Tamagawa Products for Elliptic Curves over Number Fields

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Abstract

In recent work, Griffin, Ono, and Tsai constructs an $L-$series to prove that the proportion of short Weierstrass elliptic curves over $\mathbb{Q}$ with trivial Tamagawa product is $0.5054\ldots$ and that the average Tamagawa product is $1.8183\ldots$. Following their work, we generalize their $L-$series over arbitrary number fields K to be

$$L_{\text{Tam}}(K;s) := \sum_{m=1}^{\infty} \frac{P_{\text{Tam}}(K;m)}{m^s},$$

where $P_{\text{Tam}}(K;m)$ is the proportion of short Weierstrass elliptic curves over K with Tamagawa product m. We then construct Markov chains to compute the exact values of $P_{\text{Tam}}(K;m)$ for all number fields K and positive integers m. As a corollary, we also compute the average Tamagawa product $L_{\text{Tam}}(K;-1)$. We then use these results to uniformly bound $P_{\text{Tam}}(K;1)$ and $L_{\text{Tam}}(K,-1)$ in terms of the degree of K. Finally, we show that there exist sequences of K for which $P_{\text{Tam}}(K;1)$ tends to 0 and $L_{\text{Tam}}(K;-1)$ to ∞, as well as sequences of K for which $P_{\text{Tam}}(K;1)$ and $L_{\text{Tam}}(K;-1)$ tend to 1.

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Chapter 1

Introduction and Statement of Results

It is clear that any right triangle with sides of rational length will have rational area. However, it is not true that for any $n \in \mathbb{Q}$, there exists a right triangle with area n. Thus, it is natural to wonder, for which n do such triangles exist?

Definition 1.1. For $n \in \mathbb{Q}^+$, n is congruent if there exists a right triangle with rational sides, and area n.

If we define our triangle by its side lengths, (a, b, c), with $c > a$ and $c > b$, we know that if the area of a triangle were to be n, then $n = \frac{1}{2}ab$.

Theorem 1.2. *For all $n > 0$, there exist a bijection between the following sets:*

$$A := \{(a, b, c) \in \mathbb{Q}^3 : a^2 + b^2 = c^2, ab/2 = n\} \text{ and } B := \{(x, y) \in \mathbb{Q}^2 : y^2 = x^3 - n^2x, y \neq 0\}.$$

Proof. We can show this by construction the explicit bijection. Let $f : A \to B$ and $g : B \to A$ with

$$f(a, b, c) = \left(\frac{nb}{c - q}, \frac{2n^2}{c - a}\right)$$

$$g(x, y) = \left(\frac{x^2 - n^2}{y}, \frac{2nx}{y}, \frac{x^2 + n^2}{y}\right)$$

where f and g are inverses.

□

Thus, n is congruent if and only if there exists a rational solution to $y^2 = x^3 + n^2x$, which is an example of an elliptic curve.

Definition 1.3. An *elliptic curve* is a curve of the form $E := y^2 = x^3 + ax + b$, where the right hand side has no repeated factors.

In order to discuss elliptic curves, we use Weierstrauss equations.

Definition 1.4. A *Weierstrass equation* over a number field K is defined to be

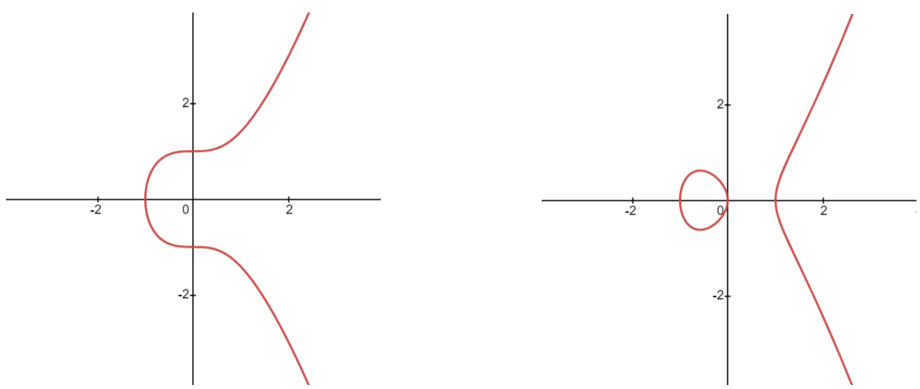
$$E : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

with $a_1, \ldots, a_6 \in K$. In our paper, we commonly use short Weierstrass form,

$$E : y^2 = x^3 + a_4x + a_6$$

.

We see some examples of elliptic curves in fig. 1-1a and fig. 1-1b.



(a) $E : y^2 = x^3 + 1$ (b) $E : y^2 = x^3 - x$

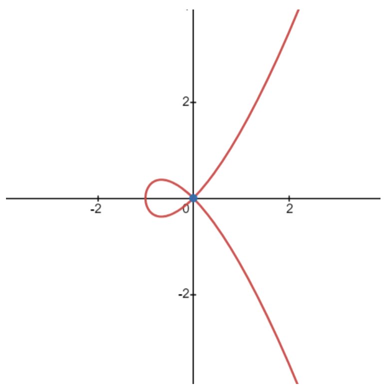
Figure 1-1: Non-singular elliptic curves.

Notably, both of these curves are smooth and do not intersect themselves. These are both examples of non-singular curves.

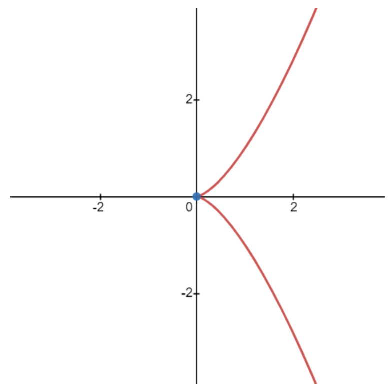
Definition 1.5. Let C be a curve over K defined by $F(x, y) = 0$. A point on C is *singular* if

$$\frac{\partial F}{\partial x}(a, b) = \frac{\partial F}{\partial y}(a, b) = 0$$

If no points on C are singular, then we call C *non-singular*. Some examples of curves with singularities are fig. 1-2a and fig. 1-2b.



(a) $E : y^2 = x^3 + x^2$ (Node)



(b) $E : y^2 = x^3$ (Cusp)

Figure 1-2: Curves with singularities at $(0, 0)$.

For our Weierstrass equations, we want to define the following quantities:

$$b_2 := a_1^2 + 4a_2, \qquad b_4 := 2a_4 + a_1a_3, \qquad b_6 := a_3^2 + 4a_6$$

$$b_8 := a_1^2a_6 + 4a_2a_6 - a_1a_3a_4 + a_2a_3^2 - a_4^2$$

$$c_4 := b_2^2 - 24b_4, \qquad c_6 = -b_2^3 + 36b_2b_4 - 216b_6$$

$$\Delta := -b_2^2b_8 - 8b_4^3 - 27b_6^2 + 9b_2b_4b_6.$$

Where Δ is called the *discriminant* of the curve.

Lemma 1.6. *For elliptic curve $E : y^2 = x^3 + a_4x + a_6$ in short Weierstrass form, the discriminant of E simplifies to $\Delta = -16(4a_4^3 + 27a_6^2)$. In fact, $\Delta \neq 0$ if an only if E is non-singular.*

Proof. If E has a singularity, there exists (x, y) with $2y = 0$ and $a_4 = -3x^2$. Thus, we see that $0 = x^3 - (3x^2)x + a_6$, so $a_6 = 2x^3$. Hence $\Delta = -16(4a_4^3 + 27a_6^2) = -16(-108x^6 + 108x^6) = 0$. To show the converse, suppose $\Delta = 0$, then $4a_4^3 = -27a_6^2$, so $-4\left(\frac{a_4}{3}\right)^3 = a_6^2$, and thus $-a_4/3$ is square. Hence, let $y = 0$ and $x^2 = -a_4/3$, then we see that E has a singularity at (x, y).

□

One important question to ask is to what degree are the Weierstrass forms of curves unique? In fact, there is only one possible change of variables that preserves the Weierstrass form of the equation, which we define as follows:

Definition 1.7. Two Weierstrass equations, $F(x, y)=0$ and $F'(x', y')=0$, define isomorphic curves over K if and only if

$$x = u^2x' + r, \qquad y = u^3y' + u^2sx' + t$$

with $u, r, s, t \in K$ and $u \neq 0$.

Under such a transformation, we get the following corresponding equations:

$$\begin{aligned}
ua_1' &= a_1 + 2s \\
u^2a_2' &= a_2 - sa_1 + 3r - s^2 \\
u^3a_3' &= a_3 + ra_1 + 2t \\
u^4a_4' &= a_4 - sa_3 + 2ra_2 - (t + rs)a_1 + 3r^2 - 2st \\
u^6a_6' &= a_6 + ra_4 + r^2a_2 + r^3 - ta_3 - t^2 - rta_1 \\
u^2b_2' &= b_2 + 12r \\
u^4b_4' &= b_4 + rb_2 + 6r^2 \\
u^6b_6' &= b_6 + 2rb_4 + r^2b_2 + 4r^3
\end{aligned}$$

$$u^8 b_8' = b_8 + 3rb_6 + 3r^2 b_4 + r^3 b_2 + 3r^4$$
$$u^4 c_4' = c_4$$
$$u^6 c_6' = c_6$$
$$u^{12}\Delta' = \Delta$$

Now we will let $K = \mathbb{Q}$, and consider how elliptic curves behave when reduced modulo primes. We let $E : y^2 = x^3 + a_4 x + a_6$ with $a_4, a_6 \in \mathbb{Z}$. Then for prime p, we define $\overline{E}_p : y^2 = x^3 + \bar{a_4} x + \bar{a_6}$, where $\bar{a_4}, \bar{a_6} \in \mathbb{F}_p$. We notice that reducing by a prime can give us a singularity. For example, take the curve $E : y^2 = x^3 - 432$. This has $\Delta = -2^{12} \cdot 3^9$, so E is non-singular. However, $\overline{E_2} : y^2 = x^3$ with $\Delta = 0$, thus $\overline{E_2}$ is singular over $\mathbb{F}_2$.

Remark. *An important fact to note is that the reduction of two curves which are isomorphic over $\mathbb{Q}$ can give curves which are not isomorphic over $\mathbb{F}_p$. For example, if we take the same curve E as above, and make the change of variables $x = 4x'$ and $y = 8y' - 4$, we get the isomorphic curve $E' : y'^2 - y' = x'^3 - 7$, which has $\Delta' = -3^9 \equiv 1 \mod 2$. Thus, $\overline{E_2'}$ is nonsingular over $\mathbb{F}_2$.*

With this, we begin to classify the different types of behavior of curves at primes. For elliptic curve E and prime p, we have the following classifications:

Definition 1.8.
1. E has **good reduction** at p if $\overline{E_p}$ is nonsingular.
2. E has **multiplicative reduction** at p if $\overline{E_p}$ has a node.
3. E has **additive reduction** at p if $\overline{E_p}$ has a cusp.

Where a node has two distinct tangent lines at the singularity (depicted in fig. 1-2a), and a cusp has exactly one (depicted in fig. 1-2b). The latter two are collectively referred to as bad reduction. We see that a curve E has bad reduction over a prime p, if $p|\Delta$. Now one might ask, can there be elliptic curves E/K with everywhere good reduction? Over an arbitrary number field, such curves do in fact exist, as shown in the works of Clemm and Trebat-Leder.

Theorem 1.9. *Over $\mathbb{Q}$ there are no elliptic curves with everywhere good reduction.*

Proof. Let E be an elliptic curve over $\mathbb{Q}$ with $\Delta = \pm 1$.

Case One: a_1 is even. Then $b_2 = a_1^2 + 4a_2 \equiv 0 \mod 4$ and $b_4 = 2a_4 + a_1 a_3 \equiv 0 \mod 2$. Thus

$$\Delta = -b_2^2 b_8 - 8b_4^3 - 27b_6^2 + 9b_2 b_4 b_6 \equiv 5b_6^2 \not\equiv \pm 1 \mod 8$$

which is a contradiction.

Case Two: a_1 is odd. Then $b_2 = a_1^2 + 4a_2 \equiv 1 \mod 4$, so $c_4 = b_2^2 - 24b_4 \equiv 1 \mod 4$, and $c_6 = -b_2^3 + 36b_2 b_4 - 216b_6 \equiv 3, 7 \mod 8$. Thus $c_6^2 \equiv 1 \mod 8$. But we know that $1728\Delta = c_4^3 - c_6^2$. Take $c_4 = u \pm 12$. Then $\pm 1728 = (u \pm 12)^3 - c_6^2$. Then we see that $c_6^2 = u(u^2 \pm 36u + 432)$. Because c_4 is odd, $2 \not| u$. Thus, for any prime $p > 3$, such that $p|u$ we see that $p^2|u$. Thus, either u is a square or three times a square. We know that $c_6^2 \equiv 1 \mod 8$, so $u^3 \pm 4u^2 \equiv 1 \mod 8$, and hence $u \equiv 5 \mod 8$. Thus, u cannot be a square, so we see that $3|u$. We take $u = 3x$, and see that then $c_4 = 9y$ where x, y are odd. Therefore $3y^2 = x(x^2 \pm 12x + 48)$. Thus, $x^3 + 4x^2 \equiv 3 \mod 8$, hence $x \equiv 7 \mod 8$. Similarly to before, we see that for a prime $p > 3$, if $p|x$ then $p^2|x$. Lastly, consider if $3|x$ such that $x = 3^k l$, then

$$3y^2 = 3^k l(3^{2k} l^2 + 4 \cdot 3^{k+1} l + 3 \cdot 16) = 3^{k+1} l(3^{2k-1} l^2 + 4 \cdot 3^k l + 16)$$

and thus k is even, and so x is an odd square. This means $x \equiv 1 \mod 8$, which is a contradiction.

Thus, elliptic curves over $\mathbb{Q}$ cannot have $\Delta = \pm 1$, and so no elliptic curves $E/\mathbb{Q}$ can have everywhere good reduction.

□

Although there are no elliptic curves $E/\mathbb{Q}$ with everywhere good reduction, for the field of p-adic numbers $\mathbb{Q}_p$, there do exist curves such that $[E(\mathbb{Q}_p) : E_0(\mathbb{Q}_p)] = 1$ for all primes p, where $E_0(\mathbb{Q}_p)$ is the subgroup consisting of the nonsingular points of $E(\mathbb{Q}_p)$ after reduction modulo p. For example, the curve

$$E/\mathbb{Q} : y^2 = x^3 + 3x + 1,$$

which has discriminant $-2^4 \cdot 3^3 \cdot 5$, has $[E(\mathbb{Q}_2) : E_0(\mathbb{Q}_2)] = [E(\mathbb{Q}_3) : E_0(\mathbb{Q}_3)] = [E(\mathbb{Q}_5) : E_0(\mathbb{Q}_5)] = 1$. We refer to such curves as *Tamagawa trivial curves.* Recent work by Griffin, Ono, and Tsai [6, Corollary 1.2] establishes that over half of the elliptic curves over $\mathbb{Q}$ are Tamagawa trivial. More precisely, the proportion of elliptic curves in short Weierstrass form that are Tamagawa trivial is $0.5054\ldots$ when the curves are ordered by height.

Definition 1.10. For every elliptic curve over $\mathbb{Q}$, we associate the *Tamagawa product*

$$\mathrm{Tam}(\mathbb{Q}; E) := \prod_{p \text{ prime}} c_p,$$

where $c_p := [E(\mathbb{Q}_p) : E_0(\mathbb{Q}_p)]$ is the *Tamagawa number* at p.

Obviously, $E/\mathbb{Q}$ is Tamagawa trivial if and only if $\mathrm{Tam}(\mathbb{Q}; E) = 1$. It is known that $\mathrm{Tam}(\mathbb{Q}; E)$ can be arbitrarily large (see, for instance, [10, Table C.15.1]), and so it is natural to ask whether there is an average Tamagawa product for $E/\mathbb{Q}$. The numerics by Balakrishnan et al. [3, Figure A.14] suggest that the average Tamagawa product over $\mathbb{Q}$ exists and is in the neighborhood of 1.82. This speculation was confirmed by Griffin et al. [6, Theorem 1.3], who constructed a new L-function and computed the exact average to be $L_{\mathrm{Tam}}(-1) = 1.8183\ldots$.

It is natural to ask about the values of such arithmetic statistics in an arbitrary number field K. To this end, we define the Tamagawa product $\mathrm{Tam}(K; E)$ for an elliptic curve E/K. We let $\mathfrak{p}$ be a prime ideal of $\mathcal{O}_K$, the ring of integers of K, that lies above a rational prime p. Recall that there is a unique extension $v := v_\mathfrak{p}$ to K corresponding to $\mathfrak{p}$. We let K_v be the completion of K with respect to v. Select a uniformizer π and denote the corresponding valuation on K_v by v_π. The Tamagawa product for elliptic curves E/K is

$$\mathrm{Tam}(K; E) := \prod_{p \text{ prime}} \prod_{\mathfrak{p}|(p)} c_\mathfrak{p}, \tag{1.1}$$

where $c_\mathfrak{p} := [E(K_v) : E_0(K_v)]$ is the Tamagawa number at $\mathfrak{p}$.

Generalizing the work of Griffin et al. [6], we compute the arithmetic statistics of Tamagawa products over arbitrary number fields K. Specifically, we compute the proportion of curves with fixed $\mathrm{Tam}(K;E)$ over short Weierstrass curves

$$E = E(a_4, a_6) : y^2 = x^3 + a_4 x + a_6, \tag{1.2}$$

where $a_4, a_6 \in \mathcal{O}_{K_v}$. To compute the proportion of curves with fixed $\mathrm{Tam}(K;E)$, we require a consistent way to count sets of elliptic curves. To do so, we order E/K by their height. Recall that the height of E/K is

$$\mathrm{ht}(K;E) := \prod_{\mathfrak{p} \in M_K} \max\left\{4|a_4|_{\mathfrak{p}}^3, 27|a_6|_{\mathfrak{p}}^2, 1\right\},$$

where M_K contains all Archimedean and non-Archimedean places on K. To count the number of E/K with height $\leq X$, we introduce:

$$\mathcal{N}(K;X) := \#\{E := E(a_4, a_6) : \mathrm{ht}(K;E) \leq X\}. \tag{1.3}$$

Similarly, to count the number of E/K with Tamagawa product m and height $\leq X$, we define

$$\mathcal{N}_m(K;X) := \#\{E := E(a_4, a_6) : \mathrm{ht}(K;E) \leq X \text{ with } \mathrm{Tam}(K;E) = m\}. \tag{1.4}$$

We now formally define the proportion of elliptic curves $E(a_4, a_6)$ with Tamagawa number m to be

$$P_{\mathrm{Tam}}(K;m) := \lim_{X \to +\infty} \frac{\mathcal{N}_m(K;X)}{\mathcal{N}(K;X)}. \tag{1.5}$$

We compute the global statistic $P_{\mathrm{Tam}}(K;m)$ by computing the local statistics of Tamagawa numbers at each $\mathfrak{p}$. Namely, we let $\delta_{K,\mathfrak{p}}(c)$ be the local proportion of elliptic curves with Tamagawa number c at $\mathfrak{p}$ when the elliptic curves are ordered by height. Using $\delta_{K,\mathfrak{p}}(c)$, we define an analogue of the $L-$function as presented in [6].

Theorem 1.11. *If K is a number field, then $P_{\mathrm{Tam}}(K;m)$ are the Dirichlet coefficients of*

$$L_{\mathrm{Tam}}(K;s) := \sum_{m=1}^{\infty} \frac{P_{\mathrm{Tam}}(K;m)}{m^s} = \prod_{p \ prime} \prod_{\mathfrak{p} | (p)} \left(\frac{\delta_{K,\mathfrak{p}}(1)}{1^s} + \frac{\delta_{K,\mathfrak{p}}(2)}{2^s} + \frac{\delta_{K,\mathfrak{p}}(3)}{3^s} + \ldots \right).$$

The exact values of $\delta_{K,\mathfrak{p}}(c)$ are given in Theorem 3.2, Theorem 4.5, and Theorem 5.7.

Remark. *Theorem 1.11 gives $P_{\mathrm{Tam}}(K;m)$ for all number fields K and every positive integer m. In particular, the theorem makes no assumption on the class number h_K, the structure of the units in $\mathcal{O}_K^\times$, as well as the possible splitting types of primes in K.*

Corollary 1.12. *If K is a number field, then the following are true.*

1. *We have*
$$P_{\mathrm{Tam}}(K;1) = \prod_{p \text{ prime}} \prod_{\mathfrak{p}|(p)} \delta_{K,\mathfrak{p}}(1) \in (0,1).$$

2. *The average Tamagawa product $L_{\mathrm{Tam}}(K;-1)$ is well-defined by absolute convergence.*

In the following example, we illustrate the results of Theorem 1.11 by computing $P_{\mathrm{Tam}}(K;1)$ and $L_{\mathrm{Tam}}(K;-1)$ for all imaginary quadratic fields $\mathbb{Q}(\sqrt{-D})$ with class number 1. For the values of $P_{\mathrm{Tam}}(\mathbb{Q}(\sqrt{-D});m)$ with $m \geq 2$, refer to Chapter 7. For further examples, also refer to Chapter 7, where we compute $P_{\mathrm{Tam}}(K;m)$ and $L_{\mathrm{Tam}}(K;-1)$ for real quadratic fields $\mathbb{Q}(\sqrt{D})$ with squarefree $D < 10^4$ and a number field with Galois group S_4.

Example 1.13. Tables 1.1 and 1.2 illustrate the convergence to $P_{\mathrm{Tam}}(\mathbb{Q}(\sqrt{-D});1)$ and $L_{\mathrm{Tam}}(\mathbb{Q}(\sqrt{-D});-1)$ for class number 1 quadratic number fields.

$\mathcal{N}_1(\mathbb{Q}(\sqrt{-D});X)/\mathcal{N}(\mathbb{Q}(\sqrt{-D});X)$									
X	$\sqrt{-1}$	$\sqrt{-2}$	$\sqrt{-3}$	$\sqrt{-7}$	$\sqrt{-11}$	$\sqrt{-19}$	$\sqrt{-43}$	$\sqrt{-67}$	$\sqrt{-163}$
10^4	0.542	0.488	0.663	0.357	0.609	0.620	0.657	0.560	0.450
10^5	0.539	0.460	0.665	0.359	0.599	0.678	0.716	0.711	0.636
10^6	0.528	0.468	0.660	0.343	0.586	0.667	0.726	0.744	0.728
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
∞	0.529	0.468	0.661	0.349	0.581	0.665	0.733	0.750	0.763

Table 1.1: Convergence to $P_{\mathrm{Tam}}(\mathbb{Q}(\sqrt{-D});1)$.

$\sum_{\mathrm{ht}(\mathbb{Q}(\sqrt{-D});E)\leq X} \mathrm{Tam}(\mathbb{Q}(\sqrt{-D})); E)/N(\mathbb{Q}(\sqrt{-D}); X)$									
X	$\sqrt{-1}$	$\sqrt{-2}$	$\sqrt{-3}$	$\sqrt{-7}$	$\sqrt{-11}$	$\sqrt{-19}$	$\sqrt{-43}$	$\sqrt{-67}$	$\sqrt{-163}$
10^4	1.751	2.054	1.589	2.570	1.850	1.718	1.535	1.698	2.017
10^5	1.720	1.979	1.538	2.417	1.763	1.537	1.393	1.403	1.612
10^6	1.708	1.946	1.508	2.418	1.723	1.519	1.361	1.333	1.372
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
∞	1.678	1.904	1.487	2.376	1.708	1.480	1.331	1.300	1.277

Table 1.2: Convergence to $L_{\mathrm{Tam}}(\mathbb{Q}(\sqrt{-D}); -1)$.

In Table 1.1, $P_{\mathrm{Tam}}(\mathbb{Q}(\sqrt{-D}); 1)$ is noticeably smaller when $D = 7$ and noticeably larger when $D = 163$. On the other hand, in Table 1.2, $L_{\mathrm{Tam}}(\mathbb{Q}(\sqrt{-D}); -1)$ is noticeably larger when $D = 7$ and noticeably smaller when $D = 163$. The variance in $P_{\mathrm{Tam}}(\mathbb{Q}(\sqrt{-D}); 1)$ and $L_{\mathrm{Tam}}(\mathbb{Q}(\sqrt{-D}); -1)$ is due to the splitting type of small primes, since $\delta_{K,\mathfrak{p}}(1)$ is smaller and $\sum_{m=1}^{\infty} \delta_{K,\mathfrak{p}}(m)m$ is larger when $\mathfrak{p}$ has small norm (see Theorem 3.2, Theorem 4.5, and Theorem 5.7). Indeed, 2 splits only in $\mathbb{Q}(\sqrt{-7})$, while $2, 3, \ldots, 37$ are all inert in $\mathbb{Q}(\sqrt{-163})$. For general number fields K, the possible splitting types of primes are determined by $d := \deg K$. It is then natural to ask whether $P_{\mathrm{Tam}}(K; 1)$ and $L_{\mathrm{Tam}}(\mathbb{Q}(\sqrt{-D}); -1)$ can be uniformly bounded as a function of d. We answer this question in the following corollary, which makes use of the Riemann zeta-function $\zeta(s)$ and Bernoulli numbers.

Corollary 1.14. *If K has degree d, then we have the following uniform bounds on $P_{\mathrm{Tam}}(K; 1)$:*

$$(0.5054)^d < P_{\mathrm{Tam}}(\mathbb{Q}; 1)^d \leq P_{\mathrm{Tam}}(K; 1) < (-1)^{d+1} \frac{2(2d)!}{B_{2d}(2\pi)^{2d}} = \frac{1}{\zeta(2d)}.$$

Moreover, we have the following uniform bounds on the average Tamagawa number $L_{\mathrm{Tam}}(K; -1)$:

$$\frac{\zeta(2d)}{\zeta(4d)} = (-1)^d \frac{B_{2d}(4d)!}{B_{4d}(2d)!(2\pi)^{2d}} < L_{\mathrm{Tam}}(K; -1) \leq L_{\mathrm{Tam}}(\mathbb{Q}; -1)^d < (1.8184)^d.$$

As $d \to \infty$ in Theorem 1.14, the given lower and upper bounds for $P_{\mathrm{Tam}}(K; 1)$ tend to 0 and 1, respectively. We can then ask whether $P_{\mathrm{Tam}}(K; 1)$ can be arbitrarily close to 0 or arbitrarily close to 1 as $d \to \infty$. More formally, we define

$$t^{-}(d) := \inf_{\deg K = d} \{P_{\mathrm{Tam}}(K;1)\} \quad \text{and} \quad t^{+}(d) := \sup_{\deg K = d} \{P_{\mathrm{Tam}}(K;1)\} \tag{1.6}$$

to be the infimum and supremum of the Tamagawa trivial proportion over number fields K with degree d. In a similar vein, one can see in Theorem 1.14 that as $d \to \infty$, the given lower and upper bounds for $L_{\mathrm{Tam}}(K;-1)$ tend to 1 and ∞, respectively. To this end, we similarly define the quantities

$$\mu^{-}(d) := \inf_{\deg K = d} \{L_{\mathrm{Tam}}(K;-1)\} \quad \text{and} \quad \mu^{+}(d) := \sup_{\deg K = d} \{L_{\mathrm{Tam}}(K;-1)\} \tag{1.7}$$

to be the infimum and supremum of the average Tamagawa product over number fields K with degree d.

