Representations of quantum groups at roots of unity and their reductions mod p to algebraic group representations

Hankyung Ko
Seoul, South Korea

B.S., Seoul National University, 2011

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Department of Mathematics

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Brian Parshall

Leonard Scott

Weiqiang Wang

Daniel Keenan

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Introduction

Let G be a semisimple simply connected algebraic group over a field k of characteristic $p > 0$. We also assume $p \neq 2$. Consider G-mod, the category of rational G-representations. Let R be the root system of G. Let U_ζ be the Lusztig quantum group associated to the root system R, over a field K of characteristic zero, specialized at a root of unity $\zeta \in K$. Consider U_ζ-mod, the category of finite dimensional weight U_ζ-modules of type 1. The two cases are closely related if the root of unity ζ has the order equals to p. The modules in the two cases have weights in the same weight lattice X; they are both highest weight categories with the poset X^+ of dominant weights (note, however, that G-mod does not have projectives and has enough (infinite dimensional) injectives, while U_ζ-mod has both enough injectives and projectives); the standard modules in two cases, indexed by the same highest weight $\gamma \in X^+$, have the same character $\chi(\gamma)$ given by Weyl's formula; we have, in both cases, the linkage principle involving the affine Weyl group action; the translation functors between two orbits are defined in the same way and share similar properties in the two cases; the standard modules in both cases have certain filtrations satisfying a sum formula (Jantzen filtration); the Frobenius kernel G_1 and the small quantum group $\mathbf{u}$ provide infinitesimal versions (G_1T-mod and $\mathbf{u}U_\zeta^0$-mod) of G-mod and U_ζ-mod. General theories including most of these can be found in [16, II] for the algebraic group representations and in [4] or [16, II.H] for the quantum group representations. In case p is large enough, the infinitesimal versions of G-mod and U_ζ-mod are even described using a common combinatorial category (Andersen-Jantzen-Soergel [2]), which implies that the multiplicities of an irreducible module in a standard module (hence the irreducible characters) in the two cases are the same if the weights involved are small.

For general p, we have a better understanding on the quantum case than on the algebraic group case. Since the Kazhdan-Lusztig correspondence (see §1.5) provides, under little restriction on p, an equivalence between U_ζ-mod and a certain subcategory of the affine Lie category O, most of what is known for the affine Lie algebra representations directly applies to the quantum case. On the affine Lie algebra side:

(1) The characters of the irreducible modules are calculated in terms of the Kazhdan-Lusztig polynomials $P_{x,y}(t)$ associated to the affine Weyl group. (Kashiwara-Tanisaki [17, 18] and Kazhdan-Lusztig [21])

(2) We have a standard Koszul grading. (Shan-Varagnolo-Vasserot [40])

By (1), the characters for the irreducible U_ζ-modules in a regular orbit are explicitly expressed as follows. Writing regular weights as $w.\lambda$ with λ in the top antidominant (p-)alcove C^- and w in the affine Weyl group W_p,

$$\mathrm{ch}\, L_\zeta(w.\lambda) = \sum_{y.\lambda\in X^+} (-1)^{l(w)-l(y)} P_{y,w}(-1)\chi(y.\lambda), \tag{0.0.1}$$

where $L_\zeta(w.\lambda)$ is the irreducible U_ζ-module of highest weight $w.\lambda \in X^+$.

The formula (0.0.1) (the Lusztig character formula) also give a homological interpretation of the Kazhdan-Lusztig polynomials as the dimensions of Ext between a standard module and an irreducible module

$$\sum_{n=0}^{\infty} \dim \mathrm{Ext}^n_{U_\zeta}(\Delta_\zeta(y.\lambda), L_\zeta(w.\lambda))t^n = t^{l(w)-l(y)} P_{y,w}(t^{-1}), \tag{0.0.2}$$

and between irreducible modules

$$\sum_{n=0}^{\infty} \dim \mathrm{Ext}^n_{U_\zeta}(L_\zeta(y.\lambda), L_\zeta(w.\lambda))t^n = \sum_{z\in W_p} t^{l(w)+l(y)-2l(z)} P_{z,w}(t^{-1})P_{z,y}(t^{-1}), \tag{0.0.3}$$

for all $y.\lambda, w.\lambda \in X^+$. (The weight λ is still regular.)

The regular condition in (0.0.1) is not a real restriction, because applying the translation functor to the formula (0.0.1) provide irreducible characters in the orbit of a singular weight μ with no difficulty. However, we cannot do the same for the homological formulas (0.0.2), (0.0.3) because the degree information adds much uncertainty in putting the regular formulas together. This can be done using the grading result (2) above. We do this in Part II. The result proves a conjecture of Parshall-Scott ([36, Conjecture III]) and is similar to the finite dimensional semisimple Lie algebra result of Soergel [41].

How about the algebraic group case? We don't have (1) in the algebraic group case in general (see §3.2 for more discussion). We need two significant restrictions, even in the regular case, when interpreting the Kazhdan-Lusztig polynomial: The weights need be small, the prime p needs be large. We don't know, in particular, the dimensions of Ext spaces between arbitrary irreducible G-modules. A possible solution is to consider the "reduction mod p" from the quantum case to the algebraic group case (see §1.3). If one replaces the irreducible modules in the homological formulas (0.0.2), (0.0.3) by appropriate reduction mod p modules (and, of course, replace Ext_{U_ζ} by Ext_G), then the formula, for all (regular) weights, is shown in [9] to be valid for $p \gg 0$. This gets rid of the "small weights" restriction. A conjecture of Parshall-Scott ([36, Conjecture II]) is that this should work for any p (under the Kazhdan-Lusztig equivalence), which will get rid of the other restriction. "One half" of this conjecture follows from Proposition 12.2.

Note here that there still remains the problem of "translating" the regular result to singular blocks. In fact, [36, Conjecture II] has no restriction on the singularity of the weights. Similarly to the quantum case, this will follow from the regular formula if we have an analogue of (2) for the algebraic group case. This is related to [36, Conjecture I], where a particular associated graded algebra of a finite dimensional (quasi-hereditary) algebra A, where A-mod is equivalent to a subcategory (truncation by a finite poset ideal) of G-mod, is conjectured to be "(standard) Q-Koszul". (We explain this sentence in §4.2.)

Now we consider a p^r-th root of unity ζ instead of a p-th root of unity. Everything in the previous paragraphs makes sense since we can still reduce mod p from the quantum side to G-mod. This reduction mod p method, in fact, is not new and was studied by Lin in [25]. Part III concerns the reduction mod p from quantum groups at p^r-th roots of unity. We check that the $r > 1$ analogues of the conjectures above are not true. We also prove that the extensions in U_ζ-mod reduces mod p to extensions in G-mod. To be more precise, the dimension of Ext^n in G between reduction mod p modules is greater than the dimension of Ext^n between the corresponding modules in U_ζ-mod. As a corollary, we obtain a result similar to that of Franklin [14] on the maps between standard modules.

The thesis starts with a preliminary part (Part I), where we explain our settings (§1 and §2.1, §2.2); review the translation functors in §2.3; discuss the Lusztig conjectures (§3.2) and the Parshall-Scott conjectures (§4) we have mentioned in the introduction.

Part I. Preliminaries

We assume that p is an odd prime.

1. The representation categories

This section introduces the main categories appearing in the thesis and explain how they are related.

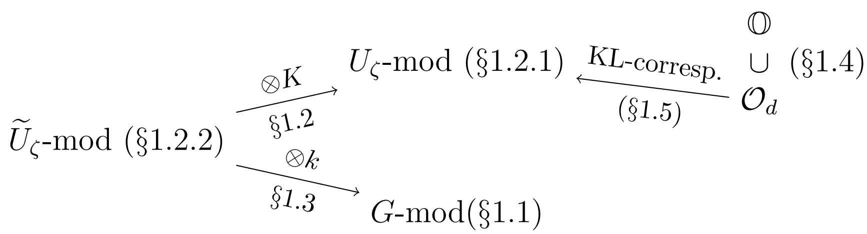

Figure 1. Character map with the two protagonists in the middle

1.1. **The algebraic group case.** Let G be a semisimple simply connected algebraic group over a field k of characteristic p. We also assume that G is defined and split over $\mathbb{F}_p \subset k$. Fix a maximal (split) torus $T \subset G$ and a Borel subgroup $B \supset T$ in G. Let R be the root system of G, Σ be the set of simple roots, R^+ the set of positive roots, $X = X(T)$ be the set of weights, $X^+ = X^+(T)$ be the set of dominant weights. A general theory for the algebraic group G and its representations is well explained in [16, II].

Our interest is on G-mod, the category of rational G-modules. The category G-mod is abelian, has enough injectives, and is a highest weight category in the sense of Cline-Parshall-Scott [6] with the infinite poset $(X^+, \uparrow)$. (See §2.1 for the ordering $\uparrow$.) For each $\gamma \in X^+$, we denote the standard object of highest weight γ by $\Delta(\gamma)$ (which is the Weyl module often denoted by $V(\gamma)$), the costandard object by $\nabla(\gamma)$ (which is the induced module and is denoted by $H^0(\gamma)$ in [16]), the simple object by $L(\gamma)$, and the tilting object by $X(\gamma)$.

1.2. **The quantum case and the integral case.** Let R be a semisimple simply connected (finite classical) root system as in §1.1. Let $(-,-)$ be a scalar product on $X(T) \otimes_{\mathbb{Z}} \mathbb{R}$ such that the smaller one among the integers (α, α) for $\alpha \in R$ is 2. The quantum enveloping algebra $U_v = U_v(R)$ is defined over the function field $\mathbb{Q}(v)$ by generators $E_\alpha, F_\alpha, K_\alpha, K_\alpha^{-1}$ $(\alpha \in \Sigma)$ and relations

$$K_\alpha K_\alpha = 1 = K_\alpha^{-1} K_\alpha, \;\; K_\alpha K_\beta = K_\beta K_\alpha,$$

$$K_\alpha E_\beta K_\alpha^{-1} = v^{(\alpha,\beta)} E_\beta,$$

$$K_\alpha F_\beta K_\alpha^{-1} = v^{-(\alpha,\beta)} F_\beta,$$

$$E_\alpha F_\beta - F_\beta E_\alpha = \delta_{\alpha\beta} \frac{K_\alpha - K_\alpha^{-1}}{v^{d_\alpha} - v^{-d_\alpha}}$$

$$\sum_{s=0}^{1-a_{\alpha\beta}} (-1)^s \begin{bmatrix} 1 - a_{\alpha\beta} \\ s \end{bmatrix}_\alpha E_\alpha^{1-a_{\alpha\beta}-s} E_\beta E_\alpha^s = 0,$$

$$\sum_{s=0}^{1-a_{\alpha\beta}} (-1)^s \begin{bmatrix} 1 - a_{\alpha\beta} \\ s \end{bmatrix}_\alpha F_\alpha^{1-a_{\alpha\beta}-s} F_\beta F_\alpha^s = 0,$$

where $d_\alpha = \frac{(\alpha,\alpha)}{2}$ (which can be $1, 2$, or 3) and $a_{\alpha\beta} = \langle \beta, \alpha^\vee \rangle = \frac{(\beta,\alpha)}{d_\alpha}$ for $\alpha, \beta \in \Sigma$. The coefficients in the last two relations are defined as follows. Set for each $\alpha \in \Sigma$ and $n \in \mathbb{Z}$

$$[n]_\alpha = \frac{v^{nd_\alpha} - v^{-nd_\alpha}}{v^{d_\alpha} - v^{-d_\alpha}},$$

and define the *Gaussian binomial coefficients* for ($n \in \mathbb{Z}$ and) $m \in \mathbb{Z}_{\geq 0}$ as

$$\begin{bmatrix} n \\ m \end{bmatrix}_\alpha = \frac{[n]_\alpha [n-1]_\alpha \cdots [n-m+1]_\alpha}{[m]_\alpha [m-1]_\alpha \cdots [1]_\alpha}$$

if $m > 0$ and

$$\begin{bmatrix} n \\ 0 \end{bmatrix}_\alpha = 1.$$

Letting U_v^0 denote the subalgebra of U_v generated by $\{K_\alpha^{\pm 1}\}$, U_v^+ the subalgebra generated by $\{E_\alpha\}$, U_v^- the subalgebra generated by the $\{F_\alpha\}$, we have the triangular decomposition

$$U_v^- \otimes_{\mathbb{Q}(v)} U_v^0 \otimes_{\mathbb{Q}(v)} U_v^+ \xrightarrow[\text{mult}]{\sim} U_v \tag{1.2.1}$$

induced by the muliplication in U_v.

The algebra U_v has an integral form $U_{\mathscr{A}}$ over $\mathscr{A} := \mathbb{Z}[v, v^{-1}]$. It is defined as the $\mathscr{A}$-subalgebra of U_v generated by the elements $K_\alpha^{\pm 1}$ and the v-divided powers

$$E_\alpha^{(n)} = \frac{E_\alpha^n}{[n]_\alpha^!}, \qquad F_\alpha^{(n)} = \frac{F_\alpha^n}{[n]_\alpha^!}$$

for $n > 0$, where $[n]_\alpha^! = [n]_\alpha [n-1]_\alpha \cdots [1]_\alpha$. It is shown in [26, 28] that $U_{\mathscr{A}}$ also has a presentation by generators and relations compatible with the presentation for U_v. In particular, the triangular decomposition (1.2.1) restricts to $U_{\mathscr{A}}$, giving

$$U_{\mathscr{A}}^- \otimes_{\mathscr{A}} U_{\mathscr{A}}^0 \otimes_{\mathscr{A}} U_{\mathscr{A}}^+ \xrightarrow{\sim} U_{\mathscr{A}}.$$

Now, given an $\mathscr{A}$-algebra $\mathscr{B}$, one can take the tensor product $U_{\mathscr{A}} \otimes_{\mathscr{A}} \mathscr{B}$ to define the quantum group over $\mathscr{B}$.

Let r be a positive integer and $\zeta \in \mathbb{C}$ be a primitive p^r-th root of unity. Then specializing v to ζ will give quantum algebras $U_{\mathscr{A}} \otimes \mathbb{Z}[\zeta]$ and $U_{\mathscr{A}} \otimes \mathbb{Q}(\zeta)$ at the root of unity ζ. For our purpose of relating the quantum group representations to G-mod, a modification on the base rings is necessary: Instead of considering the above specializations, we take the quantum algebras over a discrete valuation ring $\mathscr{O}$ with the maximal ideal (π), so that the residue field $\mathscr{O}/(\pi)$ is isomorphic to a field k of characteristic p and the quotient field of $\mathscr{O}$ is a field K of characteristic zero. Such a triple $(K, \mathscr{O}, k)$ is called a p-modular system. For example, we take the localization $\mathscr{O} := \mathbb{Z}[\zeta]_{(\zeta-1)}$ in $\mathbb{C}$. (If we don't want to start with taking some $\zeta \in \mathbb{C}$, we can set this up as follows. Consider the localization $\mathscr{A}_{(v-1,p)}$ and let $\mathscr{O} = \mathscr{A}_{(v-1,p)}/(1 + v + \cdots + v^{p^r-1})$. Then the image of v in $\mathscr{O}$ is a primitive p^r-th root of unity, which we rename to ζ. In this case, the residue field k of $\mathscr{O}$ is the prime field $\mathbb{F}_p$, and the quotient field K of $\mathscr{O}$ is contained in $\mathbb{C}$.

We denote by U_ζ the quantum group $U_v \otimes_{\mathbb{Q}(v)} K = (U_v \otimes_{v \mapsto \zeta} \mathbb{Q}(\zeta)) \otimes_{\mathbb{Q}(\zeta)} K$ thus obtained. The integral form $U_{\mathscr{O}}$ will be denoted by $\widetilde{U}_\zeta$.

1.2.1. *The quantum case.* The quantum case in this thesis refers to U_ζ-mod, the category of finite dimensional weight U_ζ-modules of type 1. We say a (weight) U_ζ-module M has type 1 if the central elements $K_\alpha^{p^r}$ act on M as an identity. The weight lattice $X(U_\zeta^0)$ for U_ζ-mod is identified with the weight lattice $X(T)$ of G, whence we write X for this common weight lattice. Then U_ζ -mod is a highest weight category with the poset X^+ of dominant weights. We denote its standard objects by $\Delta_\zeta(\gamma)$, costandard objects by $\nabla_\zeta(\gamma)$, simple objects by $L_\zeta(\gamma)$, and tilting objects by $X_\zeta(\gamma)$ where $\gamma \in X^+$. Another important point is that U_v, and hence U_ζ, is a Hopf algebra. Thus, U_ζ-mod has a tensor product. A general theory for U_ζ is developed in [4]. See also [1].

1.2.2. *The integral case.* The integral version of the category U_ζ-mod is $\widetilde{U}_\zeta$-mod, the category of finitely generated (over $\mathscr{O}$) weight $\widetilde{U}_\zeta$-modules of type 1. As in the algebraic group case and the quantum case, the highest weights of highest weight modules are indexed by the dominant weights. Also as in the two cases, we have a tensor product of $\widetilde{U}_\zeta$-modules using the Hopf algebra structure of $\widetilde{U}_\zeta$.

1.3. **Reduction mod p.** We explain how the integral case provides a direct connection between the representation theory of G and that of U_ζ. Recalling

$$\widetilde{U}_\zeta/(\{K_\alpha - 1\}_{\alpha\in\Sigma}) \otimes_{\mathscr{O}} k \cong \mathrm{Dist}(G)$$

from [28], we see that a module $\widetilde{M}$ in $\widetilde{U}_\zeta$-mod "reduces mod p" to a module $\widetilde{M} \otimes_{\mathscr{O}} k$ in G-mod. (Note here that K_α acts as 1 on the type 1 module $\widetilde{M}_k$.)

We first relate the standard and costandard modules in G-mod and U_ζ-mod. An $\mathscr{O}$-submodule $\widetilde{M}$ of a U_ζ-module M is called an admissible lattice if $\widetilde{M}$ is $\mathscr{O}$-free, $\widetilde{U}_\zeta$-invariant and K-generates M (i.e., $\widetilde{M} \otimes_{\mathscr{O}} K \cong M$). The admissible lattice $\widetilde{M}$ has a decomposition into weight $\mathscr{O}$-free modules such that $\widetilde{M}_\gamma \otimes_{\mathscr{O}} K \cong M_\gamma$ for each $\gamma \in X$. Choose a minimal admissible lattice $\widetilde{\Delta}_\zeta(\gamma)$ in $\Delta_\zeta(\gamma)$. This is done simply by picking a highest weight vector v in $\Delta_\zeta(\gamma)$ and letting $\widetilde{\Delta}_\zeta(\gamma) := \widetilde{U}_\zeta v$. For the costandard modules, we dualize[1] this to take an admissible lattice $\widetilde{\nabla}_\zeta(\gamma)$ in $\nabla_\zeta(\gamma)$ rather than dealing with the problem of what is a maximal lattice. Then we have

$$\widetilde{\Delta}_\zeta(\gamma)_K \cong \Delta_\zeta(\gamma), \qquad \widetilde{\nabla}_\zeta(\gamma)_K \cong \nabla_\zeta(\gamma).$$

and

$$\widetilde{\Delta}_\zeta(\gamma)_k \cong \Delta(\gamma), \qquad \widetilde{\nabla}_\zeta(\gamma)_k \cong \nabla(\gamma).$$

(Write $M_K := M \otimes_{\mathscr{O}} K$, $M_k := M \otimes_{\mathscr{O}} k$ for an $\mathscr{O}$-module M.) So far we don't get any new G-modules. The irreducible U_ζ-modules will give rise to the new modules of our interest. Let's do that.

Take a minimal admissible lattice $\widetilde{L}_\zeta^{\min}(\gamma)$ in $L_\zeta(\gamma)$ and its dual $\widetilde{L}_\zeta^{\max}(\gamma)$ in $L_\zeta(\gamma)$ (Note that $L_\zeta(\gamma)$ is self-dual if we take the "τ-dual". See footnote 1) Then define

$$\Delta_r^{\mathrm{red}}(\gamma) := (\widetilde{L}_\zeta^{\min}(\gamma))_k, \qquad \nabla_{\mathrm{red}}^r(\gamma) := (\widetilde{L}_\zeta^{\max}(\gamma))_k.$$

These modules are not irreducible in general. In fact, they can be pretty big, as we see in the second sentence of the following observation.

Proposition 1.1. *Let $\gamma \in X^+$. There is a surjective map $\Delta(\gamma) \to \Delta_r^{\mathrm{red}}(\gamma)$ (in G-mod) for all $r \in \mathbb{N}$. It is an isomorphism if $\Delta_\zeta(\gamma) \cong L_\zeta(\gamma)$ (in U_ζ-mod).*

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[1] We take the "τ-dual" as in [16, II.2.12]. The action of G on the dual module is twisted by the antiautomorphism $\tau : G \to G$ that swaps the positive roots and the negative roots. See [16, II.1.16] for details on τ. Alternatively, we can use a linear dual in defining the maximal lattice as follows. Since the linear dual of $\nabla(\gamma)$ is isomorphic to $\Delta(-w_0\lambda)$, where w_0 is the longest element in the finite Weyl group W, we can let $\widetilde{\nabla}(\gamma)$ to be the dual of $\widetilde{\Delta}(-w_0\gamma)$ in $\Delta_\zeta(-w_0\gamma)$.

Proof. We may assume that $\widetilde{L}_\zeta^{\min}(\gamma) = \widetilde{U}_\zeta \overline{v}$ where $\overline{v}$ is the image of a highest weight vector v in $\Delta_\zeta(\gamma)$ under the surjective map $\Delta_\zeta(\gamma) \twoheadrightarrow L_\zeta(\gamma)$. We also take $\widetilde{\Delta}_\zeta(\gamma) = \widetilde{U}_\zeta v$. Now the map

$$\widetilde{\Delta}(\gamma) \twoheadrightarrow \widetilde{L}^{\min}(\gamma)$$

given by $v \mapsto \overline{v}$ induces the desired surjection

$$\Delta(\gamma) \twoheadrightarrow \Delta_r^{\mathrm{red}}(\gamma)$$

if we apply the exact functor $- \otimes_{\mathscr{O}} k$. The second claim is trivial. □

Corollary 1.2. *For $\gamma \in X^+$, we have*

$$\Delta(\gamma) \twoheadrightarrow \Delta_r^{\mathrm{red}}(\gamma) \twoheadrightarrow L(\gamma)$$

and

$$L(\gamma) \hookrightarrow \nabla^r_{\mathrm{red}}(\gamma) \hookrightarrow \nabla(\gamma).$$

Proof. By Proposition 1.1 and its dual, it is enough to check that $\Delta_r^{\mathrm{red}}(\gamma)$ and $\nabla^r_{\mathrm{red}}(\gamma)$ are not zero. But they arise from (nonzero) $\mathscr{O}$-free lattices, hence cannot be zero. □

1.4. **The affine case.** Let $\mathfrak{g} = \mathfrak{g}_K$ be the Lie algebra associated to R. It contains a Cartan subalgebra $\mathfrak{h}$, and a Borel subalgebra $\mathfrak{b} \supset \mathfrak{h}$. The affine Kac-Moody algebra $\widehat{\mathfrak{g}}$ is defined as $\widehat{\mathfrak{g}} = (\mathfrak{g} \otimes K[t, t^{-1}]) \oplus K\mathbf{c} \oplus K\mathbf{d}$. We do not give the multiplication here but refer to [5, Ch.18]. Its (affine) Cartan is $\widehat{\mathfrak{h}} = \mathfrak{h} \oplus K\mathbf{c} \oplus K\mathbf{d}$ and the affine Borel is $\widehat{\mathfrak{b}} = \mathfrak{b} \oplus \mathfrak{g} \otimes tK[t] \oplus K\mathbf{c} \oplus K\mathbf{d}$. The category O for $\widehat{\mathfrak{g}}$ is defined to consist of the $\widehat{\mathfrak{g}}$-modules that are weight and locally $\widehat{\mathfrak{b}}$-finite. We denote this category O by $\mathbb{O}$. The algebra $\widetilde{\mathfrak{g}}$ is defined as $\widetilde{\mathfrak{g}} = [\widehat{\mathfrak{g}}, \widehat{\mathfrak{g}}]$. We have $\widetilde{\mathfrak{g}} = \mathfrak{g} \otimes K[t, t^{-1}] \oplus K\mathbf{c}$. Denoted by $\mathcal{O}$ is the category O for $\widetilde{\mathfrak{g}}$.

