Wild Ramification and Stacky Curves

Andrew Kobin

Cedarburg, Wisconsin

Bachelor of Science, Wake Forest University, 2013

Master of Arts, Wake Forest University, 2015

A Dissertation Presented to the Graduate Faculty

of the University of Virginia in Candidacy for the Degree of

Doctor of Philosophy

Department of Mathematics

University of Virginia

May, 2020

Abstract

The local structure of Deligne–Mumford stacks has been studied for decades, but most results require a tameness hypothesis that avoids certain phenomena in positive characteristic. We tackle this problem directly and classify stacky curves in characteristic $p > 0$ with cyclic stabilizers of order p using higher ramification data. Our approach replaces the local root stack structure of a tame stacky curve, similar to the local structure of a complex orbifold curve, with a more sensitive structure called an Artin–Schreier root stack, allowing us to incorporate the ramification data directly into the stack. A complete classification of the local structure of stacky curves, and more generally Deligne–Mumford stacks, will require a broader understanding of root structures, and we begin this program by introducing a higher-order version of the Artin–Schreier root stack. Finally, as an application, we compute dimensions of Riemann–Roch spaces for some examples of stacky curves in positive characteristic and suggest a program for computing spaces of modular forms using the theory of stacky modular curves.

Acknowledgments

I always knew this would be the last part of my thesis I would write – even in the final days of my journey, there are still people supporting me and helping me across the finish line. What I mean to say is, I would not be here without the generosity and encouragement of a long list of people, too many of whom I am leaving out here.

I first want to heap praise upon my parents, Jean and Peter, who taught me *nearly* everything I know. I am sure they will steadfastly deny any credit and redirect it back to me, but they are incredible people and I owe them a lot. My brothers, Danny and Billy, are likewise tireless sources of support and inspiration for me. I love how much you all dove into this with me, even though the math might have seemed daunting!

My adviser, Andrew Obus, has been a huge influence on this project and on me as a young mathematician. I struggled mightily with some of the math in the chapters that follow, but I cannot recall feeling for a single moment that I was incapable of finding my way through and that level of confidence was due to Andrew's unwavering encouragement. I specifically want to thank him for his time helping me edit this thesis, which is, shall we say, not a quick read.

I would like to thank several people who contributed in meaningful ways to the development of this project, including: David Rydh, Vaidehee Thatte, John Voight and David Zureick-Brown. Thank you also to the scores of people who listened to me spill my ideas on the floor and play around with them, including: John Berman, Brandon Boggess, Thomas Brazelton, Soumya Sankar and Libby Taylor.

The University of Virginia math department has been the perfect home the last five years and there are many people to thank for my success here. Faculty members such as Julie Bergner, Paul Bourdon, Francesco di Plinio, Valia Gazaki, Craig Huneke, Nick Kuhn, Sara Maloni, Tom Mark, Tai Melcher, Andrei Rapinchuk, Jim Rolf, Charlotte Ure and Lloyd West were all incredibly generous with their time and advice. The administrative staff – Allison Boese, Raven James, Zvezdana Kish and Robin Koontz Mongold – are simply wonderful people to interact with and made the department run smoothly at all times. And I am fortunate to have met so many extraordinary graduate students who have passed through the department over the years and enriched my life.

Beyond the walls of Kerchof Hall, the city of Charlottesville has also earned a place in my heart. I have made some dear friends here who I want to acknowledge for being great human beings, cheering me on when I was excited about something, and sympathizing when I was down. To Chris Chung, Matt Feller, Zach Gates, Eloísa Grifo, Peter Johnson, Bogdan Krstic, Chris Leonard, Kristy Mangold, Jim Phillips, Mark Schrecengost, George Seelinger and Mojdeh Tarighat: I love you all.

To the talented musicians of University Records: thanks for letting me get some musical energy out with y'all!

Finally, to B: I love you and am grateful for your support these last five years. It has not been an easy time, but I wouldn't be here without you.

Contents

Chapter 0

Introduction

0.1 Motivating Problem

Over the complex numbers, Deligne–Mumford stacks with generically trivial stabilizer can be understood by studying their underlying complex orbifold structure. This approach leads one to the following classification of stacky curves: over $\mathbb{C}$, or indeed any algebraically closed field of characteristic 0, a smooth stacky curve is uniquely determined up to isomorphism by its underlying complex curve and a finite list of numbers corresponding to the orders of the (always cyclic) stabilizer groups of the stacky points of the curve (cf. Section 4.2.2 and [GS]). This key feature of stacky curves in characteristic 0 paves the way for many applications, including the description of the canonical ring of a tame stacky curve in [VZB] (cf. Section 5.1) and a stacky approach to computing rings of modular forms (cf. 5.2 and [VZB, Ch. 6]).

Such a concise classification of stacky curves fails in positive characteristic since stabilizer groups may be nonabelian. Even in the case of stacky curves over an algebraically closed field k of characteristic $p > 0$, there exist infinitely many nonisomorphic stacky curves over k with the same underlying scheme and stacky point with stabilizer group abstractly isomorphic to $\mathbb{Z}/p\mathbb{Z}$ (cf. Remark 4.3.17). Thus any attempt at classifying stacky curves in characteristic p will require finer invariants than the order of the stabilizer group. Our main results, stated below, provide a classification of stacky curves in characteristic $p > 0$ having nontrivial stabilizers of order p, using higher ramification data at the stacky points.

A new tool called an *Artin–Schreier root stack*, denoted $\wp_m^{-1}((L, s, f)/X)$, will be constructed and studied in Section 4.3.1, but it appears in the statements of our main results below so a brief introduction is in order. The object $\wp_m^{-1}((L, s, f)/X)$ is defined using the data of a line bundle L on X and two sections $s \in H^0(X, L)$ and $f \in H^0(X, L^{\otimes m})$. It replaces the root stack $\sqrt[r]{(L, s)/X}$ from the theory of tame stacky curves. Our main results are the following theorems.

Theorem 0.1.1 (Theorem 4.3.14)**.** *Suppose $\varphi : Y \to X$ is a finite separable Galois cover of curves over an algebraically closed field k of characteristic $p > 0$ and $y \in Y$ is a ramification point with image $x \in X$ such that the inertia group $I(y \mid x)$ is $\mathbb{Z}/p\mathbb{Z}$. Then étale-locally, φ factors through an Artin–Schreier root stack $\wp_m^{-1}((L, s, f)/X)$.*

This says that cyclic p-covers of curves $Y \to X$ yield quotient stacks that have an Artin–Schreier root stack structure. By Artin–Schreier theory (Section 1.4.2), over an algebraically closed field of characteristic p there are infinitely many non-isomorphic curves Y covering $\mathbb{P}^1$ with Galois group $\mathbb{Z}/p\mathbb{Z}$, so the theorem implies we have infinitely many non-isomorphic stacky curves over $\mathbb{P}^1$ with a single stacky point of order p. This is a phenomenon that only arises in positive characteristic.

Next, if a stacky curve has an order p stacky point, then locally about this point the stacky curve is isomorphic to an Artin–Schreier root stack:

Theorem 0.1.2 (Theorem 4.3.16(1))**.** *Let $\mathcal{X}$ be a stacky curve over a perfect field k of characteristic $p > 0$. If $\mathcal{X}$ contains a stacky point x of order p, there is an open substack $\mathcal{U} \subseteq \mathcal{X}$ containing x such that $\mathcal{U} \cong \wp_m^{-1}((L, s, f)/U)$ where $(m, p) = 1$, U is an open subscheme of the coarse space X of $\mathcal{X}$ and a triple (L, s, f) on U as above.*

Moreover, if the coarse space of $\mathcal{X}$ is $\mathbb{P}^1$, then $\mathcal{X}$ is the fibre product of finitely many global Artin–Schreier root stacks:

Theorem 0.1.3 (Theorem 4.3.16(2))**.** *Suppose all the nontrivial stabilizers of $\mathcal{X}$ are cyclic of order p. If $\mathcal{X}$ has coarse space $X = \mathbb{P}^1$, then $\mathcal{X}$ is isomorphic to a fibre product of Artin–Schreier root stacks of the form $\wp_m^{-1}((L, s, f)/\mathbb{P}^1)$ for some $(m, p) = 1$ and a triple (L, s, f) on $\mathbb{P}^1$.*

If the coarse space of $\mathcal{X}$ is not $\mathbb{P}^1$ however, then Theorem 0.1.3 fails in general (see Example 4.3.19). In fact, anytime the genus of the coarse space is at least 1, there will be obstructions to sections f inducing a global Artin–Schreier root stack structure on $\mathcal{X}$.

To extend these results to the case when the stabilizers of $\mathcal{X}$ are higher-order cyclic groups, $\mathbb{Z}/p^n\mathbb{Z}$ for $n \geqslant 2$, we generalize the above approach using Artin–Schreier–Witt theory (Section 1.4.3). In particular, in Section 4.4.2, we replace the Artin–Schreier root stack $\wp_m^{-1}((L, s, f)/X)$ with an analogue $\Psi_{\bar{m}}^{-1}(\varphi/X)$. This is defined using a quotient stack $[\overline{\mathbb{W}}_n(\bar{m})/\mathbb{W}_n]$ where $\mathbb{W}_n$ is the ring scheme of length n Witt vectors and $\overline{\mathbb{W}}_n(\bar{m})$ is a new stacky compactification of $\mathbb{W}_n$ based off a construction of Garuti in [Gar] that suits our context. While we do not give proofs of higher-order analogues of Theorems 0.1.1, 0.1.2 and 0.1.3, we set the stage for future efforts at classification.

0.2 Structure of the Thesis

The main described results in Section 0.1 are a synthesis of several different ideas from number theory and algebraic geometry: wild ramification theory, covers of curves, moduli problems and algebraic stacks. Each of these is treated in great detail in the first three chapters of the thesis. Most of the necessary algebraic number theory appears in Chapter 1, in which we review the ramification theory of local fields and the theory of cyclic extensions of fields in the tame (Section 1.4.1), prime order (Section 1.4.2) and prime-power order (Section 1.4.3) cases. This trichotomy will recur throughout the thesis.

In Chapter 2, we give a survey of schemes and algebraic curves, which is necessary language for the more abstract topics in later chapters. Starting from the category of affine schemes (which is nothing but the opposite category of commutative rings) in Section 2.1.1, we can construct schemes as certain topological spaces which are covered by affine schemes (Section 2.1.2). We then describe a theory of sheaves and group actions on them by adapting module theory to this topological setting (Section 2.1.3). We also survey group schemes (Section 2.1.4) and étale morphisms (Section 2.1.5) which are important concepts in later chapters. In Section 2.2, we review the algebraic geometry of curves, including divisors (Section 2.2.1), the Riemann–Roch theorem (Section 2.2.2) and the Riemann–Hurwitz formula (Section 2.2.3), which governs the ramification theory of covers of curves.

We give an extensive treatment of algebraic stacks in Chapter 3. An algebraic stack, by definition (see Section 3.5.1), is a category fibred in groupoids $\mathcal{X}$ which satisfies a descent condition, a representability condition and has a smooth presentation $U \to \mathcal{X}$ by a scheme U. The trajectory of Chapter 3 is intentionally designed to assemble the parts of this definition one-by-one and then stitch them together in a logical way. Elsewhere in mathematics, this "strive to define" often comes at the cost of intuitive understanding, so we take time to paint a larger picture of where stacks come from. Sites and Grothendieck topologies (Section 3) are required reading for understanding stacks, but the point of this language is to extract the defining features of sheaf and cohomology theory and transport them to different geometric settings, the most important being the étale topology (Section 3.1.6). Categories fibred in groupoids (Section 3.2) are part of the definition of an algebraic stack, but more importantly, they allow us to study moduli problems in algebraic geometry that do not necessarily admit a geometric structure. In fact, such a geometric structure (a *fine moduli space*) is often obstructed by the presence of nontrivial automorphisms, an issue which is directly resolved by replacing sets with groupoids. Descent theory (Section 3.3) allows us to define the stack condition (Section 3.3.4), but more broadly, descent is a way of detecting when local data should assemble into something global, in the sense of sheaf theory. Meanwhile, over a site such as the étale site, the covering property of schemes by affine schemes gives rise to new objects called algebraic spaces (Section 3.4); it will be useful to think of algebraic spaces as "schemes in the étale topology".

Algebraic stacks, then, are a combination of these ideas (in Section 3.5 we refer to them as an "interpolation" of stacks and algebraic spaces). As stacks, they completely resolve the issues of nontrivial automorphisms in moduli spaces by incorporating groupoids and descent. As analogues to algebraic spaces, they bring scheme theory into the realm of stacks. By presenting an algebraic stack $\mathcal{X}$ with an étale morphism $U \to \mathcal{X}$, where U is a scheme, rather than a smooth morphism, we obtain a Deligne–Mumford stack (Section 3.5.2). The theory of

Deligne–Mumford stacks is in many ways more manageable than the theory of arbitrary algebraic stacks, e.g. their sheaf theory (Section 3.5.3) essentially mimics the sheaf theory of schemes and they admit a canonical morphism to a scheme called a *coarse moduli space* (Section 3.5.4).

The heart of this thesis lies in Chapter 4. In Section 4.1, we state the definition and basic properties of stacky curves, combining the approaches of Sections 2.2 and 3.5 to algebraic curves and Deligne–Mumford stacks, respectively. A key feature of tame stacky curves is that they are locally isomorphic to a root stack; we review this construction, originally due to Cadman [Cad] and, independently, to Abramovich–Graber–Vistoli [AGV], in Section 4.2. In Section 4.3, we define Artin–Schreier root stacks, describe their basic properties and use them to prove Theorems 0.1.1, 0.1.2 and 0.1.3. In Section 4.4, we generalize the Artin–Schreier root stack construction to the wild, higher-order cyclic case, completing the trichotomy established in Chapter 1.

Finally, we outline two applications in Chapter 5 which are both ongoing programs of investigation. In Section 5.1, we compute the canonical divisor and canonical ring of some stacky curves in characteristic $p > 0$, which are the first steps in a future generalization of the results in [VZB] for tame stacky curves. Then in Section 5.2 we describe a potential application of these computations to the theory of modular forms in positive characteristic.

0.3 Connections to Other Works

It is known (cf. [Ols, 11.3.1]) that under some mild hypotheses, a Deligne–Mumford stack is étale-locally a quotient stack by the stabilizer of a point.

More generally, the sequence of papers [AOV], [Alp10], [Alp13], [Alp14], [AHR1], [AHR2] establishes a structure theory for (tame) algebraic stacks which says that a large class of algebraic stacks are étale-locally a quotient stack of the form $[\operatorname{Spec} A/G_x]$ where G_x is a linearly reductive stabilizer of a point $x \in |\mathcal{X}|$. The present article can be regarded as a small first step in the direction of a structure theory for wild stacks, although it is of a different flavor than the above papers. Our approach is more akin to the one taken in [GS].

Our structure theory in Section 4.3.1 also parallels the approach to wild ramification in formal orbifolds and parabolic bundles taken in [KP], [KMa] and [Kum]. There, the authors define a *formal orbifold* by specifying a smooth projective curve over an algebraically closed field together with a *branch data* abstractly representing the ramification data present in our construction. This allows one to relate formal orbifolds and, more importantly, a suitable notion of *orbifold bundle* on a formal orbifold, to the more classical notions of *parabolic covers* of curves and *parabolic bundles* (cf. [MS]). Formal orbifolds admit a Riemann–Hurwitz formula ([KP, Thm. 2.20]) analogous to Theorem 5.1.2 and can be studied combinatorially as we did in Section 4.3.1. Moreover, they shed light on the étale fundamental group of curves in arbitrary characteristic (cf. [KP, Thm. 2.40] or [Kum, Thm. 1.1]). The perspective we take is more of an "organic" algebro-geometric view of wildly ramified stacky curves which comes naturally out of the classification problem for Deligne–Mumford stacks in dimension 1 discussed in Section 0.1.

Chapter 1

Ramification

In this chapter we gather some definitions and foundational results in the theory of Galois extensions and ramification invariants. Unless otherwise stated, the details and proofs can be found in [Ser2].

1.1 Discretely Valued Fields

The p-adic numbers, first discovered and studied by Kurt Hensel, are an algebraic analogue of the notion of power series expansions in analysis. Their applications in number theory eventually evolved into the theory of *discretely valued fields*, which we describe in this section.

The main features to recall about the ring $\mathbb{Z}_p$ of p-adic integers (where p is a prime integer) are:

- Let $\mathbb{Z}_{(p)}$ be the localization of the ring of integers $\mathbb{Z}$ at the ideal (p). Then there is a topology on $\mathbb{Z}_{(p)}$ induced by the norm $|\cdot|_p$ defined by $|x|_p = p^{-m}$ if $x = p^m \frac{a}{b}$ for $a, b \in \mathbb{Z}, p \nmid ab$.

- $\mathbb{Z}_p$ is the completion of $\mathbb{Z}_{(p)}$ with respect to this p-adic topology.
- $\mathbb{Z}_p$ is a discrete valuation ring (see below) with respect to the p-*adic valuation* ν_p defined by $\nu_p(x) = m$ if $x = p^m \frac{a}{b}$ with $p \nmid ab$.

As p ranges over all prime integers, the fraction fields $\mathbb{Q}_p = \mathrm{Frac}(\mathbb{Z}_p)$ account for all completions of $\mathbb{Q}$ with respect to any norm, save for $\mathbb{R}$. A common theme in modern number theory and arithmetic geometry, known in specific case as a "local-to-global principle" (or "Hasse principal"), is that the properties of an object defined over a field can often be recovered by knowing its properties over the various completions of the field. Therefore the p-adic fields $\mathbb{Q}_p$ and their generalizations occupy an important place in the arithmetic geometry toolkit.

To define a discretely valued field, let us begin by recalling some basic definitions.

Definition 1.1.1. *Let* K *be a field. An* **absolute value** *on* K *is a function* $|\cdot| : K \to \mathbb{R}$ *such that*

(1) $|x| \geqslant 0$ *for all* $x \in K$, *with* $|x| = 0$ *if and only if* $x = 0$.

(2) $|xy| = |x|\,|y|$ *for all* $x, y \in K$.

(3) $|x + y| \leqslant |x| + |y|$ *for all* $x, y \in K$.

Remark 1.1.2. Axiom (3) implies that $|\zeta| = 1$ for any root of unity $\zeta \in K$ such that $\zeta^n = 1$.

Definition 1.1.3. *An absolute value* $|\cdot| : K \to \mathbb{R}_{\geqslant 0}$ *is called* **nonarchimedean** *if* $|x+y| \leqslant \max\{|x|, |y|\}$ *for any* $x, y \in K$*. Otherwise* $|\cdot|$ *is called* **archimedean**.

Example 1.1.4. The trivial absolute value is defined for any field K:

$$|x|_0 = \begin{cases} 1, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

Example 1.1.5. The standard absolute value

$$|x|_\infty = \begin{cases} x, & x \geqslant 0 \\ -x, & x < 0 \end{cases}$$

is an archimedean absolute value on $\mathbb{Q}$.

Example 1.1.6. For any prime number p, the p-adic norm $|\cdot|_p$ is a nonarchimedean absolute value on $\mathbb{Q}$.

Two absolute values on K are said to be equivalent if they induce the same metric topology on K. A theorem of Ostrowski says that every nontrivial absolute value on $\mathbb{Q}$ is either equivalent to $|\cdot|_p$ for some prime p if it is nonarchimedean, or to $|\cdot|_\infty$ if it is archimedean.

The theory of nonarchimedean absolute values has a distinct flavor from that of archimedean absolute values. For instance, in the former case there is an important connection to discrete valuations on K which we explain next. Recall that an integral domain A is a *Dedekind domain* if A is integrally closed, noetherian and has Krull dimension 1 (i.e. every prime ideal is maximal).

Definition 1.1.7. *A local Dedekind domain A is called a* **discrete valuation ring** *(or DVR). Its* **residue field** *is the quotient $k = A/\mathfrak{m}$ where $\mathfrak{m}$ is the unique maximal ideal of A.*

Definition 1.1.8. *Let A be a ring. Then a* **valuation** *on A is a function $v : A \smallsetminus \{0\} \to \mathbb{Z}_{\geqslant 0}$ satisfying:*

(i) $v(xy) = v(x) + v(y)$ *for all* $x, y \in A \smallsetminus \{0\}$.

(ii) $v(x + y) \geqslant \min\{v(x), v(y)\}$ *for all* $x, y \in A \smallsetminus \{0\}$.

(iii) $v(x) = 0$ *if and only if* $x \in A^{\times}$.

A valuation v is a **discrete valuation** *if it is surjective.*

The following characterization is a standard proof in commutative algebra.

Proposition 1.1.9. *For an integral domain A, the following are equivalent:*

(1) A *is a DVR.*

(2) There is a discrete valuation v on A.

It is common to extend a valuation v on A to the field of fractions K of A by setting $v(0) = \infty$ and $v\left(\frac{a}{b}\right) = v(a) - v(b)$ to get a function $v : K \to \mathbb{Z} \cup \{\infty\}$. Then the connection between nonarchimedean absolute values and discrete valuations on K is given by:

Proposition 1.1.10. *Given a nonarchimedean absolute value* $|\cdot|$ *on* K*, setting* $\nu(x) = -\log|x|$ *for all* $x \in K^\times$ *and* $\nu(0) = \infty$ *defines a discrete valuation* $\nu : K \to \mathbb{R} \cup \{\infty\}$.

Proof. For all $x, y \in K$, we have $|xy| = |x|\,|y|$ which implies $\nu(xy) = \nu(x) + \nu(y)$. Likewise, $|x + y| = \max\{|x|, |y|\}$ implies $\nu(x + y) \geqslant \min\{\nu(x), \nu(y)\}$. □

Definition 1.1.11. *For a nonarchimedean absolute value* $|\cdot|$ *on a field* K*, define*

$$\begin{aligned}
\mathcal{O} &:= \{x \in K^\times \mid \nu(x) \geqslant 0\} \cup \{0\} = \{x \in K^\times : |x| \leqslant 1\} \cup \{0\} \\
\mathcal{O}^\times &:= \{x \in K \mid \nu(x) = 0\} = \{x \in K : |x| = 1\} \\
\mathfrak{m} &:= \{x \in K \mid \nu(x) > 0\} = \{x \in K : |x| < 1\} \\
\kappa &:= \mathcal{O}/\mathfrak{m},
\end{aligned}$$

called respectively the **valuation ring***,* **group of units***,* **valuation ideal** *and* **residue field** *of* $|\cdot|$*. We call the triple* $(K, |\cdot|, \nu)$ *a* **discretely valued field***.*

If ν is a discrete valuation on K, we can form the completion $\widehat{K}$ of K with respect to the absolute value $|\cdot| = |\cdot|_\nu$. The following properties are easy to establish.

Lemma 1.1.12. *For any valuation* ν *on* K*,*

(a) The completion $\widehat{K}$ *with respect to* $|\cdot|$ *is a field.*

(b) $|\cdot|$ *extends uniquely to an absolute value on* $\widehat{K}$*.*

(c) K *embeds as a dense subset of* $\widehat{K}$*.*

We will also denote by $|\cdot|$ the unique extension of $|\cdot|$ to $\widehat{K}$. Define the completions of the valuation ring and valuation ideal of $|\cdot|$ in $\widehat{K}$:

$$\widehat{\mathcal{O}} = \{x \in \widehat{K}^\times : |x| \leqslant 1\} \cup \{0\}$$
$$\widehat{\mathfrak{m}} = \{x \in \widehat{K}^\times : |x| < 1\}.$$

Then one can show:

Lemma 1.1.13. *For any absolute value $|\cdot|$ on K,*

(a) $\widehat{\mathcal{O}}/\widehat{\mathfrak{m}} = \mathcal{O}/\mathfrak{m}$.

(b) $\widehat{\mathcal{O}} = \varprojlim \mathcal{O}/\mathfrak{m}^n$.

(c) $\widehat{\mathcal{O}}^\times = \varprojlim (\mathcal{O}/\mathfrak{m}^n)^\times$.

A key tool in the study of discretely valued fields is Hensel's lemma, which appears in various forms in the literature. The following version can be found in (cite).

Theorem 1.1.14 (Hensel's Lemma)**.** *Assume K is a field which is complete with respect to a discrete, nonarchimedean absolute value $|\cdot|$. Suppose $f(x) \in \mathcal{O}[x]$ is a monic polynomial of degree n and $\bar{f}(x) \in \kappa[x]$ admits a factorization*

$$\bar{f}(x) = \bar{g}(x)\bar{h}(x)$$

for $\bar{g}, \bar{h}$ relatively prime, monic polynomials over κ of degrees r and $n - r$, respectively.

Then

$$f(x) = g(x)h(x)$$

for $g(x), h(x) \in \mathcal{O}[x]$ *with* $\deg g = r, \deg h = n - r, \bar{g}(x) = g(x) \mod \mathfrak{m}$ *and* $\bar{h}(x) = h(x) \mod \mathfrak{m}$.

Corollary 1.1.15. *If* $f(x) \in \mathcal{O}[x]$ *such that* $\bar{f}(x) \in \kappa[x]$ *has a simple root in* κ *then* $f(x)$ *has a simple root in* $\mathcal{O}$.

Proof. Apply Theorem 1.1.14 with $r = 1$. □

1.2 Local Fields

Definition 1.2.1. *A* **local field** *is a complete, discretely valued field with perfect residue field.*

Example 1.2.2. For any prime integer p, the p-adic field $\mathbb{Q}_p$ and the field of Laurent series $\mathbb{F}_p((t))$ are both local fields.

Remark 1.2.3. In the literature, it is sometimes required that a discretely valued field has a finite residue field to be local. In other places, the residue field is allowed to be arbitrary. Many times $\mathbb{R}$ and $\mathbb{C}$ are included in the definition of local field, for reasons related to the "local-to-global" philosophy mentioned in Section 1.1.

Theorem 1.2.4. *Every local field with finite residue field is a finite extension of* $\mathbb{Q}_p$ *or* $\mathbb{F}_p((t))$ *for some prime integer* p.

Let K be a local field with residue field κ and assume $\operatorname{char} \kappa = p > 0$ for some prime p. When $\operatorname{char} K = 0$, we call this the *mixed characteristic case*, whereas $\operatorname{char} K = p$ is called the *equal characteristic case*.

Many useful number theoretic properties of a field may be derived solely from the lifting property in Hensel's Lemma, so we may weaken the completeness assumptions in Theorem 1.1.14.

Definition 1.2.5. *A field K is* **henselian** *if there exists a nonarchimedean absolute value $|\cdot|$ on K with valuation ring $\mathcal{O}$ such that Hensel's Lemma holds for irreducible polynomials in $\mathcal{O}[x]$.*

Example 1.2.6. By Hensel's Lemma, complete, discretely valued fields are henselian.

Suppose $(K, |\cdot|, \nu)$ is a nonarchimedean field. Taking its completion $\widehat{K}$, we can consider the subextension $K \subseteq K^h \subseteq \widehat{K}$ defined by

$$K^h = \{\alpha \in \widehat{K} \mid \alpha \text{ is separable over } K\}.$$

Then ν and $|\cdot|$ extend uniquely to $\widehat{K}$ (Lemma 1.1.12); denote their restrictions to $K^h \subseteq \widehat{K}$ also by ν and $|\cdot|$. This makes K^h into a nonarchimedean field with valuation ring $\mathcal{O}^h := \mathcal{O}_{K^h}$. Note that $\mathcal{O} \subseteq \mathcal{O}^h \subseteq \widehat{\mathcal{O}}$. Since the value groups and residue fields of K and $\widehat{K}$ are the same (Lemma 1.1.13), the value group and residue field of $\mathcal{O}^h$ must coincide with these as well.

Lemma 1.2.7. K^h *is henselian.*

Proof. Factoring a monic polynomial $f(x) \in K[x]$ can be done over the algebraic closure $\overline{K}$ of K if it can be done over any extension of K. Thus Hensel's Lemma holds for $\overline{K} \cap \widehat{K} = K^{sep} \cap \widehat{K} = K^h$. $\square$

Definition 1.2.8. *For a nonarchimedean field* $(K, |\cdot|, \nu)$*, the field* $K^h \subseteq \widehat{K}$ *is called the* **henselization** *of* K*.*

Theorem 1.2.9. *If* $(K, |\cdot|)$ *is a henselian field and* L/K *is an algebraic extension, then there is a unique absolute value* $|\cdot|_L$ *on* L *extending* $|\cdot|$*. Further, if* L/K *is finite of degree* n *then*

$$|x|_L = \sqrt[n]{|N_{L/K}(x)|}$$

and L *is complete with respect to* $|\cdot|_L$ *if* K *is complete with respect to* $|\cdot|$*.*

The converse of Theorem 1.2.9 is true, that is, the property of unique extension of absolute values *characterizes* Henselian fields.

Theorem 1.2.10. *Suppose* $(K, |\cdot|, \nu)$ *is a nonarchimedean field such that* $|\cdot|$ *extends uniquely to any algebraic extension* L/K*. Then* K *is Henselian.*

Corollary 1.2.11. *Every algebraic extension of a Henselian field is Henselian. In particular, every finite extension of a Henselian field is also Henselian.*

Corollary 1.2.12. *Let* $(K, |\cdot|)$ *be a complete nonarchimedean field and* L/K *an algebraic extension. Then there is a unique absolute value* $|\cdot|_L$ *on* L *which extends* $|\cdot|$ *and is of the form* $|x|_L = \sqrt[n]{|N_{L/K}(x)|}$ *if* L/K *is finite of degree* $[L : K] = n$*. Moreover,* L *is complete with respect to this* $|\cdot|_L$*.*

Let $(K, |\cdot|, \nu)$ be a nonarchimedean field and L/K an algebraic extension.

Then the extension of absolute values to L induces an extended valuation

$$\begin{aligned} w : L^{\times} &\longrightarrow \mathbb{R} \\ \alpha &\longmapsto v(N_{L/K}(\alpha)). \end{aligned}$$

Moreover, by Theorem 1.2.9, if K is Henselian then w is the unique such valuation on L extending v.

Definition 1.2.13. *For a Henselian field* $(K, |\cdot|, v)$ *and an algebraic extension* $(L, |\cdot|_L, w)$*, the* **ramification index** *is* $e = e_{L/K} = [w(L^{\times}) : v(K^{\times})]$ *and the* **inertial degree** *is* $f = f_{L/K} = [\lambda : \kappa]$.

Notice that if v is a discrete valuation and w is its extension to L/K, we have

$$w(\pi_L^e) = ew(\pi_L) = v(\pi_K) = w(\pi_K),$$

so $(\pi_L^e) = (\pi_K)$ in $\mathcal{O}_L$, i.e. $\mathfrak{m}_L^e = \mathfrak{m}_K \mathcal{O}_L$.

Proposition 1.2.14. *Let* K *be Henselian,* L/K *a finite extension and* $e = e_{L/K}$ *and* $f = f_{L/K}$ *the ramification index and inertial degree, respectively. Then* $[L : K] \geqslant ef$ *with equality if and only if* v *is a discrete valuation and* L/K *is separable.*

Proof. Pick elements $\omega_1, \dots, \omega_f \in \mathcal{O}_L$ which reduce modulo $\mathfrak{m}_K$ to a basis of λ/κ. Also pick $\pi_0, \pi_1, \dots, \pi_{e-1} \in L^{\times}$ such that $w(\pi_0), w(\pi_1), \dots, w(\pi_{e-1})$ are representatives of $w(L^{\times})/v(K^{\times})$. It then suffices to prove the products $\omega_i \pi_j$ are linearly independent over K. Suppose $\sum_{i,j} a_{ij} \omega_i \pi_j = 0$ where $a_{ij} \in K$ are not all 0. Collecting the terms of minimal valuation in this sum, it will be enough to show that the sum of these lowest-valuation terms has the same valuation as each individually. Observe that all these terms must share the same index j, because

$$w(a_{ij}\omega_i\pi_j) = w(a_{ij}) + w(\pi_j) \equiv w(\pi_j) \mod w(K^\times),$$

so different j correspond to different valuations. Fix this j and consider

$$\sum_{i\in I} a_{ij}\omega_j\pi_j$$

where $I \subseteq \{1,\dots,f\}$ corresponds to the subset of terms of minimal valuation. Then $w(a_{ij})$ is constant over $i \in I$, say $w(a_{ij}) = a$, so $a_{ij} = \varepsilon b_{ij}$ for some $\varepsilon \in K^\times$ and b_{ij} satisfying $w(b_{ij}) = 0$. Thus

$$\varepsilon\pi_j \sum_{i\in I} b_{ij}\omega_j \not\equiv 0 \mod \mathfrak{m}_L$$

since $\bar{\omega}_1,\dots,\bar{\omega}_f$ are a basis for λ/κ. So

$$w\left(\sum_{i\in I} a_{ij}\omega_i\pi_j\right) = w(\varepsilon\pi_j) = w(a_{ij}) = a$$

and the linear independence is proved.

Now assume v is discrete and L/K is separable. Then each $\pi_j = \pi_L^j$. Define the $\mathcal{O}_L$-submodules

$$M = \sum_{i,j} \mathcal{O}_K\omega_i\pi_j = \sum_{i,j} \mathcal{O}_K\omega_i\pi_L^j \quad \text{and} \quad N = \sum_i \mathcal{O}_K\omega_i.$$

Then $M = N + \pi_L N + \ldots + \pi_L^{e-1} N$. We will show $M = \mathcal{O}_L$. Write

$$\begin{aligned}
\mathcal{O}_L &= N + \pi_L \mathcal{O}_L \\
&= N + \pi_L(N + \pi_L \mathcal{O}_L) \\
&= N + \pi_L(N + \pi_L(N + \pi_L \mathcal{O}_L)) \\
&= N + \pi_L N + \pi_L^2 N + \ldots + \pi_L^{e-1} N + \pi_L^e \mathcal{O}_L \quad \text{after } e \text{ expansions} \\
&= M + \pi_L^e \mathcal{O}_L = M + \pi_K \mathcal{O}_L.
\end{aligned}$$

Now $\mathcal{O}_K$ is a local ring (it's a DVR) and since L/K is separable, $\mathcal{O}_L$ is a finitely generated $\mathcal{O}_K$-module. Therefore by Nakayama's Lemma, $\mathcal{O}_L = M$. Hence $[L : K] = ef$. □

Remark 1.2.15. For complete fields with discrete valuations, the 'fundamental equality' in Proposition 1.2.14 holds even without the separable assumption.

1.3 Ramification Theory

Let K be a Henselian field with valuation ring $\mathcal{O}_K$, valuation ideal $\mathfrak{m}_K$, residue field κ and valuation ν as usual, and let L/K be a finite extension with extensions $\mathcal{O}_L, \mathfrak{m}_L, \lambda$ and w of the corresponding objects for K.

Definition 1.3.1. *We say* L/K *is* **unramified** *if* $f_{L/K} = [L : K]$ *and* λ/κ *is separable. Otherwise* L/K *is* **ramified***. Further, when* $\operatorname{char} \kappa = p > 0$*, the extension* L/K *is called* **tamely ramified** *(or simply* **tame***) if* λ/κ *is separable and* p *does not divide* $[L : T]$*, where* T *is the maximal unramified subextension of* L/K*.*

The ramification theory of L/K can be studied by factoring the extension into a tower

$$L \supseteq V \supseteq T \supseteq K,$$

where T/K is the maximal unramified subextension of L/K and V/K is the maximal tame subextension. We call L/K *totally ramified* if $T = K$ and *wildly ramified* if $V \neq L$. It is the last type of extension of local fields that will play a central role in this thesis.

For the remainder of this section, let L/K be a finite Galois extension of discretely valued fields with Galois group $G = \mathrm{Gal}(L/K)$. Then G acts on the set of extensions $|\cdot|_w$ of $|\cdot|_v$ to L by $\sigma(|\cdot|_w)(x) = |\sigma(x)|_w$ for all $x \in L$. One can show that this action is transitive. Fixing an extension $w \mid v$ of valuations, we define

$$\begin{aligned}
\mathcal{O}_{L,w} &= \{x \in L : |x|_w \leqslant 1\} \quad \text{(the valuation ring for } w\text{)} \\
\mathfrak{P}_{L,w} &= \{x \in L : |x|_w < 1\} \quad \text{(the valuation ideal for } w\text{)} \\
G_w &= \{\sigma \in G : |\sigma(x)|_w = |x|_w \text{ for all } x \in L\} \\
I_w &= \{\sigma \in G_w : \sigma(x) \equiv x \mod \mathfrak{P}_{L,w} \text{ for all } x \in \mathcal{O}_{L,w}\} \\
R_w &= \left\{\sigma \in G_w : \frac{\sigma(x)}{x} \equiv 1 \mod \mathfrak{P}_{L,w} \text{ for all } x \in L^\times\right\}.
\end{aligned}$$

The subgroups G_w, I_w and R_w are called, respectively, the *decomposition group*, *inertia group* and *ramification group* for $w \mid v$. In general, we have $R_w \leqslant I_w \leqslant G_w \leqslant G$. Some basic properties of these subgroups can be found in Serre ([Ser2, Ch. IV]).

Proposition 1.3.2. *Let* $\operatorname{char} \kappa = p$. *If* $p > 0$ *then* R_w *is the unique Sylow* p*-subgroup of* I_w.

The sequence of subgroups $R_w \leqslant I_w \leqslant G_w \leqslant G$ is really the beginning of a filtration of subgroups for $G = \operatorname{Gal}(L/K)$, which we construct now.

Definition 1.3.3. *For each* $s \in [-1, \infty)$, *define the* s**th higher ramification group**

$$G_s = \{\sigma \in G \mid \nu_L(\sigma(a) - a) \geqslant s + 1 \textit{ for all } a \in \mathcal{O}_L\}.$$

(These may also be referred to as the ramification groups of G *for the lower numbering.)*

Example 1.3.4. Clearly $G_{-1} = G$ and $G_0 = I = I_{\nu_L}$ is the inertia group. Moreover, if $R = R_{\nu_L}$ is the ramification group of G, we have

$$\begin{aligned} \sigma \in R &\iff \nu_L\left(\frac{\sigma(a)}{a} - 1\right) \geqslant 1 \text{ for all } a \in \mathcal{O}_L \\ &\iff \nu_L\left(\frac{\sigma(a) - a}{a}\right) \geqslant 1 \text{ for all } a \in \mathcal{O}_L. \end{aligned}$$

If $a \in \mathfrak{m}_L$, then $\nu_L\left(\frac{\sigma(a)-a}{a}\right) = \nu_L(\sigma(a) - a) - \nu_L(a)$ so $\nu_L(\sigma(a) - a) \geqslant \nu_L(a) + 1 \geqslant 2$. Likewise for $a \in \mathcal{O}_L^\times$, so $G_1 = R$ is the ramification group.

Lemma 1.3.5. G_s *is a normal subgroup of* G *for all* $s \geqslant 0$.

Proof. Take $\tau \in G_s, \sigma \in G$ and $a \in L$. Then

$$\nu_L(\sigma\tau\sigma^{-1}(a) - a) = \nu_L(\tau(\sigma^{-1}(a)) - \sigma^{-1}(a))$$

so if $\nu_L(\tau(x) - x) \geqslant s + 1$ for all $x \in \mathcal{O}_L$, then $\nu_L(\sigma\tau\sigma^{-1}(x) - x) \geqslant s + 1$ for all $x \in \mathcal{O}_L$ and vice verse, since σ acts on G by automorphisms. $\square$

The higher ramification groups G_s form a filtration of G,

$$G = G_{-1} \supseteq G_0 \supseteq G_1 \supseteq G_2 \supseteq \cdots,$$

called the *higher ramification filtration* (*in the lower numbering*) for G.

Lemma 1.3.6. G_0 *is isomorphic to a semidirect product* $P \rtimes \mathbb{Z}/m\mathbb{Z}$ *where* P *is a* p*-group and* $m \in \mathbb{Z}, p \nmid m$.

Corollary 1.3.7. G_0 *is solvable.*

Corollary 1.3.8. *If* L/K *is totally ramified and Galois, then* $\mathrm{Gal}(L/K)$ *is solvable.*

Example 1.3.9. Consider the local function field $K = \overline{\mathbb{F}}_p((t))$. Then any finite Galois extension L/K is totally ramified and hence has solvable Galois group.

Remark 1.3.10. If L/K is Galois and the residue extension λ/κ is separable, we can write $\mathcal{O}_L = \mathcal{O}_K[x]$ for some $x \in \mathcal{O}_L$. For each nontrivial $\sigma \in G = \mathrm{Gal}(L/K)$, write $i_{L/K}(\sigma) = \nu_L(\sigma(x) - x)$ and also set $i_{L/K}(1) = \infty$. In fact, $i_{L/K}(\sigma) = \min_{y \in \mathcal{O}_L}\{\nu_L(\sigma(y) - y)\}$ since for any $y \in \mathcal{O}_L$, we may write

$$y = a_0 + a_1 x + \ldots + a_n x^n$$

for $n \in \mathbb{N}, a_i \in \mathcal{O}_K$ and have $\sigma(y) - y = a_1(\sigma(x) - x) + \ldots + a_n(\sigma(x^n) - x^n)$. By a binomial expansion, each $\sigma(x^k) - x^k$ is divisible by $\sigma(x) - x$ so it follows that $\nu_L(\sigma(y) - y) \geqslant \nu_L(\sigma(x) - x)$. In particular, this implies usefully that the definition of $i_{L/K}(\sigma)$ is independent of any generator chosen for $\mathcal{O}_L$. The higher ramification groups can thus be written

$$G_s(L/K) = \{\sigma \in G \mid i_{L/K}(\sigma) \geqslant s+1\}.$$

Define the function

$$\begin{aligned} \varphi_{L/K} : [-1, \infty) &\longrightarrow [-1, \infty) \\ s &\longmapsto \int_0^s \frac{dx}{[G_0 : G_s]} \end{aligned}$$

where formally we set $[G_0 : G_{-1}] = [G : G_0]^{-1}$. Then $\varphi_{L/K}$ is piecewise-linear, nondecreasing and if $g_s = |G_s|$, then we can explicitly write

$$\varphi_{L/K}(s) = \frac{1}{g_0}(g_1 + \ldots + g_m + (s-m)g_{m+1})$$

for any $m \in \mathbb{N}$ such that $0 < m \leqslant s \leqslant m+1$. Also, $\varphi_{L/K}(s) = s$ for $-1 \leqslant s \leqslant 0$. By this reformulation, we can see that the slope of $\varphi_{L/K}(s)$ is $\frac{g_{m+1}}{g_0}$ for all s, where $m < s < m+1$, but when $s \in \mathbb{Z}$, the slope is $\frac{g_{s-1}}{g_0}$. This implies:

Lemma 1.3.11. *For any* $s \geqslant -1$, $\varphi_{L/K}(s) = \dfrac{1}{g_0} \displaystyle\sum_{\sigma \in G} \min\{i_{L/K}(\sigma), s+1\} - 1$.

Definition 1.3.12. *Let* L/K *be a Galois extension. Then the subgroups* $G^t := G_s$ *for* $t = \varphi_{L/K}(s)$ *are called the* **higher ramification groups for the upper numbering** *of* G.

The advantage of the ramification groups of upper numbering is that they are invariant under passage to a Galois subextension L'/K of L/K (corresponding to taking a quotient of the Galois group of L/K). By construction, the "jumps" in the filtration G_s can only occur at integers. However, this is not necessarily true of the ramification groups of upper numbering G^t. However, we have:

Theorem 1.3.13 (Hasse-Arf)**.** *If* L/K *is an abelian extension and* G^t *is a jump in the upper filtration of* $G = \text{Gal}(L/K)$, *then* $t \in \mathbb{Z}$*.*

1.4 Cyclic Extensions

In this section, we review the abstract algebra of cyclic Galois extensions of fields and, in the case when the ground field has characteristic $p > 0$, we describe the ramification theory of cyclic extensions of order $p^n, n \geqslant 1$. Further details can be found in [Lan, Part VI, Sec. 6].

Fix a finite Galois extension L/K of fields with group $G = \text{Gal}(L/K) = \{\sigma_1, \dots, \sigma_n\}$ and let $\overline{K}$ be an algebraic closure of K. For each $\alpha \in L$, the *norm* and *trace* of α are defined respectively by

$$N_{L/K}(\alpha) = \prod_{j=1}^{n} \sigma_j(\alpha) \qquad \text{and} \qquad T_{L/K}(\alpha) = \sum_{j=1}^{n} \sigma_j(\alpha).$$

Notice that $N_{L/K}(\alpha), T_{L/K}(\alpha) \in K$. When the extension is understood, these will be written simply as N and T. The norm and trace functions, along with the following lemma, will be key tools in classifying cyclic extensions of K.

Lemma 1.4.1. *Any set of distinct automorphisms of a field is linearly independent.*

Proof. Let $\sigma_1, \ldots, \sigma_n$ be distinct elements of $\mathrm{Aut}(L/K)$ and suppose there exist $c_1, \ldots, c_n \in K$ such that

$$c_1\sigma_1(x) + \ldots + c_n\sigma_n(x) = 0$$

for all $x \in K$. We must show $c_i = 0$ for all i. To do so, induct on n. The base case $n = 1$ is trivial, so suppose $n \geqslant 2$. Since $\sigma_1 \neq \sigma_n$, there must be some $x \in K$ with $\sigma_1(x) \neq \sigma_n(x)$. Evaluating the linear relation at xy for any $y \in F$ gives

$$c_1\sigma_1(x)\sigma_1(y) + \ldots + c_n\sigma_n(x)\sigma_n(y) = 0.$$

On the other hand,

$$c_1\sigma_n(x)\sigma_1(y) + \ldots + c_n\sigma_n(x)\sigma_n(y) = \sigma_n(x)(c_1\sigma_1(y) + \ldots + c_n\sigma_n(y)) = 0$$

and subtracting these two lines gives

$$c_1(\sigma_1(x) - \sigma_n(x))\sigma_1(y) + \ldots + c_{n-1}(\sigma_{n-1}(x) - \sigma_n(x))\sigma_{n-1}(y) = 0.$$

This is a linear relation on $\sigma_1, \ldots, \sigma_{n-1}$ evaluated at y, so by induction, $c_i(\sigma_i(x) - \sigma_n(x)) = 0$ for all $1 \leqslant i \leqslant n-1$. But $c_1(\sigma_1(x) - \sigma_n(x)) = 0$ and $\sigma_1(x) \neq \sigma_n(x)$ together imply $c_1 = 0$. Similarly, comparing $\sigma_2, \ldots, \sigma_{n-1}$ to σ_n gives $c_2 = \cdots =$

$c_{n-1} = 0$, so $c_n \sigma_n(y) = 0$ for all $y \in F$. Since σ_n is nontrivial, this implies $c_n = 0$, completing the proof. $\square$

1.4.1 Kummer Theory

Let L/K be a Galois extension with norm $N = N_{L/K}$. Then for any $\sigma \in G = \mathrm{Gal}(L/K)$, $N(\sigma(\alpha)) = N(\alpha)$ holds for all $\alpha \in L$. This implies that if $\alpha = \frac{\sigma(\beta)}{\beta}$ for some $\beta \in L^\times$, then $N(\alpha) = 1$. In fact, this characterizes the norm 1 elements in any cyclic extension.

Theorem 1.4.2 (Hilbert's Theorem 90)**.** *Let* L/K *be a cyclic extension with Galois group* $G = \langle \sigma \rangle$*. Then for all* $\alpha \in L^\times$, $N(\alpha) = 1$ *if and only if* $\alpha = \frac{\sigma(\beta)}{\beta}$ *for some* $\beta \in L^\times$.

Proof. ($\Longleftarrow$) was proven above.

($\Longrightarrow$) Let $[L : K] = n$ so that $G = \{1, \sigma, \ldots, \sigma^{n-1}\}$. Fixing $\alpha \in L$ such that $N(\alpha) = 1$, define

$$\begin{aligned} f(x) &= \alpha x + \alpha\sigma(\alpha)\sigma(x) + \alpha\sigma(\alpha)\sigma^2(\alpha)\sigma^2(x) + \ldots + \alpha\sigma(\alpha)\cdots\sigma^{n-1}(\alpha)\sigma^{n-1}(x) \\ &= \alpha x + \alpha\sigma(\alpha)\sigma(x) + \alpha\sigma(\alpha)\sigma^2(\alpha)\sigma^2(x) + \ldots + \sigma^{n-1}(x) \end{aligned}$$

since $\alpha\sigma(\alpha)\cdots\sigma^{n-1}(\alpha) = N(\alpha) = 1$. Now $\alpha \neq 0$ so the coefficients of $f(x)$ are nonzero. Thus by Lemma 1.4.1, there is some $x \in K$ such that $f(x) \neq 0$.

Applying σ, we get

$$\begin{aligned}\sigma(f(x)) &= \sigma(\alpha x + \alpha\sigma(\alpha)\sigma(x) + \alpha\sigma(\alpha)\sigma^2(\alpha)\sigma^2(x) + \ldots + \sigma^{n-1}(x)) \\ &= \sigma(\alpha)\sigma(x) + \sigma(\alpha)\sigma^2(\alpha)\sigma^2(x) + \sigma(\alpha)\sigma^2(\alpha)\sigma^3(\alpha)\sigma^3(x) + \ldots + \sigma^n(x) \\ &= \frac{f(x) - \alpha x}{\alpha} + x \quad \text{since } \sigma^n = 1 \\ &= \frac{f(x)}{\alpha} - x + x = \frac{f(x)}{\alpha}.\end{aligned}$$

Taking $\beta = \frac{1}{f(x)}$, we have $\alpha = \frac{\sigma(\beta)}{\beta}$. □

Theorem 1.4.3 (Kummer). *Suppose* K *contains a primitive* n*th root of unity. Then there is a bijection*

$$\left\{\begin{matrix}\textit{cyclic extensions } L/K \textit{ with} \\ [L:K] \textit{ dividing } n\end{matrix}\right\} \longleftrightarrow \{K(\sqrt[n]{a}) \mid a \in K^\times\}.$$

Proof. If ζ_n denotes a primitive nth root of unity in K and $\sqrt[n]{a}$ is a root of $x^n - a$, then all roots of $x^n - a$ are given by $\sqrt[n]{a}, \zeta_n\sqrt[n]{a}, \zeta_n^2\sqrt[n]{a}, \ldots, \zeta_n^{n-1}\sqrt[n]{a}$. Since each $\zeta_n^i\sqrt[n]{a} \in K(\sqrt[n]{a})$, $K(\sqrt[n]{a})/K$ is Galois. Let $G = \mathrm{Gal}(K(\sqrt[n]{a})/F)$ and define a map

$$\begin{aligned}\varphi : G &\longrightarrow \{\zeta_n^i \mid 0 \leqslant i \leqslant n-1\} \cong \mathbb{Z}/n\mathbb{Z} \\ \sigma &\longmapsto \frac{\sigma(\sqrt[n]{a})}{\sqrt[n]{a}}.\end{aligned}$$

Then φ is a group homomorphism and $\ker\varphi = \{1\}$, so G is a cyclic group of order dividing n.

Conversely, suppose L/K is cyclic of order $d \mid n$ and Galois group $\mathrm{Gal}(L/K) =$

$\langle\tau\rangle$. Then $\zeta_d = \zeta_n^{n/d}$ is a primitive dth root of unity in K and by the properties of the norm map,

$$N_{L/K}(\zeta_d) = \zeta_d^{[L:K]} = \zeta_d^d = 1.$$

So by Hilbert's Theorem 90, $\zeta_d = \frac{\tau(\beta)}{\beta}$ for some $\beta \in L^\times$, or alternatively, $\tau(\beta) = \zeta_d \beta$. Since τ generates Gal(L/K), the Galois conjugates of β are precisely $\beta, \zeta_d\beta, \zeta_d^2\beta, \ldots, \zeta_d^{d-1}\beta$ which proves $L = K(\beta)$. Also, notice that $\tau(\beta^d) = \tau(\beta)^d = \zeta_d^d \beta^d = \beta^d$, so $\beta^d \in K$. Since $d \mid n$, $\beta^n \in K$ as well and we have $L = K(\beta) = K(\sqrt[n]{a})$ where $a = \beta^n$. □

Remark 1.4.4. When K does not contain a primitive nth root of unity, one can similarly show that any cyclic extension of K of degree n is a subfield of $K(\zeta_n, \sqrt[n]{a})$ for some $a \in K^\times$.

Definition 1.4.5. *A* **Kummer extension** *is an extension* $K(\sqrt[n]{a})/K$ *for* K *a field containing a primitive* n*th root of unity and* $a \in K^\times$.

The study of Kummer extensions is called Kummer theory. The following result gives an explicit description of Kummer extensions in terms of the algebra of the base field itself. Consider the set of pairs $(L/K, \sigma)$ where L/K is a cyclic extension and σ is a generator of Gal(L/K). We call $(L/K, \sigma)$ and $(L'/K, \sigma')$ *equivalent* if there exists a K-isomorphism $L \to L'$ commuting with σ and σ'. Let $\mathrm{Cyc}_n(K)$ denote the set of equivalence classes of cyclic extensions of K of degree d with $d \mid n$.

Theorem 1.4.6. *Let* K *have a primitive* n*th root of unity. Then*

(1) *For any* $a \in K^\times$, $[K(\sqrt[n]{a}) : K]$ *is equal to the order of* a *in* $K^\times/(K^\times)^n$.

(2) *Two Kummer extensions* $K(\sqrt[n]{a})$ *and* $K(\sqrt[n]{b})$ *are isomorphic if and only if* $a = b^m c^n$ *for some* $c \in K^\times$ *and* $(m, n) = 1$.

(3) *There is a bijection between the sets* $\mathrm{Cyc}_n(K)$ *and* $K^\times/(K^\times)^n$.

1.4.2 Artin–Schreier Theory

When $\mathrm{char}\, K = p > 0$, Kummer theory fails to describe cyclic extensions of degree p^k since the only p^kth root of unity is 1 itself so there are no nontrivial extensions of the form $K(\sqrt[p]{a})/K$. However, Artin–Schreier theory (in the $\mathbb{Z}/p\mathbb{Z}$ case, and Artin–Schreier–Witt theory in general) provides a characteristic p replacement for Kummer theory. To describe this, we have the following analogue of Hilbert's Theorem 90 for the trace function.

Theorem 1.4.7 (Hilbert's Theorem 90, Additive Version)**.** *Let* L/K *be a cyclic extension with Galois group* $G = \langle\sigma\rangle$. *Then for all* $\alpha \in L$, $T_{L/K}(\alpha) = 0$ *if and only if* $\alpha = \sigma(\beta) - \beta$ *for some* $\beta \in L$.

Proof. ($\Longrightarrow$) Similarly to the norm case, the Galois hypothesis implies $T(\sigma(\alpha)) = T(\alpha)$, so if $\alpha = \sigma(\beta) - \beta$ then $T(\alpha) = 0$.

($\Longleftarrow$) Since L/K is Galois, T is a nontrivial linear combination of powers of

σ and thus is nonzero. If $[L : K] = n$ and $\alpha \in L$ such that $T(\alpha) = 0$, define

$$\begin{aligned} f(x) &= \alpha x + (\alpha + \sigma(\alpha))\sigma(x) + \ldots + (\alpha + \sigma(\alpha) + \ldots + \sigma^{n-1}(\alpha))\sigma^{n-1}(x) \\ &= \alpha x + (\alpha + \sigma(\alpha))\sigma(x) + \ldots + (\alpha + \sigma(\alpha) + \ldots + \sigma^{n-2}(\alpha))\sigma^{n-2}(x) \end{aligned}$$

since $\alpha + \sigma(\alpha) + \ldots + \sigma^{n-1}(\alpha) = T(\alpha) = 0$. Applying σ to this, we get

$$\begin{aligned} \sigma(f(x)) &= \sigma(\alpha x + (\alpha + \sigma(\alpha))\sigma(x) + \ldots + (\alpha + \sigma(\alpha) + \ldots + \sigma^{n-2}(\alpha))\sigma^{n-2}(x)) \\ &= \sigma(\alpha)\sigma(x) + (\sigma(\alpha) + \sigma^2(\alpha))\sigma^2(x) + \ldots + (\sigma^{\alpha} + \ldots + \sigma^{n-1}(\alpha))\sigma^{n-1}(x) \\ &= f(x) - \alpha x - \alpha\sigma(x) - \ldots - \alpha\sigma^{n-1}(x) \\ &= f(x) - \alpha T(x). \end{aligned}$$

Since $T \neq 0$, there is some $x \in L$ with $T(x) \neq 0$. Then setting $\beta = -\frac{f(x)}{T(x)}$ gives us $\alpha = \sigma(\beta) - \beta$. $\square$

Theorem 1.4.8 (Artin–Schreier). *Let K be a field of characteristic $p > 0$. Then there is a bijection*

$$\left\{ \begin{matrix} \textit{cyclic extensions } L/K \textit{ with} \\ [L : K] = p \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} L = K(\alpha) \textit{ for } \alpha^p - \alpha - a \in K \\ a \neq b^p - b \textit{ for } b \in K \end{matrix} \right\}.$$

Proof. Suppose α is a root of $x^p - x - a$ where $a \neq c^p - c$ for any $c \in K$. Then all roots of $x^p - x - a$ are given by $\alpha, \alpha + 1, \ldots, \alpha + p - 1$. As each of these lies in $K(\alpha)$, we see that $K(\alpha)/K$ is Galois. In particular, $\mathrm{Gal}(F(\alpha)/F) = \{1, \sigma, \ldots, \sigma^{p-1}\}$ where $\sigma^i(\alpha) = \alpha + i$ for each $0 \leqslant i \leqslant p - 1$. This shows $\mathrm{Gal}(K(\alpha)/K)$ is cyclic of order p.

Conversely, suppose L/K is cyclic of order p with Galois group $G = \langle\tau\rangle$. Then $T(1) = [L : K] \cdot 1 = p \cdot 1 = 0$, so by the additive version of Hilbert's Theorem 90, $1 = \tau(\alpha) - \alpha$ for some $\alpha \in L$. Thus $\tau(\alpha) = \alpha + 1$ and $\tau^i(\alpha) = \alpha + i$ for all $0 \leqslant i \leqslant p-1$. This shows that the minimal polynomial of α over K is

$$\prod_{i=0}^{p-1}(x - (\alpha + i)) = \prod_{i=0}^{p-1}((x-\alpha) - i) = (x-\alpha)^p - (x-\alpha) = x^p - x - (\alpha^p - \alpha).$$

Since this polynomial lies in $K[x]$, $\alpha^p - \alpha \in K$; set $a = \alpha^p - \alpha$. Then α is a root of $x^p - x - a$ and $L = K(\alpha)$ by counting dimensions. □

Definition 1.4.9. *For a prime* p, *define the* **$\wp$-function** $\wp(x) = x^p - x$. *An* **Artin–Schreier extension** *is an extension* $K(\wp^{-1}(a))/K$ *for* K *a field of characteristic* p, $a \in K$ *and* $\wp^{-1}(a)$ *any solution to* $x^p - x = a$.

The study of Artin–Schreier extensions is called Artin–Schreier theory. In analogy with Kummer theory, we can describe all Artin–Schreier extensions of a characteristic p field in terms of the structure of the field itself. As before, let $\mathrm{Cyc}_p(K)$ be the set of equivalence classes of degree p Galois extensions of K.

Theorem 1.4.10. *Let* K *be a field of characteristic* $p > 0$. *Then*

(1) *Two Artin–Schreier extensions* $K(\wp^{-1}(a))$ *and* $K(\wp^{-1}(b))$ *are isomorphic if and only if* $a = mb + c^p - c$ *for* $c \in K$ *and* $m \in \mathbb{F}_p$.

(2) *There is a bijection between the sets* $\mathrm{Cyc}_p(K)$ *and* $K/\wp(K)$.

Now let K be a local field with valuation ν.

Proposition 1.4.11. *Let* L/K *be the Artin–Schreier extension of* K *given by* $L = K(\wp^{-1}(a)) = K[x]/(x^p - x - a)$ *for* $a \in K \smallsetminus \wp(K)$*. Then the higher ramification filtration (in the lower numbering) for* $G = \mathrm{Gal}(L/K)$ *has its jump at* $m := -v(a)$*.*

Proof. We may assume $K = k((t))$ where k is algebraically closed of characteristic p and $a \in K$ is of the form $a = t^{-m}g(t)$ for some $g(t) \in k[[t]]$. Let v_L be the unique extension of v to L, which in particular satisfies $v_L(t) = p$. Pick integers α and b such that $\alpha p - bm = 1$. Set $z = y^b t^\alpha$ and note that

$$-pm = v_L(t^{-m}g) = v_L(y^p - y) = \min\{v_L(y^p), v_L(y)\} = v_L(y^p) = pv_L(y).$$

Then $v_L(y) = -m$ and $v_L(z) = -bm + \alpha p = 1$, so z is a uniformizer in L and thus $\mathcal{O}_L = \mathcal{O}_K[z]$. Thus by Remark 1.3.10, the higher ramification groups of L/K can be computed as

$$G_s = \{\sigma \in G \mid v_L(\sigma(z) - z) \geqslant s + 1\}.$$

Let $\sigma \in G$ be the generator acting by $\sigma(y) = y + 1$. Then

$$\sigma(z) = \sigma(y^b t^\alpha) = (y+1)^b t^\alpha = \sum_{i=0}^{b} \binom{b}{i} y^i t^\alpha$$

so we have

$$\begin{aligned} v_L(\sigma(z) - z) &= v_L\left(\sum_{i=0}^{b-1}\binom{b}{i} y^i t^a\right) \\ &= \min\{v_L(t^a), v_L(byt^a), \ldots, v_L(by^{b-1}t^a)\} \\ &= \min\{ap, -m + ap, \ldots, -m(b-1) + ap\} \\ &= -m(b-1) + ap = m + 1. \end{aligned}$$

It follows that $G_s = G$ for all $s \leqslant m$ and $G_s = 0$ for all $s \geqslant m + 1$. Hence the ramification filtration for G (in the lower numbering) is:

$$G = G_0 = G_1 = \ldots = G_m \supsetneq G_{m+1} = 0.$$

□

Since the ramification filtration is an isomorphism invariant of the Galois extension L/K, we see that different ramification jumps yield non-isomorphic $\mathbb{Z}/p\mathbb{Z}$-extensions of K, a phenomenon which by Kummer theory does not occur in the tame case. Moreover, the ramification jump essentially classifies all Artin–Schreier extensions of a local field (cf. [Ser2, Ch. IV]). For us, it will be the key arithmetic invariant used to classify stacky curves in Chapter 4.

1.4.3 Artin–Schreier–Witt Theory

Suppose K is a field of characteristic $p > 0$ and L/K is a Galois extension with group $G = \mathbb{Z}/p^n\mathbb{Z}$. When $n = 1$, Artin–Schreier theory said that such extensions are all of the form $L = K[x]/(x^p - x - a)$ for some $a \in K$, with isomorphism classes of such extensions determined by the ramification jump $m = -v(a)$. When $n = 2$, write L/K as a tower $L \supseteq M \supseteq K$, where L/M and M/K are both Galois extensions with group $\mathbb{Z}/p\mathbb{Z}$. Then by Artin–Schreier theory,

$$M = K[x]/(x^p - x - a) \qquad \text{and} \qquad L = M[z]/(z^p - z - b)$$

for $a \in K \setminus \wp(k)$ and $b \in M \setminus \wp(M)$ – where again, $\wp$ denotes the map $c \mapsto c^p - c$. It turns out (see [OP]) that the extension L/K itself can be defined by the equations

$$y^p - y = x \qquad \text{and} \qquad z^p - z = \frac{x^p + y^p - (x+y)^p}{p} + w$$

where both x and w lie in K. Compare this to a $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$-extension, which can be written as a tower of $\mathbb{Z}/p\mathbb{Z}$-extensions in multiple ways. The fact that L/K is cyclic is reflected in the above equations defining the extension. To see this explicitly, suppose $H = \mathrm{Gal}(M/K) \cong \langle\sigma\rangle$ where $|\sigma| = p$. Then σ acts on $M = K[x]/(x^p - x - a)$ via $\sigma(x) = x + 1$. Moreover, σ generates $G = \mathrm{Gal}(L/K)$ if and only if $\sigma^p(z) = z + 1$. It's easy to see that when L/K is Galois of order

p^2 and factors as the tower above, then $\sigma^p(z) = z+1$ occurs precisely when $G \cong \mathbb{Z}/p^2\mathbb{Z}$, while $\sigma^p(z) = z$ coincides with the case $G \cong \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$.