It is a conjecture of Ono [8] that as $d \to \infty$, the limit infimum of $t^{-}(d)$ is 0, the limit supremum of $t^{+}(d)$ is 1, the limit infimum of $\mu^{-}(d)$ is 1, and the limit supremum of $\mu^{+}(d)$ is ∞. With the following theorem, we confirm the conjecture.

Theorem 1.15. *We have*

$$\liminf_{d \to +\infty} t^{-}(d) = 0 \quad \textit{and} \quad \limsup_{d \to +\infty} t^{+}(d) = 1.$$

Moreover, we have

$$\liminf_{d \to +\infty} \mu^{-}(d) = 1 \quad \textit{and} \quad \limsup_{d \to +\infty} \mu^{+}(d) = \infty.$$

Remark. *The proof of Theorem 1.15 is constructive. Namely, we provide a family of multiquadratic fields K for which $P_{\mathrm{Tam}}(K;1) \to 0$ and $L_{\mathrm{Tam}}(K;-1) \to \infty$, and a family of cyclotomic fields K for which $P_{\mathrm{Tam}}(K;1) \to 1$ and $L_{\mathrm{Tam}}(K;-1) \to 1$. In Chapter 7, we compute $P_{\mathrm{Tam}}(K;1)$ and $L_{\mathrm{Tam}}(K;-1)$ for example fields within these families.*

The remainder of the paper is structured as follows: In Chapter 2, we introduce Tate's algorithm, a recursive procedure that computes the local invariants for elliptic curves, including the Tamagawa number of an elliptic curve E/K at $\mathfrak{p}$. Running Tate's algorithm at $\mathfrak{p} \nmid (6)$ is relatively straightforward, but additional challenges arise at $\mathfrak{p} \mid (3)$ and $\mathfrak{p} \mid (2)$ since

E/K is cubic in x and quadratic in y. Therefore, we begin by running Tate's algorithm at $\mathfrak{p} \nmid (6)$ in Chapter 3, and then run Tate's algorithm for $\mathfrak{p} \mid (3)$ and $\mathfrak{p} \mid (2)$ in Sections 4 and 5, respectively. In Chapter 6, we prove our main theorems. In Chapter 7, we compute $P_{\mathrm{Tam}}(K;m)$ and $L_{\mathrm{Tam}}(K;-1)$ for example number fields to illustrate Theorem 1.11. For a classification of non-minimal short Weierstrass models at primes above 2 and 3, refer to Appendix A. For the exact proportions of Tamagawa numbers for primes above 2 and 3 with large ramification indices, refer to Appendix B.

Chapter 2

Tate's Algorithm

Tate's algorithm (see [9, Section 4.9]) is an iterative process that returns the Kodaira type and the Tamagawa number of an elliptic curve E/K at $\mathfrak{p}$. As such, Tate's algorithm allows us to compute $\delta_{K,\mathfrak{p}}(c)$. A single iteration of the algorithm consists of 11 steps. If the model for E has minimal $v_\pi(\Delta)$ over the $\pi-$integral models for E, then the algorithm terminates during the first ten steps, which correspond to each of the ten Kodaira types. We refer to such elliptic curves as $\mathfrak{p}$-*minimal models*. However, if E is not $\mathfrak{p}-$minimal, then, at Step 11 of Tate's algorithm, we scale $(x, y) \mapsto (\pi^2 x, \pi^3 y)$. The substitutions from Step 1 through 10 of Tate's algorithm guarantee that E is $\pi-$integral after Step 11. A non-minimal model E then loops back into Step 1 of Tate's algorithm. Tate's algorithm eventually terminates as the scaling at Step 11 decreases $v_\pi(\Delta)$ by 12. At the step in which the algorithm terminates, the Tamagawa number of E/K at $\mathfrak{p}$ is determined. After determining the proportion of elliptic curves with a fixed Tamagawa number that terminate at each step, we sum these proportions over all steps to compute $\delta_{K,\mathfrak{p}}(c)$.

In [6], Griffin et al. classify elliptic curves into cases depending on $v_\pi(a_4), v_\pi(a_6)$, and $v_\pi(\Delta)$. They then apply distinct linear shifts to the curves in each case and parametrize a_4 and a_6 in terms of these shifts prior to running Tate's algorithm. Finally, they run the algorithm on each case separately. In our paper, we do not classify elliptic curves into cases before running Tate's algorithm. Instead, for each $\mathfrak{p}$, we run Tate's algorithm altogether for $\mathfrak{p}-$minimal elliptic curves in the short Weierstrass form. These $\mathfrak{p}-$minimal elliptic curves are guaranteed to terminate during the first iteration of the algorithm. But when E is non-minimal, E passes through Step 11, then loops back into Step 1. For prime ideals $\mathfrak{p} \nmid (6)$, a non-minimal $E(a_4, a_6)$ is still in the short Weierstrass form after Step 11. But when $\mathfrak{p} \mid (3)$, the coefficient a_2 may be non-zero after Step 11, since E is cubic in x. Therefore, when $\mathfrak{p} \mid (3)$, we must study Tate's algorithm over

$$E(a_2, a_4, a_6): \quad y^2 = x^3 + a_2x^2 + a_4x + a_6 \tag{2.1}$$

to understand how non-minimal curves loop back into Tate's algorithm. When $\mathfrak{p} \mid (2)$, the coefficients a_1 and a_3 may be nonzero after Step 11, since E is quadratic in y. Likewise, we must study Tate's algorithm over

$$E(a_1, a_3, a_4, a_6): \quad y^2 + a_1xy + a_3y = x^3 + a_4x + a_6. \tag{2.2}$$

Therefore, to study how Tate's algorithm acts up on non-minimal $E(a_4, a_6)$, we should study how Tate's algorithm acts upon $E(a_1, a_2, a_3, a_4, a_6)$.

We compute $\delta_{K,\mathfrak{p}}(c)$ by first studying the $\mathfrak{p}-$minimal curves and then the non-minimal curves. To study the $\mathfrak{p}-$minimal curves, we define $\delta'_{K,\mathfrak{p}}(T, c; \alpha_1, \alpha_2, \alpha_3)$ to be the proportion of $\mathfrak{p}-$minimal models $E(a_1, a_2, a_3, a_4, a_6)$ with $v_\pi(a_i) = \alpha_i$ for $i = 1, 2, 3$, Kodaira type T, and Tamagawa number c. In Theorem 3.1, Theorem 4.2, and Theorem 5.2, we compute $\delta'_{K,\mathfrak{p}}(T, c; \infty, \infty, \infty)$ for $\mathfrak{p} \nmid (6)$, $\delta'_{K,\mathfrak{p}}(T, c; \infty, \alpha_2, \infty)$ for $\mathfrak{p} \mid (3)$, and $\delta'_{K,\mathfrak{p}}(T, c; \alpha_1, \infty, \alpha_3)$ for $\mathfrak{p} \mid (2)$, respectively. These values of δ' accounts for the potentially nonzero coefficients after Step 11.

Now, we study the form of non-minimal curves after Step 11. When $\mathfrak{p} \nmid (6)$, the elliptic curves that loop back are still in the short Weierstrass form, which we visualize in a simple Markov chain in Figure 3-1. When $\mathfrak{p} \mid (3)$ (resp. $\mathfrak{p} \mid (2)$) however, a non-minimal elliptic curve after Step 11 may not be short anymore, and so the underlying Markov chain structure is more complex as in Figure 4-1 (resp. Figure 5-1). Families, which are sets of elliptic curves that act as nodes in these Markov chain, are defined in Theorem 4.1 and Theorem 5.1. The edges are determined for $\mathfrak{p} \mid (3)$ (resp. $\mathfrak{p} \mid (2)$) in Theorem 4.3 and Theorem 4.4 (resp. Lemmas 5.3, 5.4, 5.5, 5.6).

Our analysis of Tate's algorithm boils down to studying congruences in terms of the coefficients a_1, a_2, a_3, a_4, a_6 modulo bounded powers of π. We often note that, when certain quantities like a_2, a_4, or $a_6 \pmod{\pi}$ are fixed, there are a fixed number of choices for a_4 and a_6 modulo bounded powers of π. As such, an important quantity throughout our calculation is the ideal norm $q := N_{K/\mathbb{Q}}(\mathfrak{p})$, or the number of distinct residues in $\mathcal{O}_{K_v}$ modulo π. To further illustrate the connection between Tate's algorithm and these modular congruences, we generalize the results of Griffin et al. [6, Lemmas 2.2, 2.3] by classifying and counting the non-minimal short Weierstrass models at primes $\mathfrak{p} \mid (6)$ in Appendix A.

Chapter 3

Tate's Algorithm for $p \geq 5$

For reasons as described above, the required analysis of Tate's algorithm is significantly more involved for $\mathfrak{p} \mid (6)$. Therefore, we first calculate $\delta_{K,\mathfrak{p}}(c)$ for $\mathfrak{p}$ above $p \geq 5$. We realize upon running Tate's algorithm for non-minimal short Weierstrass models (see Theorem 3.1) that the coefficients a_1, a_2, and a_3 remain invariant at 0. (This confirms the converse of non-minimality for $\operatorname{char} k \neq 2, 3$ in [10].) Hence, we define $\varphi_{K,\mathfrak{p}}(T, c_\mathfrak{p}) := \delta'_{K,\mathfrak{p}}(T, c_\mathfrak{p}, \infty, \infty, \infty)$. The rest of this section is as follows: Theorem 3.1 unveils $\varphi_{K,\mathfrak{p}}(T, c_\mathfrak{p})$ and the structure of the associated Markov chain. Then, in Theorem 3.2, we calculate $\delta_{K,\mathfrak{p}}(c_\mathfrak{p})$ in Theorem 3.2 using the Markov chain and $\varphi_{K,\mathfrak{p}}(T, c_\mathfrak{p})$ from Theorem 3.1.

Lemma 3.1. *Suppose that $\mathfrak{p} \nmid (6)$ is a prime ideal in K. Consider the family of Weierstrass models $E(a_4, a_6) : y^2 = x^3 + a_4x + a_6$. Then the local densities $\varphi_{K,\mathfrak{p}}(T, c)$ are provided in Table 3.1.*

Remark. *The above local proportions $\varphi_{K,\mathfrak{p}}(T, c)$ exactly match the local proportions $\delta'_p(T, c)$ for $p \geq 5$ in Griffin et al. [6, Table 5] after replacing the rational prime p with the ideal norm q.*

Proof. We run through Tate's algorithm over $E(a_4, a_6)$ to compute $\varphi_{K,\mathfrak{p}}(T, c)$. Recall that $q = N_{K/\mathbb{Q}}(\mathfrak{p})$ is the ideal norm of $\mathfrak{p}$.

Case 1. E terminates if $\pi \nmid \Delta = -16(4a_4^3 + 27a_6^2)$ or when $(a_4, a_6) \not\equiv (-3w^2, 2w^3) \pmod{\pi}$. Therefore, we have $q^2 - q$ choices of (a_4, a_6) modulo π. Resultantly, $\varphi_{K,\mathfrak{p}}(I_0, 1) = \frac{q-1}{q}$.

Type	$c_\mathfrak{p}$	$\varphi_{K,\mathfrak{p}}(T,c_\mathfrak{p})$	Type	$c_\mathfrak{p}$	$\varphi_{K,\mathfrak{p}}(T,c_\mathfrak{p})$	Type	$c_\mathfrak{p}$	$\varphi_{K,\mathfrak{p}}(T,c_\mathfrak{p})$
I_0	1	$\frac{q-1}{q}$	I_0^*	1	$\frac{1}{3}\frac{(q^2-1)}{q^7}$	III	2	$\frac{(q-1)}{q^4}$
I_1	1	$\frac{(q-1)^2}{q^3}$	I_0^*	2	$\frac{1}{2}\frac{(q-1)}{q^6}$	III^*	2	$\frac{(q-1)}{q^9}$
I_2	2	$\frac{(q-1)^2}{q^4}$	I_0^*	4	$\frac{1}{6}\frac{(q-1)(q-2)}{q^7}$	IV	1	$\frac{1}{2}\frac{(q-1)}{q^5}$
$I_{n\geq 3}$	$\varepsilon(n)$	$\frac{1}{2}\frac{(q-1)^2}{q^{n+2}}$	$I_{n\geq 1}^*$	2	$\frac{1}{2}\frac{(q-1)^2}{q^{7+n}}$	IV	3	$\frac{1}{2}\frac{(q-1)}{q^5}$
$I_{n\geq 3}$	n	$\frac{1}{2}\frac{(q-1)^2}{q^{n+2}}$	$I_{n\geq 1}^*$	4	$\frac{1}{2}\frac{(q-1)^2}{q^{7+n}}$	IV^*	1	$\frac{1}{2}\frac{(q-1)}{q^8}$
II	1	$\frac{(q-1)}{q^3}$	II^*	1	$\frac{(q-1)}{q^{10}}$	IV^*	3	$\frac{1}{2}\frac{(q-1)}{q^8}$

Table 3.1: The $\varphi_{K,\mathfrak{p}}(T,c)$ for $\mathfrak{p} \nmid (6)$. (Note: $\varepsilon(n) := ((-1)^n + 3)/2$.)

Case 2. Suppose that E is singular at $(u,0)$ after reduction. We shift $x \mapsto x+u$; the new model is

$$y^2 = (x+u)^3 + a_4(x+u) + a_6. \tag{3.1}$$

We stop if $u \not\equiv 0 \pmod{\pi}$. We check that exactly half of the choices of u result in $T^2 - 3u$ splitting. By Hensel's lemma, these curves have a $\frac{(q-1)^2}{q^{n+2}}$ chance of satisfying $n = v_\pi(\Delta)$. Hence, we have $\delta_{K,\mathfrak{p}}(I_n, n) = \delta_{K,\mathfrak{p}}(I_n, \varepsilon(n)) = \frac{(q-1)^2}{2q^{n+2}}$. Henceforth, assume $u = 0$, which implies $\pi \mid a_4, a_6$.

Case 3. We stop if $\pi^2 \nmid a_6$. There is one choice for $a_4 \pmod{\pi}$ and $q-1$ choices for $a_6 \pmod{\pi^2}$, whence $\delta_{K,\mathfrak{p}}(II,1) = \frac{q-1}{q^3}$. Henceforth, assume $\pi^2 \mid a_6$.

Case 4. We stop when $\pi^2 \nmid a_4$. Thus, we have $q-1$ choices for $a_4 \pmod{\pi^2}$ and one choice for $a_6 \pmod{\pi^2}$, so $\delta_{K,\mathfrak{p}}(III,1) = \frac{q-1}{q^4}$. Henceforth, assume $\pi^2 \mid a_4$.

Case 5. We stop at Step 5 if $\pi^3 \nmid a_6$. Thus, we have one choice for $a_4 \pmod{\pi^2}$ and $q-1$ choices for $a_6 \pmod{\pi^3}$. The Tamagawa number is 3 if $Y^2 - (\pi^{-2}a_6)^2$ splits modulo π, and 1 otherwise. Hence, $\delta_{K,\mathfrak{p}}(IV,1) = \delta_{K,\mathfrak{p}}(IV,3) = \frac{q-1}{2q^5}$. Henceforth, assume $\pi^3 \mid a_6$.

Case 6. In Step 6 to 8, we study the polynomial $P(T) = T^3 + \frac{a_4}{\pi^2}T + \frac{a_6}{\pi^3} =: T^3 + A_4T + A_6$. Note that A_4 and A_6 are equally distributed among the residues in k. The cubic $P(T)$ has discriminant $4A_4^3 + 27A_6^2$. We stop if $P(T)$ has three distinct roots, i.e., if $4A_4^3 + 27A_6^2 \not\equiv 0 \pmod{\pi}$. This accounts for $q^2 - q$ residue pairs as $(A_4, A_6) \not\equiv (-3w^2, 2w^3)$ modulo π.

We now classify $P(T)$ based on the number of its roots in k. First, if $c_{\mathfrak{p}} = 1$, then $P(T)$ must be irreducible. The linear map $\mathrm{Tr} : \mathbb{F}_{q^3} \to \mathbb{F}_q$ is surjective, so there are $q^3/q = q^2$ traceless elements of $\mathbb{F}_{q^3}$, one of which is in $\mathbb{F}_q$. Thus, there are $\frac{q^2-1}{3}$ such cubics. Next, if $c_{\mathfrak{p}} = 2$, then $P(T)$ factors into a linear term and an irreducible quadratic. There are $\frac{q^2-q}{2}$ irreducible quadratics and upon fixing this quadratic, the linear term is fixed as $P(T)$ is traceless. Finally, if $c_{\mathfrak{p}} = 4$, then $P(T)$ has three roots in k. There are $\binom{q}{3}$ ways to select three roots and $1/q$ of them have trace 0, so we have $\frac{(q-1)(q-2)}{6}$ such $P(T)$. In all, we have $\delta_{K,\mathfrak{p}}(I_0^*, 1) = \frac{q^2-1}{3q^7}$, $\delta_{K,\mathfrak{p}}(I_0^*, 2) = \frac{q-1}{2q^6}$, and $\delta_{K,\mathfrak{p}}(I_0^*, 4) = \frac{(q-1)(q-2)}{6q^7}$.

Case 7. We stop when $4A_4^3 + 27A_6^2 \equiv 0 \pmod{\pi}$ but $(A_4, A_6) \not\equiv (0,0) \pmod{\pi}$. Let $r := \sqrt{A_4/3}$ that serves as the double root of $P(T)$. We accordingly shift $x \mapsto x + \pi r$:

$$y^2 = (x + \pi r)^3 + a_4(x + \pi r) + a_6 = x^3 + a_2' x^2 + a_6'. \tag{3.2}$$

Ultimately, the Kodaira type depends on $n := v_\pi(a_6') - 3$, which occurs with proportion $\frac{(q-1)^2}{q^{n+7}}$. The Tamagawa number depends on whether $\frac{a_6'}{\pi^{v_\pi(a_6')}}$ is a quadratic residue, which happens half of the time. Hence, $\delta_{K,\mathfrak{p}}(I_n^*, 2) = \delta_{K,\mathfrak{p}}(I_n^*, 4) = \frac{(q-1)}{2q^{n+7}}$.

Case 8. $P(T)$ is traceless. Therefore, its triple root must be 0, so henceforth $\pi^3 \mid a_4$ and $\pi^4 \mid a_6$. We stop if $Y^2 - \pi^{-4}a_6$ has distinct roots, i.e., if $\pi^{-4}a_6 \not\equiv 0 \pmod{\pi}$. We have one choice for $a_4 \pmod{\pi^3}$ and $q-1$ choices for $a_6 \pmod{\pi^5}$, half of which cause $Y^2 - \pi^{-4}a_6$ to split. Thus, $\varphi_{K,\mathfrak{p}}(IV^*, 1) = \varphi_{K,\mathfrak{p}}(IV^*, 3) = \frac{q-1}{2q^8}$.

Case 9. We terminate if $\pi^4 \nmid a_4$. There are $q-1$ choices for $a_4 \pmod{\pi^4}$ and one choice for $a_6 \pmod{\pi^5}$, whence $\varphi_{K,\mathfrak{p}}(III^*, 2) = \frac{q-1}{q^9}$.

Case 10. We stop if $\pi^6 \nmid a_6$. There is one choice for $a_4 \pmod{\pi^4}$ and $q-1$ choices for $a_6 \pmod{\pi^6}$, whence $\varphi_{K,\mathfrak{p}}(II^*, 1) = \frac{q-1}{q^{10}}$.

Case 11. For E not to be $\mathfrak{p}-$minimal, it must be that $\pi^4 \mid a_4$ and $\pi^6 \mid a_6$, so the proportion of non-minimal E is $\frac{1}{q^{10}}$. □

From Theorem 3.1, a non-minimal curve $E(a_4, a_6)$ transforms into $E'(a_4, a_6) := E(\pi^{-4}a_4, \pi^{-6}a_6)$. As we run through the non-minimal E, the family of E' is equivalent to that of E. Thus, we may rerun Theorem 3.1 on the new family E'. Figure 3-1 illustrates the resultant Markov chain structure.

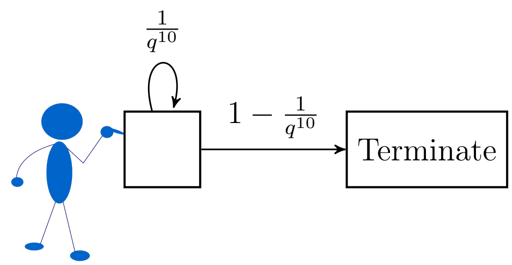


Figure 3-1: The Markov chain structure for short Weierstrass curves when $\mathfrak{p} \nmid (6)$.

Proposition 3.2. *If $\mathfrak{p} \nmid (6)$ is a prime ideal in K and $n \geq 1$, then letting $q := N_{K/\mathbb{Q}}(\mathfrak{p})$ we have*

$$\delta_{K,\mathfrak{p}}(c) = \begin{cases} 1 - \dfrac{q(6q^7+9q^6+9q^5+7q^4+8q^3+7q^2+9q+6)}{6(q+1)^2(q^8+q^6+q^4+q^2+1)} & \text{if } n=1, \\[2ex] \dfrac{q(2q^7+2q^6+q^5+q^4+2q^3+q^2+2q+2)}{2(q+1)^2(q^8+q^6+q^4+q^2+1)} & \text{if } n=2, \\[2ex] \dfrac{q^2(q^4+1)}{2(q+1)(q^8+q^6+q^4+q^2+1)} & \text{if } n=3, \\[2ex] \dfrac{q^3(3q^2-2q-1)}{6(q+1)(q^8+q^6+q^4+q^2+1)} & \text{if } n=4, \\[2ex] \dfrac{q^{10}-2q^9+q^8}{2q^n(q^{10}-1)} & \text{if } n \geq 5. \end{cases}$$

Remark. *When $K = \mathbb{Q}$, then $\mathfrak{p} = (p)$ for rational primes p and we recover the formulae over rationals found in [6]. Conversely, the expressions for the proportions over general K match that of $\mathbb{Q}$, except the rational prime p is replaced with the prime ideal norm q.*

Proof. Theorem 3.1 reveals the formulae for $\varphi_{K,\mathfrak{p}}(T, c_{\mathfrak{p}})$. Following the Markov chain as illustrated in Figure 3-1, we may explicitly compute

$$\delta_{K,\mathfrak{p}}(c) = \left(1 + \frac{1}{q^{10}} + \frac{1}{q^{20}} + \dots\right) \sum_{c_{\mathfrak{p}}=n} \varphi_{K,\mathfrak{p}}(T, c_{\mathfrak{p}}). \qquad \square \quad (3.3)$$

Chapter 4

Tate's Algorithm for $p = 3$

In this section, we derive $\delta_{K,\mathfrak{p}}(c)$ for $\mathfrak{p} \mid (3)$. Unlike in Chapter 3, Tate's algorithm may introduce a non-zero a_2-coefficient on non-minimal $E(a_4, a_6)$ that loops back into the algorithm. Therefore, to determine $\delta_{K,\mathfrak{p}}(c)$, we study the action of Tate's algorithm on a larger class of elliptic curves—namely, on $E(a_2, a_4, a_6)$, defined in Equation (2.1). Upon running Tate's algorithm, we find that the quantity $\alpha_2 := v_\pi(a_2)$ influences whether or not E passes a given step. Moreover, the short Weierstrass elliptic curves are exactly the $E(a_2, a_4, a_6)$ with $\alpha_2 = \infty$. Naturally, we group elliptic curves into families depending on α_2 as follows.

Definition 4.1. The *3-family* $F(\alpha_2)$ refers to the set of models

$$F(\alpha_2) := \{E(a_2, a_4, a_6) : v_\pi(a_2) = \alpha_2;\ a_4, a_6 \text{ integral}\}.$$

The 3-family $F(\geq \alpha_2)$ refers to the set $\bigsqcup_{\alpha \geq \alpha_2} F(\alpha)$.

For brevity, we refer to 3-families as families for this section. The rest of the section is structured as follows. In Theorem 4.2, we run Tate's algorithm to calculate $\chi_{K,\mathfrak{p}}(T, c; \alpha_2) := \delta'_{K,\mathfrak{p}}(T, c; \infty, \alpha_2, \infty)$, which is the proportion of $\mathfrak{p}-$minimal models with Kodaira type T and Tamagawa number c for each family $F(\alpha_2)$. Then, we show in Theorem 4.3 and Theorem 4.4 that the non-minimal models from a given family may themselves be viewed as a family. Finally, in Theorem 4.5, we leverage these lemmas to form a Markov chain whose nodes are families and whose edges represent the reclassification of non-minimal models, which we use to compute the local proportion $\delta_{K,\mathfrak{p}}(c)$.

Lemma 4.2. *Suppose that $\mathfrak{p} \subseteq K$ is above 3 with ramification index e. Then for $F(\alpha_2)$, the local densities $\chi_{K,\mathfrak{p}}(T, c; \alpha_2)$ is as provided in Table 4.1.*

Proof. We run Tate's algorithm over $F(\alpha_2)$ to compute $\chi_{K,\mathfrak{p}}(T, c; \alpha_2)$.

Case 1. Suppose that E terminates at Step 1. If $E \in F(0)$, then $\Delta \equiv -a_2^3 a_6 + a_2^2 a_4^2 - a_4^3 \pmod{\pi}$. Thus, for $\pi \nmid \Delta$, there are $q-1$ choices for a_6 modulo π for fixed a_2 and a_4. If $E \in F(1)$ or $E \in F(\geq 2)$, then $\Delta \equiv -a_4^3 \pmod{\pi}$. Therefore, there are $q-1$ choices for a_4 modulo π for fixed a_2 and a_6. Thus, $\chi_{K,\mathfrak{p}}(I_0, 1; 0) = \chi_{K,\mathfrak{p}}(I_0, 1; 1) = \chi_{K,\mathfrak{p}}(I_0, 1; \geq 2) = \frac{q-1}{q}$. Moving forward, for curves that pass Step 1, in $F(0)$, the units digit of a_6 is fixed for fixed a_4 and a_2, and in $F(\geq 1)$ the units digit of a_4 is 0.

