4-DIMENSIONAL 2-HANDLEBODIES AND TOPOLOGICAL INVARIANTS

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(ABSTRACT)

This document forms the basis for the new work of my Master's thesis. It will be combined with my undergraduate thesis to form a more complete path for the undergraduate reader. The first three sections constitute my undergraduate thesis to make this a more complete document. The purpose of this thesis is to study the paper Beliakova and De Renzi 2021 and explain how they construct a topological invariant of a special class of 4-manifolds using the category of representations of certain Hopf algebras.

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Contents

Chapter 1

Algebra

1.1 Module Theory

Modules are a generalization of vector spaces and show up many places in abstract algebra. They will play an important role in giving examples in the sequel and will be part of the definition of linear categories. I follow here the explanation given in Dummit and Foote 2004.

Definition 1. Let R be a ring with unit (not necessarily commutative) and M be an abelian group (with operation denoted by "+"). Then M is a ***module*** over R if there is an operation (denoted by concatenation) $R \times M \to M$ such that $\forall s, r \in R$ and $\forall m, n \in M$:

- $(rs)m = r(sm)$
- $r(m + n) = rm + rn$
- $(r + s)m = rm + sm$
- $1m = m$

Vector spaces, ideals over the base ring, quotient spaces over the base ring, and polynomials with coefficients in a ring denoted $R[x]$ are all examples of a ring. Any abelian group can be formed into a $\mathbb{Z}$ module and conversely any $\mathbb{Z}$ module is an abelian group (by definition). If we let $G = \{e\}$ be the trivial group, then it can be a module over any ring R and is called the zero module, 0, which only has one element.Roman, Axler, and Gehring 2005

Definition 2. Given two R-modules M and N, a function $\phi : M \to N$ is called ***linear*** if for all $r, s \in R$ and for all $m_1, m_2 \in M$, we have that $\phi(rm_1 + sm_2) = r\phi(m_1) + s\phi(m_2)$.

Given two modules M and N over the ring R we denote the set of linear maps from M to N as $Hom_R(M, N)$. This set $Hom_R(M, N)$ is a module over R with addition of functions and multiplication by elements of R if and only if R is commutative. The set $Hom_R(M, 0)$ (or $Hom_R(0, M)$) only has one linear map; the one which send all elements to the zero vector (respectively sends the zero vector to the zero vector of M).

Notice that these past few definitions are exactly the same as those for vector spaces and linear transformations in linear algebra. We similarly define the kernel and image of linear maps, direct product of modules, linear independence of elements, generating sets, and bases in the same way as well. However, it is not always true that modules have a basis.

Definition 3. Let R be a ring with unity and M be a R-module. Then M is called ***free*** if it has a basis.

All vector spaces are free modules because they have a basis. The module $\mathbb{Z}_2 \times \mathbb{Z}_2$ is a $\mathbb{Z}$ module that is not free, however it is free if we consider it as a $\mathbb{Z}_2$ module because it is the vector space F_2^2.

We can also construct the tensor product of two modules, a topic not covered in introductory linear algebra classes. First, to avoid complications let R be a commutative ring with unity and M, N be R-modules. Then we can define a right action of R on M by $mr = rm$ for all $r \in R$ and $m \in M$.

Definition 4. The ***tensor product*** of two R-modules is the cartesian product, but we quotient out a large number of relations. For all $r \in R$ and $m_1, m_2, m, n_1, n_2, n \in N$:

$$M \otimes_R N = \frac{M \times N}{((m_1+m_2,n)-(m_1,n)-(m_2,n),\ (m,n_1+n_2)-(m,n_1)-(m,n_2),\ (mr,n)-(m,rn))}$$

This is an R-module and satisfies the following universal property: Let P be any R-module, $\phi : M \times N \to P$ be a bilinear map into P, and $\iota : M \times N \to M \otimes_R N$ is the injection map of S into G. Then there is a unique linear map $\Phi : M \otimes_R N \to P$ such that $\phi = \Phi \circ \iota$.

In the case of vector spaces, it is easy to understand the tensor product. For two finite-dimensional vector spaces V and W over a field F, $V \cong F^n$ and $W \cong F^m$ for some $n, m \in \mathbb{N}$. Then the tensor product $V \otimes_F W \cong F^{nm}$. For any R-module M we have that $R \otimes_R M \cong M$.

The final algebraic structure we will consider is a chain of R-modules with linear maps between them.

Definition 5. Let R be a ring with unity and A, B, C be modules over R. Let $\psi : A \to B$ and $\phi : B \to C$ be linear maps. We consider the sequence of maps shown below where the maps $\iota : 0 \to A$ and $p : C \to 0$ are the trivial maps. These will be excluded in the future.

$$0 \xrightarrow{\iota} A \xrightarrow{\psi} B \xrightarrow{\phi} C \xrightarrow{p} 0$$

This sequence of modules is called ***short exact*** if the following conditions are met:

- The map ψ is injective.
- The map ϕ is surjective.
- The maps ψ and ϕ satisfy $im(\psi) = ker(\phi)$.

Equivalently, we can replace items 1 and 2 by requiring that $im(\iota) = ker(\psi)$ and $im(\phi) = ker(p)$.

One important extension is to fix a module P over R and consider the sets $Hom_R(P, A)$, $Hom_R(P, B)$, and $Hom_R(P, C)$. We can form new linear maps between these sets by taking

$$\psi' : Hom_R(P, A) \to Hom_R(P, B), \quad \psi'(f) = \psi \circ f$$
$$\phi' : Hom_R(P, B) \to Hom_R(P, C), \quad \phi'(f) = \phi \circ f$$

We can ask the question, given an exact sequence, is the following sequence exact?

$$0 \longrightarrow Hom_R(P, A) \xrightarrow{\psi'} Hom_R(P, B) \xrightarrow{\phi'} Hom_R(P, C) \longrightarrow 0$$

The answer is in fact no as we will see.

Example 6. Consider the sequence of modules over $\mathbb{Z}$:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\psi} \mathbb{Z} \xrightarrow{\phi} \mathbb{Z}_2 \longrightarrow 0$$

where $\psi(n) = 2n$ and $\phi(n) = n \bmod 2$ for all $n \in \mathbb{N}$. This is a short exact sequence because ψ in injective and ϕ is surjective. Let $P = \mathbb{Z}_2$, then the sequence

$$0 \longrightarrow Hom_R(\mathbb{Z}_2, \mathbb{Z}) \xrightarrow{\psi'} Hom_R(\mathbb{Z}_2, \mathbb{Z}) \xrightarrow{\phi'} Hom_R(\mathbb{Z}_2, \mathbb{Z}_2) \longrightarrow 0$$

is not exact because $\psi'(f) = \psi \circ f$ is the zero map for all $f \in Hom_R(\mathbb{Z}_2, \mathbb{Z})$ ($Hom_R(\mathbb{Z}_2, \mathbb{Z})$ consists of only the trivial map) and so the function ϕ' is not surjective.

There are however some special modules P such that we do satisfy this property.

Definition 7. Let R be a ring with unity, then an R module P is called ***projective*** if for all short exact sequences:

$$0 \xrightarrow{\iota} A \xrightarrow{\psi} B \xrightarrow{\phi} C \xrightarrow{p} 0$$

then the following sequence is short exact:

$$0 \longrightarrow Hom_R(P, A) \xrightarrow{\psi'} Hom_R(P, B) \xrightarrow{\phi'} Hom_R(P, C) \longrightarrow 0$$

Projective modules have many nice properties such as the direct sum of two projective modules is projective, the tensor product of two direct modules is projective (if R is commutative), and projective modules are always a direct summand of a free module. If $R = F$ is a field, then every module over R is a vector space and all vector spaces are projective.

1.2 More Algebraic Objects

While groups have one binary operation, rings and modules have two binary operations, we can keep adding operations to make higher structures. For instance, an algebra is both a ring and a vector space.

Definition 8. Let $(A, +, *)$ be a modules over some commutative ring R equipped with a bilinear product $\times : A \times A \to A$ such that $(A, +, \times)$ is a ring with 1. We will denote both "multiplications" by concatenation where the operation being used should be clear from context. Then $(A, R, +, *, \times)$ is called an ***R-algebra*** if $\forall x, y, z \in A, \forall a, b \in R$:

- $(x + y)z = xz + yz$
- $z(x + y) = zx + zy$
- $(ax)(by) = (ab)(xy)$

Furthermore, if we require $\times$ to be associative, we call A an ***associative R-algebra***. However, there is an equivalent definition of an associative algebra which will be more easily generalizable for our purposes. Let $(A, +, \mu : A \otimes A \to A)$ be a ring, let R be a commutative ring with ring homomorphism $\eta : R \to Z(A)$ from R into the center of A. Then $(A, R, +, *, \mu, \eta)$ is an associative R-algebra.

Definition 9. Let S be a finite set of letters. The ***free algebra*** generated by S over a field $\mathbb{K}$ is denoted by $\mathbb{K} < S >$ and comprises of linear combinations of words from S. If w_1, w_2 are words from S and $u, v \in \mathbb{K}$, then $(uw_1)(vw_2) = uv(w_1 w_2)$.

More specifically, if $S = \{a, b, c, d\}$ and $\mathbb{K} = \mathbb{R}$, then $\mathbb{R} < \{a, b, c, d\} >$ contains $\{2a,\ .3b,\ a^2b^3d,\ aba + a^2b + c, ...\}$. Note that letters of each word do not necessarily commute.

Other typical examples of (associative) algebras are $M_n(F)$ ($n \times n$ matrices with elements in a field F), $GL_n(F)$ (invertible $n \times n$ matrices), and $SL_n(F)$ ($n \times n$ matrices with determinant 1). These are all vector spaces with $\times$ being either matrix multiplication or function composition. There is also a dual notion to an algebra:

Definition 10. Let $(C, +, *)$ be an R-module over a commutative ring R with R-linear maps $\Delta : C \to C \otimes C$ and $\epsilon : C \to R$. Then $(C, R, +, *, \Delta, \epsilon)$ is called a ***(coassociative) coalgebra*** if:

- $(id_C \otimes \Delta)\Delta = (\Delta \otimes id_C)\Delta$
- $(id_C \otimes \epsilon)\Delta = id_C = (\epsilon \otimes id_C)\Delta$

We say that coalgebras are "dual" to algebras, because the dual of a coalgebra is an algebra and the dual of a finite-dimensional algebra is a coalgebra. Notice that we can rewrite the requirements of an algebra to say that $\mu(\mu \otimes id_A) = \mu(id_A \otimes \mu)$ and $\mu(id_A \otimes \eta) = id_A = \mu(\eta \otimes \mu)$ which are the dual version of the formulas above. We can combine an algebra and a coalgebra to get a bialgebra.

Definition 11. Let $(A, R, +, *)$ be an R-module over a commutative ring which we will abbreviate henceforth just as A. Then $(A, \mu, \eta, \Delta, \epsilon)$ is a ***bialgebra*** if (A, μ, η) is an (associative) algebra, (A, Δ, ϵ) is a (coassociative) coalgebra, and either (equivalently):

- The maps μ and η are morphisms of coalgebras.

- The maps Δ and ϵ are morphisms of algebras.

The final structure that we will add to the previous structures is called an antipode. This acts like a inverse or transpose on certain algebras.

Definition 12. Let $(H, \mu, \eta, \Delta, \epsilon)$ be a bialgebra with an additional map $S : H \to H$ such that $\mu(S \otimes id_H)\Delta = \eta\epsilon = \mu(id_H \otimes S)\Delta$. We call H a ***Hopf Algebra***.

These definitions seem very contrived, but it shows up naturally on a surprising number of structures.

Example 13. Consider to groups algebra $\mathbb{K}G$ with maps $(\forall g \in G, k \in \mathbb{K})$ μ is multiplication, $\eta(k) = k * 1$, $\Delta(g) = g \otimes g$, $\epsilon(g) = 1$, $S(g) = g^{-1}$. This is a Hopf algebra.

Example 14. The tensor algebra $T(V)$ for a vector space V/F is a Hopf algebra with maps μ multiplication, $\eta(k) = k \in F, \forall k \in F$, $\Delta(x) = 1 \otimes x + x \otimes 1, \forall x \in V$, $\epsilon(x) = 0$, $S(x) = -x, \forall x \in V$.

There are several other examples of Hopf Algebras such as the ring of symmetric functions, the cohmology of Lie Groups, and the universal enveloping algebra of a Lie Algebra. Hopf algebras also appear as tools to solve combinatorial problems.

1.3 Representation Theory

One large part of Algebra is Representation Theory which studies how you can turn elements of an algebraic object into linear endomorphisms on a vector space. Common algebraic objects of study are groups and Lie Algebras.

Definition 15. Given a group G and a vector space V, let $\phi : G \to GL(V)$, that is to say ϕ takes elements of G and sends them to invertible linear endomorphisms on V. We say that ϕ is a ***representation*** of G iff it is a group homomorphism iff $\phi(g_1 g_2) = \phi(g_1)\phi(g_2)\ \forall g_1, g_2 \in G$.

Example 16. Consider the group S_3. We can send elements of S_3 to maps on $\mathbb{R}^3$ with basis $e_1,\ e_2,\ e_3$ by $\phi(\sigma)(e_i) = e_{\sigma(i)}\ \forall \sigma \in S_3$. Alternatively, we can send $\sigma \mapsto (-1)^{sgn(\sigma)} \in \mathbb{C}$ where we note all invertible linear maps of $\mathbb{C}$, the $\mathbb{C}$-vector space are multiplication by a scalar.

Groups are not the only object of which we can study their representations. Another large part of representation theory is studying the representation of Lie Algebras. Lie Algebras are similar to associative algebras, but we replace the requirement of associativity of $\times$ with a new identity, and give it a new notation.

Definition 17. Let V be a F-vector space for some field F and let $[\,,] : V \times V \to V$ be a bilinear operation. We say that $(V, F, +, *, [\,,])$ is a ***Lie algebra*** if the following hold:

- For all $x, y \in V$, $[x, y] = -[y, x]$
- For all $x, y, z \in V$, $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$

The last point is called the ***Jacobi Identity*** and replaces our previous requirement of associativity and the new multiplication is called the Lie bracket. While previously we noted that $GL_n(F)$ and $SL_n(F)$ are associative algebras, there are similar examples of Lie Algebras denoted $\mathfrak{gl}_n(F)$ and $\mathfrak{sl}_n(F)$.

Example 18. Let $\mathfrak{gl}_n(F)$ be the vector space of all $n \times n$ matrices over F and $\mathfrak{sl}_n(F)$ be the vector space of all $n \times n$ matrices over F and zero determinant. Then these are both Lie algebras with Lie bracket $[A, B] = AB - BA$ where concatenation denotes matrix multiplication.

Example 19. Given any vector space V, the space of endomorphisms of V, denoted $\mathfrak{gl}(V)$, is a Lie algebra with Lie bracket $[\tau, \sigma] = \tau\sigma - \sigma\tau$, $\forall \tau, \sigma \in \mathfrak{gl}(V)$.

From every Lie algebra, we can form an associative algebra called the Universal Enveloping Algebra.

Definition 20. Let L be a Lie algebra over $\mathbb{K}$ with vector space basis $\{x_1, x_2, ...x_n\}$. Then the ***Universal Enveloping algebra*** of L, denoted $U(L)$ is the free algebra $\mathbb{K} < x_1, ..., x_n >$ mod the relations $< x_i x_j - x_j x_i = [x_i, x_j] >$.

In this way, we ca also form a Lie algebra from an associative algebra A such that $L = L(A)$ by defining $[x, y] = xy - yx \; \forall x, y \in A$. For any Lie algebra L, we have that $L(U(L)) = L$. We can also define representations of Lie Algebras in much the same way as for groups. Now representations must respect the extra structure of scalar multiplication and the Lie bracket.

Definition 21. Let $\mathfrak{g}$ be a Lie algebra and V a vector space over the same base field F, then $\phi : \mathfrak{g} \to \mathfrak{gl}(V)$ is called a ***Lie algebra representation*** if it is a Lie algebra homomorphism. This means $\forall g, h \in \mathfrak{g}$ and $\forall f \in F$:

- $\phi(g + h) = \phi(g) + \phi(h)$
- $\phi(fg) = f\phi(g)$
- $\phi([g, h]) = [\phi(g), \phi(h)]$

Let $\mathfrak{g}$ be a Lie algebra with representation $\phi : \mathfrak{g} \to \mathfrak{gl}(V)$. If there exists some subspace W of V such that ϕ_W is another representation of $\mathfrak{g}$, then we call ϕ_W a ***sub-representation*** of ϕ. Representations without subrepresentations are called ***simple*** representations.

If V and W are $\mathfrak{g}$-representations, there is a representation of $V \oplus W$ by $g \cdot (v, w) = (g \cdot v, g \cdot w)$, $\forall g \in \mathfrak{g}, v \in V, w \in W$. There is a representation of $V \otimes W$ by $g \cdot (v \otimes w) = (g \cdot v) \otimes w + v \otimes (g \cdot w)$, $\forall g \in \mathfrak{g}, v \in V, w \in W$.

If we wanted to find all possible representations for a Lie algebra $\mathfrak{g}$, in the best case scenario we could find all simple representations and build all other (finite-dimensional) representations out of them. We will note and take without proof that all finite-dimensional representations of $\mathfrak{sl}_2(\mathbb{C})$ are direct sums of simple representations.

Example 22. The Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ has a basis of the elements:
$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, and $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then $\mathfrak{sl}_2(\mathbb{C})$ has one simple representation of every finite dimension. Let V be a $\mathbb{C}$-vector space of dimension $n+1$ with basis $\{v_0, v_1, ...\ v_n\}$. We have that $\mathfrak{g}$ acts on V by $h \cdot v_i = (m-2i)v_i$, $f \cdot v_i = (i+1)v_{i+1}$, and $e \cdot v_i = (n-i+1)v_{i-1}$ for all $i \in \{0, 1, ..., n\}$ except $e \cdot v_0 = 0$ and $f \cdot v_n = 0$.

There one final important theorem from Representation theory that will be very useful to us later which concerns maps between representations. Let V and W be representations of a Lie algebra $\mathfrak{g}$. The map $\alpha : V \to W$ is a map between representations if $\alpha(g \cdot_V v) = g \cdot_W \alpha(v)$, $\forall v \in V, g \in \mathfrak{g}$.

Theorem 23. *(Schur's Lemma): If $\mathfrak{g}$ is a Lie algebra with irreducible representations on V and W $\mathbb{C}$-vector spaces, and α is a map between the representations. Then either $\alpha = 0$ or $V \cong W$ and $\alpha(v) = cv$ for all $v \in V$ and some fixed $c \in \mathbb{C}$.*

In all of the previous sections, we have used the internal structure of algebraic objects to discern some properties of morphisms between objects. However, we can forget the internal structure of objects, start with the most general notion of having maps between objects, and build off of properties of the maps themselves.

Chapter 2

Category Theory

2.1 Monoidal Categories

Category Theory is one of the most abstract forms of math which generalizes many branches of math. At its most basic level, it studies objects and maps between objects. The benefit of this type of study is that it allows us to prove theorems across all branches of math simultaneously by observing only a few principles. However, many of the definitions are obscure and do not seem to have motivation without many examples. We will build our examples based on the projective modules discussed in the previous section following the information given in Turaev and Virelizier 2017.

Definition 24. A ***category*** C consists of the following data:

- A class of objects with elements denoted $X \in Ob(C)$.
- For any two objects X, Y there is a (possibly empty) set of maps (also called arrows or morphisms) from X to Y denoted $Hom_C(X, Y)$.
- For any two "Hom" sets that are compatible, there is a way to compose maps from the "Hom" sets. This means for all $X, Y, Z \in C$, there is a map $\circ$: $Hom_C(X, Y) \times Hom_C(Y, Z) \to Hom_C(X, Z)$.
- For every $X \in Ob(C)$ the is a map $id_X \in Hom_C(X, X)$ such that $f \circ id_X = f$

for all $f \in Hom_C(Z, X)$ and $id_x \circ g = g$ for all $g \in Hom_C(X, Y)$.

We also require that the composition of maps be associative (which is why we do not have a different notation for $\circ$ for each choice of X, Y, Z). It is also sometimes common to be somewhat lazy and say that $X \in C$.

Some different examples of Categories are the category of groups with group homomorphisms as maps (denoted **Grp**), the category of topological spaces with continuous functions as maps (denoted **Top**), and the category of sets with maps as functions between sets (denoted **Set**). There are however categories where the maps are not special types of functions. We can consider the set of natural numbers $\mathbb{N}$ as a category with maps as $\leq$. For every number $n \in \mathbb{N}$ we have the map $n \leq n$ by reflexivity and because of transitivity we have the composition of maps.

Example 25. From now on, we will let $\mathbb{K}$ be a commutative ring with identity. A projective module is of finite type if it is a direct summand of a free module that is finitely generated. The class of projective modules of finite type with linear maps is denoted $proj_{\mathbb{K}}$.

Definition 26. A ***functor*** $F : C \to D$ is a map between categories such that:

- If $X \in Ob(C)$, then $F(X) \in Ob(D)$.
- If $f : X \to Y$ is a map between objects in C, then $F(f) : F(X) \to F(Y)$ is a map between the corresponding objects in D.
- The functor F preserves identity maps: $F(id_X) = id_{F(X)}$.
- The functor F preserve composition of maps: $F(f \circ g) = F(f) \circ F(g)$.

There is a category of categories which functors acting as the maps between them, denoted **Cat** and similarly there is a notion of maps between functors which are called natural transformations. Next, we will generalize the notion of a tensor product to a monoidal product.

Definition 27. A ***monoidal*** category C is a category with the additional items:

- A functor $\otimes : C \times C \to C$ called the monoidal product.
- A unit object $\mathbb{1} \in Ob(C)$.
- A family of isomorphisms $\{a_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)\}$ for all $X, Y, Z \in C$.
- Two more families of isomorphisms for all $X \in C$: $\{l_X : \mathbb{1} \otimes X \to X\}$ and $\{r_X : X \otimes \mathbb{1} \to X\}$.

These isomorphisms say that, for example, $X \otimes \mathbb{1} \cong X$ even if $X \otimes \mathbb{1} \neq X$. There are five additional commutative diagrams which explain the interactions of a monoidal product and are part of definition. However, these are cumbersome to write and understand. These can be summarized by the Mac Lane's Coherence Theorem.

Theorem 28. *The Mac Lane's Coherence theorem asserts that any diagram consisting of the associativity and unitality isomorphisms commute.*

What this means in practice is that any place in a sequence of functions that we see a $X \otimes \mathbb{1}$ or a $\mathbb{1} \otimes X$ we can replace it with X and we can ignore any parentheses for associativity because the isomorphisms will be implied.

Just as how functors preserve the identity maps and composition of maps in categories, we would like to have functors which preserve the properties of monoidal categories as well.

Definition 29. Let C and D be monoidal categories. Then a monoidal functor $F : C \to D$ is a functor with these additional properties:

- There is a map $F_0 : \mathbb{1}' \to F(\mathbb{1})$ in D that takes the identity object of D to the image of the identity object in C.
- There is a natural transformation F_2 (a family of morphisms): $\{F_2(X, Y) : F(X) \otimes' F(Y) \to F(X \otimes Y)\}$ for all $X, Y \in C$

It must also satisfy 3 commutative diagrams for the compatibility of the morphisms and there is an implied commutativity diagram for F_2 being a natural transformation(which we will not discuss here).

We call a monoidal functor F ***"strong"*** if F_0 and F_2 are isomorphisms and we will call F "strict" if F_0 and F_2 are identity morphisms. It is a recurring theme in category theory that it is preferrable that objects be isomorphic but it is too strong to require them to be the ssame object. For instance, recall that we require $\mathbb{1} \otimes X$ to be isomorphic to X but they need not be equal. Thus strong monoidal functors will be used frequently but not strict monoidal functors.

Our goal for the next few sections are to generalize the notion of a dual vector.

Definition 30. A ***pairing*** between two objects $X, Y \in C$ is a map $\omega : X \otimes Y \to \mathbb{1}$ and is called nondegenerate if there is another map (called the inverse of ω) $\Omega : \mathbb{1} \to Y \otimes X$ such that $(id_Y \otimes \omega)(\Omega \otimes id_Y) = id_Y$ and $(\omega \otimes id_X)(id_X \otimes \Omega) = id_X$.

Given an object X in our category C, we say that ${}^\vee X$ is a left dual of X is there is a nondegenerate pairing $\omega : {}^\vee X \otimes X \to \mathbb{1}$. We call the pairing ev_X and the inverse $coev_X$. A category in which every object has a left dual is called a left rigid category. Similarly, given an object X in our category C, we say that $X^\vee$ is a right dual of X if there is a nondegenerate pairing $\omega : X \otimes X^\vee \to \mathbb{1}$. We call the pairing $e\tilde{v}_X$ and the inverse $co\tilde{e}v_X$. A category in which every object has a left dual is called a right rigid category. A category which is both left and right rigid is called a rigid category.

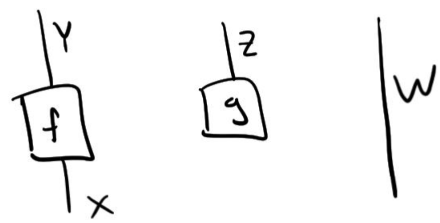
In every rigid category, we can define the left and right duals of a morphism. More specifically, if there is a morphism $f : X \to Y$, we form the map ${}^\vee f : {}^\vee Y \to {}^\vee X = (ev_Y \otimes id_{{}^\vee X})(id_{{}^\vee Y} \otimes f \otimes {}^\vee X)(id_{{}^\vee Y} \otimes coev_X)$ (and similarly for $f^\vee$). In rigid categories, we require that if we pass a map across an evaluation or coevaluation morphism, it changes to its dual. Pivotal categories build on the definition of a rigid category by combining the notions of a left and right duality.

Definition 31. A ***pivotal*** category C is a rigid category such that the left and right dualities coincide. For every $X \in C$, we have the set $\{X^*,\ ev_X : X^* \otimes X \to \mathbb{1},\ e\tilde{v}_X : X \otimes X^* \to \mathbb{1}\}$ where ev_X and $e\tilde{v}_X$ are left and right dualities respectively. We call these the left and right evaluation, and require that they coincide as monoidal functors.