For a general cyclic extension of order p^n, Artin–Schreier–Witt theory and the arithmetic of Witt vectors encode the above automorphism data in a systematic way. The basic theory can be found in various places, including [Lan, pp. 330-332], but here is an overview of what we will use later in the thesis. For a commutative ring A, we define the set of *Witt vectors* over A to be simply the set of sequences of elements of A:

$$\mathbb{W}(A) = \{(a_1, a_2, a_3, \dots) \mid a_n \in A\}.$$

This is obviously functorial, so we have defined a functor $\mathbb{W} : \mathsf{Ring} \to \mathsf{Set}$. We next define a ring structure on each $\mathbb{W}(A)$. For a Witt vector $a = (a_n) \in \mathbb{W}(A)$, an associated sequence $(a^{(n)})$ of *ghost components* (*composantes fantômes* in French) is defined by setting

$$a^{(n)} = a_0^{p^n} + pa_1^{p^{n-1}} + \dots + p^n a_n = \sum_{i=0}^{n} p^i a_i^{p^{n-i}}.$$

If A contains $\mathbb{Q}$, then the association of a Witt vector to its sequence of ghost components is bijective, but this is not true in general. Explicitly, the Witt vector associated to a sequence of ghost components $(a^{(n)})$ is given by

$$a_0 = a^{(0)} \qquad \text{and} \qquad a_n = \frac{1}{p^n} a^{(n)} - \sum_{i=0}^{n-1} \frac{1}{p^{n-i}} a_i^{p^{n-i}}.$$

In the case when $\mathbb{Q} \subseteq A$, we may define addition and multiplication operations on $\mathbb{W}(A)$ by taking $a + b$ (resp. ab) to be the Witt vector whose nth ghost component is $(a+b)^{(n)} = a^{(n)} + b^{(n)}$ (resp. $(ab)^{(n)} = a^{(n)}b^{(n)}$). Set $R = \mathbb{Q}[X_0, X_1, \ldots; Y_0, Y_1, \ldots]$ and consider the Witt vectors $X = (X_0, X_1, \ldots)$ and $Y = (Y_0, Y_1, \ldots)$ in $\mathbb{W}(R)$. Put

$$S_n(X_0, \ldots, X_n; Y_0, \ldots, Y_n) := (X+Y)_n \quad \text{and} \quad P_n(X_0, \ldots, X_n; Y_0, \ldots, Y_n) := (XY)_n.$$

Lemma 1.4.12. *For each $n \geqslant 1$, S_n and P_n are polynomials in $X_0, \ldots, X_n$ and $Y_0, \ldots, Y_n$ with integer coefficients.*

Proof. (Sketch) For any $a = (a_n) \in \mathbb{W}(A)$, define a power series

$$f_a(t) = \prod_{n=1}^{\infty} (1 - a_n t^n).$$

Then the standard and ghost components of a are related as follows:

$$-t \frac{d}{dt} \log f_a(t) = \sum_{n=1}^{\infty} a^{(n)} t^n.$$

Using this, one can show that $f_X(t) f_Y(t) = f_{X+Y}(t)$ and, with slightly more work, that

$$f_{XY}(t) = \prod_{d=0}^{\infty} \prod_{e=0}^{\infty} (1 - X_d^{m-d} Y_e^{m-e} t^m)^{d+e-m}$$

where $m = \gcd(d, e)$. These then imply that S_n and P_n have integer coefficients. □

Thus the definitions of addition and multiplication of Witt vectors can be extended to $\mathbb{Z}[X_0, X_1, \ldots; Y_0, Y_1, \ldots]$ and indeed any subring of a ring containing $\mathbb{Q}$. Finally, for an arbitrary commutative ring A, fix $a = (a_n), b = (b_n) \in \mathbb{W}(A)$, let $\varphi_{a,b}$ be the ring homomorphism

$$\begin{aligned} \varphi_{a,b} : \mathbb{Z}[X_0, X_1, \ldots; Y_0, Y_1, \ldots] &\longrightarrow A \\ X_n &\longmapsto a_n \\ Y_n &\longmapsto b_n, \end{aligned}$$

and define the addition and multiplication of a and b by

$$a + b = W(\varphi_{a,b})(X + Y) \qquad \text{and} \qquad ab = W(\varphi_{a,b})(XY).$$

Under these operations, $\mathbb{W}(A)$ is a ring with zero element $(0, 0, 0, \ldots)$ and unit $(1, 0, 0, \ldots)$, and $\mathbb{W} : \mathsf{Ring} \to \mathsf{Ring}$ is a functor. Furthermore, it is easy to check using the above description of Witt vector addition that $\mathbb{W}(A)$ is always a ring of characteristic 0. Therefore $\mathbb{W}(A) \not\cong A^{\mathbb{N}}$ in general.

Now assume for the rest of the section that K is a field of characteristic $p > 0$ and A is a K-algebra. We make the following change in notation: for each $n \geqslant 0$, replace a_n in place of a_{p^n} and write $a = (a_0, a_1, a_2, \ldots)$ for the sequence of pth power components of the original Witt vector. Call $\mathbb{W}(A)$ the set of Witt vectors over A with this new numbering convention. This obviously changes the explicit structure of $\mathbb{W}(A)$, but it is clear that the ghost components of $a = (a_0, a_1, a_2, \ldots)$ are still given by $a^{(n)} = a_0^{p^n} + pa_1^{p^{n-1}} + \ldots + p^n a_n$. Moreover, the Witt vector arithmetic defined above still defines a ring structure on $\mathbb{W}(A)$.

For each $n \geqslant 1$, let $\mathbb{W}_n(A)$ be the set of *Witt vectors of length* n over A, i.e. the image of the ring homomorphism

$$t_n : \mathbb{W}(A) \longrightarrow \mathbb{W}(A)$$
$$(a_0, \dots, a_{n-1}, a_n, a_{n+1}, \dots) \longmapsto (a_0, \dots, a_{n-1}, 0, 0, \dots).$$

We write an element of $\mathbb{W}_n(A)$ as $(a_0, \dots, a_{n-1})$. The *Verschiebung operator* defined by $V : \mathbb{W}(A) \to \mathbb{W}(A), (a_0, a_1, \dots) \mapsto (0, a_0, a_1, \dots)$ is an abelian group homomorphism, and moreover, $\mathbb{W}_n(A) \cong \mathbb{W}(A)/V^n\mathbb{W}(A)$ where $V^n = \underbrace{V \circ \cdots \circ V}_{n}$. The following results are standard (cf. [Ser2, Part II, Sec. 6]).

Lemma 1.4.13. *The Verschiebung satisfies:*

(a) Each $a = (a_n) \in \mathbb{W}(A)$ *can be written* $a = \sum_{n=0}^{\infty} V^n\{a_n\}$ *where* $\{x\} = (x, 0, 0, \dots)$.

(b) For $a = (a_n) \in \mathbb{W}(A)$ *and* $b = (0, \dots, 0, b_n, b_{n+1}, \dots) \in V^n\mathbb{W}(A)$,

$$a + b = (a_0, \dots, a_{n-1}, a_n + b_n, a_{n+1} + b_{n+1}, \dots).$$

(c) $u \in \mathbb{W}(A)$ *is a unit if and only if* u_0 *is a unit in* A.

Next, define the *Frobenius operator* $F : \mathbb{W}(A) \to \mathbb{W}(A)$ by $(a_0, a_1, \dots) \mapsto (a_0^p, a_1^p, \dots)$. Then F is a ring homomorphism which satisfies several important properties.

Lemma 1.4.14. *For* $a = (a_0, a_1, \ldots) \in \mathbb{W}(A)$*,*

(a) $FVa = VFa = pa$.

(b) $(Va)^{(n)} = pa^{(n-1)}$.

(c) $a^{(n)} = (Fa)^{(n-1)} + p^n a_n$.

(d) For all $i, j \geqslant 0$ *and* $b \in \mathbb{W}(A)$*,* $(V^i a)(V^j b) = V^{i+j}(F^{p^j} a F^{p^i} b)$.

Consider the ring of length n Witt vectors $\mathbb{W}_n(K)$. For $x \in \mathbb{W}_n(K)$, set $\wp x = Fx - x$. Then $\wp$ is an abelian group homomorphism $\mathbb{W}_n(K) \to \mathbb{W}_n(K)$ and there is an exact sequence

$$0 \to \mathbb{Z}/p^n\mathbb{Z} \xrightarrow{i} \mathbb{W}_n(K) \xrightarrow{\wp} \mathbb{W}_n(K) \to 0$$

where i is the inclusion $x \mapsto \{x\}$. Let L/K be a Galois extension with Galois group G. Then G acts on $\mathbb{W}_n(L)$ via $\sigma \cdot (x_0, \ldots, x_{n-1}) = (\sigma(x_0), \ldots, x_{n-1})$. One can mimic the proof of Hilbert's Theorem 90 (either 1.4.2 or 1.4.7) to prove the following version for Artin–Schreier–Witt theory.

Theorem 1.4.15 (Hilbert's Theorem 90, Witt Vector Version)**.** *For a Galois extension* L/K *of fields of characteristic* $p > 0$ *with Galois group* G*,* $H^1(G, \mathbb{W}_n(L)) = 0$ *for all* $n \geqslant 1$*.*

Example 1.4.16. When $K = \mathbb{F}_p$, the field of p elements, we have an isomorphism

$$\begin{aligned} \mathbb{W}_n(\mathbb{F}_p) &\longrightarrow \mathbb{Z}/p^n\mathbb{Z} \\ (x_0, x_1, \ldots, x_{n-1}) &\longmapsto \bar{x}_0 + p\bar{x}_1 + \ldots + p^{n-1}\bar{x}_{n-1} \end{aligned}$$

for all $n \geqslant 1$, where $\bar{x}_i$ denotes the image of x_i under the canonical surjection $\mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p^n\mathbb{Z}$. These commute with the natural maps $\mathbb{Z}/p^n\mathbb{Z} \to \mathbb{Z}/p^{n+1}\mathbb{Z}$, giving an isomorphism

$$\mathbb{W}(\mathbb{F}_p) \xrightarrow{\sim} \varprojlim \mathbb{Z}/p^n\mathbb{Z} = \mathbb{Z}_p.$$

This was one of the original motivations for the construction of $\mathbb{W}$: as a way to give a canonical lift of a ring in characteristic $p > 0$ to a ring in characteristic 0, just as the p-adic integers do for each finite ring $\mathbb{Z}/p^n\mathbb{Z}$. Furthermore, the natural profinite topology on $\mathbb{Z}_p$ induces a topological ring structure on $\mathbb{W}(\mathbb{F}_p)$ under which all of the previous maps in this section are continuous. Moreover, explicitly describing this topology allows one to write down a topology on $\mathbb{W}(A)$ for any ring A.

Let K^{sep} be a separable closure of K and suppose $x \in \mathbb{W}_n(K)$ and $\alpha \in \mathbb{W}_n(K^{sep})$ are Witt vectors such that $\wp(\alpha) = x$. If $\alpha = (\alpha_0, \ldots, \alpha_{n-1})$, we write $K(\wp^{-1}x) = K(\alpha_0, \ldots, \alpha_{n-1})$ as a field extension of K. The following theorem fully characterizes cyclic extensions of degree p^n of K.

Theorem 1.4.17. *Let K be a field of characteristic $p > 0$. Then for each $n \geqslant 1$, there* *is a bijection*

$$\left\{\begin{matrix} \textit{cyclic extensions } L/K \textit{ with} \\ [L:K]=p^n \end{matrix}\right\} \longleftrightarrow \mathbb{W}_n(K)/\wp(\mathbb{W}_n(K))$$

$$L = K(\wp^{-1}x) \longleftrightarrow x.$$

Proof. (Sketch) Suppose $\alpha \in \mathbb{W}_n(K^{sep})$ is a root of the Witt vector-valued polynomial $\wp x - a$, where $a \in \mathbb{W}_n(K)$ is not of the form $a = \wp b$ for any $b \in \mathbb{W}_n(K)$. Then all roots of $\wp x - a$ are given by $\alpha, \alpha + \mathbf{1}, \dots, \alpha + (p^n - 1)\mathbf{1}$ where $\mathbf{1}$ denotes the unit Witt vector $(1, 0, 0, \dots)$. In particular, this shows that $K(\wp^{-1}x)/K$ is Galois and $\mathrm{Gal}(K(\wp^{-1}x)/K) = \{1, \sigma, \dots, \sigma^{p^n-1}\}$ where $\sigma^i(\alpha) = \alpha + i\mathbf{1}$ for each $0 \leqslant i \leqslant p^n - 1$. Therefore $K(\wp^{-1}x)/K$ is cyclic of order p^n.

Conversely, suppose L/K is cyclic of order p^n with Galois group $G = \langle\tau\rangle$. Applying Galois cohomology to the short exact sequence $0 \to \mathbb{Z}/p^n\mathbb{Z} \to \mathbb{W}_n(L) \xrightarrow{\wp} \mathbb{W}_n(L) \to 0$ yields a long exact sequence

$$0 \to \mathbb{Z}/p^n\mathbb{Z} \to \mathbb{W}_n(K) \xrightarrow{\wp} \mathbb{W}_n(K) \xrightarrow{\varphi} H^1(G, \mathbb{Z}/p^n\mathbb{Z}) \to H^1(G, \mathbb{W}_n(L)) = 0$$

with the last zero coming from Hilbert's Theorem 90. The map φ sends y to the cocycle $\xi : \sigma \mapsto \sigma\beta - \beta$ where $\wp\beta = y$. Since G acts trivially on $\mathbb{W}_n(\mathbb{F}_p)$, we have $H^1(G, \mathbb{Z}/p^n\mathbb{Z}) = \mathrm{Hom}(G, \mathbb{Z}/p^n\mathbb{Z}) \cong \mathrm{End}(\mathbb{Z}/p^n\mathbb{Z}) \cong \mathbb{Z}/p^n\mathbb{Z}$. Let $x \in W_n(K)$ be a Witt vector such that φx generates $H^1(G, \mathbb{Z}/p^n\mathbb{Z})$. One finishes by showing that $L = K(\wp^{-1}x)$. $\square$

Suppose K is a complete local field of characteristic p. We will typically assume $K \cong k((t))$ for an algebraically closed field k. Then any cyclic $\mathbb{Z}/p^n\mathbb{Z}$-extension L/K is given by a set of equations

$$y_i^p - y_i = f_i(x_0, \ldots, x_i; y_0, \ldots, y_{i-1}) \text{ for } 0 \leqslant i \leqslant n-1$$

where $x_i \in K$ and f_i is a polynomial over k (see [OP] for a description of these polynomials). Let ν be the valuation on K and set $m_i = -\nu(x_i)$ for $0 \leqslant i \leqslant n-1$.

Lemma 1.4.18. *The last jump in the ramification filtration in the upper numbering for* $G = \mathrm{Gal}(L/K)$ *is* $u = \max\{p^{n-i-1}m_i\}_{i=0}^{n-1}$.

Proof. Follows from [Gar, Thm. 1.1]. □

It follows that the ramification filtration (either in the upper or lower numbering) of a cyclic $\mathbb{Z}/p^n\mathbb{Z}$-extension of complete local fields is completely determined by its Witt vector equation. For further reading, in the last section of [OP] the authors provide explicit equations describing $\mathbb{Z}/p^3\mathbb{Z}$-equations of $K = k((t))$.

Example 1.4.19. Let $K = k((t))$ and let L/K be the Artin–Schreier–Witt extension given by the equations $y^p - y = t^{-m}$ and $z^p - z = t^{-m}y$, where $(m, p) = 1$. We claim the Galois group $G = \mathrm{Gal}(L/K) \cong \mathbb{Z}/p^2\mathbb{Z}$ has ramification jumps (in the lower numbering) at m and $m(p^2+1)$. Indeed, the first jump is $m = -\nu_L(t^{-m})$ by Proposition 1.4.11 and the fact that the first jumps in the upper and lower numbering agree. On the other hand, the last jump in the upper numbering is $\max\{pm, 0\} = pm$ by Lemma 1.4.18 and applying the function $\psi_{L/K}(t) := \varphi_{L/K}^{-1}(t)$ to $t = pm$ yields $m(p^2 + 1)$, so $s = m(p^2 + 1)$ is the last jump in the lower numbering.

Chapter 2

Some Algebraic Geometry

In this chapter, we recall some of the basic concepts from algebraic geometry required to discuss algebraic curves and, in later chapters, stacky curves. A standard reference for everything in this chapter is [Hart, Chs. II and IV]. While much of this chapter might be familiar to the reader, the conceptual progression from affine schemes to schemes and from modules to quasi-coherent sheaves will be imitated in greater generality in Chapter 3. Therefore we include the details of these constructions for completeness.

2.1 Scheme Theory

2.1.1 Affine Schemes

Hilbert's Nullstellensatz is an important theorem in commutative algebra which is essentially the jumping off point for classical algebraic geometry (by which we mean the study of algebraic varieties in affine and projective space). We recall the statement here.

Theorem 2.1.1 (Hilbert's Nullstellensatz)**.** *If* k *is an algebraically closed field, then there is a bijection*

$$\mathbb{A}^n_k \longleftrightarrow \operatorname{MaxSpec} k[t_1, \ldots, t_n]$$
$$P = (\alpha_1, \ldots, \alpha_n) \longmapsto \mathfrak{m}_P = (t_1 - \alpha_1, \ldots, t_n - \alpha_n),$$

where $\mathbb{A}^n_k = k^n$ *is affine* n*-space over* k *and* MaxSpec *denotes the set of all maximal ideals of a ring.*

Further, if $f : A \to B$ is a morphism of finitely generated k-algebras then we get a map $f^* : \operatorname{MaxSpec} B \to \operatorname{MaxSpec} A$ given by $f^*\mathfrak{m} = f^{-1}(\mathfrak{m})$ for any maximal ideal $\mathfrak{m} \subset B$. Note that if A and B are arbitrary commutative rings, $f^{-1}(\mathfrak{m})$ need not be a maximal ideal of A.

Lemma 2.1.2. *Let* $f : A \to B$ *be a ring homomorphism and* $\mathfrak{p} \subset B$ *a prime ideal. Then* $f^{-1}(\mathfrak{p})$ *is a prime ideal of* A*.*

This suggests a natural replacement for $\operatorname{MaxSpec} A$, called the *prime spectrum*:

$$\operatorname{Spec} A = \{\mathfrak{p} \subset A \mid \mathfrak{p} \text{ is a prime ideal}\}.$$

Definition 2.1.3. *An* **affine scheme** *is a ringed space with underlying topological space* $X = \operatorname{Spec} A$ *for some ring* A*.*

In order to justify this definition, we need to specify the topology on $\operatorname{Spec} A$ and the sheaf of rings making it into a ringed space. For any subset $E \subseteq A$, define

$$V(E) = \{\mathfrak{p} \in \operatorname{Spec} A \mid E \subseteq \mathfrak{p}\}.$$

Lemma 2.1.4. *Let* A *be a ring and* $E \subseteq A$ *any subset. Set* $\mathfrak{a} = (E)$*, the ideal generated by* E*. Then*

(a) $V(E) = V(\mathfrak{a}) = V(\mathfrak{r}(\mathfrak{a}))$ *where* $\mathfrak{r}$ *denotes the radical of an ideal.*

(b) $V(\{0\}) = \operatorname{Spec} A$ *and* $V(A) = \varnothing$*.*

(c) For a collection of subsets $\{E_i\}$ *of* A*,* $V\left(\bigcup E_i\right) = \bigcap V(E_i)$*.*

(d) For any ideals $\mathfrak{a}, \mathfrak{b} \subset A$*,* $V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$*.*

As a result, the sets $V(E)$ for $E \subseteq A$ form the closed sets for a topology on $\operatorname{Spec} A$, called the *Zariski topology*.

Next, for any prime ideal $\mathfrak{p} \subset A$, let $A_{\mathfrak{p}}$ denote the localization at $\mathfrak{p}$. For any open set $U \subseteq \operatorname{Spec} A$, we define

$$\mathcal{O}(U) = \left\{ s : U \to \coprod_{\mathfrak{p} \in U} A_{\mathfrak{p}} \,\middle|\, \exists\, \mathfrak{p} \in V \subseteq U \text{ and } f, g \in A \text{ so that } s(\mathfrak{q}) = \frac{f}{g} \text{ for all } \mathfrak{q} \in V \right\}.$$

The main theorem describing affine schemes geometrically is the following.

Theorem 2.1.5. $(\operatorname{Spec} A, \mathcal{O})$ *is a ringed space. Moreover,*

(1) For any $\mathfrak{p} \in \operatorname{Spec} A$*,* $\mathcal{O}_{\mathfrak{p}} \cong A_{\mathfrak{p}}$ *as rings.*

(2) $\Gamma(\operatorname{Spec} A, \mathcal{O}) \cong A$ *as rings.*

(3) For any $f \in A$*, define the open set* $D(f) = \{\mathfrak{p} \in \operatorname{Spec} A \mid f \notin \mathfrak{p}\}$*. Then the* $D(f)$ *form a basis for the topology on* $\operatorname{Spec} A$ *and* $\mathcal{O}(D(f)) \cong A_f$ *as rings.*

Example 2.1.6. For any field k, $\operatorname{Spec} k$ is a single point $*$ corresponding to the zero ideal, with sheaf $\mathcal{O}(*) \cong k$.

Example 2.1.7. Let $A = k[t_1, \ldots, t_n]$ be the polynomial ring in n variables over k. Then $\operatorname{Spec} A = \mathbb{A}^n_k$, the affine n-space over k. For example, when $A = k[t]$ is the polynomial ring in a single variable, $\operatorname{Spec} k[t] = \mathbb{A}^1_k$, the affine line.

2.1.2 Schemes

Recall that a ringed space is a pair $(X, \mathcal{F})$ where X is a topological space and $\mathcal{F}$ is a sheaf of rings on X.

Definition 2.1.8. *A* **locally ringed space** *is a ringed space* $(X, \mathcal{F})$ *such that for all* $P \in X$*, there is a ring* A *such that* $\mathcal{F}_P \cong A_{\mathfrak{p}}$ *for some prime ideal* $\mathfrak{p} \subset A$*.*

Example 2.1.9. Any affine scheme $\operatorname{Spec} A$ is a locally ringed space by (1) of Theorem 2.1.5. We will sometimes denote the structure sheaf $\mathcal{O}$ by $\mathcal{O}_A$.

Definition 2.1.10. *The* **category of locally ringed spaces** *is the category whose objects are locally ringed spaces* $(X, \mathcal{F})$ *and whose morphisms are morphisms of ringed spaces* $(X, \mathcal{F}) \to (Y, \mathcal{G})$ *such that for each* $P \in X$*, the induced map* $f_P^\# : \mathcal{O}_{Y,f(P)} \to \mathcal{O}_{X,P}$ *is a morphism of local rings, i.e.* $(f_P^\#)^{-1}(\mathfrak{m}_P) = \mathfrak{m}_{f(P)}$ *where* $\mathfrak{m}_P$ *(resp.* $\mathfrak{m}_{f(P)}$*) is the maximal ideal of the local ring* $\mathcal{O}_{X,P}$ *(resp.* $\mathcal{O}_{Y,f(P)}$*).*

We now define a scheme.

Definition 2.1.11. *A* **scheme** *is a locally ringed space* $(X, \mathcal{O}_X)$ *that admits an open covering* $\{U_i\}$ *such that each* U_i *is affine, i.e. there are rings* A_i *such that* $(U_i, \mathcal{O}_X|_{U_i}) \cong (\operatorname{Spec} A_i, \mathcal{O}_{A_i})$ *as locally ringed spaces.*

The category of schemes Sch is defined to be the full subcategory of schemes in the category of locally ringed spaces. Denote the subcategory of affine schemes by AffSch. Also let $\mathsf{CommRings}$ denote the category of commutative rings with unity.

Proposition 2.1.12. *There is an isomorphism of categories*

$$\begin{aligned} \mathsf{AffSch} &\xrightarrow{\sim} \mathsf{CommRings}^{op} \\ (X, \mathcal{O}_X) &\longmapsto \mathcal{O}_X(X) \\ (\operatorname{Spec} A, \mathcal{O}) &\longleftarrow\!\shortmid A. \end{aligned}$$

Proof. (Sketch) First suppose we have a homomorphism of rings $f : A \to B$. By

Lemma 2.1.2 this induces a morphism $f^* : \operatorname{Spec} B \to \operatorname{Spec} A, \mathfrak{p} \mapsto f^{-1}(\mathfrak{p})$ which is continuous since $f^{-1}(V(\mathfrak{a})) = V(f(\mathfrak{a}))$ for any ideal $\mathfrak{a} \subset A$. Now for each $\mathfrak{p} \in \operatorname{Spec} B$, define the localization $f_\mathfrak{p} : A_{f^*\mathfrak{p}} \to B_\mathfrak{p}$ using the universal property of localization. Then for any open set $V \subseteq \operatorname{Spec} A$, we get a map

$$f^\# : \mathcal{O}_A(V) \longrightarrow \mathcal{O}_B((f^*)^{-1}(V)).$$

One checks that each is a homomorphism of rings and commutes with the restriction maps. Thus $f^\# : \mathcal{O}_A \to \mathcal{O}_B$ is defined. Moreover, the induced map on stalks is just each $f_\mathfrak{p}$, so the pair $(f^*, f^\#)$ gives a morphism $(\operatorname{Spec} B, \mathcal{O}_B) \to (\operatorname{Spec} A, \mathcal{O}_A)$ of locally ringed spaces, hence of schemes.

Conversely, take a morphism of schemes $(\varphi, \varphi^\#) : (\operatorname{Spec} B, \mathcal{O}_B) \to (\operatorname{Spec} A, \mathcal{O}_A)$. This induces a ring homomorphism $\Gamma(\operatorname{Spec} A, \mathcal{O}_A) \to \Gamma(\operatorname{Spec} B, \mathcal{O}_B)$ but by (2) of Theorem 2.1.5, $\Gamma(\operatorname{Spec} A, \mathcal{O}_A) \cong A$ and $\Gamma(\operatorname{Spec} B, \mathcal{O}_B) \cong B$ so we get a homomorphism $A \to B$. It is routine to check that the two functors described give the required isomorphism of categories. □

Example 2.1.13. We saw in Example 2.1.6 that for any field k, $\operatorname{Spec} k = *$ is a point with structure sheaf $\mathcal{O}(*) = k$. If $L_1, \ldots, L_r$ are finite separable field extensions of k, we call $A = L_1 \times \cdots \times L_r$ a *finite étale k-algebra*. Then $\operatorname{Spec} A = \operatorname{Spec} L_1 \coprod \cdots \coprod \operatorname{Spec} L_r$ is (topologically) a disjoint union of points.

Example 2.1.14. Let A be a DVR with residue field k. Then $\operatorname{Spec} A = \{0, \mathfrak{m}_A\}$, a closed point for the maximal ideal $\mathfrak{m}$ and a generic point for the zero ideal. There are two open subsets here, $\{0\}$ and $\operatorname{Spec} A$, and we have $\mathcal{O}_A(\{0\}) = k$ and

$\mathcal{O}_A(\operatorname{Spec} A) = A$.

Example 2.1.15. If k is a field and A is a finitely generated k-algebra, then the closed points of $X = \operatorname{Spec} A$ are in bijection with the closed points of an affine variety over k with coordinate ring A.

Example 2.1.16. Let $A = \mathbb{Z}$ (or any Dedekind domain). Then $\dim A = 1$ and it turns out that $\dim \operatorname{Spec} A = 1$ for some appropriate notion of dimension which we will define momentarily.

Many definitions in ring theory can be rephrased for schemes. For example:

Definition 2.1.17. *A scheme X is* **reduced** *if for all open $U \subseteq X$, $\mathcal{O}_X(U)$ has no nilpotent elements, while X is* **integral** *if for all open $U \subseteq X$, $\mathcal{O}_X(U)$ has no zero divisors.*

Definition 2.1.18. *The* **dimension** *of a scheme X (or any topological space) is*

$$\dim X = \sup\{n \in \mathbb{N}_0 \mid \exists \text{ a chain of irreducible, closed sets } X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_n \subseteq X\}.$$

Proposition 2.1.19. *Let A be a noetherian ring. Then $\dim \operatorname{Spec} A = \dim A$, the Krull dimension of A.*

Definition 2.1.20. *Let X be a scheme. Then*

- *X is* **locally noetherian** *if each stalk $\mathcal{O}_{X,P}$ is a local noetherian ring.*
- *X is* **noetherian** *if X is integral and locally noetherian.*

- *An integral scheme* X *is* **normal** *if each stalk* $\mathcal{O}_{X,P}$ *is integrally closed in its field of fractions.*

- X *is* **regular** *if each* $\mathcal{O}_{X,P}$ *is regular as a local ring, that is,* $\dim \mathcal{O}_{X,P} = \dim \mathfrak{m}_P/\mathfrak{m}_P^2$ *as* $\mathcal{O}_{X,P}/\mathfrak{m}_P$*-vector spaces.*

Definition 2.1.21. *Let* X *be a scheme. The* **category of schemes over** X*, denoted* Sch_X*, consists of objects* $Y \xrightarrow{p} X$*, where* Y *is a scheme and* p *is a morphism, and morphisms* $Y \to Z$ *making the following diagram commute:*

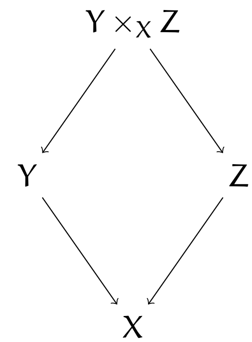
$$\begin{array}{ccc} Y & \longrightarrow & Z \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

The fibre of a topological cover $p : Y \to X$ can be interpreted as a fibre product:

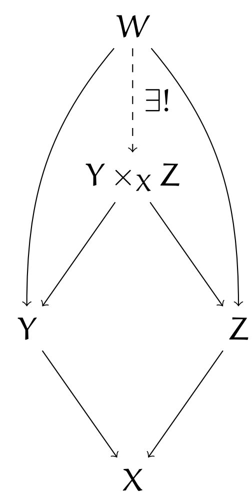
$$\begin{array}{rcl} p^{-1}(x) := \{x\} \times_X Y & \longrightarrow & Y \\ \downarrow & & \downarrow p \\ \{x\} & \hookrightarrow & X \end{array}$$

We next construct fibre products in the category Sch_X and use these to construct the algebraic analogue of a fibre.

Definition 2.1.22. *Let* X *be a scheme and* Y, Z *schemes over* X*. A* **fibre product** *of* Y *and* Z *over* X*, denoted* $Y \times_X Z$*, is a scheme over both* Y *and* Z *such that the diagram*



commutes and $Y \times_X Z$ *is universal with respect to such diagrams, i.e. for any scheme* W *over both* Y *and* Z*, the following diagram can be completed uniquely:*



Given $f : Y \to X$ and any scheme Z over X, the induced map $f_Z : Y \times_X Z \to Z$ is called the *base change* of f over Z.

Theorem 2.1.23. *For any schemes* Y, Z *over* X*, there exists a fibre product* $Y \times_X Z$ *which is unique up to unique isomorphism.*

Proof. (Sketch) When $X = \operatorname{Spec} A, Y = \operatorname{Spec} B$ and $C = \operatorname{Spec} Z$ are all affine, then the universal property of tensor products implies $Y \times_X Z := \operatorname{Spec}(B \otimes_A C)$

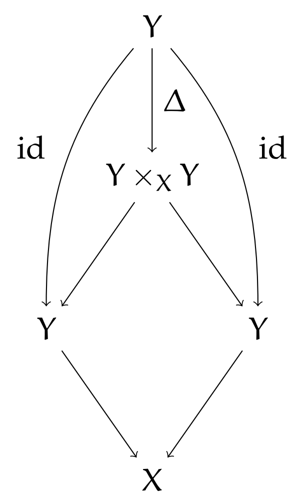
defines a fibre product. The general case is proven in [Hart, Thm. II.3.3]. □

Definition 2.1.24. *Let* $p : Y \to X$ *be a morphism of schemes,* $x \in X$ *a point and* $k(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x$ *the residue field at* x*, with natural map* $\operatorname{Spec} k(x) \hookrightarrow X$*. Then the* **fibre** *of* p *at* x *is the fibre product* $Y_x := Y \times_X \operatorname{Spec} k(x)$.

Example 2.1.25. Let A be a DVR and consider the affine scheme $X = \operatorname{Spec} A$. We saw in Example 2.1.14 that X has a closed point $\mathfrak{m} = \mathfrak{m}_A$ and a generic point (0). For any morphism $p : Y \to X$, there are two fibres:

- The *generic fibre* $Y_{(0)}$, which is an open subscheme of Y
- The *special fibre* $Y_{\mathfrak{m}}$, which is a closed subscheme of Y.

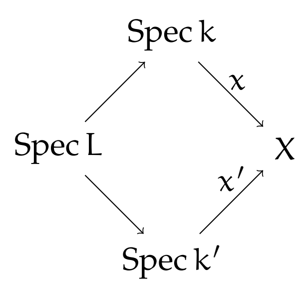
Let Y be a scheme over X and define the *diagonal map* $\Delta : Y \to Y \times_X Y$ coming from the universal property applied to the diagram



Definition 2.1.26. *A morphism* $Y \to X$ *is called* **separated** *if the diagonal* $\Delta : Y \to Y \times_X Y$ *is a closed immersion of schemes.*

Definition 2.1.27. *A morphism* $f : Y \to X$ *is of* **finite type** *if there exists an affine covering* $X = \bigcup U_i$*, with* $U_i = \operatorname{Spec} A_i$*, such that each* $f^{-1}(U_i)$ *has an open covering* $f^{-1}(U_i) = \bigcup_{j=1}^{n_i} \operatorname{Spec} B_{ij}$ *for* $n_i < \infty$ *and* B_{ij} *a finitely generated* A_i*-algebra. Further, we say* f *is a* **finite morphism** *if each* $n_i = 1$*, i.e.* $f^{-1}(U_i) = \operatorname{Spec} B_i$ *for some finitely generated* A_i*-algebra* B_i*.*

Definition 2.1.28. *The* **set of points** *of a scheme* X*, denoted* $|X|$*, is defined to be the set of equivalence classes of morphisms* $x : \operatorname{Spec} k \to X$*, where* k *is a field, and where two points* $x : \operatorname{Spec} k \to X$ *and* $x' : \operatorname{Spec} k' \to X$ *are equivalent if there exists a field* $L \supseteq k, k'$ *such that the diagram*



commutes. A **geometric point** *is a point* $\bar{x} : \operatorname{Spec} k \to \mathcal{X}$ *where* k *is algebraically closed.*

2.1.3 Sheaves of Modules

Through Proposition 2.1.12, we are able to transfer commutative ring theory to the language of affine schemes. It is also possible to translate module theory into the language of sheaves and schemes.

Definition 2.1.29. *Let* $(X, \mathcal{O}_X)$ *be a ringed space. A* **sheaf of** $\mathcal{O}_X$**-modules***, or an* $\mathcal{O}_X$*-module for short, is a sheaf of abelian groups* $\mathcal{F}$ *on* X *such that each* $\mathcal{F}(U)$ *is an* $\mathcal{O}_X(U)$*-module and for each inclusion of open sets* $V \subseteq U$*, the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{O}_X(U) \times \mathcal{F}(U) & \longrightarrow & \mathcal{F}(U) \\ \downarrow & & \downarrow \\ \mathcal{O}_X(V) \times \mathcal{F}(V) & \longrightarrow & \mathcal{F}(V) \end{array}$$

If $\mathcal{F}(U) \subseteq \mathcal{O}_X(U)$ *is an ideal for each open set* U*, then we call* $\mathcal{F}$ *a* **sheaf of ideals** *on* X*.*

Most module terminology extends to sheaves of $\mathcal{O}_X$-modules. For example,

- A *morphism of* $\mathcal{O}_X$*-modules* is a morphism of sheaves $\mathcal{F} \to \mathcal{G}$ such that each $\mathcal{F}(U) \to \mathcal{G}(U)$ is an $\mathcal{O}_X(U)$-module map. We write $\mathrm{Hom}_X(\mathcal{F}, \mathcal{G}) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ for the set of morphisms $\mathcal{F} \to \mathcal{G}$ as $\mathcal{O}_X$-modules. This defines the category of $\mathcal{O}_X$-modules, written $\mathcal{O}_X$-Mod.

- Taking kernels, cokernels and images of morphisms of $\mathcal{O}_X$-modules again give $\mathcal{O}_X$-modules.

- Taking quotients of $\mathcal{O}_X$-modules by $\mathcal{O}_X$-submodules again give $\mathcal{O}_X$-modules.

- An *exact sequence of* $\mathcal{O}_X$*-modules* is a sequence $\mathcal{F}' \to \mathcal{F} \to \mathcal{F}''$ such that each $\mathcal{F}'(U) \to \mathcal{F}(U) \to \mathcal{F}''(U)$ is an exact sequence of $\mathcal{O}_X(U)$-modules.

- Basically any functor on modules over a ring generalizes to an operation on $\mathcal{O}_X$-modules, including Hom, written $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$; direct sum $\mathcal{F} \oplus$

$\mathcal{G}$; tensor product $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$; and exterior powers $\bigwedge^n \mathcal{F}$ (sometimes after sheafification, e.g. for $\otimes$ and $\bigwedge$).

Definition 2.1.30. *An $\mathcal{O}_X$-module $\mathcal{F}$ is* **free** *(of rank r) if $\mathcal{F} \cong \mathcal{O}_X^{\oplus r}$ as $\mathcal{O}_X$-modules. $\mathcal{F}$ is* **locally free** *if X has a covering $X = \bigcup U_i$ such that each $\mathcal{F}|_{U_i}$ is free as an $\mathcal{O}_X|_{U_i}$-module. A locally free $\mathcal{O}_X$-module of rank 1 is called an* **invertible sheaf***.*

Let A be a ring, M an A-module and set $X = \operatorname{Spec} A$. To extend module theory to the language of schemes, we want to define an $\mathcal{O}_X$-module $\widetilde{M}$ on X. To start, for each $\mathfrak{p} \in \operatorname{Spec} A$, let $M_\mathfrak{p} = M \otimes_A A_\mathfrak{p}$ be the localization of the module M at $\mathfrak{p}$. Then $M_\mathfrak{p}$ is an $A_\mathfrak{p}$-module consisting of 'formal fractions' $\frac{m}{s}$ where $m \in M$ and $s \in S = A \setminus \mathfrak{p}$. For each open set $U \subseteq X$, define

$$\widetilde{M}(U) = \left\{ h : U \to \coprod_{\mathfrak{p} \in U} M_\mathfrak{p} \;\middle|\; \exists\, \mathfrak{p} \in V \subseteq U, m \in M, s \in A \text{ with } s(\mathfrak{q}) = \frac{m}{s} \text{ for all } \mathfrak{q} \in V \right\}.$$

(Compare this to the construction of the structure sheaf $\mathcal{O}_A$ on $\operatorname{Spec} A$ in Section 2.1.1. Also, note that necessarily the $s \in A$ in the definition above must lie outside of all $\mathfrak{q} \in V$.)

Proposition 2.1.31. *Let M be an A-module and $X = \operatorname{Spec} A$. Then $\widetilde{M}$ is a sheaf of $\mathcal{O}_X$-modules on X, and moreover,*

(1) For any $\mathfrak{p} \in \operatorname{Spec} A$, $\widetilde{M}_\mathfrak{p} \cong M_\mathfrak{p}$ as $A_\mathfrak{p}$-modules.

(2) $\Gamma(X, \widetilde{M}) \cong M$ as A-modules.

(3) For any $f \in A$, $\widetilde{M}(D(f)) \cong M_f = M \otimes_A A_f$ as A-modules.

Proof. The proof is similar to the proof of Theorem 2.1.5; both can be found in [Hart, Ch. II]. □

Proposition 2.1.32. *Let* $X = \operatorname{Spec} A$. *Then the association*

$$\begin{aligned} A\text{-Mod} &\longrightarrow \mathcal{O}_X\text{-Mod} \\ M &\longmapsto \widetilde{M} \end{aligned}$$

defines an exact, fully faithful functor.

Proof. Similar to the proof of Proposition 2.1.12. □

These $\widetilde{M}$ are an "affine model" for modules over a scheme X. We next define the general notion, along with an analogue of finitely generated modules over a ring.

Definition 2.1.33. *Let* $(X, \mathcal{O}_X)$ *be a scheme. An* $\mathcal{O}_X$*-module* $\mathcal{F}$ *is* **quasi-coherent** *if there is an affine covering* $X = \bigcup X_i$*, with* $X_i = \operatorname{Spec} A_i$*, and* A_i*-modules* M_i *such that* $\mathcal{F}|_{X_i} \cong \widetilde{M}_i$ *as* $\mathcal{O}_X|_{X_i}$*-modules. Further, we say* $\mathcal{F}$ *is* **coherent** *if each* M_i *is a finitely generated* A_i*-module.*

Example 2.1.34. For any scheme X, the structure sheaf $\mathcal{O}_X$ is obviously a coherent sheaf on X.

Let QCoh_X (resp. Coh_X) be the category of quasi-coherent (resp. coherent) sheaves of $\mathcal{O}_X$-modules on X.

We next identify the image of the functor $M \mapsto \widetilde{M}$ from Proposition 2.1.32.

Theorem 2.1.35. *Let* $X = \operatorname{Spec} A$*. Then there is an equivalence of categories*

$$\mathsf{A\text{-}Mod} \xrightarrow{\sim} \mathsf{QCoh}_X.$$

Moreover, if A *is noetherian, this restricts to an equivalence*

$$\mathsf{A\text{-}mod} \xrightarrow{\sim} \mathsf{Coh}_X$$

where $\mathsf{A\text{-}mod}$ *denotes the subcategory of finitely generated* A*-modules.*

Proof. (Sketch) The association $M \mapsto \widetilde{M}$ sends an A-module to a quasi-coherent sheaf on $X = \operatorname{Spec} A$ by definition of quasi-coherence. Further, one can prove that a sheaf $\mathcal{F}$ on X is a quasi-coherent $\mathcal{O}_X$-module if and only if $\mathcal{F} \cong \widetilde{M}$ for an A-module M. The inverse functor $\mathsf{QCoh}_X \to \mathsf{A\text{-}Mod}$ is given by $\mathcal{F} \mapsto \Gamma(X, \mathcal{F})$.

When A is noetherian, the above extends to say that $\mathcal{F}$ is coherent if and only if $\mathcal{F} \cong \widetilde{M}$ for a finitely generated A-module M. The rest of the proof is identical. □

Next, we construct an important example of a quasi-coherent sheaf on a scheme. As always, we begin with a construction on rings.

Definition 2.1.36. *Let* $A \to B$ *be a ring homomorphism. The* **module of relative differentials** *for* B/A *is defined to be*

$$\Omega^1_{B/A} := \mathbb{Z}\langle db \mid b \in B\rangle / N,$$

the quotient of the free B*-module generated by formal symbols* db *for all* $b \in B$ *by the* *submodule* $N = \langle da, d(b+b') - db - db', d(bb') - b(db') - (db)b' \rangle$. *This is the universal* B*-module for these three relations.*

Example 2.1.37. If $A = k$ is a field and B is a finitely generated k-algebra, write $B = k[t_1, \dots, t_n]/(f_1, \dots, f_r)$. Then B is the coordinate ring of the variety in $\mathbb{A}^n_k$ cut out by the f_i and

$$\Omega^1_{B/k} = k\langle dt_i \rangle / \left\langle \sum_{i=1}^{n} \frac{\partial f_j}{\partial t_i} dt_i \right\rangle$$

is the module of total derivatives on this variety.

Lemma 2.1.38. *Let* $A \to B$ *be a ring homomorphism. Then*

(a) For any A*-algebra* C, $\Omega^1_{B \otimes_A C/C} \cong \Omega^1_{B/A} \otimes_A C$.

(b) For any multiplicative set $S \subseteq B$, $\Omega^1_{S^{-1}B/A} \cong S^{-1}\Omega^1_{B/A} = \Omega^1_{B/A} \otimes_B S^{-1}B$.

That is, the functor $B \mapsto \Omega^1_{B/A}$ commutes with base change and localization. We now give the analogous construction for $\mathcal{O}_X$-modules, starting in the affine case.

Definition 2.1.39. *Let* $A \to B$ *be a ring homomorphism. The* **sheaf of relative differentials** *is the* $\mathcal{O}_B$*-module* $\widetilde{\Omega}_{B/A}$ *on* $\operatorname{Spec} B$ *defined by the module* $\Omega^1_{B/A}$.

Lemma 2.1.40. *Let* $A \to B$ *be a ring homomorphism. Then*

(a) $\widetilde{\Omega}_{B/A}$ *is a quasi-coherent sheaf on* $\operatorname{Spec} B$.

(b) For any element $f \in B$, $\widetilde{\Omega}_{B/A}(D(f)) \cong \Omega_{B_f/A}$ *where* B_f *is the localization of* B *at powers of* f.

Now consider the map $m : B \otimes_A B \to B, m(b_1 \otimes b_2) = b_1 b_2$. Let I be the kernel of m. Since m is surjective, this means $B \otimes_A B/I \cong B$. Since I acts trivially on I/I^2, there is an induced module action of $B \otimes_A B/I$ on I/I^2, and thus a corresponding B-module structure on I/I^2. The proof of the following fact can be found in [Bos, Sec. 8.2], among other places.

Lemma 2.1.41. $\Omega_{B/A} \cong I/I^2$.

Example 2.1.42. In Example 2.1.37, the isomorphism $I/I^2 \cong \Omega_{B/k}$ is induced by the map

$$\begin{aligned} B &\longrightarrow \Omega_{B/k} \\ t_i &\longmapsto dt_i. \end{aligned}$$

Let $Y \to X$ be a separated morphism of schemes and let $\Delta : Y \to Y \times_X Y$ be the corresponding diagonal. This induces a morphism of sheaves $\Delta^\# : \mathcal{O}_{Y \times_X Y} \to \Delta^* \mathcal{O}_X$ which has kernel sheaf $\mathcal{I}$ (a sheaf on $Y \times_X Y$). This $\mathcal{I}$ in fact defines the closed subscheme $\Delta(Y) \subseteq Y \times_X Y$.

Lemma 2.1.43. *For $Y \to X$, Δ and $\mathcal{I}$ as above,*

(a) $\mathcal{O}_{\Delta(Y)} \cong \mathcal{O}_{Y \times_X Y}/\mathcal{I}$ *as sheaves on* $\Delta(Y)$.

(b) $\mathcal{I}/\mathcal{I}^2$ *is an* $\mathcal{O}_{\Delta(Y)}$*-module.*

Identifying Y with its image $\Delta(Y)$ in the fibre product $Y \times_X Y$ allows us to define a sheaf analogue of the module of differentials by pulling back $\mathcal{I}/\mathcal{I}^2$.

Definition 2.1.44. *For a separated morphism* $Y \to X$*, the* **sheaf of relative differentials** $\Omega_{Y/X}$ *is the pullback:*

$$\begin{array}{ccc} \Omega_{Y/X} & \longrightarrow & \mathcal{I}/\mathcal{I}^2 \\ \big\downarrow & & \big\downarrow \\ Y & \xrightarrow{\sim} & \Delta(Y) \end{array}$$

Remark 2.1.45. $\Omega_{Y/X}$ is a sheaf of $\mathcal{O}_Y$-modules on Y. Moreover, on an affine patch $\operatorname{Spec} B \subseteq Y$, the sheaf of relative differentials restricts to $\widetilde{\Omega}_{B/A} \cong \widetilde{I/I^2}$ for some rings $A \to B$. In particular, $\Omega_{Y/X}$ is quasi-coherent.

2.1.4 Group Schemes

Abstractly, a group is a set G together with three maps,

$$\begin{aligned} &\mu : G \times G \longrightarrow G \quad \text{(multiplication)} \\ &e : \{e\} \hookrightarrow G \quad \text{(identity)} \\ &i : G \longrightarrow G \quad \text{(inverse)} \end{aligned}$$

satisfying associativity, identity and and inversion axioms. This generalizes to the notion of a *group object* in an arbitrary category $\mathcal{C}$. We state the definition for categories of schemes here.

The following describes an equivalent, and equally important, perspective on group schemes using the language of functors.

Proposition 2.1.46. *Let* G *be a scheme over* X*. Then a choice of group scheme structure on* G *is equivalent to a compatible choice of group structure on the sets* $\mathrm{Hom}_X(Y, G)$ *for all schemes* Y *over* X*. That is, a group scheme structure is a functor* $\mathsf{Sch}_X \to \mathsf{Groups}$ *such that the composition with the forgetful functor* $\mathsf{Groups} \to \mathsf{Sets}$ *is representable.*

Proof. This is a basic application of Yoneda's Lemma. □

Example 2.1.47. Let $X = \operatorname{Spec} A$ be affine and consider the affine line $\mathbb{A}^1_X = \operatorname{Spec} A[t]$. Then $\mathbb{A}^1_X$ is an affine group scheme over X, denoted $\mathbb{G}_a$, called the *additive group scheme* over X, with morphisms induced by the following ring homomorphisms:

$$\begin{aligned} \mu^* : A[t] &\longrightarrow A[t] \otimes A[t] \\ t &\longmapsto t \otimes 1 + 1 \otimes t \\ e^* : A[t] &\longrightarrow A \\ t &\longmapsto 0 \\ i^* : A[t] &\longrightarrow A[t] \\ t &\longmapsto -t. \end{aligned}$$

This construction generalizes to the affine line over a non-affine scheme as well.

Example 2.1.48. For $X = \operatorname{Spec} A$, the *multiplicative group scheme* over X is the group scheme $\mathbb{G}_m := \operatorname{Spec} A[t, t^{-1}]$ with morphisms induced by

$$\begin{aligned}
\mu^* : A[t, t^{-1}] &\longrightarrow A[t, t^{-1}] \otimes A[t, t^{-1}] \\
t &\longmapsto t \otimes t \\
t^{-1} &\longmapsto t^{-1} \otimes t^{-1} \\
e^* : A[t, t^{-1}] &\longrightarrow A \\
t, t^{-1} &\longmapsto 1 \\
i^* : A[t, t^{-1}] &\longrightarrow A[t, t^{-1}] \\
t &\longmapsto t^{-1} \\
t^{-1} &\longmapsto t.
\end{aligned}$$

Example 2.1.49. For $X = \operatorname{Spec} A$, the n*th roots of unity* form a group scheme defined by $\mu_n = \operatorname{Spec}(A[t, t^{-1}]/(t^n - 1))$. This is a finite group subscheme of $\mathbb{G}_m$.

Example 2.1.50. If $\operatorname{char} A = p > 0$, then $\alpha_p = \operatorname{Spec}(A[t]/(t^p))$ defines a group scheme over $\operatorname{Spec} A$ which is isomorphic as a topological space to $\operatorname{Spec} A$, but *not as a scheme!*

Definition 2.1.51. *Let* $G \xrightarrow{p} X$ *be a finite, flat group scheme. A* **left G-torsor** *is a scheme* $Y \xrightarrow{q} X$ *with* q *finite, locally free and surjective, together with a group action* $\rho : G \times_X Y \to Y$*, which satisfies:*

(1) $\rho \circ (e \times \mathrm{id}_Y)$ *is equal to the projection map* $X \times_X Y \to Y$.

(2) $\rho \circ (\mathrm{id}_G \times \rho) = \rho \circ (\mu \times \mathrm{id}_Y) : G \times_X G \times_X Y \to G \times_X Y \to Y$.

(3) $\rho \times \mathrm{id}_Y : G \times_X Y \to Y \times_X Y$ *is an isomorphism of* X*-schemes.*

Right G*-torsors are defined similarly.*

Example 2.1.52. Let k be a field and m an integer such that $\operatorname{char} k \nmid m$. Let $\mu_m = \operatorname{Spec}(k[t, t^{-1}]/(t^m - 1))$ be the group scheme of mth roots of unity over $\operatorname{Spec} k$. Take $a \in k^\times$ which is not a pth power in k for any prime p dividing m and set $L = k(\sqrt[m]{a})$, which is a finite field extension of k. We claim $Y = \operatorname{Spec} L$ is a μ_m-torsor over $\operatorname{Spec} k$. Define

$$\rho^* : k(\sqrt[m]{a}) \longrightarrow k[t, t^{-1}]/(t^m - 1) \otimes_k k(\sqrt[m]{a})$$
$$\sqrt[m]{a} \longmapsto t \otimes \sqrt[m]{a}.$$

This defines a morphism $\rho : \mu_m \times_{\operatorname{Spec} k} Y \to Y$ and one can prove that it satisfies axioms (1) and (2) of a torsor by checking the corresponding properties for ρ^*. When $\operatorname{char} k \nmid m$, μ_m is a reduced scheme over $\operatorname{Spec} k$ and it's easy to see that $\mu_m \cong (\mathbb{Z}/m\mathbb{Z})_{\operatorname{Spec} k}$, the constant group scheme on $\mathbb{Z}/m\mathbb{Z}$ over $\operatorname{Spec} k$. Moreover, by Kummer theory (Section 1.4.1), L/k is a Galois extension with $\operatorname{Gal}(L/k) \cong \mathbb{Z}/m\mathbb{Z}$, $L \otimes_k L \cong \prod_{i=1}^m L$ and this has a corresponding Galois action which induces the isomorphism $L \otimes_k L \xrightarrow{\sim} k[t, t^{-1}]/(t^m - 1) \otimes_k L$. Applying Spec again, we get the isomorphism $\mu_m \times_{\operatorname{Spec} k} Y \xrightarrow{\sim} Y \times_{\operatorname{Spec} k} Y$ so Y is indeed a μ_m-torsor. Using Kummer theory, one can show that every μ_m-torsor arises in this way, i.e. as $\operatorname{Spec} L$ for $L = k(\sqrt[m]{a})$.

Example 2.1.53. Likewise, over a field k of characteristic $p > 0$, Artin–Schreier theory (Section 1.4.2) shows that for the constant group scheme $G = \mathbb{Z}/p\mathbb{Z}$, the

G-torsors are of the form $Y = \operatorname{Spec} k(\wp^{-1}(a))$ for $a \in k \setminus \wp(k)$. More generally, Artin–Schreier–Witt theory (Section 1.4.3) says something similar for $\mathbb{Z}/p^n\mathbb{Z}$-torsors.

2.1.5 Étale Morphisms

For technical reasons, we will assume in this section that all schemes have perfect residue fields $k(x)$. The concepts of smoothness and (local) diffeomorphism from differential geometry can be phrased algebraically using the language of schemes. For example:

Definition 2.1.54. *A morphism $f : Y \to X$ between locally noetherian schemes is called* **smooth** *at a point $y \in Y$ if:*

(i) f *is of finite type at* y.

(ii) f *is flat at* y.

(iii) If $x = f(y)$, then the fibre $Y_x := Y \times_X k(x)$ is regular at y, that is, if $\mathcal{O}_{Y_x,y}$ is a regular local ring.

Otherwise f *is* **singular** *at* y. *We say* f *is a* **smooth morphism** *if it is smooth at every point* $y \in Y$, *and call* Y *a* **smooth X-scheme**. *Finally,* f *is* **smooth of relative dimension n** *if* f *is smooth and for each* x *in the image of* f, $\dim Y_x = n$.

Example 2.1.55. For a scheme X, let $\mathbb{A}^n_X = \operatorname{Spec}(\mathcal{O}_X(X)[t_1, \ldots, t_n])$. Then the ring map $\mathcal{O}_X(X) \to \mathcal{O}_X(X)[t_1, \ldots, t_n]$ induces a natural projection morphism $\pi : \mathbb{A}^n_X \to X$ which is smooth of relative dimension n.

Theorem 2.1.56. *Let* $f : Y \to X$ *be a smooth morphism of relative dimension* n. *Then*

(a) $\Omega^1_{Y/X}$ *is locally free of rank* n.

(b) For any morphism $g : Z \to Y$, *there is a short exact sequence of sheaves*

$$0 \to g^*\Omega^1_{Y/X} \to \Omega^1_{Z/X} \to \Omega^1_{Z/Y} \to 0.$$

Definition 2.1.57. *A smooth morphism of relative dimension* 0 *is called an* **étale morphism**.

There are several alternative definitions of étale morphisms, two of which are given below.

Definition 2.1.58. *A morphism* $f : Y \to X$ *of schemes which is locally of finite type is said to be* **unramified at a point** $y \in Y$ *if* $\mathfrak{m}_x \mathcal{O}_{Y,y} = \mathfrak{m}_y$, *where* $x = f(y)$, *and the extension of residue fields* $k(y)/k(x)$ *is finite and separable. Otherwise* f *is* **ramified** *at* y. *We say* f *is an* **unramified morphism** *if it is unramified at all* $y \in Y$.

Theorem 2.1.59. *Let* $f : Y \to X$ *be a morphism of schemes. Then the following are equivalent:*

(a) f *is étale.*

(b) f *is smooth and unramified.*

(c) f *is flat and unramified.*

In general, if X *and* Y *are locally noetherian and* f *is generically separable and of finite type, there is a nonempty open set* $U \subseteq Y$ *over which* f *is étale.*

Example 2.1.60. Let L/k be a finite extension of fields. The morphism $f : \operatorname{Spec} L \to \operatorname{Spec} k$ is always flat, so f is étale if and only if it is unramified, which is further equivalent to L/k being separable. For any k-algebra A, the structure morphism $f : \operatorname{Spec} A \to \operatorname{Spec} k$ is étale if and only if A is a finite étale k-algebra (cf. Example 2.1.13). If A is of the form $A = k[t]/(p(t))$, where $p(t)$ is a monic polynomial over k, then the étale condition is equivalent to $p(t)$ being separable for the same reason.

Example 2.1.61. If X and Y are nonsingular varieties over an algebraically closed field k, then $f : Y \to X$ is étale if and only if the differential $df_y : T_yY \to T_{f(y)}X$ is an isomorphism for each $y \in Y$. In particular, the fibres of f are finite and their cardinality is locally constant.

Let X and Y be schemes and suppose $\varphi : Y \to X$ is a finite étale morphism of schemes. If φ is surjective, we will call it a *finite étale cover* of schemes.

Example 2.1.62. Over any algebraically closed field k of characteristic 0, the cover $\mathbb{A}^1_k \setminus \{0\} \to \mathbb{A}^1_k \setminus \{0\}, y \mapsto y^n$ is a finite étale cover.

Proposition 2.1.63. *Let $\varphi : Y \to X$ be a finite étale cover and let $\operatorname{Aut}(Y/X)$ be the group of isomorphisms of X-schemes $Y \to Y$ commuting with φ. Then $\operatorname{Aut}(Y/X)$ is finite.*

Now let $\varphi : Y \to X$ be a finite étale cover and let G be a group scheme over X such that $Y \to X$ is a left G-torsor (see Section 2.1.4). Let Y/G be the quotient space with projection map $\pi : Y \to Y/G$. We define a sheaf on Y/G

by $\mathcal{O}_{Y/G} := (\pi_*\mathcal{O}_Y)^G$, the subsheaf of G-invariants of the pushforward of $\mathcal{O}_Y$ to Y/G along π. This makes Y/G into a ringed space.

Proposition 2.1.64. *The ringed space* $(Y/G, \mathcal{O}_{Y/G})$ *is a scheme over* X. *Moreover,* $\varphi : Y \to X$ *induces an isomorphism* $Y/G \xrightarrow{\sim} X$.

The following are analogues of the basic Galois theory of covering spaces from algebraic topology.

Proposition 2.1.65. *If* $\varphi : Y \to X$ *is a connected, finite étale cover and* $G \leqslant \mathrm{Aut}(Y/X)$ *is any finite subgroup of automorphisms, then* $\pi : Y \to Y/G$ *is a finite étale cover.*

Definition 2.1.66. *A connected, finite étale cover* $\varphi : Y \to X$ *is a* **Galois cover** *if* $\mathrm{Aut}(Y/X)$ *acts transitively on every geometric fibre of* φ.

Theorem 2.1.67. *Let* $\varphi : Y \to X$ *be a Galois cover and suppose* $\psi : Z \to X$ *is a connected, finite étale cover such that* Z *is a scheme over* Y *and the diagram*

$$\begin{array}{ccc} Y & \longrightarrow & Z \\ & {\scriptstyle\varphi}\searrow \quad \swarrow{\scriptstyle\psi} & \\ & X & \end{array}$$

commutes. Then

(1) $Y \to Z$ *is a Galois cover and* $Z \cong Y/G$ *for some subgroup* $G \leqslant \mathrm{Aut}(Y/X)$.

(2) There is a bijection

$$\{\textit{subgroups } G \leqslant \mathrm{Aut}(Y/X)\} \longleftrightarrow \{\textit{intermediate covers } Y \to Z \to X\}.$$

(3) *The correspondence is bijective on normal subgroups of* $\mathrm{Aut}(Y/X)$ *and Galois covers* $Z \to X$*, and in this case* $\mathrm{Aut}(Z/X) \cong \mathrm{Aut}(Y/X)/G$ *as groups.*

2.2 Curves

An algebraic curve X is an irreducible, projective algebraic variety X of dimension $\dim X = 1$. One of the most important features of an algebraic curve is that its local rings at nonsingular points are discrete valuation rings.

Theorem 2.2.1. *Let* X *be an algebraic curve and* $P \in X$ *a nonsingular point. Then* $\mathcal{O}_{X,P}$ *is a DVR.*

Proof. Fix $P \in X$ and let $\mathcal{O}_P = \mathcal{O}_{X,P}$ be the local ring at P, with maximal ideal $\mathfrak{m}_P$ and residue field $\kappa(P) = \mathcal{O}_P/\mathfrak{m}_P$. Then $\mathcal{O}_P$ is a regular local ring (cite). Thus $\dim_{\kappa(P)}(\mathfrak{m}_P/\mathfrak{m}_P^2) = \dim X = 1$. Let $t \in \mathfrak{m}_P$ such that $d_P t \neq 0$. Then t generates $\mathfrak{m}_P$ so for $f \in \bar{k}(X)$ with $f(P) = 0$, we have $f = t^r u$ in $\mathcal{O}_P$, for some $u \in \mathcal{O}_P^\times$. Define a map

$$\begin{aligned} \mathrm{ord}_P : \mathcal{O}_P &\longrightarrow \mathbb{Z} \\ f &\longmapsto \mathrm{ord}_P(f) = \max\{d \in \mathbb{Z} \mid f \in \mathfrak{m}_P^d\}. \end{aligned}$$

Explicitly, if $f = t^r u$ where u is a unit, then $\mathrm{ord}_P(f) = r$. Formally, we also set $\mathrm{ord}_P(f) = 0$ if $f(P) \neq 0$, to get a map on all of $\bar{k}(X)$. One then shows that ord_P is a discrete valuation with $\mathcal{O}_P$ as its valuation ring. □

Corollary 2.2.2. *For any nonsingular point* $P \in X$, $\mathcal{O}_{X,P}$ *is a PID and therefore a UFD.*

Proof. By the above, every ideal of $\mathcal{O}_{X,P}$ is of the form (t^r) where $t \in \mathfrak{m}_P$ is a generator. $\square$

An element $t \in \mathfrak{m}_P$ such that $\bar{t}$ generates $\mathfrak{m}_P/\mathfrak{m}_P^2$ is called a *uniformizer* at P.

