Constraints on Basic Classes of Lefschetz Fibrations

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"What is your ambition? What is your personality?"

—— Father, in response to "Which path shall I choose,
mathematics or actuarial science?", April 2004

"I support any decision you make."

—— Mother, in response to "Maybe I should quit.", May 2014

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Chapter 1

Introduction

1.1 Complex surfaces and symplectic 4-manifolds

Complex surfaces, which are real 4-dimensional manifolds with complex structures, are one of the more widely studied classes of smooth 4-manifolds. The Enriques-Kodaira classification is arguably one of the central results in the study of complex surfaces. We shall briefly review it to motivate our central problem; more details can be found in [11], for example.

A complex surface S can be assigned a *(holomorphic) Kodaira dimension* κ_h, where $\kappa_h(S)$ can take values within $\{-\infty, 0, 1, 2\}$. The Enriques-Kodaira classification tells us that the topology of complex surfaces S with $\kappa_h(S) = -\infty, 0$ or 1 are rather restricted, especially with the additional assumptions of simple connectedness and/or minimality (i.e. S is not the blow-up of another complex surface). To name a few of such results:

- If $\kappa_h(S) = -\infty$ and S is simply connected and minimal, then S is either biholomorphic to $\mathbb{CP}^2$ (the 2-dimensional complex projective space) or S admits a holomorphic map $\pi : S \longrightarrow C$ to a complex curve C so that $\pi^{-1}(p)$ is diffeomorphic to $\mathbb{CP}^1$ for all $p \in C$.

- If $\kappa_h(S) = 0$ and S is simply connected and minimal, then S is a $K3$-surface.
- If $\kappa_h(S) = 1$ and S is minimal, then S is an elliptic surface.

On the other hand, complex surfaces with $\kappa_h(S) = 2$, called *general type complex surfaces*, is considered the generic category. While some particular examples have been extensively studied, such as $\Sigma_g \times \Sigma_h$ or more generally certain Σ_g-fibrations over Σ_h (where Σ_g denotes the Riemann surface of genus g and $g, h \geq 2$), the $\kappa_h(S) = 2$ category as a whole is far from well understood. For example, the *geography problem*, i.e. understanding which pairs of classical topological invariants (c_1^2, χ_h) can be realized by simply connected general type complex surfaces, is not completely solved.

This thesis aims to study general type complex surfaces from the topological lens. In particular, our long term goal is to answer the following question:

Question 1.1.1. *Is it possible to describe the topology of general type complex surfaces without relying on their complex structure?*

It is a classical result [2] that a general type complex surface S holomorphically embeds into $\mathbb{CP}^n$ for some n. In particular, S has an induced Kähler form, which is a *symplectic form* compatible with the complex structure.

Definition 1.1.2. A symplectic form ω on a smooth, oriented 4-manifold X is a smooth, closed, non-degenerate 2-form which satisfies $\omega^2 > 0$.

Symplectic structures capture the global geometric-topological features of smooth 4-manifolds. On the flip side, any symplectic form is locally standard due to a Darboux-type theorem.

For a minimal Kähler surface S with symplectic form ω and the corresponding canonical coholomogy class K_ω (defined at the end of Section 2.5), it is known [29] that the signs of $K_\omega \cdot [\omega]$ and $K_\omega \cdot K_\omega$ determine the holomorphic Kodaira dimension of S. As a result one can assign a *symplectic Kodaira dimension* κ_s to a minimal symplectic 4-manifold based on the signs of $K_\omega \cdot [\omega]$ and $K_\omega \cdot K_\omega$. It follows that the notion of κ_s is an extension of the notion of κ_h for minimal Kähler surfaces.

Analogously to the Enriques-Kodaira classification, there have been ongoing schemes of classifying symplectic 4-manifolds by their symplectic Kodaira dimensions. Lists of manifolds with $\kappa_s = -\infty$ and 0, or conjectures of such lists, have been given and are similar to the lists for $\kappa_h = -\infty$ and 0, while the $\kappa_s = 1$ category starts to behave more wildly; recent progress and conjectures can be found in a survey paper by Li [17]. The category $\kappa_s = 2$, analogously named *general type symplectic 4-manifolds*, is also much less understood than the $\kappa_s = -\infty, 0, 1$ categories.

Since all minimal $\kappa_h = 2$ complex surfaces are Kähler and hence symplectic, it follows that the $\kappa_s = 2$ category is at least as wild as the $\kappa_h = 2$ category. This also makes the $\kappa_s = 2$ category a natural candidate of topologically characterizing $\kappa_h = 2$ manifolds. However, we shall see in Section 1.2 that there are $\kappa_s = 2$ manifolds whose *4-manifold invariants* do not behave like those of $\kappa_h = 2$ manifolds.

1.2 4-manifold invariants

Starting from the 1980s, powerful invariants were introduced into the study of 4-manifolds ([4], [30]). The two that are most relevant to this thesis are the *Seiberg-Witten invariant* (SW) and the *Ozsváth-Szabó 4-manifold invariant* (Φ); the latter will be discussed in detail in Section 2.8. The Seiberg-Witten invariant of a 4-manifold is defined by counting solutions of certain differential equations on the 4-manifold, and has been widely used to distinguish pairs of homeomorphic smooth 4-manifolds which in fact carry distinct smooth structures. In the same spirit, the Ozsváth-Szabó invariant is defined by counting holomorphic triangles in an associated complex manifold. It is a longstanding conjecture that the two invariants are identical.

Let X be a closed, oriented 4-manifold satisfying $b_2^+(X) \geq 1$, where $b_2^+(X)$ is the maximal rank of a positive definite subgroup of $H_2(X)$ under the intersection pairing. The invariants SW_X and Φ_X are functions from the set of spin^c structures on X (which can be identified with $H^2(X)$) to the set of integers:

$$SW_X, \Phi_X : \mathrm{Spin}^c(X) \longrightarrow \mathbb{Z}$$

We shall denote $SW_X(s)$ and $\Phi_X(s)$ by $SW_{X,s}$ and $\Phi_{X,s}$ respectively. A spin^c structure s where $SW_{X,s}$ (resp. $\Phi_{X,s}$) is non-zero is called a *Seiberg-Witten basic class* (resp. *Ozsváth-Szabó basic class*) of X. It is also a basic property that SW and Φ are invariant up to sign under conjugation.

We have the following theorem regarding the Seiberg-Witten invariant of minimal general type complex surfaces (see the end of Section 2.5 for a discussion on canonical spin^c structures):

Theorem 1.2.1. *[30] Let X be a minimal general type complex surface X satisfying $b_2^+(X) > 1$. Then X has only two Seiberg-Witten basic classes, namely the canonical structure and its conjugate.*

On the other hand, minimal general type symplectic 4-manifolds can have multiple Seiberg-Witten basic classes, and such examples were constructed by Fintushel and Stern [7].

For the Ozsváth-Szabó picture, recall that all symplectic 4-manifolds admit *Lefschetz fibrations* (see Definition 3.1.1) possibly after blow-ups, which is a consequence of a celebrated theorem by Donaldson [5]. The present work shall study symplectic 4-manifolds through Lefschetz fibrations.

For symplectic 4-manifolds whose symplectic structures are supported by a relatively minimal genus g Lefschetz fibration with $g > 1$, Ozsváth and Szabó showed that the canonical spin^c structure is always an Ozsváth-Szabó basic class, but there might be other basic classes:

Theorem 1.2.2. *([24], Theorem 5.1) Let $\pi : X \to S^2$ be a relatively minimal genus g Lefschetz fibration over the sphere with $b_2^+(X) > 1$ and $g > 1$, and denote a generic* *fiber by* Σ*. Then for the canonical spin*^{c} *structure* k*, we have*

$$\begin{aligned}\langle c_1(k), [\Sigma]\rangle &= 2-2g,\\ \Phi_{X,k} &= \pm 1.\end{aligned}$$

Moreover, for any other spin^c *structure* $s \neq k$ *with* $\Phi_{X,s} \neq 0$*, we have*

$$|\langle c_1(s), [\Sigma]\rangle| < |\langle c_1(k), [\Sigma]\rangle| = 2g-2. \tag{1.1}$$

All in all, we see that the notion of general type symplectic 4-manifolds does not capture the behaviour of minimal general type complex surfaces as far as 4-manifolds are concerned, both from Fintushel and Stern's examples and from Theorem 1.2.2 that symplectic 4-manifolds may have (non-canonical) basic classes satisfying Equation 1.1. This motivates us to ask the following question:

Question 1.2.3. *Is it possible to impose an additional geometric-topological condition on a minimal general type symplectic 4-manifold* X *which would guarantee that* X *has only one Seiberg-Witten or Ozsváth-Szabo basic class up to conjugation?*

1.3 Main results

One can see Theorem 1.2.1 as saying that if a minimal complex surface is sufficiently complicated (in the sense that it is general type), then it has the simplest possible set of Seiberg-Witten basic classes. Analogously, one can ask if there is a notion of "sufficiently complicated" Lefschetz fibration which would guarantee that the Lefschetz fibration has only the canonical spin^c structure and its conjugate as Ozsváth-Szabó basic classes.

A tool available to us is cut-and-paste. In particular, Jabuka and Mark [16] showed that if X is a 4-manifold which decomposes into U and $X - U$, where U is a codimension-0 submanifold with boundary, then under mild conditions on U, the absolute invariant Φ can be seen as a certain pairing of the *relative invariants* Ψ_U and Ψ_{X-U}; see Theorem 2.8.4 for a more precise statement. In particular, the vanishing of $\Psi_{U,s}$ for a spin^c structure s on U would force $\Phi_{X,s'}$ to vanish for any spin^c structure s' on X that restricts to s on U. Therefore, if we can find a Lefschetz fibration U with boundary so that $\Psi_{U,s} \neq 0$ only when s is the restriction of the canonical spin^c structure or its conjugate, then (with the help of Theorem 1.2.2) any closed symplectic 4-manifold X admitting a relatively minimal Lefschetz fibration and containing U as a Lefschetz subfibration must have the canonical spin^c structure and its conjugate as the only Ozsváth-Szabó basic classes.

We summarize the discussion with the formal statement of our main result:

Theorem 1.3.1. *For $g = 4$ and 5, there exists a genus g Lefschetz fibration U_g over D^2 with regular fiber Σ, so that the relative invariant $\Psi_{U_g,s}$ vanishes for all spin^c structures satisfying $|\langle c_1(s), [\Sigma]\rangle| < 2g - 2$.*

Corollary 1.3.2. *Let X be a closed, oriented symplectic 4-manifold which admits a* *minimal genus g Lefschetz fibration over S^2, with $g = 4$ or 5. If X contains U_g as a Lefschetz subfibration and $b_2^+(X - U_g) \geq 1$, then the Ozsváth-Szabo 4-manifold invariant $\Phi_{X,s}$ vanishes unless s is the canonical spin^c structure of X (or its conjugate).*

Conjecture 1.3.3. *Theorem 1.3.1 and Corollary 1.3.2 hold for all $g \geq 2$.*

A natural candidate of such a U_g would be a Lefschetz subfibration of a well-understood general type complex surface. Our choice of such complex surface is $U(g+1, n)$, defined at the beginning of Section 3.3, which is a classical example from algebraic geometry. Then $U(g+1, n)$ is a minimal complex surface of general type for all $g \geq 2$ and $n \geq 2$. Furthermore, $U(g+1, n)$ admits a genus g singular fibration which can be perturbed into a genus g Lefschetz fibration. We will choose U_g in such a manner that $U(g+1, n)$ contains the Lefschetz subfibration U_g for any $n \geq 2$.

With our choice of U_g, our main result immediately verifies the $SW = \Phi$ conjecture for $U(m, n)$ with $m = 5, 6$ and $n \geq 2$. To the best of the author's knowledge, this is the first verification of the conjecture in the case of a general type complex surface.

The organization of the thesis is as follows. Section 2 reviews the necessary background in Heegaard Floer homology, in particular defining the absolute and relative invariants. Section 3 discusses in detail the topology of $U(g+1, n)$ and U_g. Section 4 supplies the proofs of Theorem 1.3.1 and Corollary 1.3.2. Section 5 is a further application of Jabuka and Mark's pairing theorem in more general gluing situations.

Chapter 2

Background

Throughout this present work, all 3-manifolds considered are connected, closed and oriented.

2.1 Integer surgeries

A *knot* in a 3-manifold Y is a (smooth) embedding of $S^1 \hookrightarrow Y$. A *framing* of a knot $K \subset Y$ is an identification, up to homotopy, of a tubular neighourhood $nbd(K)$ of K with $S^1 \times D^2$, which is equivalent to a choice of parallel copy K' of K. If Y is the 3-sphere $S^3 = \mathbb{R}^3 \cup \{\infty\}$ (or more generally, if K is a null-homologous knot in any Y), then we can represent framings with integers which we call *framing coefficients*, annotated next to K.

Definition 2.1.1. If $K \subset S^3$ is a knot with framing determined by a parallel copy K', then the framing coefficient of K is defined to be $lk(K, K')$, the linking number of K and K', with standard sign convention in Figure 2.1. More generally, if K is a null-homologous knot in a 3-manifold Y, the framing coefficient is defined to be the intersection number of K' with any Seifert surface of K.

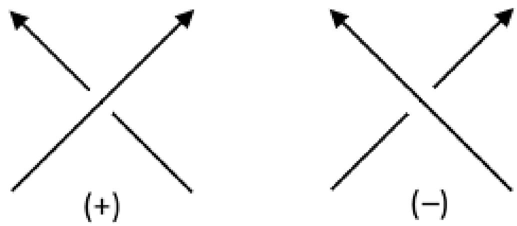

Figure 2.1: Signs of crossings in a link diagram in S^3.

Remark 2.1.2. Another natural framing coming from a knot diagram is the blackboard framing, namely the longitude determined by a unit normal vector field of K which lies entirely on the paper; see Figure 2.2. The framing coefficient of the blackboard framing determined by a diagram of K equals the writhe of the diagram of K.

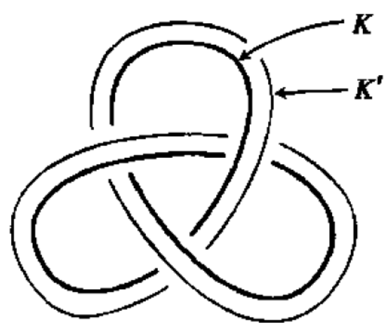

Figure 2.2: Right-handed trefoil knot K and its blackboard framing K'. Diagram is taken from [11], p.118.

Definition 2.1.3. Let K be a knot in any 3-manifold Y with framing determined by K'. Then the *surgery* along K is defined to be the 3-manifold $Y' = (Y - nbd(K)) \cup$

$(S^1 \times D^2)$, where the meridian $\{*\} \times S^1 = \{*\} \times \partial D^2$ of $S^1 \times D^2$ is glued to $\partial(Y - nbd(K))$ along K'. Whenever the framing coefficient of K is defined and equals n, the surgery is also called an n-surgery.

Definition 2.1.4. Continuing from the previous definition, the core of the surgery torus, $S^1 \times \{0\} \subset S^1 \times D^2 \subset Y'$, is called the *induced knot* in Y'.

2.2 Handles and handle diagrams

We shall discuss handles and handle diagrams for 4-manifolds. First, we may assume a connected 4-manifold has only one 0-handle, a 4-ball D^4 whose boundary is the 3-sphere S^3, where S^3 represented by the empty diagram. Then, attaching a 1-handle to a 0-handle is by definition gluing $D^1 \times D^3$ to $\partial D^4 = S^3$ along the attaching region which is a pair of 3-balls $D^3 \sqcup D^3$. We shall draw the pair of 3-balls in the diagram so that they are identified by the reflection about the plane perpendicularly bisecting the segments joining their centers (see Figure 2.3); note that the 1-handle is invisible from the picture except the attaching region.

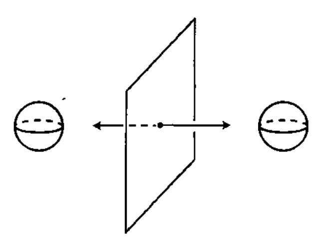

Figure 2.3: Gluing map to attach a 1-handle to D^4. Diagram is from [11], p.115.

Next, attaching a 2-handle means gluing $D^2 \times D^2$ to the union of the 0-handle and the 1-handles along $\partial D^2 \times D^2 = S^1 \times D^2$ with a certain framing. The circle $S^1 \times \{*\} \subset S^1 \times D^2$ is called the *attaching circle* and the disk $\{*\} \times D^2 \subset D^2 \times D^2$ is called *core* of the handle. We shall draw the attaching circle into the diagram, part of it possibly going through the 1-handles. We shall also extend the definition of framing coefficient by defining that the framing coefficient of the blackboard framing is the writhe (cf. Remark 2.1.2). Finally, given the union of a 0-handle, some 1-handles and some 2-handles, the extension to a closed 4-manifold by attaching 3-handles and 4-handles is unique, if such an extension exists. Therefore, we do not need to keep track of the 3-handles and 4-handles if the 4-manifold is closed (see [11], Section 4.4). See Figure 2.4 for an example.

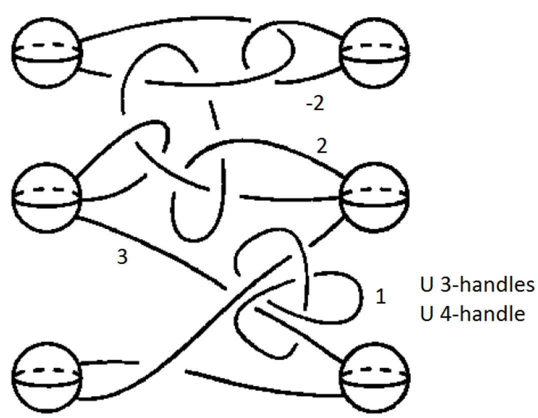

Figure 2.4: The handle diagram of a closed 4-manifold. Diagram is taken from [11], p.116 and modified.

We also have the following standard fact:

Fact 2.2.1. *A link diagram in S^3 labeled with integer coefficients simultaneously represents a handle diagram of a 4-manifold (with a single 0-handle and some 2-handles) and a surgery diagram of the boundary of the same 4-manifold.*

We recall the basics about *cobordisms* which will be ubiquitous in the present work. For an oriented 3-manifold Y, we use $-Y$ to denote Y with the opposite orientation.

Definition 2.2.2. We say that an oriented 4-manifold W is a *cobordism* from a 3-manifold Y_1 to another 3-manifold Y_2 if $\partial W = -Y_1 \sqcup Y_2$.

In particular, if Y is any 3-manifold and K is a framed knot, the 4-manifold

$$W = (Y \times I) \cup (D^2 \times D^2),$$

where the union means attaching the 2-handle $D^2 \times D^2$ to $Y \times \{1\}$ along K, will be called *the cobordism induced by attaching the 2-handle*. Then W is a cobordism from Y to Y', where Y' is the 3-manifold obtained from the surgery along K.

2.3 Surfaces and mapping class groups

This section provides the necessary preliminaries to discuss the topology of Lefschetz fibrations. Let $\Sigma_{g,r}$ denote the genus g surface with r boundary components. To simplify notations, $\Sigma_{g,0}$ will be written as Σ_g, or most frequently Σ if the genus is clear from the context. The *mapping class group* $MCG(\Sigma_{g,r})$ is the group of isotopy classes of orientation-preserving diffeomorphisms of $\Sigma_{g,r}$ fixing the boundary.

For a simple closed curve α in $\Sigma_{g,r}$, the *right-handed Dehn twist* along α is the self-diffeomorphism of Σ supported in a neighbourhood of α as shown in Figure 2.5. The right-handed Dehn twist is denoted by D_α, even though we will usually just write α if there is no possibility of confusion. A *left-handed Dehn twist* is the inverse of a right-handed Dehn twist. It is well known that $MCG(\Sigma_{g,r})$ is generated by Dehn twists.

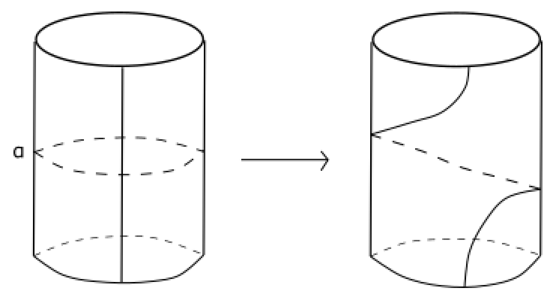

Figure 2.5: A right-handed Dehn twist about the curve α.

A *word* in $MCG(\Sigma_{g,r})$ is a sequence whose letters are right-handed Dehn twists, denoted by

$$\alpha_1\alpha_2\cdots\alpha_n.$$

The corresponding mapping class in $MCG(\Sigma_{g,r})$ is the composition of these Dehn twists, denoted by

$$\alpha_1 \circ \alpha_2 \circ \cdots \circ \alpha_n.$$

We also have the following definitions for future reference:

Definition 2.3.1. Let $\phi : \Sigma_{g,r} \longrightarrow \Sigma_{g,r}$ be a diffeomorphism. Then the *mapping* *torus* $M(\phi)$ is defined to be the quotient space

$$M(\phi) = (\Sigma_{g,r} \times [0,1]) \;/\; (\phi(x), 0) \sim (x, 1).$$

The map ϕ is called the *monodromy of* $M(\phi)$.

Definition 2.3.2. With the same notations of the previous definition, we can form the *(abstract) open book* which is the closed 3-manifold defined by

$$Y(\phi) = M(\phi) \cup_{id} (\partial\Sigma_{g,r} \times D^2),$$

i.e. the manifold obtained by gluing $M(\phi)$ and r solid tori $\partial\Sigma_{g,r} \times D^2$ along their boundary, using the identity map to identify $\partial M(\phi) = \partial\Sigma_{g,r} \times S^1$ with $\partial(\partial\Sigma_{g,r} \times D^2)$.

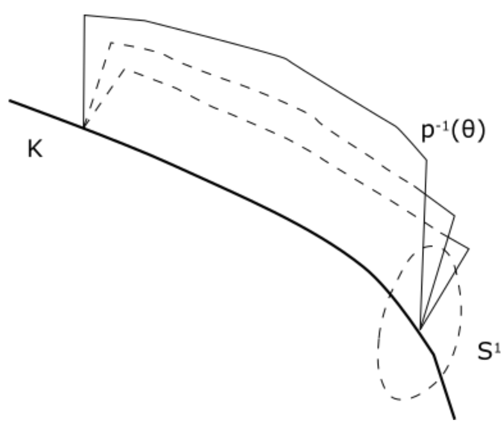

Figure 2.6: An open book near one component K of the binding.

