Bispans in Quasicategories

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Abstract

A Tambara functor is an algebraic system indexed by the subgroups of a fixed finite group, possessing additive and multiplicative inductions along subgroup inclusions as well as a twisted distributive law between the two inductions. These arise in equivariant homotopy theory and group representation theory, with examples coming from representation rings, generalized character rings, equivariant K-theory, Burnside rings, group cohomology, algebraic K-theory, and homotopy groups of equivariant E_∞-ring spectra. Tambara functors are defined using bispan categories, which simultaneously encode the inductive system and distributive law. Many Tambara functors can be defined in a compatible manner for all groups at once, suggesting the notion of a global Tambara functor. Encoding the distributivity properties for global equivariant phenomena suggests passage to bispans in the bicategory of finite groupoids, but the complicated nature of bispan composition means that an axiomatic elaboration of global Tambara functors has yet to be provided, although work of Schwede suggests a relationship to global Mackey functors with power operations. In this thesis, we provide a quasicategorical bispan construction, initiating the development of the higher categorical framework needed to study global Tambara functors.

We provide a construction of a quasicategory of bispans in a locally cartesian closed quasicategory which is compatible with the span quasicategory of Barwick. In particular, we obtain a quasicategory whose simplices consist of diagrams encoding higher composites of bispans. To this end, we develop and study quasicategorical analogues of exponential diagrams, which are the categorical construction governing composition in bispan categories.

We first generalize a theory of bispans in the category of finite sets appearing in the thesis of Cranch, creating a “decomposed” bispan quasicategory. The decomposed bispan diagrams of Cranch are sub-simplicial sets of the bispan diagrams used for the main construction, and we establish pleasant properties of these inclusions. With these results in hand, we prove that what is *a priori* a simplicial set of bispans is in fact trivially fibered over the decomposed bispan quasicategory, thus obtaining the desired quasicategory of bispans.

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Contents

Chapter 1

Introduction

1.1 Inductive systems and distributivity in the equivariant context

While the results and techniques of this thesis lie firmly within the realm of higher category theory, the motivations arise from the study of algebraic structures called *Tambara functors* arising in representation theory and equivariant homotopy theory. In the G-equivariant setting, analogues of algebraic invariants of spaces such as homotopy and homology groups gain richness and subtlety from the structure of the acting group. Instead of assigning a single group or ring to a space, the invariants instead assign to a G-space a system of objects indexed by the subgroups of G. These systems often have an intricate structure of maps between the objects assigned to the subgroups, reminiscent of the operations of induction and restriction of group representations.

The theories of Mackey functors and Tambara functors describe some of these underlying structures. An an example, let $R(G)$ denote the representation ring of a finite group G. If $K \leq H \leq G$ are subgroups, then there are operations of restriction

$R^H_K\colon R(H) \to R(K)$ and induction $T^H_K\colon R(K) \to R(H)$, with $T^H_K(V) = \bigoplus_{hK \in H/K} V$. These operations satisfy certain identities such as the Mackey double coset formula, which generalize the axioms defining R(-) as a Mackey functor for the group G. Replacing the direct sum with a tensor product defines the multiplicative induction N^H_K and a multiplicative Mackey functor structure on R(-). These operations can be depicted in a diagram as follows:

$$\begin{tikzcd} R(H) \arrow[d, bend right, "R^H_K"'] \\ R(K). \arrow[u, "N^H_K"] \arrow[u, bend right, "T^H_K"'] \end{tikzcd}$$

The maps T^H_K and N^H_K are additive and multiplicative homomorphisms, respectively.

Refining $R(-)$ to a Tambara functor reveals a twisted distributive law between the two forms of induction. Tambara's insight in [18] was to encode inductive distributivity by a construction in the category of finite G-sets. With regard to the running example, let $K_G(X)$ denote the Grothendieck ring of G-equivariant complex vector bundles over a G-set X. Then $K_G(G/H) \cong R(H)$, and if X is a disjoint union of orbits, given as $X \cong \sqcup G/H_i$, then $K_G(X)$ is the direct sum of representation rings: $K_G(X) \cong \bigoplus K_G(G/H_i)$.

A map $f\colon X \to Y$ of G-sets induces maps between the equivariant K-theory rings. For G-bundles V and W over X and Y, respectively, the *transfer, norm* and *restriction* maps T_f, N_f and R_f induced by f are given on fibers as follows:

$$T_f(V)_y = \bigoplus_{x \in f^{-1}(y)} V_x$$

$$N_f(V)_y = \bigotimes_{x \in f^{-1}(y)} V_x$$

$$R_f(W)_x = W_{f(x)}.$$

In a similar vein to the preceding discussion of representation rings, these maps fit into a diagram

$$\begin{tikzcd} K_G(Y) \arrow[d, bend right, "R_f"'] \\ K_G(X). \arrow[u, "N_f"] \arrow[u, bend right, "T_f"'] \end{tikzcd}$$

If $f\colon X \to Y$ is the collapse map $G/K \to G/H$ for a subgroup inclusion $K \leq H$, then f induces maps between the representation rings $K_G(G/K) \cong R(K)$ and $K_G(G/H) \cong R(H)$, and they satisfy the equalities $T_f = T_K^H, N_f = N_K^H$ and $R_f = R_K^H$.

Given maps of G-sets $A \xrightarrow{p} X \xrightarrow{f} Y$, one can form the composite $N_f T_p\colon K_G(A) \to K_G(Y)$, defined on fibers by

$$N_f T_p(V)_y = \bigotimes_{x \in f^{-1}(y)} \bigoplus_{a \in p^{-1}(y)} V_a.$$

The distribution of the multiplicative induction through the additive induction is via the formation of *exponential diagrams* in $\mathcal{F}in_G$. Given p and f as above, an exponential diagram is a diagram of the form

$$\begin{tikzcd} & C \arrow[r, "h"] \arrow[dl, "e"'] \arrow[d] & D \arrow[d, "v"] \\ A \arrow[r, "p"] & X \arrow[r, "f"] & Y \end{tikzcd}$$

where the square on the right is a pullback. Exponential diagrams are characterized as being terminal amongst all diagrams of such shape whose right square is a pullback and where the maps p and f are fixed. For a detailed discussion of exponential diagrams in $\mathcal{F}in_G$, see Chapters 4 and 5 in [17] and §1 of [18]. A formal definition in the general setting of quasicategories is given in Definition 3.2.2. In the category $\mathcal{F}in_G$, there is an explicit description of D as the sections of p on the fibers of f:

$$D \cong \{(y, \alpha) \mid y \in Y, \alpha\colon f^{-1}(\{y\}) \to A, p\alpha = \mathrm{id}_{f^{-1}(y)}\}.$$

The salient property of an exponential diagram such as the one above is that there is an equality $N_f T_p = T_v N_h R_\varepsilon$. In particular, the right-hand side of this equality can be described on the fibers of a bundle by

$$T_v N_h R_\varepsilon(V)_y = \bigoplus_{d \in v^{-1}(y)} \bigotimes_{c \in h^{-1}(d)} V_{\varepsilon(m)}$$

and there is an equality

$$\bigotimes_{x \in f^{-1}(y)} \bigoplus_{a \in p^{-1}(y)} V_a = \bigoplus_{d \in v^{-1}(y)} \bigotimes_{c \in h^{-1}(d)} V_{\varepsilon(c)}$$

expressing the distributivity of multiplicative induction through additive induction. In particular, given $L \leq K \leq H \leq G$, the exponential diagram above the composite $G/L \to G/K \to G/H$ can be calculated, revealing a distributive formula for $N_K^H T_L^K$, which is otherwise difficult to calculate.

1.2 Recollections on Mackey and Tambara functors

Before discussing Tambara functors in further detail, it is helpful to consider the simpler construction of Mackey functors. Lindner [8] shows that the category of Mackey functors for a finite group G is the category of product-preserving functors $\mathcal{S}pan(\mathcal{F}in_G) \to \mathcal{S}et$, where $\mathcal{S}pan(\mathcal{F}in_G)$ has objects finite G-sets, and morphisms from X to Y given by equivalence classes $[X \leftarrow A \to Y]$ of diagrams called *spans*, where two spans are equivalent if there is an isomorphism between the middle terms (called the *roofs*) compatible with the maps to X and Y:

$$\begin{tikzcd} & A \arrow[dl] \arrow[dr] \arrow[dd, "\cong"] & \\ X & & Y. \\ & A' \arrow[ul] \arrow[ur] & \end{tikzcd}$$

The composition of spans is given by pullbacks, as is illustrated in diagram below. The a representative of the equivalence class of the composite $[Y \leftarrow B \to Z] \circ [X \leftarrow A \to Y]$ is calculated by placing the spans next to each other and forming a pullback along incident arrows, and then composing with the outside legs, obtaining the span with curved legs whose roof is P:

$$\begin{tikzcd} & & P \arrow[dl] \arrow[dr] \arrow[ddll, bend right] \arrow[ddrr, bend left] & & \\ & A \arrow[dl] \arrow[dr] & & B \arrow[dl] \arrow[dr] & \\ X & & Y & & Z. \end{tikzcd} \tag{1.1}$$

Any morphism $\omega = [X \xleftarrow{f} A \xrightarrow{g} Y]$ in $\mathcal{S}pan(\mathcal{F}in_G)$ factors as a composite $\omega = T_g R_f$ of two spans of a distinct form: $T_g = [A \xleftarrow{=} A \xrightarrow{g} Y]$ and $R_f = [X \xleftarrow{f} A \xrightarrow{=} A]$, as demonstrated in the following diagram:

$$\begin{tikzcd}
 & & A \arrow[ddll, bend right, "f"'] \arrow[ddrr, bend left, "g"] \arrow[dl, equal] \arrow[dr, equal] & & \\
 & A \arrow[dl, "f"'] \arrow[dr, equal] & & A \arrow[dl, equal] \arrow[dr, "g"] & \\
X & & A & & Y.
\end{tikzcd}$$

The images of T_g and R_f under a Mackey functor $\underline{U} \colon \mathcal{S}pan(\mathcal{F}in_G) \to \mathcal{S}et$ are the *transfer* and *restriction* associated to g and f, respectively. There are functors $\mathcal{F}in_G \to \mathcal{S}pan(\mathcal{F}in_G)$ and $\mathcal{F}in_G^{\mathrm{op}} \to \mathcal{S}pan(\mathcal{F}in_G)$ which are the identity on objects and take a morphism $f \colon X \to Y$ to T_f and R_f, respectively.

While previous constructions of Mackey functors explicitly stated the axioms governing induction and restriction, Lindner's formulation lifts all the axioms to properties of the category $\mathcal{S}pan(\mathcal{F}in_G)$. Most interestingly, the rather complicated double coset formula, which is the crux of the definition of a Mackey functor, is a consequence of an interchange law arising from the interaction of spans with pullback diagrams in $\mathcal{F}in_G$. The classical definition of the double coset formula is expressed on representations as follows: if K and L are subgroups of H, then

$$R^H_K T^H_L = \bigoplus_{x \in [K \backslash H / L]} T^K_{K \cap xLx^{-1}} \circ c_x \circ R^L_{x^{-1}Kx \cap L}$$

where $[K \backslash H/L]$ is a class of double coset representatives and c_x is the isomorphism $R_G(x^{-1}Kx \cap L) \to R(K \cap xLx^{-1})$ induced by conjugation by x. Taking Lindner's point of view, let

$$\begin{tikzcd} W \arrow[r, "g"] \arrow[d, "u"'] & Y \arrow[d, "v"] \\ X \arrow[r, "f"] & Z \end{tikzcd} \tag{1.2}$$

be a pullback square in $\mathcal{F}in_G$. Then $R_vT_f = T_gR_u$ as morphisms $X \to Y$ in $\mathcal{S}pan(\mathcal{F}in_G)$, as evidenced by the span composition diagram

$$\begin{tikzcd} & & W \arrow[dl, "u"'] \arrow[dr, "g"] \arrow[ddll, bend right, "u"'] \arrow[ddrr, bend left, "g"] & & \\ & X \arrow[dl, equal] \arrow[dr, "f"'] & & Y \arrow[dl, "v"] \arrow[dr, equal] & \\ X & & Z & & Y. \end{tikzcd}$$

In particular, applying this observation to a pullback diagram

$$\begin{tikzcd} G/L \times_{G/H} G/K \arrow[r] \arrow[d] & G/L \arrow[d] \\ G/K \arrow[r] & G/H \end{tikzcd}$$

where $K, L \leq H$ recovers the usual Mackey double coset formula for a Mackey functor, once the pullback is decomposed into orbits; see Chapter 3 of [17] for further detail.

First defined by Tambara in [18] as *semi-TNR functors*, the category of *Tambara functors* for a finite group G is the category of product-preserving functors from $\mathcal{B}ispan(\mathcal{F}in_G)$ to $\mathcal{S}et$, where the objects of $\mathcal{B}ispan(\mathcal{F}in_G)$ are finite G-sets, and morphisms from X to Y are equivalence classes $\omega = [X \xleftarrow{f} A \xrightarrow{g} B \xrightarrow{h} Y]$ of diagrams called *bispans*, where two bispans from X to Y are equivalent if there are compatible isomorphisms on the intermediate G-sets, as in the diagram

$$\begin{tikzcd}
& A \arrow[r] \arrow[dl] \arrow[dd, "\sim"] & B \arrow[dr] \arrow[dd, "\sim"] & \\
X & & & Y. \\
& A' \arrow[r] \arrow[ul] & B' \arrow[ur] &
\end{tikzcd}$$

The composition of two bispans is done through the creation of a larger diagram of the following shape:

$$\begin{tikzcd}
& & E \arrow[rr] \arrow[rrrr, bend left] \arrow[ddl] \arrow[dddll, bend right] & & F \arrow[rr] \arrow[d] \arrow[ddl] \arrow[ddr] & & G \arrow[ddr] \arrow[dddrr, bend left] & & \\
& & & & P \arrow[dl] \arrow[dr] & & & & \\
& A \arrow[rr] \arrow[dl] & & B \arrow[dr] & & C \arrow[rr] \arrow[dl] & & D \arrow[dr] & \\
X & & & & Y & & & & Z.
\end{tikzcd} \tag{1.3}$$

There are three steps in forming the composite of two bispans. The first is to take the pullback P where the bispans meet. The second step is to form an exponential diagram (the pentagon $\{P, C, D, F, G\}$) and the final step is to form another another pullback, E, where the map from F to B arises by composing the map $F \to P$ from the exponential diagram with the map $P \to C$ coming from the first pullback. The curved arrows in the preceding diagram provide a representative for the composition of the bispan from Y to Z with the bispan from X to Y. When taking bispans in a category such as $\mathcal{F}in_G$, showing that the composition is well-defined with respect to the equivalence relation, as well as associative and unital, involves intricate manipulation of exponential diagrams, and establishing various naturality properties is a further difficulty (as done explicitly in Chapter 5 of [17]).

In a generalization of the behavior which allows each span to be canonically written as a composite $T_g R_f$, each equivalence class $\omega = [X \xleftarrow{f} A \xrightarrow{g} B \xrightarrow{h} Y]$ can be written as a composite $\omega = T_h N_g R_f$ of three bispans of the following special forms:

$$T_h = [B \xleftarrow{=} B \xrightarrow{=} B \xrightarrow{h} Y]$$
$$N_g = [A \xleftarrow{=} A \xrightarrow{g} B \xrightarrow{=} B]$$
$$R_f = [X \xleftarrow{f} A \xrightarrow{=} A \xrightarrow{=} A].$$

Upon application of a Tambara functor, these are the *transfer, norm* and *restriction* of the Tambara functor at h, g and f, respectively. Thus, every morphism in a Tambara functor decomposes as a composite of a restriction followed by a norm and then a transfer. Pullbacks continue to interact well with bispan composition, with the previous pullback (1.2) leading to equalities $R_v T_f = T_g R_u$ and $R_v N_f = N_g R_u$.

1.3 Global Tambara functors and bispans in quasicategories

If H is a subgroup of G and Y is a finite H-set, let $G \times_H Y$ be the induced G-set, which is the quotient of $G \times Y$ by the equivalence relation $(gh, y) \sim (g, hy)$ for g in G, h in H and y in Y. Then $K_H(Y) \cong K_G(G \times_H Y)$, and the transfer, norm and restriction for K_H agree with the corresponding operations on K_G, as expressed by the following diagram, where $f' = G \times_H f$:

$$\begin{tikzcd}
K_H(Y) \arrow[r, "\cong"] \arrow[d, bend right=60, "R_f"'] & K_G(G \times_H Y) \arrow[d, bend right=60, "R_{f'}"'] \\
K_H(X) \arrow[r, "\cong"] \arrow[u, "N_f"] \arrow[u, bend right=60, "T_f"'] & K_G(G \times_H X). \arrow[u, "N_{f'}"] \arrow[u, bend right=60, "T_{f'}"']
\end{tikzcd}$$

This behavior exemplifies the concept of a *global Tambara functor*, a collection of compatible Tambara functors for all finite groups. The theory of global Mackey functors has been studied, notably in Schwede's book [15], but the various formulations lack a categorical framework for expressing global distributivity.

The bicategory $\mathcal{F}in\mathcal{G}pd$ has objects finite groupoids, 1-morphisms functors of groupoids, and 2-morphisms natural transformations between functors, with composition of morphisms only defined up to a choice of natural isomorphism. Every finite groupoid is isomorphic to the *action groupoid* $B_G(X)$ associated to some finite group G and G-set X. The objects of $B_G(X)$ are the elements of X and morphisms are of the form $(g, x)\colon x \to gx$. This construction extends to a functor $\mathcal{F}in_G \to \mathcal{F}in\mathcal{G}pd$. A single group takes many guises in $\mathcal{F}in\mathcal{G}pd$, for the action groupoid $B_H(H/H)$ is equivalent to $B_G(G/H)$ for any G containing H, and the collapse map $B_G(G/H) \to B_H(H/H)$ is a discrete fibration. Since diagrams in $\mathcal{F}in\mathcal{G}pd$ only commute up to natural isomorphism, constructing a theory of bispans in $\mathcal{F}in\mathcal{G}pd$ naturally calls for the use of homotopical higher categorical techniques to efficiently and gracefully handle the data of the large diagrams involved in bispan composition with weakened commutativity assumption. Thus, this juncture it is reasonable to generalize and consider bispans in a suitable $(\infty, 1)$-category $\mathcal{C}$.

There are a number of predecessors besides the work done for bispans in ordinary categories by Tambara in [18] and Strickland in [17]. The problem of constructing global Tambara functors is approached by Nakaoka in [11], drawing influence from biset functors, while in [4], Gepner, Haugseng and Kock develop a related theory of polynomial functors in spaces. In [2], Barwick provides an analogous construction of the quasicategory $\mathcal{S}pan(\mathcal{C})$ in a quasicategory with pullbacks. In the earlier-referenced [19], Weber constructs a tricategory of bispans in a *strict* 2-category, but composition in the bicategory of finite groupoids is only well-defined up to natural isomorphism, leading to coherence issues. Most relevant to this work is the thesis [3] of Cranch, which provides an alternative to exponential diagrams and a compatible construction for higher bispan composition diagrams.

In this thesis, the quasicategorical model for $(\infty, 1)$-categories as developed by Joyal in [5] and [6] and Lurie in [9] provides the technical framework in which the necessary constructions can be carried out. Heuristically, this will be done in parallel with Barwick's construction in [2], where a cosimplicial sequence of diagram categories $\{\mathrm{TR}[n]\}_{n\geq 0}$ encoding n-fold composite of spans is defined, and the quasicategory $\mathcal{S}pan(\mathcal{C})$ is defined by

$$\mathrm{Hom}_{\mathcal{SS}et}(\Delta[n], \mathcal{S}pan(\mathcal{C})) = \mathrm{Hom}^{\star}_{\mathcal{SS}et}(\mathrm{TR}[n], \mathcal{C})$$

where the $\star$ indicates that certain distinguished squares in $\mathrm{TR}[n]$ are sent to pullbacks in $\mathcal{C}$. In particular, $\mathrm{TR}[0]$ is a point, $\mathrm{TR}[1]$ is a span, and $\mathrm{TR}[2]$ is a diagram of the same shape as that of diagram (1.1).

Hence, the quasicategory of bispans in a quasicategory $\mathcal{C}$ should consist of a collection of diagrams in $\mathcal{C}$ encoding n-fold composites of bispans as n varies. In Section 2.6, we will use a *twisted functor construction* to define poset categories $\mathrm{TNR}[n]$ modeling the shapes of the diagrams encoding n-fold composites of bispans. As expected, $\mathrm{TNR}[0]$ is a point and $\mathrm{TNR}[1]$ is a single bispan. The following figures depict $\mathrm{TNR}[2]$ and $\mathrm{TNR}[3]$, respectively.

$$\begin{tikzcd}
 & & 002 \arrow[r] \arrow[dl] & 012 \arrow[r] \arrow[dl] \arrow[dr] & 022 \arrow[dr] & & \\
 & 001 \arrow[r] \arrow[dl] & 011 \arrow[dr] & & 112 \arrow[r] \arrow[dl] & 122 \arrow[dr] & \\
000 & & & 111 & & & 222
\end{tikzcd}$$

$$\begin{tikzcd}
 & & & 003 \arrow[r] \arrow[dl] & 013 \arrow[r] \arrow[dl] \arrow[dr] & 023 \arrow[r] \arrow[dl] \arrow[dr] & 033 \arrow[dr] & & & \\
 & & 002 \arrow[r] \arrow[dl] & 012 \arrow[r] \arrow[dl] \arrow[dr] & 022 \arrow[dr] & 113 \arrow[r] \arrow[dl] & 123 \arrow[r] \arrow[dl] \arrow[dr] & 133 \arrow[dr] & & \\
 & 001 \arrow[r] \arrow[dl] & 011 \arrow[dr] & & 112 \arrow[r] \arrow[dl] & 122 \arrow[dr] & & 223 \arrow[r] \arrow[dl] & 233 \arrow[dr] & \\
000 & & & 111 & & & 222 & & & 333
\end{tikzcd}$$

The diagram TNR[2] thus encodes a choice of composite for two bispans. Comparing with the categorical bispan composition diagram (1.3), note that a diagram of shape TNR[2] lacks the explicit choice of auxiliary pullback (P) as made in the ordinary categorical bispan composition. Instead of taking two steps to form a pullback and an exponential diagram, composition of bispans via diagrams of shape TNR[2] involves a related universal construction which reduces these two steps to a single step, the *cromulent diagrams* of [3], appearing here as Definition 3.2.1. The cromulent diagram in question is the image of

$$\begin{tikzcd}
 & 012 \arrow[r] \arrow[dl] \arrow[dr] & 022 \arrow[dr] & \\
011 \arrow[dr] & & 112 \arrow[r] \arrow[dl] & 122 \\
 & 111. & &
\end{tikzcd}$$

Cromulent diagrams are particularly well-suited to higher bispan constructions since they encode the same data as exponential diagrams without having to form numerous explicit pullbacks in higher dimensional bispan composition diagrams. As demonstrated in Lemma 3.2.7, a diagram

$$\begin{tikzcd}
& B \arrow[r] \arrow[dl] \arrow[dr] & S \arrow[dr] & \\
R \arrow[dr, "p"'] & & A \arrow[r, "f"] \arrow[dl, "u"] & Y \\
& X & &
\end{tikzcd}$$

with the square $\{B, S, A, Y\}$ is a pullback is a cromulent diagram if and only if when one forms a pullback

$$\begin{tikzcd}
W \arrow[r] \arrow[d] & A \arrow[d, "u"] \\
R \arrow[r, "p"] & X
\end{tikzcd}$$

and then factors the above diagram as

$$\begin{tikzcd}
& B \arrow[r] \arrow[dl] \arrow[d] \arrow[dr] & S \arrow[dr] & \\
R \arrow[dr, "p"'] & W \arrow[l] \arrow[r] & A \arrow[r, "f"] \arrow[dl, "u"] & Y \\
& X & &
\end{tikzcd}$$

then the pentagon $\{W, A, Y, B, S\}$ is an exponential diagram. Cromulent diagrams are characterized by a similar universal property to that of exponential diagrams: the one above is terminal amongst all diagrams of the same shape with the right side square a pullback and with the triple (p, u, f) fixed.

In either case, a mild technical condition, that $\mathcal{C}$ is locally Cartesian closed (see Definition 2.5.3), is imposed on $\mathcal{C}$ to ensure that exponential and cromulent diagrams exist. A more precise definition of the diagrams $\mathrm{TNR}[n]$ is given in Definition 2.6.9. In particular, the method of construction of $\mathrm{TNR}[n]$ ensures they assemble into a cosimplicial sequence, and thus can be used to define the levels of a simplicial set, as follows.

Definition (Definition 4.0.1)**.** Let $\mathcal{C}$ be a locally cartesian closed quasicategory. Define the simplicial set $\mathcal{B}ispan(\mathcal{C})$ by

$$\mathrm{Hom}_{\mathcal{SS}et}(\Delta[n], \mathcal{B}ispan(\mathcal{C})) = \mathrm{Hom}^{\star}_{\mathcal{SS}et}(\mathrm{TNR}[n], \mathcal{C})$$

where the $\star$ indicates that each subdiagram of $\mathrm{TNR}[n]$ of shape

$$\begin{tikzcd} abb \arrow[d] & abc \arrow[l] \arrow[r] \arrow[d] & ab'c \arrow[d] \\ a'bb & a'bc \arrow[l] \arrow[r] & a'b'c \end{tikzcd}$$

for $0 \leq a < a' \leq b < c \leq n$ is sent to a cromulent diagram in $\mathcal{C}$ and each subdiagram of shape

$$\begin{tikzcd} abc' \arrow[r] \arrow[d] & ab'c' \arrow[d] \\ abc \arrow[r] & ab'c \end{tikzcd}$$

for $0 \leq a \leq b < b' \leq c < c' \leq n$ is sent to a pullback diagram in $\mathcal{C}$

Theorem 6.9 of Cranch's thesis [3] gives a quasicategory of bispans in sets for which taking the homotopy category recovers the ordinary category of bispans in sets. The quasicategory of bispans defined there consists of *decomposed bispans*, with n-simplices given by diagrams of shape $\mathrm{TNR}_{\mathrm{dec}}[n]$, which is a sub-simplicial set of $\mathrm{TNR}[n]$ defined by a colimit of diagrams of shape $\Delta[i] \times \mathrm{TR}[j]$. This quasicategory of decomposed bispans is denoted by $\mathcal{B}ispan_{\mathrm{dec}}(\mathcal{C})$ here, and the diagrams which define the n-simplices are decomposed into smaller diagrams upon which certain pullback and cromulence conditions are compatibly imposed. When taking bispans in an ordinary category such as $\mathcal{F}in_G$, a diagram of shape $\mathrm{TNR}_{\mathrm{dec}}[n]$ uniquely determines a diagram of shape $\mathrm{TNR}[n]$, and what Cranch's techniques accomplish is to simplify the analysis of diagrams of shape $\mathrm{TNR}[n]$ by looking at the decomposed bispan diagrams instead. In an arbitrary quasicategory, the notion of composition is only unique up to coherent homotopy, so diagrams of shape $\mathrm{TNR}_{\mathrm{dec}}[n]$ do not uniquely extend to diagrams of shape $\mathrm{TNR}[n]$, and so this thesis extends Cranch's work in two directions. The first is the generalization of the decomposed bispan construction to quasicategories, which is the main result of Chapter 2:

Theorem A (Theorem 3.4.4)**.** *Let $\mathcal{C}$ be a locally Cartesian closed quasicategory. Then $\mathcal{B}ispan_{\mathrm{dec}}(\mathcal{C})$ is a quasicategory.*

In Chapter 3, an analysis of diagrams of shapes $\mathrm{TNR}[n]$ and $\mathrm{TNR}_{\mathrm{dec}}[n]$ leads to the following comparison result.

Theorem B (Theorem 4.2.9)**.** *The restriction map $\mathcal{B}ispan(\mathcal{C}) \to \mathcal{B}ispan_{\mathrm{dec}}(\mathcal{C})$ is a trivial fibration of simplicial sets.*

An immediate corollary is the following, the primary result of the work:

Corollary C (Corollary 4.2.10)**.** *Let $\mathcal{C}$ be a locally Cartesian closed quasicategory. Then $\mathcal{B}ispan(\mathcal{C})$ is a quasicategory.*

Taking diagrams of shape $\mathrm{TNR}[n]$ instead of $\mathrm{TNR}_{\mathrm{dec}}[n]$ makes $\mathcal{B}ispan(\mathcal{C})$ compatible with Barwick's span construction for quasicategories, as discussed in Section 2.6.

If $\mathcal{C}$ is taken to be the ordinary category $\mathcal{F}in_G$ viewed as a quasicategory, then the homotopy category $\mathrm{Ho}(\mathcal{B}ispan(\mathcal{F}in_G))$ recovers the ordinary category of bispans in finite G-sets previously discussed. As a global Tambara functor should restrict to an ordinary Tambara functor for each finite group G, this should occur as a restriction induced by a functor $\mathcal{B}ispan(\mathcal{F}in_G) \to \mathcal{B}ispan(\mathcal{F}in\mathcal{G}pd)$ which is itself induced by the aforementioned action groupoid functor $\mathcal{F}in_G \to \mathcal{F}in\mathcal{G}pd$. To complete the construction of global Tambara functors with these specifications, one needs to restrict the norms and transfers to take place along *discrete fibrations* of groupoids, which are functors of groupoids corresponding to the collapse maps $G/K \to G/H$ for $H, K \leq G$. To have a proper notion of composition of two bispans with such conditions imposed on them, there must be a diagram of shape TNR[2] in $\mathcal{F}in\mathcal{G}pd$ for which the associated composite bispan satisfies the imposed conditions, and likewise for higher composites. Extending the results contained herein to this setting is the aim of future work.

Chapter 2

Technical background

This chapter consists of some background on simplicial sets and quasicategories, explaining the basic constructions necessary in the later chapters as well as providing a repository for some useful lemmas which will appear later. Lurie's book [9] is a comprehensive resource, while Rezk's notes [12] are a useful companion.

2.1 Simplicial sets and quasicategories

The model for higher categories used in this work is quasicategories, which are simplicial sets satisfying certain properties allowing for a generalized notion of compositionality up to coherent homotopy. With this in mind, it is useful to begin by briefly discussing simplicial sets and their basic properties.

Definition 2.1.1. The category Δ has objects finite totally ordered sets $[n] \coloneqq \{0 < 1 < \cdots < n\}$ for $n \geq 0$ and morphisms weakly order-preserving maps.

The notation $[n]$ will also be used to refer to the associated poset category. There are distinguished maps in Δ, called the *coface* and *codegeneracy* maps, defined as follows:

Definition 2.1.2. For $n > 0$ and $0 \leq i \leq n$, the *ith coface map* $d^i \colon [n-1] \to [n]$ is defined by

$$d^i(j) = \begin{cases} j & \text{if } j < i \\ j+1 & \text{if } j \geq i. \end{cases}$$

For $n > 0$ and $0 \leq i \leq n-1$, the *ith codegeneracy map* $s^i \colon [n] \to [n-1]$ is defined by

$$s^i(j) = \begin{cases} j & \text{if } j \leq i \\ j-1 & \text{if } j > i. \end{cases}$$

Definition 2.1.3. A *simplicial set* is a functor $K \colon \Delta^{\mathrm{op}} \to \mathcal{S}et$. The *n-simplices* of K are the set $K([n])$. Simplicial sets assemble into a category $\mathcal{SS}et$ with morphisms given by natural transformations of functors.

The convention is to write K_n for $K([n])$. Given a simplicial set K, the coface and codegeneracy maps correspond to maps $d_i \coloneqq K(d^i) \colon K_n \to K_{n-1}$ and $s_i \coloneqq K(s^i) \colon K_{n-1} \to K_n$, called *face* and *degeneracy* maps. A simplex in the image of one of the s^i is called *degenerate*.

Definition 2.1.4. A *sub-simplicial set* of a simplicial set K is a simplicial set L such that $L_n \subset K_n$ (in particular, the level-wise inclusion of simplicial sets is a morphism of simplicial sets).

Definition 2.1.5. The *n-simplex* $\Delta[n]$ is the simplicial set represented by the object $[n]$ in Δ, i.e. $\Delta[n]_m = \mathrm{Hom}_{\mathcal{SS}et}(\Delta[m], \Delta[n])$.

The following classical result, the Yoneda lemma, provides a useful interpretation for the n-simplices of a simplicial set.

Lemma 2.1.6 (Yoneda lemma)**.** *Let $\mathcal{C}$ be a category and X an object of $\mathcal{C}$, and let*

$h_X : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}et$ *be the functor represented by* X*, i.e.* $h_X(A) = \mathrm{Hom}_{\mathcal{C}}(A, X)$*. Given a functor* $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}et$*, there is a natural isomorphism* $F(X) \cong \mathrm{Hom}_{\mathcal{S}et^{\mathcal{C}^{\mathrm{op}}}}(h_X, F)$*.*

Corollary 2.1.7. *If* K *is a simplicial set, then the* n*-simplices of* K *correspond to maps of simplicial sets* $\Delta[n] \to K$*.*

Proof. In the notation of the Yoneda lemma, $\Delta[n] = h_{[n]}$ as functors $\Delta^{\mathrm{op}} \to \mathcal{S}et$, so $K_n \cong \mathrm{Hom}_{\mathcal{SS}et}(\Delta[n], K)$. □

In particular, a 0-simplex X of K corresponds to a unique map $\Delta[0] \to K$, denoted herein by $\lceil X \rceil$. The coface maps $d^i : [n-1] \to [n]$ are elements of $\Delta[n]_{n-1}$ and thus define maps of simplicial sets $d^i : \Delta[n-1] \to \Delta[n]$. The sub-simplicial set of $\Delta[n]$ given by the image of d^i can be thought of as the ith-face of $\Delta[n]$, denoted by $\partial^i \Delta[n]$.

Since $\mathcal{SS}et$ is a functor category, limits and colimits are computed levelwise, i.e. if $F : \mathcal{D} \to \mathcal{SS}et$ is a functor, then $(\lim F)_n = \lim F_n$ and $(\operatorname{colim} F)_n = \operatorname{colim} F_n$. In particular, unions are formed levelwise.