The category $\mathcal{O}$, as well as $\mathbb{O}$, is a highest weight category (as in [6]). Since the (integral) weight lattices $\widetilde{X}$, for $\widetilde{\mathfrak{g}}$ and $\widehat{X}$ for $\widehat{\mathfrak{g}}$ are different from the weight lattice X

for G and U_ζ, the highest weights for $\mathcal{O}$, $\mathbb{O}$ live in different posets $\widetilde{X}^+$, $\widehat{X}^+$. Therefore, it should not be confusing if we denote the standard, costandard, irreducible objects in $\mathcal{O}$ by $\Delta(\gamma'), \nabla(\gamma'), L(\gamma')$ with $\gamma' \in \widetilde{X}^+$, the standard, costandard, irreducible objects in $\mathbb{O}$ by $\Delta(\gamma''), \nabla(\gamma''), L(\gamma'')$ with $\gamma'' \in \widehat{X}^+$. Though the notation is not very satisfying for the moment, the distinction between the different cases becomes more transparent when we replace γ' by $\widetilde{\gamma}$ in §1.5 and γ'' by $\widehat{\gamma}$ in Part II (both defined in terms of $\gamma \in X$).

1.5. **Kazhdan-Lusztig correspondence.** We quote Tanisaki's summary in [42] of Kazhdan-Lusztig's work and refer the reader to the references therein. Consider $\mathcal{O}_d$, the category $\mathcal{O}$ for $\widetilde{\mathfrak{g}}$ at the level d, i.e., the full subcategory of $\mathcal{O}$ consisting of the modules on which the central element $\mathbf{c}$ acts as $d \in \mathbb{Q}$.

Let D be 1 for type A_n, D_n, E_n; 2 for type B_n, C_n, F_4; 3 for type G_2. Let g be the dual Coxeter number, that is, the integer $g-1$ is the sum of all coefficients of the highest short root. The KL-functor

$$\mathscr{F}_l : \mathcal{O}_{-l/2D-g} \to U_\zeta\text{-mod}$$

was defined by Kazhdan and Lusztig in [22, 23, 24]. Here $l = p^r$ is the order of ζ. It is often an equivalence of categories. In that case, $\mathscr{F}_l$ maps standard, costandard, irreducible modules to the standard, costandard, irreducible modules of the same index (see [42, Theorem 7.1]). To be more precise, note that any $\gamma \in X$ determines $\widetilde{\gamma} = \gamma + d\chi$, a weight for $\mathcal{O}_d$, where χ is the dual of the central element $\mathbf{c}$. Fixing $d = -l/2D - g$ the standard, costandard, irreducible objects in $\mathcal{O}_d$ are indexed by X^+ and denoted efficiently by $\Delta(\widetilde{\gamma}), \nabla(\widetilde{\gamma}), L(\widetilde{\gamma})$. If $\mathscr{F}_l$ is an equivalence, then we have

$$\mathscr{F}_l(\Delta(\widetilde{\gamma})) = \Delta_\zeta(\gamma), \qquad \mathscr{F}_l(\nabla(\widetilde{\gamma})) = \nabla_\zeta(\gamma), \qquad \mathscr{F}_l(L(\widetilde{\gamma})) = L_\zeta(\gamma).$$

The following terminology will be useful.

Definition 1.3. A positive integer l is *KL-good* (for R) if the KL-functor $\mathscr{F}_l$ is an equivalence of categories.

Some known conditions for l to be KL-good are found in [42]. For type A_n, there is no restriction. For other simply laced cases, l is KL-good if it is greater than or equal to 3 for D_n, 14 for E_6, 20 for E_7, and 32 for E_8. In non-simply laced cases, l is KL-good above a bound depending on the type, but they are not known. See also [27, Conjecture 2.3], which suggests there is always an equivalence between the quantum case and the affine case.

2. The linkage principle and translation functors

2.1. **Linkage on the weights.** We have identified the weight lattices for the algebraic group case, the quantum case, and the integral case. This subsection defines the affine Weyl group action and linkage classes on the common weight lattice X. Consider the $\mathbb{R}$-space $X \otimes_{\mathbb{Z}} \mathbb{R}$. For $\alpha \in R$ and $m \in \mathbb{Z}$, denote by $s_{\alpha,m}$ the reflection with respect to the hyperplane in $X \otimes_{\mathbb{Z}} \mathbb{R}$ defined by the equation $\langle \lambda, \alpha^\vee \rangle = m$. That is,

$$s_{\alpha,m}(\gamma) = \gamma - (\langle \gamma, \alpha^\vee \rangle - m)\alpha$$

for $\gamma \in X \otimes_{\mathbb{Z}} \mathbb{R}$. Let W be the finite Weyl group of R. It is the reflection group generated by the simple reflections:

$$W = \langle s_\alpha = s_{\alpha,0} \mid \alpha \in \Sigma \rangle$$

For any $l \in \mathbb{Z}$, we define the affine Weyl group W_l to be

$$W_l = \langle s_{\alpha,ml} \mid \alpha \in R,\ m \in \mathbb{Z} \rangle \cong l\mathbb{Z}R \rtimes W.$$

Remark 2.1. The affine Weyl group in the quantum case (defined in [1]) is, in fact, slightly different. Let $l_\alpha := \frac{l}{\gcd(l,d_\alpha)}$ for each $\alpha \in R$, where $d_\alpha = \frac{(\alpha,\alpha)}{2}$. Then the affine

Weyl group for the qunatum case is defined as

$$W_{D,l} = \langle s_{\alpha, ml_\alpha} \mid \alpha \in R,\ m \in \mathbb{Z} \rangle.$$

Since we assume that $l = p^r$ is odd, we have $s_{\alpha,ml_\alpha} = s_{\alpha,ml}$ except the case where $p = 3$ and α is a long root in type G_2. We denote this affine Weyl group by W_l (abusing notation in the G_2 situation above) in the thesis. See [15, §2.4.3] for a remark regarding the dual root system.

Let ρ be the sum of all fundamental weights, or equivalently, ρ is the half sum of all positive roots. We almost always shift the action of W_l on $X \otimes_Z \mathbb{R}$ by ρ, that is, we use the dot action given by

$$w.\gamma = w(\gamma + \rho) - \rho$$

for $w \in W_l, \gamma \in X \otimes_Z \mathbb{R}$.

The standard (antidominant) l-alcove is by definition

$${}^lC^- := \{\gamma \in X \otimes_{\mathbb{Z}} \mathbb{R} \mid -l < \langle \gamma + \rho, \alpha^\vee \rangle < 0 \text{ for all } \alpha \in R^+\}.$$

We call each $w.{}^lC^-$ an (l-)alcove. We call each set of the form

$$F = \{\gamma \in X \otimes_{\mathbb{Z}} \mathbb{R} \mid l(n_\alpha - 1) < \langle \gamma + \rho, \alpha^\vee \rangle < l n_\alpha \text{ for all } \alpha \in R_0^+(F),$$
$$\langle \gamma + \rho, \alpha^\vee \rangle = l n_\alpha \text{ for all } \alpha \in R_1^+(F)\}$$

an (l-)facet, where $R^+ = R_0^+(F) \sqcup R_1^+(F)$. Then the closure $\overline{w.{}^lC^-} = w.\overline{{}^lC^-}$ (for any $w \in W_l$) is a union of facets and is a fundamental domain for the W_l-action. Given $\gamma \in X \otimes_{\mathbb{Z}} \mathbb{R}$, there is a unique facet F such that γ is contained in the upper closure $\widehat{F}$, where we define the upper closure of F as

$$\widehat{F} = \{\gamma \in X \otimes_{\mathbb{Z}} \mathbb{R} \mid l(n_\alpha - 1) < \langle \gamma + \rho, \alpha^\vee \rangle \leq l n_\alpha \text{ for all } \alpha \in R_0^+(F),$$

$$\langle \gamma + \rho, \alpha^\vee \rangle = ln_\alpha \text{ for all } \alpha \in R_1^+(F)\}.$$

We write a weight γ (i.e., an element of X) as $w.\lambda$ for some $w \in W_l$ and a unique λ in $\overline{{}^lC_{\mathbb{Z}}^-} := \overline{{}^lC^-} \cap X$. (A more correct notation for this will be $\overline{{}^lC^-}_{\mathbb{Z}}$, but $\overline{{}^lC_{\mathbb{Z}}^-}$ looks better.) We call a weight $\gamma = w.\lambda$ regular if $\lambda \in {}^lC^-$. We call $\gamma \in X$ singular if it is not regular. If γ is dominant, the orbit containing γ is represented by this $\lambda \in \overline{{}^lC^-}$. ($\lambda$ is not in X^+, but it does not matter.) The choice of $w \in W_l$ is unique if and only if λ is regular. If λ is regular, this identifies $X^+ \cap W_l.\lambda$ with the subset

$$W_l^+ := \{w \in W_l \mid w.\lambda \in X^+\}$$

of W_l. For a general weight λ, we have preferred representatives. Recall that W_l is generated by the subset S_l, which we choose to correspond to the simple reflections through the walls of ${}^lC^-$. Furthermore, (W_l, S_l) is a Coxeter system which has a natural ordering and a length function $l : W_l \to \mathbb{Z}$. Let $I := \{s \in S_l \mid s.\lambda = \lambda\}$, $W_I = (W_l)_I := \{w \in W_l \mid w.\lambda = \lambda\}$, and let $W^I = (W_l)^I$ be the set of shortest coset representatives in W_l/W_I. Then for $w \in W_l^+$, we have $w \in W^I$ if and only if $w.\lambda \in \widehat{w.{}^lC^-}$. Now define

$$W_l^+(\lambda) := W^I \cap W_l^+.$$

We identify $W_l^+(\lambda)$ with the set of dominant weights in the orbit of λ. The uparrow ordering of X^+ is defined to agree with the Coxeter ordering of W_l (restricted to $W_l^+(\lambda)$) when restricted to $W_l^+(\lambda).\lambda \subset X^+$. (There is no order relation between two weights from two different W_l orbits.) See [16, II.6, 8.22] for more discussions on this.

We call γ subregular if λ belongs to a codimension one facet in $\overline{{}^lC^-}$. The existence of a regular weight is equivalent to $l \geq h$, where h is the Coxeter number. For the existence of subregular weights, we have the following elementary fact.

Proposition 2.2. [16, II.6.3] *Suppose a regular weight exists, and l is not* 30 *if the type of R is E_8; not* 12 *if F_4; not* 6 *if G_2. (These are the Coxeter numbers.) Then* *any wall of* ${}^{l}C^{-}$ *contains a weight, that is, for any* $s \in S_l$ *there exists* $\nu \in X$ *with* $\mathrm{Stab}_{W_l}(\nu) = \{e, s\}$*. This is the case, for all types, in particular, if* $l \geq h$ *is a prime power.*

2.2. **The orbit decomposition.** Now we consider the categories G-mod, U_ζ-mod, and $\widetilde{U}_\zeta$-mod. Take $l = p$ when we are in the algebraic group case. Take $l = p^r$ when we talk about the other two cases. By the linkage principle [16, II.6], [4, §8], the G-modules and U_ζ-modules decomposes into the submodules (which are summands) whose composition factors have highest weights in the same W_l-orbits. Using our notation, we can write this as the decomposition

$$G\text{-mod} = \bigoplus_{\mu \in \overline{{}^{p}C_{\mathbb{Z}}^{-}}} (G\text{-mod})[W_l^{+}(\mu).\mu]$$

and

$$U_\zeta\text{-mod} = \bigoplus_{\mu \in \overline{{}^{l}C_{\mathbb{Z}}^{-}}} (U_\zeta\text{-mod})[W_l^{+}(\mu).\mu].$$

(Given a highest weight category $\mathcal{C}$ with a poset Λ and an ideal $\Gamma \trianglelefteq \Lambda$, we set $\mathcal{C}[\Gamma]$ to be the Serre subcategory of $\mathcal{C}$ generated by the irreducibles in $\{L(\gamma)\}_{\gamma \in \Gamma}$. See §4.1 for more details.) We call the category summand $(G\text{-mod})[W_l^{+}(\mu).\mu]$ the orbit of μ in G-mod (similarly for orbits in U_ζ-mod).

2.3. **Translation functors.** We simultaneously define the translation functors on G-mod and U_ζ-mod. Denoted here by $\mathcal{C}'$ is either G-mod or U_ζ-mod. To specify and emphasize the tensor structure, we say here that we are taking either $(\mathcal{C}', \otimes) = (G\text{-mod}, \otimes_k)$ or $(\mathcal{C}', \otimes) = (U_\zeta\text{-mod}, \otimes_K)$. Let $\mathcal{C}'_\mu$ be the orbit $\mathcal{C}'[W_l^{+}(\mu).\mu]$ for each $\mu \in \overline{{}^{l}C_{\mathbb{Z}}^{-}}$. Set $l = p$ if $\mathcal{C}' = G$-mod and $l = p^r$ if $\mathcal{C}' = U_\zeta$-mod. When denoting the distinguished objects in $\mathcal{C}'$, we use the notation with not subscripts (the algebraic group notation). For example, by $\Delta(\gamma)$ we mean $\Delta_\zeta(\gamma)$ if $\mathcal{C}' = U_\zeta$-mod. We have the projection

$$\mathrm{pr}_\mu : \mathcal{C}' \to \mathcal{C}'_\mu$$

of $\mathcal{C}'$ to the orbit of μ. Now fix two weights $\lambda, \mu \in \overline{{}^lC^-_{\mathbb{Z}}}$. The translation

$$T^\mu_\lambda : \mathcal{C}'_\mu \to \mathcal{C}'_\lambda$$

is defined as

$$T^\mu_\lambda = \mathrm{pr}_\mu(- \otimes \Delta(\nu)), \tag{2.3.1}$$

where ν is the unique element in $W(\mu - \lambda) \cap X^+$. (We can of course define the translation to be an endofunctor on $\mathcal{C}'$ as $T^\mu_\lambda = \mathrm{pr}_\mu(\mathrm{pr}_\lambda(-) \otimes \Delta(\nu))$. This is what Jantzen [16, II.7] does. The translation functors in the quantum case appear in [4].)

Remark 2.3. Replacing $\Delta(\nu)$ by $L(\nu)$ or $\nabla(\nu)$ in (2.3.1) yields the same translation functor. In fact, (2.3.1) only depends on the extremal weights of $\Delta(\nu)$.

The translation functors form adjoint pairs $(T^\mu_\lambda, T^\lambda_\mu)$ and $(T^\lambda_\mu, T^\mu_\lambda)$, are exact, and preserve projectives (if exist) and injectives.

The functors T^μ_λ and T^λ_μ are better studied in case μ is in the closure of the facet containing λ. Assume that this is the case. *We keep this convention throughout the thesis.* Set

$$I = \{s \in S_l \mid s.\lambda = \lambda\}, \quad J = \{s \in S_l \mid s.\mu = \mu\}. \tag{2.3.2}$$

Then $W_I = \mathrm{Stab}_{W_l}(\lambda)$, $W_J = \mathrm{Stab}_{W_l}(\mu)$ are the Coxeter subgroups (parabolic subgroups) in W_l generated by I and J respectively. Our convention can now be expressed simply as $I \subset J$.

Proposition 2.4. *Let $y \in W_l^+(\mu)$. In particular, $y.\mu$ is in the upper closure of the facet containing $y.\lambda$.*

(1) *$T_\lambda^\mu \Delta(yx.\lambda) = \Delta(yx.\mu) = \Delta(y.\mu)$, for any $x \in W_J$.*

(2) *$T_\mu^\lambda \Delta(y.\mu)$ has a Δ-filtration whose sections are exactly $\Delta(yx.\lambda)$ where each $x \in W_J/W_I$ occurs with multiplicity one, and we have*

$$\mathrm{hd}(T_\mu^\lambda \Delta(y.\mu)) \cong L(y.\lambda).$$

(3) *$T_\lambda^\mu L(y.\lambda) = L(y.\mu)$, and $T_\lambda^\mu L(yx.\lambda) = 0$ for any nontrivial element $x \in W_J/W_I$.*

(4) *$[T_\mu^\lambda L(y.\mu) : L(y.\lambda)] = |W_J/W_I|$, and we have*

$$\mathrm{hd}(T_\mu^\lambda L(y.\mu)) \cong L(y.\lambda), \quad \mathrm{soc}(T_\mu^\lambda L(y.\mu)) \cong L(y.\lambda).$$

(5) *$T_\lambda^\mu X(yw_J.\lambda) = X(y.\mu)^{\oplus|W_J/W_I|}$, where w_J is the longest element in W_J.*

(6) *$T_\mu^\lambda X(y.\mu) = X(yw_J.\lambda)$, where w_J is the longest element in W_J.*

Proof. See [16, II.7.11, 7.13, 7.15, 7.20] for (1)-(4) and [16, II.E.11] for (5),(6). They are stated and proved in the context of algebraic groups, and some of them are less general. But all of them are proved in the same way for the quantum case and in the generality of the statement. □

3. The Lusztig conjectures

3.1. **Weyl's character formula.** Consider the group algebra $\mathbb{Z}[X]$ of X. It has a basis $\{e(\gamma)\}_{\gamma\in X}$ with the multiplication $e(\gamma)e(\gamma') = e(\gamma+\gamma')$. For $\gamma \in X$, the Weyl character

$$\chi(\gamma) := \frac{\sum_{w\in W} \det(w)e(w\gamma + w\rho)}{\sum_{w\in W} \det(w)e(w\rho)} = \frac{\sum_{w\in W} \det(w)e(w.\gamma)}{\sum_{w\in W} \det(w)e(w.0)} \tag{3.1.1}$$

is defined as an element in the fraction field of $\mathbb{Z}[X]$. This element, while written as a fraction, belongs to $\mathbb{Z}[X]$. We have for each $w \in W$ and $\gamma \in X$,

$$\chi(w\gamma) = \det(w)\chi(\gamma). \tag{3.1.2}$$

If $\gamma \in X^+$, the formula (3.1.1) gives the characters of standard and costandard modules in many module categories, including G-mod and U_ζ-mod. That is, we have

$$\operatorname{ch}\Delta(\gamma) = \operatorname{ch}\nabla(\gamma) = \operatorname{ch}\Delta_\zeta(\gamma) = \operatorname{ch}\nabla_\zeta(\gamma) = \chi(\gamma). \tag{3.1.3}$$

See, for example, [16, II.5.10].

3.2. **Lusztig's character formula.** Let $P_{x,y} \in \mathbb{Z}[t, t^{-1}]$ be the Kazhdan-Lusztig polynomial defined for each $x, y \in W_l$. The polynomial $P_{x,y}$ is, in fact, in $\mathbb{Z}[t^2]$. Assume for the moment that $\lambda \in {}^l C^-_{\mathbb{Z}}$. As conjectured by Lusztig, the characters of irreducible U_ζ-modules in the orbit of λ are given by the Lusztig character formula

$$\operatorname{ch} L_\zeta(w.\lambda) = \sum_{y \in W_l^+} (-1)^{l(w)-l(y)} P_{y,w}(-1)\chi(y.\lambda), \tag{3.2.1}$$

assuming l is KL-good. This follows from the Kazhdan-Lusztig correspondence since a similar character formula in the affine case is proved by Kazhdan-Lusztig [21], Lusztig [29] and Kashiwara-Tanisaki [17, 18]. See [16, II.H.12] for details and more references.

Lusztig has also conjectured that the characters of irreducible G-modules of small highest weights are given by the same formula, that is,

$$\operatorname{ch} L(w.\lambda) = \sum_{y \in W_l^+} (-1)^{l(w)-l(y)} P_{y,w}(-1)\chi(y.\lambda). \tag{3.2.2}$$

By "small highest weight", we mean $w.\lambda$ is in the Jantzen region

$$\Gamma_{\mathrm{Jan}} := \{\lambda' \in X^+ \mid \langle \lambda' + \rho, \alpha^\vee \rangle \le p(p-h+2),\ \forall \alpha \in R\}.$$

This condition arises from the difference between the Steinberg tensor product theorems in the quantum case (with $r = 1$) and in the algebraic group case. Compare

$$L(\gamma_0 + p\gamma_1) \cong L(\gamma_0) \otimes L(p\gamma_1) \cong L(\gamma_0) \otimes L(\gamma_1)^{[1]} \tag{3.2.3}$$

and

$$(3.2.4) \qquad L_\zeta(\gamma_0 + l\gamma_1) \cong L_\zeta(\gamma_0) \otimes L_\zeta(l\gamma_1) \cong L_\zeta(\gamma_0) \otimes V(\gamma_1)^{[1]},$$

where $\gamma_0 \in X_1 := \{\gamma \in X^+ \mid \langle \gamma + \rho, \alpha^\vee \rangle < l, \quad \forall \alpha \in \Pi\}$, $\gamma_1 \in X^+$. (Here $l = p$.) We need to explain the terms: The Frobenius twist $-^{[1]}$ for G is equivalent to the map $f \mapsto f^p$ on the coordinate algebra $k[G]$. (See [16, I.9,II.3].) The Frobenius twist $-^{[1]}$ for U_ζ (see [16, II.H.6]) is of a different nature because it is a map between U_ζ and $U(\mathfrak{g})$ the universal enveloping algebra of the Lie algebra $\mathfrak{g} = \mathfrak{g}_K$. Since the field K is of characteristic zero, the Weyl module $V(\gamma_1)$ for $\mathfrak{g}$ is irreducible and has the Weyl character $\chi(\gamma_1)$.

One now sees that the irreducible characters need to be different for the two cases when γ_1 above is such that $\operatorname{ch} L_\zeta(\gamma_1) \neq \chi(\gamma_1)(= \operatorname{ch} \Delta_\zeta(\gamma_1))$.The Jantzen region is where the weight γ_1 is in the bottom dominant alcove, which is an obvious sufficient condition for $\Delta_\zeta(\gamma_1) \cong L_\zeta(\gamma_1) \cong \nabla_\zeta(\gamma_1)$. The tensor product theorem (3.2.3), however, provides all irreducible G-characters if we know the irreducible G-characters in the region X_1.

The Lusztig conjecture for the algebraic group case is proved for $p \gg 0$ by Andersen-Jantzen-Soergel [2] (explicit but very large bounds later given by Fiebig [13]). But for smaller (yet, possibly, very large) primes, the Lusztig conjecture has many counterexamples found by Williamson [43]. Riche-Williamson [39] then formulate a new conjecture, which they prove for type A. We don't state here the new conjecture which involves the p-Kazhdan-Lusztig polynomials.

For singular weights, one can use the translation functor from a regular orbit to a singular orbit. Applying the exact functor T_λ^μ to (3.2.1), (3.2.2), Proposition 2.4 (1) provides the irreducible character formula for a general dominant weight $w.\mu$

($\mu \in \overline{{}^lC^-}$, $w \in W_l^+(\mu)$) as an alternating sum of the regular character formula:

$$\mathrm{ch}\, L_\zeta(w.\mu) = \sum_{y \in W_l^+(\mu)} \sum_{x \in W_I} (-1)^{l(w)-l(yx)} P_{yx,w}(-1)\chi(y.\mu) \tag{3.2.5}$$

$$\mathrm{ch}\, L(w.\mu) = \sum_{y \in W_p^+(\mu)} \sum_{x \in W_I} (-1)^{l(w)-l(yx)} P_{yx,w}(-1)\chi(y.\mu) \tag{3.2.6}$$

The formulas (3.2.5), (3.2.6) are, therefore, valid whenever (3.2.1), (3.2.2) are valid.

4. Finite dimensional algebras

4.1. **Highest weight categories with finite posets.** Given a highest weight category $\mathcal{C}'$ with poset Λ, a truncation $\mathcal{C} = \mathcal{C}'[\Gamma]$ by a poset ideal $\Gamma \subset \Lambda$ is defined to be the Serre subcategory of $\mathcal{C}'$ generated by $\{L(\gamma) \mid \gamma \in \Gamma\}$. Its objects are those with composition factors of the form $L(\gamma)$, $\gamma \in \Gamma$. The category $\mathcal{C}$ satisfies

$$\mathrm{Ext}^n_{\mathcal{C}}(X,Y) \cong \mathrm{Ext}^n_{\mathcal{C}'}(X,Y) \tag{4.1.1}$$

for $X, Y \in \mathcal{C}$ by the general theory of highest weight categories [6, Theorem 3.9]. This justifies our (abuse of) notation $\mathrm{Ext}^n_G(X,Y) := \mathrm{Ext}^n_{\mathcal{C}}(X,Y)$ when $X, Y \in \mathcal{C} = G\text{-mod}[\Gamma]$ for some $\Gamma \trianglelefteq X^+$, $\mathrm{Ext}^n_{U_\zeta}(X,Y) := \mathrm{Ext}^n_{\mathcal{C}}(X,Y)$ when $X, Y \in \mathcal{C} = U_\zeta\text{-mod}[\Gamma]$, etc.

