QCD Representations of Quark and Gluon Spin and Orbital Angular Momentum

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Abstract

This thesis presents the derivation of Lorentz Invariance and Equation of Motion Relations between theoretical objects that involve the partonic transeverse momentum called Generalized Transverse Momentum Distributions (GTMDs) and collinear functions known as Generalized Parton Distributions (GPDs) that describe quark and gluon orbital angular momentum. Although the GTMDs in principle define the observables for partonic orbital motion, experiments that can unambiguously detect them appear remote at present. The relations presented here provide a solution by showing how, for instance, the orbital angular momentum density is connected to directly measurable twist-three GPDs.

While experimental measurement is the only certain way to access the unknown functions that describe quarks inside a proton such as the Parton Distribution Functions (PDFs) and GPDs that characterize the nucleon, a great deal of effort has gone into evaluating these functions using theoretical techniques such as model calculations and parameterizations. Lattice QCD provides the only first principle calculations of the Mellin moments of PDFs and GPDs. The work prsented here shows how, using only a few moments, a large portion of the Fourier transform with respect to Bjorken x of the PDFs and GPDs can be mapped out. In the case of GPDs, lattice calculations provide a value for the Compton form factors that allow us to move away from the

$x = \xi$ ridge which, at present, is the only area of the phase space that has been explored experimentally. After studying how well known parameterizations of the PDFs and GPDs reproduce test functions, we demonstrate how PDFs and GPDs can be reconstructed using the moments calculated by lattice QCD.

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Chapter 1

Introduction

Numerous questions remain unanswered about the matter that surrounds us. The QCD Lagrangian is expressed using quark and gluon degrees of freedom that can not be observed directly in experiments. We are yet to understand how the partons come together to form nucleons and how these in turn combine to form nuclei. Among the many approaches to unravel this conundrum, one has been to understand the various pieces of the QCD energy momentum tensor. The diagonal elements of the tensor describe the momentum density and consequently also play a role in describing the angular momentum carried by the partons. A longstanding problem has been understanding the spin budget of the proton: how is the spin $\frac{1}{2}$ distributed between the spins and orbital angular momenta of the quarks and gluons? Another line of research has been to probe the partonic structure of nuclei. In line with the traditional picture of neutrons and protons combining to form nuclei, one can ask to what extent do protons and neutrons retain their identity in a nucleus? What, indeed, do we understand about partons in a nuclear medium? My research focuses on exploring these questions and on identifying the experimental observables that can shed light on them.

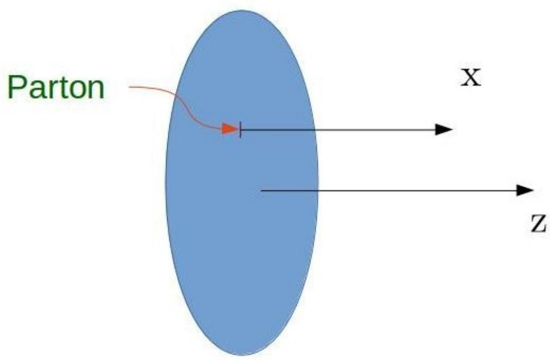


Figure 1.1: Collinear quark.

Scattering a beam of particles off a proton target can tell us about its internal structure. Since the proton is not a point like particle, its interaction with the beam can not be described from first principles in QCD/QED. As a result, by applying constraints such as Lorentz covariance and current conservation, the proton current is parameterized using unknown functions called form factors. By measuring the scattering cross section in the lab, we can extract what these functions are. In the earliest experiments that were conducted, the proton remained intact after the interaction and only received a momentum kick Δ. These kinds of experiments are referred to as elastic scattering. The form factors themselves are connected to the Fourier transforms of the spatial distribution of charge and current in the proton.

As the beam energy is increased, we are able to probe smaller and smaller distances inside the proton until at some point we start scattering off point like particles known as the quarks. The presence of free quarks is signaled by the fact that, above a certain threshold (≈ 1 GeV^2), the structure functions characterizing this process are found to be independent of the energy of the probe used. In fact, the structure functions are found to depend only on Bjorken x: the ratio of the virtual photon energy and the four vector squared of the photon momentum. This is known as scaling.

The nucleon lepton interaction via the exchange of a photon is understood as a two level procedure: one called the "hard part" in which the photon hits the quark and the other called the "soft part" which describes the probability distribution of quarks inside the proton carrying a certain helicity in a polarized or unpolarized proton as the case maybe. While the "hard part" is exactly calculable in QCD, the "soft part" can only be extracted from experiments. Switching from the lab frame to the proton infinite momentum frame, the "soft part" is parameterized using the Parton Distribution Functions that give the probability of a quark carrying a fraction x of the momentum carried by the proton. Since the quark is described traveling along a straight line, the PDFs portray a "collinear" picture of the proton. The experimental process used to extract the PDFs is called Deep Inelastic scattering. In this process, the proton disintegrates after being hit by the beam and is not detected.

The quarks had been postulated, prior to the experimental evidence pointing to their existence, as an explanation for the large number of particles that interact strongly. Collectively known as hadrons, the fermionic hadrons that undergo strong interaction are called baryons while the bosonic hadrons are known as mesons. Just as Mendeleev's table was the first step towards identifying a unique nuclear structure for every element, the proliferation of hadrons pointed to the existence of a substructure. The work of Gell-Mann and others paved the way for identification of quarks: the baryons are bound states of three quarks while the mesons consist of a quark and an antiquark. The introduction of a new quantum number called "color" and the requirement that only the totally antisymmetric wavefunction in color is permitted by nature restricted the allowed number of quark configurations to that which is ex-

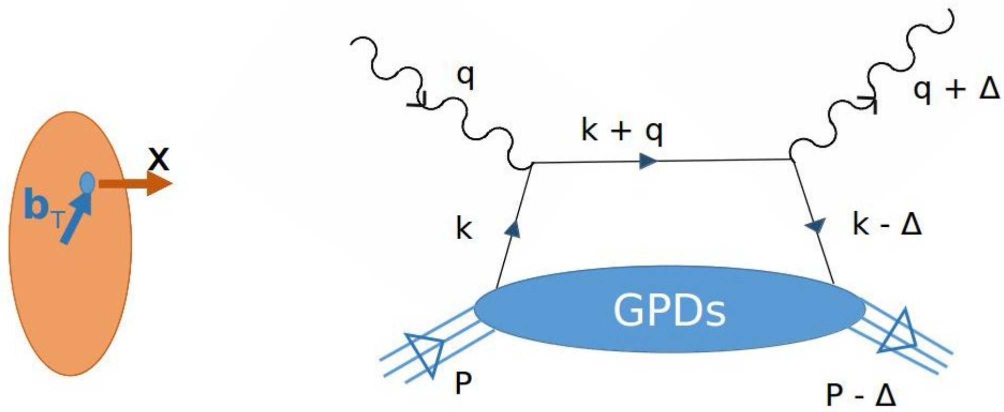


Figure 1.2: Generalized Parton Distributions

perimentally observed. The color charge in fact determines the strength of the strong interaction.

To obtain the spatial distribution of quarks inside a proton, we require a process in which the probe is able to hit the point like quark and also that the proton remains intact after the interaction. This is achieved in Deeply Virtual Compton Scattering. In this process, a high energy lepton beam interacts with the proton which then emits a photon and gets rid of the excess energy that would have caused it to disintegrate. It only retains a small momentum transfer Δ which is the Fourier conjugate to the average spatial position of the quark. The soft part in this process is described by Generalized Parton Distributions that are a function of both x and Δ. Deeply Virtual Compton Scattering (DVCS) is an example of an exclusive process because the proton in the final state is detected in the experiment. It is also “off-forward” because the proton carries a different momentum from the initial proton.

The seminal paper by X. Ji in 1997 provided a gauge invariant decomposition of the proton spin into quark and gluon spin and orbital angular momentum. The GPDs were used in this paper as a means of accessing the total quark angular momentum. Several experiments have been dedicated to collecting DVCS data such as the fixed target experiment HERMES, and the collider experiments H1 and ZEUS at the Deutsches Elektronen Synchrotron (DESY) and experiments from HALL A and CLAS collaboration at Jefferson Laboratory. Although large parts of the kinematic phase space still remain unexplored, these experiments have proven the feasibility of extraction of the GPDs. The first set of data gathered does show scaling and is an encouraging sign for using GPDs to describe the exclusive processes.

DVCS allows access to Compton Form Factors (CFFs) that are integrals in x of GPDs weighted by a Wilson coefficient function arising from the quark photon hard scattering that can be expanded order by order using perturbation theory. At variance with parton distribution functions that describe the deep-inelastic scattering process at the cross-section level, the GPDs enter the DVCS soft part at the amplitude level. The $ep \to ep\gamma$ reaction that probes DVCS also gets a contribution from the Bethe-Heitler (BH) process in which the final state photon is radiated by the incoming or scattered electron and not by the nucleon itself. As a result, the cross-section comprises a pure BH contribution, a pure DVCS contribution and a contribution that is proportional to the interfernce of the BH and DVCS amplitudes. As the BH amplitude is described by the form factors, F_1 and F_2, that have been experimentally measured to good precision, accessing the DVCS amplitude is a matter of subtracting out the contribution that comes from BH alone and disentangling it from the BH contribution in the interference term. Because we are probing the process $ep \to ep\gamma$ in a high Q^2 regime, where $Q^2 = -q^2$ and q^μ is a four vector denoting the difference in the four momentum of the final and initial electron, the virtual photon propogator in the DVCS amplitude causes it to be suppressed by a factor of $\frac{1}{\sqrt{Q^2}}$ in comparison to the BH amplitude. Hence, although it leads to a more complicated expression for the cross-section, the presence of the BH process enhances the interference term in comparison to the pure DVCS term. This is also good because the GPDs enter in a linear way in the interference term as opposed to the pure DVCS term where they occur as products of two GPDs.

Thus far, we have only considered quarks with longitudinal momentum. An additional degree of freedom is the partonic intrinsic transverse momentum k_T that allows a 3D momentum tomography of the nucleon. k_T is a unique in the sense that unlike x, it is not a scaling variable. The transverse momentum of the nucleon is zero however, the partons form a highly relativistic system inside the nucleon that can have non zero transverse momentum. In other words, by extracting the k_T structure of the nucleon, we are able to study the fundamental properties of bound states in QCD. To access the intrinsic k_T, at least one particle that is produced by the hadronization of the quark that is probed by the virtual photon needs to be detected: the momentum of the detected particle is correlated with the momentum of the quark and allows one to write the cross section using the quark transverse momentum degree of freedom. Transverse momentum dependent distributions have been measured in semi-inclusive processes such as semi-inclusive deep inelastic scattering ($ep \to hadron(P_\perp) + X$), Drell Yan ($A + B \to \mu^+\mu^-(Q_\perp) + X$) and back to back jet (dihadron) production $e^+ + e^- \to jet_1 + jet_2 + X$.

To access partonic OAM, we look at Generalized Transverse Momentum Distributions which are the off- forward transverse momentum dependent distributions. These de-

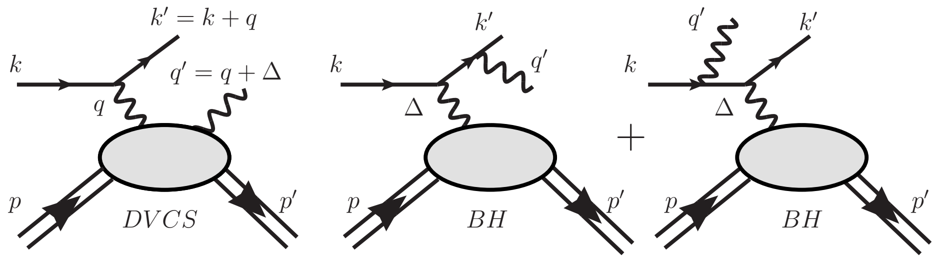
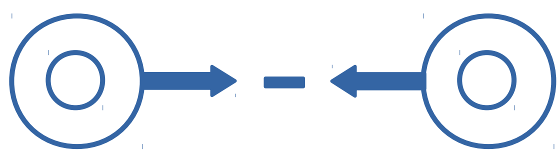


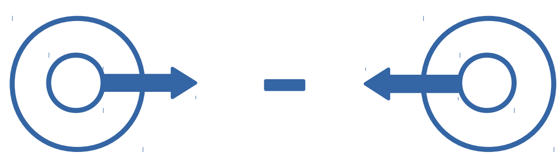
Figure 1.3: Exclusive electroproduction of a photon through the DVCS and BH processes

pend on x, k_T and Δ. To access GTMDs, not only is it required to detect a particle that results from the hadronization of the probed quark to detect k_T, but also that the nucleon remains intact after the interaction and is detected to measure the momentum transfer Δ. The experimental processes to detect GTMDs are still a topic of ongoing research. As $\mathbf{\Delta_T}$ is conjugate to the average transverse spatial position $\mathbf{b_T}$ of the quarks, GTMDs give us a way to describe longitudinal Orbital Angular Momentum $L_z = (\mathbf{b_T} \times \mathbf{k_T})_z$. The GTMD F_{14}, in the forward limit, gives the density of unpolarized quarks in a longitudinally polarized proton. As the quarks are unpolarized, the only source of quark angular momentum is quark orbital angular momentum. The average quark orbital angular momentum distribution in x is found by taking the product of $(\mathbf{b_T} \times \mathbf{k_T})_z$ and the unpolarized quark densityin a longitudinally polarized proton and integrating over k_T. In a similar fashion, the GTMD G_{11}, in the forward limit, gives the density of longitudinally polarized quarks in an unpolarized proton. This GTMD gives access to the spin orbit coupling $(\mathbf{L} \cdot \mathbf{S})_z$ of quarks.

It is interesting to note that the GTMDs F_{14} and G_{11} do not correspond to any



(a) F_{14} describes longitudinally polarized quarks in an unpolarized proton.



(b) G_{11} describes unpolarized quarks in a longitudinally polarized proton.

Figure 1.4: In the forward limit, F_{14} and G_{11} represent quark densities in a nucleon and can be interpreted by graphs similar to the ones used for TMDs in [4]. The outer circle represents the proton while the inner circle represents the quarks. The arrow shows the direction of polarization, a circle with no arrow means that the entity is unpolarized.

transverse momentum distribution in the forward limit nor any generalized parton distribution when integrated over k_T. In other words, although these functions are non zero in the forward limit and although they are non zero when integrated over k_T, they do not have a TMD or GPD counterpart: they carry completely new information that is not present in any of the predefined functions. Furthermore, these GTMDs occur with an explicit k_T coefficient in the parameterization that gives them the unique property that the k_T^2 weighted integrals of these GTMDs are connected to higher twist GPDs. F_{14} and G_{11} also occur with an explicit Δ_T which means that in the forward limit, in which these functions hold physical meaning for partonic OAM, even if the functions themselves are non zero, the quark-proton helicity amplitude structure describing them is zero.

I worked on a few main topics during my PhD. The first was deriving relations between matrix elements of light-ray operators in QCD for off-forward scattering processes. The next topic was studying the spatial distributions of partons in nucleons and nuclei obtained by taking the Fourier transform with respect to the momentum transfer to the final proton state. Lastly, I have been working on reconstructing Parton Distribution Functions taking input from lattice QCD calculations. Some of my work is published in Refs.[5], [6] and [7].

Quark-gluon interactions get significantly suppressed as the hard processes used to probe them enter the multi GeV region. These interactions involve the intrinsic transverse momentum k_T of quarks and gluons. My thesis work showed how the k_T^2 moments of the k_T distributions describing the quark and gluon longitudinal orbital angular momentum and the spin orbit correlation are related to higher twist collinear matrix elements of operators by Lorentz Invariance Relations (LIRs). Obtaining the

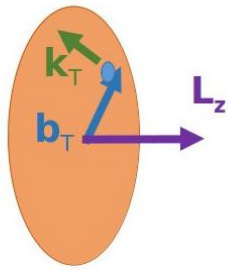


Figure 1.5: Generalized Transverse Momentum Distributions and Orbital Angular Momentum

LIRs involved looking at the substructure of the collinear and transverse momentum distributions in terms of the completely un-integrated QCD quark-quark correlator and deriving a parameterization for it. Using the LIRs along with the QCD equation of motion, we separated the Wandzura-Wilczek or the leading contribution from the genuine twist three contribution to the higher twist object. This allows us to define a strategy to directly measure the orbital angular momentum and spin orbit correlation in an experiment.

The recent prediction using the formalism of Generalized Parton Distributions (GPDs) that the neutrons core is negatively charged has completely changed our understanding of the internal structure of the neutron. GPDs are defined in the infinite momentum frame in which the transverse plane remains unaffected under boosts and allows the unambiguous interpretation of the Fourier transform as the spatial charge distribution. My work focuses on applying this principle to obtain 3D images of nuclei such as ^4He. We find that the spatial charge distributions for both u and d quarks at different Bjorken x are indeed altered as compared to those in a free nucleon.

Lattice QCD provides one of the few first principle calculations of the moments of

Parton Distribution Functions (PDFs). Unlike the PDFs and GPDs that are defined on the light-cone, the lattice is setup in Euclidean space which prevents the calculation of the distributions themselves. A recent proposition by X.Ji and A. Radyushkin is to calculate quasi-PDFs and pseudo-PDFs that are defined off the light-cone. Pseudo-PDFs converge to PDFs in the limit that z^2 goes to zero, where z^- is the Fourier conjugate to the intrinsic quark momentum and is also referred to as Ioffe time. We conducted a study using a di-quark model parametrization to calculate pseudo PDFs and studied how far we deviate from the expected PDF as we take it more off the light-cone. We also explored a way to directly use the lattice QCD moments to map out the Fourier transforms of PDFs and GPDs in Ioffe time space. The small z behavior of the Fourier transform is controlled by the moments and is written as a Taylor expansion. The large z component on the other hand is driven by Regge behavior. Once we had the z space distributions in place, we reconstructed the PDFs by performing the inverse Fourier transform. In the case of GPDs, we used the lattice calculations of the Compton form factors to generate the $x\xi$ dependence. As we increased the number of moments, we saw a marked improvement in how well the method reproduced a known function. This means that the calculation of higher moments on the lattice would really be beneficial to this method.

Chapter 2

The Definition of Twist

In this chapter we look at the different physical interpretations of higher twist and how they highlight different facets of QCD. The twist expansion gives a framework that singles out the leading contributions of the unobservable bilocal quark current operator to various deep inelastic scattering process cross-sections. We begin with the field theoretic formal definitions and move on to how observables from entirely different processes are related to one another by the QCD equation of motion which allows for the different interpretations of the concept of twist.

2.1 Deep Inelastic Scattering and the Light Cone

The kinematics in which deep inelastic scattering occurs is dominated by quarks on the light cone. We see this by studying the soft part described by the hadronic tensor in deep inelastic scattering

$$W_{\mu\nu} = \frac{1}{2\pi} \int d^4 z e^{iq\cdot z} \langle P| \left[j_\mu(z) j_\nu(0) \right] |P\rangle \tag{2.1}$$

where, q is the momentum of the photon, P the momentum of the proton and $j_\mu(z)$ describes the quark current at spatial coordinate z. This integral is dominated by regions where $|q \cdot z|$ is finite (for $|q \cdot z| \to \infty$ the exponent oscillates and the integral goes to zero).

$$q \cdot z = q^0 \left(z^0 - \sqrt{1 - \frac{q^2}{(q^0)^2}} \frac{\mathbf{q.z}}{|\mathbf{q}|} \right) \tag{2.2}$$

In the limit of $-q^2 \to \infty$ with $x = \frac{-q^2}{2P \cdot q}$, which is what we need for deep inelastic scattering, we can write a Taylor expansion for the square root. It is useful to work with the variables $r = \frac{\mathbf{q.z}}{|\mathbf{q}|}$ and $\nu = \frac{P \cdot q}{M}$. ν goes to infinity in the deep inelastic limit. In the lab frame, $\nu = q^0$ and we have,

$$q \cdot z = \nu(z^0 - r) - Mxr + O(\frac{1}{\nu}) \tag{2.3}$$

For $q \cdot z$ to be finite (a is a finite constant in the following),

$$|z_0 - r|\nu < a \qquad , \qquad |r|x < a \tag{2.4}$$

$$\Rightarrow z_0^2 < \left(r + \frac{a}{\nu} \right)^2 \simeq r^2 + a\frac{r}{\nu} < \mathbf{z^2} + \frac{a}{x\nu} \tag{2.5}$$

$$\Rightarrow z^2 \leq \frac{a}{-q^2} \tag{2.6}$$

Because of causality, the current commutator is zero for $z^2 < 0$. As $-q^2 \to \infty$, we have that $z^2 \sim 0$ [8]. Thus, quarks that are on the light cone are the main contributors in deep inelastic scattering.

2.2 Operator Product Expansion

It is well known that the product of two local operators $\hat{A}(z)$ and $B(0)$ is divergent at short distances ($z \to 0$). The operator product expansion allows one to expand this product as a series of non singular local operators multiplying c-number singular functions[9; 10].

$$\hat{A}(z)\hat{B}(0) \sim \sum_{[\alpha]} C_{[\alpha]}(z)\hat{\theta}_{[\alpha]}(z) \tag{2.7}$$

All the singular behavior in the product $\hat{A}\hat{B}$ is concentrated in the functions $C_{[\alpha]}$ known as the "Wilson coefficients". By dimensional analysis one has,

$$\left[\hat{A}\right] + \left[\hat{B}\right] = \left[C_{[\alpha]}\right] + \left[\hat{\theta}_{[\alpha]}\right]. \tag{2.8}$$

This means that,

$$C_{[\alpha]}(z) \sim \frac{1}{z^{[\hat{A}]+[\hat{B}]-[\hat{\theta}_{[\alpha]}]}}. \tag{2.9}$$

The leading contribution comes from the term that is most singular and looking at the form above, this is the term with the smallest $[\hat{\theta}_{[\alpha]}]$. Since it is the $z^2 \sim 0$ limit that applies to DIS, we can write the current commutator in terms of decreasing singularity around $z^2 = 0$ [11],

$$[J(z), J(0)] \sim \sum_{[\theta]} K_{[\theta]}(z^2) z^{\mu_1} \ldots z^{\mu_{n_\theta}} \theta_{\mu_1 \ldots \mu_{n_\theta}}(0). \tag{2.10}$$

where $K_{[\theta]}(z^2)$ are singular c-number functions that can be ordered according to their degree of singularity at $z^2 = 0$. The smaller the number $t_\theta = d_\theta - n_\theta$, the more singular the term is. This number t_θ which decides at what order the whole term contributes is known as the twist of the operator.

The product of currents in (2.10) can be simplified to bilocal operators of the form $\overline{\psi}(z)\Gamma\psi(0)$ involving the quark field ψ and a Dirac structure Γ such as γ^μ, $i\sigma^{\mu\nu}$, $\gamma^\mu\gamma^5$ etc.. This form, also known as the quark-quark correlation function, is the one that is more commonly encountered and which, when sandwiched between hadron states, is parameterized using parton distribution functions.

When considering the matrix elements of operators between hadron states with spin (such as polarized targets), the twist is dependent on the Sudakov decomposition of the spin vector and the component that is entering the description of the process. This is because the mass dimension occurs in a different way for different components of the spin vector, changing the twist.

2.3 Good and Bad Components

By using the light-cone quantized form of the QCD equation of motion, one is able to separate the dynamically independent and the dependent parts of the quark field. The so called good components are the independent propagating degrees of freedom while the bad components are constrained by the Dirac equation in terms of the good components and the transverse gluon field. In fact the bad component is treated as a quark-gluon composite. The operators that project out the good and bad components play a central role in understanding the leading order contributions in deep inelastic processes. Each bad component that enters leads to an increase in the twist [9].

In general, the quark field can be written as a superposition of good and bad components having left and right chirality,

$$\psi = \phi_L \hat{\phi}_L + \phi_R \hat{\phi}_R + \chi_L \hat{\chi}_L + \chi_R \hat{\chi}_R \tag{2.11}$$

where, the $\hat{\phi}$ and $\hat{\chi}$ are the basis vectors denoting good and bad components respectively and the subscripts L and R denote the chirality. Hence, when forming a product of the form $\bar{\psi}\Gamma\psi$, we get products of the form $\phi^{\dagger}_{R/L}\phi_{R/L}$, $\phi^{\dagger}_{R/L}\chi_{R/L}$, $\chi^{\dagger}_{R/L}\chi_{R/L}$ and their Hermitian conjugates. The first corresponds to twist 2, the second to twist 3 and the third to twist 4.

2.4 Diffrent contributions to Twist 3 observables

2.4.1 Intrinsic Twist 3

The intrinsic twist of a function is decided by the projection operator Γ that enters the quark-quark correlation function it parameterizes. This is directly related to the occurrence of bad components as discussed above. These functions are collinear i.e. only the longitudinal momentum fraction carried by the quarks enters the description enters the description [9; 12; 13; 14; 15].

2.4.2 Kinematical Twist 3

These functions involve the intrinsic transverse momentum k_T of the quark. Although one is looking at a quark-quark correlator with a Γ that, in the collinear case, only projects out only the good components, the added degree of freedom k_T introduces functions that enter the parameterization of the correlation function weighted by an explicit k_T. Upon integration over k_T, we obtain the collinear correlator in which, despite the fact that the Γ that enters the correlator connects to intrinsic twist 2, the function weighted by k_T enters the description of a collinear intrinsic twist 3 function. These functions weighted by an explicit k_T, that seemingly enter at leading twist but upon integration connect to an intrinsic twist 3 function, are referred to as kinematical twist 3. An example is the famous Sivers function $f_{1T}^{\perp}$ [16; 17] or the generalized transverse momentum distribution F_{14} that is pertinent to partonic orbital angular momentum [12; 18; 19; 14; 20; 21].

2.4.3 Dynamical/ Genuine Twist 3

These functions parameterize a correlator that involves an explicit gluon field. Since we are now looking at a three particle correlator (quark-gluon-quark), we now have a dependence on two longitudinal momentum fractions. One obtains intrinsic twist 3 functions on integrating over the gluon momentum fraction [22; 18; 21]. Using the QCD equation of motion, one can show how these connect to twist 2, kinematical twist 3 and intrinsic twist 3 as will be shown in chapter 4 [23; 18; 6].

To continue working in the setup of the free parton model, the only way to account for certain configurations, such as transverse spin in inclusive processes to describe the parton distribution function $g_T(x)$, one needs to involve the gluon field. In the language of field theory, this amounts to invoking higher twist.

Chapter 3

Spin, Helicity and Orbital Angular Momentum

Spin is a quantum mechanical property. Theoretically proposed in an attempt to explain the splitting of atomic spectral lines and anomalies in the Zeeman effect, the first suggestions of intrinsic electronic spin were ridiculed considering how fast a point particle would need to rotate to generate the required angular momentum. The calculations, however, agreed with the data beautifully once relativity was taken into account correctly and, as a result, by the late 1920s, the electron had a new property associated with itself called spin. Quantized angular momentum was not entirely a new phenomenon considering the theory of electron orbits and space quantization put forward by Neils Bohr, which despite being incorrect, was able to predict correct results for the Stern-Gerlach experiments within the range of experimental error. Hence, in some sense, space quantization was reincarnated and included in the framework in a way that was comprehensive and most importantly correct [24].

In particle physics, we often work in the infinite momentum frame in which asymptotic freedom holds and the principles of QCD apply in a direct way. In this case, a clear boost axes can be identified and the particle can be described by spinors pointing along or opposite to the direction of the boost which form the eigenstates of the helicity operator. The helicity of a massive particle depends on the frame of reference while for a massless particle such a boost that changes the helicity can not be performed as it travels at the speed of light. Both the helicity and chirality operators can be diagonalized simultaneously and depending on the component of the quark field spinor that one is looking at, the helicity and chirality of the quark are either equal or opposite to one another.

Quark orbital angular momentum has been difficult to pin down because one can not escape the fact that it is intertwined with quark gluon interactions. Historically, there has been immense debate on the formulation of OAM in a way that is consistent with QCD. There are two main descriptions, one involves the intrinsic quark transverse momentum, the other is connected to a higher twist function. While neither one of the formulations explicitly involves a gluon field, both, in effect, deviate from the leading twist parton model picture of free quarks because they involve gluon interactions, albeit in an implicit way.

Chapter 4

Lorentz Invariance Relations

A fundamental way of characterizing the internal structure of the proton is through sum rules that express how global properties of the proton are composed from corresponding quark and gluon quantities. For example, one may ask what portion of a proton's momentum is carried by either quarks or gluons; or one may ask how the spin of the proton is composed from the spins and orbital angular momenta of its quark and gluon constituents. Elucidating this latter question, the so-called proton spin puzzle [25], indeed counts among the prime endeavors of hadronic physics in the last decades. These questions can be cast in field-theoretic language by considering proton matrix elements of the energy momentum tensor, $T_{q,g}$ (q and g denote the quark and gluon sectors),

$$\int dx \, \langle p' \mid T^{0i}_{q,g} \mid p \rangle = A_{q,g} \, P^i \, \overline{U}(p') \gamma^o U(p) \tag{4.1}$$

$$\begin{aligned} \int dx \, \epsilon^{ijk} \langle p' \mid \left(x^j T^{0k} - x^k T^{0j} \right) \mid p \rangle &= A_{q,g} \, P^i \, \overline{U}(p') \gamma^o U(p) \\ &+ B_{q,g} \, P^i \, \overline{U}(p') \frac{\sigma^{o\alpha} \Delta_\alpha}{2M^2} U(p) \ , \end{aligned} \tag{4.2}$$

where p and p' describe the incoming and outgoing proton states, and $A_{q,g}(t), B_{q,g}(t)$, $(t = \Delta^2 = (p'-p)^2, P = (p'+p)/2)$ are the relevant gravitomagnetic form factors parameterizing the proton matrix elements (Refs.[26; 27], for reviews see Ref.[28; 29]). These basic constructs of the theory can be accessed experimentally owing to the connection, through the operator product expansion (OPE), of the gravitomagnetic form factors to the Mellin moments of specific parton distributions parameterizing both the forward ($p = p'$) and off-forward ($p \neq p'$) quark and gluon correlation functions. One obtains the following sum rules for momentum and angular momentum, respectively,

$$A_{q,g} = \int_0^1 dx x H_{q,g} \quad \Rightarrow \sum_{i=q,g} A_i = \epsilon_q + \epsilon_g = 1 \tag{4.3}$$

$$B_{q,g} = \int_0^1 dx x (H_{q,g} + E_{q,g}) \Rightarrow \sum_{i=q,g} (A_i + B_i) = J_q + J_g = \frac{1}{2}. \tag{4.4}$$

Eq.(4.4), the angular momentum sum rule, is also known as the Ji sum rule [27]. All of the distributions entering Eqs.(4.3) and (4.4) are observable in a wide class of experiments probing the deep inelastic structure of the proton.

$H_{q,g}(x, \xi, t)$ and $E_{q,g}(x, \xi, t)$ are the Generalized Parton Distribution (GPD) functions which depend on the longitudinal momentum transfer between the initial and final proton, represented through the skewness parameter ξ, and the four-momentum transfer squared, t, x being the light cone momentum fraction carried by the parton [30; 31]. In particular, $H_q(x, 0, 0) \equiv q(x), H_g(x, 0, 0) \equiv g(x)$, where $q(x)$ and $g(x)$ are the unpolarized quark (antiquark) and gluon distributions, or the Parton Distributions Functions (PDFs).

PDFs have been measured in decades of Deep Inelastic Scattering (DIS) experiments, with impressive accuracy and kinematical coverage, confirming to high precision the momentum sum rule, Eq.(4.3). To verify the angular momentum sum rule it is necessary to extract the GPDs from experiment, in particular, $E_{q,g}$. Sufficiently accurate values for the GPDs have just fairly recently started to become available from exclusive deeply virtual scattering experiments, namely Deeply Virtual Compton Scattering (DVCS), Deeply Virtual Meson Production (DVMP) and related processes, conducted most recently at Jefferson Lab and COMPASS (see [32] for a recent review).

DVCS experimental measurements are necessarily more involved than ones for inclusive scattering, since they require the simultaneous detection of all products of reaction. The extraction of observables, the GPDs, from experiment is also more complex owing to the increased number of kinematical variables on which they depend. An additional hurdle is present for the analysis of angular momentum in both identifying and giving a physical interpretation to the components of the sum rule (4.4): while the momentum sum rule has an immediate dynamical interpretation in terms of the average longitudinal momentum carried by the different parton components, to obtain a dynamically transparent expression for the angular momentum sum rule one has to break it down into its spin and Orbital Angular Momentum (OAM) components, while simultaneously preserving the gauge invariance of the theory. The decomposition can be performed within two different approaches, by Jaffe and Manohar (JM) [25],

$$\frac{1}{2}\Delta\Sigma_q + L_q^{JM} + \Delta G + L_g^{JM} = \frac{1}{2} \tag{4.5}$$

and by Ji [27],

$$\frac{1}{2}\Delta\Sigma_q + L_q^{Ji} + J_g^{Ji} = \frac{1}{2} \quad . \tag{4.6}$$

Longitudinal OAM distributions have been identified with parton Wigner distributions weighted by the cross product of position and momentum in the transverse plane, $b_T \times k_T$ [33; 34]. Parton Wigner distributions can be related, through Fourier transformation, to specific Generalized Transverse Momentum-Dependent Parton Distributions (GTMDs), which are off-forward TMDs. The correlation defining OAM corresponds to the GTMD F_{14} (we follow the naming scheme of Ref.[17]). In particular, the OAM distribution is described by the x-dependent k_T^2 moment of F_{14}.

The OAM term differs in the JM and Ji approaches with regard to how the gauge invariance of the theory intervenes through the gauge link in the relevant parton correlator [35]. The difference was recently explicated in the quark sector in Refs.[36; 37], where it was shown that JM OAM, L_q^{JM}, can be written as the sum of Ji's OAM, L_q^{Ji}, plus a matrix element including the gluon field. The latter was interpreted in the semi-classical picture of Ref.[37] as having the physical meaning of an integrated torque stemming from the chromodynamic force between the struck quark and the proton remnant interacting in the final state.

To summarize, in both Ji's and JM's expressions, OAM is defined through an imbalance in the distribution of the number density of quarks in longitudinally polarized proton states, when the quark's displacement in the transverse plane is simultaneously orthogonal to its intrinsic transverse motion. JM's definition includes a quark re-interaction which could be, in principle, process-dependent. How can these two pictures of the proton's angular momentum coexist, and what are experimental measurements really probing?

The work presented here [6] was motivated by the question of defining a way to test these ideas through observables that would enable direct access to OAM in experimental measurements. While $J_{q,g}$ measurements through GPDs are in progress, GTMDs, providing in principle the density distributions for OAM, remain experimentally elusive objects, since they require exclusive measurements of particles in the two distinct hadronic planes disentangling the k_T and b_T (or Δ_T) directions [38; 39; 40]. GTMDs can, however, be evaluated in ab initio calculations [41].