The core link $L = \partial\Sigma_{g,r} \times \{*\}$ of $\partial\Sigma_{g,r} \times D^2$ is called the *binding* of the open book. Each copy of $\Sigma_{g,r}$ in $M(\phi)$ extends to $\partial\Sigma_{g,r} \times D^2$ to form a *page* (or *fiber*) whose boundary is L. Thus, the fibration $(\partial\Sigma_{g,r} \times D^2) - L \longrightarrow S^1$, sending a point to the angular coordinate of the D^2-factor, extends to a locally trivial fibration $p : Y(\phi) - L \longrightarrow S^1$, so that $p^{-1}(\theta) \cup L$ is exactly a page of the open book for any $\theta \in S^1$.

In this situation, we also say that L is *fibered in* $Y(\phi)$.

It is clear from the definition that performing a page-framed surgery along each component of the binding of $Y(\phi)$ yields $M(\phi')$ where ϕ' is the image of ϕ under the map $MCG(\Sigma_{g,r}) \longrightarrow MCG(\Sigma_g)$.

2.4 Branched covers

We discuss the construction of new manifolds from old by taking *branched cover*. A branched cover $X \longrightarrow Y$ can be thought of as an honest covering map except at a codimension-2 subset, where the behavior of the map is also standard. More precisely,

Definition 2.4.1 ([11], Definition 6.3.1)**.** Let M and N be n-dimensional manifolds. A d-fold branched covering is a smooth map $f : X \longrightarrow Y$ with critical set $B \subset Y$ called the *branch locus*, such that $f|_{X-f^{-1}(B)} : X - f^{-1}(B) \longrightarrow Y - B$ is a covering map of degree d, and for each $p \in f^{-1}(B)$ there are local coordinate charts $U, V \longrightarrow \mathbb{C} \times \mathbb{R}^{n-2}_{+}$ about p and $f(p)$ on which f is given by $(z, x) \mapsto (z^m, x)$ for some positive integer m. We say that the branched cover is *cyclic* if $f|_{X-f^{-1}(B)}$ is a cyclic covering.

We shall describe a method by Akbulut and Kirby [1] which constructs an n-fold cyclic branched cover of D^4 along the Seifert surface F of a link in $\partial D^4 = S^3$ with the interior of F pushed into the interior of D^4 (while keeping the boundary of F

fixed in $\partial D^4 = S^3$). First we cut D^4 along the track of the isotopy which pushed the interior of F into the interior of D^4. The result is again D^4 with a thickened copy of F in S^3 given by

$$\overline{F} = \{(x,t) \in F \times [-1,1] | (x,t) \sim (x,t') \text{ for } x \in \partial F \text{ and all } t,t' \in [-1,1]\}.$$

Define $\overline{F}_i^{\pm} = \{(x,t) \in \overline{F} | \pm t \geq 0\}$. Then we can construct the n-fold cyclic branched cover X by gluing together n copies of D^4, namely D_i^4 (for $1 \leq i \leq n$), by the homeomorphisms $h_i : \overline{F}_i^+ \longrightarrow \overline{F}_{i+1}^-$ for $1 \leq i \leq n-1$ defined by $h_i(x,t) = (x,-t)$. The result can be schematically represented by Figure 2.7. Note that it is not necessary to glue $\overline{F}_p^+$ to $\overline{F}_1^-$ because this does not change X up to homeomorphism (for the same reason that cutting D^4 along the track of the isotopy above does not change D^4 up to homeomorphism).

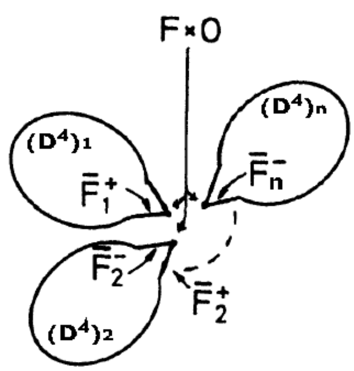

Figure 2.7: A schematic picture of the branched cover. Figure is adapted from [1], p.113.

It is clear from the construction that X is an n-fold cyclic branched cover of D^4

along $F \times 0$, and that ∂X is an n-fold cyclic branched cover of S^3 along ∂F. We further observe that if ∂F is fibered and F is a fiber of the corresponding open book $Y(\phi)$ (see Definition 2.3), then ∂X is simply the result of "stacking" the mapping tori together, and it is clear that ∂X is diffeomorphic to $Y(\phi^n)$.

In Section 3.3, we will describe Akbulut-Kirby's algorithm of producing a handle diagram of X in the specific example of interest.

2.5 Spin^c structures

There are numerous equivalent formulations of Spin^c structures, and for the purpose of working with Heegaard Floer homology, we shall adopt the ones from [23], but slightly reformulated.

Definition 2.5.1. Let Y be an oriented 3-manifold. Two oriented plane fields on Y are said to be homologous if they are homotopic in the complement of finitely many disjoint 3-balls of Y. A spin^c structure of Y is a homology class of oriented plane fields on Y, and the set of spin^c structures of Y is denoted by spin$^c(Y)$. If a spin^c structure t is represented by an oriented plane field whose (cooriented) normal vector field is v, then the conjugate spin^c structure $\bar{t}$ is defined to be the spin^c structure whose normal vector field is represented by $-v$, which is equivalent to reversing the orientation on each plane.

There is a natural map $spin^c(Y) \longrightarrow H^2(Y)$ given by $t \mapsto c_1(t)$, where $c_1(t)$ is the first Chern class of any oriented plane field that represents the spin^c structure t. If $H^2(Y)$ has no 2-torsion (which is true in all of our cases of interest), then this map is injective. We say that t is torsion if $c_1(t)$ is a torsion element in $H^2(Y)$. It also follows from the definitions that $c_1(\bar{t}) = -c_1(t)$.

To define spin^c structures on 4-manifolds, we need to recall the definition of almost-complex structures.

Definition 2.5.2. Let X be a 4-manifold and TX be the tangent bundle of X. An almost complex structure on X is a smooth, fiberwise linear map $J : TX \longrightarrow TX$ covering id_X such that $J^2 = -id_{TX}$.

Definition 2.5.3. Let X be a 4-manifold. We consider (J, P), where $P \subset X$ is a collection of finitely many points in X and J is an almost complex structure on $X - P$. We say that two pairs (J_1, P_1) and (J_2, P_2) are homologous if there is a compact 1-manifold with boundary $C \subset X$ containing $P_1 \cup P_2$ so that $J_1|_{X-C}$ and $J_2|_{X-C}$ are homotopic. A spin^c structure of X is a homology class of such pairs, and the set of spin^c structures of X is denoted by spin$^c(X)$.

Given $s \in spin^c(X)$ represented by (J, P), the first Chern class of the induced complex tangent bundle on $X - P$ canonically extends to give a class in $H^2(X)$, which we define to be the first Chern class $c_1(s)$ of s. Just like the case for 3-manifolds, there is a natural map $c_1 : s \mapsto c_1(s)$, and the map is injective if $H^2(X)$ has no 2-torsion (which is true in all of our cases of interest).

The *restriction* of s on Y is the spin^c structure on Y whose orthogonal complement is homotopic to the oriented plane field $TY \cap J(TY)$ on Y. It follows from the definitions that we have the following commutative diagram, where the vertical maps denote restrictions:

$$\begin{array}{ccc} spin^c(X) & \xrightarrow{c_1} & H^2(X) \\ \downarrow & & \downarrow \\ spin^c(Y) & \xrightarrow{c_1} & H^2(Y) \end{array}$$

We shall explain what it means by the *canonical* spin^c structure on a complex surface or a symplectic 4-manifold, which was alluded in the introductory chapter. If X is a complex manifold, then the canonical spin^c structure of X is just the one induced by its complex structure. If X is a symplectic 4-manifold with symplectic form ω, then it is known that X admits a compatible almost-complex structure J (i.e. J that satisfies $\omega(Jv_1, Jv_2) = \omega(v_1, v_2)$ for all vectors v_1, v_2), and moreover, the space of such J is contractible. Therefore we may define the canonical Spin^c structure of X to be the one induced by any such J. If X is Kähler, then the complex structure and the symplectic structure induce the same spin^c structure.

2.6 Basics of Heegaard Floer homology

Heegaard Floer homology assigns a 3-manifold Y to various flavors of abelian groups

$$\widehat{HF}(Y),\ HF^{\infty}(Y),\ HF^{+}(Y),\ HF^{-}(Y),$$

and we shall denote any one of them by $HF^\circ(Y)$. These groups are split by spin^c structures:

$$HF^\circ(Y) = \bigoplus_{t \in Spin^c(Y)} HF^\circ(Y,t).$$

These groups admit group actions by U and $H_1(Y)/Tor$ (where Tor stands for the torsion part), and so could be thought of as $(\mathbb{Z}[U] \otimes \Lambda^* H_1(Y)/Tor)$-modules, where $\mathbb{Z}[U]$ is the polynomial ring in U with integer coefficients and $\Lambda^* H_1(Y)/Tor$ is the exterior algebra generated by $H_1(Y)/Tor$. For any 3-manifold Y, $HF^+(Y,t)$ and $\widehat{HF}(Y,t)$ is non-trivial only for finitely many t. Also, $HF^\infty(Y,t)$ and $HF^-(Y,t)$ are finitely generated $\mathbb{Z}[U,U^{-1}]$-modules, and $\widehat{HF}(Y,t)$ is a finitely generated $\mathbb{Z}$-module.

We will also consider *Heegaard Floer groups with twisted coefficients.* These groups are denoted $\underline{HF}^\circ(Y,t;M)$, where M is a $\mathbb{Z}[H^1(Y)]$-module. If $M = \mathbb{Z}$ where the action of $\mathbb{Z}[H^1(Y)]$ on M is trivial, then $\underline{HF}^\circ(Y,t;M)$ is the same as the "untwisted" groups $HF^\circ(Y,t)$.

Heegaard Floer groups are invariant under conjugation of spin^c structures. In other words,

$$HF^\circ(Y,t) \cong HF^\circ(Y,\bar{t}). \tag{2.1}$$

If s is a spin^c structure on a cobordism W from Y_1 to Y_2, then there exist cobordism maps [26]

$$F^\circ_{W,s} : HF^\circ(Y_1,s_1) \to HF^\circ(Y_2,s_2),$$

where t_i is the restriction of s to Y_i. There are twisted variants $\underline{F}^\circ_{W,s}$ of these maps between the twisted groups. If the coefficient modules of $HF^\circ(Y_i,t_i)$ are both the trivial module $\mathbb{Z}$, then $\underline{F}^{\circ}_{W,s}$ is the same as $F^{\circ}_{W,s}$. There is also a variant that takes into account the $H_1(W)$-action, but we will not need it.

There are U-equivariant long exact sequences relating the different flavors of Heegaard Floer groups which respect cobordism maps:

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & HF^-(Y_1,t_1) & \xrightarrow{\iota} & HF^\infty(Y_1,t_1) & \xrightarrow{\pi} & HF^+(Y_1,t_1) & \xrightarrow{\tau} & HF^-(Y_1,t_1) & \longrightarrow & \cdots \\ & & \Big\downarrow{\scriptstyle F^-_{W,s}} & & \Big\downarrow{\scriptstyle F^\infty_{W,s}} & & \Big\downarrow{\scriptstyle F^+_{W,s}} & & \Big\downarrow{\scriptstyle F^-_{W,s}} & & \\ \cdots & \longrightarrow & HF^-(Y_2,t_2) & \xrightarrow{\iota} & HF^\infty(Y_2,t_2) & \xrightarrow{\pi} & HF^+(Y_2,t_2) & \xrightarrow{\tau} & HF^-(Y_2,t_2) & \longrightarrow & \cdots \end{array} \tag{2.2}$$

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & \widehat{HF}(Y_1,t_1) & \longrightarrow & HF^+(Y_1,t_1) & \xrightarrow{U} & HF^+(Y_1,t_1) & \longrightarrow & \widehat{HF}(Y_1,t_1) & \longrightarrow & \cdots \\ & & \Big\downarrow{\scriptstyle \widehat{F}_{W,s}} & & \Big\downarrow{\scriptstyle F^+_{W,s}} & & \Big\downarrow{\scriptstyle F^+_{W,s}} & & \Big\downarrow{\scriptstyle \widehat{F}_{W,s}} & & \\ \cdots & \longrightarrow & \widehat{HF}(Y_2,t_2) & \longrightarrow & HF^+(Y_2,t_2) & \xrightarrow{U} & HF^+(Y_2,t_2) & \longrightarrow & \widehat{HF}(Y_2,t_2) & \longrightarrow & \cdots \end{array} \tag{2.3}$$

There are analogous versions of these sequences for twisted coefficients as well.

In the long exact sequence (2.2), the map τ is the connecting homomorphism. From this we define the *reduced* Heegaard Floer groups $HF^-_{red}(Y,t) = \ker \iota$ and $HF^+_{red}(Y,t) = HF^+(Y,t)/\operatorname{im} \pi$. It follows from exactness of (2.2) that τ maps $HF^+_{red}(Y,t)$ isomorphically into $HF^-_{red}(Y,t)$. It can be shown that $\ker(\iota_*) = \ker(U^k)$ for sufficiently large k, and therefore, $HF^-_{red}(Y,t)$ (hence $HF^+_{red}(Y,t)$) is actually finitely generated over $\mathbb{Z}$.

As a fundamental example which will be useful later on, we have the following proposition concerning the Heegaard Floer groups of S^3. Notice that S^3 has only one spin^c structure, and so we drop it from the notation.

Proposition 2.6.1. *We have the following $\mathbb{Z}[U]$-module isomorphisms*

1. $\widehat{HF}(S^3) = \mathbb{Z}$
2. $HF^\infty(S^3) = \mathbb{Z}[U, U^{-1}]$
3. $HF^-(S^3) = U \cdot \mathbb{Z}[U] = \langle U, U^2, U^3, \ldots \rangle$
4. $HF^+(S^3) = \mathbb{Z}[U, U^{-1}]/U \cdot \mathbb{Z}[U] = \langle 1, U^{-1}, U^{-2}, U^{-3}, \ldots \rangle$

Furthermore, under these identifications, the maps ι and π defined in long exact sequence (2.2) are the obvious inclusion map and quotient map respectively. In particular, ι is injective, so $HF^-_{red}(S^3)$ and $HF^+_{red}(S^3)$ are trivial. Also, the map $\widehat{HF}(S^3) \longrightarrow HF^+(S^3)$ in the long exact sequence (2.3) sends the generator of $\widehat{HF}(S^3)$ to 1.

Cobordism maps obey the following composition law:

Theorem 2.6.2. *([26], Theorem 3.4) Let W_1 and W_2 be a pair of cobordisms so that $\partial W_1 = -Y_1 \cup Y_2$ and $\partial W_2 = -Y_2 \cup Y_3$, and $W = W_1 \cup_{Y_2} W_2$ be the composite cobordism. For $i = 1, 2$, let s_i be spin^c structures on W_i so that $s_1|_{Y_2} = s_2|_{Y_2}$. Then*

$$F^\circ_{W_2,s_2} \circ F^\circ_{W_1,s_1} = \sum_{\{s \in spin^c(W), s|_{W_1} = s_1, s|_{W_2} = s_2\}} \pm F^\circ_{W,s}. \tag{2.4}$$

There is again an analogous statement for twisted coefficients. The cardinality of the index of the summation can be thought of as the indeterminacy of extending s_1 on W_1 and s_2 on W_2 to s on the whole W.

We shall extensively utilize a variety of gradings on Heegaard Floer groups. In particular, if t is a torsion spin^c structure on Y, then $HF^\circ(Y, t)$ carry an absolute $\mathbb{Q}$-valued grading [26]. The grading is characterized by the following properties:

(1) The element $1 \in HF^+(S^3)$ (see Proposition 2.6.1) has grading 0.

(2) The maps ι and π in the long exact sequence (2.2) preserves grading, the map τ decreases grading by 1 and the U-action decreases grading by 2.

(3) If (W, s) is a cobordism from (Y_1, t_1) to (Y_2, t_2) and both $c_1(t_1), c_1(t_2)$ are torsion, then the map $F^\circ_{W,s}$ is homogeneous and shifts grading by

$$\frac{(c_1(s))^2 - 2\chi(W) - 3\sigma(W)}{4}, \tag{2.5}$$

where $(c_1(s))^2$ is the square of the first Chern class of s, $\chi(W)$ is the Euler characteristic of W and $\sigma(W)$ is the signature of W.

For a general spin^c structure t on Y, regardless of whether t is torsion or not, the groups $HF^\circ(Y, t)$ carries a relative $\mathbb{Z}/d\mathbb{Z}$-grading, where d is the divisibility of $c_1(t)$ (and d is 0 if $c_1(t)$ is torsion). Huang and Ramos lifted this relative grading to an absolute grading by homotopy classes of oriented plane fields [15], which will be discussed and utilized in Section 5.

There is also a pairing of Heegaard Floer groups:

$$\langle\ ,\ \rangle : HF^+(Y, t) \otimes HF^-(-Y, t) \longrightarrow \mathbb{Z}. \tag{2.6}$$

This gives us an isomorphism

$$HF^+(Y, t) \cong HF_-(-Y, t), \tag{2.7}$$

where $HF_-(-Y, t)$ denotes the Heegaard Floer cohomology group. In fact, if t is a torsion spin^c structure, the pairing gives the following refinement with respect to the $\mathbb{Q}$-valued grading:

$$HF_n^+(Y,t) \cong HF_-^{-n-2}(-Y,t), \tag{2.8}$$

where the subscript n and superscript $-n-2$ denote subgroups with those gradings. In our applications, all Heegaard Floer groups are torsion-free, so $HF_-^{-n-2}(-Y,t) \cong HF^-_{-n-2}(-Y,t)$ by universal coefficients.

2.7 Adjunction relation

It is generally difficult to calculate a cobordism map $F^\circ_{W,s}$, but we have the following criterion on a spin^c structure s which relates the maps $F^\circ_{W,s}$ and $F^\circ_{W,s'}$ for two different spin^c structures. In our application, we use this criterion heavily to derive conditions on s which guarantee that $F^\circ_{W,s}$ vanishes. The statement is simplified for our application where $H_1(W) = 0$; for the general statement, see [24].

Theorem 2.7.1 (Adjunction Relation)**.** *Let W be a cobordism from Y_1 to Y_2. If Σ is any smoothly embedded, connected, oriented genus m surface in W and s is a spin^c structure on W satisfying*

$$\langle c_1(s), [\Sigma]\rangle - [\Sigma]\cdot[\Sigma] = -2m, \tag{2.9}$$

then we have the relation:

$$F^\circ_{W,s}(\cdot) = F^\circ_{W,s+\epsilon PD[\Sigma]}(U_m \otimes \cdot), \tag{2.10}$$

where ϵ is the sign of $\langle c_1(s), [\Sigma]\rangle$.

It is worth noting that the adjunction relations holds for Σ with any genus, including genus 0.

2.8 Absolute invariant and relative invariant

The goal of this section is to define Ozsváth-Szabó 4-manifold invariant [26] mentioned in the introduction (also called the *absolute invariant*) and the *relative invariant* defined by Jabuka and Mark [16] which helps calculating the absolute invariant. For this section, similar to our discussion of cobordism maps, we shall make the simplification $H_1(W) = 0$ which will be sufficient for our applications.

First we have this definition which is useful for defining the absolute invariant:

Definition 2.8.1. Let W be a cobordism from Y_1 to Y_2 with $b_2^+(W) \geq 2$. A 3-manifold $N \subset W$ is called an *admissible cut* if N cuts W into two pieces W_1, W_2 so that

(1) $b_2^+(W_1), b_2^+(W_2) \geq 1$, and

(2) $\delta H^1(N) = 0$ in $H^2(W, \partial W)$, where δ is the connecting map in the Mayer-Vietoris sequence.

Remark 2.8.2. .

1. The condition $b_2^+(W) \geq 2$ guarantees that an admissible cut exists.
2. Some motivation of condition (2) is as follows. Given spin^c structure s_i on W_i that have common restriction on N, Mayer-Vietoris argument shows that there exists a spin^c structure s on W that restricts to s_i on W_i. However, such s is not unique in general; $\delta H^1(N)$ is exactly the indeterminacy, in the sense that the set of such spin^c structures are those of the form $s + h$ for any $h \in \delta H^1(N)$. Therefore, condition (2) is equivalent to saying that a spin^c structure s on W is uniquely determined by a pair of spin^c structures s_1, s_2 on W_1, W_2 with common restriction on N.

If $b_2^+(W_1), b_2^+(W_2) \geq 1$, then it can be shown that the image of the map

$$F^-_{W_1,s_1} : HF^-(Y_1, s_1) \longrightarrow HF^-(N, s|_N)$$

lies in $HF^-_{red}(N, s|_N)$, and the map

$$F^+_{W_2,s_2} : HF^+(N, s|_N) \longrightarrow HF^+(Y_2, s_2)$$

factors through the projection of $HF^+(N, s|_N)$ to $HF^+_{red}(N, s|_N)$. Thus we may define

$$F^{mix}_{W,N,s} : HF^-(Y_1, s|_{Y_1}) \longrightarrow HF^+(Y_2, s|_{Y_2})$$

to be the composite

$$F^+_{W_2,s_2} \circ \tau^{-1} \circ F^-_{W_1,s_1}, \tag{2.11}$$

where τ is the map defined in the exact sequence 2.2. The map $F^{mix}_{W,N,s}$ is independent of the choice of the admissible cut N and will be simply called $F^{mix}_{W,s}$.

Now given a closed 4-manifold X with $b_2^+(X) \geq 2$, we may remove two disjoint 4-balls and see X as a cobordism W from S^3 to S^3. In view of Proposition 2.6.1 and the properties of the $\mathbb{Q}$-grading, the minimal degree of $HF^+(S^3)$ is 0 and the maximal degree of $HF^-(S^3)$ is -2, so we define Θ^+ and Θ^- to be the generators of $HF_0^+(S^3) \cong \mathbb{Z}$ and $HF^-_{-2}(S^3) \cong \mathbb{Z}$ respectively.

Definition 2.8.3. The *absolute invariant* of X is defined to be the map:

$$\Phi_X : spin^c(X) \longrightarrow \mathbb{Z}/\pm 1$$

so that $\Phi_X(s)$, also denoted by $\Phi_{X,s}$, is the coefficient of $\Theta^+ \in HF^+(S^3)$ in the expression $F^{mix}_{W,s}(\Theta^-)$.

In the situation where the condition in Definition 2.8.1 (2) fails, we have seen from Remark 2.8.2 that a spin^c structure on W_1 and a spin^c structure on W_2 no longer uniquely determines a spin^c structure on W, with indeterminacy $\delta H^1(N)$. The construction above turns out to only yield one single invariant, which is the value of the sum $\sum_{h \in \delta H^1(N)} F^{mix}_{W,s+h}$ instead of the individual terms of the sum. Jabuka and Mark [16] proved the existence of a *relative invariant* which refines Definition 2.8.3 and allows us to understand the individual terms of the sum regardless of whether the condition in Definition 2.8.1 (2) holds.