Definition 2.2.3. *Fix a rational function* $f \in k(X)$ *and an integer* $r > 0$. *We say* f *has a* **pole** *of order* r *at* P *if* $\text{ord}_P(f) = -r$, *and a* **zero** *of order* r *at* P *if* $\text{ord}_P(f) = r$.

Note that a rational function $f \in k(X)$ is regular at P if and only if $\text{ord}_P(f) \geqslant 0$.

Proposition 2.2.4. *Every nonconstant, rational function* $f \in \bar{k}(X)$ *has at least one pole.*

Proof. A rational function $f \in \bar{k}(X)$ with no poles is regular everywhere on X, and therefore constant since X is projective. $\square$

Now let X and Y be curves. A nonconstant rational map $\varphi : Y \dashrightarrow X$ between curves induces a field extension $k(X) \hookrightarrow k(Y)$ and since both function fields have transcendence degree 1, this is in fact a finite extension.

Definition 2.2.5. *For curves* X *and* Y *and a rational map* $\varphi : Y \dashrightarrow X$, *define the* **degree** *of* φ *by* $\deg \varphi = [k(Y) : k(X)]$; *the* **separable degree** *of* φ *by* $\deg_s \varphi = [k(Y) : k(X)]_s$; *and the* **inseparable degree** *of* φ *by* $\deg_i \varphi = [k(Y) : k(X)]_i$. *We say* φ *is* **separable** *if* $k(Y) \supseteq k(X)$ *is a separable extension.*

For some terminology, any finitely generated field extension of k with transcendence degree 1 over k is called a *function field* of degree 1 over k.

Proposition 2.2.6. *There is an equivalence of categories*

$$\left\{\begin{array}{c}\textit{nonsingular curves over } k \\ \textit{with nonconstant, rational maps}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\textit{function fields of deg. 1 over } k \\ \textit{with } k\textit{-homomorphisms}\end{array}\right\}.$$

Proof. (Sketch) The assignment $X \mapsto k(X)$ determines one direction. Conversely, for a function field K/k, we associate an abstract algebraic curve X_K to K by putting a Zariski topology on the maximal ideals of the valuation rings $\mathcal{O} \subset K$. The structure sheaf is given by $\mathcal{O}_{X_K}(U) = \bigcap_{P \in U} \mathcal{O}_P$ where $U \subseteq X_K$ is open and $\mathcal{O}_P$ is the valuation ring corresponding to P. This determines the reverse assignment $K \mapsto X_K$. One now checks that these assignments are inverse and induce an equivalence of categories. □

2.2.1 Divisors

Definition 2.2.7. *Let X be a variety. An* **irreducible divisor** *on X is a closed, irreducible k-subvariety x of X of codimension 1.*

When X is a curve over a perfect field k, an irreducible divisor is a closed point of MaxSpec $k[X \cap U_i]$ for some affine patch U_i, or alternatively, a G_k-orbit of points in $X(\bar{k})$.

Definition 2.2.8. *The* **degree** *of an irreducible divisor x on X is the size of the G_k-orbit in $X(\bar{k})$ corresponding to x, i.e.* $\deg(x) = [\kappa(P) : k]$ *for any $P \in x$.*

Definition 2.2.9. *Let* X *be a curve over* k*. The* **divisor group** *on* X, $\mathrm{Div}(X)$*, is the free abelian group on the set of irreducible divisors on* X*:*

$$\mathrm{Div}(X) = \left\{ D = \sum_{x \in X} n_x x : n_x \in \mathbb{Z}, n_x \neq 0 \text{ for finitely many } x \right\}.$$

The elements of $\mathrm{Div}(X)$ *are called* **divisors** *on* X*. For a divisor* $D = \sum_{x \in X} n_x x \in \mathrm{Div}(X)$*, the* **degree** *of* D *is* $\deg(D) = \sum_{x \in X} n_x \deg(x)$*.*

If k is algebraically closed, the irreducible divisors are exactly the points of X, so each $D \in \mathrm{Div}(X)$ is a weighted sum of points of X: $D = \sum_{x \in X} n_x x$. The degree of such a divisor is just the sum of the weights: $\deg(D) = \sum_{x \in X} n_x$.

Now assume X is a nonsingular curve. For $f \in k(X)^*$, we can define a divisor $D(f) = \sum_{x \in X} \mathrm{ord}_x(f) x$, called the *principal divisor* of f. This defines a map

$$D : k(X)^* \longrightarrow \mathrm{Div}(X)$$

whose image is denoted $\mathrm{PDiv}(X)$, the group of principal divisors on X.

Definition 2.2.10. *The* **Picard group***, or* **divisor class group***, of* X *is the quotient group*

$$\mathrm{Pic}(X) = \mathrm{Div}(X) / \mathrm{PDiv}(X).$$

This defines an equivalence relation on divisors: $D_1 \sim D_2$ *if* $D_1 = D_2 + D(f)$ *for some* $f \in k(X)^*$*.*

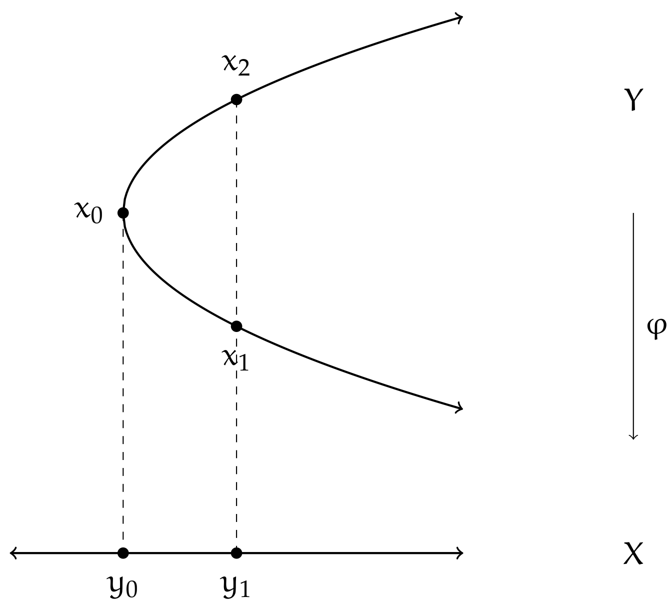
Now fix nonsingular curves X and Y over k and a finite morphism $\varphi : Y \to X$ defined over k. Then an irreducible divisor $x \in \mathrm{Div}(X)$ corresponds to a maximal ideal $\mathfrak{m}_x$ (on some affine patch) with uniformizer $t_x \in k(X)$.

Definition 2.2.11. *The* **pullback** *of* φ *is a map* $\varphi^* : \mathrm{Div}(X) \to \mathrm{Div}(Y)$ *defined on irreducible divisors by*

$$\varphi^* x = \sum_{y \in Y} \mathrm{ord}_y(\varphi^* t_x) y,$$

where t_x *is a uniformizer at* x*, and extended linearly.*

Example 2.2.12. Let Y be the plane curve defined by $y^2 - x$ and $X = \mathbb{P}^1$ the projective line, and let $\varphi : Y \to X$ be the x-coordinate projection.



Then $\varphi^* y_0 = 2x_0$ and $\varphi^* y_1 = x_1 + x_2$.

Definition 2.2.13. *Let* $\varphi : Y \to X$ *be a morphism,* $y \in Y$ *and* $x = \varphi(y) \in X$*. The number* $e_\varphi(y) = \mathrm{ord}_y(\varphi^* t_x)$ *is called the* **ramification index** *of* φ *at* y*.*

Note that if $e_\varphi(y) = 1$ and the residue field extension $\kappa(y)/\kappa(x)$ is separable,

φ is unramified at y. Otherwise, we say φ is *ramified* at y, and x is called a *branch point* of φ.

Proposition 2.2.14. *Fix a morphism* $\varphi : Y \to X$*,* $y \in Y$ *and* $x = \varphi(y) \in X$*. Then*

(1) $e_\varphi(y)$ *does not depend on the choice of uniformizer* t_x*.*

(2) For any $P \in X$*,* $\sum_{Q \in \varphi^{-1}(P)} e_\varphi(Q) = \deg \varphi$*.*

(3) All but finitely many $P \in X$ *have* $\#\varphi^{-1}(P) = \deg_s \varphi$*.*

(4) If $\psi : Z \to Y$ *is a morphism then* $e_{\varphi\psi}(x) = e_\psi(x) e_\varphi(y)$*.*

Definition 2.2.15. *Given a finite morphism* $\varphi : Y \to X$*, the* **pushforward** *of* φ *is a map* $\varphi_* : \mathrm{Div}(Y) \to \mathrm{Div}(X)$ *defined on irreducible divisors* $y \in Y$ *by*

$$\varphi_* y = [\kappa(y) : \kappa(x)] x$$

where $x = \varphi(y) \in X$*, and extended linearly.*

Proposition 2.2.16. *Let* $\varphi : Y \to X$ *be a finite morphism of curves and* $D \in \mathrm{Div}(X)$ *and* $D' \in \mathrm{Div}(Y)$ *divisors. Then*

(1) $\deg(\varphi^* D) = (\deg \varphi)(\deg D)$*.*

(2) $\varphi^*(f) = (\varphi^* f)$ *for any function* $f \in k(X)$*.*

(3) $\deg(\varphi_* D') = \deg(D')$*.*

(4) $\varphi_* \varphi^* D = (\deg \varphi) D$*.*

Corollary 2.2.17. *For any function* $f \in k(X)$ *on a curve* X, $\deg(f) = 0$.

Proof. First, if f is constant then $(f) = 0$ since X is complete. Otherwise, we may view f as a function $f : X \to \mathbb{P}^1_k$. Then $(f) = f^{-1}(0) - f^{-1}(\infty)$ for the points $0, \infty \in \mathbb{P}^1_k$, since

$$f^{-1}(P) = \sum_{Q \in X} \nu_Q(f^* t_P) Q.$$

But on the other hand, this means $\deg(f) = \deg(f^{-1}(0) - f^{-1}(\infty)) = [k(X) : k(f)] - [k(X) : k(f)] = 0$. □

Let $\mathrm{Div}^0(X)$ be the subgroup of $\mathrm{Div}(X)$ consisting of divisors of degree zero. Then Corollary 2.2.17 shows that $\mathrm{PDiv}(X) \subseteq \mathrm{Div}^0(X)$. Set

$$\mathrm{Pic}^0(X) := \mathrm{Div}^0(X) / \mathrm{PDiv}(X).$$

Then the degree map determines an exact sequence

$$0 \to k^\times \to k(X)^\times \to \mathrm{Div}^0(X) \to \mathrm{Pic}^0(X) \to 0.$$

Definition 2.2.18. *The classes* $[D] = \{D + (f) : f \in k(X)^\times\}$ *in the Picard group of* X *determines a* **linear equivalence***:* $D \sim D'$ *if there exists an* $f \in k(X)^\times$ *such that* $D + (f) = D'$.

Lemma 2.2.19. *For two divisors* $D, D' \in \mathrm{Div}(X)$*, if* $D \sim D'$ *then* $\deg(D) = \deg(D')$*. Therefore the degree map descends to a map on the Picard group,*

$$\deg : \mathrm{Pic}(X) \longrightarrow \mathbb{Z}.$$

Definition 2.2.20. *A divisor* $D = \sum n_x x$ *on* X *is called* **effective** *if* $n_x \geqslant 0$ *for all* $x \in X$*. In this case we will write* $D \geqslant 0$*. Also, if* $D_1, D_2 \in \mathrm{Div}(X)$ *and* $D_1 - D_2$ *is an effective divisor, we write* $D_1 \geqslant D_2$*. This defines a partial ordering on* $\mathrm{Div}(X)$*.*

Definition 2.2.21. *Let* D *be an effective divisor on* X*. Then the* **Riemann–Roch space** *associated to* D *is the* k*-vector space*

$$L(D) = \{f \in k(X)^\times \mid D + (f) \geqslant 0\} \cup \{0\}.$$

We denote its dimension by $\ell(D) = \dim_k L(D)$*.*

The condition that $D + (f) \geqslant 0$ can be restated as $(f) \geqslant -D$, or if $D = \sum n_x x$ then $\mathrm{ord}_x f \geqslant -n_x$ for all $x \in X$.

Example 2.2.22. Let $x \in X$ and $n > 0$. For the divisor $D = nx$, the space $L(D)$ consists of all $f \in k(X)^\times$ with no poles except possibly at x of order at most n.

Note that D is linearly equivalent to an effective divisor if and only if $\mathcal{L}(D) \neq 0$.

Theorem 2.2.23. *For any* $D \in \mathrm{Div}(X)$*,* $L(D)$ *is finite dimensional.*

Lemma 2.2.24. *If* $D_1, D_2 \in \mathrm{Div}(X)$ *are linearly equivalent, say* $D_1 - D_2 = (g)$ *for some* $g \in k(X)^\times$*, then there is an isomorphism of* k*-vector spaces*

$$L(D_1) \longrightarrow L(D_2)$$
$$f \longmapsto gf.$$

In particular, $\ell(D)$ *is a well-defined invariant of each class* $[D] \in \mathrm{Pic}(X)$.

Proposition 2.2.25. *Let* $D, D_1, D_2 \in \mathrm{Div}(X)$. *Then*

(1) $\ell(D) \leqslant \deg(D) + 1$ *if* $D \geqslant 0$.

(2) If $D_1 \leqslant D_2$ *then* $L(D_1) \subseteq L(D_2)$.

Example 2.2.26. For $X = \mathbb{P}^1$, any divisor D is linearly equivalent to $d\infty$ for some $d \in \mathbb{Z}$. Then $L(D) \cong L(d\infty) = \{f \in k[t] : \deg f \leqslant d\}$ which has dimension exactly $d + 1$. Thus the equality $\ell(D) = \deg(D) + 1$ holds for any divisor on $\mathbb{P}^1$.

Example 2.2.27. If $X \neq \mathbb{P}^1$ and D is an effective divisor, then $\ell(D) \leqslant \deg(D)$. In particular, if $\deg(D) \leqslant 0$ then $\ell(D) = 0$.

2.2.2 The Riemann–Roch Theorem

In this section, we further study divisors on an algebraic curve X by sheafifying the Riemann–Roch space $L(D)$ associated to a divisor $D \in \mathrm{Div}(X)$. For each open set $U \subseteq X$, set

$$\mathcal{L}(D)(U) = \{f \in k(X)^\times : \nu_P(f) \geqslant -n_P \text{ for all } P \in U\}.$$

Then $U \mapsto \mathcal{L}(D)(U)$ defines a sheaf of k-vector spaces on X, denoted $\mathcal{L}(D)$. In fact, $\mathcal{L}(D)$ is a subsheaf of the constant sheaf on X associated to $k(X)$ (which by abuse of notation we will also write as $k(X)$). Note that by construction, $H^0(X, \mathcal{L}(D)) = L(D)$, the Riemann–Roch space for D.

Proposition 2.2.28. *For any divisor* D *on* X, $H^i(X, \mathcal{L}(D)) = 0$ *for* $i \geqslant 2$ *and is a finite dimensional* k*-vector space for* $i = 0, 1$.

Definition 2.2.29. *The* **arithmetic genus** *of* X *is the dimension*

$$g = \dim_k H^1(X, \mathcal{L}(0)) = \dim_k H^1(X, \mathcal{O}_X).$$

Let $\Omega_X = \Omega^1_{X/k}$ be the sheaf of differentials of X. For a point $P \in X$, choose a uniformizer $t = t_P$ in $\mathcal{O}_{X,P}$. Then Ω_X is generated by dt. Hence for any $\omega \in \Omega_X$, there exists $g \in k(X)$ such that $\omega = g\,dt$.

Definition 2.2.30. *Let* $\omega \in \Omega_X$ *be a differential* 1*-form on* X. *Define the* **order** *of* ω *at* $P \in X$ *to be* $\operatorname{ord}_P(\omega) = \operatorname{ord}_P(g)$, *where* $\omega = g\,dt$. *The* **divisor associated to** ω *is then defined to be*

$$(\omega) = \sum_{P \in X} \operatorname{ord}_P(\omega) P.$$

The **canonical class** *on* X *is the class* $K_X = [(\omega)]$ *in* $\operatorname{Pic}(X)$ *for any nonzero differential form* $\omega \in \Omega_X$.

Lemma 2.2.31. *The canonical class is well-defined, i.e. does not depend on the choice of* $\omega \in \Omega_X$.

Proof. For nonzero $\omega_1, \omega_2 \in \Omega_X$, write $\omega_1 = f\omega_2$ for some $f \in k(X)^\times$. Then $(\omega_1) = (f\omega_2) = (f) + (\omega_2)$. Thus $[(\omega_1)] = [(\omega_2)]$. □

Definition 2.2.32. *We say* $\omega \in \Omega_X$ *is a* **holomorphic** *(or* **regular***)* **differential** *on* X *if* $\operatorname{ord}_P(\omega) \geqslant 0$ *for all* $P \in X$. *We denote the space of holomorphic differentials on* X *by* $\Omega[X]$.

Let K_X be the canonical class on X. By Lemma 2.2.24, for any two nonzero differential forms $\omega, \omega' \in \Omega_X$, there is an isomorphism $L((\omega)) \cong L((\omega'))$. Therefore the Riemann–Roch space $L(K_X) := L((\omega))$, $\omega \in \Omega_X$, is well-defined.

Definition 2.2.33. *The* **geometric genus** *of* X *is defined as* $g(X) := \ell(K_X)$, *the dimension of the Riemann–Roch space* $L(K_X)$ *of the canonical class.*

Lemma 2.2.34. *There is an isomorphism* $L(K_X) \cong \Omega[X]$.

Proof. The map is $f \mapsto f\omega$ for any fixed $\omega \in \Omega[X]$ defining the canonical class. □

Corollary 2.2.35. *For any curve* X, $g(X) = \dim_k \Omega[X]$.

Example 2.2.36. Let $X = \mathbb{P}^1$ and let t be a coordinate function on some affine patch U of $\mathbb{P}^1$. We claim that $(dt) = -2\infty$. Indeed, for any $\alpha \in U \cong \mathbb{A}^1$, $t - \alpha$ is a local uniformizer at α. Thus $\text{ord}_\alpha(dt) = \text{ord}_\alpha(d(t-\alpha)) = 0$. At infinity, $\frac{1}{t}$ is a local uniformizer so we can write $dt = -t^2\, d\left(\frac{1}{t}\right)$. Hence

$$\text{ord}_\infty(dt) = \text{ord}_\infty\left(-t^2\, d\left(\tfrac{1}{t}\right)\right) = \text{ord}_\infty\left(-\tfrac{1}{t^{-2}}\right) + \text{ord}_\infty\left(d\left(\tfrac{1}{t}\right)\right) = -2 + 0 = -2.$$

So $(dt) = -2\infty$ as claimed. Now for any $\omega \in \Omega_{\mathbb{P}^1}$, $\deg(\omega) = -2$ so we see that $\ell(K_{\mathbb{P}^1}) = \ell(-2\infty) = 0$. Hence the genus of the projective line is $g(\mathbb{P}^1) = 0$.

Corollary 2.2.37. *There are no holomorphic differentials on* $\mathbb{P}^1$.

Proof. By Corollary 2.2.35, $g(\mathbb{P}^1) = \dim_k \Omega[\mathbb{P}^1]$ but by the calculations above, the genus of $\mathbb{P}^1$ is zero. □

Definition 2.2.38. *For a divisor* $D \in \mathrm{Div}(X)$*, define the* **meromorphic differentials at** D *by*

$$\Omega(D) = \{\omega \in \Omega_X : (\omega) \geqslant D\}.$$

Fix a differential form $\omega_0 \in \Omega(X)$, so that $K = [(\omega_0)]$. For any $D \in \mathrm{Div}(X)$ and $\omega \in \Omega(X)$, $(\omega) \geqslant D$ is equivalent to $(f) \geqslant D - (\omega_0)$, $\omega = f\omega_0$ for $f \in k(X)$. This proves:

Lemma 2.2.39. $\omega \in \Omega(D)$ *if and only if* $f \in L(K - D)$*, where* $\omega = f\omega_0$*.*

The most important result describing the geometry of divisors on an algebraic curve X is the Riemann–Roch theorem.

Theorem 2.2.40 (Riemann–Roch)**.** *For an algebraic curve* X *with canonical divisor* K *and any divisor* $D \in \mathrm{Div}(X)$*,*

$$\ell(D) - \ell(K - D) = 1 - g + \deg(D).$$

Corollary 2.2.41. *If* K_X *is the canonical divisor on* X*, then* $\deg(K_X) = 2g - 2$*.*

Proof. Set $D = K = K_X$. Then the Riemann–Roch theorem says that

$$\ell(K) - \ell(0) = \deg(K) + 1 - g$$

but $\ell(K) = g$ by definition and $\ell(0) = 1$. Solving for $\deg(K)$ we get $\deg(K) = 2g - 2$. □

Corollary 2.2.42. *Suppose* $\deg(D) > 2g - 2$ *for some divisor* $D \in \mathrm{Div}(X)$*. Then* $\ell(D) = \deg(D) + 1 - g$*.*

Proposition 2.2.43. *Let X be an algebraic curve such that $X(k) \neq \varnothing$. Then $g(X) = 0$ if and only if $X \cong \mathbb{P}^1$.*

Proof. ($\Longleftarrow$) If $X \cong \mathbb{P}^1$ then $g(X) = g(\mathbb{P}^1) = 0$ by Example 2.2.36.

($\Longrightarrow$) Assume $g(X) = 0$. For any $x \in X(k)$, the principal divisor $D = (x)$ is a divisor of degree 1 on X. Further, Riemann–Roch implies $\ell(D) \geqslant 2$, so there exists a nonconstant function $g \in L(D)$. Now g determines a map $g : X \to \mathbb{P}^1$, under which $g^*\infty = \mathrm{ord}_\infty(g) = x$, so we must have $\deg(g) = 1$. Hence g is an isomorphism of curves. $\square$

2.2.3 The Riemann–Hurwitz Formula

Let $\varphi : Y \to X$ be a nonconstant morphism of nonsingular curves and fix $Q \in Y$. Then $e_\varphi(Q) = \mathrm{ord}_Q(\varphi^* t_{\varphi(Q)})$ where $t_{\varphi(Q)}$ is a local uniformizer. Take t to be a uniformizer at $P = \varphi(Q)$ and set $e_\varphi(Q) = e$. Then $\varphi^*(dt) = d(\varphi^* t)$. Moreover, if s is a uniformizer on Y at Q, then $\varphi^* t = us^e$ for some unit $u \in \mathcal{O}_Q^\times$. Now $d(\varphi^* t) = d(us^e) = s^e\, du + ues^{e-1}\, ds$. Write $du = g\, ds$ for a regular function $g \in \mathcal{O}_Q$ Then $d(\varphi^* t) = s^e g\, ds + eus^{e-1}\, ds$, so

$$\mathrm{ord}_Q(d(\varphi^* t)) = \mathrm{ord}_Q(s^e g + eus^{e-1}) = \min\{\mathrm{ord}_Q(s^e g), \mathrm{ord}_Q(eus^{e-1})\}.$$

If $\mathrm{char}\, k \nmid e$, then this minimum is $e - 1$; otherwise, when $\mathrm{char}\, k \mid e$ the minimum is at least e. To match the terminology from Chapter 1, we make the following definition.

Definition 2.2.44. *If* φ *is ramified and* $\operatorname{char} k \nmid e_\varphi(Q)$ *for all* $Q \in Y$*, we say* φ *is* **tamely ramified**. *Otherwise* φ *is* **wildly ramified**.

Remark 2.2.45. If φ is tamely ramified, then $\operatorname{ord}_Q(d(\varphi^* t)) = e_{\varphi(Q)} - 1$ for each Q. If φ is wildly ramified at Q, then $\operatorname{ord}_Q(d(\varphi^* t)) \geqslant e_\varphi(Q)$.

Definition 2.2.46. *For a morphism* $\varphi : Y \to X$*, define the* **ramification divisor**

$$R_\varphi = \sum_{Q \in Y} \operatorname{ord}_Q(d(\varphi^* t)) Q.$$

Now for $\omega \in \Omega_X$, the canonical classes on X and Y can be defined by $K_X = [(\omega)]$ and $K_Y = [(\varphi^* \omega)]$. On the other hand, the pullback defines a divisor $\varphi^* K_X \in \operatorname{Div}(Y)$. The Riemann–Hurwitz formula expresses a useful relation between these three divisors. The critical computation is contained in the following lemma.

Lemma 2.2.47. *If* $\varphi : Y \to X$ *is a morphism of curves, then* $K_Y = \varphi^* K_X + [R_\varphi]$*, where* R_φ *is the ramification divisor of* φ*.*

Proof. If $\omega = f\, dt \in \Omega_X$, then for any $Q \in Y$,

$$\operatorname{ord}_Q(\varphi^* \omega) = \operatorname{ord}_Q(\varphi^* f\, d(\varphi^* t)) = \operatorname{ord}_Q(\varphi^* f) + \operatorname{ord}_Q(d(\varphi^* t)),$$

so we see that $\operatorname{ord}_Q(\varphi^* \omega)$ gives the coefficient in K_Y, $\operatorname{ord}_Q(\varphi^* f)$ gives the coefficient in $\varphi^* K_Q$ and $\operatorname{ord}_Q(d(\varphi^* t))$ gives the coefficient in R_φ. Summing over $Q \in Y$ gives the desired equality. □

Taking φ to be tamely ramified, $R_\varphi = \sum_{Q \in Y}(e_\varphi(Q) - 1)Q$ so the degree function applied to the equation in Lemma 2.2.47 gives

$$\deg(K_Y) = \deg(\varphi^* K_X) + \sum_{Q \in Y}(e_\varphi(Q) - 1).$$

Let $g(X)$ and $g(Y)$ be the genera of X and Y, respectively. Since $\deg(K_Y) = 2g(Y) - 2$ by Corollary 2.2.41, this proves:

Theorem 2.2.48 (Riemann–Hurwitz Formula, Tame Version)**.** *For any tamely ramified morphism of algebraic curves* $\varphi : Y \to X$*,*

$$2g(Y) - 2 = (\deg \varphi)(2g(X) - 2) + \sum_{Q \in Y}(e_\varphi(Q) - 1).$$

Corollary 2.2.49. *For any tamely ramified morphism* $\varphi : Y \to X$*,* $g(Y) \geqslant g(X)$*.*

In the wildly ramified case, there is a more general statement of the Riemann–Hurwitz formula which follows from [Ser2, Ch. III, Prop.14 and Ch. IV, Prop. 4]. We will prove a stacky version of the most general Riemann–Hurwitz formula in Chapter 5.

2.2.4 Artin–Schreier Theory for Curves

Artin–Schreier theory (and more generally, Artin–Schreier–Witt theory) can be used to give a precise classification of Galois covers of curves $Y \to X$ with group $\mathbb{Z}/p\mathbb{Z}$. Suppose k is algebraically closed and X and Y are k-curves. If

$k(X)$ (resp. $k(Y)$) is the function field of X (resp. Y) then the completion of $k(X)$ is isomorphic to the field of Laurent series $k((x))$. The extension $k((y))/k((x))$ given by the completion of $k(Y)$ is Galois of degree p, so by Theorem 1.4.10, it is given by an equation $y^p - y = f(x)$, and the ramification jump is $m = -\nu(f)$. We can represent $f(x) = x^{-m}g(x)$ for some $g \in k[[x]]$, and after a change of formal variables, we can even arrange for $g = 1$. When k is algebraically closed, this shows there are infinitely many non-isomorphic $\mathbb{Z}/p\mathbb{Z}$-covers of any given curve X since the ramification jump is an isomorphism invariant of the field extension.

The following lemma will be useful in later arguments.

Lemma 2.2.50. *Let $K = k((x))$ be the local field of Laurent series with valuation ring $A = k[[x]]$ and let L/K be the $\mathbb{Z}/p\mathbb{Z}$-extension given by the equation $y^p - y = x^{-m}g(x)$ with $g \in A^\times$ $m \equiv -1 \pmod{p}$. Write $m+1 = pn$ for $n \in \mathbb{N}$ and let $z = x^n y$. If B denotes the integral closure of A in L, then $B = A[z]$.*

Proof. It is easy to see that z satisfies the integral equation

$$z^p - zx^{n(p-1)} = xg(x)$$

but for clarity, here's where the equation comes from. Let $\mathrm{Gal}(L/K) = \langle \sigma \rangle \cong \mathbb{Z}/p\mathbb{Z}$, where $\sigma : y \mapsto y + 1$. Then

$$\sigma(z) = x^n \sigma(y) = x^n(y+1) = z + x^n$$

and likewise, for all $0 \leqslant i \leqslant p-1$, $\sigma^i(z) = z + ix^n$. Now the (monic) minimal polynomial $h(T) \in K[T]$ for z is

$$\begin{aligned} h(T) &= \prod_{i=0}^{p-1}(T - \sigma^i(z)) = \prod_{i=0}^{p-1}(T - z - ix^n) \\ &= \prod_{i=0}^{p-1}\left(\left(\frac{T-z}{x^n}\right) - i\right)x^n = \left(\left(\frac{T-z}{x^n}\right)^p - \left(\frac{T-z}{x^n}\right)\right)x^{np} \\ &= (T-z)^p - (T-z)x^{n(p-1)} = T^p - z^p - Tx^{n(p-1)} + zx^{n(p-1)} \\ &= T^p - Tx^{n(p-1)} - (z^p - zx^{n(p-1)}) = T^p - Tx^{n(p-1)} - (y^p - y)x^{np} \\ &= T^p - Tx^{n(p-1)} - x^{np-m}g(x) = T^p - Tx^{n(p-1)} - xg(x). \end{aligned}$$

Next, note that $\nu_L(x) = p\nu_K(x) = p$ and $-mp = \nu_L(x^{-m}) = \nu_L(x^{-m}g) = \nu_L(y^p - y) = \min\{p\nu_L(y), \nu_L(y)\}$ which implies $\nu_L(y) = -m$. So $\nu_L(z) = n\nu_L(x) + \nu_L(y) = np - m = 1$. Hence z is a uniformizer of B, so $B = A[z]$ as claimed. $\square$

Remark 2.2.51. When $m \not\equiv -1 \pmod{p}$, it is not as easy to write down a normal integral equation for B/A. Write $m = pn - r$ for $1 < r < p$. Then $z = x^n y$ still satisfies the integral equation $z^p - zx^{n(p-1)} = xg(x)$, but now $\nu_L(z) = r > 1$ so we don't get a uniformizer in $A[z]$ for free. In fact, $A[z] \neq B$ in these cases. To fix this, let $c, d \in \mathbb{Z}$ be the unique integers with $0 < c < p$ such that $cr - dp = 1$ and set $u = z^c x^{-d} = x^{nc-d}y^c$. Then $\nu_L(u) = c\nu_L(z) - d\nu_L(x) = cr - dp = 1$, so $A[u] = B$. However, it is difficult to write down the minimal polynomial of u over A and one should not expect it to produce a normal equation for B/A in general. Instead, one can write down an integral basis of B/A by resolving the singularities in any of the above integral equations step-by-step (such algorithms can be found in [OP, Lem. 6.3] or [LS, Lem. 5.5]).

Chapter 3

Algebraic Stacks

In this chapter, we relax the definition of schemes from Chapter 2 rather dramatically in order to study a wider class of geometric problems. There are three common situations in which stacks can be useful. If X is an arbitrary scheme admitting a group (scheme) action $G \times X \to X$, it is often the case that a quotient scheme X/G either fails to exist or does not possess the expected universal property of a quotient. For a similar reason, if X has singularities, many important geometric results (e.g. Riemann–Roch in its nicest form) fail to hold for X. Finally and most broadly, the type of moduli problems one typically encounters in algebraic geometry often have nontrivial automorphisms that prevent the moduli problem from having a well-behaved space of solutions (i.e. a fine moduli space). Luckily, by passing to the category of algebraic spaces and stacks, we can handle each of these situations with ease and also open the door to wider applications to representation theory, algebraic topology and beyond. Unless otherwise stated, all of the statements in this chapter can be found in [Ols] (often with similar section titles).

3.1 Sites

This section covers the basic definitions and results in Grothendieck's theory of sites, a generalization of a topological space which allows for the construction of sheaves. The main motivation is to develop a working sheaf theory on schemes that can detect the features of étale morphisms and more general properties.

3.1.1 Grothendieck Topologies and Sites

To every topological space X, we can associate a category $\mathsf{Top}(X)$ consisting of the open subsets $U \subseteq X$ with morphisms given by inclusions of open sets $U \hookrightarrow V$. Then a presheaf on X is a functor $F : \mathsf{Top}(X)^{op} \to \mathsf{Set}$, i.e. a contravariant functor on the category $\mathsf{Top}(X)$. The conditions for F to be a sheaf on X can be summarized by saying that for every open set $U \in \mathsf{Top}(X)$ and every open covering $U = \bigcup U_i$, the set $F(U)$ is an equalizer in the following diagram:

$$F(U) \longrightarrow \prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \cap U_j).$$

This generalizes as follows.

Definition 3.1.1. *A* **Grothendieck topology** *on a category* $\mathcal{C}$ *is a set of collections of morphisms* $\mathrm{Cov}(X) = \{\{X_i \to X\}_i\}$ *for every objects* $X \in \mathcal{C}$*, called* **coverings**, *satisfying:*

(i) Every isomorphism $X' \to X$ *defines a covering* $\{X' \to X\}$ *in* $\mathrm{Cov}(X)$.

(ii) For any covering $\{X_i \to X\}$ *of* X *and any morphism* $Y \to X$ *in* $\mathcal{C}$*, the fibre products* $X_i \times_X Y$ *exist and the induced maps* $\{X_i \times_X Y \to Y\}$ *are a covering of* Y.

(iii) If $\{X_i \to X\}_i$ *is a covering of* X *and* $\{Y_{ij} \to X_i\}_j$ *is a covering of* X_i *for each* i*, then the compositions* $\{Y_{ij} \to X_i \to X\}_{i,j}$ *are a covering of* X.

A category equipped with a Grothendieck topology is called a **site**.

Example 3.1.2. For a topological space X, the category $\mathsf{Top}(X)$ is a site with coverings

$$\mathrm{Cov}(U) = \left\{ \{U_i \hookrightarrow U\} : U_i \subseteq U \text{ are open and } U = \bigcup_i U_i \right\}.$$

When X is a scheme with the Zariski topology, $\mathsf{Top}(X)$ is called the (*small*) *Zariski site* on X.

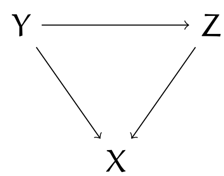
Example 3.1.3. The category Top of all topological spaces with continuous maps between them is a site, called the *big topological site*, whose coverings are defined by

$$\mathrm{Cov}(X) = \left\{ \{f_i : X_i \hookrightarrow X\} : f_i \text{ is an open embedding and } X = \bigcup_i X_i \right\}.$$

Example 3.1.4. Similarly, for a scheme X, let Sch_X be the category of X-schemes. Then Sch_X is a site, called the *big Zariski site* on X, with coverings

$$\mathrm{Cov}(Y) = \left\{ \{\varphi_i : Y_i \to Y\} : \varphi_i \text{ is an open embedding and } Y = \bigcup_i Y_i \right\}.$$

Example 3.1.5. Let $\mathcal{C}$ be a site and $X \in \mathcal{C}$ be an object. Define the *localized site* (or the *slice category*) $\mathcal{C}/X$ to be the category with objects $Y \to X \in \mathrm{Hom}_{\mathcal{C}}(Y, X)$, morphisms $Y \to Z$ in $\mathcal{C}$ such that



commutes. Then $\mathcal{C}/X$ can be equipped with a Grothendieck topology by defining

$$\mathrm{Cov}(Y \to X) = \{\{Y_i \to Y\} : Y_i \to Y \in \mathrm{Hom}_X(Y_i, Y), \{Y_i \to Y\} \in \mathrm{Cov}_{\mathcal{C}}(Y)\}.$$

Example 3.1.6. Let X be a scheme and define the *small étale site* on X to be the category $\acute{\mathrm{E}}\mathrm{t}(X)$ of X-schemes with étale morphisms $Y \to X$ and covers $\{Y_i \to Y\} \in \mathrm{Cov}(Y)$ such that $\coprod Y_i \to Y$ is surjective.

Example 3.1.7. In contrast, we can equip the slice category Sch/X with a Grothendieck topology by declaring $\{Y_i \to Y\}$ to be a covering of $Y \to X$ if each $Y_i \to Y$ is étale and $\coprod Y_i \to Y$ is surjective. The resulting site is referred to as the (*big*) *étale site* on X, written $X_{\acute{e}t}$.

Example 3.1.8. Similar constructions can be made by replacing "étale" with other properties, such as:

- The *fppf site* is the category Sch/X with coverings $\{Y_i \to Y\} \in \mathrm{Cov}(Y)$ such that $Y_i \to Y$ are flat and locally of finite presentation and $\coprod Y_i \to Y$ is surjective. This will sometimes be denoted X_{fppf}.

- The *lisse-étale site* LisÉt(X) is the category of X-schemes with smooth morphisms between them, whose coverings are $\{Y_i \to Y\} \in \mathrm{Cov}(Y)$ such that the $Y_i \to Y$ are *étale* and $\coprod Y_i \to Y$ is surjective.

- The *smooth site* $\mathsf{Sm}(X)$ is the category of X-schemes with smooth morphisms between them and surjective families of *smooth* coverings. This will sometimes be denoted X_{smooth}.

- Most generally, the *flat site* is Sch/X with surjective families of flat morphisms of finite type as coverings. This will sometimes be denoted X_{flat}.

Definition 3.1.9. *A* **continuous map between sites** $f : \mathcal{C}_1 \to \mathcal{C}_2$ *is a functor* $F : \mathcal{C}_2 \to \mathcal{C}_1$ *that preserves fibre products and takes coverings in* $\mathcal{C}_2$ *to coverings in* $\mathcal{C}_1$.

Remark 3.1.10. Notice that a continuous map between sites is a functor *in the opposite direction*. This is in analogy with the topological notion: a continuous map $f : X \to Y$ between topological spaces induces a functor $F : \mathsf{Top}(Y) \to \mathsf{Top}(X)$ given by $V \mapsto f^{-1}(V)$.

Example 3.1.11. When X is a scheme, there are continuous maps between the various sites we have defined on Sch/X. We collect some of these sites in the following table, along with their relevant features. (The arrows between sites represent continuous maps between sites, so the functors on the underlying categories go in the opposite direction. Note that when we define sheaves in the next section, sheaves will pull back in the *same direction* as these arrows.)

	X_{flat} $\to$	X_{fppf} $\to$	X_{smooth} $\to$	$X_{\text{ét}}$ $\to$	X_{Zar}
name	flat	fppf	smooth	étale	Zariski
maps	flat	flat, locally f.p.	smooth	étale	all

Example 3.1.12. Let G be a profinite group and let $\mathcal{C}_G$ be the category of all finite, discrete G-sets. Then the collections of G-homomorphisms $\{X_i \to X\}$ such that $\coprod_i X_i \to X$ is surjective form a Grothendieck topology on $\mathcal{C}_G$. When $G = \mathrm{Gal}(\bar{k}/k)$ for some field k, the category $\mathcal{C}_G$ is equivalent to $X_{\text{ét}}$ for $X = \operatorname{Spec} k$.

3.1.2 Sheaves on Sites

In this section we generalize the notions of presheaf and sheaf to a site $\mathcal{C}$.

Definition 3.1.13. *A* **presheaf** *on a site $\mathcal{C}$ is a functor $F : \mathcal{C}^{op} \to \mathsf{Set}$, that is, a contravariant functor from $\mathcal{C}$ to the category of sets. The category of presheaves on $\mathcal{C}$ (with natural transformations between them) will be denoted $\mathsf{PreSh}_{\mathcal{C}}$.*

Definition 3.1.14. *We say F is* **separated** *if for every collection of maps $\{X_i \to X\}$, the map $F(X) \to \coprod_i F(X_i)$ is injective.*

Definition 3.1.15. *A* **sheaf** *on* $\mathcal{C}$ *is a presheaf* $\mathsf{F} : \mathcal{C}^{op} \to \mathsf{Set}$ *such that for every object* $X \in \mathcal{C}$ *and every covering* $\{X_i \to X\} \in \mathrm{Cov}(X)$, *the sequence of based sets*

$$\mathsf{F}(X) \longrightarrow \prod_i \mathsf{F}(X_i) \rightrightarrows \prod_{i,j} \mathsf{F}(X_i \times_X X_j)$$

is exact, or equivalently, $\mathsf{F}(X)$ *is an equalizer in the diagram. The category of sheaves on* $\mathcal{C}$ *will be denoted* $\mathsf{Sh}_{\mathcal{C}}$.

As in topology, we can consider sheaves on $\mathcal{C}$ with values in set categories with further structure, e.g. $\mathsf{Group}, \mathsf{Ring}, \mathsf{R{-}Mod}, \mathsf{Alg}_k$.

Theorem 3.1.16 (Sheafification)**.** *The forgetful functor* $\mathsf{Sh}_{\mathcal{C}} \to \mathsf{PreSh}_{\mathcal{C}}$ *has a left adjoint* $\mathsf{F} \mapsto \mathsf{F}^a$.

Proof. First consider the forgetful functor $\mathsf{Sep}_{\mathcal{C}} \to \mathsf{PreSh}_{\mathcal{C}}$ defined on the subcategory of separated presheaves on $\mathcal{C}$. For a presheaf F on $\mathcal{C}$, let F^{sep} be the presheaf

$$X \longmapsto \mathsf{F}^{sep}(X) := \mathsf{F}(X)/\sim$$

where, for $a, b \in \mathsf{F}(X)$, $a \sim b$ if there is a covering $\{X_i \to X\}$ of X such that a and b have the same image under the map

$$\mathsf{F}(X) \to \coprod_i \mathsf{F}(X_i).$$

By construction, F^{sep} is a separated presheaf on $\mathcal{C}$ and for any other separated presheaf F', any morphism of presheaves $\mathsf{F} \to \mathsf{F}'$ factors through F^{sep} uniquely.

Hence $F \mapsto F^{sep}$ is left adjoint to the forgetful functor $\mathsf{Sep}_{\mathcal{C}} \to \mathsf{PreSh}_{\mathcal{C}}$ so it remains to construct a sheafification of every separated presheaf on $\mathcal{C}$.

For a separated presheaf F, define F^a to be the presheaf

$$X \longmapsto F^a(X) := (\{X_i \to X\}, \{\alpha_i\})/\sim$$

where $\{X_i \to X\} \in \mathrm{Cov}_{\mathcal{C}}(X)$, $\{\alpha_i\}$ is a collection of elements in the equalizer

$$\mathrm{Eq}\left(\prod_i F(X_i) \rightrightarrows \prod_{i,j} F(X_i \times_X X_j)\right),$$

and $(\{X_i \to X\}, \{\alpha_i\}) \sim (\{Y_j \to Y\}, \{\beta_j\})$ if α_i and β_j have the same image in $F(X_i \times_X Y_j)$ for all i, j. Then as above, F^a is a sheaf which is universal with respect to all morphisms of sheaves $F \to F'$. Thus $F \mapsto F^a$ defines a left adjoint to the forgetful functor $\mathsf{Sh}_{\mathcal{C}} \to \mathsf{Sep}_{\mathcal{C}}$ and composition with the first construction proves the theorem. □

Proposition 3.1.17. *For every continuous map of sites $f : \mathcal{C}' \to \mathcal{C}$, where $\mathcal{C}$ and $\mathcal{C}'$ are small categories, there exists an adjunction*

$$f^* : \mathsf{Sh}_{\mathcal{C}'} \leftrightarrows \mathsf{Sh}_{\mathcal{C}} : f_*.$$

Proof. Define f_* for each object $X' \in \mathcal{C}'$ and sheaf $F \in \mathsf{Sh}_{\mathcal{C}}$ by

$$(f_*F)(X') = F(f(X')).$$

If $\{X'_i \to X'\}$ is a covering in $\mathcal{C}'$, we have a commutative diagram

$$\begin{array}{ccccc}
(f_*F)(X') & \longrightarrow & \prod_i (f_*F)(X'_i) & \rightrightarrows & \prod_{i,j} (f_*F)(X'_i \times_{X'} X'_j) \\
= \big\downarrow & & = \big\downarrow & & \cong \big\downarrow \\
F(f(X')) & \longrightarrow & \prod_i F(f(X'_i)) & \rightrightarrows & \prod_{i,j} F(f(X'_i) \times_{f(X')} f(X'_j))
\end{array}$$

Here, the bottom row is exact since F is a sheaf and f is continuous; the right vertical arrow is an isomorphism since f is continuous; and the other vertical arrows are equalities by definition. Hence by the Five Lemma, the top row is exact, so f_*F defines a sheaf on $\mathcal{C}'$.

To define the left adjoint $f^* : \mathsf{Sh}_{\mathcal{C}'} \to \mathsf{Sh}_{\mathcal{C}}$, take an object $X \in \mathcal{C}$ and define a category I_X with objects (X', ρ) where $X' \in \mathcal{C}'$ and $\rho \in \mathrm{Hom}_{\mathcal{C}}(X, f(X'))$, and with morphisms $(X', \rho) \to (Y', \sigma)$ given by a morphism $g : X' \to Y'$ in $\mathcal{C}'$ making the following diagram commute:

$$\begin{array}{ccc}
 & & f(X') \\
 & \overset{\rho}{\nearrow} & \big\downarrow f(g) \\
X & & \\
 & \underset{\sigma}{\searrow} & \\
 & & f(Y')
\end{array}$$

Now define f^*F for a sheaf $F \in \mathsf{Sh}_{\mathcal{C}'}$ on an object $X \in \mathcal{C}$ by

$$(f^*F)(X) = \varinjlim F(X')$$

where the limit is over all objects (X', ρ) in the opposite category I_X^{op}. If $h : X \to Y$ is a morphism in $\mathcal{C}$, then there is an induced map $(f^*F)(Y) \to (f^*F)(X)$ given by the functor

$$\begin{aligned} I_Y &\longrightarrow I_X \\ (Y', \rho) &\longmapsto (Y', \rho \circ h). \end{aligned}$$

This shows that f^*F is a presheaf on $\mathcal{C}$. Moreover, the maps

$$(f^* f_* F)(X) = \varinjlim F(f(X')) \to F(X)$$

for each $F \in \mathsf{Sh}_{\mathcal{C}}, X \in \mathcal{C}$ give a natural transformation

$$\mathrm{Hom}_{\mathsf{Sh}_{\mathcal{C}'}}(F, f_* G) \longrightarrow \mathrm{Hom}_{\mathsf{Sh}_{\mathcal{C}}}(f^* F, G).$$

One can show that it is an isomorphism, which establishes that (f^*, f_*) is an adjoint pair. □

Example 3.1.18. Let $f : \mathcal{C}' \to \mathcal{C}$ be a continuous map of sites. For an object $X' \in \mathcal{C}'$, consider the presheaf represented by X':

$$\begin{aligned} h_{X'} : (\mathcal{C}')^{op} &\longrightarrow \mathsf{Set} \\ Y' &\longmapsto \mathrm{Hom}_{\mathcal{C}'}(Y', X'). \end{aligned}$$

It's easy to see that $f^* h_{X'} \cong h_{f(X')}$ as functors, that is, the adjoint pair (f^*, f_*) commutes with representable functors.

3.1.3 Cohomology

Many of the results in this section hold in greater generality than presented here – see [Ols, Sec. 2.3] for further reading. Let $\mathcal{C}$ be a site, $\mathsf{Sh}_{\mathcal{C}}$ the category of sheaves on $\mathcal{C}$ and Λ a ring in $\mathsf{Sh}_{\mathcal{C}}$. We begin by describing the category of sheaves on $\mathcal{C}$. First consider the category $\mathsf{Presh}_{\mathcal{C}}$ of presheaves on $\mathcal{C}$.

Lemma 3.1.19. $\mathsf{Presh}_{\mathcal{C}}$ *is an abelian category.*

Let $F' \to F \to F''$ be a sequence of presheaves on $\mathcal{C}$. Then this sequence is exact if and only if the sequence

$$F'(Y) \to F(Y) \to F''(Y)$$

is exact for all objects $Y \in \mathcal{C}$. Let $\mathsf{Sh}_{\mathcal{C}}$ be the full subcategory of $\mathsf{Presh}_{\mathcal{C}}$ of sheaves of abelian groups on $\mathcal{C}$. Then $\mathsf{Sh}_{\mathcal{C}}$ is an additive category; we will prove that it is abelian.

Definition 3.1.20. *A morphism of sheaves* $T : F \to F'$ *on* $\mathcal{C}$ *is* **locally surjective** *if for every* $Y \in \mathcal{C}$ *and* $s \in F'(Y)$*, there exists a covering* $\{Y_i \to Y\}$ *such that for each* i*,* $s|_{Y_i}$ *lies in the image of* $F(Y_i) \to F'(Y_i)$*.*

Proposition 3.1.21. *For a morphism of sheaves* $T : F \to F'$ *on a site* $\mathcal{C}$*, the following are equivalent:*

(a) $F \xrightarrow{T} F' \to 0$ *is exact.*

(b) T *is locally surjective.*

Proposition 3.1.22. *For a sequence of sheaves* $0 \to F' \to F \to F'' \to 0$ *in* $\mathsf{Sh}_{\mathcal{C}}$*, the following are equivalent:*

(a) The sequence is exact.

(b) For all $Y \in \mathcal{C}$*, the sequence* $0 \to F'(Y) \to F(Y) \to F''(Y) \to 0$ *is exact.*

Corollary 3.1.23. *For any site* $\mathcal{C}$*,* $\mathsf{Sh}_{\mathcal{C}}$ *is an abelian category.*

In order to define sheaf cohomology for a general site, we fix a ring object Λ in $\mathsf{Sh}_{\mathcal{C}}$. When the underlying category of $\mathcal{C}$ is a scheme category, say Sch_S for a base scheme S, we will typically choose $\Lambda = \mathcal{O}_S^{sh}$, the sheafification of the structure presheaf of S. (Note that $\mathcal{O}_S$ is in general only a presheaf in an arbitrary Grothendieck topology on Sch_S, although it is a sheaf in the étale topology; cf. Example 3.1.55.)

Definition 3.1.24. *An abelian group object* $M \in \mathsf{Sh}_{\mathcal{C}}$ *is a* **Λ-module** *if there is a morphism* $\rho : \Lambda \times M \to M$ *such that the following diagrams commute:*

$$
\begin{array}{ccc}
* \times M & \searrow & \\
\downarrow 1 & & M \\
\Lambda \times M & \xrightarrow{\ \rho\ } \!\!\nearrow &
\end{array}
\qquad\qquad
\begin{array}{ccc}
(\Lambda \times \Lambda) \times M & \xrightarrow{\ \sim\ } & \Lambda \times (\Lambda \times M) \\
\downarrow m \times \mathrm{id} & & \downarrow \mathrm{id} \times \rho \\
\Lambda \times M & & \Lambda \times M \\
& \searrow^{\rho} \quad {}^{\rho}\swarrow & \\
& M &
\end{array}
$$

Theorem 3.1.25. *Let* Mod_Λ *denote the full subcategory of* Λ*-modules in* $\mathsf{Sh}_{\mathcal{C}}$*. Then* Mod_Λ *is an abelian category with enough injectives.*

Proof. That Mod_Λ is abelian is an easy exercise, similar to the proof that the category $\mathsf{Mod}_{\mathcal{O}_X}$ of $\mathcal{O}_X$-modules on a scheme X is an abelian category. For the statement about injectives, see [SP, Tag 01DQ]. □

Fix a ring object Λ in $\mathsf{Sh}_\mathcal{C}$ and define the *global section functor* $\Gamma(\mathcal{C}, -) : \mathsf{Mod}_\Lambda \to \mathsf{Ab}$ by

$$F \longmapsto \Gamma(\mathcal{C}, F) := \mathrm{Hom}_{\mathsf{Sh}_\mathcal{C}}(\Lambda, F).$$

Also, for an object $X \in \mathcal{C}$, set $\Gamma(X, F) = \Gamma(\mathcal{C}/X, F|_X)$, called the *global sections over* X.

Proposition 3.1.26. $\Gamma(\mathcal{C}, -)$ *is left exact.*

Proof. Same as the usual proof that the global section functor of sheaves of abelian groups on any topological space is left exact. □

Definition 3.1.27. *The right derived functors of* $\Gamma(\mathcal{C}, -)$ *are called* **sheaf cohomology** *and denoted*

$$H^i(\mathcal{C}, F) := R^i\Gamma(\mathcal{C}, -) = H^i(\Gamma(\mathcal{C}, E_\bullet))$$

for any Λ*-module* F*, where* $E_\bullet$ *is an injective resolution of* F *in* Mod_Λ*.*

Theorem 3.1.28. *For every short exact sequence of* Λ*-modules* $0 \to F' \to F \to F'' \to 0$*, there is a long exact sequence*

$$0 \to \Gamma(\mathcal{C}, F') \to \Gamma(\mathcal{C}, F) \to \Gamma(\mathcal{C}, F'') \to H^1(\mathcal{C}, F') \to H^1(\mathcal{C}, F) \to H^1(\mathcal{C}, F'') \to \cdots$$

Example 3.1.29. For a scheme X, the constant sheaf $\mathbb{Z}$ is a ring in $\mathsf{Sh}_\mathcal{C}$ where

$\mathcal{C} = X_{Zar}, X_{ét}$, etc. Here, there are natural isomorphisms $\Gamma(\mathcal{C}, -) \cong \mathrm{Hom}(\mathbb{Z}, -)$ and as a result $H^i(\mathcal{C}, -) \cong \mathrm{Ext}^i(\mathbb{Z}, -)$.

As with sheaf cohomology on a topological space, sheaf cohomology can be computed with acyclic resolutions. For $X \in \mathcal{C}$, let $\mathcal{C}/X$ be the slice category defined in Example 3.1.5.

Definition 3.1.30. *A sheaf $F \in \mathsf{Mod}_\Lambda$ is* **acyclic** *if for all $X \in \mathcal{C}$ and $i > 0$, $H^i(\mathcal{C}/X, F) = 0$.*

Theorem 3.1.31. *A sheaf $F \in \mathsf{Mod}_\Lambda$ is acyclic if and only if the underlying sheaf of abelian groups is acyclic as an element of $\mathsf{Mod}_\mathbb{Z}$.*

For any object $X \in \mathcal{C}$ and sheaf $F \in \mathsf{Mod}_\Lambda$, write $\Gamma(X, F) = F(X)$. Then $\Gamma(X, -)$ is left exact and we write its right derived functors as

$$H^i(X, F) = R^i\Gamma(X, F) = H^i(\Gamma(X, E_\bullet))$$

for $E_\bullet$ an injective resolution of F in Mod_Λ.

Proposition 3.1.32. *Let $\mathcal{C}$ be a site, $X \in \mathcal{C}$ an object and $\mathcal{C}/X$ the slice category at X. Then for every $U \to X \in \mathcal{C}/X$, there are natural isomorphisms*

$$H^i(U, F) \cong H^i(\mathcal{C}/U, F|_U)$$

for all $F \in \mathsf{Mod}_\Lambda$ and $i \geqslant 0$.

3.1.4 Čech Cohomology

While the definition of sheaf cohomology for a site is packaged in an appealing way, it is usually impossible to compute from the definition. However, as with schemes, sheaf cohomology can often be computed in a combinatorial way using Čech cohomology.

Definition 3.1.33. *Let $\mathcal{C}$ be a site, Λ a ring object and suppose $\mathcal{U} = \{U_i \to X\}_{i \in I}$ is an open cover of $X \in \mathcal{C}$. For a sheaf F of Λ-modules on $\mathcal{C}$, the* **Čech complex** *of F with respect to $\mathcal{U}$ is the cochain complex $C^\bullet(\mathcal{U}, F)$ defined by*

$$C^p(\mathcal{U}, F) = \prod_{i_0, \ldots, i_p \in I} F(U_{i_0} \times_X \cdots \times_X U_{i_p})$$

with differential

$$\begin{aligned} d : C^p(\mathcal{U}, F) &\longrightarrow C^{p+1}(\mathcal{U}, F) \\ \alpha &\longmapsto \left(\sum_{k=0}^{p+1} (-1)^k \alpha_{i_0, \ldots, i_{k-1}, i_{k+1}, \ldots, i_{p+1}}|_{U_{i_0, \ldots, i_{p+1}}} \right) \end{aligned}$$

where $U_{i_0, \ldots, i_{p+1}} = U_{i_0} \times_X \cdots \times_X U_{i_{p+1}}$.

Lemma 3.1.34. $d^2 = 0$*; that is, $C^\bullet(\mathcal{U}, F)$ is a cochain complex.*

Definition 3.1.35. *The pth* **Čech cohomology with respect to an open cover** *$\mathcal{U}$ of site $\mathcal{C}$ with coefficients in a sheaf F is the pth cohomology of the Čech complex:*

$$\check{H}^p(\mathcal{U}, F) := H^p(C^\bullet(\mathcal{U}, F)).$$

Lemma 3.1.36. *For any Λ-module F and cover $\mathcal{U}$ of $X \in \mathcal{C}$, $\check{H}^0(\mathcal{U}, F) = H^0(X, F) = \Gamma(X, F)$, the global sections over X.*

Proof. By definition, $\check{H}^0(\mathcal{U}, F) = \ker(d : C^0(\mathcal{U}, F) \to C^1(\mathcal{U}, F))$. For $\alpha = (\alpha_i) \in C^0(\mathcal{U}, F)$, we have $d\alpha = (\alpha_i - \alpha_j)_{i,j}$ which is zero if and only if $\alpha_i = \alpha_j$ on $U_i \times_X U_j$ for all i, j. Thus $\ker d = \Gamma(X, F)$. □

Suppose $\mathcal{U}'$ is a refinement of $\mathcal{U}$, that is, $\mathcal{U}' = \{U_j' \to X\}_{j \in J}$ is a cover of X and there is a function $\lambda : J \to I$ such that for all $j \in J$, there is a morphism $U_j' \to U_{\lambda(j)}$. Then there is a chain map $C^\bullet(\mathcal{U}', F) \to C^\bullet(\mathcal{U}, F)$ given by

$$C^p(\mathcal{U}', F) \longrightarrow C^p(\mathcal{U}, F),\ (\alpha_{j_0,\dots,j_p}) \longmapsto (\alpha_{\lambda(j_0),\dots,\lambda(j_p)}|_{U_{j_0} \times_X \cdots \times_X U_{j_p}}).$$

This in turn induces maps on Čech cohomology:

$$\check{H}^p(\mathcal{U}', F) \longrightarrow \check{H}^p(\mathcal{U}, F)$$

for all p.

Definition 3.1.37. *The pth* **Čech cohomology** *of an object $X \in \mathcal{C}$ with coefficients in a Λ-module F is the direct limit*

$$\check{H}^p(X, F) := \varinjlim \check{H}^p(\mathcal{U}, F)$$

taken over all covers $\mathcal{U}$ of X, ordered with respect to refinement.

Fix a covering $\mathcal{U}$ of $X \in \mathcal{C}$. Then for any short exact sequence of presheaves of Λ-modules $0 \to F'' \to F \to F' \to 0$ on $\mathcal{C}$, there is a corresponding short exact sequence of complexes

$$0 \to C^\bullet(\mathcal{U}, F'') \to C^\bullet(\mathcal{U}, F) \to C^\bullet(\mathcal{U}, F') \to 0$$

Therefore there is a long exact sequence in Čech cohomology:

$$0 \to \check{H}^0(\mathcal{U}, F'') \to \check{H}^0(\mathcal{U}, F) \to \check{H}^0(\mathcal{U}, F') \to \check{H}^1(\mathcal{U}, F'') \to \check{H}^1(\mathcal{U}, F) \to \check{H}^1(\mathcal{U}, F') \to \cdots$$

Proposition 3.1.38. *The functors $\check{H}^p(\mathcal{U}, -) : \mathsf{Presh}_\Lambda \to \mathsf{Ab}$ are the derived functors of $\check{H}^0(\mathcal{U}, -)$.*

3.1.5 Direct and Inverse Image

Assume all presheaves and sheaves have values in abelian groups. Given a continuous map of sites $f : \mathcal{C}_2 \to \mathcal{C}_1$ given by a functor $T : \mathcal{C}_1 \to \mathcal{C}_2$, we have several ways of mapping sheaves on $\mathcal{C}_1$ to sheaves on $\mathcal{C}_2$ and vice versa. The simplest to define (although not always the simplest to understand) is the direct image functor.

Definition 3.1.39. *For a continuous map $f : \mathcal{C}_2 \to \mathcal{C}_1$ given by a functor $T : \mathcal{C}_1 \to \mathcal{C}_2$ and a presheaf F on $\mathcal{C}_2$, define the* **pushforward presheaf** *of F along f to be the functor $f_*F : \mathcal{C}_1^{op} \to \mathsf{Ab}$ sending an object U in $\mathcal{C}_1$ to $(f_*F)(U) := T(U)$.*

Lemma 3.1.40. *Direct image restricts to a functor $f_* : \mathsf{Sh}_{\mathcal{C}_2} \to \mathsf{Sh}_{\mathcal{C}_1}$.*

Proof. Follows from the fact that the functor T preserves coverings and fibre products. □

Example 3.1.41. When $f : Y \to X$ is a continuous map between topological spaces and $T = f^{-1} : \mathsf{Top}_X \to \mathsf{Top}_Y$ is the corresponding functor of sites, the direct image functor is the usual one for topological (pre)sheaves:

$$(f_*F)(V) = F(f^{-1}(V))$$

for any open set $V \subseteq Y$.

Proposition 3.1.42. *Suppose $\mathcal{C}_3 \xrightarrow{g} \mathcal{C}_2 \xrightarrow{f} \mathcal{C}_1$ are continuous maps of sites. Then $(f \circ g)_* = f_* \circ g_*$.*

Lemma 3.1.43. *For all continuous maps $f : \mathcal{C}_2 \to \mathcal{C}_1$, $f_* : \mathsf{Sh}_{\mathcal{C}_2} \to \mathsf{Sh}_{\mathcal{C}_1}$ is left exact.*

Proof. Direct image is exact on presheaves and sheafification is a left adjoint. □

Lemma 3.1.44. *For a continuous map of sites $f : \mathcal{C}_2 \to \mathcal{C}_1$ and any sheaf F on $\mathcal{C}_2$, f^*F is the sheafification of the presheaf $f^{-1}F$ on $\mathcal{C}_1$ which sends $V \in \mathcal{C}_1$ to*

$$f^{-1}F(V) := \varinjlim F(U)$$

where the direct limit is over all $U \in \mathcal{C}_2$ admitting a morphism $TV \to U$.