In a previous publication [7], we showed that the x-dependent k_T^2 moment of F_{14} entering Eq. (4.6) can be written in terms of a twist-three GPD, $\widetilde{E}_{2T}$ [17], as

$$\int d^2k_T \, \frac{k_T^2}{M^2} \, F_{14} = - \int_x^1 dy \, \left(\widetilde{E}_{2T} + H + E \right) \tag{4.7}$$

Here, we present several extensions of this relation, and describe the details of the derivation comprehensively. In particular, we show that a more general relation holds,

$$\int d^2k_T \, \frac{k_T^2}{M^2} \, F_{14} = - \int_x^1 dy \, \left(\widetilde{E}_{2T} + H + E + \mathcal{A}_{F_{14}} \right) \tag{4.8}$$

where $\mathcal{A}_{F_{14}}(x)$ is a term containing the gauge link dependent, or quark-gluon-quark, components of the correlation function. For a straight gauge link, $\mathcal{A}_{F_{14}}(x) = 0$, thus recovering the result displayed in Eq. (4.7). These relations are specific generalized Lorentz Invariance Relations (LIR) connecting the x-dependent k_T^2 moments of

GTMDs and GPDs. Just as in the forward case [42; 13; 20], generalized LIR are based upon the covariant decomposition of the fully unintegrated quark-quark correlation function in off-forward kinematics: the number of independent functions parameterizing the correlator is less than the total number of GTMDs and GPDs, thus inducing relations among the latter. Several LIRs have been found between forward twist-three PDFs and k_T moments of TMDs. The most remarkable example of an LIR is perhaps the relation between the TMD g_{1T} and the twist-three PDF g_T, leading to the Wandzura-Wilczek relation between the helicity distribution g_1 and $g_T = g_1 + g_2$ [15]. In the presence of a gauge link other than the straight one (*e.g.* a staple link), LIRs acquire an additional term that cannot be encoded in the available GTMD and GPD structures. As we show in the present paper, this term produces a correction to Eq. (4.7), leading eventually to the Qiu-Sterman type term of Ref. [37]. Furthermore, by combining Eqs. (4.7,4.8) with the quark field Equations of Motion (EoM), we can ascribe the difference between the integrated quark total angular momentum, J_q, and the spin, $S_q \equiv (1/2)\Delta\Sigma_q$, in Ji's description to the integral of the Wandzura-Wilczek component of the GPD combination $\widetilde{E}_{2T} + H + E$. We find that, at the unintegrated level, a quark-gluon-quark term is also present which integrates to zero consistently with Ji's sum rule. Our relation, therefore, allows one to connect the partonic sum rule originating from the dynamical definition of OAM – through the unintegrated correlation function – and the gravitomagnetic form factors which define the energy-momentum tensor (Eq. (4.4)). On the other hand, having access to relations at the unintegrated level allows us to extend the treatment to the JM case, where we obtain that the quark-gluon-quark contribution does not vanish upon integration. We show it to reproduce the Qiu-Sterman type term in [37].

In principle, 32 individual EoM relations can be constructed, associated with the 8 twist-two GTMDs in the vector and axial-vector sectors, which each feature independent real and imaginary components; an additional doubling of the number of relations is given by contracting the EoMs in the transverse plane either with the transverse momentum k_T or with the transverse momentum transfer Δ_T. However, we place a special focus in the present paper on just three further relations besides Eq. (4.7) [7] that describe spin correlations stemming from a similar operator structure as for OAM,

$$\int d^2k_T \frac{k_T^2}{M^2} G_{11} = \int_x^1 dy \left(2\widetilde{H}'_{2T} + E'_{2T} + \widetilde{H} - \mathcal{A}_{G_{11}} \right) \tag{4.9}$$

$$\frac{1}{2}\int d^2k_T \frac{k_T^2}{M^2} G_{12} = -\int_x^1 dy \left(H'_{2T} - \frac{P^2}{4M^2}\widetilde{H} + \mathcal{A}_{G_{12}} \right) \tag{4.10}$$

$$\int d^2k_T \frac{k_T^2}{M^2} F_{12}^o \equiv -f_{1T}^{\perp(1)} = -\,\mathcal{M}_{F_{12}}|_{\Delta_T=0} \tag{4.11}$$

The three additional relations presented here for the first time involve the k_T^2 moments of the following GTMDs: G_{11}, which was observed to provide information on the longitudinal part of the quark spin-orbit interaction, or the projection of quark OAM along the quark spin [33]; G_{12}, which corresponds to a transverse proton spin configuration and generalizes the TMD g_{1T} leading to the original Wandzura-Wilczek relation [15; 43], and, finally, the naive T-odd part of F_{12} which corresponds to the off-forward generalization of the Sivers function, $f_{1T}^{\perp}$ [16], which we relate to a generalized Qiu-Sterman term represented by $\mathcal{M}_{F_{12}}$ in Eq. (4.11). For G_{11}, in particular, by using the EoM we find a relation whose integral in x is consistent with the sum rule found in [44] and revisited in [45]. However, our derivation, valid for arbitrary gauge link structure, allows for a new term representing final state interactions. Furthermore, we stress the importance of the term proportional to the quark mass which appears in this relation as being generated from quark transverse spin contributions.

This chapter is organized as follows. In Section 5.1 we define the general framework: the correlation functions, the gauge link structure, the parameterization of the correlation functions which ensues, and the helicity amplitudes; in Section 4.2 we give a detailed derivation of the EoM relations, including explicit quark-gluon-quark terms; in Section 4.3 we derive the LIRs for both OAM and spin-orbit correlations. We discuss their Mellin moments to order $n = 3$; in Section 4.4 we discuss the relations for transverse proton spin configurations and their connection to the forward limit.

4.1 Formal framework and definitions

We base our treatment on the complete parameterization of the quark-quark correlation functions in the proton up to twist four given in Ref.[17]. By applying time reversal invariance, charge conjugation, parity and hermiticity one finds that, at twist two, there are three independent PDFs: f_1, g_1, in the chiral even sector, and the chiral odd h_1; eight GPDs (four chiral even and four chiral odd); eight TMDs, and sixteen GTMDs. At twist three, one has many more functions due to both the presence of additional couplings (scalar, and pseudoscalar), and to the larger number of kinematical terms in the correlation function parameterizations for the vector, axial vector and pseudoscalar couplings. Each one of the PDFs, TMDs, and GPDs corresponds to specific quark-proton helicity amplitude combinations that can be extracted from various hard inclusive, semi-inclusive and deeply virtual exclusive processes, respectively, and that represent specific polarization configurations, or spin correlations, of partons inside the proton.

It is important to distinguish between different types of twist-three objects that will be dealt with in this paper. *Canonical* twist three effects, describing quark-gluon correlations in the nucleon, appear in the OPE as coefficients of the inverse power terms in a large characteristic scale of the process, *e.g.* $\mathcal{O}(M/Q)$, M being a nonperturbative mass scale. A different class of twist three effects, *geometrical* twist three, arises from the quark field components which are not dynamically independent solutions of the equations of motion, and that can be expressed, through the equations of motion, as composites of the quark and gluon fields. These are also suppressed by inverse powers of Q (the classification of parton distributions given above concerns this type of twist three objects). The order of canonical and geometrical twist does not match beyond order two: contributions with the same power in M/Q, or same dynamical twist, can be written in terms of matrix elements of operators with different canonical twist. The Wandzura-Wilczek (WW) [15] relations between matrix elements of operators of different dynamical and same canonical twist encode this mismatch, as first exemplified for the polarized distribution functions g_1 and g_2.

A complete set of relations between twist two TMDs and twist three PDFs was presented and discussed for various correlation functions in Refs.[42; 20]. These relations are based upon the Lorentz invariant decomposition of the fully unintegrated correlation function with the two quark fields located at different space-time positions, and they necessarily involve parton transverse momentum and off-shellness both through the k_T-moments of twist-two TMDs (where k_T denotes the quark transverse momentum), and the twist-three PDFs. The different kinds of twist three functions were renamed: *intrinsic* for geometric, *dynamic* for canonical, *i.e.*, when an extra gluon field operator is directly involved in the definition, and *kinematic* which are related to k_T-moments of TMDs.

These distinctions are useful to keep in mind as we extend both the Lorentz Invariance Relations (LIRs) and the Equation of Motion relations (EoMs) to off-forward kinematics involving GTMDs and their kinematic twist-three constructs, intrinsic twist-three GPDs, and off-forward dynamical twist-three terms.

Already the construction of the aforementioned relations between TMDs and PDFs, once taken beyond a purely formal level, encounters obstacles rooted in divergences of the k_T-integrations connecting TMDs to collinear objects such as the PDFs. These divergences must be separated off to ultimately contribute to the scale evolution of the collinear quantities. Our treatment similarly relates GTMDs to GPDs through k_T-integrations, and thus inherits these issues in complete analogy. In the present paper, we do not present any further developments on this topic beyond what is given in the literature on the connection between ordinary TMDs and PDFs. In general, the precise connection of GTMDs to GPDs still requires further specification. The relations we derive can also be read purely at the GTMD level, before identifying k_T-integrals of GTMDs with GPDs. In that form, all components of our relations can be regularized on an identical footing, before identifying their collinear limits. To the extent that our relations derive from symmetries (such as Lorentz invariance), any regularization that respects these symmetries can be expected to leave the relations we derive intact. At appropriate places in our treatment, we will indicate points at which modifications of our results must be countenanced owing to issues of regularization; an example is the standard deformation of TMD gauge links off the light cone, associated with the introduction of a Collins-Soper evolution parameter. This procedure applies likewise to a proper definition of GTMDs. We will also refrain from writing explicitly the soft factors that are required [46] to regulate divergences associated with the gauge connections contained in the bilocal operators defining TMDs and GTMDs.

4.1.1 Kinematics and correlators

The completely unintegrated off forward quark-quark correlation function is defined as the matrix element between proton states with momenta and helicities p, Λ and p', Λ',

$$W^{\Gamma}_{\Lambda'\Lambda}(P,k,\Delta;\mathcal{U}) = \frac{1}{2}\int \frac{d^4z}{(2\pi)^4} e^{ik\cdot z} \langle p',\Lambda' \mid \bar{\psi}\left(-\frac{z}{2}\right)\Gamma\mathcal{U}\psi\left(\frac{z}{2}\right) \mid p,\Lambda\rangle \,, \quad (4.12)$$

where the gauge link structure $\mathcal{U}$ connecting the quark operators at positions $-z/2$ and $z/2$ is discussed in detail in the next section, Γ is a Dirac structure, $\Gamma = \mathbf{1}, \gamma^5, \gamma^\mu, \gamma^\mu\gamma^5, i\sigma_{\mu\nu}$, and the choice of four-momenta is defined with $P = (p+p')/2$ along the z-axis, $\Delta = p' - p$ as in Ref.[17],

$$P \equiv \left(P^+, \frac{\Delta_T^2 + 4M^2}{8(1-\xi^2)P^+}, 0\right) \stackrel{\xi=0}{\equiv} \left(P^+, \frac{\Delta_T^2+4M^2}{8P^+}, 0\right) \quad (4.13)$$

$$\Delta \equiv \left(-2\xi P^+, \frac{\xi(\Delta_T^2+4M^2)}{4(1-\xi^2)P^+}, \Delta_T\right) \stackrel{\xi=0}{\equiv} (0,0,\Delta_T) \quad (4.14)$$

$$k \equiv \left(xP^+, k^-, k_T\right) \quad (4.15)$$

where the initial and final quark momenta are $k-\Delta/2$ and $k+\Delta/2$, respectively. Four-vectors w^μ are represented in terms of light-cone components, $w^\mu \equiv (w^+, w^-, w_T)$; $\xi = -\Delta^+/2P^+$ is the skewness parameter, $\Delta_T \equiv (\Delta^1, \Delta^2)$, $k_T \equiv (k^1, k^2)$, and the four-momentum transfer squared is $\Delta^2 \equiv t$; we displayed the kinematics also specifically for the $\xi = 0$ case, which is the case on which we will focus in this study.

4.1.2 Gauge link structures

To ensure gauge invariance, the quark bilocal operator (4.12) requires a gauge link $\mathcal{U}$ along a path connecting the quark operator positions $-z/2$ and $z/2$. Two important choices of path are a direct straight line and a staple-shaped connection characterized by an additional vector v, cf. Fig. 4.1. These different choices will give rise to different genuine twist three contributions to the correlators.

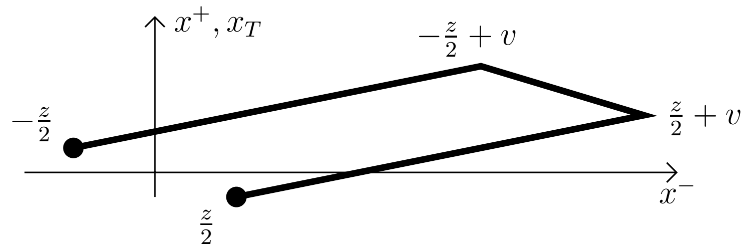


Figure 4.1: Staple-shaped gauge link path connecting quark operators located at $-z/2$ and $z/2$. The legs of the staple are described by the four-vector v. GTMDs are defined at separation $z^+ = 0$; the vector z thus deviates from the x^- axis by a transverse component z_T, i.e., $z = (0, z^-, z_T)$. On the other hand, v in general is taken to deviate from the x^- axis by a plus component v^+ in order to regulate rapidity divergences occurring if v is taken to point purely in the minus direction; i.e., $v = (v^+, v^-, 0)$. Note that, in the two-dimensional projection displayed, the x^+ and the x_T axes fall on top of one another; they are nevertheless of course distinct axes. The separation z is Fourier conjugate to the quark momentum k. Integrating over transverse momentum k_T sets $z_T = 0$, i.e., the quark operator positions then fall on the x^- axis. Nevertheless, for $v^+ \neq 0$, the path then still retains its staple shape. Only in the $v^+ = 0$ limit (staple legs become horizontal in figure) does the staple path collapse onto the x^- axis upon k_T integration, leading to a bona fide GPD limit in which all parts of the staple link cancel, except for a residual straight link directly connecting $-z/2$ to $z/2$. One can alternatively define GTMDs with a straight gauge link from the outset; in terms of the vectors defined in the figure, this simply corresponds to the limit $v = 0$.

The appropriate choice of gauge link path depends on the physical context. In the TMD limit, the staple-shaped gauge link is most relevant, since it encodes final/initial state interactions in SIDIS/DY processes. On the other hand, GPDs are defined with a straight gauge link; as discussed in more detail below and displayed in Fig. 4.1, only under certain circumstances do GTMDs with a staple-shaped gauge link have a proper GPD limit, with the staple link collapsing into a straight gauge link. In general, GTMDs defined from the outset with a straight gauge link play a separate role, and both the straight and staple-shaped gauge link choices will be treated in this work. Two specific motivations for doing so are the following:

- In the context of quark orbital angular momentum, as accessed via the GTMD F_{14} discussed in detail further below, both the straight and the staple-shaped gauge connections have a definite, distinct physical meaning [37]. A straight gauge link enters the definition of Ji quark orbital angular momentum [47], whereas a staple-shaped gauge link generates Jaffe-Manohar quark orbital angular momentum [35]. Note that F_{14} is a genuine GTMD quantity, i.e., a quantity which does not have a TMD or GPD limit.

- A central aspect of the following treatment are Lorentz invariance relations (LIRs). In the staple link case, these contain twist-three contributions (frequently referred to as "LIR violating terms", though their role is to maintain Lorentz invariance) which do not reduce to GTMDs. To ascertain their concrete physical content in terms of quark-gluon-quark correlations, it is useful to combine the staple-link LIR with the straight-link LIR (in which these contributions are absent) as well as the straight and staple-link equations of motion. The resulting information is not directly available considering the staple link case alone.

In the most basic definition of GTMDs, a staple-shaped gauge link with a staple direction vector v on the light cone is chosen [17], such that v has only a minus component, $v = (0, v^-, 0, 0)$. On the other hand, the quark operator separation z is of the form $z = (0, z^-, z_T)$, with a two-dimensional transverse vector z_T.

Note that z is Fourier conjugate to the quark momentum k, and GTMDs are defined in terms of k^--integrated correlators, setting $z^+ = 0$. Thus, when one forms the GPD limit of GTMDs by integration over the transverse momentum k_T, one sets $z_T = 0$, and v and z then lie along one common axis. In that case, the staple legs collapse onto that one common axis, the parts of the staple legs extending beyond the region in between the quark operators cancel, and one is left with a straight gauge link connecting those operators, as is appropriate for GPDs.

However, such a light-cone choice of the staple direction v meets with rapidity divergences, which, in the application to TMDs, are commonly regulated by taking v off the light cone into the space-like region [48]. Then, v is of the form $v = (v^+, v^-, 0, 0)$, and the GPD limit ceases to be straightforward; even after integration over k_T, i.e., setting $z_T = 0$, v and z do not lie on a common axis and the staple-shaped gauge link does not collapse onto a simple straight link connecting the quark operators. The k_T-integrated quantities formed in this way are not directly GPDs, but differ from GPDs by contributions which formally vanish in the $v^+ \to 0$ light-cone limit. An alternative possibility of treating this issue arising with staple links is to modify the GTMD definition such that correlators are not rigidly defined with $z^+ = 0$, but instead such that the longitudinal part of z is parallel to v for any chosen v, i.e., $z_L = (z^+, z^-)$ is parallel to $v = (v^+, v^-)$. In that case, integration over k_T does indeed lead to collapse of the staple link into a straight gauge link, but this straight gauge link now does not lie on the light cone anymore. In effect, in this way one generates quasi-GPDs in the sense discussed by Ji [49].

In the present treatment, both GTMDs defined from the beginning with straight gauge links, as well as GTMDs defined with staple-shaped gauge links will be discussed. For the latter case, the discussion will be confined to the $v^+ = 0$ limit; $v^+ \neq 0$ corrections will not be worked out explicitly. However, it should be kept in mind that these corrections may be important in future applications, and places where they arise will be pointed out as appropriate below.

4.1.3 Parameterization of unintegrated correlation function

We consider the parameterization of the completely unintegrated off-forward correlator, $W^{\Gamma}_{\Lambda\Lambda'}$ above, in terms of Generalized Parton Correlation Functions (GPCFs) for the vector, γ^μ, and axial vector, $\gamma^\mu\gamma^5$, operators. As motivated above, we are also interested in the case of a straight gauge link; the parameterization given in [17], by contrast, is constructed for a staple-shaped gauge link, and its form was chosen such that it is not straightforwardly related to the straight-link case.

In this respect, it should be noted that there is considerable freedom in constructing GPCF parameterizations. This is due to the fact that not all Lorentz structures one can write down are independent of one another; they are related by Gordon identities and other relations, as laid out in detail in [17]. After exhausting these relations, 16 GPCFs A_i^F remain to parameterize the staple-link vector correlator, and also 16 GPCFs A_i^G remain to parameterize the staple-link axial vector correlator. In the straight-link case, to be discussed in more detail below, 8 GPCFs remain in each case. The staple-link parameterizations given in [17] in neither case contain 8 GPCFs relevant for the straight-link case; some of these were instead chosen to be eliminated in favor of terms intrinsically related to a staple-link structure. The vector correlator parameterization of [17] contains only 7 GPCFs relevant for the straight-link case; one additional one therefore has to be reinstated. The axial vector correlator parameterization of [17] contains only 3 GPCFs relevant for the straight-link case, and therefore 5 have to be reinstated. Thus, one cannot simply delete the Lorentz structures containing the staple direction vector v (denoted N in [17], up to a rescaling) from the parameterizations given in [17] and already arrive at a valid straight-link parameterization. Additional terms are needed, as given below. It would be possible to construct staple-link parameterizations differing from the ones in [17], each containing a full set of 8 structures relevant for the straight-link case, and each an additional 8 structures containing the staple direction v, such that deletion of the latter 8 immediately leads to a valid straight-link parameterization. We do not pursue this here to the full extent, but only give the straight-link parameterizations.

In the case of the vector correlator, this is rather simple. The construction of the staple-link parameterization in [17] can be followed verbatim even in the straight-link case, merely omitting all structures containing the staple direction vector v, except for the very last step. In that very last step, the single missing straight-link structure, namely, $i\sigma^{k\Delta}\Delta^\mu$, is eliminated in favor of a staple-link related structure. In the straight-link case, the staple-link related structure is not available, and therefore the aforementioned straight-link structure must be kept instead. Thus, one has the straight-link vector correlator parameterization[1]

$$\begin{aligned} W^{\gamma^\mu}_{\Lambda'\Lambda} \quad = \quad & \overline{U}(p',\Lambda')\left[\frac{P^\mu}{M}A_1^F + \frac{k^\mu}{M}A_2^F + \frac{\Delta^\mu}{M}A_3^F + \frac{i\sigma^{\mu k}}{M}A_5^F + \frac{i\sigma^{\mu\Delta}}{M}A_6^F + \frac{i\sigma^{k\Delta}}{M^2}\left(\frac{P^\mu}{M}A_8^F + \right.\right. \\ & \left.\left. \frac{k^\mu}{M}A_9^F + \frac{\Delta^\mu}{M}A_{17}^F\right)\right] U(p,\Lambda) \end{aligned} \tag{4.16}$$

where the first 7 terms are identical to the ones given in [17], and the last one, con-

[1] We use the notation $\sigma^{\mu a} = \sigma^{\mu\nu}a_\nu$, and $\sigma^{ab} = \sigma^{\mu\nu}a_\mu b_\nu$.

taining the additional invariant amplitude A_{17}^F, is associated with the aforementioned missing Lorentz structure.

The case of the axial vector correlator is more involved, and the complete construction of the straight-link parameterization is given in Section 4.8. We arrive at the form

$$\begin{aligned} W_{\Lambda'\Lambda}^{\gamma^\mu\gamma^5} &= \overline{U}(p',\Lambda')\left[\frac{i\epsilon^{\mu Pk\Delta}}{M^3}A_1^G + \frac{i\sigma^{P\mu}\gamma^5}{M}A_{17}^G + \frac{i\sigma^{Pk}\gamma^5}{M^2}\left(\frac{P^\mu}{M}A_{18}^G + \frac{k^\mu}{M}A_{19}^G + \frac{\Delta^\mu}{M}A_{20}^G\right)\right. \\ &\left.\frac{i\sigma^{P\Delta}\gamma^5}{M^2}\left(\frac{P^\mu}{M}A_{21}^G + \frac{k^\mu}{M}A_{22}^G + \frac{\Delta^\mu}{M}A_{23}^G\right)\right]U(p,\Lambda) \end{aligned} \tag{4.17}$$

which in fact has only one term in common with the staple-link parameterization given in [17], namely, the one associated with the invariant amplitude A_1^G; we make choices differing from the ones in [17] even within the straight-link sector. All GPCFs in these straight-link parameterizations are functions of $k^2, k\cdot P, k\cdot\Delta, \Delta^2, P\cdot\Delta$. In staple-link parameterizations, such as the ones given in [17], the GPCFs additionally depend on all scalar products involving the additional vector v characterizing the staple link.

It is interesting to note that, for both the vector and axial vector operators, 8 GPCFs enter the parameterization for the straight gauge link case. This is the same as the total number of GPDs (including twist 2, twist 3 and twist 4). This is expected because the GPDs are defined with quarks separated only along the light cone. The number of GTMDs on the other hand is 16. Because the underlying structure functions, the GPCFs, are fewer in number, we expect the GTMDs to be connected to one another. These relations between the GTMDs are known as the Lorentz Invariance Relations and we discuss them in Sec. 4.3.

4.1.4 Generalized Transverse Momentum-Dependent Parton Distributions

The unintegrated correlator definining the Generalized Transverse Momentum-Dependent Parton Distributions (GTMDs) is given by,

$$\begin{aligned} W^{\Gamma}_{\Lambda'\Lambda}(P,x,k_T,\xi,\Delta_T;\mathcal{U}) &= \int dk^- W^{\Gamma}_{\Lambda'\Lambda}(P,k,\Delta;\mathcal{U}) \\ = \quad \frac{1}{2}\int \frac{dz^-\, d^2 z_T}{(2\pi)^3} e^{ixP^+z^- - ik_T\cdot z_T} & \langle p',\Lambda' \mid \bar{\psi}\left(-\frac{z}{2}\right)\Gamma\mathcal{U}\psi\left(\frac{z}{2}\right) \mid p,\Lambda\rangle\Big|_{z^+=0} . \end{aligned} \quad (4.18)$$

Its parameterization in terms of GTMDs, as defined in Ref.[17], reads as follows.[2] For $\Gamma = \gamma^+$, $\gamma^+\gamma^5$, $i\sigma^{i+}\gamma^5$, one has,

$$W^{\gamma^+}_{\Lambda'\Lambda} = \frac{1}{2M}\overline{U}(p',\Lambda')\left[F_{11} + \frac{i\sigma^{i+}k^i}{P^+}F_{12} + \frac{i\sigma^{i+}\Delta^i}{P^+}F_{13} + \frac{i\sigma^{ij}k^i\Delta^j}{M^2}F_{14}\right]U(p,\Lambda) \quad (4.19)$$

$$= \left[F_{11} + \frac{i\Lambda\epsilon^{ij}k^i\Delta^j}{M^2}F_{14}\right]\delta_{\Lambda'\Lambda} + \left[\frac{\Lambda\Delta^1 + i\Delta^2}{2M}(2F_{13} - F_{11}) + \frac{\Lambda k^1 + ik^2}{M}F_{12}\right]\delta_{-\Lambda'\Lambda} \quad (4.20)$$

[2]Note that the form of this GTMD parameterization, as well as the GPD parameterization exhibited further below, is independent of the choice of gauge link, contrary to the GPCF parameterization discussed above. Thus, the relations between GTMDs and GPDs given for staple-shaped gauge links in [17] remain true for straight gauge links.

$$
\begin{aligned}
W^{\gamma^+\gamma^5}_{\Lambda'\Lambda} &= \frac{1}{2M}\overline{U}(p',\Lambda')\left[-\frac{i\epsilon^{ij}k^i\Delta^j}{M^2}G_{11}+\frac{i\sigma^{i+}\gamma^5 k^i}{P^+}G_{12}\right.\\
&+ \left.\frac{i\sigma^{i+}\gamma^5\Delta^i}{P^+}G_{13}+i\sigma^{+-}\gamma^5 G_{14}\right]U(p,\Lambda) \qquad (4.21)\\
&= \left[-\frac{i(k^1\Delta^2-k^2\Delta^1)}{M^2}G_{11}+\Lambda G_{14}\right]\delta_{\Lambda'\Lambda}+\left[\frac{\Delta^1+i\Lambda\Delta^2}{M}\left(G_{13}\right.\right.\\
&+ \left.\left.\frac{i\Lambda(k^1\Delta^2-k^2\Delta^1)}{2M^2}G_{11}\right)+\frac{k^1+i\Lambda k^2}{M}G_{12}\right]\delta_{-\Lambda'\Lambda} \qquad (4.22)\\
W^{i\sigma^{i+}\gamma^5}_{\Lambda'\Lambda} &= \frac{1}{2M}\overline{U}(p',\Lambda')\left[i\epsilon^{ij}\left(\frac{k^j}{M}H_{11}+\frac{\Delta^j}{M}H_{12}\right)+\frac{Mi\sigma^{i+}\gamma^5}{P^+}H_{13}\right.\\
&+ \frac{k^i i\sigma^{k+}\gamma^5 k^k}{MP^+}H_{14}+\frac{\Delta^i i\sigma^{k+}\gamma^5 k^k}{MP^+}H_{15}+\frac{\Delta^i\, i\sigma^{k+}\gamma^5\Delta^k}{MP^+}H_{16}\\
&+ \left.\frac{k^i i\sigma^{+-}\gamma^5}{M}H_{17}+\frac{\Delta^i\, i\sigma^{+-}\gamma^5}{M}H_{18}\right]U(p,\Lambda) \qquad (4.23)\\
&= \left[i\epsilon^{ij}\left(\frac{k^j}{M}H_{11}+\frac{\Delta^j}{M}H_{12}\right)+\Lambda\left(\frac{k^i}{M}H_{17}+\frac{\Delta^i}{M}H_{18}\right)\right]\delta_{\Lambda'\Lambda}\\
&+ \left[-i\epsilon^{ij}\frac{\Lambda\Delta^1+i\Delta^2}{2M}\left(\frac{k^j}{M}H_{11}+\frac{\Delta^j}{M}H_{12}\right)+(\delta_{i1}+i\Lambda\delta_{i2})H_{13}\right.\\
&+ \left.\frac{k^1+i\Lambda k^2}{M}\left(\frac{k^i}{M}H_{14}+\frac{\Delta^i}{M}H_{15}\right)+\frac{(\Delta^1+i\Lambda\Delta^2)\Delta^i}{M^2}H_{16}\right]\delta_{-\Lambda'\Lambda} \qquad (4.24)
\end{aligned}
$$

For each correlator listed, the second equality follows once P^+ is taken to be much larger than all other mass scales. On the other hand, for $\gamma^i, \gamma^i\gamma^5$, one has

$$
\begin{aligned}
W^{\gamma^i}_{\Lambda'\Lambda} &= \frac{1}{2P^+}\overline{U}(p',\Lambda')\left[\frac{k^i}{M}F_{21}+\frac{\Delta^i}{M}F_{22}+\frac{Mi\sigma^{i+}}{P^+}F_{23}+\frac{k^i i\sigma^{k+}k^k}{MP^+}F_{24}\right.\\
&+ \left.\frac{\Delta^i i\sigma^{k+}k^k}{MP^+}F_{25}+\frac{\Delta^i i\sigma^{k+}\Delta^k}{MP^+}F_{26}+\frac{i\sigma^{ji}k^j}{M}F_{27}+\frac{i\sigma^{ji}\Delta^j}{M}F_{28}\right]U(p,\Lambda) \qquad (4.25)\\
&= \left[\frac{k^i}{P^+}F_{21}+\frac{\Delta^i}{P^+}F_{22}-i\Lambda\epsilon^{ij}\left(\frac{k^j}{P^+}F_{27}+\frac{\Delta^j}{P^+}F_{28}\right)\right]\delta_{\Lambda'\Lambda}\\
&+ \frac{1}{P^+}\left[-\frac{i\Delta^2+\Lambda\Delta^1}{2M}(k^iF_{21}+\Delta^iF_{22})+M(\Lambda\delta_{i1}+i\delta_{i2})F_{23}\right.\\
&+ \left.\frac{\Lambda k^1+ik^2}{M}(k^iF_{24}+\Delta^iF_{25})+\frac{\Delta^i}{M}(\Lambda\Delta^1+i\Delta^2)F_{26}\right]\delta_{-\Lambda'\Lambda} \qquad (4.26)
\end{aligned}
$$

$$
\begin{aligned}
W^{\gamma^i\gamma^5}_{\Lambda'\Lambda} &= \frac{1}{2P^+}\overline{U}(p',\Lambda')\left[-\frac{i\epsilon^{ji}k^j}{M}G_{21} - \frac{i\epsilon^{ji}\Delta^j}{M}G_{22} + \frac{Mi\sigma^{i+}\gamma^5}{P^+}G_{23}\right. \\
&+ \frac{k^i i\sigma^{k+}\gamma^5 k^k}{MP^+}G_{24} + \frac{\Delta^i i\sigma^{k+}\gamma^5 k^k}{MP^+}G_{25} + \frac{\Delta^i i\sigma^{k+}\gamma^5\Delta^k}{MP^+}G_{26} \\
&+ \left.\frac{k^i i\sigma^{+-}\gamma^5}{M}G_{27} + \frac{\Delta^i i\sigma^{+-}\gamma^5}{M}G_{28}\right]U(p,\Lambda) \qquad (4.27) \\
&= \left[i\epsilon^{ij}(\frac{k^j}{P^+}G_{21} + \frac{\Delta^j}{P^+}G_{22}) + \Lambda(\frac{k^i}{P^+}G_{27} + \frac{\Delta^i}{P^+}G_{28})\right]\delta_{\Lambda'\Lambda} \\
&+ \frac{1}{P^+}\left[-i\epsilon^{ij}\frac{i\Delta^2 + \Lambda\Delta^1}{2M}(k^jG_{21} + \Delta^jG_{22}) + M(\delta_{i1} + i\Lambda\delta_{i2})G_{23}\right. \\
&+ \left.\frac{(k^1 + i\Lambda k^2)}{M}(k^iG_{24} + \Delta^iG_{25}) + \frac{\Delta^i(\Delta^1 + i\Lambda\Delta^2)}{M}G_{26}\right]\delta_{-\Lambda'\Lambda} \qquad (4.28)
\end{aligned}
$$

The GTMDs considered here are complex functions of the set of kinematical variables $x, \xi, k_T^2, k_T \cdot \Delta_T, t$; in the case of a staple-shaped gauge link, they furthermore depend on the vector v characterizing the staple,

$$
X(x,\xi,k_T^2,k_T\cdot\Delta_T,t,v) = X^e(x,\xi,k_T^2,k_T\cdot\Delta_T,t,v) + iX^o(x,\xi,k_T^2,k_T\cdot\Delta_T,t,v) \quad (4.29)
$$

with $X = F_{1j}, G_{1j}, H_{1j}$, at twist two, and $X = F_{2j}, G_{2j}$, at twist three. X^e is symmetric under $v \to -v$ (T-even), while X^o reverses its sign for $v \to -v$ (T-odd). Due to Hermiticity and time reversal invariance, we have that the following GTMD components are odd for $\xi \to -\xi$, $k_T \cdot \Delta_T \to -k_T \cdot \Delta_T$,

$$
\begin{aligned}
&F^e_{12}, F^e_{22}, F^e_{23}, F^e_{24}, F^e_{26}, F^e_{27}, G^e_{13}, G^e_{21}, G^e_{25}, G^e_{28}, H^e_{11}, H^e_{15}, H^e_{18}F^o_{11}, F^o_{13}, F^o_{14}, F^o_{21}, \\
&F^o_{25}, F^o_{28}, G^o_{11}, G^o_{12}, G^o_{14}, G^o_{22}, G^o_{23}, G^o_{24}, G^o_{26}, G^o_{27}, H^o_{12}, H^o_{13}, H^o_{14}, H^o_{16}, H^o_{17}
\end{aligned}
$$

This influences which k_T-moments of these GTMDs can appear in the $\xi = 0$ case.