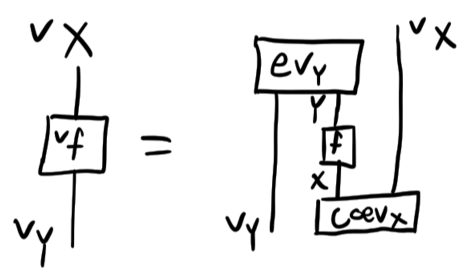
Returning to our previous example, the category $proj_{\mathbb{K}}$ is a monoidal category under the tensor product of modules briefly discussed earlier because the tensor product of two projective modules is again projective. This category is also pivotal. If we let M be a $\mathbb{K}$-module, then $M^* = Hom_{\mathbb{K}}(M, \mathbb{K})$. For any $x \in M$ and $f \in Hom_{\mathbb{K}}(M, \mathbb{K})$ we have that $ev_M(f, x) = f(x) \in \mathbb{K}$ and $e\tilde{v}_x(x, f) = f(x) \in \mathbb{K}$.

Because the composition of these functions can get very long and tedious, we will show the Penrose Calculus, a graphical representation of the objects we are studying.

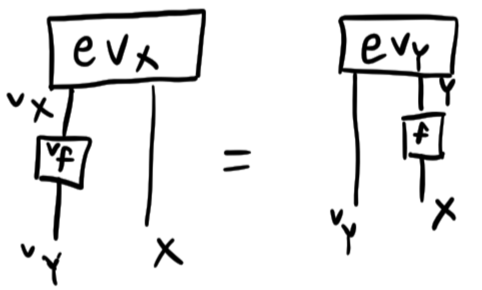
Objects of our category are represented by lines and maps are represented by boxes. Penrose Diagrams are read from the bottom to the top. The monoidal product of maps is represented by placing them next to one another and we represent the object $\mathbb{1}$ by a blank space. In the diagram below, we have three maps: $f : X \to Y$, $g : \mathbb{1} \to Z$ and the identity map id_W.

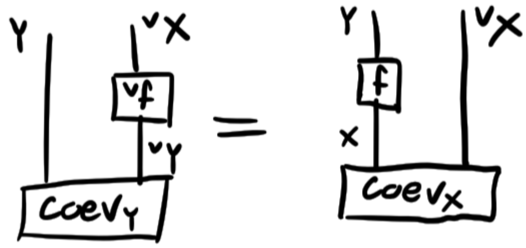


In rigid categories, the (left/right) (co)evaluation maps are represented by labeled boxes and we can easily draw the definition of a dual map:

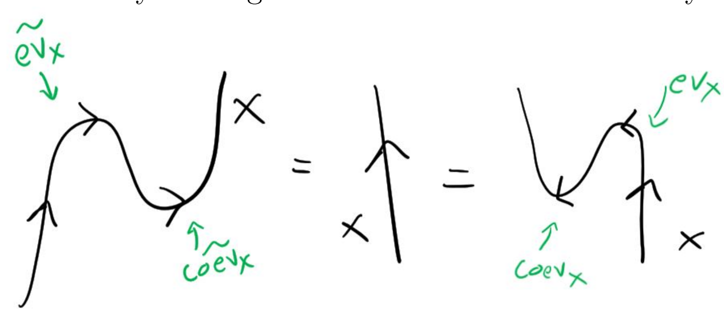


Again, in rigid categories we can draw how morphisms "commute" when we transfer them across (co)evaluation maps. If we pass a map across an evaluation or coevaluation morphism, it changes to its dual. A similar property holds for the right dual morphisms.





We can extend this calculus to representing pivotal categories. We will denote an object X by a line labeled X with a downward arrow, and we will denote X^* by a line labeled X, but with an upwards arrow. The maps ev_X, $coev_X$, $\tilde{ev}_X$, and $co\tilde{e}v_X$ will be represented by as shown below. Here we see how the Penrose Calculus can simpify the identities of the nondegenerate pairs. We can create the dual map by switching the direction of arrows. We will say two Penrose diagrams represent the same map if they can be transformed into one another by moving around boxes and continuously de-



forming lines.

Another important definition in the discussion of monoidal categories is that of a braiding which is a commutativity constraint on the monoidal product.

Definition 32. A ***braiding*** on a monoidal category is a family of isomorphisms: $\{\tau_{X,Y} : X \otimes Y \to Y \otimes X\}$ for all $X, Y \in C$. Additionally for all objects $X, Y, Z, X', Y' \in C$ with $f : X \to X'$ and $g : Y \to Y'$ we have:

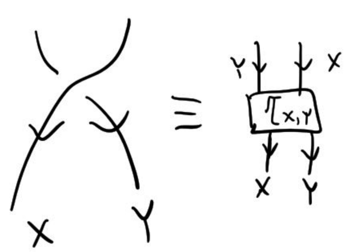
- $\tau_{X,Y\otimes Z} = (id_Y \otimes \tau_{X,Z})(\tau_{X,Y} \otimes id_Z)$

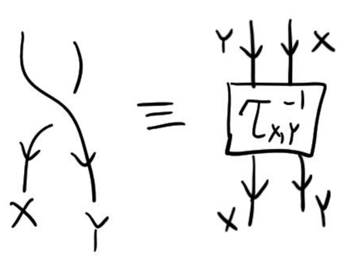
- $\tau_{X\otimes Y,Z} = (\tau_{X,Z} \otimes id_Y)(id_X \otimes \tau_{Y,Z})$
- $\tau_{X',Y'}(f \otimes g) = (g \otimes f)\tau_{X,Y}$

For instance, in the category $proj_{\mathbb{K}}$ if we have two modules M and N, then the braiding constraint is the obvious isomorphism that $M \otimes N \cong N \otimes M$. Braiding are represented by the Reidemeister Calculus which builds on the Penrose Calculus. Braidings are represented as crossings of the lines and depending on the crossing we have either τ or τ^{-1}. It is not always true that $\tau = \tau^{-1}$, but if so the category is called symmetric.

Example 33. In Physics, braided monoidal categories appear in surprising ways. Long before the notion of a category was discovered, Richard Feynman invented Feynman diagrams as a way to display how particles can evolve over time. These diagrams pictorally show representations of the Poincare group. The monoidal product of diagrams is to place two drawings next to each other. The braiding structure occurs when two particles cross without interacting.

Still further examples of braided monoidal categories occur in the formulation of loop quantum gravity. Loop quantum gravity is a modern attempt to quantize the theory of gravity. Instead of displaying an evolution over time, the objects in loop quantum gravity are fixed excited states.





While previously projective modules gave us examples of special categories, in this next section they will play a more direct role.

Definition 34. A linear category is $\mathbb{K}$***-linear*** if for all objects $X, Y \in C$, the set $Hom_C(X, Y)$ is a left-$\mathbb{K}$ module. Then compositions of maps are $\mathbb{K}$-linear. Similarly we can define $\mathbb{K}$-linear functors as those that preserve the linearity of the "Hom" sets.

While we already have a "multiplicative" structure on our categories. We would like to have an additive structure similar to the direct product. In the spirit of category theory, we look at the maps between objects rather than their internal structure.

Definition 35. Let $(X_\alpha)_{\alpha\in\Lambda}$ be a finite collection of objects in a $\mathbb{K}$-linear category C. Then an object D is a ***direct sum*** of the objects (X_α) if there are projection operators $\{p_\alpha : D \to X_\alpha\}$ and injection operators $\{q_\alpha : X_\alpha \to D\}$ such that $id_D = \Sigma_{\alpha\in\Lambda} q_\alpha p_\alpha$ and $p_\alpha q_\beta = \delta_{\alpha\beta}$ where $\delta_{\alpha\beta}$ is the Kronecker delta.

Similarly, we can define a notion of simple objects. Recall that Schur's Lemma describes how the endomorphism algebra of an irreducible representation is the field itself.

Definition 36. A ***simple object*** X in a $\mathbb{K}$-linear category C is an object where $Hom_C(X, X) \cong \mathbb{K}$ (as either $\mathbb{K}$-vector spaces or algebras). In particular, simple objects are not direct sums of other objects.

Given a general $\mathbb{K}$-linear category C with objects X and Y, we do not know a priori that the direct sum exists inside our category, but it is not hard to "close" our category under direct sum. We can take this notion to the extreme and require that the category is closed under direct sums of some collection of simple objects and require that it "identifies Schur's Lemma".

Definition 37. A ***prefusion*** $\mathbb{K}$ ***category*** C is a monoidal $\mathbb{K}$-linear category such that there is a set I of simple objects such that:

- $\mathbb{1} \in I$
- $\forall i \neq j \in I$ we have $Hom_C(i, j) = 0$
- All elements of C are direct sums of elements of I

One example of a prefusion category that we have previously seen is the category of finite-dimensional representations of $\mathfrak{sl}_2(\mathbb{C})$ where the objects are representations and the morphisms are maps between categories. The monoidal structure is given by the tensor product of representations and every finite-dimensional representation is the direct sum of simple representations. This also has a pivotal structure given by the dual of a representation.
Here there are an infinite number of simple representations, but Lie algebras over fields of characteristic p might have a finite number of simple representations.

Definition 38. A ***fusion category*** is a prefusion category with a finite number of isomorphism classes of simple objects.

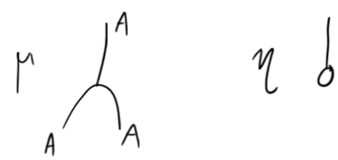
2.2 Algebraic Objects in Category Theory

Algebraic objects provide a lot of examples and motivation for Category Theory, but we can also categorize some of these objects as objects in a category. The definitions are identical to those in section 3, but now the symbol $\otimes$ represents the monoidal product in our category instead of the tensor product of modules and id_X is now the identity morphism instead of the identity map. All of the following objects will be thought of as objects in a monoidal, $\mathbb{K}$-linear category C. For example the categorical definition of an algebra is:

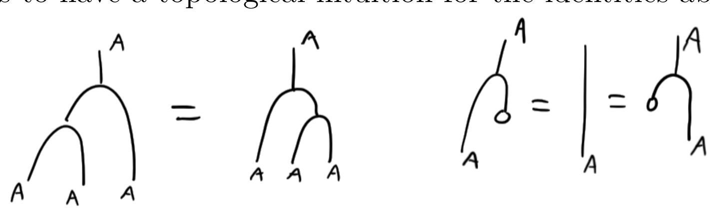
Definition 39. An ***algebra*** $A \in Ob(C)$ is given by a triple $(A, \mu : A \otimes A \to A, \eta : \mathbb{K} \to A)$ such that:

- $\mu(\mu \otimes id_A) = \mu(id_A \otimes \mu)$
- $\mu(id_A \otimes \eta) = id_A = \mu(\eta \otimes id_A)$

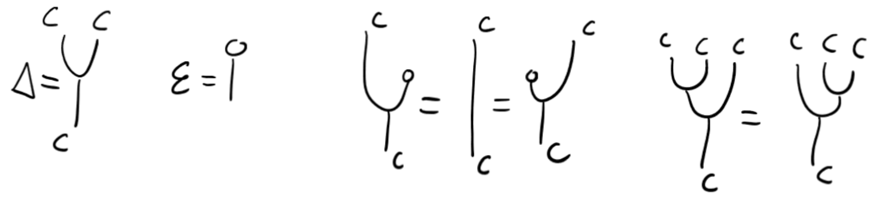
We can pictorally represent these operations:



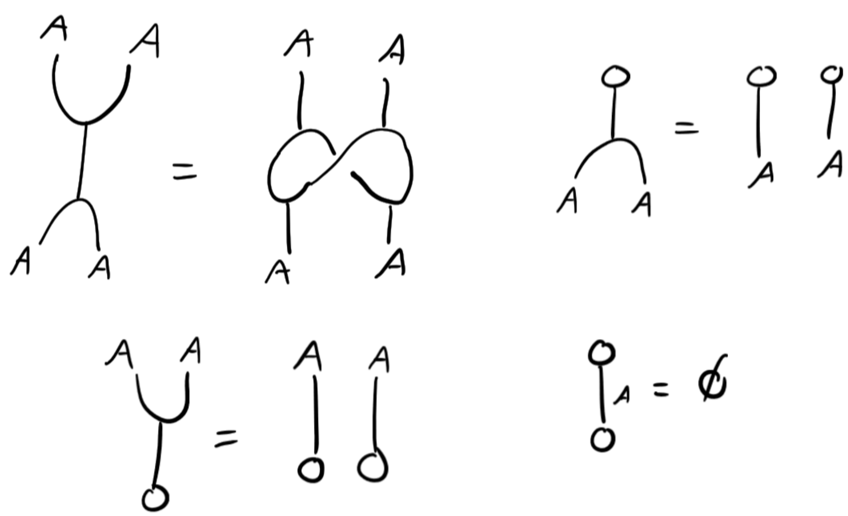
which allows us to have a topological intuition for the identities above:



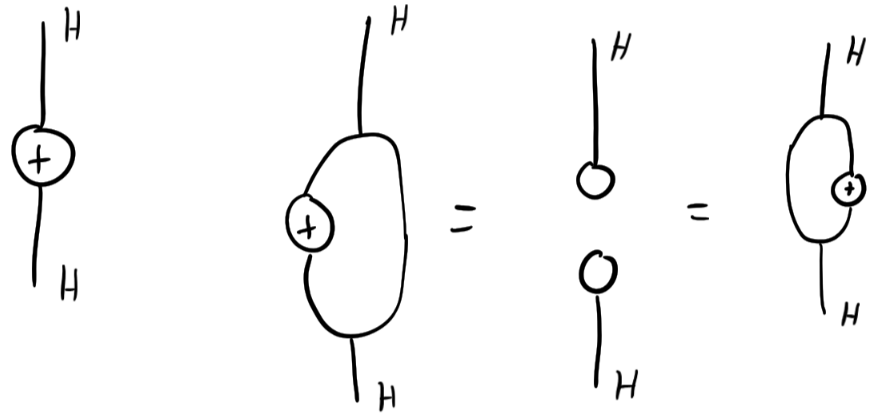
It is important to notice that we "forget" the underlying vector space requirements of an algebra. The algebraic definition of an algebra when written in full is $(A, R, +, *, \mu, \eta)$ whereas categorical algebras have the structure (A, μ, η). Coalgebras, bialgebras, and Hopf Algebras are defined in exactly the same way as their purely algebraic counterparts. The maps Δ and ϵ in a coalgebra C, as well as their identities are drawn as follows:



The categorical requirements for a bialgebra A (after some manipulation from our previous definition) can be drawn as:



Finally, the antipode of a Hopf algebra is denoted:

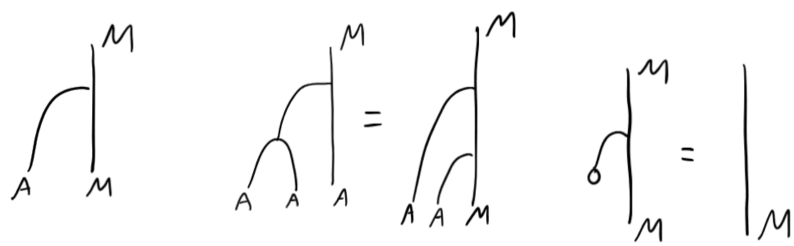


If the antipode S is invertible, then we denote it's inverse by a "-". We can also categorize the notion of a module which requires the interplay of two algebraic objects.

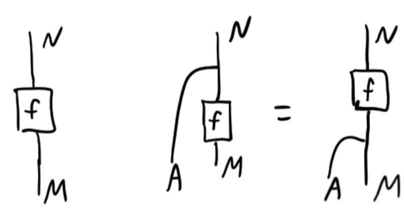
Definition 40. Given an algebra (A, μ, η), a pair (M, r) is called ***(left) A-module*** if $M \in Ob(C)$, $r : A \otimes M \to M$ such that:

- $r(\mu \otimes id_M) = r(id_A \otimes r)$
- $r(\eta \otimes id_M) = id_M$

These are drawn as:



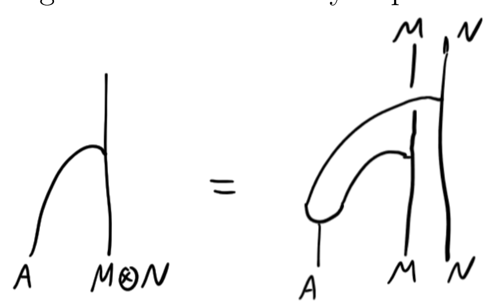
This definition aligns exactly with our algebraic definition of a left R-module except in this case, we have an algebra acting on some object without explicit internal structure in our category. If we fix an algebra $A \in Ob(C)$, then A-modules form a category denoted mod_A. Objects in this category are A-modules and morphisms are maps between objects in C. More specifically, if (M, r) and (N, s) are A-modules, then a morphism between modules is a morphism $\phi : M \to N$ such that $\phi r = s(id_A \otimes \phi)$.



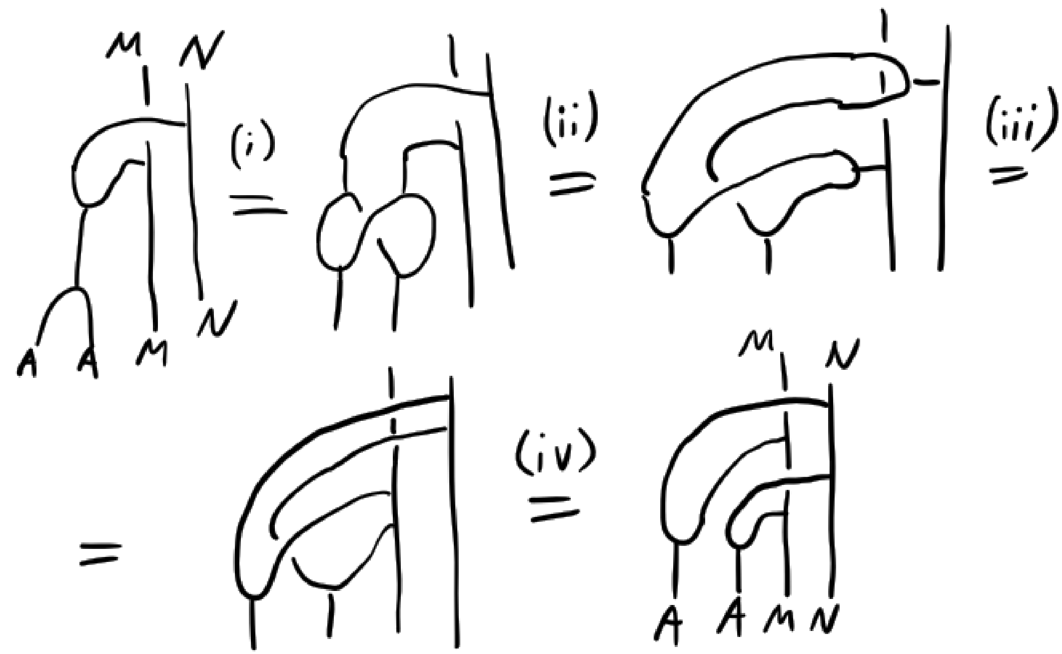
We noticed previously that when the algebra A is $\mathfrak{sl}_2(\mathbb{C}$, the category mod_A is a pivotal pre-fusion category. This begs the question, in what properties of the base algebra A will pass on properties to the category of modules?

2.3 Tannaka Duality

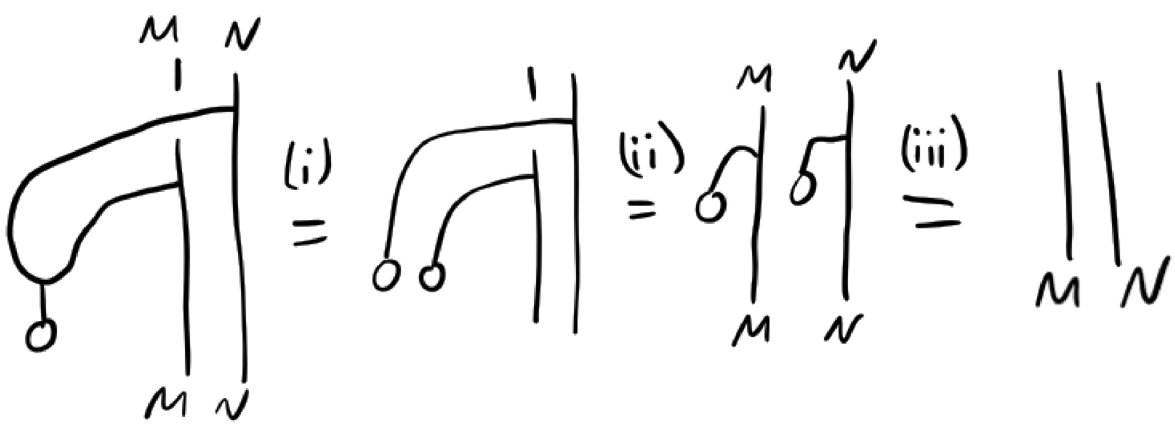
Tannaka Duality is a very general concept in Category Theory. In it's most general form, we are able to reconstruct the object A when given the category of modules mod_A. For our purposes, we will be interested in the forward direction: which properties of our object A will put properties (monoidal, linear, brading, fusion, etc.) onto our category mod_A. Because we have so many visual representations of the concept we have covered, we can figure out the necessary requirements "topologically".



If we want mod_A to have a monoidal structure, we need for A to act on $M \otimes N$. An equivalent condition is to have A act on both M and N side by side. We first "create" two copies of A using the coproduct and cross the two lines by using a braiding from our category C. Let us check that this defines an action of A on $M \otimes N$. We first check that the multiplication structure of A is compatible with our expected action. In the image below, (i) follows because A is a bialgebra and (ii), (iv) are deformations, and (iii) uses the multiplication property of A acting on M and N individually. of each other.



To check that that the identity of A is compatible with our action, in (i) we use that A is a bialgebra , (ii) is a deformation, and (iii) uses how the identity acts on M and N individually.

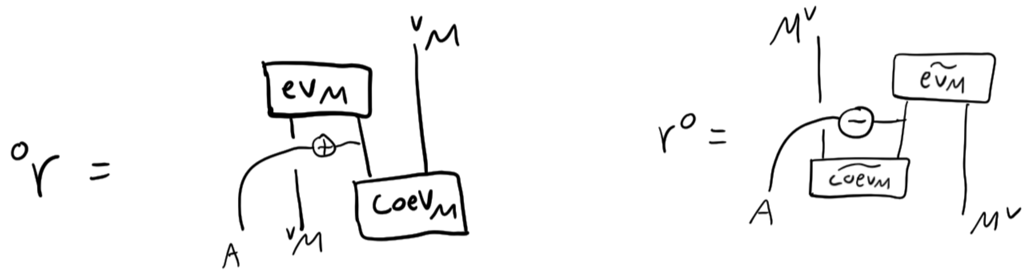


A sufficient condition for mod_A to be a monoidal category is for A to be a bialgebra and C to be a monoidal braided category. □

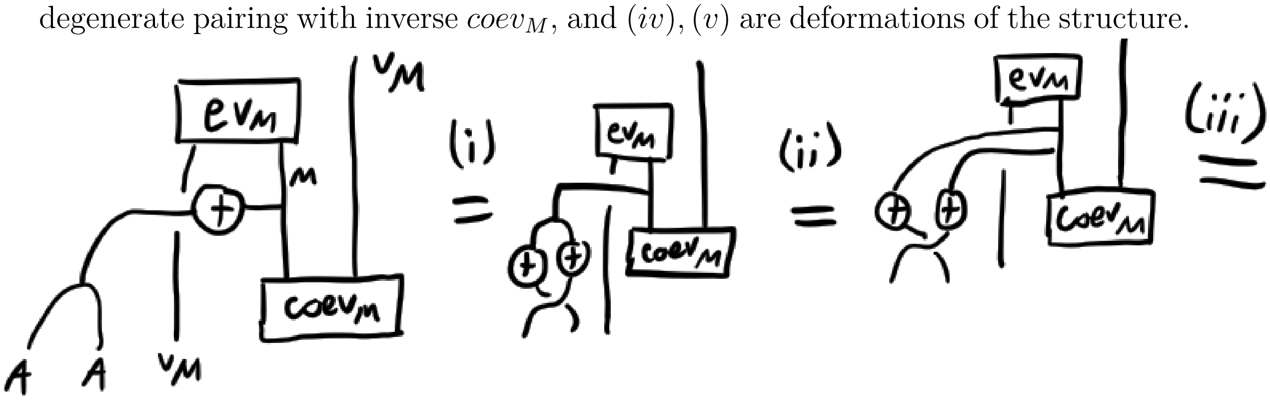
Turning mod_A into a rigid category is much more difficult. We will cite a fact from Turaev and Virelizier 2017 and prove the forward direction.