Proof. It is easy to show that f^{-1} is left adjoint to f_* on presheaves. Explicitly, for any presheaf Q on $\mathcal{C}_2$, there are bijections

$$\mathrm{Hom}_{\mathsf{Presh}_{\mathcal{C}_2}}(f^{-1}F, Q) \longleftrightarrow \mathrm{Hom}_{\mathsf{Presh}_{\mathcal{C}_1}}(F, f_*Q)$$

which are functorial in F and Q. Therefore, passing to the sheafification yields

$$\mathrm{Hom}_{\mathsf{Sh}_{\mathcal{C}_2}}((f^{-1}F)^{sh}, \mathcal{Q}) \cong \mathrm{Hom}_{\mathsf{Sh}_{\mathcal{C}_1}}(F, f_*\mathcal{Q})$$

for any sheaf $\mathcal{Q}$ on $\mathcal{C}_2$. Since left adjoints are unique up to canonical isomorphism, this proves $(f^{-1}F)^{sh} = f^*F$. □

Proposition 3.1.45. *For any continuous maps of sites* $\mathcal{C}_3 \xrightarrow{g} \mathcal{C}_2 \xrightarrow{f} \mathcal{C}_1$, $(f \circ g)^* = g^* \circ f^*$.

Proof. Both $(f \circ g)^*$ and $g^* \circ f^*$ are left adjoints to $(f \circ g)_* = f_* \circ g_*$. □

Proposition 3.1.46. *For all morphisms* $f : Y \to X$, f^* *is exact and* f_* *preserves injectives.*

Corollary 3.1.47. *For any continuous map* $f : \mathcal{C}_2 \to \mathcal{C}_1$ *and sheaf* F *on* $\mathcal{C}_2$*, there is a homomorphism*

$$H^p(\mathcal{C}_2, F) \longrightarrow H^p(\mathcal{C}_1, f_*F)$$

for all $p \geqslant 0$.

Proof. Take an injective resolution $E^\bullet$ of F in $\mathsf{Sh}_{\mathcal{C}_2}$. Then by Proposition 3.1.46, $f_*E^\bullet$ is an injective resolution of f_*F in $\mathsf{Sh}_{\mathcal{C}_1}$, so we can compute cohomology using this resolution:

$$H^p(\mathcal{C}_1, f_*F) = H^p(\Gamma(\mathcal{C}_2, f_*E^\bullet)).$$

Since each f_*E^n is defined locally by a direct limit $\varinjlim E^n(f^{-1}(V))$, there is an induced map of global sections $\Gamma(\mathcal{C}_2, E^n) \to \Gamma(\mathcal{C}_1, f_*E^n)$ for each $n \geqslant 0$. This induces the desired maps on cohomology: $H^p(\mathcal{C}_2, F) \longrightarrow H^p(\mathcal{C}_1, f_*F)$. □

3.1.6 The Étale Site

In this section, we focus on properties specific to the étale topology (from Example 3.1.7). While some statements appear in [Ols, Ch.2], a more thorough treatment can be found in [Mil1, Ch. II] or [Mil2, Secs. 4-7].

Let $X_{\text{ét}}$ denote the étale site on a scheme X. Fix a faithfully flat morphism $\varphi : Y \to X$ and a group G acting on the morphism on the right via $\alpha : G \to \text{Aut}_X(Y)$. Recall that φ is a *Galois cover* with Galois group G, or a G-*cover* for short, if the morphism

$$\begin{aligned} Y \times G &\longrightarrow Y \times_X Y \\ (y, g) &\longmapsto (y, yg) \end{aligned}$$

is an isomorphism.

Lemma 3.1.48. *$\varphi : Y \to X$ is a G-cover if and only if φ is surjective, finite, étale and* $\deg \varphi = |G|$.

Definition 3.1.49. *A Galois cover* $\varphi : Y \to X$ *is said to be* **generically Galois** *if* $k(Y)/k(X)$ *is a Galois extension of fields.*

Example 3.1.50. Let A be a ring, B an A-algebra and consider the corresponding morphism of affine schemes

$$\varphi : \operatorname{Spec} B \longrightarrow \operatorname{Spec} A.$$

Then φ is a Galois cover with Galois group G if and only if $A \to B$ is faithfully flat and G acts on B such that

$$\begin{aligned} B \otimes_A A &\longrightarrow G \times B = \prod_{g \in G} B \\ b \otimes b' &\longmapsto (bgb')_g \end{aligned}$$

is an isomorphism.

Example 3.1.51. Let k be a field and $f \in k[t]$ a monic irreducible polynomial. Set $K = k[t]/(f)$. Then $f = f_1^{e_1} \cdots f_r^{e_r}$ in $K[t]$ and by the Chinese remainder theorem,

$$K \otimes_k K \cong K[t]/(f) \cong \prod_{i=1}^{r} K[t]/(f_i^{e_i}).$$

Thus $\operatorname{Spec} K \to \operatorname{Spec} k$ is a Galois cover if and only if each f_i is linear, $f_i \neq f_j$ for any $i \neq j$ and $e_i = 1$ for all i. That is, $\operatorname{Spec} K \to \operatorname{Spec} k$ is Galois if and only if f is separable with splitting field K, just as in classical Galois theory.

Proposition 3.1.52. *Suppose* F *is a presheaf on the étale site* $X_{\text{ét}}$ *taking disjoint unions* *to products. Then* F *satisfies the sheaf condition on* $X_{ét}$ *for a given* G*-cover* $\varphi : Y \to X$ *if and only if* $F(\varphi) : F(X) \to F(Y)$ *is an isomorphism onto the fixed set* $F(Y)^G \subseteq F(Y)$.

Proof. Consider the two maps $Y \times G \rightrightarrows Y$ given by $(y, g) \mapsto y$ and $(y, g) \mapsto yg$. These fit into a commutative diagram with the two coordinate projections $Y \times_X Y \rightrightarrows Y$:

$$\begin{array}{ccccc} Y \times G & \rightrightarrows & Y & \longrightarrow & X \\ \cong \downarrow & & \downarrow \text{id} & & \downarrow \text{id} \\ Y \times_X Y & \rightrightarrows & Y & \longrightarrow & X \end{array}$$

Applying F to the diagram, we obtain a commutative diagram of sets

$$\begin{array}{ccccc} F(X) & \longrightarrow & F(Y) & \rightrightarrows & F(Y \times_X Y) \\ \text{id} \downarrow & & \downarrow \text{id} & & \downarrow \cong \\ F(X) & \longrightarrow & F(Y) & \rightrightarrows & \prod_{g \in G} F(Y) \end{array}$$

where the maps $F(Y) \rightrightarrows \prod_{g \in G} F(Y)$ are given by $s \mapsto (s)_g$ and $s \mapsto (gs)_g$. Then these maps agree precisely when $gs = s$ for all $g \in G$, i.e. $F(X)$ is the equalizer in the top row if and only if it identifies with $F(Y)^G$ in the bottom row. □

Proposition 3.1.53. *Suppose* F *is a presheaf on* $X_{ét}$ *which satisfies the condition that*

$$F(U) \longrightarrow \prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \times_U U_j)$$

is an equalizer diagram for all covers $\{U_i \to U\}$ *of* $U \in X_{Zar}$ *in the Zariski site on* X *and for all étale affine covers* $\{V \to U\}$ *of* $U \in X_{ét}$ *consisting of a single morphism in* *the étale site. Then* F *is a sheaf on* $X_{\text{ét}}$.

Proof. If $U = \coprod_i U_i$ for schemes $U_i \in X_{\text{ét}}$, then the first condition implies that $F(U) = \prod_i F(U_i)$. Thus for a covering $\{U_i \to U'\}$ in $X_{\text{ét}}$, the sequence

$$F(U') \longrightarrow \prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \times_{U'} U_j)$$

is isomorphic to the sequence

$$F(U') \longrightarrow F(U) \rightrightarrows F(U \times_{U'} U)$$

for the covering $\{U \to U'\}$, using the fact that $(\coprod_i U_i) \times_{U'} (\coprod_i U_i) = \coprod_{i,j}(U_i \times_{U'} U_j)$. Since the equalizer condition is assumed to hold for all étale affine covers, this argument shows the condition holds for all $\{U_i \to U\}_{i \in I}$ with I finite and each U_i affine.

Let $\{U_j \to U\}_j$ be a covering and set $U' = \coprod_j U_j$ and $f : U' \to U$. Write $U = \bigcup_i V_i$ for open affine subschemes $V_i \subseteq U$ and for each i, write $f^{-1}(V_i) = \bigcup_k W_{ik}$ for open affine subschemes $W_{ik} \subseteq U'$. Fix one of the V_i. Then each $f(W_{ik})$ is open in V_i, so by quasi-compactness, we may reduce to a *finite* cover $\{W_{ik} \to V_i\}_{k=1}^{K}$. Now consider the diagram

$$
\begin{array}{ccccc}
F(U) & \longrightarrow & F(U') & \rightrightarrows & F(U' \times_U U') \\
\downarrow & & \downarrow & & \downarrow \\
\prod_i F(V_i) & \longrightarrow & \prod_i \prod_k F(W_{ik}) & \rightrightarrows & \prod_i \prod_{k,\ell} F(W_{ik} \times_U W_{i\ell}) \\
\downarrow\downarrow & & \downarrow\downarrow & & \\
\prod_{i,j} F(V_i \times_U V_j) & \longrightarrow & \prod_{i,j} \prod_{k,\ell} F(W_{ik} \times_{V_i} W_{j\ell}) & &
\end{array}
$$

The two columns correspond to the coverings $\{V_i \to U\}_i$ and $\{W_{ik} \to U'\}_{i,k}$ which are all coverings in the Zariski site on X and hence these columns are exact by hypothesis. Moreover, the middle row corresponds to the coverings $\{W_{ik} \to f(W_{ik}) \subseteq V_i\}_k$ which for each i is finite and affine, so by the above paragraph this row is exact. An easy diagram chase then implies the top row of the diagram is also exact, which is what we want. □

Example 3.1.54. Let $A \to B$ be a ring homomorphism such that $\operatorname{Spec} B \to \operatorname{Spec} A$ is surjective and étale. In particular, $A \to B$ is faithfully flat and unramified. We claim that the sequence

$$0 \to A \to B \to B \otimes_A B \tag{*}$$

is exact, where the second map is $b \mapsto 1 \otimes b - b \otimes 1$. First note that the map $g : B \to B \otimes_A B, b \mapsto b \otimes 1$ has a section $s : b \otimes b' \mapsto bb'$. Then consider

$$h : (B \otimes_A B) \otimes_B (B \otimes_A B) \longrightarrow B \otimes_A B, \quad x \otimes y \longmapsto xgs(y).$$

We have $h(1 \otimes x - x \otimes 1) = gs(x) - x$, so if $1 \otimes x - x \otimes 1 = 0$, we get $x = gs(x) \in \operatorname{im} g$ and the sequence

$$0 \to B \xrightarrow{g} B \otimes_A B \to (B \otimes_A B) \otimes_B (B \otimes_A B)$$

is exact. Tensoring $(*)$ with B induces the vertical arrows in the following diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & B \otimes_A B \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & B & \longrightarrow & B \otimes_A B & \longrightarrow & (B \otimes_A B) \otimes_B (B \otimes_A B) \end{array}$$

Therefore the top row is exact as claimed.

Example 3.1.55. Let X be a scheme and $Y \to X$ an étale morphism. Set $\mathcal{O}_{X_{\text{ét}}}(Y) = \Gamma(Y, \mathcal{O}_Y)$, which defines a sheaf $\mathcal{O}_{X_{\text{ét}}}$ for the Zariski topology. To check that $\mathcal{O}_{X_{\text{ét}}}$ is a sheaf for the étale site, it suffices to check the conditions of Proposition 3.1.53, but the Zariski condition was just seen to hold. If $\{Y \to Z\}$ is an étale affine covering in $X_{\text{ét}}$, with $Y = \operatorname{Spec} B$ and $Z = \operatorname{Spec} A$, then the corresponding ring map $A \to B$ satisfies the condition of Example 3.1.54, meaning

$$0 \to A \to B \to B \otimes_A B$$

is exact and hence so is the sequence

$$\mathcal{O}_{X_{\text{ét}}}(Z) \to \mathcal{O}_{X_{\text{ét}}}(Y) \to \mathcal{O}_{X_{\text{ét}}}(Y \times_Z Y).$$

(Note that this is precisely the same as the equalizer condition since the map $B \to B \otimes_A B$ is $b \mapsto 1 \otimes b - b \otimes 1$ and $\Gamma(Y, \mathcal{O}_Y)$ are abelian groups.)

Example 3.1.56. Any scheme $Z \to X$ defines a presheaf

$$\begin{aligned} F_Z : X_{\text{ét}} &\longrightarrow \mathsf{Set} \\ Y &\longmapsto \text{Hom}_X(Y, Z). \end{aligned}$$

Then F_Z is a sheaf for X_{Zar} so once again, to show it is a sheaf for the étale topology, by Proposition 3.1.53 it will suffice to check the sheaf condition for single étale affine covers. For such a cover $\{\text{Spec}\, B \to \text{Spec}\, A\}$, the map $A \to B$ is faithfully flat with

$$0 \to A \to B \to B \otimes_A B$$

exact. Take an open affine subscheme $U = \text{Spec}\, C \hookrightarrow Z$. Applying $\text{Hom}_A(C, -)$ to the above sequence gives

$$\text{Hom}_A(C, A) \to \text{Hom}_A(C, B) \to \text{Hom}_A(C, B \otimes_A B)$$

which is exact since $\text{Hom}_A(C, -)$ is a left-exact functor. Generalizing to $\text{Hom}_X(-, Z)$ is straightforward using patching, so ultimately we conclude that F_Z is a sheaf on $X_{\text{ét}}$. As usual, by Yoneda's Lemma the assignment $Z \mapsto F_Z$ is injective so we will often write F_Z simply by Z.

Example 3.1.57. Let $n \geqslant 1$ be an integer and let μ_n be the group scheme defined (locally) by $t^n - 1 = 0$. Then for any $Y \in X_{\text{ét}}$, $\mu_n(Y)$ coincides with the set of

nth roots of unity in $\Gamma(Y, \mathcal{O}_Y)$.

Example 3.1.58. Let X be a k-scheme and let $\mathbb{G}_a$ be the affine group scheme defined by the additive group of k. Then for each $Y \in X_{\text{ét}}$, $\mathbb{G}_a(Y) = \Gamma(Y, \mathcal{O}_Y)$.

Example 3.1.59. Similarly, when $\mathbb{G}_m$ is the affine multiplicative group scheme defined by $k^\times$, then for each $Y \in X_{\text{ét}}$, $\mathbb{G}_m(Y) = \Gamma(Y, \mathcal{O}_Y)^\times$.

Example 3.1.60. When $\operatorname{char} k = p > 0$, let α_p denote the group scheme defined by $t^p = 0$. Note that α_p is *not* an étale group scheme (but it is flat). For $Y \in X_{\text{ét}}$, $\alpha_p(Y)$ corresponds to the set of nilpotent elements of order p in $\Gamma(Y, \mathcal{O}_Y)$.

Example 3.1.61. Consider the ring $k[\varepsilon] = k[t]/(t^2)$. Write $T = \operatorname{Spec} k[\varepsilon]$. The functor $T = \operatorname{Hom}_k(-, T)$ is called the (étale) *tangent space functor* since for any $Y \in X_{\text{ét}}$, $T(Y)$ is the tangent space to Y, which is locally given by $T(Y)_x = T_x Y := (\mathfrak{m}_x/\mathfrak{m}_x^2)$.

Example 3.1.62. Let R be a set and let F_R be the sheaf on $X_{\text{ét}}$ defined by

$$F_R : Y \longmapsto F_R(Y) := \prod_{\pi_0(Y)} R$$

where $\pi_0(Y)$ denotes the set of connected components of Y. Then F_R is called the *constant sheaf* on $X_{\text{ét}}$ associated to R.

Example 3.1.63. Let $\mathcal{M}$ be a sheaf of coherent $\mathcal{O}_X$-modules on the Zariski site X_{Zar}. This gives us an étale sheaf $\mathcal{M}^{\text{ét}}$ as follows. If $\varphi : Y \to X$ is an étale morphism, then $\varphi^*\mathcal{M}$ is a coherent $\mathcal{O}_Y$-module on Y_{Zar} which on affine patches $U = \operatorname{Spec} A \subseteq X, V = \operatorname{Spec} B \subseteq Y$ takes the form

$$\begin{array}{ccc} M \otimes_A B & \longleftarrow & M \\ \Big| & & \Big| \\ B & \longleftarrow & A \end{array}$$

Let $\mathcal{M}^{\text{ét}}$ be the presheaf $(Y \xrightarrow{\varphi} X) \mapsto \Gamma(Y, \varphi^*\mathcal{M})$ on $X_{\text{ét}}$. By a similar proof to the one in Example 3.1.55 for $\mathcal{O}_{X_{\text{ét}}}$, one can show that $\mathcal{M}^{\text{ét}}$ is then a sheaf on $X_{\text{ét}}$. As a special case, note that $(\mathcal{O}_{X_{Zar}})^{\text{ét}} = \mathcal{O}_{X_{\text{ét}}}$.

Example 3.1.64. Let X be a k-scheme, $\varphi : Y \to X$ a morphism and consider the exact sequence of sheaves

$$\varphi^*\Omega^1_{X/k} \to \Omega^1_{Y/k} \to \Omega^1_{Y/X} \to 0.$$

If φ is an étale morphism, then by Theorem 2.1.56, $\Omega^1_{Y/X} = 0$ so $\varphi^*\Omega^1_{X/k} \to \Omega^1_{Y/k}$ is surjective. In fact, both are locally free sheaves of the same rank, so $\varphi^*\Omega^1_{X/k} \cong \Omega^1_{Y/k}$. It follows that $(\Omega^1_{X/k})^{\text{ét}}|_{Y_{Zar}} = \Omega^1_{Y/k}$.

Example 3.1.65. Let k be a field with absolute Galois group $G = \text{Gal}(\bar{k}/k)$, set $X = \operatorname{Spec} k$ and consider the étale site $X_{\text{ét}}$. If $G\text{-Mod}^d$ denotes the category of discrete G-modules, then there is an equivalence of categories

$$\text{Sh}(X_{\text{ét}}) \longleftrightarrow G\text{-Mod}^d$$

$$F \longmapsto M_F$$

$$F_M \longleftarrow M$$

where $M_F = \varinjlim F(L)$ is the direct limit over all finite, Galois extensions L/k and F_M is the sheaf $A \mapsto \mathrm{Hom}_G(\mathrm{Hom}_k(A, k^{sep}), M)$.

Example 3.1.66. Let $n \in \mathbb{N}$ and assume n is invertible on X, i.e. $\mathrm{char}\, k \nmid n$ for any residue fields k of X. Consider the following sequence of sheaves, called the *Kummer sequence* for X:

$$0 \to \mu_n \to \mathbb{G}_m \xrightarrow{n} \mathbb{G}_m \to 0$$

where $\mathbb{G}_m \xrightarrow{n} \mathbb{G}_m$ is the morphism induced by $t \mapsto t^n$. We claim the Kummer sequence is exact (generalizing the content of Section 1.4.1). It suffices to show exactness on stalks at geometric points. For such a point $\bar{x}$, set $A = \mathcal{O}_{X,\bar{x}}$. Then $\mu_{n,\bar{x}} = \mu_n(A)$ and $\mathbb{G}_{m,\bar{x}} = A^\times$ and moreover, $0 \to \mu_n(A) \to A^\times$ is clearly exact. For the right map, notice that $t^n - a$ splits in $A[t]$ for every $a \in A^\times$, since $\frac{d}{dt}(t^n - a) = nt^{n-1} \neq 0$ when n is invertible on X. Thus every a is an nth root in $A^\times$, and it follows that the sequence

$$0 \to \mu_n(A) \to A^\times \xrightarrow{n} A^\times \to 0$$

is exact as required.

Example 3.1.67. When $\mathrm{char}\, k \mid n$ for some residue field k of X, the above example fails since étale locally, we have

$$\frac{d}{dt}(t^p - a) = pt^{p-1} = 0 \quad \text{in characteristic } p > 0.$$

Note however that the equation $t \mapsto t^p$ does define a (locally) flat covering $\mathbb{G}_m \to \mathbb{G}_m$, so the sequence

$$0 \to \mu_p \to \mathbb{G}_m \xrightarrow{p} \mathbb{G}_m \to 0$$

is exact in the flat topology on X. On the étale site, the appropriate characteristic p replacement for the Kummer sequence is the *Artin–Schreier sequence*

$$0 \to \mathbb{Z}/p\mathbb{Z} \to \mathbb{G}_a \xrightarrow{\wp} \mathbb{G}_a \to 0$$

where $\mathbb{Z}/p\mathbb{Z}$ is the constant group scheme defined by the same group and $\wp : \mathbb{G}_a \to \mathbb{G}_a$ is induced by the map $t \mapsto t^p - t$. To see that the AS sequence is exact (a generalization of Section 1.4.2), it suffices once again to check exactness on stalks at geometric points. For a geometric point $\bar{x}$, again let $A = \mathcal{O}_{X,\bar{x}}$ so that $\mathbb{Z}/p\mathbb{Z}_{\bar{x}} = \mathbb{Z}/p\mathbb{Z}(A)$ and $\mathbb{G}_{a,\bar{x}} = A$. As before, $0 \to \mathbb{Z}/p\mathbb{Z}(A) \to A$ is exact and the kernel of $t \mapsto t^p - t : A \to A$ is precisely $\mathbb{Z}/p\mathbb{Z}(A)$. For surjectivity, note that for any $a \in A$,

$$\frac{d}{dt}(t^p - t - a) = pt^{p-1} - 1 = -1 \neq 0$$

so $t^p - t - a$ splits in $A[t]$ for all $a \in A$. Thus the end of the sequence

$$0 \to \mathbb{Z}/p\mathbb{Z}(A) \to A \xrightarrow{\wp} A \to 0$$

is exact and we are done.

Example 3.1.68. For $n \geqslant 2$, consider the *Artin–Schreier–Witt sequence*

$$0 \to \mathbb{Z}/p^n\mathbb{Z} \to \mathbb{W}_n \xrightarrow{\wp} \mathbb{W}_n \to 0.$$

Then a similar argument shows this is an exact sequence of sheaves on $X_{\text{ét}}$, generalizing the results in Section 1.4.3).

3.2 Categories Fibred in Groupoids

One of the main motivations for Grothendieck's use of fibred categories in the study of algebraic spaces and stacks is to allow for the construction of universal objects. Here's an example to keep in mind. Suppose $f : X \to Y$ is a morphism in the category of topological spaces (this problem will also arise in a category of schemes). Then for any sheaf F on Y, one way to define a pullback sheaf f^*F on X is as a solution to the universal mapping problem $F \to f_*G$ of sheaves on Y, where G is a sheaf on X. This object f^*F is not unique, it is only defined up to canonical isomorphism. Similar problems occur where universal objects are present (any direct limit construction might pose a problem) so we must find a way around.

Here's a brief discussion of how nontrivial automorphisms can get in the way of solving a "moduli problem", which is loosely defined as a classification problem with some natural notion of geometry attached to the collection of all objects to be classified. A more rigorous definition is that a *moduli problem* is a functor $F : \mathsf{Sch}^{op} \to \mathsf{Set}$. If F is representable, say $F \cong \mathrm{Hom}(-, M)$ for some scheme M, we say M is a (*fine*) *moduli space* for F. In general no such space exists.

Example 3.2.1. Let k be a field. The moduli problem of finding r-dimensional vector subbundles of a rank n bundle on $X \in \mathsf{Sch}_k$ is represented by the Grassmannian variety $\mathrm{Gr}(r, n)$. In particular, $\mathbb{P}^n_k = \mathrm{Gr}(1, n)$ classifies line bundles inside rank n bundles. The moduli problem of rank n bundles themselves has the infinite Grassmannian $\mathrm{Gr}(n, \infty)$ as a topological moduli space, but this is not representable by a scheme (it is merely *pro-representable*).

Example 3.2.2. For any group G, there is a topological space BG, called the *classifying space* for G, which is a moduli space for isomorphism classes of principal G-bundles. In algebraic geometry, for any smooth algebraic group over a scheme S, there is a notion of a *classifying scheme* for principal G-bundles, namely S/G. In both cases, the classifying space/scheme only classifies G-bundles up to isomorphism, so it is merely a *coarse moduli space* for the moduli problem of principal G-bundles. Taking $G = GL_n$ recovers the previous example.

Example 3.2.3. One can think of algebraic varieties themselves (over an algebraically closed field) as moduli spaces whose points correspond to solutions to a given set of polynomial equations.

Example 3.2.4. The classical Cayley-Salmon theorem says that the moduli problem of finding lines on a smooth cubic surface is represented by a 0-dimensional scheme with 27 components. This and similar problems belong to an area known *enumerative geometry* and many of them admit scheme-theoretic moduli spaces.

Example 3.2.5. There are many interesting moduli problems in classical geometry which lie completely outside the realm of algebraic geometry. For example, the problem of classifying all circles in $\mathbb{R}^2$ is represented by $\mathbb{R}^2 \times \mathbb{R}_{>0}$, while circles up to isometry are parametrized by $\mathbb{R}_{>0}$, neither of which is algebro-geometric in the classical sense.

Example 3.2.6. In differential geometry, all complex (or equivalently, hyperbolic) structures on a surface of genus $g \geqslant 1$ are specified by *Fenchel-Nielsen coordinates*, and these form a moduli space homeomorphic to $\mathbb{R}^{6g-6}$.

Next, we introduce two of the most famous moduli problems in algebraic geometry.

Example 3.2.7. Let $\mathscr{M}_{1,1} : \mathsf{Sch}_{\mathbb{C}}^{op} \to \mathsf{Set}$ be the moduli problem parametrizing complex elliptic curves, i.e. $\mathscr{M}_{1,1}(S)$ is the set of isomorphism classes of smooth curves $E \to S$ whose geometric fibres are all complex curves of genus 1 with a marked point. For a morphism $S \to T$, we get a functor $\mathscr{M}_{1,1}(T) \to \mathscr{M}_{1,1}(S)$ defined by pullback: for a family of elliptic curves $E \to T$, $f^*(E \to T)$ is the family $E' \to S$ which is the pullback in the diagram

$$\begin{array}{ccc} E' & \longrightarrow & E \\ \downarrow & & \downarrow \\ S & \xrightarrow{f} & T \end{array}$$

Suppose $\mathscr{M}_{1,1}$ were representable, say $\mathscr{M}_{1,1} \cong \operatorname{Hom}(-, M)$ for some scheme M. Let $E_0 \in \mathscr{M}_{1,1}(M)$ correspond to the identity $\mathrm{id}_M \in \operatorname{Hom}(M, M)$. Then every elliptic curve $E \to S$ would be the pullback

$$\begin{array}{ccc} E & \longrightarrow & E_0 \\ \big\downarrow & & \big\downarrow \\ S & \longrightarrow & M \end{array}$$

by a *unique* morphism $S \to M$ – that is, $E_0 \to M$ would be a "universal" elliptic curve. However, this is impossible: every elliptic curve $E \to S$ has a nontrivial degree 2 automorphism (corresponding to the map $(x, y) \mapsto (x, -y)$ if E is locally given by $y^2 = x^3 + ax + b$) so the map $E \to E_0$ cannot be unique. (Even worse, in $\mathcal{M}_{1,1}(\mathbb{C})$ the isomorphism classes of elliptic curves with j-invariant 0 and 1728 have additional nontrivial automorphisms to worry about.) So $\mathscr{M}_{1,1}$ is not representable, but it does in fact have a *coarse* moduli space $M_{1,1}$, that is, a scheme $M_{1,1}$ and a functor $\mathscr{M}_{1,1} \to \operatorname{Hom}(-, M_{1,1})$ which is a bijection when evaluated on algebraically closed fields and such that for any scheme S and natural transformation $\mathscr{M}_{1,1} \to \operatorname{Hom}(-, S)$, there is a unique morphism $M_{1,1} \to S$ making the diagram

$$\begin{array}{ccc} \mathscr{M}_{1,1} & \longrightarrow & \operatorname{Hom}(-, M_{1,1}) \\ & \searrow \quad \swarrow & \\ & \operatorname{Hom}(-, S) & \end{array}$$

commute. In fact, $M_{1,1}$ is none other than the j-line, $\mathbb{A}^1_j = \operatorname{Spec} \mathbb{C}[j]$.

Example 3.2.8. Consider the moduli problem parametrizing Riemann surfaces of genus $g \geqslant 2$, or smooth proper complex curves X over $\mathbb{C}$ with $X(\mathbb{C}) \cong \mathbb{C}^g/\Lambda$ for a full lattice Λ. Let $\mathscr{M}_g : \mathsf{Sch}_{\mathbb{C}}^{op} \to \mathsf{Set}$ be the functor sending S to the set of isomorphism classes of smooth schemes $X \to S$ whose geometric fibres are all Riemann surfaces of genus g. Such an X is often called a family of Riemann surfaces, parametrized by the base S. As above, for a morphism $f : S \to T$ we get a pullback functor $f^* : \mathscr{M}_g(T) \to \mathscr{M}_g(S)$. By a similar proof to the elliptic curve case, $\mathscr{M}_g$ is not a representable functor – that is, there is no fine moduli space of genus g Riemann surfaces. However, as above, there is a coarse moduli space M_g for $\mathscr{M}_g$ which is of considerable complexity and has inspired much research in algebraic geometry.

The standard way to approach the issue of automorphisms in a moduli problem $F : \mathsf{Sch}^{op} \to \mathsf{Set}$ is to replace the category Set with the category Gpd of *groupoids*.

Definition 3.2.9. *A category $\mathcal{C}$ is a* **groupoid** *if every morphism in $\mathcal{C}$ is an isomorphism.*

Example 3.2.10. Let G be a group. Then G determines a category – usually also denoted by G – in which there is only one object $*$ and for each $g \in G$, there is an automorphism $* \xrightarrow{g} *$. Thus G is a groupoid.

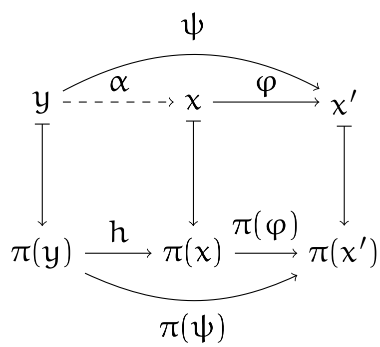
In Examples 3.2.7 and 3.2.8, the pullback functor f^* was crucial for studying the moduli problems of curves of genus g. In order to allow for pullbacks to play a role in our study of moduli spaces, we will replace a functor $\mathsf{Sch}^{op} \to \mathsf{Set}$

with a *category fibred in groupoids*. In order to make this precise, we need to introduce fibred categories.

3.2.1 Fibred Categories

Definition 3.2.11. *A* **category over a category** $\mathcal{C}$ *is a category* $\mathcal{F}$ *and a functor* $\pi : \mathcal{F} \to \mathcal{C}$*. The* **fibre** *of an object* $X \in \mathcal{C}$ *is the subcategory* $\mathcal{F}(X)$ *of objects* $x \in \mathcal{F}$ *such that* $\pi(x) = X$ *and morphisms covering* id_X*, i.e. morphisms* $\varphi : x \to x'$ *such that* $\pi(\varphi) : X \to X$ *is the identity. A* **morphism of categories over** $\mathcal{C}$ *is a functor* $T : \mathcal{F} \to \mathcal{G}$ *commuting with the functors* $\mathcal{F} \to \mathcal{C}$ *and* $\mathcal{G} \to \mathcal{C}$*.*

Definition 3.2.12. *Let* $\pi : \mathcal{F} \to \mathcal{C}$ *be a category over* $\mathcal{C}$*. A morphism* $\varphi : x \to x'$ *in* $\mathcal{F}$ *is* **cartesian** *if for any other morphism* $\psi : y \to x'$ *in* $\mathcal{F}$ *such that* $\pi(\psi) = \pi(\varphi) \circ h$ *for some* $h : \pi(y) \to \pi(x)$*, there exists a unique morphism* $\alpha : y \to x$ *covering* h *and making the diagram*



commute. Then $\pi : \mathcal{F} \to \mathcal{C}$ *is a* **fibred category** *if for every morphism* $f : X \to X'$ *in* $\mathcal{C}$ *and object* $x' \in \mathcal{F}(X')$*, there exists a cartesian morphism* $\varphi : x \to x'$ *covering* f*.*

In particular, note that if $f : X \to X'$ lifts to a cartesian morphism $\varphi : x \to x'$ in $\mathcal{F}$ then $x \in \mathcal{F}(X)$. One sometimes calls x the *pullback* of x' along f, written $x = f^*x'$.

Definition 3.2.13. *A* **morphism of fibred categories** *over* $\mathcal{C}$ *is a morphism* $T : \mathcal{F} \to \mathcal{G}$ *of categories over* $\mathcal{C}$ *that takes cartesian morphisms to cartesian morphisms. A* **base-preserving natural transformation** *between two morphisms* $S, T : \mathcal{F} \to \mathcal{G}$ *of fibred categories is a natural transformation* $\tau : S \to T$ *such that for all* $x \in \mathcal{F}$, $\tau_x : S(x) \to T(x)$ *covers the identity morphism in* $\mathcal{C}$.

The collection of all morphisms of fibred categories $\mathcal{F} \to \mathcal{G}$ over $\mathcal{C}$, together with the base-preserving natural transformations between them, forms a category denoted $\mathrm{Hom}_{\mathcal{C}}(\mathcal{F}, \mathcal{G})$. This shows that the category of all fibred categories over $\mathcal{C}$ in fact forms a 2-*category*.

Definition 3.2.14. *A* **category fibred in groupoids** *is a fibred category* $\mathcal{F} \to \mathcal{C}$ *such that for every object* $X \in \mathcal{C}$, *the fibre category* $\mathcal{F}(X)$ *is a groupoid.*

Let $\mathsf{CFG}(\mathcal{C})$ denote the full 2-subcategory of fibred categories over $\mathcal{C}$ which are fibred in groupoids.

Example 3.2.15. Suppose $\mathcal{C}$ is a category and $F : \mathcal{C}^{op} \to \mathsf{Set}$ is a presheaf on $\mathcal{C}$. Then F determines a category $\pi : \mathcal{C}_F \to \mathcal{C}$ with $\mathcal{C}_F(X) = F(X)$ for each $X \in \mathcal{C}$. There is a morphism $s \to t$ in $\mathcal{C}_F$ if $\pi(s) = X, \pi(t) = X'$ and there is a morphism $\varphi : X \to X'$ in $\mathcal{C}$ such that $F(\varphi)(F(X)) = F(X')$. A set is naturally a groupoid with only identity morphisms and it is easy to check the axioms of a fibred category hold for $\pi : \mathcal{C}_F \to \mathcal{C}$, so every presheaf on $\mathcal{C}$ naturally determines a category fibred in groupoids over $\mathcal{C}$.

Example 3.2.16. If $X \in \mathcal{C}$ is any object, then the slice category $\mathcal{C}/X$ is naturally a category fibred in groupoids over $\mathcal{C}$, where the projection $\mathcal{C}/X \to \mathcal{C}$ is just $(Y \to X) \mapsto Y$. A category $\mathcal{F}$ fibred in groupoids over $\mathcal{C}$ is called *representable* if it is equivalent (as a category fibred in groupoids) to a slice category $\mathcal{C}/X$ for some object X, in which case $\mathcal{F}$ is said to be represented by X.

Example 3.2.17. Let Sch_X be the category of X-schemes and define a category $\pi : \mathcal{F} \to \mathsf{Sch}_X$ whose objects are pullback squares

$$\begin{array}{ccc} P & \longrightarrow & Y \\ \downarrow & & \downarrow \\ T & \longrightarrow & X \end{array}$$

in Sch_X and whose morphisms are pairs of morphisms $(T' \to T, P' \to P)$ making the appropriate diagrams commute. The functor π sends the square above to $Y \to X \in \mathsf{Sch}_X$, so that for any fixed X-scheme Y, the fibre category $\mathcal{F}(Y)$ may be identified with the category of pullbacks $P = T \times_X Y$ which exist (showing the fibred condition holds for $\mathcal{F}$) and are unique up to unique isomorphism. Therefore $\mathcal{F}(Y)$ is a groupoid so $\mathcal{F}$ is a category fibred in groupoids over Sch_X.

Proposition 3.2.18. *A fibred category $\mathcal{F} \to \mathcal{C}$ is fibred in groupoids if and only if every morphism in $\mathcal{F}$ is cartesian.*

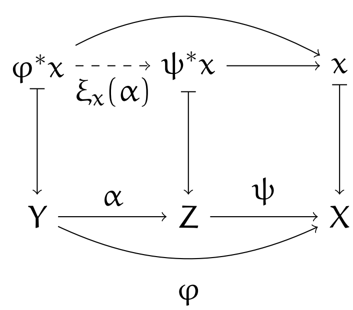
We will use liberally the following 2-categorical version of the Yoneda lemma from category theory.

Lemma 3.2.19 (2-Yoneda Lemma)**.** *For any object* $X \in \mathcal{C}$ *and fibred category* $\pi : \mathcal{F} \to \mathcal{C}$*, the functor* $\eta : \mathrm{Hom}_{\mathcal{C}}(\mathcal{C}/X, \mathcal{F}) \to \mathcal{F}(X)$ *defined by sending* $S : \mathcal{C}/X \to \mathcal{F}$ *to* $S(\mathrm{id}_X) \in \mathcal{F}(X)$ *is an equivalence of* 2*-categories.*

Proof. We define a functor $\xi : \mathcal{F}(X) \to \mathrm{Hom}_{\mathcal{C}}(\mathcal{C}/X, \mathcal{F})$ as follows. For each object $x \in \mathcal{F}(X)$ and morphism $\varphi : Y \to X$ in $\mathcal{C}$, choose a pullback $\varphi^* x \in \mathcal{F}(Y)$ and set

$$\begin{aligned} \xi_x : \mathcal{C}/X &\longrightarrow \mathcal{F} \\ \varphi &\longmapsto \varphi^* x. \end{aligned}$$

On the level of morphisms, if $\psi : Z \to X$ is another morphism in $\mathcal{C}$ and $\alpha : Y \to Z$ is a morphism over X, then by the cartesian condition in $\mathcal{F}$ there is a unique morphism $\xi_x(\alpha) : \varphi^* x \to \psi^* x$ which completes the diagram



This defines $\xi_x \in \mathrm{Hom}_{\mathcal{C}}(\mathcal{C}/X, \mathcal{F})$ for every object $x \in \mathcal{F}(X)$. If $f : x \to x'$ is a morphism in $\mathcal{F}(X)$, then for any $\varphi : Y \to X$ in $\mathcal{C}/X$, choose pullbacks $\varphi^* x$ and $\varphi^* x' \in \mathcal{F}(Y)$ of x and x', respectively. Then there is a unique morphism $\xi_f(\varphi)$ completing the diagram

$$\begin{array}{ccc} \varphi^*x & \xrightarrow{\xi_f(\varphi)} & \varphi^*x' \\ \downarrow & & \downarrow \\ x & \xrightarrow{f} & x' \end{array}$$

by the cartesian condition defining φ^*x, φ^*x'. This defines ξ on morphisms, so $\xi : \mathcal{F}(X) \to \mathrm{Hom}_{\mathcal{C}}(\mathcal{C}/X, \mathcal{F})$ is a functor.

It remains to check that η and ξ together give an equivalence of categories. On one hand, $\xi \circ \eta : \mathrm{Hom}_{\mathcal{C}}(\mathcal{C}/X, \mathcal{F}) \to \mathrm{Hom}_{\mathcal{C}}(\mathcal{C}/X, \mathcal{F})$ sends $S : \mathcal{C}/X \to \mathcal{F}$ to $\xi_{S(\mathrm{id}_X)} : (\varphi : Y \to X) \mapsto \varphi^*S(\mathrm{id}_X)$. This is canonically isomorphic to S itself since id_X is a final object in $\mathcal{C}/X$ and thus there exists a unique cartesian morphism $(\varphi : Y \to X) \to (\mathrm{id}_X : X \to X)$, which makes $S(\varphi) \to S(\mathrm{id}_X)$ also cartesian and hence $S(\varphi)$ is a pullback of $S(\mathrm{id}_X)$. This shows $\xi \circ \eta$ is naturally isomorphic to the identity functor. On the other hand, $\eta \circ \xi : \mathcal{F}(X) \to \mathcal{F}(X)$ takes $x \in \mathcal{F}(X)$ to $\mathrm{id}_X^* x$ which is canonically isomorphic to x itself, thus proving $\eta \circ \xi \simeq \mathrm{id}_{\mathcal{F}(X)}$. $\square$

Corollary 3.2.20. *For any objects $X, Y \in \mathcal{C}$, the functor*

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}}(\mathcal{C}/X, \mathcal{C}/Y) &\longrightarrow \mathrm{Hom}_{\mathcal{C}}(X, Y) \\ S &\longmapsto S(\mathrm{id}_X) \end{aligned}$$

is an equivalence of categories.

Corollary 3.2.21. *For any category $\mathcal{C}$, there is a fully faithful embedding of 2-categories*

$$\mathcal{C} \hookrightarrow \mathsf{CFG}(\mathcal{C}), X \longmapsto \mathcal{C}/X,$$

where $\mathsf{CFG}(\mathcal{C})$ is the 2-category of categories fibred in groupoids over $\mathcal{C}$. In a similar way, there is a fully faithful embedding of 2-categories

$$\mathsf{Presh}_{\mathcal{C}} \hookrightarrow \mathsf{CFG}(\mathcal{C}), F \longmapsto \mathcal{C}_F.$$

Example 3.2.22. Let Sch_k be the category of schemes over a field k. Then for each $k, n \geqslant 1$, we define a category fibred in groupoids $\mathrm{Gr}(k,n) \to \mathsf{Sch}_k$ called the (k,n)th *Grassmannian category*, by putting $\mathrm{Gr}(k,n)(X)$ as the category of vector bundles $E \to X$ of rank k, together with embeddings of bundles $E \hookrightarrow \mathbb{A}^n_X$. A morphism in $\mathrm{Gr}(k,n)$ from $E \in \mathrm{Gr}(k,n)(X)$ to $E' \in \mathrm{Gr}(k,n)(X')$ is a morphism of bundles $E \to E'$ covering a morphism of schemes $X \to X'$ and commuting with the natural map $\mathbb{A}^n_X \to \mathbb{A}^n_{X'}$.

Example 3.2.23. For each $X \in \mathsf{Sch}_k$, let $\mathscr{M}_{1,1}(X)$ be the category consisting of pairs (E, O) where $E \to X$ is a smooth curve, $O : X \to E$ is a section and for every geometric point $\bar{x} : \operatorname{Spec} \bar{k} \hookrightarrow X$, the pullback $(E_{\bar{x}}, O_{\bar{x}})$ is an elliptic curve over $\bar{k}$. Then $\mathscr{M}_{1,1} \to \mathsf{Sch}_k$ is a fibred category. Morphisms $(E, O) \to (E', O')$ in $\mathscr{M}_{1,1}$ are given by pullback diagrams

$$\begin{array}{ccc} E & \xrightarrow{F} & E' \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & X' \end{array}$$

such that $F \circ O' = O \circ f$. Since pullbacks are unique up to unique isomorphism, it follows that $\mathscr{M}_{1,1}$ is a category fibred in groupoids over Sch_k.

Example 3.2.24. Likewise for each $g = 0, g \geqslant 2$ and $X \in \mathsf{Sch}_k$, define $\mathscr{M}_g(X)$ to be the category consisting of smooth curves $C \to X$ whose geometric fibres $C_{\bar{x}}$ are smooth curves of genus g over $\bar{k}$. Then $\mathscr{M}_g$ is a fibred category over Sch_k. As above, morphisms in $\mathscr{M}_g$ are given by pullbacks, so $\mathscr{M}_g$ is a category fibred in groupoids.

Example 3.2.25. The following construction will be important in the definition of stacks. Let $\pi : \mathcal{F} \to \mathcal{C}$ be a category fibred in groupoids. For $X \in \mathcal{C}$ and $x, x' \in \mathcal{F}(X)$, define a presheaf $\mathrm{Isom}(x, x') : (\mathcal{C}/X)^{op} \to \mathsf{Set}$ by

$$\mathrm{Isom}(x, x')(\varphi : Y \to X) = \mathrm{Hom}_{\mathcal{F}(Y)}(\varphi^* x, \varphi^* x')$$

where $\varphi^* x, \varphi^* x' \in \mathcal{F}(Y)$ are pullbacks of x and x', respectively, along φ. Moreover, any other $\psi : Z \to Y$ induces a map

$$\psi^* : \mathrm{Isom}(x, x')(\varphi : Y \to X) \longrightarrow \mathrm{Isom}(x, x')(\varphi \circ \psi : Z \to X)$$

by composition of pullbacks. In particular, $\mathrm{Aut}(x) := \mathrm{Isom}(x, x)$ is a presheaf of groups on $\mathcal{C}/X$ for any $x \in \mathcal{F}(X)$.

Lemma 3.2.26. *Let* $F : \mathcal{C} \to \mathcal{D}$ *be a morphism of categories fibred in groupoids over* Sch_S. *Then* F *is an equivalence of categories fibred in groupoids if and only if for each* S*-scheme* T*, the functor* $F_T : \mathcal{C}(T) \to \mathcal{D}(T)$ *is an equivalence of categories.*

Proof. This is a special case of [SP, Tag 003Z]. □

3.2.2 Fibre Products

In this section we construct fibre products of categories fibred in groupoids. We first give the construction for individual groupoids. Fix a groupoid $\mathcal{G}$ and suppose $\mathcal{G}_1$ and $\mathcal{G}_2$ are two groupoids over $\mathcal{G}$, meaning there are functors $\pi_1 : \mathcal{G}_1 \to \mathcal{G}$ and $\pi_2 : \mathcal{G}_2 \to \mathcal{G}$. The fibre product $\mathcal{G}_1 \times_{\mathcal{G}} \mathcal{G}_2$ is defined to be the category with objects (X, Y, φ) consisting of $X \in \mathcal{G}_1$, $Y \in \mathcal{G}_2$ and an isomorphism $\varphi : \pi_1(X) \to \pi_2(Y)$ in $\mathcal{G}$. We define a morphism $(X, Y, \varphi) \to (X', Y', \varphi')$ in $\mathcal{G}_1 \times_{\mathcal{G}} \mathcal{G}_2$ to be a pair of morphisms $\alpha : X \to X'$ in $\mathcal{G}_1$ and $\beta : Y \to Y'$ in $\mathcal{G}_2$ making the diagram

$$\begin{array}{ccc} \pi_1(X) & \xrightarrow{\varphi} & \pi_2(Y) \\ {\scriptstyle \varphi(\alpha)}\downarrow & & \downarrow{\scriptstyle \varphi'(\beta)} \\ \pi_1(X') & \xrightarrow{\varphi'} & \pi_2(Y') \end{array}$$

commute. Then $\mathcal{G}_1 \times_{\mathcal{G}} \mathcal{G}_2$ is a groupoid by construction and there are functors $p_1 : \mathcal{G}_1 \times_{\mathcal{G}} \mathcal{G}_2 \to \mathcal{G}_1$ and $p_2 : \mathcal{G}_1 \times_{\mathcal{G}} \mathcal{G}_2 \to \mathcal{G}_2$ given by $p_1(X, Y, \varphi) = X$, $p_2(X, Y, \varphi) = Y$ and similar projections on morphisms. Further, the diagram

$$
\begin{array}{ccc}
\mathcal{G}_1 \times_{\mathcal{G}} \mathcal{G}_2 & \xrightarrow{p_2} & \mathcal{G}_2 \\
{\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle \pi_2} \\
\mathcal{G}_1 & \xrightarrow{\pi_1} & \mathcal{G}
\end{array}
$$

2-*commutes*, i.e. there is a natural isomorphism $\pi_1 \circ p_1 \simeq \pi_2 \circ p_2$.

Proposition 3.2.27. *For any groupoids $\pi_1 : \mathcal{G}_1 \to \mathcal{G}$ and $\pi_2 : \mathcal{G}_2 \to \mathcal{G}$, the fibre product $\mathcal{G}_1 \times_{\mathcal{G}} \mathcal{G}_2$ is universal with respect to groupoids $\mathcal{P}$ making the diagram*

$$
\begin{array}{ccc}
\mathcal{P} & \xrightarrow{q_2} & \mathcal{G}_2 \\
{\scriptstyle q_1}\downarrow & & \downarrow{\scriptstyle \pi_2} \\
\mathcal{G}_1 & \xrightarrow{\pi_1} & \mathcal{G}
\end{array}
$$

2-*commute. That is, for such a groupoid $\mathcal{P}$ there is a functor* $t : \mathcal{P} \to \mathcal{G}_1 \times_{\mathcal{G}} \mathcal{G}_2$, *unique up to unique natural isomorphism, and natural isomorphisms $\lambda_1 : q_1 \to p_1 \circ t$ and $\lambda_2 : q_2 \to p_2 \circ t$ making the diagram of functors*

$$
\begin{array}{ccc}
\pi_1 \circ q_1 & \xrightarrow{\pi_1(\lambda_1)} & \pi_1 \circ p_1 \circ t \\
{\scriptstyle \simeq}\downarrow & & \downarrow{\scriptstyle \simeq} \\
\pi_2 \circ q_2 & \xrightarrow{\pi_2(\lambda_2)} & \pi_2 \circ p_2 \circ t
\end{array}
$$

commute.

This says that $\mathcal{G}_1 \times_{\mathcal{G}} \mathcal{G}_2$ is a 2-*categorical fibre product* of $\mathcal{G}_1$ and $\mathcal{G}_2$. Now let $\mathcal{C}$ be any category and $\pi : \mathcal{F} \to \mathcal{C}, \pi_1 : \mathcal{F}_1 \to \mathcal{F}$ and $\pi_2 : \mathcal{F}_2 \to \mathcal{F}$ categories fibred in groupoids over $\mathcal{C}$. The construction of a 2-categorical fibre product $\mathcal{F}_1 \times_{\mathcal{F}} \mathcal{F}_2$

in the 2-category $\mathsf{CFG}(\mathcal{C})$ of categories fibred in groupoids over $\mathcal{C}$ is similar to the construction above, but we give it here for completeness.

Proposition 3.2.28. *There exists a category $\mathcal{F}_1 \times_{\mathcal{F}} \mathcal{F}_2$ fibred in groupoids over $\mathcal{C}$ together with morphisms of fibred categories $p_1 : \mathcal{F}_1 \times_{\mathcal{F}} \mathcal{F}_2 \to \mathcal{F}_1$ and $p_2 : \mathcal{F}_1 \times_{\mathcal{F}} \mathcal{F}_2 \to \mathcal{F}_2$ as well as an isomorphism (a base-preserving natural isomorphism) $\sigma : \pi_1 \circ p_1 \simeq \pi_2 \circ p_2$ satisfying the following universal property:*

(i) For any category $\mathcal{H} \to \mathcal{C}$ fibred in groupoids, the morphism of groupoids

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}}(\mathcal{H}, \mathcal{F}_1 \times_{\mathcal{F}} \mathcal{F}_2) &\longrightarrow \mathrm{Hom}_{\mathcal{C}}(\mathcal{H}, \mathcal{F}_1) \times_{\mathrm{Hom}_{\mathcal{C}}(\mathcal{H},\mathcal{F})} \mathrm{Hom}_{\mathcal{C}}(\mathcal{H}, \mathcal{F}_2) \\ \eta &\longmapsto (p_1 \circ \eta, p_2 \circ \eta, \sigma \circ \eta) \end{aligned}$$

is an isomorphism.

(ii) For any morphisms $q_1 : \mathcal{G}' \to \mathcal{F}_1$, $q_2 : \mathcal{G}' \to \mathcal{F}_2$ with an isomorphism $\tau : \pi_1 \circ q_1 \simeq \pi_2 \circ q_2$ making

$$\mathrm{Hom}_{\mathcal{C}}(\mathcal{H}, \mathcal{G}') \longrightarrow \mathrm{Hom}_{\mathcal{C}}(\mathcal{H}, \mathcal{F}_1) \times_{\mathrm{Hom}_{\mathcal{C}}(\mathcal{H},\mathcal{F})} \mathrm{Hom}_{\mathcal{C}}(\mathcal{H}, \mathcal{F}_2)$$

an isomorphism for all $\mathcal{H} \to \mathcal{C}$, there exists an equivalence of fibred categories $t : \mathcal{H} \to \mathcal{F}_1 \times_{\mathcal{F}} \mathcal{F}_2$, unique up to unique isomorphism, and isomorphisms $\lambda_1 : q_1 \to p_1 \circ t$ and $\lambda_2 : q_2 \to p_2 \circ t$ making the diagram of morphisms

$$\begin{array}{ccc} \pi_1 \circ q_1 & \xrightarrow{\pi_1 \circ \lambda_1} & \pi_1 \circ p_1 \circ t \\ \tau \downarrow & & \downarrow \sigma \circ t \\ \pi_2 \circ q_2 & \xrightarrow{\pi_2 \circ \lambda_2} & \pi_2 \circ p_2 \circ t \end{array}$$

commute.

Proof. (Sketch; see [Ols, 3.4.13]) Define $\mathcal{G} = \mathcal{F}_1 \times_{\mathcal{F}} \mathcal{F}_2$ to be the category with objects (x, y, φ) consisting of $x \in \mathcal{F}_1, y \in \mathcal{F}_2$ and an isomorphism $\varphi : \pi_1(x) \to \pi_2(y)$ in $\mathcal{F}$. A morphism $(x, y, \varphi) \to (x', y', \varphi')$ in $\mathcal{G}$ is a pair of morphisms $\alpha : x \to x'$ in $\mathcal{F}_1$ and $\beta : y \to y'$ in $\mathcal{F}_2$ making the diagram

$$\begin{array}{ccc} \pi_1(x) & \xrightarrow{\varphi} & \pi_2(y) \\ \varphi(\alpha) \downarrow & & \downarrow \varphi'(\beta) \\ \pi_1(x') & \xrightarrow{\varphi'} & \pi_2(y') \end{array}$$

commute. Define $p_1 : \mathcal{G} \to \mathcal{F}_1$ by $(x, y, \varphi) \mapsto x$ and the obvious projection on morphisms and $p_2 : \mathcal{G} \to \mathcal{F}_2$ similarly by $(x, y, \varphi) \mapsto y$. A natural isomorphism $\sigma : \pi_1 \circ p_1 \to \pi_2 \circ p_2$ is induced by the isomorphisms φ. One can check that $\mathcal{G}$ is a category over $\mathcal{C}$; it is clear that $\mathcal{G}(X)$ is a groupoid for each $X \in \mathcal{C}$.

(i) Suppose $\mathcal{H} \to \mathcal{C}$ is a category fibred in groupoids and (ξ_1, ξ_2, γ) is an object in the groupoid $\mathrm{Hom}_{\mathcal{C}}(\mathcal{H}, \mathcal{F}_1) \times_{\mathrm{Hom}_{\mathcal{C}}(\mathcal{H}, \mathcal{F})} \mathrm{Hom}_{\mathcal{C}}(\mathcal{H}, \mathcal{F}_2)$. This induces a morphism $\eta \in \mathrm{Hom}_{\mathcal{C}}(\mathcal{H}, \mathcal{G})$ defined on objects by $\eta(h) = (\xi_1(h), \xi_2(h), \gamma_h)$ where γ_h is the isomorphism $\pi_1 \circ \xi_1(h) \to \pi_2 \circ \xi_2(h)$ specified by γ. The definition of η on functors is similar and under the morphism

$$\mathrm{Hom}_{\mathcal{C}}(\mathcal{H},\mathcal{G}) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(\mathcal{H},\mathcal{F}_1) \times_{\mathrm{Hom}_{\mathcal{C}}(\mathcal{H},\mathcal{F})} \mathrm{Hom}_{\mathcal{C}}(\mathcal{H},\mathcal{F}_2),$$

it is clear that η goes to (ξ_1, ξ_2, γ). On the other hand, if $\eta \in \mathrm{Hom}_{\mathcal{C}}(\mathcal{H},\mathcal{G})$ then η is precisely the image under the above morphism of the triple $(p_1 \circ \eta, p_2 \circ \eta, \sigma \circ \eta)$. Therefore

$$\mathrm{Hom}_{\mathcal{C}}(\mathcal{H},\mathcal{G}) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(\mathcal{H},\mathcal{F}_1) \times_{\mathrm{Hom}_{\mathcal{C}}(\mathcal{H},\mathcal{F})} \mathrm{Hom}_{\mathcal{C}}(\mathcal{H},\mathcal{F}_2)$$

is an isomorphism of groupoids.

(ii) Now suppose $q_i : \mathcal{G}' \to \mathcal{F}_i, i = 1, 2$, are morphisms and $\tau : \pi_1 \circ q_1 \simeq \pi_2 \circ q_2$ a natural isomorphism such that

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}}(\mathcal{H},\mathcal{G}') &\longrightarrow \mathrm{Hom}_{\mathcal{C}}(\mathcal{H},\mathcal{F}_1) \times_{\mathrm{Hom}_{\mathcal{C}}(\mathcal{H},\mathcal{F})} \mathrm{Hom}_{\mathcal{C}}(\mathcal{H},\mathcal{F}_2) \\ \eta &\longmapsto (q_1 \circ \eta, q_2 \circ \eta, \tau \circ \eta) \end{aligned}$$

is an isomorphism of groupoids for any $\mathcal{H}$. Then for $\mathcal{H} = \mathcal{G} = \mathcal{F}_1 \times_{\mathcal{F}} \mathcal{F}_2$, we have an isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(\mathcal{G},\mathcal{G}') \longrightarrow \mathrm{Hom}_{\mathcal{C}}(\mathcal{G},\mathcal{F}_1) \times_{\mathrm{Hom}_{\mathcal{C}}(\mathcal{G},\mathcal{F})} \mathrm{Hom}_{\mathcal{C}}(\mathcal{G},\mathcal{F}_2).$$

Then the data (p_1, p_2, σ) on the right determines a morphism $t^{-1} : \mathcal{G} \to \mathcal{G}'$ on the left as well as natural transformations $\lambda_1^{-1} : p_1 \circ t^{-1} \to q_1$ and λ_2^{-1} :

$p_2 \circ t^{-1} \to q_2$ making the appropriate diagram of functors commute. Applying this with $\mathcal{G}$ and $\mathcal{G}'$ reversed constructs t, λ_1 and λ_2 and shows that the first is an equivalence and the second and third are natural isomorphisms. This completes the proof. □

Remark 3.2.29. When $\mathcal{H} = \mathcal{C}/X$ is the slice category over an object $X \in \mathcal{C}$, the isomorphism in (i) will be written $(\mathcal{F}_1 \times_{\mathcal{F}} \mathcal{F}_2)(X) \cong \mathcal{F}_1(X) \times_{\mathcal{F}(X)} \mathcal{F}_2(X)$.

3.3 Descent

The notion of descent can be phrased in a quite general context. Fix an object S in a category $\mathcal{C}$ with fibre products and recall that the *localized* or *slice category* at S is the category $\mathcal{C}/S$ whose objects are arrows $X \to S$ in $\mathcal{C}$ and whose morphisms are commutative triangles

$$\begin{array}{ccc} X & \longrightarrow & Y \\ & \searrow \quad \swarrow & \\ & S & \end{array}$$

Given any other object $S' \in \mathcal{C}$ and a morphism $S' \to S$, there is a *base change functor* $\mathcal{C}/S \to \mathcal{C}/S'$ given by $(X \to S) \mapsto (X' \to S')$ where X' is the fibre product

$$\begin{array}{ccc} X' = X \times_S S' & \longrightarrow & X \\ \big\downarrow & & \big\downarrow \\ S' & \longrightarrow & S \end{array}$$

This shows that an object over S always determines an object over S'. The opposite question, namely when an object over S' determines an object over S, is much harder to answer in general. In fact there are two interesting questions one might ask: given a morphism $S' \to S$ and an arrow $(X' \to S') \in \mathcal{C}/S'$, (1) is there an arrow $(X \to S) \in \mathcal{C}/S$ such that the following diagram can be completed:

$$\begin{array}{ccc} X' & \dashrightarrow & X \\ \downarrow & & \downarrow \\ S' & \longrightarrow & S \end{array}$$

and if so, (2) how many ways are there to complete the diagram? The process of answering both of these questions is known as descent, which is a crucial ingredient in the definition of a stack.

3.3.1 Galois Descent

For an extended motivation, let K/k be a field extension and obj(k) some class of objects defined over k. For example, obj(k) could be the class of k-vector spaces, or something more specific, like central simple algebras over k. Many examples like this admit morphisms over k, and so form a category over k (in technical terms, a k-*linear category*, though we will not use that terminology here).

Example 3.3.1. A *quadratic space* over k is a k-vector space V together with a quadratic form q, that is, a symmetric bilinear function $q : V \to k$. The set of quadratic spaces (V, q) may be an interesting class of objects over k. These too admit morphisms, where $(V, q) \to (V', q')$ consists of a k-linear isomorphism $V \to V'$ which commutes with q and q'.

Example 3.3.2. Important choices of obj(k) in algebraic geometry include the class of algebraic varieties or algebraic groups over k, the class of schemes over $\operatorname{Spec} k$, and more generally things like algebraic spaces and algebraic stacks over k.

Many of these examples admit a base change functor, i.e. an assignment

$$\begin{aligned} \mathrm{obj}(k) &\longrightarrow \mathrm{obj}(K) \\ X &\longmapsto X_K. \end{aligned}$$

for a field extension K/k. Most of the time this functor can be built using the tensor product or fibre product in the right category. In the pattern of the introduction, there are two natural questions we would like to answer about descending objects defined over K to objects over k:

(1) Given an object $A \in \mathrm{obj}(K)$, does there exist an object $X \in \mathrm{obj}(k)$ such that $X_K = A$?

(2) What are all the possible objects $X \in \mathrm{obj}(k)$ with $X_K = A$ or $X_K \cong A$ in $\mathrm{obj}(K)$?

When K/k is a Galois extension, this is known as *Galois descent*.

Definition 3.3.3. *An object* $X \in \mathrm{obj}(k)$ *with* $X_K \cong A$ *in* $\mathrm{obj}(K)$ *is called a* K/k**-form** *of* A.

Example 3.3.4. In some situations, descent is trivial. For example, if V_K is a K-vector space with basis $\{x_i\}$ then the same basis gives a k-vector space V_k for which $V_k \otimes_k K = V_K$.

Example 3.3.5. In other situations, descent is impossible. Let $k = \mathbb{Q}$, $K = \mathbb{Q}(\sqrt{2})$ and consider the quadratic form $q_K(x, y) = x^2 - y^2\sqrt{2}$ on $V = K^2$. Is there a quadratic form $q_{\mathbb{Q}}$ on $V_{\mathbb{Q}} = \mathbb{Q}^2$ which extends to q_K on $V_{\mathbb{Q}} \otimes_{\mathbb{Q}} K = V$? The answer in this case is no – there are some elementary conditions that are necessary for descent to be possible, and they are not satisfied in this situation.

Let K/k be a Galois extension with Galois group $\mathcal{G} = \mathrm{Gal}(K/k)$ and suppose V_k and W_k are objects over k (e.g. k-algebras) that are isomorphic over K, that is, $V_K = V_k \times_k K \cong W_k \times_k K = W_K$. Moreover, assume that under the natural Galois action on V_K and W_K, we have $V_k = V_K^{\mathcal{G}}$ and $W_k = W_K^{\mathcal{G}}$ (this is true e.g. for k-algebras). When there is a notion of maps between objects over k (e.g. k-linear algebra homomorphisms), the $\mathcal{G}$-action extends to $\mathrm{Map}_K(V_K, W_K)$ by $(\sigma f)(v) = \sigma(f(\sigma^{-1}v))$ for all $v \in V_K, \sigma \in \mathcal{G}$.

