^{\text{kh}}})]^{\mathbf{4/3}}}\sqrt{\bar{\alpha}_{\text{kh}}}\left[\frac{(\bar{\beta}_{\text{kh}}-\mathbf{1})(\mathbf{2}+\bar{\beta}_{\text{kh}}+\mathbf{3}\bar{\lambda}_{\text{kh}})}{(\bar{\alpha}_{\text{kh}}-\mathbf{2})(\bar{\beta}_{\text{kh}}+\bar{\lambda}_{\text{kh}})}\right]^{\mathbf{3/2}}$	$\mathbf{-2}$
Noncommutative	$-\frac{\mathbf{3}}{\mathbf{8}}\eta^{\mathbf{-4/5}}(\mathbf{2}\eta-\mathbf{1})\mathbf{\Lambda}_{\text{NC}}^\mathbf{2}$	$\mathbf{+4}$
Varying-G [177]	$\frac{\mathbf{5}}{\mathbf{512}}\eta_\mathbf{0}^{\mathbf{3/5}}\dot{\mathbf{G}}_{\mathbf{C,0}}\left[\mathbf{-7M_{t,0}}+(\mathbf{s_{1,0}}+\mathbf{s_{2,0}}-\delta_{\dot{G}})\mathbf{M_{t,0}}+\mathbf{13}(\mathbf{m_{1,0}s_{1,0}}+\mathbf{m_{2,0}s_{2,0}})\right]$	$\mathbf{-8}$

Table 2.2: PPE corrections to the GW amplitude $|\tilde{h}|=|\tilde{h}_{\text{GR}}|(1+\alpha_{\text{PPE}}\underline{u}^a)$ in Fourier space in various modified theories of gravity with the magnitude α_{PPE} (second column) and the exponent a (third column), and $|\tilde{h}_{\text{GR}}|$ representing the amplitude in GR. The meaning of other parameters are the same as in Table 2.1. The expressions in boldface correspond to either those derived here for the first time or corrected expressions from previous literature.

2.2 PPE Waveform

We begin by reviewing the PPE formalism. The original formalism (that we explain in detail in App. A) was developed by considering non-GR corrections to the binding energy E and GW luminosity $\dot{E}$ [86, 87]. The former (latter) correspond to conservative (dissipative) corrections. Here, we take a slightly different approach and consider corrections to the GW frequency evolution $\dot{f}$ and the Kepler's law $r_{12}(f)$, where r_{12} is the orbital separation while f is the GW frequency. This is because these two quantities directly determine the amplitude and phase corrections away from GR, and hence, the final expressions are simpler than the original ones. Moreover, non-GR corrections to $\dot{f}$ and $r_{12}(f)$ have already been derived in previous literature for many modified theories of gravity.

PPE gravitational waveform for a compact binary inspiral in Fourier domain is given by [86]

$$\tilde{h}(f) = \tilde{h}_{\rm GR}(1 + \alpha_{\rm PPE}\, \underline{u}^a)e^{i\delta\Psi_p}\,, \tag{2.1}$$

where $\tilde{h}_{\rm GR}$ is the gravitational waveform in GR. $\alpha_{\rm PPE}\, \underline{u}^a$ corresponds to the non-GR correction to the GW amplitude while $\delta\Psi_p$ is that to the GW phase with

$$\underline{u} = (\pi\mathcal{M}_{ch} f)^{\frac{1}{3}}\,. \tag{2.2}$$

$\mathcal{M}_{ch} = (m_1 m_2)^{3/5}/(m_1+m_2)^{1/5}$ is the chirp mass with component masses m_1 and m_2. u is proportional to the relative velocity of the binary components. $\alpha_{\rm PPE}$ represents the overall magnitude of the amplitude correction while a gives the velocity dependence of the correction term. In a similar manner, one can rewrite the phase correction as

$$\delta\Psi_p = \beta_{\rm PPE}\, \underline{u}^b\,. \tag{2.3}$$

$\alpha_{\rm PPE}$, $\beta_{\rm PPE}$, a, and b are called the PPE parameters. When $(\alpha_{\rm PPE}, \beta_{\rm PPE}) \equiv (0, 0)$, Eq. (2.1) reduces to the waveform in GR.

One can count the PN order of non-GR corrections in the waveform as follows. A correction term is said to be of n PN relative to GR if the *relative* correction is proportional $\underline{u}^{2n}$. Thus, the amplitude correction in Eq. (2.1) is of $a/2$ PN order. On the other hand, given that the leading GR phase is proportional to $\underline{u}^{-5}$ (see Eq. (A.12)), the phase correction in Eq. (2.3) is of $(b + 5)/2$ PN order.

As we mentioned earlier, the PPE modifications in Eq. (2.1) enter through corrections to the orbital separation and the frequency evolution. We parameterize the former as

$$r_{12} = r_{12}^{\mathrm{GR}}(1 + \gamma_r \underline{u}^{c_r})\,, \tag{2.4}$$

where γ_r and c_r are non-GR parameters which show the deviation of the orbital separation r away from the GR contribution r_{12}^{GR}. To leading PN order, r_{12}^{GR} is simply given by the Newtonian Kepler's law as $r_{12}^{\mathrm{GR}} = (M_t/\Omega^2)^{1/3}$. Here $M_t \equiv m_1 + m_2$ is the total mass of the binary while $\Omega \equiv \pi f$ is the orbital angular frequency. The above correction to the orbital separation arises purely from conservative corrections (namely corrections to the binding energy).

Similarly, we parameterize the GW frequency evolution with non-GR parameters $\gamma_{\dot f}$ and $c_{\dot f}$ as

$$\dot f = \dot f_{\mathrm{GR}} \left(1 + \gamma_{\dot f} \underline{u}^{c_{\dot f}}\right)\,. \tag{2.5}$$

Here $\dot f_{\mathrm{GR}}$ is the frequency evolution in GR which, to leading PN order, is given by [191, 192]

$$\dot f_{\mathrm{GR}} = \frac{96}{5}\pi^{8/3}\mathcal{M}_{ch}^{5/3} f^{11/3} = \frac{96}{5\pi\mathcal{M}_{ch}^2}\underline{u}^{11}\,. \tag{2.6}$$

Unlike the correction to the orbital separation, the one to the frequency evolution originates corrections from both the conservative and dissipative sectors.

Below, we will derive how the PPE parameters $(\alpha_{\mathrm{PPE}}, \beta_{\mathrm{PPE}}, a, b)$ are given in terms of (γ_r, c_r) and $(\gamma_{\dot f}, c_{\dot f})$. We will also show how the amplitude PPE parameters $(\alpha_{\mathrm{PPE}}, a)$

can be related to the phase PPE ones ($\beta_{\rm PPE}, b$) in certain cases. We will assume that non-GR corrections are always smaller than the GR contribution and keep only to leading order in such corrections at the leading PN order.

2.2.1 Amplitude corrections

Let us first look at corrections to the waveform amplitude. Within the stationary phase approximation [193, 194], the waveform amplitude for the dominant quadrupolar radiation in Fourier domain is given by

$$\tilde{\mathcal{A}}(f) = \frac{\mathcal{A}(\bar{t})}{2\sqrt{\dot{f}}} \,. \tag{2.7}$$

Here A is the waveform amplitude in the time domain while $\bar{t}(f)$ represents time at the stationary point. $\mathcal{A}(\bar{t})$ can be obtained by using the quadrupole formula for the metric perturbation in the transverse-traceless gauge given by [195]

$$h^{ij}(t) \propto \frac{G}{D_L}\frac{d^2}{dt^2}Q^{ij} \,. \tag{2.8}$$

Here D_L is the source's luminosity distance and Q^{ij} is the source's quadruple moment tensor.

For a quasi-circular compact binary, $\tilde{\mathcal{A}}$ in Eq. (2.7) then becomes

$$\tilde{\mathcal{A}}(f) \propto \frac{1}{\sqrt{\dot{f}}}\frac{G}{D_L}\mu r_{12}^2 f^2 \propto \frac{r_{12}^2}{\sqrt{\dot{f}}} \,, \tag{2.9}$$

where μ is the reduced mass of the binary. Substituting Eqs. (2.4) and (2.5) into

Eq. (2.9) and keeping only to leading order in non-GR corrections, we find

$$\tilde{\mathcal{A}}(f) = \tilde{\mathcal{A}}_{\rm GR} \left(1 + 2\gamma_r \underline{u}^{c_r} - \frac{1}{2}\gamma_{\dot{f}} \underline{u}^{c_{\dot{f}}}\right) , \tag{2.10}$$

where $\tilde{\mathcal{A}}_{\rm GR}$ is the amplitude of the Fourier waveform in GR. Notice that this expression is much simpler than that in the original formalism in Eq. (A.7).

Let us now show the expressions for the PPE parameters $\alpha_{\rm PPE}$ and a for three different cases using Eq. (2.10):

- *Dissipative-dominated Case*

  When dissipative corrections dominate, we can neglect corrections to the binary separation ($\gamma_r = 0$) and Eq. (2.10) reduces to

$$\tilde{\mathcal{A}}(f) = \tilde{\mathcal{A}}_{\rm GR} \left(1 - \frac{1}{2}\gamma_{\dot{f}} \underline{u}^{c_{\dot{f}}}\right) . \tag{2.11}$$

  Comparing this with the PPE waveform in Eq. (2.1), we find

$$\alpha_{\rm PPE} = -\frac{\gamma_{\dot{f}}}{2}, \quad a = c_{\dot{f}} . \tag{2.12}$$

- *Conservative-dominated Case*

  When conservative corrections dominate, $c_r = c_{\dot{f}}$ and there is an explicit relation between γ_r and $\gamma_{\dot{f}}$. Though finding such a relation is quite involved and one needs to go back to the original PPE formalism as explained in App. A. Non-GR corrections to the GW amplitude in such a formalism is shown in Eq. (A.14).

Setting the dissipative correction to zero, one finds

$$\alpha_{\rm PPE} = -\frac{\gamma_r}{a}(a^2 - 4a - 6)\,, \quad a = c_r = c_{\dot{f}}\,. \tag{2.13}$$

- *Comparable Dissipative and Conservative Case*

  If dissipative and conservative corrections enter at the same PN order, we can set $c_r = c_{\dot{f}}$ in Eq. (2.10). Since there is no generic relation between γ_r and $\gamma_{\dot{f}}$ in this case, one simply finds

$$\alpha_{\rm PPE} = 2\gamma_r - \frac{\gamma_{\dot{f}}}{2}\,, \quad a = c_r = c_{\dot{f}}\,. \tag{2.14}$$

Example modified theories of gravity that we study in Secs. 2.3 and 2.4 fall into either the first or third case.

2.2.2 Phase corrections

Next, let us study corrections to the GW phase. The phase Ψ_p in Fourier domain is related to the frequency evolution as [196]

$$\frac{d^2\Psi_p}{d\Omega^2} = 2\frac{dt}{d\Omega}\,, \tag{2.15}$$

which can be rewritten as

$$\frac{d^2\Psi_p}{d\Omega^2} = \frac{2}{\pi \dot{f}}\,. \tag{2.16}$$

Substituting Eq. (2.5) to the right hand side of the above equation and keeping only to leading non-GR correction, we find

$$\frac{d^2\Psi_p}{d\Omega^2} = \frac{2}{\pi \dot{f}_{\rm GR}}(1 - \gamma_{\dot{f}}\underline{u}^{c_{\dot{f}}})\,. \tag{2.17}$$

Using further Eq. (2.6) to Eq. (2.17) gives

$$\frac{d^2\Psi_p}{d\Omega^2} = \frac{5}{48}\mathcal{M}_{ch}^2\,\underline{u}^{-11}(1 - \gamma_{\dot{f}}\underline{u}^{c_{\dot{f}}})\,. \tag{2.18}$$

We are now ready to derive Ψ_p and extract the PPE parameters $\beta_{\rm PPE}$ and b. Using $\Omega = \pi f$, we can integrate Eq. (2.18) twice to find

$$\Psi_p = \Psi_{\rm GR} - \frac{15\gamma_{\dot{f}}}{16(c_{\dot{f}} - 8)(c_{\dot{f}} - 5)}\underline{u}^{c_{\dot{f}}-5} \tag{2.19}$$

for $c_{\dot{f}} \neq 5$ and $c_{\dot{f}} \neq 8$. Here we only keep to leading non-GR correction and $\Psi_{\rm GR}$ is the GR contribution given in Eq. (A.12) to leading PN order. Similar to the amplitude case, the above expression is much simpler than that in the original formalism in Eq. (A.11). Comparing this with Eqs. (2.1) and (2.3), we find

$$\beta_{\rm PPE} = -\frac{15\gamma_{\dot{f}}}{16(c_{\dot{f}} - 8)(c_{\dot{f}} - 5)}\,, \quad b = c_{\dot{f}} - 5\,. \tag{2.20}$$

The above relation is valid for all three types of corrections considered for the GW amplitude case.

In App. A, we review $\delta\Psi_p$ derived in the original PPE formalism, where we show dissipative and conservative contributions explicitly. In particular, one can use Eq. (A.15) to find $\beta_{\rm PPE}$ for all three cases separately.

2.2.3 Relations among PPE parameters

Finally, we study relations among the PPE parameters. From Eqs. (2.12)–(2.14) and (2.20), one can easily see

$$b = a - 5\,, \tag{2.21}$$

which holds in all three cases considered previously. Let us consider such three cases in turn below to derive relations between $\alpha_{\rm PPE}$ and $\beta_{\rm PPE}$.

- *Dissipative-dominated Case*

  When dissipative corrections dominate, we can use Eqs. (2.12) and (2.20) to find $\alpha_{\rm PPE}$ in terms of $\beta_{\rm PPE}$ and a as

$$\alpha_{\rm PPE} = \frac{8}{15}(a-8)(a-5)\,\beta_{\rm PPE}\,. \tag{2.22}$$

- *Conservative-dominated Case*

  When conservative corrections dominate, we can set the dissipative correction to vanish in Eq. (A.15) to find

$$\beta_{\rm PPE} = -\frac{15}{8}\frac{\gamma_r}{c_r}\frac{c_r^2 - 2c_r - 6}{(8-c_r)(5-c_r)}\,, \quad b = c_r - 5\,. \tag{2.23}$$

  Using this equation together with Eq. (2.13), we find

$$\alpha_{\rm PPE} = \frac{8}{15}\frac{(8-a)(5-a)(a^2-4a-6)}{a^2-2a-6}\beta_{\rm PPE}\,. \tag{2.24}$$

- *Comparable Dissipative and Conservative Case*

When dissipative and conservative corrections enter at the same PN order, there is no explicit relation between $\alpha_{\rm PPE}$ and $\beta_{\rm PPE}$. This is because $\alpha_{\rm PPE}$ depends both on γ_r and $\gamma_{\dot{f}}$ (see Eq. (2.14)) while $\beta_{\rm PPE}$ depends only on the latter (see Eq. (2.19)), and there is no relation between the former and the latter. Thus, one can rewrite $\gamma_{\dot{f}}$ in terms of $\beta_{\rm PPE}$ and substitute into Eq. (2.14) but cannot eliminate γ_r from the expression for $\alpha_{\rm PPE}$.

2.3 Example Theories

In this section, we consider several modified theories of gravity where non-GR corrections arise from generation mechanisms. We briefly discuss each theory, describing differences from GR and its importance. We derive the PPE parameters for each theory following the formalism in Sec. 2.2. Among the various example theories we present here, dissipative corrections dominate in scalar-tensor theories, EDGB gravity, Einstein-Æther theory, and khronometric gravity. On the other hand, dissipative and conservative corrections enter at the same PN order in DCS gravity, noncommutative gravity, and varying-G theories. We do not consider any theories where conservative corrections dominate dissipative ones, though such a situation can be realized for e.g. equal-mass and equal-spin binaries in DCS gravity, where the scalar quadrupolar radiation is suppressed and dominant corrections arise from the scalar dipole interaction and quadrupole moment corrections in the conservative sector.

2.3.1 Scalar-tensor theories

Scalar-tensor theories are one of the most well-established modified theories of gravity where at least one scalar field is introduced through a non-minimal coupling to gravity [36, 197, 198]. Such theories arise naturally from the dimensional reduction of higher dimensional theories, such Kaluza-Klein theory [199, 200] and string theories [201,202]. Scalar-tensor theories have implications to cosmology as well since they are viable candidates for accelerating expansion of our universe [168–170, 203, 204], structure formation [205], inflation [21, 167, 206], and primordial nucleosynthesis [56, 57, 207, 208]. Such theories also offer simple ways to self-consistently model possible variations in Newton's constant [21] (as we discuss in Sec. 2.4). One of the simplest scalar-tensor theories is Brans-Dicke (BD) theory, where a non-canonical scalar field is non-minimally coupled to the metric with an effective strength inversely proportional to the coupling parameter $\omega_{\rm BD}$ [166, 171]. So far the most stringent bound on the theory has been placed by the Cassini-Huygens satellite mission via Shapiro time delay measurement, which gives $\omega_{\rm BD} > 4 \times 10^4$ [209]. Another class of scalar-tensor theories that has been studied extensively is Damour-Esposito-Farèse (DEF) gravity (or sometimes called quasi Brans-Dicke theory), which has two coupling constants (α_0, β_0). This theory reduces to BD theory when β_0 is set to 0 and α_0 is directly related to $\omega_{\rm BD}$. This theory predicts nonperturbative spontaneous or dynamical scalarization phenomena for NSs [210, 211].

When scalarized NSs form compact binaries, these systems emit scalar dipole radiation that changes the orbital evolution from that in GR. Such an effect can be used to place bounds on scalar-tensor theories. For example, combining observational orbital decay results from multiple binary pulsars, the strongest upper bound on β_0 that controls the magnitude of scalarization in DEF gravity has been obtained as $\beta_0 \gtrsim -4.38$ at 90% confidence level [212]. More recently, observations of a hierarchical stellar triple system PSR J0337+1715 placed strong bounds on the Strong Equivalence Principle (SEP) violation parameter[3] as $|\Delta| \lesssim 2 \times 10^{-6}$ at 95% confidence level [213]. This bound stringently constrained the parameter space (α_0, β_0) of DEF gravity [210, 214–217].

Can BHs also possess scalar hair like NSs in scalar-tensor theories? BH no-hair theorem can be applied to many of scalar-tensor theories that prevents BHs to acquire scalar charges [218–222] including BD and DEF gravity, though exceptions exist, such as EDGB gravity [223–227] that we explain in more detail in the next subsection. On the other hand, if the scalar field cosmologically evolves as a function of time, BHs can acquire scalar charges, known as the BH miracle hair growth [228, 229] (see also [230, 231] for related works).

Let us now derive the PPE parameters in scalar tensor theories. Gravitational waveforms are modified from that in GR through the scalar dipole radiation. Using the orbital decay rate of compact binaries in scalar-tensor theories in [37, 45], one can

[3] SEP violation parameter is defined as $\Delta = m_G/m_I - 1$, where m_G and m_I are respectively the gravitational and inertial mass of a pulsar [213].

read off the non-GR corrections to $\dot{f}$ as

$$\gamma_{\dot{f}} = \frac{5}{96}\eta^{2/5}(\alpha_1 - \alpha_2)^2 \tag{2.25}$$

with $c_{\dot{f}} = -2$. Given that the leading correction to the waveform is the dissipative one in scalar-tensor theories, one can use Eq. (2.20) to derive the PPE phase correction as

$$\beta_{\rm ST} = -\frac{5}{7168}\eta^{2/5}(\alpha_1 - \alpha_2)^2 \tag{2.26}$$

with $b = -7$. Here α_A represents the scalar charge of the Ath binary component. Using further Eq. (2.22), one finds the amplitude correction as

$$\alpha_{\rm ST} = -\frac{5}{192}\eta^{2/5}(\alpha_1 - \alpha_2)^2 \tag{2.27}$$

with $a = -2$. These corrections enter at -1 PN order relative to GR.

The scalar charges α_A depend on specific theories and compact objects. For example, in situations where the BH no-hair theorem [218–220] applies, $\alpha_A = 0$. On the other hand, if the scalar field is evolving cosmologically, BHs undergo *miracle hair growth* [228] and acquire scalar charges given by [229]

$$\alpha_A = 2\, m_A\, \dot{\Phi}\, [1 + (1 - \chi_A^2)^{1/2}]\,, \tag{2.28}$$

where $\dot{\Phi}$ is the growth rate of the scalar field while m_A and χ_A are the mass and the magnitude of the dimensionless spin angular momentum of the Ath body respectively. The PPE phase parameter β for binary BHs in such a situation was derived in [27]. Another well-studied example is Brans-Dicke theory, where one can replace $(\alpha_1 - \alpha_2)^2$

in Eqs. (2.26) and (2.27) as $2(s_1 - s_2)^2/(2 + \omega_{\rm BD})$ [37]. Here s_A is the sensitivity of the Ath body and roughly equals to its compactness (0.5 for BHs and ~ 0.2 for NSs). The PPE parameters in this theory has been found in [87]. Scalar charges and the PPE parameters in generic screened modified gravity have recently been derived in [190, 232].

The phase correction in Eq. (2.26) has been used to derive current and future projected bounds with GW interferometers. Regarding the former, GW150914 and GW151226 do not place any meaningful bounds on $\dot{\Phi}$ [27]. On the other hand, by detecting GWs from BH-NS binaries, aLIGO and Virgo with their design sensitivities can place bounds that are stronger than the above binary pulsar bounds from dynamical scalarization for certain equations of state and NS mass range [91, 212, 233, 234][4]. Einstein Telescope, a third generation ground-based detector, can yield constraints on BD theory from BH-NS binaries that are 100 times stronger than the current bound [235]. Projected bounds with future space-borne interferometers, such as DECIGO, can be as large as four orders of magnitude stronger than current bounds [236], while those with LISA may not be as strong as the current bound [172, 237].

Up until now, we have focused on theories with a massless scalar field, but let us end this subsection by commenting on how the above expressions for the PPE parameters change if one considers a massive scalar field instead. In such a case, the scalar dipole radiation is present only when the mass of the scalar field m_s is smaller

[4]One needs to multiply Eq. (2.26) by a step-like function to capture the effect of dynamical scalarization.

than the orbital angular frequency $\Omega = \pi f$. Then, if the Yukawa-type correction to the binding energy is subdominant, Eqs. (2.26) and (2.27) simply acquire an additional factor of $\Theta(\Omega - m_s/\hbar)$, where Θ is the Heaviside function. For example, the gravitational waveform phase in massive BD theory is derived in [238]. The situation is similar if massive pseudo-scalars are present, such as axions [239].

2.3.2 Einstein-dilaton-Gauss-Bonnet gravity

EDGB gravity is a well-known extension of GR, which emerges naturally in the framework of low-energy effective string theories and gives one of the simplest viable high-energy modifications to GR [240, 241]. It also arises as a special case of Horndeski gravity [36, 242], which is the most generic scalar-tensor theory with at most second-order derivatives in the field equations. One obtains the EDGB action by adding a quadratic-curvature term to the Einstein-Hilbert action, where the scalar field (dilaton) is non-minimally coupled to the Gauss-Bonnet term with a coupling constant $\bar{\alpha}_{\rm EDGB}$ [243][5]. A stringent upper bound on such a coupling constant has been placed using the orbital decay measurement of a BH low-mass X-ray binary (LMXB) as $\sqrt{|\bar{\alpha}_{\rm EDGB}|} < 1.9 \times 10^5$ cm [244]. A similar upper bound has been placed from the existence of BHs [241]. Equation-of-state-dependent bounds from the maximum mass of NSs have also been derived in [245].

BHs in EDGB gravity are of particular interest since they are fundamentally different from their GR counterparts. Perturbative but analytic solutions are available

[5]We use barred quantities for coupling constants so that one can easily distinguish them from the PPE parameters.

for static [223,225,246,247] and slowly rotating EDGB BHs [248–250] while numerical solutions have been found for static [243, 251, 252] and rotating [241, 253, 254] BHs. One of the important reasons for considering BHs in EDGB is that BHs acquire scalar monopole charges [33, 173, 225, 255] while ordinary stars such as NSs do not if the scalar field is coupled linearly to the Gauss-Bonnet term in the action [173, 256]. This means that binary pulsars are inefficient to constrain the theory, and one needs systems such as BH-LMXBs [244] or BH/pulsar binaries [256] to have better probes on the theory.

We now show the expressions of the PPE parameters for EDGB gravity. The scalar monopole charge of EDGB BHs generates scalar dipole radiation, which leads to an earlier coalescence of BH binaries compared to GR. Such scalar radiation modifies the GW phase with the PPE parameters given by [27, 173]

$$\beta_{\text{EDGB}} = -\frac{5}{7168}\zeta_{\text{EDGB}}\frac{(m_1^2 \tilde{s}_2^{\text{EDGB}} - m_2^2 \tilde{s}_1^{\text{EDGB}})^2}{M_t^4 \eta^{18/5}} \tag{2.29}$$

and $b = -7$. Here, $\zeta_{\text{EDGB}} \equiv 16\pi\bar{\alpha}_{\text{EDGB}}^2/M_t^4$ is the dimensionless EDGB coupling parameter and $\tilde{s}_A^{\text{EDGB}}$ are the spin-dependent factors of the BH scalar charges given by $\tilde{s}_A^{\text{EDGB}} \equiv 2(\sqrt{1-{\chi_A}^2} - 1 + {\chi_A}^2)/{\chi_A}^2$ [33, 255][6]. In EDGB gravity, the leading order correction to the phase enters through the correction of the GW energy flux, and hence the theory corresponds to a dissipative-dominated case. We can then use

[6] $\tilde{s}_A^{\text{EDGB}}$ are zero for ordinary stars like NSs [173, 256].

Eq. (2.22) to calculate the amplitude PPE parameters as

$$\alpha_{\text{EDGB}} = -\frac{5}{192}\zeta_{\text{EDGB}}\frac{(m_1^2\tilde{s}_2^{\text{EDGB}} - m_2^2\tilde{s}_1^{\text{EDGB}})^2}{M_t^4\eta^{18/5}} \tag{2.30}$$

and $a = -2$. These corrections enter at -1 PN order.

One can use the phase correction in Eq. (2.29) to derive bounds on EDGB gravity with current [27] and future [244] GW observations. Similar to the scalar-tensor theory case, current binary BH GW events do not allow us to place any meaningful bounds on the theory. Future second- and third-generation ground-based detectors and LISA can place bounds that are comparable to current bounds from LMXBs [244]. On the other hand, DECIGO has the potential to go beyond the current bounds by three orders of magnitude.

2.3.3 Dynamical Chern-Simons gravity

DCS gravity is described by Einstein-Hilbert action with a dynamical (pseudo-)scalar field which is non-minimally coupled to the Pontryagin density with a coupling constant $\bar{\alpha}_{\text{DCS}}$ [257,258]. Similar to EDGB gravity, DCS gravity arises as an effective field theory from the compactification of heterotic string theory [259, 260]. Such a theory is also important in the context of particle physics [257,261–263], loop quantum gravity [264, 265], and inflationary cosmology [266]. Demanding that the critical length scale (below which higher curvature corrections beyond quadratic order cannot be neglected in the action) has to be smaller than the scale probed by table-top experiments, one finds $\sqrt{|\bar{\alpha}_{\text{DCS}}|} < \mathcal{O}(10^8\text{km})$ [267]. Similar constraints have been placed from measurements of the frame-dragging effect by Gravity Probe B and LAGEOS satellites [268].

We now derive the expressions of the PPE parameters for DCS gravity. While BHs in EDGB gravity possess scalar monopole charges, BHs in DCS gravity possess scalar dipole charges which induce scalar quadrupolar emission [173]. On the other hand, scalar dipole charges induce a scalar interaction force between two BHs. Each BH also acquires a modification to the quadrupole moment away from the Kerr value. All of these modifications result in both dissipative and conservative corrections entering at the same order in gravitational waveforms. For spin-aligned binaries[7], corrections to Kepler's law and frequency evolution in DCS gravity are given in [174] within the slow-rotation approximation for BHs, from which we can derive

$$\gamma_r = \frac{25}{256}\eta^{-9/5}\zeta_{\rm DCS}\chi_1\chi_2 - \frac{201}{3584}\eta^{-14/5}\zeta_{\rm DCS}\left(\frac{m_1^2}{M_t^2}\chi_2^2 + \frac{m_2^2}{M_t^2}\chi_1^2\right) \tag{2.31}$$

with $c_r = 4$, and

$$\gamma_{\dot f} = \frac{11975}{12288}\eta^{-9/5}\zeta_{\rm DCS}\chi_1\chi_2 - \frac{96305}{172032}\eta^{-14/5}\zeta_{\rm DCS}\left(\frac{m_1^2}{M_t^2}\chi_2^2 + \frac{m_2^2}{M_t^2}\chi_1^2\right) . \tag{2.32}$$

with $c_{\dot f} = 4$. Here $\zeta_{\rm DCS} = 16\pi\bar\alpha_{\rm DCS}^2/M_t^4$ is the dimensionless coupling constant. Using Eqs. (2.31) and (2.32) in Eqs. (2.14) and (2.20) respectively, one finds

$$\begin{aligned}\alpha_{\rm DCS} =& \frac{57713}{344064}\eta^{-14/5}\zeta_{\rm DCS}\left[-2\delta_m\chi_a\chi_s\right.\\ &\left.+\left(1-\frac{14976\eta}{57713}\right)\chi_a^2 + \left(1-\frac{215876\eta}{57713}\right)\chi_s^2\right] ,\end{aligned} \tag{2.33}$$

[7]See recent works [269, 270] for precession equations in DCS gravity.

with $a = 4$, and

$$\beta_{\rm DCS} = \frac{481525}{3670016}\eta^{-14/5}\zeta_{\rm DCS}\left[-2\delta_m\chi_a\chi_s + \left(1 - \frac{4992\eta}{19261}\right)\chi_a^2 + \left(1 - \frac{72052\eta}{19261}\right)\chi_s^2\right]. \tag{2.34}$$

with $b = -1$. Here $\chi_{s,a} = (\chi_1 \pm \chi_2)/2$ are the symmetric and antisymmetric combinations of dimensionless spin parameters and $\delta_m = (m_1 - m_2)/M_t$ is the fractional difference in masses relative to the total mass. The above corrections enter at 2 PN order.

Can GW observations place stronger bounds on the theory? Current GW observations do not allow us to put any meaningful bounds on DCS gravity [27] (see also [97]). However, future observations have potential to place bounds on the theory that are six to seven orders of magnitude stronger than current bounds [174]. Such stronger bounds can be realized due to relatively strong gravitational field and large spins that source the pseudo-scalar field. Measuring GWs from extreme mass ratio inspirals with LISA can also place bounds that are three orders of magnitude stronger than current bounds [271].

2.3.4 Einstein-Æther and Khronometric theory

In this section, we study two example theories that break Lorentz invariance in the gravity sector, namely Einstein-Æther and khronometric theory. Lorentz-violating theories of gravity are candidates for low-energy descriptions of quantum gravity [272, 273]. Lorentz-violation in the gravity sector has not been as stringently constrained as that in the matter sector [274–276] and several mechanisms exist that prevents percolation of the latter to the former [276, 277].

Einstein-Æther theory is a vector-tensor theory of gravity, where along with the metric, a spacetime is endowed with a dynamical timelike unit vector (Æther) field [278, 279]. Such a vector field specifies a particular rest frame at each point in spacetime, and hence breaks the local Lorentz symmetry. The amount of Lorentz violation is controlled by four coupling parameters (c_1, c_2, c_3, c_4). Einstein-Æther theory preserves diffeomorphism invariance and hence is a Lorentz-violating theory without abandoning the framework of GR [279]. Along with the spin-2 gravitational perturbation of GR, the theory predicts the existence of the spin-1 and spin-0 perturbations [280–282]. Such perturbation modes propagate at speeds that are functions of the coupling parameters c_i, and in general differ from the speed of light [281].

Khronometric theory is a variant of Einstein-Æther theory, where the ther field is restricted to be hypersurface-orthogonal. Such a theory arises as a low-energy limit of Hořava gravity, a power-counting renormalizable quantum gravity model with only spatial diffeomorphism invariance [36, 273, 283–285]. The amount of Lorentz violation in the theory is controlled by three parameters, $(\bar{\alpha}_{\rm kh}, \bar{\beta}_{\rm kh}, \bar{\lambda}_{\rm kh})$. Unlike Einstein-Æther theory, the spin-1 propagating modes are absent in khronometric theory.

Most of parameter space in Einstein-Æther and khronometric theory have been constrained stringently from current observations and theoretical requirements. Using the measurement of the arrival time difference between GWs and electromagnetic waves in GW170817, the difference in the propagation speed of GWs away from the speed of light has been constrained to be less than $\sim 10^{-15}$ [71, 286]. Such a bound can be mapped to bounds on Lorentz-violating gravity as $|c_1 + c_3| \lesssim 10^{-15}$ [287, 288] and $|\bar{\beta}_{\rm kh}| \lesssim 10^{-15}$ [289][8]. Imposing further constraints from solar system experiments [2, 291, 292], Big Bang nucleosynthesis [293] and theoretical constraints such as the stability of propagating modes, positivity of their energy density [294] and the absence of gravitational Cherenkov radiation [295], allowed regions in the remaining parameter space have been derived for Einstein-Æther [288] and khronometric [289] theory. Binary pulsar bounds on these theories were studied in [296, 297] before the discovery of GW170817, within a parameter space that is different from the allowed regions in [288, 289].

Let us now derive the PPE parameters in Einstein-Æther and khronometric theories. Propagation of the scalar and vector modes is responsible for dipole radiation and loss of angular momentum in binary systems, which increase the amount of orbital decay rate. Regarding Einstein-Æther theory, the PPE phase correction is given by [175]

$$\beta_{\rm EA} = -\frac{5}{3584}\eta^{2/5}\frac{(s_1^{\rm EA} - s_2^{\rm EA})^2}{[(1-s_1^{\rm EA})(1-s_2^{\rm EA})]^{4/3}} \times \frac{[(c_{14}-2)w_0^3 - w_1^3]}{c_{14}w_0^3 w_1^3} \tag{2.35}$$

with $b = -7$. Here w_s is the propagation speed of the spin-s modes in Einstein-Æther

[8]Such bounds are consistent with the prediction in [175] based on [290].

theory given by [279]

$$w_0^2 = \frac{(2-c_{14})c_{123}}{(2+3c_2+c_+)(1-c_+)c_{14}}\,, \tag{2.36}$$

$$w_1^2 = \frac{2c_1 - c_+c_-}{2(1-c_+)c_{14}}\,, \tag{2.37}$$

$$w_2^2 = \frac{1}{1-c_+}\,, \tag{2.38}$$

with

$$c_{14} \equiv c_1 + c_4\,, \quad c_\pm \equiv c_1 \pm c_3\,, \quad c_{123} \equiv c_1 + c_2 + c_3\,. \tag{2.39}$$

s_A in Eq. (2.35) is the sensitivity of the A-th body and has been calculated only for NSs [296,297]. Given that the leading order correction in Einstein-Æther theory arises from the dissipative sector [175], we can use Eq. (2.22) to find the PPE amplitude correction as[9]

$$\alpha_{\rm EA} = -\frac{5}{96}\eta^{2/5}\frac{(s_1^{\rm EA}-s_2^{\rm EA})^2}{[(1-s_1^{\rm EA})(1-s_2^{\rm EA})]^{4/3}} \times \frac{[(c_{14}-2)w_0^3 - w_1^3]}{c_{14}w_0^3w_1^3} \tag{2.40}$$

with $a=-2$. Similar to Einstein-Æther theory, the PPE parameters in khronometric theory is given by [175]

$$\beta_{\rm kh} = -\frac{5}{3584}\eta^{2/5}\frac{(s_1^{\rm kh}-s_2^{\rm kh})^2}{[(1-s_1^{\rm kh})(1-s_2^{\rm kh})]^{4/3}} \times \sqrt{\bar{\alpha}_{\rm kh}}\left[\frac{(\bar{\beta}_{\rm kh}-1)(2+\bar{\beta}_{\rm kh}+3\bar{\lambda}_{\rm kh})}{(\bar{\alpha}_{\rm kh}-2)(\bar{\beta}_{\rm kh}+\bar{\lambda}_{\rm kh})}\right]^{3/2} \tag{2.41}$$

with $b=-7$, and

$$\alpha_{\rm kh} = -\frac{5}{96}\eta^{2/5}\frac{(s_1^{\rm kh}-s_2^{\rm kh})^2}{[(1-s_1^{\rm kh})(1-s_2^{\rm kh})]^{4/3}}$$

[9]Eqs. (2.40) and (2.42) correct errors in [175].

$$\times \sqrt{\bar{\alpha}_{\rm kh}} \left[\frac{(\bar{\beta}_{\rm kh}-1)(2+\bar{\beta}_{\rm kh}+3\bar{\lambda}_{\rm kh})}{(\bar{\alpha}_{\rm kh}-2)(\bar{\beta}_{\rm kh}+\bar{\lambda}_{\rm kh})} \right]^{3/2} \tag{2.42}$$

with $a = -2$. These corrections enter at -1 PN order.

Above corrections to the gravitational waveform can be used to compute current and projected future bounds on the theories with GW observations, provided one knows what the sensitivities are for compact objects in binaries. Unfortunately, such sensitivities have not been calculated for BHs, and hence, one cannot derive bounds on the theories from recent binary BH merger events. Instead, Ref. [27] used the next-to-leading 0 PN correction that is independent of the sensitivities and derived bounds from GW150914 and GW151226, though such bounds are weaker than those from binary pulsar observations [296, 297]. On the other hand, Ref. [175] includes both the leading and next-to-leading corrections to the waveform and estimate projected future bounds with GWs from binary NSs. The authors found that bounds from second-generation ground-based detectors are less stringent than existing bounds even with their design sensitivities. However, third-generation ground-based ones and space-borne interferometers can place constraints that are comparable, and in some cases, two orders of magnitude stronger compared to the current bounds [175, 298].

2.3.5 Noncommutative gravity

Although the concept of nontrivial commutation relations of spacetime coordinates is rather old [299, 300], the idea has revived recently with the development of non-commutative geometry [301–305], and the emergence of noncommutative structure of spacetime in a specific limit of string theory [306, 307]. Quantum field theories on noncommutative spacetime have been studied extensively as well [308–310]. In the simplest model of noncommutative gravity, spacetime coordinates are promoted to operators, which satisfy a canonical commutation relation:

$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu} \,, \tag{2.43}$$

where $\theta^{\mu\nu}$ is a real constant antisymmetric tensor. In ordinary quantum mechanics, Planck's constant $\hbar$ measures the quantum fuzziness of phase space coordinates. In a similar manner, $\theta^{\mu\nu}$ introduces a new fundamental scale which measures the quantum fuzziness of spacetime coordinates [176].

In order to obtain stringent constraints on the scale of noncommutativity, low-energy experiments are advantageous over high-energy ones [311, 312]. Low-energy precision measurements such as clock-comparison experiments with nuclear-spin-polarized $^9\text{Be}^+$ ions [313] give a constraint on noncommutative scale as $1/\sqrt{\theta} \gtrsim 10$ TeV [311], where θ refers to the magnitude of the spatial-spatial components of $\theta^{\mu\nu}$[10]. A similar bound has been obtained from the measurement of the Lamb shift [314]. Another speculative bound is derived from the analysis of atomic experiments which is 10 orders of magnitude stronger [312, 315]. Study of inflationary observables using cosmic microwave background data from Planck gives the lower bound on the energy scale of noncommutativity as 19 TeV [316, 317].

[10]The corresponding bound on the time-spatial components of $\theta^{\mu\nu}$ is roughly six orders magnitude weaker than that on the spatial-spatial components.

Let us now review how the binary evolution is modified from that in GR in this theory. Several formulations of noncommutative gravity exist [318–323], though the first order noncommutative correction vanishes in all of them [324,325] and the leading order correction enters at second order. On the other hand, first order corrections may arise from gravity-matter interactions [325, 326]. Thus one can neglect corrections to the pure gravity sector and focus on corrections to the matter sector (i.e., energy-momentum tensor) [176]. Making corrections to classical matter source and following an effective field theory approach, expressions of energy and GW luminosity for quasi-circular BH binaries have been derived in Ref. [176], which give the correction to the frequency evolution in Eq. (2.5) as

$$\gamma_{\dot{f}} = \frac{5}{4}\eta^{-4/5}(2\eta - 1)\Lambda_{\rm NC}^2 \tag{2.44}$$

with $c_{\dot{f}} = 4$ and $\Lambda_{\rm NC}^2 = \theta^{0i}\theta_{0i}/(l_p^2 t_p^2)$ with l_p and t_p representing the Planck length and time respectively. On the other hand, modified Kepler's law in Eq. (2.4) can be found as [176]

$$\gamma_r = \frac{1}{8}\eta^{-4/5}(2\eta - 1)\Lambda_{\rm NC}^2 \tag{2.45}$$

with $c_r = 4$.

We are now ready to derive the PPE parameters in noncommutative gravity. Given that the dissipative and conservative leading corrections enter at the same PN order, one can use Eqs. (2.44) and (2.45) in Eq. (2.14) to find the PPE amplitude correction as

$$\alpha_{\rm NC} = -\frac{3}{8}\eta^{-4/5}(2\eta - 1)\Lambda_{\rm NC}^2 \tag{2.46}$$

with $a = 4$. Similarly, substituting Eq. (2.44) into Eq. (2.20) gives the PPE phase correction as

$$\beta_{\rm NC} = -\frac{75}{256}\eta^{-4/5}(2\eta - 1)\Lambda_{\rm NC}^2 \tag{2.47}$$

with $b = -1$. $\beta_{\rm NC}$ can also be read off from the phase correction derived in [176]. The above corrections enter at 2 PN order.

The above phase correction has already been used to derive bounds on noncommutative gravity from GW150914 as $\sqrt{\Lambda_{\rm NC}} \lesssim 3.5$ [176], which means that the energy scale of noncommutativity has been constrained to be the order of the Planck scale. Such a bound, so far, is the most stringent constraint on noncommutative scale and is 15 orders of magnitude stronger compared to the bounds coming from particle physics and low-energy precision measurements[11].

2.4 Varying-G Theories

Many of the modified theories of gravity that violate the strong equivalence principle [2,327,328] predict that locally measured gravitational constant (G) may vary with time [329]. Since the gravitational self-energy of a body is a function of the gravitational constant, in a theory where G is time-dependent, masses of compact bodies are also time-dependent [330]. The rate at which the mass of an object varies with time is

[11]Notice that the GW bound is on the time-spatial components of $\theta^{\mu\nu}$, while most of particle physics and low-energy precision experiments place bounds on its spatial-spatial components.

proportional to the rate of change of the gravitational coupling constant [330]. Such a variation of mass, together with the conservation of linear momentum, causes compact bodies to experience anomalous acceleration, which results in a time-evolution of the specific angular momentum [330]. Existing experiments that search for variations in G at present time (i.e., at very small redshift) include lunar laser ranging observations [331], pulsar timing observations [332,333], radar observations of planets and spacecraft [334], and surface temperature observations of PSR J0437-4715 [335]. Another class of constraints on a long-term variation of G comes from Big Bang nucleosynthesis [336, 337] and helioseismology [338]. The most stringent bound on $|\dot{G}/G|$ is of the order $\lesssim 10^{-14}\,\mathrm{yr}^{-1}$ [339].

More than one gravitational constants can appear in different areas of a gravitational theory. Here we introduce two different kinds of gravitational constant, one that arises in the dissipative sector and another in the conservative sector. The constant which enters in the GW luminosity through Einstein equations, i.e. the constant in Eq. (2.8), is the one we refer to as dissipative gravitational constant (G_D), while that enters in Kepler's law or binding energy of the binary is what we refer as the conservative one (G_C). These two constants are the same in GR, but they can be different in some modified theories of gravity. An example of such a theory is Brans-Dicke theory with a cosmologically evolving scalar field [2].

The PPE parameters for varying-G theories have previously been derived in [177] for $G_D = G_C$. Here, we improve the analysis by considering the two different types of gravitational constant and including variations in masses, which are inevitable for strongly self-gravitating objects when G varies [330]. We also correct small errors in [177]. We follow the analysis of [340] that derives gravitational waveforms from BH binary inspirals with varying mass effects from the specific angular momentum. We also present another derivation in App. B using the energy balance argument in [177].

The formalism presented in Sec. 2.2 assumes that G and the masses to be constant, and hence are not applicable to varying-G theories. Thus, we will derive the PPE parameters in varying-G theories by promoting the PPE formalism to admit time variation in the gravitational constants and masses as

$$m_A(t) \approx m_{A,0} + \dot{m}_{A,0}(t - t_0)\,, \tag{2.48}$$

$$G_C(t) \approx G_{C,0} + \dot{G}_{C,0}(t - t_0)\,, \tag{2.49}$$

$$G_D(t) \approx G_{D,0} + (1 + \delta_{\dot{G}})\dot{G}_{C,0}(t - t_0)\,, \tag{2.50}$$

where t_0 is the time of coalescence. Here we assumed that spatial variations of G_C and G_D are small compared to variations in time. $\delta_{\dot{G}}$ gives the fractional difference between the rates at which G_C and G_D vary with time, and could be a function of parameters in a theory. The subscript 0 denotes that the quantity is measured at the time $t = t_0$. Other time variations to consider are those in the specific angular momentum j and the total mass M_t:

$$j(t) \approx j_0 + \dot{j}_0(t - t_0)\,, \tag{2.51}$$

$$M_t(t) \approx M_{t,0} + \dot{M}_{t,0}(t - t_0)\,. \tag{2.52}$$

$\dot{j}_0$ and $\dot{M}_{t,0}$ can be written in terms of binary masses and sensitivities defined by

$$s_A = -\frac{G_C}{m_A}\frac{\delta m_A}{\delta G_C}\,, \tag{2.53}$$

as [330]

$$\dot{j}_0 = \frac{m_{1,0}s_{1,0} + m_{2,0}s_{2,0}}{m_{1,0} + m_{2,0}}\frac{\dot{G}_{C,0}}{G_{C,0}}j_0\,, \tag{2.54}$$

$$\dot{M}_{t,0} = -\frac{m_{1,0}s_{1,0} + m_{2,0}s_{2,0}}{m_{1,0} + m_{2,0}}\frac{\dot{G}_{C,0}}{G_{C,0}}M_{t,0}\,, \tag{2.55}$$

respectively.

Next, we explain how the binary evolution is affected by the variation of the above parameters. First, GW emission makes the orbital separation r decay with the rate given by [191]

$$\dot{r}_{12}^{\mathrm{GW}} = -\frac{64}{5}\frac{G_D G_C^2 \mu M_t^2}{r_{12}^3}\,. \tag{2.56}$$

Second, time variation of the total mass, (conservative) gravitational constant and specific angular momentum changes r_{12} at a rate of

$$\dot{r}_{12}^{\dot{G}} = -\left(\frac{\dot{G}_{C,0}}{G_C} + \frac{\dot{M}_{t,0}}{M_t} - 2\frac{\dot{j}_0}{j}\right) r_{12}\,, \tag{2.57}$$

which is derived by taking a time derivative of the specific angular momentum $j \equiv \sqrt{G_C M_t r_{12}}$. Having the evolution of r_{12} at hand, one can derive the evolution of the orbital angular frequency using Kepler's third law as

$$\dot{\Omega} = \frac{1}{2\Omega r_{12}^3}\left(M_t \dot{G}_{C,0} + \dot{M}_{t,0} G_C - 3 M_t G_C \frac{\dot{r}_{12}}{r_{12}}\right) . \tag{2.58}$$

Using the evolution of the binary separation $\dot{r}_{12} \equiv \dot{r}_{12}^{\rm GW} + \dot{r}_{12}^{\dot{G}}$ in Eq. (2.58), together with Eqs. (2.54) and (2.55), we can find the GW frequency evolution as

$$\begin{aligned} \dot{f} &= \frac{\dot{\Omega}}{\pi} \\ &= \frac{96}{5}\pi^{8/3} G_C^{2/3} G_D \mathcal{M}_{ch}^{5/3} f^{11/3} \Big\{ 1 \\ &\quad + \frac{5}{96}\frac{\dot{G}_{C,0} G_C}{G_D \eta}[2M_t - 5(m_{1,0}s_{1,0} + m_{2,0}s_{2,0})]x^{-4} \Big\} , \end{aligned} \tag{2.59}$$

where $x \equiv (\pi G_C M_t f)^{2/3}$ is the squared velocity of the relative motion. Here we only considered the leading correction to the frequency evolution entering at -4 PN order. Using Eqs. (2.48)–(2.50) and (2.55) into Eq. (2.59), one finds

$$\begin{aligned} \dot{f} = \frac{96}{5}\pi^{8/3} f^{11/3}\, \eta_0\, G_{C,0}^{2/3}\, G_{D,0}\, M_{t,0}^{5/3} \{ 1 \\ - \frac{5\, G_{C,0}\, \dot{G}_{C,0}}{768\, \eta_0\, G_{D,0}^2} [3(1+\delta_{\dot{G}}) G_{C,0} M_{t,0} - (3s_{1,0} + 3s_{2,0} \\ +14) G_{D,0} M_{t,0} + 41(m_{1,0}s_{1,0} + m_{2,0}s_{2,0}) G_{D,0}]\, x_0^{-4} \} . \end{aligned} \tag{2.60}$$

Notice that $G_{C,0}$ and $G_{D,0}$ differ only by a constant quantity, and such a difference will enter in $\dot{f}$ at 0 PN order which is much higher than the -4 PN corrections. We will thus ignore such 0 PN corrections and simply use $G_{D,0} = G_{C,0} \equiv G_0$ from now on.

Based on the above binary evolution, we now derive corrections to the GW phase. We integrate Eq. (2.60) to obtain time before coalescence $t(f) - t_0$ and the GW phase $\phi_p(f) \equiv \int 2\pi f dt = \int (2\pi f/\dot{f}) df$ as

$$\begin{aligned} t(f) &= t_0 - \frac{5}{256} G_0 \mathcal{M}_{ch,0}\, \underline{u}_0^{-8} \{1 \\ &\quad - \frac{5}{1536} \frac{\dot{G}_{C,0}}{\eta_0} [11 M_{t,0} + 3(s_{1,0} + s_{2,0} - \delta_{\dot{G}}) M_{t,0} \\ &\quad - 41(m_{1,0} s_{1,0} + m_{2,0} s_{2,0})]\, x_0^{-4}\} \,, \end{aligned} \tag{2.61}$$

$$\begin{aligned} \phi_p(f) &= \phi_{p,0} - \frac{1}{16} \underline{u}_0^{-5} \{1 \\ &\quad - \frac{25}{9984} \frac{\dot{G}_{C,0}}{\eta_0} [11 M_{t,0} + 3(s_{1,0} + s_{2,0} - \delta_{\dot{G}}) M_{t,0} \\ &\quad - 41(m_{1,0} s_{1,0} + m_{2,0} s_{2,0})]\, x_0^{-4}\} \,, \end{aligned} \tag{2.62}$$

with $\underline{u}_0 \equiv (\pi G_0 \mathcal{M}_{ch,0} f)^{\frac{1}{3}}$. The GW phase in the Fourier space is then given by

$$\begin{aligned} \Psi_p(f) =& 2\pi f t(f) - \phi(f) - \frac{\pi}{4} \\ =& 2\pi f t_0 - \phi_{p,0} - \frac{\pi}{4} + \frac{3}{128} \underline{u}_0^{-5} \left\{ 1 - \frac{25}{19968} \frac{\dot{G}_{C,0}}{\eta_0} [11 M_{t,0} \right. \\ & \left. + 3(s_{1,0} + s_{2,0} - \delta_{\dot{G}}) M_{t,0} - 41(m_{1,0} s_{1,0} + m_{2,0} s_{2,0})]\, x_0^{-4} \right\} . \end{aligned} \tag{2.63}$$

From Eq. (2.63), one finds the PPE phase parameters as $b = -13$ and

$$\begin{aligned} \beta_{\dot{G}} &= -\frac{25}{851968} \dot{G}_{C,0}\, \eta_0^{3/5} [11 M_{t,0} + 3(s_{1,0} + s_{2,0} - \delta_{\dot{G}}) M_{t,0} \\ &\quad - 41(m_{1,0} s_{1,0} + m_{2,0} s_{2,0})] \,. \end{aligned} \tag{2.64}$$

Next, we derive the PPE amplitude parameters. Using Kepler's law to Eq. (2.9), one finds

$$\tilde{\mathcal{A}}(f) \propto \frac{1}{\sqrt{\dot{f}}}\frac{G_D(t)}{D_L}\mu(t)r_{12}(t)^2 f^2$$
$$\propto \frac{1}{\sqrt{\dot{f}}}G_D(t)G_C(t)^{2/3}\mu(t)M_t(t)^{2/3}\,. \tag{2.65}$$

Using further Eqs. (2.48)–(2.50) in Eq. (2.65), we find the amplitude PPE parameters as $a = -8$ and

$$\alpha_{\dot{G}} = \frac{5}{512}\eta_0^{3/5}\dot{G}_{C,0}\left[-7M_{t,0} + (s_{1,0} + s_{2,0} - \delta_{\dot{G}})M_{t,0} + 13(m_{1,0}s_{1,0} + m_{2,0}s_{2,0})\right]\,. \tag{2.66}$$

Let us comment on how the above new PPE parameters in varying-G theories differ from those obtained previously in [177]. The latter considers $G_D = G_C$ (which corresponds to $\delta_{\dot{G}} = 0$) and $s_A = 0$ (which is only true for weakly-gravitating objects). However, the above expressions for the PPE parameters do not reduce to those in [177] under these limits. This is because Ref. [177] did not take into account the fact that the binding energy is not conserved in the absence of GW emission in varying-G theories. In App. B, we show that the correct application of the energy balance law does indeed lead to the same conclusion as in this section.

Eqs. (2.64) and (2.66) can be used to constrain varying-G theories with GW observations. Recent GW events (GW150914 and GW151226) place constraints on variation of G which are much weaker than the current constraints [27]. Projected GW bounds have been calculated in Ref. [177] (see [298] for an updated forecast of future GW bounds on $\dot{G}$) which gives $|\dot{G}_0/G_0| \lesssim 10^{-11}\,\mathrm{yr}^{-1}$, considering a single merger event. Although GW bounds are less stringent compared to the existing bounds [2], they are unique in the sense that they can provide constraints at intermediate redshifts, while the existing bounds are for very small and large redshifts [177]. Furthermore, GW constraints give $\dot{G}_0/G_0$ at the location of merger events, which means that a sufficient number of GW observations can be used to construct a 3D constraint map of $\dot{G}_0/G_0$ as a function of sky locations and redshifts [177].

2.5 Conclusions

We derived non-GR corrections to the GW phase and amplitude in various modified theories of gravity. We achieved this by revisiting the standard PPE formalism and considered generic corrections to the GW frequency evolution and Kepler's third law that have been derived in many non-GR theories. Such a formalism yields the expressions of the PPE parameters which are simpler compared to the original formalism [86, 87]. We derived the PPE amplitude parameters for the first time in EDGB, DCS and noncommutative gravity. We also corrected some errors in the expressions of the PPE amplitude parameters in Einstein-Æther and khronometric theories in previous literature [175].