4.1.5 Generalized Parton Distributions

The Generalized Parton Distributions (GPDs) are obtained by formally integrating Eq.(4.18) over the transverse parton momentum, k_T, provided that the gauge link has the appropriate form, cf. the discussion in Sec. 4.1.2,

$$F^{\Gamma}_{\Lambda'\Lambda}(x,\xi,t) = \frac{1}{2}\int \frac{dz^-}{2\pi} e^{ixP^+z^-} \langle p', \Lambda' \mid \bar{\psi}\left(-\frac{z}{2}\right) \Gamma \mathcal{U} \psi\left(\frac{z}{2}\right) \mid p, \Lambda\rangle\Big|_{z^+=0, z_T=0} . \quad (4.31)$$

For $\gamma^+, \gamma^+\gamma^5, i\sigma^{i+}\gamma^5$ one has,

$$\begin{aligned}
F^{\gamma^+}_{\Lambda'\Lambda} &= \frac{1}{2P^+}\overline{U}(p',\Lambda')\left[\gamma^+ H + \frac{i\sigma^{+\Delta}}{2M}E\right]U(p,\Lambda) &(4.32)\\
&= H\delta_{\Lambda,\Lambda'} + \frac{(\Lambda\Delta^1 + i\Delta^2)}{2M}E\delta_{-\Lambda,\Lambda'} &(4.33)\\
F^{\gamma^+\gamma^5}_{\Lambda'\Lambda} &= \frac{1}{2P^+}\overline{U}(p',\Lambda')\left[\gamma^+\gamma^5 \widetilde{H} + \frac{\Delta^+\gamma^5}{2M}\widetilde{E}\right]U(p,\Lambda) &(4.34)\\
&= \Lambda\widetilde{H}\delta_{\Lambda,\Lambda'} + \frac{(\Delta^1 + i\Lambda\Delta^2)}{2M}\xi\widetilde{E}\delta_{-\Lambda,\Lambda'} &(4.35)\\
F^{i\sigma^{i+}\gamma^5}_{\Lambda'\Lambda} &= \frac{i\epsilon^{ij}}{2P^+}\overline{U}(p',\Lambda')\left[i\sigma^{+j}H_T + \frac{\gamma^+\Delta^j - \Delta^+\gamma^j}{2M}E_T + \frac{P^+\Delta^j}{M^2}\widetilde{H}_T\right.\\
&- \left.\frac{P^+\gamma^j}{M}\widetilde{E}_T\right]U(p,\Lambda) &(4.36)\\
&= \left[\frac{i\epsilon^{ij}\Delta^j}{2M}(E_T + 2\widetilde{H}_T) + \frac{\Lambda\Delta^i}{2M}\left(\widetilde{E}_T - \xi E_T\right)\right]\delta_{\Lambda\Lambda'} + [(\delta_{i1} + i\Lambda\delta_{i2})H_T\\
&- \left.\frac{i\epsilon^{ij}\Delta^j(\Lambda\Delta^1 + i\Delta^2)}{2M^2}\widetilde{H}_T\right]\delta_{-\Lambda\Lambda'} &(4.37)
\end{aligned}$$

whereas for $\gamma^i, \gamma^i\gamma^5$,

$$
\begin{aligned}
F_{\Lambda'\Lambda}^{\gamma^i} &= \frac{M}{2(P^+)^2}\overline{U}(p',\Lambda')\left[i\sigma^{+i}H_{2T} + \frac{\gamma^+\Delta^i - \Delta^+\gamma^i}{2M}E_{2T} + \frac{P^+\Delta^i}{M^2}\widetilde{H}_{2T}\right. \\
&- \left.\frac{P^+\gamma^i}{M}\widetilde{E}_{2T}\right]U(p,\Lambda) \qquad (4.38) \\
&= \left[\frac{\Delta^i}{2P^+}E_{2T} + \frac{\Delta^i}{P^+}\widetilde{H}_{2T} + \frac{i\Lambda\epsilon^{ij}\Delta^j}{2P^+}\left(\widetilde{E}_{2T} - \xi E_{2T}\right)\right]\delta_{\Lambda\Lambda'} \\
&+ \left[\frac{-M(\Lambda\delta_{i1} + i\delta_{i2})}{P^+}H_{2T} - \frac{(\Lambda\Delta^1 + i\Delta^2)\Delta^i}{2MP^+}\widetilde{H}_{2T}\right]\delta_{\Lambda-\Lambda'} \qquad (4.39) \\
F_{\Lambda'\Lambda}^{\gamma^i\gamma^5} &= \frac{i\epsilon^{ij}M}{2(P^+)^2}\overline{U}(p',\Lambda')\left[i\sigma^{+j}H'_{2T} + \frac{\gamma^+\Delta^j - \Delta^+\gamma^j}{2M}E'_{2T} + \frac{P^+\Delta^j}{M^2}\widetilde{H}'_{2T}\right. \\
&- \left.\frac{P^+\gamma^j}{M}\widetilde{E}'_{2T}\right]U(p,\Lambda) \qquad (4.40) \\
&= \left[\frac{i\epsilon^{ij}\Delta^j}{2P^+}E'_{2T} + \frac{i\epsilon^{ij}\Delta^j}{P^+}\widetilde{H}'_{2T} - \frac{\Lambda\Delta^i}{2P^+}\left(\widetilde{E}'_{2T} - \xi E'_{2T}\right)\right]\delta_{\Lambda\Lambda'} \\
&+ \left[\frac{M(\delta_{i1} + i\Lambda\delta_{i2})}{P^+}H'_{2T} - \frac{i\epsilon^{ij}(\Lambda\Delta^1 + i\Delta^2)\Delta^j}{2MP^+}\widetilde{H}'_{2T}\right]\delta_{\Lambda-\Lambda'} \qquad (4.41)
\end{aligned}
$$

The gauge connection for GPDs is a straight link, implying that all GPDs are naive T-even. We use the GPD parameterization from Ref.[17]. As in the first parameterization introduced by Ji [27], the letter H signifies that in the forward limit these GPDs correspond to a PDF, while the ones denoted by E are completely new functions; H, E, $\widetilde{H}$, $\widetilde{E}$ parametrize the chiral-even quark operators. In the chiral-odd sector, H_T, E_T, $\widetilde{H}_T$, $\widetilde{E}_T$ describe the tensor quark operators, the subscript T signifying that the quarks flip helicity or are transversely polarized [50]. The matrix structures that enter the twist three vector (γ^i) and axial vector ($\gamma^i\gamma^5$) cases are identical to the ones occurring at the twist two level in the chiral-odd tensor sector. Hence, the GPDs have similar names: the corresponding twist three GPD, occurring with the same matrix coefficient, is named F_{2T} if parametrizing the vector case γ^i and F'_{2T} if parametrizing the axial vector case $\gamma^i\gamma^5$, with $F = H, E, \widetilde{H}, \widetilde{E}$.

4.1.6 Helicity Structure

To elucidate the helicity structure, which is needed to connect to phenomenological applications and which also serves as a heuristic tool in the construction of LIR and EoM relations below, we introduce the quark-proton helicity amplitudes, [31],

$$A_{\Lambda'\lambda',\Lambda\lambda} = \int \frac{dz^- \, d^2 z_T}{(2\pi)^3} e^{ixP^+z^- - ik_T\cdot z_T} \, \langle p', \Lambda' \mid \mathcal{O}_{\lambda'\lambda}(z) \mid p, \Lambda\rangle|_{z^+=0} \, , \tag{4.42}$$

where at twist two the bilocal quark field operators,

$$\mathcal{O}_{\pm\pm}(z) \;\; = \;\; \frac{1}{4}\bar{\psi}\left(-\frac{z}{2}\right)\gamma^+(1\pm\gamma^5)\psi\left(\frac{z}{2}\right) \equiv \phi^\dagger_\pm \phi_\pm \tag{4.43}$$

define (non flip) transitions between quark $\pm,\pm$ helicity states. Note that, in this section only, for the purpose of discussing helicity structure, we drop the gauge link in the bilocal operators to simplify notation.

The various LIRs and EoM relations that we derive in subsequent sections correspond to different helicity combinations obtained varying the initial and final proton helicity states. We obtain 8 distinct relations from the following combinations, $(+,+) \pm (-,-)$, and $(+,-) \pm (-,+)$, in the vector and axial vector sector, respectively. In what follows we derive all four spin correlations.

The correlation functions in Eqs.(4.20,4.22) can be written in terms of the quark-proton helicity amplitudes as,

$$W^{\gamma^+}_{\Lambda'\Lambda} \;\; = \;\; A_{\Lambda'+,\Lambda+} + A_{\Lambda'-,\Lambda-} \tag{4.44}$$

$$W^{\gamma^+\gamma^5}_{\Lambda'\Lambda} \;\; = \;\; A_{\Lambda'+,\Lambda+} - A_{\Lambda'-,\Lambda-}. \tag{4.45}$$

One finds the following expressions for the proton non flip terms,

$$
\begin{aligned}
F_{11} &= \frac{1}{2}(W_{++}^{\gamma^+} + W_{--}^{\gamma^+}) \\
&= \frac{1}{2}(A_{++,++} + A_{+-,+-} + A_{-+,-+} + A_{--,--}) && \text{(4.46a)} \\
i\frac{(\mathbf{k}_T \times \mathbf{\Delta}_T)_3}{M^2} F_{14} &= \frac{1}{2}(W_{++}^{\gamma^+} - W_{--}^{\gamma^+}) \\
&= \frac{1}{2}(A_{++,++} + A_{+-,+-} - A_{-+,-+} - A_{--,--}) && \text{(4.46b)} \\
G_{14} &= \frac{1}{2}(W_{++}^{\gamma^+\gamma^5} - W_{--}^{\gamma^+\gamma^5}) \\
&= \frac{1}{2}(A_{++,++} - A_{+-,+-} - A_{-+,-+} + A_{--,--}) && \text{(4.46c)} \\
-i\frac{(\mathbf{k}_T \times \mathbf{\Delta}_T)_3}{M^2} G_{11} &= \frac{1}{2}(W_{++}^{\gamma^+\gamma^5} + W_{--}^{\gamma^+\gamma^5}) \\
&= \frac{1}{2}(A_{++,++} - A_{+-,+-} + A_{-+,-+} - A_{--,--}), && \text{(4.46d)}
\end{aligned}
$$

where, because of the constraints in Eqs.(4.30), the combinations on the *rhs* of Eqs.(4.46a, 4.46c) and Eqs.(4.46b,4.46d) are purely real and imaginary, respectively.

The distributions in both transverse coordinate and momentum space corresponding to these GTMDs were analyzed in detail in Refs.[34; 33]. F_{11} describes an unpolarized quark and proton state, and it reduces to the PDF f_1 in the forward, k_T integrated, limit; G_{14} describes the quark helicity distribution, or g_1 in the forward, k_T integrated, limit. F_{14} and G_{11} do not have GPD or TMD limits. However, in the forward limit, their average over k_T weighted by k_T^2 gives [34],

$$\begin{aligned}
(L_q)_3 &= \int dx \int d^2k_T \frac{1}{2}\left(\mathbf{k}_T \times i\frac{\partial}{\partial \mathbf{\Delta}_T}\right)_3 \left(W_{++}^{\gamma^+} - W_{--}^{\gamma^+}\right) \\
&= -\int dx \int d^2k_T \frac{k_T^2}{M^2} F_{14} \qquad (4.47)\\
2(L_q)_3(S_q)_3 &= \int dx \int d^2k_T \frac{1}{2}\left(\mathbf{k}_T \times i\frac{\partial}{\partial \mathbf{\Delta}_T}\right)_3 \left(W_{++}^{\gamma^+\gamma^5} + W_{--}^{\gamma^+\gamma^5}\right) \\
&= \int dx \int d^2k_T \frac{k_T^2}{M^2} G_{11} \quad , \qquad (4.48)
\end{aligned}$$

where Eq. (4.47) represents the quark OAM along the z axis in a longitudinally polarized proton, while Eq.(4.48) gives the quark OAM along the z axis for a longitudinally polarized quark, or a spin-orbit term.

The proton spin flip terms read,

$$-\frac{i(\mathbf{k}_T \times \mathbf{\Delta}_T)_3}{M} F_{12} = \frac{1}{2}((\Delta^1 - i\Delta^2)W_{-+}^{\gamma^+} + (\Delta^1 + i\Delta^2)W_{+-}^{\gamma^+}) \qquad (4.49a)$$

$$\frac{k_T \cdot \Delta_T}{M} F_{12} + \frac{\Delta_T^2}{2M}(2F_{13} - F_{11}) = \frac{1}{2}((\Delta^1 - i\Delta^2)W_{-+}^{\gamma^+} - (\Delta^1 + i\Delta^2)W_{+-}^{\gamma^+}) \qquad (4.49b)$$

and,

$$\frac{\Delta_T^2}{M} G_{13} + \frac{k_T \cdot \Delta_T}{M} G_{12} = \frac{1}{2}((\Delta^1 - i\Delta^2)W_{-+}^{\gamma^+\gamma^5} + (\Delta^1 + i\Delta^2)W_{+-}^{\gamma^+\gamma^5}) \qquad (4.50a)$$

$$\frac{i(\mathbf{k}_T \times \mathbf{\Delta}_T)_3}{M}\left(\frac{\Delta_T^2}{2M^2} G_{11} - G_{12}\right) = \frac{1}{2}((\Delta^1 - i\Delta^2)W_{-+}^{\gamma^+\gamma^5} - (\Delta^1 + i\Delta^2)W_{+-}^{\gamma^+\gamma^5}) \qquad (4.50b)$$

At twist three, the bilocal operators can be written as the overlap of a dynamically independent quark field, ϕ (good component), and a dynamically dependent quark-gluon composite field, χ (bad component) [9],

$$\begin{aligned}
\mathcal{O}^q_{-^*+}(z) &= \frac{1}{8}\bar{\psi}\left(-\frac{z}{2}\right)(\gamma^1 - i\gamma^2)(1+\gamma^5)\psi\left(\frac{z}{2}\right) = \chi^\dagger_+\phi_+ && (4.51a)\\
\mathcal{O}^q_{+-^*}(z) &= \frac{1}{8}\bar{\psi}\left(-\frac{z}{2}\right)(\gamma^1 + i\gamma^2)(1+\gamma^5)\psi\left(\frac{z}{2}\right) = \phi^\dagger_+\chi_+ && (4.51b)\\
\mathcal{O}^q_{+^*-}(z) &= -\frac{1}{8}\bar{\psi}\left(-\frac{z}{2}\right)(\gamma^1 + i\gamma^2)(1-\gamma^5)\psi\left(\frac{z}{2}\right) = -\chi^\dagger_-\phi_- && (4.51c)\\
\mathcal{O}^q_{-+^*}(z) &= -\frac{1}{8}\bar{\psi}\left(-\frac{z}{2}\right)(\gamma^1 - i\gamma^2)(1-\gamma^5)\psi\left(\frac{z}{2}\right) = -\phi^\dagger_-\chi_- && (4.51d)
\end{aligned}$$

Notice that the $*$ on the *lhs* symbolizes the helicity of the quark within the quark-gluon composite field, χ (on the *rhs*), whose helicity is always opposite so that angular momentum is conserved [51]. As a result, one can form twice as many helicity amplitudes as compared to the twist two case [9],

$$\begin{aligned}
A^{tw3}_{\Lambda'\lambda'^*,\Lambda\lambda} &= \int \frac{dz^-\, d^2z_T}{(2\pi)^3} e^{ixP^+z^- - ik_T\cdot z_T} \left.\langle p', \Lambda' \mid \mathcal{O}_{\lambda'^*\lambda}(z) \mid p, \Lambda\rangle\right|_{z^+=0}, && (4.52a)\\
A^{tw3}_{\Lambda'\lambda',\Lambda\lambda^*} &= \int \frac{dz^-\, d^2z_T}{(2\pi)^3} e^{ixP^+z^- - ik_T\cdot z_T} \left.\langle p', \Lambda' \mid \mathcal{O}_{\lambda'\lambda^*}(z) \mid p, \Lambda\rangle\right|_{z^+=0}. && (4.52b)
\end{aligned}$$

Therefore,

$$\begin{aligned}
A^{tw3}_{\Lambda'-^*,\Lambda+} &= W^{\gamma^1}_{\Lambda'\Lambda} + W^{\gamma^1\gamma^5}_{\Lambda'\Lambda} - iW^{\gamma^2}_{\Lambda'\Lambda} - iW^{\gamma^2\gamma^5}_{\Lambda'\Lambda} && (4.53a)\\
A^{tw3}_{\Lambda'+,\Lambda-^*} &= W^{\gamma^1}_{\Lambda'\Lambda} + W^{\gamma^1\gamma^5}_{\Lambda'\Lambda} + iW^{\gamma^2}_{\Lambda'\Lambda} + iW^{\gamma^2\gamma^5}_{\Lambda'\Lambda} && (4.53b)\\
A^{tw3}_{\Lambda'+^*,\Lambda-} &= -W^{\gamma^1}_{\Lambda'\Lambda} + W^{\gamma^1\gamma^5}_{\Lambda'\Lambda} - iW^{\gamma^2}_{\Lambda'\Lambda} + iW^{\gamma^2\gamma^5}_{\Lambda'\Lambda} && (4.53c)\\
A^{tw3}_{\Lambda'-,\Lambda+^*} &= -W^{\gamma^1}_{\Lambda'\Lambda} + W^{\gamma^1\gamma^5}_{\Lambda'\Lambda} + iW^{\gamma^2}_{\Lambda'\Lambda} - iW^{\gamma^2\gamma^5}_{\Lambda'\Lambda} && (4.53d)
\end{aligned}$$

At the twist-three level, the following are the expressions for the proton helicity non-flip terms,

$$-\frac{i\epsilon^{ij}k^j}{P^+}F_{27}-\frac{i\epsilon^{ij}\Delta^j}{P^+}F_{28} = \frac{1}{2}\left(W^{\gamma^i}_{++}-W^{\gamma^i}_{--}\right) \tag{4.54a}$$
$$\frac{k^i}{P^+}F_{21}+\frac{\Delta^i}{P^+}F_{22} = \frac{1}{2}\left(W^{\gamma^i}_{++}+W^{\gamma^i}_{--}\right) \tag{4.54b}$$
$$\frac{k^i}{P^+}G_{27}+\frac{\Delta^i}{P^+}G_{28} = \frac{1}{2}\left(W^{\gamma^i\gamma^5}_{++}-W^{\gamma^i\gamma^5}_{--}\right) \tag{4.54c}$$
$$\frac{i\epsilon^{ij}k^j}{P^+}G_{21}+\frac{i\epsilon^{ij}\Delta^j}{P^+}G_{22} = \frac{1}{2}\left(W^{\gamma^i\gamma^5}_{++}+W^{\gamma^i\gamma^5}_{--}\right) \tag{4.54d}$$

As we show in subsequent sections, Eqs.(4.54a) and (4.54d) allow us to identify the twist-three GTMDs that enter the EoM relations for F_{14} and G_{11} respectively.

Writing the GTMDs that enter the proton helicity flip case one has,

$$\begin{aligned}-\frac{i\epsilon^{ij}M\Delta^j}{P^+}F_{23} &- i\frac{(\mathbf{k}_T\times\mathbf{\Delta}_T)_3}{MP^+}(k^iF_{24}+\Delta^iF_{25}) \\ &= \frac{1}{2}\left((\Delta^1-i\Delta^2)W^{\gamma^i}_{-+}+(\Delta^1+i\Delta^2)W^{\gamma^i}_{+-}\right) \qquad (4.55a) \\ -\frac{\Delta_T^2}{2MP^+}(k^iF_{21} &+ \Delta^iF_{22})+\frac{M\Delta^i}{P^+}F_{23}+\frac{k_T\cdot\Delta_T}{MP^+}(k^iF_{24}+\Delta^iF_{25})+\frac{\Delta_T^2\Delta^i}{MP^+}F_{26} \\ &= \frac{1}{2}\left((\Delta^1-i\Delta^2)W^{\gamma^i}_{-+}-(\Delta^1+i\Delta^2)W^{\gamma^i}_{+-}\right) \qquad (4.55b)\end{aligned}$$

and,

$$\begin{aligned}
\frac{M\Delta^i}{P^+}G_{23} &+ \frac{k_T\cdot\Delta_T}{MP^+}(k^iG_{24}+\Delta^iG_{25})+\frac{\Delta^i\Delta_T^2}{M}G_{26} \\
&= \frac{1}{2}\Big((\Delta^1-i\Delta^2)W_{-+}^{\gamma^i\gamma^5}+(\Delta^1+i\Delta^2)W_{+-}^{\gamma^i\gamma^5}\Big) \qquad (4.56a)\\
-\frac{i\epsilon^{ij}\Delta_T^2}{2MP^+}(k^jG_{21} &- \Delta^jG_{22})-\frac{i\epsilon^{ij}\Delta^j}{P^+}G_{23}-\frac{i(\mathbf{k}_T\times\boldsymbol{\Delta}_T)_3}{MP^+}\left(k^iG_{24}+\Delta^iG_{25}\right)\\
&= \frac{1}{2}\Big((\Delta^1-i\Delta^2)W_{-+}^{\gamma^i\gamma^5}-(\Delta^1+i\Delta^2)W_{+-}^{\gamma^i\gamma^5}\Big) \qquad (4.56b)
\end{aligned}$$

The helicity amplitude structure is preserved when going to either the GPD or the TMD limit. It plays an important role in defining the observables for the various quantities. The GTMDs defined so far are related to GPDs by integrating them over k_T and to TMDs by taking the forward limit ($\Delta \to 0$).

4.2 Equation of Motion Relations

4.2.1 Construction of Equation of Motion Relations

Equation of motion relations connect different GTMD correlators of the type defined in Eq. (4.18), in which the quark creation and annihilation operators are located at positions $-z/2$ and $z/2$. To construct them, it is useful to consider initially a somewhat more general correlator in which the quark creation and annihilation operators are located at more freely variable positions z_{in} and z_{out}, respectively. Central to the construction is the observation that, taken between physical particle states, matrix elements of operators that vanish according to the classical field equations of motion vanish in the quantum theory[3] [52]. Thus, in view of the classical quark field

[3]Note that the argument given in [52] is formulated for local operators; its extension to nonlocal operators such as considered here calls for further justification, as noted in [48].

equations of motion

$$
\begin{aligned}
(i\not{D} - m)\psi &= (i\not{\partial} + g\not{A} - m)\psi = 0, &\text{(4.57a)}\\
\bar{\psi}(i\overleftarrow{\not{D}} + m) &= \bar{\psi}(i\overleftarrow{\not{\partial}} - g\not{A} + m) = 0 &\text{(4.57b)}
\end{aligned}
$$

one has the vanishing correlation function

$$
\begin{aligned}
0 &= \int \frac{dz_{in}^{-}\, d^2 z_{in,T}}{(2\pi)^3} \int \frac{dz_{out}^{-}\, d^2 z_{out,T}}{(2\pi)^3} e^{ik(z_{out}-z_{in})+i\Delta(z_{out}+z_{in})/2} \\
&\cdot \langle p', \Lambda' \mid \bar{\psi}(z_{in}) \Big[(i\overleftarrow{\not{D}} + m)\Gamma\mathcal{U} \pm \Gamma\mathcal{U}(i\not{D} - m) \Big] \psi(z_{out}) \mid p, \Lambda \rangle \Big|_{z_{in}^{+}=z_{out}^{+}=0} \qquad (4.58)
\end{aligned}
$$

where, specifically, $\Gamma = i\sigma^{i+}\gamma^5 = \gamma^+\gamma^i\gamma^5 - \gamma^i\gamma^+\gamma^5$ with a transverse vector index $i = 1, 2$, cf. Sec. 4.1.4. Note that the $\not{D}$ and $\overleftarrow{\not{D}}$ operators act on the z_{out} and z_{in} arguments, respectively. Furthermore, no derivatives with respect to z_{in}^{+} or z_{out}^{+} appear in the square bracket; these derivatives are accompanied in the Dirac operator by a factor γ^+, implying that the terms in question vanish once multiplied by the structure Γ, which contains an additional factor γ^+. Thus, introducing the equations of motion as in (4.58) is consistent with an a priori specification $z_{in}^{+} = z_{out}^{+} = 0$.

Performing an integration by parts with respect to both z_{out} and z_{in} yields

$$
\begin{aligned}
0 &= \int \frac{dz_{in}^{-}\, d^2 z_{in,T}}{(2\pi)^3} \int \frac{dz_{out}^{-}\, d^2 z_{out,T}}{(2\pi)^3} e^{ik(z_{out}-z_{in})+i\Delta(z_{out}+z_{in})/2} \Big\{ \\
&\quad \langle p', \Lambda' \mid \bar{\psi}(z_{in}) \big[-i\not{\partial}_{in}\mathcal{U}\Gamma \mp i\Gamma\not{\partial}_{out}\mathcal{U} - g\not{A}(z_{in})\mathcal{U}\Gamma \pm \Gamma\mathcal{U}g\not{A}(z_{out}) \big] \psi(z_{out}) \mid p, \Lambda \rangle \\
&+ \langle p', \Lambda' \mid \bar{\psi}(z_{in}) \Bigg[\left(-\not{k} + \frac{\not{\Delta}}{2} \right) \mathcal{U}\Gamma \mp \Gamma\mathcal{U} \left(-\not{k} - \frac{\not{\Delta}}{2} \right) \\
&+ (m \mp m)\Gamma\mathcal{U} \Bigg] \psi(z_{out}) \mid p, \Lambda \rangle \Big\} \Big|_{z_{in}^{+}=z_{out}^{+}=0} \qquad (4.59)
\end{aligned}
$$

Two types of contributions are generated. The second line of (4.59) contains the terms in which the derivatives act on the gauge links; these terms will ultimately result in quark-gluon-quark correlators. The third line of (4.59) contains the standard terms in which the derivatives act on the exponential in the Fourier transformation; these terms result in quark-quark correlators. Proceeding by changing integration variables,

$$b = \frac{z_{in} + z_{out}}{2}, \qquad z = z_{out} - z_{in}, \tag{4.60}$$

and translating the matrix elements by $-b$, one obtains

$$\begin{aligned}
0 \;=\; & \left(\int \frac{db^- \, d^2 b_T}{(2\pi)^3} e^{ib\Delta} e^{ibp} e^{-ibp'} \right) \int \frac{dz^- \, d^2 z_T}{(2\pi)^3} e^{ikz} \Bigg\{ \\
& \langle p', \Lambda' \mid \bar{\psi}(-z/2) \left[(-i \overrightarrow{\not{\partial}} - g\not{A}) \mathcal{U}\Gamma \Big|_{-z/2} \pm \Gamma\mathcal{U}(-i \overleftarrow{\not{\partial}} + g\not{A}) \Big|_{z/2} \right] \psi(z/2) \mid p, \Lambda \rangle \\
+ \;\; & \langle p', \Lambda' \mid \bar{\psi}(-z/2) \Bigg[\left(-\not{k} + \frac{\not{\Delta}}{2} \right) \mathcal{U}\Gamma \mp \Gamma\mathcal{U} \left(-\not{k} - \frac{\not{\Delta}}{2} \right) \\
+ \;\; & (m \mp m)\Gamma\mathcal{U} \Bigg] \psi(z/2) \mid p, \Lambda \rangle \Bigg\} \Bigg|_{z^+ = b^+ = 0}
\end{aligned} \tag{4.61}$$

having taken into account the phases generated in the proton states by the translation. Thus, a δ-function which enforces momentum conservation as expected, $\delta^3(p' - p - \Delta)$, is factored out; it follows that the rest of the expression by itself must already vanish. Proceeding to simplify the Dirac structures (employing, e.g., the identity $\gamma^\mu \gamma^\rho \gamma^\nu = g^{\mu\rho}\gamma^\nu + g^{\nu\rho}\gamma^\mu - g^{\mu\nu}\gamma^\rho - i\epsilon^{\sigma\mu\nu\rho}\gamma_\sigma\gamma^5$), one can finally identify from the third line of (4.61) the GTMD correlators defined in Eq. (4.18), and one thus arrives at the equation of motion relations

$$-\frac{\Delta^+}{2}W^{\gamma^i\gamma^5}_{\Lambda'\Lambda} + ik^+\epsilon^{ij}W^{\gamma^j}_{\Lambda'\Lambda} + \frac{\Delta^i}{2}W^{\gamma^+\gamma^5}_{\Lambda'\Lambda} - i\epsilon^{ij}k^jW^{\gamma^+}_{\Lambda'\Lambda} + \mathcal{M}^{i,S}_{\Lambda'\Lambda} = 0 \tag{4.62a}$$

$$-k^+W^{\gamma^i\gamma^5}_{\Lambda'\Lambda} + \frac{i\Delta^+}{2}\epsilon^{ij}W^{\gamma^j}_{\Lambda'\Lambda} + k^iW^{\gamma^+\gamma^5}_{\Lambda'\Lambda} - i\epsilon^{ij}\frac{\Delta^j}{2}W^{\gamma^+}_{\Lambda'\Lambda} + mW^{i\sigma^{i+}\gamma^5}_{\Lambda'\Lambda} + \mathcal{M}^{i,A}_{\Lambda'\Lambda} = 0, \tag{4.62b}$$

which relate the correlation functions for different Dirac structures, $\gamma^i\gamma^5, \gamma^i, \gamma^+\gamma^5, \gamma^+$, $i\sigma^{i+}\gamma^5$, and in which the genuine/dynamic [20] twist-three terms, copied from the second line of (4.61), are given by[4]

$$\begin{aligned}\mathcal{M}^{i,S}_{\Lambda'\Lambda} &= \frac{i}{4}\int\frac{dz^-d^2z_T}{(2\pi)^3}e^{ixP^+z^- - ik_T\cdot z_T}\langle p',\Lambda' \mid \overline{\psi}\left(-\frac{z}{2}\right)\left[(\overrightarrow{\partial\!\!\!/} - igA\!\!\!/)\mathcal{U}\Gamma\Big|_{-z/2}\right.\\ &+ \left.\Gamma\mathcal{U}(\overleftarrow{\partial\!\!\!/} + igA\!\!\!/)\Big|_{z/2}\right]\psi\left(\frac{z}{2}\right)\mid p,\Lambda\rangle_{z^+=0}\end{aligned} \tag{4.63a}$$

$$\begin{aligned}\mathcal{M}^{i,A}_{\Lambda'\Lambda} &= \frac{i}{4}\int\frac{dz^-d^2z_T}{(2\pi)^3}e^{ixP^+z^- - ik_T\cdot z_T}\langle p',\Lambda' \mid \overline{\psi}\left(-\frac{z}{2}\right)\left[-(\overrightarrow{\partial\!\!\!/} - igA\!\!\!/)\mathcal{U}\Gamma\Big|_{-z/2}\right.\\ &+ \left.\Gamma\mathcal{U}(\overleftarrow{\partial\!\!\!/} + igA\!\!\!/)\Big|_{z/2}\right]\psi\left(\frac{z}{2}\right)\mid p,\Lambda\rangle_{z^+=0}\end{aligned} \tag{4.63b}$$

with $\Gamma = i\sigma^{i+}\gamma^5$. In the following, only the case of vanishing skewness, $\Delta^+ = 0$, will be considered further.

Relations (4.62a) and (4.62b) are generalizations to the off-forward case of the EoM relations involving the k_T-unintegrated correlator first introduced in [14; 18; 42]. In particular, Eq.(4.62b) leads to the relation between the polarized structure functions g_1 and g_2 first obtained in the forward limit using the same method in Refs.[14; 18]. However, notice that, at variance with [14; 18], because of the symmetrization introduced in Eqs. (4.58)-(4.61), the imaginary parts in Eq.(4.62b) appear only for

[4]Note that the expression for $\mathcal{M}^{i,S}_{\Lambda'\Lambda}$ quoted in [7] is missing an overall factor i.

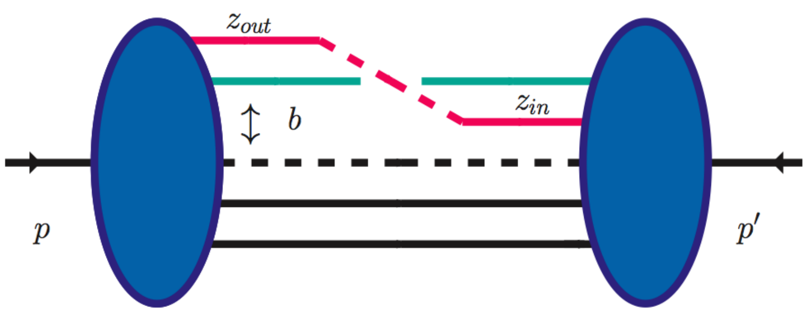


Figure 4.2: Kinematical variables for the correlation function describing a GTMD. The matrix element in the correlator is a function of $z = z_{out} - z_{in}$ where $z_{out} = z/2$ ($z_{in} = -z/2$) is the argument of ψ ($\bar{\psi}$); $b = (z_{in} + z_{out})/2$ is the Fourier conjugate of $\Delta = p' - p$.

the non forward terms (terms multiplied by Δ). As will be discussed further below, these relations represent a first step towards deriving a connection between twist-two GTMDs and twist-three GPDs using a procedure alternative to OPE that highlights the sensitivity to the quark intrinsic transverse momentum. In our case, they attain additional significance in that they provide a framework for describing partonic OAM in the proton in terms of specific distributions, thus helping to clarify possible mechanisms that generate it. A prerequisite for understanding what produces OAM in the proton is that one examines the dynamics encoded in the correlator components at the unintegrated level.

4.2.2 Gauge Link Structure and Intrinsic Twist Three Term

The form of the intrinsic twist-three terms given in Section 4.2.1 is valid for an arbitrary choice of gauge link $\mathcal{U}$. The gauge link depends parametrically on the locations of its endpoints; the derivative operators quantify those dependences.

More concrete forms are obtained by considering particular gauge link paths. An important choice is the staple-shaped gauge link path, the geometry of which was already discussed in detail in Sec. 4.1.2, with the legs of the staple described by a four-vector v; this contains also the straight gauge link path in the limit $v = 0$. Given this concrete choice, a more explicit form of the intrinsic twist-three contributions can be derived.

To establish notation, consider a staple-shaped gauge link $\mathcal{U}$ connecting the space-time points y and y' via three straight segments,

$$\begin{aligned}\mathcal{U} &= \mathcal{P}\exp\left(-ig\int_{y}^{y+v} dx^\mu A_\mu(x)\right)\mathcal{P}\exp\left(-ig\int_{y+v}^{y'+v} dx^\mu A_\mu(x)\right)\\ &\quad\cdot\mathcal{P}\exp\left(-ig\int_{y'+v}^{y'} dx^\mu A_\mu(x)\right) \equiv U_1(0,1)U_2(0,1)U_3(0,1)\ ,\end{aligned} \tag{4.64}$$

which each can be parametrized in terms of a real parameter t as

$$U_1(a,b) = \mathcal{P}\exp\left(-ig\int_a^b dt\, v^\mu A_\mu(y+tv)\right) \tag{4.65}$$

$$U_2(a,b) = \mathcal{P}\exp\left(-ig\int_a^b dt\, (y'-y)^\mu A_\mu(y+v+t(y'-y))\right) \tag{4.66}$$

$$U_3(a,b) = \mathcal{P}\exp\left(-ig\int_a^b dt\, (-v^\mu) A_\mu(y'+v-tv)\right) \tag{4.67}$$

As noted above, the four-vector v describes the legs of the staple-shaped path. The parameterization includes the special case $v = 0$, in which the staple degenerates to a straight link between y and y' given by $U_2(0,1)$, whereas $U_1 = U_3 = 1$. In the following, U_i given without an argument means $U_i \equiv U_i(0,1)$.

As shown in Section 4.5, with this parameterization, one arrives at the explicit expression

$$\begin{aligned}\left(\frac{\partial}{\partial y^\nu} - igA_\nu(y)\right)\mathcal{U} \quad &= \\ & igU_1\int_0^1 ds\, U_2(0,s)(y'-y)^\mu F_{\mu\nu}(y+v+s(y'-y))(1-s)U_2(s,1)U_3 \\ & +ig\int_0^1 ds\, U_1(0,s)v^\mu F_{\mu\nu}(y+sv)U_1(s,1)U_2U_3 \qquad (4.68)\end{aligned}$$

in which only field strength terms remain. In complete analogy, one also obtains for the adjoint term,

$$\begin{aligned}\mathcal{U}\left(\frac{\overleftarrow{\partial}}{\partial y'^\nu} + iA_\nu(y')\right) \quad &= \\ & igU_1\int_0^1 ds\, U_2(0,s)(y'-y)^\mu F_{\mu\nu}(y+v+s(y'-y))sU_2(s,1)U_3 \\ & -U_1U_2ig\int_0^1 ds\, U_3(0,s)v^\mu F_{\mu\nu}(y'+v-sv)U_3(s,1) \quad , \qquad (4.69)\end{aligned}$$

where in each integral, s parametrizes the position of the color field strength insertion along the gauge link connecting the quark positions. These forms are still completely general. In the following, in particular the k_T-integral of the genuine twist-three terms will be of interest, in which case, cf. (4.63b,4.63b), the transverse separation z_T is set to zero and z has only a minus component, $z = (0, z^-, 0, 0)$. Specializing furthermore to the case where also v has only a minus component, $v = (0, v^-, 0, 0)$, cf. the discussion in Sec.4.1.2, the staple legs collapse onto a common axis. In this case we define $U(x, x')$ to denote a straight Wilson line connecting the locations x

and x', and obtain, upon identifying the endpoints $y = -z/2$ and $y' = z/2$,

$$
\begin{aligned}
&(\overrightarrow{\partial\!\!\!/} - ig A\!\!\!/)\mathcal{U}\Big|_{-z/2} = \\
&igz^- \int_0^1 ds\,(1-s) \\
&\qquad \cdot U(-z/2, -z/2+v+sz)\gamma_\mu F^{+\mu}(-z/2+v+sz)U(-z/2+v+sz, z/2) \\
&+igv^- \int_0^1 ds\, U(-z/2, -z/2+sv)\gamma_\mu F^{+\mu}(-z/2+sv)U(-z/2+sv, z/2) \qquad (4.70) \\
&\mathcal{U}(\overleftarrow{\partial\!\!\!/} + ig A\!\!\!/)\Big|_{z/2} = \\
&igz^- \int_0^1 ds\, s \\
&\qquad \cdot U(-z/2, -z/2+v+sz)\gamma_\mu F^{+\mu}(-z/2+v+sz)U(-z/2+v+sz, z/2) \\
&-igv^- \int_0^1 ds\, U(-z/2, z/2+sv)\gamma_\mu F^{+\mu}(z/2+sv)U(z/2+sv, z/2) \qquad (4.71)
\end{aligned}
$$

Note that, in both expressions, the first line stems from the variation of the Wilson line which connects the ends of the staple legs, whereas the second line stems from the variation of the staple leg attached to the endpoint with respect to which the derivative is taken. The straight gauge link case is obtained by setting $v = 0$, i.e., only the first lines in (4.70) and (4.71) remain. This limit was already given in [7].