To state their result, let us set up some notations. Let $X = X_1 \cup_N X_2$ be a closed 4-manifold glued along N (so that X_i with a 4-ball D^4 removed is W_i). For $i = 1, 2$, let M_{X_i} be the $\mathbb{Z}[H^1(\partial X_i)]$-module defined by $M_{X_i} = \mathbb{Z}[\ker(H^2(X_i, \partial X_i) \longrightarrow H^2(X_i))]$, where $H^1(\partial X_i)$ acts on M_{X_i} by the coboundary homomorphism $H^1(\partial X_i) \longrightarrow H^2(X_i, \partial X_i)$. Analogous to $\delta H^1(N)$, we define $K(X, N) = Im[H^1(N) \longrightarrow H^2(X)]$ and $M_{X,N} = \mathbb{Z}[K(X, N)]$.

Theorem 2.8.4. *([16], Theorem 1.5) Let (X_1, s_1) and (X_2, s_2) be two spin^c 4-manifolds with spin^c boundary $\partial X_1 = -\partial X_2 = (N, t)$, and write $X = X_1 \cup_N X_2$. If $b_2^+(X_1), b_2^+(X_2) \geq 1$, then there exists relative Ozsváth-Szabó invariants*

$$\Psi_{X_1,s_1} \in \underline{HF}^-_{red}(N, t; M_{X_1}), \qquad \Psi_{X_2,s_2} \in \underline{HF}^-_{red}(-N, t; M_{X_2})$$

well-defined up to a multiplication by a unit in $\mathbb{Z}[H^1(N)]$, where Ψ_{X_i,s_i} is simply $\underline{F}^-_{X_i - D^4, s_i}(\Theta^-)$. Furthermore, there exists an $M_{X,N}$-valued pairing

$$\langle\ ,\ \rangle : \underline{HF}^-_{red}(N, t; M_{X_1}) \otimes_{\mathbb{Z}[H^1(N)]} \underline{HF}^-_{red}(-N, t; M_{X_2}) \longrightarrow M_{X,N}.$$

analogous to one given by (2.6), so that for any spin^c structure s on X restricting to s_i on

X_i, we have an equality of group ring elements

$$\sum_{h\in K(X,N)} \Phi_{X,s+h} e^h = \langle \tau^{-1}(\Psi_{X_1,s_1}), \Psi_{X_2,s_2} \rangle \tag{2.12}$$

up to multiplication by a unit in $M_{X,N}$. Here e^h is the formal variable in the group ring $M_{X,N}$ corresponding to h.

Here we emphasize that equation (2.12) is an equality of *group ring elements* in $M_{X,N} = \mathbb{Z}[K(X,N)]$, which means the right hand side of the equation determines *every individual term* of the left sum. In particular, our main theorem (Theorem 1.3.1) provides sufficient conditions for one of the relative invariants on the right side to vanish, which then forces every term on the left side to vanish.

2.9 Knot Floer homology and integer surgeries

There are *surgery exact sequences* that relate the Heegaard Floer groups of a 3-manifold Y and those of surgeries on Y, coming in both HF^+ and $\widehat{HF}$ flavors. A basic version concerns the case that Y is an integer homology sphere.

Theorem 2.9.1 (Oszváth-Szabó, [22])**.** *Let Y be an integer homology sphere, $K \subset Y$ be a knot and $Y_n(K)$ be the result of the n-surgery of K. Then there is a U-equivariant surgery exact sequence:*

$$\cdots \longrightarrow HF^+(Y) \xrightarrow{F_1} HF^+(Y_n(K)) \xrightarrow{F_2} HF^+(Y_{n+1}(K)) \xrightarrow{F_3} HF^+(Y) \longrightarrow \cdots \tag{2.13}$$

The above theorem holds with HF^+ replaced by $\widehat{HF}$. The maps F_i are equal to the sum of the cobordism maps over all spin^c structures on the cobordisms W_i where W_i are the standard cobordisms induced by the surgeries. In particular, $\text{spin}^c(W_1) \cong \mathbb{Z}$ by Mayer-Vietoris argument. Thus F_1 splits as a sum of maps:

$$F_1 = \bigoplus_{s \in spin^c(W_1)} F_{W_1,s}, \tag{2.14}$$

where F_{W_1} stands for $F^+_{W_1}$ or $\widehat{F}_{W_1}$.

While the long exact sequence (2.13) gives us information about the sum of maps, our application requires understanding each individual map. To this end, we discuss a powerful tool that enables us to do so.

Let Y be a 3-manifold, t be a spin^c structure on Y and K be a nullhomologous knot in Y. In [21], Ozsváth and Szabó defined a $\mathbb{Z} \oplus \mathbb{Z}$-bigraded chain complex $C = CFK(Y, K)$, and the grading (i, j) will be referred to as *filtration level* so as to distinguish from the homological grading of C. The subgroup of C with filtration level (i, j) is denoted by $C\{i, j\}$. In our case of interest where Y is an integer homology sphere, C has an absolute grading. For the calculations in the future, we only need to understand $CFK(Y, K)$ up to its filtered chain homotopy type.

Properties 2.9.2. *The chain complex C has the following properties:*

(1) The differential ∂ decreases homological grading by 1. Also, if $x \in C\{i, j\}$ and if a summand of ∂x has filtration level (i', j'), then $i' \leq i$ and $j' \leq j$.

(2) C has a U-action which sends $C\{i, j\}$ to $C\{i-1, j-1\}$ and decreases grading by 2.

(3) The quotient complex $C\{i = 0\} = C\{i \leq 0\}/C\{i < 0\}$ is the chain complex $\widehat{CF}(Y)$, whose homology is $\widehat{HF}(Y)$, with an extra filtration given by j. In particular, the

absolute grading on $C\{i=0\}$ *gives the absolute grading on* $\widehat{CF}(Y)$ *([21], Lemma 3.6). This property is also true if we interchange* i *and* j.

(4) *The complex* C *enjoys a lot of symmetries. In particular, up to filtered chain homotopy equivalence and assuming* $\mathbb{Z}/2\mathbb{Z}$*-coefficients, we can arrange that:*

   (a) $C\{i=0\}$ *is supported between filtration levels* $(0,k)$ *and* $(0,-k)$ *for some* $k \geq 0$, *and the ranks of* $C\{0,j\}$ *and* $C\{0,-j\}$ *are equal for all* $0 \leq j \leq k$.

   (b) C *is isomorphic to* $C\{i=0\} \otimes \mathbb{Z}[U, U^{-1}]$ *as filtered free abelian groups. In other words,* C *is generated by the generators of* $C\{i=0\}$ *followed by a power of* U. *Also,* U *respects the differential.*

The differential and absolute grading on $C\{i=0\}$ *and* $C\{j=0\}$ *determine those of the entire complex* C *through Property (4b).*

As an example, see Figure 4.1 for $CFK(S^3, T_{2,9})$ (up to filtered chain homotopy equivalence) where $T_{2,9}$ is the positive torus knot (2,9).

The rest of the section discusses Oszváth and Szabó's work [27] which proved that in the case that Y is an integer homology sphere, we can form mapping cones out of C whose homologies provide the Heegaard Floer groups of the surgeries on Y, and more importantly, certain inclusions of $\widehat{HF}(Y) \cong C\{i=0\}$ into the mapping cones can be identified with the individual cobordism maps in equation 2.14. Ozsváth-Szabó's theorem comes with flavors of HF^+ and $\widehat{HF}$, but for our application, we shall only discuss the $\widehat{HF}$ version.

Say $C\{i=0\}$ is supported between filtration levels $(0,k)$ and $(0,-k)$ for some $k \geq 0$.

We define the following quotient complexes A_s (for $s \in \mathbb{Z}$) and B.

$$A_s = \Big\{ C\{\max(i, j-s) = 0\} \Big\}, \qquad B = C\{i = 0\}. \tag{2.15}$$

There are two canonical chain maps $v_s : A_s \longrightarrow B$ and $h_s : A_s \longrightarrow B$, defined as follows. The map v_s is the projection onto $C\{i = 0\}$, while h_s is the projection onto $C\{j = s\}$, followed by the identification with $C\{j = 0\}$ (induced by the action of U^s), followed by a canonical chain homotopy equivalence ζ from $C\{j = 0\}$ to $C\{i = 0\}$.

We note that ζ preserves the absolute grading on C, but does not have to map $C\{i, j\}$ to $C\{j, i\}$ in general (even though it does in the case $(S^3, T_{2,2g+1})$).

Let $\mathbb{A} = \bigoplus_{s \in \mathbb{Z}} A_s$ and $\mathbb{B} = \bigoplus_{s \in \mathbb{Z}} B_s$ (each summand of $\mathbb{B}$ is isomorphic to B, but we included a subscript to distinguish various summands), and let $D_n : \mathbb{A} \longrightarrow \mathbb{B}$ be the chain map

$$D_n(\{a_s\}_{s \in \mathbb{Z}}) = \{b_s\}_{s \in \mathbb{Z}}$$

where

$$b_s = v_s(a_s) + h_{s-n}(a_{s-n}).$$

A visualization of a finite portion of D_n for $n = 0$ and $n = -1$ is as follows:

For $n = 0$:

$$\begin{tikzcd}
A_{-2} \arrow[d, bend right, "v_{-2}"'] \arrow[d, bend left, "h_{-2}"] & A_{-1} \arrow[d, bend right, "v_{-1}"'] \arrow[d, bend left, "h_{-1}"] & A_0 \arrow[d, bend right, "v_0"'] \arrow[d, bend left, "h_0"] & A_1 \arrow[d, bend right, "v_1"'] \arrow[d, bend left, "h_1"] & A_2 \arrow[d, bend right, "v_2"'] \arrow[d, bend left, "h_2"] \\
B_{-2} & B_{-1} & B_0 & B_1 & B_2
\end{tikzcd}$$

For $n = -1$:

$$\begin{tikzcd}
& A_{-2} \arrow[dl, "h_{-2}"] \arrow[d, "v_{-2}"] & A_{-1} \arrow[dl, "h_{-1}"] \arrow[d, "v_{-1}"] & A_0 \arrow[dl, "h_0"] \arrow[d, "v_0"] & A_1 \arrow[dl, "h_1"] \arrow[d, "v_1"] & A_2 \arrow[dl, "h_2"] \arrow[d, "v_2"] & {} \arrow[dl, "h_3"] \\
{} & B_{-2} & B_{-1} & B_0 & B_1 & B_2 &
\end{tikzcd}$$

Let $\mathbb{X}(n)$ denote the *mapping cone* of D_n, namely the chain complex whose underlying group is $\mathbb{A} \oplus \mathbb{B}$, and whose differential over $\mathbb{F}$, the field of order 2, has the form

$$\begin{pmatrix} \partial_{\mathbb{A}} & 0 \\ D_n & \partial_{\mathbb{B}} \end{pmatrix}.$$

Now we are ready to state the main theorem of this section:

Theorem 2.9.3 ([27])**.** *Let Y be an integer homology sphere. For any integer n, the homology of the mapping cone $\mathbb{X}(n)$ is isomorphic to $\widehat{HF}(Y_n(K))$. Moreover, under this identification, the map $H_*(B_s) \longrightarrow H_*(\mathbb{X}(n))$ induced by inclusion of chain complex is identified with the cobordism map $\widehat{F}_{W,t} : \widehat{HF}(Y) \longrightarrow \widehat{HF}(Y_n(K))$, where W is the cobordism induced by the 2-handle, and t is s-th spin^c structure on W (for some identification of spin$^c(W)$ with $\mathbb{Z}$).*

In [27], the main theorem (Theorem 1.1) is stated for HF^+ and the case $n \neq 0$; the analogous result for $\widehat{HF}$ as well as the case $n = 0$ are handled in Chapter 4 of the same paper.

When $n = 0$, we have $H^1(Y_0(K)) \cong \mathbb{Z}$; let T be a generator of $H^1(Y_0(K))$. Then Theorem 2.9.3 actually calculates the twisted cobordism maps

$$\underline{\widehat{F}}_{W,t} : \widehat{HF}(Y) \longrightarrow \underline{\widehat{HF}}(Y_0(K), t|_{Y_0(K)}; \mathbb{F}[T, T^{-1}])$$

as long as we replace $H_*(A_s)$ by $H_*(A_s) \otimes_{\mathbb{F}} \mathbb{F}[T, T^{-1}]$, $H_*(B_s)$ by $H_*(B_s) \otimes_{\mathbb{F}} \mathbb{F}[T, T^{-1}]$ and the maps h_s by $T \cdot h_s$ for all $s \in \mathbb{Z}$.

When $n \neq 0$, the absolute grading of $\widehat{HF}(Y_n(K))$ can be determined from the mapping cone as well. In our case of interest $n = -1$, the grading is determined by giving B_0 the grading of $C\{i = 0\}$, and then shifting the gradings on the other copies of A_s and B_s so that the v- and h-maps decrease degree by 1.

Finally, it follows from standard homological algebra that there is an exact sequence

$$0 \longrightarrow \text{coker } (D_n)_* \longrightarrow H_*(\mathbb{X}(n)) \longrightarrow \ker(D_n)_* \longrightarrow 0.$$

If the coefficient ring is $\mathbb{F}$, then the sequence splits, and so

$$H_*(\mathbb{X}(n)) = \ker(D_n)_* \oplus \text{coker}(D_n)_*. \tag{2.16}$$

Chapter 3

The topology of U_g

This chapter is devoted to the construction of the Lefschetz fibrations U_g mentioned in the statement of Theorem 1.3.1. A lot of the contents of this chapter can be found in [11], most notably Chapter 8.

3.1 Lefschetz fibrations

A Lefschetz fibration can be thought of as a generalization of products of two surfaces, where a finite number of fibers are allowed to contain singularities of the simplest kind, and away from the fibers with singularities, there is a surface bundle structure. More precisely:

Definition 3.1.1 ([11], Definition 8.1.4)**.** Let X and M be compact, connected, oriented 4- and 2-dimensional manifolds respectively. A smooth map $\pi : X \longrightarrow M$ is called a *Lefschetz fibration on* X if:

- $\pi^{-1}(\partial M) = \partial X$,

- The set of critical points $C \subset X$ of π is finite and each critical point lies in the interior of X,

- For each $p \in C$, there exist complex charts around p and $\pi(p)$, preserving orientations of X and M, so that $\pi(z_1, z_2) = z_1^2 + z_2^2$.

The preimage of any point in $\pi(C)$ is called a *singular fiber*, and the preimage of any other point in M is called a *regular fiber*. If a regular fiber is a connected surface of genus g, then $\pi : X \longrightarrow M$ is called a *genus g Lefschetz fibration.*

By perturbation, we may assume that π is injective on C.

Let D^2 be a 2-disk in M. If D^2 contains no points from $\pi(C)$, then $\pi^{-1}(D^2)$ is the tubular neighborhood of a regular fiber and is diffeomorphic to $\Sigma \times D^2$. If D^2 contains one point from $\pi(C)$, then $\pi^{-1}(D^2)$ is the tubular neighborhood of a singular fiber whose topology can be described as follows:

Proposition 3.1.2 ([11], p.292-295)**.** *$\pi^{-1}(D^2)$ is diffeomorphic to the result of attaching a 2-handle along a circle $\alpha \subset \Sigma \times \{pt.\}$ on the boundary of $\Sigma \times D^2$, where the framing of the attachment is -1 relative to the framing given by Σ. Also, $\pi^{-1}(\partial D^2)$ is the mapping torus whose monodromy is the right-handed Dehn twist D_α along α.*

More generally, a Lefschetz fibration with any number of critical points can be combinatorially described by a *monodromy representation* Ξ as follows. Choose a point q in the interior of M away from $\pi(C)$, and fix an identification of $\pi^{-1}(q)$ with Σ. Then a loop λ with $\lambda(0) = \lambda(1) = q$ induces a diffeomorphism $\phi : \pi^{-1}(\lambda(0)) =$

$\Sigma \longrightarrow \pi^{-1}(\lambda(1)) = \Sigma$. Changing λ by an isotopy only changes ϕ by isotopy, so we have a well-defined map $\Xi : \pi_1(M - \pi(C), q) \longrightarrow MCG(\Sigma)$, where π_1 denotes fundamental group and $MCG(\Sigma)$ is the mapping class group of Σ (see Section 2.3).

Based on this, we can describe a Lefschetz fibration by a *monodromy factorization* ([11], p.296). Let $C = \{p_1, p_2, \ldots, p_n\}$. Then $\pi_1(M - \pi(C), q)$ is isomorphic to the free group generated by n elements, and we can choose an ordered basis $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, where each λ_i bounds a disk D_i which contains exactly one member of $\pi(C)$, say $\pi(p_i)$. Without loss of generality, we index the $\pi(p_i)$ and λ_i so that the indices increase as we travel counter-clockwise around q; also let D_0 be a small disk containing q but none of the p_i. See Figure 3.1 for an example.

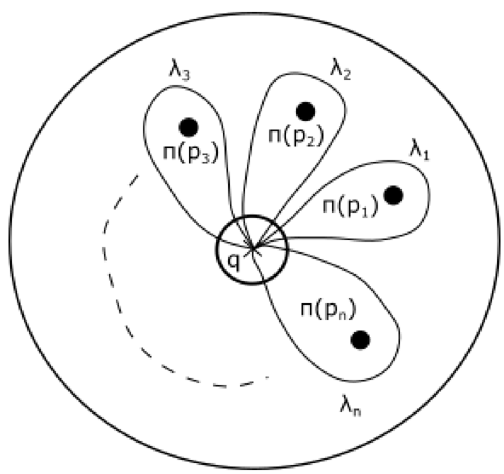

Figure 3.1: A choice of generators of $\pi_1(M - \pi(C), q)$.

It follows from Proposition 3.1.2 that X is diffeomorphic to $\pi^{-1}(D_0) \cong \Sigma \times D^2$ with n 2-handles attached along some circles $\alpha_1, \alpha_2, \ldots, \alpha_n$ in successive copies of Σ in $\Sigma \times S^1 = \partial(\Sigma \times D^2)$. Using the language of monodromy representation, we have

$\Xi(\lambda_i) = D_{\alpha_i}$. We say that the word

$$D_{\alpha_1} D_{\alpha_2} \cdots D_{\alpha_n}, \text{ or simply } \alpha_1 \alpha_2 \cdots \alpha_n,$$

is a monodromy factorization of π. Also, $\pi : \partial X \longrightarrow S^1$ is a mapping torus whose monodromy is the composition of the letters in the monodromy factorization, namely $\alpha_1 \circ \alpha_2 \circ \cdots \circ \alpha_n$. In general, the monodromy factorization depends on the choices of the identification $\pi^{-1}(q) \cong \Sigma$ and the ordered basis of $\pi_1(M - \pi(C), q)$. We shall not need to understand the dependence with depth; interested readers are referred to [11], p.297-298.

It is also clear from the construction that X contains Lefschetz subfibrations whose monodromy factorizations are subwords of that of X. (A subword of a word is a word whose letters form a subsequence of the letters of the original word.) For example, assuming $n \geq 6$, let $M' = D_0 \cup (D_1 \cup D_3 \cup D_6)$. Then $\pi^{-1}(M') \longrightarrow M'$ is a Lefschetz subfibration of X whose monodromy factorization is $\alpha_1 \alpha_3 \alpha_6$.

We shall often consider *relatively minimal* Lefschetz fibrations:

Definition 3.1.3 ([11], p.289)**.** A Lefschetz fibration X is relatively minimal if no fiber of X contains spheres of self-intersection -1.

A Lefschetz fibration is relatively minimal precisely when none of the circles $\alpha_1, \alpha_2, \ldots, \alpha_n$ are trivial in $\pi_1(\Sigma)$ ([11], p.289).

Now we consider our main case of interest, namely Lefschetz fibrations where the base space M is S^2. We may remove the neighbourhood $\Sigma \times D^2$ of a regular fiber to obtain a Lefschetz fibration over D^2 with some monodromy factorization $\alpha_1\alpha_2\cdots\alpha_n$. Since $\partial(\Sigma\times D^2)=\Sigma\times S^1$ is a bundle over S^1 whose monodromy is the identity map, it follows that $\alpha_1\circ\alpha_2\circ\cdots\circ\alpha_n$ must be the identity element in $MCG(\Sigma)$. Conversely, given a word in $MCG(\Sigma)$ in which all letters α_i are right-handed Dehn twists, one can construct a Lefschetz fibration with monodromy factorization $\alpha_1\alpha_2\cdots\alpha_n$ by attaching 2-handles to $\Sigma\times D^2$ as described before; if $\alpha_1\circ\alpha_2\circ\cdots\circ\alpha_n$ is the identity, then the Lefschetz fibration extends over S^2. The extension is unique if the genus g of Σ is at least 2 ([11], p.299).

3.2 Handle diagrams of Lefschetz fibrations

This section provides a way to construct handle diagrams of Lefschetz fibrations based on a monodromy factorization. To this end, we shall first draw the handle diagram of $\Sigma\times D^2$, the neighbourhood of a regular fiber, and then attach the 2-handles in the manner described in Section 3.1. First we give a handle diagram of $\Sigma\times D^2$:

Proposition 3.2.1. *We have the following:*

1. *A handle description of $\Sigma\times D^2$ is given by Figure 3.2(a), consisting of a 0-handle, $2g$ 1-handles and a 0-framed 2-handle. Each dashed line connects the two attaching regions belonging to the same 1-handle.*

2. *The obvious spanning disk of the 0-framed closed curve drawn in S^3 extends over the 1-handles and the core of the 2-handle to form $\Sigma\times\{pt.\}$, and the standard*

fibration of the complement of the closed curve by spanning disks gives the entire S^1-family of Σ. (See Figure 3.2(b) for the picture of Σ minus a 2-disk, whose bands lie inside the 1-handles and thus are invisible in Figure 3.2(a).)