Case 2. Suppose that the singular point of E after reduction by π is at $(u, 0)$. Then, the translation $x \mapsto x + u$ yields the new model

$$y^2 = (x+u)^3 + a_2(x+u)^2 + a_4(x+u) + a_6. \tag{4.1}$$

Curves in $F(0)$ always terminate as $\pi \nmid b_2 = a_2 + 3u$. By Hensel's lemma, exactly $\frac{q-1}{q^{n+1}}$ of curves within $F(0)$ satisfy $v_\pi(\Delta) = n$ as a_2 varies. We also check that half of these values make a_2 a quadratic residue modulo π. Thus, $\chi_{K,\mathfrak{p}}(I_n, n; 0) = \chi_{K,\mathfrak{p}}(I_n, \varepsilon(n); 0) = \frac{q-1}{2q^{n+1}}$.

On the other hand, if $E \in F(\geq 1)$, we always pass this step, so $\chi_{K,\mathfrak{p}}(I_n, n; 0) = \chi_{K,\mathfrak{p}}(I_n, \varepsilon(n); 0) = 0$. Since $(u, 0)$ lies on the curve after reduction and $v_\pi(a_4) \geq 1$ from Case 1, note $u^3 + a_6 \equiv 0 \pmod{\pi}$.

Case 3. If we stop at Step 3, $\pi^2 \nmid a_6 + a_4 u + a_2 u^2 + u^3$. From Case 2, we have that $\pi \mid a_6 + a_4 u + a_2 u^2 + u^3$. After fixing u, a_2, a_4, we have $q-1$ choices for $a_6 \pmod{\pi^2}$. As a result, $\chi(II, 1; \geq 1) = \frac{q-1}{q^2}$.

Case 4. To terminate at Step 4, we must have $\pi^3 \nmid 4(a_2 + 3u)(a_6 + a_4 u + a_2 u^2 + u^3) - (3u^2 + 3a_2 u + a_4)^2$. From Steps 2 and 3 respectively, we have that $\pi \mid a_2 + 3u$ and $\pi^2 \mid a_6 + a_4 u + a_2 u + u^3$. As such, we stop if $\pi^2 \nmid 3u^2 + 3a_2 u + a_4$. We find that for fixed a_2

		$e=1$		$e\geq 2$		
Type	$c_{\mathfrak{p}}$	$\alpha_2=0$	$\alpha_2\geq 1$	$\alpha_2=0$	$\alpha_2=1$	$\alpha_2\geq 2$
I_0	1	$(q-1)/q$	$(q-1)/q$	$(q-1)/q$	$(q-1)/q$	$(q-1)/q$
I_1	1	$(q-1)/q^2$	0	$(q-1)/q^2$	0	0
I_2	2	$(q-1)/q^3$	0	$(q-1)/q^3$	0	0
$I_{n\geq 3}$	n	$(q-1)/2q^{n+1}$	0	$(q-1)/2q^{n+1}$	0	0
$I_{n\geq 3}$	$\varepsilon(n)$	$(q-1)/2q^{n+1}$	0	$(q-1)/2q^{n+1}$	0	0
II	1	0	$(q-1)/q^2$	0	$(q-1)/q^2$	$(q-1)/q^2$
III	2	0	$(q-1)/q^3$	0	$(q-1)/q^3$	$(q-1)/q^3$
IV	1	0	$(q-1)/2q^4$	0	$(q-1)/2q^4$	$(q-1)/2q^4$
IV	3	0	$(q-1)/2q^4$	0	$(q-1)/2q^4$	$(q-1)/2q^4$
I_0^*	1	0	$(q^2-1)/3q^6$	0	$1/3q^5$	$(q-1)/3q^5$
I_0^*	2	0	$(q-1)/2q^5$	0	$(q-1)/2q^6$	$(q-1)/2q^5$
I_0^*	4	0	$(q-1)(q-2)/6q^6$	0	$(q-3)/6q^6$	$(q-1)/6q^5$
I_n^*	2	0	$(q-1)^2/2q^{6+n}$	0	$(q-1)/2q^{5+n}$	0
I_n^*	4	0	$(q-1)^2/2q^{6+n}$	0	$(q-1)/2q^{5+n}$	0
IV^*	1	0	$(q-1)/2q^7$	0	0	$(q-1)/2q^6$
IV^*	3	0	$(q-1)/2q^7$	0	0	$(q-1)/2q^6$
III^*	2	0	$(q-1)/q^8$	0	0	$(q-1)/q^7$
II^*	1	0	$(q-1)/q^9$	0	0	$(q-1)/q^8$

Table 4.1: The $\chi_{K,\mathfrak{p}}(T,c;\alpha_2)$ for $\mathfrak{p}\mid(3)$.

and u, there are $q(q-1)$ choices for $a_4 \pmod{\pi^2}$ and one choice for $a_6 \pmod{\pi^2}$. Hence, $\chi_{K,\mathfrak{p}}(III, 2; \geq 1) = \frac{q-1}{q^3}$.

Case 5. To stop at Step 5, $\pi \nmid \pi^{-2}(a_6+a_4u+a_2u^2+u^3)$. For fixed a_2 and u, we have q choices for $a_4 \pmod{\pi^2}$ and $q-1$ choices for $a_6 \pmod{\pi^3}$, half of which make $\pi^{-2}(a_6+a_4u+a_2u^2+u^3)$ a quadratic residue. Hence, $\chi_{K,\mathfrak{p}}(IV, 1; \geq 1) = \chi_{K,\mathfrak{p}}(IV, 3; \geq 1) = \frac{q-1}{2q^4}$. Moving forward, we have q choices for $a_4 \pmod{\pi^2}$ and $a_6 \pmod{\pi^3}$ for each choice of a_2, u, and the π^2-digit of a_4.

Case 6. We begin by writing $P(T) = T^3 + \frac{3u+a_2}{\pi}T^2 + \frac{3u^2+2a_2u+a_4}{\pi^2}T + \frac{u^3+a_2u^2+a_4u+a_6}{\pi^3} =: T^3 + A_2T^2 + A_4T + A_6 \pmod{\pi}$. Note that fixing $A_4, A_6 \pmod{\pi}$ and a choice of a_2, u, and the π^3-digit of a_4 uniquely determines $a_4 \pmod{\pi^3}$ and $a_6 \pmod{\pi^4}$.

Suppose that we fix A_2 and consider $P(T)$ across (A_4, A_6) modulo π. For $P(T)$ to have 3 distinct roots, $P(T)$ and $P'(T) = 2A_2x + A_4 \pmod{\pi}$ must not have a shared root. In other words, $A_4^3 + 2A_2^2A_4^2 + A_6A_2^3 \not\equiv 0 \pmod{\pi}$. As such, for a fixed A_2 modulo π, there are q choices for A_4 and $q-1$ choices of A_6 modulo π such that $P(T)$ has three distinct roots. Suppose further that all three of $P(T)$'s roots are in $\mathbb{F}_q$. If $P(T)$ is traceless, i.e., $A_2 \equiv 0 \pmod{\pi}$, there are q choices for the first root and $q-1$ choices for the second root. Then, the third root is guaranteed to be different from the first two. If $P(T)$ has a non-zero trace, however, then there are q choices for the first root and $q-3$ choices for the second root to guarantee that the fixed third root is distinct from the first two. Now, suppose that $P(T)$ has exactly one root in $\mathbb{F}_q$. Then, once we fix an irreducible quadratic, the linear term is fixed. Therefore, no matter the trace, the number of $P(T)$ with exactly one root in $\mathbb{F}_q$ across (A_4, A_6) modulo π is $\frac{q^2-q}{2}$. The last case is when $P(T)$ is irreducible. In $\mathbb{F}_{q^3}$, there are $\frac{q^2}{3}$ elements of a given trace as the linear map $\mathrm{Tr} : \mathbb{F}_{q^3} \to \mathbb{F}_q$ is surjective. The elements of $\mathbb{F}_q$ all have trace 0. Therefore, the number of irreducible $P(T)$ with zero trace is $\frac{q^2-q}{3}$, and the number of that with a non-zero trace is $\frac{q^2-q}{3}$.

We first discuss $e = 1$. Here, a fixed A_2 modulo π uniquely determines u—the units digit of a_6. We thus conclude from the aforementioned analysis that for $e = 1$, $\chi_{K,\mathfrak{p}}(I_0^*, 1; \geq 1) = \frac{q^2-1}{3q^6}$, $\chi_{K,\mathfrak{p}}(I_0^*, 2; \geq 1) = \frac{q-1}{2q^5}$, and $\chi_{K,\mathfrak{p}}(I_0^*, 4; \geq 1) = \frac{(q-1)(q-2)}{6q^6}$. We now repeat for $e \geq 2$ and $E \in F(1)$. Here, the trace is fixed and necessarily non-zero. As such, $\chi_{K,\mathfrak{p}}(I_0^*, 1; 1) = \frac{1}{3q^5}$, $\chi_{K,\mathfrak{p}}(I_0^*, 2; 1) = \frac{q-1}{2q^6}$, and $\chi_{K,\mathfrak{p}}(I_0^*, 4; 1) = \frac{q-3}{6q^6}$. Finally, we discuss $e \geq 2$ and $E \in F(\geq 2)$.

Here, the trace is necessarily zero. Thus, $\chi_{K,\mathfrak{p}}(I_0^*, 1; \geq 2) = \frac{q-1}{3q^5}$, $\chi_{K,\mathfrak{p}}(I_0^*, 2; \geq 2) = \frac{q-1}{2q^5}$, and $\chi_{K,\mathfrak{p}}(I_0^*, 4; \geq 2) = \frac{q-1}{6q^5}$.

Case 7. If E stops at Step 7, $P(T)$ has a double root that is not a triple root. This implies that $A_2 \not\equiv 0 \pmod{\pi}$, but that $A_2^3 + 2A_2^2A_4^2 + A_6A_2^3 \equiv 0 \pmod{\pi}$. For $A_2 \not\equiv 0 \pmod{\pi}$, $\alpha_2 = 1$ and $e \geq 2$, or $\alpha_2 \geq 1$ and $e = 1$. Hence, $\chi_{K,\mathfrak{p}}(I_n^*, 2; \geq 1) = \chi_{K,\mathfrak{p}}(I_n^*, 4; \geq 1) = \frac{(q-1)^2}{2q^{6+n}}$ for $e = 1$. If $E \in F(1)$ and $e \geq 2$, E terminates at Step 7. Now suppose that $E \in F(\geq 2)$ and $e = 1$. Then, for $A_2^3 + 2A_2^2A_4^2 + A_6A_2^3 \equiv 0 \pmod{\pi}$, we have $q-1$ choices for A_6 for each A_4. We then shift the double root of $P(T)$ to 0. The constant term of the shifted model is surjective over a_6. Therefore, the proportion of E with valuation $3+n$ is $\frac{q-1}{q^n}$, half of which has $n = 2$ and half of which has $n = 4$. Hence, $\chi_{K,\mathfrak{p}}(I_n^*, 2; 1) = \chi_{K,\mathfrak{p}}(I_n^*, 4; 1) = \frac{q-1}{2q^{5+n}}$ and $\chi_{K,\mathfrak{p}}(I_n^*, 2; \geq 2) = \chi_{K,\mathfrak{p}}(I_n^*, 4; \geq 2) = 0$ for $e \geq 2$.

Case 8. If E reaches Step 8, $P(T)$ has a triple root. If so, then $e \geq 1$ and $E \in F(1)$, or $e \geq 2$ and $E \in F(\geq 2)$. Let the triple root of $P(T)$ be v, so $v^3 \equiv A_6 \pmod{\pi}$. We shift the triple root of $P(T)$ to 0 via $x \mapsto x + v\pi$. Letting $s := u + v\pi$, E becomes

$$y^2 = (x+s)^3 + a_2(x+s)^2 + a_4(x+s) + a_6. \tag{4.2}$$

We stop if $\pi \nmid \pi^{-4}(s^3 + a_2s^2 + a_4s + a_6)$. We count by fixing a_2 and $s \pmod{\pi^2}$, the latter of which we have q choices for $e = 1$ and q^2 choices for $e \geq 2$. Resultantly, there is one choice for $a_4 \pmod{\pi^3}$ and $q-1$ choices for $a_6 \pmod{\pi^5}$, half of which make $\pi^{-4}(s^3 + a_2s^2 + a_4s + a_6)$ a quadratic residue. Hence, we have $\chi_{K,\mathfrak{p}}(IV^*, 1; \geq 1) = \chi_{K,\mathfrak{p}}(IV^*, 3; \geq 1) = \frac{q-1}{2q^7}$ for $e = 1$, and $\chi_{K,\mathfrak{p}}(IV^*, 1; \geq 2) = \chi_{K,\mathfrak{p}}(IV^*, 3; \geq 2) = \frac{q-1}{2q^6}$ for $e \geq 2$.

Case 9. Suppose that E reaches Step 9. We stop if $\pi^4 \nmid 3s^2 + 2a_2s + a_4$. As in Case 8, we first choose a_2 and $s \pmod{\pi^2}$, the latter of which we have q choices for $e = 1$ and q^2 choices for $e \geq 2$. These choices allow for $q-1$ choices of $a_4 \pmod{\pi^4}$ and a unique choice of a_6 $\pmod{\pi^5}$ Hence, we have $\chi_{K,\mathfrak{p}}(III^*, 2; \geq 1) = \frac{q-1}{q^8}$ for $e = 1$, and $\chi_{K,\mathfrak{p}}(III^*, 2; \geq 2) = \frac{q-1}{q^7}$ for $e \geq 2$.

Case 10. We stop at Step 10 if $\pi \nmid \pi^{-5}(s^3 + a_2s^2 + a_4s + a_6)$. Once again, we first choose a_2 and s, the latter of which we have q choices for $e = 1$ and q^2 choices for $e \geq 2$. These choices give a unique choice of $a_4 \pmod{\pi^4}$ and $q-1$ choices of $a_6 \pmod{\pi^6}$. Hence, we have $\chi_{K,\mathfrak{p}}(II^*, 1; \geq 1) = \frac{q-1}{q^9}$ for $e = 1$, and $\chi_{K,\mathfrak{p}}(II^*, 1; \geq 2) = \frac{q-1}{q^8}$ for $e \geq 2$.

Case 11. Suppose E is non-minimal. If s is fixed, then (a_4, a_6) is fixed up to modulo π^4 and π^6 respectively. When $e = 1$ and $E \in F(\geq 1)$, there are q choices of s modulo π^2. Therefore, $\frac{1}{q^9}$ of the curves are non-minimal. When $e \geq 2$ and $E \in F(\geq 2)$, there are q^2 choices for s. Hence, $\frac{1}{q^8}$ curves are non-minimal. □

We now justify how we reclassify the non-minimal models of one family as another family of curves. We first show that, for $\alpha_2 < e$, the local densities at the non-minimal models of a family $F(\alpha_2)$ exactly match the local densities at the family $F(\alpha_2 - 2)$. To do this, we establish a map which sends a non-minimal model in $F(\alpha_2)$ to another isomorphic model in $F(\alpha_2 - 2)$, induced by the transformation at Step 11.

Lemma 4.3. *Let $\alpha_2 < e$. There is a surjective, q^2-to-1 map between the set of non-minimal models in $F(\alpha_2)$ and the set $F(\alpha_2 - 2)$ that sends E to its transformation E' after passing Step 11.*

Proof. Fix a set of q^2 representatives to be the residues modulo π^2. Recall that for non-minimal $E(a_2, a_4, a_6)$, Tate's algorithm produces a unique residue $s \pmod{\pi^2}$ for which

$$y^2 = (x+s)^3 + a_2(x+s)^2 + a_4(x+s) + a_6$$

has the coefficient of x^i divisible by π^i for $i = 2, 4, 6$. Hence, each non-minimal model $E(a_2, a_4, a_6) \in F(\alpha_2)$ is sent to

$$E\left(\frac{3s + a_2}{\pi^2}, \frac{3s^2 + 2a_2 s + a_4}{\pi^4}, \frac{s^3 + a_2 s^2 + a_4 s + a_6}{\pi^6}\right) \in F(\alpha_2 - 2)$$

and the map is well-defined.

Conversely, given a model $E'(a_2', a_4', a_6') \in F(\alpha_2 - 2)$, choosing s uniquely determines a_2, a_4, a_6, and moreover $v_\pi(a_2) = \alpha_2$. Hence, E' has exactly q^2 preimages in $F(\alpha_2)$, as we had sought. □

It remains to classify the non-minimal models in $F(\geq e)$, and in particular, the non-minimal short Weierstrass models in $F(\infty)$. In fact, models in $F(\geq e)$ and $F(\infty)$ have the same local properties in the following sense. Note that linear transformations do not change the local data of an elliptic curve. As such, instead of computing local densities on short Weierstrass models $F(\infty)$, we may compute the local densities on the set

$$\{\left(E : y^2 = (x+t)^3 + a_4(x+t) + a_6\right) : t, a_4, a_6 \text{ integral}\}. \tag{4.3}$$

By the surjectivity of a_4 and a_6, the set in eq. (4.3) is precisely $F(\geq e)$. Thus, we instead begin with the curves in $F(\geq e)$ moving forward.

Now, in the manner of Theorem 4.3, we demonstrate that the local densities at the non-minimal models of $F(\geq e)$ match the local densities at $F(\geq e-2)$. The proof of the following corollary is analogous to that of Theorem 4.3.

Corollary 4.4. *There is a surjective, q^2-to-1 map between the set of non-minimal models in $F(\geq e)$ and the set $F(\geq e-2)$ that sends E to its transformation E' after passing Step 11.*

Equipped with our lemmas, we complete our classification for $\mathfrak{p} \mid (3)$ in Theorem 4.5. A key ingredient is the underlying Markov chain structure that helps us study how non-minimal curves loop back into Tate's algorithm. Note that Lemmas 4.3 and 4.4 allows us to identify the non-minimal curves of $F(0), F(1), \ldots, F(\geq e)$ to be identified with other families. In particular, we see from Theorem 4.3 that the non-minimal curves in $F(\alpha)$ for $\alpha < e$ are always transformed into curves in $F(\alpha - 2)$. We also see from Theorem 4.4 that the non-minimal curves in $F(\geq e)$ with proportion $\frac{1}{q^2}$ loop back to itself, with proportion $\frac{q-1}{q^2}$ loop to $F(e-1)$, and with proportion $\frac{q-1}{q}$ transform to $F(e-2)$. Finally, Theorem 4.2 determines the local densities at each $F(\alpha_2)$. We collate all of this information in Figure 4-1.

Proposition 4.5. *If $\mathfrak{p} \mid (3)$ is a prime ideal in K and $n \geq 1$, then letting $q := N_{K/\mathbb{Q}}(\mathfrak{p})$ and*

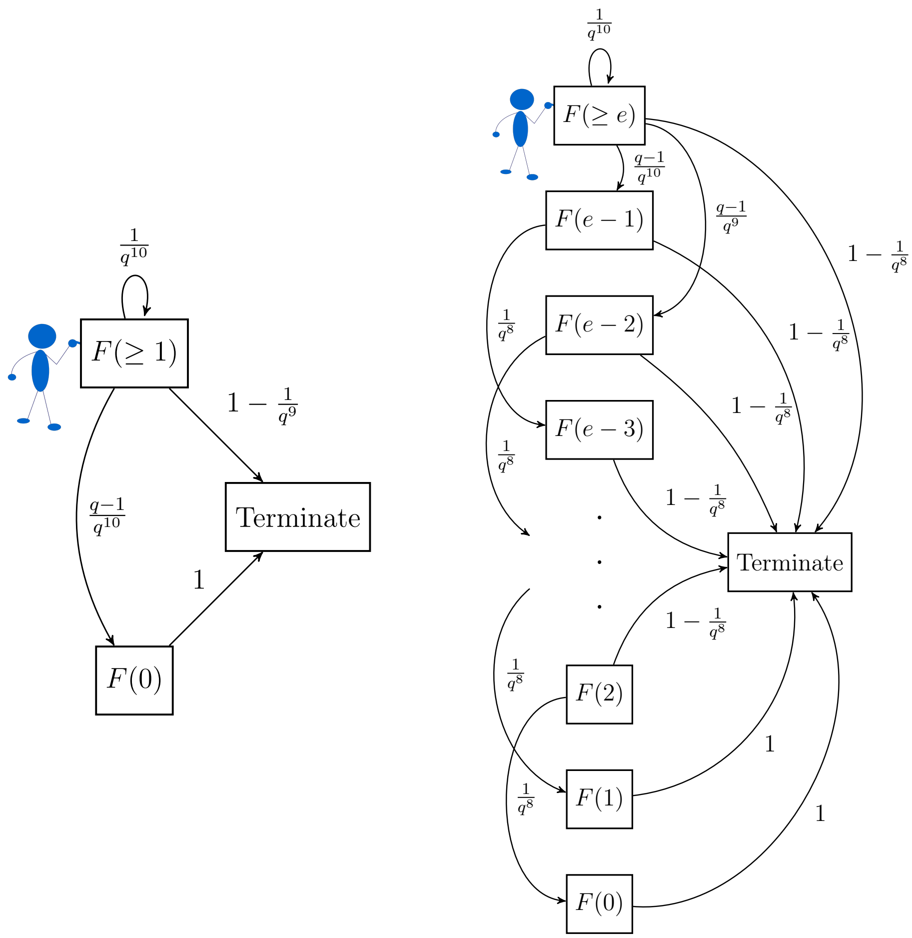


Figure 4-1: The Markov chain structure when $\mathfrak{p} \mid (3)$ for $e = 1$ (left) and $e \geq 2$ (right).

$e = 1$ we have

$$\delta_{K,\mathfrak{p}}(c) = \begin{cases} 1 - \dfrac{(q-1)(6q^{10}+9q^9+7q^8+8q^7+7q^6+9q^5+6q+3)}{6q^2(q+1)(q^{10}-1)} & \textit{if } n = 1, \\[2ex] \dfrac{(q-1)(2q^{11}+2q^{10}+q^9+2q^8+q^7+2q^6+2q^5+2q^2-1)}{2q^3(q+1)(q^{10}-1)} & \textit{if } n = 2, \\[2ex] \dfrac{(q-1)(q^{10}+q^7+q-1)}{2q^4(q^{10}-1)} & \textit{if } n = 3, \\[2ex] \dfrac{(q-1)(q^{10}+q^9+3q-3)}{6q^5(q^{10}-1)} & \textit{if } n = 4, \\[2ex] \dfrac{(q-1)^2}{2q^{n+1}(q^{10}-1)} & \textit{if } n \geq 5. \end{cases}$$

If $e \geq 2$ is even, we have

$$\delta_{K,\mathfrak{p}}(c) = \begin{cases} 1 - (q-1)\left[\dfrac{\left(6q^{14}+9q^{13}+13q^{12}+16q^{11}+22(q^{10}+q^9+q^8)\right)}{6(q+1)(q^2+1)(q^4+1)(q^{10}-1)} + O\left(\dfrac{1}{q^{10}}\right) \right] & \textit{if } n = 1, \\[3ex] \dfrac{(q-1)(2q^{13}+3q^{11}+5q^9+5q^7+5q^5+3q^3+2q)}{2(q^2+1)(q^4+1)(q^{10}-1)} + O\left(\dfrac{1}{q^{4e+3}}\right) & \textit{if } n = 2, \\[3ex] \dfrac{(q-1)(q^{4e+8}+q^{4e+6}+q^{4e+4}+q^{4e+2}+q^{4e}-q^6+q^4-q^2)}{2q^{4e-2}(q^4+1)(q^{10}-1)} & \textit{if } n = 3, \\[3ex] \dfrac{(q-1)(q^{11}+q^9+q^7+q^5+q^3)}{6(q^2+1)(q^4+1)(q^{10}-1)} + O\left(\dfrac{1}{q^{4e+4}}\right) & \textit{if } n = 4, \\[3ex] \dfrac{(q-1)^2}{2q^{4e+n-8}(q^{10}-1)} & \textit{if } n \geq 5. \end{cases}$$

If $e > 2$ is odd, we have

$$\delta_{K,\mathfrak{p}}(c) = \begin{cases} 1-(q-1)\left[\dfrac{\left(6q^{14}+9q^{13}+13q^{12}+16q^{11}+22(q^{10}+q^9+q^8)\right)}{6(q+1)(q^2+1)(q^4+1)(q^{10}-1)} + O\left(\dfrac{1}{q^{10}}\right)\right] & \textit{if } n=1, \\ \dfrac{(q-1)(2q^{13}+3q^{11}+5q^9+5q^7+5q^5+3q^3)}{2(q^2+1)(q^4+1)(q^{10}-1)} + O\left(\dfrac{1}{q^{4e-2}}\right) & \textit{if } n=2, \\ \dfrac{(q-1)(q^{10}+q^8+q^6+q^4+q^2)}{2(q^4+1)(q^{10}-1)} + O\left(\dfrac{1}{q^{4e-1}}\right) & \textit{if } n=3, \\ \dfrac{(q-1)(q^{11}+q^9+q^7+q^5+q^3)}{6(q^2+1)(q^4+1)(q^{10}-1)} + O\left(\dfrac{1}{q^{4e+1}}\right) & \textit{if } n=4, \\ \dfrac{(q-1)^2}{2q^{4e+n-3}(q^{10}-1)} & \textit{if } n \geq 5. \end{cases}$$

The exact proportions are given in Appendix B.

Proof. Refer to Figure 4-1. Because we begin with a short Weierstrass form, we start at the node $F(\geq e)$. Fix a Kodaira type T and Tamagawa number c over which we compute the local density of curves with this data. We perform casework on the family we terminate in.