It is also a general fact from Cline-Parshall-Scott [6] that the highest weight category $\mathcal{C}$ with the finite poset Γ is equivalent to A-mod, the category of finite dimensional modules over some finite dimensional algebra A. Another way to say that $\mathcal{C} = A$-mod is a highest weight category is to say that A is a quasi-hereditary algebra. We apply this to the case $\mathcal{C}' = G$-mod and U_ζ-mod. Let $\Gamma \trianglelefteq X^+$ be finite. There is a finite dimensional (quasi-hereditary) k-algebra A such that A-mod is equivalent to (G-mod)[Γ]; there is a finite dimensional K-algebra A_ζ such that A_ζ-mod is equivalent to (U_ζ-mod)[Γ]. We denote the standard, costandard, irreducible A-modules (resp.,

A_ζ-modules) by the same notation as when we are in G-mod (resp., U_ζ-mod). Truncation with a finite ideal further provides finite dimensional projective A-modules which did not exist before we truncate. We denote by $P(\gamma)$ the projective cover of $L(\gamma)$ in A-mod and by $I(\gamma)$ the injective envelope in A-mod of $L(\gamma)$ with $\gamma \in \Gamma$. Similarly, $P_\zeta(\gamma)$ and $I_\zeta(\gamma)$ denote projective and injective A_ζ-modules. (These exist already in U_ζ-mod.)

The Kazhdan-Lusztig correspondence identifies the algebra A_ζ with a certain finite dimensional algebra arising in a similar way in the affine case. In the affine case, we have standard Koszulity (of the finite dimensional algebras associated to truncations of the affine category O) as proved by Shan-Varagnolo-Vasserot [40]. The standard Koszul grading then is carried over to the quantum case to grade the algebra A_ζ (see [37, §6]).

We want to relate the two algebras A and A_ζ as we did in §1.2.2, §1.3. The integral case corresponds to an algebra $\widetilde{A}$ which is a free $\mathscr{O}$-module of finite rank. We can and do choose the algebras A, $\widetilde{A}$, A_ζ so that

$$(4.1.2) \qquad A \cong \widetilde{A} \otimes_{\mathscr{O}} k \qquad \text{and} \qquad A_\zeta \cong \widetilde{A} \otimes_{\mathscr{O}} K,$$

where the $\mathscr{O}$-algebra $\widetilde{A}$ corresponds to $\widetilde{U}_\zeta$. For example, take A_ζ as a quotient of the algebra U_ζ and let $\widetilde{A}$ be the image of $\widetilde{U}_\zeta$ in A_ζ. We also see that $\widetilde{A}$-modules reduces mod p to A-modules as $\widetilde{U}_\zeta$-modules reduced mod p to G-modules.

By §2.2, we have

$$(4.1.3) \qquad A = \bigoplus_{\mu \in \overline{{}^pC^-_{\mathbb{Z}}}} A^\mu, \qquad A_\zeta = \bigoplus_{\mu \in \overline{{}^lC^-_{\mathbb{Z}}}} A^\mu_\zeta.$$

Here it should be clear what the algebra summands are: For example, A^μ-mod is equivalent to $(G\text{-mod})[\Gamma \cap W_p.\mu]$. The translation functors in G-mod and U_ζ-mod give translation functors for A-mod and A_ζ-mod.

We may assume that Γ is chosen to make the poset ideal of A^μ-mod compatible with that of A^λ-mod for any two weights $\lambda, \mu \in \overline{{}^pC_{\mathbb{Z}}^-}$. By compatibility we mean that the translations between A^λ-mod and A^μ-mod satisfy Proposition 2.4. For example, we take Γ to satisfy $\Gamma \cap W_p.\lambda = \{w.\lambda \mid w \in W^+, w \leq vw_0\}$ for some $v \in W^+(\mu) := W^+ \cap W^J$, where w_0 is the maximal element in the finite Weyl group W. Letting λ be regular, Γ is determined by this condition, so we may assume this to be the poset ideal we use.

4.2. **Forced grading and three conjectures of Parshall-Scott.** The setting (4.1.2) enables us to define a new graded algebra associated to the algebra A. We introduce some of Parshall-Scott's works on this "forced grading" method. The forced grading comes from the natural grading on A_ζ, where ζ is a p-th root of unity. The definition goes as

$$\widetilde{\mathrm{gr}}A = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} (\widetilde{A} \cap \mathrm{rad}^n A_\zeta / \widetilde{A} \cap \mathrm{rad}^{n+1} A_\zeta)_k.$$

In other words, $\widetilde{\mathrm{gr}}A$ is the associated graded of the filtered algebra A, where the filtration is $\{F_n = (\widetilde{A} \cap \mathrm{rad}^n A_\zeta)_k\}$. We emphasize here that we do not know whether the algebra A itself is graded (i.e., $\widetilde{\mathrm{gr}}A \cong A$). By the decomposition (4.1.3), the forced graded algebra $\widetilde{\mathrm{gr}}A$ decomposes into $\widetilde{\mathrm{gr}}A^\lambda$ where λ runs through the weights in $\overline{{}^pC^-}$.

We restrict ourselves to the case $r = 1$, (i.e., ζ is a p-th root of unity) and assume that p is KL-good in this subsection. (All three conjectures from [36] does assume this.)

4.2.1. The first conjecture of Parshall-Scott in [36] expects $\widetilde{\mathrm{gr}}A$ to be standard Q-Koszul.

A positively graded finite dimensional k-algebra B is called Q-Koszul if the grade zero part B_0 is quasi-hereditary and

$$\operatorname{ext}^n_B(\Delta^0(\gamma), \nabla_0(\gamma')\langle m\rangle) = 0 \quad \text{for all } \gamma, \gamma' \in \Gamma \text{ unless } n = m, \tag{4.2.1}$$

where Γ is the poset for the highest weight category B-mod, $\operatorname{ext}_B(-,-)$ is the Ext in the category of graded B-modules, $\Delta^0(\gamma)$ (resp., $\nabla_0(\gamma)$) is the standard (resp., costandard) B_0-module of highest weight $\gamma \in \Gamma$ viewed as a graded B-module concentrated in grade 0, and $\langle m\rangle$ is the grade shift by $m \in \mathbb{Z}$. A positively graded quasi-hereditary algebra B is called standard Q-Koszul if it satisfies

$$\begin{aligned} &\operatorname{ext}^n_B(\Delta(\gamma), \nabla_0(\gamma')\langle m\rangle) = 0 \quad \text{for all } \gamma, \gamma' \in \Gamma \text{ unless } n = m, \\ &\operatorname{ext}^n_B(\Delta^0(\gamma), \nabla(\gamma')\langle m\rangle) = 0 \quad \text{for all } \gamma, \gamma' \in \Gamma \text{ unless } n = m, \end{aligned} \tag{4.2.2}$$

where $\Delta(\gamma)$ (resp., $\nabla(\gamma)$) is the standard B-module whose head (resp., socle) is in grade 0. Note that the grade zero part of a positively graded quasi-hereditary algebra is quasi-hereditary. A standard Q-Koszul algebra is Q-Koszul. A standard Koszul algebra is standard Q-Koszul; a Koszul algebra is Q-Koszul. (For Koszul and standard Koszul, We use the definitions that require the algebra to be finite dimensional.) We refer the reader to [36] for further discussion.

Assuming λ is regular and $p \gg 0$, the standard Q-Koszulity of $\widetilde{\operatorname{gr}} A^\lambda$ is proved in [35]. As another nontrivial example, the forced grading on the Schur algebra $S(5,5)$ for $p = 2$ is standard Q-Koszul [36, §6]. The standard (resp., costandard) modules in $(\widetilde{\operatorname{gr}} A^\lambda)_0$-mod are the reduction mod p modules $\Delta^{\text{red}}(\gamma)$ (resp., $\nabla_{\text{red}}(\gamma)$). More generally for $p \geq 2h-2$, [32] proves that $(\widetilde{\operatorname{gr}} A^\lambda)_0$ is quasi-hereditary (the reduced modules playing the role of standard/costandard modules in the highest weight category $(\widetilde{\operatorname{gr}} A^\lambda)_0$-mod), where λ is still regular. In fact, what [32] proves is that the algebra

$$\mathrm{gr}\widetilde{A}^\lambda = \bigoplus_{n\geq 0}(\widetilde{A}^\lambda \cap \mathrm{rad}^n\, A_\zeta^\lambda / \widetilde{A}^\lambda \cap \mathrm{rad}^{n+1}\, A_\zeta^\lambda)$$

is integral quasi-hereditary. See [38, Corollary 3.2].

4.2.2. The second conjecture [36, Conjecture II] says:

$$\begin{aligned}
\dim_K \mathrm{Ext}^n_{U_\zeta}(\Delta_\zeta(w.\lambda), L_\zeta(y.\lambda)) &= \dim_k \mathrm{Ext}^n_G(\Delta(w.\lambda), \nabla_{\mathrm{red}}(y.\lambda)),\\
\dim_K \mathrm{Ext}^n_{U_\zeta}(L_\zeta(w.\lambda), \nabla_\zeta(y.\lambda)) &= \dim_k \mathrm{Ext}^n_G(\Delta^{\mathrm{red}}(w.\lambda), \nabla(y.\lambda)),\\
\dim_K \mathrm{Ext}^n_{U_\zeta}(L_\zeta(w.\lambda), L_\zeta(y.\lambda)) &= \dim_k \mathrm{Ext}^n_G(\Delta^{\mathrm{red}}(w.\lambda), \nabla_{\mathrm{red}}(y.\lambda)),
\end{aligned} \tag{4.2.3}$$

where $\lambda \in \overline{{}^pC^-_{\mathbb{Z}}}$ and $w, y \in W_p^+(\lambda)$. Here $\Delta^{\mathrm{red}}(w.\lambda)$ is an abbreviation for $\Delta_1^{\mathrm{red}}(w.\lambda)$ and $\nabla_{\mathrm{red}}(w.\lambda)$ is an abbreviation for $\nabla^1_{\mathrm{red}}(w.\lambda)$.

The conjecture is, in fact, stated in terms of the finite dimensional algebras in [36, Conjecture II] (that is, the Ext-spaces are Ext_A, Ext_{A_ζ} in [36]), and is closely related to the forced grading on the algebra A. If $p \gg 0$, then the conjecture is proved by Cline-Parshall-Scott [9]. The condition on p, as one expects, comes from the Lusztig conjecture on algebraic groups.

4.2.3. Then, the third conjecture [36, Conjecture III] provides explicit formulas for the left hand sides of (4.2.3):

$$\sum_{n=0}^{\infty} \dim \mathrm{Ext}^n_{U_\zeta}(\Delta_\zeta(y.\lambda), L_\zeta(w.\lambda))t^n = \sum_{x\in W_J} (-1)^{l(x)} t^{l(w)-l(y)} \bar{P}_{yx,w},$$

$$\sum_{n=0}^{\infty} \dim \mathrm{Ext}^n_{U_\zeta}(\Delta_\zeta(y.\lambda), L_\zeta(w.\lambda))t^n = \sum_{x\in W_J} (-1)^{l(x)} t^{l(w)-l(y)} \bar{P}_{yx,w},$$

$$\sum_{n=0}^{\infty} \dim_K \mathrm{Ext}^n_{U_\zeta}(L_\zeta(w.\lambda), L_\zeta(y.\lambda))t^n = \sum_{\substack{z\in W^+(\lambda)\\ x,x'\in W_J}} (-1)^{l(x)+l(x')} t^{l(w)+l(y)-2l(z)} \bar{P}_{zx,w}\bar{P}_{zx',y},$$

for $y, w \in W_p^+(\lambda)$, where $\bar{P}_{y,w}(t) = P_{y,w}(t^{-1})$. Rather than discussing it further here, let us move on to Part II where we prove the conjecture.

Part II. Ext computations in the quantum case

We are now in the quantum case. Though the setting of the thesis is $l = p^r$, everything in this part works for any (KL-good) integer l.[2]

5. Kazhdan-Lusztig theory in regular blocks

We start the part with recalling some known facts in regular blocks. Suppose l is KL-good for the root system R. A consequence of §1.5 is the dimension formula for certain cohomology in a regular block.

Let $P_{x,y} \in \mathbb{Z}[t, t^{-1}]$ be the Kazhdan-Lusztig polynomial associated to $x, y \in W_l$. Take $\lambda \in {}^lC^-_{\mathbb{Z}}$. Then we have

$$\sum_{n=0}^{\infty} \dim \operatorname{Ext}^n_{U_\zeta}(\Delta_\zeta(y.\lambda), L_\zeta(w.\lambda))t^n = t^{l(w)-l(y)} \bar{P}_{y,w} \tag{5.0.1}$$

for all $y, w \in W^+(\lambda)(= W_l^+$, since λ is regular). The bar on the polynomial is the automorphism on $\mathbb{Z}[t, t^{-1}]$ that maps t to t^{-1}.

The formula (5.0.1) follows from the Lusztig character formula (3.2.1) by a chain of equivalent conditions [16, II.C], independently to the KL-good assumption.

While the character formula in singular blocks readily follows by translating from a regular block (see §3.2), the homological information does not translate easily between regular and singular orbits. This is because we cannot determine how to "sum" the formula (5.0.1). We need a certain parity vanishing property to make it work.

6. More on the translation functors

We prove some more properties of the translation functors that are important in the proof of Theorem 8.10. Most statements (all except the last proposition) in this section are valid both in the algebraic group case and in the quantum case for any

[2]Note that [4] assumes that l is an odd prime power, but the restriction is unnecessary since we have the linkage principle for all l [1]. See also [15, §2.5].

positive integer l. We drop the subscript ζ from the notation whenever that is the case.

Proposition 6.1. *Let $\lambda, \nu, \mu \in \overline{{}^lC_{\mathbb{Z}}^-}$ be such that ν is contained in the closure of the facet containing λ, and μ is contained in the closure of the facet containing ν. Then for any $y \in W^+$, $T_\mu^\lambda \Delta(y.\mu) \cong T_\nu^\lambda T_\mu^\nu \Delta(y.\mu)$.*

Proof. Let I, J as in (2.3.2). We may assume that $y \in W^J$.

Consider the tilting module $X(yw_J.\lambda)$. We check that $T_\mu^\lambda \Delta(y.\mu)$ and $T_\nu^\lambda T_\mu^\nu \Delta(y.\mu)$ are both submodules of $X(yw_J.\lambda)$. Since $\Delta(y.\mu)$ is a submodule of the tilting module $X(y.\mu)$, by exactness of translation $T_\mu^\lambda \Delta(y.\mu)$ is a submodule of $T_\mu^\lambda X(y.\mu)$. But $T_\mu^\lambda X(y.\mu)$ is isomorphic to $X(yw_J.\lambda)$ by Proposition 2.4 (6). For the same reason $T_\nu^\lambda T_\mu^\nu \Delta(y.\mu)$ is a submodule of $T_\nu^\lambda T_\mu^\nu X(y.\mu)$. The latter is isomorphic to $X(yw_J.\lambda)$, applying Proposition 2.4 (6) twice.

Now note that $T_\mu^\lambda \Delta(y.\mu)$ and $T_\nu^\lambda T_\mu^\nu \Delta(y.\mu)$ have Δ-filtrations with the same set of sections, i.e, for each $x \in W_J^I = W^I \cap W_J$ the section $\Delta(yx.\lambda)$ appears exactly once. It remains to show that there is only one submodule in $X(yw_J.\lambda)$ which has such a Δ-filtration.

We first determine which standard modules appear in a Δ-filtration of $X(yw_J.\lambda)$. The module $X(y.\mu)$ has a Δ-filtration exactly one of whose sections is isomorphic to $\Delta(y.\mu)$. Any other $\Delta(z.\mu)$ appearing in the filtration satisfies $z < y$. Translating to the λ block gives the multiplicities of all $\Delta(\gamma)$ in a Δ-filtration of $T_\mu^\lambda X(y.\mu) = X(yw_J.\lambda)$ in terms of the Δ-multiplicities of $X(y.\mu)$. By Proposition 2.4(2), the multiplicity of $\Delta(zx'.\lambda)$, for each $x' \in W_J \cap W^I$, in a Δ-filtration of $X(yw_J.\lambda)$ is the same as the multiplicity of $\Delta(z.\mu)$ in a Δ-filtration of $X(y.\mu)$. Since $\Delta(y.\mu) \ncong \Delta(z.\mu)$ implies $zW_J \cap yW_J = \emptyset$, we have in that case $\Delta(yx'.\lambda) \ncong \Delta(zx''.\lambda)$ for all $zx' \in zW_J \neq yW_J \ni zx''$. Therefore, each $\Delta(yx'.\lambda)$ for $x' \in W_J \cap W^I$ appears exactly once in the Δ-filtration of $X(yw_J.\lambda)$.

Suppose M, M' are two submodules of $X(yw_J.\lambda)$ which have Δ-filtrations with the same set of sections $\{\Delta(yx.\lambda)\}_{x\in W_J^I}$. The proposition is proved if we show $M = M'$.

The weight $yw_J.\lambda$ is maximal in M, M' and $X(yw_J.\lambda)$. Also, $yw_J.\lambda$ appears with multiplicity one in all three modules. Hence M and M' contains the unique submodule of $X(yw_J.\lambda)$ isomorphic to $\Delta(yw_J.\lambda)$. Then $M/\Delta(yw_J.\lambda)$ and $M'/\Delta(yw_J.\lambda)$ are submodules of $X(yw_J.\lambda)/\Delta(yw_J.\lambda)$. Here, each $yw_Js.\lambda$ for $s \in J$ is maximal with multiplicity one. In this way, we can show that $M \cap M'$ has a Δ-filtration with sections $\{\Delta(yx.\lambda)\}_{x\in W_J^I}$. So $M = M'$. $\square$

Composing two opposite translation functors, we get an endofunctor $T_\mu^\lambda T_\lambda^\mu : \mathcal{C}'_\lambda \to \mathcal{C}'_\lambda$. In a special case where λ is regular and μ is subregular, the functor $T_\mu^\lambda T_\lambda^\mu$ is commonly called the *s-wall crossing functor* and denoted by Θ_s, where s is the unique nontrivial stabilizer of μ.

Let λ be regular, and consider the module $T_\mu^\lambda T_\lambda^\mu \Delta(y.\lambda)$. By Proposition 2.4.(2), there is a filtration

$$T_\mu^\lambda T_\lambda^\mu \Delta(y.\lambda) = V_0 \supset V_1 \supset \cdots \supset V_n = 0$$

such that $V_i/V_{i+1} = \Delta(yx_i.\lambda)$. Then $\{x_0 = e, \cdots, x_n\} = W_J/W_I$. Since

$$\mathrm{Ext}^1_{\mathcal{C}'}(\Delta(\nu), \Delta(\nu')) = 0 \text{ for } \nu \not< \nu', \tag{6.0.1}$$

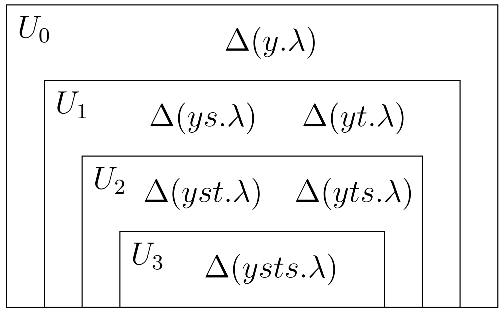
we can arrange the filtration in a way that $l(x_i) \leq l(x_{i+1})$ holds. Now consider the subfiltration

$$T_\mu^\lambda T_\lambda^\mu \Delta(y.\lambda) = U_0 \supset U_1 \supset \cdots \supset U_N = 0$$

of $\{V_i\}$ where the i-th section contains all $\Delta(yx.\lambda)$ with $l(x) = i$. Using (6.0.1) again, we have

$$U_i/U_{i+1} \cong \bigoplus_{l(x)=i, x\in W_J/W_I} \Delta(yx.\lambda).$$

For example, if W_J/W_I is the symmetric group S_3, then the filtration is illustrated in the following picture.



The filtration $\{U_i\}$ is maximal, in some sense, among the filtrations of $T_\mu^\lambda T_\lambda^\mu \Delta(y.\lambda)$ whose sections are direct sums of standard modules. To say in what sense it is so, we prove the following lemma.

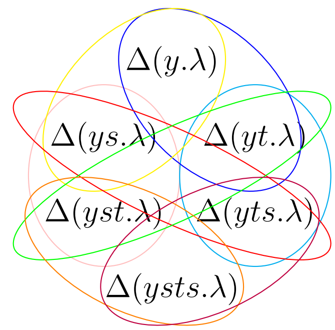
Lemma 6.2. *Let* $\lambda \in {}^l C_{\mathbb{Z}}^- = {}^l C^- \cap X$, $\mu \in \overline{{}^l C_{\mathbb{Z}}^-}$, *and* J *be as in* (2.3.2). *Then* $\Theta_s \Delta(y.\lambda)$, *whenever defined, is a subquotient of* $T_\mu^\lambda T_\lambda^\mu \Delta(y.\lambda)$ *for any* $y \in W^+$, $x \in W_J$, $s \in J$.

We actually state and prove the lemma more generally. The only difficulty it adds is notational. We generalize the s-wall crossing functors to define the *facet crossing functor* $\Theta_{J\setminus I}^I := T_\mu^\lambda T_\lambda^\mu$ with I, J as in (2.3.2). This is compatible with the wall crossing functor notation as $\Theta_s = \Theta_{\{s\}}^{\emptyset}$. This notation is useful here because there are many different facets in play. In the other sections we will go back to using $T_\mu^\lambda T_\lambda^\mu$. Note that the functor $\Theta_{J'}^I$ is defined for $J' \subset J \setminus I$ if and only if there exists a weight ν such that $\{s \in S_l \mid s.\nu = \nu\} = I \cup J'$. For the wall-crossing functors as in Lemma 6.2, this is always the case by Proposition 2.2.

Lemma 6.3. *Let* λ, μ, $I \subset J$ *as in* (2.3.2). *For any* $J' \subset J \setminus I$, $y \in W^+$, *the module* $\Theta_{J'}^I \Delta(y.\lambda)$, *whenever defined, is a subquotient of* $\Theta_{J\setminus I}^I \Delta(y.\lambda) = T_\mu^\lambda T_\lambda^\mu \Delta(y.\lambda)$.

A less formal but more illustrative way to state the lemma is to say that the facet crossings of a standard module are realized in a deeper facet crossing (of the same standard module).

We provide a simple example as another illustration. Let R be of type A, $I = \emptyset$ and $J = \{s, t\} \subset S_l$ (i.e., λ regular, μ subsubregular). Then for any $y \in W^J$, the module $T_\mu^\lambda T_\lambda^\mu \Delta(y.\lambda)$ has six Δ-sections. They are $\Delta(y.\lambda)$, $\Delta(ys.\lambda)$, $\Delta(yt.\lambda)$, $\Delta(yst.\lambda)$, $\Delta(yts.\lambda)$, $\Delta(ysts.\lambda) = \Delta(ytst.\lambda)$. The lemma shows that $\Theta_s\Delta(y.\lambda) = \Theta_s\Delta(ys.\lambda)$, $\Theta_t\Delta(y.\lambda) = \Theta_t\Delta(yt.\lambda)$, $\Theta_s\Delta(yt.\lambda) = \Theta_s\Delta(yts.\lambda)$, $\Theta_t\Delta(ys.\lambda) = \Theta_t\Delta(yst.\lambda)$, $\Theta_s\Delta(yst.\lambda) = \Theta_s\Delta(ysts.\lambda)$, $\Theta_t\Delta(yts.\lambda) = \Theta_t\Delta(ytst.\lambda)$ are subquotients of $T_\mu^\lambda T_\lambda^\mu \Delta(y.\lambda)$.



Proof of Lemma 6.3. Suppose $\Theta_{J'}^I$ is defined, that is, there is a weight ν such that $\{s \in S_l \mid s.\nu = \nu\} = I \cup J'$. Since $\Delta(y.\nu)$ is a subquotient of $T_\mu^\nu \Delta(y.\mu)$, $T_\nu^\lambda \Delta(y.\nu) = \Theta_{J'}\Delta(y.\lambda)$ is a subquotient of $T_\nu^\lambda T_\mu^\nu \Delta(y.\mu)$. But by Proposition 6.1, $T_\nu^\lambda T_\mu^\nu \Delta(y.\mu)$ is isomorphic to $T_\mu^\lambda \Delta(y.\mu) = \Theta_{J \setminus I}\Delta(y.\lambda)$. □

Let λ, μ, J be as in (2.3.2) with λ regular (that is, $I = \emptyset$).