Definition 2.1.8. The *boundary* of the n-simplex is the simplicial set $\partial\Delta[n]$, defined as the union

$$\partial\Delta[n] = \bigcup_{i=0}^{n} \partial_i \Delta[n].$$

For each k such that $0 \le k \le n$, there is a sub-simplicial set of $\partial\Delta[n]$ called a *horn*, denoted $\Lambda^k[n]$, defined as the union of all the faces of $\Delta[n]$ except the kth-face:

$$\Lambda^k[n] = \bigcup_{\substack{i=0 \\ i \ne k}}^{n} \partial_i \Delta[n].$$

If $0 < k < n$, then $\Lambda^k[n]$ is an *inner horn.*

The horns $\Lambda^0[1]$ and $\Lambda^1[1]$ are the 0-simplices $\{0\}$ and $\{1\}$ respectively. The horns

$\Lambda^0[2], \Lambda^1[2]$ and $\Lambda^2[2]$ are depicted below.

$$\Lambda^0[2]: \begin{tikzcd} & 1 & \\ 0 \arrow[ur] \arrow[rr] & & 2 \end{tikzcd} \qquad \Lambda^1[2]: \begin{tikzcd} & 1 \arrow[dr] & \\ 0 \arrow[ur] & & 2 \end{tikzcd} \qquad \Lambda^0[2]: \begin{tikzcd} & 1 \arrow[dr] & \\ 0 \arrow[rr] & & 2 \end{tikzcd}$$

Of particular interest is the inner horn $\Lambda^1[2]$, since if K is a simplicial set, and one were to think loosely of the 0-simplices of K and of the 1-simplices of K as morphisms, then an extension of a map $\Lambda^1[2] \to K$ to a map $\Delta[2] \to K$ could be thought of as giving a choice of composite of the 1-simplices chosen by the inner horn in K by seeing where the edge 02 is sent to in K.

Definition 2.1.9. A *quasicategory* is a simplicial set $\mathcal{C}$ for which every inner horn $\Lambda^k[n] \to \mathcal{C}$ has a filler $\Delta[n] \to \mathcal{C}$, depicted as the dashed arrow in the diagram

$$\begin{tikzcd} \Lambda^k[n] \arrow[r] \arrow[d] & \mathcal{C}. \\ \Delta[n] \arrow[ur, dashed] & \end{tikzcd}$$

A quasicategory can be thought of as a category where composition of morphisms (which are the 1-simplices of the quasicategory) is only defined up to coherent choice, encoded by the existence of ever-higher inner horn fillings. They are one model for the idea of an $(\infty, 1)$-category, which encapsulates the notion of a category with morphisms, morphisms between morphisms, morphisms between morphisms between morphisms, etc., with all higher morphisms being in some sense invertible. An example of such a higher category is the bicategory of groupoids, whose objects are groupoids, morphisms are functors of groupoids, and 2-morphisms are natural isomorphisms of functors of groupoids (since any natural transformation of functors of groupoids will be a natural isomorphism). Another example consists of considering the points of a topological space X as the objects of an $(\infty, 1)$-category Π_X, the paths between points as morphisms, homotopies between paths as morphisms between morphisms, and higher homotopies as higher morphisms. The book [9] develops the theory of quasicategories in parallel to existing concepts in ordinary category theory.

Example 2.1.10. If $\mathcal{C}$ is an ordinary category, there is a simplicial set $N\mathcal{C}$ called the *nerve* of $\mathcal{C}$, with n-simplices defined as composable chains of n morphisms in $\mathcal{C}$. The nerve construction defines a fully faithful functor from the category of small categories to the category of quasicategories, considered as a full subcategory of $\mathcal{SS}et$, so every category can be thought of as a quasicategory. Nerves of categories are unique amongst quasicategories in that they are characterized by a specific property, which is that each inner horn extension is *unique* in the nerve of a category. In particular, the simplicial n-simplex $\Delta[n]$ can be obtained as $N[n]$.

Definition 2.1.11. The *homotopy category* of a quasicategory $\mathcal{C}$ is a category $\mathrm{Ho}\,\mathcal{C}$ whose objects are the 0-simplices of $\mathcal{C}$, and whose morphisms are equivalence classes of 1-simplices of $\mathcal{C}$ under the following relation: if $X \xrightarrow{f} Y$ and $X \xrightarrow{g} Y$ are 1-simplices, then $[f] = [g]$ if there is a 2-simplex in $\mathcal{C}$ of either of the following forms, where id_X and id_Y denote the degenerate 1-simplices s_0X and s_0Y.

$$\begin{tikzcd} & X \arrow[dr, "g"] & \\ X \arrow[ur, "\mathrm{id}_X"] \arrow[rr, "f"] & & Y \end{tikzcd} \qquad \begin{tikzcd} & Y \arrow[dr, "\mathrm{id}_Y"] & \\ X \arrow[ur, "f"] \arrow[rr, "g"] & & Y. \end{tikzcd}$$

A pair of composable morphisms corresponds to a collection (spanned by the choices possible in the equivalence classes) of inner horns $\Lambda^1[2] \to \mathcal{C}$. Taking the equivalence class of any filling of such an inner horn gives the composition in $\mathrm{Ho}\,\mathcal{C}$:

$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, "h"] & & Z \end{tikzcd} \quad \Rightarrow [g] \circ [f] = [h].$$

The inner horn filling property of quasicategories is important to establishing that $\mathrm{Ho}\,\mathcal{C}$ is in fact a category.

Example 2.1.12. If $\mathcal{C}$ is an ordinary category, then $\mathrm{Ho}\,N\mathcal{C} \cong \mathcal{C}$.

Definition 2.1.13. A 1-simplex $X \xrightarrow{f} Y$ in a quasicategory $\mathcal{C}$ is a *quasi-isomorphism* if $[f]$ is invertible in $\mathrm{Ho}\,\mathcal{C}$, i.e. there is a 2-simplex as follows in $\mathcal{C}$:

$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, "\mathrm{id}_X"] & & X. \end{tikzcd}$$

The analogous construction for groupoids in quasicategories are Kan complexes, which are defined as follows:

Definition 2.1.14. A simplicial set $\mathcal{X}$ is a *Kan complex* if every horn $\Lambda^k[n] \to \mathcal{X}$ has a filling to an n-simplex $\Delta[n] \to \mathcal{X}$.

In particular, considering the horns $\Lambda^0[2]$ and $\Lambda^2[2]$, one can see that in a Kan complex every 1-simplex is quasi-invertible. Furthermore, Kan complexes are characterized by the following property:

Proposition 2.1.15 (Corollary 1.4 of [5])**.** *A quasicategory $\mathcal{X}$ is a Kan complex if and only if* $\mathrm{Ho}\,\mathcal{X}$ *is a groupoid.*

Given simplicial sets K and X, the set of maps of simplicial sets $K \to X$ form the 0-simplices of a mapping complex X^K, defined as follows:

Definition 2.1.16. The *mapping complex* X^K is the simplicial set whose n-simplices are given by maps $K \times \Delta[n] \to X$ of simplicial sets. A map $f\colon [m] \to [n]$ in Δ induces a map $\mathrm{id}_K \times f\colon K \times \Delta[m] \to K \times \Delta[n]$, and precomposing with such a map gives the associated map $f^*\colon (X^K)_n \to (X^K)_m$ for the simplicial set X^K.

Proposition 2.1.17. *If $\mathcal{D}$ is a quasicategory, then the mapping complex $\mathcal{D}^K$ is a quasicategory.*

If K is also a quasicategory $\mathcal{C}$, the mapping complex $\mathcal{D}^{\mathcal{C}}$ will be referred to as the *functor quasicategory*, since its objects are functors $\mathcal{C} \to \mathcal{D}$. In particular, a 1-simplex in $\mathcal{D}^{\mathcal{C}}$ is a functor $\eta\colon \mathcal{C} \times \Delta[1] \to \mathcal{D}$. Writing F for $\eta|_{\mathcal{C}\times\{0\}}$ and G for $\eta|_{\mathcal{C}\times\{1\}}$ shows η as a morphism $F \to G$ in the functor quasicategory, i.e. η is a natural transformation of functors. In the case of $\mathcal{C}$ and $\mathcal{D}$ being ordinary categories, this construction directly generalizes the bicategory of functors from $\mathcal{C} \to \mathcal{D}$ and natural transformations between them.

A further useful result is the following, enriching the adjunction

$$\mathrm{Hom}_{\mathcal{SSet}}(K \times L, X) \cong \mathrm{Hom}_{\mathcal{SSet}}(K, X^L)$$

on morphism sets to an isomorphism of mapping complexes:

Proposition 2.1.18. *Let K, L and X be simplicial sets. Then $X^{K\times L} \cong (X^K)^L$.*

2.2 Fibrations of simplicial sets and the Joyal model structure

We begin by discussing left and right lifting properties for morphisms.

Definition 2.2.1. Let $f\colon A \to B$ and $g\colon X \to Y$ be morphisms in $\mathcal{C}$. A *lifting problem* for f and g is the data of a commuative square of solid arrows of the form

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"'] & X \arrow[d, "g"] \\ B \arrow[r, "v"] \arrow[ur, dashed, "s"] & Y. \end{tikzcd}$$

A *lift* for the lifting problem is a choice of morphism $s\colon B \to X$ such that the diagram commutes, i.e $u = sf$ and $v = gs$. If every lifting problem for f and g admits a lift, then f has the *left lifting property* with respect to g and g has the *right lifting property* with respect to f. If S is a class of morphisms in a category $\mathcal{C}$, write $\mathrm{RLP}(S)$ for the class of morphisms in $\mathcal{C}$ which have the right lifting property with respect to S, and $\mathrm{LLP}(S)$ for the class of morphisms in $\mathcal{C}$ which have the left lifting property with respect to S.

Proposition 2.2.2. *Given a class S of morphisms in a category $\mathcal{C}$, the following hold:* $\mathrm{LLP}(\mathrm{RLP}(\mathrm{LLP}(S))) = \mathrm{LLP}(S)$ *and* $\mathrm{RLP}(\mathrm{LLP}(\mathrm{RLP}(S))) = \mathrm{RLP}(S)$.

The following classes of maps in $\mathcal{SS}et$ will play an important role throughout the work.

Definition 2.2.3. (a) A map $p\colon K \to L$ of simplicial sets is an *inner fibration* if it has the right lifting property against the class InnHorn of inner horn inclusions $\Lambda^k[n] \subset \Delta[n]$, i.e. every solid square as below has a dashed diagonal filler:

$$\begin{tikzcd} \Lambda^k[n] \arrow[r] \arrow[d] & K \arrow[d, "p"] \\ \Delta[n] \arrow[r] \arrow[ur, dashed] & L. \end{tikzcd}$$

(b) A *left fibration* is an inner fibration p which also has the right lifting property against the class LHorn of left horn inclusions $\mathrm{InnHorn} \cup \{\Lambda^0[n] \subset \Delta[n] \mid n \geq 1\}$, i.e. in addition to the lifts from (a), there also also lifts for every solid square as below:

$$\begin{tikzcd} \Lambda^0[n] \arrow[r] \arrow[d] & K \arrow[d, "p"] \\ \Delta[n] \arrow[r] \arrow[ur, dashed] & L. \end{tikzcd}$$

(c) A *right fibration* is an inner fibration p which also has the right lifting property against the class RHorn of right horn inclusions $\mathrm{InnHorn} \cup \{\Lambda^n[n] \subset \Delta[n] \mid n \geq 1\}$, i.e. in addition to the lifts from (a), there also also lifts for every solid square as below:

$$\begin{tikzcd} \Lambda^n[n] \arrow[r] \arrow[d] & K \arrow[d, "p"] \\ \Delta[n] \arrow[r] \arrow[ur, dashed] & L. \end{tikzcd}$$

(d) A *Kan fibration* is a map $p\colon K \to L$ which is both a left and right fibration, i.e. p has the right lifting property against the class $\mathrm{Horn} = \mathrm{RHorn} \cup \mathrm{LHorn}$.

Definition 2.2.4. A *trivial fibration* of simplicial sets is a map $p\colon K \to L$ which has the right lifting property against all boundary inclusions $\partial\Delta[n] \subset \Delta[n]$ for $n \geq 0$:

$$\begin{tikzcd} \partial\Delta[n] \arrow[r] \arrow[d] & K \arrow[d, "p"] \\ \Delta[n] \arrow[r] \arrow[ur, dashed] & L. \end{tikzcd}$$

Definition 2.2.5. Let $\mathcal{C}$ be a category with all small colimits. A *weakly saturated class* of morphisms in $\mathcal{C}$ is a class S of morphisms in $\mathcal{C}$ which satisfies the following conditions:

(a) The class S contains all isomorphisms in $\mathcal{C}$.

(b) The class S is closed under composition.

(c) The class S is closed under the formation of pushouts: if a square

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & X \arrow[d, "g"] \\ B \arrow[r] & Y \end{tikzcd}$$

is a pushout square and f is in S, then g is in S.

(d) The class S is closed under *transfinite composition*: if α is an ordinal with least element 0 and $F\colon N(\alpha) \to \mathcal{C}$ is a functor such that $\operatorname{colim}_{i<j} F(i) \to F(j)$ is in S for each $j \in \alpha$, then $F(0) \to \operatorname{colim}_{j\in\alpha} F(j)$ is in S.

(e) The class S is closed under *retracts*: if a morphism g is in S and f is a retract of g in the functor category $\mathcal{C}^{[1]}$, then f is in S.

Given a class S of morphisms in $\mathcal{C}$, the *saturation* $\overline{S}$ of S is the smallest saturated class containing S.

The role weak saturation plays in this context is in the setting of $\mathcal{C} = \mathcal{SS}et$. The following is a consequence of the small object argument (Proposition A.2.1.6 of [9]):

Proposition 2.2.6 (Corollary A.2.1.7 of [9])**.** *If S is a class of morphisms in $\mathcal{SS}et$, then $\overline{S} = \mathrm{LLP}(\mathrm{RLP}(S))$.*

Definition 2.2.7. A map $f\colon A \to B$ of simplicial sets is *inner anodyne* if it is in the class $\overline{\mathrm{InnHorn}}$, *left anodyne* if it is in the class $\overline{\mathrm{LHorn}}$, and *right anodyne* if it is in the class $\overline{\mathrm{RHorn}}$.

As a consequence of Proposition 2.2.2 and Proposition 2.2.6, inner fibrations can now be interpreted as maps which have the right lifting property with respect to inner anodyne maps, left fibrations as maps having the right lifting property with respect to left anodyne maps, and right fibrations as maps having the right lifting property with respect to right anodyne maps.

Furthermore, if Cell denotes the class of simplex boundary inclusions $\partial\Delta[n] \subset \Delta[n]$, then it can be shown that $\overline{\text{Cell}}$ is the class of monomorphisms in $\mathcal{SS}et$, so trivial fibrations of simplicial sets are characterized by having the right lifting property against monomorphisms.

The next class of fibrations are of interest due to their role in the Joyal model structure on simplicial sets, i.e. the model structure for quasicategories.

Definition 2.2.8. An inner fibration $p\colon \mathcal{C} \to \mathcal{D}$ of quasicategories is an *isofibration* if whenever X is a 0-simplex of $\mathcal{C}$ and $p(X) \xrightarrow{f} Y$ is a quasi-invertible 1-simplex of $\mathcal{D}$, f lifts to a 1-simplex g of $\mathcal{C}$ with $d^1 g = X$.

The salient aspects of the Joyal model structure on $\mathcal{SS}et$ in this work are the following:

- The cofibrations are monomorphisms of simplicial sets.
- Quasicategories are the fibrant objects.
- A functor $f\colon \mathcal{C} \to \mathcal{D}$ between quasicategories is a weak equivalence in the Joyal model structure if it admits a *categorical inverse*: a functor $g\colon \mathcal{D} \to \mathcal{C}$ such that $g \circ f$ is quasi-isomorphic to $\mathrm{id}_{\mathcal{C}}$ in the functor category $\mathcal{C}^{\mathcal{C}}$.
- Isofibrations are precisely the fibrations between quasicategories.
- An acyclic fibration in the Joyal model structure is precisely a trivial fibration of simplicial sets.
- The Joyal model structure is *Cartesian*: if $i\colon K \to L$ is a cofibration and $p\colon X \to Y$ is a fibration, then the induced map on mapping complexes $X^L \to X^K \times_{Y^K} Y^L$

is a fibration.

2.3 Joins of quasicategories and slice constructions

The first construction of this section will be that of the *join* of two quasicategories. The join of two ordinary categories $\mathcal{C}$ and $\mathcal{D}$ is the category $\mathcal{C} \star \mathcal{D}$ whose objects are given by $\mathrm{ob}\,\mathcal{C} \star \mathcal{D} = \mathrm{ob}\,\mathcal{C} \sqcup \mathrm{ob}\,\mathcal{D}$ and whose morphisms are defined as follows:

$$\mathrm{Hom}_{\mathcal{C}\star\mathcal{D}}(X,Y) = \begin{cases} \mathrm{Hom}_{\mathcal{C}}(X,Y) & X, Y \in \mathrm{ob}\,\mathcal{C} \\ \mathrm{Hom}_{\mathcal{D}}(X,Y) & X, Y \in \mathrm{ob}\,\mathcal{D} \\ * & X \in \mathrm{ob}\,\mathcal{C}, Y \in \mathrm{ob}\,\mathcal{D} \\ \emptyset & X \in \mathrm{ob}\,\mathcal{D}, Y \in \mathcal{C}. \end{cases}$$

The idea of the join of two categories is to create a new category where there is a unique morphism from each object of $\mathcal{C}$ to each object of $\mathcal{D}$, while preserving the usual morphisms of $\mathcal{C}$ and $\mathcal{D}$. If $[n]$ is thought of as a category, then $[m] \star [n] \cong [m+n+1]$ with the total ordering on $[m] \star [n]$ imposed by the join.

To define the join of two simplicial sets as a simplicial set, we will need to understand the action of the simplicial operators. A map $f\colon [m] \to [n]$ in Δ can be rewritten as $[m] \to [i] \sqcup [j]$, where $[i] \sqcup [j]$ is given a total ordering by all elements of $[i]$ being smaller than all elements of $[j]$, so that $[i] \sqcup [j] \cong [n]$. Once f is rewritten this way, taking the preimages of $[i]$ and $[j]$ allows f to be again reinterpreted as $f\colon [m_1] \sqcup [m_2] \to [i] \sqcup [j]$, leading to a decomposition $f = f_1 \sqcup f_2$.

Definition 2.3.1. Let K and L be two simplicial sets. Their *join*, denoted by $K \star L$, is the simplicial set whose n-simplices are given by

$$(K \star L)_n = K_n \cup L_n \cup \bigcup_{i+j=n-1} K_i \times L_j.$$

Given (x, y) in $K_i \times L_j$ and $f\colon [m] \to [n]$ in Δ which decomposes as $f = f_1 \sqcup f_2 \colon [m_1] \sqcup [m_2] \to [i] \sqcup [j]$, the action of f is given by $f^*(x, y) = (f_1^*(x), f_2^*(y))$.

The nerve functor preserves joins, so $\Delta[m] \star \Delta[n] \cong \Delta[m+n+1]$ as simplicial sets. A particularly useful pair of joins is the following:

Definition 2.3.2. Let K be a simplicial set. The *left* and *right cones* of K are the simplicial sets $K^{\triangleleft} := \Delta[0] \star K$ and $K^{\triangleright} := K \star \Delta[0]$, respectively.

Creating cones can be thought of as adjoining a single vertex to a simplicial set which has unique 1-simplices to or from each 0-simplex of K and which uniquely extends each n-simplex of K to an $(n+1)$-simplex. In particular,

$$\begin{aligned} \Delta[n]^{\triangleleft} &\cong \Delta[n+1] & \qquad \Delta[n]^{\triangleright} &\cong \Delta[n+1] \\ \partial\Delta[n]^{\triangleleft} &\cong \Lambda^0[n+1] & \qquad \partial\Delta[n]^{\triangleright} &\cong \Lambda^{n+1}[n+1]. \end{aligned}$$

A useful fact is that the join of quasicategories is a quasicategory (Proposition 1.2.8.3 of [9]).

The primary application of joins in this work will be towards the construction of slice quasicategories. These generalize the usual slice categories in ordinary category theory: if $\mathcal{C}$ is an ordinary category and X is an object of $\mathcal{C}$, then the slice overcategory $\mathcal{C}_{/X}$ has as objects morphisms $A \to X$ and morphisms triangles in $\mathcal{C}$ of the form

$$\begin{tikzcd} A \arrow[rr] \arrow[dr] & & B \arrow[dl] \\ & X. & \end{tikzcd}$$

Similarly, one can form the slice undercategory $\mathcal{C}_{X/}$ whose objects are morphisms $X \to A$ and morphisms are triangles

$$\begin{tikzcd} & X \arrow[dl] \arrow[dr] & \\ A \arrow[rr] & & B. \end{tikzcd}$$

To generalize these constructions to quasicategories, let X be an object in quasicategory $\mathcal{C}$. By the Yoneda lemma, X corresponds to a map $\lceil X \rceil \colon \Delta[0] \to \mathcal{C}$, and thus an object of the slice category $\mathcal{SS}et_{\Delta[0]/}$. Let $\mathcal{C}$ denote the object in $\mathcal{SS}et_{\Delta[0]/}$ corresponding to $\lceil X \rceil$. Likewise, let $\Delta[n]^{\triangleright}$ denote the object in $\mathcal{SS}et_{\Delta[0]/}$ corresponding to the inclusion of the cone point into $\Delta[n]^{\triangleright}$. Then, the morphism set $\mathrm{Hom}_{\mathcal{SS}et_{\Delta[0]/}}(\Delta[n]^{\triangleright}, \mathcal{C})$ is the set of maps $= \Delta[n] \star \Delta[0] \to \mathcal{C}$ which restrict to X on the (terminal) cone point:

$$\begin{tikzcd} & \Delta[0] \arrow[dl] \arrow[dr, "\lceil X \rceil"] & \\ \Delta[n]^{\triangleright} \arrow[rr] & & \mathcal{C}. \end{tikzcd}$$

Definition 2.3.3. Let $\mathcal{C}$ be a quasicategory, and X a 0-simplex of $\mathcal{C}$. The *slice over-quasicategory* $\mathcal{C}_{/X}$ is defined by

$$\mathrm{Hom}_{\mathcal{SS}et}(\Delta[n], \mathcal{C}_{/X}) \cong \mathrm{Hom}_{\mathcal{SS}et_{\Delta[0]/}}(\Delta[n]^{\triangleright}, \mathcal{C}).$$

In a similar vein, considering $\mathrm{Hom}_{\mathcal{SS}et_{\Delta[0]/}}(\Delta[n]^{\triangleleft}, \mathcal{C})$ yields the set of maps $\Delta[n]^{\triangleleft} = \Delta[0] \star \Delta[n] \to \mathcal{C}$ which restrict to X on the (initial) cone point:

$$\begin{tikzcd} & \Delta[0] \arrow[dl] \arrow[dr, "\lceil X \rceil"] & \\ \Delta[n]^{\triangleleft} \arrow[rr] & & \mathcal{C}. \end{tikzcd}$$

Definition 2.3.4. Let $\mathcal{C}$ be a quasicategory, and X a 0-simplex of $\mathcal{C}$. The *slice under-quasicategory* $\mathcal{C}_{X/}$ is defined by

$$\mathrm{Hom}_{\mathcal{SS}et}(\Delta[n], \mathcal{C}_{X/}) \cong \mathrm{Hom}_{\mathcal{SS}et_{\Delta[0]/}}(\Delta[n]^{\triangleleft}, \mathcal{C}).$$

The 0-simplices and 1-simplices of slice quasicategories are much as those depicted for ordinary slice categories above: an n-simplex of $C_{/X}$, say, is an $(n+1)$-simplex in $\mathcal{C}$ which restricts to X on $\{n+1\}$. The above definitions extend to slices over and under maps of simplicial sets $p\colon K \to \mathcal{C}$, as follows:

Definition 2.3.5. For a quasicategory $\mathcal{C}$ and map of simplicial sets $p\colon K \to \mathcal{C}$, the *slice quasicategories* $\mathcal{C}_{/p}$ and $\mathcal{C}_{p/}$ are defined by

$$\mathrm{Hom}_{\mathcal{SS}et}(\Delta[n], \mathcal{C}_{/p}) = \mathrm{Hom}_{\mathcal{SS}et_{K/}}(\Delta[n] \star K, \mathcal{C})$$

and

$$\mathrm{Hom}_{\mathcal{SS}et}(\Delta[n], \mathcal{C}_{p/}) = \mathrm{Hom}_{\mathcal{SS}et_{K/}}(K \star \Delta[n], \mathcal{C}),$$

respectively.

Taking $K = \Delta[0]$ recovers the previously defined slice quasicategories. The general form for slices will mostly be used to create slices over 1-simplices in $\mathcal{C}$. Proposition 1.2.9.3 of [9] ensures that the slice quasicategories $\mathcal{C}_{/p}$ and $\mathcal{C}_{p/}$ are quasicategories themselves. An alternate slice construction is often useful; this construction precisely agrees with the usual construction if $\mathcal{C}$ is the nerve of an ordinary category.

Definition 2.3.6. If $\mathcal{C}$ is a quasicategory and X is an object of $\mathcal{C}$, then the *alternate* *slice quasicategories* $\mathcal{C}^{X/}$ and $\mathcal{C}^{/X}$ are defined via the pullbacks

$$\begin{tikzcd}
\mathcal{C}^{X/} \arrow[r] \arrow[d] & \mathcal{C}^{\Delta[1]} \arrow[d, "d_1"] \\
\Delta[0] \arrow[r, "\lceil X \rceil"] & \mathcal{C}
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathcal{C}^{/X} \arrow[r] \arrow[d] & \mathcal{C}^{\Delta[1]} \arrow[d, "d_0"] \\
\Delta[0] \arrow[r, "\lceil X \rceil"] & \mathcal{C},
\end{tikzcd}$$

respectively.

Proposition 2.3.7 (Proposition 4.2.1.5 of [9])**.** *There are weak equivalences (in the Joyal model structure)* $\mathcal{C}_{/X} \xrightarrow{\sim} \mathcal{C}^{/X}$ *and* $\mathcal{C}_{X/} \xrightarrow{\sim} \mathcal{C}^{X/}$*.*

The following lemma will be useful in Chapter 3:

Lemma 2.3.8. *Let* $p\colon \mathcal{C} \to \mathcal{D}$ *be an inner fibration of quasicategories, and* X *a* 0*-simplex of* $\mathcal{C}$*. Then the induced map* $\tilde{p}\colon \mathcal{C}^{X/} \to \mathcal{D}^{p(X)/}$ *is an inner fibration.*

Proof. First, $\tilde{p}$ is induced by the following map of cospans:

$$\begin{tikzcd}
\Delta[0] \arrow[r, "\lceil X \rceil"] \arrow[d, "\mathrm{id}_{\Delta[0]}"'] & \mathcal{C} \arrow[d, "f"] & \mathcal{C}^{\Delta[1]} \arrow[l, "d_1"'] \arrow[d, "p^{\Delta[1]}"] \\
\Delta[0] \arrow[r, "\lceil p(X) \rceil"] & \mathcal{D} & \mathcal{D}^{\Delta[1]}. \arrow[l, "d_1"']
\end{tikzcd}$$

A dashed solution to a lifting problem

$$\begin{tikzcd}
\Lambda^k[n] \arrow[r] \arrow[d] & \mathcal{C}^{X/} \arrow[d, "\tilde{p}"] \\
\Delta[n] \arrow[r] \arrow[ur, dashed] & \mathcal{D}^{p(X)/}
\end{tikzcd}$$

corresponds by adjunction to a dashed solution to a lifting problem

$$\begin{tikzcd}
\Lambda^k[n] \times \Delta[1] \arrow[r, "h"] \arrow[d] & \mathcal{C} \arrow[d, "p"] \\
\Delta[n] \times \Delta[1] \arrow[r, "s"] \arrow[ur, dashed, "f"] & \mathcal{D}
\end{tikzcd}$$

where h restricted to $\Lambda^k[n] \times \{0\}$ and s restricted to $\Delta[n] \times \{0\}$ are constant on X and $p(X)$ respectively, and p restricted to $\Delta[n] \times \{0\}$ is constant on X. The vertical map on the left can be factored as

$$\Lambda^k[n] \times \Delta[1] \xrightarrow{i} (\Lambda^k[n] \times \Delta[1]) \underset{\Lambda^k[n]\times\{0\}}{\cup} \Delta[n] \times \{0\} \xrightarrow{j} \Delta[n] \times \Delta[1].$$

The inclusion i is inner anodyne since it is the pushout of an inner horn inclusion $\Lambda^k[n] \times \{0\} \subset \Delta[n] \times \{0\}$, while the composite ji is inner anodyne since inner anodyne maps are preserved by products. Therefore, j is inner anodyne by Stevenson's lemma (Lemma 4.2.3). The map $h\colon \Lambda^k[n] \times \Delta[1] \to \mathcal{C}$ can be extended to $h'\colon (\Lambda^k[n] \times \Delta[1]) \underset{\Lambda^k[n]\times\{0\}}{\cup} (\Delta[n] \times \{0\})$ by defining the extension to also be constant to X on $\Delta[n] \times \{0\}$. This then gives a new lifting problem

$$\begin{tikzcd}
(\Lambda^k[n] \times \Delta[1]) \underset{\Lambda^k[n]\times\{0\}}{\cup} \Delta[n] \times \{0\} \arrow[r, "h'"] \arrow[d, "j"'] & \mathcal{C} \arrow[d, "p"] \\
\Delta[n] \times \Delta[1] \arrow[r, "s"] \arrow[ur, dashed] & \mathcal{D}
\end{tikzcd}$$

for which a dashed lift exists because p is an inner fibration and j is inner anodyne. □

We conclude the section by defining limits in quasicategories. Let K be a simplicial set; for the sake of discussion of limits, a *diagram* of shape K in a quasicategory $\mathcal{C}$ will mean a map $p\colon K \to \mathcal{C}$ of simplicial sets.

Definition 2.3.9. An object X in a quasicategory $\mathcal{C}$ is *terminal* if every simplex boundary inclusion $f\colon \partial\Delta[n] \to \mathcal{C}$ with $f(n) = X$ has a filling $F\colon \Delta[n] \to \mathcal{C}$, i.e. every lifting problem of the form below has a dashed solution:

$$\begin{tikzcd}
\{n\} \arrow[r, hook] \arrow[rr, bend left, "\ulcorner X \urcorner"] & \partial\Delta[n] \arrow[r, "f"] \arrow[d] & \mathcal{C}. \\
 & \Delta[n] \arrow[ur, dashed, "F"'] &
\end{tikzcd}$$

Inspecting the definition, one sees that if X is terminal in $\mathcal{C}$, then for every object Y of $\mathcal{C}$ there is a (not necessarily unique) 1-simplex $Y \to X$. Given any two 1-simplices $Y \to X$, one can then construct a map $\partial\Delta[2] \to \mathcal{C}$ by adding the degenerate simplex id_Y as follows:

$$\begin{tikzcd}
Y \arrow[rr, "\mathrm{id}_Y"] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\
 & X. &
\end{tikzcd}$$

Since this is exactly the data of a lifting problem with $n = 2$ in the definition of a terminal object, there is a filling to a 2-simplex $\Delta[2] \to \mathcal{C}$. Thus, any two maps from a fixed object to a terminal object are quasi-isomorphic. In fact, more is true. This coherence data extends to higher dimensions, as evidenced by the next proposition, which is a direct consequence of the definitions.

Proposition 2.3.10. *An object X in $\mathcal{C}$ is terminal if and only if the restriction map $\mathcal{C}_{/X} \to \mathcal{C}$ is a trivial fibration.*

In an ordinary category, the full subcategory on the terminal objects is either empty or forms a contractible groupoid. This fact generalizes to quasicategories as well:

Proposition 2.3.11. *Let $\mathcal{C}^{\mathrm{term}}$ be the full sub-quasicategory of $\mathcal{C}$ spanned by the terminal objects. Then $\mathcal{C}^{\mathrm{term}}$ is either empty or a contractible Kan complex.*

With these definitions in hand, we can define limits in quasicategories in a straightforward manner.

Definition 2.3.12. Let $p\colon K \to \mathcal{C}$ be a diagram in $\mathcal{C}$. A *limit* of p is a terminal object in the slice quasicategory $\mathcal{C}_{/p}$.

For ease of notation, limits are often referred to only by their image under the forgetful functor $\mathcal{C}_{/p} \to \mathcal{C}$, with the other data implicit.

As an example let $K = \Lambda^2[2]$. A diagram of shape K in $\mathcal{C}$ is a map

$$p\colon \quad \begin{array}{ccc} & & \bullet \\ & & \downarrow \\ \bullet & \longrightarrow & \bullet \end{array} \longrightarrow \mathcal{C},$$

i.e. a diagram

$$\begin{array}{ccc} & & X \\ & & \downarrow \\ Y & \longrightarrow & Z \end{array}$$

in $\mathcal{C}$. An object of $\mathcal{C}_{/p}$ is a map $\Lambda^2[2]^{\triangleleft} \to \mathcal{C}$. With some inspection, $\Lambda^2[2]^{\triangleleft}$ is seen to be isomorphic to $\Delta[1] \times \Delta[1]$, so a map $\Lambda^2[2]^{\triangleleft} \to \mathcal{C}$ consists of a commutative square in $\mathcal{C}$ restricting to p on $\Lambda^2[2]$:

$$\begin{tikzcd} W \arrow[r] \arrow[d] \arrow[dr] & X \arrow[d] \\ Y \arrow[r] & Z. \end{tikzcd}$$

Thus, a terminal object in $\mathcal{C}_{/p}$ is a square such as the one above which admits a map restricting to the identity on $\Lambda^2[2]$ from any other square. The following diagram depicts such a map, which consists of a morphism $\Delta[1] \star \Lambda^2[2] \to \mathcal{C}$:

$$\begin{tikzcd}
A \arrow[drr] \arrow[dr, dashed] \arrow[ddr] & & \\
& W \arrow[r] \arrow[d] \arrow[dr] & X \arrow[d] \\
& Y \arrow[r] & Z.
\end{tikzcd}$$

Limits for diagrams $\Lambda^2[2] \to \mathcal{C}$ are known as *pullbacks*. Quasicategorical pullbacks in particular also satisfy the usual pullback-pasting law:

Proposition 2.3.13 (Pullback-pasting law, Lemma 4.4.2.1 of [9])**.** *Consider a diagram of shape* $\Delta[2] \times \Delta[1]$ *in* $\mathcal{C}$ *given as follows:*

$$\begin{tikzcd}
P \arrow[dr] \arrow[drr] \arrow[dd] & & \\
& W \arrow[r] \arrow[dd] & Y \arrow[dd] \\
U \arrow[dr] \arrow[drr] & & \\
& X \arrow[r] & Z.
\end{tikzcd}$$

If

$$\begin{tikzcd}
W \arrow[r] \arrow[d] & X \arrow[d] \\
Y \arrow[r] & Z
\end{tikzcd}$$

is a pullback square, then

$$\begin{tikzcd}
P \arrow[r] \arrow[d] & W \arrow[d] \\
U \arrow[r] & X
\end{tikzcd}$$

is a pullback square if and only if

$$\begin{array}{ccc} P & \longrightarrow & Y \\ \big\downarrow & & \big\downarrow \\ U & \longrightarrow & Z \end{array}$$

is a pullback square.