We also considered the PPE formalism with variable gravitational constants by extending previous work [177] in a few different ways. One difference is that we introduced two different gravitational constants, one entering in the GW luminosity (dissipative G) and the other entering in the binding energy and Kepler's law (conservative G). We also included time variations of component masses in a binary in terms of the sensitivities following [330], which is a natural consequence in varying-G theories. We further introduced the effect of non-conservation of the binding energy in the energy balance law. Such an effect arises due to an anomalous acceleration caused by time variations in G or masses [330] that was not accounted for in the original work of [177]. Including all of these, we derived the PPE amplitude and phase corrections to the gravitational waveform from compact binary inspirals.

The analytic expressions of the PPE corrections derived in this work, especially those in the amplitude, can be used to improve analyses on testing GR with observed GW events and to derive new projected bounds with future observations, since most of previous literature only include phase corrections. For example, one can reanalyze the available GW data for testing GR including amplitude corrections with a Bayesian analysis [25]. One can also carry out a similar Fisher analysis as in [27] by including amplitude corrections and mapping bounds on generic parameters to those on fundamental pillars in GR. GW amplitude corrections are also crucial for testing strong-field gravity with astrophysical stochastic GW backgrounds [97, 100]. One could further improve the analysis presented in this work by considering binaries with eccentric orbits [89] or including spin precession [88, 270].

Chapter 3

Testing Gravity with GW Amplitude Corrections

3.1 Introduction

In this chapter, we study how much impact the amplitude corrections may bring to tests of GR with GWs and provide justifications for previous studies that only considered the phase corrections. [1] To do so, we make use of PPE parameters derived in case of a few modified theories of gravity in Chapter 2. We compute constraints on those theories from both the phase and amplitude, focusing on leading PN corrections to tensorial modes only. We choose theories where the leading correction enters at a negative PN order, and sensitivities of black holes (BHs) are known. Such criteria lead us to choose Einstein-dilaton-Gauss-Bonnet (EDGB) gravity, scalar-tensor theories, and varying-G theories. We carry out Fisher analyses with Monte-Carlo simulations utilizing the parameter posterior samples of GW151226 and GW150914 released by

[1]This chapter is based on the following paper: *Testing Gravity with Gravitational Waves from Binary Black Hole Mergers: Contributions from Amplitude Corrections*; Tahura, Shammi; Yagi, Kent; Carson, Zack; Phys. Rev. D 100, 104001 (2019)

LVC [341][2]. Such analyses with actual posterior samples produce more reliable results compared to the ones with sky-averaged waveforms. In fact, when implementing such samples, we can determine the credibility of the small coupling approximation in scalar-tensor theories, which allows us to place reliable bounds on the time-evolution of the scalar field from GW observations for the first time [81].

We find that the constraints derived from the phase and the amplitude can be comparable in case of massive binary systems like GW150914. Whereas for less massive binaries with a larger number of GW cycles, the phase always yields stronger constraints. Moreover, inclusion of an amplitude correction to the waveform impacts the bound on the phase correction as well since the former can easily be related to the latter provided the dissipative and conservative corrections do not enter at the same order. The amount and direction of such effects vary with the PN order of the corrections. All such constraints in the theories under consideration are summarized in Table 1.1 in chapter 1.

The rest of the chapter is organized as follows. Section 3.2 summarizes the data analysis techniques. Section 3.3.1 is devoted to justifying our formalism against the one by LVC in massive gravity [25] while we derive constraints on EDGB, scalar-tensor, and varying-G theories in Secs. 3.3.2- 3.3.4. Section 3.4 presents a summary of our work while discussing the effects of an amplitude correction on that of phase. Appendix C compares the PhenomB and PhenomD waveforms for constraining PPE

[2] We choose GW151226 and GW150914 as representatives of low-mass and massive binaries respectively, following [27].

parameters.

3.2 Data Analysis Formalism

We adopt a Fisher analysis [191] to estimate the statistical errors of the non-GR parameters in various theories. Such an analysis is valid for GW events with sufficiently large signal-to-noise (SNR) ratios. We make the assumptions that the detector noise is Gaussian and stationary. Let us write the detector output as

$$s(t) = h(t) + n(t)\,, \tag{3.1}$$

where $h(t)$ and $n(t)$ are the GW signal and the noise respectively. Let us also define the inner product of two quantities $A(t)$ and $B(t)$ as

$$(A|B) = 4\Re\int_0^\infty df\,\frac{\tilde{A}^*(f)\tilde{B}(f)}{S_n\,(f)}\,. \tag{3.2}$$

Here $\tilde{A}(f)$ is the Fourier component of A, an asterisk $(*)$ superscript means the complex conjugate and $S_n\,(f)$ is the noise spectral density. With the above definitions, the probability distribution of the noise can be written as

$$P\,(n = n_0(t)) \propto \exp\left[-\,(n_0|n_0)\right]\,, \tag{3.3}$$

and the SNR for a given signal $h(t)$ can be defined as

$$\rho \equiv \sqrt{(h|h)}\,. \tag{3.4}$$

Under the assumptions of Gaussian and stationary noise, the posterior probability distribution of binary parameters θ^a takes the following form:

$$P\left(\theta^a|s\right) \propto p^{(0)}\left(\theta^a\right) \exp\left[-\frac{1}{2}\Gamma_{ab}\Delta\theta^a\Delta\theta^b\right], \tag{3.5}$$

where $\Delta\theta^a = \hat{\theta}^a - \theta^a$ with $\hat{\theta}^a$ being the maximum likelihood values of θ^a. $p^{(0)}\left(\theta^a\right)$ gives the probability distribution of the prior information, which we take to be in a Gaussian form for simplicity. Γ_{ab} is called the Fisher information matrix which is defined as

$$\Gamma_{ab} = \left(\partial_a h|\partial_b h\right), \tag{3.6}$$

where $\partial_b \equiv \frac{\partial}{\partial\theta^b}$. One can estimate the root-mean-square of $\Delta\theta^a$ by taking the square root of the diagonal elements of the inverse Fisher matrix $\tilde{\Sigma}^{ab}$:

$$\tilde{\Sigma}^{ab} = \left(\tilde{\Gamma}^{-1}\right)^{ab} = \left\langle\Delta\theta^a\Delta\theta^b\right\rangle, \tag{3.7}$$

where $\tilde{\Gamma}_{ab}$ is defined by

$$p^{(0)}\left(\theta^a\right)\exp\left[-\frac{1}{2}\Gamma_{ab}\Delta\theta^a\Delta\theta^b\right] = \exp\left[-\frac{1}{2}\tilde{\Gamma}_{ab}\Delta\theta^a\Delta\theta^b\right]. \tag{3.8}$$

To save computational time, we use IMRPhenomB waveform. Reference [27] showed that the difference in constraints on PPE phase parameters between IMRPhenomB and IMRPhenomD waveforms are negligible for propagation mechanisms at any PN order and for generation mechanisms at negative PN orders. In App. C, we perform a similar comparison for generation mechanism corrections in the amplitude using sky-averaged waveforms and show that the former is at least suitable for constraining generation mechanisms that enter at negative PN orders, which is what we will consider in Sec. 3.3.

We choose the following parameters as our variables for the Fisher analysis:

$$\theta^a \equiv (\ln \mathcal{M}_z, \ln \eta, \chi, \ln D_L, \ln t_0, \phi_{p,0}, \alpha_r, \delta, \psi_{\rm polar}, \iota, \theta_{\rm PPE}) \ , \tag{3.9}$$

where $\mathcal{M}_z$ is the redshifted chirp mass, $\eta \equiv m_1 m_2/(m_1+m_2)^2$ is the symmetric mass ratio and χ is the effective spin parameter[3]. $\alpha_r, \delta, \psi_{\rm polar}$, and ι are the right ascension, declination, polarization and inclination angles respectively in the detector frame. The non-GR parameter is represented by $\theta_{\rm PPE} = \alpha_{\rm PPE}$ or $\beta_{\rm PPE}$. We perform a Monte Carlo simulation by using each set of the posterior samples released by LIGO [341] for $(\mathcal{M}_z, \eta, D_L, \chi, \alpha_r, \delta, \iota)$, while we randomly sample the polarization angle $\psi_{\rm polar}$ and the coalescence phase $\phi_{p,0}$ in $[0, \pi]$ and $[0, 2\pi]$ respectively. We impose prior information such that $-1 \leq \chi \leq 1$, $-\pi \leq (\phi_{p,0}, \alpha_r, \psi_{\rm polar}) \leq \pi$, and $-\pi/2 \leq (\delta, \iota) \leq \pi/2$.

We use the detector sensitivity of Advanced LIGO (aLIGO) O1 run [29], and we consider the two detectors at Hanford and Livingston. For simplicity, we assume that the Livingston noise spectrum is identical to that of Hanford [194]. For the Fisher integration, the minimum frequency is taken to be 20 Hz while the maximum frequency is same as the cutoff frequency above which the signal power is negligible [342].

Now we are going to discuss how we compute the probability distribution of a non-GR parameter from the output of a Fisher analysis with a Monte Carlo simulation.

[3]The effective spin parameter is defined as $\chi \equiv (m_1\chi_1 + m_2\chi_2)/(m_1+m_2)$, where χ_A with $A = (1, 2)$ is the dimensionless spin of the Ath body.

We set the fiducial value of any non-GR parameter to be zero for our analysis. We perform the following integration numerically to obtain the compound probability density function[4] of any parameter ξ :

$$P\left(\xi\right)=\int P\left(\xi|\sigma_{\xi}\right)P\left(\sigma_{\xi}\right)d\sigma_{\xi}\,, \tag{3.10}$$

where $P\left(\xi\right)$ is the marginal (unconditional) probability density function of ξ. $P\left(\xi|\sigma_{\xi}\right)\propto\exp[-(\xi-\bar{\xi})^{2}/2\sigma_{\xi}^{2}]$ is the conditional probability density function of ξ which we assume to be a Gaussian distribution with a mean $\bar{\xi}$ and a standard deviation σ_{ξ}. $P(\sigma_{\xi})$ is the probability distribution of σ_{ξ} computed from the Fisher analysis for the entire posterior distribution.

Let us finish this section by explaining how we can utilize both amplitude and phase corrections to derive constraints on some theory. One can include $\alpha_{\rm PPE}$ or $\beta_{\rm PPE}$ as variables to the Fisher analysis as in Eq. (3.9) and map them to a non-GR parameter of a theory to derive constraints from the phase and amplitude independently. We refer to such constraints as the "phase-only" and "amplitude-only" bounds respectively. How can we achieve a single constraint that accommodates both of them? Recall the relations between the PPE parameters in Sec. 2.2. One can rewrite $\alpha_{\rm PPE}$ in the waveform in terms of $\beta_{\rm PPE}$ according to Eqs. (2.22) or (2.24) and eliminate the former variable from the analysis. We refer to such constraints as the "phase &

[4]If the distribution of a random variable y depends on a parameter x, and if x follows a certain distribution $P(x)$ (called the mixing or latent distribution), the marginal distribution of y is called mixture distribution or compound probability distribution and is given by $P\left(y\right)=\int P\left(y|x\right)P\left(x\right)dx$ [343].

amplitude combined" bounds[5].

3.3 Results in Example Theories

We now apply our analysis to some example theories. We begin by studying massive gravity that yields corrections in the phase through propagation mechanisms. We compare bounds from the Fisher analysis to those from LVC's Bayesian analysis to justify the former. We next study EDGB gravity, scalar-tensor theories and varying-G theories, which achieve the corrections through generation mechanisms entering at negative PN orders.

3.3.1 Validation of the Fisher analysis: massive gravity

The idea of introducing the mass to gravitons is rather old [344], and many attempts have been made to construct a feasible theory that allows one to do so [345]. Such a theory may arise in higher-dimensional setups [346] and has the potential to solve the cosmic acceleration problem [345]. Although gravitons with non-vanishing masses may have additional polarizations as well [347], we here restrict our attention to the non-GR effects on the tensor modes due to a massive dispersion relation.

We will focus on the non-GR corrections specifically to the GW phase. Thus, the purpose of this section is simply to compare our Fisher analysis with the Bayesian one performed by the LVC. Gravitons with a non-vanishing mass travel at a speed

[5] Alternatively, one can rewrite the PPE corrections in the phase and the amplitude in terms of non-GR parameters of a theory. Performing Fisher analyses with such parameters as variables lead to similar constraints as the 'phase & amplitude combined" bounds, although such an approach is not theory-agnostic.

smaller than the speed of light and the non-GR effects accumulate over the distance. Modified dispersion relation for such gravitons is given by $E^2 = p^2c^2 + m_g^2c^4$, where m_g is the mass of the graviton while E and p are the energy and the momentum respectively. The PPE phase parameters are [348]

$$\beta_{\rm MG} = \frac{\pi^2}{\lambda_g^2}\frac{\mathcal{M}_{ch}}{1+z}D\,, \qquad\qquad b = 3\,, \tag{3.11}$$

where

$$D = \frac{z}{H_0\sqrt{\Omega_M + \Omega_\Lambda}}\left[1 - \frac{z}{4}\left(\frac{3\Omega_M}{\Omega_M + \Omega_\Lambda}\right) + \mathcal{O}(z^2)\right]\,. \tag{3.12}$$

Here, Ω_M and Ω_Λ are the energy density of matter and dark energy respectively. H_0 is the Hubble constant while z is the redshift of the source. λ_g is the Compton wavelength of the graviton that is related to m_g as $\lambda_g \equiv h/(m_g c)$, where h is Planck's constant.

We compute the probability distribution of λ_g from GW150914 according to the procedure outlined in Sec. 3.2 and compare with the one obtained by the LVC [25] (Fig. 3.1). The Fisher analysis with Monte Carlo simulations yields $\lambda_g < 1.2 \times 10^{13}$ km at 90% CL, which is in a good agreement with the LVC bound of 1.0×10^{13} km and thus shows the validity of the former. The difference in the two cumulative distributions of λ_g presented in Fig. 3.1 can be attributed to the fact that the LVC used a more accurate Bayesian analysis and imposed a uniform prior on the graviton mass. The GW bound has recently been updated by combining multiple events [32]. The new bound is stronger than binary pulsar constraints [349,350] but slightly weaker

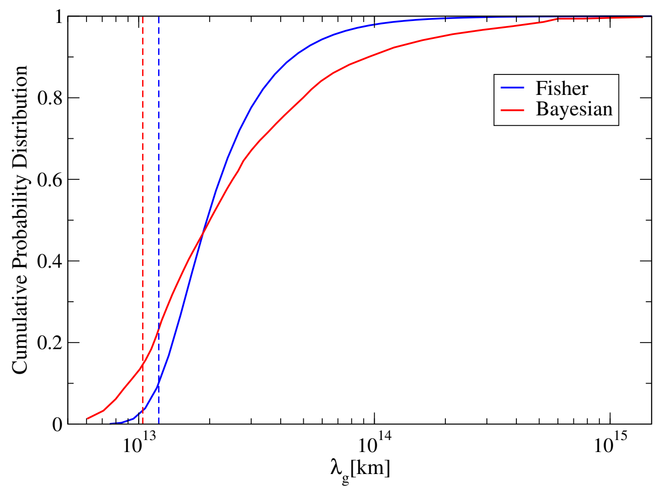

Figure 3.1: The cumulative probability distribution of the graviton Compton wavelength from GW150914. We show the ones obtained from a Fisher analysis with Monte Carlo simulations (blue solid) and from a Bayesian analysis by the LVC (red solid). Each of the vertical dashed lines corresponds to the lower bound of the distribution of the same color with 90% confidence. Observe how the two different analyses give similar bounds.

than the updated solar system bounds [351]. The bound is also weaker than the ones from the observations of galactic clusters [352–354], gravitational lensing [355], and the absence of superradiant instability in supermassive BHs [356].

3.3.2 Einstein-dilaton-Gauss-Bonnet gravity

In EDGB gravity, BHs accumulate scalar monopole charges which may generate scalar dipole radiation if they form binaries [33, 173, 225, 255]. Recall from Sec. 2.3.2, such radiation leads to an earlier coalescence of BH binaries compared to that of GR and modifies the gravitational waveform with the PPE parameters given by [27, 173]

$$\beta_{\rm EDGB} = -\frac{5}{7168}\zeta_{\rm EDGB}\frac{(m_1^2\tilde{s}_2^{\rm EDGB} - m_2^2\tilde{s}_1^{\rm EDGB})^2}{M_t^4\eta^{18/5}}\,, \tag{3.13}$$

with $b = -7$ and

$$\alpha_{\rm EDGB} = -\frac{5}{192}\zeta_{\rm EDGB}\frac{(m_1^2\tilde{s}_2^{\rm EDGB} - m_2^2\tilde{s}_1^{\rm EDGB})^2}{M_t^4\eta^{18/5}}\,, \tag{3.14}$$

with $a = -2$. Here, $\zeta_{\rm EDGB} \equiv 16\pi\bar{\alpha}_{\rm EDGB}^2/M_t^4$ is the dimensionless EDGB coupling parameter and $\tilde{s}_A^{\rm EDGB}$ are the spin-dependent factors of the BH scalar charges given by $\tilde{s}_A^{\rm EDGB} \equiv 2(\sqrt{1-{\chi_A}^2} - 1 + {\chi_A}^2)/{\chi_A}^2$ [33, 255].

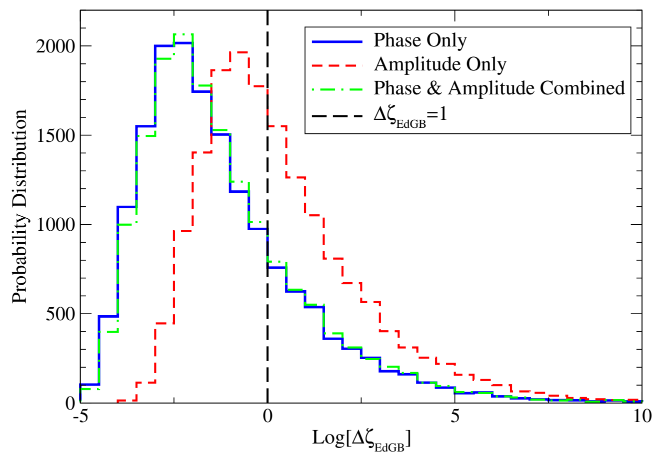

Figure 3.2: Histogram distributions of the 90% CL bounds on $\zeta_{\rm EDGB}$ from a Fisher analysis with the phase correction only (blue solid), the amplitude correction only (red dashed) and combining the two corrections (green dotted-dashed). Fiducial values are taken from the posterior samples of GW150914. The samples that lie on the left side of the vertical black dashed line satisfy the small coupling approximation with 90% CL.

We now derive constraints on EDGB gravity from GW150914 and GW151226. First, we estimate how well these events satisfy the small coupling approximation $\zeta_{\rm EDGB} < 1$. To do so, we extract the 90% CL upper bound $\Delta\zeta_{\rm EDGB}$ from each sample of the posterior distribution of a particular event. We then create histograms with all the samples (see Fig. 3.2) and calculate the fraction satisfying $\Delta\zeta_{\rm EDGB} < 1$. For

GW150914, 72% (42%) of the samples satisfy the small coupling approximation if $\Delta\zeta_{\rm EDGB}$ is derived from the phase (amplitude) correction only, while 71% of the posterior distribution satisfies such approximation if the phase and amplitude corrections are combined. A similar analysis with GW151226 gives 98% and 87% for the phase and amplitude corrections respectively while combining the two yields almost the same result as that of the phase-only case. Since the fraction of samples satisfying $\zeta_{\rm EDGB} < 1$ is much higher for GW151226 than GW150914 due to a larger number of GW cycles and slower relative velocity of the binary constituents, the former event places more reliable constraints on EDGB gravity compared to the latter one.

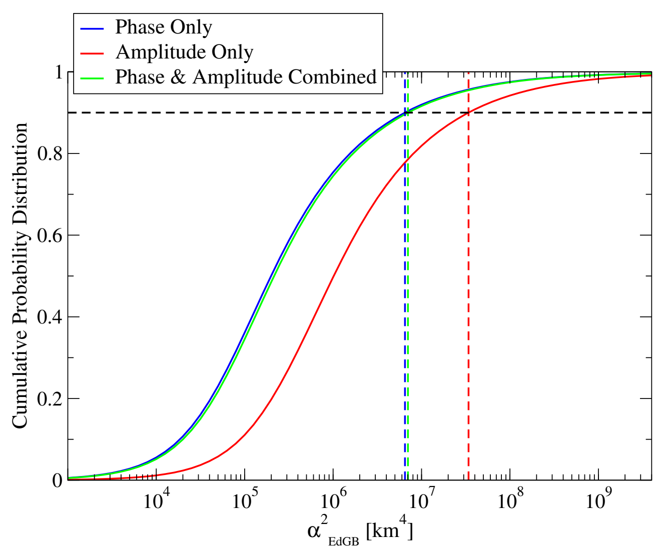

Figure 3.3: Cumulative probability distributions of $\bar{\alpha}^2_{\rm EDGB}$ obtained from GW150914 for the same three cases as in Fig. 3.2. Each vertical dashed line shows the corresponding 90% CL upper bound of a solid line of the same color.

Figure 3.3 presents cumulative probability distributions of $\bar{\alpha}^2_{\rm EDGB}$[6] for GW150914

[6]We show the distribution of $\bar{\alpha}^2_{\rm EDGB}$ instead of $\sqrt{\bar{\alpha}_{\rm EDGB}}$ as it is the former that directly enters in the waveform.

for three different cases with vertical lines representing the 90% CL of the corresponding distribution. We found the 90% CL constraints on $\sqrt{\bar{\alpha}_{\text{EDGB}}}$ from each of the phase and amplitude corrections as 50.5 km and 76.3 km respectively. Notice that these bounds have the same order of magnitude. On the other hand, combining the amplitude and phase corrections leads to an upper bound of 51.5 km, which is *weaker* than the phase-only constraint by 2% (to be discussed more in Sec. 3.4). Though above constraints may not be reliable as the 90% CL bounds on ζ_{EDGB} do not satisfy the small coupling approximation, which is shown in Table 1.1.

We now look at bounds on GW151226. We found that this event yields 4.32 km and 10.5 km respectively from the phase and amplitude corrections, while combining the two only changes the result from the phase-only case by 0.01%. These bounds are consistent with those in a recent paper [357] that utilized the LVC posterior samples including the non-GR phase corrections at -1PN order while Ref. [358] found even stronger bounds by combining multiple GW events. These GW bounds are comparable to the one obtained from low-mass X-ray binaries [244].

Although GW150914 leads to weaker constraints on EDGB gravity compared to GW151226, the effect of amplitude correction is more manifest for the former event. This is because GW150914 has a smaller number of GW cycles, and thus the amplitude contribution becomes relatively higher than GW151226.

3.3.3 Scalar-Tensor theories

Certain scalar-tensor theories predict scalarization of neutron stars [210, 211], which can also happen to BHs if the scalar field evolves with time cosmologically [228, 229]. Let us recall from Sec. 2.3.1 that compact binaries formed by such objects emit dipole radiation which modifies the GW phase with PPE parameters [37, 45, 81]

$$\beta_{\rm ST} = -\frac{5}{7168}\eta^{2/5}(\alpha_1 - \alpha_2)^2 \tag{3.15}$$

with $b = -7$, and the GW amplitude as [81]

$$\alpha_{\rm ST} = -\frac{5}{192}\eta^{2/5}(\alpha_1 - \alpha_2)^2 \tag{3.16}$$

with $a = -2$. Here α_A represents the scalar charge of the Ath binary component and depends on specific theories and the type of compact objects. If we consider a binary consisting of BHs in a theory where the scalar field Φ obeys a massless Klein-Gordon equation, α_A is given by [229]

$$\alpha_A = 2\, m_A\, \dot{\Phi}\, [1 + (1 - \chi_A^2)^{1/2}]\,, \tag{3.17}$$

where $\dot{\Phi}$ is the rate of change of Φ with time.

One can use Eqs. (3.15)–(3.17) and the numerical analysis described in Sec. 3.2 to find constraints on $\dot{\Phi}$ as long as the small coupling approximation $m_A\dot{\Phi} < 1$ is satisfied. In this regard, only 11.7% (13.5%) of the samples of GW150914 satisfies such approximation with 90% confidence level for the phase (combined) correction, while all of the samples fail to do so for the amplitude correction. Hence GW150914 cannot place any meaningful bound on scalar-tensor theories considered here. On the other hand, 90.4% of the samples from GW151226 meets the small coupling criterion for the phase-only and combined analyses, while the fraction is only 25% for the amplitude correction. Thus, we derive reliable constraints from GW151226 with the phase-only and combined analyses, with both leading to $\dot{\Phi} < 1.1 \times 10^4/\mathrm{sec}$[7]. This constraint is 10 orders of magnitude weaker than the current most stringent bound obtained from the orbital decay rate of quasar OJ287 [229], though this is the first bound obtained in the strong/dynamical regime.

3.3.4 Varying- G theories

Many metric theories of gravity that violate the strong equivalence principle [2, 327, 328] predict time variation in the gravitational coupling parameter G [329]. Scalar-tensor theories are examples where G varies as a function of the asymptotic scalar field [2], which may vary over time. As we discussed in Sec. 2.4, any such time-dependence of G leads to an alteration of the gravitational waveform through the modifications of the binary orbital evolution and the energy balance law [81].

We now show the PPE modifications due to a time variation in the gravitational constant. As pointed out in Sec. 2.4, the amount of gravitational coupling that appears in different sectors of a gravitational theory may not be unique. Einstein-Æther theory [296] and Brans-Dicke theory with a cosmologically evolving scalar

[7] A previous analysis with the sky-averaged waveform in Ref. [27] could not place a reliable bound on scalar-tensor theories. Since the posterior distributions of the GW events were not available then, one could not determine how well those events satisfied the small coupling approximation from a simple Fisher analysis.

field [2] are examples of such theories in which various gravitational constants exist. In Sec. 2.4 we studied a generic case with two distinct gravitational constants in Kepler's law (conservative sector) and GW luminosity (dissipative sector). Here we place constraints on the special case where these two constants coincide with each other. In that case, we set $G_C = G_D = G$ and Eqs. (2.64) and (2.66) change as

$$\beta_{\dot{G}} = -\frac{25}{851968}\dot{G}_0\,\eta_0^{3/5}\left[(11+3s_{1,0}+3s_{2,0})M_{t,0} - 41(s_{1,0}m_{1,0}+s_{2,0}m_{2,0})\right] \tag{3.18}$$

with $b = -13$, and

$$\alpha_{\dot{G}} = \frac{5}{512}\eta_0^{3/5}\dot{G}_0\left[-(7-s_{1,0}-s_{2,0})M_{t,0} + 13(s_{1,0}m_{1,0}+s_{2,0}m_{2,0})\right] \tag{3.19}$$

with $a = -8$ respectively.

Employing Eqs. (3.18) and (3.19), GW150914 (GW151226) imposes constraints on $|\dot{G}_0/G_0|$ from the phase-only and amplitude-only analyses as $7.30 \times 10^6\,\mathrm{yr}^{-1}$ ($2.24 \times 10^4\,\mathrm{yr}^{-1}$) and $1.37 \times 10^8\,\mathrm{yr}^{-1}$ ($3.82 \times 10^5\,\mathrm{yr}^{-1}$) respectively, with the combined analyses yielding slight improvements over the phase-only results. Unlike the EDGB and scalar-tensor cases, the amplitude-only analyses yield much worse bound than that from the phase-only cases even with GW150914. Notice also that for varying-G theories, the combined bound is slightly *stronger* than the phase-only bound (to be discussed more in the subsequent section). These bounds are much less stringent compared to the other contemporary constraints [2]. However, future space-borne detectors such as LISA [181,359] will be able to obtain constraints up to 13 orders of magnitude stronger compared to the aLIGO ones [177, 298].

3.4 Conclusion and Discussion

In this analysis, we have derived constraints on scalar-tensor, EDGB and varying-G theories from GW150914 and GW151226. To do so, we performed Fisher analyses with Monte-Carlo simulations using the posterior samples constructed by LVC. In particular, we derived reliable constraints on the time-evolution of the scalar field in scalar-tensor theories from GW observations for the first time.

We explored how amplitude corrections contribute to the constraints on such theories. We derived three sets of bounds on each theory: phase-only, amplitude-only, and from both phase and amplitude combined. We found that for binaries with large masses such as GW150914, where we have less number of cycles, the bounds from the amplitude and phase can be comparable to each other. On the other hand, combined analyses yield constraints that differ from the phase-only case at most by 3.6% for the theories under consideration. Hence, at least in theories where the leading corrections enter at negative PN orders, the phase-only analyses as done in previous literature [27,298,357,358,360] can produce sufficiently accurate constraints.

Depending on the prior information and the PN order of the non-GR correction, a combined analysis can yield stronger or weaker constraint compared to a phase-only one. With the priors mentioned in Sec. 3.2, the fractional difference between $\beta_{\rm PPE}$ for the two cases is presented in Fig. 3.4. From -4 PN to -2.5 PN correction, the

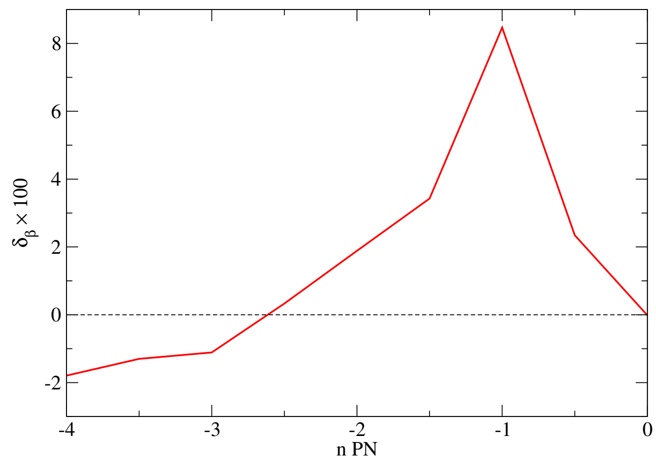

Figure 3.4: Comparison of combined and phase-only analyses at different PN orders from GW150914 with a sky-averaged phenomB waveform. We show $\delta_\beta = (\beta_{\rm comb} - \beta_{\rm phase})/\beta_{\rm phase}$, where $\beta_{\rm phase}$ and $\beta_{\rm comb}$ are bounds on $\beta_{\rm PPE}$ from phase-only and combined analyses respectively. When δ_β is positive (negative), the combined analyses yield weaker (stronger) bounds than the phase-only ones.

combined analyses give rise to slight improvements over the phase-only constraints, while for other cases, the former is weaker with a maximum deterioration of 8.5% at -1 PN order. Nonetheless, it would be safer to include both phase and amplitude corrections in the analysis as a lack of the former in the waveform may cause systematic errors on GR parameters such as luminosity distance if non-GR corrections exist in nature.

In this paper, we considered only the leading PN corrections in the inspiral part of the waveform, but how important are higher-PN corrections and modifications in the merger-ringdown portion? Reference [27] partially addressed this question by taking Brans-Dicke theory as an example whose leading correction enters at -1PN order, similar to EDGB gravity and scalar-tensor theories considered here. Appendix B of [27] shows that including higher-PN corrections only affects the bound from the leading PN correction by 10% at most for GW150914. Moreover, for EDGB gravity, including the correction to the black hole ringdown frequency and damping time only affects the bound from the leading PN corrections in the inspiral by 4.5% for GW150914 [361]. Thus, it is likely that the bounds presented here are valid as order-of-magnitude estimates.

A possible avenue for future work includes repeating the calculation presented here but with a Bayesian analysis using a more accurate waveform such as PhenomD, PhenomPv2 or effective-one-body ones. In particular, it would be interesting to investigate whether the amplitude correction contribution entering at positive PN orders is negligible like the negative PN cases reported here. It is also interesting to repeat the analysis here to all the other events in GWTC-1 [29] and study how much the bounds on each theory improve by combining these events. Another possibility is to take into account non-tensorial polarization modes following e.g. [87]. For future detectors with improved sensitivities, the ability to measure amplitude corrections may be dominated by calibration errors[8].

[8] The calibration error on the amplitude for the O2 run was 3.8% [29].

Chapter 4

Probing Compact Extra Dimensions with Gravitational Waves

4.1 Introduction

In this chapter we consider gravitational waves in a $5d$ spacetime, with the fifth dimension being compactified on a circle and with the matter constrained on a brane [1]. We work in $5d$ GR rather than in the context of the effective Einstein-Maxwell-dilaton $4d$ theory (considered in [107–110]) which arises from performing a Kaluza-Klein reduction of the $5d$ theory. Our motivation is two-fold. First motivation is simplicity: it is easier to work with a single field (the metric) than a collection of fields. Also, by not performing the Kaluza-Klein reduction (which assumes that fields are independent of the compactified dimension) allows us to account for the effect of the massive fluctuations that arise when integrating out the compact dimension. By working directly in higher-dimensional GR theory we are also able to generalize the $5d$ results to an arbitrary number of compactified extra dimensions. Second, we

[1] This chapter is based on the following paper: *Probing Compactified Extra Dimensions with Gravitational Waves*; Du, Yuchen; Tahura, Shammi; Vaman, Diana; Yagi, Kent; Phys. Rev. D 103, 044031 (2021)

hope that our 5d analysis will be useful to assess the effect of the extra dimensions on gravitational wave detection when considering other paradigms for the geometry of the extra dimensions, e.g. Randall-Sundrum (RS) models.

This chapter is organized as follows. In section 4.2 we present our conventions, notations and general framework. In section 4.3, we extract the compactified $5d$ Newtonian potential and the modified Kepler's law for a binary at a fixed position in the extra dimension. In section 4.4 we review the Kaluza-Klein reduction of the $5d$ theory and point out that $5d$ matter, when seen from a $4d$ perspective is non-minimally coupled. Section 4.5 contains our main result, namely the form of the gravitational waves sourced by the $5d$ binary. In section 4.6 we extract the luminosity of the gravitational waves, and in section 4.7 we compute the phase of the gravitational waveform as predicted by the compactified $5d$ model, and compare it with observations. Throughout the chapter we give generalizations of our formulae in the case of a comapctified D-dimensional spacetime, with four non-compact dimensions. We conclude in section 4.8.

4.2 Set-up

We will consider point-like mass sources in some higher-dimensional spacetime, and we will investigate their effect on the spacetime geometry and on the emission of gravitational waves from binaries, in perturbation theory.

To this end we will compute the metric perturbation $\tilde{h}_{\mu\nu}$ by direct integration of Einstein's equations and not from the quadrupole formula as it is customary, because the quadruple formula actually fails when there is a compactified dimension. The reason for this is that the validity of quadrupole formula relies on integration by parts. When the spatial coordinates are non-compact, the boundary terms which accompany the integration by parts are zero. However, when there are compactified extra dimensions, there are non-zero boundary terms, which are not straightforward to evaluate and which will contribute, in addition to the usual quadrupole integral. Please see Appendix D for the modified expression.

Thus, we need to solve the metric perturbation directly from the Einstein equations. We will use the relaxed Einstein equations in the harmonic gauge [178].

For simplicity in most of this chapter we will consider a five-dimensional ($5d$) spacetime with coordinates x^M, with four noncompact dimensions x^μ and a fifth dimension, $x^5 = w$, compactified on a circle of radius $\mathfrak{R}$, though we will occasionally point out how our results change in the case of additional compactified extra dimensions:

$$x^M = (x^\mu, x^5, \dots) = (t, \vec{x}, w, \dots) \sim (t, \vec{x}, w + 2\pi\mathfrak{R}, \dots), \qquad M = 0, 1, 2, 3, 5, \dots D. \tag{4.1}$$

We further denote the spatial coordinates by

$$x^I = (\vec{x}, w, \dots) = (x, y, z, w, \dots), \qquad I = 1, 2, 3, 5 \dots D\,, \tag{4.2}$$

and the spatial non-compact coordinates by

$$x^i = (x, y, z), \qquad i = 1, 2, 3\,. \tag{4.3}$$

We set the speed of light to $c = 1$.

Let us consider a perturbation of the flat spacetime

$$h_{MN} \equiv g_{MN} - \eta_{MN}\,, \qquad \eta_{MN} = \mathrm{diag}(-1, 1, 1, 1, 1, \dots), \tag{4.4}$$

and let us also define

$$\tilde{h}^{MN} \equiv \eta^{MN} - \tilde{g}^{MN} \qquad \tilde{g}^{MN} \equiv \sqrt{-g}g^{MN}, \tag{4.5}$$

where (g) is the determinant of the metric g_{MN}. As advertised, we take $\tilde{h}^{MN}$ to satisfy the Lorenz, or de Donder, or harmonic gauge condition:

$$\partial_M \tilde{h}^{MN} = 0. \tag{4.6}$$

To linear order in h_{MN}, $\tilde{h}_{MN}$ reduces to the usual trace-reversed metric perturbation:

$$\tilde{h}_{MN} \simeq h_{MN} - \frac{1}{2} h\, \eta_{MN}. \tag{4.7}$$

Then, the relaxed Einstein equations state [178]

$$\Box \tilde{h}^{MN} = -16\pi G^{(D)} \tau^{MN}\,, \tag{4.8}$$

where $G^{(D)}$ is the gravitational constant in the D-dimensional spacetime, $\Box = \partial_M \partial_N \eta^{MN}$, and where τ^{MN} is given by

$$\tau^{MN} = (-g)(T^{MN} + t^{MN}_{\mathrm{LL}}) + \frac{1}{16\pi G^{(5)}} \left(\tilde{h}^{MP}{}_{,Q}\, \tilde{h}^{NQ}{}_{,P} - \tilde{h}^{PQ} \tilde{h}^{MN}{}_{,PQ} \right). \tag{4.9}$$

Lastly, T^{MN} is the matter energy-momentum tensor while t^{MN}_{LL} is the Landau-Lifshitz

[362] gravitational energy-momentum pseudo-tensor. In a $D-$dimensional spacetime we have [363]

$$
\begin{aligned}
16\pi G^{(D)}(-g)t_{\rm LL}^{MN} &= \tilde{g}^{MN}{}_{,P}\tilde{g}^{PQ}{}_{,Q} - \tilde{g}^{MP}{}_{,P}\tilde{g}^{NQ}{}_{,Q} + \frac{1}{2}g^{MN}g_{PQ}\tilde{g}^{PR}{}_{,S}\tilde{g}^{QS}{}_{,R} \\
&- \Big(g^{MP}g_{QR}\tilde{g}^{NR}{}_{,S}\tilde{g}^{QS}{}_{,P} + g^{NP}g_{QR}\tilde{g}^{MR}{}_{,S}\tilde{g}^{QS}{}_{,P}\Big) \\
&+ g_{PQ}g^{RS}\tilde{g}^{MP}{}_{,R}\tilde{g}^{NQ}{}_{,S} + \frac{1}{4(D-2)}\left(2g^{MP}g^{NQ} - g^{MN}g^{PQ}\right) \\
&\times \left[(D-2)g_{RS}g_{R'S'} - g_{RR'}g_{SS'}\right]\tilde{g}^{RR'}{}_{,P}\tilde{g}^{SS'}{}_{,Q}\,. \qquad (4.10)
\end{aligned}
$$

From the relaxed Einstein equations (4.8) and the harmonic gauge condition (4.6) it is easy to see that τ^{MN} obeys the conservation law

$$\partial_M \tau^{MN} = \partial_M\Big((-g)(T^{MN} + t_{\rm LL}^{MN})\Big) = 0\,. \qquad (4.11)$$

4.3 Modified Kepler's Law

Let us first consider the scenario where there is one extra non-compact spatial dimension. Thus $D = 5$, the background is flat, and we assume that there is one matter source which is point-like, of mass m, at rest. Then, the energy-momentum tensor is

$$T^{MN}(x^\mu, w) = m\delta^{M0}\delta^{N0}\delta^3(\vec{x})\delta(w). \qquad (4.12)$$

The only non-trivial linearized metric fluctuation is $\tilde{h}_{\rm T}^{00}$, and it satisfies

$$\Box\tilde{h}_{\rm T}^{00} = -16\pi G^{(5)}T^{00}\,, \qquad (4.13)$$

where $G^{(5)}$ is the gravitational constant in $5d$. The solution is

$$\tilde{h}_{\mathrm{T}}^{00}(\vec{x}, w) = \frac{4G^{(5)}m}{\pi(R^2 + w^2)}\,, \qquad R^2 \equiv \vec{x}^2 = x^2 + y^2 + z^2. \tag{4.14}$$

The $5d$ linearized metric fluctuation $h_{\mathrm{T}}^{00} = (2/3)\tilde{h}_{\mathrm{T}}^{00}$ [2] corresponds to a Newtonian potential[3]

$$V^{(5)}(R, w) = -\frac{4}{3}\frac{G^{(5)}m}{\pi(R^2 + w^2)}\,. \tag{4.15}$$

If the extra dimension is not flat, but compact, with an identification $w \sim w + 2\pi\mathfrak{R}$, an observer sees a mass m at every $w = 2n\pi\mathfrak{R}$, where $\mathfrak{R}$ is the radius of compactification and $n \in \mathbb{Z}$ [364]. Summing over all such sources, the resulting linearized metric fluctuation $\tilde{h}_{\mathrm{T}}^{00}$ is periodic $\tilde{h}_{\mathrm{T}}^{00}(\vec{x}, w) \sim \tilde{h}_{\mathrm{T}}^{00}(\vec{x}, w + 2\pi n\mathfrak{R})$:

$$\tilde{h}_{\mathrm{T}}^{00}(t, \vec{x}, w) = \frac{4G^{(5)}m}{\pi}\sum_{n=-\infty}^{\infty}\frac{1}{\vec{x}^2 + (w - 2n\pi\mathfrak{R})^2}\,, \tag{4.16}$$

and, correspondingly, the Newtonian gravitational potential is given by

$$V^{(5,c)}(\vec{x}, w) = -\frac{4}{3}\frac{G^{(5)}m}{\pi}\sum_{n=-\infty}^{\infty}\frac{1}{R^2 + (w - 2n\pi\mathfrak{R})^2}\,. \tag{4.17}$$

If the observers are located at the same w coordinate as the source (think of the matter source and observer living on the same brane at $w = 0$, and ignore for simplicity the backreaction of the brane on the geometry) then we are interested in

[2] In D spacetime dimensions this relation gets modified to $h_{\mathrm{T}}^{00} = (D-3)/(D-2)\,\tilde{h}_{\mathrm{T}}^{00}$, and the linearized metric is given by $ds^2 = (-1 + (D-3)/(D-2)\,\tilde{h}_{\mathrm{T}}^{00})\,dt^2 + (1 + 1/(D-2)\,\tilde{h}_{\mathrm{T}}^{00})\,d\vec{x}\cdot d\vec{x}$.

[3] Working with a D-dimensional spacetime, (4.15) generalizes to

$$V^{(D)}(R, \rho) = -\frac{1}{2}\frac{D-3}{D-2}\tilde{h}_{\mathrm{T}}^{00} = -\frac{D-3}{(D-2)}\frac{4}{\pi^{(D-3)/2}}\Gamma(\frac{D-1}{2})\frac{G^{(D)}m}{(R^2 + \rho^2)^{D-3}}\,, \qquad \rho^2 = x^I x^I - R^2\,.$$

$V^{(5,c)}(\vec{x}, w{=}0)$[4]. Setting $w = 0$ and evaluating the sum over n in (4.17) yields

$$V^{(5,c)}(\vec{x}, w{=}0) = -\frac{4}{3}\frac{G^{(5)}m}{\ell R}\coth\left(\frac{\pi R}{\ell}\right) = -\frac{1}{3}\tilde{h}^{00}_{\mathrm{T}}(t, \vec{x}, w{=}0)\,, \tag{4.18}$$

where

$$\ell = 2\pi\mathfrak{R} \tag{4.19}$$

denotes the length of the compactified extra dimension. For a generalization of (4.18) to the case when the observer and the source are located at different positions in the compact dimension, please see Appendix E.

There are two useful limits of (4.18): one is the decompactification limit, when $\ell \gg R$, and the other is the opposite, with $\ell \ll R$. In the first case, the Newtonian potential assumes the form of a $4d$ non-compact *space*

$$V^{(5,c)}(R, w{=}0) = -\frac{4}{3}\frac{G^{(5)}m}{\pi R^2} + \mathcal{O}(R/\ell)\,, \qquad R = \vec{x}^2 = x^i x^i\,, \qquad \ell \gg R\,, \tag{4.20}$$

whereas in the second case it is equal to the Newtonian potential in a $3d$ non-compact space plus exponential corrections[5]

$$V^{(5,c)}(R, w{=}0) = -\frac{4}{3}\frac{G^{(5)}m}{\ell R}\left(1 + 2e^{-2\pi R/\ell} + \mathcal{O}(e^{-4\pi R/\ell})\right), \qquad \ell \ll R. \tag{4.21}$$

The exponential corrections look like a Yukawa potential, and can be interpreted

[4]For an investigation whether localized matter can arise in the context of effective field theory see [365]

[5]For a D-dimensional space time $R^{3,1} \times T^{D-4}$, with three non-compact spatial dimensions and the compact space being torus, the generalization of (4.21) is

$$V^{(D)}(x^I) = -\frac{2(D-3)}{D-2}\frac{G^{(D)}m}{\mathrm{Vol(Compact\ Space)}}\frac{1}{R}\left(1 + \mathcal{O}(e^{-2\pi R/\ell})\right),$$

where ℓ is the length of the largest of the cycles of the torus.

as being due to massive gravitons. From a $4d$ perspective, these massive gravitons correspond to non-uniform Fourier modes of the massless $5d$ gravitons on the circle $w \sim w + \ell$. We will have more to say about this in the following sections.

Next, let us consider a quasi-circular binary with component masses m_1 and m_2 and binary separation r_{12}, with $r_{12} \gg \ell$. The matter energy-momentum tensor is given by

$$T^{MN}(x) = \sum_{a=1,2} m_a \int d\tau \frac{\dot{x}_a^M \dot{x}_a^N}{\sqrt{-g_{PQ}(x)\dot{x}_a^P \dot{x}_a^Q}} \delta^5(x - x_a(\tau))\,, \tag{4.22}$$

where $x_a^M(\tau)$ is the trajectory of the point-like mass m_a with τ representing an affine parameter on the worldline. We can use reparametrization invariance to identify $x^0(\tau) = \tau$ and, assuming that the matter sources are located at $w_1(t) = w_2(t) = 0$ (i.e. confined to the same brane), to leading order we have

$$\vec{x}_{12}(t) = \vec{x}_1(t) - \vec{x}_2(t) = (r_{12}\cos(\Omega t), r_{12}\sin(\Omega t), 0)\,. \tag{4.23}$$

Further using (4.21) yields the effective potential of such a binary

$$V_{\text{eff}} \simeq \frac{1}{2}\mu r_{12}^2 \Omega^2 - \frac{G_{\text{N}}\mu M_t}{r_{12}}\left(1 + 2e^{-2\pi r_{12}/\ell}\right)\,, \tag{4.24}$$

where

$$M_t = m_1 + m_2 \tag{4.25}$$

is the total mass of the binary and

$$\mu = (m_1 m_2)/M_t \tag{4.26}$$

is the reduced mass, while Ω is the orbital angular frequency. G_N is the $4d$ Newton's constant, with[6]

$$G_\mathrm{N} \equiv \frac{4}{3}\frac{G^{(5)}}{\ell}\,. \tag{4.27}$$

The distance between the two sources is solved from the condition of local extremum of V_eff with respect to r_{12}. This leads to the following modification to the Kepler's law:

$$\begin{aligned} r_{12} \simeq \;& \left(\tfrac{G_\mathrm{N} M_t}{\Omega^2}\right)^{1/3}\left\{1+\tfrac{2}{3}\left(\tfrac{G_\mathrm{N} M_t}{\Omega^2}\right)^{1/3}\tfrac{2\pi}{\ell}\exp\left[-\tfrac{2\pi}{\ell}\left(\tfrac{G_\mathrm{N} M_t}{\Omega^2}\right)^{1/3}\right]\right. \\ & \left. +\tfrac{2}{3}\exp\left[-\tfrac{2\pi}{\ell}\left(\tfrac{G_\mathrm{N} M_t}{\Omega^2}\right)^{1/3}\right]\right\}\,, \end{aligned} \tag{4.28}$$

where we have retained the first order correction to the $4d$ Kepler's law.

4.4 Performing the Kaluza-Klein Reduction with $5d$ Point-like Matter Sources

One might be tempted to think that the physics of the binary system in a $5d$ spacetime is that of a binary (two point-like masses) coupled to the fields obtained via Kaluza-Klein reduction of the $5d$ metric, namely gravity, dilaton and Maxwell fields, and with the latter two being set to zero. Then, to leading order, neglecting all corrections coming from massive modes on the fifth dimensional circle, one recovers the $4d$ matter

[6]For a D-dimensional space time $R^{3,1}\times T^{D-4}$, with three non-compact spatial dimensions and the compact space being torus, the $4d$ Newton's constant is given by

$$G_\mathrm{N} = \frac{2(D-3)}{D-2}\frac{G^{(D)}}{\mathrm{Vol(Compact\ Space)}},$$

with $G^{(D)}$ the D-dimensional gravitational constant.

(the binary) plus gravity set-up. However, this is not the case. To better understand this issue, we take a quick detour and review the Kaluza-Klein reduction of the $5d$ system composed of gravity plus point-like sources. This is a self-contained section of the chapter and for the purpose of performing the Kaluza-Klein reduction we introduce the notation $\mathcal{G}_{MN}$ for the $5d$ metric and $g_{\mu\nu}$ for the $4d$ metric.

Consider five-dimensional gravity

$$\mathcal{S}_{\rm 5d} = \frac{1}{\ell\kappa}\int d^5x\sqrt{-\det\mathcal{G}_{MN}}\,\mathcal{R}[\mathcal{G}_{MN}], \tag{4.29}$$

where $\kappa = 16\pi G$ (we work in units where $c = 1$), $\mathcal{R}[\mathcal{G}_{MN}]$ is the Ricci scalar of metric $\mathcal{G}_{MN}$, and $G = G^{(5)}/\ell$. The Kaluza-Klein reduction ansatz to $4d$ is (see for example [366]):

$$\mathcal{G}_{MN} = e^{-\upsilon/3}\begin{pmatrix} g_{\mu\nu} + \kappa e^{\upsilon}A_\mu A_\nu & \sqrt{\kappa}e^{\upsilon}A_\mu \\ \sqrt{\kappa}e^{\upsilon}A_\nu & e^{\upsilon}\end{pmatrix}, \tag{4.30}$$

where all the fields in (4.30) are functions of the 4d coordinates, x^μ, only.

Substituting (4.30) into the 5d Einstein-Hilbert action yields the 4d Einstein-Maxwell-dilaton action

$$\mathcal{S}_{\rm KK} = \frac{1}{\kappa}\int d^4x\sqrt{-g}\left(\mathcal{R}[g_{\mu\nu}] - \frac{\kappa}{4}e^{\upsilon}F_{\mu\nu}F^{\mu\nu} - \frac{1}{6}\partial_\mu\upsilon\partial^\mu\upsilon\right). \tag{4.31}$$

Note that we can find solutions with a vanishing dilaton as long as the Maxwell field is pure gauge. (The dilaton equation is sourced by the Maxwell field, so setting the dilaton to zero in general would lead to an inconsistent Kaluza-Klein truncation.)

By introducing the rescaled dilaton and Maxwell field as

$$\upsilon = -2\sqrt{3}\phi\,, \quad A_\mu = \frac{2}{\sqrt{\kappa}}\bar{A}_\mu\,, \tag{4.32}$$

we can rewrite the Einstein-Maxwell-dilaton action as

$$\mathcal{S}_{\rm KK} = \frac{1}{\kappa}\int d^4x\sqrt{-g}\Big(\mathcal{R}[g_{\mu\nu}] - e^{-2\sqrt{3}\phi}\bar{F}_{\mu\nu}\bar{F}^{\mu\nu} - 2\partial_\mu\phi\partial^\mu\phi\Big)\,. \tag{4.33}$$

This agrees with (II.1) of [108]. We will use the rescaled dilaton (4.32) to make contact with [108] in Section 4.6.

Consider now adding a point-like source of mass m to the $5d$ action:

$$\mathcal{S}_{\rm matter,\,5d} = -m\int d\tau\sqrt{-\dot{x}^M(\tau)\dot{x}^N(\tau)\mathcal{G}_{MN}(x(\tau))}\,, \tag{4.34}$$

where τ is an affine parameter on the source's worldline and $\dot{x}^M = \frac{d}{d\tau}x^M$. This will source the $5d$ metric in the usual way, leading to the $5d$ perturbative analysis performed in the previous section, and continued in the next.

Here we would like to point out that the $4d$ dilaton is also being sourced by the $5d$ matter (4.34). Specifically, for a source that is not moving in the fifth dimension (note that this is a solution to its equation of motion in the context of a $5d$ metric which is independent of the fifth coordinate), the reduction of the $5d$ action (4.34) yields

$$\mathcal{S}_{\rm matter,\,KK} = -\,m\int d\tau\, e^{-\upsilon/6}\sqrt{-\dot{x}^\mu(\tau)\dot{x}^\nu(\tau)(g_{\mu\nu} + \kappa e^{\upsilon}A_\mu A_\nu)}\,, \tag{4.35}$$

where the $4d$ fields are evaluated on the worldline.

In contrast, adding a $4d$ neutral source of mass m to the Einstein-Maxwell-dilaton action is done by considering

$$\mathcal{S}_{\text{matter, 4d}} = -m \int d\tau \sqrt{-\dot{x}^{\mu}(\tau)\dot{x}^{\nu}(\tau)g_{\mu\nu}}\,. \tag{4.36}$$

The main message here is that $5d$ matter couples not only to the $4d$ graviton, but also to the dilaton and Maxwell field. Most importantly, while we can find solutions with a vanishing dilaton to the $4d$ action with a $4d$ matter source, we cannot find solutions with a vanishing dilaton to the $4d$ Kaluza-Klein reduced action when the matter source is a $5d$ point-like source; this can be understood by noticing that in (4.35) there is a linear coupling between the dilaton and the matter source, when expanding in small fluctuations about a vanishing dilaton. In effect, the $4d$ mass in the Kaluza-Klein reduced action is modulated by the dilaton,

$$m_{\text{effective}} = m \exp(-\upsilon/6)\,, \tag{4.37}$$

as evidenced by (4.35).

Lastly, in the context of Kaluza-Klein reduction of a higher-dimensional gravitational theory, the $4d$ gravitational coupling constant is $1/(16\pi G)$, as we can see from (4.31), with

$$G = G^{(5)}/\ell\,. \tag{4.38}$$

However, G and G_{N} (which shows up in the Newtonian potential and it is given in (4.27) for $D = 5$) are not equal: they differ by a factor. This is different from $4d$ GR, where G and G_{N} are equal to one another. The explanation for this mismatch stems from the fact that in an effective $4d$ theory, the interaction between two masses is not only gravitational, but there are additional contributions mediated by $4d$ scalars (e.g. dilaton) as well.

