Construction and Analysis of a Hierarchical Massless Quantum Field Theory

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Abstract

This dissertation focuses on critical phenomena in statistical mechanics and Quantum Field Theory. This involves the analysis of systems with infinitely many degrees of freedom across different length scales coupled together via interactions which can be easy to describe locally but give rise to a rich class of emergent phenomena. We adopt the framework of mathematical physics and probability where these systems are represented as measures on certain infinite dimensional spaces. The primary approach used in this dissertation is the *Renormalization Group*, a powerful and elegant framework that reveals how the collective influence of degrees of freedom manifest at different length scales within these systems. Pioneered by the physicist Kenneth Wilson, the philosophy of the RG approach is to reduce the analysis of these complex systems to the study of a "tractable" infinite dimensional dynamical system of effective potentials. The first project develops a Renormalization Group for spatially inhomogenous systems that allows one to establish a rigorous correspondence between orbits in Wilson's dynamical system and the measures one expects them to represent - this is done in the setting of a hierarchical approximation to Wilson's $4-\epsilon$ expansion. This culminates in the construction of a translation invariant, rotation invariant, and partially scale invariant generalized random field corresponding to the Wilson-Fisher fixed point. The second project leverages methods from statistical mechanics to strengthen this result and show that this generalized random field is in fact fully scale invariant.

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Contents

0.1 Introduction

This dissertation lies at the intersection of the disciplines of Constructive Field Theory, in particular the rigorous analysis of what are known as "Functional Integrals", and of probability, more specifically the study of self-similar random processes. We begin by describing the probabilistic approach to Axiomatic Quantum Field Theory.

0.1.1 Axiomatic Quantum Field Theory

We start by giving one approach to precisely mathematically defining what constitutes a "relativistic quantum field theory". More specifically we state the *Gårding-Wightman Axioms* for the case of a single Hermitean scalar field. For the axioms we give in the Minkowski space setting we take our presentation from [26]. First we list some preliminary definitions.

In follows we will use the notation $x = (x_0, x_1, \ldots, x_{d-1}) =: (x_0, \vec{x}) \in \mathbb{R}^d$. We define the Minkowski inner product on $\mathbb{R}^d$

$$\langle x, y \rangle_M \;=\; x_0 y_0 - \sum_{j=1}^{d-1} x_j y_j .$$

We say two points $x, y \in \mathbb{R}^d$ are *space-like separated* if $\langle x - y, x - y \rangle_M < 0$. We define the Lorentz group $\mathcal{L}$ to be the group of $d \times d$ matrices Λ that preserve the Minkowski inner product, i.e. $\Lambda \in \mathcal{L}$ if and only if for all $x, y \in \mathbb{R}^d$

$$\langle \Lambda x, \Lambda y \rangle_M \;=\; \langle x, y \rangle_M .$$

We define the restricted Lorentz group $\mathcal{L}_+^\uparrow$ to be the subgroup of $\mathcal{L}$ formed by those Λ with $det(\Lambda) = 1$ and with

$$\langle e_0, \Lambda e_0 \rangle_M \;>\; 0$$

where $e_0 := (1, 0, \ldots, 0) \in \mathbb{R}^d$. The restricted Poincare group $\mathcal{P}_+^\uparrow$ is the set of pairs (a, Λ) where $a \in \mathbb{R}^d$ and $\Lambda \in \mathcal{L}_+^\uparrow$ with the group operation given by

$$(a, \Lambda_1)(b, \Lambda_2) \;=\; (a + \Lambda_1, \Lambda_1\Lambda_2).$$

The natural action of $\mathcal{P}_+^\uparrow$ on $x \in \mathbb{R}^d$ is given by $(a, \Lambda)x = \Lambda x + a$.

We can now state the Gårding-Wightman Axioms (GW0) - (GW4).

(GW0) *States.* The states of the system are the unit rays of a separable complex Hilbert space $\mathcal{H}$ with a distinguished state Ω which we call the vacuum.

In what follows we denote the inner product of $\mathcal{H}$ by $(\cdot, \cdot)$.

(GW1) *Fields and temperedness.* There exists a dense subspace $D \subset \mathcal{H}$, such that for each test function f in the Schwartz space $S(\mathbb{R}^d, \mathbb{C})$ there exists a (possibly unbounded) operator $\phi(f)$ with domain D, such that:

(a) For all $\psi_1, \psi_2 \in D$, the map $f \mapsto (\psi_1, \phi(f)\psi_2)$ is a tempered distribution.

(b) For real valued f, the operator $\phi(f)$ is Hermitian.

(c) $\Omega \in D$

(d) $\phi(f)$ leaves D invariant.

(e) Finite linear combinations of vectors of the form

$$\phi(f_1) \cdots \phi(f_n)\Omega$$

with $n \geqslant 0$ and $f_1, \ldots, f_n \in S(\mathbb{R}^d, \mathbb{C})$ is dense in $\mathcal{H}$.

(GW2) *Relativistic covariance.* There is a strongly continuous unitary representation $U(a, \Lambda)$ of the restricted Poincare group $\mathcal{P}_+^\uparrow$ such that for all $(a, \Lambda) \in \mathcal{P}_+^\uparrow$

(a) $U(a, \Lambda)$ leaves D invariant

(b) $U(a, \Lambda)\Omega = \Omega$

(c) $U(a, \Lambda)\phi(f)U(a, \Lambda)^{-1} = \phi(f_{(a,\Lambda)})$ where

$$f_{(a,\Lambda)}(x) = f(\Lambda^{-1}(x - a)).$$

(GW3) *Spectral Condition.* The joint spectrum of the infinitesimal generators of the translation subgroup $U(a, \mathbf{1})$ is contained in the forward light cone $\bar{V}_+ = \{p := (p^0, \vec{p}) \in \mathbb{R}^d : p^0 \geqslant |p|\}$.

(GW4) *Locality.* If $f, g \in S(\mathbb{R}^d, \mathbb{C})$ have spacelike-separated supports, then the operators $\phi(f)$ and $\phi(g)$ commute, i.e. $[\phi(f)\phi(g) - \phi(g)\phi(f)]\,\psi = 0$ for all $\psi \in D$.

Thus a Hermitian scalar quantum field theory consists of a quadruple $(\mathcal{H}, U, D, \phi)$ that satisfies the above properties - here we see ϕ as an *operator-valued distribution*, i.e. a map from $S(\mathbb{R}^d, \mathbb{C})$ into a family of (possibly unbounded) operators on $\mathcal{H}$.

Of key interest in the above setting are the vacuum expectations of products of field operators, for $n \geqslant 1$, $f_1, \ldots, f_n \in S(\mathbb{R}^d, \mathbb{C})$ we define

$$W_n(f_1, \ldots, f_n) \;=\; (\Omega, \phi(f_1) \cdots \phi(f_n)\Omega).$$

We have that for each $n \geqslant 1$, W_n is an n-linear functional on $S(\mathbb{R}^d, \mathbb{C})$ and so by the Schwartz Kernel Theorem one can identify W_n as an element of $S'(\mathbb{R}^{nd}, \mathbb{C})$. For $n = 0$ we can define $W_0 = 1$ - seen as a linear functional on $\mathbb{C}$ which acts by multiplication.

Heuristically one can write

$$W(x^1, \ldots, x^n) \;=\; (\Omega, \phi(x^1) \cdots \phi(x^n)\Omega)$$

where $x^1, \ldots, x^n \in \mathbb{R}^d$. However the above expression is formal - in general neither side makes sense pointwise.

The *Wightman Reconstruction Theorem* gives sufficient conditions (called the *Wightman Axioms*) under which a family of such distributions $\{W_n\}_{n=0}^{\infty}$ uniquely determines a corresponding quadruple $(\mathcal{H}, U, D, \phi)$ that satisfies the the Gårding-Wightman axioms and realizes the $\{W_n\}_{n=0}^{\infty}$ as its vacuum expectation values.

We now state the Wightman Axioms (W0) - (W4). A proof for the Wightman Reconstruction Theorem can be found in [66].

(W0) *Temperedness and Hermiticity.* The $\{W_n\}_{n=0}^{\infty}$ are tempered distributions, i.e. $W_n \in S'(\mathbb{R}^{nd}, \mathbb{C})$, with $W_0 = 1$ and satisfying the hermitian condition

$$W(x^1, \ldots, x^n) \;=\; \overline{W(x^n, \ldots, x^1)}.$$

(W1) *Poincare Invariance.*

$$W_n(\Lambda x^1 + a, \ldots, \Lambda x^n + a) = W_n(x^1, \ldots, x^n)$$

for all $(a, \Lambda) \in \mathcal{P}_+^{\uparrow}$.

(W2) *Positive Definiteness.* For any almost finite sequence of test functions $\{f_n\}_{n=0}^{\infty}$ with $f_n \in S(\mathbb{R}^d, \mathbb{C})$ one has

$$\sum_{n,m} W_{n+m}(\overline{f_n} \otimes f_m) \;\geqslant\; 0$$

where for $g \in S(\mathbb{R}^l, \mathbb{C})$ and $h \in S(\mathbb{R}^k, \mathbb{C})$ we define $g \otimes h \in S(\mathbb{R}^{l+k}, \mathbb{C})$ via

$$(g \otimes h)(x, y) = g(x)h(y).$$

(W3) *Spectral Condition.* For each W_n, $n \geqslant 1$, the Fourier transform of W_n, which we formally define via

$$\widehat{W}_n(p^1, \ldots, p^n) \;=\int_{\mathbb{R}^{nd}} \mathrm{d}^d x^1 \ldots \mathrm{d}^d x^n \;\exp\left[i \sum_{j=0}^{d} \langle p^j, x^j \rangle_M \right] \;\times W_n(x^1, \ldots, x^n)$$

is a tempered distribution supported in the set

$$\left\{(p^1,\dots,p^n)|\ \sum_{j=1}^{n} p^j = 0 \text{ and } \left(\sum_{j=1}^{k} p^j\right) \in \bar{V}_+ \text{ for } k = 1,\dots,n-1\right\}$$

(W4) *Locality.* If x^j and x^{j+1} are space-like separated then one has

$$W_n(x^1,\dots,x^j,x^{j+1},\dots,x^n) \ = \ W_n(x^1,\dots,x^{j+1},x^j,\dots,x^m).$$

The Wightman Reconstruction theorem allows us to work with simpler "numeric" distributions instead of operator-valued distributions.

In [66, Theorem 3-5] it is shown that the spectral condition implies that the tempered distributions W_n can be seen as the boundary values of analytic functions in a large complex domain. In particular they can be analytically continued to imaginary time if evaluated at non-coinciding points. Given Wightman distributions satisfying the Wightman axioms one can define *Schwinger Functions*, $\{S_n\}_{n=0}^{\infty}$, with S_n an analytic function on $\mathbb{R}^{nd}_{\neq}$, via

$$S_n(x^1,\dots,x^n) \ = \ W_n\left((ix_0^1,\vec{x}^1),\dots,(ix_0^n,\vec{x}^n)\right)$$

where $\mathbb{R}^{nd}_{\neq}$ denotes the subset of $\mathbb{R}^{nd}$ given by those $(x^1,\dots,x^n)$ with $x^i \neq x^j$ for $i \neq j$. Additionally the locality properties of the W_n force the S_n to be symmetric in $x_1,\dots,x_n$. The process of going to imaginary time is called Wick rotation - it takes us from a Minkowski space-time to Euclidean space. The functions $\{S_n\}_{n=0}^{\infty}$ are also called *Euclidean Greens Functions*.

The *Osterwalder-Schrader Axioms* [51],[52] give necessary and sufficent conditions under which a set of *candidate* Schwinger functions uniquely determine a family of Wightman distributions which satisfy the conditions given by the Wightman Reconstruction theorem (one can find a recent improvement on this result in [77]).

We will not state the Osterwalder-Schrader axioms (henceforth called OS Axioms) in a form where they are equivalent to the Wightman Axioms. We instead give a more probablistic formulation of the axioms which give sufficient conditions for the Schwinger functions to uniquely determine a family Wightman distributions. The key idea here is to think of the candidate Schwinger functions as *distributions* corresponding to the *moments of a probability measure* on $S'(\mathbb{R}^d)$ - that is for $f_1,\dots,f_n \in S(\mathbb{R}^d,\mathbb{C})$ we will set

$$S_n(f_1,\dots,f_n) \ := \ \int_{S'(\mathbb{R}^d)} \mathrm{d}\mu(\phi)\ \phi(f_1)\cdots\phi(f_n). \tag{1}$$

Then the task of constructing the Wightman distributions comes down to constructing a measure μ on $S'(\mathbb{R}^d)$ with the correct properties. The probablistic approach to Euclidean quantum field theory in fact predates the Osterwalder-Schrader approach - being pioneered by Nelson [49], [50] and the even earlier work of Symanzik [69], [70].

We note that not all Wightman distributions can be specified in this way - models that allow for this probabilistic representation are called *Nelson-Symanzik positive*.

Before stating a version of the OS axioms we give some more definitions.

Let $S_+(\mathbb{R}^d)$ be the subspace of $S(\mathbb{R}^d)$ consisting of functions f with $\operatorname{supp}(f) \subset \{(x_0, \vec{x}) \in \mathbb{R}^d | \, x_0 > 0\}$, i.e. $S_+(\mathbb{R}^d)$ consists of test functions whose support is contained in the positive half-space defined with respect to the first component.

We define θ to be the linear transformation on $\mathbb{R}^d$ given by reflection across the $x_0 = 0$ hyperplane - that is for $x = (x_0, \vec{x})$ we define $\theta x = (-x_0, \vec{x})$. We define a corresponding transformation for $f \in S(\mathbb{R}^d)$ via

$$(\Theta f)(x) = f(\theta x)$$

and a transformation on $\phi \in S'(\mathbb{R}^d)$ via

$$(\Theta\phi)(f) \;=\; \phi(\Theta f).$$

We now state a probablistic formulation of the OS Axioms.

Let μ be a probability measure on $S'(\mathbb{R}^d)$ equipped with its cylinder set σ-algebra. Suppose that μ satisfies the following axioms.

(OS1) *Regularity.* There exists a seminorm $|\cdot|$ on $S(\mathbb{R}^d)$, $C > 0$, such that for all $n \geqslant 0$ and $f_1, \ldots, f_n \in S(\mathbb{R}^d)$ one has

$$\left| \int_{S'(\mathbb{R}^d)} \mathrm{d}\mu(\phi) \, \prod_{j=1}^{n} \phi(f_j) \right| \leqslant C^n \times n! \times \prod_{j=1}^{n} |f_j|.$$

(OS2) *Euclidean Invariance.* The measure μ should be invariant under translations and rotations of $\mathbb{R}^d$. In particular it is sufficient that for every $a \in \mathbb{R}^d$, $M \in O(d)$, and $f \in S(\mathbb{R}^d)$ one has that the random variables $\phi(f)$ and $\phi(f_{(a,M)})$ are equal in distribution if ϕ is distributed according to μ. Here

$$f_{(a,M)}(x) = f(M^{-1}(x-a)).$$

(OS3) *Osterwalder-Schrader Positivity* Let $A \in \mathcal{L}^2\left(S'(\mathbb{R}^d), \mu\right)$ and furthermore let A be measurable with respect to the σ-algebra $\mathcal{C}_+$ generated by the maps $\{\phi \mapsto \phi(f) | \; f \in S_+(\mathbb{R}^d)\}$. Then

$$\int_{S'(\mathbb{R}^d)} \mathrm{d}\mu(\phi) \; \overline{A(\Theta\phi)} A(\phi) \geqslant 0$$

0.1.2 ϕ^4 models

On a very formal (non-rigorous) level a ϕ^4 model is a probability measure on the space of fields $\phi : \mathbb{R}^d \to \mathbb{R}$ given by the expression

$$\frac{1}{Z} \exp\left[-\int_{\mathbb{R}^d} \mathrm{d}^d x \; \left(g\phi^4(x) + \frac{m^2}{2}\phi^2(x) + \frac{1}{2}|\nabla\phi(x)|^2 \right) \right] \prod_{x \in \mathbb{R}^d} \mathrm{d}\phi(x) \tag{2}$$

where $g > 0$, Z is a normalization factor, and $\prod_{x \in \mathbb{R}^d} d\phi(x)$ is "Lebesgue measure" on the space of functions from $\mathbb{R}^d \to \mathbb{R}$. Assuming that one could construct the correlation functions of such a measure and construct the corresponding scalar quantum field φ on Minkowski space one would expect this scalar quantum field φ

would satisfy the non-linear field equation

$$(-\Box + m^2)\varphi + 4g\varphi^3 \;=\; 0.$$

where $\Box$ is the d'Alembertian, the differential operator on Minkowski space given by

$$\Box = -\partial_0^2 + \sum_{j=1}^{d-1} \partial_j^2.$$

The first step in making (2) a little more rigorous is absorbing the $|\nabla\phi|^2$ and $m^2\phi^2$ terms of (2) into a Gaussian measure. Then one can write (2) as

$$\frac{1}{Z'} \exp\left[-\int_{\mathbb{R}^d} \mathrm{d}^d x \; g\phi^4(x)\right] \mathrm{d}\mu_C(\phi) \tag{3}$$

where $\mathrm{d}\mu_C(\phi)$ is a Gaussian measure on real valued fields ϕ on $\mathbb{R}^d$ with a covariance given by

$$(-\Delta + m^2)^{-1}.$$

where Δ is the standard Laplacian on $\mathbb{R}^d$. This means that

$$\int d\mu_C(\phi)\; \phi(x)\phi(y) \;=\; (-\Delta + m^2)^{-1}(x-y).$$

The measure $\mathrm{d}\mu_C$ is called a massive Gaussian Free Field (GFF) if $m^2 > 0$ and a massless GFF if $m^2 = 0$.

One can check that the correlation functions of the measure $\mathrm{d}\mu_C(\phi)$ satisfy the OS axioms and give rise to a scalar quantum field φ that satisfies the standard Klein-Gordon equation $(-\Box + m^2)\varphi = 0$. The measure $\mathrm{d}\mu_C(\phi)$ then corresponds to a field theory of non-interacting particles which earns it the moniker of "free field". The motivation behind the measure (3) is that it provides the simplest possible model of an *interacting* field theory.

Even the slightly less formal expression (3) is still plagued with major issues for $d \geqslant 2$. In the case $d = 1$ one is essentially in the setting one-dimensional stochastic processes ($P(\phi)_1$ proccesses) and quantum mechanics (via the Feynman - Kac Formula) - here one has a much larger arsenal.

For small $|x-y|$ the kernel $(-\Delta + m^2)^{-1}(x-y)$ goes like $|x-y|^{-d+2}$ for $d \geqslant 3$ while for $d = 2$ the divergence is logarithmic. The covariance $(-\Delta + m^2)^{-1}$ is divergent at coinciding points and so the law of the field ϕ is not supported on a space of real valued functions $\mathbb{R}^d$, instead it is more natural to see this law as living on the space of distributions S'. In particular, notions like the fields value at a point $\phi(x)$, or even worse a pointwise product $\phi(x)^n$ are completely ill-defined. This singular short range behaviour of the field ϕ is called an *ultraviolet divergence* - it is caused by the non-summability of fluctuations of the field at high Fourier modes. More concretely the covariance

$$\frac{1}{k^2 + m^2}$$

is not integrable in k at infinity for $d \geqslant 2$.

Another major issue with the quantity (3) is the fact that the integral in the exponent is over all of $\mathbb{R}^d$, even if the short-range behaviour of the field ϕ is mollifed it is certainly not clear that this integral converges, we will call this the infinite volume problem, or *infrared problem*.

When one works with a massive GFF, that is $m^2 > 0$, one expects $|(-\Delta + m^2)^{-1}(x-y)| \leqslant e^{-m|x-y|}$ for large $|x-y|$ - in other words distant regions of space should be approximately independent. In this setting one can control the infrared problem by using a family of methods called cluster expansions [15].

Fundamentally both the ultraviolet and infrared problems come from the fact that the measures (3) are meant to capture the behaviour of infinitely many degrees of freedom. In order to have an expression that is well defined one implements *cut-offs*.

A possible ultraviolet cut-off would be replacing the Gaussian measure $\mathrm{d}\mu_C$ with an alternative that is is the same for low Fourier modes but sufficiently suppresses the fluctuations of high fourier modes so that the corresponding law is supported on a space of functions defined pointwise. A simple infrared cut-off would be replacing the integral over all of $\mathbb{R}^d$ with one over a large box. The idea here is that one can use these cut-offs to define an analog of (3) that is a completely well defined measure. The central challenge is then to show that the measures without cut-offs converge to a meaningful limiting measure whose correlations satisfies the Osterwalder-Schrader axioms and so via analytic continuation give the Wightman distributions of a Minkowski quantum field theory.

In the case $d = 2$ there were major successes in the area of Constructive Field Theory when it came to the analysis of massive $P(\phi)_2$ models (here $P(\phi)_2$ means a polynomial that is bounded below and of degree at least 4, so this includes $(\phi 4)_2$-models) - see [35] and for a wider overview [62]. Since the ultraviolet divergence of the GFF becomes worse in higher dimensions the construction and analysis of the axioms for massive ϕ^4 in $d = 3$ (denoted $(\phi^4)_3$) was significantly more difficult - however this was successfully done with the functional integral approach in [23] and [25]. For $d > 5$ (and in some cases with $d = 4$) it has been proven that most attempts to define a measure corresponding to (3) will lead to a free field when one tries to remove the ultraviolet cut-off - see [5], [27], and also the discussion at the end of [26].

We will now specialize to the models that are of interest in this dissertation. For references on the functional integral approach to constructive field theory we point to [34], [56], and [11].

For the vast majority of the work above the infinite volume problem was approached via methods that are restricted to the *massive* setting. In the massless setting, that is when $m^2 = 0$, the covariance $(-\Delta)^{-1}(x-y) \sim \frac{1}{|x-y|^{-d+2}}$ for $d \geqslant 3$ (defining the massless GFF for $d = 2$ requires some technicality). Even if one ignores the divergence near the diagonal the above covariance is not summable in y for fixed x - distant regions of space remain fairly correlated. The process of taking the infinite volume limit in this setting is a much deeper problem and there are far fewer results - for example see [29] and [24] where the infinite volume limit of massless $(\phi^4)_4$ is controlled.

However while they pose some technical difficulty these massless quantum field theories are of great interest since they are are expected to coincide with scaling limits of certain models in statistical mechanics at criticality.

0.1.3 Massless ϕ^4 in three dimensions

Chapter 3 is essentially a slightly abridged version of the article [3] which was joint work with the author's advisor Abdelmalek Abdesselam, as well as Gianluca Guadagni. There we study an analog of a ϕ^4 model studied by Brydges, Mitter, and Scoppola in the article [18]- formally the model they studied, which we call the BMS model, corresponded to the measure on fields $\phi : \mathbb{R}^3 \to \mathbb{R}$ given by

$$\exp\left[- \int_{\mathbb{R}^3} g \ \phi^4(x) \ + \mu \phi^2(x) \ \mathrm{d}^3 x \right] \mathrm{d}\mu_{C_{-\infty}}(\phi) \tag{4}$$

where $g > 0$, $\mu \leqslant 0$ (we ignored this earlier but due to mass generation one expects to need $\mu < 0$ in order to construct a massless theory). Here $\mathrm{d}\mu_{C_{-\infty}}(\phi)$ is again the measure of the free field, a Gaussian measure with covariance

$$C_{-\infty} = (-\Delta)^{-\frac{3+\epsilon}{4}} \tag{5}$$

where for the time being one should take $\epsilon \in [0, 1]$. The correlations for the free field at different points in space are given by

$$C_{-\infty}(x - y) \ = \ \frac{c_\epsilon}{|x-y|^{\frac{3-\epsilon}{2}}} \ = \ \int \mathrm{d}\mu_{C_{-\infty}}(\phi) \ \phi(x)\phi(y).$$

where c_ϵ is some positive constant. Since this covariance is not summable at large distances it is natural to think of $\mu_{C_{-\infty}}$ as a *massless* generalized free field.

We remark that for the given range of ϵ the free measure $\mu_{C_{-\infty}}$ satisfies all the Osterwalder-Schrader axioms (including positivity), it is expected that the same should hold for the interacting field theory given by (4) which means the corresponding Schwinger functions could be analytically continued to give the Wightman distributions of a QFT on Minkowski space.

When $\epsilon = 1$ the measure $\mathrm{d}\mu_{C_{-\infty}}$ is just the standard massless Gaussian Free Field on $\mathbb{R}^3$ and the measure (4) would correspond to massless $(\phi^4)_3$. It is believed that massless $(\phi^4)_3$ governs the universality class which contains the critical nearest-neighbor Ising model in three dimensions, in particular folklore says that the generalized random field given by (4) is expected to be the scaling limit of the Ising models in this *universality* class. However a detailed analysis of massless $(\phi^4)_3$ and clear understanding of critical phenomena in this universality class remain, for the most part, outside the reach of current methods in mathematical physics.

We take a quick aside to give a rudimentary explanation of what is meant by the term "universality" class in the above paragraph. The term universal is applied in many ways in the context of probability theory, in a general sense it refers to a family of random processes that satisfy identical scaling behaviour for a variety of observables (such that pair correlations for example). In its stronger form the term university class refers to a family of random processes that have an identical limiting process under some aggregate averaging and scaling - the paradigmatic example being the university class of discrete sequences of i.i.d. random variables which under appropriate averaging and scaling yield a Gaussian distribution - for our purposes one can take this stronger notion of universality class.

One of the defining successes of Wilson's Renormalization Group (RG) program was the article [75] where the authors were able to apply an RG analysis in order to understand a critical ϕ^4 model in $d = 4 - \epsilon$ dimensions - they used a method called dimensional regularization to work in non-integer dimensions. In that article analysis of the RG flow when ϵ is small yields a non-trivial fixed point which corresponds to the critical ϕ^4 - one is then able to calculate expansions in ϵ for critical exponents and get an approximation for critical exponents in $d = 3$ by taking the first few terms and plugging in $\epsilon = 1$.

While dimensional regularization does not yet have a mathematical rigorous non-perturbative implementation the behaviour of RG flow in non-integer dimensions can be mimicked by working with fractional

Laplacians - here the ϵ appearing in (5) plays the same role as the ϵ of the $4-\epsilon$ expansion.

In [18] the authors defined a formal infinite volume RG transformation and were able to find and construct a fixed point in a space of effective potentials that is an analog of the Wilson-Fisher fixed point of [75]. We mention that an earlier paper [17] also simulated the $4-\epsilon$ expansion via using covariance $(-\Delta)^{-\frac{1+\epsilon}{2}}$ over $\mathbb{R}^4$ but this Gaussian measure is not Osterwalder-Schrader positive.

The work of [3] built on top of [18] to go beyond analyzing effective potentials to the construction of a concrete measure that realizes the $4-\epsilon$ model. However to simplify our analysis we worked in the *hierarchical approximation*; we replaced the underlying space-time $\mathbb{R}^3$ with an ultrametric space. Previous work on a hierarchical $4-\epsilon$ model can be found in [12], [13] ,[21], [28]; see also hierarchical work on the $\epsilon = 1$ case in [41]. However [3] has some novelties - there we developed a framework that allows one to start from an RG trajectories in a space of effective potentials and construct sufficiently many observables to rigorously obtain a measure corresponding to the given trajectory (it should for example, be able to construct the measure corresponding to the crossover trajectory in [1] connecting the BMS fixed point to the Gaussian one). An additional major new result in [3] is the construction of a measure corresponding to the composite ϕ^2 field - we will return to this later in the introduction.

The hierarchical approximation has a long and successful history of clarifying multiscale phenomena, first appearing in work by Dyson on the one dimensional Ising model with long range interactions [22]. See [2] for more context on the role of hierarchical models in clarifying RG analysis.

The benefit of the hierarchical approximation is that it allows one to study a new system that has much of the same properties as the original system but with much stronger locality properties. Ultrametric spaces are characterized by having metrics/norms that satisfy the strong triangle inequality: $|x+y| \leqslant \max(|x|,|y|)$. Instead of working over $\mathbb{R}^3$ we carried out our analysis over $\mathbb{Q}_p^3$ where $\mathbb{Q}_p$ refers to the field of p-adic numbers (note that our random fields are still real valued - $\phi : \mathbb{Q}_p^3 \to \mathbb{R}$). Our choice to use the p-adics allows us to set up a framework that mirrors the Euclidean case, in particular $\mathbb{Q}_p^d$ comes with a Fourier transform and spaces of Schwartz functions and tempered distributions S and S'. In particular we have proven theorems and used methods that are conducive to being applied to the original model over $\mathbb{R}^3$.

We now give part of the main theorem of [3], this theorem will be more precisely and completed stated as Theorem 3.1 in Chapter 3.

Theorem 0.1 ([3]). *For any p prime, $L = p^l$ large enough, and ϵ small enough there exist sequences of couplings $\{g_r, \mu_r\}_{r\in\mathbb{Z}}$ such that if one defines the measures*

$$d\nu_{r,s}(\phi) = \exp\left[-\int_{\Lambda_s} d^3x \ g_r \ \phi(x)^4 + \mu_r \ \phi(x)^2\right] d\mu_{C_r}(\phi)$$

Then there there exists a limiting measure ν_{BMS} such that one has the following convergence of measures (at the level of moments) on $\mathcal{S}'(\mathbb{Q}_p^3)$:

$$\lim_{\substack{r\to-\infty \\ s\to\infty}} \nu_{r,s} = \nu$$

Here $d\mu_{C_r}$ is a UV regularized version of a fractional Gaussian Free Field with covariance $C = (-\Delta)^{-\frac{3+\epsilon}{4}}$, we index the cutoff so $\lim_{r\to-\infty} C_r = C_{-\infty}$ is the limit where the UV cutoff is removed. Λ_s is the closed unit ball of radius L^s which serves as an IR cutoff.

The measure ν is translation invariant, rotation invariant, non-Gaussian, and has partial scale invariance: one has $L^{\kappa}\phi(L\bullet) \stackrel{d}{=} \phi(\bullet)$ for any $n \in \mathbb{Z}$ and $\kappa = \frac{3-\epsilon}{4}$.

This construction is based on getting control over the measures $\nu_{r,s}$ uniform in r and s via iteratively integrating out degrees of freedom - we tile space with blocks of side length L and integrate out short range degrees of freedom so we are left with an expression that involves only scaled averages of the field over each L-block. The distribution of these averages is again described by a functional of the form $e^{-V'(\phi)} = e^{-g'\phi^4 - \mu'\phi^2}$ where V' is a new *effective* potential.

This map $V \to V'$ (formally given by $(g, \mu) \to (g', \mu')$) is our dynamical system of effective potentials [1]. The measure constructed above corresponds to choosing the couplings so that the flow of effective potentials stays at the BMS fixed point in this system. Our scale invariance result is a direct consequence of the fact that ν_{BMS} is described by the same effective potential at all scales. We also have a mild universality result, there is an entire family of sequences of couplings which all yield the same measure ν_{BMS}.

Prior rigorous work on the RG has been mostly focused on establishing control over flows of effective potentials. For constructing a concrete limiting *measure* we must prove the convergence of the expectations of a sufficiently rich class of observables as we take $r \to -\infty,\ s \to \infty$. This is done in [3] via a generalization to a larger dynamical system that involves more complicated spatially varying effective potentials that have been influenced by the presence of observables.

We were also able to successfully construct and study a measure on S' corresponding to the ϕ^2 field, this requires additional renormalizations due to the singular behaviour of the ϕ field. Denote the suitably renormalized field $\mathcal{N}[\phi^2]$. We prove scale invariance with an anomalous exponent for this renormalized field, that is

$$L^{2\kappa-\eta}\mathcal{N}[\phi^2](L\bullet) \stackrel{d}{=} \mathcal{N}[\phi^2](\bullet)$$

where κ as defined as before and $\eta > 0$ is the anomalous part of the exponent. This gives a rigorous proof of a conjecture made by Wilson in [73]. The key point is that the anomalous scaling comes from a multiplicative renormalization that is made necessary by an anomalous eigenvalue of the BMS fixed point. For more details on this see the discussion right before section 3.8.

0.1.4 Proving Full Scale Invariance

Chapter 4 is part of [4] which is joint work with the author's advisor Abdelmalek Abdesselam and Gianluca Guadagni. We give a short overview below.

The RG transformations mentioned above yield *discrete* dynamical systems where the transformation involves iteratively "zooming" out by a fixed "demagnification" ratio L. Our generalization of the RG dynamical system allows us to prove that constant trajectories based at fixed points of this dynamical system correspond to measures with some scale invariance, in particular they are invariant with respect to the discrete scaling group generated by the ratio L. However this ratio is an artifical length parameter which is not intrinsic to the model we are studying and one expects that these partially scale invariant measures are actually fully scale invariant. In the Euclidean case the natural approach to constructing measures with full scale invariance would be defining a *continuous* renormalization group and constructing measures defined in

[1] Note that this is a simplification, the effective potentials are in fact parameterized by an infinite dimensional space and a central challenge is to define RG transformation so that this dynamical system is primarily governed by a finite set of couplings.

terms of fixed points of the corresponding continuous dynamical system. However it seems difficult to make this approach rigorous for bosonic field theories - in this case it helps to take L large in order to win the contraction of irrelevant operators in spite of various combinatorial factors.

Our results over $\mathbb{Q}_p^3$ in [3] suffer from a similar defect. Here the full scaling group is generated by powers of the underlying prime p and RG arguments allow us to prove scale invariance with respect to powers of $L = p^l$ where $l > 1$ must be a sufficiently large integer. However we overcome these difficulties in [4] and are able to show full scale invariance for the measures constructed in [3]. The proof involves defining two RG transformations defined via two length parameters L_1 and L_2 which together generate the full scaling group. These two transformations give two different dynamical systems and possible two different fixed points. If we prove that there is an non-empty intersection between the stable manifolds of the two fixed points then this implies that the measures corresponding to the two fixed points coincide and are then fully scale invariant. However the stable manifolds constructed via the RG transformations are not given explicitly and are thus hard to compare directly.

Our solution to this problem is to argue by contradiction - in particular we assume that fixed points from the two RG constructions do not coincide. We then show, via correlation inequalities, that this assumption forces there to be an open set of "critical" parameters in the (g, μ) plane. To generate the desired contradiction we combine a sharpness of transition results of Aizenman, Barsky, and Fernandez in [7] and combine this with a Gibbs variational principle in the context of superstable Gibbs measures ([58],[59],[45]) in order to show that such an open set of critical (g, μ) points cannot exist. The consequence of all of this is that the fields ϕ and $\mathcal{N}[\phi^2]$ constructed in [3] are seen to be fully scale invariant, and the anomalous exponent of the $\mathcal{N}[\phi^2]$ can be written in a way where it is seen to not depend on L. The arguments we used should have analogs in the Euclidean case where one will want to take a finite set of lengths L_i that generate the multiplicative group of the positive reals and show all these different RG transformations yield the same measure ν_{BMS}.

0.1.5 Overview of the rest of the dissertation

Sections 0.1.3 and 0.1.4 discuss the main results of this dissertation - a more comprehensive exposition of these results is given in chapters 3 and 4. In chapter 1 we give an introduction to the p-adics and the construction of measures on distributional spaces (in what follows we call such measures generalized random fields). In chapter 2 we give a result on the classification of generalized random fields with certain invariance properties.

Chapter 1

Preliminaries

1.1 Preliminaries on the p-adic rationals

1.1.1 Ultrametric Spaces

Our work over the p-adics fits into the larger framework of *hierarchical approximations* in probability and mathematical physics. These approximations have been used to study random processes indexed by some metric space where the law of the process exhibits a spatial structure. The key step of the hierarchical approximation is to replace the underlying Euclidean space with an *ultrametric space.*

Definition. *An ultrametric space* (Y, d) *is a non-empty set* Y *equipped with a distance function* $d : Y \times Y \to \mathbb{R}_{\geqslant 0}$ *that satisfies the following properties for all* $x, y, z \in Y$

(i) $d(x, y) = d(y, x)$

(ii) $d(x, y) = 0$ *if and only if* $x = y$

(iii) $d(x, z) \leqslant \max\{d(x, y), d(y, z)\}$

Condition (iii) is often called the *strong triangle inequality* in contrast with the orindary triangle inequality for metric spaces, the weaker requirement that $d(x, y) \leqslant d(x, y) + d(y, z)$. As a result ultrametric spaces have certain topological properties that can be non-intuitive.

Proposition 1.1. *Let* (Y, d) *be an ultrametric space. Then the following statements hold.*

(i) Any two balls (open or closed) in Y *are either nested within each other or disjoint.*

(ii) For any $r > 0$ *the set* Y *can be decomposed into a partition of disjoint open (resp. closed) balls of radius* r.

(iii) All open balls in (Y, d) *are closed sets and all closed balls of positive radius in* (Y, d) *are open sets.*

All three of the above statements are straightforward consequences of the the strong triangle inequality. One way to understand the second statement is that for any $r > 0$ the strong triangle inequality makes the relation $x \sim_r y \iff d(x,y) < r$ an equivalence relation (the same is true if $< r$ is replaced by $\leqslant r$) and statement (ii) is just the corresponding decomposition into equivalence classes. Additionally if $r' < r$ the the equivalence classes of $\sim_r$ consist of disjoint unions of equivalence classes of $\sim_{r'}$; this picture motivates the term "hierarchical".

We mention one more consequence: every point within an open ball can be considered the center of that open ball, that is given an open ball $B = \{y \in Y : |x-y| < r\}$ one has that for all $z \in B$, $\{y \in Y : |y-z| < r\} = B$.

1.1.2 The p-adic rationals

We will define the p-adic rationals $\mathbb{Q}_p$ as the completion of the rationals $\mathbb{Q}$ with respect to the metric induced by a particular absolute value on $\mathbb{Q}$.

Definition. *An absolute value on a field $\mathbb{F}$ is a map $|\cdot| : \mathbb{F} \to [0,\infty)$ such that for all $x, y \in \mathbb{F}$ one has*

(i) $|x| = 0 \iff x = 0$

(ii) $|xy| = |x| \cdot |y|$

(iii) $|x+y| \leqslant |x| + |y|$

For any prime p we now define the p-adic absolute value $|\cdot|_p$ on $\mathbb{Q}$. First note that for any non-zero $x \in \mathbb{Q}$ there is a unique integer k such that one can write $x = p^k \frac{a}{b}$ where a, b are integers and a, b and p coprime; for such x we define $|x|_p = p^{-k}$ and we set $|0|_p = 0$. It is immediate that $|\cdot|_p$ satisfies conditions (i) and (ii) and not hard to see that $|\cdot|_p$ satisfies a "strong triangle inequality", that is one has $|x+y|_p \leqslant \max(|x|_p, |y|_p)$. It follows that if $d(x,y) = |x-y|_p$ then $(\mathbb{Q}, d)$ is an ultrametric space.

We denote the standard absolute value on $\mathbb{Q}$ by $|\cdot|_\infty$, that is $|x|_\infty = sign(x) \cdot x$ for non-zero x and $|0|_\infty = 0$. The field $\mathbb{Q}$ is not complete (as a metric space) under any of the metrics induced by the absolute values $|\cdot|_p$ for p prime or $p = \infty$. For $p = \infty$ the corresponding completion of $\mathbb{Q}$ is of course $\mathbb{R}$. For p prime we define $\mathbb{Q}_p$ to be the completion of $\mathbb{Q}$ with respect to $|\cdot|_p$. It is not hard to check that the field structure of $\mathbb{Q}$ and the absolute value $|\cdot|_p$ naturally extends to $\mathbb{Q}_p$ making it a field with an absolute value.

We remark that when looking at sequences in $\mathbb{Q}$ there is no simple implication between being Cauchy with respect to $|\cdot|_p$ or $|\cdot|_{p'}$ for $p \neq p'$. As a consequence one cannot hope to canonically identify any of the elements in $\mathbb{Q}_p \backslash \mathbb{Q}$ with elements of $\mathbb{Q}_{p'} \backslash \mathbb{Q}$ (this includes the $p = \infty$ case, that is $\mathbb{Q}_\infty = \mathbb{R}$).

The recipe of defining an absolute value on $\mathbb{Q}$ and then forming the completion gives an easy way to construct field extensions $\mathbb{K}$ of $\mathbb{Q}$ which are equipped with an absolute value and are metrically complete. An easy way to generate a new absolute values from an existing one $|\cdot|$ is to take $|\cdot|^\alpha$ for suitable positive real numbers α. However this new absolute value will generate the same topology and the same completion of $\mathbb{Q}$ as $|\cdot|$ does. In fact a theorem of Ostrowski completely characterizes absolute values on $\mathbb{Q}$ (and in doing so essentially characterizes the normed completions of $\mathbb{Q}$).

Theorem 1.1 (Ostrowski)**.** *Let $|\cdot|$ be an absolute value on $\mathbb{Q}$, then there exists a positive real number α such that $|\cdot| = (|\cdot|_p)^\alpha$ for some p prime or $p = \infty$*

Proof: See Theorem 1.3.2 in Chapter 1 of [9].

For the rest of this section we specialize to $p \neq \infty$.

Every element of $x \in \mathbb{Q}_p$ has a unique series expansion of the form

$$x = \sum_{n=-\infty}^{\infty} a_n p^n \text{ where } a_n \in \{0, 1, \ldots, p-1\} \text{ and } \exists\ N \in \mathbb{Z} \text{ such that } a_n = 0 \text{ for } n \leqslant N \tag{1.1}$$

In other words $\mathbb{Q}_p$ is in one-to-one correspondance with Laurent series with poles of finite order and coefficients taken from $\{0, 1, \ldots, p-1\}$. Given such a representation (1.1) for an element $x \neq 0$ one can calculate the p-adic absolute value of x, in particular $|x|_p = p^{-v_p(x)}$ where $v_p(x) = \inf\{n \in \mathbb{Z} : \ a_n \neq 0\}$ (we can extend this definition to $x = 0$ with the convention that $v_p(0) = \infty$).

Note that large integer powers of p inside of $\mathbb{Q}_p$ are seen as small by the absolute value $|\cdot|_p$, that is

$$|p^N|_p = p^{-N}.$$

When reading the above equation one should remember that the p^N on the right hand side is an element of $\mathbb{Q}_p$ while the p^{-N} on the left hand side is an element of $\mathbb{R}$ which is where $|\cdot|_p$ takes its values. It is easily seen that all elements j of $\mathbb{Z} \subset \mathbb{Q}_p$ have $|j|_p \leqslant 1$. The closure of $\mathbb{Z}$ with respect to $|\cdot|_p$ will be denoted by $\mathbb{Z}_p$ and is precisely the closed unit ball centered at the origin in $\mathbb{Q}_p$. We also note that $\mathbb{Z}_p$ forms a subring of $\mathbb{Q}_p$.

We now make some remarks that will hopefully help the reader develop an accurate (but perhaps blurry) mental picture of $\mathbb{Q}_p$. One can represent "-1" , that is the additive inverse of 1 in the form (1.1):

$$-1 = \sum_{n=0}^{\infty} (p-1)p^n.$$

Arithmetic computations with p-adic numbers can be carried out in terms of these Laurent expansions using the typical "carrying rules" one uses for computation with decimal expansions in $\mathbb{R}$. We also note that unlike $\mathbb{R}$ one cannot impose a total ordering on $\mathbb{Q}_p$ that is consistent with its field structure (what we mean here is a total ordering $\preceq$ such that $x \preceq y \ \Rightarrow \ x + z \preceq y + z$ and $0 \preceq x, y \ \Rightarrow \ 0 \preceq xy$ for any $x, y, z \in \mathbb{Q}_p$). To prove such a total ordering cannot exist one can assume the existence of such an ordering and generate a contradiction by showing that one can find some $z \preceq 0$ such that z is in fact a perfect square. We remark that $\mathbb{Q}_p$ is not algebraicaly closed. With regards to its topology every closed ball in $\mathbb{Q}_p$ is homeomorphic to the Cantor set.

While $\mathbb{Q}_p$ may have properties that seem bizarre we believe that it is the ideal setting for hierarchical models. We are able to state (and prove) theorems that are formulated nearly identically to the theorems one would like to prove over $\mathbb{R}$. In particular our main constructive results describe probability measures on spaces of distributions over $\mathbb{Q}_p$. The description of this setting will be the goal of this section.

1.1.3 Norms and integration on $\mathbb{Q}_p^d$

For $d \in \mathbb{N}$ we define $\mathbb{Q}_p^d$ to be the $\mathbb{Q}_p$-vector space consisting of tuples $x = (x_1, \ldots, x_d)$ with $x_i \in \mathbb{Q}_p$ for $1 \leqslant i \leqslant d$. Vector addition is defined component-wise and scalar multiplication is defined in the typical way, that is for any $\lambda \in \mathbb{Q}_p$, $x \in \mathbb{Q}_p^d$ we define $\lambda x = (\lambda x_1, \ldots, \lambda x_d)$. We will also fix a norm on $\mathbb{Q}_p^d$.

Definition. *Let V be a $\mathbb{Q}_p$-vector space. A map $|\cdot| : V \to [0, \infty)$ is a norm on V if for all $x, y \in V$ and $\lambda \in \mathbb{Q}_p$ one has*

(i) $|x| = 0 \iff x = 0$

(ii) $|\lambda x| = |\lambda|_p \times |x|$

(iii) $|x + y| \leqslant |x| + |y|$

Definition. *We say a norm $|\cdot|$ on a vector space V satisfies the* strong triangle inequality *if for all $x, y \in V$ one has $|x + y| \leqslant \max(|x|, |y|)$*

While there are multiple choices one could make for a norm on $\mathbb{Q}_p^d$ the one we default to is $|x| = \max(|x_1|_p, \ldots, |x_d|_p)$ - this norm is natural as up to a constant it is the unique norm preserved by $GL_d(\mathbb{Z}_p^d)$ which is the unique maximal compact subgroup (up to conjugation) of $GL_d(\mathbb{Q}_p)$. Our choice here is analogous to the choice of the standard Euclidean norm on $\mathbb{R}^d$ given by $|x| = \sqrt{\sum_{j=1}^d x_j^2}$ which is the unique norm (up to a constant) which is preserved by the orthogonal group $O(d)$ which is the unique maximal compact (up to conjugation) subgroup of $GL_d(\mathbb{R})$.

It is notd hard to see that $|\cdot|$ satisfies the strong triangle inequality and it follows that $\mathbb{Q}_p^d$ is an ultrametric space when equipped with the metric induced by this norm. Before moving on we note a useful fact about norms satisfying the strong triangle inequality.

Proposition 1.2. *Let $|\cdot|$ be a norm on a vector space V that satifies the strong triangle inequality. Then for $x, y \in V$ with $|x| > |y|$ one has $|x + y| = |x|$.*

The closed unit ball in $\mathbb{Q}_p^d$ is precisely $\mathbb{Z}_p^d = \underbrace{\mathbb{Z}_p \times \cdots \times \mathbb{Z}_p}_{d \text{ times}}$. It is easy to see that $\mathbb{Q}_p^d$ splits into a disjoint union of $\mathbb{Z}_p^d$ and its translates. In particular if $a, b \in \mathbb{Q}_p$ with $|a - b| \leqslant 1$ it follows that $a + \mathbb{Z}_p^d = b + \mathbb{Z}_p^d$ (see Proposition 1.1). On the other hand if $|a - b| > 1$ it follows that $(a + \mathbb{Z}_p^d) \cap (b + \mathbb{Z}_p^d) = \varnothing$. We note that one can similarly decompose $\mathbb{Q}_p^d$ into translates of $p^{-1}\mathbb{Z}_p^d$, the closed ball of radius p containing the origin, and these larger balls individually decompose into p^d different translates of $\mathbb{Z}_p^d$, this gives us the "hierarchical" structure in $\mathbb{Q}_p^d$.

The space $\mathbb{Q}_p^d$ will be our continuum space-time but we will implement a UV cut-off which resembles the lattice UV regularization in Euclidean Field Theory where one replaces the space-time $\mathbb{R}^d$ with $\mathbb{Z}^d$. When working over $\mathbb{Q}_p^d$ the role of the unit lattice $\mathbb{L}$ will be played by a family of equivlaence classes in $\mathbb{Q}_p^d$ - the closed unit ball $\mathbb{Z}_p^d$ and its disjoint translates. More concretely we define $\mathbb{L}$ as the collection of subsets $\{a + \mathbb{Z}_p\}_{a \in \mathbb{Q}_p^d}$.

Since $\mathbb{Z}_p^d$ is an additive sub-group of $\mathbb{Q}_p^d$ one can also take $\mathbb{L} = \mathbb{Q}_p^d / \mathbb{Z}_p^d$ where the quotient operates in on the level of metric spaces and of additive groups. One particular choice of coset-representives are the p-adic vectors x whose components x_i all have representations in the sense of (1.1) with only negative powers of p. With this in mind one can immediately see that $\mathbb{L}$ is countable and discrete.

Before continuing we make the remark that while the role of "space-time" is played by $\mathbb{Q}_p^d$ the "fields" that we construct will be *real valued.* One could work with models where one has p-adic valued observables in addition to a p-adic space-time but we do work in that setting at all in this thesis.

We turn $\mathbb{Q}_p^d$ into a measurable space by equipping it with its Borel σ-algebra. For a measurable set $A \subset \mathbb{Q}_p^d$ we use $\mathbb{1}_A(\cdot)$ to denote the indicator function for the set A. Since $\mathbb{Q}_p^d$ is a locally compact abelian topological group we are guaranteed a non-trivial σ-finite Haar measure on $\mathbb{Q}_p^d$ which is unique up to normalization. We denote this measure $d^d x$ where we have fixed a normalization such that

$$\int_{\mathbb{Z}_p^d} d^d x = 1$$

By uniqueness of Haar measures we note that $d^d x$ coincides with the d-fold product of the Haar measure dx on $\mathbb{Q}_p$ normalized to give $\mathbb{Z}_p$ measure 1. We say a Borel measurable function $f : \mathbb{Q}_p^d \to \mathbb{R}$ is integrable if $\int |f| d^d x < \infty$. We say a Borel measurable function $f : \mathbb{Q}_p^d \to \mathbb{C}$ is integrable if its real and imaginary parts are both integrable.

Let $GL_d(\mathbb{Q}_p)$ be the set of invertible $d \times d$ matrices with entries in $\mathbb{Q}_p^d$. Given a $d \times d$ matrix M and $x \in \mathbb{Q}_p^d$ we write Mx to denote M's action on x corresponding to left multiplication by M where x is considered a column vector. We have the following elementary change of variable theorem:

Proposition 1.3. *Suppose that $M \in GL_d(\mathbb{Z}_p)$. Then for any Borel measurable $f : \mathbb{Q}_p \to \mathbb{R}_{\geqslant 0}$ one has*

$$\int f(Mx) d^d x = |det(M)|_p^{-1} \int f(x) \mathrm{d}^d x$$

Proposition (1.3) can be proven by noting that $\mu(A) = \int \mathbb{1}_A(Mx)\mathrm{d}^d x$ is a translation invariant measure and computing $\mu(\mathbb{Z}_p^d) = |det(M)|_p^{-1}$, the assertion then follows by appealing to the uniqueness of Haar measure (up to normalization).

We remark that the above theorem says $\int f(px) d^d x = p^d \int f(x) d^d x$ (we would expect to see a p^{-d} on the right hand side if we were working with Lebesgue measure over $\mathbb{R}^d$).

1.2 Some Distribution Theory and Fourier Analysis on $\mathbb{Q}_p^d$

The material in this section can be found in the references [71] and [9]. The Schwartz-Bruhat space of test functions over $\mathbb{Q}_p^d$, which we denote by $S(\mathbb{Q}_p^d)$, is an analog of the more familiar Schwartz space of test functions over $\mathbb{R}^d$. We start with defining what it means for a function on $\mathbb{Q}_p^d$ to be locally constant.

Definition. *A function $f : \mathbb{Q}_p^d \to \mathbb{C}$ is said to be* locally constant *if for each $x \in \mathbb{Q}_p^d$ there exists a $r(x) \in \mathbb{Z}$ such that for all y with $|y - x| \leqslant p^{r(x)}$ one has $f(y) = f(x)$.*

Local constancy will play the role of smoothness when dealing with functions on $\mathbb{Q}_p^d$. Note that it is the ultrametricity of $\mathbb{Q}_p^d$ that allows us to have non-trivial locally constant functions (a simple example being $\mathbb{1}_{\mathbb{Z}_p^d}$ which is constant over closed balls of radius 1).

We say that a function $f : \mathbb{Q}_p^d \to \mathbb{C}$ is compactly supported if there exists some $s \in \mathbb{Z}$ such that $f(x) = 0$ for all x with $|x| > p^s$. We now define $S(\mathbb{Q}_p^d)$ to be the space of all functions $f : \mathbb{Q}_p^d \to \mathbb{R}$ which are locally constant and compactly supported. Such functions will often be called *test functions*. In certain cases we will want to allow for complex valued test functions in which case we will write $S(\mathbb{Q}_p^d, \mathbb{C})$ which we view as a complex vector space.

So far $S(\mathbb{Q}_p^d)$, resp. $S(\mathbb{Q}_p^d,\mathbb{C})$, is just a real, resp. complex, vector space. We will now give a topological structure. We start with some definitions.

Definition. *Given a vector space V over $\mathbb{R}$ or $\mathbb{C}$ a map $\mathcal{N}: V \to [0,\infty)$ is a* seminorm *on V if for all $u, v \in V$ and any scalar λ one has:*

(i) $\mathcal{N}(\lambda v) = |\lambda|\, \mathcal{N}(v)$

(ii) $\mathcal{N}(u+v) \leqslant \mathcal{N}(u) + \mathcal{N}(v)$

Given a vector space V we define the *finest locally convex topology* on V to be the coarsest topology that makes **all** seminorms on V continuous. An equivalent definition is as follows:

Definition. *A subset U of a vector space V is open in the finest locally convex topology on V if and only if for every $x \in U$ there exists $\epsilon > 0$ and finitely many seminorms $\mathcal{N}_1, \ldots, \mathcal{N}_k$ such that $\cap_{j=1}^k \{y \in V : \mathcal{N}_j(y-x) < \epsilon\} \subseteq U$.*

We turn $S(\mathbb{Q}_p^d)$ into a topological vector space by equipping it with its finest locally convex topology. We give a quick proposition which says that in this setting linearity is enough for continuity.

Proposition 1.4. *Let V be a topological vector space equipped with its finest locally convex topology. Let W be a locally convex topological vector space. Then any linear map $L : V \to W$ is continuous.*

Proof: Since W is a locally convex topological vector space its topology is generated by some family of seminorms $\{\mathcal{N}_\alpha\}$. By linearity one immediately has $\mathcal{N}_\alpha(L(\cdot))$ is a seminorm on V. The result then follows by the definition of the finest locally convex topology. $\square$

We denote by $S'(\mathbb{Q}_p^d)$ the corresponding topological dual, i.e. the space of continuous linear functionals on $\mathbb{S}(\mathbb{Q}_p^d)$. Note that any linear map $\phi : S(\mathbb{Q}_p^d) \to \mathbb{R}$ is automatically continuous (in particular $|\phi(\cdot)|$ defines a seminorm on $S(\mathbb{Q}_p^d)$. It follows that $S'(\mathbb{Q}_p^d)$ coincides with the algebraic dual of $S(\mathbb{Q}_p^d)$. The notations (ϕ, f) and $\phi(f)$ denote the duality pairing of $\phi \in S'(\mathbb{Q}_p^d)$ and $f \in S(\mathbb{Q}_p^d)$. Note that every locally integrable real valued function $G(x)$ on $\mathbb{Q}_p^d$ can be seen as an element of $S'(\mathbb{Q}_p^d)$ via

$$(G, f) = \int_{\mathbb{Q}_p^d} G(x) f(x) \ \mathrm{d}^d x \text{ for } f \in S(\mathbb{Q}_p^d).$$

However not every element of $S'(\mathbb{Q}_p^d)$ is given by integration against a function. We will still sometimes write $(G(x), f(x))$ for the duality pairing, motivated by the above integral expression even if $G \in S'(\mathbb{Q}_p^d)$ is not given by integration against a function.

We turn $S'(\mathbb{Q}_p^d)$ into a topological vector space by equipping it with its cylinder set topology - this is the coarsest topology on $S'(\mathbb{Q}_p^d)$ such that for any $f_1, \ldots, f_n \in S(\mathbb{Q}_p^d)$ one has that the map $\phi \to (\phi(f_1), \ldots, \phi(f_n))$ is continuous map from $S'(\mathbb{Q}_p^d)$ to $\mathbb{R}^n$ where the latter space is given its standard topology.

We define the topological vector space $S'(\mathbb{Q}_p^d, \mathbb{C})$ as the dual of $S(\mathbb{Q}_p^d, \mathbb{C})$ in the same fashion. Note that we can canonically identify any $\phi \in S'(\mathbb{Q}_p^d)$ with an element of $S'(\mathbb{Q}_p^d, \mathbb{C})$ by having it act separately on the real and imaginary parts of any complex valued test function.

An important observation is that if f is locally constant and compactly supported then a straightforward compactness argument shows that f is "uniformly" locally constant, that is there exists $r \in \mathbb{N}$ such that

$|x-y| \leqslant p^r \ \Rightarrow f(x) = f(y)$. With this in mind we can think of $S(\mathbb{Q}_p^d)$ as a union of finite dimensional subspaces. For $r, s \in \mathbb{Z}$ we define

$$S_{r,s}(\mathbb{Q}_p^d) := \{f : \mathbb{Q}_p^d \to \mathbb{R} | \ |x| > p^s \Rightarrow f(x) = 0 \text{ and } |y-x| \leqslant p^r \Rightarrow f(x) = f(y)\}$$

Thus in the above definition r parameterizes the degree of required constancy and s parameterizes the size of the support. Clearly $r' \leqslant r \leqslant s \leqslant s' \Rightarrow S_{r,s}(\mathbb{Q}_p^d) \subseteq S_{r',s'}(\mathbb{Q}_p^d) \subset S(\mathbb{Q}_p^d)$. If $s < r$ then $S_{r,s}(\mathbb{Q}_p^d)$ contains only the 0 function while if $r \leqslant s$ then $S_{r,s}(\mathbb{Q}_p^d)$ is a $p^{d(s-r)}$ dimensional subspace of $S(\mathbb{Q}_p^d)$, one basis being the individual indicator functions for the $p^{d(s-r)}$ translates of $p^{-r}\mathbb{Z}_p^d$ contained in $p^{-s}\mathbb{Z}_p^d$. More concretely we define $\mathcal{I}_{r,s} \subset \mathbb{Q}_p^d$ as the set of vectors z whose components $(z_1, \ldots, z_d)$ are each of the following form (in the sense of 1.1):

$$z_i = \sum_{n=r+1}^{s} a_n p^{-n}$$

Then the basis in question is given explicitly by

$$\left\{ \mathbb{1}_{z+p^{-r}\mathbb{Z}_p^d}(x) \right\}_{z \in I_{r,s}} \tag{1.2}$$

It is also clear that

$$S(\mathbb{Q}_p^d) = \bigcup_{n=0}^{\infty} S_{-n,n}(\mathbb{Q}_p^d)$$

Note that we define $S_{r,s}(\mathbb{Q}_p^d, \mathbb{C})$ in the natural way and analogous statements to those above hold for the complex case.

1.2.1 The Fourier transform on $\mathbb{Q}_p^d$

Having a theory of Fourier transforms over $\mathbb{Q}_p^d$ allows the correspondance between our model and the corresponding Euclidean model to be seen much more clearly. Additionally harmonic analysis over $\mathbb{Q}_p^d$ can be more forgiving, for example we will see that functions can be compactly supported in both position and Fourier space. We will also see that analogously to the real case the Fourier transform will be a linear isomorphism on the space $S(\mathbb{Q}_p^d)$.

We first define the polar part map $\{\cdot\}_p : \mathbb{Q}_p \to \mathbb{R}$. If $u \in \mathbb{Q}_p$ has a Laurent representation

$$\sum_{n=-\infty}^{\infty} a_n p^n$$

then we set

$$\{u\}_p := \sum_{n=-\infty}^{-1} a_n p^n$$

where the right hand side of the above definition is taken as an element of $\mathbb{R}$ instead of $\mathbb{Q}_p$ - it is important to remember that this sum will have only finitely many non-zero terms.

We remark that $\{u\}_p = 0$ if and only if $u \in \mathbb{Z}_p$. We also define a dot product on $\mathbb{Q}_p^d$ in the standard way, i.e. for $x, y \in \mathbb{Q}_p^d$ we set $x \cdot y = \sum_{i=1}^{d} x_i y_i \in \mathbb{Q}_p^d$. Observe that $|x \cdot y|_p \leqslant \max 1 \leqslant i \leqslant d |x_i y_i| \leqslant |x| \cdot |y|$.

We now note that for any $z \in \mathbb{Q}_p^d$ the function $\xi_z : \mathbb{Q}_p^d \to \mathbb{C}$ given by

$$\xi_z(x) \;=\; \exp\left[2\pi i\{z \cdot x\}\right]$$

is a multiplicative map on the additive group $\mathbb{Q}_p^d$ - i.e. $\xi_z(x+y) = \xi_z(x)\xi_z(y)$ (similarly one also has $\xi_{z_1}\xi_{z_2} = \xi_{z_1+z_2}$ for any $z_1, z_2 \in \mathbb{Q}_p^d$). The key fact behind these properties is that $\{z \cdot (x+y)\}_p$ and $\{z \cdot x\}_p + \{z \cdot y\}_p$ differ by an integer. In particular $\mathbb{Q}_p^d$ can be identified as its own Pontryagin dual, with the correspondance being given by $z \to \xi_z$.

Speaking more concretely, the functions $\exp\left[2\pi i\{z \cdot x\}\right]$ will play the role that the functions $\exp[ik \cdot x]$ do in Fourier analysis over $\mathbb{R}^d$. For an integrable function $f : \mathbb{Q}_p^d \to \mathbb{C}$ we define the Fourier transform of f, $\mathcal{F}[f] : \mathbb{Q}_p^d \to \mathbb{C}$ as follows: for any $k \in \mathbb{Q}_p^d$

$$\mathcal{F}[f](k) := \int f(x) \exp[-2\pi i\{k \cdot x\}_p] \; d^d x.$$

We will also use the standard notation $\hat{f}(k)$ to denote the Fourier transform of a function $f(x)$. The next proposition gives a useful Fourier transform to know:

Proposition 1.5.

$$\mathcal{F}[\mathbb{1}_{\mathbb{Z}_p^d}] = \mathbb{1}_{\mathbb{Z}_p^d}.$$

Proof: Note that for any $a, b \in \mathbb{Q}_p$ the quantities $\{a+b\}_p$ and $\{a\}_p + \{b\}_p$ differ by an integer so one has $\exp[-2\pi i\{a+b\}_p] = \exp[-2\pi i\{a\}_p] \times \exp[-2\pi i\{b\}_p]$. We can then write

$$\int_{\mathbb{Q}_p^d} \mathbb{1}_{\mathbb{Z}_p^d}(x) \exp[-2\pi i\{k \cdot x\}_p] \; d^d x = \prod_{i=1}^{d} \left(\int_{\mathbb{Q}_p} \mathbb{1}_{\mathbb{Z}_p}(x_i) \exp[-2\pi i\{k_i \cdot x_i\}_p] \; dx_i \right).$$

This shows that it is sufficient to prove the proposition for $d = 1$ so we specialize to this case.

Suppose that $k \in \mathbb{Z}_p$, then one has $\exp[-2\pi i\{kx\}_p]\mathbb{1}_{\mathbb{Z}_p}(x) = \mathbb{1}_{\mathbb{Z}_p}(x)$ since $kx \in \mathbb{Z}_p$ for all $x \in \mathbb{Z}_p$. It follows that for $k \in \mathbb{Z}_p$ one has $\mathcal{F}[\mathbb{1}_{\mathbb{Z}_p}](k) = \int \mathbb{1}_{\mathbb{Z}_p} \; dx = 1$.

Suppose instead that $k \notin \mathbb{Z}_p$, then $|k|_p = p^j$ with $j \geqslant 1$. We then fix $\xi = (pk)^{-1} \in \mathbb{Z}_p$. It follows that

$$\begin{aligned}
\mathcal{F}[\mathbb{1}_{\mathbb{Z}_p}](k) &= \int \mathbb{1}_{\mathbb{Z}_p}(x - \xi + \xi) \exp\left[-2\pi i(\{k \cdot (x-\xi)\}_p + \{k\xi\}_p)\right] \; dx \\
&= \exp[-2\pi i p^{-1}] \int \mathbb{1}_{\mathbb{Z}_p}(x - \xi + \xi) \exp[-2\pi i\{k \cdot (x-\xi)\}_p] \; dx \\
&= \exp[-2\pi i p^{-1}] \int \mathbb{1}_{\mathbb{Z}_p}(x - \xi) \exp[-2\pi i\{k \cdot (x-\xi)\}_p] \; dx \\
&= \exp[-2\pi i p^{-1}] \int \mathbb{1}_{\mathbb{Z}_p}(y) \exp[-2\pi i\{ky\}_p] \; dy \;\; = \exp[-2\pi i p^{-1}]\mathcal{F}[\mathbb{1}_{\mathbb{Z}_p}](k).
\end{aligned}$$

In going to the third line note that since $\xi \in \mathbb{Z}_p^d$ and $\mathbb{Z}_p^d$ is an additive subgroup of $\mathbb{Q}_p^d$ it follows that $(x-\xi) + \xi \in \mathbb{Z}_p^d \iff (x-\xi) \in \mathbb{Z}_p^d$. For going to the fourth line we applied a change of variable $x - \xi \leftrightarrow y$. Since $\exp[-2\pi i p^{-1}] \neq 1$ the above computation shows $\mathcal{F}[\mathbb{1}_{\mathbb{Z}_p}](k) = 0$ for $k \notin \mathbb{Z}_p$. This finishes the proof that $\mathcal{F}[\mathbb{1}_{\mathbb{Z}_p}](k) = \mathbb{1}_{\mathbb{Z}_p}(k)$. □

We will now show that the Fourier transform leaves the space $S(\mathbb{Q}_p^d, \mathbb{C})$ invariant. First, a simple lemma:

Lemma 1.1. *Let $f : \mathbb{Q}_p^d \to \mathbb{C}$ be an integrable function. Suppose that $\lambda \in \mathbb{Q}_p$ with $\lambda \neq 0$ and $y \in \mathbb{Q}_p^d$. Then if $g(x) = f(\lambda(x-y))$ one has*

$$\mathcal{F}[g](k) = |\lambda|_p^{-d} \exp[-2\pi i\{k \cdot y\}_p]\hat{f}(\lambda^{-1}k)$$

Where again we use the notation $\hat{f}(k) = \mathcal{F}[f](k)$.

Proof: Apply a change of variable (see Proposition 1.3) in the integral defining $\mathcal{F}$. □

We now show that in addition to the position-space basis (1.2) of $S_{r,s}$ one also has a simple Fourier-mode basis of $S_{r,s}(\mathbb{Q}_p^d, \mathbb{C})$.

Proposition 1.6. *Let $r \leqslant s$. The family of functions*

$$\left\{p^{-\frac{ds}{2}} \exp[2\pi i\{k \cdot x\}_p]\mathbb{1}_{p^{-s}\mathbb{Z}_p^d}(x)\right\}_{k \in I_{-s,-r}}$$

is a basis for $S_{r,s}(\mathbb{Q}_p^d, \mathbb{C})$. In particular they are an orthonormal basis for $S_{r,s}(\mathbb{Q}_p^d, \mathbb{C})$ seen as a subspace of $\mathcal{L}^2(\mathbb{Q}_p^d, \mathbb{C})$.

Proof: The fact that the above functions are of $\mathcal{L}^2$ norm 1 is immediate.

We now check that the functions (1.6) are contained in $S_{r,s}(\mathbb{Q}_p^d, \mathbb{C})$. They are clearly supported on x with $|x| \leqslant p^s$ so we check constancy over closed balls of radius p^r. Note that the functions of (1.6) are clearly locally constant outside of Λ_s.

Fix y with $|y| \leqslant p^r$. This implies that $y \in p^{-s}\mathbb{Z}_p^d$ and it is then immediate that $\mathbb{1}_{p^{-s}\mathbb{Z}_p^d}(x) = \mathbb{1}_{p^{-s}\mathbb{Z}_p^d}(x+y)$ for all $x \in \mathbb{Q}_p^d$ ($p^{-s}\mathbb{Z}_p^d$ is an additive subgroup of $\mathbb{Q}_p^d$).

We claim that for $k \in I_{-s,-r}$, $x \in p^{-s}\mathbb{Z}_p^d$, and $y \in p^{-r}\mathbb{Z}_p^d$ one has

$$\exp[-2\pi i\{k \cdot x\}_p] = \exp[-2\pi i\{k \cdot x\}_p] \exp[-2\pi i\{k \cdot y\}_p] = \exp[-2\pi i\{k \cdot (x+y)\}_p].$$

The key point is the first equality, to see this is true observe that $|k \cdot y|_p \leqslant |k| \cdot |y| \leqslant p^{-r}p^r = 1$ so $\{k \cdot y\}_p$ is an integer.

We now check orthogonality. Observe that for $k, k' \in I_{-s,-r}$ distinct one has:

$$\begin{aligned}
&\int_{\mathbb{Q}_p^d} \exp[-2\pi i\{k \cdot x\}_p] \exp[2\pi i\{(k' \cdot x\}_p]\mathbb{1}_{p^{-s}\mathbb{Z}_p^d}(x) d^d x \\
&= \int_{\mathbb{Q}_p^d} \exp[-2\pi\{(k-k') \cdot x\}_p]\mathbb{1}_{p^{-s}\mathbb{Z}_p^d}(x) d^d x \\
&= \int_{\mathbb{Q}_p^d} \exp[-2\pi\{(k-k') \cdot x\}_p]\mathbb{1}_{\mathbb{Z}_p^d}(p^s x) d^d x \\
&= p^{ds}\mathbb{1}_{\mathbb{Z}_p^d}\left(p^{-s}(k-k')\right) \\
&= 0
\end{aligned}$$

The third equality follows from Proposition 1.5 and Lemma 1.1. The last equality comes from the fact that one has $|k-k'| \geqslant p^{-s+1}$. This shows that (1.6) must be linearly independent in $S_{r,s}(\mathbb{Q}_p^d)$ and since $|I_{-s,-r}| = p^{d(s-r)}$ the family (1.6) must span. □

It immediately follows that

Proposition 1.7. *For any $r, s \in \mathbb{Z}$ the Fourier transform $\mathcal{F}$ is a linear isomorphism of complex vector spaces taking $S_{r,s}(\mathbb{Q}_p^d, \mathbb{C})$ onto $S_{-s,-r}(\mathbb{Q}_p^d, \mathbb{C})$.*

Proof: The proposition is trivial in the case that $s < r$ so we assume $s \geqslant r$. We show that $\mathcal{F}$ takes the basis (1.2) for $S_{r,s}(\mathbb{Q}_p^d, \mathbb{C})$ onto a basis for $S_{-s,-r}(\mathbb{Q}_p^d, \mathbb{C})$. By Proposition 1.5 and Lemma 1.1 one has

$$\begin{aligned}
&\left\{\mathcal{F}\left[\mathbb{1}_{z+p^{-r}\mathbb{Z}_p^d}\right](k)\right\}_{z\in I_{r,s}} \\
&= \left\{\mathcal{F}\left[\mathbb{1}_{\mathbb{Z}_p^d}(p^r(\cdot - z))\right](k)\right\}_{z\in I_{r,s}} \\
&= \left\{p^{dr}\exp[-2\pi i\{z\cdot k\}_p]\mathbb{1}_{\mathbb{Z}_p^d}(p^{-r}k)\right\}_{z\in I_{r,s}}
\end{aligned}$$

By Proposition 1.6 the family of functions on the last line are a basis for $S_{-s,-r}(\mathbb{Q}_p^d, \mathbb{C})$. □

Corollary 1.1. *The Fourier transform $\mathcal{F}$ is a continuous linear automorphism of $S(\mathbb{Q}_p, \mathbb{C})$. In particular for $f \in S(\mathbb{Q}_p, \mathbb{C})$ one has $\mathcal{F} \circ \mathcal{F}[f](x) = f(-x)$.*

Proof: The first assertion follows immediately from the above propositions and Proposition 1.4. The second assertion is just a quick computation using the bases we gave in the above propositions. □

Proposition 1.7 can be thought of as a simple Paley-Wiener theorem for $\mathbb{Q}_p^d$ where the roles of regularity and spatial decay are played by the degree of local constancy and size of the support (respectively the r and s of $S_{r,s}$).

We also note that we have a p-adic analog of Parseval's theorem - this can be helpful for some of the calculations that will come later.

Proposition 1.8. *One has that $\mathcal{F}$ extends to a unitary map on $\mathcal{L}^2(\mathbb{Q}_p^d, \mathbb{C})$. In particular for $f, g \in L^2(\mathbb{Q}_p^d)$ one has that*

$$\int_{\mathbb{Q}_p^d} \mathrm{d}^d x\; f(x)g(x) = \int_{\mathbb{Q}_p^d} \mathrm{d}^d k\; \hat{f}(k)\hat{g}(-k)$$

Proof: This can be proven from the above propositions via standard density arguments. A proof is also given in [9, §4.8]. □

We define a Fourier transform $\mathcal{F} : S'(\mathbb{Q}_p^d, \mathbb{C}) \to S'(\mathbb{Q}_p^d, \mathbb{C})$ via duality, that is for $\phi \in S'(\mathbb{Q}_p^d, \mathbb{C})$, $f \in S(\mathbb{Q}_p^d, \mathbb{C})$ we define $\mathcal{F}\phi$ by setting $(\mathcal{F}\phi, f) = (\phi, \mathcal{F}f)$.

1.3 $S(\mathbb{Q}_p^d)$ and $S'(\mathbb{Q}_p^d)$ as spaces of sequences

In the probablistic formulation of quantum field theory the construction of the field theory corresponds to the construction of a probability measure on an appropriate distributional space which in our particular case will be $S'(\mathbb{Q}_p^d)$. The problem of specifying measures on infinite dimensional topological vector spaces takes some care. There is a great deal of literature on building measures on a class of topological vector spaces called "nuclear spaces". However in the case of the Schwartz-Bruhat spaces there is an easier method where one realizes both S and S' as spaces of sequences [61] which puts Kolmorogov's Extension Theorem at one's disposal - this approach applied to the case of reals is given in [63] and we reproduce it in section 1.4.2. The analogous construction over $\mathbb{Q}_p^d$ will be much simpler. In this section we prepare for section 1.4 by realizing both $S(\mathbb{Q}_p^d)$ and $S'(\mathbb{Q}_p^d)$ as spaces of real sequences. Along the way we will also prove assorted facts about the finest locally convex topology on $S(\mathbb{Q}_p^d)$.

Lemma 1.2. *There exists an orthonormal family of vectors $\{e_n\}_{n=0}^{\infty}$ in $\mathcal{L}^2(\mathbb{Q}_p^d)$ such that the following holds: For every $N \in \mathbb{N}_{>0}$ one has $\{e_n\}_{n=0}^{p^{2dN}-1}$ span $S_{-N,N}$ (seen as a subspace of $\mathcal{L}^2(\mathbb{Q}_p^d)$).*

Proof: Let $V_0 = S_{0,0}(\mathbb{Q}_p^d)$. We remark that V_0 is a one dimensional subspace and we set $e_0 = \mathbb{1}_{\mathbb{Z}_p^d}$. For $n \geqslant 1$ we set

$$V_n = (S_{-(n-1),(n-1)})^{\perp} \bigcap S_{-n,n}$$

where the orthogonal complement is taken in $\mathcal{L}^2$. Note that each V_n is just a $p^{2dn} - p^{2d(n-1)}$ dimensional subspace of $\mathcal{L}^2$. For $n \geqslant 1$ we choose $\{e_j\}_{j=p^{2d(n-1)}}^{p^{2dn}}$ to be an orthonormal basis of V_j. This yields the basis given in the assertion. $\square$

We denote by ℓ the vector space of almost finite real sequences:

$$\ell := \bigoplus_{i=0}^{\infty} \mathbb{R} = \{\ \{x_i\}_{i=0}^{\infty}\ : x_i \in \mathbb{R} \text{ are non-zero for only finitely many } i\}$$

We equip ℓ with its finest locally convex topology to turn it into a topological vector space. We now define a linear map $T : S(\mathbb{Q}_p^d) \to \ell$ as follows. For $f \in S(\mathbb{Q}_p^d)$ we set the sequence $Tf \in \mathbb{R}^{\mathbb{N}}$ to be given by

$$Tf := \{\langle e_i, f_i \rangle_{\mathcal{L}^2(\mathbb{Q}_p^d)}\}_{i=0}^{\infty}$$

We then have the following proposition

Proposition 1.9. *The map T is a linear homeomorphism between the topological vector spaces $S(\mathbb{Q}_p^d)$ and ℓ*

Proof: We first note that since $S(\mathbb{Q}_p^d) = \bigcup_{n=0}^{\infty} S_{-n,n}(\mathbb{Q}_p^d)$ every $f \in S(\mathbb{Q}_p^d)$ has a unique representation as a finite linear combination of the functions e_i. It then follows that T is a linear isomorphism between $S(\mathbb{Q}_p^d)$. The fact that T is a homeomorphism is immediate since any linear map between two topological vector spaces equipped with their respective finest locally convex topologies is automatically continuous (T is dominated by the seminorm $|T|$) $\square$

Unlike the Schwartz space of functions over $\mathbb{R}^d$ it turns out that the topological vector space $S(\mathbb{Q}_p^d)$ cannot be made into a Frechet space. In particular one can show that the topology on ℓ cannot be generated by a countable set of seminorms:

1.3.1 The finest locally convex topology on ℓ (and $S(\mathbb{Q}_p^d)$)

We begin with finding a concrete family of seminorms that generates the finest locally convex topology on ℓ. Given a non-negative weighting $\mu = \{\mu_i\}_{i=0}^{\infty}$ (i.e. $\mu_i \in \mathbb{R}_{\geqslant 0} \forall i$) we define a seminorm $\mathcal{N}_\mu$ on ℓ via

$$\mathcal{N}_\mu(\{x_i\}_{0=1}^{\infty}) := \sum_{i=0}^{\infty} \mu_i |x_i|$$

Proposition 1.10. *For every seminorm $\mathcal{N}$ on ℓ there exists a non-negative weight μ such that $\mathcal{N}(x) \leqslant \mathcal{N}_\mu(x)$ for all $x \in \ell$.*

Proof: For $j \in \mathbb{N}$ define $\delta^{(j)} \in \ell$ to be the sequence which is zero for all indices except the j-th one where it takes the value 1. Given a seminorm $\mathcal{N}$ on ℓ define a non-negative weighting $\mu = \{\mu_i\}_{i=1}^\infty$ by setting $\mu_i = \mathcal{N}(\delta^{(i)})$. Then for every $x \in \ell$ one has

$$\mathcal{N}(x) \leqslant \sum_{i=1}^{\infty} |x_i| \times \mathcal{N}(\delta^{(i)}) = \mathcal{N}_\mu(x)$$

We immediately have the two following corollaries.

Corollary 1.2. *The finest locally convex topology is the coarsest topology that makes the family of seminorms $\{\mathcal{N}_\mu\}$ continuous where μ ranges across all non-negative weightings.*

Corollary 1.3. *Given a sequence of sequences $\{x^{(n)}\}_{n=1}^\infty \subset \ell$ the two following statements are equivalent:*

(i) $\lim_{n\to\infty} x^{(n)} = 0 \in \ell$ *in the finest locally convex topology on ℓ.*

(ii) $\lim_{n\to\infty} \mathcal{N}_\mu(x^{(n)}) = 0$ *for every non-negative weighting μ.*

Given a sequence of real or complex numbers $x = \{x_i\}_{0=1}^\infty$ we define $\operatorname{supp}(x) = \{i \in \mathbb{N} : x_i \neq 0\}$. The next proposition says that sequential convergence of sequences in ℓ is equivalent to component-wise convergence with the condition that all the sequences are uniformly compactly supported.

Proposition 1.11. *Given a sequence of sequences $\{x^{(n)}\}_{n=1}^\infty \subset \ell$ one has that $\lim_{n\to\infty} x^{(n)} = 0$ if and only if the following statements both hold:*

(i) There exists $M \in \mathbb{N}$ such that for all n one has $\operatorname{supp}(x^{(n)}) \subset \{0, 1, \ldots, M\}$

(ii) For all $i \in \mathbb{N}$ one has $\lim_{n\to\infty} x_i^{(n)} = 0$

($\Leftarrow$) Suppose we are given $\{x^{(n)}\}_{n=1}^\infty$ satisfying statements $(i), (ii)$. Sequential convergence easily follows by the criterion of Corollary 1.3.1 since for any μ, the quantity $\mathcal{N}_\mu(x^{(n)})$ is just a sum of M terms each of which is going to 0.

($\Rightarrow$) We prove this direction by contrapositive. Suppose we are given $\{x^{(n)}\}_{n=1}^\infty$ for which statement (i) does not hold. We will construct μ such that $\mathcal{N}_\mu(x^{(n)})$ does not convergence to 0 as $n \to \infty$. By our assumptions we can find sequences of indices n_j, k_j, both strictly increasing in j, such that for all $j \in \mathbb{N}$ one has

$$x_{k_j}^{(n_j)} \neq 0.$$

Now define a non-negative weighting μ as follows:

$$\mu_i = \begin{cases} \left| x_{k_j}^{(n_j)} \right|^{-1} & \text{, if } i = k_j \text{ for some } j \in \mathbb{N} \\ 0 & \text{, otherwise} \end{cases}$$

With this choice of μ it is clear that for all $j \in \mathbb{N}$ one has $\mathcal{N}_\mu(x^{(n_j)}) \geqslant 1$. □

We can now prove the theorem mentioned earlier

Theorem 1.2. *The finest locally convex topology on ℓ cannot be generated by a countable family of seminorms.*

Proof: We proceed by contradiction and assume that $\{\mathcal{N}_j\}_{j=1}^{\infty}$ is a family of seminorms that generates the finest locally convex topology on ℓ.

Then we claim that the following is a translation invariant metric $d(\cdot,\cdot)$ on ℓ

$$d(x,y) := \sum_{j=1}^{\infty} 2^{-j} \max(1, \mathcal{N}_j(x-y)).$$

The symmetry of $d(\cdot,\cdot)$ and the fact that is satisfies the triangle inequality both follow easily from the definition above. The fact that $d(x,y) = 0 \Rightarrow x = y$ comes from the observation that the finest locally convex topology ℓ is clearly Hausdorff. This means that it must be the case that for every non-zero $x \in \ell$ one can find j such that $\mathcal{N}_j(x) > 0$.

By the assumptions made so far d must induce the finest locally convex topology on ℓ. It is clear that in this topology for every $k \in \mathbb{N}$ one has that $\epsilon\delta^{(k)}$ converges to 0 as $\epsilon \to 0$ so it must be the case that $\lim_{\epsilon\to 0} d(0, \epsilon\delta^{(k)}) = 0$ for all k. It is then possible to choose constants $\epsilon_{k,j} > 0$ such that for every $k, j \in \mathbb{N}$ one has $d(0, \epsilon_{j,k}\delta^{(k)}) < 2^{-j}$.

Now define a sequence of elements $\{y^{(j)}\}_{j=1}^{\infty} \subset \ell$ by setting $y^{(j)} = \epsilon_{j,j}\delta^{(j)}$. One must then have $\lim_{j\to\infty} d(0, y^{(j)}) = 0$ which means that $y^{(j)}$ converges to 0 in the finest locally convex topology as $j \to \infty$. However this is a contradiction; the sequence of sequences $y^{(j)}$ is not uniformly compactly supported and so by Proposition 1.11 $y^{(j)}$ does not converge to 0 in the finest locally convex topology. $\square$

We have the following as an immediate corollary.

Corollary 1.4. *The finest locally convex topology on $S(\mathbb{Q}_p^d)$ cannot be generated by a countable family of seminorms.*

For X be a topological vector space we call a map

$$L : \underbrace{X \times \cdots \times X}_{n \text{ times}} \to \mathbb{R}$$

an n-multilinear functional if it is linear in each of its components. We call the analogous map in the complex setting an n-multilinear map as well. The definitions and proposition we give below show that the topics such as the continuity of multi-linear functionals and the kernel theorem become trivial in this setting of the finest locally convex topology. The content below is stated for the case of real functionals but can easily be transferred to the complex case as well.

Definition. *An n-linear functional L on a topological vector space X is said to be jointly continuous if for every $\epsilon > 0$ there exists a n neighborhoods $N_1, \ldots, N_n \subset X$ each containing 0 such that for every choice $f_1 \in N_1, \ldots, f_n \in N_n$ one has*

$$|L(f_1, \ldots, f_n)| < \epsilon$$

We remark that if the topology on X is generated by a family of seminorms $\mathcal{SN} = \{N_\alpha\}_\alpha$ then a sufficient condition for an n multilinear functional L to be continuous is for there to exist seminorms $N_1, \ldots, N_n \in \mathcal{SN}$

such that for every $f_1, \ldots, f_n \in X$ one has

$$|L(f_1, \ldots, f_n)| \leqslant \prod_{j=1}^{n} \mathcal{N}_j(f_j). \tag{1.3}$$

With this in mind one sees that multilinearity immediately implies joint continuity.

Proposition 1.12. *Suppose that*

$$L : \underbrace{S(\mathbb{Q}_p^d) \times \cdots \times S(\mathbb{Q}_p^d)}_{n \text{ times}} \to \mathbb{R}$$

be an n-multilinear map. Then L is jointly continuous in its n arguments.

Proof: By virtue of the correspondance between $S(\mathbb{Q}_p^d)$ and ℓ it is sufficient to prove the assertion for n-multilinear functionals in the case where $X = \ell$, that is multilinear functionals

$$L : \underbrace{\ell \times \cdots \times \ell}_{n \text{ times}} \to \mathbb{R}.$$

Following our earlier remark it suffices to find seminorms for which we have a bound of the type (1.3). For $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$ we define $m_\alpha \in \mathbb{R}$ via

$$m_\alpha := \left| L\left(\delta^{(\alpha_1)}, \ldots, \delta^{(\alpha_n)} \right) \right|$$

Here for $j \in \mathbb{N}$ denote by $\delta^{(j)}$ the sequence in ℓ that is given by zeroes everywhere except for the j-th location where it takes the value 1. We define a non-negative weighting μ_k as follows:

$$\mu_k := \max\left(1, \max_{||\alpha||_\infty \leqslant k} m_\alpha\right).$$

where for a multi-index α we define $||\alpha||_\infty = \max_{1 \leqslant j \leqslant n} \alpha_j$. The key property satisfied by the sequence $\{\mu_k\}_{k=0}^{\infty}$ is that for any $\alpha \in \mathbb{N}^d$ one has

$$m_\alpha \leqslant \prod_{j=1}^{d} \mu_{\alpha_j}.$$

Let $\mathcal{N}_\mu$ be the corresponding seminorm. Now let $x^{(1)}, \ldots, x^{(n)} \in \ell$ with

$$x^{(j)} = \sum_{k=0}^{\infty} a_{j,k} \delta^{(k)}$$

for $1 \leqslant j \leqslant n$ and constants $a_{j,k}$ (which are zero for all but finitely many k). One then has

$$\left| L\left(x^{(1)}, \ldots, x^{(n)}\right) \right| \leqslant \sum_\alpha \left[\prod_{j=1}^{n} |a_{j,\alpha_j}| \right] m_\alpha \leqslant \sum_\alpha \left[\prod_{j=1}^{n} |a_{j,\alpha_j}| \mu_{\alpha_j} \right] = \prod_{j=1}^{n} \left[\sum_{k=0}^{\infty} |a_{j,k}| \mu_k \right] = \prod_{j=1}^{n} \mathcal{N}_\mu(x^{(j)}) \tag{1.4}$$

□

For $f \in S(\mathbb{Q}_p^m)$ and $g \in S(\mathbb{Q}_p^n)$ we define $f \otimes g \in S(\mathbb{Q}_p^{m+n})$ via $(f \otimes g)(x, y) = f(x)g(y)$ for $(x, y) \in \mathbb{Q}_p^m \times \mathbb{Q}_p^n$. We can now state a kernel theorem for $S(\mathbb{Q}_p^d)$.

Theorem 1.3. *Given a jointly continuous n multilinear functional L on $S(\mathbb{Q}_p^d)$ there is a unique continuous linear functional $\tilde{L} \in S'(\mathbb{Q}_p^{nd})$ such that one has*

$$L(f_1, \ldots, f_n) \;=\; \tilde{L}(f_1 \otimes \cdots \otimes f_n). \tag{1.5}$$

Proof: By the earlier proposition it is clear that we can drop the qualifier "jointly continuous" from the above theorem. The key observation for the p-adic section is that every $g \in S(\mathbb{Q}_p^{nd})$ can be written as a linear combination of functions of the form

$$f_1 \otimes \cdots \otimes f_n$$

with $f_1, \ldots, f_n \in S(\mathbb{Q}_p^d)$. To see this we remark that for $r \leqslant s$ one has that $S_{r,s}(\mathbb{Q}_p^{nd})$ is spanned by indicator functions of the sets $a + p^r\mathbb{Z}_p^{nd}$ for $a \in \mathbb{Q}_p^{nd}$ with $|a| \leqslant p^s$. However if we write $a = (a_1, \ldots, a_n)$ with $a_1, \ldots, a_n \in \mathbb{Q}_p^d$ then

$$\mathbb{1}_{a+p^r\mathbb{Z}_p^{nd}} \;=\; \mathbb{1}_{a_1+p^r\mathbb{Z}_p^d} \otimes \cdots \otimes \mathbb{1}_{a_n+p^r\mathbb{Z}_p^d}.$$

In particular each element of $S(\mathbb{Q}_p^{nd})$ can uniquely be written as a finite linear combinations in the basis

$$\left\{\otimes_{j=1}^d e_{\alpha_j}\right\}_{\alpha \in \mathbb{N}^n}$$

where the $\{e_k\}_{k=0}^{\infty}$ are the basis for $S(\mathbb{Q}_p^d)$ given by Proposition 1.2.

It is then clear that every n-multilinear functional L uniquely determines an element $\tilde{L} \in S'(\mathbb{Q}_p^{nd})$ by enforcing that (1.5) hold for $f_1, \ldots, f_n$ chosen as basis elements in $S(\mathbb{Q}_p^d)$ - $\tilde{L}$ can then be uniquely extended to all of $S'(\mathbb{Q}_p^{nd})$ via linearity. Uniqueness of such an $\tilde{L}$ is immediate. □

We will now identify $S'(\mathbb{Q}_p^d)$ with a sequence space. In particular we will identify it with the topological (and algebraic) dual of ℓ which we denote by ℓ'.

It is not hard to see that ℓ' can be realized as the space of *all* real sequences, that is $\ell' = \prod_{i\in\mathbb{N}} \mathbb{R} = \mathbb{R}^{\mathbb{N}}$. For a sequence $y = \{y_i\}_{i=0}^{\infty} \in \ell'$ and $x = \{x_i\}_{i=0}^{\infty} \in \ell$ the duality pairing (y, x) is given by the (necessarily finite) sum $\sum_{i=0}^{\infty} x_i y_i$. We view ℓ' as a topological vector space by equipping it with the product topology (which one can think of as the cylinder set topology when one keeps the duality pairing in mind).

We define the linear map $T^* : S'(\mathbb{Q}_p^d) \to \ell'$ via $T^*(\phi) = \{y_i(\phi)\}$ with $y_i(\phi) = \phi(e_i)$ where the e_i is given as in Proposition (1.2). Since both $S'(\mathbb{Q}_p^d)$ and ℓ' are equipped with their respective cylinder set topologies it is easy to see that T^* is in fact a homeomorphism. In particular T^* is the adjoint of T, i.e. $(\phi, Tx) = (T^*\phi, x)$ for $x \in \ell$ and $\phi \in S'(\mathbb{Q}_p^d)$.

1.4 Measures on S'

1.4.1 A Bochner-Minlos Theorem for $S'(\mathbb{Q}_p^d)$

We view $S'(\mathbb{Q}_p^d)$ as a measurable space by equipping it with its the Borel σ-algebra - remember that $S'(\mathbb{Q}_p^d)$ is equipped with the cylinder set topology. The corresponding Borel σ-algebra will also be called the cylinder set σ-algebra. Before describing the main result of this section we state an important definition:

Definition. *Let X be a topological vector space. We say a function $\psi : X \to \mathbb{C}$ is* positive definite *if for* *all $n \in \mathbb{N}$ and any $\zeta_1, \ldots, \zeta_n \in X$ the the $n \times n$ matrix formed by the entries $(\psi(\zeta_i - \zeta_j))_{ij}$, $1 \leqslant i, j \leqslant n$, is a Hermitian positive semidefinite matrix. Stating this condition explicitly one must have that the following two conditions hold:*

(i) $\psi(-\zeta) = \overline{\psi(\zeta)}$ *for all* $\zeta \in X$

(ii) For any $z_1, \ldots, z_n \in \mathbb{C}$ *one has*

$$\sum_{i,j=1}^{n} \bar{z}_i \psi(\zeta_i - \zeta_j) z_j \geqslant 0$$

We now state one of the main theorems of this section.

Theorem 1.4 (Bochner - Minlos Theorem for $S'(\mathbb{Q}_p^d)$)**.** *There is a one-to-one correspondance between probability measures μ on the measure space $(S'(\mathbb{Q}_p^d), \mathcal{C})$ and the set of functions $\theta : S(\mathbb{Q}_p^d) \to \mathbb{C}$ which satisfy (i) $\theta(0) = 1$, (ii) θ is continuous on $S(\mathbb{Q}_p^d)$, (iii) θ is positive definite. The correspondance $\mu \leftrightarrow \theta_\mu$ is given by*

$$\theta_\mu(f) = \int_{S'(\mathbb{Q}_p^d)} \mathrm{d}\mu(\phi)\ e^{i(\phi, f)},$$

i.e. θ_μ is the characteristic function *for the measure μ. Note that (ϕ, f) denotes the duality pairing between $S(\mathbb{Q}_p^d)$ and $S'(\mathbb{Q}_p^d)$.*

Here $\mathcal{C}$ denotes the cylinder σ-algebra. Given a topological vector space X and its topological dual X', the cylinder σ-algebra $\mathcal{C}$ on X' is the coarsest σ-algebra on X' which makes evaluation at x a measurable map from X' to $\mathbb{R}'$ for any $x \in X$.

Now using the maps T^*, T defined in the last section it follows that Theorem 1.4 is equivalent to the following theorem.

Theorem 1.5 (Bochner - Minlos Theorem for ℓ')**.** *There is a one-to-one correspondance between probability measures μ on $\ell' = \mathbb{R}^{\mathbb{N}}$ equipped with its cylinder σ-algebra (or equivalently its product σ-algebra) and the set of functions $\theta : \ell \to \mathbb{C}$ which satisfy (i) $\theta(0) = 1$, (ii) θ is continuous on ℓ, (iii) θ is positive definite. The correspondance $\mu \leftrightarrow \theta_\mu$ is given by*

$$\theta_\mu(x) = \int_{\ell'} \mathrm{d}\mu(y)\ e^{i(y,x)},$$

Note that (y, x) denotes the duality pairing between $y \in \ell'$ and $x \in \ell$

Our method of proof will involve applying Bochner's Theorem on $\mathbb{R}^d$ which we give below.

Theorem 1.6 (Bochner's Theorem for $\mathbb{R}^d$)**.** *There is a one-to-one correspondance between Borel probability measures μ on $\mathbb{R}^d$ and the set of functions $\theta : \mathbb{R}^d \to \mathbb{C}$ such that (i) $\theta(0) = 1$, (ii) θ is continuous, (iii) θ is positive definite.*

The correspondance $\mu \leftrightarrow \theta_\mu$ is given by

$$\theta_\mu(\xi) = \int_{\mathbb{R}^d} \mathrm{d}\mu(x)\ e^{ix\cdot\xi},$$

Proof: The hard direction is that conditions (i), (ii), and (iii) on θ are sufficient to guarantee the existence of a Borel probability measure μ on $\mathbb{R}^d$ such that $\theta = \theta_\mu$. For a proof of this see [55, Theorem IX.9].

For necessity of conditions (i), (ii), (iii) we note that for any μ and corresponding θ_μ condition (i) is immediately, (ii) follows from Lebesgue dominated convergence, and condition (iii) comes from observing that for any $z_1, \ldots, z_n \in \mathbb{C}$, $\xi_1, \ldots, \xi_n \in \mathbb{R}$ one has

$$0 \leqslant \int_{\mathbb{R}} \mathrm{d}\mu(x) \left| \sum_{i=1}^{n} z_i e^{ix\cdot\xi_i} \right|^2 = \int_{\mathbb{R}} \mathrm{d}\mu(x) \sum_{i,j=1}^{n} z_i \bar{z_j} e^{ix\cdot(\xi_i - \xi_j)} = \sum_{i,j=1}^{n} z_i \bar{z_j} \theta_\mu(\xi_i - \xi_j).$$

For uniqueness we suppose that for Borel probability measures μ, ν on $\mathbb{R}$ we have $\theta_\mu = \theta_\nu$. Let $f \in \mathcal{C}_c^\infty(\mathbb{R}^d)$, i.e. a smooth function of compact support. In particular one has $f, \hat{f} \in L^1(\mathbb{R}^d, \mathrm{d}^d x)$. Then by the Fourier inversion theorem and Fubini one has

$$\begin{aligned}
\int_{\mathbb{R}^d} \mathrm{d}\mu(x) f(x) &= \int_{\mathbb{R}^d} \mathrm{d}\mu(x) \left(\int_{\mathbb{R}^d} \mathrm{d}^d\xi \ e^{ix\cdot\xi} \hat{f}(\xi) \right) \\
&= \int_{\mathbb{R}^d} \mathrm{d}^d\xi \ \hat{f}(\xi) \left(\int_{\mathbb{R}^d} \mathrm{d}\mu(x) e^{ix\cdot\xi} \right) \\
&= \int_{\mathbb{R}^d} \mathrm{d}^d\xi \ \hat{f}(\xi) \ \theta_\mu(\xi) = \int_{\mathbb{R}^d} \mathrm{d}^d\xi \ \hat{f}(\xi) \ \theta_\nu(\xi) = \int_{\mathbb{R}^d} \mathrm{d}\nu(x) f(x).
\end{aligned}$$

Then by a simple approximation argument it follows that $\mu = \nu$. □

We will also need Kolmogorov's Extension Theorem but first we introduce some notation. Let I be an index set. We denote by $\mathcal{B}_I$ the product Borel σ-algebra on $\mathbb{R}^I$. For any finite set F let $\pi_{I,F} : \mathbb{R}^I \to \mathbb{R}^F$ be the canonical projection map. For any finite sets F, G with $F \subset G \subset I$ Let $\pi_{G,F}$ be the canonical projection from $\mathbb{R}^G$ to $\mathbb{R}^F$.

Theorem 1.7 (Kolmogorov's Extension Theorem)**.** *Let I be an index set. We see $\mathbb{R}^I$ as a measure space by equipping it with its product Borel σ-algebra. Suppose that we are given a family of consistent family of finite dimensional distributions - that is we are given probability measures μ_F on $(\mathbb{R}^F, \mathcal{B}_F)$ for every finite set $F \subset I$ that together satisfy the following: for any two finite sets $F \subset G \subset I$ every $A \in \mathcal{B}_F$ one has that $\mu_G\left(\pi_{G,F}^{-1}(A)\right) = \mu_F(A)$, i.e. μ_F is the pushforward of μ_G under π_F^G.*

Then it follows that there exists a unique *probability measure μ on $(\mathbb{R}^I, \mathcal{B}_I)$ such that for every finite set $F \subset I$ and $A \in \mathcal{B}_F$ one has $\mu(\pi_{I,F}^{-1}(A)) = \mu_F(A)$. In other words there is a unique probability measure μ which has the measures μ_F as its finite dimensional marginals.*

Proof: See almost any book that covers stochastic processes, for example [67, Theorem 1.1.10].

We can now prove Theorem 1.5.

Proof of Theorem 1.5: We need to show that for a given θ satisfying (i), (ii), and (iii) there exists a unique probability measure μ on the infinite product space $\ell' = (\mathbb{R}^{\mathbb{N}}, \mathcal{B}_{\mathbb{N}})$ with $\theta = \theta_\mu$.

First we focus on constructing such a measure μ from θ. For any finite set $F \subset \mathbb{N}$ and $x \in \mathbb{R}^F$ we write x_F to denote the element of ℓ with $(x_F)_i = x_i$ for $i \in F$ and $(x_F)_i = 0$ for $i \notin F$.

Now we define $\theta_F : \mathbb{R}^F \to \mathbb{C}$ via $\theta_F(x) = \theta(x_F)$. We claim that θ_F satisfies Theorem 1.6 - condition (i) and condition (iii) both follow immediately from the analogous conditions on θ. We remark that $x \to x_F$ is a continuous map between $\mathbb{R}^F$ equipped with its standard topology and ℓ with its finest locally convex topology (the standard topology $\mathbb{R}^F$ coincides with its own finest locally convex topology, so one can apply Proposition 1.4). Thus we are guaranteed a unique Borel probability measure μ_F on $\mathbb{R}^F$ with θ_F as its characteristic function. Proceeding this way for all finite subsets of $\mathbb{N}$ gives us a family of finite dimensional distributions.

We now show that this family is consistent. Suppose we have finite sets $F, G \subset \mathbb{N}$ with $F \subset G$. Let $\tilde{\mu}_F$ be the marginal of μ_G on $\mathbb{R}^F$ (i.e. $\tilde{\mu}_F$ is the pushforward of μ_G under the projection $\pi_{G,F}$). It is not hard to see that the characteristic function of $\tilde{\mu}_F$ is given by θ_F and so from the uniqueness assertion in Theorem 1.6 it follows that $\mu_F = \tilde{\mu}_F$. Thus the desired measure μ on $\mathbb{R}^{\mathbb{N}}$ can be constructed from the finite dimensional measures μ_F via Kolmogorov's Extension Theorem.

Now for any $a \in \ell$, there exists a finite set $F \subset \mathbb{N}$ and $x \in \mathbb{R}^F$ so that $x_F = a$. It then follows that

$$\int_{\ell'} \mathrm{d}\mu(y)\ e^{i(y,a)} \ = \ \int_{\mathbb{R}^F} \mathrm{d}\mu_F(y)\ e^{i(y,x)} \ = \ \theta_F(x) \ = \theta(x_F) = \theta(a)$$

so we have that $\theta = \theta_\mu$ on ℓ. We remark that if for μ and ν on ℓ' one has $\theta_\mu = \theta_\nu$ then we can again use Bochner's Theorem over $\mathbb{R}^d$ to show that μ and ν' have the same finite dimensional distributions, thus by Kolmorogorov's Extension Theorem one must have $\mu = \nu$ □

We remark that we only needed the *countable* version of Kolmogorov's Extension Theorem.

1.4.2 Bochner-Minlos Theorem for $S'(\mathbb{R}^d)$

Identification with Sequences

We start by defining Schwartz space on $\mathbb{R}^d$, denoted by $S(\mathbb{R}^d)$, and its correspond dual, the space of tempered distributions which we denote by $S'(\mathbb{R}^d)$. We make frequent use of multi-index notation. For $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$ and $x \in \mathbb{R}^d$ we define $x^\alpha = \prod_{i=1}^d x_i^{\alpha_i}$. We also define differential operators D^α acting on functions over $\mathbb{R}^d$ by defining $D^\alpha := \prod_{i=1}^d \partial_i^{\alpha_i}$.

Definition. *Schwartz space over $\mathbb{R}^d$, denoted by $S(\mathbb{R}^d)$, is defined to be the set of all smooth functions $f \in C^\infty(\mathbb{R}^d)$ such that for any multi-indices $\alpha,\ \beta \in \mathbb{N}^d$ one has*

$$||f||_{\alpha,\beta} \ := \ \sup_{x \in \mathbb{R}^d} |x^\alpha D^\alpha f(x)| \ < \ \infty$$

We turn $S(\mathbb{R}^d)$ into a topological vector space by equipping it with the topology induced by the countable family of seminorms $\{|| \cdot ||_{\alpha,\beta}\}_{\alpha,\beta \in \mathbb{N}^d}$. In particular $S(\mathbb{R}^d)$ is Frechet space with the mentioned topology coinciding with the one induced by the metric

$$d(f,g) = \sum_{k=0}^{\infty} 2^{-k} \max\left(||f-g||_k,\ 1\right)$$

where seminorms $\{||f-g||_k\}_{k=0}^\infty$ are an arbitrary enumeration of the seminorms $\{|| \cdot ||_{\alpha,\beta}\}_{\alpha,\beta \in \mathbb{N}^d}$.

In keeping with earlier conventions we will use $S(\mathbb{R}^d, \mathbb{C})$ to denote the corresponding space of complex valued test functions, in particular $S(\mathbb{R}^d, \mathbb{C})$ is just the complexification of $\mathbb{R}^d$.

Definition. *The space of tempered distributions over $\mathbb{R}^d$, denoted by $S'(\mathbb{R}^d)$, is defined to be the topological dual of the space $S(\mathbb{R}^d)$, i.e. the space of all continuous linear functionals on $S(\mathbb{Q}_p^d)$. We turn $S'(\mathbb{Q}_p^d)$ into a* *topological vector space by equipping it with its cylinder set topology.*

We similarly define $S'(\mathbb{R}^d, \mathbb{C})$ to be the topological dual of $S(\mathbb{R}^d, \mathbb{C})$. The complex case is not of interest here but will come up in Chapter 2.

Again our first step is to identify the space $S(\mathbb{R}^d)$ with a space of real sequences. As mentioned before here we follow [61]. Since $S(\mathbb{R}^d) \subset \mathcal{L}^2(\mathbb{R}^d)$ one can expand test functions in $S(\mathbb{R}^d)$ in terms of the orthonormal basis of Hermite polynomials, in other words harmonic oscillator wavefunctions.

For simplicity of notation we specialize to the case $d = 1$, the generalization to higher dimensions is straightforward. We define the n-th Hermite polynomial $\phi_n \in S(\mathbb{R})$

$$\phi_n(x) = \pi^{-\frac{1}{4}} 2^{-\frac{n}{2}} (n!)^{-\frac{1}{2}} e^{\frac{1}{2}x^2} \left[\left(\frac{d}{dx} \right)^n e^{-x^2} \right].$$

For $f \in S(\mathbb{R})$ we will identify $f \leftrightarrow \{a_n\}_{n=0}^{\infty}$ where the a_n are given by

$$a_n \;=\; \int_{\mathbb{R}} \mathrm{d}x \; f(x)\phi_n(x).$$

Clearly $||f||_{\mathcal{L}^2} = ||a||_{\ell^2} < \infty$. However one can show stronger growth conditions on the Hermite expansion coefficients $f \in S(\mathbb{R})$, in particular these coefficient sequences decay faster than any polynomial. To see this we start by defining the operator

$$H = -\frac{d^2}{dx^2} \;+\; x^2 \;+1$$

on $S(\mathbb{R})$.

Proposition 1.13. *Suppose that $f \in S(\mathbb{R})$. Then the corresponding sequence of Hermite coefficients $\{a_n\}_{n=0}^{\infty}$ satisfy*

$$\sum_{n=0}^{\infty} |a_n|^2 (n+1)^m < \infty \text{ for all } m$$

We observe that $H^m \phi_n = 2^m (n+1)^m \phi_n$ and that $H^m f \in S(\mathbb{R}) \subset \mathcal{L}^2(\mathbb{R})$. One then has

$$2^m \sum_{n=0}^{\infty} a_n (n+1)^m \;=\; \langle f, H^m f \rangle_{L^2} < \infty.$$

□

We define the norm $|\cdot|_m$ on sequences $a = \{a_n\}_{n=0}^{\infty}$ via setting

$$|a|_m = \left[\sum_{n=0}^{\infty} |a_n|^2 (n+1)^m \right]^{\frac{1}{2}}.$$

This gives us a a corresponding norm $|\cdot|_m$ on $S(\mathbb{R})$ via our linear map $f \to \{a_n\}_{n=0}^{\infty}$.

We introduce some terminology - two families of seminorms are said to be *equivalent* if for each family, each seminorm in that family can be bounded above by a finite linear combination of seminorms from the other family. In particular equivalent families of seminorms generate the same topology. One can show that the norms $|\cdot|_m$ are equivalent to the seminorms we gave when we defined Schwartz space.

Proposition 1.14. *The two families of seminorms on* $S(\mathbb{R})$

$$\{||\cdot||_{\alpha,\beta}\}_{\alpha,\beta\in\mathbb{N}^1} \text{ and } \{|\cdot|_m\}_{m=0}^{\infty}$$

on $S(\mathbb{R})$ *are equivalent*

Proof: See [61, Lemmas 6,7,8,9].

We now define the sequence space

$$\mathfrak{s} \;=\; \{a = \{a_n\} \in \mathbb{R}^{\mathbb{N}} | \; \forall \; m \in \mathbb{N}, |a|_m < \infty\}.$$

From Proposition (1.14) it follows that $f \in \mathcal{L}^2(\mathbb{R})$ satisfies $f \in S(\mathbb{R})$ if and only if $a = \{a_n\}_{n=0}^{\infty} \in \mathfrak{s}$ - so we have a linear bijection between $S(\mathbb{R})$ and $\mathfrak{s}$. Additionally if we turn $\mathfrak{s}$ into a topological vector space by equipping it with the family of seminorms $\{|\cdot|_m\}_{m=0}^{\infty}$ then $S(\mathbb{R})$ and $\mathfrak{s}$ are isomorphic as topological vector spaces.

At this point we say a little about how this construction can be carried over to higher dimensions expanding $f \in S(\mathbb{R}^d)$ as a sum of products of Hermite functions indexed by mult-indices, that is for $\alpha \in \mathbb{N}^d$ one sets

$$\phi_\alpha(x) = \prod_{i=1}^{d} \phi_{\alpha_i}(x_i).$$

Accordingly $S(\mathbb{R}^d)$ can be identified with a space of "sequences" indexed by multi-indices, the corresponding sequence space can be turned into a topological vector space by using the seminiorms similar to the seminorms used in the definition of $\mathfrak{s}$. For example for fixed sequence $a \in \mathbb{R}^{\mathbb{N}^d}$ and $m \in \mathbb{N}$ one could set

$$|a|_m = \sum_{\alpha\in\mathbb{N}^d} \left[\prod_{i=1}^{d}(\alpha_i+1)^m\right] a_\alpha.$$

In fact in [61, Theorem 9] the above identifications are used to prove the following result:

Proposition 1.15. *For any dimension* d *the spaces* $S(\mathbb{R}^d)$ *and* $S(\mathbb{R})$ *are isomorphic as topological vector spaces.*

We now continue our analysis on $S(\mathbb{R})$, keeping in mind what we do can easily be generalized to $S(\mathbb{R}^d)$.

We remark that the $|\cdot|_m$ are a *directed* family of seminorms. A family of seminorms $\{\mathcal{N}_\gamma\}_{\gamma\in I}$ is said to be directed if for every finite collection $\gamma_1, \ldots, \gamma_n \in I$ there exists $\bar{\gamma} \in I$ and $C > 0$ such that

$$\sum_{j=1}^{n} \mathcal{N}_{\gamma_j}(\cdot) \leqslant C\mathcal{N}_{\bar{\gamma}}(\cdot).$$

We also mention that the seminorms indexed by multi-indices we defined for $S(\mathbb{R}^d)$ are also directed.

We now prove the following theorem about continuous linear functionals on $S(\mathbb{R})$

Lemma 1.3. *A linear map* $T : S(\mathbb{R}) \to \mathbb{C}$ *is continuous if and only if there exists some* $C \geqslant 0$, $m \in \mathbb{N}$ *such that for all* $f \in S(\mathbb{R})$

$$|T(f)| \;\leqslant\; C|f|_m$$

Proof: Sufficiency of the given inequality is immediate, we now show necessity. Suppose that T is a continuous linear functional, then there must be a non-empty open set $U \subset S(\mathbb{R})$ containing 0 such that for all $f \in U$ one has $T(f) \leqslant 1$.

We remark that by Proposition (1.14) one has the norms $\{|\cdot|_j\}_{j=0}^{\infty}$ generate the topology on $U \subset S(\mathbb{R})$, so there must be an $\epsilon > 0$ and $N \in \mathbb{N}$ such that

$$\bigcap_{j=0}^{N} \{f \in S(\mathbb{R})|\ |f|_j < \epsilon\} \ \subset \ U$$

However since the seminorms are directed we can find $m \in \mathbb{N}$, and $A > 0$ such that

$$\sum_{j=0}^{N} |\cdot|_j \ \leqslant A \times |\cdot|_m.$$

The assertion is then proved by using this choice of M and letting $C = \frac{A}{\epsilon}$. □

We mention that it is easy to use the basic idea of Lemma 1.3 to show that tempered distributions $T \in S'(\mathbb{R}, \mathbb{C})$ are of "finite order" - i.e. $T(f)$ depends on only finitely many derivatives of f.

Now for $T \in S'(\mathbb{R})$ we identify $T \leftrightarrow \{b_n\}_{n=0}^{\infty} \in \mathbb{R}^{\mathbb{N}}$ via setting

$$b_n \ = \ T(\phi_n)$$

The next proposition shows that elements of $S(\mathbb{R})$ correspond to sequences of at most polynomial growth.

Proposition 1.16. *Suppose that $T \in S'(\mathbb{R})$ and let $\{b_n\}_{n=0}^{\infty}$ be the corresponding sequence of Hermite coefficients. Then there exists $C \geqslant 0$, $m \in \mathbb{N}$ such that for all $n \in \mathbb{N}$*

$$|b_n| \leqslant C(1+n)^m$$

Proof: We recall that $|\phi_n|_m = (n+1)^{\frac{m}{2}}$. Now let C, m be given as promised in Proposition 1.3. Then one has $|b_n| = |T(\phi_n)| \leqslant C|\phi_n|_m = C(n+1)^{\frac{m}{2}}$. □

We now define the dual sequence space

$$\mathfrak{s}' = \{b = \{b_n\}_{n=0}^{\infty} \in \mathbb{R}^{\mathbb{N}}|\ \exists m \in \mathbb{N} \text{ such that } \sup_n |b_n|(1+n)^{-m} < \infty\}$$

It then follows from Proposition 1.3 that every $T \in S(\mathbb{R})$ can be identified with a unique sequence $b \in \mathfrak{s}'$. Additionally every $b \in \mathfrak{s}'$ defines a unique $T_b \in S(\mathbb{R})$ via defining $T_b(f) = \sum_{n=0}^{\infty} b_n a_n$ where a_n are the Hermite expansion coefficients for $f \in S(\mathbb{R})$ - this is clearly a linear functional on $S(\mathbb{R})$ - continuity follows by showing the inequality stated in Lemma 1.3.

Thus our identification between elements $T \in S'(\mathbb{R})$ and sequences $b \in \mathfrak{s}'$ is a linear isomorphism of vector spaces. It is not hard to see that $\mathfrak{s}'$ is the topological dual of the Frechet space $\mathfrak{s}$ where the duality pairing is given by

$$b \cdot a = \sum_{n=0}^{\infty} b_n a_n$$

for $b \in \mathfrak{s}'$ and $a \in \mathfrak{s}'$

We turn $\mathfrak{s}'$ into a topological vector space by equipping it with its corresponding cylinder set topology. With this topology this correspondance between $S'(\mathbb{R})$ (which is equipped with its corresponding cylinder topology) and $\mathfrak{s}'$ is an isomorphism of topological vector spaces.

The characterization of Schwartz space in terms of Hermite expansions also gives a proof of the kernel theorem.

Lemma 1.4. *Let L be an n-multilinear functional on $S(\mathbb{R}^d)$. Then L is jointly continuous if and only if there exist $j_1, \ldots, j_n$ and $C > 0$ such that*

$$|L(f_1, \ldots, f_n)| \leqslant C \prod_{k=1}^{n} |f_k|_{j_k}$$

Proof: This lemma is proved just like Lemma 1.3. □

Theorem 1.8. *Given a jointly continuous n-multilinear functional L on $S(\mathbb{R}^d)$ there is a unique distribution $\tilde{L} \in S'(\mathbb{R}^{nd})$ such that*

$$L(f_1, \ldots, f_n) \ = \ \tilde{L}(f_1 \otimes \cdots \otimes f_n)$$

for any $f_1, \ldots, f_n \in S(\mathbb{R}^d)$

Proof: We sketch the proof which is quite simple in this setting, we also restrict ourselves to the case $n = 2$ - the general case follows by the same argument. The main idea is to construct $\tilde{L}$ via a Hermite expansion in Hermite polynomials over $\mathbb{R}^{nd}$.

For multi-indices $\alpha, \beta \in \mathbb{N}^d$ we define the multi-index $(\alpha, \beta) \in \mathbb{N}^{2d}$ via concatenation, then our Hermite basis over $\mathbb{R}^{nd}$ can be written

$$\{\phi_{\alpha,\beta}\}_{\alpha,\beta \in \mathbb{N}^d}$$

where for $x, y \in \mathbb{R}^d$ we have

$$\phi_{(\alpha,\beta)}(x, y) = \phi_\alpha(x)\phi_\beta(y)$$

and the ϕ_α, for $\alpha \in \mathbb{N}$ correspond to the Hermite basis for $\mathbb{R}^d$.

The Hermite expansion coefficients for our desired $\tilde{L}$ are then given by

$$\tilde{L}(\phi_{(\alpha,\beta)}) \ = \ L(\phi_\alpha, \phi_\beta).$$

Now in order for these coeffiicents to determine an element of $S'(\mathbb{R}^{2d})$ we must find some $m \in \mathbb{N}$such that

$$\sup_{\alpha,\beta \in \mathbb{N}^d} \tilde{L}(\phi_{(\alpha,\beta)}) \, ((\alpha, \beta) + 1)^{-m} \ < \ \infty$$

where for a multi-index $\gamma \in \mathbb{N}^k$

$$(\gamma + 1) \ = \ \prod_{j=1}^{k} (\gamma_j + 1)^d.$$

However this is immediate from Lemma 1.4 since one has the existence of $C > 0$ and $r, s \in \mathbb{N}$ such that

$$\tilde{L}(\phi_{(\alpha,\beta)}) \ = \ L(\phi_\alpha, \phi_\beta) \ \leqslant \ C \times |\phi_\alpha|_r \times |\phi_\beta|_s \ = \ C \times (\alpha + 1)^{\frac{r}{2}} \times (\beta + 1)^{\frac{s}{2}} \ \leqslant C \times ((\alpha, \beta) + 1)^{\frac{\max(r,s)}{2}}.$$

Thus the given Hermite coefficients $\tilde{L}(\phi_{(\alpha,\beta)})$ determine a distribution $\tilde{L} \in S'(\mathbb{R}^d)$. Uniqueness of such an $\tilde{L}$ follows by a density argument. □

We now turn our focus back to proving the Bochner Minlos Theorem for $S(\mathbb{R}^d)$. It will be more convenient to use slightly different seminorms on $S(\mathbb{R})$. For $a = \{a_n\}_{n=0}^{\infty} \in \mathbb{R}^{\mathbb{N}}$ and $m \in \mathbb{Z}$ we define the norm $||\cdot||_m$

$$||a||_m = \left[\sum_{n=0}^{\infty} (n^2+1)^m |a_n|^2\right]^{\frac{1}{2}}$$

Clearly the family of norms $\{||\cdot||_m\}_{m\in\mathbb{N}}$ and $\{|\cdot|_m\}_{m\in\mathbb{N}}$ are equivalent families for $\mathfrak{s}$ and generate the same topology on $\mathfrak{s}$.

We now define, for $m \in \mathbb{Z}$ the sequence spaces

$$\mathfrak{s}_m = \{a = \{a_n\}_{n=0}^{\infty} \in \mathbb{R}^{\mathbb{N}} |\ ||a||_m < \infty\}$$

We can then write $\mathfrak{s} = \bigcap_{m\in\mathbb{Z}} \mathfrak{s}_m$ and view $\mathfrak{s}$ as a Frechet space with the topology generated by $\{||\cdot||_m\}_{m\in\mathbb{Z}}$ (it is easy to see throwing in the negatively indexed norms doesn't change the topology). Similarly we have that $\mathfrak{s}' = \bigcup_{m\in\mathbb{Z}} \mathfrak{s}_m$.

The Bochner-Minlos Theorem for $S'(\mathbb{R}^d)$

We now follow [63]. The theorem we seek to prove is

Theorem 1.9 (Bochner - Minlos Theorem for $S'(\mathbb{R}^d)$)**.** *There is a one-to-one correspondance between probability measures μ on the measure space $(S'(\mathbb{R}^d), \mathcal{C})$ and the set of functions $\theta : S(\mathbb{R}^d) \to \mathbb{C}$ which satisfy (i) $\theta(0) = 1$, (ii) θ is continuous on ℓ, (iii) θ positive definite. The correspondance $\mu \leftrightarrow \theta_\mu$ is given by*

$$\theta_\mu(f) = \int_{S'(\mathbb{R}^d)} \mathrm{d}\mu(\phi)\ e^{i\phi(f)},$$

Note that $\phi(f)$ denotes the duality pairing between $\phi \in S'(\mathbb{R}^d)$ and $f \in S(\mathbb{R}^d)$

Owing to Proposition 1.15 and the identification of the topological vector spaces $S(\mathbb{R})$ and $S'(\mathbb{R})$ with $\mathfrak{s}$ and $\mathfrak{s}'$ it suffices to prove

Theorem 1.10 (Bochner - Minlos Theorem for $\mathfrak{s}'$)**.** *There is a one-to-one correspondance between probability measures μ on the measure space $(\mathfrak{s}', \mathfrak{c})$ and the set of functions $\theta : \mathfrak{s} \to \mathbb{C}$ which satisfy (i) $\theta(0) = 1$, (ii) θ is continuous on $\mathfrak{s}$, (iii) θ positive definite. The correspondance $\mu \leftrightarrow \theta_\mu$ is given by*

$$\theta_\mu(a) = \int_{\mathfrak{s}} \mathrm{d}\mu(b)\ \exp\left[i(b \cdot a)\right],$$

where $(b \cdot a)$ denotes the duality pairing between $b \in \mathfrak{s}'$ and $a \in \mathfrak{s}$. Here $\mathfrak{c}$ denotes the cylinder set σ-algebra on $\mathfrak{s}'$.

Proof: As before the necessity of conditions (i), (ii), and (iii) are clear.

For any finite set $F \subset \mathbb{N}$ the map $x \to x_F$ defined in the proof of Theorem 1.5 is a continuous linear map $\mathbb{R}^F \to \mathfrak{s}$. We can then proceed exactly as in the proof of Theorem 1.5 to use the function θ to define a family of consistent finite dimensional distributions which uniquely determine a measure μ on $\mathbb{R}^{\mathbb{N}}$ where $\mathbb{R}^{\mathbb{N}}$ is equipped with its product σ-algebra. In particular μ is the unique measure on $\mathbb{R}^{\mathbb{N}}$ such that

$$\theta(a) = \int_{\mathbb{R}^{\mathbb{N}}} \mathrm{d}\mu(b)\ \exp\left[i(b \cdot a)\right], \tag{1.6}$$

for all almost finite sequences a, that is for $a \in \bigoplus_{n\in\mathbb{N}}\mathbb{R}$.

Note that the sets $\mathfrak{s}_m$ and $\mathfrak{s}'$ are measurable subsets of $\mathbb{R}^{\mathbb{N}}$ and the cylinder σ-algebra on $\mathfrak{s}'$ is just the restriction of the product σ-algebra on $\mathbb{R}^{\mathbb{N}}$ to $\mathfrak{s}'$. Now we show that the constructed measure μ on $\mathbb{R}^{\mathbb{N}}$ satisfies $\mu(\mathfrak{s}') = 1$.

It suffices to show that for any $\epsilon > 0$ there exists $m \in \mathbb{Z}$ such that

$$\mu(\mathfrak{s}_m) \geqslant 1 - \epsilon.$$

Now for $\alpha > 0$, and $m, M \in \mathbb{N}$ we define the Gaussian measures $\sigma_{\alpha,m,M}$ on $\mathbb{R}^{M+1}$ via

$$\mathrm{d}\sigma_{\alpha,m,M}(y) \ = \ \prod_{j=0}^{M} \frac{1}{\sqrt{2\pi\alpha(1+j^2)^m}} \exp\left[-\frac{y_j^2}{2\alpha(1+j^2)^m}\right] \mathrm{d}y_j.$$

Using the familiar formula for the characteristic function of a Gaussian we see that for any $x = (x_0, \ldots, x_M) \in \mathbb{R}^{M+1}$ one has

$$\int_{\mathbb{R}^{M+1}} \mathrm{d}\sigma_{\alpha,m,M}(y)\ \exp\left[i(x \cdot y)\right] \ = \ \exp\left[-\frac{\alpha}{2}\sum_{j=0}^{M}(1+j^2)^m|x_j|^2\right]$$

Now by applying monotone convergence and Fubini's Theorem we see

$$\begin{aligned} \mu(\mathfrak{s}_m) \ &= \ \lim_{\alpha\to 0^+}\lim_{M\to\infty}\int_{\mathbb{R}^{\mathbb{N}}} \mathrm{d}\mu(b)\ \exp\left[-\frac{\alpha}{2}\sum_{j=0}^{M}(1+j^2)^m|b_j|^2\right] \\ &= \ \lim_{\alpha\to 0^+}\lim_{M\to\infty}\int_{\mathbb{R}^{\mathbb{N}}} \mathrm{d}\mu(b)\ \int_{\mathbb{R}^{M+1}} d\sigma_{\alpha,m,M}(y)\ \exp\left[i(b\cdot y)\right] \\ &= \ \lim_{\alpha\to 0^+}\lim_{M\to\infty}\int_{\mathbb{R}^{M+1}} \mathrm{d}\sigma_{\alpha,m,M}(y)\ \int_{\mathbb{R}^{\mathbb{N}}} \mathrm{d}\mu(b)\ \exp\left[i(b\cdot y)\right] \\ &= \ \lim_{\alpha\to 0^+}\lim_{M\to\infty}\int_{\mathbb{R}^{M+1}} \mathrm{d}\sigma_{\alpha,m,M}(y)\ \theta(y) \\ &= \ \lim_{\alpha\to 0^+}\lim_{M\to\infty}\int_{\mathbb{R}^{M+1}} \mathrm{d}\sigma_{\alpha,m,M}(y)\ \Re\left[\theta(y)\right] \end{aligned} \tag{1.7}$$

Note that above we abused notation, thinking of $y \in \mathbb{R}^{M+1}$ as an almost finite sequence where only the first $M+1$ entries are allowed to be non-vanishing.

Now let $\epsilon > 0$ be given, then since θ is continuous there exists some open set $U \subset \mathfrak{s}$ containing 0 such that for $a \in U$ one has $|\theta(a) - 1| < \epsilon$. Since the norms $||\cdot||_m$ on $\mathfrak{s}$ are directed it follows that there must be some $k \in \mathbb{Z}$ and $\delta > 0$ such that $||a||_k < \delta \ \Rightarrow a \in U$ - without loss of generality we assume $\delta < 1$.

With m given as above we claim that for all $a \in \mathfrak{s}$ one has

$$\Re\left[\theta(a)\right] \geqslant 1 - \epsilon - 2\delta^{-2}||a||_k^2. \tag{1.8}$$

We first observe that the above inequality holds if $||a||_k \leqslant \delta^2$ since for such a one has $|\theta(a) - 1| < \epsilon$. For $||a||_m > \delta^2$ the above inequality follows from the fact that one has the uniform bound $\Re[\theta] \geqslant -1$. This uniform bound comes the positive definiteness condition on θ which forces $|\theta(a)| \leqslant \theta(0) = 1$ for all $a \in \mathfrak{s}$.

We fix $m = -k - 1$ and apply the bound (1.8) in (1.7) to get

$$\begin{aligned}
\mu(\mathfrak{s}_m) &\geqslant \lim_{\alpha\to 0^+} \lim_{M\to\infty} 1 - \epsilon - 2\delta^{-2} \int_{\mathbb{R}^{M+1}} \mathrm{d}\sigma_{\alpha,m,M}(y)\ ||y||_k^2 \\
&= \lim_{\alpha\to 0^+} \lim_{M\to\infty} 1 - \epsilon - 2\delta^{-2} \int_{\mathbb{R}^{M+1}} \mathrm{d}\sigma_{\alpha,m,M}(y)\ \left(\sum_{j=0}^{M} (1+j^2)^k |y_j|^2\right) \\
&= \lim_{\alpha\to 0^+} \lim_{M\to\infty} 1 - \epsilon - 2\delta^{-2} \left(\sum_{j=0}^{M} \alpha(1+j^2)^{k+m}\right) \\
&= 1 - \epsilon - \lim_{\alpha\to 0^+} 2\alpha\delta^{-2} \left(\sum_{j=0}^{\infty} \frac{1}{1+j^2}\right) \ = \ 1 - \epsilon
\end{aligned}$$

where in going to the second to last line above we used that for $0 \leqslant j \leqslant M$ one has

$$\int_{\mathbb{R}^{M+1}} \mathrm{d}\sigma_{\alpha,m,M}(y)\ y_j^2 \ = \ \alpha(1+j^2)^m.$$

Now it follows that

$$\theta(a) = \int_{\mathbb{R}^{\mathbb{N}}} \mathrm{d}\mu(b)\ \exp\left[i(b\cdot a)\right],$$

for all almost finite sequences. However since both sides of the above equation are continuous as functions of $a \in \mathfrak{s}$ and almost finite sequences are dense in $\mathfrak{s}$ it follows that equality must hold on all of $\mathfrak{s}$. We remark that uniqueness is already taken care of (we proved it at the level of measures on $\mathbb{R}^{\mathbb{N}}$) so the proof is finished. □

1.4.3 Moment Reconstruction Theorem for $S'(\mathbb{Q}_p^d)$

Our goal for this subsection is to prove the following theorem.

Theorem 1.11. *Let $(S_n)_{n\geqslant 0}$ be a sequence of distributions with $S_n \in S'(\mathbb{Q}_p^{nd})$ which satisfies*

1. *$S_0 = 1$,*
2. *for any n, S_n is invariant by the permutation group $\mathfrak{S}_n$,*
3. *for all almost finite sequence of test functions $(h_n)_{n\geqslant 0}$ with $h_n \in S(\mathbb{Q}_p^{nd}, \mathbb{C})$ one has*

$$\sum_{n,m\geqslant 0} S_{n+m}(\overline{h_n} \otimes h_m) \ \in \ [0,\infty)\ ,$$

4. *For all finite dimensional complex subspace V of $S(\mathbb{Q}_p^d, \mathbb{C})$ there exists a semi-norm $\mathcal{N}_V$ on $S(\mathbb{Q}_p^d, \mathbb{C})$ such that for all $n \geqslant 0$ and all $f_1, \ldots, f_n$ in V one has*

$$|S_n(f_1 \otimes \cdots \otimes f_n)| \leqslant n! \times \mathcal{N}_V(f_1) \times \cdots \times \mathcal{N}_V(f_n)\ ;$$

then there exists a unique probability measure with finite moments ν on the measurable space $(S'(\mathbb{Q}_p^d), \mathcal{C})$ such that for all $f_1, \ldots, f_n \in S(\mathbb{Q}_p^d, \mathbb{C})$ we have

$$S_n(f_1 \otimes \cdots \otimes f_n) = \int_{S'(\mathbb{Q}_p^d)} \mathrm{d}\nu(\phi)\ \phi(f_1) \cdots \phi(f_n)\ .$$

We will give the proof of Theorem 1.11 at the end of this subsection. Above we are naturally identifying $S_n \in S'(\mathbb{Q}_p^{nd})$ with corresponding elements of $S'(\mathbb{Q}_p^{nd}, \mathbb{C})$. We also use the notational convention that $S(\mathbb{Q}_p^0, \mathbb{C}) = \mathbb{C}$, and $S'(\mathbb{Q}_p^0, \mathbb{C}) = \mathbb{C}$, with the corresponding duality pairing given by multiplication. Additionally for $\lambda \in S(\mathbb{Q}_p^0, \mathbb{C})$, $f \in S(\mathbb{Q}_p^{nd}, \mathbb{C})$ the notation $\lambda \otimes f$ just denotes λf.

As we did for Bochner-Minlos we will prove Theorem 1.11 by proving a similar theorem in the space ℓ'.

We will make frequent use of multi-indices in this section, in this setting some of these multi-indices will live in the set $\mathbb{N}^n$ for some positive integer n and others will live in the set $\mathcal{I} = \oplus_{i=0}^{\infty} \mathbb{N}$ - in the latter case these multindices will have infinitely many entries but only finitely many non-zero ones. For a multi-index α we write $|\alpha|$ to denote the sum of the entries, i.e. $\sum_i \alpha_i$. For $\alpha \in \mathcal{I}$ we write $\mathrm{supp}(\alpha)$ to denote the support of the multi-index, that is $\mathrm{supp}(\alpha) := \{i \in \mathbb{N} |\ \alpha_i \neq 0\}$. For any subset $F \subset \mathbb{N}$ we write $\mathcal{I}_F$ for the subset of $\mathcal{I}$ formed by multi-indices whose supports are contained in F.

The next theorem gives some conditions on a set of “candidate” moments $\{M_\alpha\}_{\alpha \in \mathcal{I}}$ that are sufficient for them to specify a unique probability measure μ on ℓ' that satisfies

$$M_\alpha = \int_{\ell'} d\mu(x) x^\alpha \tag{1.9}$$

where for $x \in \ell'$ we have $x^\alpha = \prod_{i=0}^{\infty} x_i^{\alpha_i}$.

Theorem 1.12. *Suppose that the family of real numbers M_α indexed by $\alpha \in \mathcal{I}$ satisfy the following properties*

(i) $M_0 = 1$

(ii) Positive-definiteness: For any finite subset $\mathcal{J} \subset \mathcal{I}$ and for any collection of complex numbers z_α indexed by $\alpha \in \mathcal{J}$ one has

$$\sum_{\alpha, \beta \in \mathcal{J}} z_\alpha \bar{z}_\beta M_{\alpha+\beta} \geqslant 0.$$

(iii) Exponential Summability: For any finite subset $F \subset \mathbb{N}$ there exists $C_F > 0$ such that for all $\alpha \in \mathcal{I}_F$ one has

$$|M_\alpha| \leqslant C_F^{|\alpha|} |\alpha|!.$$

Then there exists a measure μ on ℓ', equipped with its product σ-algebra, such that (1.9).

We first prove an analogous theorem that we can apply to specify measures on $\mathbb{R}^n$ from moments.

Theorem 1.13. *Let n be some fixed positive integer and suppose that the family of real numbers M_α indexed by $\alpha \in \mathbb{N}^n$ satisfy the following properties:*

(i) $M_0 = 1$

(ii) Positive-definiteness: For any collection of complex numbers z_α indexed by $\alpha \in \mathbb{N}^n$ one has

$$\sum_{\alpha,\beta} \bar{z}_\alpha z_\beta M_{\alpha+\beta} \geqslant 0.$$

(iii) Exponential Summability: There exists $C > 0$ such that

$$|M_\alpha| \leqslant C^{|\alpha|}|\alpha|!$$

Then there is a unique Borel probability measure ν on $\mathbb{R}^n$ such that

$$M_\alpha = \int_{\mathbb{R}^n} \mathrm{d}\nu(x)\ x^\alpha$$

Finding conditions on a sequence of real numbers in order to be guaranteed a measure which has these numbers as its moments is called the *Moment Problem* and has a long history as a problem of classical analysis. One can find sufficient and necessary conditions on a sequence of candidate moments to uniquely determine a measure on $\mathbb{R}^n$ that realizes those moments in the article [53]. We only concern ourselves with sufficiency here - in particular we want analytic conditions to impose on the sequence of candidate moments that can easily be checked via the RG machinery we develop. The bounds on moments we assume correspond to exponential integrability for the corresponding measures - this will greatly simplify our task.

The measure described in the assertion of Theorem 1.13 will be constructed as a spectral measure for a particular family of self-adjoint operators.

We now assume that for some n we are given a sequence of candidate moments $\{M_\alpha\}_{\alpha\in\mathbb{N}^n}$ that satisfy the assumptions of 1.13. We remark that the positive-definiteness condition on the moments gives us the "Cauchy-Schwartz" bound

$$M_\alpha^2 \leqslant M_{2\alpha}.$$

Let $\mathcal{P} = \mathbb{C}[x_1, ..., x_n]$. We are going to define a positive semidefinite sesquilinear form on $(\cdot,\cdot)_{\mathcal{P}}$ on $\mathcal{P}$ which can be thought of as a pre-inner product. Let $f, g \in \mathcal{P}$ be as follows:

$$f = \sum_\alpha g_\alpha x^\alpha,\ \ g = \sum_\beta g_\beta x^\beta$$

We then define

$$(f, g)_{\mathcal{P}} = \sum_{\alpha,\beta} \bar{f}_\alpha g_\beta M_{\alpha+\beta}$$

The fact that the form is positive semidefinite comes from condition (ii). Now for $1 \leqslant j \leqslant n$ we define linear operators A_j one $\mathcal{P}$ via

$$A_j f \ = \ x_j f$$

It is clear that all the A_j are symmetric with respect to our pre-inner product and commute on all of $\mathcal{P}$.

Let $\mathcal{Q} = \{h \in \mathcal{P}\ :\ (h,h)_{\mathcal{P}} = 0\}$. We note that $\mathcal{Q}$ is an ideal of $\mathcal{P}$. In particular our pre-inner product lifts to the complex vector space $\mathcal{P}/\mathcal{Q}$ where it becomes a positive definite inner product which we denote by $(\cdot,\cdot)$. The A_j lift to linear operators $\tilde{A}_j$ on $\mathcal{P}/\mathcal{Q}$ by virtue of the fact that $\mathcal{Q}$ is invariant under each A_j (in particular $\mathcal{Q}$ is an ideal in $\mathcal{P}$). To see this fact suppose $h \in \mathcal{Q}$. Then for any j one has

$$(A_j h, A_j h)_{\mathcal{P}} = (A_j^2 h, h)_{\mathcal{P}} \leqslant (A_j^2 h, A_j^2 h)_{\mathcal{P}}^{1/2} (h,h)_{\mathcal{P}}^{1/2} = 0.$$

We will abuse notation and continue to write A_j for $\tilde{A}_j$.

Now define $\mathcal{H}$ as the completion of $\mathcal{P}/\mathcal{Q}$ under the inner product $(\cdot,\cdot)$. The A_j are then densely defined symmetric operators on $\mathcal{H}$ with $\mathcal{D}(A_j) = \mathcal{P}/\mathcal{Q} \subset \mathcal{H}$. Our goal is to prove that the A_j have commuting self-adjoint closures $\hat{A}_j$. The measure we wish to construct will be the joint spectral measure of this family of operators.

A crucial tool in establishing the above claims is Nelson's Analytic Vector Theorem.

Definition. *Let B be a densely defined symmetric operator on a Hilbert space $\mathcal{H}$. A vector $v \in \bigcap_{n=1}^{\infty} \mathcal{D}(B^n)$ is said to be analytic for B if:*

$$\sum_{n=0}^{\infty} \frac{||B^m v||}{m!} t^m \text{ converges for some } t > 0 \tag{1.10}$$

Theorem 1.14 (Nelson's Analytic Vector Theorem)**.** *Let B be a densely defined symmetric operator. If the analytic vectors of B are dense in $\mathcal{H}$ then B is essentially self-adjoint.*

Proof: See [54, Theorem X.39]. □

We now prove that the operators A_j are essentially self-adjoint.

Lemma 1.5. *For all $1 \leqslant j \leqslant n$ we have that A_j maps $\mathcal{P}/\mathcal{Q}$ to itself. Additionally $\forall v \in \mathcal{P}/\mathcal{Q}$ one has*

$$\sum_{q=0}^{\infty} \frac{||A_j^q v||}{q!} t^q < \infty \text{ for } t \in \left[0, \frac{1}{3C}\right)$$

As a result each A_j is essentially self-adjoint.