Further constructions similar to those for ordinary categories are that terminal objects in $\mathcal{C}$ are limits of the unique diagram $\emptyset \to \mathcal{C}$, while products are limits of the diagram $\bullet \sqcup \bullet \to \mathcal{C}$. Analogously to the case for ordinary categories, limits are unique up to coherent choice of quasi-isomorphism in $\mathcal{C}_{/p}$ by virtue of their definition as terminal objects.

2.4 Cocartesian fibrations

A further technical construction is that of cocartesian morphisms and fibrations. A fuller discussion (Chapter 3 of [9]) of cocartesian fibrations interprets them as functors $p\colon \mathcal{C} \to \mathcal{D}$ for which the fibers form quasicategories and vary functorially in $\mathcal{D}$, generalizing the correspondence in ordinary category theory between Grothendieck opfibrations over a category $\mathcal{D}$ and pseudo-functors $\mathcal{D} \to \mathcal{C}at$. A deep discussion of these properties are not needed here, however. Instead, the application will be twofold, first for constructing for a cocartesian fibration $p\colon \mathcal{C} \to \mathcal{D}$ a quasicategory whose objects are the terminal objects in each of the fibers of p, leading to new constructions of cromulent and exponential diagrams (Theorem 3.2.15, Corollary 3.2.16 and Theorem 3.2.17), and second for obtaining a characterization of left fibrations in the proof of Theorem 3.5.1.

Definition 2.4.1. Let $p\colon \mathcal{C} \to \mathcal{D}$ be an inner fibration of quasicategories. A 1-simplex

$X \xrightarrow{f} Y$ in $\mathcal{C}$ is *p-cocartesian* if the natural map $\mathcal{C}_{f/} \to \mathcal{C}_{X/} \times_{\mathcal{D}_{p(X)/}} \mathcal{D}_{p(Y)/}$ is a trivial fibration.

Heuristically, f being p-cocartesian states that for every object Z in $\mathcal{C}$, 1-simplex $g\colon X \to Z$ and 2-simplex σ in $\mathcal{D}$ with $d_2\sigma = p(f)$ and $d_1\sigma = p(g)$ there is a unique, up to coherent choice, 2-simplex τ in $\mathcal{C}$ lifting σ with $d_2\tau = f$ and $d_1\tau = g$. Equivalently,

Proposition 2.4.2 (Dual to Remark 2.4.1.4 of [9])**.** *A* 1*-simplex f is p-cocartesian if and only if for each $n \geq 2$, each lifting problem of the following shape has a dashed solution, where* $01\colon \Delta[1] \to \Lambda^0[n]$ *denotes the* 1*-simplex* $0 \to 1$ *in* $\Lambda^0[n]$*:*

$$\begin{tikzcd}
\Delta[1] \arrow[d, "01"'] \arrow[dr, "f"] & \\
\Lambda^0[n] \arrow[r] \arrow[d] & \mathcal{C} \arrow[d, "p"] \\
\Delta[n] \arrow[r] \arrow[ur, dashed] & \mathcal{D}.
\end{tikzcd}$$

Definition 2.4.3. An inner fibration of quasicategories $p\colon \mathcal{C} \to \mathcal{D}$ is a *cocartesian fibration* if every 1-simplex of $\mathcal{D}$ has a p-cocartesian lift in $\mathcal{C}$.

In particular, left fibrations $p\colon \mathcal{C} \to \mathcal{D}$ are cocartesian fibrations, and are characterized by the property (dual of Lemma 2.4.2.4 of [9]) of being cocartesian fibrations with every 1-simplex of $\mathcal{C}$ being a p-cocartesian lift of its image.

2.5 Adjunctions between quasicategories

The work of Riehl and Verity in [13] offers a perspective on adjunctions for quasicategories which has slightly less theoretical overhead than that presented in [9], and more directly deals with the unit and counit natural transformations associated to an adjunction. Briefly, define the strict 2-category $q\mathcal{C}at_2$ to have objects quasicategories and hom-categories given by $\mathrm{Hom}_{q\mathcal{C}at_2}(\mathcal{C},\mathcal{D}) = \mathrm{Ho}(\mathcal{D}^{\mathcal{C}})$. The morphisms of $q\mathcal{C}at_2$ are thus functors of quasicategories, and the 2-cells are equivalence classes of natural transformations, where two natural transformations are identified if they are homotopic in the functor quasicategory $\mathcal{D}^{\mathcal{C}}$.

Definition 2.5.1 (Definition 4.0.1 of [13])**.** An *adjunction* between quasicategories is an adjunction in the 2-category $q\mathcal{C}at_2$, i.e. a pair of functors $F\colon \mathcal{C} \to \mathcal{D}$ and $U\colon \mathcal{D} \to \mathcal{C}$ and a pair of natural transformations $\eta\colon \mathrm{id}_{\mathcal{C}} \to UF$ and $\varepsilon\colon FU \to \mathrm{id}_{\mathcal{D}}$ satisfying the *triangle identities* which stipulate that the following diagrams commute in the hom-categories $\mathrm{Hom}_{q\mathcal{C}at_2}(\mathcal{C},\mathcal{D})$ and $\mathrm{Hom}_{q\mathcal{C}at_2}(\mathcal{D},\mathcal{C})$, respectively:

$$\begin{tikzcd} F \arrow[r, "F\eta"] \arrow[dr, equal] & FUF \arrow[d, "\varepsilon_F"] \\ & F \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} U \arrow[r, "\eta_U"] \arrow[dr, equal] & UFU \arrow[d, "U\varepsilon"] \\ & U. \end{tikzcd}$$

Concretely, there are 2-simplices

$$\alpha = \left(\begin{tikzcd} F \arrow[r, "F\eta"] \arrow[dr, "\mathrm{id}_F"'] & FUF \arrow[d, "\varepsilon_F"] \\ & F \end{tikzcd} \right) \quad \text{and} \quad \beta = \left(\begin{tikzcd} U \arrow[r, "\eta_U"] \arrow[dr, "\mathrm{id}_U"'] & UFU \arrow[d, "U\varepsilon"] \\ & U \end{tikzcd} \right).$$

witnessing the triangle identities in the functor quasicategories $\mathcal{D}^{\mathcal{C}}$ and $\mathcal{C}^{\mathcal{D}}$, respectively.

For a functor of quasicategories $F\colon \mathcal{C} \to \mathcal{D}$ and a 0-simplex Y of $\mathcal{D}$, define the quasicategory $F_{/Y}$ by the pullback

$$\begin{tikzcd} F_{/Y} \arrow[r] \arrow[d] & \mathcal{D}_{/Y} \arrow[d] \\ \mathcal{C} \arrow[r, "F"] & \mathcal{D} \end{tikzcd}$$

where the vertical map $\mathcal{D}_{/Y} \to \mathcal{D}$ is the forgetful map. This is an alternate formulation of the functor overcategories arising when studying adjunctions, but equivalent to other constructions, see Remark 2.4.14, Definition 3.3.15 and Lemma 4.4.6 of [13]. The application of this construction is the following pointwise universal property of the counit of an adjunction:

Proposition 2.5.2 (Proposition 4.4.8 of [13]). *Let*

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "F"] & \mathcal{D} \arrow[l, shift left, "U"] \end{tikzcd}$$

be an adjunction of quasicategories. Evaluating the counit $\varepsilon\colon FU \to \mathrm{id}_{\mathcal{D}}$ at an object Y of $\mathcal{D}$ yields a terminal object ε_Y in the quasicategory $F_{/Y}$.

If $\mathcal{C}$ is a quasicategory with pullbacks, then for every 1-simplex $X \xrightarrow{f} Y$ of $\mathcal{C}$ there is an adjunction (see discussion between 6.1.1.1 and 6.1.1.2 of [9])

$$\begin{tikzcd} \mathcal{C}_{/X} \arrow[r, shift left, "f_!"] & \mathcal{C}_{/Y} \arrow[l, shift left, "f^*"] \end{tikzcd}$$

where $f_!$ can be thought of as post-composition by f, and f^* can be thought of as pullback by f.

The suitability of a quasicategory $\mathcal{C}$ for the bispan construction is conditional on the existence of a right adjoint f_* to the pullback functor $f^*\colon \mathcal{C}_{/Y} \to \mathcal{C}_{/X}$ for certain classes of morphisms in $\mathcal{C}$.

Definition 2.5.3. A quasicategory with finite limits $\mathcal{C}$ is *locally cartesian closed* if for every 1-simplex $X \xrightarrow{f} Y$ the functor f^* has a right adjoint f_*.

The same definition can be stated for ordinary categories, where some explicit formulations can be given for f_*. The categories $\mathcal{F}in$ and $\mathcal{F}in_G$ are locally cartesian closed: if $p\colon A \to X$ is a morphism in either category, define the (G-)set $L(f,p) = \{(y,\alpha) \mid \alpha\colon f^{-1}(\{y\}) \to A, p\alpha = \mathrm{id}_{f^{-1}(\{y\})}\}$, and define $f_*(p)\colon L(f,p) \to Y$ by $f_*(p)(y,\alpha) = y$. Letting ε^f denote the counit for the adjunction $f^* \dashv f_*$, evaluating ε^f at a morphism $p\colon A \to X$ gives a map $\varepsilon^f_p\colon f^*f_*(p) \to p$, which is the data of a commutative triangle

$$\begin{tikzcd} X \times_Y L(f,p) \arrow[rr, "\varepsilon^f_p"] \arrow[dr, "f^*f_*(p)"'] & & A \arrow[dl, "p"] \\ & X. & \end{tikzcd}$$

In particular, $f^*f_*(p)(x,(y,\alpha)) = x$ and $\varepsilon^f_p(x,(y,\alpha)) = \alpha(x)$.

Definition 2.5.4 (Cf. Definition 3.2.2)**.** If $f\colon X \to Y$ is a 1-simplex in a quasicategory $\mathcal{C}$ such that the functor f^* has a right adjoint f_*, call f *exponentiable*. Given such an f and a 1-simplex $p\colon A \to X$, the canonically associated *exponential diagram* over (p,f) is given by

$$\begin{tikzcd} & X \times_Y L(f,p) \arrow[r] \arrow[dl, "\varepsilon^f_p"'] \arrow[d, "f^*f_*p"] & L(f,p) \arrow[d, "f_*p"] \\ A \arrow[r, "p"] & X \arrow[r, "f"] & Y. \end{tikzcd}$$

Considering the definition of composition of bispan diagrams, it is necessary that the horizontal legs of bispans are exponentiable morphisms. In this work, we consider quasicategories where all morphisms are exponentiable and the bispan quasicategories will not have any restrictions placed upon the bispans themselves Applications to global Tambara functors, however, will require only considering bispans in the bicategory $\mathcal{F}in\mathcal{G}pd$ of finite groupoids whose horizontal legs are *discrete fibrations* of groupoids, which are exponentiable morphisms of finite groupoids generalizing the collapse maps of finite G-sets $G/K \to G/H$ for $K \leq H \leq G$.

2.6 Twisted arrow and twisted functor constructions

The final subject of this introduction is the construction of the diagram categories $\mathrm{TR}[n]$ and $\mathrm{TNR}[n]$ governing the shapes of n-fold span and bispan compositions, respectively.

First, we consider the classical definition of the twisted arrow category, cf. Exercise IX.6.3 of [10].

Definition 2.6.1. The *twisted arrow category* of a category $\mathcal{C}$ is the category $\mathrm{Tw}(\mathcal{C})$ whose objects are the morphisms of $\mathcal{C}$ with morphisms $f \to g$ given by factorizations $g = vfu$, as follows:

$$\begin{tikzcd} A \arrow[d, "f"'] & X \arrow[l, "u"'] \arrow[d, "g"] \\ B \arrow[r, "v"] & Y. \end{tikzcd}$$

Note that a functor $F \colon \mathcal{C} \to \mathcal{D}$ induces an evident functor $\mathrm{Tw}(\mathcal{C}) \to \mathrm{Tw}(\mathcal{D})$.

Considering the category $\mathcal{C} = [n]$, we see that the objects of $\mathrm{Tw}[n]$ are pairs $ab = (a, b)$ with $0 \leq a \leq b$, while morphisms $ab \to a'b'$ consist of diagrams

$$\begin{tikzcd} a \arrow[d] & a' \arrow[l] \arrow[d] \\ b \arrow[r] & b' \end{tikzcd}$$

and it follows from the uniqueness of morphisms between objects in $[n]$ that there is a unique morphism $ab \to a'b'$ in $\mathrm{Tw}[n]$ if $a' \leq a$ and $b \leq b'$.

To construct spans using the twisted arrow construction, we need the following definition.

Definition 2.6.2. Let $\mathrm{TR}[n]$ be the category $\mathrm{Tw}[n]^{\mathrm{op}}$, so $\mathrm{TR}[n]$ has objects pairs ab

with $0 \leq a \leq b$ and a unique morphism $ab \to a'b'$ if $a \leq a'$ and $b' \leq b$.

The categories $\mathrm{TR}[n]$ provide the diagrams encoding n-fold span composition, as the following figures indicate. As one would expect, $\mathrm{TR}[0] = 00$, i.e. $\mathrm{TR}[0]$ is the category with one object and one morphism. Note that $\mathrm{TR}[1]$ gives the shape of a span diagram, while $\mathrm{TR}[2]$ is the shape defining composition of spans.

$$\mathrm{TR}[1] = \begin{tikzcd} & 01 \arrow[dl] \arrow[dr] & \\ 00 & & 11 \end{tikzcd} \qquad \mathrm{TR}[2] = \begin{tikzcd} & & 02 \arrow[dl] \arrow[dr] & & \\ & 01 \arrow[dl] \arrow[dr] & & 12 \arrow[dl] \arrow[dr] & \\ 00 & & 11 & & 22 \end{tikzcd}$$

The diagram $\mathrm{TR}[3]$ is more complicated, and expresses associativity of span composition:

$$\mathrm{TR}[3] = \begin{tikzcd} & & & 03 \arrow[dl] \arrow[dr] & & & \\ & & 02 \arrow[dl] \arrow[dr] & & 13 \arrow[dl] \arrow[dr] & & \\ & 01 \arrow[dl] \arrow[dr] & & 12 \arrow[dl] \arrow[dr] & & 23 \arrow[dl] \arrow[dr] & \\ 00 & & 11 & & 22 & & 33 \end{tikzcd}$$

Note that for $n \geq 2$ there are evident distinguished squares in the diagrams $\mathrm{TR}[n]$, of the form

$$\begin{tikzcd} & a'b' \arrow[dl] \arrow[dr] & \\ a'b \arrow[dr] & & ab' \arrow[dl] \\ & ab & \end{tikzcd}$$

for $0 \leq a' < a \leq b < b'$. Since the twisted arrow category construction is functorial, the sequence $\{\mathrm{TR}[n]\}_{n \geq 0}$ assembles into a functor $\Delta \to \mathcal{C}at$. This cosimplicial structure leads to the following definition:

Definition 2.6.3. Let $\mathcal{C}$ be a quasicategory with pullbacks. Define the simplicial set

$\mathcal{S}pan(\mathcal{C})$ by

$$\mathrm{Hom}_{\mathcal{SS}et}(\Delta[n], \mathcal{S}pan(\mathcal{C})) = \mathrm{Hom}^{\star}_{\mathcal{SS}et}(\mathrm{TR}[n], \mathcal{C})$$

where the $\star$ indicates that each distinguished square of $\mathrm{TR}[n]$ is sent to a pullback square of $\mathcal{C}$.

A priori, $\mathcal{S}pan(\mathcal{C})$ has no structure beyond that of a simplicial set. Work of [2] shows the following, and more.

Theorem 2.6.4 (Proposition 3.4 and Definition 3.6 of [2])**.** *If $\mathcal{C}$ is a quasicategory with pullbacks, then $\mathcal{S}pan(\mathcal{C})$ is a quasicategory.*

Theorem 2.6.5 (Theorem 12.2 of [2])**.** *If $p\colon \mathcal{C} \to \mathcal{D}$ is an inner fibration of quasicategories with pullbacks, then the induced functor $\mathcal{S}pan(\mathcal{C}) \to \mathcal{S}pan(\mathcal{D})$ is an inner fibration.*

Our goal is an analogous result to the first theorem for bispans in locally cartesian closed quasicategories. The first step is to define a cosimplicial sequence of diagram categories describing bispan composition. This will be due to a generalization of the twisted arrow category, due to to John Berman:

Definition 2.6.6. Let $\mathcal{C}$ and $\mathcal{D}$ be categories, with certain objects in $\mathcal{C}$ marked as "twisted". The *twisted functor category* $\mathcal{C}^{\mathcal{D}}_{\mathrm{Tw}}$ has objects functors $F\colon \mathcal{C} \to \mathcal{D}$, and a morphism $\eta\colon F \to G$ is a collection of arrows η_X between FX and GX such that if X is not a twisted object of $\mathcal{C}$, then $FX \xrightarrow{\eta_X} GX$, and if X is twisted, then $GX \xrightarrow{\eta_X} FX$, subject to the condition that if $X \xrightarrow{f} Y$ is a morphism in $\mathcal{C}$, then the square

$$\begin{array}{ccc} FX & \xrightarrow{Ff} & FY \\ \Big| \eta_X & & \Big| \eta_y \\ GX & \xrightarrow{Gf} & GY \end{array}$$

(the directions of η_X and η_Y are inferred) commutes.

Let $\mathcal{TR}$ and $\mathcal{TNR}$ be the twisted categories $T \to R^t$ and $T \to N \to R^t$, respectively, where the superscript t denotes a twisted object.

Proposition 2.6.7. *The twisted functor category* $\mathcal{C}_{\mathrm{Tw}}^{\mathcal{TR}}$ *is isomorphic to* $\mathrm{Tw}(\mathcal{C})^{\mathrm{op}}$.

Proof. An object of $\mathcal{C}_{\mathrm{Tw}}^{\mathcal{TR}}$ is a functor $F\colon \mathcal{TR} \to \mathcal{C}$, i.e. an edge $f\colon F(T) \to F(R)$ in $\mathcal{C}$. A morphism $\eta\colon F \to G$ in $\mathcal{C}_{\mathrm{Tw}}^{\mathcal{TR}}$ is then

$$\begin{tikzcd} F(T) \arrow[r, "\eta_T"] \arrow[d, "f"'] & G(T) \arrow[d, "g"] \\ F(R) & G(R). \arrow[l, "\eta_R"] \end{tikzcd}$$

This is precisely the data of a morphism $f \to g$ in $\mathrm{Tw}(\mathcal{C})^{\mathrm{op}}$, and the functoriality of this assignment is immediate. $\square$

Corollary 2.6.8. *The diagram category* $\mathrm{TR}[n]$ *is isomorphic to the twisted functor category* $[n]_{\mathrm{Tw}}^{\mathcal{TR}}$.

Definition 2.6.9. For $n \geq 0$, define the diagram categories $\mathrm{TNR}[n] \coloneqq [n]_{\mathrm{Tw}}^{\mathcal{TNR}}$.

The objects of $\mathrm{TNR}[n]$ are triples abc with a denoting the image of T, b the image of N and c the image of R, so that $0 \leq a \leq b \leq c \leq n$, and there is a unique morphism $abc \to a'b'c'$ if $a \leq a', b \leq b'$ and $c' \leq c$. Thus, $\mathrm{TNR}[0] = 000$, while

$$\mathrm{TNR}[1] = \begin{tikzcd} & 001 \arrow[r] \arrow[dl] & 011 \arrow[dr] & \\ 000 & & & 111 \end{tikzcd}$$

$$\mathrm{TNR}[2] = \begin{tikzcd}[column sep=small]
 & & 002 \arrow[r] \arrow[dl] & 012 \arrow[r] \arrow[dl] \arrow[dr] & 022 \arrow[dr] & & \\
 & 001 \arrow[r] \arrow[dl] & 011 \arrow[dr] & & 112 \arrow[r] \arrow[dl] & 122 \arrow[dr] & \\
000 & & & 111 & & & 222
\end{tikzcd}$$

and TNR[3] is the following:

$$\begin{tikzcd}[column sep=small]
 & & & 003 \arrow[r] \arrow[dl] & 013 \arrow[r] \arrow[dl] \arrow[dr] & 023 \arrow[r] \arrow[dl] \arrow[dr] & 033 \arrow[dr] & & & \\
 & & 002 \arrow[r] \arrow[dl] & 012 \arrow[r] \arrow[dl] \arrow[dr] & 022 \arrow[dr] & 113 \arrow[r] \arrow[dl] & 123 \arrow[r] \arrow[dl] \arrow[dr] & 133 \arrow[dr] & & \\
 & 001 \arrow[r, blue] \arrow[dl] & 011 \arrow[dr] & & 112 \arrow[r] \arrow[dl] & 122 \arrow[dr] & & 223 \arrow[r] \arrow[dl] & 233 \arrow[dr] & \\
000 & & & 111 & & & 222 & & & 333.
\end{tikzcd}$$

These diagrams are will be used to encode higher bispan composition, as $\mathcal{B}ispan(\mathcal{C})$ is defined as a simplicial set with

$$\mathrm{Hom}_{\mathcal{SS}et}(\Delta[n], \mathcal{B}ispan(\mathcal{C})) = \mathrm{Hom}^{\star}_{\mathcal{SS}et}(\mathrm{TNR}[n], \mathcal{C})$$

where the $\star$ indicates that certain subdiagrams of $\mathrm{TNR}[n]$ are sent to certain classes of diagrams in $\mathcal{C}$.

As a further remark, a functor $\mathcal{TR} \to \mathcal{TNR}$ induces a sequence of functors $[n]^{\mathcal{TNR}}_{\mathrm{Tw}} \to [n]^{\mathcal{TR}}_{\mathrm{Tw}}$, i.e. a sequence of functors $\mathrm{TNR}[n] \to \mathrm{TR}[n]$ agreeing with the cosimplicial structures. These then yield a functor $\mathcal{S}pan(\mathcal{C}) \to \mathcal{B}ispan(\mathcal{C})$. Define $i\colon \mathcal{TR} \to \mathcal{TNR}$ by $i(T) = T$ and $i(R) = R$, inducing $i^*\colon\ \mathrm{TNR}[n] \to \mathrm{TR}[n]$ for each n. Via the process described above, i induces $I\colon \mathcal{S}pan(\mathcal{C}) \to \mathcal{B}ispan(\mathcal{C})$, defined on n-simplices by $I_n(\omega) = \omega i^*$. Likewise, there is a functor $j\colon \mathcal{TR} \to \mathcal{TNR}$ defined by $j(T) = N$ and $j(R) = R$, inducing $J\colon \mathcal{S}pan(\mathcal{C}) \to \mathcal{B}ispan(\mathcal{C})$.

A classical Tambara functor for a finite group G restricts to a pair of additive and multiplicative Mackey functors for G, respectively, by only considering the transfers and the restrictions or the norms and the restrictions. The functors $I, J \colon \mathcal{S}pan(\mathcal{C}) \to \mathcal{B}ispan(\mathcal{C})$ allow for similar constructions when considering *homotopical* $(\mathcal{C}, \mathcal{D})$*-Tambara functors* and *homotopical* $(\mathcal{C}, \mathcal{D})$*-Mackey functors*, defined as product-preserving functors $\mathcal{B}ispan(\mathcal{C}) \to \mathcal{D}$ and $\mathcal{S}pan(\mathcal{C}) \to \mathcal{D}$, respectively. Further describing this relationship is an aim of future work.

Chapter 3

Distributive laws and bispans in quasicategories

The approach of this chapter is in the spirit of [3], generalizing aspects of the work therein to obtain a theory of bispans in quasicategories. In particular, for a longer account of distributive laws, see Chapter 6 of [3]. Let $\mathcal{SS}et$ denote the category of simplicial sets, and $s\mathcal{SS}et$ the category of bisimplicial sets. The aim is to establish the following:

Theorem A (Theorem 3.4.4)**.** *Let $\mathcal{C}$ be a locally cartesian closed quasicategory. There is a bisimplicial set $\mathbb{D}(\mathcal{C})$ and a sequence of bisimplicial sets $\{\Xi[n]\}_{n\geq 0}$ called* distributahedra *such that the simplicial set whose n-simplices are suitable maps of bisimplicial sets $\Xi^n \to \mathbb{D}(\mathcal{C})$ is an quasicategory whose n-simplices are decomposed n-fold compositions of bispans in $\mathcal{C}$.*

3.1 Distributive laws

For X and Y simplicial sets, define the bisimplicial set $X \boxtimes Y$ by $(X \boxtimes Y)_{i,j} = X_i \times Y_j$, with the bisimplicial maps coming from the simplicial maps in each variable.

Definition 3.1.1. A *bihorn* is a bisimplicial set of the form

$$\Lambda^{j,k}[m,n] = \Lambda^j[m] \boxtimes \Delta[n] \underset{\Lambda^j[m]\boxtimes\Lambda^k[n]}{\cup} \Delta[n] \boxtimes \Lambda^k[n].$$

Let $\Delta[m,n]$ denote $\Delta[m] \boxtimes \Delta[n]$, so that $\Lambda^{j,k}[m,n] \subset \Delta[m,n]$. For each $n \geq 0$, define the bisimplicial set $\Xi[n]$, called a *distributahedron* in [3], by

$$\Xi[n]_{i,j} = \mathrm{Hom}_{\mathcal{SS}et}(\Delta[i] \star \Delta[j], \Delta[n]).$$

Since every map $\Delta[i] \star \Delta[j] \to \Delta[n]$ is determined by its image on the objects of $\Delta[i] \star \Delta[j]$ and there is a total ordering on the objects of $\Delta[n]$, every such map can be written uniquely as $a_0 \cdots a_i | b_0 \cdots b_j$ with $a_0 \leq \cdots \leq a_i \leq b_0 \leq \cdots b_j$.

The idea underpinning distributahedra is to consider two classes of morphisms $\mathcal{A}$ and $\mathcal{M}$ closed under composition in a (quasi)category $\mathcal{C}$ with the property that any composite $m \circ a$ of a morphism $m \in \mathcal{M}$ with a morphism $a \in \mathcal{A}$ can be rewritten as a composition $a' \circ m'$ with $m' \in \mathcal{M}$ and $a' \in \mathcal{A}$ (but not vice-versa), and the consider the diagrams which form by freely rewriting chains $m_n \circ a_n \circ m_{n-1} \circ a_{n-1} \circ \cdots \circ m_0 \circ a_0$. Intuitively, one thinks of the morphisms in $\mathcal{M}$ as encoding a multiplicative structure and the morphisms in $\mathcal{A}$ as encoding an additive structure, so the rewriting is "distributing" multiplication over addition. In the next diagrams, red arrows depict morphisms in $\mathcal{A}$ and blue arrows depict morphisms in $\mathcal{M}$.

Im the following figures, we depict some of the $\Xi[n]$ for small values of n. The first is $\Xi[0] = 0|0$, a single $(0,0)$-simplex. In $\Xi[1]$, the non-degenerate bisimplices consist of three $(0,0)$-simplices, $0|0, 0|1$ and $1|1$, a $(0,1)$-simplex (in blue) $0|01$ and a $(1,0)$-simplex (in red) $01|1$, depicting a composite pair $a \circ m$:

$$\Xi[1] = \begin{tikzcd} & 0|1 \arrow[dr, "01|1", "a"'] & \\ 0|0 \arrow[ur, "0|01", "m"'] & & 1|1. \end{tikzcd}$$

In $\Xi[2]$, there are now non-degenerate bisimplices of bidegree $(0,2)$, $(1,1)$ and $(2,0)$, given by $0|012, 01|12$ and $012|2$, respectively. The $(0,2)$-simplex shares a $(0,1)$-simplex with the $(1,1)$-simplex, while the $(2,0)$-simplex shares a $(1,0)$-simplex with the $(1,1)$-simplex. The distributahedron $\Xi[2]$ depicts a refactorization of a composite $a_1 \circ m_1 \circ a_0 \circ m_0$ given by alternating blue and red morphisms along the upper perimeter of the diagram, from $0|0$ to $2|2$. The refactorization uses the $(1,1)$-cell $01|12$ to rewrite the composite $m_1 \circ a_0$ as the composite $a_0' \circ m_1'$, which is given by the arrows $0|12$ and $01|2$:

$$\Xi[2] = \begin{tikzcd} & & 1|1 \arrow[dr, "1|12", "m_1"'] & & \\ & 0|1 \arrow[ur, "01|1", "a_0"'] \arrow[ddr, "m_1'", "0|12"'] & 01|12 & 1|2 \arrow[ddr, "12|2", "a_1"'] & \\ & 0|012 & & 012|2 & \\ 0|0 \arrow[uur, "0|01", "m_0"'] \arrow[rr, "0|02"] & & 0|2 \arrow[uur, "a_0'", "01|2"'] \arrow[rr, "02|2"] & & 2|2. \end{tikzcd}$$

Once the refactorization through the $(1,1)$-cell is completed, the blue and red morphisms can now be composed into a single blue and a single red morphism.

By construction, the sequence $\Xi[-]$ defines a cosimplicial object in the category of bisimplicial sets, i.e. a functor $\Delta \to s\mathcal{S}et$. Thus, if $\mathbb{X}$ is a bisimplicial set, then taking morphisms level-wise defines an associated simplicial set $A(\mathbb{X}) = \mathrm{Hom}_{s\mathcal{S}et}(\Xi[-], \mathbb{X})$.

Definition 3.1.2. A bisimplicial set $\mathbb{X}$ is a *distributive law* if $A(\mathbb{X})$ is a quasicategory.

To get a sense for the definition of a distributive law, let d^i denote the usual coface map $\Delta[n-1] \to \Delta[n]$, inducing a map $d^i\colon \Xi[n-1] \to \Xi[n]$ by postcomposition, so that the sub-bisimplicial set of $\Xi[n]$ generated by the image of d^i can be thought of as the ith-face of $\Xi[n]$, denoted by $\partial_i\Xi[n]$. In analogy to the usual horns in simplicial sets, define the bisimplicial set $\Gamma^j[n]$ for $0 \le j \le n$ by

$$\Gamma^j[n] = \bigcup_{\substack{0\le i\le n\\ i\neq j}} \partial_i\Xi[n].$$

Then $\mathrm{Hom}_{\mathcal{S}Set}(\Lambda^j[n], A(\mathbb{X})) \cong \mathrm{Hom}_{s\mathcal{S}Set}(\Gamma^j[n], \mathbb{X})$. Given a map $\Lambda^j[n] \to A(\mathbb{X})$, extending it to an n-simplex $\Delta[n] \to A(\mathbb{X})$ is equivalent to extending the corresponding map $f\colon \Gamma^j[n] \to \mathbb{X}$ to a map $\Xi[n] \to \mathbb{X}$.

In particular, a horn $\Lambda^1[2] \to A(\mathbb{X})$ corresponds to a map $\Gamma^1[2] \to \mathbb{X}$. Since $\Gamma^1[2] = \partial_0\Xi[2] \cup \partial_2\Xi[2]$ can be depicted as

$$\begin{tikzcd}
 & & 1|1 \arrow[dr, "1|12"] & & \\
 & 0|1 \arrow[ur, "01|1"] & & 1|2 \arrow[dr, "12|2"] & \\
0|0 \arrow[ur, "0|01"] & & & & 2|2
\end{tikzcd}$$

where the blue arrows again depict $(0,1)$-cells and the red arrows depict $(1,0)$-cells. In the notation of the discussion of $\Xi[2]$ in the preceding, the data of the bihorn $\Gamma^1[2] \to \mathbb{X}$ then is that of a refactorization problem $a_1 \circ m_1 \circ a_0 \circ m_0$. Filling the bihorn to a diagram $\Xi[2] \to \mathbb{X}$ distributes $m_1 \circ a_0$ to $a_0' \circ m_1'$ and provides choices of composites in $\mathcal{A}$ and $\mathcal{M}$.

Lemma 3.1.3. *Let $\mathbb{X}$ be a bisimplicial set. Suppose that for every bihorn $\mathbb{X}$ depicted* *by the solid horizontal lines in the following diagrams*

$$\begin{tikzcd} \Lambda^{i,k-i}[i,n-i] \arrow[r] \arrow[d, hook] & \mathbb{X} \\ \Delta[i,n-i] \arrow[ur, dashed] & \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} \Lambda^{k,0}[l,n-l] \arrow[r] \arrow[d, hook] & \mathbb{X} \\ \Delta[l,n-l] \arrow[ur, dashed] & \end{tikzcd}$$

with $n \geq 2, 0 < k < n$ *and* $0 \leq i \leq k \leq l \leq n$*, there are dashed arrows filling the bihorns to bisimplices. Then* $\mathbb{X}$ *is a distributive law.*

Proof. Recall that an (i, j)-simplex of $\Xi[n]$ is a map $\gamma\colon \Delta[i] \star \Delta[j] \to \Delta[n]$, written as $\gamma = a_0 \ldots a_i|b_0 \ldots b_j$ with $a_0 \leq \cdots a_i \leq b_0 \leq \cdots \leq b_j$. An (i, j)-simplex is degenerate if it is degenerate in either variable, which is to say that $a_r = a_{r+1}$ for some $0 \leq r \leq i-1$ or $b_r = b_{r+1}$ for some $0 \leq r \leq j-1$. The highest-dimensional (taking the sum of the bidegrees) non-degenerate bisimplices of $\Xi[n]$ are the $(k, n-k)$-simplices $0 \cdots k|k \cdots n$ for $0 \leq k \leq n$.

Given $n > 2$ and $0 < k < n$, consider $\Gamma^k[n]$ as a subcomplex of $\Xi[n]$. There is a canonical inclusion of $\Delta[k, n-k]$ into $\Xi[n]$ given by the $(k, n-k)$-simplex $0 \cdots k|k \cdots n$. In $\Gamma^k[n]$, the highest-dimensional non-degenerate bisimplices are in bidegrees $(r, n-1-r)$ for $0 \leq r \leq n-1$, and total degree $n-1$. The non-degenerate bisimplices of highest total degree in $\Gamma^k[n]$ correspond to bisimplices $a_0 \cdots a_r|b_0 \cdots b_{n-1-r}$ with $0 \leq a_0 < \cdots < a_r = b_0 < \cdots < b_{n-1-r} \leq n$ such that either

(a) one of $a_0, \ldots, a_{r-1}$ is k,

(b) $a_r = b_0 = k$, or

(c) one of $b_1, \ldots, b_{n-1-r}$ is k.