4.5 Metric Perturbations

We now return to our main problem, namely finding the $5d$ metric fluctuations sourced by a binary in a spacetime with one compact dimension of radius $\mathfrak{R}$.

We separately find the contribution of the matter energy-momentum tensor and of the Landau-Lifshitz pseudo-tensor to the metric fluctuations: we denote these by $\tilde{h}_{\mathrm{T}}^{MN}$ and $\tilde{h}_{\mathrm{t}}^{MN}$. There is one more contribution to the metric fluctuations from the remainder of the relaxed Einstein equation source τ^{MN} (see (4.9)). However, to leading order, the extra terms in τ^{MN} contribute only to the 00-component of the metric fluctuations, $\tilde{h}^{00}$. We will compute the 0-components of the metric fluctuations not by direct integration, but by using the harmonic gauge (4.6). So, for the remaining fluctuations $\tilde{h}^{IJ} = (\tilde{h}^{ij}, \tilde{h}^{i5}, \tilde{h}^{55})$ we will evaluate first the contribution from the matter source, and then use this in the Landau-Lifshitz pseudo-tensor to evaluate the non-linear metric fluctuations. Despite $\tilde{h}_{\mathrm{t}}^{IJ}$ being non-linear, it is actually of the same order as $\tilde{h}_{\mathrm{T}}^{IJ}$ in a velocity expansion. Lastly, we add the two contributions to find the metric perturbation to second order in velocities, i.e. leading order in post-Newtonian expansion.

4.5.1 Metric perturbations: contribution from the matter sources

We begin by computing the perturbations sourced by the matter stress-energy tensor. The energy-momentum tensor T^{MN} of a system of point masses at $w = 0$ is given by

$$T^{MN}(t, \vec{x}, w) = \sum_a \frac{P_a^M P_a^N}{P_a^0} \delta^3 \left(\vec{x} - \vec{x}_a(t)\right) \delta\left(w\right) , \tag{4.39}$$

where $P_a^M = m \dot{x}_a^M / \sqrt{-g_{PQ} \dot{x}_a^P \dot{x}_a^Q}$ is the M-component of the momentum of particle a. We parametrize the particles' trajectories with $\vec{x}_a = \vec{x}_a(t)$ and take $x_a^0 = t$. For a binary source, we specifically use (4.23).

From (4.8), the linearized fluctuations are given by

$$\tilde{h}_{\mathrm{T}}^{MN}(t, \vec{x}, w) = -16\pi G^{(5)} \int dt' d^3\vec{x}' dw' \, \mathcal{G}^{(5,c)}\left(t, \vec{x}, w; t', \vec{x}', w'\right) T^{MN}\left(t', \vec{x}', w'\right) , \tag{4.40}$$

where $\mathcal{G}^{(5,c)}(t, \vec{x}, w; t', \vec{x}', w')$ is the (scalar) retarded compactified Green's function in $5d$. The retarded Green's function in flat $5d$ can be represented as [367]

$$\mathcal{G}^{(5)}\left(t, \vec{x}, w; t', \vec{x}', w'\right) = -\frac{\theta(t - t')}{4\pi^2 \mathscr{r}} \frac{\partial}{\partial \mathscr{r}} \frac{\theta\left(t - t' - \mathscr{r}\right)}{\sqrt{\left(t - t'\right)^2 - \mathscr{r}^2}} , \tag{4.41}$$

with

$$\mathscr{r}^2 = (\vec{x} - \vec{x}')^2 + (w - w')^2 , \tag{4.42}$$

and where θ denotes the Heaviside step-function. [7]

[7]For the reader accustomed to $4d$ expressions, we want to point out that even though the $5d$ retarded Green's function does not have support only on the light-cone (as opposed to the massless $4d$ retarded Green's function which has support on the light-cone only), it does have support inside the light-cone, and it is therefore causal. This is one of the peculiar features of odd dimension spacetimes.

Then, starting from (4.41), we can write the compactified $5d$ retarded Green's function, $\mathcal{G}^{(5,c)}(x,y)$:

$$\mathcal{G}^{(5,c)}\left(t,\vec{x},w;t',\vec{x}',w'\right) = -\sum_{n=-\infty}^{\infty} \frac{\theta(t-t')}{4\pi^2\, \mathscr{r}_n} \frac{\partial}{\partial \mathscr{r}_n} \frac{\theta\left(t-t'-\mathscr{r}_n\right)}{\sqrt{\left(t-t'\right)^2 - \mathscr{r}_n^2}}\,, \tag{4.43}$$

where $\mathscr{r}_n^2 = (\vec{x}-\vec{x}')^2 + (w-w'-n\ell)^2$. For practical purposes, the compactified Green's function expression given in (4.43) is not very useful.[8] Instead, we will use the equivalent representation of the compactified retarded Green's function in terms of a sum/integral over Fourier modes[9] (see also Appendix G):

$$\mathcal{G}^{(5,c)}(x^\mu,w;x'^\mu,w') = -\frac{1}{\ell}\sum_{n=-\infty}^{\infty}\int \frac{d^4p}{(2\pi)^4}\frac{e^{ip\cdot(x-x')}e^{i2\pi n(w-w')/\ell}}{-(p_0+i\epsilon)^2+\vec{p}^2+(2\pi n/\ell)^2}\,, \tag{4.44}$$

where ϵ is an infinitesimally small positive number. In (4.44) each term in the sum can be interpreted as the $4d$ retarded Green's function of a massive particle, of mass $m_s = (2\pi n/\ell)$. These massive particles are nothing else but the massive Kaluza-Klein graviton states. Thus we expect that in the limit when $\mathscr{r} \gg \ell$, and for slow moving sources, (4.44) will reduce to the $4d$ retarded Green's function of a massless particle, corresponding to $n=0$, plus exponentially suppressed corrections, with the leading order correction coming from the least massive mode, corresponding to $n=1$. Indeed, the $n=0$ term in the sum above corresponds to massless $4d$ excitations, and the retarded $4d$ Green's function $\theta(t-t')\delta(t-t'-|\vec{x}-\vec{x}'|)/(4\pi|\vec{x}-\vec{x}'|)$. The non-zero n terms are associated with massive $4d$ excitations. The retarded Green's function

[8] However, for the purpose of demonstrating how one could use (4.43) in an explicit calculation, please see Appendix F for another derivation of the Newtonian potential in $5d$ GR.

[9] The Dirac-delta function, written as a distribution on the space of periodic functions with period ℓ, is $\delta(w-w') = (1/\ell)\sum_{n=-\infty}^{\infty}\exp(i2\pi n(w-w')/\ell)$.

for a massive $4d$ scalar of mass m_s is

$$
\begin{aligned}
&-\int \frac{d^4p}{(2\pi)^4} \frac{e^{ip\cdot(x-x')}}{-(p_0+i\epsilon)^2+\vec{p}^2+m_s^2} \\
&\quad = -\frac{\theta(t-t')}{4\pi}\left[\frac{\delta(t-t'-|\vec{x}-\vec{x}'|)}{|\vec{x}-\vec{x}'|}\right. \\
&\qquad \left. -\,\theta(t-t'-|\vec{x}-\vec{x}'|)\frac{m_s J_1\left(m_s\sqrt{(t-t')^2-|\vec{x}-\vec{x}'|^2}\right)}{\sqrt{(t-t')^2-|\vec{x}-\vec{x}'|^2}}\right].
\end{aligned} \tag{4.45}
$$

Consider next the propagation of a periodic signal $e^{i\omega t'}f(\vec{x}')$, with $f(\vec{x}')$ localized near the origin (similar to the case encountered with the binary sources). In the leading multipole expansion, for $|\vec{x}-\vec{x}'|\simeq|\vec{x}|=R$ we are left with evaluating

$$
\begin{aligned}
&\int_{-\infty}^{t} dt' \left\{\frac{\delta(t-t'-R)}{R} - \theta(t-t'-R)\frac{m_s J_1\left(m_s\sqrt{(t-t')^2-R^2}\right)}{\sqrt{(t-t')^2-R^2}}\right\} e^{i\omega t'} \\
&= \frac{e^{i\omega(t-R)}}{R} - m_s e^{i\omega t} I_{\frac{1}{2}}\left[\frac{R}{2}\left(\sqrt{m_s^2-\omega^2}-i\omega\right)\right] K_{\frac{1}{2}}\left[\frac{R}{2}\left(\sqrt{m_s^2-\omega^2}+i\omega\right)\right] \\
&= \frac{e^{i\omega t}}{R} e^{-R\sqrt{m_s^2-\omega^2}}.
\end{aligned} \tag{4.46}
$$

If $m_s \gg \omega$ (which is the case for slow moving binary sources since $m_s = \frac{2\pi n}{\ell}, \ell \ll r_{12}, \Omega r_{12} \ll 1$), the approximate result from (4.46) would be simply $(1/R)\, e^{i\omega t} e^{-2\pi n \frac{R}{\ell}}$, which is the anticipated exponentially suppressed contribution.

So, putting everything together, the signal propagating from a source that is localized near the origin $f(\vec{x}', w')e^{i\omega t'}$ to a spacetime coordinate $(t,\vec{x},w)$ is

$$
\begin{aligned}
&\int dt' \int d^3\vec{x}' \int_0^{\ell} dw'\, \mathcal{G}^{(5,c)}(t,\vec{x},w;t',\vec{x}',w') f(\vec{x}',w') e^{i\omega t'} \\
&\quad \simeq -\frac{e^{i\omega(t-R)}}{4\pi\ell R} \int d^3\vec{x}' \int_0^{\ell} dw'\, f(\vec{x}',w')
\end{aligned}
$$

$$- \sum_{n, n \neq 0} \frac{e^{i\omega t}}{4\pi \ell R} e^{-R\sqrt{(2\pi n/\ell)^2 - \omega^2}} \int d^3\vec{x}' \int_0^\ell dw' \, f(\vec{x}', w') e^{2\pi i n(w-w')/\ell} . \quad (4.47)$$

In the limit of a small extra-dimension and a slow moving source (i.e. $2\pi/\ell \gg \omega$), we find that the leading contribution is

$$- \frac{1}{4\pi \ell R} \int d^3\vec{x}' \int_0^\ell dw' \, f(\vec{x}', w') e^{i\omega(t-R)} , \quad (4.48)$$

which corresponds to a signal that propagates uniformly in w and radially in the non-compact space. The massive Kaluza-Klein gravitons give an exponentially suppressed contribution of the form

$$- 2 \sum_{n=1}^{\infty} \frac{e^{i\omega t}}{4\pi \ell R} e^{-2\pi n R/\ell} \int d^3\vec{x}' \int_0^\ell dw' \, f(\vec{x}', w') \cos(2\pi n(w-w')/\ell) . \quad (4.49)$$

At large distances $R \gg \ell$ these massive states contributions can be safely ignored.

In particular, from (4.40), sourced by the binary (4.22, 4.23) energy-momentum, and using the approximations in (4.48) and (4.49) in the far-field slow motion limit, we find for example the (x, y) component as

$$\tilde{h}_{\mathrm{T}}^{xy}(t, \vec{x}, w{=}0) \simeq - \frac{3}{2} \frac{G_{\mathrm{N}}\mu}{R} r_{12}^2 \Omega^2 \left(\sin[2\Omega(t-R)] + \sum_{n=1}^{\infty} e^{-\frac{2\pi R}{\ell} n} \sin(2\Omega t) \right) , \quad (4.50)$$

where $R = |\vec{x}|$ is the $3d$ distance between the sources and the observer, and μ is the reduced mass defined in (4.26). The leading correction due to the extra compact dimension to the part of the metric fluctuation which is sourced by the matter energy-momentum tensor is given by the $n = 1$ term in the sum in (4.50), and it is an exponentially suppressed correction. Since $\ell \ll R$, the correction $\exp(-\frac{2\pi R}{\ell})$ is extremely small and can be safely ignored (after all we have already ignored corrections of order r_{12}/R, and we expect $\ell < r_{12}$).

4.5.2 Metric perturbations: the non-linear contribution from the Landau-Lifshitz pseudo-tensor

We denote the non-linear metric perturbations sourced by $t_{\rm LL}^{MN}$ in (4.8) as $\tilde{h}_{\rm t}^{MN}$. In the slow motion limit ($v \ll 1$), since $\tilde{h}_{\rm T}^{00} \sim \mathcal{O}(1)$, $\tilde{h}_{\rm T}^{0i} \sim \mathcal{O}(v)$, $\tilde{h}_{\rm T}^{ij} \simeq \mathcal{O}(v^2)$ and $\tilde{h}_{\rm T}^{M5} \simeq 0$, the leading order contribution for $I, J = 1, 2, 3, 5, \ldots D$ comes from

$$\begin{aligned}\tilde{h}_{\rm t}^{IJ}(x) &= -16\pi G^{(D)} \int d^5y\, \mathcal{G}^{(D,c)}(x,y) t_{\rm LL}^{IJ}(y) \\ &\simeq -\frac{D-3}{4(D-2)} \int d^5y\, \mathcal{G}^{(D,c)}(x,y) \partial_M \tilde{h}_{\rm T}^{00}(y) \partial_N \tilde{h}_{\rm T}^{00}(y) (2\eta^{IM}\eta^{JN} - \eta^{IJ}\eta^{MN}),\end{aligned} \tag{4.51}$$

where $\mathcal{G}^{(D,c)}(x,y)$ is the compactified retarded Green's function in D dimensions. Specializing to the case $D = 5$, we get

$$\tilde{h}_{\rm t}^{IJ}(x) \simeq -\frac{1}{6}(2\eta^{IM}\eta^{JN} - \eta^{IJ}\eta^{MN}) \int d^5y\, \mathcal{G}^{(5,c)}(x,y) \partial_M \tilde{h}_{\rm T}^{00}(y) \partial_N \tilde{h}_{\rm T}^{00}(y), \tag{4.52}$$

where $\mathcal{G}^{(5,c)}(x,y)$ was previously defined in (4.43) and (4.44). As discussed in the previous subsection, in the far field limit (when the distance to the source is much larger than the distances between sources) with the observer and the sources located at $w = 0$, and in the slow motion approximation, the compactified retarded $5d$ Green's function reduces effectively to a $4d$ retarded Green's function. The first order correction, which is proportional to $\exp(-\frac{2\pi R}{\ell})$ is negligible, and (4.52) becomes

$$\begin{aligned} \tilde{h}_{\rm t}^{IJ}(t,\vec{x},0) &\simeq \frac{1}{4\pi\ell R}\frac{1}{6}(2\eta^{IN}\eta^{JM}-\eta^{IJ}\eta^{MN}) \\ &\times \int d^3y\int_0^\ell dw\,\partial_M\tilde{h}_{\rm T}^{00}(t-R,\vec{y},w)\,\partial_N\tilde{h}_{\rm T}^{00}(t-R,\vec{y},w). \end{aligned} \quad (4.53)$$

The most striking difference in the non-linear contribution to the metric fluctuations in $5d$ with respect to the $4d$ case is the coefficient $1/6$ on the right hand side of (4.53) relative to the more familiar coefficient of $1/8$ in $4d$.

We now return to the specific case of a binary system at $w = 0$, with masses m_1, m_2 moving in the (x^1, x^2) plane. The first observation is that in (4.53), to leading order in velocities we only need to to consider the action of the spatial derivatives on $\tilde{h}_{\rm T}^{00}$, which does not contain an explicit t-dependence. Restricting now the sumation over M, N indices to spatial indices K, L, consider the term $\partial_K\tilde{h}_{\rm T}^{00}\partial_L\tilde{h}^{00}$ in (4.53). Since $\tilde{h}_{\rm T}^{00} = \tilde{h}_{\rm T,1}^{00} + \tilde{h}_{\rm T,2}^{00}$, there will be four terms. However, we are only interested in the two crossing terms because non-crossing terms will be simply regularized and effectively be dropped out. In addition, we will replace the spatial derivative on y to the derivative with respect to the position of the sources (with a minus sign). We can do so because of translation invariance of the flat background which implies that the linearized fluctuation $\tilde{h}_{\rm T,a}^{00}$ only depends on $\vec{y} - \vec{y}_a$ and $w - w_a$. We will use $\partial_K^{(a)}$ to represent partial derivatives with respect to the coordinates of the source a, $\partial/\partial y_a^K$. With the help of this little trick, we can simplify (4.53):

$$\tilde{h}_{\rm t}^{IJ}(t,\vec{x},w) \simeq \frac{(2\eta^{IK}\eta^{JL}-\eta^{IJ}\eta^{KL})(\partial_K^{(1)}\partial_L^{(2)}+\partial_K^{(2)}\partial_L^{(1)})}{24\pi\ell R}$$

$$\times \int_{\rm NZ} d^3y \int_0^\ell dw\ \tilde{h}^{00}_{\rm T,1}(t-R,\vec{y},w)\tilde{h}^{00}_{\rm T,2}(t-R,\vec{y},w)\,, \tag{4.54}$$

where $\int_{\rm NZ} d^3y$ denotes integration in the near zone (NZ) region (i.e. in the vicinity of the sources) which is the region that contributes the most to the volume integral $\int d^3y$ [178]. Due to the near-zone approximation, the wave propagation is almost instantaneous and we can use for $\tilde{h}^{00}_{\rm T,a}$ in the NZ region the result from (4.16),

$$\tilde{h}^{00}_{\rm NZ,T,a}(t,\vec{y},w) = -\frac{4}{\pi}G^{(5)}m_a\sum_n \frac{1}{(\vec{y}-\vec{x}_a(t))^2+(w-w_a+n\ell)^2}\,. \tag{4.55}$$

Substituting (4.55) into the integrand (4.54), we can use the infinite sums to extend the integration region over w first, and then we can perform the remaining sum exactly:

$$\begin{aligned}
&\int_0^\ell dw'\,\tilde{h}^{00}_{\rm NZ,T,1}(t,\vec{y},w')\tilde{h}^{00}_{\rm NZ,T,2}(t,\vec{y},w')\\
&\quad=\sum_{n_1,n_2}\int_0^l dw'\ \frac{16m_1m_2(G^{(5)})^2}{\pi^2(R_1^2+(w'-w_1+n_1\ell)^2)(R_2^2+(w'-w_2+n_2\ell)^2)}\\
&\quad=\int_{-\infty}^{+\infty}dw'\sum_{n_2}\frac{16m_1m_2(G^{(5)})^2}{\pi^2(R_1^2+w'^2)(R_2^2+(w'+w_1-w_2+n_2\ell)^2}\\
&\quad=\frac{16m_1m_2(G^{(5)})^2(R_1+R_2)}{\pi R_1R_2}\sum_{n_2}\frac{1}{(R_1+R_2)^2+(w_1-w_2+n_2\ell)^2}\\
&\quad=\ell\frac{9m_1m_2G_{\rm N}^2}{R_1R_2}\frac{\sinh\frac{2\pi(R_1+R_2)}{\ell}}{\cosh\frac{2\pi(R_1+R_2)}{\ell}-\cos\frac{2\pi(w_1-w_2)}{l}}\\
&\quad\simeq\ell\frac{9m_1m_2G_{\rm N}^2}{R_1R_2}\left(1+2e^{-\frac{2\pi(R_1+R_2)}{\ell}}\cos\frac{2\pi(w_1-w_2)}{\ell}\right)\,,
\end{aligned} \tag{4.56}$$

where $R_1=|\vec{y}-\vec{x}_1(t)|$, $R_2=|\vec{y}-\vec{x}_2(t)|$, and where $G_{\rm N}$ was previously defined in (4.27). In the last step in (4.56) we used $\ell \ll r_{12}\leq R_1+R_2$, with r_{12} the binary separation distance. It is important to keep the explicit w_i dependence because we will still have to take derivatives respect to the position of the sources in the $5d$ spacetime. Substituting (4.56) into (4.54) we obtain

$$\tilde{h}_{\rm t}^{IJ}(x) \simeq \frac{3}{2}\frac{m_1 m_2 G_{\rm N}^2}{2\pi R}(2\eta^{IK}\eta^{JL} - \eta^{IJ}\eta^{KL})(\partial_K^{(1)}\partial_L^{(2)} + \partial_K^{(2)}\partial_L^{(1)}) \times \left(-\pi r_{12} + \ell\, e^{-\frac{2\pi r_{12}}{\ell}} \cos\frac{2\pi(w_1 - w_2)}{\ell}\right). \tag{4.57}$$

For more details on how the integration was performed in (4.57), please see Appendix H.

In particular, from (4.57), for a binary at $w_1 = w_2 = 0$ as in (4.22) and (4.23), we obtain e.g. the (x, y) component of the metric perturbation as

$$\begin{aligned} \tilde{h}_{\rm t}^{xy}(t, \vec{x}, 0) &\simeq -\frac{3}{2}\frac{m_1 m_2 G_{\rm N}^2}{R}(\partial_1^{(1)}\partial_2^{(2)} + \partial_1^{(2)}\partial_2^{(1)})\left(r_{12} - \frac{\ell}{\pi}e^{-\frac{2\pi r_{12}}{\ell}}\right) \\ &\simeq -\frac{3}{2}\frac{m_1 m_2 G_{\rm N}^2}{R r_{12}}\left(1 + 2e^{-\frac{2\pi r_{12}}{\ell}} + \frac{4\pi r_{12}}{\ell}e^{-\frac{2\pi r_{12}}{\ell}}\right)\sin[2\Omega(t - R)]\,. \end{aligned} \tag{4.58}$$

We can further use the modified Kepler's law $\Omega^2 = \frac{G_{\rm N}M}{r_{12}^3}(1 + 2e^{-\frac{2\pi r_{12}}{\ell}} + \frac{4\pi r_{12}}{\ell}e^{-\frac{2\pi r_{12}}{\ell}})$ to cast it into a more familiar form:

$$\tilde{h}_{\rm t}^{xy}(t, \vec{x}, 0) \simeq -\frac{3}{2}\frac{G_{\rm N}\mu}{R}r_{12}^2\Omega^2 \sin[2\Omega(t - R)]\,. \tag{4.59}$$

4.5.3 Gravitational waves from a binary source in a 5d spacetime

Similar calculations to the ones we presented in explicit detail in the previous sections yield the following expressions for the other non-zero linearized fluctuations $\tilde{h}_{\rm T}^{IJ}$ (sourced by the matter energy-momentum tensor), as well as the leading order non-linear fluctuations, $\tilde{h}_{\rm t}^{IJ}$ (sourced by the Landau-Lifshitz pseudo-tensor):

$$\begin{aligned}
\tilde{h}_{\rm T}^{xx}(t,\vec{x},0) &\simeq 3\frac{G_{\rm N}\mu}{R}r_{12}^2\Omega^2\sin^2[\Omega(t-R)],\\
\tilde{h}_{\rm T}^{yy}(t,\vec{x},0) &\simeq 3\frac{G_{\rm N}\mu}{R}r_{12}^2\Omega^2\cos^2[\Omega(t-R)],\\
\tilde{h}_{\rm T}^{zz}(t,\vec{x},0) &\simeq 0,\\
\tilde{h}_{\rm T}^{ww}(t,\vec{x},0) &\simeq 0,\\
\tilde{h}_{\rm t}^{xx}(t,\vec{x},0) &\simeq -3\frac{G_{\rm N}\mu}{R}r_{12}^2\Omega^2\cos^2[\Omega(t-R)]\,,\\
\tilde{h}_{\rm t}^{yy}(t,\vec{x},0) &\simeq -3\frac{G_{\rm N}\mu}{R}r_{12}^2\Omega^2\sin^2[\Omega(t-R)]\,,\\
\tilde{h}_{\rm t}^{zz}(t,\vec{x},0) &\simeq 0\,,\\
\tilde{h}_{\rm t}^{ww}(t,\vec{x},0) &\simeq -3\frac{G_{\rm N}\mu}{R}r_{12}^2\Omega^2\left(1-4\frac{2\pi r_{12}}{\ell}e^{-\frac{2\pi r_{12}}{\ell}}\right)\,,
\end{aligned}\tag{4.60}$$

where we recall that μ is the reduced mass of the binary (4.26). (As a caveat, we would like to point out that we cannot set $z_1 = z_2 = w_1 = w_2 = 0$ until the derivatives in (4.57) have been taken, and the vanishing of $\tilde{h}_{\rm t}^{zz}$ is not trivial.) As advertised, both $\tilde{h}_{\rm T}^{IJ}$ and $\tilde{h}_{\rm t}^{IJ}$ are of the same order in velocities.

To second order in velocities, the non-zero metric fluctuations $\tilde{h}^{IJ}$ are obtained by adding the linearized $\tilde{h}_{\rm T}$ and non-linear $\tilde{h}_{\rm t}$:

$$\begin{aligned}
\tilde{h}^{xy}(t,\vec{x},0) &= \tilde{h}_{\rm T}^{xy}+\tilde{h}_{\rm t}^{xy} \simeq -3\frac{G_{\rm N}\mu}{R}r_{12}^2\Omega^2\sin[2\Omega(t-R)]\,,\\
\tilde{h}^{xx}(t,\vec{x},0) &= \tilde{h}_{\rm T}^{xx}+\tilde{h}_{t}^{xx} \simeq -3\frac{G_{\rm N}\mu}{R}r_{12}^2\Omega^2\cos[2\Omega(t-R)]\,,\\
\tilde{h}^{yy}(t,\vec{x},0) &= \tilde{h}_{\rm T}^{yy}+\tilde{h}_{\rm t}^{yy} \simeq 3\frac{G_{\rm N}\mu}{R}r_{12}^2\Omega^2\cos[2\Omega(t-R)]\,,\\
\tilde{h}^{ww}(t,\vec{x},0) &= \tilde{h}_{\rm T}^{ww}+\tilde{h}_{\rm t}^{ww} \simeq -3\frac{G_{\rm N}\mu}{R}r_{12}^2\Omega^2\left(1-4\frac{2\pi r_{12}}{\ell}e^{-\frac{2\pi r_{12}}{\ell}}\right)\,.
\end{aligned}\tag{4.61}$$

According to our discussion in Sections 4.5.1 (see (4.49)) and 4.5.2 (see (4.57)), the metric fluctuations $\tilde{h}^{IJ}(x^\mu, w)$ are equal to $\tilde{h}^{IJ}(x^\mu, w{=}0)$, up to exponentially suppressed corrections. However, the biggest change to the luminosity of the gravitational waves and the phase of the gravitational waveform comes from the leading order terms retained in (4.61).

The extension of the results given in (4.61) to a compactified D-dimensional spacetime is straightforward:

$$\tilde{h}^{(D)IJ} = \frac{D-2}{2(D-3)} \begin{pmatrix} \tilde{h}^{(4)ij} & 0 \\ 0 & \Big(-4\frac{G_{\rm N}\mu}{R} r_{12}^2 \Omega^2 \Big) \delta^{pq} \end{pmatrix}, \tag{4.62}$$

where $i, j = 1, 2, 3$ and $p, q = 5, 6, \ldots D$. Here $\tilde{h}^{(4)ij}$ denotes the $4d$ gravitational waves sourced by a binary with the same characteristics as ours: reduced mass μ, separation distance r_{12}, angular frequency Ω, and located at $z = 0$:

$$\tilde{h}^{(4)ij} = \begin{pmatrix} -4\frac{G_{\rm N}\mu}{R} r_{12}^2 \Omega^2 \cos[2\Omega(t-R)] & -4\frac{G_{\rm N}\mu}{R} r_{12}^2 \Omega^2 \sin[2\Omega(t-R)] & 0 \\ -4\frac{G_{\rm N}\mu}{R} r_{12}^2 \Omega^2 \sin[2\Omega(t-R)] & 4\frac{G_{\rm N}\mu}{R} r_{12}^2 \Omega^2 \cos[2\Omega(t-R)] & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Lastly, the remaining metric fluctuations $\tilde{h}^{0M}$ can be obtained either by direct integration, or more easily, by using the harmonic gauge (4.6) condition. In the next sections we will use the latter.

4.6 The Luminosity of Gravitational Waves

In this section we compute the luminosity of gravitational waves. We work in the harmonic gauge (4.6), without specializing to the more commonly used transverse-traceless gauge (for a comparison, see Appendix I).

There is one more subtlety we would like to comment on before we begin. In the D-dimensional gravitational theory, the only coupling constant is the gravitational constant $G^{(D)}$ of the Einstein-Hilbert action. After performing the Kaluza-Klein reduction, the effective $4d$ theory has the gravitational constant $G = G^{(D)}/\mathrm{Vol}(\text{Compact Space})$ and the Newton's constant is $G_{\mathrm{N}} = (2(D-3)/(D-2))G$. In contrast, in a strictly $4d$ theory of gravity coupled to matter, we would have $G = G_{\mathrm{N}}$.

In our subsequent comparisons between the predictions of the compactified higher-dimensional gravity theory and $4d$ GR we will identify the Newton's constants in the two theories.

We use the gravitational energy-momentum pseudo-tensor t_{LL}^{MN} given in (4.10). Since t_{LL}^{MN} is already second order in the metric fluctuations, we can use the linearized approximation for $\tilde{h}^{MN}$, (4.7), to obtain

$$\begin{aligned}
16\pi G^{(D)} t_{\mathrm{LL}}^{MN} \simeq\ & \tilde{h}^{MN}{}_{,P}\tilde{h}^{PQ}{}_{,Q} - \tilde{h}^{MP}{}_{,P}\tilde{h}^{NQ}{}_{,Q} + \frac{1}{2}\eta^{MN}\tilde{h}^{PR}{}_{,Q}\tilde{h}^{Q}{}_{P,R} \\
& - \left(\tilde{h}^{MP}{}_{,Q}\tilde{h}_{P}{}^{Q,N} + \tilde{h}^{NP}{}_{,Q}\tilde{h}_{P}{}^{Q,M}\right) \\
& + \tilde{h}^{MP,Q}\tilde{h}^{N}{}_{P,Q} + \frac{1}{2}\tilde{h}^{PQ,M}\tilde{h}_{PQ}{}^{,N} - \frac{1}{4}\eta^{MN}\tilde{h}^{PQ,R}\tilde{h}_{PQ,R} \\
& - \frac{1}{4(D-2)}\left(2\tilde{h}^{,M}\tilde{h}^{,N} - \eta^{MN}\tilde{h}^{,P}\tilde{h}_{,P}\right) ,
\end{aligned} \tag{4.63}$$

where the indices are raised and lowered with the Minkowski metric. Further imposing the Lorenz gauge (4.6) and performing short-wavelength averaging[10], (4.63) becomes

$$\langle (t_{\rm LL})_{MN} \rangle \simeq \frac{1}{32\pi G^{(D)}} \left\langle \partial_M \tilde{h}_{PQ} \partial_N \tilde{h}^{PQ} - \frac{1}{D-2} \partial_M \tilde{h} \partial_N \tilde{h} \right\rangle . \tag{4.64}$$

The total energy carried away by gravitational waves is given by the following volume integral:

$$E_{\rm GW} = \int d^3\vec{x} \int_0^\ell dw \, t^{00}_{\rm LL}(t, x^I) \simeq \ell \int d^3\vec{x} \, t^{00}_{\rm LL}(t, \vec{x}) \, , \tag{4.65}$$

where in the last step we used that, to leading order, the metric fluctuations propagate uniformly in w. Then the rate of change of the radiated energy is

$$\begin{aligned} \dot{E}_{\rm GW} = \frac{dE_{\rm GW}}{dt} &= \int d^3\vec{x} \int_0^\ell dw \, \partial_0 t^{00}_{\rm LL} \\ &= -\int d^3\vec{x} \int_0^\ell dw \, \partial_I t^{I0}_{\rm LL} \\ &= \oint dA \int_0^\ell dw \, (t_{\rm LL})_{0I} n^I \, , \end{aligned} \tag{4.66}$$

where we recall that our index conventions defined in (4.2) and (4.3) are: $I, J = 1, 2, 3, 5$ and $i, j = 1, 2, 3$. In (4.66), dA is the differential area element on the 2-sphere at spatial infinity and n^I is the unit vector along the direction of propagation of the gravitational waves. From (4.48) and (4.61) we saw that the gravitational waves propagate radially in the non-compact directions and uniformly in w to leading order. The non-uniform propagation along the direction of compactification is due to the massive Kaluza-Klein modes which yield exponentially suppressed corrections. So, to

[10] When performing short-wavelength averaging, integration by parts is permitted [368].

leading order, the only non-zero components of n^I are n^i. Then, the rate of change of energy in a 2-sphere at a distance R from the source becomes

$$\dot{E}_{\rm GW} = \ell \oint d\Omega \, (t_{\rm LL})_{0k} n^k R^2 \,. \tag{4.67}$$

Using repeatedly the harmonic gauge and the fact that the perturbations in the far zone depend on the retarded time, we obtain the following identities to leading order in $1/R$:

$$\begin{aligned}
\partial_k \tilde{h}_{IJ} &\simeq -\dot{\tilde{h}}_{IJ} n_k \,, \\
\partial_k \tilde{h}_{00} &\simeq -\dot{\tilde{h}}_{ij} n^i n^j n_k \,, \\
\partial_k \tilde{h}_{0I} &\simeq \dot{\tilde{h}}_{Ij} n^j n_k \,, \\
\dot{\tilde{h}}_{00} &\simeq \dot{\tilde{h}}_{ij} n^i n^j \,, \\
\dot{\tilde{h}}_{0I} &\simeq -\dot{\tilde{h}}_{Ij} n^j \,,
\end{aligned} \tag{4.68}$$

where a dot denotes a time derivative. In more detail, in writing $\partial_k \tilde{h}_{IJ} \simeq -\dot{\tilde{h}}_{IJ} n_k$, we used the fact that the metric fluctuations are spherical waves (see (4.61)), and to leading order in $1/R$, we can ignore the action of ∂_k derivative on the $1/R$ factor. Then, when acting on the periodic function of $t - R$, we can trade off ∂_k for $n_k \partial_R$ and the latter for $-n_k \partial_t$.

Substituting (4.68) into (4.64) we derive the follwing result

$$\begin{aligned}
\langle t_{0k} n^k \rangle \simeq -\frac{1}{32\pi G^{(D)}} \Big\langle & \dot{\tilde{h}}_{IJ} \dot{\tilde{h}}^{IJ} + \tfrac{D-3}{D-2} \dot{\tilde{h}}_{ij} \dot{\tilde{h}}_{kl} n^i n^j n^k n^l - 2 \dot{\tilde{h}}_{Ij} \dot{\tilde{h}}^{Ik} n^j n_k - \tfrac{1}{D-2} \dot{\tilde{h}}^I{}_I \dot{\tilde{h}}^J{}_J \\
& + \tfrac{2}{D-2} \dot{\tilde{h}}^I{}_I \dot{\tilde{h}}_{ij} n^i n^j \Big\rangle \,.
\end{aligned} \tag{4.69}$$

We are now ready to compute the luminosity of the gravitational waves from a binary source. For $D = 5$, substituting the perturbations derived in (4.61) into (4.67), and noting that to leading order we have $\partial_0 \tilde{h}^I{}_I = \partial_0 \tilde{h}^i{}_i = 0$, we find [11]

$$\dot{E}_{\rm GW}^{(5,c)} \simeq -\frac{19}{360}\frac{R^2}{G}\left\langle \dot{\tilde{h}}_{ij}\dot{\tilde{h}}^{ij}\right\rangle , \tag{4.70}$$

where we recall that G is the effective $4d$ theory gravitational constant (4.38), and we used the isotropy of the gravitational waves together with the following identities:

$$\begin{aligned} \int d^2\Omega\, n^i n^j &= \frac{4\pi}{3}\delta^{ij} , \\ \int d^2\Omega\, n^i n^j n^k n^l &= \frac{4\pi}{15}\left(\delta^{ij}\delta^{kl} + \delta^{il}\delta^{jk} + \delta^{ik}\delta^{jl}\right) . \end{aligned} \tag{4.71}$$

Substituting the metric perturbations derived earlier in (4.61) into (4.70), and keeping terms only to leading order in velocity, the compactified $5d$ GR luminosity is

$$\dot{E}_{\rm GW}^{(5,c)} \simeq -\frac{304}{45} G\mu^2 r_{12}^4 \Omega^6 = -\frac{76}{15} G_{\rm N}^{7/3}\mu^2 M_t^{4/3}\Omega^{10/3} . \tag{4.72}$$

In contrast, the luminosity of gravitational waves in a purely $4d$ gravitational theory, with the gravitational waves sourced by a binary with the same characteristics as ours, is equal to

$$\dot{E}_{\rm GW}^{(4)} \simeq -\frac{32}{5} G_{\rm N}\mu^2 r_{12}^4 \Omega^6 = -\frac{32}{5} {G_{\rm N}}^{7/3}\mu^2 M_t^{4/3}\Omega^{10/3} . \tag{4.73}$$

[11] For general D-dimensions,

$$\begin{aligned} \dot{E}_{\rm GW}^{(D,c)} \simeq -\frac{7D-16}{15(D-2)}\frac{R^2}{8G_N}\frac{2(D-3)}{D-2}\left\langle \dot{\tilde{h}}_{ij}\dot{\tilde{h}}^{ij}\right\rangle &= -\frac{7D-16}{15(D-2)}\frac{D-2}{2(D-3)} 16 G_N^{7/3}\mu^2 M_t^{4/3}\Omega^{10/3} \\ &= \frac{7D-16}{12(D-3)}\dot{E}_{\rm GW}^{(4)} , \end{aligned}$$

where $\dot{E}_{\rm GW}^{(4)}$ is defined in (4.73).

We conclude that the luminosity of gravitational waves in a $5d$ spacetime with a compact fifth dimension differs by 20.8% from the corresponding $4d$ GR luminosity.

Let us now compare the luminosity derived earlier in (4.72) with the predictions of Einstein-Maxwell-dilaton theory studied in Refs. [108–110]. For neutral matter (i.e. the electric charges are zero), the energy of a binary is dissipated via gravitational and scalar (dilaton) radiation. We refer to them as $\dot{E}_g$ and $\dot{E}_\phi$ respectively. The luminosity depends on the scalar charge through the quantity

$$\alpha_a^0 = \frac{d \ln m_a(\phi)}{d\phi} \,, \tag{4.74}$$

where the superscript "0" refers to the quantity being evaluated at ϕ_∞ (a constant corresponding to the scalar field at spatial infinity), and where $m_a(\phi)$ is the effective $4d$ mass of a source a, which may depend on the dilaton. In general, for a circular binary, the leading order term in $\dot{E}_\phi$ is dipolar and depends on the difference in scalar charges of the binary constituents [369].

In our compactified (Kaluza-Klein) higher-dimensional gravity picture, the effective $4d$ mass of source a is given by

$$m_a(\phi) = m_a e^{-\phi/\sqrt{3}} \,, \tag{4.75}$$

as in (4.37), and where we used the dilaton rescaling as in (4.32). Since in our theory masses are coupled to the dilaton universally,

$$\alpha_1^0 = \alpha_2^0 = -1/\sqrt{3} \,, \tag{4.76}$$

the dipole radiation is zero (because $\alpha_1^0 - \alpha_2^0 = 0$), and so the leading contribution in $\dot{E}_\phi$ is quadrupolar. Therefore, both $\dot{E}_g$ and $\dot{E}_\phi$ are quadrupolar, and so is their sum, which is in agreement with our earlier findings (4.72). More precisely, given the Kaluza-Klein scalar charges (4.76), the leading order contribution to the luminosity in Einstein-Maxwell-dilaton theories [109, 110] becomes

$$\dot{E}_g \simeq \left(1 + \alpha_1^0 \alpha_2^0\right)^{-1} \dot{E}_{\rm GW}^{(4)} \,, \qquad \dot{E}_\phi \simeq \frac{1}{6} \left(1 + \alpha_1^0 \alpha_2^0\right)^{-1} \alpha_1^0 \alpha_2^0 \, \dot{E}_{\rm GW}^{(4)} \,. \tag{4.77}$$

Thus, in total, we have

$$\begin{aligned} \dot{E}_g + \dot{E}_\phi &\simeq \left(1 + \alpha_1^0 \alpha_2^0\right)^{-1} \left(1 + \frac{1}{6} \alpha_1^0 \alpha_2^0\right) \dot{E}_{\rm GW}^{(4)} \\ &= \frac{19}{24} \dot{E}_{\rm GW}^{(4)} \,, \end{aligned} \tag{4.78}$$

which matches with our result in (4.72).

4.7 Constraints from Gravitational Wave Observations

In this section we compute the phase of the gravitational waveform in the frequency domain and compare it with observations. We restrict ourselves to the leading post-Newtonian contribution. We begin by deriving the frequency evolution of the gravitational waves from the energy-balance law

$$\frac{dE}{dt} = \dot{E}_{\rm GW} \,, \tag{4.79}$$

which simply states that the rate of change of the binding energy of the binary E is same as the luminosity $\dot{E}_{\rm GW}$ of the energy radiated by gravitational waves. For a circular binary, the binding energy is same as the effective potential, which is given in (4.24). However, since we are interested in calculating the leading post-Newtonian effect, we can ignore the exponentially suppressed correction. We can further use the Kepler's law to rewrite the binding energy as

$$E = -\frac{1}{2}\mu(G_N M_t \Omega)^{2/3}\,. \tag{4.80}$$

Substituting (4.72) on the right hand side of (4.79), and (4.80) into its left hand side, we find

$$\dot{f}^{(5,c)} = \frac{76}{5}\pi^{8/3} f^{11/3} G_N^{5/3} \mathcal{M}_{ch}^{5/3}\,, \tag{4.81}$$

where $f = \Omega/\pi$ is the gravitational waves frequency (this is manifest in (4.61)), and $\mathcal{M}_{ch} = (m_1 m_2)^{3/5}/(m_1 + m_2)^{1/5}$ denotes the chirp mass. On the other hand, the frequency evolution in $4d$ GR is given by

$$\dot{f}^{(4)} = \frac{96}{5}\pi^{8/3} f^{11/3} G_N^{5/3} \mathcal{M}_{ch}^{2/3}\,, \tag{4.82}$$

which differs from the compactified $5d$ result in (4.81) by a numerical factor independent of the size of the extra dimension.

We now compute the gravitational wave phase in the frequency domain. The observed waveform is given by a linear combination of the $+$ and $\times$ modes. In stationary phase approximation, the phase of gravitational waveform as a function of the frequency f is [191, 370]

$$\Psi_p(f) = 2\pi f\, t(f) - \phi_p(f) - \frac{\pi}{4}\,, \tag{4.83}$$

where

$$t(f) = t_0 + \int_\infty^f df\, \frac{dt}{df} = t_0 + \int_\infty^f df\, \frac{1}{\dot{f}}\,, \tag{4.84}$$

and

$$\phi_p(f) = \int dt\, 2\pi f = \phi_{p,0} + \int_\infty^f df\, \frac{2\pi f}{\dot{f}}\,. \tag{4.85}$$

Further using (4.81) we obtain

$$\Psi_p^{(5,c)}(f) = \frac{9}{304 G_N^{5/3} \underline{\mathrm{u}}^5} + 2\pi f t_0 - \phi_{p,0} - \frac{\pi}{4}, \tag{4.86}$$

where t_0 and $\phi_{p,0}$ are the time and phase at the coalescence respectively, and $\underline{\mathrm{u}} \equiv (\pi \mathcal{M}_{ch} f)^{1/3}$ is the effective relative velocity of the binary components. On the other hand, the $4d$ GR result for the phase of the gravitational waves in the frequency domain is [191, 370]

$$\Psi_p^{(4)}(f) = \frac{3}{128 G_N^{5/3} \underline{\mathrm{u}}^5} + 2\pi f t_0 - \phi_{p,0} - \frac{\pi}{4}\,. \tag{4.87}$$

Thus, we can rewrite (4.86) based on (4.87) as

$$\Psi_p^{(5,c)}(f) = \frac{3}{128 G_N^{5/3} \underline{\mathrm{u}}^5}(1 + \delta\hat{\phi}) + 2\pi f t_0 - \phi_{p,0} - \frac{\pi}{4}\,, \tag{4.88}$$

with

$$\delta\hat{\phi} \equiv \frac{5}{19} \sim 0.26\,. \tag{4.89}$$

We note that our results agree with those derived in the context of Einstein-Maxwell-dilaton theory discussed in Ref. [110] (with $\alpha_1 = \alpha_2 = -1/\sqrt{3}$ and the electric charges set to zero as discussed previously) to 0.3%. The difference arises due to a series expansion in luminosity in Ref. [110] which assumes that the scalar energy flux is small compared to the tensor energy flux. If one performs the calculation without making such an approximation, the result in Ref. [110] matches with ours exactly.

Let us now compare the predictions of our model, with a compactified fifth dimension with actual gravitational wave observations. From (4.89), one sees that the leading post-Newtonian term in (4.86) differs by 26% from that of (4.87), irrespective of the masses of the binary components. The LIGO/Virgo Collaborations used the events detected from the first and second observing runs and have placed upper bounds on $|\delta\hat{\phi}|$ as $\sim 15\%$ from single events and $\sim 10\%$ from combined events [32]. Hence a discrepancy of 26% is inconsistent with the gravitational wave observations, and thus we can rule out the simple compactified $5d$ GR model considered in this chapter.

We can easily generalize our previous results and compute the phase of the gravitational waves in an arbitrary number dimensions D, with four non-compact dimensions and the rest compactified (periodic). Using the the gravitational wave luminosity in a D dimensional spacetime given in footnote 11, it is straightforward to derive $\delta\hat{\phi}$ as

$$\delta\hat{\phi}^{(D)} = \frac{5(D-4)}{7D-16} \,. \tag{4.90}$$

We plot $\delta\hat{\phi}^{(D)}$ as a function of D in Figure 1.6, and notice that $\delta\hat{\phi}$ increases with D. This means that our model stays inconsistent with the LIGO/Virgo observations even if we increase the number of compact extra dimensions.

We now comment on some caveat in the above bounds. We used the bounds derived by the LIGO/Virgo Collaborations that assumed the correction to 4d GR in the phase enters only at 0PN order. Such a correction partially degenerates with the chirp mass that also enters first at 0PN order, though the mass also enters at higher PN orders and thus the degeneracy can be partially broken. In reality, higher PN corrections to 4d GR also enter in the waveform phase. This may change the amount of correlation between the chirp mass and beyond-4d-GR effects and may weaken the bound on the 0PN correction. In [25], the LIGO/Virgo Collaborations carried out another analysis for GW150914 where they included phase corrections at various PN orders in the search parameter set. This enhances the correlation significantly and the bound on the 0PN correction now becomes $|\delta\hat{\phi}| \lesssim 5$. If we quote this bound, we cannot rule out the compact extra dimension model considered here. Thus, to make a robust statement on whether one can rule out the model with gravitational-wave observations, one needs to compute corrections at higher PN orders and rederive bounds on the extra dimension effect.

Having said this, one can still rule out the model with binary pulsar observations for the following reason. A standard method for testing GR with binary pulsars is to determine the masses from at least three independent observables (such as post-Keplerian parameters including the periastron precession, Shapiro delay and orbital decay rate) assuming GR and check the consistency. The orbital decay rate $\dot{P}$ is the only post-Keplerian parameter that depends on the gravitational-wave emission. Thus, even for the compact extra dimension model considered here, one can safely use the masses obtained from other post-Keplerian parameters under the 4d GR assumption since the conservative corrections are exponentially suppressed. One can then use the measurement of $\dot{P}$ to constrain the model without having to worry about the degeneracy between the extra dimension effect and masses. Such $\dot{P}$ measurements have been mapped to a bound on $\delta\hat{\phi}$ as $|\delta\hat{\phi}| \lesssim 10^{-3}$ [371,372], which is much stronger than the gravitational-wave bound. Thus, one can rule out the compact dimension model with the binary pulsar observations[12].

4.8 Conclusions

In this work we performed an analysis of gravitational waves sourced by a binary in a D-dimensional spacetime with four non-compact dimensions and a set of compactified extra dimensions. We worked under the assumptions that the two binary sources are point-like and located on the same "brane" (i.e. at the same position in the compact

[12] A similar result was found in [373] though this reference effectively introduces matter after the KK reduction and thus is different from the setup we study here.

coordinates). For most part we took $D = 5$, but we have provided generalizations to arbitrary D throughout the chapter. We worked within the framework of GR, and in the limit of small extra dimensions. We computed the gravitational waves sourced by the binary, the luminosity of the gravitational waves and the phase of the gravitational waves, to leading order in the post-Newtonian expansion. We found that the luminosity of gravitational waves emitted in $5d$ gravity by a binary with the same characteristics (same masses and separation distance) as a $4d$ binary is 20.8% less relative to the $4d$ case, to leading post-Newtonian order. The phase of the gravitational waveform differs by 26% relative to the $4d$ case, to leading post-Newtonian order, while for general D, the fractional difference of the phase with that of $4d$ GR is $\frac{5(D-4)}{7D-16}$, which only increases with an increase in D. While there are exponential corrections which depend on the size of the extra dimensions, the leading order estimates for the gravitational wave phase we gave here are independent of size, and depend only on the number of extra dimensions. Based on a comparison with gravitational-wave observations from the LIGO/Virgo Collaborations [32] and binary pulsar observations from radio astronomy [371,372] we can rule out this class of models for compact extra dimensions. The main source of discrepancy is the higher-dimensional gravity coupling with matter, which, when seen from a $4d$ perspective, means that matter will couple not only with the $4d$ metric, but with the dilaton as well. This dilaton coupling (or scalar charge) is responsible for fifth force effects which change the phase of the gravitational waves. Our results agree with those derived in the context of $4d$ Einstein-Maxwell-dilaton theory [108–110] provided that we set the binary's scalar charges equal to one another and equal to the Kaluza-Klein value.

The same fifth force effects are responsible for the difference between the $4d$ Newton's constant $G_{\rm N}$ and the $4d$ gravitational coupling G: $G_N = \frac{2(D-3)}{D-2}G$, and for the Shapiro time-delay discrepancy with $4d$ GR. In a parametrized post-Newtonian expansion (PPN) the $4d$ "physical" metric (which is obtained by performing a rescaling of the $4d$ metric with the dilaton in such a way to eliminate the matter-dilaton coupling [2] is written as $g_{00} = -1 + 2U + \dots$, $g_{ij} = \delta_{ij}(1 + 2\gamma U + \dots)$, with $\gamma = 1$ in $4d$ GR. A measurement of the frequency shift of radio photons to and from the Cassini spacecraft as they passed near the Sun gave $\gamma = 1 + (2.1 \pm 2.3) \times 10^{-5}$ [209]. On the other hand, the "physical metric" as read off from footnote 1 has $g_{00} = -1 + \frac{2}{3}\tilde{h}_{\rm T00}$ and $g_{ij} = 1 + \frac{1}{3}\tilde{h}_{\rm T00} + \dots$, which amounts to $\gamma \sim 1/2$. Therefore this class of compactified extra dimensions models was ruled out based on Solar System measurements [374, 375].

In string theory the massless dilaton is one of the many moduli (zero mass scalars) that arise in the compactification of the higher-dimensional spacetime. Stabilzation of the moduli can be achieved, for example, by turning on fluxes for the Ramond-Ramond potentials [376]. This gives rise to a mass term for the moduli, and eliminates the large contribution of the scalar fifth force, by turning a Coulomb potential into a Yukawa potential. It would be interesting to study gravitational waves in such a set-up and place constraints on the various parameters. A somewhat simpler scenario is the Randall-Sundrum model, where the fifth dimension is large and warped. Our work here is intended as a first step in understanding how to set up the problem of solving for gravitational waves in a higher-dimensional space with compact dimensions, or warped, large extra dimensions. For example, we saw that quadrupole formula need not apply, and we had to use a direct integration of the Einstein equations. Also, when it comes to the propagation of gravitational waves in spacetimes with warped, large extra dimensions, reducing the problem to $4d$ seems less appropriate, and working in the higher-dimensional space, as we did here, presents an advantage.

Chapter 5

Brans-Dicke Theory in Bondi-Sachs Framework

5.1 Introduction

Compact binary mergers opened a new parameter space of general relativity to be tested (the region of strong curvature and high GW luminosities) which was less well probed by tests of general relativity in the Solar System or with binary pulsars. In this parameter space, there are some types of relativistic phenomena that are only likely to be measured for strongly curved and highly radiating systems. One such class of effects that has yet to be detected, but are under active investigation (see, e.g., [134, 135, 137, 377, 378]), are gravitational-wave memory effects [1].

The best known GW memory effect (sometimes referred to as *the* GW memory effect) is characterized by lasting change in the GW strain after a burst of GWs pass by a GW detector. One of the earliest explicit calculations of the GW memory ef-

[1] This chapter is based on the following paper: *Brans-Dicke theory in Bondi-Sachs form: Asymptotically flat solutions, asymptotic symmetries, and gravitational-wave memory effects*; Tahura, Shammi; Nichols, David A.; Saffer, Alexander; Stein, Leo C.; Yagi, Kent; Phys. Rev. D 103, 104026 (2021)

fect from gravitational scattering was performed in Ref. [130] (see also [379, 380]), though the possibility of a nonvanishing GW strain at late times was discussed previously (e.g., [381]). It was subsequently noted that massless (or nearly massless) fields could also produce the GW memory effect [382, 383] including the nonlinear effective stress-energy of gravitational waves themselves [131, 132]. The GW memory effect has a distinctive observational signature, in that it causes a constant, enduring displacement between nearby freely falling observers after a burst of gravitational waves have passed. A number of generalizations of the GW memory effect have been found by considering asymptotic changes in burst of other fields (such as electromagnetism [384] or massless Yang-Mills theory [385]) or in time integrals of the GW strain (e.g., [152, 156]). Other GW memories have been found from examining other kinds of lasting kinematical effects on freely falling observers (like lasting relative velocities [386, 387], relative changes in proper time [155, 388], relative rotations of parallel transported tetrads [388]) or through other types of measurement procedures [389, 390]. Also important in the discovery of new GW memory effects was the understanding of how certain GW memories are closely related to symmetries, conserved quantities, and soft theorems (see, e.g., [391]).

For understanding the relationship between memory effects and the asymptotic structure of spacetime, two approaches have been taken to study asymptotic flatness: a covariant conformal completion of spacetime [392, 393] and calculations in particular coordinate systems adapted to the spacetime geometry by Bondi, van der

Burg, and Metzner [139] and Sachs [140] or Newman and Unti [394]. We will focus on the Bondi-Sachs approach to asymptotic flatness. In this approach, coordinates are chosen that are well suited to the null hypersurfaces and the null geodesics of the spacetime. Boundary conditions can then be imposed on the metric to determine a reasonable notion of a spacetime that becomes asymptotically Minkowskian as the light rays travel an infinite distance from an isolated source. Although spacetime can be cast in an asymptotically Minkowskian form at large Bondi radius r, the asymptotic symmetry group of this spacetime does not reduce to the Poincaré group of flat spacetime; rather, it becomes the infinite-dimensional Bondi-Metzner-Sachs (BMS) group [139, 141].