First, suppose that $e = 1$. The proportion of curves that terminate in $F(\geq 1)$ with Kodaira type T and Tamagawa number c is

$$\left(1 + \frac{1}{q^{10}} + \frac{1}{q^{20}} + \dots\right)\chi_{K,\mathfrak{p}}(T, c, \geq 1).$$

On the other hand, the proportion of curves that terminate in $F(0)$ with our prescribed local data is

$$\left(1 + \frac{1}{q^{10}} + \frac{1}{q^{20}} + \dots\right)\frac{q-1}{q^{10}}\chi_{K,\mathfrak{p}}(T, c, 0).$$

Now, suppose that $e \geq 2$. The proportion of curves that terminate in $F(2), F(3), \dots, F(\geq$

e) with Kodaira type T and Tamagawa number c is

$$\left(1+\frac{1}{q^{10}}+\frac{1}{q^{20}}+\dots\right)\left(1+\sum_{k=0}^{\lfloor(e-1)/2\rfloor}\frac{q-1}{q^{8k+10}}+\sum_{k=0}^{\lfloor(e-2)/2\rfloor}\frac{q-1}{q^{8k+9}}\right)\chi_{K,\mathfrak{p}}(T,c,\geq 2).$$

Second, the proportion of curves that terminate in $F(1)$ with our prescribed local data is

$$\begin{cases}\left(1+\frac{1}{q^{10}}+\frac{1}{q^{20}}\dots\right)\frac{q-1}{q^{4e+2}}\chi_{K,\mathfrak{p}}(T,c,1) & \text{if } 2\mid e\\ \left(1+\frac{1}{q^{10}}+\frac{1}{q^{20}}\dots\right)\frac{q-1}{q^{4e-3}}\chi_{K,\mathfrak{p}}(T,c,1) & \text{if } 2\nmid e\end{cases}.$$

Finally, the proportion of curves that terminate in $F(0)$ with our prescribed local data is

$$\begin{cases}\left(1+\frac{1}{q^{10}}+\frac{1}{q^{20}}\dots\right)\frac{q-1}{q^{4e+1}}\chi_{K,\mathfrak{p}}(T,c,0) & \text{if } 2\mid e\\ \left(1+\frac{1}{q^{10}}+\frac{1}{q^{20}}\dots\right)\frac{q-1}{q^{4e+6}}\chi_{K,\mathfrak{p}}(T,c,0) & \text{if } 2\nmid e\end{cases}.$$

The values $\delta_{K,\mathfrak{p}}(T,c)$ are provided in Theorem 5.2. We sum these proportions over all Kodaira types T with Tamagawa number c to get the total density $\delta_{K,\mathfrak{p}}(c)$. □

Chapter 5

Tate's Algorithm for $p = 2$

In this section, we calculate $\delta_{K,\mathfrak{p}}(c_\mathfrak{p})$ for $\mathfrak{p} \mid (2)$. Unlike in Chapter 3 and in Chapter 4, Tate's algorithm may introduce a non-zero a_1, a_2, and a_3 coefficient on non-minimal $E(a_4, a_6)$ that loops back into the algorithm. However, even if E is transformed after Step 11 into $E'(a_1', a_2', a_3', a_4', a_6')$, as in Equation (2.2), with $a_2' \neq 0$, the translation $x \mapsto x - a_2/3$ eliminates the a_2 coefficient of E' without changing the local data of the curve. Therefore, to study how $\mathfrak{p}-$non-minimal elliptic curves loop back into Tate's algorithm, we study the action of Tate's algorithm on the larger class of elliptic curves $E = E(a_1, a_3, a_4, a_6)$, defined as in Equation (2.2). By convention, we re-eliminate the a_2 coefficient after passing Step 11 before re-running Tate's algorithm.

Upon running Tate's algorithm, we find that the sets of elliptic curves $E(a_1, a_3, a_4, a_6)$ across (a_4, a_6) with fixed a_1 and a_3 behave similarly if they have the same $\alpha_1 := v_\pi(a_1)$ and $\alpha_3 := v_\pi(a_3)$. Moreover, the short Weierstrass elliptic curve are exactly the $E(a_1, a_3, a_4, a_6)$ with $\alpha_1 = \alpha_3 = \infty$. We thus group elliptic curves entering Tate's algorithm into families depending on α_1 and α_3 as follows.

Definition 5.1. The *2-family* $F(\alpha_1, \alpha_3)$ refers to the set of models

$$F(\alpha_1, \alpha_3) := \{E(a_1, a_3, a_4, a_6) : v_\pi(a_1) = \alpha_1, v_\pi(a_3) = \alpha_3;\ a_4, a_6 \text{ integral}\}.$$

The 2-family $F(\geq \alpha_1, \geq \alpha_3)$ refers to the set $\bigsqcup_{\alpha \geq \alpha_1} \bigsqcup_{\beta \geq \alpha_3} F(\alpha, \beta)$.

For brevity, we refer to 2-families as families for the rest of this section. The rest of the section is structured similarly to Chapter 4. In Theorem 5.2, we run Tate's algorithm to calculate $\psi_{K,\mathfrak{p}}(T,c;\alpha_1,\alpha_3) := \delta'_{K,\mathfrak{p}}(T,c;\alpha_1,\infty,\alpha_3)$, which is the proportion of $\mathfrak{p}-$minimal models with Kodaira type T and Tamagawa number c for each family $F(\alpha_1,\alpha_3)$. Then, we show in Lemmas 5.3, 5.4, 5.5, and 5.6 that the non-minimal models from certain families may themselves be viewed as a family. The analysis of non-minimal models in this section is more involved than the analysis in Chapter 4 for two reasons: there are now two relevant valuations α_1 and α_3, and we have to incorporate the shift $x \mapsto x - a_2/3$ after Step 11. Finally, in Theorem 5.7, we leverage these lemmas to form a Markov chain whose nodes are families and whose edges represent the reclassification of non-minimal models, which we use to compute the local proportion $\delta_{K,\mathfrak{p}}(c)$.

Lemma 5.2. *Suppose that $\mathfrak{p} \subseteq K$ is above 2 with ramification index e. Then for $F(\alpha_1,\alpha_3)$, the local densities $\psi_{K,\mathfrak{p}}(T,c_\mathfrak{p},\alpha_1,\alpha_3)$ is as provided in Table 5.1 for $e = 1,2$ and Table 5.2 for $e \geq 3$.*

Proof. We run through Tate's algorithm to compute $\psi_{\mathfrak{p},K}(T,c_\mathfrak{p},\alpha_1,\alpha_3)$. Recall that $q = N_{K/\mathbb{Q}}(\mathfrak{p})$ is the norm of $\mathfrak{p}$.

Case 1. E terminates at Step 1 if $\pi \nmid \Delta$. By definition, $\Delta \equiv b_2^2 b_8 + b_6^2 + b_2 b_4 b_6 \pmod{\pi}$, where $b_2 \equiv a_1^2 \pmod{\pi}$, $b_4 \equiv a_1 a_3 \pmod{\pi}$, $b_6 \equiv a_3^2 \pmod{\pi}$, and $b_8 \equiv a_1^2 a_6 - a_1 a_3 a_4 - a_4^2 \pmod{\pi}$. Therefore, $\Delta \equiv a_1^6 a_6 - a_1^5 a_3 a_4 - a_1^4 a_4^2 + a_3^4 + a_1^3 a_3^3 \pmod{\pi}$. If $\alpha_1 \geq 1$, then $\Delta \equiv a_3^4 \pmod{\pi}$. As such, $\psi_{K,\mathfrak{p}}(I_0,1;\alpha_1,\alpha_3) = 1$ if $\alpha_3 = 0$ and 0 if $\alpha_3 \geq 1$. Now, suppose that $\alpha_1 = 0$, in which case Δ is then linear in terms of a_6. Therefore, for each a_1 and a_3, there is one choice of a_6 modulo π such that E terminates at Step 1. We thus have $\psi_{K,\mathfrak{p}}(I_0,1;0,\geq 0) = \frac{q-1}{q}$.

Case 2. Suppose that the singular point of E is at (s,u) after reduction by π; accordingly, we shift the singular point to $(0,0)$ by $(x,y) \mapsto (x+s, y+u)$. Our model is now:

$$(y+u)^2 + a_1(x+s)(y+u) + a_3(y+u) = (x+s)^3 + a_4(x+s) + a_6. \tag{5.1}$$

To stop at Step 2, we require that $\pi \nmid b_2 = a_1^2 + 12s$. Therefore, if $\alpha_1 = 0$, then we always stop. By Hensel's lemma, exactly $\frac{q-1}{q^{n+1}}$ of curves have $v_\pi(\Delta) = n$. Also, for exactly half of

		$e=1$			$e=2$				
Type	$c_{\mathfrak{p}}$	$\alpha_1=0$	$\alpha_1\geq 1$ $\alpha_3=0$	$\alpha_1\geq 1$ $\alpha_3\geq 1$	$\alpha_1=0$	$\alpha_1\geq 1$ $\alpha_3=0$	$\alpha_1=1$ $\alpha_3\geq 1$	$\alpha_1\geq 2$ $\alpha_3=1$	$\alpha_1\geq 2$ $\alpha_3\geq 2$
I_0	1	$\frac{q-1}{q}$	1	0	$\frac{q-1}{q}$	1	0	0	0
I_1	1	$\frac{q-1}{q^2}$	0	0	$\frac{q-1}{q^2}$	0	0	0	0
I_2	2	$\frac{q-1}{q^3}$	0	0	$\frac{q-1}{q^3}$	0	0	0	0
$I_{n\geq 3}$	n	$\frac{q-1}{2q^{n+1}}$	0	0	$\frac{q-1}{2q^{n+1}}$	0	0	0	0
$I_{n\geq 3}$	$\varepsilon(n)$	$\frac{q-1}{2q^{n+1}}$	0	0	$\frac{q-1}{2q^{n+1}}$	0	0	0	0
II	1	0	0	$\frac{q-1}{q}$	0	0	$\frac{q-1}{q}$	$\frac{q-1}{q}$	$\frac{q-1}{q}$
III	2	0	0	$\frac{q-1}{q^2}$	0	0	$\frac{q-1}{q^2}$	$\frac{q-1}{q^2}$	$\frac{q-1}{q^2}$
IV	1	0	0	$\frac{q-1}{2q^3}$	0	0	$\frac{q-1}{2q^3}$	$\frac{1}{2q^2}$	0
IV	3	0	0	$\frac{q-1}{2q^3}$	0	0	$\frac{q-1}{2q^3}$	$\frac{1}{2q^2}$	0
I_0^*	1	0	0	$\frac{q^2-1}{3q^5}$	0	0	$\frac{q^2-1}{3q^5}$	0	$\frac{q^2-1}{3q^4}$
I_0^*	2	0	0	$\frac{q-1}{2q^4}$	0	0	$\frac{q-1}{2q^4}$	0	$\frac{q-1}{2q^3}$
I_0^*	4	0	0	$\frac{(q-1)(q-2)}{6q^5}$	0	0	$\frac{(q-1)(q-2)}{6q^5}$	0	$\frac{(q-1)(q-2)}{6q^4}$
I_n*	2	0	0	$\frac{(q-1)^2}{2q^{5+n}}$	0	0	$\frac{(q-1)^2}{2q^{5+n}}$	0	$\frac{(q-1)^2}{2q^{4+n}}$
I_n*	4	0	0	$\frac{(q-1)^2}{2q^{5+n}}$	0	0	$\frac{(q-1)^2}{2q^{5+n}}$	0	$\frac{(q-1)^2}{2q^{4+n}}$
$IV*$	1	0	0	$\frac{q-1}{2q^6}$	0	0	$\frac{q-1}{2q^6}$	0	$\frac{q-1}{2q^5}$
$IV*$	3	0	0	$\frac{q-1}{2q^6}$	0	0	$\frac{q-1}{2q^6}$	0	$\frac{q-1}{2q^5}$
$III*$	2	0	0	$\frac{q-1}{q^7}$	0	0	$\frac{q-1}{q^7}$	0	$\frac{q-1}{q^6}$
$II*$	1	0	0	$\frac{q-1}{q^8}$	0	0	$\frac{q-1}{q^8}$	0	$\frac{q-1}{q^7}$

Table 5.1: The $\psi_{K,\mathfrak{p}}(T,c;\alpha_1,\infty,\alpha_3)$ for $\mathfrak{p}\mid(2)$.

$e \geq 3$								
Type	$c_{\mathfrak{p}}$	$\alpha_1 = 0$	$\alpha_1 \geq 1$ $\alpha_3 = 0$	$\alpha_1 = 1$ $\alpha_3 \geq 1$	$\alpha_1 \geq 2$ $\alpha_3 = 1$	$\alpha_1 = 2$ $\alpha_3 \geq 2$	$\alpha_1 \geq 3$ $\alpha_3 = 2$	$\alpha_1 \geq 3$ $\alpha_3 \geq 3$
I_0	1	$\frac{q-1}{q}$	1	0	0	0	0	0
I_1	1	$\frac{q-1}{q^2}$	0	0	0	0	0	0
I_2	2	$\frac{q-1}{q^3}$	0	0	0	0	0	0
$I_{n\geq 3}$	n	$\frac{q-1}{2q^{n+1}}$	0	0	0	0	0	0
$I_{n\geq 3}$	$\varepsilon(n)$	$\frac{q-1}{2q^{n+1}}$	0	0	0	0	0	0
II	1	0	0	$\frac{q-1}{q}$	$\frac{q-1}{q}$	$\frac{q-1}{q}$	$\frac{q-1}{q}$	$\frac{q-1}{q}$
III	2	0	0	$\frac{q-1}{q^2}$	$\frac{q-1}{q^2}$	$\frac{q-1}{q^2}$	$\frac{q-1}{q^2}$	$\frac{q-1}{q^2}$
IV	1	0	0	$\frac{q-1}{2q^3}$	$\frac{1}{2q^2}$	0	0	0
IV	3	0	0	$\frac{q-1}{2q^3}$	$\frac{1}{2q^2}$	0	0	0
I_0^*	1	0	0	$\frac{q^2-1}{3q^5}$	0	$\frac{q^2-1}{3q^4}$	$\frac{q^2-1}{3q^4}$	$\frac{q^2-1}{3q^4}$
I_0^*	2	0	0	$\frac{q-1}{2q^4}$	0	$\frac{q-1}{2q^3}$	$\frac{q-1}{2q^3}$	$\frac{q-1}{2q^3}$
I_0^*	4	0	0	$\frac{(q-1)(q-2)}{6q^5}$	0	$\frac{(q-1)(q-2)}{6q^4}$	$\frac{(q-1)(q-2)}{6q^4}$	$\frac{(q-1)(q-2)}{6q^4}$
I_n*	2	0	0	$\frac{(q-1)^2}{2q^{5+n}}$	0	$\frac{(q-1)^2}{2q^{4+n}}$	$\frac{(q-1)^2}{2q^{4+n}}$	$\frac{(q-1)^2}{2q^{4+n}}$
I_n*	4	0	0	$\frac{(q-1)^2}{2q^{5+n}}$	0	$\frac{(q-1)^2}{2q^{4+n}}$	$\frac{(q-1)^2}{2q^{4+n}}$	$\frac{(q-1)^2}{2q^{4+n}}$
$IV*$	1	0	0	$\frac{q-1}{2q^6}$	0	$\frac{q-1}{2q^5}$	$\frac{1}{2q^4}$	0
$IV*$	3	0	0	$\frac{q-1}{2q^6}$	0	$\frac{q-1}{2q^5}$	$\frac{1}{2q^4}$	0
$III*$	2	0	0	$\frac{q-1}{q^7}$	0	$\frac{q-1}{q^6}$	0	$\frac{q-1}{q^5}$
$II*$	1	0	0	$\frac{q-1}{q^8}$	0	$\frac{q-1}{q^7}$	0	$\frac{q-1}{q^6}$

Table 5.2: The $\psi_{K,\mathfrak{p}}(T, c; \alpha_1, \infty, \alpha_3)$ for $\mathfrak{p} \mid (2)$.

E, $T^2 + a_1T - 3s$ splits in k. Hence, $\psi_{K,\mathfrak{p}}(I_n, n; 0, \geq 0) = \psi_{K,\mathfrak{p}}(I_n, \varepsilon(n); 0, \geq 0) = \frac{q-1}{2q^{n+1}}$. If $\alpha_1 \geq 1$, then $\alpha_3 \geq 1$ by Case 1. In this case, we always pass. Thus, $\psi_{K,\mathfrak{p}}(I_n, c_{\mathfrak{p}}; \alpha_1, \alpha_3) = 0$ for all $\alpha_1, \alpha_3 \geq 1$. Henceforth, $\alpha_1, \alpha_3 \geq 1$. By taking partial derivatives of Equation (5.1), we find that $s^2 \equiv a_4 \pmod{\pi}$ and $u^2 \equiv a_6 \pmod{\pi}$.

Case 3. Suppose that E reaches Step 3. We stop if $\pi \nmid \pi^{-1}(s^3+a_4s+a_6-u^2-a_1su-a_3u)$. For fixed a_1, a_3, and a_4, there are $q(q-1)$ choices of a_6 modulo π^2. Hence, $\psi_{K,\mathfrak{p}}(II, 1; \alpha_1, \alpha_3) = \frac{q-1}{q}$ for each $\alpha_1, \alpha_3 \geq 1$.

Case 4. E terminates at this step if $\pi^3 \nmid (3s)(2u + a_1s + a_3)^2 - (3s^2 + a_4 - a_1u)^2$. By Step 2, we know that $\pi^2 \mid (3s)(2u + a_1s + a_3)^2 - (3s^2 + a_4 - a_1u)^2$. Thus, we want $\pi \nmid \pi^{-2}(3s(2u + a_1s + a_3)^2 + (3s^2 + a_4 - a_1u)^2)$. Therefore, for fixed a_1 and a_3, there are $q(q-1)$ choices for a_4 modulo π^2 and q choices for a_6 modulo π^2. Thus, $\psi_{K,\mathfrak{p}}(III, 2; \alpha_1, \alpha_3) = \frac{q-1}{q^2}$ for $\alpha_1, \alpha_3 \geq 1$.

Case 5. For an elliptic curve to terminate at Step 5, it must be that $\pi \nmid \pi^{-1}(2u + a_3 + a_1s)$. If $e = 1$, for fixed a_1 and a_3, we have q choices of a_4 modulo π^2 and $q - 1$ choices of a_6 modulo π^2. Now, the Tamagawa number depends on whether the polynomial $Y^2 + \frac{2u+a_3+a_1}{\pi}Y - \frac{a_6+a_4s+s^3-u^2-a_1su-a_3u}{\pi^2}$ modulo π factors over $\mathbb{F}_q$. To count, we will fix a_1, a_3, and a_4 and count over the $q(q-1)$ possible a_6 modulo π^3. Suppose that the polynomial has a root in $\mathbb{F}_q$ and fix one of the roots. Then, since the trace is fixed, the other root is fixed. Because $\pi^{-1}(2u + a_3 + a_1)$ is non-zero modulo π, the two roots must be distinct. Therefore, there are $\frac{q(q-1)}{2}$ choices of a_6 modulo π^3 that each results in Tamagawa number 1 and 3. Thus, $\psi_{K,\mathfrak{p}}(IV, 1; \geq 1, \geq 1) = \psi_{K,\mathfrak{p}}(IV, 3; \geq 1, \geq 1) = \frac{q-1}{2q^3}$ when $e = 1$. Now, suppose that $e \geq 2$. First, if $\alpha_1 = 1$, then we have $q - 1$ choices of a_4 modulo π^2 and q choices for a_6 modulo π^2. With the same argument as above, we conclude that for $\frac{q^2}{2}$ choices of a_6 modulo π^3, the Tamagawa number is 1, and for the same number of choices, the Tamagawa number is 3. Now, suppose that $\alpha_1 \geq 2$. If $\alpha_3 = 1$, then the elliptic curves always terminate at this step. Again, we have that for half of the choices of a_6, the Tamagawa number is 1 and that for the other half, the Tamagawa number is 3. Therefore, we have that $\psi_{K,\mathfrak{p}}(IV, 1; \geq 2, 1) = \psi_{K,\mathfrak{p}}(IV, 3; \geq 2, 1) = \frac{1}{2q^2}$. Conversely, if $\alpha_3 \geq 2$, then no curves terminate at this step. Therefore, $\psi_{K,\mathfrak{p}}(IV, 1; \geq 2, \geq 2) = \psi_{K,\mathfrak{p}}(IV, 3; \geq 2, \geq 2) = 0$.

Case 6. Let $t^2 \equiv s \pmod{\pi}$ and $\beta^2 \equiv \pi^{-2}(s^3 + a_4s + a_6 - u^2 - a_1su - a_3u) \pmod{\pi}$, and define $v := u + \beta\pi$. Following the shifts outlined in Step 6 of Tate's algorithm, we have

$$(y + tx + v)^2 + a_1(x + t^2)(y + tx + v) + a_3(y + tx + v) = (x + t^2)^3 + a_4(x + t^2) + a_6. \quad (5.2)$$

We study $P(T) = T^3 + \frac{2t^2 - a_1 t}{\pi}T^2 + \frac{3t^4 + a_4 - 2tv - a_1t^3 - a_1v - a_3t}{\pi^2}T + \frac{t^6 + a_4t^2 + a_6 - v^2 - a_1t^2v - a_3v}{\pi^3}$. Define A_2, A_4, and A_6 such that $P(Y) \equiv T^3 + A_2T^2 + A_4T + A_6 \pmod{\pi}$. Now, suppose that we fix a_4 modulo π^2 and a_6 modulo π^3. There then exists a bijective map between the π possible values of a_4 modulo π^3 and A_4 modulo π and between the π possible values of a_6 modulo π^4 and A_6 modulo π. For E to terminate at Step 6, $P(T)$ must have three distinct roots. If so, $P(T)$ and $P'(T) \equiv 3T^2 + A_4 \equiv 0 \pmod{\pi}$ should not have shared roots. Therefore, for $P(T)$ to have three distinct roots, $A_2A_4 \not\equiv A_6 \pmod{\pi}$. Thus, for each A_2 modulo π, there are $q(q-1)$ choices of (A_4, A_6) modulo π. We also note that when $P(T)$ has three distinct roots, none of the roots can be A_2 as if so, the remaining two roots must be the same—a contradiction to $P(T)$ having distinct roots.

We now fix A_2 and count the number of $P(T)$ with three distinct roots that have three, one, and no roots in $\mathbb{F}_q$ over (A_4, A_6) modulo π. Because we fix A_2 modulo π, the trace of $P(T)$ is fixed. We first count the number of $P(T)$ that have all three roots in $\mathbb{F}_q$ with fixed trace. We have $q-1$ choices for the first root, $q-2$ choices for the second root, and a fixed choice for the third root, because as long as none of the roots are congruent to A_2 modulo π, the three roots are distinct. Therefore, for fixed A_2, there are $\frac{(q-1)(q-2)}{6}$ choices of (A_4, A_6) modulo π that allows for $P(T)$ to have three distinct roots, all of which are in $\mathbb{F}_q$. We now proceed to count the number of $P(T)$ with three distinct roots with exactly one root in $\mathbb{F}_q$ with fixed a_2. We start by choosing one of $\frac{q^2-q}{2}$ irreducible quadratics. The root in $\mathbb{F}_q$ is then fixed as the trace of $P(T)$ is fixed. Therefore, there are a total of $\frac{q^2-q}{2}$ choices of $P(T)$ with three distinct roots, exactly one root of which is in $\mathbb{F}_q$. Lastly, we count the number of irreducible cubics with three distinct roots. Out of the q^3 elements in $\mathbb{F}_{q^3}$, q are in $\mathbb{F}_q$. Because the traces are equally distributed, for a fixed trace, there are $\frac{q^3-q}{q} = q^2 - 1$ elements with that fixed trace. Since $P(T)$ is a cubic, there are $\frac{q^2-1}{3}$ irreducible cubics with trace A_2.

Now, suppose that $e = 1$. From Steps 1 and 2, we have that $\alpha_1, \alpha_3 \geq 1$. Now, for each fixed $(a_4, \frac{a_1}{\pi})$ modulo π, we have $\frac{(q-1)(q-2)}{6}$ $P(T)$ with three distinct roots all in $\mathbb{F}_q$, $\frac{q(q-1)}{2}$

$P(T)$ with exactly one of the three distinct roots in $\mathbb{F}_q$, and $\frac{q^2-1}{2}$ $P(T)$ with three distinct roots, none of which are in $\mathbb{F}_q$. Thus, we have that for $\psi_{K,\mathfrak{p}}(I_0^*, 4; \geq 1, \geq 1) = \frac{(q-1)(q-2)}{6q^5}$, $\psi_{K,\mathfrak{p}}(I_0^*, 2; \geq 1, \geq 1) = \frac{(q-1)}{2q^4}$, and $\psi_{K,\mathfrak{p}}(I_0^*, 1; \geq 1, \geq 1) = \frac{q^2-1}{3q^5}$. Suppose that $e \geq 2$. If $\alpha_1 = 1$, then $\alpha_3 \geq 1$ from Step 1. Then, A_2 modulo π forms a bijective map with $\frac{a_1}{\pi}$ modulo π. Therefore, by our aforementioned counting of $P(T)$ with three distinct roots, a fixed number of which are in $\mathbb{F}_q$, we have that for $\psi_{K,\mathfrak{p}}(I_0^*, 4; 1, \geq 1) = \frac{(q-1)(q-2)}{6q^5}$, $\psi_{K,\mathfrak{p}}(I_0^*, 2; 1, \geq 1) = \frac{(q-1)}{2q^4}$, and $\psi_{K,\mathfrak{p}}(I_0^*, 1; 1, \geq 1) = \frac{q^2-1}{3q^5}$. If $\alpha_1 \geq 2$, then $\alpha_3 \geq 2$ from Step 6. Then, $A_2 \equiv 0 \pmod{\pi}$. Therefore, by our aforementioned counting of $P(T)$ with a fixed number of roots in $\mathbb{F}_q$, we have that for $\psi_{K,\mathfrak{p}}(I_0^*, 4; \alpha_1, \alpha_3) = \frac{(q-1)(q-2)}{6q^4}$, $\psi_{K,\mathfrak{p}}(I_0^*, 2; \alpha_1, \alpha_3) = \frac{(q-1)}{2q^3}$, and $\psi_{K,\mathfrak{p}}(I_0^*, 1; \alpha_1, \alpha_3) = \frac{q^2-1}{3q^4}$ for $\alpha_1, \alpha_3 \geq 2$.

Case 7. E terminates at Step 7 if $A_2A_4 \equiv A_6 \pmod{\pi}$ and $(A_4, A_6) \not\equiv (A_2^2, A_2^3) \pmod{\pi}$. We study the interaction between quadratics $R(Y) = Y^2 + a'_{3,*}Y - a'_{6,*}$ and $S(X) = a'_{2,*}X^2 + a'_{4,*}X + a'_{6,*}$, translating the curve as we move between them. Note that by varying a_6, the quantity $a'_{6,*}$ is surjective modulo π. As before, since a_2, a_3, a_4, a_6 are equidistributed, by Hensel's lemma, there are $\frac{q-1}{q^n}$ residues for which we have Kodaira type I_n, and moreover, half of these cause the quadratic in question to split. Hence, $\psi_{K,\mathfrak{p}}(I_n^*, 1, \geq 1, \geq 1) = \psi_{K,\mathfrak{p}}(I_n^*, 3, \geq 1, \geq 1) = \frac{q-1}{2q^{5+n}}$ for $e = 1$ and $\psi_{K,\mathfrak{p}}(I_n^*, 1, \geq 2, \geq 2) = \psi_{K,\mathfrak{p}}(I_n^*, 3, \geq 2, \geq 2) = \frac{q-1}{2q^{4+n}}$ for $e \geq 2$.