Proof: The statement about the A_j leaving $\mathcal{P}/\mathcal{Q}$ is clear.

It suffices to prove the second assertion for monomials $v = x^\alpha$ in $\mathcal{P}/\mathcal{Q}$. Below we use the notation δ_i to denote the multi-index in $\mathbb{N}^n$ which zero for all the entries except the i-th one where it takes the value 1.

$$\begin{aligned}
\sum_{q=0}^{\infty} \frac{||A_j^q x^\alpha||}{q!} t^q &= \sum_{q=0}^{\infty} \frac{||x^{\alpha+q\delta j}||}{q!} t^q \\
&= \sum_{q=0}^{\infty} \frac{(M_{2\alpha+2q\delta j})^{\frac{1}{2}}}{q!} t^q \\
&\leqslant \sum_{q=0}^{\infty} \frac{\left[F^{2|\alpha|+2q}(2|\alpha|+2q)!\right]^{\frac{1}{2}}}{q!} t^q
\end{aligned}$$

$$\leqslant \quad 3^{|\alpha|} C^{|\alpha|} \sqrt{|2\alpha|!} \sum_{q=0}^{\infty} 3^q C^q t^q < \infty \text{ for } t \in \left[0, \frac{1}{3C}\right)$$

In going from the third line to the fourth we used the bound:

$$\frac{(2|\alpha| + 2q)!}{q!q!(2|\alpha|)!} \leqslant 3^{2|\alpha|+2q}$$

This proves the second assertion. The claim that the A_j are essentially self-adjoint follow from the first two assertions, Theorem 1.14, and the fact that $\mathcal{P}/\mathcal{Q}$ is dense in $\mathcal{H}$. □

We now show that the self-adjoint closures $\hat{A}_j$ all commute. This is equivalent to proving that the unitary groups generated by the $\hat{A}_j$ commute. We would like to work at the level of power series expansions of the unitary groups. To facilitate this we give the following lemma which uses part of the proof of Nelson's Analytic Vector Theorem.

Lemma 1.6. *For any $1 \leqslant j \leqslant n$ and $v \in \mathcal{P}/\mathcal{Q}$ one has, for any $s \in \left(-\frac{1}{3C}, \frac{1}{3C}\right)$,*

$$e^{is\hat{A}_j} v = \sum_{q=0}^{\infty} \frac{(is)^q}{q!} A_j^q v.$$

Proof: We first remark that the quantity on the right hand side of the asserted equality is absolutely convergent by Lemma 1.5.

Now let μ_v be the spectral measure for the vector v under $\hat{A}_j$. Then

$$(v, e^{is\hat{A}_j} v) \;=\; \int_{\mathbb{R}} \mathrm{d}\mu_v(x)\; e^{isx} \;=\; \int_{\mathbb{R}} \mathrm{d}\mu_v(x)\; \left[\sum_{q=0}^{\infty} \frac{x^q}{q!} (is)^q\right]$$

Assuming that we were allowed to switch the integral and sum we would have

$$(v, e^{is\hat{A}_j} v) \;=\; \lim_{N\to\infty} \sum_{q=0}^{N} \left[\int_{\mathbb{R}} \mathrm{d}\mu_v(x)\; \frac{x^q}{q!} (is)^q\right] \;=\; \lim_{N\to\infty} \left(v, \sum_{q=0}^{N} \frac{(is)^q}{q!} \hat{A}_j^q v\right) \;=\; \left(v, \sum_{q=0}^{\infty} \frac{(is)^q}{q!} \hat{A}_j^q v\right) \qquad (1.11)$$

Now by polarization the above equality implies that for all $u, v \in \mathcal{P}/\mathcal{Q}$ one has

$$(u, e^{is\hat{A}_j} v) \;=\; \left(u, \sum_{q=0}^{\infty} \frac{(is)^q}{q!} \hat{A}_j^q v\right). \qquad (1.12)$$

Since $\mathcal{P}/\mathcal{Q}$ is dense the assertion would follow. We remark that it was important that we had a uniform "radius of analyticity" for a dense set of analytic vectors - if the domain of s for which (1.11) was valid was v-dependent then we would not have (1.12) for all $u \in \mathcal{P}/\mathcal{Q}$.

The switching of the order of integration and summation is justified by Fubini along with the bound

$$\sum_{q=0}^{\infty}\left[\int_{\mathbb{R}} \mathrm{d}\mu_v(x)\ \frac{|x|^q}{q!}|s|^q\right] \leqslant \sum_{q=0}^{\infty}\frac{|s|^q}{q!}\left(\int_{\mathbb{R}} \mathrm{d}\mu_v(x)\ |x|^{2q}\right)^{\frac{1}{2}}\frac{|s|^q}{q!}\left(\int_{\mathbb{R}} \mathrm{d}\mu_v(x)\ 1\right)^{\frac{1}{2}}$$
$$= \sum_{q=0}^{\infty}\frac{|s|^q}{q!}\left(v,\hat{A}_j^{2q}v\right)^{1/2}(v,v)^{1/2} \ = \ ||v||\sum_{q=0}^{\infty}\frac{|s|^q}{q!}||\hat{A}_j^q v|| < \infty$$

where in the last line we used Lemma 1.5.

□

More generally the proof of the above lemma shows that if one has a dense set of analytic vectors for a symmetric operator B with a uniform radius of analyticity then for any analytic vector h the quantity $e^{it\hat{B}}h$ can be represented by a power series within h's radius of analyticity. The next lemma proves that our unitary groups commute.

Lemma 1.7. *$\forall v \in \mathcal{P}/\mathcal{Q}$, $1 \leqslant j,k \leqslant n$, and $s \in \left(-\frac{1}{5C},\frac{1}{5C}\right)$ one has that*

$$e^{i\hat{A}_j s}v \in \bigcap_{m=1}^{\infty} D(\hat{A}_k^m).$$

If both $s,t \in \left(-\frac{1}{5C},\frac{1}{5C}\right)$ and $v \in \mathcal{P}/\mathcal{Q}$ then one has that

$$e^{i\hat{A}_k t}e^{i\hat{A}_j s}v \ = \ \sum_{m=0}^{\infty}\frac{(it)^m}{m!}\hat{A}_k^m e^{i\hat{A}_j s}v \ = \ \sum_{m=0}^{\infty}\sum_{q=0}^{\infty}\frac{(it)^m}{m!}\frac{(is)^q}{q!}A_k^m A_j^q v. \tag{1.13}$$

It follows that for such t,s and $v \in \mathcal{P}/\mathcal{Q}$ one has

$$e^{i\hat{A}_k t}e^{i\hat{A}_j s}v = e^{i\hat{A}_j s}e^{i\hat{A}_k t}v. \tag{1.14}$$

Finally one has that for any t,s one has that the operators $e^{i\hat{A}_k t}$ and $e^{i\hat{A}_j s}$ commute on all of $\mathcal{H}$.

Proof: The assertion that $e^{i\hat{A}_j s}v \in \bigcap_{m=1}^{\infty} D(\hat{A}_k^m)$ for $v \in \mathcal{P}/\mathcal{Q}$ follows from the fact that $\hat{A}_k$ is a closed operator and Lemma 1.6. In particular we note that for all $m \in \mathbb{N}$

$$\lim_{N\to\infty}\hat{A}_k^m\sum_{q=0}^{N}\frac{(is)^q}{q!}A_j^q v \ = \ \lim_{N\to\infty}\sum_{q=0}^{N}\frac{(is)^q}{q!}A_j^q A_k^m v$$

where the final limit exists by Lemma 1.6 since $A_k^m v \in \mathcal{P}/\mathcal{Q}$.

We now turn to proving (1.13) - without loss of generality it suffices to prove (1.13) for $v = x^\alpha$. From the proof of Lemma 1.6 and noting our remarks after that lemma we remark that in order to prove the first equality of (1.13) it suffices to show that $e^{i\hat{A}_j s}v$ is an analytic vector for $\hat{A}_k$ with radius $\frac{1}{5C}$. To see this we note that

$$\sum_{m=0}^{\infty}\frac{t^m}{m!}||\hat{A}_k^m e^{i\hat{A}_j s}x^\alpha|| \ = \ \sum_{m=0}^{\infty}\frac{t^m}{m!}\left\|\sum_{q=0}^{\infty}\frac{(is)^q}{q!}A_k^m A_j^q x^\alpha\right\|$$

$$\begin{aligned}
&\leqslant \sum_{m=0}^{\infty}\sum_{q=0}^{\infty}\frac{t^m|s|^q}{m!q!}\sqrt{M_{2(\alpha+m\delta_k+q\delta_j)}}\\
&\leqslant \sum_{m=0}^{\infty}\sum_{q=0}^{\infty}\frac{t^m|s|^q}{m!q!}\sqrt{C^{2(|\alpha|+q+m)}(2|\alpha|+2q+2m)!}\\
&\leqslant (5C)^{|\alpha|}\sqrt{(2|\alpha|)!}\sum_{m=0}^{\infty}\sum_{q=0}^{\infty}t^m|s|^q(5C)^{q+m}<\infty
\end{aligned}$$

In the second to last inequality we used the fact that

$$\frac{(2|\alpha|+2q+2m)!}{q!q!m!m!(2|\alpha|)!}\leqslant 5^{2|\alpha|+2q+2m}.$$

The last equality of (1.13) also follows. Assertion (1.14) follows for our regime of t, s since the operators A_j and A_k commute and our uniform bounds allow us to change the order of summation for the rightmost quantity in (1.13). Since the operators involved are bounded (1.14) must hold for all $v \in \mathcal{H}$ - and at that point one can use the group operation of these one parameter unitary groups to extend the commutation relation to hold for all $t, s \in \mathbb{R}$.

□

We now prove Theorems 1.13 and 1.12

Proof of Theorem 1.13: Let $R = (a_1, b_1) \times ... \times (a_n, b_n)$ be a rectangle on $\mathbb{R}^n$. Define the projection valued spectral measure of R as $\prod_{i=1}^n P_{i,(a_i,b_i)}$ where $P_{i,\lambda}$ is the projection valued measure corresponding to $\hat{A}_i$. Then $(1, \prod_{i=1}^n P_{i,(a_i,b_i)}1)$ is a premeasure on rectangles which extends to a Borel probability measure on $\mathbb{R}^n$, call this measure μ. From results proven in the appendix about the joint spectral measure we have:

$$\int_{\mathbb{R}^n} e^{it\cdot y}d\mu(y) \;=\; \left(1, \prod_{j=1}^{n} U_j(t_j)1\right)$$

where U_j is the unitary group generated by $\hat{A}_j$. Note that for sufficiently small $t_1, ..., t_n$ the left hand side is analytic in these arguments - the proof in Lemma 1.7 generalizes and one can expand the product of the U_j's if $\sup_j |t_j| < \frac{1}{(2n+1)C}$. This allows us to take partial derivatives evaluated at 0. In this way one can recover all the moments:

$$\int_{\mathbb{R}^n} y^\alpha d\mu(y) = \left(1, \prod_{i=1}^{n}\hat{A}_i^{\alpha_i}1\right) = M_\alpha.$$

We have established existence for our solution to the Moment Problem. Uniqueness comes from the fact that our moment estimates allow us to determine the characteristic function from our moments - see Theorems 5.2 and 5.3 in the appendix. □

Proof of Theorem 1.12: The assumptions of Theorem 1.12 can be used to construct for each finite subset $F \subset \mathbb{N}$ a Borel probability measure μ_F on $\mathbb{R}^F$ with moments given by M_α with $\alpha \in \mathcal{I}_F$, additionally μ_F

is the unique such measure on $\mathbb{R}^F$ with those given moments. This uniqueness allows us to prove that the family of finite dimensional distributions $\{\mu_F\}$, $F \subset \mathbb{N}$ finite, are consistent. To see this suppose that that F, G are finite subsets of $\mathbb{N}$ with $F \subset G$. Then the $\mathbb{R}^F$ marginal of the measure μ_G has the same moments as μ_F and thus the two must coincide.

This means that by Kolomogorov's Extension Theorem there exists a unique measure μ on $\mathbb{R}^{\mathbb{N}}$ (where $\mathbb{R}^{\mathbb{N}}$ is equipped with its product σ-algebra). $\square$

We can now prove Theorem 1.11.

Proof of Theorem 1.11:

Let the $\{e_n\}_{n=0}^{\infty}$ be as given in Proposition 1.2. We now define a family of candidate moments which when used as an input to Theorem 1.12 will specify a measure on ℓ. We set $M_0 := S_0 = 1$. For $\alpha \in \mathcal{I}$ with $\alpha \neq 0$ we set

$$M_\alpha \ := \ S_{|\alpha|} \left(\otimes_{j \in \mathrm{supp}(\alpha)} (\otimes_{k=1}^{\alpha_j} e_j) \right).$$

We note that the argument of $S_{|\alpha|}$ above is an element of $S(\mathbb{Q}_p^{d|\alpha|})$. In particular for non-zero $\alpha \in \mathcal{I}$ it will be useful to define

$$g_\alpha = \otimes_{j \in \mathrm{supp}(\alpha)} (\otimes_{k=1}^{\alpha_j} e_j) \in S(\mathbb{Q}_p^{d|\alpha|})$$

while for $\alpha = 0$ we define $g_0 = 1$. With these definitions we have $M_\alpha = S_{|\alpha|}(g_\alpha)$.

We now check the positive definiteness condition of Theorem 1.12. Let $\mathcal{J}$ be a finite subset of $\mathcal{I}$ and z_α be some collection of complex numbers indexed by $\alpha \in \mathcal{J}$. For $n \in \mathbb{N}$ we set $\mathcal{J}_n := \{\alpha \in \mathcal{J} |\ |\alpha| = n\}$. We then define

$$h_n \ = \ \sum_{\alpha \in \mathcal{J}_n} z_\alpha g_\alpha$$

We remark that $(h_n)_{n \geqslant 0}$ is an almost finite sequence of test functions with $h_n \in S(\mathbb{Q}_p^{nd}, \mathbb{C})$ so by assumption (3) stated in the theorem we have

$$\sum_{n,m \geqslant 0} S_{n+m}(\overline{h_n} \otimes h_m) \ \in \ [0, \infty).$$

Rewriting the summands above gives us

$$S_{n+m}(\overline{h_n} \otimes h_m) = \ S_{n+m} \left(\left[\sum_{\alpha \in \mathcal{J}_n} \overline{z_\alpha} g_\alpha \right] \otimes \left[\sum_{\beta \in \mathcal{J}_n} z_\beta g_\beta \right] \right) \ = \ \sum_{\substack{\alpha \in \mathcal{J}_n \\ \beta \in \mathcal{J}_m}} \overline{z_\alpha} z_\beta S_{n+m}(g_\alpha \otimes g_\beta).$$

Now if one writes out the definitions of g_α, g_β, and $f_{\alpha+\beta}$ it is clear that the symmetry of the S_{n+m} with respect to permutations of the $n+m$ underlying variables implies that one must have $S_{n+m}(g_\alpha \otimes g_\beta) = S_{n+m}(g_{\alpha+\beta})$ for all $\alpha \in \mathcal{J}_n$, $\beta \in \mathcal{J}_m$. It follows that

$$\sum_{n,m \geqslant 0} S_{n+m}(\overline{h_n} \otimes h_m) \ = \ \sum_{n,m \geqslant 0} \sum_{\substack{\alpha \in \mathcal{J}_n \\ \beta \in \mathcal{J}_m}} \overline{z_\alpha} z_\beta S_{n+m}(g_{\alpha+\beta}) \ = \ \sum_{n,m \geqslant 0} \sum_{\substack{\alpha \in \mathcal{J}_n \\ \beta \in \mathcal{J}_m}} \overline{z_\alpha} z_\beta M_{\alpha+\beta} \ = \ \sum_{\alpha, \beta \in \mathcal{J}} \overline{z_\alpha} z_\beta M_{\alpha+\beta},$$

so positive definiteness is proved.

The final ingredient we need for applying Theorem 1.12 are the factorial bounds on the moments $\{M_\alpha\}$.

Let $F \subset \mathbb{N}$ be a finite subset. We define V to be the subspace of $S(\mathbb{Q}_p^d, \mathbb{C})$ formed by the span of $\{e_j\}_{j \in F}$. Let $\mathcal{N}_V$ be the seminorm on V that is given by assumption (4) of the theorem. We set

$$C_F = \sup_{j \in F} \mathcal{N}_V(e_j)$$

Now choose some $\alpha \in \mathcal{I}_F$. If we set $n = |\alpha|$ then g_α is of the form $g_\alpha = f_1 \otimes \cdots \otimes f_n$ for $f_1, \ldots, f_n \in \{e_j\}_{j \in F} \subset V$. It follows that

$$\begin{aligned} |M_\alpha| \ = \ |S_n(g_\alpha)| \ &\leqslant n! \times \prod_{j=1}^{n} \mathcal{N}_V(f_j) \\ &\leqslant n! \times C_F^n \ = \ |\alpha|! \times C_F^{|\alpha|}. \end{aligned}$$

Now by Theorem 1.12 the family of candidate moments $\{M_\alpha\}$ determine a measure μ on ℓ' with

$$\int_{\ell'} \mathrm{d}\mu(y)\ y^\alpha \ = \ M_\alpha.$$

We define the measure ν on $(S'(\mathbb{Q}_p^d), \mathcal{C})$ to be the pushforward of the measure μ under the isomorphism of topological vector spaces $(T^*)^{-1} : \ell \to S'(\mathbb{Q}_p^d)$ (see the end of 1.3.1). This map's action is given by

$$y \ = \ \{y_j\}_{j=0}^{\infty} \ \longrightarrow \ \sum_{j=0}^{\infty} y_j e_j \in S'(\mathbb{Q}_p^d).$$

We also make the trivial remark that μ coincides with the pushforward of ν under $T^* : S(\mathbb{Q}_p^d) \to \ell'$.

We must now check that the measure ν is the unique measure on $S'(\mathbb{Q}_p^d)$ that satisfies

$$S_n(f_1 \otimes \cdots \otimes f_n) = \int_{S'(\mathbb{Q}_p^d)} \mathrm{d}\nu(\phi)\ \phi(f_1) \cdots \phi(f_n) \tag{1.15}$$

all $f_1, \ldots, f_n \in S(\mathbb{Q}_p^d, \mathbb{C})$ and for all n.

We fix n. By the multilinearity of both sides of (1.15) in the f_j's it suffices to show that (1.15) holds for the case where $f_1, \ldots, f_n \in \{e_j\}_{j \in \mathbb{N}}$. By symmetry of both sides of (1.15) we can assume that for $1 \leqslant j \leqslant n$

$$f_j = e_{k_j}$$

with $k_j \in \mathbb{N}$ non-decreasing in j. It follows there exists a unique $\alpha \in \mathcal{I}$ with $|\alpha| = n$ and

$$g_\alpha \ = \ f_1 \otimes \cdots \otimes f_n.$$

Thus with this choice of f_j's and α fixed as above one has that the left hand side of (1.15) is given by M_α. On the other hand if $T^*\phi = y \in \ell'$ then one has that

$$\phi(f_1) \cdots \phi(f_n) = y^\alpha.$$

Then by a change of variable we have

$$\int_{S'(\mathbb{Q}_p^d)} \mathrm{d}\nu(\phi)\ \phi(f_1)\cdots\phi(f_n)\ =\ \int_{\ell'} \mathrm{d}\mu(y)\ y^\alpha\ = M_\alpha$$

and so (1.15) is proved for our specific choice of the f_j's and the general case follows. The fact that the measure ν is the unique measure that satisfies (1.15) for all n is a direct consequence of the uniqueness result of Theorem 1.12. □

Chapter 2

Classification of translation, rotation, and scale invariant Gaussian generalized random fields

2.1 Gaussian Generalized Random Fields

It is common to learn during a first class on measure theory that there is no analog of Lebesgue measure in infinite dimensions - there are non non-zero, locally finite, translation invariant measures on infinite dimensional topological vector spaces. However thanks to the Bochner-Minlos Theorem, one can easily construct a wide variety of Gaussian measures on $S'(\mathbb{Q}_p^d)$ and $S'(\mathbb{R}^d)$. The main goal of this section is to provide a classification of Gaussian measures on these spaces which satisfy certain invariance properties. In what follows when we use the term Gaussian others might instead use the term *centered* Gaussian - that is we focus entirely on Gaussians with mean 0.

For some of this section we will work over $\mathbb{Q}_p^d$ and $\mathbb{R}^d$ simultaneously. In particular we use the notation $S(\mathbb{K}^d)$ and $S(\mathbb{K}^d, \mathbb{C})$ with the understanding that one might have $\mathbb{K} = \mathbb{Q}_p$ or $\mathbb{K} = \mathbb{R}$.

Definition. *A measure ν on $S'(\mathbb{K}^d)$ is a Gaussian Generalized Random Field over $\mathbb{K}^d$ if for ϕ distributed according to ν one has that for every $f \in S(\mathbb{K}^d)$ the distribution of the random variable $\phi(f)$ is given by a Gaussian distribution over $\mathbb{R}$.*

Gaussian measures on S' can be characterized through their covariance bilinear forms.

Definition. *We say a (jointly) continuous symmetric bilinear form $C(\cdot, \cdot)$ on $S(\mathbb{K}^d)$ is positive definite if $C(f, f) \geqslant 0$ for all $f \in S(\mathbb{K}^d)$.*

One then has the following theorem

Theorem 2.1. *There is a one - to - one correspondance between Gaussian measures μ on $S'(\mathbb{K}^d)$ and continuous symmetric bilinear forms C on $S(\mathbb{K}^d)$*

$$\mu \leftrightarrow C_\mu$$

where the correspondance is given by

$$\int_{S'(\mathbb{K}^d)} \mathrm{d}\mu(\phi) e^{i\phi(f)} = \exp\left[-\frac{1}{2}C(f,f)\right]$$

where f is an arbitrary test function in $S(\mathbb{K}^d)$.

Proof: The fact that for a given Gaussian μ, there exists a unique continuous symmetric bilinear form C_μ such that $\theta_\mu(\cdot) = \exp\left[-\frac{1}{2}C_\mu(\cdot,\cdot)\right]$ is rather straightforward and follows from the definition of Gaussian measures on $S'(\mathbb{K}^d)$ given above and the fact that any Gaussian measure on $\mathbb{R}$ is exponentially integrable.

To show that given a continuous symmetric bilinear form C on $S'(\mathbb{K}^d)$ there exists a Gaussian measure μ on $S'(\mathbb{K}^d)$ with $C_\mu = C$ we can use the corresponding Bochner-Minlos Theorems (Theorems 1.5 and 1.9). One must show that the function $\theta : S(\mathbb{K}^d) \to \mathbb{C}$ given by

$$\theta(f) \;=\; \exp\left[-\frac{1}{2}C(f,f)\right]$$

satisfies the necessary and sufficient conditions given in the Bochner-Minlos Theorem to be the characteristic function of a measure μ on $S(\mathbb{K}^d)$. $\theta(0) = 1$ and continuity of θ are immediate. For positive definiteness one needs to show that for any n and any $f_1, \ldots, f_n \in S(\mathbb{K}^d)$ the $n \times n$ matrix

$$M_{i,j} = \theta(f_i - f_j) \text{ for } 1 \leqslant i,j \leqslant n$$

is self-adjoint and positive (semi)definite on $\mathbb{C}^n$. We now note that that the $n \times n$ matrix

$$\mathring{C}_{i,j} = C(f_i, f_j) \text{ for } 1 \leqslant i,j \leqslant n$$

is a symmetric matrix with real entries and by our assumption on $C(\cdot,\cdot)$ is positive (semi)definite on $\mathbb{R}^n$ - it follows that there is an finite dimensional Gaussian measure μ_n on $\mathbb{R}^n$ with $\mathring{C}$ as its covariance matrix, i.e.

$$\int_{\mathbb{R}^n} \mathrm{d}\mu_n(x) x_i x_j = C_{i,j}$$

and the characteristic function for μ_n, denoted by $\theta_{\mu_n} : \mathbb{R}^n \to \mathbb{C}$ can easily be checked to satisfy $\theta_{\mu_n}(x_i - x_j) = M_{i,j}$ and from the positive definiteness of θ_{μ_n} we see the matrix M must be self-adjoint and positive (semi)definite. □

2.2 Classification Results

We will start by defining precisely the transformations that correspond to the invariances of interest. The most natural way to do this is to first give analogous transformations for observables, i.e. test functions in $S(\mathbb{K}^d, \mathbb{C})$

For functions $f : \mathbb{Q}_p^d \to \mathbb{C}$ (resp. $f : \mathbb{R}^d \to \mathbb{C}$) we define the following transformations:

(i) For $y \in \mathbb{Q}_p^d$ (resp. $y \in \mathbb{R}^d$) we define the translation operators τ_y via $\tau_y(f)(x) = f(x - y)$.

(ii) For $M \in GL_d(\mathbb{Z}_p)$ (resp. $M \in O(d)$) we define rotation operators R_M via $\mathbb{Q}_p^d$ via $R_M(f)(x) = f(M^{-1}x)$.

(iii) For $\lambda \in p^{\mathbb{Z}}$ (resp. $\lambda \in (0, \infty)$) we define scaling operators S_λ via $S_\lambda(f)(x) = f(\lambda^{-1}x)$

Note that in the second definition when writing $M^{-1}x$ we are using the standard action of $d \times d$ matrices on $\mathbb{K}^d$ via matrix multiplication where elements of $\mathbb{K}^d$ are seen as column vectors.

We remark that the translation, rotation, and scaling operators of (2.2) take $S(\mathbb{K}^d, \mathbb{C})$ to itself and are in fact continuous linear maps on $S(\mathbb{K}^d, \mathbb{C})$ that send real functions to real functions.

With this in mind we can define the following transformations acting on $\phi \in S'$ that agree with the transformations of (2.2) for distributions given by functions. In what follows f is an arbitrary element of $S(\mathbb{K}^d, \mathbb{C})$.

(i) For $y \in \mathbb{Q}_p^d$ (resp. $y \in \mathbb{R}^d$) we define the translation operators $\hat{\tau}$ via $\hat{\tau}_y(\phi)(f) = \phi(\tau_{-y}f)$

(ii) For $M \in GL_d(\mathbb{Z}_p)$ (resp. $M \in O(d)$) we define rotation operators $\hat{R}_M$ via $\hat{R}_M(\phi)(f) = \phi(R_{M^{-1}}(f))$

(iii) For $\lambda \in p^{\mathbb{Z}}$ (resp. $\lambda \in (0, \infty)$) we define scaling operators $\hat{S}_\lambda$ via $\hat{S}_\lambda(\phi)(f) = |\lambda|_p^d \times \phi(S_{\lambda^{-1}}(f))$ (resp. $\hat{S}_\lambda(\phi)(f) = \lambda^d \times \phi(S_{\lambda^{-1}}f)$).

The transformations given in (2.2) are measurable linear maps from $S'(\mathbb{K}^d)$ to itself. Given a measurable map $L : S'(\mathbb{K}^d) \to S'(\mathbb{K}^d)$ we denote its push-forward action on measures by $L^{\#}$.We can then define the following notions of invariance for measures on $S'(\mathbb{K}^d)$.

Definition. *Let ν be a cylinder set σ-algebra measure on $S'(\mathbb{Q}_p^d)$ or $S'(\mathbb{R}^d)$. We say that ν is* translation invariant *if $\hat{\tau}_y^{\#}\nu = \nu$ for all $y \in \mathbb{Q}_p^d$ (resp. $y \in \mathbb{R}^d$). We say that ν is* rotation invariant *if $\hat{R}_M^{\#}\nu = \nu$ for all $M \in GL_d(\mathbb{Z}_p)$ (resp. $M \in O(d)$). For $\kappa \in \mathbb{R}$ we say ν is κ scale invariant if $\left(|\lambda|_p^{-\kappa}\hat{S}_\lambda\right)^{\#} \nu = \nu$ for all $\lambda \in p^{\mathbb{Z}}$ (resp. $\left(\lambda^{-\kappa}\hat{S}_\lambda\right)^{\#} \nu = \nu$ for all $\lambda \in (0, \infty)$).*

Informally if ν is κ scale invariant this means that for ϕ distributed according to ν one has $\lambda^{-\kappa}\phi\left(\frac{\cdot}{\lambda}\right) \overset{d}{=} \phi(\cdot)$ (where the equality distribution holds in joint law).

Classifying Gaussians with the given invariances reduces to classifying continuous symmetric bilinear forms with analogous invariances.

Proposition 2.1. *Let C be a continuous symmetric bilinear form on $S(\mathbb{K}^d)$. Suppose that*

(a) C is invariant under simultaneous translations in both arguments, that is for any $f, g, \in S(\mathbb{K}^d)$ one has

$$C(\tau_z f, \tau_z g) \ = \ C(f, g)$$

for any $z \in \mathbb{K}^d$.

(b) C is invariant under rotation in both arguments, that is for any $f, g, \in S(\mathbb{K}^d)$ one has

$$C(R_M f, R_M g) \ = \ C(f, g)$$

for any $M \in O(d)$ in the case $\mathbb{K} = \mathbb{R}$ or for any $M \in GL_d(\mathbb{Z}_p)$ in the case $\mathbb{K} = \mathbb{Q}_p$.

(c) For some $\kappa \in \mathbb{R}$ the bilinear form C satisfies

$$\text{For the case } \mathbb{K} = \mathbb{Q}_p \quad |\lambda|_p^{2d} \times C\left(S_{\lambda^{-1}} f, S_{\lambda^{-1}} g\right) = |\lambda|_p^{-2\kappa} \times C(f, g) \text{ for all } \lambda \in p^{\mathbb{Z}}$$

$$\text{For the case } \mathbb{K} = \mathbb{R} \quad \lambda^{2d} \times C\left(S_{\lambda^{-1}} f, S_{\lambda^{-1}} g\right) = |\lambda|^{-2\kappa} \times C(f, g) \text{ for all } \lambda \in (0, \infty).$$

It then follows that any Gaussian measure μ on $S(\mathbb{K}^d)$ with $C_\mu = C$ is translation invariant, rotation invariant, and κ scale invariant.

Conversely if μ is translation invariant, rotation invariant, and κ scale invariant then C_μ must satisfy the conditions given above.

Proof: This follows by seeing what the consequences for the invariances are at the level of characteristic functions. Note that if the three conditions are proven for the case where $f = g$ then they extend to the cases where $f \neq g$ by polarization. □

Note that if C is a bilinear form that satisfies condition (c) above for some κ we will call C κ scale-invariant.

We now give our classification theorem.

Theorem 2.2. *Suppose that μ is a Gaussian measure on $S'(\mathbb{Q}_p^d)$ not concentrated on the 0 distribution. Then μ is translation invariant, $GL_d(\mathbb{Z}_p)$ invariant, and κ-scale invariant if and only if $\kappa \geqslant 0$ and there exists $a > 0$ such that for all $f, g \in S(\mathbb{Q}_p^d)$ one has*

$$\int_{S'(\mathbb{Q}_p^d)} \mathrm{d}\mu(\phi)\ \phi(f)\phi(g) = a \times \hat{f}(0) \times \hat{g}(0) \text{ if } \kappa = 0 \text{ and}$$

$$\int_{S'(\mathbb{Q}_p^d)} \mathrm{d}\mu(\phi)\ \phi(f)\phi(g) = a \times \int_{\mathbb{Q}_p^d} d^3k\ \frac{\hat{f}(k)\hat{g}(k)}{|k|^{d-2\kappa}} \text{ for } \kappa > 0$$

The theorem is a direct consequence of and Propositions 2.1 and 2.4. In the above theorem the $\kappa = 0$ case corresponds to ϕ being given by the constant distribution aX where X is a standard Gaussian on $\mathbb{R}$ of mean 0 and variance 1. When $d > 2$ the case $\kappa = \frac{d-2}{2}$ corresponds to the massless Gaussian Free Field (up to a constant) - we give more details.

There is no natural analog of differential operators that act on and then return functions $f : \mathbb{Q}_p^d \to \mathbb{C}$. However we can use the corresponding Fourier multiplier as a stand-in. With this convention the p-adic Laplacian can be identified with the Fourier multiplier $-|k|^2$. It then follows that for the case where $d > 2$, $\kappa = \frac{d-2}{2}$, $a = 1$, one has

$$\int_{S'(\mathbb{Q}_p^d)} d\mu(\phi)\ \phi(f)\phi(g) = \int_{\mathbb{Q}_p^d} d^3k\ \frac{\hat{f}(k)\hat{g}(k)}{|k|^2} = \left\langle f, (-\Delta)^{-1} g \right\rangle$$

where on the far right hand side $\langle \cdot, \cdot \rangle$ is the standard $\mathcal{L}^2(\mathbb{Q}_p^d)$ inner product. Generically the $\kappa > 0$ case corresponds to a fractional Gaussian Free Field, i.e. a Gaussian measure with covariance operator given by the inverse of a fractional Laplacian

$$(-\Delta)^{-\left(\frac{d-2\kappa}{2}\right)}.$$

Similarly one has the following theorem in the real case.

Theorem 2.3. *Suppose that μ is a Gaussian measure on $S'(\mathbb{R}^d)$ not concentrated on the zero distribution. Then μ is translation invariant, $O(d)$ invariant, and κ-scale invariant if and only if $\kappa \geqslant 0$ and there exists $a > 0$ such that for all $f, g \in S(\mathbb{Q}_p^d)$ one has*

$$\int_{S'(\mathbb{R}^d)} \mathrm{d}\mu(\phi)\ \phi(f)\phi(g) = a \times \hat{f}(0) \times \hat{g}(0) \text{ if } \kappa = 0 \text{ and}$$

$$\int_{S'(\mathbb{R}^d)} \mathrm{d}\mu(\phi)\ \phi(f)\phi(g) = a \times \int_{\mathbb{R}^d} d^3k\ \frac{\hat{f}(k)\overline{\hat{g}(k)}}{|k|^{d-2\kappa}} \text{ for } \kappa > 0$$

Proof: The theorem follows from Propositions 2.1 and 2.9.

2.2.1 Lemmas for the classification - p-adic case

Lemma 2.1. *For $z \in \mathbb{Q}_p^m$ define T_z acting on $f \in S(\mathbb{Q}_p^m \times \mathbb{Q}_p^n, \mathbb{C})$ via $T_z(f)(x, y) = f(x + z, y)$ (here $x \in \mathbb{Q}_p^m$ and $y \in \mathbb{Q}_p^n$).*

Suppose that $\phi \in S'(\mathbb{Q}_p^m \times \mathbb{Q}_p^n, \mathbb{C})$ satisfies the property $\phi(T_z f) = \phi(f)$ for all $z \in \mathbb{Q}_p^m$ and $f \in S(\mathbb{Q}_p^m \times \mathbb{Q}_p^n, \mathbb{C})$. Then there exists $\psi \in S'(\mathbb{Q}_p^n, \mathbb{C})$ such that for all such f one has:

$$\phi(f) = \left(\psi(y), \int_{\mathbb{Q}_p^m} d^m x\ f(x, y) \right).$$

Proof: Note that by a density argument it suffices to prove the assertion for $f(x, y) = g(x)h(y)$ where $g \in S(\mathbb{Q}_p^m, \mathbb{C})$, $h \in S(\mathbb{Q}_p^n, \mathbb{C})$. For arbitrary $h \in S(\mathbb{Q}_p^n, \mathbb{C})$ we define

$$\psi(h) = \left(\phi(x, y), \mathbb{1}_{\mathbb{Z}_p^m}(x)h(y)\right).$$

By our assumption of partial translation invariance it follows that for any $j \in \mathbb{Z}$ one has

$$\left(\phi(x, y), \mathbb{1}_{p^j\mathbb{Z}_p^m}(x)h(y)\right) = p^{-mj}\left(\phi(x, y), \mathbb{1}_{\mathbb{Z}_p^m}(x)h(y)\right) = p^{-mj}\psi(h)$$

Of course the above equation still holds if we replace $p^j\mathbb{Z}_p^m$ by any of its translates. Now we fix $j \in \mathbb{Z}$ sufficiently large such that we can write $g(x) = \sum_{i=1}^N \alpha_i \mathbb{1}_{\Delta_i}(x)$ where $\alpha_i \in \mathbb{C}$ and the $\{\Delta_i\}_{i=1}^N$ are *distinct, disjoint* translates of $p^j\mathbb{Z}_p^m$. One then has

$$\begin{aligned}(\phi(x, y), g(x)h(y)) &= \sum_{i=1}^N \alpha_i \left(\phi(x, y), \mathbb{1}_{\Delta_i}(x)h(y)\right) \\ &= \sum_{i=1}^N \alpha_i p^{-mj}\psi(h) \\ &= \left(\int_{\mathbb{Q}_p^m} d^m x\ g(x)\right)\psi(h)\end{aligned}$$

□

Proposition 2.2. *Let C be a bilinear form on $S(\mathbb{Q}_p^d, \mathbb{C})$ which satisfies $C(\tau_z f, \tau_z g) = C(f,g)$ for all $z \in \mathbb{Q}_p^d$ and $f, g \in S(\mathbb{Q}_p^d, \mathbb{C})$. Then there exists $\tilde{C} \in S'(\mathbb{Q}_p^d, \mathbb{C})$ such that for all such f, g one has*

$$C(f,g) = \tilde{C}(f \,\hat{\star}\, g)$$

where $(f \,\hat{\star}\, g)(x) = \int_{\mathbb{Q}_p^d} d^d z \; f(z) g(z-y)$

Proof: By the Theorem 1.3 there exists $F \in S'(\mathbb{Q}_p^d \times \mathbb{Q}_p^d)$ such that for all $f, g \in S(\mathbb{Q}_p^d)$ one has

$$C(f,g) = (F(x,y), f(x)g(y)) \, .$$

By invoking the translation invariance of C it immediately follows that righthand side is also equal to $(F(x,y), f(x+z)g(y+z))$ for any $z \in \mathbb{Q}_p^d$. Since products $f(x)g(y)$ span $S(\mathbb{Q}_p^d \times \mathbb{Q}_p^d)$ it then follows that for any $h \in S'(\mathbb{Q}_p^d \times \mathbb{Q}_p^d)$ one has $(F(x,y), h(x+z, y+z)) = (F(x,y), h(x,y))$. We now use a change of variable so that we are in the setting of Lemma 2.1. Define

$$\left(\tilde{F}(x,y), h(x,y)\right) := (F(x,y), h(x+y, x-y)) \, .$$

It is then easy to see that $\left(\tilde{F}(x,y), h(x,y)\right) = \left(\tilde{F}(x,y), h(x+z,y)\right)$ for all $z \in \mathbb{Q}_p^d$ and test functions h and so by Lemma 2.1 there exists $G \in S'(\mathbb{Q}_p^d)$ such that

$$\left(G(y), \int_{\mathbb{Q}_p^d} d^d x \; h(x,y)\right) = \left(\tilde{F}(x,y), h(x,y)\right) .$$

Reversing the change of variable one has

$$\begin{aligned} (F(x,y), h(x,y)) &= \left(\tilde{F}(x,y), h\left(\frac{x+y}{2}, \frac{x-y}{2}\right)\right) \\ &= \left(G(y), \int_{\mathbb{Q}_p^d} d^d x \; h\left(\frac{x+y}{2}, \frac{x-y}{2}\right)\right) \\ &= \left(G(y), |2|_p^d \int_{\mathbb{Q}_p^d} d^d z \; h\left(z, z-y\right)\right) . \end{aligned}$$

Setting $\tilde{C} = |2|_p^d G$ and $h(x,y) = f(x)g(y)$ one then has

$$C(f,g) = \left(\tilde{C}(y), \int_{\mathbb{Q}_p^d} d^d z \; f(z) g(z-y)\right) = \tilde{C}(f \,\hat{\star}\, g)$$

□

Note that $\hat{\star}$ is just a modified convolution where the second function has its argument multiplied by -1 (note that $\hat{\star}$ is not commutative).

Given a translation invariant bilinear form C on $S(\mathbb{Q}_p^d)$ we use the notation $\tilde{C}$ for the associated distribution in $S'(\mathbb{Q}_p^d)$ given by Lemma 2.2. The goal of the remainder of this subsection is to classify such distributions arising from the covariance forms of the measures μ described in Theorem 2.2.

Definition. *For* $\alpha \in \mathbb{C}$ *we say a distribution* $F \in S'(\mathbb{Q}_p^d, \mathbb{C})$ *is* homogenous of degree α *if for all* $\lambda \in p^{\mathbb{Z}}$ *one has* $\hat{S}_\lambda F = |\lambda|_p^{-\alpha} F$. *We say* F *is rotation invariant if* $\hat{R}_M F = F$ *for all* $M \in GL_d(\mathbb{Z}_p^d)$.

Above for $s \in (0, \infty)$ and $\alpha \in \mathbb{C}$ we define $s^\alpha = \exp[Log[s]\alpha]$ where Log is the principal branch of the logarithm. We will now construct a family $\{H_\alpha\}_{\alpha \in \mathbb{C}}$ of rotationally invariant generalized functions where H_α is homogenous of degree α. Following [30] we use the approach of analytic continuation to construct this family.

Definition. *Given an open domain* $D \subset \mathbb{C}$, *an* analytic generalized function *on* D *is a map* $\alpha \in D \to F_\alpha \in S'(\mathbb{Q}_p^d, \mathbb{C})$ *such that for every* $f \in S(\mathbb{Q}_p^d, \mathbb{C})$ *one has that the map from* D *to* $\mathbb{C}$ *given by* $\alpha \to F_\alpha(f)$ *is analytic on* D.

If one has two such maps $F_\alpha : U_1 \to S'(\mathbb{Q}_p^d, \mathbb{C})$ *and* $G_\alpha : U_2 \to S'(\mathbb{Q}_p^d, \mathbb{C})$ *with* $U_1 \subset U_2$ *and* $F_\alpha = G_\alpha$ *for* $\alpha \in U_1$ *we then say that* G_λ *is an* analytic continuation *of* F_λ.

For α satisfying $\Re(\alpha) > -d$ we define $\tilde{H}_\alpha \in S'(\mathbb{Q}_p^d, \mathbb{C})$ via

$$\tilde{H}_\alpha(f) = \int_{\mathbb{Q}_p^d} d^d x \; |x|^\alpha f(x) \tag{2.1}$$

For test functions f that do not vanish at the origin the assumption $\Re(\alpha) > -d$ is clearly necessary for the above integral to be well defined and it is not hard to see that $\tilde{H}_\alpha$ is an analytic generalized function in this region of the complex plane. Clearly $\tilde{H}_\alpha$ is homogenous of degree α for $\Re(\alpha) > -d$. The integral (2.1) is valid for $\Re(\alpha) \leqslant -d$ if the test function f vanishes at 0 (which by local constancy implies vanishing in a neighborhood of the origin). With this in mind we try to construct an analytic contination $\tilde{H}_\alpha$ by rewriting (2.1) as follows:

$$\begin{aligned}\tilde{H}_\alpha(f) &= \int_{\mathbb{Q}_p^d} d^d x \; |x|^\alpha \Big(f(x) - \mathbb{1}_{\mathbb{Z}_p^d}(x) f(0) \Big) + f(0) \times \int_{\mathbb{Q}_p^d} d^d x \; |x|^\alpha \mathbb{1}_{\mathbb{Z}_p^d} \\ &= \int_{\mathbb{Q}_p^d} d^d x \; |x|^\alpha \Big(f(x) - \mathbb{1}_{\mathbb{Z}_p^d}(x) f(0) \Big) + f(0) \times \frac{1 - p^{-d}}{1 - p^{-\alpha - d}}.\end{aligned} \tag{2.2}$$

In going to the second line we used that for α with $\Re(\alpha) > -d$

$$\begin{aligned}\int_{\mathbb{Z}_p^d} d^d x \; |x|^\alpha &= \sum_{\gamma=0}^{\infty} p^{-\alpha\gamma} \times vol\left(\mathbb{S}_\gamma\right) \\ &= \sum_{j=0}^{\infty} p^{-\alpha\gamma} \times \left(p^{-d\gamma} - p^{-d(\gamma+1)}\right) \;=\; \frac{1 - p^{-d}}{1 - p^{-\alpha - d}}.\end{aligned}$$

Here for $\gamma \in \mathbb{Z}$ we use the notation $\mathbb{S}_\gamma = \{x \in \mathbb{Q}_p^d : \; |x| = p^{-\gamma}\}$ and vol denotes the volume given by Haar measure to a measurable subset of $\mathbb{Q}_p^d$. In particular

$$vol(\mathbb{S}_\gamma) = vol(p^\gamma \mathbb{Z}_p^d) - vol(p^{\gamma+1} \mathbb{Z}_p^d) = p^{-d\gamma} - p^{-d(\gamma+1)}.$$

The second line of (2.2) is clearly analytic for $\alpha \in \mathbb{C}$ with the exception of $\alpha = -d$ where it has a simple pole - thus (2.2) gives an analytic extension of $\tilde{H}_\alpha$ to $\mathbb{C} \backslash \{-d\}$.

We cancel the simple pole by dividing by a non-vanishing normalizing factor that has a simple pole in the same location - we choose this normalization to be "$\Big(|x|^\alpha, \mathbb{1}_{\mathbb{Z}_p^d}(x)\Big)$". More concretely we introduce the notation

$$\Gamma_d(\alpha) = \frac{1-p^{-d}}{1-p^{-\alpha-d}}.$$

We then define the analytic generalized function $H_\alpha = \dfrac{1}{\Gamma_d(\alpha)}\tilde{H}_\alpha$ which is analytic in the whole complex plane (upon dealing with the removable singularity at $\alpha = -d$). In particular one sees that for any test function f one has

$$\lim_{\alpha \to -d} H_\alpha(f) = f(0)$$

i.e. $H_{-d} = \delta$ where $\delta \in S'(\mathbb{Q}_p^d, \mathbb{C})$ denotes the Dirac delta distribution at the origin. For $\alpha \in \mathbb{R}$, H_α is given by

$$H_\alpha(f) = \begin{cases} \dfrac{1-p^{-\alpha-d}}{1-p^{-d}} \times \int_{\mathbb{Q}_p^d} d^dx\ |x|^\alpha f(x) & \text{if } \alpha > -d \\ f(0) & \text{for } \alpha = -d \\ \dfrac{1-p^{-\alpha-d}}{1-p^{-d}} \times \int_{\mathbb{Q}_p^d} d^dx\ |x|^\alpha \Big(f(x) - \mathbb{1}_{\mathbb{Z}_p^d}(x)f(0)\Big) + f(0) & \text{for } \alpha < -d \end{cases} \tag{2.3}$$

Note that $\delta = H_{-d}$ is homogenous of degree $-d$, we also have the more general statement

Lemma 2.2. *For any $\alpha \in \mathbb{C}$ one has that $H_\alpha \in S'(\mathbb{Q}_p^d, \mathbb{C})$ is homogenous of degree α and rotation invariant.*

Proof: This can be checked by direct computation for all $\alpha \in \mathbb{C}$. One can also note that the quantity

$$\hat{S}_\lambda H_\alpha(f) - |\lambda|^{-\alpha} H_\alpha(f) \tag{2.4}$$

is entire in α for any test function f. At the same time since H_α is clearly homogenous of degree α for $\alpha \in (-d, \infty)$ so (2.4) vanishes on a non-isolated set of points which forces it to vanish for all $\alpha \in \mathbb{C}$. The argument for rotation invariance is similar. □

We now show that the bilinear forms that we are interested in correspond to rotation invariant homogenous distributions.

Lemma 2.3. *Suppose that C is a bilinear form on $S(\mathbb{Q}_p^d)$ which is translation invariant, rotation invariant, and κ-scale invariant. Then the associated distribution $\tilde{C} \in S'(\mathbb{Q}_p^d)$ is rotation invariant and homogenous of degree -2κ.*

Proof: From the assumptions it follows that for any $f, g \in S(\mathbb{Q}_p^d)$, $M \in GL_d(\mathbb{Z}_p)$, and $\lambda \in p^{\mathbb{Z}}$

$$\tilde{C}(f \hat{\star} g) = \tilde{C}\left(R_M(f) \hat{\star} R_M(g)\right) \text{ and}$$

$$\tilde{C}_\mu(f \hat{\star} g) = |\lambda|_p^{2\kappa} \times |\lambda|_p^{2d} \times \tilde{C}_\mu\left(S_{\lambda^{-1}}(f) \hat{\star} S_{\lambda^{-1}}(g)\right)$$

Quick computations using changes of variable show that

$$R_M(f) \hat{\star} R_M(g) = R_M(f \hat{\star} g) \text{ and } S_{\lambda^{-1}}(f) \hat{\star} S_{\lambda^{-1}}(g) = |\lambda|_p^{-d} \times S_{\lambda^{-1}}(f \hat{\star} g).$$

It then follows that the generalized function $\tilde{C}$ satisfies the conditions needed for rotation invariance and homogeniety when applied to test functions of the form $f \hat{\star} g$. The lemma will be proved if one shows that any $h \in S(\mathbb{Q}_p^d)$ can be written in this form but this is not hard to show. For a given h fix $j \in \mathbb{Z}$ sufficiently large such that h is locally constant over translates of $p^j \mathbb{Z}_p^d$. It is immediate that

$$h(y) = \int_{\mathbb{Q}_p^d} d^d z \; h(z) p^{-dj} \mathbb{1}_{p^j \mathbb{Z}_p^d}(z - y) = (h \hat{\star} p^{-dj} \mathbb{1}_{p^j \mathbb{Z}_p^d})(y)$$

□

While (2.3) gives examples of rotationally invariant homogenous distributions our goal is to show that these are the only possibilities (up to a constant of proportionality). The first step is to characterize the behaviour of rotationally invariant homogenous distributions away from the origin.

Lemma 2.4. *Let $F \in S'(\mathbb{Q}_p^d, \mathbb{C})$ be a rotation invariant distribution that is homogeneous of degree of α. Then there exists $K \in \mathbb{R}$ such that for all $f \in S(\mathbb{Q}_p^d, \mathbb{C})$ that satisfy $f(0) = 0$ one has*

$$F(f) = K \int_{\mathbb{Q}_p^d} d^d x \; f(x) \; |x|^{\alpha}.$$

Proof:

For any f satisfying the assumptions of the lemma we define

$$\tilde{f}(x) = \int_{GL_d(\mathbb{Z}_p)} dM \; f(M^{-1}x)$$

Here dM denotes the Haar measure on $GL_d(\mathbb{Z}_p)$ normalized to have total mass 1.

For $\gamma \in \mathbb{Z}$ we define $\mathbb{S}_\gamma = \{x \in \mathbb{Q}_p^d | \; |x| = p^\gamma\}$. Since $GL_d(\mathbb{Z}_p)$ acts transitively on $\mathbb{S}_\gamma$ for any fixed γ one has $x, y \in \mathbb{S}_\gamma \Rightarrow \tilde{f}(x) = \tilde{f}(y)$. Note that since f is locally constant the condition $f(0) = 0$ means that f vanishes in a neighborhood of the origin and since $GL_d(\mathbb{Z}_p)$ preserves the norm on $\mathbb{Q}_p^d$ it follows that the same holds for $\tilde{f}$. In particular one can find integers j, k such that the support of f is contained in $p^{-k}\mathbb{Z}_p^d \backslash p^{-j}\mathbb{Z}_p^d$ and it follows that one can then write $\tilde{f}$ in the form

$$\tilde{f}(x) = \sum_{\gamma=j}^{k} c_\gamma \mathbb{1}_{\mathbb{S}_\gamma}(x)$$

for some constants $c_\gamma \in \mathbb{C}$. Note that $\tilde{f} \in S(\mathbb{Q}_p^d, \mathbb{C})$, in particular $\mathbb{1}_{\mathbb{S}_\gamma} = \mathbb{1}_{p^\gamma \mathbb{Z}_p^d} - \mathbb{1}_{p^{\gamma+1}\mathbb{Z}_p^d}$.

We now show that $F(f) = F(\tilde{f})$. Since $F(R_M(f))$ is constant as M varies over $GL_d(\mathbb{Z}_p)$ one can write

$$F(f) = \int_{GL_d(\mathbb{Z}_p)} dM \; \big(F(x), f(M^{-1}x)\big) \; = \; \left(F(x), \int_{GL_d(\mathbb{Z}_p)} dM \; f(M^{-1}x)\right) = F(\tilde{f}).$$

We now find an explicit representation of $F(\tilde{f})$. We first claim that the constants c_γ are given by the formula

$$c_\gamma = \frac{1}{vol(\mathbb{S}_\gamma)} \int_{\mathbb{S}_\gamma} d^d x \; f(x),$$

where $vol(\mathbb{S}_\gamma) = \int_{\mathbb{S}_\gamma} d^dx = (1-p^{-d})p^{d\gamma}$. To prove this formula for c_γ we observe that by Fubini

$$\begin{aligned}\int_{\mathbb{S}_\gamma} d^dx\ \tilde{f}(x) &= \int_{\mathbb{S}_\gamma} d^dx\ \int_{GL_d(\mathbb{Z}_p)} dM\ f(M^{-1}x)\\ &= \int_{GL_d(\mathbb{Z}_p)} dM \int_{\mathbb{S}_\gamma} d^dx\ f(M^{-1}x)\\ &= \int_{GL_d(\mathbb{Z}_p)} dM \int_{\mathbb{S}_\gamma} d^dx\ f(x) = \int_{\mathbb{S}_\gamma} d^dx\ f(x).\end{aligned}$$

When going from the second to the third line we used a change of variable $M^{-1}x \to x$ which gives Jacobian of norm 1 and leaves the region of integration fixed. Now using the fact that $\tilde{f}(x)$ is constant on $\mathbb{S}_\gamma$ the calculation above shows

$$c_\gamma \times vol(\mathbb{S}_\gamma) = \int_{\mathbb{S}_\gamma} d^dx\ \tilde{f}(x) = \int_{\mathbb{S}_\gamma} d^dx\ f(x).$$

We now choose K to be the following f-independent constant.

$$K := \frac{1}{vol(S_0)} F(\mathbb{1}_{\mathbb{S}_0}) = (1-p^{-d})F(\mathbb{1}_{\mathbb{S}_0})$$

By the homogenity condition on F we have that:

$$\begin{aligned}F(\mathbb{1}_{\mathbb{S}_\gamma}) &= F\left(S_{p^{-\gamma}}(\mathbb{1}_{\mathbb{S}_0})\right)\\ &= |p^\gamma|_p^{-d} \times \hat{S}_{p^\gamma}(F)(\mathbb{1}_{S_0})\\ &= p^{d\gamma} \times |p^\gamma|_p^{-\alpha} \times F(\mathbb{1}_{\mathbb{S}_0}) = p^{\gamma(d+\alpha)} \times vol(\mathbb{S}_0) \times K\end{aligned}$$

The assertion of the lemma follows by observing

$$\begin{aligned}F(f) = F(\tilde{f}) &= \sum_{\gamma=j}^{k} c_\gamma F(\mathbb{1}_{\mathbb{S}_\gamma})\\ &= \sum_{\gamma=j}^{k} K p^{(d+\alpha)\gamma} \times \frac{vol(\mathbb{S}_0)}{vol(\mathbb{S}_\gamma)} \times \int_{\mathbb{S}_\gamma} d^dx\ f(x)\\ &= \sum_{\gamma=j}^{k} K \int_{\mathbb{S}_\gamma} d^dx\ f(x)|x|^\alpha\\ &= K \int_{\mathbb{Q}_p^d} d^dx\ f(x)|x|^\alpha.\end{aligned}$$

□

Definition. *If a distribution $F \in S'(\mathbb{Q}_p^d, \mathbb{C})$ satisfies $F(g) = 0$ for all test functions g with $g(0) = 0$ then we say F is* supported at the origin*.*

Lemma 2.5. *Let $F \in S'(\mathbb{Q}_p^d, \mathbb{C})$ be a distribution supported at the origin. Then there exists $K \in \mathbb{C}$ such that* *for all test functions f one has*

$$F(f) = K \times \delta(f) = K \times f(0)$$

Proof: Note that for any $f \in S(\mathbb{Q}_p^d, \mathbb{C})$ one has that $f - f(0)\mathbb{1}_{\mathbb{Z}_p^d}$ vanishes in a neighborhood of the origin. Thus by the support condition on F

$$F(f) - f(0)F(\mathbb{1}_{\mathbb{Z}_p^d}) = F\left(f - f(0)\mathbb{1}_{\mathbb{Z}_p^d}\right) = 0.$$

The lemma is proved if we set $K := F(\mathbb{1}_{\mathbb{Z}_p^d})$. □

We give one more lemma and then prove a proposition that classifies homogenous (of real degree), rotation invariant elements of $S'(\mathbb{Q}_p^d, \mathbb{C})$.

Lemma 2.6. *Suppose that $F_1, \ldots, F_n \in S'(\mathbb{Q}_p^d)$ are all non-zero and homogenous of distinct degrees $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ respectively. Then the F_j are linearly independent - if there exist constants $c_1, \ldots, c_n \in \mathbb{C}$ such that*

$$\sum_{j=1}^{n} c_j F_j = 0$$

it follows that one must have $c_1 = \cdots = c_n = 0$.

Proof: Without loss of generality suppose that $\alpha_n = \max_{1 \leqslant j \leqslant n} \alpha_j$. It is clear that for any $\lambda \in p^{\mathbb{Z}}$ one has

$$\sum_{j=1}^{n} c_j \hat{S}_\lambda F_j = \sum_{j=1}^{n} c_j |\lambda|_p^{-\alpha_j} F_j = 0.$$

Multiplying both sides by $|\lambda|_p^{\alpha_n}$ and taking $\lambda \to 0$ (i.e. choosing $\lambda = p^k$ and taking $k \to \infty$) one sees

$$\lim_{\lambda \to 0} \sum_{j=1}^{n} c_j |\lambda|_p^{\alpha_n - \alpha_j} F_j = c_n F_n = 0.$$

The assertion of the lemma follows by repeating this argument. □

Proposition 2.3. *Let $F \in S'(\mathbb{Q}_p^d)$ be rotation invariant and homogenous of degree $\alpha \in \mathbb{R}$. Then $F = AH_\alpha$ for some $A \in \mathbb{C}$ where H_α is given by* (2.3).

We first show that F takes the prescribed form when $\alpha \neq -d$. . In this case H_α behaves like $c|x|^\alpha$ away from the origin for some $c \neq 0$. Thus by Lemma 2.4 we can fix $A \in \mathbb{C}$ such $F - AH_\alpha$ is supported at the origin .

We can then apply Lemma 2.5 which means that $F - AH_\alpha$ must be some multiple of the delta distribution δ - that is there is some $F - AH_\alpha - K\delta = 0$ for some $K \in \mathbb{C}$. However since $F - AH_\alpha$ and δ are homogenous of different degrees it follows by Lemma 2.6 that one must have $K = 0$ which means $F = AH_\alpha$.

Now suppose F is homogenous of degree $\alpha = -d$. It is helpful to define $I_{-d} \in S'(\mathbb{Q}_p^d, \mathbb{C})$ via

$$I_{-d}(f) = \int_{\mathbb{Q}_p^d} d^d x \ \left(f(x) - f(0)\mathbb{1}_{\mathbb{Z}_p^d}(x)\right) \ |x|^{-d}.$$

Since I_{-d} behaves like $|x|^{-d}$ away from the origin it follows that there exists some $A \in \mathbb{C}$ such that $F - AI_{-d}$ is supported at the origin. As before this would force $F - AI_{-d} = K\delta$ for some $K \in \mathbb{C}$. However I_{-d} is not homogenous of degree $-d$, a simple calculation shows

$$\begin{aligned} \hat{S}_\lambda I_{-d}(f) &= |\lambda|_p^d I_{-d}(S_{\lambda^{-1}} f) \\ &= |\lambda|_p^d I_{-d}(f) + |\lambda|_p^d f(0) \int_{\mathbb{Q}_p^d} d^d x \ \left(\mathbb{1}_{\mathbb{Z}_p^d}(x) - \mathbb{1}_{\mathbb{Z}_p^d}(\lambda^{-1} x) \right) \ |x|^{-d}. \end{aligned} \tag{2.5}$$

The second term on the second line is clearly non-zero for any f not vanishing at the origin and $\lambda \neq 1$. Thus the only way for $AI_{-d} = F - K\delta$ to hold is for $A = 0$ since $F - K\delta$ is homogenous of degree $-d$. ☐

We now introduce the positivity criteria that we will use.

Lemma 2.7. *Let $C(\cdot, \cdot)$ be a real valued, translation invariant, symmetric bilinear form on $S'(\mathbb{Q}_p^d)$, let $\tilde{C} \in S'(\mathbb{Q}_p^d)$ satisfy $C(f, g) = \tilde{C}(f \hat{\star} g)$ for all $f, g \in S(\mathbb{Q}_p^d)$. Then the following conditions are equivalent:*

(i) C is positive definite

(ii) $\mathcal{F}[\tilde{C}](h) \in [0, \infty)$ for all non-negative $h \in S(\mathbb{Q}_p^d)$

Proof: C is positive definite if and only if $C(f, f) \geqslant 0$ for all (real) test functions f which is equivalent to the condition

$$\tilde{C}(f \hat{\star} f) \geqslant 0 \text{ for all } f \in S(\mathbb{Q}_p^d) \tag{2.6}$$

Viewing $\tilde{C}$ as an element of $S'(\mathbb{Q}_p^d, \mathbb{C})$ we claim that condition (2.6) is equivalent to:

$$\tilde{C}(g \hat{\star} \bar{g}) \geqslant 0 \text{ for all } g \in S(\mathbb{Q}_p^d, \mathbb{C}) \tag{2.7}$$

Clearly (2.7) $\Rightarrow$ (2.6). For the other direction we write $g = u + iv$ with $u, v \in S(\mathbb{Q}_p^d)$ and observe that

$$\begin{aligned} \tilde{C}\left((u + iv) \hat{\star} (u - iv)\right) &= \tilde{C}(u \hat{\star} u) + \tilde{C}(v \hat{\star} v) + i \left[\tilde{C}(v \hat{\star} u) - \tilde{C}(u \hat{\star} v) \right] \\ &= \tilde{C}(u \hat{\star} u) + \tilde{C}(v \hat{\star} v). \end{aligned}$$

In going to the last line we used the symmetry of C with forces $\tilde{C}(v \hat{\star} u) = \tilde{C}(u \hat{\star} v)$. We now show condition (ii) of the lemma is equivalent to (2.7). We have that

$$\begin{aligned} \tilde{C}(g \hat{\star} \bar{g}) &= \left(\mathcal{F}[\tilde{C}](k), \mathcal{F}^{-1}[g \hat{\star} \bar{g}](k) \right) \\ &= \left(\mathcal{F}[\tilde{C}](k), \mathcal{F}[g \hat{\star} \bar{g}](-k) \right) \\ &= \left(\mathcal{F}[\tilde{C}](k), \mathcal{F}[g](-k) \mathcal{F}[\bar{g}](k) \right) \\ &= \left(\mathcal{F}[\tilde{C}](k), \hat{g}(-k) \overline{\hat{g}(-k)} \right) = \left(\mathcal{F}[\tilde{C}](k), |\hat{g}(-k)|^2 \right). \end{aligned} \tag{2.8}$$

Above we used that for $a, b \in S(\mathbb{Q}_p^d, \mathbb{C})$ on has $\mathcal{F}[a \hat{\star} b](k) = \mathcal{F}\left[a \ \star R_{-1}(b)\right](k) = \hat{a}(k) \widehat{R_{-1} b}(k) = \hat{a}(k) \hat{b}(-k)$ and that $\overline{\mathcal{F}[\bar{g}](k)} = \hat{g}(-k)$.

It is clear that $\mathcal{F}[\tilde{C}](h) \in [0, \infty)$ for all non-negative $h \in S(\mathbb{Q}_p^d)$ is sufficient for condition (i) to hold.

We now show (i) $\Rightarrow$ (ii), if we assume that the last line of (2.8) is non-negative for all $g \in S(\mathbb{Q}_p^d, \mathbb{C})$ it is clear that (ii) holds for all $h \in S(\mathbb{Q}_p^d)$ with h an indicator function - just set $g = \mathcal{F}[h]$ and the result for all non-negative h then follows by linearity. □

Lemma 2.8. *Let $F \in S'(\mathbb{K}^d, \mathbb{C})$ with $\mathbb{K} = \mathbb{Q}_p$ or $\mathbb{R}^d$*

(i) If F is rotation invariant then so is $\mathcal{F}[F]$.

(ii) If F is homogenous of degree α then $\mathcal{F}[F]$ is homogenous of degree $-d-\alpha$.

Proof: Both assertions, for either choice of $\mathbb{K}$, follow immediately after computing a change of variable. In the case of assertion (ii) in the p-adic case this takes the form:

$$\begin{aligned}\left(\hat{S}_\lambda \circ \mathcal{F}\right)[F](f) &= |\lambda|_p^d \times \mathcal{F}[F]\left(S_{\lambda^{-1}}(f)\right) \\ &= |\lambda|_p^d \times F\left(\mathcal{F}[S_{\lambda^{-1}}(f)]\right) \\ &= F\left(S_\lambda\left(\mathcal{F}[f]\right)\right) \\ &= |\lambda|_p^d \times \hat{S}_{\lambda^{-1}}(F)\left(\mathcal{F}[f]\right) \\ &= |\lambda|_p^{d+\alpha} \times F\left(\mathcal{F}[f]\right) \;=\; |\lambda|_p^{d+\alpha} \times \mathcal{F}[F](f).\end{aligned}$$

□

In particular the Fourier transform leaves the class of distributions discussed Lemma 2.4 invariant and one has the following corollary:

Corollary 2.1. *For any $\alpha \in \mathbb{R}$*

$$\mathcal{F}[H_\alpha] = H_{-d-\alpha}$$

Proof: Since $\mathcal{F}[H_\alpha]$ is homogenous of degree $-d-\alpha$ and rotation invariant it follows that $\mathcal{F}[H_\alpha]$ is proportional to $H_{-d-\alpha}$ so one just needs to check the constant of proportionality is 1. We note that for arbitrary $\alpha \in \mathbb{C}$ one has

$$H_\alpha(\mathbb{1}_{\mathbb{Z}_p^d}) = 1.$$

At the same time one has

$$\mathcal{F}[H_\alpha](\mathbb{1}_{\mathbb{Z}_p^d}) \;=\; H_\alpha\left(\mathcal{F}\mathbb{1}_{\mathbb{Z}_p^d}\right) \;=\; H_\alpha\left(\mathbb{1}_{\mathbb{Z}_p^d}\right) \;= 1$$

which shows the mentioned constant of proportionality is 1. □

Proposition 2.4. *The class of symmetric, positive definite, bilinear forms C on $S(\mathbb{Q}_p^d)$ which are symmetric, translation invariant, rotation invariant, and κ - scale invariant are precisely given by the following families each parameterized by $c \geqslant 0$:*

- *For $\kappa = \dfrac{d}{2}$*

$$C(f,g) = c \times \int_{\mathbb{Q}_p^d} d^d x \; f(x)g(x) \tag{2.9}$$

$$\text{or equivalently } C(f,g) = c \times \int_{\mathbb{Q}_p^d} d^d k \; \hat{f}(k)\overline{\hat{g}(k)}$$

- *For $\kappa = 0$*

$$C(f,g) = c\left(\int_{\mathbb{Q}_p^d} d^d x\ f(x)\right) \times \left(\int_{\mathbb{Q}_p^d} d^d y\ \ g(y)\right) \tag{2.10}$$

$$\text{or equivalently } C(f,g) = c \times \hat{f}(0) \times \overline{\hat{g}(0)}$$

- *For $\kappa \in \left(0, \dfrac{d}{2}\right)$*

$$C(f,g) = c\int_{\mathbb{Q}_p^d \times \mathbb{Q}_p^d} d^d x\ d^d y\ \ f(x)g(y)|x-y|^{-2\kappa} \tag{2.11}$$

$$\text{or equivalently } C(f,g) = c \times \frac{1-p^{2\kappa}}{1-p^{-2\kappa-d}} \times \int_{\mathbb{Q}_p^d} d^d x\ \frac{\hat{f}(k)\overline{\hat{g}(k)}}{|k|^{d-2\kappa}}$$

- *For $\kappa \geqslant \dfrac{d}{2}$*

$$\begin{aligned} C(f,g) = c \times \int_{\mathbb{Q}_p^d \times \mathbb{Q}_p^d} d^d x\ d^d y \quad \Big(f(x)g(y) - \mathbb{1}_{\mathbb{Z}_p^d}(x-y)f(x)g(x)\Big)\, |x-y|^{2\kappa} \\ + \ c \times \frac{1-p^{-d}}{1-p^{-2\kappa-d}} \times \int_{\mathbb{Q}_p^d} d^d z\ f(z)g(z) \end{aligned} \tag{2.12}$$

$$\text{or equivalently } C(f,g) = c \times \frac{1-p^{-d}}{1-p^{-2\kappa-d}} \times \int_{\mathbb{Q}_p^d} d^d x\ \frac{\hat{f}(k)\overline{\hat{g}(k)}}{|k|^{d-2\kappa}}$$

In particular there are no such (non-zero) bilinear forms that are κ-scale invariant with $\kappa < 0$. Above f, g denote arbitrary elements of $S(\mathbb{Q}_p^d)$.

Proof: We first note that for $\kappa \geqslant 0$ the bilinear forms with the desired invariance properties and κ - scale invariance must be of the forms (2.9), (2.10), (2.11), or (2.12). This claim is justified via the following steps:

- Each such bilinear form C corresponds to a distribution $\tilde{C} \in S'(\mathbb{Q}_p^d)$ - Proposition 2.2
- Each such distribution $\tilde{C}$ must be rotation invariant and homogenous of degree -2κ - Lemma 2.3
- Such distributions $\tilde{C}$ are classified up to a constant of proportionality - Proposition 2.3.

Imposing positive definiteness will limit the allowed values of κ (and force $c \geqslant 0$). We now check that (2.9), (2.10), (2.11), and (2.12) satisfy positive definiteness using Lemma 2.7 as our criterion.

For $\kappa = \frac{d}{2}$ the listed class of bilinear forms are of the form $C(f,g) = \tilde{C}(f \mathbin{\hat{\star}} g)$ with $\tilde{C} = c\delta = cH_{-d}$. By Lemma 2.1 one has $\mathcal{F}[\tilde{C}] = cH_0 = c$. It immediately follows that $\mathcal{F}[\tilde{C}](h) = c\int h \geqslant 0$ for all non-negative $h \in S(\mathbb{Q}_p^d)$ which establishes positive definiteness.

Similarly for $\kappa = 0$ one has $C(f,g) = \tilde{C}(f \hat{\star} g) = c\int(f \mathbin{\hat{\star}} g)$, so $\tilde{C} = c$. In this case $\mathcal{F}[\tilde{C}](h) = c\delta(h) = ch(0)$ which is non-negative for all non-negative h.

For $\kappa \in (0,\infty) \setminus \left\{\frac{d}{2}\right\}$ we have $C(f,g) = \tilde{C}(f \mathbin{\hat{\star}} g)$ with $\tilde{C} = c \times \Gamma_d(-2\kappa)H_{-2\kappa}$. By Corollary 2.1 one has $\mathcal{F}[\tilde{C}] = c \times \Gamma_d(-2\kappa)H_{-d+2\kappa}(f)$. Then since $-d + 2\kappa > -d$ it follows that

$$\mathcal{F}[\tilde{C}](k) = c \times \frac{\Gamma_d(-2\kappa)}{\Gamma_d(-d+2\kappa)}|k|^{-d+2\kappa} \tag{2.13}$$

Since

$$\frac{\Gamma_d(-2\kappa)}{\Gamma_d(-d+2\kappa)} \;=\; \frac{1-p^{-2\kappa}}{1-p^{-d+2\kappa}} \;\geqslant\; 0 \text{ for } \kappa > 0 \tag{2.14}$$

it follows that $\mathcal{F}[\tilde{C}](h) \geqslant 0$ for any non-negative $h \in S(\mathbb{Q}_p^d)$.

We now turn to the case $\kappa < 0$ and show that any (non-zero) bilinear form C with the desired invariances cannot satisfy the positive definiteness condition. Now by classification arguments mentioned at the beginning of this proposition's proof one has that $C(f,g) = \tilde{C}(f \hat{\star} g)$ with $\tilde{C}(x) = aH_{-2\kappa}$ and $\mathcal{F}[\tilde{C}] = aH_{-d+2\kappa}$. Since $-d+2\kappa < -d$ one has $\mathcal{F}[\tilde{C}]$ is of the form

$$\mathcal{F}[\tilde{C}](f) = a\left[-\frac{p^{-2\kappa}-1}{1-p^{-d}}\int_{\mathbb{Q}_p^d} d^dx \;\left(f(x) - f(0)\mathbb{1}_{\mathbb{Z}_p^d}(x)\right)\; |x|^{-d+2\kappa} + \delta\right].$$

It is not hard to see that by Lemma 2.7 $\tilde{C}$ is positive definite only if $a = 0$. In particular positivity requires

$$\mathcal{F}[\tilde{C}](\mathbb{1}_{\mathbb{Z}_p^d}) = a \geqslant 0$$

while

$$\mathcal{F}[\tilde{C}](\mathbb{1}_{\mathbb{S}_0}) = -a \times \left(p^{-2\kappa} - 1\right) \geqslant 0.$$

which forces $a = 0$. □

2.2.2 Lemmas for the classification - real case

Lemma 2.9. *For $z \in \mathbb{R}^m$ define T_z acting on $f \in S(\mathbb{R}^m \times \mathbb{R}^n, \mathbb{C})$ via $T_z(f)(x,y) = f(x+z,y)$ (here $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$).*

Suppose that $\phi \in S'(\mathbb{R}^m \times \mathbb{R}^n, \mathbb{C})$ satisfies the property $\phi(T_z f) = \phi(f)$ for all $z \in \mathbb{R}^m$ and $f \in S(\mathbb{R}^m \times \mathbb{R}^n, \mathbb{C})$. Then there exists $\psi \in S'(\mathbb{R}^n)$ such that for all such f one has:

$$\phi(f) = \left(\psi(y), \int_{\mathbb{R}^m} d^mx \; f(x,y)\right).$$

Proof: This easily follows from the well known fact that if the derivative of a distribution is 0 then it is given by a constant, the proof of this statement is included for completeness. We give the argument for $m = 1$ which can be repeated to get the result for all m.

For any $f(x_1, \ldots, x_{n+1}) \in S(\mathbb{R}^{n+1}, \mathbb{C})$ one has

$$\begin{aligned} \lim_{u\to 0} \frac{\phi(T_u f) - \phi(f)}{u} &= \phi\left(\lim_{u\to 0}\frac{T_u f - f}{u}\right) \\ &= \phi\left(\partial_{x_1} f\right) \;=\; 0 \end{aligned}$$

Thus ϕ vanishes on any test function $g \in S(\mathbb{R}^{n+1}, \mathbb{C})$ of the form $g = \partial_{x_1} f$ for some $f \in S(\mathbb{R}^{n+1}, \mathbb{C})$.

We now state the well known fact that

$$\left\{g \in S(\mathbb{R}^{n+1}, \mathbb{C}) : \; g = \partial_{x_1} f \text{ for some } f \in S(\mathbb{R}^{n+1}, \mathbb{C})\right\} \;=\; \left\{g \in S(\mathbb{R}^{n+1}, \mathbb{C}) : \int_{\mathbb{R}} dx_1 \; g(x_1, \ldots, x_{n+1}) = 0\right\}.$$

In particular for g in the second set above it easy to check that $\frac{\hat{g}(k)}{ik_1} \in S(\mathbb{R}^{n+1}, \mathbb{C})$ and that the required f is then given by $\mathcal{F}^{-1}\left[\dfrac{\hat{g}(k)}{ik_1}\right]$.

Now choose any $h \in S(\mathbb{R}^1, \mathbb{C})$ with $\int h = 1$. Define $\psi \in S'(\mathbb{R}^n, \mathbb{C})$ via

$$(\psi(y), j(y)) \;=\; (\phi(x,y), h(x)j(y))$$

for any $j \in S(\mathbb{R}^n, \mathbb{C})$. The assertion then follows by observing that

$$\phi(f) - \left(\psi(y), \int_{\mathbb{R}} dx\ f(x,y)\right) \;=\; \left(\phi(x,y), f(x,y) - h(x)\int_{\mathbb{R}} dt\ f(t,y)\right) \;=\; 0$$

where the last equality follows since

$$\int_{\mathbb{R}} dx \left[f(x,y) - h(x)\int_{\mathbb{R}} dt\ f(t,y)\right] = 0.$$

□

The next proposition and corollary give the classification of reflection invariant homogenous distributions in $S'(\mathbb{R}\backslash\{0\}, \mathbb{C})$ which in turn characterizes the behaviour of reflection invariant homogenous distributions in $S'(\mathbb{R}, \mathbb{C})$ away from the origin.

For $U \subset \mathbb{R}^d$ which is open we define the space of test functions $S(U, \mathbb{C})$ to be closed subspace consisting of all $f \in S(\mathbb{R}^d, \mathbb{C})$ with f supported in U. We view $S(U, \mathbb{C})$ as a topological vector space with its inherited topology and denote by $S'(U, \mathbb{C})$ the corresponding topological dual.

Proposition 2.5. *Let $U \subset \mathbb{R}^d$ be an open set which is also homogenous (that is $\lambda U \subset U$ for any $\lambda \in [0, \infty)$). Suppose that $F \in S'(U, \mathbb{C})$. Then if F is homogenous of degree α it must satisfy the equation*

$$\alpha \times F(x) = \sum_{j=1}^{d} x_j \partial_j F(x). \tag{2.15}$$

Proof: First assume that F is homogenous of degree α so for every $\lambda \in (0, \infty)$ one has $\hat{S}_\lambda(F) = \lambda^{-\alpha} F$, i.e. for every $f \in S(U, \mathbb{C})$ one has

$$\lambda^d \times F\left(S_{\lambda^{-1}} f\right) = \lambda^{-\alpha} F(f)$$

Differentiating the above equation with respect to λ and then evaluating at $\lambda = 1$ gives

$$d \times F(f) + \sum_{j=1}^{n} (F(x), x_j \partial_j f(x)) \;=\; -\alpha \times F(f)$$

Equation (2.15) then follows upon observing that

$$\begin{aligned} \sum_{j=1}^{d} (F(x), x_j \partial_j f(x)) \;&=\; \sum_{j=1}^{d} -\left(\partial_j \left(x_j F(x)\right), f(x)\right) \\ &=\; -d \times F(f) - \sum_{j=1}^{d} (x_j \partial_j F, f). \end{aligned}$$

□

We remark that the converse statement to Proposition 2.5 is also true - satisfying (2.15) implies homogeniety - but we will not need that here. We have the following corollary.

Corollary 2.2. *Suppose that $F \in S'((0,\infty),\mathbb{C})$ is homogenous of degree α then F is given by the function $A|x|^{\alpha}$ for some $A \in \mathbb{C}$. The same holds if $F \in S'(\mathbb{R}\backslash\{0\},\mathbb{C})$ is homogenous of degree α and is reflection invariant.*

For arbitrary $F \in S'(\mathbb{R}\backslash\{0\},\mathbb{C})$ which are homogenous of degree α one has that there exists $A_1,\ A_2 \in \mathbb{C}$ such that F is given by the function $A_1 x_+^{\alpha} + A_2 x_-^{\alpha}$ where $x_+^{\alpha}, x_-^{\alpha} : \mathbb{R} \to \mathbb{R}$ are given by

$$\begin{aligned} x_+^{\alpha} &= |x|^{\alpha} \text{ if } x > 0,\ x_+ = 0 \text{ if } x \leqslant 0 \\ x_-^{\alpha} &= |x|^{\alpha} \text{ if } x < 0,\ x_+ = 0 \text{ if } x \geqslant 0. \end{aligned}$$

Proof: We first prove the assertion concerning $F \in S'((0,\infty),\mathbb{C})$. Applying Proposition 2.5 with $d = 1$ to F gives us that F must satisfy the following on $(0,\infty)$:

$$\partial F(x) = \alpha x^{-1} F(x)$$

It then immediately follows that derivative of the following distribution

$$\frac{F(x)}{x^{\alpha}} \tag{2.16}$$

vanishes. Then distribution (2.16) must be given by a constant which proves the assertion in question. (See Lemma 2.9).

Note that this argument does not work for homogenous $F \in S'(\mathbb{R}\backslash\{0\},\mathbb{C})$ - in particular a distribution on $\mathbb{R}\backslash\{0\}$ with vanishing distributional derivative need not be given by a constant. The proof of Lemma 2.9 breaks down in this case since a test function $f \in S(\mathbb{R}\backslash\{0\},\mathbb{C})$ with $\int f = 0$ it is not true in general that $f = \partial g$ for a test function g supported on $\mathbb{R}\backslash\{0\}$.

However the classification for homogenous $F \in S'(\mathbb{R}\backslash\{0\},\mathbb{C})$ can be proven by applying the preceding argument once for test functions supported $(0,\infty)$ and separately for test functions supported on $(-\infty,0)$ (clearly every $f \in S(\mathbb{R}\backslash 0,\mathbb{C})$ is a sum of such test functions).

The statement about reflection invariant homogenous distributions is clear consequence of the other assertions. □

We now construct examples of homogenous distributions on $\mathbb{R}$ via analytic contination following [30]. We use the same definitions of *analytic generalized function* and *analytic continuation* that were given in the p-adic setting when working over $\mathbb{R}^d$ instead .

For $\alpha \in \mathbb{C}$ with $\Re(\alpha) > -1$ it is clear that the function $|x|^{\alpha}$ yields a well defined element of $S(\mathbb{R},\mathbb{C})$ that is homogenous of degree α and reflection invariant. In particular for any test function $f \in S(\mathbb{R}\backslash\{0\})$ the quantity

$$\int_{-\infty}^{\infty} dx\ |x|^{\alpha} f(x)$$

is analytic in α for $\Re(\alpha) > -1$. We now try to rewrite the integral above so it is valid for α in a larger region of the complex plane. For any $f \in S(\mathbb{R}, \mathbb{C})$ we define $\psi_f \in S(\mathbb{R}, \mathbb{C})$ via

$$\psi_f(x) = \frac{f(x) + f(-x)}{2}.$$

Note that $f \to \psi_f$ is a continuous linear map from $S(\mathbb{R}, \mathbb{C})$ to itself.

For any $\alpha \in \mathbb{C}$ with $\Re(\alpha) > -1$ we set

$$\tilde{H}_\alpha(f) = (|x|^\alpha, \psi_f(x)).$$

For such α clearly one has $\tilde{H}_\alpha(f) = (|x|^\alpha, f(x))$. We now rewrite $\tilde{H}_\alpha$ in order to define an analytic continuation.

For any $n \in \mathbb{N}$ one can write

$$\begin{aligned} \tilde{H}_\alpha(f) &= \int_{-\infty}^{\infty} dx\ |x|^\alpha \psi_f(x) \times \mathbb{1}\{|x| > 1\} + \int_{-1}^{1} dx\ |x|^\alpha \left(\psi_f(x) - \sum_{j=0}^{n} \frac{f^{(2j)}(0)}{(2j)!} x^{2j} \right) + \sum_{j=0}^{n} \left[\frac{f^{(2j)}(0)}{(2j)!} \int_{-1}^{1} dx\ |x|^\alpha x^{2j} \right] \\ &= \int_{-\infty}^{\infty} dx\ |x|^\alpha f(x) \times \mathbb{1}\{|x| > 1\} + \int_{-1}^{1} dx\ |x|^\alpha \left(\psi_f(x) - \sum_{j=0}^{n} \frac{f^{(2j)}(0)}{(2j)!} x^{2j} \right) + \sum_{j=0}^{n} \left[\frac{f^{(2j)}(0)}{(2j)!} \times \frac{2}{\alpha + 2j + 1} \right]. \end{aligned} \tag{2.17}$$

We remark that the Taylor expansion of ψ_f about 0 reads

$$\sum_{j=0}^{\infty} \frac{f^{(2j)}(0)}{(2j)!} x^{2j}$$

so that

$$\left| \psi_f(x) - \sum_{j=0}^{2n} \frac{f^{(j)}(0)}{j!} x^j \right| \leqslant \mathcal{O}(|x|^{2n+2})$$

in the vicinity of $x = 0$. It follows that the integral appearing in the second term of the last line of (2.17) is well defined for $\Re(\alpha) > -2n - 3$. One then sees that the last line of (2.17) defines an analytic continuation of $\tilde{H}_\alpha$ to the domain

$$\{\alpha \in \mathbb{C} |\ \Re(\alpha) > -2n - 3 \text{ and } \alpha \neq -1 - 2j \text{ for } j = 0, \cdots, n\}.$$

Since $n \in \mathbb{N}$ was arbitrary clearly $\tilde{H}_\alpha$ admits an analytic continuation to $\{\alpha \in \mathbb{C} |\ \alpha \neq -1 - 2j \text{ for } j \in \mathbb{N}\}$ which we also denote by $\tilde{H}_{\alpha,1}$. At the excluded values of α one has that $\tilde{H}_\alpha$ has simple poles - the corresponding residues given by

$$\lim_{\alpha \to -1-2j} (\alpha + 1 + 2j) \tilde{H}_{\alpha,1}(f) = \frac{2}{(2j)!} f^{(2j)}(0) = \frac{2}{(2j)!} \delta^{(2j)}(f).$$

Again it is convenient to divide $\tilde{H}_\alpha$ by a suitably chosen normalization factor to cancel these poles and arrive at a an entire analytic generalized function.

For $\Re(\alpha) > -1$ one has:

$$\begin{aligned}\left(|x|^\alpha, \exp[-x^2]\right) &= 2\int_0^\infty dx\ x^\alpha \exp[-x^2] \\ &= \int_0^\infty dt\ t^{\frac{1}{2}(\alpha-1)} \exp[-t] = \Gamma\left(\frac{\alpha+1}{2}\right).\end{aligned}$$

where the Gamma function $\Gamma(z)$ is given by

$$\Gamma(z) = \int_0^\infty dt\ t^{z-1} e^{-t}$$

for $z \in \mathbb{C}$ with $\Re(z)$ but admits an analytic continuation (using the relation $\Gamma(z+1) = z\Gamma(z)$) to the entire complex plane except for simple poles at the non-positive integers. $\Gamma(z)$ is non-vanishing wherever it is defined and its residue at $z = -k$ for $k \in \mathbb{N}$ is $\frac{(-1)^k}{k!}$.

With this in mind we define

$$H_{\alpha,1} = \frac{1}{2\Gamma\left(\frac{\alpha+1}{2}\right)} \tilde{H}_{\alpha,1}$$

which yields an entire analytic generalized function - the singularities at $\alpha = -1-2j$ for $j \in \mathbb{N}$ are removable. In particular for $\alpha \in \mathbb{R}$ one has

$$\begin{aligned}H_{\alpha,1}(f) &= \frac{1}{2\Gamma\left(\frac{\alpha+1}{2}\right)} \int_{-\infty}^\infty dx\ |x|^\alpha f(x) \text{ for } \alpha > -1 \\ &= \frac{1}{2\Gamma\left(\frac{\alpha+1}{2}\right)} \Bigg(\int_{-\infty}^\infty dx\ |x|^\alpha f(x) \times \mathbb{1}\{|x| > 1\} + \int_{-1}^1 dx\ |x|^\alpha \left(\psi_f(x) - \sum_{j=0}^n \frac{f^{(2j)}(0)}{(2j)!} x^{2j} \right) \\ &\qquad + \sum_{j=0}^n \left[\frac{f^{(2j)}(0)}{(2j)!} \times \frac{2}{\alpha+2j+1} \right] \Bigg) \text{ for } \alpha \in (-2n-1, -2n-3),\ n \in \mathbb{N} \\ &= \frac{(-1)^n n!}{(2n)!} \delta^{(2n)}(f) \text{ for } \alpha = -2n-1,\ n \in \mathbb{N}\end{aligned} \tag{2.18}$$

We now define a similar construction of rotationally invariant homogenous distributions over $\mathbb{R}^d$ for $d \geqslant 2$. In this setting it will be convenient to spherically average test functions. For $f \in S(\mathbb{R}^d, \mathbb{C})$ we define the function $\psi_f : \mathbb{R} \to \mathbb{C}$ via

$$\psi_f(r) = \frac{1}{|r|^{d-1}\Omega_d} \int_{S_{|r|}} d\omega\ f(\omega) \text{ for } r \neq 0 \tag{2.19}$$

where for $r > 0$ S_r denotes the d-dimensional sphere of radius r (i.e. $S_r = \left\{x \in \mathbb{R}^d |\ \sum_{i=1}^d x_i^2 = r^2\right\}$). $d\omega$ is the standard surface area measure on the S_r.

$$\Omega_d = \frac{2\sqrt{\pi}^d}{\Gamma\left(\frac{d}{2}\right)}$$

is the surface area of the d-dimensional unit sphere S_1. Clearly we can extend $\psi_f(r)$ to $r = 0$ continuously by setting $\psi_f(0) = f(0)$.

For $d \geqslant 2$ and $\alpha \in \mathbb{C}$ with $\Re(\alpha) > -d$ we define $\tilde{H}_{\alpha,d} \in S'(\mathbb{R}^d, \mathbb{C})$ via

$$\begin{aligned} \tilde{H}_{\alpha,d}(f) &= \int_{\mathbb{R}^d} d^d x \ |x|^\alpha f(x) \\ &= \Omega_d \int_0^\infty dr \ r^{d+\alpha-1} \psi_f(r). \end{aligned} \tag{2.20}$$

The analytic continuation of $\tilde{H}_{\alpha,d}$ will be constructed using the same ideas that appeared for the $d = 1$ case. The next theorem gives the Taylor expansion of $\psi_f(r)$ about $r = 0$.

Theorem 2.4. *(Pizetti's Formula) For $f \in S(\mathbb{R}^d, \mathbb{C})$ for $d \geqslant 2$ and let ψ_f be given as in (2.19). One then has $\psi_f \in S(\mathbb{R}, \mathbb{C})$ and the derivatives of $\psi_f(r)$ at $r = 0$ are given by*

$$\psi_f^{(n)}(0) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \dfrac{(2k)!\Delta^k f(0)}{2^k k! \left(\prod_{j=1}^{k-1}(d+2j)\right)} & \text{if } n = 2k \end{cases} \tag{2.21}$$

where above Δ is the d-dimensional Laplacian, that is $\Delta = \sum_{i=1}^d \partial_i^2$.

Proof: We refer the reader to [10]. □

We remark that we don't actually need Pizetti's formula in order to perform our computations, it just makes our analytic continuation of $\tilde{H}_{\alpha,d}$ more explicit. All one really needs to know is that $\psi_f(r)$ is smooth with a Taylor expansion that only includes even powers of r - however this last fact is a direct consequence of $\psi_f(r)$ being even.

Now for any $n \in \mathbb{N}$ we can rewrite (2.20) as

$$\begin{aligned} \tilde{H}_{\alpha,d}(f) &= \Omega_d \int_0^\infty dr \ r^{d+\alpha-1} \psi_f(r) \\ &= \Omega_d \Bigg[\int_1^\infty dr \ r^{d+\alpha-1} \psi_f(r) \ + \int_0^1 dr \ r^{d+\alpha-1} \left(\psi_f(r) - \sum_{k=0}^n \frac{\Delta^k f(0)}{2^k k! \left(\prod_{j=1}^{k-1}(d+2j)\right)} r^{2k} \right) \\ &\qquad + \sum_{k=0}^n \frac{\Delta^k f(0)}{2^k k! \left(\prod_{j=1}^{k-1}(d+2j)\right)} \int_0^1 dr \ r^{d+2k+\alpha-1} \Bigg] \\ &= \Omega_d \Bigg[\int_1^\infty dr \ r^{d+\alpha-1} \psi_f(r) \ + \int_0^1 dr \ r^{d+\alpha-1} \left(\psi_f(r) - \sum_{k=0}^n \frac{\Delta^k f(0)}{2^k k! \left(\prod_{j=1}^{k-1}(d+2j)\right)} r^{2k} \right) \\ &\qquad + \sum_{k=0}^n \frac{\Delta^k f(0)}{2^k k!(d+\alpha+2k) \left(\prod_{j=1}^{k-1}(d+2j)\right)} \Bigg]. \end{aligned} \tag{2.22}$$

Now by Theorem 2.4 one has

$$\left| \psi_f(r) - \sum_{k=0}^n \frac{\Delta^k f(0)}{2^k k! \left(\prod_{j=1}^{k-1}(d+2j)\right)} r^{2k} \right| \leqslant \mathcal{O}(|r|^{2n+2})$$

in a vicinity of $r = 0$. Thus the second integral on the last line of (2.22) is valid for $\Re(\alpha) > -d - 2n - 2$. Similarly to the $d = 1$ case we can use this process to analytically continue $\tilde{H}_{\alpha,d}$ to the entire complex plane except for the values $\alpha = -d - 2k$ for $k \in \mathbb{N}$ where $\tilde{H}_{\alpha,d}$ has simple poles with the residue at $\alpha = -d - 2k$ given by

$$\frac{\Omega_d \Delta^k \delta}{2^k k! \left(\prod_{j=1}^{k-1}(d+2j)\right)}$$

where δ denotes the d-dimensional delta function.

We define, for $d \geqslant 2$ and $\alpha \in \mathbb{C}$,

$$H_{\alpha,d} := \frac{\tilde{H}_{\alpha,d}}{\Omega_d \times \Gamma\left(\frac{\alpha+d}{2}\right)} \tag{2.23}$$

In particular for $\alpha \in \mathbb{R}$ one has

$$\begin{aligned}
H_{\alpha,d} &= \frac{1}{\Omega_d \times \Gamma\left(\frac{\alpha+d}{2}\right)} \int_{\mathbb{R}^d} d^d x \ |x|^\alpha f(x) \text{ for } \alpha > -d \\
H_{\alpha,d} &= \frac{1}{\Omega_d \times \Gamma\left(\frac{\alpha+d}{2}\right)} \int_{\mathbb{R}^d} d^d x \ |x|^\alpha f(x) \times \mathbb{1}\{|x| \geqslant 1\} \\
&\quad + \frac{1}{\Gamma\left(\frac{\alpha+d}{2}\right)} \left[\int_0^1 dr \ r^{d+\alpha-1} \left(\psi_f(r) - \sum_{k=0}^{n} \frac{\Delta^k f(0)}{2^k k! \left(\prod_{j=1}^{k-1}(d+2j)\right)} r^{2k} \right) \right. \\
&\quad \left. + \sum_{k=0}^{n} \frac{\Delta^k f(0)}{2^k k! (d+\alpha+2k) \left(\prod_{j=1}^{k-1}(d+2j)\right)} \right] \text{ for } \alpha \in (-2n-2-d, -2n-d), \ n \in \mathbb{N} \\
H_{\alpha,d} &= \frac{(-1)^k \Delta^k \delta(f)}{2^k \left(\prod_{j=1}^{k-1}(d+2j)\right)} \text{ for } \alpha = -d-2k, \ k \in \mathbb{N}
\end{aligned} \tag{2.24}$$

Proposition 2.6. *Let $d \geqslant 2$ and suppose that $F \in S'(\mathbb{R}^d \backslash \{0\}, \mathbb{C})$ is rotation invariant and homogenous of degree α. Then $F(x) = A|x|^\alpha$ for some $A \in \mathbb{C}$.*

Proof: Define $L : S((0,\infty), \mathbb{C}) \to \mathbb{S}(\mathbb{R}^n \backslash \{0\}, \mathbb{C})$ via

$$(L\psi)(x) = \psi(|x|).$$

It is not hard to see that the L is a continuous linear map between the two mentioned test function spaces. Now define $G \in S'((0,\infty), \mathbb{C})$ via

$$G(\psi) = F(L\psi).$$

Using the homogeniety of F we now observe that for any $\lambda > 0$ and any $\psi \in S((0,\infty), \mathbb{C})$ one has

$$\hat{S}_\lambda G(\psi) = \lambda\left(G(r), \psi(\lambda r)\right) = \lambda\left(F(x), \psi(\lambda|x|)\right) = \lambda^{1-d-\alpha}\left(F(x), \psi(|x|)\right) = \lambda^{1-d-\alpha} G(\psi).$$

So G is homogenous of degree $\alpha + d - 1$ and by Corollary 2.2 it follows that $G(r) \in S'((0,\infty), \mathbb{C})$ is given by $Ar^{\alpha+d-1}$ for some constant $A \in \mathbb{C}$.

Now for $g \in S(\mathbb{R}^n \backslash \{0\}, \mathbb{C})$ we define $\psi_g \in S((0, \infty), \mathbb{C})$ via (2.19) for $r > 0$. Observe that

$$\begin{aligned} F(L\psi_g) = G(\psi_g) &= A \int_0^\infty dr\ r^{\alpha+d-1} \left[\frac{1}{r^{d-1}\Omega_d} \int_{S_r} d\omega\ g(\omega) \right] \\ &= \frac{A}{\Omega_d} \int_0^\infty dr \int_{S_r} d\omega\ |\omega|^\alpha g(\omega) \\ &= \frac{A}{\Omega_d} \int_{\mathbb{R}^d} d^d x\ |x|^\alpha g(x). \end{aligned}$$

The proposition will then follow if we show that for arbitrary $g \in S(\mathbb{R}^d \backslash \{0\}, \mathbb{C})$ one has $F(g) = F(L\psi_g)$. Fix such a g and observe that for any $x \in \mathbb{R}^d$ with $|x| = r > 0$ one has

$$\begin{aligned} \int_{O(d)} dM\ g(M^{-1}x) &= \frac{1}{r^{d-1}\Omega_d} \int_{S_r} d\omega\ g(\omega) \\ &= \psi_g(r) = L\psi_g(x) \end{aligned} \tag{2.25}$$

where dM refers to Haar measure on $O(d)$ normalized to have total mass 1. The first equality is just a consequence of the fact that the pushforward of the measure dM via the map $M \to M^{-1}x$ is a rotationally invariant measure on S_r of mass 1 which must then be given by the normalized surface area measure on S_r.

The proof is finished upon observing that

$$\begin{aligned} F(L\psi_g) &= F\left(\int_{O(d)} dM\ R_M g \right) \\ &= \int_{O(d)} dM\ F(R_M g) = \int_{O(d)} dM\ F(g) = F(g). \end{aligned}$$

□We now give a classification of rotationally invariant generalized functions that are supported at the origin.

Proposition 2.7. *Let $F \in S'(\mathbb{R}^d, \mathbb{C})$ be a rotation invariant (reflection invariant for $d = 1$) and satisfy $F(f) = 0$ for all $f \in S(\mathbb{R}^d, \mathbb{C})$ that vanish in some neighborhood of the origin. Then there exists $N \in \mathbb{N}$ and constants $a_0, a_1, \ldots, a_n \in \mathbb{C}$ such that*

$$F = \sum_{k=0}^{N} a_k \Delta^k \delta$$

Proof: We first give the proof for $d \geqslant 2$.

By the Paley-Wiener-Schwartz Theorem (see [38, Theorem 7.3.1]) one has that $\hat{F} := \mathcal{F}[F]$ is given by an entire function which satisfies the bound

$$\left| \hat{F}(z) \right| \leqslant (1 + |z|)^N$$

where N denotes the order of the distribution F. It follows by standard arguments that $\hat{F}(z)$ is a polynomial in the components of z of at most order N. Since for any multi-index $\alpha \in \mathbb{N}^d$ one has

$$\mathcal{F}\left[\partial^\alpha \delta\right](z) = (2\pi)^{-\frac{d}{2}} (-i)^{|\alpha|} z^\alpha$$

our assertion will follow if we prove that $\hat{F}(z)$ is in fact a polynomial in the quantity $|z|^2 = \sum_{j=1}^{d} z_j^2$ and then apply the inverse Fourier transform.

We restrict $\hat{F}(z)$ to $\mathbb{R}^d$, denoting the restriction by $\hat{F}(x) = \hat{F}(x_1, \ldots, x_d)$. We remark that $\hat{F}$ must be invariant under rotations of $\mathbb{R}^d$. In particular for any $x = (x_1, \ldots, x_d) \in \mathbb{R}^n$ with $|x| = r$ one must have $\hat{F}(x_1, \ldots, x_n) = \hat{F}(0, \ldots, r)$. Since $\hat{F}(0, \ldots, 0, r)$ is a polynomial in r our assertion would follow if we knew $\hat{F}(0, \ldots, 0, r)$ only contains even powers of r - however this is immediate upon observing that one has $\hat{F}(0, \ldots, 0, r) = \hat{F}(0, \ldots, 0, -r)$ for any $r \in \mathbb{R}$.

The $d = 1$ case is proved by a similar (and shorter) argument. □

Proposition 2.8. *Let $F \in S'(\mathbb{R}^d, \mathbb{C})$ be rotation invariant (reflection invariant for $d = 1$) and homogenous of degree $\alpha \in \mathbb{R}$. Then there exists $A \in \mathbb{C}$ such that $F = AH_{\alpha,d}$.*

Proof: The proof proceeds along the same lines as Proposition 2.3. We first treat the case $\alpha \neq -d-2k$ with $k \in \mathbb{N}$. Clearly F and $H_{\alpha,d}$ can be viewed as elements of $S'(\mathbb{R}^d \backslash \{0\}, \mathbb{C})$ and since $H_{\alpha,d}$ is given by a non-zero multiple of $|x|^\alpha$ away from the origin it follows that one can find $A \in \mathbb{C}$ such that $F - AH_{\alpha,d}$ vanishes on all elements of $S(\mathbb{R}^d \backslash \{0\}, \mathbb{C})$. In particular by Proposition (2.7) one has

$$F - A \times H_{\alpha,d} = \sum_{k=0}^{N} a_k \Delta^k \delta$$

for some $N \in \mathbb{N}$ and constants a_k. However since $\Delta^k \delta$ is homogenous of degree $-d - 2k \neq \alpha$ it follows by the easily proven real analog of Lemma 2.6 that all the constants a_k which means $F = AH_{\alpha,d}$.

Now suppose F is instead homogenous of degree $\alpha = -d - 2k$ for some $k \in \mathbb{N}$. We treat the case $d \geqslant 2$, the method for $d = 1$ is essentially the same. Define $I_{-d-2k} \in S'(\mathbb{R}^d, \mathbb{C})$ via

$$\begin{aligned} I_{-d-2k}(f) := &\int_{\mathbb{R}^d} d^d x \; |x|^{-d-2k} \psi_f(|x|) \mathbb{1}\{|x| \geqslant 1\} \\ &+ \int_{\mathbb{R}^d} d^d x \; |x|^{-d-2k} \left(\psi_f(|x|) - \sum_{j=0}^{k} \frac{\Delta^j f(0)}{2^j j! \left(\prod_{l=1}^{j-1} (d+2l) \right)} |x|^{2j} \right) \mathbb{1}\{|x| \leqslant 1\} \end{aligned}$$

where ψ_f is defined via (2.19). We remark that by Theorem 2.4 the second integral appearing above is convergent and it is not hard to see that I_{-d-2k} is rotational invariant. Additionally for $f \in S(\mathbb{R}^d \backslash \{0\}, \mathbb{C})$ one has

$$I_{-d-2k}(f) \;=\; \int_{\mathbb{R}^d} d^d x \; |x|^{-d-2k} f(x).$$

It follows by Propositions 2.6 and 2.7 that for some $K \in \mathbb{C}$ one has that $F - KI_{-d-2k}$ vanishes on all test functions $f \in S(\mathbb{R}^d \backslash \{0\}, \mathbb{C})$ and that for some $N \in \mathbb{N}$ and constants $c_0, \ldots, c_N \in \mathbb{C}$ one has

$$F - KI_{-d-2k} = \sum_{j=0}^{N} c_j \Delta^j \delta. \tag{2.26}$$

We will now try to show that one must have $K = 0$, if this is shown then the result will follow by the real analog of Lemma 2.6 since $\Delta^j \delta$ is homogenous of degree $-d - 2j$. To see this we first observe that I_{-d-2k} is not homogenous of degree $-d - 2k$. A straightforward computation shows that for any $f \in S(\mathbb{R}^d, \mathbb{C})$ and

$\lambda \in (0,\infty)$ one has

$$\lambda^{-d-2k}\hat{S}_\lambda I_{-d-2k}(f) \ = I_{-d-2k}(f) \ + \int_{\mathbb{R}^d} d^dx\ |x|^{-d-2k}\psi_f(|x|)\left[\mathbb{1}\{|x| \geqslant \lambda\} - \mathbb{1}\{|x| \geqslant 1\}\right]$$
$$+ \int_{\mathbb{R}^d} d^dx\ |x|^{-d-2k}\left[\mathbb{1}\{|x| \leqslant \lambda\} - \mathbb{1}\{|x| \leqslant 1\}\right]\left(\psi_f(|x|) - \sum_{j=0}^{k}\frac{\Delta^j f(0)}{2^j j!\left(\prod_{l=1}^{j-1}(d+2l)\right)}|x|^{2j}\right)$$

$$= I_{-d-2k}(f) \ - \ \Omega_d\sum_{j=0}^{k}\frac{\Delta^j f(0)}{2^j j!\left(\prod_{l=1}^{j-1}(d+2l)\right)}\int_1^\lambda dr\ r^{-2k+2j-1}$$
$$= I_{-d-2k}(f) \ - \Omega_d\left[\sum_{j=0}^{k-1}\frac{\Delta^j f(0)}{2^j j!\left(\prod_{l=1}^{j-1}(d+2l)\right)}\frac{(\lambda^{2j-2k}-1)}{2j-2k} + \frac{\Delta^k f(0)}{2^k k!\left(\prod_{l=1}^{k-1}(d+2l)\right)}Log(\lambda)\right]$$

Now for arbitrary $\lambda \in (0,\infty)$ we apply $\left(\lambda^{-d-2k}\hat{S}_\lambda - Id\right)$ to both sides of (2.26) and feed them both an arbitrary $f \in S(\mathbb{R}^d,\mathbb{C})$ which yields

$$K\times\Omega_d\times\left[\sum_{j=0}^{k-1}\frac{\Delta^j\delta(f)}{2^j j!\left(\prod_{l=1}^{j-1}(d+2l)\right)}\frac{(\lambda^{2j-2k}-1)}{2j-2k} + \frac{\Delta^k\delta(f)}{2^k k!\left(\prod_{l=1}^{k-1}(d+2l)\right)}Log(\lambda)\right] = \sum_{j=0}^{N} c_j\times(\lambda^{2j-2k}-1)\times\Delta^j\delta(f)$$

For fixed λ the above equation gives equality between two linear combinations of homogenous distributions and applying the real analog of Lemma 2.6 one has for all $\lambda \in (0,\infty)$

$$K \times \Omega_d \times Log(\lambda) \times \frac{1}{2^k k!\left(\prod_{l=1}^{k-1}(d+2l)\right)} = 0$$

which forces $K = 0$. □

Lemma 2.10. *Let $\alpha \in \mathbb{R}$.*

Then one has

$$\mathcal{F}[H_{\alpha,d}] = 2^{-\alpha-\frac{d}{2}} \times H_{-d-\alpha,d}.$$

Proof: We prove the assertion for $d \geqslant 2$, the one dimensional case follows by a similar argument.

By Lemma 2.8 and Proposition 2.8 it follows that

$$\mathcal{F}[H_{\alpha,d}] \ = \ KH_{-d-\alpha,d}$$

for some $K \in \mathbb{C}$, which we now calculate.

Let $\eta_d(x)$ be the density function for a standard Gaussian on $\mathbb{R}^d$, that is $\eta_d : \mathbb{R}^d \to \mathbb{R}$ is given by

$$\eta_d(x) \ = \ \frac{1}{(2\pi)^{\frac{d}{2}}}e^{-\frac{1}{2}|x|^2}.$$

Clearly $\eta_d \in S(\mathbb{R}^d)$ and with our conventions for defining the Fourier transform on $\mathbb{R}^d$ we have that

$$\mathcal{F}[\eta_d] = \eta_d.$$

We now claim that for all $\alpha \in \mathbb{C}$ one has

$$H_{\alpha,d}(\eta_d) \;=\; 2^{\frac{d+\alpha}{2}-1} \times (2\pi)^{-\frac{d}{2}}. \tag{2.27}$$

Since both sides of the above equation are analytic in α it suffices this equality for real α with $\alpha > -d$, we do this now. We note that

$$\begin{aligned}
\tilde{H}_{\alpha,d}(\eta_d) &= (2\pi)^{-\frac{d}{2}} \times \int_{\mathbb{R}^d} \mathrm{d}^d x \; |x|^{\alpha} e^{-\frac{1}{2}|x|^2} \\
&= (2\pi)^{-\frac{d}{2}} \times \Omega_d \times \int_0^{\infty} \mathrm{d}r \; r^{\alpha+d-1} e^{-\frac{1}{2}r^2} \\
&= (2\pi)^{-\frac{d}{2}} \times \Omega_d \times 2^{\frac{\alpha+d-2}{2}} \times \int_0^{\infty} \mathrm{d}s \; s^{\frac{\alpha+d-2}{2}} \times e^{-s} \\
&= (2\pi)^{-\frac{d}{2}} \times \Omega_d \times 2^{\frac{\alpha+d-2}{2}} \times \Gamma\left(\frac{\alpha+d}{2}\right).
\end{aligned}$$

Recalling the definition given in (2.23) equation (2.27) now follows.

Since

$$H_{-d-\alpha,d}(\eta_d) = 2^{-\frac{\alpha}{2}-1} \times (2\pi)^{-\frac{d}{2}}$$

and

$$\mathcal{F}[H_{\alpha,d}](\eta_d) \;=\; H_{\alpha,d}\left(\mathcal{F}\eta_d\right) \;=\; H_{\alpha,d}(\eta_d)$$

it follows that

$$K \;=\; \frac{2^{\frac{d+\alpha}{2}-1} \times (2\pi)^{-\frac{d}{2}}}{2^{-\frac{\alpha}{2}-1} \times (2\pi)^{-\frac{d}{2}}} \;=\; 2^{\alpha+\frac{d}{2}}$$

□

We now need an analog of Bochner's Theorem called the Bochner-Schwartz Theorem. Compared to Bochner's Theorem the Bochner-Schwartz Theorem has weaker conditions and a weaker conclusion. Instead of applying to continuous functions it says that any *generalized function* which satisfies a certain positive definiteness condition must be Fourier transform of a positive measure - however this measure may not be of finite mass and instead only satisfies a temperedness condition.

We say a Borel measure ν on $\mathbb{R}^d$ is tempered if

$$\int_{\mathbb{R}^d} \mathrm{d}\nu(x)\; (1+|x|)^j < \infty \text{ for some } j > 0.$$

Note that temperedness is a sufficient condition for a Borel measure to define an element of $S'(\mathbb{R}^d, \mathbb{C})$.

Theorem 2.5 (Bochner-Schwartz)**.** *Suppose that $F \in S'(\mathbb{R}^d, \mathbb{C})$. The the two following conditions are equivalent.*

1. $\forall\; g \in S(\mathbb{R}^d, \mathbb{C})$ *one has* $F[g \,\hat{\star}\, \bar{g}] \;\geqslant 0$

2. *There exists a positive Borel measure ν which is tempered such that for any $h \in S(\mathbb{R}^d, \mathbb{C})$ one has*

$$\mathcal{F}[F](h) = \int_{\mathbb{R}^d} \mathrm{d}\nu(x)\ h(x)$$

Proof: One can find a proof [31, Chapter 2, Section 3]. See also [55, Problem 20, Section IX] for an outline of a proof that resembles the one for Bochner's Theorem.

Lemma 2.11. *Let C be a continuous symmetric bilinear form on $S(\mathbb{R}^d)$ that is invariant under simultaneous translations in both arguments. Let $\tilde{C}$ be the corresponding element of $S'(\mathbb{R}^d)$ that satisfies $C(f, g) = \tilde{C}(f \hat{\star} g)$ for all $f, g \in S(\mathbb{R}^d)$. We view $\tilde{C}$ as an element of $S'(\mathbb{R}^d, \mathbb{C})$ in the natural way.*

Then C is positive definite if and only if $\mathcal{F}[\tilde{C}]$ is given by a positive tempered measure μ.

Proof: Clearly C is positive definite if and only if $\tilde{C}[f \hat{\star} f] \geqslant 0$ for all $f \in S(\mathbb{R}^d)$. This can be extended to complex functions using the symmetry of C - see Lemma 2.7. The assertion then follows by Theorem 2.5. □

Proposition 2.9. *The class of continuous, symmetric, positive definite, bilinear forms C on $S(\mathbb{R}^d)$ which are translation invariant, rotation invariant, and κ - scale invariant are precisely given by the following families each parameterized by $c \geqslant 0$:*

- *For $\kappa > 0$ one has*

$$C(f, g) \ = \ c \times \int_{\mathbb{R}^d} \mathrm{d}^d k\ \hat{f}(k) \times \overline{\hat{g}(k)} \times |k|^{-d+2\kappa} \tag{2.28}$$

- *For $\kappa = 0$ one has*

$$C(f, g) \ = \ c \times \left(\int_{\mathbb{R}^d} \mathrm{d}^d x\ f(x) \right) \times \left(\int_{\mathbb{R}^d} \mathrm{d}^d y\ g(y) \right) \tag{2.29}$$

In particular there are no such non-zero bilinear forms for $\kappa < 0$. We remark that here f, g are arbitrary test functions in $S(\mathbb{R}^d)$.

Proof: The reasoning we use here is analogous to the reasoning we used for Proposition 2.4.

First suppose that we are *given* a continuous symmetric bilinear form C that is rotation invariant, translation invariant, and κ-scale invariant with $\kappa \geqslant 0$. Then from the real analog of Lemma 2.3 and Proposition 2.8 it follows that

$$C(f, g) = K H_{-2\kappa, d}(f \hat{\star} g)$$

for some $K \in \mathbb{R}$. We remark that throughout the proof we restrict ourselves to $f, g \in S(\mathbb{R}^d)$, i.e. real valued test functions.

Now from Lemma 2.10 it follows that

$$\begin{aligned} C(f, g) \ = \ \tilde{C}(f \hat{\star} g) \ &= K \times H_{-2\kappa, d}(f \hat{\star} g) \\ &= K \times \left(\mathcal{F}[H_{-2\kappa, d}](k), \mathcal{F}^{-1}[f \hat{\star} g](k) \right) \\ &= K \times 2^{2\kappa - \frac{d}{2}} \times \left(H_{-d+2\kappa, d}](k), \mathcal{F}^{-1}[f \hat{\star} g](k) \right) \\ &= K \times 2^{2\kappa - \frac{d}{2}} \times \Big(H_{-d+2\kappa, d}](k), \hat{f}(-k) \overline{\hat{g}(-k)} \Big) . \end{aligned}$$

Specializing to the case $\kappa = 0$ we have

$$\begin{aligned} C(f,g) &= K \times 2^{-\frac{d}{2}} \times \left(H_{-d,d}(k), \hat{f}(-k)\overline{\hat{g}(-k)}\right) = K \times 2^{-\frac{d}{2}} \times \hat{f}(0)\overline{\hat{g}(0)} \\ &= K \times 2^{-\frac{d}{2}} \times \left(\int_{\mathbb{R}^d} \mathrm{d}^d x \; f(x)\right) \times \left(\int_{\mathbb{R}^d} \mathrm{d}^d y \; g(y)\right) \end{aligned}$$

where we dropped the conjugate over the g since g is real valued. In this case by inspection we see that positive definiteness holds for such the given C if and only if $K > 0$. Thus the given C must be of the form (2.29).

Now for the case $\kappa > 0$ we have

$$\begin{aligned} C(f,g) &= K \times 2^{2\kappa-\frac{d}{2}} \times \left(H_{-d+2\kappa,d}(k), \hat{f}(-k)\overline{\hat{g}(-k)}\right) \\ &= K \times 2^{2\kappa-\frac{d}{2}} \times \frac{1}{\Omega_d \times \Gamma(\kappa)} \times \int_{\mathbb{R}^d} \mathrm{d}^d k \; \hat{f}(-k)\overline{\hat{g}(-k)} \times |k|^{-d+2\kappa}. \end{aligned}$$

Here we used that $-d+2\kappa > -d$. Now by Lemma 2.11 (or by inspection) it follows that C is positive definite if and only if $K \geqslant 0$, in which case C is of the form (2.28).

On the other hand by the calculations above and the lemmas of this section it is clear that bilinear forms given by (2.28) and (2.29) are continuous, translation invariant, rotation invariant, and κ-scale invariant for the given values of κ.

Now if C is a continuous, symmetric, bilinear form on $S(\mathbb{R}^d)$ which is translation invariant, rotationally invariant, and κ scale invariant, with $\kappa < 0$, then proceeding as before we must have

$$\mathcal{F}[\tilde{C}] = K \times 2^{2\kappa-\frac{d}{2}} \times \times H_{-d+2\kappa}.$$

However it is trivial to check that unless $K = 0$ the generalized function $H_{-d+2\kappa}$ is not given by a measure so by Lemma 2.11 C cannot be positive definite. In fact it is not hard to show that for any $K \neq 0$ one can find $f \in S(\mathbb{R}^d)$ such that

$$C(f,f) = K \times 2^{2\kappa-\frac{d}{2}} \times \left(H_{-d+2\kappa,d}(k), |\hat{f}(-k)|^2\right) < 0.$$

□

Chapter 3

Construction of a massless Quantum Field Theory over $\mathbb{Q}_p^3$

3.1 Formal Description of the Model

For $\epsilon \in (0, 1]$ we define $\mu_{C_{-\infty}}$ to be the Gaussian measure on $S'(\mathbb{Q}_p^3)$ with covariance bilinear form $C_{-\infty}$ given by

$$\begin{aligned} C_{-\infty}(f,g) &= \int_{\mathbb{Q}_p^3} d^3k\ \hat{f}(k)\hat{g}(-k) \times |k|^{-\left(\frac{3+\epsilon}{2}\right)} \\ &= \frac{1-p^{-\left(\frac{3+\epsilon}{2}\right)}}{1-p^{-\left(\frac{3-\epsilon}{2}\right)}} \times \int_{\mathbb{Q}_p^3 \times \mathbb{Q}_p^3} d^3x\ d^3y\ f(x)g(y) \times |x-y|^{-\left(\frac{3-\epsilon}{2}\right)} \end{aligned} \tag{3.1}$$

for $f, g \in S(\mathbb{Q}_p^d)$. From Theorem 2.2 we know that $\mu_{C_{-\infty}}$ is (up to a constant of proportionality) the unique translation and rotation invariant Gaussian measure on $S(\mathbb{Q}_p^3)$ that is scale invariant with scaling exponent $\kappa = \frac{3-\epsilon}{4}$. In keeping with conventions of the literature we write $[\phi] := \frac{3-\epsilon}{4}$ to denote the scaling parameter, $[\phi]$ is often called the "dimension of the field". We remark that for our purposes $[\phi]$ is a constant that only depends on ϵ (not some function of ϕ).

We denote by $C_{-\infty}(x)$ the kernel corresponding to $C_{-\infty}(\cdot,\cdot)$.

$$C_{-\infty}(x) = \frac{\chi_\epsilon}{|x|^{\frac{3-\epsilon}{2}}},\ \text{where}\ \chi_\epsilon := \frac{1-p^{-\left(\frac{3+\epsilon}{2}\right)}}{1-p^{-\left(\frac{3-\epsilon}{2}\right)}}$$

and as an operator

$$C_{-\infty} = (-\Delta)^{-\frac{3+\epsilon}{4}}.$$

In particular for $\epsilon = 1$ the measure $\mu_{C_{-\infty}}$ is a p-adic analog of the three dimensional massless Gaussian Free Field. One of the central results of [3] is the construction and analysis of a translation, rotation, and scale invariant *non-Gaussian* measure ν on $S'(\mathbb{Q}_p^3)$ which is defined (in a formal sense) via perturbing $\mu_{C_{-\infty}}$ by the following Radon-Nikodym derivative:

$$\exp\left[-\int_{\mathbb{Q}_p^3} d^3x\ g\phi^4(x) + \mu\phi^2(x)\right]. \tag{3.2}$$

All the results we outline will apply to the *small ϵ regime* and are in the spirit of the Wilson-Fisher $4-\epsilon$ expansion of [75]. The choice of the covariance 3.1 is, as we mentioned in the introduction, inspired by the choice made by Brydges, Mitter, and Scopolla in [18], there the authors used the covariance $(-\Delta)^{-\frac{3+\epsilon}{4}}$ over $\mathbb{R}^3$ to mimic working in non-integer dimensions. Much of our analysis is based on the methods of [18] and so we will call our model the "p-adic BMS model". We mention that an earlier paper [17] also simulated the $4-\epsilon$ expansion via using covariance $(-\Delta)^{-\frac{1+\epsilon}{2}}$ over $\mathbb{R}^4$ but this Gaussian measure is not Osterwalder-Schrader positive.

As mentioned before, the singular nature of the expression (3.2) is due to our Radon-Nikodym derivative involving non-summable interactions between $\mu_{C_{-\infty}}$'s infinitely many degrees of freedom. To deal with these singularities in the form of our pertubation we implement *cut-offs* - phrased differently we will essentially replace the underlying Gaussian $\mu_{C_{-\infty}}$ with a *finite dimensional marginal* and study the consequences of the perturbation there.

3.2 Slicing the covariance $C_{-\infty}$

We now remind the reader of the well-known fact that decompositions of a covariance form into a sum of covariance forms yields a decomposition of a Gaussian process as a sum of independent Gaussian Processes. Concretely if the Gaussian process X is given by a covariance C and one $C = \Gamma_1 + \Gamma_2$ for Γ_1, Γ_2 both covariance forms then it follows that the process X can be realized as the sum of two independent Gaussian process Y_1 and Y_2 with Y_i distributed according to Γ_i for $i = 1, 2$.

With this in mind we start to decompose the $C_{-\infty}$ in order to see it as a multitude of degrees of freedom across different scales. We have

$$\widehat{C}_{-\infty}(k) = \sum_{j=-\infty}^{\infty} \mathbb{1}\left\{p^{-j-1} < |k| \leqslant p^{-j}\right\} |k|^{-\left(\frac{3+\epsilon}{2}\right)}.$$

Writing the same decomposition in terms of position variables gives

$$\begin{aligned} C_{-\infty}(x) &= \sum_{j=-\infty}^{\infty} p^{-2j[\phi]}\left[\mathbb{1}_{\mathbb{Z}_p^3}(p^j x) - p^{-3}\mathbb{1}_{\mathbb{Z}_p^3}(p^{j+1}x)\right] \\ &= \sum_{j=0}^{\infty} p^{-2j[\phi]}\ \tilde{\Gamma}(p^j x) \end{aligned} \tag{3.3}$$

where we have defined

$$\tilde{\Gamma}(x) := \mathbb{1}_{\mathbb{Z}_p^3}(x) - p^{-3}\mathbb{1}_{\mathbb{Z}_p^3}(px).$$

We remark $\tilde{\Gamma}(x-y)$ is also a covariance kernel - symmetry is immediate and positive definiteness is a consequence of the fact that

$$\hat{\tilde{\Gamma}}(k) = \mathbb{1}\left\{p^{-1} < |k| \leqslant 1\right\} |k|^{-\left(\frac{3+\epsilon}{2}\right)} \geqslant 0$$

Let $\mu_{\tilde{\Gamma}}$ be the corresponding Gaussian measure on $S'(\mathbb{Q}_p^3)$. We will later see that $\mu_{\tilde{\Gamma}}$ is in fact supported on a space of locally constant *functions* $f : \mathbb{Q}_p^3 \to \mathbb{R}$ lying within $S'(\mathbb{Q}_p^3)$. This follows from $\tilde{\Gamma}$ being locally constant - see Proposition 3.1.

We also remark that $\tilde{\Gamma}$ is an example of a *finite-range* covariance, i.e. it vanishes at some distance away from the diagonal. It is easy to see that $\tilde{\Gamma}(x)$ is supported on $p^{-1}\mathbb{Z}_p^3$. It follows that for ξ distributed according to μ_Γ one has

$$|x-y| \ > \ p \ \Rightarrow \ \mathbb{E}[\xi(x)\xi(y)] \ = 0$$

this means for such x, y the Gaussian random variables $\xi(x)$ and $\xi(y)$ are independent.

Now if ϕ is a generalized random field distributed according to $\mu_{C_{-\infty}}$ we can realize ϕ as a bi-infinite sum

$$\phi(x) = \sum_{j=-\infty}^{\infty} p^{-j[\phi]} \xi_j(p^j x) \tag{3.4}$$

where the random fields $\{\xi_j\}_{j\in\mathbb{Z}}$ are independently and identically distributed according to $\mu_{\tilde{\Gamma}}$. Note that while each summand on the right hand side makes sense point-wise in x, the sum as whole only makes sense as a distribution. This presents the UV divergence as the non-summability of fluctuations at arbitrarily small length scales.

The index j parameterizes our length scales logarithmically (base p) where $j \to -\infty$ corresponds to short distance (high fourier mode) behaviour involving rougher fields with short range correlations and $j \to \infty$ corresponds to long-range (low fourier mode) behaviour involving smoother fields with long distance correlations.

3.3 Implementation of Cut-offs

At this step we introduce an artificial scaling factor $L = p^l$ for l some positive integer. L will determine our step-size in the multiscale analysis that follows - i.e. how many degrees of freedom we integrate out in each iteration. We will see that analytic control over the RG map will require taking l quite large. A key remark to make is that L is not an intrinsic length scaling factor for our system - that role is played by p. This is one difference in our setting compared to some previous RG work on hierarchical models. A failure to make this distinction can lead to critical exponents and universality classes that depend on L - see [47, Section 5.2].

We define

$$\Gamma(x) = \sum_{j=0}^{l-1} p^{-2j[\phi]} \left[\mathbb{1}_{\mathbb{Z}_p^3}(p^j x) - p^{-3}\mathbb{1}_{\mathbb{Z}_p^3}(p^{j+1} x)\right] \tag{3.5}$$

so that

$$C_{-\infty}(x) \ = \ \sum_{j=-\infty}^{\infty} L^{-2j[\phi]}\Gamma(L^j x). \tag{3.6}$$

In the rest of this chapter our scale indices will be given in terms of L instead of p as before. For any

$r \in \mathbb{Z}$ we define the covariance with UV cut-off at scale r to be given by

$$C_r(x) \ = \ \sum_{j=r}^{\infty} L^{-2j[\phi]}\Gamma(L^j x). \tag{3.7}$$

Equivalently we could have defined C_r via $\hat{C}_r(k) = \mathbb{1}\{|k| \leqslant L^{-r}\}\,\hat{C}_{-\infty}(k)$. We remark that $C_r(x)$ is locally constant over the translates of $L^{-r}\mathbb{Z}_p^3$.

As we alluded to before locally constant covariances will give rise to measures on locally constant field. In preparation for that we introduce some notation. We set $\mathbb{L}_q = \mathbb{Q}_p^3/(L^{-q}\mathbb{Z}_p^3)$, i.e. $\mathbb{L}_q$ is the lattice of translates of $L^{-q}\mathbb{Z}_p^3$ which we individually call blocks. The unit lattice $\mathbb{L}_0$ will simply be denoted by $\mathbb{L}$. We will typically denote an element of the latter by Δ (not to be confused with the Laplacian). We will call such an element a unit cube, a unit block or simply a box. We will call the elements of $\mathbb{L}_1$ L-blocks, and note that every L-block is of the form $L^{-1}\Delta$ for some $\Delta \in \mathbb{L}$. We remark that every L-block can be partitioned into L^3 distinct unit blocks. For an L-block $\square \in \mathbb{L}_1$ we write $[\square]$ for the set of L^3 unit boxes Δ contained in $\square$.

The following proposition shows that the Gaussian measure μ_{C_r} on $S'(\mathbb{Q}_p^3)$ is in fact supported on functions that are constant over the blocks of $\mathbb{L}_r$.

Proposition 3.1. *Let $\Omega(x,y) : \mathbb{Q}_p^d \times \mathbb{Q}_p^d \to \mathbb{R}$ be a covariance kernel and μ_Ω be the corresponding measure on $S'(\mathbb{Q}_p^d)$. Suppose that there exists some $r \in \mathbb{Z}$ such that for any $z_1, z_2 \in \mathbb{Q}_p^d$ with $|z_1|, |z_2| \leqslant L^r$ one has*

$$\Omega(x + z_1, y + z_2) = \Omega(x,y)$$

Then μ_Ω is supported on the set

$$\{F \in S'(\mathbb{Q}_p^d)|\ F(x) = \sum_{\Delta \in \mathbb{L}_r} \alpha_\Delta \mathbb{1}_\Delta(x) \text{ where } \alpha_\Delta \in \mathbb{R}\}$$

where for $\Delta \in \mathbb{L}_r$ we write $\mathbb{1}_\Delta(x)$ for the corresponding characteristic function.

Proof: In some sense this is a p-adic analog of the Kolmogorov Continuity Theorem which links regularity of a covariance to regularity of sample paths/fields. However in the p-adic case there is almost nothing to show. Due to the local constancy of Ω one can view it as a map $\Gamma : \mathbb{L}_r \times \mathbb{L}_r \to \mathbb{R}$ which satisfies the necessary symmetry and positive definiteness to determine (via the Kolmogorov Extension Theorem) a Gaussian measure $\tilde{\mu}$ on the direct product

$$\prod_{\Delta \in \mathbb{L}_r} \mathbb{R} = \{\{\alpha_\Delta\}_{\Delta \in \mathbb{L}_r}\}$$

where $\mathbb{E}_{\tilde{\mu}}[\alpha_\Delta \alpha_{\Delta'}] = \Omega(\Delta, \Delta')$. The assertion then follows from noting that the map

$$\{\alpha_\Delta\}_{\Delta \in \mathbb{L}_r} \to \sum_{\Delta \in \mathbb{L}_r} \alpha_\Delta \mathbb{1}_\Delta$$

is a measurable map from $\prod_{\Delta \in \mathbb{L}_r} \mathbb{R}$ to $S'(\mathbb{Q}_p^d)$ where both spaces are equipped with the cylinder σ-algebras. The measure μ_Ω can then be realized as the pushforward of $\tilde{\mu}$ under the above map whose codomain is clearly the support set given in our assertion. $\square$

We remark that our UV cut-off has left some of the invariances of $\mu_{C_{-\infty}}$ intact - μ_{C_r} is still translation and rotation invariant but we have lost scale invariance.

The infrared behaviour of $C_{-\infty}$ is not severe enough to *require* regularizing the sum (3.6) for large j (in contrast with trying to define the Gaussian Free Field with covariance $(-\Delta)^{-1}$ for $d = 1$ or 2). However the integral over all of $\mathbb{Q}_p^3$ in (3.2) must be regularized - otherwise an argument via Jensen's inequality shows that this aspiring Radon-Nikodym derivative is μ_{C_r} almost surely 0.

For $s \in \mathbb{Z}$ we set $\Lambda_s := \{x \in \mathbb{Q}_p^3 | \; |x| \leqslant L^s\}$, i.e. $\Lambda_s = L^{-s}\mathbb{Z}_p^3$. Now for any $r, s \in \mathbb{Z}$ the functional

$$\exp\left[-\int_{\Lambda_s} d^3x \; g \; \phi^4(x) + \mu \; \phi^2(x)\right] \tag{3.8}$$

is strictly positive, bounded, and basically well-defined on the support of $\mu_{C_r}(\phi)$ within $S'(\mathbb{Q}_p^3)$ - thus modulo a (finite) normalization the functional (3.8) is the Radon-Nikodym derivative with respect to μ_{C_r} for some probability measure which for now we call $\nu_{r,s}$. We remark that if one reduces to marginals corresponding to our field's values within Λ_s the measures μ_{C_r} and $\nu_{r,s}$ are measures on the finite dimensional space $\mathbb{R}^N$ where $N = L^{3(s-r)}$ (here we assume assume that $s \geqslant r$).

The construction and analysis of ν entails control over the removal of the cut-offs, i.e. we want to show the measures $\nu_{r,s}$ converge in some sense as we take $r \to -\infty$ and $s \to \infty$ to a non-Gaussian measure with the desired invariance properties.

In order for this to be successful we will see in what follows that it will be necessary to replace the fixed parameters g and μ inside the integrand in (3.8) with some parameters $\tilde{g}_r$, $\tilde{\mu}_r$ that are allowed to depend on the ultraviolet cut-off r - these are called "bare couplings". As a matter of convenience we will also change the role of these parameters by rewriting the integrand as a linear combination of "Hermite" (Wick) polynomials which we define now.

Definition. *Let Ω be some set and C be a function $C : \Omega \times \Omega \to \mathbb{R}$ which is symmetric (i.e. $C(x, y) = C(y, x)$ for all $x, y \in \Omega$).*

Let $\mathcal{P}$ be the set of polynomials in commuting indeterminates $\{\phi(x)\}_{x \in \Omega}$ with coefficients in $\mathbb{R}$, i.e.

$$\mathcal{P} := \mathbb{R}\left[\{\phi(x)\}_{x\in\Omega}\right].$$

We define a map $: \bullet :_C$: taking $\mathcal{P} \to \mathcal{P}$ as follows. For $P \in \mathcal{P}$ we set

$$: P(\phi) :_C \;\; := \;\; e^{-\Delta_C} P(\phi)$$

where $\Delta_C : \mathcal{P} \to \mathcal{P}$ is the differential operator defined via

$$\Delta_C \;\; := \;\; \frac{1}{2} \sum_{x,y\in\Omega} C(x, y) \times \frac{\delta}{\delta\phi(x)} \frac{\delta}{\delta\phi(y)}$$

and $e^{-\Delta_C} : \mathcal{P} \to \mathcal{P}$ is then defined via

$$e^{-\Delta_C} := \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (\Delta_C)^n.$$

We will choose Ω to be index set for some Gaussian process on a product space and C is chosen to be the corresponding covariance. In this setting the map $P(\phi) \to : P(\phi) :_C$ replaces a linear combination of monomials with the same linear combination of corresponding partially "orthogonalized " monomials in $L^2(\mathbb{R}^\Omega, \mu_C)$.

Lemma 3.1. *Let $\{\phi(x)\}_{x\in\Omega}$ be a Gaussian process indexed by Ω with covariance C.*

Then for any $x_1, \ldots, x_m, y_1, \ldots, y_n \in \Omega$ (not necessarily distinct) one has

$$\mathbb{E}_{\mu_C}\left[\left(:\prod_{i=1}^{m}\phi(x_i):_C\right)\left(:\prod_{j=1}^{n}\phi(y_j):_C\right)\right] = \begin{cases} 0 & \text{if } m \neq m \\ \sum_{\sigma\in S_n}\left(\prod_{j=1}^{n}\mathbb{E}_{\mu_C}[\phi(x_i)\phi(y_{\sigma(i)})]\right) & \text{if } m = n \end{cases}$$

In the bottom sum we are summing over permutations on n-elements - the resulting expression is similar to Wick's rule for Gaussian expectations except that Wick contractions internal to a single Wick ordered monomial are not allowed. For the case Ω a singleton and $C = 1$ one has that $: \phi^n :_C$ is just the n-th Hermite polynomial in ϕ. In the more general context the transformation $: \bullet :_C$ is called *Wick* ordering.

For our Wick ordering we set $\Omega = \mathbb{Q}_p^3$ (or equivalently $\mathbb{L}_r$) and $C(x, y) = C_r(x - y)$. Some of the key Wick monomials for us will be

$$: \phi^4(x) :_C = \phi^4(x) - 6C_r(0)\phi^2(x) + 3C_r(0)^2 \tag{3.9}$$

and

$$: \phi^2(x) :_C = \phi^2(x) - C_r(0). \tag{3.10}$$

We can now give a precise formula for our cut-off measures. Given a sequence of bare couplings $\{\tilde{g}_r, \tilde{\mu}_r\}_{r\in\mathbb{Z}}$ (where $\tilde{g}_r > 0$, $\tilde{\mu}_r \in \mathbb{R}$) we define, for $r, s \in \mathbb{Z}$, the measures $\nu_{r,s}$ on $\mathcal{S}'(\mathbb{Q}_p^3)$ via

$$d\nu_{r,s}(\phi) \;=\; \frac{1}{\mathcal{Z}_{r,s}}\exp\left[-\int_{\Lambda_s} d^3x\ \{\tilde{g}_r\ : \phi^4(x) :_{C_r}\ +\tilde{\mu}_r\ : \phi^2(x) :_{C_r}\}\right] d\mu_{C_r}(\phi)$$

where the normalization factor $\mathcal{Z}_{r,s}$ is chosen to make $\nu_{r,s}$ a probability measure, i.e.

$$\mathcal{Z}_{r,s} \;=\; \int_{S'(\mathbb{Q}_p^3)} d\mu_{C_r}(\phi)\exp\left[-\int_{\Lambda_s} d^3x\ \{\tilde{g}_r\ : \phi^4(x) :_{C_r}\ +\tilde{\mu}_r\ : \phi^2(x) :_{C_r}\}\right].$$

We will also choose $\tilde{g}_r, \tilde{\mu}_r$ to be of the form

$$\tilde{g}_r = L^{-\epsilon r}g\ , \tilde{\mu}_r = L^{-(\frac{3+\epsilon}{2})r}\mu \text{ for some } g > 0, \mu \in \mathbb{R}. \tag{3.11}$$

The choice (3.11) corresponds to an attempt to construct a measure that looks the same at all scales (using appropriate units at each scale).

It is not hard to see that $\mathcal{Z}_{r,s} \in (0, \infty)$. Introducing the short hand

$$\tilde{V}_{r,s}(\phi) \;=\; \int_{\Lambda_s} d^3x\ \{\tilde{g}_r\ : \phi^4(x) :_{C_r}\ +\tilde{\mu}_r\ : \phi^2(x) :_{C_r}\}$$

by Lemma 3.1 one has that

$$\int_{S'(\mathbb{Q}_p^3)} d\mu_{C_r}(\phi)\ \tilde{V}_{r,s}(\phi) = 0.$$

Since e^{-t} is convex in t it follows by Jensen's inequality that $\mathcal{Z}_{r,s} \geqslant 1$. On the other hand since $\tilde{g}_r > 0$ and recalling definitions (3.10) and (3.9) it is immediate that $\tilde{V}_{r,s}(\phi)$ is bounded below which gives an upper bound on $\mathcal{Z}_{r,s}$ for fixed r, s.

3.4 Sketch of general strategy and main results

We start by introducing the main quantities we will achieve analytic control over in order to prove our main results. The measure ν on $S'(\mathbb{Q}_p^3)$ will be constructed via solving a moment problem, the candidates for the moments will be the limits of corresponding moments of $\nu_{r,s}$. Denoting $\langle\cdot\rangle$ for "expectation with respect to ν" and $\langle\cdot\rangle_{r,s}$ for expectation with respect to $\nu_{r,s}$ for $f_1,\ldots,f_n \in S(\mathbb{Q}_p^3)$ we define candidate moments for ν via

$$\left\langle \prod_{j=1}^{n} \phi(f_j) \right\rangle := \lim_{\substack{r\to-\infty \\ s\to\infty}} \left\langle \prod_{j=1}^{n} \phi(f_j) \right\rangle_{r,s}. \tag{3.12}$$

With our methods the order of removing the UV-cutoffs and IR cut-offs does not need to be prescribed. The key qualities needed for our candidate moments on the left hand side of (3.12) to be actual moments are symmetry in the f_j's, a positive definiteness condition, and certain $n!$ bounds (see Theorem 1.12).