The structure of the BMS group is in some ways similar to the Poincaré group: it contains the Lorentz transformations, but rather than having an additional four spacetime translations as the remaining group elements, it has an infinite-dimensional commutative group called the supertranslations [141] (the usual Poincaré translations are a normal finite subgroup of the supertranslations). It is possible to associate charges conjugate to these asymptotic symmetries (see, e.g., [148–151]). These charges are conserved in the sense that the difference in the charges between two times is equal to the flux of the quantity between these two times. Associated with the Lorentz symmetries are the six components of the relativistic angular momentum [which can be divided into center-of-mass (CM) and spin parts] and corresponding to the supertranslations are conserved quantities called supermomenta. Note that there also have been proposals to extend the Lorentz part of the symmetry algebra to include all conformal Killing vectors on the 2-sphere called superrotations [142–144] (see also [145]) or all smooth vector fields on the 2-sphere [146, 147] (sometimes called super-Lorentz symmetries [395]). The additional charges of these extended BMS algebras are the super CM and super spin charges [153] or the super-angular momentum [152].

The connection between asymptotic symmetries, conserved quantities, and GW memory can now be more clearly stated with the nomenclature now set. Changes in the supermomentum charges, generated by both massive particles and massless fields, induce a nonzero GW memory effect; in addition, when the GW memory effect is present, the final state of the system is supertranslated from a certain canonical asymptotic rest frame for the system (see, e.g., [153]). Changes in the super-angular momentum charges can induce two additional types of GW memory effects called spin [156] and CM [152] memory. These memory effects are not necessarily related to a spacetime that has been superrotated or super-Lorentz transformed from a certain canonical frame, since such solutions often are not asymptotically flat in the usual sense [395, 396].

While GW memory effects and their analogues for other matter fields have now been much more carefully studied in a number of contexts, they have not been studied as systematically in modified theories of gravity. Modified theories can have additional GW polarizations [2, 87, 165], which could allow for additional types of GW memory effects (see, e.g., [179, 180, 397, 398]). In addition, as far as we are aware, there is not a standard definition of asymptotic flatness in these theories, nor is the set of asymptotic symmetries of these solution clearly understood. It is not obvious, *a priori*, that modified theories of gravity generically have the same asymptotic properties as in general relativity, or that their memory effects would be related to symmetries and conserved quantities as in general relativity. A main aim of this chapter is to develop a better understanding of these relationships in a relatively simple modification of general relativity known as Brans-Dicke theory [166].

Brans-Dicke theory is one example of a scalar-tensor theory, i.e., a theory in which there is a scalar field that couples to gravity nonminimally (see, e.g., the review [36]). Scalar-tensor theories have appeared in the contexts of string theory, inflation [21, 167], and the accelerated expansion of the Universe [168–170]. In our work, we will focus on Brans-Dicke theory, with a massless scalar field. It is known from calculations in linearized gravity and post-Newtonian (PN) theory, the scalar field generates an additional polarization of gravitational waves sometimes called a "breathing mode" [2, 399, 400] (it produces a transverse uniform expansion and contraction of a ring of freely falling test masses). It was also noted (from the 1.5PN and 2PN calculations in [179,180]) that the GW memory effect differs in scalar-tensor theory from in general relativity.[2] It was also shown in [180] that the scalar, breathing

[2]Specifically, the energy radiated from the dipole moment of the scalar field gives rise to a formally 1.5PN-order effect in the tensor gravitational waveform that would appear at Newtonian order in the waveform for nonspinning compact binaries, which are inspiraling because of the emission of dipole radiation. This is analogous to how the energy radiated in gravitational waves gives rise to a 2.5PN-order effect that appears at Newtonian order in the waveform for nonspinning compact-binary

polarization of the GWs does not have a nonlinear-type memory effect at 1.5PN order. Finally, it was observed that there is a new type of nonhereditary, nonlinear term in the tensor waveform arising from the scalar field that took on an analogous form to the nonhereditary and nonoscillatory term found in [401] (and discussed in [164]), which was shown to be related to the spin memory effect in [154]. Our calculations in Brans-Dicke theory in Bondi-Sachs coordinates allow us to compute the memory effects using the fully nonlinear field equations. This will provide us with the framework to understand the presence (and absence) of the memory effects computed at 2PN order in [179] and 1.5PN order in [180] (though we leave the explicit calculations for future work) and to determine the relevant radiative and nonradiative data needed to compute these effects.

Scalar-tensor theories are frequently studied in two different conformal frames, called the Jordan and Einstein frames, respectively. We find that the Einstein frame is more convenient for determining the asymptotic boundary conditions on the scalar field and metric, because the field equations have the same form as the Einstein-Klein-Gordon equations for a massless scalar field. The statement of stress-energy conservation is more complicated in the Einstein frame, however, because the stress-energy tensor of all matter fields besides the scalar field is no longer divergence free, but equals a nontrivial right-hand side involving gradients of the scalar field. Consequently, test particles follow accelerated curves in the Einstein frame (with an acceleration related

sources in GR [163,164]. Because stationary black holes in Brans-Dicke theory do not support scalar fields [218,220], the compact binary can have at most one black hole to have this new scalar-dipole-sourced GW memory effect.

to the gradients of the scalar field in this frame) rather than following the geodesics of the Einstein-frame metric. In the Jordan frame, the modified Einstein equations are more complicated than in the Einstein frame, but the stress-energy tensor of all matter fields besides the scalar field is divergence free, and thus test particles follow the geodesics of the Jordan-frame metric. It is therefore much simpler to compute the response of a gravitational-wave detector to any impinging gravitational waves in the Jordan frame. Flanagan [402] has argued that all classical physical predictions (such as gravitational-wave memory effects) are conformal-frame invariants. This allows us to compute the memory effects in the Jordan frame, in which the computation is simpler, but to obtain a result that is independent of the choice of conformal frame (after properly identifying any potentially different conventions between the frames, as discussed further in [402]).

The rest of the chapter is organized as follows: In Sec. 5.2, we describe the conditions we use to define asymptotic flatness in Brans-Dicke theory, by examining the theory in both Einstein and Jordan frames [199]. This includes deriving the field equations of the theory in Bondi-Sachs coordinates. In Sec. 5.3, we compute the asymptotic symmetries that preserve our definition of asymptotic flatness in the previous part. We describe how the functions in the metric must transform to maintain the Bondi gauge conditions and the asymptotically flat boundary conditions. In Sec. 5.4, we describe how the memory effects can be measured through geodesic deviation and how the changes in the charges related to (extended) BMS symmetries constrain the different GW memory effects in Brans-Dicke theory. We discuss our results and some future directions in Sec. 5.5.

Throughout this chapter, we use units in which $c = 1$, and we use the conventions for the metric and curvature tensors given in [185]. Greek indices $(\mu, \nu, \alpha, \dots)$ represent four-dimensional spacetime indices, and uppercase Latin indices $(A, B, C, \dots)$ represent indices on the 2-sphere. Indices with a circumflex diacritic (e.g., $\hat{\alpha}$) represent those of an orthonormal tetrad.

While we were completing this work, there appeared a closely related pre-print [403] investigating asymptotically flat solutions and GW memory effects in scalar-tensor theories. Our work and that of [403] agree in the boundary conditions used to define asymptotically flat solutions in Brans-Dicke theory and the leading-order symmetry vectors that preserve these conditions and our gauge choices (though not subleading corrections to extend these symmetries into the spacetime). Our works differ in the choices of gauges, the classes of spacetimes in which we compute memory effects, and the procedures by which we compute the scalar-type memory effect. We will comment in more detail on the similarities and differences between our works at a few points throughout the text.

5.2 Bondi-Sachs Framework

In this section, we impose the Bondi-Sachs coordinate conditions in Brans-Dicke theory, and we solve the field equations in both the Einstein and the Jordan frames. We begin with the Einstein frame, where it is easier to identify a set of asymptotic boundary conditions that can be imposed on the scalar field and on the metric that we use to define an asymptotically flat solution in Brans-Dicke theory. We next perform conformal transformation to the Jordan frame (in which the stress-energy tensor of all other matter fields besides the scalar field is divergence free), and we find the corresponding boundary conditions on the scalar field and metric. We then solve the field equations of Brans-Dicke theory in this frame. Our notation and conventions for the Bondi-Sachs framework will parallel the ones used in Ref. [404], which treats general relativity.

5.2.1 Einstein frame

We begin by investigating the Brans-Dicke theory in the Einstein frame. The action in the Einstein frame in the absence of additional matter fields is given by [369]

$$S = \int d^4x\sqrt{-\tilde{g}}\left[\frac{\tilde{\mathcal{R}}}{16\pi} - \frac{1}{2}\tilde{g}^{\rho\sigma}\left(\tilde{\nabla}_\rho\Phi\right)\left(\tilde{\nabla}_\sigma\Phi\right)\right], \tag{5.1}$$

where $\tilde{g}$ is the metric in the Einstein frame, $\tilde{\mathcal{R}} = \tilde{R}^\mu_\mu$ is the Ricci scalar and Φ is a real scalar field. We also use units where the gravitational constant in the Einstein frame G_E satisfies $G_E = 1$. We use $\tilde{\nabla}_\mu$ to denote the covariant derivative compatible with $\tilde{g}_{\mu\nu}$. Varying the action with respect to the metric and the scalar field leads to the following equations of motion for the theory:

$$\tilde{\mathcal{E}}_{\mu\nu} \equiv \tilde{R}_{\mu\nu} - \frac{1}{2}\tilde{\mathcal{R}}\tilde{g}_{\mu\nu} - 8\pi\tilde{T}^{(\Phi)}_{\mu\nu} = 0\,, \tag{5.2a}$$

$$\tilde{\nabla}_\mu\tilde{\nabla}^\mu\Phi = 0\,. \tag{5.2b}$$

The quantity $\tilde{T}^{(\Phi)}_{\mu\nu}$ is the stress-energy tensor for the scalar field, which is given by

$$\tilde{T}^{(\Phi)}_{\mu\nu} = \tilde{\nabla}_\mu \Phi \tilde{\nabla}_\nu \Phi - \tilde{g}_{\mu\nu} \left[\frac{1}{2} \tilde{g}^{\rho\sigma} \tilde{\nabla}_\rho \Phi \tilde{\nabla}_\sigma \Phi \right] . \tag{5.2c}$$

The field equations, therefore, have the same form as in Einstein-Klein-Gordon theory for a real scalar field Φ, so their solutions will also have the same form as in Einstein-Klein-Gordon theory in general relativity. We will review the solutions of these equations in Bondi coordinates next.

5.2.1.1 Bondi gauge and field equations

First, we introduce Bondi-Sachs coordinates $\tilde{x}^\mu = (\tilde{u}, \tilde{r}, \tilde{x}^A)$. The quantity $\tilde{u}$ is the retarded time, $\tilde{r}$ is an areal coordinate (and $\vec{\partial}_{\tilde{r}}$ is a null vector field), and $\tilde{x}^A$ are coordinates on the 2-sphere (with $A = 1, 2$) [139, 404]. The conditions that define Bondi gauge are given by [139, 404]

$$\tilde{g}_{\tilde{r}A} = \tilde{g}_{\tilde{r}\tilde{r}} = 0, \quad \det\left[\tilde{g}_{AB}\right] = \tilde{r}^4 q\left(\tilde{x}^C\right) . \tag{5.3}$$

The function q is the determinant of a metric on the 2-sphere, $q_{AB}(x^C)$, which is restricted to be independent of $\tilde{u}$ and $\tilde{r}$. The Bondi gauge conditions fix four of the ten functions in the metric, leaving six free functions. It is conventional to write these six degrees of freedom as follows:

$$\tilde{g}_{\mu\nu} d\tilde{x}^\mu d\tilde{x}^\nu = -\frac{\tilde{V}}{\tilde{r}} e^{2\tilde{\beta}} d\tilde{u}^2 - 2e^{2\tilde{\beta}} d\tilde{u} d\tilde{r} + \tilde{r}^2 \tilde{h}_{AB} \left(d\tilde{x}^A - \tilde{U}^A d\tilde{u} \right) \left(d\tilde{x}^B - \tilde{U}^B d\tilde{u} \right) . \tag{5.4}$$

The functions $\tilde{V}$, $\tilde{\beta}$, $\tilde{U}^A$ and $\tilde{h}_{AB}$ here depend on all four Bondi coordinates $\tilde{x}^\mu = (\tilde{u}, \tilde{r}, \tilde{x}^A)$.

The modified Einstein equations (5.2a) and the scalar-field equation (5.2b) satisfy an interesting hierarchy in Bondi coordinates [139, 404], which we will now further elaborate. The functions $\tilde{V}$, $\tilde{\beta}$, and $\tilde{U}^A$ satisfy the so-called "hypersurface equations." The equations were given this name because they do not involve derivatives with respect to $\tilde{u}$, which in turn allows the functions $\tilde{V}$, $\tilde{\beta}$, and $\tilde{U}^A$ to be determined on surfaces of constant $\tilde{u}$ in terms of the 2-metric $\tilde{h}_{AB}$, the scalar field Φ, and "functions of integration" (i.e., functions of $\tilde{u}$ and $\tilde{x}^A$ that will be constrained by $\tilde{u}\tilde{u}$ and $\tilde{u}\tilde{A}$ components of the modified Einstein equations). The concrete form of the hypersurface equations can be obtained from substituting the metric (5.4) into the field equations in Eq. (5.2a), using the definition of the stress-energy tensor in Eq. (5.2c), and considering the appropriate components of the modified Einstein equations. The $\tilde{r}\tilde{r}$ component yields the equation

$$\partial_{\tilde{r}}\tilde{\beta} - \frac{\tilde{r}}{16}\tilde{h}^{AC}\tilde{h}^{BD}\partial_{\tilde{r}}\tilde{h}_{AB}\partial_{\tilde{r}}\tilde{h}_{CD} = 2\pi\tilde{r}\partial_{\tilde{r}}\Phi\partial_{\tilde{r}}\Phi\,. \tag{5.5a}$$

where $\tilde{h}^{AB}$ is the inverse of $\tilde{h}_{AB}$. Once $\tilde{\beta}$ is determined in terms of $\tilde{h}_{AB}$ (and its inverse), Φ, and their derivatives, then it is also possible to use the $\tilde{r}A$ components of the field equations to solve for $\tilde{U}^A$ in terms of the same quantities from the following equation:

$$\partial_{\tilde{r}} \left[\tilde{r}^4 e^{-2\tilde{\beta}} \tilde{h}_{AB} \partial_{\tilde{r}} \tilde{U}^B\right] - 2\tilde{r}^4 \partial_{\tilde{r}} \left(\frac{1}{\tilde{r}^2} \tilde{D}_A \tilde{\beta}\right) + \tilde{r}^2 \tilde{h}^{BC} \tilde{D}_B \partial_{\tilde{r}} \tilde{h}_{AC} - 16\pi \tilde{r}^2 \partial_{\tilde{r}} \Phi \partial_A \Phi = 0\,. \tag{5.5b}$$

where $\tilde{D}_A$ is the covariant derivative compatible with the 2-metric $\tilde{h}_{AB}$. Finally, from the trace of the AB components of the field equations, it is then possible to solve for $\tilde{V}$ in terms of the same data:

$$2e^{-2\tilde{\beta}} \left(\partial_{\tilde{r}} \tilde{V}\right) - \tilde{\mathscr{R}} - \frac{e^{-2\tilde{\beta}}}{\tilde{r}^2} \tilde{D}_A \left[\partial_{\tilde{r}} \left(\tilde{r}^4 \tilde{U}^A\right)\right] + 2\tilde{h}^{AB} \left[\tilde{D}_A \tilde{D}_B \tilde{\beta} - \left(\tilde{D}_A \tilde{\beta}\right)\left(\tilde{D}_B \tilde{\beta}\right)\right]$$
$$+ \frac{1}{2} \tilde{r}^4 e^{-4\tilde{\beta}} \tilde{h}_{AB} \left(\partial_{\tilde{r}} \tilde{U}^A\right)\left(\partial_{\tilde{r}} \tilde{U}^B\right) - 8\pi \tilde{h}^{AB} \partial_A \Phi \partial_B \Phi = 0\,. \tag{5.5c}$$

Here $\tilde{\mathscr{R}}$ is the Ricci scalar of 2-metric $\tilde{h}_{AB}$. The remaining two independent components of the modified Einstein equations are called the evolution equations, and they arise from the trace-free (with respect to h_{AB}) part of $\tilde{\mathcal{E}}_{AB}$. It is convenient to write this expression using a complex polarization dyad composed of $\tilde{m}^A = \delta^A{}_\mu \tilde{m}^\mu$ (which satisfies $\tilde{m}^\mu \tilde{\nabla}_\mu \tilde{u} = 0$) and $\bar{\tilde{m}}^A$ (the complex conjugate of $\tilde{m}^A$). The evolution is given by $\tilde{m}^A \tilde{m}^B \tilde{\mathcal{E}}_{AB} = 0$, which can be written in terms of the metric functions as

$$\tilde{m}^A \tilde{m}^B \left\{ \tilde{r} \partial_{\tilde{r}} \left[\tilde{r} \left(\partial_{\tilde{u}} \tilde{h}_{AB}\right)\right] - \frac{1}{2} \partial_{\tilde{r}} \left[\tilde{r} \tilde{V} \left(\partial_{\tilde{r}} \tilde{h}_{AB}\right)\right] + \tilde{h}_{CA} \tilde{D}_B \left[\partial_{\tilde{r}} \left(\tilde{r}^2 \tilde{U}^C\right)\right] \right.$$
$$- \frac{1}{2} \tilde{r}^4 e^{-2\tilde{\beta}} \tilde{h}_{AC} \tilde{h}_{BD} \left(\partial_{\tilde{r}} \tilde{U}^C\right)\left(\partial_{\tilde{r}} \tilde{U}^D\right) + \frac{\tilde{r}^2}{2} \left(\partial_{\tilde{r}} \tilde{h}_{AB}\right)\left(\tilde{D}_C \tilde{U}^C\right) + \tilde{r}^2 \tilde{U}^C \tilde{D}_C \left(\partial_{\tilde{r}} \tilde{h}_{AB}\right)$$
$$\left. - \tilde{r}^2 \left(\partial_{\tilde{r}} \tilde{h}_{AC}\right) \tilde{h}_{BE} \left(\tilde{D}^C \tilde{U}^E - \tilde{D}^E \tilde{U}^C\right) - 8\pi e^{2\tilde{\beta}} \partial_A \Phi \partial_B \Phi - 2e^{\tilde{\beta}} \tilde{D}_A \tilde{D}_B e^{\tilde{\beta}} \right\} = 0\,. \tag{5.5d}$$

We will discuss the evolution equations in more detail in Sec. 5.2.2 on the Jordan frame.

In vacuum general relativity, once the metric functions $\tilde{\beta}$, $\tilde{U}^A$ and $\tilde{V}$ are determined on a hypersurface of constant $\tilde{u}$, they can be used in the evolution equation for the 2-metric $\tilde{h}_{AB}$, to evolve h_{AB} to the next hypersurface; the hypersurface equations can then be solved again in an iterative process. In Brans-Dicke theory, however, one must jointly evolve the evolution equation for $\tilde{h}_{AB}$ with the scalar field equation to obtain the data $\tilde{h}_{AB}$ and Φ needed to solve the hypersurface equations. For convenience, we give the scalar wave equation (5.2b) when written in terms of the Bondi metric functions below:

$$
\begin{aligned}
2\partial_{\tilde{u}}\partial_{\tilde{r}}\Phi + \tilde{D}_A(\tilde{U}^A\partial_{\tilde{r}}\Phi) + \partial_{\tilde{r}}(\tilde{U}^A\tilde{D}_A\Phi) - \frac{1}{\tilde{r}}\left(-2\tilde{U}^A\tilde{D}_A\Phi - 2\partial_{\tilde{u}}\Phi + \partial_{\tilde{r}}\tilde{V}\partial_{\tilde{r}}\Phi + \tilde{V}\partial_{\tilde{r}}\partial_{\tilde{r}}\Phi\right) \\
- \frac{1}{\tilde{r}^2}\left[e^{2\tilde{\beta}}\tilde{h}^{AB}\left(2\tilde{D}_A\tilde{\beta}\tilde{D}_B\Phi + \tilde{D}_B\tilde{D}_A\Phi\right) + \tilde{V}\left(\partial_{\tilde{r}}\Phi\right)\right] = 0\,.
\end{aligned}
\tag{5.5e}
$$

Aside from the additional complication that the scalar-wave equation and evolution equation for $\tilde{h}_{AB}$ must be solved as a coupled system, the form and the hierarchy of the modified Einstein and scalar field equations in the Einstein frame is similar to that of the Einstein equations in vacuum general relativity.

5.2.1.2 Conditions for asymptotic flatness

We next study the asymptotic behavior of the metric and the scalar field at large Bondi radius r. Because $\tilde{V}$, $\tilde{\beta}$, and $\tilde{U}^A$ are determined by $\tilde{h}_{AB}$ and Φ, we must posit boundary conditions on $\tilde{h}_{AB}$ and Φ; we can then deduce the remaining conditions on the metric from the hypersurface equations (5.5a)–(5.5c) up to functions of integration. There are well-established definitions for asymptotic flatness for the Einstein equations [139, 140]. For the scalar field, we will assume that it satisfies the following scaling as $\tilde{r} \to \infty$:

$$\Phi\left(\tilde{u}, \tilde{r}, \tilde{x}^A\right) = \Phi_0 + \frac{\Phi_1\left(\tilde{u}, \tilde{x}^A\right)}{\tilde{r}} + O(\tilde{r}^{-2})\,, \tag{5.6}$$

where Φ_0 is a constant.[3]

In GR, the action for a massless scalar field (and hence the stress-energy tensor and equations of motion) is independent of the value of Φ_0 in Eq. (5.6). Thus, there is no loss of generality by requiring that the constant value of the scalar field is zero. The boundary condition on the massless scalar field as r goes to infinity can then be given by "Sommerfeld's radiation condition": $\lim_{r\to\infty} r\Phi$ is finite. In Brans-Dicke theory, the constant value of the scalar field is related to the asymptotic value of Newton's constant G. Setting Φ_0 to zero, therefore, does have a physical effect in Brans-Dicke theory (note, however, that the precise value of the constant does not affect the stress-energy tensor for the scalar field, nor does it affect the equation of motion for the scalar field in vacuum). We thus require a nonzero Φ_0 in Eq. (5.6), and we do not employ Sommerfeld's radiation condition to write the limit of the scalar field as r goes to infinity.

[3]With the expansion of $\tilde{h}_{AB}$ in Eq. (5.7) and with a polynomial expansion of $\tilde{\beta}, \tilde{U}^A$ and $\tilde{V}$ in $\tilde{r}^{-1}$ consistent with Eq. (5.8), one can prove from the $\tilde{r}^{-1}$ piece of Eq. (5.5e) that Φ_0 is independent of $\tilde{u}$; from the $\tilde{r}^{-2}$ piece of Eq. (5.5e), one can show that Φ_0 is independent of $\tilde{x}^A$.

Similarly, we adopt the same expansion of the 2-metric $\tilde{h}_{AB}$ as $\tilde{r} \to \infty$ as in GR:

$$\tilde{h}_{AB} = q_{AB}(\tilde{x}^C) + \frac{\tilde{c}_{AB}(\tilde{u}, \tilde{x}^C)}{\tilde{r}} + O(\tilde{r}^{-2})\,. \tag{5.7}$$

The determinant condition of Bondi gauge requires that $q^{AB}\tilde{c}_{AB} = 0$. It is then convenient to define a covariant derivative operator compatible with q_{AB}, which will be denoted by $\eth_A$. In addition, it is also helpful to raise (or lower) capital Latin indices on 2-spheres of constant $\tilde{u}$ and $\tilde{r}$ with the 2-metric q^{AB} (or q_{AB}).

Next, we assume the functions $\tilde{\beta}$, $\tilde{U}^A$, $\tilde{V}$ and $\tilde{h}_{AB}$ have the following limits as $\tilde{r}$ approaches infinity:[4]

$$\lim_{\tilde{r}\to\infty} \tilde{\beta} = \lim_{\tilde{r}\to\infty} \tilde{U}^A = 0, \quad \lim_{\tilde{r}\to\infty} \frac{\tilde{V}}{\tilde{r}} = 1, \quad \lim_{\tilde{r}\to\infty} \tilde{h}_{AB} = q_{AB}\,. \tag{5.8}$$

The metric thus reduces to Minkowski spacetime in inertial Bondi coordinates in this limit. Hou and Zhu independently proposed similar conditions in [403]. Imposing these conditions and radially integrating the hypersurface equations in Eqs. (5.5a)–(5.5c) further, we then arrive at the solutions[5]

$$\tilde{\beta} \;=\; -\frac{1}{32\tilde{r}^2}\tilde{c}^{AB}\tilde{c}_{AB} - \frac{1}{\tilde{r}^2}\pi\Phi_1^2 + O(\tilde{r}^{-3})\,, \tag{5.9a}$$

[4]We assume that it is possible to impose these conditions at all retarded times u. Given the structure of the Bondi-Sachs equations as described in Sec. 5.2.1.1, these conditions can be imposed on an initial hypersurface $u =$ const., but they will not necessarily be preserved under evolution to future hypersurfaces. It is possible to construct coordinate transformations that reimpose the conditions Eq. (5.8) after evolution (see, e.g., [405, 406] for more details).

[5]Note that in the expression for U^A in Eq. (5.9b), the remainder contains a term of order $\tilde{r}^{-3}\log\tilde{r}$. The coefficient of the term that scales as $\tilde{r}^{-3}\log\tilde{r}$ is proportional to $\eth_B\tilde{D}^{AB}$, where we have denoted the $\tilde{r}^{-2}$ the part of $\tilde{h}_{AB}$ that is trace-free with respect to the metric q^{AB} by $\tilde{D}_{AB}$. The order $1/\tilde{r}$ part of the Einstein equation (5.5d) imposes that $\tilde{D}_{AB}$ satisfies $\partial_{\tilde{u}}\tilde{D}_{AB} = 0$ (i.e., that it is nondynamical, as the analogous quantity in general relativity is). This will not be true of the analogous function in the Jordan frame, as we discuss in Sec. 5.2.2.

$$\tilde{U}^A = -\frac{1}{2\tilde{r}^2}\eth_B \tilde{c}^{AB} + O(\tilde{r}^{-3}\log\tilde{r})\,, \tag{5.9b}$$

$$\tilde{V} = \tilde{r} - 2\tilde{M} + O(\tilde{r}^{-1})\,. \tag{5.9c}$$

The function $\tilde{M}(\tilde{u}, \tilde{x}^A)$ is called the Bondi mass aspect and is one of the functions of integration that arises from integrating the hypersurface equations.

5.2.2 Jordan frame

Having determined the asymptotic fall-off conditions in the Einstein frame, we now consider the asymptotic properties of the solutions in the Jordan frame, in which it is more straightforward to understand the response of a detector to the gravitational waves emitted from an isolated system (because test particles follow geodesics of the Jordan-frame metric). A solution in the Jordan frame can be found from one in the Einstein frame by performing a conformal transformation [407]

$$g_{\mu\nu} = \frac{1}{\lambda}\tilde{g}_{\mu\nu}\,, \tag{5.10}$$

where

$$\lambda = \exp\left(\Phi/\mathcal{W}\right), \qquad \mathcal{W} \equiv \sqrt{\frac{2\omega_{\rm BD}+3}{16\pi}}\,. \tag{5.11}$$

The scalar field is called λ in this frame, and $\omega_{\rm BD}$ is the Brans-Dicke parameter. In the limits in which $\omega_{\rm BD} \to \infty$ and λ becomes nondynamical, general relativity is recovered. The Brans-Dicke action in the Jordan frame is given by [166]

$$S = \int d^4x\sqrt{-g}\left[\frac{\lambda}{16\pi}\mathcal{R} - \frac{\omega_{\rm BD}}{16\pi}g^{\mu\nu}\frac{(\partial_\mu\lambda)\,(\partial_\nu\lambda)}{\lambda}\right], \tag{5.12}$$

where $\mathcal{R}$ is the Ricci scalar of the Jordan-frame metric $g_{\mu\nu}$. The field equations are given by

$$G_{\mu\nu} = \frac{1}{\lambda}\left(8\pi T^{(\lambda)}_{\mu\nu} + \nabla_\mu \nabla_\nu \lambda - g_{\mu\nu}\Box\lambda\right) , \tag{5.13a}$$

$$\Box\lambda = 0 , \tag{5.13b}$$

where $G_{\mu\nu}$ is the Einstein tensor, $\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu$ is the covariant wave operator, and

$$T^{(\lambda)}_{\mu\nu} = \frac{\omega_{\rm BD}}{8\pi\lambda}\left(\nabla_\mu\lambda\nabla_\nu\lambda - \frac{1}{2}g_{\mu\nu}\nabla^\alpha\lambda\nabla_\alpha\lambda\right) . \tag{5.13c}$$

is the stress-energy tensor of the scalar field. It is also convenient to define a tensor $\mathcal{E}_{\mu\nu}$ by

$$\mathcal{E}_{\mu\nu} \equiv G_{\mu\nu} - \frac{1}{\lambda}\left(8\pi T^{(\lambda)}_{\mu\nu} + \nabla_\mu \nabla_\nu \lambda - g_{\mu\nu}\Box\lambda\right) , \tag{5.14}$$

which vanishes when the equations of motion are satisfied.

5.2.2.1 Bondi gauge and asymptotic boundary conditions

We would now like to compute a metric in Bondi-Sachs coordinates in the Jordan frame that is consistent with our definition of asymptotic flatness in the Einstein frame. The transformation in Eq. (5.11) implies that λ admits an expansion in $1/\tilde{r}$, in which the leading-order term is constant: i.e.,

$$\lambda(\tilde{u},\tilde{r},\tilde{x}^A) = \exp\left(\frac{\Phi_0}{\mathcal{W}}\right)\left(1 + \frac{\Phi_1}{\mathcal{W}}\frac{1}{\tilde{r}}\right) + O(\tilde{r}^{-2}) , \tag{5.15}$$

The conformal transformation of the metric in Eq. (5.10) preserves the Bondi gauge conditions $g_{rr} = g_{rA} = 0$, but the determinant condition becomes $\det[g_{AB}] = \tilde{r}^4\lambda^{-2}q(x^C)$.

Consequently, the final condition of Bondi gauge will not be satisfied in general (i.e., $\tilde{r}$ is not an areal radius in the Jordan frame). It is possible to work in a set of coordinates that do not impose the determinant condition in Bondi gauge (as was done in [403]); however, when λ is positive (as it is expected to be far from an isolated source, since λ is related to the gravitational constant [2]), it is also possible to redefine the radial coordinate so as to make it an areal coordinate. The transformation that effects this change is

$$u = \frac{\tilde{u}}{\sqrt{\lambda_0}}\,, \quad r = \tilde{r}\lambda^{-1/2}\,, \quad x^A = \tilde{x}^A\,, \tag{5.16}$$

where we have introduced the notation $\lambda_0 = \exp(\Phi_0/\mathcal{W})$. The retarded time $\tilde{u}$ is rescaled by λ_0 so that the metric coefficient $-g_{ur}$ becomes one as $r \to \infty$. In the coordinates (u, r, x^A), the metric takes the Bondi form,

$$g_{\mu\nu}dx^\mu dx^\nu = -\frac{V}{r}e^{2\beta}du^2 - 2e^{2\beta}dudr + r^2 h_{AB}\left(dx^A - U^A du\right)\left(dx^B - U^B du\right)\,, \tag{5.17}$$

where V, β, U^A and h_{AB} are functions of coordinates $x^\mu = (u, r, x^A)$. The metric satisfies all the Bondi gauge conditions

$$g_{rA} = g_{rr} = 0\,, \quad \det\left[g_{AB}\right] = r^4 q\left(x^C\right)\,. \tag{5.18}$$

By performing the conformal and coordinate transformation on the solutions of the field equations in the Einstein frame [Eqs. (5.9a)–(5.9c)], we find that the functions

β, U^A, and V should have the following forms:

$$\beta = -\frac{1}{2r}\frac{\Phi_1}{\mathcal{W}\sqrt{\lambda_0}} + O\left(r^{-2}\right) \,, \tag{5.19a}$$

$$U^A = -\frac{1}{2\sqrt{\lambda_0} r^2}\left(\eth_B c^{AB} - \eth^A \Phi_1\right) + O\left(r^{-3}\log r\right) \,, \tag{5.19b}$$

$$V = \left(1 + \frac{\partial_u \Phi_1}{\mathcal{W}}\right) r + O\left(r^0\right) . \tag{5.19c}$$

Interestingly, in the limit as r goes to infinity, V/r goes as $1+\partial_u\Phi_1/\mathcal{W}$ (i.e., when Φ_1 is dynamical, the leading-order Minkowski part of the metric is expressed in noninertial coordinates when the Bondi gauge conditions are imposed). This occurs because the component of the Ricci tensor, R_{uu}, scales as $1/r$ when $\partial_u\Phi_1$ in nonvanishing, as we discuss in more detail below and in Sec. 5.4. In addition, β scales as $O\left(r^{-1}\right)$ instead of $O\left(r^{-2}\right)$, as in the Einstein frame (or in general relativity). Based on these considerations, we expect that the metric functions will have the following scaling with r in the Jordan frame:

$$\beta = O\left(r^{-1}\right) \,, \quad V = O\left(r\right) \,, \quad U^A = O\left(r^{-2}\right) \,. \tag{5.20}$$

We explicitly verify this by solving the field equations in the next part.

5.2.2.2 Asymptotically flat solutions

The Bondi-Sachs field equations for Brans-Dicke theory in the Jordan frame have a similar hierarchy as in the Einstein frame. The trace-free part of h_{AB} satisfies an evolution equation, and the scalar field satisfies the curved-space wave equation (also an evolution equation). The remaining metric functions β, U^A, and V can be solved from hypersurface equations on surfaces of constant u in terms of h_{AB}, λ, and functions of integration known as the Bondi mass aspect and angular momentum aspect. The mass and angular momentum aspects satisfy the conservation equations. The full expressions for these equations are rather lengthy, though we give the expressions for the scalar wave equation and the hypersurface equations in Appendix J. Thus, we will focus on determining the metric functions and the evolution equations satisfied by these functions when these quantities are expanded in a series in $1/r$.

As in the Einstein frame, it is necessary to assume an expansion of the 2-metric h_{AB} and the scalar field λ as series in $1/r$, and the expansions of the remaining quantities will follow from the field equations and boundary conditions in Eq. (5.20). For the scalar field, we have

$$\lambda(u,r,x^A) = \lambda_0 + \frac{\lambda_1\left(u,x^A\right)}{r} + \frac{\lambda_2\left(u,x^A\right)}{r^2} + \frac{\lambda_3\left(u,x^A\right)}{r^3} + O(r^{-4})\,, \tag{5.21}$$

The constant λ_0 is related to the gravitational constant[6], and λ_1 is the leading-order non-constant part of the scalar field, which is closely connected to the additional polarization of the gravitational waves in Brans-Dicke theory. That λ in Eq. (5.21) has a similar expansion in $1/r$ as Φ in Eq. (5.6) follows from the relation between λ and Φ in Eq. (5.11).

[6]The relation between the gravitational constant and the scalar field in Brans-Dicke theory is given by $G(\lambda) = \frac{4+2\omega_{\rm BD}}{3+2\omega_{\rm BD}}\frac{1}{\lambda}$. If one assumes the experimentally measured value of G at infinity to be 1, λ_0 can be written in terms of the Brans-Dicke parameter $\omega_{\rm BD}$ as $\lambda_0 = \frac{3+2\omega_{\rm BD}}{4+2\omega_{\rm BD}}$.

For the two-metric, we take the expansion to be

$$h_{AB} = q_{AB}(x^C) + \frac{c_{AB}(u, x^C)}{r} + \frac{d_{AB}(u, x^C)}{r^2} + \frac{e_{AB}(u, x^C)}{r^3} + O\left(r^{-4}\right) . \tag{5.22a}$$

The determinant condition in Bondi gauge fixes the part of d_{AB} and e_{AB} that is proportional to q_{AB}. Thus, we write them as

$$d_{AB} = D_{AB} + \frac{1}{4} c_{FG} c^{FG} q_{AB} \,, \tag{5.22b}$$

$$e_{AB} = E_{AB} + \frac{1}{2} c_{FG} D^{FG} q_{AB} \,, \tag{5.22c}$$

where c_{AB}, D_{AB}, and E_{AB} are trace-free with respect to q_{AB}. The tensor $c_{AB}(u, x^A)$ is closely related to the shear of outgoing null geodesics at large r, and is thus also related to the gravitational waves.

We now substitute the expansion of λ and h_{AB} in Eqs. (5.21) and (5.22a) into the field equations, solve order by order in r^{-1}, and compute the metric functions and their corresponding evolution equations. We begin with the curved-space, scalar wave equation in Eq. (5.13b). The explicit forms, in Bondi coordinates, of Eq. (5.13b) and the hypersurface equations in Eq. (5.13a) are given in Appendix J. We find that the assumption of λ_0 =constant is consistent with these field equations; at $O(r^{-2})$, the wave equation reduces to the expression $\partial_r(\partial_u \lambda_1) = 0$, which implies that $\partial_u \lambda_1 = N_{(\lambda)}(u, x^A)$ is an arbitrary function. An analogous equation arises for the evolution of the tensor c_{AB}, which leads to $\partial_u c_{AB}$ being unconstrained (and equal to an arbitrary symmetric, trace-free tensor that gets called the Bondi news tensor, which is defined below). To obtain higher-order terms in the wave equation, we need to first solve for some functions in the Bondi metric.

Next, integrating the rr, rA, and the trace of the AB components of the modified Einstein equations presented in Appendix J, we find

$$\beta = -\frac{\lambda_1}{2\lambda_0 r} - \frac{1}{r^2}\left[\frac{1}{32}c^{AB}c_{AB} + \frac{\omega_{\text{BD}}-1}{8\lambda_0^2}\lambda_1^2 + \frac{3\lambda_2}{4\lambda_0}\right) + O\left(r^{-3}\right) , \tag{5.23a}$$

$$U^A = -\frac{1}{2r^2}\left(\eth_F c^{AF} - \frac{\eth^A\lambda_1}{\lambda_0}\right) + \frac{1}{3r^3}\left[c^{AD}\eth^F c_{DF} - \frac{1}{\lambda_0}c^{AD}\eth_D\lambda_1 + \frac{\lambda_1}{\lambda_0}\eth_B c^{AB} - \frac{\lambda_1}{\lambda_0^2}\eth^A\lambda_1 + \mathcal{U}^A\left(1+3\log r\right) + 6L^A\right] + O\left(r^{-4}\right) , \tag{5.23b}$$

$$V = \left(1 + \frac{\partial_u\lambda_1}{\lambda_0}\right) r - 2M + O\left(r^{-1}\right) , \tag{5.23c}$$

respectively. Here $M(u, x^A)$ is a function of integration. While an analogous quantity is defined to be the Bondi mass aspect in the Einstein frame or in GR, here we find it convenient to define a slightly different quantity to be the mass aspect (which is defined shortly below). The second function of integration, the angular-momentum aspect L^A, can be obtained from the expression

$$L_A(u, x^A) = -\frac{1}{6}\lim_{r\to\infty}\left(r^4 e^{-2\beta}h_{AB}\partial_r U^B - r\eth^B c_{AB} + r\frac{\eth_A\lambda_1}{\lambda_0} + 3\mathcal{U}_A\log r\right) . \tag{5.24}$$

The integration procedure allows for a term proportional to $\log r/r^3$ in U^A. The term $\mathcal{U}^A$ is given by

$$\mathcal{U}_A = -\frac{2}{3}\eth^B\left(D_{AB} + \frac{1}{2\lambda_0}\lambda_1 c_{AB}\right) . \tag{5.25}$$

We will only consider solutions with $\mathcal{U}^A = 0$ for reasons which we discuss below Eq. (5.30).

We can now return to the scalar wave equation to solve for the higher-order terms. The $O(r^{-3})$ and $O(r^{-4})$ parts of the scalar wave equation determine that λ_2 and λ_3 evolve via the equations

$$\partial_u \lambda_2 = -\frac{1}{2}\eth^2 \lambda_1 \,, \tag{5.26a}$$

$$\begin{aligned}\partial_u \lambda_3 = &-\frac{1}{2\lambda_0}\partial_u(\lambda_1\lambda_2) + \frac{1}{2}M\lambda_1 - \frac{1}{4}(\eth^2 + 2)\lambda_2 + \frac{1}{2}\eth_B c^{AB}\eth_A\lambda_1 \\ &- \frac{1}{8\lambda_0}\lambda_1\eth^2\lambda_1 + \frac{1}{4}c^{AB}\eth_A\eth_B\lambda_1 + \frac{1}{8}\lambda_1\eth_A\eth_B c^{AB} \,.\end{aligned} \tag{5.26b}$$

To simplify the notation slightly, we have introduced the quantity $\eth^2 = \eth_A\eth^A$ to denote the Laplacian on the 2-sphere.

The evolution equations for h_{AB} come from the trace-free part of the AB components of the field equations

$$\mathcal{E}_{AB} - \frac{1}{2}g_{AB}g^{CD}\mathcal{E}_{CD} = 0 \,. \tag{5.27}$$

Because we have already imposed the field equation $h^{CD}\mathcal{E}_{CD} = 0$ to determine V, the term proportional to g_{AB} in Eq. (5.27) does not contribute. As a practical computational matter, it can be more convenient to contract Eq. (5.27) into a complex polarization dyad $m^A = \delta^A{}_\mu m^\mu$ (and its complex conjugate) where m^μ satisfies $m^\mu\nabla_\mu u = 0$ [404, 408, 409] (a similar procedure was performed in the Einstein frame). Then the two degrees of freedom in the evolution equation can be recast in terms of a single complex equation

$$m^A m^B \mathcal{E}_{AB} = 0 \,. \tag{5.28}$$

The $O(r^0)$ part of Eq. (5.28) reduces to the equation proportional to $\partial_r(\partial_u c_{AB}) = 0$. This implies that

$$\partial_u c_{AB} = N_{AB}\,, \tag{5.29}$$

where N_{AB} is an arbitrary symmetric trace-free tensor, called the news tensor. In GR, spacetimes with a vanishing news tensor contain no gravitational waves [148]. The $O(r^{-1})$ terms of Eq. (5.28) lead to the equation

$$\partial_u \left(D_{AB} + \frac{1}{2\lambda_0}\lambda_1 c_{AB} \right) = 0\,. \tag{5.30}$$

By taking ∂_u of Eq. (5.25) and $\eth_A$ of Eq. (5.30), then one can see that one must have $\partial_u \mathcal{U}^A = 0$. Thus the choice $\mathcal{U}^A = 0$ made above will not affect the dynamics of D_{AB}, but it does impose a constraint on the allowed initial data for the quantity $D_{AB} + \lambda_1 c_{AB}/(2\lambda_0)$. The $O\left(r^{-2}\right)$ part of Eq. (5.28) is a significantly more complicated expression, which we give below:

$$\begin{aligned}
\partial_u E_{AB} \;=\; & -\frac{1}{2}D_{AB} + \frac{1}{2}c_{AB}\mathcal{M} - \eth_{(B}L_{A)} + \frac{1}{2}q_{AB}\eth_C L^C + \frac{1}{4}c_{AB}c^{CD}N_{CD} \\
& + \frac{1}{32}\left(\eth_A\eth_B - \frac{1}{2}q_{AB}\eth^2\right)\left(c^{ED}c_{ED}\right) + \frac{1}{8}\epsilon_{C(A}c_{B)}{}^{C}\left(\epsilon^{DE}\eth_E\eth^F c_{DF}\right) \\
& + \frac{1}{6}\left[\eth_{(B}\left(c^C_{A)}\eth^D c_{CD}\right) - \frac{1}{2}q_{AB}\eth^E\left(c^C_E\eth^D c_{CD}\right)\right] \\
& - \frac{1}{12\lambda_0^2}\left(3\omega_{\rm BD}+7\right)\left(\eth_A\lambda_1\eth_B\lambda_1 - \frac{1}{2}q_{AB}\eth^C\lambda_1\eth_C\lambda_1\right) - \frac{1}{2\lambda_0}\lambda_2 N_{AB} \\
& + \frac{1}{12\lambda_0^2}\left(3\omega_{\rm BD}+2\right)\lambda_1\left(\eth_B\eth_A - \frac{1}{2}q_{AB}\eth^2\right)\lambda_1 - \frac{1}{3\lambda_0}\lambda_1 c_{AB} \\
& + \frac{1}{12\lambda_0}c_{AB}\eth^2\lambda_1 + \frac{3\lambda_1^2}{8\lambda_0^2}N_{AB} + \frac{1}{4\lambda_0}\left(\eth_B\eth_A - \frac{1}{2}q_{AB}\eth^2\right)\lambda_2 \\
& - \frac{1}{6\lambda_0}\left(\eth^C\lambda_1\eth_{(B}c_{A)C} - \frac{1}{2}q_{AB}\eth_C c^{CD}\eth_D\lambda_1\right) + \frac{1}{12\lambda_0}\eth^C\lambda_1\eth_C c_{AB}
\end{aligned}$$

$$+\frac{1}{24\lambda_0}\lambda_1 \text{Ð}^2 c_{AB} - \frac{1}{4\lambda_0^2}\left(4\lambda_0 D_{AB} - \lambda_1 c_{AB}\right)\partial_u \lambda_1 \,. \tag{5.31}$$

We use this expression to understand the properties of the angular momentum aspect L_A in nonradiative regions of spacetime in Sec. 5.4.

To complete our treatment of the field equations, we must consider the conservation equations in $\mathcal{E}_{uu}$ and $\mathcal{E}_{uA}$. These equations result in conservation equations for the mass and angular-momentum aspects. The equation for the mass aspect comes from the $O(r^{-2})$ part of $\mathcal{E}_{uu}$, and it is given by

$$\partial_u \mathcal{M} = -\frac{1}{8}N_{AB}N^{AB} + \frac{1}{4}\eth_A\eth_B N^{AB} - (3+2\omega_{\text{BD}})\frac{1}{4\lambda_0^2}\left(\partial_u\lambda_1\right)^2 + \frac{1}{4\lambda_0}\partial_u \text{Ð}^2\lambda_1 \,, \tag{5.32a}$$

where we have defined

$$\mathcal{M}\left(u,x^A\right) = M\left(u,x^A\right) - \frac{1}{4\lambda_0^2}\lambda_1\partial_u\lambda_1 \,, \tag{5.32b}$$

to be the Bondi mass aspect in the Jordan frame. With this definition of $\mathcal{M}$ the average of the right-hand side of Eq. (5.32a) over the 2-sphere is a non-positive number: i.e., the average value of $\mathcal{M}$ is a strictly decreasing quantity. This makes $\mathcal{M}$ more closely analogous to the Bondi mass aspect in general relativity, in which the average value of mass aspect gives rise to the well known Bondi mass-loss formula [139]. Note that M would not necessarily satisfy this property, because $\lambda_1\partial_u\lambda_1 = \partial_u(\lambda_1^2/2)$ is not necessarily a decreasing quantity. The calculations of symplectic fluxes and charges in Sec. 5.4 would suggest one might also include the $\text{Ð}^2\lambda_1$ term in the definition of the mass aspect, though we do not do that above.

Finally, from the $O(r^{-2})$ part of $\mathcal{E}_{uA}$, the angular momentum aspect satisfies a conservation equation of the form

$$\begin{aligned}
-3\partial_u L_A = \eth_A\mathcal{M} &- \frac{1}{4}\eth^E\left(\eth_E\eth^F c_{AF} - \eth_A\eth^F c_{EF}\right) + \frac{1}{16}\eth_A\left(c_{EF}N^{EF}\right) - \frac{1}{2}\eth_C\left(c^{CF}N_{AF}\right)\\
&+ \frac{1}{4}c^{EF}\left(\eth_A N_{EF}\right) + \frac{1}{8\lambda_0}\eth_A\text{Ð}^2\lambda_1 - \frac{1}{4\lambda_0^2}\left(2+3\omega_{\text{BD}}\right)\eth_A\lambda_1\partial_u\lambda_1\\
&+ \frac{\lambda_1}{4\lambda_0^2}\left(4+\omega_{\text{BD}}\right)\eth_A\partial_u\lambda_1 + \frac{1}{4\lambda_0}\partial_u\left(c_{AB}\eth^B\lambda_1 - \lambda_1\eth^B c_{AB}\right). \qquad (5.33)
\end{aligned}$$

To summarize, the structure of the Einstein equations is very much like the Bondi-Sachs formalism for general relativity [404] (though with an additional massless field). There are unconstrained functions $N_{AB} = \partial_u c_{AB}$ and $N_{(\lambda)} = \partial_u\lambda_1$ that determine the evolution of the different functions in the expansion of the metric and scalar field. Then initial data must be given for λ_1, λ_2, λ_3, $\mathcal{M}$, L_A, c_{AB}, and E_{AB}. Initial data also must be given for D_{AB}, but because we did not allow log terms in our expansion, this initial data is not independent of that of λ_1 and c_{AB}. Our field equations have a slightly different form than those given in [403], because of the different gauge conditions that we use (note also that [403] did not compute the evolution equations for E_{AB} or λ_3).

5.3 Asymptotic Symmetries

We now turn to computing the infinitesimal vector fields $\vec{\xi}$ that preserve the Bondi gauge conditions and the asymptotic form of the metric and the scalar field in Brans-

Dicke theory. Our treatment parallels that given in [143] for general relativity. The scalar field is Lie dragged along the generators of these asymptotic symmetries $\vec{\xi}$, so it transforms as $\lambda \to \lambda + \mathcal{L}_\xi \lambda$ (where we use $\mathcal{L}_\xi$ to denote the Lie derivative along $\vec{\xi}$). To preserve the Bondi gauge conditions, the following components of the metric must be left invariant when the metric is Lie dragged along $\vec{\xi}$:

$$\mathcal{L}_\xi g_{rr} = 0, \quad \mathcal{L}_\xi g_{rA} = 0, \quad g^{AB}\mathcal{L}_\xi g_{AB} = 0\,. \tag{5.34}$$

The four differential equations in Eq. (5.34) constrain the four components of $\vec{\xi}$. Because these conditions rely only upon Bondi gauge and not the underlying theory, we can expand the solution in Eq. (4.7) of [143] using our solutions for β, U^A, and h_{AB} in the Jordan frame of Brans-Dicke theory, which were computed in the Sec. 5.2. The results are that

$$\xi^u = f\left(u, x^A\right)\,, \tag{5.35a}$$

$$\begin{aligned}\xi^r =& -\frac{1}{2}r\eth_A Y^A + \frac{1}{2}\eth^A\eth_A f - \frac{1}{4r}\bigg(c^{AB}\eth_B\eth_A f \\ &+ 2\eth_A f\eth_B c^{AB} + \frac{\lambda_1}{\lambda_0}\text{Đ}^2 f\bigg) + O\left(r^{-2}\right)\,,\end{aligned} \tag{5.35b}$$

$$\begin{aligned}\xi^A =& Y^A\left(u, x^A\right) - \frac{1}{r}\eth^A f + \frac{1}{2r^2}\left(c^{AB}\eth_B f + \frac{1}{\lambda_0}\lambda_1\eth^A f\right) \\ &+ \frac{1}{r^3}\bigg[\frac{1}{3}D^{AB}\eth_B f - \frac{1}{16}c^{BC}c_{BC}\eth^A f - \frac{\lambda_1}{3\lambda_0}c^{AB}\eth_B f \\ &+ \frac{\lambda_2}{2\lambda_0}\eth^A f + \frac{\lambda_1^2}{12\lambda_0^2}\left(\omega_{\text{BD}} - 3\right)\eth^A f\bigg] + O\left(r^{-4}\right)\,.\end{aligned} \tag{5.35c}$$

The functions of integration $f(u, x^A)$ and $Y^A(u, x^A)$ come from radially integrating Eq. (5.34).

To maintain the asymptotic fall-off conditions that we have determined, we require that the remaining metric components transform as follows:

$$\mathcal{L}_\xi g_{ur} = O(r^{-1})\,, \quad \mathcal{L}_\xi g_{uA} = O(r^0)\,,$$
$$\mathcal{L}_\xi g_{AB} = O(r)\,, \quad \mathcal{L}_\xi g_{uu} = O(r^0)\,. \tag{5.36}$$

Note that in GR $\mathcal{L}_\xi g_{uu} = O(r^{-1})$; however, because in Brans-Dicke theory in the Jordan frame g_{uu} is given by $g_{uu} = -1 + \partial_u\lambda_1/\lambda_0 + O(r^{-1})$, we allow a change in g_{uu} at $O(r^0)$ (which occurs from the change in $\partial_u\lambda_1$). To express the conditions that we use to constrain $\vec{\xi}$ and the change in the metric coefficients, it is convenient to expand $\mathcal{L}_\xi g_{\mu\nu}$ in a series in r as

$$\mathcal{L}_\xi g_{\mu\nu} = \sum_n r^n l^{(n)}_{\mu\nu}\,, \tag{5.37}$$

where n can be an integer, and the coefficients $l^{(n)}_{\mu\nu}$ in the expansion are functions of u and x^A. Then one can show from $l^{(2)}_{uA} = 0$ that Y^A is independent of u, and from $l^{(2)}_{AB} = 0$ that it is a conformal Killing vector on a 2-sphere: i.e.,

$$\eth_A Y_B + \eth_B Y_A = \psi q_{AB}\,, \tag{5.38}$$

where $\psi = \eth_A Y^A$. The coefficient $l^{(0)}_{ur} = 0$ restricts f to be given by

$$f = \frac{1}{2}u\psi + \alpha\left(x^A\right)\,. \tag{5.39}$$

The functions f and Y^A have the same form as in general relativity. Thus, the different fall-off conditions of components of the the metric in Brans-Dicke theory do not alter the symmetries of the spacetime. The interpretation of Y^A and α will, therefore, be the same as in GR: the globally defined Y^A span a six-parameter algebra isomorphic to the proper, isochronous Lorentz algebra (and the locally defined Y^A will be the infinite-dimensional group of super-rotation symmetries [142]) and α span the infinite-dimensional commutative algebra of supertranslations [139, 141]. How the asymptotic Killing vectors $\vec{\xi}$ are extended into the interior of the spacetime from future null infinity is different in GR from in Brans-Dicke theory in the Jordan frame. This will lead to the functions in the metric transforming differently between the two theories.