Case 8. Suppose that E reaches Step 8. Then $(A_2, A_4, A_6) \equiv (A_2, A_2^2, A_2^3) \pmod{\pi}$. Perform $x \mapsto x + \pi A_2 = x + (2t^2 - a_1t)$ and let $v' := v + 2t^3 - a_1t^2$ to get the penultimate model

$$(y+tx+v')^2+a_1(x+3t^2-a_1t)(y+tx+v')+a_3(y+tx+v') = (x+3t^2-a_1t)^3+a_4(x+3t^2-a_1t)+a_6. \tag{5.3}$$

We stop if $\pi \nmid \pi^{-2}(2v' + 3a_1t^2 - a_1^2t + a_3)$. We notice that if we fix a_1, a_4, and a_6, then a_3 modulo π^2 is fixed such that $2v' + 3a_1t^2 - a_1^2t + a_3$ is a multiple of π^2. We then notice that $\pi^{-2}(2v' + 3a_1t^2 - a_1^2t + a_3)$ modulo π forms a bijective map with the q possible values of a_3 modulo π^3.

First, suppose that $e = 1$. From Steps 1 and 2, we have that $\alpha_1, \alpha_3 \geq 1$. For each a_3 modulo π^2, we see that for $\frac{q}{2}$ possible values of a_3 modulo π^3, E terminates with Tamagawa number 1 and that for $\frac{q}{2}$ choices for a_3 moudlo π^3, E terminates with Tamagawa number 3. Therefore, when $e \geq 1$, $\psi_{K,\mathfrak{p}}(IV^*, 1; \geq 1, \geq 1) = \psi_{K,\mathfrak{p}}(IV^*, 3; \geq 1, \geq 1) = \frac{q-1}{2q^6}$. For the same reasons, we conclude that when $e = 2$ and $\alpha_1 = 1$ and $\alpha_3 \geq 1$, $\psi_{K,\mathfrak{p}}(IV^*, 1; 1, \geq 1) = \psi_{K,\mathfrak{p}}(IV^*, 3; 1, \geq 1) = \frac{q-1}{2q^6}$. We also see in the same way that $\psi_{K,\mathfrak{p}}(IV^*, 1; \geq 2, \geq 2) = \psi_{K,\mathfrak{p}}(IV^*, 3; \geq 2, \geq 2) = \frac{q-1}{2q^5}$.

Now, suppose that $e \geq 3$. When $\alpha_1 = 1$ and $\alpha_3 \geq 1$, we conclude as we did in the previous paragraph that $\psi_{K,\mathfrak{p}}(IV^*, 1; 1, \geq 1) = \psi_{K,\mathfrak{p}}(IV^*, 3; 1, \geq 1) = \frac{q-1}{2q^6}$. We also see in the same way that $\psi_{K,\mathfrak{p}}(IV^*, 1; \geq 2, \geq 2) = \psi_{K,\mathfrak{p}}(IV^*, 3; \geq 2, \geq 2) = \frac{q-1}{2q^6}$. Similarly, when $\alpha_1 = 2$ and $\alpha_3 \geq 2$, we have that $\psi_{K,\mathfrak{p}}(IV^*, 1; 2, \geq 2) = \psi_{K,\mathfrak{p}}(IV^*, 3; 2, \geq 2) = \frac{q-1}{2q^5}$. But when $\alpha_1 \geq 3$ and $\alpha_3 = 2$, then E necessarily terminates at Case 8. Then depending on $\frac{a_3}{\pi^2}$, E has Tamagawa number 1 and 3 with equal proportions. Therefore, we have that $\psi_{K,\mathfrak{p}}(IV^*, 1; \geq 3, 2) = \psi_{K,\mathfrak{p}}(IV^*, 3; \geq 3, 2) = \frac{1}{2q^4}$. If $\alpha_1, \alpha_3 \geq 3$, however, no E terminates at this step. Therefore, $\psi_{K,\mathfrak{p}}(IV^*, 1; \geq 3, 2) = \psi_{K,\mathfrak{p}}(IV^*, 3; \geq 3, 2) = 0$.

Case 9. Let $w^2 \equiv \pi^{-4}((3t^2 - a_1t)^3 + a_4(3t^2 - a_1t) + a_6 - v'^2 + a_1^2tv' - a_3v') \pmod{\pi}$ and let $w := \pi^2 w' + v'$. Then, we have the final model

$$(y+tx+w)^2+a_1(x+3t^2-a_1t)(y+tx+w)+a_3(y+tx+w) = (x+3t^2-a_1t)^3+a_4(x+3t^2-a_1t)+a_6. \tag{5.4}$$

We terminate at this case if $\pi^4 \nmid -2tw - a_1w - a_1^2t^2 - a_3t + 3(3t^2 - a_1t)^2 + a_4$. From Step 8, we have that $\pi^3 \mid -2tw - a_1w - a_1^2t^2 - a_3t + 3(3t^2 - a_1t)^2 + a_4$. Therefore, we want $\pi \mid \pi^{-3}(-2tw - a_1w - a_1^2t^2 - a_3t + 3(3t^2 - a_1t)^2 + a_4)$. For fixed a_4 modulo π^3, $\pi^{-3}(-2tw - a_1w - a_1^2t^2 - a_3t + 3(3t^2 - a_1t)^2 + a_4)$ forms a bijective map with the q possible values of a_4 modulo π^4. When $e = 1$, $\alpha_1, \alpha_3 \geq 1$ from Steps 1 and 2. Therefore, $\psi_{K,\mathfrak{p}}(III^*, 2; \geq 1, \geq 1) = \frac{q-1}{q^7}$. When $e = 2$, we have from Steps 1, 2, and 5 that either $\alpha_1 = 1$ and $\alpha_3 \geq 1$ or $\alpha_1, \alpha_3 \geq 2$. We thus conclude, $\psi_{K,\mathfrak{p}}(III^*, 2; 1, \geq 1) = \frac{q-1}{q^7}$ and $\psi_{K,\mathfrak{p}}(III^*, 2; \geq 2, \geq 2) = \frac{q-1}{q^6}$. Lastly, when $e \geq 3$, we have that either $\alpha_1 = 1$ and $\alpha_3 \geq 1$, $\alpha_1 = 2$ and $\alpha_3 \geq 2$, and $\alpha_1 \geq 3$ and $\alpha_3 \geq 3$. We similarly conclude that $\psi_{K,\mathfrak{p}}(III^*, 2; 1, \geq 1) = \frac{q-1}{q^7}$, $\psi_{K,\mathfrak{p}}(III^*, 2; 2, \geq 2) = \frac{q-1}{q^6}$, and $\psi_{K,\mathfrak{p}}(III^*, 2; \geq 3, \geq 3) = \frac{q-1}{q^5}$.

Case 10. E terminates at Step 10 if $\pi^6 \nmid -w^2 + a_1^2tw - a_3w + (3t^2 - a_1t)^3 - a_1a_4t + a_6$. From Step 9, we have that $\pi^5 \mid -w^2 + a_1^2tw - a_3w + (3t^2 - a_1t)^3 - a_1a_4t + a_6$. Therefore, we want that $\pi \mid \pi^{-5}(-w^2 + a_1^2tw - a_3w + (3t^2 - a_1t)^3 - a_1a_4t + a_6)$. Fix a_6 modulo π^5. Then, note that $\pi^{-5}(-w^2 + a_1^2tw - a_3w + (3t^2 - a_1t)^3 - a_1a_4t + a_6)$ modulo π forms a bijective map with the q possible values of a_6 modulo π^6. For $q-1$ of the q possible values of a_6 modulo π^6, E terminates at Case 10. When $e=1$, $\alpha_1, \alpha_3 \geq 1$ from Steps 1 and 2. Therefore, $\psi_{K,\mathfrak{p}}(II^*, 1; \geq 1, \geq 1) = \frac{q-1}{q^8}$. When $e = 2$, we have from Steps 1, 2, and 5 that either $\alpha_1 = 1$ and $\alpha_3 \geq 1$ or $\alpha_1, \alpha_3 \geq 2$. We thus conclude, $\psi_{K,\mathfrak{p}}(II^*, 1; 1, \geq 1) = \frac{q-1}{q^8}$ and $\psi_{K,\mathfrak{p}}(II^*, 2; \geq 2, \geq 2) = \frac{q-1}{q^7}$. Lastly, when $e \geq 3$, we have that either $\alpha_1 = 1$ and $\alpha_3 \geq 1$, $\alpha_1 = 2$ and $\alpha_3 \geq 2$, and $\alpha_1 \geq 3$ and $\alpha_3 \geq 3$. We similarly conclude that $\psi_{K,\mathfrak{p}}(II^*, 1; 1, \geq 1) = \frac{q-1}{q^8}$, $\psi_{K,\mathfrak{p}}(II^*, 1; 2, \geq 2) = \frac{q-1}{q^7}$, and $\psi_{K,\mathfrak{p}}(II^*, 1; \geq 3, \geq 3) = \frac{q-1}{q^6}$.

Case 11. For E to reach Step 11, it must not have terminated at a previous step. Therefore, we check that when $e = 1$, $\alpha_1, \alpha_3 \geq 1$, the proportion of non-minimal curves is $\frac{1}{q^8}$, when $e = 2$, $\alpha_1 = 1$, and $\alpha_3 \geq 1$, the proportion of non-minimal curves is $\frac{1}{q^8}$ as well, and that when $e = 2$ and $\alpha_1, \alpha_3 \geq 2$, the proportion of non-minimal curves is $\frac{1}{q^7}$. When $e \geq 3$, the proportion of non-minimal curves equal $\frac{1}{q^8}$ when $\alpha_1 = 1$ and $\alpha_3 \geq 1$, $\frac{1}{q^7}$ when $\alpha_1 = 2$ and $\alpha_3 \geq 2$, and $\frac{1}{q^6}$ when $\alpha_1, \alpha_3 \geq 3$. $\square$

We now show how we reclassify the non-minimal models of one family as another family of curves. We first show that, for $\alpha_3 < \alpha_1 < e$, the local densities at the non-minimal models of a family $F(\alpha_1, \alpha_3)$ exactly match the local densities at the family $F(\alpha_1 - 1, \alpha_3 - 3)$. To do this, we establish a map which sends a non-minimal model in $F(\alpha_1, \alpha_3)$ to another isomorphic model in $F(\alpha_1 - 1, \alpha_3 - 3)$, induced by the transformation at Step 11 followed by the shift $x \mapsto x - a_2/3$.

Lemma 5.3. *Let $\alpha_3 < \alpha_1 < e$. There is a surjective, q^4-to-1 map between the set of non-minimal models in $F(\alpha_1, \alpha_3)$ and the set $F(\alpha_1 - 1, \alpha_3 - 3)$ that sends E to its transformation E' after passing Step 11.*

Proof. Recall from Step 11 of Tate's algorithm that for non-minimal $E(a_1, a_2, a_3, a_4, a_6)$, Tate's algorithm produces a unique residue $t \pmod{\pi}$ and $w \pmod{\pi^3}$ for which

$$\begin{aligned}\widehat{E}(\widehat{a_1}, \widehat{a_2}, \widehat{a_3}, \widehat{a_4}, \widehat{a_6}) &:= (y + tx + w)^2 + a_1(x + 3t^2 - a_1 t)(y + tx + w) + a_3(y + tx + w) \\ &= (x + 3t^2 - a_1 t)^3 + a_4(x + 3t^2 - a_1 t) + a_6\end{aligned}$$

has the coefficient of y and xy divisible by π and π^3, respectively, and the coefficient of x^i

divisible by π^i for $i = 2, 4, 6$. Hence, each non-minimal model $E(a_1, a_2, a_3, a_4, a_6) \in F(\alpha_1, \alpha_3)$ is sent to

$$\widehat{E}\left(\frac{a_1 + 2t}{\pi}, \frac{8t^2 - 4a_1 t}{\pi^2}, \frac{2w + 3a_1 t^2 - a_1^2 t + a_3}{\pi^3}, \frac{-2tw - a_1 w - 3a_1 t^3 + a_1^2 t^2 - a_3 t + 3(3t^2 - a_1 t^2) + a_4}{\pi^4}, \frac{-w^2 - a_1 w(3t^2 - a_1 t) - a_3 w + (3t^2 - a_1 t)^3 + a_4(3t^2 - a_1 t) + a_6}{\pi^6}\right),$$

with $v_\pi(2t + a_1) = v_\pi(a_1) = \alpha_1$ and $v_\pi(2w + 3a_1 t^2 - a_1^2 t + a_3) = v_\pi(a_3) = \alpha_3$. Now, we perform $x \to x - \frac{\widehat{a_2}}{3}$ to $\widehat{E}$ and transform $\widehat{E}$ to E' :

$$E'(a_1', a_3', a_4', a_6') := y^2 + \widehat{a_1}\left(x - \frac{\widehat{a_2}}{3}\right) y + \widehat{a_3} y = \left(x - \frac{\widehat{a_2}}{3}\right)^3 + \widehat{a_2}\left(x - \frac{\widehat{a_2}}{3}\right)^2 + \widehat{a_4}\left(x - \frac{\widehat{a_2}}{3}\right) + \widehat{a_6}$$

We now have

$$E'\left(\widehat{a_1}, -\frac{\widehat{a_1}\widehat{a_2}}{3} + \widehat{a_3}, \frac{-\widehat{a_2}^2}{3} + \widehat{a_4}, \frac{2\widehat{a_2}^3}{27} - \frac{\widehat{a_2}\widehat{a_4}}{3} + \widehat{a_6}\right) \in F(\alpha_1 - 1, \alpha_3 - 3).$$

The two transformations are well-defined, so the map is also well-defined.

Conversely, given a model $E'(a_1', a_3', a_4', a_6') \in F(\alpha_1 - 1, \alpha_3 - 3)$, pick a pair of residues $t \pmod{\pi}$ and $w \pmod{\pi^3}$. Then, there is a unique choice of $\widehat{a_2}$ for which

$$\widehat{E}(\widehat{a_1}, \widehat{a_2}, \widehat{a_3}, \widehat{a_4}, \widehat{a_6}) : y^2 + a_1'\left(x + \frac{\widehat{a_2}}{3}\right) y + a_3' y = \left(x + \frac{\widehat{a_2}}{3}\right)^3 + a_4'\left(x - \frac{\widehat{a_2}}{3}\right) + a_6'$$

and $(\widehat{a_1}, \widehat{a_2}) = \left(\frac{a_1 + 2t}{\pi}, \frac{8t^2 - 4a_1 t}{\pi^2}\right)$ for some a_1; this ensures that $\widehat{E}$ has some preimage $E \in F(\alpha_1, \alpha_3)$. In fact, there is a unique preimage E for which $E \mapsto \widehat{E}$ after Step 11. Hence, after varying t and w across a set of q and q^3 representatives, respectively, we have shown the aforementioned map is q^4-to-1, as we had sought. □

On the other hand, in the case of $\alpha_1 < e$ but $\alpha_3 \geq \alpha_1$, we may reclassify the non-minimal models of $F(\alpha, \geq \alpha)$ as exactly the models of $F(\alpha - 1, \geq \alpha - 3)$. Thus, the local densities of curves for each of the two sets are the same. The proof is analogous to that of Theorem 5.3 and is thus omitted.

Lemma 5.4. *Let $\alpha < e$. There is a surjective, q^4-to-1 map between the set of non-minimal models in $F(\alpha, \geq \alpha)$ and the set $F(\alpha - 1, \geq \alpha - 3)$ that sends E to its transformation E' after passing Step 11.*

Similarly, in the case of $\alpha_1 \geq e$ but $\alpha_3 < e$, we may reclassify the non-minimal models of $F(\geq e, \alpha_3)$ as exactly the models of $F(\geq e - 1, \alpha_3 - 3)$. The proof is analogous to that of Theorem 5.3 and is thus omitted.

Lemma 5.5. *Let $\alpha < e$. There is a surjective, q^4-to-1 map between the set of non-minimal models in $F(\alpha, \geq \alpha)$ and the set $F(\alpha - 1, \geq \alpha - 3)$, sending E to its transformation E' after passing Step 11.*

Finally, it remains to classify the non-minimal models in the case of $\alpha_1, \alpha_3 \geq e$, and in particular, the case of short Weierstrass models $F(\infty, \infty)$. As in Chapter 4, it turns out the local data of the family $F(\infty, \infty)$ exactly matches that of $F(\geq e, \geq e)$ in the following sense. Note, once again, that linear transformations do not change the local data of an elliptic curve. Therefore, without loss of generality, instead of computing the local density on short Weierstrass forms $F(\infty, \infty)$, we can compute the local density at the set

$$\left\{\left(E : (y + tx + s)^2 = \left(x + \frac{t^2}{3}\right)^3 + a_4\left(x + \frac{t^2}{3}\right) + a_6\right) : s, t, a_4, a_6 \text{ integral}\right\}. \tag{5.5}$$

The set in (5.5) is exactly the set $F(\geq e, \geq e)$, up to equal multiplicity at each model, so we work with these models instead. We show that the non-minimal models of $F(\geq e, \geq e)$ are exactly the models of $F(\geq e - 1, \geq e - 3)$ by establishing the aforementioned map. The proof is analogous to that of Theorem 5.3 and is thus omitted.

Lemma 5.6. *There is a surjective, q^4-to-1 map between the non-minimal models in $F(\geq e, \geq e)$ and the set $F(\geq e - 1, \geq e - 3)$, sending E to its transformation E' after passing Step 11.*

We finish by using our lemmas to compute $\delta_{K,\mathfrak{p}}(c)$ by forming a Markov chain amongst families of curves. Lemmas 5.3, 5.4, 5.5, and 5.6 establish the edges between these families, while Lemma 5.2 establishes the local densities at each node. This information is collated in Figures 5-1, 5-2, and 5-3.

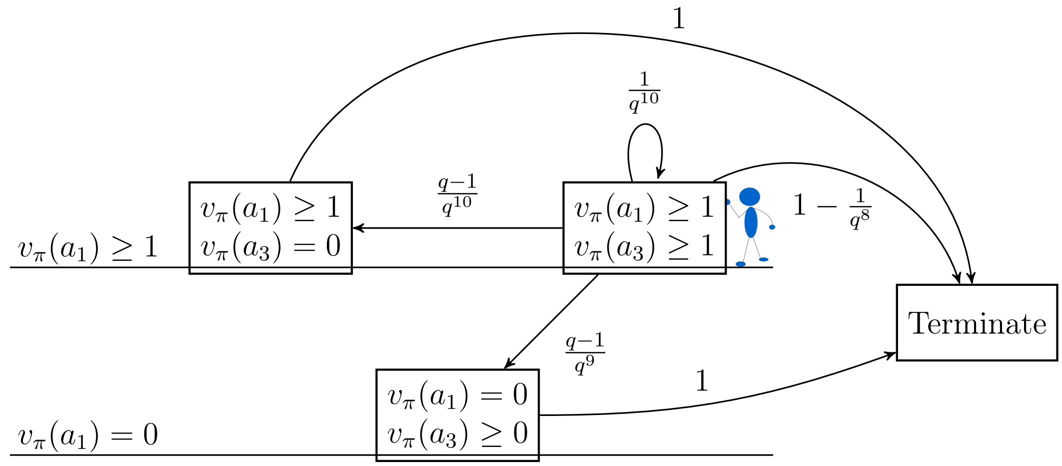


Figure 5-1: The Markov Chain structure when $\mathfrak{p} \mid (2)$ for $e = 1$.

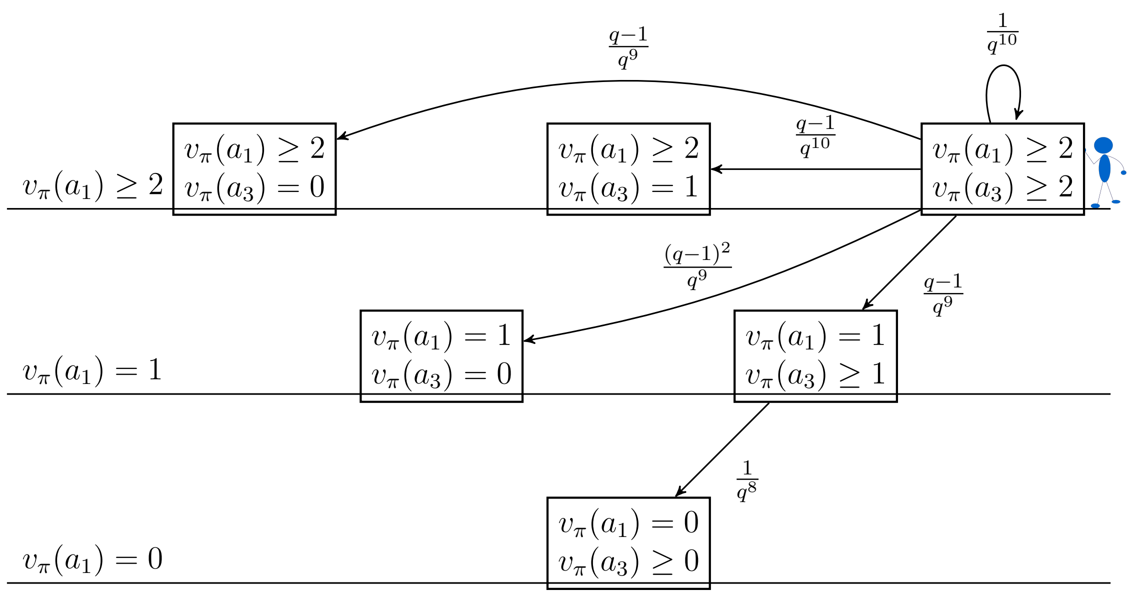


Figure 5-2: The Markov Chain structure when $\mathfrak{p} \mid (2)$ for $e = 2$.

Proposition 5.7. *For $\mathfrak{p} \mid (2)$ is a prime ideal in K and $n \geq 1$, let $q := N_{K/\mathbb{Q}}(\mathfrak{p})$. If $e = 1$,* *we have*

$$\delta_{K,\mathfrak{p}}(c) = \begin{cases} 1 - \dfrac{(q-1)(6q^{10}+9q^9+7q^8+8q^7+7q^6+9q^5+6q^4+6q+3)}{6q(q+1)(q^{10}-1)} & \textit{if } c = 1, \\[2ex] \dfrac{(q-1)(2q^{11}+2q^{10}+q^9+2q^8+q^7+2q^6+2q^5+2q^2-1)}{2q^2(q+1)(q^{10}-1)} & \textit{if } c = 2, \\[2ex] \dfrac{(q-1)(q^{10}+q^7+q-1)}{2q^3(q^{10}-1)} & \textit{if } c = 3, \\[2ex] \dfrac{(q-1)(q^{10}+q^9+3q-3)}{6q^4(q^{10}-1)} & \textit{if } c = 4, \\ \dfrac{(q-1)^2}{2q^c(q^{10}-1)} & \textit{if } c \geq 5. \end{cases}$$

If $e = 2$, we have

$$\delta_{K,\mathfrak{p}}(c) = \begin{cases} 1 - \dfrac{(q-1)(6q^{18}+10q^{17}+8q^{16}+7q^{15}+9q^{14}+6q^{13}+6q^{10}+9q^9)}{6q^9(q+1)(q^{10}-1)} + O\left(\dfrac{1}{q^{11}}\right) & \textit{if } c = 1, \\[2ex] \dfrac{(q-1)(2q^{19}+3q^{18}+2q^{17}+q^{16}+2q^{15}+2q^{14}+2q^{11}+2q^{10}))}{2q^{10}(q+1)(q^{10}-1)} + O\left(\dfrac{1}{q^{11}}\right) & \textit{if } c = 2, \\[2ex] \dfrac{(q-1)(q^2+1)(q^4-q^2+1)(q^{10}+q-1)}{2q^{11}(q^{10}-1)} & \textit{if } c = 3, \\[2ex] \dfrac{(q-1)(q^{19}+q^{18}+q^{10}-q^8+3q-3)}{6q^{12}(q^{10}-1)} & \textit{if } c = 4, \\[2ex] \dfrac{(q-1)^2}{2q^{8+c}(q^{10}-1)} & \textit{if } c \geq 5. \end{cases}$$

The exact proportions are given in Appendix B. For $e \geq 3$, refer to Table B.4 and Table B.5. These values are not written here due to their length and complexity.

Proof. The proof is very similar to Theorem 4.5: we compute the proportion of curves which reach each of the $(e+1)(e+2)/2$ non-terminal nodes, then sum and scale the proportions by $\frac{q^{10}}{q^{10}-1}$ to account for curves which initially loop back to $F(\geq e, \geq e)$. For sake of brevity, the exact computations and proportions are provided in Table B.4 and Table B.5. $\square$

Case	$v(a_1) = 0$	$v(a_1) \geq 1$ $v(a_3) = 0$	$v(a_1) = 1$ $v(a_3) \geq 1$	$v(a_1) \geq 2$ $v(a_3) = 1$	$v(a_1) = 2$ $v(a_3) \geq 2$	$v(a_1) \geq 3$ $v(a_3) = 2$	$v(a_1) \geq 3$ $v(a_3) \geq 3$
Color							

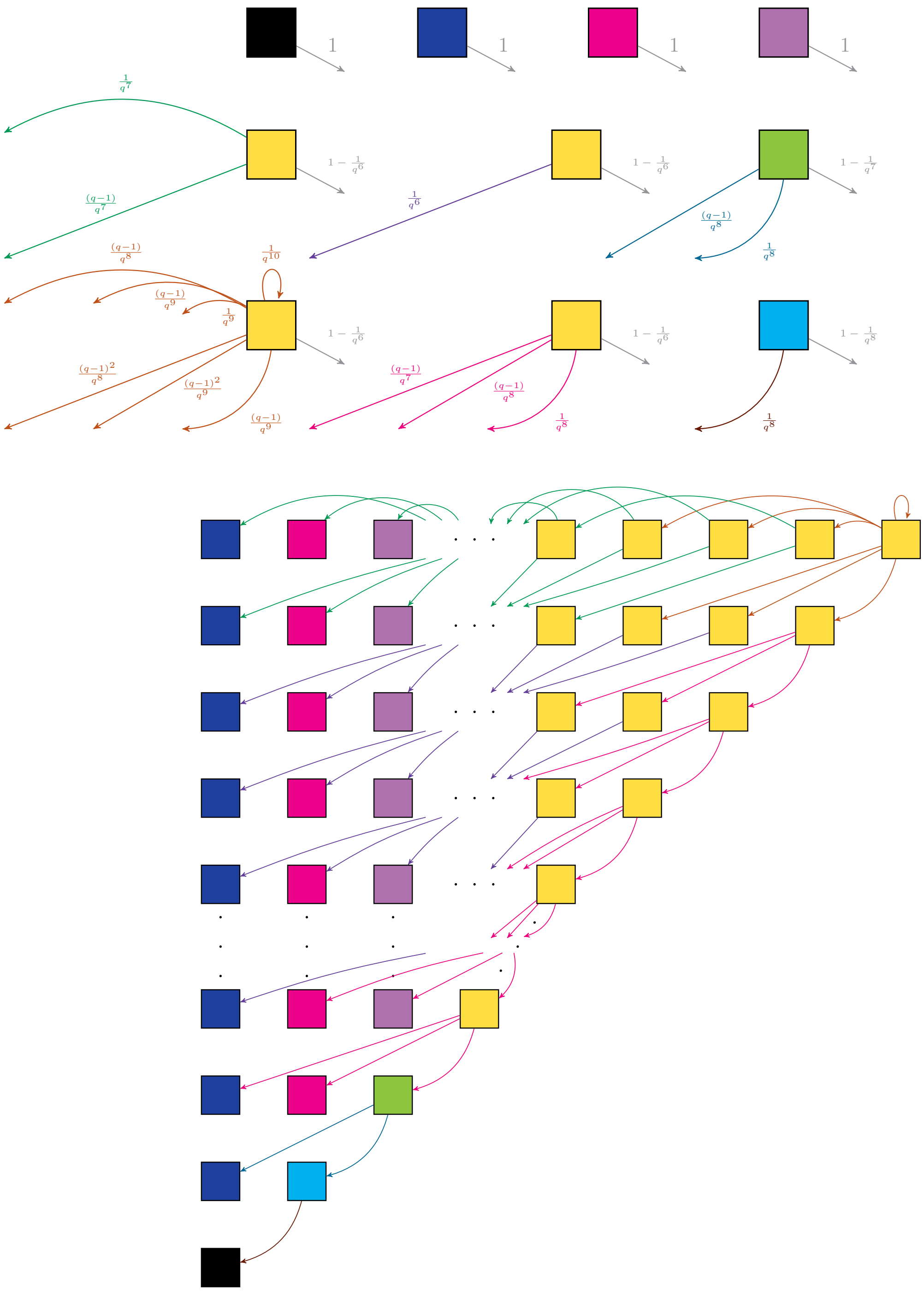


Figure 5-3: The Markov Chain structure when $\mathfrak{p} \mid (2)$ for $e \geq 3$.