The first two of these immediately follow after one has established convergence of the right hand side of (3.12). With regards to the $n!$ bounds one needs to show that for *each finite dimensional subspace* $U \subset S(\mathbb{Q}_p^3)$ there exists K_U such that for any $n \in \mathbb{N}$, and some seminorm $\mathcal{N}_U$ on $S(\mathbb{Q}_p^3)$ one has

$$\left| \left\langle \prod_{j=1}^{n} \phi(f_j) \right\rangle \right| \leqslant K_U^n \times n! \times \prod_{j=1}^{n} \mathcal{N}_U(f_j) \tag{3.13}$$

for all choices of $f_1,\ldots,f_n \in U$. Clearly it is sufficient to establish the estimate (3.13) for all U of the form S_{q_-,q_+} for $q_-, q_+ \in \mathbb{Z}$ where

$$S_{q_-,q_+}(\mathbb{Q}_p^3) := \left\{ f \in S(\mathbb{Q}_p^3) :\ f \text{ constant on translates of } L^{-q_-}\mathbb{Z}_p^3 \text{ and } \operatorname{supp}(f) \subset L^{-q_+}\mathbb{Z}_p^3 \right\}.$$

This conflicts with earlier notation of section 1.2 but is consistent with scale indices being given in terms of L, not p. $S_{q_-,q_+}(\mathbb{Q}_p^3, \mathbb{C})$ will denote the corresponding space of complex valued test functions. We will see that analyticity methods will be used to establish *both* the convergence of the moments and the estimates (3.13) for any subspace S_{q_-,q_+}. For $f \in S(\mathbb{Q}_p^3, \mathbb{C})$ we define

$$\mathcal{Z}_{r,s}(f) = \int_{S'(\mathbb{Q}_p^3)} d\mu_{C_r}(\phi) \exp\left[-\int_{\Lambda_s} d^3x\ \left(L^{-\epsilon r}\ g : \phi^4(x) :_{C_r} + L^{-\frac{(3+\epsilon)}{2} r} \mu\ : \phi^2(x) :_{C_r} - \phi(x) f(x) \right) \right]$$

so that

$$\mathcal{S}_{r,s}(f) := \frac{\mathcal{Z}_{r,s}(f)}{\mathcal{Z}_{r,s}(0)} = \langle \exp[\phi(f)] \rangle_{r,s}$$

is the moment generating function for the measure $\nu_{r,s}$. It is not hard to see that for any $q_-, q_+ \in \mathbb{Z}$ one has that $\mathcal{S}_{r,s}(\cdot)$ seen as a function on the finite dimensional complex vector space $\mathcal{S}_{q_-,q_+}(\mathbb{Q}_p^3, \mathbb{C})$ is entire. In particular we can express n-th moments of $\nu_{r,s}$ as n-th order functional derivatives at $f = 0$:

$$\left\langle \prod_{j=1}^{n} \phi(f_j) \right\rangle_{r,s} = \left(\prod_{j=1}^{n} \frac{\partial}{\partial t_j} \right) \Bigg|_{t_1 = \cdots = t_n = 0} \mathcal{S}_{r,s} \left(\sum_{j=1}^{n} t_j f_j \right).$$

We equip the finite dimensional complex vector space $S_{q_-,q_+}(\mathbb{Q}_p^3, \mathbb{C})$ with the supremum norm. The convergence of moments and required estimate (3.13) would follow if one establishes the following result

For every $q_- \leqslant q_+ \in \mathbb{Z}$ there exist $\delta_{q_-,q_+} > 0$ such that the functions $\mathcal{S}_{r,s}(\cdot)$ converge uniformly to a limiting function $\mathcal{S}(\cdot)$ on the ball $\mathcal{B}(0, \delta_{q_-,q_+}) \subset S_{q_-,q_+}(\mathbb{Q}_p^3, \mathbb{C})$

The convergence of the moments is immediate and the bound (3.13) follows by a Cauchy estimate. Our RG analysis will establish this uniform convergence as a consequence of establishing a particular expansion across scales for $\mathcal{S}_{r,s}(f)$ where each term corresponds to the contributions from the fluctuations assigned to one of the $r - s$ length scales.

However our methods can show the validity and uniform convergence of these expansions only for sufficiently small f (where the degree of smallness) will depend on the length scales that f lives on - i.e. how small and large q_- and q_+ must be so that $f \in S_{q_-,q_+}(\mathbb{Q}_p^3, \mathbb{C})$). This smallness requirement is the reason that we use a moment argument to construct the measure ν (as opposed to constructing the characteristic function).

Note that ν automatically inherits both translation and rotation invariance - this is a consequence of the fact that the measure $\nu_{r,s}$ is rotation invariant and due to ultrametricity is invariant under translations τ_z for $|z| \leqslant L^s$. The measure ν will also be partially scale-invariant, in particular one will have $\left(\lambda^{[\phi]} \hat{S}_\lambda\right)^{\#} \nu = \nu$ for all $\lambda \in L^{\mathbb{Z}}$. This follows from the forms we have chosen for the $\nu_{r,s}$ (in particular our choices of $\tilde{g}_r$ and $\tilde{\mu}_r$). It can also be seen as a shift in the expansion we establish for $\mathcal{S}(f)$.

The RG approach is also suited to the construction of a second measure on $S'(\mathbb{Q}_p^3)$ which heuristically corresponds to the law of "ϕ^2" for ϕ distributed according to ν - we will denote this measure by ν_{ϕ^2} and will sometimes denote the original measure ν by ν_ϕ. Higher powers of the field ϕ are called "composite fields" or "operator products" [72]. Since ν_ϕ will not be be supported on actual functions making sense of pointwise products like "ϕ^2" is far from clear. In order to see how to overcome this problem it is instructive to look at how one can make sense of of "ϕ^2" for ϕ distributed according to $\nu_{C_{-\infty}}$.

The strategy we describe is to implement cut-offs and make substractions. Note that as $r \to -\infty$ the quantity $\mathbb{E}_{\mu_{C_r}}[\phi^2(x)] = C_r(0)$ diverges like $L^{-2[\phi]r}$. Wick ordering can then be seen as a method of UV regularization by subtracting divergent counter terms coming from self-contractions. If ϕ is distributed according to μ_{C_r} then $:\phi^2(\cdot):_{C_r}$, seen as a random element of $\mathcal{S}'(\mathbb{Q}_p^3)$, converges in the sense of a moments to a random generalized function which we denote $:\phi^2:_{C_{-\infty}}$ (see [62, §V.1] for details). Writing $\mathbb{E}$ for the expectation with respect to the law of $:\phi^2:_{C_{-\infty}}$ then one has for any $f, g \in S(\mathbb{Q}_p^3)$

$$\mathbb{E}\left[:\phi^2:_{C_{-\infty}}(f) \ :\phi^2:_{C_{-\infty}}(g) \right] = \int_{\mathbb{Q}_p^3 \times \mathbb{Q}_p^3} \mathrm{d}^3x \ \mathrm{d}^3y \ f(x) C_{-\infty}(x-y)^2 g(y),$$

in particular $:\phi^2:_{C_{-\infty}}$ is a non-degenerate process. $:\phi^2:_{C_{-\infty}}$ is called a "normal ordered field" and we see that in case of the free field normal ordering is essentially given by Wick ordering. In fact when defining perturbations of the form (3.2) in dimension $d = 2$ with respect to the massive Gaussian Free Field (the Gaussian measure on S' with covariance $(-\Delta + m^2)^{-1}$ for some $m^2 > 0$) the problem of ultraviolet singularities is reduced to Wick ordering - one can define any power of the field similarly to how we defined $:\phi^2:_{C_{-\infty}}$ and this makes a perturbation of the form (3.2) well defined (when in finite volume) with respect to the Gaussian Free Fields *without any UV regularization* when ϕ^4 and ϕ^2 are replaced by their Wick ordered counterparts. However the ultraviolet divergence in higher dimensions tends to be more severe - from $d = 3$ up until $4 - \epsilon$ dimensions this can only be done for the ϕ^2 and for $d \geqslant 4$ one does not expect to be able to define any Wick ordered powers of the Gaussian Free Field as random elements of S'.

Our approach to defining the normal ordered ϕ^2 with respect to the ν measure involves both an additive and multiplicative renormalization. For ϕ distributed according to μ_{C_r} we define $N_r[\phi^2](x)$ via

$$N_r[\phi^2](x) \;=\; Z_2^r Y_2 :\phi^2(x):_{C_r} - Y_0 Z_0^r$$

where Z_2, Y_2, Z_0, and Y_0 are parameters which will be chosen based on the RG analysis. Constructing the normal ordered ϕ^2 with respect to our measure ν_ϕ now corresponds to showing convergence of the law of $N_r[\phi^2]$ as $r \to -\infty$ to a limiting measure ν_{ϕ^2} on $S'(\mathbb{Q}_p^3)$ - the corresponding limiting process will be denoted by $N[\phi^2]$.

We will simultaneously construct the measures ν_ϕ and ν_{ϕ^2} by applying our yet to be defined RG map to generate expansions, uniform bounds, and convergence for the *joint* moment generating functions:

$$\mathcal{S}_{r,s}(\tilde{f},\tilde{j}) \;:=\; \frac{\mathcal{Z}_{r,s}(\tilde{f},\tilde{j})}{\mathcal{Z}_{r,s}(0,0)} = \left\langle \exp[\phi(\tilde{f}) + N_r(\tilde{j})] \right\rangle_{r,s} \tag{3.14}$$

where

$$\mathcal{Z}_{r,s}(\tilde{f},\tilde{j}) = \int_{S'(\mathbb{Q}_p^3)} d\mu_{C_r}(\tilde{\phi}) \exp\left[-\int_{\Lambda_s} d^3x \left\{ L^{-\epsilon r}\, g :\tilde{\phi}^4(x):_{C_r} + L^{-\frac{(3+\epsilon)}{2} r} \mu :\tilde{\phi}^2(x):_{C_r} - \tilde{\phi}(x)\tilde{f}(x) - N_r[\tilde{\phi}^2](x)\tilde{j}(x) \right\} \right] \tag{3.15}$$

and $\tilde{f}, \tilde{j} \in S(\mathbb{Q}_p^3, \mathbb{C})$ (the introduction of tildes is mostly vacuous and only preparation for an upcoming changes of variable).

The main theorem of [3] is given below:

Theorem 3.1.

$\exists \rho > 0$, $\exists L_0$, $\forall L \geqslant L_0$, $\exists \epsilon_0 > 0$, $\forall \epsilon \in (0, \epsilon_0]$, one can find $\eta_{\phi^2} > 0$ and functions $\mu(g), Y_0(g), Y_2(g)$ of g in the interval $(\bar{g}_* - \rho\epsilon^{\frac{3}{2}}, \bar{g}_* + \rho\epsilon^{\frac{3}{2}})$, where

$$\bar{g}_* = \frac{p^\epsilon - 1}{36 L^\epsilon (1 - p^{-3})}, \tag{3.16}$$

such that if one sets $\mu = \mu(g)$, $Z_2 = L^{-\frac{1}{2}\eta_{\phi^2}}$, $Z_0 = Z_2 L^{-2[\phi]}$, $Y_0 = Y_0(g)$ and $Y_2 = Y_2(g)$ in the previous definitions, then for all collections of test functions $f_1, \ldots, f_n, j_1, \ldots, j_m$, the limits

$$\left\langle \phi(f_1)\cdots\phi(f_n) N[\phi^2](j_1)\cdots N[\phi^2](j_m) \right\rangle_{r,s} := \lim_{\substack{r\to-\infty \\ s\to\infty}} \left\langle \phi(f_1)\cdots\phi(f_n) N_r[\phi^2](j_1)\cdots N_r[\phi^2](j_m) \right\rangle_{r,s}$$

exist and do not depend on the order in which the $r \to -\infty$ and $s \to \infty$ limits are taken. Moreover, the candidate moments (which we call correlators) satisfy the following properties:

1) They are invariant by any translations or rotations of all the test functions $f_1, \ldots, f_m, j_1, \ldots, j_m$.

2) They satisfy the partial scale invariance property

$$\left\langle \phi\left(S_\lambda f_1\right) \cdots \phi\left(S_\lambda f_n\right)\ N[\phi^2]\left(S_\lambda j_1\right) \cdots N[\phi^2]\left(S_\lambda j_m\right)\right\rangle =$$

$$|\lambda|_p^{(3-[\phi])n+(3-2[\phi]-\frac{1}{2}\eta_{\phi^2})m} \left\langle \phi(f_1) \cdots \phi(f_n)\ N[\phi^2](j_1) \cdots N[\phi^2](j_m)\right\rangle$$

for all $\lambda \in L^{\mathbb{Z}}$.

3) They satisfy the nontriviality conditions

$$\langle \phi(\mathbb{1}_{\mathbb{Z}_p^3})^4 \rangle - 3\langle \phi(\mathbb{1}_{\mathbb{Z}_p^3})^2 \rangle < 0\ ,$$

$$\langle N[\phi^2](\mathbb{1}_{\mathbb{Z}_p^3})^2 \rangle = 1\ .$$

4) The pure ϕ correlators are the moments of a unique probability measure ν_ϕ on $S'(\mathbb{Q}_p^3)$ with finite moments. This measure is translation and rotation invariant. It is also partially scale invariant with scaling parameter $[\phi]$ with respect to the scaling subgroup $L^{\mathbb{Z}}$ - i.e. $\left(|\lambda|_p^{-[\phi]}\hat{S}_\lambda\right)^{\#}\nu_\phi = \nu_\phi$ for all $\lambda \in L^Z$

5) The pure $N[\phi^2]$ correlators are the moments of a unique probability measure ν_{ϕ^2} on $S'(\mathbb{Q}_p^3)$ with finite moments. This measure is translation and rotation invariant. It is also partially scale invariant with scaling parameter $2[\phi] + \frac{1}{2}\eta_{\phi^2}$ with respect to the scaling subgroup $L^{\mathbb{Z}}$ - i.e. $\left(|\lambda|_p^{-(2[\phi]+\frac{1}{2}\eta_{\phi^2})}\hat{S}_\lambda\right)^{\#}\nu_{\phi^2} = \nu_{\phi^2}$ for all $\lambda \in L^{\mathbb{Z}}$.

6) The measures ν_ϕ and ν_{ϕ^2} satisfy a mild form of universality: they do not depend on g in the above-mentioned interval.

Our first remark is that a major weakness of the above result is the restriction of the scaling group to $L^{\mathbb{Z}}$ instead of the full scaling group $p^{\mathbb{Z}}$. Unfortunately it is expected that $L_0 > p$ so different techniques are required to upgrade the scale invariance given by the Theorem above - this is the focus of Chapter 4.

The parameter η_{ϕ^2} is called the anomalous dimension of the composite field $N[\phi^2]$, it represents the discrepancy in scaling between $N[\phi^2]$ and the corresponding power of the free field $:\phi^2:_{C_{-\infty}}$. The law of the elementary field ν_ϕ has the same scaling parameter as the free field $\mu_{C_{-\infty}}$, that is we have $\eta_\phi = 0$. The absence of an anomalous dimension for the elementary field ϕ and the presence of the anomalous dimension for the composite field $N[\phi^2]$ agree with predictions made for the real BMS model and for an analogous model [73]. In fact the value of $\eta_{\phi^2} = \frac{2}{3}\epsilon + o(\epsilon)$ agrees with the calculation Wilson made in [73].

However in certain computations in the $4-\epsilon$ expansion one sees anomalous dimension for elementary field ϕ of order ϵ^2 [76, p. 133] while both the p-adic and real BMS models are not able to capture this behaviour. This is a shortcoming of using fractional Laplacians to mimic non-integer space-times. One expects such an anomalous dimension to develop due to wave-function renormalizations, i.e. the flow of a $|\nabla\phi|^2$ term which are comparable to the quantity $(\phi, -\Delta\phi)$ which is responsible for the Gaussian measure. This flow is dealt with by multiplicative renormalizations of ϕ which cumulatively lead to a change in scaling behaviour.

The p-adic models, like other implementations of hierarchical RGs, are not expected to manifest wave function renormalization. One does not expect appearance of derivatives in the flow, in fact derivatives are usually not natural to define on these spaces. Additionally the usual mechanism for the generation of derivatives, that is writing the difference of nonlocal relevant terms with local subtractions, is not applicable to the hierarchical case because there we keep locality.

However even the RG flow for the real BMS models does not generate wave function renormalizations - here the Gaussian measure comes from a term $(\phi, (-\Delta)^{\frac{3+\epsilon}{4}}\phi)$ and no terms like this will be produced by the RG flow. Additionally if one puts in a gradient squared term by hand one will see, by power counting when $\epsilon \in (0,1)$, that such a term is an irrelevant operator (it will be washed out by the iteration of the RG transformation).

3.4.1 Heuristic overview of the RG transformation and associated expansion

In this subsection we sketch how the RG allows us to expand the partition functions $\mathcal{Z}_{r,s}(\tilde{f},\tilde{j})$. Applying this separately to the numerator and denominator of (3.14) then yields the expansion of $\mathcal{S}_{r,s}(\tilde{f},\tilde{j})$. We assume that $\tilde{f},\tilde{j} \in S_{q_-,q_+}(\mathbb{Q}_p^3,\mathbb{C})$ for some fixed q_- and q_+ and we assume that $r \leqslant q_-$ and $s \geqslant q_+$. Again we remark that we want to control the $r \to -\infty$, $s \to \infty$ limit so one should think of r as being a negative integer of large magnitude and s being a positive integer of large magnitude.

Our first step is to perform a change of variable so we replace our very rough field distributed according to μ_{C_r} with a smoother one distributed according to μ_{C_0}.

An important fact is that our covariances with different UV cut offs are related via scale transformations - for any $r \in \mathbb{Z}$ one has $C_r(x-y) = L^{-2r[\phi]}C_0(L^r x - L^r y)$. As a direct consequence of this if $\tilde{\phi}$ is a random field distributed according to C_r and ϕ is a random field distributed according to C_0 then $\tilde{\phi}(\cdot) \overset{d}{=} L^{-r[\phi]}\phi(L^r\cdot)$.

By applying this change of variables in ϕ and also applying a change of variable $L^r x \leftrightarrow x$ for the integral over $\mathbb{Q}_p^3$ we arrive at

$$\mathcal{Z}_{r,s}(\tilde{f},\tilde{j}) = \exp\left(-Y_0 Z_0^r L^{2[\phi]r}\int_{\mathbb{Q}_p^3} j(x)\ \mathrm{d}^3x\right) \times \int_{S'(\mathbb{Q}_p^3)} \mathrm{d}\mu_{C_0}(\phi)\exp\left(-V_{r,s}(\phi) + \phi(f) + Y_2 Z_2^r :\phi^2:_{C_0}(j)\right) \tag{3.17}$$

where

$$f(x) = L^{(3-[\phi])r}\tilde{f}(L^{-r}x) \text{ and } j(x) = L^{(3-2[\phi])r}\tilde{j}(L^{-r}x)$$

and

$$V_{r,s}(\phi) = \int_{\Lambda_{s-r}} \mathrm{d}^3x \left[g:\phi^4(x):_{C_0} + \mu:\phi^2(x):_{C_0}\right] \ .$$

One can imagine this rescaling as zooming in our field and so we trade dealing with a multitiude of short-range degrees of freedom (the UV problem) in exchange for having a larger box (exacerbating the IR problem). We also note that $3-[\phi],\ 3-2[\phi] > 0$ so our observables have been scaled in such a way that makes their values smaller in magnitude while their supports have grown larger. In particular since $\tilde{f}$ and $\tilde{j}$ were constant over the blocks of $\mathbb{L}_{q_-}$ and $r < q_-$ the functions f and j will be constant over the blocks of $\mathbb{L}$. We introduce some notation related to the scaling of observables, we write

$$\begin{aligned} f_{\to q}(x) &= L^{-(3-[\phi])q} f(L^q x) \\ j_{\Rightarrow q}(x) &= L^{-(3-2[\phi])q} j(L^q x) \end{aligned} \tag{3.18}$$

With the above notation one then has $f = \tilde{f}_{\to -r}$ and $j = \tilde{j}_{\Rightarrow -r}$.

In our change of variable the quantities with tildes can be imagined as living at their native (true) scale while those without tildes are living at what we call unit scale. The RG will traverse small length scales to large length scales but will always take as input and produce as output quantities at unit scale.

Before continuing our discussion we quickly describe a few more conventions. For an arbitrary function or field $\psi : \mathbb{Q}_p^3 \to \mathbb{C}$ which is constant over the blocks of $\mathbb{L}$ we often identify ψ with a $\mathbb{L}$-indexed complex vector, i.e. we identify $\psi \leftrightarrow \{\psi_\Delta\}_{\Delta \in \mathbb{L}} \in \mathbb{C}^{\mathbb{L}}$ with $\psi(x) = \psi_{\Delta(x)}$ for all $x \in \mathbb{Q}_p^3$ - here $\Delta(x)$ denotes the unique $\Delta \in \mathbb{L}$ that contains x.

For the time being we ignore the field independent prefactor appearing in (3.17). Since a field ϕ distributed according to C_0 is almost surely constant on blocks of $\mathbb{L}$ we can realize the integrand as a product of functions of a single real variable, each evaluated on our field's value on a particular block Δ:

$$\int_{S'(\mathbb{Q}_p^3)} \mathrm{d}\mu_{C_0}(\phi) \exp\left(-V_{r,s}(\phi) + \phi(f) + Y_2 Z_2^r :\phi^2:_{C_0}(j)\right) = \int_{S'(\mathbb{Q}_p^3)} \mathrm{d}\mu_{C_0}(\phi) \prod_{\substack{\Delta \in \mathbb{L} \\ \Delta \subset \Lambda_{s-r}}} F_{0,\Delta}(\phi_\Delta) \tag{3.19}$$

where for a given $\Delta \in \mathbb{L}$ the functions $F_\Delta : \mathbb{R} \to \mathbb{R}$ are given by

$$F_{0,\Delta}(\phi_\Delta) = \exp\left[-g :\phi_\Delta^4:_C - (\mu - Y_2 Z_2^r j_\Delta) :\phi_\Delta^2:_C + f_\Delta \phi_\Delta\right]. \tag{3.20}$$

Note that the above definition of $F_{0,\Delta}(\phi_\Delta)$ makes sense for all $\Delta \in \mathbb{L}$, even those not part of the product above. Referring back to the definitions (3.5) and (3.7) one has the covariance decomposition

$$C_0 = \Gamma + C_1$$

which immediately means if ϕ is distributed according to C_0 one can write

$$\phi = \phi_1 + \zeta$$

where the random fields ϕ_1 and ζ are independent with ϕ_1 distributed according C_1 and ζ distributed according to Γ. The field ζ is called the *fluctuation* field while ϕ_1 is called the *background* field.

The function $\Gamma(\cdot)$ is constant over the blocks of $\mathbb{L}$ and is also of finite range, being supported on $L^{-1}\mathbb{Z}_p^3$. This means that the ζ can be seen as a random field indexed by the lattice $\mathbb{L}$ whose values in different L-blocks are independent, i.e. $\mathbb{E}_\Gamma[\zeta(x)\zeta(y)] = 0$ if $|x - y| > L$. The function $C_1(\cdot)$ is constant over $\Delta \in \mathbb{L}_1$ and so the field ϕ_1 is almost-surely constant over L-blocks. In particular since $C_1(\cdot) = L^{-[\phi]} C_0(L\,\cdot)$ it follows that if ϕ is distributed according to C_0 one has

$$L^{-[\phi]}\phi(L\,\cdot) \overset{d}{=} \phi_1(\cdot). \tag{3.21}$$

The RG expansion of (3.19) comes out of careful iterative partial integration which we now describe. First we realize ϕ as the sum of independent fields $\zeta + \phi_1$ and integrates out the fluctuation field ζ. Afterwards we performs a change of variable corresponding to (3.21) to again arrive at an integrand that is being integrated with respect to the measure $\mathrm{d}\mu_{C_0}$. We now write this process out, assuming $r < s$ (note that we will sometimes refrain from writing $S'(\mathbb{Q}_p^3)$ under our integrals to lighten notation):

$$
\begin{aligned}
\int \mathrm{d}\mu_{C_0}(\phi) \prod_{\substack{\Delta \in \mathbb{L} \\ \Delta \subset \Lambda_{s-r}}} F_{0,\Delta}(\phi_\Delta) &= \int \mathrm{d}\mu_{C_1}(\phi_1) \int \mathrm{d}\mu_\Gamma(\zeta) \prod_{\substack{\square \in \mathbb{L}_1 \\ \square \subset \Lambda_{s-r}}} \prod_{\substack{\Delta \in \mathbb{L} \\ \Delta \subset \square}} F_{0,\Delta}(\phi_{1,\Delta} + \zeta_\Delta) \\
&= \int \mathrm{d}\mu_{C_1}(\phi_1) \prod_{\substack{\square \in \mathbb{L}_1 \\ \square \subset \Lambda_{s-r}}} \left[\int \mathrm{d}\mu_\Gamma(\zeta) \prod_{\substack{\Delta \in \mathbb{L} \\ \Delta \subset \square}} F_{0,\Delta}(\phi_{1,\square} + \zeta_\Delta) \right] \\
&= \int \mathrm{d}\mu_{C_0}(\phi) \prod_{\substack{\Delta' \in \mathbb{L} \\ \Delta' \subset \Lambda_{s-r-1}}} \left[\int \mathrm{d}\mu_\Gamma(\zeta) \prod_{\substack{\Delta \in \mathbb{L} \\ \Delta \subset L^{-1}\Delta'}} F_{0,\Delta}(\phi_{1,L^{-1}\Delta'} + \zeta_\Delta) \right] \qquad (3.22) \\
&= \int \mathrm{d}\mu_{C_0}(\phi) \prod_{\substack{\Delta' \in \mathbb{L} \\ \Delta' \subset \Lambda_{s-r-1}}} \left[\int \mathrm{d}\mu_\Gamma(\zeta) \prod_{\substack{\Delta \in \mathbb{L} \\ \Delta \subset L^{-1}\Delta'}} F_{0,\Delta}(L^{-[\phi]}\phi_{\Delta'} + \zeta_\Delta) \right] \\
&= \int \mathrm{d}\mu_{C_0}(\phi) \prod_{\substack{\Delta' \in \mathbb{L} \\ \Delta' \subset \Lambda_{s-r-1}}} \mathring{F}_{1,\Delta'}(\phi_{\Delta'})
\end{aligned}
$$

where

$$
\mathring{F}_{1,\Delta'}(\phi_{\Delta'}) = \int \mathrm{d}\mu_\Gamma(\zeta) \prod_{\substack{\Delta \in \mathbb{L} \\ \Delta \subset L^{-1}\Delta'}} F_{0,\Delta}(L^{-[\phi]}\phi_{\Delta'} + \zeta_\Delta). \qquad (3.23)
$$

In the first line of (3.22) we decomposed our field and we broke up the product over unit blocks within Λ_{s-r} into two products - the first product being over the L-blocks contained in Λ_{s-r} and then the second product being over the L^3 unit boxes contained within each fixed L-block.

In going to the second line of (3.22) the interchange of integral and product is allowed since for any two distinct L-blocks $\square_1, \square_2$ one has that the collections of random variables $\{\zeta_\Delta\}_{\Delta \subset \square_1}$ and $\{\zeta_\Delta\}_{\Delta \subset \square_2}$ are independent. Here the fact that we are working over an ultrametric space is quite important - for $\Delta_1 \subset \square_1$ and $\Delta_2 \subset \square_2$ one must have $|\Delta_1 - \Delta_2| > L$. We also use the notation $\phi_{1,\square}$ since ϕ_1 is constant over L-blocks $\square$.

In the third line of (3.22) we have just re-parameterized the product over L-blocks, noting that there is a one-to-one correspondance between L-blocks contained in Λ_{s-r} and unit blocks Δ' in Λ_{s-r-1} via $L^{-1}\Delta' \leftrightarrow \Delta'$. In going to the third line we have applied the change of variable corresponding to (3.21).

This transforms our integral expressed as a product of local functionals of the field into another integral which is also given by a product of local functionals but this time the product involves a smaller volume. We remark that this process is much cleaner than what happens in the Euclidean case where locality cannnot be perfectly perserved, in particular in step 2 one would have to worry that unit blocks in distinct but adjacent L-blocks can be close. However this can be dealt with in Brydges' RG formalism which works with a similar *approximate* factorization involving functionals with weaker locality properties (polymer activities) - this was first developed in [19], also see [16] for a pedagogical introduction to an updated version in the situation where one uses finite range fluctuation covariances.

While the steps of (3.22) have reduced the number of degrees of freedom in our original integral we have only won some understanding of the quantity we're trying to calculate if we have a sufficiently good understanding of the functionals $\{\mathring{F}_{1,\Delta}\}_{\Delta \subset \Lambda_{s-r-1}}$ that appear in our new integral. Calculating the integral by iterating this partial integration requires understanding the dynamical system generated by the iteration of this transformation on functionals.

Note that while constant functions are easy to track under this transformation they are very unstable, if for all $\Delta \in \mathbb{L}$ we set $F_\Delta = c$ for some $c \in \mathbb{R}$ then one will have $\mathring{F}_\Delta = c^{L^3}$. A remedy to this is to extract a carefully chosen constant at each step - our transformation on functionals will actually take the form

$$F_{k+1,\Delta'}(\phi_{\Delta'}) \;=\; e^{-\delta b[F_k]_{\Delta'}} \int \mathrm{d}\mu_\Gamma(\zeta) \prod_{\substack{\Delta \in \mathbb{L} \\ \Delta \subset L^{-1}\Delta'}} F_{k,\Delta}(L^{-[\phi]}\phi_{\Delta'} + \zeta_\Delta). \tag{3.24}$$

One can think of the constants $\delta b[F_k]_{\Delta'}$, called vacuum renormalizations, as being chosen as functions of $\{F^k_\Delta\}_{\Delta \subset L^{-1}\Delta'}$ in order to guarantee some approximate normalization condition, for example one could ask that for all Δ and k one has $F_{k,\Delta}(0) = a \approx 1$. We will see in later subsections that the collection of vacuum renormalizations across all scales will in fact contain the computation of $\mathcal{S}$.

We also remark that just as the initial data $\{F_{0,\Delta}\}_{\Delta \in \mathbb{L}}$ was defined in infinite volume (all of $\mathbb{L}$) we can also define our transformation $\{F_{k,\Delta}\}_{\Delta \in \mathbb{L}} \to \{F_{k+1,\Delta}\}_{\Delta \in \mathbb{L}}$ in infinite volume - each iteration of the RG transformation boils down to parallel (that is independent) transformations on each L-block.

3.4.2 Relevant and irrelevant operators

At this point it is instructive to turn our focus to how this process of iterative integration looks when being applied to the partition function $\mathcal{Z}_{r,s}(0,0)$. Since in (3.20) one will have $f = j = 0$ our initial functions are spatially homogenous, i.e $F_{k,\Delta}$ is given by an expression independent of Δ when $k = 0$. Due to the translation invariance of our covariances and the lack of boundary effects this property will be preserved for all values of k. Thus we only need to keep track of a transformation acting on some space of functions of a single variable which we write $F_k \to F_{k+1}$. We refer to the spatially homogenous evolution as the "bulk" flow.

While this is simpler then the spatially inhomogenous case we are still dealing with an infinite dimensional dynamical system. The RG approach involves introducing a coordinate system to parameterize the state space of this dynamical system which allows us to establish analytic control of the associated flow by studying how the flow acts on just finitely many coordinates - these coordinates representing the "relevant operators" of [74].

Working formally suppose that we used a coordinate system $(\beta_{k,n})_{n=0}^\infty \leftrightarrow F_k$ where the correspondance is given by

$$F_k(\phi_\Delta) := \exp\left[-V_k(\phi)\right] := \exp\left[-\sum_{n=1}^\infty \beta_{k,n} :\phi_\Delta^n:_{C_0}\right].$$

We note that this expression is formal but it includes the original form of our functional F_0, it can also be seen as the most general way to parameterize our functions as exponentials. We investigate the first order (linear) part of the flow $(\beta_{k,n})_{n=0}^{\infty} \to (\beta_{k+1,n})_{n=0}^{\infty}$.

$$\begin{aligned}
\int \mathrm{d}\mu_\Gamma(\zeta) \prod_{\substack{\Delta\in\mathbb{L} \\ \Delta\subset L^{-1}\Delta'}} \exp\left[-V_k(\phi_{1,\Delta}+\zeta_\Delta)\right] &= \int \mathrm{d}\mu_\Gamma(\zeta) \exp\left[- \sum_{\substack{\Delta\in\mathbb{L} \\ \Delta\subset L^{-1}\Delta'}} V_k(\phi_{1,\Delta}+\zeta_\Delta)\right] \\
&\approx \int \mathrm{d}\mu_\Gamma(\zeta) \left[1 - \sum_{\substack{\Delta\in\mathbb{L} \\ \Delta\subset L^{-1}\Delta'}} V_k(\phi_{1,\Delta}+\zeta_\Delta)\right] \\
&= 1 - \int \mathrm{d}\mu_\Gamma(\zeta) \sum_{\substack{\Delta\in\mathbb{L} \\ \Delta\subset L^{-1}\Delta'}} \left[\sum_{n=1}^{\infty} \beta_{k,n} :(\phi_{1,\Delta}+\zeta_\Delta)^n:_{C_0}\right] \\
&= 1 - \int \mathrm{d}\mu_\Gamma(\zeta) \sum_{\substack{\Delta\in\mathbb{L} \\ \Delta\subset L^{-1}\Delta'}} \left[\sum_{n=1}^{\infty} \beta_{k,n} \left(\sum_{m=0}^{n} \binom{n}{m} :\phi_{1,\Delta}^m:_{C_1} \times :\zeta_\Delta^{n-m}:_\Gamma\right)\right] \\
&= 1 - \sum_{\substack{\Delta\in\mathbb{L} \\ \Delta\subset L^{-1}\Delta'}} \sum_{n=1}^{\infty} \beta_{k,n} :\phi_{1,\Delta}^n:_{C_1} \\
&\to 1 - \sum_{\substack{\Delta\in\mathbb{L} \\ \Delta\subset L^{-1}\Delta'}} \sum_{n=1}^{\infty} L^{-n[\phi]}\beta_{k,n} :\phi_{\Delta'}^n:_{C_0} \\
&\approx \exp\left[-\sum_{n=1}^{\infty} L^{3-n[\phi]}\beta_{k,n} :\phi_{\Delta'}^n:_{C_0}\right] = e^{-V_{k+1}(\phi_{\Delta'})}
\end{aligned}$$

While many of the steps and expressions above were formal, in going to the fourth line we used a binomial identity for Wick powers

$$:(\phi_{1,\Delta}+\zeta_\Delta)^n:_{C_0} = \sum_{m=0}^{n} \binom{n}{m} :\phi_{1,\Delta}^m:_{C_1} \times :\zeta_\Delta^{n-m}:_\Gamma$$

and in going to the fifth line we used the fact that

$$\int \mathrm{d}\mu_\Gamma(\zeta) :\zeta_\Delta^l:_\Gamma = \delta_{l,0}.$$

In going to the sixth line we just rescaled from ϕ_1 to ϕ. We see that under this linear approximation of the RG flow $\beta_{k+1,n} = L^{3-n[\phi]}\beta_{k+1,n}$. Remembering that $[\phi] = \frac{3+\epsilon}{4}$ we see that for $n \geqslant 5$ one has that $L^{3-n[\phi]} < 1$. Thus the coefficients of the terms $:\phi_\Delta^n:_{C_0}$ appearing in (3.4.2) are all contracting for $n \geqslant 5$ - these terms are called *irrelevant*. However for $1 \leqslant n \leqslant 4$ one has the $L^{3-n[\phi]} > 1$ and the corresponding terms $:\phi_\Delta^1:_{C_0}, \ldots, :\phi_\Delta^4:_{C_0}$ are called *relevant*. The picture one expects to hold is that the irrelevant parameters should stay small because they are contracting at first order and the important qualitiative and quantitative features of the dynamical system should be recoverable through studying the flow of just the relevant parameters.

Clearly $F = 1$, or $\beta_n = 0$ for all $n \geqslant 1$ corresponds to a fixed point of our dynamical system which we call the Gaussian fixed point. The observations above suggest that the gaussian fixed point has an infinite dimensional stable manifold and a finite dimensional unstable manifold. Obsterve the the coefficient of the term $:\phi_\Delta^4:_{C_0}$ expands weakly, being scaled by L^ϵ. When $\epsilon = 0$ (in which case our model should resemble massless ϕ^4 in four dimensions) the $:\phi_\Delta^4:_{C_0}$ is *marginal*. The picture put forth by [75] is that if one takes $\epsilon = 0$ and slowly increases ϵ one will see a new fixed point emerge out of the Gaussian fixed point in the ϕ^4 direction - ϵ then plays the role of a bifurcation parameter. When one takes ϵ all the way to $\epsilon = 1$ one gets a model of great interest - massless ϕ^4 in 3 dimensions. However the at $\epsilon = 1$ the non-trivial fixed point is no longer a small perturbation of the Gaussian fixed point which puts it outside the purview of our RG machinery which depends on working close to the Gaussian fixed point.

3.4.3 The bulk flow in second order perturbation theory

In order to see the emergence of the non-trivial fixed point one must work at second order in g. In light of the above discussion we will only use two coordinates to parameterize our functionals - setting $F_k(\phi_\Delta) = \exp\left[-g_k :\phi_\Delta^4:_{C_0} - \mu_k :\phi_\Delta^2:_{C_0}\right]$ (we won't include wick monomials of order 3 or 1, one expects that under the bulk RG flow the functions F_k will stay even). We will expand $e^{-V} \approx 1 - V + \frac{V^2}{2}$ and ignore terms that are higher than order 2 in g and μ or irrelevant.

$$
\begin{aligned}
&\int \mathrm{d}\mu_\Gamma(\zeta) \prod_{\substack{\Delta\in\mathbb{L}\\ \Delta\subset L^{-1}\Delta'}} F_{k,\Delta}(\phi_{1,\Delta}+\zeta_\Delta)\\
&= \int \mathrm{d}\mu_\Gamma(\zeta) \exp\left[\sum_{\substack{\Delta\in\mathbb{L}\\ \Delta\subset L^{-1}\Delta'}} -V_k(\phi_{1,\Delta}+\zeta_\Delta)\right]\\
&\approx \int \mathrm{d}\mu_\Gamma(\zeta) \left[1 - \sum_{\substack{\Delta\in\mathbb{L}\\ \Delta\subset L^{-1}\Delta'}} V_k(\phi_{1,\Delta}+\zeta_\Delta) + \frac{1}{2}\sum_{\substack{\Delta_1,\Delta_2\in\mathbb{L}\\ \Delta_1,\Delta_2\subset L^{-1}\Delta'}} V_k(\phi_{1,\Delta}+\zeta_{\Delta_1})V_k(\phi_{1,\Delta}+\zeta_{\Delta_1})\right]\\
&\to 1 + \delta b_{k+1} - g_{k+1} :\phi_{\Delta'}^4:_{C_0} - \mu_{k+1} :\phi_{\Delta'}^2:_{C_0}\\
&\approx e^{\delta b_{k+1}} \exp\left[-g_{k+1} :\phi_{\Delta'}^4:_{C_0} - \mu_{k+1} :\phi_{\Delta'}^2:_{C_0}\right]
\end{aligned}
$$

where the evolution $(g_k, \mu_k) \to (g_{k+1}, \mu_{k+1})$ is given by

$$
\begin{aligned}
g_{k+1} &= L^\epsilon g_k - A_1 g_k^2\\
\mu_{k+1} &= L^{\frac{3+\epsilon}{2}}\mu_k - A_2 g_k^2 - A_3 g_k \mu_k
\end{aligned}
\tag{3.25}
$$

with the constants above given by

$$
\begin{aligned}
A_1 &= 36L^{3-4[\phi]} \int_{\mathbb{Q}_p^3} \Gamma(x)^2 \, \mathrm{d}^3x\\
A_2 &= 48L^{3-2[\phi]} \left(\int_{\mathbb{Q}_p^3} \Gamma(x)^3 \, \mathrm{d}^3x\right) + 144L^{3-4[\phi]} C_0(0) \left(\int_{\mathbb{Q}_p^3} \Gamma(x)^2 \, \mathrm{d}^3x\right)
\end{aligned}
$$

$$A_3 \quad = \quad 12L^{3-2[\phi]}\int_{\mathbb{Q}_p^3}\Gamma(x)^2\ \mathrm{d}^3x\ .$$

We then see that there is a fixed point for the flow of g given by

$$\bar{g}_* = \frac{L^\epsilon - 1}{A_1} = \frac{p^\epsilon - 1}{36L^\epsilon(1-p^{-3})} > 0$$

where we have used Lemma 5.5 which gives that

$$A_1 = 36L^\epsilon \times \frac{(1-p^{-3})(L^\epsilon - 1)}{p^\epsilon - 1}.$$

We remark that for any fixed value of $L = p^l \geqslant 1$ one has

$$\frac{\bar{g}_*}{\epsilon} \longrightarrow \frac{\log p}{36(1-p^{-3})}$$

as $\epsilon \to 0$ so when working in the small ϵ regime we have that $\bar{g}_*$ is of $O(\epsilon)$.

In the what follows we will define an exact renormalization group transformation that does not disregard higher order terms or irrelevant operators. The exact flow equations for g and μ will be given by (3.25) with additional $O(\epsilon^3)$ corrections. Our functions F_k will take the form

$$F_k(\phi_\Delta) = \exp\left[-g_k :\phi_\Delta^4:_{C_0} -\mu_k :\phi^2:_{C_0}\right] + R_k(\phi_\Delta)$$

where at each stage k the function $R_k : \mathbb{R} \to \mathbb{R}$ will live in an appropriately chosen infinite dimensional Banach space and should be thought of as containing all the irrelevant operators. We will not have explicit formulas for R_k's but only show that they stay small and that the corrections they induce in (3.25) are small.

The above discussion focused entirely on the bulk, in the presence of observables we will have spatially inhomogenous collection of functions $F_{k,\Delta}$ which will have a more complicated preserved functional form:

$$F_\Delta(\phi_\Delta) \ = \ e^{f_\Delta \phi_\Delta} \times \left(\exp\left[-\sum_{k=1}^{4}\beta_{k,\Delta} :\phi_\Delta^k:_{C_0}\right] \times \left(1 + W_{5,\Delta} :\phi_\Delta^5:_{C_0} + W_{6,\Delta} :\phi_\Delta^6:_{C_0}\right) + R_\Delta(\phi_\Delta)\right)$$

where $\beta_{1,\Delta}, \ldots, \beta_{4,\Delta}, f, W_{5,\Delta}, W_{6,\Delta}$ are numeric parameters and R_Δ again is a function of one real variable which we track in a Banach space.

3.5 Defining the Extended RG transformation

3.5.1 The state space of the Extended RG transformation

The functions R_Δ mentioned above will lie in the space $C^9_{\mathrm{bd}}(\mathbb{R}, \mathbb{C})$, namely, the space of nine times continuously differentiable functions from $\mathbb{R}$ to $\mathbb{C}$ which, together with their derivatives up to order nine, are bounded. We again remark this functional space is parameterizing an infinitude of irrelevant coordinates when it comes to our *RG* - we won't have an explicit understanding of the form of R at each step but we want to show that it stays small in the *RG* flow and that it's influence on the flow of the relevant parameters is negligible.

We will use seminorms $||\cdot||_{\partial\phi,\psi,\theta}$ defined for $K \in C^9_{\rm bd}(\mathbb{R},\mathbb{C})$ by

$$||K(\phi)||_{\partial\phi,\psi,\theta} = \sum_{j=0}^{9} \frac{\theta^j}{j!} \left| \frac{{\rm d}^j K}{{\rm d}\phi^j}(\psi) \right| \ .$$

Here $\partial\phi$ is merely a symbol which indicates the variable with respect to which the derivatives are taken. This will be especially useful when the function may depend on several such variables. By contrast, ψ is an argument of the seminorm. The derivatives are evaluated at $\phi = \psi$ and therefore the result depends on ψ. Finally $\theta \in [0,\infty)$ is a parameter used to properly calibrate this seminorm. We will mainly use two values for this parameter denoted by h and h_* to be specified later. As an example of use of the previous notation, we have $||\phi^2||_{\partial\phi,\psi,\theta} = |\psi|^2 + 2\theta|\psi| + \theta^2$. In the important special case where $\psi = 0$, we will abbreviate the notation as

$$|K(\phi)|_{\partial\phi,\theta} = ||K(\phi)||_{\partial\phi,0,\theta} \ .$$

We will often call the above seminorm a "kernel" seminorm. A nice property of all of these seminorms is multiplicativity. Indeed for any two functions K_1, K_2 in $C^9_{\rm bd}(\mathbb{R},\mathbb{C})$ we have

$$||K_1(\phi)K_2(\phi)||_{\partial\phi,\psi,\theta} \leqslant ||K_1(\phi)||_{\partial\phi,\psi,\theta} \times ||K_2(\phi)||_{\partial\phi,\psi,\theta}$$

which is an easy consequence of the Leibniz rule and the choice of $\frac{1}{j!}$ weights.

To a parameter $\bar{g} > 0$ called a calibrator we associate a norm $|||\cdot|||_{\bar{g}}$ on the complex Banach space $C^9_{\rm bd}(\mathbb{R},\mathbb{C})$ defined by

$$|||K|||_{\bar{g}} = \max\left\{ |K(\phi)|_{\partial\phi,h_*}, \bar{g}^2 \sup_{\phi\in\mathbb{R}} ||K(\phi)||_{\partial\phi,\phi,h} \right\} \ .$$

While the coefficient of ϕ^4 in our problem will be a dynamical variable which generically does not stay fixed we expect to work in a regime where it stays in the neighborhood of $\bar{g}_*$. We will set $\bar{g} = \bar{g}_*$, thinking of $\bar{g}$ as something of order ϵ that we use to measure the size of certain quantities appearing in our RG analysis. In particular it serves as an upper bound (with an $O(1)$ constant) on the size of the relevant couplings $\beta_4, \beta_3, \beta_2, \beta_1$.

We will choose $h = c_1\bar{g}^{-\frac{1}{4}}$ and $h_* = c_2 L^{\frac{3+\epsilon}{4}}$ for some specifically chosen fixed constants $c_1, c_2 > 0$. We will try to give some motivation for having two seminorms with two different values of θ.

Evaluating $R(\phi)$ at $\phi = 0$ in some sense measures the cumulative size of all the "irrelevant couplings" inside of R - the kernel seminorm will be our tool for keeping track of this. Later we will see that the kernel seminorm also directly determines the magnitude of the correction generated by the (non-explicit) irrelevant couplings to the approximate flow equations for the relevant couplings (3.25). However control of a function $R(\phi)$ at $\phi = 0$ does not immediately translate to good control of $\int d\mu_\Gamma(\zeta)\ R(\phi+\zeta)$ at $\phi = 0$. We need to understand R across the distribution of ζ. The kernel seminorm of R tries to capture a helpful amount information of R's behaviour near 0 with an eye towards helping control the fluctuation integral - the quantity h_* in some sense is an order of magnitude estimate for $|\zeta|$ see Lemmas 3.3 and 3.4 for example. However the kernel seminorm by itself is not enough to control the fluctuation integration, we will need some control at the tails of the distribution of ζ.

The norm $||R(\phi)||_{\partial\phi,\phi,h}$ is used to give more uniform control in ϕ. At the price of consuming the decay factors $e^{-g\phi^4}$ with $g > 0$ one will be able to bound $|\phi|$ by a factor of $g^{-\frac{1}{4}}$ - this motivates our choice of h.

The g^2 appearing within the definition of $|||\cdot|||_{\bar{g}}$ fixes the discrepancy between the two norms we use to measure the R. Since R involves terms formed by three vertices we expect the kernel seminorm of R to be approximately order g^3 but in order to establish a contractive bound we'll want an exponent drop and impose that the kernel seminorm of R is of order $g^{\frac{11}{4}}$. Now when using the h-norm to measure R we have to pay a price to bound powers of the field ϕ - the worst vertices come with a power of ϕ^3 and a single power of g - thus we expect the h-norm of K to be approximately of order $(g \times g^{-\frac{3}{4}})^3 = \bar{g}^{\frac{3}{4}}$. Thus we need an extra g^2 so the h-norm is of the same magnitude as the kernel seminorm. This $\bar{g}^2$ discrepancy in turn effects the number of derivatives appearing in our seminorms - see in particular Lemma 3.11.

The space of collections of functions $\{F_\Delta\}_{\Delta\in\mathbb{L}}$ will be parameterized by the Banach space $\mathcal{E}_{\text{ex}}$. An element of the space $\mathcal{E}_{ex}$ is an indexed family

$$\vec{V} = (V_\Delta)_{\Delta\in\mathbb{L}}$$

where

$$V_\Delta = (\beta_{4,\Delta}, \beta_{3,\Delta}, \beta_{2,\Delta}, \beta_{1,\Delta}, W_{5,\Delta}, W_{6,\Delta}, f_\Delta, R_\Delta) \in \mathbb{C}^7 \times C^9_{\text{bd}}(\mathbb{R},\mathbb{C}).$$

We define the norm

$$||V_\Delta|| = \max\Big\{|\beta_{4,\Delta}|\bar{g}^{-\frac{3}{2}}, |\beta_{3,\Delta}|\bar{g}^{-1}, |\beta_{2,\Delta}|\bar{g}^{-1}, |\beta_{1,\Delta}|\bar{g}^{-1},$$
$$|W_{5,\Delta}|\bar{g}^{-2}, |W_{6,\Delta}|\bar{g}^{-2}, |f_\Delta|L^{(3-[\phi])}, |||R_\Delta|||_{\bar{g}}\ \bar{g}^{-\frac{21}{8}}\Big\}.$$

We also define

$$||\vec{V}|| = \sup_{\Delta\in\mathbb{L}} ||V_\Delta||$$

and

$$\mathcal{E}_{\text{ex}} = \left\{\vec{V} \in \prod_{\Delta\in\mathbb{L}} \left(\mathbb{C}^7 \times C^9_{\text{bd}}(\mathbb{R},\mathbb{C})\right)\ \mid ||\vec{V}|| < \infty\right\}.$$

Now the correspondence between vectors $\vec{V}$ and integrands is given by defining (for t a non-negative integer):

$$\mathcal{I}_t[\vec{V}](\phi) := \prod_{\substack{\Delta\in\mathbb{L}\\ \Delta\subset\Lambda_t}} \mathcal{I}_\Delta[\vec{V}](\phi)$$

where

$$\mathcal{I}_\Delta[\vec{V}](\phi) := e^{f_\Delta\phi_\Delta} \times \{\exp\left[-\beta_{4,\Delta} :\phi^4_\Delta:_{C_0} -\beta_{3,\Delta} :\phi^3_\Delta:_{C_0} -\beta_{2,\Delta} :\phi^2_\Delta:_{C_0} -\beta_{1,\Delta} :\phi_\Delta:_{C_0}\right] \times$$
$$\left(1 + W_{5,\Delta} :\phi^5_\Delta:_{C_0} + W_{6,\Delta} :\phi^6_\Delta:_{C_0}\right) + R_\Delta(\phi_\Delta)\}.$$

The most general of the Renormalization Group transformations we define will be a map $RG_{\text{ex}}[\cdot] : \mathcal{E}_{\text{ex}} \to \mathcal{E}_{\text{ex}}$. We call this map the *extended* RG map and the dynamical system generated by $\vec{V} \to RG_{\text{ex}}[\vec{V}]$ is called the extended RG flow. Along with the extended RG transformation we will also define associated vacuum renormalization maps $\delta b_\Delta[\cdot] : \mathcal{E}_{\text{ex}} \to \mathbb{C}$ for each $\Delta \in \mathbb{L}$. We remark that we will be able to establish analyticity and our core estimates for RG_{ex} only within a rather small open set $U \subset \mathcal{E}_{\text{ex}}$.

The algebraic definition of this map will be quite involved and we leave that to the next subsection. The key identity that RG_{ex} and vacuum renormalizations satisfy which will yield the expansion across scales described in Section 3.4 is:

$$\mathcal{I}_{\Delta'}\left[RG_{\text{ex}}[\vec{V}]\right] = \exp\left[\frac{1}{2}(f,\Gamma f)_{L^{-1}\Delta'} + \delta b_{\Delta'}[\vec{V}]\right] \times \int_{S'(\mathbb{Q}_p^3)} \mathrm{d}\mu_{C_0}(\phi) \prod_{\Delta \subset L^{-1}\Delta'} \mathcal{I}_{\Delta}[\vec{V}](\phi) \tag{3.26}$$

where we write

$$(f,\Gamma f)_X = \int_{X^2} \mathrm{d}^3x\ \mathrm{d}^3y\ f(x)\ \Gamma(x-y)\ f(y)$$

for any measurable subset X of $\mathbb{Q}_p^3$ and by f we refer to the corresponding component of $\vec{V}$. We remark that in (3.26) we have broken up the vacuum renormalization, the first part corresponding to what one would get from just the Gaussian measure. Linking back to our earlier discussion (3.26) is just a way of writing (3.24). Applying (3.27) for an integrand over a box Λ_t with $t \geqslant 1$ yields:

$$\int_{S'(\mathbb{Q}_p^3)} \mathrm{d}\mu_{C_0}(\phi)\ \mathcal{I}_t[\vec{V}](\phi) = \exp\left[\frac{1}{2}(f,\Gamma f)_{\Lambda_t} + \sum_{\substack{\Delta \in \mathbb{L} \\ \Delta \subset \Lambda_{t-1}}} \delta b_{\Delta}[\vec{V}]\right] \times \int_{S'(\mathbb{Q}_p^3)} \mathrm{d}\mu_{C_0}(\phi)\ \mathcal{I}_{t-1}\left[RG_{\text{ex}}[\vec{V}]\right](\phi). \tag{3.27}$$

It is useful to compactly write an element $\vec{V} \in \mathcal{E}_{ex}$ in the form

$$\vec{V} = (\beta_4, \beta_3, \beta_2, \beta_1, \beta_1, W_5, W_6, f, R)$$

where each entry on the righthand side itself corresponds to a collection of $\mathbb{L}$ - indexed quantities, for example:

$$\beta_4 = (\beta_{4,\Delta})_{\Delta \in \mathbb{L}} \quad \text{where } \beta_{4,\Delta} \in \mathbb{C}$$

or equivalently we can imagine β_4 as being a function $\beta_4(x)$ on $\mathbb{Q}_p^3$ constant on unit blocks. Now the vector $\vec{V}^{(r,r)}(\tilde{f},\tilde{j})$ is the initial data for the RG flow involved in the computation of (3.19):

$$\vec{V}^{(r,r)}(\tilde{f},\tilde{j}) = (\beta_4, \beta_3, \beta_2, \beta_1, W_5, W_6, f, R) \tag{3.28}$$

where

$$\begin{aligned}
\beta_3 &= 0 \\
\beta_1 &= 0 \\
W_5 &= 0 \\
W_6 &= 0 \\
R &= 0 \\
f &= L^{(3-[\phi])r}\tilde{f}(L^{-r}x) \text{ for all } x \\
\beta_4(x) &= g \text{ for all } x \\
\beta_2(x) &= \mu - Y_2 Z_2^r\ L^{(3-2[\phi])r}\ \tilde{j}(L^{-r}x) \text{ for all } x
\end{aligned} \tag{3.29}$$

so

$$\mathcal{Z}_{r,s}(\tilde{f},\tilde{j}) = \exp\left(-Y_0 Z_0^r L^{2[\phi]r}\int_{\mathbb{Q}_p^3} j(x)\ \mathrm{d}^3x\right)\times\int \mathrm{d}\mu_{C_0}(\phi)\ \mathcal{I}_{s-r}[\vec{V}^{(r,r)}(\tilde{f},\tilde{j})](\phi).$$

We remark that $\vec{V}^{(r,r)}(\tilde{f},\tilde{j})$ specifies the integrand for any value of s, it in fact represents a global configuration of functionals for each $\Delta \in \mathbb{L}$. The map RG_{ext} will generate a sequence

$$\vec{V}^{(r,r)}(\tilde{f},\tilde{j}) \to \vec{V}^{(r,r+1)}(\tilde{f},\tilde{j}) \to \vec{V}^{(r,r+2)}(\tilde{f},\tilde{j}) \to \cdots$$

where for every scale $q \geqslant r$ one has $\vec{V}^{(r,q)}(\tilde{f},\tilde{j}) := RG_{\mathrm{ex}}^{q-r}\left[\vec{V}^{(r,r)}(\tilde{f},\tilde{j})\right]$. Now iterating the identity (3.27) gives

$$\mathcal{Z}_{r,s}(\tilde{f},\tilde{j}) = \exp\left(-Y_0 Z_0^r L^{2[\phi]r}\int_{\mathbb{Q}_p^3} j(x)\ \mathrm{d}^3x\right)$$

$$\times\exp\left[\sum_{r\leqslant q<s}\left(\frac{1}{2}(f^{(r,q)},\Gamma f^{(r,q)})_{\Lambda_{s-q}} + \sum_{\substack{\Delta\in\mathbb{L}\\ \Delta\subset\Lambda_{s-q-1}}}\delta b_{\Delta}[\vec{V}^{r,q}(\tilde{f},\tilde{j})]\right)\right]\times\int \mathrm{d}\mu_{C_0}(\phi)\ \mathcal{I}_0[\vec{V}^{(r,s)}(\tilde{f},\tilde{j})](\phi)$$

for every scale t with $r \leqslant t < s$. here $f^{(r,q)}$ denotes the f component of $\vec{V}^{(r,q)}(\tilde{f},\tilde{j})$. We note that the last integral on the right hand side is just a Gaussian integral on $\mathbb{R}$. We will write

$$\partial\mathcal{Z}_{r,s}(\tilde{f},\tilde{j}) := \int \mathrm{d}\mu_{C_0}(\phi)\ \mathcal{I}_0[\vec{V}^{(r,s)}(\tilde{f},\tilde{j})](\phi).$$

3.5.2 Algebraic Definition of the extended RG map

The full RG transformation $\vec{V} \to \vec{V}' = RG_{\mathrm{ex}}[\vec{V}]$ will be defined by specifying

$$\vec{V}' = (\beta_4', ..., \beta_1', W_5', W_6', f', R')$$

starting from the analogous unprimed quantities. We will also give formulas for the corresponding vacuum renormalizations $\{\delta b[\vec{V}]\}_{\Delta\in L}$.

We introduce the short hand

$$V_{\Delta}(\phi_{\Delta}) = \exp\left[-\sum_{k=1}^{4}\beta_{k,\Delta} : \phi_{\Delta}^k :_{C_0}\right].$$

Note that there is a namespace collision with the V_{Δ} mentioned in the definition $\mathcal{E}_{\mathrm{ex}}$ but the difference should be clear from context (the $V_{\Delta}(\cdot)$ above is a function of one real variable determined by V_{Δ} of the last section.). We also define

$$Q(\phi_{\Delta}) \ = \ \left(W_{5,\Delta} : \phi_{\Delta}^5 :_{C_0} + W_{6,\Delta} : \phi_{\Delta}^6 :_{C_0}\right)$$

and

$$K_{\Delta}(\phi_{\Delta}) \ = \ Q(\phi_{\Delta})e^{-V_{\Delta}(\phi_{\Delta})} + R_{\Delta}(\phi_{\Delta}) \tag{3.30}$$

so that

$$\mathcal{I}_{\Delta}[\vec{V}](\phi_{\Delta}) \ = \ e^{f_{\Delta}\phi_{\Delta}} \times \left(e^{-V_{\Delta}(\phi_{\Delta})} + K_{\Delta}(\phi_{\Delta})\right) \tag{3.31}$$

where one should think of the two terms within the parentheses on the right hand side of (3.31) as respectively the relevant part and irrelevant part of our functional form. In the definition (3.30) we see the irrelevant part K is further decomposed into an explicit part Qe^{-V} which should be thought of as containing irrelevant terms coming from V^2 terms and the R which contains terms that are of order 3 and higher in V. By V here we mean vertices.

By our earlier discussion we have

$$\begin{aligned}\int \mathrm{d}\mu_{C_0}(\phi)\ \mathcal{I}_t[\vec{V}](\phi) &= \int \mathrm{d}\mu_{C_0}(\phi) \prod_{\substack{\Delta\in\mathbb{L} \\ \Delta\subset\Lambda_t}} \left\{ e^{f_\Delta \phi_\Delta} \times \left[e^{-V_\Delta(\phi_\Delta)} + K_\Delta(\phi_\Delta) \right] \right\} \\ &= \int \mathrm{d}\mu_{C_0}(\phi) \int \mathrm{d}\mu_\Gamma(\zeta) \prod_{\substack{\Delta\in\mathbb{L} \\ \Delta\subset\Lambda_t}} \left\{ e^{f_\Delta \phi_{1,\Delta} + f_\Delta \zeta_\Delta} \times \left[e^{-V_\Delta(\phi_{1,\Delta}+\zeta_\Delta)} + K_\Delta(\phi_{1,\Delta}+\zeta_\Delta) \right] \right\}.\end{aligned}$$

For the above formula and what follows the ϕ_1 should be thought of as function of the random field ϕ, in particular ϕ_1 is constant over the blocks of $\mathbb{L}$ for all $\Delta \in \mathbb{L}$ we set $\phi_{1,\Delta} = L^{-[\phi]}\phi_{\Delta'}$ where $\Delta' \in \mathbb{L}$ is the unique unit block with $\Delta \subset [L^{-1}\Delta']$. We remark that with this convention ϕ being distributed according to C_0 means that ϕ_1 is distributed according to C_1.

We then organize the product according to the L-blocks containing Δ and use the independence of the ζ random variables living in different L-blocks to obtain

$$\begin{aligned}\int \mathrm{d}\mu_{C_0}(\phi)\ \mathcal{I}_t[\vec{V}](\phi) &= \int \mathrm{d}\mu_{C_0}(\phi) \prod_{\substack{\Delta'\in\mathbb{L} \\ \Delta'\subset\Lambda_{t-1}}} \Bigg(\int \mathrm{d}\mu_\Gamma(\zeta) \\ &\qquad \prod_{\Delta\in[L^{-1}\Delta']} \left\{ e^{f_\Delta \phi_{1,\Delta} + f_\Delta \zeta_\Delta} \times \left[e^{-V_\Delta(\phi_{1,\Delta}+\zeta_\Delta)} + K_\Delta(\phi_{1,\Delta}+\zeta_\Delta) \right] \right\} \Bigg) \\ &= \int \mathrm{d}\mu_{C_0}(\phi) \prod_{\substack{\Delta'\in\mathbb{L} \\ \Delta'\subset\Lambda_{t-1}}} \left(e^{f'_{\Delta'}\phi_{\Delta'}} \times \mathcal{B}_{\Delta'} \right)\end{aligned}$$

where

$$\mathcal{B}_{\Delta'} = \int \mathrm{d}\mu_\Gamma(\zeta) \prod_{\Delta\in[L^{-1}\Delta']} \left\{ e^{f_\Delta \zeta_\Delta} \times \left[e^{-V_\Delta(\phi_{1,\Delta}+\zeta_\Delta)} + K_\Delta(\phi_{1,\Delta}+\zeta_\Delta) \right] \right\} .$$

and

$$f'_{\Delta'} = L^{3-[\phi]} \operatorname*{avg}_{\Delta\in[L^{-1}\Delta']} f_\Delta \tag{3.32}$$

where "avg" means average. The formula (3.32) gives the evolution of the f component and we remark that it evolves autonomously of all the other parameters.

Now the linear flow of the relevant parameters is given by

$$\hat{\beta}_{k,\Delta'} = L^{3-k[\phi]} \underset{\Delta\in[L^{-1}\Delta']}{\text{avg}} \beta_{k,\Delta} \tag{3.33}$$

for $1 \leqslant k \leqslant 4$.

With a slight abuse of notation we define

$$\tilde{V}_\Delta(\phi_1) = \sum_{k=1}^{4} \beta_{k,\Delta} :\phi_1^k:_{C_1}$$

and

$$\hat{V}_{\Delta'}(\phi) = \sum_{k=1}^{4} \hat{\beta}_{k,\Delta'} :\phi^k:_{C_0} \ .$$

Note that $\sum_{\Delta\in[L^{-1}\Delta']} \tilde{V}_\Delta(\phi_1) = \hat{V}_{\Delta'}(\phi)$ where ϕ is in fact the component $\phi_{\Delta'}$ of the field but we suppressed this from the notation. Now define

$$p_\Delta = p_\Delta(\phi_1,\zeta) = V_\Delta(\phi_1+\zeta) - \tilde{V}_\Delta(\phi_1)$$

namely

$$p_\Delta = \sum_{a,b} \mathbb{1}\left\{ \begin{array}{c} a+b \leqslant 4 \\ a \geqslant 0\ ,\ b \geqslant 1 \end{array} \right\} \frac{(a+b)!}{a!\ b!}\ \beta_{a+b,\Delta} :\phi_{1,\Delta}^a:_{C_1} \ \times\ :\zeta_\Delta^b:_\Gamma \ .$$

The terms within p_Δ are what we were refering to as "vertices" in our earlier discussion. Now let

$$P_\Delta(\phi_1,\zeta) = e^{-V_\Delta(\phi_1+\zeta)} - e^{-\tilde{V}_\Delta(\phi_1)} \ .$$

We expand $\mathcal{B}_{\Delta'}$ by writing the factors as

$$e^{-V_\Delta(\phi_{1,\Delta}+\zeta_\Delta)} + K_\Delta(\phi_{1,\Delta}+\zeta_\Delta) = e^{-\tilde{V}_\Delta(\phi_1)} + P_\Delta(\phi_1,\zeta) + K_\Delta(\phi_{1,\Delta}+\zeta_\Delta) \ .$$

This results in

$$\mathcal{B}_{\Delta'} = e^{\frac{1}{2}(f,\Gamma f)_{L^{-1}\Delta'} - \hat{V}_{\Delta'}(\phi)} + \hat{K}_{\Delta'}(\phi) \tag{3.34}$$

where

$$\hat{K}_{\Delta'}(\phi) = \sum_{Y_P,Y_K} \int \mathrm{d}\mu_\Gamma(\zeta)\ e^{\int_{L^{-1}\Delta'} f\zeta} \times \prod_{\substack{\Delta\in[L^{-1}\Delta'] \\ \Delta\notin Y_P\cup Y_K}} \left[e^{-\tilde{V}_\Delta(\phi_1)}\right] \times \prod_{\Delta\in Y_P} [P_\Delta(\phi_1,\zeta)] \times \prod_{\Delta\in Y_K} [K_\Delta(\phi_1+\zeta)]$$

where the sum is over pairs of disjoint subsets Y_P, Y_K of $[L^{-1}\Delta']$ such that at least one of them is nonempty.

One can interpret (3.34) as the new functional we would get for the unit block Δ' if we performed no extraction, that is if we just let the relevant parameters flow according to linear flow. While $\hat{K}_{\Delta'}$ would aspire to be the irrelevant part of our new functional it is not irrelevant or even approximately irrelevant. The process of removing relevant operators from $\hat{K}$ by "moving" them to $\hat{V}$ will be called *extraction*. We now describe this process.

Assume that we are given collections of numbers $\delta\beta_{k,\Delta'}$ for $0 \leqslant k \leqslant 4$ and $\Delta' \in \mathbb{L}$. For $1 \leqslant k \leqslant 4$ these will represent the corrections to the linear flow given by (3.33) arising from the extraction process. The $k = 0$ quantity corresponds to the vacuum renormalization - we will write $\delta b_{\Delta'} = \delta\beta_{0,\Delta'}$. We remark that the vacuum renormalization will be implemented differently then the other extractions. We have the trivial identity

$$\begin{aligned}\int \mathrm{d}\mu_{C_0}(\phi)\ \mathcal{I}_t[\vec{V}](\phi) \quad &= \quad \exp\left(\frac{1}{2}(f,\Gamma f)_{\Lambda_t} + \sum_{\substack{\Delta'\in\mathbb{L} \\ \Delta'\subset\Lambda_{t-1}}} \delta b_{\Delta'}\right)\times \\ &\int \mathrm{d}\mu_{C_0}(\phi) \prod_{\Delta'\in\mathbb{L}} \left\{ e^{f'_{\Delta'}\phi_{\Delta'}} \times \left[e^{-\hat{V}_{\Delta'}(\phi_{\Delta'})-\delta b_{\Delta'}} + \hat{K}_{\Delta'}(\phi_{\Delta'}) e^{-\delta b_{\Delta'} - \frac{1}{2}(f,\Gamma f)_{L^{-1}\Delta'}} \right]\right\}.\end{aligned}$$

Define

$$\delta V_{\Delta'}(\phi) = \sum_{k=0}^{4} \delta\beta_{k,\Delta'} : \phi^k :_{C_0} .$$

Post-extraction our "new V_Δ" will be denoted by $V'_{\Delta'}$ and will be given by

$$V'_{\Delta'}(\phi) = \sum_{k=1}^{4} (\hat{\beta}_{k,\Delta'} - \delta\beta_{k,\Delta'}) : \phi^k :_{C_0} .$$

In particular

$$V'_{\Delta'}(\phi) = \hat{V}_{\Delta'}(\phi) - \delta V_{\Delta'}(\phi) + \delta b_{\Delta'} .$$

Now one can check

$$\begin{aligned}\int \mathrm{d}\mu_{C_0}(\phi)\ \mathcal{I}_t[\vec{V}](\phi) \quad &= \quad \exp\left(\frac{1}{2}(f,\Gamma f)_{\Lambda_t} + \sum_{\substack{\Delta'\in\mathbb{L} \\ \Delta'\subset\Lambda_{t-1}}} \delta b_{\Delta'}\right)\times \\ &\int \mathrm{d}\mu_{C_0}(\phi) \prod_{\substack{\Delta'\in\mathbb{L} \\ \Delta'\subset\Lambda_{t-1}}} \left\{ e^{f'_{\Delta'}\phi_{\Delta'}} \times \left[e^{-V'_{\Delta'}(\phi_{\Delta'})} + K'_{\Delta'}(\phi_{\Delta'}) \right]\right\}\end{aligned}$$

where

$$K'_{\Delta'}(\phi) = e^{-\delta b_{\Delta'} - \frac{1}{2}(f,\Gamma f)_{L^{-1}\Delta'}} \times \left\{ \hat{K}_{\Delta'}(\phi) - e^{-\hat{V}_{\Delta'}(\phi) + \frac{1}{2}(f,\Gamma f)_{L^{-1}\Delta'}} \left(e^{\delta V_{\Delta'}(\phi)} - 1 \right) \right\} .$$

$K'_{\Delta'}$ will be the irrelevant part of the new functional we output - extraction will involve choosing the constants $\delta\beta_{k,\Delta'}$ so that particular relevant terms do not appear in $K'_{\Delta'}$.

We will remove from $\hat{K}$ the following types of relevant terms

(i) Relevant terms appearing from a contraction of a term in V with the observable f - the corresponding counter terms will be explicit and of order $\bar{g}$.

(ii) Relevant terms coming from a V^2 terms or from Q either of which could involve contractions with our observables. These counterterms will be explicit and of order $\bar{g}^2$.

(iii) Relevant terms coming from R that are first order in R. These will not be explicit, and will mostly be specified by a linear functional of R. These counter terms are morally of order $\bar{g}^3$.

Note that *we do not extract all* relevant parts of $\hat{K}$, only enough to maintain analytic control of the RG flow. In particular we don't immediately extract relevant terms that are second order or higher in Q or R or of order 3 or higher in V - these will be buried in the new R' with enough powers of $\bar{g}$ to keep them quiet for the time being.

The $\bar{g}^2$ counterms appearing in our flow equations for relevant parameters will need to be explicit - this is necessary to clearly see that the bulk flow of g and μ has a non-trivial fixed point. It will be useful to include a complex parameter λ into many of our expressions that keeps track of how many V's (and thus how many $\bar{g}$'s) are contained in the various parts of K' - we call this process λ-deformation. In particular the form of our extractions and the functionals we output will be explicit up to order λ^2. The final λ-dependent expressions we produce will define our extended RG map when we fix $\lambda = 1$. We define

$$r_{1,\Delta} = r_{1,\Delta}(\phi_1,\zeta) = e^{-\tilde{V}_\Delta(\phi_1)}\left[e^{-p_\Delta} - 1 + p_\Delta - \frac{1}{2}p_\Delta^2\right]$$

and let

$$P_\Delta(\lambda,\phi_1,\zeta) = e^{-\tilde{V}_\Delta(\phi_1)}\left[-\lambda p_\Delta + \frac{\lambda^2}{2}p_\Delta^2\right] + \lambda^3 r_{1,\Delta}(\phi_1,\zeta)$$

so that

$$P_\Delta(\lambda,\phi_1,\zeta)|_{\lambda=1} = P_\Delta(\phi_1,\zeta)\ .$$

We also define

$$K_\Delta(\lambda,\phi_1,\zeta) = \lambda^2 Q_\Delta(\phi_1+\zeta)\ e^{-\tilde{V}_\Delta(\phi_1)} + \lambda^3\left[Q_\Delta(\phi_1+\zeta)\left(e^{-p_\Delta}-1\right)e^{-\tilde{V}_\Delta(\phi_1)} + R_\Delta(\phi_1+\zeta)\right]$$

so that

$$K_\Delta(\lambda,\phi_1,\zeta)|_{\lambda=1} = K_\Delta(\phi_1+\zeta).$$

We use the same expansion formula as before in order to define the λ-deformation

$$\hat{K}_{\Delta'}(\lambda,\phi) = \sum_{Y_P,Y_K}\int \mathrm{d}\mu_\Gamma(\zeta)\ e^{\int_{L^{-1}\Delta'} f\zeta} \times \prod_{\substack{\Delta\in[L^{-1}\Delta'] \\ \Delta\notin Y_P\cup Y_K}}\left[e^{-\tilde{V}_\Delta(\phi_1)}\right] \times \prod_{\Delta\in Y_P}\left[P_\Delta(\lambda,\phi_1,\zeta)\right] \times \prod_{\Delta\in Y_K}\left[K_\Delta(\lambda,\phi_1,\zeta)\right]\ . \tag{3.35}$$

This is a polynomial expression in λ with no constant term. We can write it as

$$\hat{K}_{\Delta'}(\lambda,\phi) = \mathsf{A}\lambda + \mathsf{B}\lambda^2 + \mathsf{C}\lambda^3 + \hat{K}^{\geqslant 4}_{\Delta'}(\lambda,\phi)$$

where $\hat{K}^{\geqslant 4}_{\Delta'}(\lambda,\phi)$ contains the terms of order 4 or more in λ.

We now implement a λ grading for our proposed counterterms - we assume that there are numbers $\delta\beta_{k,j,\Delta'}$ for $0 \leqslant k \leqslant 4$, $1 \leqslant j \leqslant 3$ and $\Delta' \in \mathbb{L}$ such that

$$\delta\beta_{k,\Delta'} = \delta\beta_{k,1,\Delta'} + \delta\beta_{k,2,\Delta'} + \delta\beta_{k,3,\Delta'}$$

and we define

$$\delta\beta_{k,\Delta'}(\lambda) = \lambda\ \delta\beta_{k,1,\Delta'} + \lambda^2\delta\beta_{k,2,\Delta'} + \lambda^3\delta\beta_{k,3,\Delta'}\ .$$

In particular this defines $\delta b_{\Delta'}(\lambda) = \delta\beta_{0,\Delta'}(\lambda)$. We also let

$$\delta V_{\Delta'}(\lambda,\phi) = \sum_{k=0}^{4} \delta\beta_{k,\Delta'}(\lambda) : \phi^k :_{C_0}\ .$$

Using the same formula as before for $K'_{\Delta'}$, we define the corresponding λ-deformation:

$$K'_{\Delta'}(\lambda,\phi) = e^{-\delta b_{\Delta'}(\lambda)-\frac{1}{2}(f,\Gamma f)_{L^{-1}\Delta'}} \times \left\{\hat{K}_{\Delta'}(\lambda,\phi) - e^{-\hat{V}_{\Delta'}(\phi)+\frac{1}{2}(f,\Gamma f)_{L^{-1}\Delta'}}\left(e^{\delta V_{\Delta'}(\lambda,\phi)}-1\right)\right\}\ . \tag{3.36}$$

We again expand this in λ up to order 3:

$$K'_{\Delta'}(\lambda,\phi) = \mathsf{A}'\lambda + \mathsf{B}'\lambda^2 + \mathsf{C}'\lambda^3 + O(\lambda^4)\ .$$

We will give an explicit formula for A'. Note that the quantity in the braces of (3.36) contains terms of order at least order 1 in λ. The order 1 term of $\hat{K}_{\Delta'}(\lambda,\phi)$, which was denoted by A, is given by the terms (3.35) with Y_P a singleton and Y_K empty (since $K_\Delta(\phi,\lambda)$ is of order 2 in λ) where for the Y_P singleton we take only the first order part of P_Δ.

$$\begin{aligned}
\mathsf{A}\ &=\ \sum_{\Delta\in[L^{-1}\Delta']} \int d\mu_\Gamma(\zeta)\ e^{\int_{L^{-1}\Delta'} f\zeta} \times \left(\prod_{\substack{\Delta_1\in[L^{-1}\Delta'] \\ \Delta_1\neq\Delta}} \left[e^{-\tilde{V}_{\Delta_1}(\phi_1)}\right]\right) \times \left(e^{-\tilde{V}_{\Delta}(\phi_1)}\ p_\Delta\right) \\
&= e^{-\hat{V}_{\Delta'}(\phi)} \int d\mu_\Gamma(\zeta)\ e^{\int_{L^{-1}\Delta'} f\zeta} \times \left[\sum_{\Delta\in[L^{-1}\Delta']} p_\Delta\right] \\
&= e^{-\hat{V}_{\Delta'}(\phi)} \sum_{a=0}^{4} L^{-a[\phi]} : \phi^a_{\Delta'} :_{C_0}\ \sum_b \mathbb{1}\left\{\begin{array}{c} a+b\leqslant 4 \\ b\geqslant 1 \end{array}\right\} \frac{(a+b)!}{a!\ b!} \\
&\qquad\qquad \times \sum_{\Delta\in[L^{-1}\Delta']} \int d\mu_\Gamma(\zeta)\ e^{\int_{L^{-1}\Delta'} f\zeta} \times \beta_{a+b,\Delta} \times : \zeta^b_\Delta :_\Gamma \\
&= \sum_{a=0}^{4} L^{-a[\phi]} : \phi^a_{\Delta'} :_{C_0}\ \sum_b \mathbb{1}\left\{\begin{array}{c} a+b\leqslant 4 \\ b\geqslant 1 \end{array}\right\} \frac{(a+b)!}{a!\ b!} \\
&\qquad\qquad \times \int_{(L^{-1}\Delta')^{b+1}} d^3x\ d^3y_1\cdots d^3y_b\ \beta_{a+b}(x) \times \prod_{i=1}^{b} \left[\Gamma(x-y_i)\ f(y_i)\right].
\end{aligned} \tag{3.37}$$

We note that the final integral could also be written in the form

$$\sum_{\Delta,\Delta_1,\ldots,\Delta_b\in[L^{-1}\Delta']} \beta_{a+b,\Delta} \times \prod_{i=1}^{b} \left[\Gamma(\Delta,\Delta_i)\ f_{\Delta_i}\right]$$

where $\Gamma(\Delta,\Delta_i) = \int d\mu_\Gamma\ \zeta_\Delta\zeta_{\Delta_i}$.

These terms represent all the contractions of a single vertex from V with external legs coming from f

along with an overall factor of $\exp\left[\frac{1}{2}(f,\Gamma f)_{L^{-1}\Delta'}\right]$.

We take a short detour to explain how one can compute the Gaussian integral with respect to μ_Γ above. One way to do this is via the translation trick which we now explain. Suppose that H is a function on $\mathbb{R}^{[L^{-1}\Delta']}$ such that

$$\int \mathrm{d}\mu_\Gamma(\zeta)\,|H(\zeta)| < \infty.$$

Then one has

$$\int \mathrm{d}\mu_\Gamma(\zeta) e^{\int_{L^{-1}\Delta'} f\zeta} H(\zeta) \;=\; e^{\frac{1}{2}(f,\Gamma f)_{L^{-1}\Delta'}} \int \mathrm{d}\mu_\Gamma(\zeta)\; H(\zeta+\Gamma f)$$

where $(\Gamma f)(x) = \int_{\mathbb{Q}_p^3} \Gamma(x-y) f(y)$ - we remark that Γf is also constant over blocks of $\mathbb{L}$.

This identity essentially boils down to "completing the square" or equivalently a change of variable $\zeta \leftrightarrow \zeta - \Gamma f$ in the left hand integral. One has to take some care to be precise however since Γ is not positive definite but this can be handled easily by using a change of variable taking one to the subspace of $\mathbb{R}^{[L^{-1}\Delta']}$ where μ_Γ is supported. To apply this to compute (3.37) we observe that for $\Delta \in [L^{-1}\Delta']$ one has

$$\begin{aligned}
&\int \mathrm{d}\mu_\Gamma(\zeta)\; e^{\int_{L^{-1}\Delta'} f\zeta} \times \beta_{a+b,\Delta} \times :\zeta_\Delta^b:_\Gamma \\
&= e^{\frac{1}{2}(f,\Gamma f)_{L^{-1}\Delta'}} \int \mathrm{d}\mu_\Gamma(\zeta)\; \beta_{a+b,\Delta} \times :(\zeta_\Delta + (\Gamma f)_\Delta)^b :_\Gamma \\
&= e^{\frac{1}{2}(f,\Gamma f)_{L^{-1}\Delta'}} \int \mathrm{d}\mu_\Gamma(\zeta)\; \beta_{a+b,\Delta} \times \sum_{l=0}^{b} \binom{b}{l} :\zeta_\Delta^l:_\Gamma \times (\Gamma f)_\Delta^{b-l} \\
&= e^{\frac{1}{2}(f,\Gamma f)_{L^{-1}\Delta'}} \times (\Gamma f)_\Delta^b \;=\; \sum_{\Delta_1,\ldots,\Delta_b \in [L^{-1}\Delta']} \prod_{i=1}^{b} [\Gamma(\Delta,\Delta_i)\; f_{\Delta_i}]
\end{aligned}$$

which yields the last line of (3.37).

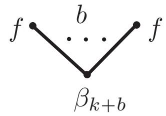
We now end the detour and write out A' which denoted the second order part of $K'_{\Delta'}(\phi,\lambda)$ - one has

$$\mathsf{A}' = e^{-\hat{V}_{\Delta'}(\phi)} \times \left\{ e^{-\frac{1}{2}(f,\Gamma f)_{L^{-1}\Delta'}} \times \left(\int \mathrm{d}\mu_\Gamma(\zeta)\; e^{\int_{L^{-1}\Delta'} f\zeta} \times \left[\sum_{\Delta \in [L^{-1}\Delta']} p_\Delta \right] \right) - e^{\frac{1}{2}(f,\Gamma f)_{L^{-1}\Delta'}} \sum_{k=0}^{4} \delta\beta_{k,1,\Delta'} :\phi_{\Delta'}^k:_{C_0} \right\}.$$

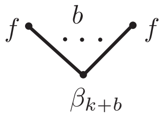
We note that the quantity A', apart from an overall factor of $e^{-\hat{V}_{\Delta'}(\phi)}$, is just some polynomial in $\phi_{\Delta'}$ or equivalently a linear combination of the $:\phi_{\Delta'}^j:_{C_0}$ - note that one will have $0 \leqslant j \leqslant 4$ so all these terms are relevant. We now specify a choice of the counterterms $\delta\beta_{k,1,\Delta'}$ to make A' vanish.

Namely, for any k, $0 \leqslant k \leqslant 4$, we let

$$\delta\beta_{k,1,\Delta'} = -\sum_b \mathbb{1}\left\{ \begin{array}{c} k+b \leqslant 4 \\ b \geqslant 1 \end{array} \right\} \frac{(k+b)!}{k!\ b!}\ L^{-k[\phi]} \tag{3.38}$$



where



$$= \int_{(L^{-1}\Delta')^{b+1}} \mathrm{d}^3x\ \mathrm{d}^3y_1 \cdots \mathrm{d}^3y_b\ \beta_{k+b}(x) \times \prod_{i=1}^{b} \left[\Gamma(x-y_i)\ f(y_i)\right] .$$

We remark that these counterterms correspond to item (i) on the list of the types of relevant terms we will extract.

We now describe the order 2 in λ part of the $\hat{K}'_{\Delta'}(\phi,\lambda)$ given in (3.36), i.e. B'. First we investigate the order 2 portion of $\hat{K}_{\Delta'}(\phi,\Lambda)$, i.e. B. The quantity B involves terms where in (3.35) where $|Y_P| = 2$ and $Y_K = \varnothing$, $|Y_P| = 1$ and $Y_K = \varnothing$, and $Y_P = \varnothing$ and $|Y_K| = 1$. Writing this out one has

$$\begin{aligned} \mathsf{B} \ = \int \mathrm{d}\mu_\Gamma(\zeta)\ e^{\int_{L^{-1}\Delta'} f\zeta} \times \Bigg[& \sum_{\substack{\Delta_1,\Delta_2 \in [L^{-1}\Delta'] \\ \Delta_1 \neq \Delta_2}} \prod_{\substack{\Delta \in [L^{-1}\Delta'] \\ \Delta \neq \Delta_1 \mathrm{or} \Delta_2}} \left[e^{-\tilde{V}_\Delta(\phi_1)}\right] \times \left(e^{-\tilde{V}_{\Delta_1}(\phi_1)}\ p_{\Delta_1}\right) \times \left(e^{-\tilde{V}_{\Delta_2}(\phi_1)}\ p_{\Delta_2}\right) \\ & + \sum_{\Delta_3 \in [L^{-1}\Delta']} \prod_{\substack{\Delta \in [L^{-1}\Delta'] \\ \Delta \neq \Delta_3}} \left[e^{-\tilde{V}_\Delta(\phi_1)}\right] \times \left(e^{-\tilde{V}_{\Delta_3}(\phi_1)}\ \frac{1}{2} p_{\Delta_3}^2\right) \\ & + \sum_{\Delta_4 \in [L^{-1}\Delta]} \prod_{\substack{\Delta \in [L^{-1}\Delta'] \\ \Delta \neq \Delta_4}} \left[e^{-\tilde{V}_\Delta(\phi_1)}\right] \times Q_{\Delta_4}(\phi_1) \Bigg]. \end{aligned} \tag{3.39}$$

From the third term we see that the only way the old K can contribute to B is through its explicit Q part - this is due to the imposed λ-grading in our definition of the λ-deformed $K_\Delta(\phi,\lambda)$. The first two terms in the formula are precisely the content of second order pertubation theory, that is the V^2 terms. Simplifying the above expression gives

$$\mathsf{B} \ = e^{-\hat{V}_{\Delta'}(\phi)} \times \int \mathrm{d}\mu_\Gamma(\zeta)\ e^{\int_{L^{-1}\Delta'} f\zeta} \times \left[\frac{1}{2} \left(\sum_{\Delta \in [L^{-1}\Delta']} p_\Delta \right)^2 + Q_{\Delta_4}(\phi_1) e^{-\tilde{V}_{\Delta_4}(\phi_1)} \right].$$

Along with an overall factor $e^{-\frac{1}{2}(f,\Gamma f)_{L^{-1}\Delta'}}$ the quantity B will contain the terms described in item (ii) of our list of relevant terms that will be extracted. It also contains products of relevant terms from item (i) on our list - however our previous counterterms $\delta\beta_{k,1,\Delta'}$ will precisely cancel these products. Finally B also contains $:\phi_\Delta^5:_{C_0}$ and $:\phi_\Delta^6:_{C_0}$ terms but we do not extract these - we will leave these terms to form the our new $Q'_{\Delta'}$. We make this more precise: the order 2 in λ part of $K'_{\Delta'}(\lambda, \phi)$ is given by

$$\begin{aligned} \mathsf{B}' &= e^{-\frac{1}{2}(f,\Gamma f)_{L^{-1}\Delta'}} \times \left(-\delta\beta_{0,1,\Delta'} \times \left[\mathsf{A} - e^{-\hat{V}_{\Delta'}(\phi)} \sum_{k=0}^{\infty} \delta\beta_{k,1,\Delta'} :\phi_{\Delta'}^{k}:_{C_0} \right] \right) \\ &+ \left[\mathsf{B} - \sum_{k=0} \delta\beta_{k,2,\Delta'} :\phi_{\Delta'}^{k}:_{C_0} - \left(\sum_{k=0} \delta\beta_{k,1,\Delta'} :\phi_{\Delta'}^{k}:_{C_0} \right)^2 \right]. \end{aligned} \tag{3.40}$$

The first vanishes immediately since that is precisely how the $\delta\beta_{k,1,\Delta'}$ we chosen. We remark that the second line of the above expression is what we refering to in the earlier paragraph. We will choose the yet to be defined order 2 counterterms $\delta\beta_{k,2,\Delta'}$ so that

$$\mathsf{B}' = e^{-\hat{V}_{\Delta'}(\phi)} Q'_{\Delta'}(\phi)$$

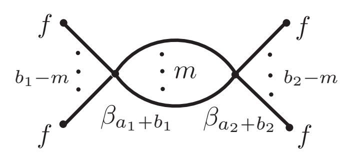
where $Q'_{\Delta'}(\phi)$ is of the form

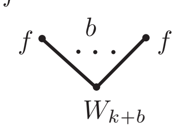
$$Q'_{\Delta'}(\phi) = W'_{5,\Delta'} :\phi_{\Delta'}^{5}:_{C_0} + W'_{6,\Delta'} :\phi_{\Delta'}^{6}:_{C_0}$$

and $W'_{5,\Delta'}$, $W'_{6,\Delta'}$ are new coefficients.

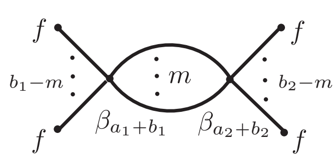
In particular we choose the counterterms $\delta\beta_{k,2,\Delta'}$ so that when writing out the quantity within brackets on the second line of (3.40) in terms of the powers $:\phi_{\Delta'}^{k}:_{C_0}$ there are no such terms with $k = 0, 1, 2, 3, 4$. This fixes the choice

$$\delta\beta_{k,2,\Delta'} = \sum_{a_1,a_2,b_1,b_2,m} \mathbb{1}\left\{ \begin{array}{c} a_i + b_i \leqslant 4 \\ a_i \geqslant 0 \ , \ b_i \geqslant 1 \\ 1 \leqslant m \leqslant \min(b_1, b_2) \end{array} \right\} \frac{(a_1+b_1)!\ (a_2+b_2)!}{a_1!\ a_2!\ m!\ (b_1-m)!\ (b_2-m)!}$$

$$\times \frac{1}{2} C(a_1, a_2|k) \times L^{-(a_1+a_2)[\phi]} \times C_0(0)^{\frac{a_1+a_2-k}{2}} \times$$ 

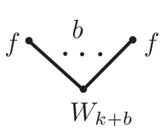
$$+ \sum_{b} \mathbb{1}\left\{ \begin{array}{c} k+b = 5 \text{ or } 6 \\ b \geqslant 0 \end{array} \right\} \frac{(k+b)!}{k!\ b!}\ L^{-k[\phi]}$$  (3.41)

where



$$= \int_{(L^{-1}\Delta')^{b_1+b_2-2m+2}} \mathrm{d}^3x_1\ \mathrm{d}^3x_2\ \mathrm{d}^3y_1 \cdots \mathrm{d}^3y_{b_1-m}\ \ \mathrm{d}^3z_1 \cdots \mathrm{d}^3z_{b_2-m}$$

$$\beta_{a_1+b_1}(x_1)\ \beta_{a_2+b_2}(x_2)\ \Gamma(x_1-x_2)^m \times \prod_{i=1}^{b_1-m} [\Gamma(x_1-y_i)\ f(y_i)] \times \prod_{i=1}^{b_2-m} [\Gamma(x_2-z_i)\ f(z_i)]$$



$$= \int_{(L^{-1}\Delta')^{b+1}} \mathrm{d}^3x\ \mathrm{d}^3y_1 \cdots \mathrm{d}^3y_b\ W_{k+b}(x) \times \prod_{i=1}^{b} [\Gamma(x-y_i)\ f(y_i)]$$

and where $C(a_1, a_2|k)$ are connection coefficients for Hermite polynomials. More precisely

$$C(a_1,a_2|k) = \mathbb{1}\left\{ \begin{array}{c} |a_1-a_2| \leqslant k \leqslant a_1+a_2 \\ a_1+a_2+k \in 2\mathbb{Z} \end{array} \right\} \times \frac{a_1!\ a_2!}{\left(\frac{a_1+a_2-k}{2}\right)!\ \left(\frac{a_1+k-a_2}{2}\right)!\ \left(\frac{a_2+k-a_1}{2}\right)!}\ .$$

These satisfy the property

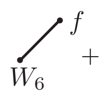
$$:\phi_{\Delta'}^{a_1}:_{C_0}\ \times\ :\phi_{\Delta'}^{a_2}:_{C_0} = \sum_k C(a_1,a_2|k)\ C_0(0)^{\frac{a_1+a_2-k}{2}}\ :\phi_{\Delta'}^{k}:_{C_0}\ .$$

Now considering the $\delta\beta_{k,2,\Delta'}$ given as above we reading the coefficents of $:\phi_{\Delta'}^5:_{C_0}$ and $\phi_{\Delta}^6:_{C_0}$ terms that are left in (3.40) to give us our formulas for $W'_{5,\Delta'}$ and $W_{6,\Delta'}$ -

$$W'_{6,\Delta'} = L^{3-6[\phi]} \underset{\Delta\in[L^{-1}\Delta']}{\text{avg}} W_{6,\Delta} + 8L^{-6[\phi]}$$

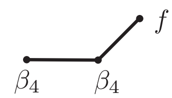
and

$$W'_{5,\Delta'} = L^{3-5[\phi]} \underset{\Delta\in[L^{-1}\Delta']}{\text{avg}} W_{5,\Delta} + 6L^{-5[\phi]}$$



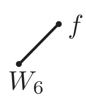
$$+\ 12L^{-5[\phi]}$$

$$+\ 48L^{-5[\phi]}$$



The Feynman diagrams are given explicitly by

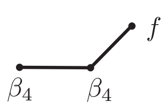
$$= \int_{(L^{-1}\Delta')^2} \mathrm{d}^3x\ \mathrm{d}^3y\ \beta_4(x)\ \Gamma(x-y)\ \beta_4(y)$$



$$= \int_{(L^{-1}\Delta')^2} \mathrm{d}^3x\ \mathrm{d}^3y\ W_6(x)\ \Gamma(x-y)\ f(y)$$

$$= \int_{(L^{-1}\Delta')^2} \mathrm{d}^3x\ \mathrm{d}^3y\ \beta_4(x)\ \Gamma(x-y)\ \beta_3(y)$$

and



$$= \int_{(L^{-1}\Delta')^3} \mathrm{d}^3x\ \mathrm{d}^3y\ \mathrm{d}^3z\ \beta_4(x)\ \Gamma(x-y)\ \beta_4(y)\ \Gamma(y-z)\ f(z).$$

The order 3 counterterms $\delta\beta_{k,3,\Delta'}$ will be defined as $(\vec{\beta}, f)$-dependent linear functions of R. This is a bit lengthy so we need a few preparatory steps before we can give the explicit formulas for these counterterms.

We split the third order part of $\hat{K}_{\Delta'}(\phi,\lambda)$, which we denoted by the quantity C, as $\mathsf{C} = \mathsf{C}_0 + \mathsf{C}_1$. Here C_0 contains the three vertex terms coming from products involving p_Δ's and Q_Δ's while C_1 contains the terms containing R.

$$\mathsf{C}_0 \quad = \quad -\frac{1}{6} \sum_{\substack{\Delta_1,\Delta_2,\Delta_3\in[L^{-1}\Delta'] \\ \text{distinct}}} e^{-\hat{V}_{\Delta'}(\phi)} \int \mathrm{d}\mu_\Gamma(\zeta)\ e^{\int_{L^{-1}\Delta'} f\zeta}\ p_{\Delta_1}\ p_{\Delta_2}\ p_{\Delta_3}$$

$$-\frac{1}{2}\sum_{\substack{\Delta_1,\Delta_2\in[L^{-1}\Delta'] \\ \text{distinct}}} e^{-\hat{V}_{\Delta'}(\phi)}\int d\mu_\Gamma(\zeta)\ e^{\int_{L^{-1}\Delta'} f\zeta}\ p_{\Delta_1}\ p_{\Delta_2}^2$$

$$-\sum_{\substack{\Delta_1,\Delta_2\in[L^{-1}\Delta'] \\ \text{distinct}}} e^{-\hat{V}_{\Delta'}(\phi)}\int d\mu_\Gamma(\zeta)\ e^{\int_{L^{-1}\Delta'} f\zeta}\ p_{\Delta_1}\ Q_{\Delta_2}(\phi_1+\zeta)$$

$$+\sum_{\Delta_1\in[L^{-1}\Delta']} e^{-\hat{V}_{\Delta'}(\phi)}\int d\mu_\Gamma(\zeta)\ e^{\int_{L^{-1}\Delta'} f\zeta}\ Q_{\Delta_1}(\phi_1+\zeta)\left(e^{-p_{\Delta_1}}-1\right)$$

$$+\sum_{\Delta_1\in[L^{-1}\Delta']}\left(\prod_{\substack{\Delta\in[L^{-1}\Delta'] \\ \Delta\neq\Delta_1}} e^{-\tilde{V}_\Delta(\phi_1)}\right)\times\int d\mu_\Gamma(\zeta)\ e^{\int_{L^{-1}\Delta'} f\zeta}\ r_{1,\Delta_1}$$

and

$$\mathsf{C}_1=\sum_{\Delta_1\in[L^{-1}\Delta']}\left(\prod_{\substack{\Delta\in[L^{-1}\Delta'] \\ \Delta\neq\Delta_1}} e^{-\tilde{V}_\Delta(\phi_1)}\right)\times\int d\mu_\Gamma(\zeta)\ e^{\int_{L^{-1}\Delta'} f\zeta}\ R_{\Delta_1}(\phi_1+\zeta)\ .$$

Note that we will not need the detailed evaluation of C_0, but we simply need the remark that it is R-independent. We remark that we will not extract any terms from C_0. Note that since each R_Δ comes with a λ^3 the quantity C_1 is linear in the R_Δ's.

We now define

$$\delta V_{j,\Delta'}(\phi)=\sum_{k=0}^{4}\delta\beta_{k,j,\Delta'}:\phi^k:_{C_0}$$

for $1\leqslant j\leqslant 3$ so that

$$\delta V_{j,\Delta'}(\phi,\lambda)=\sum_{j=1}^{3}\delta V_{j,\Delta'}(\phi)\lambda^j.$$

Then the λ^3 coefficient of $K'_{\Delta'}(\lambda,\phi)$ is given by $\mathsf{C}'=\mathsf{C}'_0+\mathsf{C}'_1$ where

$$\mathsf{C}'_0=e^{-\frac{1}{2}(f,\Gamma f)_{L^{-1}\Delta'}}\ \mathsf{C}_0-e^{-\hat{V}_{\Delta'}(\phi)}\left(\frac{1}{6}\delta V_{1,\Delta'}(\phi)^3+\delta V_{1,\Delta'}(\phi)\ \delta V_{2,\Delta'}(\phi)\right)-e^{-\hat{V}_{\Delta'}(\phi)}\ Q'_{\Delta'}(\phi)\ \delta\beta_{0,1,\Delta'}$$

and

$$\mathsf{C}'_1=e^{-\frac{1}{2}(f,\Gamma f)_{L^{-1}\Delta'}}\ \mathsf{C}_1-e^{-\hat{V}_{\Delta'}(\phi)}\ \delta V_{3,\Delta'}(\phi)\ .$$

The order 3 counterterms $\delta\beta_{k,3,\Delta'}$ in $\delta V_{3,\Delta'}(\phi)$ will be chosen to eliminate the zeroth through fourth order terms in the Taylor expansion for C'_1 around $\phi=0$ - this is what the removal of relevant terms looks like at this order. Due to the spatial inhomogeniety of the terms R_Δ we arrive at our order 3 counterms by defining counterterms corresponding to each of the unit blocks $\Delta\in[L^{-1}\Delta]$. More concrerely we will define $\delta\beta_{k,3,\Delta',\Delta_1}$ for $0\leqslant k\leqslant 4$, $\Delta'\in\mathbb{L}$ and $\Delta_1\in[L^{-1}\Delta']$ such that

$$\delta\beta_{k,3,\Delta'}=\sum_{\Delta_1\in[L^{-1}\Delta']}\delta\beta_{k,3,\Delta',\Delta_1}\ .$$

We then have

$$\mathsf{C}_1' = \sum_{\Delta_1 \in [L^{-1}\Delta']} \left(\prod_{\substack{\Delta \in [L^{-1}\Delta'] \\ \Delta \neq \Delta_1}} e^{-\tilde{V}_\Delta(\phi_1)} \right) \times J_{\Delta',\Delta_1}(\phi)$$

where

$$\begin{aligned} J_{\Delta',\Delta_1}(\phi) &= e^{-\frac{1}{2}(f,\Gamma f)_{L^{-1}\Delta'}} \times \int \mathrm{d}\mu_\Gamma(\zeta)\ e^{\int_{L^{-1}\Delta'} f\zeta}\ R_{\Delta_1}(\phi_1+\zeta) \\ &- \left(\sum_{k=0}^{4} \delta\beta_{k,3,\Delta',\Delta_1} :\phi^k:_{C_0} \right) \times e^{-\tilde{V}_{\Delta_1}(\phi_1)} \ . \end{aligned} \tag{3.42}$$

The quantities $\delta\beta_{k,3,\Delta',\Delta_1}$ are uniquely determined by imposing the following normalization conditions on the derivatives up to order 4:

$$J^{(\nu)}_{\Delta',\Delta_1}(0) = 0$$

for all $\Delta' \in \mathbb{L}$, $\Delta_1 \in [L^{-1}\Delta']$ and ν such that $0 \leqslant \nu \leqslant 4$.

Write $J_{\Delta',\Delta_1}(\phi) = J_+(\phi) - J_-(\phi)$ where

$$J_+(\phi) = e^{-\frac{1}{2}(f,\Gamma f)_{L^{-1}\Delta'}} \times \int \mathrm{d}\mu_\Gamma(\zeta)\ e^{\int_{L^{-1}\Delta'} f\zeta}\ R_{\Delta_1}(\phi_1+\zeta)$$

and

$$J_-(\phi) = \left(\sum_{k=0}^{4} \delta\beta_{k,3,\Delta',\Delta_1} :\phi^k:_{C_0} \right) \times e^{-\tilde{V}_{\Delta_1}(\phi_1)} \ .$$

For any ν, $0 \leqslant \nu \leqslant 4$, we have

$$J_+^{(\nu)}(0) = L^{-\nu[\phi]}\ e^{-\frac{1}{2}(f,\Gamma f)_{L^{-1}\Delta'}} \times \int \mathrm{d}\mu_\Gamma(\zeta)\ e^{\int_{L^{-1}\Delta'} f\zeta}\ R^{(\nu)}_{\Delta_1}(\zeta) \ .$$

Whereas

$$J_-(\phi) = u(\phi)\ e^{v(\phi)}$$

with

$$u(\phi) = u_4\phi^4 + u_3\phi^3 + u_2\phi^2 + u_1\phi + u_0$$

and

$$v(\phi) = v_4\phi^4 + v_3\phi^3 + v_2\phi^2 + v_1\phi + v_0$$

with coefficients explicitly given by

$$\begin{aligned} u_4 &= \delta\beta_4 \\ u_3 &= \delta\beta_3 \\ u_2 &= \delta\beta_2 - 6C\delta\beta_4 \\ u_1 &= \delta\beta_1 - 3C\delta\beta_3 \\ u_0 &= \delta\beta_0 - C\delta\beta_2 + 3C^2\delta\beta_4 \end{aligned}$$

and

$$
\begin{aligned}
v_4 &= -L^{-4[\phi]}\beta_4 \\
v_3 &= -L^{-3[\phi]}\beta_3 \\
v_2 &= -L^{-2[\phi]}\beta_2 + 6CL^{-4[\phi]}\beta_4 \\
v_1 &= -L^{-[\phi]}\beta_1 + 3CL^{-3[\phi]}\beta_3 \\
v_0 &= CL^{-2[\phi]}\beta_2 - 3C^2L^{-4[\phi]}\beta_4 \ .
\end{aligned}
$$

Note that we used the abbreviated notation $\delta\beta_k = \delta\beta_{k,3,\Delta',\Delta_1}$, $\beta_k = \beta_{k,\Delta_1}$ and $C = C_0(0)$. Using Maple we found for the Taylor expansion of $J_-(\phi)$ up to order 4:

$$
\begin{aligned}
J_-(\phi) \quad = \quad & e^{v_0} \times \left\{ u_0 + (u_0\, v_1 + u_1)\,\phi + \left(u_1\, v_1 + u_0\, v_2 + \frac{1}{2}\, u_0\, {v_1}^2 + u_2 \right) \phi^2 \right.\\
& + \left(u_1\, v_2 + \frac{1}{2}\, u_1\, {v_1}^2 + u_2\, v_1 + u_0\, v_3 + u_0\, v_1\, v_2 + \frac{1}{6}\, u_0\, {v_1}^3 + u_3 \right) \phi^3 \\
& + \left(u_4 + u_1\, v_3 + u_1\, v_1\, v_2 + \frac{1}{6}\, u_1\, {v_1}^3 + u_0\, v_4 + u_0\, v_1\, v_3 \right.\\
& \left.\left. + \frac{1}{2}\, u_0\, {v_2}^2 + \frac{1}{2}\, u_0\, v_2\, {v_1}^2 + \frac{1}{24}\, u_0\, {v_1}^4 + u_2\, v_2 + \frac{1}{2}\, u_2\, {v_1}^2 + u_3\, v_1 \right) \phi^4 \right\} + O(\phi^5) \ .
\end{aligned}
$$

Write $a_\nu = e^{-v_0} J_+^{(\nu)}(0)$. We therefore have to solve for $u_0, \ldots, u_4$ in the triangular polynomial system

$$
\begin{aligned}
a_0 &= u_0 \\
a_1 &= u_1 + u_0\, v_1 \\
\frac{1}{2}\, a_2 &= u_2 + u_1\, v_1 + u_0\, v_2 + \frac{1}{2}\, u_0\, {v_1}^2 \\
\frac{1}{6}\, a_3 &= u_3 + u_1\, v_2 + \frac{1}{2}\, u_1\, {v_1}^2 + u_2\, v_1 + u_0\, v_3 + u_0\, v_1\, v_2 + \frac{1}{6}\, u_0\, {v_1}^3 \\
\frac{1}{24}\, a_4 &= u_4 + u_1\, v_3 + u_1\, v_1\, v_2 + \frac{1}{6}\, u_1\, {v_1}^3 + u_0\, v_4 + u_0\, v_1\, v_3 \\
& \quad + \frac{1}{2}\, u_0\, {v_2}^2 + \frac{1}{2}\, u_0\, v_2\, {v_1}^2 + \frac{1}{24}\, u_0\, {v_1}^4 + u_2\, v_2 + \frac{1}{2}\, u_2\, {v_1}^2 + u_3\, v_1 \ .
\end{aligned}
$$

This is straightforward but leads to complicated intermediate formulas which we skip. We then replace the $v's$ by their expressions in terms of the β's. Finally we use the obtained formulas for the u's in order to get

$$
\begin{aligned}
\delta\beta_4 &= u_4 \\
\delta\beta_3 &= u_3 \\
\delta\beta_2 &= u_2 + 6Cu_4 \\
\delta\beta_1 &= u_1 + 3Cu_3 \\
\delta\beta_0 &= u_0 + Cu_2 + 3C^2u_4 \ .
\end{aligned}
$$

The final result, obtained with the help of Maple, can be found on [3, p. 23].

Using the notation

$$
\begin{aligned}
a_i \quad &= \quad \exp\left[-CL^{-2[\phi]}\beta_2 + 3C^2L^{-4[\phi]}\beta_4 - \frac{1}{2}(f,\Gamma f)_{L^{-1}\Delta'}\right] \\
&\quad \times L^{-i[\phi]} \times \int \mathrm{d}\mu_\Gamma(\zeta)\ e^{\int_{L^{-1}\Delta'} f\zeta}\ R^{(i)}_{\Delta_1}(\zeta)\ . \qquad (3.43)
\end{aligned}
$$

The final formulas are of the form

$$
\delta\beta_k = \sum_{i=0}^{4} M_{k,i}\ a_i
$$

where the matrix elements $M_{k,i}$ are given by finite sums of the form

$$
M_{k,i} = \sum\ \#\ C^j L^{-(l_1+\cdots+l_n)[\phi]}\ \beta_{l_1}\cdots\beta_{l_n} \qquad (3.44)
$$

with $j \geqslant 0$, $n \geqslant 0$, and $1 \leqslant l_m \leqslant 4$ for every m, $1 \leqslant m \leqslant n$. Here the symbol # stands for some purely numerical constants. Furthermore, the terms which appear satisfy the homogeneity constraint

$$
l_1 + \cdots + l_n - 2j = k - i\ . \qquad (3.45)
$$

We also have a limitation on the range of allowed n's:

$$
n \leqslant (k-i) + 2\left\lfloor \frac{4-k}{2} \right\rfloor\ .
$$

This completes the definition of the $\delta\beta_{k,3,\Delta',\Delta_1}$ and therefore of the order 3 counterterms

$$
\delta\beta_{k,3,\Delta'} = \sum_{\Delta_1 \in [L^{-1}\Delta']} \delta\beta_{k,3,\Delta',\Delta_1}\ .
$$

We then define

$$
\mathcal{L}^{(\vec{\beta},f)}_{\Delta'}(R) = \mathsf{C}'_1
$$

with the previous choices for the $\delta\beta_{k,3,\Delta',\Delta_1}$. This makes $\mathcal{L}^{(\vec{\beta},f)}$ a $(\vec{\beta},f)$-dependent linear operator on the space where R lives (in particular R and both the left and right sides of the equation above are seen as $\mathbb{L}$ indexed vectors of elements in $\mathcal{C}^9_{\mathrm{bd}}(\mathbb{R},\mathbb{C})$). We have arrived at a complete definition of

$$
K'_{\Delta'}(\lambda,\phi) = \lambda^2 e^{-\hat{V}_{\Delta'}(\phi)} Q'_{\Delta'}(\phi) + \lambda^3\mathsf{C}'_0 + \lambda^3\mathsf{C}'_1 + O(\lambda^4)\ .
$$

Looking to the desired output of the RG map we remark that the new couplings $\beta'_{k,\Delta'}$ as well as the quantities $\delta b_{\Delta'}$ are fully defined. We just need the new R. It is given by

$$
R'_{\Delta'} = \mathcal{L}^{(\vec{\beta},f)}_{\Delta'}(R) + \xi_{R,\Delta'}(\vec{V})
$$

where the formula for remainder term is

$$\xi_{R,\Delta'}(\vec{V})(\phi) = \frac{1}{2\pi i}\oint_{\gamma_0} \frac{\mathrm{d}\lambda}{\lambda^4} K'_{\Delta'}(\lambda,\phi)\big|_{R=0} \tag{3.46}$$

$$+\frac{1}{2\pi i}\oint_{\gamma_{01}} \frac{\mathrm{d}\lambda}{\lambda^4(\lambda-1)} K'_{\Delta'}(\lambda,\phi) \tag{3.47}$$

$$+\left(e^{-\hat{V}_{\Delta'}(\phi)} - e^{-V'_{\Delta'}(\phi)}\right) Q'_{\Delta'}(\phi) \tag{3.48}$$

where γ_0 is any positively oriented contour around $\lambda = 0$, and γ_{01} is any positively oriented contour which encloses both $\lambda = 0$ and $\lambda = 1$. With these choices our new K' takes the form of our old K, that is

$$K'_{\Delta'}(\lambda,\phi)\big|_{\lambda=1} = Q'_{\Delta'}(\phi)e^{V'_{\Delta'}(\phi)} + R'_{\Delta'}.$$

In [18] the three terms for the remainder (3.46) are respectively denoted by R_{main}, R_3, and R_4. It is important to note that in the first term we set $R = 0$, which means that all $\delta\beta_{k,3,\Delta'}$ are set equal to zero. Also note that $\mathcal{L}^{(\vec{\beta},f)}(R)$ corresponds to the R_{linear} notation in [18].

To finish setting up the notation we write for $1 \leqslant k \leqslant 4$

$$\xi_{k,\Delta'}(\vec{V}) = -\delta\beta_{k,3,\Delta'}$$

whereas

$$\xi_{0,\Delta'}(\vec{V}) = \delta\beta_{0,3,\Delta'}.$$

In this way the RG evolution for the couplings is

$$\beta'_{k,\Delta'} = \hat{\beta}_{k,\Delta'} - \delta\beta_{k,1,\Delta'} - \delta\beta_{k,2,\Delta'} + \xi_{k,\Delta'}(\vec{V})$$

for $1 \leqslant k \leqslant 4$. and the vacuum renormalizations are given by

$$\delta b_{\Delta'}[\vec{V}] = \delta\beta_{0,1,\Delta'} + \delta\beta_{0,2,\Delta'} + \xi_{0,\Delta'}(\vec{V}).$$

3.6 Estimates on the extended RG map

While we won't directly use the following lemma to prove the main theorem of this section (Theorem 3.2) it will be crucial to leveraging the results of that theorem - in particular it allows us to use analyticity and strong uniform bounds to get crucial Lipschitz estimates that will be used when analyzing the RG flow. For a reference on the theory of analytic maps in the complex Banach space context see [20]. Below we use the notation $B(x_0, r)$ for the open ball of radius r centered at x_0. We likewise use $\bar{B}(x_0, r)$ to denote the corresponding closed ball.

Lemma 3.2. *Let X and Y be two complex Banach spaces. Suppose $r_1 > 0$ and $r_2 \geqslant 0$. Let $x_0 \in X$ and $y_0 \in Y$, and let f be an analytic map*

$$f: \quad B(x_0, r_1) \quad \longrightarrow \quad \bar{B}(y_0, r_2) \quad .$$

Let $\nu \in (0, \frac{1}{2})$, then for any $x_1, x_2 \in \bar{B}(x_0, \nu r_1)$

$$||f(x_1) - f(x_2)|| \leqslant \frac{r_2(1-\nu)}{r_1(1-2\nu)} \ ||x_1 - x_2|| \ .$$

Proof: Suppose $x_1 \neq x_2$ satisfy the hypothesis of the proposition. For $z \in \mathbb{C}$ define

$$g(z) = f\left(\frac{x_1 + x_2}{2} + z\frac{x_1 - x_2}{2}\right) - y_0 \ .$$

We first find a bound on $|z|$ which guarantees that the argument of f is in the ball $B(x_0, r_1)$. Since $\nu < \frac{1}{2}$, we have

$$2r_1(1-\nu) > 2\nu r_1 \geqslant ||x_1 - x_0|| + ||x_2 - x_0|| \geqslant ||x_1 - x_2|| \ .$$

Therefore

$$R_{\max} = \frac{2r_1(1-\nu)}{||x_1 - x_2||} > 1 \ .$$

Now the open interval $(1, R_{\max})$ is nonempty, and for any R in this interval as well as for any z with $|z| \leqslant R$ we have

$$||\frac{x_1 + x_2}{2} + z\frac{x_1 - x_2}{2} - x_0|| \leqslant \nu r_1 + \frac{R}{2}||x_1 - x_2|| < r_1 \ .$$

Let γ be the circle of radius R around the origin in the complex plane. For such an $R \in (1, R_{\max})$ we have by Cauchy's Theorem

$$f(x_1) - f(x_2) = g(1) - g(-1) = \frac{1}{\pi i} \oint_\gamma \frac{g(z)}{z^2 - 1} \ dz \ .$$

Hence

$$||f(x_1) - f(x_2)|| \leqslant \frac{1}{\pi} \times 2\pi R r_2 \times \max_{|z|=R} \frac{1}{|z^2 - 1|} = \frac{2Rr_2}{R^2 - 1} \ .$$

We now minimize this bound with respect to $R \in (1, R_{\max})$. Since $R \mapsto \frac{2R}{R^2-1}$ is decreasing on $(1, \infty)$,

$$\inf_{R \in (1, R_{\max})} \frac{2R}{R^2 - 1} = \frac{2R_{\max}}{R_{\max}^2 - 1} \ .$$

Inserting the formula for $R_{\max}$ in the upper bound for $||f(x_1)-f(x_2)||$ and simplifying the resulting expression gives the desired Lipschitz estimate. □

3.6.1 Exponential Bounds and Stability Estimates, and comments on the Functional norms

The background field ϕ and fluctuation field ζ that appear in expressions will be estimated by exponential bounds. For the background field ϕ this exponential bound will taking advantage of factors $e^{-\beta_4 \phi^4}$ while for the ζ we will steal from the Gaussian measure μ_Γ and so we will use exponential estimates using $e^{\kappa \zeta^2}$ for some κ sufficiently small (see Lemma 3.4).

At this point we remark that for certain stability estimates it will be crucial that the possibly complex ϕ^4 couplings β_4 are "mostly positive real", by the assumptions of Theorem 3.2, they will sit in an open ball of the form $|\beta_4 - \bar{g}| < \frac{1}{2}\bar{g}$ with $\bar{g} > 0$. By elementary trigonometry it easily follows that $\frac{\Re\beta_4}{|\beta_4|} \geqslant \frac{\sqrt{3}}{2}$. We of course also have $\frac{1}{2} < \frac{\Re\beta_4}{\bar{g}} < \frac{3}{2}$.

Lemma 3.3. *[Lemmas 8,9 of [3]]*

$\forall j \in \mathbb{N}$, $\forall\kappa > 0$, $\forall\zeta \in \mathbb{R}$ we have

$$|\zeta|^j \leqslant \left(\frac{j}{2e}\right)^{\frac{j}{2}} \kappa^{-\frac{j}{2}} e^{\kappa\zeta^2}$$

while $\forall\bar{g} > 0$, $\forall\gamma > 0$, $\forall\beta_4 \in \mathbb{C}$ such that $|\beta_4 - \bar{g}| < \frac{1}{2}\bar{g}$, $\forall\phi \in \mathbb{R}$ we have

$$|\phi|^j \leqslant \left(\frac{j}{4e}\right)^{\frac{j}{4}} (\gamma\Re\beta_4)^{-\frac{j}{4}} e^{\gamma(\Re\beta_4)\phi^4} \leqslant \left(\frac{j}{2e}\right)^{\frac{j}{4}} (\gamma\bar{g})^{-\frac{j}{4}} e^{\gamma(\Re\beta_4)\phi^4}$$

Above we use the convention $j^j = 1$ if $j = 0$.

Proof: The first assertion follows from noting that the function $u^{\frac{j}{2}} e^{u}$ for $u \geqslant 0$ is maximized when $u = \frac{j}{2}$ and then applying thus fact with $u = \kappa\zeta^2$. For the second asserition one notes that the function $u^{\frac{j}{4}} e^{-u}$ for $u \geqslant 0$ is maximized when $u = \frac{j}{4}$. Now simply apply this to $u = \gamma(\Re\beta_4)\phi^4$ and use $\frac{1}{2} < \frac{\Re\beta_4}{\bar{g}}$ for the second inequality. ☐

Lemma 3.4. *Let Δ' be a block in $\mathbb{L}$. Let the real parameter α satisfy $0 \leqslant \alpha \leqslant \frac{\sqrt{2}}{4} L^{-(3-2[\phi])}$. If f is a real-valued function on $L^{-1}\Delta'$ which is constant on unit cubes and such that $||f||_{L^\infty} \leqslant \frac{1}{2} L^{-\frac{1}{2}(3-2[\phi])}$, then for any finite set $X \subset [L^{-1}\Delta']$ we have the bound*

$$\int \mathrm{d}\mu_\Gamma(\zeta)\ e^{\int_{L^{-1}\Delta'} f\zeta} \prod_{\Delta \in X} e^{\alpha\zeta_\Delta^2} \leqslant 2^{|X|} e^{\frac{1}{2}(f,\Gamma f)_{L^{-1}\Delta'}}\ .$$

Proof:

First note that one can view the integral we would like to bound, I, as an expectation with respect to the centered Gaussian vector $(\zeta_\Delta)_{\Delta \in [L^{-1}\Delta']}$ in $\mathbb{R}^{L^3}$ with covariance $\mathbf{E}(\zeta_{\Delta_1}\zeta_{\Delta_2}) = \Gamma_{\Delta_1,\Delta_2} = \Gamma(x_1 - x_2)$ where x_1 is any point in Δ_1 and likewise for x_2 in Δ_2. Let $u_1, \ldots, u_{L^3}$ be an orthonormal basis which diagonalizes Γ (seen as an $L^3 \times L^3$ matrix). Let $\lambda_1, \ldots, \lambda_{L^3}$ be the corresponding eigenvalues and suppose we arranged the numbering so that $\lambda_1 \geqslant \lambda_2 \geqslant \cdots$. Note that the matrix Γ is singular and therefore only positive semi-definite, because of the property that $\int_{L^{-1}\Delta'} \zeta = 0$ almost surely. We therefore introduce $m = \max\{i | \lambda_i > 0\}$. We now have that ζ has the same law as $\sum_{i=1}^m a_i u_i$ where the a_i's are independent centered Gaussian random variables with variance λ_i. Thus

$$I = \prod_{i=1}^m (2\pi\lambda_i)^{-\frac{1}{2}} \times \int_{\mathbb{R}^m} \mathrm{d}a_1 \ldots \mathrm{d}a_m\ \exp\left[-\frac{1}{2}\sum_{i=1}^m \frac{a_i^2}{\lambda_i} + \sum_{\substack{\Delta \in [L^{-1}\Delta'] \\ 1 \leqslant i \leqslant m}} f_\Delta a_i u_{i,\Delta} + \alpha \sum_{\Delta \in X} \left(\sum_{i=1}^m a_i u_{i,\Delta}\right)^2\right]\ .$$

Since $X \subset [L^{-1}\Delta']$

$$\sum_{\Delta \in X} \left(\sum_{i=1}^m a_i u_{i,\Delta}\right)^2 \leqslant \sum_{\Delta \in [L^{-1}\Delta']} \left(\sum_{i=1}^m a_i u_{i,\Delta}\right)^2 = \sum_{i=1}^m a_i^2$$

because of the orthonormality of the u's. Therefore a sufficient condition for the convergence of the integral is that $2\alpha\lambda_i < 1$ for all i, $1 \leqslant i \leqslant m$. Granting this condition for now, we define $\tilde{f}_i = \sum_{\Delta \in [L^{-1}\Delta']} f_\Delta u_{i,\Delta}$ and use the standard 'completing the square' trick by writing

$$-\frac{1}{2}\sum_{i=1}^{m} \frac{a_i^2}{\lambda_i} + \sum_{i=1}^{m} a_i \tilde{f}_i = -\frac{1}{2}\sum_{i=1}^{m} \frac{1}{\lambda_i}(a_i - \lambda_i \tilde{f}_i)^2 + \frac{1}{2}\sum_{i=1}^{m} \lambda_i \tilde{f}_i^2$$

and changing variables to $a_i - \lambda_i \tilde{f}_i$. Hence

$$I = \prod_{i=1}^{m} (2\pi\lambda_i)^{-\frac{1}{2}} \times \int_{\mathbb{R}^m} \mathrm{d}a_1 \ldots \mathrm{d}a_m \ \exp\left[-\frac{1}{2}\sum_{i=1}^{m} \frac{a_i^2}{\lambda_i} + \frac{1}{2}\sum_{i=1}^{m} \lambda_i \tilde{f}_i^2 + \alpha \sum_{\Delta \in X} \left(\sum_{i=1}^{m} (a_i + \lambda_i \tilde{f}_i) u_{i,\Delta} \right)^2 \right] .$$

Note that

$$\begin{aligned}
\sum_{i=1}^{m} \lambda_i \tilde{f}_i^2 &= \sum_{i=1}^{m} \sum_{\Delta_1, \Delta_2 \in [L^{-1}\Delta']} \lambda_i f_{\Delta_1} f_{\Delta_2} u_{i,\Delta_1} u_{i,\Delta_2} \\
&= \sum_{\Delta_1, \Delta_2 \in [L^{-1}\Delta']} f_{\Delta_1} f_{\Delta_2} \Gamma_{\Delta_1, \Delta_2} \\
&= (f, \Gamma f)_{L^{-1}\Delta'}
\end{aligned}$$

by construction of the u's. We also have

$$\begin{aligned}
\sum_{i=1}^{m} (a_i + \lambda_i \tilde{f}_i) u_{i,\Delta} &= \zeta_\Delta + \sum_{i=1}^{m} \sum_{\Delta_1 \in [L^{-1}\Delta']} \lambda_i f_{\Delta_1} u_{i,\Delta_1} u_{i,\Delta} \\
&= \zeta_\Delta + \sum_{\Delta_1 \in [L^{-1}\Delta']} \Gamma_{\Delta,\Delta_1} f_{\Delta_1} \\
&= \zeta_\Delta + (\Gamma f)_\Delta
\end{aligned}$$

where we reverted to the use of the ζ_Δ variables of integration which have the same law as the quantities $\sum_{i=1}^{m} a_i u_{i\Delta}$, and where $(\Gamma f)(x)$ denotes $\int_{\mathbb{Q}_p^3} \mathrm{d}^3 y \ \Gamma(x-y) f(y)$. By the finite range property of Γ we have, for $x \in \Delta \in [L^{-1}\Delta']$, $(\Gamma f)(x) = (\Gamma f)_\Delta = \sum_{\Delta_1 \in [L^{-1}\Delta']} \Gamma_{\Delta,\Delta_1} f_{\Delta_1}$. As a result of the previous calculations

$$I = e^{\frac{1}{2}(f,\Gamma f)_{L^{-1}\Delta'}} \times \int \mathrm{d}\mu_\Gamma(\zeta) \ e^{\alpha \sum_{\Delta \in X} ((\Gamma f)_\Delta + \zeta_\Delta)^2} .$$

We now expand the square in the last exponential and we also introduce the covariance matrix Γ_X for the marginal random vector $\zeta|_X = (\zeta_\Delta)_{\Delta \in X}$ in order to write

$$I = e^{\frac{1}{2}(f,\Gamma f)_{L^{-1}\Delta'} + \alpha(\Gamma f, \Gamma f)_X} \times \int \mathrm{d}\mu_{\Gamma_X}(\zeta|_X) \ e^{\alpha\langle \zeta|_X, \zeta|_X \rangle + 2\alpha \langle \Gamma f|_X, \zeta|_X \rangle}$$

where the inner products are the ones of $l^2(X)$, namely $\langle w, w' \rangle = \sum_{\Delta \in X} w_\Delta w'_\Delta$ for vectors in $l^2(X)$ which are indexed by boxes in the finite set X.

Let $(v_i)_{1 \leqslant i \leqslant |X|}$ be an orthonormal basis diagonalizing the symmetric positive semi-definite matrix Γ_X, with eigenvalues μ_i arranged so that $\mu_1 \geqslant \mu_2 \geqslant \cdots$ and let $n = \max\{i | \mu_i > 0\}$. As before, we have that the random vector $\zeta|_X$ has the same law as $\sum_{i=1}^n b_i v_i$ where the b_i are independent centered Gaussian random variables with variance μ_i. Following this change of variables of integration $\langle \zeta|_X, \zeta|_X \rangle$ becomes $\sum_{i=1}^n b_i^2$ whereas $\langle \Gamma f|_X, \zeta|_X \rangle$ becomes $\sum_{i=1}^n g_i b_i$ with $g_i = \sum_{\Delta \in X} (\Gamma f)_\Delta v_{i,\Delta}$. Hence

$$\begin{aligned}
\int \mathrm{d}\mu_{\Gamma_X}(\zeta|_X)\ e^{\alpha\langle \zeta|_X, \zeta|_X\rangle + 2\alpha\langle \Gamma f|_X, \zeta|_X\rangle} &= \prod_{i=1}^n \left[\frac{1}{\sqrt{2\pi\mu_i}} \int_{\mathbb{R}} \mathrm{d}b_i\ e^{-\frac{b_i^2}{2\mu_i} + \alpha b_i^2 + 2\alpha g_i b_i} \right] \\
&= \prod_{i=1}^n \left[\frac{1}{\sqrt{2\pi\mu_i}} \times \sqrt{2\pi \left(\frac{1}{\mu_i} - 2\alpha\right)^{-1}} \times e^{\frac{1}{2}\left(\frac{1}{\mu_i} - 2\alpha\right)^{-1}(2\alpha g_i)^2} \right] \\
&= \prod_{i=1}^n \left[\frac{1}{\sqrt{1-2\alpha\mu_i}} e^{2\alpha^2 \frac{\mu_i}{1-2\alpha\mu_i} g_i^2} \right]
\end{aligned}$$

provided $2\alpha\mu_i < 1$ for all i, $1 \leqslant i \leqslant n$.

Now $\mu_i \leqslant ||\Gamma_X||$ where the latter quantity is the operator norm of Γ_X induced by the norm on $l^2(X)$ coming from the inner product $\langle \cdot, \cdot \rangle$. For v a real vector in $l^2(X)$, we have $||\Gamma_X v||^2 = \sum_{\Delta \in X} (\Gamma_X v)_\Delta^2 = \sum_{\Delta \in X} (\Gamma w)_\Delta^2$ where $w \in l^2([L^{-1}\Delta'])$ is the extension of v by zero outside X. Thus

$$||\Gamma_X v||^2 \leqslant \sum_{\Delta \in [L^{-1}\Delta']} (\Gamma_X w)_\Delta^2 = ||\Gamma w||^2 \leqslant ||\Gamma||^2 ||w||^2 = ||\Gamma||^2 ||v||^2 .$$

As a result $||\Gamma_X|| \leqslant ||\Gamma||$ where the latter is the operator norm of the matrix Γ coming from the inner product norm of $l^2([L^{-1}\Delta'])$. However we have the bound $||\Gamma|| \leqslant ||\Gamma||_{L^1} = \int_{\mathbb{Q}_p^3} |\Gamma(x)| \mathrm{d}^3x$. Indeed, given $w \in l^2([L^{-1}\Delta'])$ which we can identify with a function $w(x)$ on $\mathbb{Q}_p^3$ with support in $L^{-1}\Delta'$ and which is constant on unit blocks, we have

$$\begin{aligned}
||\Gamma w||^2 &= \int_{\mathbb{Q}_p^3} [(\Gamma w)(x)]^2\ \mathrm{d}^3x \\
&= \int_{\mathbb{Q}_p^{3\times 3}} \Gamma(x-y)\Gamma(x-z) w(y) w(z)\ \mathrm{d}^3x\ \mathrm{d}^3y\ \mathrm{d}^3z \\
&\leqslant \int_{\mathbb{Q}_p^{3\times 3}} |\Gamma(x-y)|\ |\Gamma(x-z)|\ |w(y)|\ |w(z)|\ \mathrm{d}^3x\ \mathrm{d}^3y\ \mathrm{d}^3z \\
&\leqslant \int_{\mathbb{Q}_p^{3\times 3}} |\Gamma(x-y)|\ |\Gamma(x-z)|\ \left(\frac{1}{2}|w(y)|^2 + \frac{1}{2}|w(z)|^2\right)\ \mathrm{d}^3x\ \mathrm{d}^3y\ \mathrm{d}^3z \\
&= 2 \times \frac{1}{2} \times ||\Gamma||_{L^1}^2 ||w||_{L^2}^2 \ .
\end{aligned}$$

Therefore from Corollary 5.1 we get $||\Gamma|| \leqslant ||\Gamma||_{L^1} < \frac{1}{\sqrt{2}} L^{3-2[\phi]}$. Since the λ_i are bounded by $||\Gamma||$ (the case where $X = [L^{-1}\Delta']$), the hypothesis $\alpha \leqslant \frac{\sqrt{2}}{4} L^{-3+2[\phi]}$ implies that the previous convergence requirement $2\alpha\lambda_i < 1$ is satisfied and also that not only $2\alpha\mu_i < 1$ holds but so does the stronger inequality $2\alpha\mu_i \leqslant \frac{1}{2}$. From the latter we have $\frac{\mu_i}{1-2\alpha\mu_i} \leqslant 2\mu_i$ and thus

$$\int \mathrm{d}\mu_{\Gamma_X}(\zeta|_X)\ e^{\alpha\langle \zeta|_X, \zeta|_X\rangle + 2\alpha\langle \Gamma f|_X, \zeta|_X\rangle} \leqslant \prod_{i=1}^n \left(\sqrt{2} e^{4\alpha^2 \mu_i g_i^2} \right)$$

$$\leqslant\ 2^{\frac{|X|}{2}} \exp\left(\frac{\sqrt{2}}{4} L^{-(3-2[\phi])} \sum_{i=1}^{n} g_i^2\right)$$

where we used $n \leqslant |X|$, $\alpha \leqslant \frac{\sqrt{2}}{4} L^{-(3-2[\phi])}$ and $\mu_i < \frac{1}{\sqrt{2}} L^{3-2[\phi]}$. Besides, $g_i = \sum_{\Delta\in X} (\Gamma f)_\Delta v_{i,\Delta} = \langle v_i, (\Gamma f)|_X\rangle$ and therefore

$$\sum_{i=1}^{n} g_i^2 \leqslant \sum_{i=1}^{|X|} \langle v_i, (\Gamma f)|_X\rangle^2 = \langle (\Gamma f)|_X, (\Gamma f)|_X\rangle = (\Gamma f, \Gamma f)_X\ .$$

But $(\Gamma f, \Gamma f)_X = \sum_{\Delta\in X} (\Gamma f)_\Delta^2$ and clearly $|(\Gamma f)_\Delta| \leqslant ||\Gamma||_{L^1} ||f||_{L^\infty}$ so $(\Gamma f, \Gamma f)_X \leqslant |X|\ ||\Gamma||_{L^1}^2 ||f||_{L^\infty}^2$.

Putting all the previous bounds together we see that the desired inequality holds provided

$$\exp\left[\frac{\sqrt{2}}{4} L^{3-2[\phi]} ||f||_{L^\infty}^2\right] \leqslant \sqrt{2}$$

which is true since, by hypothesis, $||f||_{L^\infty} \leqslant \frac{1}{2} L^{-\frac{1}{2}(3-2[\phi])}$ and $\frac{4}{\sqrt{2}} \times \frac{1}{2}\log 2 \simeq 0.980\ldots > \frac{1}{4}$. □

Now we state the key stability bound that allows us to use $e^{-V(\phi)}$ to control growth in the field ϕ when using our norm $||\cdot||_{\partial\phi,\phi,h}$.

Lemma 3.5. *Let $U(\phi) = a_4\phi^4 + a_3\phi^3 + a_2\phi^2 + a_1\phi + a_0$ where the possibly complex coefficients $a_0, \ldots, a_4$ satisfy $|a_4| > 0$, $\Re a_4 \geqslant \frac{\sqrt{3}}{2}|a_4|$, $|a_4| \leqslant \frac{1}{3}\log\left(\frac{1+\sqrt{2}}{2}\right)|a_k|^{\frac{k}{4}}$ for $k = 1, 2, 3$, and $|a_0| \leqslant \log 2$. Then*

1. *the condition*

$$0 \leqslant \theta \leqslant \frac{\sqrt{2}-1}{4} e^{-918785} \times |a_4|^{-\frac{1}{4}}$$

implies

$$||e^{-U(\phi)}||_{\partial\phi,\phi,\theta} \leqslant 2e^{-\frac{1}{2}(\Re a_4)\phi^4}$$

for all $\phi \in \mathbb{R}$;

2. *the condition*

$$0 \leqslant \theta \leqslant \frac{(\sqrt{2}-1)^2}{e} \times |a_4|^{-\frac{1}{4}}$$

implies

$$|e^{-U(\phi)}|_{\partial\phi,\theta} \leqslant 2\ .$$

Proof: It follows from the definition of our seminorms that

$$||e^{-U(\phi)}||_{\partial\phi,\phi,\theta} = e^{-\Re U(\phi)} + \sum_{n=1}^{9} \frac{\theta^n}{n!} \left|D^n e^{-U(\phi)}\right|$$

where D denotes the differentiation operator $\frac{\mathrm{d}}{\mathrm{d}\phi}$. An easy induction provides the following explicit formula of Faa di Bruno type for the derivatives of functions of the form $e^{f(\phi)}$:

$$D^n e^{f(\phi)} = \sum_{k\geqslant 0} \frac{1}{k!} \sum_{\substack{m_1,\ldots,m_k\geqslant 1\\ \Sigma m_i = n}} \frac{n!}{m_1!\cdots m_k!} \left(\prod_{i=1}^{k} D^{m_i} f(\phi)\right) e^{f(\phi)}\ . \tag{3.49}$$

This will be used in order to bound the quantities $\left|D^n e^{-U(\phi)}\right|$. First, let us introduce the notation $\alpha = \frac{\sqrt{3}}{2}$

and $r=\frac{1}{3}\log\left(\frac{1+\sqrt{2}}{2}\right)$. We have

$$\begin{aligned}-\Re U(\phi) &= -\sum_{k=0}^{4}(\Re a_k)\phi^k \\ &\leqslant -\frac{1}{2}(\Re a_4)\phi^4-\frac{\alpha}{2}|a_4|\phi^4+\left(\sum_{k=1}^{3}|a_k|\ |\phi|^k\right)+|a_0|\end{aligned}$$

from the hypothesis $\Re a_4\geqslant \alpha|a_4|$. Using the assumption $|a_k|\leqslant r|a_4|^{\frac{k}{4}}$ we then obtain

$$-\Re U(\phi)\leqslant -\frac{1}{2}(\Re a_4)\phi^4+\Omega_1(|a_4|^{\frac{1}{4}}|\phi|)+|a_0|$$

where $\Omega_1(x)=-\frac{\alpha}{2}x^4+r(x^3+x^2+x)$. We note that with our numeric values of r,α one has $\sup_{x\geqslant 0}\Omega_1(x)<3r$. As a result

$$e^{-\Re U(\phi)}\leqslant e^{-\frac{1}{2}(\Re a_4)\phi^4+3r+|a_0|}\ .$$

We now use the formula (3.49) and write, for $1\leqslant n\leqslant 9$,

$$D^n e^{-U(\phi)}=e^{-U(\phi)}\times\sum_{k=1}^{n}\frac{1}{k!}\sum_{\substack{1\leqslant m_1,\ldots,m_k\leqslant 4\\ \Sigma m_i=n}}\frac{n!}{m_1!\cdots m_k!}\times\prod_{i=1}^{k}(-D^{m_i}U(\phi))\ .$$

Using the condition $\Sigma m_i=n$ for handling the θ exponents we get the bound

$$\frac{\theta^n}{n!}\left|D^n e^{-U(\phi)}\right|\leqslant e^{-\Re U(\phi)}\times\sum_{k=1}^{n}\frac{1}{k!}\sum_{\substack{1\leqslant m_1,\ldots,m_k\leqslant 4\\ \Sigma m_i=n}}\prod_{i=1}^{k}\left[\frac{\theta^{m_i}\,|D^{m_i}U(\phi)|}{m_i!}\right]\ . \qquad (3.50)$$

We now assume $\theta\leqslant\gamma_1|a_4|^{-\frac{1}{4}}$ for some suitable $\gamma_1\geqslant 0$ to be specified later. We insert this inequality in (3.50) and pull out ${\gamma_1}^{\Sigma m_i}={\gamma_1}^n$ before throwing away the constraint $\Sigma m_i=n$ which results in

$$\begin{aligned}\frac{\theta^n}{n!}\left|D^n e^{-U(\phi)}\right| &\leqslant e^{-\Re U(\phi)}{\gamma_1}^n\times\sum_{k=1}^{n}\frac{1}{k!}\left(\sum_{m=1}^{4}\frac{|a_4|^{-\frac{m}{4}}}{m!}\,|D^mU(\phi)|\right)^k \\ &\leqslant {\gamma_1}^n\ \exp\left[-\Re U(\phi)+\sum_{m=1}^{4}\frac{|a_4|^{-\frac{m}{4}}}{m!}\,|D^mU(\phi)|\right]\ .\end{aligned}$$

The individual quantities in the last exponential are bounded in terms of $x=|a_4|^{\frac{1}{4}}|\phi|$ as follows:

$$\begin{aligned}|a_4|^{-\frac{1}{4}}\,|DU(\phi)| &= |a_4|^{-\frac{1}{4}}\times|4a_4\phi^3+3a_3\phi^2+2a_2\phi+a_1| \\ &\leqslant 4x^3+3rx^2+2rx+r\ ,\end{aligned}$$

$$\begin{aligned}\frac{|a_4|^{-\frac{2}{4}}}{2}\left|D^2U(\phi)\right| &= |a_4|^{-\frac{1}{2}}\times|6a_4\phi^2+3a_3\phi+a_2| \\ &\leqslant 6x^2+3rx+r\ ,\end{aligned}$$

$$\begin{aligned}\frac{|a_4|^{-\frac{3}{4}}}{3!}\left|D^3U(\phi)\right| &= |a_4|^{-\frac{3}{4}} \times |4a_4\phi + a_3| \\ &\leqslant 4x + r \ ,\end{aligned}$$

whereas

$$\frac{|a_4|^{-\frac{4}{4}}}{4!}\left|D^4U(\phi)\right| = 1 \ .$$

Therefore

$$\sum_{m=1}^{4} \frac{|a_4|^{-\frac{m}{4}}}{m!} |D^mU(\phi)| \leqslant 4x^3 + (3r+6)x^2 + (5r+4)x + (3r+1)$$

and

$$-\Re U(\phi) + \sum_{m=1}^{4} \frac{|a_4|^{-\frac{m}{4}}}{m!} |D^mU(\phi)| \leqslant -\frac{1}{2}(\Re a_4)\phi^4 + \Omega_2(|a_4|^{\frac{1}{4}}|\phi|) + |a_0|$$

where

$$\Omega_2(x) = -\frac{\alpha}{2}x^4 + (r+4)x^3 + (4r+6)x^2 + (6r+4)x + (3r+1) \ .$$

We remark that $\sup_{x\geqslant 0}\Omega_2(x) < 918785$. We denote the latter numerical constant by M. The previous considerations now give

$$\begin{aligned}||e^{-U(\phi)}||_{\partial\phi,\phi,\theta} &\leqslant e^{-\frac{1}{2}(\Re a_4)\phi^4+3r+|a_0|} \\ &\quad + \sum_{n=1}^{9} {\gamma_1}^n \ \exp\left[-\frac{1}{2}(\Re a_4)\phi^4 + M + |a_0|\right] \\ &\leqslant e^{-\frac{1}{2}(\Re a_4)\phi^4} \times e^{|a_0|} \times \left[e^{3r} + e^M \times \frac{\gamma_1}{1-\gamma_1}\right]\end{aligned}$$

provided $\gamma_1 < 1$. If one requires the stronger condition $\gamma_1 \leqslant \frac{1}{2}$ then $e^{3r} + e^M \times \frac{\gamma_1}{1-\gamma_1} \leqslant e^{3r} + 2e^M\gamma_1$. From our choice for r we have $e^{3r} = \frac{1+\sqrt{2}}{2}$. If we now set $\gamma_1 = \frac{\sqrt{2}-1}{4}e^{-M}$ which clearly is less than $\frac{1}{2}$ then $e^{3r} + 2e^M\gamma_1 = \sqrt{2}$. On the other hand, by assumption on a_0 we have $e^{|a_0|} \leqslant \sqrt{2}$. The statement in 1) is therefore proved.