Before we compute the transformation of the metric functions, it is necessary to determine how the functions λ_1 and λ_2 in the expansion of the scalar field transform as they are Lie dragged along $\vec{\xi}$. We denote this transformation as $\delta_\xi \lambda_1$ and $\delta_\xi \lambda_2$ and they can be computed from the $O(r^{-1})$ and $O(r^{-2})$ of $\mathcal{L}_\xi \lambda$, respectively. The result is

$$\delta_\xi \lambda_1 = \frac{1}{2}\lambda_1 \psi + Y^A \eth_A \lambda_1 + f \partial_u \lambda_1 \,, \tag{5.40}$$

$$\delta_\xi \lambda_2 = \lambda_2 \psi - \frac{1}{2}\lambda_1 \text{Ð}^2 f - \eth^C f \eth_C \lambda_1 + Y^D \eth_D \lambda_2 + f \partial_u \lambda_2 \,. \tag{5.41}$$

Next, we can compute how the functions c_{AB}, D_{AB}, $\mathcal{M}$, and L_A transform when Lie dragged along $\vec{\xi}$ given in Eq. (5.35). We denote these quantities $\delta_\xi c_{AB}$ and similarly for the other three functions. The term $\delta_\xi c_{AB}$ can be obtained directly from the appropriate coefficients and components of $l^{(n)}_{\mu\nu}$, but other terms require also removing the transformation of combinations of $\delta_\xi \lambda_1$ and $\delta_\xi c_{AB}$ that appear at the same order in the metric. The expressions used to compute these quantities are given below:

$$\delta_\xi c_{AB} = l^{(1)}_{AB} \tag{5.42a}$$

$$\delta_\xi \mathcal{M} = \frac{1}{2} l^{(-1)}_{uu} - \frac{1}{2}\left[\frac{\delta_\xi \lambda_1}{\lambda_0} + \frac{3}{2\lambda_0^2}\delta_\xi\left(\lambda_1 \partial_u \lambda_1\right)\right] \tag{5.42b}$$

$$\delta_\xi D_{AB} = l^{(0)}_{AB} - \frac{1}{4} q_{AB} \delta_\xi \left(c^{CD} c_{CD}\right) \tag{5.42c}$$

$$\begin{aligned}\delta_\xi L_A = &-\frac{1}{2} l^{(-1)}_{uA} + \frac{1}{12}\delta_\xi\left(c_{AB}\eth_C c^{BC}\right) - \frac{1}{6\lambda_0}\delta_\xi\left(\lambda_1 \eth_B c^B_A\right)\\ &+ \frac{1}{6\lambda_0^2}\delta_\xi\left(\lambda_1 \eth_A \lambda_1\right) - \frac{1}{12\lambda_0}\delta_\xi\left(c_{AB}\eth^B \lambda_1\right)\end{aligned} \tag{5.42d}$$

Thus, we can compute $\delta_\xi \mathcal{M}$ and $\delta_\xi c_{AB}$ from the relevant $l^{(n)}_{\mu\nu}$ and $\delta_\xi \lambda_1$ to be

$$\delta_\xi c_{AB} = \mathcal{L}_Y c_{AB} + f N_{AB} - \frac{1}{2}\psi c_{AB} - 2\eth_A \eth_B f + q_{AB} Ð^2 f\,, \tag{5.43a}$$

$$\begin{aligned}\delta_\xi \mathcal{M} = &f\partial_u \mathcal{M} + \frac{3}{2}\mathcal{M}\psi + Y^A \eth_A \mathcal{M} + \frac{1}{8}c^{AB}\eth_A\eth_B\psi + \frac{1}{2}\eth_A f \eth_B N^{AB} + \frac{1}{4}N^{AB}\eth_A\eth_B f\\ &+ \frac{1}{4\lambda_0}\eth_A\psi\eth^A\lambda_1 + \frac{1}{4\lambda_0}Ð^2 f\partial_u\lambda_1 + \frac{1}{2\lambda_0}\eth^A f\eth_A\partial_u\lambda_1 - \frac{\lambda_1\psi}{4\lambda_0}\,.\end{aligned} \tag{5.43b}$$

Then with the expression for $\delta_\xi c_{AB}$, it is possible to compute the remaining two terms for $\delta_\xi D_{AB}$ and $\delta_\xi L_A$. They are given by

$$\delta_\xi D_{AB} = \mathcal{L}_Y D_{AB} + \frac{\lambda_1}{\lambda_0}\left(\eth_A\eth_B - \frac{1}{2}q_{AB}Ð^2\right) f - \frac{1}{2\lambda_0} f\partial_u\left(\lambda_1 c_{AB}\right)\,, \tag{5.43c}$$

for $\delta_\xi D_{AB}$ and

$$\begin{aligned}\delta_\xi L_A \;=\;& f\partial_u L_A + \mathcal{L}_Y L_A + L_A\psi + \frac{1}{96}c^{CD}c_{CD}\eth_A\psi + \frac{1}{6}D_{AB}\eth^B\psi - \mathcal{M}\eth_A f\\ &-\frac{1}{8}\eth_A\left(c^{BC}\eth_B\eth_C f\right) + \frac{1}{4}\left(\eth^D\eth_C c_{AD} - \eth_A\eth^B c_{BC}\right)\eth^C f - \frac{1}{6}c_{AB}\eth^B f\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{6}c^{BC}N_{AB}\eth_C f - \frac{1}{12\lambda_0^2}\left[c_{AB}\lambda_0\eth^B f + \lambda_1\left(\omega_{\rm BD}+4\right)\eth_A f\right]\partial_u\lambda_1 \\
&-\frac{1}{12\lambda_0}\left(2\lambda_1 + 3\partial_u\lambda_2\right)\eth_A f - \frac{5}{24\lambda_0}\eth_A\left(\lambda_1\eth_C\eth^C f\right) + \frac{\lambda_1}{12\lambda_0}N_{AB}\eth^B f \\
&+\frac{1}{12}\left(\eth^C\eth_C f\eth^B c_{AB} - c_{AB}\eth^B\eth^C\eth_C f\right) - \frac{1}{6}\eth_B\eth_A f\eth_C c^{BC} - \frac{5}{48}c^{BC}N_{BC}\eth_A f \\
&+\frac{1}{24\lambda_0^2}\left[6\lambda_0\lambda_2 + \left(\omega_{\rm BD}-1\right)\lambda_1^2\right]\eth_A\psi - \frac{1}{12\lambda_0}\left(\eth_B\eth_A f\eth^B\lambda_1 + 3\eth_B\eth_A\lambda_1\eth^B f\right)\,,
\end{aligned}
\tag{5.43d}
$$

for $\delta_\xi L_A$. In deriving the expression for $\delta_\xi\mathcal{M}$, we used the properties $\eth^2\psi = -2\psi$ and $\eth^2 Y^A = -Y^A$. To derive $\delta_\xi L_A$, we also used the identities in [143]

$$
2c_{C(A}\eth_{B)}\eth^C f - q_{AB}c^{CD}\eth_C\eth_D f - c_{AB}\eth^2 f = 0\,, \tag{5.44a}
$$

$$
2\eth^C c_{C(A}\eth_{B)}f + 2\eth_{(A}c_{B)C}\eth^C f - 2\eth_C c_{AB}\eth^C f - 2q_{AB}\eth_C f\eth_D c^{CD} = 0\,. \tag{5.44b}
$$

The expressions in Eq. (5.43) will be useful for understanding the properties of metric in nonradiative regions, which we discuss in Sec. 5.4 soon hereafter. The GR limit of our expressions agrees with the equivalent results in [144] after taking into account differences in conventions. Our results are similar to those in [403], but not identical, because of the different gauge conditions that we use.

Before concluding this part, we note that because the scalar field appears in the metric, we can check whether the transformation of the metric is consistent with requiring that the scalar field is Lie dragged along $\vec{\xi}$. We can obtain $\delta_\xi\lambda_1$ from $2\lambda_0 l_{ur}^{(-1)}$, and we find that it agrees with Eq. (5.40). We can also obtain $\delta_\xi\left(\partial_u\lambda_1\right)$ from $-\lambda_0 l_{uu}^{(0)}$ and we find that it is equivalent to $\partial_u(\delta_\xi\lambda_1)$, as it should be.

We can also explicitly compute the quantities $\delta_\xi(c^{AB}c_{AB})$ and $\delta_\xi(\eth_B c^{AB})$ from Lie dragging the metric. The relevant expressions for computing this are

$$\delta_\xi\left(c^{AB}c_{AB}\right) = 16 l_{ur}^{(-2)} + \frac{4}{\lambda_0^2}\left(3-\omega_{\text{BD}}\right)\delta_\xi(\lambda_1)^2 - \frac{24}{\lambda_0}\delta_\xi\lambda_2 \tag{5.45a}$$

$$\delta_\xi\left(\eth^B c_{AB}\right) = 2 l_{uA}^{(0)} + \frac{1}{\lambda_0}\delta_\xi\left(\eth_A\lambda_1\right) \tag{5.45b}$$

Not surprisingly, we find that

$$\delta_\xi\left(c^{AB}c_{AB}\right) = \delta_\xi c^{AB}c_{AB} + c^{AB}\delta_\xi c_{AB}\,, \tag{5.46a}$$

$$\delta_\xi\left(\eth^B c_{AB}\right) = \eth^B\left(\delta_\xi c_{AB}\right)\,, \tag{5.46b}$$

as the latter relation was proven in GR in [143]. For completeness, we give the expressions for these terms here

$$\delta_\xi\left(c^{AB}c_{AB}\right) = 2fN_{AB}c^{AB} + c_{AB}c^{AB}\psi + 2c^{BC}Y^A\eth_A c_{BC} - 4c^{AB}\eth_A\eth_B f\,, \tag{5.47a}$$

$$\delta_\xi\left(\eth^B c_{AB}\right) = -\,\eth_A(\text{Đ}^2+2)f - \frac{1}{2}c_{AB}\eth^B\psi + \frac{1}{2}\psi\eth^B c_{AB} + \mathcal{L}_Y\eth^C c_{AC} + \eth^B\left(fN_{AB}\right)\,. \tag{5.47b}$$

It does not seem possible to verify these types of relationships with all of the terms that appear in Eq. (5.42). Thus, for example, for the term $\delta_\xi\left(c_{AB}\eth_C c^{BC}\right)$, we assumed it can be written as the sum of $\delta_\xi c_{AB}\eth_C c^{BC}$ and $c_{AB}\delta_\xi\eth_C c^{BC}$.

5.4 Gravitational-wave Memory Effects

Gravitational-wave (GW) memory effects are commonly defined for bursts of gravitational waves of finite duration between two nonradiative regions before and after the burst; they are also defined for sources of gravitational waves that become asymptotically nonradiative in the limits as $u \to \pm\infty$ at large Bondi radius r. In either case, discussing GW memory effects requires a notion of a nonradiative region of spacetime. In this section, we first describe the properties of nonradiative regions in Brans-Dicke theory, we then discuss the measurement of GW memory effects through geodesic deviation, and we finally discuss how the conservation equations constrain the GW memory effects (thereby allowing them to be computed approximately).

5.4.1 Nonradiative and stationary regions

Nonradiative regions For general relativity, it is typical to consider regions of vanishing Bondi news N_{AB}, and vanishing stress-energy tensor. In the context of Brans-Dicke theory, we will instead consider regions where $N_{AB} = 0$, $\partial_u \lambda_1 = 0$, and any other stress-energy from matter fields vanishes. These two equations imply that λ_1 and c_{AB} must be independent of u. Integrating Eqs. (5.26), (5.25), (5.32a) and (5.33), we can then show that λ_2, D_{AB}, $\mathcal{M}$, L_A, and λ_3 are given by

$$\lambda_1 = \lambda_1^{(0)}(x^A)\,, \tag{5.48a}$$

$$c_{AB} = c_{AB}^{(0)}(x^C)\,, \tag{5.48b}$$

$$\lambda_2 = -\frac{u}{2}\eth^2\lambda_1^{(0)} + \lambda_2^{(0)}(x^B)\,, \tag{5.48c}$$

$$D_{AB} = -\frac{1}{2\lambda_0}\lambda_1^{(0)} c_{AB}^{(0)}\,, \tag{5.48d}$$

$$\mathcal{M} = \mathcal{M}^{(0)}(x^A)\,, \tag{5.48e}$$

$$L_A = -\frac{u}{3}\eth_A\mathcal{M}^{(0)} - \frac{u}{24\lambda_0}\eth_A\eth^2\lambda_1^{(0)} + \frac{u}{12}\eth^D(\eth_D\eth^B c_{AB}^{(0)}$$

$$- \eth_A \eth^B c^{(0)}_{DB}) + L^{(0)}_A(x^E)\,, \tag{5.48f}$$

$$\begin{aligned}\lambda_3 = & \frac{u^2}{16\lambda_0}(Ð^2+2)Ð^2\lambda_1^{(0)} + \frac{u}{2}\Big[-\frac{1}{2}(Ð^2+2)\lambda_2^{(0)} \\ & + \mathcal{M}^{(0)}\lambda_1^{(0)} + \frac{1}{4\lambda_0}\lambda_1^{(0)}Ð^2\lambda_1^{(0)} + \frac{1}{2}c^{AB}_{(0)}\eth_A\eth_B\lambda_1^{(0)} \\ & + \frac{1}{4}\lambda_1^{(0)}\eth_A\eth_B c^{AB}_{(0)} + \eth_B c^{AB}_{(0)}\eth_A\lambda_1^{(0)}\Big] + \lambda_3^{(0)}(x^C)\,. \end{aligned} \tag{5.48g}$$

In a nonradiative region, E_{AB} has the form

$$E_{AB} = u^2 E^{(2)}_{AB} + uE^{(1)}_{AB} + E^{(0)}_{AB}(x^C)\,, \tag{5.49a}$$

where the coefficients $E^{(2)}_{AB}$ and $E^{(1)}_{AB}$ are given by

$$\begin{aligned} E^{(2)}_{AB} = & \frac{1}{6}\left(\eth_A\eth_B - \frac{1}{2}q_{AB}Ð^2\right)\left(\mathcal{M}^{(0)} - \frac{1}{4\lambda_0}Ð^2\lambda_1^{(0)}\right) \\ & - \frac{1}{24}\eth_{(A}\epsilon_{B)C}\eth^C(\epsilon^{DE}\eth_E\eth^F c^{(0)}_{DF})\,, \end{aligned} \tag{5.49b}$$

$$\begin{aligned} E^{(1)}_{AB} = & -\eth_{(A}L^{(0)}_{B)} + \frac{1}{2}q_{AB}\eth^C L^{(0)}_C + \frac{1}{2}\mathcal{M}^{(0)}c^{(0)}_{AB} + \frac{1}{8}\epsilon^C{}_{(A}c^{(0)}_{B)C}(\epsilon^{DE}\eth_E\eth^F c^{(0)}_{DF}) \\ & + \frac{1}{6}\eth_{(A}\left(c^{(0)}_{B)C}\eth_D c^{DC}_{(0)}\right) - \frac{1}{12}q_{AB}\eth^D(c^{(0)}_{DC}\eth_E c^{EC}_{(0)}) \\ & + \frac{1}{32}\left(\eth_A\eth_B - \frac{1}{2}q_{AB}Ð^2\right)(c^{(0)}_{CD}c^{CD}_{(0)}) - \frac{1}{12\lambda_0}\lambda_1^{(0)}c^{(0)}_{AB} \\ & + \frac{1}{24\lambda_0}\lambda_1^{(0)}Ð^2 c^{(0)}_{AB} + \frac{1}{12\lambda_0}Ð^2\lambda_1^{(0)}c^{(0)}_{AB} + \frac{1}{12\lambda_0}\eth^C\lambda_1^{(0)}\eth_C c^{(0)}_{AB} \\ & - \frac{1}{6\lambda_0}\eth_{(A}c^{(0)}_{B)C}\eth^C\lambda_1^{(0)} + \frac{1}{12\lambda_0}q_{AB}\eth^D c^{(0)}_{DC}\eth^C\lambda_1^{(0)} \\ & + \frac{2+3\omega_{\rm BD}}{12(\lambda_0)^2}\lambda_1^{(0)}\left(\eth_A\eth_B - \frac{1}{2}q_{AB}Ð^2\right)\lambda_1^{(0)} + \frac{1}{4\lambda_0}\left(\eth_A\eth_B - \frac{1}{2}q_{AB}Ð^2\right)\lambda_2^{(0)} \\ & - \frac{3\omega_{\rm BD}+7}{12(\lambda_0)^2}\left(\eth_A\lambda_1^{(0)}\eth_B\lambda_1^{(0)} - \frac{1}{2}q_{AB}\eth_C\lambda_1^{(0)}\eth^C\lambda_1^{(0)}\right)\,, \end{aligned} \tag{5.49c}$$

$$E^{(0)}_{AB} = E^{(0)}_{AB}(x^C)\,. \tag{5.49d}$$

This expression will simplify considerably in some more restrictive classes of nonradiative solutions that we discuss next.

Stationary regions and the canonical frame In general relativity, there are frames for stationary regions in which the Bondi metric functions are independent of u. We next discuss how the metric functions and scalar field in nonradiative regions in Brans-Dicke theory [given in Eq. (5.48)] simplify when there exist such frames in the Jordan frame. To discuss this, it is useful to recall that a vector field, such as L_A, on the 2-sphere can be decomposed into divergence- and curl-free parts as follows

$$L_A = \eth_A \rho + \epsilon_{AB} \eth^B \sigma \,. \tag{5.50}$$

Similarly, a symmetric trace-free tensor like c_{AB} can be decomposed as [381, 410]

$$c_{AB} = \left(\eth_A \eth_B - \frac{1}{2} q_{AB} \eth_C \eth^C \right) \Theta + \epsilon_{C(A} \eth_{B)} \eth^C \Psi \,. \tag{5.51}$$

The terms without the antisymmetric tensor ϵ_{AB} in the last two equations are often called the "electric (parity)" part and the terms with ϵ_{AB} are called the "magnetic (parity)" part.

If we require that the scalar field is also independent of u in these regions, then the expression for λ_2 in Eq. (5.48c) requires that $\text{Ð}^2 \lambda_1 = 0$, or namely λ_1 is constant. The expression for L_A in Eq. (5.48f) shows that the magnetic part of $c^{(0)}_{AB}$, Ψ, vanishes (see, e.g., [153]). Together with the fact that $\lambda^{(0)}_1$ is constant, it also follows from Eq. (5.48f) that $\mathcal{M}^{(0)} = M^{(0)}$ is a constant. Because c_{AB} is an electric-parity tensor field, then it can be set to zero by performing a supertranslation with $\alpha = \Theta/2$ [see Eq. (5.43a)]. With $c_{AB} = 0$ as well as $\lambda_1^{(0)}$ and $\mathcal{M}^{(0)}$ being constant, then by requiring λ_3 is independent of time Eq. (5.48g) gives the following condition on $\lambda_2^{(0)}$:

$$(\eth^2 + 2)\lambda_2^{(0)} = 2\mathcal{M}^{(0)}\lambda_1^{(0)} \,. \tag{5.52}$$

This nonhomogeneous linear elliptic equation can be written as the sum of the particular solution $\lambda_2^{(0)} = \mathcal{M}^{(0)}\lambda_1^{(0)}$ and a linear combination of the solutions to the homogeneous equation

$$(\eth^2 + 2)\lambda_2^{(0)} = 0 \,. \tag{5.53}$$

The solution of the homogeneous equation is a superposition of $\ell = 1$ spherical harmonics. Finally, with these conditions on the metric functions, this greatly simplifies the form of E_{AB} in Eq. (5.49a). That $\lambda_1^{(0)}$ and $\mathcal{M}^{(0)}$ are constants and that $c_{AB}^{(0)}$ vanishes cause the coefficient in Eq. (5.49b) to vanish; similarly, the lengthy expression in Eq. (5.49c) reduces to the following much simpler equation:

$$\eth_{(A}L_{B)}^{(0)} - \frac{1}{2}q_{AB}\eth^C L_C^{(0)} = 0 \,. \tag{5.54}$$

To have smooth solutions $L_A^{(0)}$, then it must be a superposition of the six electric-parity and magnetic-parity $\ell = 1$ vector spherical harmonics. The electric part of $L_A^{(0)}$ can be set to zero by performing a translation with $\alpha = \kappa/[\mathcal{M}^{(0)} - \lambda_1^{(0)}/(4\lambda_0)]$ The magnetic part could be chosen to align with a particular axis by performing a rotation if desired.

Like in general relativity, this class of stationary regions in Brans-Dicke theory admit a "canonical" frame, in which $\mathcal{M}$ and λ_1 are constant, $c_{AB} = 0$, and L_A is composed of $\ell = 1$ magnetic-parity vector harmonics. Furthermore, the scalar-field function λ_1 is also constant, and the function λ_2 is equal to the constant $\mathcal{M}\lambda_1$ plus a superposition of $l = 1$ spherical harmonics. For bursts of gravitational and scalar radiation, there can be transitions between such stationary regions where the initial stationary region is in the canonical frame, but the final stationary region is supertranslated from its canonical frame, so that c_{AB} is nonzero. This nonzero c_{AB} at late times is, in essence, the GW memory effect (see e.g., [155]); thus, transitions between these stationary regions provide a sufficiently general arena in which to study certain types of GW memory effects in general relativity (these transitions were called "BMS vacuum transitions in [155]). Note that these types of transitions also do not allow "ordinary" memory [411], so they do not admit memory effects of full generality (see, e.g., [412]).

However, in Brans-Dicke theory, because λ_1 must be a constant in both stationary regions in the canonical frames, such a transition would significantly restrict the types of possible memory effects that could occur. For the memory effects related to the scalar radiation (discussed in greater detail in the next part) such a transition would only allow these scalar-type memory effects to have a uniform sky pattern. As a result, considering only these types of transitions between these frames will not be sufficiently general to explore the full range of possible memory effects in Brans-Dicke theory. A slight generalization would be to consider transitions between stationary regions in which one of the regions is both boosted and supertranslated from the canonical frame. However, this still seems to be a somewhat restrictive scenario, as it does not seem to admit solutions that are superpositions of boosted massive bodies with a scalar charge. As a result, we will next focus on a slightly more general set of frames, that is still somewhat simpler than the nonradiative regions without restrictions.

Nonradiative regions with vanishing magnetic-parity shear For a slightly more general set of solutions, though which lack the full generality of the nonradiative regions, we will consider regions of spacetime with vanishing stress-energy (not including the scalar field), N_{AB}, $\partial_u \lambda_1$, and Ψ (the magnetic-parity part of the shear). Like the stationary regions, it will again be possible to set $c_{AB} = 0$ by a supertranslation to produce a "semi-canonical" frame; however, the mass-aspect and scalar-field functions $\mathcal{M}^{(0)}$ and $\lambda_1^{(0)}$ will no longer be constants, and will remain arbitrary functions of x^A as in Eq. (5.48) in this frame. This will imply that λ_2 depends linearly on u as in Eq. (5.48c), D_{AB} will vanish in this semi-canonical frame, and the electric part of L_A will depend linearly on u, whereas the part independent of u will contain both electric and magnetic parts. Thus, transitions between nonradiative regions of this type should be sufficiently general to capture both the usual tensor-type and scalar-type memory effects, which will be discussed in greater detail below. This was also the scenario considered by [403].

5.4.2 Geodesic deviation and GW memory effects

GW memory effects are frequently described by their effects on families of nearby freely falling observers at large distances r from a source of gravitational waves [130, 413, 414]. The deviation vector $\vec{X}$ between a geodesic with tangent $\vec{u}$ and a nearby geodesic satisfies the equation of geodesic deviation

$$u^\gamma \nabla_\gamma (u^\beta \nabla_\beta X^\alpha) = -R_{\beta\gamma\delta}{}^\alpha u^\beta X^\gamma u^\delta \,, \tag{5.55}$$

to linear order in the deviation vector X^α. It is then useful to expand the vector X^α in terms of an orthonormal triad $e^\alpha_{\hat{i}}$ with $e^\alpha_{\hat{i}} u_\alpha = 0$ that is parallel transported along the geodesic with tangent u^α. If u^α is denoted by $e^\alpha_{\hat{0}}$, then $e^\alpha_{\hat{\mu}} = \{e^\alpha_{\hat{0}}, e^\alpha_{\hat{i}}\}$ forms an orthonormal tetrad. It is also convenient to introduce a dual triad $e^{\hat{j}}_\alpha$ with $e^\alpha_{\hat{i}} e^{\hat{j}}_\alpha = \delta^{\hat{j}}_{\hat{i}}$. The vector can then be written in the form $X^\alpha = X^{\hat{i}}(\tau) e^\alpha_{\hat{i}}$, where τ is the proper time along the geodesic worldline. The equation of geodesic deviation then reduces to the expression

$$\ddot{X}^{\hat{i}} = -R_{\hat{0}\hat{j}\hat{0}}{}^{\hat{i}} X^{\hat{j}} \tag{5.56}$$

where the dot denotes $d/d\tau$. Given a set of tetrad coefficients $X^{\hat{i}}_0 = X^{\hat{i}}(\tau_0)$ and $\dot{X}^{\hat{i}}_0 = \dot{X}^{\hat{i}}(\tau_0)$ that represent the initial separation and relative velocity of the nearby geodesics, it is possible to solve for the change in the final values of the tetrad coefficients of the separation vector, which we denote by

$$\Delta X^{\hat{i}} = X^{\hat{i}}(\tau_f) - X^{\hat{i}}(\tau_0) \,. \tag{5.57}$$

We then expand this vector in a series to linear order in the Riemann tensor as

$$\Delta X^{\hat{i}} = \Delta X^{\hat{i}}_{(0)} + \Delta X^{\hat{i}}_{(1)} \,, \tag{5.58}$$

where the value $\Delta X^{\hat{i}}_{(0)}$ is identical to the expected result in flat spacetime

$$\Delta X^{\hat{i}}_{(0)} = (\tau_f - \tau_0)\dot{X}^{\hat{i}}_0 \,. \tag{5.59}$$

The correction $\Delta X^{\hat{i}}_{(1)}$ to the deviation vector to linear order in the Riemann tensor is given by [390]

$$\Delta X^{\hat{i}}_{(1)} = - X^{\hat{j}}_0 \int_{\tau_0}^{\tau_f} d\tau \int_{\tau_0}^{\tau} d\tau' R_{\hat{0}\hat{j}\hat{0}}{}^{\hat{i}} - \dot{X}^{\hat{j}}_0 \int_{\tau_0}^{\tau_f} d\tau \int_{\tau_0}^{\tau} d\tau' \int_{\tau'}^{\tau} d\tau'' R_{\hat{0}\hat{j}\hat{0}}{}^{\hat{i}} \,. \tag{5.60}$$

Note that in the triple integral, the limits of integration on the innermost integral run from τ' to τ.

To compute $\Delta X^{\hat{i}}$ associated with a burst of gravitational waves at large distances r from a source of GWs (and thereby compute the GW memory effects), it will be necessary to compute the leading $1/r$ parts of the Riemann tensor components $R_{\hat{0}\hat{j}\hat{0}}{}^{\hat{i}}$, the geodesic with tangent u^α, the infinitesimal element of proper time $d\tau$, and the orthonormal triad $e^\alpha_{\hat{i}}$. In Bondi coordinates, with V given by Eq. (5.23c), a vector $\vec{u} = \vec{e}_{\hat{0}}$ that is tangent to a timelike geodesic to leading order in $1/r$ is given by

$$\vec{u} = \vec{\partial}_u - \frac{1}{2\lambda_0}\dot{\lambda}_1 \vec{\partial}_r + O(r^{-1}) \,. \tag{5.61a}$$

The retarded time u is the proper time τ along the geodesic at this order. A useful triad is given by

$$\vec{e}_{\hat{r}} = \vec{\partial}_u - \left(1 + \frac{1}{2\lambda_0}\dot{\lambda}_1\right)\vec{\partial}_r + O(r^{-1})\,, \tag{5.61b}$$

$$\vec{e}_{\hat{A}} = \frac{1}{r}\vec{\mathsf{e}}_{\hat{A}} + O(r^{-2})\,, \tag{5.61c}$$

where $\vec{\mathsf{e}}_{\hat{A}}$ is an orthonormal dyad associated with the metric q_{AB}. The nonzero tetrad components of the Riemann tensor at $O(r^{-1})$ are given by

$$R_{\hat{0}\hat{A}\hat{0}\hat{B}} = -\frac{1}{2r}\ddot{c}_{\hat{A}\hat{B}} + \frac{1}{2\lambda_0 r}\delta_{\hat{A}\hat{B}}\ddot{\lambda}_1 + O\left(r^{-2}\right)\,. \tag{5.62}$$

Note that if the Riemann tensor is decomposed into its Weyl and Ricci parts, the relevant nonzero components are given by

$$C_{\hat{0}\hat{A}\hat{0}\hat{B}} = -\frac{1}{2r}\ddot{c}_{\hat{A}\hat{B}} + O\left(r^{-2}\right)\,, \tag{5.63a}$$

$$R_{\hat{0}\hat{0}} = \frac{1}{\lambda_0 r}\ddot{\lambda}_1 + O\left(r^{-2}\right)\,. \tag{5.63b}$$

It then follows that the Ricci scalar, R, satisfies $R = O\left(r^{-2}\right)$.

Putting these results together, we find that the $O(r^0)$ part of $\Delta X^{\hat{i}}$ is the same as the flat-space result in Eq. (5.59), and the $O(r^{-1})$ part is given by

$$\begin{aligned}\Delta X^{(1)}_{\hat{A}} = {} & \frac{1}{2r}\left(\Delta c_{\hat{A}\hat{B}} - \frac{1}{\lambda_0}\Delta\lambda_1\delta_{\hat{A}\hat{B}}\right)X_0^{\hat{B}} - \frac{1}{r}\left(\Delta\mathcal{C}_{\hat{A}\hat{B}} - \frac{1}{\lambda_0}\Delta\Lambda_1\delta_{\hat{A}\hat{B}}\right)\dot{X}_0^{\hat{B}} \\ & + \frac{1}{2r}\Delta\left[u c_{\hat{A}\hat{B}}(u) - \frac{1}{\lambda_0}u\lambda_1(u)\delta_{\hat{A}\hat{B}}\right]\dot{X}_0^{\hat{B}} + \frac{\Delta u}{2r}\left[c_{\hat{A}\hat{B}}(u_0) - \frac{1}{\lambda_0}\lambda_1(u_0)\delta_{\hat{A}\hat{B}}\right]\dot{X}_0^{\hat{B}} \\ & - \frac{u_0}{2r}\left(\Delta c_{\hat{A}\hat{B}} - \frac{1}{\lambda_0}\Delta\lambda_1\delta_{\hat{A}\hat{B}}\right)\dot{X}_0^{\hat{B}}\,. \end{aligned} \tag{5.64}$$

We have defined $\Delta u = u_f - u_0$,

$$\Delta\mathcal{C}_{\hat{A}\hat{B}} = \int_{u_0}^{u_f} du\, c_{\hat{A}\hat{B}}\,, \quad \text{and} \quad \Delta\Lambda_1 = \int_{u_0}^{u_f} du\, \lambda_1\,; \tag{5.65}$$

in the third line of Eq. (5.64), the Δ of the quantity in square brackets means to take the difference of the quantity within the brackets at $u = u_f$ and $u = u_0$. Equation (5.64) contains (in addition to initial and final data) six memory effects, which we will now discuss in greater detail (or in the language of [390, 415] six persistent observables, three of which are memory effects).

The first two collections of effects, $\Delta c_{\hat{A}\hat{B}}$ and $\Delta\mathcal{C}_{\hat{A}\hat{B}}$, have the same type of effect on nearby freely falling observers as GW memory effects in GR: namely, they produce a shearing (transverse to the propagation direction of the gravitational waves) of an initially circular congruence of geodesics after a burst of GWs pass. The tensor $\Delta c_{\hat{A}\hat{B}}$ was the first type of GW memory effect identified in calculations, and it produces a lasting change in the deviation vector between initially comoving observers. When $\Delta c_{\hat{A}\hat{B}}$ is nonvanishing, then the tensor $\Delta\mathcal{C}_{\hat{A}\hat{B}}$ will be the sum of a term that grows with u after the burst passes and a term $\Delta\mathcal{C}^{(0)}_{\hat{A}\hat{B}}$ that is independent of u. For observers with an initial relative velocity, this will cause $\Delta X^{(1)}_{\hat{A}}$ to have a shearing part that grows linearly with u after the GWs pass (this effect is also sometimes called the "subleading displacement memory"). The electric- and magnetic-parity parts of the tensor $\Delta\mathcal{C}^{(0)}_{\hat{A}\hat{B}}$ are closely related to the spin and center-of-mass (CM) GW memory effects, that were more recently identified. The tensor $\Delta c_{\hat{A}\hat{B}}$ was frequently described as being of electric parity, but it was shown that there are sources of stress-energy that can produce a magnetic-parity $\Delta c_{\hat{A}\hat{B}}$ [410, 412, 416].

The second two terms, $\Delta\lambda_1$ and $\Delta\Lambda_1$, are memory effects related to the passage of the scalar field. These effects cause an initially circular congruence of geodesics to undergo a relative uniform expansion (or contraction) in the direction transverse to the propagation direction of the scalar radiation.[7] Thus, for initially comoving observers, a nonzero $\Delta\lambda_1$ would cause a uniform, transverse change in $\Delta X^{(1)}_{\hat{A}}$. When $\Delta\lambda_1$ is nonvanishing, then $\Delta\Lambda_1$ would also be a sum of term that grows with u after the burst of scalar field and a term $\Delta\Lambda_1^{(0)}$ that is independent of u; thus, the deviation vector $\Delta X^{(1)}_{\hat{A}}$ would have an expanding (or contracting) part that grows linearly with u for observers with an initial relative velocity. The scalar-field memory effect $\Delta\lambda_1$ had been discussed in the context of post-Newtonian theory in [179] or in gravitational collapse in [397, 398], for example. The quantity $\Delta\Lambda_1^{(0)}$ is the scalar-field analog of the CM memory, and it seems to have not been discussed previously.

We turn in the next part of this section to how the different memory scalars and tensors—$\Delta c_{\hat{A}\hat{B}}$, $\Delta\mathcal{C}_{\hat{A}\hat{B}}$, $\Delta\lambda_1$ and $\Delta\Lambda_1$—are constrained (or not constrained) by the asymptotic field equations of Brans-Dicke theory and the properties of the nonradia-

[7] It is possible to define a suitably adapted Newman-Penrose tetrad [417] with $l_\mu = \nabla_\mu u$ and with a complex dyad chosen to have only its 2-sphere indices nonvanishing and to be normalized to one: $m_A\bar{m}^A = 1$. The spin coefficient $\rho = -m^\mu\bar{m}^\nu\nabla_\nu l_\mu$ then can be expanded at large Bondi radius r in this tetrad as

$$\rho = -\frac{1}{r^2}m^A\bar{m}^B\nabla_B\nabla_A u = \frac{1}{r} + \frac{\lambda_1}{2\lambda_0 r^2} + O(r^{-3})\,. \tag{5.66}$$

As ρ is one of the "optical scalars" and its real part corresponds to the expansion of a congruence to which l^μ is tangent, this provides a second geometrical viewpoint on how λ_1 causes a type of expansion at large r.

tive regions before and after the passages of the gravitational waves and the radiative scalar field.

5.4.3 Constraints on GW memory effects from fluxes of conserved quantities

Memory effects were defined in [390] to be the subset of the persistent observables that are associated with symmetries and conserved quantities at spacetime boundaries, like null infinity. A commonly used procedure for computing these conserved quantities related to symmetries is due to Wald and Zoupas [151], who computed the "conserved" quantities associated with BMS symmetries at null infinity in vacuum general relativity. The word "conserved" is used in quotes, because these quantities (also called "charges") are not constant along cross-sections (or "cuts") of null infinity, but change so that the difference of the charges between the cuts is equal to the flux of the charge integrated over the region of null infinity between the cuts. The flux had been computed previously by Ashtekar and Streubel [149], and it is consistent with the result in [151]. In Bondi coordinates and in general relativity, the change in the charges, ΔQ_ξ, can be concisely expressed by the expression

$$\Delta Q_\xi = -\frac{1}{32\pi G}\int du\, d^2\Omega\, N^{AB}\delta_\xi c_{AB}\,, \tag{5.67}$$

where $\delta_\xi c_{AB}$ is given in Eq. (5.43a) and $d^2\Omega = \sqrt{q}dx^1dx^2$ is the two-dimensional volume element associated with the metric q_{AB} (see, e.g. [153]). The charge is given by the Komar formula [418], with an additional prescription needed to make the charge integrable in radiative regions.

The formalism for computing conserved quantities outlined in [151] can be applied to a large class of gravitational theories that can be derived from a Lagrangian, such as Brans-Dicke theory. In the Einstein frame, the action has the form of the Einstein-Klein-Gordon theory. Wald and Zoupas noted in [151] that having a minimally coupled scalar field causes stress-energy terms to be added to the flux, but will otherwise not greatly change the charges. However, they posited that $r\Phi$ has a finite limit to null infinity, which would require that $\Phi_0 = 0$. We checked whether having a constant Φ_0 that is nonzero would alter the flux, and because this nonzero Φ_0 is constant, and we found that it did not. Wald and Zoupas also mentioned in [151] that a conformally coupled scalar field, such as in Brans-Dicke theory in the Jordan frame, would also only add terms to the flux. However, they did not specify whether the kinetic term for the scalar field must have the canonical form (as in the Einstein frame), which it does not in the Jordan frame. Consequently, we computed the flux of the charges associated with a BMS symmetry in the Jordan frame in Bondi coordinates. We found that the integral of the flux over a region of future null infinity is given by

$$\Delta Q_{\vec{\xi}} = -\frac{\lambda_0}{32\pi}\int du\, d^2\Omega \left[N^{AB}\delta_\xi c_{AB} + \frac{6+4\omega_{\text{BD}}}{(\lambda_0)^2}\partial_u\lambda_1\delta_\xi\lambda_1\right]. \tag{5.68}$$

Note that by combining Eqs. (5.2c) and (5.11), expanding λ as in Eq. (5.21), and using Eq. (5.40), then we find that the term $(3+2\omega_{\text{BD}})\partial_u\lambda\delta_\xi\lambda/(16\pi)$ in Eq. (5.68) is the $O(r^{-2})$ part of $\lambda_0 T^{(\Phi)}_{u\nu}\xi^\nu$. Thus, the result is consistent with the expectations of Wald and Zoupas for a conformally coupled scalar field, despite the noncanonical form of the kinetic term for λ (and the flux is then conformally invariant as required in [151]).

5.4.3.1 Displacement memory and electric-parity part of Δc_{AB}

For computing the GW memory effect connected with the electric-parity part of $\Delta c_{\hat{A}\hat{B}}$, we should specialize the flux expression for a supertranslation vector field $\vec{\xi} = \alpha(x^A)\vec{\partial}_u$. Restricting Eqs. (5.43a) and (5.40) to a supertranslation, and integrating by parts for terms involving $\eth_A$ (there are no boundary terms on the 2-sphere), we can show that the expression in Eq. (5.68) can be written as

$$\Delta Q_{(\alpha)} = -\frac{\lambda_0}{32\pi}\int du\, d^2\Omega\, \alpha\left[N_{AB}N^{AB} - 2\eth_A\eth_B N^{AB} + \frac{6+4\omega_{\rm BD}}{(\lambda_0)^2}(\partial_u\lambda_1)^2\right]. \quad (5.69)$$

It will next be useful to make a few definitions and to relate some of the quantities in Eq. (5.69) to quantities that we have computed earlier in the chapter.

The term proportional to $\eth_A\eth_B N^{AB}$ depends only on the electric part of N_{AB} (and thus the electric part of Δc_{AB}, when the integral with respect to u is performed). With the definition of c_{AB} in Eq. (5.51) and of the news tensor in Eq. (5.29), we can write the term $\eth_A\eth_B N^{AB}$ as

$$2\eth_A\eth_B N^{AB} = \text{Đ}^2(\text{Đ}^2+2)\partial_u\Theta\,. \quad (5.70)$$

With the equation for the Bondi mass aspect (5.32a), it is possible to show that the supertranslation charge (i.e., the supermomentum) needed to satisfy Eq. (5.69) is given by

$$Q_{(\alpha)} = \frac{\lambda_0}{4\pi} \int d^2\Omega\, \alpha \left(\mathcal{M} - \frac{1}{4\lambda_0} \eth^2 \lambda_1 \right) . \tag{5.71}$$

Note that when $\alpha = 1$, this corresponds to a time translation, and the associated conserved charge is the energy. Solutions of physical interest have non-negative energy. Because $\eth^2\lambda_1$ vanishes when integrated over the 2-sphere, this implies that the integral of $\mathcal{M}$ over S^2 must be non-negative, or the 2-sphere integral of M must be greater than or equal to the same integral of $\frac{1}{4\lambda_0^2}\lambda_1 \partial_u \lambda_1$. Finally, it will be helpful to define α times the change in the energy radiated by the gravitational waves and the scalar field λ as

$$\Delta\mathscr{E}_{(\alpha)} = \frac{\lambda_0}{32\pi} \int du\, d^2\Omega\, \alpha \left[N_{AB} N^{AB} + \frac{6 + 4\omega_{\text{BD}}}{(\lambda_0)^2} (\partial_u \lambda_1)^2 \right] . \tag{5.72}$$

Then Eq. (5.69) can be written as

$$\int d^2\Omega\, \alpha\, \eth^2(\eth^2 + 2)\Delta\Theta = \frac{32\pi}{\lambda_0} \left(\Delta\mathscr{E}_{(\alpha)} + \Delta Q_{(\alpha)} \right) . \tag{5.73}$$

The supertranslations α are allowed to be any smooth function on the 2-sphere. By choosing for α an appropriate basis of functions that span this space of smooth functions on S^2 (e.g., spherical harmonics), it is then possible to use Eq. (5.73) to determine the coefficients of $\Delta\Theta$ expanded in these basis functions in terms of the expansion coefficients of the energy flux and the change in the supermomentum charges. In other words, supposing that the energy flux $\Delta\mathscr{E}_{(\alpha)}$ is known for some given radiative data N_{AB} and $\partial_u\lambda_1$, and that the early- and late-time nonradiative data through

$\Delta\mathcal{M}$ and $\Delta\lambda_1$ are also known, then it is possible to determine the corresponding electric-parity memory effect in Δc_{AB}. The computation of this memory effect is not substantially different from in general relativity; the main difference is that it is necessary to provide both radiative ($\partial_u\lambda_1$) and nonradiative ($\Delta\lambda_1$) data for the scalar field, in addition to the radiative (N_{AB}) and nonradiative ($\Delta\mathcal{M}$) gravitational data. We will use this procedure to calculate the memory effect from compact binaries in Brans-Dicke theory in future work.

The two types of sources of GW memory in Eq. (5.73)—i.e., $\Delta\mathscr{E}_{(\alpha)}$ and $\Delta Q_{(\alpha)}$—are often called "null" and "ordinary" memory, respectively in general relativity [411]. The word "null" refers to the fact that it is sourced by massless fields (including gravitational waves), and the word "ordinary" refers to the fact that it is sourced by "ordinary" massive particles (and fields). The specific components of the spacetime curvature and matter stress-energy tensor responsible for producing the ordinary and null memory are distinct and distinguishable in GR. How to classify the contributions of a scalar field to the ordinary and null memory is not as immediately obvious in Brans-Dicke theory as it is in GR, because (i) massive objects can have "scalar charges" (nontrivial stationary scalar field configurations of the massless scalar) in Brans-Dicke theory, and (ii) the radiative and the static parts of the scalar field both appear at leading order in $1/r$. While in GR all terms involving the scalar field would be treated as null memory, in Brans-Dicke theory, we will consider one part of the scalar field to contribute to the null memory and another part to contribute to the ordinary memory. Specifically, because the term quadratic in $\partial_u \lambda_1$ in the energy flux $\Delta\mathscr{E}_{(\alpha)}$ has the form of a flux of energy per solid angle, we will consider it to be null memory. The term proportional to $\eth^2 \Delta\lambda_1$ enters in the charge $\Delta Q_{(\alpha)}$ and is linear in λ_1, so we treat it as a source of ordinary memory for the shearing GW memory Δc_{AB}. [8]

Because the right-hand side of (5.73) is determined by the changes in $\mathcal{M}$ and $\Delta\lambda_1$ (for the term $\Delta Q_{(\alpha)}$) and the change in the flux of tensor and scalar waves (for the term $\Delta\mathcal{E}_{(\alpha)}$), then we can solve for $\Delta\Theta$ in Eq. (5.73) in terms of a sum of these two contributions. We will then write this solution for the total potential as a sum of two terms

$$\Delta\Theta = \Delta\Theta_{(\mathrm{n})} + \Delta\Theta_{(\mathrm{o})} \,, \tag{5.74}$$

which correspond to the solutions for the null and ordinary parts, separately. This splitting will be useful in discussing the CM memory effect in the next part.

Lastly, note that no constraints on the magnetic-parity part of Δc_{AB} are found from supermomentum conservation. Thus, it would be classified as a persistent observable rather than a memory effect in the language of [390].

[8] There is a second possibility that one might have considered the change in $\eth^2\Delta\lambda_1$ to be a scalar GW memory that is constrained at the same time as the tensor-type memory through the flux $\Delta\mathcal{E}_{(\alpha)}$ and the change in the integral of α times the mass aspect $\mathcal{M}$. However, this is not a viable option, because to specify the properties of the initial and final nonradiative states, one has to specify the nonradiative value of the scalar field λ_1 (analogously to how one has to specify the value of the mass aspect $\mathcal{M}$. Thus, there is no freedom to constrain the value of λ_1 through the memory equation (5.73). This does have the noteworthy consequence that to determine the tensor-type memory $\Delta\Theta$, one needs to know the scalar memory $\Delta\lambda_1$ to be able to compute the term $\Delta\eth^2\lambda_1$ that enters into the ordinary memory $\Delta Q_{(\alpha)}$.

5.4.3.2 Subleading displacement memory and $\Delta\mathcal{C}_{AB}$

In the BMS group, there are also symmetries parameterized by the vector field on the 2-sphere, Y^A. This vector field is required to be a conformal Killing vector on the 2-sphere from Eq. (5.38); the space of such vector fields that are globally defined form a six-dimensional algebra, which is isomorphic to the Lorentz algebra of 3+1 dimensional Minkowski spacetime. There have also been proposals to consider extensions of the BMS algebra that enlarge the symmetry algebra by including either all the conformal Killing vectors on the 2-sphere that have complex-analytic singularities [142, 143] or all smooth vector fields on the 2-sphere [146, 147]. When the Wald-Zoupas prescription was applied to these extended BMS algebras, it was found that there needed to be an additional term to the flux (or the change in the charges) to maintain that the difference in the charges was equal to the flux [153]. For the smooth vector fields, it was shown that this related term could be absorbed into the definition of the charges [395]. This new term was closely related to a new type of GW memory effect called GW spin memory [156]. There was also a second type of new GW memory related to these extended symmetries called GW CM memory [152]. These two new memory effects are related to the electric- and magnetic-parity parts of the subleading displacement memory in $\Delta\mathcal{C}_{AB}$. We now discuss the computation of these effects in Brans-Dicke theory.

First, we write the change in the charges associated with an extended BMS algebra element $\vec{\xi} = Y^A\vec{\partial}_A$ for a smooth vector field Y^A. Starting from Eq. (5.68) and integrating by parts to simplify the expression, we find

$$\Delta Q_{(Y)} = -\frac{\lambda_0}{32\pi}\int du\, d^2\Omega\, Y^A \Bigg\{ \frac{u}{2}\eth_A\left[2\eth_B\eth_C N^{BC} - N_{BC}N^{BC} - \frac{6+4\omega_{\rm BD}}{(\lambda_0)^2}(\partial_u\lambda_1)^2\right] + \frac{1}{2}\eth_A(c_{BC}N^{BC}) + N^{BC}\eth_A c_{BC} - 2\eth_B(c_{AC}N^{BC}) + \frac{2\omega_{\rm BD}+3}{(\lambda_0)^2}(\partial_u\lambda_1\eth_A\lambda_1 - \lambda_1\eth_A\partial_u\lambda_1)\Bigg\} + \Delta\mathcal{F}_{(Y)}\,, \tag{5.75}$$

where $\Delta\mathcal{F}_{(Y)}$ is the additional term needed to relate the change in the charges to the flux integral. It is given by

$$\Delta\mathcal{F}_{(Y)} = \frac{\lambda_0}{64\pi}\int d^2\Omega\, Y^A \epsilon_{AB}\eth^B \text{Đ}^2(\text{Đ}^2+2)\Delta\Sigma\,, \tag{5.76}$$

where we introduced the notation of [154] for the u integral of Ψ

$$\Delta\Sigma = \int du\,\Psi\,, \tag{5.77}$$

and where Ψ determines the magnetic-parity part of c_{AB} in Eq. (5.51).[9] The GW spin memory effect is related to the quantity $\Delta\Sigma$, which determines the magnetic-parity part of $\Delta\mathcal{C}_{AB}$. In the absence of magnetic-parity displacement memory Δc_{AB}, the spin memory will be independent of u, and given by just the magnetic-parity part of $\Delta\mathcal{C}^{(0)}_{AB}$.

Let us now make a few additional definitions. Note that in Eq. (5.75), there is a term that is linear in the news tensor N_{AB}, like the term that gives rise to the

[9] The modification to the charge defined in [395] is similar to the quantity $\Delta\mathcal{F}_{(Y)}$, but instead of $\Delta\Sigma$, a term proportional to $u\Psi$ was used instead.

displacement memory; however, the term in (5.75) is multiplied by u. When this term is integrated over u, the resulting quantity has dimensions or strain multiplied by time, like the GW spin memory. It was argued in [152] that a quantity related to this term is responsible for a new type of GW memory called GW center-of-mass (CM) memory. Specifically, consider the quantity defined by u times the u integral of $\partial_u\Theta$, with the part of $\partial_u\Theta$ responsible for the ordinary memory $\Delta\Theta_{(\mathrm{o})}$; i.e.,

$$\Delta\mathcal{K} = \int du\, u\partial_u(\Theta - \Theta_{(\mathrm{n})})\,. \tag{5.78}$$

Then the integral of the term in square brackets in Eq. (5.68) can be written in the form

$$\Delta\mathcal{C}_{(Y)} = -\frac{\lambda_0}{64\pi}\int d^2\Omega\, Y^A \eth_A \text{Đ}^2(\text{Đ}^2+2)\Delta\mathcal{K}\,. \tag{5.79}$$

Finally, define the remaining terms in Eq. (5.75) to be

$$\begin{aligned}\Delta\mathcal{J}_{(Y)} = \frac{\lambda_0}{64\pi}\int du\, d^2\Omega\, Y^A\bigg[&\eth_A(c_{BC}N^{BC}) + 2N^{BC}\eth_A c_{BC} - 4\eth_B(c_{AC}N^{BC}) \\ &+ \frac{4\omega_{\mathrm{BD}}+6}{(\lambda_0)^2}(\partial_u\lambda_1\eth_A\lambda_1 - \lambda_1\eth_A\partial_u\lambda_1)\bigg]\,,\end{aligned} \tag{5.80}$$

which are the moments of the change in the super angular momentum with respect to the vector field Y^A. With these definitions, Eq. (5.75) reduces to the expression

$$\Delta Q_{(Y)} = -\Delta\mathcal{J}_{(Y)} + \Delta\mathcal{C}_{(Y)} + \Delta\mathcal{F}_{(Y)} \tag{5.81}$$

Using the evolution equation for the Bondi mass aspect (5.33), we can show that the definition of the charge needed to satisfy Eq. (5.75) is given by

$$
\begin{aligned}
Q_{(Y)} =& \frac{\lambda_0}{8\pi}\int d^2\Omega\, Y^A\bigg[-u\,\eth_A\Big(\mathcal{M}-\frac{1}{4\lambda_0}\text{Đ}^2\lambda_1\Big)-3L_A+\frac{1}{32}\eth_A(c_{BC}c^{BC}) \\
&+\frac{1}{4\lambda_0}\eth_A\Big(3\lambda_2+\frac{\omega_{\text{BD}}-1}{2\lambda_0}(\lambda_1)^2\Big)-\frac{1}{4\lambda_0}(c_{AB}\eth^B\lambda_1-\lambda_1\eth^B c_{AB})\bigg]. \qquad (5.82)
\end{aligned}
$$

Next, it is useful to consider decomposing the vector field Y^A into gradient and curl parts via the expression

$$
Y^A = \eth^A\beta(x^C)+\epsilon^{AB}\eth_B\gamma(x^C)\,, \qquad (5.83)
$$

(for smooth functions β and γ) and to treat the case of divergence- and curl-free vector fields Y^A separately. This will allow us to isolate the GW spin and CM memory effects.

CM memory and electric-parity Y^A Let us first specialize to $Y^A=\eth^A\beta$. The term $\Delta\mathcal{F}_{(Y)}$ vanishes for vector fields Y^A of this type. After integrating by parts, this means that we can determine the CM memory through the equation

$$
\int d^2\Omega\,\beta\,\text{Đ}^4(\text{Đ}^2+2)\Delta\mathcal{K}=\frac{64\pi}{\lambda_0}\left(\Delta\mathcal{J}_{(\beta)}+\Delta Q_{(\beta)}\right)\,. \qquad (5.84)
$$

In the above equation, we have defined $\text{Đ}^4=(\text{Đ}^2)^2$, and we have let $\Delta Q_{(\beta)}$ and $\Delta\mathcal{J}_{(\beta)}$ given in Eqs. (5.82) and (5.80) be the change in the charges and in a part of the flux associated with the vector field $Y^A=\eth^A\beta$. The procedure for computing the CM memory works similarly to that for computing the standard GW memory described by the potential $\Delta\Theta$: (i) first pick a basis of functions for the smooth function β on

S^2 to determine the coefficients of $\Delta\mathcal{K}$ expanded in this basis (perhaps most usefully, spherical harmonics); (ii) then provide radiative (and some nonradiative) data in the functions λ_1, c_{AB}, and their u derivatives to evaluate the basis-function coefficients of the flux term $\Delta\mathcal{J}_{(\beta)}$; (iii) next specify the nonradiative data in $\mathcal{M}$, L_A, c_{AB}, λ_1, and λ_2 to evaluate the coefficients of the change in the charges $\Delta Q_{(\beta)}$; (iv) finally, solve for the relevant coefficients of $\Delta\mathcal{K}$ by acting on it with the elliptic operator $\eth^4(\eth^2+2)$ and performing the integral. Some interesting differences from the standard GW memory are that the flux term involves c_{AB} and λ_1 in addition to $\partial_u\lambda_1$ and N_{AB}, and the charge involves c_{AB}, L_A, and λ_2 in addition to λ_1 and $\mathcal{M}$.

Spin memory and magnetic-parity Y^A Next we shall discuss vector fields given by $Y^A = \epsilon^{AB}\eth_B\gamma$. In this case, it is the term $\Delta\mathcal{C}_{(Y)}$ that vanishes, and one can solve for the spin memory through the equation

$$\int d^2\Omega\,\gamma\,\eth^4(\eth^2+2)\Delta\Sigma = -\frac{64\pi}{\lambda_0}\left(\Delta\mathcal{J}_{(\gamma)} + \Delta Q_{(\gamma)}\right) . \tag{5.85}$$

The prescription used to determine the coefficients of the potential $\Delta\Sigma$ when expanded in a basis of functions on S^2 works nearly identically to that for the expansion of $\Delta\mathcal{K}$ for the spin memory. The main difference is that less nonradiative data is needed to determine the spin memory. Specifically, because the quantities $\eth_A\lambda_2$ and $\eth_A\mathcal{M}$ enter into the charge as gradients, then these terms will vanish for a magnetic-parity vector field of the form $Y^A = \epsilon^{AB}\eth_B\gamma$. Thus, computing the spin memory does not require knowledge of the functions λ_2 and $\mathcal{M}$.