Corollary 6.4. *Let $y \in W^+(\mu)$. Then $\Theta_J\Delta(y.\lambda) = T_\mu^\lambda T_\lambda^\mu \Delta(y.\lambda)$ has a filtration each of whose sections is isomorphic to $\Theta_s\Delta(yx.\lambda)$ for some $s \in J$, $x \in W_J$.*

Proof. By Proposition 2.2, for any $s \in J$ the functor Θ_s is defined on $\mathcal{C}'_\lambda$. We can construct a desired filtration using Lemma 6.2. □

The following corollary explains the "maximality" of the filtration U_i.

Corollary 6.5. *We have for all i*

$$\operatorname{hd} U_i = \bigoplus_{l(x)=i,\ x \in W_J} L(yx.\lambda). \tag{6.0.2}$$

Proof. By construction, the head of U_i contains all $L(yx.\lambda)$ for $l(x) = i, x \in W_J$. This shows the "$\supset$" part. Since the head of any $\Theta_s\Delta(\gamma)$ is irreducible, Lemma 6.2 shows that it does not contain anything other than those irreducibles. This shows that the inclusion "$\supset$" is an equality. □

In the quantum case when l is KL-good, we further have the following. It requires the Lusztig conjecture and its consequences.

Proposition 6.6. *For each i, we have*

(1) $U_i \subset \operatorname{rad}^i T_\mu^\lambda T_\lambda^\mu \Delta_\zeta(y.\lambda)$;
(2) $\operatorname{rad} U_i = \operatorname{rad}^{i+1} T_\mu^\lambda T_\lambda^\mu \Delta_\zeta(y.\lambda) \cap U_i$.

In other words, the submodule U_i of $U_0 = T_\mu^\lambda T_\lambda^\mu \Delta_\zeta(y.\lambda)$ has its head in the i-th radical layer $\operatorname{rad}^i T_\mu^\lambda T_\lambda^\mu \Delta_\zeta(y.\lambda)/\operatorname{rad}^{i+1} T_\mu^\lambda T_\lambda^\mu \Delta_\zeta(y.\lambda)$ of $T_\mu^\lambda T_\lambda^\mu \Delta_\zeta(y.\lambda)$.

Proof. This is clear by Corollary 6.5 and the fact that the Δ_ζ-sections in $T_\mu^\lambda T_\lambda^\mu \Delta_\zeta(y.\lambda)$ extends at their heads, that is,

$$\begin{aligned} \operatorname{Ext}^1_{U_\zeta}(\Delta_\zeta(yxs.\lambda), \Delta_\zeta(yx.\lambda)) &\overset{\cong}{\leftarrow} \operatorname{Ext}^1_{U_\zeta}(L_\zeta(yxs.\lambda), \Delta_\zeta(yx.\lambda)) \\ &\overset{\cong}{\rightarrow} \operatorname{Ext}^1_{U_\zeta}(L_\zeta(yxs.\lambda), L_\zeta(yx.\lambda)), \end{aligned} \tag{6.0.3}$$

where $s \in J$, $xs < x \in W_J$. Here the first isomorphism is induced by the nonzero map $\Delta_\zeta(yxs.\lambda) \to L_\zeta(yxs.\lambda)$ and is a consequence of the Lusztig character formula. See [7, Theorem 4.3]. The second isomorphism is induced by the nonzero map $\Delta_\zeta(yx.\lambda) \to L_\zeta(yx.\lambda)$ and is a general fact, which also tells us that the Ext spaces in (6.0.3) are one dimensional. See for example [16, II.7.19 (d)]. Jantzen's proof for G-modules works the same for U_ζ-modules.

We provide, nevertheless, a more formal proof. We prove (1), (2) together by induction on i. If $i = 0$, then

(1) $U_0 = T_\mu^\lambda T_\lambda^\mu \Delta_\zeta(y.\lambda) = \operatorname{rad}^0 T_\mu^\lambda T_\lambda^\mu \Delta_\zeta(y.\lambda)$;
(2) $\operatorname{rad} U_0 = \operatorname{rad} T_\mu^\lambda T_\lambda^\mu \Delta_\zeta(y.\lambda) = \operatorname{rad} T_\mu^\lambda T_\lambda^\mu \Delta_\zeta(y.\lambda) \cap U_0$.

Now suppose (1), (2) is true for $i-1$. By Corollary 6.5, we have $U_i \subset \operatorname{rad} U_{i-1}$. And induction hypothesis $U_{i-1} \subset \operatorname{rad}^i T_\mu^\lambda T_\lambda^\mu \Delta_\zeta(y.\lambda)$ implies $\operatorname{rad} U_{i-1} \subset \operatorname{rad}^{i+1} T_\mu^\lambda T_\lambda^\mu \Delta_\zeta(y.\lambda)$. Thus (1) holds for U_i. The inclusion $\operatorname{rad} U_i \subset \operatorname{rad}^{i+1} T_\mu^\lambda T_\lambda^\mu \Delta_\zeta(y.\lambda) \cap U_i$ in (2) now follows from $U_i \subset \operatorname{rad}^i T_\mu^\lambda T_\lambda^\mu \Delta_\zeta(y.\lambda)$.

For the other inclusion in (2), suppose for contradiction that

$$\operatorname{rad} U_i \not\supset \operatorname{rad}^{i+1} T_\mu^\lambda T_\lambda^\mu \Delta_\zeta(y.\lambda) \cap U_i.$$

This means that there is a surjective map $f : U_i \to \Delta_\zeta(yx.\lambda)$ for some $x \in W_J$ whose restriction to $U_i \cap \operatorname{rad}^{i+1} T_\mu^\lambda T_\lambda^\mu \Delta_\zeta(y.\lambda)$ is still surjective. We call the restriction f'. Now recall the Δ_ζ-filtration $\{V_j\}$ of $T_\mu^\lambda T_\lambda^\mu \Delta_\zeta(y.\lambda)$. Take j to be such that $V_j = U_i$. Pick $s \in J$ with $xs < x$. We may assume that (switching the order of the filtration if necessary) there is a short exact sequence

$$0 \to U_i = V_j \to V_{j-1} \to \Delta_\zeta(yxs.\lambda) \to 0,$$

and by Lemma 6.2 there is a surjective map $g : V_{j-1} \to \Theta_s \Delta_\zeta(yx.\lambda)$ whose restriction to U_i is the map f. By (6.0.3), there is a map $h : \Theta_s \Delta_\zeta(yx.\lambda) \twoheadrightarrow N$, where N represents a nontrivial element in $\operatorname{Ext}^1_{U_\zeta}(L_\zeta(yxs.\lambda), L_\zeta(yx.\lambda))$, and the restriction of h to the submodule $\Delta_\zeta(yx.\lambda) \subset \Theta_s \Delta_\zeta(yx.\lambda)$ has image (isomorphic to) $L_\zeta(yx.\lambda)$. Thus $h \circ g$ is surjective and $h \circ f$, $h \circ f'$ have image $L_\zeta(yx.\lambda) \subset N$. But this implies that the map $h \circ g$ induces the following two surjective maps

$$V_{j-1} \cap \operatorname{rad}^i T_\mu^\lambda T_\lambda^\mu \Delta_\zeta(y.\lambda) \to N/L_\zeta(yx.\lambda)$$

and

$$V_{j-1} \cap \operatorname{rad}^{i+1} T_\mu^\lambda T_\lambda^\mu \Delta_\zeta(y.\lambda) \to N/L_\zeta(yx.\lambda),$$

which is a contradiction. This proves (2) for U_i and completes the induction step. $\square$

7. Grading and parity vanishing

This section is devoted to proving some lemmas in a more general setting of graded and ungraded highest weight categories and their derived categories. To apply these lemmas to U_ζ-mod, just let the length of the weight $w.\mu$ ($w \in W_l^+, \mu \in \overline{{}^lC_{\mathbb{Z}}^-}$) be the integer $l(w)$. We call $w.\mu$ even if $l(w)$ is even, odd if $l(w)$ is odd.

7.1. **Parity vanishing.** Let $\mathcal{D}$ be a triangulated category.

Definition 7.1. Let $\mathcal{A}, \mathcal{B}$ be classes of objects in $\mathcal{D}$.

(1) We say $\mathcal{A}$ is *(left) $\mathcal{B}$-even* (respectively, *$\mathcal{B}$-odd*) if $\mathrm{Hom}^n_{\mathcal{D}}(X, Y) = 0$ for all odd (resp., even) n and all $X \in \mathcal{A}, Y \in \mathcal{B}$. Then $\mathcal{A}$ is said to have *(left) $\mathcal{B}$-parity* if it is either (left) $\mathcal{B}$-even or $\mathcal{B}$-odd.

(2) We say $\mathcal{A}$ is *right $\mathcal{B}$-even* (resp., *$\mathcal{B}$-odd*) if $\mathrm{Hom}^n_{\mathcal{D}}(Y, X) = 0$ for all odd (resp., even) n and all $X \in \mathcal{A}, Y \in \mathcal{B}$. Then $\mathcal{A}$ is said to have *right $\mathcal{B}$-parity* if it is either right $\mathcal{B}$-even or right $\mathcal{B}$-odd.

Note that $\mathcal{A}$ is $\mathcal{B}$-even if and only if $\mathcal{B}$ is right $\mathcal{A}$-even. In case $\mathcal{A} = \{X\}$, $\mathcal{B} = \{Y\}$, we simply say that X is *Y-even* if $\mathrm{Hom}^n_{\mathcal{D}}(X, Y) = 0$ for all odd n.

Proposition 7.2. *Let*

$$X' \to X \to X'' \to$$

be a distinguished triangle in $\mathcal{D}$. If X' and X'' are Y-even, then X is Y-even. If X' and X'' are Y-odd, then X is Y-odd. The same is true for right Y-parity.

Proof. This is obvious applying $\mathrm{Hom}(-, Y)$ and $\mathrm{Hom}(Y, -)$ to the distinguished triangle. □

Definition 7.3. Let $\mathcal{A}$ be a class of objects in $\mathcal{D}$. We define the *even closure* of $\mathcal{A}$ as

$$^{\mathsf{E}}\mathcal{A} := \{X \in \mathcal{D} \mid X \text{ is } \mathcal{A}\text{-even}\}.$$

Similarly we define the *right even closure* as

$$\mathcal{A}^{\mathsf{E}} := \{X \in \mathcal{D} | \ \mathcal{A} \text{ is } X\text{-even}\}.$$

For an object $Y \in \mathcal{D}$, we set ${}^{\mathsf{E}}Y := {}^{\mathsf{E}}\{Y\}$ and $Y^{\mathsf{E}} := \{Y\}^{\mathsf{E}}$.

Remark 7.4. We identify a class $\mathcal{A}$ with the full subcategory of $\mathcal{D}$ with objects in $\mathcal{A}$. Though ${}^{\mathsf{E}}-$, $-^{\mathsf{E}}$ are not functors, their images are to be seen as full subcategories of $\mathcal{D}$. By definition the closures are strict subcategories (i.e., a subcategory such that all objects isomorphic to one of its objects belong to it) that contains 0. By Proposition 7.2, they are also closed under extension.

Proposition 7.5. *Let $\mathcal{A} \subset \mathcal{B}$ be classes of objects in $\mathcal{D}$. We have*

(1) $\mathcal{A}^{\mathsf{E}} \supset \mathcal{B}^{\mathsf{E}}$.

(2) $\mathcal{D}^{\mathsf{E}} = 0$, $0^{\mathsf{E}} = \mathcal{D}$.

(3) $({}^{\mathsf{E}}\mathcal{A})^{\mathsf{E}} \supset \mathcal{A}$.

(4) ${}^{\mathsf{E}}(({}^{\mathsf{E}}\mathcal{A})^{\mathsf{E}}) = {}^{\mathsf{E}}\mathcal{A}$.

The same relations hold for the right closure.

Proof. (1), (2), (3) are immediate from the definition, and (4) follows from (3). □

It is not true in general ${}^{\mathsf{E}}(\mathcal{A}^{\mathsf{E}}) = ({}^{\mathsf{E}}\mathcal{A})^{\mathsf{E}}$. An easy example is found when $\mathcal{D}$ a derived category of a highest weight category: Take $\mathcal{A}$ to consist of a single standard object.

The proof of the following proposition is left to the reader.

Proposition 7.6. *Let $\mathcal{D}$, $\mathcal{D}'$ be triangulated categories and $\mathcal{A}$ be a class of objects in $\mathcal{D}$, $\mathcal{B}$ be a class of objects in $\mathcal{D}'$. Let $L : \mathcal{D} \to \mathcal{D}'$ be a functor and $R : \mathcal{D}' \to \mathcal{D}$ be its right adjoint. Then*

(1) $(L\mathcal{A})^{\mathsf{E}} = R^{-1}(\mathcal{A}^{\mathsf{E}})$.

(2) ${}^{\mathsf{E}}(R\mathcal{B}) = L^{-1}({}^{\mathsf{E}}\mathcal{B})$.

Example 7.7. Consider $\mathcal{D} = \mathcal{D}^b(U_\varsigma\text{-mod})$. Let F be a facet in $\overline{{}^lC^-}$ and $\lambda \in F \cap X$. Suppose μ is a weight in $\overline{F} \setminus F$. Let $M \in (U_\varsigma\text{-mod})[W_l^+.\mu] \subset \mathcal{D}$. If $T_\mu^\lambda M \in \mathcal{E}^R$ (defined in §7.2), then $M = 0$.

This is proved as follows. By Proposition 7.6 and Proposition 7.10 below, we have

$$(T_\lambda^\mu \mathcal{E}_0^L)^{\mathsf{E}} = (T_\mu^\lambda)^{-1}((\mathcal{E}_0^L)^{\mathsf{E}}) = (T_\mu^\lambda)^{-1}\mathcal{E}^R.$$

But since

$$T_\lambda^\mu \Delta(y.\lambda)[l(y) + 2m] = \Delta(y.\mu)[l(y) + 2m],$$

$$T_\lambda^\mu \Delta(ys.\lambda)[l(y) + 1 + 2m] = \Delta(y.\mu)[l(y) + 1 + 2m]$$

for $y \in W^J$, $s \in J \setminus I$, $m \in \mathbb{Z}$, all shifts of $\Delta(y.\mu)$ for all (dominant) $y.\mu$ belong to $T_\lambda^\mu \mathcal{E}_0^L$. So if $T_\mu^\lambda M \in \mathcal{E}^R$, then $\mathrm{Hom}^n(\Delta(y.\mu), M) = 0$ for all n, which implies $M = 0$.

7.2. **Parity vanishing in a highest weight category.** Let $\mathcal{C}$ be a highest weight category with a finite poset Λ of weights. It has standard objects $\Delta(\lambda)$, costandard objects $\nabla(\lambda)$, irreducible objects $L(\lambda)$ for $\lambda \in \Lambda$. We sometimes call the objects in $\mathcal{C}$ modules. Let us also assume that $\mathrm{End}_{\mathcal{C}}(L(\lambda))$ is one dimensional for all $\lambda \in \Lambda$. Take the bounded derived category $\mathcal{D}^b(\mathcal{C})$. An object in $\mathcal{C}$ is identified via the obvious inclusion $\mathcal{C} \to \mathcal{D}^b(\mathcal{C})$ with an object in $\mathcal{D}^b(\mathcal{C})$ concentrated in degree 0. Note that for $X, Y \in \mathcal{C}$, we have $\mathrm{Ext}^n_{\mathcal{C}}(X, Y) = \mathrm{Hom}_{\mathcal{D}^b(\mathcal{C})}(X, Y[n])$. We omit the subscripts and use the notation $\mathrm{Hom}^n(-,-) = \mathrm{Hom}_{\mathcal{D}^b(\mathcal{C})}(-, -[n])$.

We further assume that the set Λ is equipped with a length function $l : \Lambda \to \mathbb{Z}$. Set $\mathcal{E}_0$ to be the full subcategory of $\mathcal{D}^b(\mathcal{C})$ whose objects are the direct sums of $\nabla(\lambda)[l(\lambda) + 2m]$ for $\lambda \in \Lambda$, $m \in \mathbb{Z}$. Then $\mathcal{E}_i$ is defined inductively as the full subcategory of $\mathcal{D}^b(\mathcal{C})$ such that

$$X \in \mathcal{E}_i \Leftrightarrow \exists \text{ a distinguished triangle } X' \to X \to X'' \to \text{ with } X' \in \mathcal{E}_{i-1}, X'' \in \mathcal{E}_0.$$

Set $\mathcal{E}$ to be the union $\bigcup_i \mathcal{E}_i$. This is by construction a subcategory of $\mathcal{E}^R$ defined in [7], whose defining condition is

$$X \in \mathcal{E}_i \Leftrightarrow \exists \text{ a distinguished triangle } X' \to X \to X'' \to \text{ with } X', X'' \in \mathcal{E}_{i-1}^R,$$

with $\mathcal{E}_0 = \mathcal{E}_0^R$. In fact, it is implicit in (the proof of) the recognition theorem [7, (2.4) Theorem] that $\mathcal{E}^R = \mathcal{E}$. We make it explicit.

Proposition 7.8. *Let $\mathcal{A}$ be a class of objects in $\mathcal{D}^b(\mathcal{C})$. Then the following conditions are equivalent.*

(1) $\mathcal{A} \subset \mathcal{E}^R$.
(2) $\mathcal{A} \subset \mathcal{E}$.
(3) *For each $X \in \mathcal{A}$, we have $\mathrm{Hom}^n(\Delta(\lambda), X) = 0$ for all $\lambda \in \Lambda$ and all integers $n \not\equiv l(\lambda)$ mod 2.*

Proof. It is enough to consider the case in which $\mathcal{A}$ consists of a single object X. The implications (2) $\Rightarrow$ (1) $\Rightarrow$ (3) are clear. (3) $\Rightarrow$ (2) is the only nontrivial step. Although it is proved in the proof of [7, (2.4) Theorem], we provide a full proof because it contains an important construction.

Suppose $\mathrm{Hom}^n(\Delta(\lambda), X) = 0$ for $n \not\equiv l(\lambda)$ mod 2. Let $Y_0 = X$. We show that we can construct $Y_0, \cdots, Y_i \in \mathcal{D}^b(\mathcal{C})$ inductively. It is enough to show that we can find a distinguished triangle $Y_{i+1} \to Y_i \to \nabla(\lambda_i)[n_i] \to$ such that (i) $n_i \equiv l(\lambda_i)$ mod 2; (ii) the cohomology $H^\bullet(Y_{i+1})$ has composition factors with lower highest weights compared to the composition factors in $H^\bullet(Y_i)$ (the meaning of this condition will become clearer in the course of the proof); (iii) $\mathrm{Hom}^n(\Delta(\lambda), Y_{i+1}) = 0$ for $n \equiv l(\lambda)+1$ mod 2. Pick a maximal weight λ_i among the highest weights of the composition factors in $H^\bullet(Y_i)$. Say it is in $H^{n_i}(Y_i)$. Since λ_i is maximal, by universal property of $\nabla(\lambda_i)$, there is a nonzero map from $H^{n_i}(Y_i)$ to $\nabla(\lambda_i)$. This map lifts to a morphism from Y_i to $\nabla(\lambda_i)[n_i]$ in the derived category $\mathcal{D}^b(\mathcal{C})$. So we get a distinguished triangle

$Y_{i+1} \to Y_i \to \nabla(\lambda_i)[n_i] \to$. Since we took a map to $\nabla(\lambda_i)$ whose preimage contains a composition factor of $H^\bullet(Y_i)$ isomorphic to $L(\lambda_i)$, we have

$$[H^\bullet(Y_{i+1}) : L(\lambda_i)] < [H^\bullet(Y_i) : L(\lambda_i)],$$

and all the other differences between $H^\bullet(Y_i)$ and $H^\bullet(Y_{i+1})$ involve only the composition factors in $\nabla(\lambda_i)/L(\lambda_i)$ which only has weights lower than λ_i. Thus we have the condition (ii). Since $\mathrm{Hom}^n(\Delta(\lambda_i), Y_i) = 0$ for $n \equiv l(\lambda_i) + 1 \bmod 2$, the n_i should satisfy the condition (i). Finally (i) and the right $\Delta(\lambda)$-parity of Y_i implies (iii). $\square$

Remark 7.9.

(1) In fact, the construction of the distinguished triangle in the proof does not use the right $\Delta(\lambda)$-parity of X. The same induction in the proof works removing the conditions (i), (iii). This shows that all complexes are filtered by shifts of costandard modules. A complex belongs to the category $\mathcal{E}$ when there appear the "correct shifts" only. For example, let $\mathcal{C}$ be (a truncation of) G-mod or U_ζ-mod with $l \geq h$. So 0 is a regular weight, and $L(0) = \Delta(0) = \nabla(0)$. Denoting by s the reflection through the upper wall of C, we have short exact sequences $0 \to L(0) \to \Delta(s.0) \to L(s.0) \to 0$ and $0 \to L(s.0) \to \nabla(s.0) \to L(0) \to 0$ of U_ζ-modules in the orbit of the weight 0. Then $\Delta(s.0)$ is not in $\mathcal{E}^R$, even up to shifts, because both $\nabla(0)$ and $\nabla(0)[-1]$ appear when one applies the above construction of distinguished triangles:

$$\nabla(0) \oplus \nabla(0)[-1] = L(0) \oplus L(0)[-1] \cong Y_1 \to Y_0 = \Delta(s.0) \to \nabla(s.0) \to,$$

$$\nabla(0)[-1] \cong Y_2 \to Y_1 \to \nabla(0) \to,$$

$$0 = Y_3 \to Y_2 \to \nabla(0)[-1] \to .$$

(2) If the Y_i, λ_i, n_i are as in the proof, the character of X is given by $\Sigma_i(-1)^{n_i}[\nabla(\lambda_i)]$. By (1) this is true for any $X \in \mathcal{D}^b(\mathcal{C})$. Then X is in $\mathcal{E}^R$ if and only if there is no

cancellation in the character formula. In the example above, $\nabla(0)$ and $\nabla(0)[-1]$ cancel each other in characters, hence are invisible in the character formula.

(3) The construction of "realizations" in the proof can be used in describing birth and death of extensions. Suppose $N \in \mathcal{E}^R$ and $Y_\bullet$ is a sequence of distinguished triangles as above that realizes N. Then the "difference" between each adjacent terms in the sequence $\{\mathrm{Hom}^\bullet(M, Y_i)\}_i$ is $\mathrm{Hom}^\bullet(M, \nabla(\lambda_i)[m_i])$. Further description involves many uncertainties, since usually do not know $\mathrm{Ext}^\bullet(M, \nabla(\lambda'))$ or the induced maps between Ext spaces. But, for example, if $M = \Delta(\lambda)$, then we know everything: $\{\mathrm{Hom}^\bullet(M, Y_i)\}_i$ increase at i in degree m_i when $\lambda_i = \lambda$. More generally, suppose M has a Δ-filtration such that no $\Delta(\lambda)$ appears in the filtration more than once. (We are thinking of the wall-crossing module $T_\mu^\lambda T_\lambda^\mu \Delta(\lambda)$ in G-mod or in U_ζ-mod.) Then $\mathrm{Hom}^\bullet(M, \nabla(\lambda_i)[m_i])$ is zero except in degree 0 where it is either zero or one dimensional. Once we know these homomorphisms, we can find $\mathrm{Hom}^\bullet(M, Y_m)$, $\cdots$, $\mathrm{Hom}^\bullet(M, Y_0)$ successively. The first terms $\mathrm{Hom}^\bullet(M, Y_i)$ are 0 until we reach the first $i = i_0$ such that $\mathrm{Hom}(M, \nabla(\lambda_i)) = \mathrm{Hom}^0(M, \nabla(\lambda_i)) = \mathrm{Hom}^{m_i}(M, \nabla(\lambda_i)[-m_i])$ is nonzero (we know that $\lambda_{i_0} = y.\lambda$); then the nonzero map from V_i to $\nabla(\lambda_{i_0})$ adds a dimension to $\mathrm{Hom}^\bullet(M, Y_{i_0})$ at degree m_{i_0}; then $\mathrm{Hom}^\bullet(M, Y_i)$ is isomorphic to $\mathrm{Hom}^\bullet(M, Y_{j+1})$ until we reach the second $i = i_1$ such that $\mathrm{Hom}(M, \nabla(\lambda_i)) \neq 0$; this time the nonzero map from V_i to $\nabla(\lambda_{i_1})$ either adds a dimension to $\mathrm{Hom}^\bullet(M, Y_{i_1})$ at degree m_{i_1} or subtract a dimension from $\mathrm{Hom}^\bullet(M, Y_{i_1})$ at degree $m_{i_1} + 1$; and it goes on. That is, $\mathrm{Hom}^\bullet(M, Y_i)$ changes, by dimension one, in one degree, precisely at such i's. Whether it adds an extension or it kills one depends on the maps in the long exact sequence

$$\to \mathrm{Hom}^{m_i}(M, Y_i) \to \mathrm{Hom}^{m_i}(M, \nabla(\lambda_i)[-m_i]) \to \mathrm{Hom}^{m_i+1}(M, Y_{j+1}) \to .$$

If the second map is nonzero and the third map zero, $\mathrm{Hom}^\bullet(M, Y_i)$ increases at degree m_i. If the third map is nonzero then $\mathrm{Hom}^\bullet(M, Y_i)$ decreases at degree $m_i + 1$ (we are going from $i + 1$ to i). These two are the only possibilities since $\mathrm{Hom}^{m_i}(M, \nabla(\lambda_i)[-m_i])$ cannot have dimension more than one.