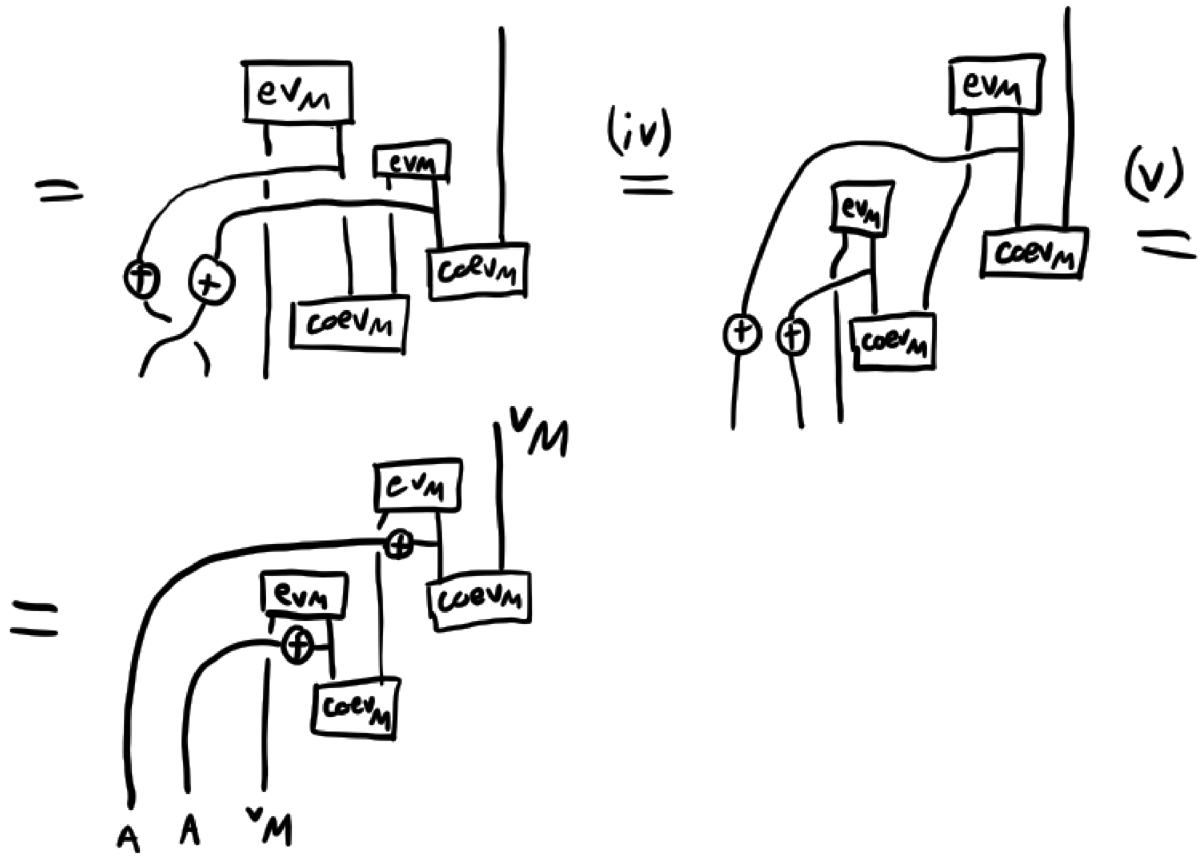
Theorem 41. *Let A be a bialgebra in a braided rigid category C. The monoidal category mod_A is rigid if and only if A is a Hopf algebra.*

Proof: If we fix dualities $\{({}^\vee X, ev_X)\}$ and $\{(X^\vee, \tilde{ev}_X)\}$ of our category C, then duals of (M, r) are $({}^\vee M, {}^\circ r)$ and $(M^\vee, r^\circ)$ where we draw ${}^\circ r$ and r° as:

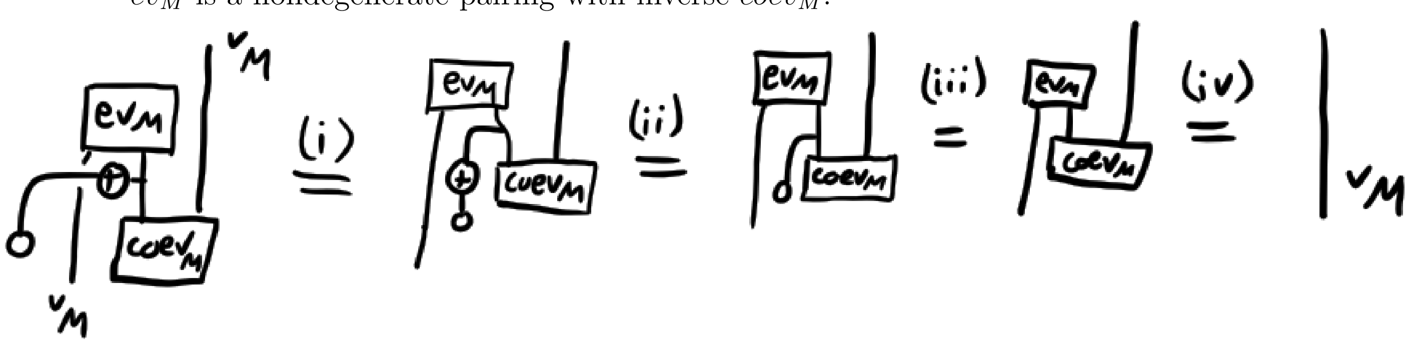


Again, there are two properties of the modules structure to check. To check the multiplication compatibility, in (i) we use that A is a Hopf algebra, (ii) uses the compatibility of multiplication with the module structure on M, (iii) uses that ev_M is a non-degenerate pairing with inverse $coev_M$, and $(iv), (v)$ are deformations of the structure.





Nest we prove the compatibility of the unit on the module structure. In (i) we deform the image, in (ii) we use a property of A as a Hopf algebra, in (iii) we use the compatibility of the unit on the usual module structure, and in (iv) we use that ev_M is a nondegenerate pairing with inverse $coev_M$.



Thus we have show that $^\circ r$ is a well-defined action on $^\vee M \in mod_A$ and the proof of r° is similar. The final step is to check that this in fact defines a left dual structure but this follows from the fact that ${}^{\vee}M$ is a left dual of M in our category C $\square$.

We will list some other examples of Tannaka duality below, some beyond the scope of this paper:

- C linear $\to mod_A$ is linear
- C is braided and pivotal, A is an involutory Hopf algebra $\to mod_A$ is pivotal
- C is also spherical, then mod_A is spherical
- Hopf algebras A where mod_A has a braiding are called quasitriangular Hopf algebras.

In the next section, we will look at two examples of Hopf algebras and an example of a quasitriangular Hopf algebra.

Chapter 3

Examples of Braided Monoidal Categories

3.1 Quantum Groups

In Quantum Mechanics, to quantize a theory means to change the measurement values (x, p) or the fields themselves from ordinary numbers into operators. As a side effect, certain values (such as the position and momentum operators) no longer commute. This process motivate the ideas behind quantum groups. Instead of forcing values to become operators, one way of producing quantum groups is to "deform" the commutation bracket of a Lie Algebra. More concretely, this means to insert some value q into the commutation relation, where if $q = 1$ we recover the original equations Kassel 2012.

In these next few problems, we will "quantize" the plane of the vector space of $\mathbb{K} < x, y >$ where instead of having $xy = yx$, we let $yx = qxy$ for some number q.

Definition 42. We define the quantum group $M_q(2)$ as the free $\mathbb{C}$-algebra $\mathbb{C} < a, b, c, d >$ quotient the relations $\{ba = qab,\ cb = qbc,\ db = qbd,\ ca = qca,\ bc = cb,\ ad - da = (q^{-1} - q)bc\}$.

These relations look very contrived, but in fact these relations are equivalent to the conditions that $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\begin{pmatrix} x'' \\ y'' \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ satisfy the relations $y'x' = qx'y'$ and $y''x'' = qx''y''$. The quantum group $M_q(2)$ consists of all 2×2 matrices which preserve the quantum plane (as well as their transposes) and is another example of an algebra.

We can define maps on $M_q(2)$ as natural operations of matrices $\Delta : M_q(2) \to M_q(2) \otimes M_q(2)$ and $\epsilon : M_q(2) \to \mathbb{C}$. We define these maps on the variables a, b, c, d which generate $M_q(2)$. We abbreviate these equations as follows:

$$\Delta \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } \epsilon \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Theorem 43. *The algebra $M_q(2)$ is a bialgebra under the operations of multiplication, unit, Δ, and ϵ as defined above.*

Proof: The calculations to confirm the bialgebra properties are straightforward and require little explanation.

$$(\Delta \otimes id)\Delta \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)$$

$$= (id \otimes \Delta)\Delta \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$(\epsilon \otimes id)\Delta \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} =$$

$$(id \otimes \epsilon)\Delta \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

These two equations show that $M_q(2)$ is a coalgebra. To complete the proof, we can show that μ and η are coalgebra morphims, Δ and ϵ are algebra morphisms, or using the diagrams in (2.2) as a reference:

$$\Delta\mu(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} f & g \\ h & j \end{pmatrix}) = \begin{pmatrix} af+bh & ag+bj \\ cf+dh & cg+dj \end{pmatrix} \otimes \begin{pmatrix} af+bh & ag+bj \\ cf+dh & cg+dj \end{pmatrix} =$$

$$(\mu \otimes \mu)(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} f & g \\ h & j \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} f & g \\ h & j \end{pmatrix}) =$$

$$(\mu \otimes \mu)(id \otimes \tau \otimes id)(\Delta \otimes \Delta)(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} f & g \\ h & j \end{pmatrix})$$

$$\Delta\eta(1) = \Delta(1) = (\eta \otimes \eta)(1 \otimes 1) = (\eta \otimes \eta)1$$

Here we used the natural isomorphism of $\mathbb{K} \otimes \mathbb{K} \cong \mathbb{K}$ where in this case $\mathbb{K} = \mathbb{C}$.

$$\epsilon\mu(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} f & g \\ h & j \end{pmatrix}) = \epsilon(\begin{pmatrix} af+bh & ag+bj \\ cf+dh & cg+dj \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = (\epsilon \otimes \epsilon)(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} f & g \\ h & j \end{pmatrix})$$

$$\epsilon\eta(1) = \epsilon(1) = 1$$

These final four equations complete the proof that $M_q(2)$ is a bialgebra $\square$.

Another useful quantity to define is called the *quantum determinant* of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ as the value $det_q = ad - q^{-1}bc = da - qbc$.

Definition 44. The quantum group $SL_q(2) = M_q(2)/(det_q - 1)$ is defined by setting det_q equal to 1 in $M_q(2)$.

Theorem 45. *This additional structure gives us a Hopf algebra with* $S\begin{pmatrix} a & b \\ c & d \end{pmatrix} =$

$det_q^{-1}\begin{pmatrix} d & -qb \\ -q^{-1}c & a \end{pmatrix}$ *which is the quantum version of the inverse of a matrix.*

Proof: There is a single equation to check for this proof:

$\mu(S\otimes id)\Delta\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \mu(S\otimes id)(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\otimes\begin{pmatrix} a & b \\ c & d \end{pmatrix})$

$= \mu(det_q^{-1}\begin{pmatrix} d & -qb \\ -q^{-1}c & a \end{pmatrix}\otimes\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = det_q^{-1}\begin{pmatrix} da-qbc & db-qbd \\ q^{-1}ca+ac & q^{-1}cb+ad \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} =$ $\mu(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\otimes det_q^{-1}\begin{pmatrix} d & -qb \\ -q^{-1}c & a \end{pmatrix}) = \mu(id\otimes S)(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\otimes\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = \mu(id\otimes S)\Delta\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\square$

In addition to quantizing $SL(2)$, we can also quantize the universal enveloping algebra of $\mathfrak{sl}(2)$.

Definition 46. The quantum group $SL_q(2) = M_q(2)/(det_q - 1)$ is defined by setting det_q equal to 1 in $M_q(2)$.

As an example of another quantization of a "classical" algebraic object, we will provide the quantization of $\mathfrak{sl}(2)$ over $\mathbb{C}$.

Definition 47. We define the quantization of $\mathfrak{sl}(2)$, denoted $U_q(\mathfrak{sl}(2))$, as the free algebra

$\mathbb{C} < E, F, K, K^{-1} >$ mod the relations:

$$KK^{-1} = 1 = K^{-1}K,\ KEK^{-1} = q^2E,\ KFK^{-1} = q^{-2}F,\ [E,F] = \frac{K-K^{-1}}{q-q^{-1}}$$

As a ring, $U_q(\mathfrak{sl}(2))$ is Notherian, has no zero divisors, and in the case that q is not a root of unity, it has a basis of $\{E^iF^jK^l\}_{i,j\in\mathbb{N},\, l\in\mathbb{Z}}$. All of these facts are proven using Ore extensions; noncommutative way of adjoining a free variable to a ring. In particular, we can obtain a noncommutative version of the Hilbert basis theorem, but we will not talk about this here. If q is a root of unity, we obtain a finite dimensional vector space:

Definition 48. Let q be a root of unity and e be the smallest integer such that $q^{e-1}+q^{e-3}+...+q^{-e+1}=0$. Then E^e, F^e, K^e are all in the center of U_q and define $\bar{U_q} = U_q/(E^e, F^e, K^e-1)$. Then $\bar{U_q}$ is a Noetherian ring with no zero divisors. As a vector space, this has the basis $\{E^iF^kK^l\}_{0\leq i,j,l\leq e-1}$.

Theorem 49. *The algebra $U_q(\mathfrak{sl}(2))$ with multiplication and unit, can be made into a Hopf algebra with the maps Δ, ϵ, S such that:*
$\Delta(E) = 1\otimes E + E\otimes K, \quad \Delta(F) = K^{-1}\otimes F + F\otimes 1, \quad \Delta(K) = K\otimes K, \quad \Delta(K^{-1}) = K^{-1}\otimes K^{-1}$
$\epsilon(E) = \epsilon(F) = 0, \qquad \epsilon(K) = \epsilon(K^{-1}) = 1$
$S(E) = -K^{-1}E, \quad S(F) = -KF, \quad S(K) = K^{-1}, \quad S(K^{-1}) = K$

Proof: To prove that $U_q(\mathfrak{sl}(2))$ is a Hopf algebra, we must show that Δ is a morphism of algebras and is coassociative, ϵ is a morphism of algebras and satisfies the counit axiom, and S is a morphism of algebras and is an antipode (with all of the relevant interactions between S and $\mu, \eta, \Delta, \epsilon$). More specifically, we show a map is a morphism of algebras for $U_q(\mathfrak{sl}(2))$ if all of the relations of $U_q(\mathfrak{sl}(2))$ are preserved under the operation. However, the calculations are tedious and mostly not insightful. Here we will check the properties of the comultiplication Δ only.

1. $\Delta(K)\Delta(K^{-1}) = (K\otimes K)(K^{-1})\otimes K^{-1}) = 1\otimes 1 = 1 = (K^{-1}\otimes K^{-1})(K\otimes K) = \Delta(K^{-1})\Delta(K)$

2. $\Delta(K)\Delta(E)\Delta(K^{-1}) = (KK^{-1}) \otimes (KEK^{-1}) + (KEK^{-1}) \otimes (KKK^{-1}) = 1 \otimes q^2E + q^2E \otimes K = q^2\Delta(E)$

3. $\Delta(K)\Delta(F)\Delta(K^{-1}) = (KK^{-1}K^{-1}) \otimes (KFK^{-1}) + (KFK^{-1}) \otimes (KK^{-1}) = K^{-1} \otimes q^{-2}F + q^{-2}F \otimes 1 = q^{-2}\Delta(F)$

4. $[\Delta(E), \Delta(F)] = (1 \otimes E + E \otimes K, K^{-1} \otimes F)(K^{-1} \otimes F + F \otimes 1) - (K^{-1} \otimes F + F \otimes 1)(1 \otimes E + E \otimes K, K^{-1} \otimes F) = K^{-1} \otimes EF + F \otimes E + EK^{-1} \otimes KF + EF \otimes K - K^{-1} \otimes FE - K^{-1}E \otimes FK - F \otimes E - FE \otimes K = K^{-1} \otimes [E,F] + [E,F] \otimes K = \frac{K^{-1} \otimes (K-K^{-1}) + (K-K^{-1}) \otimes K}{q-q^{-1}} = \frac{\Delta(K) - \Delta(K^{-1})}{q-q^{-1}}$

This shows that Δ is a morphism of algebras because it preserves all 4 identities of the algebra structure on $U_q(\mathfrak{sl}(2))$. The final property needed to show that Δ satisfies the property coassociativity for $U_q(\mathfrak{sl}(2))$ to be a coalgebra is:

1. $(id \otimes \Delta)\Delta(E) = (id \otimes \Delta)(1 \otimes E + E \otimes K) = (1 \otimes 1 \otimes E + 1 \otimes E \otimes K) + E \otimes K \otimes K = 1 \otimes 1 \otimes E + (1 \otimes E \otimes K + E \otimes K \otimes K) = (\Delta \otimes id)(1 \otimes E + E \otimes K) = (\Delta \otimes id)\Delta(E)$

2. $(id \otimes \Delta)\Delta(F) = (id \otimes \Delta)(K^{-1} \otimes F + F \otimes 1) = (K^{-1} \otimes K^{-1} \otimes F + K^{-1} \otimes F \otimes 1) + F \otimes 1 \otimes 1 = K^{-1} \otimes K^{-1} \otimes F + (K^{-1} \otimes F \otimes 1 + F \otimes 1 \otimes 1) = (\Delta \otimes id)(K^{-1} \otimes F + F \otimes 1) = (\Delta \otimes id)\Delta(F)$

3. $(id \otimes \Delta)\Delta(K^{-1}) = K^{-1} \otimes K^{-1} \otimes K^{-1} = (\Delta \otimes id)\Delta(K^{-1})$

4. $(id \otimes \Delta)\Delta(K) = K \otimes K \otimes K = (\Delta \otimes id)\Delta(K)$

Because Δ, ϵ, S are linear, these formulas on the generators extend to all of $U_q(\mathfrak{sl}(2))$.

Remark: This is a Hopf algebra which is neither commutative nor cocommutative.

Remark: For any Hopf algebra H, we can construct the Drinfeld double; a Hopf Algebra structure on $(H^{op})^* \otimes H$ and denote it $D(H)$. If we return to the case of $\bar{U}_q$

(q is a root of unity), then define $B_q = \{E^i K^l\}_{0 \leq\ i,l\ \leq e-1}$. It is possible to define a braiding on $mod_{D(B_q)}$ which can be surjected onto $\bar{U}_q$. Thus $\bar{U}_q$ is an example of a quasitriangular Hopf algebra.

3.2 Extension to Physics

In this paper, we have looked at many algebraic objects and seen how they generalize in Category Theory. Furthermore, we have seen how morphisms in monoidal categories can be drawn and used for intuition to build more complicated structures. Monoidal Categories are a very modern branch of mathematics. The formal definition of a braided monoidal category was introduced by a paper by Joyal and Street in 1985, but examples have exists since Feynman diagrams were invented in 1948. Monoidal categories are the basis of more general physics theories such as Topological Quantum Field Theories. In TQFTs, the objects of the category are the initial and final states of particle(s) with morphisms representing the evolution of particles. Even today, TQFTs are an area of active research in an attempt to prove the ADS/CFT conjecture in physics. Finally, braided monoidal categories appear in Loop Quantum Gravity in an attempt to combine General Relativity and quantum mechanics. In this theory, the morphisms no longer represent evolutions in time, but instead the quantized states of gravity.

Chapter 4

4-Manifolds

One topic of interest in topology is to determine which manifolds are equivalent to one another and if possible to describe all equivalence classes of manifolds. There are several different definitions of "equivalent". For smooth manifolds, we typically mean two manifolds are equivalent if they are diffeomorphic; there exists a C^∞ map with C^∞ inverse. For more general manifolds, we might ask if two manifolds are homeomorphic; a continuous map between the manifolds with continuous inverse. In particular, the extra structure on a manifold (such as smoothness) motivates new notions of equivalence.

For our purposes, we are interested in the topology of a class of 4 dimensional manifolds called handlebodies. These are obtained by gluing "handles" onto S^4. We will describe how we can visualise handlebodies by drawing the attachment sites of the handles onto the surface of the ball. Finally, we will introduce a class of 4-Manifolds which are diffeomorphic to S^4, but may not satisfy the stronger condition of 2-equivalence.

4.1 Handlebodies

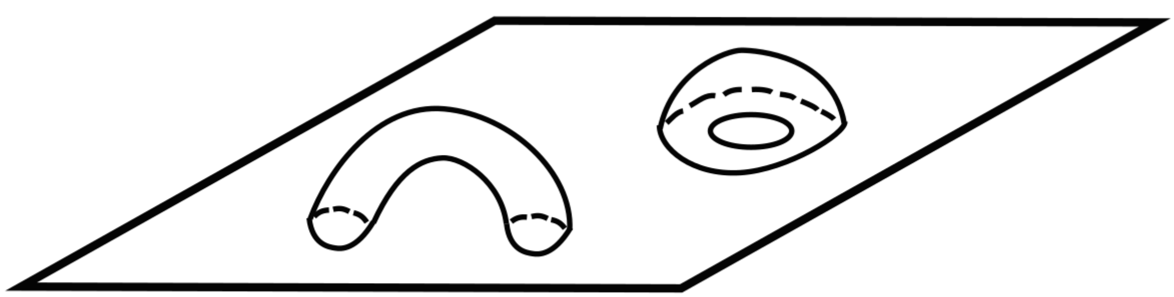
Handlebodies are a type of manifold that are constructed in a similar way to CW complexes. Handlebodies are constructed by successively attaching handles of increasing dimension to the base $D^n \simeq I^n$. To add handles to this ball, we begin with a handle $I^n = I^k \times I^{n-k} \simeq D^k \times D^{n-k}$ and attach it to the boundary of the ball via its attaching map $\phi : \partial D^k \times D^{n-k} \to \partial D^n$. This is called a n-dimensional k-handle.

Definition 50. Given such a handle described above, we call $D^k \times \{0\}$ the **core** of the handle and $\{0\} \times D^{n-k}$ the **cocore**. Thus we can think of attaching the boundary of the core D^k onto D^n, and then thickening the core by the cocore D^{n-k}. A **handlebody** X is built as a filtration:

$$D^n = X^{-1} \subset X^0 \subset X^1 \subset ... \subset X^n = X$$

where X^i is constructed from X^{i-1} by attaching a finite number of n-dimensional i-handles. Note this means a j-handle can be attached to the boundary of an i-handle if $j > i$. We say that a manifold is a n-dimensional k-handlebody if it is a n-dimensional handlebody with $X = X^k$.

In the figure below, we have a 3-dimensional 1-handlebody (left) and a 3-dimensional 2-handlebody (right).



Note, in the general case, we do not need to start with D^n and attach handles, we can instead begin with an $n-1$ dimensional manifold Y and successively attach handles to $Y \times I$. Handlebodies also have their own special type of equivalences.

Definition 51. Let $l \leq n$ be an integer, then a **l-deformation** of an n-dimensional k-handlebody is a finite sequence of moves of the following type (for some i; $0 \leq i \leq l$):

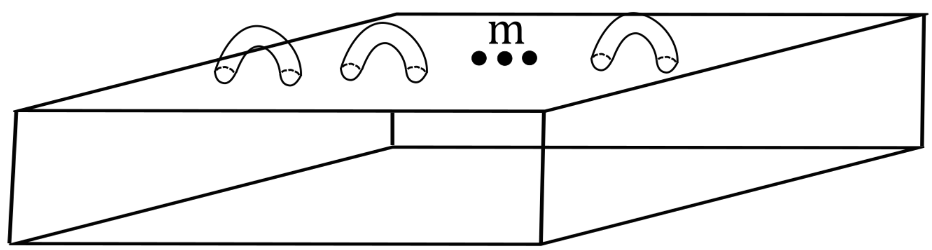
1. isotopy of the attaching map of an i-handle inside of the boundary of X^{i-1}
2. sliding an i-handle over another i-handle
3. creating/ removing a canceling pair of $(i-1)/i$-handles

For example, in the diagram above, if we were to glue half of the 2-handle onto the 1-handle and the other half onto $\partial(Y \times I)$, then this pair can annihilate by folding the "cup" like handle into $Y \times I$.

As noted in Gompf and Stipsicz 1999, there is a canonical way to smooth handlebodies so we take them to be smooth manifolds and thus l-deformations are smooth. This raises the question: if two handlebodies are diffeomorphic, are they l-equivalent? In the case of $(n-1)$ equivalence, this is known to be the same as a diffeomorphism. However, this is an open question for $(n-2)$-equivalence. The paper in study for this thesis, Beliakova and De Renzi 2021, has constructed a topological invariant which may allow us to distinguish between 4-dimensional 2-manifolds that are not 2-equivalent. We now introduce the category of 4-dimensional 2-handlebodies.

Definition 52. The category of 4-dimensional 2-handlebodies, denoted **4HB** has as objects integers $m \in \mathbb{Z}$ and morphisms are 2-deformation classes of 4-dimensional 2-handlebodies. Take a representative handlebody that lies in $Hom(m, m')$. If we slice the handlebody at the top or bottom, we obtain a cube with m or m' 3-dimensional

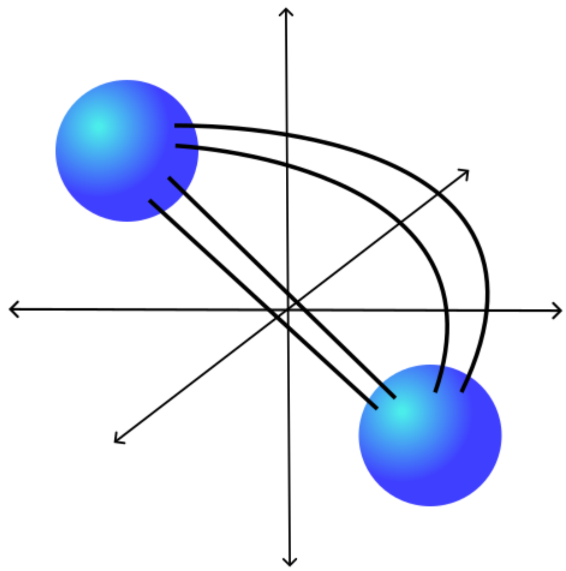
1-handlebodies (see below) denoted H_m. Thus the representatives can be thought of as a way that H_m evolves into $H_{m'}$ over time $t \in [0, 1]$. Then we have composition of handle bodies by vertically stacking them ($m \to m' \to m''$). This is a monoidal category by $m \otimes m' = m + m'$ and we place handlebodies side by side.



4.2 Kirby Tangles

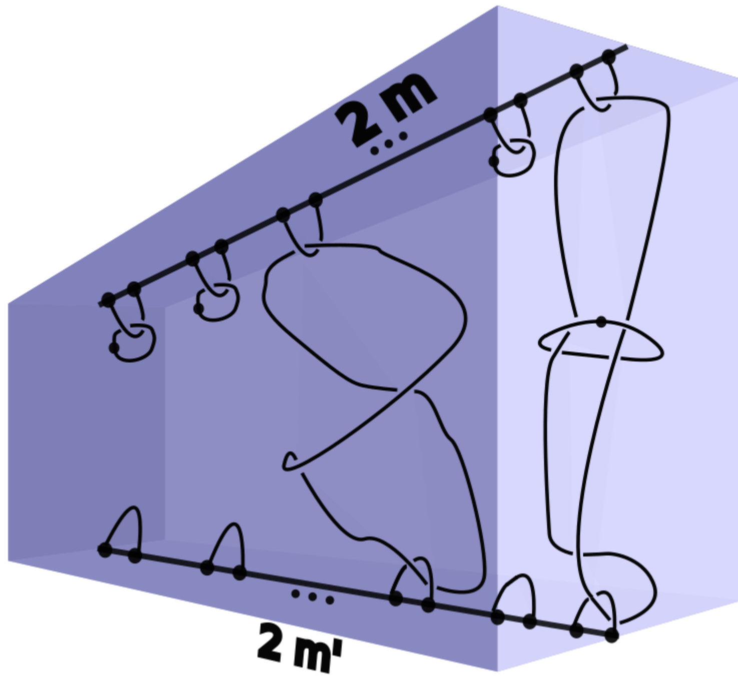
A Kirby diagram represents a 4-dimensional handlebody with a diagram in R^3. We start with a unique 0-handle D^4 with boundary $S^3 = \mathbb{R}^3 \cup \{\infty\}$. Instead of trying to visualize the entire 4-dimensional manifold, we only focus on drawing the attaching points of the handles.