5.4.4 Summary and discussion

To summarize, in asymptotically flat general relativity in Bondi coordinates, there are four types of memory effects that are encoded in the electric- and magnetic-parity parts of the tensors Δc_{AB} and $\Delta \mathcal{C}_{AB}$. All four memory effects can be measured through geodesic deviation, and they produce a type of shearing of a family of deviation vectors pointing from some fiducial timelike worldline far from a source of gravitational waves. The memory effects encoded in Δc_{AB} are related to the dependence of the final deviation vector on the initial deviation vector; the memory effects encapsulated in $\Delta \mathcal{C}_{AB}$ are connected to the dependence of the final deviation on the initial relative velocity of the deviation vector. Three of the four memory effects were constrained by conservation laws for charges associated with the (extended) BMS algebra. Specifically, the electric-parity part of Δc_{AB} is constrained through the statement of supermomentum conservation associated with the supertranslation symmetries of the BMS group. The electric- and magnetic-parity parts of $\Delta \mathcal{C}_{AB}$ were determined through the conservation of super-angular momentum conjugate to the super-Lorentz symmetries of the extended BMS algebra. The magnetic-parity part of Δc_{AB} does not seem to have any conservation equation that constrains its value (and thus might be classified as just a persistent observable).

In our treatment of asymptotically flat solutions of Brans-Dicke theory in Bondi coordinates, we observed that there were a total of six types of memory effects: the four that exist in general relativity, and two more that are related to the leading-order dynamical part of the scalar field, λ_1. The two new memory effects also could be measured through geodesic deviation, though they would produce an expansion (or contraction) of the family of deviation vectors pointing orthogonally away from a given worldline (a so-called "breathing" mode). The memory effect $\Delta\lambda_1$ was related to the amplitude of this effect which depends on the initial deviation vector, and the effect in $\Delta\Lambda_1$ corresponded to the scale of the effect depending on the initial relative velocity of the nearby worldlines. The quantities $\Delta\lambda_1$ and $\Delta\Lambda_1$ were not constrained by any conservation laws associated with conserved quantities in asymptotically flat spacetimes in Brans-Dicke theory (so they would also just be persistent observables). Rather, because the symmetries of asymptotically flat solutions of Brans-Dicke theory are the same as those of general relativity, the same three types of memory effects are constrained by the fluxes of conserved quantities as in general relativity. Because the definition of the flux and charges includes additional radiative and non-radiative data (namely, $\partial_u\lambda_1$, λ_1, and λ_2), the precise expressions used for computing the memory effects and the data necessary to compute these effects differs in Brans-Dicke theory from the expressions used in general relativity.

5.5 Conclusions

We investigated asymptotically flat solutions of Brans-Dicke theory in Bondi-Sachs coordinates. We solved the field equations of this theory, and we found that they have a similar structure to the Bondi-Sachs form of the Einstein equations in general relativity. The expansions of the metric and the Ricci tensor in series in $1/r$ (r

being the areal radius) have somewhat different forms from the equivalent quantities in general relativity. Specifically, the Ricci tensor in Bondi coordinates scales like $1/r$, which allows for a scalar (or breathing-mode) gravitational-wave polarization not present in general relativity; other coefficients in the metric also fall off more slowly with $1/r$ in Brans-Dicke theory than in general relativity to accommodate this additional GW polarization. Interestingly, this different "peeling" property of the Ricci tensor does not affect the asymptotic symmetry group in Brans-Dicke theory, which remains the Bondi-Metzner-Sachs group (though the way in which these symmetries are extended into the interior of the spacetime in Brans-Dicke theory differs from the related extension in general relativity). We also computed the properties of nonradiative and stationary regions of spacetime in Brans-Dicke theory, which is important for computing and understanding GW memory effects.

We found six types of memory effects generated after a burst of the scalar field and tensorial gravitational waves pass by an observer's location. Four of these effects are also present in GR: namely, they are the electric- and magnetic-parity parts of displacement and subleading displacement memories. These effects produce the familiar, lasting shearing of a ring of freely falling test masses, with the displacement part depending on the initial separation of the test masses, and the subleading displacement part depending on the initial relative velocity of the masses (the electric- and magnetic-parity parts refer to the parity properties of the sky pattern of the memory effect over the anti-celestial sphere). The amplitude of the memory effects in

Brans-Dicke and in GR will differ, because in Brans-Dicke theory, there are additional contributions from the fluxes of energy and angular momentum per solid angle from the scalar field. The two additional GW memory effects in Brans-Dicke theory are related to the breathing-mode polarization of the gravitational waves, and they could also be classified into leading and subleading displacement terms. These memory effects cause a ring of freely falling test masses to have an enduring, uniform expansion (or contraction) of a circular congruence of geodesics transverse to GW propagation. The leading part that depends upon the initial displacement of the masses had been previously considered, but the subleading part, which depends on the initial relative velocity of the masses appears not to have been. The latter can be thought of as the scalar analog of the center-of-mass memory effect.

Half of these memory effects are constrained by fluxes of conserved quantities associated with the extended BMS group (these are the electric-parity displacement memory, the spin memory, and the center-of mass memory). The other half (the magnetic-parity displacement memory, and both breathing-mode memory effects) are not, and would be described as being persistent observables in the nomenclature of [390, 415]. For all the memory effects, but particularly for the persistent-observable types, understanding the properties of the nonradiative regions before and after the burst of the scalar field and gravitational waves is important for understanding the set of possible memory effects. For example, in general relativity, stationary-to-stationary transitions in which the two stationary regions differ by only a supertranslation allow for a wide range of possible electric-parity displacement memory effects; however, in Brans-Dicke theory, such transitions would only allow for scalar-type memory effects with constant sky pattern. More general types of nonradiative regions at early and late times are necessary to have less trivial memory effects.

Let us conclude with a few comments on future applications and directions for our work. It would be interesting to explore the post-Newtonian limit of our results for compact binary systems, so as to make contact with some existing results computed by Lang [179, 180]. Another potential direction is to explore a broader set of modified gravity theories. We note that our formalism can easily be extended to more general massless scalar-tensor theories, such as those proposed by Damour and Esposito-Farèse [210, 369]. It would be interesting to understand whether there are similar relationships between symmetries and memory effects in theories where additional polarizations are present, such as the scalar-vector-tensor theories [419]. (A generic theory of gravity can have up to six polarizations, and there would typically be additional GW memory effects associated with all such polarizations.) Other viable theories of gravity such as the higher curvature theories [36] (e.g., dynamical Chern-Simon gravity [258] and Einstein-dilaton-Gauss-Bonnet gravity [240, 241]) and massive scalar-tensor theories would also be useful to explore.

Chapter 6

Gravitational Wave Memories in Brans-Dicke Theory in Post-Newtonian Approximation

6.1 Introduction

In this chapter, we apply the results of chapter 5 to construct the GW memory waveform from compact binary systems [1]. We will focus on the memory effects that appear in the tensor polarizations of the GWs, because we can use the BMS flux balance laws to construct nonlinear memory effects based on linearized (or nonlinear) waveforms that do not include the memory effects. The procedure in Brans-Dicke theory is closely analogous to that used in GR [154, 154, 157–162]. The two new memory effects in the scalar polarizations of the GWs are related to shifts in the scalar field and its time integral. It was recently shown in [420] that the scalar memory effects are closely related to the large gauge symmetries of 2-form theory that was shown in [421] to be dual to the scalar field theory. The symplectic flux

[1] This chapter is based on the following paper: *Gravitational-wave memory effects in Brans-Dicke theory: Waveforms and effects in the post-Newtonian approximation*; Tahura, Shammi; Nichols, David A.; Yagi, Kent; arXiv:2107.02208 (*Submitted to Physical Review D*)

of the scalar field is linear in the field and in the large gauge transformation; as a result, we cannot use the flux balance laws to construct a nonlinear memory of the scalar waves as one can for the tensor waves via the BMS flux-balance laws (one must instead solve the scalar field equation directly to determine the scalar memory effect). Since our focus is on the application of the BMS balance laws in BD theory to determine the GW memory effects, we will focus here on computing the tensor memory effects in BD theory, which differ from those of GR due to the scalar dipole radiation.

In BD theory, tensor GW memory effects are sourced by both the scalar and tensor energy and angular momentum fluxes. Because solar-system experiments [209] and pulsar observations [37, 213] have constrained the amount of scalar radiation in BD theory, we assume that the scalar radiation leads to energy and angular momentum fluxes that are small compared to the leading quadrupole fluxes of tensor GWs in GR. Note, however, that the scalar fluxes appear at a lower post-Newtonian (PN) than the tensor fluxes do (see, e.g., [195] for a review of the post-Newtonian, as well as the multipolar post-Minkowskian, expansion). For a fixed value of the small (dimensionless) inverse coupling parameter in BD theory, there is thus a smallest PN parameter at which our approximation of small scalar fluxes holds. To compute GW memory effects in BD theory at Newtonian order, we will need to include higher-PN-order terms (in the frequency evolution and Kepler's law, for example) than we would need to go to Newtonian order in the calculation in GR. In addition, we will also truncate our results at a finite, but smallest PN parameter, which is the smallest value for which our approximation holds (unlike in GR, in which we can take the PN parameter to zero).[2]

We computed our memory effects using the oscillatory waveforms computed in, e.g., [179, 180, 399] after verifying that we can relate the waveforms computed in harmonic coordinates in these references to our Bondi-Sachs quantities. The memory effects that we compute in BD theory, have small terms (proportional to the small BD parameter) that appear at a PN order less than the leading Newtonian order. We can relate part of our results to a part of the waveform computed by Lang in [179,180] using the direct integration of the relaxed Einstein equations for the scalar and tensor waveforms up to 1.5PN and 2PN orders, respectively. Lang found no scalar GW memory effects, but he computed a (hereditary) tensor GW memory effect formally at 1.5PN order that arises from the flux of energy radiated in the scalar waves. Upon integrating this 1.5PN term for compact-binary source in our approximation, this term leads to a memory effect that depends logarithmically on the PN parameter (this is analogous to how a formally 2.5PN order term in GR, when integrated for compact binaries, leads to a Newtonian-order effect in the waveform [163,164]). If we compare our result in the Bondi-Sachs framework with Lang's harmonic-coordinate

[2]Note, of course, that we could also compute the memory effects from a PN parameter of zero up to the small PN parameter at which the scalar and tensor fluxes have comparable magnitudes, if we assume that the radiated fluxes are dominated by the scalar emission. This, in fact, is the approximation used in [179,180], for example. However, because memory effects are most important when the fluxes are large, this early-time (or small-PN-parameter) regime is not expected to produce a significant GW memory effect, and we do not compute it in this chapter.

expression, the two terms agree. The BMS flux balance laws are not as helpful for verifying the absence of scalar memory effects through 1.5PN order.

Our BMS flux-balance approach allows us to compute the Newtonian-order tensor waveforms—which have not been computed before, as far as we are aware—that should appear if the work of Lang [179] were extended to 2.5PN order. We find that because of the dipole emission, the Newtonian-order GW memory effects sourced by the tensor GW energy flux has contributions from current quadrupole and mass octopole and hexadecapole moments. These higher multipole moments produce GW memory waveforms that have a different dependence on the inclination angle than the tensor GW memory effect in GR at the equivalent PN order. The Newtonian GW memory effect generated by the scalar field's energy flux also has a different dependence on inclination angle from that sourced by the tensor GWs. The inclination-angle dependence of the GW memory effect has been shown to be something that can be tested with second- and third-generation ground-based GW detectors [184].

The rest of the chapter is organized as follows. In Sec. 6.2, we present a few elements of BD theory in harmonic and Bondi coordinates. Section 6.3 lists the oscilliatory radiative mass and current multipole moments for a quasi-circular, nonspinning compact-binary inspiral (Sec. 6.3.1); reviews the derivation of Kepler's law, the evolution of the orbital frequency, and the phase of GWs in BD theory at the necessary PN orders in our approximation (Sec. 6.3.2); and presents scalar multipole moments generated by an inspiraling quasi-circular, nonspinning compact binary (Sec. 6.3.3).

In Sec. 6.4 we compute nonlinear displacement and spin GW memory waveforms in BD theory from the BMS fluxes. We conclude in Sec. 6.5. We give additional results in two appendices where we show the coordinate transformations that relate the Bondi coordinates to harmonic coordinates including relations between the metric functions in two coordinates systems (Appendix K), and we argue that the ordinary parts of the GW memory effects are subleading compared to null memory effects in BD theory (Appendix L).

Throughout this chapter, we use units in which $c = 1$, and we also set the asymptotic value of the gravitational constant in BD theory to 1. We use Greek indices $(\mu, \nu, \ldots)$ to denote four-dimensional spacetime indices, and uppercase Latin indices $(A, B, C, \ldots)$ for indices on the 2-sphere, and lowercase Latin indices $(i, j, k, \ldots)$ for the spatial indices in quasi-Cartesian harmonic coordinates.

6.2 Waveform in Harmonic and Bondi Coordinates

In this section, we discuss briefly the Bondi-Sachs framework [139, 140] and the harmonic-gauge waveform in post-Newtonian theory, both of which we will use to compute the GW memory waveform. Specifically, in Sec. 6.2.1, we discuss BD theory in harmonic coordinates and decompose the GW strain into radiative multipole moments. In Sec. 6.2.2, we present BD theory in the Bondi-Sachs framework and relate the shear tensor in Bondi coordinates to the radiative mass and current multipole moments found in harmonic coordinates. We also relate the scalar waveform in Bondi coordinates to that of the harmonic coordinates.

We require both coordinate systems and frameworks, because the GW memory effects are straightforward to compute through the BMS balance laws in the Bondi approach, but it is more challenging to relate the Bondi-Sachs framework to a specific solution of a Cauchy initial-value problem. In the harmonic-gauge PN approach the scalar and tensor GW waveforms already have been computed generally and for specific compact-binary sources in, e.g., [179, 180]; however, the GW memory effects are of a sufficiently high PN order in PN theory that they have not been fully computed in Brans-Dicke theory. After relating the harmonic-gauge waveform to the shear in the Bondi-Sachs framework, we can then determine the GW memory waveforms using the balance laws (and thereby avoiding high PN-order calculations).

Throughout this chapter, we treat Brans-Dicke theory in the Jordan frame [166], in which the action takes the form

$$S = \int d^4x \sqrt{-g} \left[\frac{\lambda}{16\pi} \mathcal{R} - \frac{\omega_{\rm BD}}{16\pi} g^{\mu\nu} \frac{(\partial_\mu \lambda)(\partial_\nu \lambda)}{\lambda} \right] . \tag{6.1}$$

Here $g_{\mu\nu}$ is the Jordan-frame metric, $\mathcal{R}$ is the Ricci scalar of $g_{\mu\nu}$, λ is a massless scalar field with a nonminimal coupling to gravity, and $\omega_{\rm BD}$ is a coupling constant called the Brans-Dicke parameter. In this section and subsequent ones, we set the gravitational constant at infinity to unity, i.e.,

$$G_0 = \frac{4 + 2\omega_{\rm BD}}{3 + 2\omega_{\rm BD}} \frac{1}{\lambda_0} = 1 \, , \tag{6.2}$$

where λ_0 is the constant value that λ approaches in the limit of infinite distances from an isolated source.

6.2.1 Waveform in harmonic coordinates

We will denote our quasi-Cartesian harmonic-gauge coordinates as X^μ, and we will use the notation $X^0 = t$ for the time coordinate and X^i (for $i = 1, 2, 3$) for the spatial coordinates. We will denote the Euclidean distance from the origin at fixed t by $R = \sqrt{X^i X^j \delta_{ij}}$. The tensor GWs in Brans-Dicke theory are described by the transverse-tracelesss (TT) components of the metric perturbation, $\tilde{h}_{ij}^{\rm TT}$, and the scalar GWs are encapsulated in the scalar field λ. Both fields can be obtained from the metric at order $1/R$ in an expansion in R, from the spatial components of the spacetime metric g_{ij}. The metric is more conveniently written in terms of the metric perturbation $\tilde{h}_{ij}$ and its trace $\tilde{h}$ rather than the TT part. For extracting the GWs, we need only the part of the spacetime metric that is linear in the fields λ and $\tilde{h}_{ij}$, which can be obtained, e.g., from the results in [179] to be

$$g_{ij} = \delta_{ij} + \tilde{h}_{ij} - \frac{1}{2}\tilde{h}\delta_{ij} - \left(\frac{\lambda}{\lambda_0} - 1\right)\delta_{ij} \, . \tag{6.3}$$

The scalar GWs are present in the $1/R$ part of λ, which we expand as

$$\lambda = \lambda_0 + \frac{\Xi(\tilde{u}, y^A)}{R} + \mathcal{O}\left(\frac{1}{R^2}\right) \, . \tag{6.4}$$

We have written the scalar field in terms of $\tilde{u} = t - R$, the retarded time in harmonic coordinates, and the angles $y^A \equiv (\iota, \varphi)$. The angle ι is the polar angle and φ is the azimuthal angle of a spherical polar coordinate system.[3] We expand the TT projection of $\tilde{h}_{ij}$ in terms of second-rank electric-parity and magnetic-parity tensor spherical harmonics ($T_{ij}^{(e),lm}$ and $T_{ij}^{(b),lm}$, respectively; see, e.g., [422]) as [195]

$$\tilde{h}_{ij}^{\mathrm{TT}} = \frac{1}{R}\sum_{l,m}\left[U_{lm}(\tilde{u})T_{ij}^{(e),lm} + V_{lm}(\tilde{u})T_{ij}^{(b),lm}\right] . \tag{6.5}$$

The sum runs over integer values of l and m with $l \geq 2$ and $-l \leq m \leq l$. The coefficients U_{lm} and V_{lm} are two sets of radiative multipole moments which are called the mass and current moments, respectively. Because $\tilde{h}_{ij}^{\mathrm{TT}}$ is real, the mass and current moments satisfy the following properties under complex conjugation:

$$\bar{U}_{lm} = (-1)^m U_{l,-m}, \quad \bar{V}_{lm} = (-1)^m V_{l,-m} . \tag{6.6}$$

We use an overline to denote the complex conjugate.

We will also use the complex gravitational waveform h which is composed of the plus and cross polarizations as follows:

$$h = h_+ - ih_\times . \tag{6.7}$$

We use the conventions for the polarization tensors e_{ij}^+ and $e_{ij}^\times$ given in [164] or [423] to construct the polarizations $h_+ = h_{\mathrm{TT}}^{ij}e_{ij}^+$ and $h_\times = h_{\mathrm{TT}}^{ij}e_{ij}^\times$. We expand h as in

[3] For compact binary sources, ι is the inclination angle between the orbital angular momentum of the binary (assumed to be in the Z direction) and φ is the azimuthal angle as measured from the X direction.

terms of spin-weighted spherical harmonics ${}_sY_{lm}$ with spin weight $s = -2$:

$$h = \sum_{l,m} h_{lm}(\tilde{u})({}_{-2}Y_{lm})\,. \tag{6.8}$$

For a non-spinning planar binary, the modes h_{lm} are related to the mass and current multipole moments by [195]

$$h_{lm} = \begin{cases} \frac{1}{\sqrt{2}R} U_{lm} & (l+m \text{ is even})\,, \\ -\frac{i}{\sqrt{2}R} V_{lm} & (l+m \text{ is odd})\,. \end{cases} \tag{6.9}$$

6.2.2 Waveform and metric in Bondi coordinates

We use (u, r, x^A) to denote Bondi coordinates. The coordinate u is the retarded time, r is an areal radius, and x^A are coordinates on a 2-sphere cross sections of constant u and r (where $A = 1,\, 2$). We expand λ as a series in $1/r$ as

$$\lambda(u, r, x^A) = \lambda_0 + \frac{\lambda_1\left(u, x^A\right)}{r} + \frac{\lambda_2\left(u, x^A\right)}{r^2} + O(r^{-3})\,. \tag{6.10}$$

The metric in Bondi gauge satisfies the conditions $g_{rr} = 0$, $g_{rA} = 0$, and the determinant of the metric on the 2-sphere cross sections scaled by r^{-4} is independent of r (and u). We imposed a set of aysmptotic boundary conditions on the nonzero components of the metric in Bondi gauge in [82] and postulated a Taylor series expansion of the scalar field and metric on the 2-sphere cross sections in $1/r$. This allowed us to solve the field equations of Brans-Dicke theory to obtain the following solution for the line element [82]:

$$
\begin{aligned}
ds^2 \quad &= -\left[1 + \frac{\dot{\lambda}_1}{\lambda_0} - \frac{1}{r}\left(2\mathcal{M} + \frac{\lambda_1}{\lambda_0} + \frac{3\lambda_1}{2\lambda_0^2}\dot{\lambda}_1\right)\right] du^2 \\
&-2\left(1 - \frac{\lambda_1}{\lambda_0 r}\right) dudr + r^2\left(q_{AB} + \frac{1}{r}c_{AB}\right) dx^A dx^B \\
&+\left\{\eth_F c^{AF} - \frac{\eth^A\lambda_1}{\lambda_0} + \frac{1}{r}\left[-4L_A + \frac{1}{3}c_{AB}\eth_C c^{BC}\right.\right. \\
&\left.\left.- \frac{1}{3\lambda_0}(2\lambda_1\eth^B c_{AB} + c_{AB}\eth^B\lambda_1 - \frac{1}{\lambda_0}\eth_A\lambda_1^2)\right]\right\} dudx^A + \dots . \qquad (6.11)
\end{aligned}
$$

The ellipsis at the end of the equation indicates higher order terms in powers of $1/r$ that we are neglecting (the terms are of order $1/r^2$ except for the term proportional to $dx^A dx^B$, which is of order unity, because of the r^2 term multiplying the expression). In Eq. (6.11), we have introduced $\mathcal{M}$ and L_A which are (related to) functions of integration in Brans-Dicke theory that are the analogues of the Bondi mass aspect and angular momentum aspect in GR [82]. The two-dimension metric q_{AB} is the unit-sphere metric and $\eth^A$ is the covariant derivative compatible with q_{AB}. We will raise and lower 2-sphere indices (such as A and B) with the metrics q^{AB} and q_{AB}, respectively. The overhead dot means a derivative with respect to the retarded time u. The symmetric trace-free tensor c_{AB} is called the shear tensor, and is related to the GW strain. The time-derivative of c_{AB} is a symmetric trace-free tensor known as the news tensor:

$$
N_{AB} = \partial_u c_{AB}\,. \qquad (6.12)
$$

It is not constrained by the asymptotic field equations in Brans-Dicke theory, and it contains information about the tensor GWs. If the news tensor vanishes it means the corresponding region of spacetime contains no GWs [148].

We will also expand c_{AB} in spherical harmonics as

$$c_{AB} = \sum_{l,m} \left(c_{(e),lm} T_{AB}^{(e),lm} + c_{(b),lm} T_{AB}^{(b),lm} \right) . \tag{6.13}$$

The tensor spherical harmonics can be defined from the scalar spherical harmonics

$$T_{AB}^{(e),lm} = \frac{1}{2}\sqrt{\frac{2(l-2)!}{(l+2)!}} \left(2\eth_A \eth_B - q_{AB} \text{Đ}^2\right) Y_{lm} , \tag{6.14a}$$

$$T_{AB}^{(b),lm} = \sqrt{\frac{2(l-2)!}{(l+2)!}} \epsilon_{C(A} \eth_{B)} \eth^C Y_{lm} , \tag{6.14b}$$

or instead in terms of spin-weighted spherical harmonics and a complex null dyad on the unit 2-sphere of m^A and its complex conjugate $\bar{m}^A$ (see, e.g., [422]):

$$T_{AB}^{(e),lm} = \frac{1}{\sqrt{2}} \left({}_{-2}Y_{lm} m_A m_B + {}_{2}Y_{lm} \bar{m}_A \bar{m}_B \right) , \tag{6.15a}$$

$$T_{AB}^{(b),lm} = -\frac{i}{\sqrt{2}} \left({}_{-2}Y_{lm} m_A m_B - {}_{2}Y_{lm} \bar{m}_A \bar{m}_B \right) . \tag{6.15b}$$

The dyad is normalized such that $m^A \bar{m}_A = 1$.

6.2.3 Relation between Bondi- and harmonic-gauge quantities

We construct a coordinate transformation between harmonic and Bondi gauges in App. K, at the first order beyond the Minkowski background in a series in $1/R$. The procedure used is similar to that recently outlined in [424], but it is adapted to Brans-Dicke theory (rather than GR) and it is accurate only to the first nontrivial order in $1/R$. This coordinate transformation leads to a simple relationships between c_{AB} and

$\tilde{h}^{\rm TT}_{ij}$, first, and λ_1 and Ξ, second:

$$c_{AB}(u, x^A) = R\,\tilde{h}^{\rm TT}_{ij}(t-R, y^A)\partial_A n^i \partial_B n^j\,, \tag{6.16}$$

$$\lambda_1(u, x^A) = \Xi(t-R, y^A)\,. \tag{6.17}$$

The spatial vector $\vec{n}$ is the unit vector pointing radially outward at fixed t in harmonic coordinates (i.e., in the direction of propagation for outgoing GWs). The full transformation between the spherical polar coordinates R and $y^A = (\iota, \varphi)$ constructed from the quasi-Cartesian harmonic coordinates and Bondi coordinates is given in App. K; we list here the relevant leading-order parts of the transformation needed to relate u to $t - R$, r to R, and x^A to y^A in Eq. (6.16):

$$u = t - R - \frac{2M_t}{\lambda_0}\log(R) + O(R^{-1})\,, \tag{6.18a}$$

$$x^A = y^A + O(r^{-2})\,, \qquad r = R + O(R^0)\,, \tag{6.18b}$$

where M_t is a constant which we will later consider as the total mass of the system.

The second-rank tensor spherical harmonics on the unit 2-sphere in spherical and Cartesian coordinates are related by the following transformation:

$$T^{(e),lm}_{AB} = T^{(e),lm}_{ij}\partial_A n^i \partial_B n^j\,, \tag{6.19a}$$

$$T^{(b),lm}_{AB} = T^{(b),lm}_{ij}\partial_A n^i \partial_B n^j\,. \tag{6.19b}$$

Combining the expressions (6.5), (6.13), and (6.16), we find that the multipole moments of the strain and shear are related by

$$c_{(e),lm} = U_{lm}, \quad c_{(b),lm} = V_{lm}\,, \tag{6.20}$$

as was given in [154] (though there the relationship was derived through a different argument involving the Riemann tensor in linearized gravity). The relation (6.20) allows us to express the multipole moments of the shear tensor in terms of multipole moments of the harmonic-gauge TT strain tensor, once the difference between the retarded times in harmonic and Bondi coordinates in Eq. (6.18a) is taken into account.

6.3 Post-Newtonian Radiative Multipole Moments

In this section, we compute expressions for the radiative multipole moments U_{lm} and V_{lm}, as well as the scalar multipole moments which we will define herein. We obtain the moments for nonspinning, quasicircular binaries. We denote the total mass by $M_t = m_1 + m_2$ (where m_1 and m_2 are the individual masses), the symmetric mass-ratio by $\eta = m_1 m_2/M_t^2$, and orbital separation by r_{12}. We also introduce the parameters $\xi = 1/(2 + \omega_{\text{BD}})$ and $x = (\pi M_t f)^{2/3}$, where f is the GW frequency. We will work in the approximation in which $\xi \ll x$, which corresponds to assuming that BD modifications to the waveforms and the dynamics are small corrections to the corresponding quantities in GR. Given that the Shapiro delay measurement in the solar system bounds the BD parameter to be $\omega_{\text{BD}} > 4 \times 10^4$ [209] (a similar bound has been derived from the pulsar triple system PSR J0337+1715 [213]), this implies that our approximation is valid when $x \gg 2.5 \times 10^{-5}$. In this work, we will compute GW memory waveforms at Newtonian order, and we keep BD terms that are leading order in ξ; we then retain only the terms in the radiative multipole moments at the appropriate powers of x and ξ to obtain Newtonian-order-accurate memory waveforms.

We will need radiative mass and current multipole moments at a higher PN order than those required for computing 0PN memory effects in GR. Specifically, in GR, one only requires the mass quadrupole moment at 0PN or $\mathcal{O}(x)$ to obtain Newtonian-order memory effects. In BD theory, we will need the BD correction to the mass quadrupole moment. We will also need 1PN or $\mathcal{O}(x^{3/2})$ GR terms of the mass quadrupole moment since it can multiply -1PN terms present in GW phase or frequency evolution to contribute to Newtonian-order memory effects. Other radiative multipole moments that contain $\mathcal{O}(x^{3/2})$ terms are also required, which include current quadrupole, mass octupole, and mass hexadecapole moments. BD corrections to $\mathcal{O}(x^{3/2})$ terms in radiative multipole moments are not required, as Newtonian-order memory terms produced them would be higher order in ξ. As for scalar multipole moments, depending on which combinations of moments can produce Newtonian-order memories, we will need terms at $\mathcal{O}\left(x^{3/2}\right)$ at most.

6.3.1 Radiative mass and current multipole moments

First, we present the real GW polarizations for $l = 2$, $m = \pm 2$ that is needed to compute the radiative mass multipole moment U_{22}. For GWs generated by a quasi-circular nonspinning compact binary, the plus and cross polarizations of the leading-order quadrupolar GWs were computed in Ref. [87]. Specifically, they computed the real GW polarizations from a superposition of $l = 2$, $m = \pm 2$ spin-weighted spherical harmonic modes which are given by:

$$h_+^{(2\pm2)} = \left(1 - \frac{\xi}{2}\right) \mathcal{G}^{2/3} \mathcal{A}^{(22)}_{\text{GR},+}(x, \iota) \cos[\phi_p(x)] \,, \tag{6.21a}$$

$$h_\times^{(2\pm2)} = \left(1 - \frac{\xi}{2}\right) \mathcal{G}^{2/3} \mathcal{A}^{(22)}_{\text{GR},\times}(x, \iota) \sin[\phi_p(x)] \,, \tag{6.21b}$$

The notation $h_+^{(2\pm2)}$ and $h_\times^{(2\pm2)}$ means the plus and cross polarizations associated with only the $l = 2$, $m = \pm 2$ modes of the waveform. The quantities $\mathcal{A}^{(22)}_{\text{GR},+}$ and $\mathcal{A}^{(22)}_{\text{GR},\times}$ are the amplitudes of plus and cross polarizations in GR for the $l = 2$, $m = \pm 2$ modes (the amplitudes for the $m = \pm 2$ modes are equal, which is why we drop the superscript $\pm$). The modified gravitational constant in BD theory is denoted by

$$\mathcal{G} = 1 - \xi(s_1 + s_2 - 2s_1 s_2) \,, \tag{6.22}$$

where s_1 and s_2 are the sensitivities (see, e.g., [399]) of the binary components. Because we work to linear order in ξ throughout this chapter, we find it convenient to also write

$$\mathcal{G} = 1 - \mathscr{G} \,, \quad \text{with} \quad \mathscr{G} = \xi(s_1 + s_2 - 2s_1 s_2) \,, \tag{6.23}$$

as we will linearize powers of $\mathcal{G}$ in ξ (or equivalently $\mathscr{G}$). The GW phase is denoted $\phi_p(x)$, and it differs from the phase of the $l = 2$, $m = \pm 2$ modes of the waveform in GR; we give the expression for the phase in Eq. (6.37) below.

We now give an expression for U_{22} at 1PN and including the Newtonian-order

Brans-Dicke correction linear in ξ. The complex strain $h_+^{(2\pm2)} - i h_\times^{(2\pm2)}$ from Eqs. (6.21a) and (6.21b) is proportional to the sum of the Newtonian order moment U_{22} [times ${}_{-2}Y_{22}(\iota, 0)$] plus $U_{2,-2}$ [times ${}_{-2}Y_{2,-2}(\iota, 0)$]. One can then obtain the mode U_{22} by performing the overlap integral of the complex strain with a spin weighted spherical harmonic. This gives the Newtonian-order U_{22} mode in BD theory. However, we will also need the 1PN corrections to the amplitude of U_{22} in the GR limit (we do not need the BD terms linear in ξ at 1PN). We use the 1PN GR terms from the review article [195]. Putting these two results together, we have the following expression for U_{22}:

$$U_{22} = -8\sqrt{\frac{2\pi}{5}}\eta M_t x \, e^{-i\phi_p} \left[\left(1 - \frac{\xi}{2} - \frac{2}{3}\mathscr{G}\right) + \left(\frac{55\eta}{42} - \frac{107}{42}\right) x \right] . \tag{6.24a}$$

The term proportional to x in the square bracket is the 1PN GR term taken from [195].

As we will explain in section 6.4, to compute the GW memory waveform at Newtonian order, we need the radiative current quadrupole moment and a subset of the radiative mass octopole and hexadecapole moments. To work to linear order in ξ, we can use the GR amplitudes of the moments (though we use the BD phase). This allows us to take the amplitudes from the expressions given, e.g., in the review [195]:

$$V_{21} = \frac{8}{3}\sqrt{\frac{2\pi}{5}}\eta \delta m \, x^{3/2} e^{-i\phi_p/2} , \tag{6.24b}$$

$$U_{33} = 6i\sqrt{\frac{3\pi}{7}}\eta \delta m \, x^{3/2} e^{-3i\phi_p/2} , \tag{6.24c}$$

$$V_{32} = i\frac{8}{3}\sqrt{\frac{\pi}{14}} M_t \eta (1 - 3\eta) x^2 e^{-i\phi_p} , \tag{6.24d}$$

$$U_{31} = -\frac{2i}{3}\sqrt{\frac{\pi}{35}}\eta\delta m\, x^{3/2}e^{-i\phi_p/2}\,, \tag{6.24e}$$

$$U_{42} = -\frac{8}{63}\sqrt{2\pi}M_t\eta(1-3\eta)x^2e^{-i\phi_p}\,, \tag{6.24f}$$

We use the notation $\delta m = (m_1 - m_2)$.

6.3.2 Kepler's law, frequency evolution, and GW phase

Let us first give an expression for Kepler's law, which we will need to compute the scalar multipole moments and the frequency evolution. To obtain a Newtonian-order accurate GW memory waveform, it is necessary to have an expression for Kepler's law at 1PN order. This higher order is needed, because when evaluating the integrals involved in computing the GW memory effect, there are -1PN terms arising from dipole radiation in the energy flux, which multiply 1PN terms in Kepler's law that give rise to Newtonian-order terms in the waveform. The two-body equations of motion of non-spinning compact objects in Brans-Dicke theory has been computed in Ref. [425]. For circular orbits, the relative acceleration is proportional to the orbital frequency squared, Ω^2, and the relative separation to 1PN order. Working to linear order in ξ, the results of Eqs. (1.4) and (1.5a) of [425] show that Kepler's law in BD theory (in this approximation) is

$$\Omega^2 = \frac{M_t}{r_{12}^3}\left[1 - \mathscr{G} - \frac{M_t}{r_{12}}\left(1-2\mathscr{G}\right)\left(3-\eta\right) - \frac{M_t}{r_{12}}\mathcal{G}\bar{\gamma}\right] + \mathcal{O}(\xi^2)\,. \tag{6.25}$$

We have introduced the parameter

$$\bar{\gamma} = -\mathcal{G}^{-1}\xi\left(1-2s_1\right)\left(1-2s_2\right)\,, \tag{6.26}$$

in the equation above and $\mathcal{G}$ is defined in Eq. (6.22).

Let us next compute the evolution of the GW frequency, $\dot{f}$. We again need a 1PN-order-accurate expression, which turns out to contain many terms. Because we work to linear order in ξ, the only 1PN terms that we need to obtain a Newtonian-order expression for the GW memory waveforms are the 1PN GR terms in $\dot{f}$ (i.e., the 1PN terms without ξ).[4] We can then write the expression for $\dot{f} = df/dt$ in the following form:

$$\dot{f} = \dot{f}_0 + \dot{f}_{1,\mathrm{GR}}\,, \tag{6.27}$$

where $\dot{f}_0$ is the BD expression to Newtonian order and linear order in ξ, and $\dot{f}_{1,\mathrm{GR}}$ is $\xi = 0$ (or GR) limit of the 1PN terms. We first compute $\dot{f}_0$ using results from [425], and then we add to it the terms $\dot{f}_{1,\mathrm{GR}}$ taken from [426]. To compute $\dot{f}_0$, we first use the binding energy of a binary in BD theory from Eq. (6.14) of [425], which is valid to 1PN order:

$$\begin{aligned} E_b = {} & \frac{1}{2}\mu v^2 - \mu\frac{\mathcal{G}M_t}{r_{12}} + \frac{3}{8}\mu(1-3\eta)v^4 + \frac{1}{2}\mu\frac{\mathcal{G}M_t}{r_{12}}(3+2\bar{\gamma}+\eta)v^2 \\ & + \frac{1}{2}\mu(1-2\mathscr{G})\left(\frac{M_t}{r_{12}}\right)^2 + \mathcal{O}(\xi^2)\,. \end{aligned} \tag{6.28}$$

We linearized the expression in ξ and we used $\mu = \eta M_t$ to denote the reduced mass. We will next express the binding energy in terms of the PN parameter x; to do this

[4]The $-1PN$ term that multiplies the 1PN term in this calculation is linear in ξ, which implies that the BD modification to $\dot{f}$ at 1PN enters at higher order in ξ in the GW memory waveforms.

it is useful to have the expressions for M_t/r_{12} and v^2 written in terms of x:

$$\frac{M_t}{r_{12}} = x\left[1+\frac{1}{3}\mathscr{G}+\left(1-\frac{1}{3}\mathscr{G}\right)\left(1-\frac{1}{3}\eta\right)x+\frac{1}{3}\mathcal{G}\bar{\gamma}x\right]+O(\xi^2,x^3)\,, \tag{6.29a}$$

$$v = \sqrt{x}\left[1-\frac{1}{3}\mathscr{G}-(1-\mathscr{G})\left(1-\frac{1}{3}\eta\right)x-\frac{1}{3}\mathcal{G}\bar{\gamma}x\right]+O(\xi^2,x^{5/2})\,, \tag{6.29b}$$

We can then substitute Eq. (6.29) into Eq. (6.28) to obtain

$$E_b = -\frac{1}{2}\mu x\left[1-\frac{2}{3}\mathscr{G}-\frac{1}{12}\left(1-\frac{4}{3}\mathscr{G}\right)(9+\eta)x-\frac{2}{3}\mathcal{G}\bar{\gamma}x\right]+O(\xi^2,x^3)\,. \tag{6.30}$$

The rate of change of energy radiated in GWs in BD theory through Newtonian order is given by a -1PN term plus a Newtonian term. If we define

$$S = s_1 - s_2 \tag{6.31}$$

and we make use of expressions (6.16) and (6.19) given in [425], then linearizing their expression in ξ, we have

$$\dot{E}_{\rm GW} = \frac{32}{5}\eta^2x^5\left[\frac{5\xi S^2}{48x}+1-\frac{7}{3}\mathscr{G}+\frac{5}{12}\mathcal{G}\bar{\gamma}-\frac{5}{72}\xi S^2(3+2\eta)\right]+O(\xi^2,x^6)\,. \tag{6.32}$$

Imposing energy balance $\dot{E}_b = -\dot{E}_{\rm GW}$ (the change in the binding energy is equal to the energy radiated by the GWs) and using the chain rule to write $\dot{f} = (df/dE_b)\dot{E}_b$, we can write the Newtonian-order frequency derivative $\dot{f}_0$ as a function of the PN parameter x as

$$\dot{f}_0 = \frac{96\eta}{5\pi M_t^2}x^{11/2}\left[1+\xi\left(\frac{5S^2}{48x}+\mathscr{F}\right)\right]+\mathcal{O}(\xi^2)\,, \tag{6.33}$$

where

$$\mathscr{F} = -\frac{5}{12} - \frac{5}{6}(s_1 + s_2) + \frac{5}{144}(51 + 7\eta)s_1 s_2 - \frac{5}{288}(3 + 7\eta)(s_1^2 + s_2^2)\,. \tag{6.34}$$

Finally, including the GR frequency evolution at 1PN [426] to $\dot{f}_0$, we find

$$\dot{f} = \frac{96\eta x^{11/2}}{5\pi M_t^2}\left[1 + \xi\left(\frac{5S^2}{48x} + \mathscr{F}\right) - \left(\frac{743}{336} + \frac{11}{4}\eta\right)x\right]. \tag{6.35}$$

We previously introduced a waveform phase variable ϕ_p, which we will now compute explicitly. For computing the GW memory waveforms, we will again need an expression for the GW phase through 1PN order; however, because we are working to linear order in ξ, we will only need the terms without ξ at 1PN order in the phase (analogously to our calculation of $\dot{f}$). The GW phase is typically obtained by integrating the GW frequency with respect to time from some appropriate starting time. For calculations of the GW memory waveform, it is more useful to write the phase as a function of $x = (\pi M_t f)^{2/3}$; then by using the chain rule, we can write the time integral of the frequency in terms of an integral with respect to the PN parameter x as follows:

$$\phi_p(x) = 2\pi \int_{x_i}^{x} \frac{f}{\dot{f}}\frac{df}{dx'}dx'\,, \tag{6.36}$$

where the frequency f and the derivatives $\dot{f}$ and df/dx are functions of x. We have also introduced an initial PN parameter x_i that should be greater than ξ, so that our approximation of $\xi \ll x$ holds. From $\dot{f}$ in Eq. (6.35), the GW phase at the desired order can be computed as

$$\phi_p(x) - \phi_{p,0} = -\frac{1}{16\eta x^{5/2}}\left[1 - \xi\left(\mathscr{F} + \frac{25S^2}{336x}\right) + \frac{5}{3}\left(x - \frac{1}{8}\xi S^2\right)\left(\frac{743}{336} + \frac{11}{4}\eta\right)\right] + O(\xi^2, x^{-1/2})\,. \tag{6.37}$$

We defined a constant $\phi_{p,0}$, the phase at coalescence, which is chosen such that the phase at x_i vanishes [i.e., $\phi_p(x_i) = 0$]. The terms in the second line of Eq. (6.37) come from the product of a -1PN term multiplying a $+1$PN term, which produces a Newtonian-order effect on the phase (specifically, this arises because of dipole radiation in BD theory, which allows for -1PN order effects).

6.3.3 Scalar Multipole Moments

We will expand λ_1 in terms of the scalar spherical harmonics and the corresponding multipole moments $\lambda_{1(l,m)}$:

$$\lambda_1 = \sum_{l,m} \lambda_{1(lm)} Y^{lm}\,. \tag{6.38}$$

Specifically, we give expressions for the scalar moments $\lambda_{1(1,1)}$, $\lambda_{1(2,2)}$ and $\lambda_{1(3,1)}$ in terms of the PN parameter x, which we will then use to derive the tensor GW memory effects sourced by the fluxes of the scalar field. These three moments are those needed to compute the GW memory effects at Newtonian order. The relevant part of the scalar field λ_1 had been computed previously in [180, 399], and we give the expression to 1PN order above the leading dipole radiation, and to linear order in ξ:

$$\lambda_1 = \eta M_t \lambda_0 \xi \left\{ \left[-2S + \frac{M_t}{2r_{12}} \left(3\Gamma \frac{\delta m}{M_t} + 10S\eta \right) \right] v_i n^i + \Gamma \left[(v_i n^i)^2 - \frac{M_t}{r_{12}} (n_{12}^i n_i)^2 \right] - (\Gamma \frac{\delta m}{M_t} + 2\eta S) \times \left[(v_i n^i)^3 - \frac{7}{2} \frac{M_t}{r_{12}} (v_i n^i)(n_{12}^i n_i)^2 \right] \right\} . \tag{6.39}$$

Above we introduced the quantity

$$\Gamma = 1 - 2(m_1 s_2 + m_2 s_1)/M_t \, , \tag{6.40}$$

the unit vector n^i pointing radially outward in the direction of the GW's propagation, the unit separation vector n_{12}^i between the binary's components, and the relative velocity vector $v^i = v_1^i - v_2^i$ of the binary's masses. In terms of ι and φ (the polar and the azimuthal angles, respectively, in the center-of-mass frame of the binary) and the GW phase ϕ_p, the two unit vectors and the relative velocity vector take the form

$$n^i = (\sin\iota \cos\varphi, \sin\iota \sin\varphi, \cos\iota) \, , \tag{6.41a}$$

$$n_{12}^i = \{\cos[\phi_p(u)/2], \sin[\phi_p(u)/2], 0\} \, , \tag{6.41b}$$

$$v^i = \{-v \sin[\phi_p(u)/2], v\cos[\phi_p(u)/2], 0\} \, . \tag{6.41c}$$

For the magnitude of the velocity, v, we need only the GR expression (zeroth-order in ξ) at 1PN order in x, because λ_1 is already linear in ξ:

$$v = \sqrt{x} \left[1 - \frac{1}{3} (3 - \eta) \, x \right] . \tag{6.42}$$

We wrote the phase as $\phi_p(u)$ [short for $\phi_p[x(u)]$, which is given in Eq. (6.37), to emphasize its retarded time dependence. The multipole moments can be extracted through the overlap integral

$$\lambda_{1(lm)} = \int d^2\Omega\, \lambda_1 Y^*_{lm}(\iota,\varphi)\,. \tag{6.43}$$

Using Eqs. (6.38)–(6.41) and Kepler's law in Eq. (6.25), we find that the harmonic components of λ_1 can be written as

$$\lambda_{1(11)} = -\,2i\sqrt{\frac{2\pi}{3}}\lambda_0\xi\eta M_t\sqrt{x}e^{-i\phi_p/2}\left\{S-\frac{x}{15}\left[12\Gamma\frac{\delta m}{M_t}+S\left(15+34\eta\right)\right]\right\}\,, \tag{6.44a}$$

$$\lambda_{1(22)} = -\,2\sqrt{\frac{2\pi}{15}}\lambda_0\xi\Gamma\eta M_t x e^{-i\phi_p}\,, \tag{6.44b}$$

$$\lambda_{1(31)} = -\,\frac{i}{10}\sqrt{\frac{\pi}{21}}\lambda_0\xi\left(\Gamma\frac{\delta m}{M_t}+2\eta S\right)\eta M_t x^{3/2}e^{-i\phi_p/2}\,. \tag{6.44c}$$

Equations (6.24) and (6.44) are all the sets of radiative moments that we will need to compute the GW memory effects in the next section.

6.4 Memory Effects

In this section, we compute the displacement and spin GW memory effects produced by a quasi-circular compact-binary inspiral. The displacement and spin memory effects are both constructed from the shear tensor c_{AB}, and they have sky patterns with different parities. It is then useful to first decompose the shear tensor into electric- and magnetic-parity parts as follows:

$$c_{AB} = \frac{1}{2}(2\eth_A\eth_B - q_{AB}\text{Đ}^2)\Theta + \epsilon_{C(A}\eth_{B)}\eth^C\Psi\,, \tag{6.45}$$

where $\eth^2 \equiv \eth^A \eth_A$ is the Laplacian on the unit 2-sphere. We compute the displacement and spin GW memory effects using the BMS flux-charge balance laws that were computed in Brans-Dicke theory in [82]. We focus on the nonlinear GW memory effects and the null memory associated with the stress-energy tensor of the scalar waves. These effects can be computed using low PN-order oscillatory waveforms and the BMS balance laws, whereas if one were to try to compute them directly through the relaxed Einstein equations in harmonic gauge, one would need to compute the gravitational waveform at a higher PN order in BD theory than has been completed thus far. We also argue that the ordinary parts of the GW memory effects are of a higher PN order than the nonlinear and null parts in Appendix L.

6.4.1 Spherical harmonics and angular integrals

We will compute multipole moments of the GW memory effects, starting from the oscillatory tensor and scalar waves expanded in terms of the multipole moments in Eqs. (6.24) and (6.44), respectively. Evaluating these multipole moments involves computing angular integrals involving products of three spherical harmonics of different types (scalar, vector, and tensor). We instead follow the strategy in, e.g., [152,154], in which the vector and tensor harmonics are recast in terms of spin-weighted spherical harmonics. The angular integrals then involve products of three spin-weighted spherical harmonics (we use the conventions for the spherical harmonics in [154]). We also use the notation for the integral of three spin-weighted spherical harmonics in [152]

$$\mathcal{B}_l(s',l',m';s'',l'',m'') \equiv \int d^2\Omega\, ({}_{s'}Y_{l'm'})({}_{s''}Y_{l''m''})({}_{s'+s''}\bar{Y}_{l(m'+m'')})\,, \tag{6.46}$$

which can be written in terms of Clebsch-Gordan coefficients (denoted by $\langle l', m'; l'', m''|l, m'+m''\rangle$) as was shown, e.g., in [427] (though using the conventions of [154]):

$$\mathcal{B}_l(s',l',m';s'',l'',m'') = (-1)^{l+l'+l''}\sqrt{\frac{(2l'+1)(2l''+1)}{4\pi(2l+1)}} \times\langle l', s'; l'', s''|l, s'+s''\rangle\langle l', m'; l'', m''|l, m'+m''\rangle\,. \tag{6.47}$$

The multipolar expansion of the nonlinear memory effects in terms of the radiative moments U_{lm} and V_{lm} have the same form as in GR, which is given in [154]. However, we will need to perform a new multipolar expansion of the null memory effects from the stress-energy tensor of the scalar field. For this expansion, we will need the vector spherical harmonics

$$T_A^{(e),lm} = \frac{1}{\sqrt{l(l+1)}}\eth_A Y_{lm}\,, \tag{6.48a}$$

$$T_A^{(b),lm} = \frac{1}{\sqrt{l(l+1)}}\epsilon_{AB}\eth^B Y_{lm}\,. \tag{6.48b}$$

In terms of the spin-weighted spherical harmonics and a complex dyad m^A on the unit 2-sphere, we can write the vector spherical harmonics as

$$T_A^{(e),lm} = \frac{1}{\sqrt{2}}\left({}_{-1}Y_{lm}m_A - {}_1Y_{lm}\bar{m}_A\right)\,, \tag{6.49a}$$

$$T_A^{(b),lm} = \frac{i}{\sqrt{2}}\left({}_{-1}Y_{lm}m_A + {}_1Y_{lm}\bar{m}_A\right)\,, \tag{6.49b}$$

where $m^A m_A = \bar{m}^A \bar{m}_A = 0$ and $m^A \bar{m}_A = 1$.

6.4.2 Displacement memory effects

Supermomentum conservation requires that the change in the "potential" Θ that produces the electric part of the shear tensor, $\Delta\Theta$, must have its change between two retarded times and satisfy the following relationship [82]:

$$\int d^2\Omega\, \alpha\, \eth^2(\eth^2+2)\Delta\Theta = \int du\, d^2\Omega\, \alpha \left[N_{AB}N^{AB} + \frac{6+4\omega_{\rm BD}}{(\lambda_0)^2}(\partial_u\lambda_1)^2\right] + 8\int d^2\Omega\, \alpha \left(\Delta\mathcal{M} - \frac{1}{4\lambda_0}\eth^2\Delta\lambda_1\right) . \tag{6.50}$$

The supermomentum is conserved charge conjugate to a supertranslation BMS symmetry, and $\alpha(x^A)$ is the function that parametrizes the supertranslation symmetry. The first two terms inside the square brackets on the right-hand side of Eq. (6.50) produce the null memory (i.e., the memory sourced by massless fields) and the first term is the nonlinear (Christodoulou) memory. Both $\Delta\mathcal{M}$ and $\eth^2\Delta\lambda_1$ generate ordinary memory [82], but we argue in App. L that the ordinary memory is a higher PN-order effect. We will then focus on only the null memory, and we will derive separately the contributions of the tensor energy flux and the scalar energy flux. We denote the nonlinear (tensor) part by $\Delta\Theta^{\rm T}$ and the null part from the scalar field by $\Delta\Theta^{\rm S}$; the full memory effect is then the sum of the two components:

$$\Delta\Theta = \Delta\Theta^{\rm T} + \Delta\Theta^{\rm S} . \tag{6.51}$$

While $\Delta\Theta$ is the quantity most straightforwardly constrained by supermomentum conservation, it is the change in the strain Δh that is the more commonly used gravitational-wave observable. It is thus useful to relate the potential $\Delta\Theta$ to the strain. To do this, we will first introduce the following notation for just the electric part of the change in the shear, Δc_{AB}:

$$\Delta c_{AB,(e)} = \frac{1}{2}(2\eth_A\eth_B - q_{AB}\text{Đ}^2)\Delta\Theta\,. \tag{6.52}$$

We expand $\Delta\Theta$ in scalar spherical harmonics

$$\Delta\Theta = \sum_{l,m}\Delta\Theta_{lm}Y^{lm}(\iota,\varphi)\,, \tag{6.53}$$

where $l \geq 2$ (and $-l \leq m \leq l$), because the $l \leq 1$ harmonics are in the kernel of the operator $2\eth_A\eth_B - q_{AB}\text{Đ}^2$. By substituting (6.53) into Eq. (6.52) and using the definition of $T^{(e),lm}_{AB}$ in Eq. (6.14a), we can relate $\Delta\Theta_{lm}$ to $\Delta c_{(e),lm}$ via Eq. (6.13),

$$\Delta c_{(e),lm} = \sqrt{\frac{(l+2)!}{2(l-2)!}}\Delta\Theta_{lm}\,. \tag{6.54}$$

The above equation will be necessary when we construct the waveform from $\Delta\Theta_{lm}$. Specifically, we can compute the waveform by combining Eqs. (6.9), (6.20), and (6.54) in Eq. (6.8) to obtain

$$\Delta h^{(\text{disp})} = \frac{1}{\sqrt{2}R}\sum_{l,m}\sqrt{\frac{(l+2)!}{2(l-2)!}}\Delta\Theta_{lm}\;{}_{-2}Y_{lm}\,. \tag{6.55}$$

We will denote the memory waveform Δh^{disp} as a sum of the tensor-sourced, $\Delta h^{\text{disp,T}}$, and scalar-sourced $\Delta h^{\text{disp,S}}$ constributions as follows:

$$\Delta h^{\text{disp}} = \Delta h^{\text{disp,T}} + \Delta h^{\text{disp,S}}\,. \tag{6.56}$$

We first compute $\Delta h^{\mathrm{disp,T}}$ followed by $\Delta h^{\mathrm{disp,S}}$.

6.4.2.1 Displacement memory effect sourced by the tensor energy flux

The contribution to $\Delta\Theta$ from the tensor energy flux in Eq. (6.50) has the same form as in GR,

$$\eth^2(\eth^2+2)\Delta\Theta^{\mathrm{T}} = \int_{u_i}^{u_f} du\, \alpha(x^C) N_{AB} N^{AB} \,, \tag{6.57}$$

but there is a subtlety related to the limits of integration (u_i and u_f) in the retarded-time integral over u. Because we work in an approximation in which $\xi \ll x$, the lower limit u_i must start at a PN parameter x_i for which $x_i \gg \xi$. This differs from the corresponding convention in GR, in the limit $u_i \to -\infty$ is often taken. The upper limit, u_f is a retarded time at which the corresponding PN parameter x_f, is sufficiently large that the PN approximation (at the order at which we work) becomes inaccurate.