We are mostly interested in the case in which $\mathcal{A}$ in Proposition 7.8 is the set $\{L(\lambda)[l(\lambda)] \mid \lambda \in \Lambda\}$. We say that $M \in \mathcal{C}$ has parity if it has L-parity for any irreducible $L \in \mathcal{C}$. This is equivalent to M having a parity projective resolution, i.e., a projective resolution $P_\bullet$ such that

$$P(\lambda) \text{ is a direct summand of } P_i \Rightarrow i \equiv l(\lambda) + \epsilon \bmod 2,$$

where ϵ is either 0 or 1 (depending on M). Then $\{L(\lambda)[l(\lambda)] \mid \lambda \in \Lambda\} \subset \mathcal{E}^L \cap \mathcal{E}^R$ if and only if all standard modules have parity. (The ϵ in a parity projective resolution of $\Delta(\lambda)$ is determined by the equality $\epsilon \equiv l(\lambda) \bmod 2$.) Following [7], we say that $\mathcal{C}$ has a *Kazhdan-Lusztig theory* if the set $\{L(\lambda)[l(\lambda)] \mid \lambda \in \Lambda\}$ is contained in $\mathcal{E}^R$ (and $\mathcal{E}^L$, but the two conditions are the same under duality).

In the case of U_ζ-modules, each $L(w.\lambda)[l(w)]$ for $\lambda \in {}^lC^-_\mathbb{Z}$ does belong to $\mathcal{E}^L \cap \mathcal{E}^R$. (The length function we use in defining $\mathcal{E}^L$ and $\mathcal{E}^R$ is, of course, the usual length function on W_l.) This follows from Proposition 7.8 and (5.0.1) (and its dual), since $P_{x,y}$ is a polynomial on t^2.

Letting $\mathcal{D} = \mathcal{D}^b(\mathcal{C})$, the recognition theorem can be formulated in our notation from §7.1 as follows.

Proposition 7.10. *We have*

$$(\mathcal{E}^L_0)^{\mathsf{E}} = \mathcal{E}^R \text{ and } {}^{\mathsf{E}}(\mathcal{E}^R_0) = \mathcal{E}^L.$$

An immediate consequence of this (and Proposition 7.5) is that $\mathcal{E}^R, \mathcal{E}^L$ are *closed* in the sense that $({}^{\mathsf{E}}(\mathcal{E}^R))^{\mathsf{E}} = \mathcal{E}^R$ and ${}^{\mathsf{E}}((\mathcal{E}^L)^{\mathsf{E}}) = \mathcal{E}^L$.

7.3. **Linearity and parity.** In this section, we consider positively graded highest weight categories. Let $\mathcal{C}$ be a highest weight category as in §7.2. Identify $\mathcal{C}$ with the category of (finite dimensional) A-modules for some (finite dimensional quasi-hereditary) algebra A. What we assume now is that A is a positively graded algebra and A_0 is semisimple. We let $\widehat{\mathcal{C}}$ be the category of graded A-modules. So we have the "forget the grading" functor $F : \widehat{\mathcal{C}} \to \mathcal{C}$ with $F\langle 1\rangle \cong F$. Here $\langle 1\rangle$ is the grade shift defined by $(M\langle 1\rangle)^i = M^{i-1}$ where M^i denotes the grade i component of $M \in \widehat{\mathcal{C}}$.

We call a graded module $\widehat{M} \in \widehat{\mathcal{C}}$ a *(graded) lift* of $M \in \mathcal{C}$ if $F(\widehat{M}) \cong M$. For any irreducible $L(\lambda) \in \mathcal{C}$, let $\widehat{L}(\lambda) \in \widehat{\mathcal{C}}$ be the irreducible of highest weight λ concentrated in grade 0, let $\widehat{\Delta}(\lambda)$ be the lift of $\Delta(\lambda)$ whose head is $\widehat{L}(\lambda)$, let $\widehat{\nabla}(\lambda)$ be the lift of $\nabla(\lambda)$ whose socle is $\widehat{L}(\lambda)$, let $\widehat{P}(\lambda)$ be the projective cover of $\widehat{L}(\lambda)$ in $\widehat{\mathcal{C}}$, and let $\widehat{I}(\lambda)$ be the injective envelope of $\widehat{L}(\lambda)$ in $\widehat{\mathcal{C}}$. Of course, $\widehat{P}(\lambda)$ lifts $P(\lambda)$ and $\widehat{I}(\lambda)$ lifts $I(\lambda)$.

Recall that $M \in \widehat{\mathcal{C}}$ is called *linear* if it has a projective resolution $P = P_\bullet$ such that the head of P_{-i} is homogeneous of grade i, in other words, $\operatorname{ext}^n(M, \widehat{L}(\lambda)\langle i\rangle) = 0$ unless $i = n$ for any $\lambda \in \Lambda$. We call such a projective resolution a *linear projective resolution*. By definition, $\widehat{\mathcal{C}}$ is *Koszul* if each irreducible $\widehat{L}(\lambda)$ is linear for any $\lambda \in \Lambda$. It is *standard Koszul* if each standard module $\widehat{\Delta}(\lambda)$ for $\lambda \in \Lambda$ is linear and each costandard module is *colinear*, i.e., has an injective resolution $I_\bullet$ such that the socle of I_i is homogeneous of grade $-i$. If $\mathcal{C}$ has a duality, then the condition on costandard modules follows from the one on standard modules.

Compare the following with Proposition 7.2.

Proposition 7.11. *Suppose there is a short exact sequence*

$$0 \to M \to M' \to M'' \to 0$$

in $\widehat{\mathcal{C}}$. Suppose M', M'' are linear. If M is concentrated in grades ≥ 1, then $M\langle -1\rangle$ is linear.

Proof. Let $P, P', P'' \in \mathcal{D}^b(\widehat{\mathcal{C}})$ be minimal projective resolutions of M, M', M'' respectively. Automatically, P', P'' are linear. There is a distinguished triangle

$$P \to P' \to P'' \to P[1] \to .$$

Positivity of grading and the assumption on M implies that the degree n term of P_n of P is generated by grade $n+1$ or greater. By linearity, the kernel of $P' \to P''$ in degree n should be generated by grade n, but the image of $P \to P'$ is in grades $n+1$ or greater. This shows that the map $P \to P'$ is zero (in each degree). So we have a short exact sequence

$$0 \to P' \to P'' \to P[1] \to 0.$$

It follows that $P[1]$ is linear, and so is $P\langle -1\rangle = P[1][-1]\langle -1\rangle$. Hence $M\langle -1\rangle$ is linear. □

Corollary 7.12. *Suppose there is a short exact sequence*

$$0 \to M \to M' \to M'' \to 0$$

in $\widehat{\mathcal{C}}$, and M', M'' linear. If M is concentrated in grades ≥ 2, then M is 0.

Proof. By Proposition 7.11, there is a surjective map $P_0 \to M$ where $P_0 \in \widehat{\mathcal{C}}$ is generated by its components in grade 1. Since M is concentrated in grades ≥ 2, the image of the map $P_0 \to M$ is zero, and hence $M = 0$. □

There are analogues of the categories $\mathcal{E}^R, \mathcal{E}^L$ for $\mathcal{D}^b(\widehat{\mathcal{C}})$. The category $\widehat{\mathcal{E}}^R$ (denoted by $\mathcal{E}^R$ in [36]) is defined as the union of $\widehat{\mathcal{E}}_i^R$ where $\widehat{\mathcal{E}}_i^R$ is defined inductively as follows. Set $\widehat{\mathcal{E}}_0^R$ to be the full subcategory of $\mathcal{D}^b(\widehat{\mathcal{C}})$ whose objects are the direct sums of $\widehat{\nabla}(\lambda)\{m\}$ for $\lambda \in \Lambda$, $m \in \mathbb{Z}$. Here $\{-\}$ is the shift defined as $\{1\} = \langle 1\rangle[1]$. Then we define $\widehat{\mathcal{E}}_i^R$ to be the full subcategory of $\mathcal{D}^b(\widehat{\mathcal{C}})$ such that

$$X \in \widehat{\mathcal{E}}_i^R \Leftrightarrow \exists \text{ a distinguished triangle } X' \to X \to X'' \to \text{ with } X' \in \widehat{\mathcal{E}}_{i-1}^R, X'' \in \widehat{\mathcal{E}}_0^R.$$

The dual category $\widehat{\mathcal{E}}^L$ is defined dually. There is also a version of the recognition theorem (Proposition 7.8), which is proved in a similar way.

Proposition 7.13. [36, Theorem 3.3] *Let* $X \in \mathcal{D}^b(\widehat{\mathcal{C}})$. *Then*

$$X \in \widehat{\mathcal{E}}^R \Leftrightarrow \operatorname{Hom}^n_{\mathcal{D}^b(\widehat{\mathcal{C}})}(\widehat{\Delta}(\lambda), X\langle m\rangle) \neq 0 \textit{ implies } m = n \textit{ (for all } \lambda \in \Lambda).$$

Thus, standard Koszulity (and its dual) is equivalent to that $\widehat{\mathcal{E}}^R$ (and $\widehat{\mathcal{E}}^L$) contains all irreducibles in $\mathcal{D}^b(\widehat{\mathcal{C}})$. We can combine $\widehat{\mathcal{E}}^R$ and $\mathcal{E}^R$ to define a category studied in [8]. We will call it $\mathcal{E}^R_{gr}$, following [8, §1.3]. Let $\mathcal{E}^R_{\mathrm{gr},0} := \widehat{\mathcal{E}}_0 \cap \mathcal{E}^R_0$, where we view $\mathcal{E}^R_0$ a subcategory of $\mathcal{D}^b(\widehat{\mathcal{C}})$, pulling back via the forgetful functor. Thus $\mathcal{E}^R_{\mathrm{gr},0}$ consists of direct sums of $\widehat{\nabla}(\lambda)\{l(\lambda) + 2m\}$, $m \in \mathbb{Z}$, $\lambda \in \Lambda$. The category $\mathcal{E}^R_{\mathrm{gr}}$ is the union of all $\mathcal{E}^R_{\mathrm{gr},i}$, where $\mathcal{E}^R_{\mathrm{gr},i}$ is inductively defined as

$$X \in \mathcal{E}^R_{\mathrm{gr},i} \Leftrightarrow \exists \text{ a distinguished triangle } X' \to X \to X'' \to$$
$$\text{with } X' \in \mathcal{E}^R_{\mathrm{gr},i-1}, X'' \in \mathcal{E}^R_{\mathrm{gr},0}.$$

Using this, the notion of a graded Kazhdan-Lusztig theory is introduced in [8]: $\mathcal{C}$ is said to have a *graded Kazhdan-Lusztig theory* if $\mathcal{E}^R_{\mathrm{gr}}$ contains $\{L(\lambda)\{l(\lambda) + 2m\} \mid \lambda \in \Lambda, m \in \mathbb{Z}\}$.

We have the third recognition theorem.

Proposition 7.14. [8, Theorem 1.3.1] *Let* $X \in \mathcal{D}^b(\widehat{\mathcal{C}})$. *Then*

$$X \in \mathcal{E}^R_{\mathrm{gr}} \Leftrightarrow \operatorname{Hom}^n_{\mathcal{D}^b(\widehat{\mathcal{C}})}(\widehat{\Delta}(\lambda), X\langle m\rangle) \neq 0 \textit{ implies } m = n \textit{ and } n \equiv l(\lambda) \textit{ (for all } \lambda \in \Lambda).$$

This shows that $\mathcal{E}^R_{\mathrm{gr}} = F^{-1}\mathcal{E}^R \cap \widehat{\mathcal{E}}^R$, where F is the forgetful functor from $\mathcal{D}^b(\widehat{\mathcal{C}})$ to $\mathcal{D}^b(\mathcal{C})$ induced by the forgetful functor from $\widehat{\mathcal{C}}$ to $\mathcal{C}$. Therefore, $\mathcal{C}$ has a graded Kazhdan-Lusztig theory if and only if $\mathcal{C}$ has a Kazhdan-Lusztig theory and is standard Koszul.

We conclude the section by presenting a relation between linearity and parity. It will apply to the quantum case.

Proposition 7.15. *Suppose we have* $\mathrm{Ext}^1_{\mathcal{C}}(L(\lambda_1), L(\lambda_2)) = 0$ *whenever* $l(\lambda_1) \equiv l(\lambda_2)$ *mod 2. If* $M \in \mathcal{C}$ *has a linear lift* $\widehat{M} \in \widehat{\mathcal{C}}$*, then* M *has parity. In particular, standard Koszulity implies a Kazhdan-Lusztig theory.*

Proof. Let $P_\bullet$ be a linear projective resolution of $\widehat{M}$. Then $P_i \to P_{i+1}$ maps the head of P_i, which is in grade $-i$, to the second radical layer of P_{i+1}. Then by Lemma 8.7 below, P_i and P_{i+1} have opposite parity. In other word, $P_\bullet$ is a parity resolution of $\widehat{M}$. Let L be any irreducible object in $\mathcal{C}$. Then $\mathrm{Ext}^n_{\mathcal{C}}(M, L) = \mathrm{Hom}_{\mathcal{C}}(P_{-n}, L)$ can be nonzero only when P_{-n} and L have the same parity, thus M has L-parity. The claim follows. The last sentence of the Proposition is obtained by taking M to be a standard module. □

8. Koszulity and singular Kazhdan-Lusztig theory

Let for $J \subset S_l$ and $y, w \in W^J$

$$P^J_{y,w} := \sum_{x \in W_J} (-1)^{l(x)} P_{yx,w}.$$

This is called a *parabolic Kazhdan-Lusztig polynomial* [10, 20].

Our goal is to show that the formula

$$\sum_{n=0}^{\infty} \dim \mathrm{Ext}^n_{U_\zeta}(\Delta_\zeta(y.\mu), L_\zeta(w.\mu)) t^n = t^{l(w)-l(y)} \bar{P}^J_{y,w} \tag{8.0.1}$$

holds for all $\mu \in \overline{{}^l C^-_{\mathbb{Z}}}$, $y, w \in W^+(\mu)$, where $J = \{s \in S_l \mid s.\mu = \mu\}$. Assuming that l is KL-good, it is enough to prove the formula (8.0.1) in $\mathcal{O}$ at the negative level $d = -l/2D - g$. Recall that we let $\widetilde{\gamma} = \gamma + d\chi$. Identifying the affine Weyl group for U_ζ with the one for $\widetilde{\mathfrak{g}}$ as in §1.5, we have $w.\widetilde{\mu} = \widetilde{w.\mu}$.

To apply [40] more easily, we bring the formula into $\mathbb{O}$. Given a weight $\widetilde{\mu} = \mu + k\chi$ for $\widetilde{\mathfrak{g}}$, we fix a weight

$$\widehat{\mu} := \mu + k\chi + b\delta$$

for $\widehat{\mathfrak{g}}$, where δ is the imaginary part and b is some number we don't care as long as it makes $\widehat{\mu}$ lie out of the critical hyperplanes. Recall that the integral Weyl group of $\xi \in \widehat{\mathfrak{h}}^*$ (resp., $\widetilde{\mathfrak{h}}^*$) is defined to be generated by the simple reflections corresponding to the simple roots α such that (α, α) divides $2(\xi + \rho, \alpha)$, where $(-,-)$ is a non-degenerate bilinear form on $\widehat{\mathfrak{h}}^*$ (resp., $\widetilde{\mathfrak{h}}^*$) extending one on $\mathfrak{h}^*$. Since $\widehat{\mu}$ lies out of the critical hyperplanes, the integral Weyl group of $\widehat{\mu}$ is isomorphic to W_l as a Coxeter group, and is denoted by W_l for convenience. (We can also use the (Coxeter) ordering on $W^+(\mu)$ as the poset ordering in the affine cases. See [33, Appendix I].) By [33, Corollary 3.2] and the preceding footnote in [33], the orbit of $\widetilde{\mu}$ in $\mathcal{O}$ (i.e., the truncation $\mathcal{O}[W_l.\widetilde{\mu}]$ of $\mathcal{O}$, which is a direct summand) is isomorphic to the orbit of $\widehat{\mu}$ in $\mathbb{O}^+$. Here $\mathbb{O}^+$ is the full subcategory of $\mathbb{O}$ consisting of the modules whose composition factors are of integral dominant highest weight (dominant for the subalgebra $\mathfrak{g}$).

In this setting, the formula (8.0.1) is equivalent to

$$\sum_{n=0}^{\infty} \dim \operatorname{Ext}^n_{\mathbb{O}^+}(\Delta(y.\widehat{\mu}), L(w.\widehat{\mu}))t^n = t^{l(w)-l(y)} \bar{P}^J_{y,w} \tag{8.0.2}$$

for $\mu \in \overline{{}^l C^-_{\mathbb{Z}}}$, $y, w \in W^+(\mu)$.

Applying the truncation of highest weight category (§4.1) to $\mathcal{C}' = \mathbb{O}^+$, it is enough to prove (8.0.2) in $\mathcal{C} = \mathbb{O}^+[\Gamma]$ for a finite ideal Γ containing $y.\widehat{\mu}, w.\widehat{\mu}$.

8.1. **Koszul grading and parity vanishing.** We assume in this subsection that the level d is an integer. This is in order to use the result of [40]. We also assume that $l > h$. We see in (the proof of) Theorem 8.10 below that these restrictions are not necessary for our result.

Let $\mathcal{C}_{\widehat{\lambda}}$, $\mathcal{C}_{\widehat{\mu}}$ be truncations of $\widehat{\lambda}$ and $\widehat{\mu}$ orbits as in [40, §3.4]. That is, there is some $v \in W_l$, which we do not keep track of, such that $\mathcal{C}_{\widehat{\lambda}} = \mathbb{O}^+[\Lambda]$ where $\Lambda = \{w.\widehat{\lambda} \in W^+.\widehat{\lambda} \mid w \leq v\}$. And $\mathcal{C}_{\widehat{\mu}}$ is similarly defined. (In the notation of [40], $\mathcal{C}_{\widehat{\lambda}} = {}^v\mathbf{O}^{\emptyset}_{I,-}$ and $\mathcal{C}_{\widehat{\mu}} = {}^v\mathbf{O}^{\emptyset}_{J,-}$.) We assume that $\widehat{\lambda}$ is regular.[3] Then we have the following result of Shan-Varagnolo-Vasserot.

Theorem 8.1. [40, Theorem 3.12, Lemma 5.10] *The categories $\mathcal{C}_{\widehat{\lambda}}$, $\mathcal{C}_{\widehat{\mu}}$ are standard Koszul. Letting $\widehat{\mathcal{C}_{\widehat{\lambda}}}$, $\widehat{\mathcal{C}_{\widehat{\mu}}}$ be the corresponding categories of graded modules, there is a graded translation functor $\widehat{T^{\mu}_{\lambda}} : \widehat{\mathcal{C}_{\widehat{\lambda}}} \to \widehat{\mathcal{C}_{\widehat{\mu}}}$ which lifts the (ungraded) translation functor $T^{\mu}_{\lambda} : \mathcal{C}_{\widehat{\lambda}} \to \mathcal{C}_{\widehat{\mu}}$. (See* [40, Proposition 4.36] *and the remark below.) That is, $F \circ \widehat{T^{\mu}_{\lambda}} \cong T^{\mu}_{\lambda} \circ F$ where F is the functor (on both $\widehat{\mathcal{C}_{\widehat{\lambda}}}$ and $\widehat{\mathcal{C}_{\widehat{\mu}}}$) that forgets the grading. The functor $\widehat{T^{\mu}_{\lambda}}$ satisfies $\widehat{T^{\mu}_{\lambda}}\widehat{L}(w.\widehat{\lambda}) = \widehat{L}(w.\widehat{\mu})$ for $w \in W^J$.*

Remark 8.2. The condition "$d+N > f$" in [40, Lemma 5.10] or a similar condition in [40, Proposition 4.36] says that the difference between the level of $\widehat{\mu}$ and the level of $\widehat{\lambda}$ is less than the dual Coxeter number g. (The dual Coxeter number is denoted by N in [40]. The numbers d, f in [40] are such that $-d-N$ and $-f-N$ are the levels of the weights.) But to use the translation in [19], as the beginning of the proof of [40, Proposition 4.36] does, a different assumption on the weights is required: Given two integral affine weights ν, ξ of (not necessarily the same) negative levels, the translation

$$T^{\xi}_{\nu} : \mathbb{O}_{\nu} \to \mathbb{O}_{\xi}$$

from the orbit of ν in $\mathbb{O}$ (called $\mathbb{O}_{\nu}$) to the orbit of ξ in $\mathbb{O}$ (called $\mathbb{O}_{\xi}$) as in [19, §3] is defined as

$$T^{\xi}_{\nu} = \mathrm{pr}_{\xi}(- \otimes L(\omega))$$

[3]We need neither fix the level d nor assume $l > h$, as the translation functors can move the level. But we make this assumption anyway, because it is easy to take care of the restriction on d altogether when we treat the case of non-integer d. See the proof of Theorem 8.10.

when there exist a weight $\omega \in P^+ \cap W_a(\xi - \nu)$ where P^+ is the set of integral dominant (affine) weights for $\widehat{\mathfrak{g}}$ and W_a is the (affine) Weyl group of $\widehat{\mathfrak{g}}$ (see [19, §2 ,3]). Since $\omega \in P^+$, the irreducible module $L(\omega)$ is integrable, and we are in the situation very similar to the algebraic group case or the quantum case. (See also Remark 2.3.) The requirement, which is equivalent to $\xi - \nu \in W_a P^+$, is different from and not implied by the condition "$d + N > f$".

We can instead construct the desired translation in two steps as follow. As in [40], it is enough to define a translation $T_\nu^\xi : \mathbb{O}_\nu \to \mathbb{O}_\xi$ where ν is a regular (antidominant integral) weight. Then we can restrict, as usual, to truncated categories (having finite poset ideals) to view the functor as $T_\nu^\xi : \mathcal{C}_\nu \to \mathcal{C}_\xi$ and take $T_\xi^\nu : \mathcal{C}_\xi \to \mathcal{C}_\nu$ to be its left adjoint. Let $\widehat{\rho} := \rho + g\chi$ be the "affine ρ". Then, given any integral weight ξ in the closure of the antidominant alcove, the weights $\xi + n\widehat{\rho}$, $\nu + n\widehat{\rho}$ are integral for each $n \in \mathbb{Z}$. They are dominant if n is sufficiently large. Take such an n. Now $\xi - (-n\widehat{\rho}), \nu - (-n\widehat{\rho}) \in P^+ \subset W_a P^+$ defines the translations $T^\xi_{-n\widehat{\rho}}$ and $T^\nu_{-n\widehat{\rho}}$. Note that ν and $-n\widehat{\rho}$ are in the same facet, the antidominant alcove. This implies the translation functor $T^\nu_{-n\widehat{\rho}}$ is an equivalence (see for example [19, Propositions 3.6, 3.8] and the comparison theorem [31, Theorem 5.8], or see [33, §6]). We fix an inverse and call it $T_\nu^{-n\widehat{\rho}}$. Since $T_\nu^{-n\widehat{\rho}}$ is an inverse of a translation functor, it behaves just like a classical translation functor. Finally, let $T_\nu^\xi := T^\xi_{-n\widehat{\rho}} \circ T_\nu^{-n\widehat{\rho}}$. The functor T_ν^ξ has all the properties that the classical translations have. Therefore, the rest of [40, Proposition 4.36, Lemma 5.10] works.