For example, were we to attach a 3-dimensional 1-handle to D^3, the intersection with the boundary would be a pair of disjoint circles. The corresponding 4-dimensional 1-handle is a pair of spheres in $\mathbb{R}^3$. Then 4-dimensional 2-handles can be drawn given some strand $S : [0, 1] \to \mathbb{R}^3$ as $S \times D^2$. However, it is simply easier to draw S in $\mathbb{R}^3$.



There is a related notion to the Kirby diagram called the Kirby tangle. Kirby tangles are another way to describe 4 dimensional 2-handlebodies, but vary slightly from Kirby diagrams. Kirby tangles are tangles of string inside I^3, connecting $2m$ points to $2m'$ points, as well as consisting of dotted unknots. We can denote the category of Kirby tangles as $KTan$.

Definition 53. The category of Kirby tangles **KTan** is category whose objects are integers $m \in \mathbb{Z}$ and morphisms are 2-deformation classes of Kirby tangles. Composition is again given by vertical stacking. The monoidal product is given by $m \otimes m' = m + m'$, morphisms are placed side by side.



The categories 4HB and KTan are equivalent categories. We send objects $m \mapsto m$ and for each Kirby tangle, we turn strands into 4-dimensional 1-handles.

4.3 Potential Applications

As motivation for the machinery we are about to build up, we include some conjectures these could help answer. The first conjecture appears as Conjecture 1.4 in Beliakova and De Renzi 2021.

Conjecture 1: Every 4-dimensional 2-handlebody that is diffeomorphic to D^4 is 2-equivalent to D^4.

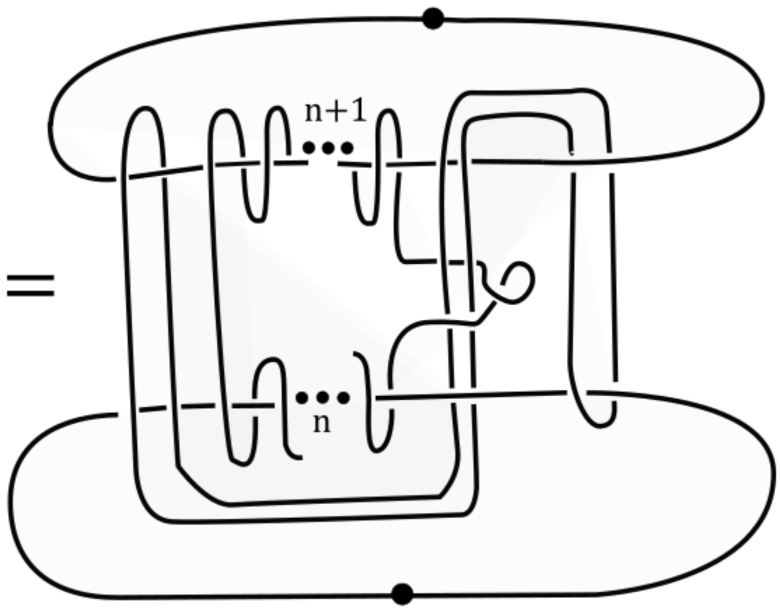
This is the most immediate consequence of Beliakova and De Renzi 2021 and this work. All of the 4-balls Δ_n shown in Section 4.4 are diffeomorphic to D^4, but they are not know to be 2-equivalent. The constructed invariants have the potential to distinguish between non 2-equivalent manifolds so long as we can find two 4-balls and a topological invariant on which they differ.

Conjecture 2: (Andrews-Curtis Conjecture) Every balanced presentation of the trivial can be transformed into the trivial group using Nielsen moves.

In the next subsection we will display the Kirby tangles of these 4-balls and explain their significance.

4.4 Examples

The class of examples of 4 manifolds we will study are the family of 4-balls labeled $\{\Delta_n : n \in \mathbb{N}\}$. These were introduced in Gompf and Stipsicz 1999, are diffeomorphic to D^4, and are conjectured to provide a counterexample to Conjecture 1.4 in Beliakova and De Renzi 2021. Furthermore, they have fundamental group $\pi_1(\Delta_n) =< x, y \,|\, xyx = yxy, x^n = y^{n+1} >$ which is expected to give counterexamples to the Andrews-Curtis Conjecture for $n \geq 3$.

$\Delta_n =$ 

Chapter 5

Topological Invariant

As we have seen previously, one powerful method for determining the equivalence of two manifolds is to use an algebraic invariant. These processes assign to each manifold an algebraic object (such as the Fundamental Group or Homology Groups), a polynomial (such as the Jones poynomial), or some noncommuting polynomial expression (Reshetikhin-Turaev invariants and the invariants of Beliakova and De Renzi 2021). Most of the work of these techniques is in showing that the algebraic value is preserved under equivalence of manifolds. For our purposes, we will follow the algorithm proved in Beliakova and De Renzi 2021.

5.1 More Algebra and Category Theory

Recall from section 1.2 that a Hopf algebra over a field k is a (finite-dimensional) vector space H equipped with five additional maps $(H, \mu, \eta, \Delta, \epsilon, S)$ satisfying certain relations. Many examples of Hopf Algebras, such as those that appear in Algebraic Geometry are cocommutative.

Definition 54. Choose a basis for the Hopf algebra H as a vector space: $H = < v_1, \dots, v_n >$ so that for all $x \in H$, x is written $w = \Sigma_i a_i v_i$ and $\Delta(x) = \Sigma_{i,j}\, b_{ij}\, v_i \otimes v_j$. We say that H is **cocommutative** if $b_{ij} = b_{ji}$ for all i, $j = 1, \dots, n$.

One example of a Hopf algebra that is neither commutative nor cocommutative is the quantum group $U_q\mathfrak{sl}_2$ from Section 3.1. In Section 6 we will explain much more general constructions of quantum groups which provides a large class of neither commutative nor cocommutative Hopf algebras. However, they all satisfy a weaker property: they are cocommutative up to conjugation of an element of $H \otimes H$.

Definition 55. Let $(H, \mu, \eta, \Delta, \epsilon, S)$ be a Hopf algebra. We say that H is **quasicocommutative** if there is an invertible element R of $H \otimes H$ such that $\Delta^{op}(x) = R\Delta(x)R^{-1}$ for all $x \in H$ where if $\Delta(x) = \Sigma_{i,j}\ b_{ij}\ v_i \otimes v_j$, $\Delta^{op}(x) = \Sigma_{i,j}\ b_{ji}\ v_i \otimes v_j$. We call the element R a **(universal) R-matrix** Kassel 2012 and denote $R = \Sigma_i\ s_i \otimes t_i$.

As we will see, many quantum groups satisfy a slightly stronger condition.

Definition 56. Let $(H, \mu, \eta, \Delta, \epsilon, S)$ be a quasicocommutative Hopf algebra with universal R-matrix R. We say that H is **quasitriangular** or a **braided Hopf algebra** if the universal R-matrix satisfies $(\Delta \otimes id_H)(R) = R_{13}R_{23}$ and $(id_H \otimes \Delta)(R) = R_{13}R_{12}$. Here by R_{jk} we mean $R_{jk} = \Sigma_i\ R_i^{(1)} \otimes R_i^{(2)} \otimes R_i^{(3)}$ where $R_i^{(j)} = s_i$, $R_i^{(k)} = t_i$, and $R_i^{(l)} = 1$ for $l \neq j, k$.

A Hopf algebra H being quasitriangular is actually a very desirable property. The proofs of these properties occur in Chapter VIII of Kassel 2012, but they are listed here.

- The inverse of the R-matrix is given $R^{-1} = (S \otimes id_H)(R)$.
- The category of modules of a quasitriangular Hopf algebra is braided.
- In a cocommutative Hopf algebra, $S^2 = id_H$.
- In a Hopf algebra with invertible antipode and universal R-matix $R = \Sigma_i\ s_i \otimes t_i$. Let $u = \Sigma_i\ S(t_i)\, s_i$ which is invertible in H, $u^{-1} = \Sigma_i\ S^{-1}(t_i)\, S(s_i)$ and

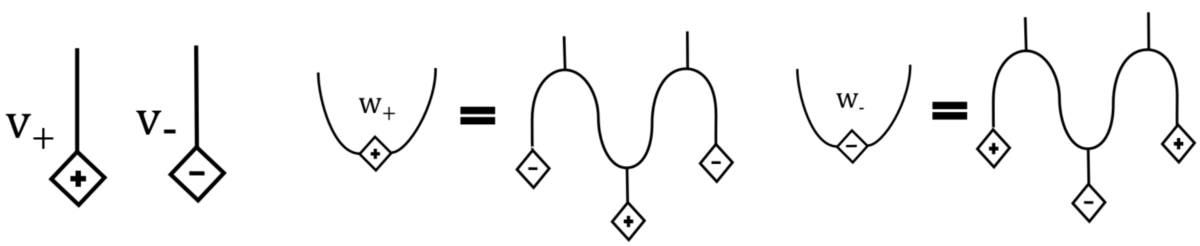
- In the above case, $S^2(x) = uxu^{-1}$ for all $x \in H$ and $S(u)u = uS(u)$ is central in H.

The paper Beliakova and De Renzi 2021 introduces these additional definitions. They are not central to the implementation of the algorithm, but are necessary to understand the underlying machinery explained in Section 8.

Definition 57. A **finite** category C is a k-linear category such that every object X has a projective cover, that is an object P_X with projection morphism $\epsilon_X : P_X \to X$. (Any finite category is equivalent to mod_A, the category of finite dimensionals left-A modules for an algebra A over k.)

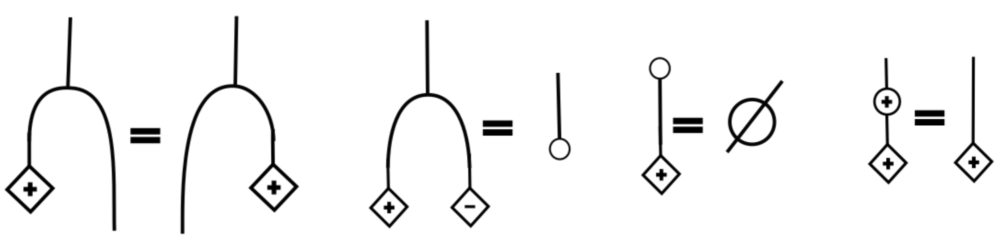
Definition 58. A rigid category C is **unimodular** if it is finite and for the identity element of the category 1, its projective cover satisfies $P_1^* \equiv P_1$.

Definition 59. A **BP-ribbon Hopf algebra** is a Hopf algebra H with two additional morphisms v_+, $v_- \in Hom(1, H)$ and we define the copairings w_+, and w_- as follows:



The morphisms $v_\pm$ are called ribbon elements and must satisfy several conditions:

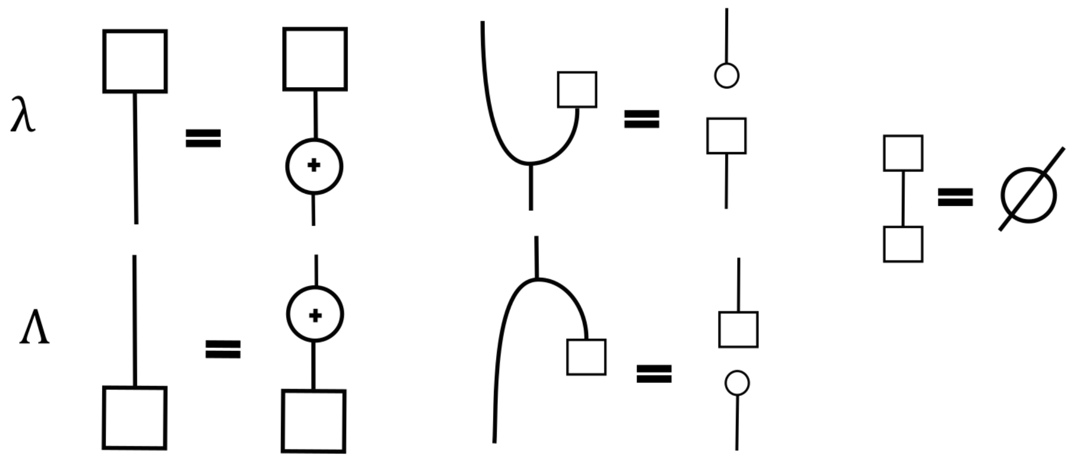
$v+$ is central, invertible, normalized, and antipode invariant.



They also satisfy (see Beliakova and De Renzi 2021):

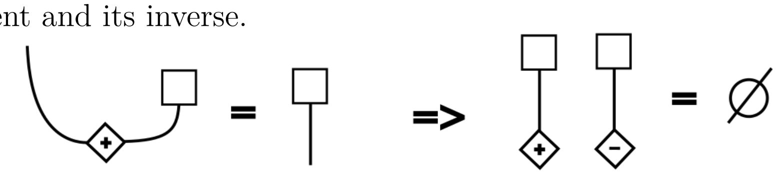
1. w_+ is a Hopf copairing. 2. w_+ is compatible with the coproduct and the copairing.

Definition 60. A **BP-unimodular Hopf algebra** is a braided Hopf algebra with two additional morphisms $\lambda \in C(H, 1)$ and $\Lambda \in C(1, H)$ called the integral and cointegral. They satisfy the following conditions:



Definition 61. A Hopf algebra that is both BP-ribbon and BP-unimodular is called a **BPH algebra** (or **4-modular** or **premodular**, the authors have changed language several times).

Definition 62. A BPH algebra is (**3-modular** or **modular**) **factorizable** if w_+ is nondegenerate. This implies the cancelation of the product of the integrals of the ribbon element and its inverse.



Definition 63. A factorizable BPH algebra is **anomaly-free** if the integral cancels ribbon element.

Definition 64. A **transparent** object X of a category C is an element that lies in its Muger center. That is, for all other objects Y in C, the braiding satisfies $c_{Y,X} \circ c_{X,Y} = id_{X \otimes Y}$.

Definition 65. A **factorizable** category is one in which every element of its Muger center is a direct sum of copies of the tensor unit 1.

Definition 66. An **end** in a category consists of an object $\epsilon = \int_{X \in C} X \otimes X^*$ and dinatural transformation $\{j_X : \epsilon \to X \otimes X^* \ : \ \forall X \in C\}$. Recall that a **dinatural transformation** satisfies for all $f : X \to Y$, we have $(f \otimes id_{X^*}) \circ j_X = (id_Y \otimes f^*) \circ j_Y$. The notation $\int_{X \in C}$ means to take the direct sum over the simple objects in the category.

Every finite, rigid, monoidal category has an end $\epsilon = \int_{X \in C} X \otimes X^*$ and universal dinatural transformation $(\_ \otimes \_^*) : C \otimes C^{op} \to C$ which sends $(X, Y) \mapsto X \otimes Y^*$. We call it "universal" because for any other such dinatural transformation $(\eta, \{k_X\})$ splits by a map $\phi : \epsilon \to \eta$ such that for all X, $k_X = j_X \circ \phi$. as As an aside, there is also a dual notion called a coend.

Definition 67. A coend is an object D with universal dinatural transformation $\{\rho_X : X^* \otimes X \to D\}$ such that for any other dinatural transformation $(E, \{\sigma_X\})$, there is a unique $\psi : E \to D$ such that for all X, $\sigma_X = \psi \circ \rho_X$.

Facts about ends from Beliakova and De Renzi 2021 and coends from Turaev and Virelizier 2017:

- The end of a unimodular ribbon category is a BPH algebra.
- The end of a factorizable ribbon category is factorizable.
- In a left rigid category, the coend (D, ρ) is a coalgebra.
- In a braided rigid category, the coend (D, ρ) is a Hopf algebra.
- We have that the category of D-modules is isomorphic to the Drinfeld center of a category $mod_D \simeq Z(C)$.
- For additive pivotal fusion k-categories, the coend is given by $\bigoplus_{i \in I} i^* \otimes i$.
- For additive pivotal **braided** fusion k-categories D has a left integral and cointegral.
- If the underlying category is also a ribbon category, D has ribbon elements.

Definition 68. Given a unimodular ribbon Hopf algebra H, and mod_H denotes the unimodular ribbon category of finite-dimensional left H-modules. We define the **transmutation** $\underline{H} \in mod_H$ where we can give the adjoint representation of H lots of structure (sometimes denoted ad). It is a fact that mod_H is a unimodular ribbon category. We equip the vector space H with the adjoint left H-action $h \triangleright x = h_{(1)} x S(h_{(2)})$.

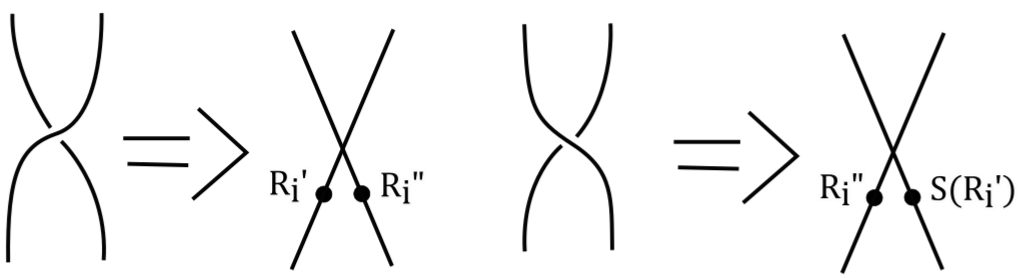
We construct $\underline{H}$ as the end of mod_H. It is given by an object $\underline{H} = \int_{V \in mod_H} V \otimes V^*$ with structure morphism $j_X : \underline{H} \to X \otimes X^*$ which sends $x \mapsto \Sigma_i (x \cdot v_i) \otimes f_i$ where $\{v_i\}$ and $\{f_i\}$ are dual bases for X, X^* respectively. This has a braiding $c_{\underline{H},\underline{H}}(x \otimes y) = (R'' \triangleright y) \otimes (R' \triangleright x)$ where R is the R-matrix of H.

Finally, $\underline{H}$ inherits its own Hopf algebra, ribbon, integral morphisms from those of H. They are given in Beliakova and De Renzi 2021, but we will include a few here for clarity: $\underline{\mu}(x \otimes y) = xy$, $\underline{\eta}(1) = 1$, $\underline{\Delta}(x) = x_{(1)}S(R''_i) \otimes (R'_i \triangleright x_{(2)})$, $\underline{\epsilon}(x) = \epsilon(x)$, ... etc.

Most of these formalities are not necessary to understand the algorithm. However, we will use some of these terms to explain the algorithm more fully in Section 8.

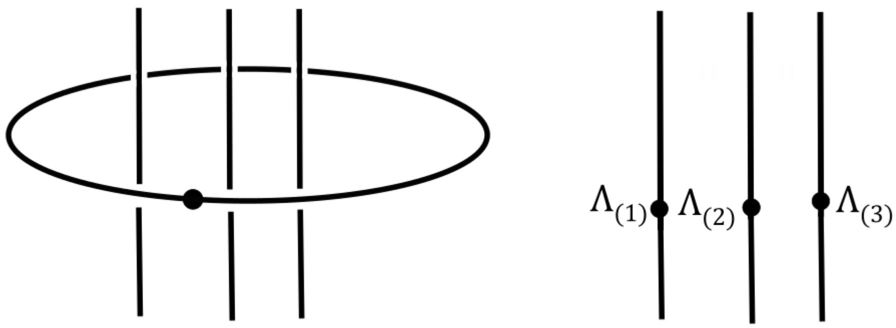
5.2 The Algorithm Described

For our purposes, the algorithm in Beliakova and De Renzi 2021 will take in the Kirby tangle of Δ_n and produce a map $J_4(\Delta_n) = \Phi_n : \mathbb{C} \to \mathbb{C}$. The first step of the algorithm is to identify all places where nondotted strands cross over one another. Depending on which strand is on top, we introduce a pair of beads and remove the crossing.

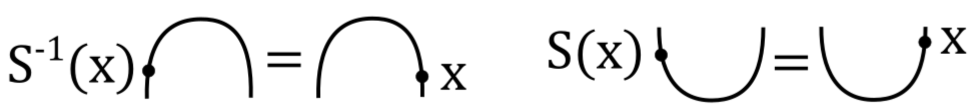


Each crossing that is removed will introduce components of our R matrix, each which a new subscript. Keep in mind that because $R = \Sigma R'_i \otimes R''_i$, there is an implicit sum for each new subscript. For brevity we will not write these sums until the very end. The second step is to remove the dotted components and replace their encircled strands with beads labeled by Λ.

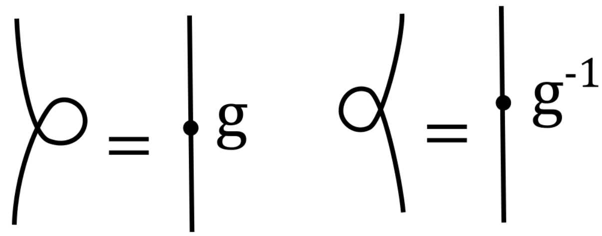
These dots should be thought of as being "beads" on a string. They are allowed to move over flattened crossings, but can not move over caps and cups until later. The second step of the algorithm is to remove dotted components at the cost of adding additional beads. At every strand encircled by a dotted component, we add a bead labeled by Λ_i.



Using Sweedler notation, these beads mean that $\Delta(\Lambda) = \Sigma\ \Lambda_1 \otimes \Lambda_2$, $(\Delta \otimes id)\Delta(\Lambda) = \Sigma\ \Lambda_1 \otimes \Lambda_2 \otimes \Lambda_3$ and so on. It does not matter to which component we apply Δ by coassociativity. The third step of the algorithm is to pull apart any remaining loops. Because any remaining crossings are "flattened", there is no trouble with translating the loops away from one another (this is visualized in Section 5.3). The fourth step is to move all of the beads to the leftmost side of their loop. Beads are allowed to slide over caps or under cups at the cost of picking up an S or S^{-1}.



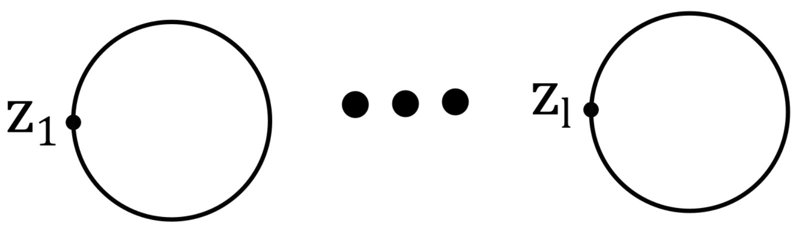
Any remaining twists in the loops are turned into beads labeled by the pivotal element and moved to the leftmost edge.



Finally, the loops are homotopic to circles and so we simplify the loops. For our purposes, we can use the simplified equation from Beliakova and De Renzi 2021:

$$J_4(T)(1) = \Pi_{k=1}^{l} \lambda(z_k)$$

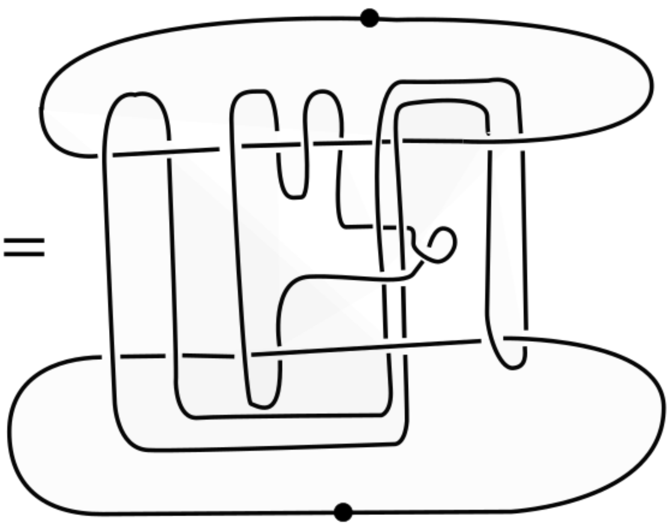
where the z_k are beads corresponding to each circle. In the simplified diagram, the z_k are obtained by reading the column of beads from top to bottom.



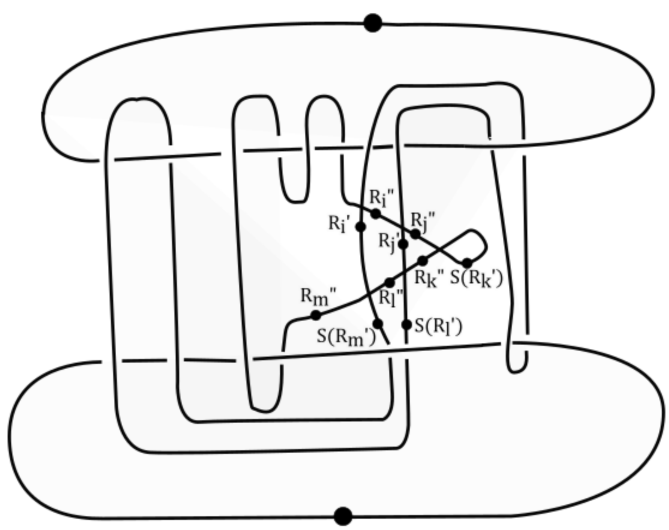
More specifically, we will see that for Δ_n the procedure will leave only two circles.

5.3 Some Computed Examples

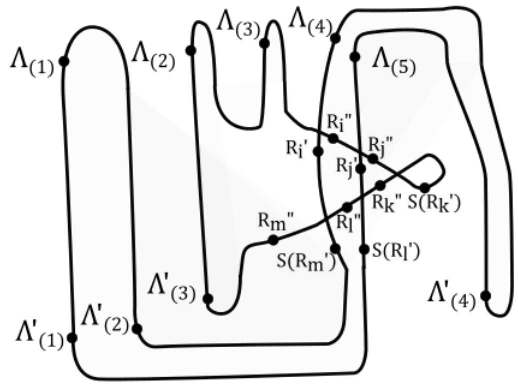
We calculate the topological invariant for Δ_n in the symbols λ, Λ, S, R, and g before giving them meaning in Section 6. There we discuss more general quantum groups. Finally, in Section 7 we will discuss some implications of our results. We first focus on the ball Δ_1:

$\Delta_1 =$ 

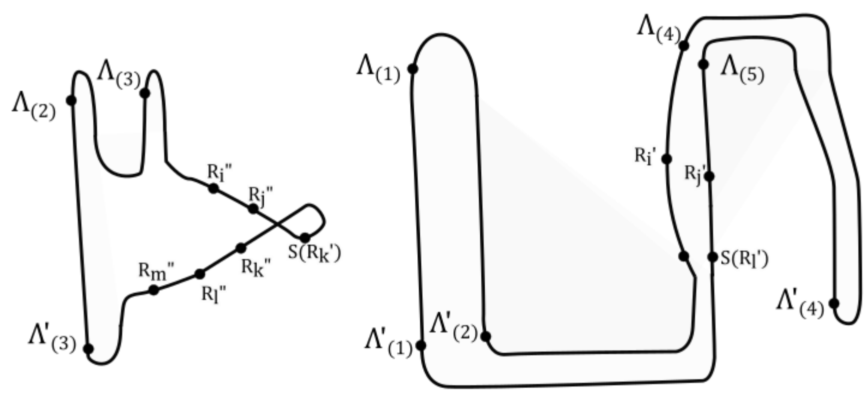
As described in 5.2, the first step is to replace crossings of undotted strings with dotted, flattened tangles.