Chapter 6

Proofs of Main Results

In this section, we make use of the computed local densities to prove our main results. We prove Theorem 1.11 by analyzing the convergence of $P_{\mathrm{Tam}}(K; m)$, thus establishing the absolute convergence of $L_{\mathrm{Tam}}(K; -1)$. For Theorem 1.14, we use the formulae from Theorem 3.2, Theorem 4.5, and Theorem 5.7 to uniformly bound $P_{\mathrm{Tam}}(K; 1)$ and $L_{\mathrm{Tam}}(K; -1)$. Finally, we prove Theorem 1.15 by constructing a family of multiquadratic fields and a family of cyclotomic fields.

Proof of Theorem 1.11. All of the counts in this paper assume that the elliptic curves are ordered by height. But the congruence conditions are over bounded powers of π which are pairwise relatively prime for different prime ideals, so we can compute the L-series with a simple multiplicativity argument. Namely, we have that

$$L_{\mathrm{Tam}}(K; s) := \sum_{m=1}^{\infty} \frac{P_{\mathrm{Tam}}(K; m)}{m^s} = \prod_{p \text{ prime}} \prod_{\mathfrak{p}|p} \left(\frac{\delta_{K,\mathfrak{p}}(1)}{1^s} + \frac{\delta_{K,\mathfrak{p}}(2)}{2^s} + \frac{\delta_{K,\mathfrak{p}}(3)}{3^s} + \dots \right).$$

It is sufficient to verify the convergence of the Dirichlet coefficients $P_{\mathrm{Tam}}(K; m)$ to show that they are well-defined. Thus, we make use of Corollary 3.2 (3) for a prime ideal $\mathfrak{p}$ such that $\mathfrak{p} \nmid (6)$. In particular, we note that $1 - \frac{1}{q^2} < \delta_{K,\mathfrak{p}}(1) < 1$, where $q := N_{K/\mathbb{Q}}(\mathfrak{p})$. Therefore, we see that convergence follows with the comparison to the Dedekind zeta function associated with the number field K at $s = 2$, making use of the well-known fact that $\zeta_K(s)$ is convergent for $\mathrm{Re}(s) > 1$.

To find the “average value” of $\mathrm{Tam}(K;E)$, given by

$$L_{\mathrm{Tam}}(K;-1) := \sum_{m=1}^{\infty} P_{\mathrm{Tam}}(K;m)m = \prod_{p \text{ prime}} \prod_{\mathfrak{p}|p} \left(\delta_{K,\mathfrak{p}}(1) + 2\delta_{K,\mathfrak{p}}(2) + 3\delta_{K,\mathfrak{p}}(3) + \dots\right), \quad (6.1)$$

we once again use Corollary 3.2. For prime ideal $\mathfrak{p}$ with $\mathfrak{p} \nmid (6)$ and $q := \mathcal{N}_{K/\mathbb{Q}}(\mathfrak{p})$, we have $\delta_{K,\mathfrak{p}}(1) = 1 - \frac{1}{q^2} + o(1/q^2)$ and $0 < n\delta_{K,\mathfrak{p}}(c) < \frac{n}{q^n}$ for $n \geq 2$. Since $\sum_{n=1}^{\infty} \frac{n}{q^n} = \frac{q}{(q-1)^2}$, we obtain

$$\sum_{n=1}^{\infty} n\delta_{K,\mathfrak{p}}(c) = 1 - \frac{1}{q^2} + o\left(\frac{1}{q^2}\right).$$

For $\mathfrak{p} \mid (2)$ and $\mathfrak{p} \mid (3)$, letting $q := N_{K/\mathbb{Q}}(\mathfrak{p})$, we have

$$\sum_{n\geq 1} n\delta_{K,\mathfrak{p}}(c) \leq \delta_{K,\mathfrak{p}}(1) + \sum_{k=1}^{\infty}(4+k)(1-\delta_{K,\mathfrak{p}}(1))\frac{q-1}{q^k} = 5 - 4\delta_{K,\mathfrak{p}}(1) + \frac{(1-\delta_{K,\mathfrak{p}}(1))}{q-1}.$$

The explicit values of $\delta_{K,\mathfrak{p}}(1)$ can be found in Theorem 4.5 and Theorem 5.7. By multiplicativity, (6.1) must converge, and we are thus done. □

Proof of Theorem 1.14. We first note that for $\mathfrak{p} \mid (2)$ and $\mathfrak{p} \mid (3)$, the value $\delta_{K,\mathfrak{p}}(1)$ when $\mathfrak{p}$ ramifies is greater than the square of the value when $\mathfrak{p}$ splits completely, as can be seen in Theorem 4.5 and Theorem 5.7. For $\mathfrak{p} \nmid (6)$, $\delta_{K,\mathfrak{p}}(1)$ is similarly minimized when $\mathfrak{p}$ splits, as can be seen from the formulae in Theorem 3.2.

We now prove the uniform bounds of $P_{\mathrm{Tam}}(K;1)$. Recall

$$P_{\mathrm{Tam}}(K;1) = \prod_{p \text{ prime}} \prod_{\mathfrak{p}|(p)} \delta_{K,\mathfrak{p}}(1).$$

To bound $P_{\mathrm{Tam}}(K;1)$ from below, we consider the case where each $\mathfrak{p}$ splits completely; specifically, there are d distinct prime ideals lying above it, where each prime ideal has ramification index and inertial degree 1. Then, $q = \mathcal{N}_{K/\mathbb{Q}}(\mathfrak{p}) = p$ and we have

$$\prod_{p \text{ prime}} \delta_{\mathbb{Q},p}(1)^d = P_{\mathrm{Tam}}(\mathbb{Q};1) \leq P_{\mathrm{Tam}}(K;1).$$

We make use of the computation $P_{\text{Tam}}(\mathbb{Q};1) = 0.5054\ldots$ in [6] to obtain the bound

$$(0.5054)^d < P_{\text{Tam}}(\mathbb{Q};1)^d \leq P_{\text{Tam}}(K;1)^d,$$

where equality only holds when $d = 1$.

We now bound $P_{\text{Tam}}(K;1)$ from above by supposing each $\mathfrak{p}$ is inert, meaning $\mathfrak{p}$ has ramification index 1 and inertial degree d. In this case, $q := N_{K/\mathbb{Q}}(\mathfrak{p}) = p^d$. Using Table 3.1 and Table 4.1, we see $\delta_{K,\mathfrak{p}}(1) \leq 1 - \frac{1}{q^2} + \frac{1}{q^3} = 1 - \frac{1}{p^{2d}} + \frac{1}{p^{3d}}$ for every $\mathfrak{p} \mid (3)$ and $\mathfrak{p} \nmid (6)$. From Table 5.1 for $\mathfrak{p} \mid (2)$, we see $\delta_{K,\mathfrak{p}}(1) \leq 1 - \frac{1}{q} + \frac{1}{q^2} = 1 - \frac{1}{2^d} + \frac{1}{2^{2d}}$. Hence, we have

$$P_{\text{Tam}}(K;1) \leq \left(1 - \frac{1}{2^d} + \frac{1}{2^{2d}}\right) \prod_{p \geq 3 \text{ prime}} \left(1 - \frac{1}{p^{2d}} + \frac{1}{p^{3d}}\right).$$

We wish to prove the following inequality:

$$\left(1 - \frac{1}{2^d} + \frac{1}{2^{2d}}\right) \prod_{p \geq 3 \text{ prime}} \left(1 - \frac{1}{p^{2d}} + \frac{1}{p^{3d}}\right) \leq \prod_{p \text{ prime}} \left(1 - \frac{1}{p^{2d}}\right) = \frac{1}{\zeta(2d)},$$

which can be rewritten as

$$\prod_{p \geq 3 \text{ prime}} \left(1 + \frac{1}{p^{3d} - p^d}\right) \leq 1 + \frac{2^d - 2}{2^{2d} - 2^d + 1}.$$

We proceed as follows:

$$\prod_{p \geq 3 \text{ prime}} \left(1 + \frac{1}{p^{3d} - p^d}\right) \leq \prod_{p \geq 3 \text{ prime}} \left(1 + \frac{1}{p^{3d-1}}\right) \leq \zeta(3d-1).$$

We then have the bounds

$$\zeta(3d-1) \leq 1 + \frac{1}{2^{3d-1}} + \int_2^\infty \frac{1}{x^{3d-1}}\,dx = 1 + \frac{1}{2^{3d-1}} + \frac{1}{3d \cdot 2^{3d-2} - 2^{3d-1}} \leq 1 + \frac{2^d - 2}{2^{2d} - 2^d + 1}.$$

We have therefore shown that

$$P_{\text{Tam}}(K;1) \leq \frac{1}{\zeta(2d)} = (-1)^{d+1} \frac{2(2d)!}{B_{2d}(2\pi)^{2d}}.$$

The last equality is a well-known relationship due to Euler between the Riemann zeta function $\zeta(2n)$ and the Bernoulli numbers B_{2n}. Thus, we have bounded $P_{\mathrm{Tam}}(K;1)$ depending on $d = [K:\mathbb{Q}]$, as desired.

In a similar fashion as above, we prove the uniform bounds for $L_{\mathrm{Tam}}(K;-1)$. Recall

$$L_{\mathrm{Tam}}(K;-1) = \prod_{p \text{ prime}} \prod_{\mathfrak{p}|(p)} \sum_{m=1}^{\infty} \delta_{K,\mathfrak{p}}(m)m.$$

In order to bound $L_{\mathrm{Tam}}(K;-1)$ from below, we must again consider the case where every $\mathfrak{p}$ is inert, meaning $q := N_{K/\mathbb{Q}} = p^d$. We first note $\sum_{m=1}^{\infty} \delta_{K,\mathfrak{p}}(m)m \geq \delta_{K,\mathfrak{p}}(1) + 2(1-\delta_{K,\mathfrak{p}}(1)) = 2 - \delta_{K,\mathfrak{p}}(1)$. Recalling the previous bounds of $\delta_{K,\mathfrak{p}}(1)$ for $\mathfrak{p}$ inert, we have

$$\prod_{p \text{ prime}} \prod_{\mathfrak{p}|(p)} (2 - \delta_{K,\mathfrak{p}}(1)) \geq \left(1 + \frac{1}{2^d} - \frac{1}{2^{2d}}\right) \prod_{p \geq 3 \text{ prime}} \left(1 + \frac{1}{p^{2d}} - \frac{1}{p^{3d}}\right).$$

We wish to prove that

$$\left(1 + \frac{1}{2^d} - \frac{1}{2^{2d}}\right) \prod_{p \geq 3 \text{ prime}} \left(1 + \frac{1}{p^{2d}} - \frac{1}{p^{3d}}\right) \geq \prod_{p \text{ prime}} \left(1 + \frac{1}{p^{2d}}\right) = \frac{\zeta(2d)}{\zeta(4d)},$$

which can be restated as

$$\prod_{p \geq 3 \text{ prime}} \left(1 - \frac{1}{p^{3d} + p^d}\right) \geq 1 - \frac{2^d - 2}{2^{2d} + 2^d - 1}.$$

We proceed by bounding as follows:

$$\prod_{p \geq 3 \text{ prime}} \left(1 - \frac{1}{p^{3d} + p^d}\right) \geq \prod_{p \geq 3 \text{ prime}} \left(1 - \frac{1}{p^{3d-1}}\right) \geq \frac{1}{\zeta(3d-1)}.$$

We make use of our previous upper bound for $\zeta(3d-1)$ to see

$$\frac{1}{\zeta(3d-1)} \geq \frac{1}{1 + \frac{1}{2^{3d-1}} + \frac{1}{2^{3d-2}(3d-2)}} = 1 - \frac{3d}{2^{3d-1}(3d-2) + 3d} \geq 1 - \frac{2^d - 2}{2^{2d} + 2^d - 1},$$

as desired. Thus, using the aforementioned relationship between the Riemann zeta function

$\zeta(2n)$ and the Bernoulli numbers B_{2n}, we have the lower bound

$$\frac{\zeta(2d)}{\zeta(4d)} = (-1)^d \frac{B_{2d}(4d)!}{B_{4d}(2d)!(2\pi)^{2d}} \leq L_{\mathrm{Tam}}(K;-1).$$

To now bound $L_{\mathrm{Tam}}(K;-1)$ from above, we once again consider the case where each $\mathfrak{p}$ splits completely. Thus, $q := N_{K/\mathbb{Q}}(\mathfrak{p}) = p$ and we have

$$L_{\mathrm{Tam}}(K;-1) \leq \prod_{p \text{ prime}} \sum_{m=1}^{\infty} \delta_{\mathbb{Q},p}(m)m = L_{\mathrm{Tam}}(\mathbb{Q};-1).$$

The value of $L_{\mathrm{Tam}}(\mathbb{Q};-1)$ is computed in [6] to be $1.8184\ldots$. Hence, we obtain the bound

$$L_{\mathrm{Tam}}(K;-1) \leq L_{\mathrm{Tam}}(\mathbb{Q};-1) < (1.8184)^d,$$

and we are therefore done. □

Proof of Theorem 1.15. To prove this theorem, we explicitly construct a family of multiquadratic fields whose Tamagawa trivial proportions tend to 0 and a family of cyclotomic fields whose proportions tend to 1.

We begin by proving $\liminf_{d\to+\infty} t^-(d) = 0$. Let $p_1 < p_2 < p_3 < \ldots$ be an infinite sequence of $1 \pmod 8$ primes. Consider the sequence of multiquadratic field extensions

$$\begin{aligned} K_1 &= \mathbb{Q}(\sqrt{p_1}), \\ K_2 &= \mathbb{Q}(\sqrt{p_1}, \sqrt{p_2}), \\ K_3 &= \mathbb{Q}(\sqrt{p_1}, \sqrt{p_2}, \sqrt{p_3}), \\ &\ \vdots \end{aligned}$$

Fix a field K_i. Since 2 is a quadratic residue modulo p_i for $1 \leq j \leq i$, (2) splits completely in the number fields $\mathbb{Q}(\sqrt{p_j})$. Hence the ideal (2) also splits completely in the composite field K_i. In particular, there are 2^i distinct prime ideals above (2), each with inertial degree and ramification index 1. Recall from Theorem 5.7 that for norm 2 unramified prime ideals $\mathfrak{p}$ we have $\delta_{K,\mathfrak{p}}(1) = 241/396$. Hence, we have an upper bound

$$t^-(2^i) \leq P_{\text{Tam}}(K_i;1) = \prod_{p \text{ prime}} \prod_{\mathfrak{p}|(p)} \delta_{K_i,\mathfrak{p}}(1) \leq \prod_{\mathfrak{p}|(2)} \delta_{K_i,\mathfrak{p}}(1) = (241/396)^{2^i}$$

which tends to 0 as i grows large. Thus, $\liminf_{d\to+\infty} t^-(d) = 0$.

In fact, this sequence of fields also show that $\limsup_{d\to+\infty} \mu^+(d) = \infty$. For each $\mathfrak{p}$ above 2, we may compute $\sum_{m=1}^{\infty} \delta_{K,\mathfrak{p}}(m)m > 1.49$, whence

$$\mu^+(2^i) \geq \prod_{p \text{ prime}} \prod_{\mathfrak{p}|(p)} \sum_{m=1}^{\infty} \delta_{K_i,\mathfrak{p}}(m)m \geq \prod_{\mathfrak{p}|(2)} \sum_{m=1}^{\infty} \delta_{K_i,\mathfrak{p}}(m)m > (1.49)^{2^i}.$$

The right-hand side tends to infinity as $i \to \infty$, from which the claim follows.

We now prove $\limsup_{d\to+\infty} t^+(d) = 1$. Let a be an arbitrary prime and $K = \mathbb{Q}(\zeta_a)$ be the a-th cyclotomic field. For primes $p \neq a$, the ideal (p) factors in K as the product of $(a-1)/\operatorname{ord}_a(p)$ prime ideals of norm $p^{\operatorname{ord}_a(p)}$. On the other hand, $(a) = \mathfrak{a}^{a-1}$ where $\mathfrak{a} = (1+\zeta_a)$.

By Propositions 3.2, 4.5, and 5.7, a prime ideal $\mathfrak{p}$ satisfies the bound $\delta_{K,\mathfrak{p}}(1) \geq 1 - (N_{K/\mathbb{Q}}(\mathfrak{p}))^{-1}$. Hence, we have the lower bound

$$t^+(a) \geq P_{\text{Tam}}(K;1) = \prod_{p \text{ prime}} \prod_{\mathfrak{p}|(p)} \delta_{K,\mathfrak{p}}(1) \geq \delta_{K,\mathfrak{a}}(1) \cdot \prod_{p\neq a \text{ prime}} \left(1 - p^{-\operatorname{ord}_a(p)}\right)^{\frac{a-1}{\operatorname{ord}_a(p)}}.$$

We claim that as a grows large, the right-hand side converges to 1. This is because we have $\delta_{\mathfrak{a},K}(1) \to 1$, and moreover by the naive bound $\operatorname{ord}_a(p) \geq \max\{2, \log_p(a)\}$ we have

$$1 \geq \lim_{a\to\infty} \prod_{p \text{ prime}} \left(1 - p^{-\operatorname{ord}_a(p)}\right)^{\frac{a-1}{\operatorname{ord}_a(p)}} \geq \lim_{a\to\infty} \prod_{p \text{ prime}} \left(1 - \max\{a, p^2\}^{-1}\right)^{\frac{a-1}{\max\{2,\log_p(a)\}}} = 1.$$

Thus, $\limsup_{d\to+\infty} t^+(d) = 1$.

We finish by showing $\liminf_{d\to+\infty} \mu^-(d) = 1$. Borrow the fields $K = \mathbb{Q}(\zeta_a)$ from the above proof. Then by Propositions 3.2, 4.5, and 5.7, we have the loose bound $\sum_{m=1}^{\infty} \delta_{K,\mathfrak{p}}(m)m \leq$

$1 + 100(N_{K/\mathbb{Q}}(\mathfrak{p}))^{-1}$. Thus,

$$\mu^-(a) \leq \prod_{p \text{ prime}} \prod_{\mathfrak{p}|(p)} \sum_{m=1}^{\infty} \delta_{K,\mathfrak{p}}(m)m \leq \sum_{m=1}^{\infty} \delta_{K,\mathfrak{a}}(m)m \cdot \prod_{p \neq a \text{ prime}} \left(1 + 100p^{-\operatorname{ord}_a(p)}\right)^{\frac{a-1}{\operatorname{ord}_a(p)}}.$$

As $a \to \infty$, the right-hand side converges to 1. This is because $\sum_{m=1}^{\infty} \delta_{K,\mathfrak{a}}(m)m \to 1$, and moreover by the naive bound $\operatorname{ord}_a(p) \geq \max\{\log_p(a), 2\}$ we have

$$1 \leq \lim_{a\to\infty} \prod_{p \text{ prime}} \left(1 + 100p^{-\operatorname{ord}_a(p)}\right)^{\frac{a-1}{\operatorname{ord}_a(p)}} \leq \lim_{a\to\infty} \prod_{p \text{ prime}} (1 + 100 \max\{a, p^2\}^{-1})^{\frac{a-1}{\max\{\log_p(a),2\}}} = 1.$$

We have proved both the limit infimum and limit supremum bounds, and we are thus done.

□

Chapter 7

Numerical Examples

In this section, we offer numerical examples which illustrate the results in this paper.

Example 7.1. There are finitely many imaginary quadratic fields. As briefly discussed in the introduction, in Table 7.1 we calculated the following values for their proportion of Tamagawa trivial curves. Then we have similar Tables 7.2 and 7.3, showing the convergence to $P_{Tam}(\mathbb{Q}(\sqrt{-D}), 2)$ and $P_{Tam}(\mathbb{Q}(\sqrt{-D}), 3)$ respectively. Lastly, in Table 7.4 we have average Tamagawa product for each of the imaginary quadratic fields, as shown below. Note that in Table 7.1, $P_{Tam}(\mathbb{Q}(\sqrt{-D}, 1)$ is noticeably smaller when $D = 7$, and noticeably larger when $D = 163$. This is due to the fact that 2 splits over $\mathbb{Q}(\sqrt{-7})$, and primes up to 37 are inert in $\mathbb{Q}(\sqrt{-163})$. Similarly in Table 7.4, notice that $\mathbb{Q}(\sqrt{-7})$ has a noticeably high average Tamagawa product, which is related to the noticeably low percentage of Tamagawa trivial curves. On the opposite end, we see that $\mathbb{Q}(\sqrt{-163})$ has a noticeably low average Tamagawa product, which corresponds to its high percentage of Tamagawa trivial curves.

$\mathcal{N}_1(\mathbb{Q}(\sqrt{-D});X)/\mathcal{N}(\mathbb{Q}(\sqrt{-D});X)$									
X	$\sqrt{-1}$	$\sqrt{-2}$	$\sqrt{-3}$	$\sqrt{-7}$	$\sqrt{-11}$	$\sqrt{-19}$	$\sqrt{-43}$	$\sqrt{-67}$	$\sqrt{-163}$
10^4	0.542	0.488	0.663	0.357	0.609	0.620	0.657	0.560	0.450
10^5	0.539	0.460	0.665	0.359	0.599	0.678	0.716	0.711	0.636
10^6	0.528	0.468	0.660	0.343	0.586	0.667	0.726	0.744	0.728
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
∞	0.529	0.468	0.661	0.349	0.581	0.665	0.733	0.750	0.763

Table 7.1: Convergence to $P_{\text{Tam}}(\mathbb{Q}(\sqrt{-D});1)$.

$\mathcal{N}_2(X,K)/\mathcal{N}(X,K)$									
X	$\sqrt{-1}$	$\sqrt{-2}$	$\sqrt{-3}$	$\sqrt{-7}$	$\sqrt{-11}$	$\sqrt{-19}$	$\sqrt{-43}$	$\sqrt{-67}$	$\sqrt{-163}$
10^4	0.353	0.351	0.248	0.354	0.255	0.278	0.260	0.284	0.283
10^5	0.360	0.387	0.253	0.369	0.277	0.231	0.220	0.218	0.207
10^6	0.377	0.382	0.266	0.382	0.298	0.256	0.228	0.211	0.215
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
∞	0.378	0.384	0.264	0.370	0.299	0.265	0.226	0.216	0.206

Table 7.2: Convergence to $P_{\text{Tam}}(\mathbb{Q}(\sqrt{-D});2)$.

$\mathcal{N}_3(X,K)/\mathcal{N}(X,K)$									
X	$\sqrt{-1}$	$\sqrt{-2}$	$\sqrt{-3}$	$\sqrt{-7}$	$\sqrt{-11}$	$\sqrt{-19}$	$\sqrt{-43}$	$\sqrt{-67}$	$\sqrt{-163}$
10^4	0.008	0.023	0.015	0.074	0.018	0.027	0.019	0.069	0.100
10^5	0.015	0.026	0.029	0.078	0.035	0.029	0.035	0.040	0.089
10^6	0.017	0.025	0.027	0.078	0.035	0.024	0.021	0.025	0.031
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
∞	0.018	0.026	0.032	0.082	0.038	0.028	0.024	0.024	0.024

Table 7.3: Convergence to $P_{\text{Tam}}(\mathbb{Q}(\sqrt{-D});3)$.

$\sum_{\mathrm{ht}(\mathbb{Q}(\sqrt{-D});E)\leq X} \mathrm{Tam}(\mathbb{Q}(\sqrt{-D})); E)/N(\mathbb{Q}(\sqrt{-D}); X)$									
X	$\sqrt{-1}$	$\sqrt{-2}$	$\sqrt{-3}$	$\sqrt{-7}$	$\sqrt{-11}$	$\sqrt{-19}$	$\sqrt{-43}$	$\sqrt{-67}$	$\sqrt{-163}$
10^4	1.751	2.054	1.589	2.570	1.850	1.718	1.535	1.698	2.017
10^5	1.720	1.979	1.538	2.417	1.763	1.537	1.393	1.403	1.612
10^6	1.708	1.946	1.508	2.418	1.723	1.519	1.361	1.333	1.372
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
∞	1.678	1.904	1.487	2.376	1.708	1.480	1.331	1.300	1.277

Table 7.4: Convergence to $L_{\mathrm{Tam}}(\mathbb{Q}(\sqrt{-D}); -1)$.

Example 7.2. We computed the Tamagawa trivial proportions for some real quadratic fields. Below we have Figure 7-1 and Figure 7-2 demonstrating the spread of $P_{\mathrm{Tam}}(\mathbb{Q}(\sqrt{D}), 1)$ a, for square-free $2 \leq D < 10^4$.