Given a particular value of k, the restrictions can be described more precisely, but they are not relevant here. Simply put, these bisimplices correspond to choosing n

values between 0 and n with one fixed to be k and then writing them in order with one of the values repeated.

In particular, the bisimplices $0\cdots\hat{j}\cdots k|k\cdots n$ for $0 \le j < k$ and $0\cdots k|k\cdots\widehat{\ell}\cdots n$ for $k < \ell \le n$ describe a canonical inclusion of

$$\Lambda^{k,0}[k,n-k] = \Lambda^k[k] \boxtimes \Delta[n-k] \underset{\Lambda^k[k]\boxtimes\Lambda^0[n-k]}{\cup} \Delta[k] \boxtimes \Lambda^0[n-k]$$

inside $\Gamma^k[n]$ considered as a subcomplex of $\Xi[n]$. Furthermore, by the considerations on the bisimplices of $\Gamma^k[n]$ in the preceding,

$$\Gamma^k[n] \cap \Delta[k,n-k] = \Lambda^{k,0}[k,n-k]$$

inside $\Xi[n]$ since the bisimplices of total degree $n-1$ in $\Delta[k,n-k]$ are precisely those in $\Lambda^{k,0}[k,n-k]$. Thus, define $\Gamma^k[n]^{\triangledown}_0$ as the pushout $\Gamma^k[n] \underset{\Lambda^{k,0}[k,n-k]}{\cup} \Delta[k,n-k]$, so that the following is a pushout square:

$$\begin{tikzcd}
\Lambda^{k,0}[k,n-k] \arrow[r] \arrow[d] & \Gamma^k[n] \arrow[d] \\
\Delta[k,n-k] \arrow[r] & \Gamma^k[n]^{\triangledown}_0 .
\end{tikzcd}$$

Consider the intersection $\Delta[k-1,n-(k-1)] \cap \Gamma^k[n]^{\triangledown}_0$ inside $\Xi[n]$. Rewriting the latter term, we obtain

$$\begin{aligned}
&\Delta[k-1,n-(k-1)] \cap (\Delta[k,n-k] \cap \Gamma^k[n]) \\
&= (\Delta[k-1,n-(k-1)] \cap \Delta[k,n-k]) \cup (\Delta[k-1,n-(k-1)] \cap \Gamma^k[n]).
\end{aligned}$$

The intersection $\Delta[k-1,n-(k-1)] \cap \Delta[k,n-k]$ in $\Xi[n]$ is the intersection of the bisimplices $0\cdots(k-1)|(k-1)\cdots n$ and $0\cdots k|k\cdots n$, which is the bisimplex

$0\cdots(k-1)|k\cdots n$ of total degree $n-1$, (which by construction does not appear in $\Gamma^k[n]$). The intersection $\Delta[k-1,n-(k-1)]\cap\Gamma^k[n]$ is obtained by intersecting $0\cdots(k-1)|(k-1)\cdots n$ with the bisimplices of $\Gamma^k[n]$ described above. In particular, the highest-degree non-degenerate bisimplices in the intersection are of the form $0\cdots\hat{j}\cdots(k-1)|(k-1)\cdots n$ with $0\leq j<k-1$ and $0\cdots(k-1)|(k-1)\cdots\widehat{\ell}\cdots n$ with $k-1<\widehat{\ell}\leq n$ and $\widehat{\ell}\neq k$. Altogether, the intersections combine to recover the canonical copy of $\Lambda^{k-1,1}[k-1,n-(k-1)]$ contained in $\Xi[n]$.

With this observation, let $1\leq s\leq k$, and suppose that for $0\leq s'<s$ there are pushouts

$$\Gamma^k[n]^{\triangledown}_{s'}=\Gamma^k[n]^{\triangledown}_{s'-1}\underset{\Lambda^{k-s',s'}[k-s',n-(k-s')]}{\cup}\Delta[k-s',n-(k-s')],$$

so that

$$\Gamma^k[n]^{\triangledown}_{s'}=\Gamma^k[n]\cup\Delta[k,n-k]\cup\Delta[k-1,n-(k-1)]\cup\cdots\cup\Delta[k-s',n-(k-s')].$$

This was shown above for $s=1$. Consider the intersection

$$\Gamma^k[n]^{\triangledown}_{s-1}\cap\Delta[k-s,n-(k-s)]$$

in $\Xi[n]$. By construction,

$$\Gamma^k[n]^{\triangledown}_{s-1}=\Gamma^k[n]\cup\Delta[k,n-k]\cup\Delta[k-1,n-(k-1)]\cup\cdots\cup\Delta[k-(s-1),n-(k-(s-1))]$$

so it suffices to consider the intersections

$$\Gamma^k[n]\cap\Delta[k-s,n-(k-s)]$$

and, for $0 \leq s' < s$,

$$\Delta[k-s', n-(k-s')] \cap \Delta[k-s, n-(k-s)].$$

In the first instance, the intersection $\Gamma^k[n] \cap \Delta[k-s, n-(k-s)]$ has highest-degree non-degenerate bisimplices of the form $0 \cdots \hat{j} \cdots (k-s)|(k-s) \cdots n$ with $0 \leq j < k-s$ and $0 \cdots (k-s)|(k-s) \cdots \widehat{\ell} \cdots n$ with $k-s < \ell \leq n$ and $\ell \neq k$. The intersection $\Delta[k-s', n-(k-s')] \cap \Delta[k-s, n-(k-s)]$ in $\Xi[n]$ is the intersection $0 \cdots (k-s)|(k-s') \cdots n$ of the bisimplices $0 \cdots (k-s')|(k-s') \cdots n$ and $0 \cdots (k-s)|(k-s) \cdots n$. Note that each of these intersections as s' varies is contained in $0 \cdots (k-s)|(k-s+1) \cdots n$, which is the intersection $\Delta[k-(s-1), n-(k-(s-1))] \cap \Delta[k-s, n-(k-s)]$. Putting the intersections together precisely recovers the bihorn $\Lambda^{k-s,s}[k-s, n-(k-s)]$ in $\Xi[n]$.

We can also consider adjoining the bisimplices $\Delta[k+t, n-(k+t)]$ to $\Gamma^k[n]$. Define $\Gamma^k[n]^{\vartriangle}_0 = \Gamma^k[n]^{\triangledown}_0$, the same as before. Analysis entirely analogous to that in the preceding shows that

$$\Gamma^k[n]^{\vartriangle}_0 \cap \Delta[k+1, n-(k+1)] = \Lambda^{k,0}[k+1, n-(k+1)]$$

and for $1 \leq t \leq n-k$, one can iteratively produce pushouts

$$\begin{tikzcd}
\Lambda^{k,0}[k+t, n-(k+t)] \arrow[r] \arrow[d] & \Gamma^k[n]^{\vartriangle}_{t-1} \arrow[d] \\
\Delta[k+t, n-(k+t)] \arrow[r] & \Gamma^k[n]^{\vartriangle}_t .
\end{tikzcd}$$

By construction, $\Xi[n] = \Gamma^k[n]^{\triangledown}_k \underset{\Gamma^k[n]}{\cup} \Gamma^k[n]^{\vartriangle}_{n-k}$. Given a map $f\colon \Gamma^k[n] \to \mathbb{X}$, the hypotheses on $\mathbb{X}$ ensure that the bihorn $f|_{\Lambda^{k,0}[k,n-k]}$ extends a map $\Delta[k, n-k] \to \mathbb{X}$.

Using the universal property of the pushout, this induces $\widetilde{f}_0 \colon \Gamma^k[n]_0^{\triangledown} = \Gamma^k[n]_0^{\vartriangle} \to \mathbb{X}$. Now $\widetilde{f}_0$ can be restricted to $\Gamma^{k-1,1}[k-1, n-(k-1)]$ and $\Gamma^{k,0}[k+1, n-(k+1)]$ and then extended to maps $\Delta[k-1, n-(k-1)] \to \mathbb{X}$ and $\Delta[k+1, n-(k+1)] \to \mathbb{X}$, inducing maps $\widetilde{f}_0^{\triangledown} \colon \Gamma^k[n]_1^{\triangledown} \to \mathbb{X}$ and $\widetilde{f}_0^{\vartriangle} \colon \Gamma^k[n]_1^{\vartriangle} \to \mathbb{X}$. Continuing to restrict to bihorns and extending to bisimplices using the hypotheses on $\mathbb{X}$ ensures that f can be extended compatibly to $\Gamma^k[n]_k^{\triangledown}$ and $\Gamma^k[n]_{n-k}^{\vartriangle}$, and thus to all of $\Xi[n]$, verifying that $A(\mathbb{X})$ is a quasicategory. $\square$

3.2 Cromulent and exponential diagrams in quasicategories

This section contains rigorous definitions of cromulent and exponential diagrams in quasicategories and some useful properties for their manipulation which will be crucial in latter sections. In this section, as well as the next, $\mathcal{C}$ is assumed to be a locally Cartesian closed quasicategory throughout.

Let $\mathrm{Pb}(\mathcal{C})$ denote the quasicategory of pullback squares in $\mathcal{C}$, regarded as a subcategory of $\mathrm{Sq}(\mathcal{C}) \coloneqq \mathcal{C}^{\Delta[1]\times\Delta[1]}$, and consider the pullback $\mathrm{Sq}(\mathcal{C}) \times_{\mathcal{C}^{\Delta[1]}} \mathrm{Pb}(\mathcal{C})$ consisting of diagrams of shape $\Delta[1] \times \mathrm{TR}[1]$ in $\mathcal{C}$, i.e. diagrams of the following shape

$$\begin{tikzcd}
(0,00) & (0,01) \arrow[l] \arrow[r] \arrow[d] & (0,11) \arrow[d] \\
(1,00) \arrow[from=u] & (1,01) \arrow[l] \arrow[r] & (1,11)
\end{tikzcd}$$

where the square on the right is a pullback square. The inclusion of the subdiagram

$$\Delta[1]\cup\mathrm{TR}[1] = (\Delta[1]\times\{00\})\cup(\{1\}\times\mathrm{TR}[1]) = \left(\begin{tikzcd} (0,00) \arrow[d] & & \\ (1,00) & (1,01) \arrow[l] \arrow[r] & (1,11) \end{tikzcd}\right)$$

into $\Delta[1]\times\mathrm{TR}[1]$ induces a restriction map $\mathrm{Sq}(\mathcal{C})\times_{\mathcal{C}^{\Delta[1]}}\mathrm{Pb}(\mathcal{C})\to\mathcal{C}^{\Delta[1]\cup\mathrm{TR}[1]}$.

Definition 3.2.1. Given a functor $(p,u,f)\colon \Delta[1]\cup\mathrm{TR}[1]\to\mathcal{C}$, depicted as a diagram

$$\begin{tikzcd} X' \arrow[d, "p"'] & & \\ X & A \arrow[l, "u"'] \arrow[r, "f"] & Y \end{tikzcd}$$

in $\mathcal{C}$, a *cromulent diagram* (with respect to p,u and f) in $\mathcal{C}$) is a terminal object of the pullback quasicategory $(\mathrm{Sq}(\mathcal{C})\times_{\mathcal{C}^{\Delta[1]}}\mathrm{Pb}(\mathcal{C}))\times_{\mathcal{C}^{\Delta[1]\cup\mathrm{TR}[1]}}\{(p,u,f)\}$. Let $\mathcal{CromDiag}(p,u,f)$ refer to the (contractible) category of cromulent diagrams above (p,u,f). The diagram (p,u,f) will be referred to as the *base* of the cromulent diagram.

Cromulent diagrams appear for the first time in [3] as a reformulation of *exponential diagrams*, which we now describe. Let $A\xrightarrow{p}X$ and $X\xrightarrow{f}Y$ be 1-simplices in $\mathcal{C}$. Let $\mathrm{Pb}(\mathcal{C})_f$ denote the subcategory of $\mathrm{Pb}(\mathcal{C})$ defined by the pullback

$$\begin{tikzcd} \mathrm{Pb}(\mathcal{C})_f \arrow[r] \arrow[d] & \mathrm{Pb}(\mathcal{C}) \arrow[d] \\ \Delta[0] \arrow[r, "\lceil f\rceil"] & \mathcal{C}^{\Delta[1]} \end{tikzcd}$$

where the map $\mathrm{Pb}(\mathcal{C})\to\mathcal{C}^{\Delta[1]}$ is induced by the inclusion $\Delta[1]\hookrightarrow\Delta[1]\times\Delta[1]$ whose image is the 1-simplex $(0,1)\to(1,1)$. Thus, $\mathrm{Pb}(\mathcal{C})_f$ has objects pullback squares in $\mathcal{C}$

of the form

$$\begin{tikzcd} M \arrow[r] \arrow[d] & N \arrow[d] \\ X \arrow[r, "f"] & Y. \end{tikzcd}$$

There is a restriction map $\mathrm{Pb}(\mathcal{C})_f \to \mathcal{C}^{/X}$ induced by the inclusion of the 1-simplex $(0,0) \to (0,1)$ into $\Delta[1] \times \Delta[1]$, recovering $M \to X$ from the diagram above.

Given a 1-simplex p of $\mathcal{C}$, let $F(p)$ denote the pullback

$$\begin{tikzcd} F(p) \arrow[r] \arrow[d] & \mathcal{C}^{\Delta[1]^{\triangleleft}} \arrow[d] \\ \Delta[0] \arrow[r, "\lceil p \rceil"] & \mathcal{C}^{\Delta[1]} \end{tikzcd}$$

so that $F(p)$ is the subcategory of the functor category $\mathcal{C}^{\Delta[1]^{\triangleleft}}$ whose objects are triangles

$$\begin{tikzcd} & M \arrow[dl] \arrow[d] \\ A \arrow[r, "p"] & X. \end{tikzcd}$$

Letting $*$ denote the cone point of $(\Delta[1])^{\triangleleft}$, the inclusion of the 1-simplex $* \to 1$ into $\Delta[1]^{\triangleleft}$ induces a restriction map $F(p) \to \mathcal{C}^{/X}$, which recovers $M \to X$ in the diagram above.

Definition 3.2.2. Let $A \xrightarrow{p} X$ and $X \xrightarrow{f} Y$ be 1-simplices in $\mathcal{C}$. An *exponential diagram* (with respect to p and f) is a terminal object in $F(p) \times_{C^{/X}} \mathrm{Pb}(C)_f$, i.e. a terminal object in the space of diagrams of the form

$$\begin{tikzcd} & M \arrow[r] \arrow[dl] \arrow[d] & N \arrow[d] \\ A \arrow[r, "p"] & X \arrow[r, "f"] & Y \end{tikzcd}$$

where M and N can vary and the right hand square is a pullback. Let $\mathcal{E}xp\mathcal{D}iag(p, f)$

denote the (contractible) quasicategory of exponential diagrams above the *base* (p, f).

The following technical lemma is useful for constructing quasicategories equivalent to slice quasicategories.

Lemma 3.2.3. *Let $\mathcal{C}$ be a quasicategory and K a simplicial set, with a map $p\colon K \to \mathcal{C}$. For any other simplicial set X, there is a map $K \to X \star K$ inducing a restriction map $\mathcal{C}^{X\star K} \to \mathcal{C}^K$. Viewing p as a map $\Delta[0] \to \mathcal{C}^K$, define the quasicategory $F(p, X)$ as the pullback*

$$\begin{tikzcd} F(p,X) \arrow[r] \arrow[d] & \mathcal{C}^{X\star K} \arrow[d] \\ \Delta[0] \arrow[r, "\lceil p \rceil"] & \mathcal{C}^K. \end{tikzcd}$$

Then $F(p, X) \simeq (\mathcal{C}_{/p})^X$ over $\mathcal{C}^X$.

Proof. This is Corollary 47.9 of [12]. □

Corollary 3.2.4. *If $A \xrightarrow{p} X \xrightarrow{f} Y$ are 1-simplices in $\mathcal{C}$, then $F(p) \times_{C^{/X}} \mathrm{Pb}(\mathcal{C})_f \simeq \mathcal{C}_{/p} \times_{\mathcal{C}_{/X}} ((\mathcal{C}^{\Delta[1]})_{/f})_{\mathrm{Pb}}$, where $((\mathcal{C}^{\Delta[1]})_{/f})_{\mathrm{Pb}}$ is the subcategory of $(\mathcal{C}^{\Delta[1]})_{/f}$ spanned by the pullback squares of $\mathcal{C}$.*

Proof. This is shown through multiple applications of Lemma 3.2.3. First, $F(p) = F(p, \Delta[0])$, and so $F(p) \simeq \mathcal{C}_{/p}$ over $\mathcal{C}$. Meanwhile, $\mathrm{Sq}(\mathcal{C})_f$ can be constructed as a pullback

$$\begin{tikzcd} \mathrm{Sq}(\mathcal{C})_f \arrow[r] \arrow[d] & (\mathcal{C}^{\Delta[1]})^{\Delta[0]^{\lhd}} \arrow[d] \\ \Delta[0] \arrow[r, "\lceil f \rceil"] & (\mathcal{C}^{\Delta[1]})^{\Delta[0]} \end{tikzcd}$$

and applying Lemma 3.2.3 establishes $\mathrm{Sq}(\mathcal{C})_f = F(f, \Delta[0]) \simeq (\mathcal{C}^{\Delta[1]})_{/f}$ from which it follows that $\mathrm{Pb}(\mathcal{C})_f \simeq ((\mathcal{C}^{\Delta[1]})_{/f})_{\mathrm{Pb}}$. The same result also shows that $\mathcal{C}_{/X} \simeq \mathcal{C}^{/X}$

compatibly, in the sense that there is a diagram

$$\begin{tikzcd}
\mathcal{C}_{/p} \arrow[r] \arrow[d, "\sim"] & \mathcal{C}_{/X} \arrow[d, "\sim"] & ((\mathcal{C}^{\Delta[1]})_{/f})_{\mathrm{Pb}} \arrow[l] \arrow[d, "\sim"] \\
F(p) \arrow[r] & \mathcal{C}^{/X} & \mathrm{Pb}(\mathcal{C})_f. \arrow[l]
\end{tikzcd}$$

The map $\mathcal{C}_{/p} \to \mathcal{C}_{/X}$ is a fibration in the Joyal model structure since it is a right fibration, which follows from the fact that the inclusion $\{1\} \hookrightarrow \Delta[1]$ is right anodyne. Meanwhile, the map $F(p) \to \mathcal{C}^{/X}$ is a retract of the right fibration $\mathcal{C}^{/p} \to \mathcal{C}^{/X}$ and thus itself also a fibration in the Joyal model structure. Therefore, there is an induced equivalence $F(p) \times_{\mathcal{C}^{/X}} \mathrm{Pb}(\mathcal{C})_f \simeq \mathcal{C}_{/p} \times_{\mathcal{C}_{/X}} ((\mathcal{C}^{\Delta[1]})_{/f})_{\mathrm{Pb}}$. □

Remark 3.2.5. Considering a 1-simplex $A \xrightarrow{p} X$ in $\mathcal{C}$, the inclusion $\{0\} \hookrightarrow \Delta[1]$ is left anodyne, so the restriction $\mathcal{C}_{/p} \to \mathcal{C}_{/A}$ is a trivial fibration, and thus $F(p) \simeq \mathcal{C}_{/A}$. Therefore, the triangle on the left side of an exponential diagram above p and f is determined up to equivalence by its representative in $\mathcal{C}_{/A}$.

Exponential and cromulent diagrams can be characterized as arising from the counit of the adjunction $f^* \dashv f_*$ as in the following lemmas.

Lemma 3.2.6. *Let $\mathcal{C}$ be locally cartesian closed quasicategory, and let $A \xrightarrow{p} X$ and $X \xrightarrow{f} Y$ be 1-simplices in $\mathcal{C}$. Then, $\mathcal{E}xp\mathcal{D}iag(p, f) \simeq ({f^*}_{/p})^{\mathrm{term}}$*

Proof. There is a pullback square with vertical arrows trivial fibrations as follows:

$$\begin{tikzcd}
\mathrm{Pb}(C)_f \arrow[r] \arrow[d, two heads, "\sim"'] & \mathrm{Pb}(C) \arrow[d, two heads, "\sim"] \\
\mathcal{C}^{/Y} \arrow[r] & \mathcal{C}^{\Lambda^2[2]},
\end{tikzcd}$$

where the map $\mathcal{C}^{/Y} \to \mathcal{C}^{\Lambda^2[2]}$ comes from forming horns along the 1-simplex $X \xrightarrow{f} Y$.

There is also a square

$$\begin{tikzcd}
((\mathcal{C}^{\Delta[1]})_{/f})_{\mathrm{Pb}} \arrow[r, "\sim"] \arrow[d] & \mathrm{Pb}(\mathcal{C})_f \arrow[d, "\sim"] \\
\mathcal{C}_{/Y} \arrow[r, "\sim"] & \mathcal{C}^{/Y}.
\end{tikzcd}$$

The various equivalences appearing here between slice constructions allow us to interpolate between the exponential and cromulent diagrams as constructed in this chapter for application to establishing a distributive law for bispans and more elementary formulations of exponential diagrams more directly analogous to those from ordinary category theory, as well as their interpretation as counits for the adjunction $f^* \dashv f_*$. By the 2-out-of-3 property for equivalences, the projection $((\mathcal{C}^{\Delta[1]})_{/f})_{\mathrm{Pb}} \to \mathcal{C}_{/Y}$ is an equivalence. The pullback functor $f^* \colon \mathcal{C}_{/Y} \to \mathcal{C}_{/X}$ can then be thought of as taking the inverse of this equivalence and then composing with the projection $((\mathcal{C}^{\Delta[1]})_{/f})_{\mathrm{Pb}} \to \mathcal{C}_{/X}$. Then, there is a diagram as follows:

$$\begin{tikzcd}
\mathcal{C}_{/p} \arrow[r] \arrow[d, "\cong"] & \mathcal{C}_{/X} \arrow[d, equal] & ((\mathcal{C}^{\Delta[1]})_{/f})_{\mathrm{Pb}} \arrow[l] \arrow[d, "\sim"] \\
(\mathcal{C}_{/X})_{/p} \arrow[r] & \mathcal{C}_{/X} & \mathcal{C}_{/Y}. \arrow[l, "f^*"']
\end{tikzcd}$$

Since the map $C_{/p} \to \mathcal{C}_{/X}$ is a fibration in the Joyal model structure, the induced map of pullbacks $C_{/p} \times_{\mathcal{C}_{/X}} ((\mathcal{C}^{\Delta[1]})_{/f})_{\mathrm{Pb}} \to {f^*}_{/p}$ is an equivalence, and equivalences preserve terminal objects. In particular, exponential diagrams over (p, f) can be identified with ε_p^f, the counit for the adjunction $f^* \dashv f_*$ evaluated at p in $\mathcal{C}_{/X}$. $\square$

Lemma 3.2.7. *Let $\mathcal{C}$ be locally cartesian closed quasicategory. Consider a functor $(p, u, f) \colon \Delta[1] \cup \mathrm{TR}[1] \to \mathcal{C}$, encoding a diagram*

$$\begin{tikzcd}
X' \arrow[d, "p"'] & & \\
X & A \arrow[l, "u"'] \arrow[r, "f"] & Y.
\end{tikzcd}$$

Then a cromulent diagram with base (p, u, f) *is equivalent to an exponential diagram above* $(u^*(p), f)$*, i.e.* $\mathcal{C}rom\mathcal{D}iag(p, u, f) \simeq \mathcal{E}xp\mathcal{D}iag(u^*(p), f) \simeq ({f^*}_{/u^*(p)})^{\mathrm{term}}$

Proof. The analysis only depends on understanding the squares on the left side of cromulent diagrams. As $\Delta[1] \times \Delta[1] \cong \Lambda^2[2]^{\triangleleft}$, it follows from Lemma 3.2.3 that if $\mathrm{Sq}(\mathcal{C})_{p,u}$ is defined by the pullback

$$\begin{tikzcd} \mathrm{Sq}(\mathcal{C})_{p,u} \arrow[r] \arrow[d] & \mathrm{Sq}(C) \arrow[d] \\ \Delta[0] \arrow[r, "\ulcorner K \urcorner"] & \mathcal{C}^{\Lambda^2[2]} \end{tikzcd}$$

where $K\colon \Lambda^2[2] \to \mathcal{C}$ denotes the horn defined by u and p, then $\mathrm{Sq}(\mathcal{C})_{p,u} \simeq \mathcal{C}_{/K} \simeq \mathcal{C}_{/u^*p} \simeq F(u^*p)$, from which the result follows, since $\mathcal{C}rom\mathcal{D}iag(p, u, f)$ is the subcategory spanned by the terminal objects in $\mathrm{Sq}(\mathcal{C})_{p,u} \times_{\mathcal{C}^{/A}} \mathrm{Pb}(\mathcal{C})_f$. □

The final important result of this section, Corollary 3.2.16, gives another construction for cromulent diagrams as terminal objects in the fibers of a particular fibration. Some preliminary constructions must first be carried out. Let $\mathcal{C}^{\Delta[1]}_{\mathrm{Pb}}$ denote the subcategory of $\mathcal{C}^{\Delta[1]}$ spanned by simplices whose 1-simplices are pullback squares in $\mathcal{C}$. The collapse map $\Lambda^2[2] \to \Delta[0]$ provides an induced functor $f\colon \mathcal{C}^{\Delta[1]}_{\mathrm{Pb}} \to \mathcal{C}^{\Lambda^2[2]} \times_{\mathcal{C}} \mathcal{C}^{\Delta[1]}_{\mathrm{Pb}}$ whose target is the pullback

$$\begin{tikzcd} \mathcal{C}^{\Lambda^2[2]} \times_{\mathcal{C}} \mathcal{C}^{\Delta[1]}_{\mathrm{Pb}} \arrow[r] \arrow[d] & \mathcal{C}^{\Delta[1]}_{\mathrm{Pb}} \arrow[d, "d^1"] \\ \mathcal{C}^{\Lambda^2[2]} \arrow[r] & \mathcal{C} \end{tikzcd}$$

with the map $\mathcal{C}^{\Lambda^2[2]} \to \mathcal{C}$ induced by the inclusion $\{0\} \hookrightarrow \Lambda^2[2]$. The 0-simplices of

$\mathcal{C}^{\Lambda^2[2]} \times_{\mathcal{C}} \mathcal{C}_{\mathrm{Pb}}^{\Delta[1]}$ are thus diagrams of shape

$$\begin{tikzcd} \bullet \arrow[d, red] & & \\ \bullet & \bullet \arrow[l, red] \arrow[r, blue] & \bullet \end{tikzcd}$$

where the red arrows describe a map $\Lambda^2[2] \to \mathcal{C}$ and the blue arrow describes a map $\Delta[1] \to \mathcal{C}$. The following lemma will be used to construct a cocartesian fibration that will be the first step in producing a quasicategory of cromulent diagrams in $\mathcal{C}$ with varying bases.

Lemma 3.2.8 (Lemma 2.4.7.12 of [9])**.** *Let $f\colon \mathcal{C} \to \mathcal{D}$ be a functor of quasicategories. Consider the pullback square*

$$\begin{tikzcd} \mathcal{D}^{\Delta[1]} \times_{\mathcal{D}} \mathcal{C} \arrow[r] \arrow[d] & \mathcal{C} \arrow[d, "f"] \\ \mathcal{D}^{\Delta[1]} \arrow[r, "d^1"] & \mathcal{D}. \end{tikzcd}$$

The map $d^0\colon \mathcal{D}^{\Delta[1]} \to \mathcal{D}$ induces a cocartesian fibration $p\colon \mathcal{D}^{\Delta[1]} \times_{\mathcal{D}} \mathcal{C} \to \mathcal{D}$.

Applying the lemma to the functor $f\colon \mathcal{C}_{\mathrm{Pb}}^{\Delta[1]} \to \mathcal{C}^{\Lambda^2[2]} \times_{\mathcal{C}} \mathcal{C}_{\mathrm{Pb}}^{\Delta[1]}$ described above yields a cocartesian fibration

$$p\colon (\mathcal{C}^{\Lambda^2[2]} \times_{\mathcal{C}} \mathcal{C}_{\mathrm{Pb}}^{\Delta[1]})^{\Delta[1]} \times_{\mathcal{C}^{\Lambda^2[2]} \times_{\mathcal{C}} \mathcal{C}_{\mathrm{Pb}}^{\Delta[1]}} \mathcal{C}_{\mathrm{Pb}}^{\Delta[1]} \to \mathcal{C}^{\Lambda^2[2]} \times_{\mathcal{C}} \mathcal{C}_{\mathrm{Pb}}^{\Delta[1]}$$

which on objects is the restriction to the solid arrows in diagrams of the following form in $\mathcal{C}$, where the subdiagram on A' consists entirely of degenerate simplices:

$$\begin{tikzcd}
 & A' \arrow[dl, dashed] \arrow[d, dashed] \arrow[ddl, dashed] & & \\
X' \arrow[d] & A' \arrow[dl, dashed] & A' \arrow[l, dashed] \arrow[r, dashed] \arrow[d, dashed] \arrow[dl, dashed] \arrow[dll, dashed] & Y' \arrow[dl, dashed] \\
X & A \arrow[l] \arrow[r] & Y. &
\end{tikzcd}$$

Since diagrams of this sort will provide an intermediate step in the analysis of cromulent diagrams at a large scale, define the quasicategory $\overline{\mathcal{C}rom\mathcal{S}hape}(\mathcal{C})$ to be $(\mathcal{C}^{\Lambda^2[2]} \times_{\mathcal{C}} \mathcal{C}_{\mathrm{Pb}}^{\Delta[1]})^{\Delta[1]} \times_{\mathcal{C}^{\Lambda^2[2]} \times_{\mathcal{C}} \mathcal{C}_{\mathrm{Pb}}^{\Delta[1]}} \mathcal{C}_{\mathrm{Pb}}^{\Delta[1]}$.

Define the simplicial set S as the quotient $\Delta[2]/(01)$, where 01 denotes the 1-simplex $0 \to 1$ of $\Delta[2]$. Thus, S is the pushout

$$\begin{tikzcd}
\Delta[1] \arrow[r] \arrow[d, "01"'] & \Delta[0] \arrow[d] \\
\Delta[2] \arrow[r] & S.
\end{tikzcd}$$

Lemma 3.2.9. *The edge* 12 *of* S *describes an inner anodyne map* $\Delta[1] \to S$*.*

Proof. The simplicial set S can be alternately described as a pushout

$$\begin{tikzcd}
\Lambda^1[2] \arrow[r] \arrow[d, hook] & \Delta[1] \arrow[d, "12"] \\
\Delta[2] \arrow[r] & S
\end{tikzcd}$$

where the map $\Lambda^1[2] \to \Delta[1]$ is given on 0-simplices by $0 \mapsto 0, 1 \mapsto 0, 2 \mapsto 1$. Since inner anodyne maps are closed under pushouts, it follows that $\Delta[1] \xrightarrow{12} S$ is inner anodyne. $\square$

It follows that $12\colon \Delta[1] \to S$ is an equivalence, with inverse given by the quotient map $S \to \Delta[1]$ which maps the equivalence class of 0 and 1 to 0 and maps 2 to 1.

Next, define a simplicial set K by a pushout square

$$\begin{tikzcd} \Delta[1] \arrow[r, "02"] \arrow[d, "02"'] & S \arrow[d] \\ \Delta[2] \arrow[r] & K \end{tikzcd}$$

so that K consists of a copy of S glued to a 2-simplex along the 02-edges of each. In particular, K is a quotient of $\Delta[1] \times \Delta[1]$, with an edge collapsed.

Lemma 3.2.10. *There is an equivalence* $K \xrightarrow{\sim} \Delta[2]$.

Proof. The diagram

$$\begin{tikzcd} \Delta[2] \arrow[d, "\mathrm{id}_{\Delta[2]}"'] & \Delta[1] \arrow[l, hook', "02"'] \arrow[r, hook, "02"] \arrow[d, "\mathrm{id}_{\Delta[1]}"] & S \arrow[d, "\sim"] \\ \Delta[2] & \Delta[1] \arrow[l, hook', "02"'] \arrow[r, "\mathrm{id}_{\Delta[1]}"] & \Delta[1] \end{tikzcd}$$

induces an equivalence $K \xrightarrow{\sim} \Delta[2]$ between the pushouts along the spans. ∎

Considering K as a quotient of $\Delta[1] \times \Delta[1]$, gluing two copies of K along the collapsed edge yields a new simplicial set $K \cup_{\Delta[1]} K$ which can also be thought of as a quotient of $\Lambda^2[2] \times \Delta[1]$, with $\Lambda^2[2] \times \{0\}$ collapsed. From the construction, it follows that $\overline{\mathcal{C}rom\mathcal{S}hape}(\mathcal{C}) \cong \mathcal{C}^{K \cup_{\Delta[1]} K} \times_{\mathcal{C}^{\Delta[1]}} \mathrm{Pb}(\mathcal{C})$, where the restriction to the edge $A' \to A$ in the diagram preceding the definition of $\overline{\mathcal{C}rom\mathcal{S}hape}(\mathcal{C})$ provides the map $\mathcal{C}^{K \cup_{\Delta[1]} K} \to \mathcal{C}^{\Delta[1]}$ for the pullback. The next lemma establishes an equivalence for $K \cup_{\Delta[1]} K$ that will lead to an equivalence $\overline{\mathcal{C}rom\mathcal{S}hape}(\mathcal{C}) \to \mathcal{C}rom\mathcal{S}hape(\mathcal{C})$.

Lemma 3.2.11. *There is an equivalence* $K \cup_{\Delta[1]} K \xrightarrow{\sim} \Delta[1] \times \Delta[1]$.

Proof. Note that $\Delta[1] \times \Delta[1] \cong \Delta[2] \cup_{\Delta[1]} \Delta[2]$. The diagram

$$\begin{tikzcd}
K \arrow[d, "\sim"'] & \Delta[1] \arrow[l, hook'] \arrow[r, hook] \arrow[d, "\mathrm{id}_{\Delta[2]}"] & K \arrow[d, "\sim"] \\
\Delta[2] & \Delta[1] \arrow[l, hook'] \arrow[r, hook] & \Delta[2]
\end{tikzcd}$$

induces the desired equivalence between the pushouts along the spans. $\square$

Now, let $\mathcal{C}rom\mathcal{S}hape(\mathcal{C})$ denote the quasicategory $\mathrm{Sq}(\mathcal{C}) \times_{\mathcal{C}^{\Delta[1]}} \mathrm{Pb}(\mathcal{C})$. This is the quasicategory of diagrams in $\mathcal{C}$ which have the same shape and pullback condition as cromulent diagrams, i.e. they are diagrams of cromulent shape. The terminal objects of the fibers of the restriction to the base, as described earlier in this chapter, are precisely the cromulent diagrams of $\mathcal{C}$. The analysis of $\overline{\mathcal{C}rom\mathcal{S}hape}(\mathcal{C})$ is leveraged to establish the construction of cromulent diagrams in $\mathcal{C}$ varying in their bases.