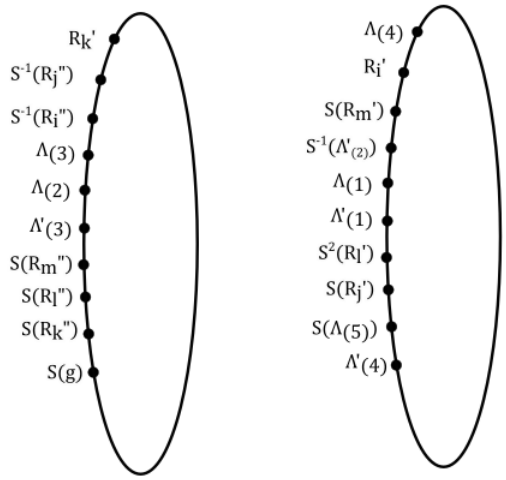
We remove the dotted components and add beads to the ensnared strings.



In this flattened version of our tangle, we can translate the two loops away from each other.



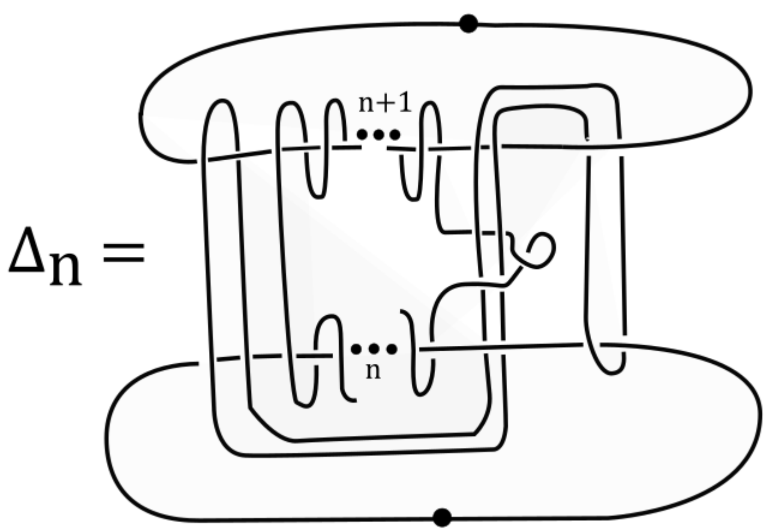
Finally, we can move beads along to one side of the loop up to action by S. These loops are homotopic to circles.



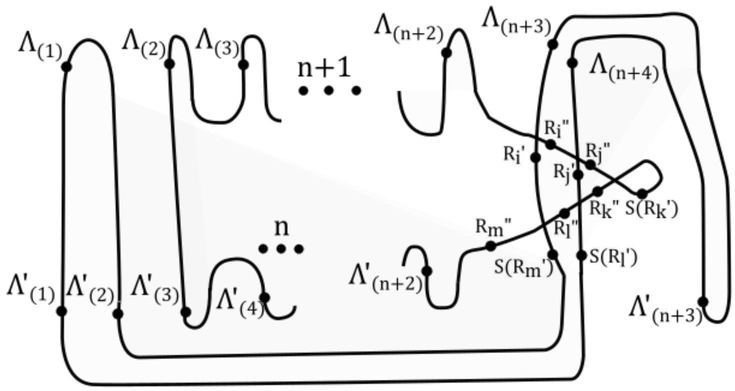
Finally, we can apply the formula as included in Section 5.2.

$$J_4(\Delta_1) = \lambda(R'_k\ S^{-1}(R''_j)\ S^{-1}(R''_i)\ \Lambda_{(3)}\ \Lambda_{(2)}\ \Lambda'_{(3)}\ S(g\ R''_k\ R''_l\ R''_m))$$
$$*\lambda(\Lambda_{(4)}\ R'_i\ S(R'_m)\ S^{-1}(\Lambda'_{(2)})\ \Lambda_{(1)}\ \Lambda'_{(1)}\ S^2(R'_l)\ S(R'_j)\ S(\Lambda_{(5)})\ \Lambda'_{(4)})$$

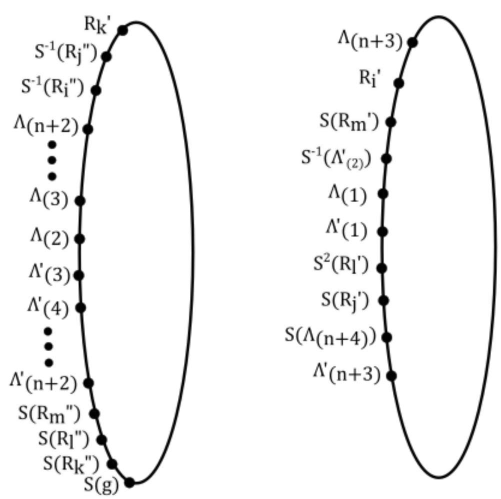
The good news is that the case of general n is not much more complicated. We again start with the Kirby tangle.



Just as before, we remove crossings and dotted components at the cost of adding labeled beads. The number of crossings remains constant so the only difference is that every increase in n by 1 adds another two Λ beads.



Finally, we can translate the beads to one side as in the Δ_1 case.



The final formula is:

$$J_4(\Delta_n) = \lambda(R'_k\ S^{-1}(R''_j)\ S^{-1}(R''_i)\ \Lambda_{(n+2)}\ ...\ \Lambda_{(3)}\ \Lambda_{(2)}\ \Lambda'_{(3)}\ \Lambda'_{(4)}\ ...\ \Lambda'_{(n+2)}\ S(g\ R''_k\ R''_l\ R''_m))$$
$$*\lambda(\Lambda_{(n+3)}\ R'_i\ S(R'_m)\ S^{-1}(\Lambda'_{(2)})\ \Lambda_{(1)}\ \Lambda'_{(1)}\ S^2(R'_{(1)})\ S(R'_j)\ S(\Lambda_{(n+4)})\ \Lambda'_{(n+3)})$$

For different Hopf algebras, these invariants will change as the R-matrix, Λ, λ etc. change. In the next section, we will show how to compute these for different quantum groups.

Chapter 6

General Quantum Group

We can construct general quantum groups from a Lie algebra. This process is outlined in the papers Beliakova and De Renzi 2021 and De Renzi, Geer, and Patureau-Mirand 2020.

Let $\mathfrak{g}$ be a Lie algebra of rank n and $\mathfrak{h}$ be a Cartan Lie subalgebra of $\mathfrak{g}$ (note that $dim(\mathfrak{h}) = n$). We choose a base for Φ_+, the positive roots of the adjoint representation of $\mathfrak{g}$, denoted $\{\alpha_1, ..., \alpha_n\}$. We will let q be a formal parameter before selecting a particular $q \in \mathbb{C}$. Using the data from the Cartan matrices (a_{ij}) and the length of α_i, we will construct $U_q\mathfrak{g}$. For each $\alpha \in \Phi_+$, define $q_\alpha = q^{d_\alpha} = q^{<\alpha,\alpha>/2}$. Here and in future equations we will denote α_i by i. The fundamental dominant weights ω_i are defined by $< \alpha_i,\ \omega_j >= \delta_{ij} d_i$. We define the quantum numbers for all $k \geq l \in \mathbb{N}$:

$$\{k\}_\alpha = q_\alpha^k - q_\alpha^{-k} \qquad [k]_\alpha = \{k\}_\alpha/\{1\}_\alpha \qquad [k]_\alpha! = [k]_\alpha[k-1]\alpha...[1]_\alpha \qquad \left[\begin{smallmatrix} k \\ l \end{smallmatrix}\right]_\alpha = \frac{[k]_\alpha!}{[l]_\alpha![k-l]_\alpha!}$$

Then the quantum group is a free $\mathbb{C}(q)$ algebra

$U_q\ \mathfrak{g} = \mathbb{C}(q) < E_i,\ F_i,\ K_i,\ K_i^{-1} : 1 \leq i \leq n >$, mod the relations for all $1 \leq i,\ j \leq n$:

$$K_iK_i^{-1} = 1 = K_i^{-1}K_i \quad [K_i,\ K_j] = 0 \quad K_iE_jK_i^{-1} = q_i^{a_{ij}}E_j \quad K_iF_jK_i^{-1} = q_i^{-a_{ij}}F_j$$

$[E_i,\ F_j] = \delta_{ij}\frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}$ and for all $i \neq j$:

$$\textstyle\sum_{k=0}^{1-a_{ij}} (-1)^k \left[\begin{smallmatrix} 1-a_{ij} \\ k \end{smallmatrix}\right]_i E_i^k\ E_j\ E_i^{1-a_{ij}-k} = 0$$

$$\textstyle\sum_{k=0}^{1-a_{ij}} (-1)^k \left[\begin{smallmatrix} 1-a_{ij} \\ k \end{smallmatrix}\right]_i F_i^k\ F_j\ F_i^{1-a_{ij}-k} = 0$$

Some useful identities are the following:

$$[0]_i! = [1]_i! = 1 \qquad \begin{bmatrix} k \\ k \end{bmatrix}_i = \begin{bmatrix} k \\ 0 \end{bmatrix}_i = 1 \qquad \begin{bmatrix} 2 \\ 1 \end{bmatrix}_i = [2]_i! = [2]_i = q_i + q_i^{-1}$$

We can make these algebras into Hopf algebras by defining:

$$\Delta(K_i) = K_i \otimes K_i \qquad \Delta(E_i) = E_i \otimes K_i + 1 \otimes E_i \quad \Delta(F_i) = F_i \otimes 1 + K_i^{-1} \otimes F_i$$

$$\epsilon(K_i) = 1 \qquad \epsilon(E_i) = 0 = \epsilon(F_i)$$

$$S(K_i) = K_i^1 \qquad S(E_i) = -E_i K_i^{-1} \quad S(F_i) = -K_i F_i$$

Now, if we are given some element of the root lattice $\mu = m_1\alpha_1 + ... + m_n\alpha_n \in \Lambda_R$, define $K_\mu = \Pi_{i=1}^n K_i^{m_i} \in U_q\ \mathfrak{g}$.

Given the Weyl group of our Lie algebra, fix a representation of the longest word $w_0 = s_{i_1} \circ s_{i_2} \circ ... \circ s_{i_N}$ where each $1 \leq i_j \leq N$. This induces an ordering on $\Phi_+ = \{\alpha_{i_1},\ s_{i_1}(\alpha_{i_2}),\ s_{i_1} \circ s_{i_2}(\alpha_{i_3}),\ \dots, (s_{i_1} \circ ... \circ s_{i_{N-1}})(\alpha_{i_N})\} = \{\beta_1, ..., \beta_N\}$.

From this, we can define E_{β_k} and F_{β_k}, but the process is more complicated than for K_μ. For this, we refer to [DGP 18]...

To construct the small quantum groups, we first need to fix an $r > 2\ max\{d_1,\ \dots, d_n\}$ and set $q = e^{2\pi i/r}$. As motivation, consider the map $\Lambda_R \times \Lambda_W \to \mathbb{C}^\times$ given by $(\mu,\ \nu) = q^{<\mu,\,\nu>}$ so that $\Lambda_R^\perp = \{\nu \in \Lambda_W : q^{<\mu,\,\nu>} = 1\ \ \forall \mu \in \Lambda_R\} = \{\frac{r}{gcd(r,2d_\alpha)}\ \omega_i : 1 \leq i \leq n\}$. This motivates defining $r_\alpha = \frac{r}{gcd(r,2d_\alpha)} > 1\ \ \forall \alpha \in \Phi_+$. We define the small quantum group $u_q\mathfrak{g}$ to be $U_q\mathfrak{g}$ mod the relations:

$$\mu \in \Lambda_R \cap \Lambda_R^\perp \implies K_\mu = 1 \text{ and } \alpha \in \Phi_+.$$

Because $U_q\mathfrak{g}$ is a Hopf algebra, so is $u_q\mathfrak{g}$ and has a PBW basis:

$$\{(\Pi_{k=1}^N F_{\beta_k}^{c_k})\ K_\mu\ (\Pi_k^N = 1 E_{\beta_k}^{b_k}) : \mu \in \Lambda_R/(\Lambda_R \cap \Lambda_R^\perp)\ \ 0 \leq b_k, c_k \leq r_{\beta_k}\}.$$

Furthermore, $u_q\mathfrak{g}$ is a quasicommutative Hopf algebra. For instance, if we define

$$D = \frac{1}{\Lambda_R/(\Lambda_R \cap\ Lambda_R^\perp)} \sum_{\mu,\mu' \in \Lambda_R/(\Lambda_R \cap \Lambda_R^\perp)} q^{-<\mu,\mu'>} K_\mu \otimes K_{\mu'}$$

and

$$\Theta = \sum_{b_1=0}^{r_{\beta_1}-1} \dots \sum_{b_N=0}^{r_{\beta_N}-1} (\Pi_{k=1}^{N} \frac{\{1\}_{\beta_k}^{b_k}}{[b_k]_{\beta_k}!} q^{\frac{b_k(b_k-1)}{2}})(\Pi_{k=1}^{N} E_{\beta_k}^{b_k}) \otimes (\Pi_{k=1}^{N} F_{\beta_k}^{b_k})$$

then $R = D\Theta$ is an R-matrix, i.e. $\Delta^{op} \circ R = R \circ \Delta$. If we define $\rho = \frac{1}{2}\sum_{k=1}^{N} \beta_k$, then $g := K_{2\rho}$ is a pivotal element. Recall from section 5.2 of [BR 21] ... that $R = R'_i \otimes R''_i$ gives us the Drinfeld element $u = S(R''_i)R'_i$, the M-matrix $M = R''_j R'_i \otimes R'_j R''_i$, and the inverse element v_- in the equation $g = uv_-$. We can calculate the ribbon element v_+ by $v_+ u^{-1} = g^{-1}$.

Every nonzero left integral λ is given (up to choice of $\xi \in \mathbb{C}^\times$):

$$\lambda(K_\mu(\Pi_{k=1}^{N} F_{\beta_k}^{c_k})\ (\Pi_{k=1}^{N} E_{\beta_k}^{b_k})) = \xi\delta_{\mu,\,-2\rho}(\Pi_{k=1}^{N}\delta_{b_k,\,r_{\beta_k}-1})\ (\Pi_{k=1}^{N}\delta_{\beta_k,\,r_{\beta_k}-1}).$$

We have that $u_q\mathfrak{g}$ is unimodular and every two-sided cointegral Λ such that $\lambda(\Lambda) = 1$ is given by:

$$\Lambda = \xi^{-1} \sum_{\mu,\mu' \in \Lambda_R/(\Lambda_R \cap \Lambda_R^{\perp})} K_\mu(\Pi_{k=1}^{N} F_{\beta_k}^{r_{\beta_k}-1})(\Pi_{k=1}^{N} E_{\beta_k}^{r_{\beta_k}-1}).$$

Finally, we know that $u_q\mathfrak{g}$ is factorizable iff $\{K_{2\mu} \in u_q\mathfrak{g} : \mu \in \Lambda_R\} = \{K_\mu \in u_q\mathfrak{g} : \mu \in \Lambda_R\}$.

6.1 Algebras of A_l and B_l

Before we move onto some full computed examples of quantum groups, it is useful to consider just the underlying algebras. This displays why the quantum groups are so difficult to calculate, but how they may have the potential to form powerful topological invariants.

6.1.1 The Algebra of A_l

We consider the general case of the algebra A_l for $l \in \mathbb{N}$ which corresponds to the quantum group $U_q\,\mathfrak{sl}_{l+1}$. The dynkin diagram of $\mathfrak{sl}_{l+1}$ is $\underset{\alpha_1}{\bullet} - \underset{\alpha_2}{\bullet} \cdots\cdots \underset{\alpha_{l-1}}{\bullet} - \underset{\alpha_l}{\bullet}$. For each $1 \le i \le l$, we have that $d_i = <\alpha_i,\ \alpha_i>/2 = 1$ so $q_i = q$. The Cartan matrix of $\mathfrak{sl}_{l+1}$ is:

$$(a_{ij}) = \begin{bmatrix} 2 & -1 & 0 & 0 & \dots & & 0 \\ -1 & 2 & -1 & 0 & \dots & & 0 \\ 0 & -1 & 2 & -1 & \dots & & 0 \\ 0 & 0 & -1 & 2 & \dots & & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & & \\ & & & & & 2 & -1 \\ 0 & 0 & 0 & 0 & & -1 & 2 \end{bmatrix}$$

The algebra for $U_q\,\mathfrak{sl}_{l+1}$ is given by $U_q\,\mathfrak{sl}_{l+1} = \mathbb{C}(q) < E_i,\ F_i,\ K_i,\ K_i^{-1} : 1 \le i \le l >$

mod the relations for all $1 \le i,\ j \le l$:

$$K_i K_i^{-1} = 1 = K_i^{-1} K_i \quad [K_i,\ K_j] = 0$$

$$K_i E_i K_i^{-1} = q^2 E_i \quad K_i E_{i\pm1} K_i^{-1} = q^{-1} E_{i\pm1} \quad K_i E_j K_i^{-1} = E_j \text{ otherwise}$$

$$K_i F_i K_i^{-1} = q^{-2} F_i \quad K_i F_{i\pm1} K_i^{-1} = q F_{i\pm1} \quad K_i F_j K_i^{-1} = F_j \text{ otherwise}$$

$$[E_i,\ F_j] = \delta_{ij} \tfrac{K_i - K_i^{-1}}{q_i - q_i^{-1}} \text{ and for all } i \ne j:$$

$$\text{If } j = i \pm 1,\ \ E_j\,E_i^2 - (q + q^{-1}) E_i\,E_j\,E_i + E_i^2\,E_j = 0, \text{ otherwise } [E_i,\ E_j] = 0.$$

$$\text{If } j = i \pm 1,\ \ F_j\,F_i^2 - (q + q^{-1}) F_i\,F_j\,F_i + F_i^2\,F_j = 0, \text{ otherwise } [F_i,\ F_j] = 0.$$

In the case that $l = 1$, we get the more common example of a quantum group

$U_q\,\mathfrak{sl}_2 = \mathbb{C}(q) < E,\ F,\ K,\ K^{-1} >$ mod the relations:

$$KEK^{-1} = q^2E \quad KFK^{-1} = q^{-2}F \quad [E, F] = (K - K^{-1})/(q - q^{-1}).$$

6.1.2 The Algebra of B_l

We consider the general case of the algebra B_l for $l \in \mathbb{N},\ l \geq 2$ which corresponds to the quantum group $U_q\,\mathfrak{so}_{2l+1}$. The dynkin diagram of $\mathfrak{so}_{2l+1}$ is 

.
For each $1 \leq i \leq l-1$, we have that $d_i =< \alpha_i,\, \alpha_i > /2 = 1$ so $q_i = q$. However, now $d_l = 1/2$ so $q_l = q^{1/2}$. The Cartan matrix of $\mathfrak{so}_{2l+1}$ is:

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 & \dots & & & 0 \\ -1 & 2 & -1 & 0 & \dots & & & 0 \\ 0 & -1 & 2 & -1 & \dots & & & 0 \\ 0 & 0 & -1 & 2 & \dots & & & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & & & \\ & & & & & 2 & -1 & 0 \\ & & & & & -1 & 2 & -2 \\ 0 & 0 & 0 & 0 & & 0 & -1 & 2 \end{bmatrix}$$

The algebra for $U_q\,\mathfrak{so}_{2l+1}$ is given by $U_q\,\mathfrak{sl}_{l+1} = \mathbb{C}(q) < E_i,\ F_i,\ K_i,\ K_i^{-1} : 1 \leq i \leq l >$ mod certain relation. If $1 \leq i,\ j \leq l$ and we do not have $i, j = l$, the relations are the same as in the case of A_l. If i or j are equal to l, we have the new relations:

We still have $K_iK_i^{-1} = 1 = K_i^{-1}K_i \quad [K_i,\ K_j] = 0$ for all i, j.

$$K_lE_{l-1}K_l^{-1} = q^{-1/2}E_{l-1} \quad K_{l-1}E_lK_{l-1}^{-1} = q^{-2}E_l \quad K_lE_lK_l^{-1} = q^2E_l.$$

$$K_lF_{l-1}K_l^{-1} = q^{1/2}F_{l-1} \quad K_{l-1}F_lK_{l-1}^{-1} = q^2F_l \quad K_lF_lK_l^{-1} = q^{-2}F_l.$$

$$[E_l,\ F_j] = \delta_{lj}\frac{K_l - K_l^{-1}}{q_i^{1/2} - q_i^{-1/2}} \text{ for all } j.$$

$$E_{l-1}\,E_l^2 - (q^{1/2} + q^{-1/2})E_l\,E_{l-1}\,E_l + E_l^2\,E_{l-1} = 0.$$

$$F_{l-1}\, F_l^2 - (q^{1/2} + q^{-1/2}) F_l\, F_{l-1}\, F_l + F_l^2\, F_{l-1} = 0.$$
$$E_l\, E_{l-1}^3 - (q^2 + 1 + q^{-2})\, E_{l-1}\, E_l\, E_{l-1}^2 + (q^2 + 1 + q^{-2})\, E_{l-1}^2\, E_l\, E_{l-1} - E_{l-1}^3\, E_l = 0.$$
$$F_l\, F_{l-1}^3 - (q^2 + 1 + q^{-2})\, F_{l-1}\, F_l\, F_{l-1}^2 + (q^2 + 1 + q^{-2})\, F_{l-1}^2\, F_l\, F_{l-1} - F_{l-1}^3\, F_l = 0.$$

6.2 Full Example A_1

Given the algebra $U_q\,\mathfrak{sl}_2$, we can make this into a Hopf algebra simply by defining the coproduct, counit, and antipode:

$$\Delta(K) = K \otimes K \qquad \Delta(E) = E \otimes K + 1 \otimes E \qquad \Delta(F) = F \otimes 1 + K^{-1} \otimes F$$
$$\epsilon(K) = 1 \qquad \epsilon(E) = 0 = \epsilon(F)$$
$$S(K) = K^{-1} \qquad S(E) = -EK^{-1} \qquad S(F) = -KF$$

We define have that $\Phi_+ = \{\alpha\}$ and that $K_\alpha = K$, $E_\alpha = E$, and $F_\alpha = F$. These details for $U_q\,\mathfrak{sl}_2$ are trivial and will be explored further in later cases. If we pick an $r > 2\,max\{d_\alpha\} = 2$, set $q = e^{2\pi i/r}$ and $r_\alpha = r/gcd(r,\,2d_\alpha) = r/gcd(r,\,2)$. Note that $\Lambda_R^\perp =< r\alpha >$. Thus to obtain the small quantum group $u_q\,\mathfrak{sl}_2$, we add the relations: $E^{r_\alpha} = F^{r_\alpha} = 0$ and $K^{r_\alpha} = 1$. A PBW basis of $u_q\,\mathfrak{sl}_2$ is $\{F^c K^a E^b : 0 \leq a,\, b,\, c \leq r_\alpha - 1\}$. To obtain the R-matrix, we first calculate:

$$D = \frac{1}{r_\alpha} \sum_{0 \leq \mu,\, \mu' \leq r_\alpha - 1} q^{<\mu\alpha,\, \mu'\alpha>} K_{\mu\alpha} \otimes K_{\mu'\alpha} = \frac{1}{r_\alpha} \sum_{0 \leq \mu,\, \mu' \leq r_\alpha - 1} q^{-2\mu\,\mu'} K^{\mu} \otimes K^{\mu'}$$

$$\Theta = \sum_{b=0}^{r_\alpha - 1} \Big(\frac{\{1\}_\alpha^b}{[b]_\alpha!}\, q^{b(b-1)/2}\Big) E_\alpha^b \otimes F_\alpha^b = \sum_{b=0}^{r_\alpha - 1} \frac{q^{b(b-1)/2}) E^b \otimes F^b}{(q^b - q^{-b})...(q^1 - q^{-1})} = \sum_{b=0}^{r_\alpha - 1} \frac{1}{1 - q^{-2b}} \cdots \frac{1}{1 - q^{-2}}\; q^{-b}\; E^b \otimes F^b$$

$$R = D\Theta = \frac{1}{r_\alpha} \sum_{0 \le \mu, \mu' \le r_\alpha - 1} \sum_{b=0}^{r_\alpha - 1} q^{-2\mu\mu'} K^\mu \otimes K^{\mu'} \frac{1}{1-q^{-2b}} \cdots \frac{1}{1-q^{-2}}\, q^{-b}\, E^b \otimes F^b$$

More compactly,

$$R = \frac{1}{r_\alpha} \sum_{\alpha, \beta, b=0}^{r_\alpha - 1} q^{-2\alpha\beta - b}\, \frac{1}{1-q^{-2b}} \cdots \frac{1}{1-q^{-2}}\, K^\alpha E^b \otimes K^\beta F^b.$$

Then

$$u = S(R_i'')R_i' = \frac{1}{r_\alpha} \sum_{\alpha, \beta, b=0}^{r_\alpha - 1} q^{-2\alpha\beta - b}\, \frac{1}{1-q^{-2b}} \cdots \frac{1}{1-q^{-2}}\, S(K^\beta F^b) K^\alpha E^b$$

We can simplify this by $S(K^\beta F^b) = S(F)^b S(K)^\beta = (-KF)^b K^{-\beta} = (-1)^b (KF)^b K^{-\beta} = (-1)^b q^{-b(b+1)} F^b K^{b+\beta}$ so that:

$$u = \frac{1}{r_\alpha} \sum_{\alpha, \beta, b=0}^{r_\alpha - 1} (-1)^b q^{-2\alpha\beta - b(b+2)}\, \frac{1}{1-q^{-2b}} \cdots \frac{1}{1-q^{-2}}\, F^b K^{b+\beta+\alpha} E^b$$

A left integral is given (for any chosen $\xi \in \mathbb{C}^\times$): $\lambda(K^a F^c E^b) = \xi \delta_{a, r_\alpha - 1} \delta_{b, r_\alpha - 1} \delta_{c, r_\alpha - 1}$ and the given cointegral is given by $\Lambda = \xi^{-1} \sum_{a=0}^{r_\alpha - 1} K^a F^{r_\alpha - 1} E^{r_\alpha - 1}$.