Let

$$\widehat{T_\mu^\lambda} : \widehat{\mathcal{C}}_{\widehat{\mu}} \to \widehat{\mathcal{C}}_{\widehat{\lambda}}$$

be a left adjoint of $\widehat{T_\lambda^\mu}$. Its existence follows from the adjoint functor theorem because we are dealing with finite number of irreducible objects and $\mathrm{End}(L) = K$ for each irreducible L.

We want the translation functors in Theorem 8.1 for $\widehat{\mathfrak{g}}$ (restricted to $\mathcal{O}_k$) to agree with the translation functors for U_ζ-mod via the Kazhdan-Lusztig correspondence. To avoid discussing this problem, we redefine the translation $T_\lambda^\mu : \mathcal{C}_\lambda^\zeta \to \mathcal{C}_\mu^\zeta$ to be $\mathscr{F}_l(T_\lambda^\mu)$ and $T_\mu^\lambda : \mathcal{C}_\mu^\zeta \to \mathcal{C}_\lambda^\zeta$ to be $\mathscr{F}_l(T_\mu^\lambda)$, where the category $\mathcal{C}_\lambda^\zeta$ is the truncation of U_ζ-mod by the ideal corresponding to the poset of $\mathcal{C}_{\widehat{\lambda}}$ and $\mathcal{C}_\mu^\zeta$ is the truncation by the ideal corresponding to the poset of $\mathcal{C}_{\widehat{\mu}}$. Then everything we need from §6 is still true by the same proof using the basic properties in [40, Proposition 4.36]. We denote $\mathrm{Ext}^n_{\widehat{\mathcal{C}}}(-,-)$ by $\mathrm{ext}^n_{\mathcal{C}}(-,-)$ and $\mathrm{Hom}_{\widehat{\mathcal{C}}}(-,-)$ by $\hom_{\mathcal{C}}(-,-)$.

Corollary 8.3. *The module $\widehat{T_\mu^\lambda}\widehat{T_\lambda^\mu}\widehat{\Delta}(y.\widehat{\lambda})$ is linear for any $y \in W^J$.*

Proof. Adjunction gives for all n, i

$$\begin{aligned}\mathrm{ext}^n_{\mathcal{C}_{\widehat{\lambda}}}(\widehat{T_\mu^\lambda}\widehat{T_\lambda^\mu}\widehat{\Delta}(y.\widehat{\lambda}), \widehat{L}(w.\widehat{\lambda})\langle i\rangle) &\cong \mathrm{ext}^n_{\mathcal{C}_{\widehat{\lambda}}}(\widehat{T_\lambda^\mu}\widehat{\Delta}(y.\widehat{\lambda}), \widehat{T_\lambda^\mu}\widehat{L}(w.\widehat{\lambda})\langle i\rangle)\\ &\cong \mathrm{ext}^n_{\mathcal{C}_{\widehat{\mu}}}(\widehat{\Delta}(y.\widehat{\mu}), \widehat{L}(w.\widehat{\mu})\langle i\rangle),\end{aligned}$$

which is 0 unless $n = i$ by standard Koszulity of $\mathcal{C}_{\widehat{\mu}}$. Thus $\widehat{T_\mu^\lambda}\widehat{T_\lambda^\mu}\widehat{\Delta}(y.\widehat{\lambda})$ is linear. □

Remark 8.4. In fact, a linear projective resolution of $\widehat{T_\mu^\lambda}\widehat{T_\lambda^\mu}\widehat{\Delta}(y.\widehat{\lambda}) = \widehat{T_\mu^\lambda}\widehat{\Delta}(y.\widehat{\mu})$ is obtained by applying the translation to a linear projective resolution of $\widehat{\Delta}(y.\widehat{\mu})$. Let $P_\bullet$ be one. It is obvious that $\widehat{T_\mu^\lambda}P_\bullet$ is a projective resolution of $\widehat{T_\mu^\lambda}\widehat{\Delta}(y.\widehat{\mu})$. For linearity, we check

$$\widehat{T_\mu^\lambda}\widehat{P}(w.\widehat{\mu}) \cong \widehat{P}(w.\widehat{\lambda}).$$

This is true up to grading shift by [16, II.7.16], and we only need to check that the head of $\widehat{T_\mu^\lambda}\widehat{P}(w.\widehat{\mu})$ is in grade 0. But this is the case because

$$\hom_{\mathcal{C}_{\widehat{\lambda}}}(\widehat{T_\mu^\lambda}\widehat{P}(w.\widehat{\mu}), \widehat{L}(z.\widehat{\lambda})\langle i\rangle) \cong \hom_{\mathcal{C}_{\widehat{\mu}}}(\widehat{P}(w.\widehat{\mu}), \widehat{L}(z.\widehat{\mu})\langle i\rangle)$$

is zero unless $i = 0$.

Fix $y, w \in W^J$ where J is associated to $\mu \in \overline{{}^l C_{\mathbb{Z}}^-}$. Recall the filtration U_i from §6. We still denote by U_i the $\widehat{\mathfrak{g}}$-module $\mathscr{F}_l^{-1} U_i$ embedded in $\mathbb{O}$. Our new definition of the quantum translation gives $U_0 = T_\mu^\lambda \Delta(y.\widehat{\lambda})$. Lemma 6.2, 6.3 and Corollary 6.5 are still valid. Using the graded translations, we construct a graded lift of U_i starting from $\widehat{U}_0 = \widehat{T_\mu^\lambda} \widehat{T_\lambda^\mu} \widehat{\Delta}(y.\widehat{\lambda})$. We have

$$0 \to \widehat{U}_{i+1} \to \widehat{U}_i \to \bigoplus_{x \in W_J,\ l(x)=i} \widehat{\Delta}(yx.\widehat{\lambda})\langle n_x \rangle \to 0,$$

for some $n_x \in \mathbb{Z}$ depending on x. In fact, we know what the shifts n_x are.

Proposition 8.5. *The filtration* $\{\widehat{U}_i\}$ *of* $\widehat{T_\mu^\lambda} \widehat{T_\lambda^\mu} \widehat{\Delta}(y.\widehat{\lambda})$ *satisfies the short exact sequences*

$$0 \to \widehat{U}_{i+1} \to \widehat{U}_i \to \bigoplus_{x \in W_J,\ l(x)=i} \widehat{\Delta}(yx.\widehat{\lambda})\langle i \rangle \to 0$$

for all i.

Proof. Since $\widehat{U}_0$ has an irreducible head, its radical filtration agrees with its grading filtration by Koszulity. So this follows from Proposition 6.6. □

Corollary 8.6. *For all* i, $\widehat{U}_i\langle -i \rangle \in \widetilde{\mathcal{C}}_{\widehat{\lambda}}$ *is linear.*

Proof. It follows by induction on i. The base case is proven in Corollary 8.3, and Propositions 8.5, 7.11 does the induction step. □

We need the following in order to apply Proposition 7.15.

Lemma 8.7. *For* $\mu \in \overline{{}^l C_{\mathbb{Z}}^-}$ *and* $y, z \in W^+(\mu)$ *with* $l(y) \equiv l(z)$ *mod* 2, *we have*

$$\mathrm{Ext}^1_{\mathcal{C}_{\widehat{\mu}}}(L(y.\widehat{\mu}), L(z.\widehat{\mu})) = 0.$$

Proof. First note that the statement is true for a regular weight λ. (For example, it follows from (5.0.1), its dual, and [7, Corollary (3.6)].) Also, Koszulity implies that the radical filtration and the grade filtration of a standard module $\Delta(y.\widehat{\lambda})$ are the same. So the grade filtration of $\Delta(y.\widehat{\lambda})$ has alternating parity. Since $\widehat{T_\lambda^\mu}$ is exact and preserves the parity of irreducibles, the module $\widehat{T_\lambda^\mu}\widehat{\Delta}(y.\widehat{\lambda}) = \widehat{\Delta}(y.\widehat{\mu})$ also has a grade filtration with alternating parity. Hence $\Delta(y.\widehat{\mu})$ has a radical filtration with alternating parity. Now suppose

$$0 \to L(z.\widehat{\mu}) \to M \to L(y.\widehat{\mu}) \to 0$$

represents a non-trivial element in $\mathrm{Ext}^1_{\mathcal{C}_{\widehat{\mu}}}(L(y.\widehat{\mu}), L(z.\widehat{\mu}))$. The linkage principle rules out any possibilities other than the cases $z > y$ or $y > z$. We may assume $y > z$ by duality. Then there is a surjective map from $\Delta(y.\widehat{\mu})$ to M. This contradicts the assumption that z and y are of the same parity and that $\Delta(y.\widehat{\mu})$ has a radical filtration with alternating parity. □

We now obtain a key property of the modules $U_i \in \mathcal{C}_{\widehat{\lambda}}$. Recall that (for a general highest weight category $\mathcal{C}$) an object $M \in \mathcal{C}$ is said to have N-parity if $\mathrm{Ext}^{2n+1}_{\mathcal{C}}(M, N) = 0$ for all $n \in \mathbb{Z}$ and is said to have parity if it has L-parity for all irreducible $L \in \mathcal{C}$.

Corollary 8.8. *For each i, U_i has parity.*

Proof. This is an immediate corollary of Corollary 8.6, Lemma 8.7, and Proposition 7.15. □

Example 8.9. Consider the quotient $\widehat{U}_i' := \widehat{U}_0/\widehat{U}_i$ of $\widehat{U}_0$. We have

$$\widehat{T_\mu^\lambda}\widehat{T_\lambda^\mu}\Delta(y.\widehat{\lambda}) = \widetilde{U}_N' \twoheadrightarrow \widetilde{U}_{N-1}' \twoheadrightarrow \cdots \twoheadrightarrow \widetilde{U}_1' \twoheadrightarrow \widetilde{U}_0' = 0,$$

where $N = l(w_J)$. By Corollary 7.12 $\widehat{U}_i'$ is not linear, even up to shift, for $1 < i < N$, while $U_i' = F(\widetilde{U}_i')$ has parity if i is odd. (If i is odd, then U_i has L-parity opposite of U_0 with respect to any irreducible L. Lemma 7.2 shows that U_i' has L-parity.)

8.2. **Cohomology in singular blocks.** We are ready to prove our main theorem using that U_i has parity. Note that the statement of Corollary 8.8 does not involve any grading. We now forget the grading and prove our main theorem. Recall the definition

$$\bar{P}^J_{y,w} = \sum_{x \in W_J} (-1)^{l(x)} \bar{P}_{yx.w}.$$

Theorem 8.10. [36, Conjecture III] *Suppose l is KL-good for the root system R. Let $\mu \in \overline{{}^lC^-_{\mathbb{Z}}}$ and $J = \{s \in S_l \mid s.\mu = \mu\}$. We have*

$$\sum_{n=0}^{\infty} \dim \mathrm{Ext}^n_{U_\zeta}(\Delta_\zeta(y.\mu), L_\zeta(w.\mu))t^n = t^{l(w)-l(y)} \bar{P}^J_{y,w}$$

for $y, w \in W^J$.

Proof. As we discussed in the beginning of this section (§8), this follows if we show

$$\sum_{n=0}^{\infty} \dim \mathrm{Ext}^n_{\widehat{\mathfrak{g}}}(\Delta(y.\widehat{\mu}), L(w.\widehat{\mu}))t^n = t^{l(w)-l(y)} \bar{P}^J_{y,w}$$

for $y, w \in W^J$.

We first reduce the statement to the case where the assumptions in §8.1 are satisfied. If we pick a large integer $l' \geq h$ that is divisible by $2D$, there is a regular weight $\widehat{\lambda}$ and a weight $\widehat{\nu}$ of level k' with $k' = -l'/2D - g \in \mathbb{Z}$ such that the integral Weyl group of $\widehat{\lambda}, \widehat{\nu}$ are both isomorphic to $W_{l'}$ and $\mathrm{Stab}_{W_{l'}}(\widehat{\nu})$ is isomorphic to $\mathrm{Stab}_{W_l}(\widehat{\mu})$ under the Coxeter group isomorphism $(W_l, S_l) \xrightarrow{\sim} (W_{l'}, S_{l'})$. By Fiebig's combinatorial description [12, Theorem 11], it is enough to prove the theorem for $\widehat{\nu}$ instead of $\widehat{\mu}$. The problem of the full category $\mathbb{O}$ in [12] and the categories of [40] being different is treated in [37].[4] So we may assume that we are in the situation in §8.1.

Let $\widehat{\lambda}$ be a regular weight. We translate from $\widehat{\lambda}$ to $\widehat{\mu}$ as in §8.1. Corollary 8.6 and Proposition 7.15 show that each U_i has parity. In particular it has $L = L(w.\widehat{\lambda})$-parity, that is, $\mathrm{Ext}^n_{\widehat{\mathfrak{g}}}(U_i, L)$ is zero in every other degree. To be more precise, U_i is L-even

[4] In [37], it is similarly shown that U_ζ-mod is Koszul. Using that we could have worked entirely in the quantum case to prove the theorem. But then, if $l < h$, there is no regular weight we can translate from, and we will anyway have to use the affine category $\mathcal{O}$ to obtain our result for small (KL-good) l.

(resp., odd), if and only if $\bigoplus_{l(x)=i, x\in W_J} \Delta(yx.\widehat{\lambda})$ is L-even (resp., odd), if and only if U_{i+1} is L-odd (resp., even). Therefore, half the terms in the long exact sequence induced by applying $\mathrm{Hom}_{\widehat{\mathfrak{g}}}(-, L)$ to each short exact sequence

$$0 \to U_{i+1} \to U_i \to \bigoplus_{l(x)=i,\ x\in W_J} \Delta(yx.\widehat{\lambda}) \to 0$$

vanish, and the sequence splits into the short exact sequences

$$0 \to \mathrm{Ext}^{n-1}_{\widehat{\mathfrak{g}}}(U_{i+1}, L) \to \mathrm{Ext}^n_{\widehat{\mathfrak{g}}}(\bigoplus_{l(x)=i,\ x\in W_J} \Delta(yx.\widehat{\lambda}), L) \to \mathrm{Ext}^n_{\widehat{\mathfrak{g}}}(U_i, L) \to 0.$$

They give

$$\begin{aligned}\sum_{n=0}^{\infty} \dim \mathrm{Ext}^n_{\widehat{\mathfrak{g}}}(U_i, L)t^n = &\sum_{n=0}^{\infty} \dim \mathrm{Ext}^n_{\widehat{\mathfrak{g}}}(\oplus_{l(x)=i}\Delta(yx.\widehat{\lambda}), L)t^n \\ &- t\sum_{n=0}^{\infty} \dim \mathrm{Ext}^n_{\widehat{\mathfrak{g}}}(U_{i+1}, L)t^n\end{aligned}$$

for all n.

Putting them together, we get

$$\begin{aligned}\sum_{n=0}^{\infty} \dim \mathrm{Ext}^n_{\widehat{\mathfrak{g}}}(\Delta(y.\widehat{\mu}), L(w.\widehat{\mu}))t^n &= \sum_{n=0}^{\infty} \dim \mathrm{Ext}^n_{\widehat{\mathfrak{g}}}(T^\lambda_\mu T^\mu_\lambda \Delta(y.\widehat{\lambda}), L(w.\widehat{\lambda}))t^n \\ &= \sum_i (-t)^i \sum_{n=0}^{\infty} \dim \mathrm{Ext}^n_{\widehat{\mathfrak{g}}}(\oplus_{l(x)=i}\Delta(yx.\widehat{\lambda}), L(w.\widehat{\lambda}))t^n \\ &= \sum_i (-t)^i \sum_{l(x)=i} \sum_{n=0}^{\infty} \dim \mathrm{Ext}^n_{\widehat{\mathfrak{g}}}(\Delta(yx.\widehat{\lambda}), L(w.\widehat{\lambda}))t^n \\ &= \sum_{x\in W_J} (-t)^{l(x)} \sum_{n=0}^{\infty} \dim \mathrm{Ext}^n_{\widehat{\mathfrak{g}}}(\Delta(yx.\widehat{\lambda}), L(w.\widehat{\lambda}))t^n \\ &= \sum_{x\in W_J} (-t)^{l(x)} t^{l(w)-l(yx)} \bar{P}_{yx,w}\end{aligned}$$

$$= t^{l(w)-l(y)} \sum_{x \in W_J} (-1)^{l(x)} \bar{P}_{yx,w}$$

$$= t^{l(w)-l(y)} \bar{P}^J_{y,w},$$

and we are done. □

For the next corollary, we make statements in U_ζ-mod rather than in $\mathbb{O}$ or in $\mathcal{O}$ in order to simplify the notation. In particular, U_i is in U_ζ-mod again. In Theorem 8.10, we computed the dimensions of $\mathrm{Ext}^n_{U_\zeta}(U_i, L(w.\lambda))$, for $w \in W^+(\mu)$. But we don't need w to be in $W^+(\mu)$:

Corollary 8.11. *Fix an integer i. We have for $y \in W^+(\mu)$ and $w \in W^+$,*

$$\sum_{n=0}^{\infty} \dim \mathrm{Ext}^n_{U_\zeta}(U_i, L_\zeta(w.\lambda))t^n = t^{l(w)-i} \sum_{x \in W_J,\ l(x) \geq i} (-1)^{l(x)-i} P_{yx,w}.$$

In particular, this polynomial has non-negative coefficients.

Proof. Since all U_j have $L_\zeta(w.\lambda)$-parity, we obtain the formula as in the proof of Theorem 8.10. □

If $w \notin W^J$, then $T^\lambda_\mu L_\zeta(w.\mu)$ is 0 and $\mathrm{Ext}^n_{U_\zeta}(U_0, L_\zeta(w.\lambda))$ is 0. This shows an identity in Kazhdan-Lusztig polynomials (which might have been known for any $y \in W_l$ and $w \notin W^J$).

Corollary 8.12. *If $w \in W^+ \setminus W^+(\mu)$ and $y \in W^+(\mu)$, then*

$$\sum_{x \in W_J} (-1)^{l(x)} P_{yx,w} = 0.$$

8.3. **Graded enriched Grothendieck groups.** We present another proof of Theorem 8.10. We are still in the setting of §8.1. In particular, $w \in W^J$. Our plan is to apply the translation functor $\widehat{T^\mu_\lambda} : \widehat{\mathcal{C}}_{\widehat{\lambda}} \to \widehat{\mathcal{C}}_{\widehat{\mu}}$ to a sequence of distinguished triangles that realizes $\widehat{L}(w.\widehat{\lambda})$ in $\widehat{\mathcal{E}}^R(\widehat{\mathcal{C}}_{\widehat{\lambda}})$. Recall the construction in the proof of

Proposition 7.8. Replacing $\nabla(\widehat{\lambda}_i)[n_i]$ by $\widehat{\nabla}(\lambda_i)\{n_i\}$, we obtain the graded complexes $\widehat{L}(w.\widehat{\lambda}) = Y_0, \cdots, Y_N = 0$ in $\widehat{\mathcal{E}}^R$. Writing $\lambda_i = w_i.\widehat{\lambda}$, there is a distinguished triangle

$$Y_{i+1} \to Y_i \to \widehat{\nabla}(w_i.\widehat{\lambda})\{n_i\} \to$$

for each $0 \leq i \leq N$. We know by Lemma 8.7 and Proposition 7.15 that $n_i \equiv l(w) - l(w_i) \mod 2$. Since the translation functors are exact, applying $\widehat{T_\lambda^\mu}$ to the sequence $Y_0, \cdots, Y_N$ produce the sequence $\widehat{L}(w.\mu) = \widehat{T_\lambda^\mu}Y_0, \cdots, \widehat{T_\lambda^\mu}Y_N = 0$ of objects in $\mathcal{D}^b(\widehat{\mathcal{C}}_{\widehat{\mu}})$ and distinguished triangles

$$\widehat{T_\lambda^\mu}Y_{i+1} \to \widehat{T_\lambda^\mu}Y_i \to \widehat{T_\lambda^\mu}\widehat{\nabla}(w_i.\widehat{\lambda})\{n_i\} \to$$

in $\mathcal{D}^b(\widehat{\mathcal{C}}_{\widehat{\mu}})$.

Proposition 8.13. *We have*

$$\widehat{T_\lambda^\mu}\widehat{\nabla}(yx.\widehat{\lambda}) \cong \widehat{\nabla}(y.\widehat{\mu})\langle -l(x)\rangle, \quad \widehat{T_\lambda^\mu}\widehat{\Delta}(yx.\widehat{\lambda}) \cong \widehat{\Delta}(y.\widehat{\mu})\langle l(x)\rangle$$

for $y \in W^J, x \in W_J$.

Proof. We show only the assertion for $\widehat{\Delta}(yx.\widehat{\lambda})$. Let $l(x) = i$. Recall that

$$\widehat{T_\lambda^\mu}\widehat{L}(yx.\widehat{\lambda}) \cong \delta_{yx,y}\widehat{L}(y.\widehat{\lambda}).$$

Since $T_\lambda^\mu \Delta(yx.\widehat{\lambda}) \cong \Delta(y.\widehat{\mu})$ and $\Delta(y.\widehat{\mu})$ has only one composition factor isomorphic to $L(y.\widehat{\mu})$, it is enough to show that $\widehat{\Delta}(yx.\widehat{\lambda})$ has $\widehat{L}(y.\widehat{\lambda})\langle i\rangle$ as its composition factor. By the Brauer-Humphreys reciprocity, this is equivalent to $\widehat{\Delta}(yx.\widehat{\lambda})\langle i\rangle$ appearing in a $\widehat{\Delta}$-filtration of $\widehat{P}(y.\widehat{\lambda})$. But we saw in Proposition 6.6 that this is true for $\widehat{U}_0$ instead of $\widehat{P}(y.\widehat{\lambda})$, because Koszulity implies that the radical filtration of $\widehat{U}_0$ agrees with the grading filtration. Since $\widehat{P}(y.\widehat{\lambda}) \twoheadrightarrow \widehat{U}_0$, and since the kernel of this map has a Δ-filtration, this is enough. □

Writing $w_i = y_i x_i$ with $y_i \in W^J, x_i \in W_J$ uniquely, Proposition 8.13 tells us that the distinguished triangles are

$$\widehat{T^\mu_\lambda} Y_{i+1} \to \widehat{T^\mu_\lambda} Y_i \to \widehat{\nabla}(y_i.\widehat{\mu})\{n_i\}\langle -l(x_i)\rangle = \widehat{\nabla}(y_i.\widehat{\mu})[l(x_i)]\{n_i - l(x_i)\} \to .$$

These are not distinguished triangles in $\widehat{\mathcal{E}}^R$. But we know by Theorem 8.1 that there exists a sequence $\widehat{L}(w.\widehat{\mu}) = X_0, \cdots, X_{N'} = 0$ in $\widehat{\mathcal{E}}^R$ with distinguished triangles

$$X_{j+1} \to X_j \to \widehat{\nabla}(z_j.\widehat{\mu})\{m_j\} \to .$$

Let us compare these two sequence to determine the (unordered) multiset $\{(z_j, m_j)\}$.

Consider the enriched Grothendieck group $K^R = K_0^R(\mathcal{C}_{\widehat{\mu}})$ and the graded enriched Grothendieck group $\widehat{K}^R = K_0^R(\widehat{\mathcal{C}}_{\widehat{\mu}})$ defined in [7]. The two sequences provide two expressions of $[\widehat{L}(w.\widehat{\mu})] \in \widehat{K}^R$ with respect to the $\mathbb{Z}[v, v^{-1}]$-basis $\{[\widehat{\nabla}(y.\widehat{\mu})]\}_{y \in W^J}$. The sequence $\widehat{T^\mu_\lambda} Y_i$ provides

$$\sum_{0 \le i \le N} (-1)^{l(x_i)} t^{n_i - l(x_i)} [\widehat{\nabla}(y_i.\widehat{\mu})], \tag{8.3.1}$$

and the sequence X_j provides

$$\sum_{0 \le j \le N'} t^{m_j} [\widehat{\nabla}(z_j.\widehat{\mu})].$$

Let $c_{y,n}$ be the $\mathbb{Z}$-coefficient of $t^{-n}[\widehat{\nabla}(y.\widehat{\mu})]$ in the expression, thus

$$c_{y,n} = |\{j \in [0, N'] \mid z_j = y,\ -m_j = n\}|.$$

(Recall that m_j are negative integers.) This is the dimension of $\mathrm{ext}^n_{C_\mu}(\widehat{\Delta}(y.\widehat{\mu})\langle -n\rangle, \widehat{L}(w.\widehat{\mu}))$ which is the same as $\mathrm{Ext}^n_{C_\mu}(\Delta(y.\widehat{\mu}), L(w.\widehat{\mu}))$ by standard Koszulity.