For example, let f be an equivalence $V_K \xrightarrow{f} W_K$, e.g. a K-algebra isomorphism. Set $\xi(\sigma) = f^{-1} \circ \sigma f \in G_K := \mathrm{Aut}_K(V_K)$. Notice that if $\xi(\sigma) = 1$ for all $\sigma \in \mathcal{G}$, then $\sigma f = f\sigma$ so f descends to an isomorphism $f_k : V_k = V_K^{\mathcal{G}} \xrightarrow{\sim} W_K^{\mathcal{G}} = W_k$. That is, $\xi = 1$ is a necessary condition for two (K/k)-forms of V_K to be isomorphic over k. In general, the automorphism $\xi(\sigma) = f^{-1} \cdot \sigma f$ satisfies

$$\xi(\sigma\tau) = f^{-1} \cdot (\sigma\tau)f = (f^{-1} \cdot \sigma f)((\sigma f)^{-1}\sigma\tau f) = \xi(\sigma)\sigma(\xi(\tau)).$$

Such a map $\xi : \mathcal{G} \to G_K$ is called a 1-*cocycle*, and the set of all 1-cocycle is denoted $Z^1(\mathcal{G}, G_K)$. If K/k is an infinite extension, we are guaranteed to have $f(V_k) \subseteq W_k \otimes_k \ell \subseteq W_k \otimes_k K = W_K$ for some finite extension ℓ/k. If $\sigma \in \mathcal{H} := \mathrm{Aut}(K/\ell)$, then for all $v \in V_k$,

$$(\sigma f)(v) = \sigma(f(\sigma^{-1}v)) = \sigma\sigma^{-1}f(v) = f(v)$$

since $f(v) \in W_k \otimes_k \ell$. Thus $\sigma f = f$, so $\xi(\sigma) = 1$ for all $\sigma \in \mathcal{H}$. Said another way, the 1-cocycle ξ is constant on cosets of $\mathcal{H}$, which means ξ is in fact a *continuous* 1-*cocycle* on

$$\mathcal{G} = \varinjlim \mathrm{Aut}(\ell/k)$$

over all finite extensions ℓ/k, when this direct limit is equipped with the Krull topology induced by discrete topologies on each $\mathrm{Aut}(\ell/k)$.

Turning back to our K-isomorphism $f : V_K \xrightarrow{\sim} W_K$, the definition of ξ may depend on this f, but suppose f' were a different isomorphism $V_K \xrightarrow{\sim} W_K$. Then $f' = f \circ g$ for some $g \in G_K$ and the cocycle ξ' defined by f' has the form

$$\xi'(\sigma) = (f')^{-1} \cdot \sigma f' = (f \circ g)^{-1} \cdot \sigma(f \circ g) = g^{-1}f^{-1} \cdot (\sigma f)(\sigma g) = g^{-1}\xi(\sigma)\sigma g.$$

Definition 3.3.6. *Two 1-cocycles $\xi, \xi' : \mathcal{G} \to G_K$ are* **equivalent cocycles** *if there* *is some* $g \in G_K$ *such that* $\xi'(\sigma) = g^{-1}\xi(\sigma)\sigma g$ *for all* $\sigma \in \mathcal{G}$. *The set of equivalence classes of* 1*-cocycles is denoted* $H^1(\mathcal{G}, G_K)$.

Fix an object V_K over K, set $G_K = \mathrm{Aut}_K(V_K)$ and let $F(K/k, V_K)$ be the set of k-isomorphism classes of (K/k)-forms of V_k. Then our work above shows that there is a map

$$\Theta : F(K/k, V_K) \longrightarrow H^1(\mathcal{G}, G_K).$$

The theory of Galois descent comes down to deciding when Θ is a bijection. For example, when V_k is a k-algebra, Θ is a bijection.

For another example, suppose G is a linear algebraic group over k on which $\mathcal{G}$ acts continuously. Then

$$H^1(\mathcal{G}, G) = \varinjlim H^1(\mathcal{G}/\mathcal{U}, G(K^{\mathcal{U}}))$$

where the limit is over all finite index open subgroups $\mathcal{U} \subseteq \mathcal{G}$ and $K^{\mathcal{U}}$ is the subfield of K fixed by $\mathcal{U}$. Hilbert's Theorem 90 (Theorems 1.4.2 and 1.4.7) says that $H^i(\mathcal{G}, K) = 0$ for all $i \geqslant 1$ and $H^1(\mathcal{G}, K^\times) = 1$. In this sense, Kummer theory and Artin–Schreier theory can be viewed as specific examples of Galois descent for $\mathcal{G} = \mu_n$ and $\mathbb{Z}/p\mathbb{Z}$, respectively.

Here is a general strategy for descent: define a new Galois action on the object V_K over K and take W_k to be the fixed points of V_K under this action.

3.3.2 Faithfully Flat Descent

Switching to a geometric context, we can find basic examples of descent in the theory of quasicoherent sheaves on a scheme X. Fix a Zariski open cover $\{U_i\}_{i\in I}$ of X. Then it is a standard fact that a quasicoherent sheaf F on X is equivalent to the data of a collection of quasicoherent sheaves $(F_i)_{i\in I}$ on each U_i and isomorphisms $(\sigma_{ij})_{i,j\in I}$, $\sigma_{ij} : F_i|_{U_i\cap U_j} \to F_j|U_i \cap U_j$ such that for all $i, j, k \in I$, $\sigma_{ii} = id_{F_i}$ and $\sigma_{ik} = \sigma_{jk} \circ \sigma_{ij}$ over $U_i \cap U_j \cap U_k$. This extends to an equivalence of suitable categories.

Replacing the Zariski topology with a more general topology results in more interesting categories of sheaves. In this section, we outline so-called *faithfully flat descent* for sheaves on X in the fppf topology. Let X be a scheme. Our goal is to show that the functor $h_X = \text{Hom}(-, X)$ is a sheaf in the fppf topology on Sch. More generally, this will imply that for any base scheme S and S-scheme $X \to S$, h_X is a sheaf on S_{fppf}.

First, let $f : A \to B$ be a ring homomorphism and M an A-module. Set $M_B = M \otimes_A B$ and let f also denote the map $M \to M_B$.

Proposition 3.3.7. *If $f : A \to B$ is faithfully flat, then for any A-module M, the sequence*

$$M \xrightarrow{f} M_B \rightrightarrows M_{B\otimes_A B}$$

is exact, where the two maps $M_B \to M_{B\otimes_A B} = M \otimes_A (B \otimes_A B)$ are induced by $B \to B \otimes_A B, b \mapsto b \otimes 1$ and $b \mapsto 1 \otimes b$.

Corollary 3.3.8. *Let $\varphi : V \to U$ be a faithfully flat morphism of affine schemes. Then* *for any affine scheme* X*, the sequence of sheaves*

$$h_X(U) \xrightarrow{\varphi^*} h_X(V) \rightrightarrows h_X(V \times_U V)$$

is exact.

Lemma 3.3.9. *Let* $F : \mathsf{Sch}^{op} \to \mathsf{Set}$ *be a sheaf in the Zariski topology. Then* F *is a sheaf in the fppf topology if and only if for every faithfully flat morphism* $Y \to X$ *of schemes which is locally of finite presentation, the sequence*

$$F(X) \longrightarrow F(Y) \rightrightarrows F(Y \times_X Y)$$

is exact.

Proof. ($\Longrightarrow$) is immediate.

($\Longleftarrow$) We must show that for a scheme X and a cover $\{X_i \to X\} \in \mathrm{Cov}(X)$,

$$F(X) \longrightarrow \prod_i F(X_i) \rightrightarrows \prod_{i,j} F(X_i \times_X X_j)$$

is exact. Set $Y = \coprod X_i$ with the canonical projection $Y \to X$, which is faithfully flat, and consider the diagram

$$\begin{array}{ccccc} F(X) & \longrightarrow & F(Y) & \rightrightarrows & F(Y \times_X Y) \\ \downarrow {\scriptstyle =} & & \downarrow {\scriptstyle \cong} & & \downarrow {\scriptstyle \cong} \\ F(X) & \longrightarrow & \prod_i F(X_i) & \rightrightarrows & \prod_{i,j} F(X_i \times_X X_j) \end{array}$$

Since F is a sheaf in the Zariski topology, the vertical maps are isomorphisms, and by hypothesis, the top row is exact. It follows that the bottom row is exact.

□

Lemma 3.3.10. *Suppose* F *is a sheaf in the Zariski topology on* Sch *and for all fppf morphisms of affine schemes* $V \to U$*, the sequence*

$$F(U) \to F(V) \to F(V \times_U V)$$

is exact. Then F *is a sheaf for the fppf topology.*

Theorem 3.3.11. *For any scheme* X*, the presheaf* h_X *is a sheaf in the fppf topology.*

Corollary 3.3.12. *For any scheme* X*,* h_X *is a sheaf in the smooth and étale topologies.*

3.3.3 Effective Descent Data

Let $\mathcal{F}$ be a fibred category over a site $\mathcal{C}$. For each morphism $Y \to X$ in $\mathcal{C}$, we define a category $\mathcal{F}(Y \to X)$ whose objects are pairs (E, σ) where $E \in \mathcal{F}(Y)$ and $\sigma : E|_{Y \times_X Y} \to E|_{Y \times_X Y}$ is an isomorphism between the two different pullbacks of E along the two projections $Y \times_X Y \to Y$, which agrees on each map from the triple product $Y \times_X Y \times_X Y \to Y \times_X Y$. A morphism $(E', \sigma') \to (E, \sigma)$ in $\mathcal{F}(Y \to X)$ is taken to be a morphism $g : E' \to E$ in $\mathcal{F}(Y)$ such that $\sigma g|_{Y \times_X Y} = g|_{Y \times_X Y} \sigma' : E'|_{Y \times_X Y} \to E|_{Y \times_X Y}$. Call σ the *descent data* for E.

More generally, for a collection of morphisms $\{Y_i \to X\}$ in $\mathcal{C}$, set $Y_{ij} = Y_i \times_X Y_j, Y_{ijk} = Y_i \times_X Y_j \times_X Y_k$, etc. Then one can define the category $\mathcal{F}(\{Y_i \to X\})$

to have objects $(\{E_i\},\{\sigma_{ij}\})$ with $E_i \in \mathcal{F}(Y_i)$ and isomorphisms $\sigma_{ij} : E_i|_{Y_{ij}} \to E_j|_{Y_{ij}}$ such that $\sigma_{ik} = \sigma_{jk}\sigma_{ij}$ over Y_{ijk}. Morphisms $(\{E_i'\},\{\sigma_{ij}'\}) \to (\{E_i\},\{\sigma_{ij}\})$ in $\mathcal{F}(\{Y_i \to X\})$ are taken to be $g_i : E_i' \to E_i$ in $\mathcal{F}(Y_i)$ such that $\sigma_{ij}g = g\sigma_{ji}$ over Y_{ij}. The isomorphisms $\{\sigma_{ij}\}$ are called the *descent data* for $\{E_i\}$.

Next, define a functor $\Phi : \mathcal{F}(X) \to \mathcal{F}(\{Y_i \to X\})$ by sending an object $E \in \mathcal{F}(X)$ to $\Phi(E) = (\{E_i\},\{\sigma_{ij}\})$ where $E_i = E|_{Y_i}$ and σ_{ij} is the canonical isomorphism $\left(E|_{Y_i}\right)|_{Y_{ij}} \tilde{\to} \left(E|_{Y_j}\right)|_{Y_{ij}}$. For a morphism $E' \to E$ in $\mathcal{F}(X)$, $\Phi(E' \to E) = \{g_i : E_i' \to E_i\}$ is defined via the universal property of pullbacks.

Definition 3.3.13. *Any collection of morphisms* $\{Y_i \to X\}$ *has* **effective descent** *for the fibred category* $\mathcal{F}$ *if* Φ *is an equivalence of categories. An* **effective descent data** *is a descent data* $\{\sigma_{ij}\}$ *for* $\{E_i\}$ *such that* $(\{E_i\},\{\sigma_{ij}\}) \cong \Phi(E)$ *for some* $E \in \mathcal{F}(X)$.

In the case of a single morphism $Y \to X$, we often say the morphism has effective descent (for $\mathcal{F}$) if $\mathcal{F}(X) \to \mathcal{F}(Y \to X)$ is an equivalence of categories.

Example 3.3.14. If $f : Y \to X$ admits a section $s : X \to Y$, then f has effective descent. Indeed, a quasi-inverse to $\Phi : \mathcal{F}(X) \to \mathcal{F}(Y \to X)$ is given by $(E, \sigma) \mapsto s^*E$.

Let $\mathcal{C}$ be a site with representable finite limits and define a category Sh with objects given by pairs (X, E) where $X \in \mathcal{C}$ and $E \in \mathcal{C}/X$ (the slice category; see Example 3.1.5), and with morphisms $(X, E) \to (Y, F)$ corresponding to a morphism $f : X \to Y$ in $\mathcal{C}$ and a morphism $\eta : E \to f^*F$ in $\mathcal{C}/X$. Then Sh is a fibred category over $\mathcal{C}$ via the projection $(X, E) \mapsto X$. A generalization of the standard gluing lemma for sheaves on a topological space proves the following.

Proposition 3.3.15. *Let $\mathcal{C}$ be a site with representable finite limits. Then any covering* $(Y \to X) \in \mathcal{C}$ *is an effective descent morphism for the fibred category* Sh.

Now let $\mathcal{C} = \mathsf{Sch}$ be the category of schemes equipped with the fppf topology. Define QCoh to be the category whose objects are pairs (T, E) where T is a scheme and E is a quasicoherent sheaf on T with the Zariski topology. Morphisms $(T', E') \to (T, E)$ in QCoh are morphisms of schemes $f : T' \to T$ and quasicoherent sheaves $E' \to f^*E$ on T'. Then $(T, E) \mapsto T$ makes QCoh into a fibred category over Sch.

Theorem 3.3.16. *If* $f : Y \to X$ *is a cover in the fppf (resp. smooth, étale) topology on* Sch, *then* f *has effective descent for* QCoh.

Example 3.3.17. Let Open be the fibred category over Sch consisting of pairs (X, U) where X is a scheme and $U \subseteq X$ is an open subscheme. Then any fppf (resp. smooth, étale) cover $f : Y \to X$ has effective descent for Open.

Example 3.3.18. Let Aff be the fibred category of affine morphisms of schemes $Y \to X$ over Sch. Then any fppf/smooth/étale cover has effective descent for Aff. Combining this with the previous example, one can also show effective descent for the fibred category of quasi-affine morphisms.

3.3.4 Stacks

Definition 3.3.19. *Let $\mathcal{C}$ be a site. A* **stack** *over $\mathcal{C}$ is a category fibred in groupoids* $\mathcal{F} \to \mathcal{C}$ *such that for every object* $X \in \mathcal{C}$ *and any cover* $\{X_i \to X\} \in \mathrm{Cov}(X)$, *the functor* $\mathcal{F}(X) \to \mathcal{F}(\{X_i \to X\})$ *is an equivalence of categories.*

Let $X \in \mathcal{C}$ and for any $x, y \in \mathcal{F}(X)$, recall the presheaf $\mathrm{Isom}(x, y)$ on $\mathcal{C}/X$ defined in Example 3.2.25.

Proposition 3.3.20. *A category fibred in groupoids $\mathcal{F} \to \mathcal{C}$ is a stack if and only if*

(1) For any object $X \in \mathcal{C}$ and any $x, y \in \mathcal{F}(X)$, the presheaf $\mathrm{Isom}(x, y)$ is a sheaf on $\mathcal{C}/X$ with respect to the induced Grothendieck topology.

(2) For any covering $\{X_i \to X\} \in \mathrm{Cov}(X)$, every descent data $\{\sigma_{ij}\}$ for $\{E_i \in \mathcal{F}(X_i)\}$ is effective for $\mathcal{F}$.

Proof. Suppose $\mathcal{F}$ is a stack over $\mathcal{C}$ and take a covering $\{X_i \to X\}$ in $\mathcal{C}$. Since $\mathcal{F}(X) \to \mathcal{F}(\{X_i \to X\})$ is an equivalence of categories, it is immediate that any descent data is effective. Next, take an object $(Y \xrightarrow{\varphi} X) \in \mathcal{C}/X$ and a covering $\{Y_i \to Y\}$ in $\mathcal{C}/X$. By definition,

$$\mathrm{Isom}(x, y)(Y_i \times_Y Y_j) \cong \mathrm{Hom}_{\mathcal{F}(Y_i \times_Y Y_j)}(\varphi^* x|_{Y_i}, \varphi^* y|_{Y_j})$$

so the sequence

$$\mathrm{Isom}(x, y)(Y) \longrightarrow \prod \mathrm{Isom}(x, y)(Y_i) \rightrightarrows \prod \mathrm{Isom}(x, y)(Y_i \times_Y Y_j)$$

is isomorphic to

$$\mathrm{Hom}_{\mathcal{F}(Y)}(\varphi^* x, \varphi^* y) \longrightarrow \prod \mathrm{Hom}_{\mathcal{F}(Y_i)}(\varphi^* x|_{Y_i}, \varphi^* y|_{Y_i}) \rightrightarrows \prod \mathrm{Hom}_{\mathcal{F}(Y_{ij})}(\varphi^* x|_{Y_{ij}}, \varphi^* y|_{Y_{ij}}).$$

This latter sequence being exact is precisely equivalent to $\mathcal{F}(Y) \to \mathcal{F}(\{Y_i \to Y\})$

being fully faithful.

Conversely, suppose (1) and (2) hold. Note that (2) says that $\Phi : \mathcal{F}(X) \to \mathcal{F}(\{X_i \to X\})$ is essentially surjective. On the other hand, as noted above, (1) is equivalent to Φ being fully faithful, so Φ is an equivalence. □

Remark 3.3.21. A category fibred in groupoids is called a *prestack* if condition (1) holds.

Definition 3.3.22. *A* **morphism of stacks** $\mathcal{F} \to \mathcal{G}$ *over* $\mathcal{C}$ *is a morphism of the underlying categories fibred in groupoids, i.e. a natural transformation* $T : \mathcal{F} \to \mathcal{G}$ *over* $\mathcal{C}$ *taking cartesian morphisms to cartesian morphisms.*

If $\mathcal{F}_1, \mathcal{F}_2$ and $\mathcal{F}$ are categories fibred in groupoids over $\mathcal{C}$ and $\mathcal{F}_1 \to \mathcal{F}$ and $\mathcal{F}_2 \to \mathcal{F}$ are morphisms of categories fibred in groupoids, let $\mathcal{F}_1 \times_{\mathcal{F}} \mathcal{F}_2$ denote the category fibred in groupoids constructed in Proposition 3.2.28.

Proposition 3.3.23. *If* $\mathcal{F}_1, \mathcal{F}_2$ *and* $\mathcal{F}$ *are stacks over* $\mathcal{C}$ *and* $\mathcal{F}_1 \to \mathcal{F}$ *and* $\mathcal{F}_2 \to \mathcal{F}$ *are morphisms of stacks, then the fibre product* $\mathcal{F}_1 \times_{\mathcal{F}} \mathcal{F}_2$ *is a stack.*

Proof. For any object $X \in \mathcal{C}$, Proposition 3.2.28(i) says there is an equivalence of groupoids

$$(\mathcal{F}_1 \times_{\mathcal{F}} \mathcal{F}_2)(X) \xrightarrow{\sim} \mathcal{F}_1(X) \times_{\mathcal{F}(X)} \mathcal{F}_2(X).$$

Likewise, for any covering $\{X_i \to X\}$ in $\mathcal{C}$, there is an equivalence of groupoids

$$(\mathcal{F}_1 \times_{\mathcal{F}} \mathcal{F}_2)(\{X_i \to X\}) \xrightarrow{\sim} \mathcal{F}_1(\{X_i \to X\}) \times_{\mathcal{F}(\{X_i \to X\})} \mathcal{F}_2(\{X_i \to X\}).$$

This implies the proposition. □

Proposition 3.3.20 suggests that stacks are a generalization of sheaves; more colloquially, stacks are "sheaves valued in categories". Just as with presheaves and the sheafification construction, there is notion of "stackification" of any category fibred in groupoids. This is described in the next theorem.

Theorem 3.3.24. *For any category fibred in groupoids $\mathcal{F} \to \mathcal{C}$, there exists a stack $\mathcal{F}^{st} \to \mathcal{C}$ and a morphism of fibred categories $\mathcal{F} \to \mathcal{F}^{st}$ such that for any stack $\mathcal{G}$ over $\mathcal{C}$, the functor*

$$\mathrm{Hom}_{\mathcal{C}}(\mathcal{F}^{st}, \mathcal{G}) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(\mathcal{F}, \mathcal{G})$$

is an equivalence of categories.

Corollary 3.3.25. *The stack $\mathcal{F}^{st}$ is unique up to a canonical equivalence categories.*

Proposition 3.3.26. *A morphism $f : \mathcal{F} \to \mathcal{G}$ of stacks over $\mathcal{C}$ is an isomorphism of stacks if and only if for every $X \in \mathcal{C}$, the functor $f_X : \mathcal{F}(X) \to \mathcal{G}(X)$ is an equivalence of categories.*

Definition 3.3.27. *A* **substack** *of a stack $\mathcal{F} \to \mathcal{C}$ is a subcategory fibred in groupoids $\mathcal{H} \subseteq \mathcal{F}$ which satisfies the stack condition.*

Example 3.3.28. The category fibred in groupoids $\mathsf{Sh} \to \mathcal{C}$ constructed in Section 3.3.3 is a stack over $\mathcal{C}$.

Example 3.3.29. Every scheme can be viewed as a category over Sch via its functor of points: $X \in \mathsf{Sch}$ defines a fibred category $X \to \mathsf{Sch}$ by taking $X(T) = \mathrm{Hom}(T, X)$. Morphisms in $\mathrm{Hom}(T, X)$ are just the identity morphisms $T \to T$ and the forgetful functor $(T \to X) \mapsto T$ makes X into a category fibred in groupoids over Sch. We claim that X is a stack in the Zariski or the étale topology on Sch. Note that for any $f, g \in \mathrm{Hom}(T, X)$, $\mathrm{Isom}(f, g)$ is locally constant or empty: for a scheme $U \to T$, $\mathrm{Isom}(f, g)(U) = \{\mathrm{id}_U\}$ if $f|_U = g|_U$ or $\mathrm{Isom}(f, g)(U) = \varnothing$ otherwise. In particular, $\mathrm{Isom}(f, g)$ is a sheaf (in any topology on Sch/T). Next, suppose $\{T_i \to T\}$ is a covering either in the Zariski topology or the étale topology. By Example 3.1.56, we know $X(-) = \mathrm{Hom}(-, X)$ is a sheaf in both topologies. Therefore a descent data $\delta = (\{\varphi_i : T_i \to X\}, \{\sigma_{ij} : \varphi_i|_{T_{ij}} \xrightarrow{\sim} \varphi_j|_{T_{ij}}\})$ determines a morphism $\varphi : T \to X$ by gluing the φ_i. In particular, δ is the image under $\Phi : \mathrm{Hom}(T, X) \to \mathrm{Hom}(\{T_i \to T\}, X)$ of $\varphi : T \to X$, so δ is effective. This proves $X(-) = \mathrm{Hom}(-, X)$ is a stack in the Zariski and étale topologies.

Example 3.3.30. The same argument shows that for a base scheme S and any S-scheme X, the functor of points $X(-) = \mathrm{Hom}_S(-, X)$ is a stack in the Zariski and étale topologies on Sch_S.

This shows that the 2-Yoneda embedding $S_{\text{ét}} \hookrightarrow \mathsf{CFG}(S_{\text{ét}})$ from Corollary 3.2.21 actually lands in stacks: $S_{\text{ét}} \hookrightarrow \mathsf{Stack}(S_{\text{ét}})$.

Definition 3.3.31. *A stack $\mathcal{F}$ is* **representable** *(by a scheme) if there exists an isomorphism of stacks $\mathcal{F} \xrightarrow{\sim} h_X = \mathrm{Hom}(-, X)$ for some scheme X, that is, if $\mathcal{F}$ is in the essential image of the 2-Yoneda embedding.*

Example 3.3.32. For $\mathcal{C} = \mathsf{Sch}$ the category of schemes in the fppf/smooth/étale topology, the subcategory $\mathsf{QCoh} \subseteq \mathsf{Sh}$ constructed in Section 3.3.3 is a substack. Likewise, there are substacks of Sh corresponding to some of the usual notions for $\mathcal{O}_X$-modules, e.g. locally free sheaves of finite rank and invertible sheaves (the latter is sometimes called the *Picard stack* of $\mathcal{C}$). One way of saying all of these things is that the moduli problem of quasicoherent sheaves/locally free sheaves of finite rank/invertible sheaves is *representable by a stack*. We will see that many such moduli problems are represented by stacks, and ultimately algebraic stacks, in Section 3.5.

Example 3.3.33. The moduli problem $\mathscr{M}_{1,1}$ from Examples 3.2.7 corresponds to a category fibred in groupoids by 3.2.23, which is a stack in the fppf (resp. smooth, étale) topology on Sch.

Example 3.3.34. Likewise, the moduli problem $\mathscr{M}_g$ from Example 3.2.8 defines a category fibred in groupoids over Sch (Example 3.2.24) which is a stack in the fppf/smooth/étale topologies.

Example 3.3.35. Let X be a scheme, G a group (or an algebraic group) acting on X and for every scheme T, set $G_T := G \times T$. Let $\{X/G\}(T)$ be the groupoid whose objects are pairs (P, g) where P is a G_T-torsor over T (as in Section 2.1.4) and $g : P \to X \times T$ is a G_T-equivariant morphism. A morphism $(P', g') \to (P, g)$ in $\{X/G\}(T)$ is a pair of compatible morphisms $\varphi : T' \to T$ (of schemes) and $P' \to \varphi^*P$ (of $G_{T'}$-torsors). Varying T, this defines a category fibred in groupoids $\{X/G\}$ over Sch, though in general, this need not be a stack in the various topologies we put on Sch. Denote its stackification $\{X/G\}^{st}$ by $[X/G]$, called the *quotient stack* of X by G. If X/G exists in the category of schemes, then the natural morphism $[X/G] \to X/G$ sending $P \mapsto P/G$ is a morphism of stacks.

Definition 3.3.36. *A* **representable morphism** *of stacks is a morphism* $\mathcal{F} \to \mathcal{G}$ *such that for all morphisms of stacks* $X \to \mathcal{G}$ *where* X *is a scheme, the product* $\mathcal{F} \times_{\mathcal{G}} X$ *is a scheme.*

Example 3.3.37. All of $\mathsf{Sh}, \mathsf{QCoh}, \mathsf{LFS}, \mathsf{Pic}, \mathscr{M}_{1,1}$ and $\mathscr{M}_g$ are representable by stacks, but not by schemes.

Lemma 3.3.38. *Let* X *and* Y *be schemes, viewed as stacks via the* 2*-Yoneda embedding. Then any morphism of stacks* $f : X \to Y$ *is representable.*

Proof. The 2-Yoneda lemma says that $\mathrm{Hom}_{\mathsf{Sch}}(X, Y) \to \mathrm{Hom}_{\mathsf{Stack}}(X, Y)$ is an isomorphism, so f corresponds to a unique morphism $\bar{f} : X \to Y$ in the category of schemes. Likewise, for any scheme X' and morphism of stacks $X' \to X$, there is a corresponding morphism in the category of schemes. Then taking the fibre product $X \times_Y X'$ with respect to the two morphisms of schemes of course yields a scheme (by Theorem 2.1.23 if you like). Hence f is representable. □

This proof seems a bit silly: the 2-Yoneda lemma ensures that any morphism of stacks between two schemes automatically comes from a morphism of schemes. However, this perspective of representability will be useful when we define algebraic spaces and algebraic stacks in the next two sections. In a word, we want to identify those functors (sheaves or categories fibred in groupoids) which are locally representable by schemes in the étale topology, much as schemes themselves are exactly the functors that can be covered by affine schemes in the Zariski topology.

3.4 Algebraic Spaces

Classically, a scheme is defined to be a locally ringed space that can be covered by affine schemes, which are themselves built from commutative rings. Any scheme X in turn defines a presheaf (its functor of points)

$$h_X : \mathsf{AffSch}^{op} \longrightarrow \mathsf{Set}, \quad T \mapsto h_X(T) := \mathrm{Hom}(T, X)$$

and the Yoneda Lemma says that the functor $\mathsf{Sch} \to \mathsf{Presh}_{\mathsf{AffSch}}, X \mapsto h_X$ is fully faithful. Further, Example 3.1.56 shows h_X is a sheaf in the Zariski topology on Sch, so we may view Sch as a full subcategory of $\mathsf{Sh}_{\mathsf{AffSch}}$.

Amazingly, it is also possible to define schemes intrinsically, using only the notion of sheaves on $\mathsf{AffSch} = \mathsf{CommRing}^{op}$. The following proposition characterizes the subcategory of $\mathsf{Sh}_{\mathsf{AffSch}}$ corresponding to the sheaves represented by schemes.

Proposition 3.4.1. *A presheaf* $F : \mathsf{AffSch}^{op} \to \mathsf{Set}$ *is represented by a scheme if and only if:*

(1) F *is a sheaf in the Zariski topology on* AffSch*.*

(2) The diagonal $\Delta_F : F \to F \times F$ *is representable by separated schemes, i.e. schemes whose diagonals are closed immersions.*

(3) F *can be "covered by affine schemes", i.e. there is a collection of objects* $\{X_i\}$ *in* AffSch *and open embeddings* $\{h_{X_i} \to F\}$ *such that the map of Zariski sheaves* $\coprod h_{X_i} F$ *is surjective.*

Notice that this alternative definition of schemes is not entirely intrinsic: one first needs the notion of separated scheme. However, this too can be defined intrinsically, resulting in a sequence of full embeddings

$$\mathsf{CommRings}^{op} = \mathsf{AffSch} \hookrightarrow \mathsf{Sch}^{sep} \hookrightarrow \mathsf{Sch} \hookrightarrow \mathsf{Sh}_{\mathsf{AffSch}} \subseteq \mathsf{Presh}_{\mathsf{AffSch}}.$$

Remark 3.4.2. The previous discussion can be modified to intrinsically construct the category of S-schemes Sch_S for any scheme S, specializing to the above case when $S = \operatorname{Spec} \mathbb{Z}$. We now pass to the relative perspective for the remainder of the chapter, since it will be essential in our later discussions of algebraic stacks and stacky curves.

Let S be a scheme. In Example 3.1.56, we showed that a representable presheaf h_X is a sheaf in the étale topology on Sch_S. The natural question then is: which sheaves on $S_{\text{ét}}$ can be covered by representable sheaves h_X?

$$\mathsf{AffSch} \hookrightarrow \mathsf{Sch} \hookrightarrow ??? \hookrightarrow \mathsf{Sh}_{S_{\text{ét}}} \subseteq \mathsf{Presh}_{S_{\text{ét}}}.$$

3.4.1 Definitions and Properties

Let S be a scheme. As in Section 3.3.4, we say a morphism $F \to G$ of presheaves on Sch_S is representable (by schemes) if for all morphisms of presheaves $h_Z \to G$, where Z is an S-scheme, the fibre product $F \times_G h_Z$ is representable.

Lemma 3.4.3. *Suppose F is a sheaf in the étale topology on Sch_S such that $\Delta_F : F \to F \times_S F$ is representable by schemes. Then for any S-scheme Z, every morphism $h_Z \to F$*

is representable.

As a consequence, it makes sense to talk about certain properties of morphisms between schemes also in the context of sheaves on $S_{ét}$, e.g. proper, separated, surjective, smooth, unramified, étale.

Definition 3.4.4. *An* **algebraic space** *over* S *is a presheaf* $X : \mathsf{Sch}_S^{op} \to \mathsf{Set}$ *such that:*

(1) X *is a sheaf in the étale topology on* Sch_S*.*

(2) The diagonal morphism $\Delta_X : X \to X \times_S X$ *is representable by schemes.*

(3) There exists a surjective étale morphism $h_U \to X$ *where* U *is an* S*-scheme.*

Lemma 3.4.5. *For every* S*-scheme* X*, the presheaf* h_X *is an algebraic space over* S*.*

Proof. The only nontrivial axiom to verify is (1), which was shown in Example 3.1.56. □

Let AS_S denote the full subcategory of $S_{ét}$ of algebraic spaces.

Remark 3.4.6. In analogy with the category of S-schemes, the category AS_S is equivalent to the category of pairs (X, φ) where X is an algebraic space (over $\operatorname{Spec} \mathbb{Z}$) and $\varphi : X \to S$ is a morphism of algebraic spaces.

Next, we describe algebraic spaces as explicit quotients of schemes, via so-called *étale equivalence relations*. As before, all of these constructions specialize to the case when $S = \operatorname{Spec} \mathbb{Z}$, giving an absolute notion of étale equivalence relation.

Definition 3.4.7. *Let* U *be an* S*-scheme. An* **étale equivalence relation** *on* U *is a subscheme* $R \subseteq U \times_S U$ *satisfying:*

(1) *For any* S*-scheme* T*, the subset* $R(T) \subseteq U(T) \times U(T)$ *defines an equivalence relation on* $U(T)$*.*

(2) *Let* $s, t : U \times_S U \to U$ *be the first and second projections, respectively. Then their restrictions* $s, t : R \rightrightarrows U$ *are étale.*

An étale equivalence relation R on U will sometimes be denoted $s, t : R \rightrightarrows U$, dropping the s and t when these are understood.

Proposition 3.4.8. *Let* U *be an* S*-scheme,* $R \subseteq U \times_S U$ *an étale equivalence relation and* U/R *the sheafification of the presheaf* $T \mapsto U(T)/R(T)$ *on* $S_{\text{ét}}$*. Then* U/R *is an algebraic space over* S *and moreover, every such algebraic space arises as the sheaf quotient* $X = U/R$ *for some scheme* U *and the étale equivalence relation* $R = U \times_X U$*.*

An equivalent way to present an algebraic space is by a surjective étale morphism $U \to X$, where U is a scheme (this is $U \to U/R$ in the statement of Proposition 3.4.8). Such a morphism $U \to X$ is called an *étale presentation* of X.

Example 3.4.9. Let G be a discrete group acting freely on an S-scheme X. The freeness assumption says that the map

$$G \times_S X \to X \times_S X, \quad (g, x) \mapsto (x, g \cdot x)$$

is injective, so $R := G \times_S X$ is an étale equivalence relation on X. By Proposition 3.4.8, the quotient sheaf X/G is an algebraic space. However, in general

X/G need not be representable by a scheme (cf. [Ols, Exs. 5.3.2–5.3.6]).

Let P be a property of morphisms of sheaves in the étale topology preserved under pullback. We say a morphism $f : Y \to X$ of algebraic spaces has property P if there exists an étale presentation $U \to X$ where U is a scheme and $U \times_X Y \to U$ has property P. Examples of such a property include: normal, proper, of relative dimension n, dominant, (closed/open) embedding, flat, smooth, unramified, étale, surjective, locally of finite type, etc.

Definition 3.4.10. *An algebraic space X over S is called* **quasi-separated** *(resp.* **locally separated**, **separated***) if the diagonal $\Delta_X : X \to X \times_S X$ is quasi-compact (resp. injective, a closed embedding).*

Proposition 3.4.11. *Let X, Y_1 and Y_2 be algebraic spaces over S with morphisms $Y_1 \to X$ and $Y_2 \to X$. Then the fibre product $Y_1 \times_X Y_2$ is an algebraic space over S.*

Example 3.4.12. For any algebraic space X, if there exists an étale morphism $X \to \operatorname{Spec} k$ with a field k, then X is a scheme and we even have $X = \coprod \operatorname{Spec} L_i$ where L_i/k are separable field extensions.

Example 3.4.13. Conversely, if X is a quasi-separated algebraic space and $\operatorname{Spec} k \to X$ is surjective, then $X = \operatorname{Spec} k'$ for a field k'.

Theorem 3.4.14. *Let X be a quasi-separated algebraic space over S. Then there exists a dense open embedding $V \hookrightarrow X$ where V is a scheme.*

3.4.2 Sheaves on Algebraic Spaces

The construction of the étale site $X_{ét}$ on a scheme X in Section 3.1.1 can be extended to any algebraic space X. Concretely, let $X_{ét}$ be the site whose objects are morphisms of algebraic spaces $Y \to X$, whose morphisms are X-morphisms and whose coverings are collections $\{Y_i \to Y\}$ of étale morphisms of algebraic spaces such that $\coprod Y_i \to Y$ is surjective.

Lemma 3.4.15. *Let X be an algebraic space and let $\mathcal{C}$ be the full subcategory of $X_{ét}$ consisting of objects $Y \to X$ where Y is a scheme. Then when $\mathcal{C}$ is equipped with the induced Grothendieck topology, there is an equivalence of categories $\mathsf{Sh}_{\mathcal{C}} \xrightarrow{\sim} \mathsf{Sh}_{X_{ét}}$.*

Proof. Let $i : \mathcal{C} \hookrightarrow X_{ét}$ be the natural inclusion of sites. By Proposition 3.1.17, there is an adjunction $i^* : \mathsf{Sh}_{\mathcal{C}} \rightleftarrows \mathsf{Sh}_{X_{ét}} : i_*$ which one can check is an equivalence in this case. □

Definition 3.4.16. *Let X be an algebraic space and let $\mathcal{C}$ be the category defined in Lemma 3.4.15. The* **structure sheaf of an algebraic space** *X is the sheaf (of rings) $\mathcal{O}_X$ on $X_{ét}$ corresponding to the sheaf on $\mathcal{C}$ defined by $(Y \to X) \mapsto \Gamma(Y_{Zar}, \mathcal{O}_Y)$ where Y is a scheme.*

Lemma 3.4.15 allows us to build sheaves on an algebraic space X using an étale presentation $U \to X$. Let $R \rightrightarrows U$ be the étale equivalence relation from Proposition 3.4.8 such that $X = U/R$.

Lemma 3.4.17. *Let X be an algebraic space with presentation $s, t : R \rightrightarrows U$ where U is a scheme. Then*

(a) *There is an equivalence of categories between* $\mathsf{Sh}_{X_{ét}}$ *and the category of pairs* (F, η) *where* F *is a sheaf on* $U_{ét}$ *and* $\eta : s^*F \to t^*F$ *is an isomorphism of sheaves on* R *such that the pullbacks of* η *along the three different morphisms* $U \times_X U \times_X U \to U \times_X U = R$ *agree (i.e.* η *satisfies a cocycle condition).*

(b) *Similarly, the category of* $\mathcal{O}_X$*-modules on* $X_{ét}$ *is equivalent to the category of pairs* (M, η) *where* M *is an* $\mathcal{O}_U$*-module on* $U_{ét}$ *and* $\eta : s^*M \to t^*F$ *is an isomorphism of* $\mathcal{O}_U$*-modules satisfying the same cocycle condition as in (a).*

Thus we can make the following definitions for sheaves on algebraic spaces.

Definition 3.4.18. *Let* X *be an algebraic space and* M *an* $\mathcal{O}_X$*-module. Then* M *is* **quasi-coherent** *if there is an étale presentation* $U \to X$ *such that the corresponding* $\mathcal{O}_U$*-module* M_U *is quasi-coherent. Further, if* X *is locally noetherian, then* M *is* **coherent** *if there exists such a* $U \to X$ *such that* M_U *is coherent.*

To ensure that these notions are independent of the choice of presentation $U \to X$, we have:

Lemma 3.4.19. *Let* M *be an* $\mathcal{O}_X$*-module. Then*

(a) M *is quasi-coherent if and only if for every étale presentation* $V \to X$*, the corresponding* $\mathcal{O}_V$*-module* M_V *is quasi-coherent.*

(b) *If* X *is locally noetherian, then* M *is coherent if and only if for every étale presentation* $V \to X$*,* M_V *is coherent.*

Proof. Suppose M is quasi-coherent (resp. coherent) and let $U \to X$ be an étale presentation such that M_U is quasi-coherent (resp. coherent). Then for any scheme V and étale map $V \to X$, M_U and M_V both pull back to the same sheaf on $V \times_X U$, which must be quasi-coherent (resp. coherent). But the base change $V \times_X U \to V$ is étale, so M_V itself must be quasi-coherent (resp. coherent). $\square$

Definition 3.4.20. *Let $f : Y \to X$ be a morphism of algebraic spaces and M an $\mathcal{O}_X$-module. Define the* **pullback** *of M to be the $\mathcal{O}_Y$-module*

$$f^*M = f^{-1}M \otimes_{f^{-1}\mathcal{O}_X} \mathcal{O}_Y.$$

Lemma 3.4.21. *If $f : Y \to X$ is a morphism of algebraic spaces and M is a quasi-coherent sheaf on X, then f^*M is a quasi-coherent sheaf on Y.*

The categories QCoh_X and Coh_X of quasi-coherent and coherent sheaves, resp., on an algebraic space X possess many of the same properties as the corresponding categories of sheaves over a scheme. In particular, their cohomology theory is well-behaved (cf. [Ols, Sec. 7.5]).

3.5 Algebraic Stacks

We have now assembled enough material to define algebraic stacks. They are, in an informal way, an interpolation of the concepts of stack and algebraic space: stacks were a generalization of moduli spaces that incorporated both groupoids and descent data, while algebraic spaces generalized the notion of a scheme to the étale topology. In this chapter we give a precise definition of an algebraic stack and explore various properties inherited from both its stack and algebraic space lineages.

3.5.1 Definitions and Properties

Let $\mathcal{C}$ be the étale site on a scheme S. From Section 3.3.4, recall that a stack over $\mathcal{C}$ is a category fibred in groupoids $\mathcal{X} \to \mathcal{C}$ such that for every S-scheme T and every cover $\{T_i \to T\}$, there is an equivalence of categories $\mathcal{X}(T) \xrightarrow{\sim} \mathcal{X}(\{T_i \to T\})$. Going forward, we will called such $\mathcal{X}$ a stack over S, sometimes writing $\mathcal{X} \to S$. Now that we are using the étale topology, we redefine the notions of representable (for stacks and for morphisms of stacks) from Section 3.3.4.

Example 3.5.1. A modification of the argument in Example 3.3.29 shows that every algebraic space X determines a stack $h_X \to S$ given by $h_X(T) = \mathrm{Hom}(T, X)$. In particular, by Lemma 3.4.5, every S-scheme X is a stack over S via its functor of points. We will abuse notation and write X for both the algebraic space (or scheme) and the stack.

Definition 3.5.2. *A stack $\mathcal{X}$ is* **representable** *if there exists an isomorphism of stacks $\mathcal{X} \xrightarrow{\sim} h_X = \mathrm{Hom}(-, X)$ for some algebraic space X.*

Definition 3.5.3. *A* **representable morphism of stacks** *is a morphism $\mathcal{Y} \to \mathcal{X}$ such that for every scheme U and morphism of stacks $U \to \mathcal{X}$, the fibre product $\mathcal{Y} \times_{\mathcal{X}} U$ is an algebraic space.*

Lemma 3.5.4. *Suppose $f : \mathcal{Y} \to \mathcal{X}$ is a representable morphism of stacks. Then for any morphism $V \to \mathcal{X}$ where V is an algebraic space, the fibre product $V \times_{\mathcal{X}} \mathcal{Y}$ is an algebraic space.*

Definition 3.5.5. *An* **algebraic stack** *is a stack $\mathcal{X}$ over S satisfying:*

(1) The diagonal $\Delta_{\mathcal{X}} : \mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is representable.

(2) There exists a smooth surjective morphism of stacks $U \to \mathcal{X}$ where U is a scheme.

Remark 3.5.6. In axiom (2), one can replace "scheme" with "algebraic space". Indeed, as we will see below, it makes sense to talk about a morphism of stacks $\mathcal{Y} \to \mathcal{X}$ being smooth (among other things) if the diagonal of $\mathcal{X}$ is representable.

Let Stack_S be the 2-category of stacks over $S_{\text{ét}}$ and let $\mathsf{AlgStack}_S$ denote the full sub-2-category of algebraic stacks in Stack_S. That is, a morphism of algebraic stacks is just a morphism of categories fibred in groupoids over $S_{\text{ét}}$. In particular, for any two algebraic stacks $\mathcal{X}$ and $\mathcal{Y}$ over S, $\mathrm{Hom}_S(\mathcal{X}, \mathcal{Y}) := \mathrm{Hom}_{\mathsf{AlgStack}_S}(\mathcal{X}, \mathcal{Y}) = \mathrm{Hom}_{\mathsf{Stack}_S}(\mathcal{X}, \mathcal{Y})$ is a category.

Lemma 3.5.7. *Let $\mathcal{X}$ be an algebraic stack and T a scheme over S. Then every morphism $T \to \mathcal{X}$ is representable.*

Proof. Suppose U is a scheme and $U \to \mathcal{X}$ is a morphism of stacks. Then $T \times_{\mathcal{X}} U$ is isomorphic to the fibre product of the diagram

$$\begin{array}{ccc} & & T \times_S U \\ & & \downarrow \\ \mathcal{X} & \xrightarrow{\Delta_{\mathcal{X}}} & \mathcal{X} \times_S \mathcal{X} \end{array}$$

Since the diagonal $\Delta_{\mathcal{X}}$ is representable, $T \times_{\mathcal{X}} U$ is isomorphic to an algebraic space. □

Proposition 3.5.8. *Let $\mathcal{X}$ be a stack over S. Then*

(1) The diagonal $\Delta_{\mathcal{X}} : \mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is representable if and only if for every algebraic space U and $u, v \in \mathcal{X}(U)$, the sheaf $\mathrm{Isom}(u, v)$ on $U_{\text{ét}}$ is an algebraic space.

(2) Suppose $\mathcal{X}$ is an algebraic stack and $X_1 \to \mathcal{X}$ and $X_2 \to \mathcal{X}$ are morphisms of stacks where X_1 and X_2 are algebraic spaces. Then the fibre product $X_1 \times_{\mathcal{X}} X_2$ is an algebraic space.

Proof. (1) If U is a scheme, this follows from the fact that the following diagram defining $\mathrm{Isom}(u, v)$ is cartesian:

$$\begin{array}{ccc} \mathrm{Isom}(u,v) & \longrightarrow & U \\ \downarrow & & \downarrow (u,v) \\ \mathcal{X} & \xrightarrow{\Delta_{\mathcal{X}}} & \mathcal{X} \times_S \mathcal{X} \end{array}$$

More generally, if U is an algebraic space then $\mathrm{Isom}(u, v)$ is an algebraic space if and only if it pulls back to an algebraic space along some (or equivalently any) étale morphism $V \to U$ where V is a scheme. Hence the first sentence applies.

(2) Let $x_1 : X_1 \to \mathcal{X}$ and $x_2 : X_2 \to \mathcal{X}$ be morphisms from algebraic spaces. Then $X_1 \times_{\mathcal{X}} X_2$ is isomorphic to $\mathrm{Isom}(p_1^* x_1, p_2^* x_2)$ where $p_i : X_1 \times_S X_2 \to X_i$, $i = 1, 2$, are the canonical projections. Note that by Proposition 3.4.11, $X_1 \times_S X_2$ is an algebraic space. Then by (1), $\mathrm{Isom}(p_1^* x_1, p_2^* x_2)$ is an algebraic space as required. □

Corollary 3.5.9. *Any morphism from an algebraic space to an algebraic stack is representable.*

Definition 3.5.10. *The* **set of points** *of a stack* $\mathcal{X}$*, denoted* $|\mathcal{X}|$*, is defined to be the set of equivalence classes of morphisms* $x : \operatorname{Spec} k \to \mathcal{X}$*, where* k *is a field, and where two points* $x : \operatorname{Spec} k \to \mathcal{X}$ *and* $x' : \operatorname{Spec} k' \to \mathcal{X}$ *are equivalent if there exists a field* $L \supseteq k, k'$ *such that the diagram*

$$\begin{array}{ccccc} & & \operatorname{Spec} k & & \\ & \nearrow & & \searrow^{x} & \\ \operatorname{Spec} L & & & & \mathcal{X} \\ & \searrow & & \nearrow_{x'} & \\ & & \operatorname{Spec} k' & & \end{array}$$

commutes. Then the **stabilizer group** *(or* **automorphism group***) of a point* $x \in |\mathcal{X}|$ *is taken to be the pullback* G_x *in the following diagram:*

$$\begin{array}{ccc} G_x & \longrightarrow & \operatorname{Spec} k \\ \downarrow & & \downarrow{\scriptstyle (x,x)} \\ \mathcal{X} & \xrightarrow{\Delta_{\mathcal{X}}} & \mathcal{X} \times_S \mathcal{X} \end{array}$$

That is, $G_x = \operatorname{Aut}(x) = \operatorname{Isom}(x, x)$.

As in Section 2.1.2, a *geometric point* is a point $\bar{x} : \operatorname{Spec} k \to \mathcal{X}$ where k is algebraically closed.

Example 3.5.11. Let X be an algebraic space over S and suppose G is a smooth group scheme acting on X. Let $[X/G]$ denote the quotient stack constructed in

Example 3.3.35. Explicitly, the objects of $[X/G]$ are triples (T, P, π), where T is an S-scheme, P is a torsor under $G_T := G \times_S T$ for the étale site $T_{\acute{e}t}$ and $\pi : P \to X \times_S T$ is a G_T-equivariant morphism. Morphisms $(T', P', \pi') \to (T, P, \pi)$ in $[X/G]$ are given by compatible morphisms of S-schemes $\varphi : T' \to T$ and $G_{T'}$-torsors $\psi : P' \to \varphi^* P$ such that $\varphi^*\pi \circ \psi = \pi'$. We claim $[X/G]$ is an algebraic stack. The *tautological* G-*torsor* $X \to [X/G]$ given by $(T \to X) \mapsto (T, X \times_S T, \mathrm{id}_{X \times_S T})$ is a smooth presentation. So we need only check that the diagonal map $\Delta = \Delta_{[X/G]}$ is representable. To do this, we use Proposition 3.5.8(1).

Let T be an S-scheme and for two objects $t_1 = (P_1, \pi_1)$ and $t_2 = (P_2, \pi_2) \in [X/G](T)$, consider the sheaf $\mathrm{Isom}(t_1, t_2)$ over $T_{\acute{e}t}$. As in Proposition 3.5.8, $\mathrm{Isom}(t_1, t_2)$ is an algebraic space if and only if pulls back to an algebraic space along some étale morphism $f : V \to T$ where V is a scheme. We may choose V so that f^*P_1 and f^*P_2 are both trivial G_V-torsors on $V_{\acute{e}t}$. To make notation easier, replace T by V and fix isomorphisms $\varphi_i : P_i \to G_T$ for $i = 1, 2$. Then $\mathrm{Isom}(t_1, t_2)$ is the sheaf sending

$$U \longmapsto \{g \in G(U) \mid \pi_1(e) = \pi_2(g)\}$$

where $e \in G_T(T)$ is the identity element. In other words, $\mathrm{Isom}(t_1, t_2)$ is isomorphic to the fibre product in the diagram

$$\begin{array}{ccc} & & G_T \\ & & \big\downarrow {\scriptstyle (\pi_1(e), \pi_2)} \\ X_T & \xrightarrow{\Delta_{X_T}} & X_T \times_T X_T \end{array}$$

where $X_T := X \times_S T$. It follows that $\mathrm{Isom}(t_1, t_2)$ is a scheme and therefore an algebraic space by Lemma 3.4.5.

Example 3.5.12. The *classifying stack* of a smooth group scheme G over S is the quotient stack $BG = [S/G]$, where G acts trivially on S.

Proposition 3.5.13. *Let $\mathcal{X}, \mathcal{Y}_1$ and $\mathcal{Y}_2$ be algebraic stacks over S and suppose $\mathcal{Y}_1 \to \mathcal{X}$ and $\mathcal{Y}_2 \to \mathcal{X}$ are morphisms of stacks. Then $\mathcal{Y}_1 \times_{\mathcal{X}} \mathcal{Y}_2$ is an algebraic stack over S.*

Example 3.5.14. The stabilizer groups $G_x = \mathrm{Aut}(x)$ are part of a larger structure on an algebraic stack $\mathcal{X}$. The *inertia stack* of $\mathcal{X}$ is the pullback in the following diagram

$$\begin{array}{ccc} \mathcal{I}_{\mathcal{X}} & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \Delta_{\mathcal{X}} \\ \mathcal{X} & \xrightarrow{\Delta_{\mathcal{X}}} & \mathcal{X} \times_S \mathcal{X} \end{array}$$

Explicitly, the objects of $\mathcal{I}_{\mathcal{X}}$ are pairs (T, x, g) where T is a scheme, $x \in \mathcal{X}(T)$ and $g \in \mathrm{Aut}_{\mathcal{X}(T)}(x)$. Morphisms $(T', x', g') \to (T, x, g)$ only exist when $T' = T$ and consist of a morphism $f : x' \to x$ in $\mathcal{X}(T)$ such that $gf = fg'$. For any scheme T and morphism of stacks $T \to \mathcal{X}$, the fibre product $\mathcal{I}_{\mathcal{X}} \times_{\mathcal{X}} T$ is equal to the algebraic space Aut_T on $T_{\acute{e}t}$. For example, when $T = \mathrm{Spec}\, k$ and $x : \mathrm{Spec}\, k \to \mathcal{X}$ is a point of $\mathcal{X}$, $\mathcal{I}_{\mathcal{X}} \times_{\mathcal{X}} \mathrm{Spec}\, k \cong G_x$.

Let P be a property of S-schemes in the smooth topology on Sch_S preserved under pullback. We say an algebraic stack $\mathcal{X}$ over S has property P if there is a smooth presentation $U \to \mathcal{X}$ where U is a scheme with property P. Examples of such properties include: (locally) noetherian, regular, normal, locally of finite type or finite presentation, n-dimensional, etc. Likewise, we say a morphism $f : \mathcal{Y} \to \mathcal{X}$ has property P if there exists a smooth presentation $U \to \mathcal{X}$ by a scheme U such that $U \times_{\mathcal{X}} \mathcal{Y} \to U$ has property P. Examples include: normal, proper, (closed/open) embedding, flat, smooth of relative dimension n, unramified, étale, surjective, affine, etc.

Definition 3.5.15. *An algebraic stack $\mathcal{X}$ over S is* **quasi-separated** *(resp.* **separated***) if the diagonal $\Delta_{\mathcal{X}} : \mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is quasi-compact and quasi-separated (resp. proper).*

Example 3.5.16. An important example of a property of morphisms of schemes that is available for algebraic stacks is the notion of formally unramified. Its definition (given in [EGA IV, Ch. 17]) is equivalent to the following: a morphism of schemes $Y \to X$ is *formally unramified* if and only if $\Omega^1_{Y/X} = 0$. As this vanishing condition is preserved under pullbacks and is local in the smooth (and étale) topology, we also have a notion of a formally unramified morphism of algebraic stacks.

Let $\mathcal{X}$ be a locally noetherian algebraic stack over S with smooth presentation $U \to \mathcal{X}$. As with algebraic spaces (cf. Proposition 3.4.8), $\mathcal{X}$ can also be presented by a pair of morphisms of algebraic spaces $s, t : R \rightrightarrows U$ where $R = U \times_{\mathcal{X}} U$. There is also a canonical morphism $e : U \to R$. (This is an example of a *groupoid presentation*, cf. [SP, Tag 04T3].)

Definition 3.5.17. *Let $x : \operatorname{Spec} k \to \mathcal{X}$ be a point of $\mathcal{X}$ and $u \in U$ a point mapping to*

x *along* $U \to \mathcal{X}$*. The* **dimension** *of* $\mathcal{X}$ *at* x *is the difference*

$$\dim_x(\mathcal{X}) = \dim_u(U) - \dim_{e(u)}(R_u) \in \mathbb{Z} \cup \{\infty\}$$

where R_u *is the fibre of* u *along* $s : R \to U$*. The* **dimension of the stack** $\mathcal{X}$ *is*

$$\dim(\mathcal{X}) = \sup_{x \in |\mathcal{X}|} \dim_x(\mathcal{X}).$$

Example 3.5.18. Let G be a group scheme of finite type acting on a scheme X. Then the quotient stack $[X/G]$ has dimension $\dim(X) - \dim(G)$, so in particular, stacks can have negative dimension. For example, if G is a group scheme over a field k, then $\dim(BG) = \dim([\operatorname{Spec} k/G]) = -\dim G$.

For a locally noetherian scheme X over S, the *relative normalization* of X is an S-scheme X^ν together with an S-morphism $X^\nu \to X$ uniquely determined by the following properties:

(1) $X^\nu \to X$ is integral, surjective and induces a bijection on irreducible components.

(2) $X^\nu \to X$ is terminal among morphisms of S-schemes $Z \to X$ where Z is normal.

Equivalently, X^ν is the normalization of X in its total ring of fractions (cf. [SP, Tag 035E]).

Lemma 3.5.19 ([SP, Tag 07TD])**.** *If* $Y \to X$ *is a smooth morphism of locally noetherian* S*-schemes and* X^ν *is the relative normalization of* X*, then* $X^\nu \times_X Y$ *is the relative* *normalization of* Y*.*

As a consequence we can extend the definition of normal/normalization to an algebraic stack. The existence and uniqueness of the following construction are proven in [AB, Lem. A.4].

Definition 3.5.20. *Let* $\mathcal{X}$ *be a locally noetherian algebraic stack over* S*. Then* $\mathcal{X}$ *is* **normal** *if there is a smooth presentation* $U \to \mathcal{X}$ *where* U *is a normal scheme. The* **relative normalization** *of* $\mathcal{X}$ *is an algebraic stack* $\mathcal{X}^\nu$ *and a representable morphism of stacks* $\mathcal{X}^\nu \to \mathcal{X}$ *such that for any smooth morphism* $U \to \mathcal{X}$ *where* U *is a scheme,* $U \times_{\mathcal{X}} \mathcal{X}^\nu$ *is the relative normalization of* $U \to S$*.*

Lemma 3.5.21 ([AB, Lem. A.5])**.** *For a locally noetherian algebraic stack* $\mathcal{X}$*, the relative normalization* $\mathcal{X}^\nu$ *is uniquely determined by the following two properties:*

(1) $\mathcal{X}^\nu \to \mathcal{X}$ *is an integral surjection which induces a bijection on irreducible components.*

(2) $\mathcal{X}^\nu \to \mathcal{X}$ *is terminal among morphisms of algebraic stacks* $\mathcal{Z} \to \mathcal{X}$*, where* $\mathcal{Z}$ *is normal, which are dominant on irreducible components.*

For more properties of normalizations of stacks, see [AB, Appendix A].

Definition 3.5.22. *Let* $\mathcal{X}, \mathcal{Y}$ *and* $\mathcal{Z}$ *be algebraic stacks and suppose there are morphisms* $\mathcal{Y} \to \mathcal{X}$ *and* $\mathcal{Z} \to \mathcal{X}$*. Define the* **normalized pullback** $\mathcal{Y} \times^\nu_{\mathcal{X}} \mathcal{Z}$ *to be the relative normalization of the fibre product* $\mathcal{Y} \times_{\mathcal{X}} \mathcal{Z}$*.*

We will write the normalized pullback as a diagram

$$\begin{array}{ccc} \mathcal{Y} \times^{\nu}_{\mathcal{X}} \mathcal{Z} & \longrightarrow & \mathcal{Z} \\ \downarrow & \lrcorner\,\nu & \downarrow \\ \mathcal{Y} & \longrightarrow & \mathcal{X} \end{array}$$

3.5.2 Deligne–Mumford Stacks

We have so neglected a key property of algebraic spaces when generalizing them to algebraic stacks: the étale presentation (or equivalently by Proposition 3.4.8, the étale equivalence relation). In many ways, the *true* hybrid between a stack and an algebraic space is a Deligne–Mumford stack.

Definition 3.5.23. *A* **Deligne–Mumford stack** *is an algebraic stack $\mathcal{X}$ over S which admits an étale surjection $U \to \mathcal{X}$ where U is a scheme.*

Proposition 3.5.24. *Let $\mathcal{X}$ be a stack over S. Then*

(1) $\mathcal{X}$ is algebraic if and only if it has a presentation $s, t : R \rightrightarrows U$, with R and U schemes and s and t smooth.

(2) $\mathcal{X}$ is Deligne–Mumford if and only if it has a presentation $s, t : R \rightrightarrows U$, with R and U schemes and s and t étale.

Proof. (1) follows from [SP, Tags 04T5 and 04TK] and (2) can be obtained by a similar proof. See also [LMB, Sec. 4.3]. □

Theorem 3.5.25. *An algebraic stack $\mathcal{X}$ over S is Deligne–Mumford if and only if the diagonal $\Delta_{\mathcal{X}} : \mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is formally unramified.*

Proof. This is long; see [DM, Thm. 4.21], [LMB, Thm. 8.1] or [Ols, Thm. 8.3.3]. □

Proposition 3.5.26. *Assume $\mathcal{X}$ is an algebraic stack whose diagonal $\Delta_{\mathcal{X}}$ is finitely presented. Then $\Delta_{\mathcal{X}}$ is formally unramified if and only if for every geometric point $\bar{x} : \operatorname{Spec} k \to \mathcal{X}$, the scheme $\operatorname{Aut}_{\bar{x}}$ is a reduced, finite group scheme over k.*

Proof. See [Ols, Rmk. 8.3.4]. □

The punchline of these last two results is that we can think of a Deligne–Mumford stack as an algebraic stack with no "infinitesimal automorphism groups". Colloquially, we will even say that a Deligne–Mumford stack has "finite stabilizers", as a reduced, finite group scheme over an algebraically closed field is precisely the constant group scheme determined by a finite group.

Corollary 3.5.27. *If $\mathcal{X}$ is an algebraic stack such that for every scheme U and every object $x \in \mathcal{X}(U)$, the group Aut_x is trivial, then $\mathcal{X}$ is representable by an algebraic space.*

Proof. The hypothesis implies that the diagonal $\Delta_{\mathcal{X}} : \mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is representable by *injective* morphisms of algebraic spaces, so in particular $\mathcal{X}$ is Deligne–Mumford. Let $U \to \mathcal{X}$ be an étale presentation by a scheme U. Then $R := U \times_{\mathcal{X}} U$ is an étale equivalence relation on U and $\mathcal{X} \cong U/R$ which is an algebraic space by Proposition 3.4.8. □

Remark 3.5.28. A stronger version of Corollary 3.5.27 says that $\mathcal{X}$ is an algebraic space if the automorphism group of any geometric point is trivial; cf. [Ols, Rmk. 8.3.6].

We next give a criterion for when a quotient stack (Example 3.5.11) is Deligne–Mumford.

Theorem 3.5.29 ([Ols, Cor. 8.4.2]). *Let X be a scheme and G a smooth group scheme acting on X. Then $[X/G]$ is Deligne–Mumford if and only if for every geometric point $\bar{y} : \operatorname{Spec} k \to [X/G]$, the stabilizer $G_{\bar{y}}$ is étale.*

This and Corollary 3.5.27 show that algebraic spaces are precisely those Deligne–Mumford stacks with trivial stabilizer groups. So in particular, schemes are Deligne–Mumford stacks with trivial stabilizers.

Example 3.5.30. Famously, the stacks $\mathscr{M}_{1,1}$ and $\mathscr{M}_g$, $g \geqslant 2$, of Examples 3.2.23 and 3.2.24 are Deligne–Mumford stacks. This was proven by Deligne and Mumford in [DM]; see also [Ols, Sec. 8.4.3].