In the paper Beliakova and De Renzi 2021 they define $r' = r_\alpha = r/gcd(r, 2)$ and $r'' = r/gcd(r, 4)$. They define

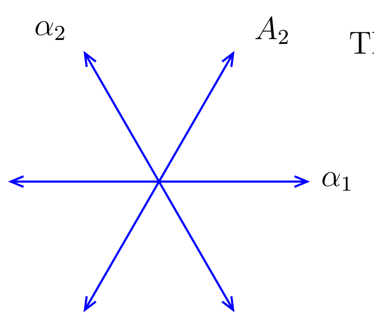
$$\lambda(E^a F^b K^c) = \frac{\sqrt{r''}[r'-1]!}{\{1\}^{r'-1}} \delta_{a, r'-1}\, \delta_{b, r'-1}\, \delta_{c, r'-1}$$

$$\Lambda = \frac{\{1\}^{r'-1}}{\sqrt{r''}[r'-1]!}\sum_{a=0}^{r'-1} E^{r'-1}\, F^{r'-1}\, K^a$$

$$R^{-1} = \frac{1}{r'}\sum_{a,b,c=0}^{r'-1} \frac{\{-1\}^a}{[a]!} q^{-a(a-1)/2+2bc}\, E^a\, K^c \otimes F^b\, K^c$$

6.3 More Computations of E_α and F_α

There are 4 cases of Lie algebras of rank 2: $A_1 \times A_1$, A_2, B_2, and G_2. We include some additional computations for the words E_alpha and F_α to illustrate the potential additional complexity from using more complicated quantum groups.



The longest word of A_2 is $s_1 s_2 s_1$.

The matrix is $\begin{matrix} 2 & -1 \\ -1 & 2. \end{matrix}$

$E_{\beta_1} = E_{\alpha_1} = E_1 \quad F_{\beta_1} = F_{\alpha_1} = F_1$

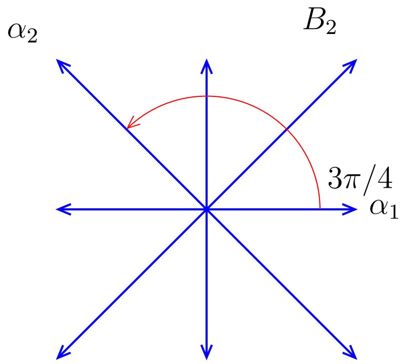
$E_{\beta_2} = E_{\alpha_1+2\alpha_2} = T_1(E_2) = q^{-1}E_2E_1 - E_1E_2$

$F_{\beta_2} = F_{\alpha_1+2\alpha_2} = T_1(F_2) = -F_2F_1 + qF_1F_2$

$E_{\beta_3} = E_{\alpha_1+\alpha_2} = T_1(T_2(E_1)) = -q^{-2}F_1K_1E_2E_1 + q^{-1}F_1K_1E_1E_2 + q^{-1}E_2E_1F_1K_1 - E_1E_2F_1K_1$

$F_{\beta_3} = F_{\alpha_1+\alpha_2} = T_1(T_2(F_1)) = -K_1^{-1}E_1F_2F_1 + qK_1^{-1}E_1F_1F_2 + qF_2F_1K_1^{-1}E_1 - q^2F_1F_2K_1^{-1}E_1$

Next, we have the Lie algebra B_2 which has longest word $s_1s_2s_1s_2$.



$E_{\beta_1} = E_{\alpha_1} = E_1 \quad F_{\beta_1} = F_{\alpha_1} = F_1$

$E_{\beta_2} = E_{2\alpha_1+\alpha_2} = T_1(E_2) = q^{-2}E_2E_1^2 - \frac{q^{-1}}{q+q^{-1}}E_1E_2E_1 + E_1^2E_2$

$F_{\beta_2} = F_{2\alpha_1+\alpha_2} = T_1(F_2) = F_2F_1^2 - \frac{q}{q+q^{-1}}F_1F_2F_1 + q^2F_1^2F_2$

$E_{\beta_3} = E_{\alpha_1+\alpha_2} = T_1(T_2(E_1)) = T_1(q^{-2}E_1E_2 - E_2E_1) =$

$-q^{-2}F_1K_1(q^-2E_2E_1^2 - \frac{q^{-1}}{q+q^{-1}}E_1E_2E_1 + E_1^2E_2) + q^-2E_2E_1^2 - \frac{q^{-1}}{q+q^{-1}}E_1E_2E_1 + E_1^2E_2)F_1K_1$

$F_{\beta_3} = F_{\alpha_1+\alpha_2} = T_1(T_2(F_1)) = T_1(-F_1F_2 + q^2F_2F_1) =$

$K_1^{-1}E_1(F_2F_1^2 - \frac{q}{q+q^{-1}}F_1F_2F_1 + q^2F_1^2F_2) - q^2(F_2F_1^2 - \frac{q}{q+q^{-1}}F_1F_2F_1 + q^2F_1^2F_2)K_1^{-1}E_1$

$E_{\beta_4} = E_{\alpha_2} = T_1(T_2(T_1(E_2))) = T_1(T_2(q^{-2}E_2E_1^2 - \frac{q^{-1}}{q+q^{-1}}E_1E_2E_1 + E_1^2E_2))$

$E_{\beta_4} = -q^{-2}(F_2F_1^2 - \frac{q}{q+q^{-1}}F_1F_2F_1 + q^2F_1^2F_2)K_2K_1^2(-q^{-2}F_1K_1(q^{-2}E_2E_1^2 - \frac{q^{-1}}{q+q^{-1}}E_1E_2E_1 + E_1^2E_2) + (q^{-2}E_2E_1^2 - \frac{q^{-1}}{q+q^{-1}}E_1E_2E_1 + E_1^2E_2)F_1K_1)^2 + \frac{q^{-1}}{q+q^{-1}}(-q^{-2}F_1K_1(q^{-2}E_2E_1^2 - \frac{q^{-1}}{q+q^{-1}}E_1E_2E_1 + E_1^2E_2) + (q^{-2}E_2E_1^2 - \frac{q^{-1}}{q+q^{-1}}E_1E_2E_1 + E_1^2E_2)F_1K_1)(F_2F_1^2 - \frac{q}{q+q^{-1}}F_1F_2F_1 + q^2F_1^2F_2)K_2K_1^2(-q^{-2}F_1K_1*$

$(q^{-2}E_2E_1^2 - \frac{q^{-1}}{q+q^{-1}}E_1E_2E_1 + E_1^2E_2) + (q^{-2}E_2E_1^2 - \frac{q^{-1}}{q+q^{-1}}E_1E_2E_1 + E_1^2E_2)F_1K_1) - (-q^{-2}F_1K_1* (q^{-2}E_2E_1^2 - \frac{q^{-1}}{q+q^{-1}}E_1E_2E_1 + E_1^2E_2) + (q^{-2}E_2E_1^2 - \frac{q^{-1}}{q+q^{-1}}E_1E_2E_1 + E_1^2E_2)F_1K_1)^2(F_2F_1^2 - \frac{q}{q+q^{-1}}F_1F_2F_1 + q^2F_1^2F_2)K_2K_1^2$

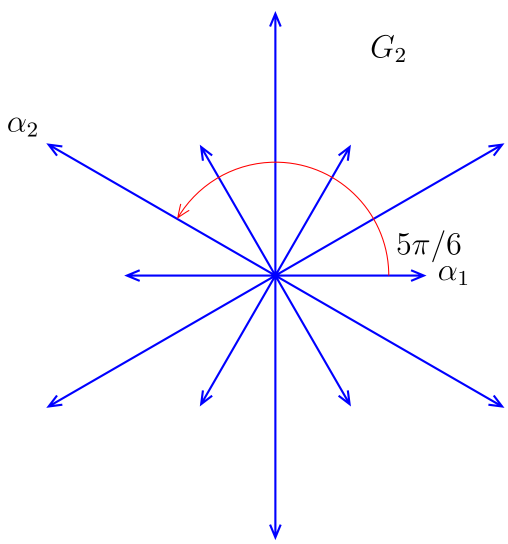
$$F_{\beta_4} = F_{\alpha_2} = T_1(T_2(T_1(F_2))) = T_1(T_2(F_2F_1^2 - \tfrac{q}{q+q^{-1}}F_1F_2F_1 + q^2F_1^2F_2))$$

$$F_{\beta_4} = -K_2^{-1}K_1^{-2}(q^{-2}E_2E_1^2 - \tfrac{q^{-1}}{q+q^{-1}}E_1E_2E_1 + E_1^2E_2)(K_1^{-1}E_1(F_2F_1^2 - \tfrac{q}{q+q^{-1}}F_1F_2F_1 + q^2F_1^2F_2) - q^2(F_2F_1^2 - \tfrac{q}{q+q^{-1}}F_1F_2F_1 + q^2F_1^2F_2)K_1^{-1}E_1)^2$$

$$+\tfrac{q}{q+q^{-1}}(K_1^{-1}E_1(F_2F_1^2 - \tfrac{q}{q+q^{-1}}F_1F_2F_1 + q^2F_1^2F_2) - q^2(F_2F_1^2 - \tfrac{q}{q+q^{-1}}F_1F_2F_1 + q^2F_1^2F_2)K_1^{-1}E_1) * K_2^{-1}K_1^{-2}(q^{-2}E_2E_1^2 - \tfrac{q^{-1}}{q+q^{-1}}E_1E_2E_1 + E_1^2E_2)(K_1^{-1}E_1(F_2F_1^2 - \tfrac{q}{q+q^{-1}}F_1F_2F_1 + q^2F_1^2F_2) - q^2(F_2F_1^2 - \tfrac{q}{q+q^{-1}}F_1F_2F_1 + q^2F_1^2F_2)K_1^{-1}E_1)$$

$$-(K_1^{-1}E_1(F_2F_1^2 - \tfrac{q}{q+q^{-1}}F_1F_2F_1 + q^2F_1^2F_2) - q^2(F_2F_1^2 - \tfrac{q}{q+q^{-1}}F_1F_2F_1 + q^2F_1^2F_2)K_1^{-1}E_1)K_2^{-1}K_1^{-2} * (q^{-2}E_2E_1^2 - \tfrac{q^{-1}}{q+q^{-1}}E_1E_2E_1 + E_1^2E_2)$$

The next Lie algebra is G_2 with longest word $s_1s_2s_1s_2s_1s_2$.



$E_{\beta_1} = E_{\alpha_1} = E_1 \quad F_{\beta_1} = F_{\alpha_1} = F_1$

$E_{\beta_2} = E_{3\alpha_1+\alpha_2} = T_1(E_2)q^{-1} = E_2E_1 - E_1E_2$

$F_{\beta_2} = F_{3\alpha_1+\alpha_2} = T_1(F_2) = -F_2F_1 + qF_1F_2 \quad ...$

It still remains to calculate E_{β_3}, E_{β_4}, E_{β_5}, E_{β_6}, F_{β_3}, F_{β_4}, F_{β_5}, F_{β_6}, but it is obvious that the complexity increases with each iteration. This calculation will not be used in later sections.

Chapter 7

Partial Results

This section is dedicated to trying to apply the case of $U_q\mathfrak{sl}_2$ to Δ_1. Recall the invariant formula for Δ_n:

$$J_4(\Delta_n) = \lambda(R'_k\; S^{-1}(R''_j)\; S^{-1}(R''_i)\; \Lambda_{(n+2)} \ldots \Lambda_{(3)}\; \Lambda_{(2)}\; \Lambda'_{(3)}\; \Lambda'_{(4)} \ldots \Lambda'_{(n+2)}\; S(g\; R''_k\; R''_l\; R''_m))$$
$$*\lambda(\Lambda_{(n+3)}\; R'_i\; S(R'_m)\; S^{-1}(\Lambda'_{(2)})\; \Lambda_{(1)}\; \Lambda'_{(1)}\; S^2(R'_{(1)})\; S(R'_j)\; S(\Lambda_{(n+4)})\; \Lambda'_{(n+3)})$$

Recall that our R-matrix ($\in H \otimes H$) is written $R = \Sigma_i R'_i \otimes R''_i$. Recall also that when we write a subscript $\Lambda_(A)$, it means it is the A^{th} index of of $\Delta(\Delta(\ \ldots\Delta(\Lambda)\ldots))$. It will be useful to perform some basic calculations before plugging in the Λ's and $S^{\pm 1}(R)$ into the formulas.

7.1 Useful Identities

Here are a collection of identities which are useful in the proceeding calculations. Here recall $E,\ F,\ K \in U_q\mathfrak{sl}_2$, $q \in \mathbb{C}^{\times}$, and let $a,\ b,\ c,\ \alpha,\ \beta \in \mathbb{N}$.

$$K^a E^b = q^{2ab} E^b K^a \qquad K^a F^b = q^{-2ab} F^b K^a$$
$$S(K^\alpha E^b) = S(E)^b S(K)^\alpha = (-EK^{-1})^b K^{-\alpha} = (-1)^b q^{b(\alpha+1)} K^{-\alpha-b} E^b$$
$$S(K^\beta F^b) = S(F)^b S(K)^\beta = (-KF)^b K^{-\beta} = (-1)^b q^{-b(b+1)} F^b K^{b-\beta}$$

$$S^{-1}(E) = S^{-1}((-EK^{-1})(-K)) = S^{-1}(-S(E)K) = -K^{-1}E$$
$$S^{-1}(F) = S^{-1}(-K^{-1}(-KF)) = S^{-1}(-K^{-1}S(F)) = -FK$$

$$S^{-1}(K) = K^{-1}$$

$$S^2(K^\alpha E^b) = q^{b(\alpha+2-b)} E^b K^\alpha$$

$$S(g) = S(K) = K^{-1}$$

$$S^{-1}(K^\beta F^b) = (-1)^b q^{b(1-b)} F^b K^{b-\beta}$$

7.2 Λ_A Calculations

Let $A_0 = \frac{\{1\}^{r'-1}}{\sqrt{r''}[r'-1]!}$, then $\Lambda = A_0 \, \Sigma_{a=0}^{r'-1} \, E^{r'-1} \, F^{r'-1} \, K^a$. For notational brevity we introduce Δ^m defined inductively and $\Delta^0 = \Lambda$. Because Δ is an algebra morphism:

$$\Delta^1 := \Delta(\Lambda) = A_0 \, \Sigma_{a=0}^{r'-1} \, \Delta(E)^{r'-1} \, \Delta(F)^{r'-1} \, \Delta(K)^a$$

$$= A_0 \, \Sigma_{a=0}^{r'-1} \, (E \otimes K + 1 \otimes E)^{r'-1} \, (F \otimes 1 + K^{-1} \otimes F)^{r'-1} \, (K \otimes K)^a.$$

This is messy and even if we had a fixed r' we wouldn't want to expand this at this point. Luckily, we can keep it in this form at least for now. Note we are implicitly using the coassociativity of Δ to apply Δ to the first term.

$$\Delta^2 := \Delta(\Delta^1(1)) \otimes \Delta^1_{(2)} = A_0 \, \Sigma_{a=0}^{r'-1} \, (E \otimes K \otimes K + 1 \otimes E \otimes K + 1 \otimes 1 \otimes E)^{r'-1}$$

$$(F \otimes 1 \otimes 1 + K^{-1} \otimes F \otimes 1 + K^{-1} \otimes K^{-1} \otimes F)^{r'-1} \, (K \otimes K \otimes K)^a.$$

This pattern repeats and we can continue to the general case:

$$\Delta^m := \Delta(\Delta^{m-1}_{(1)}) \otimes \Delta^{m-1}_{(2)} \otimes \ \ldots \ \Delta^{m-1}_{(m-1)} =$$

$$A_0 \, \Sigma_{a=0}^{r'-1} \, (E \otimes K \otimes K \otimes \ \ldots \ \otimes K + 1 \otimes E \otimes K \otimes \ \ldots \ \otimes K + \ \ldots \ + 1 \otimes 1 \otimes \ \ldots \ \otimes E)^{r'-1}$$

$$(F \otimes 1 \otimes \ \ldots \ \otimes 1 + K^{-1} \otimes F \otimes 1 \otimes \ \ldots \ \otimes 1 + \ \ldots \ + K^{-1} \otimes K^{-1} \otimes \ \ldots \ \otimes F)^{r'-1} \, (K^a \otimes \ \ldots \ \otimes K^a).$$

Then $\Delta^m_{(k)}$ has the relatively simpler form:

$$\Delta^m_{(k)} = A_0 \, \Sigma_{a=0}^{r'-1} \, (E + (k-1)K + (m-k))^{r'-1} (F + (m-k)K^{-1} + (k-1))^{r'-1} K^a$$

Note that the value of $\Delta^m_{(k)}$ depends on m. This is why we distinguish between Λ and Λ' in the algorithm process.

Finally, it will be useful to have this final calculation to use in the next section:

$$S((E+(k-1)K+(m-k))^{r'-1}(F+(m-k)K^{-1}+(k-1))^{r'-1}K^a) =$$

$$K^{-a}\ S(F+(m-k)K^{-1}+(k-1))^{r'-1}\ S(E+(k-1)K+(m-k))^{r'-1} =$$

$$K^{-a}\ [S(F)+(m-k)S(K^{-1})+(k-1)]^{r'-1}\ [S(E)+(k-1)S(K)+(m-k)]^{r'-1} =$$

$$K^{-a}\ [-KF+(m-k)K+k-1]^{r'-1}\ [-EK^{-1}+(k-1)K^{-1}+m-k]^{r'-1}.$$

Additionally,

$$S^{-1}((E+(k-1)K+(m-k))^{r'-1}(F+(m-k)K^{-1}+(k-1))^{r'-1}K^a) =$$

$$K^{-a}\ [S^{-1}(F)+(m-k)S^{-1}(K^{-1})+(k-1)]^{r'-1}\ [S^{-1}(E)+(k-1)S^{-1}(K)+(m-k)]^{r'-1} =$$

$$K^{-a}\ [-FK+(m-k)K+(k-1)]^{r'-1}\ [-K^{-1}E+(k-1)K^{-1}+(m-k)]^{r'-1}$$

7.2.1 How to Commute E and F

The two variables E and F do not commute in the quantum group $U_q\mathfrak{sl}_2$, but are related by the equation: $FE = EF + \frac{K^{-1}-K}{q-q^{-1}}$. A major problem with computing the invariants of different knots is finding a formula for how we can commute F^nE^m. This is a very difficult problem which we weren't able to solve in general, but here we include the progress that can be easily made. Begin by noting the following equations:

$$FE = EF + \frac{K^{-1}-K}{q-q^{-1}}$$

$$F^2E = FEF + F\frac{K^{-1}-K}{q-q^{-1}} = EF^2 + F\frac{K^{-1}-K}{q-q^{-1}} + \frac{K^{-1}-K}{q-q^{-1}}F$$

$$= EF^2 + \frac{F}{q-q^{-1}}((1+q^2)K^{-1}-(1+q^{-2})K) = EF^2 + \frac{F}{q-q^{-1}}((1+q^2)K^{-1}-(1+q^{-2})K)$$

A simple exercise by induction shows that for $n \geq 1$:

$$F^nE = EF^n + \frac{F^{n-1}}{q-q^{-1}}(\frac{1-q^{2n}}{1-q^2}K^{-1} - \frac{1-q^{-2n}}{1-q^{-2}}K).$$

The general case is considerably more difficult. For $n \geq 2$:

$$F^nE^2 = E^2F^n + \frac{EF^{n-1}}{q-q^{-1}}(\frac{1-q^{2n}}{1-q^2}(1+q^{-2})K^{-1} - \frac{1-q^{-2n}}{1-q^{-2}}(1+q^2)K)$$

$$+\frac{F^{n-2}}{(q-q^{-1})^2}(\frac{1-q^{2n-2}}{1-q^2}K^{-1} - \frac{1-q^{-2n+2}}{1-q^{-2}}K)(\frac{1-q^{2n}}{1-q^2}q^2K^{-1} - \frac{1-q^{-2n}}{1-q^{-2}}q^{-2}K)$$

$$= E^2F^n + \sum_{k=1}^{2} E^{2-k}\frac{F^{n-k}}{(q-q^{-1})^k}\prod_{\lambda=0}^{k-1}(\frac{q^{-2\lambda}-q^{2n+2-2k+2\lambda}}{1-q^2}\frac{1-q^{-6+2k}}{1-q^{-2}}K^{-1} - \frac{q^{2\lambda}-q^{-2n-2+2k-2\lambda}}{1-q^{-2}}\frac{1-q^{6-2k}}{1-q^2}K$$

For $n \geq 3$:

$$F^nE^3 = E^2F^nE + \frac{EF^{n-1}E}{q-q^{-1}}(\frac{1-q^{2n}}{1-q^2}\frac{1-q^{-4}}{1-q^{-2}}q^2K^{-1} - \frac{1-q^{-2n}}{1-q^{-2}}\frac{1-q^4}{1-q^2}q^{-2}K)$$

$$+\frac{F^{n-2}E}{(q-q^{-1})^2}(\frac{1-q^{2n-2}}{1-q^2}q^2K^{-1} - \frac{1-q^{-2n+2}}{1-q^{-2}}q^{-2}K)(\frac{1-q^{2n}}{1-q^2}q^4K^{-1} - \frac{1-q^{-2n}}{1-q^{-2}}q^{-4}K)$$

$$= E^3F^n + \frac{E^2F^{n-1}}{q-q^{-1}}(\frac{1-q^{2n}}{1-q^2}K^{-1} - \frac{1-q^{-2n}}{1-q^{-2}}K)$$

$$+\frac{E}{q-q^{-1}}[EF^{n-1} + \frac{F^{n-2}}{q-q^{-1}}(\frac{1-q^{2n}}{1-q^2}K^{-1} - \frac{1-q^{-2n}}{1-q^{-2}}K)](\frac{1-q^{2n}}{1-q^2}(1+q^{-2})q^2K^{-1} - \frac{1-q^{-2n}}{1-q^{-2}}(1+q^2)q^{-2}K$$

$$+\frac{1}{(q-q^{-1})^2}(\frac{1-q^{2n-2}}{1-q^2}q^2K^{-1} - \frac{1-q^{-2n+2}}{1-q^{-2}}q^{-2}K)(\frac{1-q^{2n}}{1-q^2}q^4K^{-1} - \frac{1-q^{-2n}}{1-q^{-2}}q^{-4}K)$$

$$[EF^{n-2}+\frac{F^{n-3}}{q-q^{-1}}(\frac{1-q^{2n}}{1-q^2}K^{-1}-\frac{1-q^{-2n}}{1-q^{-2}}K)]$$

$$=E^3F^n+\frac{E^2F^{n-1}}{q-q^{-1}}(\frac{1-q^{2n}}{1-q^2}(2+q^2)K^{-1}-\frac{1-q^{-2n}}{1-q^{-2}}(2+q^{-2})K)$$

$$+\frac{EF^{n-2}}{(q-q^{-1})^2}(\frac{1-q^{2n}}{1-q^2}K^{-1}-\frac{1-q^{-2n}}{1-q^{-2}}K)(\frac{1-q^{2n}}{1-q^2}(1+q^2)K^{-1}-\frac{1-q^{-2n}}{1-q^{-2}}(1+q^{-2})K)$$

$$+\frac{EF^{n-2}}{(q-q^{-1})^2}(\frac{q^2-q^{2n}}{1-q^2}K^{-1}-\frac{q^{-2}-q^{-2n}}{1-q^{-2}}K)(\frac{q^4-q^{2n+4}}{1-q^2}K^{-1}-\frac{q^{-4}-q^{-2n-4}}{1-q^{-2}}K)$$

$$+\frac{F^{n-3}}{(q-q^{-1})^3}(\frac{q^2-q^{2n}}{1-q^2}K^{-1}-\frac{q^{-2}-q^{-2n}}{1-q^{-2}}K)(\frac{q^4-q^{2n+4}}{1-q^2}K^{-1}-\frac{q^{-4}-q^{-2n-4}}{1-q^{-2}}K)(\frac{1-q^{2n}}{1-q^2}K^{-1}-\frac{1-q^{-2n}}{1-q^{-2}}K)$$

To make the next simplification, examine the two middle terms:

$$(\frac{1-q^{2n}}{1-q^2}K^{-1}-\frac{1-q^{-2n}}{1-q^{-2}}K)(\frac{1-q^{2n}}{1-q^2}(1+q^2)K^{-1}-\frac{1-q^{-2n}}{1-q^{-2}}(1+q^{-2})K)$$
$$+(\frac{q^2-q^{2n}}{1-q^2}K^{-1}-\frac{q^{-2}-q^{-2n}}{1-q^{-2}}K)(\frac{q^4-q^{2n+4}}{1-q^2}K^{-1}-\frac{q^{-4}-q^{-2n-4}}{1-q^{-2}}K)=$$

$$(\frac{1-q^{2n}}{1-q^2}K^{-1}-\frac{1-q^{-2n}}{1-q^{-2}}K)(\frac{1-q^{2n}}{1-q^2}(1+q^2)K^{-1}-\frac{1-q^{-2n}}{1-q^{-2}}(1+q^{-2})K)$$
$$-(\frac{q^2-q^{2n}}{1-q^2}K^{-1}-\frac{q^{-2}-q^{-2n}}{1-q^{-2}}K)(\frac{1-q^{2n}}{1-q^2}(1+q^2)K^{-1}-\frac{1-q^{-2n}}{1-q^{-2}}(1+q^{-2})K)$$
$$+(\frac{q^2-q^{2n}}{1-q^2}K^{-1}-\frac{q^{-2}-q^{-2n}}{1-q^{-2}}K)(\frac{1-q^{2n}}{1-q^2}(1+q^2)K^{-1}-\frac{1-q^{-2n}}{1-q^{-2}}(1+q^{-2})K)$$
$$+(\frac{q^2-q^{2n}}{1-q^2}K^{-1}-\frac{q^{-2}-q^{-2n}}{1-q^{-2}}K)(\frac{q^4-q^{2n+4}}{1-q^2}K^{-1}-\frac{q^{-4}-q^{-2n-4}}{1-q^{-2}}K)=$$

$$(K^{-1}-K)(\frac{1-q^{2n}}{1-q^2}(1+q^2)K^{-1}-\frac{1-q^{-2n}}{1-q^{-2}}(1+q^{-2})K)$$

$$+(\frac{q^2-q^{2n}}{1-q^2}K^{-1}-\frac{q^{-2}-q^{-2n}}{1-q^{-2}}K)(\frac{1-q^{2n}}{1-q^2}(1+q^2+q^4)K^{-1}-\frac{1-q^{-2n}}{1-q^{-2}}(1+q^{-2}+q^{-4})K)=$$

$$(\frac{1-q^{2n}}{1-q^2}K^{-1}-\frac{1-q^{-2n}}{1-q^{-2}}K)(\frac{1-q^{2n}}{1-q^2}(1+q^2+q^4)K^{-1}-\frac{1-q^{-2n}}{1-q^{-2}}(1+q^{-2}+q^{-4})K)$$

$$-(K^{-1}-K)(\frac{1-q^{2n}}{1-q^2}(q^4)K^{-1}-\frac{1-q^{-2n}}{1-q^{-2}}(q^{-4})K)$$