Lemma 3.2.12. *There is an equivalence* $\overline{\mathcal{C}rom\mathcal{S}hape}(\mathcal{C}) \xrightarrow{h} \mathcal{C}rom\mathcal{S}hape(\mathcal{C})$.

Proof. In the following diagram, the left horizontal maps are isofibrations of quasicategories, and thus fibrations in the Joyal model structure:

$$\begin{tikzcd}
\mathcal{C}^{K \cup_{\Delta[1]} K} \arrow[r] \arrow[d, "\sim"'] & \mathcal{C}^{\Delta[1]} \arrow[d, "\mathrm{id}_{\mathcal{C}^{\Delta[1]}}"] & \mathrm{Pb}(\mathcal{C}), \arrow[l] \arrow[d, "\mathrm{id}_{\mathrm{Pb}(\mathcal{C})}"] \\
\mathrm{Sq}(\mathcal{C}) \arrow[r] & \mathcal{C}^{\Delta[1]} & \mathrm{Pb}(\mathcal{C}). \arrow[l]
\end{tikzcd}$$

Since all objects involved in the diagram are fibrant, the induced map $\overline{\mathcal{C}rom\mathcal{S}hape}(\mathcal{C}) \to \mathcal{C}rom\mathcal{S}hape(\mathcal{C})$ between the pullbacks of the cospans is an equivalence. $\square$

Restricting to the base for cromulent diagrams in $\mathcal{C}rom\mathcal{S}hape(\mathcal{C})$ gives a functor $\mathcal{C}rom\mathcal{S}hape(\mathcal{C}) \to \mathcal{C}^{\Lambda^2[2]} \times_{\mathcal{C}} \mathcal{C}^{\Delta[1]}_{\mathrm{Pb}}$.

Lemma 3.2.13. *The restriction* $\mathcal{C}rom\mathcal{S}hape(\mathcal{C}) \to \mathcal{C}^{\Lambda^2[2]} \times_{\mathcal{C}} \mathcal{C}^{\Delta[1]}_{\mathrm{Pb}}$ *is an isofibration.*

Proof. Since the Joyal model structure on simplicial sets is cartesian, the restriction map $\mathrm{Sq}(\mathcal{C}) \times_{\mathcal{C}^{\Delta[1]}} \mathrm{Sq}(\mathcal{C}) \to \mathcal{C}^{\Lambda^2[2]} \times_{\mathcal{C}} \mathcal{C}^{\Delta[1]}$ is an isofibration. Pullback pasting ensures that this map restricted to the relevant subspaces remains an isofibration. $\square$

To complete the comparison, the following result is needed:

Proposition 3.2.14 (Corollary 5.1.17 in [14])**.** *Let*

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\sim"] \arrow[d, "q"'] & \mathcal{E} \arrow[d, "p"] \\ \mathcal{A} \arrow[r, "\sim"] & \mathcal{B} \end{tikzcd}$$

be a commutative diagram of quasicategories with p and q isofibrations. Then p is a cocartesian fibration if and only if q is.

Applying the above proposition to the triangle

$$\begin{tikzcd} \overline{\mathcal{C}rom\mathcal{S}hape}(\mathcal{C}) \arrow[rr, "\sim"] \arrow[dr] & & \mathcal{C}rom\mathcal{S}hape(\mathcal{C}) \arrow[dl] \\ & \mathcal{C}^{\Lambda^2[2]} \times_{\mathcal{C}} \mathcal{C}^{\Delta[1]}_{\mathrm{Pb}} & \end{tikzcd}$$

yields the following:

Theorem 3.2.15. *The restriction $\mathcal{C}rom\mathcal{S}hape(\mathcal{C}) \to \mathcal{C}^{\Lambda^2[2]} \times_{\mathcal{C}} \mathcal{C}^{\Delta[1]}_{\mathrm{Pb}}$ is a cocartesian fibration.*

In particular, applying Proposition 2.4.4.9 of [9] yields the following result:

Corollary 3.2.16. *Let $\mathcal{C}rom\mathcal{D}iag(\mathcal{C})$ be the full simplicial subset of $\mathcal{C}rom\mathcal{S}hape(\mathcal{C})$ spanned by the terminal objects of each fiber of the restriction $\mathcal{C}rom\mathcal{S}hape(\mathcal{C}) \to \mathcal{C}^{\Lambda^2[2]} \times_{\mathcal{C}} \mathcal{C}^{\Delta[1]}_{\mathrm{Pb}}$. Then $\mathcal{C}rom\mathcal{D}iag(\mathcal{C}) \to \mathcal{C}^{\Lambda^2[2]} \times_{\mathcal{C}} \mathcal{C}^{\Delta[1]}_{\mathrm{Pb}}$ is a trivial fibration.*

As suggested by the notation, the objects of $\mathcal{C}rom\mathcal{D}iag(\mathcal{C})$ are indeed the cromulent diagrams in $\mathcal{C}$, and this quasicategory provides a home for all these diagrams at once. Analogous arguments show that if $\mathcal{E}xp\mathcal{S}hape(\mathcal{C})$ is defined as the quasicategory $\mathcal{C}^{\Delta[2]} \times_{\mathcal{C}^{\Delta[1]}} \mathrm{Pb}(\mathcal{C})$ which on objects produces diagrams which have the pentagonal shape of exponential diagrams in $\mathcal{C}$, then two similar results hold:

Theorem 3.2.17. *The restriction to the base of diagrams of exponential shape is a cocartesian fibration $\mathcal{E}xp\mathcal{S}hape(\mathcal{C}) \to \mathcal{C}^{\Delta[1]} \times_{\mathcal{C}} \mathcal{C}^{\Delta[1]}_{\mathrm{Pb}(\mathcal{C})}$. If $\mathcal{E}xp\mathcal{D}iag(\mathcal{C})$ is the full simplicial subset of $\mathcal{E}xp\mathcal{S}hape(\mathcal{C})$ spanned by the terminal objects of each fiber of said restriction, then $\mathcal{E}xp\mathcal{D}iag(\mathcal{C}) \to \mathcal{C}^{\Delta[1]} \times_{\mathcal{C}} \mathcal{C}^{\Delta[1]}_{\mathrm{Pb}(\mathcal{C})}$ is a trivial fibration.*

3.3 Lemmas for manipulation of exponential diagrams

This section contains some lemmas which regulate how exponential diagrams interact with the notion of composition in a quasicategory, as well as a Beck-Chevalley-type equivalence for the adjunction $f^* \dashv f^*$.

First, a lemma which translates the data of an exponential diagram into a functor between slice categories. This point of view is explored in more detail in the study of *polynomial functors*, e.g. in [4] for higher categories and in a broad literature for ordinary categories.

Lemma 3.3.1. *Let*

$$\begin{tikzcd}
 & M \arrow[r, "h"] \arrow[dl, "e"'] \arrow[d] & N \arrow[d, "v"] \\
A \arrow[r, "p"] & X \arrow[r, "f"] & Y
\end{tikzcd}$$

be an exponential diagram in $\mathcal{C}$. Then $f_* p_! \simeq v_! h_* \varepsilon^*$ as functors $\mathcal{C}_{/A} \to \mathcal{C}_{/Y}$

Proof. The proof of Proposition 2.1.7 of [4] works. □

For the first of the two lemmas regarding compositionality, consider a 2-simplex in $\mathcal{C}$ as follows:

$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, "f"'] & & Z. \\ & Y \arrow[ur, "g"'] & \end{tikzcd}$$

There are adjunctions

$$\begin{tikzcd} \mathcal{C}_{/Z} \arrow[r, shift left, "g^*"] \arrow[rr, bend left, "h^*"] & \mathcal{C}_{/Y} \arrow[l, shift left, "g_*"] \arrow[r, shift left, "f^*"] & \mathcal{C}_{/X} \arrow[l, shift left, "f_*"] \arrow[ll, bend left, "h_*"] \end{tikzcd}$$

whose counits are denoted by $\varepsilon^f, \varepsilon^g$ and ε^h. We will analyze the counit ε^h in terms of ε^g and ε^f and interpret this analysis in the context of exponential diagrams. A consequence of the definition of adjunctions is that the counit ε^h is quasi-isomorphic in the slice quasicategory $(\mathcal{C}_{/X})^{\mathcal{C}_{/X}})_{/\,\mathrm{id}_{\mathcal{C}_{/X}}}$ to any choice of composite of ε^f and $f^*\varepsilon^g f_*$:

Lemma 3.3.2. *If ζ is a natural transformation $\mathrm{id}_{\mathcal{C}_{/X}} \to \mathrm{id}_{\mathcal{C}_{/X}}$ with $[\zeta] = [\varepsilon^f] \circ [f^*\varepsilon^g f_*]$ in $\mathrm{Ho}(\mathcal{C}_{/X})^{\mathcal{C}_{/X}}$ (i.e. ζ is in the same equivalence class of 2-morphisms in $q\mathcal{C}at_2$ as the composite of $f^*\varepsilon^g f_*$ and ε^f), then ζ is quasi-isomorphic in $((\mathcal{C}_{/X})^{\mathcal{C}_{/X}})_{/\,\mathrm{id}_{\mathcal{C}_{/X}}}$ to ε^h.*

Proof. This is a property of adjunctions. □

Diagrammatically, the lemma is depicted by

$$\begin{tikzcd} h^*h_* \arrow[d, "\sim"'] \arrow[drr, "\varepsilon^h"] & & \\ f^*g^*g_*f_* \arrow[rr, "\zeta"] \arrow[dr, "f^*\varepsilon^g f_*"'] & & \mathrm{id}_{\mathcal{C}_{/X}}\,. \\ & f^*f_* \arrow[ur, "\varepsilon^f"'] & \end{tikzcd}$$

To interpret this relationship between the counits in terms of exponential diagrams, recall that the counit ε^h arises from the adjunction

$$\begin{tikzcd}
\mathcal{C}_{/Z} \arrow[r, shift left, "h^*"] & \mathcal{C}_{/X} \arrow[l, shift left, "h_*"]
\end{tikzcd}$$

and when evaluated at an object $p\colon A \to X$ in $\mathcal{C}_{/X}$, the morphism ε^h_p is terminal in ${h^*}_{/p}$. For such a p, we can first form an exponential diagram

$$\begin{tikzcd}
& K(p,f) \arrow[r] \arrow[dl, "\varepsilon^f_p"'] \arrow[d, "f^*f_*(p)"] & L(p,f) \arrow[d, "f_*(p)"] \\
A \arrow[r, "p"] & X \arrow[r, "f"] & Y.
\end{tikzcd}$$

As $g\colon Y \to Z$, we form another exponential diagram above (f_*p, g):

$$\begin{tikzcd}
& K(f_*(p),g) \arrow[r] \arrow[dl, "\varepsilon^g_{f_*(p)}"'] \arrow[d, "g^*g_*f_*(p)"] & L(f_*(p),g) \arrow[d, "g_*f_*(p)"] \\
L(p,f) \arrow[r, "f_*(p)"] & Y \arrow[r, "g"] & Z.
\end{tikzcd}$$

These two exponential diagrams can be combined, as in

$$\begin{tikzcd}
& & & K(f_*(p),g) \arrow[r] \arrow[dl, "\varepsilon^g_{f_*(p)}"'] \arrow[ddl, "g^*g_*f_*(p)"] & L(f_*(p),g) \arrow[dd, "g_*f_*(p)"] \\
& K(p,f) \arrow[r] \arrow[dl, "\varepsilon^f_p"'] \arrow[d, "f^*f_*(p)"] & L(p,f) \arrow[d, "f_*(p)"'] & & \\
A \arrow[r, "p"] & X \arrow[r, "f"] & Y \arrow[rr, "g"] & & Z.
\end{tikzcd}$$

As $\varepsilon^g_{f_*(p)}$ is a morphism $g^*g_*f_*(p) \to f_*(p)$ in $\mathcal{C}_{/Y}$, applying $f^*\colon \mathcal{C}_{/Y} \to \mathcal{C}_{/X}$ obtains

$f^*\varepsilon^g_{f_*(p)}\colon f^*g^*g_*f_*(p) \to f_*(p)$:

$$\begin{tikzcd}
 & & W(p,f,g) \arrow[rr] \arrow[dl, "f^*\varepsilon^g_{f_*(p)}"'] \arrow[ddl, "f^*g^*g_*f_*(p)"] & & K(f_*(p),g) \arrow[dl, "\varepsilon^g_{f_*(p)}"'] \arrow[ddl, "g^*g_*f_*(p)"] \\
 & K(p,f) \arrow[rr] \arrow[dl, "\varepsilon^f_p"'] \arrow[d, "f^*f_*(p)"'] & & L(p,f) \arrow[d, "f_*(p)"] & \\
A \arrow[r, "p"] & X \arrow[rr, "f"] & & Y. &
\end{tikzcd}$$

In the notation of the lemma, there are a pair of 2-simplices in $\mathcal{C}_{/X}$ meeting on ζ_p as follows:

$$\begin{tikzcd}
h^*h_*p \arrow[dr, "\varepsilon^h_p"] & \\
f^*g^*g_*f_*(p) \arrow[u, "\sim"] \arrow[r, "\zeta_p"] \arrow[d, "f^*\varepsilon^g_{f_*(p)}"'] & p. \\
f_*f^*(p) \arrow[ur, "\varepsilon^f_p"'] &
\end{tikzcd}$$

Thus, if

$$\begin{tikzcd}
f^*g^*g_*f_*(p) \arrow[r, "\gamma"] \arrow[d, "f^*\varepsilon^g_{f_*(p)}"'] & p \\
f_*f^*(p) \arrow[ur, "\varepsilon_p"'] &
\end{tikzcd}$$

is a 2-simplex in $\mathcal{C}_{/X}$ expressing a choice of composite of ε_p with $f^*\varepsilon^g_{f^*(p)}$ in $\mathcal{C}_{/X}$, it follows that γ is quasi-isomorphic in $(\mathcal{C}_{/X})_{/p} \cong \mathcal{C}_{/p}$ to ε^h_p. Since $h_* \simeq g_*f_*$, it follows that there is an exponential diagram

$$\begin{tikzcd}
 & W(p,f,g) \arrow[r] \arrow[dl, "\zeta"'] \arrow[d, "f^*g^*g_*f_*(p)"] & K(f_*(p),g) \arrow[d, "g_*f_*(p)"] \\
A \arrow[r, "p"] & X \arrow[r, "h"] & Z.
\end{tikzcd}$$

Now, the exponential diagrams constructed above are the ones canonically associated to the pairs (p, f) and (p, g) once the choices of functors (which are only determined up to equivalence) f_*, f^*, g_* and g^* are made. Consider another pair of exponential diagrams

$$\begin{tikzcd}
 & M \arrow[r] \arrow[dl, "e_p^f"'] \arrow[d, "u"] & N \arrow[d, "v"] \\
A \arrow[r, "p"] & X \arrow[r, "f"] & Y
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & K \arrow[r] \arrow[dl, "e_{f_*(p)}^g"'] \arrow[d, "s"] & L \arrow[d, "t"] \\
N \arrow[r, "v"] & Y \arrow[r, "g"] & Z
\end{tikzcd}$$

which assemble into

$$\begin{tikzcd}
 & & & K \arrow[r] \arrow[dl, "e_v^g"'] \arrow[ddl, "s"] & L \arrow[dd, "t"] \\
 & M \arrow[r] \arrow[dl, "e_p^f"'] \arrow[d, "u"] & N \arrow[d, "v"] & & \\
A \arrow[r, "p"] & X \arrow[r, "f"] & Y \arrow[rr, "g"] & & Z.
\end{tikzcd}$$

Taking a pullback W forms

$$\begin{tikzcd}
 & & W \arrow[r] \arrow[dl, "r"'] \arrow[ddl, "s'"] & K \arrow[r] \arrow[dl, "e_v^g"'] \arrow[ddl, "s"] & L \arrow[dd, "t"] \\
 & M \arrow[r] \arrow[dl, "e_p^f"'] \arrow[d, "u"] & N \arrow[d, "v"] & & \\
A \arrow[r, "p"] & X \arrow[r, "f"] & Y \arrow[rr, "g"] & & Z
\end{tikzcd}$$

and if

$$\begin{tikzcd} s' \arrow[r, "\gamma'"] \arrow[d, "r"'] & p \\ u \arrow[ur, "e_p^f"'] & \end{tikzcd}$$

is a 2 simplex in $\mathcal{C}_{/X}$, then γ' is quasi-isomorphic in $(\mathcal{C}_{/X})_{/p} \cong \mathcal{C}_{/p}$ to ε_p^h and there is an exponential diagram

$$\begin{tikzcd} & W \arrow[r] \arrow[dl, "\gamma'"'] \arrow[d, "s'"] & L \arrow[d, "t"] \\ A \arrow[r, "p"] & X \arrow[r, "h"] & Z. \end{tikzcd}$$

The Beck-Chevalley-type equivalence is the following:

Lemma 3.3.3. *Let*

$$\begin{tikzcd} U \arrow[r, "g"] \arrow[d, "u"'] & V \arrow[d, "v"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

be a pullback square in $\mathcal{C}$*. Then* $v^*f_* \simeq g_*u^*$ *as functors* $\mathcal{C}_{/X} \to \mathcal{C}_{/V}$*.*

Proof. Lemma 2.1.6 of [4]. □

This lemma has the following consequence for the interpretation of exponential diagrams. Given $p\colon A \to X$ in $\mathcal{C}_{/X}$ and $q \simeq u^*p$ in $\mathcal{C}_{/U}$, then the lemma states that if $\{A, X, Y, M, N\}$ and $\{B, U, V, K, L\}$ are exponential diagrams over (p, f) and (q, g), respectively, then there is a larger diagram in $\mathcal{C}$ as follows:

$$\begin{tikzcd} & & & K \arrow[rr] \arrow[dl, "w"] \arrow[dd, "t"] \arrow[ddll, bend right, "e_q^g"'] & & L \arrow[dl] \arrow[dd, "s"] \\ & & M \arrow[rr] \arrow[dd] \arrow[ddll, bend right, "e_p^f"'] & & N \arrow[dd, "r"] & \\ & B \arrow[rr, "q"] \arrow[dl] & & U \arrow[rr, "g"] \arrow[dl, "u"] & & V \arrow[dl, "v"] \\ A \arrow[rr, "p"] & & X \arrow[rr, "f"] & & Y. & \end{tikzcd}$$

By construction of q, there is a quasi-isomorphism $B \simeq U \times_X A$, so e_q is determined as a morphism to a pullback by the pair $(t, (e_p^f)_!(w))$.

Stating the second lemma regarding compositionality requires brief setup. Let

$$\begin{tikzcd} B \arrow[rr, "i"] \arrow[dr, "q"'] & & A \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

be a 2-simplex in $\mathcal{C}$ (in particular, it is also a 1-simplex in $\mathcal{C}_{/X}$) and suppose that the following is an exponential diagram:

$$\begin{tikzcd} & M \arrow[r, "h"] \arrow[dl, "e"'] \arrow[d, "u"] & N \arrow[d, "v"] \\ A \arrow[r, "p"] & X \arrow[r, "f"] & Y. \end{tikzcd}$$

Given a pullback square

$$\begin{tikzcd} W \arrow[r, "e^*(i)"] \arrow[d, "\tilde{e}"'] & M \arrow[d, "e"] \\ B \arrow[r, "i"] & A \end{tikzcd}$$

there is an exponential diagram

$$\begin{tikzcd} & K(e^*(i), h) \arrow[r] \arrow[dl, "\varepsilon^h_{e^*(i)}"'] \arrow[d, "h^* h_* e^*(i)"] & L(e^*(i), h) \arrow[d, "h_* e^*(i)"] \\ W \arrow[r, "e^*(i)"] & M \arrow[r, "h"] & N \end{tikzcd}$$

and also a diagram

$$\begin{tikzcd} & W \arrow[r, "e^*(i)"] \arrow[dl, "\tilde{e}"'] & M \arrow[r, "h"] \arrow[dl, "e"'] \arrow[d, "u"] & N \arrow[d, "v"] \\ B \arrow[r, "i"] \arrow[rr, bend right, "q"'] & A \arrow[r, "p"] & X \arrow[r, "f"] & Y. \end{tikzcd}$$

Lemma 3.3.4. *The counit ε_q^f is quasi-isomorphic in $\mathcal{C}_{/B}$ to $\tilde{e}_!(\varepsilon_e^h)$.*

Proof. The natural isomorphism $v_!h_*e^* \to f_*p_!$ is given by (cf. Proposition 2.1.7 of [4])

$$v_!h_*e^* \longrightarrow v_!h_*e^*p^*p_! \xrightarrow{\sim} v_!h_*u^*p_! \xrightarrow{\sim} v_!v^*f_*p_! \longrightarrow f_*p_!$$

using the unit $\mathrm{id}_{\mathcal{C}_{/A}} \to p^*p_!$ and counit $\zeta^v\colon v_!v^* \to \mathrm{id}_{\mathcal{C}_{/Y}}$ of the adjunctions $p_! \dashv p^*$ and $v_! \dashv v^*$, respectively and the Beck-Chevalley equivalence $h_*u^* \to v^*f_*$. Letting ζ^e denote the counit $e_!e^* \to \mathrm{id}_{\mathcal{C}_{/A}}$, there is a diagram of natural transformations as follows:

$$\begin{tikzcd}
f^*v_!h_*e^* \arrow[r] \arrow[d, "\sim"'] \arrow[rrrr, bend left, "\sim"] & f^*v_!h_*e^*p^*p_! \arrow[r, "\sim"] \arrow[d, "\sim"] & f^*v_!h_*u^*p_! \arrow[r, "\sim"] & f^*v_!v^*f_*p_! \arrow[r, "f^*\zeta^v f_*"] \arrow[ddl, "\sim"'] \arrow[ddddl, "(3)"] & f^*f_*p_! \arrow[ddddll, "\varepsilon^f p_!"] \\
u_!h^*h_*e^* \arrow[r] \arrow[d, "\sim"'] & u_!h^*h_*e^*p^*p_! \arrow[d, "\sim"] & & & \\
p_!e_!h^*h_*e^* \arrow[r, "(4)"] \arrow[d, "p_!e_!\varepsilon^h e^*"'] \arrow[ddrr, "\gamma"] & p_!e!h^*h_*e^*p^*p_! \arrow[r, "\sim"] \arrow[ddr, "(1)"] & u_!h^*h_*u^*p_! \arrow[dd, "(2)"] & & \\
p_!e_!e^* \arrow[drr, "p_!\zeta^e"'] & & & & \\
 & & p_!. & &
\end{tikzcd}$$

The adjoint pairs $p_!e_!h^* \dashv h_*e^*p^*$, $u!h^* \dashv h_*u^*$ and $f^*v_! \dashv v^*f_*$ are all adjunctions

$$\mathcal{C}_{/X} \rightleftarrows \mathcal{C}_{/N}$$

and all have equivalent functors on their respective left and right sides, so their counits are also equivalent. Given an adjunction

$$\mathcal{A} \underset{U}{\overset{F}{\rightleftarrows}} \mathcal{B}$$

with counit $\varepsilon\colon FU \to \mathrm{id}_{\mathcal{B}}$ there is an induced adjunction on mapping complexes

$$\mathcal{A}^K \underset{\widetilde{U}}{\overset{\widetilde{F}}{\rightleftarrows}} \mathcal{B}^K$$

whose counit $\tilde{\varepsilon}\colon \widetilde{F}\widetilde{U} \to \mathrm{id}_{\mathcal{B}^K}$ evaluates at $P\colon K \to \mathcal{B}$ as $\tilde{\varepsilon}_P = \varepsilon P$. Taking $K = \mathcal{C}_{/A}$ and $\mathcal{A} = \mathcal{C}_{/X}, \mathcal{B} = \mathcal{C}_{/N}$, we obtain induced adjunctions

$$(\mathcal{C}_{/X})^{\mathcal{C}_{/A}} \rightleftarrows (\mathcal{C}_{/N})^{\mathcal{C}_{/A}}.$$

The maps (1), (2) and (3) are then each the counits evaluated at $p!$ for the induced adjunctions $\widetilde{p}_!\widetilde{e_!h^*} \dashv \widetilde{h_*e^*}\widetilde{p}^*$, $\widetilde{u}!\widetilde{h^*} \dashv \widetilde{h_*}\widetilde{u^*}$ and $\widetilde{f^*}\widetilde{v}_! \dashv \widetilde{v^*}\widetilde{f_*}$, and thus all three counits are equivalent over $p!$.

The arrow (4) arises as follows: any choice of γ as a composite of $p_!e_!\varepsilon^h e^*$ with $p_!\zeta^e$ will be an object of $(\widetilde{p}_!\widetilde{e_!h^*})_{/p!}$ and thus have a unique, up to coherent homotopy, map to the counit (1).

Thus, there is a 2-simplex in $(\mathcal{C}_{/X})^{\mathcal{C}_{/A}}$

$$\begin{tikzcd}
f^*f_*p_! \arrow[r, "\varepsilon^f p_!"] \arrow[d, "\sim"'] & p_! \\
p_!e_!h^*h_*e^* \arrow[ur, "\zeta"'] &
\end{tikzcd}$$

which when evaluated at $i\colon B \to A$ yields

$$\begin{tikzcd}
f^*f_*p_!(i) \arrow[r, "\varepsilon^f_{p_!(i)}"] \arrow[d, "\sim"'] & p_!(i). \\
p_!e_!h^*h_*e^*(i) \arrow[ur, "\zeta_i"'] &
\end{tikzcd}$$

Thus, in $\mathcal{C}_{/p_!(i)}$ there is a quasi-isomorphism $\varepsilon^f_{p_!i} \simeq \zeta_i$. By construction of ζ, there is a a 2-simplex in $\mathcal{C}_X$ as follows:

$$\begin{tikzcd}
 & & K(e^*(i), h) \arrow[dl, "\varepsilon^h_{e^*(i)}"'] \arrow[dd, "p_!e_!h^*h_*e^*(i)"] \arrow[ddll, bend right, "\zeta_i"'] \\
 & W \arrow[dl, "\tilde{e}"'] \arrow[dr, "p_!e_!e^*(i)"] & \\
B \arrow[rr, "p_!(i)"] & & X.
\end{tikzcd}$$

In particular, $\tilde{e}$ induces a functor $\tilde{e}_!\colon (\mathcal{C}_{/X})_{/p_!e_!e^*(i)} \to (\mathcal{C}_{/X})_{/p_!(i)}$ and so ζ_i is quasi-isomorphic to $\tilde{e}_!(\varepsilon^h_{e^*i})$ in $(\mathcal{C}_{/X})_{/p_!(i)} \cong \mathcal{C}_{/p_!(i)}$, from which the quasi-isomorphism $\varepsilon^f_{p_!i} \simeq \tilde{e}_!(\varepsilon^h_{e^*i})$ (in both $\mathcal{C}_{/p_!(i)}$ and $\mathcal{C}_{/B}$) follows. Since $q \simeq p_!(i)$, there is a quasi-isomorphism $\varepsilon^f_q \simeq \varepsilon^f_{p_!(i)}$ in $\mathcal{C}_{/B}$, so $\varepsilon^f_q \simeq \tilde{e}_!(\varepsilon^h_{e^*(i)})$ in $\mathcal{C}_{/B}$.

$\square$

Since W is defined as pullback

$$\begin{tikzcd}
W \arrow[r, "e^*(i)"] \arrow[d, "\tilde{e}"'] & M \arrow[d, "e"] \\
B \arrow[r, "i"] & A
\end{tikzcd}$$

and $i_!(\varepsilon^f_{p_!(i)}) \simeq i_!\tilde{e}_!(\varepsilon^h_{h_*e^*(i)}) \simeq e_!h^*h_*e^*(i)$, there is a square

$$\begin{tikzcd} K \arrow[r, "h^*h_*e^*(i)"] \arrow[d, "\varepsilon^h_{h_*e^*(i)}"'] & M \arrow[d, "e"] \\ B \arrow[r, "i"] & A \end{tikzcd}$$

and by the lemma, the induced morphism from K to W is quasi-isomorphic to $\varepsilon^f_{p_!(i)}$ and hence ε^f_q. Thus, ε^f_q is determined, up to quasi-isomorphism in $\mathcal{C}_{/W}$, by the pair of morphisms $\varepsilon^h_{e^*(i)}$ and $h^*h_*e^*(i)$.

3.4 A distributive law for bispans in quasicategories

In this section, we define a bisimplicial set $\mathbb{D}(\mathcal{C})$ whose associated simplicial set is the simplicial set of decomposed bispans in $\mathcal{C}$. After some elaboration, we prove the main theorem of the chapter, showing that $\mathbb{D}(\mathcal{C})$ is a distributive law, from which it follows that $A(\mathbb{D}(\mathcal{C}))$ is a quasicategory — the quasicategory of decomposed bispans in $\mathcal{C}$. The results of this section are dependent on the results of §2.5.

Definition 3.4.1. Define the bisimplicial set $\mathbb{D}(\mathcal{C})$ to have (m, n)-simplices the set of all maps $\Delta[m] \times \mathrm{TR}[n] \to \mathcal{C}$ which satisfy the following conditions:

(a) Every subdiagram of $\Delta[m] \times \mathrm{TR}[n]$ of the form

$$\begin{tikzcd} (a, bc) \arrow[d] & (a, bc') \arrow[l] \arrow[r] \arrow[d] & (a, b'c') \arrow[d] \\ (a', bc) & (a', bc') \arrow[l] \arrow[r] & (a', b'c') \end{tikzcd}$$

with $a < a', b < b'$ and $c < c'$ is sent to a cromulent diagram in $\mathcal{C}$.

(b) Every subdiagram of $\Delta[m] \times \mathrm{TR}[n]$ of the form

$$\begin{tikzcd}
(a, bd) \arrow[r] \arrow[d] & (a, cd) \arrow[d] \\
(a, bc) \arrow[r] & (a, cc)
\end{tikzcd}$$

with $b < c < d$ is sent to a pullback diagram in $\mathcal{C}$.

Definition 3.4.2. Let $\mathcal{B}ispan_{\mathrm{dec}}(\mathcal{C}) \coloneqq A(\mathbb{D}(\mathcal{C}))$ be the simplicial set of *decomposed bispans* in $\mathcal{C}$.

Depictions of the simplices of $\mathcal{B}ispan_{\mathrm{dec}}(\mathcal{C})$ follow:

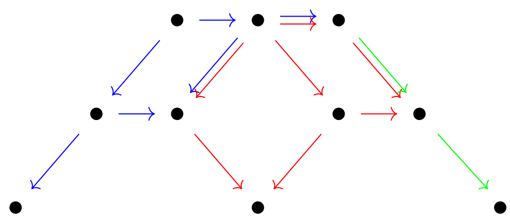
0-simplices These are objects of $\mathcal{C}$.

1-simplices These are maps of simplicial sets from $(\Delta[0] \times \mathrm{TR}[1]) \cup \Delta[1] \times \mathrm{TR}[0]$, viewed as a $(0,1)$-simplex (blue) of $\mathbb{D}(\mathcal{C})$ attached to a $(1,0)$-simplex (red) of $\mathbb{D}(\mathcal{C})$:

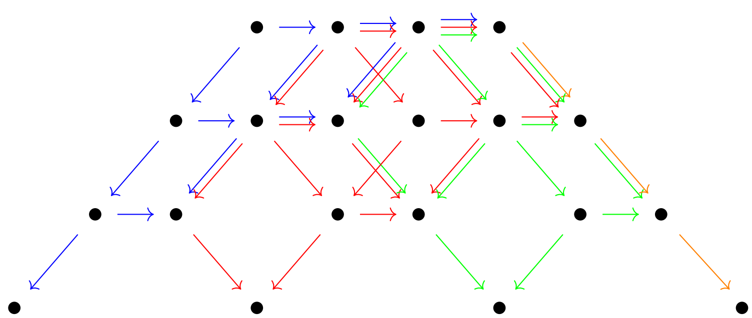
$$\begin{tikzcd}
 & \bullet \arrow[r, blue] \arrow[dl, blue] & \bullet \arrow[dr, red] & \\
\bullet & & & \bullet.
\end{tikzcd}$$

It is important to note that there is no choice of a 2-simplex which contains both a blue and red arrow.

2-simplices These are maps of simplicial sets from $(\Delta[0] \times \mathrm{TR}[2]) \cup (\Delta[1] \times \mathrm{TR}[1]) \cup \Delta[2] \times \mathrm{TR}[0])$ to $\mathcal{C}$, expressing a $(0,2)$-simplex (blue) and a $(2,0)$-simplex (green) of $\mathbb{D}(\mathcal{C})$ attached to a $(1,1)$-simplex of $\mathbb{D}(\mathcal{C})$ (red). The cromulence condition means that the image of $\Delta[1] \times \mathrm{TR}[1]$ is a cromulent diagram in $\mathcal{C}$. Again, it is important to note that a 2-simplex of $\mathcal{B}ispan_{\mathrm{dec}}(\mathcal{C})$ is not a priori a functor $\mathrm{TNR}[2] \to \mathcal{C}$, but instead a map defined on a subcomplex of $\mathrm{TNR}[2]$, given by the union of the diagrams which are distinguished by their color below:



3-simplices These are maps of simplicial sets from $(\Delta[0]\times \mathrm{TNR}[3])\cup(\Delta[1]\times \mathrm{TNR}[2])\cup(\Delta[2]\times \mathrm{TNR}[1])\cup(\Delta[3]\times \mathrm{TNR}[0])$ to $\mathcal{C}$, satisfying pullback and cromulence conditions, where the domain is a subcomplex of TNR[3], formed by taking the union of the complexes given by the four colors. The simplicial set $\mathrm{TNR}_{\mathrm{dec}}[3]$ is depicted below:



In general, the n-simplices of $\mathcal{B}ispan_{\mathrm{dec}}(\mathcal{C})$ are functors from a simplicial set $\mathrm{TNR}_{\mathrm{dec}}[n]$ to $\mathcal{C}$ satisfying pullback and cromulence conditions.