Particularly compact expressions are obtained if one further integrates over the longitudinal momentum fraction x, in which case $z = 0$ altogether, cf. (4.63b,4.63b). For $z = 0$, the first lines in (4.70) and (4.71) vanish, *i.e.*, the genuine twist-three terms integrate to zero for a straight gauge link. On the other hand, in the general staple link case, the second lines remain, and give identical contributions up to a relative minus sign. Combining with the Dirac structure Γ and assembling the complete genuine twist-three expressions, one has in the completely integrated limit,

$$\int dx \int d^2k_T\, \mathcal{M}^{i,S}_{\Lambda'\Lambda} = \\ i\epsilon^{ij} g v^- \frac{1}{2P^+} \int_0^1 ds\, \langle p', \Lambda' | \bar{\psi}(0)\gamma^+ U(0,sv) F^{+j}(sv) U(sv,0)\psi(0)|p,\Lambda\rangle \tag{4.72}$$

$$\int dx \int d^2k_T\, \mathcal{M}^{i,A}_{\Lambda'\Lambda} = \\ -g v^- \frac{1}{2P^+} \int_0^1 ds\, \langle p', \Lambda' | \bar{\psi}(0)\gamma^+ \gamma^5 U(0,sv) F^{+i}(sv) U(sv,0)\psi(0)|p,\Lambda\rangle \tag{4.73}$$

Note that the ϵ^{ij} in (4.72) can be absorbed into the dual field strength $\widetilde{F}^{+i} = -\epsilon^{ij} F^{+j}$, useful for the analysis within instanton models [44], in which $\widetilde{F} = \pm F$. On the other hand, compact expressions also for the second Mellin moments result if one specializes to the straight link case. A weighting by a factor x can be generated by taking a derivative with respect to z^-, cf. (4.63b,4.63b); in the limit $z = 0$, only the contributions from the derivative acting on either of the z^- prefactors in the first lines of (4.70) and (4.71) remain. Thus, one arrives at

$$\int dx\, x \int d^2k_T\, \mathcal{M}^{i,S}_{\Lambda'\Lambda} = \frac{ig}{4(P^+)^2} \langle p', \Lambda' | \bar{\psi}(0)\gamma^+\gamma^5 F^{+i}(0)\psi(0)|p,\Lambda\rangle \tag{4.74}$$

$$\int dx\, x \int d^2k_T\, \mathcal{M}^{i,A}_{\Lambda'\Lambda} = \frac{g}{4(P^+)^2} \epsilon^{ij} \langle p', \Lambda' | \bar{\psi}(0)\gamma^+ F^{+j}(0)\psi(0)|p,\Lambda\rangle \tag{4.75}$$

for straight gauge links. Note that one can obtain, *e.g.*, the right-hand side of (4.75) by evaluating the v^--derivative of (4.72) at $v^- = 0$, and multiplying by a factor $-i/(2P^+)$. In other words, we uncover a connection between straight-link quark-gluon-quark correlators such as (4.75) and v^--derivatives of Qiu-Sterman type terms such as (4.72), where the latter can be accessed using Lattice QCD TMD data [53], such as given in [54; 55; 56].

4.2.3 EoM Relations involving Orbital Angular Momentum

Altogether, Eqs. (4.62a) and (4.62b) generate 32 individual relations between GT-MDs, obtained by inserting the parameterizations (4.20),(4.22),(4.26),(4.28): each of the two relations is a two-component equation in the transverse plane; furthermore, the resulting 4 individual component relations are complex, *i.e.*, each comprises a relation for the real (T-even) and the imaginary (T-odd) parts of GTMDs. The resulting 8 relations finally each contain 4 possible helicity combinations, as discussed in Section 4.1.6, for the proton helicity conserving, Eqs.(4.46), and the helicity flip, Eqs.(4.49) combinations, respectively. We refrain from quoting all 32 of these relations. They can be specialized to the $\Delta = 0$ TMD limit and to the k_T-integrated GPD limit. In the TMD limit, a number of known TMD relations [23] is reproduced, including explicit expressions for the genuine twist-3 parts in terms of quark-gluon-quark correlators, encoded in the $\mathcal{M}^{i,S}_{\Lambda'\Lambda}$ and $\mathcal{M}^{i,A}_{\Lambda'\Lambda}$ terms. For the k_T-integrated case, we focus on purely transverse momentum transfer, *i.e.*, vanishing skewness, $\xi = 0$. In this case, there are potentially 8 relations: Of the original 32, 16 are ξ-odd, and of course only the T-even relations are relevant for the GPD limit. Among these 8 relations, we discuss in detail three which involve exclusively k_T^2 moments of GTMDs and GPDs. These three are moreover singled out by the fact that they are also accompanied by three corresponding LIRs.

In this section, we present, in particular, the EoM relations describing the quark OAM and spin-orbit contributions. These involve F_{14}, which is obtained for the helicity configuration (4.46b) describing an unpolarized quark in a longitudinally polarized proton, and a relation for G_{11}, obtained for the helicity configuration (4.46d), describing a longitudinally polarized quark in an unpolarized proton. These configurations are obtained by taking the helicity combinations $(\Lambda'\Lambda) = (++) \pm (--)$, in Eqs. (4.62a) and (4.62b), respectively. The relations we obtain constitute x-dependent identities tying the definitions, respectively, of partonic OAM, L_z, and the longitudinal contribution to the spin-orbit coupling $L \cdot S$, to directly observable twist-three distributions. We present the third EoM relation, which instead involves transverse polarization, in Section 4.4. As we show below, after taking $\Delta^+ = 0$ (without loss of generality in the angular momentum sum rule), we obtain the following EoM relations from Eqs. (4.62a) and (4.62b), respectively,

$$x\widetilde{E}_{2T}(x) = -\widetilde{H}(x) + F_{14}^{(1)}(x) - \mathcal{M}_{F_{14}} \tag{4.76}$$

$$x\left[2\widetilde{H}'_{2T}(x) + E'_{2T}(x)\right] = -H(x) + \frac{m}{M}(2\widetilde{H}_T(x) + E_T(x)) - G_{11}^{(1)}(x) - \mathcal{M}_{G_{11}} \tag{4.77}$$

where we defined

$$X^{(1)} = 2\int d^2k_T \frac{k_T^2}{M^2}\frac{k_T^2\Delta_T^2 - (k_T\cdot\Delta_T)^2}{k_T^2\Delta_T^2} X(x,0,k_T^2,k_T\cdot\Delta_T,\Delta_T^2) \tag{4.78}$$

Note that, in the forward limit, this reduces to the standard k_T^2-moment,

$$X^{(1)}\big|_{\Delta_T=0} = \int d^2k_T \frac{k_T^2}{M^2} X(x,0,k_T^2,0,0) \tag{4.79}$$

The genuine twist three contributions are defined as[5]

$$\mathcal{M}_{F_{14}}(x) = \int d^2k_T \frac{\Delta^i}{\Delta_T^2} \left(\mathcal{M}^{i,S}_{++} - \mathcal{M}^{i,S}_{--} \right) \tag{4.80}$$

$$\mathcal{M}_{G_{11}}(x) = \int d^2k_T i\epsilon^{ij} \frac{\Delta^j}{\Delta_T^2} \left(\mathcal{M}^{i,A}_{++} + \mathcal{M}^{i,A}_{--} \right) \quad , \tag{4.81}$$

where the expressions for $\mathcal{M}^{i,S(A)}_{\Lambda\Lambda'}$ given in Eqs.(4.63b,4.63b,4.72,4.73), can be interpreted as quantifying the quark-gluon-quark interaction experienced by a quark of specific x, in the given helicity configuration.

Eqs.(4.76, 4.77) are the equation of motion relations involving the OAM and the longitudinal part of the spin-orbit $L \cdot S$ distributions, defined through F_{14} and G_{11}, in Eqs. (4.47) and (4.48), respectively. They are particularly important among the various GTMD EoM relations that we can write because they allow us to define observables other than the GTMDs to measure the OAM distribution in the proton.

All of the distributions in the EoM relations are defined according to the scheme of Ref. [17] (see Section 4.2): H and $\widetilde{H}$ are twist two GPDs, in the vector and axial vector sector respectively; $\widetilde{E}_{2T}$ is a twist three GPD in the vector sector, $\widetilde{H}'_{2T}$ and E'_{2T} are axial vector twist three GPDs.

Eq. (4.76) relates an *intrinsic* [20] twist three GPD, $\widetilde{E}_{2T}$, on the *lhs* [7], to a twist two GPD, $\widetilde{H}$, the k_T-moment of the GTMD, F_{14}, Eq.(4.78), and a *genuine* twist three term, $\mathcal{M}_{F_{14}}$. It is obtained by contracting Eq. (4.62a) with Δ^i/Δ_T^2, forming the $(\Lambda'\Lambda) = (++) - (--)$ combination of helicity components, and inserting the GTMD

[5] Note that in [7], the first of these relations was quoted with an erroneous additional normalization factor $2M$.

parameterizations of the correlators, yielding

$$0 = -2x\left(\frac{k_T\cdot\Delta_T}{\Delta_T^2}F_{27}+F_{28}\right)+G_{14}-2\frac{k_T^2\Delta_T^2-(k_T\cdot\Delta_T)^2}{M^2\Delta_T^2}F_{14}+\frac{\Delta^i}{\Delta_T^2}\left(\mathcal{M}^{i,S}_{++}-\mathcal{M}^{i,S}_{--}\right) \tag{4.82}$$

Integrating over k_T and identifying the resulting GPDs [17] gives

$$0 = x\widetilde{E}_{2T}+\widetilde{H}-F_{14}^{(1)}+\int d^2k_T\frac{\Delta^i}{\Delta_T^2}\left(\mathcal{M}^{i,S}_{++}-\mathcal{M}^{i,S}_{--}\right), \tag{4.83}$$

i.e., one obtains Eq. (4.76). Recalling the discussion in Sec. 4.1.2, in the case of a staple-shaped gauge link, this requires that the legs of the staple properly collapse upon k_T-integration such as to produce GPDs with straight gauge link.

Eq. (4.77) was derived in a similar way. It relates the twist-three GPD combination, $2\widetilde{H}'_{2T}(x)+E'_{2T}(x)$, to the GPD H, the k_T-moment of the GTMD, G_{11}, which describes the longitudinal part of the parton spin-orbit distribution, and a *genuine* twist three term. Notice the appearance of a quark mass term proportional to the GPD $2\widetilde{H}_T+E_T$ in the chiral odd sector [57]. Contracting Eq. (4.62b) with $i\epsilon^{ij}\Delta^j/\Delta_T^2$, forming the $(\Lambda'\Lambda)=(++)+(--)$ combination of helicity components, cf. Eq. (4.46d), and inserting the GTMD parameterizations of the correlators yields

$$\begin{aligned} 0 &= 2x\left(\frac{k_T\cdot\Delta_T}{\Delta_T^2}G_{21}+G_{22}\right)+F_{11}+2\frac{k_T^2\Delta_T^2-(k_T\cdot\Delta_T)^2}{M^2\Delta_T^2}G_{11} \\ &- 2\frac{m}{M}\left(\frac{k_T\cdot\Delta_T}{\Delta_T^2}H_{11}+H_{12}\right)+i\epsilon^{ij}\frac{\Delta^j}{\Delta_T^2}\left(\mathcal{M}^{i,A}_{++}+\mathcal{M}^{i,A}_{--}\right) \end{aligned} \tag{4.84}$$

Integrating over k_T and identifying the resulting GPDs gives

$$\begin{aligned} 0 &= x\left(2\widetilde{H}'_{2T} + E'_{2T}\right) + H + G_{11}^{(1)} - \frac{m}{M}\left(2\widetilde{H}_T + E_T\right) \\ &+ \int d^2k_T i\epsilon^{ij}\frac{\Delta^j}{\Delta_T^2}\left(\mathcal{M}_{++}^{i,A} + \mathcal{M}_{--}^{i,A}\right) \end{aligned} \tag{4.85}$$

i.e., Eq. (4.77).

4.3 Generalized Lorentz Invariance Relations

The underlying Lorentz structure of the unintegrated correlator, Eqs. (4.16,4.17) allows one to find relations between the x-dependent k_T-moments of GTMDs and GPDs. As stated before, this is due to the fact that, for the straight gauge link case, the total number of GPCFs is less than the number of GTMDs. Similar relations connecting the various TMDs, in the forward limit, were derived in Refs.[13; 14]. These equations are a consequence of the covariant definition of the correlation function, and they are therefore referred to as Lorentz Invariance Relations (LIRs).

The following LIRs, which we derive further below, involve the k_T-moments of the GTMDs respectively describing the OAM and longitudinal spin-orbit terms which also enter the EoMs derived in Section 4.2, Eqs.(4.76,4.77),

$$\frac{dF_{14}^{(1)}}{dx} = \widetilde{E}_{2T} + H + E \quad \Rightarrow \quad F_{14}^{(1)} = -\int_x^1 dy\left[\widetilde{E}_{2T} + H + E\right] \tag{4.86}$$

$$\frac{dG_{11}^{(1)}}{dx} = -\left(E'_{2T} + 2\widetilde{H}'_{2T} + \widetilde{H}\right) \quad \Rightarrow \quad G_{11}^{(1)} = \int_x^1 dy\left[2\widetilde{H}'_{2T} + E'_{2T} + \widetilde{H}\right] \tag{4.87}$$

On the left hand side, we have k_T^2-moments of twist two GTMDs. These GTMDs are unique in that, in the limit $t = 0$, they carry the physical meaning of parton longitudinal OAM distribution, $F_{14}^{(1)}$, and longitudinal parton spin-orbit distribution, $G_{11}^{(1)}$. On the right hand side, the integral expressions for the intrinsic twist three GPDs $\widetilde{E}_{2T} + H + E$, and $2\widetilde{H}'_{2T} + E'_{2T} + \widetilde{H}$, allow us to access both OAM and the longitudinal spin-orbit term directly from deeply virtual exclusive measurements as these GPDs enter as coefficients of specific azimuthal angular modulations of the cross section. Note that these x-dependent relations are valid also for $\Delta_T \neq 0$.

Note also that, at variance with previous work [58; 44; 36; 49], Eq (4.86) allows us to obtain directly information on the OAM distribution because its form is not integrated in x (it occurs at the k_T-integrated level). Eq.(4.87) is new: it allows us to connect the longitudinal spin-orbit x-distribution, $G_{11}^{(1)}$, to a specific twist three GPD combination, $2\widetilde{H}'_{2T} + E'_{2T}$ that uniquely appears in off-forward processes.

If we were to work with a staple gauge link, the number of GPCFs would increase to 16 for both the vector and the axial vector case. In this scenario, since the number of GTMDs is the same as the number of GPCFs, we do not expect there to be any LIRs connecting exclusively GTMDs (or their GPD limits). Indeed, if we do try to write these relations, we find that extra terms appear that consist of GPCFs that cannot be combined to form either GPDs or GTMDs. These extra terms, which are required in order to properly encode Lorentz invariance in the relations, have been termed LIR breaking terms [43]. For example, (4.86) is modified to read

$$\frac{dF_{14}^{(1)}}{dx} = \widetilde{E}_{2T} + H + E + \mathcal{A}_{F_{14}} \ , \tag{4.88}$$

with,

$$\mathcal{A}_{F_{14}} = v^- \frac{(2P^+)^2}{M^2} \int d^2k_T \int dk^- \left[\frac{k_T \cdot \Delta_T}{\Delta_T^2}(A_{11}^F + xA_{12}^F) \right.$$
$$\left. + \quad A_{14}^F + \frac{k_T^2 \Delta_T^2 - (k_T \cdot \Delta_T)^2}{\Delta_T^2} \left(\frac{\partial A_8^F}{\partial(k \cdot v)} + x \frac{\partial A_9^F}{\partial(k \cdot v)} \right) \right] \tag{4.89}$$

where the 4-vector $v = (0, v^-, 0, 0)$ describes the direction of the staple, which here is taken to extend along the light cone. The amplitudes A_i^F are the ones appearing in the parameterization given in [17], appropriate for a staple link structure, up to a rescaling stemming from the fact that the staple vector v used here and the analogous vector N used in [17] are related by a rescaling. Note that, if one were to take v off the light cone, $v = (v^+, v^-, 0, 0)$, cf. the discussion in Sec. 4.1.2, additional terms would appear in (4.89) that formally vanish as $v^+ \to 0$; examples of such terms in the case of TMD LIRs have been given in [43]. Of course, the GPCFs themselves then also depend on v^+.

In what follows, we work with straight gauge links, where terms such as (4.89) are absent; in Sec. 4.3.3, we return to the staple link case and obtain a concrete expression for (4.89) in terms of quark-gluon-quark correlators, by combining the LIR with the corresponding EoM.

4.3.1 Construction of Lorentz invariance relations

The general structure of the unintegrated correlation function was written in terms of all the independent Lorentz structures multiplied by scalar functions, A_i^F, A_i^G in Section 4.1.3. The correlation function integrated in k^- (and k_T dependent) was parametrized in terms of GTMDs in Ref.[17] (Section 4.1.4). GTMDs can, therefore, be expressed through k^- integrals of the scalar functions A_i^F, A_i^G. These expressions are given in Section 4.7. As was shown in Section 4.1.3, the total number of independent functions in the unintegrated corrrelator is 8 for the vector and 8 for the axial vector sectors. The total number of twist two plus twist three GTMDs is 12 vector and 12 axial vector [17]. Since this number exceeds the number of A_i^F, A_i^G functions, the GTMDs will be related to one another. This type of relation that is just originating from the parameterization in terms of Lorentz covariant structures is called a LIR.

In the following, we describe the procedure used to derive LIRs between the k_T-moments of the twist two GTMDs listed in Section 4.1.4, Eqs.(4.46b, 4.46d), and the twist three GPDs listed in Section 4.1.5. It is based on the following integral relation for amplitudes A depending on the integration variable k via Lorentz invariants as $A \equiv A(k \cdot P, k^2, k \cdot \Delta)$,

$$\begin{aligned} &\frac{d}{dx} \int d^2k_T \int dk^- \frac{k_T^2 \Delta_T^2 - (k_T \cdot \Delta_T)^2}{\Delta_T^2} \; \mathcal{X}[A;x] \;\; = \\ &\int d^2k_T \int dk^- (k \cdot P - xP^2) \mathcal{X}[A;x] + \int d^2k_T \int dk^- \frac{k_T^2 \Delta_T^2 - (k_T \cdot \Delta_T)^2}{\Delta_T^2} \frac{\partial \mathcal{X}}{\partial x}[A;x] \end{aligned} \tag{4.90}$$

where $\mathcal{X}[A;x]$ is a linear combination of amplitudes A in which the coefficients, aside from containing the invariants $k \cdot P$, k^2 and $k \cdot \Delta$, may have an explicit x-dependence. This is an off-forward extension of relations used previously in the analysis of TMD LIRs [43; 14; 59; 18]; here, the presence of the additional invariant $k \cdot \Delta$ must be properly accounted for. In view of this complication, it is worth laying out the elements of the derivation of (4.90); this is presented in Section 4.6.

4.3.2 Relating k_T^2 moments of GTMDs to GPDs

Since a generic GTMD X can be expressed in the form $X = \int dk^- \mathcal{X}[A; x]$, as given in Section 4.7, one can use (4.90) to cast the x-derivative of its k_T-moment, $X^{(1)}$, cf. (4.78), in terms of A amplitudes. In particular,

$$\begin{aligned}\frac{d}{dx}F_{14}^{(1)} &= \\ &\frac{4P^+}{M^2}\int d^2k_T \int dk^- \left[(k\cdot P - xP^2)(A_8^F + xA_9^F) + \frac{k_T^2\Delta_T^2 - (k_T\cdot\Delta_T)^2}{\Delta_T^2}A_9^F\right] \qquad (4.91)\\ \frac{d}{dx}G_{11}^{(1)} &= \\ &\frac{4P^+}{M^2}\int d^2k_T \int dk^- \left[(k\cdot P - xP^2)\left(A_1^G + \frac{A_{18}^G + xA_{19}^G}{2}\right) + \frac{k_T^2\Delta_T^2 - (k_T\cdot\Delta_T)^2}{\Delta_T^2}\frac{A_{19}^G}{2}\right] \qquad (4.92)\end{aligned}$$

To complete the LIRs, one constructs the appropriate combinations of GPDs which yield the right-hand sides. The relevant combinations, cf. Section 4.7, are[6]

$$\begin{aligned}H + E &= \\ &2P^+\int d^2k_T \int dk^-\ 2\left(\frac{k_T\cdot\Delta_T}{\Delta_T^2}A_5^F + A_6^F + \frac{P\cdot k - xP^2}{M^2}(A_8^F + xA_9^F)\right) \qquad (4.93)\\ \widetilde{E}_{2T} &= \\ &2P^+\int d^2k_T \int dk^-\ (-2)\left(\frac{k_T\cdot\Delta_T}{\Delta_T^2}A_5^F + A_6^F + \frac{(k_T\cdot\Delta_T)^2 - k_T^2\Delta_T^2}{M^2\Delta_T^2}A_9^F\right) \qquad (4.94)\end{aligned}$$

[6]Note that one could equally quote the left-hand sides of (4.93)-(4.96) in terms of k_T-integrals of GTMDs instead of quoting directly their GPD limits, cf. Section 4.7. This would facilitate a consistent regularization of the obtained LIRs at the level of k_T-integrals of GTMDs.

$$\widetilde{H} = 2P^+ \int d^2k_T \int dk^- \left(-A_{17}^G + \frac{xP^2 - k \cdot P}{M^2} (A_{18}^G + xA_{19}^G) \right) \tag{4.95}$$

$$E'_{2T} + 2\widetilde{H}'_{2T} = 2P^+ \int d^2k_T \int dk^- \left(2\,\frac{xP^2 - k \cdot P}{M^2} A_1^G + A_{17}^G + \frac{(k_T \cdot \Delta_T)^2 - k_T^2 \Delta_T^2}{M^2 \Delta_T^2} A_{19}^G \right) \tag{4.96}$$

To construct the appropriate combinations completing the LIRs, we examine the expression for the proton helicity combination associated with the GTMD appearing on the left hand side of (4.91), (4.92), and find the twist-three GPDs corresponding to that same helicity structure. The GPCF substructure of the twist-three GPDs need not, in general, completely match the GPCF combination of the x-derivative of the k_T^2 moment of the GTMD. One may need to add a twist-two GPD with the appropriate GPCF substructure.

In particular, F_{14} describes an unpolarized quark in a longitudinally polarized proton, Eq.(4.46b) for $\Gamma = \gamma^+$; the twist-three GPD with a similar proton helicity combination is $\widetilde{E}_{2T}$. Comparing their GPCF decompositions, we see that if we add $H + E$, we arrive at the LIR,

$$\frac{dF_{14}^{(1)}}{dx} = \widetilde{E}_{2T} + H + E. \tag{4.97}$$

Similarly, G_{11} describes a longitudinally polarized quark in an unpolarized proton, Eq.(4.46d) with $\Gamma = \gamma^+\gamma^5$. The corresponding twist-three combination with the same proton helicity combination is $2\widetilde{H}'_{2T}+E'_{2T}$, with their GPCF substructure given above.

By adding the GPD $\widetilde{H}$, this gives us,

$$\frac{dG_{11}^{(1)}}{dx} = -\left(2\widetilde{H}_{2T}' + E_{2T}'\right) - \widetilde{H}. \tag{4.98}$$

As already noted further above, in the case of a staple link, these Lorentz Invariance Relations acquire LIR violating terms that we introduce as, cf. (4.88),

$$\frac{dF_{14}^{(1)}}{dx} = \widetilde{E}_{2T} + H + E + \mathcal{A}_{F_{14}} \tag{4.99}$$

$$\frac{dG_{11}^{(1)}}{dx} = -\left(2\widetilde{H}_{2T}' + E_{2T}'\right) - \widetilde{H} + \mathcal{A}_{G_{11}} \tag{4.100}$$

These relations are a central result of our paper: they give a connection valid point by point in the kinematical variables x and $t = -\Delta_T^2$ among the k_T moments of GTMDs that define dynamically OAM and longitudinal spin-orbit coupling, specific twist three GPDs, and LIR violating terms that can be expressed in terms of genuine twist three contributions; the latter connection will be elucidated using the example of $\mathcal{A}_{F_{14}}$, cf. (4.99), in the next section.

4.3.3 Intrinsic twist three contributions

Lorentz invariance relations (LIRs) derived in the presence of a staple-shaped gauge link generally include additional terms beyond those found for straight gauge links, as exemplified by (4.88),(4.89) in comparison to (4.86). Whereas the staple LIR by itself does not yield the concrete physical content of these terms, considering it in the context of the straight-link LIR as well as staple and straight link EoMs provides more detailed insight into their meaning. To illustrate this, it is useful to pursue the case of the LIRs (4.86) and (4.88), relevant for the description of quark orbital angular momentum in the nucleon, further. Subtracting the former LIR from the latter yields

$$\begin{aligned}
\mathcal{A}_{F_{14}}(x) &\equiv v^{-}\frac{(2P^{+})^2}{M^2}\int d^2k_T\int dk^{-}\left[\frac{k_T\cdot\Delta_T}{\Delta_T^2}(A_{11}+xA_{12})+A_{14}\right.\\
&+\left.\frac{k_T^2\Delta_T^2-(k_T\cdot\Delta_T)^2}{\Delta_T^2}\left(\frac{\partial A_8}{\partial(k\cdot v)}+x\frac{\partial A_9}{\partial(k\cdot v)}\right)\right]\\
&= \frac{dF_{14}^{(1)}}{dx}-\left.\frac{dF_{14}^{(1)}}{dx}\right|_{v=0} \qquad (4.101)
\end{aligned}$$

giving a concrete expression for $\mathcal{A}_{F_{14}}$ in terms of the GTMD F_{14}. Note that, here, the discussion given in Sec. 4.1.2 should be kept in mind: Formulating the LIRs (as well as the EoMs below) in terms of GPDs assumes that, in the staple-link case, the legs of the staples have properly collapsed upon k_T-integration such as to produce GPDs with their straight gauge link structures. This requires the staple link vector v to lie on the light cone, $v=(0,v^{-},0,0)$. Corrections to the above relation would arise from several sources if one were to take the staple vector v off the light cone, $v=(v^{+},v^{-},0,0)$. On the one hand, the cancellation between the straight and staple link GTMD precursors of the GPDs in (4.86) and (4.88) would be incomplete; there would be residual terms corresponding to the difference between the two cases (unless one opts for the alternative quasi-GPD scheme also mentioned in Sec. 4.1.2). On the other hand, as already noted in connection with eq. (4.89), additional amplitudes would enter the GPCF expression.

Now, the difference of GTMD k_T-moments in (4.101) can also be extracted from the EoMs: subject again to the above caveats, the GPD terms in the EoM (4.76) are identical for a straight link and a staple link, and subtracting an instance of (4.76) with a straight link from an instance with a staple link yields

$$F_{14}^{(1)} - F_{14}^{(1)}\Big|_{v=0} = \mathcal{M}_{F_{14}} - \mathcal{M}_{F_{14}}|_{v=0} \tag{4.102}$$

Thus, the additional terms in the staple LIR (4.88) are associated with quark-gluon-quark correlations,

$$\mathcal{A}_{F_{14}}(x) = \frac{d}{dx}\left(\mathcal{M}_{F_{14}} - \mathcal{M}_{F_{14}}|_{v=0}\right) \tag{4.103}$$

Therefore, we see that, comparing the genuine twist-three terms entering the staple link LIR and the staple link EoM, these encode independent information: the EoM contains $\mathcal{M}_{F_{14}}$ alone, whereas the LIR contains the difference of $\mathcal{M}_{F_{14}}$ and $\mathcal{M}_{F_{14}}|_{v=0}$.

As was shown in Refs. [35; 37], in the forward limit, and integrated over momentum fraction x, the quantity $-F_{14}^{(1)}$ corresponds to Jaffe-Manohar quark orbital angular momentum in the staple link case, whereas it corresponds to Ji quark orbital angular momentum in the $v = 0$ straight link case. Using (4.72), we obtain a concrete expression for the difference between the two,

$$\begin{aligned} -\int dx \left(F_{14}^{(1)} - F_{14}^{(1)}\Big|_{v=0}\right)\Bigg|_{\Delta_T=0} &= \\ &- \frac{\partial}{\partial\Delta^i} i\epsilon^{ij} g v^- \frac{1}{2P^+}\int_0^1 ds\, \langle p', +|\bar{\psi}(0)\gamma^+ U(0,sv) F^{+j}(sv) U(sv,0)\psi(0)|p,+\rangle\Bigg|_{\Delta_T=0}, \end{aligned} \tag{4.104}$$

where it has been used that $2(\Delta^i/\Delta_T^2)f^i = (\partial/\partial\Delta^i)f^i$ in the limit $\Delta_T \to 0$ for a vector function f which vanishes at least linearly in that limit (this is clear if one decomposes f using Δ_T, $f^i = \Delta^i f^{||} + \epsilon^{ij}\Delta^j f^{\perp}$); note that the function on which the Δ_T-derivative acts in (4.104) satisfies this requirement since the left-hand side is regular at $\Delta_T = 0$. In deriving (4.104), it has furthermore been used that, once one is considering Δ_T-derivatives, the $(++)$ and $(--)$ helicity combinations contribute equally to quark orbital angular momentum. Eq. (4.104) can be interpreted in terms of the accumulated torque experienced by the struck quark in a deep inelastic scattering process as a result of final state interactions [37]. The genuine twist-three term $\mathcal{A}_{F_{14}}(x)$ entering the staple link LIR thus rather directly encodes information about this torque, via repeated integration in x. Eq. (4.104) reproduces[7] the expression for the torque given in [37].

Analogous considerations apply to the staple link version of the other LIR derived in section 4.3.2. For the spin-orbit sum rule, one has

$$\mathcal{A}_{G_{11}} = \frac{d}{dx}\left(G_{11}^{(1)} - G_{11}^{(1)}\Big|_{v=0}\right) = -\frac{d}{dx}\left(\mathcal{M}_{G_{11}} - \mathcal{M}_{G_{11}}|_{v=0}\right) \tag{4.105}$$

and in the completely integrated, forward limit,

$$\begin{aligned} \int dx \left(G_{11}^{(1)} - G_{11}^{(1)}\Big|_{v=0}\right)\Big|_{\Delta_T=0} &= \\ -\frac{\partial}{\partial\Delta^i} i\epsilon^{ij} g v^- \frac{1}{2P^+}\int_0^1 ds\, \langle p', +|\bar{\psi}(0)\gamma^+\gamma^5 U(0,sv)F^{+j}(sv)U(sv,0)\psi(0)|p,+\rangle\Big|_{\Delta_T=0} & \end{aligned} \tag{4.106}$$

This term is analogous to Eq. (4.104), the only difference being in $\gamma^+ \to \gamma^+\gamma^5$.

[7] To see the correspondence, it is useful to reinstate into the expression given in [37] a small momentum transfer, and to translate the matrix element such that the quark operators are located at the origin, as they are in (4.104). Taking into account the resulting phases stemming from the proton states, one can then identify $\int d^3r\, r^i \exp(i\vec{\Delta}\vec{r}) = i(2\pi)^3\delta^3(\Delta)\partial/\partial\Delta^i$. In view of the standard normalization of states $\langle p' + |p+\rangle = 2P^+(2\pi)^3\delta^3(p'-p)$, the correspondence becomes evident.

4.3.4 Eliminating GTMD moments from LIR and EoM relations

We now merge the information from the LIR, Eqs.(4.99, 4.100), and EoM relations Eqs. (4.76,4.77) such as to eliminate the GTMD moments. By eliminating $F_{14}^{(1)}$ between Eqs.(4.99) and (4.76), and $G_{11}^{(1)}$ between Eqs. (4.100) and (4.77), respectively, we obtain relations involving only twist two and twist three GPDs including their corresponding genuine twist terms. Considering again separately the vector and axial vector cases one has,

$$\begin{aligned} \widetilde{E}_{2T} &= \\ -\int_x^1 \frac{dy}{y}(H+E) - &\left[\frac{\widetilde{H}}{x} - \int_x^1 \frac{dy}{y^2}\widetilde{H}\right] - \left[\frac{1}{x}\mathcal{M}_{F_{14}} - \int_x^1 \frac{dy}{y^2}\mathcal{M}_{F_{14}}\right] - \int_x^1 \frac{dy}{y}\mathcal{A}_{F_{14}} \end{aligned} \tag{4.107}$$

$$\begin{aligned} 2\widetilde{H}'_{2T} + E'_{2T} &= \\ &-\int_x^1 \frac{dy}{y}\widetilde{H} - \left[\frac{H}{x} - \int_x^1 \frac{dy}{y^2}H\right] + \frac{m}{M}\left[\frac{1}{x}(2\widetilde{H}_T + E_T) - \int_x^1 \frac{dy}{y^2}(2\widetilde{H}_T + E_T)\right] \\ &\qquad - \left[\frac{1}{x}\mathcal{M}_{G_{11}} - \int_x^1 \frac{dy}{y^2}\mathcal{M}_{G_{11}}\right] + \int_x^1 \frac{dy}{y}\mathcal{A}_{G_{11}} \end{aligned} \tag{4.108}$$

These relations are valid for either a staple or a straight gauge link structure (with staple vector v on the light cone in the former case), keeping in mind that $\mathcal{A}_{F_{14}} \equiv 0$ and $\mathcal{A}_{G_{11}} \equiv 0$ in the straight-link case. Since the GPDs in these relations by definition are identical in the staple and straight link cases, subtracting a straight-link instance of (4.107) from a staple-link instance again yields the relation (4.103) between quark-gluon-quark terms (upon differentiation with respect to x), and one likewise obtains the analogous relation for $\mathcal{A}_{G_{11}}$. A converse way of stating this is that the terms containing $\mathcal{M}_{F_{14}}$ and $\mathcal{A}_{F_{14}}$ always conspire such that only a straight-link quark-gluon-quark contribution remains, even if (4.107) is formally written for the staple-link case; the same is true for (4.108).