Example 3.5.31. Let m and n be positive integers and let $\mathbb{P}(m, n) = [(\mathbb{A}^2 \setminus \{0\})/\mathbb{G}_m]$ where $\mathbb{G}_m$ acts on $\mathbb{A}^2$ by $\lambda \cdot (x, y) = (\lambda^m x, \lambda^n y)$. Then $\mathbb{P}(m, n)$ is a Deligne–Mumford stack called the *weighted projective line* with weights (m, n). When m and n are coprime, $\mathbb{P}(m, n)$ has a trivial generic stabilizer, so by Corollary 3.5.27, it contains a dense open algebraic space which is in fact a scheme (isomorphic to $\mathbb{P}^1 \setminus \{0, \infty\} = \mathbb{A}^1 \setminus \{0\}$). If m and n have greatest common divisor d, then $\mathbb{P}(m, n)$ has generic stabilizer $\mathbb{Z}/d\mathbb{Z}$.

Example 3.5.32. More generally, for a sequence of positive integers $(m_0, \ldots, m_k)$, the *weighted projective space* with these weights is the Deligne–Mumford stack $\mathbb{P}(m_0, \ldots, m_k) = [(\mathbb{A}^{k+1} \setminus \{0\})/\mathbb{G}_m]$ where $\mathbb{G}_m$ acts on $\mathbb{A}^{k+1}$ by $\lambda \cdot (x_0, \ldots, x_k) =$

$(\lambda^{m_0}x_0, \ldots, \lambda^{m_k}x_k)$. As above, if $\gcd(m_0, \ldots, m_k) = d$ then $\mathbb{P}(m_0, \ldots, m_k)$ has a generic stabilizer $\mathbb{Z}/d\mathbb{Z}$.

Example 3.5.33. Let G be a group scheme acting on a scheme X. If G is not a finite group scheme, then $[X/G]$ need not be Deligne–Mumford. For instance,

- $B\mathbb{G}_m = [S/\mathbb{G}_m]$ is not Deligne–Mumford (it has a single $\mathbb{G}_m$-stabilizer).
- More generally, if G itself is infinite, then BG is not Deligne–Mumford.
- If $\mathbb{G}_m$ acts on $\mathbb{A}^1$ by $\lambda \cdot x = \lambda x$, then $[\mathbb{A}^1/\mathbb{G}_m]$ is not Deligne–Mumford, since it has a $\mathbb{G}_m$-stabilizer at one closed point (the image of $0 \in \mathbb{A}^1$). However, the generic point has a trivial stabilizer.
- If $\mathbb{G}_m$ acts on $\mathbb{A}^1$ by $\lambda \cdot x = \lambda^n x$, then $[\mathbb{A}^1/\mathbb{G}_m]$ is slightly worse: it has a $\mathbb{G}_m$ at the closed point and a generic μ_n.

3.5.3 Sheaves on Deligne–Mumford Stacks

Fix an algebraic stack $\mathcal{X}$ over S and let $\mathsf{AS}/\mathcal{X}$ be the category whose objects are morphisms of stacks $X \xrightarrow{p} \mathcal{X}$, where X is an algebraic space over S, and whose morphisms are 2-commutative diagrams

$$\begin{array}{ccc} X' & \xrightarrow{\quad f \quad} & X \\ & {\scriptstyle p'}\searrow \;\Rightarrow\; \swarrow{\scriptstyle p} & \\ & \mathcal{X} & \end{array}$$

That is, a morphism $(X' \xrightarrow{p'} \mathcal{X}) \to (X \xrightarrow{p} \mathcal{X})$ is a pair (f, η) where $f : X' \to X$ is a morphism of algebraic spaces and $\eta : p' \Rightarrow p \circ f$ is a natural isomorphism of functors. We call $\mathsf{AS}/\mathcal{X}$ the category of $\mathcal{X}$-spaces. Let $\mathsf{Sch}/\mathcal{X}$ be the full subcategory of $\mathcal{X}$-schemes, i.e. where objects are morphisms $X \to \mathcal{X}$ with X an S-scheme. Note that if X is an algebraic space (or a scheme), then AS/X (or Sch/X) is the usual slice category.

For technical reasons, we can only construct the étale site of a stack $\mathcal{X}$ (in analogy with Section 3.4.2) when $\mathcal{X}$ is a Deligne–Mumford stack; cf. [Ols, Sec. 9.1] for further details, including the construction of the more general *lisse-étale site* on an algebraic stack. For a Deligne–Mumford stack $\mathcal{X}$, let $\mathcal{X}_{\text{ét}}$ be the site with underlying category $\mathsf{Sch}_{\mathcal{X}}$ and coverings given by collections $\{(f_i, \eta_i) : X_i \to X\}$ where each f_i is étale.

Remark 3.5.34. There are a number of slight variations on the definition of $\mathcal{X}_{\text{ét}}$ that yield the same category of sheaves $\mathsf{Sh}_{\mathcal{X}_{\text{ét}}}$, for example using $\mathsf{AS}/\mathcal{X}$ in place of $\mathsf{Sch}/\mathcal{X}$; cf. [Ols, Sec. 9.1].

Definition 3.5.35. *The* **structure sheaf of a Deligne–Mumford stack** $\mathcal{X}$ *is the sheaf*

$$\mathcal{O}_{\mathcal{X}} := \mathcal{O}_{\mathcal{X}_{\text{ét}}} : \mathcal{X}_{\text{ét}}^{\text{op}} \longrightarrow \mathsf{Ring}, \quad T \longmapsto \Gamma(T, \mathcal{O}_T).$$

Definition 3.5.36. *A sheaf of $\mathcal{O}_{\mathcal{X}}$-modules $\mathcal{M}$ is* **quasi-coherent** *if for any object $(T \to \mathcal{X}) \in \mathcal{X}_{\text{ét}}$, the restriction $\mathcal{M}_T$ is quasi-coherent. Further, if $\mathcal{X}$ is locally noetherian, then $\mathcal{M}$ is* **coherent** *if for any such $T \to \mathcal{X}$, $\mathcal{M}_T$ is coherent.*

Example 3.5.37. Let BG be the classifying stack (Example 3.5.12) of a finite group G acting trivially on a scheme S. Then there is an equivalence of categories between quasi-coherent sheaves on BG and quasi-coherent sheaves on S with a left G-action. If $S = \operatorname{Spec} k$ for a field k, then a quasi-coherent sheaf $\mathcal{E}$ on BG is equivalent to a G-representation V over k. The cohomology of the algebraic stack BG is then given by group cohomology for G:

$$H^i(BG, \mathcal{E}) \cong H^i(G, V) \quad \text{for all } i \geqslant 0.$$

Remark 3.5.38. Concretely, a quasi-coherent (resp. coherent) sheaf on a Deligne–Mumford stack $\mathcal{X}$ is the data of a quasi-coherent (resp. coherent) sheaf $\mathcal{M}_T$ for all schemes $T \to \mathcal{X}$ and compatible isomorphisms of sheaves $\varphi^*\mathcal{M}_T \xrightarrow{\sim} \mathcal{M}_{T'}$ for any $\mathcal{X}$-morphism $\varphi : T' \to T$. Write $\Gamma(\mathcal{X}, \mathcal{M})$ for the space of *global sections* of $\mathcal{M}$, i.e. choices of sections $s_T \in \Gamma(T, \mathcal{M}_T)$ for all schemes $T \to \mathcal{X}$ that are compatible with pullbacks.

Lemma 3.5.39. *The structure sheaf $\mathcal{O}_{\mathcal{X}}$ is given by $\mathcal{O}_{\mathcal{X},T} = \mathcal{O}_T$ for any scheme $T \to \mathcal{X}$.*

Example 3.5.40. Generalizing the previous example, a (quasi-)coherent sheaf on a quotient stack $\mathcal{X} = [X/G]$, where X is a scheme and G is a finite group acting on X, is the same thing as a G-equivariant (quasi-)coherent sheaf on X. Then for any such sheaf $\mathcal{M}$, $\Gamma([X/G], \mathcal{M}) = \Gamma(X, \mathcal{M}_X)^G$, the space of invariant global sections of $\mathcal{M}$ over X.

Definition 3.5.41. *Let $f : \mathcal{Y} \to \mathcal{X}$ be a representable morphism of Deligne–Mumford* *stacks and $\mathcal{M}$ a quasi-coherent sheaf on $\mathcal{X}$. Then the* **pullback** *of $\mathcal{M}$ along f is the sheaf*

$$f^*\mathcal{M} = f^{-1}\mathcal{M} \otimes_{f^{-1}\mathcal{O}_{\mathcal{X}}} \mathcal{O}_{\mathcal{Y}}$$

where $f^{-1}\mathcal{M}$ is the sheaf sending $(T \xrightarrow{p} \mathcal{Y})$ to the quasi-coherent sheaf assigned to T by the map $f \circ p : T \to \mathcal{X}$.

Lemma 3.5.42. *Let $f : \mathcal{Y} \to \mathcal{X}$ be a representable morphism of Deligne–Mumford stacks. If $\mathcal{M}$ is a quasi-coherent sheaf on $\mathcal{X}$, then $f^*\mathcal{M}$ is a quasi-coherent sheaf on $\mathcal{Y}$.*

3.5.4 Coarse Moduli Spaces

Let us return to the perspective of Section 3.2, in which we considered a moduli problem as a presheaf $F : \mathsf{Sch}^{op} \to \mathsf{Set}$. The key to understanding many moduli problems is to exploit the underlying geometry of F, in the case F is represented by a scheme M (a fine moduli space). However, as we saw in several examples in Section 3.2, no such scheme exists in general, so we embarked on a long and arduous journey to replace presheaves $\mathsf{Sch}^{op} \to \mathsf{Set}$ with categories fibred in groupoids and ultimately algebraic stacks. We can think of an algebraic stack $\mathcal{X}$ as an assignment of a groupoid $\mathcal{X}(T)$ to each S-scheme T which parametrizes the solutions to some moduli problem over T *while remembering all isomorphisms between solutions*. So in this sense, we have solved the issue of nontrivial automorphisms in moduli problems by passing to stacks, just as promised.

One might ask what happens when we pass back from groupoids to sets by identifying all isomorphic objects in $\mathcal{X}(T)$. This is made precise by the notion of a coarse moduli space.

Definition 3.5.43. *Let $\mathcal{X}$ be an algebraic stack over a scheme S. A* **coarse moduli space** *for $\mathcal{X}$ is an algebraic space X over S and a morphism of stacks $\pi : \mathcal{X} \to X$ satisfying:*

(1) π is initial among all maps $\mathcal{X} \to X'$, where X' is an algebraic space over S. That is, if $g : \mathcal{X} \to X'$ is such a map, then there is a unique morphism of algebraic spaces $f : X \to X'$ such that $g = f \circ \pi$.

(2) The map $|\mathcal{X}(k)| \to X(k)$ is bijective for any algebraically closed field k, where $|\mathcal{X}(k)|$ denotes the set of isomorphism classes in the groupoid $\mathcal{X}(k)$.

The main theorem governing coarse moduli spaces was assumed for a long time, but not proven until work of Keel and Mori ([KM]) in the 1990s; see also [Ols, Sec. 11.2].

Theorem 3.5.44 (Keel–Mori)**.** *Suppose $\mathcal{X}$ is an algebraic stack over S which is locally of finite presentation and has finite diagonal $\Delta_{\mathcal{X}}$. Then $\mathcal{X}$ admits a coarse moduli space $\pi : \mathcal{X} \to X$ which is locally of finite type.*

Remark 3.5.45. (Comparing fine and coarse moduli spaces) We say a moduli problem F (now considered as a category fibred in groupoids over Sch or Sch_S) has a *fine moduli space* if F is representable by an algebraic space (or, as is usually desired, by a scheme), say M. This means that for any scheme T, $F(T) \cong \mathrm{Hom}(T, M)$ and this identification is functorial in T. In particular, the identity map $\mathrm{id}_M \in \mathrm{Hom}(M, M)$ determines a "universal family" $U \in F(M)$, i.e. a solution to the moduli problem F over M such that for any T, every family of solutions to F over T is obtained by pulling back U to T. In contrast, a coarse moduli space only possesses a map $F(T) \to \mathrm{Hom}(T, M)$ for each T (plus the universal property in axiom (2)), but not a universal family, since $\mathrm{id}_M \in \mathrm{Hom}(M, M)$ need not lift to an element of $F(M)$.

Example 3.5.46. Let G be a finite group scheme acting on a quasi-projective scheme X. Then the quotient stack $[X/G]$ has coarse moduli space X/G with coarse moduli map $[X/G] \to X/G$ as constructed in Example 3.3.35. Note that when X is affine, say $X = \operatorname{Spec} A$, then $X/G = \operatorname{Spec}(A^G)$, where A^G is the ring of G-invariants of A.

Example 3.5.47. Let $\mathscr{M}_{1,1}$ denote the moduli stack of elliptic curves from Example 3.2.23, which is Deligne–Mumford by [DM]. Denote the coarse moduli space of $\mathscr{M}_{1,1}$ by $M_{1,1}$. Classically, the j-invariant identifies $M_{1,1} \cong \mathbb{A}^1 = \operatorname{Spec} k[j]$ via the map $(E, O) \mapsto j(E)$; this is proven in detail in [Ols, Sec. 13.1], for example.

Example 3.5.48. The coarse moduli space of the moduli stack $\mathscr{M}_g$ of curves of genus $g \geqslant 2$ is likewise denoted M_g and is a subject of much study in algebraic geometry. Deligne and Mumford famously introduced a compactification $\overline{\mathscr{M}}_g$ of $\mathscr{M}_g$ which is a proper Deligne–Mumford stack (even over $\operatorname{Spec} \mathbb{Z}$). Both $\mathscr{M}_g$ and $\overline{\mathscr{M}}_g$, as well as their coarse spaces, have dimension $3g - 3$ when $g \geqslant 2$, so the complexity of these spaces grows with g.

The Keel–Mori theorem guarantees that every Deligne–Mumford stack (locally of finite presentation and with finite diagonal) has a coarse moduli space. This allows us to give an explicit description of the local structure of such a Deligne–Mumford stack in terms of its coarse space.

Proposition 3.5.49. *Let $\mathcal{X}$ be a Deligne–Mumford stack which is locally of finite presentation and has finite diagonal $\Delta_{\mathcal{X}}$ and coarse moduli space $\pi : \mathcal{X} \to X$ and fix a geometric point $\bar{x} : \operatorname{Spec} k \to \mathcal{X}$ with stabilizer $G_{\bar{x}}$. Then there is an étale neighborhood $U \to X$ of $x := \pi \circ \bar{x}$ and a finite morphism of schemes $V \to U$ such that $G_{\bar{x}}$ acts on V and $\mathcal{X} \times_X U \cong [V/G_{\bar{x}}]$ as stacks.*

Proof. This is [Ols, Thm. 11.3.1]. □

Chapter 4

Stacky Curves

In this chapter we explore the central topic of this thesis: stacky curves. In Section 4.1, we give the basic definitions and constructions on stacky curves which will be used to classify them in later sections. Following a running theme of tame vs. wild, we divide up the our classification efforts into the tame case (Section 4.2) and the wild case (Sections 4.3 and 4.4). This includes proofs of the main results (Theorems 0.1.1, 0.1.2 and 0.1.3) stated in Section 0.1.

4.1 Basics

4.1.1 Definitions

Fix a field k of characteristic $p \geqslant 0$ and let $\mathcal{X}$ be a Deligne–Mumford stack over the étale site $(\operatorname{Spec} k)_{\text{ét}}$. Recall that by Proposition 3.5.26, the stabilizer group of any point $x \in |\mathcal{X}|$ is finite.

Definition 4.1.1. *We say $\mathcal{X}$ is a* **tame stack** *if the order of each stabilizer group G_x is coprime to p. If p divides $|G_x|$ for any $x \in |\mathcal{X}|$, we say $\mathcal{X}$ is a* **wild stack**.

Example 4.1.2. Every Deligne–Mumford stack over a field of characteristic 0 is tame.

Tame stacks in general are better understood because they are more well-behaved, e.g. if $\pi : \mathcal{X} \to X$ is a coarse moduli map then the pushforward functor on quasi-coherent sheaves $\pi_* : \mathsf{QCoh}_{\mathcal{X}} \to \mathsf{QCoh}_X$ is an equivalence of categories (cf. [Ols, Prop. 11.3.4]).

Definition 4.1.3. *A* **stacky curve** *is a smooth, separated, connected, one-dimensional Deligne–Mumford stack $\mathcal{X}$ which is generically a scheme, i.e. there exists an open subscheme U of the coarse moduli space X of $\mathcal{X}$ such that the induced map $\mathcal{X} \times_X U \to U$ is an isomorphism.*

Proposition 3.5.49 says that that every stacky curve is, étale locally, a quotient stack $[U/G]$ for U a scheme and $G = G_{\bar{x}}$ the stabilizer of a geometric point (and thus a constant group scheme). By ramification theory (cf. [Ser2, Ch. IV, Cor. 1]), if $\mathcal{X}$ is a tame stacky curve then every stabilizer group of $\mathcal{X}$ is cyclic. Consequently, with some hypotheses (cf. [GS]) a tame stacky curve can be described completely by specifying its coarse moduli space and a finite list of numbers corresponding to the finitely many points with nontrivial stabilizers of those orders. In contrast, if $\mathcal{X}$ is wild, it may have higher ramification data and even nonabelian stabilizers. One of the main goals of this thesis is to describe how wild stacky curves can still be classified, which will be done in Sections 4.3 and 4.4.

Let $\mathcal{X}$ be a stacky curve and suppose $\bar{x} : \operatorname{Spec} k \to \mathcal{X}$ is a geometric point with image x and stabilizer G_x.

Definition 4.1.4. *The* **residue gerbe** *at $\bar{x}$ is the unique stack $\mathcal{G}_x$ satisfying:*

(1) $\mathcal{G}_x$ is a reduced algebraic stack.

(2) $|\mathcal{G}_x|$ consists of a single point.

(3) There is a monomorphism of stacks $\mathcal{G}_x \hookrightarrow \mathcal{X}$ whose image is x.

Existence and uniqueness follow from [SP, Tags 06UH and 06ML]. We say x is a *stacky point* of $\mathcal{X}$ if $G_x \neq 1$. As a substack of $\mathcal{X}$, this $\mathcal{G}_x$ may be regarded as a "fractional point", in the sense that $\deg \mathcal{G}_x = \frac{1}{|G_x|}$.

4.1.2 Divisors and Vector Bundles

Let $\mathcal{X}$ be a normal Deligne–Mumford stack with coarse space X. As in scheme theory (cf. Section 2.2.1), we make the following definitions:

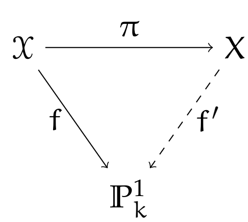
- An **irreducible (Weil) divisor** on $\mathcal{X}$ is an irreducible, closed substack of $\mathcal{X}$ of codimension 1.
- The **(Weil) divisor group** of $\mathcal{X}$, denoted $\operatorname{Div} \mathcal{X}$, is the free abelian on the irreducible divisors of $\mathcal{X}$; its elements are called **(Weil) divisors** on $\mathcal{X}$.

- A **principal (Weil) divisor** on $\mathcal{X}$ is a divisor $\operatorname{div}(f)$ associated to a morphism $f : \mathcal{X} \to \mathbb{P}^1_k$ (equivalently, a rational section f of the structure sheaf $\mathcal{O}_{\mathcal{X}}$) given by
$$\operatorname{div}(f) = \sum_Z \nu_Z(f) Z$$
where $\nu_Z(f)$ is the valuation of f in the local ring $\mathcal{O}_{X,Z}$.

- Two divisors $D, D' \in \operatorname{Div} \mathcal{X}$ are **linearly equivalent** if $D = D' + \operatorname{div}(f)$ for some morphism $f : \mathcal{X} \to \mathbb{P}^1_k$.

- The subgroup of principal divisors in $\operatorname{Div} \mathcal{X}$ is denoted $\operatorname{PDiv} \mathcal{X}$. The **divisor class group** of $\mathcal{X}$ is the quotient group $\operatorname{Cl}(\mathcal{X}) = \operatorname{Div} \mathcal{X} / \operatorname{PDiv} \mathcal{X}$.

Lemma 4.1.5. *Let $\mathcal{X}$ be a stacky curve over k with coarse space morphism $\pi : \mathcal{X} \to X$. Then for any nonconstant map $f : \mathcal{X} \to \mathbb{P}^1_k$, $\operatorname{div}(f) = \pi^* \operatorname{div}(f')$, where $f' : X \to \mathbb{P}^1_k$ is the unique map making the diagram*



commute.

Proof. See [VZB, 5.4.4]. □

For a scheme X, a *rational divisor* on X is a formal sum $E = \sum_{Z \subset X} r_Z Z$ where $r_Z \in \mathbb{Q}$ for each irreducible divisor $Z \subset X$. Denote the abelian group of rational divisors on X by $\mathbb{Q} \operatorname{Div} X$.

Proposition 4.1.6 ([Beh, Thm. 1.187])**.** *Let $\mathcal{X}$ be a stacky curve with stacky points $P_1, \ldots, P_n$ whose stabilizers have orders $m_1, \ldots, m_n$, respectively, and let X be the coarse space of $\mathcal{X}$. Then there is a one-to-one correspondence between divisors $D \in \operatorname{Div} \mathcal{X}$ and rational divisors $E = \sum_P r_P P \in \mathbb{Q} \operatorname{Div} X$ such that $m_i r_{P_i} \in \mathbb{Z}$ for each $1 \leqslant i \leqslant n$.*

Definition 4.1.7. *The* **degree** *of a divisor $D = \sum_P n_P P$ on a stacky curve is the formal sum $\deg(D) = \sum_P n_P \deg P$, where $\deg P = \deg \mathcal{G}_P = \frac{1}{|G_P|}$ is the degree of the residue gerbe $\mathcal{G}_P$ at P.*

Remark 4.1.8. The degree of a divisor on a higher dimensional normal Deligne–Mumford stack is also defined, though we will not need it here. In general, a divisor on $\mathcal{X}$ need not have integer degree or coefficients, as the example below illustrates. However, Lemma 4.1.5 shows that for a morphism $f : \mathcal{X} \to \mathbb{P}^1_k$ (i.e. a rational section of $\mathcal{O}_\mathcal{X}$), the principal divisor $\operatorname{div}(f)$ always does.

Example 4.1.9. Let $n \geqslant 1$ and consider the quotient stack $\mathcal{X} = [\mathbb{A}^1_\mathbb{C}/\mu_n]$, where μ_n is the finite group scheme of nth roots of unity acting on $\mathbb{A}^1_\mathbb{C}$ by multiplication. Then $\mathcal{X}$ has coarse space $X = \mathbb{A}^1_\mathbb{C}/\mu_n = \mathbb{A}^1_\mathbb{C}$; let $\pi : \mathcal{X} \to X$ be the coarse map. Consider the morphism of sheaves $\pi^*\Omega_\mathcal{X} \to \Omega_\mathcal{X}$, where $\Omega_\mathcal{X}$ is the sheaf of differentials (defined below). Here, $\Omega_{\mathbb{A}^1_\mathbb{C}}$ is freely generated by dt where $\mathbb{A}^1_\mathbb{C} = \operatorname{Spec} \mathbb{C}[t]$. Let z be a coordinate on the upstairs copy of $\mathbb{A}^1_\mathbb{C}$, so that the cover $\mathbb{A}^1_\mathbb{C} \to [\mathbb{A}^1_\mathbb{C}/\mu_n] \to \mathbb{A}^1_\mathbb{C}$ is given by $t \mapsto z^n$. Since $\mathbb{A}^1_\mathbb{C} \to [\mathbb{A}^1_\mathbb{C}/\mu_n]$ is étale, we can identify $\Omega_\mathcal{X}$ with $\Omega_{\mathbb{A}^1_\mathbb{C}}$ generated by dz. Then the section s of $\Omega_\mathcal{X} \otimes \pi^*\Omega_X^\vee$ given by $\pi^* dt \mapsto d(z^n) = nz^{n-1}\,dt$ has divisor

$\text{div}(s) = (n-1)P$, where P is the stacky point over the origin in $\mathbb{A}^1_{\mathbb{C}}$. Thus $\deg(\text{div}(s)) = (n-1)\deg(P) = \frac{n-1}{n}$, so divisors of sections of nontrivial line bundles can have non-integer coefficients.

Definition 4.1.10. *A* **Cartier divisor** *on a normal Deligne–Mumford stack $\mathcal{X}$ is a Weil divisor which is (étale-)locally of the form (f) for a rational section f of $\mathcal{O}_{\mathcal{X}}$.*

Lemma 4.1.11. *If $\mathcal{X}$ is a smooth Deligne–Mumford stack, then every Weil divisor is a Cartier divisor.*

Proof. Pass to a smooth presentation and use [GS, Lem. 3.1]. □

For the most part in this chapter, we will consider stacky curves $\mathcal{X}$, which are by definition smooth and Deligne–Mumford, so we can identify the group of Weil divisors on $\mathcal{X}$ with the group of Cartier divisors on $\mathcal{X}$.

Lemma 4.1.12 ([VZB, 5.4.5])**.** *Every line bundle L on a stacky curve $\mathcal{X}$ is isomorphic to $\mathcal{O}_{\mathcal{X}}(D)$ for some divisor $D \in \text{Div}(\mathcal{X})$. Moreover, $\mathcal{O}_{\mathcal{X}}(D) \cong \mathcal{O}_{\mathcal{X}}(D')$ if and only if D and D' are linearly equivalent.*

Remark 4.1.13. Lemma 4.1.12 says that for a stacky curve $\mathcal{X}$, the homomorphism $\text{Cl}(\mathcal{X}) \to \text{Pic}(\mathcal{X}), D \mapsto \mathcal{O}_{\mathcal{X}}(D)$ is an isomorphism. This isomorphism holds more generally for locally factorial schemes which are reduced (cf. [SP, Tag 0BE9]), and it should similarly hold for reduced, locally factorial stacks, though we were not able to find a reference.

Lemma 4.1.14. *Let $\mathcal{X}$ be a Deligne–Mumford stack and $D \in \operatorname{Div} \mathcal{X}$. Then there is an isomorphism of sheaves (on X)*

$$\pi_* \mathcal{O}_{\mathcal{X}}(D) \cong \mathcal{O}_X(\lfloor D \rfloor)$$

where $\lfloor D \rfloor$ is the floor divisor obtained by rounding down all $\mathbb{Q}$-coefficients of D, considered as an integral divisor on X.

Proof. For an étale morphism $U \to X$, define the map

$$\begin{aligned} \mathcal{O}_X(\lfloor D \rfloor)(U) &\longrightarrow \pi_* \mathcal{O}_{\mathcal{X}}(D)(U) \\ f &\longmapsto f \circ p_2 \end{aligned}$$

where $p_2 : \mathcal{X} \times_X U \to U$ is the canonical projection. A section $f \in \mathcal{O}_X(\lfloor D \rfloor)(U)$ is a rational function on $\mathcal{X}$ satisfying $\lfloor D \rfloor + \operatorname{div}(f) \geqslant 0$, or equivalently, $D + \operatorname{div}(f \circ p_2) \geqslant 0$. Indeed, by Lemma 4.1.5, $\pi^* \operatorname{div}(f) = \operatorname{div}(f \circ p_2)$ and the floor function does not change effectivity. Moreover, it is clear that $f \mapsto f \circ p_2$ is injective. For a map $g : \mathcal{X} \times_X U \to \mathbb{P}^1$, there is a map $f : U \to \mathbb{P}^1$ making the diagram

$$\begin{array}{ccc} \mathcal{X} \times_X U & \xrightarrow{p_2} & U \\ & {\scriptstyle g} \searrow \quad \swarrow {\scriptstyle f} & \\ & \mathbb{P}^1 & \end{array}$$

commute by the universal property of p_2. Therefore $f \mapsto f \circ p_2$ is surjective, so we get the desired isomorphism $\mathcal{O}_X(\lfloor D \rfloor) \xrightarrow{\sim} \pi_* \mathcal{O}_{\mathcal{X}}(D)$. ☐

Definition 4.1.15. *For any morphism of Deligne–Mumford stacks $f : \mathcal{X} \to \mathcal{Y}$, define the* **sheaf of relative differentials** $\Omega_{\mathcal{X}/\mathcal{Y}}$ *to be the sheaf which takes an étale map $U \to \mathcal{X}$ to $\Omega_{U/k}(U)$, the vector space of k-differentials over U. Set $\Omega_{\mathcal{X}} := \Omega_{\mathcal{X}/\operatorname{Spec} k}$.*

Equivalently, $\Omega_{\mathcal{X}/\mathcal{Y}}$ can be defined as the sheafification of the presheaf $U \mapsto \Omega_{\mathcal{O}_{\mathcal{X}}(U)/f^{-1}\mathcal{O}_{\mathcal{Y}}(U)}$.

Lemma 4.1.16. *For a morphism of Deligne–Mumford stacks $f : \mathcal{X} \to \mathcal{Y}$, let $\mathcal{I} \subseteq \mathcal{O}_{\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}}$ be the ideal sheaf corresponding to the diagonal morphism $\Delta : \mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$. Then $\Omega_{\mathcal{X}/\mathcal{Y}} \cong \Delta^* \mathcal{I}/\mathcal{I}^2$.*

Proof. See [Bos, Sec. 8.2] for the proof in the category of schemes, which is easily generalized to Deligne–Mumford stacks using étale presentations. ☐

Corollary 4.1.17. *Let $\mathcal{X}$ be a stacky curve over k. Then $\Omega_{\mathcal{X}}$ is a line bundle.*

Proof. For any étale map $U \to \mathcal{X}$, $f^*\Omega_{\mathcal{X}} \cong \Omega_U$ is a line bundle. Moreover, $\Omega_{\mathcal{X}}$ is locally free and rank of locally free sheaves is preserved under f^*, so $\Omega_{\mathcal{X}}$ is itself a line bundle. ☐

As a result, Lemma 4.1.12 shows that $\Omega_{\mathcal{X}} \cong \mathcal{O}_{\mathcal{X}}(K_{\mathcal{X}})$ for some divisor $K_{\mathcal{X}} \in \operatorname{Div} \mathcal{X}$, called a *canonical divisor* of $\mathcal{X}$. Note that a canonical divisor is unique up to linear equivalence.

Proposition 4.1.18 (Stacky Riemann–Hurwitz – Tame Case). *For a tame stacky curve $\mathcal{X}$ with coarse moduli space $\pi : \mathcal{X} \to X$, the formula*

$$K_{\mathcal{X}} = \pi^* K_X + \sum_{x \in \mathcal{X}(k)} (|G_x| - 1)x$$

defines a canonical divisor $K_{\mathcal{X}}$ on $\mathcal{X}$.

Proof. This is [VZB, Prop. 5.5.6], but the technique will be useful in a later generalization so we paraphrase it here. First assume $\mathcal{X}$ has a single stacky point P. Since π is an isomorphism away from the stacky points, we may assume by Lemma 3.5.49 that $\mathcal{X}$ is of the form $\mathcal{X} = [U/\mu_r]$ for a scheme U and μ_r the group scheme of rth roots of unity, where r is coprime to $\operatorname{char} k$. Set $X = U/\mu_r$ and consider the étale cover $f : U \to [U/\mu_r]$. Then $f^*\Omega_{\mathcal{X}/X} \cong \Omega_{U/X}$ so the stalk of $\Omega_{\mathcal{X}/X}$ at the stacky point has length $r - 1$. In general, since $\mathcal{X} \to X$ is an isomorphism away from the stacky points, the above calculation at each stacky point implies that the given formula for $K_{\mathcal{X}}$ defines a canonical divisor globally. □

For a tame stacky curve $\mathcal{X}$, define its *(topological) Euler characteristic* $\chi(\mathcal{X}) = -\deg(K_{\mathcal{X}})$ and define its *genus* $g(\mathcal{X})$ by the equation $\chi(\mathcal{X}) = 2 - 2g(\mathcal{X})$.

Corollary 4.1.19. *For a tame stacky curve $\mathcal{X}$ with coarse space X,*

$$g(\mathcal{X}) = g(X) + \frac{1}{2} \sum_{x \in \mathcal{X}(k)} \left(1 - \frac{1}{|G_x|}\right) \deg \pi(x).$$

where $\pi : \mathcal{X} \to X$ is the coarse map.

4.1.3 Line Bundles and Sections

Recall that in the category of topological spaces, for any group G there is a principal G-bundle functor

$$X \longmapsto \widetilde{\mathrm{Bun}}_G(X) = \{\text{principal } G\text{-bundles } P \to X\}/\text{iso}.$$

which is represented by a classifying space BG, i.e. there is a natural isomorphism

$$\widetilde{\mathrm{Bun}}_G(-) \cong [-, BG]$$

where $[X, BG]$ denotes the set of homotopy classes of maps $X \to BG$. Topologically, BG is the base of a universal bundle $EG \to BG$ with contractible total space EG, so informally, we can think of the classifying space as $BG = \bullet/G$.

The algebraic version of this is the classifying stack $BG = [\operatorname{Spec} k/G]$, together with the tautological bundle $EG = \operatorname{Spec} k \to BG$ (Example 3.5.12). Additionally, it is often desirable to have a classification of *all* principal G-bundles over X, rather than just isomorphism classes of bundles. (From here on, BG will denote the classifying stack; there should be no confusion with the topological space discussed above.) For a fixed k-scheme X, the assignment $T \mapsto \mathrm{Bun}_G(X)(T) = \{\text{principal } G\text{-bundles over } T \times_k X\}$ defines a stack ([SGA I, VIII, Thm. 1.1 and Prop. 1.10]), and we have:

Proposition 4.1.20. *For all* k*-schemes* X*, there is an isomorphism of stacks*

$$\mathrm{Bun}_G(X) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{Stacks}}(X, BG)$$

where $\mathrm{Hom}_{\mathsf{Stacks}}(-,-)$ *denotes the internal* Hom *stack in the category of* k*-stacks.*

Proof. Follows from the definition of $BG = [\mathrm{Spec}\, k/G]$ in Example 3.5.12. □

Example 4.1.21. When $G = GL_n(k)$, principal G-bundles are in one-to-one correspondence with rank n vector bundles (locally free sheaves) on X. Any line bundle $L \to X$ determines a $\mathbb{G}_m$-bundle $L_0 = L \setminus s_0(X)$, where $s_0 : X \to L$ is the zero section. (Equivalently, L_0 is the frame bundle of L.) Conversely, a $\mathbb{G}_m$-bundle $P \to X$ determines a line bundle by the associated bundle construction:

$$L = P \times^{\mathbb{G}_m} \mathbb{A}^1 := \{(y, \lambda) \in P \times \mathbb{A}^1\}/(y \cdot g, \lambda) \sim (y, g\lambda) \text{ for } g \in \mathbb{G}_m.$$

(We will use the convention that a group acts on principal bundles on the right.) One can check that $L \mapsto L_0$ and $P \mapsto P \times^{\mathbb{G}_m} \mathbb{A}^1$ are mutual inverses. In particular, principal $\mathbb{G}_m$-bundles are identified with line bundles on X, and this correspondence extends to an isomorphism of stacks $\mathrm{Pic}(X) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{Stacks}}(X, B\mathbb{G}_m)$, where $\mathrm{Pic}(X)$ is the Picard stack on X (cf. [SP, Tag 0372]).

For a scheme X, let $\mathfrak{Div}^{[1]}(X)$ denote the category whose objects are pairs (L, s), with $L \to X$ a line bundle and $s \in H^0(X, L)$ is a global section. A morphism $(L, s) \to (M, t)$ in $\mathfrak{Div}^{[1]}(X)$ is given by a bundle isomorphism

$$\begin{array}{ccc} L & \xrightarrow{\varphi} & M \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

under which $\varphi(s) = t$. The notation $\mathfrak{Div}^{[1]}(X)$ is adapted from the notation $\mathfrak{Div}^{+}(X)$ used in some places in the literature, e.g. [Ols]. We make the change to allow for a generalization in Section 4.3.1.

By Example 4.1.21, $B\mathbb{G}_m$ classifies line bundles, but to classify pairs (L, s), we need to add a little "fuzz" to $B\mathbb{G}_m$. The next result shows how to do this with a quotient stack that is a "thickened" version of $B\mathbb{G}_m$.

Proposition 4.1.22. *There is an isomorphism of categories fibred in groupoids*

$$\mathfrak{Div}^{[1]} \cong [\mathbb{A}^1/\mathbb{G}_m].$$

Proof. Let $[\mathbb{A}^1/\mathbb{G}_m] \to B\mathbb{G}_m$ be the "forgetful map", sending an X-point

$$\left(\begin{array}{ccc} P & \xrightarrow{f} & \mathbb{A}^1 \\ {\scriptstyle \pi}\downarrow & & \\ X & & \end{array} \right) \in [\mathbb{A}^1/\mathbb{G}_m](X)$$

to the $\mathbb{G}_m$-bundle $(P \xrightarrow{\pi} X) \in B\mathbb{G}_m(X)$. By the universal property of pullbacks, a map $X \to [\mathbb{A}^1/\mathbb{G}_m]$ is equivalent to the choice of a map $g : X \to B\mathbb{G}_m$ and a section s of $X \times_{B\mathbb{G}_m} [\mathbb{A}^1/\mathbb{G}_m] \to X$. Further, g is equivalent to a $\mathbb{G}_m$-bundle $E = g^*E\mathbb{G}_m \to X$, where $E\mathbb{G}_m \to B\mathbb{G}_m$ is the universal $\mathbb{G}_m$-bundle, and $L = X \times_{B\mathbb{G}_m} [\mathbb{A}^1/\mathbb{G}_m]$ is the line bundle on X corresponding to E (as in Example 4.1.21), so $[\mathbb{A}^1/\mathbb{G}_m](X)$ is in bijection with $\mathfrak{Div}^{[1]}(X)$. All of the above choices are natural, so we have constructed an equivalence of categories on fibres $\mathfrak{Div}^{[1]}(X) \xrightarrow{\sim} [\mathbb{A}^1/\mathbb{G}_m](X)$ for each X. By definition, $\mathfrak{Div}^{[1]}$ is a category fibred in groupoids (over Sch), so by Lemma 3.2.26, this defines an isomorphism of categories fibred in groupoids $\mathfrak{Div}^{[1]} \xrightarrow{\sim} [\mathbb{A}^1/\mathbb{G}_m]$. □

Corollary 4.1.23. *$\mathfrak{Div}^{[1]}$ is an algebraic stack.*

We can similarly classify sequences of pairs $(L_1, s_1), \ldots, (L_n, s_n)$ by a single quotient stack.

Lemma 4.1.24. *For all $n \geqslant 2$, $[\mathbb{A}^n/\mathbb{G}_m] \cong \underbrace{[\mathbb{A}^1/\mathbb{G}_m] \times_{B\mathbb{G}_m} \cdots \times_{B\mathbb{G}_m} [\mathbb{A}^1/\mathbb{G}_m]}_{n}$.*

Proof. An X-point of the n-fold product $[\mathbb{A}^1/\mathbb{G}_m] \times_{B\mathbb{G}_m} \cdots \times_{B\mathbb{G}_m} [\mathbb{A}^1/\mathbb{G}_m]$ is the same thing as a $\mathbb{G}_m$-bundle $P \to X$ with a collection of equivariant maps $f_1, \ldots, f_n : P \to \mathbb{A}^1$, but this data is equivalent to the same bundle $P \to X$ with a single map $(f_1, \ldots, f_n) : P \to \mathbb{A}^n$, i.e. an X-point of the quotient stack $[\mathbb{A}^n/\mathbb{G}_m]$. □

4.2 Tame Stacky Curves

In this section, we give the construction of a (tame) root stack, originally defined in [Cad] and [AGV], and use it to classify tame stacky curves, following the article [GS].

4.2.1 Root Stacks

Suppose L is a line bundle on a scheme X. A natural question to ask is whether there exists another line bundle, say E, such that $E^r := E^{\otimes r} = L$ for a given integer $r \geqslant 1$. The following construction, originally found in [Cad] and [AGV], produces a stacky version of X called a root stack on which objects like $L^{1/r}$ live. An important application for our purposes is that every tame stacky curve over an algebraically closed field is a root stack. The authors in [VZB] use this to give a complete description of the canonical ring of tame stacky curve, which will be the subject of Section 5.1.

The question of when an rth root of a line bundle exists can alternatively be phrased in terms of cyclic G-covers (when the characteristic of the ground field does not divide $|G|$), and Kummer theory gives a natural answer to the question. Recall from Section 1.4.1 that a Kummer extension of fields is a Galois extension L/K with group $G = \mathbb{Z}/r\mathbb{Z}$. We will assume $(r, p) = 1$ when $\operatorname{char} K = p > 0$. By Theorem 1.4.6, every such extension is of the form

$$L = K[x]/(x^r - s) \quad \text{for some } s \in K^\times$$

when K contains all rth roots of unity, and the general case has a similar form. To understand cyclic extensions in the language of stacks, we have the following construction due independently to [Cad] and [AGV].

Definition 4.2.1. *For* $r \geqslant 1$, *the* **universal Kummer stack** *is the cover of stacks*

$$r : [\mathbb{A}^1/\mathbb{G}_m] \longrightarrow [\mathbb{A}^1/\mathbb{G}_m]$$

induced by $x \mapsto x^r$ *on both* $\mathbb{A}^1$ *and* $\mathbb{G}_m$.

Definition 4.2.2. *For a scheme* X, *a line bundle* $L \to X$ *with section* s *and an integer* $r \geqslant 1$, *the* r*th* **root stack** *of* X *along* (L, s), *written* $\sqrt[r]{(L,s)/X}$, *is defined to be the pullback*

$$\begin{array}{ccc} \sqrt[r]{(L,s)/X} & \longrightarrow & [\mathbb{A}^1/\mathbb{G}_m] \\ \downarrow & & \downarrow r \\ X & \longrightarrow & [\mathbb{A}^1/\mathbb{G}_m] \end{array}$$

where the bottom row is the morphism corresponding to (L, s) *via Proposition 4.1.22.*

Remark 4.2.3. Explicitly, for a test scheme T, the category $\sqrt[r]{(L,s)/X}(T)$ consists of tuples $(T \xrightarrow{\varphi} X, M, t, \psi)$ where $M \to T$ is a line bundle with section t and $\psi : M^r \xrightarrow{\sim} \varphi^* L$ is an isomorphism of line bundles such that $\psi(t^r) = \varphi^* s$.

To take iterated roots, we need to extend our definition of $\mathfrak{Div}^{[1]}$ to Deligne–Mumford stacks. This is implicitly used in the literature but let us carefully spell things out here. For a Deligne–Mumford stack $\mathcal{X}$, let $\mathfrak{Div}^{[1]}(\mathcal{X})$ be the collection of pairs $(\mathcal{L}, s)$ where $\mathcal{L}$ is a line bundle on $\mathcal{X}$ and s is a *section of* $\mathcal{L}$. As with schemes, there is a natural notion of morphisms $(\mathcal{L}, s) \to (\mathcal{L}', s')$ in $\mathfrak{Div}^{[1]}(\mathcal{X})$. A direct consequence of Proposition 4.1.22 is the following.

Corollary 4.2.4. *Let $\mathcal{X}$ be a Deligne–Mumford stack. There is an equivalence of categories*

$$\mathfrak{Div}^{[1]}(\mathcal{X}) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{Stacks}}(\mathcal{X}, [\mathbb{A}^1/\mathbb{G}_m]).$$

This extends the definition of a root stack to a stacky base: for a line bundle $\mathcal{L} \to \mathcal{X}$ over a stack $\mathcal{X}$ with section s, let $\sqrt[r]{(\mathcal{L}, s)/\mathcal{X}}$ be the pullback

$$\begin{array}{ccc} \sqrt[r]{(\mathcal{L}, s)/\mathcal{X}} & \longrightarrow & [\mathbb{A}^1/\mathbb{G}_m] \\ \downarrow & & \downarrow r \\ \mathcal{X} & \longrightarrow & [\mathbb{A}^1/\mathbb{G}_m] \end{array}$$

where the bottom row comes from Corollary 4.2.4. The following basic results and examples may be found in [Cad].

Lemma 4.2.5. *For any morphism of stacks $h : \mathcal{Y} \to \mathcal{X}$ and line bundle $\mathcal{L} \to \mathcal{X}$ with section s, there is an isomorphism of root stacks*

$$\sqrt[r]{(h^*\mathcal{L}, h^*s)/\mathcal{Y}} \xrightarrow{\sim} \sqrt[r]{(\mathcal{L}, s)/\mathcal{X}} \times_{\mathcal{X}} \mathcal{Y}.$$

Proof. Follows easily from either the definition of root stack as a pullback, or from the description of its T-points above (Remark 4.2.3). □

Example 4.2.6. Let X be the affine line $\mathbb{A}^1 = \operatorname{Spec} k[x]$. Then $L = \mathcal{O} = \mathcal{O}_X$ is a line bundle and the coordinate x gives a section of L. Choose $r \geqslant 1$ that is coprime to $\operatorname{char} k$. We claim that $\sqrt[r]{(\mathcal{O}, x)/\mathbb{A}^1} \cong [\mathbb{A}^1/\mu_r]$. By the comments above, for a test scheme T the category $\sqrt[r]{(\mathcal{O},x)/\mathbb{A}^1}(T)$ consists of tuples

$$(T \xrightarrow{\varphi} \mathbb{A}^1, M, t, \psi)$$

where $M \to T$ is a line bundle with section t and $\psi : M^r \xrightarrow{\sim} \varphi^*\mathcal{O}_X$ sending $t^r \mapsto \varphi^*x$. Note that $\varphi^*\mathcal{O}_X = \mathcal{O}_T$ so that φ^*x corresponds to a section $f \in H^0(T, \mathcal{O}_T)$ and thus determines a map $f : T \to \mathbb{A}^1$. On the other hand, $[\mathbb{A}^1/\mu_r](T)$ consists of principal μ_r-bundles $P \to T$ together with μ_r-equivariant morphisms $P \to \mathbb{A}^1$. Define a map $\sqrt[r]{(\mathcal{O},x)/\mathbb{A}^1}(T) \to [\mathbb{A}^1/\mu_r](T)$ by sending

$$(T \to X, M, t, \psi) \longmapsto \left(\begin{array}{l} M_0 \xrightarrow{h} \mathbb{A}^1 \\ \pi \big\downarrow \\ \;\; T \end{array} \right)$$

where $M_0 \to T$ and h are obtained as follows. The Kummer sequence

$$0 \to \mu_r \to \mathbb{G}_m \xrightarrow{r} \mathbb{G}_m \to 0$$

induces an exact sequence

$$H^1(T, \mu_r) \to H^1(T, \mathbb{G}_m) \xrightarrow{r} H^1(T, \mathbb{G}_m).$$

By exactness, the choice of trivialization $\psi : M^r \xrightarrow{\sim} \mathcal{O}_T$ determines a lift of M to a μ_r-bundle $M_0 \to T$. Then h is the map which takes a root of unity ζ in the fibre over $q \in T$ to the value $\zeta\sqrt[r]{f(q)} \in \mathbb{A}^1$, where $\sqrt[r]{f(q)}$ is a fixed rth root of

$f(q)$ which is determined after choosing an explicit trivialization of $\varphi^*\mathcal{O}_X = \mathcal{O}_T$. (So in each fibre, $1 \mapsto \sqrt[m]{f(q)}$.) This extends to an isomorphism of stacks

$$\sqrt[r]{(\mathcal{O}, x)/\mathbb{A}^1} \xrightarrow{\sim} [\mathbb{A}^1/\mu_r].$$

Example 4.2.7. More generally, when $X = \operatorname{Spec} A$ and $L = \mathcal{O}_X$ with any section s, we have

$$\sqrt[r]{(\mathcal{O}_X, s)/X} \cong [\operatorname{Spec} B/\mu_r] \quad \text{where } B = A[x]/(x^r - s).$$

To see this, note that $(\mathcal{O}_X, s)$ induces a morphism $\operatorname{Spec} A \to [\mathbb{A}^1/\mathbb{G}_m]$ by Proposition 4.1.22. Lifting along the $\mathbb{G}_m$-cover $\mathbb{A}^1 \to [\mathbb{A}^1/\mathbb{G}_m]$, we get a map $F : X \to \mathbb{A}^1$ making the diagram

$$\begin{array}{ccc} & & \mathbb{A}^1 \\ & \overset{F}{\nearrow} & \downarrow \\ X & \longrightarrow & [\mathbb{A}^1/\mathbb{G}_m] \end{array}$$

commute and such that $\mathcal{O}_X = F^*\mathcal{O}_{\mathbb{A}^1}$ and $s = F^*x$. By Lemma 4.2.5,

$$\begin{aligned} \sqrt[r]{(\mathcal{O}_X, s)/X} &\cong \sqrt[r]{(\mathcal{O}_{\mathbb{A}^1}, x)/\mathbb{A}^1} \times_{\mathbb{A}^1} X \\ &\cong [\mathbb{A}^1/\mu_r] \times_{\mathbb{A}^1} X \quad \text{by Example 4.2.6} \\ &\cong [\operatorname{Spec} B/\mu_r] \end{aligned}$$

for $B = k[x]/(x^r - s) \otimes_k A = A[x]/(x^r - s)$. In general, any root stack $\sqrt[r]{(L, s)/X}$

may be covered by such "affine" root stacks $[\operatorname{Spec} B/\mu_r]$:

Proposition 4.2.8. *Let* $\mathcal{X} = \sqrt[r]{(L,s)/X}$ *be an* r*th root stack of a scheme* X *along a pair* (L,s)*, with coarse map* $\pi : \mathcal{X} \to X$*. Then for any point* $\bar{x} : \operatorname{Spec} k \to \mathcal{X}$*, there is an affine étale neighborhood* $U = \operatorname{Spec} A \to X$ *of* $x = \pi \circ \bar{x}$ *such that* $U \times_X \mathcal{X} \cong [\operatorname{Spec} B/\mu_r]$*, where* $B = A[x]/(x^r - s)$*.*

Proof. This is an easy consequence of Lemma 4.2.5 and Example 4.2.7. □

Theorem 4.2.9. *If* $\mathcal{X}$ *is a Deligne–Mumford stack with line bundle* $\mathcal{L}$ *and section* s *and* r *is invertible on* $\mathcal{X}$*, then* $\sqrt[r]{(\mathcal{L},s)/\mathcal{X}}$ *is a Deligne–Mumford stack.*

Proof. See [Cad, 2.3.3]. The technique of this proof will be used to prove an analogous result for Artin–Schreier root stacks (Theorem 4.3.13). □

Example 4.2.10. If s is a nonvanishing section of a line bundle $L \to X$ over a scheme, then $\sqrt[r]{(L,s)/X} \cong X$ as stacks. To see this, note that the following statements are equivalent:

(a) s is nonvanishing.

(b) L is trivial.

(c) $L_0 \cong X \times \mathbb{G}_m$ as principal bundles over X.

(d) The induced map $X \to [\mathbb{A}^1/\mathbb{G}_m]$ factors through $X \to [\mathbb{G}_m/\mathbb{G}_m] = \operatorname{Spec} k$.

Further, (d) implies that $\sqrt[r]{(L,s)/X} \to X$ is an isomorphism, so (a) does as well. So for any pair (L,s), the stacky structure of $\sqrt[r]{(L,s)/X}$ occurs precisely at the vanishing locus of s.

Example 4.2.11. Let L be a line bundle on a scheme X and consider the pull-back:

$$\begin{array}{ccc} X \times_{B\mathbb{G}_m} B\mathbb{G}_m & \longrightarrow & B\mathbb{G}_m \\ \downarrow & & \downarrow r \\ X & \xrightarrow{\;L\;} & B\mathbb{G}_m \end{array}$$

Here, the bottom row is induced from the line bundle L, using that $B\mathbb{G}_m$ classifies line bundles. For the zero section 0 of L, we can view the root stack $\sqrt[r]{(L,0)/X}$ as an infinitesimal thickening of this fibre product $X \times_{B\mathbb{G}_m} B\mathbb{G}_m$. To see this explicitly, note that by Proposition 4.2.8, $\sqrt[r]{(L,0)/X}$ may be covered by root stacks of the form $[\operatorname{Spec} B/\mu_r]$, where $B = A[x]/(x^r - 0) = A[x]/(x^r) = A \otimes_k k[x]/(x^r)$ for some $\operatorname{Spec} A \subseteq X$. Thus $[\operatorname{Spec} B/\mu_r] \cong \operatorname{Spec} A \times B\mu_r$, which is indeed an infinitesimal thickening of $\operatorname{Spec} A \times_{B\mathbb{G}_m} B\mathbb{G}_m$. Alternatively: $X \times_{B\mathbb{G}_m} B\mathbb{G}_m$ is the closed substack of $\sqrt[r]{(L,0)/X}$ whose T-points for a scheme T are given by

$$(X \times_{B\mathbb{G}_m} B\mathbb{G}_m)(T) = \{(T \to X, M, t, \psi) \mid \psi(t^r) = 0\}.$$

In some places in the literature, the notation $\sqrt[r]{L/X}$ is used for the stack $X \times_{B\mathbb{G}_m} B\mathbb{G}_m$, in which case $\sqrt[r]{L/X} \hookrightarrow \sqrt[r]{(L,0)/X}$ is an intuitive notation for this infinitesimal thickening.

Note that if p divides r, the root stack construction of [Cad] and [AGV] is still well-defined, but Theorem 4.2.9 fails. Thus to be able to study pth order (and more general) stacky structure in characteristic p, we must work with a different notion of root stack.

4.2.2 Classification Results

In [GS], the authors prove that, with some hypotheses, a tame Deligne–Mumford stack of finite type over k may be obtained from its coarse space by iterating two operations: (i) the root stack construction and (ii) the canonical stack construction. In plain terms, this means that such a stack can be determined from its singularities and its stacky structure. We will not discuss canonical stacks here, but for stacky curves, the main theorem in [GS] reduces to the following.

Theorem 4.2.12. *Let $\mathcal{X}$ be a tame stacky curve with coarse map $\pi : \mathcal{X} \to X$. Then $\mathcal{X}$ is isomorphic to the fibre product*

$$\sqrt[r_1]{(L_1, s_1)/X} \times_X \cdots \times_X \sqrt[r_n]{(L_n, s_n)/X}$$

where the ramification locus of π is $\{x_1, \ldots, x_n\}$ and for $1 \leqslant i \leqslant n$, the pair (L_i, s_i) corresponds to the effective divisor x_i.

Corollary 4.2.13 ([VZB, Lem. 5.3.10])**.** *Let $\mathcal{X}$ be a tame stacky curve. Then*

(a) The isomorphism class of $\mathcal{X}$ is completely determined by the coarse space, the stacky locus and the set of stabilizer groups of the stacky points of $\mathcal{X}$.

(b) For each stacky point $x \in |\mathcal{X}|$, the stabilizer group G_x is isomorphic to μ_n for some $n \geqslant 2$.

(c) Zariski-locally, X is isomorphic to V/G_x for a scheme V.

We will see in Section 4.3.2 that such a clean classification fails in the wild case, but Artin–Schreier theory can be used to tackle the classification problem at least for wild stacky curves with $\mathbb{Z}/p\mathbb{Z}$-stabilizer groups.

4.3 Wild Stacky Curves: The $\mathbb{Z}/p\mathbb{Z}$ Case

In this section we classify stacky curves in characteristic $p > 0$ with cyclic stabilizers of order p using higher ramification data in the sense of Chapter 1. This approach replaces the local root stack structure of a tame stacky curve with a more sensitive structure called an Artin–Schreier root stack, allowing us to incorporate this ramification data directly into the stack. We rigorously define Artin–Schreier root stacks in Section 4.3.1 and use their properties to classify wild stacky curves with $\mathbb{Z}/p\mathbb{Z}$ stabilizers in Section 4.3.2.

4.3.1 Artin–Schreier Root Stacks

When $\operatorname{char} k = p > 0$ and we want to compute a pth root of a line bundle, the Frobenius immediately presents problems. Specifically, the cover $[\mathbb{A}^1/\mathbb{G}_m] \to [\mathbb{A}^1/\mathbb{G}_m]$ induced by $x \mapsto x^p$ is *not* étale. To remedy this, we can once again rephrase the question in terms of cyclic G-covers. This time, we will make use of Artin–Schreier theory (Section 1.4.2) when $G = \mathbb{Z}/p\mathbb{Z}$ and Artin–Schreier–Witt theory (Section 1.4.3) in the general case.

Suppose we have a line bundle $L \to X$ and a section $s \in H^0(X, L)$ of which we would like to find a pth root, i.e. a pair (E, t) with $(E^{\otimes p}, t^p) \cong (L, s)$. By Proposition 4.1.22, such pairs (L, s) are classified by X-points of the quotient stack $[\mathbb{A}^1/\mathbb{G}_m]$. For our purposes, the algebraic stack $[\mathbb{A}^1/\mathbb{G}_m]$ is not sensitive enough to keep track of the extra information present in Artin–Schreier theory, namely the ramification jump.

Instead, let $\mathbb{P}(1, m)$ be the weighted projective line for an integer $m \geqslant 1$ which is coprime to p, as in Example 3.5.31. Note that some authors view $\mathbb{P}(1, m)$ as a scheme, in which case it is the projective line with homogeneous coordinates $[x, y]$ corresponding to the graded ring $k[x_0, x_1]$, but with a generator x_0 in degree 1 and a generator x_1 in degree m. However, it is more natural to view $\mathbb{P}(1, m)$ as a stack with a single nontrivial stabilizer group $\mathbb{Z}/m\mathbb{Z}$ at the point $\infty = [0, 1]$. In particular, $\mathbb{P}(1, m)$ is a stacky curve and the natural morphism $\mathbb{P}(1, m) \to \mathbb{P}^1$ given by sending $[x, y] \mapsto [x, z]$, where $z = y^m$, is a coarse moduli map.

Lemma 4.3.1. *For any $m \geqslant 1$ coprime to p, there is an isomorphism of stacks*

$$\mathbb{P}(1, m) := [\mathbb{A}^2 \smallsetminus \{0\}/\mathbb{G}_m] \cong \sqrt[m]{(\mathcal{O}(1), s_\infty)/\mathbb{P}^1}$$

where $\mathbb{G}_m$ acts on $\mathbb{A}^2 \smallsetminus \{0\}$ with weights $(1, m)$ and s_∞ is the section whose divisor is the point $[0, 1]$. Furthermore, $\mathbb{P}(1, m)$ is a Deligne–Mumford stack.

Proof. This is clear from the structure of $\mathbb{P}(1, m)$ on the standard covering by affine opens, along with Proposition 4.2.8. Then Theorem 4.2.9 implies $\mathbb{P}(1, m)$ is Deligne–Mumford. □

Fix $m \geqslant 1$ coprime to p and let $\mathfrak{Div}^{[1,m]}(X)$ be the category consisting of triples (L, s, f) with $L \in \mathrm{Pic}(X)$ and sections $s \in H^0(X, L)$ and $f \in H^0(X, L^m)$ that don't vanish simultaneously. Morphisms $(L, s_L, f_L) \to (M, s_M, f_M)$ in $\mathfrak{Div}^{[1,m]}(X)$ are given by bundle isomorphisms

$$\begin{array}{ccc} L & \xrightarrow{\varphi} & M \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

under which $\varphi(s_L) = s_M$ and $\varphi^{\otimes m}(f_L) = f_M$. Then $\mathfrak{Div}^{[1,m]}$ is a category fibred in groupoids over Sch_k. In analogy with Proposition 4.1.22, we have:

Proposition 4.3.2. *For each* $m \geqslant 1$, *there is an isomorphism of categories fibred in groupoids* $\mathfrak{Div}^{[1,m]} \cong \mathbb{P}(1, m)$.

Proof. A map $X \to [\mathbb{A}^2 \smallsetminus \{0\}/\mathbb{G}_m]$ is equivalent to a map $X \to [\mathbb{A}^2/\mathbb{G}_m]$ avoiding $(0, 0)$, where $\mathbb{G}_m$ acts on $\mathbb{A}^2$ with weights $(1, m)$. Let $[\mathbb{A}^2/\mathbb{G}_m] \to B\mathbb{G}_m$ be the forgetful map. Then by the universal property of pullbacks, the map $X \to [\mathbb{A}^2/\mathbb{G}_m]$ is equivalent to the choice of a map $g : X \to B\mathbb{G}_m$ and a section σ of the line bundle $L = X \times_{B\mathbb{G}_m} [\mathbb{A}^2/\mathbb{G}_m] \to X$. The proof of Lemma 4.1.24 carries through when $\mathbb{G}_m$ acts on $\mathbb{A}^2$ with any weights, giving $[\mathbb{A}^2/\mathbb{G}_m] \cong [\mathbb{A}^1/\mathbb{G}_m] \times_{B\mathbb{G}_m} [\mathbb{A}^1/\mathbb{G}_m]$ with weights $(1, m)$. Then σ really corresponds to a section s of L, the line bundle associated to $g^*E\mathbb{G}_m$, and a section f of L^m, the line bundle associated to $(g^*E\mathbb{G}_m)^m$. All of these choices are natural, so we have constructed an equivalence of categories $\mathfrak{Div}^{[1,m]}(X) \xrightarrow{\sim} \mathbb{P}(1, m)(X)$ for each X. As in the proof of Proposition 4.1.22, Lemma 3.2.26 guarantees that this extends to an isomorphism $\mathfrak{Div}^{[1,m]} \xrightarrow{\sim} \mathbb{P}(1,m)$ of categories fibred in groupoids. $\square$

Corollary 4.3.3. $\mathfrak{Div}^{[1,m]}$ *is a Deligne–Mumford stack.*

Remark 4.3.4. It also follows from Lemma 4.3.1 and Proposition 4.3.2 that $\mathfrak{Div}^{[1,m]}$ is a root stack, namely $\sqrt[m]{(\mathcal{O}(1), s_\infty)/\mathbb{P}^1}$ but it will be useful for later arguments to exhibit this isomorphism directly, which we do now. For any scheme X, a functor

$$\Upsilon : \sqrt[m]{(\mathcal{O}(1), s_\infty)/\mathbb{P}^1}(X) \longrightarrow \mathfrak{Div}^{[1,m]}(X)$$

can be built in the following way. For an object $\mathcal{B} = (X \xrightarrow{\varphi} \mathbb{P}^1, L, t, L^m \xrightarrow{\sim} \varphi^*\mathcal{O}(1))$ of the root stack, the morphism φ induces two sections $\varphi^* s_0, \varphi^* s_\infty \in H^0(X, \varphi^*\mathcal{O}(1))$ and under the isomorphism $L^m \cong \varphi^*\mathcal{O}(1)$, $\varphi^* s_\infty$ may be identified with t^m. Set $\Upsilon(\mathcal{B}) = (L, s, f)$ where $s = t = s_\infty^{1/m}$ and $f = s_0$. Naturality of Υ is clear from the definitions of morphisms in each category. One can check this gives the same isomorphism of stacks as Proposition 4.3.2.

As in Section 4.2.1, we extend the definition of $\mathfrak{Div}^{[1,m]}$ to a stacky base $\mathcal{X}$ by taking $\mathfrak{Div}^{[1,m]}(\mathcal{X})$ to be the category of triples $(\mathcal{L}, s, f)$ where $\mathcal{L}$ is a line bundle on $\mathcal{X}$ and s and f are sections on $\mathcal{L}$ and $\mathcal{L}^m$, respectively. Then Proposition 4.3.2 implies the following.