3. *The standard chain $\gamma_1, \gamma_2, \ldots, \gamma_{2g}$ on Σ in Figure 3.2(c) corresponds to the chains $\gamma_1, \gamma_2, \ldots, \gamma_{2g}$ in Figures 3.2(a) and 3.2(b).*

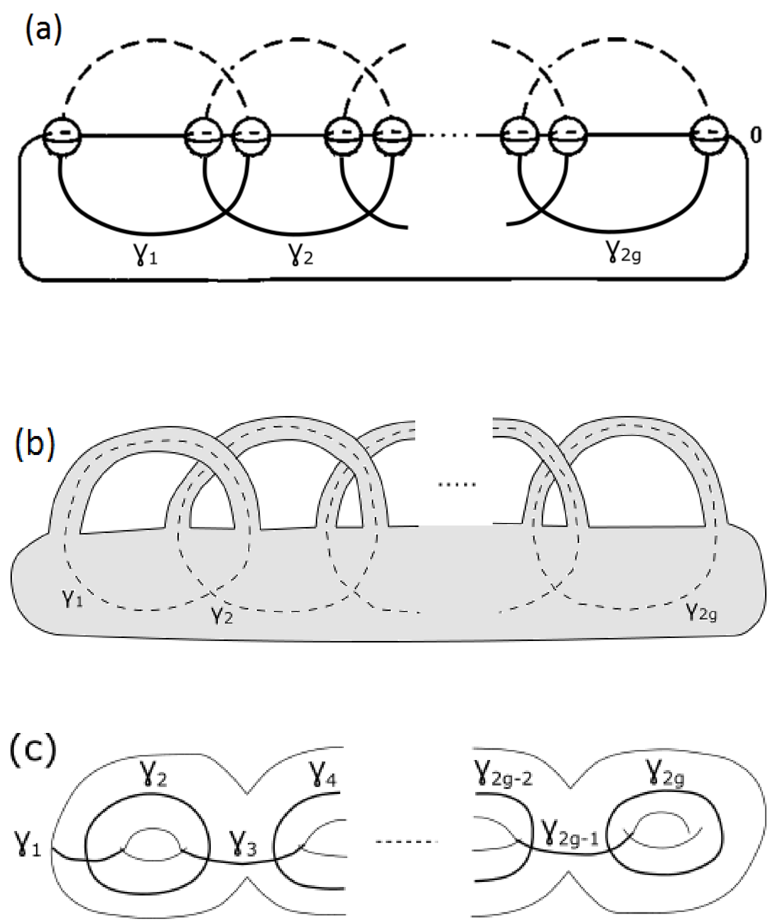

Figure 3.2: A handle description of $\Sigma \times D^2$. Diagram (a) is taken from [11], p.321 and modified.

The first two parts of the proposition are a direct generalization of [11], Exercise 4.6.6(a); see also Exercise 8.4.1. The third part can be visualized by starting with Figure 3.2(b) and gluing a disk at its boundary to recover Σ as in Figure 3.2(c).

Given a monodromy factorization $\alpha_1\alpha_2\cdots\alpha_n$, we can form a handle description of the corresponding Lefschetz fibration over D^2 by attaching the -1-framed 2-handles along successive copies of Σ in $\Sigma\times S^1$. By Proposition 3.2.1(2), these successive copies of Σ can be visualized by starting with the obvious spanning disk, and then gradually pushing the interior of the disk out of the paper (while keeping the boundary fixed). Therefore, the handle diagram for the Lefschetz fibration can be drawn by starting with the diagram of $\Sigma\times D^2$, then add 2-handles whose attaching circles consist of the curve α_1 on the obvious spanning disk, then α_2 on a spanning disk pushed slightly out of paper thus overcrossing α_1, then α_3 on a further pushed spanning disk thus overcrossing α_1 and α_2, and so on. All framings of α_i will be -1 relative to the framing from Σ. If the writhe of α_i is 0, for example, when α_i is one of the γ-curves in Proposition 3.2.1, then the framing coefficient would be -1 (see Section 2.2). See Figure 3.3 in Section 3.3 for the particular example of our interest.

3.3 The definition of U_g

In this section, we shall define the Lefschetz fibration U_g mentioned in the statement of Theorem 1.3.1.

We begin by defining a classical complex manifold $U(g+1,n)$. Consider the holomorphic line bundle $L_k \longrightarrow \mathbb{CP}^1$ with Euler number k. Then the Hirzebruch surface $\mathbb{F}_k$ is defined to be the fiberwise projectivization of the $\mathbb{C}^2$-bundle $L_k \oplus \underline{\mathbb{C}} \longrightarrow \mathbb{CP}^1$, where $\underline{\mathbb{C}}$ denotes the trivial line bundle. Thus $\mathbb{F}_k$ is a $\mathbb{CP}^1$-bundle over $\mathbb{CP}^1$ with Euler number k, and in fact, $\mathbb{F}_k$ is diffeomorphic to $S^2 \times S^2$ when k is even.

Definition 3.3.1 ([11], p.263)**.** We define $U(g+1,n)$ be the desingularization of the double branched cover of the Hirzebruch surface $\mathbb{F}_{2n}$ branched along the union of $2g+1$ affine sections of self-intersection $2n$ and one of self-intersection $-2n$.

We are mainly interested in the Lefschetz fibration structure on $U(g+1,n)$ and thus omit details of the algebro-geometric construction; interested readers can find more facts about the Hirzebruch surface in, for example, [11], p.87-88, and more discussion of the double branched cover construction in [11], Chapter 7.3.

It is known that $U(g+1,n)$ is a minimal general type complex surface for $g \geq 2$ and $n \geq 2$ ([11], p.320 and [28], Proposition 1.3), and $U(g+1,n)$ admits a natural genus g singular fibration that becomes a Lefschetz fibration after perturbation. More precisely, Fuller [8] showed that the handle diagram of $U(g+1,n)$ is given by Figure 3.3.

By Proposition 3.2.1 and the subsequent discussion, $U(g+1,n)$ contains a genus g Lefschetz fibration over D^2 with monodromy factorization

$$(\gamma_1\gamma_2\cdots\gamma_{2g})^{(4g+2)n}, \tag{3.1}$$

whose handle description consists of the 0-handle, the $2g$ 1-handles, the 0-framed

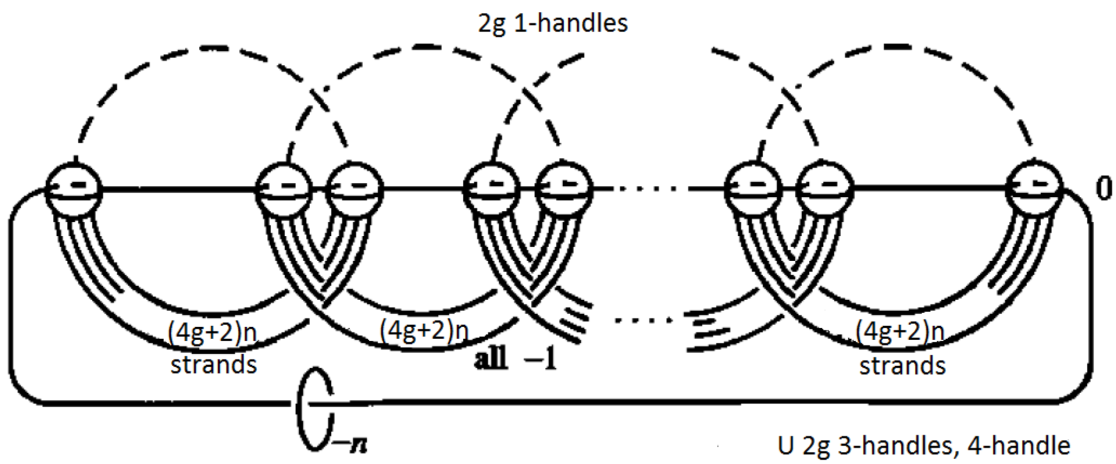

Figure 3.3: The handle diagram of $U(g+1,n)$. Diagram is taken from [11], p.321 and modified.

2-handle and the $(4g+2)n$ 2-handles in each half-ring. Because of the well-known relation $(\gamma_1 \circ \gamma_2 \circ \cdots \circ \gamma_{2g})^{4g+2} = 1$ in $MCG(\Sigma)$, we see that this Lefschetz fibration over D^2 extends to one over S^2 by gluing a copy of $\Sigma \times D^2$. In fact, the $-n$-framed 2-handle, the $2g$ 3-handles and the 4-handle in the diagram correspond to such a copy of $\Sigma \times D^2$ ([11], p.321). This makes the whole $U(g+1,n)$ a Lefschetz fibration over S^2 with monodromy given by (3.1).

Definition 3.3.2. We define U_g to be the genus g Lefschetz fibration over D^2 with monodromy factorization $(\gamma_1\gamma_2\cdots\gamma_{2g})^{4g+3}$.

Since $(\gamma_1\gamma_2\cdots\gamma_{2g})^{4g+3}$ is a subword of $(\gamma_1\gamma_2\cdots\gamma_{2g})^{(4g+2)n}$ for all $n \geq 2$, it follows from the discussion preceding Definition 3.1.3 that $U(g+1,n)$ contains U_g as a Lefschetz subfibration for all $n \geq 2$. The handle diagram of U_g is given by Figure 3.3

except with only the 0-handle, the 1-handles and $4g+3$ strands in each half-ring. Fuller [8] showed that after cancelling one 2-handle from each half-ring with the 1-handles and performing isotopies on the 0-framed knot, we obtain a handle diagram of U_g with only a 0-handle and $2g(4g+2)$ 2-handles, each with framing coefficient -2, shown in Figure 3.4. The box with the number -1 denotes a full negative twist.

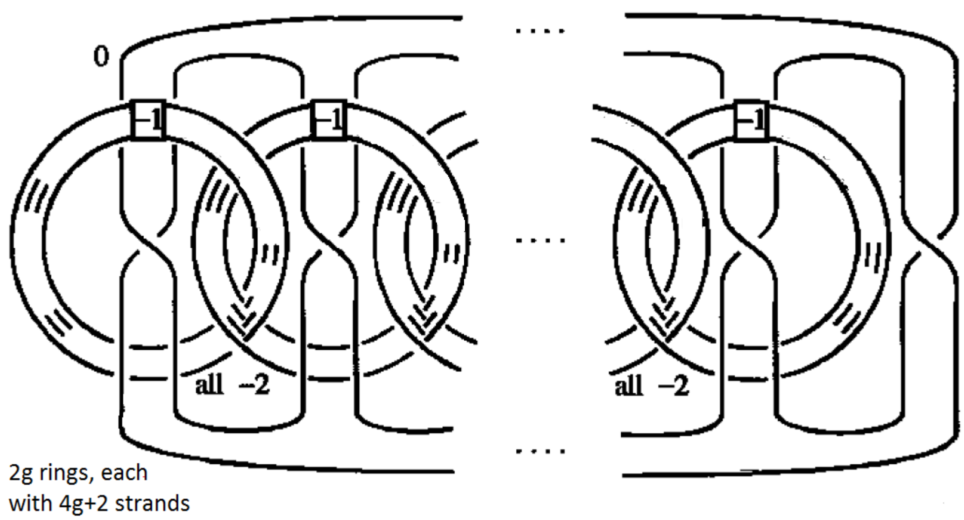

Figure 3.4: A handle description of U_g without 1-handles. Diagram is taken from [11], p.322 and modified.

The fibration U_g contains *Milnor fibers*, for which we shall recall some definitions and facts. For positive integers p, q, r, a Milnor fiber $M(p,q,r)$ associated to the complex singularity $\{x^p + y^q + z^r = 0\}$ is defined to be the set $\{(x,y,z) \in \mathbb{C}^3 | x^p + y^q + z^r = \epsilon\}$, where $\epsilon \in \mathbb{C} - \{0\}$, and we shall also define $V(p,q,r) = M(p,q,r) \cap D^6$ where D^6 is a 6-ball with a sufficiently small radius centered at the origin. Milnor [18] showed that $V(p,q,r)$ is the r-fold cyclic branched cover of $D^4 \subset \mathbb{C}^2$ branched along the surface $B = \{(x, y)|x^p + y^q = \epsilon\}$ where B is obtained by pushing the canonical Seifert surface of the positive torus link $T_{p,q}$ into D^4. We are most interested in the case $p = 2$ and $q = 2g + 1$; see Figure 3.5 for two isotopic drawings of $T_{2,2g+1}$ with the standard Seifert surface. Also, $T_{p,q}$ is a fibered link in S^3 and the standard Seifert surface, whose genus is $\frac{(p-1)(q-1)}{2}$, is a page of the open book, so the genus of the standard Seifert surface of $T_{2,2g+1}$ is g.

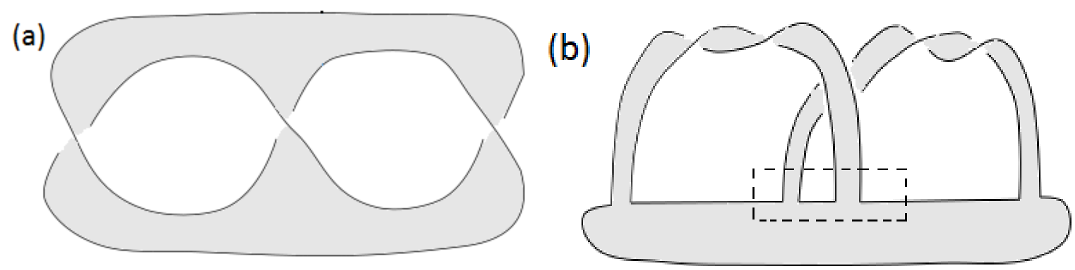

Figure 3.5: The torus knot $T_{2,2g+1}$ (with $g = 1$) and the canonical Seifert surface. We can visualize the isotopy from (b) to (a) as follows: push the two feet in the boxed region into contact, push the box up to the top of the diagram (creating a "thick band" in the middle), then perform some untwisting.

As promised in Section 2.4, we shall describe Akbulut-Kirby's algorithm [1] in this particular situation to produce a handle diagram of $V(2, q, r)$. First start with a drawing of the Seifert surface F which consists of a single 0-handle and some 1-handles, namely Figure 3.5(b). Visualize r copies of thickened copies $\overline{F}_i$ of F (as defined in Section 2.4) in S^3 like the slices of a loaf of bread, with index increasing from left to right, and so that $\overline{F}_i^-$ (resp. $\overline{F}_i^+$) is on the left (resp. right) of the slice.

The gluing of $\overline{F}_i^+$ and $\overline{F}_{i+1}^-$ is equivalent to attaching 2-handles whose attaching circles are the union of the cores of the 1-handles in $\overline{F}_i^+$ and $\overline{F}_{i+1}^-$, and the framing coefficient of each 2-handle is equal to twice the number of full twists in the corresponding 1-handle. The result is given by Figure 3.6(a). Also, Figure 3.6(b) shows how F (as a subset of $\overline{F}_1$) sits inside the boundary relative to the arcs in $\overline{F}_1^+$ which are the leftmost arcs in Figure 3.6(a).

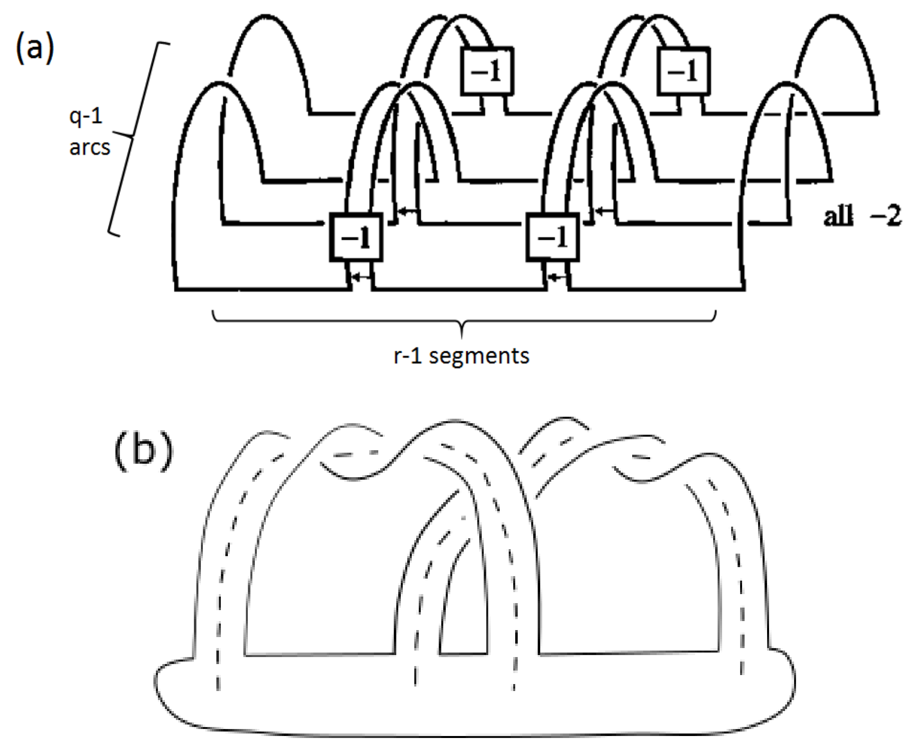

Figure 3.6: The handle diagram of $V(2,q,r)$ (where $q=3$ and $r=4$), and how ∂F sits inside it. Diagram (a) is taken from [11], p.233 and modified.

Akbulut and Kirby also described a series of isotopies in Figure 3.6(a) collapsing the arches to the ring picture (Figure 3.4), except with $q-1$ strands in each of the $r-1$ rings and without the 0-framed knot. In other words, the Akbulut-Kirby construction allows us to decompose U_g into the Milnor fiber $V(2, 2g+1, 4g+3)$ and a 2-handle attachment.

Definition 3.3.3. We define V_g to be $V(2, 2g+1, 4g+3)$. We also define W_g to be $U_g - \mathrm{int}(V_g)$, which is the cobordism from ∂V_g to ∂U_g induced by attaching the 0-framed 2-handle in Figure 3.4.

As a result, we can see $U_g - D^4$ as the composition of the cobordism $V_g - D^4$ from S^3 to ∂V_g and the cobordism W_g from ∂V_g to ∂U_g.

We further observe that if one keeps track of F in Akbulut's and Kirby's isotopy, then F is actually isotopic to the obvious surface bounded by the 0-framed knot in Figure 3.4 through the previously described isotopy from Figure 3.5(b) to 3.5(a) (and shifting the rings slightly so as to not intersect the obvious surface). As a result, we do not only understand ∂V_g as an open book abstractly, but also the binding and a page of the open book relative to our surgery description of ∂V_g. Based on this observation, we shall give an alternative description of W_g, which will be more suitable for the Heegaard Floer calculation later on. Let $S^3_n(K)$ denote the 3-manifold resulted from performing an n-surgery on $K \subset S^3$.

Proposition 3.3.4. *The cobordism W_g is diffeomorphic to:*

(1) The cobordism induced by attaching a page-framed 2-handle along the binding

of a genus g open book with monodromy $(\gamma_1 \circ \gamma_2 \circ \cdots \circ \gamma_{2g})^{4g+3} \in MCG(\Sigma_{g,1})$.

(2) *The cobordism induced by the 0-surgery on the induced knot in $S^3_{-1}(T_{2,2g+1})$ (see Definition 2.1.4).*

Proof. We first prove that W_g is diffeomorphic to (1). Our discussion has established that ∂V_g is a genus g open book which is a $(4g+3)$-fold cyclic branched cover of S^3 along the fibered knot $T_{2,2g+1}$, and the obvious surface bounded by the 0-framed knot in Figure 3.4 is a page of the open book on ∂V_g. Since the monodromy of the corresponding open book on S^3 is $\gamma_1 \circ \gamma_2 \circ \cdots \circ \gamma_{2g}$, we know that the monodromy of the open book on ∂V_g is $(\gamma_1 \circ \gamma_2 \circ \cdots \gamma_{2g})^{4g+3}$ (see the end of Section 2.4). It is also clear from the diagram that the page framing does have framing coefficient 0, so it follows from definition that W_g is diffeomorphic to (1).

Next we prove that (1) and (2) are diffeomorphic. Let's start over with the open book structure on $T_{2,2g+1} \subset S^3$. If we let h be the right-handed Dehn twist along the boundary of $\Sigma_{g,1}$, then $S^3_{-1}(T_{2,2g+1})$ is an open book where the induced knot is the binding and the monodromy is $(\gamma_1 \circ \gamma_2 \circ \cdots \circ \gamma_{2g}) \circ h$ (see Etnyre's lecture notes [6], Theorem 5.7), which equals $(\gamma_1 \circ \gamma_2 \circ \cdots \circ \gamma_{2g})^{4g+3}$ due to the chain relation $(\gamma_1 \circ \gamma_2 \circ \cdots \circ \gamma_{2g})^{4g+2} = h$ in $MCG(\Sigma_{g,1})$. On the binding, the page framing is the same as the Seifert framing (or 0-framing). It follows that (1) and (2) are diffeomorphic. $\square$

By performing a handleslide, one can show that the 0-surgery on the induced knot in $S^3_{-1}(T_{2,2g+1})$ is actually $S^3_0(T_{2,2g+1})$. By the discussion at the end of Section 2.3,

$S^3_0(T_{2,2g+1})$ is also a mapping torus.

3.4 The algebraic topology of V_g

We shall record some information of the classical invariants of V_g for future application. In the following lemma, $\chi(V_g)$ denotes the Euler characteristics of V_g. Also, $b_2^+(V_g)$, $b_2^-(V_g)$ and $\sigma(V_g)$ denote the rank of the maximal positive definite subspace, rank of the maximal negative definite subspace and the signature of the intersection form of $H_2(V_g)$ respectively.

Lemma 3.4.1. *Let $g \geq 1$. The rank of $H_2(V_g)$ is $8g^2+4g$, and $\chi(V_g) = 8g^2+4g+1$. Also, $b_2^+(V_g) = 2g^2$, $b_2^-(V_g) = -6g^2-4g$ and $\sigma(V_g) = -4g^2-4g$.*

Proof. The handlebody diagram of V_g has one 0-handle and $2g \cdot (4g+2) = 8g^2+4g$ 2-handles, so the statements concerning rank of $H_2(V_g)$ and $\chi(V_g)$ immediately follow. To prove the rest, we use the fact (see [19] Remark 4.6, for example) that $b_2^+(V(p,q,r))$ is equal to twice the number of lattice points in the open tetrahedron spanned by $(0,0,0), (p,0,0), (0,q,0)$ and $(0,0,r)$.

Claim 3.4.2. *The number of lattice points in the open tetrahedron spanned by $(0,0,0)$, $(2,0,0), (0,2g+1,0)$ and $(0,0,4g+3)$ is equal to g^2.*

To prove the claim, first observe that the set of lattice points described in the claim is the same as the set of lattice points in the open triangle spanned by $(0,0)$, $\left(\dfrac{2g+1}{2},0\right)$

and $\left(0, \dfrac{4g+3}{2}\right)$ in the $x = 1$ plane. Then it is easy to check that a lattice point (y, z) lies inside the open triangle if and only if $1 \leq y \leq g$ and $1 \leq z \leq 2g + 1 - 2y$. Thus the total number of lattice points is $\sum_{y=1}^{g}(2g + 1 - 2y)$ which equals g^2.