If one disregards the quark-gluon-quark contributions, and the quark mass term in Eq. (4.108), one obtains generalizations of the relation derived by Wandzura and Wilczek (WW) in Ref. [15],

$$\widetilde{E}_{2T}^{WW} = -\int_x^1 \frac{dy}{y}(H+E) - \left[\frac{\widetilde{H}}{x} - \int_x^1 \frac{dy}{y^2}\widetilde{H}\right] \tag{4.109}$$

$$(2\widetilde{H}'_{2T} + E'_{2T})^{WW} = -\int_x^1 \frac{dy}{y}\widetilde{H} - \left[\frac{H}{x} - \int_x^1 \frac{dy}{y^2}H\right] \tag{4.110}$$

isolating the twist-two components of $\widetilde{E}_{2T}$ and $(2\widetilde{H}'_{2T} + E'_{2T})$. We can then re-express Eqs.(4.107, 4.108) as,

$$\widetilde{E}_{2T} = \widetilde{E}_{2T}^{WW} + \widetilde{E}_{2T}^{(3)} + \widetilde{E}_{2T}^{LIR} \tag{4.111}$$

$$\overline{E}'_{2T} = \overline{E}'^{WW}_{2T} + \overline{E}'^{(3)}_{2T} + \overline{E}'^{LIR}_{2T} + \overline{E}'^{m}_{2T} \tag{4.112}$$

where we defined

$$\overline{E}'_{2T} = 2\widetilde{H}'_{2T} + E'_{2T} \ .$$

Here, $\widetilde{E}_{2T}^{LIR}$ and $\overline{E}'^{LIR}_{2T}$ are the LIR violating terms containing $\mathcal{A}_{F_{14}}$ and $\mathcal{A}_{G_{11}}$, respectively, $\widetilde{E}_{2T}^{(3)}$ and $\overline{E}'^{(3)}_{2T}$ are the genuine twist three terms containing $\mathcal{M}_{F_{14}}$ and $\mathcal{M}_{G_{11}}$, and $\overline{E}'^{m}_{2T}$ is the quark mass dependent term.

4.3.5 x^0, x and x^2 Moments

We now consider the x moments for the twist three GPDs entering Eqs.(4.107,4.108). Integral relations for twist three GPDs were first obtained in Ref.[44; 58] directly from the OPE while in this paper we derive them by integrating the x-dependent expressions found from the LIR and EoM.[8] It is therefore important to check how the two approaches correspond to one another. For the vector case we have,

$$\int dx \widetilde{E}_{2T} = -\int dx(H+E) \qquad \Rightarrow \int dx \left(\widetilde{E}_{2T} + H + E\right) = 0 \tag{4.113a}$$

$$\int dx x \widetilde{E}_{2T} = -\frac{1}{2}\int dx x(H+E) - \frac{1}{2}\int dx \widetilde{H} \tag{4.113b}$$

$$\int dx x^2 \widetilde{E}_{2T} = -\frac{1}{3}\int dx x^2(H+E) - \frac{2}{3}\int dx x\widetilde{H} - \frac{2}{3}\int dx\, x\, \mathcal{M}_{F_{14}}\Big|_{v=0} \tag{4.113c}$$

where one can see that the contributions from $\mathcal{A}_{F_{14}}$ and $\mathcal{M}_{F_{14}}$ cancel in the first two expressions integrating by parts; it is assumed that the integrands are sufficiently well behaved at the boundaries for all such integrations. Notice that Eq.(4.113a) is an extension of the Burkhardt-Cottingham sum rule to the off-forward case. Eq.(4.113b), taken in the forward limit, is a sum rule for Ji quark angular momentum,

$$J_q^{Ji} = \frac{1}{2}\Delta\Sigma_q + L_q^{Ji} \tag{4.114}$$

as can be seen by identifying the terms,

$$J_q^{Ji} = \frac{1}{2}\int dx\, x(H+E), \quad \Delta\Sigma_q = \int dx \widetilde{H}, \quad L_q^{Ji} = \int dx\, x(\widetilde{E}_{2T} + H + E)\ . \tag{4.115}$$

Finally, Eq.(4.113c) is the only one containing a genuine twist three contribution. It should be noticed that this contribution was surmised to be the same for all helicity configurations in Ref.[44], while here we see that they are distinct terms.

[8] Notice the notation difference between Refs.[58; 44; 36] and the classification scheme followed in this paper [17]: $\int dx\, xG_2 = -\int dx\, x(\widetilde{E}_{2T} + H + E)$.

In order to gauge the size of the OAM component, one can use data on the twist two GPDs contributing to the WW definition, and simultaneously extract the twist three GPDs. Detailed comparisons between the two sets of measurements will allow us to constrain this quantity.

The axial vector moments are given by,

$$\int dx \left(E'_{2T} + 2\widetilde{H}'_{2T}\right) = -\int dx \widetilde{H} \qquad \Rightarrow \int dx \left(E'_{2T} + 2\widetilde{H}'_{2T} + \widetilde{H}\right) = 0 \tag{4.116a}$$

$$\int dxx \left(E'_{2T} + 2\widetilde{H}'_{2T}\right) = -\frac{1}{2}\int dxx\widetilde{H} - \frac{1}{2}\int dxH + \frac{m}{2M}\int dx(E_T + 2\widetilde{H}_T) \tag{4.116b}$$

$$\int dx\, x^2 \left(E'_{2T} + 2\widetilde{H}'_{2T}\right) = -\frac{1}{3}\int dxx^2\widetilde{H} - \frac{2}{3}\int dxxH + \frac{2m}{3M}\int dxx(E_T + 2\widetilde{H}_T) - \frac{2}{3}\int dxx\mathcal{M}_{G_{11}}\Big|_{v=0} \tag{4.116c}$$

Eqs.(4.116a,4.116b,4.116c) are also consistent with those found in Ref.[44] and revisited in Ref.[60]. In particular, Eq.(4.116a) is an extension of the Burkhardt-Cottingham sum rule to the off-forward case. Similarly to the vector case, the various terms in Eq.(4.116b) can be rearranged so as to single out the second moment of a twist-three GPD, namely the combination $2\widetilde{H}'_{2T} + E'_{2T} + \widetilde{H}$, which in the forward limit can be interpreted through the LIR in Eq. (4.87) as the longitudinal contribution to the parton spin-orbit interaction $(L_z S_z)_q$, cf. (4.48),

$$2(L_z S_z)_q = \int dxx \left(E'_{2T} + 2\widetilde{H}'_{2T} + \widetilde{H}\right). \tag{4.117}$$

One then has, in the forward limit,

$$\frac{1}{2}\int dx x\widetilde{H} + \frac{m_q}{2M}\kappa_T^q = \int dx x(2\widetilde{H}'_{2T} + E'_{2T} + \widetilde{H}) + \frac{1}{2}e_q \tag{4.118}$$

corresponding to the sum rule

$$2(J_z S_z)_q = 2(L_z S_z)_q + 2(S_z S_z)_q \tag{4.119}$$

where the transverse anomalous magnetic moment, κ_T^q, and the quark number, e_q,

$$\kappa_T^q = \int dx\,(E_T + 2\widetilde{H}_T)\;, \qquad e_q = \int dx\, H \tag{4.120}$$

have been defined.

The quark mass-dependent term which appears in Eq.(4.116b), technically through the equations of motion, is due to transverse angular momentum components that are present for non-zero quark mass. Note that this term is chiral even, being given by the product of two chiral-odd quantities. We thus find the following partitioning of the terms representing total angular momentum,

$$2(J_z S_z)_q \equiv 2[(J\cdot S)_q - (J_T\cdot S_T)_q] = \frac{1}{2}\int dx x\widetilde{H} + \frac{m_q}{2M}\kappa_T^q\;. \tag{4.121}$$

In the chiral limit, only the longitudinal polarization component is available to the quarks, and the correlation $(J_z S_z)$ is then quantified correctly by helicity-weighting the correlator yielding J_z, cf. (4.115), which converts $H+E$ into $\widetilde{H}$. No contribution from $\widetilde{E}$ appears due to time reversal invariance. In the presence of a non-zero quark mass, this is modified by the transverse anomalous magnetic moment term, which accounts for the fact that also transverse polarization components are available to massive quarks. Note that one does not have to polarize the proton to observe these correlations between quark spin and angular momentum.

4.4 LIR and EoM relations involving transverse spin configurations

The main results of this paper are given by the EoM relations in Eqs.(4.76,4.77), the LIR relations in Eqs.(4.86,4.87), and the WW relations in Section 4.3, which were obtained for longitudinal proton polarization at $\xi = 0$. Most of the LIRs [20], however, including the original ones [14; 18], were originally derived for the proton helicity flip case, or for transversely polarized proton configurations. It is therefore interesting to study the extension to the off-forward case for these helicity configurations. We obtain the following EoM result for the axial-vector GTMD,

$$\frac{1}{2}G_{12}^{(1)} = x\left[H'_{2T} - \frac{\Delta_T^2}{4M^2}E'_{2T}\right] - \frac{\Delta_T^2}{4M^2}(H+E) - \frac{m}{M}\left[H_T - \frac{\Delta_T^2}{4M^2}E_T\right] - \mathcal{M}_{G_{12}}, \quad (4.122)$$

where the genuine twist-three term

$$\mathcal{M}_{G_{12}} = -\int d^2k_T i\epsilon^{ij}\frac{\Delta^j}{\Delta_T^2}\left[\frac{\Delta^1 + i\Delta^2}{2M}\mathcal{M}_{+-}^{i,A} + \frac{-\Delta^1 + i\Delta^2}{2M}\mathcal{M}_{-+}^{i,A} - \frac{\Delta_T^2}{4M^2}\mathcal{M}_{++}^{i,A} - \frac{\Delta_T^2}{4M^2}\mathcal{M}_{--}^{i,A}\right] \quad (4.123)$$

has been defined.

Our derivation proceeds in analogy to the steps used in the longitudinally polarized case, with a few important differences, as follows. Multiplying the $(\Lambda'\Lambda) = (+-)$ component of (4.62b) with $(\Delta^1 + i\Delta^2)$ and the $(\Lambda'\Lambda) = (-+)$ component of (4.62b) with $(\Delta^1 - i\Delta^2)$, subtracting these two component equations and contracting with $i\epsilon^{ij}\Delta^j/(2M\Delta_T^2)$ yields, upon inserting the parameterizations in terms of GTMDs,

$$\begin{aligned}
0 \;=\;& \frac{k_T\cdot\Delta_T}{2M^2}F_{12} + \frac{\Delta_T^2}{2M^2}\left(F_{13} - \frac{F_{11}}{2}\right) + \frac{k_T^2\Delta_T^2 - (k_T\cdot\Delta_T)^2}{M^2\Delta_T^2}\left(G_{12} - \frac{\Delta_T^2}{2M^2}G_{11}\right) \\
& -x\left(\frac{k_T\cdot\Delta_T}{2M^2}G_{21} + \frac{\Delta_T^2}{2M^2}G_{22} + G_{23} + \frac{k_T^2\Delta_T^2 - (k_T\cdot\Delta_T)^2}{M^2\Delta_T^2}G_{24}\right) \\
& +\frac{m}{M}\left(\frac{k_T\cdot\Delta_T}{2M^2}H_{11} + \frac{\Delta_T^2}{2M^2}H_{12} + H_{13} + \frac{k_T^2\Delta_T^2 - (k_T\cdot\Delta_T)^2}{M^2\Delta_T^2}H_{14}\right) \\
& -\frac{i\epsilon^{ij}\Delta^j}{2M\Delta_T^2}\left((\Delta^1 + i\Delta^2)\mathcal{M}^{i,A}_{+-} + (-\Delta^1 + i\Delta^2)\mathcal{M}^{i,A}_{-+}\right)
\end{aligned} \tag{4.124}$$

and, upon integration with respect to k_T and identifying the corresponding GPDs,

$$\begin{aligned}
0 \;=\;& \frac{\Delta_T^2}{4M^2}E + \frac{1}{2}G_{12}^{(1)} - \frac{\Delta_T^2}{4M^2}G_{11}^{(1)} - x\left(H'_{2T} + \frac{\Delta_T^2}{2M^2}\widetilde{H}'_{2T}\right) + \frac{m}{M}\left(H_T + \frac{\Delta_T^2}{2M^2}\widetilde{H}_T\right) \\
& -\frac{i\epsilon^{ij}\Delta^j}{2M\Delta_T^2}\int d^2k_T\left((\Delta^1 + i\Delta^2)\mathcal{M}^{i,A}_{+-} + (-\Delta^1 + i\Delta^2)\mathcal{M}^{i,A}_{-+}\right)
\end{aligned} \tag{4.125}$$

Finally, eliminating $G_{11}^{(1)}$ using Eq. (4.77), we obtain Eq. (4.122).

Eq. (4.122) is a direct, off-forward GPD extension of the well-known relation involving the polarized twist three PDF g_T, the k_T-moment of the TMD g_{1T} and transversity, h_1 [14], as can be seen by identifying, in the forward limit, $H'_{2T} \to g_T$, $H_T \to h_1$, and $G_{12} \to g_{1T}$,

$$0 = \frac{1}{2}g_{1T}^{(1)}(x) - xg_T(x) + \frac{m}{M}h_1(x) + x\widetilde{g}_T(x) \tag{4.126}$$

Note that our definition of the k_T-moment $X^{(1)}$ differs from the one in Refs. [43; 14] by a factor of 2. The intrinsic twist-three contribution can be given explicitly in terms of quark-gluon-quark correlators as,

$$x\widetilde{g}_T(x) = \frac{1}{4M}\int d^2k_T \left(\mathcal{M}^{1,A}_{+-} + i\mathcal{M}^{2,A}_{+-} + \mathcal{M}^{1,A}_{-+} - i\mathcal{M}^{2,A}_{-+}\right)\Big|_{\Delta_T=0} = \mathcal{M}_{G_{12}}|_{\Delta_T=0} \tag{4.127}$$

This simplified form for $\mathcal{M}_{G_{12}}$ in the $\Delta_T = 0$ limit is obtained from (4.123) by considering approaches to the $\Delta_T = 0$ limit along both the Δ^1 and the Δ^2 axes.

The EoM relation (4.122) is accompanied by a corresponding LIR, which one obtains in complete analogy to the treatment in Sec. 4.3.2 by considering the appropriate decompositions into amplitudes A^G, cf. Section 4.7. For straight gauge links, one has

$$\begin{aligned} \frac{d}{dx}G_{12}^{(1)} &= \frac{4P^+}{M^2}\int d^2k_T \int dk^- \frac{P^2}{M^2}\Big[(k\cdot P - xP^2)(A_{18}^G + xA_{19}^G) \\ &+ \frac{k_T^2\Delta_T^2 - (k_T\cdot\Delta_T)^2}{\Delta_T^2}A_{19}^G\Big] \end{aligned} \tag{4.128}$$

as well as

$$\widetilde{H} = 2P^+\int d^2k_T\int dk^-\left(-A_{17}^G + \frac{xP^2 - k\cdot P}{M^2}(A_{18}^G + xA_{19}^G)\right) \tag{4.129}$$

$$H'_{2T} - \frac{\Delta_T^2}{4M^2}E'_{2T} = 2P^+\int d^2k_T\int dk^-\frac{P^2}{M^2}\left(-A_{17}^G - \frac{(k_T\cdot\Delta_T)^2 - k_T^2\Delta_T^2}{M^2\Delta_T^2}A_{19}^G\right) \tag{4.130}$$

leading to the LIR

$$\frac{1}{2}\frac{dG_{12}^{(1)}}{dx} = H'_{2T} - \frac{\Delta_T^2}{4M^2}E'_{2T} - \left(1 + \frac{\Delta_T^2}{4M^2}\right)\widetilde{H} + \mathcal{A}_{G_{12}} \tag{4.131}$$

where $\mathcal{A}_{G_{12}} \equiv 0$ in the straight-link case, but in the presence of a staple link with staple direction v on the light cone, cf. the analogous discussion in Sec. 4.3.3, one has the genuine twist-three contribution

$$\mathcal{A}_{G_{12}} = -\frac{d}{dx}\left(\mathcal{M}_{G_{12}} - \mathcal{M}_{G_{12}}|_{v=0}\right) \tag{4.132}$$

This LIR is likewise an off-forward generalization of a well-known structure function relation; setting $\Delta_T = 0$ in (4.131) directly yields

$$\frac{1}{2}\frac{d}{dx}g_{1T}^{(1)}(x) = g_T(x) - g_1(x) - \widehat{g}_T(x) \tag{4.133}$$

in which the genuine twist-three contribution is given in terms of quark-gluon-quark correlators as

$$\widehat{g}_T(x) = \left.\frac{d}{dx}\left(\mathcal{M}_{G_{12}} - \mathcal{M}_{G_{12}}|_{v=0}\right)\right|_{\Delta_T=0} \tag{4.134}$$

By using the same techniques as in Section 4.3, eliminating the term containing G_{12}, we obtain the following relation,

$$\begin{aligned} H'_{2T} - \frac{\Delta_T^2}{4M^2}E'_{2T} \;\; = & \;\; \left(1 + \frac{\Delta_T^2}{4M^2}\right)\int_x^1 \frac{dy}{y}\widetilde{H} + \frac{m}{M}\left[\frac{1}{x}\left(H_T - \frac{\Delta_T^2}{4M^2}E_T\right)\right. \\ & \left. - \int_x^1 \frac{dy}{y^2}\left(H_T - \frac{\Delta_T^2}{4M^2}E_T\right)\right] + \frac{\Delta_T^2}{4M^2}\left[\frac{1}{x}(H+E) - \int_x^1 \frac{dy}{y^2}(H+E)\right] \\ & + \left[\frac{\mathcal{M}_{G_{12}}}{x} - \int_x^1 \frac{dy}{y^2}\mathcal{M}_{G_{12}}\right] - \int_x^1 \frac{dy}{y}\mathcal{A}_{G_{12}}. \end{aligned} \tag{4.135}$$

Notice that this relation reduces in the forward limit to the one for the polarized structure functions, g_1 and g_T [43], namely, taking $H'_{2T} \to g_T = g_1 + g_2$, $\widetilde{H} \to g_1$, and

$H_T \to h_1$, as well as taking into account (4.127), (4.132) and (4.134),

$$g_T = \int_x^1 \frac{dy}{y} g_1 + \frac{m}{M}\left(\frac{1}{x}h_1 - \int_x^1 \frac{dy}{y^2} h_1\right) + \left(\widetilde{g}_T - \int_x^1 \frac{dy}{y}\widetilde{g}_T\right) + \int_x^1 \frac{dy}{y}\widehat{g}_T \quad (4.136)$$

Taking moments of (4.135) in x, one obtains,

$$\int dx \left(H'_{2T} - \frac{\Delta_T^2}{4M^2}E'_{2T}\right) = \left(1 + \frac{\Delta_T^2}{4M^2}\right)\int dx \widetilde{H} \overset{\Delta_T \to 0}{\Rightarrow} \int dx \left(H'_{2T} - \widetilde{H}\right) \equiv \int dx\, g_2 = 0 \quad (4.137a)$$

$$\int dx\, x \left(H'_{2T} - \frac{\Delta_T^2}{4M^2}E'_{2T}\right) = \frac{1}{2}\left(1 + \frac{\Delta_T^2}{4M^2}\right)\int dx x \widetilde{H} + \frac{\Delta_T^2}{8M^2}\int dx (H+E) + \frac{m}{2M}\int dx \left(H_T - \frac{\Delta_T^2}{4M^2}E_T\right) \quad (4.137b)$$

$$\int dx\, x^2 \left(H'_{2T} - \frac{\Delta_T^2}{4M^2}E'_{2T}\right) = \frac{1}{3}\left(1 + \frac{\Delta_T^2}{4M^2}\right)\int dx x^2 \widetilde{H} + \frac{\Delta_T^2}{6M^2}\int dx x (H+E) + \frac{2m}{3M}\int dx x \left(H_T - \frac{\Delta_T^2}{4M^2}E_T\right) + \frac{2}{3}\int dx\, x\, \mathcal{M}_{G_{12}}\Big|_{v=0} . \quad (4.137c)$$

Eq.(4.137a) is the off-forward generalization of the original Burkhardt-Cottingham sum rule; similarly, Eq.(4.137b) is the generalization of the Efremov-Leader-Teryaev sum rule [61],

$$\int dx\, x \left[g_T(x) - \frac{1}{2}g_1(x)\right] = 0 \quad , \quad (4.138)$$

which is valid in the chiral limit, $m \to 0$, whereas Eq.(4.137c) in the forward and chiral limits reduces to[9],

$$\int dx\, x^2 \left[g_T(x) - \frac{1}{3}g_1(x)\right] = \frac{2}{3}\int dx\, x\, \mathcal{M}_{G_{12}}\Big|_{v=0,\Delta_T=0} = \frac{1}{3}d_2 \quad . \quad (4.139)$$

[9]Note that definitions of d_2 in the literature vary by a factor of 2.

d_2, which incorporates quark-gluon-quark correlations, is one of the few quantities where these effects can be obtained unambiguously from inclusive polarized scattering experiments ([62] and references therein). One can write explicitly the helicity structure of $\mathcal{M}_{G_{12}}$ as,

$$d_2 = \frac{1}{2M}\int dx\, x \int d^2k_T\, \left(\mathcal{M}^{1,A}_{+-} + i\mathcal{M}^{2,A}_{+-} + \mathcal{M}^{1,A}_{-+} - i\mathcal{M}^{2,A}_{-+}\right)\Big|_{v=0,\Delta_T=0} \tag{4.140}$$

A relation involving the GTMD F_{12}, the imaginary part of which in the forward limit is (minus) the Sivers function $f_{1T}^{\perp}$ [17], is obtained by considering the combination $(\Delta^1 - i\Delta^2)$ times the $(\Lambda'\Lambda) = (-+)$ helicity component added to $(\Delta^1 + i\Delta^2)$ times the $(\Lambda'\Lambda) = (+-)$ helicity component of Eq. (4.62a) and multiplying by $\Delta^i/M\Delta_T^2$,

$$\begin{aligned} - &\quad x\left(F_{23} + \frac{k_T^2\Delta_T^2 - (k_T\cdot\Delta_T)^2}{M^2\Delta_T^2}F_{24}\right) + \frac{1}{2M^2}\left(\Delta_T^2 G_{13} + k_T\cdot\Delta_T G_{12}\right) \\ + &\quad \frac{k_T^2\Delta_T^2 - (k_T\cdot\Delta_T)^2}{M^2\Delta_T^2}F_{12} + \frac{\Delta^i}{2M\Delta_T^2}\left((\Delta^1 - i\Delta^2)\mathcal{M}^{i,S}_{-+} + (\Delta^1 + i\Delta^2)\mathcal{M}^{i,S}_{+-}\right) = 0. \end{aligned} \tag{4.141}$$

In the forward limit, Eq. (4.141) is a relation between purely imaginary quantities. This follows from the fact that, for $\Delta = 0$, the real parts of all GTMDs entering Eq. (4.141) except for G_{12} vanish [17]; on the other hand, G_{12} is multiplied by Δ_T. As a consequence, also the real part of the genuine twist three term vanishes for $\Delta = 0$. Turning therefore to the imaginary part, in the forward limit, one has the TMD identifications $F_{23}^o = f_T'$ and $F_{24}^o = f_T^{\perp}$ [17]. Integrated over k_T, the term in the first parenthesis thus combines to $\int d^2k_T f_T = 0$ [23]. One is therefore left with the following k_T-integrated relation involving the Sivers function $f_{1T}^{\perp}$ in the forward limit,

$$f_{1T}^{\perp(1)} = -F_{12}^{o(1)} = \mathcal{M}_{F_{12}}|_{\Delta_T=0} \tag{4.142}$$

where the quark-gluon-quark term

$$\mathcal{M}_{F_{12}} = -2i\frac{\Delta^i}{2M\Delta_T^2}\int d^2k_T \left((\Delta^1 - i\Delta^2)\mathcal{M}_{-+}^{i,S} + (\Delta^1 + i\Delta^2)\mathcal{M}_{+-}^{i,S}\right) \tag{4.143}$$

has been introduced. Eq. (4.142) indicates a correspondence of $\mathcal{M}_{F_{12}}$ to the well-known Qiu-Sterman term $T_q(x,x)$ in the forward limit. Indeed, in analogy to the discussion surrounding Eq. (4.104), a compact expression for $\mathcal{M}_{F_{12}}$ can be obtained in the fully integrated case. Approaching the $\Delta_T = 0$ limit either along the $\Delta^2 = 0$ axis or the $\Delta^1 = 0$ axis, one has

$$\mathcal{M}_{F_{12}}|_{\Delta_T=0} = -i\frac{1}{M}\int d^2k_T(\mathcal{M}_{+-}^{1,S} + \mathcal{M}_{-+}^{1,S}) = \frac{1}{M}\int d^2k_T(\mathcal{M}_{+-}^{2,S} - \mathcal{M}_{-+}^{2,S}) \tag{4.144}$$

Considering, for example, the form given in terms of $\mathcal{M}_{\Lambda'\Lambda}^{1,S}$, and integrating with respect to x, one can insert (4.72) and obtain the Sivers shift

$$\begin{aligned} \langle k_2 \rangle &= M\frac{1}{2}\int dx f_{1T}^{\perp(1)} \\ &= gv^- \frac{1}{2P^+}\int_0^1 ds \langle P, S_1|\bar{\psi}(0)\gamma^+ U(0,sv)F^{+2}(sv)U(sv,0)\psi(0)|P, S_1\rangle \end{aligned} \tag{4.145}$$

after having converted the states from the helicity basis to a spin quantization axis in 1-direction, and having used a rotation by π in the transverse plane to combine terms associated with spin in the ± 1-directions. The case of spin in the 2-direction can be treated analogously. One thus obtains the standard Qiu-Sterman form [37] in the forward limit. For $\Delta_T \neq 0$, $\mathcal{M}_{12}$ is an off-forward/generalized analogue of the Qiu-Sterman $T_q(x,x)$ term.

The EoM relations presented so far in either the longitudinal or transverse proton polarization cases allow us to decompose specific twist-three GPDs into a linear combination of a twist-two GPD, a quark-gluon-quark correlation, the k_T^2 moment of a twist-two GTMD and a mass term in the axial-vector case. The k_T^2 moment of the GTMD can be eliminated using the LIRs. The resulting relations, when integrated over x, are analogous to the relations provided by Kiptily and Polyakov in [44]. Note, however, that not all EoM relations are of this form; in general, also other GTMD moments besides k_T^2 moments appear in the EoM relations that we have not discussed in detail in this work. For instance, one finds a relation in which the GTMD G_{13} contributes to the EoM weighted by $(k_T \cdot \Delta_T)$. Moreover, that EoM relation cannot have a LIR counterpart within the twist two and twist three sectors, since G_{13} is the only GTMD in those sectors which contains the invariant amplitude A_{21}^G, cf. Section 4.7.

4.5 Explicit form of quark-gluon-quark terms

Consider a staple-shaped gauge link $\mathcal{U}$ connecting the space-time points y and y' via three straight segments,

$$\begin{aligned}
\mathcal{U} \;&=\; \mathcal{P}\exp\left(-ig\int_{y}^{y+v} dx^\mu A_\mu(x)\right)\mathcal{P}\exp\left(-ig\int_{y+v}^{y'+v} dx^\mu A_\mu(x)\right) \\
&\quad \cdot \mathcal{P}\exp\left(-ig\int_{y'+v}^{y'} dx^\mu A_\mu(x)\right) && (4.146) \\
&\equiv\; U_1(0,1)U_2(0,1)U_3(0,1)\ , && (4.147)
\end{aligned}$$

which each can be parametrized in terms of a real parameter t as

$$U_1(a,b) = \mathcal{P}\exp\left(-ig\int_a^b dt\, v^\mu A_\mu(y+tv)\right) \tag{4.148}$$

$$U_2(a,b) = \mathcal{P}\exp\left(-ig\int_a^b dt\, (y'-y)^\mu A_\mu(y+v+t(y'-y))\right) \tag{4.149}$$

$$U_3(a,b) = \mathcal{P}\exp\left(-ig\int_a^b dt\, (-v^\mu) A_\mu(y'+v-tv)\right) \tag{4.150}$$

The four-vector v describes the legs of the staple-shaped path. The parameterization includes the special case $v=0$, in which the staple degenerates to a straight link between y and y' given by $U_2(0,1)$, whereas $U_1 = U_3 = 1$. In the following, U_i given without an argument means $U_i \equiv U_i(0,1)$.

The goal of the following treatment is to evaluate

$$\left(\frac{\partial}{\partial y^\nu} - igA_\nu(y)\right)\mathcal{U} = \left(\frac{\partial U_1}{\partial y^\nu} - igA_\nu(y)U_1\right)U_2U_3 + U_1\frac{\partial U_2}{\partial y^\nu}U_3 \tag{4.151}$$

(note that U_3 is independent of y). Consider first $\partial U_1/\partial y^\nu$. The derivative of the path-ordered exponential, cf. (4.148), is

$$\frac{\partial U_1}{\partial y^\nu} = \int_0^1 ds\, U_1(0,s)\left[-igv^\mu \partial_\nu A_\mu(y+sv)\right]U_1(s,1) \tag{4.152}$$

This can be recast by the following integration by parts. Noting that

$$\frac{d}{ds}U_1(0,s) = U_1(0,s)(-ig)v^\mu A_\mu(y+sv) \tag{4.153}$$

$$\frac{d}{ds}U_1(s,1) = igv^\mu A_\mu(y+sv)U_1(s,1) \tag{4.154}$$

it follows that

$$
\begin{aligned}
U_1 igA_\nu(y+v) &- igA_\nu(y)U_1 \\
&= \int_0^1 ds \frac{d}{ds}\left[U_1(0,s)igA_\nu(y+sv)U_1(s,1)\right] \qquad (4.155) \\
&= \int_0^1 ds\, [U_1(0,s)(-ig)v^\mu A_\mu(y+sv)igA_\nu(y+sv)U_1(s,1) \\
&\qquad +U_1(0,s)igA_\nu(y+sv)igv^\mu A_\mu(y+sv)U_1(s,1) \\
&\qquad +U_1(0,s)igv^\mu \partial_\mu A_\nu(y+sv)U_1(s,1)] \\
&= igv^\mu \int_0^1 ds\, U_1(0,s)\left[F_{\mu\nu}(y+sv)+\partial_\nu A_\mu(y+sv)\right]U_1(s,1)
\end{aligned}
$$

having introduced the field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]$. Adding the left- and right-hand sides of (4.152) to the initial and final expressions in (4.155), respectively, as well as subtracting $U_1 igA_\nu(y+v)$ from both sides, finally yields

$$
\left(\frac{\partial}{\partial y^\nu} - igA_\nu(y)\right)U_1 = igv^\mu \int_0^1 ds\, U_1(0,s)F_{\mu\nu}(y+sv)U_1(s,1) - U_1 igA_\nu(y+v) \qquad (4.156)
$$

The term $\partial U_2/\partial y^\nu$ can be treated analogously; the resulting expressions are slightly more involved, since, in this case, also the line element in U_2 depends explicitly on y, cf. (4.149):

$$
\begin{aligned}
\frac{\partial U_2}{\partial y^\nu} &= \\
&\int_0^1 dsU_2(0,s)(-ig)\left[-A_\nu(y+v+s(y'-y))\right. \\
&\qquad \left.+(y'-y)^\mu \partial_\nu A_\mu(y+v+s(y'-y))(1-s)\right]U_2(s,1) \qquad (4.157)
\end{aligned}
$$

Noting, in analogy to above,

$$\begin{aligned}\frac{d}{ds}U_2(0,s) &= U_2(0,s)(-ig)(y'-y)^\mu A_\mu(y+v+s(y'-y)) & (4.158)\\ \frac{d}{ds}U_2(s,1) &= ig(y'-y)^\mu A_\mu(y+v+s(y'-y))U_2(s,1) & (4.159)\end{aligned}$$

one has

$$\begin{aligned}& -igA_\nu(y+v)U_2 \\ = & \int_0^1 ds\frac{d}{ds}\left[U_2(0,s)igA_\nu(y+v+s(y'-y))(1-s)U_2(s,1)\right] \qquad (4.160)\\ = & \int_0^1 ds\left[U_2(0,s)(-ig)(y'-y)^\mu A_\mu(y+v+s(y'-y))\right. \\ & \cdot igA_\nu(y+v+s(y'-y))(1-s)U_2(s,1) \\ + & U_2(0,s)igA_\nu(y+v+s(y'-y))(1-s)ig(y'-y)^\mu A_\mu(y+v+s(y'-y))U_2(s,1) \\ + & U_2(0,s)ig(y'-y)^\mu\partial_\mu A_\nu(y+v+s(y'-y))(1-s)U_2(s,1) \\ - & \left.U_2(0,s)igA_\nu(y+v+s(y'-y))U_2(s,1)\right] \\ = & ig\int_0^1 ds\,U_2(0,s)\left[(1-s)(y'-y)^\mu\left(F_{\mu\nu}(y+v+s(y'-y))\right.\right. \\ + & \left.\left.\partial_\nu A_\mu(y+v+s(y'-y))\right)-A_\nu(y+v+s(y'-y))\right]U_2(s,1)\end{aligned}$$

Adding the left- and right-hand sides of (4.157) to the initial and final expressions in (4.160), respectively, as well as adding $igA_\nu(y+v)U_2$ to both sides, finally leaves

$$\frac{\partial U_2}{\partial y^\nu} = ig\int_0^1 ds\,U_2(0,s)(1-s)(y'-y)^\mu F_{\mu\nu}(y+v+s(y'-y))U_2(s,1)+igA_\nu(y+v)U_2 \qquad (4.161)$$

Inserting (4.161) and (4.156) on the right-hand side of (4.151), one finally obtains an expression in which only field strength terms remain,

$$\begin{aligned}\left(\frac{\partial}{\partial y^\nu} - igA_\nu(y)\right)\mathcal{U} \;\; = & \\ & igU_1 \int_0^1 ds\, U_2(0,s)(y'-y)^\mu F_{\mu\nu}(y+v+s(y'-y))(1-s)U_2(s,1)U_3 \\ & + ig\int_0^1 ds\, U_1(0,s)v^\mu F_{\mu\nu}(y+sv)U_1(s,1)U_2U_3 \end{aligned} \tag{4.162}$$

In complete analogy, one obtains for the adjoint term,

$$\begin{aligned}\mathcal{U}\left(\overleftarrow{\frac{\partial}{\partial y'^\nu}} + iA_\nu(y')\right) \;\; = & \\ & igU_1 \int_0^1 ds\, U_2(0,s)(y'-y)^\mu F_{\mu\nu}(y+v+s(y'-y))sU_2(s,1)U_3 \\ & -U_1U_2 ig\int_0^1 ds\, U_3(0,s)v^\mu F_{\mu\nu}(y'+v-sv)U_3(s,1) \end{aligned} \tag{4.163}$$

4.6 Integral relation for the construction of LIRs

The construction of LIRs is based on the integral relation (4.90) for amplitudes A depending on the integration variable k via Lorentz invariants as $A \equiv A(k\cdot P, k^2, k\cdot\Delta)$,

$$\begin{aligned}\frac{d}{dx}\int d^2k_T \int dk^- \frac{k_T^2\Delta_T^2 - (k_T\cdot\Delta_T)^2}{\Delta_T^2}\;\mathcal{X}[A;x] \;\; = & \\ \int d^2k_T\int dk^-(k\cdot P - xP^2)\mathcal{X}[A;x] + \int d^2k_T \int dk^- & \frac{k_T^2\Delta_T^2 - (k_T\cdot\Delta_T)^2}{\Delta_T^2}\frac{\partial\mathcal{X}}{\partial x}[A;x] \end{aligned} \tag{4.164}$$

where $\mathcal{X}[A;x]$ is a linear combination of amplitudes A in which the coefficients, aside from containing the invariants $k\cdot P$, k^2 and $k\cdot\Delta$, may have an explicit x-dependence.