The expression (8.3.1) determines $c_{y,n}$. It remains to write down the relation explicitly. We have

$$c_{y,n} = |\{i \in [0,N]|\ y_i = y,\ n_i - l(x_i) = -n,\ l(x_i) \text{ even}\}| \\ - |\{i \in [0,N]|\ y_i = y,\ n_i - l(x_i) = -n,\ l(x_i) \text{ odd}\}|.$$

Letting $c^x_{y,n} := |\{i \in [0,N]\ |y_i = y,\ x_i = x,\ -n_i = n\}|$, we can write

$$c_{y,n} = \sum_{x \in W_J} (-1)^{l(x)} c^x_{y,n-l(x)}.$$

Note also that

$$c^x_{y,n} = |\{i \in [0,N]\ |w_i = yx,\ -n_i = n\}|.$$

Since we started from the realization Y_i of $\widehat{L}(w.\widehat{\lambda})$, the number $c^x_{y,n}$ is the dimension of

$$\operatorname{ext}^n_{C_{\widehat{\lambda}}}(\widehat{\Delta}(yx.\widehat{\lambda})\langle -n\rangle, \widehat{L}(w.\widehat{\lambda})) \cong \operatorname{Ext}^n_{C_{\widehat{\lambda}}}(\Delta(yx.\widehat{\lambda}), L(w.\widehat{\lambda})).$$

Combining all this, we obtain the identity

$$\dim \operatorname{Ext}^n_{C_{\widehat{\lambda}}}(\Delta(y.\widehat{\mu}), L(w.\widehat{\mu})) = \sum_{x \in W_J} \dim \operatorname{Ext}^{n-l(x)}_{C_{\widehat{\lambda}}}(\Delta(yx.\widehat{\lambda}), L(w.\widehat{\lambda})).$$

This is equivalent to the formula (8.0.1) by the formula (5.0.1). Finally, we transfer this to the quantum case as in the first proof.

8.4. **Ext-groups between irreducibles.** Dualizing Theorem 8.10, we obtain

$$\sum_{n=0}^{\infty} \dim \operatorname{Ext}^n_{U_\zeta}(L_\zeta(w.\mu), \nabla_\zeta(y.\mu)) t^n = t^{l(w)-l(y)} \bar{P}^J_{y,w}$$

for $y, w \in W^J$. Then [7, Corollary (3.6)] combined with the fact that $P^J_{y,w}$ is a polynomial on t^2 shows that the dimension for $\operatorname{Ext}^\bullet_{U_\zeta}(L_\zeta(w.\mu), L_\zeta(z.\mu))$ is given as

$$\dim \operatorname{Ext}^n_{U_\zeta}(L_\zeta(w.\mu), L_\zeta(z.\mu))$$

$$= \sum_{i+j=n,\ y\in W^+(\mu)} \dim \mathrm{Ext}^i_{U_\zeta}(L_\zeta(w.\mu), \nabla_\zeta(y.\mu)) \dim \mathrm{Ext}^j_{U_\zeta}(\Delta_\zeta(y.\mu), L_\zeta(z.\mu)).$$

This is a finite sum as the right hand side is 0 unless $y \leq w, z$. We have proved the following.

Theorem 8.14. *Suppose l is KL-good. Let $\mu \in \overline{{}^lC^-_{\mathbb{Z}}}$, $J = \{s \in S_l \mid s.\mu = \mu\}$, and $w, z \in W^+(\mu)$. Then we have*

$$\sum_{n=0}^{\infty} \dim \mathrm{Ext}^n_{U_\zeta}(L_\zeta(w.\mu), L_\zeta(z.\mu))t^n = \sum_{y\in W^+(\mu)} t^{l(w)+l(z)-2l(y)} \bar{P}^J_{y,w} \bar{P}^J_{y,z}.$$

9. Cohomology for q-Schur algebras

The above results provide calculations of Ext-groups between irreducible modules for important families of finite dimensional algebras associated to quantum enveloping algebras.

Consider first the type A quantum groups $U_\zeta(\mathfrak{sl}_n)$. Any positive integer l is KL-good in this case. As explained in [34, §9], a classical q-Schur algebra over K (or $\mathbb{C}$) with $q = \zeta^2$ arises as a truncation of $U_\zeta(\mathfrak{sl}_n)$-mod by a certain ideal Γ of dominant weights. Thus, Theorem 8.10 and Theorem 8.14 compute the corresponding cohomology for q-Schur algebras.

A generalized q-Schur algebra arises in a similar way. In fact, they are the algebras $A_\zeta = A_\zeta(\Gamma)$ that appear in §4.1 when identifying $(U_\zeta\text{-mod})[\Gamma]$ with A_ζ-mod (for finite ideals $\Gamma \trianglelefteq X^+$). The algebra $A_\zeta(\Gamma)$ is only determined up to Morita equivalence. But, by abuse of language, it is often called "the generalized q-Schur algebra" associated to Γ. This defines the generalized q-Schur algebras for all other types as well. Now, in any type (assuming l is KL-good), Theorem 8.10 and Theorem 8.14 provide the corresponding cohomology dimension for the generalized q-Schur algebras.

Part III. Reduction mod p to the algebraic group case

Let us revisit the conjectures from [36, Conjecture I, Conjecture II] stated in §4.2 above. Since [36, Conjecture III] is proved in Theorem 8.10 for arbitrary KL-good integers $l = p^r$, it is natural to ask whether [36, Conjecture I, Conjecture II], or a part of them, work in this generality. There is, at least, no difficulty in generalizing the statements: The forced grading is defined using U_ζ (or A_ζ) where ζ is a p^r-th root of unity; the reduced modules $\Delta^{\mathrm{red}}(\gamma)$, $\nabla_{\mathrm{red}}(\gamma)$ are simply replaced by $\Delta_r^{\mathrm{red}}(\gamma)$, $\nabla_{\mathrm{red}}^r(\gamma)$.

We examine the higher reduced modules $\Delta_r^{\mathrm{red}}(\gamma)$, $\nabla_{\mathrm{red}}^r(\gamma)$ (and $\Delta(\gamma), \nabla(\gamma)$!) in this part and explore the question above. We answer positively one direction of [36, Conjecture II] (see §4.2.2) in Proposition 12.2. Other than this, however, we provide examples that disprove the $r > 1$-analogue of the second conjecture as well as some other $r = 1$ result in [35].

In this part, ζ is always a primitive p^r-th root of unity for some positive integer r and an odd prime p.

10. Reducing modules modulo p

Recall from §1.3 the reduced modules

$$\widetilde{\Delta}(\gamma)_k \cong \Delta(\gamma), \qquad \widetilde{\nabla}(\gamma)_k \cong \nabla(\gamma)$$

and

$$\Delta_r^{\mathrm{red}}(\gamma) := (\widetilde{L}_\zeta^{\min}(\gamma))_k, \qquad \nabla_{\mathrm{red}}^r(\gamma) := (\widetilde{L}_\zeta^{\max}(\gamma))_k.$$

Also recall

$$\Delta(\gamma) \twoheadrightarrow \Delta_r^{\mathrm{red}}(\gamma) \twoheadrightarrow L(\gamma) \tag{10.0.1}$$

and

$$L(\gamma) \hookrightarrow \nabla^r_{\mathrm{red}}(\gamma) \hookrightarrow \nabla(\gamma). \tag{10.0.2}$$

By Proposition 1.1, the first epimorphism and the last monomorphism are isomorphisms if $\gamma \in \overline{{}^{p^r}C^+_{\mathbb{Z}}}$, where ${}^{p^r}C^+_{\mathbb{Z}}$ is the bottom dominant p^r-alcove

$${}^{p^r}C^+ := \{\gamma \in X \otimes_{\mathbb{Z}} \mathbb{R} \mid 0 < \langle \gamma + \rho, \alpha^\vee \rangle < p^r \text{ for all } \alpha \in R^+\}.$$

We can further compare the reduction mod p modules with another class of well-known modules. Recall first the Steinberg tensor product theorems for the two cases (see §3.2 for the $r = 1$ case):

$$L(\gamma_0 + p^r\gamma_1) \cong L(\gamma_0) \otimes L(\gamma_1)^{[r]} \tag{10.0.3}$$

$$L_\zeta(\gamma_0 + p^r\gamma_1) \cong L_\zeta(\gamma_0) \otimes V(\gamma_1)^{[1]}, \tag{10.0.4}$$

where $\gamma_0 \in X_r := \{\gamma \in X^+ \mid \langle \gamma + \rho, \alpha^\vee \rangle < p^r, \quad \forall \alpha \in \Pi\}$, $\gamma_1 \in X^+$. (Recall also that $V(\gamma_1)$ denotes the Weyl module for $\mathfrak{g}$ and has the Weyl character $\chi(\gamma_1)$.) The $-^{[r]}$ is the composition of the Frobenius twist $-^{[1]}$ r times. We have the third Steinberg tensor product theorem, regarding reduction mod p modules:

Proposition 10.1. [25, Theorem 2.7] *For $\gamma_0 \in X_r$ and $\gamma_1 \in X^+j$, there are isomorphisms of G-modules*

$$\Delta_r^{\mathrm{red}}(\gamma_0 + p^r\gamma_1) \cong \Delta_r^{\mathrm{red}}(\gamma_0) \otimes \Delta(\gamma_1)^{[r]}, \qquad \nabla^r_{\mathrm{red}}(\gamma_0 + p^r\gamma_1) \cong \nabla^r_{\mathrm{red}}(\gamma_0) \otimes \nabla(\gamma_1)^{[r]}.$$

From now on, we always write $\gamma \in X^+$ as $\gamma = \gamma_0 + p^r\gamma_1$ (uniquely) with $\gamma_0 \in X_r$ and $\gamma_1 \in X^+$. Define

$$\Delta^{p^r}(\gamma) := L(\gamma_0) \otimes \Delta(\gamma_1)^{[r]}$$

and

$$\nabla_{p^r}(\gamma) := L(\gamma_0) \otimes \nabla(\gamma_1)^{[r]}.$$

Then Proposition 10.1 gives the following.

Corollary 10.2. *For* $\gamma \in X^+$, *we have*

$$\Delta_r^{\mathrm{red}}(\gamma) \twoheadrightarrow \Delta^{p^r}(\gamma), \qquad \nabla_{p^r}(\gamma) \hookrightarrow \nabla_{\mathrm{red}}^r(\gamma).$$

The modules $\Delta^{p^r}(\gamma)$ for different $r \geq 1$ form a descending chain

$$\Delta^p(\gamma) \twoheadrightarrow \Delta^{p^2}(\gamma) \twoheadrightarrow \cdots \twoheadrightarrow \Delta^{p^r}(\gamma) \twoheadrightarrow \cdots .$$

This can be seen from the definition and the tensor product theorem (10.0.3) as follows. Write

$$\gamma = \gamma_0 + p^{r-1}(\gamma_1 + p\gamma_2) = (\gamma_0 + p^{r-1}\gamma_1) + p^r\gamma_2$$

with $\gamma_0 \in X_{r-1}$, $\gamma_1 \in X_1$, and $\gamma_2 \in X^+$. Then we have

$$\begin{aligned} \Delta^{p^{r-1}}(\gamma) &= L(\gamma_0) \otimes \Delta(\gamma_1 + p\gamma_2)^{[r-1]} \\ &\twoheadrightarrow L(\gamma_0) \otimes (L(\gamma_1) \otimes \Delta(\gamma_2)^{[1]})^{[r-1]} \\ &\cong L(\gamma_0) \otimes L(\gamma_1)^{[r-1]} \otimes \Delta(\gamma_2)^{[r]} \\ &\cong L(\gamma_0 + p^{r-1}\gamma_1) \otimes \Delta(\gamma_2)^{[r]} \\ &= \Delta^{p^r}(\gamma) \end{aligned}$$

for each $r > 1$. The second line follows from combining (10.0.1) and Corollary 10.2. The fourth line follows from (10.0.3).

There is no obvious relation between the $\Delta_r^{\mathrm{red}}(\gamma)$ for $r \geq 1$. Instead, we know the characters of the modules $\Delta_r^{\mathrm{red}}(\gamma)$ in most (possibly all) cases since $\operatorname{ch} \Delta_r^{\mathrm{red}}(\gamma) =$

$\operatorname{ch} L_\zeta(\gamma)$. Let p^r be KL-good. For $\lambda \in \overline{{}^{p^r}C_{\mathbb{Z}}^-}, w \in W_{p^r}^+$,

$$\operatorname{ch} \Delta_r^{\mathrm{red}}(\gamma) = \sum_{y \in W_{p^r}, y \in W_{p^r}^+(\lambda)} (-1)^{l(w)-l(y)} P_{y,w}^J(-1)\chi(y.\lambda)$$

where $J = \{s \in S_{p^r} \mid s.\lambda = \lambda\}$ and $\gamma = w.\lambda$ (see §3.2).

We can use the characters to actually show that there is no map between $\Delta_r^{\mathrm{red}}(\gamma)$ and $\Delta_{r'}^{\mathrm{red}}(\gamma)$ in general for $r \neq r' \in \mathbb{Z}$. For $\chi, \chi' \in \mathbb{Z}[X]$, we say $\chi \geq \chi'$ if $\chi - \chi' \in \mathbb{Z}_{\geq 0}[X]$ and $\chi \ngeq \chi'$ otherwise. Whether we require $r < r'$ or we require $r' < r$, we can easily find a case that $\operatorname{ch} \Delta_r^{\mathrm{red}}(\gamma) \ngeq \operatorname{ch} \Delta_{r'}^{\mathrm{red}}(\gamma)$. (See §13.) Since they have the same irreducible head $L(\gamma)$ and

$$[\Delta_r^{\mathrm{red}}(\gamma) : L(\gamma)] = [\Delta_{r'}^{\mathrm{red}}(\gamma) : L(\gamma)] = 1,$$

any nonzero map between $\Delta_r^{\mathrm{red}}(\gamma)$ and $\Delta_{r'}^{\mathrm{red}}(\gamma)$ is surjective. So $\operatorname{ch} \Delta_r^{\mathrm{red}}(\gamma) \ngeq \operatorname{ch} \Delta_{r'}^{\mathrm{red}}(\gamma)$ implies

$$\operatorname{Hom}_G(\Delta_r^{\mathrm{red}}(\gamma), \Delta_{r'}^{\mathrm{red}}(\gamma)) = 0.$$

11. Comparing the Jantzen sum formulas

The Jantzen filtration on standard modules $\Delta(\gamma) \in G$-mod is fully discussed in Jantzen's book [16, II.8]. An important consequence of the filtration is the Jantzen sum formula we state below. We need first to introduce a notation. Let ν_p be the p-adic valuation on $\mathbb{Z}$. That is, if $n \in \mathbb{Z}$ has the form $n = p^r d$ with p, d relatively prime, then $\nu_p(n) = r$. Recall for $\alpha \in R$ and $m \in \mathbb{Z}$ the (affine) reflection on $X \otimes_{\mathbb{Z}} \mathbb{R}$ given by

$$s_{\alpha,m}(\gamma) = \gamma - (\langle \gamma, \alpha^\vee \rangle - m)\alpha$$

for $\gamma \in X \otimes_{\mathbb{Z}} \mathbb{R}$.

Proposition 11.1. [16, II.8.19] *Let $\gamma \in X^+$. There is a filtration*

$$\Delta(\gamma) = V^0 \supset V^1 \supset \cdots$$

of $\Delta(\gamma)$ in G-mod such that

$$\sum_{i>0} \operatorname{ch} V^i = \sum_{\alpha\in R^+} \sum_{0<mp<\langle\gamma+\rho,\alpha^\vee\rangle} \nu_p(mp)\chi(s_{\alpha,mp}.\gamma) \tag{11.0.1}$$

where $\chi(s_{\alpha,mp}.\gamma)$ is the Weyl character (see §3.1) and

$$\Delta(\gamma)/V^1 \cong L(\gamma).$$

Let's denote the right hand side of (11.0.1) by $\chi_J(\gamma)$, and put

$$\chi_J(\gamma, p^r) := \sum_{\alpha\in R^+} \sum_{0<mp^r<\langle\gamma+\rho,\alpha^\vee\rangle} \chi(s_{\alpha,mp^r}.\gamma).$$

We now rewrite the formula (11.0.1) as

$$\begin{aligned} \chi_J(\gamma) &= \sum_{\alpha\in R^+} \sum_{0<mp<\langle\gamma+\rho,\alpha^\vee\rangle} \nu_p(mp)\chi(s_{\alpha,mp}.\gamma) \\ &= \sum_{\alpha\in R^+} \Big(\sum_{0<mp<\langle\gamma+\rho,\alpha^\vee\rangle} \chi(s_{\alpha,mp}.\gamma) + \sum_{0<mp^2<\langle\gamma+\rho,\alpha^\vee\rangle} \chi(s_{\alpha,mp^2}.\gamma) + \cdots \Big) \\ &= \sum_r \chi_J(\gamma, p^r). \end{aligned} \tag{11.0.2}$$

The reason we do this is that the Jantzen sum formula works for the quantum case, with a different formula:

Proposition 11.2. [4, §10] *Let $\gamma \in X^+$, $r \geq 1$. There is a filtration of the U_ζ-module $\Delta_\zeta(\gamma)$*

$$\Delta_\zeta(\gamma) = V_\zeta^0 \supset V_\zeta^1 \supset \cdots$$

such that

$$\sum_{i>0} \operatorname{ch} V_\zeta^i = \chi_J(\gamma, p^r) = \sum_{\alpha\in R^+} \sum_{0<mp^r<\langle\gamma+\rho,\alpha^\vee\rangle} \chi(s_{\alpha,mp^r}.\gamma) \tag{11.0.3}$$

and

$$\Delta_\zeta(\gamma)/V_\zeta^1 \cong L_\zeta(\gamma).$$

We can draw an observation from these formulas.

Proposition 11.3. *For $\gamma, \gamma' \in X^+$, we have*

$$[\Delta_\zeta(\gamma) : L_\zeta(\gamma')] \neq 0 \Rightarrow [\Delta(\gamma) : L(\gamma')] \neq 0.$$

Proof. Suppose $[\Delta_\zeta(\gamma) : L_\zeta(\gamma')] \neq 0$. Since $[\Delta_\zeta(\gamma) : L_\zeta(\gamma)] = [\Delta(\gamma) : L(\gamma)] = 1$, we may assume that $\gamma > \gamma'$.

By Proposition 11.1, $[\Delta(\gamma) : L(\gamma')] \neq 0$ if and only if $\operatorname{ch} L(\gamma')$ has a nonzero coefficient when we write $\chi_J(\gamma)$ as a ($\mathbb{Z}$-)linear combination of the characters of the irreducible G-modules (with non-negative coefficients). In (11.0.2), which says

$$\chi_J(\gamma) = \sum_r \chi_J(\gamma, p^r),$$

each $\chi_J(\gamma, p^r)$ is also a non-negative sum of irreducible G-characters by Proposition 11.2, since all U_ζ-characters are also G-characters. Thus, the claim is proved if we check that $\operatorname{ch} L(\gamma')$ has a nonzero coefficient when we write $\chi_J(\gamma, p^e)$ as a linear combination of the characters of the irreducible G-modules, where e is such that ζ is a primitive p^e-th root of unity. By Proposition 11.2 and the assumption $[\Delta_\zeta(\gamma) : L_\zeta(\gamma')] \neq 0$, the character $\chi_J(\gamma, p^e)$ has a positive coefficient for $\operatorname{ch} L_\zeta(\gamma')$ when we write it as a non-negative sum of irreducible U_ζ-characters. But $\operatorname{ch} L_\zeta(\gamma')$, when written as a sum of irreducible G-characters, has a nonzero $\operatorname{ch} L(\gamma')$ term. $\square$

Corollary 11.4. *For $\gamma, \gamma' \in X^+$ and $r \geq 1$, the module $\Delta(\gamma)$ has a composition factor $L(\gamma')$ provided that $\gamma' < \gamma$ and γ, γ' are mirror images under the reflection through a wall of the p^r-facet containing γ.*

Proof. We take the integer r as in the statement. That is, we have $\gamma' = s_{\beta,np^r}.\gamma \in X^+$ for an appropriate positive root β and an integer n. Now observe in

$$\chi_J(\gamma, p^r) = \sum_{\alpha \in R^+} \sum_{0 < mp^r < \langle \gamma + \rho, \alpha^\vee \rangle} \chi(s_{\alpha, mp^r}.\gamma)$$

that only $\chi(s_{\beta,np^r}.\gamma)$, among the Weyl characters appearing, has a nonzero ch $L(s_{\beta,np^r}.\gamma)$-term when it is written as a sum of irreducible G-characters. Necessarily, the multiplicity $[\Delta_\zeta(\gamma) : L_\zeta(\gamma')]$ is nonzero. Proposition 11.3 gives the corollary. $\square$

12. Reducing morphisms modulo p

We use the reduction mod p procedure to construct many nontrivial elements in Hom and Ext^n spaces for G-mod.

Proposition 12.1. *Let $M, N \in U_\zeta$-mod and $\widetilde{M}, \widetilde{N} \in \widetilde{U}_\zeta$-mod be admissible lattices of M, N respectively. Then for all $n \geq 0$,*

$$\dim_k \mathrm{Ext}^n_G(\widetilde{M}_k, \widetilde{N}_k) \geq \dim_K \mathrm{Ext}^n_{U_\zeta}(M, N).$$

Proof. The short exact sequence

$$0 \to \widetilde{N} \xrightarrow{\pi} \widetilde{N} \to \widetilde{N}_k \to 0$$

of $\widetilde{U}_\zeta$-modules induces the long exact sequence

$$\begin{aligned} 0 \to &\mathrm{Hom}_{\widetilde{U}_\zeta}(\widetilde{M}, \widetilde{N}) \xrightarrow{\pi} \mathrm{Hom}_{\widetilde{U}_\zeta}(\widetilde{M}, \widetilde{N}) \to \mathrm{Hom}_{\widetilde{U}_\zeta}(\widetilde{M}, \widetilde{N}_k) \to \cdots \\ \to &\mathrm{Ext}^n_{\widetilde{U}_\zeta}(\widetilde{M}, \widetilde{N}) \xrightarrow{\pi} \mathrm{Ext}^n_{\widetilde{U}_\zeta}(\widetilde{M}, \widetilde{N}) \to \mathrm{Ext}^n_{\widetilde{U}_\zeta}(\widetilde{M}, \widetilde{N}_k) \to \\ \to &\mathrm{Ext}^{n+1}_{\widetilde{U}_\zeta}(\widetilde{M}, \widetilde{N}) \xrightarrow{\pi} \mathrm{Ext}^{n+1}_{\widetilde{U}_\zeta}(\widetilde{M}, \widetilde{N}) \to \cdots \end{aligned} \tag{12.0.1}$$

of $\mathscr{O}$-modules, where the map π is multiplication by the generator of the maximal ideal of $\mathscr{O}$.

We also have

$$\mathrm{Ext}^n_{\widetilde{U}_\zeta}(\widetilde{M}, \widetilde{N}) \otimes_{\mathscr{O}} K \cong \mathrm{Ext}^n_{U_\zeta}(M, N),$$

by [11, (2.9), Theorem 3.2]. Let d_n be the (K-)dimension of this space. Thus, $\mathrm{Ext}^n_{\widetilde{U}_\zeta}(\widetilde{M}, \widetilde{N}) = \mathscr{O}^{\oplus d_n} \oplus T_n$, where T_n is a torsion $\mathscr{O}$-module. So the sequence (12.0.1) is of the form

$$\begin{aligned} \cdots \to \mathscr{O}^{\oplus d_n} \oplus T_n \xrightarrow{\pi} \mathscr{O}^{\oplus d_n} \oplus T_n \to \mathrm{Ext}^n_{\widetilde{U}_\zeta}(\widetilde{M}, \widetilde{N}_k) \to \\ \to \mathscr{O}^{\oplus d_{n+1}} \oplus T_{n+1} \xrightarrow{\pi} \mathscr{O}^{\oplus d_{n+1}} \oplus T_{n+1} \to \cdots . \end{aligned} \tag{12.0.2}$$

Since $\mathscr{O}/\pi\mathscr{O} \cong k$, we have

$$\mathrm{Ext}^n_{\widetilde{U}_\zeta}(\widetilde{M}, \widetilde{N}_k) \cong k^{\oplus d_n} \oplus T_n/\pi T_n \oplus \ker(T_{n+1} \xrightarrow{\pi} T_{n+1}).$$

The proof is complete using [11, (2.9), Theorem 3.2], which says

$$\mathrm{Ext}^n_{\widetilde{U}_\zeta}(\widetilde{M}, \widetilde{N}_k) \cong \mathrm{Ext}^n_G(\widetilde{M}_k, \widetilde{N}_k).$$

□

The following is an immediate consequence.