Therefore, $b_2^+(V_g)$ equals $2g^2$. Also, by the same remark ([19], Remark 4.6), the triple $(2, 2g + 1, 4g + 3)$ being pairwise relatively prime implies that all eigenvalues of the intersection form are either positive or negative, so the rank of $H_2(V_g)$ equals $b_2^+(V_g) + b_2^-(V_g)$. This implies $b_2^-(V_g) = 6g^2 + 4g$, and it follows that $\sigma(V_g) = b_2^+(V_g) - b_2^-(V_g) = -4g^2 - 4g$. $\square$

Since the handle description of V_g (Figure 3.4 without the 0-framed 2-handle) involves only a 0-handle and some 2-handles, we know that $H_2(V_g)$ is free abelian generated by the cores of the 2-handles union their cap-offs in D^4, which is in bijection to the link components of the handle diagram. We choose an ordered basis of $H_2(V_g)$ based on Figure 3.4. First start with the outermost strand of the leftmost ring, then the second outermost strand, and so on, until we finish the ring. Next we move on to the strands in the second leftmost ring, again going from the outermost strand to the innermost, and we shall repeat with all the rings from left to right. We can choose the orientations of the basis elements to our convenience; our choice would be equivalent to orienting all link components counterclockwise. Under these choices, and keeping track of the criss-crosses carefully, we obtain the following linking matrix which is also the matrix representation Q of the intersection form:

$$Q = \begin{bmatrix} A & B & & & & \\ B^t & A & B & & & \\ & B^t & A & & & \\ & & & \ddots & & \\ & & & & A & B \\ & & & & B^t & A \end{bmatrix}, \tag{3.2}$$

where there are $2g$ copies of A, and both A and B are $(4g+2)$-by-$(4g+2)$ matrices, defined as follows:

- the diagonal entries of A are -2 and the non-diagonal entries of A are all -1,
- all diagonal and lower triangular entries of B are 1 and the remaining entries of B are zero,

and all remaining entries of Q are 0.

Chapter 4

Proof of Main Results

This chapter is devoted to proving the main results (Theorem 1.3.1 and Corollary 1.3.2) of the present work, which we restate here for convenience:

Theorem 4.0.1. *For $g = 4$ and 5, there exists a genus g Lefschetz fibration U_g over D^2 with regular fiber Σ, so that the relative invariant $\Psi_{U_g,s}$ vanishes for all spin^c structures satisfying $|\langle c_1(s), [\Sigma]\rangle| < 2g - 2$.*

Corollary 4.0.2. *Let X be a closed, oriented symplectic 4-manifold which admits a relatively minimal genus g Lefschetz fibration over S^2, with $g = 4$ or 5. If X contains U_g as a Lefschetz subfibration and $b_2^+(X - U_g) \geq 1$, then the Ozsváth-Szabo 4-manifold invariant $\Phi_{X,s}$ vanishes unless s is the canonical spin^c structure of X (or its conjugate).*

Let $S^3_n(K)$ denote the result of the n-surgery along a knot K in S^3. Since $\partial V_g = S^3_{-1}(T_{2,2g+1})$ is an integer homology sphere, we have $spin^c(U_g - D^4) \cong spin^c(V_g - D^4) \oplus spin^c(W_g)$. The Composition Law (Equation 2.4 and the subsequent discussion) tells us that

$$\Psi_{U_g,s} = \underline{F}^-_{U_g-D^4,s}(\Theta^-) = \underline{F}^-_{W_g,s|_{W_g}} \circ F^-_{V_g-D^4,s|_{V_g-D^4}}(\Theta^-); \tag{4.1}$$

note that ∂V_g is an integer homology sphere, so $\underline{F}^-_{V_g-D^4}$ is the same as $F^-_{V_g-D^4}$.

We will devote this chapter to prove the two key properties of $\underline{F}^-_{W_g}$ and $F^-_{V_g-D^4}$ by some intricate calculations (Proposition 4.1.1 and Proposition 4.2.1). The calculation of $F^-_{V_g-D^4}$ will be performed for all $g \geq 2$. The calculation of $\underline{F}^-_{W_g}$ is done for $g = 4$ and $g = 5$, although the author anticipates that the calculation generalizes to all $g \geq 2$. Once these calculations are in place, Theorem 4.0.1 and Corollary 4.0.2 will quickly follow (Section 4.3).

4.1 Calculation of $F^-_{V_g-D^4}$

We shall prove the following:

Proposition 4.1.1. *For all $g \geq 2$, the element $F^-_{V_g-D^4,s}(\Theta^-) \in HF^-_{red}(S^3_{-1}(T_{2,2g+1}))$ is trivial if $s \neq s_0$, where s_0 is the trivial spin^c structure.*

The proof is a combinatorial argument using the adjunction relation (Theorem 2.7.1).

Recall from Section 3.3 that the handle diagram of V_g consists of $2g$ rings with $4g+2$ strands in each ring. Continuing from our discussion of choosing a basis for $H_2(V_g)$ at the end of Section 3.4, for all $1 \leq i \leq 2g$ and $1 \leq j \leq 4g+2$, we use $L_{i,j}$ to denote the generator of $H_2(V_g)$ represented by the j-th strand in the i-th ring. We shall apply the adjunction relation on V_g with the following collection of homology classes in $H_2(V_g)$:

- Singletons, i.e. $L_{i,j}$ for some i and j;
- Doubletons, i.e. $L_{i,j} + L_{i+1,j'}$, where $1 \leq i \leq 2g-1$ and $j \geq j'$.

Notice that all these homology classes can be represented by smooth spheres and have square -2. Indeed, we can represent each singleton by taking the union of the core of the 2-handle and the disk that the attaching sphere bounds in S^3. For the doubletons, the condition $j \geq j'$ implies that $L_{i,j} \cdot L_{i+1,j'} = +1$ (see the linking matrix Q at Equation 3.2), and in fact, the smooth spheres representing $L_{i,j}$ and $L_{i+1,j'}$ intersect geometrically once. Thus we can splice the two spheres at the intersection point to yield one single smooth sphere, and it is easy to check that $(L_{i,j} + L_{i+1,j'})^2 = -2$. An application of the adjunction relation to these smooth surfaces yields:

Lemma 4.1.2. *Let s be a spin^c structure on V_g and $[\Sigma]$ be any of the homology classes defined above.*

(1) If $\langle c_1(s), [\Sigma]\rangle = -2$, then $F^-_{V_g-D^4,s} = F^-_{V_g-D^4,s-PD[\Sigma]}$.

(2) If $\langle c_1(s), [\Sigma]\rangle \leq -4$, then $F^-_{V_g-D^4,s} = U^m \cdot F^-_{V_g-D^4,s-PD[\Sigma]}$ for some $m > 0$.

Proof of Lemma. For (1), take a smooth sphere that represents $[\Sigma]$. Then the condition of the adjunction relation (Equation 2.9) holds for $m = 0$:

$$\langle c_1(s), [\Sigma]\rangle - [\Sigma]^2 = -2 - (-2) = 0 = -2m.$$

So adjunction relation applies and $F^-_{V_g-D^4,s} = U^0 \cdot F^-_{V_g-D^4,s-PD[\Sigma]} = F^-_{V_g-D^4,s-PD[\Sigma]}$.

For (2), since $\langle c_1(s), [\Sigma]\rangle - [\Sigma]^2 \leq -4 - (-2) = -2$, we may add $m > 0$ null-homologous handles to the smooth sphere that represents $[\Sigma]$ to make the condition of adjunction relation hold. The result follows. $\square$

The statement of Lemma 4.1.2 can be paraphrased as: if the condition in (1) is met, then transforming s to $s - PD[\Sigma]$ does not change the $F^-_{V_g-D^4}$ map; if the condition in (2) is met, then transforming s to $s - PD[\Sigma]$ multiplies $F^-_{V_g-D^4}$ by a positive power of U.

For any spin^c structure s in $V_g - D^4$, let $l_{i,j} = \frac{1}{2} \cdot \langle c_1(s), L_{i,j}\rangle$. We shall record these values in the form of a table:

	$L_{1,1}$	$\cdots$	$L_{1,4g+2}$	$L_{2,1}$	$\cdots$	$L_{2,4g+2}$	$\cdots$	$L_{2g,1}$	$\cdots$	$L_{2g,4g+2}$
s	$l_{1,1}$	$\cdots$	$l_{1,4g+2}$	$l_{2,1}$	$\cdots$	$l_{2,4g+2}$	$\cdots$	$l_{2g,1}$	$\cdots$	$l_{2g,4g+2}$

Now we describe the effect of applying Lemma 4.1.2 (1) taking $[\Sigma]$ to be a singleton $L_{i,j}$. If the corresponding entry $l_{i,j}$ equals -1, then $\langle c_1(s), L_{i,j}\rangle = 2l_{i,j} = -2$, so Lemma 4.1.2 (1) applies, and the transformation from s to $s - PD(L_{i,j})$ preserves $F^-_{V_g-D^4}$. Utilizing the intersection matrix Q and the facts that $\langle c_1(s), L_{i,j}\rangle = PDc_1(s) \cdot L_{i,j}$ and $PDc_1(s - PD(L_{i,j})) = PDc_1(s) - 2L_{i,j}$, we see that the transformation from s to $s - PD(L_{i,j})$ changes the entries in the following way:

- $l_{i,j}$ increases by 2;

- $l_{i,j'}$ increases by 1 for all $j' \neq j$;
- if $i \geq 2$, then $l_{i-1,j'}$ decreases by 1 for all $j' \geq j$;
- if $i \leq 2g-1$, then $l_{i+1,j'}$ decreases by 1 for all $j' \leq j$;
- all other entries are preserved.

Definition 4.1.3. We call the above transformation of s to $s - PD(L_{i,j})$ (through applying Lemma 4.1.2(1)) an *augmentation by* $L_{i,j}$.

Here is an augmentation expressed in tabular form (assuming $2 \leq i \leq 2g-1$), broken into three rows due to editorial constraints:

	$L_{i-1,j}$	$\cdots$	$L_{i-1,4g+2}$
s	$l_{i-1,j}$	$\cdots$	$l_{i-1,4g+2}$
Aug. by $L_{i,j}$	$l_{i-1,j}-1$	$\cdots$	$l_{i-1,4g+2}-1$

$L_{i,1}$	$\cdots$	$L_{i,j-1}$	$L_{i,j}$	$L_{i,j+1}$	$\cdots$	$L_{i,4g+2}$
$l_{i,1}$	$\cdots$	$l_{i,j-1}$	-1	$l_{i,j+1}$	$\cdots$	$l_{i,4g+2}$
$l_{i,1}+1$	$\cdots$	$l_{i,j-1}+1$	1	$l_{i,j+1}+1$	$\cdots$	$l_{i,4g+2}+1$

$L_{i+1,1}$	$\cdots$	$L_{i+1,j}$
$l_{i+1,1}$	$\cdots$	$l_{i+1,j}$
$l_{i+1,1}-1$	$\cdots$	$l_{i+1,j}-1$

Since the first Chern class of s reverses sign under conjugation (i.e. $c_1(\overline{s}) = -c_1(s)$), it is clear that conjugation of spin^c structure switches the signs of all entries.

Claim 4.1.4. *For any non-trivial spin^c structure s on $V_g - D^4$, there exists a finite sequence of augmentations and conjugations, ending with another spin^c structure s' for which at least one entry is -2 or less.*

Proof of Proposition 4.1.1 assuming Claim 4.1.4. For any non-trivial spin^c structure s on $V_g - D^4$, let s' be a spin^c structure produced by the claim; we have observed that $F^-_{V_g-D^4,s} = F^-_{V_g-D^4,s'}$. If $l_{i,j}$ is -2 or less, then $\langle c_1(s'), L_{i,j}\rangle = 2l_{i,j} \leq -4$, so we may apply Lemma 4.1.2 (2) to find $m_1 > 0$ and another spin^c structure s'' so that $F^-_{V_g-D^4,s'} = U^{m_1} \cdot F^-_{V_g-D^4,s''}$. If $s'' = s_0$, then $F^-_{V_g-D^4,s'} = 0$ since $F^-_{V_g-D^4,s_0}(\Theta^-)$ belongs to the kernel of U (to be proved in Lemma 4.2.2). If $s'' \neq s_0$, then we apply the claim and Lemma 4.1.2 (2) again to obtain another spin^c structure s''' so that $F^-_{V_g-D^4,s''} = U^{m_2} \cdot F^-_{V_g-D^4,s'''}$ for some $m_2 > 0$. Since the image of $F^-_{V_g-D^4,s}$ lies in the reduced group $HF^-_{red}(\partial V_g)$ which by definition vanishes under sufficiently high power of U, this process must yield $F^-_{V_g-D^4,s} = 0$ in finitely many steps. □

We shall spend the remainder of this section to prove Claim 4.1.4.

Proof of Claim 4.1.4. In the case where any entry has absolute value at least 2, the claim follows, by a conjugation if necessary. Therefore, it suffices to consider the case where all entries are $-1, 0, 1$. Observe that:

(*) The intersection matrix Q has determinant ± 1 because ∂V_g is an integer homology sphere. Since s is a non-trivial spin^c structure, we know that at least one entry is non-zero.

(**) If there is doubleton $L_{i,j} + L_{i+1,j'}$ (with $j' \leq j$) defined before Lemma 4.1.2 whose corresponding entries $l_{i,j}$ and $l_{i+1,j'}$ are both -1, then we may augment s by $L_{i,j}$ which makes $l_{i+1,j'}$ equal -2 (or augment s by $L_{i+1,j'}$ making $l_{i,j} = -2$), and the claim follows.

Therefore, we will prove the claim by describing an algorithm to augment any non-zero sequence $\{l_{i,j}\}$ into one that falls into situation (**). We may assume all entries are $-1, 0$ and 1 after each step of augmentation. Also, it is possible to augment by $L_{i,j}$ whenever $l_{i,j}$ is non-zero, just that if $l_{i,j} = 1$, then we just have to conjugate s beforehand in order to change $l_{i,j}$ to -1. Such conjugations will often be implicit in the proof. Therefore, to simplify our language, a generator $L_{i,j}$ will be called *permissible* (for the purpose of augmentation) if $l_{i,j} \neq 0$.

The algorithm is as follows. For a fixed value of i, the collection of generators $\{L_{i,j}\}_{1 \leq j \leq 4g+2}$ will be called the L_i-block, or simply L_i. We say that a block L_i is non-zero if at least one corresponding entry $l_{i,j}$ is non-zero.

Step 1. Augment so that we have two non-zero blocks with exactly one block in between (it does not matter whether the block in between is zero or not).

Any sequence that are not already in the desired form falls into one of the following five cases:

1. There are two or more non-zero blocks, and any two non-zero blocks have more than one zero block in between.

2. There are exactly two non-zero blocks L_i and L_{i+1}, and neither $i = 1$ nor $i + 1 = 2g$.

3. There are exactly two non-zero blocks L_i and L_{i+1}, and either $i = 1$ or $i+1 = 2g$.

4. There is exactly one non-zero block L_i with $i \neq 1, 2g$.

5. There is exactly one non-zero block L_i with $i = 1$ or $2g$.

- Case 1. Choose a pair of non-zero blocks L_i and L_i' (where $i' \geq i + 3$) with only zero blocks in between. Now augment by any permissible generator in $L_{i'}$ to make $L_{i'-1}$ non-zero, then augment by any permissible generator in $L_{i'-1}$ to make $L_{i'-2}$ non-zero, and so on. Repeat this process until L_{i+2} is non-zero.

- Case 2. Let j be the smallest value so that $l_{i,j} \neq 0$ and j' be the largest value so that $l_{i+1,j'} \neq 0$.

  - Case 2(a): If $j < j'$, then augment by $L_{i+1,j'}$ to make the L_{i+2}-block non-zero. Since $j < j'$, the augmentation preserves $l_{i,j}$, and so the L_i-block is non-zero.

- Case 2(b): If $j > j'$, then augment by $L_{i+1,j'}$ to make the L_{i+2}-block non-zero. Since $j > j'$, the entry $l_{i,j'}$ changes from 0 to -1, so the L_i-block remains non-zero.

- Case 2(c): If $j = j'$, then notice that if any entry between $l_{i,j}$ and $l_{i+1,j}$ is non-zero, then we may just augment by any generator strictly between them. Otherwise, if every entry between $L_{i,j}$ and $L_{i+1,j}$ is zero, then we consider either $l_{i,j}$ if $j < 4g+2$, or $l_{i+1,j}$ if $j > 1$. In the first case, assume $l_{i,j} = -1$ (possibly by a conjugation), augment by $L_{i,j}$ so that $l_{i-1,j} = -1$ and $l_{i,4g+2} = 1$, then conjugate and augment by $L_{i,4g+2}$ so that $l_{i+1,4g+2} = -1$ while $l_{i-1,j}$ is preserved. As a result, blocks L_{i-1} and L_{i+1} become non-zero. Expressed in a table where entries that do not matter are left blank:

	$\cdots$	$L_{i-1,j}$	$\cdots$	$\cdots$	$L_{i,j}$	$\cdots$	$L_{i,4g+2}$	$\cdots$	$L_{i+1,4g+2}$
Start		0			-1		0		0
Aug. by $L_{i,j}$		-1					1		0
Conj.		1					-1		0
Aug. by $L_{i,4g+2}$		1							-1

The argument of picking $l_{i+1,j}$ in the case of $j > 1$ is completely symmetric.

- <u>Case 3.</u> By symmetry, we may assume that only L_1 and L_2 are non-zero. Then we may augment by any permissible generator in L_2, so that L_3 becomes non-zero. If L_1 remains non-zero, then we are done; if L_1 becomes zero, see Case 2.

- Case 4. If L_i (with $i \neq 1, 2g$) is the only non-zero block, then we may augment by any permissible generator in L_i so that L_{i-1} and L_{i+1} are non-zero.

- Case 5. By symmetry, we may assume that L_1 is the only non-zero block; then augment by any permissible generator in L_1 so that L_2 becomes non-zero. If L_1 remains non-zero, see Case 3; if L_1 becomes zero, see Case 4.

Step 2. With the two non-zero blocks L_{i-1} and L_{i+1} ($2 \leq i \leq 2g-1$) constructed in Step 1, augment so that $l_{i-1,4g+2} = l_{i+1,1} = -1$.

If $L_{i-1,4g+2}$ is not already ± 1, we may augment by any permissible generator in L_{i-1} so that $l_{i-1,4g+2} = 1$ while preserving the absolute value of $l_{i+1,1}$; similarly, if $L_{i+1,1}$ is not already ± 1, we may augment by any permissible generator in L_{i+1} while preserving the absolute value of $l_{i-1,4g+2}$. Thus we have arranged that $l_{i-1,4g+2}$ and $l_{i+1,1}$ are both ± 1. Then:

- If $l_{i-1,4g+2} = l_{i+1,1} = -1$, then we are done.

- If $l_{i-1,4g+2} = l_{i+1,1} = 1$, then conjugate.

- If $l_{i-1,4g+2} = 1$ and $l_{i+1,1} = -1$, then augment by $L_{i+1,1}$ and conjugate; if $l_{i-1,4g+2} = -1$ and $l_{i+1,1} = 1$, then augment by $L_{i-1,4g+2}$ and conjugate.

Step 3. Exhaust the remaining cases. Assume we have arranged that $l_{i-1,4g+2} = l_{i+1,1} = -1$ for some $2 \leq i \leq 2g-1$, based on Step 2. We make the following observations:

(1) Since augmenting by both $L_{i-1,4g+2}$ and $L_{i+1,1}$ decreases all entries in L_i by 2 and the entries are among $-1, 0$ and 1 to start with, it suffices to prove the claim in the case where all entries in L_i are equal to 1.

(2) If $l_{i-1,4g+1} = 1$, then we may augment by $L_{i-1,4g+2}$ so that $l_{i-1,4g+1}$ becomes 2. Likewise, if $l_{i+1,2} = 1$, then we may augment by $L_{i+1,1}$ so that $l_{i+1,2}$ becomes 2.

Given these observations, we are left with the following four explicit cases, based on whether $l_{i-1,4g+1}$ and $l_{i+1,2}$ are 0 or -1, which can be handled easily by brute force (tables only show entries that matter)

- Case 1: $l_{i-1,4g+1} = 0$ and $l_{i+1,2} = 0$.

	$L_{i-1,4g+1}$	$L_{i-1,4g+2}$	$L_{i,1}$	$L_{i,4g+1}$	$L_{i+1,1}$	$L_{i+1,2}$
	0	-1	1	1	-1	0
Conj.	0	1	-1	-1	1	0
Aug. by $L_{i,1}$	-1	0	1	0	0	-1
Aug. by $L_{i-1,4g+1}$				-1		-1

This puts us into situation (**).

- Case 2: $l_{i-1,4g+1} = 0$ and $l_{i+1,2} = -1$.

	$L_{i-1,4g+1}$	$L_{i-1,4g+2}$	$L_{i,1}$	$L_{i,4g+1}$	$L_{i,4g+2}$	$L_{i+1,1}$	$L_{i+1,2}$
	0	-1	1	1	1	-1	-1
Conj.	0	1	-1	-1	-1	1	1
Aug. by $L_{i,1}$	-1	0	1	0	0	0	1
Aug. by $L_{i-1,4g+1}$	1	1	0	-1	0	0	1
Conj.	-1	-1	0	1	0	0	-1
Aug. by $L_{i-1,4g+2}$					-1		-1

  Thus we are in situation (**) again.

- Case 3: $l_{i-1,4g+1} = -1$ and $l_{i+1,2} = 0$. Augmenting by both $L_{i-1,4g+2}$ and $L_{i+1,1}$ then conjugating, we go back to Case 2.

- Case 4: $l_{i-1,4g+1} = -1$ and $l_{i+1,2} = -1$. Augmenting by both $L_{i-1,4g+2}$ and $L_{i+1,1}$ then conjugating, we go back to Case 1.

This finishes the proof of the claim. $\square$

4.2 Calculation of $\underline{F}^{-}_{W_g}$ for $g = 4, 5$

This section analyzes the behavior of $F^{-}_{V_g - D^4}(\Theta^-)$ under the map $\underline{F}^{-}_{W_g}$ in the special cases $g = 4$ and $g = 5$.

Recall from Proposition 3.3.4 and the subsequent discussion that W_g is a cobordism from $S^3_{-1}(T_{2,2g+1})$ to $S^3_0(T_{2,2g+1})$ which can be described as the cobordism induced by performing a 0-surgery on the induced knot K in $S^3_{-1}(T_{2,2g+1})$. By Mayer-Vietoris argument, we know that $\mathrm{spin}^c(W_g) \cong \mathbb{Z}$ and maps isomorphically onto $\mathrm{spin}^c(S^3_0(T_{2,2g+1}))$ by restriction. Let $[\Sigma]$ be a generator of $H_2(S^3_0(T_{2,2g+1}))$, and for each integer s, let t_s be the unique spin^c structure on $S^3_0(T_{2,2g+1})$ so that $\langle c_1(t_s), [\Sigma]\rangle = 2s$, and $\mathcal{S}_s$ be the spin^c structure on W_g that extends t_s. In particular, the canonical spin^c structure k of $U(g+1,n)$ restricts to $\mathcal{S}_{1-g}$ on W_g and satisfies $\langle c_1(k), [\Sigma]\rangle = -(2g-2)$.