Corollary 4.3.5. *Let* $\mathcal{X}$ *be a Deligne–Mumford stack. For each* $m \geqslant 1$, *there is an* *equivalence of categories*

$$\mathfrak{Div}^{[1,m]}(\mathcal{X}) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{Stacks}}(\mathcal{X}, \mathbb{P}(1,m)).$$

The main feature of $\mathbb{P}(1,m)$ that makes it valuable to our program of study is the fact that the cyclic order p isogeny $\wp : \mathbb{G}_a \to \mathbb{G}_a, \alpha \mapsto \alpha^p - \alpha$ extends to a ramified cyclic p-cover $\Psi : \mathbb{P}(1,m) \to \mathbb{P}(1,m)$ given by $[u,v] \mapsto [u^p, v^p - vu^{m(p-1)}]$. Meanwhile, $\mathbb{G}_a$ acts on $\mathbb{P}(1,m)$ via $\alpha \cdot [u,v] = [u, v + \alpha u^m]$ and it is easy to check this action commutes with Ψ, so we get an induced morphism on the quotient stack $[\mathbb{P}(1,m)/\mathbb{G}_a]$. We now use $[\mathbb{P}(1,m)/\mathbb{G}_a]$ to construct a characteristic p analogue of the root stack in Section 4.2.1.

Definition 4.3.6. *Let* $m \geqslant 1$ *be coprime to* p*. The* **universal Artin–Schreier cover with ramification jump** m *is the cover of stacks*

$$\wp_m : [\mathbb{P}(1,m)/\mathbb{G}_a] \longrightarrow [\mathbb{P}(1,m)/\mathbb{G}_a]$$

induced by $[u,v] \mapsto [u^p, v^p - vu^{m(p-1)}]$ *on* $\mathbb{P}(1,m)$ *and* $\alpha \mapsto \alpha^p - \alpha$ *on* $\mathbb{G}_a$.

The following definition is inspired by a short article [Ryd] by D. Rydh and email correspondence between him and the author. The work in [Ryd] ultimately dates back to discussions between Rydh and A. Kresch in October 2010, and this portion of the thesis might be viewed as a realization of some of the questions about wildly ramified stacks that first arose then.

Definition 4.3.7. *For a stack* $\mathcal{X}$*, a line bundle* $\mathcal{L} \to \mathcal{X}$ *and sections* s *of* $\mathcal{L}$ *and* f *of*

$\mathcal{L}^m$, *the* **Artin–Schreier root stack** $\wp_m^{-1}((\mathcal{L}, s, f)/\mathcal{X})$ *is defined to be the normalized pullback of the diagram*

$$\begin{array}{ccc} \wp_m^{-1}((\mathcal{L}, s, f)/\mathcal{X}) & \longrightarrow & [\mathbb{P}(1, m)/\mathbb{G}_a] \\ \downarrow & \lrcorner^{\nu} & \downarrow \wp_m \\ \mathcal{X} & \longrightarrow & [\mathbb{P}(1, m)/\mathbb{G}_a] \end{array}$$

where the bottom row is the composition of the morphism $\mathcal{X} \to \mathbb{P}(1, m)$ *corresponding to* $(\mathcal{L}, s, f)$ *by Corollary 4.3.5 and the quotient map* $\mathbb{P}(1, m) \to [\mathbb{P}(1, m)/\mathbb{G}_a]$.

That is, $\wp_m^{-1}((\mathcal{L}, s, f)/\mathcal{X}) = \mathcal{X} \times^{\nu}_{[\mathbb{P}(1,m)/\mathbb{G}_a]} [\mathbb{P}(1, m)/\mathbb{G}_a]$ with respect to the universal Artin–Schreier cover $\wp_m : [\mathbb{P}(1, m)/\mathbb{G}_a] \to [\mathbb{P}(1, m)/\mathbb{G}_a]$.

Remark 4.3.8. Proposition 4.3.2 showed that for a scheme X, the X-points of $\mathbb{P}(1, m)$ are given by triples (L, s, f) for a line bundle $L \to X$ and two non-simultaneously vanishing sections $s \in H^0(X, L)$ and $f \in H^0(X, L^m)$. However, the classifying map used to define an Artin–Schreier root stack construction has target $[\mathbb{P}(1, m)/\mathbb{G}_a]$, so a global section of L^m is sometimes more than what is necessary. In fact, the X-points $[\mathbb{P}(1, m)/\mathbb{G}_a](X)$ by definition are $\mathbb{G}_a$-torsors $P \to X$ together with $\mathbb{G}_a$-equivariant maps $P \to \mathbb{P}(1, m)$. Assuming s is a *regular section* of L (i.e. a nonzero divisor in each stalk), let $D = \operatorname{div}(s)$, so that $L = \mathcal{O}_X(D)$, and consider the short exact sequence of sheaves

$$0 \to \mathcal{O}_X \xrightarrow{s^m} \mathcal{O}_X(mD) \to \mathcal{O}_X(mD)|_{mD} \to 0.$$

This induces a long exact sequence in sheaf cohomology:

$$0 \to H^0(X, \mathcal{O}_X) \to H^0(X, \mathcal{O}_X(mD)) \to H^0(mD, \mathcal{O}_X(mD)|_{mD}) \xrightarrow{\delta} H^1(X, \mathcal{O}_X) \to \cdots .$$

So one way to produce a $\mathbb{G}_a$-torsor on X is to take the image under the connecting homomorphism δ of a section $f \in H^0(mD, \mathcal{O}_X(mD)|_{mD})$. That is, $P = \delta(f)$ where f is a section of *the restriction of* L^m *to the divisor* mD. One can show that this is bijective: the X-points of $[\mathbb{P}(1, m)/\mathbb{G}_a]$ are in one-to-one correspondence with triples (L, s, f) with (L, s) as usual and f a section of L^m supported on the divisor mD.

Now we can give an explicit description of the points of $\wp_m^{-1}((L, s, f)/X)$ in the style of Remark 4.2.3. If X is a scheme, let $\mathcal{V} = X \times_{[\mathbb{P}(1,m)/\mathbb{G}_a]} [\mathbb{P}(1, m)/\mathbb{G}_a]$ be the actual pullback of the diagram in Definition 4.3.7. Then for a test scheme T the category $\mathcal{V}(T)$ consists of tuples $(T \xrightarrow{\varphi} X, M, t, g, \psi)$ where $M \to T$ is a line bundle with section $t \in H^0(T, M)$, $g \in H^0(mE, M^m|_{mE})$ (where $E = (t)$) and $\psi : M^p \xrightarrow{\sim} \varphi^*L$ is an isomorphism of line bundles such that

$$\psi(t^p) = \varphi^* s \quad \text{and} \quad \psi_{mpE}(g^p - t^{m(p-1)}g) = \varphi^*_{mD} f$$

where $(-)_{mpE}$ denotes the restriction to mpE and likewise for φ^*_{mD}. By [AB, Prop. A.7], the T-points of $\wp_m^{-1}((L, s, f)/X) = \mathcal{V}^\nu$ has the same description when T is a normal scheme. In general, the defining equations on t and g are more complicated. There is a similar description of the T-points of the Artin–Schreier root stack $\wp_m^{-1}((\mathcal{L}, s, f)/\mathcal{X})$ when $\mathcal{X}$ is a stack.

For our purposes, namely when X (resp. $\mathcal{X}$) is a curve (resp. stacky curve), we will not need this level of control over the sections f. Indeed, étale-locally, $H^1(X, \mathcal{O}_X) = 0$ so any $f \in H^0(mD, L^m|_{mD})$ as above lifts to a section $F \in H^0(X, L^m)$. Therefore, étale-locally the T-points of $\wp_m^{-1}((L, s, f)/X)$ are given by (φ, M, t, g, ψ) where $g \in H^0(T, M^m)$ and the rest are as above (when T is normal). While some of our results require global sections, the computations happen locally so this technical point is not a significant issue in the present article.

Lemma 4.3.9. *For any morphism of stacks $h : \mathcal{Y} \to \mathcal{X}$ and line bundle $\mathcal{L} \to \mathcal{X}$ with sections s of $\mathcal{L}$ and f of $\mathcal{L}^m$, there is an isomorphism of algebraic stacks*

$$\wp_m^{-1}((h^*\mathcal{L}, h^*s, h^*f)/\mathcal{Y}) \xrightarrow{\sim} \wp_m^{-1}((\mathcal{L}, s, f)/\mathcal{X}) \times_{\mathcal{X}}^{\nu} \mathcal{Y}.$$

Proof. As before, this is an immediate consequence of the definition or the explicit description in Remark 4.3.8. □

4.3.2 Classification Results

We start with a key computation.

Example 4.3.10. Let $X = \mathbb{P}^1 = \operatorname{Proj} k[x_0, x_1]$ and suppose $Y \to X$ is the smooth projective model of the one-point cover given by the affine Artin–Schreier equation $y^p - y = x^{-m}$. Then Y admits an additive $\mathbb{Z}/p\mathbb{Z}$-action such that $Y \to X$ is a Galois cover with group $\mathbb{Z}/p\mathbb{Z}$. There is an isomorphism of stacks

$$\wp_m^{-1}((\mathcal{O}(1), x_0, x_1^m)/\mathbb{P}^1) \cong [Y/(\mathbb{Z}/p\mathbb{Z})]$$

which we describe now. As in Remark 4.3.8, let $\mathcal{V}$ be the pullback $\mathbb{P}^1 \times_{[\mathbb{P}(1,m)/\mathbb{G}_a]} [\mathbb{P}(1,m)/\mathbb{G}_a]$ so that $\wp_m^{-1}((\mathcal{O}(1), x_0, x_1^m)/\mathbb{P}^1) = \mathcal{V}^\nu$ is the normalized pullback (Definition 3.5.22). Also let $Y_0 \to \mathbb{P}^1$ be the projective closure of the affine curve given by the equation $y^p x^m - y x^m = 1$, which is in general not normal. Then by [AB, Prop. A.7], it is enough to construct an isomorphism $\mathcal{V}(T) \cong [Y_0/(\mathbb{Z}/p\mathbb{Z})](T)$ for any *normal* test scheme T. By Remark 4.3.8, assuming T is "local enough", the category $\mathcal{V}(T)$ consists of tuples $(T \xrightarrow{\varphi} \mathbb{P}^1, L, s, f, \psi)$ where $\psi : L^p \xrightarrow{\sim} \varphi^*\mathcal{O}(1)$ is an isomorphism such that $\psi(s^p) = \varphi^* x_0$ and $\psi(f^p - f s^{m(p-1)}) = \varphi^* x_1^m$. Define a mapping

$$\begin{aligned} \mathcal{V}(T) &\longrightarrow [Y_0/(\mathbb{Z}/p\mathbb{Z})](T) \\ (\varphi, L, s, f, \psi) &\longmapsto \left(\begin{array}{ccc} \widetilde{L} & \longrightarrow & Y_0 \\ \downarrow & & \\ T & & \end{array} \right) \end{aligned}$$

where $\widetilde{L} \to T$ is the $\mathbb{Z}/p\mathbb{Z}$-bundle obtained by first constructing a $\mathbb{G}_a$-bundle $P \to T$ and showing its transition maps actually take values in $\mathbb{Z}/p\mathbb{Z} \subseteq \mathbb{G}_a$.

Fix a cover $\{U_i \to T\}$ over which $L \to T$ is trivial; we may choose the U_i small enough so that on each, either s or f is nonzero. Then P can be constructed by specifying transition functions $\varphi_{ij} : U_i \cap U_j \to \mathbb{G}_a$ for any pair

U_i, U_j in the cover. Let $s_i = s|_{U_i}$ and $f_i = f|_{U_i}$. There are three cases to consider. First, if s is nonzero on both U_i and U_j, then we let

$$\varphi_{ij} : t \longmapsto \frac{f_i(t)}{s_i(t)^m} - \frac{f_j(t)}{s_j(t)^m} \in \mathbb{G}_a.$$

If s vanishes somewhere on U_i and U_j, then set

$$\varphi_{ij} : t \longmapsto \frac{s_i(t)^m}{f_i(t)} - \frac{s_j(t)^m}{f_j(t)}.$$

Finally, if s does not vanish on U_i but does vanish somewhere on U_j, we let

$$\varphi_{ij} : t \longmapsto \frac{f_i(t)}{s_i(t)^m} - \frac{s_j(t)^m}{f_j(t)}.$$

It is easy to see that if s is nonzero on three charts U_i, U_j, U_k, then the transition maps satisfy the additive cocycle relation $\varphi_{ij} + \varphi_{jk} + \varphi_{ki} = 0$. Similarly, for s vanishing on any combination of U_i, U_j, U_k, the same cocycle relation holds, e.g. if s vanishes on U_i but not on U_j or U_k, then

$$\varphi_{ij} + \varphi_{jk} + \varphi_{ki} = \frac{s_i(t)^m}{f_i(t)} - \frac{f_j(t)}{s_j(t)^m} + \frac{f_j(t)}{s_j(t)^m} - \frac{f_k(t)}{s_k(t)^m} + \frac{f_k(t)}{s_k(t)^m} - \frac{s_i(t)^m}{f_i(t)} = 0.$$

Therefore (φ_{ij}) defines a $\mathbb{G}_a$-bundle $P \to T$ which is trivial over the cover $\{U_i\}$. One can show that the total space of P is $(L \setminus \{0\} \times L^m)/\mathbb{G}_m$ over the locus where s vanishes, while over the locus where f vanishes, the total space is $(L \times L^m \setminus \{0\})/\mathbb{G}_m$, where $\mathbb{G}_m$ acts on fibres with weights $(1, m)$ in both cases.

Over any of the U_i, P is trivialized by

$$\begin{aligned} \varphi_i : U_i \times \mathbb{G}_a &\longrightarrow P|_{U_i} \\ (t, \alpha) &\longmapsto [s(t), f(t) + \alpha s(t)^m]. \end{aligned}$$

Alternatively, P can be defined as in Remark 4.3.8. By construction, the transition maps are all $\mathbb{Z}/p\mathbb{Z}$-valued since s and f satisfy the equation $f^p - fs^{m(p-1)} = \varphi^* x_1^m$, which over each type of trivialization looks like one of

$$\left(\frac{f}{s^m}\right)^p - \frac{f}{s^m} = \left(\frac{\varphi^* x_1}{s^p}\right)^m \qquad \text{or} \qquad \left(\frac{s^m}{f}\right)^p - \frac{s^m}{f} = -\frac{(s\varphi^* x_1)^m}{f^{p+1}}.$$

Thus (φ_{ij}) actually determine a $\mathbb{Z}/p\mathbb{Z}$-bundle $\widetilde{L} \to T$. One can also consider the Artin–Schreier sequence

$$0 \to \mathbb{Z}/p\mathbb{Z} \to \mathbb{G}_a \xrightarrow{\wp} \mathbb{G}_a \to 0.$$

This induces an exact sequence

$$H^1(T, \mathbb{Z}/p\mathbb{Z}) \to H^1(T, \mathbb{G}_a) \xrightarrow{\wp} H^1(T, \mathbb{G}_a).$$

Then $\wp(P)$ is the $\mathbb{G}_a$-bundle with transition functions $\wp(\varphi_{ij}) = \varphi_{ij}^p - \varphi_{ij}$, so using the defining equation for s and f, one can again see that $\wp(\varphi_{ij}) \in Z^1(T, \mathbb{G}_a)$ is trivial in $H^1(T, \mathbb{G}_a)$. Therefore by exactness, P lifts to $\widetilde{L} \in H^1(T, \mathbb{Z}/p\mathbb{Z})$.

Next, there is a map $\widetilde{L} \to Y_0$ given by sending

$$(t,0) \in \widetilde{L}_t \longmapsto \left(\frac{s(t)}{\varphi^* x_1(t)}, \frac{f(t)}{s(t)^m}\right)$$

and extending by the $\mathbb{Z}/p\mathbb{Z}$-action on the fibre $\widetilde{L}_t$. That is, the entire fibre over t is mapped to the Galois orbit of a corresponding point on Y_0. The resulting morphism

$$\mathcal{V}(T) \to [Y_0/(\mathbb{Z}/p\mathbb{Z})](T)$$

is an isomorphism for all T, and it is easy to check this is functorial in T. This proves that for any sufficiently local test scheme T (in the sense of Remark 4.3.8) which is normal, there is an isomorphism $\mathcal{V}(T) \xrightarrow{\sim} [Y_0/(\mathbb{Z}/p\mathbb{Z})](T)$. The argument can be extended to all normal schemes T, so by [AB, Prop. A.7], we get an isomorphism of stacks $\wp_m^{-1}((\mathcal{O}(1), x_0, x_1^m)/\mathbb{P}^1) \cong [Y/(\mathbb{Z}/p\mathbb{Z})]$.

A similar analysis shows that for an arbitrary curve X and for any $F \in k(X) \setminus \wp(k(X))$, there is an isomorphism

$$\wp_m^{-1}((L,\sigma,\tau)/X) \xrightarrow{\sim} [Y_F/(\mathbb{Z}/p\mathbb{Z})]$$

where (L,σ) corresponds to the divisor $\mathrm{div}(F) \in \mathrm{Div}(X)$, τ is a section of L^m such that σ restricts to a local parameter at any zero of τ, Y_F is the Galois cover of X with birational Artin–Schreier equation $y^p - y = F(x)$.

Example 4.3.11. When $m \equiv -1 \pmod{p}$, we even have $\wp_m^{-1}((\mathcal{O}(1), x_0, x_1^m)/\mathbb{P}^1) \cong \mathcal{V}$, i.e. the pullback of $\mathbb{P}^1$ along the universal Artin–Schreier cover $\wp_m : [\mathbb{P}(1,m)/\mathbb{G}_a] \to$

$[\mathbb{P}(1,m)/\mathbb{G}_a]$ is already a normal stack. To show this, we need to write down an integral equation for $\wp_m^{-1}((\mathcal{O}(1),x_0,x_1)/\mathbb{P}^1)\times_{\mathbb{P}^1} T$ for any sufficiently local test scheme T and show that it can be mapped to an integral equation for $Y\times_{\mathbb{P}^1} T\to\mathbb{P}^1_T$. In fact, the equation $f^p - fs^{m(p-1)} = \varphi^* x_1^m$ on the level of sections can be written

$$\left(\frac{fs}{\varphi^* x_1^n}\right)^p - \left(\frac{fs}{\varphi^* x_1^n}\right)\left(\frac{s^p}{\varphi^* x_1}\right)^{n(p-1)} = \frac{s^p}{\varphi^* x_1}$$

where, as in the notation of Lemma 2.2.50, $m+1=pn$. This pulls back to a similar equation on each $\wp_m^{-1}((\mathcal{O}(1),x_0,x_1^m)/\mathbb{P}^1)\times_{\mathbb{P}^1} T$ which then maps to $z^p - zx^{n(p-1)} = x$ on $Y\times_{\mathbb{P}^1} T$ by sending

$$\left(\frac{fs}{\varphi^* x_1^n}\right)\longmapsto z \quad\text{and}\quad \left(\frac{s^p}{\varphi^* x_1}\right)\longmapsto x.$$

By Lemma 2.2.50, $z^p - zx^{n(p-1)} = x$ is an integral equation for $Y\times_{\mathbb{P}^1} T\to\mathbb{P}^1_T$ in this case since $m = pn-1$.

In general, every Artin–Schreier root stack $\wp_m^{-1}((L,s,f)/X)$ can be covered in the étale topology by "elementary" Artin–Schreier root stacks of the form $[Y/(\mathbb{Z}/p\mathbb{Z})]$ as above – this is completely analogous to the Kummer case described by Proposition 4.2.8 but here is the formal statement.

Proposition 4.3.12. *Let* $\mathcal{X} = \wp_m^{-1}((L,s,f)/X)$ *be an Artin–Schreier root stack of a scheme* X *with jump* m *along a triple* (L,s,f) *and let* $\pi:\mathcal{X}\to X$ *be the coarse map. Then for any point* $\bar{x}:\operatorname{Spec} k\to\mathcal{X}$*, there is an étale neighborhood* U *of* $x=\pi\circ\bar{x}$

such that $U \times_X \mathcal{X} \cong [Y/(\mathbb{Z}/p\mathbb{Z})]$, *where* Y *is the Galois* $\mathbb{Z}/p\mathbb{Z}$*-cover of* U *given by an affine Artin–Schreier equation* $y^p - y = F(x)$.

Also, we have the following analogue of Theorem 4.2.9 for Artin–Schreier root stacks:

Theorem 4.3.13. *If* $\mathcal{X}$ *is a Deligne–Mumford stack over a perfect field* k *of characteristic* $p > 0$, m *is an integer relatively prime to* p *and* $\mathcal{L} \to \mathcal{X}$ *is a line bundle with sections* s *of* $\mathcal{L}$ *and* f *of* $\mathcal{L}^m$, *then* $\wp_m^{-1}((\mathcal{L}, s, f)/\mathcal{X})$ *is a Deligne–Mumford stack.*

Proof. Take an étale map $p : U \to \mathcal{X}$ such that $p^*\mathcal{L}$ is trivial on the scheme U. It suffices to show $\wp_m^{-1}((\mathcal{L}, s, f)/\mathcal{X}) \times_{\mathcal{X}} U$ is Deligne–Mumford. The corresponding map $U \to \mathbb{P}(1, m)$ induced by $(p^*\mathcal{L}, p^*s, p^*f)$ lifts the composition $U \to \mathcal{X} \to [\mathbb{P}(1, m)/\mathbb{G}_a]$ along the quotient map $\mathbb{P}(1, m) \to [\mathbb{P}(1, m)/\mathbb{G}_a]$. Consider the composition $U \to \mathbb{P}(1, m) \to \mathbb{P}^1$. By Lemma 4.3.9,

$$\wp_m^{-1}((\mathcal{L}, s, f)/\mathcal{X}) \times_{\mathcal{X}} U \cong \wp_m^{-1}((p^*\mathcal{L}, p^*s, p^*f)/U) \cong \wp_m^{-1}((\mathcal{O}(1), x_0, x_1^m)/\mathbb{P}^1) \times_{\mathbb{P}^1} U.$$

Then Example 4.3.10 shows that $\wp_m^{-1}((\mathcal{O}(1), x_0, x_1^m)/\mathbb{P}^1) \cong [Y/(\mathbb{Z}/p\mathbb{Z})]$ where Y is a smooth scheme and $\mathbb{Z}/p\mathbb{Z}$ acts on $Y \to \mathbb{P}^1$. Since $\mathbb{Z}/p\mathbb{Z}$ is an étale group scheme, $[Y/(\mathbb{Z}/p\mathbb{Z})]$ is Deligne–Mumford (cf. [Ols, Cor. 8.4.2]). Therefore $[Y/(\mathbb{Z}/p\mathbb{Z})] \times_{\mathbb{P}^1} U$ is Deligne–Mumford, so $\wp_m^{-1}((\mathcal{L}, s, f)/\mathcal{X}) \times_{\mathcal{X}} U$ is Deligne–Mumford as required. ☐

Next, we give a new characterization of $\mathbb{Z}/p\mathbb{Z}$-covers of curves in characteristic p using Artin–Schreier root stacks.

Theorem 4.3.14. *Let k be an algebraically closed field of characteristic $p > 0$ and suppose $Y \to X$ is a finite separable Galois cover of curves over k and $y \in Y$ is a ramification point with image $x \in X$ such that the inertia group $I(y \mid x)$ is $\mathbb{Z}/p\mathbb{Z}$. Then there are étale neighborhoods $V \to Y$ of y and $U \to X$ of x such that $V \to U$ factors through an Artin–Schreier root stack*

$$V \longrightarrow \wp_m^{-1}((L, s, f)/U) \to U$$

for some m.

Proof. Let $\mathcal{O}_{Y,y}$ and $\mathcal{O}_{X,x}$ be the local rings at y and x, respectively. Passing to completions, we may assume the extension $\widehat{\mathcal{O}}_{Y,y}/\widehat{\mathcal{O}}_{X,x}$ is isomorphic to $\widehat{A}/k[[t]]$ where $\widehat{A} = k[[t, u]]/(u^p - u - t^m g)$ for some $g \in k[[t]]$. Per an earlier discussion, after a change of formal coordinates, we may assume $g = 1$. By Artin approximation ([SP, Tag 0CAT]), the extension of henselian rings $\mathcal{O}^h_{Y,y}/\mathcal{O}^h_{X,x}$ also has this form, but since the henselization of a local ring is a direct limit over the finite étale neighborhoods of the ring, there must be finite étale neighborhoods $V_0 = \operatorname{Spec} B \to Y$ of y and $U_0 = \operatorname{Spec} A \to X$ of x such that the corresponding ring extension is of the form $B/A = B/k[t]$, where $B = k[t, u]/(u^p - u - t^m)$. Furthermore, by results in section 2 of [Harb], there is an étale neighborhood U of x such that:

(i) $U_0 = U \smallsetminus \{x\}$ is an affine curve with local coordinate t, and

(ii) the cover $V := U \times_X Y \to U$ is isomorphic to the one-point Galois cover of U defined by the birational Artin–Schreier equation $u^p - u = t^m$ over

U_0.

Then $\mathbb{Z}/p\mathbb{Z}$ acts (as an étale group scheme) on V via the usual action on this Artin–Schreier extension and by Example 4.3.10, $[V/(\mathbb{Z}/p\mathbb{Z})] \cong \wp_m^{-1}((L, s, f)/U)$ for some (L, s, f), so it follows immediately that $V \to U$ factors through this Artin–Schreier root stack. $\square$

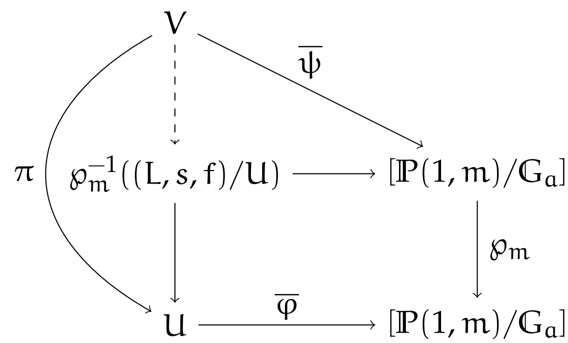
Remark 4.3.15. An alternate proof of Theorem 4.3.14 in the spirit of [Gar] goes as follows. Since the morphism $\pi : Y \to X$ has abelian inertia group at x, by geometric class field theory (cf. [Ser2, Ch. IV, Sec. 2, Prop. 9]) there is a modulus $\mathfrak{m}$ on X with support contained in the branch locus of π and a rational map $\varphi : X \to J_{\mathfrak{m}}$ to the generalized Jacobian of X with respect to this modulus, such that $Y \cong \varphi^* J'$ for a cyclic isogeny $J' \to J_{\mathfrak{m}}$ of degree p. Meanwhile, every such isogeny $J' \to J_{\mathfrak{m}}$ is a pullback of the Artin–Schreier isogeny $\wp : \mathbb{G}_a \to \mathbb{G}_a, \alpha \mapsto \alpha^p - \alpha$:

$$\begin{array}{ccccc} J' & \longrightarrow & \mathbb{G}_a & \hookrightarrow & \mathbb{P}(1, m) \\ \downarrow & & \downarrow \wp & & \downarrow \wp_m \\ J_{\mathfrak{m}} & \longrightarrow & \mathbb{G}_a & \hookrightarrow & \mathbb{P}(1, m) \end{array}$$

Let U' be an étale neighborhood of X on which φ is defined and set $U = U' \cup \{x\}$. Then over U, φ can be extended to a morphism $\overline{\varphi} : U \to \mathbb{P}(1, m)$ such that x maps to the stacky point at infinity, where $m = \operatorname{ord}_x \mathfrak{m} - 1$ (This m is also equal to $\operatorname{cond}_{Y/X}(x) - 1$, the conductor of the extension minus 1.) By Proposition 4.3.2, $\overline{\varphi}$ may also be defined by specifying a triple (L, s, f) on U: the pair (L, s) corresponds to the effective divisor $D = x$ and f is a section of

$\mathcal{O}(mD) = L^{\otimes m}$ determined from the affine equation expressing the pullback of the isogeny $J' \to J_{\mathfrak{m}}$ to U.

Let $V = \pi^* U$ which is an étale neighborhood of Y. Pulling back (L, s, f) to V locally defines $\overline{\psi}$ in the following diagram:



By the universal property of $\wp_m^{-1}((L, s, f)/U)$, there is a map $V \to \wp_m^{-1}((L, s, f)/U)$ factoring π locally as required.

Theorem 4.3.16. *Let $\mathcal{X}$ be a stacky curve over a perfect field k of characteristic $p > 0$. Then*

(1) If $\mathcal{X}$ contains a stacky point x of order p, there is an open substack $\mathcal{U} \subseteq \mathcal{X}$ containing x such that $\mathcal{U} \cong \wp_m^{-1}((L, s, f)/U)$ where $(m, p) = 1$, U is an open subscheme of the coarse space X of $\mathcal{X}$ and $(L, s, f) \in \mathfrak{Div}^{[1,m]}(U)$.

(2) Suppose all the nontrivial stabilizers of $\mathcal{X}$ are cyclic of order p. If $\mathcal{X}$ has coarse space $\mathbb{P}^1$, then $\mathcal{X}$ is isomorphic to a fibre product of Artin–Schreier root stacks of the form $\wp_m^{-1}((L, s, f)/\mathbb{P}^1)$ for $(m, p) = 1$ and $(L, s, f) \in \mathfrak{Div}^{[1,m]}(\mathbb{P}^1)$.

Proof. (1) Let m be the unique positive integer such that for any étale presentation $Y \to \mathcal{X}$ and any point $y \in Y$ mapping to x, the ramification jump of the induced cover $Y \to X$ at y is m. Let $U \subseteq X$ be a subscheme such that x is the only stacky point of $\mathcal{U} := \pi^{-1}(U) \subseteq \mathcal{X}$, where $\pi : \mathcal{X} \to X$ is the coarse map. Let $L = \mathcal{O}_U(x)$. Then for any étale map $\varphi : T \to \mathcal{U}$, there is a canonical Artin–Schreier root of the line bundle $\varphi^*\pi^*L$. Indeed, since $\pi \circ \varphi : T \to U$ is a one-point cover of curves with inertia group $\mathbb{Z}/p\mathbb{Z}$ and ramification jump m at x, Theorem 4.3.14 says there are sections s and f such that $\pi \circ \varphi$ factors as $T \to \wp_m^{-1}((L, s, f)/U) \to U$. On the other hand, by the universal property of $\wp_m^{-1}((L, s, f)/U)$, there is a canonical morphism $\mathcal{U} \to \wp_m^{-1}((L, s, f)/U)$, which is therefore an isomorphism.

(2) Let $B = \{x_1, \ldots, x_r\}$ be the finite set of points in $\mathbb{P}^1$ covered by points in $\mathcal{X}$ with nontrivial automorphism groups. Since $k(\mathbb{P}^1) = k(t)$, we can choose $F_i \in k(t)$ having a pole of any desired order at x_i for each $1 \leqslant i \leqslant r$. For instance, if m_i is the ramification jump at x_i as defined in (1), then we can arrange for $\mathrm{ord}_{x_i}(F_i) = -m_i$ for each i. Again by section 2 of [Harb], there is a proper curve $Y_i \to \mathbb{P}^1$ with affine Artin–Schreier equation $y^p - y = F_i(t)$, corresponding to the function field $k[t, y]/(y^p - y - F_i)$ as an extension of $k(t)$. We claim

$$\mathcal{X} \cong \mathcal{Y} := [Y_1/(\mathbb{Z}/p\mathbb{Z})] \times_{\mathbb{P}^1} \cdots \times_{\mathbb{P}^1} [Y_r/(\mathbb{Z}/p\mathbb{Z})]$$

which will prove (2) after applying Example 4.3.10 to each $[Y_i/(\mathbb{Z}/p\mathbb{Z})]$. We construct a map $\mathcal{X} \to \mathcal{Y}$ as follows. Let T be an arbitrary k-scheme. Since both coarse maps $\mathcal{X} \to \mathbb{P}^1$ and $\mathcal{Y} \to \mathbb{P}^1$ are isomorphisms away from B, we only need to specify the image of each stacky point $Q \in \mathcal{X}(T)$. Note that $\pi(Q) \in B$, so $\pi(Q) = x_i$ for some $1 \leqslant i \leqslant r$. If $T = \operatorname{Spec} k$, then such a Q is represented by a gerbe $B(\mathbb{Z}/p\mathbb{Z}) \hookrightarrow \mathcal{X}$, so send $Q \in \mathcal{X}(k)$ to the point of $[Y_i/(\mathbb{Z}/p\mathbb{Z})](k)$ corresponding to

$$\begin{array}{ccc} \operatorname{Spec} k & \xrightarrow{g_Q} & Y_i \\ \downarrow & & \\ B(\mathbb{Z}/p\mathbb{Z}) & & \end{array}$$

where $\operatorname{Spec} k \to B(\mathbb{Z}/p\mathbb{Z})$ is the universal $\mathbb{Z}/p\mathbb{Z}$-bundle and g_Q has image $x_i \in Y_i$. Extending this to any scheme T is easy: replace the above diagram with

$$\begin{array}{ccc} T & \xrightarrow{g_Q} & Y_i \times_k T \\ \downarrow & & \\ B(\mathbb{Z}/p\mathbb{Z}) \times_k T & & \end{array}$$

and define g_Q by $t \mapsto (x_i, t)$. This defines an equivalence of categories $\mathcal{X}(T) \to \mathcal{Y}(T)$ for any scheme T, so we can invoke Lemma 3.2.26 to ensure that these fibrewise equivalences assemble into an equivalence of stacks $\mathcal{X} \xrightarrow{\sim} \mathcal{Y}$. □

Remark 4.3.17. As illustrated in in the proof of Theorem 4.3.16, the ramification data at a stacky $\mathbb{Z}/p\mathbb{Z}$-point $x \in \mathcal{X}$ (i.e. the ramification jump m) can be determined from the stack itself, namely, by which étale covers are allowed at x. This is unique to the positive characteristic case; in characteristic 0, one only needs to know the order of the stabilizer at every stacky point to understand the entire stacky structure. In contrast, for each fixed m there is a *family* of non-isomorphic stacky curves, with coarse space $\mathbb{P}^1$ and order p stabilizers at any prescribed points, parametrized by the possible choices of f.

Remark 4.3.18. The local structure of a stacky curve in characteristic p is separable since the stack is by definition Deligne–Mumford and generically a scheme. Thus around a wildly ramified point, one does not have structures like $[U/\mu_p]$ or $[U/\alpha_p]$ which would be more problematic, but will be interesting to have a description of in the future.

Example 4.3.19. When the coarse space is not $\mathbb{P}^1$, Theorem 4.3.16(2) is false. For example, let k be an algebraically closed field of characteristic $p > 0$ and let E be an elliptic curve over k with a rational point P. By Riemann–Roch, $h^0(E, \mathcal{O}(P)) = 1$ so every global section of $\mathcal{O}(P)$ vanishes at P and we can exploit this limitation to construct a counterexample. Let $\mathcal{E}$ be a stacky curve with coarse space E and a stacky point of order p at P with ramification jump 1. This can be achieved, by Theorem 4.3.16(1), through taking a local Artin–Schreier root stack $\wp_1^{-1}((\mathcal{O}(P), s_P, f_P)/U)$ where U is an étale neighborhood of P, $(\mathcal{O}(P), s_P)$ is the pair corresponding to the effective divisor P (in U_P) and f_P is a section of $\mathcal{O}(P)|_{U_P}$ not vanishing at P. Then if $\mathcal{E}$ were a global Artin–Schreier root stack over E, there would be some $s \in H^0(E, \mathcal{O}(P))$ vanishing at P and some $f \in H^0(E, \mathcal{O}(P))$ not vanishing at P such that $\mathcal{E} \cong \wp_1^{-1}((\mathcal{O}(P), s, f)/E)$ but such an f does not exist by the reasoning above. Variants of this counterexample can be constructed for single stacky points with ramification jump $m > 1$, as well as multiple stacky points with the same ramification jump which cannot be obtained through a fibre product of global Artin–Schreier root stacks. This argument also works for higher genus curves, so the $\mathbb{P}^1$ case is quite special.

4.4 Wild Stacky Curves: The Higher-Order Cyclic Case

In this section, we begin extending the techniques in Section 4.3 to the higher-order cyclic case, i.e. when a stacky curve has a point with stabilizer group $\mathbb{Z}/p^n\mathbb{Z}$. As in the tame and prime-cyclic wild case, this comes down to being able to take p^nth roots of line bundles on the coarse space. To study the general case, we will replace $[\mathbb{P}(1,m)/\mathbb{G}_a]$ with $[\overline{\mathbb{W}}_n(1,m_1,\ldots,m_n)/\mathbb{W}_n]$ where $\overline{\mathbb{W}}_n(1,m_1,\ldots,m_n)$ is a new stacky equivariant compactification of the ring scheme $\mathbb{W}_n$ of Witt vectors of length n. This construction will be equal to $\mathbb{P}(1,m)$ in the $n=1$ case, thus capturing the case studied in Section 4.3.1. Before constructing this compactification in general, we introduce the compactification $\overline{\mathbb{W}}_n$ in the category of schemes, due to Garuti [Gar], which will be the $m_1 = \ldots = m_n = 1$ case of the stacky construction.

In future work, we plan to give an explicit description of the points of the stacks $\overline{\mathbb{W}}_n(1,m_1,\ldots,m_n)$ and use a certain order p^n map

$$\Psi : [\overline{\mathbb{W}}_n(1,m_1,\ldots,m_n)/\mathbb{W}_n] \to [\overline{\mathbb{W}}_n(1,m_1,\ldots,m_n)/\mathbb{W}_n],$$

constructed below, to classify stacky curves with order p^n cyclic stacky structure.

4.4.1 Garuti's Compactification

For a vector bundle $E \to X$, let $\mathbb{P}(E) \to X$ denote the *projective bundle* associated to E. By definition, $\mathbb{P}(E) = \mathrm{Proj}_X(\mathrm{Sym}(E))$ which comes equipped with a tautological bundle $\mathcal{O}_{\mathbb{P}}(1)$. Set $\mathcal{O}_{\mathbb{P}}(m) = \mathcal{O}_{\mathbb{P}}(1)^{\otimes m}$ for any $m \in \mathbb{Z}$, where $\mathcal{O}_{\mathbb{P}}(-1) = \mathcal{O}_{\mathbb{P}}(1)^*$ by convention. Define ringed spaces $(\overline{\mathbb{W}}_n, \mathcal{O}_{\overline{\mathbb{W}}_n}(1))$ inductively by

$$\begin{aligned} &(\overline{\mathbb{W}}_1, \mathcal{O}_{\overline{\mathbb{W}}_1}(1)) = (\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)) \\ \text{and} \quad &(\overline{\mathbb{W}}_n, \mathcal{O}_{\overline{\mathbb{W}}_n}(1)) = (\mathbb{P}(\mathcal{O}_{\overline{\mathbb{W}}_{n-1}} \oplus \mathcal{O}_{\overline{\mathbb{W}}_{n-1}}(p)), \mathcal{O}_{\mathbb{P}}(1)) \quad \text{for } n \geqslant 2, \end{aligned}$$

where $\mathcal{O}_{\mathbb{P}}(1)$ is the tautological bundle of the projective bundle in that step. There is a morphism

$$r : \overline{\mathbb{W}}_n \longrightarrow \overline{\mathbb{W}}_{n-1}$$

for all $n \geqslant 1$ which has fibres $\mathbb{P}^1$ by definition. Note that $r_*\mathcal{O}_{\overline{\mathbb{W}}_n}(1) = \mathcal{O}_{\overline{\mathbb{W}}_{n-1}} \oplus \mathcal{O}_{\overline{\mathbb{W}}_{n-1}}(p)$. For each $n \geqslant 2$, there is a canonical section of r corresponding to the zero section of the bundle $\mathbb{P}(\mathcal{O}_{\overline{\mathbb{W}}_{n-1}} \oplus \mathcal{O}_{\overline{\mathbb{W}}_{n-1}}(p))$ over $\overline{\mathbb{W}}_{n-1}$. Let Z_n be the divisor associated to the zero locus of this section. On the other hand, for any vector bundle E there is an isomorphism $\mathbb{P}(E^\vee) \cong \mathbb{P}(E)$ where $E^\vee$ denotes the dual bundle. So in our case, there is an isomorphism

$$\mathbb{P}(\mathcal{O}_{\overline{\mathbb{W}}_{n-1}} \oplus \mathcal{O}_{\overline{\mathbb{W}}_{n-1}}(p)) \cong \mathbb{P}(\mathcal{O}_{\overline{\mathbb{W}}_{n-1}}(-p) \oplus \mathcal{O}_{\overline{\mathbb{W}}_{n-1}})$$

which induces another section of r, corresponding via this isomorphism to the zero section of $\mathbb{P}(\mathcal{O}_{\overline{\mathbb{W}}_{n-1}}(-p) \oplus \mathcal{O}_{\overline{\mathbb{W}}_{n-1}})$. We call this the "infinity section" and denote its divisor by Σ_n.

Proposition 4.4.1 ([Gar, Prop. 2.4])**.** *There is a system of open immersions of schemes* $j_n : \mathbb{W}_n \hookrightarrow \overline{\mathbb{W}}_n$ *such that* $j_n(\mathbb{W}_n) = \overline{\mathbb{W}}_n \smallsetminus B_n$ *where* B_n *is the zero locus of a section of* $\mathcal{O}_{\overline{\mathbb{W}}_n}(1)$*, given by*

$$B_1 = \Sigma_1 \quad \textit{and} \quad B_n = \Sigma_n + pr^* B_{n-1} \textit{ for } n \geqslant 2.$$

Proof. Let j_1 be the embedding $\mathbb{G}_a = \mathbb{A}^1 \hookrightarrow \mathbb{P}^1$ taking $0 \mapsto [0,1]$. Now induct. For $n \geqslant 2$, set $U_n = j_n(\mathbb{W}_n) = \overline{\mathbb{W}}_n \smallsetminus B_n$. Then $\mathcal{O}_{\overline{\mathbb{W}}_n}(1)$ is trivial over U_n, so $r^{-1}(U_n) = \mathbb{P}^1_{U_n}$. By divisor theory, $B_{n+1} := \Sigma_{n+1} + pr^* B_n$ is a divisor corresponding to the zero locus of some section of $\mathcal{O}_{\overline{\mathbb{W}}_n}(1)$. Take $U_{n+1} = \overline{\mathbb{W}}_{n+1} \smallsetminus B_{n+1}$. We have $U_{n+1} = \mathbb{A}^1_{U_n} \hookrightarrow \mathbb{P}^1_{U_n} = r^{-1}(U_n)$ under which $r^{-1}(0) \cap U_{n+1} = \mathbb{A}^1$. Then there is a diagram

$$\begin{array}{ccccc} \mathbb{G}_a & \xhookrightarrow{V^n} & \mathbb{W}_{n+1} & \xrightarrow{r} & \mathbb{W}_n \\ {\scriptstyle j_1}\downarrow{\scriptstyle \sim} & & \vdots & & \downarrow{\scriptstyle j_n} \\ \mathbb{A}^1 & \hookrightarrow & U_{n+1} & \xrightarrow{r} & U_n \end{array}$$

where V^n is the n-fold composition of the Verschiebung $V : \mathbb{W}_r \to \mathbb{W}_{r+1}$, $(x_0, \dots, x_r) \mapsto (0, x_0, \dots, x_r)$. This defines $j_{n+1} : \mathbb{W}_{n+1} \hookrightarrow U_{n+1} \subseteq \overline{\mathbb{W}}_{n+1}$ and the rest of the statement follows. □

Corollary 4.4.2 ([Gar, Cor. 2.5]). *For all* $n \geqslant 2$,

$$B_n = \sum_{i=1}^{n} p^{n-i}(r^{n-i})^* \Sigma_i.$$

We next show that $\overline{\mathbb{W}}_n$ is a compactification of $\mathbb{W}_n$ which is equivariant with respect to the action of $\mathbb{W}_n$ on itself.

Lemma 4.4.3 ([Gar, Lem. 2.7]). *Let* $\mathcal{O}_{\overline{\mathbb{W}}_n}(1)$ *be the tautological bundle on* $\overline{\mathbb{W}}_n$. *Then*

(1) $\mathcal{O}_{\overline{\mathbb{W}}_n}(1)$ *is generated by global sections.*

(2) For any $m \geqslant 0$, *there is an isomorphism of rings*

$$H^0(\overline{\mathbb{W}}_n, \mathcal{O}_{\overline{\mathbb{W}}_n}(m)) \xrightarrow{\sim} \mathrm{Sym}^m(H_{p^{n-1}})$$

where H_d *denotes the* d*th graded piece of the graded ring*

$$H = \mathbb{F}_p[t, y_0, y_1, \ldots,], \quad \text{with } \deg(t) = 1 \text{ and } \deg(y_i) = p^i.$$

(3) Under this isomorphism, y_{n-1} *and* $t^{p^{n-1}}$ *define principal divisors*

$$(y_{n-1}) = \sum a_P P \qquad \text{and} \qquad (t^{p^{n-1}}) = \sum b_P P$$

such that $\sum_{a_P \geqslant 0} a_P P = Z_n$ *and* $\sum_{b_P \geqslant 0} b_P P = B_n$.

This allows us to construct the action of $\mathbb{W}_n$ on $\overline{\mathbb{W}}_n$.

Proposition 4.4.4 ([Gar, Prop. 2.8]). *The action of $\mathbb{W}_n$ on itself by Witt-vector translation extends to an action on $\overline{\mathbb{W}}_n$ which stabilizes $\mathcal{O}_{\overline{\mathbb{W}}_n}(1)$.*

Proof. As mentioned in the introduction to Section 4.3.1, for $n = 1$, the translation action of $\mathbb{W}_1 = \mathbb{G}_a$ on itself by $\lambda \cdot x = x + \lambda$ extends to an action on $\mathbb{P}^1 = \overline{\mathbb{W}}_1$ by $\lambda \cdot [x, y] = [x + \lambda y, y]$. Since this fixes $\infty = [1, 0]$, the action stabilizes $\mathcal{O}(1) = \mathcal{O}(1 \cdot \infty)$. To induct, suppose the action of $\mathbb{W}_n$ on $\overline{\mathbb{W}}_n$ has been constructed. Recall that $\mathbb{W}_{n+1}$ acts on itself by

$$a \cdot x = x + a := (S_0(x; a), S_1(x; a), \ldots, S_n(x; a))$$

where $S_i(X; Y) \in \mathbb{Z}[X_0, X_1, \ldots; Y_0, Y_1, \ldots]$ are the polynomials defined by

$$\Phi_n(S_0, \ldots, S_n) = \Phi_n(X_0, \ldots, X_n) + \Phi_n(Y_0, \ldots, Y_n)$$

for the polynomial $\Phi_n(X_0, \ldots, X_n) = \sum_{j=0}^{n} p^j X_j^{p^{n-j}}$. We may alternatively write

$$S_i(X; Y) = X_i + Y_i + c_i(X; Y)$$

for some $c_i \in \mathbb{Z}[X_0, \ldots, X_{i-1}; Y_0, \ldots, Y_{i-1}]$. Fix an $\mathbb{F}_p$-algebra A. By induction, $\mathbb{W}_n$ acts on $\overline{\mathbb{W}}_n$ and fixes $\mathcal{O}_{\overline{\mathbb{W}}_n}(1)$, so the ring $\mathbb{W}_n(A)$ acts on the A-module $H^0(\overline{\mathbb{W}}_n(A), \mathcal{O}_{\overline{\mathbb{W}}_n}(p))$. Since $\mathbb{W}_n(A)$ is a quotient of $\mathbb{W}_{n+1}(A)$, the latter also acts A-linearly on $H^0(\overline{\mathbb{W}}_n(A), \mathcal{O}_{\overline{\mathbb{W}}_n}(p))$. By Lemma 4.4.3,

$$\begin{aligned} & H^0(\overline{\mathbb{W}}_n(A), \mathcal{O}_{\overline{\mathbb{W}}_n}(p)) \cong \mathrm{Sym}^p(H_{p^{n-1}}) \otimes A = H_{p^{n-1}} \otimes A \\ \text{and} \quad & H^0(\overline{\mathbb{W}}_{n+1}(A), \mathcal{O}_{\overline{\mathbb{W}}_{n+1}}(p)) \cong \mathrm{Sym}^p(H_{p^n}) \otimes A \cong (H_{p^{n-1}} \otimes A) \oplus A[y_n]. \end{aligned}$$

Define the action of $\mathbb{W}_{n+1}(A)$ on $H^0(\overline{\mathbb{W}}_{n+1}(A), \mathcal{O}_{\overline{\mathbb{W}}_{n+1}}(p))$ by the above action on $H_{p^{n-1}} \otimes A$ and by the following on y_n:

$$a \cdot y_n := y_n + a_n t^{p^n} + t^{p^n} c_n \left(\frac{y_0}{t}, \ldots, \frac{y_{n-1}}{t^{p^{n-1}}}; a \right) = t^{p^n} S_n \left(\frac{y_0}{t}, \ldots, \frac{y_n}{t^{p^n}}; a \right).$$

The line bundle $\mathcal{O}_{\overline{\mathbb{W}}_{n+1}}(1)$ corresponds to the blowup of the point $[0, 0, \ldots, 0, 1] \in \mathbb{P}^{n+1}_A \cong \mathrm{Proj}(H_{p^n} \otimes A)$ and one can check that this is fixed by the defined action. This completes the proof. $\square$

Proposition 4.4.5 ([Gar, Prop. 2.9]). *The isogeny $\wp : \mathbb{W}_n \to \mathbb{W}_n$ extends to a cyclic cover of degree p^n,*

$$\Psi_n : \overline{\mathbb{W}}_n \longrightarrow \overline{\mathbb{W}}_n$$

which is defined over $\mathbb{F}_p$, commutes with the maps $r : \overline{\mathbb{W}}_n \to \overline{\mathbb{W}}_{n-1}$ and has branch locus B_n, with $\Psi_n^* B_n = p B_n$.

Proof. The $n = 1$ case is outlined in Section 4.3.1 and is in any case well-known. To induct, consider the fibre product

$$\begin{array}{ccc} P & \xrightarrow{\pi} & \overline{\mathbb{W}}_{n+1} \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle r} \\ \overline{\mathbb{W}}_n & \xrightarrow{\Psi_n} & \overline{\mathbb{W}}_n \end{array}$$

Then $\pi : P \to \overline{\mathbb{W}}_{n+1}$ is a cyclic p^n-cover given explicitly by

$$P = \mathbb{P}(\mathcal{O}_{\overline{\mathbb{W}}_n}, \mathcal{O}_{\overline{\mathbb{W}}_n}(p^2))$$

since $\Psi_n^* B_n = pB_n$ and $\mathcal{O}_{\mathbb{P}}(B_n) = \mathcal{O}_{\overline{\mathbb{W}}_n}(p)$. Using Lemma 4.4.3, it is possible to construct a finite, flat morphism

$$\varphi : \overline{\mathbb{W}}_{n+1} \longrightarrow P$$

over $\overline{\mathbb{W}}_n$ using the equation

$$\begin{aligned} &t^{p^{n+1}} S_n\left(\frac{y_0^p}{t^p}, \ldots, \frac{y_n^p}{t^{p^{n+1}}}; -\frac{y_0}{t}, \ldots, -\frac{y_n}{t^{p^n}}\right) \\ &\qquad = y_n^p - y_n t^{p^n(p-1)} + t^{p^{n+1}} c_n\left(\frac{y_0^p}{t^p}, \ldots, \frac{y_n^p}{t^{p^{n+1}}}; -\frac{y_0}{t}, \ldots, -\frac{y_n}{t^{p^n}}\right). \end{aligned}$$

This defines Ψ_{n+1} as the composition

$$\Psi_{n+1} : \overline{\mathbb{W}}_{n+1} \xrightarrow{\varphi} P \xrightarrow{\pi} \overline{\mathbb{W}}_{n+1}$$

which is then finite, flat and extends $\wp : \mathbb{W}_{n+1} \to \mathbb{W}_{n+1}$ by construction. It is easy to check that the Ψ_n and r commute. Finally, (3) of Lemma 4.4.3 tells us that B_{n+1} is the effective part of the principal divisor (t^{p^n}), so $\Psi_{n+1}^* B_n = pB_{n+1}$ follows from the fact that $\Psi_1^*(t) = (t^p)$. $\square$

Remark 4.4.6. By construction, the sequence of $\overline{\mathbb{W}}_n$ form what is known as a *Bott tower*, which originally appears in [GK]. In particular, each $\overline{\mathbb{W}}_n$ is a smooth, projective toric variety (cf. [CS]).

4.4.2 Artin–Schreier–Witt Root Stacks

Next we turn to the construction of the stacky compactification $\overline{\mathbb{W}}_n(1, m_1, \ldots, m_n)$ of the Witt scheme $\mathbb{W}_n$ for $n > 1$. We begin by setting $\overline{\mathbb{W}}_1(1, m) := \mathbb{P}(1, m)$, our stacky compactification of $\mathbb{W}_1 = \mathbb{A}^1$. The key insight for generalizing this is to use the fact (Lemma 4.3.1) that $\mathbb{P}(1, m)$ is itself a root stack over $\mathbb{P}^1$:

$$\begin{array}{ccc} \mathbb{P}(1,m) & \longrightarrow & [\mathbb{A}^1/\mathbb{G}_m] \\ \downarrow & & \downarrow m \\ \mathbb{P}^1 & \xrightarrow{(\mathcal{O}_{\mathbb{P}^1}(1), \Sigma_1)} & [\mathbb{A}^1/\mathbb{G}_m] \end{array}$$

Pulling back $\mathbb{P}(1, m) = \overline{\mathbb{W}}_1(1, m)$ along the sequence

$$\cdots \to \overline{\mathbb{W}}_3 \xrightarrow{r} \overline{\mathbb{W}}_2 \xrightarrow{r} \overline{\mathbb{W}}_1 = \mathbb{P}^1$$

defines $\overline{\mathbb{W}}_n(1, m, 1, \ldots, 1)$ for each $n > 1$. Each of these is a root stack over $\overline{\mathbb{W}}_n$ with stacky structure at (the pullback of) Σ_1; for example, $\overline{\mathbb{W}}_2(1, m, 1) = r^*\overline{\mathbb{W}}_2(1, m)$ is a root stack over $\overline{\mathbb{W}}_2$:

$$\begin{array}{ccc} \overline{\mathbb{W}}_2(1,m,1) & \longrightarrow & [\mathbb{A}^1/\mathbb{G}_m] \\ \downarrow & & \downarrow m \\ \overline{\mathbb{W}}_2 & \xrightarrow{(r^*\mathcal{O}_{\mathbb{P}^1}(1),\, r^*\Sigma_1)} & [\mathbb{A}^1/\mathbb{G}_m] \end{array}$$

For a pair of positive integers (m_1, m_2), the compactification $\overline{\mathbb{W}}_2(1, m_1, m_2)$ of $\mathbb{W}_2$ is defined by a second root stack, $\overline{\mathbb{W}}_2(1, m_1, m_2) := \sqrt[m_2]{(\mathcal{O}(1), \Sigma_2)/\overline{\mathbb{W}}_2(1, m_1, 1)}$:

$$\begin{array}{ccc} \overline{\mathbb{W}}_2(1,m_1,m_2) & \longrightarrow & [\mathbb{A}^1/\mathbb{G}_m] \\ \downarrow & & \downarrow m_2 \\ \overline{\mathbb{W}}_2(1,m_1,1) & \xrightarrow{(\mathcal{O}(1),\, \Sigma_2)} & [\mathbb{A}^1/\mathbb{G}_m] \end{array}$$

Here, $\mathcal{O}(1)$ denotes the pullback of the line bundle $\mathcal{O}_{\overline{\mathbb{W}}_2}$ to $\overline{\mathbb{W}}_2(1, m_1, 1)$ along the coarse map. Now we proceed inductively. Let $n \geqslant 2$.

Definition 4.4.7. *For a sequence of positive integers $(m_1, \ldots, m_n)$, define the* **compactified Witt stack** $\overline{\mathbb{W}}_n(1, m_1, \ldots, m_n)$ *to be the root stack in the following diagram:*

$$\begin{array}{ccc} \overline{\mathbb{W}}_n(1,m_1,\ldots,m_n) & \longrightarrow & [\mathbb{A}^1/\mathbb{G}_m] \\ \downarrow & & \downarrow m_n \\ \overline{\mathbb{W}}_n(1,m_1,\ldots,m_{n-1},1) & \xrightarrow{(\mathcal{O}(1),\, \Sigma_n)} & [\mathbb{A}^1/\mathbb{G}_m] \end{array}$$

where $\overline{\mathbb{W}}_n(1, m_1, \ldots, m_{n-1}, 1) = r^*\overline{\mathbb{W}}_{n-1}(1, m_1, \ldots, m_{n-1})$, the pullback along r of the compactified Witt stack $\overline{\mathbb{W}}_{n-1}(1, m_1, \ldots, m_{n-1})$ over $\overline{\mathbb{W}}_{n-1}$, and $\mathcal{O}(1)$ is pulled back inductively as explained above.

We will continue writing r for the maps $\overline{\mathbb{W}}_n(1, m_1, \ldots, m_n) \to \overline{\mathbb{W}}_{n-1}(1, m_1, \ldots, m_{n-1})$.

Proposition 4.4.8. *For each $n \geqslant 1$, the cyclic p^n-cover $\Psi_n : \overline{\mathbb{W}}_n \to \overline{\mathbb{W}}_n$ induces a morphism of stacks*

$$\Psi = \Psi_{m_1,\ldots,m_n} : \overline{\mathbb{W}}_n(1, m_1, \ldots, m_n) \to \overline{\mathbb{W}}_n(1, m_1, \ldots, m_n)$$

which commutes with the maps $\overline{\mathbb{W}}_n(1, m_1, \ldots, m_n) \to \overline{\mathbb{W}}_n$ and $r : \overline{\mathbb{W}}_n(1, m_1, \ldots, m_n) \to \overline{\mathbb{W}}_{n-1}(1, m_1, \ldots, m_{n-1})$ and satisfies $\Psi^* B_n = pB_n$.

Proof. For $n = 1$, $\Psi_1 : \mathbb{P}^1 \to \mathbb{P}^1$ is the extension of $\wp(x) = x^p - x$ from $\mathbb{A}^1$ to $\mathbb{P}^1$. As explained in Section 4.3.1, this extends naturally to $\overline{\mathbb{W}}_1(1, m) = \mathbb{P}(1, m)$ as $[x, y] \mapsto [x^p - xy^{m(p-1)}, y^p]$. Then by construction $\Psi^* \Sigma_1 = p\Sigma_1$. To induct, suppose $\Psi : \overline{\mathbb{W}}_{n-1}(1, m_1, \ldots, m_{n-1}) \to \overline{\mathbb{W}}_{n-1}(1, m_1, \ldots, m_{n-1})$ has been constructed. Then pulling back along r extends Ψ to a cover $\overline{\mathbb{W}}_n(1, m_1, \ldots, m_{n-1}, 1) \to \overline{\mathbb{W}}_n(1, m_1, \ldots, m_{n-1}, 1)$. Since the root stack construction commutes with pullback by Lemma 4.2.5, this induces a morphism $\Psi : \overline{\mathbb{W}}_n(1, m_1, \ldots, m_n) \to \overline{\mathbb{W}}_n(1, m_1, \ldots, m_n)$. By construction this commutes with $r : \overline{\mathbb{W}}_n(1, m_1, \ldots, m_n) \to \overline{\mathbb{W}}_{n-1}(1, m_1, \ldots, m_{n-1})$ and we can compute

$$\begin{aligned}
\Psi^* B_n &= \Psi^*(\Sigma_n + pr^* B_{n-1}) \quad \text{by Proposition 4.4.1} \\
&= \Psi^* \Sigma_n + pr^*(\Psi^* B_{n-1}) \quad \text{since } \Psi \text{ and } r \text{ commute} \\
&= p\Sigma_n + pr^*(pB_{n-1}) \quad \text{by induction} \\
&= p(\Sigma_n + pr^* B_{n-1}) = pB_n.
\end{aligned}$$

This proves the statements for all $n \geqslant 1$ by induction. $\square$

To further study wild structures on stacky curves, it will be necessary to understand the category of $\mathcal{X}$-points $\overline{\mathbb{W}}_n(1, m_1, \ldots, m_n)(\mathcal{X})$ for any stack $\mathcal{X}$. Then one can extend the definition of an Artin–Schreier root stack as follows.

Definition 4.4.9. *For a stack $\mathcal{X}$, sequence of positive integers $(m_1, \ldots, m_n)$ and morphism $\varphi : \mathcal{X} \to \overline{\mathbb{W}}_n(1, m_1, \ldots, m_n)$, the* **Artin–Schreier–Witt root stack** *of $\mathcal{X}$ along φ is defined to be the normalized pullback $\Phi^{-1}(\varphi/\mathcal{X})$ of the diagram*

$$\begin{array}{ccc} \Psi^{-1}(\varphi/\mathcal{X}) & \longrightarrow & [\overline{\mathbb{W}}_n(1, m_1, \ldots, m_n)/\mathbb{W}_n] \\ \big\downarrow \;\; \nu\lrcorner & & \big\downarrow \Psi \\ \mathcal{X} & \longrightarrow & [\overline{\mathbb{W}}_n(1, m_1, \ldots, m_n)/\mathbb{W}_n] \end{array}$$

Our eventual goal is to use Artin–Schreier–Witt theory of covers of curves to generalize the classification theorems in Section 4.3.2 to the higher-order cyclic case. Ultimately, we believe it is possible to use this framework to describe a unified theory of tame and wild stacky curves and their local structure, in the style of [GS].

Chapter 5

Applications

In the final chapter, we outline two directions of application for our theory of wild stacky curves. One direction (Section 5.1), the original inspiration for this project, concerns canonical rings of stacky curves, à la Voight and Zureick-Brown [VZB]. The other direction (Section 5.2), is a program for computing modular forms in characteristic $p > 0$, à la Katz [Kat], by treating modular curves as stacky curves with potentially wild ramification. We plan to continue investigating applications in both settings in future work.

5.1 Canonical Rings

Let X be a compact complex manifold and $\Omega_X = \Omega_{X(\mathbb{C})}$ the sheaf of holomorphic differential 1-forms on X. Then in many situations X can be outfitted with the structure of a complex projective *variety* by explicitly embedding it into projective space using different tensor powers of Ω_X. For example, if X is a Riemann surface of genus $g \geqslant 2$ that is not hyperelliptic, the canonical map $X \to |\mathcal{O}_X| \cong \mathbb{P}^{g-1}$ is an embedding. More generally, any Riemann surface X of genus $g \geqslant 2$ can be recovered as $\mathrm{Proj}\, R(X)$ where

$$R(X) := \bigoplus_{r=0}^{\infty} H^0(X, \Omega_X^r)$$

is the *canonical ring* of X. In general, the isomorphism $X \cong \mathrm{Proj}\, R(X)$ may not hold since Ω_X need not be ample, but it is still a useful gadget for studying the algebraic properties of X.

In [VZB], the authors extend this strategy to tame stacky curves by giving explicit generators and relations for the canonical ring

$$R(\mathcal{X}) = \bigoplus_{r=0}^{\infty} H^0(\mathcal{X}, \Omega_{\mathcal{X}}^r)$$

of a stacky curve $\mathcal{X}$, where $\Omega_{\mathcal{X}} = \Omega_{\mathcal{X}/k}$ is the sheaf of differentials defined in Section 4.1.2. This has numerous applications, perhaps most importantly to the computation of the ring of modular forms for a given congruence subgroup of $SL_2(\mathbb{Z})$ (see Section 5.2), and holds in all characteristics provided the stack $\mathcal{X}$ has no wild ramification. However, numerous (stacky) modular curves one would like to study in characteristic p, such as $X_0(N)$ with $p \mid N$, have wild ramification and therefore fall outside the scope of results like [VZB, Thm. 1.4.1].