The multipolar expansion of $\Delta\Theta^{\mathrm{T}}$ proceeds exactly as in GR (and we note just a few features of the calculation here; see [154] for further details). We can first replace the function $\alpha(x^C)$ with the scalar spherical harmonic $\bar{Y}_{lm}$ and then use Eqs. (6.12), (6.13), (6.15), and (6.53) and the form of the result for the moments $\Delta\Theta^{\mathrm{T}}_{lm}$ in terms of the radiative moments $\dot{U}_{lm}$ and $\dot{V}_{lm}$ is the same as that derived in GR in [154]:

$$\begin{aligned}\Delta\Theta^{\mathrm{T}}_{lm} &= \frac{1}{2}\frac{(l-2)!}{(l+2)!}\sum_{l',l'',m',m''}\mathcal{B}_l\left(-2,l',m';2,l'',m''\right)\\ &\times \int_{u_i}^{u_f} du\left[s^{l,(+)}_{l';l''}\left(\dot{U}_{l'm'}\dot{U}_{l''m''}+\dot{V}_{l'm'}\dot{V}_{l''m''}\right)+2is^{l,(-)}_{l';l''}\dot{U}_{l'm'}\dot{V}_{l''m''}\right].\end{aligned} \tag{6.58}$$

We however, introduced the coefficients

$$s^{l,(\pm)}_{l';l''} = 1 \pm (-1)^{l+l'+l''} \tag{6.59}$$

that were used in [152] to make the notation more compact. As in [154], the sum runs over $l', l'' \geq 2$ and l must be in the range $|l' - l''| \leq l \leq |l' + l''|$ so that the coefficients $\mathcal{B}_l(-2, l', m'; 2, l'', m'')$ given in Eq. (6.47) are nonzero. One must also have $m = m' + m''$ for the coefficients $\mathcal{B}_l(-2, l', m'; 2, l'', m'')$ to be nonzero.

Because we focus on the leading GW memory effects in the non-oscillatory ($m = 0$) part of the waveform, this will further restrict m' and m'' to have equal magnitudes and opposite signs: $m' = -m''$. While the abstract expression for $\Delta\Theta^{\mathrm{T}}_{lm}$ in terms of radiative multipole moments has exactly the same form as that in GR, the time-derivatives of the radiative multipole moments $\dot{U}_{l'm'}$ and $\dot{V}_{l'm'}$ in BD theory differ from the corresponding moments in GR. This leads to a number of order ξ terms in the expression for the GW memory effect that we will give below.

Next, we will summarize how we compute the memory waveforms, including which radiative multipoles we need and at what PN-order accuracy we require these multipole moments. For concreteness, let us first focus on products of the mass moments $\dot{U}_{l'm'}\dot{U}_{l''m''}$ in Eq. (6.58). We perform the integral over u by using the chain rule to recast the integral over u in terms of an integral over x

$$\int du \dot{U}_{l'm'}\dot{U}_{l''m''} = \int \frac{d}{dx}U_{l'm'}\frac{d}{dx}U_{l''m''}\dot{x}dx\,, \tag{6.60}$$

as was outlined in, e.g., [164] [the integrals for the other products of radiative moments in Eq. (6.58) are evaluated similarly]. In GR, the Newtonian-order memory waveform can be calculated from just $U_{l'm'} = U_{22}$ and $U_{l''m''} = U_{2,-2}$ (and similarly $U_{l'm'} = U_{2,-2}$ and $U_{l''m''} = U_{22}$), with U_{22} evaluated at Newtonian order, as well. In BD theory, however, both $\dot{x}$ and $\frac{d\phi_p}{dx}$ (the latter term coming from $\frac{dU_{lm}}{dx}$) have contributions from the dipole moments of the scalar field, and these effects enter at a −1PN order relative to the GR result (and they are proportional to ξ). To obtain the full result at the Newtonian order requires the 1PN contributions to U_{22}, $\dot{x}$ and $\frac{d\phi_p}{dx}$. Because the −1PN terms of $\dot{x}$ and $\frac{d\phi_p}{dx}$ are proportional to ξ, we only need the parts of the 1PN terms in U_{22}, $\dot{x}$ and $\frac{d\phi_p}{dx}$ that are independent of ξ to compute $\Delta\Theta^{\mathrm{T}}_{lm}$. Hence, the BD corrections to the 1PN terms in Eqs. (6.24a), (6.35) and (6.37) were not given. The −1PN term in $\dot{x}$ also requires that we compute the part of the GW memory waveform sourced by products of the other radiative multipole moments in Eq. (6.24) (specifically V_{21}, U_{31}, U_{33}, U_{42} and their complex conjugates) to obtain a Newtonian-order-accurate result; in GR, these moments all source higher PN corrections to the GW memory effect.

Considering the radiative multipoles described above, we can then compute the GW memory moments $\Delta\Theta^{\mathrm{T}}_{l0}$ from Eq. (6.58). Because the memory is electric-type, only even l moments are nonvanishing. When written in terms of the relevant radiative moments U_{lm} and V_{lm}, the $\Delta\Theta^{\mathrm{T}}_{l0}$ are given by

$$\Delta\Theta^{\mathrm{T}}_{20} = \frac{1}{168}\sqrt{\frac{5}{\pi}}\int_{u_i}^{u_f} du\left(2|\dot{U}_{22}|^2 - |\dot{V}_{21}|^2 + \sqrt{7}\Im\left[\sqrt{2}\dot{\bar{U}}_{31}\dot{V}_{21} + \sqrt{5}\dot{\bar{U}}_{22}\dot{V}_{32}\right]\right.$$
$$\left.+\sqrt{5}\Re\left[\dot{\bar{U}}_{42}\dot{U}_{22}\right]\right) , \tag{6.61a}$$

$$\Delta\Theta^{\rm T}_{40} = \frac{1}{23760\sqrt{\pi}}\int_{u_i}^{u_f} du \left(\frac{11}{7}|\dot{U}_{22}|^2 - \frac{44}{7}|\dot{V}_{21}|^2 - 21|\dot{U}_{33}|^2 - 7|\dot{U}_{31}|^2 \right.$$
$$\left. + \frac{324\sqrt{5}}{7}\Re\left[\dot{\bar{U}}_{42}\dot{U}_{22}\right] + \frac{22}{\sqrt{7}}\Im\left[2\sqrt{5}\dot{\bar{U}}_{22}\dot{V}_{32} + 5\sqrt{2}\dot{\bar{V}}_{21}\dot{U}_{31}\right]\right), \quad (6.61b)$$

$$\Delta\Theta^{\rm T}_{60} = -\frac{1}{36960\sqrt{13\pi}}\int_{u_i}^{u_f} du \left(|\dot{U}_{33}|^2 + 15|\dot{U}_{31}|^2 - 4\sqrt{5}\Re\left[\dot{\bar{U}}_{42}\dot{U}_{22}\right]\right). \quad (6.61c)$$

Next, we use Eqs. (6.24a), (6.24b)–(6.24f), and (6.35)–(6.37) to perform the integral over u and to write the moments in terms of x. With the identity $(\delta m/M_t)^2 = 1-4\eta$, we can write the moments as

$$\Delta\Theta^{\rm T}_{20} = \frac{2\sqrt{5\pi}}{21}M_t\eta\Delta x\left\{1 - \frac{4}{3}\mathscr{G} - \frac{5\xi S^2}{48\Delta x}\ln\left(\frac{x_f}{x_i}\right)\right.$$
$$\left. -\xi\left[1 + \mathscr{F} + S^2\left(\frac{1915}{96768} + \frac{665\eta}{1152}\right)\right]\right\}, \quad (6.62a)$$

$$\Delta\Theta^{\rm T}_{40} = \frac{\sqrt{\pi}}{1890}M_t\eta\Delta x\left\{1 - \frac{4}{3}\mathscr{G} - \frac{5\xi S^2}{48\Delta x}\ln\left(\frac{x_f}{x_i}\right)\right.$$
$$\left. -\xi\left[1 + \mathscr{F} + S^2\left(-\frac{737045}{709632} + \frac{143395\eta}{25344}\right)\right]\right\}, \quad (6.62b)$$

$$\Delta\Theta^{\rm T}_{60} = -\sqrt{\frac{\pi}{13}}\frac{5M_t\eta S^2\xi\Delta x}{178827264}(-839 + 3612\eta). \quad (6.62c)$$

We use the notation $\Delta x = x_f - x_i$, where x_i and x_f correspond to the PN parameter x evaluated at an early time u_i and a final time u_f during the inspiral, respectively. The terms outside the curly braces in Eq. (6.62) in the expressions for $\Delta\Theta^{\rm T}_{20}$ and $\Delta\Theta^{\rm T}_{40}$ are equal to the equivalent results in GR.

Finally, we will construct the displacement memory waveform from the $\Delta\Theta_{l0}$ in Eq. (6.62). To do this, it is helpful to have the expressions for the spin-weighted spherical harmonics

$$ {}_{-2}Y_{20}(\iota,\varphi) = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\iota\,, \tag{6.63a} $$

$$ {}_{-2}Y_{40}(\iota,\varphi) = \frac{3}{16}\sqrt{\frac{10}{\pi}}\sin^2\iota(7\cos^2\iota-1)\,, \tag{6.63b} $$

$$ {}_{-2}Y_{60}(\iota,\varphi) = \frac{1}{64}\sqrt{\frac{1365}{\pi}}\sin^2\iota(33\cos^4\iota-18\cos^2\iota+1)\,. \tag{6.63c} $$

Substituting Eqs. (6.62) and (6.63) into Eq. (6.55), we obtain the displacement memory waveform due to the tensor energy flux in BD theory. We find the waveform only contains the plus polarization, and at Newtonian order, it is given by

$$ \begin{aligned} \Delta h_+^{(\mathrm{disp,T})} = {} & \frac{\eta M_t\Delta x}{48R}\sin^2\iota(17+\cos^2\iota)\left[1-\frac{4}{3}\mathscr{G}-\frac{5\xi S^2}{48\Delta x}\ln\left(\frac{x_f}{x_i}\right)-\xi(1+\mathscr{F})\right. \\ & \left.-\xi S^2\left(\frac{81145}{73728}-\frac{65465}{18432}\eta\right)+\xi S^2\left(\frac{20975}{172032}-\frac{1075}{2048}\eta\right)\cos^2\iota\right] \\ & +\frac{M_t\eta S^2\xi\Delta x}{R}\sin^2\iota\left(\frac{783875}{2064384}-\frac{35575}{24576}\eta\right)\,. \end{aligned} \tag{6.64} $$

The expression in front of the square brackets on the first line of Eq. (6.64) is the same as the Newtonian-order waveform for the memory effect in GR. The terms within the square bracket highlight a number of corrections introduced into the memory waveform amplitude in BD theory. These include effects related to the change in the amplitude of the $l=2$, $m=\pm2$ modes (the ξ and $\mathscr{G}$ terms) and changes in the frequency evolution (the $\mathscr{F}$ term). In particular, there is a change in the scaling of the memory with the PN parameter that is proportional to $\ln(x_f/x_i)$, which arises because of scalar dipole radiation. At the end of the first and on the second line of Eq. (6.64) are a number of terms arising from 1PN-order products of multipole moments coupling to the -1PN term in the frequency evolution (or $\dot{x}$); the terms on the second line lead to a small (order ξ) difference to the sky pattern of the memory effect between BD theory and GR.

6.4.2.2 Displacement memory effect sourced by the scalar energy flux

The displacement memory effect also has a contribution from the integral of the energy density radiated through scalar radiation. Its effect can be computed from the terms proportional to $(\partial_u \lambda_1)^2$ in Eq. (6.50):

$$\eth^2(\eth^2 + 2)\Delta\Theta^{\rm S} = \frac{6 + 4\omega_{\rm BD}}{(\lambda_0)^2} \int_{u_i}^{u_f} (\partial_u \lambda_1)^2 du \,. \tag{6.65}$$

Expanding λ_1 and $\Delta\Theta^{\rm S}$ in scalar spherical harmonics as in Eqs. (6.38) and (6.53), respectively, we can determine the multipole moments $\Delta\Theta^{\rm S}_{lm}$ in terms of the multipole moments $\lambda_{1(lm)}$ and the integrals of three spherical harmonics defined in Eq. (6.46). The result is

$$\Delta\Theta^{\rm S}_{lm} = \frac{(l-2)!}{(l+2)!} \frac{6 + 4\omega_{\rm BD}}{(\lambda_0)^2} \sum_{l',m',l'',m''} B_l(0, l', m'; 0, l'', m'') \int_{u_i}^{u_f} du \, \dot{\lambda}_{1(l'm')} \dot{\lambda}_{1(l''m'')} , \tag{6.66}$$

where $l \geq 2$ and $l', l'' \geq 1$ (and m, m', and m'' must satisfy $m + m' + m'' = 0$). Because λ_1 is proportional to ξ, one might be concerned that $\Delta\Theta^{\rm S}$ will be an $O(\xi^2)$ effect and be negligible in our approximation. Note, however, that $3 + 2\omega_{\rm BD} = 2/\xi - 1$, which implies that $\Delta\Theta^{\rm S}$ is an $\mathcal{O}(\xi)$ effect, which means that the integrand can be one order higher in ξ and still produce an effect at linear order in ξ.

We will now discuss which scalar multipole moments contribute to the displacement memory waveform and the accuracies at which we need the different moments to obtain a Newtonian-order accurate memory waveform at linear order in ξ. The 1PN scalar multipole moments $\lambda_{1(lm)}$ that are computed from Eq. (6.39) are at least $\mathcal{O}(\xi)$; thus, to linear order in ξ, we can treat $6 + 4\omega_{\text{BD}}$ as $4/\xi$. We will evaluate the integral over u in Eq. (6.66) by converting it to an integral over x (as was done in Sec. 6.4.2.1), but unlike in Sec. 6.4.2.1 we need to keep only the GR contribution in $\dot{x}$, which scales as x^5. Similarly, when computing $d\phi_p/dx$, we again need to retain just the GR contribution that goes as $x^{-7/2}$. The scalar field has a radiative dipole moment, which from Eq. (6.44a), goes as $x^{-1/2}$. The leading-order part of the integrand (proportional to $\dot{\lambda}_{1(11)}\dot{\lambda}_{1(1,-1)}$) scales as $1/x$ rather than $O(x^0)$ as in GR; in this sense, the integrand is a -1PN order.[5] This product of dipole terms will also contribute to the waveform at 0PN order because of the $O(x^{3/2})$ terms in $\lambda_{1(11)}$; see Eq. (6.44a). To work to linear order in ξ, we do not need to go to a higher PN order for $\lambda_{1(11)}$. Similar arguments show that the remaining scalar moments in Eq. (6.44) (namely, $\lambda_{1(22)}$ and $\lambda_{1(31)}$) are the ones that are needed to compute Newtonian-order accurate moments of $\Delta\Theta^{\text{S}}_{l0}$.

We then first list the integrals of the relevant moments $\lambda_{1(lm)}$ that contribute to

[5]Note, however, when the integrand is integrated, it will again give rise to a logarithm in x rather than being proportional to x, as in GR. We will refer to this effect sometimes as a -1PN term, since it comes from such an effect in the integrand, and since log terms do not enter into the PN order counting of a term.

the scalar-sourced part of the tensor memory at Newtonian order:

$$\Delta\Theta^{\mathrm{S}}_{20} = -\frac{1}{42\sqrt{5\pi}\lambda_0^2\xi}\int_{u_i}^{u_f} du\left(7|\dot{\lambda}_{1(11)}|^2 + 10|\dot{\lambda}_{1(22)}|^2 - 6\sqrt{14}\Re\left[\dot{\lambda}_{1(11)}\dot{\bar{\lambda}}_{1(31)}\right]\right), \tag{6.67a}$$

$$\Delta\Theta^{\mathrm{S}}_{40} = \frac{1}{630\sqrt{\pi}\lambda_0^2\xi}\int_{u_i}^{u_f} du\left(|\dot{\lambda}_{1(22)}|^2 - 2\sqrt{14}\Re\left[\dot{\lambda}_{1(11)}\dot{\bar{\lambda}}_{1(31)}\right]\right). \tag{6.67b}$$

As we did with the the memory sourced by the tensor energy flux, we first substitute the expressions for the scalar moments in Eq. (6.44) into Eq. (6.67) and evaluate the integrals in terms of x by using Eqs. (6.35)–(6.37). This gives the following results

$$\Delta\Theta^{\mathrm{S}}_{20} = -\frac{M_t\eta\xi\sqrt{5\pi}}{144}\left\{S^2\ln\left(\frac{x_f}{x_i}\right) + \Delta x\left[\frac{8}{7}\Gamma^2 - \frac{23}{14}S\Gamma\frac{\delta m}{M_t} + S^2\left(\frac{71}{336} - \frac{157}{84}\eta\right)\right]\right\}, \tag{6.68a}$$

$$\Delta\Theta^{\mathrm{S}}_{40} = \frac{M_t\eta\xi\sqrt{\pi}}{30240}\left(8\Gamma^2 - \Gamma\frac{\delta m}{M_t} + 2S^2\eta\right). \tag{6.68b}$$

We then substitute the equations into Eq. (6.55) and with the expressions for the spin-weighted spherical harmonics in Eq. (6.63), we arrive at the following equation for the displacement memory waveform sourced by the scalar energy flux:

$$\begin{aligned}\Delta h_+^{(\mathrm{disp,S})} = &-\frac{5\eta M_t\xi}{192R}\sin^2\iota\left\{S^2\ln\left(\frac{x_f}{x_i}\right) + \Delta x\left[\frac{6}{5}\Gamma^2 - \frac{33}{20}S\Gamma\frac{\delta m}{M_t}\right.\right.\\ &\left.\left.+S^2\left(\frac{71}{336} - \frac{113}{60}\eta\right) - \frac{1}{20}\left(8\Gamma^2 - S\Gamma\frac{\delta m}{M_t} - 2S^2\eta\right)\cos^2\iota\right]\right\}.\end{aligned} \tag{6.69}$$

There are terms in Eq. (7.2e) of [179] which, after performing the integration over time in our approximation, produce a -1PN term in the waveform; this term agrees with the first line of Eq. (6.69). The Newtonian-order terms require going to a higher PN order than was computed in [179], but the BMS balance laws allow us to determine this expressions.

Because the total GW memory Δh^{disp} is a sum of the scalar and tensor contributions, as given in Eq. (6.56), then the scalar-sourced contribution will produce an additional correction to the amplitude and the sky pattern beyond the corrections given in Eq. (6.64) for the tensor-sourced part of the memory effect.

6.4.3 Spin GW memory effect

The spin memory effect is a lasting offset in the time integral of the magnetic part of the shear tensor. It can be constrained through the evolution equation for the Bondi angular momentum aspect, or equivalently the magnetic part of the flux of the super Lorentz charges. To compute the spin memory effect, it is helpful to denote the time integral of the potential Ψ, which gives rise to the magnetic part of c_{AB} in Eq. (6.45), as $\Delta\Sigma$:

$$\Delta\Sigma = \int \Psi du \,. \tag{6.70}$$

We leave off the limits of integration for convenience here, though we will restore these limits below when we compute the spin memory in the PN limit. The generalized BMS balance law for the super angular momentum was given in [82], and analogously to the computation in GR (see, e.g., [153]), a term involving Eq. (6.70) was needed to ensure the balance law was satisfied and the charge was constructed from fields defined locally in time on a cut of constant Bondi coordinate u. The form of the balance law can be written as follows:

$$\int d^2\Omega\, \gamma \eth^2(\eth^2+2)\Delta\Sigma = -\frac{64\pi}{\lambda_0}\left(\Delta Q_{(\gamma)} + \Delta\mathcal{J}_{(\gamma)}\right) , \tag{6.71a}$$

where $Y^A = \epsilon^{AB}\eth_B\gamma$ is a smooth, magnetic-parity vector field on the 2-sphere and where we have defined

$$\Delta\mathcal{J}_{(\gamma)} = \frac{\lambda_0}{64\pi}\int du\, d^2\Omega\, \epsilon^{AD}\eth_D\gamma\Big[\eth_A(c_{BC}N^{BC}) + 2N^{BC}\eth_A c_{BC} - 4\eth_B(c_{AC}N^{BC}) + \frac{4\omega_{\mathrm{BD}}+6}{(\lambda_0)^2}(\partial_u\lambda_1\eth_A\lambda_1 - \lambda_1\eth_A\partial_u\lambda_1)\Big] , \tag{6.71b}$$

$$\Delta Q_{(\gamma)} = \frac{\lambda_0}{8\pi}\int d^2\Omega\, \epsilon^{AD}\eth_D\gamma\left[-3\Delta L_A - \frac{1}{4\lambda_0}\Delta(c_{AB}\eth^B\lambda_1 - \lambda_1\eth^B c_{AB})\right] . \tag{6.71c}$$

The net flux is denoted by $\Delta\mathcal{J}_{(\gamma)}$, the change in the charges is denoted by $\Delta Q_{(\gamma)}$, and the left-hand side of Eq. (6.71a) (which is related to the spin memory effect) is the additional term required for the balance law to be satisfied. Analogously to the displacement memory, the contribution of $\Delta Q_{(\gamma)}$ to Eq. (6.71a) is referred to as ordinary spin memory, and $\Delta J_{(\gamma)}$ is the null spin memory (which contains a nonlinear part). We will focus on the null contribution to Eq. (6.71a) here, as we argue in App. L that the ordinary contribution to the spin memory is likely to be a higher PN effect than the null memory is.

As we did with the change in $\Delta\Theta$ related to the displacement memory effect, we will split $\Delta\Sigma$ into a sum of its contributions from the the tensor- and scalar-sourced parts of the angular momentum flux. We denote these two contributions by

$$\Delta\Sigma = \Delta\Sigma^{\mathrm{T}} + \Delta\Sigma^{\mathrm{S}} \,. \tag{6.72}$$

We compute these two contributions separately below.

In addition, while the spin-weight-zero quantity $\Delta\Sigma$ is the most convenient quantity to compute from the balance law (6.71a), the shear c_{AB} or strain h are the more commonly used quantities in gravitational waveform modeling and data analysis. We thus relate $\Delta\Sigma$ to a time integral of the shear; specifically, we denote the change in the time integral of the magnetic-parity part of the shear tensor by

$$\int c_{AB,(b)} du = \epsilon_{C(A}\eth_{B)}\eth^{C}\Delta\Sigma \,. \tag{6.73}$$

Expanding $\Delta\Sigma$ in terms of scalar spherical harmonics

$$\Delta\Sigma = \sum_{l,m} \Delta\Sigma_{lm} Y^{lm}(\iota, \varphi) \tag{6.74}$$

(with $l \geq 2$ and $-l \leq m \leq l$), then we can relate the multipole moments $\Delta\Sigma_{lm}$ to the time integrals of the radiative current moments V_{lm} via Eqs. (6.73), (6.14b), and (6.13). The result is that

$$\int V_{lm} du = \sqrt{\frac{(l+2)!}{2(l-2)!}} \Delta\Sigma_{lm} \,. \tag{6.75}$$

Equation (6.75) allows us to compute the time integral of the strain from $\Delta\Sigma_{lm}$.

6.4.3.1 Spin memory effect sourced by the tensor angular momentum flux

The null part of the spin memory effect in Eqs. (6.71a) and (6.71b), that is sourced by the tensor GWs can be computed from the following expression:

$$\int d^2\Omega\, \gamma \eth^2(\eth^2+2)\Delta\Sigma^{\mathrm{T}} = \int_{u_i}^{u_f} du\, d^2\Omega\, \epsilon^{AD}\eth_D\gamma\Big[\eth_A(c_{BC}N^{BC}) + 2N^{BC}\eth_A c_{BC} - 4\eth_B(c_{AC}N^{BC})\Big]. \tag{6.76}$$

It has the same form as the analogous expression in GR. It can then be recast into the same form as in Eq. (3.23) of [153] by using the identities in Appendix C of [153]. This expression was then the starting point for the multipolar expansion of the spin memory effect given in [154]. We reproduce the result from [154] below; however, we first introduce, in addition to $s^{l,(\pm)}_{l';l''}$ in Eq. (6.59), the following coefficient (defined in [152]) to make the expression more concise:

$$c^l_{l'm';l''m''} = 3\sqrt{(l'-1)(l'+2)}\mathcal{B}_l(-1,l',m';2,l'',m'') + \sqrt{(l''-2)(l''+3)}\mathcal{B}_l(-2,l',m';3,l'',m'')\,. \tag{6.77}$$

The expression for the moments $\Delta\Sigma_{lm}$ can then be derived through a lengthy calculation outlined in [154], and the result is given by

$$\Delta\Sigma^{\mathrm{T}}_{lm} = \frac{1}{4\sqrt{l(l+1)}}\frac{(l-2)!}{(l+2)!} \times \sum_{\substack{l',l'',\\ m',m''}} c^l_{l'm';l''m''}\int_{u_i}^{u_f} du\Big[is^{l,(-)}_{l';l''}\Big(U_{l'm'}\dot{U}_{l''m''} - \dot{U}_{l'm'}U_{l''m''} + V_{l'm'}\dot{V}_{l''m''}$$

$$-\dot{V}_{l'm'}V_{l''m''}\Big) - s^{l,(+)}_{l';l''}\left(U_{l'm'}\dot{V}_{l''m''} + \dot{V}_{l'm'}U_{l''m''} - \dot{U}_{l'm'}V_{l''m''} - V_{l'm'}\dot{U}_{l''m''}\right)\Big] \,. \tag{6.78}$$

To compute the spin memory effect to Newtonian order, we need precisely the same set of radiative multipole moments U_{lm} and V_{lm} that were used to compute the displacement memory effect in Sec. 6.4.2.1. We then compute the spin memory effect following the same procedure in Sec. 6.4.2.1 by first writing the needed moments of $\Delta\Sigma_{lm}$ in terms of integrals of the relevant radiative moments U_{lm} and V_{lm}:

$$\begin{aligned}\Delta\Sigma^{\mathrm{T}}_{30} = -\frac{1}{720\sqrt{7\pi}}\int_{u_i}^{u_f} du \Bigg\{ &\Im\Big[9\left(\dot{U}_{22}U_{22} - 2\dot{V}_{21}V_{21}\right) + 7\left(\dot{U}_{31}U_{31} - \dot{U}_{33}U_{33}\right) \\ &+11\sqrt{5}\left(\dot{U}_{22}U_{42} + \dot{U}_{42}U_{22}\right)\Big] + \Re\Big[5\sqrt{35}\left(\dot{V}_{32}U_{22} - \dot{U}_{22}V_{32}\right) \\ &-5\sqrt{\frac{7}{2}}\left(\dot{V}_{21}U_{31} - \dot{U}_{31}V_{21}\right)\Big]\Bigg\} \,, \end{aligned} \tag{6.79a}$$

$$\begin{aligned}\Delta\Sigma^{\mathrm{T}}_{50} = \frac{1}{5040\sqrt{11\pi}}\int_{u_i}^{u_f} du \Bigg\{ &\Im\Big[5\left(\dot{U}_{33}U_{33} + 5\dot{U}_{31}U_{31}\right) - \frac{38}{\sqrt{5}}\left(\dot{U}_{22}U_{42} + \dot{U}_{42}U_{22}\right)\Big] \\ &+\Re\Big[2\sqrt{\frac{7}{5}}\left(\dot{U}_{22}V_{32} - \dot{V}_{32}U_{22}\right) + 2\sqrt{14}\left(\dot{V}_{21}U_{31} - \dot{U}_{31}V_{21}\right)\Big]\Bigg\} \,. \end{aligned} \tag{6.79b}$$

As in Sec. 6.4.2.1, we then use Eqs. (6.24a), (6.24b)–(6.24f), and (6.35)–(6.37) to evaluate the integrals in Eqs. (6.79a)–(6.79b) and to write the expression for the moments $\Delta\Sigma_{30}$ and $\Delta\Sigma_{50}$ in terms of x. Unlike in Sec. 6.4.2.1, the integrand does not depend on $\dot{x}$, when written as an integral over x, because there is only one time derivative of the radiative multipole moments. The result of this integration is given by

$$\Delta\Sigma_{30}^{\rm T} = \sqrt{\frac{\pi}{7}}\frac{\eta M_t^2}{10}\left\{-\frac{5\xi S^2}{144}\Delta(x^{-3/2}) + \Delta(x^{-1/2})\left[1 - \frac{4}{3}\mathscr{G} - (1+\mathscr{F})\xi + \frac{5(47796\eta + 5003)S^2\xi}{435456}\right]\right\}, \tag{6.80a}$$

$$\Delta\Sigma_{50}^{\rm T} = -\sqrt{\frac{\pi}{11}}\frac{\eta M_t^2\xi S^2}{12192768}(21588\eta - 4117)\Delta(x^{-1/2}), \tag{6.80b}$$

where we have defined $\Delta(x^{-1/2}) = x_f^{-1/2} - x_i^{-1/2}$ and similarly for $\Delta(x^{-3/2})$.

In GR, the only term that appears in the Newtonian-order spin-memory waveform is $\Delta(x^{-1/2})$ times the coefficient outside the curly braces in Eq (6.79a). The remaining terms in $\Delta\Sigma_{30}$ and the entire expression for $\Delta\Sigma_{50}$ appear in the BD-theory waveform at Newtonian order because of the -1 PN term in $d\phi_p/dx$.

Finally, we construct the time-integrated strain from the moments $\Delta\Sigma_{l0}$ in Eqs. (6.80a)–(6.80b). Using Eqs. (6.8), (6.9), (6.20), and (6.75), we can write the relation between the time integral of h and a general $\Delta\Sigma_{lm}$ as

$$\int_{u_i}^{u_f} h^{(\rm spin)} du = \frac{-i}{2R}\sum_{l,m}\sqrt{\frac{(l+2)!}{(l-2)!}}\Delta\Sigma_{lm}\ {}_{-2}Y_{lm}\,. \tag{6.81}$$

Because the modes $\Delta\Sigma_{l0}$ that produce the time-integrated strain $h^{(\rm spin)}$ in Eq. (6.80a) and (6.80b) are real, then from Eq. (6.7) it follows that the spin memory enters in the cross mode polarization of gravitational waves (as it does in GR [154]). Finally substituting Eqs. (6.80a)–(6.80b) into (6.81), and using the expressions for the spin-weighted spherical harmonics

$${}_{-2}Y_{30}(\iota,\varphi) = \frac{1}{4}\sqrt{\frac{105}{2\pi}}\sin^2\iota\cos\iota\,, \tag{6.82a}$$

$$
{}_{-2}Y_{50}(\iota,\varphi) = \frac{1}{8}\sqrt{\frac{1155}{2\pi}}\sin^2\iota\cos\iota(3\cos^2\iota - 1)\,, \tag{6.82b}
$$

we obtain for the time-integrated strain

$$
\begin{aligned}
\int_{u_i}^{u_f} du h_\times^{(\mathrm{spin,T})} = \frac{3\eta M_t^2}{8R}\sin^2\iota\cos\iota\Bigg\{ & -\frac{5\xi S^2}{144}\Delta(x^{-3/2}) + \Bigg[1 - \frac{4}{3}\mathscr{G} - (1+\mathscr{F})\xi \\
& - \xi S^2\left(\frac{3365}{6912}\eta + \frac{1915}{27648}\right)\Bigg]\Delta(x^{-1/2}) \\
& + \xi S^2\left(\frac{20585}{580608} - \frac{1285}{6912}\eta\right)\Delta(x^{-1/2})\cos^2\iota\Bigg\}\,.
\end{aligned} \tag{6.83}
$$

The expression for the u integral of $h_\times^{(\mathrm{spin,T})}$ is written such that the angular dependence and coefficient $3\eta M^2/(8R)$ outside of the curly braces coincides with the expression in GR at Newtonian order. With the curly braces there are several sorts of terms: (i) the first term proportional to $\Delta(x^{-3/2})$ is a -1PN term arising from the dipole term in the phase, but which have the same angular dependence as the spin memory effect in GR; (ii) the terms in the square bracket (aside from the factor of unity that reproduces the GR expression for the spin memory) are small BD corrections (proportional to ξ) that modify the amplitude of the waveform without changing its x or ι dependence; (iii) the final terms on the second line are those which have the same x dependence, but a different angular dependence from the GR expression (and are again proportional to ξ).

6.4.3.2 Spin memory effect sourced by the scalar angular momentum flux

The angular momentum flux per solid angle sourced by the scalar radiation produces a second contribution to the spin memory effect. Its contribution can be obtained from the scalar field terms in the balance law in Eq. (6.71a), which are given by

$$\int d^2\Omega \gamma \eth^4(\eth^2+2)\Delta\Sigma^{\mathrm{S}} = -\frac{6+4\omega_{\mathrm{BD}}}{(\lambda_0)^2}\int_{u_i}^{u_f} d^2\Omega du \epsilon^{AB}\eth_B\gamma(\dot{\lambda}_1\eth_A\lambda_1 - \lambda_1\eth_A\dot{\lambda}_1)\,. \tag{6.84}$$

The multipolar expansion of $\Delta\Sigma$ can be obtained by assuming γ is equal to the spherical harmonic $\bar{Y}_{lm}$, and then using the multipolar expansion of λ_1 in Eq. (6.38). After relating the gradients and curls of spherical harmonics in this expansion to the electric- and magnetic-parity vector harmonics in Eqs. (6.48a)–(6.48b) and then employing the relationship between these vector harmonics and the spin-weighted spherical harmonics in Eqs. (6.49a)–(6.49b), we can derive the moments $\Delta\Sigma_{lm}$ in terms of moments $\lambda_{1(lm)}$ (and their time derivatives and complex conjugates) and the coefficients $\mathcal{B}_l(s',l',m';s'',l'',m'')$ given in Eq. (6.46). The resulting expression is given below:

$$\Delta\Sigma^{\mathrm{S}}_{lm} = \frac{i(2\omega_{\mathrm{BD}}+3)}{\lambda_0^2\sqrt{l(l+1)}}\frac{(l-2)!}{(l+2)!}\sum_{l',m',l'',m''} s^{l,(-)}_{l';l''}\sqrt{l''(l''+1)}\mathcal{B}_l(0,l',m';1,l'',m'')$$
$$\times\int_{u_i}^{u_f} du(\dot{\lambda}_{1(l'm')}\lambda_{1(l''m'')} - \lambda_{1(l'm')}\dot{\lambda}_{1(l''m'')})\,. \tag{6.85}$$

The coefficient $s^{l,(-)}_{l';l''}$ is defined in Eq. (6.59).

The moments of $\lambda_{1(lm)}$ that contribute to the Newtonian-order spin memory effect are the same moments needed to compute the scalar-sourced displacement memory effect in Sec. 6.4.2.2. The only nonvanishing moment of $\Delta\Sigma$ at this order then is $\Delta\Sigma_{30}$, and to linear order in ξ it is given by

$$\Delta\Sigma^{\mathrm{S}}_{30} = -\frac{1}{30\xi(\lambda_0)^2}\frac{1}{\sqrt{7\pi}}\int_{u_i}^{u_f} du\, \Im\left[\dot{\lambda}_{1(22)}\bar{\lambda}_{1(22)} - \sqrt{\frac{7}{2}}\left(\dot{\lambda}_{1(11)}\bar{\lambda}_{1(31)} - \lambda_{1(11)}\dot{\bar{\lambda}}_{1(31)}\right)\right]. \tag{6.86}$$

We can then use the expressions for the moments in Eq. (6.44) to evaluate the integrals and write them in terms of x (analogously to what was done for the tensor-sourced part of the spin memory effect), and we find it is given by

$$\Delta\Sigma^{\mathrm{S}}_{30} = \frac{\sqrt{\pi}\eta M_t^2\xi}{1440\sqrt{7}}\left(2\eta S^2 + \Gamma S\frac{\delta m}{M_t} - 8\Gamma^2\right)\Delta\left(x^{-1/2}\right)\,, \tag{6.87}$$

It is then straightforward to use Eq. (6.81) to write the retarded-time integral of the spin memory waveform generated by scalar angular momentum flux as

$$\int_{u_i}^{u_f} du h_\times^{(\mathrm{spin,S})} = \frac{\eta M_t^2\xi}{384R}\left(2\eta S^2 + \Gamma\frac{\delta m}{M_t}S - 8\Gamma^2\right)\Delta\left(x^{-1/2}\right)\sin^2\iota\cos\iota\,. \tag{6.88}$$

The full retarded-time integral of the spin memory waveform $h_\times^{(\mathrm{spin})} = h_\times^{(\mathrm{spin,S})} + h_\times^{(\mathrm{spin,T})}$ then has an additional small correction linear in ξ to the GR expression that changes the amplitude but does not alter the x or ι dependence of the effect.

6.5 Conclusions

In this chapter, we computed the displacement and spin GW memory effects generated by nonprecessing, quasi-circular binaries in BD theory. We worked in the PN approximation, and the expression we computed are accurate to leading Newtonian order in the PN parameter x, and they include the leading-order corrections in the BD parameter ξ. Our calculations relied upon using the BMS balance laws associated with the asymptotic symmetries in BD theory (which are the same as in GR) computed recently. These balance laws permit us to determine the tensor memory effects, but the scalar GW memory effects associated with the breathing polarization are not constrained through these balance laws. We further focused on null contributions to the tensor memory effects, because we estimated that the ordinary (linear) memory effects (which are functions of both gravitational and scalar data) would contribute to the memory effects at a higher PN order than the null (including the nonlinear) memory effects.

Using the BMS balance laws has an advantage that it can determine the memory effects to a higher PN order than the direct integration of the relaxed Einstein equations in harmonic gauge. This allowed us to compute the memory effects at a higher PN order than had been previously computed. However, the balance laws take as input radiative and nonradiative data at large Bondi radius and this data must be obtained through some other method. Specifically, in the context of this chapter in PN theory, we needed to take as input the scalar and tensor gravitational waves computed in harmonic gauge in BD theory in [180]. This required us to compute a coordinate transformation between harmonic and Bondi gauges at leading order in the inverse distance to the source, so that we could relate the radiative GW data in these two different coordinate choices (and formalisms). There were relatively simple transformations that allowed us to relate the Bondi shear tensor to the transverse-traceless components of GW strain, and these relations were particularly simple when expressed in terms of the radiative multipole moments of the shear and strain tensors. There were similar expressions relating the scalar field at leading order in inverse distance in the two coordinate systems and the corresponding multipole moments of the scalar field. With these relationships between the multipolar expansions of the scalar field and the shear tensor in harmonic and Bondi gauges, we could then use the BMS balance laws to compute the GW memory effects.

The tensor GW memory effects in BD theory have several noteworthy differences from the corresponding effects in GR at the equivalent PN order. First, because of scalar dipole radiation in BD theory, there are relative −1PN-order terms in the memory effects in BD theory. The −1PN term in the displacement memory waveform comes from two sources in the supermomentum balance law: (i) directly from the energy flux of scalar radiation and (ii) indirectly from the energy flux in tensor radiation (specifically through dipole contributions to the frequency evolution and the GW phase). The spin memory waveform, however, has a relative −1PN correction from GR arising from only the energy flux of the tensor waves (the scalar-sourced part of the energy flux gives rise to a contribution at the same leading order as in GR, and it comes from products of dipole and octupole moments, as well as quadrupole moments with themselves). The absence and presence of the −1PN term, respectively, in the spin and displacement memory effects arises because of the different lowest multipole term in the sky pattern of the effects: the spin memory effect begins with the $l = 3$, $m = 0$ mode, whereas the displacement memory has an $l = 2$, $m = 0$ mode.

A second noteworthy feature is that the computation of Newtonian-order memory effects in BD theory requires radiative multipole moments of the strain at higher PN orders than are required in GR (in which the leading-order effects can be computed from just the $l = 2$, $m = 2$ radiative mass quadrupole moment and its complex conjugate). Because there are -1PN terms in the GW phase and the evolution of the GW frequency, computing the Newtonian GW memory effects requires higher-PN-order radiative multipole moments (including the current quadrupole, the mass and current octupoles, and the mass hexadecapole). These higher-order mass and current multipole moments also give rise to a sky pattern of the GW memory effect in BD theory that differs from the sky pattern of the effect in GR. In addition, the presence of dipole radiation required using that to compute the memory effects to Newtonian order, for which we needed to include 1PN corrections to the GW phase and frequency time derivative in BD theory (though we only required the GR limit of these 1PN corrections when working to linear order in the BD parameter ξ).

Finally, let us conclude by commenting on potential applications of the calculations of GW memory effects given in this chapter. Because the calculations herein have shown that the memory effect in BD theory differs from that in GR, it is natural to ask if these differences could be detected. Given the challenges of detecting the memory effect in GR with LIGO and Virgo [134, 135, 428] and the fact that the PN corrections are small, it would be more natural to consider whether next-generation gravitational-wave detectors—such as the space-based detector LISA [181] or ground-based detectors like the Einstein Telescope or Cosmic Explorer [182, 183]—could constrain BD theory through a measurement of the GW memory effect. The constraints could come from searching for differences in the leading-order amplitude of the effect, in the time dependence of the accumulation of the memory effect (through the different dependence on the PN parameter x), or in the sky pattern of memory effects in BD theory (the latter being similar to the hypothesis test described in [184]). Because memory effects accumulate most rapidly during the merger of compact binaries, having waveforms that go beyond the PN approximation and include the merger and ringdown would be important for producing the most accurate constraints.

Appendix

Appendix A

Original PPE Formalism

In this appendix, we review the original PPE formalism. In particular, we will show how the amplitude and phase corrections depend on conservative and dissipative corrections, where the former are corrections to the effective potential of a binary while the latter are those to the GW luminosity. We will mostly follow the analysis in [87].

First, let us introduce conservative corrections. We modify the reduced effective potential of a binary as

$$V_{\text{eff}} = \left(-\frac{M_t}{r_{12}} + \frac{L_z^2}{2\mu^2 r_{12}^2} \right) \left[1 + A \left(\frac{M_t}{r_{12}} \right)^p \right] , \tag{A.1}$$

where L_z is the z-component of the angular momentum. A and p show the magnitude and exponent of the non-GR correction term respectively. Such a modification to the effective potential also modifies Kepler's law. Taking the radial derivative of V_{eff} in Eq. (A.1) and equating it to zero gives modified Kepler's law as

$$\Omega^2 = \frac{M_t}{r_{12}^3} \left[1 + \frac{1}{2} A\, p \left(\frac{M_t}{r_{12}} \right)^p \right] . \tag{A.2}$$

The above equation further gives the orbital separation as

$$r_{12} = r_{12}^{\rm GR} \left[1 + \frac{1}{6} A\, p\, \eta^{-\frac{2p}{5}}\, \underline{u}^{2p} \right] \,, \tag{A.3}$$

where to leading PN order, $r_{12}^{\rm GR}$ is given by Kepler's law as $r_{12}^{\rm GR} = (M_t/\Omega^2)^{1/3}$. For a circular orbit, radial kinetic energy does not exist and the effective potential energy is same as the binding energy of the binary. Using Eq. (A.3) in Eq. (A.1) and keeping only to leading order in non-GR corrections, the binding energy becomes

$$E = -\frac{1}{2}\eta^{-2/5}\underline{u}^2 \left[1 - \frac{1}{3} A(2p-5)\eta^{-\frac{2p}{5}}\, \underline{u}^{2p} \right] \,. \tag{A.4}$$

Next, let us introduce dissipative corrections. Such corrections to the GW luminosity can be parameterized by

$$\dot{E} = \dot{E}_{\rm GR} \left[1 + B \left(\frac{M_t}{r_{12}} \right)^q \right] \,, \tag{A.5}$$

where $\dot{E}_{GR}$ is the GR luminosity which is proportional to $v^2(M_t/r_{12})^4$ with $v = r_{12}\,\Omega = (\pi M_t f)^{1/3}$ representing the relative velocity of binary components[1].

Let us now derive the amplitude corrections. First, using Eqs. (A.4) and (A.5) and applying the chain rule, the GW frequency evolution is given by

$$\begin{aligned} \dot{f} &= \frac{df}{dE}\frac{dE}{dt} \\ &= \dot{f}_{GR} \left[1 + B\eta^{-\frac{2q}{5}}\, \underline{u}^{2q} + \frac{1}{3} A(2p^2 - 2p - 3)\eta^{-\frac{2p}{5}}\, \underline{u}^{2p} \right] \,, \end{aligned} \tag{A.6}$$

where $\dot{f}_{\rm GR}$ is given by Eq. (2.6). Next, using Eqs. (A.3) and (A.6) to Eq. (2.9) and

[1] If we assume $\dot{E}_{\rm GR}$ to be proportional to $r_{12}^4\,\Omega^6$ which directly follows from the quadrupole formula without using Kepler's law, we will find slightly different expressions for $\dot{f}$ and the waveform [87]

keeping only to leading order in non-GR corrections, the GW amplitude in Fourier domain becomes

$$\tilde{\mathcal{A}}(f) = \tilde{\mathcal{A}}_{GR}\left[1 - \frac{B}{2}\eta^{-\frac{2q}{5}}\underline{u}^{2q} - \frac{1}{6}A(2p^2 - 4p - 3)\eta^{-\frac{2p}{5}}\underline{u}^{2p}\right] . \tag{A.7}$$

Next, we move onto deriving phase corrections. One can derive the GW phase in Fourier domain by integrating Eq. (2.18) twice. Equivalently, one can use the following expression

$$\Psi_p(f) = 2\pi f t(f) - \phi_p(f) - \frac{\pi}{4} , \tag{A.8}$$

where $t(f)$ gives the relation between time and frequency and can be obtained by integrating (A.6) as

$$\begin{aligned} t(f) =& \int \frac{dt}{df}\, df \\ =& t_0 - \frac{5\mathcal{M}_{ch}}{256\underline{u}^8}\left[1 + \frac{4}{3}A\frac{(2p^2 - 2p - 3)}{(p-4)}\eta^{-\frac{2p}{5}}\underline{u}^{2p}\right. \\ &\left. + \frac{4}{q-4}B\eta^{-\frac{2q}{5}}\underline{u}^{2q}\right] , \end{aligned} \tag{A.9}$$

with t_0 representing the time of coalescence and keeping only the Newtonian term and leading order non-GR corrections. On the other hand, $\phi_p(f)$ in Eq. (A.8) corresponds to the GW phase in time domain and can be calculated from Eq. (A.6) as

$$\begin{aligned} \phi_p(f) =& \int 2\pi f dt = \int \frac{2\pi f}{\dot{f}} df \\ =& \phi_{p,0} - \frac{1}{16\underline{u}^5}\left[1 + \frac{5}{3}A\frac{(2p^2 - 2p - 3)}{(2p-5)}\eta^{-\frac{2p}{5}}\underline{u}^{2p}\right. \\ &\left. + \frac{5}{2q-5}B\eta^{-\frac{2q}{5}}\underline{u}^{2q}\right] , \end{aligned} \tag{A.10}$$

with $\phi_{p,0}$ representing the coalescence phase. Using Eqs. (A.9) and (A.10) into (A.8) and writing $\Psi_p(f)$ as $\Psi_{\rm GR}(f) + \delta\Psi_p(f)$, non-GR modifications to the phase can be found as

$$\begin{aligned} \delta\Psi_p(f) = &- \frac{5}{32} A \frac{2p^2 - 2p - 3}{(4-p)(5-2p)} \eta^{-\frac{2p}{5}} \underline{u}^{2p-5} \\ &- \frac{15}{32} B \frac{1}{(4-q)(5-2q)} \eta^{-\frac{2q}{5}} \underline{u}^{2q-5}\,, \end{aligned} \tag{A.11}$$

with $\Psi_{\rm GR}$ to leading PN order is given by [192]

$$\Psi_{\rm GR} = 2\pi f t_0 - \phi_{p,0} - \frac{\pi}{4} + \frac{3}{128} \underline{u}^{-5}\,. \tag{A.12}$$

We can easily rewrite the above expressions using γ_r and c_r. Comparing Eq. (A.3) with Eq. (2.4), we find

$$A = \frac{12\gamma_r}{c_r} \eta^{\frac{c_r}{5}}\,, \quad p = \frac{c_r}{2}\,. \tag{A.13}$$

Using this, we can rewrite the GW amplitude in Eq. (A.7) as

$$\tilde{\mathcal{A}}(f) = \tilde{\mathcal{A}}_{GR} \left[1 - \frac{B}{2} \eta^{-\frac{2q}{5}} \underline{u}^{2q} - \frac{\gamma_r}{c_r} (c_r^2 - 4c_r - 6) \underline{u}^{c_r} \right]. \tag{A.14}$$

Similarly, one can rewrite the correction to the GW phase in Eq. (A.11) as

$$\begin{aligned} \delta\Psi_p(f) = &- \frac{15}{8} \frac{\gamma_r}{c_r} \frac{c_r^2 - 2c_r - 6}{(8-c_r)(5-c_r)} \underline{u}^{c_r - 5} \\ &- \frac{15}{32} B \frac{1}{(4-q)(5-2q)} \eta^{-\frac{2q}{5}} \underline{u}^{2q-5}\,. \end{aligned} \tag{A.15}$$

On the other hand, rewriting the above expressions further in terms of $\gamma_{\dot f}$ and $c_{\dot f}$ is not so trivial in general since corrections to the frequency evolution in Eq. (A.6) involves two independent terms instead of one.

Appendix B

Frequency Evolution From Energy Balance Law

In this appendix, we show an alternative approach to find $\dot{f}$ in varying-G theories in Eq. (2.59) by correcting and applying the energy balance law used in Ref. [177]. We begin by considering the total energy of a binary given by $E = -(G_C \mu M_t)/2r_{12}$. In order to calculate the leading order correction to the frequency evolution due to the time-varying gravitational constants, we use Kepler's law to rewrite the binding energy as

$$E(f, G_C, m_1, m_2) = -\frac{1}{2}\mu(G_C M_t \Omega)^{2/3}\,, \tag{B.1}$$

where $\Omega = \pi f$ is the orbital angular frequency. Taking a time derivative of the above expression and using Eqs. (2.48)–(2.50) in Eq. (B.1), the rate of change of the binding energy becomes

$$\frac{dE}{dt} = \frac{\pi^{2/3}}{6 f^{1/3} G_C^{1/3} M_t^{4/3}} \left[-3 f G_C M_t (\dot{m}_{1,0} m_2 + m_1 \dot{m}_{2,0}) \right.$$
$$\left. -2 M_t^3 \eta (G_C \dot{f} + f \dot{G}_C) + M_t^2 f G_C \eta \dot{M}_t \right]\,. \tag{B.2}$$

We can use the following energy balance argument to derive $\dot{f}$. In GR, the time variation in the binding energy is balanced with the GW luminosity $\dot{E}_{\rm GW}$ emitted from the system given by

$$\dot{E}_{\rm GW} = \frac{1}{5} G_D \left\langle \dddot{Q}_{ij} \dddot{Q}_{ij} - \frac{1}{3} (\dddot{Q}_{kk})^2 \right\rangle = \frac{32}{5} r_{12}^4 G_D \mu^2 \Omega^6 \,. \tag{B.3}$$

In varying-G theories, there is an additional contribution $\dot{E}_{\dot{G}}$ due to variations in G, masses, and the specific angular momentum. Namely, the binding energy is not conserved even in the absence of GW emission and the energy balance law is modified as

$$\frac{dE}{dt} = -\dot{E}_{\rm GW} + \dot{E}_{\dot{G}} \,. \tag{B.4}$$

To estimate such an additional contribution, we rewrite the binding energy in terms of the specific angular momentum as

$$E(G_C, m_1, m_2, j) = -\frac{G_C^2 \, \mu \, M_t^2}{2j^2} \,. \tag{B.5}$$

Taking the time variation of this leads to

$$\dot{E}_{\dot{G}} = \frac{\partial E}{\partial j} \dot{j}_0 + \frac{\partial E}{\partial m_1} \dot{m}_{1,0} + \frac{\partial E}{\partial m_2} \dot{m}_{2,0} + \frac{\partial E}{\partial G_C} \dot{G}_{C,0} \,, \tag{B.6}$$

where $\dot{j}_0$ is given by Eq. (2.54) and originates purely from the variation of G_C (i.e. no GW emission).

We are now in a position to derive the frequency evolution. Using Eqs. (B.3), (B.5) and (B.6) in Eq. (B.4), one finds

$$\frac{dE}{dt} = -\frac{32}{5}\pi^{10/3} f^{10/3} \eta^2 G_C^{4/3} G_D M_t^{10/3} \left[1 + \frac{5 G_C^2 \eta^{3/5} M_t}{64 G_D} \times \left(\frac{\dot{M}_{t,0}}{M_t} + \frac{\dot{m}_{1,0}}{m_1} + \frac{\dot{m}_{2,0}}{m_2} - 2\frac{\dot{j}_0}{j} + 2\frac{\dot{G}_{C,0}}{G_C}\right) \underline{u}^{-8}\right], \tag{B.7}$$

where $\underline{u} = (\pi G_C \mathcal{M}_{ch} f)^{1/3}$. Substituting this further into Eq. (B.2) and solving for $\dot{f}$, one finds the frequency evolution as

$$\dot{f} = \frac{96}{5}\pi^{8/3} G_C^{2/3} G_D \mathcal{M}_{ch}^{5/3} f^{11/3} \left\{1 + \frac{5}{96}\frac{G_C}{G_D}\dot{G}_{C,0}\eta^{3/5}[2M_t - 5(m_{1,0}s_{1,0} + m_{2,0}s_{2,0})]\underline{u}^{-8}\right\}, \tag{B.8}$$

in agreement with Eq. (2.59).

Along with the constancy of masses, the second term in Eq. (B.4) was also missing in [177]. Consequently, our PPE parameters in Eqs. (2.64) and (2.66) do not agree with Ref. [177] even when we take the limit of no time variation in masses. Difference in $\beta_{\dot{G}}$ is smaller than 20% while $\alpha_{\dot{G}}$ differs by a factor of 7. Despite the discrepancy, we expect the projected bounds on $\dot{G}_0/G_0$ calculated in Ref. [177] to be qualitatively correct. This is because a matched filtering analysis is more sensitive to phase corrections than to amplitude ones, where the difference between our results and [177] is small.

Appendix C

Bounds on PPE parameters: PhenomB and PhenomD Waveforms

Even though the PhenomD waveform produces more accurate results, we utilized the PhenomB one in this paper throughout because the latter is simpler and saves computational time when performing Monte Carlo simulations. In this appendix, we compare constraints on the PPE parameter α_{PPE} from both waveforms to justify our method.

Let us discuss the distinct features of the two waveforms first. Both PhenomB and PhenomD waveforms are spin-aligned (non-precessing) frequency-domain phenomenological models of gravitational waveforms [342, 429]. The PhenomB waveform is calibrated for mass ratios up to $m_1/m_2 = 4$ and spin components of $\chi_i \in [-0.85, 0.85]$ are unified into a single effective spin. On the other hand, the PhenomD waveform covers a larger region of the parameter space with mass ratios upto 18 and spins of $\chi_i \in [-0.95, 0.95]$, with both spins introduced independently. The waveform contains a much higher order in PN terms in the inspiral than the PhenomB waveform and further introduces an intermediate phase connecting the inspiral and merger-ringdown portions, which make such waveforms more reliable than the PhenomB ones.