For the statement in 2) concerning the bound on $|e^{-U(\phi)}|_{\partial\phi,\theta} = ||e^{-U(\phi)}||_{\partial\phi,0,\theta}$, with derivatives taken at zero, we follow the same steps. However, the situation simplifies considerably. Indeed,

$$|e^{-U(\phi)}|_{\partial\phi,\theta} = e^{-\Re U(0)} + \sum_{n=1}^{9} \frac{\theta^n}{n!} \left| D^n e^{-U(\phi)}\Big|_{\phi=0} \right|$$

can be bounded as we did before, under the new hypothesis $\theta \leqslant \gamma_2|a_4|^{-\frac{1}{4}}$ for suitable $\gamma_2 \in [0,1)$, by the estimate

$$|e^{-U(\phi)}|_{\partial\phi,\theta} \leqslant e^{-\Re a_0} + \frac{\gamma_2}{1-\gamma_2} \times \exp\left[-\Re U(0) + \sum_{m=1}^{4} \frac{|a_4|^{-\frac{m}{4}}}{m!} |D^mU(0)|\right] \ .$$

Now

$$\sum_{m=1}^{4} \frac{|a_4|^{-\frac{m}{4}}}{m!} |D^mU(0)| = \sum_{m=1}^{4} |a_4|^{-\frac{m}{4}} |a_m| \leqslant 3r + 1 \ .$$

If one imposes the condition $\gamma_2 \leqslant \frac{1}{2}$, then

$$|e^{-U(\phi)}|_{\partial\phi,\theta} \leqslant e^{|a_0|} \times \left[1 + 2\gamma_2 e^{3r+1}\right] \ .$$

Because of the chosen value of r, one will have $1 + 2\gamma_2 e^{3r+1} = \sqrt{2}$ if one now sets $\gamma_2 = \frac{(\sqrt{2}-1)^2}{e} \simeq 0.0631\ldots$ which is less than $\frac{1}{2}$. The statement in 2) then follows easily. □

We will now specify the numeric parameters

$$c_1 = 2^{-\frac{9}{4}}(\sqrt{2}-1)e^{-918785} \qquad \text{and} \qquad c_2 = 2^{\frac{3}{4}}$$

which are used to calibrate the parameters

$$h = c_1 \bar{g}^{-\frac{1}{4}} \qquad \text{and} \qquad h_* = c_2 L^{\frac{3+\epsilon}{4}}$$

for the seminorms we use.

With these choices the norm

$$|||R|||_{\bar{g}} = \max\left\{ |R(\phi)|_{\partial\phi,h_*}, \bar{g}^2 \sup_{\phi\in\mathbb{R}} ||R(\phi)||_{\partial\phi,\phi,h} \right\}$$

is now unambiguously defined in terms of the calibrator $\bar{g}$. We give one more lemma before stating the main estimates theorem.

Lemma 3.6. *For all unit cube Δ' and for all subset $Y_0 \subset [L^{-1}\Delta']$ we have*

$$\forall \phi \in \mathbb{R} \ , \quad \left|\left| \prod_{\Delta\in Y_0} e^{-\tilde{V}_\Delta(\phi_1)} \right|\right|_{\partial\phi,\phi,h} \leqslant 2$$

as well as

$$\left| \prod_{\Delta\in Y_0} e^{-\tilde{V}_\Delta(\phi_1)} \right|_{\partial\phi,h_*} \leqslant 2 \ .$$

If $|Y_0| \geqslant \frac{L^3}{2}$ (which holds if $|Y_0| = L^3$ or $L^3 - 1$ because $L \geqslant 2$) then we have the improved bound

$$\forall \phi \in \mathbb{R} \ , \quad \left|\left| \prod_{\Delta\in Y_0} e^{-\tilde{V}_\Delta(\phi_1)} \right|\right|_{\partial\phi,\phi,h} \leqslant 2e^{-\frac{\bar{g}}{16}\phi^4} \ .$$

Here ϕ_1 denotes the rescaled field $L^{-[\phi]}\phi$.

Proof: One needs to unwrap the definitions of the quantities above and apply Lemma 3.5, keeping in mind that we have $\phi_1 = L^{-[\phi]}\phi$ but derivatives are with respect to ϕ.

3.6.2 Main Estimates Theorem

Below we give a version of [3, Theorem 4] which gives the fundamental estimates on the single iteration of the map RG_{ex}

Theorem 3.2. *$\exists B_{R\mathcal{L}} \geqslant 0$, $\forall L = p^l > 1$, $\exists B_0, \ldots, B_4, B_{R\xi} \geqslant 0$, $\exists \epsilon_0 > 0$, $\forall \epsilon \in (0, \epsilon_0]$, and $\Delta' \in \mathbb{L}$, then on the domain*

$$\forall \Delta \in [L^{-1}\Delta'], \quad \begin{cases} |\beta_{4,\Delta} - \bar{g}_*| & < & \frac{1}{2}\bar{g} & & \\ |\beta_{k,\Delta}| & < & \bar{g} & \text{for} & k = 1,2,3 \\ |W_{k,\Delta}| & < & \bar{g}^2 & \text{for} & k = 5,6 \\ |f_\Delta| & < & L^{-(3-[\phi])} & & \\ |||R_\Delta|||_{\bar{g}} & < & \bar{g}^{\frac{21}{8}} & & \end{cases}$$

the maps $\xi_{0,\Delta'}, \ldots, \xi_{4,\Delta'}$, $\mathcal{L}_{\Delta'}$ and $\xi_{R,\Delta'}$ are well-defined, analytic, send real data to real data and satisfy the bounds

$$|\xi_{k,\Delta'}(\vec{V})| \leqslant B_k \max_{\Delta \in [L^{-1}\Delta']} |||R_\Delta|||_{\bar{g}} \qquad \text{for } k = 0, \ldots, 4 \ ,$$

$$|||\mathcal{L}_{\Delta'}^{\vec{\beta},f}(R)|||_{\bar{g}} \leqslant B_{R\mathcal{L}} L^{3-5[\phi]} \max_{\Delta \in [L^{-1}\Delta']} |||R_\Delta|||_{\bar{g}} \ ,$$

and

$$|||\xi_{R,\Delta'}(\vec{V})|||_{\bar{g}} \leqslant B_{R\xi} \bar{g}^{\frac{11}{4}} \ .$$

We remark the above theorem is formulated completely locally - it gives estimates $RG_{ext}[\vec{V}]_{\Delta'}$ that are depend on hypothesises that depend only on those components of $\vec{V}$ that lie in $L^{-1}\Delta'$, i.e. $\{V_\Delta\}_{\Delta \in [L^{-1}\Delta']}$.

Some important facts about the content of the above theorem and how we use it are

(i) In the statement of the above theorem L is an arbitrary positive integer power of p - ϵ will then need to be taken sufficiently small depending on how large L is.

(ii) While this is not done right now we remark that L will be taken large sufficently large and then fixed to beat the L independent combinatorial factor $B_{R\mathcal{L}}$ to guarantee that the linear map $\mathcal{L}_{\Delta'}^{\vec{\beta},f}(\cdot)$ is a contraction.

(iii) The higher order contributions in R denoted by $\xi_{R,\Delta'}(\vec{V})$ are of order $B_{R\xi}\bar{g}^{\frac{11}{4}}$ - what's important is by taking ϵ small we can make this negligible relative to the linear flow since $B_{R\xi}$ depends only on L and so the remainder comes with an extra factor of $\bar{g}^{\frac{1}{8}}$ - then by making ϵ sufficiently small this term can be made negligible compared to the linear term.

(iv) The non-explicit counterterms $\{\zeta_{k,\Delta'}[\vec{V}]\}_{k=1}^4$ that will appear in the exact flow equations of the coupling constants $\{\beta_{k.\Delta'}\}_{k=1}^4$ are of order $\bar{g}^{\frac{21}{8}}$ and since $\frac{21}{8} > 2$ these exact flows will be primarily governed by the approximate flows given by second order perturbation theory.

We abuse notation and will write $\bar{g}_*$ for our approximate fixed point in $\mathcal{E}_{\text{ex}}$, using the notation $(\beta_4, \beta_3, \beta_2, \beta_1, W_5, W_6, f, R)$ for an element of $\mathcal{E}_{\text{ex}}$, then by $\bar{g}_* \in \mathcal{E}_{\text{ex}}$ we mean the element with $\beta_4(x) = \bar{g}_*$ as specified in (4.1) and all other entries set to 0. The estimates above then show that RG_{ex} is an analytic map on any open ball of radius less than 1 around $\bar{g}_*$ in $\mathcal{E}_{\text{ex}}$.

Analyticity methods are crucial to our approach. As we mentioned before complex analyticity in f and j of the moment generating functions is what allows us to recover moments with the necessary $n!$ bounds. Additionally complex analyticity of the various maps involved in our RG analysis will be crucial to prove

Lipschitz estimates for certain maps that are given by contractive linear part and a higher-order non-explicit terms that we only have uniform bounds on. This is why we work with complex Banach spaces for all of our parameters even though in our final application we are mostly concerned with real data.

In [3] the proof of Theorem 3.2 involves around 54 different lemmas. Our strategy for proving Theorem 3.2 (and the nature of the actual theorem) closely follows what is done in [18]. Our work is more general in the sense that we work in a setting where our RG data is not translation invariant, on the other hand since we work in the easier hierarchical setting we don't have to worry about polymer activities (non-local functionals of the field).

We do not prove Theorem 3.2 here, all the steps are given in explicit detail in [3]. For the exposition here we will restrict ourselves to explaining how one proves $\mathcal{L}^{\vec{\beta},f}_{\Delta'}(\cdot)$ is contractive.

Proving $\mathcal{L}^{\vec{\beta},f}$ is contractive

First we give a simple estimate helpful for working with one of our seminorms:

Lemma 3.7. *For any $0 \leqslant k < 9$ and $\gamma \in (0,1]$ one has*

$$||\psi^k||_{\partial\psi,\psi,h} \leqslant 27 \times \gamma^{-\frac{k}{4}} \bar{g}^{-\frac{k}{4}} e^{\gamma(\Re\beta_{4,\Delta})\phi_1^4}$$

Proof: We note that

$$||\psi^k||_{\partial\psi,\psi,h} = (h + |\psi|)^k = \sum_{n=0}^{k} \begin{pmatrix} k \\ n \end{pmatrix} (c_1 \bar{g}^{-\frac{1}{4}})^{k-n} |\psi|^n \ .$$

We use Lemma 3.3 to write, for $\gamma \in (0,1]$,

$$|\psi|^n \leqslant \left(\frac{n}{2e}\right)^{\frac{n}{4}} [\gamma\bar{g}]^{-\frac{n}{4}} e^{\gamma(\Re\beta_{4,\Delta})\psi^4}$$

which proves the assertion. □

Throughout this whole section ϕ_1 should be seen as a shorthand for $L^{-[\phi]}\phi$ - this corresponds to the rescaling of the field.

Lemma 3.8. *For any $K \in \mathcal{C}^9_{\mathrm{bd}}(\mathbb{R},\mathbb{C})$*

$$||K(\phi_1)||_{\partial\phi,\phi_1,h} = ||K(\psi)||_{\partial\psi,\phi_1,L^{-[\phi]}h} \leqslant ||K(\psi)||_{\partial\psi,\phi_1,h}$$

$$|K(\phi_1)|_{\partial\phi,h_*} = |K(\psi)|_{\partial\psi,L^{-[\phi]}h_*} \leqslant |K(\psi)|_{\partial\psi,h_*}$$

For $K \in \mathcal{C}^9_{\mathrm{bd}}(\mathbb{R},\mathbb{C})$ such that for some n one has $\frac{d^j}{d\psi^j}K(\psi)\Big|_{\psi=0} = 0$ for $0 \leqslant j < n$

$$|K(\phi_1)|_{\partial\phi,h_*} \leqslant L^{-n[\phi]}|K(\psi)|_{\partial\psi,h_*}$$

Proof: The equalities in the first two assertions follows from definition of the seminorms and the chain rule, the inequalities in the first two assertions follow since $L^{-[\phi]} \leqslant 1$.

For the third assertion we assume that $n \leqslant 9$ since the inequality is trivial otherwise. Now one only needs to observe that

$$
\begin{aligned}
|K(\phi_1)|_{\partial\phi,0,h_*} = |K(\psi)|_{\partial\psi,L^{-[\phi]}h_*} &= \sum_{j=0}^{9} \frac{(L^{-[\phi]}h_*)^j}{j!} \left| \frac{d^j}{d\psi^j} K(0) \right| \\
&= \sum_{j=n}^{9} \frac{(L^{-[\phi]}h_*)^j}{j!} \left| \frac{d^j}{d\psi^j} K(0) \right| \\
&\leqslant L^{-n[\phi]} \sum_{j=n}^{9} \frac{(h_*)^j}{j!} \left| \frac{d^j}{d\psi^j} K(0) \right| = L^{-n[\phi]} |K(\psi)|_{\partial\psi,h_*}
\end{aligned}
$$

□

We remark that we similarly have the bound

$$
|K(\phi)|_{\partial\phi,0,L^{[\phi]}h_*} \leqslant L^{-n[\phi]} |K(\phi)|_{\partial\phi,0,h_*}
$$

for $K(\phi)$ with $K^{(j)}(0) = 0$ for $0 \leqslant j \leqslant n$. This is the form of the third assertion that will appear in our lemmas.

The key point of the above lemma is that if our function has its first few derivatives at 0 vanish then rescaling gives us a strict contraction in the kernel seminorm. The contractivity $\mathcal{L}^{\vec{\beta},f}_{\Delta'}$ will come from the "normalization conditions" we implemented in its definition which require the vanishing of the first four derivatives - this will give a factor of $L^{-5[\phi]}$ after rescaling which is enough to beat the volume factor L^3 that will appear in the linear flow for R. However we are subjecting the kernel seminorm to both a rescaling and a fluctuation integral - as we discussed earlier controlling the integration requires more than the kernel seminorm. This is why we have the supremum norm inside of $|||\cdot|||_{\bar{g}}$. However this adds a new challenge, as now one must prove a contractive estimate in terms of $|||\cdot|||_{\bar{g}}$.

The main tool for dealing with the above problems are the following two lemmas which relate our seminorms.

Lemma 3.9. *For all $K \in C^9_{\mathrm{bd}}(\mathbb{R},\mathbb{C})$ and for all $\sigma \in \mathbb{R}$ we have*

$$
||K(\psi)||_{\partial\psi,\sigma,h_*} \leqslant \mathcal{O}_1 e^{h_*^{-2}\sigma^2} \times \left[|K(\psi)|_{\partial\psi,h_*} + h_*^9 h^{-9} \sup_{\psi\in\mathbb{R}} ||K(\psi)||_{\partial\psi,\psi,h_*} \right]
$$

where

$$
\mathcal{O}_1 = 1 + 511 \times \max_{0\leqslant j\leqslant 9} \left(\frac{j}{2e} \right)^{\frac{j}{2}} .
$$

Proof: Recall that by definition

$$
||K(\psi)||_{\partial\psi,\sigma,h_*} = \sum_{n=0}^{9} \frac{h_*^n}{n!} |K^{(n)}(\sigma)| .
$$

The term with $n = 9$ is bounded by writing

$$
\frac{h_*^9}{9!} |K^{(9)}(\sigma)| = h_*^9 h^{-9} \times \frac{h^9}{9!} |K^{(9)}(\sigma)| \leqslant h_*^9 h^{-9} \times \sup_{\psi\in\mathbb{R}} ||K(\psi)||_{\partial\psi,\psi,h_*} .
$$

For terms with $0 \leqslant n \leqslant 8$ we use a Taylor expansion around zero of order $8-n$ so that the integral remainder involves $(9-n)$-th derivatives of $K^{(n)}$, i.e., 9-th derivatives of the original function K. Indeed, one can write

$$K^{(n)}(\sigma) = \sum_{m=0}^{8-n} \frac{\sigma^m}{m!} K^{(n+m)}(0) + \frac{1}{(8-n)!} \int_0^1 (1-s)^{8-n} \sigma^{9-n} K^{(9)}(s\sigma)\ \mathrm{d}s$$

and therefore

$$|K^{(n)}(\sigma)| \leqslant \sum_{m=0}^{8-n} \frac{|\sigma|^m}{m!} (n+m)!\ h_*^{-(n+m)} |K(\psi)|_{\partial\psi, h_*}$$

$$+ \frac{1}{(8-n)!} |\sigma|^{9-n} \times 9!\ h^{-9} \left(\sup_{\psi \in \mathbb{R}} ||K(\psi)||_{\partial\psi,\psi,h} \right) \int_0^1 (1-s)^{8-n}\ \mathrm{d}s\ .$$

We use Lemma 3.3 with $\kappa = h_*^{-2}$ in order to bound powers of $|\sigma|$ by

$$|\sigma|^m \leqslant \left(\frac{m}{2e}\right)^{\frac{m}{2}} \times h_*^m e^{h_*^{-2}\sigma^2}$$

which inserted in the previous inequality gives

$$\frac{h_*^n}{n!} |K^{(n)}(\sigma)| \leqslant \left(\max_{0 \leqslant j \leqslant 9} \left(\frac{j}{2e}\right)^{\frac{j}{2}} \right) \times e^{h_*^{-2}\sigma^2}$$

$$\times \left[\sum_{m=0}^{8-n} \frac{(n+m)!}{n!m!} |K(\psi)|_{\partial\psi,h_*} + \frac{9!\ h_*^{9-n}}{n!(9-n)!} h_*^n h^{-9} \sup_{\psi \in \mathbb{R}} ||K(\psi)||_{\partial\psi,\psi,h} \right] .$$

Putting together the bounds for the different values of n we obtain

$$||K(\psi)||_{\partial\psi,\sigma,h_*} \leqslant h_*^9 h^{-9} \sup_{\psi\in\mathbb{R}} ||K(\psi)||_{\partial\psi,\psi,h}$$

$$+ e^{h_*^{-2}\sigma^2} \left(\max_{0\leqslant j\leqslant 9} \left(\frac{j}{2e}\right)^{\frac{j}{2}} \right) \sum_{n=0}^{8} \left[\left(\sum_{m=0}^{8-n} \frac{(n+m)!}{n!m!} \right) |K(\psi)|_{\partial\psi,h_*} + \frac{9!}{n!(9-n!)} h_*^9 h^{-9} \sup_{\psi\in\mathbb{R}} ||K(\psi)||_{\partial\psi,\psi,h} \right] .$$

The result as well as the given value of $\mathcal{O}_1$ then follow since

$$\sum_{n=0}^{8} \sum_{m=0}^{8-n} \frac{(n+m)!}{n!m!} = \sum_{n=0}^{8} \frac{9!}{n!(9-n!)} = 2^9 - 1 = 511.$$

□

Lemma 3.10. *For all* $K \in C^9_{\mathrm{bd}}(\mathbb{R}, \mathbb{C})$, $\beta_4 \in \mathbb{C}$ *such that* $|\beta_4 - \bar{g}| < \frac{1}{2}\bar{g}$, $\gamma \in (0,1]$ *and* $\phi \in \mathbb{R}$ *we have*

$$||K(\phi)||_{\partial\phi,\phi,h} \leqslant \mathcal{O}_2 \gamma^{-\frac{9}{4}} e^{\gamma(\Re\beta_4)\phi^4} \left[|K(\psi)|_{\partial\psi,h} + L^{-9[\phi]} \sup_{\psi\in\mathbb{R}} ||K(\psi)||_{\partial\psi,\psi,L^{[\phi]}h} \right]$$

with

$$\mathcal{O}_2 = 1 + \left((1+c_1^{-1})^9 - 1\right) \times \max_{0\leqslant j\leqslant 9} \left(\frac{j}{2e}\right)^{\frac{j}{4}} .$$

Proof: We proceed as in the proof of the previous lemma and write

$$\frac{h^9}{9!}|K^{(9)}(\phi)| = L^{-9[\phi]} \times \frac{(L^{[\phi]}h)^9}{9!}|K^{(9)}(\phi)| \leqslant L^{-9[\phi]} \times \sup_{\psi\in\mathbb{R}} ||K(\psi)||_{\partial\psi,\psi,L^{[\phi]}h}$$

in order to handle the $n = 9$ term in the sum defining $||K(\phi)||_{\partial\phi,\phi,h}$. For the other terms with $0 \leqslant n \leqslant 8$ one has, as before,

$$|K^{(n)}(\phi)| \leqslant \sum_{m=0}^{8-n} \frac{|\phi|^m}{m!}(n+m)!\ h^{-(n+m)}|K(\psi)|_{\partial\psi,h}$$

$$+\frac{1}{(9-n)!}|\phi|^{9-n} \times 9!(L^{[\phi]}h)^{-9} \sup_{\psi\in\mathbb{R}} ||K(\psi)||_{\partial\psi,\psi,L^{[\phi]}h} \ .$$

We this time use Lemma 3.3 in order to bound powers of $|\phi|$ by

$$|\phi|^m \leqslant \left(\frac{m}{2e}\right)^{\frac{m}{4}} \gamma^{-\frac{m}{4}} \bar{g}^{-\frac{m}{4}} e^{\gamma(\Re\beta_4)\phi^4} \ .$$

Note that $\gamma^{-\frac{m}{4}} \leqslant \gamma^{-\frac{9}{4}}$ since $0 < \gamma \leqslant 1$, $0 \leqslant n \leqslant 8$ and $0 \leqslant m \leqslant 9-n$. Besides $\bar{g}^{-\frac{m}{4}} = (c_1^{-1}h)^m$ and therefore

$$\frac{h^n}{n!}|K^{(n)}(\phi)| \leqslant \left(\max_{0\leqslant j\leqslant 9} \left(\frac{j}{2e}\right)^{\frac{j}{4}}\right) \times \gamma^{-\frac{9}{4}} \times e^{\gamma(\Re\beta_4)\phi^4}$$

$$\times \left[\sum_{m=0}^{8-n} \frac{h^m c_1^{-m}}{m!}\frac{h^n}{n!}(n+m)!h^{-(n+m)}|K(\psi)|_{\partial\psi,h} + \frac{9!}{n!(9-n)!}h^n h^{9-n} c_1^{-(9-n)}(L^{[\phi]}h)^{-9} \sup_{\psi\in\mathbb{R}} ||K(\psi)||_{\partial\psi,\psi,L^{[\phi]}h}\right] .$$

Altogether this gives the estimate

$$||K(\phi)||_{\partial\phi,\phi,h} \leqslant L^{-9[\phi]} \sup_{\psi\in\mathbb{R}} ||K(\psi)||_{\partial\psi,\psi,L^{[\phi]}h}$$

$$+\left(\max_{0\leqslant j\leqslant 9} \left(\frac{j}{2e}\right)^{\frac{j}{4}}\right) \times \gamma^{-\frac{9}{4}} \times e^{\gamma(\Re\beta_4)\phi^4} \times \left\{\left(\sum_{m=0}^{8-n} \binom{n+m}{m} c_1^{-m}\right)|K(\psi)|_{\partial\psi,h}\right.$$

$$\left.+\binom{9}{n} c_1^{-(9-n)} L^{-9[\phi]} \sup_{\psi\in\mathbb{R}} ||K(\psi)||_{\partial\psi,\psi,L^{[\phi]}h}\right\} .$$

The result with the given value for $\mathcal{O}_2$ follows from this last inequality since

$$\sum_{n=0}^{8}\sum_{m=0}^{8-n} \binom{n+m}{m} c_1^{-m} = c_1\left[(1+c_1^{-1})^9 - 1\right] < (1+c_1^{-1})^9 - 1 = \sum_{n=0}^{8}\binom{9}{n} c_1^{-(9-n)} \ .$$

□

Now we recall that from our definition of the extended RG map one has

$$\mathcal{L}_{\Delta'}^{\vec{\beta},f}(R) \ = \ \sum_{\Delta_1\in[L^{-1}\Delta']} \left(\prod_{\substack{\Delta\in[L^{-1}\Delta'] \\ \Delta\neq\Delta_1}} e^{-\tilde{V}_\Delta(\phi_1)}\right) \times J_{\Delta',\Delta_1}(\phi) \qquad (3.51)$$

where for $\Delta_1 \in [L^{-1}\Delta']$ we have

$$J_{\Delta',\Delta_1}(\phi) \;=\; J_+(\phi) - J_-(\phi) \tag{3.52}$$

with

$$\begin{aligned} J_+(\phi) \;&= e^{-\frac{1}{2}(f,\Gamma f)_{L^{-1}\Delta'}} \times \int \mathrm{d}\mu_\Gamma(\zeta)\; e^{\int_{L^{-1}\Delta'} f\zeta}\; R_{\Delta_1}(\phi_1+\zeta) \\ J_-(\phi) \;&=\; \left(\sum_{k=0}^{4} \delta\beta_{k,3,\Delta',\Delta_1} :\phi^k:_{C_0}\right) \times e^{-\tilde{V}_{\Delta_1}(\phi_1)} \;. \end{aligned} \tag{3.53}$$

Note that the notation above suppresses a lot of dependences, in particular the numeric quantities $\delta\beta_{k,3,\Delta',\Delta_1}$ are defined as linear functions of R_{Δ_1} - see (3.43) and (3.44).

The next lemma gives a kernel seminorm bound on J_+, using Lemma 3.9 to control the fluctuation integral inside of J_+.

A notational note: to realize the rescaling contraction we get bounds on quantities of interest with respect to a "shifted" kernel seminorm $|\cdot|_{\partial\phi,L^{[\phi]}h_*}$, then the rescaling contraction will appear when we unshift $L^{[\phi]}h_*$ to h_*.

Lemma 3.11. *For all $\Delta' \in \mathbb{L}$ and $\Delta_1 \in [L^{-1}\Delta']$ the quantity $J_+(\phi)$ satisfies the bound*

$$|J_+(\phi)|_{\partial\phi,L^{[\phi]}h_*} \leqslant \mathcal{O}_3 |||R_{\Delta_1}|||_{\bar{g}}$$

where $\mathcal{O}_3 = 4\mathcal{O}_1 \times \exp\left(2^{-\frac{3}{2}}\right)$.

Proof: From the definition of $J_+(\phi)$ we immediately have:

$$|J_+(\phi)|_{\partial\phi,L^{[\phi]}h_*} \leqslant e^{-\frac{1}{2}\Re(f,\Gamma f)_{L^{-1}\Delta'}} \int \mathrm{d}\mu_\Gamma(\zeta)\; e^{\int_{L^{-1}\Delta'}(\Re f)\zeta} |R_{\Delta_1}(\phi_1+\zeta)|_{\partial\phi,L^{[\phi]}h_*} \;.$$

By the definitions of the seminorms and the chain rule one has

$$|R_{\Delta_1}(\phi_1+\zeta)|_{\partial\phi,L^{[\phi]}h_*} = ||R_{\Delta_1}(\phi_1+\zeta)||_{\partial\phi,0,L^{[\phi]}h_*} = ||R_{\Delta_1}(\psi+\zeta)||_{\partial\psi,0,h_*} = ||R_{\Delta_1}(\psi)||_{\partial\psi,\zeta,h_*} \;.$$

From Lemma 3.9 we then derive

$$\begin{aligned} |R_{\Delta_1}(\phi_1+\zeta)|_{\partial\phi,L^{[\phi]}h_*} \;&\leqslant\; \mathcal{O}_1 e^{h_*^{-2}\zeta_{\Delta_1}^2} \left[|R_{\Delta_1}(\psi)|_{\partial\psi,h_*} + h_*^9 h^{-9} \sup_{\psi\in\mathbb{R}} ||R_{\Delta_1}(\psi)||_{\partial\psi,\psi,h} \right] \\ &\leqslant\; \mathcal{O}_1 e^{h_*^{-2}\zeta_{\Delta_1}^2} |||R_{\Delta_1}|||_{\bar{g}} \left(1 + h_*^9 h^{-9} \bar{g}^{-2}\right) \\ &\leqslant\; 2\mathcal{O}_1 e^{h_*^{-2}\zeta_{\Delta_1}^2} |||R_{\Delta_1}|||_{\bar{g}}. \end{aligned}$$

In going to the last line we remark that $h_*^9 h^{-9}\bar{g}^{-2} = c_2^9 c_1^{-9} \bar{g}^{\frac{9}{4}-2}$, now since $\bar{g}$ is $\mathcal{O}(\epsilon)$ we can make this term arbitrary sufficiently small. As a result

$$|J_+(\phi)|_{\partial\phi,L^{[\phi]}h_*} \leqslant e^{-\frac{1}{2}\Re(f,\Gamma f)_{L^{-1}\Delta'}} \times 2\mathcal{O}_1 \times |||R_{\Delta_1}|||_{\bar{g}} \times \int \mathrm{d}\mu_\Gamma(\zeta)\; e^{\int_{L^{-1}\Delta'}(\Re f)\zeta} e^{h_*^{-2}\zeta_{\Delta_1}^2} \;.$$

Now by our choice of c_2 we can use Lemma 3.4 with $\alpha = h_*^{-2}$ to the effect that

$$|J_+(\phi)|_{\partial\phi, L^{[\phi]}h_*} \leqslant 4\mathcal{O}_1 \times |||R_{\Delta_1}|||_{\bar{g}} \times \exp\left\{-\frac{1}{2}\Re(f,\Gamma f)_{L^{-1}\Delta'} + \frac{1}{2}(\Re f, \Gamma \Re f)_{L^{-1}\Delta'}\right\}$$

holds. Note that

$$\Re(f,\Gamma f)_{L^{-1}\Delta'} = (\Re f, \Gamma\Re f)_{L^{-1}\Delta'} - (\Im f, \Gamma \Im f)_{L^{-1}\Delta'}$$

and thus

$$|J_+(\phi)|_{\partial\phi, L^{[\phi]}h_*} \leqslant 4\mathcal{O}_1 \times |||R_{\Delta_1}|||_{\bar{g}} \times \exp\left\{\frac{1}{2}(\Im f, \Gamma\Im f)_{L^{-1}\Delta'}\right\} \ .$$

But

$$\begin{aligned} |(\Im f, \Gamma \Im f)_{L^{-1}\Delta'}| &\leqslant \int_{(L^{-1}\Delta')^2} \mathrm{d}^3x\mathrm{d}^3y \ |\Gamma(x-y)| \ |\Im f(x)| \ |\Im f(y)| \\ &\leqslant ||f|_{L^{-1}\Delta'}||^2_{L^\infty} \times L^3 \times ||\Gamma||_{L^1} \\ &\leqslant L^{-2(3-[\phi])} \times L^3 \times \frac{L^{3-2[\phi]}}{\sqrt{2}} \ \leqslant \ \frac{1}{\sqrt{2}} \end{aligned}$$

because of our assumptions on $||f||$ in Theorem 3.2, the finite range property of Γ and the bound in Corollary 5.1. Inserting this last inequality in the previous estimate for J_+ gives the wanted bound. □

In order to estimate $J_-(\phi)$ we will need estimates on the third order counterterms which due to the consequences of the enforced normalization condition boil down to using the estimate of (3.12) along with some knowledge of the general structure of the counter-terms.

Lemma 3.12. *For all $\Delta' \in \mathbb{L}$, $\Delta_1 \in [L^{-1}\Delta']$ and integer k such that $0 \leqslant k \leqslant 4$ the $\delta\beta$ quantities defined in §3.5.2 satisfy*

$$|\delta\beta_{k,3,\Delta',\Delta_1}| \leqslant \mathcal{O}_4 \times \left(L^{[\phi]}h_*\right)^{-k} \times |||R_{\Delta_1}|||_{\bar{g}}$$

and

$$|\delta\beta_{k,3,\Delta'}| \leqslant \mathcal{O}_4 \times L^{3-k[\phi]} \times \max_{\Delta_1 \in [L^{-1}\Delta']} |||R_{\Delta_1}|||_{\bar{g}}$$

with

$$\mathcal{O}_4 = 48 \times \mathcal{O}_3 \times \sum_{i=0}^{4}\sum_{j,n,l} |\#_{k,i,j,n,l}|\, 2^j \left(\frac{3}{2}\right)^n$$

where $\#_{k,i,j,n,l}$ denote the numerical coefficients in the explicit formulas produced by Maple from §3.5.2.

Proof: Recall that

$$\delta\beta_{k,3,\Delta',\Delta_1} = \sum_{i=0}^{4} M_{k,i}a_i$$

where

$$\begin{aligned} a_i &= \exp\left[-C_0(0)L^{-2[\phi]}\beta_{2,\Delta_1} + 3C_0(0)^2L^{-4[\phi]}\beta_{4,\Delta_1} - \frac{1}{2}(f,\Gamma f)_{L^{-1}\Delta'}\right] \\ &\quad \times L^{-i[\phi]} \times \int \mathrm{d}\mu_\Gamma(\zeta)\ e^{\int_{L^{-1}\Delta'} f\zeta} R^{(i)}_{\Delta_1}(\zeta) \\ &= \exp\left[-C_0(0)L^{-2[\phi]}\beta_{2,\Delta_1} + 3C_0(0)^2L^{-4[\phi]}\beta_{4,\Delta_1}\right] \times J_+^{(i)}(0) \end{aligned}$$

and

$$M_{k,i} = \sum_{j,n,l} \#_{k,i,j,n,l} C_0(0)^j L^{-(l_1+\cdots+l_n)[\phi]} \beta_{l_1,\Delta_1} \cdots \beta_{l_n,\Delta_1} \ .$$

Using $|\beta_{2,\Delta_1}| < \bar{g}$, $|\beta_{4,\Delta_1}| < \frac{3}{2}\bar{g}$, $C_0(0) < 2$, and $L^{-[\phi]} \leqslant 1$ gives the bound

$$|a_i| \leqslant |J_+^{(i)}(0)| \times e^{20\bar{g}} \leqslant 2|J_+^{(i)}(0)| \ .$$

for ϵ taken sufficiently small. By definition of the seminorms

$$|J_+^{(i)}(0)| \leqslant i!(L^{[\phi]}h_*)^{-i} |J_+(0)|_{\partial\phi, L^{[\phi]}h_*} \ .$$

Since $i \leqslant 4$ we then get from the last inequality

$$|a_i| \leqslant 48(L^{[\phi]}h_*)^{-i} |J_+(0)|_{\partial\phi, L^{[\phi]}h_*} \ .$$

Now recall that the sum expressing the $M_{k,i}$ is quantified over $j \geqslant 0$, $n \geqslant 0$ and $l = (l_1, \ldots, l_n) \in \{1, \ldots, 4\}^n$. For the numerical coefficients $\#_{k,i,j,n,l}$ to be nonzero the constraint

$$l_1 + \cdots + l_n - 2j = k - i$$

must be satisfied. The β_{l_ν,Δ_1} are bounded by a worst case scenario of $\frac{3}{2}\bar{g}$. We can thus write

$$|M_{k,i}| \leqslant \sum_{j,n,l} |\#_{k,i,j,n,l}| 2^j L^{-(l_1+\cdots+l_n)[\phi]} \times \left(\frac{3}{2}\bar{g}\right)^n \ .$$

We now consider two different cases in order to continue estimating the $|M_{k,i}|$.

1st case: Suppose $i \geqslant k$. Since the l's are positive, we have $L^{-(l_1+\cdots+l_n)[\phi]} \leqslant 1$. We use the coarse bound $\bar{g} \leqslant 1$ and then simply write

$$|M_{k,i}| \leqslant \sum_{j,n,l} |\#_{k,i,j,n,l}| 2^j \times \left(\frac{3}{2}\right)^n \ .$$

2nd case: Suppose $i < k$. Since $j \geqslant 0$, the previous constraint implies

$$l_1 + \cdots + l_n = 2j + k - i \geqslant k - i$$

and therefore $L^{-(l_1+\cdots+l_n)[\phi]} \leqslant L^{-(k-i)[\phi]}$. One can also infer that $n \geqslant 1$ since $l_1 + \cdots + l_n \geqslant k - i > 0$. The bound on $|M_{k,i}|$ which results from these remarks can be reorganized as

$$|M_{k,i}| \leqslant \sum_{j,n,l} |\#_{k,i,j,n,l}| 2^j \times \left(h_* L^{[\phi]}\right)^{-(k-i)} \times \left(\frac{3}{2}\right)^n \bar{g} h_*^{k-i} \ .$$

Since $0 \leqslant i < k \leqslant 4$, $h_* \geqslant 1$ and $\epsilon \leqslant 1$ we have

$$h_*^{k-i} \leqslant h_*^4 = \left(2^{\frac{3}{4}} L^{\frac{3+\epsilon}{4}}\right)^4 \leqslant 8L^4 \ .$$

By taking ϵ sufficiently small (dependent on L) we can assume $8\bar{g}L^4 < 1$ so

$$|M_{k,i}| \leqslant \left(h_* L^{[\phi]}\right)^{-(k-i)} \times \sum_{j,n,l} |\#_{k,i,j,n,l}| 2^j \left(\frac{3}{2}\right)^n$$

which is the wanted bound for $|M_{k,i}|$ in this second case.

We now combine the previous consideration and get

$$|\delta\beta_{k,3,\Delta',\Delta_1}| \leqslant \sum_{i=k}^{4} |M_{k,i}|\ |a_i| + \sum_{0\leqslant i<k} |M_{k,i}|\ |a_i|$$

$$\leqslant \sum_{i=k}^{4} 48(L^{[\phi]}h_*)^{-i} |J_+(0)|_{\partial\phi, L^{[\phi]}h_*} \times \sum_{j,n,l} |\#_{k,i,j,n,l}| 2^j \times \left(\frac{3}{2}\right)^n$$

$$+ \sum_{0\leqslant i<k} 48(L^{[\phi]}h_*)^{-k} |J_+(0)|_{\partial\phi, L^{[\phi]}h_*} \times \sum_{j,n,l} |\#_{k,i,j,n,l}| 2^j \times \left(\frac{3}{2}\right)^n .$$

Since $L^{[\phi]}$ and h_* are greater than 1 we have $(L^{[\phi]}h_*)^{-i} \leqslant (L^{[\phi]}h_*)^{-k}$ when $i \geqslant k$. We can then more conveniently write

$$|\delta\beta_{k,3,\Delta',\Delta_1}| \leqslant 48(L^{[\phi]}h_*)^{-k} |J_+(0)|_{\partial\phi, L^{[\phi]}h_*} \times \sum_{i=0}^{4} \sum_{j,n,l} |\#_{k,i,j,n,l}| 2^j \times \left(\frac{3}{2}\right)^n$$

from which the desired follows thanks to Lemma 3.11. Finally the second bound on $|\delta\beta_{k,3,\Delta'}|$ follows simply by summing over $\Delta_1 \in [L^{-1}\Delta']$ and discarding the factors $h_*^{-k} \leqslant 1$. □

Lemma 3.13. *For all $\Delta' \in \mathbb{L}$ and $\Delta \in [L^{-1}\Delta']$ we have*

$$|J_{\Delta',\Delta_1}(\phi)|_{\partial\phi, L^{[\phi]}h_*} \leqslant \mathcal{O}_5 |||R_{\Delta_1}|||_{\bar{g}}$$

where $\mathcal{O}_5 = \mathcal{O}_3 + 250\mathcal{O}_4$.

Proof: By definition

$$J_{\Delta',\Delta_1}(\phi) = J_+(\phi) - J_-(\phi)$$

where

$$J_+(\phi) = e^{-\frac{1}{2}(f,\Gamma f)_{L^{-1}\Delta'}} \times \int d\mu_\Gamma(\zeta) e^{\int_{L^{-1}\Delta'} f\zeta} R_{\Delta_1}(\phi_1 + \zeta)$$

and

$$J_-(\phi) = \left(\sum_{k=0}^{4} \delta\beta_{k,3,\Delta',\Delta_1} : \phi^k :_{C_0}\right) \times e^{-\tilde{V}_{\Delta_1}(\phi_1)} .$$

By Lemma 3.11 we have

$$|J_+(\phi)|_{\partial\phi, L^{[\phi]}h_*} \leqslant \mathcal{O}_3 |||R_{\Delta_1}|||_{\bar{g}} .$$

By Lemma 3.12 we also have

$$|\delta\beta_{k,3,\Delta',\Delta_1}| \leqslant \mathcal{O}_4 L^{-k[\phi]} h_*^{-k} |||R_{\Delta_1}|||_{\bar{g}} .$$

We have the bound

$$| : \phi^k :_{C_0} |_{\partial\phi, L^{[\phi]}h_*} \leqslant 25 \max_{0\leqslant a\leqslant k} |\phi^a|_{\partial\phi, L^{[\phi]}h_*} \leqslant 25 L^{k[\phi]} h_*^k$$

since $|\phi^a|_{\partial\phi, L^{[\phi]}h_*} = (L^{[\phi]}h_*)^a$ and $L^{[\phi]}h_* \geqslant 1$. Finally, by the chain rule

$$|e^{-\tilde{V}_{\Delta_1}(\phi_1)}|_{\partial\phi, L^{[\phi]}h_*} = |e^{-\tilde{V}_{\Delta_1}(\psi)}|_{\partial\psi, h_*} \leqslant 2$$

by Lemma 3.6. As result we easily arrive at

$$|J_-(\phi)|_{\partial\phi, L^{[\phi]}h_*} \leqslant 250\mathcal{O}_4 |||R_{\Delta_1}|||_{\bar{g}} \ .$$

The latter as well as the previous inequality for J_+ imply the desired estimate. □

That finishes our kernel norm estimates, we then have

Lemma 3.14. *For all $\Delta' \in \mathbb{L}$, $\Delta \in [L^{-1}\Delta']$ and $\phi \in \mathbb{R}$ we have*

$$||J_{\Delta',\Delta_1}(\phi)||_{\partial\phi,\phi,L^{[\phi]}h} \leqslant \mathcal{O}_6 \bar{g}^{-2} |||R_{\Delta_1}|||_{\bar{g}}$$

where

$$\mathcal{O}_6 = \exp\left(\frac{\sqrt{2}}{2}\right) + 155\mathcal{O}_4 \ .$$

Proof: Clearly, we have

$$||J_{\Delta',\Delta_1}(\phi)||_{\partial\phi,\phi,L^{[\phi]}h} \leqslant ||J_+(\phi)||_{\partial\phi,\phi,L^{[\phi]}h} + ||J_-(\phi)||_{\partial\phi,\phi,L^{[\phi]}h}$$

and both terms will be bounded as follows. We first write

$$||J_+(\phi)||_{\partial\phi,\phi,L^{[\phi]}h} \leqslant e^{-\frac{1}{2}\Re(f,\Gamma f)_{L^{-1}\Delta'}} \times \int \mathrm{d}\mu_\Gamma(\zeta)\ e^{\int_{L^{-1}\Delta'}(\Re f)\zeta} ||R_{\Delta_1}(\phi_1+\zeta)||_{\partial\phi,\phi,L^{[\phi]}h}$$

and then use the chain rule as well as the definition of the $|||\cdot|||_{\bar{g}}$ norm in order to derive

$$||R_{\Delta_1}(\phi_1+\zeta)||_{\partial\phi,\phi,L^{[\phi]}h} = ||R_{\Delta_1}(\psi+\zeta)||_{\partial\psi,\phi_1,h} = ||R_{\Delta_1}(\psi)||_{\partial\psi,\phi_1+\zeta,h} \leqslant \bar{g}^{-2} |||R_{\Delta_1}|||_{\bar{g}} \ .$$

Besides, as shown before $|(f,\Gamma f)_{L^{-1}\Delta'}| \leqslant \frac{1}{\sqrt{2}}$. Hence

$$||J_+(\phi)||_{\partial\phi,\phi,L^{[\phi]}h} \leqslant \exp[2^{-\frac{3}{2}}]\bar{g}^{-2} |||R_{\Delta_1}|||_{\bar{g}} \int \mathrm{d}\mu_\Gamma(\zeta)\ e^{\int_{L^{-1}\Delta'}(\Re f)\zeta}$$

$$\leqslant \exp[2^{-\frac{3}{2}}]\bar{g}^{-2} |||R_{\Delta_1}|||_{\bar{g}} e^{\frac{1}{2}(\Re f,\Gamma\Re f)_{L^{-1}\Delta'}}$$

by Lemma 3.4 with $X = \varnothing$ or simply exact computation. Again one easily gets that $|(\Re f, \Gamma \Re f)_{L^{-1}\Delta'}| \leqslant \frac{1}{\sqrt{2}}$ which results in

$$||J_+(\phi)||_{\partial\phi,\phi,L^{[\phi]}h} \leqslant \exp[2^{-\frac{1}{2}}]\bar{g}^{-2} |||R_{\Delta_1}|||_{\bar{g}} \ .$$

From the definition of $J_-(\phi)$ we immediately get

$$||J_-(\phi)||_{\partial\phi,\phi,L^{[\phi]}h} \leqslant \sum_{k=0}^{4} |\delta\beta_{k,3,\Delta',\Delta_1}| \times || :\phi^k:_{C_0} ||_{\partial\phi,\phi,L^{[\phi]}h} ||e^{-\tilde{V}_{\Delta_1}(\phi_1)}||_{\partial\phi,\phi,L^{[\phi]}h} \ .$$

By the chain rule and Lemma 3.6

$$||e^{-\tilde{V}_{\Delta_1}(\phi_1)}||_{\partial\phi,\phi,L^{[\phi]}h} = ||e^{-\tilde{V}_{\Delta_1}(\psi)}||_{\partial\psi,\phi_1,h} \leqslant 2e^{-\frac{1}{2}(\Re\beta_{4,\Delta_1})\phi_1^4} \ .$$

Again by undoing the Wick ordering we have

$$|| :\phi^k:_{C_0} ||_{\partial\phi,\phi,L^{[\phi]}h} \leqslant 25 \max_{0\leqslant a\leqslant k} ||\phi^a||_{\partial\phi,\phi,L^{[\phi]}h} \ .$$

But

$$||\phi^a||_{\partial\phi,\phi,L^{[\phi]}h} = (L^{[\phi]}h + |\phi|)^a \leqslant (L^{[\phi]}h + |\phi|)^k$$

since $L^{[\phi]}h \geqslant 1$. Now

$$|| :\phi^k:_{C_0} ||_{\partial\phi,\phi,L^{[\phi]}h} \leqslant 25(L^{[\phi]}h + |\phi|)^k = 25 \sum_{n=0}^{k} \binom{k}{n} \left(L^{[\phi]}c_1\bar{g}^{-\frac{1}{4}}\right)^{k-n} |\phi|^n$$

$$\leqslant 25 \sum_{n=0}^{k} \binom{k}{n} \left(L^{[\phi]}c_1\bar{g}^{-\frac{1}{4}}\right)^{k-n} \left(\frac{n}{2e}\right)^{\frac{n}{4}} (\gamma\bar{g})^{-\frac{n}{4}} e^{\gamma(\Re\beta_{4,\Delta_1})\phi^4}$$

by Lemma 3.3 and for any $\gamma > 0$. Here we choose $\gamma = \frac{1}{2}L^{-4[\phi]}$ which entails

$$|| :\phi^k:_{C_0} ||_{\partial\phi,\phi,L^{[\phi]}h} \leqslant 25 \times \left(\max_{0\leqslant n\leqslant 4}\left(\frac{n}{2e}\right)^{\frac{n}{4}}\right) \times e^{\frac{1}{2}(\Re\beta_{4,\Delta_1})\phi_1^4}$$

$$\times \sum_{n=0}^{k} \binom{k}{n} \left(L^{[\phi]}c_1\bar{g}^{-\frac{1}{4}}\right)^{k-n} \left(\frac{1}{2}L^{-4[\phi]}\bar{g}\right)^{-\frac{n}{4}} \ .$$

As a result of the previous considerations we arrive at

$$||J_-(\phi)||_{\partial\phi,\phi,L^{[\phi]}h} \leqslant 50 \times \left(\max_{0\leqslant n\leqslant 4}\left(\frac{n}{2e}\right)^{\frac{n}{4}}\right)$$

$$\times \sum_{k=0}^{4} |\delta\beta_{k,3,\Delta',\Delta_1}| \times L^{k[\phi]}\bar{g}^{-\frac{k}{4}} \left(\sum_{n=0}^{k}\binom{k}{n} c_1^{k-n} 2^{\frac{n}{4}}\right) \ .$$

Since $n \leqslant 4$ we simply bound $\frac{n}{2e}$ by 1. We also use Lemma 3.12 in order to write

$$||J_-(\phi)||_{\partial\phi,\phi,L^{[\phi]}h} \leqslant 50 \times \mathcal{O}_4 |||R_{\Delta_1}|||_{\bar{g}} \times \sum_{n=0}^{k} \bar{g}^{-\frac{k}{4}} h_*^{-\frac{k}{4}} (c_1 + 2^{\frac{1}{4}})^k \ .$$

Now we bound h_*^{-1} by 1, $\bar{g}^{-\frac{k}{4}}$ by the worst case scenario $\bar{g}^{-1} \leqslant \bar{g}^{-2}$ and finally $c_1 + 2^{\frac{1}{4}}$ by 2. Since

$1+2+\cdots+2^4=31$ we then obtain

$$||J_-(\phi)||_{\partial\phi,\phi,L^{[\phi]}h} \leqslant 50\times 31\times \mathcal{O}_4\bar{g}^{-2}|||R_{\Delta_1}|||_{\bar{g}}\ .$$

The latter inequality, combined with the previous one for J_+, gives us the desired result. □

Finally we prove the contractive bound. We mention that Lemma (3.10) is essential for getting the contractive estimate for both parts of the triple norm $|||\cdot|||_{\bar{g}}$:

Lemma 3.15. *For all unit cube $\Delta'\in\mathbb{L}$ we have*

$$|||\mathcal{L}^{(\vec{\beta},f)}_{\Delta'}(R)|||_{\bar{g}} \leqslant \mathcal{O}_7\times L^{3-5[\phi]}\times \max_{\Delta_1\in[L^{-1}\Delta']}|||R_{\Delta_1}|||_{\bar{g}}$$

where

$$\mathcal{O}_7 = 2^{10}\times\mathcal{O}_2\times(\mathcal{O}_5+\mathcal{O}_6)\ .$$

Proof: Recall that

$$\mathcal{L}^{(\vec{\beta},f)}_{\Delta'}(R) = \sum_{\Delta_1\in[L^{-1}\Delta']}\left(\prod_{\substack{\Delta\in[L^{-1}\Delta']\\ \Delta\neq\Delta_1}} e^{-\tilde{V}_\Delta(\phi_1)}\right)\times J_{\Delta',\Delta_1}(\phi)\ .$$

Hence

$$|\mathcal{L}^{(\vec{\beta},f)}_{\Delta'}(R)|_{\partial\phi,h_*} \leqslant \sum_{\Delta_1\in[L^{-1}\Delta']}\left|\prod_{\substack{\Delta\in[L^{-1}\Delta']\\ \Delta\neq\Delta_1}} e^{-\tilde{V}_\Delta(\phi_1)}\right|_{\partial\phi,h_*}\times|J_{\Delta',\Delta_1}(\phi)|_{\partial\phi,h_*}\ .$$

Now by Lemma 3.6 with $Y_0=[L^{-1}\Delta']\backslash\{\Delta_1\}$ we have

$$\left|\prod_{\substack{\Delta\in[L^{-1}\Delta']\\ \Delta\neq\Delta_1}} e^{-\tilde{V}_\Delta(\phi_1)}\right|_{\partial\phi,h_*}\leqslant 2\ .$$

Now by the construction in §3.5.2, the derivatives $J^{(n)}_{\Delta',\Delta_1}(0)$ vanish when $0\leqslant n\leqslant 4$. As a result

$$|J_{\Delta',\Delta_1}(\phi)|_{\partial\phi,h_*} = \sum_{n=5}^{9}\frac{h_*^n}{n!}|J^{(n)}_{\Delta',\Delta_1}(0)| \leqslant L^{-5[\phi]}|J_{\Delta',\Delta_1}(\phi)|_{\partial\phi,L^{[\phi]}h_*}$$

and thus by Lemma 3.13 we have

$$\begin{aligned}
|\mathcal{L}^{(\vec{\beta},f)}_{\Delta'}(R)|_{\partial\phi,h_*} &\leqslant 2L^{-5[\phi]}\sum_{\Delta_1\in[L^{-1}\Delta']}\mathcal{O}_5|||R_{\Delta_1}|||_{\bar{g}}\\
&\leqslant 2\mathcal{O}_5L^{3-5[\phi]}\max_{\Delta_1\in[L^{-1}\Delta']}|||R_{\Delta_1}|||_{\bar{g}}\ .
\end{aligned}$$

Likewise, we have

$$||\mathcal{L}_{\Delta'}^{(\vec{\beta},f)}(R)||_{\partial\phi,\phi,h} \leqslant \sum_{\Delta_1\in[L^{-1}\Delta']} \left|\left| \prod_{\substack{\Delta\in[L^{-1}\Delta'] \\ \Delta\neq\Delta_1}} e^{-\tilde{V}_\Delta(\phi_1)} \right|\right|_{\partial\phi,\phi,h} \times ||J_{\Delta',\Delta_1}(\phi)||_{\partial\phi,\phi,h} \ .$$

If we let $Y_0 = [L^{-1}\Delta']\backslash\{\Delta_1\}$, then $|Y_0| \geqslant \frac{L^3}{2}$ and by Lemma 3.6 we have

$$\left|\left| \prod_{\substack{\Delta\in[L^{-1}\Delta'] \\ \Delta\neq\Delta_1}} e^{-\tilde{V}_\Delta(\phi_1)} \right|\right|_{\partial\phi,\phi,h} \leqslant 2e^{-\frac{\bar{g}}{16}\phi^4} \ .$$

By Lemma 3.10 with $\beta_4 = \bar{g}$ and $\gamma = \frac{1}{16}$ one has

$$||J_{\Delta',\Delta_1}(\phi)||_{\partial\phi,\phi,h} \leqslant \mathcal{O}_2 16^{\frac{9}{4}} e^{\frac{\bar{g}}{16}\phi^4} \left[|J_{\Delta',\Delta_1}(\phi)|_{\partial\phi,h} + L^{-9[\phi]} \sup_{\psi\in\mathbb{R}} |J_{\Delta',\Delta_1}(\psi)|_{\partial\psi,\psi,L^{[\phi]}h} \right] \ .$$

By the same argument utilizing the vanishing of the first few derivatives at the origin as before, with h instead of h_*, we get

$$|J_{\Delta',\Delta_1}(\phi)|_{\partial\phi,h} \leqslant L^{-5[\phi]} |J_{\Delta',\Delta_1}(\phi)|_{\partial\phi,L^{[\phi]}h} \ .$$

Now by Lemma 3.13

$$|J_{\Delta',\Delta_1}(\phi)|_{\partial\phi,L^{[\phi]}h} \leqslant \mathcal{O}_5 |||R_{\Delta_1}|||_{\bar{g}}$$

whereas, by Lemma 3.14, one has

$$\sup_{\psi\in\mathbb{R}} |J_{\Delta',\Delta_1}(\psi)|_{\partial\psi,\psi,L^{[\phi]}h} \leqslant \mathcal{O}_6 \bar{g}^{-2} |||R_{\Delta_1}|||_{\bar{g}} \ .$$

We then arrive at the estimate

$$||J_{\Delta',\Delta_1}(\phi)||_{\partial\phi,\phi,h} \leqslant \mathcal{O}_2 \times 2^9 \times e^{\frac{\bar{g}}{16}\phi^4} |||R_{\Delta_1}|||_{\bar{g}} \times \left[L^{-5[\phi]}\mathcal{O}_5 + L^{-9[\phi]}\bar{g}^{-2}\mathcal{O}_5 \right] \ .$$

Using $L^{-5[\phi]}\bar{g}^{-2}$ as a common bound of $L^{-9[\phi]}\bar{g}^{-2}$ and $L^{-5[\phi]}$ we immediately get

$$||\mathcal{L}_{\Delta'}^{(\vec{\beta},f)}(R)(\phi)||_{\partial\phi,\phi,h} \leqslant \sum_{\Delta_1\in[L^{-1}\Delta']} 2^{10}\mathcal{O}_2(\mathcal{O}_5+\mathcal{O}_6) L^{-5[\phi]}\bar{g}^{-2} |||R_{\Delta_1}|||_{\bar{g}}$$

and hence

$$\bar{g}^2 \times ||\mathcal{L}_{\Delta'}^{(\vec{\beta},f)}(R)(\phi)||_{\partial\phi,\phi,h} \leqslant 2^{10}\mathcal{O}_2(\mathcal{O}_5+\mathcal{O}_6) L^{3-5[\phi]} \max_{\Delta_1\in[L^{-1}\Delta']} |||R_{\Delta_1}|||_{\bar{g}} \ .$$

The latter inequality, combined with the previous one for the $|\cdot|_{\partial\phi,h_*}$ seminorm, give

$$|||\mathcal{L}_{\Delta'}^{(\vec{\beta},f)}(R)|||_{\bar{g}} \leqslant L^{3-5[\phi]} \left(\max_{\Delta_1\in[L^{-1}\Delta']} |||R_{\Delta_1}|||_{\bar{g}} \right) \times \max\left[2\mathcal{O}_5, 2^{10}\mathcal{O}_2(\mathcal{O}_5+\mathcal{O}_6)\right] \ .$$

Since clearly $\mathcal{O}_2 > 1$, the last maximum reduces to the second term, i.e., the given value of $\mathcal{O}_7$. □

3.7 The bulk RG and analysis of the non-trivial fixed point

3.7.1 The bulk RG

As mentioned in section §3.4.3 we call the RG flow associated with calculation of $\mathcal{Z}_{r,s}(0,0)$ the bulk RG flow. Refering back to the definition for the initial RG_{ex} data $\vec{V}^{(r,r)}(\tilde{f},\tilde{j})$ corresponding to $\mathcal{Z}_{r,s}(\tilde{f},\tilde{j})$ we see that in the case where $\tilde{f} = \tilde{j} = 0$ the initial data is spatially homogenous and it is clear from the definition of RG_{ext} that property will continue to hold for $\vec{V}^{(r,q)}(0,0)$ for all $q \geqslant r$. In the discussion at the end of §3.4.3 we also claimed that the preserved functional form for the bulk flow is simpler, in what follows we set up a simple Banach space in which the bulk flow will live.

We define $\mathcal{E} = \mathbb{C}^2 \times C^9_{\text{bd,ev}}(\mathbb{R},\mathbb{C})$ where $C^9_{\text{bd,ev}}(\mathbb{R},\mathbb{C})$ denotes the closed subspace of even functions in $C^9_{\text{bd}}(\mathbb{R},\mathbb{C})$, i.e. elements $K(\phi)$ that satisfy $K(\phi) = K(-\phi)$. We can write an element $V \in \mathcal{E}$ in the form $V = (g,\mu,R)$ where $g,\mu \in \mathbb{C}$ and $R \in C^9_{\text{bd}}(\mathbb{R},\mathbb{C})$. For now we equip $\mathcal{E}$ with the norm

$$||V|| = ||(g,\mu,R)|| = \max\left(|g|\bar{g}^{-\frac{3}{2}},\ |\mu|\bar{g}^{-1}, |||R|||_{\bar{g}}\bar{g}^{-\frac{21}{8}}\right).$$

However later we will shift coordinates (and with it the norm) for $\mathcal{E}$. It is clear that $\mathcal{E}$ can be identified with a subspace of $\mathcal{E}_{\text{ex}}$ via an isometric map $V \to \vec{V}$ with $\vec{V} \in \mathcal{E}$ of the form

$$\vec{V} = (\beta_{4,\Delta}, \beta_{3,\Delta}, \beta_{2,\Delta}, \beta_{1,\Delta}, W_{5,\Delta}, W_{6,\Delta}, f_\Delta, R_\Delta)_{\Delta\in\mathbb{L}}$$

with the above parameters given by

$$\begin{aligned}
\beta_{4,\Delta} &= g\\
\beta_{3,\Delta} &= 0\\
\beta_{2,\Delta} &= \mu\\
\beta_{1,\Delta} &= 0\\
W_{5,\Delta} &= 0\\
W_{6,\Delta} &= 0\\
f_\Delta &= 0\\
R_\Delta &= R
\end{aligned}$$

for all unit cubes Δ.

Using this identification the next proposition claims that space $\mathcal{E}$ is invariant under the extended RG map RG_{ex}. Note that in what follows we drop Δ indices from many quantities due to spatial homogeneity. We also write $\mathcal{L}^{(g,\mu)}$ instead of the notation $\mathcal{L}^{(\vec{\beta},f)}$ we used earlier.

Proposition 3.2. *The space $\mathcal{E}$ is invariant by the map $RG_{\rm ex}$. The restricted transformation*

$$\begin{array}{cccc} RG: & \mathcal{E} & \longrightarrow & \mathcal{E} \\ & (g,\mu,R) & \longmapsto & (g',\mu',R') \end{array}$$

which we call the bulk RG is given by

$$\begin{aligned} g' &= L^{\epsilon} g - A_1 g^2 + \xi_4(g,\mu,R) \\ \mu' &= L^{\frac{3+\epsilon}{2}}\mu - A_2 g^2 - A_3 g\mu + \xi_2(g,\mu,R) \\ R' &= \mathcal{L}^{(g,\mu)}(R) + \xi_R(g,\mu,R) \end{aligned} \tag{3.54}$$

where

$$\begin{aligned} A_1 &= 36 L^{3-4[\phi]} \int_{\mathbb{Q}_p^3} \Gamma(x)^2 \ {\rm d}^3x \\ A_2 &= 48 L^{3-2[\phi]} \left(\int_{\mathbb{Q}_p^3} \Gamma(x)^3 \ {\rm d}^3x \right) + 144 L^{3-4[\phi]} C_0(0) \left(\int_{\mathbb{Q}_p^3} \Gamma(x)^2 \ {\rm d}^3x \right) \\ A_3 &= 12 L^{3-2[\phi]} \int_{\mathbb{Q}_p^3} \Gamma(x)^2 \ {\rm d}^3x \ . \end{aligned} \tag{3.55}$$

In addition, the vacuum counter-term $\delta b = \delta b(g,\mu,R)$ is given by

$$\delta b = A_4 g^2 + A_5 \mu^2 + \xi_0(g,\mu,R)$$

where

$$\begin{aligned} A_4 &= 12L^3 \left(\int_{\mathbb{Q}_p^3} \Gamma(x)^4 \ {\rm d}^3x \right) + 48 L^{3-2[\phi]} C_0(0) \left(\int_{\mathbb{Q}_p^3} \Gamma(x)^3 \ {\rm d}^3x \right) + 72 L^{3-4[\phi]} C_0(0)^2 \left(\int_{\mathbb{Q}_p^3} \Gamma(x)^2 \ {\rm d}^3x \right) \\ A_5 &= L^3 \int_{\mathbb{Q}_p^3} \Gamma(x)^2 \ {\rm d}^3x \ . \end{aligned}$$

Partial Proof: This is Proposition 2 in [3] and there the assertion is carefully checked step-by-step by studying the transformation $\vec{V} \mapsto \vec{V}'$ defined in §3.5.2. Instead of giving the details here we remark that proving the simplified form the bulk flow boils down to parity considerations - if one starts with even terms for all the inputs then then $RG_{\rm ex}$ produces even outputs.

However the continued vanishing of the W_6 term has a different cause which we now describe. Clearly since $f_\Delta = 0$, the new $f'_{\Delta'}$'s defined in (3.32) are identically zero. Likewise, since the W_6 are zero the equation for the new one reduces to

$$W'_{6,\Delta'} = 8L^{-6[\phi]} \int_{(L^{-1}\Delta')^2} {\rm d}^3x \ {\rm d}^3y \ \beta_4(x) \ \Gamma(x-y) \ \beta_4(y) = 8L^{-6[\phi]} g^2 \int_{(L^{-1}\Delta')^2} {\rm d}^3x \ {\rm d}^3y \ \Gamma(x-y) \ .$$

But for $x \in L^{-1}\Delta'$, by a simple change of variables $z = x - y$,

$$\int_{L^{-1}\Delta'} \mathrm{d}^3 y\ \Gamma(x-y) = \int_{L^{-1}\Delta(0)} \mathrm{d}^3 z\ \Gamma(z) = \int_{\mathbb{Q}_p^3} \mathrm{d}^3 z\ \Gamma(z) = \widehat{\Gamma}(0) = 0$$

because of the finite range property and the vanishing property at zero momentum. Therefore $W'_{6,\Delta'}$ vanishes identically. □

3.7.2 The infrared fixed point and local analysis of the bulk RG

In this section we will find a non-zero fixed point for the flow given by (3.54); we will call this the *non-trivial infrared fixed point*. As we mentioned in §3.4.3 the equations (3.54) are in close correspondance to the flow equations (3.25) given by second order perturbation theory with the addition of some remainder terms and a flow equation for the irrelevant part R. We will now fix a choice of L, once and for all, so

$$B_{R\mathcal{L}} L^{3-5[\phi]} \leqslant \frac{1}{2} \tag{3.56}$$

holds. We remark that $B_{R\mathcal{L}}$ was a purely numeric constant independent of both L and ϵ. Note that $3-5[\phi] = -\frac{3}{4} + \frac{5}{4}\epsilon$. If we add the harmless condition $\epsilon \leqslant \frac{1}{5}$ which we now assume, then $3-5[\phi] \leqslant -\frac{1}{2}$. Now we pick L large enough so that $B_{R\mathcal{L}} L^{-\frac{1}{2}} \leqslant \frac{1}{2}$ and therefore (3.56) holds. Now that L is considered fixed the only free parameter in our construction is ϵ.

We now apply Theorem 3.2 with the choices just mentioned and in concert with Proposition 3.2 to obtain that, provided ϵ is small enough, the bulk RG transformation is well-defined and analytic on the domain

$$|g-\bar{g}| < \frac{1}{2}\bar{g}\ ,\ |\mu| < \bar{g}\ ,\ |||R|||_{\bar{g}} < \bar{g}^{\frac{21}{8}}$$

and therein satisfies

$$\begin{aligned} |\xi_4(g,\mu,R)| &\leqslant B_4 |||R|||_{\bar{g}} \\ |\xi_2(g,\mu,R)| &\leqslant B_2 |||R|||_{\bar{g}} \\ |||\xi_R(g,\mu,R)|||_{\bar{g}} &\leqslant B_{R\xi} \bar{g}^{\frac{11}{4}} \\ |||\mathcal{L}^{(g,\mu)}|||_{\bar{g}} &\leqslant \frac{1}{2} \end{aligned}$$

where $|||\mathcal{L}^{(g,\mu)}|||_{\bar{g}}$ is the operator norm of the linear operator $\mathcal{L}^{(g,\mu)}$ (with respect to the R variable) corresponding to the norm $|||\cdot|||_{\bar{g}}$. Note that the statement on analyticity applies not only to the full map RG but also to the constituent pieces such as ξ_4, ξ_2, ξ_R and $\mathcal{L}^{(g,\mu)}(R)$.

In our search for a non-zero fixed point to (3.54) we will change our coordinate system so that we write our data as a perturbation of the approximate fixed point $(\bar{g}, 0, 0)$ - when the flow (3.54) is studied in this coordinate system the perturbation in the g direction will be contracting. Concretely we change from (g, μ, R) to $(\delta g, \mu, R)$ where $\delta g = g - \bar{g}$. In this new coordinate system, the bulk RG transformation, still denoted by RG for simplicity, becomes $(\delta g,\mu,R)\longmapsto RG(\delta g,\mu,R)=(\delta g',\mu',R')$ with

$$\begin{aligned}\delta g' &= (2-L^{\epsilon})\delta g+\tilde{\xi}_4(\delta g,\mu,R)\\ \mu' &= L^{\frac{3+\epsilon}{2}}\mu+\tilde{\xi}_2(\delta g,\mu,R)\\ R' &= \tilde{\mathcal{L}}^{(\delta g,\mu)(R)}+\tilde{\xi}_R(\delta g,\mu,R)\end{aligned} \tag{3.57}$$

where

$$\begin{aligned}\tilde{\xi}_4(\delta g,\mu,R) &= -A_1\delta g^2+\xi_4(\bar{g}+\delta g,\mu,R)\\ \tilde{\xi}_2(\delta g,\mu,R) &= -A_2(\bar{g}+\delta g)-A_3(\bar{g}+\delta g)\mu+\xi_2(\bar{g}+\delta g,\mu,R)\\ \tilde{\xi}_R(\delta g,\mu,R) &= \xi_R(\bar{g}+\delta g,\mu,R)\\ \tilde{\mathcal{L}}^{(\delta g,\mu)}(R) &= \mathcal{L}^{(\bar{g}+\delta g,\mu)(R)}\end{aligned}$$

as follows from an easy computation using the relation $A_1\bar{g}=L^{\epsilon}-1$. We remark that $2-L^{\epsilon}<1$. We will commit a similar abuse of notation for the function δb. Namely, we will write $\delta b(\delta g,\mu,R)$ for what in fact is $\delta b(\bar{g}+\delta g,\mu,R)$. We will also translate the norms we use for $v=(\delta g,\mu,R)\in\mathcal{E}$, namely,

$$||v||=\max\left\{|\delta g|\bar{g}^{-\frac{3}{2}},|\mu|\bar{g}^{-1},|||R|||_{\bar{g}}\bar{g}^{-\frac{21}{8}}\right\}\ .$$

The following Lipschitz estimates are crucial ingredients for our analysis near the nontrivial infrared fixed point. In particular these estimates show that the expansion and contraction rates of our δg, μ, and R flow are not significantly influenced by the presence of each of the corresponding remainder terms.

Lemma 3.16. *For ϵ small enough we have for all $v=(\delta g,\mu,R)$, $v'=(\delta g',\mu',R')$ in $\mathcal{E}$ such that $||v||$, $||v'||\leqslant\frac{1}{8}$,*

$$|\xi_4(\bar{g}+\delta g,\mu,R)-\xi_4(\bar{g}+\delta g',\mu',R')|\leqslant 2B_4\bar{g}^{\frac{21}{8}}||v-v'||\ ,$$

$$|\xi_2(\bar{g}+\delta g,\mu,R)-\xi_4(\bar{g}+\delta g',\mu',R')|\leqslant 2B_2\bar{g}^{\frac{21}{8}}||v-v'||\ ,$$

$$|||\mathcal{L}^{(\bar{g}+\delta g,\mu)}(R)-\mathcal{L}^{(\bar{g}+\delta g',\mu')}(R')|||_{\bar{g}}\leqslant\frac{3}{4}\bar{g}^{\frac{21}{8}}||v-v'||$$

and

$$|||\xi_R(\bar{g}+\delta g,\mu,R)-\xi_R(\bar{g}+\delta g',\mu',R')|||_{\bar{g}}\leqslant 3B_{R\xi}\bar{g}^{\frac{11}{4}}||v-v'||\ .$$

Proof: If $||v||<\frac{1}{2}$, then since $\bar{g}\leqslant 1$ for ϵ small we have

$$\begin{aligned}|\delta g| &< \tfrac{1}{2}\bar{g}^{\frac{3}{2}} \leqslant \tfrac{1}{2}\bar{g}\\ |\mu| &< \tfrac{1}{2}\bar{g} \leqslant \tfrac{1}{2}\bar{g}\\ |||R|||_{\bar{g}} &< \tfrac{1}{2}\bar{g}^{\frac{1}{8}} \leqslant \tfrac{1}{2}\bar{g}^{\frac{21}{8}}\ .\end{aligned}$$

Hence, by Theorem 3.2

$$|\xi_4(\bar{g}+\delta g,\mu,R)|\leqslant B_4|||R|||_{\bar{g}}\leqslant\frac{1}{2}B_4\bar{g}^{\frac{1}{8}}\ .$$

Therefore the analytic map $v\mapsto\xi_4(\bar{g}+\delta g,\mu,R)$ satisfies the hypotheses of Lemma 3.2 with $r_1=\frac{1}{2}$ and

$r_2 = \frac{1}{2}B_4\bar{g}^{\frac{1}{8}}$. We pick $\nu = \frac{1}{4}$ which results in

$$\frac{r_2(1-\nu)}{r_1(1-2\nu)} = \frac{3}{2}B_4\bar{g}^{\frac{1}{8}} .$$

With these choices, Lemma 3.2 implies the desired Lipschitz estimate where we replaced the numerical factor $\frac{3}{2}$ by 2 for a simpler looking formula. The proof of the Lipschitz estimate for ξ_2 is exactly the same apart from changing ξ_4, B_4 to ξ_2, B_2 respectively.

We now do the same for the analytic map $v \mapsto \mathcal{L}^{(\bar{g}+\delta g,\mu)}(R)$. For $||v|| < \frac{1}{2} = r_1$ we obtain, as before from Theorem 3.2 and from the choice we made when fixing L,

$$|||\mathcal{L}^{(\bar{g}+\delta g,\mu)}(R)|||_{\bar{g}} \leqslant \frac{1}{2}|||R|||_{\bar{g}} \leqslant \frac{1}{2}||v||\bar{g}^{\frac{1}{8}} \leqslant r_2$$

with $r_2 = \frac{1}{4}\bar{g}^{\frac{1}{8}}$. Lemma 3.2 with $\nu = \frac{1}{4}$ now immediately implies the wanted estimate.

Finally, for ξ_R we again note that $||v|| < \frac{1}{2} = r_1$ implies

$$|||\xi_R(\bar{g}+\delta g, \mu, R)|||_{\bar{g}} \leqslant r_2$$

with $r_2 = B_{R\xi}\bar{g}^{\frac{11}{4}}$. Again, Lemma 3.2 with $\nu = \frac{1}{4}$ does the rest. □

In what follows we will construct the nontrivial fixed point v_* and patches of the corresponding local *stable and unstable manifolds*. Before giving the details we give a pedagogical explanation.

From the flow equations and our estimates on the remainders it is not very difficult to show that if there is a nontrivial infrared fixed point $v_* = (\delta g_*, \mu_*, R_*)$ with $||v_*|| \leqslant \mathcal{O}(1)$ (i.e. close to our approximate fixed point) then it must be a hyperbolic fixed point. In particular it would have two contracting directions (corresponding to δg and R) and one expanding direction (corresponding to μ). The linearization of the RG at the v_* should have no eigenvalues on the unit circle.

Now assume that one wants to find a $v = (\delta g, \mu, R)$ close to v_* with $\lim_{n\to\infty} RG^n[v]$. Then the value of v's component in the expanding direction, i.e. μ, must be tuned carefully in a way dependent on δg and R in order for v to be driven to v_* - this value $\mu_{\mathrm{s}}(\delta g, R)$ is called the *critical mass* corresponding to $(\delta g, R)$. In what follows we will construct this function μ_{s} for a particular $(\delta g, R)$ domain.

The graph of the function μ_{s} , i.e v of the form $(\delta g, \mu_{\mathrm{s}}(\delta g, R), R)$, will correspond to a piece of the stable manifold of v_*. We remind the reader that the stable manifold of the fixed point v_* is the set of all v in the domain of the RG such that $\lim_{n\to\infty} RG^n[v] \to v_*$.

Our argument will proceed in an order that is reverse to the above explanation. In order to find the nontrivial fixed point v_* our first step will be to find a function μ_{s} defined for some non-empty open set of $(\delta g, R)$ such that any point $(\delta g, \mu_{\mathrm{s}}(\delta g, R), R)$ remains within our domain after arbitrarily many iterations of the RG map. We will then show that the RG map restricted to a portion of the graph of μ_{s} is a contraction mapping - this will yield a fixed point v_*. It then follows that this portion of the graph of μ_{s} is a patch of the stable manifold of v_* containing v_* . A heuristic explanation of why this works is that if for some fixed $\delta g, R$ one chooses μ either lower or higher than the critical mass then one expects that our system will be driven to the high temperature or low temperature fixed point both of which are far outside of the domain of where we defined the RG - (this picture would assume that there is no intermediate phase - that topic will be taken up in Chapter 4).

Our next step will then be to construct a local patch of the unstable manifold. The unstable manifold of a fixed point v_* is the set of all v such that we can find a backward sequence of preimages of v under the RG map which converge to v_*. More concretely the unstable manifold is the set of all v such that one can find $\{v_n\}_{-\infty<n\leqslant -1}$ such that $RG^j[v^{-j}] = v$ and $\lim_{j\to\infty} v^{-j} = v_*$. Just as the stable manifold can be parameterized via coordinates for the contracting directions we will construct the local patch of the unstable manifold as the graph of a function $\mu \to (\delta g_u(\mu), R_u(\mu))$, i.e. it will be parameterized by the expanding direction.

3.7.3 The local stable manifold

We now proceed with the first step which is the construction of the stable manifold also using the Banach Fixed Point Theorem in a space of one-sided sequences, in the spirit of Irwin's method [39]. This method is based on the same idea that is used for the the construction solutions to finite dimensional ODEs - we reformulate the problem as a fixed point problem and applying the contraction mapping theorem.

The general idea is as follows: given a starting δg and R we will define a map $\mathfrak{m}$ that acts on the Banach space of all bounded sequences $\{(\delta g_n, \mu_n, R_n)\}_{n=0}^{\infty}$ such that a fixed point of $\mathfrak{m}$ in this space of sequences will be a sequence that is consistent with the flow equations (3.57) and satisfies particular boundary conditions at $n = 0$ and $n = \infty$ - these conditions being that $\delta g_0 = \delta g$, $R_0 = R$, and that μ_n does not blow up as $n \to \infty$.

We give the precise implementation of the above strategy now. Let $\mathcal{B}_+$ be the Banach space of sequences

$$\vec{u} = (\mu_0, (\delta g_1, \mu_1, R_1), (\delta g_2, \mu_2, R_2), \ldots) \in \mathbb{C} \times \prod_{n\geqslant 1} \left[\mathbb{C}^2 \times C^9_{\mathrm{bd,ev}}(\mathbb{R}, \mathbb{C})\right]$$

which have finite norm given by

$$||\vec{u}|| = \sup\left\{|\delta g_j|\bar{g}^{-\frac{3}{2}} \text{ for } j \geqslant 1;\ |\mu_j|\bar{g}^{-1} \text{ for } j \geqslant 0; |||R_j|||_{\bar{g}}\bar{g}^{-\frac{21}{8}} \text{ for } j \geqslant 1\right\}\ .$$

We will define a map $\mathfrak{m}$ on this space of sequences which depends on parameters $\delta g_0, R_0$ serving as boundary conditions. Given δg_0 and R_0, the image $\vec{u}' = \mathfrak{m}(\vec{u})$ is defined as follows. For $n \geqslant 1$, we let

$$\delta g'_n = (2 - L^{\epsilon})^n \delta g_0 + \sum_{j=0}^{n-1} (2 - L^{\epsilon})^{n-1-j} \tilde{\xi}_4(\delta g_j, \mu_j, R_j)$$

and

$$\begin{aligned} R'_n &= \tilde{\mathcal{L}}^{(\delta g_{n-1}, \mu_{n-1})} \circ \cdots \circ \tilde{\mathcal{L}}^{(\delta g_0, \mu_0)}(R_0) \\ &\quad + \sum_{j=0}^{n-1} \tilde{\mathcal{L}}^{(\delta g_{n-1}, \mu_{n-1})} \circ \cdots \circ \tilde{\mathcal{L}}^{(\delta g_{j+1}, \mu_{j+1})}\left(\tilde{\xi}_R(\delta g_j, \mu_j, R_j)\right)\ . \end{aligned}$$

For $n \geqslant 0$, we let

$$\mu'_n = -\sum_{j=n}^{\infty} L^{-(j-n+1)\left(\frac{3+\epsilon}{2}\right)} \tilde{\xi}_2(\delta g_j, \mu_j, R_j)\ .$$

As an aside we make a remark to motivate the last formula above. Here we are propagating the μ boundary condition backwards. If one writes the μ evolution of (3.57) in reverse one has

$$\mu_k \ = \ L^{-\frac{3+\epsilon}{2}} \left[\mu_{k+1} - \tilde{\xi}_2(\delta g_k, \mu_k, R_k)\right]$$

Now if one sets a boundary condition μ_q for some scale $q \geqslant 0$, then propagating this backwards to scale $n \leqslant q$ would give an equation

$$\mu'_n = L^{-\frac{3+\epsilon}{2}(q-n)} - \sum_{j=n}^{q} L^{-(j-n+1)\left(\frac{3+\epsilon}{2}\right)} \tilde{\xi}_\mu(\delta g_j, \mu_j, R_j) \ .$$

This equation agrees with the earlier one given when we take $q \to \infty$ assuming the μ_q are bounded.

The next Proposition shows that given a sufficiently small $\rho > 0$ this map $\mathfrak{m}$ is well defined and analytic on the open ball $B(\vec{0}, \rho) \in \mathcal{B}_+$ in the regime of small ϵ (made small after fixing ρ).

Proposition 3.3. *If $0 < \rho < \frac{1}{12}$, $|\delta g_0| < \frac{\rho}{12}\bar{g}^{\frac{3}{2}}$ and $|||R_0|||_{\bar{g}} < \frac{\rho}{8}\bar{g}^{\frac{21}{8}}$ then the map $\mathfrak{m}$ is well defined, analytic on $B(\vec{0}, \rho)$ and takes its values in the closed ball $\bar{B}(\vec{0}, \frac{\rho}{4})$, provided ϵ is made sufficiently small after fixing ρ. Moreover, $\mathfrak{m}$ is jointly analytic in $\vec{u}$ and the implicit variables δg_0 and R_0.*

Proof: We do not reproduce the proof here, this is Proposition 3 of [3].

Using Lemma 3.2 with $r_1 = \rho$, $r_2 = \frac{\rho}{4}$ and $\nu = \frac{1}{3}$ so that

$$\frac{r_2(1-\nu)}{r_1(1-2\nu)} = \frac{1}{2}$$

we immediately see that, under the hypotheses of Proposition 3.3, the closed ball $\bar{B}\left(\vec{0}, \frac{\rho}{3}\right)$ is stable by $\mathfrak{m}$ and is a contraction. More precisely, for any $\vec{u}_1$ and $\vec{u}_2$ in that ball, we have

$$||\mathfrak{m}(\vec{u}_1) - \mathfrak{m}(\vec{u}_2)|| \leqslant \frac{1}{2}||\vec{u}_1 - \vec{u}_2|| \ .$$

By the Banach Fixed Point Theorem we then have the existence of a unique fixed point denoted by $\vec{u}_*$ for the map $\mathfrak{m}$ in the ball $\bar{B}\left(\vec{0}, \frac{\rho}{3}\right)$. Using the representation of this fixed point as

$$\vec{u}_* = \sum_{n=0}^{\infty} \left[\mathfrak{m}^{n+1}(\vec{0}) - \mathfrak{m}^n(\vec{0})\right]$$

and by uniform absolute convergence, it is easy to see that $\vec{u}_*$ is analytic in the implicit data $(\delta g_0, R_0)$. In particular we will define $\mu_{\mathrm{s}}(\delta g_0, R_0)$ as the μ_0 component of the sequence $\vec{u}_*$ and remark that $\mu_{\mathrm{s}}(\delta g_0, R_0)$ is analytic on the domain given by $|\delta g_0| < \frac{\rho}{12}\bar{g}^{\frac{3}{2}}$ and $|||R_0|||_{\bar{g}} < \frac{\rho}{8}\bar{g}^{\frac{21}{8}}$.

As we mentioned before the graph $(\delta g, \mu_{\mathrm{s}}(\delta g, R), R)$ will be a local patch of the stable manifold of the sought after nontrivial infrared fixed point. We will denote this patch by $W^{\mathrm{s,loc}}$, we give a concrete definitions below (here the patch is determined in terms of a radius ρ which must be chosen satisfy the hypothesis of Proposition 3.3)

$$W^{\mathrm{s,loc}} = \left\{(\delta g, \mu, R) \in \mathcal{E} | \ |\delta g| \leqslant \frac{\rho}{13}\bar{g}^{\frac{3}{2}}, |||R|||_{\bar{g}} \leqslant \frac{\rho}{13}\bar{g}^{\frac{21}{8}}, \mu = \mu_{\mathrm{s}}(\delta g, R)\right\} \ .$$

We then have the following proposition which claims that $W^{\mathrm{s,loc}}$ is all of the local stable manifold.

Proposition 3.4. *For fixed $\rho \in (0, \frac{1}{12})$ and for ϵ small enough, an equivalent description of $W^{\mathrm{s,loc}}$ is as the set of triples $(\delta g, \mu, R) \in \mathcal{E}$ that satisfy all of the following properties:*

- $|\delta g| \leqslant \frac{\rho}{13}\bar{g}^{\frac{3}{2}}$,
- $|||R|||_{\bar{g}} \leqslant \frac{\rho}{13}\bar{g}^{\frac{21}{8}}$,
- *there exists a sequence $(\delta g_n, \mu_n, R_n)_{n\geqslant 0}$ in $\mathcal{E}$ such that $\delta g_0 = \delta g$, $\mu_0 = \mu$, $R_0 = R$, $\forall n \geqslant 1$, $|\delta g_n| \leqslant \frac{\rho}{3}\bar{g}^{\frac{3}{2}}$ and $|||R_n|||_{\bar{g}} \leqslant \frac{\rho}{3}\bar{g}^{\frac{21}{8}}$, $\forall n \geqslant 0$, $|\mu_n| \leqslant \frac{\rho}{3}\bar{g}$, and $\forall n \geqslant 0$, $(\delta g_{n+1}, \mu_{n+1}, R_{n+1}) = RG(\delta g_n, \mu_n, R_n)$.*

Proof: We don't give a full proof but the steps are straightforward - one can find details in [3, Proposition 4]. When showing that $W_{\mathrm{s,loc}}$ is contained in the set described in the assertion one just needs to check that the fixed point $\vec{u}_* = \mathfrak{m}(\vec{u}_*)$ satisfies the third condition - i.e. a sequence that satisfies the flow equations. The other conditions are automatic.

Showing the reverse inclusion involves observing that a sequence $\vec{u}$ that satisfies the conditions of the assertion must be a fixed point of $\mathfrak{m}$ in the ball $B(0, \frac{\rho}{3})$ - then one must have $\vec{u} = \vec{u}_*$ due to the uniquess of the fixed point delivered by the contraction mapping theorem. In particular the μ_0 component of $\vec{u}$ must coincide with the μ_0 component of $\vec{u}_*$ which is given by $\mu_{\mathrm{s}}(\delta g, R)$.

A dichotomy lemma

We now state an important lemma which gives quantitative growth and decay estimates which will establish separation between expanding and contracting directions. This will be important in the actual construction of the non-trivial fixed point and for analysis of the unstable manifold and the composite field.

We first perform a crude splitting of $\mathcal{E}$ into contracting and expanding directions, writing $\mathcal{E} = \mathcal{E}_1 \oplus \mathcal{E}_2$ where

$$\mathcal{E}_1 = \{(\delta g, 0, R) | \delta g \in \mathbb{C}, R \in C^9_{\mathrm{bd,ev}}(\mathbb{R}, \mathbb{C})\}$$

and

$$\mathcal{E}_2 = \{(0, \mu, 0) | \mu \in \mathbb{C}\} \ .$$

We denote by v_1 and v_2 the pieces of the unique decomposition $v = v_1 + v_2$ of an element $v \in \mathcal{E}$. Note that we will commit a slight abuse of notation by writing $v_1 = (\delta g, R)$ and $v_2 = \mu$ if $v = (\delta g, \mu, R)$. We then have

$$||v_1|| = \max\left[|\delta g|\bar{g}^{-\frac{3}{2}}, |||R|||_{\bar{g}}\bar{g}^{-\frac{21}{8}}\right] \quad \text{and} \quad ||v_2|| = |\mu|\bar{g}^{-1} \ .$$

Finally if v is in the domain of definition for the map RG we write $RG_1(v) = [RG(v)]_1$ and $RG_2(v) = [RG(v)]_2$ for better readability. Our dichotomy lemma, in the spirit of [39, Lemma 2.2] is the following result.

Lemma 3.17. *There exists $\epsilon_0 > 0$ and functions $c_1(\epsilon)$, $c_2(\epsilon)$, $c_3(\epsilon)$, $c_4(\epsilon)$, on $(0, \epsilon_0)$ which satisfy $0 < c_1(\epsilon) < 1$, $L^{\frac{3}{4}} \geqslant c_2(\epsilon) > 1$, $2L^{\frac{3}{2}} \geqslant c_3(\epsilon) \geqslant L^{\frac{3}{2}}$ and $0 < c_4(\epsilon) < 1$ (in fact $\lim_{\epsilon\to 0} c_4(\epsilon) = 0$) on that interval such that for all $v, v' \in \bar{B}\left(0, \frac{1}{8}\right) \subset \mathcal{E}$ the following statements hold:*

1. *unconditionally, $||RG_1(v) - RG_1(v')|| \leqslant c_1(\epsilon)||v - v'||$;*

2. *if* $L^{\frac{3}{4}}||v_2 - v_2'|| \geqslant ||v_1 - v_1'||$ *then* $||RG_2(v) - RG_2(v')|| \geqslant c_2(\epsilon)||v - v'||$*;*

3. *unconditionally,* $||RG_2(v) - RG_2(v')|| \leqslant c_3(\epsilon)||v - v'||$*;*

4. *unconditionally,*

$$||RG_2(v) - RG_2(v') - L^{\frac{3+\epsilon}{2}}(v_2 - v_2')|| \leqslant c_4(\epsilon)||v - v'|| \ .$$

where the $c(\epsilon)$ *functions are given by the formulas*

$$\begin{aligned}
c_1(\epsilon) &= \max\left[1 - \frac{3}{4}(L^\epsilon - 1) + 2B_4\bar{g}^{\frac{21}{8}-\frac{3}{2}}, \frac{3}{4} + 3B_{R\xi}\bar{g}^{\frac{11}{4}-3\eta-\eta_R}\right] \\
c_2(\epsilon) &= L^{\frac{3}{4}} - \frac{9}{4}A_{2,\max}\bar{g}^{\frac{3}{2}-1+1} - \frac{5}{4}A_{3,\max}\bar{g} - 2B_2\bar{g}^{\frac{21}{8}-1} \\
c_3(\epsilon) &= L^{\frac{3+\epsilon}{2}} + \frac{9}{4}A_{2,\max}\bar{g}^{\frac{3}{2}-1+1} + \frac{5}{4}A_{3,\max}\bar{g} + 2B_2\bar{g}^{\frac{21}{8}-1} \\
c_4(\epsilon) &= \frac{9}{4}A_{2,\max}\bar{g}^{\frac{3}{2}-1+1} + \frac{5}{4}A_{3,\max}\bar{g} + 2B_2\bar{g}^{\frac{21}{8}-1}
\end{aligned}$$

with

$$\begin{aligned}
A_{1,\min} &= 35(1 - p^{-3})l, \ \ A_{1,\max} \ = \ 37(1 - p^{-3})l \\
A_{2,\max} &= 2 \times [4 + 144] \times \frac{1}{36} \times A_{1,\max} \\
A_{3,\max} &= 12L^2 \times \frac{1}{36} \times A_{1,\max}.
\end{aligned} \tag{3.58}$$

Proof: For a full proof see [3, Proposition 5]. We remark that the constants (3.58) are just uniform bounds on the constants (3.55). A key ingredient for these estimates is Lemma (3.16).

3.7.4 The infrared RG fixed point

We now show how one can use Lemma (3.17) to show that the RG map restricted to $W^{\mathrm{s,loc}}$ is a strict contraction.

The idea of the next lemma is as follows. If one has two points $v, v' \in \mathcal{E}$ with full forward RG trajectories that stay in our domain then at every RG step the magnitude of their difference between the images of the two points in the contracting directions must dominate the difference in the expanding direction. If at *any* stage the difference in the expanding directions dominates the difference in the contracting direction then $||RG^n[v] - RG^n[v']||$ will blow up as $n \to \infty$.

Lemma 3.18. *If* $v \neq v'$ *belong to* $W^{\mathrm{s,loc}}$ *then* $||v_1 - v_1'|| > L^{\frac{3}{4}}||v_2 - v_2'||$.

Proof: Note that by the prevailing assumptions we have $\rho < \frac{1}{12} < \frac{1}{8}$ and thus Lemma 3.17 is applicable to all elements of $W^{\mathrm{s,loc}}$ and their RG iterates by stability of that set. We proceed by contradiction and suppose that $||v_1 - v_1'|| \leqslant L^{\frac{3}{4}}||v_2 - v_2'||$. Then by Lemma 3.17 Part 1) and 2)

$$||RG_1(v) - RG_1(v')|| \leqslant c_1(\epsilon)||v - v'|| \leqslant c_1(\epsilon)c_2(\epsilon)^{-1}||RG_2(v) - RG_2(v')|| \ .$$

From the bounds we have on $c_1(\epsilon)$ and $c_2(\epsilon)$ we trivially get $c_1(\epsilon)c_2(\epsilon)^{-1} < L^{\frac{3}{4}}$ and therefore

$$||RG_1(v) - RG_1(v')|| \leqslant L^{\frac{3}{4}}||RG_2(v) - RG_2(v')|| \ ,$$

i.e., the first iterates $RG(v)$ and $RG(v')$ satisfy the same hypothesis as v and v'. By an easy induction we then have

$$\forall n \geqslant 0, \quad ||RG_1^n(v) - RG_1^n(v')|| \leqslant L^{\frac{3}{4}}||RG_2^n(v) - RG_2^n(v')||$$

for the higher iterates where $RG_1^n(\cdot)$ means $(RG^n(\cdot))_1$ and likewise for the second components. By Lemma 3.17 Part 2) we obtain, for all $n \geqslant 0$,

$$||RG_2^{n+1}(v) - RG_2^{n+1}(v')|| \geqslant c_2(\epsilon)||RG^n(v) - RG^n(v')|| \geqslant c_2(\epsilon)||RG_2^n(v) - RG_2^n(v')|| \ .$$

Again by a trivial induction we get, for all $n \geqslant 0$,

$$||RG_2^n(v) - RG_2^n(v')|| \geqslant c_2(\epsilon)^n||v_2 - v_2'|| \ .$$

But $c_2(\epsilon) > 1$, so if $||v_2 - v_2'|| > 0$ we have

$$\lim_{n\to\infty} ||RG_2^n(v) - RG_2^n(v')|| = \infty$$

which contradicts the stability and boundedness of the set $W^{\rm s,loc}$. Therefore $||v_2 - v_2'|| = 0$ which also entails $||v_1 - v_1'|| = 0$ by the assumtion made at the beginning of this proof. This therefore leads to $v = v'$ which is the desired contradiction. □

This then yields the contractive bound which is given in the next lemma.

Lemma 3.19. *For all $v, v' \in W^{\rm s,loc}$ we have $||RG(v) - RG(v')|| \leqslant c_1(\epsilon)||v - v'||$.*

Proof: By the previous lemma and the stability of $W^{\rm s,loc}$ we have

$$||RG_2(v) - RG_2(v')|| \leqslant L^{-\frac{3}{4}}||RG_1(v) - RG_1(v')|| \leqslant ||RG_1(v) - RG_1(v')||$$

and therefore

$$||RG(v) - RG(v')|| = ||RG_1(v) - RG_1(v')|| \ .$$

As a result, the desired conclusion follows from Lemma 3.17 Part 1). □

By applying the contraction mapping theorem one immediately has

Proposition 3.5. *The map RG is a contraction when restricted to $W^{\rm s,loc}$ and thus has a unique fixed point $v_* = (\delta g_*, \mu_*, R_*)$ in that set. In fact v_* belongs to the interior of $W^{\rm s,loc}$ which we denote $W^{\rm s,loc}_{\rm int}$.*

Proof: Note that $W^{\rm s,loc}$ is a closed subset of the Banach space $\mathcal{E}$. Indeed, $\mu_{\rm s}$ is analytic and thus continuous on an open domain containing that given by the condition $||(\delta g, R)|| \leqslant \frac{\rho}{13}$. Since $W^{\rm s,loc}$ is therefore a complete metric space for the distance coming from the $||\cdot||$ norm, and since RG restricted to this set is a contraction as follows form Lemma 3.19 and $c_1(\epsilon) < 1$, the Banach Fixed Point Theorem establishes the present lemma. The fixed point is in $W^{\rm s,loc}_{\rm int}$ since v_* is its own image by application of the stronger conclusion of Proposition 3.17. □

3.7.5 The unstable manifold

We now construct the local unstable manifold following a procedure similar to that of §3.7.3 but give some exposition before getting down to details. We remind the reader that the unstable manifold will turn out to be the graph of a function $\mu \to (\delta g_{\mathrm{u}}(\mu), R_{\mathrm{u}}(\mu))$. The situation here is somewhat dual to the situation in §3.7.3 - for fixed μ we want to find $\delta g_{\mathrm{u}}(\mu), R_{\mathrm{u}}(\mu)$ such that one can find a backwards sequence of RG *pre-images* of $(\delta g_{\mathrm{u}}(\mu), \mu, R_{\mathrm{u}}(\mu))$ denoted $\{(\delta g_n, \mu_n, R_n)\}_{-\infty<n<0}$

$$\text{i.e. } RG^{-n}[(\delta g_n, \mu_n, R_n)] = (\delta g_{\mathrm{u}}(\mu), \mu, R_{\mathrm{u}}(\mu))$$

with δg_n and R_n bounded as $n \to -\infty$. Essentially the roles of expanding and contracting directions are reversed since we are working with backward trajectories. Later on we will see that such a trajectory will in fact satisfy $\lim_{n\to-\infty}(\delta g_n, \mu_n, R_n) = (\delta g_*, \mu_*, R_*)$.

For a fixed μ we will define a map $\mathfrak{n}$ on this space of backwards sequences whose fixed point will correspond to a backwards RG trajectory $\{(\delta g_n, \mu_n, R_n)\}_{-\infty<n\leqslant 0}$ which satisfies the boundary condition $\mu_0 = \mu$ and δg_n and R_n bounded as $n \to -\infty$.

We now give the precise implementation. Let $\mathcal{B}_-$ be the Banach space of backwards sequences

$$\vec{u} = (\ldots, (\delta g_{-2}, \mu_{-2}, R_{-2}), (\delta g_{-1}, \mu_{-1}, R_{-1}), \delta g_0, R_0) \in \prod_{n\leqslant -1} \left[\mathbb{C}^2 \times C^9_{\mathrm{bd,ev}}(\mathbb{R},\mathbb{C})\right] \times \mathbb{C} \times C^9_{\mathrm{bd,ev}}(\mathbb{R},\mathbb{C})$$

which have finite norm given by

$$||\vec{u}|| = \sup\left\{|\delta g_j|\bar{g}^{-\frac{3}{2}} \text{ for } j \leqslant 0; |\mu_j|\bar{g}^{-1} \text{ for } j \leqslant -1; |||R_j|||\bar{g}^{-\frac{21}{8}} \text{ for } j \leqslant 0\right\} .$$

We define the map $\mathfrak{n}$ on this space of sequences which depends on the parameter μ_0. Given μ_0, the image $\vec{u}' = \mathfrak{n}(u)$ is defined as follows. For $n \leqslant 0$, we let

$$\delta g'_n = \sum_{j\leqslant n-1} (2 - L^\epsilon)^{n-1-j} \tilde{\xi}_4(\delta g_j, \mu_j, R_j)$$

and

$$R'_n = \sum_{j\leqslant n-1} \tilde{\mathcal{L}}^{(\delta g_{n-1},\mu_{n-1})} \circ \cdots \circ \tilde{\mathcal{L}}^{(\delta g_{j+1},\mu_{j+1})} \left(\tilde{\xi}_R(\delta g_j, \mu_j, R_j)\right) .$$

Again in the two equations above we are propagating the δg and R boundary conditions backwards.

For $n \leqslant -1$, we let

$$\mu'_n = L^{n\left(\frac{3+\epsilon}{2}\right)}\mu_0 - \sum_{j=n}^{-1} L^{-(j-n+1)\left(\frac{3+\epsilon}{2}\right)} \tilde{\xi}_\mu(\delta g_j, \mu_j, R_j) .$$

As before for $\rho' > 0$ one can show that this map is well defined and analytic on the open ball $B(\vec{0}, \rho') \in \mathcal{B}_-$ in the regime of small ϵ (made small after fixing ρ').

Proposition 3.6. *If* $0 < \rho' \leqslant \frac{1}{8}$, $|\mu_0| < \frac{\rho'}{8}\bar{g}$ *then the map* $\mathfrak{n}$ *is well defined, analytic on* $B(\vec{0}, \rho')$ *and takes its values in the closed ball* $\bar{B}(\vec{0}, \frac{\rho'}{4})$*, provided* ϵ *is made sufficiently small after fixing* ρ'*. Moreover,* $\mathfrak{n}$ *is jointly analytic in* $\vec{u}$ *and the implicit variable* μ_0*.*

Proof: See [3, Proposition 7].

Again using Lemma 3.2 with $r_1 = \rho'$, $r_2 = \frac{\rho'}{4}$ and $\nu = \frac{1}{3}$ so that

$$\frac{r_2(1-\nu)}{r_1(1-2\nu)} = \frac{1}{2}$$

we see that, under the hypotheses of Proposition 3.6, the closed ball $\bar{B}\left(\vec{0}, \frac{\rho'}{3}\right)$ is stable by $\mathfrak{n}$ and is a contraction. More precisely, for any $\vec{u}_1$ and $\vec{u}_2$ in that ball, we have

$$||\mathfrak{n}(\vec{u}_1) - \mathfrak{n}(\vec{u}_2)|| \leqslant \frac{1}{2}||\vec{u}_1 - \vec{u}_2|| \ .$$

By the Banach Fixed Point Theorem we have the existence of a unique fixed point which we again denote by $\vec{u}_*$ for the map $\mathfrak{n}$ in the ball $\bar{B}\left(\vec{0}, \frac{\rho'}{3}\right)$. Using the representation of this fixed point as

$$\vec{u}_* = \sum_{n=0}^{\infty}\left[\mathfrak{n}^{n+1}(\vec{0}) - \mathfrak{n}^{n}(\vec{0})\right]$$

and by uniform absolute convergence, we see that $\vec{u}_*$ is analytic in the implicit data μ_0. In particular the δg_0, R_0 components of the sequence $\vec{u}_*$ which we will denote by $\delta g_{\mathrm{u}}(\mu_0)$, $R_{\mathrm{u}}(\mu_0)$ respectively are analytic on the domain given by $|\mu_0| < \frac{\rho'}{8}\bar{g}$.

We now let

$$W^{\mathrm{u,loc}} = \left\{(\delta g, \mu, R) \in \mathcal{E}|\ |\mu| < \frac{\rho'}{8}\bar{g}, \delta g = \delta g_{\mathrm{u}}(\mu), R = R_{\mathrm{u}}(\mu)\right\} \ .$$

Analogously to the case of the stable manifold we have

Proposition 3.7. *For fixed $\rho' \in (0, \frac{1}{8}]$ and for ϵ small enough, an equivalent description of $W^{\mathrm{u,loc}}$ is as the set of triples $(\delta g, \mu, R) \in \mathcal{E}$ that satisfy all of the following properties:*

- $|\mu| < \frac{\rho'}{8}\bar{g}$,
- *there exists a sequence $(\delta g_n, \mu_n, R_n)_{n\leqslant 0}$ in $\mathcal{E}$ such that $\delta g_0 = \delta g$, $\mu_0 = \mu$, $R_0 = R$, $\forall n \leqslant 0$, $|\delta g_n| \leqslant \frac{\rho'}{3}\bar{g}^{\frac{3}{2}}$ and $|||R_n|||_{\bar{g}} \leqslant \frac{\rho'}{3}\bar{g}^{\frac{21}{8}}$, $\forall n \leqslant -1$, $|\mu_n| \leqslant \frac{\rho'}{3}\bar{g}$, and $\forall n \leqslant -1$, $(\delta g_{n+1}, \mu_{n+1}, R_{n+1}) = RG(\delta g_n, \mu_n, R_n)$.*

Proof: See [3, Proposition 8].

It follows form the precise characterization of $W^{\mathrm{u,loc}}$ given in the last lemma that the fixed point $(\delta g_*, \mu_*, R_*)$ must be an element of $W^{\mathrm{u,loc}}$.

Lemma 3.20. *Provided ρ and ρ' are chosen so that $\rho < \frac{3}{8}\rho'$, we have $v_* \in W^{\mathrm{u,loc}}$ as well as the equations*

$$\mu_* = \mu_{\mathrm{s}}(\delta g_*, R_*) \ , \ \delta g_* = \delta g_{\mathrm{u}}(\mu_*) \ , \ R_* = R_{\mathrm{u}}(\mu_*) \ .$$

Proof: This is [3, Lemma 61].

3.7.6 Study of the differential of the RG at the nontrivial infrared fixed point

We now give results about the differential $D_{v_*}RG$ of the map RG at the fixed point v_* in relation to the invariant linear subspaces $\mathcal{E}^{\mathrm{s}}$ and $\mathcal{E}^{\mathrm{u}}$ corresponding to the tangent spaces to the stable and unstable manifolds at the fixed point respectively. We define $\mathcal{E}^{\mathrm{s}}$ as the kernel of the $\mathbb{C}$-linear form

$$(\delta g, \mu, R) \mapsto \mu - D_{v_{*,1}}\mu_{\mathrm{s}}[\delta g, R]$$

where $D_{v_{*,1}}\mu_{\mathrm{s}}$ is the differential of μ_s at $v_{*,1} = (\delta g_*, R_*)$. This linear form is clearly nonzero. It is also continuous by analyticity of μ_s. Therefore $\mathcal{E}^{\mathrm{s}}$ is a closed complex hyperplane in $\mathcal{E}$.

We likewise define $\mathcal{E}^{\mathrm{u}}$ as the kernel of the $\mathbb{C}$-linear map

$$\begin{cases} \mathcal{E} & \longrightarrow \quad \mathcal{E}_1 \\ (\delta g, \mu, R) & \longmapsto \quad (\delta g - D_{v_{*,2}}\delta g_{\mathrm{u}}[\mu], R - D_{v_{*,2}}R_{\mathrm{u}}[\mu]) \end{cases}$$

in terms of the differentials at $v_{*,2} = \mu_*$ of the analytic maps δg_{u} and R_{u}. Again, $\mathcal{E}^{\mathrm{u}}$ is a closed subspace of $\mathcal{E}$. In fact, it is easy to see that $\mathcal{E}^{\mathrm{u}}$ is equal to the complex line $\mathbb{C}e_{\mathrm{u}}$ with

$$e_{\mathrm{u}} = (D_{v_{*,2}}\delta g_{\mathrm{u}}[1], 1, D_{v_{*,2}}R_{\mathrm{u}}[1]) \ .$$

In the following lemma $D_{v_*}RG$ denotes the differential of the bulk RG map at $v_* = (\delta g_*, \mu_*, R_*)$

Lemma 3.21. *One has the direct sum decomposition $\mathcal{E} = \mathcal{E}^{\mathrm{s}} \bigoplus \mathcal{E}^{\mathrm{u}}$. Additionally both the subspaces $\mathcal{E}^{\mathrm{s}}$ and $\mathcal{E}^{\mathrm{u}}$ are left invariant by $D_{v_*}RG$.*

Proof: See [3, Lemmas 63, 64, and 65].

We will denote by P_{s} and P_{u} the projection operators from $\mathcal{E}$ to $\mathcal{E}_{\mathrm{s}}$ and $\mathcal{E}_{\mathrm{u}}$ respectively.

We remark that since $\mathcal{E}^{\mathrm{u}}$ is a one dimensional space $D_{v_*}RG$'s action on $\mathcal{E}^{\mathrm{u}}$ must just be multiplication by some scalar α_{u}. The next lemma gives more detail about the action of $D_{v_*}RG$ on each of the tangent spaces.

Lemma 3.22. *The restriction $D_{v_*}RG\big|_{\mathcal{E}^{\mathrm{u}}}$ is the multiplication by an eigenvalue α_{u} which is real and greater than 1. One also has the more precise estimate*

$$|\alpha_{\mathrm{u}} - L^{\frac{3+\epsilon}{2}}| \leqslant c_4(\epsilon)$$

where $c_4(\epsilon)$ has been defined in Lemma 3.17.

The restriction $D_{v_*}RG\big|_{\mathcal{E}^{\mathrm{s}}}$ is a contraction on the subspace $\mathcal{E}^{\mathrm{s}}$. More precisely, for every $v \in \mathcal{E}^{\mathrm{s}}$, we have $D_{v_*}RG[v] \in \mathcal{E}^{\mathrm{s}}$ and

$$||D_{v_*}RG[v]|| \leqslant c_1(\epsilon)||v||$$

where $c_1(\epsilon) \in (0,1)$ has been defined in Lemma 3.17.

Proof: See [3, Lemmas 66, 67].

We give a few remarks on the importance of α_{u}. One should imagine e_{u} as an $O(\epsilon)$ perturbation of $e_{\phi^2} \in \mathcal{E}$ where $e_{\phi^2} = (0, 1, 0)$ (using the $(\delta g, \mu, R)$ notation for elements of $\mathcal{E}$). The vector e_{ϕ^2} is an eigenvector of the differential of the RG at the Gaussian fixed point corresponding to the RG's most strongly expanding direction there - the corresponding eigenvalue is given by $L^{3-2[\phi]} = L^{\frac{3+\epsilon}{2}}$.

Later we will see that $\alpha_{\mathrm{u}} < L^{\frac{3+\epsilon}{2}}$, that is the eigenvalues of e_{u} and e_{ϕ^2} differ by a quantity of order ϵ. The fact that α_{u}, not $L^{\frac{3+\epsilon}{2}}$, governs the expansion of mass perturbations off of the stable manifold of the non-trivial fixed point is what makes the multiplicative renormalization Z_2 necessary in order to end up with a non-degenerate $N[\phi^2]$ field, this also leads to the anomalous dimension for $N[\phi^2]$. In particular we now fix

$$Z_2 \;=\; \alpha_{\mathrm{u}} L^{-\frac{3+\epsilon}{2}}. \tag{3.59}$$

3.7.7 Some context for the preceding results

We close this section by putting our analysis of the nontrivial infrared fixed point in context. After we constructed the local stable manifold and the nontrivial infrared fixed point we finished all the work necessary to control the evolution of the bulk potentials $\vec{V}^{(r,q)}(0,0)$. Denote by $\vec{V}_*$ the element of $\mathcal{E}_{\mathrm{ext}}$ corresponding to the nontrivial infrared fixed point $v_* = (\delta g_*, \mu_*, R_*)$.

The function $\mu(g)$ referenced in Theorem 3.1 will be set as $\mu(g) := \mu_{\mathrm{s}}(g - \bar{g}, 0)$ - i.e. we pick our initial bulk data to lie on the stable manifold. It then follows by the analysis of this section that for any integer r one will have $\lim_{q\to\infty} \vec{V}^{(r,q)}(0,0) = \vec{V}_*$ (we remark that one will also have $\lim_{r\to-\infty} \vec{V}^{(r,q)} = \vec{V}^*$ for fixed q).

We will establish control over the flow of the potentials $\vec{V}^{r,q}(\tilde{f},\tilde{j})$ for non-zero $\tilde{f}$ or $\tilde{j}$ by treating their flow as a perturbation of the flow of $\vec{V}^{(r,q)}(0,0)$. We will employ different strategies to do this, each strategy being used in one of three different scale regimes - the ultraviolet regime where $q < q_-$, the middle regime where $q_- \leqslant q < q_+$ and the infrared regime where $q \geqslant q_+$.

The core of the work to analyze the composite field is careful analysis of the influence of $\tilde{j}$ in the ultraviolet regime. When $r << q_-$ one has that the quantity $j_{\Rightarrow -r}$ (see (3.18) for the definition) appearing in $\tilde{V}_{r,s}(f,j)$ is spatially spread out and acts like a bulk perturbation in the μ direction which lifts us off the stable manifold. Our analysis of the local *unstable* manifold and the partial diagonalization of the linearized RG at the non-trivial infrared fixed point are just the first steps in understanding the behaviour of these bulk mass perturbations. The central ingredient for controlling this perturbations is the partial analytic linearization theorem of the next section.

3.8 Partial Analytic Linearization

3.8.1 Some preliminaries and intermediate estimates

Before talking about the main results of this section we give a concrete example which should clarify the issues of the ultraviolet regime for the composite field. We take $\tilde{j} = \mathbb{1}_{\mathbb{Z}_p^3}$, $\tilde{f} = 0$ - thus we can fix $q_- = q_+ = 0$. In this scenario one should imagine $r << 0$. The inital RG data relevant to $\mathcal{Z}_{r,s}(\tilde{j},0)$ is then given by

$$\vec{V}^{(r,r)}(0,\tilde{j}) = (\beta_{4,\Delta}, \beta_{3,\Delta}, \beta_{2,\Delta}, \beta_{1,\Delta}, W_{5,\Delta}, W_{6,\Delta}, f_\Delta, R_\Delta)_{\Delta\in\mathbb{L}}$$

with

$$\beta_{4,\Delta} \quad = \quad g$$

$$\begin{aligned}\beta_{3,\Delta} &= 0\\ \beta_{1,\Delta} &= 0\\ W_{5,\Delta} &= 0\\ W_{6,\Delta} &= 0\\ f_{\Delta} &= 0\\ R_{\Delta} &= 0\end{aligned}$$

for all unit cubes Δ, while

$$\beta_{2,\Delta} = \begin{cases}\mu_{\rm s}(g-\bar{g},0) & \text{if } \Delta \not\subset \Lambda_{-r}\\ \mu_{\rm s}(g-\bar{g},0) - Y_2 Z_r^r L^{-(3-2[\phi])r} & \text{if } \Delta \subset \Lambda_{-r}\end{cases}.$$

Since $\vec{V}^{(r,r)}(0,\tilde{j})$ is not spatially homogenous it cannot be directly identified with a single element of the bulk RG space $\mathcal{E}$. However it can be identified as an amalgamation of *two* bulk elements. The element $v = (g-\bar{g}, \mu_{\rm s}(g-\bar{g},0),0) \in \mathcal{E}$ specifies $\vec{V}^{r,r}_{\Delta}(0,\tilde{j})$ for $\Delta \not\subset \Lambda_{-r}$. while the element $v - Y_2 Z_r^r L^{-(3-2[\phi])r} e_{\phi^2} \in \mathcal{E}$ specifies $\vec{V}^{r,r}_{\Delta}(0,\tilde{j})$ for $\Delta \subset \Lambda_{-r}$.

Due to the strict locality of extended RG map one can view $RG_{\rm ex}$ as a direct product of maps each acting on a single L-block. When we look at the transformation $RG_{\rm ex}$ acting on $\vec{V}^{r,q}_{\Delta}(0,\tilde{j})$ for $r \leqslant q < 0$ the individual local RG transformations occurring on each L-block are given by the bulk RG map acting on one of the two mentioned bulk data points.

We make this more concrete. Let $\iota(\bullet) : \mathcal{E} \to \mathcal{E}_{\rm ex}$ be the map that takes an element of $\mathcal{E}$ to the corresponding spatially homogenous element of $\mathcal{E}_{\rm ex}$. Then for $r \leqslant q < 0$ and $\Delta \in \mathbb{L}$ one has that

$$\vec{V}^{r,q}_{\Delta}(0,\tilde{j}) \;=\; \left(RG^{q-r}_{\rm ex}\left[\vec{V}^{r,r}(0,\tilde{j})\right]\right)_{\Delta} \;=\begin{cases}\left(\iota\left(RG^{q-r}(v)\right)\right)_{\Delta} & \text{if } \Delta \not\subset \Lambda_{-q}\\ \left(\iota\left(RG^{q-r}(v - Y_2 Z_r^r L^{-(3-2[\phi])r} e_{\phi^2})\right)\right)_{\Delta} & \text{if } \Delta \subset \Lambda_{-q}\end{cases}$$

where the RG without the subscript ex again refers to the bulk RG analyzed in the previous section. The central problem of understanding the total ultraviolet regime contribution of an arbitrary $\tilde{j}$ is controlling where we end up at scale $q = q_-$ as we take $r \to -\infty$. In the particular example we took above this involves understanding

$$\lim_{r\to-\infty} RG^{-r}(v - Y_2 Z_r^r L^{-(3-2[\phi])r} e_{\phi^2})$$

We note that we have chosen Z_2 in (3.59) precisely so $Z_r^r L^{-(3-2[\phi])r} = \alpha_{\rm u}^r$. In general we will want to control a limit of the form

$$\lim_{n\to\infty} RG^n(v + \alpha_{\rm u}^{-n} w) \tag{3.60}$$

for $v, w \in \mathcal{E}$ where v lies on v_*'s stable manifold. We will denote the quantity (3.60), when it exists, as $\Psi(v,w)$. In our concrete example $w = -Y_2 e_{\phi^2}$ and for the purpose of constructing the composite field we will be interested in w's that are some multiple of e_{ϕ^2}, that is w pointing in an expanding direction.

Iterating the RG infinitely many times from a point that is off the stable manifold clearly is a recipe for disaster, however in the above quantity the perturbation w is being deamplified by $\alpha_{\rm u}^{-n}$ which we hope will precisely balance the expansion that will occur from RG^n in the $n \to \infty$ limit.

Theorems 3.3 and 3.4, the central results of this section, will state that for v on the stable manifold $W^{\mathrm{s,loc}}$ and suitably small w the limit (3.60) exists, is analytic in w (this is essential, as it gives analyticity in $\tilde{j}$ which is needed for moment bounds), and is in some sense non-degenerate. What we mean by non-degeneracy is that $\Psi(v,w)$ should not be constant in w (i.e. one shouldn't have $\Psi(v,w) = v_*$ for all w). If $\Psi(v,w)$ is constant in w this means that the presence of $\tilde{j}$ will be rubbed out completely in the ultraviolet regime - the functional derivatives in $\tilde{j}$ will all be 0 and the corresponding normal ordered field $N[\phi^2]$ will be the 0 field.

In particular the deamplification has to be chosen precisely or else the resulting normal ordered field $N[\phi^2]$ will be degenerate. It follows from Theorem 3.3 that if one chooses $\alpha > \alpha_{\mathrm{u}}$ and takes the limit

$$\lim_{n\to\infty} RG^n(v + \alpha^{-n} w)$$

then one will just get v_* for all w - the w will be washed out. This shows that one must have $Z_2 = \alpha_{\mathrm{u}} L^{-\frac{3+\epsilon}{2}}$ (and by a slightly longer argument this idea shows that the anomalous dimension η_{ϕ^2} is independent of L).

The function $\Psi(v,w)$ can be considered to be a partial linearization for the RG map in the vicinity in v_*. A linearization for a dynamical system at a particular fixed point is a change of coordinate system for an open set around that fixed point which makes the action of the dynamical system linear.

Koenig's theorem of holomorphic dynamics (see [48]) states that this can always be done for one dimensional holomorphic dynamical systems. Concretely the theorem states that for a dynamical system given by an analytic function $f : \mathbb{C} \to \mathbb{C}$ and a fixed point at 0 and $f'(0) = \lambda$ with $|\lambda| \neq 1$ then there exists an an analytic map φ defined in a neighborhood of U of 0, and satisfying $\varphi(0) = 0$ and

$$\left(\varphi^{-1} \circ f \circ \varphi\right)(z) = \lambda z$$

for all $z \in U$.

Rearranging this one sees ϕ satisfies an "intertwining" relation, i.e. $(f \circ \varphi)(z) = \phi(\lambda z)$. If $\Psi(v,w)$ exists then a similar intertwining relation holds just by appealing to the continuity of the RG map. Observe that

$$\begin{aligned} RG(\Psi(v,w)) &= RG\left(\lim_{n\to\infty} RG^n(v + \alpha_{\mathrm{u}}^{-n} w)\right) \\ &= \lim_{n\to\infty} RG^{n+1}(v + \alpha_{\mathrm{u}}^{-(n+1)} \alpha_{\mathrm{u}} w) \\ &= \Psi(v, \alpha_{\mathrm{u}} w). \end{aligned}$$

One can view the construction of $\Psi(v,w)$ as mimicing the construction of a curvilinear coordinate for the unstable direction in which the RG takes a linear form.

The hope for controlling an object like (3.60) is that the large n limit action of $RG^n(v + \alpha_{\mathrm{u}}^{-n} \bullet)$ should be comparable to $v_* + \alpha_{\mathrm{u}}^{-n}(D_{v_*}RG)^n \bullet$. However comparing these maps is not trivial for two reasons (i) in general one will have $v \neq v_*$ and (ii) the RG map does not coincide with its differential.

However ignoring these issues and observing that (by Lemma (3.22)) one has

$$\alpha_{\mathrm{u}}^{-n}(D_{v_*}RG)^n \underset{n\to\infty}{\to} P_{\mathrm{u}}$$

our intuition would be that $\lim_{n\to\infty} RG^n(v + \alpha_{\mathrm{u}}^{-n} w) \approx v_* + P_{\mathrm{u}} w$ for tiny w. Our main theorem states that one will actually have $\Psi(v,w) \in W^{\mathrm{u,loc}}$.

Both issues (i) and (ii) must be dealt with in order to prove Theorem 3.3. For issue (ii) we remark that it is is straightforward to show an $O(1)$ bound on the second order differential of the map RG in the small ϵ-regime (see [3, Lemma 69]). One then has quantities like $RG(v+w)$ can be replaced by $RG[v] + D_v RG[w] + O(1)||w||^2$.

Issue (i) is much deeper, it requires performing a "parallel transport" along the RG orbit connecting v to v_*. For each $v \in W^{\mathrm{s,loc}}$ we define a linear operator $T_n(v)[\bullet]$ via

$$T_n(v)[w] = \alpha_{\mathrm{u}}^{-n} D_v RG^n[w].$$

We remark that $D_v RG^j$ denotes the differential at v of the j-fold iteration of the RG map. One can imagine as $T_n(v)$ corresponding to a parallel transport from v to $RG^n(v)$ with regards to the unstable direction. The following lemma gives the key facts about these parallel transports. When we write $||T_n(v)||$ the notation $||\cdot||$ denotes the operator norm for bounded linear operators on $\mathcal{E}$.

Lemma 3.23.

- *For all $v \in W^{\mathrm{s,loc}}$ and all $n \geqslant 0$ we have*

$$||T_n(v)|| \leqslant\ 10 \times \mathcal{C}_1(\epsilon),$$

  where $\mathcal{C}_1(\epsilon) = \exp\left[\dfrac{85}{\alpha_{\mathrm{u}}(1-c_1(\epsilon))}\right]$.

- *For $v \in W^{\mathrm{s,loc}}$ and $n \geqslant 0$*

$$||T_{n+1}(v) - T_n(v)|| \leqslant 10 \times \mathcal{C}_3(\epsilon) c_1(\epsilon)^{\frac{n}{2}}$$

  where $\mathcal{C}_3(\epsilon) = \mathcal{C}_1(\epsilon)\left[85 + (1+\alpha_{\mathrm{u}}^{-1} c_1(\epsilon))\left[1 + \dfrac{85\mathcal{C}_1(\epsilon)}{c_1(\epsilon)(1-c_1(\epsilon))}\right]\right]$.

- *There exists bounded operators $T_\infty(v)$ such that*

$$\lim_{n\to\infty} T_n(v) = T_\infty(v)$$

  in the operator norm. One has $||T_\infty(v)|| \leqslant \mathcal{C}_1(\epsilon)$ and additionally $P_{\mathrm{s}} T_\infty(v) = 0$, in particular the linear operator $T_\infty(v)$ is a multiple of P_{u}.

Proof: See [3, Lemmas 70,71,72,73]

We remark that some of these estimates become singular when $\epsilon \to 0$. The main culprit here is the slow convergence of $RG^n(v)$to v_* for v on the stable manifold. Our best estimate for this is $||RG^n(v) - v_*|| \leqslant c_1(\epsilon)$ with $c_1(\epsilon) < 1$ for $\epsilon > 0$. However as $\epsilon \to 0$ one has $c_1(\epsilon) \to 1$ as (morally $c_1(\epsilon) = 2 - L^\epsilon$, the linear contraction rate of δg, slowest of the contracting directions.) As a result the quantity $\mathcal{C}_1(\epsilon)$ appearing in this section blows up as $e^{\frac{1}{\epsilon}}$ as $\epsilon \to 0$.

Sketch of the proof of Theorems 3.3 and 3.4

We will now *sketch* some of the ideas of the proof of Theorems 3.3 and 3.4. The idea is to control the limit (3.60) by controlling the telescoping series

$$\sum_{n=0}^{\infty} RG^{n+1}(v+\alpha^{-n-1}w) - RG^{n}(v+\alpha^{-n}w). \tag{3.61}$$

In particular the strategy is to try to establish a geometric in n bound on

$$||RG^{n+1}(v+\alpha^{-n-1}w) - RG^{n}(v+\alpha^{-n}w)||. \tag{3.62}$$

For the proof of Theorem 3.3 we will further break up (3.62) before estimating it. For $0 \leqslant k \leqslant n$ we write

$$\begin{aligned}
&||RG^{n+1}(v+\alpha^{-n-1}w) - RG^{n}(v+\alpha^{-n}w)|| \\
&\leqslant \left|\left|RG^{n+1}\left(v+\alpha^{-n-1}w\right) - RG^{k}\left(RG^{n+1-k}(v) + D_v RG^{n+1-k}[\alpha^{-n-1}w]\right)\right|\right| \\
&\quad + \left|\left|RG^{k}\left(RG^{n+1-k}(v) + D_v RG^{n+1-k}[\alpha^{-n-1}w]\right) - RG^{k}\left(RG^{n-k}(v) + D_v RG^{n-k}[\alpha^{-n}w]\right)\right|\right| \\
&\qquad + \left|\left|RG^{k}\left(RG^{n-k}(v) + D_v RG^{n-k}[\alpha^{-n}w]\right) - RG^{n}\left(v+\alpha^{-n}w\right)\right|\right|
\end{aligned} \tag{3.63}$$

In the above estimate we simply added and subtracted the two terms appearing in the third line of (3.63). Another telescoping argument will be used for both the second and fourth lines while the third line will be estimated via repeated Lipschitz estimates. After arriving at a bound for(3.63) one then optimizes the choice of k in order to for the bound on (3.63) to be summable in n. In particular k will be chosen dependently on n - for the proof of Theorem 3.3 it turns out to be sufficient to take $k = \lfloor \sigma n \rfloor$ with

$$\sigma = \frac{1}{2} \times \frac{-\log\left(c_1(\epsilon)\right)}{\log\left(c_3(\epsilon)\right) - \log\left(c_1(\epsilon)\right)} \in \left(0, \frac{1}{2}\right).$$

We now show the second telescoping argument that will be used for the quantities on the second and fourth lines of (3.63). One expands and bounds

$$\begin{aligned}
&||RG^{n}(v+\alpha^{-n}w) - RG^{k}(RG^{n-k}(v) + D_v RG^{n+1-k}[\alpha^{-n}w])|| \\
&\leqslant \sum_{j=k}^{n-1} \left|\left|RG^{j+1}\left(RG^{n-j-1}(v) + D_v RG^{n-j-1}[\alpha_{\mathrm{u}}^{-n}w]\right) - RG^{j}\left(RG^{n-j}(v) + D_v RG^{n-j}[\alpha_{\mathrm{u}}^{-n}w]\right)\right|\right|
\end{aligned} \tag{3.64}$$

Now for $k \leqslant j \leqslant n-1$ one can just apply the most brutal Lipschitz estimate we have on the bulk RG map given to us in Lemma 3.17 j-times to get

$$\begin{aligned}
&||RG^{j+1}\left(RG^{n-j-1}(v)+D_vRG^{n-j-1}[\alpha_{\mathrm{u}}^{-n}w]\right)-RG^{j}\left(RG^{n-j}(v)+D_vRG^{n-j}[\alpha_{\mathrm{u}}^{-n}w]\right)||\\
\leqslant& c_3(\epsilon)^j||RG\left(RG^{n-j-1}(v)+D_vRG^{n-j-1}[\alpha_{\mathrm{u}}^{-n}w]\right)-RG^{n-j}(v)-D_vRG^{n-j}[\alpha_{\mathrm{u}}^{-n}w]||\\
=&c_3(\epsilon)^j||RG\left(RG^{n-j-1}(v)+D_vRG^{n-j-1}[\alpha_{\mathrm{u}}^{-n}w]\right)-RG^{n-j}(v)-D_{RG^{n-j-1}(v)}RG\left[D_vRG^{n-j-1}[\alpha_{\mathrm{u}}^{-n}w]\right]||\\
\leqslant& c_3(\epsilon)^j\times O(1)\times||D_vRG^{n-j-1}[\alpha_{\mathrm{u}}^{-n}w]||^2\\
=&c_3(\epsilon)^j\times O(1)\times\left[\alpha_{\mathrm{u}}^{-(j+1)}||T_{n-j-1}(v)[w]||\right]^2\\
\leqslant& O(1)\mathcal{C}_1(\epsilon)^2||w||^2\times\alpha_{\mathrm{u}}^{-2}\times\left[c_3(\epsilon)\alpha_{\mathrm{u}}^{-2}\right]^j\ .
\end{aligned} \tag{3.65}$$

In going to the third line of (3.65) we just used the chain rule, that is $D_v(A\circ B)=D_{B(v)}\circ D_vB$. Now the quantity on the third line of (3.65) can be estimated as a second order Taylor remainder which is how we go the fourth line. In going to the last line we used Lemma 3.23.

We remark on why this estimate requires w to be very small (which is why this requirement is propagated to Theorem 3.3). Our control of the RG transformation is limiting to a ball of $O(1)$ radius in the $||\cdot||$ norm in $\mathcal{E}$ - the crude Lipschitz estimates we used require that quantities like

$$RG^a\left(RG^b(v)+D_vRG^b[\alpha_{\mathrm{u}}^{-n}w]\right)\ \text{where}\ a+b\leqslant n$$

stay within this ball. Estimating the size of the above quantities using inductive telescoping arguments very similar to (3.65), and these produce factors like $\mathcal{C}_1(\epsilon)^2||w||^2$ which must be $O(1)$ so $||w||$ must be tiny in the small ϵ regime. The root of these singular estimates come from our singular bounds on the $T_n[v]$. However in the case $v=v_*$ this estimate is trivial. Observe that

$$T_n[v_*]=\alpha_{\mathrm{u}}^{-n}D_{v_*}RG^n=\left(\alpha_{\mathrm{u}}^{-1}D_{v_*}RG\right)^n$$

where we used the chain rule and the fact that $RG(v_*)=v_*$. Since $||\alpha_{\mathrm{u}}^{-1}D_{v_*}||\leqslant 1$ one has $||T_n[v_*]||\leqslant 1$. In particular for $w\in\mathcal{E}^{\mathrm{u}}$ one has $T_n(v_*)[w]=w$. This is the main reason for the difference between the conditions on w in Theorem 3.3 and Theorem 3.4, the latter theorem specializes to $\Psi(v_*,w)$ for $w\in\mathcal{E}^{\mathrm{u}}$.

We remark that the estimate (3.65) is summable in j since $(\alpha_{\mathrm{u}}^{-2}c_3(\epsilon))<1$ - morally both $c_3(\epsilon)$ and α_{u} are within $O(\epsilon)$ of $L^{\frac{3+\epsilon}{2}}$ (see [3, pg 101]). In particular, since the sum starts at $j=k$ then for sufficiently tiny $||w||$, i.e. $||w||\leqslant O(1)\mathcal{C}_1(\epsilon)^{-1}$, one will have

$$||RG^n(v+\alpha^{-n}w)-RG^k(RG^{n-k}(v)+D_vRG^{n+1-k}[\alpha^{-n}w])||\ \leqslant\ \mathcal{O}(1)\left[c_3(\epsilon)\alpha_{\mathrm{u}}^{-2}\right]^k. \tag{3.66}$$

This gives the estimate for the fourth line of (3.63), one can get the same estimate for the second line of (3.63). Note the bound above will have to be summable in n, here k is some function of n. To continue our proof of Theorem 3.3 we must now look at the third line of (3.63) where we will directly encounter the discrepancy between v and v_* - this estimate is what necessitates a careful choice of k dependent on n. We note that for $v=v_*$ and $w\in\mathcal{E}^{\mathrm{u}}$ - i.e. the circumstances of Theorem 3.4 - the third line (3.63) vanishes.

To estimate the third line (3.63) we again start by applying crude Lipschitz estimates to get

$$
\begin{aligned}
&\left|\left|RG^k\left(RG^{n+1-k}(v)+D_vRG^{n+1-k}[\alpha_{\mathrm{u}}^{-(n+1)}w]\right)-RG^k\left(RG^{n-k}(v)+D_vRG^{n-k}[\alpha_{\mathrm{u}}^{-n}w]\right)\right|\right|\\
\leqslant&c_3(\epsilon)^k||RG^{n+1-k}(v)+D_vRG^{n+1-k}[\alpha_{\mathrm{u}}^{-(n+1)}w]-RG^{n-k}(v)-D_vRG^{n-k}[\alpha_{\mathrm{u}}^{-n}w]||\\
\leqslant&c_3(\epsilon)^k\left[||RG^{n+1-k}(v)-RG^{n-k}(v)||+||D_vRG^{n+1-k}[\alpha_{\mathrm{u}}^{-(n+1)}w]-D_vRG^{n-k}[\alpha_{\mathrm{u}}^{-n}w]||\right]\\
=&c_3(\epsilon)^k\left[||RG^{n-k}\left(RG(v)\right)-RG^{n-k}(v)||+\alpha_{\mathrm{u}}^{-k}||T_{n-k+1}(v)[w]-T_{n-k}(v)[w]||\right]\\
\leqslant&c_3(\epsilon)^k\left[c_1(\epsilon)^{n-k}||RG(v)-v||+\alpha_{\mathrm{u}}^{-k}\times||T_{n-k+1}(v)-T_{n-k}(v)||\times||w||\right]\\
\leqslant&O(1)c_3(\epsilon)^kc_1(\epsilon)^{n-k}+c_3(\epsilon)^k\alpha_{\mathrm{u}}^{-k}\times 10\times\mathcal{C}_3(\epsilon)c_1(\epsilon)^{\frac{n-k}{2}}\times||w||.
\end{aligned}
\tag{3.67}
$$

In going to the fifth line from the fourth line we used Lemma 3.19 - since both v and $RG(v)$ are in $W^{\mathrm{s,loc}}$ we now that their distance is contracting under the RG flow with rate $c_1(\epsilon)$. In going from the fifth line to the last line we used $||RG(v)-v||\leqslant||RG(v)||+||v||\leqslant O(1)$ when bounding the first term. For the second term we used Lemma 3.23.

The final steps to controling the limit (3.60) are applying the estimates (3.65) and (3.67) to (3.63) and choosing $k=\lfloor\sigma n\rfloor$ for $\sigma\in(0,\frac{1}{2})$ and summing in n. The main theorems are given by

Theorem 3.3. *For $v\in W^{\mathrm{s,loc}}$ and $||w||\leqslant\dfrac{1}{240\mathcal{C}_1(\epsilon)}$ the quantity*

$$
\Psi(v,w)=\lim_{n\to\infty}RG^n(v+\alpha_{\mathrm{u}}^{-n}w)\text{ exists in }\mathcal{E}
$$

and defines a function of (v,w) with the following properties:

1. *Ψ is continuous in the domain $v\in W^{\mathrm{s,loc}}$ and $||w||\leqslant\frac{1}{240\mathcal{C}_1(\epsilon)}$. Over this set one has the uniform bound $||\Psi(v,w)||\leqslant\dfrac{1}{8}$.*
2. *Ψ is jointly analytic in v_1 and w in the domain $||v_1||<\dfrac{\rho}{13}$, $||w||<\dfrac{1}{240\mathcal{C}_1(\epsilon)}$ where we have implied the use of the parameterization*

$$
v_1\mapsto v=(v_1,v_2)=(v_1,\mu_{\mathrm{s}}(v_1))\text{ of }W^{\mathrm{s,loc}}_{\mathrm{int}}\,.
$$

3. *For all $v\in W^{\mathrm{s,loc}}$, w such that $||w||\leqslant\dfrac{1}{240\mathcal{C}_1(\epsilon)\alpha_{\mathrm{u}}}$ we have the intertwining relation*

$$
RG(\Psi(v,w))=\Psi(v,\alpha_{\mathrm{u}}w).
$$

4. *For all $v\in W^{\mathrm{s,loc}}$, w such that $||w||\leqslant\dfrac{1}{2400\mathcal{C}_1(\epsilon)^2}$, and all integers $q\geqslant 0$, we have*

$$
\Psi(v,w)=\Psi(RG^q(v),T_q(v)[w]).
$$

5. *For all* $v \in W^{\mathrm{s,loc}}$ *and* w *such that* $||w|| \leqslant \dfrac{1}{2400\mathcal{C}_1(\epsilon)^2}$, *we have*

$$\Psi(v, w) = \Psi(v_*, T_\infty(v)[w]).$$

Proof: See [3, Theorem 5]. Analyticity again is a simple consequence of the absolute uniform convergence of the telescoping series defining $\Psi(v, w)$. The intertwining relation of Part 3) is an immediate consequence of the definition of $\Psi(v, w)$ as we mentioned earlier. The argument for Part 4) is as follows:

$$\begin{aligned}\Psi(v, w) &= \lim_{n\to\infty} RG^{n+q}[v + \alpha_{\mathrm{u}}^{-(n+q)} w] \\ &= \lim_{n\to\infty} RG^{n}\left[RG^{q}(v) + \alpha_{\mathrm{u}}^{-n}\left(\alpha_{\mathrm{u}}^{-q} D_v RG^{q}[w]\right)\right] \\ &=\Psi(RG^{q}(v), T_q(v)[w])\end{aligned}$$

where the second equality is shown by taking the difference of the two quantities at finite n, writing it as a telescoping series similar to (3.64), and showing it vanishes as $n \to \infty$. Part 5) follows from Part 4). Note that the above theorem does not show non-degeneracy of $\Psi(v, \cdot)$ but Part 5) will let one pass the buck to showing $\Psi(v_*, T_\infty(v)\cdot)$ is non-degenerate.