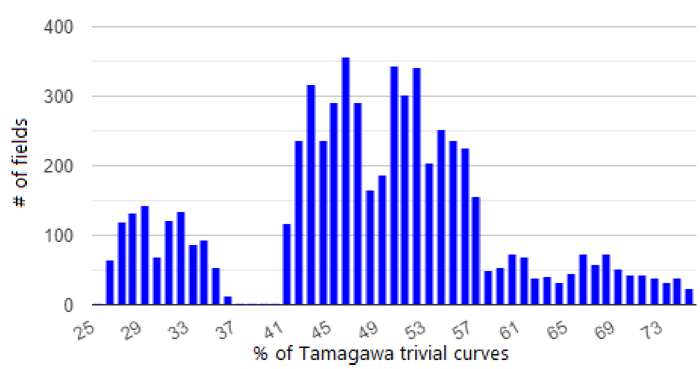


Figure 7-1: Distribution of $P_{\mathrm{Tam}}(K; 1)$ for real quadratic number fields.

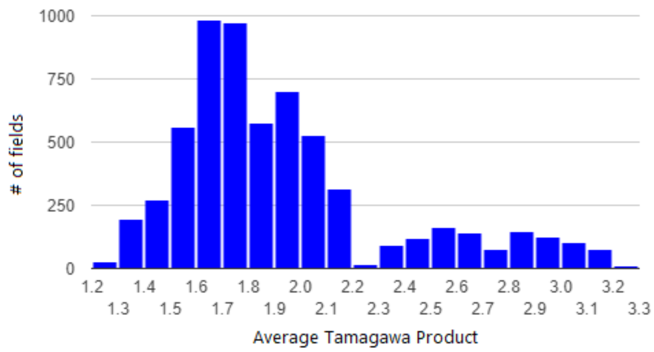


Figure 7-2: Distribution of $L_{\text{Tam}}(K;-1)$ for real quadratic number fields.

In Figure 7-1, one can see that $P_{\text{Tam}}(\mathbb{Q}(\sqrt{D}),1)$ does not have a normal distribution, nor does it resemble a skewed normal distribution. Instead, there appear to be three distinct sections, from ≈ 0.26 to ≈ 0.36, from ≈ 0.41 to ≈ 0.57, and from ≈ 0.58 to ≈ 0.75. These three distinct regions correspond to how 2 behaves at any particular field. The left-most section corresponds to fields where 2 splits, the middle section corresponds to fields where 2 ramifies, and the right-most section corresponds to fields where 2 is inert. This leads us to many possible questions. Is the distribution uniform or random among each section? Are the proportions dense on any interval? How might these generalize over different degree fields? Are there always gaps between the sections? In Figure 7-2, we see that $L_{\text{Tam}}(Q(\sqrt{D}),-1)$ also does not have a normal distribution, but instead has two distinct sections. This leads to further questions, such as how do these sections relate to those for $P_{\text{Tam}}(Q(\sqrt{D}),1)$? Though we do not answer these questions in our paper, they are a noteworthy topic for future study.

Example 7.3. From Theorem 1.15 we have

$$\liminf_{d\to+\infty} t^-(d) = 0 \quad \text{and} \quad \limsup_{d\to+\infty} t^+(d) = 1.$$

Moreover, we have

$$\liminf_{d\to+\infty} \mu^-(d) = 1 \quad \text{and} \quad \limsup_{d\to+\infty} \mu^+(d) = \infty.$$

Below in Table 7.5, Figure 7-3, and Figure 7-4 we have the trivial Tamagawa proportions and average Tamagawa products for sequences of fields that demonstrate these limits.

K	$\mathbb{Q}(\sqrt{17})$	$\mathbb{Q}(\sqrt{17}, \sqrt{41})$	$\mathbb{Q}(\sqrt{17}, \sqrt{41}, \sqrt{73})$	$\mathbb{Q}(\sqrt{17}, \sqrt{41}, \sqrt{73}, \sqrt{89})$
$P_{\text{Tam}}(K;1)$	0.35585	0.13273	0.01778	0.00031
$L_{\text{Tam}}(K;-1)$	2.32335	5.14423	26.22779	686.87874

Table 7.5: Tamagawa trivial proportions of multiquadratic fields.

In Table 7.5 we have an example of a sequence of fields with Tamagawa trivial proportion decreasing to zero, and the corresponding average Tamagawa products which head off to infinity. The sequence is constructed by the methods described in the proof of Theorem 1.15.

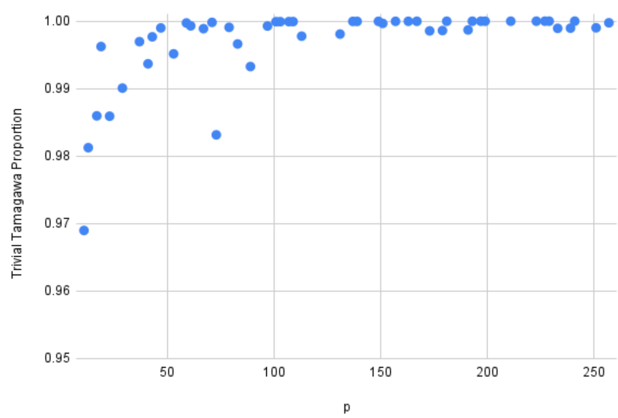


Figure 7-3: $P_{\text{Tam}}(\mathbb{Q}(\zeta_p), 1)$, for cyclotomic fields with $p < 258$.

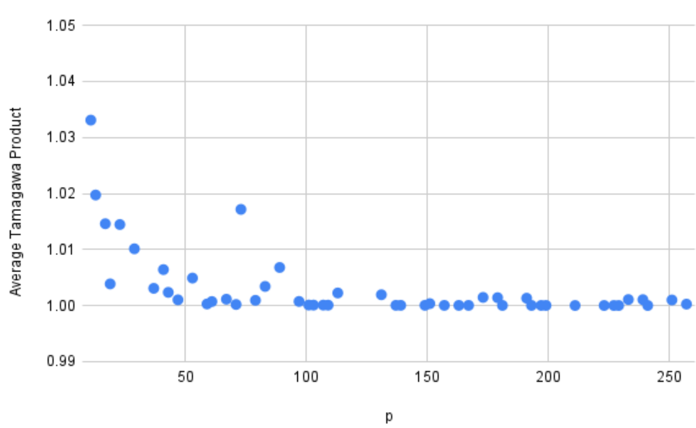


Figure 7-4: $L_{\text{Tam}}(\mathbb{Q}(\zeta_p), -1)$, for cyclotomic fields with $p < 258$.

Similarly, in Figure 7-3, we have an example of a sequence of fields with Tamagawa trivial proportion tending to 1. In Figure 7-4 we have the corresponding average Tamagawa products similarly converging to 1. The sequence is created by taking $\mathbb{Q}(\zeta_p)$ for primes p. Note that in both Figure 7-3 and Figure 7-4 the data points from $p = 5, 7, 31$, and 127 have been omitted due to being outliers.

p	5	7	31	127
$P_{\text{Tam}}(\mathbb{Q}(\zeta_p), 1)$	$0.867\ldots$	$0.753\ldots$	$0.827\ldots$	$0.868\ldots$
$L_{\text{Tam}}(\mathbb{Q}(\zeta_p), -1)$	$1.155\ldots$	$1.309\ldots$	$1.205\ldots$	$1.151\ldots$

Table 7.6: Omitted values for $P_{\text{Tam}}(\mathbb{Q}(\zeta_p), 1)$ and $L_{\text{Tam}}(\mathbb{Q}(\zeta_p), -1)$

Example 7.4. Many of the above examples are of simpler fields, however in our paper we are able to calculate trivial Tamagawa proportions and average Tamagawa products for all fields, regardless of class number or degree. Thus, even for a more complicated field such as $\mathbb{Q}(x^4 + 5x^2 - 6x + 3)$ with Galois group S_4, we can even determine the trivial Tamagawa proportion. Note that $\mathbb{Q}(x^4 + 5x^2 - 6x + 3)$ has $\Delta = 32880$, and thus the primes $2, 3, 5$, and 137 all ramify. More specifically, we have that $2 = (-8\alpha^3 - \alpha^2 - 37\alpha + 47)^2$, $3 = (-\alpha)^2(2\alpha^3 + 9\alpha - 13)(\alpha - 1)$, $5 = (-\alpha^3 + \alpha - 1)(-2\alpha^2 + 2\alpha - 1)^2$, and $137 = (44\alpha^3 + 33\alpha^2 + 235\alpha - 100)(-\alpha^3 - 4\alpha^2 - 9\alpha - 17)^2$. By knowing how these primes ramify, we can calculate that $P_{\text{Tam}}(\mathbb{Q}(x^4 + 5x^2 - 6x + 3), 1) = 0.526\ldots$.

Appendix A

Appendix A

In this section, we classify the non-minimal short Weierstrass models at prime ideals $\mathfrak{p} \mid (3)$ and $\mathfrak{p} \mid (2)$. These results generalize the work of Griffin et al. [6, Lemmas 2.2, 2.3], who classify the non-minimal short Weierstrass rational elliptic curves for primes $p = 2, 3$. The conditions for non-minimality can be written as a set of modular equations for bounded powers of π, which allows for a parametrization for the non-minimal curves. Since we are working with primes modulo powers of π, our results depend on the size of e.

Lemma 1.1. *Let $\mathfrak{p} \subseteq K$ have ramification index e over (3). The curve $E(a_4, a_6)$ is not $\mathfrak{p}-$minimal if and only if there exist residues $r \pmod{\pi^{\min\{2,e\}}}$ and $w \pmod{\pi^2}$ for which*

$$a_4 \equiv -3\pi^{\max\{0,4-2e\}}r^2 + \pi^4 w \pmod{\pi^6}, \quad a_6 \equiv 2\pi^{\max\{0,6-3e\}}r^3 - \pi^{\max\{4,6-e\}}rw \pmod{\pi^6}.$$

Moreover, across a_4 modulo π^4 and a_6 modulo π^6 such that $E(a_4, a_6)$ is non-minimal, the choice of (r, w) from their respective residue classes is unique, i.e., there are exactly q^3 (resp. q^4) classes $(a_4, a_6) \pmod{\pi^6}$ of non-minimal models for $e = 1$ (resp. $e \geq 2$).

Proof. Suppose that E/K is not $\mathfrak{p}-$minimal. Throughout Step 1 to 10 of Tate's algorithm, we potentially translate (x, y) in the original curve to $(x+R, y+Vx+U)$. If the starting curve is non-minimal, we must reach Step 11, and a_i should be divisible by π^i for $i = 1, 2, 3, 4, 6$.

Translating this into equations, the restrictions on a_4, a_6, R, V, U are as follows:

$$2V \equiv 0 \pmod{\pi} \tag{A.1}$$

$$3R - V^2 \equiv 0 \pmod{\pi^2} \tag{A.2}$$

$$2U \equiv 0 \pmod{\pi^3} \tag{A.3}$$

$$3R^2 + a_4 - 2UV \equiv 0 \pmod{\pi^4} \tag{A.4}$$

$$R^3 + a_4 R + a_6 - U^2 \equiv 0 \pmod{\pi^6} \tag{A.5}$$

Regardless of $\mathfrak{p}$, A.1 and A.3 imply that $V \equiv 0 \pmod{\pi}$ and $U \equiv 0 \pmod{\pi^3}$. As such, U and V vanish from the remaining equations.

From A.2, we have that $R \equiv 0 \pmod{\pi^{\max\{0,2-e\}}}$. Therefore, suppose that $R = \pi^{\max\{0,2-e\}} r$ for some $\pi-$adic integer r. Then, from A.4, we have that $a_4 \equiv -3\pi^{\max\{0,4-2e\}} r^2 \pmod{\pi^4}$. We therefore write $a_4 = -3\pi^{\max\{0,4-2e\}} r^2 + \pi^4 w$. Then, A.5 is equivalent to $a_6 \equiv -R^3 - a_4 R \equiv 2\pi^{\max\{0,6-3e\}} r^3 - \pi^{\max\{4,6-e\}} rw \pmod{\pi^6}$.

To determine (a_4, a_6) up to $\pmod{\pi^6}$, r should be determined up to $\pmod{\pi^{\max\{3,e+1\}}}$ and w should be determined up to π^2. Yet, we contend, in order for the map between $(a_4, a_6) \pmod{\pi^6}$ and (r, w) to be bijective, the residues r and w must be selected modulo $\pi^{\min\{2,e\}}$ and modulo π^2, respectively. To show injectivity, we note that the resulting $(a_4, a_6) \pmod{\pi^6}$ from (r, w) and $(r + k\pi^{\min\{2,e\}}, w + \frac{6}{\pi^{\min\{2,e\}}}\alpha k + 3k^2)$ are equivalent. To show surjectivity, suppose that for some (r, w) and (r', w'), the resulting (a_4, a_6) are equivalent $\pmod{\pi^6}$, i.e.,

$$-3\pi^{\max\{0,4-2e\}} r^2 + \pi^4 w \equiv -3\pi^{\max\{0,4-2e\}} r'^2 + \pi^4 w' \pmod{\pi^6}; \tag{A.6}$$

$$2\pi^{\max\{0,6-3e\}} r^3 - \pi^{\max\{4,6-e\}} rw \equiv 2\pi^{\max\{0,6-3e\}} r'^3 - \pi^{\max\{4,6-e\}} r'w' \pmod{\pi^6}. \tag{A.7}$$

From A.7, $r \equiv r' \pmod{\pi^{\min\{2,e\}}}$. Then, from A.6, $w \equiv w' \pmod{\pi^2}$ as we had sought. □

Lemma 1.2. *Let $\mathfrak{p}$ have ramification index e over (2). The curve $E(a_4, a_6)$ is not $\mathfrak{p}$-minimal* *if and only there exist residues* $u \pmod{\pi^{\min\{3,e\}}}$, $v \pmod{\pi}$, *and* $w \pmod{\pi^2}$ *for which*

$$a_4 \equiv 2\pi^{\max\{0,3-e\}}uv - 3v^4 + \pi^4 w \pmod{\pi^6}, \quad a_6 \equiv \pi^{\max\{0,6-2e\}}u^2 - v^6 - a_4 v^2 \pmod{\pi^6}.$$

For each (a_4, a_6)*, the choice of* (u, v, w) *from their respective residue classes is unique, i.e., there are exactly* q^4 *(resp.* q^5 *and* q^6*) classes* $(a_4, a_6) \pmod{\pi^6}$ *of non-minimal models for* $e = 1$ *(resp.* $e = 2$ *and* $e \geq 3$*).*

Proof. Suppose that E is not $\mathfrak{p}-$minimal in K. Following the same steps as in the proof of Theorem 1.1, we have A.1, A.2, A.3, A.4, A.5 as restrictions on R, U, V and a_4, a_6. From here, we check that (R, U, V) and $(R+k\pi^2, U+k\pi^2 V, V)$ give rise to the same (a_4, a_6) modulo π^6. Hence, by choosing a suitable value of k, we assume $R = -V^2$.

To begin, condition A.3 yields $U \equiv 0 \pmod{\pi^{\max\{0,3-e\}}}$. Therefore, we suppose that $U = \pi^{\max\{0,3-e\}}u$ for some π-adic integer u. From A.4, we get $a_4 = 2\pi^{\max\{0,3-e\}}uV - 3V^4$, whence we write $a_4 = (2\pi^{\max\{0,3-e\}}uV - 3V^4) + \pi^4 w$ for some π-adic integer w. Finally, eq. (A.5) gives $a_6 \equiv \pi^{\max\{0,6-2e\}}u^2 - V^6 - V^2 a_4 \pmod{\pi^6}$.

By the analogous reasoning as in the proof of Theorem 1.1, it can be shown that selecting u, v, w as representatives modulo $\pi^{\min\{3,e\}}$, π, and π^2 respectively forms a bijective map between u, v, w and (a_4, a_6) as we had sought. $\square$

Appendix B

Appendix B

Type	$c_{\mathfrak{p}}$	$e=1$	Type	$c_{\mathfrak{p}}$	$e=1$	Type	$c_{\mathfrak{p}}$	$e=1$
I_0	1	$\frac{(q^{10}+q-1)(q-1)}{q(q^{10}-1)}$	I_0^*	1	$\frac{1}{3}\frac{q^4(q^2-1)}{q^{10}-1}$	III	2	$\frac{q^7(q-1)}{q^{10}-1}$
I_1	1	$\frac{(q-1)^2}{q^2(q^{10}-1)}$	I_0^*	2	$\frac{1}{2}\frac{q^5(q-1)}{q^{10}-1}$	III^*	2	$\frac{q^2(q-1)}{q^{10}-1}$
I_2	2	$\frac{(q-1)^2}{q^3(q^{10}-1)}$	I_0^*	4	$\frac{1}{6}\frac{q^4(q-1)(q-2)}{q^{10}-1}$	IV	1	$\frac{1}{2}\frac{q^6(q-1)}{q^{10}-1}$
$I_{n\geq 3}$	$\varepsilon(n)$	$\frac{1}{2}\frac{(q-1)^2}{q^{n+1}(q^{10}-1)}$	$I_{n\geq 1}^*$	2	$\frac{1}{2}\frac{(q-1)}{q^{-5+n}(q^{10}-1)}$	IV	3	$\frac{1}{2}\frac{q^6(q-1)}{q^{10}-1}$
$I_{n\geq 3}$	n	$\frac{1}{2}\frac{(q-1)^2}{q^{n+1}(q^{10}-1)}$	$I_{n\geq 1}^*$	4	$\frac{1}{2}\frac{(q-1)}{q^{-5+n}(q^{10}-1)}$	IV^*	1	$\frac{1}{2}\frac{q^3(q-1)}{q^{10}-1}$
II	1	$\frac{q^8(q-1)}{q^{10}-1}$	II^*	1	$\frac{q(q-1)}{q^{10}-1}$	IV^*	3	$\frac{1}{2}\frac{q^3(q-1)}{q^{10}-1}$

Table B.1: The $\delta'_{K,\mathfrak{p}}(T,c)$ for $\mathfrak{p}\nmid(6)$ and $e=1$. (Note. $\varepsilon(n):=((-1)^n+3)/2$.)

Proposition 2.1. *For $\mathfrak{p} \mid (3)$ and $e \geq 2$ even, we have*

$$\delta_{K,\mathfrak{p}}(c) = \begin{cases} 1-(q-1)\Bigg[\dfrac{(6q^{4e}+6q^{4e+13}+9q^{4e+12}+13q^{4e+11}+16q^{4e+10})}{6q^{4e-1}(q+1)(q^2+1)(q^4+1)(q^{10}-1)} \\ +\dfrac{(22(q^{4e+9}+q^{4e+8}+q^{4e+7}+q^{4e+6}+q^{4e+5}+q^{4e+4})+16q^{4e+3}+13q^{4e+2}+9q^{4e+1})}{6q^{4e-1}(q+1)(q^2+1)(q^4+1)(q^{10}-1)} & \text{if } n=1, \\ +\dfrac{(2q^{11}+2q^{10}+q^9+q^8+10q^7+7q^6+4q^5-2q^4+q^3-2q^2)}{6q^{4e-1}(q+1)(q^2+1)(q^4+1)(q^{10}-1)}\Bigg] \\ \dfrac{(q-1)(2q^{4e+12}+3q^{4e+10}+5q^{4e+8}+5q^{4e+6}+5q^{4e+4}+3q^{4e+2}+2q^{4e})}{2q^{4e-1}(q^2+1)(q^4+1)(q^{10}-1)} \\ +\dfrac{(q-1)(q^{11}-q^{10}+q^9+2q^6-q^3+2q^2-q)}{2q^{4e-1}(q^2+1)(q^4+1)(q^{10}-1)} & \text{if } n=2, \\ \dfrac{(q-1)(q^{4e+8}+q^{4e+6}+q^{4e+4}+q^{4e+2}+q^{4e}-q^6+q^4-q^2)}{2q^{4e-2}(q^4+1)(q^{10}-1)} & \text{if } n=3, \\ \dfrac{(q-1)(q^{4e+9}+q^{4e+7}+q^{4e+5}+q^{4e+3}+q^{4e+1})}{6q^{4e-2}(q^2+1)(q^4+1)(q^{10}-1)} \\ +\dfrac{(q-1)(2q^9-3q^8+7q^7-6q^6+7q^5-6q^4+7q^3-6q^2+4q-3)}{6q^{4e-2}(q^2+1)(q^4+1)(q^{10}-1)} & \text{if } n=4, \\ \dfrac{(q-1)^2}{2q^{4e+n-8}(q^{10}-1)} & \text{if } n \geq 5. \end{cases}$$

For $\mathfrak{p} \mid (3)$ and $e > 2$ odd, we have

$$
\delta_{K,\mathfrak{p}}(c) = \begin{cases}
1-(q-1)\Bigg[\dfrac{(6q^{4e}+6q^{4e+13}+9q^{4e+12}+13q^{4e+11}+16q^{4e+10})}{6q^{4e-1}(q+1)(q^2+1)(q^4+1)(q^{10}-1)} \\
+\dfrac{(22(q^{4e+9}+q^{4e+8}+q^{4e+7}+q^{4e+6}+q^{4e+5}+q^{4e+4})+16q^{4e+3}+13q^{4e+2}+9q^{4e+1})}{6q^{4e-1}(q+1)(q^2+1)(q^4+1)(q^{10}-1)} & \textit{if } n=1, \\
-\dfrac{(6q^{19}+12q^{18}+6q^{17}+3q^{16}+3q^{15}+4q^{14}+5q^{13}+2q^{12})}{6q^{4e-1}(q+1)(q^2+1)(q^4+1)(q^{10}-1)} \\
+\dfrac{(4q^{11}+16q^{10}+4q^9+4q^8+3q^7+7q^6+3q^5+6q^4+9q^3+6q^2+3q)}{6q^{4e-1}(q+1)(q^2+1)(q^4+1)(q^{10}-1)}\Bigg] \\
\\
\dfrac{(q-1)(2q^{4e+12}+3q^{4e+10}+5q^{4e+8}+5q^{4e+6}+5q^{4e+4}+3q^{4e+2})}{2q^{4e-1}(q^2+1)(q^4+1)(q^{10}-1)} \\
+\dfrac{(q-1)(2q^{16}+3q^{13}-2q^{12}+5q^{11}-4q^{10}+5q^9-8q^8)}{2q^{4e-1}(q^2+1)(q^4+1)(q^{10}-1)} & \textit{if } n=2, \\
+\dfrac{(q-1)(7q^7-6q^6+4q^5-5q^4+2q^3-q^2+2q-1)}{2q^{4e-1}(q^2+1)(q^4+1)(q^{10}-1)} \\
\\
\dfrac{(q-1)(q^{4e+10}+q^{4e+8}+q^{4e+6}+q^{4e+4}+q^{4e+2}}{2q^{4e}(q^4+1)(q^{10}-1)} \\
+\dfrac{(q-1)(q^{14}-q^{12}+q^{11}-q^8+q^7-2q^6+q^5-q^4+q-1)}{2q^{4e}(q^4+1)(q^{10}-1)} & \textit{if } n=3, \\
\\
\dfrac{(q-1)(q^{4e+12}+q^{4e+10}+q^{4e+8}+q^{4e+6}+q^{4e+4})}{6q^{4e+1}(q^2+1)(q^4+1)(q^{10}-1)} \\
+\dfrac{(q-1)(5q^{15}-4q^{14}+5q^{13}-4q^{12}+5q^{11}-4q^{10}+5q^9-5q^8)}{6q^{4e+1}(q^2+1)(q^4+1)(q^{10}-1)} & \textit{if } n=4, \\
+\dfrac{3(q-1)(q^7-q^6+q^5-q^4+q^3-q^2+q-1)}{6q^{4e+1}(q^2+1)(q^4+1)(q^{10}-1)} \\
\\
\dfrac{(q-1)^2}{2q^{4e+n-3}(q^{10}-1)} & \textit{if } n\geq 5.
\end{cases}
$$

Proposition 2.2. *For $\mathfrak{p} \mid (2)$ and $e = 2$, we have*

$$\delta_{K,\mathfrak{p}}(c) = \begin{cases} 1 - \dfrac{(q-1)(6q^{18}+10q^{17}+8q^{16}+7q^{15}+9q^{14}+6q^{13}+6q^{10}+9q^{9})}{6q^{9}(q+1)(q^{10}-1)} & \\ -\dfrac{(q-1)(q^{8}-2q^{7}-q^{6}+2q^{5}-3q^{4}-6q^{3}+6q+3)}{6q^{9}(q+1)(q^{10}-1)} & \text{if } n = 1, \\ \dfrac{(q-1)(2q^{19}+3q^{18}+2q^{17}+q^{16}+2q^{15}+2q^{14}+2q^{11}+2q^{10}))}{2q^{10}(q+1)(q^{10}-1)} & \\ -\dfrac{(q-1)(q^{9}+q^{8}+q^{7}-q^{6}+2q^{4}-2q^{2}+1)}{2q^{10}(q+1)(q^{10}-1)} & \text{if } n = 2, \\ \dfrac{(q-1)(q^{2}+1)(q^{4}-q^{2}+1)(q^{10}+q-1)}{2q^{11}(q^{10}-1)} & \text{if } n = 3, \\ \dfrac{(q-1)(q^{19}+q^{18}+q^{10}-q^{8}+3q-3)}{6q^{12}(q^{10}-1)} & \text{if } n = 4, \\ \dfrac{(q-1)^{2}}{2q^{8+n}(q^{10}-1)} & \text{if } n \geq 5. \end{cases}$$