Now we are ready to state the main proposition of this section:

Proposition 4.2.1. *For $g = 4$ or 5, the element $F^-_{V_g - D^4}(\Theta^-)$ vanishes under $\underline{F}^-_{W_g,\mathcal{S}_s}$ if $|\langle c_1(\mathcal{S}_s), [\Sigma]\rangle| < 2g-2$.*

The proof is an application of the mapping cone theorem (Theorem 2.9.3) with Y being the integer homology sphere $S^3_{-1}(T_{2,2g+1})$ and K being the induced knot. The major steps of the proof and the necessary setup are outlined as follows.

1. *Convert Proposition 4.2.1 into one involving $\widehat{HF}$.* An upcoming lemma (Lemma 4.2.2) shows that $F^-_{V_g - D^4, s_0}(\Theta^-) \in HF^-_{red}(S^3_{-1}(T_{2,2g+1}))$ is non-trivial (so that Proposition 4.2.1 is actually essential), has degree $-g^2+g-2$ and lies in the kernel of U. It follows from the discussion after exact sequence (2.2) that $\tau^{-1}F^-_{V_g - D^4, s_0}(\Theta^-)$ is an element in $HF^+_{red}(S^3_{-1}(T_{2,2g+1}))$ with degree $-g^2+g-1$. By exact sequence (2.3), we may lift $\tau^{-1}F^-_{V_g - D^4, s_0}(\Theta^-)$ to an element in $\widehat{HF}(S^3_{-1}(T_{2,2g+1}))$ with the same degree. Therefore, to prove the proposition, it suffices to show the following statement:

Proposition 4.2.1′. *The subgroup of* $\widehat{HF}(S^3_{-1}(T_{2,2g+1}))$ *with degree* $-g^2+g-1$ *becomes trivial under the map* $\underline{\widehat{F}}_{W_g,\mathcal{S}_i}$ *whenever* $|\langle c_1(\mathcal{S}_i), [\Sigma]\rangle| < 2g-2$.

2. *Calculate the chain complex* $CFK(S^3_{-1}(T_{2,2g+1}), K))$ *(up to filtered chain homotopy equivalence).* In general, it is difficult to calculate the CFK complex if the background 3-manifold is not S^3. However, in our special situation that K is the induced knot in a surgery of S^3, one can introduce a new filtration on the mapping cone that calculates $\widehat{HF}(S^3_{-1}(T_{2,2g+1}))$ to arrive at the desired chain complex $CFK(S^3_{-1}(T_{2,2g+1}), K)$. This method is due to a work in preparation by Matt Hedden and Adam Levine [12] which extends the results in [14].

3. *For* $CFK(S^3_{-1}(T_{2,2g+1}), K)$*, extract information of the chain homotopy equivalence* ζ *in the definition of the h-map (see the paragraph following equation 2.15).* We shall do so by studying the mapping cone for the (+1)-surgery on K, utilizing the absolute grading on $CFK(S^3_{-1}(T_{2,2g+1}), K)$ as well as the fact that the (+1)-surgery on K results in S^3.

4. *Use the information from Step 3 to study the mapping cone for the 0-surgery on* K. We will show that the inclusion of $\widehat{HF}(S^3_{-1}(T_{2,2g+1}))$ into B_s is trivial if $|s| < g-1$, thus finishing the proof of the proposition.

Some steps will be done for general values of g and some steps will be specialized to the $g = 4$ and 5. After the complete proof for the case $g = 4$ is done, we will perform the calculation for $g = 5$ with less detail.

4.2.1 Step 1

Here is the lemma promised by the proof outline.

Lemma 4.2.2. *For all $g \geq 2$, the element $F^-_{V_g - D^4, s_0}(\Theta^-) \in HF^-_{red}(S^3_{-1}(T_{2,2g+1}))$ is non-trivial, lies in the kernel of U and has absolute grading $-g^2 + g - 2$, which is the minimal grading in $HF^-_{red}(S^3_{-1}(T_{2,2g+1}))$.*

Proof. To prove that $F^-_{V_g - D^4, s_0}(\Theta^-)$ is non-trivial, first observe that ∂V_g is an admissible cut of $U(g+1, n)$ for $n \geq 3$ (see Definition 2.8.1). Indeed, we have seen that $b_2^+(V_g) > 0$, and $U(g+1, n) - V_g$ also contains a copy of V_g implying that $b_2^+(U(g+1, n)) > 0$ as well. Also, ∂V_g is an integer homology sphere, so the second requirement of admissibie cut is automatically satisfied.

The canonical spin^c structure k on $U(g+1, n)$ restricts to s_0 on V_g. Then it follows from Theorem 1.2.2 and the definition of absolute invariant (Definition 2.8.3) that

$$\pm 1 = \Phi_{U(g+1,n),k} = F^+_{U(g+1,n) - V_g - D^4, k'} \circ \tau^{-1} \circ F^-_{V_g - D^4, s_0}(\Theta^-).$$

It follows that $F^-_{V_g - D^4, s_0}(\Theta^-)$ is non-trivial (and in fact, primitive).

To prove the remaining assertions, we first set up the notations for some $\mathbb{Z}[U]$-modules.

- $\mathcal{T}^-$ denotes $\mathbb{Z}[U]$ which is $\langle 1, U, U^2, \ldots \rangle$.
- $\mathcal{T}^-(n)$ denotes $\mathbb{Z}[U]/(U^n \cdot \mathbb{Z}[U])$ which is $\langle 1, U, U^2, \ldots, U^{n-1} \rangle$.
- $\mathcal{T}^+$ denotes $\mathbb{Z}[U, U^{-1}]/(U \cdot \mathbb{Z}[U])$ which is $\langle 1, U^{-1}, U^{-2}, \ldots \rangle$.

- $\mathcal{T}^+(n)$ denotes the kernel of the action of U^n on $\mathcal{T}^+$, which is $\langle 1, U^{-1}, U^{-2}, \ldots, U^{-(n-1)} \rangle$.

- A subscript of $\mathcal{T}^+$, $\mathcal{T}^+(n)$, $\mathcal{T}^-$ and $\mathcal{T}^-(n)$ means that the module is graded in such a way that the absolute grading of the element 1 equals the subscript, and U shifts degree by -2.

By Borodzik and Némethi ([3], Proposition 6.1),

$$HF^+(-S^3_{-1}(T_{2,2g+1})) = \mathcal{T}^+_0 \oplus \left(\bigoplus_{k=0}^{g-2} \mathcal{T}^+_{(k+1)(k+2)} \left(g - \lfloor \frac{g+k+1}{2} \rfloor \right)^{\oplus 2} \right) \oplus \mathcal{T}^+_0 \left(g - \lfloor \frac{g}{2} \rfloor \right). \tag{4.2}$$

By duality (Equation 2.8) and universal coefficients, we have

$$HF^-(S^3_{-1}(T_{2,2g+1})) = \mathcal{T}^-_{-2} \oplus \left(\bigoplus_{k=0}^{g-2} \mathcal{T}^-_{-(k+1)(k+2)-2} \left(g - \lfloor \frac{g+k+1}{2} \rfloor \right)^{\oplus 2} \right) \oplus \mathcal{T}^-_{-2} \left(g - \lfloor \frac{g}{2} \rfloor \right). \tag{4.3}$$

We claim that for all $g \geq 2$, the minimal degree of $HF^-_{red}(S^3_{-1}(T_{2,2g+1}))$ is $-g^2+g-2$, attained uniquely by the summand with index $k = g-2$. Indeed, from Equation 4.3, the minimal degree corresponding to the summand with index k is

$$f(k) := -k^2 - 3k - 2 - 2g + 2\lfloor \frac{g+k+1}{2} \rfloor,$$

and the minimal degree corresponding to the last summand of Equation 4.3 is $-2g + 2 \cdot \lfloor \frac{g}{2} \rfloor$. Then one can check that $f(k+1) < f(k)$ for all $k \geq 0$, and $f(g-2) = -g^2 + g - 2 < -2g + 2 \cdot \lfloor \frac{g}{2} \rfloor$ for all $g \geq 2$.

The summand corresponding to $k = g-2$ is the $\mathbb{Z}[U]$-module $\mathcal{T}^-_{-g^2+g-2}(1) \oplus \mathcal{T}^-_{-g^2+g-2}(1)$ which is clearly in the kernel of U. Therefore, the proof of the lemma will be complete once we show that the degree of $F^{-}_{V_g-D^4,s_0}(\Theta^{-})$ must be $-g^2+g-2$. But it quickly follows from Lemma 3.4.1 and the degree shift formula (Equation 2.5) that the degree shift of $F^{-}_{V_g-D^4,s_0}$ is given by

$$\begin{aligned} & \frac{c_1^2(s_0) - 2\chi(V_g - D^4) - 3\sigma(V_g - D^4)}{4} \\ = & \frac{0 - 2(8g^2+4g) - 3(-4g^2-4g)}{4} \\ = & -g^2+g. \end{aligned}$$

Since Θ^{-} has degree -2, it follows that $F^{-}_{V_g-D^4,s_0}(\Theta^{-})$ has degree $-g^2+g-2$. $\square$

4.2.2 Step 2

Starting from this step, we shall specialize to the case $g=4$. We shall write down the mapping cone for calculating $\widehat{HF}(S^3_{-1}(T_{2,9}))$ in S^3 and calculate $CFK(S^3_{-1}(T_{2,9}), K)$ (up to filtered chain homotopy equivalence) using Hedden-Levine's work.

Recall that $T_{2,2g+1}$ is an alternating knot with symmetrized Alexander polynomial $\sum_{i=-g}^{g}(-1)^j T^j$ and signature $-2g$. Let $C = CFK(S^3, T_{2,2g+1})$. A classical result of Ozsváth and Szabó ([20], Theorem 1.3) says that up to filtered chain homotopy equivalence, the rank of $C\{0,j\}$ in $C = CFK(S^3, T_{2,2g+1})$ equals the absolute value of the coefficient of T_i in the symmetrized Alexander polynomial, and $C\{0,j\}$ is supported in absolute grading $j-g$. Since the homology of $C\{i=0\}$ must be $\widehat{HF}(S^3)=\mathbb{Z}$ with absolute grading 0 (Property 2.9.2 (3)), the homology of $C\{i=0\}$ is generated by $C\{0,g\}$. Then the differential ∂ of C is completely determined by the fact that ∂

decreases grading by 1: if x_j is the generator of $C\{0,j\}$, then $\partial x_g = 0$ and $\partial x_j = x_{j-1}$ for all $j = g-1, g-3, \ldots, -(g-3), -(g-1)$. The differential of C on $C\{j=0\}$ can be deduced by the same argument. Then Properties 2.9.2 determine the entire chain complex C. See Figure 4.1 for part of the chain complex in the case $g = 4$.

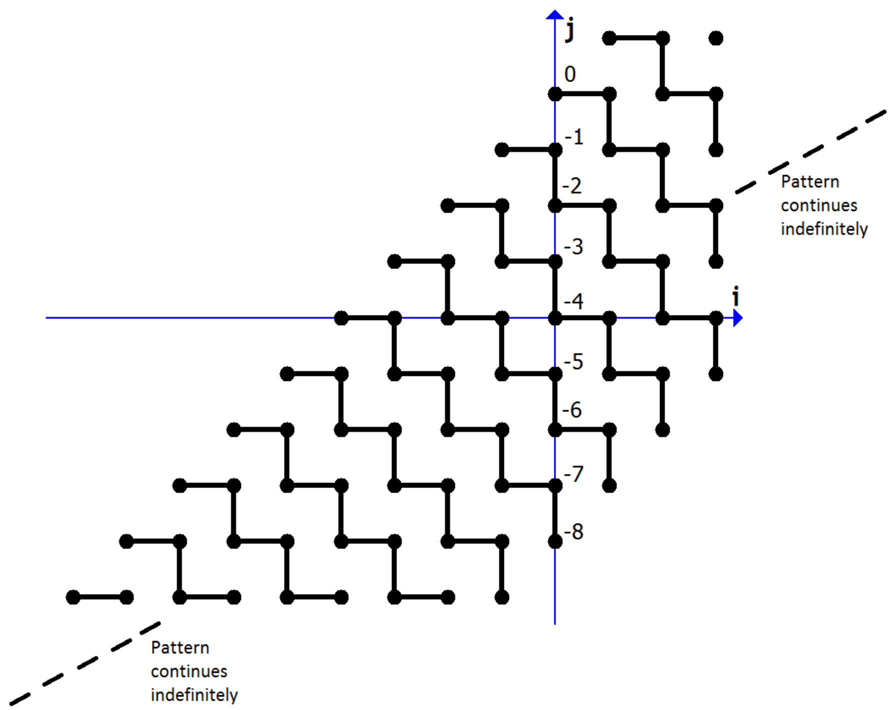

Figure 4.1: The chain complex $C = CFK(S^3, T_{2,9})$. Each solid dot represents $\mathbb{Z}$ and a segment between two dots represents the differential. The absolute grading of C is only labeled on $C\{i=0\}$, and the grading for the rest of C is determined by U-action.

Now we can form the mapping cone $\mathbb{X}(-1)$ of C which would calculate $\widehat{HF}(S^3_{-1}(T_{2,9}))$; see Section 2.9. Due to limitation of Theorem 2.9.3, we will pass to $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$ coefficients from now on. For the remainder of this step, consult Figure 4.2.

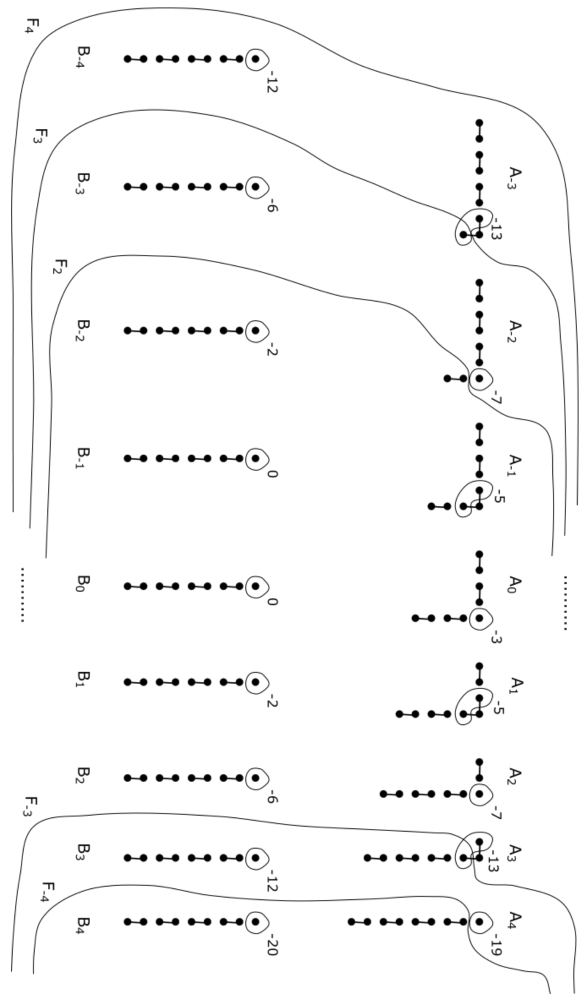

Figure 4.2: Part of $\mathbb{X}(-1)$ of $C = CFK(S^3, T_{2,9})$.

A description of the complexes A_s and B_s are as follows:

- For $-4 \leq s \leq 4$, the quotient complex A_s consists of the generators in the inverted L-shaped elbow going from $(0,-4)$ up to $(0,s)$, then going left from $(0,s)$ to $(s-4,s)$.

- For $s \geq 5$, A_s is $C\{i=0\}$.

- For $s \leq -5$, A_s is the horizontal complex $C\{j = s\}$.

- For all s, B_s is just $C\{i = 0\}$.

By Equation 2.16 at the discussion after Theorem 2.9.3, we have to calculate ker $(D_{-1})_*$ and coker $(D_{-1})_*$; to do so, we have to first determine the chain homotopy equivalence $\zeta : C\{i = 0\} \longrightarrow C\{j = 0\}$ in the definition of the h-map (see the paragraph following equation 2.15). This is easy in this scenario: since filtration levels $(0, k)$ and $(k, 0)$ are supported by a single absolute grading $k - 4$ for any $-4 \leq k \leq 4$, the fact that ζ must preserve absolute grading dictates that ζ maps $C\{0, k\}$ to $C\{k, 0\}$. Then it is easy to see that $(D_{-1})^*$ maps $H_*(A_s)$ isomorphically onto $H_*(B_s)$ for all $s \geq 5$ and $H_*(A_s)$ isomorphically onto $H_*(B_{s-1})$ for all $s \leq -4$; Figure 4.3 shows the v-maps and h-maps that justify the statement. Therefore, only A_s for $-3 \leq s \leq 4$ and B_s for $-4 \leq s \leq 4$, which is exactly the portion of $\mathbb{X}(-1)$ drawn in Figure 4.2, can contribute to $H_*(\mathbb{X}(-1))$.

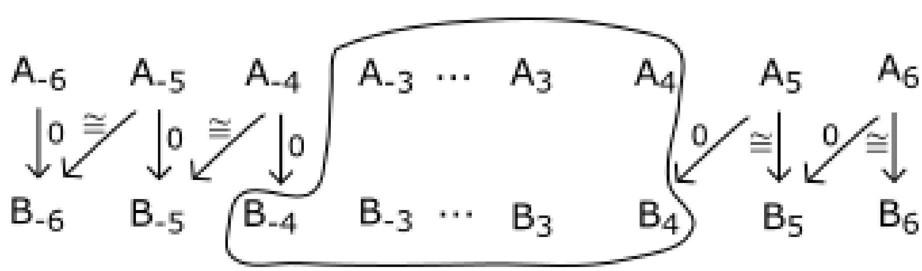

Figure 4.3: A schematic diagram of the mapping cone $\mathbb{X}(-1)$. The circled part is the only part that can contribute to the homology of $\mathbb{X}(-1)$.

Each small circle in Figure 4.2 contains one generator of $H_*(\mathbb{A}) \oplus H_*(\mathbb{B})$. One can check that $(D_{-1})_*$ is trivial on $H_*(\mathbb{A})$ except on $H_*(A_4)$, where the circled generator is mapped to the circled generator in $H_*(B_4)$. As a result, $H_*(\mathbb{X}(-1))$ is generated by all the circled generators except those in $H_*(A_4)$ and $H_*(B_4)$. The absolute grading of $\widehat{HF}(S^3_{-1}(T_{2,9}))$ can be determined by the method described after the statement of Theorem 2.9.3; they are labeled next to the generators. To summarize, we calculated that

$$\widehat{HF}(S^3_{-1}(T_{2,9})) = \left(\mathbb{F}_{(-3)} \oplus \mathbb{F}^2_{(-5)} \oplus \mathbb{F}^2_{(-7)} \oplus \mathbb{F}^2_{(-13)}\right) \oplus \left(\mathbb{F}^2_{(0)} \oplus \mathbb{F}^2_{(-2)} \oplus \mathbb{F}^2_{(-6)} \oplus \mathbb{F}^2_{(-12)}\right). \quad (4.4)$$

The same result, except in $\mathbb{Z}$-coefficients, can also be obtained from Equation 4.2 and the exact sequence 2.3.

We now follow Hedden and Levine to define a new filtration on $\mathbb{X}(-1)$ which will give $C'\{i=0\}$ up to filtered chain homotopy equivalence, where $C' = CFK(S^3_{-1}(T_{2,9}), K)$ and K is the induced knot in $S^3_{-1}(T_{2,9})$. For $-4 \leq k \leq 4$, define $\widetilde{A}_k = A_k \cap C\{j < k\}$; in other words, $\widetilde{A}_k$ is the vertical portion of A_k excluding the corner. Then we define a nested sequence of subcomplexes F_k of $\mathbb{X}(-1)$:

$$F_k = \widetilde{A}_{-k} \oplus B_{-k} \oplus \left(\bigoplus_{s>-k} A_s \oplus B_s\right).$$

Then it is clear that $F_k \supset F_l$ if $k \geq l$. Some of the subcomplexes are drawn in Figure 4.2 as well. Hedden and Levine's work shows that under the identification of $S^3_{-1}(T_{2,2g+1})$ with $\mathbb{X}(-1)$, the induced knot K in $S^3_{-1}(T_{2,2g+1})$ induces the filtration F_k on $\mathbb{X}(-1)$. In particular, for each j, the associated complex F_j/F_{j-1} generates $C'\{0,j\}$. Therefore, we have the following:

Lemma 4.2.3. *Let $C' = CFK(S^3_{-1}(T_{2,9}), K)$. Then, up to chain homotopy equivalence, $C'\{i=0\}$ at various filtration levels $(0,j)$ are as follows:*

$$
\begin{array}{ll}
j = 4: & \mathbb{F}_{(-12)} \\
j = 3: & \mathbb{F}_{(-13)} \oplus \mathbb{F}_{(-7)} \oplus \mathbb{F}_{(-6)} \\
j = 2: & \mathbb{F}_{(-2)} \\
j = 1: & \mathbb{F}_{(-5)} \oplus \mathbb{F}_{(-3)} \oplus \mathbb{F}_{(0)} \\
j = 0: & \mathbb{F}_{(0)} \\
j = -1: & \mathbb{F}_{(-7)} \oplus \mathbb{F}_{(-5)} \oplus \mathbb{F}_{(-2)} \\
j = -2: & \mathbb{F}_{(-6)} \\
j = -3: & \mathbb{F}_{(-19)} \oplus \mathbb{F}_{(-13)} \oplus \mathbb{F}_{(-12)} \\
j = -4: & \mathbb{F}_{(-20)}
\end{array}
$$

and there is a unique differential $\mathbb{F}_{(-19)} \longrightarrow \mathbb{F}_{(-20)}$ in the spectral sequence associated to the filtration of $C'\{i=0\}$ given by j.

This lemma and Property 2.9.2 then determine the entire C'.

4.2.3 Step 3

The goal of this step is to extract information of the chain homotopy equivalence ζ for C' (see the paragraph following equation (2.15)) by considering $\mathbb{X}'(1)$, the mapping cone for the $(+1)$-surgery on K. Instead of drawing the entire $\mathbb{X}'(1)$ like Figure 4.2, we will first make some observations that will reduce the number of generators that we have to handle. Overall, this step and the next will utilize the grading of C'

heavily.

(1) The v-map clearly preserves grading of C'. Also, the h-map shifts the grading of C' by a multiple of 2 because the action by U^s shifts grading by $-2s$ and ζ preserves grading. Therefore, we may split $H_*(\mathbb{A}') \oplus H_*(\mathbb{B}')$ into a subcomplex with odd gradings of C' and a subcomplex with even gradings of C'. We use $H_*(\mathbb{A}')_{odd}, H_*(\mathbb{A}')_{even}, H_*(\mathbb{B}')_{odd}$ and $H_*(\mathbb{B}')_{even}$ to denote the corresponding subgroups of $H_*(\mathbb{A}')$ and $H_*(\mathbb{B}')$.