Theorem 3.4. *On the domain* $||w|| < \frac{1}{24}$ *of the one-dimensional space* $\mathcal{E}^{\mathrm{u}}$ *the limit*

$$\lim_{n\to\infty} RG^{n}(v_* + \alpha_{\mathrm{u}}^{-n} w)$$

exists and defines an analytic function of w *which will be denoted by* $\Psi(v_*, w)$ *since it coincides with previous one given for* $\Psi(\cdot, \cdot)$ *on the common domain of definition. On the domain* $B\left(0, \frac{1}{24}\right) \cap \mathcal{E}^{\mathrm{u}}$, *this function satisfies the bound*

$$||\Psi(v_*, w)|| \leqslant \frac{1}{8}$$

as well as

$$||\Psi(v_*, w) - v_* - w|| \leqslant \frac{17}{8}||w||^2 \ .$$

In particular, the differential with respect to w *at* $w = 0$ *is the identity on* $\mathcal{E}^{\mathrm{u}}$. *On the domain* $B\left(0, \frac{1}{24}\right) \cap \mathcal{E}^{\mathrm{u}}$ *we also have the intertwining relation*

$$RG(\Psi(v_*, \alpha_{\mathrm{u}}^{-1} w)) = \Psi(v_*, w) \ .$$

For w *small enough in* $\mathcal{E}^{\mathrm{u}}$ *we have* $\Psi(v_{*,w}) \in W^{\mathrm{u,loc}}$

Proof: See [3, Theorem 6].

For the non-degeneracy, i.e. the second inequality of the above assertion, one can use the estimates of

(3.65) with $k = 0$, which gives

$$||RG^n(v_* + \alpha_{\mathrm{u}}^{-n} w) - (RG^n(v_*) + D_{v_*}RG[\alpha_{\mathrm{u}}^{-n} w])|| \leqslant \sum_{j=0}^{n-1} c_3(\epsilon)^j \times \frac{17}{2}\left[\alpha_{\mathrm{u}}^{-(j+1)}||T_{n-j-1}(v_*)[w]||\right]^2$$

but we can simply rewrite the righthand side

$$||RG^n(v_* + \alpha_{\mathrm{u}}^{-n} w) - v_* - w|| \leqslant \sum_{j=0}^{n-1} c_3(\epsilon)^j \times \frac{17}{2}\left[\alpha_{\mathrm{u}}^{-(j+1)}||w||\right]^2$$

from which the wanted estimate follows by taking $n \to \infty$. The intertwining relation follows as in Part 3) of Theorem 3.3. By the intertwining relation one has that $\{\Psi(v_*, \alpha_{\mathrm{u}}^n w)\}_{-\infty < n \leqslant 0}$ is a backwards trajectory emanating form the fixed point v_* (clearly $\lim_{n\to-\infty} \Psi(v_*, \alpha_{\mathrm{u}}^n w) = \Psi(v_*, 0) = v_*$.) Thanks to the criterion in Proposition 3.7 we see that $\Psi(v_*, w)$ is then on the local unstable manifold $W^{\mathrm{u,loc}}$. □

3.9 Control of the deviation from the bulk

3.9.1 Algebraic considerations

We now pick up the thread from §3.5.2 where we consider for test functions $\tilde{f}, \tilde{j} \in S_{q_-,q_+}(\mathbb{Q}_p^3, \mathbb{C})$ the quantity

$$\mathcal{S}_{r,s}(\tilde{f}, \tilde{j}) = \frac{\mathcal{Z}_{r,s}(\tilde{f}, \tilde{j})}{\mathcal{Z}_{r,s}(0,0)}$$

which is the moment generating function with UV and IR cutoffs r and s respectively.

Introduce

$$\begin{aligned}\mathcal{S}^{\mathrm{T}}_{r,s}(\tilde{f}, \tilde{j}) = & -Y_0 Z_0^r \int_{\mathbb{Q}_p^3} \tilde{j}(x)\ d^3x + \frac{1}{2}\sum_{r\leqslant q<s}\left(f^{(r,q)}, \Gamma f^{(r,q)}\right)_{\Lambda_{s-q}} \\ & + \sum_{r\leqslant q<s}\ \sum_{\substack{\Delta\in\mathbb{L} \\ \Delta\subset\Lambda_{s-q-1}}} \left(\delta b_\Delta\left[\vec{V}^{(r,q)}(\tilde{f},\tilde{j})\right] - \delta b_\Delta\left[\vec{V}^{(r,q)}(0,0)\right]\right) \\ & + \mathrm{Log}\left(\frac{\partial\mathcal{Z}_{r,s}(\tilde{f},\tilde{j})}{\partial\mathcal{Z}_{r,s}(0,0)}\right)\end{aligned}$$

where Log is the principal logarithm with argument in $(-\pi, \pi]$.

We will show that it is indeed a well defined quantity which boils down to making sure all the RG iterates $\vec{V}^{(r,q)}$ are in the domain of definition and analyticity for RG_{ex} provided by Theorem 3.2. One also needs to check that $\frac{\partial\mathcal{Z}_{r,s}(\tilde{f},\tilde{j})}{\partial\mathcal{Z}_{r,s}(0,0)}$ is well defined and nonzero.

Once this is verified then it immediately follows from the considerations in §3.5.2 that

$$\mathcal{S}_{r,s}(\tilde{f},\tilde{j}) = \exp\left(\mathcal{S}^{\mathrm{T}}_{r,s}(\tilde{f},\tilde{j})\right) \ .$$

In particular $\mathcal{S}^{\mathrm{T}}_{r,s}(\tilde{f},\tilde{j})$ generates truncated correlation functions.

The brunt of the remaining work is controlling the $r \to -\infty$ and $s \to \infty$ limits of the log-moment generating function $\mathcal{S}^{\mathrm{T}}_{r,s}(\tilde{f},\tilde{j})$.

Recall that for the denominator, i.e. when $\tilde{f},\tilde{j} = 0$, the initial condition for the RG_{ex} iterations is

$$\vec{V}^{(r,r)}(0,0) = (g, 0, \mu_{\mathrm{c}}(g), 0, 0, 0, 0, 0)$$
$$\text{with } \mu_{\mathrm{c}}(g) = \mu_{\mathrm{s}}(g - \bar{g}, 0) \text{ by definition.}$$

If ι is the affine isometric injection $\mathcal{E} \to \mathcal{E}_{\mathrm{ex}}$ which sends $(\delta g, \mu, R)$ to the vector

$$\vec{V} = (\beta_{4,\Delta}, \beta_{3,\Delta}, \beta_{2,\Delta}, \beta_{1,\Delta}, W_{5,\Delta}, W_{6,\Delta}, f_{\Delta}, R_{\Delta})_{\Delta\in\mathbb{L}}$$

where for all $\Delta \in \mathbb{L}$

$$\begin{aligned}
\beta_{4,\Delta} &= \bar{g} + \delta g \\
\beta_{3,\Delta} &= 0 \\
\beta_{2,\Delta} &= \mu \\
\beta_{1,\Delta} &= 0 \\
W_{5,\Delta} &= 0 \\
W_{6,\Delta} &= 0 \\
f_{\Delta} &= 0 \\
R_{\Delta} &= R
\end{aligned}$$

then $\vec{V}^{r,r}(0,0) = \iota(v)$ with $v = (\delta g, \mu_{\mathrm{s}}(\delta g, 0), 0)$ where $\delta g = g - \bar{g}$.

By construction $v \in W^{\mathrm{s,loc}}$ and therefore all of its iterates are well defined and we have

$$\vec{V}^{(r,q)}(0,0) = \iota\left(RG^{q-r}(v)\right) \longrightarrow \iota(v_*) \text{ where } r \to -\infty \text{ with } q \text{ fixed.}$$

The purpose of this section is to derive estimates which control the deviations from this bulk trajectory due to the test functions $\tilde{f}$ and $\tilde{j}$. An important fact is that when comparing $\vec{V}^{(r,q)}(\tilde{f},\tilde{j})$ to the corresponding bulk $\vec{V}^{(r,q)}(0,0)$ any differences between $\vec{V}^{(r,q)}_{\Delta}(\tilde{f},\tilde{j})$ and $\vec{V}^{(r,q)}_{\Delta}(0,0)$ must be constrainted to those $\Delta \subset \Lambda_{\min(0,q_+-q)}$. The disturbances caused by observables can't propagate any further due to the strict locality of the RG map. One immediate consequence is that $\delta b_{\Delta'}[\vec{V}^{(r,q)}(\tilde{f},\tilde{j})] = \delta b_{\Delta'}[\vec{V}^{(r,q)}(0,0)]$ for $\Delta' \not\subset \Lambda_{\min(0,q_+-q-1)}$ and so at a fixed scale q we will get contributions from only finitely many blocks Δ.

We will break up the log-moment generating function into five pieces which will be analyzed separately.

Namely, we write

$$\begin{aligned}\mathcal{S}^{\mathrm{T}}_{r,s}(\tilde{f},\tilde{j}) =&\mathcal{S}^{\mathrm{T,FR}}_{r,s}(\tilde{f},\tilde{j}) + \mathcal{S}^{\mathrm{T,UV}}_{r,s}(\tilde{f},\tilde{j}) + \mathcal{S}^{\mathrm{T,MD}}_{r,s}(\tilde{f},\tilde{j})\\ &+ \mathcal{S}^{\mathrm{T,IR}}_{r,s}(\tilde{f},\tilde{j}) + \mathcal{S}^{\mathrm{T,BD}}_{r,s}(\tilde{f},\tilde{j})\end{aligned}$$

where

$$\mathcal{S}^{\mathrm{T,FR}}_{r,s}(\tilde{f},\tilde{j}) =\frac{1}{2}\sum_{r\leqslant q<s}\left(f^{(r,q)},\Gamma f^{(r,q)}\right)_{\Lambda_{s-q}}$$

$$\mathcal{S}^{\mathrm{T,UV}}_{r,s}(\tilde{f},\tilde{j}) =-\mathrm{Y}_0 Z_0^r\int_{\mathbb{Q}_p^3}\tilde{j}(x)\ d^3x + \sum_{r\leqslant q<q_-}\sum_{\substack{\Delta\in\mathbb{L}\\ \Delta\subset\Lambda_{s-q-1}}}\left(\delta b_\Delta[\vec{V}^{(r,q)}(\tilde{f},\tilde{j})]-\delta b_\Delta[\vec{V}^{(r,q)}(0,0)]\right)$$

$$\mathcal{S}^{\mathrm{T,MD}}_{r,s}(\tilde{f},\tilde{j}) =\sum_{q_-\leqslant q<q_+}\sum_{\substack{\Delta\in\mathbb{L}\\ \Delta\subset\Lambda_{s-q-1}}}\left(\delta b_\Delta[\vec{V}^{(r,q)}(\tilde{f},\tilde{j})]-\delta b_\Delta[\vec{V}^{(r,q)}(0,0)]\right)$$

$$\mathcal{S}^{\mathrm{T,IR}}_{r,s}(\tilde{f},\tilde{j}) =\sum_{q_+\leqslant q<s}\sum_{\substack{\Delta\in\mathbb{L}\\ \Delta\subset\Lambda_{s-q-1}}}\left(\delta b_\Delta[\vec{V}^{(r,q)}(\tilde{f},\tilde{j})]-\delta b_\Delta[\vec{V}^{(r,q)}(0,0)]\right)$$

and

$$\mathcal{S}^{\mathrm{T,BD}}_{r,s}(\tilde{f},\tilde{j}) =\mathrm{Log}\left(\frac{\partial\mathcal{Z}_{r,s}(\tilde{f},\tilde{j})}{\partial\mathcal{Z}_{r,s}(0,0)}\right).$$

The subscript "FR" stands for the free contribution. Indeed, an easy exercise shows that

$$\lim_{\substack{r\rightarrow-\infty\\ s\rightarrow\infty}}\mathcal{S}^{\mathrm{T,FR}}_{r,s}(\tilde{f},\tilde{j}) = \frac{1}{2}\left(\tilde{f},C_{-\infty}\tilde{f}\right)$$

which corresponds to the free massless measure without cut-offs, i.e., the Gaussian measure with covariance $C_{-\infty}$.

The quantity $\mathcal{S}^{\mathrm{T,UV}}_{r,s}(\tilde{f},\tilde{j})$ collects the ultraviolet contributions while $\mathcal{S}^{\mathrm{T,IR}}_{r,s}(\tilde{f},\tilde{j})$ contains the infrared contributions. Most of the influence of the test functions is felt in the middle regime $q_-\leqslant q<q_+$, hence the abbreviation "MD". Finally $\mathcal{S}^{\mathrm{T,BD}}_{r,s}(\tilde{f},\tilde{j})$ corresponds to the a boundary term left after the RG iterations have shrunk the confining volume Λ_s down to a single unit cube.

The analysis will make use of the following observations which are of an algebraic or combinatorial nature. Since the RG runs from UV scales to IR scales we will first have a closer look at the terms featuring in $\mathcal{S}^{\mathrm{T,UV}}_{r,s}(\tilde{f},\tilde{j})$.

From the definition of RG_{ex} in §3.5.2 one sees that this map is given by a collection of independent operations performed locally.

Indeed the output $\left(\beta'_{4,\Delta'},\ldots,\beta'_{1,\Delta'},W'_{5,\Delta'},W'_{6,\Delta'},f'_{\Delta'},R_{\Delta'}\right)$ as well as the output $\delta b_{\Delta'}$ produced for a cube Δ' only involves the data $(\beta_{4,\Delta},\ldots,\beta_{1,\Delta},W_{5,\Delta},W_{6,\Delta},f_\Delta,R_\Delta)_{\Delta\in[L^{-1}\Delta']}$.

We define $\mathcal{E}_{1B}$ to be the "one block space", i.e $\mathcal{E}_{\rm ex}=\prod_{\Delta\in\mathbb{L}}\mathcal{E}_{1B}$. With those notation $RG_{\rm ex}$ is made up of independent copies of a map $(\mathcal{E}_{\rm 1B})^{\times L^3}\longrightarrow\mathcal{E}_{\rm 1B}$.

Let $\widetilde{\Delta}\in\mathbb{L}_{q_-}$ so that $\tilde{f}$ and $\tilde{j}$ are constant on $\widetilde{\Delta}$ taking the values $\tilde{f}_{\widetilde{\Delta}}$ and $\tilde{j}_{\widetilde{\Delta}}$ respectively. If $\widetilde{\Delta}\notin\Lambda_{q_+}$ then $\tilde{f}_{\widetilde{\Delta}}=\tilde{j}_{\widetilde{\Delta}}=0$.

First let us see what happens for the first iteration, i.e., $q=r$.

If a unit cube Δ is in $\Lambda_{s-r}\backslash\Lambda_{q_+-r}$ then the Δ component of $\vec{V}^{(r,r)}(\tilde{f},\tilde{j})$ of $\vec{V}^{(r,r)}(\tilde{f},\tilde{j})$ is exactly the same as that of the bulk $\vec{V}^{(r,r)}(0,0)=\iota(\delta g,\mu,0)$ with $\mu=\mu_{\rm s}(\delta g,0)$.

If $\Delta\in\Lambda_{q_+-r}$ then there is a unique $\widetilde{\Delta}\in\mathbb{L}_{q_-}$, $\widetilde{\Delta}\subset\Lambda_{q_+}$ such that $\Delta\subset L^r\widetilde{\Delta}$

In this case:

$$\vec{V}^{(r,r)}(\tilde{f},\tilde{j})=(g,0,\mu-Y_2Z_2^rL^{(3-2[\phi])r}\tilde{j}_{\widetilde{\Delta}},0,0,0,L^{(3-[\phi])r}\tilde{f}_{\widetilde{\Delta}},0).$$

Now since we chose $Z_2=\alpha_{\rm u}L^{-(3-2[\phi])}$ and thus

$$\vec{V}^{(r,r)}_{\Delta}(\tilde{f},\tilde{j})=(g,0,\mu-Y_2\alpha_{\rm u}^r\tilde{j}_{\widetilde{\Delta}},0,0,0,L^{(3-[\phi])r}\tilde{f}_{\widetilde{\Delta}},0).$$

If $q=r<q_-$ then all immediate neighbors Δ carry the same data. Here by neighbors we mean the L^3-1 other unit cubes contained in the same L-block $L^{-1}\Delta'$ as Δ. The fact that our data is constant over L-blocks make this situation reminiscent of the bulk RG.

We claim that the computation producing $\delta b_{\Delta'}[\vec{V}^{(r,r)}(\tilde{f},\tilde{j})]$ as well as $\vec{V}^{(r,r+1)}_{\Delta'}(\tilde{f},\tilde{j})$ is the same as the RG acting on the space $\mathcal{E}$, except for the presence of the f-component $L^{(3-[\phi])r}\tilde{f}_{\widetilde{\Delta}}$ which evolves by averaging without influencing or being influenced by the other variables.

This again results from the property that $\int_{L^{-1}\Delta'}\Gamma(x-y){\rm d}^3y=0$ for all $x\in L^{-1}\Delta'$, as in the proof of Proposition 3.2.

Indeed for the explicit diagrams in the RG transformation the possible effect of f is through legs attached to f-vertices of valence 1 which precisely contribute a factor of the type $\int_{L^{-1}\Delta'}\Gamma(x-y){\rm d}^3y=0$ because f is constant over the L-block $L^{-1}\Delta'$.

For the other $\mathcal{L}$ or ξ terms, observe that one has $e^{\int_{L^{-1}\Delta'}f\zeta}=1$ because f is constant on $L^{-1}\Delta'$ and $\int_{L^{-1}\Delta'}\zeta=0$ almost surely by the property of the fluctuation covariance Γ.

As a result

$$\vec{V}_{\Delta'}^{(r,r+1)}(\tilde{f},\tilde{j}) = (g', 0, \mu', 0, 0, 0, L^{(3-[\phi])(r+1)}\tilde{f}_{\widetilde{\Delta}}, R')$$

where

$$(g' - \bar{g}, \mu', R') = RG(g - \bar{g}, \mu - \alpha_{\mathrm{u}}^{r} Y_2 \tilde{j}_{\widetilde{\Delta}}, 0)$$

and also

$$\delta b_{\Delta'}[\vec{V}^{(r,r)}(\tilde{f},\tilde{j})] = \delta b(g - \bar{g}, \mu - \alpha_{\mathrm{u}}^{r} Y_2 \tilde{j}_{\widetilde{\Delta}}, 0).$$

The same decoupling applies to subsequent iterates $\vec{V}^{(r,q+1)}(\tilde{f},\tilde{j}) = RG_{\mathrm{ex}}[\vec{V}^{(r,q)}(\tilde{f},\tilde{j})]$ as long as $q < q_-$, i.e. as long as $f^{(r,q)}$ is constant over each individual L-block.

Hence, in the quantity

$$\sum_{r \leqslant q < q_-} \sum_{\substack{\Delta \in \mathbb{L} \\ \Delta \subset \Lambda_{s-q-1}}} \left(\delta b_{\Delta}[\vec{V}^{(r,q)}(\tilde{f},\tilde{j})] - \delta b_{\Delta}[\vec{V}^{(r,q)}(0,0)] \right)$$

appearing in $\mathcal{S}_{r,s}^{\mathrm{T,UV}}(\tilde{f},\tilde{j})$, only boxes $\Delta \subset \Lambda_{q_+-r-1}$ will contribute and these can be organized according to $\widetilde{\Delta} \in \mathbb{L}_{q_-}$, $\widetilde{\Delta} \subset \Lambda_{q_+}$ such that $L^{q+1}\widetilde{\Delta}$ contains Δ. All $L^{3(q_- - q - 1)}$ boxes Δ which satisfy that condition for given $\widetilde{\Delta}$ produce the same contribution.

In other words, the previous expression can be rewritten as

$$\sum_{\substack{\widetilde{\Delta} \in \mathbb{L}_{q_-} \\ \widetilde{\Delta} \subset \Lambda_{q_+}}} \sum_{r \leqslant q < q_-} L^{3(q_- - q - 1)} \left(\delta b \left[RG^{q-r} \left(v - \alpha_{\mathrm{u}}^{r} Y_2 \tilde{j}_{\widetilde{\Delta}} e_{\phi^2} \right) \right] - \delta b \left[RG^{q-r}(v) \right] \right)$$

$$\text{where } v = (\delta g, \mu_{\mathrm{s}}(\delta g, 0), 0) \text{ with } \delta g = g - \bar{g}$$
$$\text{and } e_{\phi^2} = (0, 1, 0) \in \mathcal{E}.$$

Here e_{ϕ^2} gives the direction of pure $:\phi^2:$ perturbations in the bulk.

We are thus reduced to $L^{3(q_+ - q_-)}$ separate and independent bulk RG trajectories as considered in §3.7.1, one for each $\widetilde{\Delta}$. Also note that the effect of $\tilde{f}$ is completely absent form the UV regime contribution.

By also organizing the explicit extra linear term in $\tilde{j}$ according to boxes $\widetilde{\Delta}$ of size L^{q_-} we can write

$$\mathcal{S}_{r,s}^{\mathrm{T,UV}}(\tilde{f},\tilde{j}) = \sum_{\substack{\widetilde{\Delta} \in \mathbb{L}_{q_-} \\ \widetilde{\Delta} \subset \Lambda_{q_+}}} \mathcal{K}_{\widetilde{\Delta}}$$

with

$$\mathcal{K}_{\tilde{\Delta}} = -Y_0 Z_0^r L^{3q_-} \tilde{j}_{\tilde{\Delta}} + \sum_{r \leqslant q < q_-} L^{3(q_- - q - 1)} \left(\delta b \left[RG^{q-r} \left(v - \alpha_{\mathrm{u}}^r Y_2 \tilde{j}_{\tilde{\Delta}} e_{\phi^2} \right) \right] - \delta b \left[RG^{q-r}(v) \right] \right) \ .$$

That concludes our organization of UV contributions.

As soon as $q \geqslant q_-$ and we enter the middle regime we are not guaranteed that our RG data is constant over L-blocks - we may have to deal with inhomogenieties within L-blocks and so we must use the extended RG map RG_{ex}. We will want to take the $L^{q_+ - q_-}$ different bulk data point points we tracked in the UV regime along with the f-component that has been flowing by averaging and amalagmate all of this into the corresponding data point in $\mathcal{E}_{\mathrm{ex}}$.

We now give the notation used to describe the above process which at scale $q = q_-$ will take us from bulk data to data in $\mathcal{E}_{\mathrm{ex}}$.

For $m \geqslant 0$ we introduce the reinjection map

$$\mathcal{J}_m : S_{0,m}(\mathbb{Q}_p^3, \mathbb{C}) \times \left(\prod_{\substack{\Delta \in \mathbb{L} \\ \Delta \subset \Lambda_m}} \mathcal{E} \right) \longrightarrow \mathcal{E}_{\mathrm{ex}}$$

$$\left(F, (\delta g_\Delta, \mu_\Delta, R_\Delta)_{\substack{\Delta \in \mathbb{L} \\ \Delta \subset \Lambda_m}}, (\delta g, \mu, R) \right) \mapsto \vec{V}' = \left(\beta'_{4,\Delta}, \ldots, \beta'_{1,\Delta}, W'_{5,\Delta}, W'_{6,\Delta}, f'_{\Delta'}, R'_{\Delta'} \right)_{\Delta \in \mathbb{L}}$$

defined as follows.

We let

$$\beta'_{4,\Delta} = \begin{cases} \bar{g} + \delta g_\Delta & \text{if } \Delta \subset \Lambda_m \\ \bar{g} + \delta g & \text{if } \Delta \not\subset \Lambda_m \end{cases}$$

$$\beta'_{2,\Delta} = \begin{cases} \mu_\Delta & \text{if } \Delta \subset \Lambda_m \\ \mu & \text{if } \Delta \not\subset \Lambda_m \end{cases}$$

$$R'_\Delta = \begin{cases} R_\Delta & \text{if } \Delta \subset \Lambda_m \\ R & \text{if } \Delta \not\subset \Lambda_m \end{cases}$$

$$\beta'_{3,\Delta} = \beta'_{1,\Delta} = W'_{5,\Delta} = W'_{6,\Delta} = 0$$

and finally f'_Δ is defined by

$$f'_{\Delta(x)} = F(x) \text{ for all } x \in \mathbb{Q}_p^3.$$

Recall that since $F \in S_{0,m}(\mathbb{Q}_p^3, \mathbb{C})$ is assumed constant on unit cubes and with support contained in Λ_m.

Now it is easy to see from the previous considerations that

$$\vec{V}^{(r,q_-)}(\tilde{f},\tilde{j}) = \mathcal{J}_{q_+-q_-}\left(\tilde{f}_{\to(-q_-)}, \left(RG^{q_--r}\left(v-\alpha_{\mathrm{u}}^r Y_2 \tilde{j}_{L^{-q_-}\Delta} e_{\phi^2}\right)\right)_{\substack{\Delta\in\mathbb{L}\\ \Delta\subset\Lambda_{q_+-q_-}}}, RG^{q_--r}(v)\right).$$

Note in particular that $f^{(r,q_-)} = \left(\tilde{f}_{\to(-r)}\right)_{\to(q_--r)} = \tilde{f}_{\to(-q_-)}$.

We also have the special case

$$\begin{aligned}\vec{V}^{(r,q_-)}(0,0) =& \mathcal{J}_{q_+-q_-}\left(0, \left(RG^{q_--r}(v)\right)_{\substack{\Delta\in\mathbb{L}\\ \Delta\subset\Lambda_{q_+-q_-}}}, RG^{q_--r}(v)\right)\\ =& \iota\left(RG^{q_--r}(v)\right).\end{aligned}$$

We now look at the middle regime and note that

$$\mathcal{S}_{r,s}^{\mathrm{T,MD}}(\tilde{f},\tilde{j}) = \sum_{q_-\leqslant q<q_+}\ \sum_{\substack{\Delta\in\mathbb{L}\\ \Delta\subset\Lambda_{q_+-q-1}}} \left(\delta b_\Delta[\vec{V}^{(r,q)}(\tilde{f},\tilde{j})] - \delta b_\Delta[\vec{V}^{(r,q)}(0,0)]\right).$$

Here we replaced the s that appeared earlier with q_+ when describing the summation over boxes Δ. Indeed, if $\Delta\subset\Lambda_{s-q-1}$ is outside the rescaling Λ_{q_+-q-1} of the set Λ_{q_+} containing the supports of the $\tilde{f}$ and $\tilde{j}$, then the effect of Δ is nil.

Finally we turn to the infrared regime. Here we have

$$\mathcal{S}_{r,s}^{\mathrm{T,IR}}(\tilde{f},\tilde{j}) = \sum_{q_+\leqslant q<s}\ \sum_{\substack{\Delta\in\mathbb{L}\\ \Delta\in\Lambda_{s-q-q}}} \left(\delta b_\Delta\left[RG^{q-q_+}\left(\vec{V}^{(r,q_+)}(\tilde{f},\tilde{j})\right)\right] - \delta b_\Delta\left[RG^{q-q_+}\left(\vec{V}^{(r,q_+)}(0,0)\right)\right]\right)$$

$$\text{where } \vec{V}^{(r,q_+)}(\tilde{f},\tilde{j}) = RG^{q_+-q_-}\left(\vec{V}^{(r,q_-)}(\tilde{f},\tilde{j})\right).$$

Since $\vec{V}^{(r,q_-)}(\tilde{f},\tilde{j})$ agrees with $\vec{V}^{(r,q_-)}(0,0)$ on all unit cubes $\Delta\not\subset\Lambda_{q_+-q_-}$, it is easy to see that

$$RG^{q_+-q_-}\left(\vec{V}^{(r,q_-)}(\tilde{f},\tilde{j})\right) \text{ agrees with } RG^{q_+-q_-}\left(\vec{V}^{(r,q_-)}(0,0)\right)$$

$$\text{on all unit cubes } \Delta\not\subset\Lambda_0=\Delta(0) \text{ the unit cube containing the origin.}$$

We denote by $\mathcal{E}_{pt}$ those elements of $\mathcal{E}_{\mathrm{ex}}$ that are supported on $\Delta(0)$. Thus

$$\vec{V}^{(r,q_+)}(\tilde{f},\tilde{j}) - \vec{V}^{(r,q_+)}(0,0)\in\mathcal{E}_{\mathrm{pt}}$$

or

$$\vec{V}^{(r,q_+)}(\tilde{f},\tilde{j})\in\iota(\mathcal{E})\oplus\mathcal{E}_{\mathrm{pt}}\subset\mathcal{E}_{\mathrm{bk}}\oplus\mathcal{E}_{\mathrm{pt}}$$

where $\mathcal{E}_{\rm bk}$ is just the space of spatially homogenous elements of $\mathcal{E}_{\rm ex}$. This property remains true for the next iterates since the only difference with the bulk now only happens in $\Delta(0)$.

Therefore no summation over Δ is needed in the formula for $\mathcal{S}^{\rm T,IR}_{r,s}(\tilde{f},\tilde{j})$ which thus reduces to

$$\mathcal{S}^{\rm T,IR}_{r,s}(\tilde{f},\tilde{j}) = \sum_{q_+\leqslant q<s}\left(\delta b_{\Delta(0)}\left[RG^{q-q_+}\left(\vec{V}^{(r,q_+)}(\tilde{f},\tilde{j})\right)\right] - \delta b_{\Delta(0)}\left[RG^{q-q_+}\left(\vec{V}^{(r,q_+)}(0,0)\right)\right]\right).$$

After these prepatory steps we can now address the estimates needed in order to take the $r\to-\infty$ and $s\to\infty$ limits.

3.9.2 The ultraviolet regime

We now resume the analysis of the expression for $\mathcal{S}^{T,UV}_{r,s}(\tilde{f},\tilde{j})$ derived in the last section.

Adding and subtracting terms linear in $\tilde{j}_{\tilde{\Delta}}$ we write

$$\mathcal{K}_{\tilde{\Delta}} = \tilde{j}_{\tilde{\Delta}}\left[-Y_0 Z_0^r L^{3q_-} + \sum_{r\leqslant q<q_-} L^{3(q_--q-1)} D_v\left(\delta b\circ RG^{q-r}\right)\left[-\alpha_{\rm u}^r Y_2 e_{\phi^2}\right]\right] + \sum_{r\leqslant q<q_-} L^{3(q_--q-1)}\mathcal{K}_{\tilde{\Delta},q}$$

$$\text{where } \mathcal{K}_{\tilde{\Delta},q} = \delta b\left[RG^{q-r}\left(v-\alpha_{\rm u}^r Y_2\tilde{j}_{\tilde{\Delta}} e_{\phi^2}\right)\right] - \delta b\left[RG^{q-r}(v)\right] + D_v\left(\delta b\circ RG^{q-r}\right)\left[\alpha_{\rm u}^r Y_2\tilde{j}_{\tilde{\Delta}} e_{\phi^2}\right] .$$

Now $\mathcal{K}_{\tilde{\Delta},q} = \mathcal{K}'_{\tilde{\Delta},q} + \mathcal{K}''_{\tilde{\Delta},q}$ where

$$\mathcal{K}'_{\tilde{\Delta},q} = \delta b\left[RG^{q-r}\left(v-\alpha_{\rm u}^r Y_2\tilde{j}_{\tilde{\Delta}} e_{\phi^2}\right)\right] - \delta b\left[RG^{q-r}(v)\right] - D_{RG^{q-r}(v)}\delta b\left[RG^{q-r}\left(v-\alpha_{\rm u}^r Y_2\tilde{j}_{\tilde{\Delta}} e_{\phi^2}\right) - RG^{q-r}(v)\right]$$

and

$$\mathcal{K}''_{\tilde{\Delta},q} = D_{RG^{q-r}(v)}\delta b\left[RG^{q-r}\left(v-\alpha_{\rm u}^r Y_2\tilde{j}_{\tilde{\Delta}} e_{\phi^2}\right) - RG^{q-r}(v) + D_v RG^{q-r}\left[\alpha_{\rm u}^r Y_2\tilde{j}_{\tilde{\Delta}} e_{\phi^2}\right]\right] .$$

The quantities above are similar to some of the terms estimated in section 3.8.1. We remark that the first and second differentials of δb can be bounded with an $O(1)$ estimate in a straightforward way as long as we stay within the domain of the RG. In particular the two terms immediately above are quadratic terms which are fairly easily to estimate as long as one has control over the term

$$RG^{q-r}\left(v-\alpha_{\rm u}^r Y_2\tilde{j}_{\tilde{\Delta}} e_{\phi^2}\right)$$

for $q<q_-$.

An analysis similar to that used in section 3.8.1 will show that this term can be controlled uniformly in r if

$$||\alpha_{\rm u}^q Y_2\tilde{j}_{\tilde{\Delta}} e_{\phi^2}|| \leqslant \frac{1}{240\mathcal{C}_1(\epsilon)} \text{ for all } q<q_-.$$

We guarantee this by enforcing that that

$$||\alpha_{\mathrm{u}}^{q_{-}-1} Y_2 \tilde{j}_{\tilde{\Delta}} e_{\phi^2}|| \leqslant \frac{1}{240\mathcal{C}_1(\epsilon)}.$$

Proceeding in this way one can show $||\mathcal{K}''_{\tilde{\Delta},q}|| \leqslant \alpha_{\mathrm{u}}^{-2(q_{-}-q-1)}$ and $||\mathcal{K}'_{\tilde{\Delta},q}|| \leqslant 2\alpha_{\mathrm{u}}^{-2(q_{-}-q-1)}$ in which case one has the the estimate $||\mathcal{K}_{\tilde{\Delta},q}|| \leqslant 3\alpha_{\mathrm{u}}^{-2(q_{-}-q-1)}$ for simplicity.

Y_2 is a strictly positive quantity that will be fixed later and we have that $||e_{\phi^2}|| = ||(0,1,0)|| = \bar{g}^{-1}$. So the previous construction and bounds work if

$$||\tilde{j}||_{L^\infty} \leqslant \left[240\mathcal{C}_1(\epsilon)\alpha_{\mathrm{u}}^{q_{-}-1} Y_2 \bar{g}^{-1}\right]^{-1} \ . \tag{3.68}$$

We will later also show $L^3\alpha_{\mathrm{u}}^{-2} < 1$ which will imply that $\sum_{r\leqslant q<q_{-}} L^{3(q_{-}-q-1)}||\mathcal{K}_{\tilde{\Delta},q}||$ is summable with uniform bounds with respect to the UV cut-off r.

We now analyze the more dangerous linear term in $\tilde{j}_{\tilde{\Delta}}$, that is the quantity

$$\Omega_r = -Y_0 Z_0^r L^{3q_{-}} + \sum_{r\leqslant q<q_{-}} L^{3(q_{-}-q-1)} D_v \left(\delta b \circ RG^{q-r}\right)\left[-\alpha_{\mathrm{u}}^r Y_2 e_{\phi^2}\right].$$

We change the summation index to $n = q - r$ and rewrite the differential using the chain rule and get

$$\begin{aligned}
\Omega_r =& L^{3q_{-}}\left(-Y_0 Z_0^r - Y_2 \sum_{n=0}^{q_{-}-r-1} L^{-3(n+r+1)}\alpha_{\mathrm{u}}^r D_{RG^n(v)}\delta b\left[D_v RG^n[e_{\phi^2}]\right]\right)\\
=& L^{3q_{-}}\left(-Y_0 Z_0^r - Y_2 \sum_{n=0}^{q_{-}-r-1} L^{-3(n+r+1)}\alpha_{\mathrm{u}}^{r+n} D_{RG^n(v)}\delta b\left[T_n(v)[e_{\phi^2}]\right]\right)\\
=& L^{3q_{-}}\left(-Y_0 Z_0^r - Y_2 L^{-3}(L^{-3}\alpha_{\mathrm{u}})^r \sum_{n=0}^{q_{-}-r-1} (L^{-3}\alpha_{\mathrm{u}})^n \Xi_n\right)\\
&\text{with } \Xi_n = D_{RG^n(v)}\delta b\left[T_n(v)[e_{\phi^2}]\right] \ .
\end{aligned}$$

Remembering that $L^{-3}\alpha_{\mathrm{u}} < 1$, and applying Lemma 3.23 one can show

$$\begin{aligned}
\left|D_{RG^n(v)}\delta b\left[T_n(v)[e_{\phi^2}]\right]\right| \leqslant& ||D_{RG^n(v)}\delta b|| \times ||T_n(v)|| \times ||e_{\phi^2}||\\
\leqslant& 10\mathcal{C}_1(\epsilon)\bar{g}^{-1} \ .
\end{aligned}$$

We then see that Ξ_n is bounded uniformly with respect to n. Hence

$$\Upsilon = \sum_{n=0}^{\infty} (L^{-3}\alpha_{\rm u})^n \ \Xi_n \text{ converges}$$

and we can write

$$\Omega_r = L^{3q_-}\left(-Y_0 Z_0^r - Y_2 L^{-3}(L^{-3}\alpha_{\rm u})^r \Upsilon + Y_2 L^{-3}(L^{-3}\alpha_{\rm u})^r \sum_{n=q_- - r}^{\infty} (L^{-3}\alpha_{\rm u})^n \ \Xi_n \right) \ .$$

Since $L^{-3}\alpha_{\rm u} < 1$ and $r \to -\infty$ we choose Y_0, Y_2, and Z_0 so that the dangerous first two terms cancel.

Namely, we set:

$$\begin{aligned} Z_0 =& L^{-3}\alpha_{\rm u} \ , \\ Y_0 =& -L^{-3} Y_2 \Upsilon \ . \end{aligned}$$

Then

$$\begin{aligned} \Omega_r =& L^{3q_-} Y_2 L^{-3}(L^{-3}\alpha_{\rm u})^r \sum_{n=q_- - r}^{\infty} (L^{-3}\alpha_{\rm u})^n \ \Xi_n \\ =& Y_2 L^{-3} \alpha_{\rm u}^{q_-} \sum_{k=0}^{\infty} (L^{-3}\alpha_{\rm u})^k \ \Xi_{k+q_- - r} \end{aligned}$$

after changing the summation index to $k = n - q_- + r$.

Provided one shows that $\lim\limits_{n\to\infty} \Xi_n = \Xi_\infty$ exists, the discrete dominated convergence theorem will immediately imply

$$\lim_{r\to -\infty} \Omega_r = \frac{Y_2 L^{-3}\alpha_{\rm u}^{q_-} \Xi_\infty}{1 - L^{-3}\alpha_{\rm u}} \ .$$

Now

$$\begin{aligned} \left|\Xi_n - D_{v_*}\delta b\left[T_\infty(v)[e_{\phi^2}]\right]\right| \leqslant& \left|D_{RG^n(v)}\delta b\left[T_n(v)[e_{\phi^2}]\right] - D_{v_*}\delta b\left[T_n(v)[e_{\phi^2}]\right]\right| + \left|D_{v_*}\delta b\left[T_n(v)[e_{\phi^2}] - T_\infty(v)[e_{\phi^2}]\right]\right| \\ \leqslant& 2||RG^n(v) - v_*|| \times 10\mathcal{C}_1(\epsilon)||e_{\phi^2}|| + ||T_n(v) - T_\infty(v)|| \times ||e_{\phi^2}|| \ . \end{aligned}$$

Above we used Lemma 3.23. Finally, Proposition 3.5 and Lemma 3.23 ensure that the limit of the Ξ_n exists and is given by $\Xi_\infty = D_{v_*}\delta b\left[T_\infty(v)[e_{\phi^2}]\right]$.

As a consequence of the previous considerations and Theorem 3.3 we see that

$$\lim_{\substack{r\to-\infty \\ s\to\infty}} \mathcal{S}^{\rm T,UV}_{r,s}(\tilde{f},\tilde{j}) = \mathcal{S}^{\rm T,UV}(\tilde{f},\tilde{j}) \text{ with}$$

$$
\mathcal{S}^{\mathrm{T,UV}}(\tilde{f},\tilde{j}) = \sum_{\substack{\tilde{\Delta}\in\mathbb{L}_{q_-} \\ \tilde{\Delta}\subset\Lambda_{q_+}}} \Bigg\{ \tilde{j}_{\tilde{\Delta}} \frac{Y_2\alpha_{\mathrm{u}}^{q_-}}{L^3-\alpha_{\mathrm{u}}} D_{v_*}\delta b\left[T_\infty(v)[e_{\phi^2}]\right]
$$

$$
+ \sum_{q<q_-} L^{3(q_--q-1)} \left(\delta b\left(\Psi(v, -\alpha_{\mathrm{u}}^q Y_2 \tilde{j}_{\tilde{\Delta}} e_{\phi^2})\right) - \delta b(v_*) + \alpha_{\mathrm{u}}^q Y_2 \tilde{j}_{\tilde{\Delta}} D_{v_*}\delta b[T_\infty(v)(e_{\phi^2})]\right) \Bigg\} .
$$

The latter is easily seen to be analytic in $\tilde{j}$ in the domain $||\tilde{j}||_{L^\infty} < \left[240\mathcal{C}_1(\epsilon)\alpha_{\mathrm{u}}^{q_--1}Y_2\bar{g}^{-1}\right]^{-1}$ of $S_{q_-,q_+}(\mathbb{Q}_p^3,\mathbb{C})$.

Note that there is no dependence on $\tilde{f}$ for this piece. In fact the finite cut-off quantity $\mathcal{S}^{\mathrm{T,UV}}_{r,s}(\tilde{f},\tilde{j})$ does not depend on $\tilde{f}$ nor s.

3.9.3 The middle regime

We introduce the notation $\bar{V}$ for the approximate fixed point in $\mathcal{E}_{\mathrm{bk}}$. Namely we set $\bar{V}_\Delta = (\bar{g}, 0, \ldots, 0)$ for all $\Delta \in \mathbb{L}$. We note that RG_{ex} is well defined and analytic on $B(\bar{V}, \frac{1}{2})$.

We will next establish some very coarse bounds on the expansion of deviations which will be enough for the control of the middle regime.

The main idea is that the middle regime will only be a finite number scales, in particular $q_+ - q_-$ scales. We will establish a brutal expansion bound in this regime on the size of deviations from the bulk. After the middle regime comes the infrared regime in which we expect that deviations, once small enough, should contract away. The role of this brutal expansion bound in the middle regime is to tell us how small we must choose our initial deviations (our observables $\tilde{j}, \tilde{f}$) so the deviations they induce are sufficiently small enough at the end of the middle regime to guarantee that they will contract away in the infrared regime.

To establish our brutal expansion bound on the extended RG we establish a uniform bound on the output of RG_{ex} on the ball $B(\bar{V}, \frac{1}{2})$. The only bound that takes some care is the β_4 evolution which we include below.

Lemma 3.24. *Suppose that $\vec{V}$ in $B(\bar{V}, \frac{1}{2})$. Then one has the following bound for the β_4' component of $\vec{V}' = RG_{\mathrm{ex}}[\vec{V}]$: For all $\Delta' \in \mathbb{L}$*

$$
\left|\beta'_{4,\Delta'} - \bar{g}\right| \bar{g}^{-\frac{3}{2}} \leqslant \mathbf{O}_3
$$

where $\mathbf{O}_3 = 434 + \mathcal{O}_{26}$ with $\mathcal{O}_{26}$ defined in the statement of [3, Lemma 38].

Proof: Due to our assumption on $\vec{V}$ for any $\Delta \in \mathbb{L}$ we can write $\beta_{4,\Delta} = \bar{g} + \delta g_\Delta$ where $|\delta g_\Delta| < \frac{1}{2}\bar{g}^{\frac{3}{2}}$. We substitute this into the flow equation to get the following:

$$
\beta'_{4,\Delta'} = L^{3-4[\phi]}\bar{g} + L^{-4[\phi]} \sum_{\Delta\in[L^{-1}\Delta']} \delta g_\Delta - \delta\beta_{4,2,\Delta'}[\vec{V}] + \xi_{4,\Delta'}[\vec{V}]
$$

$$
= L^{3-4[\phi]}\bar{g} - 36L^{-4[\phi]}
$$

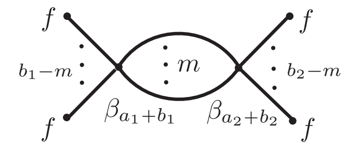
$\bar{g}+\delta g \quad \bar{g}+\delta g$

$$
- \widetilde{\delta\beta}_{4,2,\Delta'}[\vec{V}] + \xi_{4,\Delta'}[\vec{V}] + L^{-4[\phi]} \sum_{\Delta\in[L^{-1}\Delta']} \delta g_\Delta \ . \tag{3.69}
$$

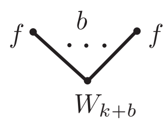
We have used the fact that $\delta\beta_{4,1,\Delta}[\vec{V}] = 0$. In the formula above $\widetilde{\delta\beta}_{4,2,\Delta'}[\vec{V}]$ is defined to be $\delta\beta_{4,2,\Delta'}[\vec{V}]$ with the graph that we have made explicit removed:

$$\widetilde{\delta\beta}_{4,2,\Delta'}\left[\vec{V}\right] := \sum_{a_1,a_2,b_1,b_2,m} \mathbb{1}\left\{\begin{array}{c} a_i + b_i \leqslant 4 \\ a_i \geqslant 0\ ,\ b_i \geqslant 1 \\ m = 1 \end{array}\right\} \frac{(a_1+b_1)!\,(a_2+b_2)!}{a_1!\,a_2!\,m!\,(b_1-m)!\,(b_2-m)!}$$

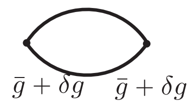
$$\times \frac{1}{2} C(a_1, a_2|4) \times L^{-(a_1+a_2)[\phi]} \times C_0(0)^{\frac{a_1+a_2-4}{2}} \times$$



$$+ \sum_b \mathbb{1}\left\{\begin{array}{c} 4+b = 5 \text{ or } 6 \\ b \geqslant 0 \end{array}\right\} \frac{(k+b)!}{k!\,b!} L^{-k[\phi]}$$



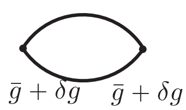
$$= \delta\beta_{4,2,\Delta'}[\vec{V}] - \frac{1}{2} L^{-4[\phi]} \frac{4!4!}{2!2!2!}$$

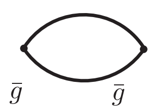


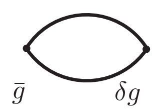
.

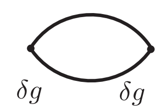
Indeed, first note that there is no graph with $m = 3$. This is because this would imply $a_1, a_2 \leqslant 1$ which contradicts $a_1 + a_2 \geqslant 4$ imposed by the nonvanishing of the connection coefficient $C(a_1, a_2|4)$. Also the removed graph is the only one with $m = 2$. This is because $b_1, b_2 \geqslant 2$ implies $a_1, a_2 \leqslant 4 - 2 = 2$, but the connection coefficient requires $a_1 + a_2 \geqslant 4$ so we are forced to have $a_1 = a_2 = 2$ which implies $b_1, b_2 \leqslant 2$ and therefore $b_1 = b_2 = 2$.

We note that we can decompose the graph above as follows:



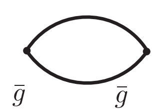
$=$


$+\,2$


$+$


We now use the fact that $\bar{g}$ is an approximate fixed point:

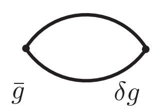
$$\bar{g} = L^{\epsilon}\bar{g} - A_1\bar{g}^2 = L^{3-4[\phi]}\bar{g} - 36L^{-4[\phi]}$$

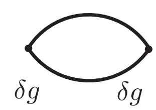


Using this we can write:

$$\beta'_{4,\Delta'} = \bar{g} + L^{-4[\phi]} \sum_{\Delta \in [L^{-1}\Delta']} \delta g_{\Delta}$$

$$- 36L^{-4[\phi]} \Bigg(2 \qquad\qquad (3.70)$$

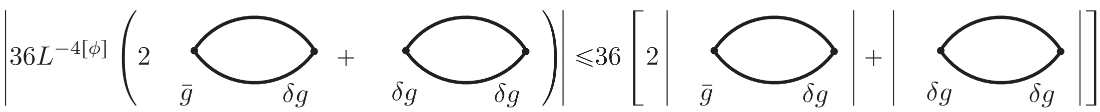


$+$


$\Bigg)$

$$- \widetilde{\delta\beta}_{4,2,\Delta'}[\vec{V}] + \xi_{4,\Delta'}[\vec{V}]\ .$$

We now describe how to bound the second and third lines of (3.70). By the same arguments as used in [3, Lemma 38] the contribution of the two graphs on the second line can each be bounded by $4L^5\bar{g}^2$ as follows from the very coarse bounds $\bar{g} \leqslant \bar{g}$ and $|\delta g| \leqslant \bar{g}$. This gives us:



$$\leqslant 36 \times 3 \times 4 L^5 \bar{g}^2 = 432 L^5 \bar{g}^2 \ .$$

Note that in the first line we dropped the factor of $L^{-4[\phi]}$. The quantity $\widetilde{\delta\beta}_{4,2,\Delta'}[\vec{V}]$ on the third line of (3.70) can be bounded by $\mathcal{O}_{26} L^5 \bar{g}^2$ as in [3, Lemma 38] where $\mathcal{O}_{26}$ is a purely numeric constant(we are overestimating since we are summing over fewer graphs). We combine this with the estimate on $\xi_{4,\Delta'}[\vec{V}]$ from Theorem 3.2 to get

$$\begin{aligned}
\left|\beta'_{4,\Delta'} - \bar{g}\right| \bar{g}^{-\frac{3}{2}} &\leqslant \frac{1}{2} L^{3-4[\phi]} + (432 + \mathcal{O}_{26})\, L^5 \bar{g}^{2-\frac{3}{2}} + B_4 \bar{g}^{\frac{21}{8}-\frac{3}{2}} \\
&\leqslant \frac{1}{2} L^{3-4[\phi]} + (432 + \mathcal{O}_{26}) + 1 \\
&\leqslant 1 + (432 + \mathcal{O}_{26}) + 1
\end{aligned}$$

In going to the third we used the bound $L^{3-4[\phi]} = L^{\epsilon} \leqslant 2$ for ϵ small. □

Lemma 3.25. *$RG_{\rm ex}$ is well defined and analytic on $B(\bar{V}, \frac{1}{2})$. Additionally one has the following uniform bound for $\vec{V} \in B(\bar{V}, \frac{1}{2})$:*

$$||RG_{\rm ex}[\vec{V}] - \bar{V}|| \leqslant \mathbf{O}_5 L^{\frac{5}{2}} \tag{3.71}$$

for a purely numeric constant $\mathbf{O}_5$.

Proof: This is [3, Lemma 87]

Proposition 3.8. *For any $\vec{V}^1, \vec{V}^2 \in \bar{B}(\bar{V}, \frac{1}{6})$ one has:*

$$||RG_{\rm ex}[\vec{V}^1] - RG_{\rm ex}[\vec{V}^2]|| \leqslant \mathbf{O}_6 L^{\frac{5}{2}} ||\vec{V}^1 - \vec{V}^2||,$$

where $\mathbf{O}_6 = 4\mathbf{O}_5$.

Proof:

By Lemma 3.25 we know that $RG_{\rm ex}$ is an analytic map taking $B(\bar{V}, \frac{1}{2})$ into $\bar{B}(\bar{V}, \mathbf{O}_5 L^{\frac{5}{2}})$. We get the desired inequality by applying Lemma 3.2 with the choice $\nu = \frac{1}{3}$. □

After the previous estimates we now return to the analysis of the $r \to -\infty$ and $s \to \infty$ limits of $\mathcal{S}^{\rm T,MD}_{r,s}(\tilde{f}, \tilde{j})$ which in fact does not depend on s such that $s \geqslant q_+$. Since the summation range $q_- \leqslant q < q_+$ is fixed and finite, all we need is to show that $RG_{\rm ex}$ remain in the domains of definition and analyticity, despite the temporary expansion with rate controlled by Lemma 3.25 and Proposition 3.8.

The quantity of interest, as delivered by §3.9.1, is

$$\mathcal{S}^{\mathrm{T,MD}}_{r,s}(\tilde{f},\tilde{j}) = \sum_{q_- \leqslant q < q_+} \sum_{\substack{\Delta\in\mathbb{L} \\ \Delta\subset\Lambda_{q_+-q-1}}} \left(\delta b_\Delta[\vec{V}^{(r,q)}(\tilde{f},\tilde{j})] - \delta b_\Delta[\vec{V}^{(r,q)}(0,0)]\right)$$

where

$$\vec{V}^{(r,q)}(\tilde{f},\tilde{j}) = RG_{\mathrm{ex}}^{q-q_-}\left(\vec{V}^{(r,q_-)}(\tilde{f},\tilde{j})\right)$$

with

$$\vec{V}^{(r,q_-)}(\tilde{f},\tilde{j}) = \mathcal{J}_{q_+-q_-}\left(\tilde{f}_{\to(-q_-)}, \left(RG^{q_--r}\left(v-\alpha_{\mathrm{u}}^r Y_2 \tilde{j}_{L^{-q_-}\Delta} e_{\phi^2}\right)\right)_{\substack{\Delta\in\mathbb{L} \\ \Delta\subset\Lambda_{q_+-q_-}}}, RG^{q_--r}(v)\right).$$

It follows from our definitions for the norms and the reinjection map $\mathcal{J}$ that

$$||\vec{V}^{(r,q_-)}(\tilde{f},\tilde{j}) - \vec{V}^{(r,q_-)}(0,0)||$$

$$= \max\left\{ ||\tilde{f}_{\to(-q_-)}||_{L^\infty}, \max_{\substack{\Delta\in\mathbb{L} \\ \Delta\subset\Lambda_{q_+-q_-}}} ||RG^{q_--r}(v-\alpha_{\mathrm{u}}^r Y_2 \tilde{j}_{L^{-q_-}\Delta} e_{\phi^2}) - RG^{q_--r}(v)|| \right\} .$$

We also have

$$||\tilde{f}_{\to(-q_-)}||_{L^\infty} = L^{(3-[\phi])q_-}||\tilde{f}||_{L^\infty} .$$

We slightly strengthen the requirement in (3.68) by imposing

$$||\tilde{j}||_{L^\infty} \leqslant [240\mathcal{C}_1(\epsilon)\alpha_{\mathrm{u}}^{q_-} Y_2 \bar{g}^{-1}]^{-1}$$

which implies

$$||-\alpha_{\mathrm{u}}^{q_-} Y_2 \tilde{j}_{L^{-q_-}\Delta} e_{\phi^2}|| \leqslant \frac{1}{240\mathcal{C}_1(\epsilon)}$$

for all $\Delta\in\mathbb{L}$ such that $\Delta\subset\Lambda_{q_+-q_-}$. Proceeding as in section 3.8.1 one can prove the bound

$$\begin{aligned} ||RG^{q_--r}(v-\alpha_{\mathrm{u}}^r Y_2 \tilde{j}_{L^{-q_-}\Delta} e_{\phi^2}) - RG^{q_--r}(v)|| &\leqslant 11\mathcal{C}_1(\epsilon)||-\alpha_{\mathrm{u}}^{q_-} Y_2 \tilde{j}_{L^{-q_-}\Delta} e_{\phi^2}|| \\ &\leqslant 11\mathcal{C}_1(\epsilon)\alpha_{\mathrm{u}}^{q_-} Y_2 \bar{g}^{-1} \times ||\tilde{j}||_{L^\infty} \end{aligned}$$

and therefore

$$||\vec{V}^{(r,q_-)}(\tilde{f},\tilde{j}) - \vec{V}^{(r,q_-)}(0,0)|| \leqslant \max\left\{ L^{(3-[\phi])q_-}||\tilde{f}||_{L^\infty}, 11\mathcal{C}_1(\epsilon)\alpha_{\mathrm{u}}^{q_-} Y_2 \bar{g}^{-1} \times ||\tilde{j}||_{L^\infty} \right\} .$$

On the other hand, minding the $\bar{g}$ shift for β_4 components only, we easily see that

$$||\vec{V}^{(r,q_-)}(0,0) - \bar{V}|| = ||\iota(RG^{q_--r}(v)) - \bar{V}|| = ||RG^{q_--r}(v)||$$

where the latter quantity can be computed as in section §3.7.2, i.e., via the norm inherited by $\mathcal{E}$ from $\mathcal{E}_{\mathrm{ex}}$ and expressed in $(\delta g, \mu, R)$ coordinates.

By construction of $W^{\mathrm{s,loc}}$, $||RG^{q_- - r}(v)|| \leqslant \frac{\rho}{3}$ with $\rho \in \left(0, \frac{1}{12}\right)$ as yet unspecified. We thus have

$$||\vec{V}^{(r,q_-)}(0,0) - \bar{V}|| \leqslant \frac{1}{12} \ .$$

Provided we also have

$$\left(\mathbf{O}_6 L^{\frac{5}{2}}\right)^{q_+ - q_-} \times \max\left\{L^{(3-[\phi])q_-}||\tilde{f}||_{L^\infty}, 11\mathcal{C}_1(\epsilon)\alpha_{\mathrm{u}}^{q_-} Y_2 \bar{g}^{-1} \times ||\tilde{j}||_{L^\infty}\right\} \leqslant \frac{1}{12}$$

then a trivial inductive application of Proposition 3.8 will guarantee that for all q, $q_- \leqslant q \leqslant q_+$,

$$||\vec{V}^{(r,q_-)}(\tilde{f},\tilde{j}) - \bar{V}|| \leqslant \frac{1}{12}$$

so one remains, throughout the iterations, in the domain of definition and analyticity of RG_{ex} as well as the δb functions.

As a result of Theorem 3.3 we then immediately obtain, regardless of the order of limits,

$$\lim_{\substack{r \to -\infty \\ s \to \infty}} \mathcal{S}^{T,MD}_{r,s}(\tilde{f},\tilde{j}) = \mathcal{S}^{T,MD}(\tilde{f},\tilde{j})$$

where

$$\mathcal{S}^{T,MD}(\tilde{f},\tilde{j}) = \sum_{q_- \leqslant q < q_+} \sum_{\substack{\Delta \in \mathbb{L} \\ \Delta \subset \Lambda_{q_+ - q - 1}}} \left(\delta b_\Delta[\vec{V}^{(-\infty,q)}(\tilde{f},\tilde{j})] - \delta b_\Delta[\iota(v_*)]\right)$$

with

$$\vec{V}^{(-\infty,q)}(\tilde{f},\tilde{j}) = RG_{\mathrm{ex}}^{q-q_-}\left(\vec{V}^{(-\infty,q_-)}(\tilde{f},\tilde{j})\right)$$

for

$$\vec{V}^{(-\infty,q_-)}(\tilde{f},\tilde{j}) = \mathcal{J}_{q_+ - q_-}\left(\tilde{f}_{\to(-q_-)}, \left(\Psi v, -\alpha_{\mathrm{u}}^{q_-} Y_2 \tilde{j}_{L^{-q_-}\Delta} e_{\phi^2}\right)_{\substack{\Delta \in \mathbb{L} \\ \Delta \subset \Lambda_{q_+ - q_-}}}, v_*\right). \tag{3.72}$$

Analyticity of $\mathcal{S}^{T,MD}(\tilde{f},\tilde{j})$ is also immediate.

For the purposes of the next section we also note that $\vec{V}^{(r,q_+)}(\tilde{f},\tilde{j})$ satisfies the bound

$$||\vec{V}^{(r,q_+)}(\tilde{f},\tilde{j}) - \vec{V}^{(r,q_+)}(0,0)|| \leqslant \left(\mathbf{O}_6 L^{\frac{5}{2}}\right)^{q_+ - q_-} \times \max\left\{L^{(3-[\phi])q_-}||\tilde{f}||_{L^\infty}, 11\mathcal{C}_1(\epsilon)\alpha_{\mathrm{u}}^{q_-} Y_2 \bar{g}^{-1} \times ||\tilde{j}||_{L^\infty}\right\} \ . \tag{3.73}$$

3.9.4 The infrared regime

In this section we are concerned with showing that the differential of RG_{ex} at any suitable $\vec{V}_{\mathrm{bk}} \in \mathcal{E}_{\mathrm{bk}}$ in any direction $\dot{V} \in \mathcal{E}_{\mathrm{pt}}$ is a contraction.

We will introduce new notation to facilitate the lemmas below. For $\vec{V}_{\mathrm{bk}} \in \mathcal{E}_{\mathrm{bk}}$ we write:

$$\vec{V}_{\mathrm{bk}} = \{V_{\mathrm{bk}}\}_{\Delta \in \mathbb{L}} = \{(\beta_{4,\mathrm{bk}}, \ldots, \beta_{1,\mathrm{bk}}, W_{5,\mathrm{bk}}, W_{6,\mathrm{bk}}, f_{\mathrm{bk}}, R_{\mathrm{bk}})\}_{\Delta \in \mathbb{L}} \ .$$

Note that we do need to burden the notation with Δ subscripts since the quantities above are independent of the box Δ by definition of being in $\mathcal{E}_{\text{bk}}$.

Similarly for $\dot{V} \in \mathcal{E}_{\text{pt}}$ we write:

$$\dot{V} = \left\{ \dot{V}_\Delta \right\}_{\Delta \in \mathbb{L}} = \left\{ (\dot{\beta}_{4,\Delta}, \ldots, \dot{\beta}_{1,\Delta}, \dot{W}_{5,\Delta}, \dot{W}_{6,\Delta}, \dot{f}_\Delta, \dot{R}_\Delta) \right\}_{\Delta \in \mathbb{L}} .$$

Note that $\dot{V}_\Delta = 0$ for $\Delta \neq \Delta(0)$. We also recall that $RG_{\text{ex}}[\vec{V}_{\text{bk}} + \dot{V}] - RG_{\text{ex}}[\vec{V}_{\text{bk}}] \in \mathcal{E}_{\text{pt}}$ and so in our estimates we are only concerned with the $\Delta(0)$ component of $RG_{\text{ex}}[\vec{V}_{\text{bk}} + \dot{V}] - RG_{\text{ex}}[\vec{V}_{\text{bk}}] \in \mathcal{E}_{\text{pt}}$.

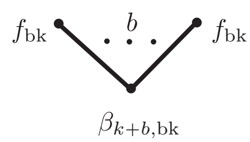
The key estimates proved in this section are Lemmas 3.28 and 3.9. We have to be more careful here than in our treatment of the middle regime - in order to carefully track the flow of deviations from the bulk we decompose the vertices at graphs appearing in the flow equations into bulk and deviation components. The fact that the fluctuation covariance Γ vanishes at 0 momentum (or equivalently, has integral one) causes many of the terms that appear in our analysis to vanish. Lemmas 3.26 and 3.27 give some examples of how this works out, but we have not included the other details. We remark that the R deviations, which we don't explicitly address here, are already guaranteed to contract due to the estimates of Theorem 3.2 and an easy non-bulk generalization of the Lipschitz estimate on the R remainder in Lemma 3.16.

Lemma 3.26. *Let $\vec{V}_{\text{bk}} \in B(\bar{V}, \frac{1}{4}) \cap \mathcal{E}_{\text{bk}}$ and $\dot{V} \in B(0, \frac{1}{4}) \cap \mathcal{E}_{\text{pt}}$. Then one has the bound for $k = 1, 2, 3, 4$ and all $\Delta' \in \mathbb{L}$:*

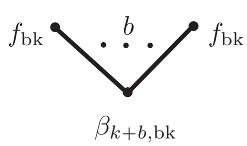
$$\left| \delta\beta_{k,1,\Delta(0)} \left[\vec{V}_{\text{bk}} + \dot{V} \right] - \delta\beta_{k,1,\Delta(0)} \left[\vec{V}_{\text{bk}} \right] \right| \bar{g}^{-e_k} \leqslant \mathbb{1}\{1 \leqslant k \leqslant 3\} \mathbf{O}_7 L^{-\frac{9}{4}} ||\dot{V}||^2,$$

where $\mathbf{O}_7 = \left(6 + 21 \times 2^{\frac{3}{2}} \right)$.

Proof: We again note that the vanishing for $k = 4$ follows by inspection of the definition of $\delta\beta_{k,1,\Delta(0)}$. We now observe that $\delta\beta_{k,1,\Delta(0)}[\vec{V}_{\text{bk}}]$ vanishes. Indeed, by definition we have

$$\delta\beta_{k,1,\Delta(0)}[\vec{V}_{\text{bk}}] = -\sum_b \mathbb{1} \left\{ \begin{array}{c} k + b \leqslant 4 \\ b \geqslant 1 \end{array} \right\} \frac{(k+b)!}{k!\ b!} \ L^{-k[\phi]}$$ 

However one has that



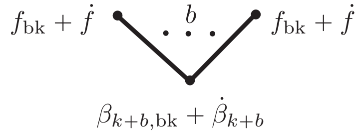
$= 0$.

This is because we have at least one integration vertex of degree 1 which has been assigned a coupling f_{bk} which is constant over the integration region $L^{-1}\Delta(0)$. Using ultrametricity and the fact that Γ integrates to 0 allows one to show that after integrating any of the f_{bk} vertices the entire integral vanishes. So

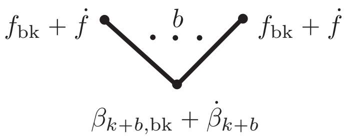
$$\delta\beta_{k,1,\Delta(0)}[\vec{V}_{\text{bk}}] = 0 .$$

We now turn to $\delta\beta_{k,1,\Delta(0)}[\vec{V}_{\text{bk}} + \dot{V}]$. From the definition we have:

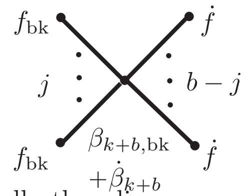
$$\delta\beta_{1,k,\Delta(0)}[\vec{V}_{\rm bk}+\dot{V}] = -\sum_b \mathbb{1}\left\{\begin{array}{c} k+b\leqslant 4 \\ b\geqslant 1\end{array}\right\}\frac{(k+b)!}{k!\ b!}\ L^{-k[\phi]}$$



Under the assumption that $b\geqslant 1$ we have:

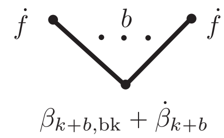


$$=\sum_{j=0}^{b}\binom{b}{j}$$

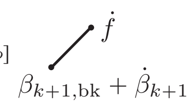


In the sum above only the $j=0$ term can be non-vanishing, all other diagrams will have at least one integration vertex of degree 1 with a bulk variable assigned to it. We substitute this back into our formula for $\delta\beta_{k,1,\Delta(0)}$ and perform more manipulations:

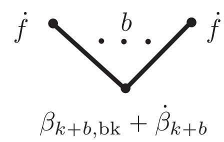
$$\delta\beta_{k,1,\Delta(0)}[\vec{V}_{\rm bk}+\dot{V}] = -\sum_b \mathbb{1}\left\{\begin{array}{c} k+b\leqslant 4 \\ b\geqslant 1\end{array}\right\}\frac{(k+b)!}{k!\ b!}\ L^{-k[\phi]} \tag{3.74}$$



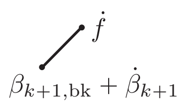
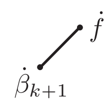
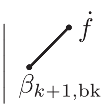
$$= -(k+1)L^{-k[\phi]}$$

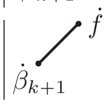


$$-\sum_b \mathbb{1}\left\{\begin{array}{c} k+b\leqslant 4 \\ b\geqslant 2\end{array}\right\}\frac{(k+b)!}{k!\ b!}\ L^{-k[\phi]} \tag{3.75}$$



where we have isolated the $b=1$ term. Note that for $k=3$ the sum on the last line is empty. We now bound the diagrams appearing above:

$$\Bigg|$$  $$\Bigg| \leqslant \Bigg|$$  $$\Bigg| + \Bigg|$$  $$\Bigg|$$

$$= \Bigg|$$  $$\Bigg|$$

$$\begin{aligned}
&= \left|\dot{f}(0)\times\Gamma(0)\times\dot{\beta}_{k+1,\Delta(0)}\right| \\
&\leqslant 2\left(L^{-(3-[\phi])}||\dot{V}||\right)\left(||\dot{V}||\bar{g}^{e_{k+1}}\right) \\
&\leqslant 2L^{-\frac{9}{4}}||\dot{V}||^2\bar{g}^{e_{k+1}}\ .
\end{aligned} \tag{3.76}$$

In going to the third to last line we used local constancy at unit scale and the fact that all the couplings were supported at $\Delta(0)$ so we did not really do any integration. In going to the second to last line we used the bound $|\Gamma(0)|\leqslant 2$ which comes from Lemma 5.4. In going to the last line we used the bound $-(3-[\phi])\leqslant -\frac{9}{4}$.

For $k = 3$ we immediately have the bound:

$$\begin{aligned}\left|\delta\beta_{3,1,\Delta(0)}\right|\bar{g}^{-1} \leqslant & 4L^{-3[\phi]} \times 2L^{-\frac{9}{4}}||\dot{V}||^2\bar{g}^{\frac{3}{2}}\bar{g}^{-1} \\ \leqslant & 8L^{-\frac{9}{4}}||\dot{V}||^2 .\end{aligned}$$

Note that in going to the last line we dropped the factor of $L^{-3[\phi]}$ and used $e_4 \geqslant e_3$. This proves the lemma for the case $k = 3$. We now bound the remaining diagrams to prove the lemma for the cases $k = 1$ and $k = 2$. Before note that in these two cases $k + b = 3$ or 4 because we also assume $b \geqslant 2$.

If $k + b = 4$ then, because of the domain hypotheses for our lemma and noting the $\bar{g}$ shift for the β_4 component of the bulk, we must have

$$|\beta_{k+b,\mathrm{bk}}| + |\dot{\beta}_{b+k}| \leqslant \bar{g} + \frac{1}{4}\bar{g}^{e_4} + \frac{1}{4}\bar{g}^{e_4} \leqslant \frac{3}{2}\bar{g} \leqslant \frac{3}{2}\bar{g}^{e_k} \ .$$

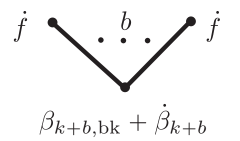
This is because of our assumptions $e_1, e_2 \leqslant 1 \leqslant e_4$.

If $k + b = 3$ then

$$|\beta_{k+b,\mathrm{bk}}| + |\dot{\beta}_{b+k}| \leqslant \frac{1}{4}\bar{g}^{e_3} + \frac{1}{4}\bar{g}^{e_3} \leqslant \frac{3}{2}\bar{g}^{e_k}$$

because of the assumption $e_1 \leqslant e_2 \leqslant e_3$. So in all relevent cases we can use $\frac{3}{2}\bar{g}^{e_k}$ as a bound, as we do next.

$$\mathbb{1}\left\{\begin{array}{c} k+b \leqslant 4 \\ b \geqslant 2 \end{array}\right\}\Bigg|$$



$$\begin{aligned}\Bigg| \leqslant & \mathbb{1}\left\{\begin{array}{c} k+b \leqslant 4 \\ b \geqslant 2 \end{array}\right\} \times \left|\dot{f}(0)\right|^b \\ & \times \left(|\beta_{b+k,\mathrm{bk}}| + \left|\dot{\beta}_{b+k}\right|\right) \times \int_{\mathbb{Q}_p^3} \mathrm{d}^3x \ |\Gamma(x)|^b \qquad (3.77) \\ \leqslant & \left(L^{-(3-[\phi])}||\dot{V}||\right)^2 \times \frac{3}{2}\bar{g}^{e_k} \times 2^{5/2}L^{3-2[\phi]} \\ \leqslant & 3 \times 2^{3/2}L^{-3}||\dot{V}||^2\bar{g}^{e_k} \ .\end{aligned}$$

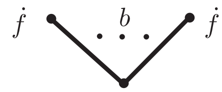
For the bound on the first line we used the fact that all the $\dot{f}$ vertices are pinned to the origin and the only integration occurs at the $\beta_{b+k,\mathrm{bk}} + \dot{\beta}_{b+k}$ vertex which has been left with b copies of the fluctuation covariance.

In going to the second to last line we used the bound $|\dot{f}(0)|^b \leqslant |\dot{f}(0)|^2$ since $b \geqslant 2$ and $|\dot{f}(0)| \leqslant 1$. For that same line we also used the following bound which is valid for $2 \leqslant b \leqslant 4$:

$$\begin{aligned}\int_{\mathbb{Q}_p^3} \mathrm{d}^3x \ |\Gamma(x)|^b \leqslant & ||\Gamma||_{L^1}||\Gamma||_{L^\infty}^{b-1} \\ \leqslant & \left(\frac{1}{\sqrt{2}}L^{3-2[\phi]}\right)2^{b-1} \\ \leqslant & 2^{5/2}L^{3-2[\phi]} \ .\end{aligned}$$

Note that we have used fluctuation covariance bounds of Corollary 5.1 and Lemma 5.4. Thus we can use (3.76) to get the following bound for $k = 1$ and $k = 2$:

$$\left|\sum_b \mathbb{1}\left\{\begin{array}{c} k+b \leqslant 4 \\ b \geqslant 2 \end{array}\right\} \frac{(k+b)!}{k!\, b!}\, L^{-k[\phi]}\right.$$



$$\beta_{k+b,\mathrm{bk}} + \dot{\beta}_{k+b}$$

$$\left| \leqslant \sum_b \mathbb{1}\left\{\begin{array}{c} k+b \leqslant 4 \\ b \geqslant 2 \end{array}\right\} \frac{(k+b)!}{k!\, b!} \times 3 \times 2^{3/2} L^{-3} ||\dot{V}||^2 \bar{g}^{e_k} \leqslant 21 \times 2^{3/2} L^{-3} ||\dot{V}||^2 \bar{g}^{e_k}\ . \right. \tag{3.78}$$

Note in going to the last line we dropped the factors of $L^{-k[\phi]}$ and used that

$$\max_{k=1,2} \sum_b \mathbb{1}\left\{\begin{array}{c} k+b \leqslant 4 \\ b \geqslant 2 \end{array}\right\} \frac{(k+b)!}{k!\, b!} = 7\ .$$

Finally by inserting the bound (3.76) and (3.78) into (3.74) we get the following bound for $k = 1$ and $k = 2$:

$$\begin{aligned} \left|\delta\beta_{k,1,\Delta(0)}[\vec{V}_{\mathrm{bk}} + \dot{V}]\right| \bar{g}^{-e_k} &\leqslant (k+1) \times 2L^{-\frac{9}{4}} ||\dot{V}||^2 + 21 \times 2^{3/2} L^{-3} ||\dot{V}||^2 \\ &\leqslant \left(6 + 21 \times 2^{\frac{3}{2}}\right) L^{-\frac{9}{4}} ||\dot{V}||^2\ . \end{aligned}$$

In going to the last line we simply bounded L^{-3} by $L^{-\frac{9}{4}}$. This proves the lemma for $k = 1$ and $k = 2$ which finishes the proof. □

Given $\vec{V}_{\mathrm{bk}} \in B(\bar{V}, \frac{1}{4}) \cap \mathcal{E}_{\mathrm{bk}}$ and $\dot{V} \in B(0, \frac{1}{4}) \cap \mathcal{E}_{\mathrm{pt}}$ we define:

$$RG_{\mathrm{dv}}[\vec{V}_{\mathrm{bk}}, \dot{V}] = RG_{\mathrm{ex}}[\vec{V}_{\mathrm{bk}} + \dot{V}] - RG_{\mathrm{ex}}[\vec{V}_{\mathrm{bk}}].$$

Note that, as a subspace of $\mathcal{E}_{\mathrm{ex}}$, the space $\mathcal{E}_{\mathrm{bk}} \oplus \mathcal{E}_{\mathrm{pt}}$ is invariant by RG_{ex}. Since $\vec{V}_{\mathrm{bk}} + \dot{V} \in \mathcal{E}_{\mathrm{bk}} \oplus \mathcal{E}_{\mathrm{pt}}$ one has a unique decomposition $RG_{\mathrm{ex}}[\vec{V}_{\mathrm{bk}} + \dot{V}] = \vec{V}'_{\mathrm{bk}} + \dot{V}'$ with $\vec{V}'_{\mathrm{bk}} \in \mathcal{E}_{\mathrm{bk}}$ and $\dot{V}' \in \mathcal{E}_{\mathrm{pt}}$. Using the locality of RG_{ex} it is not hard to see that $\vec{V}'_{\mathrm{bk}} = RG_{\mathrm{ex}}[\vec{V}_{\mathrm{bk}}]$ and $\dot{V}' = RG_{\mathrm{dv}}[\vec{V}_{\mathrm{bk}}, \dot{V}]$. In particular $RG_{\mathrm{dv}}[\bullet, \bullet]$ takes values in $\mathcal{E}_{\mathrm{pt}}$.

Lemma 3.27. *Suppose that $\vec{V}_{\mathrm{bk}} \in B(\bar{V}, \frac{1}{4}) \cap \mathcal{E}_{\mathrm{bk}}$ and $\dot{V} \in B(0, \frac{1}{4}) \cap \mathcal{E}_{\mathrm{pt}}$. Let $\dot{V}' = RG_{\mathrm{dv}}[\vec{V}_{\mathrm{bk}}, \dot{V}]$ and for $k = 5, 6$ let $\dot{W}'_k$ be the corresponding components of $\dot{V}'$.*

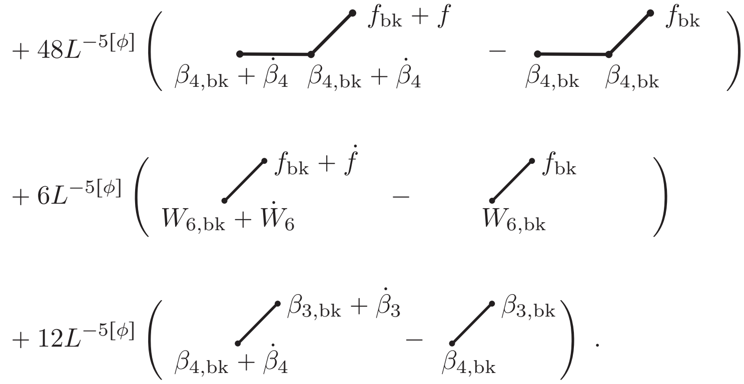
We then have the following bound for $k = 5, 6$

$$\left|\dot{W}'_{k,\Delta(0)}\right| \bar{g}^{-2} \leqslant 2^{-\frac{5}{2}} ||\dot{V}|| + \mathbf{O}_8 ||\dot{V}||^2,$$

where $\mathbf{O}_8 = \left(18 + \frac{9}{\sqrt{2}}\right)$.

Proof: For $k = 5$ we have:

$$\dot{W}'_{5,\Delta(0)} = L^{-5[\phi]} \sum_{\Delta \in [L^{-1}\Delta(0)]} \dot{W}_{5,\Delta}$$

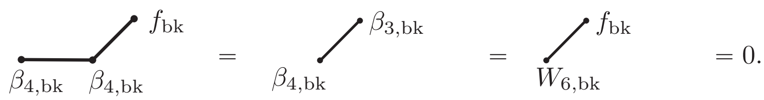


(3.79)

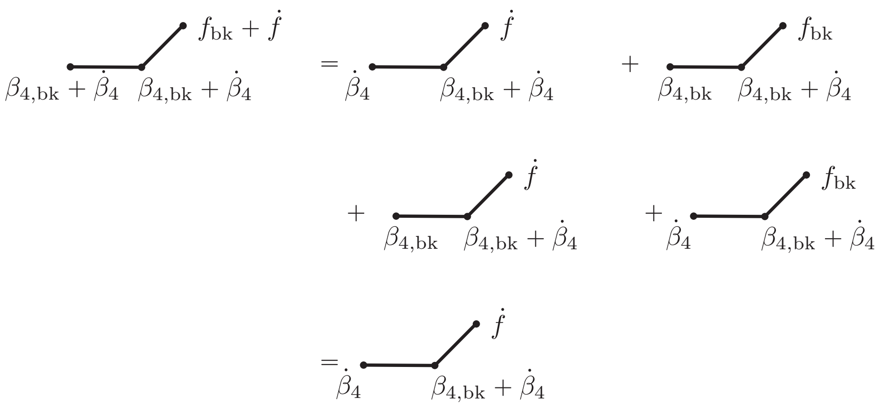
As before using that $\dot{W}_{5,\Delta}$ is supported on $\Delta = \Delta(0)$ gives us the bound:

$$\left| L^{-5[\phi]} \sum_{\Delta \in [L^{-1}\Delta(0)]} \dot{W}_{5,\Delta} \right| \leqslant L^{-5[\phi]} \bar{g}^2 ||\dot{V}|| \ .$$

We now bound the various graphs appearing in (3.79). We again note that when a graph has an integration vertex of degree one that has been assigned a bulk variable the graph will vanish. This tells us that:

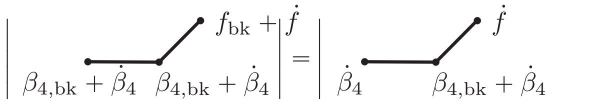


We use this same observation to break up the non-vanishing graphs and show that their contribution is second order in $||\dot{V}||$. For example:



after expanding the two outer vertices of valence one.

We then have

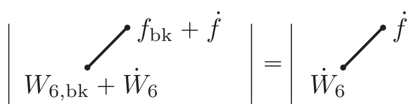


$$
\begin{aligned}
&\leqslant \left|\dot f(0)\right| \times \left|\dot\beta_{4,\Delta(0)}\right| \times \left(|\beta_{4,\rm bk}| + \left|\dot\beta_{4,\Delta(0)}\right|\right) \times \int_{\mathbb{Q}_p^3} \mathrm{d}^3x \ \ |\Gamma(x)|^2 \\
&\leqslant L^{-(3-[\phi])}||\dot V|| \times ||\dot V||\bar g^{\frac{3}{2}} \times \frac{3}{2}\bar g \times \left(2^{\frac{1}{2}} L^{3-2[\phi]}\right) \\
&\leqslant 3 \times 2^{-\frac{1}{2}} L^{-[\phi]} \bar g^{\frac{5}{2}} ||\dot V||^2 \\
&\leqslant 3 \times 2^{-\frac{1}{2}} \bar g^{\frac{5}{2}} ||\dot V||^2 \ .
\end{aligned}
\tag{3.80}
$$

Note that in going to the second to last line we again used the bound:

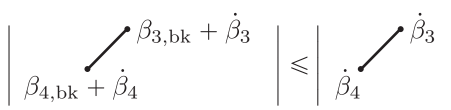
$$
\int_{\mathbb{Q}_p^3} \mathrm{d}^3x \, |\Gamma(x)|^n \leqslant 2^{n-\frac{3}{2}} L^{3-2[\phi]} \ .
$$

Proceeding similarly for the other graphs we have:



$$
\begin{aligned}
&= \left|\dot f(0)\right| \times \left|\dot W_{6,\Delta(0)}\right| \times |\Gamma(0)| \\
&\leqslant L^{-(3-[\phi])} \bar g^2 \, |\Gamma(0)| \times ||\dot V||^2 \\
&\leqslant 2\bar g^2 ||\dot V||^2 \ .
\end{aligned}
\tag{3.81}
$$

In going to the last line we used the bound $|\Gamma(0)| \leqslant 2$ which is a consequence of Corollary 5.1. We also dropped the factor of $L^{-(3-[\phi])} \leqslant L^{-\frac{9}{4}} \leqslant 1$. We continue to the last graph we need to bound for $\dot W_5'$:



$$
\begin{aligned}
&= \left|\dot\beta_{4,\Delta(0)}\right| \times \left|\dot\beta_{3,\Delta(0)}\right| \times |\Gamma(0)| \\
&\leqslant 2||\dot V||^2 \bar g^{\frac{5}{2}} \ .
\end{aligned}
\tag{3.82}
$$

Using the bounds (3.80), (3.81), and (3.82) in (3.79) gives us the bound:

$$
\begin{aligned}
\left|\dot W_{5,\Delta(0)}\right| \bar g^{-2} &\leqslant L^{-5[\phi]}||\dot V|| + L^{-5[\phi]} \left[48 \times 3 \times 2^{-\frac{1}{2}} \bar g^{\frac{1}{2}} + 6 \times 2 + 12 \times 2\bar g^{\frac{1}{2}}\right] ||\dot V||^2 \\
&\leqslant 2^{-\frac{5}{2}} ||\dot V|| + 2^{-\frac{5}{2}} \left[48 \times 3 \times 2^{-\frac{1}{2}} + 6 \times 2 + 12 \times 2\right] ||\dot V||^2 \\
&= 2^{-\frac{5}{2}} ||\dot V|| + \left(18 + \frac{9}{\sqrt{2}}\right) ||\dot V||^2 \ .
\end{aligned}
$$

In going to the second line we used the fact that $\epsilon \leqslant 1$ and $L \geqslant 2$ to bound $L^{-5[\phi]} \leqslant 2^{-\frac{5}{2}}$. This proves the lemma for $k = 5$.

For $k = 6$ we have:

$$
\begin{aligned}
\dot{W}'_{6,\Delta(0)} =& L^{-6[\phi]} \sum_{\Delta \in [L^{-1}\Delta(0)]} \dot{W}_{6,\Delta} \\
&+ 8L^{-6[\phi]} \left(\underset{\beta_{4,\mathrm{bk}} + \dot{\beta}_4}{\bullet} \diagup \overset{\beta_{4,\mathrm{bk}} + \dot{\beta}_4}{\bullet} - \underset{\beta_{4,\mathrm{bk}}}{\bullet} \diagup \overset{\beta_{4,\mathrm{bk}}}{\bullet} \right) .
\end{aligned} \tag{3.83}
$$

Proceeding as last time we see:

$$
\left| L^{-6[\phi]} \sum_{\Delta \in [L^{-1}\Delta(0)]} \dot{W}_{6,\Delta} \right| \leqslant L^{-6[\phi]} ||\dot{V}|| \bar{g}^2
$$

and

$$
\underset{\beta_{4,\mathrm{bk}}}{\bullet} \diagup \overset{\beta_{4,\mathrm{bk}}}{\bullet} = 0, \qquad \underset{\beta_{4,\mathrm{bk}} + \dot{\beta}_4}{\bullet} \diagup \overset{\beta_{4,\mathrm{bk}} + \dot{\beta}_4}{\bullet} = \underset{\dot{\beta}_4}{\bullet} \diagup \overset{\dot{\beta}_4}{\bullet}
$$

which simplifies the right-hand side of (3.83). We now bound the contributing graph:

$$
\begin{aligned}
\left| \underset{\dot{\beta}_4}{\bullet} \diagup \overset{\dot{\beta}_4}{\bullet} \right| =& |\dot{\beta}_{4,\Delta(0)}|^2 \times |\Gamma(0)| \\
\leqslant& 2||\dot{V}||^2 \bar{g}^3 \ .
\end{aligned} \tag{3.84}
$$

Inserting (3.84) along with the our earlier bound into (3.83) gives us:

$$
\begin{aligned}
\left| \dot{W}'_{6,\Delta(0)} \right| \bar{g}^{-2} \leqslant& L^{-6[\phi]} ||\dot{V}|| + 8L^{-6[\phi]} \times 2||\dot{V}||^2 \bar{g} \\
\leqslant& 2^{-3} ||\dot{V}|| + 2||\dot{V}||^2 \ .
\end{aligned}
$$

In going to the last line we used our assumption that $\epsilon \leqslant 1$ and $L \geqslant 2$ to bound $L^{-6[\phi]} \leqslant 2^{-3}$. This proves the bound of our lemma for the case $k = 6$ which finishes the proof. □

Lemma 3.28. *Suppose that* $\vec{V}_{\mathrm{bk}} \in \bar{B}(\bar{V}, \frac{1}{40}) \cap \mathcal{E}_{\mathrm{bk}}$ *and* $\dot{V} \in \bar{B}(0, \frac{1}{40}) \cap \mathcal{E}_{\mathrm{pt}}$. *Let* $\dot{V}' = RG_{\mathrm{dv}}[\vec{V}_{\mathrm{bk}}, \dot{V}]$. *Then one has the following bound:*

$$
||\dot{V}'|| \leqslant \frac{27}{32} ||\dot{V}|| + \mathbf{O}_9 ||\dot{V}||^2
$$

where $\mathbf{O}_9$ *is a purely numeric constant.*

Proof: See [3, Lemma 93]

Proposition 3.9. *Suppose that* $\vec{V}_{\mathrm{bk}} \in \bar{B}(\bar{V}, \frac{1}{40}) \cap \mathcal{E}_{\mathrm{bk}}$ *and* $\dot{V} \in \bar{B}(0, \mathbf{O}_{10}) \cap \mathcal{E}_{\mathrm{pt}}$ *where* $\mathbf{O}_{10} = \min(\frac{1}{40}, \frac{3}{32}\mathbf{O}_9^{-1})$. *Let* $\dot{V}' = RG_{\mathrm{dv}}[\vec{V}_{\mathrm{bk}}, \dot{V}]$ *Then one has the following bound:*

$$||\dot{V}'|| \leqslant \frac{15}{16}||\dot{V}|| \ .$$

Proof: This proposition is a direct consequence of Lemma 3.28.

□

We remark that it is important that the contractive bound for deviations holds for $\dot{V}$ that are $O(1)$ small - if $\dot{V}$ needed to be less than some power of g then our neighborhood of analyticity in $\tilde{f}$ might be too small to easily prove non-triviality.

For the control of the infrared contributions to the log-moment generating function we will finally need a very coarse Lipschitz estimate on the δb functions.

Lemma 3.29. *For all* $\vec{V}^1$, $\vec{V}^2$ *in* $\bar{B}\left(\bar{V}, \frac{1}{6}\right)$ *we have*

$$|\delta b_{\Delta(0)}[\vec{V}^1] - \delta b_{\Delta(0)}[\vec{V}^2]| \leqslant 4||\vec{V}^1 - \vec{V}^2|| \ .$$

Proof: See [3, Lemma 94]. □

Now recall from §3.9.1 that

$$\mathcal{S}^{\mathrm{T,IR}}_{r,s}(\tilde{f},\tilde{j}) = \sum_{q_+ \leqslant q < s} \left(\delta b_{\Delta(0)}\left[\vec{V}^{(r,q)}(\tilde{f},\tilde{j})\right] - \delta b_{\Delta(0)}\left[\vec{V}^{(r,q)}(0,0)\right]\right)$$

where

$$\vec{V}^{(r,q)}(\tilde{f},\tilde{j}) = RG_{\mathrm{ex}}^{q-q_+}\left(\vec{V}^{(r,q_+)}(\tilde{f},\tilde{j})\right) \ .$$

With a view to lighten the notation we write

$$\vec{V}^{(r,q)}(\tilde{f},\tilde{j}) = \vec{V}^{(r,q)}_{\mathrm{bk}} + \dot{V}^{(r,q)}$$

where

$$\vec{V}^{(r,q)}_{\mathrm{bk}} = \vec{V}^{(r,q)}(0,0) = \iota(RG^{q-r}(v)) \in \mathcal{E}_{\mathrm{bk}}$$

and

$$\dot{V}^{(r,q)} = \vec{V}^{(r,q)}(\tilde{f},\tilde{j}) - \vec{V}^{(r,q)}(0,0) \in \mathcal{E}_{\mathrm{pt}} \ .$$

We will control the latter via Proposition 3.9.

First note that

$$||\vec{V}^{((r,q))}_{\mathrm{bk}} - \bar{V}|| = ||RG^{q-r}(v)|| \leqslant \frac{\rho}{3} \ .$$

To make this at most $\frac{1}{40}$ we add the new requirement on ρ:

$$\rho \leqslant \frac{3}{40} \ .$$

If we can ensure that $||\dot{V}^{(r,q_+)}|| \leqslant \mathbf{O}_{10}$ then a trivial inductive use of Proposition 3.9 will imply that

$$||\dot{V}^{(r,q)}|| \leqslant \mathbf{O}_{10} \times \left(\frac{15}{16}\right)^{q-q_+}$$

for all q, such that $q_+ \leqslant q \leqslant s$. We again include the value s although it does not belong to what we called the infrared regime in order to pass the baton to the next section about controlling the boundary term. In view of (3.73), we now impose the new domain condition

$$\left(\mathbf{O}_6 L^{\frac{5}{2}}\right)^{q_+-q_-} \times \max\left\{L^{(3-[\phi])q_-}||\tilde{f}||_{L^\infty}, 11\mathcal{C}_1(\epsilon)\alpha_{\mathrm{u}}^{q_-} Y_2 \bar{g}^{-1} \times ||\tilde{j}||_{L^\infty}\right\} \leqslant \mathbf{O}_{10} \ . \tag{3.85}$$

Now Proposition 3.9 followed by Lemma 3.29 imply that for any q with $q_+ \leqslant q < s$ we have

$$\left|\delta b_{\Delta(0)}\left[\vec{V}^{(r,q)}(\tilde{f},\tilde{j})\right] - \delta b_{\Delta(0)}\left[\vec{V}^{(r,q)}(0,0)\right]\right| \leqslant 4\mathbf{O}_{10} \times \left(\frac{15}{16}\right)^{q-q_+} \ .$$

Hence we get the uniform absolute convergence of the sum over q needed to say

$$\lim_{\substack{r\to-\infty \\ s\to\infty}} \mathcal{S}_{r,s}^{T,IR}(\tilde{f},\tilde{j}) = \mathcal{S}^{T,IR}(\tilde{f},\tilde{j})$$

with

$$\mathcal{S}^{T,IR}(\tilde{f},\tilde{j}) = \sum_{q=q_+}^{\infty}\left(\delta b_{\Delta(0)}[\vec{V}^{(-\infty,q)}(\tilde{f},\tilde{j})] - \delta b_{\Delta(0)}[\iota(v_*)]\right)$$

where

$$\vec{V}^{(-\infty,q)}(\tilde{f},\tilde{j}) = RG_{\mathrm{ex}}^{q-q_-}\left(\vec{V}^{(-\infty,q_-)}(\tilde{f},\tilde{j})\right)$$

and $\vec{V}^{(-\infty,q_-)}(\tilde{f},\tilde{j})$ has been defined in (3.72). The limit $\mathcal{S}^{T,IR}(\tilde{f},\tilde{j})$ is analytic and the order of the $r\to-\infty$, $s\to\infty$ limits is immaterial.

3.9.5 The boundary term

Let $\vec{V} \in \mathcal{E}_{\mathrm{ex}}$ and simply denote by

$$(\beta_4, \beta_3, \beta_2, \beta_1, W_5, W_6, f, R) \in \mathbb{C}^7 \times C_{\mathrm{bd}}^9(\mathbb{R}, \mathbb{C})$$

its component at $\Delta = \Delta(0)$. We let

$$\begin{aligned}\partial\mathcal{Z}[\vec{V}] = \int \mathrm{d}\mu_{C_0}(\phi)\ e^{f\phi} \times \{&\exp\left(-\beta_4 :\phi^4:_{C_0} - \beta_3 :\phi^3:_{C_0} - \beta_2 :\phi^2:_{C_0} - \beta_1 :\phi:_{C_0}\right) \\ &\times(1 + W_5 :\phi^5:_{C_0} + W_6 :\phi^6:_{C_0}) + R(\phi)\}\end{aligned}$$

which reduces to an integral over a single real variable still denoted by ϕ. Let $\partial\mathcal{Z}_* = \partial\mathcal{Z}[\iota(v_*)]$ which is the value at the infrared fixed point. We have

$$\partial\mathcal{Z}_* = \int \mathrm{d}\mu_{C_0}(\phi)\ \left\{\exp\left(-g_* :\phi^4:_{C_0} - \mu_* :\phi^2:_{C_0}\right) + R_*(\phi)\right\}$$

with $g_* = \bar{g} + \delta g_*$. Recall that g_*, μ_*, R_* are real. Note that by Jensen's inequality and the basic properties of Wick ordering one has the lower bound

$$\int \mathrm{d}\mu_{C_0}(\phi)\ \exp\left(-g_* :\phi^4:_{C_0} - \mu_* :\phi^2:_{C_0}\right) \geqslant \exp\left(-\int \mathrm{d}\mu_{C_0}(\phi)\ \left(g_* :\phi^4:_{C_0} + \mu_* :\phi^2:_{C_0}\right)\right) = 1\ .$$

Besides

$$\begin{aligned}\left|\int \mathrm{d}\mu_{C_0}(\phi)\ R_*(\phi)\right| &\leqslant \sup_{\phi\in\mathbb{R}} |R_*(\phi)| \leqslant \sup_{\phi\in\mathbb{R}} ||R_*(\phi)||_{\partial\phi,\phi,h} \\ &\leqslant \bar{g}^{-2}|||R_*|||_{\bar{g}} \leqslant \bar{g}^{\frac{5}{8}}\frac{\rho}{13}\ .\end{aligned}$$

Since $\bar{g} \leqslant 1$ and $\rho < \frac{3}{40}$, we clearly have $\partial\mathcal{Z}_* \geqslant \frac{1}{2}$.

Now if $||\vec{V} - \bar{V}|| < \frac{1}{2}$ it is easy to see that $|\partial\mathcal{Z}[\vec{V}]| \leqslant \mathcal{C}_5(\epsilon)$ with

$$\begin{aligned}\mathcal{C}_5(\epsilon) = &\int \mathrm{d}\mu_{C_0}(\phi)\ e^{\frac{1}{2}L^{3-[\phi]}|\phi|}\times \\ &\left\{\exp\left[-\frac{1}{2}\bar{g}\phi^4 + \frac{3}{4}\bar{g}\left(|\phi|^3 + 13\phi^2 + 7|\phi| + 14\right)\right]\right. \\ &\times\left(1 + \frac{1}{2}\bar{g}^2\left(|\phi|^5 + 20|\phi|^3 + 60|\phi|\right) + \frac{1}{2}\bar{g}^2\left(\phi^6 + 30\phi^4 + 180\phi^2 + 120\right)\right) \\ &\left. + \frac{1}{2}\bar{g}^{\frac{21}{8}-2}\right\}\ .\end{aligned}$$

Indeed, by undoing the Wick ordering

$$-\Re\left[\beta_4 :\phi^4:_{C_0} + \beta_3 :\phi^3:_{C_0} + \beta_2 :\phi^2:_{C_0} + \beta_1 :\phi:_{C_0}\right] = -\bar{g}\phi^4 - Y(\phi)$$

with

$$\begin{aligned}Y(\phi) =& \Re(\beta_4 - \bar{g})\phi^4 \\ &+ (\Re\beta_3)\phi^3 \\ &+ (\Re\beta_2 - 6C_0(0)\Re\beta_4)\phi^2 \\ &+ (\Re\beta_1 - 3C_0(0)\Re\beta_3)\phi \\ &+ (-C_0(0)\Re\beta_2 + 3C_0(0)^2\Re\beta_4)\ .\end{aligned}$$

Using $|\Re(\beta_4 - \bar{g})| < \frac{1}{2}\bar{g}^{\frac{3}{2}} \leqslant \frac{1}{2}\bar{g}$ for the fourth degree monomial and $|\Re\beta_k| \leqslant \frac{3}{2}\bar{g}^{1-\eta}$ for $k = 1,2,3,4$ when bounding the lower degree monomials, and finally using $C_0(0) \leqslant 2$ we obtain

$$|Y(\phi)| \leqslant \frac{1}{2}\bar{g}\phi^4 + \frac{3}{4}\bar{g}\left(|\phi|^3 + 13\phi^2 + 7|\phi| + 14\right)\ .$$

The bounds on $W_k :\phi^k:_{C_0}$, for $k = 5,6$ are similar.

Since $\partial\mathcal{Z}[\vec{V}]$ is clearly analytic in the domain $||\vec{V} - \bar{V}|| < \frac{1}{2}$, Lemma 3.2 with $\nu = \frac{1}{3}$ tell us that for all

$\vec{V}^1$, $\vec{V}^2$ in $\bar{B}\left(\bar{V}, \frac{1}{6}\right)$ one has the Lipschitz estimate

$$|\partial\mathcal{Z}[\vec{V}^1] - \partial\mathcal{Z}[\vec{V}^2]| \leqslant 4\mathcal{C}_5(\epsilon)||\vec{V}^1 - \vec{V}^2|| \ .$$

We now have, using the outcome of the discussion for the infrared regime

$$\begin{array}{rcl} |\partial\mathcal{Z}_{r,s}(\tilde{f},\tilde{j}) - \partial\mathcal{Z}_*| & = & |\partial\mathcal{Z}[\vec{V}^{(r,s)}(\tilde{f},\tilde{j})] - \partial\mathcal{Z}[\iota(v_*)]| \\ & \leqslant & 4\mathcal{C}_5(\epsilon) \times \left[||\vec{V}^{(r,s)}(\tilde{f},\tilde{j}) - \vec{V}^{(r,s)}(0,0)|| + ||\vec{V}^{(r,s)}(0,0) - \iota(v_*)||\right] \\ & \leqslant & 4\mathcal{C}_5(\epsilon) \times \left[||\dot{V}^{(r,s)}|| + ||RG^{s-r}(v) - v_*||\right] \\ & \leqslant & 4\mathcal{C}_5(\epsilon) \times \left[\mathbf{O}_{10} \times \left(\frac{15}{16}\right)^{s-q_+} + c_1(\epsilon)^{s-r}||v - v_*||\right] \ . \end{array}$$

One of course has a similar and simpler estimate for the quantity $\partial\mathcal{Z}_{r,s}(0,0)$ appearing in the denominator of the boundary ratio. Namely, the $\mathbf{O}_{10}$ term is absent. Bounding $c_1(\epsilon)^{s-r}$ by $c_1(\epsilon)^{s-q_+}$ and using the previous lower bound $\partial\mathcal{Z}_* \geqslant \frac{1}{2}$ we see that

$$\frac{\partial\mathcal{Z}_{r,s}(\tilde{f},\tilde{j})}{\partial\mathcal{Z}_{r,s}(0,0)} \longrightarrow 1$$

when $s \to \infty$, uniformly in $r \leqslant q_-$. Therefore the boundary term $\mathcal{S}^{\mathrm{T,BD}}$ disappears when $r \to -\infty$, $s \to \infty$ regardless of the order of limits.

3.10 Construction of the limit measures and invariance properties

As a consequence of what we have shown in the previous section we see that

$$\mathcal{S}_{r,s}(\tilde{f},\tilde{j}) = \exp\left(\mathcal{S}^{\mathrm{T}}_{r,s}(\tilde{f},\tilde{j})\right)$$

converges uniformly to the analytic function

$$\mathcal{S}(\tilde{f},\tilde{j}) = \exp\left(\mathcal{S}^{\mathrm{T}}(\tilde{f},\tilde{j})\right)$$

in a suitable neighborhood of $\tilde{f} = \tilde{j} = 0$ in $S_{q_-,q_+}(\mathbb{Q}_p^3, \mathbb{C})$, when $r \to -\infty$ and $s \to \infty$. Using the multivariate Cauchy formula it is immediate that the cut-off correlators

$$\left\langle \tilde{\phi}(\tilde{f}_1)\cdots\tilde{\phi}(\tilde{f}_n)\ N_r[\tilde{\phi}^2](\tilde{j}_1)\cdots N_r[\tilde{\phi}^2](\tilde{j}_m)\right\rangle_{r,s} =$$

$$\frac{1}{(2i\pi)^{n+m}}\oint\cdots\oint\prod_{j=1}^{n}\frac{\mathrm{d}z_j}{z_j^2}\ \prod_{k=1}^{m}\frac{\mathrm{d}u_k}{u_k^2}\ \mathcal{S}_{r,s}(z_1\tilde{f}_1 + \cdots + z_n\tilde{f}_n, u_1\tilde{j}_1 + \cdots + u_m\tilde{j}_m)$$

converge to the similar integrals with $\mathcal{S}$ instead of $\mathcal{S}_{r,s}$. The contours of integration are governed by the domain condition (3.85). We define our mixed correlators by

$$\left\langle \tilde{\phi}(\tilde{f}_1)\cdots\tilde{\phi}(\tilde{f}_n)\ N[\tilde{\phi}^2](\tilde{j}_1)\cdots N[\tilde{\phi}^2](\tilde{j}_m)\right\rangle =$$

$$\frac{1}{(2i\pi)^{n+m}}\oint\cdots\oint\prod_{j=1}^{n}\frac{\mathrm{d}z_j}{z_j^2}\ \prod_{k=1}^{m}\frac{\mathrm{d}u_k}{u_k^2}\ \mathcal{S}(z_1\tilde{f}_1+\cdots+z_n\tilde{f}_n,u_1\tilde{j}_1+\cdots+u_m\tilde{j}_m)$$

which are multilinear in the $\tilde{f}$'s and $\tilde{j}$'s. Because of the uniform bounds on $\mathcal{S}^{\mathrm{T}}_{r,s}$, and therefore on $\mathcal{S}^{\mathrm{T}}$, proved in the last section and thanks to Cauchy's formula, it is immediate that the pure $\tilde{\phi}$ or $N[\tilde{\phi}^2]$ correlators will satisfy Condition 4) in Theorem 1.11. The other conditions are satisfied by the cut-off correlators $\langle\cdots\rangle_{r,s}$ as joint moments of random variables obtained from the probability measures $\nu_{r,s}$. As these properties are preserved in the limit $r\rightarrow-\infty$ and $s\rightarrow\infty$ we can use Theorem 1.11 to affirm the existence and uniqueness of the measures ν_ϕ and ν_{ϕ^2} mentioned in Theorem 3.1. By the uniqueness part of Theorem 1.11, the invariance properties of the measures ν_ϕ and ν_{ϕ^2} follow from those of the moments. Hence it is enough to show Parts 1) and 2) of Theorem 3.1. These are easier to prove from the functional integral definitions of the cut-off correlators.

Indeed, one can trivially check that for $M\in GL_3(\mathbb{Z}_p)$ one has

$$\left\langle\tilde{\phi}(R_M\tilde{f}_1)\cdots\tilde{\phi}(R_M\tilde{f}_n)\ N_r[\tilde{\phi}^2](R_M\tilde{j}_1)\cdots N_r[\tilde{\phi}^2](R_M\tilde{j}_m)\right\rangle_{r,s}=$$

$$\left\langle\tilde{\phi}(\tilde{f}_1)\cdots\tilde{\phi}(\tilde{f}_n)\ N_r[\tilde{\phi}^2](\tilde{j}_1)\cdots N_r[\tilde{\phi}^2](\tilde{j}_m)\right\rangle_{r,s}$$

because $\mathrm{d}\mu_{C_r}$ is invariant by rotation and the rotation M takes the volume Λ_s to Λ_s.

Also if $y\in\mathbb{Q}_p^3$ with $|y|\leqslant L^s$ then

$$\left\langle\tilde{\phi}(\tau_y\tilde{f}_1)\cdots\tilde{\phi}(\tau_y\tilde{f}_n)\ N_r[\tilde{\phi}^2](\tau_y\tilde{j}_1)\cdots N_r[\tilde{\phi}^2](\tau_y\tilde{j}_m)\right\rangle_{r,s}=$$

$$\left\langle\tilde{\phi}(\tilde{f}_1)\cdots\tilde{\phi}(\tilde{f}_n)\ N_r[\tilde{\phi}^2](\tilde{j}_1)\cdots N_r[\tilde{\phi}^2](\tilde{j}_m)\right\rangle_{r,s}$$

because Λ_s is unchanged by this translation as results from ultrametricity.

Finally, by changing variables from $\tilde{\phi}$ to $\tilde{\phi}_{\rightsquigarrow 1}:=L^{-[\phi]}\tilde{\phi}(L\cdot)$, one has

$$\left\langle\tilde{\phi}(S_L\tilde{f}_1)\cdots\tilde{\phi}(S_L\tilde{f}_n)\ N_r[\tilde{\phi}^2](S_L\tilde{j}_1)\cdots N_r[\tilde{\phi}^2](S_L\tilde{j}_m)\right\rangle_{r,s}=$$

$$\left\langle\tilde{\phi}(\tilde{f}_1)\cdots\tilde{\phi}(\tilde{f}_n)\ N_r[\tilde{\phi}^2](\tilde{j}_1)\cdots N_r[\tilde{\phi}^2](\tilde{j}_m)\right\rangle_{r+1,s+1}\times\left[L^{-(3-[\phi])}\right]^n\times\left[L^{-(3-2[\phi])}Z_2^{-1}\right]^m\ .$$

Noting that $|L|=L^{-1}$ and $Z_2=L^{-\frac{1}{2}\eta_{\phi^2}}$ by definition of η_{ϕ^2}, and from the existence of the $r\rightarrow-\infty$, $s\rightarrow\infty$ limits, we see that the property in Part 3) of Theorem 3.1 holds for $\lambda=L^{-1}$. Thus it holds for the subgroup $L^{\mathbb{Z}}$ it generates.

A trivial consequence of these invariance properties is that

$$\langle N[\tilde{\phi}^2](\tilde{j})\rangle=0$$

identically. Namely, the one-point function vanishes. Indeed, it is enough to show this for $\tilde{j}=\mathbb{1}_{\mathbb{Z}_p^3}$. In that case, by translation invariance followed by scale invariance

$$\langle N[\tilde{\phi}^2](\mathbb{1}_{\mathbb{Z}_p^3})\rangle\ \ =\ \ L^3\langle N[\tilde{\phi}^2](\mathbb{1}_{(L\mathbb{Z}_p)^3})\rangle$$

$$
\begin{aligned}
&= L^3 \times L^{-3+2[\phi]+\frac{1}{2}\eta_{\phi^2}} \times \langle N[\tilde{\phi}^2](\mathbb{1}_{\mathbb{Z}_p^3})\rangle \\
&= L^3 \alpha_{\mathrm{u}}^{-1} \times \langle N[\tilde{\phi}^2](\mathbb{1}_{\mathbb{Z}_p^3})\rangle \ .
\end{aligned}
$$

By Lemma 3.22 it is clear that $L^3\alpha_{\mathrm{u}}^{-1} > 1$ for ϵ small and the vanishing follows.

3.11 Nontriviality and proof of existence of anomalous dimension

3.11.1 The two-point and four-point functions of the elementary field

We have constructed the generalized random field $\tilde{\phi}$ via constructing and proving the analyticity of $\mathcal{S}^{\mathrm{T}}(\tilde{f},0)$, the cumulant generating function. We now show that the process $\tilde{\phi}$ is not Gaussian. In particular we show that in the small ϵ regime one has

$$
\frac{d^4}{dz^4}\Big|_{z=0} \mathcal{S}^{\mathrm{T}}(z\mathbb{1}_{\mathbb{Z}_p^3},0) = \langle \tilde{\phi}(\mathbb{1}_{\mathbb{Z}_p^3})^4\rangle - 3\langle \tilde{\phi}(\mathbb{1}_{\mathbb{Z}_p^3})^2\rangle < 0 \ .
$$

We establish the inequality above by expanding $\mathcal{S}^{\mathrm{T}}(z\mathbb{1}_{\mathbb{Z}_p^3},0)$ and isolating a part that explicitly contains first order pertubation theory. We will calculate the derivative by hand for this explicit part and use Cauchy bounds to estimate the contribution of the remainder. From now on we will drop the tildes from the notation for the fields $\tilde{\phi}$ and $N[\tilde{\phi}^2]$ but we will still use tildes for test functions if needed.

Since $z\mathbb{1}_{\mathbb{Z}_p^3} \in S_{0,0}(\mathbb{Q}_p^3,\mathbb{C})$ we can set $q_- = q_+ = 0$. From section §3.9 and in particular the domain condition (3.85) we know that $\mathcal{S}^T(z\mathbb{1}_{\mathbb{Z}_p^3},0)$ is an analytic function for z such that $|z| < \mathbf{O}_{10}$. This condition is assumed throughout this section. We will repeatedly make use of the fact that for z in this domain $|z| \leqslant 1$ which follows from $\mathbf{O}_{10} \leqslant \frac{1}{40}$. In particular for z in that domain we have

$$
\mathcal{S}^{\mathrm{T}}(z\mathbb{1}_{\mathbb{Z}_p^3},0) = \mathcal{S}^{\mathrm{T,FR}}(z\mathbb{1}_{\mathbb{Z}_p^3},0) + \mathcal{S}^{\mathrm{T,UV}}(z\mathbb{1}_{\mathbb{Z}_p^3},0) + \mathcal{S}^{\mathrm{T,MD}}(z\mathbb{1}_{\mathbb{Z}_p^3},0) + \mathcal{S}^{\mathrm{T,IR}}(z\mathbb{1}_{\mathbb{Z}_p^3},0) \ .
$$

For our choice of test function we have:

$$
\begin{aligned}
\mathcal{S}^{\mathrm{T,FR}}(z\mathbb{1}_{\mathbb{Z}_p^3},0) &= \frac{1}{2}z^2\left(\mathbb{1}_{\mathbb{Z}_p^3}, C_{-\infty}\mathbb{1}_{\mathbb{Z}_p^3}\right) \\
\mathcal{S}^{\mathrm{T,UV}}(z\mathbb{1}_{\mathbb{Z}_p^3},0) &= 0 \text{ since } \tilde{j} = 0 \\
\mathcal{S}^{\mathrm{T,MD}}(z\mathbb{1}_{\mathbb{Z}_p^3},0) &= 0 \text{ since } q_- = q_+ = 0 \\
\mathcal{S}^{\mathrm{T,IR}}(z\mathbb{1}_{\mathbb{Z}_p^3},0) &= \sum_{q=0}^{\infty}\left(\delta b_{\Delta(0)}\left[\vec{V}^{(-\infty,q)}(z\mathbb{1}_{\mathbb{Z}_p^3},0)\right] - \delta b_{\Delta(0)}[\vec{V}_*]\right)
\end{aligned}
$$

where $\vec{V}_* = \iota(v_*) = \vec{V}^{(-\infty,q)}(0,0)$.

By previous considerations we know that up to scale $q_- = 0$ the test function $\tilde{f} = z\mathbb{1}_{\mathbb{Z}_p}$ does not influence the evolution of the other parameters, thus for scales $q \leqslant q_- = 0$ all components of $\vec{V}^{(-\infty,q)}(z\mathbb{1}_{\mathbb{Z}_p^3},0)$ other than the f component take their fixed point value. Additionally we know that for scales $q \geqslant q_+ = 0$ the vector $\vec{V}^{(-\infty,q)}$ deviates from $\vec{V}_*$ only at $\Delta = \Delta(0)$.

We write

$$\vec{V}^{(-\infty,q)}(z\mathbb{1}_{\mathbb{Z}_p^3},0) = \left((\beta_{4,\Delta}^{(q)},\ldots,\beta_{1,\Delta}^{(q)},W_{5,\Delta}^{(q)},W_{6,\Delta}^{(q)},f_{\Delta}^{(q)},R_{\Delta}^{(q)})\right)_{\Delta\in\mathbb{L}} .$$

Keeping our previous observations in mind for $k = 1,2,3,4$ we decompose $\beta_{k,\Delta}^{(q)}$ as follows:

$$\beta_{4,\Delta}^{(q)} = \begin{cases} g_* + \beta_4^{(q,\mathrm{exp})} + \beta_4^{(q,\mathrm{imp})} & \text{if } \Delta = \Delta(0) \\ g_* & \text{if } \Delta \neq \Delta(0) \end{cases}$$

$$\beta_{3,\Delta}^{(q)} = \begin{cases} \beta_3^{(q,\mathrm{exp})} + \beta_3^{(q,\mathrm{imp})} & \text{if } \Delta = \Delta(0) \\ 0 & \text{if } \Delta \neq \Delta(0) \end{cases}$$

$$\beta_{2,\Delta}^{(q)} = \begin{cases} \mu_* + \beta_2^{(q,\mathrm{exp})} + \beta_2^{(q,\mathrm{imp})} & \text{if } \Delta = \Delta(0) \\ \mu_* & \text{if } \Delta \neq \Delta(0) \end{cases}$$

$$\beta_{1,\Delta}^{(q)} = \begin{cases} \beta_1^{(q,\mathrm{exp})} + \beta_1^{(q,\mathrm{imp})} & \text{if } \Delta = \Delta(0) \\ 0 & \text{if } \Delta \neq \Delta(0) \end{cases}$$

Here "exp" and "imp" are abbreviations for explicit and implicit. The quantities $\beta_k^{(q,\mathrm{exp})}$ and $\beta_k^{(q,\mathrm{imp})}$ will be defined inductively starting from $q = 0$. We start with the following intital condition:

$$\text{for } k = 1,2,3,4 \text{ we set } \beta_k^{(0,\mathrm{exp})} = \beta_k^{(0,\mathrm{imp})} = 0.$$

Now we prepare to give the inductive part of the definition. Recall that for $k = 1,2,3,4$ the evolution of our couplings is given by

$$\begin{aligned}\beta_{k,\Delta(0)}^{(q+1)} =& L^{-k[\phi]}\left(\sum_{\Delta\in[L^{-1}\Delta(0)]}\beta_{k,\Delta}^{(q)}\right) - \delta\beta_{k,1,\Delta(0)}\left[\vec{V}^{(-\infty,q)}(z\mathbb{1}_{\mathbb{Z}_p^3},0)\right] \\ &- \delta\beta_{k,2,\Delta(0)}\left[\vec{V}^{(-\infty,q)}(z\mathbb{1}_{\mathbb{Z}_p^3},0)\right] + \xi_{k,\Delta(0)}\left[\vec{V}^{(-\infty,q)}(z\mathbb{1}_{\mathbb{Z}_p^3},0)\right] .\end{aligned}$$

We introduce some more short hand. For $k = 1,2,3,4$ we define β_k^* to be the corresponding component of $\vec{V}_* \in \mathcal{E}_{\mathrm{bk}}$. In particular $\beta_4^* = g_*$, $\beta_3^* = 0$, $\beta_2^* = \mu_*$, and $\beta_1^* = 0$. These are also seen as constant vectors in $\mathbb{C}^{\mathbb{L}}$.

We now use the fact that $\vec{V}_*$ is a fixed point of RG_{ex} to arrive at the following formula:

$$
\begin{aligned}
\beta_{k,\Delta(0)}^{(q+1)} =& \beta_k^* + L^{-k[\phi]}\left(\beta_k^{(q,\exp)} + \beta_k^{(q,\mathrm{imp})}\right) \\
&- \delta\beta_{k,1,\Delta(0)}\left[\vec{V}^{(-\infty,q)}(z\mathbb{1}_{\mathbb{Z}_p^3},0)\right] \\
&+ \left(\delta\beta_{k,2,\Delta(0)}\left[\vec{V}_*\right] - \delta\beta_{k,2,\Delta(0)}\left[\vec{V}^{(-\infty,q)}(z\mathbb{1}_{\mathbb{Z}_p^3},0)\right]\right) \\
&- \left(\xi_{k,\Delta(0)}\left[\vec{V}_*\right] - \xi_{k,\Delta(0)}\left[\vec{V}^{(-\infty,q)}(z\mathbb{1}_{\mathbb{Z}_p^3},0)\right]\right).
\end{aligned}
\tag{3.86}
$$

Above we have used the fact that $\delta b_{k,1,\Delta}\left[\vec{V}_*\right] = 0$. We now decompose $\delta\beta_{k,1,\Delta(0)}\left[\vec{V}^{(-\infty,q)}(z\mathbb{1}_{\mathbb{Z}_p^3},0)\right]$. For $0 \leqslant k < l \leqslant 4$ and $\beta, f \in \mathbb{C}^{\mathbb{L}}$ define

$$
F_{k,l}\left[\beta,f\right] = \binom{l}{k}\int_{(L^{-1}\Delta(0))^{l-k}} \mathrm{d}^3a\ \mathrm{d}^3b_1\cdots\mathrm{d}^3b_{l-k}\ \beta(a)\times\prod_{i=1}^{l-k}\left[\Gamma(a-b_i)f(b_i)\right].
$$

With this notation we have:

$$
\delta\beta_{k,1,\Delta(0)}\left[\vec{V}^{(-\infty,q)}(z\mathbb{1}_{\mathbb{Z}_p^3},0)\right] = -\sum_{l=k+1}^{4} L^{-k[\phi]}F_{k,l}\left[\beta_l^{(q)}, f^{(q)}\right].
$$

We define the evolution for $\beta_k^{(q,\exp)}$ and $\beta_k^{(q,\mathrm{imp})}$ as follows:

$$
\beta_k^{(q+1),\exp} = L^{-k[\phi]}\beta_k^{(q,\exp)} + \sum_{l=k+1}^{4} L^{-k[\phi]}F_{k,l}\left[\beta_l^* + \beta_l^{(q,\exp)}\mathbb{1}_{\Delta(0)}, f^{(q)}\right] \tag{3.87}
$$

$$
\begin{aligned}
\beta_k^{(q+1),\mathrm{imp}} &= L^{-k[\phi]}\beta_k^{(q,\mathrm{imp})} + \sum_{l=k+1}^{4} L^{-k[\phi]}F_{k,l}\left[\beta_l^{(q,\mathrm{imp})}\mathbb{1}_{\Delta(0)}, f^{(q)}\right] \\
&+ \left(\delta\beta_{k,2,\Delta(0)}[\vec{V}_*] - \delta\beta_{k,2,\Delta(0)}[\vec{V}^{(-\infty,q)}(z\mathbb{1}_{\mathbb{Z}_p^3},0)]\right) + \left(\xi_{k,\Delta(0)}[\vec{V}_*] - \xi_{k,\Delta(0)}[\vec{V}^{(-\infty,q)}(z\mathbb{1}_{\mathbb{Z}_p^3},0)]\right) .
\end{aligned}
\tag{3.88}
$$

Here we have designated $\mathbb{1}_{\Delta(0)} : \mathbb{L} \to \mathbb{C}$ as the indicator function of $\{\Delta(0)\}$.

We also impose a splitting of the difference of vacuum renormalizations at $\Delta(0)$. For $q \geqslant 0$ we have:

$$
\delta b_{\Delta(0)}\left[\vec{V}^{(-\infty,q)}(z\mathbb{1}_{\mathbb{Z}_p^3},0)\right] - \delta b_{\Delta(0)}[\vec{V}_*] = \delta b^{(q,\exp)} + \delta b^{(q,\mathrm{imp})} .
$$

We define

$$
\delta b^{(q,\exp)} = -\sum_{l=1}^{4} F_{0,l}\left[\beta_l^* + \beta_l^{(q,\exp)}\mathbb{1}_{\Delta(0)}, f^{(q)}\right] , \tag{3.89}
$$

$$
\begin{aligned}
\delta b^{(q,\mathrm{imp})} = & -\sum_{l=1}^{4} F_{0,l}\left[\beta_l^{(q,\mathrm{imp})}\mathbb{1}_{\Delta(0)}, f^{(q)}\right] \\
& + \left(\delta\beta_{0,2,\Delta(0)}[\vec{V}^{(-\infty,q)}(z\mathbb{1}_{\mathbb{Z}_p^3},0)] - \delta\beta_{0,2,\Delta(0)}[\vec{V}_*]\right) \\
& + \left(\xi_{0,\Delta(0)}[\vec{V}^{(-\infty,q)}(z\mathbb{1}_{\mathbb{Z}_p^3},0)] - \xi_{0,\Delta(0)}[\vec{V}_*]\right) \ .
\end{aligned} \tag{3.90}
$$

We now derive explicit formulas for $\beta_k^{(q,\exp)}$ and $\delta b^{(q,\exp)}$.

Lemma 3.30. *Given the previous inductive definitions for $\beta_k^{(q,\exp)}$ for $q \geqslant 0$ and $k = 1,2,3,4$ we have the following explicit formulas:*

$$
\begin{aligned}
\beta_4^{(q,\exp)} =& 0 \\
\beta_3^{(q,\exp)} =& 0 \\
\beta_2^{(q,\exp)} =& 6qL^{-2q[\phi]}z^2 g_* ||\Gamma||_{L^2}^2 \\
\beta_1^{(q,\exp)} =& z^3 g_* L^{-q[\phi]}\left[4\frac{1-L^{-2q[\phi]}}{1-L^{-2[\phi]}}\left(\int_{\mathbb{Q}_p^3} \mathrm{d}^3x\ \Gamma(x)^3\right) + 12\left(\sum_{n=0}^{q-1} nL^{-2n[\phi]}\right)||\Gamma||_{L^2}^2 \times \Gamma(0)\right] \ .
\end{aligned}
$$

For $q \geqslant 0$ we also have

$$
\begin{aligned}
\delta b^{(q,\exp)} = & -z^4 g_*\Bigg[L^{-4q[\phi]}\left(\int_{\mathbb{Q}_p^3}\mathrm{d}^3x\ \Gamma(x)^4\right) + 6L^{-4q[\phi]}q||\Gamma||_{L^2}^2\Gamma(0)^2 + 12L^{-2q[\phi]}\left(\sum_{n=0}^{q-1} nL^{-2n[\phi]}\right)||\Gamma||_{L^2}^2\Gamma(0)^2 \\
& + 4L^{-2q[\phi]}\frac{1-L^{-2q[\phi]}}{1-L^{-2[\phi]}}\Gamma(0)\left(\int_{\mathbb{Q}_p^3}\mathrm{d}^3x\ \Gamma(x)^3\right)\Bigg] - z^2\mu_* L^{-2[\phi]q}||\Gamma||_{L^2}^2 \ .
\end{aligned}
$$

Proof: We first note that below one often sees expressions of the form $\int_{L^{-1}\Delta(0)} \Gamma(x)^n$. In the statement of the theorem we extended the integration to all of $\mathbb{Q}_p^3$, we can do this since Γ is supported on $L^{-1}\Delta(0)$.

For $\beta_4^{(q,\exp)}$ the result is immediate after recalling that $\beta_4^{(0,\exp)} = 0$ and noticing the evolution for this parameter reduces to multiplication by $L^{-4[\phi]}$.

For $\beta_3^{(q,\exp)}$ we have

$$\begin{aligned}\beta_3^{(q,\exp)} &= \sum_{n=0}^{q-1} L^{-3[\phi](q-n)} F_{3,4}[\beta_4^* + \beta_4^{(n,\exp)} \mathbb{1}_{\Delta(0)}, f^{(n)}] \\ &= \sum_{n=0}^{q-1} L^{-3[\phi](q-n)} F_{3,4}[g_*, f^{(n)}] \\ &= \sum_{n=0}^{q-1} 0 \ .\end{aligned}$$

The last line follows from ultrametricity and the fact that Γ integrates to 0. In particular $F_{j,j+1}\left[\beta_j^*, f^{(\cdot)}\right]$ will always vanish.

For $\beta_2^{(q,\exp)}$ we have

$$\begin{aligned}\beta_2^{(q,\exp)} &= \sum_{n=0}^{q-1} L^{-2(q-n)[\phi]} \left(F_{2,4}\left[\beta_4^* + \beta_4^{(n,\exp)} \mathbb{1}_{\Delta(0)}, f^{(n)}\right] + F_{2,3}\left[\beta_3^* + \beta_3^{(n,\exp)} \mathbb{1}_{\Delta(0)}, f^{(n)}\right]\right) \\ &= \sum_{n=0}^{q-1} L^{-2(q-n)[\phi]} F_{2,4}\left[g_*, f^{(n)}\right] \\ &= \sum_{n=0}^{q-1} L^{-2(q-n)[\phi]} 6 \left(\int_{(L^{-1}\Delta(0))^3} \mathrm{d}^3 a\ \mathrm{d}^3 b_1\ \mathrm{d}^3 b_2\ g_* \prod_{i=1,2} \left[\Gamma(a-b_i) L^{-n[\phi]} z \mathbb{1}_{\mathbb{Z}_p}(b_i)\right]\right) \\ &= \sum_{n=0}^{q-1} L^{-2(q-n)[\phi]} 6 z^2 g_* L^{-2n[\phi]} \left(\int_{L^{-1}\Delta(0)} \mathrm{d}^3 a\ \Gamma(a)^2\right) \\ &= \sum_{n=0}^{q-1} L^{-2q[\phi]} 6 z^2 g_* ||\Gamma||_{L^2}^2\end{aligned}$$

from which the formula for $\beta_2^{(q,\exp)}$ follows. Note that above we used the fact that $f^{(n)} = L^{-n[\phi]} \mathbb{1}_{\Delta(0)}$ as a vector in $\mathbb{C}^{\mathbb{L}}$ or $L^{-n[\phi]} \mathbb{1}_{\mathbb{Z}_p^3}$ as function on $\mathbb{Q}_p^3$.

For $\beta_1^{(q,\exp)}$ we have

$$\begin{aligned}\beta_1^{(q,\exp)} &= \sum_{n=0}^{q-1} L^{-(q-n)[\phi]} \left(F_{1,4}\left[\beta_4^* + \beta_4^{(n,\exp)} \mathbb{1}_{\Delta(0)}, f^{(n)}\right] + F_{1,3}\left[\beta_3^* + \beta_3^{(n,\exp)} \mathbb{1}_{\Delta(0)}, f^{(n)}\right]\right. \\ &\qquad \left. +\ F_{1,2}\left[\beta_2^* + \beta_2^{(n,\exp)} \mathbb{1}_{\Delta(0)}, f^{(n)}\right]\right) \\ &= \sum_{n=0}^{q-1} L^{-(q-n)[\phi]} \left(F_{1,4}\left[g_*, f^{(n)}\right] + F_{1,2}\left[\mu_* + \beta_2^{(n,\exp)} \mathbb{1}_{\Delta(0)}, f^{(n)}\right]\right) .\end{aligned}$$

Looking at the terms involved one sees

$$F_{1,4}\left[g_*, f^{(n)}\right] = 4g_* z^3 L^{-3n[\phi]}\left(\int_{L^{-1}\Delta(0)} \mathrm{d}^3x\ \Gamma(x)^3\right)$$

and

$$\begin{aligned}F_{1,2}\left[\mu_* + \beta_2^{(n,\exp)}\mathbb{1}_{\Delta(0)}, f^{(n)}\right] =& F_{1,2}\left[\mu_*, f^{(n)}\right] + F_{1,2}\left[\beta_2^{(n,\exp)}\mathbb{1}_{\Delta(0)}, f^{(n)}\right]\\ =& F_{1,2}\left[\beta_2^{(n,\exp)}\mathbb{1}_{\Delta(0)}, f^{(n)}\right]\\ =& 2L^{-n[\phi]}z\Gamma(0) \times \left(6nL^{-2n[\phi]}z^2 g_* ||\Gamma||^2_{L^2}\right).\end{aligned}$$

The formula for $\beta_1^{(q,\exp)}$ then follows.

We now move on to $\delta b^{(q,\exp)}$. To keep things lighter we have left out terms with a vanishing contribution:

$$\delta b^{(q,\exp)} = -F_{0,4}\left[g_*, f^{(q)}\right] - F_{0,2}\left[\beta_2^{(q,\exp)}\mathbb{1}_{\Delta(0)}, f^{(q)}\right] - F_{0,2}\left[\mu_*, f^{(q)}\right] - F_{0,1}\left[\beta_1^{(q,\exp)}\mathbb{1}_{\Delta(0)}, f^{(q)}\right] \ .$$

We calculate each of the terms appearing above:

$$\begin{aligned}F_{0,4}\left[g_*, f^{(q)}\right] =& z^4 g_* L^{-4q[\phi]}\left(\int_{L^{-1}\Delta(0)} \mathrm{d}^3x\ \Gamma(x)^4\right)\\ F_{0,2}\left[\beta_2^{(q,\exp)}\mathbb{1}_{\Delta(0)}, f^{(q)}\right] =& z^2 L^{-2q[\phi]}\Gamma(0)^2 \times \left[6qL^{-2q[\phi]}z^2 g_* ||\Gamma||^2_{L^2}\right]\\ F_{0,2}\left[\mu_*, f^{(q)}\right] =& z^2 L^{-2q[\phi]}||\Gamma||^2_{L^2}\mu_*\\ F_{0,1}\left[\beta_1^{(q,\exp)}\mathbb{1}_{\Delta(0)}, f^{(q)}\right] =& zL^{-q[\phi]}\Gamma(0) \times \left\{z^3 g_* L^{-q[\phi]}\left[4\frac{1-L^{-2q[\phi]}}{1-L^{-2[\phi]}}\left(\int_{\mathbb{Q}_p^3} \mathrm{d}^3x\ \Gamma(x)^3\right)\right.\right.\\ &\left.\left. +12\left(\sum_{n=0}^{q-1} nL^{-2[\phi]n}\right)||\Gamma||^2_{L^2} \times \Gamma(0)\right]\right\} \ .\end{aligned}$$

This proves the formula for $\delta b^{(q,\exp)}$.

□

We now calculate running bounds for the $\beta_k^{(q,\mathrm{imp})}$.

Lemma 3.31. *In the small ϵ regime one has the following bounds for $q \geqslant 0$*

$$|\beta_4^{(q,\mathrm{imp})}| \leqslant \mathbf{O}_{11} \times q \times L^8 \bar{g}^2 \left(\frac{15}{16}\right)^q$$

$$|\beta_3^{(q,\mathrm{imp})}| \leqslant 17 \times \mathbf{O}_{11} \times q \times L^8 \bar{g}^2 \left(\frac{15}{16}\right)^q$$

$$|\beta_2^{(q,\mathrm{imp})}| \leqslant 253 \times \mathbf{O}_{11} \times q \times L^8 \bar{g}^2 \left(\frac{15}{16}\right)^q$$

$$|\beta_1^{(q,\mathrm{imp})}| \leqslant 2497 \times \mathbf{O}_{11} \times q \times L^8 \bar{g}^2 \left(\frac{15}{16}\right)^q$$

$$|\delta b^{(q,\mathrm{imp})}| \leqslant \mathbf{O}_{12} \times L^8 \times \bar{g}^2 \left(\frac{15}{16}\right)^q$$

where $\mathbf{O}_{11} = (4\mathcal{O}_{26} + 1)$ *and* $\mathbf{O}_{12} = 319617 \times \mathbf{O}_{11}$.

Proof: We note that for all $q \geqslant 0$ one has $\vec{V}^{(-\infty,q)}(z\mathbb{1}_{\mathbb{Z}_p^3}, 0), \vec{V}_* \in \bar{B}(0, \frac{1}{6})$. Thus by the proof of [3, Lemma 89] we have the following bounds for all $q \geqslant 0$ and for $k = 0, 1, 2, 3, 4$.

$$\begin{aligned}
\left|\delta\beta_{k,2,\Delta(0)}\left[\vec{V}_*\right] - \delta\beta_{k,2,\Delta(0)}\left[\vec{V}^{(-\infty,q)}(z\mathbb{1}_{\mathbb{Z}_p^3}, 0)\right]\right| &\leqslant 4\mathcal{O}_{26} L^5 \bar{g}^2 ||\vec{V}^{(-\infty,q)}(z\mathbb{1}_{\mathbb{Z}_p^3}, 0) - \vec{V}_*|| \\
\left|\xi_{k,\Delta(0)}\left[\vec{V}_*\right] - \xi_{k,\Delta(0)}\left[\vec{V}^{(-\infty,q)}(z\mathbb{1}_{\mathbb{Z}_p^3}, 0)\right]\right| &\leqslant 2B_k \bar{g}^{\frac{21}{8}} ||\vec{V}^{(-\infty,q)}(z\mathbb{1}_{\mathbb{Z}_p^3}, 0) - \vec{V}_*|| \ .
\end{aligned}$$

We also note that by applying the bound of Proposition 3.9 q-times one has:

$$\begin{aligned}
||\vec{V}^{(-\infty,q)}(z\mathbb{1}_{\mathbb{Z}_p^3}, 0) - \vec{V}_*|| &= ||\vec{V}^{(-\infty,q)}(z\mathbb{1}_{\mathbb{Z}_p^3}, 0) - \vec{V}^{(-\infty,q)}(0,0)|| \\
&\leqslant \left(\frac{15}{16}\right)^q ||\vec{V}^{(-\infty,0)}(z\mathbb{1}_{\mathbb{Z}_p^3}, 0) - \vec{V}^{(-\infty,0)}(0,0)|| \\
&\leqslant \left(\frac{15}{16}\right)^q \ .
\end{aligned}$$

Now in the ϵ small regime one has:

$$\begin{aligned}
&\left|\delta\beta_{k,2,\Delta(0)}[\vec{V}_*] - \delta\beta_{k,2,\Delta(0)}[\vec{V}^{(-\infty,q)}(z\mathbb{1}_{\mathbb{Z}_p^3}, 0)]\right| + \left|\xi_{k,\Delta(0)}[\vec{V}_*] - \xi_{k,\Delta(0)}[\vec{V}^{(-\infty,q)}(z\mathbb{1}_{\mathbb{Z}_p^3}, 0)]\right| \\
&\leqslant (4\mathcal{O}_{26} + 1)\, L^5 \bar{g}^2 ||\vec{V}^{(-\infty,q)}(z\mathbb{1}_{\mathbb{Z}_p^3}, 0) - \vec{V}_*|| \\
&= \mathbf{O}_{11} L^5 \bar{g}^2 ||\vec{V}^{(-\infty,q)}(z\mathbb{1}_{\mathbb{Z}_p^3}, 0) - \vec{V}_*|| \\
&\leqslant \mathbf{O}_{11} L^5 \bar{g}^2 \left(\frac{15}{16}\right)^q \ .
\end{aligned}$$

We start with estimating $\beta_4^{(q,\mathrm{imp})}$:

$$
\begin{aligned}
|\beta_4^{(q,\mathrm{imp})}| \leqslant & L^{4[\phi]} \sum_{n=0}^{q-1} L^{-4(q-n)[\phi]} \left(\left| \delta\beta_{4,2,\Delta(0)}[\vec{V}_*] - \delta\beta_{4,2,\Delta(0)}[\vec{V}^{(-\infty,n)}(z\mathbb{1}_{\mathbb{Z}_p^3},0)] \right| \right. \\
& \left. + \left| \xi_{4,\Delta(0)}[\vec{V}_*] - \xi_{4,\Delta(0)}[\vec{V}^{(-\infty,n)}(z\mathbb{1}_{\mathbb{Z}_p^3},0)] \right| \right) \\
\leqslant & L^{4[\phi]} \mathbf{O}_{11} \times L^5 \times \bar{g}^2 \sum_{n=0}^{q-1} L^{-4(q-n)[\phi]} \left(\frac{15}{16} \right)^n \\
\leqslant & L^{4[\phi]} \mathbf{O}_{11} \times L^5 \bar{g}^2 \sum_{n=0}^{q-1} \left(\frac{15}{16} \right)^{(q-n)} \left(\frac{15}{16} \right)^n \\
\leqslant & \mathbf{O}_{11} \times q L^{5+4[\phi]} \bar{g}^2 \left(\frac{15}{16} \right)^q .
\end{aligned}
$$

In going to the second to last line we used the fact that for $L \geqslant 2$ and $\epsilon \leqslant 1$ we have the following inequality : $L^{-4[\phi]} \leqslant L^{-[\phi]} \leqslant 2^{-\frac{1}{2}} < \left(\frac{15}{16}\right)$. Then by bounding $L^{5+4[\phi]} \leqslant L^8$ we get the desired bound for $\beta_4^{(q,\mathrm{imp})}$.