5.1.1 The Tame Case

Let $\mathcal{X}$ be a tame stacky curve over an algebraically closed field k and let $\chi(\mathcal{X}) = -\deg(K_{\mathcal{X}})$ be the Euler characteristic of $\mathcal{X}$. Then we say $\mathcal{X}$ is:

- **spherical** if $\chi(\mathcal{X}) > 0$;
- **euclidean** if $\chi(\mathcal{X}) = 0$; and
- **hyperbolic** if $\chi(\mathcal{X}) < 0$.

In analogy with the theory of orbifolds in differential geometry, over $\mathbb{C}$ these correspond to a suitable analytification of $\mathcal{X}$ (cf. [VZB, Sec. 6.1]) having universal orbifold cover equal to

- a "football" $F(m, n)$: a stacky $\mathbb{P}^1$ obtained by gluing $[\mathbb{A}^1/\mu_m]$ and $[\mathbb{A}^1/\mu_n]$, if $\chi(\mathcal{X}) > 0$;
- $\mathbb{A}^1$ if $\chi(\mathcal{X}) = 0$; or
- the complex upper half-plane $\mathfrak{h}$ if $\chi(\mathcal{X}) < 0$.

The canonical ring of $\mathcal{X}$ is isomorphic to k when $\mathcal{X}$ is spherical and a polynomial ring in a single variable when $\mathcal{X}$ is euclidean. The main theorem in [VZB] computes the canonical ring of $\mathcal{X}$ in the hyperbolic case.

Theorem 5.1.1 ([VZB, Cor. 1.4.2]). *Let $\mathcal{X}$ be a hyperbolic tame stacky curve over a perfect field k with stacky points $x_1, \ldots, x_n$ having cyclic stabilizers of orders $e_1, \ldots, e_n$,* *respectively. Then the canonical ring*

$$R(\mathcal{X}) = \bigoplus_{r=0}^{\infty} H^0(\mathcal{X}, \Omega_{\mathcal{X}}^r)$$

is a k*-algebra generated in degree at most* $e = \max\{1, e_1, \ldots, e_n\}$ *with relations in degree at most* $2e$*.*

This gives formulas (in some cases new formulas) for rings of modular forms, since these can be interpreted as canonical rings of suitable stacky curves corresponding to modular curves like $X_0(N)$. We do not at present have such a concise result for canonical rings of wild stacky curves. However, in the next few sections, we outline plans to approach this problem in the future.

5.1.2 The Wild Case

In this section, we will show how the Artin–Schreier root stack construction can be used to study canonical rings. First, we give a generalization of the Riemann–Hurwitz formula (Proposition 4.1.18) for stacky curves with arbitrary (finite) stabilizers. The upshot is that we can then compute the canonical divisor from knowledge of the canonical divisor on the coarse space and the ramification filtrations at the stacky points of $\mathcal{X}$.

Let $\mathcal{X}$ be a stacky curve with coarse moduli space X and suppose $x \in \mathcal{X}(k)$ is a wild stacky point, i.e. a stacky point with stabilizer G_x such that p divides $|G_x|$. Then by Proposition 3.5.49, $\mathcal{X}$ is locally given by a quotient stack of the form $[U/G_x]$ where U is a scheme. Consider the composite $U \to [U/G_x] \xrightarrow{\pi} X$

where $\pi : \mathcal{X} \to X$ is the coarse map. Let W denote the schematic image of U in X, so that $U \to W$ is a one-point cover of curves with Galois group G_x. Then G_x has a higher ramification filtration (for the lower numbering), say $(G_{x,i})_{i \geqslant 0}$. It is easy to check this filtration does not depend on the choice of U, so we call $(G_{x,i})_{i \geqslant 0}$ the *higher ramification filtration at the stacky point* x (for the lower numbering). In the tame case, it is common to put $G_{x,0} = G_x$ and $G_{x,i} = 1$ for $i > 0$, so the Riemann–Hurwitz formula in the tame case is subsumed by the following formula.

Proposition 5.1.2 (Stacky Riemann–Hurwitz)**.** *For a stacky curve $\mathcal{X}$ with coarse moduli space $\pi : \mathcal{X} \to X$, the formula*

$$K_{\mathcal{X}} = \pi^* K_X + \sum_{x \in \mathcal{X}(k)} \sum_{i=0}^{\infty} (|G_{x,i}| - 1)x$$

defines a canonical divisor $K_{\mathcal{X}}$ on $\mathcal{X}$.

Proof. As before, begin by assuming $\mathcal{X}$ has a single stacky point P. The tame case was proven in Proposition 4.1.18 but this proof subsumes it, so in theory P could have any stabilizer group $G = G_P$. Proposition 3.5.49 says that locally, $\mathcal{X}$ is of the form $[U/G]$ for a scheme U, but since the computation is local, we may assume $\mathcal{X} \cong [U/G]$. If $f : U \to [U/G]$ is the quotient map, then $f^*\Omega_{\mathcal{X}/X} \cong \Omega_{U/X}$ so the stalk of $\Omega_{\mathcal{X}/X}$ at P has length equal to the different of the local ring extension $\widehat{\mathcal{O}}_{\mathcal{X},P}/\widehat{\mathcal{O}}_{X,\pi(P)}$. This is precisely $\sum_{i=0}^{\infty}(|G_{P,i}| - 1)$, by [Ser2, Ch. IV, Prop. 4] for example, so the formula defines a canonical divisor on $[U/G]$. The local argument extends to the general case once again because $\mathcal{X} \to X$ is an isomorphism away from the stacky locus. $\square$

Example 5.1.3. Consider the Artin–Schreier cover $Y \to \mathbb{P}^1$ of Example 4.3.10 defined birationally by the Artin–Schreier equation $y^p - y = f(x)$, where f is a degree m polynomial. The resulting quotient stack $\mathcal{X} = [Y/(\mathbb{Z}/p\mathbb{Z})]$ has a single stacky point Q lying above ∞. Let $(G_i)_{i\geqslant 0}$ be the higher ramification filtration of the inertia group at Q. Then the coarse space of $\mathcal{X}$ is $Y/(\mathbb{Z}/p\mathbb{Z}) = \mathbb{P}^1$, with coarse map $\pi : \mathcal{X} \to \mathbb{P}^1$, and

$$\pi^* K_{\mathbb{P}^1} + \sum_{i=0}^{\infty}(|G_i| - 1)Q = -2H + \sum_{i=0}^{m}(p-1)Q = -2H + (m+1)(p-1)Q$$

where H is a hyperplane, i.e. a point, in $\mathbb{P}^1$ which we take to be distinct from ∞. Thus we can take $K_{\mathcal{X}} = -2H + (m+1)(p-1)Q$. More generally, if f is a rational function with poles at $x_1, \ldots, x_r$ of orders $m_1, \ldots, m_r$, respectively, then

$$K_{\mathcal{X}} = -2H + (m+1)(p-1)Q + \sum_{j=1}^{r}(m_j+1)(p-1)Q_j$$

where Q_j is the stacky point lying above x_j, Q is again the stacky point above ∞ and $m = \deg(f)$.

As in the tame case, we may define the *Euler characteristic* $\chi(\mathcal{X}) = -\deg(K_{\mathcal{X}})$ and the *genus* $g(\mathcal{X})$ via $\chi(\mathcal{X}) = 2 - 2g(\mathcal{X})$.

Corollary 5.1.4. *For a stacky curve $\mathcal{X}$ over an algebraically closed field k, with coarse space X,*

$$g(\mathcal{X}) = g(X) + \frac{1}{2}\sum_{x\in\mathcal{X}(k)}\sum_{i=0}^{\infty}\left(\frac{1}{[G_x : G_{x,i}]} - \frac{1}{|G_x|}\right)$$

where $\pi : \mathcal{X} \to X$ is the coarse map.

Example 5.1.5. If $Y \to \mathbb{P}^1$ is the Artin–Schreier cover given by $y^p - y = f(x)$ with $f \in k[x]$ of degree $(m, p) = 1$ and $\mathcal{X} = [Y/(\mathbb{Z}/p\mathbb{Z})]$, then

$$g(\mathcal{X}) = \frac{(m+1)(p-1)}{2p}$$

where $m = -\operatorname{ord}_\infty(f)$. For instance, when $m = 1$, $\mathcal{X} = [\mathbb{P}^1/(\mathbb{Z}/p\mathbb{Z})]$ has canonical divisor $K_{\mathcal{X}} = -2H + 2(p-1)Q$ and genus $g(\mathcal{X}) = \frac{p-1}{p} = 1 - \frac{1}{p}$, similar to the tame case, where $[\mathbb{P}^1/\mu_r]$ has genus $1 - \frac{1}{r}$. However, if $m > 0$, the genus formula again illustrates that we have infinitely many non-isomorphic stacky $\mathbb{P}^1$'s.

Example 5.1.6. Let $\operatorname{char} k = 3$ and let $E \to \mathbb{P}^1$ be the Artin–Schreier cover defined by the equation $y^2 = x^3 - x$. In general, the genus of an Artin–Schreier curve in characteristic p with jump m is $\frac{(p-1)(m-1)}{2}$ so in this case we have $g(E) = 1$. Thus E is an elliptic curve and it is well-known (and easy to check) that $\omega = \frac{dx}{2y}$ is a differential form on E. Since $\dim H^0(E, \Omega^1_E) = g(E) = 1$, ω is, up to multiplication by an element of $k^\times$, the only nonzero, holomorphic differential form on E.

Meanwhile, the stack $\mathcal{X} = [E/(\mathbb{Z}/3\mathbb{Z})]$ also has genus $g(\mathcal{X}) = \frac{(3-1)(2+1)}{6} = 1$, so we might call it a "stacky elliptic curve". Notice that the group action of $\mathbb{Z}/3\mathbb{Z}$ on E, which is induced by $x \mapsto x + 1$, leaves ω invariant. Therefore ω generates the vector space of differential forms on $\mathcal{X}$, which is equivalently the space of $\mathbb{Z}/3\mathbb{Z}$-invariant differential forms on E. However, the coarse space here is $\mathbb{P}^1$ which has $H^0(\mathbb{P}^1, \Omega_{\mathbb{P}^1}) = 0$.

In general, if $\mathcal{X}$ has coarse space X and admits a presentation by a scheme $f : Y \to \mathcal{X}$, then these three cohomology groups, $H^0(Y, f^*\Omega_{\mathcal{X}}), H^0(\mathcal{X}, \Omega_{\mathcal{X}})$ and $H^0(X, \Omega_X)$, need not be the same. The genus may even *increase* along f, although this is already true in the characteristic 0 case.

Example 5.1.7. Let $Y \to \mathbb{P}^1$ be the Artin–Schreier–Witt cover given by the equations

$$y^p - y = \frac{1}{x^m} \quad \text{and} \quad z^p - z = \frac{y}{x^m}.$$

This cover is ramified at the point Q lying over ∞ with group $G = \mathbb{Z}/p^2\mathbb{Z}$ and ramification jumps m and $m(p^2+1)$ (by Example 1.4.19), so by the Riemann–Hurwitz formula, the quotient stack $\mathcal{X} = [Y/G]$ has canonical divisor

$$\begin{aligned} K_{\mathcal{X}} &= -2H + \sum_{i=0}^{m} (p^2 - 1)Q + \sum_{i=m+1}^{m(p^2+1)} (p-1)Q \\ &= -2H + ((m+1)(p^2-1) + mp^2(p-1))Q \\ &= -2H + (mp^3 + p^2 - m - 1)Q \end{aligned}$$

Therefore the genus of $\mathcal{X}$ is

$$g(\mathcal{X}) = \frac{mp^3 + p^2 - m - 1}{2p^2}.$$

Once we have our hands on the canonical divisor, the next step in trying to compute the canonical ring of a stacky curve is to apply a suitable version of Riemann–Roch to $K_{\mathcal{X}}$ and count dimensions. When $\mathcal{X}$ is a tame stacky curve, we do this as follows. For a divisor D on $\mathcal{X}$, Lemma 4.1.14 implies that $H^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(D)) \cong H^0(X, \mathcal{O}_X(\lfloor D \rfloor)$ where $\lfloor D \rfloor$ denotes the floor divisor. Then at any tame stacky point x, the stabilizer group G_x is cyclic and by the tame Riemann–Hurwitz formula (Proposition 4.1.18), the coefficient of the canonical divisor at x is $\frac{|G_x|-1}{|G_x|}$. As a consequence, for any divisor D on $\mathcal{X}$, $\lfloor K_{\mathcal{X}} - D \rfloor = K_X - \lfloor D \rfloor$. This yields the following version of Riemann–Roch for $\mathcal{X}$:

$$\begin{aligned} h^0(\mathcal{X}, D) - h^0(\mathcal{X}, K_{\mathcal{X}} - D) &= h^0(X, \lfloor D \rfloor) - h^0(X, \lfloor K_{\mathcal{X}} - D \rfloor) \\ &= h^0(X, \lfloor D \rfloor) - h^0(X, K_X - \lfloor D \rfloor) \\ &= \deg(\lfloor D \rfloor) - g(X) + 1. \end{aligned}$$

(This appears as Corollary 1.189 in [Beh], for example.)

Example 5.1.8. Let a and b be relatively prime integers that are not divisible by $\operatorname{char} k$ and consider the weighted projective line $\mathcal{X} = \mathbb{P}(a, b)$. Then $\mathcal{X}$ is a stacky $\mathbb{P}^1$ with two stacky points P and Q having cyclic stabilizers $\mathbb{Z}/a\mathbb{Z}$ and $\mathbb{Z}/b\mathbb{Z}$, respectively. Thus $\lfloor K_{\mathcal{X}} \rfloor = K_{\mathbb{P}^1} = -2H$ so $h^0(\mathcal{X}, rK_{\mathcal{X}}) = 0$ for all $r \geqslant 1$. In this case the canonical ring is trivial: $R(\mathcal{X}) = H^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \cong k$. This agrees with Example 5.6.9 in [VZB]: $\mathbb{P}(a, b)$ is *hyperbolic*, i.e. $\deg K_{\mathcal{X}} < 0$, so $R(\mathcal{X}) \cong k$.

Example 5.1.9. Assume $\operatorname{char} k \neq 2$. Let $\mathcal{X}$ be a stacky curve with a single stacky point Q of order 2 and with coarse space X of genus 1. Then $K_X = 0$ so $K_{\mathcal{X}} = \frac{1}{2}Q$.

Thus by Riemann–Roch, for any $n \geqslant 1$ we have

$$h^0(\mathcal{X}, nK_{\mathcal{X}}) = \begin{cases} 1, & n = 0, 1 \\ \lfloor \frac{n}{2} \rfloor, & n \geqslant 2. \end{cases}$$

Example 5.7.1 in [VZB] gives an explicit description of $R(\mathcal{X})$ in terms of generators and relations, but for now, the dimension count is the essential information.

The subtle point above is that for a divisor D on a stacky curve $\mathcal{X}$ with coarse space X, $\lfloor K_{\mathcal{X}} - D \rfloor = K_X - \lfloor D \rfloor$ need only hold when $\mathcal{X}$ is *tame*. We have seen that this is not the case in the wild case. For example the stack $\mathcal{X} = [Y/(\mathbb{Z}/p\mathbb{Z})]$ from Example 5.1.3 has canonical divisor $K_{\mathcal{X}} = -2H + (m+1)(p-1)Q$, so $\lfloor K_{\mathcal{X}} \rfloor \neq K_{\mathbb{P}^1} = -2H$ for most choices of m. However, we can still apply Riemann–Roch to obtain new information in the wild case.

Example 5.1.10. Already for a wild $\mathbb{P}^1$ in characteristic p we have new behavior compared to the tame case (see Example 5.1.8). Let $\mathcal{X}$ be the quotient stack $[Y/(\mathbb{Z}/p\mathbb{Z})]$ given by Artin–Schreier equation $y^p - y = x^m$, as in Example 5.1.3. We computed $K_{\mathcal{X}} = -2H + (m+1)(p-1)Q$, where Q is the single stacky point over ∞ of order p. Then by Lemma 4.1.14 and Riemann–Roch,

$$h^0(\mathcal{X}, nK_{\mathcal{X}}) = h^0(\mathbb{P}^1, \lfloor nK_{\mathcal{X}} \rfloor) = \deg(\lfloor nK_{\mathcal{X}} \rfloor) + 1 + h^0(\mathbb{P}^1, K_{\mathbb{P}^1} - \lfloor nK_{\mathcal{X}} \rfloor).$$

For $n = 1$, this is already a new formula:

$$\lfloor K_{\mathcal{X}} \rfloor = -2H + \left(m - \left\lfloor \frac{m}{p} \right\rfloor\right) \infty.$$

Therefore $\deg(\lfloor K_{\mathcal{X}} \rfloor) = m - \left\lfloor \frac{m}{p} \right\rfloor - 2$. This also shows that $K_{\mathbb{P}^1} - \lfloor K_{\mathcal{X}} \rfloor$ is non-effective when $m \geqslant 2$, so we get

$$h^0(\mathcal{X}, K_{\mathcal{X}}) = \max\left\{m - \left\lfloor \frac{m}{p} \right\rfloor - 1, 1\right\}.$$

More generally, since $\left\lfloor \frac{k(p-1)}{p} \right\rfloor = k - \left\lfloor \frac{k}{p} \right\rfloor - 1$, we can compute

$$\lfloor nK_{\mathcal{X}} \rfloor = -2nH + \left(n(m+1) - \left\lfloor \frac{n(m+1)}{p} \right\rfloor - 1\right) \infty.$$

So $\deg(\lfloor nK_{\mathcal{X}} \rfloor) = n(m-1) - \left\lfloor \frac{n(m+1)}{p} \right\rfloor - 1$ and again, when $m \geqslant 2$, $K_X - \lfloor K_{\mathcal{X}} \rfloor$ is non-effective. By Riemann–Roch,

$$h^0(\mathcal{X}, nK_{\mathcal{X}}) = \max\left\{n(m-1) - \left\lfloor \frac{n(m+1)}{p} \right\rfloor, 1\right\}.$$

Example 5.1.11. Let $\mathcal{X} = [Y/(\mathbb{Z}/p^2\mathbb{Z})]$ be the Artin–Schreier–Witt quotient from Example 5.1.7. For the cases when $m < p^2$, we have

$$\lfloor K_{\mathcal{X}} \rfloor = -2H + \left\lfloor \frac{mp^3 + p^2 - m - 1}{p^2} \right\rfloor \infty = -2H + mp\infty$$

so by Riemann–Roch, $h^0(\mathcal{X}, K_{\mathcal{X}}) = mp$. There's not such a clean formula for the global sections of $nK_{\mathcal{X}}$, but one still has

$$h^0(\mathcal{X}, nK_{\mathcal{X}}) = -2n + \left\lfloor \frac{n(mp^3 + p^2 - m - 1)}{p^2} \right\rfloor + 1 = n(mp-1) + \left\lfloor \frac{-n(m+1)}{p^2} \right\rfloor.$$

When $m \geqslant p^2$, the formulas are even more complicated, reflecting the importance of the ramification jumps in the geometry of these wild stacky curves.

5.2 Modular Forms

Modular forms are ubiquitous objects in modern mathematics, appearing in a startling number of places, such as: the study of the Riemann zeta function and more general L-functions; the proof of the Modularity Theorem and related results by Wiles, Taylor, et al.; the representation theory of finite groups; sphere packing problems; and quantum gravity in theoretical physics. They are a critical tool in the Langlands Program and in the study of the Birch and Swinnerton-Dyer Conjecture. For an overview of the modern theory, see [BvdGHZ], and for a longer survey, see [Ono]. A useful feature of modular forms is that they fall into finite dimensional vector spaces and therefore possess linear relations among their coefficients that encode a wealth of number theoretic data.

In the next few sections, we will recall the classical definition of modular forms, define modular forms geometrically (following [Kat]) and suggest a program for proving dimension formulas for spaces of modular forms coming from geometry. Theoretically, this can be done using the stacky version of the modular curves $X(N), X_0(N), X_1(N)$, etc., but will be saved for future work.

5.2.1 Background

Let $\mathfrak{h} = \{z \in \mathbb{C} : \mathrm{im}(z) > 0\}$ be the complex upper half-plane and define the *completed upper half-plane* $\mathfrak{h}^* = \mathfrak{h} \cup \{\infty\} \cup \mathbb{Q}$, where ∞ is considered as the point "$i\infty$" and $\mathbb{Q}$ is viewed as a subset of the real axis in $\mathbb{C}$. Classically, the modular group $\Gamma = \Gamma(1) = SL_2(\mathbb{Z})$ acts on $\mathfrak{h}$ by fractional linear transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}.$$

The quotient space $Y = Y(1) = \mathfrak{h}/\Gamma$, our first example of a(n affine) modular curve, is isomorphic to $\mathbb{C} = \mathbb{P}^1_{\mathbb{C}} \smallsetminus \{\infty\}$, the once-punctured Riemann sphere. Its compactification $X = X(1) = \overline{Y(1)}$ can also be described as a quotient: $x = \mathfrak{h}^*/PSL_2(\mathbb{Z})$, where $PSL_2(\mathbb{Z})$ acts on $\{\infty\} \cup \mathbb{Q}$ by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \frac{p}{q} = \frac{ap+bq}{cp+dq} \quad \text{and} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \infty = \frac{a}{c}.$$

The modular curve X is a proper Riemann surface isomorphic to $\mathbb{P}^1_{\mathbb{C}}$. Recall that a (weakly) *modular function of weight* $2k$ is a holomorphic function $f : \mathfrak{h} \to \mathbb{C}$

such that for all $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$,

$$f(z) = (cz + d)^{-2k} f(gz).$$

Let $q = e^{2\pi i z}$. If the q-expansion $f(q) = \sum_{n=-\infty}^{\infty} a_n q^n$ of a modular function f is holomorphic at ∞, i.e. $a_n = 0$ for all $n < 0$, then f is a *modular form* of the same weight. *Cusp forms* are those modular forms whose q-expansions have no constant term, i.e. $a_0 = 0$. For each $k \in \mathbb{Z}$, let $\mathcal{M}_k$ (resp. $\mathcal{S}_k$) denote the $\mathbb{C}$-vector space of modular forms of weight 2k (resp. cusp forms of weight 2k). For a modular form $f \in \mathcal{M}_k$, we define a (holomorphic) differential k-form on $\mathfrak{h}$ by

$$\omega_f := f(z)\, dz^k \in \Omega^k_{\mathfrak{h}/\mathbb{C}}(\mathfrak{h}) = H^0(\mathfrak{h}, \Omega^k_{\mathfrak{h}/\mathbb{C}}).$$

Then for every $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we have

$$\begin{aligned} g^*\omega_f = f(gz)\, d(gz)^k &= (cz+d)^{2k} f(z) \left(\frac{d}{dz}\left(\frac{az+b}{cz+d}\right)\right)^k dz^k \\ &= (cz+d)^{2k}\left(\frac{a(cz+d) - c(az+b)}{(cz+d)^2}\right)^k f(z)\, dz^k \\ &= (ad - bc)^k \omega_f = \omega_f \end{aligned}$$

since $g \in \Gamma = SL_2(\mathbb{Z})$. Hence ω_f is Γ-invariant so it descends to a differential k-form on Y, which we will still write as $\omega_f \in H^0(Y, \Omega^k_{Y/\mathbb{C}})$. This computation is also compatible with the quotient structure $X = \mathfrak{h}^*/\mathrm{PSL}_2(\mathbb{Z})$, so we can identify modular forms $f \in \mathcal{M}_k$ with certain sections $\omega_f \in H^0(X, \Omega^k_{X/\mathbb{C}})$ – the question is, which sections?

Let $\varphi : \mathfrak{h}^* \to X$ be the quotient map and fix points $P \in \mathfrak{h}^*$ and $Q = \varphi(P) \in X$. It is a standard fact that φ is a finite branched cover of X with branch locus $\{i, \rho, \infty\}$, where $\rho = e^{2\pi i/3}$.

Lemma 5.2.1. *If* $f : \mathfrak{h} \to \mathbb{C}$ *is a modular form of weight* $2k$*, then the order of vanishing of* ω_f *at a point* $Q \in X$ *is given by*

$$\mathrm{ord}_Q(\omega_f) = \begin{cases} \frac{1}{2}(\mathrm{ord}_i(f) - k), & P = i \\ \frac{1}{3}(\mathrm{ord}_\rho(f) - 2k), & P = \rho \\ \mathrm{ord}_\infty(f) - k, & P = \infty \\ \mathrm{ord}_P(f), & P \notin \{i, \rho, \infty\}. \end{cases}$$

Proof. First assume $P = i$ and z is a local parameter at i. Then φ is locally given by $z \mapsto z^2$ so

$$\begin{aligned} \varphi^*\omega_f &= \varphi^*(f(z)\,dz^k) = f(z^2)\,d(z^2)^k \\ &= f(z^2)(2z\,dz)^k = 2f(z^2)z^k\,dz^k. \end{aligned}$$

Thus $\mathrm{ord}_i(f) = 2\,\mathrm{ord}_Q(\omega_f) + k$. Similarly, if $P = \rho$ and z is a local parameter at

ρ, then φ is locally of the form $z \mapsto z^3$ and the computation becomes:

$$\varphi^* \omega_f = f(z^3)(3z^2\, dz)^k = 3f(z^3) z^{2k}\, dz^k.$$

So $\operatorname{ord}_\rho(f) = 3 \operatorname{ord}_Q(\omega_f) + 2k$. When $P = \infty$, a local parameter is $q = e^{2\pi i z}$ and φ sends $z \mapsto q$, so if $\omega_f = q^n\, dq^k$, we get

$$\varphi^* \omega_f = \varphi^*(q^n\, d(e^{2\pi i z})^k) = q^n (2\pi i e^{2\pi i z}\, dz)^k = (2\pi i)^k q^{n+k}\, dz^k.$$

Thus $\operatorname{ord}_\infty(f) = n + k = \operatorname{ord}_Q(\omega_f) + k$. Finally, φ is an isomorphism away from the branch locus $\{i, \rho, \infty\}$, so the order of vanishing remains constant at any point outside the branch locus. □

Remark 5.2.2. In general, if a differential form $\omega = f(z)\, dz^k$ is pulled back (locally) along $z \mapsto z^n$, then the orders of vanishing of ω and f are related by the equation

$$\operatorname{ord}_P(f) = n \operatorname{ord}_Q(\omega) + k(n-1).$$

This follows directly from the Riemann–Hurwitz formula for the branched cover $\mathfrak{h}^* \to X$ and adapts easily to the modular curve $X(\Gamma)$ when $\Gamma \leqslant SL_2(\mathbb{Z})$ is any congruence subgroup. To handle cusps like ∞, one must be careful with the choice of coordinate q about the cusp, but this can be done. Later, we will use this approach together with the wild versions of Riemann–Hurwitz and Riemann–Roch to compute dimensions of spaces of modular forms in characteristic p.

Corollary 5.2.3. *For any modular form* $f : \mathfrak{h} \to \mathbb{C}$ *of weight* $2k$*,*

$$\frac{1}{2}\operatorname{ord}_i(f) + \frac{1}{3}\operatorname{ord}_\rho(f) + \operatorname{ord}_\infty(f) + \sum_{z \in D \setminus \{i,\rho\}} \operatorname{ord}_z(f) = \frac{k}{6}$$

where $D = \left\{z \in \mathfrak{h} : |z| \geqslant 1, |Re(z)| \leqslant \frac{1}{2}\right\}$ *is the standard fundamental domain for the action of* Γ *on* $\mathfrak{h}$*.*

Proof. Since $\omega_f = f(z)\,dz$ is a nonzero holomorphic differential 1-form on X, the Riemann–Hurwitz formula implies $\deg(\omega_f) = k(2g(X) - 2) = -2k$. On the other hand, Lemma 5.2.1 gives us

$$\begin{aligned}
\deg(\omega_f) &= \sum_{Q \in X} \operatorname{ord}_Q(\omega_f) = \operatorname{ord}_i(\omega_f) + \operatorname{ord}_\rho(\omega_f) + \operatorname{ord}_\infty(\omega_f) + \sum_{Q \notin \{i,\rho,\infty\}} \operatorname{ord}_Q(\omega_f) \\
&= \frac{1}{2}(\operatorname{ord}_i(f) - k) + \frac{1}{3}(\operatorname{ord}_\rho(f) - 2k) + (\operatorname{ord}_\infty(f) - k) + \sum_{P \notin \{i,\rho,\infty\}} \operatorname{ord}_P(f) \\
&= -\frac{13k}{6} + \frac{1}{2}\operatorname{ord}_i(f) + \frac{1}{3}\operatorname{ord}_\rho(f) + \operatorname{ord}_\infty(f) + \sum_{z \in D \setminus \{i,\rho\}} \operatorname{ord}_z(f).
\end{aligned}$$

Combining these gives the right formula. □

This allows us to identify the image of the map $\mathcal{M}_k \to H^0(X, \Omega^k_{X/\mathbb{C}})$.

Corollary 5.2.4. *For each* $k \in \mathbb{Z}$*,*

$$\mathcal{M}_k = \left\{ \omega \in H^0(X, \Omega^k_{X/\mathbb{C}}) \,\middle|\, \operatorname{ord}_i(\omega) \geqslant -\frac{k}{2}, \operatorname{ord}_\rho(\omega) \geqslant -\frac{2k}{3}, \operatorname{ord}_\infty(\omega) \geqslant -k \right\}.$$

Remark 5.2.5. Consider the $\mathbb{Q}$-divisor $D = \frac{1}{2}i + \frac{2}{3}\rho + \infty$ on X. Then Corollary 5.2.4 shows we can interpret $\mathcal{M}_k$ as the space of global sections $H^0(X, \Omega^k_{X/\mathbb{C}}(\lfloor kD \rfloor))$.

Further, Proposition 4.1.6 suggests we may instead view D as a true divisor on some stack $\mathcal{X}$ whose coarse space is X. Indeed, taking $\mathcal{X}$ to be a stacky $X = \mathbb{P}^1_{\mathbb{C}}$ with stacky points at i and ρ of orders 2 and 3, respectively, we then obtain a stacky interpretation of the formula in Corollary 5.2.4 by way of Lemma 4.1.14.

Corollary 5.2.6. *Let $k \in \mathbb{Z}$ and let $\Delta \in \mathcal{S}_6$ be the cusp form $\Delta = (60G_2)^3 - 27(140G_3)^2$, where G_k denotes the kth Eisenstein series. Then*

(a) For $k < 0$ and $k = 1$, $\mathcal{M}_k = 0$.

(b) $\Delta \neq 0$.

(c) Multiplication by Δ gives an isomorphism $\mathcal{M}_k \to \mathcal{S}_{k+6}$ for all $k \in \mathbb{Z}$.

(d) For $k = 0, 2, 3, 4, 5$, $\dim_k \mathcal{M}_k = 1$. Explicitly, $\mathcal{M}_0 = \mathbb{C}[1]$ and for $k = 2, 3, 4, 5$, $\mathcal{M}_k = \mathbb{C}[G_k]$.

(e) For any $k \geqslant 0$,

$$\dim \mathcal{M}_k = \begin{cases} \left\lfloor \dfrac{k}{6} \right\rfloor, & k \equiv 1 \pmod 6 \\ \left\lfloor \dfrac{k}{6} \right\rfloor + 1, & k \not\equiv 1 \pmod 6. \end{cases}$$

Proof. (a) Every $f \in \mathcal{M}_k$ is holomorphic, so for $k < 0$ there is no way for the order formula in Corollary 5.2.3 to be satisfied unless $f \equiv 0$. Likewise, when $k = 1$ the right-hand side of the formula is $\frac{1}{6}$ and there are no positive integers a, b, c, d satisfying $a + \frac{1}{2}b + \frac{1}{3}c + d = \frac{1}{6}$. Hence $\mathcal{M}_1 = 0$.

(b) Since $G_2 \in \mathcal{M}_2$, the formula in Corollary 5.2.3 has $\frac{1}{3}$ on the right, so $\text{ord}_i(G_2) = 0, \text{ord}_\rho(G_2) = 1$ and hence $G_2(i) \neq 0$ and $G_2(\rho) = 0$. Similarly for $G_3 \in \mathcal{M}_3$, we have $\text{ord}_i(G_3) = 1, \text{ord}_\rho(G_3) = 0$ and therefore $G_3(i) = 0$ and $G_3(\rho) \neq 0$. Since Δ is a linear combination of G_2^3 and G_3^2, this shows that $\Delta(i)$ and $\Delta(\rho)$ are both nonzero. In particular, Δ is nontrivial.

(c) The order formula also shows that Δ has a simple zero at ∞. If $f \in \mathcal{S}_{k+6}$ is a cusp form, then $f(\infty) = 0$ so $\frac{f}{\Delta}$ is holomorphic and hence $\frac{f}{\Delta} \in \mathcal{M}_k$. As $\Delta \neq 0$, this clearly establishes the isomorphism $\mathcal{M}_k \to \mathcal{S}_{k+6}, g \mapsto g\Delta$.

(d) In general, if $k - 6 < 0$ then by (a), $\mathcal{M}_{k-6} = 0$. By (c), this implies $\mathcal{S}_k = 0$, so there are no cusp forms in $\mathcal{M}_k$. In other words, the map $\mathcal{M}_k \to \mathbb{C}$ sending $f \mapsto f(\infty)$ is injective, so it follows that for $k < 6$, $\dim \mathcal{M}_k \leqslant 1$. Since Eisenstein series exist and are nontrivial for $k = 2, 3, 4, 5$, we therefore have $\dim \mathcal{M}_k = 1$ for each of these k and $\mathcal{M}_k = \mathbb{C}[G_k]$.

(e) This obviously holds for $k < 6$ by (a) and (d), but then (c) implies it for all k by induction, since $\mathcal{M}_k \cong \mathcal{S}_k \oplus \mathbb{C}$. □

In the next section we will give a more geometric proof of (e) that adapts well to other settings.

5.2.2 Geometric Modular Forms

To describe modular forms geometrically, let $\mathcal{R}$ be the set of lattices in $\mathbb{C}$. Each $\Lambda \in \mathcal{R}$ can be written $\Lambda = [\omega_1, \omega_2] := \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$ for two linearly independent $\omega_1, \omega_2 \in \mathbb{C}$. Note that $\mathbb{C}^\times$ acts on $\mathcal{R}$ by scaling. Each $\Lambda \in \mathcal{R}$ also determines an elliptic curve $E_\Lambda := \mathbb{C}/\Lambda$. We will write $E_{\omega_1,\omega_2} = \mathbb{C}/[\omega_1, \omega_2]$ and for a complex number $\tau \in \mathfrak{h}$, $E_\tau = E_{\tau,1}$. Let Ell be the set of isomorphism classes of complex elliptic curves.

Proposition 5.2.7. *There is a bijective correspondence*

$$\begin{aligned} \mathcal{R}/\mathbb{C}^\times &\longleftrightarrow \mathsf{Ell} \\ [\Lambda] &\longmapsto [E_\Lambda]. \end{aligned}$$

Proof. Standard (cf. [DS, Prop. 1.4.1]). □

Fix an elliptic curve E. A lattice $\Lambda \subset \mathbb{C}$ such that $E_\Lambda \cong E$ is called a *lattice of periods* for E. Explicitly, such a lattice is given by

$$\Lambda = \left\{ \int_\gamma \omega : \gamma \in H_1(E; \mathbb{Z}) \right\}$$

where $\omega \in H^1_{dR}(E)$ is a nonzero differential 1-form on E, called an *invariant differential*. On the other hand, we have:

Proposition 5.2.8. *There is an analytic isomorphism*

$$\mathcal{R}/\mathbb{C}^\times \xrightarrow{\sim} Y(1) = \mathfrak{h}/\Gamma.$$

Proof. The bijection is given by

$$\begin{aligned} \mathcal{R}/\mathbb{C}^\times &\longleftrightarrow \mathfrak{h}/\Gamma \\ [\omega_1, \omega_2] &\longmapsto \frac{\omega_1}{\omega_2} + \Gamma \\ [\tau, 1] &\longleftarrow\!\shortmid \tau + \Gamma. \end{aligned}$$

It is routine to check that both maps are analytic. □

Corollary 5.2.9. *The modular curve* $Y(1)$ *is a moduli space for isomorphism classes of complex elliptic curves.*

Further, we can identify modular forms of weight $2k$ with certain lattice functions as follows. For a function $F : \mathcal{R} \to \mathbb{C}$, we say F is *modular of weight* $2k$ if for all $\Lambda \in \mathcal{R}$ and $\alpha \in \mathbb{C}^\times$, we have

$$F(\alpha\Lambda) = \alpha^{-2k} F(\Lambda).$$

Any modular lattice function $F : \mathcal{R} \to \mathbb{C}$ determines a (weakly) modular function $f : \mathfrak{h} \to \mathbb{C}$ of the same weight given by $f(z) = F([z, 1])$. Conversely, given a (weakly) modular function f, the formula $F([\omega_1, \omega_2]) = \omega_2^{2k} f\left(\frac{\omega_1}{\omega_2}\right)$ defines a modular lattice function F. Under this correspondence, modular forms correspond to modular lattice functions with the appropriate condition at ∞.

Next, observe that the set of lattices $\mathcal{R}$ is a $\mathbb{G}_m$-bundle on $\mathcal{R}/\mathbb{C}^\times$. Propositions 5.2.7 and 5.2.8 suggest that we should be able to construct an analogous bundle $\mathcal{E} \to \mathsf{Ell}$ performing a similar role. This would allow us to interpret modular forms as functions on E just as we did for $\mathcal{R}$. First, define

$$\mathcal{E} = \{(E, \omega) \mid E \in \mathsf{Ell}, \omega \in H^0(E, \Omega^1_{E/\mathbb{C}}), \omega \neq 0\}/\sim$$

where $\sim$ denotes the following equivalence: $(E, \omega) \sim (E', \omega')$ if there exists an isomorphism of elliptic curves $\varphi : E \to E'$ such that $\varphi^*\omega' = \omega$. Since $H^0(E, \Omega^1_{E/\mathbb{C}})$ has dimension $g(E) = 1$ for any elliptic curve $E/\mathbb{C}$, such a ω is unique up to scaling, so the forgetful functor $\mathcal{E} \to \mathsf{Ell}, [E, \omega] \mapsto [E]$ is a $\mathbb{G}_m$-bundle. Moreover, there is a map $\mathcal{R} \to \mathcal{E}$ which sends $[\Lambda]$ to $[E_\Lambda, dz]$.

Lemma 5.2.10. *The diagram*

$$\begin{array}{ccc} \mathcal{R} & \longrightarrow & \mathcal{E} \\ \downarrow & & \downarrow \\ \mathcal{R}/\mathbb{C}^\times & \xrightarrow{\sim} & \mathsf{Ell} \end{array}$$

commutes and preserves the $\mathbb{G}_m$*-bundle structures on* $\mathcal{R}$ *and* $\mathcal{E}$.

We say a function $G : \mathcal{E} \to \mathbb{C}$ is *modular of weight* $2k$ if for all $[E, \omega] \in \mathcal{E}$ and $\alpha \in \mathbb{C}^\times$,

$$G([E, \alpha\omega]) = \alpha^{-2k} G([E, \omega]).$$

Such a G determines a (weakly) modular function $g : \mathfrak{h} \to \mathbb{C}$ by setting $g(\tau) = G([E_\tau, dz])$ and conversely. We know from classical modular form theory that the holomorphicity conditions defining a holomorphic form on $\mathfrak{h}$ can be phrased in terms of holomorphic differential forms on the compactified modular curve

$X(1)$. To translate such conditions to the bundle $\mathcal{E} \to \mathsf{Ell}$, we introduce the following construction.

For each $\tau \in \mathfrak{h}$, set $q(\tau) = e^{2\pi i\tau}$. Then $E_\tau = \mathbb{C}/[\tau, 1]$ is isomorphic to $\mathbb{C}^\times/\langle q(\tau)\rangle$ via the analytic map $z \mapsto q(z) = e^{2\pi i z}$. The elliptic curve $\mathbb{C}^\times/\langle q(\tau)\rangle$ is called the *Tate curve*, written $\mathrm{Tate}_{\mathbb{C}}(q(\tau))$ or just $\mathrm{Tate}_{\mathbb{C}}(q)$. Explicitly, $\mathrm{Tate}_{\mathbb{C}}(q)$ is an elliptic curve over $\mathbb{Z}((q))$ given by the affine equation

$$\begin{aligned} & y^2 + xy = x^3 + A(q)x + B(q) \\ \text{where} \quad & A(q) = -5\sum_{n=1}^{\infty} \sigma_3(n)q^n = \frac{1 - E_4(q)}{48} \\ \text{and} \quad & B(q) = -\frac{1}{12}\sum_{n=1}^{\infty}(5\sigma_3(n) + 7\sigma_5(n))q^n = \frac{1}{12}\left(\frac{1 - E_4(q)}{48} + \frac{1 - E_6(q)}{72}\right). \end{aligned}$$

This allows us to define the Tate curve over an arbitrary ring.

Definition 5.2.11. *The* **Tate curve over a ring** R *is the base change*

$$\mathrm{Tate}_R(q) := \mathrm{Tate}_{\mathbb{C}}(q) \times_{\mathbb{Z}} R \; (= \mathrm{Tate}_{\mathbb{C}}(q) \times_{\mathrm{Spec}\,\mathbb{Z}} \mathrm{Spec}\, R)$$

which is an elliptic curve over $R \otimes_{\mathbb{Z}} \mathbb{Z}((q))$.

The following geometric version of modular forms is due to Katz [Kat]. Let R be a ring and $p : E \to \mathrm{Spec}\, R$ an elliptic curve, i.e. a morphism of schemes admitting a section $O : \mathrm{Spec}\, R \to E$ such that each geometric fibre is an elliptic curve with basepoint given by O. The sheaf $\omega_{E/R} := p_*\Omega^1_{E/R}$ is called the *Katz canonical sheaf* of E.

Definition 5.2.12. *A* **geometric modular function of weight** $2k$ *over a base ring* R *is an assignment* F *of a section* $F(E/A) \in \omega_{E/A}^{k}$ *for every* R*-algebra* A *and every elliptic curve* $E \to \operatorname{Spec} A$ *which satisfies:*

(1) $F(E/A)$ *is constant on the isomorphism class of* E/A*.*

(2) If $\varphi : A \to B$ *is a morphism of* R*-algebras and* E *is an elliptic curve over* A *with base change* $E' = E \times_{\operatorname{Spec} A} \operatorname{Spec} B$*, then* $F(E'/B) = \varphi(F(E/A))$*.*

Proposition 5.2.13. *The data of a geometric modular function of weight* $2k$ *over* R *is equivalent to the assignment of an element* $f(E/A, \omega) \in A$ *to every* R*-algebra* A*, elliptic curve* $E \to \operatorname{Spec} A$ *and nonzero element* $\omega \in H^0(E, \Omega^1_{E/A})$ *such that:*

(1) $f(E/A, \omega)$ *is constant on the isomorphism class of* E/A*.*

(2) For all $\alpha \in A^\times$*,* $f(E/A, \alpha\omega) = \alpha^{-2k} f(E/A, \omega)$*.*

(3) If $\varphi : A \to B$ *is a morphism of* R*-algebras and* E/A *is an elliptic curve with base change* $E' = E \times_{\operatorname{Spec} A} \operatorname{Spec} B$ *and compatible sections* $\omega \in H^0(E, \Omega^1_{E/A})$ *and* $\omega' \in H^0(E', \Omega^1_{E'/B})$*, then* $f(E'/B, \omega') = \varphi(f(E/A, \omega))$*.*

Proof. For any such f, define $F(E/A) = \omega^{2k} f(E/A, \omega)$ for any nonzero $\omega \in H^0(E, \Omega^1_{E/A})$. By (2), $F(E/A)$ is well-defined and axioms (1) and (2) are easy to verify from the other conditions on f. Conversely, for a geometric modular form F, define $f(E/A, \omega) = \omega^{-2k} F(E/A)$. □

The Tate curve allows us to define q-expansions geometrically.

Definition 5.2.14. *The* **q-expansion of a geometric modular function** F *over* R *is the section* $F(\mathrm{Tate}_R(q)/R \otimes_{\mathbb{Z}} \mathbb{Z}((q)))$ *in* $\omega_{\mathrm{Tate}_R(q)/R \otimes_{\mathbb{Z}} \mathbb{Z}((q))}$.

Remark 5.2.15. If $f(E/A, \omega) = \omega^{-2k} F(E/A)$ as in Proposition 5.2.13, then the q-expansion of F can be thought of as an element $f(\mathrm{Tate}_R(q)/R \otimes_{\mathbb{Z}} \mathbb{Z}((q)), \omega_{can}) \in R \otimes_{\mathbb{Z}} \mathbb{Z}((q))$, where $\omega_{can} = \frac{dx}{x+2y}$ is the canonical differential form on $\mathrm{Tate}_R(q)$.

Definition 5.2.16. *A* **geometric modular form of weight** $2k$ *over* R *is a geometric modular function* F *of weight* $2k$ *whose* q*-expansion* $f(\mathrm{Tate}_R(q)/R \otimes_{\mathbb{Z}} \mathbb{Z}((q)), \omega_{can})$ *lies in* $R \otimes_{\mathbb{Z}} \mathbb{Z}[[q]]$. *Further, we say* F *is a* **geometric cusp form** *if its* q*-expansion lies in* $R \otimes_{\mathbb{Z}} q\mathbb{Z}[[q]]$.

Example 5.2.17. When $R = \mathbb{C}[j]$, the curve $Y(1) = \operatorname{Spec} \mathbb{C}[j]$ is the affine j-line parametrizing complex elliptic curves. Any geometric modular form F over $\mathbb{C}[j]$ determines a classical modular form $f : \mathfrak{h} \to \mathbb{C}$ of the same weight by setting

$$f(\tau) = F(E_\tau/\mathbb{C}[j]).$$

Suppose there were an elliptic curve **E** over $Y(1)$ which is a *fine moduli space* for isomorphism classes of complex elliptic curves. (It turns out that such a curve does exist for $Y(N)$ with level structure $N > 2$, but not for $N = 1, 2$.) Set $\omega = \omega_{\mathbf{E}/Y(1)}$. Then a modular function of weight 2k would be the same thing as a global section of $\omega^{\otimes k}$ on $Y(1)$ and if ω can be extended to $X(1) = \overline{Y(1)}$, then a modular form of weight 2k would just be a global section of $\omega^{\otimes k}$ on $X(1)$. The fact that no such **E** exists over $Y(1)$ should not deter us – in fact, the formula

$$\mathcal{M}_k = \left\{ \omega \in H^0(X, \Omega^k_{X/\mathbb{C}}) \middle| \operatorname{ord}_i(\omega) \geqslant -\frac{k}{2}, \operatorname{ord}_\rho(\omega) \geqslant -\frac{2k}{3}, \operatorname{ord}_\infty(\omega) \geqslant -k \right\}$$

from Corollary 5.2.4 strongly suggests we should be able to interpret $\mathcal{M}_k$ as global sections over something...

This leads us to the orbifold interpretation of classical modular forms. After developing this, we will unite the geometric approaches and begin to tackle modular forms in characteristic p. Let $\mathcal{Y} = \mathcal{Y}(1) = [\mathfrak{h}/\Gamma]$ be the *modular orbifold curve* of level $\Gamma = SL_2(\mathbb{Z})$, which is a stacky curve over $\mathbb{C}$. For each $k \in \mathbb{Z}$, there is a line bundle $\mathcal{L}_k \to \mathcal{Y}$ whose total space is the quotient stack $\mathcal{L}_k = [(\mathfrak{h} \times \mathbb{C})/\Gamma]$, where Γ acts on $\mathfrak{h} \times \mathbb{C}$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (\tau, z) = \left(\frac{a\tau + b}{c\tau + d}, (c\tau + d)^{2k} z \right).$$

Lemma 5.2.18. *For all $k \in \mathbb{Z}$, the vector space of classical (weakly) modular functions on $\mathfrak{h}$ of weight $2k$ is isomorphic to $H^0(\mathcal{Y}, \mathcal{L}_k)$.*

Proof. Every (weakly) modular function f of weight $2k$ defines a section of the projection $\mathfrak{h} \times \mathbb{C} \to \mathfrak{h}$ given by $\tau \mapsto (\tau, f(\tau))$. By construction, this section commutes with the Γ-action on both $\mathfrak{h}$ and $\mathfrak{h} \times \mathbb{C}$, so it descends to a section of $\mathcal{L}_k \to \mathcal{Y}$ and one can show every section of $\mathcal{L}_k \to \mathcal{Y}$ arises this way. $\square$

Further, there is an orbifold compactification $\mathcal{X} = \mathcal{X}(1) = [\mathfrak{h}^*/PSL_2(\mathbb{Z})]$ of $\mathcal{Y}$ which can be constructed from $\mathcal{Y}$ by adding a stacky point at infinity with stabilizer abstractly isomorphic to $\mathbb{Z}/2\mathbb{Z}$. Alternatively, one can construct $\mathcal{X}$ as an orbifold curve by gluing the affine orbifold curves $\mathcal{Y}$ and $[D^2/\mu_2]$ along $[\mathfrak{h}/\langle -I, T\rangle] \cong [D^2 \smallsetminus \{0\}/\mu_2]$, where D^2 is the complex unit disk and $\langle -I, T\rangle$ is the subgroup of $SL_2(\mathbb{Z})$ generated by

$$-I = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \text{and} \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Lemma 5.2.19. *There is an extension of $\mathcal{L}_k$ to $\mathcal{X}$, which we will also denote by $\mathcal{L}_k$, whose global holomorphic sections are $H^0(\mathcal{X}, \mathcal{L}_k) \cong \mathcal{M}_k$, the space of modular forms of weight $2k$.*

Proof. Take $\mathcal{L}_k|_{[D^2/\mu_2]} = \mathcal{O}$, the trivial line bundle on $[D^2/\mu_2]$, and glue with the same data used to define $\mathcal{X} = \mathcal{Y} \cup [D^2/\mu_2]$. Then $H^0([D^2/\mu_2], \mathcal{L}_k) = H^0([D^2/\mu_2], \mathcal{O}) \cong \mathbb{C}[[q]]$, so the restriction of any section $f \in H^0(\mathcal{X}, \mathcal{L}_k)$ to $[D^2/\mu_2]$ is the q-expansion of the corresponding modular function $f|_{\mathcal{Y}} \in H^0(\mathcal{Y}, \mathcal{L}_k)$ and it is a power series. □

Proposition 5.2.20. *There is an isomorphism of stacks $\mathcal{X} \cong \mathbb{P}(4, 6)$, where $\mathbb{P}(4, 6)$ is the weighted projective line considered as a 1-dimensional stack with generic μ_2 stabilizer, under which $\mathcal{L}_k$ is identified with $\mathcal{O}(k)$.*

Proof. This is outlined in [Beh, 1.154]. □

Corollary 5.2.21. *For each $k \in \mathbb{Z}$,* $\dim H^0(\mathcal{X}, \mathcal{L}_k) = \#\{(a, b) \in \mathbb{N}_0^2 \mid 4a + 6b = k\}$.

Proof. This follows from the standard fact that for any weights $m, n \in \mathbb{N}$,

$$H^0(\mathbb{P}(m, n), \mathcal{O}(k)) \cong \bigoplus_{\substack{(a,b) \in \mathbb{N}_0^2 \\ ma+nb=k}} \mathbb{C} x^a y^b$$

which can be obtained from the computation of $H^\bullet(\mathbb{P}^1, \mathcal{O}(k))$ (cf. [Hart, Ch. III, Sec. 5]) after a grading shift. □

Notice that this confirms the dimension formula in Corollary 5.2.6(e). More importantly, it suggests a method for counting dimensions of spaces of modular forms in much greater generality.

5.2.3 Modular Forms in Characteristic p

In his article [Ser1], Serre began investigating modular forms over different fields than $\mathbb{C}$ (or $\mathbb{Q}$) by defining p-adic modular forms to be formal p-adic power series whose coefficients are p-adic limits of classical modular forms. This approach made sense given the number of arithmetic congruences that had been discovered among the coefficients of classical modular forms. However, the spaces of such p-adic modular forms were not as well-behaved as the classical ones (e.g. the weights of the classical modular forms defining a p-adic modular form coefficient-wise were not always the same).

As we have seen, modular forms can also be defined geometrically, that is, as global sections of certain line bundles over the moduli spaces representing moduli problems of elliptic curves. Taken literally, this viewpoint leads to the definition of geometric modular forms in Section 5.2.2 and this was indeed how Katz approached the problem in [Kat]. Over the base ring $R = \mathbb{Q}_p$, Katz's definition gives a competing notion of p-adic modular form. Furthermore, either notion of p-adic modular can be reduced modulo p to yield *mod-*

ular forms mod p: Serre's p-adic modular forms with $\mathbb{Z}_p$-coefficients may be reduced coefficient-wise, while Katz's geometric modular forms may be constructed directly over $\mathbb{F}_p$. The question is: how do these compare?

Theorem 5.2.22. *For all weights* $k \geqslant 2$*, levels* $N \geqslant 1$ *and primes* $p \neq 2, 3$ *with* $p \nmid N$*, there are isomorphisms*

$$\mathcal{M}_k(\Gamma_1(N); \mathbb{F}_p) \cong \mathcal{M}_k(\Gamma_1(N); \mathbb{Q}_p)_{\mathrm{Serre}} \otimes_{\mathbb{Q}_p} \mathbb{F}_p.$$

Proof. This follows from [Edi, Lem. 1.9]. □

That is, in most cases Serre's and Katz's modular forms mod p agree (and cusp forms also agree in these cases). However, there is a modular form $A \in \mathcal{M}_{p-1}(\Gamma(1); \mathbb{F}_p)$ for $p = 2, 3$ called the *Hasse invariant* which is not the reduction of any p-adic modular form.

Question 1. *Can one compute the space* $\mathcal{M}_k(N; K)$ *of geometric modular forms of weight* k *and level* N *over an algebraically closed field* K *of characteristic* $p > 0$*? In particular, can this be done when the corresponding stacky modular curve has wild ramification?*

For example, in [Kat], Katz observes that if A is the Hasse invariant and Δ is the modular discriminant, then $A\Delta$ is a cusp form of weight 13 over $\mathbb{F}_2$ (resp. of weight 14 over $\mathbb{F}_3$) but this cannot be the reduction mod 2 (resp. mod 3) of a modular form over $\mathbb{Z}$ (or $\mathbb{Z}_p$). More generally, we expect the stacky structures of $X_K(N), X_{K,0}(N)$ and $X_{K,1}(N)$ to play a role in the determination of $\mathcal{M}_k(N; K)$.

When $\operatorname{char} K = p > 0$, the covers of curves

$$X_K(N) \to X_{K,1}(N) \to X_{K,0}(N) \to X_K(1) = \mathbb{P}^1_K$$

may have points with wild ramification. In fact, one such example is pointed out in [VZB, Rmk. 5.3.11], which in turn cites an article [BCG] that describes the ramification of the full cover $X_{\mathbb{F}_p}(\ell) \to X_{\mathbb{F}_p}(1) = \mathbb{P}^1_{\mathbb{F}_p}$ for primes $\ell \neq p$. These covers can have nonabelian inertia groups – something that only happens in finite characteristic – and they can then be decomposed and studied more carefully in the tower $X_{\mathbb{F}_p}(\ell) \to X_{\mathbb{F}_p,1}(\ell) \to X_{\mathbb{F}_p,0}(\ell) \to \mathbb{P}^1_{\mathbb{F}_p}$. The author plans to explore this problem more in the future.

Bibliography

[AGV] Abramovich, D., Graber, T. and Vistoli, A. "Gromov–Witten theory of Deligne–Mumford stacks". American Journal of Mathematics, 130 (2008), no. 5, 1337 - 1398.

[AOV] Abramovich, D., Olsson, M. and Vistoli, A. "Tame stacks in positive characteristic". Annales de l'Institute Fourier, 58 (2008), no. 4, 1057 - 1091.

[AV] Abramovich, D. and Vistoli, A. "Compactifying the space of stable maps". Journal of the AMS, 15 (2002), no. 1, 27 - 75.

[Alp10] Alper, J. "On the local quotient structure for Artin stacks". Journal of Pure and Applied Algebra, 214 (2010), no. 9, 1576 - 1591.

[Alp13] Alper, J. "Good moduli spaces for Artin stacks". Annales de l'Institute Fourier, 63 (2013), no. 6, 2349 - 2402.

[Alp14] Alper, J. "Adequate moduli spaces and geometrically reductive group schemes". Algebraic Geometry 1, 4 (2014), 498 - 531.

[AHR1] Alper, J., Hall, J. and Rydh, D. "The étale local structure of algebraic stacks". Preprint (2019).

[AHR2] Alper, J., Hall, J. and Rydh, D. "A Luna étale slice theorem for algebraic stacks". Preprint (2015), arXiv:1504.06467.

[Art] Artin, M. "Versal deformations and algebraic stacks". Inventiones Mathematicae, 27 (1974), 165 - 189.

[AB] Ascher, K. and Bejleri, D. "Moduli of weighted stable elliptic surfaces and invariance of log plurigenera". Preprint (2018), arXiv:1702.06107.

[Beh] Behrend, K. "Introduction to algebraic stacks". *Moduli Spaces*, ed. Brambila-Paz, et al. Cambridge University Press (2014), 1 - 131.

[BCG] Bending, P., Camina, A. and Guralnick, R. “Automorphisms of the modular curve”. Progress in Galois Theory, *Developments in Mathematics*, 12 (2005), 25 - 37.

[Bis] Biswas, I. “Parabolic bundles as orbifold bundles”. Duke Mathematical Journal, 88 (1997), no. 2, 305 - 325.

[Bos] Bosch, S. *Algebraic Geometry and Commutative Algebra*. Springer (2013).

[BvdGHZ] Bruinier, J.H., van der Geer, G., Harder, G. and Zagier, D. “The 1-2-3 of modular forms”. Bulletin of the American Mathematical Society, 46 (2009), no. 3, 527 - 533.

[Cad] Cadman, C. “Using stacks to impose tangency conditions on curves”. American Journal of Mathematics, 129 (2007), no. 2, 405 - 427.

[CS] Choi, S. and Suh, D.Y. “Properties of Bott manifolds and cohomological rigidty”. Algebraic & Geometric Topology, 11 (2011), no. 2, 1053 - 1076.

[DM] Deligne, P. and Mumford, D. “The irreducibility of the space of curves of given genus”. Publications Mathématiques de l’I.H.É.S., 36 (1969). pp. 75 - 109.

[DS] Diamond, F. and Shurman, J. *A First Course in Modular Forms*. Springer (2005).

[Edi] Edixhoven, B. “Serre’s conjecture”. *Modular Forms and Fermat’s Last Theorem (Boston, MA, 1995)*. Springer (1997), 209 - 242.

[EGA IV] Grothendieck, A. and Dieudonné, J. *Éléments de géometrie algébrique: IV. Étude locale des schémas et des morphismes des schémas*. Publications Mathématiques de l’I.H.É.S., 32 (1967), 5 - 361.

[SGA I] Grothendieck, A., Raynaud, M., et al. *Revêtements étales et groupe fondamental*. *Lecture Notes in Mathematics*, 224. Springer-Verlag (1971).

[Gar] Garuti, M. “Linear systems attached to cyclic inertia”. *Arithmetic Fundamental Groups and Noncommutative Algebra*, Proceedings of Symposia in Pure Mathematics, 70 (2002), 377 - 386.

[GS] Geraschenko, A. and Satriano, M. “A ‘bottom-up’ characterization of smooth Deligne–Mumford stacks”. International Mathematics Research Notices (2017), issue 21, 6469 - 6483.

[GK] Grossberg, M. and Karshon, Y. "Bott towers, complete integrability, and the extended character of representations". Duke Mathematical Journal (1994), vol. 76, no. 1, 23 - 58.

[Harb] Harbater, D. "Moduli of p-covers". Commutative Algebra, 8 (1980), 1095 - 1122.

[Hart] Hartshorne, R. *Algebraic Geometry*. Springer (1997).

[Kat] Katz, N.M. "p-adic properties of modular forms and modular schemes". *Modular functions of one variable III*, ed. Serre and Kuijk. Springer-Verlag (1973), 69 - 190.

[KM] Keel, S. and Mori, S. "Quotients by groupoids". Annals of Mathematics 2, 145 (1997), no. 1, 193 - 213.

[Kum] Kumar, M. "Ramification theory and formal orbifolds in arbitrary dimension". Proceedings of the Indian Academy of Sciences - Mathematical Sciences, 129 (2019), no. 3, article 38.

[KMa] Kumar, M. and Majumder, S. "Parabolic bundles in positive characteristic". J. Ramanujan Mathematical Society, 33 (2018), no. 1, 1 - 36.

[KP] Kumar, M. and Parameswaran, A.J. "Formal orbifolds and orbifold bundles in positive characteristic". International Journal of Mathematics (2019).

[Lan] Lang, S. *Algebra*, 3rd ed. Springer-Verlag (2002).

[LMB] Laumon, G. and Moret-Bailly, L. *Champs algébriques*. Springer-Verlag (2000).

[LS] Lorenzini, D. and Schröer, S. "Moderately ramified actions in positive characteristic". Preprint (2019), arXiv:1904.08371.

[MS] Mehta, V.B. and Seshadri, C.S. "Moduli of vector bundles on curves with parabolic structures". Mathematische Annalen, 248 (1980), issue 3, 205 - 239.

[Mil1] Milne, J.S. *Étale Cohomology*. Princeton University Press (1980).

[Mil2] Milne, J.S. *Lectures on Étale Cohomology*, v2.21 (2013). Available at www.jmilne.org/math/.

[OP] Obus, A. and Pries, R. "Wild cyclic-by-tame-extensions". Journal of Pure and Applied Algebra, 214 (2010), no. 5, 565 - 573.

[Ols] Olsson, M. *Algebraic Spaces and Stacks*. AMS Colloquium Publications (2016).

[Ono] Ono, K. *The Web of Modularity: Arithmetic of the Coefficients of Modular Forms and* q*-Series*. CBMS Conference Series, vol. 102. American Mathematical Society (2004).

[Ryd] Rydh, D. "Tame and wild ramification via stacks". Oberwolfach Reports, 9 (2012), issue 3, 2342 - 2344. DOI: 10.4171/OWR/2012/38

[Ser1] Serre, J.P. "Congruences et formes modulaires". *Séminaire Bourbaki*, vol. 1971/72, exposés 400-417. Springer-Verlag (1973), 319 - 338.

[Ser2] Serre, J.P. *Local Fields*. Springer (1979).

[SP] The Stacks Project. https://stacks.math.columbia.edu/

[SGA IV] Verdier, J.L., Artin, M. and Grothendieck, A. *Theorie de topos et cohomologie étale des schemas*. *Lecture Notes in Mathematics*, vols. 269, 270 and 305. Springer-Verlag (1971).

[VZB] Voight, J. and Zureick-Brown, D. "The canonical ring of a stacky curve". To appear in *Memoirs of the AMS*. Preprint (2018), arXiv:1501.04657.

[Wit] Witt, E. "Zyklische Körper und Algebren der Charakteristik p vom Grad p^n. Struktur diskret bewerteter perfekter Körper mit vollkommenem Restklassenkörper der Characteristik p^n". Journal für die reine und angewandte Mathematik, 176 (1936), 126 - 140.