(2) Recall from Proposition 4.2.1$'$ that our ultimate goal is to understand how the degree -13 $(= -4^2 + 4 - 1)$ subgroup of $\widehat{HF}(S^3_{-1}(T_{2,9}))$ behaves under the cobordism map $\widehat{F}$, or by Theorem 2.9.3, the inclusion into the various B'_s of a corresponding mapping cone. Since the inclusion respects grading, we only need to focus on the odd complex $H_*(\mathbb{A})_{odd} \oplus H_*(\mathbb{B})_{odd}$ in terms of analyzing ζ.

For the next two observations, a schematic diagram of the A'_s-complexes and the B'_s-complexes (Figure 4.4) will be useful. In the diagram, each box represents one filtration level. Solid dots represent generators with odd gradings and hollow dots represent generators with even gradings. Line segments represent differentials; note how A'_s contains one differential for $|s| \geq 4$ but two differentials for $|s| \leq 3$. The whole A'_{-1} is drawn explicitly to clarify the pattern suggested by A'_s for $-3 \leq s \leq 3$.

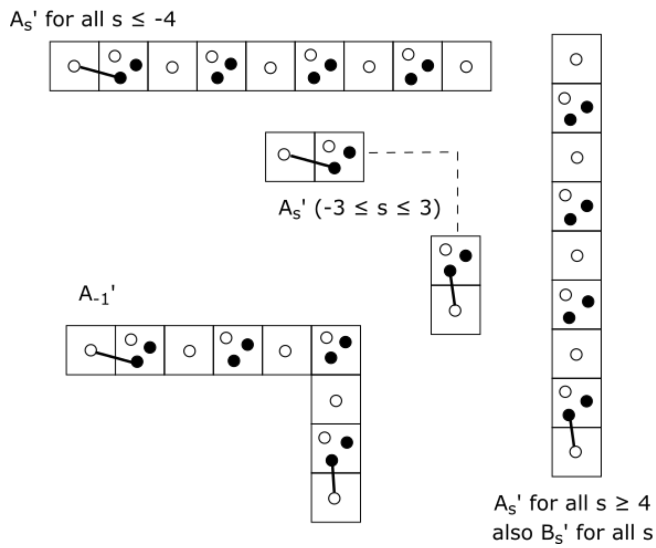

Figure 4.4: A schematic diagram of $\mathbb{X}'(1)$.

(3) Observe that:

- $(v_s)_* : H_*(A'_s) \longrightarrow H_*(B'_s)$ is trivial if $s \leq -4$ and is an isomorphism if $s \geq 5$. The case $|s| \geq 5$ is clear from the definition of v_s. The case $s = -4$ holds because the only generator of A'_{-4} that survives the vertical projection is in $C'\{0,-4\}$ (the rightmost box), whose image is the generator of B'_{-4} in $C'\{0,-4\}$ (the bottom box) which is trivial in $H_*(B'_{-4})$.

- $(h_s)_* : H_*(A'_s) \longrightarrow H_*(B'_{s+1})$ is trivial if $s \geq 4$ and is an isomorphism if $s \leq -5$. The case $|s| \geq 5$ is again clear from the definition of h_s. For $s = 4$, the only generator in A'_4 that survives the horizontal projection is $C'\{0,4\} = \mathbb{F}_{(-12)}$ (the top box, also see Lemma 4.2.3 for the degree). By observation (1), h_4 shifts degree by -8, so the image of $C'\{0,4\}$ under h_4

is contained in $\mathbb{F}_{(-20)}$ (the bottom box by Lemma 4.2.3) which is trivial in $H_*(B_5)$.

Figure 4.5 summarizes the above assertions on the v_*-maps and h_*-maps.

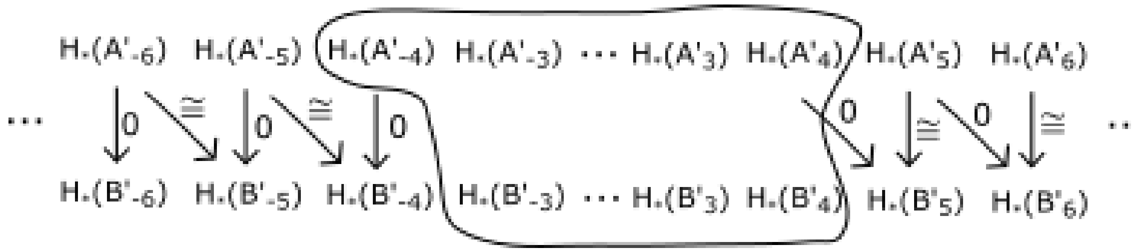

Figure 4.5: A schematic diagram of $H_*(\mathbb{X}'(1))$.

Therefore, only $H_*(A'_s)$ for $-4 \leq s \leq 4$ and $H_*(B'_s)$ for $-3 \leq s \leq 4$ (circled in the figure) can contribute to $H_*(\mathbb{X}'(1))$, and from now on, we shall restrict the groups defined in the previous observation to these $H_*(A'_s)$ and $H_*(B'_s)$.

(4) Now we count the ranks of $H_*(\mathbb{A}')_{odd}, H_*(\mathbb{A}')_{even}, H_*(\mathbb{B}')_{odd}$ and $H_*(\mathbb{B}')_{even}$, restricted to relevant values of s based on the previous observation.

- For $\mathbb{A}'$, we see that $H_*(A'_s)$ has 6 odd-degree generators and 7 even-degree generators for $-3 \leq s \leq 3$. However, $H_*(A'_4)$ and $H_*(A'_{-4})$ each has 7 generator and 8 even-degree generator (one extra for each parity) because A'_{-4} and A'_4 have one fewer differential than A_s for $-3 \leq s \leq 3$. As a result, $rk(H_*(\mathbb{A}')_{odd}) = 6\times 7+7\times 2 = 56$ and $rk(H_*(\mathbb{A}')_{even}) = 7\times 7+8\times 2 = 65$.
- For $\mathbb{B}'$, we see that $H_*(B'_s)$ has 7 odd-degree generators and 8 even-degree

generators for all s, just like $H_*(A'_4)$. Therefore, $rk(H_*(\mathbb{B}')_{odd}) = 7 \times 8 = 56$ and $rk(H_*(\mathbb{B}')_{even}) = 8 \times 8 = 64$.

Now, the 3-manifold resulting from the (+1)-surgery is S^3, and Theorem 2.9.3 tells us that $H_*(\mathbb{X}'(1)) \cong \widehat{HF}(S^3) \cong \mathbb{Z}$. It follows from the rank count that the generator of $H_*(\mathbb{X}'(1))$ is contained in $H_*(\mathbb{A}')_{even}$ (which is consistent with the fact that $\widehat{HF}(S^3)$ has absolute grading 0). In particular, $H_*(\mathbb{X}'(1))$ restricted on the odd-degree complex is trivial.

With these observations in place, we are ready to analyze the chain homotopy equivalence ζ. We define $x_{i,m}$ to be the generator of $C'\{i, 0\}$ with grading m and $y_{j,n}$ be the generator of $C'\{0, j\}$ with grading n (so $x_{0,m}$ and $y_{0,m}$ coincide).

Claim 4.2.4. *The chain homotopy equivalence ζ for C' maps $x_{-3,-13}$ to $y_{3,-13}$.*

Proof. Part of $\mathbb{X}'(1)$ is drawn in Figure 4.6. The bottom grid of each $H_*(A'_s)$ and $H_*(B'_s)$ (which is empty) represents the filtration level $(0, -4)$.

The grading convention in the figure is as follows. The generators in the vertical portions of each $H_*(A'_s)$ and $H_*(B'_s)$ are labeled with gradings from C' (numbers not contained in parentheses). The generators of the horizontal portions in each $H_*(A'_s)$ are labeled with gradings from C' but shifted by $-2s$ (numbers contained in parentheses), which is the degree shift by U^s. The generators in the corners of each $H_*(A'_s)$ are labeled both gradings. Under this convention, the v-maps preserve the gradings not contained in parentheses, and the h-maps preserve the gradings contained in parentheses. This grading convention is convenient for inspecting the behavior of v- and h-maps, and will be useful in Section 4.2.4 as well.

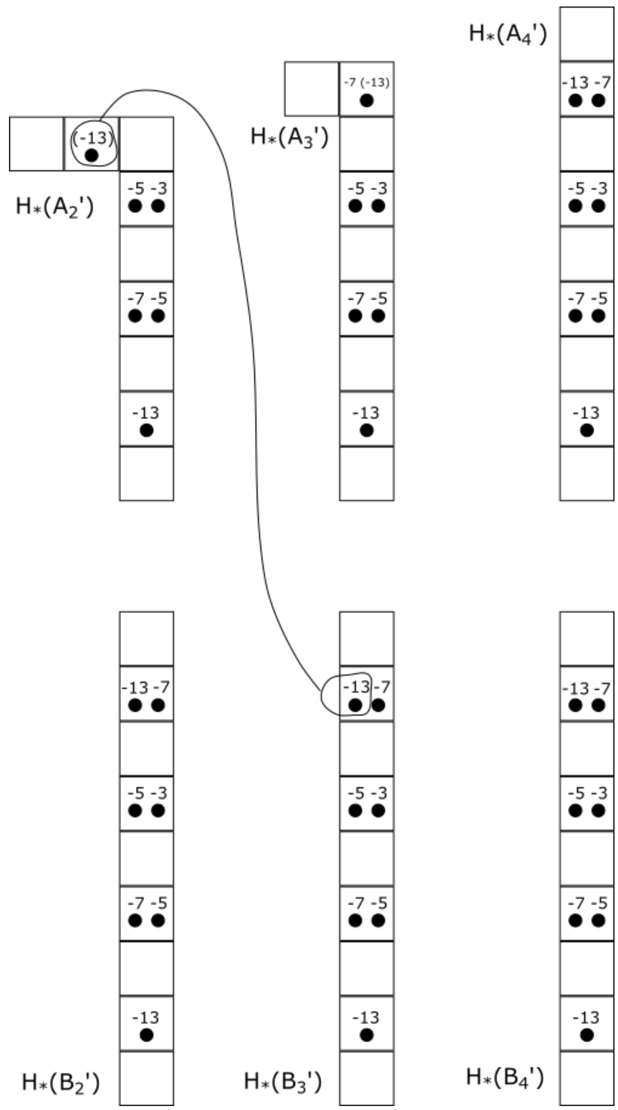

Figure 4.6: Part of $\mathbb{X}'(1)$, showing only the odd-degree generators.

Now observe that the generator $y_{3,-13}$ in B_3' is not in the image of v_3. We have seen before that $H_*(\mathbb{X}'(1))$ restricted to the odd complex is trivial, so the $y_{3,-13}$ in B_3'

must be in the image of h_2. Since h_2 is only non-trivial at the generator in filtration level $(-1,2)$ (circled in A'_2) whose image under U^2 is $x_{-3,-13}$, the claim follows. $\square$

4.2.4 Step 4

It remains to utilize Claim 4.2.4 to study $\mathbb{X}'(0)$, the mapping cone for the 0-surgery on K, allowing us to calculate the *twisted* cobordism map $\underline{\widehat{F}}_{W_g}$ based on the discussion after Theorem 2.9.3. In particular, our goal is to prove that the degree (-13) subgroup of $\widehat{HF}(S^3_{-1}(T_{2,9}))$, generated by $\{y_{-3,-13}, y_{3,-13}\}$, is trivial under the map $\widehat{HF}(S^3_{-1}(T_{2,9})) \longrightarrow H_*(B'_s) \otimes_{\mathbb{F}} \mathbb{F}[T,T^{-1}]$ if $|s| < 3$. It is the same as proving the following:

Claim 4.2.5. *The subgroup* $\langle y_{-3,-13}, y_{3,-13}\rangle \otimes_{\mathbb{F}} \mathbb{F}[T,T^{-1}]$ *in* $H_*(B'_s) \otimes_{\mathbb{F}} \mathbb{F}[T,T^{-1}]$ *is trivial in* $H_*(\mathbb{X}'(0))$ *if* $|s| < 3$.

Proof. Now $\mathbb{X}'(0)$ splits as a sum of mapping cones indexed by s (see the visualization of the mapping cones preceding Theorem 2.9.3). The value s corresponds to the spin^c structure $\mathcal{S}_s$ on the cobordism satisfying $\langle c_1(\mathcal{S}_s), [\Sigma]\rangle = 2s$; denote the restriction of $\mathcal{S}_s$ on $S^3_0(T_{2,9})$ by t_s. Since $\underline{\widehat{HF}}(S^3_0(T_{2,9}), t_s; \mathbb{F}[T,T^{-1}])$ vanishes whenever $|s| > 3$ (by adjunction inequality, Proposition 7.1 of [22]), we only need to consider $|s| \leq 3$. While the statement of the claim only requires us to consider $|s| < 3$, we will also discuss the case $|s| = 3$ for the sake of completeness.

See Figure 4.7 for a diagram of $\mathbb{X}'(0)$, where each dot now represents $\mathbb{F}[T,T^{-1}]$ instead of $\mathbb{F}$. The complexes $H_*(A'_s) \otimes_{\mathbb{F}} \mathbb{F}[T,T^{-1}]$ and $H_*(B'_s) \otimes_{\mathbb{F}} \mathbb{F}[T,T^{-1}]$ are drawn horizontally corresponding to Figure 4.6 in the following way: going from left to right in Figure 4.7 corresponds to going from upper left corner to the lower right corner for the A'-complexes and going from up to down for the B'-complexes in Figure 4.6. The generators are labeled with gradings as explained in the proof of Claim 4.2.4.

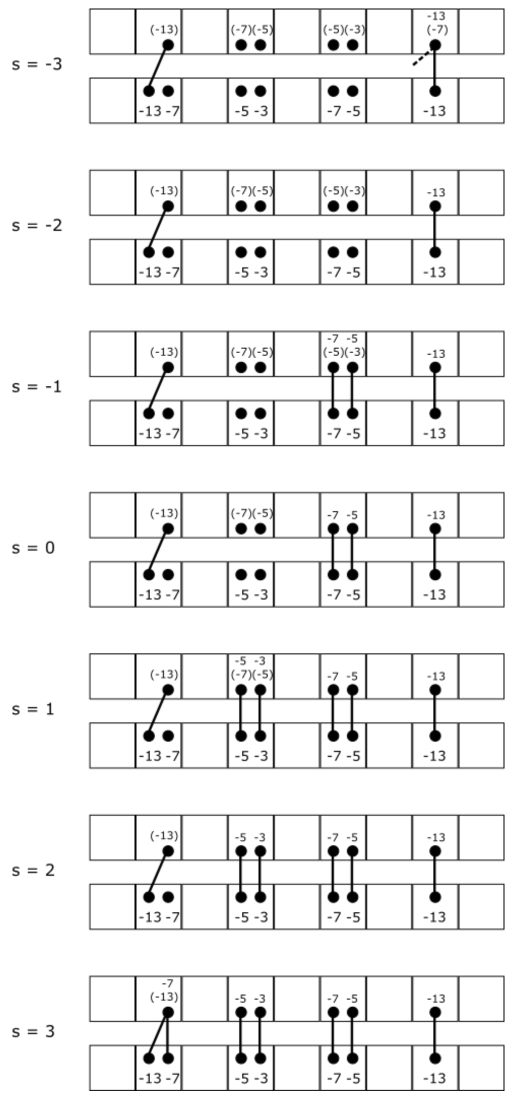

Figure 4.7: A picture of $\mathbb{X}'(0)$.

Now, for $|s| < 3$, we exhibit a subgroup of $H_*(A'_s)$ that surjects onto $\langle y_{-3,-13}, y_{3,-13}\rangle \otimes_{\mathbb{F}}$

$\mathbb{F}[T, T^{-1}]$ under $(D'_0)_* = v_s^* + T \cdot h_s^*$:

- Let P denote the submodule represented by the leftmost dot in $H_*(A'_s) \otimes_{\mathbb{F}} \mathbb{F}[T, T^{-1}]$. Then P is mapped to $\langle x_{-3,-13} \rangle \otimes_{\mathbb{F}} \mathbb{F}[T, T^{-1}]$ under the horizontal projection and the multiplication by U^s. From Claim 4.2.4, ζ sends $\{x_{-3,-13}\} \otimes 1$ to $\{y_{3,-13}\} \otimes 1$ (represented by the slanted segments in Figure 4.7). Combining, h_s^* maps P isomorphically onto $\langle y_{3,-13} \rangle \otimes_{\mathbb{F}} \mathbb{F}[T, T^{-1}]$, and the same is true for $T \cdot h_s^*$ since T is a unit in $\mathbb{F}[T, T^{-1}]$. Also, it is clear that $v_s^*(P)$ is trivial. Therefore, $(D'_0)^*(P) = \langle y_{3,-13} \rangle \otimes_{\mathbb{F}} \mathbb{F}[T, T^{-1}]$.

- Let Q denote the submodule represented by rightmost dot in $H_*(A'_s) \otimes_{\mathbb{F}} \mathbb{F}[T, T^{-1}]$. Obviously, v_s^* maps Q isomorphically onto $\langle y_{-3,-13} \rangle \otimes_{\mathbb{F}} \mathbb{F}[T, T^{-1}]$ (represented by the rightmost vertical segments). Also, $T \cdot h_s^*(Q)$ is trivial because Q vanishes under the horizontal projection. Therefore, $(D'_0)^*(Q) = \langle y_{-3,-13} \rangle \otimes_{\mathbb{F}} \mathbb{F}[T, T^{-1}]$.

Therefore, the image of $P \oplus Q$ under $(D'_0)^*$ is $\langle y_{-3,-13}, y_{3,-13} \rangle \otimes_{\mathbb{F}} \mathbb{F}[T, T^{-1}]$. This proves that the latter is trivial in $H_*(\mathbb{X}'(0))$ and finishes the proof of the claim. $\square$

This completes Step 4 and the proof of Proposition 4.2.1.

Remark 4.2.6. .

- For $|s| < 3$, it is also true that the images under $(D'_0)^*$ of the submodules represented by the dots other than P and Q in $H_*(A'_s)$ has no component in $\langle y_{-3,-13}, y_{3,-13} \rangle \otimes_{\mathbb{F}} \mathbb{F}[T, T^{-1}]$, because those submodules have the wrong gradings.

- For $|s| = 3$, at least one of the elements $y_{-3,-13}$ and $y_{3,-13}$ must be non-trivial in $H_*(\mathbb{X}'(0))$. (For $s = -3$, the rightmost generator of A'_s must map non-trivially to some degree (-7) under h'_3.) This is consistent with the fact that the relative invariant $\Psi_{U_g,k|_{U_g}}$ must be non-trivial (which is a consequence of Theorem 1.2.2).

4.2.5 Sketch of proofs for the case $g = 5$

The calculations for the case $g = 5$ is very similar to $g = 4$. Step 2 has given a description of $CFK(S^3, T_{2,2g+1})$ for general g. Then we can apply Hedden and Levine's method to calculate $CFK(S^3_{-1}(T_{2,11}), K)$:

Lemma 4.2.7. *Let $C' = CFK(S^3_{-1}(T_{2,11}), K)$. Then up to filtered chain homotopy equivalence, $C'\{i = 0\}$ at various filtration levels $(0, j)$ are as follows:*

$$
\begin{aligned}
j = 5 : &\quad \mathbb{F}_{(-20)} \\
j = 4 : &\quad \mathbb{F}_{(-21)} \oplus \mathbb{F}_{(-13)} \oplus \mathbb{F}_{(-12)} \\
j = 3 : &\quad \mathbb{F}_{(-6)} \\
J = 2 : &\quad \mathbb{F}_{(-9)} \oplus \mathbb{F}_{(-5)} \oplus \mathbb{F}_{(-2)} \\
j = 1 : &\quad \mathbb{F}_{(0)} \\
j = 0 : &\quad \mathbb{F}_{(-5)} \oplus \mathbb{F}_{(-5)} \oplus \mathbb{F}_{(0)} \\
j = -1 : &\quad \mathbb{F}_{(-2)} \\
j = -2 : &\quad \mathbb{F}_{(-13)} \oplus \mathbb{F}_{(-9)} \oplus \mathbb{F}_{(-6)} \\
j = -3 : &\quad \mathbb{F}_{(-12)} \\
j = -4 : &\quad \mathbb{F}_{(-29)} \oplus \mathbb{F}_{(-21)} \oplus \mathbb{F}_{(-20)} \\
j = -5 : &\quad \mathbb{F}_{(-30)}
\end{aligned}
$$

and there is a unique differential $\mathbb{F}_{(-29)} \longrightarrow \mathbb{F}_{(-30)}$ *in the spectral sequence associated to the filtration of* $C'\{i=0\}$ *given by* j.

The observations and setup in Step 3 apply verbatim except numerical changes. We still consider $\mathbb{X}'(1)$, the mapping cone for the $(+1)$-surgery on K, since the result of the surgery is still S^3.

- Only $H_*(A'_s)$ for $-5 \leq s \leq 5$ and $H_*(B'_s)$ for $-4 \leq s \leq 5$ can contribute to $H_*(\mathbb{X}'(1))$.

- We are interested in the subgroup of $\widehat{HF}(S^3_{-1}(T_{2,11}))$ with degree $-5^2+5-1=-21$ instead of -13, so again, we only need to focus on the odd complex $H_*(\mathbb{A})_{odd} \oplus H_*(\mathbb{B})_{odd}$.

- The rank count is now:

  - $rk(H_*(\mathbb{A}')_{odd}) = 8 \times 9 + 9 \times 2 = 90$, and $rk(H_*(\mathbb{B}')_{odd}) = 9 \times 10 = 90$.

  - $rk(H_*(\mathbb{A}')_{even}) = 9 \times 9 + 10 \times 2 = 101$, and $rk(H_*(\mathbb{B}')_{even}) = 10 \times 10 = 100$.

  So $H_*(\mathbb{X}'(1))$ restricted to the odd-degree complex is still trivial.

Then we can form the odd complex $H_*(\mathbb{A})_{odd} \oplus H_*(\mathbb{B}_{odd})$, which behave similarly to that for $g=4$. In particular, we observe that the two cases $g=4$ and $g=5$ have the following in common:

- $H_*(B'_s)_{odd}$ is supported at filtration levels $(0,j)$ where $j = g-1, g-3, \ldots, -(g-3), -(g-1)$ and has rank 2 at each of these filtration levels.

- The degree-$(-g^2+g-1)$ subgroup of $H_*(B'_s)_{odd}$ has rank 1 at filtration levels $(0, \pm(g-1))$. Using the notations preceding Claim 4.2.4, the generators are $y_{-(g-1),-g^2+g-1}$ and $y_{g-1,-g^2+g-1}$.

- The generator $y_{-g^2+g-1,g-1}$ in $H_*(B'_{g-1})$ is not in the image of v_{g-1}, which must then be $h_{g-2}(p)$, where p is the generator at filtration level $(-1, g-2)$ in $H_*(A'_{g-2})$.

- $x_{-g^2+g-1,-(g-1)}$ equals $U^{g-2} \cdot p$.