For $\beta_3^{(q,\mathrm{imp})}$ we have

$$
\begin{aligned}
|\beta_3^{(q,\mathrm{imp})}| \leqslant & L^{3[\phi]} \left[\sum_{n=0}^{q-1} L^{-3(q-n)[\phi]} \left(\left| \delta\beta_{3,2,\Delta(0)}[\vec{V}_*] - \delta\beta_{3,2,\Delta(0)}[\vec{V}^{(-\infty,n)}(z\mathbb{1}_{\mathbb{Z}_p^3},0)] \right| \right. \right. \\
& \left. \left. + \left| \xi_{3,\Delta(0)}[\vec{V}_*] - \xi_{3,\Delta(0)}[\vec{V}^{(-\infty,n)}(z\mathbb{1}_{\mathbb{Z}_p^3},0)] \right| \right) \right] \\
& + \left[\sum_{n=0}^{q-1} L^{-3(q-n)[\phi]} \left| F_{3,4}\left[\beta_4^{(n,\mathrm{imp})} \mathbb{1}_{\Delta(0)}, f^{(n)} \right] \right| \right] \\
\leqslant & L^{3[\phi]} \mathbf{O}_{11} L^5 \bar{g}^2 \left[\sum_{n=0}^{q-1} L^{-3(q-n)[\phi]} \left(\frac{15}{16} \right)^n \right] + \left[\sum_{n=0}^{q-1} L^{-3(q-n)[\phi]} \left| F_{3,4}\left[\beta_4^{(n,\mathrm{imp})} \mathbb{1}_{\Delta(0)}, f^{(n)} \right] \right| \right] \\
\leqslant & \mathbf{O}_{11} q L^{5+3[\phi]} \bar{g}^2 \left(\frac{15}{16} \right)^q + \left[\sum_{n=0}^{q-1} L^{-3(q-n)[\phi]} \left| F_{3,4}\left[\beta_4^{(n,\mathrm{imp})} \mathbb{1}_{\Delta(0)}, f^{(n)} \right] \right| \right] .
\end{aligned}
$$

In the above expressions the first term was bounded just as it was for $\beta_4^{(q,\mathrm{imp})}$. We now try to estimate the summands appearing inside of the second term. We will use $|z| \leqslant 1$.

$$
\begin{aligned}
\left| F_{3,4}\left[\beta_4^{(n,\mathrm{imp})} \mathbb{1}_{\Delta(0)}, f^{(n)} \right] \right| \leqslant & 4 L^{-n[\phi]} |\beta_4^{(n,\mathrm{imp})}| \times |\Gamma(0)| \\
\leqslant & 8 L^{-n[\phi]} \mathbf{O}_{11} n L^8 \bar{g}^2 \left(\frac{15}{16} \right)^n \\
\leqslant & 16 \mathbf{O}_{11} L^8 \bar{g}^2 \left(\frac{15}{16} \right)^n .
\end{aligned}
$$

In going to the second to last line we used the bound $|\Gamma(0)| \leqslant ||\Gamma||_{L^\infty} \leqslant 2$. In going to the last line note that for $\epsilon \leqslant 1$ and $L \geqslant 2$ one has $nL^{-n[\phi]} \leqslant n2^{-\frac{n}{2}} \leqslant \dfrac{2}{e \times \log(2)} \leqslant 2$. Inserting this into our previous inequality gives us

$$
\begin{aligned}
|\beta_3^{(q,\mathrm{imp})}| \leqslant & \mathbf{O}_{11} q L^{5+3[\phi]} \bar{g}^2 \left(\frac{15}{16}\right)^q + 16 \mathbf{O}_{11} L^8 \bar{g}^2 \sum_{n=0}^{q-1} \left[L^{-3(q-n)[\phi]} \left(\frac{15}{16}\right)^n \right] \\
\leqslant & \mathbf{O}_{11} q L^{5+3[\phi]} \bar{g}^2 \left(\frac{15}{16}\right)^q + 16 \mathbf{O}_{11} q L^8 \bar{g}^2 \left(\frac{15}{16}\right)^q \\
\leqslant & 17 \mathbf{O}_{11} q L^8 \bar{g}^2 \left(\frac{15}{16}\right)^q .
\end{aligned}
$$

Not that in going to the second line we used the bound $L^{-3(q-n)[\phi]} \leqslant \left(\frac{15}{16}\right)^{(q-n)}$.

We start on $\beta_2^{(q,\mathrm{imp})}$ by making the following estimates:

$$
\begin{aligned}
\left| F_{2,4}\left[\beta_4^{(n,\mathrm{imp})} \mathbb{1}_{\Delta(0)}, f^{(n)}\right] \right| \leqslant & 6 \times \left| \beta_4^{(n,\mathrm{imp})} \right| \times \Gamma(0)^2 \times L^{-2n[\phi]} \\
\leqslant & 24 \times \mathbf{O}_{11} n L^8 \bar{g}^2 \left(\frac{15}{16}\right)^n L^{-2n[\phi]} \\
\leqslant & 48 \times \mathbf{O}_{11} L^8 \bar{g}^2 \left(\frac{15}{16}\right)^n .
\end{aligned}
$$

Similarly one gets the bound

$$
\left| F_{2,3}\left[\beta_3^{(n,\mathrm{imp})} \mathbb{1}_{\Delta(0)}, f^{(n)}\right] \right| \leqslant 204 \times \mathbf{O}_{11} L^6 \bar{g}^2 \left(\frac{15}{16}\right)^n .
$$

The bound for $\beta_2^{(q,\mathrm{imp})}$ then proceeds along familiar lines. One uses the same arguments to prove the estimate for $\beta_1^{(q,\mathrm{imp})}$. In particular

$$
\begin{aligned}
\left| F_{1,4}\left[\beta_4^{(n,\mathrm{imp})} \mathbb{1}_{\Delta(0)}, f^{(n)}\right] \right| \leqslant & 64 \times \mathbf{O}_{11} L^8 \bar{g}^2 \left(\frac{15}{16}\right)^n \\
\left| F_{1,3}\left[\beta_3^{(n,\mathrm{imp})} \mathbb{1}_{\Delta(0)}, f^{(n)}\right] \right| \leqslant & 408 \times \mathbf{O}_{11} L^8 \bar{g}^2 \left(\frac{15}{16}\right)^n \\
\left| F_{1,2}\left[\beta_2^{(n,\mathrm{imp})} \mathbb{1}_{\Delta(0)}, f^{(n)}\right] \right| \leqslant & 2024 \times \mathbf{O}_{11} L^8 \bar{g}^2 \left(\frac{15}{16}\right)^n .
\end{aligned}
$$

To bound $\delta b^{(q,\mathrm{imp})}$ we first make the following estimate. For $k = 1, 2, 3, 4$ one has:

$$\left|F_{0,k}\left[\beta_k^{(q,\mathrm{imp})}\mathbb{1}_{\Delta(0)}, f^{(q)}\right]\right| \leqslant L^{-kq[\phi]} \times \Gamma(0)^k \times |\beta_k^{(q,\mathrm{imp})}|$$
$$\leqslant L^{-q[\phi]} \times 2^4 \times 2497 \times \mathbf{O}_{11} \times qL^8\bar{g}^2\left(\frac{15}{16}\right)^q$$
$$\leqslant 79904 \times \mathbf{O}_{11}L^8\bar{g}^2\left(\frac{15}{16}\right)^q .$$

We then have

$$|\delta b^{(q,\mathrm{imp})}| \leqslant \left|\delta\beta_{0,2,\Delta(0)}[V_*] - \delta\beta_{0,2,\Delta(0)}[\vec{V}^{(-\infty,q)}(z\mathbb{1}_{\mathbb{Z}_p^3},0)]\right|$$
$$+ \left|\xi_{0,\Delta(0)}[V_*] - \xi_{0,\Delta(0)}[\vec{V}^{(-\infty,q)}(z\mathbb{1}_{\mathbb{Z}_p^3},0)]\right|$$
$$+ \left[\sum_{k=1}^{4}\left|F_{0,k}\left[\beta_k^{(q,\mathrm{imp})}\mathbb{1}_{\Delta(0)}, f^{(q)}\right]\right|\right]$$
$$\leqslant \mathbf{O}_{11} \times L^5\bar{g}^2\left(\frac{15}{16}\right)^q + 4 \times 79904 \times \mathbf{O}_{11} \times L^8\bar{g}^2\left(\frac{15}{16}\right)^q$$
$$\leqslant 319617 \times \mathbf{O}_{11} \times L^8\bar{g}^2\left(\frac{15}{16}\right)^q .$$

This gives the desired bound. □

Lemma 3.32. *In the ϵ small regime and on the domain $\{z \in \mathbb{C} \mid |z| < \mathbf{O}_{10}\}$ one has the decomposition*

$$\mathcal{S}^{\mathrm{T}}(z\mathbb{1}_{\mathbb{Z}_p^3},0) = S^{\mathrm{T,exp}}(z) + S^{\mathrm{T,imp}}(z) .$$

All three of the above functions are analytic on the above domain. Additionally, over this domain one has the following explicit formula

$$\mathcal{S}^{\mathrm{T,exp}}(z) = -\sum_{q=0}^{\infty}\Bigg\{z^4 g_*\Bigg[L^{-4q[\phi]}\left(\int_{\mathbb{Q}_p^3}\mathrm{d}^3x\ \Gamma(x)^4\right) + 6L^{-4q[\phi]}q||\Gamma||_{L^2}^2 \times \Gamma(0)^2$$
$$+ 12L^{-2q[\phi]}\left(\sum_{n=0}^{q-1} nL^{-2n[\phi]}\right)||\Gamma||_{L^2}^2 \times \Gamma(0)^2$$
$$+ 4L^{-2q[\phi]}\frac{1-L^{-2q[\phi]}}{1-L^{-2[\phi]}}\Gamma(0)\left(\int_{\mathbb{Q}_p^3}\mathrm{d}^3x\ \Gamma(x)^3\right)\Bigg] + z^2\mu_* L^{-2q[\phi]}||\Gamma||_{L^2}^2\Bigg\}$$
$$+ \frac{z^2}{2}\left(\mathbb{1}_{\mathbb{Z}_p^3}, C_{-\infty}\mathbb{1}_{\mathbb{Z}_p^3}\right)$$

and the following uniform bound

$$|\mathcal{S}^{\mathrm{T,imp}}(z)| \leqslant \mathbf{O}_{13}L^8\bar{g}^2.$$

where $\mathbf{O}_{13} = 16 \times \mathbf{O}_{12}$.

Proof: From earlier definitions we have that

$$\mathcal{S}^{\mathrm{T}}(z\mathbb{1}_{\mathbb{Z}_p^3}, 0) = \frac{z^2}{2}\left(\mathbb{1}_{\mathbb{Z}_p^3}, C_{-\infty}\mathbb{1}_{\mathbb{Z}_p^3}\right) + \sum_{q=0}^{\infty}\left(\delta b^{(q,\exp)} + \delta b^{(q,\mathrm{imp})}\right) .$$

We define

$$\mathcal{S}^{\mathrm{T},\exp}(z) = \frac{z^2}{2}\left(\mathbb{1}_{\mathbb{Z}_p^3}, C_{-\infty}\mathbb{1}_{\mathbb{Z}_p^3}\right) + \sum_{q=0}^{\infty}\delta b^{(q,\exp)} .$$

$$\mathcal{S}^{\mathrm{T},\mathrm{imp}}(z) = \sum_{q=0}^{\infty}\delta b^{(q,\mathrm{imp})} .$$

The explicit formula given for $\mathcal{S}^{\mathrm{T},\exp}(z)$ comes from substitution of the explicit formula for the $\delta b^{(q,\exp)}$ from Lemma 3.30. Since $[\phi] > 0$ for $\epsilon \in (0,1]$ it is not hard to see that the infinite sum in the expression for $\mathcal{S}^{\mathrm{T},\exp}(z)$ is uniformly absolutely summable on our domain. Analyticity follows from the explicit formula.

On the other hand we have

$$\begin{aligned}|\mathcal{S}^{\mathrm{T},\mathrm{imp}}(z)| \leqslant& \sum_{q=0}^{\infty}|\delta b^{(q,\mathrm{imp})}|\\ \leqslant& \mathbf{O}_{12} \times L^6 \times \bar{g}^2 \sum_{q=0}^{\infty}\left(\frac{15}{16}\right)^q\\ \leqslant& 16 \times \mathbf{O}_{12} \times L^8 \times \bar{g}^2 .\end{aligned}$$

We have then proved the desired uniform bound and we have uniform absolute convergence yielding analyticity as well. □

Lemma 3.33. *In the small* ϵ *regime one has*

$$\left|\frac{d^2}{dz^2}\Big|_{z=0}\mathcal{S}^{\mathrm{T}}(z\mathbb{1}_{\mathbb{Z}_p^3}, 0) - U_2\right| \leqslant \mathbf{O}_{14}L^8\bar{g}^2$$

where

$$U_2 = \left(\mathbb{1}_{\mathbb{Z}_p^3}, C_{-\infty}\mathbb{1}_{\mathbb{Z}_p^3}\right) - 2||\Gamma||_{L^2}^2 \times \frac{1}{1 - L^{-2[\phi]}} \times \mu_*$$

and

$$\left|\frac{d^4}{dz^4}\Big|_{z=0}\mathcal{S}^{\mathrm{T}}(z\mathbb{1}_{\mathbb{Z}_p^3}, 0) - U_4\right| \leqslant \mathbf{O}_{15}L^8\bar{g}^2$$

where

$$U_4 = -24 g_* \sum_{q=0}^{\infty} \left[L^{-4q[\phi]} \left(\int_{\mathbb{Q}_p^3} \mathrm{d}^3x\ \Gamma(x)^4 \right) + 6L^{-4q[\phi]} q ||\Gamma||_{L^2}^2 \Gamma(0)^2 + 12 L^{-2q[\phi]} \left(\sum_{n=0}^{q-1} n L^{-2n[\phi]} \right) ||\Gamma||_{L^2}^2 \Gamma(0)^2 \right.$$
$$\left. + 4L^{-2q[\phi]} \frac{1-L^{-2q[\phi]}}{1-L^{-2[\phi]}} \Gamma(0) \left(\int_{\mathbb{Q}_p^3} \mathrm{d}^3x\ \Gamma(x)^3 \right) \right] .$$

Here we have used the following numerical constants: $\mathbf{O}_{14} = 8 \times \mathbf{O}_{10}^{-2} \mathbf{O}_{13}$ *and* $\mathbf{O}_{15} = 384 \times \mathbf{O}_{10}^{-4} \mathbf{O}_{13}$.

Proof: We note that for $j = 2, 4$ we have that $U_j = \frac{d^j}{dz^j}\Big|_{z=0} \mathcal{S}^{\mathrm{T,exp}}(z)$.

By the previous lemma the bounds above will follow if we have the necessary bounds on $\left| \frac{d^j}{dz^j}\Big|_{z=0} \mathcal{S}^{\mathrm{T,imp}}(z) \right|$. By Cauchy's formula we have

$$\frac{d^j}{dz^j}\Big|_{z=0} \mathcal{S}^{\mathrm{T,imp}}(z) = \frac{j!}{2i\pi} \oint \frac{d\lambda}{\lambda^{j+1}}\ \mathcal{S}^{\mathrm{T,imp}}(\lambda)$$

Here we are integrating around the contour $|\lambda| = \frac{1}{2}\mathbf{O}_{10}$. Utilizing the uniform bound on $\mathcal{S}^{\mathrm{T,imp}}(z)$ from the previous lemma we get the estimate:

$$\left| \frac{d^j}{dz^j}\Big|_{z=0} \mathcal{S}^{\mathrm{T,imp}}(z) \right| \leqslant j! \times 2^j \mathbf{O}_{10}^{-j} \times \mathbf{O}_{13} \times L^8 \times \bar{g}^2\ .$$

This proves the lemma. □

Proposition 3.10. *In the small* ϵ *regime*

$$\frac{d^4}{dz^4}\Big|_{z=0} \mathcal{S}^{\mathrm{T}}(z \mathbb{1}_{\mathbb{Z}_p^3}, 0) \leqslant -\frac{1}{4}\bar{g} < 0\ .$$

Proof: We observe that since $\hat{\Gamma}(k) \geqslant 0$ one has

$$\Gamma(0) = \int_{\mathbb{Q}_p^3} \mathrm{d}^3k\ \hat{\Gamma}(k) \geqslant 0$$
$$\int_{\mathbb{Q}_p^3} \mathrm{d}^3x\ \Gamma(x)^3 = \left(\hat{\Gamma} * \hat{\Gamma} * \hat{\Gamma} \right)(0) \geqslant 0\ .$$

In the above expression $*$ denotes convolution. It then follows by only keeping the first $q = 0$ term that

$$
\begin{aligned}
U_4 \leqslant & -24 g_* \int_{\mathbb{Q}_p^3} \mathrm{d}^3x \ \Gamma(x)^4 \\
\leqslant & -24 g_* \int_{\mathbb{Z}_p^3} \mathrm{d}^3x \ \Gamma(x)^4 \\
= & -24 g_* \Gamma(0)^4 \\
= & -24 g_* \times \left[\frac{1-p^{-3}}{1-p^{-2[\phi]}} \left(1 - L^{-2[\phi]}\right) \right]^4 .
\end{aligned}
$$

In going to the last line we used Lemma 5.1. Now we note that $p, L \geqslant 2$ and $\epsilon \leqslant 1$ implies that $-2[\phi] \leqslant -1$

$$
\begin{aligned}
U_4 \leqslant & -24 g_* \times \left[\frac{1-\frac{1}{2^3}}{1} \times \left(1 - \frac{1}{2}\right) \right]^4 \\
= & -24 \left(\frac{7}{16}\right)^4 g_* \\
\leqslant & -12 \left(\frac{7}{16}\right)^4 \bar{g} \\
\leqslant & -\frac{1}{3} \bar{g} .
\end{aligned}
$$

Note that in going to the third line we used that $g_* > \frac{1}{2}\bar{g}$. Now using the previous lemma we have:

$$
\begin{aligned}
\frac{d^4}{dz^4}\Big|_{z=0} \mathcal{S}^{\mathrm{T}}(z\mathbb{1}_{\mathbb{Z}_p^3}, 0) \leqslant & U_4 + \mathbf{O}_{15} L^8 \bar{g}^2 \\
\leqslant & -\frac{1}{3}\bar{g} + \mathbf{O}_{15} L^8 \bar{g}^2 .
\end{aligned}
$$

We can take ϵ sufficiently small to guarantee that $\mathbf{O}_{15} L^8 \bar{g}^2 \leqslant \frac{1}{12}\bar{g}$. This proves the proposition. □

3.11.2 The two-point function for the composite field

We now study the ϕ^2 correlation when smeared with the characteristic function of $\mathbb{Z}_p^3$, i.e., the quantity

$$
\begin{aligned}
\frac{d^2}{dz^2}\Big|_{z=0} \mathcal{S}^T(0, z\mathbb{1}_{\mathbb{Z}_p^3}) = & \langle N[\phi^2](\mathbb{1}_{\mathbb{Z}_p^3})^2 \rangle - \langle N[\phi^2](\mathbb{1}_{\mathbb{Z}_p^3}) \rangle^2 \\
= & \langle N[\phi^2](\mathbb{1}_{\mathbb{Z}_p^3})^2 \rangle
\end{aligned}
$$

since the one-point function is identically zero. Our main goal is to show the quantity above is non-zero so that $N[\phi^2]$ is non-trivial. The key strategy used here is show that the UV contribution to the above quantity, which we will have to calculate somewhat explicitly, diverges as $\epsilon \to 0$. Combining this with more uniform upper bounds on the size of the IR contribution of the above quantity will give our desired quantity.

Here $q_- = q_+ = 0$ so there is no contribution from the middle regime. Thus

$$\langle N[\phi^2](\mathbb{1}_{\mathbb{Z}_p^3})^2\rangle = \langle N[\phi^2](\mathbb{1}_{\mathbb{Z}_p^3})^2\rangle^{\mathrm{UV}} + \langle N[\phi^2](\mathbb{1}_{\mathbb{Z}_p^3})^2\rangle^{\mathrm{IR}}$$

$$\text{where } \langle N[\phi^2](\mathbb{1}_{\mathbb{Z}_p^3})^2\rangle^{\mathrm{UV}} = \frac{d^2}{dz^2}\Big|_{z=0} \mathcal{S}^{\mathrm{T,UV}}(0, z\mathbb{1}_{\mathbb{Z}_p^3})$$

$$\text{and } \langle N[\phi^2](\mathbb{1}_{\mathbb{Z}_p^3})^2\rangle^{\mathrm{IR}} = \frac{d^2}{dz^2}\Big|_{z=0} \mathcal{S}^{\mathrm{T,IR}}(0, z\mathbb{1}_{\mathbb{Z}_p^3}).$$

Clearly, since we can derive term-by-term in the sum over q and since the constant and linear parts disappear

$$\begin{aligned}\langle N[\phi^2](\mathbb{1}_{\mathbb{Z}_p^3})^2\rangle^{\mathrm{UV}} =& \frac{d^2}{dz^2}\Big|_{z=0} \sum_{q<0} L^{-3(q+1)} \delta b \left[\Psi(v, -\alpha_{\mathrm{u}}^q Y_2 z e_{\phi^2})\right] \\ &= Y_2^2 \times \left(\sum_{q<0} L^{-3(q+1)} \alpha_{\mathrm{u}}^{2q} \right) \times \frac{d^2}{dz^2}\Big|_{z=0} \delta b \left[\Psi(v, z e_{\phi^2})\right]\end{aligned}$$

by the chain rule. This also uses $L^3\alpha_{\mathrm{u}}^{-2} < 1$ which will be proved shortly.

We will use the more convenient notation $\Psi_v(w)$ instead of $\Psi(v, w)$.

Now for w small we have by Theorem 3.3

$$\Psi_v(w) = \Psi_{v_*}\left(T_\infty(v)[w]\right).$$

By the remark following Lemma 3.23

$$P_s T_\infty(v)[e_{\phi^2}] = 0$$

i.e. $T_\infty(v)[e_{\phi^2}]$ is in $\mathcal{E}^{\mathrm{u}}$ and therefore is proportional to e_{u}.

We define $\varkappa_{\phi^2}$ as the proportionality constant, i.e., by

$$T_\infty(v)[e_{\phi^2}] = \varkappa_{\phi^2} e_{\mathrm{u}}.$$

Hence

$$\Psi(v, z e_{\phi^2}) = \Psi_{v_*}(z \varkappa_{\phi^2} e_{\mathrm{u}})$$

and as a result

$$\langle N[\phi^2](\mathbb{1}_{\mathbb{Z}_p^3})^2\rangle^{\mathrm{UV}} = Y_2^2 \varkappa_{\phi^2}^2 \langle N[\phi^2](\mathbb{1}_{\mathbb{Z}_p^3})^2\rangle^{\mathrm{UV}}_{\mathrm{reduced}}$$

with

$$\langle N[\phi^2](\mathbb{1}_{\mathbb{Z}_p^3})^2\rangle^{\rm UV}_{\rm reduced} = \frac{1}{\alpha_{\rm u}^2 - L^3} \times D_0^2(\delta b \circ \Psi_{v_*})[e_{\rm u}, e_{\rm u}].$$

We will show that $\frac{1}{\alpha_{\rm u}^2 - L^3}$ diverges as $\epsilon \to 0$ and that the other factor above is non-zero. For the infrared contribution we have

$$\langle N[\phi^2](\mathbb{1}_{\mathbb{Z}_p^3})^2\rangle^{\rm IR} = \sum_{q \geqslant 0} \frac{d^2}{dz^2}\Big|_{z=0} \delta b_{\Delta(0)} \left[RG^q_{\rm ex}\left(\vec{V}^{(-\infty,0)}(0, z\mathbb{1}_{\mathbb{Z}_p^3})\right)\right]$$

where

$$\vec{V}^{(-\infty,0)}(0, z\mathbb{1}_{\mathbb{Z}_p^3}) = \mathcal{J}_0\left(0, (\Psi_v(-Y_2 z e_{\phi^2})), v_*\right).$$

We define the affine isometric map $\varpi : \mathcal{E} \to \mathcal{E}_{\rm pt}$ which sends $v = (\delta g, \mu, R)$ to $\vec{V} = (V_\Delta)_{\Delta \in \mathbb{L}} = \varpi(v)$ such that

$$V_\Delta = (\beta_{4,\Delta}, \beta_{3,\Delta}, \beta_{2,\Delta}, \beta_{1,\Delta}, W_{5,\Delta}, W_{6,\Delta}, f_\Delta, R_\Delta)$$

is zero for $\Delta \neq \Delta(0)$ and equal to

$$(\delta g - \delta g_*, 0, \mu - \mu_*, 0, 0, 0, 0, R - R_*)$$

for $\Delta = \Delta(0)$.

It easily follows from the definitions that

$$\begin{aligned} \vec{V}^{(-\infty,0)}(0, z\mathbb{1}_{\mathbb{Z}_p^3}) &= \iota(v_*) + \varpi \circ \Psi_v(-Y_2 z e_{\phi^2}) \\ &= \iota(v_*) + \varpi \circ \Psi_{v_*}(-Y_2 \varkappa_{\phi^2} z e_{\rm u}) \end{aligned}$$

for z small.

Hence by the chain rule

$$\langle N[\phi^2](\mathbb{1}_{\mathbb{Z}_p^3})^2\rangle^{\rm IR} = Y_2^2 \varkappa_{\phi^2}^2 \langle N[\phi^2](\mathbb{1}_{\mathbb{Z}_p^3})^2\rangle^{\rm IR}_{\rm reduced}$$

where

$$\langle N[\phi^2](\mathbb{1}_{\mathbb{Z}_p^3})^2\rangle^{\rm IR}_{\rm reduced} = \sum_{q \geqslant 0} \frac{d^2}{dz^2}\Big|_{z=0} \delta b_{\Delta(0)} \left[\iota(v_*) + RG^q_{{\rm dv},\iota(v_*)} \circ \varpi \circ \Psi_{v_*}(z e_{\rm u})\right]$$

where we introduced the more convenient notation $RG_{{\rm dv},\vec{V}_{\rm bk}}[\dot{V}]$ for $RG_{\rm dv}[\vec{V}_{\rm bk}, \dot{V}]$ of section §3.9.4.

In what follows we will show that when $\epsilon \to 0$, $\langle N[\phi^2](\mathbb{1}_{\mathbb{Z}_p^3})^2\rangle^{\rm IR}_{\rm reduced}$ remains bounded while $\langle N[\phi^2](\mathbb{1}_{\mathbb{Z}_p^3})^2\rangle^{\rm UV}$

blows up.

We first introduce the subspace $\mathcal{E}_{\mathrm{ex,ev}}$ of $\mathcal{E}_{\mathrm{ex}}$.

It is the space of vectors

$$(\beta_{4,\Delta},\beta_{3,\Delta},\beta_{2,\Delta},\beta_{1,\Delta},W_{5,\Delta},W_{6,\Delta},f_{\Delta},R_{\Delta})_{\Delta\in\mathbb{L}}$$

such that for all $\Delta\in\mathbb{L}$,

$$\beta_{3,\Delta}=\beta_{1,\Delta}=W_{5,\Delta}=W_{6,\Delta}=f_{\Delta}=0$$
$$\text{and } R_{\Delta}\in C^{9}_{\mathrm{bd,ev}}(\mathbb{R},\mathbb{C}).$$

Using the same line of reasoning as in the proof of Proposition 3.2 or in §3.9.1 it is easy to see that $\mathcal{E}_{\mathrm{ex,ev}}$ is invariant by RG_{ex}.

Lemma 3.34. *In the small ϵ regime and for $\vec{V}\in B(\bar{V},\frac{1}{2})\cap\mathcal{E}_{\mathrm{ex,dv}}$ we have for all $\Delta'\in\mathbb{L}$*

$$|\delta b_{\Delta'}[\vec{V}]|\leqslant \mathbf{O}_{16}L^{5}\bar{g}^{4}$$

where

$$\mathbf{O}_{16}=1+9\sum_{a_1,a_2,m}\mathbb{1}\left\{\begin{array}{c}a_i+m=2\text{ or }4\\ a_i\geqslant 0,\ m\geqslant 1\end{array}\right\}\times C(a_1,a_2|0)\times 2^{\frac{a_1+a_2}{2}}.$$

Proof: Recall that

$$\delta b_{\Delta'}[\vec{V}]=\delta\beta_{0,1,\Delta'}+\delta\beta_{0,2,\Delta'}+\xi_{0,\Delta'}(\vec{V}).$$

Since there are no f's we have $\beta_{0,1,\Delta'}=0$. Similarly the $\delta\beta_{0,2,\Delta'}$ contirbution reduces to

$$\delta\beta_{0,2,\Delta'}=\sum_{a_1,a_2,m}\mathbb{1}\left\{\begin{array}{c}a_i+m=2\text{ or }4\\ a_i\geqslant 0,\ m\geqslant 1\end{array}\right\}\frac{(a_1+m)!(a_2+m)!}{a_1!a_2!m!}\times\frac{1}{2}C(a_1,a_2|0)$$
$$\times L^{-(a_1+a_2)[\phi]}C_0(0)^{\frac{a_1+a_2}{2}}\times\int_{(L^{-1}\Delta')^2}\mathrm{d}^3x_1\mathrm{d}^3x_2\ \beta_{a_1+m}(x_1)\beta_{a_2+m}(x_2)\ \Gamma(x_1-x_2)^m.$$

We use the bound

$$\left|\int_{(L^{-1}\Delta')^2}\mathrm{d}^3x_1\mathrm{d}^3x_2\ \beta_{a_1+m}(x_1)\beta_{a_2+m}(x_2)\ \Gamma(x_1-x_2)^m\right|\leqslant L^3||\Gamma||_{L^\infty}^{m-1}\times||\Gamma||_{L^1}$$
$$\times\sup_{x\in L^{-1}\Delta'}|\beta_{a_1+m}(x)|\times\sup_{x\in L^{-1}\Delta'}|\beta_{a_2+m}(x)|.$$

We bound the supremums by noting that β_{a_1+m} can only be β_2 or β_4. Since $\bar{V}$ has no β_2 component

$$|\beta_2(x)| \leqslant ||\vec{V} - \bar{V}||\bar{g}^2 \leqslant \frac{1}{2}\bar{g}^2.$$

On the other hand

$$|\beta_4(x)| \leqslant \bar{g} + ||\vec{V} - \bar{V}||\bar{g}^{\frac{3}{2}} \leqslant \frac{3}{2}\bar{g}$$

As a result the previous integral is bounded by

$$L^3||\Gamma||_{L^\infty}^{m-1} \times ||\Gamma||_{L^1} \times \frac{9}{4}\bar{g}^4 \leqslant L^3 \times 2^{m-1} \times \frac{1}{\sqrt{2}}L^{3-2[\phi]} \times \frac{9}{4}\bar{g}^4 \leqslant 18L^5\bar{g}^4$$

where we used $\epsilon \leqslant 1$ so $3-2[\phi] \leqslant 2$, and $m \leqslant 4$ while dropping $\sqrt{2}$. Finally $|\xi_{0,\Delta'}(\vec{V})| \leqslant \frac{1}{2}B_4\bar{g}^{\frac{21}{8}}$ by Theorem 3.2. Noting that $\frac{1}{2}B_4\bar{g}^{\frac{5}{8}} \leqslant 1$ for ϵ small the lemma follows. □

Lemma 3.35. *For $\vec{V}^1, \vec{V}^2 \in \bar{B}(\bar{V}, \frac{1}{6}) \cap \mathcal{E}_{\text{ex,dv}}$, we have the Lipschitz estimate*

$$\left|\delta b_{\Delta(0)}[\vec{V}^1] - \delta b_{\Delta(0)}[\vec{V}^2]\right| \leqslant 4\mathbf{O}_{16}L^5\bar{g}^4||\vec{V}^1 - \vec{V}^2|| \ .$$

Proof: This is an immediate consequence of the previous lemma and Lemma 3.2 with $\nu = \frac{1}{2}$. □

Since we are computing second derivatives there is no harm in writing

$$\langle N[\phi^2](\mathbb{1}_{\mathbb{Z}_p^3})^2\rangle_{\text{reduced}}^{\text{IR}} = \sum_{q\geqslant 0} \frac{d^2}{dz^2}\Big|_{z=0} \left\{\delta b_{\Delta(0)}\left[\iota(v_*) + RG^q_{\text{dv},\iota(\text{v}_*)} \circ \varpi \circ \Psi_{v_*}(ze_{\text{u}})\right] - \delta b_{\Delta(0)}\left[\iota(v_*)\right]\right\}.$$

If z is small enough so that

$$||\Psi(ze_{\text{u}}) - v_*|| \leqslant \mathbf{O}_{10}$$

which is the same as saying that $||\varpi \circ \Psi_{v_*}(ze_{\text{u}})|| \leqslant \mathbf{O}_{10}$, then Proposition 3.9 along with the last lemma will imply

$$\left|\delta b_{\Delta(0)}\left[\iota(v_*) + RG^q_{\text{dv},\iota(\text{v}_*)} \circ \varpi \circ \Psi_{v_*}(ze_{\text{u}})\right] - \delta b_{\Delta(0)}\left[\iota(v_*)\right]\right| \leqslant 4\mathbf{O}_{16}L^5\bar{g}^4\left(\frac{15}{16}\right)^q \times \mathbf{O}_{10}.$$

Let $z_{\max} > 0$ be such that $|z| \leqslant z_{\max}$ implies $||\Psi_{v_*}(ze_{\text{u}}) - v_*|| \leqslant \mathbf{O}_{10}$. Then by extracting the derivatives with Cauchy's formula we easily arrive at the bound

$$\left|\langle N[\phi^2](\mathbb{1}_{\mathbb{Z}_p^3})^2\rangle_{\text{reduced}}^{\text{IR}}\right| \leqslant 4\mathbf{O}_{10}\mathbf{O}_{16}L^5\bar{g}^4 \times \frac{1}{1-\frac{15}{16}} \times 2! \times z_{\max}^{-2} \ .$$

Now from Theorem 3.4 $||ze_{\text{u}}|| < \frac{1}{24}$ implies

$$\begin{aligned}||\Psi_{v_*}(ze_{\rm u})-v_*|| \leqslant & ||ze_{\rm u}||\left(1+\frac{17}{18}\times\frac{1}{24}\right)\\ \leqslant & 2||ze_{\rm u}||\end{aligned}$$

for simplicity. So $z_{\rm max} = \frac{1}{2}\mathbf{O}_{10}||e_{\rm u}||^{-1}$ works because $\frac{1}{2}\mathbf{O}_{10} \leqslant \frac{1}{80} < \frac{1}{24}$. Also we have that $||e_{\rm u}|| = \bar{g}^{-2}$. Hence in the small ϵ regime we have the bound

$$\left|\langle N[\phi^2](\mathbb{1}_{\mathbb{Z}_p^3})^2\rangle^{\rm IR}_{\rm reduced}\right| \leqslant 512\mathbf{O}_{10}^{-1}\times\mathbf{O}_{16}\times L^5\ .$$

Namely, the infrared contribution remains finite when $\epsilon\to 0$.

We now examine the ultraviolet contribution more closely. From Theorem 3.4 the small z expansion of $\Psi_{v_*}(ze_{\rm u})$ is of the form

$$\Psi_{v_*}(ze_{\rm u}) = v_* + ze_{\rm u} + z^2\Theta + O(z^3) \tag{3.91}$$

for some vector Θ to be determined shortly. Now we can decompose

$$D_0^2(\delta b\circ\Psi_{v_*})[e_{\rm u},e_{\rm u}] = \frac{d^2}{dz^2}\Big|_{z=0}\delta b^{\rm explicit}\left(\Psi_{v_*}(ze_{\rm u})\right) + \frac{d^2}{dz^2}\Big|_{z=0}\delta b^{\rm implicit}\left(\Psi_{v_*}(ze_{\rm u})\right).$$

If $|z|\leqslant\dfrac{1}{30}\bar{g}^2$ then as before we get

$$\begin{aligned}||\Psi_{v_*}(ze_{\rm u})|| \leqslant & ||v_*|| + 2||ze_{\rm u}||\\ \leqslant & \frac{1}{40}+\frac{1}{15}<\frac{1}{2}.\end{aligned}$$

So by Theorem 3.2

$$\left|\delta b^{\rm implicit}\left(\Psi_{v_*}(ze_{\rm u})\right)\right| \leqslant \frac{1}{2}B_0\bar{g}^{\frac{21}{8}}.$$

Cauchy's formula then immediately implies

$$\left|\frac{d^2}{dz^2}\Big|_{z=0}\delta b^{\rm implicit}\left(\Psi_{v_*}(ze_{\rm u})\right)\right| \leqslant 2!\left(\frac{1}{30}\bar{g}^2\right)^{-2}\times\frac{1}{2}B_0\bar{g}^{\frac{21}{8}}.$$

So we have

$$\lim_{\epsilon\to 0}\frac{d^2}{dz^2}\Big|_{z=0}\delta b^{\rm implicit}\left(\Psi_{v_*}(ze_{\rm u})\right) = 0.$$

Now recall that

$$\delta b^{\rm explicit} = A_4\bar{g}^2 + \delta b^{\rm explicit}_{\rm I}(\delta g,\mu,R) + \delta b^{\rm explicit}_{\rm II}(\delta g,\mu,R)$$

$$\text{where } \delta b^{\rm explicit}_{\rm I}(\delta g,\mu,R) = 2A_4\bar{g}\delta g + A_4\delta g^2$$

$$\text{and } \delta b_{\mathrm{II}}^{\mathrm{explicit}}(\delta g, \mu, R) = A_5 \mu^2 \ .$$

Note that the $A_4 \bar{g}^2$ term disappears in the computation of derivatives while $\delta b_{\mathrm{I}}^{\mathrm{explicit}}$ can be treated as we treated $\delta b^{\mathrm{implicit}}$. Indeed by Cauchy's formula and Theorem 3.2

$$\left| \frac{d^2}{dz^2}\Big|_{z=0} \delta b_{\mathrm{I}}^{\mathrm{explicit}} \left(\Psi_{v_*}(z e_{\mathrm{u}})\right) \right| \leqslant 2! \left(\frac{1}{30}\bar{g}^2\right)^{-2} \times A_{4,\max} \left[2 \times \bar{g} \times \frac{1}{2}\bar{g}^{\frac{3}{2}} + \left(\frac{1}{2}\bar{g}^{\frac{3}{2}}\right)^2 \right].$$

So we have

$$\lim_{\epsilon \to 0} \frac{d^2}{dz^2}\Big|_{z=0} \delta b_{\mathrm{I}}^{\mathrm{explicit}} \left(\Psi_{v_*}(z e_{\mathrm{u}})\right) = 0.$$

As a result of the formula $e_{\mathrm{u}} = (\delta g_{\mathrm{u}}'(\mu_*), 1, R_{\mathrm{u}}'(\mu_*))$ and the expansion (3.91) we easily compute

$$\frac{d^2}{dz^2}\Big|_{z=0} \delta b_{\mathrm{II}}^{\mathrm{explicit}} \left(\Psi_{v_*}(z e_{\mathrm{u}})\right) = 2A_5 \left(1 + 2\mu_* \Theta_\mu\right)$$

where Θ_μ is the μ component of $\Theta \in \mathcal{E}$.
We determine the latter using the intertwining relation in Theorem 3.4 for small z.
We have by an easy calculation using (3.91)

$$RG\left(\Psi_{v_*}(z e_{\mathrm{u}})\right) = v_* + D_{v_*} RG[e_{\mathrm{u}}] + z^2 \left(D_{v_*} RG[\Theta] + \frac{1}{2} D_{v_*}^2 RG[e_{\mathrm{u}}, e_{\mathrm{u}}] \right) + O(z^3) \ .$$

But this is the same as

$$\Psi_{v_*}(\alpha_{\mathrm{u}} z e_{\mathrm{u}}) = v_* + z \alpha_{\mathrm{u}} e_{\mathrm{u}} + z^2 \alpha_{\mathrm{u}}^2 \Theta + O(z^3).$$

Thus

$$\alpha_{\mathrm{u}}^2 \Theta = D_{v_*} RG[\Theta] + \frac{1}{2} D_{v_*}^2 RG[e_{\mathrm{u}}, e_{\mathrm{u}}]. \tag{3.92}$$

On the other hand $\Psi_{v_*} \in W^{\mathrm{u,loc}}$ for z small and therefore

$$\left[\Psi_{v_*}(z e_{\mathrm{u}})\right]_{\delta g} = \delta g_{\mathrm{u}} \left(\left[\Psi_{v_*}(z e_{\mathrm{u}})\right]_\mu \right)$$

and

$$\left[\Psi_{v_*}(z e_{\mathrm{u}})\right]_R = R_{\mathrm{u}} \left(\left[\Psi_{v_*}(z e_{\mathrm{u}})\right]_\mu \right)$$

where $[\cdots]_{\delta g}$, $[\cdots]_\mu$, and $[\cdots]_R$ refer to the δg, μ, and R components respectively.

Expanding these relations up to second order imply

$$\Theta = (\Theta_{\delta g}, \Theta_\mu, \Theta_R) = \Theta_\mu e_{\mathrm{u}} + \frac{1}{2} c_{\mathrm{u}}$$
$$\text{where } c_{\mathrm{u}} = (\delta g_{\mathrm{u}}''(\mu_*), 0, R_{\mathrm{u}}''(\mu_*)).$$

Taking the μ component of (3.92) we see that

$$\alpha_{\mathrm{u}}^2\Theta_\mu = \Theta_\mu\alpha_{\mathrm{u}} + \frac{1}{2}\left[D_{v_*}RG[c_{\mathrm{u}}]\right]_\mu + \frac{1}{2}\left[D^2_{v_*}RG[e_{\mathrm{u}},e_{\mathrm{u}}]\right]_\mu$$

where we have used $[e_{\mathrm{u}}]_\mu = 1$ and $D_{v_*}RG[e_{\mathrm{u}}] = \alpha_{\mathrm{u}}e_{\mathrm{u}}$.
Since α_{u} we have

$$\Theta_\mu = \frac{1}{2\alpha_{\mathrm{u}}(\alpha_{\mathrm{u}}-1)}\left\{\left[D_{v_*}RG[c_{\mathrm{u}}]\right]_\mu + \left[D^2_{v_*}RG[e_{\mathrm{u}},e_{\mathrm{u}}]\right]_\mu\right\} .$$

Now $|\mu-\mu_*| < \rho''\bar{g}^2$ implies $|\delta g_{\mathrm{u}}(\mu)| \leqslant \frac{\rho'}{3}\bar{g}^{\frac{3}{2}}$ and $|||R_{\mathrm{u}}(\mu)|||_{\bar{g}} \leqslant \frac{\rho'}{3}\bar{g}^{\frac{21}{8}}$. Using $|\mu-\mu_*| = \frac{1}{2}\rho''\bar{g}^2$ as a contour of integration, Cauchy's formula implies the following estimates:

$$|\delta g'_{\mathrm{u}}(\mu_*)| \leqslant \frac{2\rho'}{3\rho''}\bar{g}^{\frac{1}{2}}$$

$$|\delta g''_{\mathrm{u}}(\mu_*)| \leqslant \frac{8\rho'}{3(\rho'')^2}\bar{g}^{-\frac{1}{2}}$$

$$|||R'_{\mathrm{u}}(\mu_*)|||_{\bar{g}} \leqslant \frac{2\rho'}{3\rho''}\bar{g}^{\frac{13}{8}}$$

$$|||R''_{\mathrm{u}}(\mu_*)|||_{\bar{g}} \leqslant \frac{8\rho'}{3(\rho'')^2}\bar{g}^{\frac{5}{8}}.$$

As a result

$$||c_{\mathrm{u}}|| = \max\left\{|\delta g''_{\mathrm{u}}(\mu_*)|\bar{g}^{-\frac{3}{2}}, |||R''_{\mathrm{u}}(\mu_*)|||_{\bar{g}}\bar{g}^{-\frac{21}{8}}\right\} \leqslant \frac{8\rho'}{3(\rho'')^2}\bar{g}^{-4} .$$

Now one can write

$$\left[D_vRG[v']\right]_\mu = L^{\frac{3+\epsilon}{2}}\mu' - 2A_2(\bar{g}+\delta g)\delta g' - A_3(\bar{g}+\delta g)\mu' - A_3\mu\ \delta g' + \left[D_vRG^{\mathrm{implicit}}[v']\right]_\mu . \qquad (3.93)$$

For $v = v_*$ and $v' = c_{\mathrm{u}}$ this gives

$$\left[D_{v_*}RG[v']\right]_\mu = -2A_2(\bar{g}+\delta g_*)\delta g''_{\mathrm{u}}(\mu_*) - A_3\mu_*\delta g''_{\mathrm{u}}(\mu_*) + \left[D_{v_*}RG^{\mathrm{implicit}}[c_{\mathrm{u}}]\right]_\mu .$$

The infinitesimal version of the ξ_2 Lipschitz estimate in Lemma 3.16 immediately implies

$$||\left[D_{v_*}RG^{\mathrm{implicit}}\right]_\mu|| \leqslant 2B_2\bar{g}^{\frac{21}{8}}$$

for the operator norm induced on linear maps from $\mathcal{E}$ to $\mathbb{C}$ by the norm $||\cdot||$ on $\mathcal{E}$ and the modulus on $\mathbb{C}$. As a result we have

$$\left|\left[D_{v_*}RG[c_{\mathrm{u}}]\right]_\mu\right| \leqslant \left(2A_{2,\max}\times\frac{3}{2}\bar{g} + A_{3,\max}\times\frac{1}{2}\bar{g}^2\right)\times\frac{8\rho'}{3(\rho'')^2}\bar{g}^{-\frac{1}{2}} + 2B_2\bar{g}^{\frac{21}{8}}\times\frac{8\rho'}{3(\rho'')^2}\bar{g}^{-4} .$$

So we have

$$\lim_{\epsilon\to 0}\left[D_{v_*}RG[c_{\rm u}]\right]_\mu = 0\ .$$

Uniform bounds on the second differential of the μ component of the RG give

$$\left[D^2_{v_*}RG[e_{\rm u},e_{\rm u}]\right]_\mu = -2A_2\delta g'_{\rm u}(\mu_*)^2 - 2A_3\delta g'_{\rm u}(\mu_*) + \left[D^2_{v_*}RG^{\rm implicit}[e_{\rm u},e_{\rm u}]\right]_\mu$$

where $D^2_{v_*}RG^{\rm implicit}$ corresponds to the second differential of the remainder term ξ_2 and an easy estimate gives

$$\left|\left[D^2_{v_*}RG[e_{\rm u},e_{\rm u}]\right]_\mu\right| \leqslant 2A_{2,\max}\left(\frac{2\rho'}{3\rho''}\bar{g}^{\frac{1}{2}}\right)^2 + 2A_{3,\max}\left(\frac{2\rho'}{3\rho''}\bar{g}^{\frac{1}{2}}\right) + \left|\left[D^2_{v_*}RG^{\rm implicit}[e_{\rm u},e_{\rm u}]\right]_\mu\right|.$$

Now one can easily estimate μ component of $RG^{\rm implicit}$

$$||D^2_{v_*}RG^{\rm implicit}_\mu|| \leqslant 32B_2\bar{g}^{\frac{13}{8}}$$

for the norm of the second differential.

Since $||e_{\rm u}|| = \bar{g}^{-2}$ we obtain

$$\left|\left[D^2_{v_*}RG^{\rm implicit}[e_{\rm u},e_{\rm u}]\right]_\mu\right| \leqslant 32B_2\bar{g}^{-\frac{3}{8}}.$$

So we have

$$\lim_{\epsilon\to 0}\mu_*\left[D^2_{v_*}RG^{\rm implicit}[e_{\rm u},e_{\rm u}]\right]_\mu = 0.$$

Since also $\lim_{\epsilon\to 0}\mu_* = 0$ and $\lim_{\epsilon\to 0}\alpha_{\rm u} = L^{\frac{3}{2}} > 1$ we have enough to affirm

$$\lim_{\epsilon\to 0}\mu_*\Theta_\mu = 0.$$

Thus

$$\lim_{\epsilon\to 0}\frac{d^2}{dz^2}\Big|_{z=0}D^2_0(\delta b\circ\Psi_{v_*})[e_{\rm u},e_{\rm u}] = 2\lim_{\epsilon\to 0}A_5 = 2L^3(1-p^{-3})\times l > 0 \text{ by Lemma 5.5.}$$

We now study the $\epsilon\to 0$ asymptotics of $\alpha_{\rm u}$ more closely. One way to get a precise hold on this eigenvalue is to note that

$$\alpha_{\rm u} = \left[D_{v_*}RG[e_{\rm u}]\right]_\mu .$$

Then by the formula in (3.93) we have

$$\alpha_{\mathrm{u}} = L^{\frac{3+\epsilon}{2}} - 2A_2(\bar{g}+\delta g_*)\delta g'_{\mathrm{u}}(\mu_*) - A_3(\bar{g}+\delta g_*) - A_3\mu_*\delta g'_{\mathrm{u}}(\mu_*) + \left[D_{v_*}RG^{\mathrm{implicit}}[e_{\mathrm{u}}]\right]_{\mu}$$

since $e_{\mathrm{u}} = (\delta g'_{\mathrm{u}}(\mu_*), 1, R'_{\mathrm{u}}(\mu_*))$.

As before

$$\begin{aligned}\left|\left[D_{v_*}RG^{\mathrm{implicit}}[e_{\mathrm{u}}]\right]_{\mu}\right| \leqslant & ||D_{v_*}RG^{\mathrm{implicit}}|| \times ||e_{\mathrm{u}}|| \\ \leqslant & 2B_2\bar{g}^{\frac{21}{8}} \times \bar{g}^{-2}.\end{aligned}$$

But $\bar{g}$ is of order ϵ so

$$\left[D_{v_*}RG^{\mathrm{implicit}}[e_{\mathrm{u}}]\right]_{\mu} = o(\epsilon).$$

We have

$$\left|-2A_2(\bar{g}+\delta g_*)\delta g'_{\mathrm{u}}(\mu_*)\right| \leqslant 2A_{2,\max} \times \frac{3}{2}\bar{g} \times \frac{2\rho'}{3\rho''}\bar{g}^{\frac{1}{2}}$$

so this is an $o(\epsilon)$ term.

Likewise

$$|-A_3\delta g_*| \leqslant A_{3,\max} \times \frac{1}{2}\bar{g}^{\frac{3}{2}}$$

so this is $o(\epsilon)$.

Finally,

$$\left|-A_3\mu_*\delta g'_{\mathrm{u}}(\mu_*)\right| \leqslant A_{3,\max} \times \frac{1}{2}\bar{g}^2 \times \frac{2\rho'}{3\rho''}\bar{g}^{\frac{1}{2}}$$

so this is an $o(\epsilon)$ term too.

As a result we have

$$\begin{aligned}\alpha_{\mathrm{u}} =& L^{\frac{3+\epsilon}{2}} - A_3\bar{g} + o(\epsilon) \\ =& L^{\frac{3+\epsilon}{2}} - 12 \times L^{\frac{3+\epsilon}{2}} \times \frac{A_1}{36L^{\epsilon}}\bar{g} + o(\epsilon) \\ =& L^{\frac{3+\epsilon}{2}}\left(1 - \frac{1}{3}\left(\frac{L^{\epsilon}-1}{L^{\epsilon}}\right)\right) + o(\epsilon)\end{aligned}$$

from the relations between A_3, A_1, and $\bar{g}$.

It is now a simple calculus exercise to derive

$$\eta_{\phi^2} = \frac{2}{3}\epsilon + o(\epsilon)$$

where η_{ϕ^2} is defined by

$$L^{\frac{1}{2}\eta_{\phi^2}} = Z_2^{-1} = L^{\frac{3+\epsilon}{2}} \alpha_{\mathrm{u}}^{-1}.$$

We also easily get

$$L^3 \alpha_{\mathrm{u}}^{-2} = 1 - \frac{\log(L)}{3}\epsilon + o(\epsilon)$$

which proves the earlier statement that

$$L^3 \alpha_{\mathrm{u}}^{-2} < 1$$

in the small ϵ regime which was crucial for the convergence and analyticity in the ultraviolet regime.

Another byproduct is

$$\frac{1}{\alpha_{\mathrm{u}}^2 - L^3} \sim \frac{3L^{-3}}{\log(L)} \times \frac{1}{\epsilon}$$

and therefore

$$\langle\langle N[\phi^2](\mathbb{1}_{\mathbb{Z}_p^3})^2 \rangle_{\mathrm{reduced}}^{\mathrm{UV}} \sim \frac{6(1-p^{-3})}{\log(p)} \times \frac{1}{\epsilon}$$

when $\epsilon \to 0$.

Since $\langle N[\phi^2](\mathbb{1}_{\mathbb{Z}_p^3})^2 \rangle_{\mathrm{reduced}}^{\mathrm{IR}}$ remains bounded, the quantity

$$\langle N[\phi^2](\mathbb{1}_{\mathbb{Z}_p^3})^2 \rangle_{\mathrm{reduced}} = \langle N[\phi^2](\mathbb{1}_{\mathbb{Z}_p^3})^2 \rangle_{\mathrm{reduced}}^{\mathrm{UV}} + \langle N[\phi^2](\mathbb{1}_{\mathbb{Z}_p^3})^2 \rangle_{\mathrm{reduced}}^{\mathrm{IR}}$$

is strictly positive for ϵ small enough.

Provided $\varkappa_{\phi^2} \neq 0$ we can then impose by definition

$$Y_2 = |\varkappa_{\phi^2}|^{-1} \times \left\{ \langle N[\phi^2](\mathbb{1}_{\mathbb{Z}_p^3})^2 \rangle_{\mathrm{reduced}} \right\}^{-\frac{1}{2}}$$

and thus force the normalization

$$\langle N[\phi^2](\mathbb{1}_{\mathbb{Z}_p^3})^2 \rangle = 1.$$

We now address the issue of showing $\varkappa_{\phi^2} \neq 0$. While most of the proof so far relied on quantitative estimates, here we had to use a more qualitative approach. This is because of the slow convergence to the fixed point on the stable manifold and the fact that we do not have much freedom of choice for our starting point v. The latter has to be on the $R = 0$ bare surface and therefore we cannot choose it as close to v_* as we would like to.

Recall that $W^{\rm s,loc}_{\rm int}$ is parametrized as

$$v_1 \mapsto (v_1, \mu_{\rm s}(v_1))$$

for $||v_1|| < \dfrac{\rho}{13}$ in $\mathcal{E}_1$. For $v \in W^{\rm s,loc}_{\rm int}$ we consider the tangent space $T_v W^{\rm s}$ defined as the kernel of the linear form

$$(w_1, w_2) \mapsto w_2 - D_{v_1}\mu_{\rm s}[w_1]$$

via the identification $\mathcal{E}_2 \simeq \mathbb{C}$.

This linear form is continuous and does not vanish identically, so $T_v W^{\rm s}$ is a closed complex hyperplane in $\mathcal{E}$. If $w \in \mathcal{E}$ satisfies $w \notin T_v W^{\rm s}$ then we have a direct sum decomposition $\mathcal{E} = \mathbb{C} \oplus T_v W^{\rm s}$.

We have the following infinitesimal version of Parts 1) and Parts 2) of Lemma 3.17 and Lemma 3.18.

Lemma 3.36. *For all $v \in W^{\rm s,loc}_{\rm int}$ we have:*

1) for all $w \in \mathcal{E}$,

$$||(D_v RG[w])_1|| \leqslant c_1(\epsilon)||w||$$

2) for all $w \in \mathcal{E}$, such that $L^{\frac{3}{4}}||w_2|| \geqslant ||w_1||$,

$$||(D_v RG[w])_2|| \geqslant c_2(\epsilon)||w||$$

3) for all $w \in T_v W^{\rm s}$,

$$||w_1|| \geqslant L^{\frac{3}{4}}||w_2|| \ .$$

Proof: Consider the complex curve $\gamma(t) = v + tw$ for t small which ensures that $\Gamma(t) \in \bar{B}\left(0, \frac{1}{8}\right)$. Lemma 3.17 Part 1) implies

$$||RG_1(\gamma(t)) - RG_1(\gamma(0))|| \leqslant c_1(\epsilon)||tw||.$$

Dividing by $|t|$ and taking $t \to 0$ we immediately get $||(D_v(RG[w])_1|| \leqslant c_1(\epsilon)||w||$.

Now if $L^{\frac{3}{4}}||w_2|| \geqslant ||w_1||$ then we have

$$L^{\frac{3}{4}}||\gamma(t)_2 - \gamma(0)_2|| \geqslant ||\gamma(t)_1 - \gamma(0)_1||$$

and thus

$$||RG_2(\gamma(t)) - RG_2(\gamma(0))|| \geqslant c_2(\epsilon)||tw||$$

by Lemma 3.17 Part 2). Taking the $t \to 0$ limit as before we obtain

$$||(D_v RG[w])_2|| \geqslant c_2(\epsilon)||w||.$$

For the third part we use Lemma 3.18 to write

$$||(v_1 + tw_1) - v_1|| \geqslant L^{\frac{3}{4}}||\mu_{\mathrm{s}}(v_1 + tw_1) - \mu_{\mathrm{s}}(v_1)||$$

for t small. Dividing by $|t|$ and taking $t \to 0$ gives

$$||w_1|| \geqslant L^{\frac{3}{4}}||D_{v_1}\mu_{\mathrm{s}}[w_1]|| = L^{\frac{3}{4}}||w_2||$$

since $w \in T_v W^{\mathrm{s}}$. □

Lemma 3.37. *For all $v \in W^{\mathrm{s,loc}}_{\mathrm{int}}$ and $w \in \mathcal{E}$ we have the implication*

$$L^{\frac{3}{4}}||w_2|| > ||w_1|| \Rightarrow D_v RG[w] \notin T_{RG(v)}W^{\mathrm{s}} \ .$$

Proof: We proceed by contradiction. Suppose

$$L^{\frac{3}{4}}||w_2|| > ||w_1|| \text{ and } D_v RG[w] \in T_{RG(v)}W^{\mathrm{s}}.$$

Then by Lemma 3.36 Parts 1), 2), 3) we have

$$c_1(\epsilon)||w|| \geqslant ||(D_v RG[w])_1||,$$

$$||(D_v RG[w])_2|| \geqslant c_2(\epsilon)||w||$$

and

$$||(D_v RG[w])_1|| \geqslant L^{\frac{3}{4}}||(D_v RG[w])_2||$$

respectively. As a result

$$c_1(\epsilon)||w|| \geqslant L^{\frac{3}{4}}c_2(\epsilon)||w||.$$

But $c_1(\epsilon) < 1 < L^{\frac{3}{4}}c_2(\epsilon)$ so $||w|| = 0$ which contradicts the strict inequality $L^{\frac{3}{4}}||w_2|| > ||w_1||$. □

Lemma 3.38. *For all $v \in W^{\mathrm{s,loc}}_{\mathrm{int}}$ and $w \in T_v W^{\mathrm{s}}$*

$$T_1(v)[w] \in T_{RG(v)}W^{\mathrm{s}}$$

and

$$T_\infty(v)[w] \in T_{v_*} W^{\rm s} \ .$$

Proof: Consider the curve $t \mapsto (v_1 + tw_1, \mu_{\rm s}(v_1 + tw_1))$ in $W^{\rm s,loc}_{\rm int}$ for t small. Using the fact that RG maps $W^{\rm s,loc}$ into $W^{\rm s,loc}$ and the parametrization of $W^{\rm s,loc}_{\rm int}$ we have

$$RG_2(v_1 + tw_1, \mu_{\rm s}(v_1 + tw_1)) = \mu_{\rm s}\left(RG_1(v_1 + tw_2, \mu_{\rm s}(v_1 + tw_1))\right).$$

Differentiating this at $t = 0$ gives

$$(D_v RG[(w_1, D_{v_1}\mu_{\rm s}[w_1])])_2 = D_{RG_1(v)}\mu_{\rm s}\left[(D_v RG[(w_1, D_{v_1}\mu_{\rm s}[w_1])])_1\right],$$

i.e.,

$$(D_v RG[w])_2 = D_{RG_1(v)}\mu_{\rm s}\left[(D_v RG[w])_1\right].$$

Hence $D_v RG[w]$ belongs to $T_{RG(v)} W^{\rm s}$ and so does $T_1(v)[w] = \alpha_{\rm u}^{-1} D_v RG[w]$.

By iteration this immediately implies

$$T_n(v)[w] \in T_{RG^n(v)} W^{\rm s}$$

for all integer $n \geqslant 0$.

Namely, we have

$$(T_n(v)[w])_2 = D_{(RG^n(v))_1}\mu_{\rm s}\left[(T_n(v)[w])_1\right].$$

Using continuity, the remark following Lemma 3.23, and the fact that $RG^n(v) \to v_*$, we can take the $n \to \infty$ limit in the previous equality and obtain

$$(T_\infty(v)[w])_2 = D_{v_{*,1}}\mu_{\rm s}\left[(T_\infty(v)[w])_1\right].$$

This proves $T_\infty(v)[w] \in T_{v_*} W^{\rm s} = \mathcal{E}^{\rm s}$ by definition of $\mathcal{E}^{\rm s}$. □

Lemma 3.39. *For all $v \in W^{\rm s,loc}_{\rm int}$ and $w \in T_v W^{\rm s}$*

$$D_0 \Psi_v[w] = 0,$$

where the differential is with respect to the w variable at $w = 0$ for the function $\Psi_v(\bullet) = \Psi(v, \bullet)$.

Proof: By Theorem 3.3 Part 5)

$$\Psi_v = \Psi_{v_*} \circ T_\infty(v)$$

and thus by the chain rule

$$D_0\Psi_v[w] = D_0\Psi_{v_*}\left[T_\infty(v)[w]\right].$$

However by the previous lemma $T_\infty(v)[w] \in \mathcal{E}^{\rm s}$ so $P_{\rm u}T_\infty(v)[w] = 0$. But we also have $P_s T_\infty(v)[w] = 0$ as a follow up to Lemma 3.23.

As a result, $T_\infty(v)[w] = 0$ and consequently $D_0\Psi_v[w] = 0$. □

Lemma 3.40. *For all $v \in W^{\rm s,loc}_{\rm int}$, if $D_0\Psi_v = 0$ then $D_0\Psi_{RG(v)} = 0$.*

Proof: By Theorem 3.3 Part 4)

$$\Psi_v = \Psi_{RG(v)} \circ T_1(v)$$

near $w = 0$. Differentiating at zero gives

$$D_0\Psi_v = D_0\Psi_{RG(v)} \circ T_1(v). \tag{3.94}$$

Pick some vector $u \in \mathcal{E}$ satisfying the hypothesis of lemma 3.37. For instance e_{ϕ^2} works since $L^{\frac{3}{4}}||e_{\phi^2,2}|| = L^{\frac{3}{4}}\bar{g}^{-2} > ||e_{\phi^2,1}|| = 0$. By the same lemma $T_1(v)[u] \notin T_{RG(v)}W^{\rm s}$ and therefore $\mathcal{E} = \mathbb{C}T_1(v)[u] \oplus T_{RG(v)}W^{\rm s}$.

Let $w \in \mathcal{E}$. We decompose it as $w = \lambda T_1(v)[u] + w'$ with $w' \in T_{RG(v)}W^{\rm s}$. Then by (3.94):

$$\begin{aligned} D_0\Psi_{RG(v)}[w] =& \lambda D_0\Psi_v[u] + D_0\Psi_{RG(v)}[w'] \\ =& 0 \end{aligned}$$

by the hypothesis and the previous lemma for $RG(v)$ instead of v. Hence the differential $D_0\Psi_{RG(v)}$ vanishes. □

Iterating the last lemma we see that if $D_*\Psi_v = 0$ then $D_0\Psi_{RG^n(v)} = 0$ for all $n \geqslant 0$. By the joint analyticity in Theorem 3.3 we can take the $n \to \infty$ limit which gives $D_0\Psi_{v_*} = 0$ and therefore

$$\left.\frac{d}{dz}\right|_{z=0} \Psi_{v_*}(ze_{\rm u}) = 0$$

which contradicts (3.91) and $e_{\rm u} \neq 0$.

We have proved $D_0\Psi_v \neq 0$ for all $v \in W^{\rm s,loc}_{\rm int}$. Now since e_{ϕ^2} satisfies $L^{\frac{3}{4}}||e_{\phi^2,2}|| > ||e_{\phi^2,1}||$ we know that $e_{\phi^2} \notin T_vW^{\rm s}$ by Lemma 3.36. Thus $\mathcal{E} = \mathbb{C}e_{\phi^2} \oplus T_vW^{\rm s}$.

Recall that $D_0\Psi_v = D_0\Psi_{v_*} \circ T_\infty(v)$ so $D_0\Psi_v[e_{\phi^2}] = \varkappa_{\phi^2} D_0\Psi_{v_*}[e_{\rm u}]$ by definition of $\varkappa_{\phi^2}$. If the latter vanishes then $D_0\Psi_v$ vanishes on $\mathbb{C}e_{\phi^2}$ and therefore on all of $\mathcal{E}$ by Lemma 3.39. This contradicts $D_0\Psi_v \neq 0$. We have now finally proved $\varkappa_{\phi^2} \neq 0$.

The remaining item to be settled is the mini-universality result but this should be clear at this point: the generating function $\mathcal{S}^T(\tilde{f},\tilde{j})$ does not depend on the starting point $v = (g - \bar{g}, \mu_{\rm c}(g), 0) \in W^{\rm s,loc}_{\rm int}$ for the RG iterations. Indeed using $\Psi_v = \Psi_{v_*} \circ T_\infty(v)$ we see that the effect of v is entirely in the multiplying factor $\varkappa_{\phi^2}$

which however always comes in the combination $Y_2\varkappa_{\phi^2}$. By our choice of normalization, $Y_2\varkappa_{\phi^2}$ is defined in terms of the reduced $N[\phi^2]$ two-point function which only involves data at the fixed point v_*.

Chapter 4

Proving Full Scale Invariance

4.1 General Strategy for Proving Scale Invariance

The goal of this section is to prove a stronger version of Theorem 3.1 where we are promised full scale invariance for our constructed measures ν_ϕ and ν_{ϕ^2} instead of scale invariance with respect to powers of L - in particular we would want statement 2) of that theorem to hold for all $\lambda \in p^{\mathbb{Z}}$ instead of just for $\lambda \in L^{\mathbb{Z}}$ (here we see $p^{\mathbb{Z}}$ and $L^{\mathbb{Z}}$ as subsets of $\mathbb{Q}_p$).

There are two main limitations of our RG approach in Chapter 3 that prevent it from establishing full scale invariance: [i] The granularity of the scale invariance that the RG analysis gives us is directly determined by the range of length scales we integrate in a single RG step (which is what L represents) , [ii] L governs the contraction of irrelevant parameters and must be taken sufficiently large to defeat various combinatorial factors, i.e. we can't expect to be able to take $L = p$ in the previous RG construction.

To prove a stronger scale invariance property we proceed somewhat indirectly - we will show that the measures ν_ϕ and ν_{ϕ^2} produced by the RG construction do not actually depend on one's choice of L.

Throughout this section p will return to its rightful place as the fundamental length scale and accordingly our scale indices will be given in terms of p. Given $\epsilon > 0$ we define $C_{-\infty}$ as in Chapter 3,

$$\hat{C}_{-\infty}(k) = \frac{1}{|k|^{\frac{3+\epsilon}{2}}}.$$

and for $r \in \mathbb{Z}$ we (re)-define the covariance C_r with UV cut-off at scale r by

$$\hat{C}_r(k) = \frac{\mathbb{1}\{|k| \leqslant p^{-r}\}}{|k|^{\frac{3+\epsilon}{2}}}$$

. We remark that μ_{C_r} is supported on a subspace of functions inside $S(\mathbb{Q}_p^3)$, in particular a subspace of functions constant over the translates of $p^{-r}\mathbb{Z}_p^3$.

We denote by Λ_s the set $p^{-s}\mathbb{Z}_p^3$, i.e. all those $x \in \Lambda_s$ with $|x| \leqslant p^s$. For given parameters $g > 0$ and

$\mu, Z_0, Y_0, Z_2, Y_2, \in \mathbb{R}$ we define

$$\mathcal{Z}_{r,s}(\tilde{f},\tilde{j}) = \int\limits_{S'(\mathbb{Q}_p^3)} \mathrm{d}\mu_{C_r}(\tilde{\phi}) \exp\left[-\int_{\Lambda_s} d^3x\ p^{-\epsilon r} g : \tilde{\phi}^4(x) :_{C_r} + p^{-\frac{(3+\epsilon)}{2} r} \mu : \tilde{\phi}^2(x) :_{C_r} - \tilde{\phi}(x)\tilde{f}(x) - N_r[\tilde{\phi}^2](x)\tilde{j}(x)\right]$$

where $\tilde{f}, \tilde{j} \in S(\mathbb{Q}_p^3, \mathbb{C})$ and we have set

$$N_r[\phi^2](x) \ = \ Z_2^r Y_2 : \phi^2(x) :_{C_r} - Y_0 Z_0^r.$$

We also define the moment generating functions

$$\mathcal{S}_{r,s}(\tilde{f},\tilde{j}) := \frac{\mathcal{Z}_{r,s}(\tilde{f},\tilde{j})}{\mathcal{Z}_{r,s}(0,0)}.$$

In Chapter 3 we fixed $L = p^l$ for l a positive integer and established control over the limit

$$\lim_{\substack{r\to-\infty \\ s\to\infty}} \mathcal{S}_{lr,ls}(\tilde{f},\tilde{j})$$

for appropriately small $\tilde{f}$, $\tilde{j}$.

By paying attention to the quantifiers used in the statement of Theorem 3.1 it is not hard to see that given various distinct values of L (each sufficently large) - it is possible to find a small enough ϵ_0 such that for every fixed $\epsilon \in (0, \epsilon_0]$ we can apply Theorem 3.1 for this value of ϵ simultaneously for two distinct values l_1 and l_2 for l, we state this corollary of Theorem 3.1 below.

Corollary 4.1.

$\exists \rho > 0$, $\exists l_0$ a positive integer, such that $\forall$ positive integers $l_1, l_2 \geqslant l_0$, $\exists \epsilon_0 > 0$, $\forall \epsilon \in (0, \epsilon_0]$, such that for $i = 1, 2$ one can find $\eta_{\phi^2,i} > 0$ and functions $\mu_i(g), Y_{0,i}(g), Y_{2,i}(g)$ of g in the intervals $(\bar{g}_{*,i} - \rho\epsilon^{\frac{3}{2}}, \bar{g}_{*,i} + \rho\epsilon^{\frac{3}{2}})$, where

$$\bar{g}_{*,i} = \frac{p^\epsilon - 1}{36 p^{l_i \epsilon}(1 - p^{-3})}, \tag{4.1}$$

such that if one sets $\mu = \mu_i(g)$, $Z_2 = p^{-\frac{1}{2}\eta_{\phi^2,i}}$, $Y_0 = Y_{0,i}(g)$ and $Y_2 = Y_{2,i}(g)$ in the previous definitions, then for all collections of test functions $f_1, \ldots, f_n, j_1, \ldots, j_m$, the limits

$$\begin{aligned}&\left\langle \phi(f_1) \cdots \phi(f_n) N[\phi^2](j_1) \cdots N[\phi^2](j_m) \right\rangle_i \\ &\qquad := \lim_{\substack{r\to-\infty \\ s\to\infty}} \left\langle \phi(f_1) \cdots \phi(f_n) N_r[\phi^2](j_1) \cdots N_r[\phi^2](j_m) \right\rangle_{\underline{r},\underline{s},i}\end{aligned}$$

exist and do not depend on the order in which the $r \to -\infty$ and $s \to \infty$ limits are taken. Here for an integer $n \in \mathbb{Z}$ we set $\underline{n} = n \times l_1 \times l_2$. The correlators $\langle \bullet \rangle_{r,s,i}$ correspond to those given by functional derivatives of the quantity $\mathcal{S}_{r,s,i}(\tilde{f},\tilde{j})$ where this quantity is defined with the parameters with subscript i.

Moreover, the the limiting correlators satisfy the following properties:

1) They are left invariant by any translations or rotations of all the test functions $f_1, \ldots, f_m, j_1, \ldots, j_m$.

2) They satisfy the partial scale invariance property

$$\left\langle \phi\left(S_\lambda f_1\right) \cdots \phi\left(S_\lambda f_n\right)\ N[\phi^2]\left(S_\lambda j_1\right) \cdots N[\phi^2]\left(S_\lambda j_m\right) \right\rangle_i =$$

$$|\lambda|_p^{(3-[\phi])n+(3-2[\phi]-\frac{1}{2}\eta_{\phi^2})m} \left\langle \phi(f_1) \cdots \phi(f_n)\ N[\phi^2](j_1) \cdots N[\phi^2](j_m) \right\rangle_i$$

for all $\lambda \in p^{l_i \mathbb{Z}}$.

3) They satisfy the nontriviality conditions

$$\langle \phi(\mathbb{1}_{\mathbb{Z}_p^3})^4 \rangle_i - 3\langle \phi(\mathbb{1}_{\mathbb{Z}_p^3})^2 \rangle_i < 0\ ,$$

$$\langle N[\phi^2](\mathbb{1}_{\mathbb{Z}_p^3})^2 \rangle_i = 1\ .$$

4) The pure ϕ *correlators with subscript* i *are the moments of a unique probability measure* $\nu_{\phi,i}$ *on* $S'(\mathbb{Q}_p^3)$ *with finite moments. This measure is translation and rotation invariant. It is also partially scale invariant with scaling parameter* $[\phi]$ *with respect to the scaling subgroup* $p^{l_i \mathbb{Z}}$ *- i.e.* $\left(|\lambda|_p^{-[\phi]} \hat{S}_\lambda\right)^{\#} \nu_{\phi,i} = \nu_{\phi,i}$ *for all* $\lambda \in p^{l_i \mathbb{Z}}$

5) The pure $N[\phi^2]$ *correlators with subscript* i *are the moments of a unique probability measure* $\nu_{\phi^2,i}$ *on* $S'(\mathbb{Q}_p^3)$ *with finite moments. This measure is translation and rotation invariant. It is also partially scale invariant with scaling parameter* $2[\phi]+\frac{1}{2}\eta_{\phi^2}$ *with respect to the scaling subgroup* $p^{l_i \mathbb{Z}}$ *- i.e.* $\left(|\lambda|_p^{-(2[\phi]+\frac{1}{2}\eta_{\phi^2})} \hat{S}_\lambda\right)^{\#} \nu_{\phi^2,i} = \nu_{\phi^2,i}$ *for all* $\lambda \in p^{l_i \mathbb{Z}}$

6) For each i *the measures* $\nu_{\phi,i}$ *and* $\nu_{\phi^2,i}$ *satisfy a mild form of universality: they do not depend on the choice of* g *in the above-mentioned interval.*

Note that in the above corollary we introduced the notation $\underline{\cdot}$ for scale indices so we can work along a subsequence of cut-offs for which we are guaranteed convergence for both RG constructions.

If we make a choice of the form $l_1 = l > l_0$ and $l_2 = l + 1$ in Corollary 4.1 then proving that

$$\nu_{\phi,1} \;=\; \nu_{\phi,2} \text{ and } \nu_{\phi^2,1} = \nu_{\phi^2,2} \tag{4.2}$$

would show that the above measures are fully scale invariant - this follows since the subgroups $p^{l\mathbb{Z}}$ and $p^{(l+1)\mathbb{Z}}$ together generate the full scaling group $p^{\mathbb{Z}}$.

A crucial fact for our approach to proving this is that with the given choices of l_1 and l_2 one has

$$\begin{aligned} |\bar{g}_{*,2} - \bar{g}_{*,1}| \;&=\; \frac{p^{-(l-1)\epsilon}}{36(1-p^{-3})}\left(1-p^{-\epsilon}\right)^2 \\ &\leqslant O(\epsilon^2) \end{aligned}$$

so for ϵ sufficently small one has non-empty interval of intersection of domains for g:

$$(\bar{g}_{*,1} - \rho\epsilon^{\frac{3}{2}}, \bar{g}_{*,1} + \rho\epsilon^{\frac{3}{2}}) \cap (\bar{g}_{*,2} - \rho\epsilon^{\frac{3}{2}}, \bar{g}_{*,2} + \rho\epsilon^{\frac{3}{2}}) = \;I_{\epsilon,l}\;\neq \varnothing.$$

Our method of proving (4.2) hinges on showing that the functions $\mu_1(g)$ and $\mu_2(g)$ must coincide on the interval $I_{\epsilon,l}$ - i.e. the bare slice of the two stable manifolds delivered by each RG constructions must agree on their common domain of definition.

We show that $\mu_1(g) = \mu_2(g)$ immediately implies that $\nu_{\phi,1} = \nu_{\phi,2}$. If one chooses $g \in I_{\epsilon,l}$ then it is immediate that for all test functions $\tilde{f}$ and $r, s \in \mathbb{Z}$

$$\mathcal{S}_{\underline{r},\underline{s},1}(\tilde{f},0) \; = \; \mathcal{S}_{\underline{r},\underline{s},2}(\tilde{f},0)$$

and so

$$\langle \phi(f_1)\cdots\phi(f_n)\rangle_{\underline{r},\underline{s},1} = \langle \phi(f_1)\cdots\phi(f_n)\rangle_{\underline{r},\underline{s},2}\,.$$

The assertion follows by taking the limit $r \to -\infty$, $s \to \infty$.

We now show how the equality $\mu_1(g) = \mu_2(g)$ implies the coinciding of measures $\nu_{\phi^2,1} = \nu_{\phi^2,2}$. The key observation here is that the multiplicative renormalizations $\mathcal{Z}_{2,i}$ must be chosen precisely in order to avoid having a degenerate law for either composite field. To see this it is convenient to look at $N_r[\phi^2]$ cumulants for the cut-off measures. Observe that

$$\langle N[\phi^2](\mathbb{1}_{\mathbb{Z}_p^3}),\ N[\phi^2](\mathbb{1}_{\mathbb{Z}_p^3})\rangle^T_{\underline{r},\underline{s},1} \;=\; \left(\frac{Y_{2,1}(g)}{Y_{2,2}(g)}\right)^2 \times \left(\frac{Z_{2,1}}{Z_{2,2}}\right)^{2rl(l+1)} \langle N[\phi^2](\mathbb{1}_{\mathbb{Z}_p^3})^2\rangle_{\underline{r},\underline{s},2}.$$

One must have $Z_{2,1},\ Z_{2,2},\ Y_{2,1}(g),\ Y_{2,2}(g) \neq 0$, otherwise this would lead to a degenerate law for a composite field. In particular the RG construction fixes $Z_{2,i} = L^{-\frac{3+\epsilon}{2}}\alpha_{\mathrm{u},i} \neq 0$ and in section 3.11.2 it is shown we choose the $Y_{2,i}$ in a way that guarantees they are non-vanishing. Now remembering that

$$\lim_{\substack{r\to-\infty\\ s\to\infty}} \langle N[\phi^2](\mathbb{1}_{\mathbb{Z}_p^3}),\ N[\phi^2](\mathbb{1}_{\mathbb{Z}_p^3})\rangle^T_{\underline{r},\underline{s},1} = \lim_{\substack{r\to-\infty\\ s\to\infty}} \langle N[\phi^2](\mathbb{1}_{\mathbb{Z}_p^3}),\ N[\phi^2](\mathbb{1}_{\mathbb{Z}_p^3})\rangle^T_{\underline{r},\underline{s},2} = 1$$

one sees that it must be the case that

$$\lim_{r\to-\infty} \left(\frac{Y_{2,1}(g)}{Y_{2,2}(g)}\right)^2 \times \left(\frac{Z_{2,1}}{Z_{2,2}}\right)^{2rl(l+1)} = 1$$

from which it follows that $Y_{2,1}(g) = Y_{2,2}(g)$ and $Z_{2,1} = Z_{2,2}$. With this in hand it is immediate that all cumulants of order higher than 1 for the cut-off measures coincide (the point here being that the choice of parameters $Z_{0,i}, Y_{0,i}(g)$ only influences the order 1 cumulants, i.e. the first moment). Since Theorem 3.1 asserts that the first moment of our constructed composite fields must vanish it follows that *all* cumulants of $\nu_{\phi^2,1}$ and $\nu_{\phi^2,2}$ coincide. We remark that this shows equality of the anomalous dimensions, i.e $\eta_{\phi^2,1} = \eta_{\phi^2,2}$.

4.2 Formulation as Statistical Mechanics

We accomplish proving that $\mu_1(g) = \mu_2(g)$ for $g \in I_{\epsilon,l}$ by recasting this as a problem of statistical mechanics. In what follows we will sometimes make the dependence of the measures $\nu_{r,s}$ on g, μ explicit by writing $\nu_{r,s}[g,\mu]$.

In Theorem 3.1 the measure ν_ϕ was realized as a limit in the sense of moments of measures $\nu_{jr,js}[g,\mu(g)]$ on $S(\mathbb{Q}_p^3)$ for some $L = p^j$ where we took $r \to -\infty, s \to \infty$, removing the UV cut-off and the IR cut-off respectively. A key point is that our RG machinery, without any real changes, can show the convergence of the measures $\lim_{s\to\infty} \nu_{0,js}[g,\mu(g)]$ to a limiting measure $\nu_{0,\infty}[g,\mu(g)]$.

When we keep $r = 0$ fixed the measures $\nu_{0,s}$ are supported on functions constant over the the blocks of $\mathbb{L}$. Equivalently we can see the measures $\nu_{0,s}$ as living on $\mathbb{R}^{\mathbb{L}}$ equipped with its cylinder set σ-algebra. Remembering that $\mathbb{L}$ can be thought of as the "integer lattice" corresponding to $\mathbb{Q}_p^3$ one can interpret the elements of $\mathbb{R}^{\mathbb{L}}$ as lattice field configurations $\{\phi_x\}_{x\in\mathbb{L}}$. Later on we will focus more on the finite volume marginal of $\nu_{0,s}$ on $\mathbb{R}^{\Lambda_s}$ where we overload notation and define $\Lambda_s := \{x \in \mathbb{L} \mid |x| \leqslant p^s\}$.

We will view the measures $\nu_{0,s}$ as models of statistical mechanics, in particular Ising models with unbounded spins. The limit $s \to \infty$ with r fixed to be 0 corresponds to taking what is sometimes called a *thermodynamic limit* in statistical mechanics. Taking the limit $r \to -\infty$ afterwards then correspond to a *scaling limit* or *continuum limit*. This latter fact is a consequence of the observation that if ϕ is distributed according to $\nu_{r,s}$ then $p^{-[\phi]}\phi(p\cdot)$ is distributed according to $\nu_{r-1,s-1}$.

4.3 Ising Models and some fundamental correlation inequalities

We now introduce some definitions, followed by proving some fundamental correlation inequalities that will be needed later.

We also introduce some new notation: for any set X we denote by $X^{(2)}$ the set of all two element subsets of X.

Definition. *For any finite set Λ a* ***classical Ising model*** *on Λ is a measure on the space of spin configurations $\sigma_\Lambda = \{\sigma_x\}_{x\in\Lambda} \in \{-1,1\}^\Lambda$ of the following form:*

$$\langle \sim \rangle = \frac{1}{\mathcal{Z}} \sum_{\sigma_\Lambda \in \{-1,1\}^\Lambda} \sim \exp\left[\sum_{\{x,y\}\in\Lambda^{(2)}} J_{\{x,y\}} \sigma_x \sigma_y + \sum_{x\in\Lambda} h_x \sigma_x \right]$$

We will call $\{J_{\{x,y\}}\}_{\{x,y\}\in\Lambda^{(2)}}$ the interaction and $\{h_x\}_{x\in\Lambda}$ the external field. If in the above definition $J_{\{x,y\}} \geqslant 0$ for all $\{x,y\} \in \Lambda^{(2)}$ and $h_x \geqslant 0$ for all $x \in \Lambda$ we call the system a ***classical Ising ferromagnet****.*

Definition. *For any finite set Λ a* ***generalized Ising model*** *on Λ is a measure on the space of spin configurations $\phi_\Lambda = \{\phi_x\}_{x\in\Lambda} \in \mathbb{R}^\Lambda$ of the following form:*

$$\langle \sim \rangle = \frac{1}{\mathcal{Z}} \int_{\mathbb{R}^\Lambda} \sim \exp\left[\sum_{\{x,y\}\in\Lambda^{(2)}} J_{x,y} \phi_x \phi_y + \sum_{x\in\Lambda} h_x \phi_x \right] \prod_{x\in\Lambda} d\rho(\phi_x)$$

We require that the single site spin measure $d\rho$ be even, not have an atom at 0, and to also satisfy the following integrability condition:

$$\int_{\mathbb{R}} e^{\alpha s^2} d\rho(s) < \infty \text{ for any } \alpha \in \mathbb{R} \tag{4.3}$$

We will use the notation $d\rho_\Lambda(\phi_\Lambda)$ to represent the product measure $\prod_{x\in\Lambda} d\rho(\phi_x)$.

If in the above definition $J_{\{x,y\}} \geqslant 0$ for all $\{x,y\} \in \Lambda^{(2)}$ and $h_x \geqslant 0$ for all $x \in \Lambda$ we call the system a ***generalized Ising ferromagnet****.*

Note that with the definitions above any classical ferromagnetic Ising model is also a generalized ferromagnetic Ising model.

In most versions of the above definitions one sees prefactor $\beta \in [0, \infty)$ for the interaction $\{J_{\{x,y\}}\}_{\{x,y\}\in\Lambda^{(2)}}$ which is called the *inverse temperature* but for now we absorb it into our definition of the interaction, however we will re-introduce it as a parameter later.

We can also absorb "boundary condition" prescriptions into the above definition, absorbing them into the external field. Note that a ferromagnetic system under the influence of sufficiently summable non-negative boundary configuration is again ferromagnetic - the boundary spins give a non-negative contributions to the external field $\{h'_x\}_{x\in\Lambda} = \{\sum_{y\notin\Lambda} J_{x,y} S_y\}_{x\in\Lambda}$ where S_{Λ^c} represents the vector of external spins.

The first correlation inequalities we give are the first and second Griffiths' Inequalities which apply to all generalized ferromagnetic Ising models.

The following lemma is taken from [44] and is based on the approach of [33].

Lemma 4.1. *Let $d\rho$ be an even measure on $\mathbb{R}$ with all moments finite. Suppose that $f_\alpha(s)$, $\alpha = 1, ..., n$ are odd monotone non-decreasing polynomially bounded functions of $s \in \mathbb{R}$. Let $Q(s, s')$ be a bounded symmetric, even, non-negative function of $s, s' \in \mathbb{R}$ (that is $Q(s, s') = Q(s', s) = Q(-s, -s') \geqslant 0$)*

Then for any collection of non-negative integers k_α, l_α one has the following inequality:

$$M = \int_{\mathbb{R}} d\rho(s) \int_{\mathbb{R}} d\rho(s') \prod_{\alpha=1}^{n} \left[f_\alpha(s) - f_\alpha(s')\right]^{k_\alpha} \left[f_\alpha(s) + f_\alpha(s')\right]^{l_\alpha} Q(s, s') \geqslant 0. \tag{4.4}$$

Proof: Case 1: Suppose both $\sum_\alpha k_\alpha$ and $\sum_\alpha l_\alpha$ are even.

By the assumption of monotonicity if $s \geqslant s'$ then for all α the quantity $f_\alpha(s) - f_\alpha(s') \geqslant 0$, while if $s \leqslant s'$ then the quantity $f_\alpha(s) - f_\alpha(s') \leqslant 0$. In either case we have that

$$\prod_{\alpha=1}^{n} \left[f_\alpha(s) - f_\alpha(s')\right]^{k_\alpha} \geqslant 0.$$

This follows since the product above is either zero or has an even number of terms of the same sign. Using the fact that the f_α are odd we can use the same reasoning when looking at $f_\alpha(s) + f_\alpha(s') = f_\alpha(s) - f_\alpha(-s')$, the terms are of the same sign when $s \geqslant -s'$ or $s \leqslant -s$. Hence the entire integrand is non-negative.

Case 2: Suppose $\sum_\alpha k_\alpha$ is odd. Then by changing variables $s \leftrightarrow s'$ one sees that $M = (-1)^{\sum_\alpha k_\alpha} M$ so M vanishes.

Case 3: Suppose $\sum_\alpha k_\alpha$ is even but $\sum_\alpha l_\alpha$ is odd. Then by changing variables $s \leftrightarrow -s$ and $s' \leftrightarrow -s'$ one has $M = (-1)^{\sum_\alpha (k_\alpha + l_\alpha)} M$ so again M vanishes. □

Theorem 4.1. *For any generalized ferromagnetic Ising model on Λ one has the following inequalities:*

For any multi-index A supported on Λ

$$\left\langle \phi^A \right\rangle \geqslant 0. \tag{Griffiths I}$$

For any multi-indices A and B supported on Λ one has:

$$\left\langle \phi^A \phi^B \right\rangle - \left\langle \phi^A \right\rangle \left\langle \phi^B \right\rangle \geqslant 0. \tag{Griffiths II}$$

Proof: We start with Griffiths I. Note that

$$\int_{\mathbb{R}^\Lambda} d\mu_\Lambda(\phi_\Lambda)\phi^A = \mathcal{Z}^{-1} \int_{\mathbb{R}^\Lambda} \left(\sum_{n=0}^{\infty} \frac{1}{n!} \left[\left(\sum_{\{x,y\} \in \Lambda^{(2)}} J_{x,y} \phi_x \phi_y \right) + \left(\sum_{x \in \Lambda} h_x \phi_x \right) \right]^n \right) \phi^A d\rho_\Lambda(\phi_\Lambda).$$

One can expand the various products and sums and interchange the summation with the integration over ϕ_Λ (this interchange is allowed due to the integrability condition on $d\rho$). One will then have an expression of the form:

$$\sum_{B \in \mathbb{N}^\Lambda} c_B \int_{\mathbb{R}^\Lambda} \phi^B d\rho_\Lambda(\phi_\Lambda).$$

The coefficients c_B will all be non-negative (this follows because $J_{\{\cdot,\cdot\}}, h_\cdot \geqslant 0$, $Z > 0$). Since $d\rho$ is even all the above moments will be positive or vanish. Thus sum must be non-negative - this proves Griffiths I.

To prove Griffiths II we introduce duplicate sets of variables ϕ_Λ and ψ_Λ each distributed according to μ_Λ. Griffiths II will follow if we show:

$$\int_{\mathbb{R}^\Lambda} d\mu_\Lambda(\phi_\Lambda) \int_{\mathbb{R}^\Lambda} d\mu_\Lambda(\psi_\Lambda) \left(\phi^A - \psi^A\right) \left(\phi^B - \psi^B\right) \geqslant 0.$$

We rewrite the left hand side:

$$\mathcal{Z}^{-2} \int_{\mathbb{R}^\Lambda} d\rho_\Lambda(\phi_\Lambda) \int_{\mathbb{R}^\Lambda} d\rho_\Lambda(\psi_\Lambda) \left(\sum_{n=0}^{\infty} \frac{1}{n!} \left[\left(\sum_{\{x,y\} \in \Lambda^{(2)}} J_{x,y} \left(\phi_x \phi_y + \psi_x \psi_y\right) \right) + \left(\sum_{x \in \Lambda} h_x \left(\phi_x + \psi_x\right) \right) \right]^n \times \left(\phi^A - \psi^B\right) \times \left(\phi^B - \psi^B\right) \right). \tag{4.5}$$

We now make use of the following identity:

$$\left(a_i a_j + b_i b_j\right) = \frac{1}{2} \left[\left(a_i + b_i\right)\left(a_j - b_j\right) + \left(a_i - b_i\right)\left(a_j + b_j\right) \right]. \tag{4.6}$$

Using this one can write:

$$J_{x,y} \left(\phi_x \phi_y + \psi_x \psi_y\right) = \frac{1}{2} J_{x,y} \left[\left(\phi_x + \psi_x\right)\left(\phi_y - \psi_y\right) + \left(\phi_x - \psi_x\right)\left(\phi_y + \psi_y\right) \right].$$

We insert this expression into (4.5), then interchange the summation over n with the integration over $\mathbb{R}^{\mathbb{L}} \times \mathbb{R}^{\mathbb{L}}$ (valid by Fubini-Tonelli), and also expand out the n-fold product. We will be left with an infinite sum of integrals. Every integral in the sum will have a non-negative coefficient since $J_{\{\cdot,\cdot\}}, h_\cdot \geqslant 0$ and $\mathcal{Z} > 0$.

The integrals in the sum will have an integrand built up out of products of the form $(\phi_x - \psi_x)$ and $(\phi_y + \psi_y)$. Thus each of the integrals appearing in this sum can be factorized into a product of pairs of double integrals - each pair corresponding to a particular lattice site - these factors will take the following form:

$$\int_{\mathbb{R}} d\rho(\phi_x) \int_{\mathbb{R}} d\rho(\psi_x) \ (\phi_x + \psi_x)^k (\phi_x - \psi_x)^l \,.$$

These double integrals are of the form (4.4) where $Q(\cdot, \cdot) = 1$ and $f_\alpha(s) = s$. Thus each of the integral factors is non-negative which means (4.5) is non-negative. □

We note that Griffiths II tells us that for Ising ferromagnets the expectation of moments is non-decreasing under an increase in the interaction or the external field. More concretely if μ is an Ising ferromagnet on Λ then for any $\{x, y\} \in \Lambda^{(2)}$ one has:

$$\frac{\partial}{\partial J_{\{x,y\}}} \langle \phi^A \rangle_{\mu(\vec{J},\vec{h})} = \langle \phi^A \phi_x \phi_y \rangle_{\mu(\vec{J},\vec{h})} - \langle \phi^A \rangle_{\mu(\vec{J},\vec{h})} \langle \phi_x \phi_y \rangle_{\mu(\vec{J},\vec{h})} \geqslant 0.$$

In particular Griffiths II proves that for a fixed interaction the expectations of moments are non-decreasing as one takes a larger volume. For example let $\Lambda_1 \subset \Lambda_2$ and let $\{J_{\{x,y\}}\}_{\{x,y\} \in \Lambda_2^{(2)}}$ be a ferromagnetic interaction. Suppose we have two models, $\langle \sim \rangle_{\Lambda_2}$ defined over Λ_2 with the given interaction and $\langle \sim \rangle_{\Lambda_1}$ defined over Λ_1 with the restriction of the same interaction to bonds in Λ_1. However $\langle \sim \rangle_{\Lambda_1}$ can be seen a modified version of $\{J_{\{x,y\}}\}_{\{x,y\} \in \Lambda_2^{(2)}}$ where the interaction has been set to zero for bonds that don't have both endpoints within Λ_1. Thus by Griffiths II one has $\langle \phi^A \rangle_{\Lambda_1} \leqslant \langle \phi^A \rangle_{\Lambda_2}$. This will allow us to use Griffiths II to help us establish the existence of infinite volume limits of finite volume Ising models.

The next three results we give hold for classical Ising ferromagnets and a proper subset of generalized Ising ferromagnets. We will state these results for classical Ising ferromagnets and refer to the literature for their proofs. After explaining the Griffiths-Simons approximation we will show that these carry over to the continuous spin models of interest.

Theorem 4.2. *For any classical Ising ferromagnet on Λ and any $i, j, k \in \Lambda$ one has*

$$\begin{aligned} \langle \sigma_i, \sigma_j, \sigma_k \rangle^T :=& \langle \sigma_i \sigma_j \sigma_k \rangle - \langle \sigma_i \rangle \langle \sigma_j \sigma_k \rangle - \langle \sigma_j \rangle \langle \sigma_i \sigma_k \rangle - \langle \sigma_k \rangle \langle \sigma_i \sigma_j \rangle + 2 \langle \sigma_i \rangle \langle \sigma_j \rangle \langle \sigma_k \rangle \\ \leqslant& \, 0 \end{aligned} \tag{GHS}$$

Proof: This is known as the Griffiths-Hurst-Sherman Inequality, the original proof can be found in [36]. □

Theorem 4.3. *Let Λ be a finite set and let $\{J_{\{x,y\}}\}_{\{x,y\} \in \Lambda^{(2)}}$ be a ferromagnetic interaction, that is $J_{\{\cdot,\cdot\}} \geqslant 0$.*

Define the function $\mathcal{Z}_\Lambda(h_\Lambda)$ as follows:

$$\mathcal{Z}_\Lambda(h) = \sum_{\sigma_\Lambda \in \{-1,1\}^\Lambda} \exp\left[\sum_{\{x,y\} \in \Lambda^{(2)}} J_{\{x,y\}} \sigma_x \sigma_y + \sum_{x \in \Lambda} h_x \sigma_x \right].$$

Then if $\mathcal{Z}_\Lambda(h_\Lambda)$ is viewed as a function of $h_\Lambda \in \mathbb{C}^\Lambda$ one has that $\mathcal{Z}_\Lambda(h_\Lambda)$ does not vanish if $\Re(h_x) \neq 0$ for any $x \in \Lambda$.

Equivalently if $\langle \sim \rangle$ is the expectation for a classical Ising ferromagnet over a finite set Λ with zero external field then the following expression

$$\left\langle \exp\left(\sum_{x\in\Lambda} h_x \sigma_x\right)\right\rangle$$

does not vanish if $\Re(h_x) \neq 0$ for any $x \in \Lambda$.

Proof : This is the Lee-Yang Theorem which originated in the paper [46]. The second statement is equivalent to the first since $\left\langle \exp\left(\sum_{x\in\Lambda} h_x \sigma_x\right)\right\rangle = \dfrac{\mathcal{Z}_\Lambda(h_\Lambda)}{Z_\Lambda(0)}$ and $Z_\Lambda(0) > 0$. □

4.3.1 Griffiths-Simon Approximation

We now narrow our attention to the family of generalized Ising models that will be of main interest to us.

Definition: *A ϕ^4 model is on Λ is a generalized Ising model on Λ with a single site spin measure of the following form:*

$$d\rho(s) = \exp\left[-gs^4 - bs^2\right] \text{ with } g > 0,\ s \in \mathbb{R}.$$

If the interaction and the field are non-negative then we call the model a ferromagnetic ϕ^4 model.

The correlation inequalities of the previous subsection which were stated just for classical ferromagnetic Ising models can be extended to ferromagnetic ϕ^4 models via an approximation technique due to Griffiths and Simon [65]. The method involves approximating the distribution $d\rho$ with the distribution of the magnetization (scaled) of a carefully chosen "mean-field" classical Ising model.

We give a few more details on structure of the approximation. Fix N to be a large positive integer. Each point $x \in \Lambda$ will have a corresponding family of classical Ising spins $\{\sigma_{(x,\alpha)}\}_{\alpha=1}^N$, we call this family a block. Suppose that this family of spins are distributed according to the following measure:

$$\frac{1}{\mathcal{Z}} \sum_{\{\sigma_{(x,\alpha)}\}\in\{-1,1\}^N} \cdots \exp\left[d_N \sum_{\alpha,\delta=1}^{N} \sigma_{(x,\alpha)}\sigma_{(x,\delta)}\right]$$

where we defined $d_N = (2N)^{-1}\left[1 - b\,(3gN)^{-\frac{1}{2}}\right]$ and b, g are given as above.

Define $c_N = \left(\dfrac{N}{6g}\right)^{\frac{1}{4}} N^{-1}$. Then [65] shows the random variable $\phi_x = c_N \sum_{\alpha=1}^N \sigma_{(x,\alpha)}$ will weakly converge to a random variable distributed according to $d\rho(s) = \exp\left[-gs^4 - bs^2\right]$ (modulo normalization). In fact the analysis of [65] proves a much stronger statement which we now describe. Below we use the notation

$[N] := \{1, 2, \ldots, N-1, N\}$.

For any function F in the components of $\phi_\Lambda = \{\phi_x\}_{x\in\Lambda} \in \mathbb{R}^\Lambda$ we define $\theta_N(F)$, a function of spin variables $\{\sigma_{(x,\alpha)}\}_{(x,\alpha)\in\Lambda\times[N]}$ as follows:

$$\theta_N(F)(\sigma_{\Lambda\times[N]}) = F\left(\left\{\sum_{\alpha=1}^{N} c_N \sigma_{(x,\alpha)}\right\}_{x\in\Lambda}\right).$$

For example if P is a polynomial in the components of $\phi_\Lambda \in \mathbb{R}^\Lambda$:

$$P(\phi_\Lambda) = \sum_{A\in\mathbb{N}^\Lambda} P_A \prod_{x\in\Lambda} \left[\phi_x^{A(x)}\right]$$

then

$$\theta_N(P)(\sigma_{\Lambda\times[N]}) = \sum_{A\in\mathbb{N}^\Lambda} P_A \prod_{x\in\Lambda} \left[\left(c_N \sum_{\alpha=1}^{N} \sigma_{(x,\alpha)}\right)^{A(x)}\right].$$

With this notation in hand we state the Griffiths-Simon approximation theorem.

Theorem 4.4 ([65]). *Suppose one is given a ϕ^4 model μ_Λ on Λ defined in terms of an appropriate $\{J_{\{x,y\}}\}_{\{x,y\}\in\Lambda^{(2)}}$, $\{h_x\}_{x\in\Lambda}$, g, and b.*

We define the measure μ_Λ^N on $\mathbb{R}^{\Lambda\times[N]}$ as follows: For any function H on $\mathbb{R}^{\Lambda\times[N]}$ we have

$$\begin{aligned}\langle H\rangle_{\mu_\Lambda^N} =& \frac{1}{\mathcal{Z_N}} \sum_{\sigma_{\Lambda\times[N]}\in\{-1,1\}^{\Lambda\times[N]}} H(\sigma_{\Lambda\times[N]}) \exp\left[\sum_{x\in\Lambda}\left(d_N \sum_{\alpha,\delta=1}^{N} \sigma_{(x,\alpha)}\sigma_{(x,\delta)}\right)\right. \\ &\left. +\beta \sum_{\{x,y\}\in\Lambda^{(2)}} J_{\{x,y\}} \left(c_N \sum_{\alpha=1}^{N} \sigma_{(x,\alpha)}\right)\left(c_N \sum_{\alpha=1}^{N} \sigma_{(y,\alpha)}\right) + \sum_{x\in\Lambda} h_x \left(c_N \sum_{\alpha=1}^{N} \sigma_{(x,\alpha)}\right)\right].\end{aligned} \tag{4.7}$$

Suppose G is a measurable function on $\mathbb{R}^\Lambda$ which can be dominated pointwise by a Gaussian, that is:

$$\sup_{\phi_\Lambda\in\mathbb{R}^\Lambda} \left|G(\phi_\Lambda) e^{-t\sum_{x\in\Lambda}\phi_x^2}\right| < \infty$$

for some $t \geqslant 0$. Then one has

$$\lim_{N\to\infty} \langle\theta_N(G)\rangle_{\mu_\Lambda^N} = \langle G\rangle_{\mu_\Lambda}. \tag{4.8}$$

Proof: See Theorem 1 in [65]. □.

Note that if $\langle\sim\rangle_{\mu_\Lambda}$ is a ϕ^4 ferromagnet then for sufficiently large N the model μ_Λ^N is also ferromagnetic. Whenever this approximation is applied in this paper we assume that N has been taken sufficiently large for the above implication to hold.

Following [65] we immediately have the GHS inequality and Lee-Yang theorem for ϕ^4 ferromagnets.

Theorem 4.5. *[GHS Inequality for ϕ^4 ferromagnets] Let $\langle\cdot\rangle_{\mu_\Lambda}$ be a ferromagnetic ϕ^4 model over some set Λ. Then for any $x, y, z \in \Lambda$ one has*

$$\langle \phi_x, \phi_y, \phi_z \rangle^T_{\mu_\Lambda} \leqslant 0$$

Proof:

The proof follows easily from the multi-linearity of the inequality, Theorem 4.4, and the knowledge that this inequality holds for classical Ising ferromagnets (Theorem 4.2). We note that

$$\langle \theta_N(\phi_x), \theta_N(\phi_y), \theta_N(\phi_z) \rangle^T_{\mu^N_\Lambda} = c_N^3 \sum_{\alpha,\delta,\gamma=1}^{N} \langle \sigma_{(x,\alpha)}, \sigma_{(y,\delta)}, \sigma_{(z,\gamma)} \rangle^T_{\mu^N_\Lambda} \leqslant 0.$$

The desired statement follows by taking the the $N \to \infty$ limit. □.

Theorem 4.6. *[Lee-Yang Theorem for ϕ^4 ferromagnets]*

The function $\mathcal{Z}_\Lambda(g, b, \beta, h)$ does not vanish for $h \in \mathbb{C}$ with $\Re(h) \neq 0$

Equivalently if $\langle \sim \rangle_{\mu_\Lambda}$ is the expectation for a ferromagnetic ϕ^4 model over a finite set Λ with zero external field then the following expression

$$\left\langle \exp\left[\sum_{x\in\Lambda} h_x \phi_x\right]\right\rangle_{\mu_\Lambda} \tag{4.9}$$

does not vanish if $\Re(h_x) \neq 0$ for any $x \in \Lambda$.

In particular $\mathcal{Z}_\Lambda(g, b, \beta, h)$ does not vanish for $h \in \mathbb{C}$ with $\Re(h) \neq 0$

Proof:

Note that if μ_Λ has then the same is true of its approximating classical Ising models μ^N_Λ - this gives us the equivalence mentioned above by using the same argument used in Theorem 4.3.

For any $h_\Lambda \in \mathbb{C}^\Lambda$

$$f_N(h_\Lambda) := \left\langle \theta_N\left(e^{\sum_{x\in\Lambda} h_x\phi_x}\right)\right\rangle_{\mu^N_\Lambda}$$

We remark that the f_N are analytic on $\mathbb{C}^\Lambda$.

By Theorem 4.4 we know that the f_N converge to a limiting function f on $\mathbb{C}^\Lambda$ as $N \to \infty$ with f coinciding with (4.9) for constant external fields $h_\Lambda = h$.

Now suppose that $h_\Lambda \in \mathbb{C}^\Lambda$ with $\Re(h_x) > 0$ for some $x \in \Lambda$. Then by Theorem 4.3 one has:

$$\begin{aligned} f_N(h_\Lambda) &= \left\langle \theta_N\left(e^{\sum_{x\in\Lambda} h_x\phi_x}\right)\right\rangle_{\mu^N_\Lambda} \\ &= \left\langle \exp\left[\sum_{(x,\alpha)\in\Lambda\times[N]} c_N \times h_x \sigma_{(x,\alpha)}\right]\right\rangle_{\mu^N_\Lambda} > 0. \end{aligned}$$

Thus the functions f_N are all non-vanishing on the open set $U \subset \mathbb{C}^\Lambda$ consisting of all those $h_\Lambda \in \mathbb{C}^\Lambda$ which satisfy $h_x > 0$ for some $x \in \Lambda$. We first check th

We must now show f is non-vanishing on U. For this we will use Theorems 5.4 and 5.5 from the appendix. For Theorem 5.4 all that remains to be shown is local uniform boundedness of the f_N on U can be checked fairly easily - for example using the bound

$$\left|\left\langle \theta_N \left(e^{\sum_{x \in \Lambda} h_x \phi_x} \right) \right\rangle_{\mu_\Lambda^N} \right| \leqslant \left\langle \theta_N \left(e^{\sum_{x \in \Lambda} |h_x| \times |\phi_x|} \right) \right\rangle_{\mu_\Lambda^N}$$

and then observing that the righthand side converges to a finite value as $N \to \infty$ by Theorem 4.4.

Theorem 5.5 then implies that the f is non-vanishing on U or $f = 0$ on U. We now show $f \neq 0$ on U, we show this now. Fix some arbitrary $z \in \Lambda$ and let $\tilde{h}_\Lambda \in U$ be given by $\tilde{h}_z = 1$ and $\tilde{h}_x = 0$ for $x \neq z$. Then we have

$$f(\tilde{h}_\Lambda) \;=\; \left\langle e^{\phi_z} \right\rangle_{\mu_\Lambda} \;=\; \sum_{n=0}^{\infty} \frac{1}{n!} \langle \phi_z^n \rangle_{\mu_\Lambda} \;\geqslant 1. \tag{4.10}$$

The power series expansion is valid by Fubini-Tonelli and our integrability condition on the single site measures $\mathrm{d}\rho$, the last inequality then follows by Griffiths First Inequality. $\square$.

4.4 Rewriting our measures as Ising models

4.4.1 Viewing the measures $\nu_{0,\infty}$ as critical Ising Models

A key difference between how one approaches classical spin systems in statistical mechanics versus models in Euclidean Quantum Field Theory is that in the former the "reference measure" completely factorizes over space and one proceeds to perturb this by some kind of interaction which couples different spins in regions of space. In Euclidean Quantum Field Theory the reference measure is a Gaussian measure which couples together fields in different regions of space and then one perturbs this by a product of local self-interactions.

Our first step in going from the latter setting to the former is writing our Gaussian as a two body interaction. The covariance $C_0 : \mathbb{Q}_p^3 \times \mathbb{Q}_p^3 \to \mathbb{R}$ is given by

$$C_0(x, y) \;=\; \int_{\mathbb{Q}_p^3} \mathrm{d}^3 k \; \frac{\exp[2\pi i \{k \cdot (x-y)\}_p]}{|k|^{3-2[\phi]}} \mathbb{1}\{|k| \leqslant 1\}$$

for $x, y \in \mathbb{Q}_p^3$.

Since $C_0(x, y)$ is locally constant over the blocks of $\mathbb{L}$ we can see it as function $C_0 : \mathbb{L} \times \mathbb{L} \to \mathbb{R}$ and as mentioned before C_0 is also the covariance for a Gaussian measure μ_{C_0} on the product space $\mathbb{R}^{\mathbb{L}}$.

By restriction C_0 defines a $p^{3s} \times p^{3s}$ matrix indexed by the elements of Λ_s which we call M, where the matrix entries for $x, y \in \Lambda_s$ are given by

$$M_{x,y} \;=\; C_0(x - y).$$

The matrix $M_{x,y}$ is clearly symmetric, we now analyze it further.

We note that the complex vector space $\mathbb{C}^{\Lambda_s}$ can be canonically identified with $S_{0,s}(\mathbb{Q}_p^3, \mathbb{C})$ of section 1.2 and one can check that M can be seen as a linear operator $S_{0,s}(\mathbb{Q}_p^3, \mathbb{C}) \to S_{0,s}(\mathbb{Q}_p^3, \mathbb{C})$. The standard basis of $\mathbb{C}^{\Lambda_s}$ can be identified with indicator functions of translates of $\mathbb{Z}_p^3$ in Proposition 1.6 we also give a Fourier basis of $S_{0,s}(\mathbb{Q}_p^3, \mathbb{C})$. One can easily check M is diagonal in that Fourier basis. In particular for fixed $k \in \mathcal{I}_{-s,0}$ the function

$$p^{-\frac{3s}{2}} \mathbb{1}_{p^{-s}\mathbb{Z}_p^3}(x) \exp\left[2\pi i \{k \cdot x\}_p\right]$$

is an eigenvector for M with eigenvalue

$$p^{3s} \quad \times \quad \int_{k+p^s\mathbb{Z}_p^3} \mathrm{d}^3k' \; |k'|^{-(3-2[\phi])}. \tag{4.11}$$

For this computation all one needs to do is write $M(x-y) = C(x-y) \times \mathbb{1}_{p^{-s}\mathbb{Z}_p^3}(x) \times \mathbb{1}_{p^{-s}\mathbb{Z}_p^3}(y)$ and use standard facts about convolutions and Fourier transforms. In particular for $k \in \mathcal{I}_{-s,0}$ one has

$$\begin{aligned}
&\int_{\mathbb{Q}_p^3} \mathrm{d}^3y \; M(x-y) p^{\frac{3s}{2}} \mathbb{1}_{p^{-s}\mathbb{Z}_p^3}(y) \exp\left[2\pi i \{k \cdot y\}_p\right] \\
&= p^{-\frac{3s}{2}} \mathbb{1}_{p^{-s}\mathbb{Z}_p^3}(x) \;\times\; \int_{\mathbb{Q}_p^3} \mathrm{d}^3y \; C(x-y) \mathbb{1}_{p^{-s}\mathbb{Z}_p^3}(y) \exp\left[2\pi i \{k \cdot y\}_p\right] \\
&= p^{-\frac{3s}{2}} \mathbb{1}_{p^{-s}\mathbb{Z}_p^3}(x) \;\times\; \int_{\mathbb{Q}_p^3} \mathrm{d}^3k' \; \exp\left[2\pi i \{k' \cdot x\}_p\right] \times |k'|^{-(3-2[\phi])} \times p^{3s} \mathbb{1}_{k+p^s\mathbb{Z}_p^3}(k') \\
&= p^{\frac{-3s}{2}} \mathbb{1}_{p^{-s}\mathbb{Z}_p^3}(x) \;\times\; \exp\left[2\pi i \{k \cdot x\}_p\right] \;\times\; p^{3s} \int_{\mathbb{Q}_p^3} \mathrm{d}^3k' \; |k'|^{-(3-2[\phi])} \mathbb{1}_{k+p^s\mathbb{Z}_p^3}(k')
\end{aligned} \tag{4.12}$$

In going from the second to third line we rewrote the convolution of functions f, g as the inverse Fourier transform (with the fourier variable being k') of $\hat{f} \times \hat{g}$. In particular for $k \in \mathcal{I}_{-s,0}$ and any $k' \in \mathbb{Q}_p^3$ one has

$$\int_{\mathbb{Q}_p^3} \mathrm{d}^3y \; \exp\left[-2\pi i \{k \cdot y\}_p\right] \times \mathbb{1}_{p^{-s}\mathbb{Z}_p^3}(y) \exp\left[2\pi i \{k \cdot y\}_p\right] \;=\; p^{3s} \mathbb{1}_{\mathbb{Z}_p^d}\left(p^{-s}(k-k')\right) \;= p^{3s} \mathbb{1}_{k+p^s\mathbb{Z}_p^d}(k').$$

In going to the last line from the second to last line of (4.12) we remark that if $x \in p^{-s}\mathbb{Z}_p^3$, and $|k'-k| \leqslant p^{-s}$ then $\{(k-k') \cdot x\}_p$ is an integer which means

$$\exp\left[2\pi i \{k' \cdot x\}_p\right] \;=\; \exp\left[2\pi i \{k \cdot x\}_p\right].$$

One way to think about the integral in (4.11) is that when working in a finite box $p^{-s}\mathbb{Z}_p^d$ the fourier modes $\exp\left[2\pi i \{k \cdot x\}_p\right]$ get replaced with averaged (in k) modes - each average occuring over a translate of $p^s\mathbb{Z}_p^d$. In some sense this means makes the infrared cut-off "dual" to the UV cut-off - in the UV cut-off we are essentially replacing our field with one locally averaged in position space while for the infrared cut-off we are in some sense locally averaging in momentum space. We now end this aside and pick up from where we left off.

This computation of eigenvalues and eigenvectors makes it clear that M is positive definite and invertible - it also gives us a method of computing the inverse of M which we denote by A. After some computations one sees the matrix entry $A_{x,y}$ for $x, y \in \Lambda_s$ is given by

$$A_{x,y}^{(s)} \;=\; -\frac{(p^{3-2[\phi]}-1)^2}{(1-p^{-3})(p^{6-2[\phi]}-1)} \;\times p^{-s(6-2[\phi])} \;+\; G(x-y) \tag{4.13}$$

where G is a function $G : \mathbb{Q}_p^3 \to \mathbb{R}$ which is constant on translates over the blocks of Δ given by

$$G(x) = \int_{\mathbb{Q}_p^3} \mathrm{d}^3k \ \exp[2\pi i\{k\cdot x\}_p] \times \mathbb{1}\{|k| \leqslant 1\} \times |k|^{\frac{3-2[\phi]}{2}}.$$

In particular

$$\begin{aligned} G(0) &= \frac{p^{3-2[\phi]}(p^3-1)}{p^{6-2[\phi]}-1} > 0 \\ G(x) &= -\frac{p^{3-2[\phi]}}{1-p^{-(6-2[\phi])}} \times \frac{1}{|x|^{6-2[\phi]}} \text{ for } x \neq 0 \end{aligned} \tag{4.14}$$

Note that in (4.13) the local constancy of G makes the function $G(x,y) = G(x-y)$ well defined for $x, y \in \mathbb{L}$. G can be thought of as a formal matrix inverse for C_0.

M is the covariance matrix for the marginal of μ_{C_0} on $\mathbb{R}^{\Lambda_s}$, this marginal is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^{\Lambda_s}$ and one has

$$\mathrm{d}\mu_{C_0}(\phi_{\Lambda_s}) = \frac{1}{\sqrt{(2\pi)^{|\Lambda_s|}\det(M)}} \exp\left[-\frac{1}{2}\sum_{x,y\in\Lambda_s} A^{(s)}_{x,y}\phi_x\phi_y\right] \prod_{x\in\Lambda_s} d\phi_x.$$

Here ϕ_{Λ_s} denotes the vector of components $\{\phi_x\}_{x\in\Lambda_s}$ and $d\phi_x$ denotes Lebesgue measure. Note that $A^{(s)}_{x,y}$ is invariant under translation by elements in Λ_s and accordingly we will sometimes write $A^{(s)}(x-y) = A^{(s)}_{x,y}$. Now for given parameters $g,\ \mu$ one has that the $\mathbb{R}^{\Lambda_s}$ marginal of $\nu_{0,s}[g,\mu]$ is given by

$$\begin{aligned} \mathrm{d}\nu_{0,s}(\phi_{\Lambda s}) &= \frac{1}{Z}\left(\prod_{x\in\Lambda_s} \exp\left[-g : \phi_x^4 :_{C_0} -\mu : \phi_x^2 :_{C_0}\right]\right) \mathrm{d}\mu_{C_0}(\phi_{\Lambda_s}) \\ &= \frac{1}{Z'}\exp\left[-\frac{1}{2}\sum_{\substack{x,y\in\Lambda_s\\ x\neq y}} A^{(s)}(x-y)\phi_x\phi_y\right]\left(\prod_{x\in\Lambda_s}\exp\left[-g\phi_x^4 - b\phi_x^2\right]\mathrm{d}\phi_x\right) \end{aligned} \tag{4.15}$$

where we have undone the Wick ordering and absorbed the diagonal part of the Gaussian into the single site measure- and above we have set

$$b = -6C_0(0)g + \mu + \frac{1}{2}A^{(s)}(0)$$

and Z and Z' are just normalization factors. This realizes $\nu_{0,s}(\phi_{\Lambda s})[g,\mu]$ as the law of a generalized Ising ferromagnet in zero external field- for distinct $x, y \in \Lambda_s$ one has $J_{\{x,y\}} = -A^s(x-y) \geqslant 0$ and $\mathrm{d}\rho(\phi_x) = \exp\left[-a\phi_x^4 - b\phi_x^2\right]\mathrm{d}\phi_x$.

We refer back to Corollary 4.1 specializing to the scenario where $l_1 = l$ and $l_2 = l+1$ in a small ϵ regime where $I_{\epsilon,l}$ is non-empty. We then have, as a consequence of our RG analysis, the convergence of measures (in the sense of moments)

$$\lim_{s\to\infty} \nu_{0,s}[g,\mu_i(g)] := \nu_{0,\infty}[g,\mu_i(g)] \tag{4.16}$$

for $g \in I_{\epsilon,l}$.

We remark that for $i = 1, 2$ the measures $\nu_{0,\infty}[g,\mu_i(g)]$ on $\mathbb{R}^{\mathbb{L}}$ are translation invariant, rotation invariant, invariant under global spin flips $\{\phi_x\}_{x\in\mathbb{L}} \leftrightarrow \{-\phi_x\}_{x\in\mathbb{L}}$, and have exponentially integrable finite dimensional marginals - the last fact being another consequence of the RG analysis. We also remark that since $\nu_{0,\infty}[g,\mu_i(g)]$ can be realized as the infinite volume limit of ising ferromagnets the Griffith's inequality hold for these measures.

The key observation is that the infinite volume measures $\nu_{0,\infty}[g,\mu_i(g)]$ both corresponding to Ising models "at criticality" - that is they have pair correlations that decay to 0 but are not summable. These facts follow from the fact that these Ising models have non-degenerate scaling limits. We give more details on the last comment.

We can (re)identify the measures $\nu_{0,\infty}[g,\mu_i(g)]$ on $\mathbb{R}^{\mathbb{L}}$ as measures on $S'(\mathbb{Q}_p^3)$ by identifying $\{\phi_x\}_{x\in\mathbb{L}}$ with the element of $S'(\mathbb{Q}_p^3)$ given by the function

$$\phi(y) = \sum_{\Delta\in\mathbb{L}} \phi_\Delta \mathbb{1}_\Delta(y) \tag{4.17}$$

where $\mathbb{1}_\Delta : \mathbb{Q}_p^3 \to \mathbb{R}$ is the indicator function for the unit block Δ. With this convention if ϕ is a random element of $S'(\mathbb{Q}_p^3)$ distributed according to the measure $\nu_{0,\infty}[g,\mu_i(g)]$ then one has that the law of

$$\lim_{r\to-\infty} p^{-[\phi]\underline{r}}\phi(p^{\underline{r}}\ \bullet)$$

converges in the sense of moments to the law of $\nu_{i,\phi}$.

In particular the law of $p^{-[\phi]\underline{r}}\phi(p^{\underline{r}}\ \bullet)$ with ϕ distributed according to $\nu_{0,\infty}[g,\mu_i(g)]$ is the same as $\nu_{\underline{r},\infty}[g,\mu_i(g)] := \lim_{s\to\infty} \nu_{\underline{r},\underline{s}}$ where $\nu_{\underline{r},\underline{s}}$ is defined with the parameters g and $\mu_i(g)$. With this in mind we have the following lemma (note that below we write ν_i to denote $\nu_{i,\phi}$).

Lemma 4.2. *The measures $\nu_{0,\infty}[g,\mu_i(g)]$, with $i = 1$ or 2 both satisfy the following conditions*

$$\sum_{x\in\mathbb{L}} \langle\phi_0\phi_x\rangle_{\nu_{0,\infty}[g,\mu_i(g)]} = \infty \tag{4.18}$$

and

$$\inf_{x\in\mathbb{L}} \langle\phi_0\phi_x\rangle_{\nu_{0,\infty}[g,\mu_i(g)]} = 0. \tag{4.19}$$

Proof:

We first prove (4.18). By Corollary 4.1 one has

$$c_i := \int_{S'(\mathbb{Q}_p^3)} \mathrm{d}\nu_i(\phi)\ \phi(\mathbb{1}_{\mathbb{Z}_p^3})^2 > 0$$

Now using that ν_i can be realized as scaling limits of the $\nu_{0,\infty}[g,\mu_i(g)]$ one has

$$\begin{aligned} c_i &= \lim_{n\to\infty} \int_{S'(\mathbb{Q}_p^3)} \mathrm{d}\nu_{0,\infty}[g,\mu_i(g)](\phi) \ \left(\int_{\mathbb{Q}_p^3} \mathrm{d}^3y \ p^{[\phi]l(l+1)n} \phi(p^{-n(l+1)l} y) \ \mathbb{1}_{\mathbb{Z}_p^3}(y) \right)^2 \\ &= \lim_{n\to\infty} p^{(-6+2[\phi])l(l+1)n} \int_{S'(\mathbb{Q}_p^3)} \mathrm{d}\nu_{0,\infty}[g,\mu_i(g)](\phi) \ \left(\int_{\mathbb{Q}_p^3} \mathrm{d}^3y \ \phi(y) \ \mathbb{1}_{p^{-n(l+1)l}\mathbb{Z}_p^3}(y) \right)^2 \\ &= \lim_{n\to\infty} p^{(6-2[\phi])l(l+1)n} \left\langle \left(\sum_{x\in\Lambda_{n(l+1)l}} \phi_x \right)^2 \right\rangle_{\nu_{0,\infty}[g,\mu_i(g)]} . \end{aligned} \tag{4.20}$$

Now using the fact that $\nu_{0,\infty}[g,\mu_i(g)]$ is translation invariant and the fact that $\Lambda_{nl(l+1)} \subset \mathbb{L}$ is closed as an additive group it follows that

$$\begin{aligned} \left\langle \left(\sum_{x\in\Lambda_{n(l+1)l}} \phi_x \right)^2 \right\rangle_{\nu_{0,\infty}[g,\mu_i(g)]} &= \left\langle \left(\sum_{z\in\Lambda_{n(l+1)l}} \phi_z \right) \times \left(\sum_{x\in\Lambda_{n(l+1)l}} \phi_x \right) \right\rangle_{\nu_{0,\infty}[g,\mu_i(g)]} \\ &= p^{3n(l+1)l} \left\langle \phi_0 \times \left(\sum_{x\in\Lambda_{n(l+1)l}} \phi_x \right) \right\rangle_{\nu_{0,\infty}[g,\mu_i(g)]} . \end{aligned}$$

Inserting this into (4.20) gives

$$c_i = \lim_{n\to\infty} p^{(-3+2[\phi])l(l+1)n} \sum_{x\in\Lambda_{n(l+1)l}} \langle \phi_0 \phi_x \rangle_{\nu_{0,\infty}[g,\mu_i(g)]} \, . \tag{4.21}$$

Now since $c_i > 0$ and $-3 + 2[\phi] = -\frac{3+\epsilon}{2} < 0$ it follows that

$$\lim_{n\to\infty} \sum_{x\in\Lambda_{n(l+1)l}} \langle \phi_0 \phi_x \rangle_{\nu_{0,\infty}[g,\mu_i(g)]} = \sum_{x\in\mathbb{L}} \langle \phi_0 \phi_x \rangle_{\nu_{0,\infty}[g,\mu_i(g)]} = \infty.$$

This proves statement (4.18).

We now prove (4.19) by contradiction. We remark that by Griffiths' first inequality the infinum of (4.19) must be non-negative so if we assume it is non-zero we must have

$$\inf_{x\in\mathbb{L}} \langle \phi_0 \phi_x \rangle_{\nu_{0,\infty}[g,\mu_i(g)]} = \delta > 0.$$

However this implies that

$$\sum_{x\in\Lambda_{nl(l+1)}} \langle \phi_0 \phi_x \rangle_{\nu_{0,\infty}[g,\mu_i(g)]} \geqslant \delta \times p^{3nl}. \tag{4.22}$$

However by (4.21)

$$\lim_{n\to\infty} \frac{1}{p^{(-3+2[\phi])l(l+1)n}} \sum_{x\in\Lambda_{n(l+1)l}} \langle \phi_0 \phi_x \rangle_{\nu_{0,\infty}[g,\mu_i(g)]} = c_i < \infty \tag{4.23}$$

where the fact $c_i < \infty$ comes from the fact that all moments of ν_i are finite. Observing that $3 - 2[\phi] < 3$ we see that (4.22) conflicts with (4.23) so (4.19) is proved. □

4.4.2 Switching Boundary Conditions

Proving full scale invariance will involve investigating the phase diagram of a particular class of Ising ferromagnets, as such we will need a way to take infinite volume limits which gives us much more freedom when it comes to our choice of parameters, for example including a non-zero external field.

The cleanest method for establishing convergence of an infinite volume limit for an arbitrary Ising ferromagnet is by working in a scenario where we can appeal to Griffiths' Second Inequality to show that the moments $\langle \phi^A \rangle_{\mu_\Lambda}$ are increasing in Λ and then combine this with $n!$ moment bounds uniform in the volume - we will then have some infinite volume measure $\mu_{\mathbb{L}}$ with $\lim_{s\to\infty} \mu_{\Lambda_s} = \mu_L$ in the sense of moments.

However this fails when we use an interaction J as given in the last line of (4.15) - this interaction is becoming less ferromagnetic as $s \to \infty$. The solution is to change boundary conditions on our Gaussian measure so that the corresponding interaction is no longer dependent on the volume.

The boundary conditions for the measures $\nu_{0,s}$ would be called *Free Boundary Conditions* in the *field theory* literature - the self interaction was implemented as a finite volume perturbation of an infinite volume Gaussian Field and the finite volume marginals are given by the same finite volume perturbation against a finite volume marginal of the same Gaussian measure.

If the covariance C_0 is thought of as a formal $\mathbb{L} \times \mathbb{L}$ matrix then G (given by (4.14)) can be thought of as its formal matrix inverse. The key issue is that matrix restriction doesn't commute with taking matrix inverses. In particular if for $\Lambda \subset \mathbb{L}$ we write $\bullet|_\Lambda$ for the restriction of an $\mathbb{L} \times \mathbb{L}$ matrix to a $\Lambda \times \Lambda$ matrix then

$$\left(C|_\Lambda \right)^{-1} \neq G|_\Lambda \, .$$

What would be called free boundary conditions *in statistical mechanics* are more akin to Dirichlet boundary conditions in field theory.

In particular instead of having the interaction depend on the volume we will have the entries of the covariance depend on the volume - we define the Dirichlet covariance in the volume Λ as $C_{D,\Lambda} := \left(G|_\Lambda \right)^{-1}$.

A computation similar in flavor to the one for (4.13) shows that

$$C_{D,\Lambda_s} \;=\; C_0|_{\Lambda_s} + B^{(s)}$$

where $B^{(s)}$ is a rank 1 $\Lambda_s \times \Lambda_s$ matrix. More explicitly for every $x, y \in \Lambda_s$ one has

$$B^{(s)}_{x,y} \;=\; p^{-2[\phi]s} \times \frac{(1 - p^{-2[\phi]})^2}{(1 - p^{-3})(1 - p^{-6+2[\phi]})} := \sigma_s^2 > 0$$

In particular B^s is the covariance for a Gaussian field ψ_s on Λ_s which is constant on Λ_s, i.e. for all $x \in \Lambda_s$ one has $\psi_x = \psi_0 \sim \mathcal{N}(0, \sigma_s^2)$.

Instead of working with the measures $\nu_{0,s}$ defined with respect to C_0 we will instead work with the "Half-Dirichlet" [37] boundary conditions - "Half-Dirichlet" refers to the fact that we will use C_{D,Λ_s} as our background Gaussian measure but we will continue to Wick order with respect to the covariance C_0 . The reason to do this is that if we also used C_{D,Λ_s} for our Wick ordering then for fixed g, μ our single site measure $\mathrm{d}\rho$ would have volume dependence. We will denote these measures by $\nu_{0,s}^{HD}$.

It is not technically difficult to use our RG machinery to tackle infinite volume limits taken with Half-Dirichlet boundary. We assume that we are running the RG with fixed $L = p^l$. We give an informal explanation of how the RG can control the limit $s \to \infty$ of

$$\mathcal{S}_{0,ls}^{T,HD}(\tilde{f}) \ := \ Log\left[\frac{\mathcal{Z}_{0,ls}^{HD}(\tilde{f})}{\mathcal{Z}_{0,ls}^{HD}(0)}\right] \tag{4.24}$$

where

$$\mathcal{Z}_{0,ls}^{HD}(\tilde{f}) = \int_{S'(\mathbb{Q}_p^3)} \mathrm{d}\mu_{C^D_{\Lambda_{ls}}}(\mathring{\phi}) \exp\left[-\int_{\Lambda_{ls}} d^3x \ g : \mathring{\phi}^4(x) :_{C_0} \ + \mu \ : \mathring{\phi}^2(x) :_{C_0} \ - \mathring{\phi}(x)\tilde{f}(x)\right].$$

The random field ϕ distributed according to C_0 restricted to Λ_{ls} can be decomposed as a sum of independent random fields

$$\phi(x) \ = \ \sum_{j=0}^{s-1} \zeta_j(x) + \phi_s(x)$$

where the ζ_j are the first j fluctuation fields, that is the ζ_j are distributed according to the scaled fluctuation covariance $p^{-2[\phi]lj}\Gamma(p^{lj}\bullet)\big|_{\Lambda_{ls}}$. The field ϕ_s is distributed according to $p^{-2[\phi]ls}C_0(p^{ls}\cdot)\big|_{\Lambda_{ls}}$. In particular ϕ_s is constant on Λ_{ls}. The RG map iteratively integrates out the ζ_j's for $0 \leqslant j \leqslant s-1$. The integration of ϕ_s occurs at the final step when the volume Λ_{ls} has been shrunk down to a single block (earlier we called the contribution from this step the "boundary" term of our RG analysis).

We can similarly decompose the random field $\mathring{\phi}$ on Λ_{ls} distributed according to $C_{D,\Lambda_{ls}}$ as a sum of independent random fields

$$\mathring{\phi}(x) \ = \ \sum_{j=0}^{s-1} \zeta_j(x) + \phi_s(x) + \psi_{sl}(x)$$

Here both ϕ_s and ψ_{sl} are constant on Λ_{ls}. We can apply the same RG map for this scenario which will generate the same flow, the only difference being that in the final step we our final integrand will be a function of both $\phi_s + \psi_{sl}$. In particular when we include rescaling the"boundary" is given by

$$Log\left[\frac{\int_{\mathbb{R}} \mathrm{d}\tilde{\mu}(\varphi)\mathcal{I}_{\Delta(0)}[\vec{V}^{0,ls}(\tilde{f},0)](\varphi)}{\int_{\mathbb{R}} \mathrm{d}\tilde{\mu}(\varphi)\mathcal{I}_{\Delta(0)}[\vec{V}^{0,ls}(0,0)](\varphi)}\right].$$

where $\tilde{\mu}$ is a Gaussian measure on $\mathbb{R}$, in particular it is the law of $p^{ls[\phi]}\left(\phi_l(0) + \psi_{ls}(0)\right)$ (the prefactor coming from rescaling) which is a centered Gaussian random variable with variance

$$C_0(0) + p^{2[\phi]ls}\sigma_{ls}^2 \ = C_0(0) + \frac{(1-p^{-2[\phi]})^2}{(1-p^{-3})(1-p^{-6+2[\phi]})}.$$

Observe that this final one variable Gaussian integrals has a density independent of s and so this term can be controlled just like the boundary term studied in the RG analysis.

We define the measures $\nu^{HD}_{0,s}[g,\mu]$ via the log moment generating function $\mathcal{S}^{T,HD}_{0,s}(\tilde{f})$.By proceeding along the lines above one can prove the following theorem which states that for our infinite volume limits constructed via RG we will get the same limits whether we use free boundary conditions or Half Dirichlet boundary conditions. We state this result as the following theorem.

Theorem 4.7. *For $\epsilon > 0$, $L = p^l$, $g > 0$, and $\mu(g)$ as in Theorem 3.1 one has that for $\tilde{f} \in S(\mathbb{Q}_p^3, \mathbb{C})$ sufficiently small*

$$\lim_{s\to\infty} \mathcal{S}^{T,HD}_{0,ls}(\tilde{f}) = \lim_{s\to\infty} \mathcal{S}^{T}_{0,ls}(\tilde{f},0)$$

and consequently

$$\lim_{s\to\infty} \nu^{HD}_{0,ls}[g,\mu(g)] = \lim_{s\to\infty} \nu_{0,ls}[g,\mu(g)] = \nu_{0,\infty}[g,\mu(g)].$$

For any non-negative integer s we have

$$\mathrm{d}\mu_{C_{D,\Lambda_s}}(\phi) = \frac{1}{\sqrt{(2\pi)^{|\Lambda|_s} det\,(G|_{\Lambda_s})}} \exp\left[-\frac{1}{2}\sum_{x,y\in\Lambda_s} G(x-y)\phi_x\phi_y\right] \prod_{x\in\Lambda_s} d\phi_x.$$

We write the Λ_s marginal of the measure $\nu^{HD}_{0,s}[g,\mu]$ as an generalized Ising model.

$$\begin{aligned}
\nu^{HD}_{0,s}(\phi_{\Lambda s}) &= \frac{1}{Z}\left(\prod_{x\in\Lambda_s} \exp\left[-g :\phi_x^4:_{C_0} - \mu :\phi_x^2:_{C_0}\right]\right) \mathrm{d}\mu_{C^D_{\Lambda_s}}(\phi_{\Lambda_s}) \\
&= \frac{1}{Z'}\exp\left[-\frac{1}{2}\sum_{\substack{x,y\in\Lambda_s\\ x\neq y}} G(x-y)\phi_x\phi_y\right]\left(\prod_{x\in\Lambda_s}\exp\left[-g\phi_x^4 - b\phi_x^2\right]\mathrm{d}\phi_x\right)
\end{aligned} \tag{4.25}$$

where we set

$$b = -6C_0(0)g + \mu + \frac{1}{2}G(0)$$

and G is given as in (4.14). For generic $g > 0$, $b \in \mathbb{R}$ we define $\mu[\Lambda_s, g, b]$ to be the probability measure on $\mathbb{R}^{\Lambda_s}$ given by the measure denoted on the last line of (4.25). We will also write $\nu[\Lambda_s, g, b]$ for the corresponding measure $\mathbb{R}^{\mathbb{L}}$ given by $\mu[\Lambda_s, g, b] \otimes \delta_{\mathbb{L}\setminus\Lambda_s}$ where $\delta_{\mathbb{L}\setminus\Lambda_s}$ is the measure concentrated on the zero element of $\mathbb{R}^{\mathbb{L}\setminus\Lambda_s}$.

On the domain of $\mu_i(g)$ we define the function

$$b_{i,\mathrm{crit}}(g) = -6C_0(0)g + \mu_i(g) + \frac{1}{2}G(0).$$

In particular for g in the given domain the $\mathbb{R}^{\Lambda_s}$ marginals of $\nu\Lambda_s, g, b_{i,\mathrm{crit}}(g)]$ and $\nu^{HD}_{0,s}[g,\mu_i(g)]$ agree. It follows from the Theorem 4.7 one has convergence in the sense of moments of the measures

$$\lim_{s\to\infty} \nu[\Lambda_{\underline{s}}, g, b_{i,\mathrm{crit}}(g)] \;=:\; \nu[\mathbb{L}, g, b_{i,\mathrm{crit}}(g)] \;=\; \nu_{0,\infty}[g,\mu(g)] \tag{4.26}$$

Now by Griffiths Second Inequality one also has that for fixed g, b the moments $\langle\phi^A\rangle_{\nu[\Lambda_s,g,b]}$ are non-decreasing in s. It follows that the convergence in (4.26) doesn't need to be taken along the particular subsequence of volumes $\Lambda_{\underline{s}}$ but could be taken along the Λ_s. This also allows us to construct infinite volume measures for arbitrary $g > 0$, $b \in \mathbb{R}$ modulo finding sufficiently strong uniform upper bounds on the moments.

4.5 Criticality and the Intermediate Phase

One expects that the generalized Ising ferromagnets $\nu[\mathbb{L}, g, b]$ should undergo a phase transition just as classical nearest neighbor interaction ferromagnetic Ising models on $\mathbb{Z}^d$ do for $d \geqslant 2$ - the role of the temperature now being played by the mass b.

In particular for fixed $g > 0$ the measure $\nu[\mathbb{L}, g, b]$ should have dramatically different behaviour for different regimes of b - for b sufficently large one expects the measures $\nu[\mathbb{L}, g, b]$ to have pair correlations that decay quick enough for

$$\sum_{x \in \mathbb{L}} \langle \phi_0 \phi_x \rangle_{\nu[\mathbb{L}, g, b]} < \infty \tag{4.27}$$

where $\langle \cdot \rangle_\mu$ denotes expectation with respect to the measure μ. We call the quantity on the left hand side above the *susceptability* (denoted by ξ). This regime of b is called the *single phase regime* or *disordered phase* or the *high temperature regime*.

On the other hand one expects that there is a regime of sufficently small b where correlations do not decay to zero, that is

$$\inf_{x \in \mathbb{L}} \langle \phi_0 \phi_x \rangle^T_{\nu[\mathbb{L}, g, b]} > 0. \tag{4.28}$$

The failure of correlations to vanish at long distances is called *long range order* and the parameter regime where one finds this is called the *two phase regime* or the *low temperature regime*. The values of the mass b at which one transitions from one regime of behaviour to another are called *critical*.