To see this relation, it is useful to handle the dependences of the amplitudes A on the invariants $k \cdot P$, k^2 and $k \cdot \Delta$ by introducing new variables embodying these invariants,

$$\sigma \equiv 2k \cdot P = xP^2 + 2k^- P^+ \Rightarrow k^- = \frac{1}{2P^+}\left(\sigma - xP^2\right) \tag{4.165}$$

$$\tau \equiv k^2 = x\sigma - x^2 P^2 - k_T^2 \tag{4.166}$$

$$\sigma' \equiv k \cdot \Delta = -k_T \cdot \Delta_T = -|k_T||\Delta_T| \cos\phi \tag{4.167}$$

Note again that the present treatment is for vanishing skewness, in which case $P^2 = M^2 + \Delta_T^2/4$. Examining the two terms on the right-hand side of (4.164), they take the form

$$\begin{aligned} I_1 &= \int d^2k_T \int dk^-(k \cdot P - xP^2)\mathcal{X}[A(2k \cdot P, k^2, k \cdot \Delta); x] \qquad (4.168) \\ &= \frac{1}{8P^+} \int d\sigma d\tau d\sigma' \int_0^\infty dk_T^2 \int_0^{2\pi} d\phi\, \delta(\tau - x\sigma + x^2P^2 + k_T^2)\delta(\sigma' + k_T \cdot \Delta_T) \\ &\qquad \cdot(\sigma - 2xP^2)\mathcal{X}[A(\sigma, \tau, \sigma'); x] \\ &= \frac{1}{4P^+} \int d\sigma d\tau d\sigma'\, \theta(x\sigma - \tau - x^2P^2)\theta(\Delta_T^2(x\sigma - \tau - x^2P^2) - \sigma'^2). \\ &\qquad \cdot \frac{\sigma - 2xP^2}{\sqrt{\Delta_T^2(x\sigma - \tau - x^2P^2) - \sigma'^2}}\mathcal{X}[A(\sigma, \tau, \sigma'); x] \end{aligned}$$

$$\begin{aligned} I_2 &= \int d^2k_T \int dk^- \frac{k_T^2\Delta_T^2 - (k_T \cdot \Delta_T)^2}{\Delta_T^2} \frac{\partial \mathcal{X}}{\partial x}[A(2k \cdot P, k^2, k \cdot \Delta); x] \qquad (4.169) \\ &= \frac{1}{4P^+} \int d\sigma d\tau d\sigma' \int_0^\infty dk_T^2 \int_0^{2\pi} d\phi\, \delta(\tau - x\sigma + x^2P^2 + k_T^2)\delta(\sigma' + k_T \cdot \Delta_T) \\ &\qquad \cdot k_T^2 \sin^2\phi \frac{\partial \mathcal{X}}{\partial x}[A(\sigma, \tau, \sigma'); x] \\ &= \frac{1}{2P^+} \int d\sigma d\tau d\sigma'\, \theta(x\sigma - \tau - x^2P^2)\theta(\Delta_T^2(x\sigma - \tau - x^2P^2) - \sigma'^2) \\ &\qquad \cdot \sqrt{\frac{x\sigma - \tau - x^2P^2}{\Delta_T^2} - \frac{\sigma'^2}{\Delta_T^4}} \frac{\partial \mathcal{X}}{\partial x}[A(\sigma, \tau, \sigma'); x] \end{aligned}$$

where in each case, in the first step, the integration variable k^- has been substituted by σ according to (4.165), and two representations of unity have been introduced enforcing the identifications (4.166) and (4.167); also the k_T-integration has been cast in polar coordinates. In the second step, the angular integrations have been carried out using

$$\int_0^{2\pi} d\phi\, \delta(\sigma' + k_T \cdot \Delta_T) \sin^2\phi = \frac{2}{k_T^2 \Delta_T^2} \sqrt{k_T^2\Delta_T^2 - \sigma'^2}\, \theta(k_T^2\Delta_T^2 - \sigma'^2) \quad (4.170)$$

$$\int_0^{2\pi} d\phi\, \delta(\sigma' + k_T \cdot \Delta_T) = \frac{2}{\sqrt{k_T^2\Delta_T^2 - \sigma'^2}}\, \theta(k_T^2\Delta_T^2 - \sigma'^2) \quad (4.171)$$

followed by the integration over k_T^2. Consider now the left-hand side of (4.164). It is of the same form as (4.169), except for containing $\mathcal{X}$ instead of $\partial\mathcal{X}/\partial x$, and for the overall derivative with respect to x. Thus, in view of the last line of (4.169), it reads

$$\begin{aligned} I &= \frac{d}{dx}\frac{1}{2P^+}\int d\sigma d\tau d\sigma'\, \theta(x\sigma - \tau - x^2P^2)\theta(\Delta_T^2(x\sigma - \tau - x^2P^2) - \sigma'^2) \\ &\qquad \cdot\sqrt{\frac{x\sigma - \tau - x^2P^2}{\Delta_T^2} - \frac{\sigma'^2}{\Delta_T^4}}\mathcal{X}[A;x] \quad (4.172) \\ &= \frac{1}{2P^+}\int d\sigma d\tau d\sigma'\, \delta(x\sigma - \tau - x^2P^2)\theta(\Delta_T^2(x\sigma - \tau - x^2P^2) - \sigma'^2) \\ &\qquad \cdot(\sigma - 2xP^2)\sqrt{\frac{x\sigma - \tau - x^2P^2}{\Delta_T^2} - \frac{\sigma'^2}{\Delta_T^4}}\mathcal{X}[A;x] \\ &\quad + \frac{1}{2P^+}\int d\sigma d\tau d\sigma'\, \theta(x\sigma - \tau - x^2P^2)\delta(\Delta_T^2(x\sigma - \tau - x^2P^2) - \sigma'^2) \\ &\qquad \cdot(\sigma - 2xP^2)\sqrt{\Delta_T^2(x\sigma - \tau - x^2P^2) - \sigma'^2}\mathcal{X}[A;x] \\ &\quad + \frac{1}{2P^+}\int d\sigma d\tau d\sigma'\, \theta(x\sigma - \tau - x^2P^2)\theta(\Delta_T^2(x\sigma - \tau - x^2P^2) - \sigma'^2) \\ &\quad \cdot\left[\frac{1}{2}\frac{\sigma - 2xP^2}{\sqrt{\Delta_T^2(x\sigma - \tau - x^2P^2) - \sigma'^2}}\mathcal{X}[A;x] + \sqrt{\frac{x\sigma - \tau - x^2P^2}{\Delta_T^2} - \frac{\sigma'^2}{\Delta_T^4}}\frac{\partial\mathcal{X}}{\partial x}[A;x]\right] \\ &\qquad (4.173) \end{aligned}$$

The last term corresponds to $I_1 + I_2$; to see (4.164), it thus remains to argue that the first two lines in (4.173) yield no contribution. In the first term, the δ-function sets $x\sigma - \tau - x^2P^2 = 0$, and therefore the rest of the integrand is proportional to $\theta(-\sigma'^2)\sqrt{-\sigma'^2}$, which vanishes for any σ'. In the second line in (4.173), the δ-function sets the quantity in the square root to zero, $\Delta_T^2(x\sigma - \tau - x^2P^2) - \sigma'^2 = 0$. It should be emphasized that these properties hinge on the $\sin^2\phi$ weighting of the k_T-integral, cf. (4.169). Without this weighting, it is not clear that the two terms do not contribute, and the LIR could potentially be modified by boundary terms. The possibility of corrections through boundary terms in LIRs not weighted by $\sin^2\phi$ has also been noted in [43]. Note furthermore that no pathology arises in the limit $\Delta_T \to 0$; this limit merely generates $\delta(\sigma')$ distributions in the integrands, as is clear from inspecting (4.170) and (4.171), which are the source of the superficially singular dependences on Δ_T^2. One can retrace the above derivation analogously in the $\Delta_T = 0$ limit, with (4.164) continuing to hold.

4.7 GTMDs in terms of A amplitudes

To relate GTMDs to A amplitudes, one equates the k^- integrals of the GPCF parameterizations (4.16) and (4.17), for $\mu = +$ and $\mu = i$, a transverse vector index, to the corresponding GTMD parameterizations (4.19), (4.21), (4.25), (4.27). Complete correspondence between the structures is achieved by eliminating terms in the GPCF parameterizations containing σ^{i-}. This can be effected using the Gordon identity

$$0 = \overline{U}(p', \Lambda')\left(\frac{\Delta^\mu}{2} + i\sigma^{\mu\nu}P_\nu\right)U(p, \Lambda) \ . \tag{4.174}$$

For purely longitudinal P and transverse Δ, this allows one to substitute

$$\overline{U}(p', \Lambda')i\sigma^{i-}P^+U(p, \Lambda) = \overline{U}(p', \Lambda')\left(-\frac{i\sigma^{i+}P^2}{2P^+} - \frac{\Delta^i}{2}\right)U(p, \Lambda) \tag{4.175}$$

and furthermore implies $\overline{U}\sigma^{+-}U = 0$; moreover, in combination with $i\sigma^{\mu\nu}\gamma^5 = -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\sigma_{\rho\sigma}$ it also yields

$$\overline{U}(p',\Lambda')i\sigma^{i-}\gamma^5 P^+ U(p,\Lambda) = \overline{U}(p',\Lambda')\left(\frac{i\sigma^{i+}\gamma^5 P^2}{2P^+} - i\epsilon^{ij}\frac{\Delta^j}{2}\right)U(p,\Lambda) \ . \qquad (4.176)$$

In addition, it is useful to contract the twist-three equations, which carry a transverse vector index, with the two available transverse vectors k_T and Δ_T in order to extract the full information from the equations. The following relations result:

For the twist-two vector GTMDs as functions of the A^F amplitudes, one obtains:

$$F_{11} = 2P^+ \int dk^- \left[A_1^F + xA_2^F - \frac{x\Delta_T^2}{2M^2}(A_8^F + xA_9^F)\right] \qquad (4.177)$$

$$F_{12} = 2P^+ \int dk^- \left[A_5^F\right] \qquad (4.178)$$

$$F_{13} = 2P^+ \int dk^- \left[A_6^F + \frac{P\cdot k - xP^2}{M^2}(A_8^F + xA_9^F)\right] \qquad (4.179)$$

$$F_{14} = 2P^+ \int dk^- \left[A_8^F + xA_9^F\right] \qquad (4.180)$$

For the twist-three vector GTMDs as functions of the A^F amplitudes:

$$F_{21} = 2P^+ \int dk^- \left[A_2^F - \frac{x\Delta_T^2}{2M^2}A_9^F\right] \quad (4.181)$$

$$F_{22} = 2P^+ \int dk^- \left[A_3^F - \frac{x}{2}A_5^F - \frac{x\Delta_T^2}{2M^2}A_{17}^F\right] \quad (4.182)$$

$$F_{23} = 2P^+ \int dk^- \frac{P\cdot k - xP^2}{M^2}\left[A_5^F + \frac{(k_T\cdot\Delta_T)^2 - k_T^2\Delta_T^2}{M^2(k_T\cdot\Delta_T)}A_9^F\right] \quad (4.183)$$

$$F_{24} = 2P^+ \int dk^- \frac{P\cdot k - xP^2}{M^2}\frac{\Delta_T^2}{k_T\cdot\Delta_T}\left[A_9^F\right] \quad (4.184)$$

$$F_{25} = 2P^+ \int dk^- \frac{xP^2 - P\cdot k}{M^2}\left[A_9^F\right] \quad (4.185)$$

$$F_{26} = 2P^+ \int dk^- \frac{P\cdot k - xP^2}{M^2}\left[\frac{k_T^2}{k_T\cdot\Delta_T}A_9^F + A_{17}^F\right] \quad (4.186)$$

$$F_{27} = 2P^+ \int dk^- \left[A_5^F + \frac{k_T\cdot\Delta_T}{M^2}A_9^F + \frac{\Delta_T^2}{M^2}A_{17}^F\right] \quad (4.187)$$

$$F_{28} = 2P^+ \int dk^- \left[A_6^F - \frac{k_T^2}{M^2}A_9^F - \frac{k_T\cdot\Delta_T}{M^2}A_{17}^F\right] \quad (4.188)$$

For the twist-two axial vector GTMDs as functions of the A^G amplitudes:

$$G_{11} = 2P^+ \int dk^- \left[A_1^G + \frac{A_{18}^G + xA_{19}^G}{2}\right] \quad (4.189)$$

$$G_{12} = 2P^+ \int dk^- \frac{P^2}{M^2}[A_{18}^G + xA_{19}^G] \quad (4.190)$$

$$G_{13} = 2P^+ \int dk^- \frac{P^2}{M^2}\left[A_{21}^G + xA_{22}^G\right] \quad (4.191)$$

$$G_{14} = 2P^+ \int dk^- \left[-A_{17}^G + \frac{xP^2 - k\cdot P}{M^2}(A_{18}^G + xA_{19}^G)\right] \quad (4.192)$$

$$(4.193)$$

For the twist-three axial vector GTMDs as functions of the A^G amplitudes:

$$G_{21} = 2P^+ \int dk^- \left[\frac{k_T \cdot \Delta_T}{2M^2} A^G_{19} + \frac{\Delta_T^2}{2M^2} A^G_{20} \right] \tag{4.194}$$

$$G_{22} = 2P^+ \int dk^- \left[\frac{xP^2 - P \cdot k}{M^2} A^G_1 + \frac{1}{2} A^G_{17} - \frac{k_T^2}{2M^2} A^G_{19} - \frac{k_T \cdot \Delta_T}{2M^2} A^G_{20} \right] \tag{4.195}$$

$$G_{23} = 2P^+ \int dk^- \frac{P^2}{M^2} \left[-A^G_{17} + \frac{(k_T \cdot \Delta_T)^2 - k_T^2 \Delta_T^2}{M^2 (k_T \cdot \Delta_T)} A^G_{22} \right] \tag{4.196}$$

$$G_{24} = 2P^+ \int dk^- \frac{P^2}{M^2} \left[A^G_{19} + \frac{\Delta_T^2}{k_T \cdot \Delta_T} A^G_{22} \right] \tag{4.197}$$

$$G_{25} = 2P^+ \int dk^- \frac{P^2}{M^2} \left[A^G_{20} - A^G_{22} \right] \tag{4.198}$$

$$G_{26} = 2P^+ \int dk^- \frac{P^2}{M^2} \left[A^G_{23} + \frac{k_T^2}{k_T \cdot \Delta_T} A^G_{22} \right] \tag{4.199}$$

$$G_{27} = 2P^+ \int dk^- \frac{xP^2 - P \cdot k}{M^2} \left[A^G_{19} \right] \tag{4.200}$$

$$G_{28} = 2P^+ \int dk^- \frac{xP^2 - P \cdot k}{M^2} \left[A^G_{20} \right] \tag{4.201}$$

Combining these relations with ones expressing GPDs in terms of k_T-integrals over GTMDs, as given in [17], one can also obtain the GPD combinations relevant for the developments in this work in terms of the A amplitudes. In particular,

$$\begin{aligned}
H + E &= \int d^2k_T\ 2\left(\frac{k_T\cdot\Delta_T}{\Delta_T^2}F_{12} + F_{13}\right) && (4.202)\\
&= 2P^+\int d^2k_T\int dk^- \\
&\qquad \cdot 2\left(\frac{k_T\cdot\Delta_T}{\Delta_T^2}A_5^F + A_6^F + \frac{P\cdot k - xP^2}{M^2}(A_8^F + xA_9^F)\right) && (4.203)\\
\widetilde{H} &= \int d^2k_T\ G_{14} && (4.204)\\
&= 2P^+\int d^2k_T\int dk^-\left(-A_{17}^G + \frac{xP^2 - k\cdot P}{M^2}(A_{18}^G + xA_{19}^G)\right) && (4.205)\\
\widetilde{E}_{2T} &= \int d^2k_T\ (-2)\left(\frac{k_T\cdot\Delta_T}{\Delta_T^2}F_{27} + F_{28}\right) && (4.206)\\
&= 2P^+\int d^2k_T\int dk^- \\
&\qquad \cdot(-2)\left(\frac{k_T\cdot\Delta_T}{\Delta_T^2}A_5^F + A_6^F + \frac{(k_T\cdot\Delta_T)^2 - k_T^2\Delta_T^2}{M^2\Delta_T^2}A_9^F\right) && (4.207)\\
E'_{2T} + 2\widetilde{H}'_{2T} &= \int d^2k_T\ 2\left(\frac{k_T\cdot\Delta_T}{\Delta_T^2}G_{21} + G_{22}\right) && (4.208)\\
&= 2P^+\int d^2k_T\int dk^- \\
&\qquad \cdot\left(2\ \frac{xP^2 - k\cdot P}{M^2}A_1^G + A_{17}^G + \frac{(k_T\cdot\Delta_T)^2 - k_T^2\Delta_T^2}{M^2\Delta_T^2}A_{19}^G\right) && (4.209)\\
H'_{2T} - \frac{\Delta_T^2}{4M^2}E'_{2T} &= \int d^2k_T\left(G_{23} - \frac{\Delta_T^2}{M^2}\frac{(k_T\cdot\Delta_T)^2 - k_T^2\Delta_T^2}{(\Delta_T^2)^2}G_{24}\right) && (4.210)\\
&= 2P^+\int d^2k_T\int dk^-\frac{P^2}{M^2}\left(-A_{17}^G - \frac{(k_T\cdot\Delta_T)^2 - k_T^2\Delta_T^2}{M^2\Delta_T^2}A_{19}^G\right) && (4.211)
\end{aligned}$$

4.8 The axial vector parameterization

We outline here the steps used to obtain the GPCFs that parametrize the completely unintegrated quark-quark correlation function for a straight-line gauge link in the axial vector case. They parallel the steps followed in [17]. We use the Gordon identities:

$$\overline{U}\gamma^\mu U = \overline{U}\left[\frac{P^\mu}{M} + \frac{i\sigma^{\mu\Delta}}{2M}\right]U \tag{4.212}$$

$$0 = \overline{U}\left[\frac{\Delta^\mu}{2M} + \frac{i\sigma^{\mu P}}{M}\right]U \tag{4.213}$$

$$\overline{U}\gamma^\mu\gamma^5 U = \overline{U}\left[\frac{\Delta^\mu\gamma^5}{2M} + \frac{i\sigma^{\mu P}\gamma^5}{M}\right]U \tag{4.214}$$

$$0 = \overline{U}\left[\frac{P^\mu\gamma^5}{M} + \frac{i\sigma^{\mu\Delta}\gamma^5}{2M}\right]U \tag{4.215}$$

The ϵ identity :

$$g^{\alpha\beta}\epsilon^{\mu\nu\rho\sigma} = g^{\mu\beta}\epsilon^{\alpha\nu\rho\sigma} + g^{\nu\beta}\epsilon^{\mu\alpha\rho\sigma} + g^{\rho\beta}\epsilon^{\mu\nu\alpha\sigma} + g^{\sigma\beta}\epsilon^{\mu\nu\rho\alpha} \tag{4.216}$$

The σ identity :

$$i\sigma^{\mu\nu}\gamma^5 = -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\sigma_{\rho\sigma} \tag{4.217}$$

A complete parameterization of the axial vector Dirac bilinear can be obtained by treating all possible Dirac currents one after another:

1. Vector current $[\overline{U}(p',\Lambda')\gamma^\mu U(p,\Lambda)]$: Using the Gordon identity in eq.(4.212) all vector currents can be replaced by scalar and tensor currents.

2. Axial vector current $[\overline{U}(p',\Lambda')\gamma^\mu\gamma^5 U(p,\Lambda)]$: Using the Gordon identity in

eq.(4.214) all axial vector currents can be replaced by pseudoscalar and pseudotensor currents.

3. Pseudoscalar current $[\overline{U}(p',\Lambda')\gamma^5 U(p,\Lambda)]$: Using eq.(4.215) and contracting with P^μ all pseudoscalar currents can be replaced by pseudotensor currents.

4. Tensor current $[\overline{U}(p',\Lambda')\sigma^{\mu\nu}U(p,\Lambda)]$: Using the σ identity in eq.(4.217) all tensor currents can be replaced by pseudotensor currents.

5. Pseudotensor current $[\overline{U}(p',\Lambda')\sigma^{\mu\nu}\gamma^5 U(p,\Lambda)]$: All possible pseudotensor currents are of the form

$$\overline{U}(p',\Lambda')\sigma^{\mu a}\gamma^5 U(p,\Lambda), \qquad \overline{U}(p',\Lambda')a^\mu\sigma^{bc}\gamma^5 U(p,\Lambda) \tag{4.218}$$

where a, b and c can be any of the vectors P, k and Δ.

6. Scalar current $[\overline{U}(p',\Lambda')U(p,\Lambda)]$: There is only one possible scalar current

$$\frac{\overline{U}U}{M^3} i\epsilon^{\mu P k \Delta} \tag{4.219}$$

A useful relation that can be derived by multiplying the Gordon identity eq.(4.213) by the ϵ identity eq.(4.216) and using eq.(4.217) is

$$0 = \overline{U}\left[\frac{P^\mu}{M}i\sigma^{\nu\rho}\gamma^5 + \frac{P^\nu}{M}i\sigma^{\rho\mu}\gamma^5 + \frac{P^\rho}{M}i\sigma^{\mu\nu}\gamma^5 - i\frac{\epsilon^{\mu\nu\rho\Delta}}{2M}\right]U \tag{4.220}$$

Contracting with $P^\nu k^\rho$, $P^\nu \Delta^\rho$ and $k^\nu \Delta^\rho$:

$$0 = \overline{U}\left[\frac{P^\mu}{M} i\sigma^{Pk}\gamma^5 + \frac{P^2}{M} i\sigma^{k\mu}\gamma^5 + \frac{k\cdot P}{M} i\sigma^{\mu P}\gamma^5 - i\frac{\epsilon^{\mu Pk\Delta}}{2M}\right] U \quad (4.221)$$

$$0 = \overline{U}\left[\frac{P^\mu}{M} i\sigma^{P\Delta}\gamma^5 + \frac{P^2}{M} i\sigma^{\Delta\mu}\gamma^5 + \frac{P\cdot \Delta}{M} i\sigma^{\mu P}\gamma^5\right] U \quad (4.222)$$

$$0 = \overline{U}\left[\frac{P^\mu}{M} i\sigma^{k\Delta}\gamma^5 + \frac{P\cdot k}{M} i\sigma^{\Delta\mu}\gamma^5 + \frac{P\cdot \Delta}{M} i\sigma^{\mu k}\gamma^5\right] U \quad (4.223)$$

Using these relations, we can eliminate currents $\sigma^{k\mu}\gamma^5$, $\sigma^{\Delta\mu}\gamma^5$. On the other hand, contracting with $P^\mu k^\nu \Delta^\rho$:

$$0 = \frac{P^2}{M}\overline{U} i\sigma^{k\Delta}\gamma^5 U + \frac{P\cdot k}{M}\overline{U} i\sigma^{\Delta P}\gamma^5 U + \frac{P\cdot \Delta}{M}\overline{U} i\sigma^{Pk}\gamma^5 U \quad (4.224)$$

which allows us to eliminate $\sigma^{k\Delta}\gamma^5 a^\mu$. The parameterization can thus be written as :

$$\begin{aligned} W^{\gamma^\mu\gamma^5} &= \frac{\overline{U}U}{M^3} i\epsilon^{\mu Pk\Delta} A_1^G + \frac{\overline{U} i\sigma^{P\mu}\gamma^5 U}{M} A_{17}^G + \frac{\overline{U} i\sigma^{Pk}\gamma^5 U}{M^3}\left(P^\mu A_{18}^G + k^\mu A_{19}^G + \Delta^\mu A_{20}^G\right) \\ &\quad + \frac{\overline{U} i\sigma^{P\Delta}\gamma^5 U}{M^3}\left(P^\mu A_{21}^G + k^\mu A_{22}^G + \Delta^\mu A_{23}^G\right) \end{aligned} \quad (4.225)$$

4.9 The Forward Limit

To derive the decomposition of the distribution functions in terms of GPCFs, we come across terms like $\frac{(k_1\Delta_2 - k_2\Delta_1)^2}{\Delta_T^2 M^2} B$. We show below how, in the forward limit, this term transforms into $\frac{k_T^2}{M^2} B$. This is also useful for deriving the equation of motion relations at the k_T integrated level in the forward case.

In the off forward case,

$$\int d^2k_T dk^- \left[k^i (k_1\Delta_2 - k_2\Delta_1) B\right] \overset{\times \epsilon^{ij}\Delta_T^j}{\rightarrow} \int d^2k_T dk^- \left[(k_1\Delta_2 - k_2\Delta_1)^2 B\right] \quad (4.226)$$

For the forward case, first we Fourier transform (FT) to b_T space,

$$\int d^2k_T dk^- \left[k^i(k_1\Delta_2 - k_2\Delta_1)B\right] \overset{FT}{\rightarrow} \int d^2k_T dk^- \left[k^i\left(k^1\frac{\partial}{\partial b^2} - k^2\frac{\partial}{\partial b^1}\right)\mathcal{B}\right] \quad (4.227)$$

where,

$$\mathcal{B} = \int \frac{d^2\Delta_T}{(2\pi)^2} e^{i\Delta_T\cdot b_T} B \quad (4.228)$$

and multiplying by $\epsilon^{ij}b_T^j$ and integrate over b_T

$$\int d^2b_T \left[\epsilon^{ij}b_T^j k_T^i\left(k^1\frac{\partial}{\partial b^2} - k^2\frac{\partial}{\partial b^1}\right)\mathcal{B}\right] \;\; = \;\; k_T^2 B \quad (4.229)$$

The integral in b_T is a Fourier transform back into momentum space with $\Delta_T \rightarrow 0$. This trick is also useful for taking deriving the EOM relations in the forward limit involving GPDs with an explicit Δ_T coefficient. Consider a GPD F. In the off forward case, the manipulation looks like,

$$\epsilon^{ij}\Delta_T^j F \rightarrow \epsilon^{im}\epsilon^{ij}\Delta_T^m\Delta_T^j F = \Delta_T^2 F \quad (4.230)$$

In the forward case, we take the Fourier transform with respect to Δ_T, multiply by $\epsilon^{im}\epsilon^{ij}b_T^m$ and integrate over b_T which is equivalent to an inverse Fourier transform back to momentum space with $\Delta_T = 0$.

$$\epsilon^{ij}\Delta_T^j F \rightarrow \epsilon^{im}\epsilon^{ij}b_T^m\frac{\partial}{\partial b_T^i}\mathcal{F} \rightarrow F \quad (4.231)$$

4.10 The Chiral Odd Sector

The study presented in the previous sections can be extended to the chiral odd sector as well. Unlike the chiral even case, there are no GTMDs such as F_{14} or G_{11} that contain entirely new information which is not already available in a GPD or TMD. In other words, all leading twist GTMD either transform to a TMD in the forward limit or enter the description of a GPD when integrated over k_T. The only LIR we find is an off forward extension of the previously known LIR between $h_{1L}^{\perp(1)}$, h_1 and h_L. The parameterization of GPCFs in the straight gauge link case used for deriving the LIR is as follows,

$$\begin{aligned} \mathcal{W}_{\Lambda\Lambda'}^{\left[i\sigma^{\mu\nu}\gamma^5\right]} &= (\delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\nu\rho}\delta_{\mu\sigma})\,\overline{U}(p',\Lambda')\left[\frac{i\epsilon^{\rho\sigma P\Delta}}{M^2}A_1^H + \frac{i\epsilon^{\rho\sigma Pk}}{M^2}A_2^H + \frac{i\epsilon^{\rho\sigma k\Delta}}{M^2}A_3^H\right. \\ &+ i\sigma^{\rho\sigma}\gamma^5 A_4^H + \frac{i\sigma^{\sigma P}\gamma^5}{M}\left(\frac{k^\rho}{M}A_5^H + \frac{\Delta^\rho}{M}A_6^H\right) + \frac{i\sigma^{Pk}\gamma^5}{M^2}\left(\frac{P^\rho k^\sigma}{M^2}A_7^H\right. \\ &+ \left.\frac{P^\rho\Delta^\sigma}{M^2}A_8^H + \frac{k^\rho\Delta^\sigma}{M^2}A_9^H\right) + \frac{i\sigma^{P\Delta}\gamma^5}{M^2}\left(\frac{P^\rho k^\sigma}{M^2}A_{10}^H + \frac{P^\rho\Delta^\sigma}{M^2}A_{11}^H\right. \\ &+ \left.\left.\frac{k^\rho\Delta^\sigma}{M^2}A_{12}^H\right)\right]U(p,\Lambda). \end{aligned} \tag{4.232}$$

The possible equations of motion for the chiral odd case that involve a k_T^2 moment are,

$$-x\tilde{H}_2' - \int d^2k_T \frac{(k_T \times \Delta_T)^2}{M^2\Delta_T^2} H_{17} + \frac{m}{M}\tilde{H} - \frac{1}{2M}\int d^2k_T \left(\mathcal{M}_{++}^{\gamma^+\gamma^5 A} - \mathcal{M}_{--}^{\gamma^+\gamma^5 A}\right) = 0 \tag{4.233}$$

$$xH_2' + \int d^2k_T \frac{(k_T \times \Delta_T)^2}{M^2\Delta_T^2} H_{11} - \frac{1}{2M}\int d^2k_T \left(\mathcal{M}_{++}^{\gamma^+ S} + \mathcal{M}_{--}^{\gamma^+ S}\right) = 0 \tag{4.234}$$

$$xE_2' - \int d^2k_T \frac{(k_T \times \Delta_T)^2}{M^2\Delta_T^2} H_{11} - \frac{1}{\Delta_T^2}\int d^2k_T \left[(\Delta_1 - i\Delta_2)\mathcal{M}_{-+}^{\gamma^+ S} - (\Delta_1 + i\Delta_2)\mathcal{M}_{+-}^{\gamma^+ S}\right] = 0. \tag{4.235}$$

In the above, $\mathcal{M}_{\Lambda\Lambda'}^{\Gamma A,S}$ are defined similar to (4.63a) and (4.63b). In order to obtain the equations of motion in the chiral odd sector, Γ is taken as γ^+ or $\gamma^+\gamma^5$ unlike the chiral even case where it was taken as $i\sigma^{i+}\gamma^5$. In the forward limit, $H_{11}^{(o)}$ is the Boer Mulders function. Like in the case of the Sivers function, for $\xi = 0$, $H_{11}^{(e)}$ does not correspond to an LIR. H_{17} in the forward limit is $h_{1L}^{\perp}$ for which the LIR between $h_{1L}^{\perp(1)}$ and the PDFs h_1 and h_L has been derived previously (Mulders and Tangerman)

$$\frac{dh_{1L}^{\perp(1)}}{dx} = h_1 - h_L \tag{4.236}$$

h_1 is the chiral odd twist two PDF describing transversely polarized quarks in an unpolarized proton whereas h_L is the chiral odd twist three PDF describing the quark operator $i\sigma^{+-}\gamma^5$ in a longitudinally polarized proton.$h_{1L}^{\perp}$ is a TMD describing transversely polarized quarks in a longitudinally polarized proton. The LIR in the off-forward case is

$$\frac{dH_{17}^{(1)}}{dx} = H_T - \frac{\Delta_T^2}{4M^2}E_T - \tilde{H}_2' \tag{4.237}$$

$$H_{17} \xrightarrow{\Delta\to 0} h_{1L}^{\perp} \qquad \tilde{H}_2' \xrightarrow{\Delta\to 0} h_L \qquad H_T \xrightarrow{\Delta\to 0} h_1 \tag{4.238}$$

To derive the LIR, we write the GTMD H_{17} in terms of GPCFs

$$\begin{aligned} H_{17}^{(1)} &= \int d^2k_T dk^- \frac{(k_T \times \Delta_T)^2}{M^2\Delta_T^2} H_{17} \\ &= \frac{2P^+}{M} \int d^2k_T dk^- \frac{(k_T \times \Delta_T)^2}{M^2\Delta_T^2} \left(A_5^H + \frac{k\cdot P - xP^2}{M^2} A_7^H \right) \\ &= \int d\sigma d\tau d\sigma' \frac{M^3\Delta_T}{2} \sqrt{x\sigma - \tau - \frac{x^2P^2}{M^2} - \left(\frac{\sigma'\Delta_T}{M}\right)^2} \left(A_5^H - x\frac{P^2}{M^2} A_7^H \right) \end{aligned} \tag{4.239}$$

$$\begin{aligned} \Rightarrow \frac{dH_{17}^{(1)}}{dx} &= \int d\sigma d\tau d\sigma' \frac{M^3\Delta_T}{2\sqrt{x\sigma - \tau - \frac{x^2P^2}{M^2} - \left(\frac{\sigma'\Delta_T}{M}\right)^2}} \left[\left(\frac{\sigma}{2} - \frac{xP^2}{M^2} \right) A_5^H \right. \\ &+ \left. \left(\left(\frac{\sigma}{2} - \frac{xP^2}{M^2} \right)^2 - \left(x\sigma - \tau - \frac{x^2P^2}{M^2} - \left(\frac{\sigma'\Delta_T}{M}\right)^2 \right) \frac{P^2}{M^2} \right) A_7^H \right] \end{aligned} \tag{4.240}$$

Writing $\tilde{H}_2'$ in terms of the GPCFs given in (4.251) using σ, τ and σ' we have

$$\tilde{H}_2' = \int d\sigma d\tau d\sigma' \frac{M^3\Delta_T}{2\sqrt{x\sigma - \tau - \frac{x^2P^2}{M^2} - \left(\frac{\sigma'\Delta_T}{M}\right)^2}} \left[2A_4^H - \frac{\sigma}{2} A_5^H - \left(\frac{\sigma}{2} - \frac{xP^2}{M^2} \right) A_7^H \right] \tag{4.241}$$

Comparing equations (4.240) and (4.241), we see we need to cancel out A_4^H and add components to A_5^H and A_7^H. Looking at equations (4.245) - (4.248) we see that the combination $H_T - \frac{\Delta_T^2}{4M^2} E_T$ works. This is obtained by subtracting $\Delta_T^2/4M^2$ times

(4.245) from (4.247). Hence, the LIR is,

$$\frac{dH_{17}^{(1)}}{dx} = H_T - \frac{\Delta_T^2}{4M^2}E_T - \tilde{H}_2'. \tag{4.242}$$

The GPCF expansion for the twist 2 case is,

$$\begin{aligned}
\mathcal{W}_{\Lambda\Lambda'}^{\left[i\sigma^{i+}\gamma^5\right]} &= \frac{P^+}{M}\Bigg[4i\epsilon^{ij}\left(\Delta_T^j A_1^H + k^j A_2^H + x\Delta^j A_3^H\right) + \left(2\Lambda k_T^i + ix\epsilon^{ij}\Delta_T^j\right)A_5^H \\
&+ 2\Lambda\Delta_T^i A_6^H - \left(\frac{2\Lambda\left(xP^2 - k\cdot P\right) - ik_T\times\Delta_T}{M^2}\right)\left(k_T^i A_7^H + \Delta_T^i A_8^H\right. \\
&+ \left. x\Delta_T^i A_9^H\right)\Bigg]\delta_{\Lambda\Lambda'} \\
&+ \frac{P^+}{M}\Bigg[-\frac{2i\epsilon^{ij}\left(\Lambda\Delta_1 + i\Delta_2\right)}{M^2}\left(\Delta_T^j A_1^H + k_T^j A_2^H + x\Delta_T^j A_3^H\right) \\
&+ 4M(\delta_{i1} + i\Lambda\delta_{i2})A_4^H - x\left(\frac{2P^2(\delta_{i1} + i\Lambda\delta_{i2})}{M} + \frac{i\epsilon^{ij}\Delta^j(\Lambda\Delta_1 + i\Delta_2)}{2M}\right)A_5^H \\
&- \left(\frac{2P^2(k_1 + i\Lambda k_2)}{M^3} + \frac{ik_T\times\Delta_T(\Lambda\Delta_1 + i\Delta_2)}{2M^3}\right) \\
&\qquad\qquad\qquad\qquad \cdot\left(k_T^i A_7^H + \Delta_T^i A_8^H + x\Delta_T^i A_9^H\right) \\
&- \left(k_T^i A_{10}^H + \Delta_T^i A_{11}^H + x\Delta_T^i A_{12}^H\right)\frac{2P^2(\Delta_1 + i\Lambda\Delta_2)}{M^3}\Bigg]\delta_{-\Lambda\Lambda'}.
\end{aligned} \tag{4.243}$$