The aim of this section is to establish that $\mathcal{B}ispan_{\mathrm{dec}}(\mathcal{C})$ is a quasicategory by showing that $\mathbb{D}(\mathcal{C})$ is a distributive law. To this end, there are some useful applications of the adjunctions involving $\boxtimes$ which simplify the analysis of bihorns in $\mathbb{D}(\mathcal{C})$. As discussed in §2 of [7], the functors $X\boxtimes -$ and $-\boxtimes Y$ from $\mathcal{SS}et$ to $s\mathcal{SS}et$ have right adjoints, denoted $X\backslash -$ and $-/Y$, respectively. If $\mathbb{X}$ is a bisimplicial set thought of as a simplicial object $\mathbb{X}_{i*}$ in $\mathcal{SS}et$, with each $\mathbb{X}_{i*}$ forming a column of $\mathbb{X}$, then $\Delta[m]\backslash\mathbb{X}$ is the mth column $\mathbb{X}_{m*}$ of $\mathbb{X}$, while $\mathbb{X}/\Delta[n]$ is the nth row $\mathbb{X}_{*n}$ of $\mathbb{X}$.

In the following, the symbol $\star$ indicates the restriction to maps which upon application of the various adjunctions correspond to maps of simplicial sets $K\times\Delta[n]\to\mathcal{C}$ for $K\subset\Delta[m]$ satisfying condition (a) of Definition 3.4.1. In particular, there is the following definition:

Definition 3.4.3. Let $\mathcal{C}$ be a locally cartesian closed quasicategory, and $m\geq 1$. For a subsimplicial set $K\subset\Delta[m]$, the n-simplices of $\mathcal{S}pan(\mathcal{C}^K)$ are diagrams of the form $K\times\mathrm{TR}[n]$ in $\mathcal{C}$. Let $\mathcal{S}pan^\star(\mathcal{C}^K)$ be the subsimplicial set of $\mathcal{S}pan(\mathcal{C}^K)$ spanned by the simplices satisfying condition (a) of Definition 3.4.1.

With this notation in hand,

$$\begin{aligned}
\mathrm{Hom}_{s\mathcal{S}\mathcal{S}et}(\Lambda^j[m]\boxtimes\Delta[n],\mathbb{D}(\mathcal{C})) &\cong \mathrm{Hom}_{\mathcal{S}\mathcal{S}et}(\Lambda^j[m],\mathbb{D}(\mathcal{C})/\Delta[n])\\
&= \mathrm{Hom}_{\mathcal{S}\mathcal{S}et}(\Lambda^j[m],\mathrm{Hom}^\star_{\mathcal{S}\mathcal{S}et}(\Delta[-]\times\mathrm{TR}[n],\mathcal{C}))\\
&= \mathrm{Hom}^\star_{\mathcal{S}\mathcal{S}et}(\Lambda^j[m],\mathcal{C}^{\mathrm{TR}[n]})\\
&\cong \mathrm{Hom}^\star_{\mathcal{S}\mathcal{S}et}(\Lambda^j[m]\times\mathrm{TR}[n],\mathcal{C}).
\end{aligned}$$

For $0\leq i\leq n$, the set $[n]\setminus\{i\}$ retains the total order from $[n]$, so can still be interpreted as a category. Define the category

$$\partial_i\,\mathrm{TR}[n]\coloneqq([n]\setminus\{i\})^{\mathcal{TR}}_{\mathrm{Tw}}$$

to be the full subcategory of $\mathrm{TR}[n]$ on $[n]\setminus\{i\}$. If $\mathcal{C}$ is a category with pullbacks, the ith face of an n-simplex in $\mathcal{S}pan(\mathcal{C})$ encoded by a map $\mathrm{TR}[n]\to\mathcal{C}$ is obtained by restricting the map defining the n-simplex along the inclusion $\partial_i\,\mathrm{TR}[n]\hookrightarrow\mathrm{TR}[n]$. Considering horns in $\mathcal{S}pan(\mathcal{C})$ leads to the definition of the simplicial set $\Lambda^k\,\mathrm{TR}[n]$ for

$0 \leq k \leq n$ by

$$\Lambda^k \operatorname{TR}[n] = \bigcup_{\substack{0 \leq i \leq n \\ i \neq k}} \partial_i \operatorname{TR}[n].$$

Then in a similar vein to the preceding,

$$\begin{aligned}
\mathrm{Hom}_{s\mathcal{S}\mathcal{S}et}(\Delta[m] \boxtimes \Lambda^k[n], \mathbb{D}(\mathcal{C})) &\cong \mathrm{Hom}_{\mathcal{S}\mathcal{S}et}(\Lambda^k[n], \Delta[m] \backslash \mathbb{D}(\mathcal{C})) \\
&= \mathrm{Hom}_{\mathcal{S}\mathcal{S}et}(\Lambda^k[n], \mathrm{Hom}^{\star}_{\mathcal{S}\mathcal{S}et}(\Delta[m] \times \operatorname{TR}[-], \mathcal{C})) \\
&\cong \mathrm{Hom}_{\mathcal{S}\mathcal{S}et}(\Lambda^k[n], \mathrm{Hom}^{\star}_{\mathcal{S}\mathcal{S}et}(\operatorname{TR}[-], \mathcal{C}^{\Delta[m]})) \\
&= \mathrm{Hom}_{\mathcal{S}\mathcal{S}et}(\Lambda^k[n], \mathcal{S}pan^{\star}(\mathcal{C}^{\Delta[m]})) \\
&= \mathrm{Hom}^{\star}_{\mathcal{S}\mathcal{S}et}(\Lambda^k \operatorname{TR}[n], \mathcal{C}^{\Delta[m]}) \\
&\cong \mathrm{Hom}^{\star}_{\mathcal{S}\mathcal{S}et}(\Delta[m] \times \Lambda^k \operatorname{TR}[n], \mathcal{C}).
\end{aligned}$$

Thus, bihorns in $\mathbb{D}(\mathcal{C})$ may be reinterpreted in the following manner:

$$\begin{aligned}
&\mathrm{Hom}_{s\mathcal{S}\mathcal{S}et}(\Lambda^{j,k}[m,n], \mathbb{D}(\mathcal{C})) \\
&\cong \mathrm{Hom}^{\star}_{\mathcal{S}\mathcal{S}et}((\Lambda^j[m] \times \operatorname{TR}[n]) \underset{\Lambda^j[m] \times \Lambda^k \operatorname{TR}[n]}{\cup} (\Delta[m] \times \Lambda^k \operatorname{TR}[n]), \mathcal{C}).
\end{aligned}$$

If $A \to B$ and $K \to L$ are maps of simplicial sets, then extensions of a map of simplicial sets $(A \times L) \cup_{A \times K} (B \times K) \to \mathcal{C}$ to a map $B \times L \to \mathcal{C}$ are in bijection with dashed fillings in the square below:

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & \mathcal{C}^L \arrow[d] \\
B \arrow[r] \arrow[ur, dashed] & \mathcal{C}^K.
\end{tikzcd}$$

Therefore, extending a bihorn $\Lambda^{j,k}[m,n] \to \mathbb{D}(\mathcal{C})$ to an (m,n)-bisimplex $\Delta[m,n] \to \mathbb{D}(\mathcal{C})$ can be interpreted as the data of a dashed filling in either of the following squares:

$$
\begin{tikzcd}
\Lambda^j[m] \arrow[r, "\star"] \arrow[d] & \mathcal{C}^{\mathrm{TR}[n]} \arrow[d] \\
\Delta[m] \arrow[r, "\star"] \arrow[ur, dashed, "\star"] & \mathcal{C}^{\Lambda^k \mathrm{TR}[n]}
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\Lambda^k \mathrm{TR}[n] \arrow[r, "\star"] \arrow[d] & \mathcal{C}^{\Delta[m]} \arrow[d] \\
\mathrm{TR}[n] \arrow[r, "\star"] \arrow[ur, dashed, "\star"] & \mathcal{C}^{\Lambda^j[m]}.
\end{tikzcd}
\tag{3.1}
$$

In particular, a dashed filling of the latter square corresponds to a dashed filling in the square

$$
\begin{tikzcd}
\Lambda^k[n] \arrow[r] \arrow[d] & \mathcal{S}pan^\star(\mathcal{C}^{\Delta[m]}) \arrow[d] \\
\Delta[n] \arrow[r] \arrow[ur, dashed] & \mathcal{S}pan^\star(\mathcal{C}^{\Lambda^j[m]}).
\end{tikzcd}
\tag{3.2}
$$

With these preliminaries complete, we arrive at the main theorem of the chapter.

Theorem 3.4.4. *If $\mathcal{C}$ is a locally cartesian closed quasicategory, the $\mathbb{D}(\mathcal{C})$ is a distributive law, and so $\mathcal{B}ispan_{\mathrm{dec}}(\mathcal{C})$ is a quasicategory.*

A remark before the proof, which holds for the proofs of results of the next section as well: the pullback condition (b) from the definition of $\mathbb{D}(\mathcal{C})$ is automatically satisfied by all fillings which are generated by the properties of span quasicategories, since they all take spans in functor categories and pullbacks in functor categories are computed pointwise.

Proof. To show that $\mathbb{D}(\mathcal{C})$ is a distributive law, recall that fillings are required along bihorns $\Lambda^{i,k-i}[i, n-i] \to \mathbb{D}(\mathcal{C})$ and $\Lambda^{k,0}[l, n-l] \to \mathbb{D}(\mathcal{C})$ for $n \geq 2, 0 < k < n$ and $0 \leq i \leq k \leq l \leq n$.

By the diagram (3.2), filling a bihorn $\Lambda^{i,k-i}[i, n-i] \to \mathbb{D}(\mathcal{C})$ corresponds to a dashed filling of the square

$$\begin{tikzcd}
\Lambda^{k-i}[n-i] \arrow[r] \arrow[d] & \mathcal{S}pan^{\star}(\mathcal{C}^{\Delta[i]}) \arrow[d, "\rho"] \\
\Delta[n-i] \arrow[r] \arrow[ur, dashed] & \mathcal{S}pan^{\star}(\mathcal{C}^{\Lambda^i[i]}).
\end{tikzcd} \tag{3.3}$$

If $i = 0$, then the square becomes the triangle

$$\begin{tikzcd}
\Lambda^{k}[n] \arrow[r] \arrow[d] & \mathcal{S}pan(\mathcal{C}) \\
\Delta[n] \arrow[ur, dashed] &
\end{tikzcd} \tag{3.4}$$

and a dashed filling exists because $\mathcal{S}pan(\mathcal{C})$ is a quasicategory. If $i > 0$, then since $0 \leq k - i < n - i$, the square (3.3) has a dashed filling because ρ is a left fibration, by Theorem 3.5.1.

Considering the second set of bihorn fillings, extending a bihorn $\Lambda^{k,0}[l, n-l] \to \mathbb{D}(\mathcal{C})$ to a map $\Delta[l, n-l] \to \mathbb{D}(\mathcal{C})$ corresponds, by the rewriting diagram (3.1), to a dashed filling

$$\begin{tikzcd}
\Lambda^{k}[l] \arrow[r, "\star"] \arrow[d] & \mathcal{C}^{\mathrm{TR}[n-l]} \arrow[d] \\
\Delta[l] \arrow[r, "\star"] \arrow[ur, dashed, "\star"] & \mathcal{C}^{\Lambda^0\,\mathrm{TR}[n-l]}.
\end{tikzcd} \tag{3.5}$$

The extreme cases $l = k$ and $l = n$ are dealt with first. If $l = k$, the diagram (3.5) is equivalent via the diagrams (3.1) and (3.2) to a dashed filling

$$\begin{tikzcd}
\Lambda^{0}[n-k] \arrow[r] \arrow[d] & \mathcal{S}pan^{\star}(\mathcal{C}^{\Delta[k]}) \arrow[d] \\
\Delta[n-k] \arrow[r] \arrow[ur, dashed] & \mathcal{S}pan^{\star}(\mathcal{C}^{\Lambda^k[k]})
\end{tikzcd}$$

and the filling is guaranteed again by Theorem 3.5.1. If $l = n$, then the square (3.5) becomes a triangle

$$\begin{tikzcd}
\Lambda^k[n] \arrow[r] \arrow[d] & \mathcal{S}pan(\mathcal{C}) \\
\Delta[n] \arrow[ur, dashed] &
\end{tikzcd}$$

and a filling exists because $\mathcal{S}pan(\mathcal{C})$ is a quasicategory.

If $k < l < n$, then the map $\mathcal{C}^{\mathrm{TR}[n-l]} \to \mathcal{C}^{\Lambda^0 \mathrm{TR}[n-l]}$ is an inner fibration since the Joyal model structure on simplicial sets is cartesian and $\Lambda^0 \mathrm{TR}[n-l] \hookrightarrow \mathrm{TR}[n-l]$. Therefore, a dashed filling of the square (3.5) exists, but a priori it may not satisfy the necessary cromulence conditions. For dimensional reasons as discussed in the beginning of the proof of Lemma 3.5.2, the pullback and cromulence conditions are satisfied except for the cases $n = 3, k = 1, l = 2$ and $n = 4, k = 1, l = 2$, which correspond to bihorns $\Lambda^{1,0}[2,1] \to \mathbb{D}(\mathcal{C})$ and $\Lambda^{1,0}[2,2] \to \mathbb{D}(\mathcal{C})$, respectively.

Consider first the case of extending a bihorn $\Lambda^{1,0}[2,1] \to \mathbb{D}(\mathcal{C})$ to a map $\Delta[2,1] \to \mathbb{D}(\mathcal{C})$, which amounts to verifying that any extension $\Delta[2] \times \mathrm{TR}[1] \to \mathcal{C}$ of a map $(\Lambda^1[2] \times \mathrm{TR}[1]) \cup (\Delta[2] \times \Lambda^0 \mathrm{TR}[1]) \to \mathcal{C}$ satisfying the cromulence conditions will itself satisfy the cromulence conditions. Letting (a, bc) denote the 0-simplices of $\Delta[2] \times \mathrm{TR}[1]$, let X_{bc}, Y_{bc} and Z_{bc} denote the images of $(0, bc), (1, bc)$ and $(2, bc)$ in the map $\Delta[2] \times \mathrm{TR}[1] \to \mathcal{C}$. In this case, it needs to be shown that the diagram

$$\begin{tikzcd}
X_{00} \arrow[d, "q"'] & X_{01} \arrow[l] \arrow[r] \arrow[d] & X_{11} \arrow[d] \\
Z_{00} & Z_{01} \arrow[l, "u"'] \arrow[r, "f"] & Z_{11}
\end{tikzcd}$$

in $\mathcal{C}$ is cromulent. The initial data includes the information that the diagrams

$$\begin{tikzcd}
X_{00} \arrow[d, "i"'] & X_{01} \arrow[l] \arrow[r] \arrow[d, "s"] & X_{11} \arrow[d] \\
Y_{00} & Y_{01} \arrow[l] \arrow[r, "h"] & Y_{11}
\end{tikzcd} \quad \text{and} \quad \begin{tikzcd}
Y_{00} \arrow[d, "p"'] & Y_{01} \arrow[l] \arrow[r, "h"] \arrow[d] & Y_{11} \arrow[d] \\
Z_{00} & Z_{01} \arrow[l] \arrow[r, "f"] & Z_{11}
\end{tikzcd} \tag{3.6}$$

are cromulent, with

$$\begin{tikzcd}
X_{00} \arrow[dd, "q"'] \arrow[dr, "i"] & \\
& Y_{00} \arrow[dl, "p"] \\
Z_{00} &
\end{tikzcd}$$

a 2-simplex in $\mathcal{C}$. By pullback pasting,

$$\begin{tikzcd}
X_{01} \arrow[r] \arrow[d] & X_{11} \arrow[d] \\
Z_{01} \arrow[r] & Z_{11}
\end{tikzcd}$$

is a pullback square in $\mathcal{C}$, so it only remains to be shown that the left hand square in the diagram of interest has the correct form, which is determined as follows: The left hand square induces a 1-simplex $X_{01} \xrightarrow{e} B$, where B is formed by a pullback

$$\begin{tikzcd}
X_{00} \arrow[d, "q"'] & B \arrow[l] \arrow[d, "\tilde{q}"] \\
Z_{00} & Z_{01}. \arrow[l, "u"']
\end{tikzcd}$$

The correct form for the square is for the map $X_{01} \to B$ to be equivalent in $\mathcal{C}_{/B}$ to $\varepsilon^{f}_{\tilde{q}}$. One can form auxiliary pullbacks in $\mathcal{C}$ and a larger diagram as follows, where each square is a pullback (omitted but implicit are arrows $W \to X_{00}$ and $Y_{01} \to Y_{00}$ completing the pullback in the upper row):

$$\begin{tikzcd}
X_{00} \arrow[d, "i"] \arrow[dd, bend right, "q"'] & B \arrow[l] \arrow[d, "\tilde{i}"'] \arrow[dd, bend left, "\tilde{q}"] & W \arrow[l] \arrow[d, "j"] \\
Y_{00} \arrow[d, "p"] & A \arrow[l] \arrow[d, "\tilde{p}"'] & Y_{01} \arrow[l, "e_{\tilde{p}}"'] \\
Z_{00} & Z_{01}. \arrow[l, "u"'] &
\end{tikzcd}$$

In particular, since the diagram on the left of (3.6) is cromulent, the induced map $X_{01} \to W$ is equivalent in $\mathcal{C}_{/W}$ to ε_j^h, and is induced by the pair (s, e). Since $s \simeq h^* h_*(j)$, it follows from Lemma 3.3.4 that $e \simeq \varepsilon_{\tilde{q}}^f$, as desired.

Showing that a filling of a bihorn $\Lambda^{1,0}[2,2] \to \mathbb{D}(\mathcal{C})$ to a map $\Delta[2,2] \to \mathbb{D}(\mathcal{C})$ exists consists of verifying that any extension $\Delta[2] \times \mathrm{TR}[2] \to \mathcal{C}$ of the map $(\Lambda^1[2] \cup \mathrm{TR}[2]) \times (\Delta[2] \times \Lambda^0 \,\mathrm{TR}[2])$ corresponding to the bihorn $\Lambda^{1,0}[2,2] \to \mathbb{D}(\mathcal{C})$ will satisfy the cromulence conditions. As in the preceding case, let X_{bc}, Y_{bc} and Z_{bc} denote the images of $(0, bc), (1, bc)$ and $(2, bc)$. The $(1,1)$-simplex of $\Delta[2,2]$ missing from $\Lambda^{1,0}[2,2]$ is $(02, 12)$, so the diagram in $\mathcal{C}$ obtained by the filling which needs to be checked is cromulent is

$$\begin{tikzcd} X_{11} \arrow[d] & X_{12} \arrow[l] \arrow[r] \arrow[d] & X_{22} \arrow[d] \\ Z_{11} & Z_{12} \arrow[l] \arrow[r] & Z_{22}. \end{tikzcd}$$

By assumption, the diagrams

$$\begin{tikzcd} X_{11} \arrow[d] & X_{12} \arrow[l] \arrow[r] \arrow[d] & X_{22} \arrow[d] \\ Y_{11} & Y_{12} \arrow[l] \arrow[r] & Y_{22} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} Y_{11} \arrow[d] & Y_{12} \arrow[l] \arrow[r] \arrow[d] & Y_{22} \arrow[d] \\ Z_{11} & Z_{12} \arrow[l] \arrow[r] & Z_{22} \end{tikzcd}$$

are cromulent, and so an analysis identical to that of the preceding case establishes the desired cromulence for the filled in diagram.

$\square$

3.5 Fibrations between quasicategories of spans with cromulence conditions

This section leads to an important technical result (Theorem 3.5.1) which is crucial to the proof of Theorem 3.4.4. Before proving Theorem 3.5.1, some preliminary lemmas are established.

Theorem 3.5.1. *The restriction map* $\rho\colon \mathcal{S}pan^{\star}(\mathcal{C}^{\Delta[m]}) \to \mathcal{S}pan^{\star}(\mathcal{C}^{\Lambda^m[m]})$ *is a left fibration for* $m \geq 1$*.*

The idea behind the proof of Theorem 3.5.1 is to first establish that ρ is a cocartesian fibration such that every 1-simplex of $\mathcal{S}pan^{\star}(\mathcal{C}^{\Delta[m]})$ is a cocartesian lift of its image in $\mathcal{S}pan^{\star}(\mathcal{C}^{\Lambda^m[m]})$, from which it follows from Lemma 2.4.2.4 of [9] that ρ is a left fibration. A requisite condition for ρ being a cocartesian fibration is that ρ is an inner fibration, which is established in the following lemma:

Lemma 3.5.2. *The restriction map* $\rho\colon \mathcal{S}pan^{\star}(\mathcal{C}^{\Delta[m]}) \to \mathcal{S}pan^{\star}(\mathcal{C}^{\Lambda^m[m]})$ *is an inner fibration for* $m \geq 1$*.*

Proof. Since the restriction on ordinary spans $\mathcal{S}pan(\mathcal{C}^{\Delta[m]}) \to \mathcal{S}pan(\mathcal{C}^{\Lambda^m[m]})$ is an inner fibration by Theorem 2.6.5, dotted fillings exist of the extended diagrams

$$\begin{tikzcd}
\Lambda^k[n] \arrow[r] \arrow[d] & \mathcal{S}pan^{\star}(\mathcal{C}^{\Delta[m]}) \arrow[r, hook] \arrow[d] & \mathcal{S}pan(\mathcal{C}^{\Delta[m]}) \arrow[d] \\
\Delta[n] \arrow[r] \arrow[ur, dashed] \arrow[urr, dotted] & \mathcal{S}pan^{\star}(\mathcal{C}^{\Lambda^m[m]}) \arrow[r, hook] & \mathcal{S}pan(\mathcal{C}^{\Lambda^m[m]}).
\end{tikzcd} \tag{3.7}$$

Recall that a filling

$$\begin{tikzcd}
\Lambda^k[n] \arrow[r] \arrow[d] & \mathcal{S}pan^{\star}(\mathcal{C}^{\Delta[m]}) \arrow[d] \\
\Delta[n] \arrow[r] \arrow[ur, dashed] & \mathcal{S}pan^{\star}(\mathcal{C}^{\Lambda^m[m]})
\end{tikzcd} \tag{3.8}$$

corresponds to an extension

$$\begin{tikzcd}
\Lambda^{m,k}[m,n] \arrow[r] \arrow[d] & \mathbb{D}(\mathcal{C}) \\
\Delta[m,n] \arrow[ur, dashed] &
\end{tikzcd}$$

where, in particular, the $(1,1)$-bisimplices of $\Lambda^{m,k}[m,n]$ and $\Delta[m,n]$ are sent to cromulent diagrams in $\mathcal{C}$.

If either of m or n is at least 3, then $\Lambda^{m,k}[m,n]$ will contain all the $(1,1)$-bisimplices of $\Delta[m,n]$. This in true for bihorns in general, so we can consider $\Lambda^{j,k}[m,n]$, and without a loss of generality, assume that $m \geq 3$. Then $\Lambda^{j,k}[m,n]_{1,1} = (\Lambda^j[m]_1 \times \Delta[n]_1) \cup (\Delta[m]_1 \times \Lambda^k[n]_1)$. Since $m \geq 3$, the horn $\Lambda^j[m]$ has all the 1-simplices of $\Delta[m]$, and so $\Lambda^j[m]_1 \times \Delta[n]_1 = \Delta[m]_1 \times \Delta[n]_1 = \Delta[m,n]_{1,1}$. Therefore, if $m, n \geq 3$, then a dotted filling in the diagram (3.7) restricts to a dashed filling, since all the cromulence information is already contained in the map $\Lambda^k[n] \to \mathcal{S}pan^\star(\mathcal{C}^{\Delta[m]})$. Thus, to show that ρ is an inner fibration, it only remains to construct bihorn fillings for maps $\Lambda^{1,1}[1,2] \to \mathbb{D}(\mathcal{C})$ and $\Lambda^{2,1}[2,2] \to \mathbb{D}(\mathcal{C})$.

The extension of a map $\Lambda^{1,1}[1,2] \to \mathbb{D}(\mathcal{C})$ to a map $\Delta[1,2] \to \mathbb{D}(\mathcal{C})$ is the data of the filling of diagram $(\Lambda^1[1] \times \mathrm{TR}[2]) \cup (\Delta[1] \times \Lambda^1 \mathrm{TR}[2]) \xrightarrow{\star} \mathcal{C}$ to a diagram $\Delta[1] \times \mathrm{TR}[2] \xrightarrow{\star} \mathcal{C}$. The initial data of the bihorn is depicted in the following by the diagram on the solid lines, and the extension by the red 0-simplex X_{02} and the red dashed lines. Higher dimensional simplices are omitted from the diagram, but they are also part of the data.

$$
\begin{tikzcd}
 & & {\color{red}X_{02}} \arrow[dl, dashed, red] \arrow[d, dashed, red, "s"] \arrow[dr, dashed, red] & & \\
 & X_{01} \arrow[dl] \arrow[d] \arrow[dr] & Y_{02} \arrow[dl, "w"'] \arrow[dr, "h"] & X_{12} \arrow[dl] \arrow[d] \arrow[dr] & \\
X_{00} \arrow[d, "p"'] & Y_{01} \arrow[dl, "u"'] \arrow[dr, "f"] & X_{11} \arrow[d, "q"] & Y_{12} \arrow[dl, "v"'] \arrow[dr, "g"] & X_{22} \arrow[d, "r"] \\
Y_{00} & & Y_{11} & & Y_{22}
\end{tikzcd}
$$

Let z and k in $\mathcal{C}$ be the 1-simplices omitted in the earlier large diagram as follows:

$$
\begin{tikzcd}
 & & Y_{02} \arrow[dl, "w"'] \arrow[dr, "h"] \arrow[ddll, bend right, "z"'] \arrow[ddrr, bend left, "k"] & & \\
 & Y_{01} \arrow[dl, "u"'] & & Y_{12} \arrow[dr, "g"] & \\
Y_{00} & & & & Y_{22}.
\end{tikzcd}
$$

Given the bihorn, there is a filling to a diagram $\Delta[1] \times \mathrm{TR}[2] \to \mathcal{C}$ because $\mathcal{S}pan(\mathcal{C}^{\Delta[1]}) \to \mathcal{S}pan(\mathcal{C}^{\Lambda^1[1]})$ is an inner fibration, so what remains to be seen is that the diagram

$$
\begin{tikzcd}
X_{00} \arrow[d, "p"'] & X_{02} \arrow[l] \arrow[r] \arrow[d, "s"] & X_{22} \arrow[d, "r"] \\
Y_{00} & Y_{02} \arrow[l, "z"'] \arrow[r, "k"] & Y_{22}
\end{tikzcd}
\tag{3.9}
$$

is cromulent. From the definition of $\mathcal{S}pan(\mathcal{C}^{\Delta[1]})$, the squares

$$
\begin{tikzcd}
X_{02} \arrow[r] \arrow[d] & X_{12} \arrow[d] \\
X_{01} \arrow[r] & X_{11}
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
Y_{02} \arrow[r, "h"] \arrow[d, "w"'] & Y_{12} \arrow[d, "v"] \\
Y_{01} \arrow[r, "f"] & Y_{11}
\end{tikzcd}
$$

are pullbacks, and because the initial diagram is a bihorn in $\mathbb{D}(\mathcal{C})$, the square

$$\begin{tikzcd} X_{01} \arrow[r] \arrow[d] & X_{11} \arrow[d] \\ Y_{01} \arrow[r] & Y_{11} \end{tikzcd}$$

is a pullback. Therefore, by the pullback pasting property,

$$\begin{tikzcd} X_{02} \arrow[r] \arrow[d, "s"'] & X_{12} \arrow[d] \\ Y_{02} \arrow[r, "h"] & Y_{12} \end{tikzcd}$$

is a pullback square. Applying pullback pasting a second time reveals that

$$\begin{tikzcd} X_{02} \arrow[r] \arrow[d, "s"'] & X_{22} \arrow[d, "r"] \\ Y_{02} \arrow[r, "k"] & Y_{22} \end{tikzcd}$$

is a pullback square.

Applying Lemma 3.3.3,

$$r \simeq g_* v^*(q) \simeq g_* v^* f_* u^*(p) \simeq g_* h_* w^* u^*(p) \simeq k_* z^*(p)$$

and so $s \simeq k^* k_* z^*(p)$. Therefore, to establish that the desired diagram 3.9 is cromulent, all that remains to be seen is that the left square

$$\begin{tikzcd} X_{00} \arrow[d, "p"'] & X_{02} \arrow[l] \arrow[d, "s"] \\ Y_{00} & Y_{02} \arrow[l, "z"'] \end{tikzcd}$$

of (3.9) is of the correct form, which is that if $B \simeq X_{00} \times_{Y_{00}} Y_{02}$, then the induced

1-simplex $(X_{02} \to B) \simeq \varepsilon^k_{z^*p}$ in $\mathcal{C}_{/B}$.

Since

$$h_* z^*(p) \simeq h_* w^* u^*(p) \simeq v^* f_* u^*(p) \simeq v^*(q),$$

if A and B are pullbacks below, then, by Lemma 3.3.3, $\varepsilon^h_{z^*p}$ is determined up to quasi-isomorphism in $\mathcal{C}_{/Y_{02}}$ by the pair $(s, (e_1)_!(w'))$, where $e_1 \simeq e^f_{u^*(p)}$, as seen in the following diagram, where each square in the cube is a pullback:

$$\begin{tikzcd}
 & & & & K \arrow[rr] \arrow[dl, "w'"'] \arrow[dd, "h^*h_*z^*(p)"] \arrow[ddll, bend right, "\varepsilon^h_{z^*(p)}"'] & & L \arrow[dl] \arrow[dd, "h_*z^*(p)"] \\
 & & & X_{01} \arrow[rr] \arrow[dd] \arrow[ddll, bend right, "e_1"'] & & X_{11} \arrow[dd, "q"] & \\
 & & B \arrow[rr, "z^*(p)"] \arrow[dl] & & Y_{02} \arrow[rr, "h"] \arrow[dl, "w"] & & Y_{12} \arrow[dl, "v"] \\
 & A \arrow[rr, "u^*(p)"] \arrow[dl] & & Y_{01} \arrow[rr, "f"] \arrow[dl, "u"] & & Y_{11} & \\
X_{00} \arrow[rr, "p"] & & Y_{00}. & & & &
\end{tikzcd}$$

Since

$$t \simeq g^*(r) \simeq g^* g_* v^*(q) \simeq g^* g_* v^* f_* u^*(p) \simeq g^* g_* h_* w^* u^*(p) \simeq g^* g_* h_* z^*(p)$$

there is an exponential diagram

$$\begin{tikzcd}
 & X_{12} \arrow[r] \arrow[dl, "e_2"'] \arrow[d, "t"] & X_{22} \arrow[d, "r"] \\
L \arrow[r, "h_*z^*(p)"] & Y_{12} \arrow[r, "g"] & Y_{22}.
\end{tikzcd}$$

The triangle in the diagram above extends to a prism as follows, with each square a pullback in $\mathcal{C}$, with $e_3 \simeq h^*(e_2)$ in particular:

$$\begin{tikzcd}
 & & X_{02} \arrow[r] \arrow[dl, "e_3"'] \arrow[ddll, "s"] & X_{12} \arrow[dl, "e_2"'] \arrow[ddl, "t"] \\
 & K \arrow[r] \arrow[d, "h^*h_*z^*(p)"'] & L \arrow[d, "h_*z^*(p)"'] & \\
 & Y_{02} \arrow[r, "h"] & Y_{12}. &
\end{tikzcd}$$

Therefore, the induced simplex $X_{02} \to B$ is as desired, since, by Lemma 3.3.2,

$$X_{02} \to B \simeq (\varepsilon^h_{z_*(p)})_!(e_3) \simeq (\varepsilon^h_{z_*(p)})_! h^*(e_2) \simeq (\varepsilon^h_{z_*(p)})_! h^*(\varepsilon^g_{h_*z^*(p)}) \simeq \varepsilon^k_{z^*(p)}.$$

The desired filling of the bihorn $\Lambda^{1,1}[1,2] \to \mathbb{D}(\mathcal{C})$ has thus been achieved.

A similar procedure will guarantee the existence of fillings for bihorns $\Lambda^{2,1}[2,2] \to \mathbb{D}(\mathcal{C})$. As before, unwinding the definition leads to filling a diagram $(\Lambda^2[2] \times \mathrm{TR}[2]) \times (\Delta[2] \times \Lambda^1 \mathrm{TR}[2]) \xrightarrow{\star} \mathcal{C}$ to a diagram $\Delta[2] \times \mathrm{TR}[2] \xrightarrow{\star} \mathcal{C}$. A filling for which all relevant subdiagrams are not a priori cromulent exists because $\mathcal{S}pan(\mathcal{C}^{\Delta[2]}) \to \mathcal{S}pan(\mathcal{C}^{\Lambda^2[2]})$ is an inner fibration by Theorem 2.6.5, and it will be shown any such filling does satisfy the cromulence condition.