We could not find a suitable simplification or a way to proceed with this induction. Nevertheless, we will keep our work here in the hope that we can make further progress. As it stands, this is the largest obstacle to calculating the invariants that follow. As a final comment, we will include the general formula to calculate inductively. If we have $n \geq m$ and a formula:

$$F^n\,E^{m-1}=E^{m-1}\,F^n+E^{m-2}\,F^{n-1}\,\gamma_1(K)+E^{m-3}\,F^{n-2}\,\gamma_2(K)+...$$

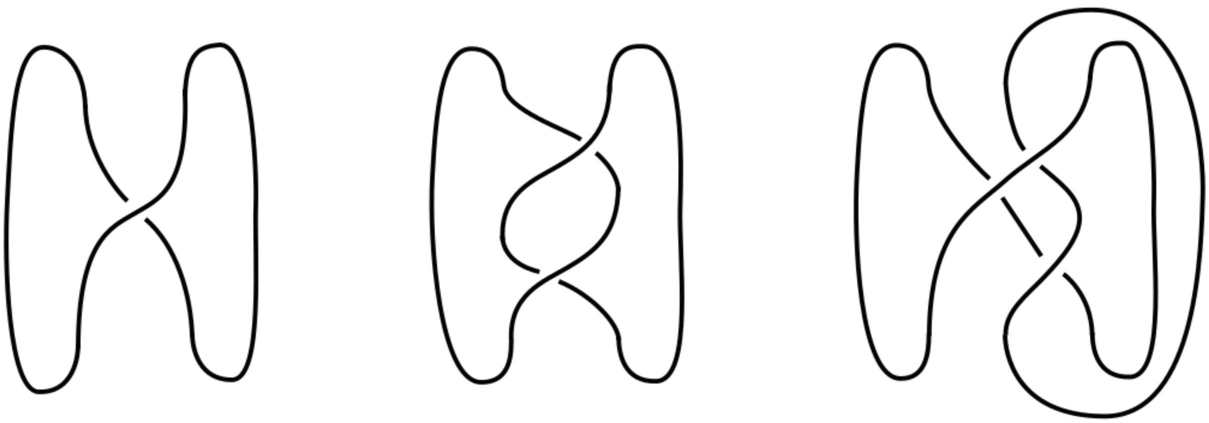
, then if we denote $\tilde{\gamma}_i$ by $E\,\tilde{\gamma}_i=\gamma_i\,E$:

$$F^n\,E^m=E^m\,F^n+E^{m-1}\,F^{n-1}\,(\tilde{\gamma}_1(K)+\frac{1}{q-q^{-1}}(\frac{1-q^{2n}}{1-q^2}K^{-1}-\frac{1-q^{-2n}}{1-q^{-2}}K))$$

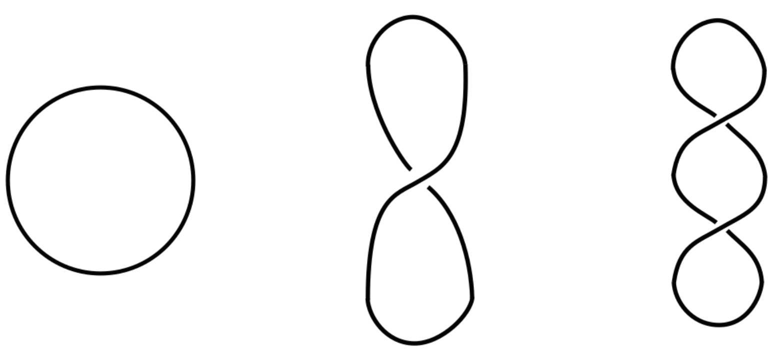
$$+E^{m-2}\,F^{n-2}\,(\tilde{\gamma}_2+\tilde{\gamma}_1\,\tfrac{1}{q-q^{-1}}(\tfrac{1-q^{2n-2}}{1-q^2}K^{-1}-\tfrac{1-q^{-2n+2}}{1-q^{-2}}K))+...$$
$$+F^{n-m}\,\tilde{\gamma}_{m-1}\,\tfrac{1}{q-q^{-1}}(\tfrac{1-q^{2n-2m+2}}{1-q^2}K^{-1}-\tfrac{1-q^{-2n+2m-2}}{1-q^{-2}}K)$$

7.3 Hopf Link Invariants

A Hopf Link is a type of knot, formed from iterating the longest element of a braid group some number of times. Then closing up the braid by attaching each strand to the strand directly above/ below it. For example, we have the following:



We begin with the simplest case, the unknot. In this case, there are no crossings, beaded components, or loops. Thus we have $J_4(\text{Unknot}) = 0$. We can complicate this case slightly by adding a twist or two to the unknot. These are still homotopic to the unknot if we just untwist, and thus we expect to get an invariant of 0.



Lets call these knots O_0, O_1, and O_2. In the case of a single twist O_1, we have a single crossing, and after shifting beads, we have a single loop to untwist. Thus we have that:

$$J_4(O_1) = \sum_i \lambda(R_i'' S^{-1}(R_i') S(g)) = \frac{1}{r'} \sum_{\alpha,\,\beta,\,b=0}^{r'-1} q^{-2\alpha\beta - b} (\frac{1}{1-q^{-2b}})...(\frac{1}{1-q^{-2}})\; \lambda(K^{\beta} F^{b} S^{-1}(K^{\alpha} E^{b}) K^{-1}$$

$$= \frac{1}{r'} \sum_{\alpha,\,\beta,\,b=0}^{r'-1} q^{-2\alpha\beta - b} (\frac{1}{1-q^{-2b}})...(\frac{1}{1-q^{-2}})\; \lambda(K^{\beta} F^{b} (-1)^{b} q^{b(1-b)} F^{b} K^{b-\beta-1}) = 0$$

This is already a very complicated invariant for a very simple knot. Luckily, we can exploit the fact that $\lambda(E^a F^b K^c)\, \alpha\, \delta_{a=r'-1}\delta_{b=r'-1}\delta_{c=r'-1}$. Because the expression in the line above contains no E's, the entire expression must be 0. We are not so lucky with the case of O_2. Here, we have two crossings. Thus

$$J_4(O_2) = \lambda(R_i''\, R_j'\, S(R_j'')\, S(R_i'))$$

$$= \frac{1}{(r')^2} \sum_{\alpha_i,\,\beta_i,\,b_i,\,\alpha_j,\,\beta_j,\,b_j=0}^{r'-1} q^{-2\alpha_i\beta_i - b_i - 2\alpha_j\beta_j - b_j} (\frac{1}{1-q^{-2b_i}})...(\frac{1}{1-q^{-2}})(\frac{1}{1-q^{-2b_j}})...(\frac{1}{1-q^{-2}})$$

$$\lambda(K^{\beta_i}\, F^{b_i}\, K^{\alpha_j}\, E^{b_j}\, S(K^{\beta_j}\, F^{b_j})\, S(K^{\alpha_i}\, E^{b_i}))$$

$$= \frac{1}{(r')^2} \sum_{\alpha_i,\,\beta_i,\,b_i,\,\alpha_j,\,\beta_j,\,b_j=0}^{r'-1} (-1)^{b_i+b_j}\, (\frac{1}{1-q^{-2b_i}})...(\frac{1}{1-q^{-2}})(\frac{1}{1-q^{-2b_j}})...(\frac{1}{1-q^{-2}})$$

$$q^{-2\alpha_i\beta_i - 2\alpha_j\beta_j - 2\beta_i b_i + b_i\alpha_i - b_j(b_j+2) + 2b_i(b_j - \beta_j - \alpha_i - b_i)}\; \lambda(F^{b_i}\, E^{b_j}\, F^{b_j}\, E^{b_i}\, K^{\alpha_j + b_j - \alpha_i - b_i})$$

Unfortunately, we are stuck here as we do not have a nice formula to commute E and F in generality, though in principle it is possible to calculate for a selected r'. It is tempting to conclude here, but there is still more progress we can make on a general method. Assume that we had a general formula for commuting $F^n E^m$, $n \geq m$, then it would be of the form: $F^n\, E^m = E^m\, F^n + E^{m-1}\, F^{m-1}\, \gamma_1(K) + E^{m-2}\, F^{m-2}\, \gamma_2(K) + ...$ for some functions $\gamma_i(K)$which can be "computed" inductively from the formula for $F^n\, E^{m-1}$. Then we have:

$$E^{b_j}\, F^{b_j}\, E^{b_i} = E^{b_i+b_j}\, F^{b_j} + E^{b_i+b_j-1}\, F^{b_j-1}\, \gamma_1(K) + E^{b_i+b_j-2}\, F^{b_j-2}\, \gamma_2(K) + ...$$

But recall that $E^{r'} = 0$ so we can simplify this as:

$$E^{b_j}\, F^{b_j}\, E^{b_i} = E^{r'-1}\, F^{r'-1-b_i}\, \gamma_{b_i+b_j-r'-1}(K) + ...$$

Note that we only wrote the first term. This is because we know that $\lambda(E^m F^n K^l)$ is proportional to $\delta_{n,\, r'-1}\ \delta_{m,\, r'-1}\ \delta_{l,\, r'-1}$ so we will only select the first term anyway. Thus we have for $b_j \geq b_i$:

$$\lambda(F^{b_i}\, E^{b_j}\, F^{b_j}\, E^{b_i}\, K^{\alpha_j+b_j-\alpha_i-b_i}) = \lambda(E^{r'-1}\, F^{r'-1}\, \gamma_{b_i+b_j-r'-1}(K)\ K^{\alpha_j+b_j-\alpha_i-b_i})$$

In particular, if $b_i + b_j < r' - 1$: $\lambda(F^{b_i}\, E^{b_j}\, F^{b_j}\, E^{b_i}\, K^{\alpha_j+b_j-\alpha_i-b_i}) = 0$. Furthermore, if we have that $b_i + b_j = r' - 1$: $\lambda(F^{b_i}\, E^{b_j}\, F^{b_j}\, E^{b_i}\, K^{\alpha_j+b_j-\alpha_i-b_i}) = \delta_{r'-1,\, \alpha_j+b_j-\alpha_i-b_i}$. There will be slightly different formulas for the case when $b_j \leq b_i$.

7.4 The Case of Δ_1

We need to plug in the entries from Section 6.2 and simplify the expression to obtain an understandable result. For the case of Δ_1 we begin with the general expression:

$$J_4(\Delta_1) = \lambda(R'_k\ S^{-1}(R''_j)\ S^{-1}(R''_i)\ \Lambda_{(3)}\ \Lambda_{(2)}\ \Lambda'_{(3)}\ S(g\ R''_k\ R''_l\ R''_m))$$
$$*\lambda(\Lambda_{(4)}\ R'_i\ S(R'_m)\ S^{-1}(\Lambda'_{(2)})\ \Lambda_{(1)}\ \Lambda'_{(1)}\ S^2(R'_l)\ S(R'_j)\ S(\Lambda_{(5)})\ \Lambda'_{(4)})$$

First we recognize the implicit sums that were hidden in the indices of the R-matrices. We may pull them out of the equation because λ is linear:

$$J_4(\Delta_1) = \Sigma_{i,j,k,l,m}\ \lambda(R'_k\ S^{-1}(R''_j)\ S^{-1}(R''_i)\ \Lambda_{(3)}\ \Lambda_{(2)}\ \Lambda'_{(3)}\ \Lambda'_{(4)}\ \Lambda'_{(3)}\ S(g\ R''_k\ R''_l\ R''_m))$$
$$*\lambda(\Lambda_{(4)}\ R'_i\ S(R'_m)\ S^{-1}(\Lambda'_{(2)})\ \Lambda_{(1)}\ \Lambda'_{(1)}\ S^2(R'_l)\ S(R'_j)\ S(\Lambda_{(5)})\ \Lambda'_{(4)})$$

Recall the following formulas:

$$R = \frac{1}{r_\alpha}\sum_{\alpha,\beta,b=0}^{r_\alpha-1} q^{-2\alpha\beta-b}\ \frac{1}{1-q^{-2b}}\cdots\frac{1}{1-q^{-2}}\ K^\alpha E^b \otimes K^\beta F^b \qquad g = K$$
$$\Delta^m_{(k)} = A_0\ \Sigma_{a=0}^{r'-1}\ (E + (k-1)K + (m-k))^{r'-1}(F + (m-k)K^{-1} + (k-1))^{r'-1}K^a$$

and remember $r' = r_\alpha$ for $U_q\mathfrak{sl}_2$. The work for this section is included in Section 9 for brevity. We will include the final formula here, but first we need to introduce some new notation to make the problem tractable. We'll introduce the notation for some k such that $0 \le k \le r' - 1$:

$$\sum_{\Xi=k}[f(\Xi)] := \sum_{k_1=0}^{k_0+1} f(k_1) \sum_{k_2=0}^{k_1} f(k_2)\ \ldots \sum_{k_{r'-1-k_0}=0}^{k_{r'-2-k_0}} f(k_{r'-1-k_0})$$

Then for instance:

$$(E + aK + b)^{r'-1} = \sum_{k=0}^{r'-1} E^{k_0} \sum_{\Xi=k}(aq^{2\Xi}K + b)$$

Furthermore, there are many sums over for instance $\alpha_1,\ \alpha_2,\ \alpha_3,\ \ldots$ from 0 to $r'-1$ so instead we'll just write $\overline{\alpha}$ to sum from 0 to $r'-1$. Because there are several sums of $\Xi_l = k_l$, $\Lambda_l = j_l$, $\Phi_l = k_l$ so we abbreviate these as $\overline{\Xi}$, $\overline{\Lambda}$, $\overline{\Phi}$

$$J_4(\Delta_1) = \frac{A_0^2}{r'^5} \sum_{\overline{\alpha},\overline{\beta},\overline{b},\overline{c},\overline{d},\overline{i},\overline{j},\overline{k}=0}^{r'-1} \sum_{\overline{\Xi}=\overline{i}} \sum_{\overline{\Lambda}=\overline{j}} \sum_{\overline{\Phi}=\overline{k}} \binom{r'-1}{k_4}\binom{r'-1}{k_5}\binom{r'-1}{k_6}\binom{r'-1}{k_7}$$

$$(3q^{2\Xi_5})(4q^{2\Lambda_5})(3q^{2\Lambda_6})(4q^{-2\Phi_3})4^{r'-1-k_5}\,3^{r'-1-k_6}\,3^{r'-1-k_7}$$

$$q^{-2\alpha_i\beta_i-b_i-2\alpha_j\beta_j-2\alpha_k\beta_k-b_k-2\alpha_l\beta_l-b_l-2\alpha_m\beta_m+b_j\alpha_j+b_m\alpha_m-k_1(k_1-1)-k_2(k_2+1)-k_3(k_3-1)-k_4(k_4+1)}$$

$$\left(\tfrac{1}{1-q^{-2b_i}}\cdots\tfrac{1}{1-q^{-2}}\right)\left(\tfrac{1}{1-q^{-2b_j}}\cdots\tfrac{1}{1-q^{-2}}\right)\left(\tfrac{1}{1-q^{-2b_k}}\cdots\tfrac{1}{1-q^{-2}}\right)\left(\tfrac{1}{1-q^{-2b_l}}\cdots\tfrac{1}{1-q^{-2}}\right)\left(\tfrac{1}{1-q^{-2b_m}}\cdots\tfrac{1}{1-q^{-2}}\right)$$

$$(-1)^{b_i+b_j+b_k+b_m+b_l}q^{b_i(1-b_i)+b_j(1-b_j)-b_k(b_k+1)+b_l(\alpha_l+2-b_l)-b_m(b_m+1)-b_l(b_l+1)+b_j+b_m+k_1+k_2+k_3+k_4}$$

$$q^{2(b_k(\beta_l-b_l)+(\beta_m-b_m)(b_l+b_k)-d_3(b_m+b_l+b_k)+c_2(i_3-j_3-b_m-b_l-b_k)+c_3(i_2-j_2+i_3-j_3-b_m-b_l-b_k))}$$

$$q^{2(-k_7(r'-1-i_5)+(-r'+1)(i_5-k_7)+k_4(k_3+i_5-k_7)-c_5(-k_4+k_3+i_5-k_7)+(\alpha_l-\alpha_j-b_j)(b_j-k_4+k_3+i_5-k_7))}$$

$$q^{2((-r'+1+j_6+d_1)(b_l+b_j-k_4+k_3+i_5-k_7)+(-r'+1+j_5+c_1)(k_6-j_6+b_l+b_j-k_4+k_3+i_5-k_7))}$$

$$q^{2((b_i-\beta_i)(i_1-j_1+i_2-j_2+i_3-j_3-b_m-b_l-b_k)+(b_j-\beta_j)(-b_i+i_1-j_1+i_2-j_2+i_3-j_3-b_m-b_l-b_k)+\alpha_k(b_k-b_j-b_i+i_1-j_1+i_2-j_2+i_3-j_3-b_m-b_l-b_k))}$$

$$q^{2(k_2(-k_5+j_5-k_6+j_6-b_l-b_j+k_4-k_3-i_5+k_7)+k_1(k_2+k_5-j_5+k_6-j_6+b_l+b_j-k_4+k_3+i_5-k_7))}$$

$$q^{2(d_2(k_1-k_2-k_5+j_5-k_6+j_6-b_l-b_j+k_4-k_3-i_5+k_7)}$$

$$q^{2((\alpha_m+b_m)(-b_m+k_1-k_2-k_5+j_5-k_6+j_6-b_l-b_j+k_4-k_3-i_5-k_7)+(c_4+\alpha_i)(b_i+b_m-k_1+k_2+k_5-j_5+k_6-j_6+b_l+b_j-k_4+k_3+i_5-k_7))}$$

$$\lambda(E^{b_k}F^{b_j+b_i}E^{i_1}F^{j_1}E^{i_2}F^{j_2}E^{i_3}F^{j_3+b_m+b_l+b_k}(2q^{2\Xi_1+2(-j_1+i_2-j_2+i_3-j_3-b_m-b_l-b_k)}K+2)$$

$$(2q^{2\Lambda_1+2(-i_2+j_2-i_3+j_3+b_m+b_l+b_k)}K^{-1}+2)(q^{2\Xi_2+2(i_3-j_2-j_3-b_m-b_l-b_k)}K+3)(3q^{2\Lambda_2+2(-i_3+j_3+b_m+b_l+b_k)}K^{-1}+1)$$

$$(2q^{2\Xi_3-2(j_3+b_m+b_l+b_k)}K+1)(q^{2\Lambda_3+2(b_m+b_l+b_k)}K^{-1}+2)K^{\alpha_k+b_j-\beta_j+b_i-\beta_i+c_3+c_2+d_3+b_m-\beta_m+b_l-\beta_l+b_k-\beta_k-1})$$

$$*\lambda(E^{i_4}F^{j_4}E^{b_i+b_m}F^{k_1}E^{k_2+k_5}F^{j_5}E^{k_6}F^{j_6}E^{b_l+b_j}F^{k_4}E^{k_3+i_5}F^{k_7}$$

$$3q^{2\Xi_4+2(j_4-b_i-b_m+k_1-k_2-k_5+j_5-k_6+j_6-b_l-b_j+k_4-k_3-i_5-k_7)}K+1)(q^{2\Lambda_4+2(-b_i-b_m+k_1-k_2-k_5+j_5-k_6+j_6-b_l-b_j+k_4-k_3-i_5-k_7)}K^{-1}+3)$$

$$(2q^{-2\Phi_1+2(k_2+k_5-j_5+k_6-j_6+b_l+b_j-k_4+k_3+i_5-k_7)}K+1)(q^{-2\Phi_2+2(-k_5+j_5-k_6+j_6-b_l-b_j+k_4-k_3-i_5+k_7)}K^{-1}+2)$$

$$K^{c_4+\alpha_i-\alpha_m-b_m-d_2+k_1-k_2+j_5+c_1-2r'+2+j_6+d_1+\alpha_l-\alpha_j-b_j-c_5+k_4-i_5+d_4})$$

Obviously much more work is needed before this invariant can be made manageable.

The are $25+14r'$ nested sums where recall that $q=e^{2\pi i/r}$ and $r'=r/gcd(r,\,2)$.

Chapter 8

Additional Facts about the J_4 Functor

This section is dedicated to providing additional insight into the paper Beliakova and De Renzi 2021. First, we denote for a BPH algebra H_4, $4Alg$, the braided monoidal category freely generated by H_4 (tensor and direct products of H_4). Formally, they construct a functor $J_4 : 4Alg \to mod_{H_4}$. The codomain of J_4 is the category of (left) H_4-modules. Acting on objects, we only need to define that $J_4(H_4) = \underline{H}$ meaning J_4 sends H_4 to its transmutation $\underline{H}$, which is the end in the category of H_4 modules. What is more interesting is how J_4 sends morphisms.

Fundamentally, what the algorithm is doing is to first identify any kirby tangle as a morphism from $H_4^{\otimes m}$ to $H_4^{\otimes m'}$. Then by applying J_4 we obtain a morphism from $\underline{H}^{\otimes m} \to \underline{H}^{\otimes m'}$. More basically, J_4 sends the structure morphisms of H_4 (μ, η, R, ...) to structure morphisms of $\underline{H}$. Because the three categories $4HB$, $KTan$, and $4Alg$ are all equivalent, we abuse notation to use J_4 as a topological invariant on 4-dimensional 2-handlebodies. For instance $J_4(\lambda) = \underline{\lambda}$, $J_4(\mu) = \underline{\mu}$, ... etc. The main work done by this paper was to verify the existence and uniqueness of such a functor. That is to say, they verified that applying the algorithm described in Section 5.2 sends the structure morphisms in $KTan$ to the appropriate structure morphisms in mod_{H_4}.

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Appendices

Appendix A

Extra Calculations for Δ_1

We begin again with the equation for $J_4(\Delta_1)$.

$$J_4(\Delta_1) = \lambda(R'_k\ S^{-1}(R''_j)\ S^{-1}(R''_i)\ \Lambda_{(3)}\ \Lambda_{(2)}\ \Lambda'_{(3)}\ S(g\ R''_k\ R''_l\ R''_m))$$

$$*\lambda(\Lambda_{(4)}\ R'_i\ S(R'_m)\ S^{-1}(\Lambda'_{(2)})\ \Lambda_{(1)}\ \Lambda'_{(1)}\ S^2(R'_l)\ S(R'_j)\ S(\Lambda_{(5)})\ \Lambda'_{(4)})$$

First we recognize the implicit sums that were hidden in the indices of the R-matrices. We may pull them out of the equation because λ is linear:

$$J_4(\Delta_1) = \Sigma_{i,j,k,l,m}\ \lambda(R'_k\ S^{-1}(R''_j)\ S^{-1}(R''_i)\ \Lambda_{(3)}\ \Lambda_{(2)}\ \Lambda'_{(3)}\ \Lambda'_{(4)}\ \Lambda'_{(3)}\ S(g\ R''_k\ R''_l\ R''_m))$$

$$*\lambda(\Lambda_{(4)}\ R'_i\ S(R'_m)\ S^{-1}(\Lambda'_{(2)})\ \Lambda_{(1)}\ \Lambda'_{(1)}\ S^2(R'_l)\ S(R'_j)\ S(\Lambda_{(5)})\ \Lambda'_{(4)})$$

$$J_4(\Delta_1) = \frac{A_0^2}{r'^5} \sum_{\alpha_i,\beta_i,b_i,\alpha_j,\beta_j,b_j,\alpha_k,\beta_k,b_k,\alpha_l,\beta_l,b_l,\alpha_m,\beta_m,b_m=0}^{r'-1}\ \sum_{c_1,c_2,c_3,c_4,c_5,d_1,d_2,d_3,d_4=0}^{r'-1}$$

$$q^{-2\alpha_i\beta_i-b_i-2\alpha_j\beta_j-b_j-2\alpha_k\beta_k-b_k-2\alpha_l\beta_l-b_l-2\alpha_m\beta_m-b_m}$$

$$\left(\tfrac{1}{1-q^{-2b_i}}\cdots\tfrac{1}{1-q^{-2}}\right)\left(\tfrac{1}{1-q^{-2b_j}}\cdots\tfrac{1}{1-q^{-2}}\right)\left(\tfrac{1}{1-q^{-2b_k}}\cdots\tfrac{1}{1-q^{-2}}\right)\left(\tfrac{1}{1-q^{-2b_l}}\cdots\tfrac{1}{1-q^{-2}}\right)\left(\tfrac{1}{1-q^{-2b_m}}\cdots\tfrac{1}{1-q^{-2}}\right)$$

$$\lambda(K^{\alpha_k}E^{b_k}\ S^{-1}(K^{\beta_j}F^{b_j})\ S^{-1}(K^{\beta_i}F^{b_i})\ (E+2K+2)^{r'-1}(F+2K^{-1}+2)^{r'-1}K^{c_3}$$

$$(E+K+3)^{r'-1}(F+3K^{-1}+1))^{r'-1}K^{c_2}\ (E+2K+1)^{r'-1}(F+K^{-1}+2)^{r'-1}K^{d_3}$$

$$S(K^{\beta_m}F^{b_m})S(K^{\beta_l}F^{b_l})\ S(K^{\beta_k}F^{b_k})\ K^{-1})$$

$$*\lambda((E+3K+1)^{r'-1}(F+K^{-1}+3)^{r'-1}K^{c_4}\ K^{\alpha_i}E^{b_i}$$

$$S(K^{\alpha_m}E^{b_m})\ S^{-1}((E+K+2)^{r'-1}(F+2K^{-1}+1)^{r'-1}K^{d_2})$$
$$(E+4)^{r'-1}(F+4K^{-1})^{r'-1}K^{c_1}\ (E+3)^{r'-1}(F+3K^{-1})^{r'-1}K^{d_1}$$
$$S^2(K^{\alpha_l}E^{b_l})\ S(K^{\alpha_j}E^{b_j})\ S((E+4K)^{r'-1}(F+4)^{r'-1}K^{c_5}))\ (E+3K)^{r'-1}(F+3)^{r'-1}K^{d_4})$$