The GPCF expansion for the twist 3 case is,

$$
\begin{aligned}
\mathcal{W}_{\Lambda\Lambda'}^{\left[i\sigma^{+-}\gamma^5\right]} &= \Bigg[-\frac{4ik_T\times\Delta_T}{M}A_3^H + 4\Lambda M A_4^H - \frac{2\Lambda k\cdot P}{M}A_5^H \\
&+ \frac{k\cdot P - xP^2}{M}\left(\frac{2\Lambda\left(xP^2 - k\cdot P\right)}{M^2} - \frac{ik_T\times\Delta_T}{M^2}\right)A_7^H\Bigg]\delta_{\Lambda\Lambda'} \\
&+ \Bigg[\frac{2i(\Lambda\Delta_1 + i\Delta_2)k_T\times\Delta_T}{M^2}A_3^H + \Bigg(\frac{2P^2(k_1 + i\Lambda k_2)}{M^3} \\
&+ \frac{ik_T\times\Delta_T(\Lambda\Delta_1 + i\Delta_2)}{2M^4}\Bigg)\frac{k\cdot P - xP^2}{M}A_7^H + \frac{2P^2(\Delta_1 + i\Lambda\Delta_2)}{M^2}A_{10}^H\Bigg]\delta_{-\Lambda\Lambda'}.
\end{aligned}
\tag{4.244}
$$

The twist two GPDs in terms of GPCFs are,

$$
\begin{aligned}
E_T + 2\tilde{H}_T &= \\
& 2P^+\int d^2k_T dk^- 4\left(A_1^H + \frac{k_T\cdot\Delta_T}{\Delta_T^2}A_2^H + xA_3^H\right) + xA_5^H + \frac{(k_T\times\Delta_T)^2}{M^2\Delta_T^2}A_7^H
\end{aligned}
\tag{4.245}
$$

$$
\begin{aligned}
\tilde{E}_T &= \\
& 4P^+\int d^2k_T dk^- \frac{k_T\cdot\Delta_T}{\Delta_T^2}A_5^H + A_6^H + \frac{k\cdot P - xP^2}{M^2}\left(\frac{k_T\cdot\Delta_T}{\Delta_T^2}A_7^H + A_8^H + xA_9^H\right)
\end{aligned}
\tag{4.246}
$$

$$
\begin{aligned}
H_T + \frac{\Delta_T^2}{2M^2}\tilde{H}_T &= \\
& 2P^+\int d^2k_T dk^- \frac{\Delta_T^2}{M^2}\left(A_1^H + \frac{k_T\cdot\Delta_T}{\Delta_T^2}A_2^H + xA_3^H\right) + 2A_4^H - xA_5^H - \frac{(k_T\times\Delta_T)^2}{M^2\Delta_T^2}A_7^H
\end{aligned}
\tag{4.247}
$$

$$
\begin{aligned}
H_T &= \\
2P^+ \int d^2k_T dk^- 2A_4^H &- \frac{xP^2}{M^2}A_5^H - \frac{k_T\cdot\Delta_T P^2}{M^4}\left(A_7^H + \frac{k_T\cdot\Delta_T}{\Delta_T^2}A_8^H + xA_9^H\right) \\
&- \frac{P^2}{M^2}\left(A_{10}^H + \frac{k_T\cdot\Delta_T}{\Delta_T^2}A_{11}^H + xA_{12}^H\right). \qquad (4.248)
\end{aligned}
$$

The twist three GPDs in terms of GPCFs are,

$$
H_2' = 2P^+ \int d^2k_T dk^- \frac{xP^2 - k\cdot P}{M^2}A_2^H - \frac{k_T\cdot\Delta_T}{\Delta_T^2}A_5^H - A_6^H + \frac{(k_T\times\Delta_T)^2}{M^4}A_9^H \qquad (4.249)
$$

$$
E_2' = -4P^+ \int d^2k_T dk^- \frac{xP^2 - k\cdot P}{M^2}A_2^H + \frac{k_T\cdot\Delta_T}{\Delta_T^2}A_5^H + A_6^H - \frac{(k_T\times\Delta_T)^2}{M^4}A_9^H \qquad (4.250)
$$

$$
\tilde{H}_2' = 2P^+ \int d^2k_T dk^- 2A_4^H - \frac{k\cdot P}{M^2}A_5^H - \frac{xP^2 - k\cdot P}{M^2}A_7^H \qquad (4.251)
$$

$$
\tilde{E}_2' = 4P^+ \int d^2k_T dk^- \frac{P^2(xP^2 - k\cdot P)}{M^4}\left(\frac{k_T\cdot\Delta_T}{\Delta_T^2}A_7^H + A_{10}^H\right). \qquad (4.252)
$$

4.11 Equation of Motion relations in the Chiral Odd Sector

We derive the possible equation of motion relations connecting twist two GTMDs with twist three GPDs and a genuine twist three term $\mathcal{M}$, all chiral odd (except the mass term which gets multiplied by a chiral even function). Unlike the chiral even case where we obtained two possible cases, symmetric and anti-symmetric, by sandwiching the equation of motion multiplied with $\Gamma = \sigma^{i+}\gamma^5$ between the quark fields, in the chiral odd case, we have two options for the operator we use in the correlator $\Gamma = \gamma^+\gamma^5$ and $\Gamma = \gamma^+$ and two options, symmetric and anti-symmetric, for each. To obtain the realtions, we start with the template written in terms of $W^{[i\sigma^{+-}\gamma^5]}_{\Lambda'\Lambda}, W^{[i\sigma^{i+}\gamma^5]}_{\Lambda'\Lambda}, W^{[\gamma^5]}_{\Lambda'\Lambda}, W^{[i\sigma^{12}\gamma^5]}_{\Lambda'\Lambda}, W^{[1]}_{\Lambda'\Lambda}$, the mass term, in the anti symmetric cases, multiplying $W^{[\gamma^+\gamma^5]}_{\Lambda'\Lambda}$ or $W^{[\gamma^+]}_{\Lambda'\Lambda}$ and $\mathcal{M}^{\gamma^+ A,S}_{\Lambda\Lambda'}$ or $\mathcal{M}^{\gamma^+\gamma^5 A,S}_{\Lambda\Lambda'}$. Next, we take the proton helicity combinations that lead transverse momentum dependent functions which are meaningful when integrated over k_T, that is they connect to a GPD or are of the form of a k_T^2 moment of a twist two GTMD. The combinations that we use are denoted as $\frac{1}{2}((++) \pm (--))$ which means we are taking the sum or the difference of the template equation involving $W^{\Gamma}_{\Lambda\Lambda'}$ for $(\Lambda = \Lambda' = -)$ case and the $(\Lambda = \Lambda' = +)$ case. For the proton helicity flip case, we need to multiply by a factor of the form $\Delta_1 \pm i\Delta_2$ before taking the sum and difference. For instance $\frac{1}{2}\left[(\Delta_1 - i\Delta_2)W_{-+} + (\Delta_1 + i\Delta_2)W_{+-}\right]$ denotes that we take the template equation for the case $(\Lambda = + = -\Lambda')$, multiply it by $\Delta_1 - i\Delta_2$ and add it to the $(\Lambda = - = -\Lambda')$ case multiplied by $\Delta_1 + i\Delta_2$. We list the possible relations below.

4.11.1 $\gamma^+\gamma^5$ Anti Symmetric

$$-k^+ W^{[i\sigma^{+-}\gamma^5]}_{\Lambda'\Lambda} - k_T^i W^{[i\sigma^{i+}\gamma^5]}_{\Lambda'\Lambda} + \frac{\Delta^+}{2} W^{[\gamma^5]}_{\Lambda'\Lambda} + m W^{[\gamma^+\gamma^5]}_{\Lambda'\Lambda} - \mathcal{M}^{\gamma^+\gamma^5 A}_{\Lambda'\Lambda} = 0 \tag{4.253}$$

Taking the difference of proton helicities $\frac{1}{2}((++) - (--))$

$$-xH_{28} - \frac{k_T^2}{M^2} H_{17} + \frac{k_T \cdot \Delta_T}{\Delta_T^2} H_{18} + \frac{m}{M} G_{14} - \frac{1}{2M}\left(\mathcal{M}^{\gamma^+\gamma^5 A}_{++} - \mathcal{M}^{\gamma^+\gamma^5 A}_{--}\right) = 0 \tag{4.254}$$

Integrating over k_T,

$$\Rightarrow -x\tilde{H}_2' - \int d^2k_T \frac{(k_T \times \Delta_T)^2}{M^2\Delta_T^2} H_{17} + \frac{m}{M}\tilde{H}$$
$$-\frac{1}{2M}\int d^2k_T \left(\mathcal{M}_{++}^{\gamma^+\gamma^5 A} - \mathcal{M}_{--}^{\gamma^+\gamma^5 A}\right) = 0. \quad (4.255)$$

Proton helicity flip, $\frac{1}{2}\left[(\Delta_1 - i\Delta_2)W_{-+} + (\Delta_1 + i\Delta_2)W_{+-}\right]$

$$-x(k_T\cdot\Delta_T H_{26} + \Delta_T^2 H_{27}) - k_T\cdot\Delta_T\left(H_{13} + \frac{k_T^2}{M^2}H_{14} + \frac{k_T\cdot\Delta_T}{M^2}H_{15} + \frac{\Delta_T^2}{M^2}H_{16}\right)$$
$$+\frac{m}{M}(\Delta_T^2 G_{13} + k_T\cdot\Delta_T G_{12}) + \frac{1}{2}\left[(\Delta_1 - i\Delta_2)\mathcal{M}_{-+}^{\gamma^+\gamma^5 A} + (\Delta_1 + i\Delta_2)\mathcal{M}_{+-}^{\gamma^+\gamma^5 A}\right] = 0$$
$$(4.256)$$

Integrating over k_T,

$$\Rightarrow x\tilde{E}_2' + \frac{m}{M}\xi\tilde{E} + \frac{1}{\Delta_T^2}\int d^2k_T\left[(\Delta_1 - i\Delta_2)\mathcal{M}_{-+}^{\gamma^+\gamma^5 A} + (\Delta_1 + i\Delta_2)\mathcal{M}_{+-}^{\gamma^+\gamma^5 A}\right] = 0 \quad (4.257)$$

4.11.2 γ^+ Symmetric

$$ik^+W_{\Lambda'\Lambda}^{[i\sigma^{12}\gamma^5]} + i\epsilon^{ij}k_T^i W_{\Lambda'\Lambda}^{[i\sigma^{j+}\gamma^5]} + \frac{\Delta^+}{2}W_{\Lambda'\Lambda}^1 - \mathcal{M}_{\Lambda'\Lambda}^{\gamma^+ S} = 0 \quad (4.258)$$

Summing proton helicities $\frac{1}{2}((++) + (--))$

$$x\left(H_{21} + \frac{ik_T\times\Delta_T}{M^2}H_{24}\right) + \frac{k_T\cdot\Delta_T}{M^2}\left(\frac{k_T\cdot\Delta_T}{\Delta_T^2}H_{11} + H_{12}\right)$$
$$+\frac{k_T\times\Delta_T}{M^2\Delta_T^2}H_{11} - \frac{1}{2M}\left(\mathcal{M}_{++}^{\gamma^+ S} + \mathcal{M}_{--}^{\gamma^+ S}\right) = 0 \quad (4.259)$$

Integrating over k_T ,

$$xH_2' + \int d^2k_T \frac{(k_T \times \Delta_T)^2}{M^2\Delta_T^2} H_{11} - \frac{1}{2M}\int d^2k_T \left(\mathcal{M}_{++}^{\gamma^+ S} + \mathcal{M}_{--}^{\gamma^+ S}\right) = 0 \tag{4.260}$$

Proton helicity flip, $\frac{1}{2}\left[(\Delta_1 - i\Delta_2)W_{-+} - (\Delta_1 + i\Delta_2)W_{+-}\right]$

$$\begin{aligned} x\left(-\frac{\Delta_T^2}{2}H_{21} + k_T\cdot\Delta_T H_{22} + \Delta_T^2 H_{23}\right) - \frac{\Delta_T^2}{2M^2}(k_T^2 H_{11} + k_T\cdot\Delta_T H_{12}) \\ -\frac{1}{2}\left[(\Delta_1 - i\Delta_2)\mathcal{M}_{-+}^{\gamma^+ S} - (\Delta_1 + i\Delta_2)\mathcal{M}_{+-}^{\gamma^+ S}\right] = 0 \end{aligned} \tag{4.261}$$

$$xE_2' - \int d^2k_T \frac{(k_T \times \Delta_T)^2}{M^2\Delta_T^2} H_{11} - \frac{1}{\Delta_T^2}\int d^2k_T \left[(\Delta_1 - i\Delta_2)\mathcal{M}_{-+}^{\gamma^+ S} - (\Delta_1 + i\Delta_2)\mathcal{M}_{+-}^{\gamma^+ S}\right] = 0 \tag{4.262}$$

4.11.3 $\gamma^+\gamma^5$ Symmetric

$$-k^+ W_{\Lambda'\Lambda}^{[\gamma^5]} + \frac{\Delta_T^i}{2} W_{\Lambda'\Lambda}^{[i\sigma^{i+}\gamma^5]} - \mathcal{M}_{\Lambda'\Lambda}^{\gamma^+\gamma^5 S} = 0 \tag{4.263}$$

Taking the difference of proton helicities $\frac{1}{2}((++) - (--))$

$$-xE_{28} + \frac{k_T\cdot\Delta_T}{2M^2} H_{17} \frac{\Delta_T^2}{2M^2} H_{18} - \frac{1}{2M}\left(\mathcal{M}_{++}^{\gamma^+\gamma^5 S} - \mathcal{M}_{--}^{\gamma^+\gamma^5 S}\right) = 0 \tag{4.264}$$

Integrating over k_T

$$-x\tilde{H}_2 + \frac{\Delta_T^2}{4M^2}\tilde{E}_T - \frac{1}{2M}\int d^2k_T \left(\mathcal{M}_{++}^{\gamma^+\gamma^5 S} - \mathcal{M}_{--}^{\gamma^+\gamma^5 S}\right) = 0 \tag{4.265}$$

Proton helicity flip, $\frac{1}{2}\left[(\Delta_1 - i\Delta_2)W_{-+} + (\Delta_1 + i\Delta_2)W_{+-}\right]$

$$\begin{aligned}&-x(k_T\cdot\Delta_T E_{26} + \Delta_T^2 E_{27}) + \frac{\Delta_T^2}{2}\left(H_{13} + \frac{(k_T\cdot\Delta_T)^2}{M^2\Delta_T^2}H_{14} + \frac{k_T\cdot\Delta_T}{M^2}H_{15} + \frac{\Delta_T^2}{M^2}H_{16}\right)\\ &-\frac{1}{2}\left[(\Delta_1 - i\Delta_2)\mathcal{M}_{-+}^{\gamma^+\gamma^5 S} + (\Delta_1 + i\Delta_2)\mathcal{M}_{+-}^{\gamma^+\gamma^5 S}\right] = 0\end{aligned} \tag{4.266}$$

Integrating over k_T,

$$x\tilde{E}_2 + H_T - \frac{1}{2\Delta_T^2}\int d^2k_T \left[(\Delta_1 - i\Delta_2)\mathcal{M}_{-+}^{\gamma^+\gamma^5 S} + (\Delta_1 + i\Delta_2)\mathcal{M}_{+-}^{\gamma^+\gamma^5 S}\right] = 0 \tag{4.267}$$

4.11.4 γ^+ Anti symmetric

$$-k^+W_{\Lambda'\Lambda}^{[1]} - i\frac{\Delta_T^+}{2}W_{\Lambda'\Lambda}^{[i\sigma^{12}\gamma^5]} - i\epsilon^{ij}\frac{\Delta_T^i}{2}W_{\Lambda'\Lambda}^{[i\sigma^{j+}\gamma^5]} + mW^{[\gamma^+]} - \mathcal{M}_{\Lambda'\Lambda}^{\gamma^+ A} = 0 \tag{4.268}$$

Summing proton helicities $\frac{1}{2}((++) + (--))$

$$-xE_{21} + \frac{1}{2}\left(\frac{k_T\cdot\Delta_T}{M^2}H_{11} + \frac{\Delta_T^2}{M^2}H_{12}\right) + \frac{m}{M}F_{11} - \frac{1}{2M}\left(\mathcal{M}_{++}^{\gamma^+ A} + \mathcal{M}_{--}^{\gamma^+ A}\right) = 0 \tag{4.269}$$

Integrating over k_T,

$$-xH_2 + \frac{\Delta_T^2}{M^2}\left(\tilde{H}_T + \frac{E_T}{2}\right) + \frac{m}{M}H - \frac{1}{2M}\int d^2k_T \left(\mathcal{M}_{++}^{\gamma^+ A} + \mathcal{M}_{--}^{\gamma^+ A}\right) = 0 \quad (4.270)$$

Proton helicity flip, $\frac{1}{2}\left[(\Delta_1 - i\Delta_2)W_{-+} - (\Delta_1 + i\Delta_2)W_{+-}\right]$

$$\begin{aligned}
&-x\left(-E_{21} + 2\left(\frac{k_T\cdot\Delta_T}{\Delta_T^2}E_{22} + E_{23}\right)\right) + \frac{1}{2M^2}\left(k_T\cdot\Delta_T H_{11} + \Delta_T^2 H_{12}\right.\\
&\left.+2\frac{(k_T\times\Delta_T)^2}{\Delta_T^2}H_{14} + 2M^2 H_{13}\right) + \frac{m}{M}\left(-F_{11} + 2\left(\frac{k_T\cdot\Delta_T}{\Delta_T^2}F_{12} + F_{13}\right)\right)\\
&-\frac{1}{2\Delta_T^2}\left[(\Delta_1 - i\Delta_2)\mathcal{M}_{-+}^{\gamma^+ A} - (\Delta_1 + i\Delta_2)\mathcal{M}_{+-}^{\gamma^+ A}\right] = 0 \qquad (4.271)
\end{aligned}$$

Integrating over k_T,

$$\begin{aligned}
&-xE_2 + \frac{1}{2}\left(\frac{\Delta_T^2}{M^2}\tilde{H}_T + 2H_T\right) + \frac{m}{M}E\\
&\qquad -\frac{1}{\Delta_T^2}\int d^2k_T\left[(\Delta_1 - i\Delta_2)\mathcal{M}_{-+}^{\gamma^+ A} - (\Delta_1 + i\Delta_2)\mathcal{M}_{+-}^{\gamma^+ A}\right] = 0 \qquad (4.272)
\end{aligned}$$

Chapter 5

Reconstructing Parton Distribution Functions from Lattice QCD moments

Electromagnetic interactions probe the Fourier transform of the quark-quark correlation function measuring the light cone distance traveled by the struck quark during the interaction. The small distance behavior of the latter can be determined by the first few moments calculated on the lattice while the large distance behavior, exceeding the proton size, is regulated by Regge behavior.We use this characteristic dependence of the correlation function to extract the x dependence of the u and d quark distributions functions in the proton.

PDFs and GPDs have, so far, not been computed directly in lattice QCD because they involve non-local light-cone correlation functions evaluated at real time (Minkowski time): in a nutshell lattice QCD calculations of PDFs are hampered by the breaking of rotational symmetry which is introduced when using discretized

Euclidean space-time.

Local operators giving the Mellin moments of PDFs and GPDs can be evaluated on the lattice. However, calculations are practically feasible only for the first few moments. For the higher moments operator mixing with lower dimension operators occurs which introduces divergences enhancing the errors associated with the renormalization procedure [63]. Therefore, a reconstruction of PDFs from its moments is ridden with uncertainty due to the moments series truncation at $n \approx 2, 3$ [1; 64; 65].

An alternative method was proposed by Ji, Large Momentum Effective Theory (LaMET), where the correlation functions are evaluated at equal-time, or for space-like intervals [49]. These functions, named *quasi-PDFs*, can then be computed directly on the lattice. In the limit of infinite proton momentum ($P_z \to \infty$) the light-cone PDFs are, in principle, recovered through a matching procedure, where a matching factor is derived which connects the regularization schemes for the quasi-PDF obtained on the lattice and the light cone PDF, respectively. The matching factor is calculated perturbatively. While the calculation of quasi-PDFs has been addressed by several lattice groups [66; 67; 68] for both the unpolarized and transversely polarized distributions, various complications have emerged concerning the treatment of the matching factor (see discussion in [69; 70]), and, more recently, the possible role in the lattice evaluation, of power divergent operator mixing terms which would require a specific non-perturbative renormalization [71].

A more recent idea introduces *pseudo-PDFs* $P(x, z^2)$ that generalize the light-front PDFs onto space-like intervals [72]. Pseudo-PDFs correspond to Fourier transforms of "Generalized" Ioffe-Time [73] Distributions (GITDs) $\mathcal{M}((kz), z^2)$, which are obtained by taking the original Ioffe-time distributions (ITDs) [74], $\mathcal{M}((kz), 0) \equiv \mathcal{M}(z^-, 0)$ off the light-cone. The quantity z^2 is chosen to be the interval $z_3^2 < 0$, so as to enable direct lattice evaluations of the GITDs and their connection to PDFs [75]. Pseudo-PDFs present the advantage that they have the same $1 \leq x \leq 1$ support as PDFs, and that their z_3^2 dependence for small z_3^2 obeys standard renormalization group equations, thus providing a substantial simplification of the matching procedure. In the hypothesis that the GITD's (kz) and z_3^2 dependence factorizes, the matching onto observable PDFs, given by the Fourier conjugates of the ITDs, $\mathcal{M}((kz), 0))$, is then obtained by taking the ratio $\mathcal{M}((pz), z_3^2)/\mathcal{M}(0, z_3^2)$. $\mathcal{M}(0, z_3^2)$ is the rest frame distribution. A clear connection between the infinite momentum frame (light-cone) PDFs and rest frame PDFs has been shown in [75] by numerically displaying the apparent (kz) and z_3^2 factorization of available lattice data.

Spurred by both the general question of evaluating the impact of lattice QCD on global PDF and GPD analyses [76], and by the pivotal new role Ioffe-time might play, in this paper we present a quantitative analysis of the spatial dependence of the correlation function merging information from lattice QCD moments and phenomenology.

ITDs have been introduced and evaluated in Ref.[74]. A proof of principle that ITDs could be used to evince both nucleon and nuclear PDFs from lattice QCD moments and other phenomenological information was provided in Refs. [77; 78; 79]. In particular, it was shown that ITDs are characterized by two distinct regimes, for small and large longitudinal distances, z^-, respectively. At small z^-, an almost linear behavior ensues that reflects the PDFs Fourier transform's behavior, Taylor expanded around $z^- = 0$, whereas the large z^- behavior is dominated by the

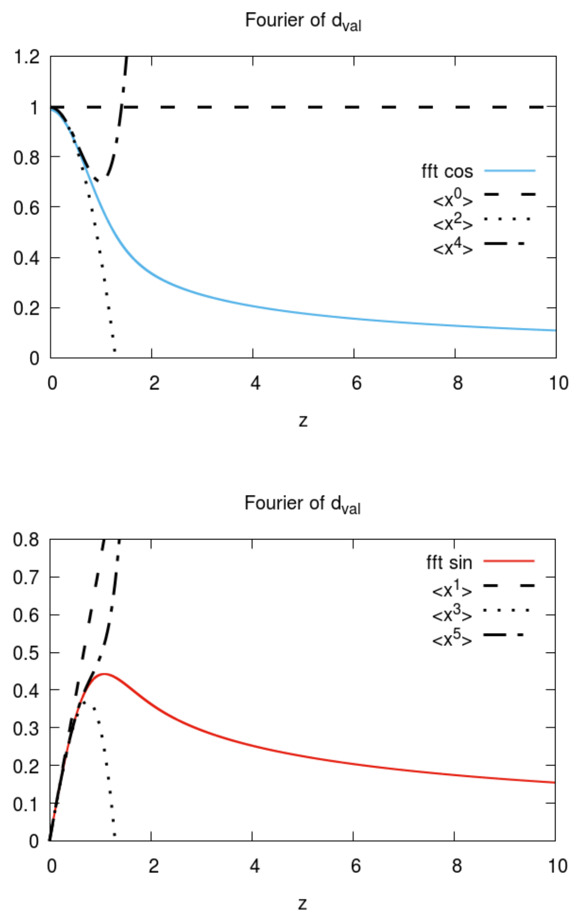
Fourier transform of the low x, power-like, Regge-type behavior of the PDF. These two distinct behaviors in z^- can be smoothly matched onto one another in the intermediate region of z^-. The ITDs characteristic features are even more striking in the description of the nuclear structure functions, from the low x nuclear shadowing region to the EMC effect [78; 79].

While at the time of the analysis in Refs.[74; 77] it was only possible to make an educated guess, although insightful, of the PDFs behavior, it is now possible to conduct a fully quantitative analysis based on the availability of different sets of lattice results for both PDFs and GPDs [64; 3; 80; 81; 82]. Similarly the analysis of Ref.[77] we reconstruct the PDFs using different sets of lattice moments and Regge behavior. We are, however, able to provide quantitative limits for our analysis given by error bands that reflect both the error on lattice moments and the theoretical error due to extrapolations in the intermediate z^- regions.

After showing that a quantitative reconstruction of PDFs is feasible, if moments up to at least $n = 3$ are known, we extend our Ioffe-time analysis to GPDs. An ITDs analysis presents several appealing features: most remarkably, it reflects physical features concerning both the skewness and t dependence of GPDs that can now be evaluated fully quantitatively, with a controllable error, rather than guessed at.

5.1 Reconstructing PDFs and GPDs

The off forward quark-quark correlation function in momentum space is defined as the Fourier transform of the quark fields matrix element between proton states with



(a)

(b)

Figure 5.1: The Fourier transform to z^- space of the d valence CT10 PDF. We see that adding more moments allows us to predict the Fourier transform over a larger range in z^- space.

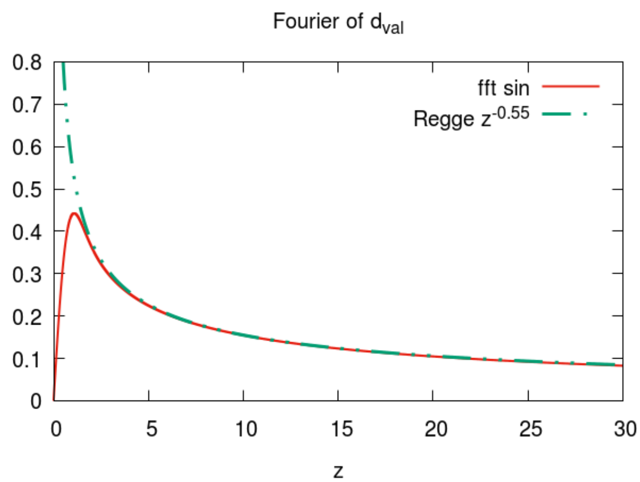


Figure 5.2: The large z^- behavior of the distributions is determined primarily by the Regge part of the PDF.

momenta and helicities p, Λ and p', Λ',

$$
\begin{aligned}
F^{\Gamma}_{\Lambda'\Lambda}(x, \zeta, t) &= \frac{1}{2} \int \frac{dz^-}{2\pi} e^{ixp^+z^-} \\
&\times \quad \langle p', \Lambda' \mid \bar{\psi}\,(0)\, \Gamma\, \mathcal{U}(0, z) \psi\,(z) \mid p, \Lambda \rangle \big|_{z^+=0, z_T=0}\,,
\end{aligned}
\tag{5.1}
$$

where, $\Gamma = \gamma^+, \gamma^+\gamma_5, i\sigma^{+i}$, and the choice of four-momenta is defined with p along the $z = z_3$ axis, $\Delta = p' - p$; the quark light-cone momentum is $k^+ = xp^+$, and we omitted the scale dependence for simplicity (see *e.g.* Ref.[17]).

We take the gauge link equal to one in light cone gauge, γ^+, and the unpolarized forward case for simplicity. This defines the distribution f_1,

$$f_1(x) = \frac{1}{2}\int \frac{dz^-}{2\pi} e^{ixp^+z^-} \langle p \mid \bar{\psi}(0)\,\gamma^+\,\psi\left(z^-\right) \mid p\rangle \,,$$
$$= \frac{1}{2}\int \frac{dz^-}{2\pi} e^{ixp^+z^-} \tilde{f}_1(z^-) \qquad (5.2)$$

The spatial coordinates take on a physical interpretation once we consider Eq.(5.2) in the context of the deep inelastic scattering process described in terms of Feynman diagrams, where the struck quark undergoes a hard scattering.

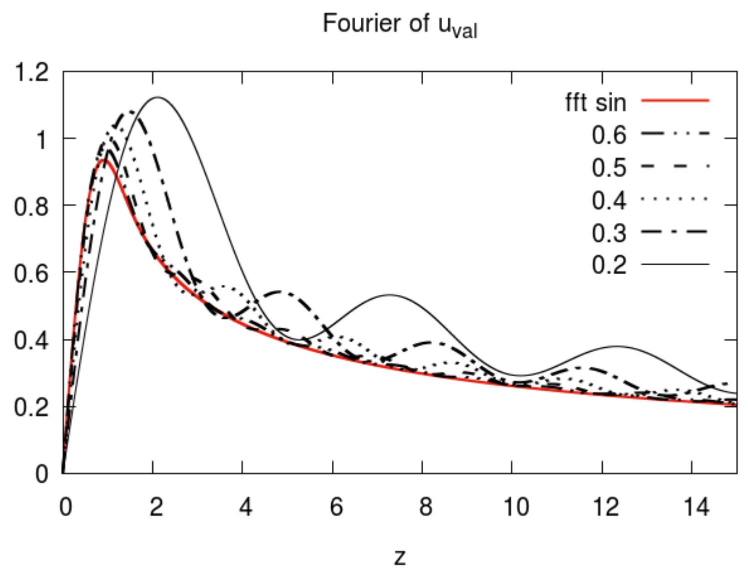
$$\gamma + q \rightarrow \gamma' + q'.$$

The numerator of the quark propagator in the reaction is described by $(\not{k} + \not{q})$, where q is the large momentum transfer. The struck quark (at an initial distance $z = 0$), with large momentum q, travels along the light-cone direction for a distance, z, and is then reabsorbed in the target proton. The distance between these two events is what characterizes the deep inelastic scattering process. One has to be precise about the coordinates, namely there are two longitudinal distances that enter the hadronic tensor, given by

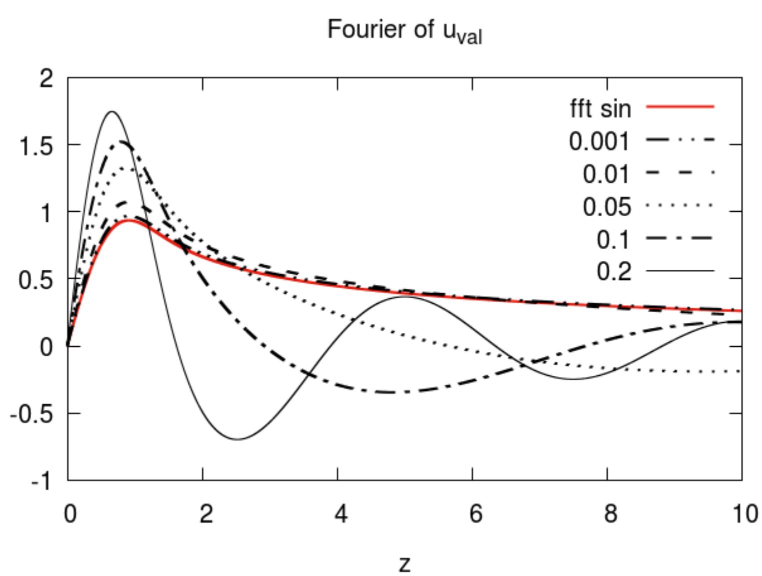
$$(qz) = q^+z^- + q^-z^+ - q_T \cdot z_T$$

In a DIS process, we can choose $q^- \rightarrow \infty$ large corresponding to the small distance scale $z^+ \approx 0$, and simultaneously finite $q^+ = xP^+$ and z^-.

Figure 5.1 illustrates the various features of the coordinate space sine and cosine components. As we can see, we get fairly close match with the actual Fourier transform of the PDF for small z using only a few moments. As shown in Figure5.2,

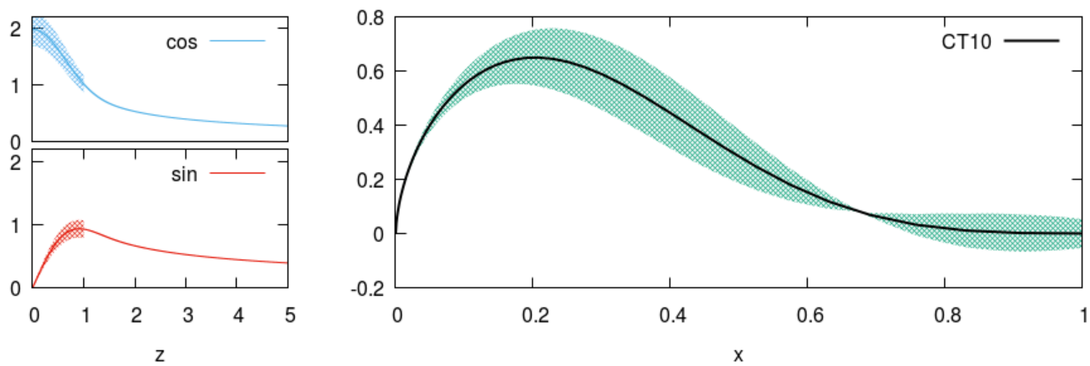


(a)

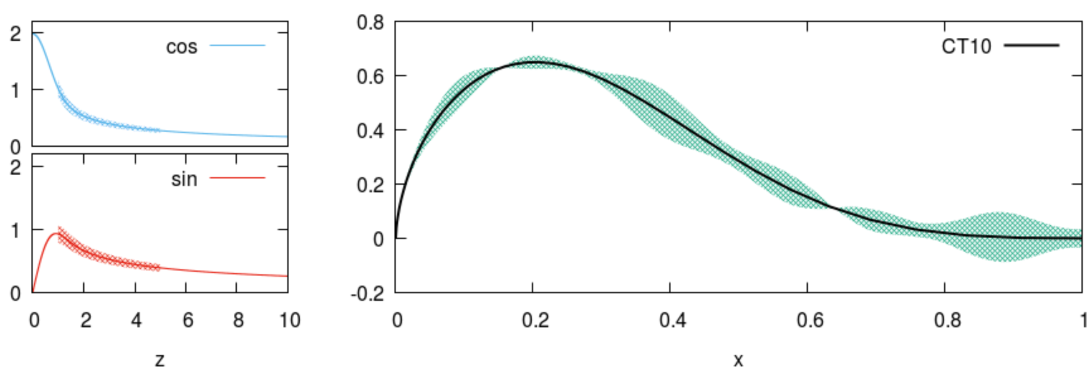


(b)

Figure 5.3: The z^- distributions for different (a) upper and (b) lower cut off values of x.



(a)



(b)

Figure 5.4: We introduce a 15% error in the z^- distributions (left) for (a) $0 < z^- < 1$ and (b) $1 < z^- < 5$. The resulting error on the PDF is shown on the right.