Considering the bihorn $\Lambda^{2,1}[2,2]$, the missing $(1,1)$-cell in $\mathbb{D}(\mathcal{C})$ is the image of the $(1,1)$-cell

$$\begin{tikzcd}
(0,00) \arrow[d] & (0,02) \arrow[l] \arrow[r] \arrow[d] & (0,22) \arrow[d] \\
(1,00) & (1,02) \arrow[l] \arrow[r] & (1,22).
\end{tikzcd}$$

Graphically, the bihorn $\Lambda^{2,1}[2,2] \to \mathbb{D}(\mathcal{C})$ can be visualized as a pair of diagrams of the following shapes in $\mathcal{C}$ which agree on their intersections and have the necessary cromulent subdiagrams, as the following figures show.

$$
\begin{tikzcd}
 & & X_{02} \arrow[ddl] \arrow[ddr] \arrow[dd, bend right] & & \\
 & & Y_{02} \arrow[d] \arrow[ddl] \arrow[ddr] & & \\
 & X_{01} \arrow[ddl] \arrow[dd, bend right] \arrow[ddr] & Z_{02} \arrow[ddl] \arrow[ddr] & X_{12} \arrow[ddl] \arrow[dd, bend left] \arrow[ddr] & \\
 & Y_{01} \arrow[d] \arrow[ddl] \arrow[ddr] & & Y_{12} \arrow[d] \arrow[ddl] \arrow[ddr] & \\
X_{00} \arrow[dd, bend right] & Z_{01} \arrow[ddl] \arrow[ddr] & X_{11} \arrow[dd, bend left] & Z_{12} \arrow[ddl] \arrow[ddr] & X_{22} \arrow[dd, bend left] \\
Y_{00} \arrow[d] & & Y_{11} \arrow[d] & & Y_{22} \arrow[d] \\
Z_{00} & & Z_{11} & & Z_{22}
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & X_{01} \arrow[ddl] \arrow[d] \arrow[dd, bend right] \arrow[ddr] & & X_{12} \arrow[ddl] \arrow[d] \arrow[dd, bend left] \arrow[ddr] & \\
 & Y_{01} \arrow[d] \arrow[ddl] \arrow[ddr] & & Y_{12} \arrow[d] \arrow[ddl] \arrow[ddr] & \\
X_{00} \arrow[d] \arrow[dd, bend right] & Z_{01} \arrow[ddl] \arrow[ddr] & X_{11} \arrow[d] \arrow[dd, bend left] & Z_{12} \arrow[ddl] \arrow[ddr] & X_{22} \arrow[d] \arrow[dd, bend left] \\
Y_{00} \arrow[d] & & Y_{11} \arrow[d] & & Y_{22} \arrow[d] \\
Z_{00} & & Z_{11} & & Z_{22}
\end{tikzcd}
$$

Filling the bihorn to obtain a map $\Delta[2,2] \to \mathbb{D}(\mathcal{C})$ can be visualized via the following diagram, with 1-simplex in $\mathcal{C}$ missing from the bihorn shown in red. Higher simplices in $\mathcal{C}$ are omitted from the diagram, but are present and part of the filling data.

$$\begin{tikzcd}
 & & X_{02} \arrow[ddl] \arrow[ddr] \arrow[d, dashed, red, "k"] \arrow[dd, bend right] & & \\
 & & Y_{02} \arrow[d] \arrow[ddl, "v"'] \arrow[ddr, "g"] & & \\
 & X_{01} \arrow[ddl] \arrow[d] \arrow[dd, bend right] \arrow[ddr] & Z_{02} \arrow[ddl] \arrow[ddr] & X_{12} \arrow[ddl] \arrow[d] \arrow[dd, bend left] \arrow[ddr] & \\
 & Y_{01} \arrow[d] \arrow[ddl] \arrow[ddr] & & Y_{12} \arrow[d] \arrow[ddl] \arrow[ddr] & \\
X_{00} \arrow[d, "i"] \arrow[dd, bend right, "q"'] & Z_{01} \arrow[ddl] \arrow[ddr] & X_{11} \arrow[d] \arrow[dd, bend left] & Z_{12} \arrow[ddl] \arrow[ddr] & X_{22} \arrow[d, "j"'] \arrow[dd, bend left, "s"] \\
Y_{00} \arrow[d, "p"] & & Y_{11} \arrow[d] & & Y_{22} \arrow[d, "r"'] \\
Z_{00} & & Z_{11} & & Z_{22}
\end{tikzcd}$$

The aim is to show that the diagram

$$\begin{tikzcd}
X_{00} \arrow[d, "i"'] & X_{02} \arrow[l] \arrow[r] \arrow[d, "k"] & X_{22} \arrow[d, "j"] \\
Y_{00} & Y_{02} \arrow[l, "v"'] \arrow[r, "g"] & Y_{22}
\end{tikzcd} \tag{3.10}$$

is cromulent. By assumption,

$$\begin{aligned}
j &\simeq (Y_{12} \to Y_{22})_*(Y_{12} \to Y_{11})^*(X_{11} \to Y_{11}) \\
&\simeq (Y_{12} \to Y_{22})_*(Y_{12} \to Y_{11})^*(Y_{01} \to Y_{11})_*(Y_{01} \to Y_{00})^*(i) \\
&\simeq (Y_{12} \to Y_{22})_*(Y_{02} \to Y_{12})_*(Y_{02} \to Y_{01})^*(Y_{01} \to Y_{00})^*(i) \\
&\simeq (Y_{02} \to Y_{00})_*(Y_{02} \to Y_{22})^*(i) \\
&= g_*v^*(i).
\end{aligned}$$

Meanwhile the right square of the diagram (3.10) is a pullback square, by pullback pasting. Therefore, it only remains to be seen that left square of (3.10) is of the correct form, i.e. that if

$$\begin{tikzcd}
X_{00} & W \arrow[l] \arrow[d, "v^*i"] \\
Y_{00} \arrow[from=1-1, d, "i"'] & Y_{02} \arrow[l, "v"']
\end{tikzcd}$$

is a pullback in $\mathcal{C}$, then the induced 1-simplex $X_{02} \to W$ is quasi-isomorphic in $\mathcal{C}_{/W}$ to $\varepsilon^g_{v^*i}$. Begin by forming the pullbacks A, B as follows, which fit into a diagram with W from above:

$$\begin{tikzcd}
 & & & W \arrow[dlll] \arrow[dl] \arrow[dd, "v^*i"] \\
X_{00} \arrow[dd, "i"'] \arrow[dddr, "q"] & & B \arrow[ll] \arrow[dd, "i'"] \arrow[dddr, "u^*(q)"] & \\
 & & & Y_{02} \arrow[dlll, "v"'] \arrow[dl] \\
Y_{00} \arrow[dr, "p"'] & & A \arrow[ll] \arrow[dr, "u^*(p)"] & \\
 & Z_{00} & & Z_{02}. \arrow[ll, "u"']
\end{tikzcd}$$

In particular, by pullback pasting, $W \simeq B \times_A Y_{02}$, so $X_{02} \to W$ is determined, up to quasi-isomorphism, by $X_{02} \to B$ and $X_{02} \to Y_{02}$.

Since the diagram

$$\begin{tikzcd}
X_{00} \arrow[d, "q"'] & X_{02} \arrow[l] \arrow[r] \arrow[d] & X_{22} \arrow[d, "s"] \\
Z_{00} & Z_{02} \arrow[l, "u"'] \arrow[r, "f"] & Z_{22}
\end{tikzcd}$$

is cromulent, the induced diagram

$$\begin{tikzcd}
 & X_{02} \arrow[dl, "e^f_{u^*(q)}"'] \arrow[r] \arrow[d] & X_{22} \arrow[d, "s"] \\
B \arrow[r, "u^*q"] & Z_{02} \arrow[r, "f"] & Z_{22}
\end{tikzcd}$$

is exponential. Note that $u^*(q) \simeq (u^*p)_!(i')$, so $e^f_{u^*(q)} \simeq \varepsilon^f_{(u^*(p))_!(i')}$. Therefore, $X_{02} \to W$ is induced by $k \simeq g^*(j) \simeq g^*g_*v^*(i)$ and $e^f_{u^*(q)} \simeq \varepsilon^f_{(u_*(p))_!(i')}$.

The diagram

$$\begin{tikzcd}
Y_{00} \arrow[d, "p"'] & Y_{02} \arrow[l] \arrow[r] \arrow[d] & Y_{22} \arrow[d, "r"] \\
Z_{00} & Z_{02} \arrow[l, "u"'] \arrow[r, "f"] & Z_{22}
\end{tikzcd}$$

is also cromulent, so the induced diagram

$$\begin{tikzcd}
 & Y_{02} \arrow[r] \arrow[d] \arrow[dl, "e^f_{u^*(p)}"'] & Y_{22} \arrow[d, "s"] \\
A \arrow[r, "u^*(p)"] & Z_{02} \arrow[r, "f"] & Z_{22}
\end{tikzcd}$$

is exponential. Applying Lemma 3.3.4, it follows that the induced map $X_{02} \to W$ is quasi-isomorphic in $\mathcal{C}_{/W}$ to $\varepsilon^g_{v^*(i)}$, as desired. Therefore, ρ is an inner fibration. $\square$

With the the preceding lemma in hand, the proof of the following lemma analyzes the structure of $\mathcal{S}pan^\star(\mathcal{C}^{\Lambda^m[m]})$ and establishes that both $\mathcal{S}pan^\star(\mathcal{C}^{\Lambda^m[m]})$ and $\mathcal{S}pan^\star(\mathcal{C}^{\Delta[m]})$ are quasicategories. This fact is necessary in the proof of Theorem 3.5.1 for the application of the Lemma 3.2.2.(1(a)) of [1], which provides an alternate characterization of cocartesian fibrations between quasicategories.

Lemma 3.5.3. *Let $m \geq 1$. Then the simplicial sets $\mathcal{S}pan^\star(\mathcal{C}^{\Delta[m]})$ and $\mathcal{S}pan^\star(\mathcal{C}^{\Lambda^m[m]})$ are quasicategories.*

Proof. By Lemma 3.5.2, it suffices to show that $\mathcal{S}pan^\star(\mathcal{C}^{\Lambda^m[m]})$ is a quasicategory. An inner horn in $\mathcal{S}pan^\star(\mathcal{C}^{\Lambda^m[m]})$ is also an inner horn in the quasicategory $\mathcal{S}pan(\mathcal{C}^{\Lambda^m[m]})$, so the dotted filling in the diagram

$$\begin{tikzcd}
\Lambda^k[n] \arrow[r] \arrow[d] & \mathcal{S}pan^\star(\mathcal{C}^{\Lambda^m[m]}) \arrow[r, hook] & \mathcal{S}pan(\mathcal{C}^{\Lambda^m[m]}) \\
\Delta[n] \arrow[ur, dashed] \arrow[urr, dotted] & &
\end{tikzcd}$$

always exists, but it remains to be seen that any such dotted filling in fact restricts to a dashed one. If $m = 1$, so that $\mathcal{C}^{\Lambda^1[1]} \cong \mathcal{C}$, then no nontrivial cromulence conditions are imposed and $\mathcal{S}pan^\star(\mathcal{C}^{\Lambda^1[1]}) \cong \mathcal{S}pan(\mathcal{C})$, so it is only necessarily to consider the cases where $m \geq 2$. The cases under consideration can be further reduced to be those with $n = 2$ and $m \geq 2$, as $\Lambda^m[m] \boxtimes \Lambda^k[n]$ contains all the $(1,1)$-simplices of $\Lambda^m[m] \boxtimes \Delta[n]$ if $n \geq 3$, and the current filling problem can be be phrased the original setting of bisimplicial sets as extending a map $\Lambda^m[m] \boxtimes \Lambda^k[n] \to \mathbb{D}(\mathcal{C})$ to a map $\Lambda^m[m] \boxtimes \Delta[n] \to \mathbb{D}(\mathcal{C})$. Thus, the problem reduces to showing the existence of dashed fillings

$$\begin{tikzcd} \Lambda^1[2] \arrow[r] \arrow[d] & \mathcal{S}pan^\star(\mathcal{C}^{\Lambda^m[m]}) \arrow[r, hook] & \mathcal{S}pan(\mathcal{C}^{\Lambda^m[m]}) \\ \Delta[2] \arrow[ur, dashed] \arrow[urr, dotted] & & \end{tikzcd}$$

for $m \geq 2$, which can be reinterpreted as showing the existence of dashed fillings

$$\begin{tikzcd} \Lambda^m[m] \times \Lambda^1\,\mathrm{TR}[2] \arrow[r, "\star"] \arrow[d] & \mathcal{C}. \\ \Lambda^m[m] \times \Delta[2] \arrow[ur, dashed, "\star"'] & \end{tikzcd}$$

A filling which does not necessarily satisfy the cromulence conditions exists, as noted previously. Verifying the cromulence conditions in this case reduces to showing that each subdiagram

$$\begin{tikzcd} (a,00) \arrow[d] & (a,02) \arrow[l] \arrow[r] \arrow[d] & (a,22) \arrow[d] \\ (b,00) & (b,02) \arrow[l] \arrow[r] & (b,22) \end{tikzcd}$$

of $\Lambda^m[m] \times \Delta[2]$, where $a \to b$ is a 1-simplex of $\Lambda^m[m]$, is sent to a cromulent diagram in

$\mathcal{C}$. The initial data of the filling problem includes the assumption that the subdiagrams

$$\begin{array}{ccccc} (a,00) & \longleftarrow & (a,01) & \longrightarrow & (a,11) \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ (b,00) & \longleftarrow & (b,01) & \longrightarrow & (b,11) \end{array} \quad \text{and} \quad \begin{array}{ccccc} (a,11) & \longleftarrow & (a,12) & \longrightarrow & (a,22) \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ (b,11) & \longleftarrow & (b,12) & \longrightarrow & (b,22) \end{array}$$

of $\Lambda^m[m] \times \mathrm{TR}[2]$ are sent to cromulent diagrams, and now a verification nearly identical to that for extending maps $\Lambda^{1,1}[1,2] \to \mathbb{D}(\mathcal{C})$ to $\Delta[1,2]$ in the proof of Lemma 3.5.2 establishes the desired cromulence conditions. $\square$

Before proceeding to the proof of Theorem 3.5.1, the following two lemmas provide some useful isomorphisms and equivalences.

Lemma 3.5.4. *Let $\mathcal{C}$ be a quasicategory with pullbacks. Then there is an isomorphism $\mathcal{S}pan(\mathcal{C})^{\Delta[1]} \cong \mathcal{S}pan(\mathcal{C}^{\mathrm{TR}[1]})$.*

Proof. In the following, let $\mathrm{Hom}'_{\mathcal{SS}et}(\mathrm{TR}[n] \times \mathrm{TR}[1], \mathcal{C})$ denote the set of maps $\mathrm{TR}[n] \times \mathrm{TR}[1]$ of simplicial sets which send each subdiagram

$$\begin{array}{ccc} (ac,de) & \longrightarrow & (bc,de) \\ \big\downarrow & & \big\downarrow \\ (ab,de) & \longrightarrow & (bb,de) \end{array}$$

of $\mathrm{TR}[n] \times \mathrm{TR}[1]$ to a pullback in $\mathcal{C}$, for $0 \le a < b < c \le n$ and $0 \le d \le e \le 1$. Similarly, $\mathrm{Hom}'_{\mathcal{SS}et}(\mathrm{TR}[n], \mathcal{C}^{\mathrm{TR}[1]})$ denotes the set of maps $\mathrm{TR}[n] \to \mathcal{C}^{\mathrm{TR}[1]}$ sending each subdiagram

$$\begin{array}{ccc} ac & \longrightarrow & bc \\ \big\downarrow & & \big\downarrow \\ ab & \longrightarrow & bb \end{array}$$

of $\mathrm{TR}[n]$ with $0 \le a < b < c \le n$ to a pullback in $\mathcal{C}^{\mathrm{TR}[1]}$. Since pullbacks in functor categories are evaluated pointwise, these are the same, and this observation is crucial in the following:

$$\begin{aligned}\mathrm{Hom}_{\mathcal{SS}et}(\Delta[n], \mathcal{S}pan(\mathcal{C})^{\Delta[1]}) &\cong \mathrm{Hom}_{\mathcal{SS}et}(\Delta[n] \times \Delta[1], \mathcal{S}pan(\mathcal{C})) \\ &\cong \mathrm{Hom}'_{\mathcal{SS}et}(\mathrm{TR}[n] \times \mathrm{TR}[1], \mathcal{C}) \\ &\cong \mathrm{Hom}'_{\mathcal{SS}et}(\mathrm{TR}[n], \mathcal{C}^{\mathrm{TR}[1]}) \\ &\cong \mathrm{Hom}_{\mathcal{SS}et}(\Delta[n], \mathcal{S}pan(\mathcal{C}^{\mathrm{TR}[1]})).\end{aligned}$$

□

Lemma 3.5.5. *Let $\mathcal{X}$ be a quasicategory, and x an object of $\mathcal{X}$. The quasicategories $(\mathcal{X}_{/x})^{\Delta[n]}$ and $\mathcal{X}^{\Delta[n+1]}_{n+1\mapsto x}$ are categorically equivalent, where the latter is defined as the pullback*

$$\begin{tikzcd} \mathcal{X}^{\Delta[n+1]}_{n+1\mapsto x} \arrow[r] \arrow[d] & \mathcal{X}^{\Delta[n+1]} \arrow[d, "(d_0)^n"] \\ \Delta[0] \arrow[r, "\lceil x \rceil"] & \mathcal{X}. \end{tikzcd}$$

Proof. Apply Lemma 3.2.3 with $X = \Delta[n]$ and $p\colon S \to \mathcal{C}$ given by $\lceil x \rceil\colon \Delta[0] \to \mathcal{X}$. □

The next lemma provides a useful way to identify trivial fibrations between contractible Kan complexes.

Lemma 3.5.6. *Let $p\colon \mathcal{X} \to \mathcal{Y}$ be an inner fibration between contractible Kan complexes. If p is surjective on objects, then p is a trivial fibration.*

Proof. Since $\mathcal{X}$ and $\mathcal{Y}$ are contractible, p is an equivalence, so it suffices to show that p is an isofibration. As all 1-simplices in $\mathcal{Y}$ are quasi-invertible, it is in fact enough to show that p has the left lifting property against the inclusion $\{0\} = \Lambda^0[1] \to \Delta[1]$, i.e. that every square of the following form has a diagonal filler:

$$\begin{tikzcd}
\{0\} \arrow[r, "x"] \arrow[d] & \mathcal{X} \arrow[d, "p"] \\
\Delta[1] \arrow[r, "f"] \arrow[ur, dashed] & \mathcal{Y}.
\end{tikzcd}$$

Let y be the target of f, so that f can be depicted as $p(x) \xrightarrow{f} y$. Since p is surjective on 0-simplices, there is an x' in $\mathcal{X}_0$ such that $p(x') = y$. Since $\mathcal{X}$ is contractible, there is a 1-simplex $x \xrightarrow{\sim} x'$, from which a map $\Lambda^1[2] \to \mathcal{X}$ can be formed by the diagram

$$\begin{tikzcd}
 & x' \arrow[dr, "\mathrm{id}_{x'}"] & \\
x \arrow[ur, "\sim"] & & x'.
\end{tikzcd}$$

Letting $p(x) \xrightarrow{g} y$ be the image of $x \xrightarrow{\sim} x'$, applying p to the above diagram yields a diagram in $\mathcal{Y}$ as follows:

$$\begin{tikzcd}
 & y \arrow[dr, "p(\mathrm{id}_{x'})"] & \\
p(x) \arrow[ur, "g"] & & y.
\end{tikzcd}$$

Filling in the missing 1-simplex by f produces a map $\partial\Delta[2] \to \mathcal{Y}$ defined by

$$\begin{tikzcd}
 & y \arrow[dr, "p(\mathrm{id}_{x'})"] & \\
p(x) \arrow[ur, "g"] \arrow[rr, "f"] & & y.
\end{tikzcd}$$

Since $\mathcal{Y}$ is contractible, there is an extension to $\Delta[2] \to \mathcal{Y}$, leading to a lifting problem

$$\begin{tikzcd}
\Lambda^1[2] \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "p"] \\
\Delta[2] \arrow[r] \arrow[ur, dashed] & \mathcal{Y}
\end{tikzcd}$$

for which a dashed lift exists because p is an inner fibration. The image of the edge 02 in $\Delta[2]$ under the lift provides the required lift of f. $\square$

With these comparisons in hand, we may complete the proof of Theorem 3.5.1.

Proof of Theorem 3.5.1. Let Σ be a 0-simplex of $\mathcal{S}pan^{\star}(\mathcal{C}^{\Delta[m]})$, $\sigma = \rho(\Sigma)$ a 0-simplex of $\mathcal{S}pan^{\star}(\mathcal{C}^{\Lambda^m[m]})$ and ω a 0-simplex of $\mathcal{S}pan^{\star}(\mathcal{C}^{\Lambda^m[m]})^{/\sigma}$. Define the simplicial set $\mathcal{D}[\Sigma, \sigma, \omega]$ as the pullback (note the use of the alternate slice here)

$$\begin{tikzcd} \mathcal{D}[\Sigma,\sigma,\omega] \arrow[r] \arrow[d] & \mathcal{S}pan^{\star}(\mathcal{C}^{\Delta[m]})^{\Sigma/} \arrow[d, "\rho'"] \\ \Delta[0] \arrow[r, "\lceil\omega\rceil"] & \mathcal{S}pan^{\star}(\mathcal{C}^{\Lambda^m[m]})^{\sigma/}. \end{tikzcd}$$

If $\mathcal{D}[\Sigma, \sigma, \omega]$ is contractible, then it follows from Lemma 3.2.2(1(a)) of [1] that every lift of ω at F is ρ-cocartesian., If each $\mathcal{D}[\Sigma, \sigma, \omega]$ is nonempty, then ρ is a cocartesian fibration such every 1-simplex in $\mathcal{S}pan^{\star}(\mathcal{C}^{\Delta[m]})$ is ρ-cocartesian, and so ρ is a left fibration, by the dual of Lemma 2.4.2.4 of [9].

If $m = 1$, write $X' \xrightarrow{p} X$ for Σ, so that the pullback under consideration is

$$\begin{tikzcd} \mathcal{D}[p,X,\omega] \arrow[r] \arrow[d] & \mathcal{S}pan^{\star}(\mathcal{C}^{\Delta[1]})^{p/} \arrow[d, "(d_1)'_*"] \\ \Delta[0] \arrow[r, "\lceil\omega\rceil"] & \mathcal{S}pan(\mathcal{C})^{X/}. \end{tikzcd}$$

By definition, there is a pullback square

$$\begin{tikzcd} \mathcal{S}pan^{\star}(\mathcal{C}^{\Delta[1]})^{p/} \arrow[r] \arrow[d] & \mathcal{S}pan^{\star}(\mathcal{C}^{\Delta[1]})^{\Delta[1]} \arrow[d, "d_1"] \\ \Delta[0] \arrow[r, "\lceil p\rceil"] & \mathcal{S}pan^{\star}(\mathcal{C}^{\Delta[1]}). \end{tikzcd}$$

Write $\mathcal{S}pan^{\star}(\mathcal{C}^{\Delta[1]\times \mathrm{TR}[1]})$ for the subcategory of $\mathcal{S}pan(\mathcal{C}^{\Delta[1]\times \mathrm{TR}[1]})$ which is identified with the subcategory $\mathcal{S}pan^\star(\mathcal{C}^{\Delta[1]})^{\Delta[1]}$ of $\mathcal{S}pan(\mathcal{C}^{\Delta[1]})^{\Delta[1]}$ under the isomorphism of Lemma 3.5.4. The 0-simplices of $\mathcal{S}pan^\star(\mathcal{C}^{\Delta[1]\times \mathrm{TR}[1]})$ are cromulent diagrams, while n-simplices for $n \geq 1$ are n-fold spans of cromulent diagrams which, upon restriction to n-fold spans in $\mathcal{C}^{\Delta[1]\times\{ab\}}$ for $ab \in \mathrm{TR}[1]$ are n-simplices of $\mathcal{S}pan^\star(\mathcal{C}^{\Delta[1]})$. Therefore, $\mathcal{S}pan^\star(\mathcal{C}^{\Delta[1]})^{p/}$ can be identified as the subcategory of $\mathcal{S}pan^\star(\mathcal{C}^{\Delta[1]\times \mathrm{TR}[1]})$ consisting of simplices which, upon restriction to $\mathcal{S}pan^\star(\mathcal{C}^{\Delta[1]\times\{00\}})$, are degenerate on the 0-simplex p of $\mathcal{C}^{\Delta[1]}$.

In a similar analysis, the pullback $\mathcal{D}[p, X, \omega]$ is the subcategory of $\mathcal{S}pan^\star(\mathcal{C}^{\Delta[1]})^{p/}$ consisting of simplices which, when restricted to simplices of $\mathcal{S}pan^\star(\mathcal{C}^{\{0\}\times \mathrm{TR}[1]})$, are degenerate on the 0-simplex ω of $\mathcal{C}^{\mathrm{TR}[1]}$. Let $\mathcal{C}rom\mathcal{D}iag(f, \omega)$ denote the quasicategory of cromulent diagrams above f and ω, viewed as a subcategory of $\mathcal{C}^{\Delta[1]\times\Delta[1]}$. It has thus been established that $\mathcal{D}[p, X, \omega] \cong \mathcal{S}pan(\mathcal{C}rom\mathcal{D}iag(f, \omega))$. Since $\mathcal{C}rom\mathcal{D}iag(f, \omega)$ is contractible, $\mathcal{D}[p, X, \omega]$ is as well.

Now, let $m \geq 2$. To show that $\mathcal{D}[\Sigma, \sigma, \omega]$ is contractible, $\mathcal{D}[\Sigma, \sigma, \omega]$ will be constructed via an intermediate pullback diagram:

$$\begin{tikzcd}
\mathcal{D}[\Sigma,\sigma,\omega] \arrow[r] \arrow[d] & (\mathcal{S}pan^\star(\mathcal{C}^{\Delta[m]})^{\Sigma/})_{(u_m,f_m)} \arrow[r, hook] \arrow[d] & \mathcal{S}pan^\star(\mathcal{C}^{\Delta[m]})^{\Sigma/} \arrow[d] \\
\Delta[0] \arrow[r, "\ulcorner\omega\urcorner"] & (\mathcal{S}pan^\star(\mathcal{C}^{\Lambda^m[m]})^{\sigma/})_{(u_m,f_m)} \arrow[r, hook] & \mathcal{S}pan^\star(\mathcal{C}^{\Lambda^m[m]})^{\sigma/}.
\end{tikzcd} \tag{3.11}$$

Then the map $(\mathcal{S}pan^\star(\mathcal{C}^{\Delta[m]})^{\Sigma/})_{(u_m,f_m)} \to (\mathcal{S}pan^\star(\mathcal{C}^{\Lambda^m[m]})^{\sigma/})_{(u_m,f_m)}$ will be shown to be a trivial fibration of contractible Kan complexes, from which the contractibility of $\mathcal{D}[\Sigma, \sigma, \omega]$ follows.

Let $X_i = \Sigma(i) = \sigma(i)$, and let ω be denoted as a span by

$$\omega = \begin{tikzcd}
& \alpha \arrow[dl, "u"'] \arrow[dr, "f"] & \\
\sigma & & \tau
\end{tikzcd}$$

with $A_i = \alpha(i)$ and $Y_i = \tau(i)$, while u and f are 1-simplices in $\mathcal{C}^{\Lambda^m[m]}$, given component wise as $A_i \xrightarrow{u_i} X_i$ and $A_i \xrightarrow{f_i} Y_i$. The inclusions $\{m\} \hookrightarrow \Lambda^m[m] \subset \Delta[m]$ induce functors $\mathcal{C}^{\Delta[m]} \to \mathcal{C}^{\Lambda^m[m]} \to \mathcal{C}$, which in turn induce functors $\mathcal{S}pan^\star(\mathcal{C}^{\Delta[m]}) \to \mathcal{S}pan^\star(\mathcal{C}^{\Lambda^m[m]}) \to \mathcal{S}pan(\mathcal{C})$. Taking further induced functors between slice categories, there is a sequence

$$\mathcal{S}pan^\star(\mathcal{C}^{\Delta[m]})^{\Sigma/} \longrightarrow \mathcal{S}pan^\star(\mathcal{C}^{\Lambda^m[m]})^{\sigma/} \longrightarrow \mathcal{S}pan(\mathcal{C})^{X_m/}.$$

Considering the pair (u_m, f_m) as a 0-simplex ω_m of $\mathcal{S}pan(\mathcal{C})^{X_m/}$ and taking fibers along the above restriction functors leads to the following, where each square is a pullback:

$$\begin{tikzcd}
(\mathcal{S}pan^\star(\mathcal{C}^{\Delta[m]})^{\Sigma/})_{\omega_m} \arrow[r] \arrow[d] & \mathcal{S}pan^\star(\mathcal{C}^{\Delta[m]})^{\Sigma/} \arrow[d] \\
(\mathcal{S}pan^\star(\mathcal{C}^{\Lambda^m[m]})^{\sigma/})_{\omega_m} \arrow[r] \arrow[d] & \mathcal{S}pan^\star(\mathcal{C}^{\Lambda^m[m]})^{\sigma/} \arrow[d] \\
\Delta[0] \arrow[r, "\lceil \omega_m \rceil"] & \mathcal{S}pan(\mathcal{C})^{X_m/}.
\end{tikzcd}$$

In particular, applying Lemma 2.3.8 to the inner fibration $\rho\colon \mathcal{S}pan^\star(\mathcal{C}^{\Delta[m]}) \to \mathcal{S}pan^\star(\mathcal{C}^{\Lambda^m[m]})$ ensures that the vertical maps in the upper square are both inner fibrations. Considering the diagram (3.11) it thus suffices to consider $\mathcal{D}[\Sigma, \sigma, \omega]$ as the fiber of an inner fibration as follows:

$$\begin{tikzcd}
\mathcal{D}[\Sigma, \sigma, \omega] \arrow[r] \arrow[d] & (\mathcal{S}pan^\star(\mathcal{C}^{\Delta[m]})^{\Sigma/})_{\omega_m} \arrow[d, "\tilde{\rho}"] \\
\Delta[0] \arrow[r] & (\mathcal{S}pan^\star(\mathcal{C}^{\Lambda^m[m]})^{\sigma/})_{\omega_m}.
\end{tikzcd}$$

Applying Lemma 3.5.6 it suffices to show that the Kan complexes $(\mathcal{S}pan^\star(\mathcal{C}^{\Delta[m]})^{\Sigma/})_{\omega_m}$ and $(\mathcal{S}pan^\star(\mathcal{C}^{\Lambda^m[m]})^{\sigma/})_{\omega_m}$ are contractible and that $\tilde{\rho}$ is surjective on 0-simplices.

The surjectivity of $\tilde{\rho}$ will be shown first. A 0-simplex ω in $(\mathcal{S}pan^\star(\mathcal{C}^{\Lambda^m[m]})^{\sigma/})_{\omega_m}$

consists of a map $\Lambda^m[m] \times \mathrm{TR}[1] \xrightarrow{\star} \mathcal{C}$ which restricts to ω_m on $\{m\} \times \mathrm{TR}[1]$. The pair $\omega_m = (u_m, f_m)$ induces a map $\mathcal{C}_{/X_m} \to \mathcal{C}^{\Lambda^2[2]} \times_{\mathcal{C}} \mathcal{C}^{\Delta[1]}_{\mathrm{Pb}}$, where the target is the base of the trivial fibration defining $\mathcal{C}rom\mathcal{D}iag(\mathcal{C})$. On 0-simplices, this map takes a slice over X_m and produces a base for a cromulent diagram by pasting it to ω_m. Pulling back the trivial fibration $\mathcal{C}rom\mathcal{D}iag(\mathcal{C}) \to \mathcal{C}^{\Lambda^2[2]} \times_{\mathcal{C}} \mathcal{C}^{\Delta[1]}_{\mathrm{Pb}}$ of Corollary 3.2.16 along the map $\mathcal{C}_{/X_m} \to \mathcal{C}^{\Lambda^2[2]} \times_{\mathcal{C}} \mathcal{C}^{\Delta[1]}_{\mathrm{Pb}}$ yields a trivial fibration $\mathcal{C}rom\mathcal{D}iag(\mathcal{C})_{\omega_m} \to \mathcal{C}_{/X_m}$:

$$\begin{tikzcd}
\mathcal{C}rom\mathcal{D}iag(\mathcal{C})_{\omega_m} \arrow[r] \arrow[d] & \mathcal{C}rom\mathcal{D}iag(\mathcal{C}) \arrow[d] \\
\mathcal{C}_{/X_m} \arrow[r] & \mathcal{C}^{\Lambda^2[2]} \times_{\mathcal{C}} \mathcal{C}^{\Delta[1]}_{\mathrm{Pb}}.
\end{tikzcd}$$

In particular, a map $\Lambda^m[m] \times \mathrm{TR}[1] \xrightarrow{\star} \mathcal{C}$ restricting to ω_m on $\{m\} \times \mathrm{TR}[1]$ is the same data as a map $\partial\Delta[m-1] \to \mathcal{C}rom\mathcal{D}iag(\mathcal{C})_{\omega_m}$, while Σ and σ correspond to compatible maps

$$\begin{tikzcd}
\partial\Delta[m-1] \arrow[d, hook] \arrow[dr, "\sigma"] & \\
\Delta[m-1] \arrow[r, "\Sigma"] & \mathcal{C}_{/X_m}
\end{tikzcd}$$

since $\Lambda^m[m] \cong \partial\Delta[m-1] \star \Delta[0]$ and $\Delta[m] \cong \Delta[m-1] \star \Delta[0]$. Therefore, extending a map $\Lambda^m[m] \times \mathrm{TR}[1] \xrightarrow{\star} \mathcal{C}$ to $\Delta[m] \times \mathrm{TR}[1] \xrightarrow{\star} \mathcal{C}$ corresponds to providing a dashed filling in the square

$$\begin{tikzcd}
\partial\Delta[m-1] \arrow[r] \arrow[d] & \mathcal{C}rom\mathcal{D}iag(\mathcal{C})_{\omega_m} \arrow[d] \\
\Delta[m-1] \arrow[r] \arrow[ur, dashed] & \mathcal{C}_{/X_m}
\end{tikzcd}$$

which exists because the right hand map is a trivial fibration. Therefore, the surjectivity of $\tilde{\rho}$ on 0-simplices has been established.

The contractibility of $(\mathcal{S}pan^\star(\mathcal{C}^{\Delta[m]})^{\Sigma/})_{\omega_m}$ and $(\mathcal{S}pan^\star(\mathcal{C}^{\Lambda^m[m]})^{\sigma/})_{\omega_m}$ follows from identifying these as precisely the quasicategories of spans of contractible quasicategories given by the following pullbacks:

$$\begin{tikzcd}
((\mathcal{C}rom\mathcal{D}iag(\mathcal{C})_{\omega_m})^{\Delta[m-1]})_{\Sigma} \arrow[r] \arrow[d] & (\mathcal{C}rom\mathcal{D}iag(\mathcal{C})_{\omega_m})^{\Delta[m-1]} \arrow[d] \\
\Delta[0] \arrow[r, "\lceil\Sigma\rceil"] & (\mathcal{C}_{/X_m})^{\Delta[m-1]}
\end{tikzcd}$$

$$\begin{tikzcd}
((\mathcal{C}rom\mathcal{D}iag(\mathcal{C})_{\omega_m})^{\partial\Delta[m-1]})_{\sigma} \arrow[r] \arrow[d] & (\mathcal{C}rom\mathcal{D}iag(\mathcal{C})_{\omega_m})^{\partial\Delta[m-1]} \arrow[d] \\
\Delta[0] \arrow[r, "\lceil\sigma\rceil"] & (\mathcal{C}_{/X_m})^{\partial\Delta[m-1]}.
\end{tikzcd}$$

Hence $\mathcal{D}[\Sigma, \sigma, \omega]$ is contractible since it is the fiber of a trivial fibration between contractible Kan complexes, and the proof is complete.

□

Chapter 4

Equivalence of bispan constructions

In this chapter, we prove the main theorem of the thesis, Theorem 4.2.9, showing that diagrams of shape $\mathrm{TNR}[n]$ for varying n in a locally cartesian closed quasicategory $\mathcal{C}$, subject to pullback and cromulence conditions, assemble into a quasicategory, $\mathcal{B}ispan(\mathcal{C})$, defined below. The argument is fairly technical, and is dependent upon a thorough analysis in §3.2 of the relationship between the diagrams $\mathrm{TNR}[n]$ and the diagrams $\mathrm{TNR}_{\mathrm{dec}}[n]$ involved in the decomposed bispan construction from Chapter 2.