This expression is very complicated, but remember that by applying λ, we will only be summing the coefficients of the maximal degree terms. We can apply some identities from 7.1 to further obtain:

$$J_4(\Delta_1) = \frac{A_0^2}{r'^5} \sum_{\alpha_i,\beta_i,b_i,\alpha_j,\beta_j,b_j,\alpha_k,\beta_k,b_k,\alpha_l,\beta_l,b_l,\alpha_m,\beta_m,b_m,c_1,c_2,c_3,c_4,c_5,d_1,d_2,d_3,d_4=0}^{r'-1}$$
$$q^{-2\alpha_i\beta_i-b_i-2\alpha_j\beta_j-b_j-2\alpha_k\beta_k-b_k-2\alpha_l\beta_l-b_l-2\alpha_m\beta_m-b_m+b_j(\alpha_j+1)+b_m(\alpha_m+1)}$$
$$\left(\tfrac{1}{1-q^{-2b_i}}\cdots\tfrac{1}{1-q^{-2}}\right)\left(\tfrac{1}{1-q^{-2b_j}}\cdots\tfrac{1}{1-q^{-2}}\right)\left(\tfrac{1}{1-q^{-2b_k}}\cdots\tfrac{1}{1-q^{-2}}\right)\left(\tfrac{1}{1-q^{-2b_l}}\cdots\tfrac{1}{1-q^{-2}}\right)\left(\tfrac{1}{1-q^{-2b_m}}\cdots\tfrac{1}{1-q^{-2}}\right)$$
$$(-1)^{b_i+b_j+b_k+b_m+b_l}q^{b_i(1-b_i)+b_j(1-b_j)-b_k(b_k+1)+b_l(\alpha_l+2-b_l)-b_m(b_m+1)-b_l(b_l+1)+b_j+b_m}$$
$$\lambda(K^{\alpha_k}E^{b_k}\ F^{b_j}K^{b_j-\beta_j}\ F^{b_i}K^{b_i-\beta_i}\ (E+2K+2)^{r'-1}(F+2K^{-1}+2)^{r'-1}K^{c_3}$$
$$(E+K+3)^{r'-1}(F+3K^{-1}+1)^{r'-1}K^{c_2}\ (E+2K+1)^{r'-1}(F+K^{-1}+2)^{r'-1}K^{d_3}$$
$$F^{b_m}K^{b_m-\beta_m}F^{b_l}K^{b_l-\beta_l}\ F^{b_k}K^{b_k-\beta_k}\ K^{-1})$$
$$*\lambda((E+3K+1)^{r'-1}(F+K^{-1}+3)^{r'-1}K^{c_4}\ K^{\alpha_i}E^{b_i}$$
$$K^{-\alpha_m-b_m}E^{b_m}\ S^{-1}((E+K+2)^{r'-1}(F+2K^{-1}+1)^{r'-1}K^{d_2})$$
$$(E+4)^{r'-1}(F+4K^{-1})^{r'-1}K^{c_1}\ (E+3)^{r'-1}(F+3K^{-1})^{r'-1}K^{d_1}$$
$$E^{b_l}K^{\alpha_l}\ K^{-\alpha_j-b_j}E^{b_j}\ S((E+4K)^{r'-1}(F+4)^{r'-1}K^{c_5}))\ (E+3K)^{r'-1}(F+3)^{r'-1}K^{d_4})$$

We can remove the remaining instances of S by applying the final formulas in 7.2 (we also make a minor simplification to the exponent of q).

$$J_4(\Delta_1) = \frac{A_0^2}{r'^5} \sum_{\alpha_i,\beta_i,b_i,\alpha_j,\beta_j,b_j,\alpha_k,\beta_k,b_k,\alpha_l,\beta_l,b_l,\alpha_m,\beta_m,b_m,c_1,c_2,c_3,c_4,c_5,d_1,d_2,d_3,d_4=0}^{r'-1}$$

$$q^{-2\alpha_i\beta_i-b_i-2\alpha_j\beta_j-2\alpha_k\beta_k-b_k-2\alpha_l\beta_l-b_l-2\alpha_m\beta_m+b_j\alpha_j+b_m\alpha_m}$$

$$\left(\tfrac{1}{1-q^{-2b_i}}\cdots\tfrac{1}{1-q^{-2}}\right)\left(\tfrac{1}{1-q^{-2b_j}}\cdots\tfrac{1}{1-q^{-2}}\right)\left(\tfrac{1}{1-q^{-2b_k}}\cdots\tfrac{1}{1-q^{-2}}\right)\left(\tfrac{1}{1-q^{-2b_l}}\cdots\tfrac{1}{1-q^{-2}}\right)\left(\tfrac{1}{1-q^{-2b_m}}\cdots\tfrac{1}{1-q^{-2}}\right)$$

$$(-1)^{b_i+b_j+b_k+b_m+b_l}q^{b_i(1-b_i)+b_j(1-b_j)-b_k(b_k+1)+b_l(\alpha_l+2-b_l)-b_m(b_m+1)-b_l(b_l+1)+b_j+b_m}$$

$$\lambda(K^{\alpha_k}E^{b_k}\ F^{b_j}K^{b_j-\beta_j}\ F^{b_i}K^{b_i-\beta_i}\ (E+2K+2)^{r'-1}(F+2K^{-1}+2)^{r'-1}K^{c_3}$$

$$(E+K+3)^{r'-1}(F+3K^{-1}+1)^{r'-1}K^{c_2}\ (E+2K+1)^{r'-1}(F+K^{-1}+2)^{r'-1}K^{d_3}$$

$$F^{b_m}K^{b_m-\beta_m}F^{b_l}K^{b_l-\beta_l}\ F^{b_k}K^{b_k-\beta_k}\ K^{-1})$$

$$*\lambda((E+3K+1)^{r'-1}(F+K^{-1}+3)^{r'-1}K^{c_4}\ K^{\alpha_i}E^{b_i}$$

$$K^{-\alpha_m-b_m}E^{b_m}\ K^{-d_2}\ [-FK+2K+1]^{r'-1}\ [-K^{-1}E+K^{-1}+2]^{r'-1}$$

$$(E+4)^{r'-1}(F+4K^{-1})^{r'-1}K^{c_1}\ (E+3)^{r'-1}(F+3K^{-1})^{r'-1}K^{d_1}$$

$$E^{b_l}K^{\alpha_l}\ K^{-\alpha_j-b_j}E^{b_j}\ K^{-c_5}\ [-KF+4]^{r'-1}\ [-EK^{-1}+4K^{-1}]^{r'-1}\ (E+3K)^{r'-1}(F+3)^{r'-1}K^{d_4})$$

There are 18 terms taken to the power: $(\ \ldots\ )^{r'-1}$. Ordinarily we could use the binomial formula to simplify these, however we do not have commutativity. Instead, we can recognize how closely the terms are to commuting and perform a sort of "weighted stars and bars" method. We will try to bring the arguments of the λ into the form $E^\alpha F^\beta K^\gamma$. First, recall that $KE = q^2EK \qquad KF = q^{-2}FK$. This allows us to compute:

$$(aK+b)E = E(aq^2K+b) \qquad (aK^{-1}+b)F = F(aq^2K^{-1}+b)$$

$$(2K+1)(-FK) = (-FK)(2q^{-2}K+1) \qquad 4K^{-1}(-EK^{-1}) = (-EK^{-1})(4q^{-2}K^{-1})$$

$$(K^{-1}+2)(-K^{-1}E) = (-K^{-1}E)(q^{-2}K^{-1}+2)$$

The following formulas are also useful which we recall:

$$(-FK)^{k_0} = (-1)^{k_0}F^{k_0}K^{k_0}q^{-k_0(k_0-1)}$$

$$(-EK^{-1})^{k_0} = (-1)^{k_0} E^{k_0} K^{-k_0} q^{-k_0(k_0-1)}$$
$$(-K^{-1}E)^{k_0} = (-1)^{k_0} E^{k_0} K^{-k_0} q^{-k_0(k_0+1)}$$

Then we can define the following lengthy formulas (note that the number of sums depends on the first index k_0):

$$(E+aK+b)^{r'-1} = \sum_{k_0=0}^{r'-1} E^{k_0} \sum_{k_1=0}^{k_0+1}(aq^{2k_1}K+b) \sum_{k_2=0}^{k_1}(aq^{2k_2}K+b) \ ... \sum_{k_{r'-1-k_0}=0}^{k_{r'-2-k_0}} (aq^{2k_{r'-1-k_0}}K+b)$$

$$(E+b)^{r'-1} = \sum_{k=0}^{r'-1} \binom{r'-1}{k} E^k b^{r'-1-k}$$

$$(F+b)^{r'-1} = \sum_{k=0}^{r'-1} \binom{r'-1}{k} F^k b^{r'-1-k}$$

$$(F+aK^{-1}+b)^{r'-1} = \sum_{k_0=0}^{r'-1} F^{k_0} \sum_{k_1=0}^{k_0+1}(aq^{2k_1}K^{-1}+b) \sum_{k_2=0}^{k_1}(aq^{2k_2}K^{-1}+b) \ ... \sum_{k_{r'-1-k_0}=0}^{k_{r'-2-k_0}} (aq^{2k_{r'-1-k_0}}K^{-1}+b)$$

$$(-FK+2K+1)^{r'-1} = \sum_{k_0=0}^{r'-1}(-1)^{k_0} q^{-k_0(k_0-1)} F^{k_0} K^{k_0} \sum_{k_1=0}^{k_0+1}(2q^{-2k_1}K+1) \ ... \sum_{k_{r'-1-k_0}=0}^{k_{r'-2-k_0}} (2q^{-2k_{r'-1-k_0}}K+1)$$

$$(-KF+4)^{r'-1} = \sum_{k=0}^{r'-1}(-1)^k \binom{r'-1}{k}^{-k(k+1)} F^k K^k$$

$$(-EK^{-1}+4K^{-1})^{r'-1} = \sum_{k_0=0}^{r'-1}(-1)^{k_0}q^{-k_0(k_0-1)}E^{k_0}K^{-k_0}\sum_{k_1=0}^{k_0+1}4q^{-2k_1}K^{-1}\ \dots \sum_{k_{r'-1-k_0}=0}^{k_{r'-2-k_0}}4q^{-2k_{r'-1-k_0}}K^{-1}$$

$$(-K^{-1}E+K^{-1}+2)^{r'-1} = \sum_{k_0=0}^{r'-1}(-1)^{k_0}q^{-k_0(k_0+1)}E^{k_0}K^{-k_0}\sum_{k_1=0}^{k_0+1}(q^{-2k_1}K^{-1}+2)\ \dots \sum_{k_{r'-1-k_0}=0}^{k_{r'-2-k_0}}(q^{-2k_{r'-1-k_0}}K^{-1}+2)$$

These formulas are extremely cumbersome and we'll need to introduce some new notation to make them easier to work with. We'll introduce the notation

$$\sum_{\Xi=k}[f(\Xi)] := \sum_{k_1=0}^{k_0+1} f(k_1)\sum_{k_2=0}^{k_1} f(k_2)\ \dots \sum_{k_{r'-1-k_0}=0}^{k_{r'-2-k_0}} f(k_{r'-1-k_0})$$

Then for instance:

$$(E+aK+b)^{r'-1} = \sum_{k=0}^{r'-1}E^{k_0}\sum_{\Xi=k}(aq^{2\Xi}K+b)$$

We'll also suppress the sum of α_i, α_j, α_k, , α_l, α_m into just a sum over $\overline{\alpha}$ (for β, b, c, d too). In this new notation we have:

$$J_4(\Delta_1) = \frac{A_0^2}{r'^5}\sum_{\overline{\alpha},\overline{\beta},\overline{b},\overline{c},\overline{d},i_1,i_2,i_3,i_4,i_5,j_1,j_2,j_3,j_4,j_5,j_6,k_1,k_2,k_3,k_4,k_5,k_6,k_7=0}^{r'-1}$$

$$\sum_{\Xi_1=i_1}\sum_{\Xi_2=i_2}\sum_{\Xi_3=i_3}\sum_{\Xi_4=i_4}\sum_{\Xi_5=i_5}\sum_{\Lambda_1=j_1}\sum_{\Lambda_2=j_2}\sum_{\Lambda_3=j_3}\sum_{\Lambda_4=j_4}\sum_{\Lambda_5=j_5}\sum_{\Lambda_6=j_6}\sum_{\Phi_1=k_1}\sum_{\Phi_2=k_2}\sum_{\Phi_3=k_3}$$

$$q^{-2\alpha_i\beta_i-b_i-2\alpha_j\beta_j-2\alpha_k\beta_k-b_k-2\alpha_l\beta_l-b_l-2\alpha_m\beta_m+b_j\alpha_j+b_m\alpha_m}$$

$$\left(\tfrac{1}{1-q^{-2b_i}}\cdots\tfrac{1}{1-q^{-2}}\right)\left(\tfrac{1}{1-q^{-2b_j}}\cdots\tfrac{1}{1-q^{-2}}\right)\left(\tfrac{1}{1-q^{-2b_k}}\cdots\tfrac{1}{1-q^{-2}}\right)\left(\tfrac{1}{1-q^{-2b_l}}\cdots\tfrac{1}{1-q^{-2}}\right)\left(\tfrac{1}{1-q^{-2b_m}}\cdots\tfrac{1}{1-q^{-2}}\right)$$

$$(-1)^{b_i+b_j+b_k+b_m+b_l}q^{b_i(1-b_i)+b_j(1-b_j)-b_k(b_k+1)+b_l(\alpha_l+2-b_l)-b_m(b_m+1)-b_l(b_l+1)+b_j+b_m}$$

$$\lambda(K^{\alpha_k}E^{b_k}\ F^{b_j}K^{b_j-\beta_j}\ F^{b_i}K^{b_i-\beta_i}\ E^{i_1}(2q^{2\Xi_1}K+2)F^{j_1}(2q^{2\Lambda_1}K^{-1}+2)K^{c_3}$$

$$E^{i_2}(q^{2\Xi_2}K+3)F^{j_2}(3q^{2\Lambda_2}K^{-1}+1)K^{c_2}\ E^{i_3}(2q^{2\Xi_3}K+1)F^{j_3}(q^{2\Lambda_3}K^{-1}+2)K^{d_3}$$

$$F^{b_m}K^{b_m-\beta_m}F^{b_l}K^{b_l-\beta_l}\ F^{b_k}K^{b_k-\beta_k}\ K^{-1})$$

$$*\lambda(E^{i_4}(3q^{2\Xi_4}K+1)F^{j_4}(q^{2\Lambda_4}K^{-1}+3)K^{c_4}\ K^{\alpha_i}E^{b_i}$$

$$K^{-\alpha_m-b_m}E^{b_m}\ K^{-d_2}\ (-1)^{k_1}F^{k_1}K^{k_1}q^{-k_1(k_1-1)}(2q^{-2\Phi_1}K+1)\ (-1)^{k_2}E^{k_2}K^{-k_2}q^{-k_2(k_2+1)}(q^{-2\Phi_2}K^{-1}+2)$$

$$\tbinom{r'-1}{k_5}4^{r'-1-k_5}E^{k_5}F^{j_5}(4q^{2\Lambda_5}K^{-1})K^{c_1}\ \tbinom{r'-1}{k_6}3^{r'-1-k_6}E^{k_6}F^{j_6}(3q^{2\Lambda_6}K^{-1})K^{d_1}$$

$$E^{b_l}K^{\alpha_l}\ K^{-\alpha_j-b_j}E^{b_j}\ K^{-c_5}\ (-1)^{k_4}q^{-k_4(k_4+1)}\tbinom{r'-1}{k_4}F^{k_4}K^{k_4}$$

$$(-1)^{k_3}E^{k_3}K^{-k_3}q^{-k_3(k_3-1)}(4q^{-2\Phi_3}K^{-1})\ E^{i_5}(3q^{2\Xi_5}K)\tbinom{r'-1}{k_7}3^{r'-1-k_7}F^{k_7}K^{d_4})$$

This can be mildly cleaned up:

$$J_4(\Delta_1)=\frac{A_0^2}{r'^5}\sum_{\overline{\alpha},\overline{\beta},\overline{b},\overline{c},\overline{d},\overline{i},\overline{j},\overline{k}=0}^{r'-1}\ \sum_{\overline{\Xi}=\overline{i}}\ \sum_{\overline{\Lambda}=\overline{j}}\ \sum_{\overline{\Phi}=\overline{k}}\binom{r'-1}{k_4}\binom{r'-1}{k_5}\binom{r'-1}{k_6}\binom{r'-1}{k_7}$$

$$(3q^{2\Xi_5})(4q^{2\Lambda_5})(3q^{2\Lambda_6})(4q^{-2\Phi_3})4^{r'-1-k_5}\,3^{r'-1-k_6}\,3^{r'-1-k_7}$$

$$q^{-2\alpha_i\beta_i-b_i-2\alpha_j\beta_j-2\alpha_k\beta_k-b_k-2\alpha_l\beta_l-b_l-2\alpha_m\beta_m+b_j\alpha_j+b_m\alpha_m-k_1(k_1-1)-k_2(k_2+1)-k_3(k_3-1)-k_4(k_4+1)}$$

$$\left(\tfrac{1}{1-q^{-2b_i}}\cdots\tfrac{1}{1-q^{-2}}\right)\left(\tfrac{1}{1-q^{-2b_j}}\cdots\tfrac{1}{1-q^{-2}}\right)\left(\tfrac{1}{1-q^{-2b_k}}\cdots\tfrac{1}{1-q^{-2}}\right)\left(\tfrac{1}{1-q^{-2b_l}}\cdots\tfrac{1}{1-q^{-2}}\right)\left(\tfrac{1}{1-q^{-2b_m}}\cdots\tfrac{1}{1-q^{-2}}\right)$$

$$(-1)^{b_i+b_j+b_k+b_m+b_l}q^{b_i(1-b_i)+b_j(1-b_j)-b_k(b_k+1)+b_l(\alpha_l+2-b_l)-b_m(b_m+1)-b_l(b_l+1)+b_j+b_m+k_1+k_2+k_3+k_4}$$

$$\lambda(K^{\alpha_k}E^{b_k}\ F^{b_j}K^{b_j-\beta_j}\ F^{b_i}K^{b_i-\beta_i}\ E^{i_1}(2q^{2\Xi_1}K+2)F^{j_1}(2q^{2\Lambda_1}K^{-1}+2)K^{c_3}$$

$$E^{i_2}(q^{2\Xi_2}K+3)F^{j_2}(3q^{2\Lambda_2}K^{-1}+1)K^{c_2}\ E^{i_3}(2q^{2\Xi_3}K+1)F^{j_3}(q^{2\Lambda_3}K^{-1}+2)K^{d_3}$$

$$F^{b_m}K^{b_m-\beta_m}F^{b_l}K^{b_l-\beta_l}\ F^{b_k}K^{b_k-\beta_k-1})$$

$$*\lambda(E^{i_4}(3q^{2\Xi_4}K+1)F^{j_4}(q^{2\Lambda_4}K^{-1}+3)K^{c_4+\alpha_i}E^{b_i}$$

$$K^{-\alpha_m-b_m}E^{b_m}\ K^{-d_2}\ F^{k_1}K^{k_1}(2q^{-2\Phi_1}K+1)\ E^{k_2}K^{-k_2}(q^{-2\Phi_2}K^{-1}+2)$$
$$E^{k_5}F^{j_5}K^{-r'+1+j_5+c_1}\ E^{k_6}F^{j_6}K^{-r'+1+j_6+d_1}$$
$$E^{b_l}K^{\alpha_l-\alpha_j-b_j}E^{b_j}\ K^{-c_5}\ F^{k_4}K^{k_4}E^{k_3}K^{-r'+1}\ E^{i_5}K^{r'-1-i_5}F^{k_7}K^{d_4})$$

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where we have moved as many constants as possible outside of the λ. The next big step will be to commute around the terms, picking up instances of q. where we have moved as many constants as possible outside of the λ. The next big step will be to commute around the terms, picking up instances of q.

$$J_4(\Delta_1)=\frac{A_0^2}{r'^5}\sum_{\overline{\alpha},\overline{\beta},\overline{b},\overline{c},\overline{d},\overline{i},\overline{j},\overline{k}=0}^{r'-1}\ \sum_{\overline{\Xi}=\overline{i}}\ \sum_{\overline{\Lambda}=\overline{j}}\ \sum_{\overline{\Phi}=\overline{k}}\binom{r'-1}{k_4}\binom{r'-1}{k_5}\binom{r'-1}{k_6}\binom{r'-1}{k_7}$$

$$(3q^{2\Xi_5})(4q^{2\Lambda_5})(3q^{2\Lambda_6})(4q^{-2\Phi_3})4^{r'-1-k_5}\ 3^{r'-1-k_6}\ 3^{r'-1-k_7}$$
$$q^{-2\alpha_i\beta_i-b_i-2\alpha_j\beta_j-2\alpha_k\beta_k-b_k-2\alpha_l\beta_l-b_l-2\alpha_m\beta_m+b_j\alpha_j+b_m\alpha_m-k_1(k_1-1)-k_2(k_2+1)-k_3(k_3-1)-k_4(k_4+1)}$$
$$\left(\tfrac{1}{1-q^{-2b_i}}\cdots\tfrac{1}{1-q^{-2}}\right)\left(\tfrac{1}{1-q^{-2b_j}}\cdots\tfrac{1}{1-q^{-2}}\right)\left(\tfrac{1}{1-q^{-2b_k}}\cdots\tfrac{1}{1-q^{-2}}\right)\left(\tfrac{1}{1-q^{-2b_l}}\cdots\tfrac{1}{1-q^{-2}}\right)\left(\tfrac{1}{1-q^{-2b_m}}\cdots\tfrac{1}{1-q^{-2}}\right)$$
$$(-1)^{b_i+b_j+b_k+b_m+b_l}q^{b_i(1-b_i)+b_j(1-b_j)-b_k(b_k+1)+b_l(\alpha_l+2-b_l)-b_m(b_m+1)-b_l(b_l+1)+b_j+b_m+k_1+k_2+k_3+k_4}$$
$$q^{2(b_k(\beta_l-b_l)+(\beta_m-b_m)(b_l+b_k)-d_3(b_m+b_l+b_k)+c_2(i_3-j_3-b_m-b_l-b_k)+c_3(i_2-j_2+i_3-j_3-b_m-b_l-b_k))}$$
$$q^{2(-k_7(r'-1-i_5)+(-r'+1)(i_5-k_7)+k_4(k_3+i_5-k_7)-c_5(-k_4+k_3+i_5-k_7)+(\alpha_l-\alpha_j-b_j)(b_j-k_4+k_3+i_5-k_7))}$$
$$q^{2((-r'+1+j_6+d_1)(b_l+b_j-k_4+k_3+i_5-k_7)+(-r'+1+j_5+c_1)(k_6-j_6+b_l+b_j-k_4+k_3+i_5-k_7))}$$
$$q^{2((b_i-\beta_i)(i_1-j_1+i_2-j_2+i_3-j_3-b_m-b_l-b_k)+(b_j-\beta_j)(-b_i+i_1-j_1+i_2-j_2+i_3-j_3-b_m-b_l-b_k)+\alpha_k(b_k-b_j-b_i+i_1-j_1+i_2-j_2+i_3-j_3-b_m-b_l-b_k))}$$
$$q^{2(k_2(-k_5+j_5-k_6+j_6-b_l-b_j+k_4-k_3-i_5+k_7)+k_1(k_2+k_5-j_5+k_6-j_6+b_l+b_j-k_4+k_3+i_5-k_7))}$$
$$q^{2(d_2(k_1-k_2-k_5+j_5-k_6+j_6-b_l-b_j+k_4-k_3-i_5+k_7)}$$
$$q^{2((\alpha_m+b_m)(-b_m+k_1-k_2-k_5+j_5-k_6+j_6-b_l-b_j+k_4-k_3-i_5-k_7)+(c_4+\alpha_i)(b_i+b_m-k_1+k_2+k_5-j_5+k_6-j_6+b_l+b_j-k_4+k_3+i_5-k_7))}$$
$$\lambda(E^{b_k}F^{b_j+b_i}E^{i_1}F^{j_1}E^{i_2}F^{j_2}E^{i_3}F^{j_3+b_m+b_l+b_k}(2q^{2\Xi_1+2(-j_1+i_2-j_2+i_3-j_3-b_m-b_l-b_k)}K+2)$$
$$(2q^{2\Lambda_1+2(-i_2+j_2-i_3+j_3+b_m+b_l+b_k)}K^{-1}+2)(q^{2\Xi_2+2(i_3-j_2-j_3-b_m-b_l-b_k)}K+3)(3q^{2\Lambda_2+2(-i_3+j_3+b_m+b_l+b_k)}K^{-1}+1)$$

$$(2q^{2\Xi_3-2(j_3+b_m+b_l+b_k)}K+1)(q^{2\Lambda_3+2(b_m+b_l+b_k)}K^{-1}+2)K^{\alpha_k+b_j-\beta_j+b_i-\beta_i+c_3+c_2+d_3+b_m-\beta_m+b_l-\beta_l+b_k-\beta_k-1})$$

$$*\lambda(E^{i_4}F^{j_4}E^{b_i+b_m}F^{k_1}E^{k_2+k_5}F^{j_5}E^{k_6}F^{j_6}E^{b_l+b_j}F^{k_4}E^{k_3+i_5}F^{k_7}$$

$$(3q^{2\Xi_4+2(j_4-b_i-b_m+k_1-k_2-k_5+j_5-k_6+j_6-b_l-b_j+k_4-k_3-i_5-k_7)}K+1)(q^{2\Lambda_4+2(-b_i-b_m+k_1-k_2-k_5+j_5-k_6+j_6-b_l-b_j+k_4-k_3-i_5-k_7)}K^{-1}+3$$

$$(2q^{-2\Phi_1+2(k_2+k_5-j_5+k_6-j_6+b_l+b_j-k_4+k_3+i_5-k_7)}K+1)(q^{-2\Phi_2+2(-k_5+j_5-k_6+j_6-b_l-b_j+k_4-k_3-i_5+k_7)}K^{-1}+2)$$

$$K^{c_4+\alpha_i-\alpha_m-b_m-d_2+k_1-k_2+j_5+c_1-2r'+2+j_6+d_1+\alpha_l-\alpha_j-b_j-c_5+k_4-i_5+d_4})$$

This is the final equation that results. We have a few ideas for how we can make progress on this, but everything ultimately relies on how we can commute E and F. We will probably have to examine the recurrence relation and try to work out a formula from a two-variable recurrence relation.