With this in mind we can define at least two notions of "criticality" for our mass parameter b:

$$b_{\mathrm{LRO}}(g) = \sup \left\{ b \in \mathbb{R} | \ \inf_{x \in \mathbb{L}} \langle \phi_0 \phi_x \rangle_{\nu[\mathbb{L}, g, b]} > 0 \right\}$$

and

$$b_{\chi}(g) = \inf \left\{ b \in \mathbb{R} | \ \sum_{x \in \mathbb{L}} \langle \phi_0 \phi_x \rangle_{\nu[\mathbb{L}, g, b]} < \infty \right\}.$$

Now by the Griffith's inequalities we see that for fixed $g > 0$ the quantity $\langle \phi_0 \phi_x \rangle_{\nu[\mathbb{L}, g, b]}$ (or more generally any moment of spin variables) is non-negative and should be non-increasing in the parameter b. Both inequalities are applied in finite volume and then carry over to the infinite volume limit - for the second assertion we note that

$$\frac{\partial}{\partial b} \langle \phi_0 \phi_x \rangle_{\mu_{\Lambda_s}[g,b]} \ = \ - \sum_{z \in \Lambda s} \langle \phi_0 \phi_x, \phi_z \rangle_{\mu_{\Lambda_s}[g,b]} \ \leqslant \ 0$$

It then follows that one must have $b_{\mathrm{LRO}}(g) \leqslant b_{\chi}(g)$, Additionally for fixed $g > 0$ it follows if $b < b_{\mathrm{LRO}}(g)$ then b must lie in the two phase regime, similarly if $b > b_{\chi}$ then b must lie in the one phase regime, in particular both these regimes are semi-infinite intervals in b (again this description holds for fixed g). The behaviour of the susceptability and decay of pair correlations at criticality is a more subtle issue which we don't address here.

Now if $b_{\mathrm{LRO}}(g) < b_{\chi}(g)$ one would have a third regime $b_{\mathrm{LRO}}(g) < b < b_{\xi}(g)$ where susceptability χ is infinite but one does not have long range order. However Aizenman, Barsky, and Fernandez, in [7], proved that this was impossible for ferromagnetic Ising models - any transition between the one phase and two phase regimes must be *sharp*. More precisely a sharp transition in b means that $b_{\mathrm{LRO}}(g) = b_{\chi}(g)$, i.e. no intermediate phase in b for fixed g. Before we continue we make the important remark that there are some differences between how the result in [7] was stated and our current situation - much of our work for proving full scale invariance is overcoming those differences.

The basic idea of our argument is to leverage the main result of [7] so that our main result follows via a proof by contradiction. By Theorem 4.7 and Lemma 4.2 one has that $b_{\mathrm{LRO}}(g) \leqslant b_{i,\mathrm{crit}}(g) \leqslant b_{\chi}(g)$. This means that if $\mu_1(\mathring{g}) \neq \mu_2(\mathring{g})$ (or equivalently $b_{1,\mathrm{crit}}(\mathring{g}) \neq b_{2,\mathrm{crit}}(\mathring{g})$) for some $\mathring{g} \in I_{\epsilon,l}$ then this would imply the existence of an intermediate phase in the mass parameter b when $g = \mathring{g}$. We remark that for the purposes of our proof we can assume that $\mu_{1,\mathrm{crit}}(\mathring{g}) < \mu_{2,\mathrm{crit}}(\mathring{g})$, the possibility of the other inequality can be handled in an identical way.

With this assumption we can already construct an intermediate phase not just in the mass parameter b but one in the (g,b) plane.

Proposition 4.1. *Suppose that there exists $\mathring{g} \in I_{\epsilon,l}$ for which $\mu_1(\mathring{g}) > \mu_2(\mathring{g})$. Then there exists a non-empty open set $U \subset (0,\infty) \times \mathbb{R}$ such that for every $(g,b) \in U$ one has the following*

The measures $\nu[\Lambda_s, g, b]$ converge (in the sense of moments) in the infinite volume limit

$$\lim_{s \to \infty} \nu[\Lambda_s, g, b] = \nu[\mathbb{L}, g, b]$$

and additionally one has

$$\begin{aligned} \sum_{x \in \mathbb{L}} \langle \phi_0 \phi_x \rangle_{\nu[\mathbb{L},g,b]} &= \infty \\ \inf_{x \in \mathbb{L}} \langle \phi_0 \phi_x \rangle_{\nu[\mathbb{L},g,b]} &= 0. \end{aligned} \tag{4.29}$$

Proof: We remark that via our RG analysis (i.e. analyticity of the stable manifold) the functions $\mu_1(\cdot)$ and $\mu_2(\cdot)$ are continuous on $I_{\epsilon,l}$. It follows thats there exists $\delta > 0$ such that for all $g \in (\mathring{g} - \delta, \mathring{g} + \delta) \subset I_{\epsilon,l}$ one has $\mu_1(g) > \mu_2(g)$ and so one also has $b_{1,\mathrm{crit}}(g) > b_{2,\mathrm{crit}}(g)$ for such g.

We set

$$U = \{(g,b) \in (0,\infty) \times \mathbb{R} |\ g \in (\mathring{g} - \delta, \mathring{g} + \delta),\ b \in (b_{2,\mathrm{crit}}(g), b_{1,\mathrm{crit}}(g))\}.$$

Since the $b_{i,\mathrm{crit}}(g)$ are continuous it is clear that the above set is open.

Additionally for any (g,b), any non-negative integer s, and any moment ϕ^A we have by Griffiths' Second Inequality

$$\langle \phi^A \rangle_{\nu[\Lambda_s, g, b_{1,\mathrm{crit}}(g)]} \leqslant \langle \phi^A \rangle_{\nu[\Lambda_s, g, b]} \leqslant \langle \phi^A \rangle_{\nu[\Lambda_s, g, b_{2,\mathrm{crit}}(g)]} \tag{4.30}$$

Also by Griffiths' second inequality the terms above are all increasing in s, and the last term converges to $\langle \phi^A \rangle_{\nu[\mathbb{L}, g, b_{2,\mathrm{crit}}(g)]}$ as $s \to \infty$. Since the measure $\nu[\mathbb{L}, g, b_{2,\mathrm{crit}}(g)]$ is exponentially integrable for any of the spin variables one has that the moments $\langle \phi^A \rangle_{\nu[\Lambda_s,g,b]}$ converge to the moments of a measure we denote by $\nu[\mathbb{L}, g, b]$. The assertions of (4.30) then follow by taking $s \to \infty$ in (4.30) and observing that both assertions of (4.29) hold for the measures $\nu[\mathbb{L}, g, b_{1,\mathrm{crit}}(g)]$ and $\nu[\mathbb{L}, g, b_{2,\mathrm{crit}}(g)]$ □

The above discussion described the setting of our stat mech approach for proving full scale invariance. We now precisely state the main theorem of this chapter, this theorem when combined with Proposition 4.1

will establish that $\mu_1(g)$ and $\mu_2(g)$ agree on $I_{\epsilon,l}$. Below we denote by $\mathcal{B}$ the product σ-algebra on $\mathbb{R}^{\mathbb{L}}$. We also use the notation "$\Subset$" to denote finite subset.

Theorem 4.8. *Let $J(\cdot)$ be a fixed function $\mathbb{L}\backslash\{0\} \to (0,\infty)$.*

Define $\Psi : \{0, p, p^2, \dots\} \to [0,\infty)$ as follows:

$$\Psi(0) = 0$$

$$\text{For } j \geqslant 1,\ \ \Psi(p^j) = \sup_{\substack{x\in\mathbb{L} \\ |x|=p^j}} J(x)$$

We require that J satisfy the following integrability property: there must exist $\eta > 0$ such that

$$\sum_{x\in\mathbb{L}} \Psi(|x|)|x|^{\eta} < \infty \tag{4.31}$$

For any $g \in (0,\infty)$, $\beta \in [0,\infty)$, $b \in \mathbb{R}$, $h \in [0,\infty)$, and $\Lambda \Subset \mathbb{L}$ we define a Borel probability measure $\mu_\Lambda[g,b,\beta,h]$ on $\mathbb{R}^\Lambda$ as follows:

$$d\mu_\Lambda[g,b,\beta,h](\phi_\Lambda) = \frac{1}{\mathcal{Z}_\Lambda[\beta,g,b,h]} \exp\left[\frac{\beta}{2}\sum_{\substack{x,y\in\Lambda \\ x\neq y}} J(x-y)\phi_x\phi_y\right] \exp\left[-\sum_{x\in\Lambda}\left(g\phi_x^4 + b\phi_x^2\right)\right] d\phi_\Lambda$$

Above the quantity $\mathcal{Z}_\Lambda[g,b,\beta,h]$ is defined to make μ_Λ a probability measure. We define the probability measure $\nu[\Lambda,g,b,\beta,h]$ on the measure space $(\mathbb{R}^{\mathbb{L}},\mathcal{B})$ as the product measure $\mu_\Lambda[\beta,g,b,h]\otimes\delta_{\mathbb{L}\backslash\Lambda}$. Here $\delta_{\mathbb{L}\backslash\Lambda}$ denotes the Dirac delta measure concentrated on the zero element of $\mathbb{R}^{\mathbb{L}\backslash\Lambda}$.

With the above assumptions and definitions one has the following results.

(i) For any non-negative integer j let $\Lambda_j = \{x \in \mathbb{L} : \ ||x|| \leqslant p^j\}$. There exists a translation invariant probability measure measure $\nu[\mathbb{L},\beta,g,b,h]$ on $\mathbb{R}^{\mathbb{L}}$ such that the measures $\nu[\Lambda_j,\beta,g,b,h]$ converge in the sense of moments to the measure $\nu[\mathbb{L},\beta,g,b,h]$.

(ii) For any $\Lambda \Subset \mathbb{L}$ the measure $\nu[\mathbb{L},g,b,\beta,h]$ has a marginal on $\mathbb{R}^\Lambda$ which is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^\Lambda$. We denote the corresponding Radon-Nikdoym derivative as $\frac{d\nu^\Lambda}{d\phi_\Lambda}(\phi_\Lambda)$. For any fixed $g > 0$ and $b \in \mathbb{R}$ and any compact set of $(\beta,h) \in [0,\infty)^2$ there exists $\delta \in \mathbb{R}$ such that one has the bound

$$\left|\frac{d\nu^\Lambda}{d\phi_\Lambda}(\phi_\Lambda)\right| \leqslant \exp\left[-\sum_{x\in\Lambda}\left(-\frac{g}{4}\phi_x^4 + \delta\right)\right].$$

(iii) Define

$$M(\beta, g, b, h) \;=\; \langle \phi_0 \rangle_{\nu[\mathbb{L},\beta,g,b,h]}$$

Then the limit $\lim_{h \to 0^+} M(\beta, g, b, h) =: M^+(\beta, g, b)$ *exists.*

(iv) For any fixed choices of g, b *there exists a countable set* D *such that if* $\beta \in [0, \infty)\backslash D$ *then*

$$\inf_{x \in \mathbb{L}} \langle \phi_0 \phi_x \rangle_{\nu[\mathbb{L},\beta,g,b,0]} \geqslant \left(M^+(\beta, g, b)\right)^2$$

(v) Suppose there exists $\hat{g}$*,* $\hat{b}$*,* $\hat{\beta}$ *such that*

$$\sum_{x \in \mathbb{L}} \langle \phi_0 \phi_x \rangle_{\nu[\mathbb{L},\hat{g},\hat{b},\hat{\beta},0]} = \infty.$$

Then for any $\beta > \hat{\beta}$ *one has:*

$$\inf_{x \in \mathbb{L}} \langle \phi_0 \phi_x \rangle_{\nu[\mathbb{L},\hat{g},\hat{b},\beta,0]} \;> 0$$

and so for any $\lambda > 1$*:*

$$\inf_{x \in \mathbb{L}} \langle \phi_0 \phi_x \rangle_{\nu[\mathbb{L},\lambda^{-2}\hat{g},\lambda^{-1}\hat{b},\hat{\beta},0]} \;>\; 0.$$

In our case of interest one has $J(\cdot) = G(\cdot)$. it is easy to check that this choice satisfies the requirements on J made above. Statements (i), (ii), (iii), and (iv) above all follow from the work of Section 4.7. The first part of statement (v) which contains the sharpness result is proved via combining results from sections 4.7 and 4.6, the second assertion follows by a simply scaling argument in the variables ϕ. Section 4.7 is independent of section 4.6 but not vice-versa - however we still put section 4.6 first in our exposition since that is the key element of our argument.

Once Theorem 4.8 is proved the combined with Proposition 4.29 we then have that $\mu_1(g) = \mu_2(g)$ for all $g \in I_{\epsilon,l}$ - otherwise Proposition 4.29 would give us an open set $U \subset (0, \infty) \times \mathbb{R}$ such that for all $(g, \mu) \in U$ the measures $\nu[\mathbb{L}, g, \mu, 1, 0]$ have both infinite susceptiability and an absence of long range order. However this contradicts statement (v). For $(g, \mu) \in U$ one can find $\lambda > 1$ such that $(\lambda g, \lambda\mu) \in U$ and so the last assertion of statement (v) states that $\nu[\mathbb{L}, \lambda g, \lambda\mu, 1, 0]$ should exhibit long range order.

All that is left to prove full scale invariance then is to prove Theorem 4.8.

4.6 The Sharpness of Transition result of Aizenman, Barsky, and Fernandez

We first remark that there is nothing new in this section, this material follows expositions from [5], [8], [6], and [7] although we give more details on some of the steps.

For now we take the results of section 4.7 for granted and assume that for any $g \in (0, \infty)$, $\beta \in [0, \infty)$, $b \in \mathbb{R}$, $h \in [0, \infty)$ one has the convergence of the infinite volume limit in the sense of moments

$$\lim_{n\to\infty} \nu[\Lambda_n, g, b, \beta, h] \;=\; \nu[\mathbb{L}, g, b, \beta, h].$$

Once such a limit exists it is clear the limiting measure is automatically translation invariant. We remark that the analysis of section 4.7 will also show that the limiting measure is exponentially integrable in the spin variables.

The key phase diagram parameters used in [7] for their analysis of critical Ising models are the inverse temperature β and the (uniform) external field h. Their main results concern properties of the magnetizations

$$M(\beta, g, b, h) \;=\; \langle \phi_0 \rangle_{\nu[\mathbb{L},g,b,\beta,h](\phi)}. \tag{4.32}$$

and

$$M^+(\beta, g, b) \;=\; \lim_{h\to 0^+} M(\beta, g, b, h). \tag{4.33}$$

We quickly mention some important facts about the above quantities.

Theorem 4.9. *Let $g > 0$, $b \in \mathbb{R}$, and $\beta, h \in [0, \infty)$. Then $M(g, b, \beta, h)$ and $M^+(g, b, \beta)$ as defined above are well defined and finite. Additionally one has the following:*

1. *$M(\beta, g, b, h)$ is a concave function of h for $h \geqslant 0$*
2. *For $h > 0$ the function $M(\beta, g, b, h)$ is real analytic in h.*

Proof: The first item is a direct consequence of GHS, the second comes from the Lee Yang Theorem. The detailed proof of this theorem is in the appendix, section 5.3.

If one has $M^+(\beta, g, b) > 0$ then the Ising models with parameters β, g, b are said to undergo *spontaneous magnetization*, this represents another criteria which one can use to define the two phase regime - although the equivalence of these two criteria is not obvious (and a central issue for us).

We define the two following notionts of critical inverse temperature:

$$\beta_\chi(g, b) \;=\; \inf\left\{\beta \geqslant 0 |\ \sum_{x\in\mathbb{L}} \langle \phi_0 \phi_x \rangle_{\nu[\mathbb{L},\beta,g,b,0]} < \infty\right\}$$

and

$$\beta_M(g, b) \;=\; \sup\left\{\beta \geqslant 0 |\ M^+(\beta, g, b) > 0\right\}.$$

We remark that by Griffiths II one has that pair correlations and $M^+(\beta, g, b)$ are increasing in β for fixed g, b. Thus $\beta_\chi(g, b)$ and $\beta_M(g, b)$ are the endpoints of semi-infinite intervals in β.

An important fact is that the susceptability χ can be identified as the derivative of the magnetization at $h = 0$

Lemma 4.3. *For any $g \in (0, \infty)$, $b \in \mathbb{R}$, and $\beta \geqslant 0$ one has*

$$\lim_{h\to 0^+} \frac{M(g,b,\beta,h)}{h} = \sum_{x\in\mathbb{L}} \langle \phi_0\phi_x \rangle_{\nu[\mathbb{L},g,b,\beta,0]}$$

Proof: This is a simple example of what is called a fluctuation dissipation relation. See appendix 5.3.

It follows from this that one must have $\beta_\chi(g,b) \leqslant \beta_M(g,b)$. Here the question of sharpness of transition is whether $\beta_\chi(g,b) = \beta_M(g,b)$.

The main result of [7] transcribed to the context of our models reads as following.

Theorem 4.10 ([7])**.** *Fix $g \geqslant 0$, and $b \in \mathbb{R}$. Suppose there exists a $\beta_0 \geqslant 0$ such that:*

$$\sum_{x\in\mathbb{L}} \langle \phi_0\phi_x \rangle_{\nu[\mathbb{L},\beta_0,g,b,0]} < \infty = \infty \tag{4.34}$$

Then for any $\beta > \beta_0$ one has $M^+(g,b,\beta) > 0$. In particular $\beta_\chi(g,b) = \beta_M(g,b)$.

There are two difference between Theorem 4.10 and the result we want. The first is that the intermediate phase ruled out by the above theorem is for β with fixed g, b while the intermediate regime we constructed in section 4.5 was an open set in (g,b) for fixed $\beta = 1$. However by scaling ϕ we can use our intermediate regime to construct an intermediate phase in β for fixed g, b.

The second more serious issue is that the above theorem uses spontaneous magnetization instead of long range order to characterize the two phase regime. In [68, V.4] it is shown that for generalized Ising models satisfying the GHS inequality the presence of Long Range Order implies spontaneous magnetization. This establishes $\beta_M(g,b) \leqslant \beta_{LRO}(g,b)$ where $\beta_{LRO}(g,b)$ represents the onset of long range order. However for our purposes we need the opposite inequality for our generalized Ising model which is is the central goal of section 4.7.

The analysis and results of [7] all apply for fixed g and b so we will often drop these parameters in our notation. The approach utilized in [7] is to establish certain partial differential inequalities for the finite volume magnetization in the presence of a uniform and positive external field. For $n \in \mathbb{N}$ we define the function $M_n : [0,\infty) \times (0,\infty) \to (0,\infty)$ as follows:

$$M_n(\beta,h) := \langle \phi_0 \rangle_{\mu[\Lambda_n,g,b,\beta,h]} = \frac{1}{|\Lambda_n|} \sum_{x\in\Lambda_n} \langle \phi_x \rangle_{\mu[\Lambda_n,g,b,\beta,h]} \tag{4.35}$$

We've included the second equality above to make it clear that M_n is in fact the average magnetization. A convenient simplification for our model is that for any $n \in \mathbb{N}$ the measure $\nu_{\Lambda_n}[g,b,\beta,h]$ is invariant by translations by $x \in \Lambda_n$. This will be enough translation invariance for the key correlation inequalities to be identified as the desired partial differential inequality for M_n. Over the lattice $\mathbb{Z}^d$ one would have to switch to periodic boundary conditions or perform more work to establish the desired result for Dirichlet boundary conditions. The main partial differential inequality proved in [7] is the following.

Theorem 4.11 ([7])**.** *For any $n \in \mathbb{N}$*

$$M_n \leqslant h\frac{\partial M_n}{\partial h} + hM_n^2 + \beta\|J\|_{L^1} M_n^3 + \beta M_n \left(h + \beta\|J\|_{L^1} M_n\right) \frac{\partial M_n}{\partial \beta}. \tag{4.36}$$

In section 4.6.4 we will see how this establishes Theorem 4.10. The main techniques of [7] are the Random Current Representation (RCR) along with an associated Random Walk Representation (RWR) which provide combinatorial representations of correlation functions. Both the RCR and RWR apply to any ferromagnetic classical Ising model without referencing any geometric or algebraic structure on the underlying finite set Λ or the interaction J - until we specialize further one should imagine Λ as an abstract set.

We start by giving some general results on the RCR. Afterwards we will describe how the Griffiths-Simon approximation [65] allows one use the RCR and RWR to analyze ferromagnetic ϕ^4 models.

4.6.1 Overview of Random Current techniques

We start by quickly reviewing the definition of the Random Current Representation for an arbitrary ferromagnetic classical Ising model over some finite set Λ. We restrict ourselves to the case where the external field is uniform. In particular we can represent our measure as follows: Let A be subset of Λ and for $\sigma \in \{-1,1\}^{\Lambda}$ define $\sigma^A = \prod_{x \in A} \sigma_x$. Then one has:

$$\langle \sigma^A \rangle = \frac{2^{-|\Lambda|}}{Z} \sum_{\sigma \in \{-1,1\}^{\Lambda}} \sigma^A \exp\left[\beta \sum_{\{x,y\} \in \Lambda^{(2)}} J_{\{x,y\}} \sigma_x \sigma_y + h \sum_{x \in \Lambda} \sigma_x \right]$$

where

$$Z = 2^{-|\Lambda|} \sum_{\sigma \in \{-1,1\}^{\Lambda}} \exp\left[\beta \sum_{\{x,y\} \in \Lambda^{(2)}} J_{\{x,y\}} \sigma_x \sigma_y + h \sum_{x \in \Lambda} \sigma_x \right].$$

Recall that $\Lambda^{(2)}$ is the set of two element subsets of Λ. We add a new ghost site g to the finite set giving us the "enhanced" vertex set $\bar{\Lambda} = \Lambda \cup \{g\}$. We will try to always use g to denote the ghost vertex, when quantifying over vertices the variables x, y, u, etc. will represent an vertices in Λ unless we specifically allow for them to represent the ghost site.

The set of possible bonds on $\bar{\Lambda}$ is denoted by $\bar{\Lambda}^{(2)}$. Correlation functions will be represented by sums over "current configurations" $\mathbf{n}$, vectors with non-negative integer components indexed by bonds of the enhanced vertex set: $\mathbf{n} = \{n_b\}_{b \in \bar{\Lambda}^{(2)}}$. The value of an individual component n_b is referred to as the flux through bond b.

We now give a representation for the partition function. First we introduce the following notation:

$$\bar{J}_b = \begin{cases} \beta J_{\{x,y\}} & \text{if } b = \{x,y\} \text{ with } x, y \in \Lambda \\ h & \text{if } b = \{x,g\} \text{ with } x \in \Lambda \end{cases}.$$

Starting from the definition of Z and using the above notation one has the following:

$$
\begin{aligned}
Z &= 2^{-|\Lambda|} \sum_{\sigma \in \{-1,1\}^{\Lambda}} \left(\prod_{\{x,y\} \in \Lambda^{(2)}} \exp\left[\beta J_{\{x,y\}} \sigma_x \sigma_y\right] \right) \left(\prod_{x \in \Lambda} \exp\left[h \sigma_x\right] \right) \\
&= 2^{-|\Lambda|} \sum_{\sigma \in \{-1,1\}^{\Lambda}} \prod_{b \in \bar{\Lambda}^{(2)}} \left[\sum_{n_b=0}^{\infty} \frac{\bar{J}_b^{n_b}}{n_b!} \left(\prod_{i \in \Lambda \cap b} \sigma_i \right)^{n_b} \right] \\
&= \sum_{\mathbf{n}} \left[w(\mathbf{n}) \times 2^{-|\Lambda|} \left(\sum_{\sigma \in \{-1,1\}^{\Lambda}} \prod_{x \in \Lambda} \sigma_x^{\left(\sum_{b \ni x} n_b\right)} \right) \right]
\end{aligned}
\tag{4.37}
$$

The sum $\sum_{b \ni x}$ is a sum over bonds $b \in \bar{\Lambda}^{(2)}$ such that b contains the site x. We've also defined a weight system $w(\bullet)$ on current configurations given by:

$$
w(\mathbf{n}) = \prod_{b \in \bar{\Lambda}^{(2)}} \frac{\bar{J}_b^{n_b}}{n_b!}
$$

The key observation is that the sum over σ on the final line of (4.37) vanishes if $\sum_{b \ni x} n_b$ is odd for any site $x \in \Lambda$ while if $\sum_{b \ni x} n_b$ is even for every $x \in \Lambda$ then the sum is equal to $2^{|\Lambda|}$. This motivates the following definition: the "sources" $\partial \mathbf{n}$ of a current configuration $\mathbf{n}$ is the set of vertices with an an odd flux coordination number:

$$
\partial \mathbf{n} = \{x \in \bar{\Lambda} \mid \sum_{\substack{b \in \bar{\Lambda}^{(2)} \\ b \ni x}} n_b \text{ is odd } \}.
$$

It is not hard to see that the set $\partial \mathbf{n}$ must be of even cardinality. With this notation one can now write:

$$
Z = \sum_{\partial \mathbf{n} = \varnothing} w(\mathbf{n}).
$$

Above we are summing over all current configurations $\mathbf{n}$ with $\partial \mathbf{n} = \varnothing$. The presence of an observable σ^A flips our parity requirement on the flux coordination of the sites of A, thus one has the following identities:

$$
\langle \sigma_A \rangle = \begin{cases} \dfrac{1}{Z} \displaystyle\sum_{\partial \mathbf{n} = A} w(\mathbf{n}) & \text{if } |A| \text{ is even} \\ \dfrac{1}{Z} \displaystyle\sum_{\partial \mathbf{n} = A \cup \{g\}} w(\mathbf{n}) & \text{if } |A| \text{ is odd} \end{cases}
\tag{4.38}
$$

In what follows we will often suppress the dependence on the volume Λ, all the identities hold for an arbitrary but fixed finite volume.

We quickly cover some additional notation. $A \Delta B$ denotes the symmetric difference between the sets A and B, that is $(A \cup B) \setminus (A \cap B)$.

One defines the support of current configuration $\mathbf{n}$ as follows: $\text{supp}(\mathbf{n}) = \{b : n_{1,b} > 0\}$. It is also useful to define component-wise addition on the set of current configurations, that is $\mathbf{n_1} + \mathbf{n_2} = \{n_{1,b} + n_{2,b}\}_{b \in \bar{\Lambda}^{(2)}}$. In general one does not have $w(\mathbf{n_1} + \mathbf{n_2}) = w(\mathbf{n_1}) \times w(\mathbf{n_2})$, however this factorization does hold if $\text{supp}(\mathbf{n_1}) \cap \text{supp}(\mathbf{n_2}) = \varnothing$.

We say that the event $x \leftrightarrow y$ occurs under the flux configuration $\mathbf{n}$ if there exists a path of bonds linking x and y with $n_b \neq 0$ for each bond along this path. We denote the negation of that event by $x \nleftrightarrow y$. We adopt the convention that every site x is connected to itself, that is the event $x \leftrightarrow x$ occurs under all flux configurations.

Expressions will often involve indicator functions (defined on current configurations) for particular events. As an example the indicator function for the event $x \leftrightarrow y$ will be denoted $\text{I}[\mathbf{n} : x \leftrightarrow y]$. Additionally $\text{I}[\mathbf{n} : x \leftrightarrow x] = 1$ regardless of the choice of $\mathbf{n}$.

An essential identity for proving the neccessary bounds is the switching lemma which we state below.

Lemma 4.4 (**Switching Lemma**). *Let f be a function on current configurations. Then one has the following identity:*

$$\sum_{\substack{\partial \mathbf{n_1} = A \\ \partial \mathbf{n_2} = \{x,y\}}} w(\mathbf{n_1}) w(\mathbf{n_2}) f(\mathbf{n_1} + \mathbf{n_2}) = \sum_{\substack{\partial \mathbf{n_1} = A \Delta \{x,y\} \\ \mathbf{n_2} = \varnothing}} w(\mathbf{n_1}) w(\mathbf{n_2}) f(\mathbf{n_1} + \mathbf{n_2}) \times \text{I}[\mathbf{n_1} + \mathbf{n_2} : x \leftrightarrow y]$$

Proof: See Lemma 3.2 of [5]. □

The switching lemma can be used to produce useful representations of truncated correlation functions. Two such identities that will be used are given below.

Corollary 4.2.

$$\langle \sigma_x, \sigma_y \rangle^T = Z^{-2} \sum_{\substack{\partial \mathbf{n_1} = \{x\} \Delta \{y\} \\ \partial \mathbf{n_2} = \varnothing}} w(\mathbf{n_1}) w(\mathbf{n_2}) \times \text{I}[\mathbf{n_1} + \mathbf{n_2} : x \nleftrightarrow y] \tag{4.39}$$

$$\langle \sigma_x, \sigma_u \sigma_v \rangle^T = Z^{-2} \sum_{\substack{\partial \mathbf{n_1} = \{x,g\} \Delta \{u\} \Delta \{v\} \\ \partial \mathbf{n_2} = \varnothing}} w(\mathbf{n_1}) w(\mathbf{n_2}) \times \text{I}[\mathbf{n_1} + \mathbf{n_2} : x \nleftrightarrow g] \tag{4.40}$$

Proof: This is Corollary 3.5 of [8] but we quickly give the proof of 4.39 here. First observe that

$$\begin{aligned}\langle\phi_x\rangle\langle\phi_y\rangle &= Z^{-2}\sum_{\substack{\partial\mathbf{n_1}=\{y,g\}\\ \partial\mathbf{n_2}=\{x,g\}}} w(\mathbf{n_1})w(\mathbf{n_2})\\ &= Z^{-2}\sum_{\substack{\partial\mathbf{n_1}=\{x\}\Delta\{y\}\\ \partial\mathbf{n_2}=\varnothing}} w(\mathbf{n_1})w(\mathbf{n_2})\times \mathrm{I}[\mathbf{n_1}+\mathbf{n_2}:\mathrm{x}\leftrightarrow \mathrm{g}].\end{aligned}$$

In going to the last line above we used the switching lemma and the fact that $\{y,g\}\Delta\{x,g\}=\{x\}\Delta\{y\}$. Now by inserting a dummy sum over $\mathbf{n_2}$ and another factor of Z^{-1} we have

$$\langle\phi_x\phi_y\rangle = Z^{-2}\sum_{\substack{\partial\mathbf{n_1}=\{x\}\Delta\{y\}\\ \partial\mathbf{n_2}=\varnothing}} w(\mathbf{n_1})w(\mathbf{n_2})$$

Thus one has

$$\langle\phi_x,\phi_y\rangle^T = \langle\phi_x\phi_y\rangle - \langle\phi_x\rangle\langle\phi_y\rangle = Z^{-2}\sum_{\substack{\partial\mathbf{n_1}=\{x\}\Delta\{y\}\\ \partial\mathbf{n_2}=\varnothing}} w(\mathbf{n_1})w(\mathbf{n_2})\times(1-\mathrm{I}[\mathbf{n_1}+\mathbf{n_2}:\mathrm{x}\leftrightarrow\mathrm{g}])$$

.

This gives (4.39). One proves (4.40) similarly.

□

Another tool repeatedly used in [7] is conditioning on clusters. For a flux configuration $\mathbf{n}$ define the bond cluster of the site x as follows:

$$\mathbf{C_n}(x) = \{\{u,v\}\in\Lambda^{(2)}|\ \mathbf{n}: u\leftrightarrow x \text{ or } \mathbf{n}: v\leftrightarrow x\}\cup\{\{u,g\}|\mathbf{n}: u\leftrightarrow x\}$$

Note that $\mathbf{C_n}(x)$ may contain bonds b for which $n_b=0$. In particular $\mathbf{C_n}(x)$ is never empty and at least contains all bonds b with $b\ni x$ (since $x\leftrightarrow x$). We also remark that for a given $\mathbf{n}$ the question of whether $\mathbf{C_n}(x)=\mathbf{C}$ for some $\mathbf{C}$ is independent of $\mathbf{n}$'s flux numbers for bonds outside of $\mathbf{C}$. We say a bond is in the boundary of $\mathbf{C}$ if one of its endpoints is also contained in a bond that is not within $\mathbf{C}$.

We now introduce notation for taking expectations under modified interactions. Let $\mathbf{C}$ be a subset of bonds, then set

$$\langle\sigma^A\rangle_{\mathbf{C}} = \begin{cases}\dfrac{1}{Z_{\mathbf{C}}}\displaystyle\sum_{\substack{\partial\mathbf{n}=A\\ \mathrm{supp}(\mathbf{n})\subseteq\mathbf{C}}} w(\mathbf{n}) & \text{if } |A| \text{ is even}\\ \dfrac{1}{Z_{\mathbf{C}}}\displaystyle\sum_{\substack{\partial\mathbf{n}=A\cup\{g\}\\ \mathrm{supp}(\mathbf{n})\subseteq\mathbf{C}}} w(\mathbf{n}) & \text{if } |A| \text{ is odd}\end{cases} \tag{4.41}$$

where

$$Z_{\mathbf{C}} = \sum_{\substack{\partial \mathbf{n}=\varnothing \\ \mathrm{supp}(\mathbf{n})\subseteq \mathbf{C}}} w(\mathbf{n}). \tag{4.42}$$

It is not hard to see that $\langle \sim \rangle_{\mathbf{C}}$ an expectation taken with a modified version of our classical Ising ferromagnet where bonds outside of the set $\mathbf{C}$ have had their couplings $\bar{J}_b$ set to 0. We note that by Griffiths II one has $\langle \sigma^A \rangle_{\mathbf{C}} \leqslant \langle \sigma^A \rangle$ for any $A \subseteq \Lambda$, $C \subseteq \bar{\Lambda}^{(2)}$.

By conditioning on clusters within sums over current configurations one can prove the following identities:

Lemma 4.5.

$$\begin{aligned} \langle \sigma_x, \sigma_u \sigma_v \rangle^T =& Z^{-2} \sum_{\substack{\partial \mathbf{n_1}=\{x\}\Delta\{u\} \\ \partial \mathbf{n_2}=\varnothing}} w(\mathbf{n_1}) w(\mathbf{n_2}) \langle \sigma_v \rangle_{(\mathbf{C}_{\mathbf{n_1}+\mathbf{n_2}}(x))^c} \\ &+ Z^{-2} \sum_{\partial \mathbf{n_1}=\{x\}\Delta\{v\},\ \partial \mathbf{n_2}=\varnothing} w(\mathbf{n_1}) w(\mathbf{n_2}) \langle \sigma_u \rangle_{(\mathbf{C}_{\mathbf{n_1}+\mathbf{n_2}}(x))^c} \end{aligned} \tag{4.43}$$

$$\begin{aligned} \langle \sigma_x, \sigma_u \sigma_v \rangle^T =& Z^{-2} \sum_{\substack{\partial \mathbf{n_1}=\{u,g\} \\ \partial \mathbf{n_2}=\varnothing}} w(\mathbf{n_1}) w(\mathbf{n_2}) \langle \sigma_x \sigma_v \rangle_{(\mathbf{C}_{\mathbf{n_1}+\mathbf{n_2}}(g))^c} \times \mathrm{I}[\mathbf{n_1}+\mathbf{n_2} : \mathrm{x} \not\leftrightarrow \mathrm{g}] \\ &+ Z^{-2} \sum_{\substack{\partial \mathbf{n_1}=\{v,g\} \\ \partial \mathbf{n_2}=\varnothing}} w(\mathbf{n_1}) w(\mathbf{n_2}) \langle \sigma_x \sigma_u \rangle_{(\mathbf{C}_{\mathbf{n_1}+\mathbf{n_2}}(g))^c} \times \mathrm{I}[\mathbf{n_1}+\mathbf{n_2} : \mathrm{x} \not\leftrightarrow \mathrm{g}] \end{aligned} \tag{4.44}$$

$$\langle \sigma_x, \sigma_y \rangle^T = Z^{-2} \sum_{\substack{\partial \mathbf{n_1}=\varnothing \\ \partial \mathbf{n_2}=\varnothing}} w(\mathbf{n_1}) w(\mathbf{n_2}) \langle \sigma_x \sigma_y \rangle_{(\mathbf{C}_{\mathbf{n_1}+\mathbf{n_2}}(g))^c} \times \mathrm{I}[\mathbf{n_1}+\mathbf{n_2} : x \not\leftrightarrow g] \tag{4.45}$$

Proof. We start with proving (4.43). We observe that if $u = v$ then both sides of the equation vanish. The fact that the left hand side vanishes in this case is immediate. For the first term on the right hand side note that since x and u are sources for $\mathbf{n_1}$ then all the bonds with $v = u$ as an endpoint are contained in the bond cluster $C_{\mathbf{n_1}+\mathbf{n_2}}(x)$ so $\langle \sigma_v \rangle_{[C_{\mathbf{n_1}+\mathbf{n_2}}(x)]^c} = 0$ (with those bonds suppressed σ_v is mean zero bernoulli random variable). The second term on the right hand side vanishes by the same argument. Thus the equation holds if $u = v$. □

We now work under the assumption that u and v are distinct. Note that by Corollary 4.2 we can write

$$\langle \sigma_x, \sigma_u \sigma_v \rangle^T = Z^{-2} \sum_{\substack{\partial \mathbf{n_1}=\{x,g\}\Delta\{u\}\Delta\{v\} \\ \partial \mathbf{n_2}=\varnothing}} w(\mathbf{n_1}) w(\mathbf{n_2}) \times \mathrm{I}[\mathbf{n_1}+\mathbf{n_2} : x \not\leftrightarrow g] \times (\mathrm{I}[\mathbf{n_1}+\mathbf{n_2} : v \leftrightarrow g] + \mathrm{I}[\mathbf{n_1}+\mathbf{n_2} : u \leftrightarrow g]) \tag{4.46}$$

Since u, v are distinct the sum of the two indicator functions at the end of the RHS will be precisely 1 under the source constraints and the indicator function forcing $x \not\leftrightarrow g$. Either u or v must be connected to the ghost site g since x cannot be. However if u is connected to g then v cannot be and vice-versa.

We now work on the last expression with just the first of the last two indicator functions above and condition on the bond cluster of x:

$$\begin{aligned}
&\sum_{\substack{\partial\mathbf{n_1}=\{x,g\}\Delta\{u\}\Delta\{v\}\\ \partial\mathbf{n_2}=\varnothing}} w(\mathbf{n_1})w(\mathbf{n_2}) \times \mathrm{I}[\mathbf{n_1}+\mathbf{n_2} : x \nleftrightarrow g] \times \mathrm{I}[\mathbf{n_1}+\mathbf{n_2} : v \leftrightarrow g] \\
&= \sum_{\mathbf{C}\subseteq\bar{\Lambda}^{(2)}} \sum_{\substack{\partial\mathbf{n_1}=\{x,g\}\Delta\{u\}\Delta\{v\}\\ \partial\mathbf{n_2}=\varnothing}} \Big[w(\mathbf{n_1})w(\mathbf{n_2}) \times \mathrm{I}[\mathbf{n_1}+\mathbf{n_2} : x \nleftrightarrow g] \\
&\qquad\qquad \times \mathrm{I}[\mathbf{n_1}+\mathbf{n_2} : v \leftrightarrow g] \times \mathrm{I}[\mathbf{n_1}+\mathbf{n_2} : \mathbf{C}_{\mathbf{n_1}+\mathbf{n_2}}(x) = \mathbf{C}] \Big] \\
&= \sum_{\mathbf{C}\subseteq\bar{\Lambda}^{(2)}} \sum_{\substack{\partial\mathbf{n'_1}=\{x\}\Delta\{u\},\partial\mathbf{n''_1}=\{v,g\}\\ \partial\mathbf{n'_2}=\varnothing,\partial\mathbf{n''_2}=\varnothing}} w(\mathbf{n'_1})w(\mathbf{n''_1})w(\mathbf{n'_2})w(\mathbf{n''_2}) \times \mathrm{I}[\mathbf{n'_1}+\mathbf{n'_2} : \mathbf{C}_{\mathbf{n'_1}+\mathbf{n'_2}}(x) = \mathbf{C}] \\
&\qquad\qquad \times \prod_{i=1,2} \left(\mathrm{I}[\mathbf{n'_i} : \mathrm{supp}(\mathbf{n'_i}) \subseteq \mathbf{C}] \times \mathrm{I}[\mathbf{n''_i} : \mathrm{supp}(\mathbf{n''_i}) \subseteq \mathbf{C}^c] \right)
\end{aligned}$$

To go to the bottom expression one splits $\mathbf{n_i} = \mathbf{n'_i} + \mathbf{n''_i}$ for $i = 1,2$ where $\mathbf{n'_i}$ is supported on $\mathbf{C}$ and $\mathbf{n''_i}$ is supported on $\mathbf{C}^c$. By virtue of their supports being disjoint we have the factorization of weights $w(\mathbf{n'_i} + \mathbf{n''_i}) = w(\mathbf{n'_i})w(\mathbf{n''_i})$. Since $\mathbf{C} = \mathbf{C}_{\mathbf{n_1}+\mathbf{n_2}}(x) = \mathbf{C}_{\mathbf{n'_1}+\mathbf{n'_2}}(x)$ we know that $\mathbf{n'_1} + \mathbf{n'_2}$ vanishes on the boundary of $\mathbf{C}$. Thus the flux configuration $\mathbf{n'_1}$ must have x and u as sources while $\mathbf{n''_1}$ must have v and g as sources (in other words $\mathbf{n'_1}$ and $\mathbf{n'_2}$ don't need to work together to allow $\mathbf{n_1}$ to satisfy its source condition.

Carrying out the summation over $\mathbf{n''_1}$ and $\mathbf{n''_2}$ gives

$$\begin{aligned}
&\sum_{\mathbf{C}\subseteq\bar{\Lambda}^{(2)}} \sum_{\substack{\partial\mathbf{n'_1}=\{x\}\Delta\{u\}\\ \partial\mathbf{n'_2}=\varnothing}} Z^2_{\mathbf{C}^c} \langle\sigma_v\rangle_{\mathbf{C}^c} \, w(\mathbf{n'_1})w(\mathbf{n'_2}) \times \mathrm{I}[\mathbf{n'_1}+\mathbf{n'_2} : \mathbf{C}_{\mathbf{n'_1}+\mathbf{n'_2}}(x) = \mathbf{C}] \prod_{i=1,2} \left(\mathrm{I}[\mathbf{n'_i} : \mathrm{supp}(\mathbf{n'_i}) \subseteq \mathbf{C}] \right) \\
&= \sum_{\mathbf{C}\subseteq\bar{\Lambda}^{(2)}} \sum_{\substack{\partial\mathbf{n'_1}=\{x\}\Delta\{u\}\\ \partial\mathbf{n'_2}=\varnothing}} \langle\sigma_v\rangle_{\mathbf{C}^c} \, w(\mathbf{n'_1})w(\mathbf{n'_2}) \mathrm{I}[\mathbf{n'_1}+\mathbf{n'_2} : \mathbf{C}_{\mathbf{n'_1}+\mathbf{n'_2}}(x) = \mathbf{C}] \\
&= \sum_{\substack{\partial\mathbf{n'_1}=\{x\}\Delta\{u\}\\ \partial\mathbf{n'_2}=\varnothing}} \langle\sigma_v\rangle_{(\mathbf{C}_{\mathbf{n'_1}+\mathbf{n'_2}}(x))^c} \, w(\mathbf{n'_1})w(\mathbf{n'_2})
\end{aligned} \tag{4.47}$$

The first equality above has yet to be justified, we will do so below. The last expression is the first term on the RHS of (4.43). If we can just justify the first equality then we will get the second term on the RHS of (4.43) by choosing the other indicator function at the bottom of (4.46).

We now justify the first equality of (4.47). Let $\mathbf{D}$ be a set of bonds and let f be a function on flux configurations that does not depend on the fluxes assigned to bonds in $\mathbf{D}^c$. Then one has:

$$
\begin{aligned}
&Z_{\mathbf{D}^c} \sum_{\mathbf{n_1}} w(\mathbf{n_1}) f(\mathbf{n_1}) \times \mathrm{I}[\mathbf{n_1} : \mathrm{supp}(\mathbf{n_1}) = \mathbf{D}] \\
&= \sum_{\mathbf{n_1},\mathbf{n_2}} w(\mathbf{n_1}) w(\mathbf{n_2}) f(\mathbf{n_1}) \times \mathrm{I}[\mathbf{n_1} : \mathrm{supp}(\mathbf{n_1}) = \mathbf{D}] \times \mathrm{I}[\mathbf{n_1} : \mathrm{supp}(\mathbf{n_1}) = \mathbf{D}^c] \\
&= \sum_{\mathbf{n_1},\mathbf{n_2}} w(\mathbf{n_1} + \mathbf{n_2}) f(\mathbf{n_1} + \mathbf{n_2}) \times \mathrm{I}[\mathbf{n_1} : \mathrm{supp}(\mathbf{n_1}) = \mathbf{D}] \times \mathrm{I}[\mathbf{n_1} : \mathrm{supp}(\mathbf{n_1}) = \mathbf{D}^c] \\
&= \sum_{\mathbf{n}} w(\mathbf{n}) f(\mathbf{n}).
\end{aligned}
$$

Since $\mathrm{I}[\mathbf{n_1'} : \partial \mathbf{n_1'} = \{x, u\}] \times \mathrm{I}[\mathbf{n_1'} + \mathbf{n_2'} : \mathbf{C}_{\mathbf{n_1'}+\mathbf{n_2'}}(x) = \mathbf{C}]$ does not depend on bonds outside of $\mathbf{C}$ this justifies the first equality in (4.47). We have now finished the proof of (4.43).

The proof of (4.44) is nearly the same, we again starts with (4.46) but this time we condition on $\mathbf{C}_{\mathbf{n_1}+\mathbf{n_2}}(g)$. To prove (4.45) one starts with (4.39) and then conditions on $\mathbf{C}_{\mathbf{n_1}+\mathbf{n_2}}(g)$. □

Lemma 4.6. *Let $x, y, u, v \in \Lambda$ with $v \neq y$ and $v \neq u$. Then one has:*

$$
\begin{aligned}
&\sum_{\substack{\partial \mathbf{n_1} = \{v,g\} \\ \partial \mathbf{n_2} = \{y,g\}}} w(\mathbf{n_1}) w(\mathbf{n_2}) \langle \sigma_x \sigma_u \rangle_{(\mathbf{C}_{\mathbf{n_1}+\mathbf{n_2}}(g))^c} \times \mathrm{I}[\mathbf{n_1} + \mathbf{n_2} : \mathrm{x} \nleftrightarrow \mathrm{g}] \\
\leqslant &\langle \sigma_y \rangle \sum_{\substack{\partial \mathbf{n_1} = \{x\} \Delta \{u\} \\ \partial \mathbf{n_2} = \varnothing}} w(\mathbf{n_1}) w(\mathbf{n_2}) \langle \sigma_v \rangle_{(\mathbf{C}_{\mathbf{n_1}+\mathbf{n_2}}(x))^c} \times \mathrm{I}[\mathbf{n_1} + \mathbf{n_2} : \mathrm{x} \nleftrightarrow \mathrm{g}]
\end{aligned}
\tag{4.48}
$$

Proof: If $y = x$ then the inequality is trivial since the first expression vanishes - the source constraints of $\mathbf{n_2}$ conflicts with the indicator function disallowing x to be connected to g. For what follows we assume that $\underline{x \neq y}$.

Now if $u = y$ then again then again the first expression vanishes. If $u = y$ then n_2 must connect u and g which means one has that $(\mathbf{C}_{\mathbf{n_1}+\mathbf{n_2}}(g))$ contains all the bonds touching u, thus $\langle \sigma_x \sigma_u \rangle_{(\mathbf{C}_{\mathbf{n_1}+\mathbf{n_2}}(g))^c} = 0$. Thus in what follows we work under the assumptions that $\underline{u \neq v,\ x \neq y}$. With these assumptions in place we first prove the following claim:

$$
\begin{aligned}
&Z^{-2} \sum_{\substack{\partial \mathbf{n_1} = \{v,g\} \\ \partial \mathbf{n_2} = \{u,y,g\} \Delta \{x\}}} w(\mathbf{n_1}) w(\mathbf{n_2}) \times \mathrm{I}[\mathbf{n_1} + \mathbf{n_2} : \mathrm{x} \nleftrightarrow \mathrm{g}] \times \mathrm{I}[\mathbf{n_1} + \mathbf{n_2} : \mathrm{u} \nleftrightarrow \mathrm{g}] \\
= &Z^{-2} \sum_{\substack{\partial \mathbf{n_1} = \{v,g\} \\ \partial \mathbf{n_2} = \{y,g\}}} w(\mathbf{n_1}) w(\mathbf{n_2}) \langle \sigma_x \sigma_u \rangle_{(\mathbf{C}_{\mathbf{n_1}+\mathbf{n_2}}(g))^c} \times \mathrm{I}[\mathbf{n_1} + \mathbf{n_2} : \mathrm{x} \nleftrightarrow \mathrm{g}]
\end{aligned}
\tag{4.49}
$$

We start with the expression on the top and condition on the cluster $\mathbf{C}_{\mathbf{n_1}+\mathbf{n_2}}(g)$ to get the expression on the bottom. We have that

$$\sum_{\mathbf{C}\subseteq\bar{\Lambda}^{(2)}}\sum_{\substack{\partial\mathbf{n_1}=\{v,g\}\\ \partial\mathbf{n_2}=\{u,y,g\}\Delta\{x\}}} w(\mathbf{n_1})w(\mathbf{n_2})\times\mathrm{I}[\mathbf{n_1}+\mathbf{n_2}:\mathrm{x}\not\leftrightarrow\mathrm{g}]\times\mathrm{I}[\mathbf{n_1}+\mathbf{n_2}:\mathrm{u}\not\leftrightarrow\mathrm{g}]\times\mathrm{I}[\mathbf{n_1}+\mathbf{n_2}:\mathbf{C}_{\mathbf{n_1}+\mathbf{n_2}}(\mathrm{g})=\mathbf{C}]$$

$$=\sum_{\mathbf{C}\subseteq\bar{\Lambda}^{(2)}}\sum_{\substack{\partial\mathbf{n}'_1=\{v,g\},\partial\mathbf{n}''_1=\varnothing\\ \partial\mathbf{n}'_2=\{y,g\},\partial\mathbf{n}''_2=\{x\}\Delta\{u\}}}\Bigg[w(\mathbf{n}'_1)w(\mathbf{n}''_1)w(\mathbf{n}'_2)w(\mathbf{n}''_2)\times\mathrm{I}[\mathbf{n}'_1+\mathbf{n}'_2:\mathrm{x}\not\leftrightarrow\mathrm{g}]\times\mathrm{I}[\mathbf{n}'_1+\mathbf{n}'_2:\mathrm{u}\not\leftrightarrow\mathrm{g}]$$

$$\times\mathrm{I}[\mathbf{n}'_1+\mathbf{n}'_2:\mathbf{C}_{\mathbf{n}'_1+\mathbf{n}'_2}(g)=\mathbf{C}]\times\prod_{i=1,2}\big(\mathrm{I}[\mathbf{n}'_\mathbf{i}:\mathrm{supp}(\mathbf{n}'_\mathbf{i})\subseteq\mathbf{C}]\times\mathrm{I}[\mathbf{n}''_\mathbf{i}:\mathrm{supp}(\mathbf{n}''_\mathbf{i})\subseteq\mathbf{C}^c]\big)\Bigg]$$

As in Lemma 4.5 we have split each of the current configurations $\mathbf{n_1}$ and $\mathbf{n_2}$ into two pieces - one living on the cluster we're conditioning on and one living on that cluster's complement. Since we force n'_1 and n'_2 to have $\mathbf{C}_{\mathbf{n}'_1+\mathbf{n}'_2}(x)=\mathbf{C}$ we have that the n'_i are supported on a set of bonds smaller than $\mathbf{C}$ and cannot touch any site that is also visited by a bond in $\mathbf{C}^c$. Thus $\mathbf{n}'_1$ will have $\{v,g\}$ as sources . The restriction that both x and g are not connected to g by $\mathbf{n_1}+\mathbf{n_2}$ means that $\mathbf{n}''_2$ will have $\{x\}\Delta\{u\}$ as its sources instead of $\mathbf{n}'_2$. We now carry out the sum over the $\mathbf{n}''_\mathbf{i}$ and then proceed to undo the conditioning:

$$\sum_{\mathbf{C}\subseteq\bar{\Lambda}^{(2)}}\sum_{\substack{\partial\mathbf{n}'_1=\{v,g\}\\ \mathbf{n}'_2=\{y,g\}}}\Bigg[w(\mathbf{n}'_1)w(\mathbf{n}'_2)\times\mathrm{I}[\mathbf{n}'_1+\mathbf{n}'_2:\mathrm{x}\not\leftrightarrow\mathrm{g}]\times\mathrm{I}[\mathbf{n}'_1+\mathbf{n}'_2:\mathrm{u}\not\leftrightarrow\mathrm{g}]\times\mathrm{Z}^2_{\mathbf{C}^c}\times\langle\sigma_\mathrm{x}\sigma_\mathrm{u}\rangle_{\mathbf{C}^c}$$

$$\times\mathrm{I}[\mathbf{n}'_1+\mathbf{n}'_2:\mathbf{C}_{\mathbf{n}'_1+\mathbf{n}'_2}(\mathrm{g})=\mathbf{C}]\times\mathrm{I}[\mathbf{n}'_1:\mathrm{supp}(\mathbf{n}'_\mathbf{i})\subseteq\mathbf{C}]\times\mathrm{I}[\mathbf{n}'_2:\mathrm{supp}(\mathbf{n}'_\mathbf{i})\subseteq\mathbf{C}]\Bigg]$$

$$=\sum_{\mathbf{C}\subseteq\bar{\Lambda}^{(2)}}\sum_{\substack{\partial\mathbf{n}'_1=\{v,g\}\\ \mathbf{n}'_2=\{y,g\}}}\Bigg[w(\mathbf{n}'_1)w(\mathbf{n}'_2)\times\mathrm{I}[\mathbf{n}'_1+\mathbf{n}'_2:\mathrm{x}\not\leftrightarrow\mathrm{g}]$$

$$\times\mathrm{I}[\mathbf{n}'_1+\mathbf{n}'_2:\mathrm{u}\not\leftrightarrow\mathrm{g}]\times\langle\sigma_\mathrm{x}\sigma_\mathrm{u}\rangle_{\mathbf{C}^c}\times\mathrm{I}[\mathbf{n}'_1+\mathbf{n}'_2:\mathbf{C}_{\mathbf{n}'_1+\mathbf{n}'_2}(\mathrm{g})=\mathbf{C}]\Bigg]$$

$$=\sum_{\substack{\partial\mathbf{n}'_1=\{v,g\}\\ \mathbf{n}'_2=\{y,g\}}}\Big[w(\mathbf{n}'_1)w(\mathbf{n}'_2)\times\mathrm{I}[\mathbf{n}'_1+\mathbf{n}'_2:\mathrm{x}\not\leftrightarrow\mathrm{g}]\times\mathrm{I}[\mathbf{n}'_1+\mathbf{n}'_2:\mathrm{u}\not\leftrightarrow\mathrm{g}]\times\langle\sigma_\mathrm{x}\sigma_\mathrm{u}\rangle_{(\mathbf{C}_{\mathbf{n}'_1+\mathbf{n}'_2}(\mathrm{g}))^c}\Big]$$

$$=\sum_{\substack{\partial\mathbf{n}'_1=\{v,g\}\\ \mathbf{n}'_2=\{y,g\}}} w(\mathbf{n}'_1)w(\mathbf{n}'_2)\times\mathrm{I}[\mathbf{n}'_1+\mathbf{n}'_2:\mathrm{x}\not\leftrightarrow\mathrm{g}]\times\langle\sigma_\mathrm{x}\sigma_\mathrm{u}\rangle_{(\mathbf{C}_{\mathbf{n}'_1+\mathbf{n}'_2}(\mathrm{g}))^c}$$

For the first equality above we used the two factors of $Z_{\mathbf{C}^c}$ to remove the support condition on $\mathbf{n}'_1$ and $\mathbf{n_2}'$ - for more details look at the proof of Lemma 4.5 and note that the value of the product of functions:

$$\mathrm{I}[\mathbf{n}'_1+\mathbf{n}'_2:\mathrm{x}\not\leftrightarrow\mathrm{g}]\times\mathrm{I}[\mathbf{n}'_1+\mathbf{n}'_2:\mathrm{u}\not\leftrightarrow\mathrm{g}]\times\langle\sigma_\mathrm{x}\sigma_\mathrm{u}\rangle_{\mathbf{C}^c}\times\mathrm{I}[\mathbf{n}'_1+\mathbf{n}'_2:\mathbf{C}_{\mathbf{n}'_1+\mathbf{n}'_2}(\mathrm{g})=\mathbf{C}]$$

is in fact independent of flux numbers of bonds in $\mathbf{C}^c$ (the last indicator function prevents flux numbers in

$\mathbf{C}^c$ from influencing the connectivity of x and u to g).

To see how we dropped the indicator function in the last equality note that the functions $\mathrm{I}[\mathbf{n_1'} + \mathbf{n_2'} : \mathrm{x} \nleftrightarrow \mathrm{g}] \times \mathrm{I}[\mathbf{n_1'} + \mathbf{n_2'} : \mathrm{u} \nleftrightarrow \mathrm{g}]$ are actually completely extraneous if x and u are distinct - in this case $\langle \sigma_x \sigma_u \rangle_{(\mathbf{C}_{\mathbf{n_1'}+\mathbf{n_2'}}(g))^c}$ vanishes if x or u are connected to g. The indicator function is not extraneous if $x = u$ but in this case we only need one of them. Relabeling the variables $\mathbf{n_i'}$ as $\mathbf{n_i}$ proves (4.49).

By proceeding in almost exactly the same way as above but this time conditioning on $\mathbf{C}_{\mathbf{n_1}+\mathbf{n_2}}(x)$ one can prove the follow claim:

$$\begin{aligned}&\sum_{\substack{\partial \mathbf{n_1} = \{v,g\} \\ \partial \mathbf{n_2} = \{u,y,g\}\Delta\{x\}}} w(\mathbf{n_1}) w(\mathbf{n_2}) \times \mathrm{I}[\mathbf{n_1} + \mathbf{n_2} : \mathrm{x} \nleftrightarrow \mathrm{g}] \times \mathrm{I}[\mathbf{n_1} + \mathbf{n_2} : \mathrm{u} \nleftrightarrow \mathrm{g}] \\ = &\sum_{\substack{\partial \mathbf{n_1} = \{x\}\Delta\{u\} \\ \partial \mathbf{n_2} = \varnothing}} w(\mathbf{n_1}) w(\mathbf{n_2}) \langle \sigma_y \rangle_{\left(\mathbf{C}_{\mathbf{n_1}+\mathbf{n_2}}(x)\right)^c} \langle \sigma_v \rangle_{\left(\mathbf{C}_{\mathbf{n_1}+\mathbf{n_2}}(x)\right)^c} \times \mathrm{I}[\mathbf{n_1} + \mathbf{n_2} : \mathrm{x} \nleftrightarrow \mathrm{g}]\end{aligned} \tag{4.50}$$

Combining (4.49), (4.50), and the observation that $\langle \sigma_y \rangle_{\left(\mathbf{C}_{\mathbf{n_1}+\mathbf{n_2}}(x)\right)^c} \leqslant \langle \sigma_y \rangle$ by Griffiths II will finish the proof of the lemma.

$\square$.

The random current representation in its basic form is particular to classical Ising ferromagnets. In order to prove the desired partial differential inequalities for ϕ^4 ferromagnets one must use Griffiths-Simon approximation.

Recall that in the Griffiths-Simons approximation one replaces each ϕ^4 spin ϕ_x on the lattice Λ with a block of N microscopic classical Ising spins $\sigma_{(x,\cdot)}$ - the spin ϕ_x is well approximated (in the $N \to \infty$ limit) by the scaled average $\theta_N(\phi_x) = c_N \sum_{\alpha=1}^{N} \sigma_{(x,\alpha)}$. One then applies the random current representation to the classical Ising ferromagnetic system $\{\sigma_{(x,\alpha)}\}_{(x,\alpha)\in\Lambda\times[N]}$ to try to get the necessary inequalities for correlation functions of the block spin variables $\theta_N(\phi_\cdot)$ which are uniform in N.

In doing this one implements connectivity conditions on current configurations that involve blocks in addition to the previously mentioned conditions involving individual sites. The following versions of the switching lemma are useful tools for working with such block connectivity conditions.

First we give some more notation: given a collection of lattice sites B and a single lattice site z one says that the event $B \leftrightarrow z$ occurs under a current configuration $\mathbf{n}$ if there exists an $x \in B$ such that $x \leftrightarrow z$ occurs with $\mathbf{n}$.

Lemma 4.7. *Let B be a collection lattice sites and let z be a site not contained in B. Let f be a function on current configurations that is decreasing in each flux number. Then one has*

$$\sum_{\substack{\partial\mathbf{n_1}=\emptyset \\ \partial\mathbf{n_2}=\emptyset}} w(\mathbf{n_1})w(\mathbf{n_2})f(\mathbf{n_1}+\mathbf{n_2}) \times I[\mathbf{n_1}+\mathbf{n_2}:\ B \leftrightarrow z]$$

$$\leqslant \sum_{\substack{x\in B \\ y\in B^c}} \bar{J}_{\{x,y\}} \left[\sum_{\substack{\partial\mathbf{n_1}=\{x,z\} \\ \partial\mathbf{n_2}=\{y\}\Delta\{z\}}} w(\mathbf{n_1})w(\mathbf{n_2})f(\mathbf{n_1}+\mathbf{n_2}) \right]$$

Note that above we allow y or z to be the ghost site g.

Proof. This is Lemma 4.1 in [7]. □

4.6.2 Random Walk Expansion

The random walk expansion represents correlation functions and their truncated counterparts as sums over sequences of walks on the enhanced lattice $\bar{\Lambda}$.

A walk γ is a finite non-empty sequence of oriented bonds: $\gamma = ((u_0, u_1), \ldots, (u_k, u_{k+1}))$ where the (u_{2j}, u_{2j+1}) are ordered pairs that represent oriented bonds (they are non-diagonal elements of $\bar{\Lambda} \times \bar{\Lambda}$. We will call these oriented bonds "steps". It is required that consecutive steps have common endpoints: if (u_{2j}, u_{2j+1}), (u_{2j+2}, u_{2j+3}) are consecutive steps in a walk ω then one must have $u_{2j+1} = u_{2j+2}$.

A backbone ω is a sequence of walks $\omega = (\gamma_0, \cdots, \gamma_n)$. The boundary of a backbone ω is denoted $\partial\omega$ and is defined to be those sites that the backbone visits an odd number of times during its sequence of walks. There is a concatenation operating defined on backbones, $\omega_1 \circ \omega_2$ is a new sequence of walks formed by concatenating the sequences ω_1 and ω_2.

The symbol ω will be used to represent both walks and backbones. A single walk can be viewed as a backbone with a one element sequence.

The random walk expansion is given by a sum over backbones, each backbone corresponding to the collective contribution of a group of current configurations $\mathbf{n}$. Let B be a non-empty subset of $\bar{\Lambda}$ with $|B|$ even. If $\mathbf{n}$ is a current configuration with $\partial\mathbf{n} = B$ then one must be able to find a collection of disjoint paths of odd flux bonds which connect the the elements of B in pairs. In [8], [7] one is given a special consistent way to pick out such a collection of paths for every current configuration $\mathbf{n}$ with $\partial n = B$ and then order/orient these paths to give rise to a backbone ω with $\partial\omega = B$.

This is done via defining a map $\Omega_B : \{\mathbf{n}|\ \partial\mathbf{n} = B\} \to \{\omega|\partial\omega = B\}$. We now give the definition for this map given in [7]. First choose a total ordering on sites of Λ (instead of describing sites as smaller or larger we'll describe them as earlier or later). The method of assigning backbones is not unique because of this initial choice of ordering. Extend this ordering to $\bar{\Lambda}$ by making the ghost site g the earliest site. Now for a fixed current figuration $\partial n = B$ carry out the following algorithm to generate a sequence of walks. This algorithm will also generate a collection of non-oriented bonds (called "cancelled" bonds) $\tilde{\omega}(\mathbf{n})$.

- The first step of this first walk ω_1 starts with the earliest site of $B\backslash\{g\}$ which we denote u_1. The first step ends on the earliest site among those sites v with $n_{\{u_1,v\}}$ odd. The bond $\{u_1, u_2\}$ that is traversed in this step is added to the set of cancelled bonds $\tilde{\omega}(\mathbf{n})$, along with all bonds $\{u_1, z\}$ with z earlier than u_2. The walk stops if it reaches any site in B or the ghost site

- Every subsequent step (u_i, u_{i+1}) of ω_1 (if the walk hasn't terminated) is chosen the same way: u_{i+1} is the earliest among those sites v for which: (i) $n_{\{u_i,v\}}$ is odd, and (ii) $\{u_i, v\}$ is not yet among the set of bonds that have been canceled up to now. The bond $\{u_i, u_{i+1}\}$ chosen for the walk is added to the collection of cancelled bonds, along with all bonds $\{u_i, z\}$ with z earlier than u_{i+1} in the site ordering. This process continues until ω_1 reaches a site in B or the ghost site g.

- Once the walks $\omega_1, \ldots \omega_j$ have been generated one starts the walk ω_{j+1} from the earliest site in $B\backslash\{g\}$ that has not been visited by any previous walk. Every subsequent step (u_i, u_{i+1}) of ω_{j+1} is chosen as above (proceeding to the earliest vertex connected via a bond with odd flux and avoiding any bond cancelled by previous steps of this walk or earlier walks). One continues to update the set of cancelled bonds as before. ω_{j+1} terminates when it reaches a site in B that has not been visited by any walk or the ghost site g.

- This process is continued until the set B has been exhausted, one will be left with a sequence of walks $(\omega_1, \omega_2, \ldots, \omega_k)$ with $\partial\omega = B$.

We also adopt the following convention: if $\partial\mathbf{n} = \varnothing$ then $\Omega_\varnothing(\mathbf{n}) = \varnothing$, that is all sourceless current configurations are assigned to the empty backbone.

The above algorithm defines the desired map $\Omega_B(\cdot)$ for every set $B \subset \Lambda$ with $|B|$ even. This map is then used to reorganize the sum over current configurations. For a backbone ω define

$$\rho(\omega) = Z^{-1} \sum_{\mathbf{n} \text{ such that } \partial\mathbf{n} = \partial\omega} w(\mathbf{n}) \times I[\Omega_{\partial\omega}(\mathbf{n}) = \omega].$$

It then immediately follows that one has the following representation for correlation functions. For any $A \subseteq \Lambda$:

$$\langle \sigma^A \rangle = \begin{cases} \sum_{\partial\omega = A} \rho(\omega) & \text{if } |A| \text{ is even} \\ \sum_{\partial\omega = A \cup \{g\}} \rho(\omega) & \text{if } |A| \text{ is odd }. \end{cases} \tag{4.51}$$

We sometimes abuse notation and will sometimes use the notation ω to represent the set of non-oriented bonds traversed by that backbone.

We now give more details (taken from [8]) on the properties of the weights of the random walk representation.

We call a backbone ω consistent if there exists a current configuration $\mathbf{n}$ with $\Omega_{\partial\mathbf{n}}(\mathbf{n}) = \omega$. Note that if ω is not consistent then $\rho(\omega) = 0$. In particular, ω is consistent if and only if the current configuration $\mathbf{n}$ defined by $n_b = \mathbb{1}[b \in \omega]$ satisfies $\Omega_{\partial\mathbf{n}}(\mathbf{n}) = \omega$.
We also remark that the set of cancelled bonds $\tilde{\omega}(\mathbf{n})$ for a given current configuration $\mathbf{n}$ is determined entirely by $\mathbf{n}$'s backbone, that is by $\Omega_{\partial\mathbf{n}}(\mathbf{n})$. In other words for every consistent backbone ω there exists a unique $\tilde{\omega}$ such that for all $\mathbf{n}$ with $\Omega_{\partial\mathbf{n}}(\mathbf{n}) = \omega$ one has $\tilde{\omega}(\mathbf{n}) = \tilde{\omega}$.
We note that the definition of consistency used here, along with the definitions of $\Omega_{\bullet}(\bullet)$, $\rho(\bullet)$, and $\tilde{\omega}$ are all dependent on the initial choice of ordering on Λ.

With this in mind we now give following lemma from [8].

Lemma 4.8. *Let ω be a consistent backbone and let $\tilde{\omega}$ be its associated set of cancelled bonds $\tilde{\omega}$. Then for any current configurations $\mathbf{n}$ one has $\Omega_{\partial\mathbf{n}}(\partial\mathbf{n}) = \omega$ if and only if the following three conditions are all met.*

(a) $\mathbf{n}$ *is odd on all of the bonds traversed by ω.*

(b) $\mathbf{n}$ *is even on all of the bonds in $\tilde{\omega}\backslash\omega$ (or else the backbone* $\mathbf{n}$ *would have traversed them in place of some bond in ω).*

(c) $\mathbf{n}$ *restricted to the bonds in ω^c is sourceless.*

Proof: Clear from the definition of the backbone map and the definition of consistency. □

As a corollary one has:

Corollary 4.3. *One has the following representation for backbone weights:*

$$\begin{aligned}\rho(\omega) =& \mathrm{I}[\ \omega \text{ is consistent }] \left(\prod_{b\in\omega}\tanh(\bar{J}_b)\right)\left(\prod_{b\in\tilde{\omega}}\cosh(\bar{J}_b)\right)\left(\frac{Z_{\tilde{\omega}^c}}{Z}\right)\\ =& \mathrm{I}[\ \omega \text{ is consistent }] \left(\prod_{b\in\omega}\tanh(\bar{J}_b)\right)\times\frac{1}{Z}\left(\sum_{\partial\mathbf{n}=\varnothing} w(\mathbf{n})\times\mathrm{I}[\mathbf{n} : n_b \text{ is even for all } b\in\tilde{\omega}]\right)\end{aligned} \tag{4.52}$$

Proof:

Suppose that ω is consistent. Then when summing over all $\mathbf{n}$ satisfying the conditions of Lemma 4.8 the sum over $\{n_b\}_{b\in\bar{\Lambda}^{(2)}}$ factors into sums over the odd flux numbers of each of the bonds in ω, sums over even flux numbers for each of the bonds in $\tilde{\omega}\backslash\omega$, and a sum over sourceless current configurations living on $\bar{\Lambda}^{(2)}\backslash\tilde{\omega}$. This, along with the fact that $\sum_{n\geqslant 0 \text{ even}}\frac{t^n}{n!} = \cosh(t)$ and $\sum_{n\geqslant 0 \text{ odd}}\frac{t^n}{n!} = \sinh(t)$ tells us that

$$\rho(\omega) = \frac{1}{Z}\times\mathrm{I}[\ \omega \text{ is consistent }]\left(\prod_{b\in\omega}\sinh(\bar{J}_b)\right)\left(\prod_{b\in\tilde{\omega}\backslash\omega}\cosh(\bar{J}_b)\right)\left(\sum_{\partial\mathbf{n}=\varnothing} w(\mathbf{n})\times\mathrm{I}[\mathbf{n} : \mathrm{supp}(\mathbf{n})\subseteq\tilde{\omega}^c]\right).$$

This establishes the first equality of the lemma, the second equality follows immediately. □

If $\mathbf{C}$ is a set of bonds we use the notation $\rho_{\mathbf{C}}(\cdot)$ to denote the modified normalized weighting that comes from setting $\bar{J}_b = 0$ for $b \notin \mathbf{C}$. In particular

$$\rho_{\mathbf{C}}(\omega) = \frac{1}{Z_{\mathbf{C}}} \sum_{\mathbf{n} \text{ such that } \partial\mathbf{n}=\partial\omega} w(\mathbf{n}) \times \mathrm{I}[\mathbf{n} : \mathrm{supp}(\mathbf{n}) \subseteq \mathbf{C}].$$

Observe that the formula (4.51) holds with expectations $\langle \sim \rangle_{\mathbf{C}}$ and weights given by $\rho_{\mathbf{C}}(\cdot)$.

The following lemma gives some useful properties of these backbone weights:

Lemma 4.9. *(a) Let ω_1 and ω_2 be two backbones such that $\omega_1 \circ \omega_2$ is consistent. Then one has*

$$\rho(\omega_1 \circ \omega_2) = \rho(\omega_1)\rho_{\tilde{\omega}_1^c}(\omega_2). \tag{4.53}$$

(b) Suppose ω is a backbone that does not traverse any bond within $\mathbf{A}$. Then one has

$$\rho(\omega) \leqslant \rho_{\mathbf{A}^c}(\omega). \tag{4.54}$$

Proof:

A larger list of properties of the backbone weights are proved in Proposition 4.4 of [6].

We first give the proof of statement (a). Since the backbone $\omega_1 \circ \omega_2$ is consistent one has that (i) ω_1 and ω_2 are each consistent by themselves and (ii) no step of ω_2 uses a step taken or canceled by ω_1 - i.e. $\omega_2 \cap \tilde{\omega_1} = \varnothing$.

Now note that $\mathbf{n}$ satifies $\Omega_{\partial\mathbf{n}}(\mathbf{n})$ if and only if $\mathbf{n} = \mathbf{n_1} + \mathbf{n_2}$ with $\mathrm{supp}(\mathbf{n_1}) \subseteq \tilde{\omega_1}$, $\mathrm{supp}(\mathbf{n_2}) = \tilde{\omega_1}^c$ and $\Omega_{\partial\mathbf{n}_i}(\mathbf{n}_i) = \omega_i$ for $i = 1, 2$. With this observation we can now condition on $\tilde{\omega}_1$:

$$\rho(\omega_1 \circ \omega_2) = \frac{1}{Z} \sum_{\mathbf{n}=\partial(\omega_1 \circ \omega_2)} w(\mathbf{n}) \times \mathrm{I}[\mathbf{n} : \Omega_{\partial\mathbf{n}}(\mathbf{n}) = \omega_1 \circ \omega_2]$$

$$= \frac{1}{Z}\left(\sum_{\substack{\partial\mathbf{n_1}=\partial\omega_1 \\ \mathrm{supp}(\mathbf{n_1}) \subseteq \tilde{\omega}_1}} w(\mathbf{n}_1) \times \mathrm{I}[\mathbf{n_1} : \Omega_{\partial\omega_1}(\mathbf{n_1}) = \omega_1] \right)\left(\sum_{\substack{\partial\mathbf{n_2}=\partial\omega_2 \\ \mathrm{supp}(\mathbf{n_2}) \subseteq \tilde{\omega}_1^c}} w(\mathbf{n}_2) \times \mathrm{I}[\mathbf{n_2} : \Omega_{\partial\omega_2}(\mathbf{n_2}) = \omega_2] \right)$$

Now, on account of Lemma 4.8 we have

$$\frac{1}{Z}\sum_{\substack{\partial\mathbf{n_1}=\partial\omega_1\\ \mathrm{supp}(\mathbf{n_1})\subseteq\tilde{\omega}_1}} w(\mathbf{n}_1)\times \mathrm{I}[\mathbf{n_1}:\Omega_{\partial\omega_1}(\mathbf{n_1})=\omega_1]$$

$$=\frac{1}{Z}\times\frac{1}{Z_{\tilde{\omega}_1^c}}\sum_{\substack{\partial\mathbf{n_1}=\partial\omega_1,\ \partial\mathbf{n'_1}=\varnothing\\ \mathrm{supp}(\mathbf{n_1})\subseteq\tilde{\omega}_1,\ \mathrm{supp}(\mathbf{n'_1})\subseteq\tilde{\omega}_1^c}} w(\mathbf{n}_1)\times \mathrm{I}[\mathbf{n_1}+\mathbf{n'_1}:\Omega_{\partial\mathbf{n_1}+n'_1}(\mathbf{n_1}+\mathbf{n'_1})=\omega_1]$$

$$=\frac{1}{Z_{\tilde{\omega}_1^c}}\rho(\omega_1).$$

On the other hand

$$\left(\sum_{\substack{\partial\mathbf{n_2}=\partial\omega_2\\ \mathrm{supp}(\mathbf{n_2})\subseteq\tilde{\omega}_1^c}} w(\mathbf{n}_2)\times \mathrm{I}[\mathbf{n_2}:\Omega_{\partial\omega_2}(\mathbf{n_2})=\omega_2]\right)=Z_{\tilde{\omega}_1^c}\times\rho_{\tilde{\omega}_1^c}(\omega_2)$$

This proves statement (a). We now give the proof of statement (b). Fix a set of bonds $\mathbf{A}$ and suppose that ω is a backbone with $\omega\cap\mathbf{A}=\varnothing$. By Corollary 4.3 one has:

$$\rho(\omega)\leqslant \mathrm{I}[\ \omega \text{ is consistent }]\left(\prod_{b\in\omega}\tanh(\bar{J}_b)\right)\times\frac{1}{Z}\left(\sum_{\partial\mathbf{n}=\varnothing} w(\mathbf{n})\times \mathrm{I}[\mathbf{n}:n_b \text{ is even for all } b\in\tilde{\omega}\cap\mathbf{A}^c]\right)$$

$$=\mathrm{I}[\ \omega \text{ is consistent }]\left(\prod_{b\in\omega}\tanh(\bar{J}_b)\right)\left(\prod_{b\in\tilde{\omega}\cap\mathbf{A}^c}\cosh(\bar{J}_b)\right)\times\frac{1}{Z}\left(\sum_{\substack{\partial\mathbf{n}=\varnothing\\ \mathrm{supp}(\mathbf{n})\subseteq\tilde{\omega}^c\cup\mathbf{A}}} w(\mathbf{n})\right)$$

$$=\mathrm{I}[\ \omega \text{ is consistent }]\left(\prod_{b\in\omega}\tanh(\bar{J}_b)\right)\left(\prod_{b\in\tilde{\omega}\cap\mathbf{A}^c}\cosh(\bar{J}_b)\right)\times\frac{Z_{\tilde{\omega}^c\cup\mathbf{A}}}{Z}$$

The first inequality comes from the fact that $\mathrm{I}[\mathbf{n}:n_b \text{ is even for all } b\in\tilde{\omega}]\leqslant \mathrm{I}[\mathbf{n}:n_b \text{ is even for all } b\in\tilde{\omega}\cap\mathbf{A}^c]$. Now by arguments identical to those used in Lemma 4.8 and Corollary 4.3 one has

$$\rho_{\mathbf{A}^c}(\omega)=\frac{1}{Z_{\mathbf{A}^c}}\mathrm{I}[\ \omega \text{ is consistent }]\left(\prod_{b\in\omega}\tanh(\bar{J}_b)\right)\left(\prod_{b\in\tilde{\omega}\cap\mathbf{A}^c}\cosh(\bar{J}_b)\right)$$

$$\times\left(\sum_{\partial\mathbf{n}=\varnothing} w(\mathbf{n})\times \mathrm{I}[\mathbf{n}:\mathrm{supp}(\mathbf{n})\subseteq\tilde{\omega}^c]\times \mathrm{I}[\mathbf{n}:\mathrm{supp}(\mathbf{n})\subseteq\mathbf{A}^c]\right)$$

$$=\mathrm{I}[\ \omega \text{ is consistent }]\left(\prod_{b\in\omega}\tanh(\bar{J}_b)\right)\left(\prod_{b\in\tilde{\omega}\cap\mathbf{A}^c}\cosh(\bar{J}_b)\right)\times\left(\frac{Z_{\tilde{\omega}^c\cap\mathbf{A}^c}}{Z_{\mathbf{A}^c}}\right)$$

Therefore one has

$$\rho(\omega) \leqslant \frac{Z_{\mathbf{A}^c}}{Z_{\tilde{\omega}^c \cap \mathbf{A}^c}} \times \frac{Z_{\tilde{\omega}^c \cup \mathbf{A}}}{Z} \rho_{\mathbf{A}^c}(\omega)$$
$$\leqslant \frac{Z_{\mathbf{A} \cup (\mathbf{A}^c \cap \tilde{\omega}^c)}}{Z_{(\mathbf{A}^c \cap \tilde{\omega}^c)}} \times \frac{Z_{\mathbf{A}^c}}{Z} \rho_{\mathbf{A}^c}(\omega)$$

Statement (b) will follows if the prefactor in the bottom line above is less than 1, this fact follows from applying Lemma 4.10 below with the choice $\mathbf{B} = \mathbf{A}^c \cap \tilde{\omega}^c$.

□

Lemma 4.10. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be disjoint sets of bonds. Then*

$$\frac{Z_{\mathbf{A} \cup \mathbf{B}}}{Z_{\mathbf{B}}} \leqslant \frac{Z}{Z_{\mathbf{A}^c}}$$

Proof:

This fact follows from the observation that

$$\frac{Z_{\mathbf{A} \cup \mathbf{B}}}{Z_{\mathbf{B}}} = \left\langle \exp\left[\sum_{\{x,y\} \in \mathbf{A}} \bar{J}_{\{x,y\}} \sigma_x \sigma_y\right] \right\rangle_{\mathbf{B}} \leqslant \left\langle \exp\left[\sum_{\{x,y\} \in \mathbf{A}} \bar{J}_{\{x,y\}} \sigma_x \sigma_y\right] \right\rangle_{\mathbf{A}^c} = \frac{Z}{Z_{\mathbf{A}^c}}$$

For the middle inequality note that by Griffiths II one has the following for any $D \subset \Lambda$:

$$\langle \sigma^D \rangle_{\mathbf{B}} \leqslant \langle \sigma^D \rangle_{\mathbf{A}}.$$

The middle inequality then follows by expanding both exponentials. □

The notation $\sum_{\omega : x \to y}$ will represent a sum over all backbones consisting of a single walk starting at x and terminating at y. For example, if x precedes y in the ordering we have imposed on the lattice then one has:

$$\langle \sigma_x \sigma_y \rangle = \sum_{\omega : x \to y} \rho(\omega) + \sum_{\substack{\omega_1 : x \to g \\ \omega_2 : y \to g}} \rho(\omega_1 \circ \omega_2)$$

We note that if one is taking a modified expectation where all h-bonds have been suppressed then the second sum above can be dropped.

4.6.3 Derivation of the Partial Differential Inequality

We now specialize to our model of interest. Let $g > 0$ and $b \in \mathbb{R}$ be fixed for the remainder of this section and the section 4.6.4. For $n \in \mathbb{N}$ let $M_n(\beta, h)$ be defined as in (4.35). We will establish (4.11) for M_n by approximating the (Λ_n marginal of the) ϕ^4 measure $d\nu_{\Lambda_n}[g, b, \beta, h]$ with the Griffiths-Simons approximation - that is via measures $d\mu^N_{\Lambda_n}$ as given in Theorem 4.4.

The measures $d\mu^N_{\Lambda_n}$ correspond to classical Ising ferromagnets on a lattice $\Lambda_n \times [N]$. We will apply the Random Current Representation to this system keeping both n and N fixed and prove analagous correlation inequalities for block spins $\theta_N(\phi.)$ (defined as in 4.4 with out fixed values of g, b, and N). These inequalities will be uniform in n and N. In what follows below we set $\Lambda = \Lambda_n \times [N]$ for fixed n and N, and $\bar{\Lambda} = \Lambda \cup \{g\}$.

The bond parameters for bonds in $\bar{\Lambda}^{(2)}$ are as follows:

$$\bar{J}_{\{(x,\alpha),(y,\delta)\}} = \begin{cases} \beta c_N^2 J_{\{x,y\}} & \text{if } x \neq y \\ d_N & \text{if } x = y \end{cases}$$

$$\bar{J}_{\{(x,\alpha),g\}} = c_N h$$

For $x \in \Lambda_n$ we denote the corresponding block within Λ as B_x, that is $B_x = \{(x,\alpha)\}_{\alpha=1}^N$. With these definitions in hand we now mention a corollary of Lemma 4.4:

Corollary 4.4. *If f is a function on current configurations which is decreasing in each flux number, then*

$$\sum_{\substack{\partial \mathbf{n_1} = \varnothing \\ \partial \mathbf{n_2} = \varnothing}} w(\mathbf{n_1}) w(\mathbf{n_2}) \times f(\mathbf{n_1} + \mathbf{n_2}) \times \mathrm{I}\left[\mathbf{n_1} + \mathbf{n_2} : \mathrm{B_v} \leftrightarrow \mathrm{g}\right]$$
$$\leqslant \sum_{\alpha=1}^{N} \Bigg[c_N h \sum_{\substack{\partial \mathbf{n_1} = \{(v,\alpha),g\} \\ \partial \mathbf{n_2} = \varnothing}} w(\mathbf{n_1}) w(\mathbf{n_2}) \times f(\mathbf{n_1} + \mathbf{n_2})$$
$$+ \; c_N^2 \times \beta \sum_{y \in \Lambda_n \setminus \{v\}} \sum_{\delta=1}^{N} J_{\{v,y\}} \left(\sum_{\substack{\partial \mathbf{n_1} = \{(v,\alpha),g\} \\ \partial \mathbf{n_2} = \{(y,\delta),g\}}} w(\mathbf{n_1}) w(\mathbf{n_2}) \times f(\mathbf{n_1} + \mathbf{n_2}) \right) \Bigg]$$

Proof: The result follows from applying Lemma 4.7 with $B = B_v$ and $z = g$. The sum has been split into two pieces corresponding to the site outside of B_v being the ghost site or a normal lattice site. $\square$.

For many quantities given below we will suppress the dependence of various quantities on n and N. In particular the expectations $\langle \bullet \rangle$ represents expectations over the approximating system $\{\sigma_{(x,\alpha)}\}_{(x,\alpha) \in \Lambda_n \times [N]}$ with measure $d\mu^N_{\Lambda_n}$. We will also use the function θ_N defined in Theorem 4.4 using the desired values of g, b, and N.

We will establish a partial differential inequality for the quantity

$$M_{n,N} = \langle \theta_N(\phi_0) \rangle$$

First note that one has:

$$
\begin{aligned}
M_{n,N} =& c_N \sum_{\alpha=1}^{N} \left\langle \sigma_{(0,\alpha)} \right\rangle \\
=& c_N \sum_{\alpha=1}^{N} Z^{-1} \left[\sum_{\partial \mathbf{n_1} = \{(0,\alpha),g\}} w(\mathbf{n_1}) \right] \\
=& c_N \sum_{\alpha=1}^{N} Z^{-3} \left(\sum_{\substack{\partial \mathbf{n_1} = \{(0,\alpha),g\} \\ \partial \mathbf{n_2} = \varnothing \\ \partial \mathbf{n_3} = \varnothing}} w(\mathbf{n_1}) w(\mathbf{n_2}) w(\mathbf{n_3}) \right)
\end{aligned}
\tag{4.55}
$$

In the last line two duplicate current configurations were inserted. For each $(0,\alpha)$ the sum over current configurations $\mathbf{n_1}$, $\mathbf{n_2}$, $\mathbf{n_3}$ will be split into three different pieces. This decomposition will depend on the RWR mentioned earlier. Note that the Random Walk Representation depends on an initial choice of an order on the sites of $\Lambda_n \times [N]$. For each $\alpha \in [1, \ldots, N]$ we will use a different ordering on lattice sites before applying the RWR, in particular we will enforce that $(0,\alpha)$ be the earliest site. This will not cause any problems since the RWR is always manipulated for just one value of α of the time. For now we will write $\Omega^{\alpha}_{\bullet}(\bullet)$ to denote the corresponding backbone maps and $\rho^{\alpha}(\bullet)$ for the corresponding backbone weights.

The decomposition for the sum in the last line of (4.55) is determined by three different cases for the answer to the following question:

What is the first block B_v visited by $\Omega^{\alpha}_{\{(0,\alpha),g\}}(\mathbf{n_1})$ that is also connected to g via the current configuration $\mathbf{n_2} + \mathbf{n_3}$ (i.e. $B_v \leftrightarrow g$ under $\mathbf{n_2} + \mathbf{n_3}$)?

1. There is no block B_v such that $\Omega^{\alpha}_{\{(0,\alpha),g\}}(\mathbf{n_1})$ visits B_v and $B_v \leftrightarrow g$ under $\mathbf{n_2} + \mathbf{n_3}$.
2. The first block visited by $\Omega^{\alpha}_{\{(0,\alpha),g\}}(\mathbf{n_1})$ that is connected to g via the current configuration $\mathbf{n_2} + \mathbf{n_3}$ is B_0.
3. The first block visited by $\Omega^{\alpha}_{\{(0,\alpha),g\}}(\mathbf{n_1})$ that is connected to g via the current configuration $\mathbf{n_2} + \mathbf{n_3}$ is some B_v with $B_v \neq B_0$.

Corresponding to these different cases we have the following decomposition:

$$
M_{n,N} = T + R_0 + \sum_{\alpha=1}^{N} \sum_{\substack{v \in \Lambda_n \\ v \neq 0}} \sum_{\substack{u \in \Lambda_n \\ u \neq v}} \sum_{\delta,\gamma=1}^{N} R_{\{\alpha,(u,\delta),(v,\gamma)\}} \tag{4.56}
$$

The contribution from the first case is given by:

$$
T = c_N \sum_{\alpha=1}^{N} Z^{-3} \left(\sum_{\substack{\partial \mathbf{n_1} = \{(0,\alpha),g\} \\ \partial \mathbf{n_2} = \partial \mathbf{n_3} = \varnothing}} w(\mathbf{n_1}) w(\mathbf{n_2}) w(\mathbf{n_3}) \times \mathrm{I} \left[\begin{array}{c} \Omega^{\alpha}_{\{(0,\alpha),g\}}(\mathbf{n_1}) \text{ doesn't visit any } B_v \\ \text{with } B_v \leftrightarrow g \text{ via } \mathbf{n_2} + \mathbf{n_3} \end{array} \right] \right)
$$

The contribution from the second case is:

$$R_0 = c_N \sum_{\alpha=1}^{N} Z^{-3} \left(\sum_{\substack{\partial \mathbf{n_1} = \{(0,\alpha),g\} \\ \partial \mathbf{n_2} = \varnothing \\ \partial \mathbf{n_3} = \varnothing}} w(\mathbf{n_1}) w(\mathbf{n_2}) w(\mathbf{n_3}) \times I[\mathbf{n_2} + \mathbf{n_3} : B_0 \leftrightarrow g] \right)$$

The third case is broken down further by summing over the first step $((u,\delta),(v,\gamma))$ of $\Omega^{\alpha}_{\{(0,\alpha),g\}}(\mathbf{n_1})$ which has endpoint in B_v. For any $\alpha \in [N]$, and $\{(u,\delta),(v,\gamma)\} \in \bar{\Lambda}$ with $v \neq 0$ and $u \neq v$ we define

$$\begin{aligned} R_{\alpha,(u,\delta),(v,\gamma)} =& c_N Z^{-3} \Bigg(\sum_{\substack{\partial \mathbf{n_1} = \{(0,\alpha),g\} \\ \partial \mathbf{n_2},\ \partial \mathbf{n_3} = \varnothing}} w(\mathbf{n_1}) w(\mathbf{n_2}) w(\mathbf{n_3}) \\ & \times \mathrm{I} \left[\begin{array}{c} B_v \text{ is the first block connected to } g \text{ via } \mathbf{n_2} + \mathbf{n_3} \\ \text{that is visited by } \Omega^{\alpha}_{\{(0,\alpha),g\}}(\mathbf{n_1}) \end{array} \right] \\ & \times \mathrm{I} \left[\mathbf{n_1} : \begin{array}{c} ((u,\delta),(v,\gamma)) \text{ is the first bond in} \\ \Omega^{\alpha}_{\{(0,\alpha),g\}}(\mathbf{n_1}) \text{ with an endpoint in } B_v \end{array} \right] \\ & \times \mathrm{I}\left[\mathbf{n_2} + \mathbf{n_3} : \mathrm{B}_0 \not\leftrightarrow \mathrm{g}\right] \Bigg) \end{aligned}$$

We now bound each of these contributions.

Lemma 4.11. *For any $n, N \in \mathbb{N}$ one has the bound:*

$$T \leqslant h \sum_{x \in \Lambda_n} \langle \theta_N(\phi_0), \theta_N(\phi_x) \rangle^T$$

Proof:

One starts by classifying the current configurations $\mathbf{n_1}$ summed over in T by the last site $(x,\delta) \in \Lambda$ that $\Omega^{\alpha}_{\{(0,\alpha),g\}}$ visited before reaching the ghost site g.

One can then write $T = \sum_{(x,\delta) \in \Lambda} T_{(x,\delta)}$ where

$$\begin{aligned} T_{(x,\delta)} = c_N \sum_{\alpha=1}^{N} Z^{-3} \Bigg(& \sum_{\substack{\partial \mathbf{n_1} = \{(0,\alpha),g\} \\ \partial \mathbf{n_2} = \partial \mathbf{n_3} = \varnothing}} w(\mathbf{n_1}) w(\mathbf{n_2}) w(\mathbf{n_3}) \times \mathrm{I} \left[\begin{array}{c} \Omega^{\alpha}_{\{(0,\alpha),g\}}(\mathbf{n_1}) \text{ doesn't visit any } B_v \\ \text{with } B_v \leftrightarrow g \text{ via } \mathbf{n_2} + \mathbf{n_3} \end{array} \right] \\ & \times \mathrm{I} \left[\mathbf{n_1} : \begin{array}{c} (x,\delta) \text{ is the last site visited by } \Omega_{\{(0,\alpha),g\}}(\mathbf{n_1}) \\ \text{before reaching } g \end{array} \right] \Bigg) \end{aligned}$$

Note that that for any current configuration $\mathbf{n}_1$ satisfying the above indicator functions and source constraints one has that $n_{1,\{(x,\delta),g\}}$ is odd. In particular this is the only constraint on $n_{1,\{(x,\delta),g\}}$. That is to say that if one modified $\mathbf{n}_1$ by changing $n_{1,\{(x,\delta),g\}}$ to another odd number then the resulting current configuration would again satisfy the same indicator functions and source constraints so this flux number can be summed over all odd non-negative integers independently of the other flux numbers.

We now do a change of variable in the sum to flip this constraint, that is we change $\mathbf{n_1} \to \mathbf{n_1}'$ where $n'_{1,b} = n_{1,b}$ for $b \neq \{(x,\delta), g)\}$ and $n'_{1,b} = n_{1,b} - 1$ for $b = \{(x,\delta), g)\}$. Now we have the constraint that $n'_{1,b}$ be even, however just as before this is the only constraint and the sum over this flux number will be independent of the others.

This change of variable will have three consequences for our expression for $T_{(x,\delta)}$. First, the sources constraint for the current configurations $\mathbf{n'_1}$ will now be changed to $\{(0,\alpha)\}\Delta\{(x,\delta)\}$. Second, the backbone $\Omega^{\alpha}_{\{(0,\alpha)\}\Delta\{(x,\delta)\}}(\mathbf{n'_1}$ will be determined by truncated the step $((x,\delta), g)$ from the backbone of the corresponding $\mathbf{n_1}$ - $\Omega^{\alpha}_{\{(0,\alpha),g\}}(\mathbf{n_1}$ (here we use the fact that $(0,\alpha)$ is the earliest site of Λ - this guarantees that if the backbone is non-empty $(0,\alpha)$ remains the starting point of the backbone). Thus we can keep the desired non-intersection constraint for the new backbone. Third we will get an overall factor of

$$\tanh\left(\bar{J}_b\right) = \frac{\displaystyle\sum_{n_{1,b}\ \text{odd}} \frac{(\bar{J}_b)^{n_{1,b}}}{n_{1,b}!}}{\displaystyle\sum_{n'_{1,b}\ \text{even}} \frac{(\bar{J}_b)^{n'_{1,b}}}{n'_{1,b}!}}$$

with $b = \{(x,\delta), g\}$ accompanying the weights $w(\mathbf{n'_1})$. With these observations, and dropping the prime from $\mathbf{n'_1}$ we get the following expression for $T_{(x,\delta)}$:

$$\begin{aligned} T_{(x,\delta)} = \tanh\left(\bar{J}_{\{(x,\delta),g\}}\right) \times c_N \sum_{\alpha=1}^{N} Z^{-3} \Bigg(& \sum_{\substack{\partial\mathbf{n_1} = \{(0,\alpha)\}\Delta\{(x,\delta)\} \\ \partial\mathbf{n_2} = \partial\mathbf{n_3} = \varnothing}} w(\mathbf{n_1}) w(\mathbf{n_2}) w(\mathbf{n_3}) \\ & \times \mathrm{I}[\mathbf{n_1} : \mathrm{n}_{1,\{(\mathrm{x},\delta),\mathrm{g}\}} \text{ is even }] \\ & \times \mathrm{I}\left[\begin{array}{c} \Omega^{\alpha}_{\{(0,\alpha)\}\Delta\{(x,\delta)\}}(\mathbf{n_1}) \text{ doesn't visit any } B_v \\ \text{with } B_v \leftrightarrow g \text{ via } \mathbf{n_2} + \mathbf{n_3} \end{array} \right] \\ & \times \mathrm{I}\left[\mathbf{n_2} + \mathbf{n_3} : \mathrm{B}_0 \nleftrightarrow \mathrm{g}\right] \Bigg) \end{aligned}$$

The last indicator function comes from dealing with the case that $(x,\delta) = (0,\alpha)$ - in this case the backbone $\Omega^{\alpha}_{\{(0,\alpha)\}\Delta\{(x,\delta)\}}(\mathbf{n_1})$ is empty so the first indicator function is vacuous but we still inherit a connectivity restriction from the expression with the earlier flux constraint.

We now loosen the constraints of the two indicator functions to get:

$$T_{(x,\delta)} \leqslant \tanh\left(\bar{J}_{\{(x,\delta),g\}}\right) \times c_N \sum_{\alpha=1}^{N} Z^{-3} \Bigg(\sum_{\substack{\partial \mathbf{n_1} = \{(0,\alpha)\}\Delta\{(x,\delta)\} \\ \partial \mathbf{n_2} = \partial \mathbf{n_3} = \varnothing}} w(\mathbf{n_1}) w(\mathbf{n_2}) w(\mathbf{n_3})$$

$$\times \mathrm{I}\left[\begin{array}{c} \Omega^{\alpha}_{\{(0,\alpha)\}\Delta\{(x,\delta)\}}(\mathbf{n_1}) \text{ doesn't visit any site } (v,\gamma) \\ \text{with } (v,\gamma) \leftrightarrow g \text{ via } \mathbf{n_1} + \mathbf{n_2} \end{array} \right]$$

$$\times \mathrm{I}\left[\mathbf{n_2} + \mathbf{n_3} : (0,\alpha) \nleftrightarrow \mathrm{g}\right] \Bigg)$$

$$= \tanh\left(\bar{J}_{\{(x,\delta),g\}}\right) \times c_N \sum_{\alpha=1}^{N} Z^{-3} \Bigg(\sum_{\substack{\partial \mathbf{n_1} = \{(0,\alpha),g\} \\ \partial \mathbf{n_2} = \partial \mathbf{n_3} = \varnothing}} w(\mathbf{n_1}) w(\mathbf{n_2}) w(\mathbf{n_3})$$

$$\times \mathrm{I}\left[\Omega^{\alpha}_{\{(0,\alpha)\}\Delta\{(x,\delta)\}}(\mathbf{n_1}) \cap \mathbf{C}_{\mathbf{n_2}+\mathbf{n_3}}(g) = \varnothing\right] \times \mathrm{I}\left[\mathbf{n_2} + \mathbf{n_3} : (0,\alpha) \nleftrightarrow \mathrm{g}\right] \Bigg)$$

$$= \tanh\left(\bar{J}_{\{(x,\delta),g\}}\right) \times c_N \sum_{\alpha=1}^{N} Z^{-2} \Bigg(\sum_{\partial \mathbf{n_2} = \partial \mathbf{n_3} = \varnothing} w(\mathbf{n_2}) w(\mathbf{n_3}) \times \mathrm{I}\left[\mathbf{n_2} + \mathbf{n_3} : (0,\alpha) \nleftrightarrow \mathrm{g}\right]$$

$$\times \left[\sum_{\omega:(0,\alpha)\to(x,\delta)} \rho^{\alpha}(\omega) \mathrm{I}\left[\omega \cap \mathbf{C}_{\mathbf{n_2}+\mathbf{n_3}}(g) = \varnothing\right] \right] \Bigg).$$

In going to second inequality it is important to remember that the bond cluster $\mathbf{C}_{\mathbf{n_1}+\mathbf{n_2}}(g)$ contains dangling bonds.

For any fixed $\mathbf{n_2} + \mathbf{n_3}$ one can use the inequality (4.54) to get that

$$\begin{aligned} \sum_{\omega:(0,\alpha)\to(x,\delta)} \rho^{\alpha}(\omega) \mathrm{I}\left[\omega \cap \mathbf{C}_{\mathbf{n_2}+\mathbf{n_3}}(g) = \varnothing\right] &\leqslant \sum_{\omega:(0,\alpha)\to(x,\delta)} \rho^{\alpha}_{\left(\mathbf{C}_{\mathbf{n_2}+\mathbf{n_3}}(g)\right)^c}(\omega) \mathrm{I}\left[\omega \cap \mathbf{C}_{\mathbf{n_2}+\mathbf{n_3}}(g) = \varnothing\right] \\ &\leqslant \sum_{\omega:(0,\alpha)\to(x,\delta)} \rho^{\alpha}_{\left(\mathbf{C}_{\mathbf{n_2}+\mathbf{n_3}}(g)\right)^c}(\omega) \\ &= \langle \sigma_{(0,\alpha)} \sigma_{(x,\delta)} \rangle_{\left(\mathbf{C}_{\mathbf{n_2}+\mathbf{n_3}}(g)\right)^c}. \end{aligned}$$

Note that in the last equality we didn't have to sum over walks traveling to the ghost since we have suppressed all bonds to the ghost site g. Inserting this into the earlier bound for $T_{(x,\delta)}$ one gets

$$\begin{aligned}
T_{(x,\delta)} \leqslant & \tanh\left(\bar{J}_{\{(x,\delta),g\}}\right) \\
& \times c_N \sum_{\alpha=1}^{N} Z^{-2} \Bigg(\sum_{\partial \mathbf{n_2}=\partial \mathbf{n_3}=\varnothing} w(\mathbf{n_2}) w(\mathbf{n_3}) \\
& \times \mathrm{I}\left[\mathbf{n_2}+\mathbf{n_3} : (0,\alpha) \leftrightarrow \mathrm{g}\right] \times \left\langle \sigma_{(0,\alpha)} \sigma_{(\mathrm{x},\delta)} \right\rangle_{\left(\mathbf{C}_{\mathbf{n_2}+\mathbf{n_3}}(\mathrm{g})\right)^{c}} \Bigg) \\
\leqslant & \, c_N h \times c_N \sum_{\alpha=1}^{N} \langle \sigma_{(0,\alpha)}, \sigma_{(x,\delta)} \rangle^{T}.
\end{aligned}$$

Above we used that $\tanh(\bar{J}_{\{(x,\delta),g\}}) = \tanh(c_N h) \leqslant c_N h$. Inserting this into the expression for T gives

$$\begin{aligned}
T &\leqslant \sum_{(x,\delta)\in\Lambda} T_{(x,\delta)} \\
&\leqslant h \times c_N^2 \sum_{\alpha=1}^{N} \sum_{(x,\delta)\in\Lambda} \langle \sigma_{(0,\alpha)}, \sigma_{(x,\delta)} \rangle^{T} \\
&= h \sum_{x\in\Lambda_n} \langle \theta_N(\phi_0), \theta_N(\phi_x) \rangle^{T}.
\end{aligned}$$

$\square$.

Lemma 4.12.

$$R_0 \leqslant h M_{n,N}^2 + \beta ||J||_{L^1} M_{n,N}^3.$$

Proof: First observe that the summation over α and $\mathbf{n_1}$ is independent of the summation over $\mathbf{n_2}$ and $\mathbf{n_3}$. Thus one has

$$R_0 = M_{n,N} Z^{-2} \left(\sum_{\substack{\partial \mathbf{n_2}=\varnothing \\ \partial \mathbf{n_3}=\varnothing}} w(\mathbf{n_2}) w(\mathbf{n_3}) \times I[\mathbf{n_2}+\mathbf{n_3} : B_0 \leftrightarrow g] \right).$$

Note that by applying Lemma 4.4 with respect to the sum over $\mathbf{n_2}$ and $\mathbf{n_3}$ one gets

$$\begin{aligned}
R_0 \leqslant & M_{n,N} Z^{-2} \sum_{\alpha=1}^{N} \left(c_N \times h \sum_{\substack{\partial \mathbf{n_2}=\{(0,\alpha),g\} \\ \mathbf{n_3}=\varnothing}} w(\mathbf{n_2}) w(\mathbf{n_3}) + \sum_{y\in\Lambda_n, y\neq 0} c_N^2 \beta J_{\{0,y\}} \sum_{\delta=1}^{N} \sum_{\substack{\partial \mathbf{n_2}=\{(0,\alpha),g\} \\ \mathbf{n_3}=\{(y,\delta),g\}}} w(\mathbf{n_2}) w(\mathbf{n_3}) \right) \\
= & M_{n,N} \left[h \times M_{n,N} + \beta \times M_{n,N} \left(\sum_{y\in\Lambda_n, y\neq 0} J_{\{0,y\}} \langle \theta_N(\phi_y) \rangle \right) \right]
\end{aligned}$$

The result now follows by using translation invariance: the original interaction $J_{\cdot}$ on Λ_n was translation invariant which means the approximating classical ising measure on $\Lambda = \Lambda_n \times [N]$ is invariant under translation induced transformations on the set of blocks. In particular for all $y \in \Lambda_n$ one has $\langle \theta_N(\phi_y) \rangle = \langle \theta_N(\phi_0) \rangle = M_{n,N}$. □.

Lemma 4.13.

$$\begin{aligned}
&\sum_{\alpha=1}^{N} \sum_{\substack{v\in\Lambda_n \\ v\neq 0}} \sum_{\substack{u\in\Lambda_n \\ u\neq v}} \sum_{\delta,\gamma=1}^{N} R_{\{\alpha,(u,\delta),(v,\gamma)\}} \\
&\qquad \leqslant \beta M_{n,N} \left(h + M_{n,N} \|J\|_{L^1} \beta\right) \sum_{\{u,v\}\in\Lambda_n^{(2)}} J_{\{u,v\}} \langle \theta_N(\phi_0), \theta_N(\phi_u)\theta_N(\phi_v)\rangle^T
\end{aligned} \tag{4.57}$$

Proof: This proof is quite involved and combines methods used for bounding the previous two terms. We refer the reader to [7]. □.

Putting the three bounds together gives us the following theorem:

Proposition 4.2. *For any $n, N \in \mathbb{N}$ one has*

$$\begin{aligned}
M_{n,N} \leqslant & h \sum_{x\in\Lambda_n} \langle \theta_N(\phi_0), \theta_N(\phi_x)\rangle^T + h M_{n,N}^2 + \beta ||J||_{L^1} M_{n,N}^3 \\
& + \beta M_{n,N} \left(h + \beta M_{n,N} \|J\|_{L^1}\right) \sum_{\{u,v\}\in\Lambda_n^{(2)}} J_{\{u,v\}} \langle \theta_N(\phi_0), \theta_N(\phi_u)\theta_N(\phi_v)\rangle^T
\end{aligned}$$

Proof : The statement follows immediately from (4.56), Lemma 4.11, Lemma 4.12, and Lemma 4.13. □.

We close this subsection with finishing the proof of Theorem 4.11.

Proof of Theorem 4.11 We note that by Theorem 4.4 one has the following in the $N \to \infty$ limit:

$$\begin{aligned}
M_{n,N} &\longrightarrow M_n \\
\sum_{x\in\Lambda_n} \langle \theta_N(\phi_0), \theta_N(\phi_x)\rangle^T &\longrightarrow \sum_{x\in\Lambda_n} \langle \phi_0, \phi_x \rangle^T_{\mu[\Lambda_n,g,b,\beta,h]} = \frac{\partial M_n}{\partial h} \\
\sum_{\{u,v\}\in\Lambda_n^{(2)}} J_{\{u,v\}} \langle \theta_N(\phi_0), \theta_N(\phi_u)\theta_N(\phi_v)\rangle^T &\longrightarrow \sum_{\{u,v\}\in\Lambda_n^{(2)}} J_{\{u,v\}} \langle \phi_0, \phi_u\phi_v \rangle^T_{\mu[\Lambda_n,g,b,\beta,h]} = \frac{\partial M_n}{\partial \beta}
\end{aligned}$$

Combining the above with Proposition 4.2 immediately gives 4.11. □

4.6.4 Consequences of the Partial Differential Inequality

In this subsection we explain how one goes from Theorem 4.11 to Theorem 4.10. The first lemma alows one to trade factors of h for factors of M_n

Lemma 4.14. *Define*

$$S = \sqrt{\frac{\int_{\mathbb{R}} dt\ t^2 e^{-gt^4+bt^2}}{\int_{\mathbb{R}} dt\ e^{-gt^4+bt^2}}}$$

$$\beta_{MF} = (\|J\|_{L^1} S^2)$$

Let $\epsilon > 0$. Then for any $h \in \left[0, \frac{\epsilon}{S}\right]$, any $n \in \mathbb{N}$ one has

$$h \leqslant \left[\frac{\epsilon \times \beta_{MF}}{\tanh(\epsilon) \times \|J\|_{L^1}}\right] M_n$$

Proof: This is Lemma 5.2 in [7]. □

Proposition 4.3. *For all $A_\beta > 0$, $A_h > 0$ there exist constants B_1 and B_2 such that for all $\beta \in [0, A_\beta]$ and for all $h \in [0, A_h]$ one has*

$$M_n \leqslant h\frac{\partial M_n}{\partial h} + B_1 M_n^3 + B_2 M_n^2 \frac{\partial M_n}{\partial \beta}$$

Proof: This is a consequence of Theorem 4.11 and an application of Lemma 4.14 with $\epsilon = A_h S$. One can then choose

$$B_1 = \left[\frac{A_h S \times \beta_{MF}}{\tanh(A_h S) \times \|J\|_{L^1}}\right] + A_\beta \|J\|_{L^1},$$
$$B_2 = A_\beta \times \left[\frac{A_h S \times \beta_{MF}}{\tanh(A_h S) \times \|J\|_{L^1}}\right] + A_\beta^2 \|J\|_{L^1}.$$

This proves the proposition. □

Lemma 4.15. *For any $n \in \mathbb{N}$ one has*

$$\frac{\partial M_n}{\partial \beta} \leqslant M_n \|J\|_{L^1} \frac{\partial M_n}{\partial h}$$

Proof:

By the GHS inequality we have that $\langle \phi_0, \phi_u \phi_v \rangle^T_{\Lambda_n} \leqslant 0$ for any $u, v \in \Lambda_n$. This is equivalent to

$$\begin{aligned}\langle \phi_0 \phi_u \phi_v \rangle_{\Lambda_n} - \langle \phi_0 \rangle_{\Lambda_n} \langle \phi_u \phi_v \rangle_{\Lambda_n} \leqslant\ & \langle \phi_u \rangle_{\Lambda_n} \left[\langle \phi_0 \phi_v \rangle_{\Lambda_n} - \langle \phi_0 \rangle_{\Lambda_n} \langle \phi_v \rangle_{\Lambda_n}\right] \\ & + \langle \phi_v \rangle_{\Lambda_n} \left[\langle \phi_0 \phi_u \rangle_{\Lambda_n} - \langle \phi_0 \rangle_{\Lambda_n} \langle \phi_u \rangle_{\Lambda_n}\right]\end{aligned}$$

We then have

$$
\begin{aligned}
\frac{\partial M_n}{\partial \beta} =& \frac{1}{2} \sum_{\substack{u,v\in\Lambda_n \\ u\neq v}} J_{\{u,v\}} \langle \phi_0, \phi_u\phi_v\rangle^T_{\Lambda_n} \\
\leqslant& \frac{1}{2} M_n \sum_{\substack{u,v\in\Lambda_n \\ u\neq v}} J_{\{u,v\}} \Bigg([\langle\phi_0\phi_v\rangle_{\Lambda_n} - \langle\phi_0\rangle_{\Lambda_n}\langle\phi_v\rangle_{\Lambda_n}] + [\langle\phi_0\phi_u\rangle_{\Lambda_n} - \langle\phi_0\rangle_{\Lambda_n}\langle\phi_u\rangle_{\Lambda_n}] \Bigg) \\
\leqslant& M_n \|J\|_{L^1} \frac{\partial M_n}{\partial h}.
\end{aligned}
$$

Lemma 4.16 ([6])**.** *Let* $\{\bar{M}_n(\beta,h)\}_{n\in\mathbb{N}}$ *be a sequence of non-negative functions defined for* $(\beta,h)\in[0,\infty)\times(0,\infty)$*, increasing and differentiable in both* β *and* h*.*

Suppose that:

1. $\bar{M}_n(\beta,h)$ *converge pointwise as* $n\to\infty$ *for* $(\beta,h)\in[0,\infty)\times(0,\infty)$ *to a function* $\bar{M}(\beta,h)$*.*

2. $\bar{M}(\beta,h)$ *can be continuously extended to* $[0,\infty)\times[0,\infty)$*. We use* $\bar{M}$ *to denote this extension.*

3. $\bar{M}(\beta,h)$ *is differentiable in* h *for* $(\beta,h)\in[0,\infty)\times(0,\infty)$ *and on this set one has* $\dfrac{\partial\bar{M}_n}{\partial h}\to\dfrac{\partial\bar{M}}{\partial h}$*.*

4. *There exists* $\theta\in(0,\infty)$ *such that for any* $A_\beta>0$ *one can find* $a_1,a_2\geqslant 0$ *and a non-negative continuous function of a single variable* f *satisfying the following conditions*

   (a)
$$\lim_{x\to 0^+} f(x) = 0$$

   (b)
$$\int_0^1 dx\ \frac{f(x)}{x} < \infty$$

   such that for any $(\beta,h)\in[0,A_\beta)\times(0,1)$ *the functions* $\{\bar{M}_n(\beta,h)\}$ *all satisfy the following partial differential inequalities:*

$$\bar{M}_n \leqslant\ h\frac{\partial\bar{M}_n}{\partial h} + \bar{M}_n f(\bar{M}_n) + a_1\bar{M}_n^{\theta}\partial\bar{M}_n\partial\beta \tag{4.58}$$

$$\frac{\partial\bar{M}_n}{\partial\beta} \leqslant a_2\bar{M}_n\frac{\partial\bar{M}_n}{\partial h}. \tag{4.59}$$

Under all the above conditions one has the following:

If there exists some $\beta_0\geqslant 0$ *such that*

$$\lim_{h\to 0^+} \frac{\bar{M}(\beta_0, h)}{h} = \infty$$

then

$$\liminf_{h\to 0^+} \frac{\bar{M}(\beta_0, h)}{h^{\frac{1}{1+\theta}}} > 0$$

and for any $\beta > \beta_0$

$$\bar{M}(\beta, 0) := \lim_{h\to 0^+} \bar{M}(\beta, h) > 0$$

Proof: See Lemmas 5.1 and 4.1 of [6]. □.

The essential results needed for proving Theorem 4.10 have all been stated so we now finish the proof of that theorem to finish the section.

Proof of Theorem 4.10: We now apply Lemma 4.16 by setting $\bar{M}_n(\beta, h) = M_n(\beta, h)$ and setting $\bar{M}(\beta, h) = M(\beta, h)$ for $(\beta, h) \in [0, \infty) \times (0, \infty)$. What remains is checking that these choices satisfy the conditions of Lemma 4.16. Conditions (1) and (2) are immediate. From Theorem 4.9 we have the necessary differentiability of $M(\beta, h)$. Convergence of the associated derivatives is a consequence of general properties of concave functions.

We note that by Proposition 4.3 and Lemma 4.15 we have that condition (4) holds with $\theta = 2$ and $f(x) = B_2 x^2$. We then have that the consequences of Lemma 4.16 hold for $M(\beta, h)$.

The proof of 4.10 is complete if we show condition (4.34) implies that $\lim_{h\to 0^+} \frac{M(\beta_0, h)}{h} = \infty$. This follows from Lemma 4.3 in the appendix. □.

4.7 Superstable Gibbs Measures

4.7.1 Overview of Section

In this section we will (i) establish full control over the infinite volume limits of the measures described in Theorem 4.8 and to (ii) show that the presence of spontaneous magnetization implies the presence of long range order.

The key for establishing infinite volume limits is Lemma 4.64 - this lemma establishes strong estimates on the finite volume marginals of the measures $\nu[\Lambda_j, \beta, g, b, h]$ that are uniform in j. This gives us compactness (tightness) that allows us to prove the existence of subsequential limits. However in the case where we don't have 0 boundary conditions we can apply Griffiths Second inequality and use monotonicity of moments with respect to the volume to drop the need to take subsequences.

With regards to item (ii) the intuition between the equivalence of long range order and spontaneous magnetization is that they both signal the existence of multiple "phases" for our Ising models - more precisely the existence of multiple (translation invariant) Gibbs measures. A clear reference for the discussion that follows is [32].

Suppose that we fix some choice g, b, β for which $M^+(g, b, \beta) > 0$, that is is $\beta > \beta_M(g, b)$. Then there are at least two translation invariant Gibbs measures, one given by

$$\langle \bullet \rangle_+ \ := \lim_{h \to 0^+} \langle \bullet \rangle_{\nu[\mathbb{L}, g, b, \beta, h]}$$

and the other by

$$\langle \bullet \rangle_0 \ := \langle \bullet \rangle_{\nu[\mathbb{L}, g, b, \beta, 0]}$$

The first measure has a positive first moment and the second, by symmetry, having a zero first moment.

Moreover if one defines $\langle \bullet \rangle_-$ via the pushforward under a global spin flip of the measure $\langle \bullet \rangle_-$ it is expected that

$$\langle \bullet \rangle_0 \ = \ \frac{1}{2} \left[\langle \bullet \rangle_- \ + \langle \bullet \rangle_+ \ \right]. \tag{4.60}$$

Since *even* moments are unchanged when flipping spins all the measures appearing in (4.60) have the same even moments, in particular for all $x \in \mathbb{L}$

$$\langle \phi_0 \phi_x \rangle_0 \ = \ \langle \phi_0 \phi_x \rangle_+. \tag{4.61}$$

On the other hand applying Griffiths' second inequality one has

$$\langle \phi_0, \phi_x \rangle_+^T \ = \ \langle \phi_0 \phi_x \rangle_+ - \langle \phi_0 \rangle_+^2 \ > \ 0,$$

where we've used translation invariance.

Now the assumption of spontaneous magnetization means that $\langle \phi_0 \rangle_+^2 > 0$ so

$$\inf_{x \in \mathbb{L}} \langle \phi_0 \phi_x \rangle_0 \ = \ \inf_{x \in \mathbb{L}} \langle \phi_0 \phi_x \rangle_+ \ > 0.$$

Looking back we see that establishing (4.61) is sufficient to show that spontaneous magnetization implies long range order.

The statement (4.60) is of course stronger than (4.61). In particular one expects that above the critical value of β all translation invariant Gibbs measures are given by convex combinations of the measures $\langle \bullet \rangle_+$ and $\langle \bullet \rangle_-$. This says that while there are infinitely many Gibbs measures there are only two pure ones.

The first (partial) proof of this statement was given by Lebowitz in [43] for classical Ising ferromagnets. The key idea there is to use a variational principle to show the equality of spin-spin correlations across all Gibbs measures - if $p(\beta)$ denotes the free energy per unit volume (the pressure) then one expects

$$\frac{\partial p}{\partial \beta} \ = \ \sum_{\substack{x \in \mathbb{L} \\ x \neq 0}} J(x) \langle \phi_0 \phi_x \rangle$$

which should hold for all Gibbs measures for the given values of g, b, β, h. See [32] for more discussion on the variational principle - we remark that it is not completely trivial to show that derivatives of the pressure correspond to expectation values for Gibbs measures but for the classical Ising model this a well established result.

Combining the above variational argument with a clever correlation inequality [43] inductively shows that n-th order moments of all Gibbs measures must all agree if all the lower order moments agree. However this approach does not work for all $\beta > \beta_M$ - the variational principle depends on the differentiability of $p(\beta)$ with respect to β. This is differentiability is expected to hold but a *direct* proof of this is out of the reach of current methods. However $p(\beta)$ is convex in β so this differentiabilty can fail at most countably many values of β. We remark that for the classical nearest neighbor Ising ferromagnet a full characterization of translation invariant Gibbs measures for $\beta > \beta_M$ was proved more recently in [14].

We need an analog of Lebowitz's result for our ϕ^4 ferromagnets. Analogs of the results of [43] (in particular the necessary correlational inequalities) were transferred to this setting in [44]. A variational principle in this setting (which needs a corresponding notion of Gibbs measures) is much harder than the case of the classical Ising model. However one can formulate these ideas within the setting of superstable Gibbs measures for spin systems, introduced in [45]. A variational principle in this setting was in fact proved in [42]. One then has that for almost every $\beta > \beta_c$ there are only two pure Gibbs states corresponding to the $+$ and $-$ measures. In particular for almost every β one has (4.61) and so for such β the presence of long range order implies spontaneous magetization. By a simple argument using Griffiths Second inequality this implication between long range order and spontaneous magnetization holds for all β except for perhaps $\beta = \beta_M$. However this is certainly sufficient for our purposes.

Below we go through the proof of the above mentioned result which is certainly not new - however we are able to dramatically simplify many of the steps in the ultrametric setting and at the same time be more explicit in our calculation. In particular we believe there is a mistake in the proof of the main superstablity estimate in [45] - however this mistake disappears in the ultrametric setting. We also give a full presentation here because the earlier exposition of the superstability estimates is spread across multiple papers in slightly different settings ([45], [58], [59]). Additionally instead of trying to apply the more general variational principle [42] we proceed along a more direct route to get exactly the one variational principle we need.

4.7.2 Preliminaries for Superstable Gibbs Measures

We now start studying the measures mentioned in Theorem 4.8 in the context of Gibbs measures on $(\mathbb{R}^{\mathbb{L}}, \mathcal{B})$.

Before presenting the main material we give some notation for various σ-algebras and define certain modes of convergence for measures.

For a subset $A \subseteq \mathbb{L}$ we define $\mathcal{B}(A)$ to be the smallest sigma-algebra of sets that makes the collection of projections $\{\phi \mapsto \phi_x \,| x \in A\}$ measurable. We define $\bar{\mathcal{B}}(A)$ to be those sets that depend on only finitely many sites in A, that is they must be members of $\mathcal{B}(\Lambda)$ for some $\Lambda \Subset A$.

Definition. *Let μ_n be a sequence of Borel probability measures on $\mathbb{R}^{\mathbb{L}}$. We say μ_n converges locally weakly to a Borel probability measure μ if for every $\Gamma \Subset \mathbb{L}$ the corresponding finite-dimensional marginals $\mu_{n,\Gamma}$ converge in the topology of weak converge to μ_Γ.*

Definition. *Let μ_n be a sequence of Borel probability measures on $\mathbb{R}^{\mathbb{L}}$. We say μ_n converges locally set-wise to a Borel probability measure μ if for every set $B \in \bar{\mathcal{B}}(\mathbb{L})$ one has $\lim_{n\to\infty} \mu_n(B) = \mu(B)$.*

We remark that if μ_n converges locally set-wise to μ then μ_n also converge locally weakly to μ (although the converse may not hold).

When working with unbounded spin systems it is often necessary to impose a temperedness condition on our space of field configurations so that the Gibbs interaction terms are well defined. For our model we study Gibbs random fields supported on configurations of at most logarithmic growth. In what follows we use a regularized logarithm function: $\log_{+}(r) := \max(\log(r), 1)$.

We note that all of the analysis of this section takes place for arbitrary but fixed $g > 0$ and $b \in \mathbb{R}$.

Definition. *Let $s > 0$. We define the following sets of field configurations:*

$$X_s := \left\{ S \in \mathbb{R}^{\mathbb{L}} \mid |S_x|^2 \leqslant s \log_{+}(\|x\|) \forall x \in \mathbb{L} \right\},$$

$$\bar{X}_s := \left\{ S \in \mathbb{R}^{\mathbb{L}} \mid \text{ There exists a finite set } \Lambda(S) \text{ such that for any } x \notin \Lambda(S) \text{ one has } |S_x|^2 \leqslant s^2 \log_{+}(\|x\|) \right\},$$

$$X_\infty = \bigcup_{n=1}^{\infty} X_n.$$

Note that one has $\bar{X}_s \subset X_\infty$. If a probability measure on $(\mathbb{R}^{\mathbb{L}}, \mathcal{B})$ is supported on X_∞ then we call it a *tempered* measure.

For any finite Λ we define the following energy function on configurations ϕ_Λ on $\mathbb{R}^\Lambda$:

$$U(\phi_\Lambda) = \sum_{x \in \Lambda} \left(g\phi_x^4 + b\phi_x^2 - h\phi_x \right) - \frac{1}{2}\beta \sum_{\substack{x,y \in \Lambda \\ x \neq y}} J(x-y)\phi_x\phi_y$$

We define an associated interaction energy function as follows. Let $A \Subset \mathbb{L}$ and $B \subset \mathbb{L}$, then we define:

$$W(\phi_A|\psi_B) = -\beta \sum_{x \in A,\ y \in B} J(x-y)\phi_x\psi_y$$

In cases where B is infinite we note that if ψ_B can be written as restrictions of field configurations in X_∞ then $W(\phi_A|\psi_B)$ is finite. This interaction energy satisfies the following relationship with the previously mentioned energy functions - if Λ_1 and Λ_2 are two disjoint finite sets then one has the following equality for any $\phi_{\Lambda_1 \cup \Lambda_2} \in \mathbb{R}^{\Lambda_1 \cup \Lambda_2}$

$$U\left(\phi_{\Lambda_1 \cup \Lambda_2}\right) = U\left(\phi_{\Lambda_1}\right) + U\left(\phi_{\Lambda_2}\right) + W\left(\phi_{\Lambda_1} \middle| \phi_{\Lambda_2}\right)$$

A simple but essential estimate for this section is the following:

Lemma 4.17. *For any compact $K \subset (0, \infty) \times [0, \infty)$ there exists a constant $\mathbf{O}_1$ such that for all $(\beta, h) \in K$, and all $\Lambda \Subset \mathbb{L}$ one has:*

$$U(\phi_\Lambda) \geqslant \sum_{x\in\Lambda} \left(\frac{g}{2}\phi_x^4 - \mathbf{O}_1\right) \tag{4.62}$$

Proof: Immediate from inspection of the definition of $U(\phi_\Lambda)$. □

We define $\phi_A \wedge \psi_B$ to be the element of $\mathbb{R}^{A\cup B}$ defined via

$$(\phi_A \wedge \psi_B)_x = \begin{cases} \phi_x, & \text{if } x \in A \\ \psi_x, & \text{if } y \in B \end{cases}$$

We are now ready to define our Gibbsian specification, a family of measure kernels $\pi_{\Lambda,\beta,h}(\cdot|\cdot) : \mathcal{B} \times \mathbb{R}^{\mathbb{L}} \to [0,1]$. For $A \in \mathcal{B}$ and $\psi \in \mathbb{R}^{\mathbb{L}}$ we set:

$$\pi_{\Lambda,\beta,h}(A|\psi) = \begin{cases} \mathcal{Z}(\Lambda|\psi)^{-1} \int_{\mathbb{R}^\Lambda} d\phi_\Lambda \; \mathbb{1}_A\left(\phi_\Lambda \wedge \psi_{\Lambda^c}\right) \exp\left[-U(\phi_\Lambda) - W(\phi_\Lambda|\psi_{\Lambda^c})\right], & \text{if } \psi \in X_\infty \\ 0, & \text{if } \psi \notin X_\infty \end{cases}$$

In the definition above we have defined $\mathcal{Z}(\Lambda|\psi)$ to be a normalizing factor when $\psi \in X_\infty$, that is:

$$\mathcal{Z}(\Lambda|\psi) = \int_{\mathbb{R}^\Lambda} d\phi_\Lambda \;\; \exp\left[-U(\phi_\Lambda) - W(\phi_\Lambda|\psi_{\Lambda^c})\right] \;\text{ for } \psi \in X_\infty$$

It is not hard to check (see [32] for more details) that the family of measure kernels $\{\pi_{\Lambda,\beta,h}\}_{\Lambda\Subset\mathbb{L}}$ is consistent for fixed β and h, that is if $\Lambda_1 \subset \Lambda_2$ then $\pi_{\Lambda_2,\beta,h}\Big(\pi_{\Lambda_1,\beta,h}(A|\cdot)\Big|\psi\Big) = \pi_{\Lambda_2,\beta,h}(A|\psi)$ for all $A \in \mathcal{B}$. If one restricts these measure kernels to X_∞ then they become actual probability kernels.

If f is a $\mathcal{B}$-measurable function on $\mathbb{R}^{\mathbb{L}}$ we will write

$$\pi_\Lambda(f|\psi) := \int_{\mathbb{R}^{\mathbb{L}}} d\pi_\Lambda(\cdot|\psi) f(\cdot)$$

We now give the core definition of this section:

Definition. *A probability measure μ on $(\mathbb{R}^{\mathbb{L}}, \mathcal{B})$ is said to be a tempered Gibbs measure with respect to the interaction $U(\beta, h)$ if for all $A \in \mathcal{B}$ we have:*

$$\int_{\mathbb{R}^{\mathbb{L}}} d\mu(\phi)\pi_{\Lambda,\beta,h}(A|\phi) = \mu(A) \text{ for all } \Lambda \Subset \mathbb{L} \tag{4.63}$$

Note that a measure μ is supported on X_∞ automatically if it satisfies the consistency condition above. Also note that because of ultrametricity the Gibbs consistency condition implies translation invariance. We define $\mathcal{G}(\beta, h)$ to be the set of all Gibbs interactions with respect to the interaction $U(\beta, h)$.

4.7.3 Superstability Estimates and existence of Gibbs Measures

For $\Gamma \subset \Lambda \Subset \mathbb{L}$ we denote the Radon-Nikodym derivative (with respect to Lebesgue measure) of the marginal of $\pi_\Lambda(\cdot|S)$ onto $\mathbb{R}^\Gamma$ by $\rho_\Lambda^\Gamma(\phi_\Gamma|S)$. For $S \in X_\infty$ one has:

$$\rho_{\Lambda}^{\Gamma}(\phi_{\Gamma}|S) := \frac{1}{\mathcal{Z}(\Lambda|S)} \int_{\mathbb{R}^{\Lambda\setminus\Gamma}} d\phi_{\Lambda\setminus\Gamma} \exp\left[-U(\phi_{\Lambda}) - W\left(\phi_{\Lambda}|S_{\Lambda^c}\right)\right]$$

The next lemma gives the core superstability estimate for our model following the ideas of [58] and [59]..

Lemma 4.18. *$\forall g \in [0,\infty)$, $\forall b \in \mathbb{R}$*
$\forall A_{\beta} > 0$, $\forall A_h > 0$
$\exists \delta > 0$ such that
$\forall \beta \in [0, A_{\beta}]$, $\forall h \in [0, A_h]$
$\forall \Gamma \Subset \mathbb{L}$, $\forall k \in \mathbb{N}$ with $\Lambda_k \supseteq \Gamma$, one has the following bound:

$$\left|\rho_{\Lambda_k}^{\Gamma}(\phi_{\Gamma}|S)\right| \leqslant \exp\left[\sum_{x\in\Gamma} -\frac{g}{4}\phi_x^4 + \delta\right] \tag{4.64}$$

Proof. In appendix.

Exponential bounds of the form above will be crucial for establishing that sequences of finite volume measures have cluster points and for showing that these cluster points have the appropriate properties.

Definition. *A probability measure λ on $(\mathbb{R}^{\mathbb{L}}, \mathcal{B})$ will be called* regular *if and only if for every Λ its marginal restricted to Λ has a Radon-Nikodym derivative with respect to Lebesgue measure on $\mathbb{R}^{\Lambda}$ (which we denote $g(\phi_{\Lambda}|\lambda)$) that satisfies the following bound:*

$$|g(\phi_{\Lambda}|\lambda)| \leqslant \exp\left[\sum_{x\in\Lambda} -\frac{g}{4}\phi_x^4 + \delta\right] \tag{4.65}$$

We now show that being a regular measure is a stronger statement than being a tempered measure.

Lemma 4.19. *Suppose λ is a regular measure, then:*

1. *If $s > 0$ then $\lambda\left(\bar{X}_s\right) = 1$.*
2. *λ is a tempered measure.*
3. *There exists $\mathbf{O}_2$ such that for all $s > 0$ one has $\lambda\left(X_n^c\right) \leqslant \mathbf{O}_2 \exp\left[-\frac{a}{8}s^2\right]$.*

Proof:
All three statements come from proving appropriate bounds on:

$$\sum_{x\in\mathbb{L}} \lambda\left(\left\{\phi_x^2 > s\log_+(\|x\|)\right\}\right) \tag{4.66}$$

Statements (1) and (2) will follow from Borel-Cantelli if we show that the above sum is finite.

$$
\begin{aligned}
\lambda\left(\left\{\phi_x^2 > s\log_+(\|x\|)\right\}\right) &\leqslant 2e^{\delta}\int_{\sqrt{s\log_+(\|x\|)}}^{\infty} dt\ e^{-gt^4/4} \\
&\leqslant K_{s,g}\int_{\sqrt{s\log_+(\|x\|)}}^{\infty} dt\ \exp\left[-\frac{d+2}{s}t^2\right] \\
&\leqslant K_{s,g}e^{-(d+1)\log_+(\|x\|)}\int_{-\infty}^{\infty} dt\ e^{-t^2} \leqslant K_{s,g}\times\sqrt{\pi}\times e^{-(d+1)\log_+(\|x\|)}
\end{aligned}
$$

In the inequality on the first line we used (4.64). In going to the second line we defined $K_{s,g} = 2\exp\left[\delta + \frac{4}{g}\times\left(\frac{d+2}{s}\right)^2\right]$ and used the fact that $-c_1t^4 \leqslant -c_2t^2 + \dfrac{c_2^2}{c_1}$. Note that the last line is summable over $x \in \mathbb{L}$ so the first two statements are proven.

To prove statement (3) we first assume that $s > 1$. Then we have:

$$
\begin{aligned}
\lambda\left(\left\{\phi_x^2 > s\log_+(\|x\|)\right\}\right) &\leqslant 2e^{\delta}\int_{\sqrt{s\log_+(\|x\|)}}^{\infty} dt\ e^{-gt^4/4} \\
&\leqslant 2e^{\delta}e^{-gs^2/8}\int_{\sqrt{s\log_+(\|x\|)}}^{\infty} dt\ e^{-gt^4/8} \\
&\leqslant 2e^{\delta}e^{-gs^2/8}\int_{\sqrt{\log_+(\|x\|)}}^{\infty} dt\ e^{-gt^4/8} \\
&\leqslant K_g e^{-gs^2/8}\int_{\sqrt{\log_+(\|x\|)}}^{\infty} dt\ e^{-(d+2)t^2} \\
&\leqslant K_g e^{-gs^2/8}e^{-(d+1)\log_+(||x||)}\int_{\sqrt{\log_+(\|x\|)}}^{\infty} dt\ e^{-t^2} \leqslant K_g e^{-gs^2/8}e^{-(d+1)\log_+(||x||)}\times\sqrt{\pi}
\end{aligned}
$$

Here we have defined $K_g = 2\exp\left[\delta + \frac{8}{g}(d+1)^2\right]$. We then have for $s > 1$:

$$
\lambda(X_s^c) \leqslant \sum_{x\in\mathbb{L}}\lambda\left(\left\{\phi_x^2 > s\log_+(\|x\|)\right\}\right) \leqslant \left[\sum_{x\in\mathbb{L}} e^{-(d+1)\log_+(||x||)}\right]\times K_g\times\sqrt{\pi}\times e^{-gs^2/8} < \infty
$$

Statement (iii) then holds for all $s > 0$ by choosing $\mathbf{O}_2 = \max\left(\left[\sum_{x\in\mathbb{L}} e^{-(d+1)\log_+(||x||)}\right]\times K_g\times\sqrt{\pi}, e^{\frac{g}{4}}\right)$.
□

Next we will prove some regularity results for the conditional probabilities π_Λ. For any $S \in \mathbb{R}^{\mathbb{L}}$ and $A \Subset \mathbb{L}$ we define $S(A) := S_A \wedge 0_{A^c}$.

Lemma 4.20. *(i) Let Λ be a cube centered at the origin, that is $\Lambda = \Lambda_N$ for some N. Then for any $S \in X_\infty$ one has:*

$$
\lim_{j\to\infty}\pi_{\Lambda,\beta,h}\left(A|S\left(\Lambda_j\right)\right) = \pi_{\Lambda,\beta,h}\left(A|S\right)
$$

The above convergence is uniform over $A \in \mathcal{B}(\Lambda)$ and $S \in X_n$ for fixed n.

(ii) For any $\epsilon > 0$ there exists $m \in \mathbb{N}$ such that for all $S \in X_1$ one has:

$$\sup_j \pi_{\Lambda_j,\beta,h}(X_m^c|S) < \epsilon$$

Proof:

Fix $n \geqslant 1$ and suppose that $S \in X_n$. From the definition one has:

$$\pi_{\Lambda,\beta,h}\left(A|S\left(\Lambda_j\right)\right) = \frac{\int_A d\phi_\Lambda \exp\left[-U(\phi_\Lambda) - W\left(\phi_\Lambda|S\left(\Lambda_j\right)_{\Lambda^c}\right)\right]}{\int_{\mathbb{R}^\Lambda} d\phi_\Lambda \exp\left[-U(\phi_\Lambda) - W\left(\phi_\Lambda|S\left(\Lambda_j\right)_{\Lambda^c}\right)\right]}$$

Note that above we have abused notation and are using the symbol A to denote the projection of the original set $A \in \mathbb{R}^{\mathbb{L}}$ onto $\mathbb{R}^\Lambda$. By arguments identical to those used in Sub-Lemma 5.2 one can show that

$$\inf_{S \in X_n} \int_{\mathbb{R}^\Lambda} d\phi_\Lambda \exp\left[-U(\phi_\Lambda) - W\left(\phi_\Lambda|S\left(\Lambda_j\right)_{\Lambda^c}\right)\right] > 0$$

Thus it suffices to prove that for arbitrary $B \subset \mathbb{R}^\Lambda$

$$\lim_{j\to\infty} \int_B d\phi_\Lambda \exp\left[-U(\phi_\Lambda) - W\left(\phi_\Lambda|S\left(\Lambda_j\right)_{\Lambda^c}\right)\right] = \int_B d\phi_\Lambda \exp\left[-U(\phi_\Lambda) - W\left(\phi_\Lambda|S_{\Lambda^c}\right)\right]$$

where the convergence above is uniform as we quantify over $B \in \mathbb{R}^\Lambda$ and $S \in X_n$. We establish a pointwise in ϕ_Λ bound on the integrand on the left hand side.

$$\begin{aligned}
&\left|\exp\left[-U(\phi_\Lambda) - W\left(\phi_\Lambda|S\left(\Lambda_j\right)_{\Lambda^c}\right)\right] - \exp\left[-U(\phi_\Lambda) - W\left(\phi_\Lambda|S_{\Lambda^c}\right)\right]\right| \\
=& e^{-U(\phi_\Lambda)}\left|\exp\left[-W\left(\phi_\Lambda|S\left(\Lambda_j\right)_{\Lambda^c}\right)\right] - \exp\left[-W\left(\phi_\Lambda|S_{\Lambda^c}\right)\right]\right| \\
\leqslant& e^{-U(\phi_\Lambda)} \exp\left[\sum_{x\in\Lambda}\sum_{y\in\Lambda^c} \bar{J}(|x-y|)\left(\phi_x^2 + S_y^2\right)\right] \left|W\left(\phi_\Lambda|S(\Lambda_j\backslash\Lambda)_{\Lambda^c}\right)\right| \\
\leqslant& e^{-U(\phi_\Lambda)} \exp\left[\sum_{x\in\Lambda}\left(||\bar{J}||_{L^1}\phi_x^2 + n||\bar{J}\log_+||_{L^1}\right)\right] \sum_{x\in\Lambda}\left(||\bar{J}\mathbb{1}_{\Lambda_j^c}||_{L^1}\phi_x^2 + n||\bar{J}\mathbb{1}_{\Lambda_j^c}\log_+||_{L^1}\right) \\
\leqslant& ||\bar{J}\mathbb{1}_{\Lambda_j^c}\log_+||_{L^1} e^{-U(\phi_\Lambda)} \exp\left[\sum_{x\in\Lambda}\left(||\bar{J}||_{L^1}\phi_x^2 + n||\bar{J}\log_+||_{L^1}\right)\right] \sum_{x\in\Lambda}\left(\phi_x^2 + n\right)
\end{aligned}$$

We then have:

$$\begin{aligned}
&\int_B d\phi_\Lambda \left|\exp\left[-U(\phi_\Lambda) - W\left(\phi_\Lambda|S\left(\Lambda_j\right)_{\Lambda^c}\right)\right] - \exp\left[-U(\phi_\Lambda) - W\left(\phi_\Lambda|S_{\Lambda^c}\right)\right]\right| \\
\leqslant& ||\bar{J}\mathbb{1}_{\Lambda_j^c}\log_+||_{L^1} \int_{\mathbb{R}^\Lambda} d\phi_\Lambda e^{-U(\phi_\Lambda)} \exp\left[\sum_{x\in\Lambda}\left(||\bar{J}||_{L^1}\phi_x^2 + n||\bar{J}\log_+||_{L^1}\right)\right] \sum_{x\in\Lambda}\left(\phi_x^2 + n\right) \\
\leqslant& ||\bar{J}\mathbb{1}_{\Lambda_j^c}\log_+||_{L^1} \left(\int_{\mathbb{R}^\Lambda} d\phi_\Lambda \exp\left[\sum_{x\in\Lambda} -\frac{g}{2}\phi_x^4 + \mathbf{O}_1 + ||\bar{J}||_{L^1}\phi_x^2 + n||\bar{J}\log_+||_{L^1}\right] \sum_{x\in\Lambda}\left(\phi_x^2 + n\right)\right)
\end{aligned}$$

The integral above can obviously be bounded uniformly in B, j and $S \in X_n$, part (i) now follows since we have $\lim_{j\to\infty} ||\bar{J}\mathbb{1}_{\Lambda_j^c} \log_+ ||_{L^1} = 0$.