Definition 4.0.1. Let $\mathcal{C}$ be a locally cartesian closed quasicategory. Define the simplicial set $\mathcal{B}ispan(\mathcal{C})$ by

$$\mathrm{Hom}_{\mathcal{SS}et}(\Delta[n], \mathcal{B}ispan(\mathcal{C})) = \mathrm{Hom}^{\star}_{\mathcal{SS}et}(\mathrm{TNR}[n], \mathcal{C})$$

where the $\star$ indicates that each subdiagram of $\mathrm{TNR}[n]$ of shape

$$\begin{tikzcd} abb \arrow[d] & abc \arrow[l] \arrow[r] \arrow[d] & ab'c \arrow[d] \\ a'bb & a'bc \arrow[l] \arrow[r] & a'b'c \end{tikzcd}$$

for $0 \leq a < a' \leq b < b' \leq c \leq n$ is sent to a cromulent diagram in $\mathcal{C}$ and each subdiagram of shape

$$\begin{tikzcd} abc' \arrow[r] \arrow[d] & ab'c' \arrow[d] \\ abc \arrow[r] & ab'c \end{tikzcd}$$

for $0 \leq a \leq b < b' \leq c < c' \leq n$ is sent to a pullback diagram in $\mathcal{C}$

The conditions imposed by $\star$ are precisely those which ensure there is a restriction map $\mathcal{B}ispan(\mathcal{C}) \to \mathcal{B}ispan_{\mathrm{dec}}(\mathcal{C})$ which is induced by the inclusion $\mathrm{TNR}_{\mathrm{dec}}(\mathcal{C}) \hookrightarrow \mathrm{TNR}[n]$.

With this definition in hand, the aim is to show the following theorem and its corollary:

Theorem B (Theorem 4.2.9)**.** *The restriction map $\mathcal{B}ispan(\mathcal{C}) \to \mathcal{B}ispan_{\mathrm{dec}}(\mathcal{C})$ is a trivial fibration of simplicial sets.*

Corollary C (Corollary 4.2.10)**.** *Let $\mathcal{C}$ be a locally Cartesian closed quasicategory. Then, $\mathcal{B}ispan(\mathcal{C})$ is a quasicategory.*

Prior to proving Theorem 4.2.9, we establish some results in §3.1 regarding posets, which are useful for showing Lemma 4.2.1 from which the remaining work in proving the main theorem ensues.

4.1 Inner anodyne maps and nerves of posets

This section proves an important lemma, Lemma 4.1.4, which is used in the aforementioned analysis of the decomposed bispan construction.

Definition 4.1.1. For $S \subset [n]$, define the *generalized horn* $\Lambda^S[n] \subset \Delta[n]$ as

$$\Lambda^S[n] = \bigcup_{i \in S} \partial_i \Delta[n].$$

A generalized horn $\Lambda^S[n]$ is an *inner generalized horn* if there exist $s < t < s'$ with $s, s' \in S$ and $t \notin S$.

Lemma 4.1.2. *Inner generalized horn inclusions are inner anodyne.*

Proof. Prop 2.12 of [6]. □

Definition 4.1.3. A subset U of a poset X is *downward closed* if whenever $u \in U$ and $x \leq u$, then $x \in U$. A subset V of a poset X is *upward closed* if whenever $v \in V$ and $v \leq x$, then $x \in V$.

Lemma 4.1.4. *Let X be a finite poset such that X is the union $P \cup Q$ of subposets P and Q. If P is downward closed in X and Q is upward closed in X, then the inclusion $NP \cup NQ \hookrightarrow NX$ is an inner anodyne map of simplicial sets.*

Proof. Let $\mathcal{A}$ be the subset of maximal chains in X of the form

$$\sigma = (p_0 < p_1 < \cdots < p_r < x_0 < x_1 < \cdots < x_s < q_0 < q_1 < \cdots < q_t)$$

with $p_i \in P \setminus Q, x_i \in P \cap Q$ and $q_i \in Q \setminus P$, respectively, for all i. There are finitely many such σ, and for each such, a sequence of simplices will be attached along generalized inner horns to recover the nerve of X. It follows from the assumptions on P and Q if $p \in P$ and $q \in Q$ with $p \geq q$, then $p, q \in P \cap Q$.

The order in which the sequence of simplices will be attached for each σ is as follows: Given $\sigma \in \mathcal{A}$, either

(a) no simplex has yet been attached along a generalized inner horn to realize a subchain containing all the x_i as well as at least one of both the p_i and q_i as a simplex in NX, or

(b) there is a subchain $(p_i < \cdots < p_r < x_0 < \cdots x_s < q_0 < \cdots < q_j)$ and an $(r-i)+s+j$ simplex has already been attached to realize the subchain as a simplex of NX.

In case (a), attach a $(s+2)$-simplex realizing the chain $(p_r < x_0 < \cdots < x_s < q_0)$ by forming a inner generalized horn $\Lambda^{\{0,s+2\}}[s+2]$ with the $(s+1)$-simplices $(p_r < x_0 < \cdots < x_s)$ and $(x_0 < \cdots < x_s < q_0)$. Continue this process, next attaching along the inner generalized horn $\Lambda^{\{0,s+3\}}[s+3]$ with faces $(p_r < x_0 < \cdots < x_s < q_0)$ and $(x_0 < \cdots < x_s < q_0 < q_1)$, then attaching along the inner generalized horn $\Lambda^{\{0,s+4\}}[s{+}4]$ with faces $(p_r < x_0 < \cdots < x_s < q_0 < q_1)$ and $(x_0 < \cdots < x_s < q_0 < q_1 < q_2)$, and so on, until a simplex realizing the chain $(p_r < x_0 < \cdots < x_s < q_0 < \cdots < q_t)$ is attached.

Then, repeat the process, starting by attaching along an inner generalized horn $\Lambda^{\{0,s+3\}}[s+3]$ with faces $(p_{r-1} < p_r < x_0 < \cdots < x_s)$ and $(p_r < x_0 < \cdots < x_s < q_0)$, and proceeding as before until a simplex is attached realizing the chain $(p_{r-1} < p_r < x_0 < \cdots < x_s < q_0 < \cdots < q_t)$. Repeat again, realizing chains starting with p_{r-2}, then p_{r-3} until the simplex realizing the chain $(p_0 < \cdots < p_r < x_0 < \cdots < x_s < q_0 < \cdots q_t)$ is attached.

Case (b) proceeds similarly, but begins by attaching an inner generalized horn along faces $(p_{i-1} < p_i < \cdots < p_r < x_0 < \cdots < x_s)$ and $(p_i < \cdots < p_r < x_0 < \cdots < x_s < q_0)$ to realize the chain $(p_{i-1} < p_i < \cdots p_r < x_0 < \cdots < x_s < q_0)$, then realizes $(p_{i-1} < p_i < \cdots p_r < x_0 < \cdots < x_s < q_0 < q_1)$ by attaching a simplex along an inner generalized horn, and so on, until the chain $(p_{i-1} < p_i < \cdots p_r < x_0 < \cdots < x_s < q_0 < \cdots < q_t)$ is realized. Then, repeat the process starting with p_{i-2}, and so on, until $(p_0 < \cdots < p_r < x_0 < \cdots < x_s < q_0 < \cdots q_t)$ is attached.

Throughout this process, each filling of an inner generalized horn is along a pair of faces which contain all the possible lower-dimensional simplices in NX realizing subchains which do not include the first or last element of the chain; in particular, the simplicial set obtained at the conclusion of the process has unique spine extensions, and is thus the nerve of a category. From the 1-simplices, it follows that this simplicial set is in fact NX. $\square$

4.2 Equivalence of bispan constructions

In the following, let $\mathrm{TR}[i,n]$ denote the sub-simplicial set of $\mathrm{TR}[n]$ spanned by the objects ab for $i \leq a \leq b \leq n$.

Let $P^i[n]$ denote the underlying poset of $\Delta[i] \times \mathrm{TR}[i,n]$, so that $\Delta[i] \times \mathrm{TR}[i,n] = NP^i[n]$. Recall that $\mathrm{TNR}_{\mathrm{dec}}[n]$ is formed as a union of simplicial sets, each of which is the nerve of a poset, as follows:

$$\mathrm{TNR}_{\mathrm{dec}}[n] \coloneqq \bigcup_{0 \leq i \leq n} \Delta[i] \times \mathrm{TR}[i,n].$$

For $0 \leq j \leq n$, define the posets

$$\mathrm{tnr}^{\leq j}_{\mathrm{dec}}[n] \coloneqq \bigcup_{0 \leq i \leq j} P^i[n]$$

so that $\mathrm{tnr}^{\leq j}_{\mathrm{dec}}[n] = \mathrm{tnr}^{\leq j-1}_{\mathrm{dec}}[n] \cup P^j[n]$.

Lemma 4.2.1. *Let $0 \leq j \leq n-1$.*

(a) The poset $\mathrm{tnr}^{\leq j-1}_{\mathrm{dec}}[n]$ is downward closed in $\mathrm{tnr}^{\leq j}_{\mathrm{dec}}[n]$.

(b) The poset $P^j[n]$ is upward closed in $\mathrm{tnr}^{\leq j}_{\mathrm{dec}}[n]$.

Proof. (a) Let $abc \in \mathrm{tnr}^{\leq j-1}_{\mathrm{dec}}[n]$, so that $0 \leq a \leq b \leq c \leq n$ and $a \leq j-1$. Thus, $abc \in P^a[n] \subset \mathrm{tnr}^{\leq j-1}_{\mathrm{dec}}[n]$. Consider an element $a'b'c'$ in $\mathrm{tnr}^{\leq j}_{\mathrm{dec}}[n]$ with $a'b'c' \to abc$, which factors as

$$a'b'c' \xrightarrow{(iii)} ab'c' \xrightarrow{(ii)} abc' \xrightarrow{(i)} abc.$$

Examining each of (i)-(iii) will show that $a'b'c' \in \mathrm{tnr}_{\mathrm{dec}}^{\leq j-1}[n]$.

(i) $0 \leq a \leq b \leq c' \leq c \leq n$, so $abc' \in P^a[n] \subset \mathrm{tnr}_{\mathrm{dec}}^{\leq j-1}[n]$.

(ii) $0 \leq a \leq b' \leq b \leq c' \leq n$, so $ab'c' \in P^a[n] \subset \mathrm{tnr}_{\mathrm{dec}}^{\leq j-1}[n]$.

(iii) $0 \leq a' \leq a \leq b' \leq c' \leq n$, so $a'b'c' \in P^{a'}[n]$, which is in $\mathrm{tnr}_{\mathrm{dec}}^{\leq j-1}[n]$ since $0 \leq a' \leq a$.

(b) Let $abc \in P^j[n]$, so that $0 \leq a \leq j \leq b \leq c \leq n$. Consider an element $a'b'c' \in \mathrm{tnr}_{\mathrm{dec}}^{\leq j}[n]$ with $abc \to a'b'c'$, which factors as

$$a'b'c' \xrightarrow{(i)} a'bc \xrightarrow{(ii)} a'b'c \xrightarrow{(iii)} a'b'c'.$$

Examining each of (i)-(iii) will show that $a'b'c' \in P^j[n]$.

(i) $0 \leq a \leq a' \leq j \leq b \leq c \leq n$, so $a'bc \in P^j[n]$.

(ii) $0 \leq a' \leq j \leq b \leq b' \leq c \leq n$, so $a'b'c \in P^j[n]$.

(iii) $0 \leq a' \leq j \leq b' \leq c' \leq c \leq n$, so $a'b'c' \in P^j[n]$.

□

Lemma 4.2.2. *The inclusion* $\mathrm{TNR}_{\mathrm{dec}}[n] \hookrightarrow \mathrm{TNR}[n]$ *is inner anodyne.*

Proof. By construction,

$$\mathrm{TNR}_{\mathrm{dec}}[n] = \bigcup_{0 \leq i \leq n} \Delta[i] \times \mathrm{TR}[i, n] = \bigcup_{0 \leq i \leq n} NP^i[n]$$

and

$$\mathrm{TNR}[n] = N\left(\bigcup_{0\le i\le n} P^i[n]\right).$$

By repeated application of Lemma 4.1.4 and Lemma 4.2.1, the result follows. □

Lemma 4.2.3. *Let $X \xrightarrow{i} Y \xrightarrow{j} Z$ be a diagram of simplicial sets. If ji and i are inner anodyne, then j is inner anodyne.*

Proof. Theorem 1.5 of [16]. □

Let

$$\partial\,\mathrm{TNR}[n] = \bigcup_{0\le i\le n} \partial_i\,\mathrm{TNR}[n] = \bigcup_{0\le i\le n} ([n]\setminus\{i\})^{\mathcal{TNR}}_{\mathrm{Tw}}$$

and

$$\partial\,\mathrm{TNR}_{\mathrm{dec}}[n] = \bigcup_{0\le i\le n} \partial_i\,\mathrm{TNR}_{\mathrm{dec}}[n-1]$$

where, letting $\mathrm{TR}[i,\dots,\hat{j},\dots,n]$ denote the sub-simplicial set of $\mathrm{TR}[i,n]$ spanned by the objects ab with $a,b\neq j$,

$$\partial_j\,\mathrm{TNR}_{\mathrm{dec}}[n] = \left(\bigcup_{0\le i<j} \Delta[i]\times\mathrm{TR}[i,\dots,\hat{j},\dots,n])\right)\cup\left(\bigcup_{j<i\le n}\partial_j\Delta[i]\times\mathrm{TR}[i,n]\right).$$

Let

$$\partial\,\mathrm{TNR}_{\le j}[n] = \left(\bigcup_{0\le i\le j}\partial_i\,\mathrm{TNR}[n]\right)\cup\left(\bigcup_{j<i\le n}\partial_i\,\mathrm{TNR}_{\mathrm{dec}}[n]\right)\subset\partial\,\mathrm{TNR}[n]$$

so that $\mathrm{TNR}[n] = \partial\,\mathrm{TNR}_{\le n}[n]$.

Lemma 4.2.4. *The inclusion $\partial\,\mathrm{TNR}_{\le j}[n]\hookrightarrow\partial\,\mathrm{TNR}_{\le j+1}[n]$ is inner anodyne for $0\le j\le n-1$.*

Proof. By definition, there is a pushout diagram

$$\begin{tikzcd}
\partial_{j+1} \mathrm{TNR}[n] \cap \partial \mathrm{TNR}_{\leq j}[n] \arrow[r, hook] \arrow[d, hook] & \partial \mathrm{TNR}_{\leq j}[n] \arrow[d, hook] \\
\partial_{j+1} \mathrm{TNR}[n] \arrow[r, hook] & \partial \mathrm{TNR}_{\leq j+1}[n].
\end{tikzcd}$$

Therefore, it suffices to show that the inclusion $\partial_{j+1} \mathrm{TNR}[n] \cap \partial \mathrm{TNR}_{\leq j}[n] \hookrightarrow \partial_{j+1} \mathrm{TNR}[n]$ is inner anodyne. From Lemma 4.2.3, it suffices to show that $\partial_{j+1} \mathrm{TNR}_{\mathrm{dec}}[n] \hookrightarrow \partial_{j+1} \mathrm{TNR}[n] \cap \partial \mathrm{TNR}_{\leq j}[n]$ is inner anodyne, as there is a commutative diagram

$$\begin{tikzcd}
\partial_{j+1} \mathrm{TNR}_{\mathrm{dec}}[n] \arrow[d, hook] \arrow[dr, hook] & \\
\partial_{j+1} \mathrm{TNR}[n] \cap \partial \mathrm{TNR}_{\leq j}[n] \arrow[r, hook] & \partial_{j+1} \mathrm{TNR}[n]
\end{tikzcd}$$

and the inclusion $\partial_{j+1} \mathrm{TNR}_{\mathrm{dec}}[n] \hookrightarrow \partial_{j+1} \mathrm{TNR}[n]$ is inner anodyne by Lemma 4.2.2.

Now,

$$\begin{aligned}
&\partial_{j+1} \mathrm{TNR}[n] \cap \partial \mathrm{TNR}_{\leq j}[n] \\
&= \left(\bigcup_{0 \leq i \leq j} \partial_{j+1} \mathrm{TNR}[n] \cap \partial_i \mathrm{TNR}[n] \right) \cup \left(\bigcup_{j < i \leq n} \partial_{j+1} \mathrm{TNR}[n] \cap \partial_i \mathrm{TNR}_{\mathrm{dec}}[n] \right) \\
&= \left(\bigcup_{0 \leq i \leq j} \partial_i \partial_{j+1} \mathrm{TNR}[n] \right) \cup \partial_{j+1} \mathrm{TNR}_{\mathrm{dec}}[n].
\end{aligned}$$

Let $A^{j+1,0}[n] = \partial_{j+1} \mathrm{TNR}_{\mathrm{dec}}[n] \cap \partial_0 \partial_{j+1} \mathrm{TNR}[n]$, and define

$$D^{j+1,0}[n] = \partial_{j+1} \mathrm{TNR}_{\mathrm{dec}}[n] \cup_{A^{j,0}[n]} \partial_0 \partial_{j+1} \mathrm{TNR}[n].$$

For $1 \leq i_1 \leq j$, let $A^{j+1,i_1}[n] = D^{j+1,i_1-1}[n] \cap \partial_{i_1} \partial_{j+1} \mathrm{TNR}[n]$, and define $D^{j+1,i_1}[n]$ as the pushout

$$\begin{tikzcd} A^{j+1,i_1}[n] \arrow[r] \arrow[d] & D^{j+1,i_1-1}[n] \arrow[d] \\ \partial_{i_1}\partial_{j+1}\operatorname{TNR}[n] \arrow[r] & D^{j+1,i_1}[n]. \end{tikzcd}$$

By construction,

$$D^{j+1,j}[n] = \left(\bigcup_{0\leq i\leq j} \partial_i\partial_{j+1}\operatorname{TNR}[n]\right) \cup \partial_{j+1}\operatorname{TNR}_{\mathrm{dec}}[n].$$

Therefore, the inclusion $\partial_{j+1}\operatorname{TNR}[n] \cap \partial\operatorname{TNR}_{\leq j}[n] \hookrightarrow \partial_{j+1}\operatorname{TNR}[n]$ decomposes as a sequence of monomorphisms induced by cobase change, in the following manner:

$$\partial_{j+1}\operatorname{TNR}_{\mathrm{dec}}[n] \hookrightarrow D^{j+1,0}[n] \hookrightarrow D^{j+1,1}[n] \hookrightarrow \cdots \hookrightarrow D^{j+1,j-1}[n] \hookrightarrow D^{j+1,j}[n].$$

It then suffices to show that each inclusion $D^{j,i_1-1}[n] \hookrightarrow D^{j,i_1}[n]$ is inner anodyne, which follows if each inclusion $A^{j+1,i_1}[n] \hookrightarrow \partial_{i_1}\partial_{j+1}\operatorname{TNR}[n]$ is inner anodyne for $0 \leq i_1 \leq j$.

If $i_1 = 0$, then

$$A^{j+1,0}[n] = \partial_{j+1}\operatorname{TNR}_{\mathrm{dec}}[n] \cap \partial_0\partial_{j+1}\operatorname{TNR}[n] = \partial_0\partial_{j+1}\operatorname{TNR}_{\mathrm{dec}}[n]$$

and $\partial_0\partial_{j+1}\operatorname{TNR}_{\mathrm{dec}}[n] \hookrightarrow \partial_0\partial_{j+1}\operatorname{TNR}_{\mathrm{dec}}[n]$ is inner anodyne by Lemma 4.2.2. In general, the map $A^{j+1,i_1}[n] \hookrightarrow \partial_{i_1}\partial_{j+1}\operatorname{TNR}[n]$ fits into a commutative diagram of monomorphisms as follows:

$$\begin{tikzcd} \partial_{i_1}\partial_{j+1}\operatorname{TNR}_{\mathrm{dec}}[n] \arrow[d] \arrow[dr] & \\ A^{j+1,i_1}[n] \arrow[r] & \partial_{i_1}\partial_{j+1}\operatorname{TNR}[n]. \end{tikzcd} \tag{4.1}$$

By Lemma 4.2.3, the inclusion $A^{j,i_1}[n] \hookrightarrow \partial_{i_1}\partial_{j+1}\,\mathrm{TNR}[n]$ is inner anodyne if the vertical map in (4.1) is.

For every sequence $j \geq i_1 > i_2 > \cdots > i_{r-1} > 0$, define

$$\begin{aligned} A^{j+1,i_1,\ldots,i_{r-1},0}[n] &= \partial_{i_{r-1}}\cdots\partial_{i_1}\partial_{j+1}\,\mathrm{TNR}_{\mathrm{dec}}[n] \cap \partial_0\partial_{i_{r-1}}\cdots\partial_{i_1}\partial_{j+1}\,\mathrm{TNR}[n] \\ &= \partial_0\partial_{i_{r-1}}\cdots\partial_{i_1}\partial_{j+1}\,\mathrm{TNR}_{\mathrm{dec}}[n]. \end{aligned}$$

Then, construct the pushout of monomorphisms

$$\begin{tikzcd} A^{j+1,i_1,\ldots,i_{r-1},0}[n] \arrow[r] \arrow[d] & \partial_{i_{r-1}}\cdots\partial_{i_1}\partial_{j+1}\,\mathrm{TNR}_{\mathrm{dec}}[n] \arrow[d] \\ \partial_0\partial_{i_{r-1}}\cdots\partial_{i_1}\partial_{j+1}\,\mathrm{TNR}[n] \arrow[r] & D^{j+1,i_1,\ldots,i_{r-1},0}[n] \end{tikzcd} \tag{4.2}$$

and iterate this construction for $1 \leq i_r < i_{r-1}$, defining

$$A^{j+1,i_1,\ldots,i_{r-1},i_r}[n] = D^{j+1,i_1,\ldots,i_{r-1},i_r-1}[n] \cap \partial_{i_r}\partial_{i_{r-1}}\cdots\partial_{i_1}\partial_{j+1}\,\mathrm{TNR}[n]$$

and pushout squares

$$\begin{tikzcd} A^{j+1,i_1,\ldots,i_{r-1},i_r}[n] \arrow[r] \arrow[d] & D^{j+1,i_1,\ldots,i_{r-1},i_r-1}[n] \arrow[d] \\ \partial_{i_r}\partial_{i_{r-1}}\cdots\partial_{i_1}\partial_{j+1}\,\mathrm{TNR}[n] \arrow[r] & D^{j+1,i_1,\ldots,i_{r-1},i_r}[n]. \end{tikzcd} \tag{4.3}$$

The vertical map of (4.1) then factors as a chain of inclusions

$$\partial_{i_1}\partial_{j+1}\,\mathrm{TNR}_{\mathrm{dec}}[n] \longrightarrow D^{j+1,i_1,0}[n] \longrightarrow D^{j+1,i_1,1}[n] \longrightarrow \cdots \longrightarrow D^{j+1,i_1,i_1-1}[n] = A^{j+1,i_1}[n]$$

where the final equality holds since

$$\begin{aligned}
D^{j+1,i_1,i_1-1}[n] &= \partial_{i_1}\partial_{j+1}\,\mathrm{TNR}_{\mathrm{dec}}[n] \cup \partial_0\partial_{i_1}\partial_{j+1}\,\mathrm{TNR}[n] \cup \cdots \cup \partial_{i_1-1}\partial_{i_1}\partial_{j+1}\,\mathrm{TNR}[n] \\
&= (\partial_{j+1}\,\mathrm{TNR}_{\mathrm{dec}}[n] \cup \partial_0\partial_{j+1}\,\mathrm{TNR}[n] \cup \cdots \cup \partial_{i_1-1}\partial_{j+1}\,\mathrm{TNR}[n]) \\
&\quad \cup \partial_{i_1}\partial_{j+1}\,\mathrm{TNR}[n] \\
&= D^{j+1,i_1-1}[n] \cap \partial_{i_1}\partial_{j+1}\,\mathrm{TNR}[n] \\
&= A^{j+1,i_1}[n].
\end{aligned}$$

Each inclusion in the chain is induced by cobase change as in (4.3), and thus is inner anodyne if the left vertical map in each diagram of the form (4.3) is inner anodyne. Each such inclusion

$$A^{j+1,i_1,\ldots,i_{r-1},i_r}[n] \hookrightarrow \partial_{i_r}\partial_{i_r-1}\partial_{i_{r-1}}\cdots\partial_{i_1}\partial_{j+1}\,\mathrm{TNR}[n] \tag{4.4}$$

fits into a commutative diagram as follows:

$$\begin{tikzcd}
\partial_{i_r}\partial_{i_{r-1}}\cdots\partial_{i_1}\partial_{j+1}\,\mathrm{TNR}_{\mathrm{dec}}[n] \arrow[d] \arrow[dr] & \\
A^{j+1,i_1,\ldots,i_{r-1},i_r}[n] \arrow[r] & \partial_{i_r}\partial_{i_{r-1}}\cdots\partial_{i_1}\partial_{j+1}\,\mathrm{TNR}[n].
\end{tikzcd} \tag{4.5}$$

The diagonal map is inner anodyne by Lemma 4.2.2, and so (4.4) is inner anodyne if the vertical map in (4.5) is. Each of these vertical maps further decomposes as a chain of inclusions induced by cobase change as in the preceding:

$$\begin{tikzcd}
\partial_{i_r}\cdots\partial_{i_1}\partial_{j+1}\,\mathrm{TNR}_{\mathrm{dec}}[n] \arrow[r] & D^{j+1,i_1,\ldots,i_r,0}[n] \arrow[r] & D^{j+1,i_1,\ldots,i_r,1}[n)] \arrow[dll, rounded corners] \\
\cdots \arrow[r] & D^{j+1,i_1,\ldots,i_r,i_r-1}[n] \arrow[r, equal] & A^{j+1,i_1,\ldots,i_r}[n].
\end{tikzcd}$$

These factorizations are along strictly longer sequences $j \geq i_1 > i_2 > \cdots i_r \geq 0$, with each factorization increasing the length r by 1, until i_r is forced to be 0. Then, diagram (4.2) produces a map

$$\partial_0 \partial_{i_{r-1}} \cdots \partial_{i_1} \partial_{j+1} \operatorname{TNR}_{\mathrm{dec}}[n] \hookrightarrow \partial_0 \partial_{i_{r-1}} \cdots \partial_{i_1} \partial_{j+1} \operatorname{TNR}[n]$$

which is inner anodyne, ensuring that all vertical maps in all triangles of the form (4.5) are inner anodyne. □

Lemma 4.2.5. *The inclusion* $\partial \operatorname{TNR}_{\mathrm{dec}}[n] \hookrightarrow \partial \operatorname{TNR}[n]$ *is inner anodyne.*

Proof. The inclusion $\partial \operatorname{TNR}_{\mathrm{dec}}[n] \hookrightarrow \partial \operatorname{TNR}[n]$ factors as

$$\partial \operatorname{TNR}_{\mathrm{dec}}[n] \subset \partial \operatorname{TNR}_{\leq 0}[n] \subset \partial \operatorname{TNR}_{\leq 1}[n] \subset \cdots \subset \partial \operatorname{TNR}_{\leq n-1}[n] \subset \partial \operatorname{TNR}_{\leq 0}[n]$$

and $\partial \operatorname{TNR}_{\leq 0}[n] = \operatorname{TNR}[n]$. The inclusion $\partial \operatorname{TNR}_{\mathrm{dec}}[n] \hookrightarrow \partial \operatorname{TNR}_{\leq 0}[n]$ attaches $\partial_0 \operatorname{TNR}[n]$ along $\partial_0 \operatorname{TNR}_{\mathrm{dec}}[n]$, and is therefore inner anodyne, while the subsequent inclusions are inner anodyne by Lemma 4.2.4. □

Lemma 4.2.6. *The inclusion* $\operatorname{TNR}_{\mathrm{dec}}[n] \hookrightarrow \partial \operatorname{TNR}[n] \cup_{\partial \operatorname{TNR}_{\mathrm{dec}}[n]} \operatorname{TNR}_{\mathrm{dec}}[n]$ *is inner anodyne.*

Proof. The inclusion is the pushout of an inner anodyne map, by Lemma 4.2.5:

$$\begin{tikzcd}
\partial \operatorname{TNR}_{\mathrm{dec}}[n] \arrow[r] \arrow[d] & \operatorname{TNR}_{\mathrm{dec}}[n] \arrow[d] \\
\partial \operatorname{TNR}[n] \arrow[r] & \partial \operatorname{TNR}[n] \cup_{\partial \operatorname{TNR}_{\mathrm{dec}}[n]} \operatorname{TNR}_{\mathrm{dec}}[n].
\end{tikzcd}$$

□

Definition 4.2.7. Define the simplicial sets $\mathcal{B}ispan_{\mathrm{un}}(\mathcal{C})$ and $\mathcal{B}ispan_{\mathrm{un,dec}}(\mathcal{C})$ of *un-*

restricted and unrestricted decomposed bispan diagrams in $\mathcal{C}$ as follows:

$$\mathrm{Hom}_{\mathcal{SSet}}(\Delta[n], \mathcal{B}ispan_{\mathrm{un}}(\mathcal{C})) = \mathrm{Hom}_{\mathcal{SSet}}(\mathrm{TNR}[n], \mathcal{C})$$

$$\mathrm{Hom}_{\mathcal{SSet}}(\Delta[n], \mathcal{B}ispan_{\mathrm{un,dec}}(\mathcal{C})) = \mathrm{Hom}_{\mathcal{SSet}}(\mathrm{TNR}_{\mathrm{dec}}[n], \mathcal{C}).$$

Then, the main result will follow from the next lemma:

Lemma 4.2.8. *The restriction* $\mathcal{B}ispan_{\mathrm{un}}(\mathcal{C}) \to \mathcal{B}ispan_{\mathrm{un,dec}}(\mathcal{C})$ *is a trivial fibration of simplicial sets.*

Proof. Consider the following lifting problem:

$$\begin{tikzcd} \partial\Delta[n] \arrow[r] \arrow[d] & \mathcal{B}ispan_{\mathrm{un}}(\mathcal{C}) \arrow[d] \\ \Delta[n] \arrow[r] \arrow[ur, dashed] & \mathcal{B}ispan_{\mathrm{un,dec}}(\mathcal{C}). \end{tikzcd}$$

Providing a dashed lift is equivalent to showing the surjectivity of the set map

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{SSet}}(\Delta[n], \mathcal{B}ispan_{\mathrm{un}}(\mathcal{C})) \arrow[d] \\ \mathrm{Hom}_{\mathcal{SSet}}(\partial\Delta[n], \mathcal{B}ispan_{\mathrm{un}}(\mathcal{C})) \underset{\mathrm{Hom}_{\mathcal{SSet}}(\partial\Delta[n], \mathcal{B}ispan_{\mathrm{un,dec}}(\mathcal{C}))}{\times} \mathrm{Hom}_{\mathcal{SSet}}(\Delta[n], \mathcal{B}ispan_{\mathrm{un,dec}}(\mathcal{C})). \end{tikzcd}$$

Unwinding the definitions, this is equivalent to the surjectivity of

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{SSet}}(\mathrm{TNR}[n], \mathcal{C}) \arrow[d] \\ \mathrm{Hom}_{\mathcal{SSet}}(\partial\,\mathrm{TNR}[n], \mathcal{C}) \times_{\mathrm{Hom}_{\mathcal{SSet}}(\partial\,\mathrm{TNR}_{\mathrm{dec}}[n], \mathcal{C})} \mathrm{Hom}_{\mathcal{SSet}}(\mathrm{TNR}_{\mathrm{dec}}[n], \mathcal{C}). \end{tikzcd}$$

The right hand side is isomorphic as a simplicial set to $\mathrm{Hom}_{\mathcal{SSet}}(\partial\,\mathrm{TNR}[n] \cup_{\partial\,\mathrm{TNR}_{\mathrm{dec}}[n]}$

$\mathrm{TNR}_{\mathrm{dec}}[n], \mathcal{C})$ and so the objective is now to show the surjectivity of the restriction

$$\mathrm{Hom}_{\mathcal{SS}et}(\mathrm{TNR}[n], \mathcal{C}) \to \mathrm{Hom}_{\mathcal{SS}et}(\partial\,\mathrm{TNR}[n] \cup_{\partial\,\mathrm{TNR}_{\mathrm{dec}}[n]} \mathrm{TNR}_{\mathrm{dec}}[n], \mathcal{C}).$$

This final result follows from the surjectivity on 0-simplices of the trivial fibration

$$\mathcal{C}^{\mathrm{TNR}[n]} \xrightarrow{\sim} \mathcal{C}^{\partial\,\mathrm{TNR}[n] \cup_{\partial\,\mathrm{TNR}_{\mathrm{dec}}[n]} \mathrm{TNR}_{\mathrm{dec}}[n]}$$

which is obtained from Lemma 4.2.3 applied to the sequence of inclusions

$$\mathrm{TNR}_{\mathrm{dec}}[n] \xhookrightarrow{i} \partial\,\mathrm{TNR}[n] \cup_{\partial\,\mathrm{TNR}_{\mathrm{dec}}[n]} \mathrm{TNR}_{\mathrm{dec}}[n] \xhookrightarrow{j} \mathrm{TNR}[n].$$

□

Theorem 4.2.9. *The restriction $\mathcal{B}ispan(\mathcal{C}) \to \mathcal{B}ispan_{\mathrm{dec}}(\mathcal{C})$ is a trivial fibration of simplicial sets.*

Proof. Observe the pullback diagram:

$$\begin{tikzcd}
\mathcal{B}ispan(\mathcal{C}) \arrow[r] \arrow[d, two heads, "\sim"'] & \mathcal{B}ispan_{\mathrm{un}}(\mathcal{C}) \arrow[d, two heads, "\sim"] \\
\mathcal{B}ispan_{\mathrm{dec}}(\mathcal{C}) \arrow[r] & \mathcal{B}ispan_{\mathrm{un,dec}}(\mathcal{C}).
\end{tikzcd}$$

□

Corollary 4.2.10. *$\mathcal{B}ispan(\mathcal{C})$ is a quasicategory.*

Proof. As $\mathcal{B}ispan^{\mathrm{dec}}(\mathcal{C})$ is a quasicategory and $\mathcal{B}ispan(\mathcal{C}) \to \mathcal{B}ispan_{\mathrm{dec}}(\mathcal{C})$ is a trivial fibration, it follows immediately that $\mathcal{B}ispan(\mathcal{C})$ is also a quasicategory. □

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