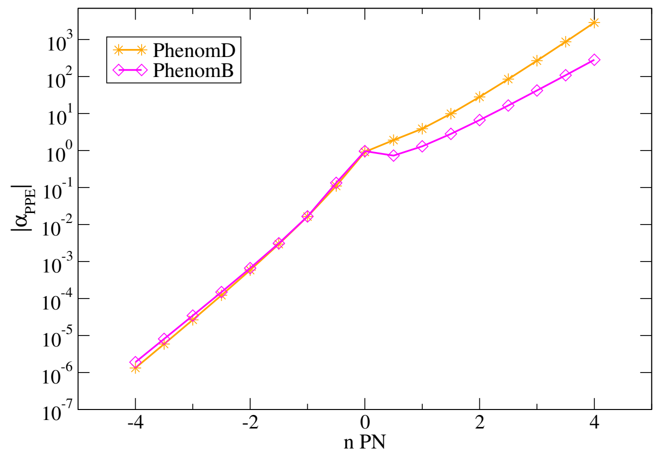

Figure C.1: Comparison of 90% confidence constraints on $\alpha_{\rm PPE}$ from GW1501914 with the PhenomB and PhenomD waveforms for generation effects.

We now estimate the constraints on the PPE amplitude modification from the two waveforms. Since modifications to propagation mechanisms used for massive gravity in Sec. 3.3.1 do not give rise to amplitude corrections, we here focus on modifications to generation mechanisms. We performed Fisher analyses with sky-averaged PhenomB and PhenomD waveforms and derived upper bounds on $\alpha_{\rm PPE}$ at different PN orders. As shown in Fig. C.1, the results from the two waveforms agree very well at negative PN corrections but deviate from each other at the positive ones. On the other hand, truncating the Fisher analyses at the end of the inspiral phase show significant agreement between the two waveforms at positive PN orders (Fig. C.2), suggesting that the deviation in Fig. C.1 originates mainly from the intermediate/merger-ringdown portion.

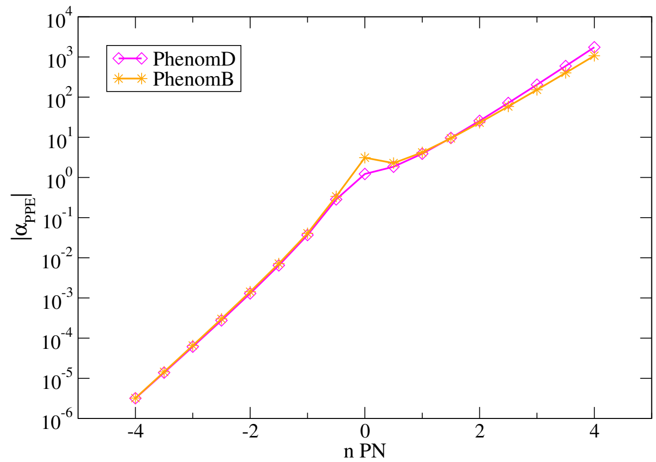

Figure C.2: Similar to Fig. C.1 but with inspiral signals only. The Fisher analyses are truncated at 104Hz which is corresponding to the transition frequency between the inspiral and merger portions of the PhenomB waveform, and we use the inspiral portion of the PhenomD waveform all the way up to this cutoff frequency.

The example theories considered in this paper acquire leading non-GR corrections either from propagation effects or from the generation effects with the latter entering in negative PN orders. A comparison between the PhenomB and PhenomD results for constraining $\beta_{\rm PPE}$ presented in Ref. [27] reveals consistency on constraining modifications to propagation mechanisms at both positive and negative PN orders, while the two waveforms show agreement only at negative PN orders for constraining modifications to generation mechanisms. Together with the results on amplitude corrections discussed above confirms that the results of this paper should not change significantly if one utilizes the PhenomD waveform instead. On the other hand, for constraining theories like dynamical Chern-Simons or noncommutative gravity where the leading correction enters at a positive PN order, the PhenomB waveform is not expected to produce reliable results.

Appendix D

Quadrupole Formula in the Presence of Compact Dimensions

Here we explicitly point out why the usual quadrupole formula

$$\int d^3\vec{x}\,\tau^{ij} = \frac{1}{2}\partial_0^2 \int d^3\vec{x}\,\tau^{00} x^i x^j\,, \tag{D.1}$$

cannot be directly extended to higher dimensions if there are compact dimensions.

Consider the expression $\frac{1}{2}\partial_0^2 \int d^3\vec{x} \int_0^\ell dw\, \tau^{00} x^I x^J$, then use repeatedly the conservation law for τ^{MN} and integrate by parts, while paying attention to the boundary terms:

$$\begin{aligned}\tfrac{1}{2}\partial_0^2 \int d^3\vec{x} \int_0^\ell dw\, \tau^{00} x^I x^J &= \int d^3\vec{x} \int_0^\ell dw\, \tau^{IJ} \\ &+ \tfrac{1}{2}\int d^3\vec{x} (\partial_K \tau^{5K} x^I x^J - \tau^{5I} x^J - \tau^{5J} x^I)\Big|_{w=0}^{w=\ell}\,. \end{aligned} \tag{D.2}$$

For $I, J = i, j = 1, 2, 3$, the quadrupole formula applies because of the periodicity of the metric fluctuations in w. However if either I or J are along the compact dimension, then there are extra terms relative to those expected based on the quadrupole formula. These terms are not straightforward to evaluate, and for this reason we rely on direct integration of the Einstein equations.

Appendix E

Hidden Brane Scenario

We generalize the Newtonian potential by having the observer located at $w = 0$, and the static mass source m on a hidden brane at $w = w_1$:

$$T_{00}(x^\mu, w) = m\delta^3(\vec{x})\delta(w - w_1)\,. \tag{E.1}$$

The corresponding Newtonian potential evaluated by the observer is

$$V^{(5,c)}(R, w_1) = -\frac{4}{3\pi} G^{(5)} m \sum_{n=-\infty}^{\infty} \frac{1}{R^2 + (w_1 + n\ell)^2}\,. \tag{E.2}$$

Using Poisson summation, the above sum can be evaluated to give the following result:

$$\begin{aligned} V^{(5,c)}(R, w_1) &= -\frac{4}{3}\frac{G^{(5)} m}{\ell R} \tanh\frac{2\pi R}{\ell}\left(1 + \operatorname{sech}\frac{2\pi R}{\ell}\cos\frac{2\pi w_1}{\ell}\right) \\ &\simeq -\frac{4}{3}\frac{G^{(5)} m}{\ell R}\left(1 + 2e^{-2\pi R/\ell}\cos\frac{2\pi w_1}{\ell}\right)\,, \end{aligned} \tag{E.3}$$

where in the last step we used $R \gg \ell$ and $R \gg w_1$. When $w_1 = 0$, (E.3) reproduces the Newtonian potential in Section 4.3 in the limit of $2\pi R \gg \ell$.

Appendix F

Newtonian Potential from $5d$ Compactified Green's Function

In this appendix we offer an alternative derivation of the Newtonian potential obtained in Section 4.3, by using the retarded $5d$ compactified Green's function. A static source of mass m located at the origin has

$$T_{00}(x^\mu, w) = m\delta^3(\vec{x})\delta(w)\,. \tag{F.1}$$

Substituting (4.43) and (F.1) into (4.40) we obtain

$$\tilde{h}_{\mathrm{T},00}\left(x^\mu, w\right) = -\frac{4}{\pi}G^{(5)}m\sum_{n=-\infty}^{\infty}\frac{1}{\mathscr{r}_n}\int_0^\infty dt'\,\frac{\partial}{\partial\mathscr{r}_n}\frac{\theta\left(t-t'-\mathscr{r}_n\right)}{\sqrt{\left(t-t'\right)^2-\mathscr{r}_n^2}}\,, \tag{F.2}$$

where $\mathscr{r}_n = \sqrt{x^2+y^2+z^2+(w-n\ell)^2}$. The integral in (F.2) can be evaluated by re-expressing the $\partial_{\mathscr{r}_n}$ derivative in terms of a time-derivative using

$$\frac{\partial}{\partial\mathscr{r}_n}\left[\frac{\theta\left(\Delta t-\mathscr{r}_n\right)}{\left(\Delta t^2-\mathscr{r}_n^2\right)^{1/2}}\right] = -\frac{\partial}{\partial\Delta t}\left[\frac{\theta\left(\Delta t-\mathscr{r}_n\right)}{\left(\Delta t^2-\mathscr{r}_n^2\right)^{1/2}}\right] + \frac{\theta\left(\Delta t-\mathscr{r}_n\right)}{\left[(\Delta t)^2-\mathscr{r}_n^2\right]^{3/2}}\left(\mathscr{r}_n-\Delta t\right)\,, \tag{F.3}$$

where $\Delta t = t - t'$. Substituting (F.3) into (F.2) we obtain

$$\tilde{h}_{\mathrm{T}}^{00}(x^\mu, w) = -\frac{4}{\pi} G^{(5)} m \sum_{n=-\infty}^{\infty} \frac{1}{\mathscr{r}_n} \int_{\mathscr{r}_n}^{\infty} d\Delta t \frac{\mathscr{r}_n - \Delta t}{[(\Delta t)^2 - \mathscr{r}_n^2]^{3/2}} = \frac{4}{\pi} G^{(5)} m \sum_{n=-\infty}^{\infty} \frac{1}{\mathscr{r}_n^2}\,, \tag{F.4}$$

and the corresponding Newton's potential $V^{(5,c)} \equiv (-1/3)\tilde{h}_{\mathrm{T},00}$ matches with the one in Eq. (4.17).

Appendix G

Retarded Green's Function

The massless scalar D-dimensional flat space Euclidean Green's function $\mathcal{G}_{\mathrm{E}}(x, x') = \mathcal{G}_{\mathrm{E}}(x - x')$ is the inverse of the D-dimensional Laplacian

$$\Delta_{(D)}\mathcal{G}^{(D)}E(x - x') = \delta^D(x - x'), \qquad \mathcal{G}_{\mathrm{E}}^{(D)}(x - x') = -\int \frac{d^D p}{(2\pi)^D}\frac{e^{ip\cdot x}}{p \cdot p}. \tag{G.1}$$

Starting from the unique Euclidean Green's function, in Minkowski signature the retarded Green's function is obtained via the analytical continuation $p^0_{\mathrm{E}} \longrightarrow -i(p^0 + i\epsilon)$. The Euclidean metric δ^{MN} gets replaced by the Minkowski metric η^{MN}, and the $i\epsilon$ prescription yields the *retarded* Green's function:

$$\mathcal{G}^{(D)}(x - x') = -\int \frac{d^D p}{(2\pi)^D}\,\frac{e^{ip\cdot x}}{-(p^0 + i\epsilon)^2 + p^i p^i}. \tag{G.2}$$

The location of the poles is in the lower half-plane, when viewed as a function of p^0 as a complex variable. To evaluate the integral one integrates over p^0, using Cauchy's theorem, and if $t - t' > 0$, one picks up the contribution from the two poles, otherwise the retarded Green's function is zero. If $D = 4$, we recover the familiar expression

$$\mathcal{G}^{(4)}(x-x') = -\theta(t-t') \int \frac{d^3p}{(2\pi)^3} \frac{\sin(p(t-t'))}{p} e^{i\vec{p}\cdot(\vec{x}-\vec{x}')} = -\theta(t-t') \frac{1}{4\pi r} \delta((t-t')-|\vec{x}-\vec{x}'|), \tag{G.3}$$

where we used p to denote the magnitude of the spatial vector $\vec{p}$, i.e. $p = |\vec{p}|$. If $D = 5$, then

$$\begin{aligned}
\mathcal{G}^{(5)}(x-x') &= -\theta(t-t') \int \frac{d^4p}{(2\pi)^4} \frac{\sin(p(t-t'))}{p} e^{i\vec{p}\cdot(\vec{x}-\vec{x}')} \\
&= -\theta(t-t') \frac{4\pi}{(2\pi)^4} \int_0^\infty dp\, p^3 \int_0^\pi d\theta\, \sin^2\theta\, \frac{\sin(p(t-t'))}{p} e^{ip|\vec{x}-\vec{x}'|\cos\theta} \\
&= -\theta(t-t') \frac{1}{4\pi^2 |\vec{x}-\vec{x}'|} \int_0^\infty dp\, p \sin(p(t-t')) J_1(p|\vec{x}-\vec{x}'|) \\
&= -\theta(t-t') \frac{1}{4\pi^2 |\vec{x}-\vec{x}'|} \frac{\partial}{\partial |\vec{x}-\vec{x}'|} \int_0^\infty dp\, \sin(p(t-t')) J_0(p|\vec{x}-\vec{x}'|) \\
&= -\theta(t-t') \frac{1}{4\pi^2 |\vec{x}-\vec{x}'|} \frac{\partial}{\partial |\vec{x}-\vec{x}'|} \frac{\theta(t-t'-|\vec{x}-\vec{x}'|)}{\sqrt{(t-t')^2-(\vec{x}-\vec{x}')^2}}\,.
\end{aligned} \tag{G.4}$$

Appendix H

Direct Integration vs. Quadrupole Formula

Here we use post-Newtonian order counting to explain a somewhat subtle aspect of our calculations. For simplicity's sake, in this appendix we restrict ourselves to $4d$ GR. Specifically, we will show that if in computing the spatial metric fluctuations $\tilde{h}^{ij}$, one uses the quadrupole formula, then one can safely neglect nonlinear source terms in the relaxed Einstein equations. However, if one directly integrates the relaxed Einstein equations, then the nonlinear terms cannot be neglected already at the leading post-Newtonian order. Since our goal is only to highlight the dependence on velocities, we will write our equations with squiggle lines, signaling that we are imprecise about numerical factors.

The relaxed Einstein equations in the harmonic gauge are given in (4.8), whose $4d$ solution is given by

$$\tilde{h}^{\mu\nu} \sim \frac{4}{R} \int d^3\vec{x}\, \tau^{\mu\nu}(\vec{x}, t - R)\,, \tag{H.1}$$

where we assumed that we are working in the far-field approximation, R is the distance from the source to the field point, and the right hand side is evaluated at the retarded time.

H.1 Direct Integration

We are interested in extracting the order of magnitude in a post-Newtonian (PN) expansion estimates of the spatial components $\tilde{h}^{ij}$ for a compact binary. We will be somewhat careless about the indices and numerical factors since we only care about counting the PN order, namely powers of the relative velocity of the binary constituents, v. Eq. (H.1) should receive contributions from both T^{ij} and t^{ij}_{LL}. To leading PN order, the contribution from T^{ij} is roughly given by

$$\tilde{h}^{ij}_{\mathrm{T}} \sim \frac{\mu}{R} v^i v^j \sim \frac{\mu}{R} v^2 \,, \tag{H.2}$$

where μ is the reduced mass of the binary. To consider the contribution of t^{ij}_{LL}, let us substitute $\tilde{g}^{ij} = \eta^{ij} - \tilde{h}^{ij}$ into (4.10) and look at one term (inside t^{ij}_{LL}), for example

$$(-g) t^{ij}_{\mathrm{LL}} \sim \tilde{h}^{00,i} \tilde{h}_{00}{}^{,j} \sim \tilde{h}_{00,i} \tilde{h}_{00,j} \,, \tag{H.3}$$

so that

$$\tilde{h}^{ij}_{\mathrm{t}} \sim \frac{1}{R} \int d^3x \, \tilde{h}_{00,i} \tilde{h}_{00,j} \,. \tag{H.4}$$

To compute (H.4), let us consider partitioning the spacetime into a near zone (NZ) and a far zone (FZ)[1] relative to the location of the sources. NZ is the region centered around the source with the size of the gravitational wavelength while FZ is the region

[1] FZ is also called the radiation zone or the wave zone.

exterior to it [178]. Within the NZ, the gravitational fields can be considered as almost instantaneous and retardation can be neglected. The integral in (H.4) can be decomposed into the NZ and FZ integrals. It turns out that the former dominates the latter, so what we need is the NZ solution for $\tilde{h}_{00}$. For a compact binary, the leading NZ solution is given by [430]

$$h_{00}^{\mathrm{NZ}} \sim \frac{2m_1}{r_1} + \frac{2m_2}{r_2}\,, \tag{H.5}$$

$$h_{ij}^{\mathrm{NZ}} \sim \left(\frac{2m_1}{r_1} + \frac{2m_2}{r_2}\right)\delta_{ij}\,, \tag{H.6}$$

with $r_a \equiv |\vec{x} - \vec{x}_a|$ and where $a = 1, 2$ corresponding to one of the two sources. Thus, $h^{\mathrm{NZ}} \sim 4m_1/r_1 + 4m_2/r_2$ and

$$\tilde{h}_{00}^{\mathrm{NZ}} \sim h_{00}^{\mathrm{NZ}} - \frac{1}{2}h^{\mathrm{NZ}}\eta_{00} \sim \frac{4m_1}{r_1} + \frac{4m_2}{r_2}\,. \tag{H.7}$$

We now substitute the above equation into the right hand side of (H.4). Those terms that only depend on r_1 or r_2 will diverge and be dropped upon regularization, so what matters is the cross term between sources 1 and 2. Ignoring numerical factors, we find

$$\begin{aligned}\tilde{h}_{\mathrm{t}}^{ij} &\sim \frac{m_1 m_2}{R}\int_{\mathrm{NZ}} d^3x\,\partial_i\left(\frac{1}{r_1}\right)\partial_j\left(\frac{1}{r_2}\right) + (1 \leftrightarrow 2) \\ &\sim \frac{m_1 m_2}{R}\int_{\mathrm{NZ}} d^3x\,\partial_i^{(1)}\left(\frac{1}{r_1}\right)\partial_j^{(2)}\left(\frac{1}{r_2}\right) + (1 \leftrightarrow 2) \\ &\sim \frac{m_1 m_2}{R}\partial_i^{(1)}\partial_j^{(2)}\int_{\mathrm{NZ}} d^3x\,\frac{1}{r_1 r_2} + (1 \leftrightarrow 2)\,,\end{aligned} \tag{H.8}$$

where we changed the partial derivatives to source-derivatives $\partial_i^{(a)} \equiv \partial/\partial x_a^i$ so that derivatives can be taken outside of the integral. The remaining integral, is given by [431]

$$\int_{\rm NZ} d^3x \, \frac{1}{r_1 r_2} = -2\pi r_{12} \, , \tag{H.9}$$

where $r_{12} = |\vec{x}_1 - \vec{x}_2|$ is the binary separation. Thus,

$$\begin{aligned} \tilde{h}^{ij}_{\rm t} &\sim \frac{m_1 m_2}{R} \partial^{(1)}_i \partial^{(2)}_j r_{12} + (1 \leftrightarrow 2) \\ &\sim \frac{m_1 m_2}{R} \frac{1}{r_{12}} \\ &\sim \frac{\mu}{R} \frac{M_t}{r_{12}} \\ &\sim \frac{\mu}{R} v^2 \, , \end{aligned} \tag{H.10}$$

where M_t is the total mass and we used the Kepler's law in the last equation. This scaling could in fact be easily obtained from the first line of (H.8) by replacing all the length scales inside the integral with r_{12}. Notice that $\tilde{h}^{ij}_{\rm t}$ is of the same PN order as $\tilde{h}^{ij}_{\rm T}$. This means that the contribution from $t^{ij}_{\rm LL}$ cannot be neglected even at the leading order.

H.2 Quadrupole Formula

So far we have seen that by using direct integration of the relaxed Einstein equations, where $\tilde{h}^{ij}$ is sourced by τ^{ij}, both the linearized $\tilde{h}^{ij}_{\rm T}$ and second order in fluctuation $\tilde{h}^{ij}_{\rm t}$ are of the same order in a velocity expansion. However, if we replace our starting point for the derivation of $\tilde{h}^{ij}$ with the quadrupole formula (see Appendix D)

$$\int \tau^{ij} d^3x = \frac{1}{2} \frac{d^2}{dt^2} \int d^3x \, \tau^{00} x^i x^j \, , \tag{H.11}$$

then we would be using τ^{00} in extracting $\tilde{h}^{ij}$. In this case, the contribution from $t^{00}_{\rm LL}$ is subleading to that of T^{00}, as we will now show.

Using (H.11) and (H.1) one derives

$$\tilde{h}^{ij} \sim \frac{2}{R}\frac{d^2}{dt^2}\int d^3x\, \tau^{00} x^i x^j\,. \tag{H.12}$$

The contribution from T^{00} has the same scaling as in (H.2), i.e.

$$\tilde{h}^{ij}_{(T^{00})} \sim \frac{\mu}{R} v^2\,. \tag{H.13}$$

On the other hand, the contribution from t^{00} is given by

$$\begin{aligned}
\tilde{h}^{ij}_{(t^{00})} &\sim \frac{1}{R}\frac{d^2}{dt^2}\int_{\rm NZ} d^3x\, \tilde{h}^{\rm NZ}_{00}{}_{,i}\tilde{h}^{\rm NZ}_{00}{}_{,j}\, x^i x^j \\
&\sim \frac{1}{R}\frac{d^2}{dt^2}\int_{\rm NZ} d^3x\, \partial_i\left(\frac{m_1}{r_1}\right)\partial_j\left(\frac{m_2}{r_2}\right) x^i x^j \\
&\sim \frac{1}{R}\Omega^2\frac{m_1}{r_{12}^2}\frac{m_2}{r_{12}^2} r_{12}^2 r_{12}^3 \\
&\sim \frac{m_1 m_2}{R} r_{12}\Omega^2 \\
&\sim \frac{\mu}{R}\frac{M_t}{r_{12}}(r_{12}\Omega)^2 \\
&\sim \frac{\mu}{R} v^4\,,
\end{aligned} \tag{H.14}$$

where Ω is the binary angular frequency and we replaced all the length scale by r_{12} in the third line as noted in Sec. H.1. Notice that $\tilde{h}^{ij}_{(\rm t^{00})}$ is of higher order in velocities than $\tilde{h}^{ij}_{(\rm T^{00})}$. Thus, once the expression for $\tilde{h}^{ij}$ is turned into the quadrupole formula, the contribution from t^{00} becomes subdominant and one only needs to consider T^{00} to leading order.

Appendix I

GW Luminosity in Transverse-Traceless (TT) Gauge

The transverse traceless (TT) gauge for linearized gravitational fluctuations about a flat spacetime imposes the following conditions:[1]

$$\tilde{h}^{\mathrm{TT}}_{0M} = 0\,, \qquad (\tilde{h}^{\mathrm{TT}})^{I}{}_{I} = 0\,, \qquad \partial_J (\tilde{h}^{\mathrm{TT}})^{IJ} = 0\,. \tag{I.1}$$

Starting from the trace-reversed metric fluctuations, $\tilde{h}_{IJ}$, one can show that the transverse traceless components are obtained by simply acting with a transverse-tracelss projector

$$\Lambda_{IJKL} = P_{IK}P_{JL} - \frac{1}{D-2}P_{IJ}P_{KL}\,, \tag{I.2}$$

[1]The TT gauge uses the residual gauge freedom of the Lorenz gauge to impose the additional conditions: $(\tilde{h}^{\mathrm{TT}})^{M}{}_{M} = 0$, $\tilde{h}^{\mathrm{TT}}_{0M} = 0$. Take n_I to be pointing in the direction of propagation of the waves, assuming they are plane waves: $\tilde{h}^{\mathrm{TT}}_{MN} = \tilde{h}^{\mathrm{TT}}_{MN}(t - n_I x^I)$. Then, from the harmonic gauge one finds that $n_I (\tilde{h}^{\mathrm{TT}})^{IJ} = \partial_I (\tilde{h}^{\mathrm{TT}})^{IJ} = 0, (\tilde{h}^{\mathrm{TT}})^{I}{}_{I} = 0$. For spherical waves these relations remain true to leading order in $1/R$.

and where P_{IK} are projectors orthogonal to n^I, with n^I the direction of propagation of the waves and $n^I n^I = 1$:

$$P_{IJ} = \delta_{IJ} - n_I n_J \,, \qquad P_{IJ} n^J = 0 \,. \tag{I.3}$$

Therefore, the non-vanishing (same as the trace-reversed) metric fluctuations in the TT gauge are

$$\tilde{h}^{\rm TT}_{IJ} = \Lambda_{IJKL} \tilde{h}^{KL} \,, \qquad (\tilde{h}^{\rm TT})^I{}_I = 0 \,, \qquad n^I \tilde{h}^{\rm TT}_{IJ} = \tilde{h}^{\rm TT}_{IJ} n^J = 0 \,. \tag{I.4}$$

It is easy to verify that

$$\begin{aligned} \tilde{h}^{\rm TT}_{IJ} (\tilde{h}^{\rm TT})^{IJ} = \; & \tilde{h}_{IJ} \tilde{h}^{IJ} - 2 n^I n^J \delta^{KL} \tilde{h}_{IK} \tilde{h}_{JL} + \left(\frac{D-3}{D-2} \right) n^I n^J n^K n^L \tilde{h}_{IJ} \tilde{h}_{KL} \\ & - \left(\frac{1}{D-2} \right) (\tilde{h}^I{}_I)^2 + \left(\frac{2}{D-2} \right) n^K n^L \tilde{h}_{KL} \tilde{h}^I{}_I \,. \end{aligned} \tag{I.5}$$

In particular, working in the TT gauge, the equation (4.64) becomes simply

$$t_{0K} n^K = - \frac{1}{32\pi G^{(D)}} \left\langle \dot{\tilde{h}}^{\rm TT}_{IJ} (\dot{\tilde{h}}^{\rm TT})^{IJ} \right\rangle \,. \tag{I.6}$$

We can nevertheless recover (4.69), starting from the TT gauge expression (I.6) and substituting (I.5), which is to be expected given that we are computing a gauge-invariant quantity.

Appendix J

Brans-Dicke Theory: Field Equations in Jordan Frame

In this appendix, we present scalar wave equation and the hypersurface equations in the Jordan frame. The *rr, rA,* and the trace of the *AB* components of the modified Einstein equations in Eq. (5.13a) give

$$\left(\tfrac{4}{r} + \tfrac{2\partial_r\lambda}{\lambda}\right)\partial_r\beta - \tfrac{1}{\lambda}\partial_r\partial_r\lambda - \tfrac{\omega_{\rm BD}}{\lambda^2}(\partial_r\lambda)^2 - \tfrac{1}{4}h^{AB}h^{CD}\partial_r h_{AC}\partial_r h_{BD} = 0\,, \tag{J.1a}$$

$$\begin{aligned}\tfrac{1}{2r^2}\partial_r(r^4e^{-2\beta}h_{AB}\partial_rU^B) - r^2\partial_r\left(\tfrac{1}{r^2}D_A\beta\right) + \tfrac{1}{2}h^{BC}D_B\left(\partial_r h_{AC}\right) + \tfrac{D_A\lambda}{\lambda r} - \tfrac{\omega_{\rm BD}}{\lambda^2}\partial_r\lambda D_A\lambda \\ +\tfrac{1}{\lambda}D_A\beta\partial_r\lambda + \tfrac{1}{2\lambda}h^{BC}D_B\lambda\partial_r h_{AC} + \tfrac{1}{2\lambda}e^{-2\beta}h_{AB}r^2\partial_r\lambda\partial_rU^B - \tfrac{1}{\lambda}\partial_rD_A\lambda = 0\,,\end{aligned} \tag{J.1b}$$

$$\begin{aligned}2h^{AB}\left(D_AD_B\beta + D_A\beta D_B\beta\right) - \mathscr{R} - \tfrac{1}{r^2}e^{-2\beta}D_A\partial_r(r^4U^A) + \tfrac{1}{2}r^4e^{-4\beta}h_{AB}\partial_rU^A\partial_rU^B \\ +2e^{-2\beta}\partial_rV + \tfrac{r^2}{\lambda}\Box\lambda + \tfrac{r^2}{\lambda}g^{AB}\nabla_A\nabla_B\lambda + \tfrac{\omega_{\rm BD}r^2}{\lambda^2}g^{AB}\nabla_A\lambda\nabla_B\lambda = 0\,,\end{aligned} \tag{J.1c}$$

respectively. On the other hand, the scalar wave equation, $\Box\lambda = 0$, is given by

$$2\partial_u\partial_r\lambda + D_A(U^A\partial_r\lambda) + \partial_r(U^A D_A\lambda) - \frac{1}{r}\left(-2U^A D_A\lambda - 2\partial_u\lambda + \partial_r V\partial_r\lambda + V\partial_r\partial_r\lambda\right)$$
$$-\frac{1}{r^2}\left[e^{2\beta}h^{AB}\left(2D_A\beta D_B\lambda + D_B D_A\lambda\right) + V\left(\partial_r\lambda\right)\right] = 0\,. \quad \text{(J.2)}$$

Appendix K

Coordinate Transformations: Harmonic Gauge to Bondi Gauge

In this appendix, we construct a coordinate transformation between harmonic coordinates to first order in $1/R$ and Bondi coordinates at an equivalent order in $1/r$ for a radiating spacetime in Brans-Dicke theory. The procedure is similar to that recently described in [424] in GR, but we do not work to all orders in $1/r$ as in [424]. Rather, we only work to an order in $1/r$ so that we can relate the radiative data in harmonic gauge ($\tilde{h}_{ij}^{\rm TT}$ and Ξ) to the corresponding radiative data in Bondi gauge (c_{AB} and λ_1) and compute the GW memory effects from PN waveforms in harmonic gauge .

We will denote the harmonic-gauge metric by $g_{\mu\nu}^{(\rm H)}$ which we write in quasi-Cartesian coordinates $X^\mu = (X^0, X^i)$, where $X^0 = t$ and $X^i = (X, Y, Z)$. However, we find it convenient to define $R = \sqrt{X^i X^j \delta_{ij}}$ to be the harmonic-gauge distance from the origin, and $y^A = (\iota, \varphi)$ to be spherical polar coordinates with $\cos\iota = Z/R$ and $\tan\varphi = Y/X$. The components of the metric can then be written in the form

$$g_{00}^{(H)} = -1 + \frac{2aM_t}{R} + \frac{1}{R} H_{00}(t - R, y^A)\,, \tag{K.1a}$$

$$g_{0i}^{(H)} = \frac{1}{R} H_{0i}(t-R, y^A)\,, \tag{K.1b}$$

$$g_{ij}^{(H)} = \delta_{ij} + \frac{2bM_t}{R}\delta_{ij} + \frac{1}{R} H_{ij}(t-R, y^A)\,. \tag{K.1c}$$

We have written the metric in the center-of-mass rest frame of the system, and we have split the $O(1/R)$ part of the metric in terms of the constant mass monopole moment M_t and all the higher-order multipole moments in $H_{\mu\nu}$ which are time dependent. We have also introduced constants a and b (defined below) which are needed so that the metric satisfies the modified Einstein's equations in the static limit (at the leading nontrivial order in $1/R$). To specify a solution, we also need an expression for the scalar field, which we give in the Jordan frame, and which we denote by λ. We will again expand it in terms of a static multipole moment and higher-order multipole moments that are time dependent as follows:

$$\lambda = \lambda_0 \left[1 + \frac{\psi(t-R, y^A) + 2cM_t}{R} \right]. \tag{K.2}$$

The monopole term is the $O(1/R)$ piece proportional to $2\lambda_0 c M_t$ and ψ contains the higher-order, time-dependent multipole moments. The third coefficient c is again needed to satisfy the field equations in the static limit. The expressions for the constants a, b, and c were previously determined in [179, 399, 432] and are given by

$$a = \frac{\omega_{\text{BD}} + 2 - \kappa}{\omega_{\text{BD}} + 2}\,, \tag{K.3a}$$

$$b = \frac{\omega_{\text{BD}} + 1 + \kappa}{\omega_{\text{BD}} + 2}\,, \tag{K.3b}$$

$$c = \frac{1 - 2\kappa}{2\omega_{\text{BD}} + 4}\,. \tag{K.3c}$$

We introduced the notation $\kappa = (m_1 s_1 + m_2 s_2)/M_t$. The coefficients also satisfy the relationships $a - b = 2c$ and $a + b = 2/\lambda_0$.

The harmonic gauge conditions lead to relationships between the components of the quantity $H_{\mu\nu}$. These relationships are more conveniently expressed in terms of a quantity $\tilde{H}_{\mu\nu}$, which is related to $H_{\mu\nu}$ by

$$H_{\mu\nu} = \tilde{H}_{\mu\nu} - \frac{1}{2}\tilde{H}\eta_{\mu\nu} - \psi\eta_{\mu\nu} \,, \tag{K.4}$$

and they are given by

$$\tilde{H}_{00} = n^i n^j \tilde{H}_{ij} \,, \qquad \tilde{H}_{0i} = -n^j \tilde{H}_{ij} \,. \tag{K.5}$$

We have defined $n^i = X^i/R = \partial_i R$ above, and it reduces to the expression $\vec{n} \equiv (\sin\iota\cos\varphi, \sin\iota\sin\varphi, \cos\iota)$ when written in terms of ι and φ.[1]

Our procedure for transforming from harmonic gauge to Bondi gauge follows some aspects of Ref. [424], in which Bondi coordinates were determined in terms of harmonic-gauge quantities in GR. In our case, however, we work in BD theory, work only to linear order in $1/R$ in harmonic coordinates, and we determine the corresponding Bondi coordinates in an expansion in $1/R$, such that Bondi gauge is imposed to one order in $1/r$ beyond the leading-order metric. To perform the coordinate transformation, it is useful to work with the components of the inverse metric.

[1] The conditions in Eq. (K.5) can be derived by first making the definitions given, e.g., in [399,425], of a conformally rescaled metric $\tilde{g}_{\mu\nu} = \lambda g_{\mu\nu}$, then defining $\tilde{h}^{\mu\nu}$ to be $\tilde{h}^{\mu\nu} = \eta^{\mu\nu} - \sqrt{-\tilde{g}}\tilde{g}^{\mu\nu}$, and imposing the harmonic gauge condition $\partial_\nu \tilde{h}^{\mu\nu} = 0$. When the harmonic gauge condition is imposed at leading order in $1/R$, then the conditions in (K.5) can be obtained (up to integration constants that we set to zero, so as to maintain the static solution of Einstein's equations in Eqs. (K.1) and (K.2)).

In harmonic gauge, it is given by

$$g^{\mu\nu}_{(\mathrm{H})} = \eta^{\mu\nu}_{(\mathrm{H})} - \frac{1}{R}\left[H^{\mu\nu} + 2M_t(a\delta^\mu_0\delta^\nu_0 + b\delta^\mu_i\delta^\nu_j\delta^{ij})\right] + O(R^{-2})\,, \tag{K.6}$$

where $H^{\mu\nu}$ is related to $H_{\mu\nu}$ by raising indices with $\eta^{\mu\nu}_{(\mathrm{H})}$. We aim to put the metric in Bondi form, in which the nonzero components of the inverse Bondi metric are given by

$$g^{ur}_{(\mathrm{B})} = -1 - \frac{\lambda_1}{\lambda_0 r} + O\left(r^{-2}\right)\,, \tag{K.7a}$$

$$g^{rr}_{(\mathrm{B})} = 1 + \frac{\partial_u\lambda_1}{\lambda_0} + \frac{1}{r}\left(\frac{\lambda_1}{\lambda_0} - 2\mathcal{M} + \frac{\lambda_1\partial_u\lambda_1}{2\lambda_0^2}\right) + \mathcal{O}\left(r^{-2}\right)\,, \tag{K.7b}$$

$$g^{rA}_{(\mathrm{B})} = \frac{1}{2r^2}\left(\eth_B c^{AB} - \frac{\eth^A\lambda_1}{\lambda_0}\right) + O\left(r^{-3}\right)\,, \tag{K.7c}$$

$$g^{AB}_{(\mathrm{B})} = \frac{1}{r^2}q^{AB} - \frac{1}{r^3}c^{AB} + O(r^{-4})\,, \tag{K.7d}$$

(and where $g^{uu}_{(\mathrm{B})} = g^{uA}_{(\mathrm{B})} = 0$). For simplicity, we will summarize Eq. (K.7) as

$$g^{\mu\nu}_{(\mathrm{B})} = \eta^{\mu\nu}_{(\mathrm{B})} - \frac{1}{r}h^{\mu\nu}_{(\mathrm{B})}\,, \tag{K.8}$$

where the quantity $\eta^{\mu\nu}_{(\mathrm{B})}$ consists of the $O(r^0)$ pieces of $g^{ur}_{(\mathrm{B})}$ and $g^{rr}_{(\mathrm{B})}$, the $O(r^{-1})$ part of $g^{rA}_{(\mathrm{B})}$ (which vanishes) and the $O(r^{-2})$ part of $g^{AB}_{(\mathrm{B})}$; the quantity $h^{\mu\nu}_{(\mathrm{B})}$ consists of the coefficients of the relative $1/r$ corrections to the components of $\eta^{\mu\nu}_{(\mathrm{B})}$.

We perform the coordinate transformation in two stages for illustrative purposes (one could perform it in one stage as in [424], but the two-stage process here allows us to highlight the different roles of the different terms in the transformation more easily). The first stage imposes the gauge conditions on the inverse Bondi metric that

$g^{uu}_{(\mathrm{B})}$ and $g^{uA}_{(\mathrm{B})}$ vanish to the accuracy in $1/r$ at which we work, it also makes r an areal radius, and finally, it relates the Bondi-coordinate retarded time u to the harmonic-gauge retarded time $t - R$. It can be thought of as a "finite" gauge transformation, in the sense that it is needed to relate the background metrics $\eta^{\mu\nu}_{(\mathrm{B})}$ and $\eta^{\mu\nu}_{(\mathrm{H})}$, which differ by a large (not perturbative in $1/r$) coordinate transformation. The second stage, which can be treated as perturbative in $1/r$, then sets the metric following the first transformation into a Bondi-gauge metric that satisfies the modified Einstein equations of BD theory.

In the first stage, the finite part of the coordinate transformation expresses a set of coordinates $x^\alpha = (u, r, x^A)$ in terms of harmonic gauge coordinates $X^\alpha = (t, X^i)$. Although it is expressed more easily in terms of the spherical polar coordinates (t, R, y^A) constructed from harmonic coordinates as follows:

$$u = t - R - M_t(a+b)\ln R\,, \tag{K.9a}$$

$$r = R + M_t b - \frac{\psi}{2}\,, \tag{K.9b}$$

$$x^A = y^A\,. \tag{K.9c}$$

The coordinates (u, r, x^A) resemble, but are not precisely Bondi coordinates at the order in $1/r$ at which we work, because they do not enforce all of the required properties of the Bondi-gauge metric. We will thus write this "intermediate" metric as $g^{\mu\nu}_{(\mathrm{I})}$, and it can be computed from the harmonic-gauge metric using the usual tensor transformation law

$$g^{\mu\nu}_{(\mathrm{I})} = g^{\alpha\beta}_{(\mathrm{H})} \frac{\partial x^\mu}{\partial X^\alpha} \frac{\partial x^\nu}{\partial X^\beta} \,. \tag{K.10}$$

A somewhat lengthy, but otherwise straightforward calculation shows that this metric can be written in the form

$$g^{\mu\nu}_{(\mathrm{I})} = \eta^{\mu\nu}_{(\mathrm{B})} - \frac{1}{r} h^{\mu\nu}_{(\mathrm{I})} \,, \tag{K.11}$$

where $\eta^{(\mathrm{B})}_{\mu\nu}$ is the leading order part of the inverse Bondi metric in Eq. (K.7). The coefficients of the relative order $1/r$ corrections to $\eta^{\mu\nu}_{(\mathrm{B})}$ we denoted by $h^{\mu\nu}_{(\mathrm{I})}$ and they are given by

$$\begin{aligned} h^{rr}_{(\mathrm{I})} &= (a+b)M_t(\partial_u\psi)^2 - \left(\frac{\tilde{H}}{2} - 2bM_t + \psi\right)\partial_u\psi \\ &\quad + H_{ij}n^i n^j + 2bM_t + \mathcal{O}\left(\frac{1}{r}\right) , \end{aligned} \tag{K.12a}$$

$$h^{ur}_{(\mathrm{I})} = -\frac{1}{2}(a+b)M_t\partial_u\psi + \frac{\Xi}{\lambda_0} + \frac{\tilde{H}}{2} + \mathcal{O}\left(\frac{1}{r}\right) , \tag{K.12b}$$

$$r\, h^{rA}_{(\mathrm{I})} = -\left(H_{0i}\eth^A n^i - \frac{1}{2}\eth^A\psi\right) + \mathcal{O}\left(\frac{1}{r}\right) , \tag{K.12c}$$

$$h^{uu}_{(I)} = \mathcal{O}\left(\frac{1}{r}\right) , \tag{K.12d}$$

$$h^{uA}_{(\mathrm{I})} = \mathcal{O}\left(\frac{1}{r^2}\right) , \tag{K.12e}$$

$$r^2 h^{AB}_{(\mathrm{I})} = H_{ij}\eth^A n^i \eth^B n^j + \psi q^{AB} + \mathcal{O}\left(\frac{1}{r}\right) . \tag{K.12f}$$

To arrive at Eq. (K.12), we used the conditions in Eq. (K.5) and we expressed $\partial_t\psi$ in terms of $\partial_u\psi$ (and other terms) using the derivatives of the first two lines in Eq. (K.9) and the chain rule:

$$\partial_t \psi = \partial_u \psi - \frac{1}{2r}(a+b)M_t(\partial_u \psi)^2 \,. \tag{K.13}$$

Note that nonlinear terms involving in ψ appear here and elsewhere in $h^{rr}_{(I)}$, because the coordinate transformations in (K.9a)–(K.9c) involve ψ at leading order in $1/r$.

The metric is similar to a Bondi-Sachs form to the order in $1/r$ at which we work, in the sense that the Bondi gauge conditions $g^{uu}_{(\mathrm{I})} = g^{uA}_{(\mathrm{I})} = 0$ are satisfied at this order and the right-hand side of Eq. (K.12f) is traceless with respect to q_{AB} (thereby being consistent with the determinant condition of Bondi gauge). Note, however, that the ur, rr, and rA components of $h^{\mu\nu}_{(I)}$ do not satisfy the modified Einstein equations in Bondi-Sachs gauge, as they are not consistent with the form of the inverse metric in Eq. (K.7). The metric can be put into Bondi gauge with a perturbative (in $1/r$) coordinate transformation, as we describe next.

We will parametrize the perturbative coordinate transformation in terms of a vector ξ^μ which effects the coordinate transformation $x^\mu \to x^\mu + \xi^\mu$. This coordinate transformation will take the part of the metric $h^{\mu\nu}_{(\mathrm{I})}/r$ and bring it to the Bondi-Sachs form $h^{\mu\nu}_{(\mathrm{B})}/r$, through the transformation

$$\frac{1}{r}h^{\mu\nu}_{(\mathrm{B})} = \frac{1}{r}h^{\mu\nu}_{(\mathrm{I})} + \mathcal{L}_{\vec{\xi}}\eta^{\mu\nu}_{(B)} \,, \tag{K.14}$$

where $\mathcal{L}_{\vec{\xi}}$ is the Lie derivative along $\vec{\xi}$. To solve for the perturbative gauge vector that generates this transformation, one can write the components of ξ^μ as

$$\xi^\mu = \frac{1}{r}(\xi^u_{(1)}, \xi^r_{(1)}, \xi^A_{(1)}/r) \,, \tag{K.15}$$

where the functions $\xi^\mu_{(1)}$ depend on u and x^A (not r). These lead to a set of partial differential equations for the $\xi^\mu_{(1)}$ that can be integrated by requiring that $h^{\mu\nu}_{(\mathrm{I})}$ be transformed into Bondi-Sachs form. Before giving the full details of this procedure, it is worth noting that given the form of the components of the vector ξ^μ in Eq. (K.15), the radiative data λ_1 and c_{AB} can be related to harmonic-gauge data and the finite coordinate transformation (K.9) without needing the full expression for ξ^μ in (K.15).

For the scalar field, because in both harmonic and Bondi gauges, the field has an expansion of the form $\lambda = \lambda_0 + O(r^{-1})$ (where λ_0 is constant and the coefficient of the $O(r^{-1})$ term is denoted $\Xi(t-R, y^A)$ in harmonic coordinates and $\lambda_1(u, x^A)$ in Bondi coordinates), then the transformation parametrized by ξ^μ in Eq. (K.15) will not change Ξ or λ_1. Thus, one must have that

$$\lambda_1(u, x^A) = \Xi[t-R, y^A]\,, \tag{K.16}$$

where u is related to $t-R$ (and x^A to y^A) by the transformations in Eq. (K.9). For c_{AB}, a direct calculation shows that $\mathcal{L}_{\vec{\xi}}\eta^{AB}_{(B)}$ is of order r^{-4} for ξ^μ in (K.15). This implies that

$$c^{AB} = H_{ij}\eth^A n^i \eth^B n^j + \psi q^{AB}\,. \tag{K.17}$$

Using the definition of $\tilde{H}_{\mu\nu}$ in Eq. (K.4), the harmonic gauge conditions in (K.5), and the fact that $q^{AB} = \delta_{ij}\eth^A n^i \eth^B n^j$, this equation can be recast as

$$c^{AB} = \tilde{H}_{ij}\eth^A n^i \eth^B n^j - \frac{1}{2}\tilde{H} q^{AB}\,. \tag{K.18}$$

After using Eq. (K.5) again, it follows that c^{AB} is related to just the transverse-traceless (TT) part of $\tilde{H}_{ij}$ as

$$c^{AB}(u, x^A) = \tilde{H}_{ij}^{\mathrm{TT}}(t - R, y^A)\eth^A n^i \eth^B n^j \,, \tag{K.19}$$

where again u is related to $t - R$ and x^A to y^A by the transformations in (K.9).

For our purposes of relating the radiative data in Bondi coordinates to that in harmonic coordinates, Eqs. (K.16) and (K.19) provide the solution. However, for completeness we do compute the form of the required gauge vector ξ^μ needed to complete the transformation from harmonic to Bondi coordinates. The components of the gauge vector in Eq. (K.15) can be constrained from the ur, rA, and rr components of Eq. (K.14), which state, respectively,

$$\partial_u \xi^u_{(1)} = h^{ru}_{(\mathrm{B})} - h^{ru}_{(\mathrm{I})} \,, \tag{K.20a}$$

$$\partial_u \xi^A_{(1)} = r\left(h^{rA}_{(\mathrm{B})} - h^{rA}_{(\mathrm{I})}\right) \,, \tag{K.20b}$$

$$2\partial_u \xi^r_{(1)} + \xi^u_{(1)}(\partial_u)^2(\lambda_1/\lambda_0) = h^{rr}_{(\mathrm{B})} - h^{rr}_{(\mathrm{I})} \,, \tag{K.20c}$$

By substituting the relationships in Eqs. (K.16) and (K.19) into Eq. (K.7), extracting the relevant components of $h^{\mu\nu}_{(\mathrm{B})}$, and using the expressions for $h^{\mu\nu}_{(\mathrm{I})}$ in Eq. (K.12), it is straightforward to integrate the first two lines in Eq. (K.20) to obtain expressions for $\xi^u_{(1)}$ and $\xi^A_{(1)}$. Once $\xi^u_{(1)}$ has been determined, integrating the final line of Eq. (K.20) to determine $\xi^r_{(1)}$ is also, in principle, straightforward. There is one subtlety in this procedure: $h^{rr}_{(\mathrm{B})}$ involves the Bondi mass aspect $\mathcal{M}$, which has not yet been determined from metric quantities in harmonic gauge. Because the mass aspect satisfies the conservation equation [82]

$$\partial_u \mathcal{M} = -\frac{1}{8} N_{AB} N^{AB} + \frac{1}{4} \eth_A \eth_B N^{AB} - (3 + 2\omega_{\text{BD}}) \frac{1}{4\lambda_0^2} (\partial_u \lambda_1)^2 + \frac{1}{4\lambda_0} \text{Đ}^2 \partial_u \lambda_1 \,, \quad \text{(K.21)}$$

it is possible to integrate this equation and express $\mathcal{M}$ in terms of harmonic-gauge quantities using Eqs. (K.16) and (K.19). To simplify the notation, however, we will not write this out in detail below, and we will write this quantity just as $\mathcal{M}$. The result of performing these integrations and using the harmonic gauge conditions in (K.5) is that the components of the vector ξ^μ are given by

$$\xi^u = -\frac{1}{2r} \int dt \tilde{H} + \frac{M}{\lambda_0^2 r} \Xi \,, \quad \text{(K.22a)}$$

$$\xi^r = -\frac{1}{2r} \int dt \left[\frac{\xi^u_{(1)}}{\lambda_0} \partial_u^2 \Xi + \frac{2M}{\lambda_0^3} (\partial_u \Xi)^2 - 2\mathcal{M} + \tilde{H}_{00} \right.$$
$$\left. - \frac{1}{2} \frac{\Xi}{\lambda_0^2} \partial_u \Xi - \left(\frac{1}{2} \tilde{H} - \frac{2M}{\lambda_0} \right) \left(1 + \frac{\partial_u \Xi}{\lambda_0} \right) \right] \,, \quad \text{(K.22b)}$$

$$\xi^A = -\frac{1}{r^2} \int dt \left[\frac{1}{2} \eth_B \left(\tilde{H}^{\text{TT}}_{ij} \eth^A n^i \eth^B n^j \right) + H_{ij} n^i \eth^A n^j \right] . \quad \text{(K.22c)}$$

This transformation, along with the finite transformation in Eq. (K.9), brings the metric into the form in Eq. (K.7), in which λ_1 and c_{AB} are related to the harmonic-gauge quantities Ξ and $\tilde{H}^{\text{TT}}_{ij}$ by Eqs. (K.16) and (K.19).

Appendix L

Estimates of Ordinary Memory Effects in Brans-Dicke Theory

L.1 Ordinary Displacement Memory Effect

The contribution to the ordinary memory effect comes from the "charge" rather than the "flux" terms in Eq. (6.50), i.e.:

$$\int d^2\Omega\, \alpha\, \eth^2(\eth^2+2)\Delta\Theta^{\mathrm{O}} = 8\Delta \int d^2\Omega\, \alpha \left(\mathcal{M} - \frac{1}{4\lambda_0}\eth^2\lambda_1\right) . \tag{L.1}$$

Expanding $\Delta\mathcal{M}$, $\Delta\Theta^{\mathrm{O}}$ and $\Delta\lambda_1$ in spherical harmonics, the moments $\Delta\Theta^{\mathrm{O}}$ are given by

$$\Delta\Theta^{\mathrm{O}}_{lm} = \frac{(l+2)!}{(l-2)!}\left[8\Delta\mathcal{M}_{lm} + \frac{2}{\lambda_0}l(l+1)\Delta\lambda_{1(lm)}\right] . \tag{L.2}$$

We would like to estimate if the quantity $\Delta\Theta^{\mathrm{O}}_{lm}$ is of a similar PN order to the nonlinear and null parts of the memory $\Delta\Theta^{\mathrm{T}}_{lm}$ and $\Delta\Theta^{\mathrm{S}}_{lm}$ that were computed in Sec. 6.4.2 for any of the specific values of $l = 2$, 4, or 6 and $m = 0$. To do so, we will focus on the moment $\Delta\Theta^{\mathrm{O}}_{20}$ for simplicity (the other three moments will have the same, or a higher PN order).

One natural way to compute the ordinary memory would be to directly evaluate the moments $\lambda_{1(20)}$ and $\Delta\mathcal{M}_{20}$. Using Eq. (6.39), we can show that $\lambda_{1(20)}$ is at least $\mathcal{O}(x^2)$; thus the scalar field's contribution to the memory is a higher PN order than the Newtonian order at which we work. Although, we do not have an independent expression for $\Delta\mathcal{M}_{20}$ that would allow us to directly compute $\Delta\Theta^{\mathrm{O}}_{20}$, we already verified that $\lambda_{1(20)}$ is of a higher PN order. Thus, we can compute Θ^{O}_{20} from the waveform that were already computed in Eqs. (7.1) and (7.2a) of [179] to verify that $\Delta\mathcal{M}_{20}$ would not contribute at Newtonian order. Specifically, we contract the Newtonian-order expression with the polarization tensors $e^{ij}_{+} - ie^{ij}_{\times}$, multiply by the spin-weighted spherical harmonic ${}_{-2}\bar{Y}_{20}$ to obtain U_{20} and then rescale it to obtain $\Delta\Theta^{\mathrm{O}}_{20}$. We find that the Newtonian-order result vanishes, and there is thus no Newtonian-order ordinary displacement memory.

L.2 Ordinary Spin Memory Effect

The ordinary part of the spin memory effect can be computed from Eq. (6.71a) with just the term (6.71c) on the right-hand side:

$$\int d^2\Omega\, \gamma \eth^2(\eth^2+2)\Delta\Sigma^0 = -8\Delta\int d^2\Omega\, \epsilon^{AD}\eth_D\gamma\left[-3L_A - \frac{1}{4\lambda_0}(c_{AB}\eth^B\lambda_1 - \lambda_1\eth^B c_{AB})\right]. \tag{L.3}$$

The terms $c_{AB}\eth^B\lambda_1 - \lambda_1\eth^B c_{AB}$ on second line of the equation will not contribute at Newtonian order for the spin memory effect (i.e, at order $x^{-1/2}$), because both c_{AB}

and λ_1 involve non-negative powers of x in the PN expansion, so their product will not be a negative power of x. The only term that could contribute to the spin memory effect comes from the change in the angular momentum aspect, L_A.

We do not have an expression for L_A in terms of harmonic-gauge metric functions, which (analogously to the case of the mass aspect and ordinary displacement memory effect) prohibits a direct calculation of the ordinary spin memory effect. In addition, it is not possible to directly check the time integral of the waveform. This is because the Newtonian-order terms in the spin memory effect arise from formally 2.5PN-order terms in the waveform that are then integrated with respect to retarded time; however, the waveform has only been computed to 2PN order in Ref [179]. While we cannot then be certain that the ordinary memory terms do not contribute at the same order because of additional nonlinear terms in the near zone, we can estimate the size of the effect in linearized theory.

The ordinary spin memory effect would arise at the lowest PN order from changes in $\Delta\Sigma^{\mathrm{O}}_{30}$, which is proportional to the retarded time integral of the radiative moment V_{30}. Because at leading PN order, V_{30} is related to three time derivatives of the source current octopole J_{30}, then $\Delta\Sigma^{\mathrm{O}}_{30}$ should be proportional to $\ddot{J}_{30}$. By dimensional analysis, J_{30} is proportional to $M_t v r_{12}^3$ (or see, e.g., [195]); thus, $\ddot{J}_{30}$ scales as $M_t v r_{12} \dot{r}_{12}^2$. This scales with the PN parameter as $\xi x^{9/2} + x^{11/2}$, which would be a 6PN correction to the nonlinear and null effects. We thus anticipate from these arguments in linearized theory that the ordinary part of the spin memory will be small.

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