Part (ii) follows easily via applying the superstability estimate of Lemma 4.18 for $\pi_{\Lambda_j,\beta,h}(\cdot|S)$ with $\Gamma = \Lambda_j$. We note that for $m > 1$ we have:

$$\begin{aligned}\pi_{\Lambda_j,\beta,h}(X_m^c|S) &\leqslant \sum_{x\in\Lambda_j} \pi_{\Lambda_j,\beta,h}\left(\mathbb{1}\{\phi_x^2 > m\log_+(\|x\|)\}|S\right)\\ &\leqslant \sum_{x\in\Lambda}\left(2e^{\delta}\int_{\sqrt{m\log_+(\|x\|)}}^{\infty} dt\; e^{-gt^4/4}\right)\\ &\leqslant \mathbf{O}_2 e^{-\frac{gm^2}{8}}.\end{aligned}$$

The last line of the estimate above is proven in the same way we proved statement (iii) of Lemma 4.19. Statement (ii) of the current lemma now follows by making m sufficiently large. □

We now give a definition followed by a quick lemma which will help us establish that we have cluster points as we take infinite volume limits.

Definition. *A family of Borel measures $\{\mu_i\}_{i\in I}$ on $\mathbb{R}^n$ are said to be uniformly absolutely continuous with respect to a Borel measure ν on $\mathbb{R}^n$ if for every $\epsilon > 0$ there exists a $\delta > 0$ such that for any Borel set one has*

$$\nu(A) < \delta \;\Rightarrow \mu_i(A) < \epsilon \text{ for all } i \in I.$$

Lemma 4.21. *Let μ_n be a sequence of Borel probability measures on $\mathbb{R}^{\mathbb{L}}$ converging locally weakly to a measure μ. Suppose furthermore that for every $\Lambda \Subset \mathbb{L}$ there exists an N such that the sequence of $\mathbb{R}^{\Lambda}$ marginals $\{\mu_{n,\Lambda}\}_{n\geqslant N}$ are uniformaly absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^{\Lambda}$. Then the measures μ_n converge locally set-wise to the measure μ.*

Proof: In appendix.

Theorem 4.12. *Let $S \in X_1$, and let $\{n_j\}_{j=1}^{\infty}$ be an increasing sequence of natural numbers with $\lim_{j\to\infty} n_j = \infty$. Then the sequence of measures $\{\pi_{\Lambda_{n_j},\beta,h}(\cdot|S)\}_{j=1}^{\infty}$ has a convergent subsequence (in the topology of local set-wise convergence).*

The limit points of these subequences are tempered, translation invariance, and satisfy all the statements of Lemma 4.19.

Proof: The superstability bound of Lemma 4.18 establishes that for any $N \in p^{\mathbb{Z}_+}$ the marginals of the $\{\pi_{\Lambda_{n_j},\beta,h}(\cdot|S)\}_{n_j}$ on $\mathbb{R}^{\Lambda_N}$ are tight (relatively compact in the topology of weak convergence). Thus by a diagonalization argument in j and N we can find a subsequence $\{n'_k\}_{k=1}^{\infty}$ such that for any $\Lambda \Subset \mathbb{L}$ the marginals of $\pi_{\Lambda_{n'_k},\beta,h}(\cdot|S)$ on $\mathbb{R}^{\Lambda}$ converge in the topology of weak convergence to a limiting measure ν_Λ. It is not hard to see that $\{\nu_\Lambda\}_{\Lambda\Subset\mathbb{L}}$ is a family of consistent finite dimensional marginals. By the Kolmogorov extension theorem they uniquely define a measure ν on $\mathbb{R}^{\mathbb{L}}$. By construction one has that $\lim_{k\to\infty} \pi_{\Lambda_{n'_k},\beta,h}(\cdot|S) = \nu$ in the topology of weak local convergence.

Since the measures $\pi_{\Lambda_{n'_k},\beta,h}(\cdot|S)$ satisfy the conditions of Lemma 4.21. Thus the $\pi_{\Lambda_{n'_k},\beta,h}(\cdot|S)$ converges to ν locally set-wise. Additionally since the measure $\pi_{\Lambda_j,\beta,h}(\cdot|S)$ is invariant under translations of norm less then p^j we have that any infinite volume limit point is in fact completely translation invariant.

Now suppose that ν is the limit in the topology of local weak convergence of a sequence of finite volume Gibbs measures $\pi_{\Lambda_{m_k},\beta,h}(\cdot|S)$. Then by Lemma 4.18 the measures $\pi_{\Lambda_{m_k},\beta,h}(\cdot|S)$ satisfy the requirements of Lemma 4.21 so these measures converge to ν locally setwise. In particular for every $x \in \mathbb{L}$ one has

$$\nu\left(\left\{\phi_x^2 > s\log_+(\|x\|)\right\}\right) = \lim_{k\to\infty} \pi_{\Lambda_{m_k},\beta,h}\left(\left\{\phi_x^2 > s\log_+(\|x\|)\right\}|S\right)$$

On the other hand as soon as m_k is large enough for $\Lambda_{m_k} \ni x$ then we have the superstability estimate for the one point marginal of $\pi_{\Lambda_{m_k},\beta,h}(\cdot|S)$ at x , this means that:

$$\nu\left(\left\{\phi_x^2 > s\log_+(\|x\|)\right\}\right) \leqslant 2e^{\delta}\int_{\sqrt{s\log_+(\|x\|)}}^{\infty} dt\; e^{-gt^4/4}.$$

We then have the necessary estimate to show that statements (i), (ii), and (iii) of Lemma 4.19 hold for ν. $\square$

Theorem 4.13. *Suppose that for some n_k one has $\lim_{k\to\infty}\pi_{\Lambda^{n_k},\beta,h}(\cdot|S) = \nu$ (where $S \in X_1$). Then ν is a tempered Gibbs measure with respect to the interaction $U(\beta,h)$.*

Proof:

We use the notation $\nu_k = \pi_{\Lambda_{n_k},\beta,h}(\cdot|S)$.

Our goal is to show that for any any $\Gamma \Subset \mathbb{L}$ and for any $A \in \mathcal{B}$ and one has:

$$\int_{\mathbb{R}^{\mathbb{L}}} d\nu(S)\;\pi_{\Gamma,\beta,h}(A|S) =: \nu\left(\pi_{\Gamma,\beta,h}(A|\cdot)\right) = \nu(A)$$

We first prove the above equality for A of the form $A_1 \cap A_2$ where $A_1 \in \mathcal{B}(\Gamma),\; A_2 \in \bar{\mathcal{B}}(\Gamma^c)$. This establishes the above equality for all of $\mathcal{B}$ since sets of this form generate the sigma-algebra $\mathcal{B}$.

Since the specifications $\{\pi_{\Lambda,\beta,h}\}_{\Lambda\Subset\mathbb{L}}$ are consistent among themselves it suffices to prove the equality above for lattice sets Γ of the form $\Gamma = \Lambda_k$ for some k.

Let $\epsilon > 0$ be arbitrary, we will show that

$$\left|\nu\left(\pi_{\Gamma,\beta,h}(A|\cdot)\right) - \nu\left(A\right)\right| < 6\epsilon$$

.

By Lemma 4.20 Part (ii) we can find an $m \in \mathbb{N}$ such that:

$$\sup_k \nu_k(X_m^c) < \epsilon$$

We next note that for any k with $\Lambda_k \supset \Gamma$ and for any $j \in \mathbb{N}$ one has:

$$
\begin{aligned}
|\nu\left(\pi_{\Gamma,\beta,h}(A|\cdot)\right) - \nu\left(A\right)| &\leqslant |\nu\left(\pi_{\Gamma,\beta,h}(A|\cdot)\right) - \nu_k\left(\pi_{\Gamma,\beta,h}(A|\cdot)\right)| + |\nu_k\left(A\right) - \nu(A)| \\
&= |\nu\left(\mathbb{1}_{A_2}(\cdot)\pi_{\Gamma,\beta,h}(A_1|\cdot)\right) - \nu_k\left(\mathbb{1}_{A_2}(\cdot)\pi_{\Gamma,\beta,h}(A_1|\cdot)\right)| + |\nu_k\left(A\right) - \nu(A)| \\
&\leqslant \left|\int_{\mathbb{R}^{\mathbb{L}}} d\nu(S)\mathbb{1}_{A_2}(S)\left[\pi_{\Gamma,\beta,h}\left(A|S\right) - \pi_{\Gamma,\beta,h}\left(A|S(\Lambda_j)\right)\right]\right| \\
&+ \left|\int_{\mathbb{R}^{\mathbb{L}}} d\nu(S)\mathbb{1}_{A_2}(S)\pi_{\Gamma,\beta,h}\left(A|S(\Lambda_j)\right) - \int_{\mathbb{R}^{\mathbb{L}}} d\nu_k(S)\mathbb{1}_{A_2}(S)\pi_{\Gamma,\beta,h}\left(A|S(\Lambda_j)\right)\right| \\
&+ \left|\int_{\mathbb{R}^{\mathbb{L}}} d\nu_k(S)\mathbb{1}_{A_2}(S)\left[\pi_{\Gamma,\beta,h}\left(A|S(\Lambda_j)\right) - \pi_{\Gamma,\beta,h}\left(A|S\right)\right]\right| \\
&+ |\nu_k\left(A\right) - \nu(A)|
\end{aligned}
$$

In the first line we use the consistency of our specification, that is $\nu_k\left(\pi_{\Gamma,\beta,h}(A|\cdot)\right) = \nu_k\left(A\right)$. In the next line we have used that $\pi_{\Gamma,\beta,h}(A_1 \cap A_2|S) = \mathbb{1}_{A_2}(S)\pi_{\Gamma,\beta,h}(A_1|S)$ (this comes from the fact that our specification is proper). We now bound the quantities on the third, fourth, fifth, and sixth lines.

For bounding the quantity on the third line we use Lemma 4.20 part (i) and the bounded convergence theorem so that for j sufficiently large one has:

$$
\left|\int_{\mathbb{R}^{\mathbb{L}}} d\nu(S)\left[\pi_{\Gamma,\beta,h}\left(A|S\right) - \pi_{\Gamma,\beta,h}\left(A|S(\Lambda_j)\right)\right]\right| < \epsilon
$$

We bound the fifth line for j sufficiently large, uniformly in k :

$$
\begin{aligned}
\int_{\mathbb{R}^{\mathbb{L}}} d\nu_k(S)\Big|\pi_{\Gamma,\beta,h}\left(A|S(\Lambda_j)\right) - \pi_{\Gamma,\beta,h}\left(A|S\right)\Big| &\leqslant \int_{X_m} d\nu_k(S)\mathbb{1}_{X_m}(S)\Big|\pi_{\Gamma,\beta,h}\left(A|S(\Lambda_j)\right) - \pi_{\Gamma,\beta,h}\left(A|S\right)\Big| + \epsilon \\
&\leqslant 2\epsilon
\end{aligned}
$$

In the two lines immediately above we used the fact that we have uniform convergence of the probability kernels in j on the set X_m, along with control over measure X_m^c uniform in k.

After fixing j we can bound the fourth line of our main bound by ϵ if we take k sufficiently large since we have local set-wise convergence of the v_k and $\mathbb{1}_{A_2}(S)\pi_{\Gamma,\beta,h}\left(A|S(\Lambda_j)\right)$ is a function of finitely many spin variables, the same is true for bounding the sixth line. □

Theorem 4.14. *Let μ be a tempered Gibbs measure with respect to the interaction $U(\beta,h)$ for appropriate β and h. Then μ is also regular*

Proof: What we must show is that for arbitrary $\Gamma \Subset \mathbb{L}$ the marginal of μ corresponding to $\mathbb{R}^{\Gamma}$ (as before, denoted μ_{Γ}) is absolutely continuous with respect to Lesbesgue measure on $\mathbb{R}^{\Lambda}$. Additionally we must show that the corresponding Radon Nikodym derivative, which we will denote $g_{\Gamma}(\phi_{\Gamma}|\mu)$, satisfies the bound (4.65).

There are multiple ways one can show the mentioned absolutely continuity, however we will try to directly construct the Radon-Nikodym derivative and use our explicit formulas to establish our the estimate (4.65). In particular we will use the same bounds used in the proof of Lemma 4.18 to establish the necessary bounds on $g(\phi_\Gamma|\mu)$.

Below we use notation defined in the section of the appendix that proves Lemma 4.18. Fix some $z \in \Gamma$. We note that it is not hard to see that $\{\hat{R}^z_q\}_{q=-1}^{\infty}$ is a partition of X_∞. Let $A \in \mathcal{B}(\Gamma)$. Since μ is tempered we have that

$$\int_{\mathbb{R}^{\mathbb{L}}} d\mu(S)\mathbb{1}_A(S) = \sum_{q=1}^{\infty}\int_{\mathbb{R}^{\mathbb{L}}} d\mu(S)\mathbb{1}_A(S)\mathbb{1}_{\hat{R}^z_q}(S).$$

For any $q \in \mathbb{N} \cup \{-1\}$ and $\Lambda_j \supset \Gamma$ by Gibbs consistency we have that:

$$\begin{aligned}\int_{\mathbb{R}^{\mathbb{L}}} d\mu(S)\mathbb{1}_A(S)\mathbb{1}_{\hat{R}_q}(S) &= \int_{\mathbb{R}^{\mathbb{L}}} d\mu(S)\rho_{\Lambda_j}\left(\mathbb{1}_A(\phi_\Gamma)\mathbb{1}_{\hat{R}^z_q}\left(\phi_\Gamma \wedge \phi_{\Lambda_j\setminus\Gamma} \wedge S_{\Lambda_j^c}\right)|S\right)\\ &= \int_{\mathbb{R}^\Gamma} d\phi_\Gamma\ \mathbb{1}_A(\phi_\Gamma)\int_{\mathbb{R}^{\mathbb{L}}} d\mu(S)\frac{1}{\mathcal{Z}(\Lambda_j|S)}\int_{\mathcal{R}_q(\phi_\Gamma,\Lambda_k,S)} d\phi_{\Lambda_j\setminus\Gamma}\exp\left[-U(\phi_{\Lambda_j}) - W(\phi_{\Lambda_j}|S)\right]\end{aligned}$$

Define $k(n) : \mathbb{N} \cup \{-1\} \to \mathbb{N}$ so that $\Lambda_{k(n)} \supset \Gamma \cup \left(z + \Lambda_{\max(q,0)}\right)$. We now define

$$g_\Gamma(\phi_\Gamma|\mu) := \sum_{q=-1}^{\infty}\left(\int_{\mathbb{R}^{\mathbb{L}}} d\mu(S)\left[\mathcal{Z}(\Lambda_{k(q)}|S)^{-1}\int_{\mathcal{R}_q(\phi_\Gamma,\Lambda_{k(q)},S)} d\phi_{\Lambda_{k(q)}\setminus\Gamma}\exp\left[-U(\phi_{\Lambda_{k(q)}}) - W(\phi_{\Lambda_{k(q)}}|S)\right]\right]\right) \tag{4.67}$$

Since the each of the summands on the RHS of (4.67) is non-negative function on $\mathbb{R}^\Gamma$ it is clear that one has point-wise convergence to a measurable function on $\mathbb{R}^\Gamma$. In particular by monotone convergence theorem and Fubini-Tonelli one has

$$\int_{\mathbb{R}^\Gamma} \mathrm{d}\phi_\Gamma\ \mathbb{1}_A(\phi_\Gamma)g_\Gamma(\phi_\Gamma|\mu)\ =\ \mu(A)$$

Thus absolute continuity is established. Now we claim that for any $S \in X_\infty$:

$$\begin{aligned}&\frac{1}{\mathcal{Z}(\Lambda_{k(-1)}|S)}\int_{\mathcal{R}_{-1}(\phi_\Gamma,\Lambda_{k(-1)},S)} d\phi_{\Lambda_{k(-1)}\setminus\Gamma}\exp\left[-U(\phi_{\Lambda_{k(-1)}}) - W(\phi_{\Lambda_{k(-1)}}|S)\right]\\ &\qquad\leqslant\ \exp\left[-\frac{g}{2}\phi_z^4 + A_\beta\|J\|_{L^1}\phi_z^2 + \mathcal{O}_3\right] \times \rho_{\Lambda_k}^{\Gamma\setminus\{z\}}(\phi_{\Gamma\setminus\{z\}}|S).\end{aligned} \tag{4.68}$$

And for $q \geqslant 0$:

$$\begin{aligned}&\frac{1}{\mathcal{Z}(\Lambda_{k(q)}|S)}\int_{\mathcal{R}_q(\phi_\Gamma,\Lambda_{k(q)},S)} d\phi_{\Lambda_{k(q)}\setminus\Gamma}\exp\left[-U(\phi_{\Lambda_{k(q)}}) - W(\phi_{\Lambda_{k(q}}|S)\right]\\ &\qquad\leqslant\ \exp\left[\left(\mathcal{O}_5 - \psi_q\right)|\Lambda_q|\right] \times \exp\left[\sum_{x\in(z+\Lambda_q)\cap\Gamma}\left(-\frac{g}{2}\phi_x^4 + \left(A_\beta\|J\|_{L^1} + 1\right)\phi_x^2 + \mathcal{O}_3\right)\right]\\ &\qquad\quad\times \rho_{\Lambda_k}^{\Gamma\setminus(z+\Lambda_q)}(\phi_{\Gamma\setminus(z+\Lambda_q)}|S)\end{aligned} \tag{4.69}$$

First we prove (4.69). Note that $k(q) \geqslant q$. Now if $S(\Lambda^c_{k(q)}) \in \hat{R}^z_q$ then the estimate follows by sublemma (5.4) in the appendix (note we have the assumption of condition (b) in this case).

We now turn to the other case: suppose that $S(\Lambda^c_{k(q)}) \notin \hat{R}^z_q$. Then there is some non-negative integer r such that $r > q$ and

$$\sum_{x \in (z+\Lambda_r)} S(\Lambda_{k(q)})^2_x \geqslant \psi_r |\Lambda_r|.$$

Now let $\phi_\Gamma \in \mathbb{R}^\Gamma$ and $\phi_{\Lambda_{k(q)}} \in \mathbb{R}^{\Lambda_{k(q)}}$ be arbitrary and define $\hat{\phi} = \phi_\Gamma \wedge \phi_{\Lambda_{k(q)}} \wedge S_{\Lambda^c_{k(q)}}$. Then we have that

$$\sum_{x \in (z+\Lambda_r)} \hat{\phi}^2_x \geqslant \sum_{x \in (z+\Lambda_r)} S(\Lambda_{k(q)})^2_x \geqslant \psi_r |\Lambda_r|$$

In particular we have that $\hat{\phi} \notin \hat{R}^z_q$. From this observation we see that for all $\phi_\Gamma \in \mathbb{R}^\Gamma$ the set $\mathcal{R}_q(\phi_\Gamma, \Lambda_{k(q)}, S)$ is the empty set and the integral on the top line of (4.69) vanishes. One can prove (4.68) in an analagous way, this time using sub-lemma (5.3) of the appendix.

We then have

$$\begin{aligned}
g(\phi_\Gamma|\mu) \leqslant{} & \exp\left[-\frac{g}{2}\phi^4_z + A_\beta \|J\|_{L^1}\phi^2_z + \mathcal{O}_3\right] \int_{\mathbb{L}} d\mu(S) \rho^{\Gamma\setminus\{z\}}_{\Lambda_k}(\phi_{\Gamma\setminus\{z\}}|S) \\
& + \sum_{q=0}^{\infty} \Bigg(\exp\left[\left(\mathcal{O}_5 - \psi_q\right)|\Lambda_q|\right] \times \exp\left[\sum_{x \in (z+\Lambda_q)\cap\Gamma} \left(-\frac{g}{2}\phi^4_x + \left(A_\beta\|J\|_{L^1} + 1\right)\phi^2_x + \mathcal{O}_3\right)\right] \\
& \qquad \times \int_{\mathbb{L}} d\mu(S) \rho^{\Gamma\setminus(z+\Lambda_q)}_{\Lambda_k}(\phi_{\Gamma\setminus(z+\Lambda_q)}|S) \Bigg) \\
={} & \exp\left[-\frac{g}{2}\phi^4_z + A_\beta\|J\|_{L^1}\phi^2_z + \mathcal{O}_3\right] g(\phi_{\Gamma\setminus\{z\}}|\mu) \\
& + \sum_{q=0}^{\infty} \left(\exp\left[\left(\mathcal{O}_5 - \psi_q\right)|\Lambda_q|\right] \exp\left[\sum_{x \in (z+\Lambda_q)\cap\Gamma} \left(-\frac{g}{2}\phi^4_x + \left(A_\beta\|J\|_{L^1} + 1\right)\phi^2_x + \mathcal{O}_3\right)\right] g(\phi_{\Gamma\setminus(z+\Lambda_q)}|\mu) \right)
\end{aligned}$$

Note in the final equality we used Gibbs consistency. We also use the convention that $g(\phi_\varnothing|\mu) = 1$. The regularity bound for $g(\phi_\Gamma|\mu)$ now follows by arguments identical to those used in the final steps of proving Lemma 4.18. □.

4.7.4 Properties of various Gibbs measures

Theorem 4.15. *For any $\beta, h \geqslant 0$ the measures $\pi_{\Lambda_n,\beta,h}(\cdot|0)$ converge to a tempered measure $\nu[\beta,h]$ locally setwise. This measure is a Gibbs measure with respect to the interaction $U(\beta,h)$.*

Proof :

All that needs to be proved is that the sequence $\pi_{\Lambda_n,\beta,h}(\cdot|0)$ converges locally weakly.

By Theorem 4.12 we have that there exists a convergent subsequence $\pi_{\Lambda_{n_k},\beta,h}(\cdot|0)$. We denote this measure by $\nu[\beta,h]$. Note that for any $\Gamma \Subset \Lambda$ the Γ-marginals of $\pi_{\Lambda_n,\beta,h}(\cdot|0)$ and $\nu[\beta,h,0]$ are all uniformly exponentially integrable - they have entire moment generating functions. Thus we can prove convergence by just showing convergence of all moments. We also have that the moments of $\pi_{\Lambda_{n_k},\beta,h}(\cdot|0)$ converge to the moments of $\nu[\beta,h,0]$.

Now for any multi-index $A \in \mathbb{N}^{\mathbb{L}}$ with $|A| \leqslant \infty$ we claim that the sequence $\pi_{\Lambda_n,\beta,h}(\phi^A|0)$ is monotone increasing - this is a consequence of Griffiths II. Since this sequence has a convergence subsequence the entire sequence must converge to the same limit. □

Lemma 4.22. *The limit* $\lim_{h\to 0^+} \nu[\beta,h] := \nu[\beta,0^+]$ *exists in the topology of weak local convergence. The measure* $\nu[\beta,0^+]$ *is a Gibbs measure with respect to the interaction* $U(\beta,0)$

Proof:
First we show convergence, again it suffices to show convergence of moments. Let $h_2 \geqslant h_1 \geqslant 0$. Now by Griffiths I and II we have the following inequality for any moment ϕ^A and any n:

$$\pi_{\Lambda_n,\beta,h_2}(\phi^A|0) \geqslant \pi_{\Lambda_n,\beta,h_1}(\phi^A|0) \geqslant 0$$

Taking the $n \to \infty$ limit we see that $\langle \phi^A \rangle_{\nu[\beta,h_2} \geqslant \langle \phi^A \rangle_{\nu[\beta,h_1]}$ so the quantity $\langle \phi^A \rangle_{\nu[\beta,h]}$ is monotone decreasing in h for $h \geqslant 0$ and bounded below by 0, this establishes convergence.

We note that by applying Lemma 4.18 with the choices $A_h = 1$ and $A_\beta = \beta$ we get a bound (4.18) which lets us prove statement (iii) of Lemma 4.19 for $\nu[\beta,0^+]$ via the same arguments used in Lemma 4.19.

We now turn to proving Gibbs consistency with respect to $U(\beta,0)$. Let $\{h_n\} \subset [0,1]$ be a sequence converging to 0 from above. To lighten notation for the proof we denote $\nu[\beta,h_n] := \nu_n$, and $\nu[\beta,0] := \nu$. We pick some $\Gamma = \Lambda_k$ for some k and pick $A \in \bar{\mathcal{B}}(\mathcal{L})$ which is of the form $A = A_1 \cap A_2$ where $A_1 \in \mathcal{B}(\Gamma)$ and $A_2 \in \bar{\mathcal{B}}(\Gamma^c)$. We want to show

$$\nu\left(\pi_{\Gamma,\beta,0}(A|\cdot)\right) = \nu(A).$$

We will prove that

$$\lim_{n\to\infty} \nu_n\left(\pi_{\Gamma,\beta,h_n}(A|\cdot)\right) = \nu\left(\pi_{\Gamma,\beta,0}(A|\cdot)\right). \tag{4.70}$$

Since ν_n is a Gibbs interaction with respect to the interaction $U(\beta,h_n)$ we can apply Gibbs consistency: $\lim_{n\to\infty} \nu_n\left(\pi_{\Gamma,\beta,h_n}(A|\cdot)\right) = \lim_{n\to\infty} \nu_n(A) = \nu(A)$, thus proving (4.70) suffices.

For any $n, j \in \mathbb{N}$ we have:

$$
\begin{aligned}
&\left|\nu_n\left(\pi_{\Gamma,\beta,h_n}(A|\cdot)\right) - \nu\left(\pi_{\Gamma,\beta,0}(A|\cdot)\right)\right| \\
=&\left|\int_{\mathbb{R}^{\mathcal{L}}} d\nu_n(S)\mathbb{1}_{A_2}(S)\pi_{\Gamma,\beta,h_n}(A_1|S) - \int_{\mathbb{R}^{\mathcal{L}}} d\nu(S)\mathbb{1}_{A_2}(S)\pi_{\Gamma,\beta,0}(A_1|S)\right| \\
\leqslant&\left|\int_{\mathbb{R}^{\mathcal{L}}} d\nu_n(S)\mathbb{1}_{A_2}(S)\left[\pi_{\Gamma,\beta,h_n}(A_1|S) - \pi_{\Gamma,\beta,h_n}(A_1|S(\Lambda_j))\right]\right| \\
&+\left|\int_{\mathbb{R}^{\mathcal{L}}} d\nu_n(S)\mathbb{1}_{A_2}(S)\pi_{\Gamma,\beta,h_n}(A_1|S(\Lambda_j)) - \int_{\mathbb{R}^{\mathcal{L}}} d\nu(S)\mathbb{1}_{A_2}(S)\pi_{\Gamma,\beta,0}(A_1|S(\Lambda_j))\right| \\
&+\left|\int_{\mathbb{R}^{\mathcal{L}}} d\nu(S)\mathbb{1}_{A_2}(S)\left[\pi_{\Gamma,\beta,0}(A_1|S(\Lambda_j)) - \pi_{\Gamma,\beta,0}(A_1|S)\right]\right|.
\end{aligned}
$$

Let $\epsilon > 0$. Since our superstability estimates are uniform in h one can find $m \in \mathbb{N}$ such that

$$\sup_n \nu_n(X_m^c) < \epsilon, \ \nu(X_m^c) < \epsilon.$$

Therefore one can use the same argument as used in Theorem 4.13 to bound the second and fourth lines uniformly in n for sufficently large j.

For fixed j the third line can be made small for large enough n, this follows from the fact that we have the pointwise (in S) convergence of the following (uniformly) local functions:

$$\lim_{n\to\infty} \mathbb{1}_{A_2}(S)\pi_{\Gamma,\beta,h_n}(A_1|S(\Lambda_j)) = \mathbb{1}_{A_2}(S)\pi_{\Gamma,\beta,0}(A_1|S(\Lambda_j))$$

The pointwise convergence above follows via a dominated convergence argument applied to integrals over $\mathbb{R}^{\Gamma}$.

Since the ν_n converge locally setwise to ν we can use the generalized dominated convergence theorem given below to make the third line arbitrarily small for large enough n. □

Theorem 4.16. *Let $(\Omega, \mathcal{B})$ be a measurable space and suppose that the measures λ_n converge setwise to the measure λ. Suppose furthermore that one has a sequence of measurable functions f_n on Ω such that functions f_n converge pointwise to a function f. Furthermore suppose that there exist a sequence of measurable functions g_n converging pointwise to a function g with $|f_n| \leqslant g_n$ for all n.*

Then if

$$\lim_{n\to\infty}\int_\Omega d\lambda_n \ g_n = \int_\Omega d\lambda \ g < \infty$$

One has

$$\lim_{n\to\infty}\int_\Omega d\lambda_n \ f_n = \int_\Omega d\lambda \ f$$

Proof:

See [57][Chapter 11, Section 4]. □

Lemma 4.23. *For any moment ϕ^A one has that*

$$\langle \phi^A \rangle_{\nu[\beta,0^+]} \geqslant \langle \phi^A \rangle_{\nu[\beta,0]}$$

This follows by the arguments made in the first part of Lemma 4.22. □

Definition:

The pressure in a finite volume Λ with deterministic boundary condition $S \in \bar{X}_1$ is defined as:

$$p(\Lambda, \beta, h, S) = \frac{1}{|\Lambda|} \log \left[\mathcal{Z}(\Lambda|S) \right]$$

The next theorem states that the infinite volume limit of the pressure is independent of boundary condition:

Theorem 4.17. *There is a convex function $p(\beta, h)$ such that for any $S \in \bar{X}_1$ one has:*

$$\lim_{k \to \infty} p(\Lambda_k, \beta, h, S) = p(\beta, h).$$

Proof: See [45, Lemma 2.6, Theorem 3.1]

The next lemmas set up a variational principle argument which will show equality of certain expectations across different Gibbs measures. The general approach of combining bounds uniform in volume and the choice of the boundary condition bounds with Lebesgue's dominated convergence theorem was something that we saw in [40].

Lemma 4.24. *For $\beta \in [0, A_\beta]$, $h \in [0, A_h]$ we have the following bounds:*

$$\int_{\mathbb{R}^{\Lambda_k}} d\pi_{\Lambda_k,\beta,h} \left(\phi_{\Lambda_k} | S \right) \Bigg| \sum_{\substack{x,y \in \Lambda_k \\ x \neq y}} J(x-y) \phi_x \phi_y \Bigg| \leqslant \|J\|_{L^1} \times \left[\mathbf{O}_3 A_\beta + 2n(S) \mathcal{O}_1 + \mathbf{O}_4 \right] |\Lambda_k|. \tag{4.71}$$

$$\int_{\mathbb{R}^{\Lambda_k}} d\pi_{\Lambda_k,\beta,h} \left(\phi_{\Lambda_k} | S \right) \Bigg| \sum_{\substack{x \in \Lambda_k \\ y \in \Lambda_k^c}} J(x-y) \phi_x S_y \Bigg| \leqslant \|J\|_{L^1} \times \left[\mathbf{O}_3 A_\beta + 3n(S) \mathcal{O}_1 + \mathbf{O}_4 \right] |\Lambda_k|. \tag{4.72}$$

Here we have set

$$\mathbf{O}_3 = \int_{\mathbb{R}} ds \ e^{-\frac{g}{4} s^4 + \delta} s^2.$$

$$\mathbf{O}_4 = \int_{\mathbb{R}} ds \ e^{-\frac{g}{4} s^4 + \delta} e^{\left(1 + \frac{A_\beta}{2} \|J\|_{L^1}\right) s^2}.$$

Proof: First note that

$$\sum_{\substack{x,y\in\Lambda_k\\ x\neq y}} |J(x-y)\phi_x\phi_y| \leqslant ||J|| \sum_{x\in\Lambda_k} \phi_x^2.$$

Now by Jensen's inequality

$$\begin{aligned}
\int_{\mathbb{R}^{\Lambda_k}} d\pi_{\Gamma,\beta,h}\left(\phi_{\Lambda_k}|S\right) \sum_{x\in\Lambda_k}\phi_x^2 \leqslant\ & \log\left(\int_{\mathbb{R}^{\Lambda_k}} d\pi_{\Lambda_k,\beta,h}\left(\phi_{\Lambda_k}|S\right) \exp\left[\sum_{x\in\Lambda_k}\phi_x^2\right]\right)\\
=\ & \log\left(\int_{\mathbb{R}^{\Lambda_k}} d\pi_{\Lambda_k,\beta,h}\left(\phi_{\Lambda_k}|0\right) \exp\left[\sum_{x\in\Lambda_k}\phi_x^2 - W\left(\phi_{\Lambda_k}|S_{\Lambda_k^c}\right)\right]\right)\\
& - \log\left(\int_{\mathbb{R}^{\Lambda_k}} d\pi_{\Lambda_k,\beta,h}\left(\phi_{\Lambda_k}|0\right) \exp\left[-W\left(\phi_{\Lambda_k}|S_{\Lambda_k^c}\right)\right]\right).
\end{aligned}$$

Using Jensen's inequality again to bound the last line of the estimate gives

$$\begin{aligned}
- \log\left(\int_{\mathbb{R}^{\Lambda_k}} d\pi_{\Lambda_k,\beta,h}\left(\phi_{\Lambda_k}|0\right) \exp\left[-W\left(\phi_{\Lambda_k}|S_{\Lambda_k^c}\right)\right]\right) &\leqslant \int_{\mathbb{R}^{\Lambda_k}} d\pi_{\Lambda_k,\beta,h}\left(\phi_{\Lambda_k}|0\right) W\left(\phi_{\Lambda_k}|S_{\Lambda_k^c}\right)\\
&\leqslant \int_{\mathbb{R}^{\Lambda_k}} d\pi_{\Lambda_k,\beta,h}\left(\phi_{\Lambda_k}|0\right) \sum_{\substack{x\in\Lambda_k\\ y\notin\Lambda_k}} \frac{1}{2}\bar{J}(x-y)\left(\phi_x^2+S_y^2\right)\\
&\leqslant \left(\mathbf{O}_3\|\bar{J}\|_{L^1} + n(S)\mathcal{O}_1\right)|\Lambda_k|.
\end{aligned}$$

Here we used the estimate

$$\sum_{\substack{x\in\Lambda_k\\ y\notin\Lambda_k}} \bar{J}(x-y)S_y^2 \leqslant n(S) \sum_{\substack{x\in\Lambda_k\\ y\notin\Lambda_k}} \mathcal{J}(\|y\|)\log_+(\|y\|) \leqslant n(S)\mathcal{O}_1.$$

The quantity $\mathcal{O}_1$ is defined in Sublemma 5.1. The other contribution satisfies the bound

$$\begin{aligned}
& \log\left(\int_{\mathbb{R}^{\Lambda_k}} d\pi_{\Lambda_k,\beta,h}\left(\phi_{\Lambda_k}|0\right) \exp\left[\sum_{x\in\Lambda_k}\phi_x^2 - W\left(\phi_{\Lambda_k}|S_{\Lambda_k^c}\right)\right]\right)\\
\leqslant\ & \log\left(\int_{\mathbb{R}^{\Lambda_k}} d\pi_{\Lambda_k,\beta,h}\left(\phi_{\Lambda_k}|0\right) \exp\left[\left(1+\frac{\|\bar{J}\|_{L_1}}{2}\right)\sum_{x\in\Lambda_k}\phi_x^2 + n(S)\mathcal{O}_1|\Lambda_k|\right]\right)\\
\leqslant\ & n(S)\mathcal{O}_1|\Lambda_k| + \log\left(\int_{\mathbb{R}^{\Lambda_k}} d\pi_{\Lambda_k,\beta,h}\left(\phi_{\Lambda_k}|0\right) \exp\left[\left(1+\frac{\|\bar{J}\|_{L^1}}{2}\right)\sum_{x\in\Lambda_k}\phi_x^2\right]\right)\\
\leqslant\ & n(S)\mathcal{O}_1|\Lambda_k| \ + \log\left(\mathbf{O}_4^{|\Lambda_k|}\right)
\end{aligned}$$

This proves statement (i) of the lemma.

For statement (ii) note that

$$\Big|\sum_{\substack{x\in\Lambda_k\\ y\in\Lambda_k^c}} J(x-y)\phi_x S_y\Big| \leqslant \frac{1}{2}\|J\|_{L^1}\left(\sum_{x\in\Lambda_k}\phi_x^2\right)+\frac{1}{2}n(S)\mathcal{O}_1|\Lambda_k|$$

We throw away the factors of $\frac{1}{2}$ and bound the integral of the first term just as we did for statement (i) - this proves statement (ii).

□

Lemma 4.25. *Suppose that at some $\beta_0 \in (0,\infty)$ one has that $p(\beta,0)$ is differentiable in β at $\beta=\beta_0$. Then for any distinct $x,y\in\mathbb{L}\backslash\{0\}$ such that $J(x-y)>0$ one has:*

$$\langle\phi_x\phi_y\rangle_{\nu[\beta_0,0]} = \langle\phi_x\phi_y\rangle_{\nu[\beta_0,0^+]}$$

Proof:

We start by choosing A_β large enough so that $\beta_0\in[0,A_\beta)$ and by choosing $A_h=1$.

Let $S\in X_\infty$. We have that the sequence of convex functions $p(\Lambda_k,\beta,0,S)$ (for any $S\in X_\infty$) and their pointwise limit $\lim_{k\to\infty} p(\Lambda_k,\beta,0,S)=p(\beta,h)$ are all differentiable in β at $\beta=\beta_0$. Then by standard facts about convex functions (see §I.3 in [64]) we immediately have the convergence of corresponding derivatives:

$$\lim_{k\to\infty}\frac{\partial p}{\partial\beta}(\Lambda_k,\beta_0,0,S)=\frac{\partial p}{\partial\beta}(\beta_0,0).$$

Note that the right hand side of the equation above does not depend on S.

We also have the bound

$$\begin{aligned}\left|\frac{\partial p}{\partial\beta}(\Lambda_k,\beta_0,0,S)\right| &\leqslant \frac{1}{|\Lambda_k|}\int_{\mathbb{R}^{\Lambda_k}} d\pi_{\Lambda_k,\beta,h}\left(\phi_{\Lambda_k}|S\right)\Big|\sum_{\substack{x,y\in\Lambda_k\\ x\neq y}} J(x-y)\phi_x\phi_y+\sum_{\substack{x\in\Lambda_k\\ y\in\Lambda_k^c}} J(x-y)\phi_x S_y\Big| \\ &\leqslant 2\times\|J\|_{L^1}\times\left[\mathbf{O}_3 A_\beta+3n(S)\mathcal{O}_1+\mathbf{O}_4\right].\end{aligned}$$

Above we have used statements (i) and (ii) of Lemma 4.24.

Define

$$\theta(S)=2\times\|J\|_{L^1}\left[\mathbf{O}_3 A_\beta+3n(S)\mathcal{O}_1+\mathbf{O}_4\right]$$

We note that $\theta(S)$ is integrable under any regular measure μ. In particular arguments similar to those used in the proof of statement (ii) of Lemma 4.19 show that

$$\int_{\mathbb{R}^{\mathcal{L}}} d\mu(S)n(S) \leqslant 1 + \sum_{k=1}^{\infty} \mu(\{X_k^c\})$$
$$\leqslant 1 + \mathbf{O}_2 \sum_{k=1}^{\infty} e^{-\frac{g}{8}k^2} < \infty.$$

Thus by Lebesgue Dominated Convergence theorem for any regular measure μ

$$\lim_{k\to\infty} \int_{\mathbb{R}^{\mathbb{L}}} d\mu(S)\ \frac{\partial p}{\partial \beta}(\Lambda_k, \beta_0, 0, S) = \int_{\mathbb{R}^{\mathbb{L}}} d\mu(S)\ \lim_{k\to\infty} \frac{\partial p}{\partial \beta}(\Lambda_k, \beta_0, 0, S)$$
$$= \int_{\mathbb{R}^{\mathbb{L}}} d\mu(S)\ \frac{\partial p}{\partial \beta}(\beta_0, 0) = \frac{\partial p}{\partial \beta}(\beta_0, 0).$$

It follows that for any two regular measures μ, μ'

$$\lim_{k\to\infty} \int_{\mathbb{R}^{\mathbb{L}}} d\mu(S)\ \frac{\partial p}{\partial \beta}(\Lambda_k, \beta_0, 0, S) = \lim_{k\to\infty} \int_{\mathbb{R}^{\mathbb{L}}} d\mu'(S)\ \frac{\partial p}{\partial \beta}(\Lambda_k, \beta_0, 0, S) \tag{4.73}$$

We now apply the above equality with the regular measures $\nu[\beta_0, 0]$ and $\nu[\beta_0, 0^+]$. Since $\nu[\beta_0, 0^+]$ is a Gibbs measure with respect to the interaction $U(\beta_0, 0)$ one has

$$\lim_{k\to\infty} \int_{\mathbb{R}^{\mathbb{L}}} d\nu[\beta_0, 0^+](S)\ \frac{\partial p}{\partial \beta}(\Lambda_k, \beta_0, 0, S)$$
$$= \lim_{k\to\infty} \frac{1}{|\Lambda_k|} \int_{\mathbb{R}^{\mathbb{L}}} d\nu[\beta_0, 0^+](S) \left[\int_{\mathbb{R}^{\Lambda_k}} d\pi_{\Lambda_k,\beta,h}\,(\phi_{\Lambda_k}|S) \left(\sum_{\substack{x,y\in\Lambda_k \\ x\neq y}} J(x-y)\phi_x\phi_y + \sum_{\substack{x\in\Lambda_k \\ y\notin\Lambda_k}} J(x-y)\phi_x S_y \right) \right]$$
$$= \frac{1}{|\Lambda_k|} \sum_{x\in\Lambda_k} \sum_{y\in\mathbb{L}\setminus\{x\}} J(x-y)\langle\phi_x\phi_y\rangle_{\nu[\beta_0,0^+]}$$
$$= \sum_{y\in\mathbb{L}\setminus\{0\}} J(y)\langle\phi_0\phi_y\rangle_{\nu[\beta_0,0^+]}.$$

In going to the third line we used Gibbs consistency and in going to the fourth line we used translation invariance. Note that the interchange of summation and integration is allowed since $\langle\phi_x\phi_0\rangle_{\nu[\beta_0,0^+]}$ is uniformly bounded in x by Lemma 4.14 and because $|J|$ is summable.

By identical arguments one has:

$$\lim_{k\to\infty} \int_{\mathbb{R}^{\mathbb{L}}} d\nu[\beta_0, 0](S)\ \frac{\partial p}{\partial \beta}(\Lambda_k, \beta_0, 0, S) = \sum_{y\in\mathbb{L}\setminus\{0\}} J(y)\langle\phi_0\phi_y\rangle_{\nu[\beta_0,0]}$$

Applying (4.73) then gives

$$\sum_{y\in\mathbb{L}\setminus\{0\}} J(x-y)\langle\phi_x\phi_y\rangle_{\nu[\beta_0,0]} = \sum_{y\in\mathbb{L}\setminus\{0\}} J(y)\langle\phi_0\phi_y\rangle_{\nu[\beta_0,0^+]} \tag{4.74}$$

Note that by Lemma 4.23 one has that for all $y \in \mathbb{L}$ the inequality $\langle\phi_0\phi_y\rangle_{\nu[\beta_0,0^+]} \geqslant \langle\phi_0\phi_y\rangle_{\nu[\beta_0,0]}$. However for $y \neq 0$ and $J(y) \neq 0$ then it is impossible for that inequality to be strict, otherwise (4.74) could not hold. Thus if $J(y) \neq 0$ then it follows that $\langle\phi_0\phi_y\rangle_{\nu[\beta_0,0^+]} = \langle\phi_0\phi_y\rangle_{\nu[\beta_0,0]}$. The lemma then follows by translation invariance of both measures and the interaction. □

Chapter 5

Appendix

5.1 Lemmas for Section 1.4

Theorem 5.1. *Let μ be a joint spectral measure of a vector $\psi \in \mathcal{H}$ on $\mathbb{R}^n$, that is μ is determined by the distribution function $F(\lambda_1, ..., \lambda_n) = (1, \prod_{j=1}^n P_{j,(-\infty,\lambda_i]}1)$ where the P_j are the commuting projection valued measures corresponding to self-adjoint operators $\hat{A}_j$.*

Then we have

$$\int_{\mathbb{R}^n} \mathrm{d}\mu(x) e^{i\vec{t}\cdot\vec{x}} = (\psi, \prod_{j=1}^{\infty} e^{i\hat{A}_j t_j}\psi)$$

Proof: Let $f(x)$ be a bounded continuous function on $\mathbb{R}$. We first prove that:

$$\lim_{m\to\infty} \sum_{i\in\mathbb{Z}} f\left(\frac{i}{m}\right) P_{j,[\frac{i}{m},\frac{i+1}{m})} = f(A_j)$$

were the convergence is in the strong operator topology on $\mathcal{H}$. This fact is part of the spectral theorem but we prove it again here. For any $h \in \mathcal{H}$ write $\nu_{j,h}$ for the spectral measure of h under A_j. We have that

$$\begin{aligned}
\lim_{m\to\infty}\left(\sum_{i\in\mathbb{Z}} f\left(\frac{i}{m}\right) P_{j,[\frac{i}{m},\frac{i+1}{m})}h, \sum_{i\in\mathbb{Z}} f\left(\frac{i}{m}\right) P_{j,[\frac{i}{m},\frac{i+1}{m})}h\right) &= \lim_{m\to\infty}\left(h, \sum_{i\in\mathbb{Z}} \bar{f}(\frac{i}{m}) f\left(\frac{i}{m}\right) P_{j,[\frac{i}{m},\frac{i+1}{n})}h\right) \\
&= \int_{\mathbb{R}} \bar{f}(x) f(x) d\nu_{j,v}(x) \text{ by Bounded Convergence Theorem} \\
&= \left(h, \bar{f}(A_j) f(A_j) h\right) \\
&= (f(A_j)h, f(A_j)h)\,.
\end{aligned}$$

Now we study the μ integral:

$$\begin{aligned}\int_{\mathbb{R}^n} \prod_{j=1}^{n} f_j(x_j) d\mu(\vec{x}) &= \lim_{m\to\infty} \int_{\mathbb{R}^n} \sum_{i_1,\ldots,i_n\in\mathbb{Z}} \prod_{j=1}^{n} f_j\left(\frac{i_j}{m}\right) d\mu(\vec{x}) \\ &= \lim_{m\to\infty} \left(\psi, \sum_{i_1,\ldots,i_n\in\mathbb{Z}} \prod_{j=1}^{n} f_j\left(\frac{i_j}{m}\right) P_{j,[\frac{i_j}{m},\frac{i_j+1}{m})}\psi\right)\end{aligned}$$

Exploiting the commutativity of the projection operators we get that (with everything in terms of strong operator topology limits):

$$\begin{aligned}\lim_{m\to\infty} \prod_{j=1}^{n} \sum_{i_1,\ldots,i_n\in\mathbb{Z}} f_j\left(\frac{i_j}{m}\right) P_{j,[\frac{i_j}{m},\frac{i_j+1}{m})} &= \lim_{m\to\infty} \prod_{j=1}^{n} \left[\sum_{i_j\mathbb{Z}} f_j\left(\frac{i_j}{m}\right) P_{j,[\frac{i_j}{m},\frac{i_j+1}{m})}\right] \\ &= \prod_{j=1}^{\infty} f_j(A_j)\end{aligned}$$

In the last line we used the fact that the multiplication of operators is jointly continous in the strong operator topology. Thus we have that

$$\lim_{m\to\infty} \left(\psi, \sum_{i_1,\ldots,i_n\in\mathbb{Z}} \prod_{j=1}^{n} f_j\left(\frac{i_j}{m}\right) P_{j,[\frac{i_j}{m},\frac{i_j+1}{m})}\psi\right) = (\psi, \prod_{j=1}^{\infty} f_j(A_j)\psi)$$

□

Theorem 5.2. *Let μ be a measure on $\mathbb{R}^n$ with moments given by $\{M_\alpha\}_{\alpha\in\mathbb{N}^d}$ Also suppose that there exists $C > 0$ such that for every $\alpha \in \mathbb{N}^d$*

$$|M_\alpha| \leqslant C^{|\alpha|}|\alpha|!$$

Then there exists $\delta > 0$ such that for all $t \in \mathbb{R}^d$ with $|t| \leqslant \delta$ one has

$$\int_{\mathbb{R}^d} \mathrm{d}\mu(x)\ e^{t\cdot x} < \infty$$

Proof: Let $t \in \mathbb{R}^n$ with $\max_{1\leqslant j\leqslant d} |t_j| \leqslant \sigma$. Then one has

$$\begin{aligned}
\int_{\mathbb{R}^d} \mathrm{d}\mu(x)\ e^{t\cdot x} &= \int_{\mathbb{R}^d} \mathrm{d}\mu(x)\ \left[\sum_{n=0}^{\infty} \frac{1}{n!}\left(\sum_{\substack{\alpha\in\mathbb{N}^d\\|\alpha|=n}} \binom{n}{\alpha} t^\alpha x^\alpha\right)\right] \\
&\leqslant \int_{\mathbb{R}^d} \mathrm{d}\mu(x)\ \left[\sum_{n=0}^{\infty} \frac{1}{n!}\left(\sum_{\substack{\alpha\in\mathbb{N}^d\\|\alpha|=n}} \binom{n}{\alpha} |t^\alpha| \times |x^\alpha|\right)\right] \\
&\leqslant \sum_{n=0}^{\infty} \frac{1}{n!} \times \sigma^n \left(\sum_{\substack{\alpha\in\mathbb{N}^d\\|\alpha|=n}} \binom{n}{\alpha} \int_{\mathbb{R}^d} \mathrm{d}\mu(x)\ |x^\alpha|\right) \\
&\leqslant \sum_{n=0}^{\infty} \frac{1}{n!}\sigma^n \left(\sum_{\substack{\alpha\in\mathbb{N}^d\\|\alpha|=n}} \binom{n}{\alpha} M_{2\alpha}^{\frac{1}{2}}\right) \\
&\leqslant \sum_{n=0}^{\infty} \frac{1}{n!}\sigma^n \times C^n \sqrt{(2n)!} \sum_{\substack{\alpha\in\mathbb{N}^d\\|\alpha|=n}} \binom{n}{\alpha} \\
&\leqslant \sum_{n=0}^{\infty} \frac{1}{n!}\sigma^n \times C^n \times \sqrt{(2n)!} \times d^n \ \leqslant \ \sum_{n=0}^{\infty} (\sigma \times C \times 2 \times d)^n
\end{aligned}$$

Clearly the last line is finite for sufficiently small σ. Note that above we used $\binom{n}{\alpha}$ to refer to a multinomial coefficient, in particular

$$\binom{n}{\alpha} = \frac{n!}{\prod_{j=1}^{d}(\alpha_j!)}.$$

□

Theorem 5.3. *Let μ and ν be measures on $\mathbb{R}^d$ with the same family of moments $\{M_\alpha\}_{\alpha\in\mathbb{N}^d}$. Furthermore suppose the two measures have moment generating functions that exist in a neighborhood of* 0*, that is there* $\exists\delta > 0$ *such that for all $t \in \mathbb{R}^d$ with $|t| \leqslant \delta$ one has that*

$$\int_{\mathbb{R}^d} \mathrm{d}\mu(x)\ e^{t\cdot x} < \infty$$

$$\int_{\mathbb{R}^d} \mathrm{d}\nu(x)\ e^{t\cdot x} < \infty$$

Then μ, ν have the same characteristic functions and hence coincide

Proof: For a d-dimensional random vector X a one dimensional marginal of X is a random variable of form $l \cdot X$ for $l \in \mathbb{R}^d$. We remark that the the law of a random vector is characterized by the collection of laws of its one dimensional marginals - this follows from the the fact that the law of a random vector is characterized by its characteristic function. Thus it suffices to prove that the one dimensional marginals of μ and ν agree, in particular we only need to prove the assertion of the theorem for the $d = 1$ so we specialize to this case.

We first prove that the characteristic functions $\theta_\mu(t), \theta_\nu(t)$ are actually analytic in a strip about the real axis. Let $t \in \mathbb{C}$ with $|\Im(t)| < \delta$. It follows that

$$\begin{aligned} \int_{\mathbb{R}} \mathrm{d}\mu(x) \left|e^{itx}\right| &= \int_{\mathbb{R}} \mathrm{d}\mu(x) \left|e^{i\Re(t)x}\right| \times \left|e^{\Im(t)x}\right| \\ &\leqslant \int_{\mathbb{R}} \mathrm{d}\mu(x) \left[e^{-\delta x} + e^{\delta x}\right] < \infty \end{aligned}$$

Since we have a uniform bound on this integral for all t in our strip we can apply a Fubini/Morrera argument to establish analyticity. Now that both $\theta_\mu(t)$ and $\theta_\nu(t)$ are analytic in the strip we use the fact that the family of moments $\{M_\alpha\}_{\alpha \in \mathbb{N}^1}$ are precisely all the derivatives of both characteristic functions evaluated at $t = 0$. Since all derivatives at zero match the two functions must coincide on the entire strip. □

5.2 Estimates on Covariances

This appendix gives needed estimates for Chapter 3. Here we prove some properties satisfied by the covariances C_0 and Γ for some fixed $L = p^l$ where l is an integer $l > 0$.

Lemma 5.1. *The covariance Γ can be expressed pointwise as follows.*

1. *If $|x| \leqslant 1$ then*
$$\Gamma(x) = \frac{1 - p^{-3}}{1 - p^{-2[\phi]}}(1 - L^{-2[\phi]}) \ .$$

2. *If $|x| = p^i$ with $1 \leqslant i \leqslant l$, then*
$$\Gamma(x) = -p^{-3+2[\phi]}p^{-2l[\phi]} + \frac{1 - p^{-3+2[\phi]}}{1 - p^{-2[\phi]}}(p^{-2i[\phi]} - p^{-2l[\phi]}) \ .$$

3. *If $|x| > L$ then $\Gamma(x) = 0$.*

Proof: Recall that
$$\Gamma(x) = \sum_{j=0}^{l-1} p^{-2j[\phi]} \left(\mathbb{1}_{\mathbb{Z}_p^3}(p^j x) - p^{-3}\mathbb{1}_{\mathbb{Z}_p^3}(p^{j+1}x)\right) \ .$$

By Abel summation, or discrete integration by parts, this can be rewritten as

$$\Gamma(x) = \mathbb{1}_{\mathbb{Z}_p^3}(x) - p^{-3-2(l-1)[\phi]}\mathbb{1}_{\mathbb{Z}_p^3}(p^l x) + \sum_{j=1}^{l-1} p^{-2j[\phi]}(1 - p^{-3+2[\phi]})\mathbb{1}_{\mathbb{Z}_p^3}(p^j x) \ . \tag{5.1}$$

Now we also have
$$\mathbb{1}_{\mathbb{Z}_p^3}(p^j x) = \mathbb{1}\{|p^j x| \leqslant 1\} = \mathbb{1}\{|x| \leqslant p^j\} = \sum_{i \leqslant j} \mathbb{1}\{|x| = p^i\} \ .$$

We insert the last expression into the sum in (5.1) and get after commuting the sums over i and j that

$$\Gamma(x) = \mathbb{1}_{\mathbb{Z}_p^3}(x) - p^{-3-2(l-1)[\phi]}\mathbb{1}_{\mathbb{Z}_p^3}(p^l x) + \sum_{i\in\mathbb{Z}} U_i \mathbb{1}\{|x| = p^i\}$$

where

$$U_i = \sum_{j\in\mathbb{Z}} \mathbb{1}\left\{ \begin{array}{c} 1 \leqslant j \leqslant l-1 \\ i \leqslant j \end{array} \right\} p^{-2j[\phi]} \ .$$

Now note that if $i \geqslant l$ then $U_i = 0$. Also, if $i \leqslant 0$ then

$$U_i = \frac{p^{-2[\phi]} - p^{-2l[\phi]}}{1 - p^{-2[\phi]}} \ .$$

Finally, if $1 \leqslant i \leqslant l-1$ then

$$U_i = \frac{p^{-2i[\phi]} - p^{-2l[\phi]}}{1 - p^{-2[\phi]}} \ .$$

As a result we have

$$\Gamma(x) = \mathbb{1}\{|x| \leqslant 1\} - p^{-3+2[\phi]}p^{-2l[\phi]}\mathbb{1}\{|x| \leqslant p^l\} + \frac{1 - p^{-3+2[\phi]}}{1 - p^{-2[\phi]}} \sum_{i\leqslant l-1} \mathbb{1}\{|x| = p^i\} \left(p^{-2[\phi]\max(i,1)} - p^{-2l[\phi]}\right)$$

from which the result follows by specialization to the different cases mentioned. □

As a result of the previous lemma we have a precise control over the sign of the function Γ.

Lemma 5.2.

1. *If* $|x| < p^l$ *then* $\Gamma(x) > 0$.
2. *If* $|x| = p^l$ *then* $\Gamma(x) < 0$.
3. *If* $|x| > p^l$ *then* $\Gamma(x) = 0$.

Proof: Recall that $\epsilon \in (0, 1]$ and therefore $[\phi] = \frac{3-\epsilon}{4} \in \left[\frac{1}{2}, \frac{3}{4}\right)$. We also have $l \geqslant 1$ and of course the prime number p is at least 2. From Lemma 5.1), we then readily get that $\Gamma(x) > 0$ if $|x| \leqslant 1$. The case $|x| > p^l$ has already been considered. For $|x| = p^l$ the formula in Lemma 5.1) reduces to $\Gamma(x) = -p^{-3+2[\phi]}p^{-2l[\phi]} < 0$. Finally when $|x| = p^i$, $2 \leqslant i \leqslant l-1$ then the formula in Lemma 5.1) shows that $\Gamma(x)$ decreases with i in that range. We only need to look at the case $i = l-1$ where one has

$$\Gamma(x) = p^{-2(l-1)[\phi]}\left[1 - p^{-3} - p^{-3+2[\phi]}\right] \ .$$

Simply using $p^{-3} \leqslant \frac{1}{8}$ and $3 - 2[\phi] > \frac{3}{2}$, which implies $p^{-3+2[\phi]} < 2^{-\frac{3}{2}}$, we get $1 - p^{-3} - p^{-3+2[\phi]} > 0$ and thus $\Gamma(x) > 0$. □

Corollary 5.1. *The fluctuation covariance satisfies the* L^1 *bound*

$$||\Gamma||_{L^1} < \frac{1}{\sqrt{2}} L^{3-2[\phi]} \ .$$

Proof: Indeed, by $\Gamma = C_0 - C_1$ and the definitions of the C_r covariances we have that $\int_{\mathbb{Q}_p^3} \mathrm{d}^3x\ \Gamma(x) = \widehat{\Gamma}(0) = 0$. In other words the positive part exactly cancels the negative part which is easy to compute since it only involves x's with $|x| = p^l$. Therefore

$$\begin{aligned} ||\Gamma||_{L^1} &= -2\int_{\mathbb{Q}_p^3} \mathrm{d}^3x\ \Gamma(x)\ \mathbb{1}\{|x| = p^l\} \\ &= 2(1-p^{-3})p^{-3+2[\phi]}L^{3-2[\phi]}\ . \end{aligned}$$

We use $1 - p^{-3} < 1$ and again $p^{-3+2[\phi]} < 2^{-\frac{3}{2}}$ to conclude. □

As for the unit cut-off covariance C_0, the following easy property will be useful in the sequel.

Lemma 5.3. *When $\epsilon \in (0,1]$, we have $1 < C_0(0) < 2$.*

Proof: Recall that

$$C_0(0) = \frac{1-p^{-3}}{1-p^{-2[\phi]}} = \frac{1-p^{-3}}{1-p^{-\left(\frac{3-\epsilon}{2}\right)}}\ .$$

Only using $p \geqslant 1$ and the given range for ϵ we get

$$p^{-\frac{3}{2}} \leqslant p^{-\left(\frac{3-\epsilon}{2}\right)} \leqslant p^{-1} \leqslant \frac{1}{2}\ .$$

Hence

$$1 < \frac{1-p^{-3}}{1-p^{-1}} \leqslant C_0(0) \leqslant \frac{1-p^{-3}}{1-p^{-\frac{3}{2}}} = 1 + p^{-\frac{3}{2}} < 2\ .$$

□

We will also need some information on the L^∞ and L^2 norms of Γ which are provided by the following two easy lemmas.

Lemma 5.4. *We have the simple estimate*

$$||\Gamma||_{L^\infty} \leqslant 2\ .$$

Proof: If $|x| \leqslant 1$, it follows from Lemmas 5.1 and 5.3 that $0 < \Gamma(x) < 2$. If $|x| > L$, then $\Gamma(x) = 0$. If $|x| = L$, then

$$|\Gamma(x)| = |-p^{-(3-2[\phi])}L^{-2[\phi]}| \leqslant 1\ .$$

Finally if $|x| = p^i$ with $1 \leqslant i \leqslant l-1$, then by Lemma 5.2

$$|\Gamma(x)| = \Gamma(x) = -p^{-3+2[\phi]}p^{-2l[\phi]} + \frac{1-p^{-3+2[\phi]}}{1-p^{-2[\phi]}}(p^{-2i[\phi]} - p^{-2l[\phi]})$$

$$\leqslant \frac{1-p^{-3+2[\phi]}}{1-p^{-2[\phi]}}(p^{-2i[\phi]} - p^{-2l[\phi]}) \leqslant \frac{1-p^{-3+2[\phi]}}{1-p^{-2[\phi]}} \leqslant \frac{1-p^{-3}}{1-p^{-2[\phi]}} = C_0(0) < 2\ .$$

This shows $|\Gamma(x)| \leqslant 2$ in all cases. □

Lemma 5.5. *We have*

$$\int_{\mathbb{Q}_p^3} |\Gamma(x)|^2 \ \mathrm{d}^3x = \frac{(1-p^{-3})(L^{\epsilon}-1)}{p^{\epsilon}-1} \longrightarrow (1-p^{-3}) \times l$$

when $\epsilon \to 0$*, with* l *defined by* $L = p^l$ *and the limit taken with* L *fixed.*

Proof: By the Plancherel formula over the p-adics

$$\int_{\mathbb{Q}_p^3} |\Gamma(x)|^2 \ \mathrm{d}^3x = \int_{\mathbb{Q}_p^3} |\widehat{\Gamma}(k)|^2 \ \mathrm{d}^3k \ .$$

But

$$\widehat{\Gamma}(k) = \widehat{C}_0(k) - \widehat{C}_1(k) = \frac{\mathbb{1}\{L^{-1} < |k| \leqslant 1\}}{|k|^{3-2[\phi]}}$$

and therefore

$$\begin{aligned}
\int_{\mathbb{Q}_p^3} |\Gamma(x)|^2 \ \mathrm{d}^3x &= \int_{\mathbb{Q}_p^3} \frac{\mathbb{1}\{L^{-1} < |k| \leqslant 1\}}{|k|^{6-4[\phi]}} \mathrm{d}^3k \\
&= \sum_{j=0}^{l-1} \int_{\mathbb{Q}_p^3} \frac{\mathbb{1}\{|k| = p^{-j}\}}{(p^{-j})^{6-4[\phi]}} \mathrm{d}^3k \\
&= \sum_{j=0}^{l-1} (1-p^{-3}) p^{-3j} p^{j(6-4[\phi])} \ .
\end{aligned}$$

The result follows since $3 - 4[\phi] = \epsilon$ and of course the $\epsilon \to 0$ limit is trivial. □

5.3 Properties of the Magnetization

5.3.1 Proof of Theorem 4.9

We first state two standard results of complex analysis:

Theorem 5.4 (Vitali - Porter Convergence Theorem)**.** *Let* $f_n(z)$ *be a sequence of analytic functions on a domain* $\Omega \subseteq \mathbb{C}$*. Suppose that this sequence is locally uniformly bounded (that is for any compact* $K \subset \Omega$ *one has* $\sup_{z \in K} \sup_n |f_n(z)| < \infty$ *). Also suppose that this sequence converges pointwise on some set* $E \subset \Omega$ *where* E *has an accumulation point in* Ω*. Then there exists an analytic function* f *on* Ω *such that the sequences* f_n *converges to* f *uniformly on any compact subset of* Ω*.*

Proof: See §2.4 of [60]. □

Theorem 5.5. *Let* $f_n(z)$ *be a sequence of analytic functions on a domain* $G \subseteq \mathbb{C}$ *such that the following hold: (i) for each* n *the function* $f_n(z)$ *is non-zero on* G*, (ii) the functions* $f_n(z)$ *converge uniformly on compact sets to a function* $f(z)$*. Then the function* $f(z)$ *is either non-zero on* G *or it completely vanishes.*

Proof: See §3.8 of [60]. □

Proof of Theorem 4.9

The fact that $M(g, b, \beta, h)$ is well-defined and finite comes from the the exponential integrability of the measure $\mu[\mathbb{L}, g, b, \beta, h]$ which follows from (4.14), the fact that $M^+(g, b, \beta)$ is well defined and finite comes from Lemma 4.22.

We now fix $g, b,$ and β. We will often suppress them from the notation. To prove statement (i) we first introduce some new notation, set

$$M_k(h) := M_k(g, b, \beta, h) = \langle \phi_0 \rangle_{\mu[\Lambda_k, g, b, \beta, h]}.$$

Note that $M_k(g, b, \beta, h)$ is concave as a consequence of (4.5) - in particular observe that

$$\frac{\partial^2}{\partial h^2} M_k(h) = \langle \phi_0, \phi_0, \phi_0 \rangle^T_{\mu[\Lambda_k, g, b, \beta, h]} \leqslant 0.$$

Statement (i) is now proved since the point-wise limit of concave functions is itself concave and

$$\lim_{k \to \infty} M_k(g, b, \beta, h) = M(g, b, \beta, h).$$

We now move on to statement (ii). Define $\Omega = \{z \in \mathbb{C} | \ \Re(z) > 0\}$. For $h \in \Omega$ define:

$$\mathcal{Z}_{\Lambda_k}(h) := \mathcal{Z}_{\Lambda_k}(g, b, \beta, h) := \int_{\mathbb{R}^{\Lambda_k}} \exp\left[\frac{\beta}{2} \sum_{\substack{x, y \in \Lambda_k \\ x \neq y}} J(x - y)\phi_x \phi_y\right] \exp\left[- \sum_{x \in \Lambda_k} \left(g\phi_x^4 + b\phi_x^2 - h\phi_x\right)\right] d\phi_{\Lambda_k}.$$

Note that by Theorem 4.6 one has that $\mathcal{Z}_{\Lambda_k}(h) \neq 0$ for $h \in \Omega$. We use the notation $\mathcal{Z}'_{\Lambda_k}(h)$ to denote the derivative in h of $\mathcal{Z}_{\Lambda_k}(h)$. We now define the pressure corresponding to a volume Λ_k and external field $h \in \Omega$:

$$p_{\Lambda_k}(h) := \frac{1}{|\Lambda_k|}\left[\mathrm{Log}\mathcal{Z}_{\Lambda_k}(1) + \int_1^h \frac{\mathcal{Z}'_{\Lambda_k}(h)}{\mathcal{Z}_{\Lambda_k}(h)} dz\right].$$

Above the integral is taken over any path in Ω connecting 1 and h and Log is the principal branch of the complex logarithm. Since Ω is simply connected and the integrand above is analytic on Ω the choice of path doesn't matter. Additionally $p_{\Lambda_k}(h)$ is analytic on Ω.

This definition satisfies $\exp\left[|\Lambda_k| p_{\Lambda_k}(z)\right] = \mathcal{Z}_{\Lambda_k}(h)$. It is also agrees with the standard definition $p_{\Lambda_k}(h) = \frac{1}{|\Lambda_k|} \log\left(\mathcal{Z}_{\Lambda_k}(h)\right)$ for $h \in \Omega \cap \mathbb{R}$.

We define functions on $f_k : \Omega \mapsto \mathbb{C}$ via

$$f_k(h) := \exp\left[p_{\Lambda_k}(h)\right].$$

By Theorem 4.17 we have the pointwise convergence $\lim_{k \to \infty} p_{\Lambda_k}(h) = p(h)$ for $h \in \Omega \cap \mathbb{R}$ and so the functions

$f_k(h)$ converge pointwise to a function $f(h) = e^{p(h)}$ for $h \in \Omega \cap \mathbb{R}$.

We now show that the functions $f_k(h)$ are locally uniformly bounded. First observe that

$$|f_k(h)| = \exp\left[\Re\left(p_{\Lambda_k}(h)\right)\right] \leqslant \exp\left[p_{\Lambda_k}(|h|)\right].$$

To see why the last inequality is valid we first note that $|\mathcal{Z}_{\Lambda_k}(h)| \leqslant \mathcal{Z}_{\Lambda_k}(|h|)$ which means

$$\Re(p_{\Lambda_k}(h)) = \frac{1}{|\Lambda_k|}\log\left(|\mathcal{Z}_{\Lambda_k}(h)|\right) \leqslant \frac{1}{|\Lambda_k|}\log\left(\mathcal{Z}_{\Lambda_k}(|h|)\right) = p_{\Lambda_k}(|h|).$$

Thus to show that the $f_n(h)$ are locally uniformly bounded it suffices to show that for any $K \subset \Omega$ compact one has $\sup_k \sup_{h \in K} p_{\Lambda_k}(|h|) < \infty$.

Set $h_K = \sup_{h \in K}|h|$. Then one has $\sup_{h \in K} p_{\Lambda_k}(|h|) = p_{\Lambda_k}(h_K)$. This follows from the fact that $p_{\Lambda_k}(h)$ restricted to the non-negative real line is increasing in h (this last fact itself follows from noticing that for such h one has $\frac{\partial}{\partial h}p_{\Lambda_k}(h) = M_k(h) \geqslant 0$ where the last inequality is a consequence of Griffiths I). Since $\lim_{k\to\infty} p_{\Lambda_k}(h_K) = p(h_K)$ we have that $\sup_k \sup_{h \in K} p_{\Lambda_k}(|h|) = \sup_k p_{\Lambda_k}(h_k) < \infty$. Thus the $f_k(h)$ are locally uniformly bounded on Ω.

It then follows by the Vitali-Porter convergence theorem that the $f_k(h)$ converge uniformly on compact sets of Ω to an analytic extension of $f(h)$. Since all the $f_k(h)$ are non-vanishing we can use Theorem 5.5 to infer that $f(h)$ is non-vanishing as well (note that we already knew $f(h)$ is non-zero on $\Omega \cap \mathbb{R}$ by Theorem 4.17). In particular we can define an analytic extension of $p(h)$ to all of Ω by setting

$$p(h) = \log\left(f(1)\right) + \int_1^h \frac{f'(z)}{f(z)}dz$$

Above $f'(h) = \frac{\partial}{\partial h}f(h)$ and the integral is along any path within Ω connecting 1 and h. Now statement (ii) will be proved if we show that for $h \in \Omega \cap \mathbb{R}$ one has $M(h) = \frac{\partial}{\partial h}p(h)$. Note that for such h one has that $p(h)$ is convex, so by general facts about convex functions we have the point-wise convergence $\frac{\partial}{\partial h}p_{\Lambda_k}(h) \to \frac{\partial}{\partial h}p(h)$. However we have that $\frac{\partial}{\partial h}p_{\Lambda_k}(h) = M_k(h)$ (we remark that we are using translation invariance here). Statement (ii) is then proved. □

5.3.2 Proof of Lemma 4.3

We again sometimes suppress dependence on g and b.

$$\lim_{n\to\infty}\lim_{h\to 0^+}\frac{M_n(\beta,h)}{h} = \lim_{n\to\infty}\frac{\partial}{\partial h}M_n(\beta,h)\Big|_{h=0}$$
$$= \lim_{n\to\infty}\sum_{x\in\Lambda_n}\langle\phi_0\phi_x\rangle_{\mu[\Lambda_n,g,b,\beta,0]}$$
$$= \sum_{x\in\mathbb{L}}\langle\phi_0\phi_x\rangle_{\nu[\mathbb{L},g,b,\beta,0]}$$

On the second to last line the individual integrals are non-negative and increasing in n so the convergence to the last line is monotone.

Now by previous arguments (see Theorem 4.9) we have that $M_n(\beta,h)$ is concave for $h \geqslant 0$. This means that $\frac{M_n(\beta,h)}{h}$ is increasing as we take h down to 0. On the other hand we have that $M_n(\beta,h)$ is increasing in n for $h \geqslant 0$. We can then interchange limits to see that:

$$\lim_{h\to 0^+}\frac{M(\beta,h)}{h} = \lim_{h\to 0^+}\lim_{n\to\infty}\frac{M_n(\beta,h)}{h} = \lim_{n\to\infty}\lim_{h\to 0^+}\frac{M_n(\beta,h)}{h} = \sum_{x\in\mathbb{L}}\langle\phi_0\phi_x\rangle_{\nu[\mathbb{L},g,b,\beta,0]}$$

□

5.4 Additional Proofs for Section 4.7

5.4.1 Proof of Lemma 4.18

This proof is technical. We will define constants so that the estimates can be used for proving both Lemmas 4.18 and 4.14. We are following [58], [59], and [45] with some differences and simplifications coming from working in the ultrametric setting. After proving some sublemmas we will give a description of what the approach for the main bound will be.

We begin by introducing some notation.

Recall that for any non-negative integer j we set $\Lambda_j := \{x \in \mathbb{L} \mid |x| \leqslant p^j\}$.

We define a decreasing function $\mathcal{J} : \{0, p, p^2, p^3, ...\} \to (0,\infty)$ as follows: for $k \in \mathbb{N}$ we set $\mathcal{J} = A_\beta \Psi$, recall that Ψ is defined in Theorem 4.8.

We also define: $\mathcal{J}_{q,k} := \sup_{\substack{x\in\Lambda_{q+1}\\ y\notin\Lambda_{q+k+1}}} \mathcal{J}(\|x-y\|) = \mathcal{J}(p^{q+k+2})$.

Let $\psi(t) = 2 \times \log_+(t) := 2 \times \max(\log(t), 1)$ (we could set $\psi(t) = b\log_+(t)$ for any $b > 1$). We use the notation $\psi_j := \psi(p^j)$. We list some properties of $\psi(t)$:

$$\psi \geqslant 1,\ \lim_{t\to\infty}\psi(t) = \infty,$$
$$\sum_{x\in\mathbb{L}}\mathcal{J}(\|x\|)\psi(\|x\|) < \infty. \tag{5.2}$$

The sub-lemmas involved in the proof of Lemma 4.18 will involve a certain decomposition of field configurations which we now describe. For each $\phi \in \bar{X}_1$ one has that

$$\exists\ N \in \mathbb{N} \text{ such that } \forall\ r > N,\ \sum_{x\in\Lambda_r} \phi_x^2 < \psi_r|\Lambda_r|.$$

For any $z \in \mathbb{L}$, one can take r sufficiently large so that one has $z + \Lambda_r = \Lambda_r$. Therefore the following condition also holds for $\phi \in \bar{X}_1$:

$$\forall z \in \mathbb{L}, \exists\ N \in \mathbb{N} \text{ such that } \forall\ r > N,\ \sum_{x\in(z+\Lambda_r)} \phi_x^2 < \psi_r|\Lambda_r|.$$

For any $\phi \in \bar{X}_1$ and $z \in \mathbb{L}$ we define $q_\phi^z \in \mathbb{N}$ as follows:

$$q_\phi^z \text{ is the largest non-negative integer } q \text{ for which } \sum_{x\in(z+\Lambda_q)} \phi_x^2 \geqslant \psi_q|\Lambda_q|. \tag{5.3}$$

If $\sum_{x\in(z+\Lambda_q)} \phi_x^2 < \psi_q|\Lambda_q|$ for all non-negative integers q then we set $q_\phi^z = -1$.

For any $q \in \mathbb{N} \cup \{-1\}$ we define $\hat{R}_q^z$ to be the set of those $\phi \in \bar{X}_1$ for which $q_\phi^z = q$. We note that for any fixed z the family of sets $\{\hat{R}_q^z\}_{q=-1}^{\infty}$ form a partition of $\bar{X}_1$.

Sub-Lemma 5.1. *For any q and for any $\hat{\phi} \in \hat{R}_q^z$ one has:*

$$\sum_{x\notin(z+\Lambda_m)} \mathcal{J}(||z-x||)\hat{\phi}_x^2 \leqslant \mathcal{O}_1 \text{ for any non-negative integer} m \geqslant q \tag{5.4}$$

For any $A \subseteq (z+\Lambda_q)$ and $B \subseteq (z+\Lambda_q)^c$

$$\left|W(\hat{\phi}_A|\hat{\phi}_B)\right| \leqslant \frac{1}{2}\sum_{x\in A}\left(\|\bar{J}\|_{L^1}\hat{\phi}_x^2 + \mathcal{O}_1\right) \tag{5.5}$$

Where we have set $\mathcal{O}_1 := \dfrac{1}{1-p^{-d}}\sum_{x\in\mathbb{L}} \mathcal{J}(\|x\|)\psi(\|x\|)$.

Proof:

We start by proving (5.4). Note that

$$
\begin{aligned}
\sum_{x\notin(z+\Lambda_q)} \mathcal{J}(\|z-x\|)\hat{\phi}_x^2 =& \sum_{j=q+1}^{\infty} \mathcal{J}(\|z-x\|)\left[\sum_{x\in(z+\Lambda_j)\setminus(z+\Lambda_{j-1})} \mathcal{J}(\|z-x\|)\hat{\phi}_x^2\right] \\
=& \sum_{j=q+1}^{\infty} \mathcal{J}(p^j)\left[\sum_{x\in(z+\Lambda_j)\setminus(z+\Lambda_{j-1})} \hat{\phi}_x^2\right] \\
\leqslant& \sum_{j=q+1}^{\infty} \mathcal{J}(p^j)\left[\sum_{x\in(z+\Lambda_j)} \hat{\phi}_x^2\right] \\
<& \sum_{j=q+1}^{\infty} \mathcal{J}(p^j)\psi_j|\Lambda_j| \\
=& \frac{1}{1-p^{-d}} \sum_{j=q+1}^{\infty} \mathcal{J}(p^j)\psi_j|\Lambda_j\setminus\Lambda_{j-1}| \\
\leqslant& \frac{1}{1-p^{-d}} \sum_{x\in\mathbb{L}} \mathcal{J}(\|x\|)\psi(\|x\|)
\end{aligned}
$$

In the strict inequality above we used condition (5.3). This proves (5.4).

For (5.5) we note that

$$
\begin{aligned}
\left|W(\hat{\phi}_A|\hat{\phi}_B)\right| \leqslant& \sum_{x\in A}\sum_{y\notin(z+\Lambda_q)} \bar{J}(x-y)|\hat{\phi}_x\hat{\phi}_y| \\
\leqslant& \frac{1}{2}\sum_{x\in A}\sum_{y\notin(z+\Lambda_q)} \bar{J}(x-y)\hat{\phi}_x^2 + \frac{1}{2}\sum_{x\in A}\sum_{y\notin(z+\Lambda_q)} \mathcal{J}(\|x-y\|)\hat{\phi}_y^2.
\end{aligned}
$$

Now for the first sum note that

$$
\sum_{x\in A}\sum_{y\in\Lambda_k^c} \bar{J}(x-y)\phi_x^2 \leqslant ||\bar{J}||_{L^1}\sum_{x\in A}\phi_x^2.
$$

For the second sum we have

$$
\begin{aligned}
\sum_{x\in A}\sum_{y\notin(z+\Lambda_q)} \mathcal{J}(\|x-y\|)\hat{\phi}_y^2 =& \sum_{x\in A}\sum_{y\notin(z+\Lambda_q)} \mathcal{J}(\|z-y\|)\hat{\phi}_y^2 \\
\leqslant& |A|\sum_{y\notin(z+\Lambda_q)} \mathcal{J}(\|z-y\|)\hat{\phi}_y^2 \\
\leqslant& |A|\mathcal{O}_1
\end{aligned}
$$

In the manipulations above the first equality used ultrametricity: since $x, z \in (z+\Lambda_k)$ and $y \notin (z+\Lambda_q)$ one has $\|x-y\| = \|z-y\|$. In going to the last line we used (5.4). We have now proved (5.5). $\square$.

We now describe the general approach for establishing a pointwise bound for:

$$\rho^{\Gamma}_{\Lambda_k}(\phi_\Gamma|S) := \frac{1}{\mathcal{Z}(\Lambda_k|S)} \int\limits_{\mathbb{R}^{\Lambda_k\setminus\Gamma}} d\phi_{\Lambda_k\setminus\Gamma} \exp\left[-U(\phi_{\Lambda_k}) - W\left(\phi_{\Lambda_k}|S_{\Lambda_k^c}\right)\right]$$

We are establishing a pointwise bound in ϕ_Γ for fixed S - it is important to keep in mind that many of our sets of partial spin configurations we define below depend on $\phi_\Gamma \in \mathbb{R}^\Gamma$ so it is helpful to think of ϕ_Γ as being fixed as well.

First fix $z \in \Gamma$. We then define a partition of $\mathbb{R}^{\Lambda_k\setminus\Gamma}$ which we denote $\{R^z_q\}_{q=0}^{\infty}$. This partition is dependent on both ϕ_Γ and $S_{\Lambda_k^c}$ but we will sometimes suppress this from the notation.

For any $\phi_{\Lambda_k\setminus\Gamma} \in \mathbb{R}^{\Lambda_k\setminus\Gamma}$ we define the corresponding corresponding full lattice field configuration

$$\hat{\phi} = \phi_\Gamma \wedge \phi_{\Lambda_k\setminus\Gamma} \wedge S_{\Lambda_k^c}$$

.

Note that $\hat{\phi} \in \bar{X}_1$

We now define the sets of our partition: for $q \in \mathbb{N} \cup \{-1\}$; we set $R^z_q(\phi_\Gamma, \Lambda_k, S) := \{\phi_{\Lambda_k\setminus\Gamma} \in \mathbb{R}^{\Lambda_k\setminus\Gamma} \mid \hat{\phi} \in \hat{R}^z_q\}$.

To prove our estimate we first decompose

$$\rho^{\Gamma}_{\Lambda_k}(\phi_\Gamma|S) = \frac{1}{\mathcal{Z}(\Lambda_k|S)} \sum_{q=-1}^{\infty} \left[\int\limits_{R^z_q(\phi_\Gamma,\Lambda_k,S)} d\phi_{\Lambda_k\setminus\Gamma} \exp\left[-U(\phi_{\Lambda_k}) - W(\phi_{\Lambda_k}|S_{\Lambda_k^c})\right] \right]$$

We give one more sublemma before beginning to tackle the main bound.

Sub-Lemma 5.2. *For any non-negative integer k let $A \subseteq \Lambda_k$ and suppose that S satisfies one of two conditions: (a) $S \in X_1$ or (b) for some $z \in A$ one has $S(\Lambda_k^c) \in \hat{R}^z_n$ for some $n \leqslant k$.*

(i)

$$\left|W(\phi_A|S_{\Lambda_k^c})\right| \leqslant \frac{1}{2}||\bar{J}|| \sum_{x\in A} \left(\phi_x^2 + \mathcal{O}_1\right)$$

(ii) There exists $\lambda \in (0,1]$ and a a bounded Borel set $\Sigma \subseteq \mathbb{R}$ such that one has:

$$\int\limits_{\Sigma^{|A|}} d\phi_A \ \exp\left[-U(\phi_A) - W(\phi_A|S_{\Lambda_k^c})\right] \geqslant \lambda^{|A|}$$

Proof:

To prove statement (i) we first note that both conditions (a) and (b) imply that

$$\sum_{x\in\Lambda_j} S(\Lambda_k^c)^2_x < \psi_j|\Lambda_j| \text{ for all } j > k \tag{5.6}$$

For condition (b) this follows from ultrametricity: $z + \Lambda_j = \Lambda_j$ for $j > k$ since $z \in \Lambda_k$. For condition (a) we note that $\psi_j = \psi(p^j) = 2 \times \log_+(p^j)$. Thus if S satifies the bound $S_x^2 \leqslant \log_+(\|x\|)$ then S certainly satisfies (5.6). Therefore we can just assume (5.6). We now start bounding the expression of interest:

$$
\begin{aligned}
|W(\phi_A|S_{\Lambda_k^c})| \leqslant& \frac{1}{2} \sum_{\substack{x\in A \\ y\notin\Lambda_k}} \bar{J}(x-y)(\phi_x^2 + S_y^2) \\
\leqslant& \frac{1}{2}\|\bar{J}\|_{L^1} \sum_{x\in A} \phi_x^2 + \frac{1}{2} \sum_{x\in A} \sum_{y\notin\Lambda_k} \mathcal{J}(\|x-y\|) S_y^2
\end{aligned}
$$

Now for the second term we have

$$
\begin{aligned}
\sum_{x\in A} \sum_{y\notin\Lambda_k} \mathcal{J}(\|x-y\|) S_y^2 =& \sum_{x\in A} \sum_{y\notin\Lambda_k} \mathcal{J}(\|y\|) S_y^2 \\
=& |A| \sum_{y\notin\Lambda_k} \mathcal{J}(\|y\|) S_y^2 \\
=& |A| \sum_{y\notin\Lambda_k} \mathcal{J}(\|y\|) S(\Lambda_k^c)_y^2 \\
\leqslant& |A| \sum_{j=k+1}^{\infty} \mathcal{J}(p^{k+1}) \left[\sum_{y\in\Lambda_{k+1}} S(\Lambda_k^c)_y^2 \right] \\
<& |A| \sum_{j=k+1}^{\infty} \mathcal{J}(p^{k+1}) \psi_{k+1} |\Lambda_{k+1}| \leqslant |A| \mathcal{O}_1
\end{aligned}
$$

The final bound is by the same argument as used in the proof of (5.4). This proves statement (i). We now prove statement (ii). Using statement (i) and the definition of U we have the bound:

$$
\begin{aligned}
U(\phi_k) + W(\phi_k|S_{\Lambda_n^c}) \leqslant& \sum_{x\in A} \left[g\phi_x^4 + |b|\phi_x^2 + \frac{1}{2}||\bar{J}||\phi_x^2 + \mathcal{O}_1 \right] + \frac{1}{2} \sum_{\substack{x,y\in A \\ x\neq y}} \bar{J}(x-y)|\phi_x\phi_y| \\
\leqslant& \sum_{x\in A} \left[g\phi_x^4 + |b|\phi_x^2 + A_\beta \|J\|_{L^1} \phi_x^2 + \mathcal{O}_1 \right] \\
\leqslant& \sum_{x\in A} f(\phi_x), \text{ where we have defined: } f(s) = gs^4 + (|b| + A_\beta \|J\|_{L^1})\, s^2 + \mathcal{O}_1
\end{aligned}
$$

In going to the second line we used the bound :

$$
\frac{1}{2} \sum_{\substack{x,y\in A \\ x\neq y}} \bar{J}(x-y)|\phi_x\phi_y| \leqslant \frac{1}{4} \sum_{\substack{x,y\in A \\ x\neq y}} \bar{J}(x-y) \left(\phi_x^2 + \phi_y^2\right) \leqslant \frac{A_\beta}{2} \|J\|_{L^1} \left(\sum_{x\in A} \phi_x^2 \right).
$$

The claim now follows from noticing that one can pick appropriate λ and bounded Borel Σ such that:

$$\int_{\Sigma} ds\ e^{-f(s)} \geqslant \lambda$$

Along with the fact that:

$$\int_{\Sigma^{|A|}} d\phi_A\ \exp\left[-U(\phi_k) - W(\phi_k|S_{\Lambda_k^c})\right] \geqslant \int_{\Sigma^{|A|}} d\phi_A\ \exp\left[-\sum_{x\in A} f(\phi_x)\right] = \left(\int_{\Sigma} ds\ e^{-f(s)}\right)^{|A|}$$

. □

In the bounds of the next three sub-lemmas we take $\rho^{\varnothing}_{\Lambda_k}(\phi_{\varnothing}|S) := 1$.

Sub-Lemma 5.3. *Let k be a non-negative integer. Suppose that $z \in \Gamma \subseteq \Lambda_k$ and S satisfies at least one of two conditions: (a) $S \in X_1$ or (b) One has $S(\Lambda_k^c) \in \hat{R}^z_n$ for some $n \leqslant k$.*

$$\frac{1}{\mathcal{Z}(\Lambda_k|S)} \int_{R^z_{-1}(\phi_\Gamma,\Lambda_k,S)} d\phi_{\Lambda_k\setminus\Gamma} \exp\left[-U(\phi_{\Lambda_k}) - W\left(\phi_{\Lambda_k}|S_{\Lambda_k^c}\right)\right]$$
$$\leqslant \exp\left[-\frac{g}{2}\phi_z^4 + A_\beta\|J\|_{L^1}\phi_z^2 + \mathcal{O}_3\right] \times \rho^{\Gamma\setminus\{z\}}_{\Lambda_k}(\phi_{\Gamma\setminus\{z\}}|S)$$

where we set

$$\mathcal{O}_2 := \sup_{s\in\Sigma}\left(\exp\left[\frac{A_\beta}{2}||J||_{L^1}s^2\right]\right).$$

$$\mathcal{O}_3 := -\log(\lambda) + \log(\mathcal{O}_2) + \frac{3}{2}\mathcal{O}_1 + \mathbf{O}_1.$$

Proof:

We manipulate the integrand appearing above and insert a dummy variable $\tilde{\phi}_z$.

$$\begin{aligned}
&\int_{R^z_{-1}(\phi_\Gamma,\Lambda_k,S)} d\phi_{\Lambda_k\setminus\Gamma} \exp\left[-U(\phi_{\Lambda_k}) - W(\phi_{\Lambda_k}|S_{\Lambda_k^c})\right] \\
=&\int_{R^z_{-1}(\phi_\Gamma,\Lambda_k,S)} d\phi_{\Lambda_k\setminus\Gamma} \exp\left[-U(\phi_{\Lambda_k\setminus\{z\}}) - U(\phi_z) - W(\phi_{\Lambda_k\setminus\{z\}}|S_{\Lambda_k^c}) - W(\phi_z|S_{\Lambda_k^c})\right] \\
=&\int_{R^z_{-1}(\phi_\Gamma,\Lambda_k,S)} d\phi_{\Lambda_k\setminus\Gamma} \exp\left[-U(\phi_{\Lambda_k\setminus\{z\}}) - W(\tilde{\phi}_z|\phi_{\Lambda_k}) - W(\phi_{\Lambda_k\setminus\{z\}}|S_{\Lambda_k^c})\right] \\
&\qquad\times \exp\left[-U(\phi_z) - W(\phi_z|S_{\Lambda_k^c}) + W(\tilde{\phi}_z|\phi_{\Lambda_k})\right]
\end{aligned} \tag{5.7}$$

Now by Lemma 4.17 we have

$$\exp\left[-U(\phi_z)\right] \leqslant \exp\left[-\frac{g}{2}\phi_z^4 + \mathbf{O}_1\right]$$

Now observe that as a consequence of $\phi_{\Lambda_k\setminus\Gamma} \in R^z_{-1}(\phi_\Gamma, \Lambda_k, S)$ we have

$$\begin{aligned}\left|W(\tilde{\phi}_z|\phi_{\Lambda_k\setminus\{z\}})\right| \leqslant & \frac{1}{2}\sum_{y\in\Lambda_k\setminus\{z\}} \bar{J}(y-z)\left(\tilde{\phi}_z^2 + \hat{\phi}_y^2\right) \\ \leqslant & \frac{1}{2}||\bar{J}||_{L^1}\tilde{\phi}_z^2 + \frac{1}{2}\sum_{y\in\Lambda_k\setminus\{z\}} \mathcal{J}(\|y-z\|)\hat{\phi}_y^2 \\ \leqslant & \frac{1}{2}\left(||\bar{J}||_{L^1}\tilde{\phi}_z^2 + \mathcal{O}_1\right)\end{aligned}$$

To bound the second sum on the second line we can proceed just as we did in the proof of (5.3) - one has that $\sum_{z+\Lambda_j}\hat{\phi}_y^2 < \psi_j|\Lambda_j|$ for all $j \geqslant 0$ since $\hat{\phi} \in \hat{R}^z_{-1}$

An identical argument gives

$$\left|W(\phi_z|\phi_{\Lambda\setminus\{z\}})\right| \leqslant \frac{1}{2}\left(||\bar{J}||_{L^1}\phi_z^2 + \mathcal{O}_1\right)$$

Sub-lemma 5.2 gives us that

$$|W(\phi_z|S^c_\Lambda)| \leqslant \frac{1}{2}\left(\|\bar{J}\|_{L^1}\phi_z^2 + \mathcal{O}_1\right)$$

Inserting this into our previous expression we have the following inequality valid for all $\tilde{\phi}_z \in \mathbb{R}$:

$$\begin{aligned}&\int\limits_{R^z_{-1}(\phi_\Gamma,\Lambda_k,S)} d\phi_{\Lambda_k\setminus\Gamma}\exp\left[-U(\phi_{\Lambda_k}) - W(\phi_{\Lambda_k}|S_{\Lambda_k^c})\right] \\ \leqslant\ & \exp\left[-\frac{g}{2}\phi_z^4 + \|\bar{J}\|_{L^1}\phi_z^2 + \mathbf{O}_1 + \frac{3}{2}\mathcal{O}_1\right] \\ & \times \int\limits_{R^z_{-1}(\phi_\Gamma,\Lambda_k,S)} \exp\left[-U(\phi_{\Lambda_k\setminus\{z\}}) - W(\tilde{\phi}_z|\phi_{\Lambda\setminus z}) - W\left(\phi_{\Lambda\setminus\{z\}}|S_{\Lambda^c}\right) + \frac{1}{2}\|\bar{J}\|_{L^1}\tilde{\phi}_z^2\right]\end{aligned} \tag{5.8}$$

Now we integrate both sides of the bound with respect to $\tilde{\phi}_z$ weighted by the probability measure

$$\frac{\int_\Sigma d\tilde{\phi}_z\cdots\exp\left[-U(\tilde{\phi}_z) - W(\tilde{\phi}_z|S_{\Lambda_k^c})\right]}{\int_\Sigma d\tilde{\phi}_z\exp\left[-U(\tilde{\phi}_z) - W(\tilde{\phi}_z|S_{\Lambda_k^c})\right]}$$

Below we introduce some new notation: ϕ^*_Λ to denote $\phi_{\Lambda_k\setminus\{z\}} \wedge \phi_z$

$$
\begin{aligned}
&\exp\left[-\frac{g}{2}\phi_z^4 + \|\bar{J}\|_{L^1}\phi_z^2 + \mathbf{O}_1 + \frac{3}{2}\mathcal{O}_1\right]\left(\int_\Sigma d\tilde{\phi}_z \exp\left[-U(\tilde{\phi}_z) - W(\tilde{\phi}_z|S_{\Lambda_k^c})\right]\right)^{-1} \\
&\quad\times \int_\Sigma d\tilde{\phi}_z \int_{R^z_{-1}(\phi_\Gamma,\Lambda_k,S)} d\phi_{\Lambda_k\setminus\Gamma} \exp\left[-U(\tilde{\phi}_z) - W(\tilde{\phi}_z|S_{\Lambda_k^c})\right] \\
&\qquad\qquad \times \exp\left[-U(\phi_{\Lambda_k\setminus\{z\}}) - W(\tilde{\phi}_z|\phi_{\Lambda_k\setminus\{z\}}) - W\left(\phi_{\Lambda_k\setminus\{z\}}|S_{\Lambda_k^c}\right) + \frac{1}{2}\|\bar{J}\|_{L^1}\tilde{\phi}_z^2\right] \\
=\ &\exp\left[-\frac{g}{2}\phi_z^4 + \|\bar{J}\|_{L^1}\phi_z^2 + \mathbf{O}_1 + \frac{3}{2}\mathcal{O}_1\right]\left(\int_\Sigma d\tilde{\phi}_z \exp\left[-U(\tilde{\phi}_z) - W(\tilde{\phi}_z|S_{\Lambda_k^c})\right]\right)^{-1} \\
&\quad\times \int_\Sigma d\tilde{\phi}_z \int_{R^z_{-1}(\phi_\Gamma,\Lambda_k,S)} d\phi_{\Lambda_k\setminus\Gamma} \exp\left[-U(\phi^*_{\Lambda_k}) - W\left(\phi^*_{\Lambda_k}|S_{\Lambda_k^c}\right)\right] \exp\left[\frac{1}{2}\|\bar{J}\|_{L^1}\tilde{\phi}_z^2\right] \\
\leqslant\ &\lambda^{-1} \times \sup_{s\in\Sigma}\left(\exp\left[\frac{1}{2}||\bar{J}||_{L^1}s^2\right]\right) \times \exp\left[-\frac{g}{2}\phi_z^4 + \|\bar{J}\|_{L^1}\phi_z^2 + \mathbf{O}_1 + \frac{3}{2}\mathcal{O}_1\right] \\
&\quad\times \int_\Sigma d\tilde{\phi}_z \int_{R^z_{-1}(\phi_\Gamma,\Lambda_k,S)} d\phi_{\Lambda_k\setminus\Gamma} \exp\left[-U(\phi^*_{\Lambda_k}) - W(\phi^*_{\Lambda_k}|S^c_{\Lambda_k})\right]
\end{aligned}
$$

Note that in the last inequality above we use statement (ii) of Sub-lemma 5.2. We then have the bound:

$$
\begin{aligned}
&\frac{1}{\mathcal{Z}(\Lambda_k|S)} \int_{R^z_{-1}(\phi_\Gamma,\Lambda_k,S)} d\phi_{\Lambda_k\setminus\Gamma} \exp\left[-U(\phi_{\Lambda_k}) - W\left(\phi_{\Lambda_k}|S_{\Lambda_k^c}\right)\right] \\
\leqslant\ &\lambda^{-1} \times \mathcal{O}_2 \times \exp\left[-\frac{g}{2}\phi_z^4 + \|\bar{J}\|_{L^1}\phi_z^2 + \mathbf{O}_1 + \frac{3}{2}\mathcal{O}_1\right] \\
&\qquad \times \frac{1}{\mathcal{Z}(\Lambda_k|S)} \int_\Sigma d\tilde{\phi}_z \int_{R^z_{-1}(\phi_\Gamma,\Lambda_k,S)} d\phi_{\Lambda_k\setminus\Gamma} \exp\left[-U(\phi^*_{\Lambda_k}) - W(\phi^*_{\Lambda_k}|S^c_{\Lambda_k})\right] \\
\leqslant\ &\lambda^{-1} \times \mathcal{O}_2 \times \exp\left[-\frac{g}{2}\phi_z^4 + \|\bar{J}\|_{L^1}\phi_z^2 + \mathbf{O}_1 + \frac{3}{2}\mathcal{O}_1\right] \\
&\qquad \times \frac{1}{\mathcal{Z}(\Lambda_k|S)} \int_{\mathbb{R}} d\tilde{\phi}_z \int_{\mathbb{R}^{\Lambda_k\setminus\Gamma}} d\phi_{\Lambda_k\setminus\Gamma} \exp\left[-U(\phi^*_{\Lambda_k}) - W(\phi^*_{\Lambda_k}|S^c_{\Lambda_k})\right] \\
\leqslant\ &\lambda^{-1} \times \mathcal{O}_2 \times \exp\left[-\frac{g}{2}\phi_z^4 + \|\bar{J}\|_{L^1}\phi_z^2 + \mathbf{O}_1 + \frac{3}{2}\mathcal{O}_1\right] \times \rho^{\Gamma\setminus\{z\}}_{\Lambda_k}(\phi_{\Gamma\setminus\{z\}}|S)
\end{aligned}
$$

This proves the sublemma.

□

Sub-Lemma 5.4. *Suppose that $q \geqslant 0$. Let k be a non-negative integer. Suppose that $\Gamma \subseteq \Lambda_k$ and S satisfies at least one of two conditions: (a) $S \in X_1$ or (b) $S(\Lambda_k^c) \in \hat{R}^z_n$ for some $n \leqslant k$. Also suppose that $z + \Lambda_q \subseteq \Lambda_k$. Then one has the following bound:*

$$\frac{1}{\mathcal{Z}(\Lambda_k|S)} \int\limits_{R_q^z(\phi_\Gamma,\Lambda_k,S)} d\phi_{\Lambda_k\setminus\Gamma} \exp\left[-U(\phi_{\Lambda_k}) - W\left(\phi_{\Lambda_k}|S_{\Lambda_k^c}\right)\right]$$

$$\leqslant \exp\left[\left(\mathcal{O}_5 - \psi_q\right)|\Lambda_q|\right] \exp\left[\sum_{x\in(z+\Lambda_q)\cap\Gamma} \left(-\frac{g}{2}\phi_x^4 + \left(A_\beta\|J\|_{L^1}+1\right)\phi_x^2 + \mathcal{O}_3\right)\right] \rho_{\Lambda_k}^{\Gamma\setminus(z+\Lambda_q)}(\phi_{\Gamma\setminus(z+\Lambda_q)}|S)$$

where we have set

$$\mathcal{O}_4 := \max\left(\int\limits_{\mathbb{R}} ds\ \exp\left[-\frac{g}{2}s^4 + \left(A_\beta\|J\|_{L^1}+1\right)s^2 + \frac{3}{2}\mathcal{O}_1 + \mathbf{O}_1\right], 1\right)$$

$$\mathcal{O}_5 := -\log(\lambda) + \log(\mathcal{O}_4) + \log(\mathcal{O}_2)$$

Proof:

We start by introducing a vector of dummy variables $\tilde{\phi}_{\Lambda_k\setminus(z+\Lambda_q)}$ into the expression we're trying to bound.

$$\int\limits_{R_q^z(\phi_\Gamma,\Lambda_k,S)} d\phi_{\Lambda_k\setminus\Gamma} \exp\left[-U(\phi_{\Lambda_k}) - W\left(\phi_{\Lambda_k}|S_{\Lambda_k^c}\right)\right]$$

$$= \int\limits_{R_q^z(\phi_\Gamma,\Lambda_k,S)} d\phi_{\Lambda_k\setminus\Gamma} \exp\Big[-U(\phi_{\Lambda_k\setminus(z+\Lambda_q)}) - U(\phi_{(z+\Lambda_q)}) - W(\phi_{(z+\Lambda_q)}|\phi_{\Lambda_k\setminus(z+\Lambda_q)})$$

$$- W(\phi_{\Lambda_k\setminus(z+\Lambda_q)}|S_{\Lambda_k^c}) - W(\phi_{(z+\Lambda_q)}|S_{\Lambda_k^c})\Big]$$

$$= \int\limits_{R_q^z(\phi_\Gamma,\Lambda_k,S)} d\phi_{\Lambda_k\setminus\Gamma} \exp\left[-U(\phi_{\Lambda_k\setminus(z+\Lambda_q)}) - W(\tilde{\phi}_{(z+\Lambda_q)}|\phi_{\Lambda_k\setminus(z+\Lambda_q)}) - W(\phi_{\Lambda_k\setminus(z+\Lambda_q)}|S_{\Lambda_k^c})\right]$$

$$\times \exp\left[-U(\phi_{(z+\Lambda_q)})\right] \exp\left[-W(\phi_{(z+\Lambda_q)}|\phi_{\Lambda_k\setminus(z+\Lambda_q)}) - W(\phi_{(z+\Lambda_q)}|S_{\Lambda_k^c}) + W(\tilde{\phi}_{(z+\Lambda_q)}|\phi_{\Lambda_k\setminus(z+\Lambda_q)})\right]$$

Once again by 4.17 we have that

$$\exp\left[-U(\phi_{(z+\Lambda_q)})\right] \leqslant \exp\left[\sum_{x\in(z+\Lambda_q)} \left(-\frac{g}{2}\phi_x^4 + \mathbf{O}_1\right)\right]$$

Since we enforce that $\phi^{\Lambda_k\setminus\Gamma} \in R_q^z(\phi_\Gamma, \Lambda_k, S)$ we have that the corresponding $\hat{\phi}$ lies in $\hat{R}_q$. We then get the following bound via (5.5):

$$\left|W(\phi_{(z+\Lambda_q)}|\phi_{\Lambda_k\setminus(z+\Lambda_q)})\right| = \left|W(\hat{\phi}_{(z+\Lambda_q)}|\hat{\phi}_{\Lambda_k\setminus(z+\Lambda_q)})\right|$$
$$\leqslant \frac{1}{2}\sum_{x\in(z+\Lambda_k)} \left(\|\bar{J}\|_{L^1}\phi_x^2 + \mathcal{O}_1\right).$$

By Sub-lemma 5.2 we have:

$$\left|W(\phi_{(z+\Lambda_q)}|S_{\Lambda_k^c})\right| \leqslant \frac{1}{2}\sum_{x\in(z+\Lambda_q)}\left(\|\bar{J}\|_{L^1}\phi_x^2+\mathcal{O}_1\right).$$

We also have

$$\begin{aligned}\left|W(\tilde{\phi}_{(z+\Lambda_q)}|\phi_{\Lambda_k\setminus(z+\Lambda_q)})\right| \leqslant & \frac{1}{2}\sum_{x\in(z+\Lambda_q)}\sum_{y\notin(z+\Lambda_q)}\bar{J}(x-y)\left(\tilde{\phi}_x^2+\hat{\phi}_y^2\right)\\ \leqslant & \frac{1}{2}\sum_{x\in(z+\Lambda_q)}\|\bar{J}\|_{L^1}\tilde{\phi}_x^2+\frac{1}{2}\sum_{x\in(z+\Lambda_q)}\sum_{y\notin(z+\Lambda_q)}\mathcal{J}(\|x-y\|)\\ = & \frac{1}{2}\sum_{x\in(z+\Lambda_q)}\|\bar{J}\|_{L^1}\tilde{\phi}_x^2+\frac{1}{2}\sum_{x\in(z+\Lambda_q)}\sum_{y\notin(z+\Lambda_q)}\mathcal{J}(\|z-y\|)\\ \leqslant & \frac{1}{2}\sum_{x\in(z+\Lambda_q)}\left(\|\bar{J}\|_{L^1}+\mathcal{O}_1\right)\end{aligned}$$

In going to the second line we used ultrametricity to note that $\|x-y\|=\|z-y\|$. In going to the last line we used (5.3) and the fact that $\hat{\phi}\in\hat{R}^z_{q+1}$.

With these bounds in mind we have the following inequality which is valid for any $\tilde{\phi}_{(z+\Lambda_q)}\in\mathbb{R}^{(z+\Lambda_k)}$.

$$\begin{aligned}&\int_{R_q^z(\phi_\Gamma,\Lambda_k,S)} d\phi_{\Lambda_k\setminus\Gamma}\exp\left[-U(\phi_{\Lambda_k})-W\left(\phi_{\Lambda_k}|S_{\Lambda_k^c}\right)\right]\\ \leqslant & \int_{R_q^z(\phi_\Gamma,\Lambda_k,S)} d\phi_{\Lambda_k\setminus\Gamma}\exp\left[-U(\phi_{\Lambda_k\setminus(z+\Lambda_q)})-W(\tilde{\phi}_{(z+\Lambda_q)}|\phi_{\Lambda_k\setminus(z+\Lambda_q)})-W(\phi_{\Lambda_k\setminus(z+\Lambda_q)}|S_{\Lambda_k^c})\right]\\ &\qquad\times\exp\left[\sum_{x\in(z+\Lambda_q)}\left(-\frac{g}{2}\phi_x^4+\|\bar{J}\|_{L^1}\phi_x^2+\frac{1}{2}\|\bar{J}\|_{L^1}\tilde{\phi}_x^2+\frac{3}{2}\mathcal{O}_1+\mathbf{O}_1\right)\right]\\ \leqslant & \exp\left[-\psi_q|\Lambda_q|\right]\int_{R_q^z(\phi_\Gamma,\Lambda_k,S)} d\phi_{\Lambda_k\setminus\Gamma}\exp\left[-U(\phi_{\Lambda_k\setminus(z+\Lambda_q)})-W(\tilde{\phi}_{(z+\Lambda_q)}|\phi_{\Lambda_k\setminus(z+\Lambda_q)})-W(\phi_{\Lambda_k\setminus(z+\Lambda_q)}|S_{\Lambda_k^c})\right]\\ &\qquad\times\exp\left[\sum_{x\in(z+\Lambda_q)}\left(-\frac{g}{2}\phi_x^4+\left(\|\bar{J}\|_{L^1}+1\right)\phi_x^2+\frac{1}{2}\|\bar{J}\|_{L^1}\tilde{\phi}_x^2+\frac{3}{2}\mathcal{O}_1+\mathbf{O}_1\right)\right]\end{aligned} \tag{5.9}$$

In going to the last line we used that within our above integral the requirement that $\phi_\Lambda\in R_q^z(\phi_\Gamma,\Lambda_k,S)$ means that

$$\exp\left[\sum_{x\in(z+\Lambda_q)}\phi_x^2\right]\exp\left[-\psi_q|\Lambda_q|\right]\geqslant 1$$

We now enlarge the domain of integration on the last line of (5.9) and continue to make more estimates:

$$
\begin{aligned}
&\int\limits_{R_q^z(\phi_\Gamma,\Lambda_k,S)} d\phi_{\Lambda_k\setminus\Gamma} \exp\left[-U(\phi_{\Lambda_k}) - W\left(\phi_{\Lambda_k}|S_{\Lambda_k^c}\right)\right] \\
\leqslant\ & \exp\left[-\psi_q|\Lambda_q|\right] \int\limits_{\mathbb{R}^{\Lambda_k\setminus\Gamma}} d\phi_{\Lambda_k\setminus\Gamma} \exp\left[-U(\phi_{\Lambda_k\setminus(z+\Lambda_q)}) - W(\tilde{\phi}_{(z+\Lambda_q)}|\phi_{\Lambda_k\setminus(z+\Lambda_q)}) - W(\phi_{\Lambda_k\setminus(z+\Lambda_q)}|S_{\Lambda_k^c})\right] \\
&\qquad \times \exp\left[\sum_{x\in(z+\Lambda_q)}\left(-\frac{g}{2}\phi_x^4 + \left(\|\bar{J}\|_{L^1}+1\right)\phi_x^2 + \frac{1}{2}\|\bar{J}\|_{L^1}\tilde{\phi}_x^2 + \frac{3}{2}\mathcal{O}_1 + \mathbf{O}_1\right)\right] \\
=\ & \exp\left[-\psi_q|\Lambda_q|\right] \exp\left[\sum_{x\in(z+\Lambda_q)\cap\Gamma}\left(-\frac{g}{2}\phi_x^4 + \left(\|\bar{J}\|_{L^1}+1\right)\phi_x^2 + \frac{3}{2}\mathcal{O}_1 + \mathbf{O}_1\right)\right] \\
&\times \left(\int\limits_{\mathbb{R}^{(z+\Lambda_q)\setminus\Gamma}} d\phi_{(z+\Lambda_q)\setminus\Gamma} \exp\left[\sum_{x\in(z+\Lambda_q)\setminus\Gamma}\left(-\frac{g}{2}\phi_x^4 + \left(\|\bar{J}\|_{L^1}+1\right)\phi_x^2 + \frac{3}{2}\mathcal{O}_1 + \mathbf{O}_1\right)\right]\right) \\
&\times \exp\left[\sum_{x\in(z+\Lambda_q)}\frac{1}{2}\|\bar{J}\|_{L^1}\tilde{\phi}_x^2\right] \int\limits_{\mathbb{R}^{\Lambda_k\setminus((z+\Lambda_q)\cup\Gamma)}} d\phi_{\Lambda_k\setminus((z+\Lambda_q)\cup\Gamma)} \exp\Bigg[-U(\phi_{\Lambda_k\setminus(z+\Lambda_q)}) - W(\tilde{\phi}_{(z+\Lambda_q)}|\phi_{\Lambda_k\setminus(z+\Lambda_q)}) \\
&\qquad\qquad - W(\phi_{\Lambda_k\setminus(z+\Lambda_q)}|S_{\Lambda_k^c})\Bigg] \\
\leqslant\ & \exp\left[-\psi_q|\Lambda_q|\right] \times \mathcal{O}_4^{|(z+\Lambda_q)\setminus\Gamma|} \times \exp\left[\sum_{x\in(z+\Lambda_q)\cap\Gamma}\left(-\frac{g}{2}\phi_x^4 + \left(\|\bar{J}\|_{L^1}+1\right)\phi_x^2 + \frac{3}{2}\mathcal{O}_1 + \mathbf{O}_1\right)\right] \\
&\times \exp\left[\sum_{x\in(z+\Lambda_q)}\frac{1}{2}\|\bar{J}\|_{L^1}\tilde{\phi}_x^2\right] \int\limits_{\mathbb{R}^{\Lambda_k\setminus((z+\Lambda_q)\cup\Gamma)}} d\phi_{\Lambda_k\setminus((z+\Lambda_q)\cup\Gamma)} \exp\Bigg[-U(\phi_{\Lambda_k\setminus(z+\Lambda_q)}) - W(\tilde{\phi}_{(z+\Lambda_q)}|\phi_{\Lambda_k\setminus(z+\Lambda_q)}) \\
&\qquad\qquad - W(\phi_{\Lambda_k\setminus(z+\Lambda_q)}|S_{\Lambda_k^c})\Bigg]
\end{aligned}
\tag{5.10}
$$

We will integrate the bound of (5.10) with respect to $\tilde{\phi}_{(z+\Lambda_k)}$ weighted by the probability measure

$$
\frac{\int_{\Sigma^{(z+\Lambda_q)}} d\tilde{\phi}_{(z+\Lambda_q)} \cdots \exp\left[-U(\tilde{\phi}_{(z+\Lambda_q)}) - W(\tilde{\phi}_{(z+\Lambda_q)}|S_{\Lambda_k^c})\right]}{\int_{\Sigma^{(z+\Lambda_q)}} d\tilde{\phi}_{(z+\Lambda_q)} \exp\left[-U(\tilde{\phi}_{(z+\Lambda_q)}) - W(\tilde{\phi}_{(z+\Lambda_q)}|S_{\Lambda_k^c})\right]}
$$

To keep expressions shorter we just work with the very last line of (5.10).

$$
\begin{aligned}
&\left(\int_{\Sigma^{(z+\Lambda_q)}} d\tilde{\phi}_{(z+\Lambda_q)} \exp\left[-U(\tilde{\phi}_{(z+\Lambda_q)}) - W(\tilde{\phi}_{(z+\Lambda_q)}|S_{\Lambda_k^c})\right]\right)^{-1} \\
&\times \int_{\Sigma^{(z+\Lambda_q)}} d\tilde{\phi}_{(z+\Lambda_q)} \int_{\mathbb{R}^{\Lambda_k\setminus((z+\Lambda_q)\cup\Gamma)}} d\phi_{\Lambda_k\setminus((z+\Lambda_q)\cup\Gamma)} \exp\left[-U(\tilde{\phi}_{(z+\Lambda_q)}) - W(\tilde{\phi}_{(z+\Lambda_q)}|S_{\Lambda_k^c})\right] \\
&\qquad \times \exp\left[\sum_{x\in(z+\Lambda_q)} \frac{1}{2}\|\bar{J}\|_{L^1}\tilde{\phi}_x^2\right] \exp\left[-U(\phi_{\Lambda_k\setminus(z+\Lambda_q)}) - W(\tilde{\phi}_{(z+\Lambda_q)}|\phi_{\Lambda_k\setminus(z+\Lambda_q)}) - W(\phi_{\Lambda_k\setminus(z+\Lambda_q)}|S_{\Lambda_k^c})\right] \\
&\leqslant \lambda^{-|\Lambda_q|} \times \left(\sup_{s\in\Sigma}\left\{\exp\left[\frac{1}{2}||\bar{J}||_{L^1}s^2\right]\right\}\right)^{|\Lambda_q|} \\
&\qquad \times \int_{\Sigma^{(z+\Lambda_q)}} d\tilde{\phi}_{(z+\Lambda_q)} \int_{\mathbb{R}^{\Lambda_k\setminus((z+\Lambda_q)\cup\Gamma)}} d\phi_{\Lambda_k\setminus((z+\Lambda_q)\cup\Gamma)} \exp\left[-U(\tilde{\phi}_{(z+\Lambda_q)}) - W(\tilde{\phi}_{(z+\Lambda_q)}|S_{\Lambda_k^c})\right] \\
&\qquad\qquad \times \exp\left[-U(\phi_{\Lambda_k\setminus(z+\Lambda_q)}) - W(\tilde{\phi}_{(z+\Lambda_q)}|\phi_{\Lambda_k\setminus(z+\Lambda_q)}) - W(\phi_{\Lambda_k\setminus(z+\Lambda_q)}|S_{\Lambda_k^c})\right] \\
&\leqslant \lambda^{-|\Lambda_q|} \times \mathcal{O}_2^{|\Lambda_q|} \times \int_{\Sigma^{(z+\Lambda_q)}} d\tilde{\phi}_{(z+\Lambda_q)} \int_{\mathbb{R}^{\Lambda_k\setminus((z+\Lambda_q)\cup\Gamma)}} d\phi_{\Lambda_k\setminus((z+\Lambda_q)\cup\Gamma)} \exp\left[-U(\phi^*_{\Lambda_k}) - W(\phi^*_{\Lambda_k}|S_{\Lambda_k^c})\right]
\end{aligned} \tag{5.11}
$$

Above we use the notation $\phi^*_{\Lambda_k} = \phi_{\Lambda_k\setminus(z+\Lambda_q)} \wedge \tilde{\phi}_{(z+\Lambda_q)}$. We end up with the following bound:

$$
\begin{aligned}
&\frac{1}{\mathcal{Z}(\Lambda_k|S)} \int_{R_q^z(\phi_\Gamma,\Lambda_k,S)} d\phi_{\Lambda_k\setminus\Gamma} \exp\left[-U(\phi_{\Lambda_k}) - W\left(\phi_{\Lambda_k}|S_{\Lambda_k^c}\right)\right] \\
&\leqslant \exp\left[-\psi_q|\Lambda_q|\right] \times \mathcal{O}_4^{|(z+\Lambda_q)\setminus\Gamma|} \times \lambda^{-|\Lambda_q|} \times \mathcal{O}_2^{|\Lambda_q|} \times \exp\left[\sum_{x\in(z+\Lambda_q)\cap\Gamma}\left(-\frac{g}{2}\phi_x^4 + \left(\|\bar{J}\|_{L^1}+1\right)\phi_x^2 + \frac{3}{2}\mathcal{O}_1 + \mathbf{O}_1\right)\right] \\
&\qquad \times \frac{1}{\mathcal{Z}(\Lambda_k|S)} \int_{\Sigma^{(z+\Lambda_q)}} d\tilde{\phi}_{(z+\Lambda_q)} \int_{\mathbb{R}^{\Lambda_k\setminus((z+\Lambda_q)\cup\Gamma)}} d\phi_{\Lambda_k\setminus((z+\Lambda_q)\cup\Gamma)} \exp\left[-U(\phi^*_{\Lambda_k}) - W(\phi^*_{\Lambda_k}|S_{\Lambda_k^c})\right] \\
&\leqslant \exp\left[-\psi_q|\Lambda_q|\right] \times \mathcal{O}_4^{|\Lambda_q|} \times \lambda^{-|\Lambda_q|} \times \mathcal{O}_2^{|\Lambda_q|} \times \exp\left[\sum_{x\in(z+\Lambda_q)\cap\Gamma}\left(-\frac{g}{2}\phi_x^4 + \left(\|\bar{J}\|_{L^1}+1\right)\phi_x^2 + \frac{3}{2}\mathcal{O}_1 + \mathbf{O}_1\right)\right] \\
&\qquad \times \rho_{\Lambda_k}^{\Gamma\setminus(z+\Lambda_q)}(\phi_{\Gamma\setminus(z+\Lambda_q)}|S).
\end{aligned}
$$

This finishes the proof of the lemma. □

The next lemma handles the case where $(z+\Lambda_q) \nsubseteq \Lambda_k$ - we will then have to require $S \in X_1$

Sub-Lemma 5.5. *Suppose that $q \geqslant 0$. Let k be a non-negative integer. Suppose that $\Gamma \subseteq \Lambda_k$ and that $S \in X_1$. Also suppose that $(z+\Lambda_q) \nsubseteq \Lambda_k$. Then one has:*

$$\frac{1}{\mathcal{Z}(\Lambda_k|S)} \int\limits_{R_q^z(\phi_\Gamma,\Lambda_k,S)} d\phi_{\Lambda_k\backslash\Gamma} \exp\left[-U(\phi_{\Lambda_k}) - W\left(\phi_{\Lambda_k}|S_{\Lambda_k^c}\right)\right]$$

$$\leqslant \exp\left[\left(\mathcal{O}_5 - \frac{1}{2}\psi_q\right)|\Lambda_q|\right] \exp\left[\sum_{x\in(z+\Lambda_q)\cap\Gamma}\left(-\frac{g}{2}\phi_x^4 + (A_\beta\|J\|_{L^1}+1)\,\phi_x^2 + \mathcal{O}_3\right)\right] \rho_{\Lambda_k}^{\Gamma\backslash(z+\Lambda_q)}(\phi_{\Gamma\backslash(z+\Lambda_q)}|S)$$

Proof:

We start by introducing dummy variables $\tilde{\phi}_{\Lambda_k\cap(z+\Lambda_q)}$. We then have

$$\begin{aligned}
&\int\limits_{R_q^z(\phi_\Gamma,\Lambda_k,S)} d\phi_{\Lambda_k\backslash\Gamma} \exp\left[-U(\phi_{\Lambda_k}) - W\left(\phi_{\Lambda_k}|S_{\Lambda_k^c}\right)\right] \\
&= \int\limits_{R_q^z(\phi_\Gamma,\Lambda_k,S)} d\phi_{\Lambda_k\backslash\Gamma} \exp\left[-U(\phi_{\Lambda_k\backslash(z+\Lambda_q)}) - W(\tilde{\phi}_{\Lambda_k\cap(z+\Lambda_q)}|\phi_{\Lambda_k\backslash(z+\Lambda_q)}) - W(\phi_{\Lambda_k\backslash(z+\Lambda_q)}|S_{\Lambda_k^c})\right] \\
&\qquad \times \exp\left[-U(\phi_{\Lambda_k\cap(z+\Lambda_q)})\right] \\
&\qquad \times \exp\left[-W(\phi_{\Lambda_k\cap(z+\Lambda_q)}|\phi_{\Lambda_k\backslash(z+\Lambda_q)}) - W(\phi_{\Lambda_k\cap(z+\Lambda_q)}|S_{\Lambda_k^c}) + W(\tilde{\phi}_{\Lambda_k\cap(z+\Lambda_q)}|\phi_{\Lambda_k\backslash(z+\Lambda_q)})\right]
\end{aligned}$$

By arguments analagous to those used in Sub-lemma 5.4 to get to the first bound of (5.9) we have that

$$\begin{aligned}
&\int\limits_{R_q^z(\phi_\Gamma,\Lambda_k,S)} d\phi_{\Lambda_k\backslash\Gamma} \exp\left[-U(\phi_{\Lambda_k}) - W\left(\phi_{\Lambda_k}|S_{\Lambda_k^c}\right)\right] \\
&\leqslant \int\limits_{R_q^z(\phi_\Gamma,\Lambda_k,S)} d\phi_{\Lambda_k\backslash\Gamma} \exp\left[-U(\phi_{\Lambda_k\backslash(z+\Lambda_q)}) - W(\tilde{\phi}_{\Lambda_k\cap(z+\Lambda_q)}|\phi_{\Lambda_k\backslash(z+\Lambda_q)}) - W(\phi_{\Lambda_k\backslash(z+\Lambda_q)}|S_{\Lambda_k^c})\right] \\
&\qquad \times \exp\left[\sum_{x\in\Lambda_k\cap(z+\Lambda_q)}\left(-\frac{g}{2}\phi_x^4 + \|\bar{J}\|_{L^1}\phi_x^2 + \frac{1}{2}\|\bar{J}\|_{L^1}\tilde{\phi}_x^2 + \frac{3}{2}\mathcal{O}_1 + \mathbf{O}_1\right)\right]
\end{aligned} \tag{5.12}$$

Now note that over our domain of integration $\hat{\phi} := \phi_\Gamma \cap \phi_{\Lambda_k\backslash\Gamma} \wedge S_{\Lambda_k^c} \in \hat{R}_q^z$ with $q \geqslant 0$. In this case we have that

$$\sum_{x\in\Lambda_k\cap(z+\Lambda_q)} \phi_x^2 + \sum_{x\in(z+\Lambda_q)\backslash\Lambda_k} S_x^2 = \sum_{x\in(z+\Lambda_q)} \hat{\phi}_x^2 \geqslant \psi_q|\Lambda_q| \tag{5.13}$$

On the other hand since $S \in X_1$ one has

$$\begin{aligned}\sum_{x\in(z+\Lambda_q)\setminus\Lambda_k} S_x^2 &\leqslant \sum_{x\in(z+\Lambda_q)} \log_+(\|x\|)\\ &\leqslant \sum_{x\in(z+\Lambda_q)} \log_+(p^q) \qquad (5.14)\\ &= \frac{1}{2}\psi_q|\Lambda_q|\end{aligned}$$

In the above bound we used the claim $\|x\| \leqslant p^q$ for all $x \in (z + \Lambda_q)$ which we quickly justify now. Note that by the ultrametric property any two closed balls in $\mathbb{L}$ are either disjoint or one is completely contained in the other. In particular since $(z + \Lambda_q) \cap \Lambda_k \supseteq \{z\}$ we must have that Λ_k is a proper subset of $(z + \Lambda_q)$ (remember that this lemma assumes $(z + \Lambda_q) \nsubseteq \Lambda_k$).

This means there exists a $y' = z + y$ with $y \in \Lambda_q$ and $\|y'\| > p^k$. However $\|y'\| \leqslant \max(\|y\|, \|z\|) \leqslant \max(p^k, p^q)$. Thus it must be the case that $p^q > p^k$ which means $z \in \Lambda_q$. Since Λ_q is closed under addition we have that $z + \Lambda_q = \Lambda_q$. This proves the claim.

We can then combine (5.13) and (5.14) to see that

$$\exp\left[\left(\sum_{x\in\Lambda_k\cap(z+\Lambda_q)} \phi_x^2\right) - \frac{1}{2}\psi_q|\Lambda_q|\right] \geqslant 1$$

Inserting this into (5.12) gives us

$$\begin{aligned}&\int\limits_{R_q^z(\phi_\Gamma,\Lambda_k,S)} d\phi_{\Lambda_k\setminus\Gamma} \exp\left[-U(\phi_{\Lambda_k}) - W\left(\phi_{\Lambda_k}|S_{\Lambda_k^c}\right)\right]\\ &\leqslant \exp\left[-\frac{1}{2}\psi_q|\Lambda_q|\right] \int\limits_{R_q^z(\phi_\Gamma,\Lambda_k,S)} d\phi_{\Lambda_k\setminus\Gamma} \exp\left[-U(\phi_{\Lambda_k\setminus(z+\Lambda_q)}) - W(\tilde{\phi}_{\Lambda_k\cap(z+\Lambda_q)}|\phi_{\Lambda_k\setminus(z+\Lambda_q)}) - W(\phi_{\Lambda_k\setminus(z+\Lambda_q)}|S_{\Lambda_k^c})\right]\\ &\qquad \times \exp\left[\sum_{x\in\Lambda_k\cap(z+\Lambda_q)} \left(-\frac{g}{2}\phi_x^4 + \left(\|\bar{J}\|_{L^1}+1\right)\phi_x^2 + \frac{1}{2}\|\bar{J}\|_{L^1}\tilde{\phi}_x^2 + \frac{3}{2}\mathcal{O}_1 + \mathbf{O}_1\right)\right]\end{aligned} \tag{5.15}$$

We now integrate the bound of (5.15) with respect to $\tilde{\phi}_{\Lambda_k\cap(z+\Lambda_k)}$ weighted by the probability measure:

$$\frac{\int_{\Sigma^{\Lambda_k\cap(z+\Lambda_q)}} d\tilde{\phi}_{\Lambda_k\cap(z+\Lambda_q)} \cdots \exp\left[-U(\tilde{\phi}_{\Lambda_k\cap(z+\Lambda_q)}) - W(\tilde{\phi}_{\Lambda_k\cap(z+\Lambda_q)}|S_{\Lambda_k^c})\right]}{\int_{\Sigma^{\Lambda_k\cap(z+\Lambda_q)}} d\tilde{\phi}_{\Lambda_k\cap(z+\Lambda_q)} \exp\left[-U(\tilde{\phi}_{\Lambda_k\cap(z+\Lambda_q)}) - W(\tilde{\phi}_{\Lambda_k\cap(z+\Lambda_q)}|S_{\Lambda_k^c})\right]}.$$

We can then perform the same estimates we performed in (5.11) which will give us the bound:

$$
\begin{aligned}
&\frac{1}{\mathcal{Z}(\Lambda_k|S)} \int_{R_q^z(\phi_\Gamma,\Lambda_k,S)} d\phi_{\Lambda_k\setminus\Gamma} \exp\left[-U(\phi_{\Lambda_k}) - W\left(\phi_{\Lambda_k}|S_{\Lambda_k^c}\right)\right] \\
\leqslant\ & \exp\left[-\frac{1}{2}\psi_q|\Lambda_q|\right] \times \mathcal{O}_4^{|(\Lambda_k\cap(z+\Lambda_q))\setminus\Gamma|} \times \lambda^{-|\Lambda_k\cap(z+\Lambda_q)|} \times \mathcal{O}_2^{|\Lambda_k\cap(z+\Lambda_q)|} \\
& \times \exp\left[\sum_{x\in(z+\Lambda_q)\cap\Gamma} \left(-\frac{g}{2}\phi_x^4 + \left(\|\bar{J}\|_{L^1}+1\right)\phi_x^2 + \frac{3}{2}\mathcal{O}_1 + \mathbf{O}_1\right)\right] \\
& \times \frac{1}{\mathcal{Z}(\Lambda_k|S)} \int_{\Sigma^{\Lambda_k\cap(z+\Lambda_q)}} d\tilde{\phi}_{\Lambda_k\cap(z+\Lambda_q)} \int_{\mathbb{R}^{\Lambda_k\setminus((z+\Lambda_q)\cup\Gamma)}} d\phi_{\Lambda_k\setminus((z+\Lambda_q)\cup\Gamma)} \exp\left[-U(\phi^*_{\Lambda_k}) - W(\phi^*_{\Lambda_k}|S_{\Lambda_k^c})\right].
\end{aligned}
$$

Note that here we use the notation $\phi^*_{\Lambda_k} := \tilde{\phi}_{\Lambda_k\cap(z+\Lambda_q)} \wedge \phi_{\Lambda_k\setminus(z+\Lambda_q)}$. The lemma now follows. □.

We can now prove Lemma 4.18.

Proof of Lemma 4.18:

Define

$$
\begin{aligned}
\mathcal{O}_6 :=& \sup_{s\in\mathbb{R}} \left(-\frac{g}{4}s^4 + (A_\beta\|J\|_{L^1}+1)s^2 + \mathcal{O}_3\right) \\
\delta :=& \mathcal{O}_6 + \log\left(1 + \sum_{q=0}^{\infty} \exp\left[\left(\mathcal{O}_5 - \frac{1}{2}\psi_q\right)p^{3q}\right]\right)
\end{aligned}
$$

Note that $\lim_{q\to\infty}\left(\mathcal{O}_5 - \frac{1}{2}\psi_q\right) = -\infty$ so the sum in the definition of δ is finite. We also note that $\mathcal{O}_6$, $\delta > 0$.

We prove the statement of the lemma by inducation on the cardinality of Γ. For the base case first assume that $\Gamma = \{z\}$. Then by sub-lemmas 5.3 -5.5 we have that

$$
\begin{aligned}
\rho^{\{z\}}_{\Lambda_k}(\phi_z|S) =& \frac{1}{\mathcal{Z}(\Lambda_k|S)} \sum_{q=-1}^{\infty} \left[\int_{R_q^z(\phi_z,\Lambda_k,S)} d\phi_{\Lambda_k\setminus\{z\}} \exp\left[-U(\phi_{\Lambda_k}) - W(\phi_{\Lambda_k}|S_{\Lambda_k^c})\right]\right] \\
\leqslant& \exp\left[-\frac{g}{4}\phi_z^2 + \mathcal{O}_6\right] + \sum_{q=0}^{\infty}\left(\exp\left[\left(\mathcal{O}_5 - \frac{1}{2}\psi_q\right)p^{3q}\right]\exp\left[-\frac{g}{4}\phi_z^2 + \mathcal{O}_6\right]\right) \\
=& \exp\left[-\frac{g}{4}\phi_z^4 + \delta\right].
\end{aligned}
$$

Thus the statement of the lemma holds for Γ with $|\Gamma| = 1$. Suppose that the statement of the lemma holds for all Γ' with $|\Gamma'| = n$. Let Γ have cardinality $n+1$. Fix some $z \in \Gamma$. Then by sub-lemmas 5.3 -5.5 we have that

$$
\begin{aligned}
\rho^{\Gamma}_{\Lambda_k}(\phi_z|S) =& \frac{1}{\mathcal{Z}(\Lambda_k|S)} \sum_{q=-1}^{\infty} \left[\int\limits_{R^z_q(\phi_\Gamma,\Lambda_k,S)} d\phi_{\Lambda_k\setminus\Gamma} \exp\left[-U(\phi_{\Lambda_k}) - W(\phi_{\Lambda_k}|S_{\Lambda_k^c})\right] \right] \\
&\leqslant \exp\left[-\frac{g}{4}\phi_z^2 + \mathcal{O}_6\right] \rho^{\Gamma\setminus\{z\}}_{\Lambda_k}(\phi_{\Gamma\setminus\{z\}}|S) \\
&\quad + \sum_{q=0}^{\infty} \left(\exp\left[\left(\mathcal{O}_5 - \frac{1}{2}\psi_q\right) p^{3q}\right] \exp\left[\sum_{x\in\Gamma\cap(z+\Lambda_q)} \left(-\frac{g}{4}\phi_x^2 + \mathcal{O}_6\right)\right] \rho^{\Gamma\setminus(z+\Lambda_q)}_{\Lambda_k}(\phi_{\Gamma\setminus(z+\Lambda_q)}|S) \right) \\
&\leqslant \exp\left[-\frac{g}{4}\phi_z^2 + \mathcal{O}_6\right] \exp\left[\sum_{x\in\Gamma\setminus\{z\}} \left(-\frac{g}{4}\phi_x^2 + \delta\right)\right] \\
&\quad + \sum_{q=0}^{\infty} \left(\exp\left[\left(\mathcal{O}_5 - \frac{1}{2}\psi_q\right) p^{3q}\right] \exp\left[\sum_{x\in\Gamma\cap(z+\Lambda_q)} \left(-\frac{g}{4}\phi_x^2 + \mathcal{O}_6\right)\right] \exp\left[\sum_{x\in\Gamma\setminus(z+\Lambda_q)} \left(-\frac{g}{4}\phi_x^2 + \delta\right)\right] \right) \\
&\leqslant \exp\left[\sum_{x\in\Gamma} \left(-\frac{g}{4}\phi_x^2 + \delta\right)\right]
\end{aligned}
$$

This finishes the proof of Lemma 4.18. □

5.4.2 Proof of Lemma 4.21

:

Fix $\Lambda \Subset \mathbb{L}$ and let N be such that the marginals $\{\mu_{n,\Lambda}\}_{n\geqslant N}$ are uniformly absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^\Lambda$. We will prove that for any Borel set $B \subset \mathbb{R}^\Lambda$ one has

$$
\lim_{n\to\infty} \mu_{n,\Lambda}(B) = \mu_\Lambda(B).
$$

We start by proving the claim for compact sets $C \subset \mathbb{R}^\Lambda$. We first fix such a C and note that it is possible to find a family of continuous real valued functions $\{f_j\}$ on $\mathbb{R}^\Lambda$ such that:

1. For all n one has $0 \leqslant f_j \leqslant 1$.
2. The functions f_j decrease (in j) to $\mathbb{1}_C$ pointwise where $\mathbb{1}_C$ is the indicator function of the set C.
3. There is a compact set $K \supset C$ such that for all j one has that the supports of the f_j lie within K.

For example, one can define $g_n : [0,\infty) \to \mathbb{R}$ as $g_n(s) = min(1 - \frac{s}{n}, 0)$ and then set $f_n(x) = g_n(d(x,C))$ where $d(x,C) = \inf_{y\in C}|x-y|$.

Now let $\epsilon > 0$. Note that by assumptions (b) and (c) above and by the assumption of uniform absolute continuity we can find M such that for all $m > M$ one has $\sup_{n\geqslant N} \mu_{n,\Lambda}(\{x \in \mathbb{R}^\Lambda : \ f_m(x) \neq \mathbb{1}_C(x)\}) < \epsilon$. Then for any $n \geqslant N$ and sufficiently large m we have

$$|\mu_{n,\Lambda}(C) - \mu_\Lambda(C)| \leqslant \int\limits_{\mathbb{R}^\Lambda} d\mu_{n,\Lambda} |\mathbb{1}_C - f_m| + \left| \int\limits_{\mathbb{R}^\Lambda} d\mu_{n,\Lambda}\ f_m - \int\limits_{\mathbb{R}^\Lambda} d\mu_\Lambda\ f_m \right| + \int\limits_{\mathbb{R}^\Lambda} d\mu_\Lambda |f_m - \mathbb{1}_C|$$

By our assumptions the first term above is uniformly bounded by ϵ, the last term can be made small by bounded convergence theorem by taking m larger, and the second term can be made arbitrarily small by taking n sufficiently large by local weak convergence.

We have proved (5.4.2) for compact sets. Now let B be an arbitrary Borel subset of $\mathbb{R}^\Lambda$ and let $\epsilon > 0$. Then since μ_Λ is a regular Borel measure there exists compact $C_1 \subset B$ such that $\mu_\Lambda(B \backslash C_1) < \epsilon$. By the regularity of Lebesgue measure on $\mathbb{R}^\Lambda$ and the uniform absolute continuity of the measures $\{\mu_{n,\Lambda}\}_{n \geqslant N}$ one can find compact $C_2 \subset B$ such that for all for all $n > N$ one has $\mu_{n,\Lambda}(B \backslash C_2) < \epsilon$. Then one has

$$|\mu_{n,\Lambda}(B) - \mu_\Lambda(B)| \leqslant |\mu_{n,\Lambda}(B \backslash (C_1 \cup C_2))| + |\mu_\Lambda(B \backslash (C_1 \cup C_2))| + |\mu_{n,\Lambda}(C_1 \cup C_2) - \mu_\Lambda(C_1 \cup C_2)|$$

The first two terms above can each be bounded by ϵ and the third term vanishes as $n \to \infty$ since we proved (5.4.2) for compact sets. □

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