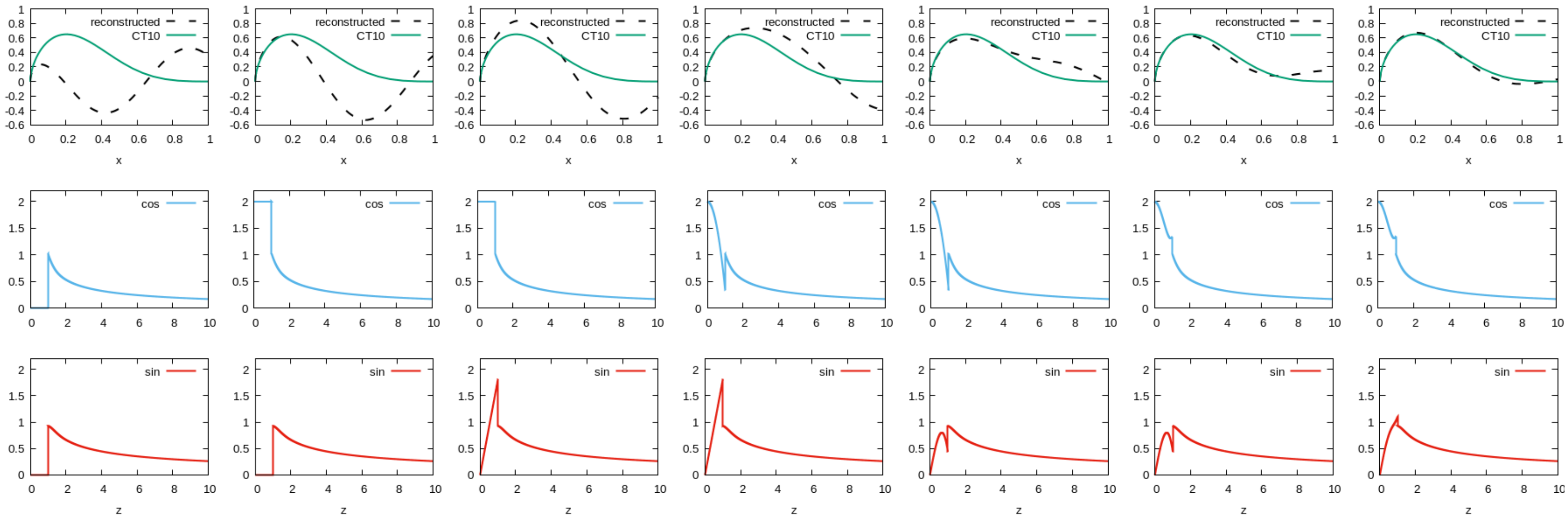
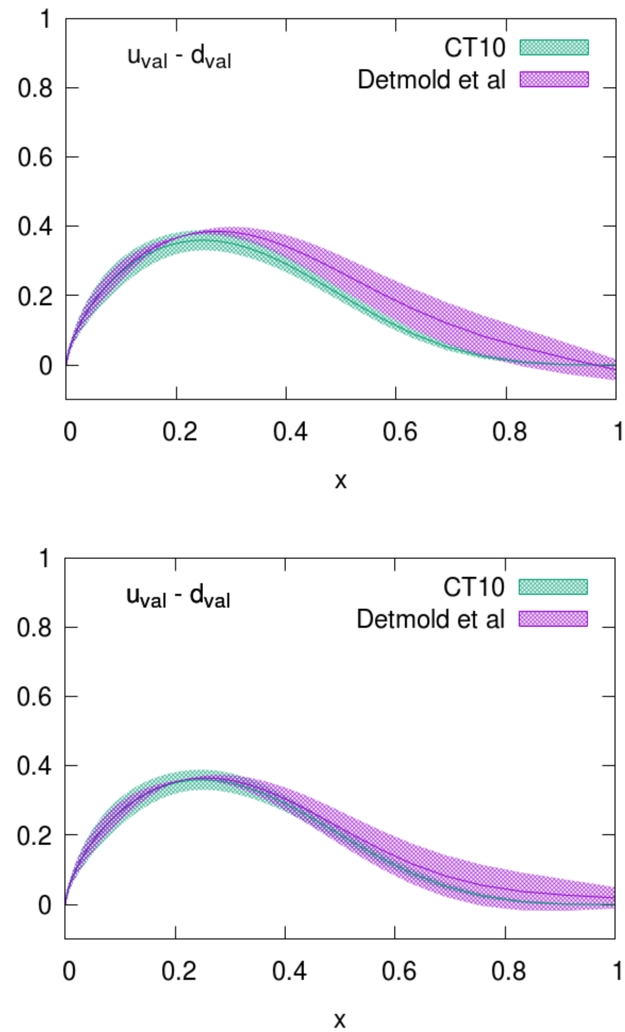


Figure 5.5: Treating the $z^{-} < 1$ region as unknown, we construct the Fourier transforms (the bottom two rows) by successively adding moments moving from left to right. The PDF is reconstructed by taking the inverse Fourier transform of these distributions. The first plot uses no moments while the last uses 6.



(a)

(b)

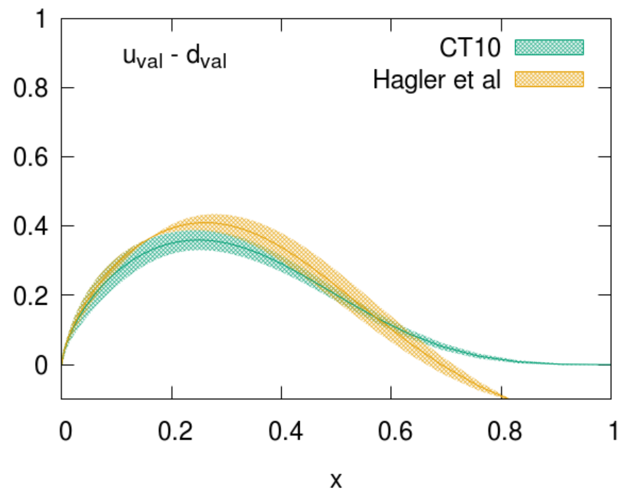
Figure 5.6: PDFs using the first three lattice moments calculated by Detmold et al. (a) and using first four moments (b). Adding more moments brings the reconstructed curve closer to the actual PDF.

the large z portion of the Fourier transform is fixed by a function of the form $z^{-\alpha}$ and is primarily governed by the small x Regge behavior. This type of coordinate space behavior was first pointed out in the Braun Mankiewicz paper.

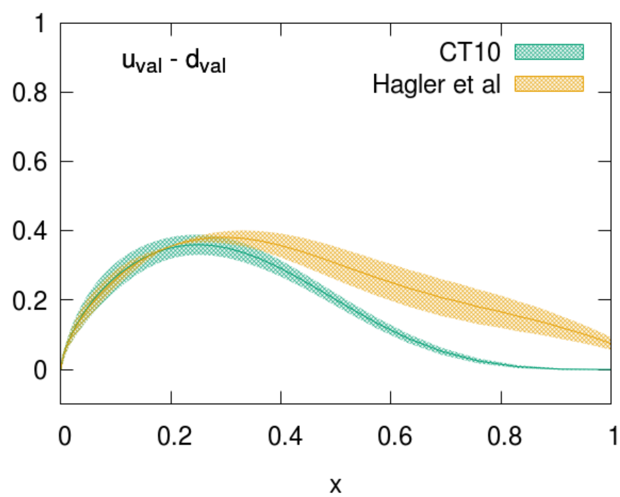
To understand more about which range of x impacts which part of the Fourier transform in z we introduced cut-offs in x above or below which the PDF was taken to be zero and performed the Fourier transform of the cropped function and studied the behavior in z space. Our results are presented in Fig.5.3. To study the same concept, we try a slightly different approach in Fig.5.4. By introducing an error in only certain regions in z, we study how the error propagates in the inverse Fourier transform where we obtain the PDF. We see that while uncertainity in the transition region between the small z and large z regions might produce a significant effect, the error in the large z mainly leads to oscillations.

As a proof of principle, we show in Fig.5.5 the process of reconstructing the PDF from a known parameterization. The column on the left most side shows the reconstruction based on no moments and using only the large z behavior. As we add more and more moments (calculated using the known parameterization), the reconstruction gets closer and closer to the actual function shown as a black dashed curve.

Based on this, we are now ready to use Lattice data to reconstruct the PDF. We can take the first few moments from Lattice QCD calculations and assume Regge behavior as suggested from the DIS data to obtain the PDF in coordinate space. Fig.5.6 demonstrates one such reconstruction using moments from [1] and also illustrates the error generated by using just a few of the moments as compared to the actual Fourier



(a)



(b)

Figure 5.7: In (a) the PDF is reconstructed using the first two lattice moments calculated by Hagler et al. [3]. In (b) we use the third moment as well, however it is not extrapolated to the actual pion mass.

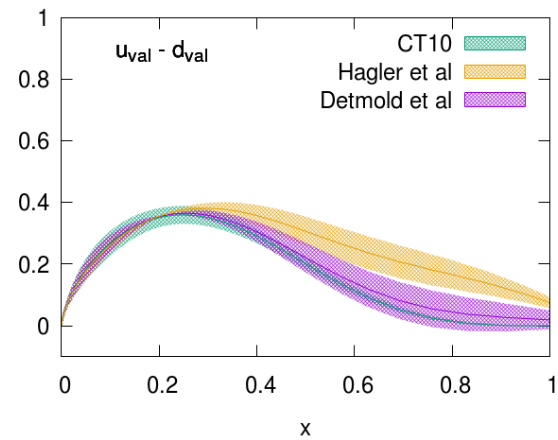


Figure 5.8: A comparison of the two reconstructions using moments from Detmold et al. and Hagler et al.

transform. Fig.5.7 shows a reconstruction using moments from [3]. We provide a comparison between the reconstructions using moments from Detmold et al. and Hagler et al. in Fig.5.8.

Points for the analysis:

– we reconstruct using the first few moments from Lattice and fitting the Regge behavior to the PDF parameterizations

– the functional form in coordinate space is easier because at low z it is given by the Taylor expansion terms. This does not exist in momentum space. We are using the fact that it is easier to reconstruct the PDF in coordinate space knowing the moments than in momentum space.

5.1.1 Extending to GPDs

The moments of GPDs in x give us polynomials in ξ with coefficients known as Compton Form Factors. Hence, using the Lattice QCD calculations of the Compton Form Factors one is able to recreate the z distribution at every ξ. Performing the inverse Fourier transform, one is able to generate the $x\,\xi$ surface of GPDs.

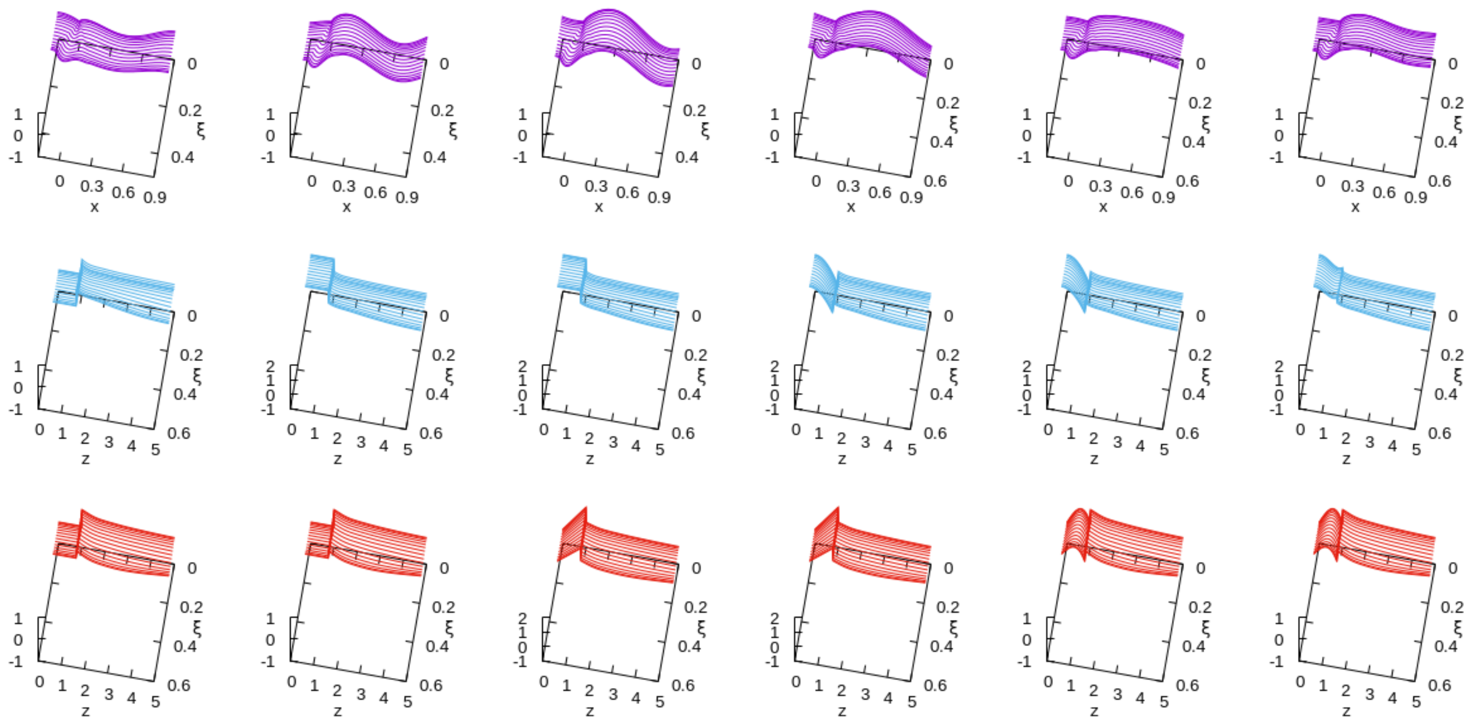


Figure 5.9: Reconstruction of the GPD H from its own moments. The functional form of H to obtain the large z behavior is taken from the diquark model parameterization in[2]

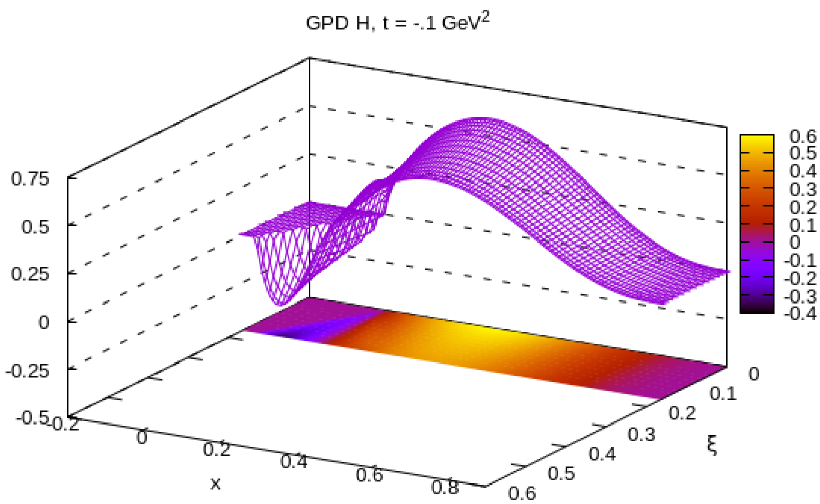


Figure 5.10: The GPD H for u valence quarks at $t = 0.1GeV^2$ using the diquark model parameterization in [2].

Like in the case of PDFs, we first see how well the reconstruction works using a known functional form of the GPD H (depicted in Fig.5.10). The reconstruction is shown in Fig.5.9. Next, we use Compton Form Factor calculations from to reconstruct H. This result is shown in Fig.5.11.

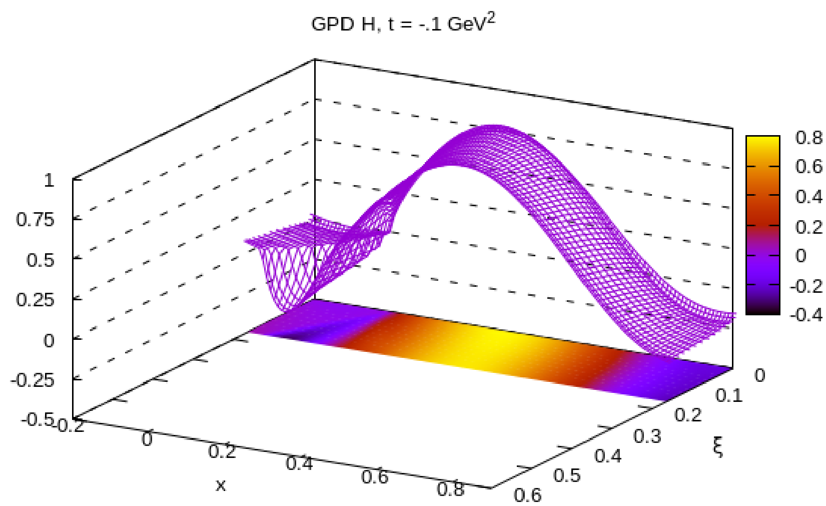


Figure 5.11: Reconstruction of the GPD H for the valence u quarks using Hagler's moments from [3].

Chapter 6

Conclusions and Outlook

We presented the derivation of a set of relations connecting k_T^2-moments of GTMDs and twist-two as well as twist-three GPDs, known as Lorentz Invariance Relations (LIRs) and Equation of Motion (EoM) relations. LIRs stem from the Lorentz structure of the off-forward correlation function. By examining their gauge link structure, we find that two different types of relations exist: one obtained by considering a staple-shaped gauge link, where an explicit quark-gluon-quark contribution appears, and one for the straight gauge link, where this term is instead absent. On the other hand, the QCD equations of motion yield complementary relations containing explicit quark-gluon-quark contributions that have a different structure than the ones in the LIRs. By inserting the LIRs in the equations of motion we can eliminate the k_T^2-moments of GTMDs, and obtain relations directly between twist-two and twist-three GPDs. In the absence of genuine twist-three terms, these relations represent off-forward generalizations of the original Wandzura-Wilzcek relations connecting twist-two and twist-three PDFs.

Within our general scheme of constructing LIRs, we focus particularly on ones involving the k_T^2-moments of the GTMDs F_{14} and G_{11}, which describe the x-density distributions of the quark OAM, L_z, and longitudinal spin-orbit interaction, $L_z S_z$. Our detailed study of the k_T-dependence of these OAM-related observables provides physical insight that buttresses previous suggestions in the literature, stemming from OPE-based integral relations, that partonic OAM is described by twist-three GPDs.

Our results, therefore, represent a step forward in comprehending parton OAM in the proton, on two accounts. On the one hand, the obtained relations are key to accessing information from experiment on the missing piece in the proton's angular momentum budget: we obtain the x-dependent distribution of OAM through the GPDs $\widetilde{E}_{2T}$, H and E, which can be readily measured from various azimuthal angular modulations in DVCS and related processes.
The new x-dependent expressions written in terms of twist-three GPDs including the genuine quark-gluon-quark terms bring, for the first time, partonic OAM within experimental grasp.
On the other hand, taking integrals in x, and using the QCD equations of motion, one recovers the sum rule relating the second Mellin-Barnes moment of a specific twist-three GPD combination, here called $\widetilde{E}_{2T} + H + E$, to the moments of twist-two GPDs yielding the combination $J_q - S_q$. Our result is therefore not only consistent with previous findings hinting at a twist-three nature of OAM [58; 44; 36; 47]: it goes beyond these predictions by providing a physical link, missing from earlier work, which explains how OAM is described at twist-three through its connection with the k_T^2-moment of a GTMD.

The, perhaps, most distinguishing merit of these new relations lies in that they provide a handle on the dynamical underpinnings of the parton correlations through which OAM is generated. OAM is present because of the transverse motion of partons when they are displaced from the origin. This is described in QCD by a twist-three parton correlation; the correlation is generated by the Lorentz invariant structure of the proton matrix elements appearing in the QCD equations of motion.

The LIRs will allow us to directly connect, on the one hand, twist-three GPD measurements of OAM and spin-orbit correlations, and on the other hand, Lattice QCD evaluations of GTMDs. The k_T^2-moment of F_{14} has already been accessed in a preliminary Lattice QCD calculation [41]: GTMD k_T^2-moments can be obtained by generalizing the proton matrix elements of quark bilocal operators used to study TMDs, namely, by supplementing the transverse momentum information with transverse position information through the introduction of an additional nonzero momentum transfer. The calculation in Ref. [41] also includes the gauge connection between the quarks in the quark bilocal operators, enabling the evaluation of both the staple gauge link path used in TMD calculations, characterizing Jaffe-Manohar (JM) OAM, and the straight path yielding Ji OAM. Although this exploration was performed at the pion mass $m_\pi = 518$ MeV, its results suggest a sizable difference between the two definitions.

Our findings provide a perspective for accessing experimentally all terms appearing in both the JM and the Ji definitions: Ji OAM is given by the Wandzura-Wilczek component of $\widetilde{E}_{2T}$, which is described in terms of twist-two GPDs, while JM OAM is given by the sum of these terms and the genuine/intrinsic twist-three contribution, which we identified as an integral over $\mathcal{A}_{F_{14}}$, technically a Lorentz invariance relation violating term. Such a term may be obtained by a careful analysis of DVCS type experiments (see *e.g.* an analogous term in the forward case for the axial vector components $g_1(x)$ and $g_2(x)$, [43]).

Our findings extend to other GTMDs: here, we have treated specifically G_{11}, encoding spin-orbit correlations, and G_{12}, the off-forward extension of g_{1T}, leading to a direct measurement of the color force between quarks.

Understanding the role of GTMDs and twist-three GPDs in quark OAM has initiated a fruitful interaction between phenomenology, theory and Lattice QCD which we intend to pursue further. In particular, the structure of the underlying QCD matrix element suggests the study of experimental processes containing two hadronic reaction planes, one associated with the hadron momentum transfer, and one associated with the transverse momentum of the hadronized ejected quark. We envisage developing the description of such two-jet processes to underpin future experimental efforts to access quark OAM directly from GTMDs. Investigations of experimental hard scattering processes/observables that measure OAM have started, and the opportunity to measure OAM using deeply virtual multiple coincidence exclusive processes will be soon within reach at the new Jlab upgrade and, even more promisingly, at an upcoming Electron Ion Collider (EIC). Having understood the mechanisms that regulate quark OAM in the proton paves the way for future studies of the gluon sector which will be crucial to understand the spin of hadrons.

PDFs and GPDs provide the best window to non perturbative QCD dynamics that can be extracted in a precise and detailed way from experiment. It is important to therefore start answering the question of whether lattice QCD can provide an accurate enough determination of PDFs. This includes understanding both the question of whether the theoretical errors pertaining to the various extraction methods are under control, and whether the accuracy that is (or is predicted to soon be) attainable in lattice calculations can yield results comparable to the experimental errors. It is also important to have physical insight on the way the various lattice components, whether Mellin moments or quasi-PDFs, contribute to PDFs. In this work we attempted to use the existing lattice calculations of PDF moments and GPD moments to reconstruct these functions. Why it is that we are able to describe the gross properties of the proton with such few moments is an open question and a topic of future work.

Bibliography

[1] W. Detmold, W. Melnitchouk, and A. W. Thomas, Mod. Phys. Lett. **A18**, 2681 (2003), arXiv:hep-lat/0310003 [hep-lat] .

[2] G. R. Goldstein, J. O. G. Hernandez, and S. Liuti, (2010), 10.1103/PhysRevD.84.034007, arXiv:1012.3776 .

[3] P. Hagler *et al.* (LHPC), Phys. Rev. **D77**, 094502 (2008), arXiv:0705.4295 [hep-lat] .

[4] M. Boglione and P. J. Mulders, Phys. Rev. **D60**, 054007 (1999), arXiv:hep-ph/9903354 [hep-ph] .

[5] A. Courtoy, G. R. Goldstein, J. O. Gonzalez-Hernandez, S. Liuti, and A. Rajan, Phys. Lett. **B731**, 141 (2014), arXiv:1310.5157 [hep-ph] .

[6] A. Rajan, M. Engelhardt, and S. Liuti, (2017), arXiv:1709.05770 [hep-ph] .

[7] A. Rajan, A. Courtoy, M. Engelhardt, and S. Liuti, Phys. Rev. **D94**, 034041 (2016), arXiv:1601.06117 [hep-ph] .

[8] T. Muta, *Foundations of Quantum Chromodynamics*, 2nd ed. (World Scientific, 1998).

[9] R. L. Jaffe, Lect.Notes Phys. **496**, 178 (1997), arXiv:hep-ph/9602236 [hep-ph] .

[10] K. G. Wilson, Phys. Rev. **179**, 1499 (1969).

[11] D. J. Gross and S. B. Treiman, Phys. Rev. **D4**, 1059 (1971).

[12] R. Jakob, P. J. Mulders, and J. Rodrigues, Nucl. Phys. **A626**, 937 (1997), arXiv:hep-ph/9704335 [hep-ph] .

[13] K. Goeke, A. Metz, and M. Schlegel, Phys. Lett. **B618**, 90 (2005), arXiv:hep-ph/0504130 [hep-ph] .

[14] P. Mulders and R. Tangerman, Nucl. Phys. **B461**, 197 (1996), arXiv:hep-ph/9510301 [hep-ph] .

[15] S. Wandzura and F. Wilczek, Phys. Lett. **B72**, 195 (1977).

[16] D. W. Sivers, Phys. Rev. **D41**, 83 (1990).

[17] S. Meissner, A. Metz, and M. Schlegel, JHEP **0908**, 056 (2009), arXiv:0906.5323 [hep-ph] .

[18] R. D. Tangerman and P. J. Mulders, (1994), arXiv:hep-ph/9408305 [hep-ph] .

[19] D. Boer, P. Mulders, and F. Pijlman, Nucl. Phys. **B667**, 201 (2003), arXiv:hep-ph/0303034 [hep-ph] .

[20] K. Kanazawa, Y. Koike, A. Metz, D. Pitonyak, and M. Schlegel, Phys. Rev. **D93**, 054024 (2016), arXiv:1512.07233 [hep-ph] .

[21] D. Boer, P. J. Mulders, and O. V. Teryaev, Phys. Rev. **D57**, 3057 (1998), arXiv:hep-ph/9710223 [hep-ph] .

[22] R. Jaffe and X.-D. Ji, Phys. Rev. Lett. **67**, 552 (1991).

[23] A. Bacchetta, M. Diehl, K. Goeke, A. Metz, P. J. Mulders, *et al.*, JHEP **0702**, 093 (2007), arXiv:hep-ph/0611265 [hep-ph] .

[24] B. Friedrich and D. Herschbach, Daedalus **127**, 165 (1998).

[25] R. Jaffe and A. Manohar, Nucl. Phys. **B337**, 509 (1990).

[26] X.-D. Ji, Phys. Rev. Lett. **74**, 1071 (1995), arXiv:hep-ph/9410274 [hep-ph] .

[27] X.-D. Ji, Phys. Rev. Lett. **78**, 610 (1997), arXiv:hep-ph/9603249 [hep-ph] .

[28] M. Wakamatsu, Int. J. Mod. Phys. **A29**, 1430012 (2014), arXiv:1402.4193 [hep-ph] .

[29] K.-F. Liu and C. Lorcé, Eur. Phys. J. **A52**, 160 (2016), arXiv:1508.00911 [hep-ph] .

[30] A. Belitsky and A. Radyushkin, Phys.Rept. **418**, 1 (2005), arXiv:hep-ph/0504030 [hep-ph] .

[31] M. Diehl, Phys. Rept. **388**, 41 (2003), arXiv:hep-ph/0307382 [hep-ph] .

[32] K. Kumericki, S. Liuti, and H. Moutarde, Eur. Phys. J. **A52**, 157 (2016), arXiv:1602.02763 [hep-ph] .

[33] C. Lorcé and B. Pasquini, Phys. Rev. **D84**, 014015 (2011), arXiv:1106.0139 [hep-ph] .

[34] C. Lorcé, B. Pasquini, X. Xiong, and F. Yuan, Phys. Rev. **D85**, 114006 (2012), arXiv:1111.4827 [hep-ph] .

[35] Y. Hatta, Phys. Lett. **B708**, 186 (2012), arXiv:1111.3547 [hep-ph] .

[36] Y. Hatta and S. Yoshida, JHEP **1210**, 080 (2012), arXiv:1207.5332 [hep-ph] .

[37] M. Burkardt, Phys. Rev. **D88**, 014014 (2013), arXiv:1205.2916 [hep-ph] .

[38] S. Bhattacharya, A. Metz, and J. Zhou, Phys. Lett. **B771**, 396 (2017), arXiv:1702.04387 [hep-ph] .

[39] Y. Hagiwara, Y. Hatta, and T. Ueda, Phys. Rev. **D94**, 094036 (2016), arXiv:1609.05773 [hep-ph] .

[40] S. Liuti, in *Workshop on High-Intensity Photon Sources (HIPS2017) Mini-Proceedings* (2017) pp. 48–50.

[41] M. Engelhardt, Phys. Rev. **D95**, 094505 (2017), arXiv:1701.01536 [hep-lat] .

[42] K. Goeke, A. Metz, P. V. Pobylitsa, and M. V. Polyakov, Phys. Lett. **B567**, 27 (2003), arXiv:hep-ph/0302028 [hep-ph] .

[43] A. Accardi, A. Bacchetta, W. Melnitchouk, and M. Schlegel, JHEP **0911**, 093 (2009), arXiv:0907.2942 [hep-ph] .

[44] D. Kiptily and M. Polyakov, Eur. Phys. J. **C37**, 105 (2004), arXiv:hep-ph/0212372 [hep-ph] .

[45] C. Lorcé, Phys. Lett. **B735**, 344 (2014), arXiv:1401.7784 [hep-ph] .

[46] M. G. Echevarria, A. Idilbi, K. Kanazawa, C. Lorcé, A. Metz, B. Pasquini, and M. Schlegel, Phys. Lett. **B759**, 336 (2016), arXiv:1602.06953 [hep-ph] .

[47] X. Ji, X. Xiong, and F. Yuan, Phys. Rev. Lett. **109**, 152005 (2012), arXiv:1202.2843 [hep-ph] .

[48] J. Collins, *Foundations of perturbative QCD* (2011).

[49] X. Ji, Phys. Rev. Lett. **110**, 262002 (2013), arXiv:1305.1539 [hep-ph] .

[50] M. Diehl, Eur. Phys. J. **C19**, 485 (2001), arXiv:hep-ph/0101335 [hep-ph] .

[51] J. B. Kogut and D. E. Soper, Phys. Rev. **D1**, 2901 (1970).

[52] H. D. Politzer, Nucl. Phys. **B172**, 349 (1980).

[53] W. Armstrong, M. Engelhardt, S. Liuti, and A. Rajan, in preparation .

[54] B. U. Musch, P. Hägler, M. Engelhardt, J. W. Negele, and A. Schäfer, Phys. Rev. **D85**, 094510 (2012), arXiv:1111.4249 [hep-lat] .

[55] M. Engelhardt, P. Hägler, B. Musch, J. Negele, and A. Schäfer, Phys. Rev. **D93**, 054501 (2016), arXiv:1506.07826 [hep-lat] .

[56] B. Yoon, M. Engelhardt, R. Gupta, T. Bhattacharya, J. R. Green, B. U. Musch, J. W. Negele, A. V. Pochinsky, A. Schäfer, and S. N. Syritsyn, (2017), arXiv:1706.03406 [hep-lat] .

[57] G. R. Goldstein, J. O. Gonzalez-Hernandez, and S. Liuti, J.Phys. **G39**, 115001 (2012), arXiv:1201.6088 [hep-ph] .

[58] M. Penttinen, M. V. Polyakov, A. Shuvaev, and M. Strikman, Phys. Lett. **B491**, 96 (2000), arXiv:hep-ph/0006321 [hep-ph] .

[59] A. Bacchetta and M. Radici, Phys. Rev. **D69**, 074026 (2004), arXiv:hep-ph/0311173 [hep-ph] .

[60] A. Bhoonah and C. Lorcé, (2017), arXiv:1703.08322 [hep-ph] .

[61] A. V. Efremov, O. V. Teryaev, and E. Leader, Phys. Rev. **D55**, 4307 (1997), arXiv:hep-ph/9607217 [hep-ph] .

[62] D. Flay *et al.* (Jefferson Lab Hall A), Phys. Rev. **D94**, 052003 (2016), arXiv:1603.03612 [nucl-ex] .

[63] H.-W. Lin, *Proceedings, 34th International Symposium on Lattice Field Theory (Lattice 2016): Southampton, UK, July 24-30, 2016*, PoS **LATTICE2016**, 005 (2016), arXiv:1612.09366 [hep-lat] .

[64] W. Detmold, W. Melnitchouk, and A. W. Thomas, Eur. Phys. J.direct **3**, 13 (2001), arXiv:hep-lat/0108002 [hep-lat] .

[65] W. Detmold, W. Melnitchouk, and A. W. Thomas, *Lattice hadron physics. Proceedings, 2nd Topical Workshop, LHP 2003, Cairns, Australia, July 22-30, 2003*, Nucl. Phys. Proc. Suppl. **128**, 148 (2004), [,148(2004)].

[66] H.-W. Lin, J.-W. Chen, S. D. Cohen, and X. Ji, Phys. Rev. **D91**, 054510 (2015), arXiv:1402.1462 [hep-ph] .

[67] J.-W. Chen, S. D. Cohen, X. Ji, H.-W. Lin, and J.-H. Zhang, Nucl. Phys. **B911**, 246 (2016), arXiv:1603.06664 [hep-ph] .

[68] C. Alexandrou, K. Cichy, V. Drach, E. Garcia-Ramos, K. Hadjiyiannakou, K. Jansen, F. Steffens, and C. Wiese, Phys. Rev. **D92**, 014502 (2015), arXiv:1504.07455 [hep-lat] .

[69] C. E. Carlson and M. Freid, Phys. Rev. **D95**, 094504 (2017), arXiv:1702.05775 [hep-ph] .

[70] R. A. Briceno, J. J. Dudek, and R. D. Young, (2017), arXiv:1706.06223 [hep-lat] .

[71] G. C. Rossi and M. Testa, (2017), arXiv:1706.04428 [hep-lat] .

[72] A. V. Radyushkin, (2017), arXiv:1705.01488 [hep-ph] .

[73] B. L. Ioffe, Phys. Lett. **B30**, 123 (1969).

[74] V. Braun, P. Gornicki, and L. Mankiewicz, Phys. Rev. **D51**, 6036 (1995), arXiv:hep-ph/9410318 [hep-ph] .

[75] K. Orginos, A. Radyushkin, J. Karpie, and S. Zafeiropoulos, (2017), arXiv:1706.05373 [hep-ph] .

[76] PDFLattice2017, (22-24 Mar 2017. Oxford, United Kingdom), arXiv:C17-03-22 .

[77] L. Mankiewicz and T. Weigl, Phys. Lett. **B380**, 134 (1996), arXiv:hep-ph/9604382 [hep-ph] .

[78] P. Hoyer and M. Vanttinen, Z. Phys. **C74**, 113 (1997), arXiv:hep-ph/9604305 [hep-ph] .

[79] M. Vanttinen, G. Piller, L. Mankiewicz, W. Weise, and K. J. Eskola, Eur. Phys. J. **A3**, 351 (1998), arXiv:hep-ph/9808330 [hep-ph] .

[80] A. Abdel-Rehim *et al.*, Phys. Rev. **D92**, 114513 (2015), [Erratum: Phys. Rev.D93,no.3,039904(2016)], arXiv:1507.04936 [hep-lat] .

[81] J. Green, M. Engelhardt, S. Krieg, J. Negele, A. Pochinsky, *et al.*, (2012), arXiv:1209.1687 [hep-lat] .

[82] G. Bali, S. Collins, M. Gckeler, R. Rdl, A. Schfer, and A. Sternbeck, *Proceedings, 33rd International Symposium on Lattice Field Theory (Lattice 2015): Kobe, Japan, July 14-18, 2015*, PoS **LATTICE2015**, 118 (2016), arXiv:1601.04818 [hep-lat] .