Proposition 12.2. *Let $\gamma, \gamma' \in X^+$. We have*

(1) $\dim_k \mathrm{Ext}^n_G(\Delta(\gamma), \Delta(\gamma')) \geq \dim_K \mathrm{Ext}^n_{U_\zeta}(\Delta_\zeta(\gamma), \Delta_\zeta(\gamma'))$;
(2) $\dim_k \mathrm{Ext}^n_G(\Delta(\gamma), \Delta^{\mathrm{red}}_r(\gamma')) \geq \dim_K \mathrm{Ext}^n_{U_\zeta}(\Delta_\zeta(\gamma), L_\zeta(\gamma'))$;
(3) $\dim_k \mathrm{Ext}^n_G(\Delta(\gamma), \nabla^r_{\mathrm{red}}(\gamma')) \geq \dim_K \mathrm{Ext}^n_{U_\zeta}(\Delta_\zeta(\gamma), L_\zeta(\gamma'))$;
(4) $\dim_k \mathrm{Ext}^n_G(\Delta^{\mathrm{red}}_r(\gamma), \nabla^r_{\mathrm{red}}(\gamma')) \geq \dim_K \mathrm{Ext}^n_{U_\zeta}(L_\zeta(\gamma), L_\zeta(\gamma'))$;
(5) $\dim_k \mathrm{Ext}^n_G(\Delta^{\mathrm{red}}_r(\gamma), \Delta^{\mathrm{red}}_r(\gamma')) \geq \dim_K \mathrm{Ext}^n_{U_\zeta}(L_\zeta(\gamma), L_\zeta(\gamma'))$;

and similar inequalities for the dual modules (replace "Δ" by "∇").

The right hand sides of Proposition 12.2 (2)-(5) are known by Theorems 8.10, 8.14. The "$r \geq 1$"-analogues of (§4.2.2) would say that the "$\geq$" are actually "$=$" for (3), (4). We give some examples in §13 below where this is a strict inequality for $r > 1$.

Unlike in the other inequalities in Proposition 12.2, the right hand side in Proposition 12.2 (1) is not known in general. (A result on Ext between two (co)standard modules in special cases can be found in [30].) Another difference between this case and the rest is that the left hand side in Proposition 12.2 (1) does not depend on r. Considering all the cases $r \geq 1$ together, we obtain

(12.0.3)

$$\dim_k \operatorname{Ext}^n_G(\Delta(\gamma), \Delta(\gamma')) \geq \max\{\dim_K \operatorname{Ext}^n_{U_\zeta}(\Delta_\zeta(\gamma), \Delta_\zeta(\gamma')) \mid \zeta^{p^r} = 1, r \geq 1\}$$

We explore some special cases where we can say something about the dimensions of $\operatorname{Ext}^n_{U_\zeta}(\Delta_\zeta(\gamma), \Delta_\zeta(\gamma'))$ for the rest of this subsection.

It will be convenient to employ the following convention when writing weights. Recall that we identify the weight lattices for G and for U_ζ, for any root of unity ζ. Now write a G-weight γ as $\gamma = w.\lambda$ where $\lambda \in \overline{{}^{p^r}C^-} \cap X$ and $w \in W^+_{p^r}(\lambda) \subset W_{p^r} \subset W_p$. The following two corollaries have a large intersection with Franklin's results [14].

Corollary 12.3. *Let $r \geq 1$ be such that $p^r \geq h$. Let $\mu \in \overline{{}^{p^r}C^-} \cap X$, $w \in W^+_{p^r} \subset W_p$ and $s \in S_{p^r} \subset W_p$ (So s may not be in S_p). If $ws > w$, then*

$$\operatorname{Hom}_G(\Delta(w.\mu), \Delta(ws.\mu)) \neq 0.$$

In other words, there is a nonzero map

$$\Delta(\gamma) \to \Delta(\gamma')$$

if $\gamma' > \gamma \in X^+$ and γ' is the reflection image of γ through a wall of the p^r-facet containing γ.

Proof. First consider the case where $\mu \in {}^{p^r}C^- \cap X$ is regular. The condition on p^r ensures the existence of (a regular and) a subregular weight for U_ζ with ζ a primitive p^r-th root of unity. (See Proposition 2.2.) We can, thus, apply the translation argument [16, II.7.19] to obtain

$$\mathrm{Hom}_{U_\zeta}(\Delta_\zeta(w.\mu), \Delta_\zeta(ws.\mu)) \cong K$$

in the quantum case. The corollary follows from Proposition 12.2 (1).

Now we treat the general weight $\mu \in \overline{{}^{p^r}C^-} \cap X$. Again, by Proposition 12.2 (1), it is enough to obtain

$$\mathrm{Hom}_{U_\zeta}(\Delta_\zeta(w.\mu), \Delta_\zeta(ws.\mu)) \neq 0.$$

Pick a regular weight $\lambda \in {}^{p^r}C^- \cap X$ (possible since $p^r \geq h$) and consider the translation functor T_λ^μ in U_ζ-mod. We may assume that $w \in W^J$. We have

$$\begin{aligned}\mathrm{Hom}_{U_\zeta}(\Delta_\zeta(w.\mu), \Delta_\zeta(ws.\mu)) &\cong \mathrm{Hom}_{U_\zeta}(T_\lambda^\mu \Delta_\zeta(w.\lambda), T_\lambda^\mu \Delta_\zeta(ws.\lambda)) \\ &\cong \mathrm{Hom}_{U_\zeta}(T_\mu^\lambda T_\lambda^\mu \Delta_\zeta(w.\lambda), \Delta_\zeta(ws.\lambda))\end{aligned}$$

But the surjection

$$T_\mu^\lambda T_\lambda^\mu \Delta_\zeta(w.\lambda) \twoheadrightarrow \Delta_\zeta(w.\lambda)$$

(see page 30) induces an inclusion

$$\mathrm{Hom}_{U_\zeta}(\Delta_\zeta(w.\lambda), \Delta_\zeta(ws.\lambda)) \hookrightarrow \mathrm{Hom}_{U_\zeta}(T_\mu^\lambda T_\lambda^\mu \Delta_\zeta(w.\lambda), \Delta_\zeta(ws.\lambda)).$$

The left hand side is nonzero by the regular case done in the first paragraph. □

Remark 12.4. Since $\mathrm{hd}\,\Delta(\gamma) \cong L(\gamma)$, if there is a nonzero map from $\Delta(\gamma)$ to $\Delta(\gamma')$ then $L(\gamma)$ is a composition factor of $\Delta(\gamma')$. Thus, Corollary 12.3 implies Corollary 11.4. The same remark applies to Corollary 12.5 below.

In fact, we know many more morphisms between standard modules in the quantum case, from which we can reduce mod p.

Corollary 12.5. *Let $r \geq 1$ be such that $p^r \geq h$. Let $\lambda \in \overline{{}^{p^r}C^-} \cap X$, $s \in S_{p^r} \setminus I$, where $I = \{s \in S_{p^r} \mid s.\lambda = \lambda\}$ For $w \in W_{p^r}^{(S_{p^r}\setminus\{s\})} \cap W_{p^r}^+$ and $x < y \in W_{(S_{p^r}\setminus\{s\})}(= (W_{p^r})_{(S_{p^r}\setminus\{s\})})$, we have*

$$\mathrm{Hom}_G(\Delta(wx.\lambda), \Delta(wy.\lambda)) \neq 0.$$

Proof. A nonzero morphism in the quantum case

$$\Delta_\zeta(wx.\lambda) \to \Delta_\zeta(wy.\lambda)$$

when λ is regular is well known or explained in [3, Remark 3.6, Proposition 3.7]. It is a composition of maps obtained in Corollary 12.3 and also uses translation functors in showing that it is nonzero. The proof of Corollary 12.3 gives the singular case. Finally, use Proposition 12.2. □

A very similar proof gives the next corollary.

Corollary 12.6. *In the situation of Corollary 12.3, we have*

$$\mathrm{Ext}^1_G(\Delta(w.\mu), \Delta(ws.\mu)) \neq 0.$$

Proof. By Proposition 12.2 (1), it is enough to show that $\mathrm{Ext}^1_{U_\zeta}(\Delta_\zeta(w.\mu), \Delta_\zeta(ws.\mu)) \neq 0$.

The regular case is done by the argument in [16, II.7.19]. For the singular case, take a regular weight λ in ${}^{p^r}C^-$ and consider the translation functor T_λ^μ of U_ζ-modules. Then,

$$\mathrm{Ext}^1_{U_\zeta}(\Delta_\zeta(w.\mu), \Delta_\zeta(ws.\mu)) \cong \mathrm{Ext}^1_{U_\zeta}(T_\mu^\lambda T_\lambda^\mu \Delta_\zeta(w.\lambda), \Delta_\zeta(ws.\lambda))$$

Assuming $w \in W^J$, there is a short exact sequence

$$0 \to M \to T_\mu^\lambda T_\lambda^\mu \Delta_\zeta(w.\lambda) \to \Delta_\zeta(w.\lambda) \to 0,$$

where M has a Δ-filtration with sections $\Delta_\zeta(wx.\lambda)$, $x \in W^J \backslash \{e\}$. Taking $\mathrm{Hom}_{U_\zeta}(-, \Delta_\zeta(ws.\lambda))$, we have (a part of) a long exact sequence

$$\begin{aligned} &\to \mathrm{Hom}_{U_\zeta}(M, \Delta_\zeta(ws.\lambda)) \\ &\to \mathrm{Ext}^1_{U_\zeta}(\Delta_\zeta(w.\lambda), \Delta_\zeta(ws.\lambda)) \to \mathrm{Ext}^1_{U_\zeta}(T_\mu^\lambda T_\lambda^\mu \Delta_\zeta(w.\lambda), \Delta_\zeta(ws.\lambda)) \to . \end{aligned} \tag{12.0.4}$$

Since the head of M is a direct sum of irreducibles of highest weight wt with $t \in J$, and since $wt \not< ws$ for $t \neq s \in S_{p^r}$, the first term $\mathrm{Hom}_{U_\zeta}(M, \Delta_\zeta(ws.\lambda))$ in (12.0.4) is zero. The second term $\mathrm{Ext}^1_{U_\zeta}(\Delta_\zeta(w.\lambda), \Delta_\zeta(ws.\lambda))$ in (12.0.4) is nonzero by the regular case considered above. Therefore $\mathrm{Ext}^1_{U_\zeta}(T_\mu^\lambda T_\lambda^\mu \Delta_\zeta(w.\lambda), \Delta_\zeta(ws.\lambda))$ is also not zero, which proves the claim. □

Remark 12.7. The condition $p^r \geq h$ in Corollary 12.3, 12.5, 12.6 can be replaced by the KL-good condition if we transfer the assertions in U_ζ-mod to the affine case via the Kazhdan-Lusztig correspondence, use the result of Fiebig [12] to move the level, and then transfer the assertion back to the quantum case. See the proof of Theorem 8.10 where we do this.

13. Type A_1 examples

Let $G = SL_2$. We provide some examples of the $r = 2$ case which shows that many nice results for the $r = 1$ case does not generalize to $r > 1$. The $r \geq 1$ versions of the "nice results" that are disproved in this section are the following.

(1) If $p \gg 0$ and $\lambda \in {}^pC_{\mathbb{Z}}^-$, then $\Delta(w.\lambda)$ for each $w \in W_p^+$ has a filtration with sections of the form $\Delta_r^{\mathrm{red}}(\gamma)$. (The $r = 1$ case is proved in [35].)

(2) If $p \gg 0$ and $\lambda \in {}^pC_{\mathbb{Z}}^-$, then $\Delta_r^{\mathrm{red}}(w.\lambda)$ for $w \in W_p^+$ has left parity with respect to all irreducible G-modules, or equivalently, $\Delta_r^{\mathrm{red}}(w.\lambda) \in \mathcal{E}^L$, or equivalently,

we have

$$\mathrm{Ext}^n_G(\Delta^{\mathrm{red}}_r(w.\lambda), \nabla(y.\lambda)) = 0 \;\text{ if } l(w) - l(y) \not\equiv n \bmod 2.$$

(The $r = 1$ case is proved in [9]: See §4.2.2.)

We take $p = 3$ to have concrete numbers, but all the examples here work for a larger p. Note that the condition "$p \gg 0$" means, by our convention, "the Lusztig conjecture for G is true and $p \geq 2h - 2$". Thus, $3 \gg 0$ for SL_2.

In this case, the dominant weights are identified with the integers $n \in \mathbb{Z}_{\geq 0}$. The Jantzen region in this notation is defined by the condition $n \leq 8$. On the quantum side, we have $U_\zeta = U_\zeta(\mathfrak{sl}_2)$ with ζ a primitive 9th root of unity and U_{ζ^3} the corresponding quantum group at a 3rd root of unity.

Let's consider the regular orbit containing $0 \in {}^pC^-_{\mathbb{Z}}$. The highest weights are $0, 4, 6, 10, 12,\ 16, 18, \cdots$. We express the radical filtration of a G-module via the following notation.

$$M = \begin{matrix} M/\operatorname{rad} M = \operatorname{hd} M \\ \operatorname{rad} M/\operatorname{rad}^2 M \\ \operatorname{rad}^2 M/\operatorname{rad}^3 M \\ \cdots \end{matrix}$$

It is easy to check that

$$\Delta(6) = \Delta^{\mathrm{red}}_2(6) = \begin{matrix} L(6) \\ L(4) \end{matrix}$$

and

$$\Delta^{\mathrm{red}}_2(10) = \; L(10),$$

while the structure of $\Delta(10)$ is either

(Case 1)
$$\Delta(10) = \begin{matrix} L(10) \\ L(4) \\ L(6) \end{matrix}$$

or

(Case 2)
$$\Delta(10) = \begin{matrix} L(10) \\ L(4) \oplus L(6) \end{matrix}.$$

(We cannot have
$$\Delta(10) = \begin{matrix} L(10) \\ L(6) \\ L(4) \end{matrix}$$
because $\Delta(10) \twoheadrightarrow \Delta^{\mathrm{red}}(10) = \begin{matrix} L(10) \\ L(4) \end{matrix}$.)

In either case, $\Delta(10)$ does not have a filtration with sections of the form $\Delta_2^{\mathrm{red}}(\gamma)$. We actually know which is the case. Suppose (Case 2) is true. Then, we have

$$\mathrm{Ext}^1_G(L(10), L(6)) \neq 0.$$

Since the weight 10 is the only one out of the Jantzen region among the weights $0, 4, 6, 10$, we have

$$\mathrm{Ext}^1_G(L(a), L(b)) \neq 0$$

for every pair of weights $a, b \in \{0, 4, 6, 10\}$ in two adjacent alcoves. Applying [7, Theorem 5.3] to the category $(G\text{-mod})[0, 4, 6, 10]$, we have $\mathrm{ch}\, L(10) = \mathrm{ch}\, L_\zeta(10)$. This contradicts $\mathrm{ch}\, L_\zeta(10) = \mathrm{ch}\, L(10) + \mathrm{ch}\, L(4)$. Thus, (Case 1) is the case.

Now we check that $\Delta_2^{\mathrm{red}}(10)$ is not in the category $\mathcal{E}^L$ (see Proposition 7.8). A sequence of distinguished triangle constructions of $\Delta_2^{\mathrm{red}}(10)$ with the standard modules (see the proof of Proposition 7.8 and Remark 7.9) is as follows.

$$\Delta(10) \to Y_0 = \Delta_2^{\mathrm{red}}(10) \to Y_1 = \begin{matrix} L(4) \\ L(6) \end{matrix} [1] \to,$$

$$\Delta(6)[1] \to Y_1 = \begin{matrix} L(4) \\ L(6) \end{matrix} [1] \to Y_2 = L(4)[1] \oplus L(4)[2] \to,$$

$$\Delta(4)[1] \oplus \Delta(4)[2] \to Y_2 = L(4)[1] \oplus L(4)[2] \to Y_4 = L(0)[2] \oplus L(0)[3] \to,$$

$$\Delta(0)[2] \oplus \Delta(0)[3] \xrightarrow{\cong} Y_4 = L(0)[2] \oplus L(0)[3] \to 0 \to,$$

using

$$\Delta(0) = L(0), \qquad \Delta(4) = \begin{matrix} L(4) \\ L(0) \end{matrix} .$$

We see that the "wrong" shifts $\Delta(4)[1]$ and $\Delta(0)[2]$ appear. In view of the recognition theorem (Proposition 7.8), the sequence above of distinguished triangles show that $\mathrm{Ext}_G^i(\Delta_2^{\mathrm{red}}(10), \nabla(4))$ has dimension one at $i = 1, 2$ and that $\mathrm{Ext}_G^i(\Delta_2^{\mathrm{red}}(10), \nabla(0))$ has dimension one at $i = 2, 3$. In particular, the p^2-analogue of [36, Conjecture II] (§4.2.2) is not true.

References

[1] H. H. Andersen. The strong linkage principle for quantum groups at roots of 1. *J. Algebra*, 260(1):2–15, 2003. Special issue celebrating the 80th birthday of Robert Steinberg.

[2] H. H. Andersen, J. C. Jantzen, and W. Soergel. Representations of quantum groups at a pth root of unity and of semisimple groups in characteristic p: independence of p. *Astérisque*, (220):321, 1994.

[3] H. H. Andersen and M. Kaneda. Rigidity of tilting modules. *Mosc. Math. J.*, 11(1):1–39, 181, 2011.

[4] H. H. Andersen, P. Polo, and K. X. Wen. Representations of quantum algebras. *Invent. Math.*, 104(1):1–59, 1991.

[5] R. W. Carter. *Lie algebras of finite and affine type*, volume 96 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2005.

[6] E. Cline, B. Parshall, and L. Scott. Finite-dimensional algebras and highest weight categories. *J. Reine Angew. Math.*, 391:85–99, 1988.

[7] E. Cline, B. Parshall, and L. Scott. Abstract Kazhdan-Lusztig theories. *Tohoku Math. J. (2)*, 45(4):511–534, 1993.

[8] E. Cline, B. Parshall, and L. Scott. The homological dual of a highest weight category. *Proc. London Math. Soc. (3)*, 68(2):294–316, 1994.

[9] E. Cline, B. Parshall, and L. Scott. Reduced standard modules and cohomology. *Trans. Amer. Math. Soc.*, 361(10):5223–5261, 2009.

[10] V. V. Deodhar. On some geometric aspects of Bruhat orderings. II. The parabolic analogue of Kazhdan-Lusztig polynomials. *J. Algebra*, 111(2):483–506, 1987.

[11] J. Du and L. Scott. Lusztig conjectures, old and new. I. *J. Reine Angew. Math.*, 455:141–182, 1994.

[12] P. Fiebig. The combinatorics of category $\mathscr{O}$ over symmetrizable Kac-Moody algebras. *Transform. Groups*, 11(1):29–49, 2006.

[13] P. Fiebig. An upper bound on the exceptional characteristics for Lusztig's character formula. *J. Reine Angew. Math.*, 673:1–31, 2012.

[14] J. Franklin. Homomorphisms between Verma modules in characteristic p. *J. Algebra*, 112(1):58–85, 1988.

[15] T. Hodge, P. Karuppuchamy, and L. Scott. Remarks on the ABG induction theorem. *arXiv:1603.05699*, 2016.

[16] J. C. Jantzen. *Representations of algebraic groups*, volume 107 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, second edition, 2003.

[17] M. Kashiwara and T. Tanisaki. Kazhdan-Lusztig conjecture for affine Lie algebras with negative level. *Duke Math. J.*, 77(1):21–62, 1995.

[18] M. Kashiwara and T. Tanisaki. Kazhdan-Lusztig conjecture for affine Lie algebras with negative level. II. Nonintegral case. *Duke Math. J.*, 84(3):771–813, 1996.

[19] M. Kashiwara and T. Tanisaki. Characters of irreducible modules with non-critical highest weights over affine Lie algebras. In *Representations and quantizations (Shanghai, 1998)*, pages 275–296. China High. Educ. Press, Beijing, 2000.

[20] M. Kashiwara and T. Tanisaki. Parabolic Kazhdan-Lusztig polynomials and Schubert varieties. *J. Algebra*, 249(2):306–325, 2002.

[21] D. Kazhdan and G. Lusztig. Schubert varieties and Poincaré duality. In *Geometry of the Laplace operator (Proc. Sympos. Pure Math., Univ. Hawaii, Honolulu, Hawaii, 1979)*, Proc. Sympos. Pure Math., XXXVI, pages 185–203. Amer. Math. Soc., Providence, R.I., 1980.

[22] D. Kazhdan and G. Lusztig. Tensor structures arising from affine Lie algebras. I, II. *J. Amer. Math. Soc.*, 6(4):905–947, 949–1011, 1993.

[23] D. Kazhdan and G. Lusztig. Tensor structures arising from affine Lie algebras. III. *J. Amer. Math. Soc.*, 7(2):335–381, 1994.

[24] D. Kazhdan and G. Lusztig. Tensor structures arising from affine Lie algebras. IV. *J. Amer. Math. Soc.*, 7(2):383–453, 1994.

[25] Z. Lin. Highest weight modules for algebraic groups arising from quantum groups. *J. Algebra*, 208(1):276–303, 1998.

[26] G. Lusztig. Finite-dimensional Hopf algebras arising from quantized universal enveloping algebra. *J. Amer. Math. Soc.*, 3(1):257–296, 1990.

[27] G. Lusztig. On quantum groups. *J. Algebra*, 131(2):466–475, 1990.

[28] G. Lusztig. Quantum groups at roots of 1. *Geom. Dedicata*, 35(1-3):89–113, 1990.

[29] G. Lusztig. Monodromic systems on affine flag manifolds. *Proc. Roy. Soc. London Ser. A*, 445(1923):231–246, 1994.

[30] A. E. Parker. On the good filtration dimension of Weyl modules for a linear algebraic group. *J. Reine Angew. Math.*, 562:5–21, 2003.

[31] B. Parshall and L. Scott. Derived categories, quasi-hereditary algebras, and algebraic groups. *Carlton University Mathematical notes*, 3:1–104, 1988.

[32] B. Parshall and L. Scott. Forced gradings in integral quasi-hereditary algebras with applications to quantum groups. In *Recent developments in Lie algebras, groups and representation theory*, volume 86 of *Proc. Sympos. Pure Math.*, pages 247–276. Amer. Math. Soc., Providence, RI, 2012.

[33] B. Parshall and L. Scott. A semisimple series for q-Weyl and q-Specht modules. In *Recent developments in Lie algebras, groups and representation theory*, volume 86 of *Proc. Sympos. Pure Math.*, pages 277–310. Amer. Math. Soc., Providence, RI, 2012.

[34] B. Parshall and L. Scott. A new approach to the Koszul property in representation theory using graded subalgebras. *J. Inst. Math. Jussieu*, 12(1):153–197, 2013.

[35] B. Parshall and L. Scott. New graded methods in the homological algebra of semisimple algebraic groups. *arXiv:1304.1461*, 2013.

[36] B. Parshall and L. Scott. Q-Koszul algebras and three conjectures. *arXiv:1405.4419*, 2014.

[37] B. Parshall and L. Scott. From forced gradings to Q-Koszul algebras. *arXiv:1502.06927*, 2015.

[38] B. Parshall and L. Scott. On p-filtrations of Weyl modules. *J. Lond. Math. Soc. (2)*, 91(1):127–158, 2015.

[39] S. Riche and G. Williamson. Tilting modules and the p-canonical basis. *arXiv:1512.08296*, 2015.

[40] P. Shan, M. Varagnolo, and E. Vasserot. Koszul duality of affine Kac-Moody algebras and cyclotomic rational double affine Hecke algebras. *Adv. Math.*, 262:370–435, 2014.

[41] W. Soergel. $\mathfrak{n}$-cohomology of simple highest weight modules on walls and purity. *Invent. Math.*, 98(3):565–580, 1989.

[42] T. Tanisaki. Character formulas of Kazhdan-Lusztig type. In *Representations of finite dimensional algebras and related topics in Lie theory and geometry*, volume 40 of *Fields Inst. Commun.*, pages 261–276. Amer. Math. Soc., Providence, RI, 2004.

[43] G. Williamson. Schubert calculus and torsion explosion. *arXiv:1309.5055*, 2013.