Type	$c_{\mathfrak{p}}$	e even	e odd
I_0	1	$\left[\left(\frac{(q-1)}{q^{4e+1}}\right)\left(\frac{q-1}{q}\right)+\left(\frac{(q-1)}{q^{4e+2}}\right)\left(\frac{q-1}{q}\right)+\left(1+\frac{(q^2-1)(q^8-q^{16-4e})}{q^{10}(q^8-1)}\right)\left(\frac{q-1}{q}\right)\right]\frac{q^{10}}{q^{10}-1}$	$\left[\left(\frac{(q-1)}{q^{4e+6}}\right)\left(\frac{q-1}{q}\right)+\left(\frac{(q-1)}{q^{4e-3}}\right)\left(\frac{q-1}{q}\right)+\left(1+\frac{(q^2-1)(q^8-q^{12-4e})}{q^{10}(q^8-1)}+\frac{q-1}{q^{4e-2}}\right)\left(\frac{q-1}{q}\right)\right]\frac{q^{10}}{q^{10}-1}$
I_1	1	$\left[\left(\frac{(q-1)}{q^{4e+1}}\right)\left(\frac{q-1}{q^2}\right)\right]\frac{q^{10}}{q^{10}-1}$	$\left[\left(\frac{(q-1)}{q^{4e+6}}\right)\left(\frac{q-1}{q^2}\right)\right]\frac{q^{10}}{q^{10}-1}$
I_2	2	$\left[\left(\frac{(q-1)}{q^{4e+1}}\right)\left(\frac{q-1}{q^3}\right)\right]\frac{q^{10}}{q^{10}-1}$	$\left[\left(\frac{(q-1)}{q^{4e+6}}\right)\left(\frac{q-1}{q^3}\right)\right]\frac{q^{10}}{q^{10}-1}$
$I_n\geq 3$	n	$\left[\left(\frac{(q-1)}{q^{4e+1}}\right)\left(\frac{q-1}{2q^{n+1}}\right)\right]\frac{q^{10}}{q^{10}-1}$	$\left[\left(\frac{(q-1)}{q^{4e+6}}\right)\left(\frac{q-1}{2q^{n+1}}\right)\right]\frac{q^{10}}{q^{10}-1}$
$I_n\geq 3$	$\varepsilon(c)$	$\left[\left(\frac{(q-1)}{q^{4e+1}}\right)\left(\frac{q-1}{2q^{n+1}}\right)\right]\frac{q^{10}}{q^{10}-1}$	$\left[\left(\frac{(q-1)}{q^{4e+6}}\right)\left(\frac{q-1}{2q^{n+1}}\right)\right]\frac{q^{10}}{q^{10}-1}$
II	1	$\left[\left(\frac{(q-1)}{q^{4e+2}}\right)\left(\frac{q-1}{q^2}\right)+\left(1+\frac{(q^2-1)(q^8-q^{16-4e})}{q^{10}(q^8-1)}\right)\left(\frac{q-1}{q^2}\right)\right]\frac{q^{10}}{q^{10}-1}$	$\left[\left(\frac{(q-1)}{q^{4e-3}}\right)\left(\frac{q-1}{q^2}\right)+\left(1+\frac{(q^2-1)(q^8-q^{12-4e})}{q^{10}(q^8-1)}+\frac{q-1}{q^{4e-2}}\right)\left(\frac{q-1}{q^2}\right)\right]\frac{q^{10}}{q^{10}-1}$
III	2	$\left[\left(\frac{(q-1)}{q^{4e+2}}\right)\left(\frac{q-1}{q^3}\right)+\left(1+\frac{(q^2-1)(q^8-q^{16-4e})}{q^{10}(q^8-1)}\right)\left(\frac{q-1}{q^3}\right)\right]\frac{q^{10}}{q^{10}-1}$	$\left[\left(\frac{(q-1)}{q^{4e-3}}\right)\left(\frac{q-1}{q^3}\right)+\left(1+\frac{(q^2-1)(q^8-q^{12-4e})}{q^{10}(q^8-1)}+\frac{q-1}{q^{4e-2}}\right)\left(\frac{q-1}{q^3}\right)\right]\frac{q^{10}}{q^{10}-1}$
IV	1	$\left[\left(\frac{(q-1)}{q^{4e+2}}\right)\left(\frac{q-1}{2q^4}\right)+\left(1+\frac{(q^2-1)(q^8-q^{16-4e})}{q^{10}(q^8-1)}\right)\left(\frac{q-1}{2q^4}\right)\right]\frac{q^{10}}{q^{10}-1}$	$\left[\left(\frac{(q-1)}{q^{4e-3}}\right)\left(\frac{q-1}{2q^4}\right)+\left(1+\frac{(q^2-1)(q^8-q^{12-4e})}{q^{10}(q^8-1)}+\frac{q-1}{q^{4e-2}}\right)\left(\frac{q-1}{2q^4}\right)\right]\frac{q^{10}}{q^{10}-1}$
IV	3	$\left[\left(\frac{(q-1)}{q^{4e+2}}\right)\left(\frac{q-1}{2q^4}\right)+\left(1+\frac{(q^2-1)(q^8-q^{16-4e})}{q^{10}(q^8-1)}\right)\left(\frac{q-1}{2q^4}\right)\right]\frac{q^{10}}{q^{10}-1}$	$\left[\left(\frac{(q-1)}{q^{4e-3}}\right)\left(\frac{q-1}{2q^4}\right)+\left(1+\frac{(q^2-1)(q^8-q^{12-4e})}{q^{10}(q^8-1)}+\frac{q-1}{q^{4e-2}}\right)\left(\frac{q-1}{2q^4}\right)\right]\frac{q^{10}}{q^{10}-1}$
I_0^*	1	$\left[\left(\frac{(q-1)}{q^{4e+2}}\right)\left(\frac{1}{3q^5}\right)+\left(1+\frac{(q^2-1)(q^8-q^{16-4e})}{q^{10}(q^8-1)}\right)\left(\frac{q-1}{3q^5}\right)\right]\frac{q^{10}}{q^{10}-1}$	$\left[\left(\frac{(q-1)}{q^{4e-3}}\right)\left(\frac{1}{3q^5}\right)+\left(1+\frac{(q^2-1)(q^8-q^{12-4e})}{q^{10}(q^8-1)}+\frac{q-1}{q^{4e-2}}\right)\left(\frac{q-1}{3q^5}\right)\right]\frac{q^{10}}{q^{10}-1}$
I_0^*	2	$\left[\left(\frac{(q-1)}{q^{4e+2}}\right)\left(\frac{q-1}{2q^6}\right)+\left(1+\frac{(q^2-1)(q^8-q^{16-4e})}{q^{10}(q^8-1)}\right)\left(\frac{q-1}{2q^5}\right)\right]\frac{q^{10}}{q^{10}-1}$	$\left[\left(\frac{(q-1)}{q^{4e-3}}\right)\left(\frac{q-1}{2q^6}\right)+\left(1+\frac{(q^2-1)(q^8-q^{12-4e})}{q^{10}(q^8-1)}+\frac{q-1}{q^{4e-2}}\right)\left(\frac{q-1}{2q^5}\right)\right]\frac{q^{10}}{q^{10}-1}$
I_0^*	4	$\left[\left(\frac{(q-1)}{q^{4e+2}}\right)\left(\frac{q-3}{6q^6}\right)+\left(1+\frac{(q^2-1)(q^8-q^{16-4e})}{q^{10}(q^8-1)}\right)\left(\frac{q-1}{6q^5}\right)\right]\frac{q^{10}}{q^{10}-1}$	$\left[\left(\frac{(q-1)}{q^{4e-3}}\right)\left(\frac{q-3}{6q^6}\right)+\left(1+\frac{(q^2-1)(q^8-q^{12-4e})}{q^{10}(q^8-1)}+\frac{q-1}{q^{4e-2}}\right)\left(\frac{q-1}{6q^5}\right)\right]\frac{q^{10}}{q^{10}-1}$
I_n^*	2	$\left[\left(\frac{(q-1)}{q^{4e+2}}\right)\left(\frac{q-1}{2q^{5+n}}\right)\right]\frac{q^{10}}{q^{10}-1}$	$\left[\left(\frac{(q-1)}{q^{4e-3}}\right)\left(\frac{q-1}{2q^{5+n}}\right)\right]\frac{q^{10}}{q^{10}-1}$
I_n^*	4	$\left[\left(\frac{(q-1)}{q^{4e+2}}\right)\left(\frac{q-1}{2q^{5+n}}\right)\right]\frac{q^{10}}{q^{10}-1}$	$\left[\left(\frac{(q-1)}{q^{4e-3}}\right)\left(\frac{q-1}{2q^{5+n}}\right)\right]\frac{q^{10}}{q^{10}-1}$
IV^*	1	$\left[\left(1+\frac{(q^2-1)(q^8-q^{16-4e})}{q^{10}(q^8-1)}\right)\left(\frac{q-1}{2q^6}\right)\right]\frac{q^{10}}{q^{10}-1}$	$\left[\left(1+\frac{(q^2-1)(q^8-q^{12-4e})}{q^{10}(q^8-1)}+\frac{q-1}{q^{4e-2}}\right)\left(\frac{q-1}{2q^6}\right)\right]\frac{q^{10}}{q^{10}-1}$
IV^*	3	$\left[\left(1+\frac{(q^2-1)(q^8-q^{16-4e})}{q^{10}(q^8-1)}\right)\left(\frac{q-1}{2q^6}\right)\right]\frac{q^{10}}{q^{10}-1}$	$\left[\left(1+\frac{(q^2-1)(q^8-q^{12-4e})}{q^{10}(q^8-1)}+\frac{q-1}{q^{4e-2}}\right)\left(\frac{q-1}{2q^6}\right)\right]\frac{q^{10}}{q^{10}-1}$
III^*	2	$\left[\left(1+\frac{(q^2-1)(q^8-q^{16-4e})}{q^{10}(q^8-1)}\right)\left(\frac{q-1}{q^7}\right)\right]\frac{q^{10}}{q^{10}-1}$	$\left[\left(1+\frac{(q^2-1)(q^8-q^{12-4e})}{q^{10}(q^8-1)}+\frac{q-1}{q^{4e-2}}\right)\left(\frac{q-1}{q^7}\right)\right]\frac{q^{10}}{q^{10}-1}$
II^*	1	$\left[\left(1+\frac{(q^2-1)(q^8-q^{16-4e})}{q^{10}(q^8-1)}\right)\left(\frac{q-1}{q^8}\right)\right]\frac{q^{10}}{q^{10}-1}$	$\left[\left(1+\frac{(q^2-1)(q^8-q^{12-4e})}{q^{10}(q^8-1)}+\frac{q-1}{q^{4e-2}}\right)\left(\frac{q-1}{q^8}\right)\right]\frac{q^{10}}{q^{10}-1}$

Table B.2: $\delta_{K,\mathfrak{p}}(T, c_{\mathfrak{p}})$ for $e \geq 2$ and $\mathfrak{p} \mid (3)$.

Type	$c_{\mathfrak{p}}$	$e=1$	$e=2$
I_0	1	$\frac{q(q-1)}{q^{10}-1}$	$\frac{(q-1)(q^{10}+q-1)}{q^8(q^{10}-1)}$
I_1	1	$\frac{(q-1)^2}{q(q^{10}-1)}$	$\frac{(q-1)^2}{q^9(q^{10}-1)}$
I_2	2	$\frac{(q-1)^2}{q^2(q^{10}-1)}$	$\frac{(q-1)^2}{q^{10}(q^{10}-1)}$
$I_{n\geq 3}$	n	$\frac{1}{2}\frac{(q-1)^2}{q^n(q^{10}-1)}$	$\frac{1}{2}\frac{(q-1)^2}{q^{8+n}(q^{10}-1)}$
$I_{n\geq 3}$	$\varepsilon(c)$	$\frac{1}{2}\frac{(q-1)^2}{q^n(q^{10}-1)}$	$\frac{1}{2}\frac{(q-1)^2}{q^{8+n}(q^{10}-1)}$
II	1	$\frac{q^9(q-1)}{q^{10}-1}$	$\frac{(q-1)(q^{10}+q^2-1)}{q(q^{10}-1)}$
III	2	$\frac{q^8(q-1)}{q^{10}-1}$	$\frac{(q-1)(q^{10}+q^2-1)}{q^2(q^{10}-1)}$
IV	1	$\frac{1}{2}\frac{q^7(q-1)}{q^{10}-1}$	$\frac{1}{2}\frac{(q-1)}{q(q^{10}-1)}$
IV	3	$\frac{1}{2}\frac{q^7(q-1)}{q^{10}-1}$	$\frac{1}{2}\frac{(q-1)}{q(q^{10}-1)}$
I_0^*	1	$\frac{1}{3}\frac{q^5(q^2-1)}{q^{10}-1}$	$\frac{1}{3}\frac{(q^2-1)(q^{10}+q-1)}{q^4(q^{10}-1)}$
I_0^*	2	$\frac{1}{2}\frac{q^6(q-1)}{q^{10}-1}$	$\frac{1}{2}\frac{(q-1)(q^{10}+q-1)}{q^3(q^{10}-1)}$
I_0^*	4	$\frac{1}{6}\frac{q^5(q-1)(q-2)}{q^{10}-1}$	$\frac{1}{6}\frac{(q-1)(q-2)(q^{10}+q-1)}{q^4(q^{10}-1)}$
I_n^*	2	$\frac{1}{2}\frac{(q-1)^2}{q^{-5+n}(q^{10}-1)}$	$\frac{1}{2}\frac{(q-1)^2(q^{10}+q-1)}{q^{4+n}(q^{10}-1)}$
I_n^*	4	$\frac{1}{2}\frac{(q-1)^2}{q^{-5+n}(q^{10}-1)}$	$\frac{1}{2}\frac{(q-1)^2(q^{10}+q-1)}{q^{4+n}(q^{10}-1)}$
IV^*	1	$\frac{1}{2}\frac{q^4(q-1)}{q^{10}-1}$	$\frac{1}{2}\frac{(q-1)(q^{10}+q-1)}{q^5(q^{10}-1)}$
IV^*	3	$\frac{1}{2}\frac{q^4(q-1)}{q^{10}-1}$	$\frac{1}{2}\frac{(q-1)(q^{10}+q-1)}{q^5(q^{10}-1)}$
III^*	2	$\frac{q^3(q-1)}{q^{10}-1}$	$\frac{(q-1)(q^{10}+q-1)}{q^6(q^{10}-1)}$
II^*	1	$\frac{q^2(q-1)}{q^{10}-1}$	$\frac{(q-1)(q^{10}+q-1)}{q^7(q^{10}-1)}$

Table B.3: The $\delta_{K,\mathfrak{p}}(T,c)$ for $\mathfrak{p} \mid (2)$ and $e=1,2$. (Note $\varepsilon(n) := ((-1)^n+3)/2$.)

Type	$c_{\mathfrak{p}}$	$e \equiv 0 \mod 3$, $e = 3k$	$e \equiv 1 \mod 3$, $e = 3k+1$	$e \equiv 2 \mod 3$, $e = 3k+2$
I_0	1	$\frac{(q-1)}{q}A + B_0$	$\frac{(q-1)}{q}A + B_1$	$\frac{(q-1)}{q}A + B_2$
I_1	1	$\frac{(q-1)}{q^2}A$	$\frac{(q-1)}{q^2}A$	$\frac{(q-1)}{q^2}A$
I_2	2	$\frac{(q-1)}{q^3}A$	$\frac{(q-1)}{q^3}A$	$\frac{(q-1)}{q^3}A$
$I_{n\geq 3}$	n	$\frac{(q-1)}{2q^{n+1}}A$	$\frac{(q-1)}{2q^{n+1}}A$	$\frac{(q-1)}{2q^{n+1}}A$
$I_{n\geq 3}$	$\epsilon(c)$	$\frac{(q-1)}{2q^{n+1}}A$	$\frac{(q-1)}{2q^{n+1}}A$	$\frac{(q-1)}{2q^{n+1}}A$
II	1	$\frac{(q-1)}{q}\left(C + D_0 + E + F_0 + G_0\right)$	$\frac{(q-1)}{q}\left(C + D_1 + E + F_1 + G_1\right)$	$\frac{(q-1)}{q}\left(C + D_2 + E + F_2 + G_2\right)$
III	2	$\frac{(q-1)}{q^2}\left(C + D_0 + E + F_0 + G_0\right)$	$\frac{(q-1)}{q^2}\left(C + D_1 + E + F_1 + G_1\right)$	$\frac{(q-1)}{q^2}\left(C + D_2 + E + F_2 + G_2\right)$
IV	1	$\frac{(q-1)}{2q^3}C + \frac{1}{2q^2}D_0$	$\frac{(q-1)}{2q^3}C + \frac{1}{2q^2}D_1$	$\frac{(q-1)}{2q^3}C + \frac{1}{2q^2}D_2$
IV	3	$\frac{(q-1)}{2q^3}C + \frac{1}{2q^2}D_0$	$\frac{(q-1)}{2q^3}C + \frac{1}{2q^2}D_1$	$\frac{(q-1)}{2q^3}C + \frac{1}{2q^2}D_2$
I_0^*	1	$\frac{(q^2-1)}{3q^4}\left(\frac{1}{q}C + E + F_0 + G_0\right)$	$\frac{(q^2-1)}{3q^4}\left(\frac{1}{q}C + E + F_1 + G_1\right)$	$\frac{(q^2-1)}{3q^4}\left(\frac{1}{q}C + E + F_2 + G_2\right)$
I_0^*	2	$\frac{(q-1)}{2q^3}\left(\frac{1}{q}C + E + F_0 + G_0\right)$	$\frac{(q-1)}{2q^3}\left(\frac{1}{q}C + E + F_1 + G_1\right)$	$\frac{(q-1)}{2q^3}\left(\frac{1}{q}C + E + F_2 + G_2\right)$
I_0^*	4	$\frac{(q-1)(q-2)}{6q^4}\left(\frac{1}{q}C + E + F_0 + G_0\right)$	$\frac{(q-1)(q-2)}{6q^4}\left(\frac{1}{q}C + E + F_1 + G_1\right)$	$\frac{(q-1)(q-2)}{6q^4}\left(\frac{1}{q}C + E + F_2 + G_2\right)$
I_n^*	2	$\frac{(q-1)^2}{2q^{4+n}}\left(\frac{1}{q}C + E + F_0 + G_0\right)$	$\frac{(q-1)^2}{2q^{4+n}}\left(\frac{1}{q}C + E + F_1 + G_1\right)$	$\frac{(q-1)^2}{2q^{4+n}}\left(\frac{1}{q}C + E + F_2 + G_2\right)$
I_n^*	4	$\frac{(q-1)^2}{2q^{4+n}}\left(\frac{1}{q}C + E + F_0 + G_0\right)$	$\frac{(q-1)^2}{2q^{4+n}}\left(\frac{1}{q}C + E + F_1 + G_1\right)$	$\frac{(q-1)^2}{2q^{4+n}}\left(\frac{1}{q}C + E + F_2 + G_2\right)$
IV^*	1	$\frac{(q-1)}{2q^6}C + \frac{(q-1)}{2q^5}E + \frac{1}{q^4}F_0$	$\frac{(q-1)}{2q^6}C + \frac{(q-1)}{2q^5}E + \frac{1}{q^4}F_1$	$\frac{(q-1)}{2q^6}C + \frac{(q-1)}{2q^5}E + \frac{1}{q^4}F_2$
IV^*	3	$\frac{(q-1)}{2q^6}C + \frac{(q-1)}{2q^5}E + \frac{1}{q^4}F_0$	$\frac{(q-1)}{2q^6}C + \frac{(q-1)}{2q^5}E + \frac{1}{q^4}F_1$	$\frac{(q-1)}{2q^6}C + \frac{(q-1)}{2q^5}E + \frac{1}{q^4}F_2$
III^*	2	$\frac{(q-1)}{q^5}\left(\frac{1}{q^2}C + \frac{1}{q}E + G_0\right)$	$\frac{(q-1)}{q^5}\left(\frac{1}{q^2}C + \frac{1}{q}E + G_1\right)$	$\frac{(q-1)}{q^5}\left(\frac{1}{q^2}C + \frac{1}{q}E + G_2\right)$
II^*	1	$\frac{(q-1)}{q^6}\left(\frac{1}{q^2}C + \frac{1}{q}E + G_0\right)$	$\frac{(q-1)}{q^6}\left(\frac{1}{q^2}C + \frac{1}{q}E + G_1\right)$	$\frac{(q-1)}{q^6}\left(\frac{1}{q^2}C + \frac{1}{q}E + G_2\right)$

Table B.4: The $\delta'_{K,\mathfrak{p}}(T, c)$ for $\mathfrak{p} \mid (2)$ and $e \geq 3$.

A	$\frac{q^{10}}{(q^{10}-1)}\left(\frac{(q-1)}{q^{8e+1}}\right)$
B_0	$\frac{q^{10}}{(q^{10}-1)}\left(\frac{(q-1)(q^{17k+1}+q^{18k+1}-q^{18k}+q^{10}-q^{9})}{q^{24k+2}}+\sum_{i-0}^{k-2}\frac{(q-1)^2(q^{17i+17}+q^{9}+1)}{q^{6k+18i+20}}\right)$
B_1	$\frac{q^{10}}{(q^{10}-1)}\left(\frac{q^{17k+6}+q^{23k+1}-q^{23k}+q^{25}-2q^{24}+q^{23}+q^{16}-2q^{15}+q^{14}}{q^{24k+15}}+\sum_{i-0}^{k-2}\frac{(q-1)(q^{17i+9}+q^{10}-q^{9}+q-1)}{q^{6k+18i+19}}\right)$
B_2	$\frac{q^{10}}{(q^{10}-1)}\left(\frac{(q-1)(q^{17k+1}+q^{18k+1}-q^{18k}+q^{10}-q^{9}+q-1)}{q^{24k+9}}+\sum_{i-0}^{k-2}\frac{(q-1)(q^{17i+18}-q^{17i+17}+q^{10}-q^{9}+1)}{q^{6k+18i+27}}\right)$
C	$\frac{q^2(q^8-1)}{(q^{10}-1)}\left(\frac{(q-1)}{q^{8e-7}}\right)$
D_0	$\frac{q^{10}}{(q^{10}-1)}\left(\frac{(q-1)(q^{k+1}-q^{k}+q)}{q^{7k+3}}+\sum_{i=0}^{k-2}\frac{(q-1)(q^{17i+18}-q^{17i+17}+q^{10}-q^{9}+1)}{q^{6k+18i+21}}\right)$
D_1	$\frac{q^{10}}{(q^{10}-1)}\left(\frac{(q-1)(q^{17k+1}+q^{18k+1}-q^{18k}+q^{10}-q^{9})}{q^{24k+2}}+\sum_{i=0}^{k-2}\frac{(q-1)(q^{17i+18}-q^{17i+17}+q^{10}-q^{9}+1)}{q^{6k+18i+20}}\right)$
D_2	$\frac{q^{10}}{(q^{10}-1)}\left(\frac{1}{q^{7k+9}}+\sum_{i=0}^{k-1}\frac{(q-1)(q^{17i+10}-q^{17i+9}+q^{10}-q^{9}+1)}{q^{6k+18i+19}}\right)$
E	$\frac{q^3(q^7-1)}{(q^{10}-1)}\left(\frac{(q-1)}{q^{8e-15}}\right)$
F_0	$\frac{q^{10}}{(q^{10}-1)}\left(\frac{1}{q^{7k+2}}+\sum_{i=0}^{k-2}\frac{(q-1)(q^{17i+9}+q^{10}-q^{9}+q-1)}{q^{6k+18i+13}}\right)$
F_1	$\frac{q^{10}}{(q^{10}-1)}\left(\frac{(q-1)(q^{k+1}-q^{k}+q)}{q^{7k+3}}+\sum_{i=0}^{k-2}\frac{(q-1)(q^{17i+18}-q^{17i+17}+q^{10}-q^{9}+1)}{q^{6k+18i+21}}\right)$
F_2	$\frac{q^{10}}{(q^{10}-1)}\left(\frac{(q-1)(q^{17k+1}+q^{18k+1}-q^{18k}+q^{10}-q^{9}}{q^{24k+2}}+\sum_{i=0}^{k-2}\frac{(q-1)(q^{17i+10}-q^{17i+9}+q^{10}-q^{9}+1)}{q^{6k+18i+13}}\right)$
G_0	$\frac{q^4(q^6-1)}{(q^{10}-1)}\Bigg(\frac{(q-1)(q^{8}-q^{32-24k})}{q^{17}-q^{9}}+\sum_{i=0}^{k-2}\frac{(q^{i+3}-q^{i+2}-q^{i+1}+q^{i}+q^{3}-q^{2}+q)}{q^{7i+9}}$ $+\sum_{i=1}^{k-2}\frac{(q^3-2q+1)}{q^{17}}\left[\sum_{j=1}^{i}\left(\frac{1}{q^6}\right)^{j-1}\left(\frac{1}{q^7}\right)^{k-2-i}\right]+\sum_{i=1}^{k-1}\left[\frac{(q-1)^2}{q^{24i-7}}\sum_{j=0}^{k-i-1}\frac{1}{q^{6j}}\right]$ $+\sum_{i=1}^{k-3}\left[\frac{(q-1)^2}{q^{24i+16}}\sum_{j=0}^{k-i-3}\frac{1}{q^{6j}}\right]+\sum_{i=1}^{k-2}\left[\frac{(q-1)^2(q^9+q^8+q+1)}{q^{24i+9}}\sum_{j=0}^{k-i-2}\frac{1}{q^{6j}}\right]\Bigg)$
G_1	$\frac{q^4(q^6-1)}{(q^{10}-1)}\Bigg(\frac{(q-1)(q^{8}-q^{32-24k})}{q^{17}-q^{9}}+\frac{q^{k+2}-2q^{k+1}+q^{k}+q}{q^{7k+3}}+\sum_{i=0}^{k-2}\frac{(q^{i+3}-q^{i+2}-q^{i+1}+q^{i}+q^{3}-q^{2}+q)}{q^{7i+9}}$ $+\frac{(q-1)}{q^{16}}\sum_{j=1}^{k-1}\left(\frac{1}{q^6}\right)^{j-1}\left(\frac{1}{q^7}\right)^{k-2-i}+\sum_{i=1}^{k-2}\left[\frac{(q^3-2q+1)}{q^{17}}\sum_{j=1}^{i}\left(\frac{1}{q^6}\right)^{j-1}\left(\frac{1}{q^7}\right)^{k-2-i}\right]$ $+\sum_{i=1}^{k-1}\left[\frac{(q-1)^2(q^8+1)}{q^{24i+1}}\sum_{j=0}^{k-1-i}\left(\frac{1}{q^6}\right)^{j}\right]+\sum_{i=1}^{k-2}\left[\frac{(q-1)^2(q^{16}+q^8+q^7+1)}{q^{24i+16}}\sum_{j=0}^{k-2-i}\left(\frac{1}{q^6}\right)^{j}\right]\Bigg)$
G_2	$\frac{q^4(q^6-1)}{(q^{10}-1)}\Bigg(\frac{(q-1)(q^{8}-q^{32-24k})}{(q^{17}-q^{9})}+\frac{(q^{k+3}-q^{k+2}-q^{k+1}+q^{k}+q^{2})}{q^{7k+3}}+\sum_{i=0}^{k-2}\frac{(q^{i+3}-q^{i+2}-q^{i+1}+q^{i}+q^{3}-q^{2}+q)}{q^{7i+9}}$ $+\frac{(q-1)}{q^{15}}\sum_{j=1}^{k-1}\left[\left(\frac{1}{q^6}\right)^{j-1}\left(\frac{1}{q^7}\right)^{k-2-i}\right]+\sum_{i=1}^{k-2}\left[\frac{(q^3-2q+1)}{q^{17}}\sum_{j=1}^{i}\left(\frac{1}{q^6}\right)^{j-1}\left(\frac{1}{q^7}\right)^{k-2-i}\right]$ $+\sum_{i=1}^{k-1}\left[\frac{(q-1)^2(q^{16}+q^8+1)}{q^{24i+9}}\sum_{j=0}^{k-1-i}\left(\frac{1}{q^6}\right)^{j}\right]+\sum_{i=1}^{k-2}\left[\frac{(q-1)^2(q^{16}+q^8+1)}{q^{24i+16}}\sum_{j=0}^{k-2-i}\left(\frac{1}{q^6}\right)^{j}\right]\Bigg)$

Table B.5: Formulae for variables in Table B.4.

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