Thus we have established a similar claim to Claim 4.2.4:

Claim 4.2.8. *The chain homotopy equivalence ζ for C' maps $x_{-4,-21}$ to $y_{4,-21}$.*

Then we can form the mapping cone $\mathbb{X}'(0)$. The argument for Step 4 applies verbatim except obvious numerical changes.

4.3 Proofs of Theorem 1.3.1 and Corollary 1.3.2

Now we are ready to prove Theorem 1.3.1 and Corollary 1.3.2.

Proof of Theorem. Recall from the beginning of the chapter that

$$\Psi_{U_g,s} = \underline{F}^-_{W_g,s|_{W_g}} \circ F^-_{V_g-D^4,s|_{V_g-D^4}}(\Theta^-). \tag{4.5}$$

Proposition 4.1.1 tells us that $F^-_{V_g-D^4,s|_{V_g-D^4}}(\Theta^-)$ vanishes except when $s|_{V_g-D^4}$ is the trivial spin^c structure s_0, and Lemma 4.2.2 tells us that $x := F^-_{V_g-D^4,s_0}(\Theta^-)$

has degree $-g^2 + g - 1$. Then, Proposition 4.2.1 tells us that if $g = 4$ or 5, then $\underline{F}^-_{W_g, s|_{W_g}}(x)$ is trivial if $|\langle c_1(s|_{W_g}), [\Sigma]\rangle| < 2g - 2$. Combining, $\Psi_{U_g,s}(\Theta^-)$ is trivial if $|\langle c_1(s), [\Sigma]\rangle| < 2g - 2$. □

Proof of Corollary. Let Σ be a generic fiber of X and s be any spin^c structure on X. If $|\langle c_1(s), [\Sigma]\rangle| < 2g-2$, then Theorem 1.3.1 tells us that $\Psi_{U_g, s|_{U_g}}$ is trivial. Then the right hand side of Equation (2.12) in Theorem 2.8.4 is trivial. Since Equation (2.12) is an equality of *group ring elements*, every individual term on the sum on the left side is trivial. In particular, $\Phi_{X,s}$ is trivial.

Thus we have shown that $\Phi_{X,s}$ is trivial unless $|\langle c_1(s), [\Sigma]\rangle| = 2g - 2$. The result immediately follows from Theorem 1.2.2. □

Chapter 5

Additional Results

Suppose $X = Z_1 \cup_Y Z_2$ is the oriented closed 4-manifold formed by gluing Z_1 and Z_2 along their common boundary Y. If we take $Y = \Sigma_g \times S^1$, then this would include the situation of taking *fiber sum* of two genus g Lefschetz fibrations (i.e. removing the neighborhood of a regular fiber of each Lefschetz fibration and gluing them by a fiber-preserving, orientation-reversing diffeomorphism $\Sigma_g \times S^1 \longrightarrow \Sigma_g \times S^1$). The groups $HF^+(\Sigma_g \times S^1, t)$ at all spin^c structures t are well understood, and as mentioned in Section 2.6, while these groups do not admit absolute $\mathbb{Q}$-gradings when t is non-torsion, they do admit absolute plane field gradings, which shall be reviewed shortly. We will convert these plane field gradings to Gompf's Θ-grading [11] and prove a degree shift formula analogous to Equation (2.5). Based on that, we apply Jabuka and Mark's pairing theorem (Theorem 2.8.4) in the same manner as the proof of Theorem 1.3.1 to obtain a numerical degree shift criterion that Ozsváth-Szabó basic classes must satisfy. In principle, we can derive degree shift criteria with other choices of Y provided that we can understand the Θ-gradings of $HF^+(Y)$.

In this chapter, all plane fields considered are oriented.

5.1 Absolute plane field grading and Θ-grading

Let ξ be an oriented plane field on a 3-manifold Y, $[\xi]$ be the homotopy class of ξ, t_ξ be the spin^c structure induced by ξ, and $\bar{\xi}$ be the conjugate of ξ (defined by reversing the orientation of each plane). Then Huang and Ramos [15] defined an absolute grading by homotopy classes of oriented plane fields on various flavors of HF-groups which satisfies these properties (HF° stands for $\widehat{HF}, HF^+, HF^-, HF^\infty$ and HF_{red}):

- $HF^\circ_{[\xi]}(Y) \subset HF^\circ(Y, t_\xi)$, where $HF^\circ_{[\xi]}(Y)$ denotes the subgroup of $HF^\circ(Y)$ with degree $[\xi]$.
- $HF^\circ_{[\xi]}(Y, t_\xi) \cong HF^\circ_{[\bar{\xi}]}(Y, t_{\bar{\xi}})$. This statement can be seen as a refinement of $HF^\circ(Y, t_\xi) \equiv HF^\circ(Y, t_{\bar{\xi}})$ (Equation (2.1)).
- The absolute plane field grading lifts the relative $\mathbb{Z}/d(c_1(t_\xi))$-grading mentioned in Section 2.6.

The cobordism map $F^\circ_{W,s}$ respects the absolute plane field grading in the following sense:

Theorem 5.1.1 ([15], Theorem 1.1(d))**.** *If x is a homogeneous element in $HF^\circ_{[\xi_1]}(Y_1, t_1)$ and y is a non-trivial homogeneous component of $F^\circ_{W,s}(x)$ lying in $HF^\circ_{[\xi_2]}(Y_2, t_2)$, then there exists an almost complex structure J on W that induces s on W and induces ξ_i on Y_i for $i = 1, 2$ i.e. $\xi_1 = TY_1 \cap J(TY_1)$ and $\xi_2 = TY_2 \cap J(TY_2)$.*

We move on to numerical gradings of oriented plane fields. If $c_1(t_\xi)$ is torsion, then there is a standard $\mathbb{Q}$-valued 3-dimensional invariant (see [11] Chapter 11, for example):

$$d_3([\xi]) = \frac{c_1^2(X,J) - 2\chi(X) - 3\chi(X)}{4}, \tag{5.1}$$

where (X, J) is any almost-complex 4-manifold bounded by (Y, ξ).

Remark 5.1.2. The plane field grading $[\xi]$ is equivalent to $d_3([\xi]) + \frac{1}{2}$ in the absolute $\mathbb{Q}$-grading defined in Section 2.6.

However, if $c_1(t_\xi)$ is non-torsion, then the above definition does not make sense because there is no canonical way to calculate the square $c_1^2(X, J)$. To this end, Gompf ([10], Chapter 4) devised the following method to calculate such a square. We start with choosing a smooth 2-cycle z representing $PDc_1(X, J) \in H_2(X, \partial X)$, so that ∂z is a smooth 1-cycle on ∂X. If we choose a framing ν on ∂z, then we can attach a 2-handle along ∂z with framing ν to form a new manifold $\widehat{X}$, and extend z over the core of the 2-handle to form $\widehat{z}$ which is a smooth 2-cycle in $\widehat{X}$. Then we can define our square to be $\epsilon = [\widehat{z}]^2$ using the intersection pairing in $H_2(\widehat{X})$.

Now we explain the dependence of ϵ on the choices that we have made. First of all, if we change z within its class in $H_2(X, \partial X)$ keeping ∂z fixed, then $[\widehat{z}]$ changes by a class in $H_2(\partial X)$, so ϵ changes by an even multiple of $d(\partial z)$, where $d(\partial z)$ denotes the divisibility of ∂z. Secondly, if y is another smooth 1-cycle on ∂X in the same class as ∂z, then notice that there exists a framing ν' on y so that ∂z and y are *framed cobordant*, i.e. there exists a surface W in $\partial X \times [0, 1]$ so that $W \cap (\partial X \times \{0\}) = \partial z$,

$W \cap (\partial X \times \{1\}) = y$, and a parallel pushoff of W induces the framings ν, ν' on ∂z and y respectively. It is then clear that ϵ is independent of the choice of representative in a fixed framed cobordism class. Finally, fixing ∂z, Gompf ([10], Prop. 4.1) showed that the set of framed cobordism classes supporting ∂z is a $\mathbb{Z}/2d(\partial z)$-set (with $+1$ action meaning adding a right twist to the framing ν of ∂z). Since adding a right twist to ν increases ϵ by 1, it follows that adding $2d(\partial z)$ right twists to ν, while preserving the framed cobordism class, increases ϵ by $2d(\partial z)$.

Therefore, overall, ϵ is well-defined modulo $\mathbb{Z}/2d(\xi)$ for any fixed choice of framed cobordism class. (To simplify notations, $d(\xi)$ means $d(c_1(t_\xi))$.) Gompf defined *a framing f on a class $x \in H_1(\partial X)$* to be a choice of framed cobordism class supporting x, i.e. a smooth 1-chain that represents x and a framing (in the ordinary sense) on that 1-chain. Then ϵ is denoted by $Q_f(PDc_1(X, J))$.

With this in place, we are ready to define Gompf's Θ-invariant ([11], Definition 4.2).

Definition 5.1.3. Let ξ be an oriented 2-plane field on a closed, oriented 3-manifold M (not necessarily connected). Let (X, J) be an almost complex 4-manifold so that $\partial(X, J) = (M, \xi)$, i.e. $\partial X = M$ and J induces ξ on M. Then for any framing f on $PDc_1(\xi)$, we define

$$\Theta_f([\xi]) = Q_f(PDc_1(X, J)) - 2\chi(X) - 3\sigma(X) \in \mathbb{Z}/2d(\xi).$$

Fix a spin^c structure t on Y. It is well-known that (see [11], Proposition 4.1 for example) the set $P(Y,t)$ of homotopy classes of plane fields ξ inducing t has cardinality $d(\xi)$. Moreover, $P(Y,t)$ is a $\mathbb{Z}/d(\xi)$-set with action defined as follows. Fix a metric on Y and trivialization $TY \cong Y \times \mathbb{R}^3$; then by taking normal vectors, an oriented plane field ξ corresponds to a map $\xi^\tau : Y \longrightarrow S^2$. By the Pontryagin-Thom construction, homotopy classes of such maps correspond to cobordism classes of framed links in Y. The action of $1 \in \mathbb{Z}/d(\xi)$ on $[\xi]$, which we shall denote by $[\xi]+1$, is given by adding a *left* twist to the framing of some component of a link representing $[\xi]$. Note that this action implicitly depends on the orientation of Y.

By [11] Proposition 4.16, the Θ-grading satisfies

$$\Theta_f([\xi]+1) = \Theta_f([\xi]) + 4. \tag{5.2}$$

Remark 5.1.4. Gompf defined the $+1$ action to be adding a *right* twist instead. In his convention, $\Theta_f([\xi]+1) = \Theta_f([\xi]) - 4$. Our convention is more customary in the Heegaard Floer context.

Since Θ takes value in $\mathbb{Z}/2d(\xi)$, it follows from Equation 5.2 that $\Theta_f([\xi]+\frac{d(\xi)}{2}) = \Theta_f([\xi])$ for all $[\xi] \in P(Y,t)$; in particular, Θ is a two-to-one map from $P(Y,t)$ to their Θ-values. Gompf defined in the same paper a 2-fold lift $\widetilde{\Theta}$ of Θ which takes value in $\mathbb{Z}/4d(t)$, so that $\widetilde{\Theta}$ maps bijectively from $P(Y,t)$ to their $\widetilde{\Theta}$-values. We will not utilize $\widetilde{\Theta}$ in the present work, although it might be possible to strengthen the results in this chapter by utilizing $\widetilde{\Theta}$ instead of Θ.

5.2 Degree shift formula

We are now ready to state and prove the degree shift formula for the Θ-grading.

Proposition 5.2.1. *Let (X_1, J_1) be an almost complex cobordism from Y_0 to Y_1 so that $\partial(X_1, J_1) = (-Y_0, \xi_0) \sqcup (Y_1, \xi_1)$. Let s_1 be the spin^c structure determined by J_1. For $i = 0, 1$, let t_i be the restriction of s_1 to Y_i, d_i be the divisibility of $c_1(t_i)$ and f_i be a framing on $PDc_1(t_i)$. Then*

$$\Theta_{f_0}([\xi_0]) + Q_{f_0,f_1}(PDc_1(s_1)) - 2\chi(X_1) - 3\sigma(X_1) \equiv \Theta_{f_1}([\xi_1]) \mod \gcd(2d_0, 2d_1). \tag{5.3}$$

Proof. Let (X_0, J_0) be an almost complex manifold so that $\partial(X_0, J_0) = (Y_0, \xi_0)$, and let s_0 be the spin^c structure determined by J_0. Then we may form $(X_0 \cup X_1, J_0 \cup J_1)$ with the induced spin^c structure $s_0 \cup s_1$. By definition, we have

$$\begin{aligned}
\Theta_{f_0}(\xi_0) &= Q_{f_0}(PDc_1(s_0)) - 2\chi(X_0) - 3\sigma(X_0) \in \mathbb{Z}/2d_0;\\
\Theta_{f_1}(\xi_1) &= Q_{f_1}(PDc_1(s_0 \cup s_1)) - 2\chi(X_0 \cup X_1) - 3\sigma(X_0 \cup X_1) \in \mathbb{Z}/2d_1.
\end{aligned}$$

By additivity of Euler characteristic and signature, we have

$$\Theta_{f_0}(\xi_0) + [Q_{f_1}(PDc_1(s_0 \cup s_1)) - Q_{f_0}(PDc_1(s_0))] - 2\chi(X_1) - 3\sigma(X_1) \equiv \Theta_{f_1}(\xi_1) \mod \gcd(2d_0, 2d_1).$$

Thus the proposition follows once we show that

$$Q_{f_1}(PDc_1(s_0 \cup s_1)) \equiv Q_{f_0}(PDc_1(s_0)) + Q_{f_0,f_1}(PDc_1(s_1)) \mod \gcd(2d_0, 2d_1). \tag{5.4}$$

To prove the above equality, let z be a smooth 2-cycle representing $PDc_1(s_0 \cup s_1) \in H_2(X_0 \cup X_1, Y_1)$, so that ∂z is a smooth 1-cycle γ in Y_1. Then by definition,

$$Q_{f_1}(PDc_1(s_0 \cup s_1)) \equiv [\widehat{z}]^2 \mod 2d_1$$

where $\widehat{z}$ is the cap-off defined in the discussion preceding Definition 5.1.3. On the other hand, we may assume Y_0 intersects z at a smooth 1-cycle, which divides z into z_0 and z_1, and we can similarly obtain cap-offs $\widehat{z_0}$ and $\widehat{z_1}$. And so

$$Q_{f_0}(PDc_1(s_0)) \equiv [\widehat{z_0}]^2 \mod 2d_0, \qquad Q_{f_0,f_1}(PDc_1(s_1)) \equiv [\widehat{z_1}]^2 \mod \gcd(2d_0, 2d_1).$$

To analyze $[\widehat{z_0}]^2$ and $[\widehat{z_1}]^2$, we consider pushoffs of $\widehat{z_0}$ and $\widehat{z_1}$. If we arrange the pushoffs of $\widehat{z_0}$ and $\widehat{z_1}$ to be identical near Y_0, then we can glue the two pushoffs to form a pushoff of $\widehat{z}$. Then it is clear that

$$[\widehat{z_0}]^2 + [\widehat{z_1}]^2 = [\widehat{z}]^2,$$

and Equation (5.4) follows. □

5.3 Contact geometry of $\Sigma_g \times S^1$

We are going to apply Proposition 5.2.1 on cobordisms from S^3 to $\Sigma_g \times S^1$ and $-\Sigma_g \times S^1$. While classical results give us a lot of understanding of the algebraic structure of $HF^+(\pm\Sigma_g \times S^1)$, we will need to fill in the Θ-gradings of these groups. To this end, we shall:

(1) Construct plane fields on $\Sigma_g \times S^1$ which are *contact structures* induced by *Stein structures*.

(2) Identify those plane fields with special gradings in the Heegaard Floer groups.

(3) Combine (1) and (2) to calculate the Θ-gradings of the Heegaard Floer groups.

This section will achieve the first item on the agenda and the next section will achive the rest. All material in this section are classical results; more detail can be found in, for example, [10], and also [11], Chapter 11.

Definition 5.3.1. Let Y be an oriented 3-manifold. A (positive) contact structure is a maximally non-integrable plane field ξ on Y, so that ξ is the kernel of a 1-form α satisfying $\alpha \wedge d\alpha > 0$. A knot $K \subset Y$ is called Legendrian if the tangent vectors of K all lie in ξ.

$S^3 = \mathbb{R}^3 \cup \{\infty\}$ has a standard contact structure ξ_{std} which on $\mathbb{R}^3$ can be expressed by the form $dz + xdy$. A Legendrian knot $(x(t), y(t), z(t))$ in (S^3, ξ_{std}) (assuming it misses $\{\infty\}$) can be represented by its *front projection* $(y(t), z(t))$ where the positive x-axis points out of the paper. Since a Legendrian knot in (S^3, ξ_{std}) has to satisfy $z' + xy' = 0$, we see that such projections must not have vertical tangencies, but in general have horizontal tangencies and cusps. Furthermore, away from cusp points, one may recover $x(s)$ by using $x(s) = -\dfrac{z'(s)}{y'(s)}$, which implies that at any self-crossing, the curve with a more negative slope always crosses in front. See Figure 5.1 for an example.

Just like how we can construct new smooth 3-manifolds from old by surgery along knots, we can construct new contact 3-manifolds from old by surgery along Legendrian

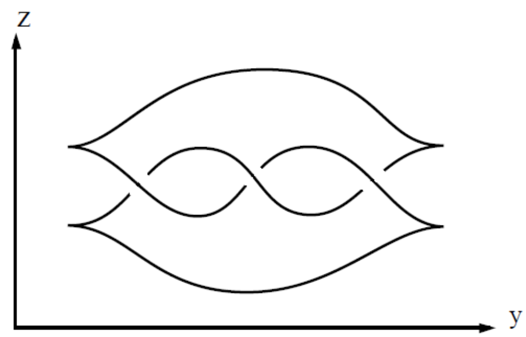

Figure 5.1: The front projection of a Legendrian representative of the right-handed trefoil. Diagram is adapted from [10], Figure 1.

knots. To this end, we shall recall the following definition of a classical invariant of null-homologous Legendrian knots L:

Definition 5.3.2. Let L be a null-homologous Legendrian knot in (Y, ξ). The Thurston-Bennequin number $tb(L)$ is the coefficient of the framing determined by the normal of ξ along L.

It is a classical fact that if we perform a $(tb(L) - 1)$-surgery on L by removing $\mathcal{N}$ and gluing back in a solid torus, then the resulting manifold has a unique contact structure which extends the one on $Y - \mathcal{N}$. Such a surgery is called a *Legendrian surgery*.

Contact structures naturally arise from *Stein surfaces*, i.e. complex surfaces that admit proper holomorphic embeddings into $\mathbb{C}^N$ for sufficiently large N. A complex manifold is Stein if and only if it admits an "exhausting strictly plurisubharmonic function", which is essentially characterized as being a proper function $f : X \longrightarrow \mathbb{R}$

that is bounded below and can be assumed a Morse function, whose generic level sets $f^{-1}(c)$ are "strictly pseudoconvex". Let S be the complex manifold $f^{-1}(-\infty, c]$; if S has real dimension 4, then strict pseudoconvexity implies that the induced plane field $T(\partial S) \cap iT(\partial S)$ is a contact structure on ∂S. Contact structures arisen in this manner are called *Stein fillable*. For example, the standard complex structure on the complex 2-ball induces the standard contact structure on S^3.

Eliashberg (see [10] Theorem 1.3) provides a method to construct 4-manifolds made of 0-, 1- and 2-handles which carry Stein structures. We shall give the union of the 0-handle and the 1-handles the standard complex structure of $\natural^n(S^1 \times D^3)$, which is a Stein structure that extends the standard one on D^4. Then Gompf considers a *Legendrian link diagram in standard form* ([10], Definition 2.1), which consists of:

1. A rectangular box in $\mathbb{R}^2$,

2. n 1-handles with attaching spheres drawn on the vertical edges, where the identification of the two attaching spheres of a 1-handle is given by the reflection about the vertical line in the middle, and

3. a Legendrian tangle inside the box.

The left side of Figure 5.2 is a general illustration. The right side of the same figure is the example of our particular interest, with the only component of the Legendrian tangle denoted by K.

Eliashberg showed that if we perform a Legendrian surgery along each component

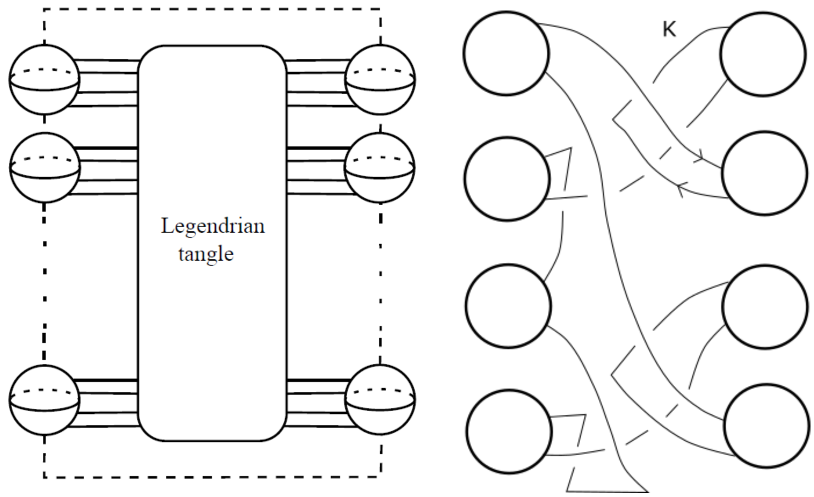

Figure 5.2: A generic Legendrian link diagram as well as a special example. The left side of the figure is taken from [10], Figure 8.

K_i of the Legendrian tangle, then the cobordism induced by the surgery carries a Stein structure that extends the one on $\natural^n(S^1 \times D^3)$.

Now we go back to our example of interest. Gompf showed that the tb of a Legendrian knot in standard form can be calculated from their front projections:

$$tb(K) = \text{writhe of } K - \text{number of left cusps of } K.$$

Thus one can check from the diagram that $tb(K) = 2g - 1$.

Our goal is to produce Stein fillable contact structures on $\Sigma_g \times S^1$ induced from Stein structures on $\Sigma_g \times D^2$. One can check that attaching a 0-framed 2-handle along K in the right side of Figure 5.2 (illustrated with $g = 2$) would produce a handle description of $\Sigma_g \times D^2$ (which is isotopic to the left side of Figure 3.2). Therefore, $tb(K)$ is $2g - 2$ more than what is required to extend the Stein structure to the 2-

handle. To remedy this, we may add $2g-2$ zig-zags to K. See Figure 5.3 for the two ways to add zig-zags, depending on whether the two extra cusps point upward or downward. Addition of either type of zig-zag decreases $tb(K)$ by 1, so adding $2g-2$ zig-zags in any combination of the two types would change $tb(K)$ to 1, which guarantees that the Stein structure extends to the 2-handle.

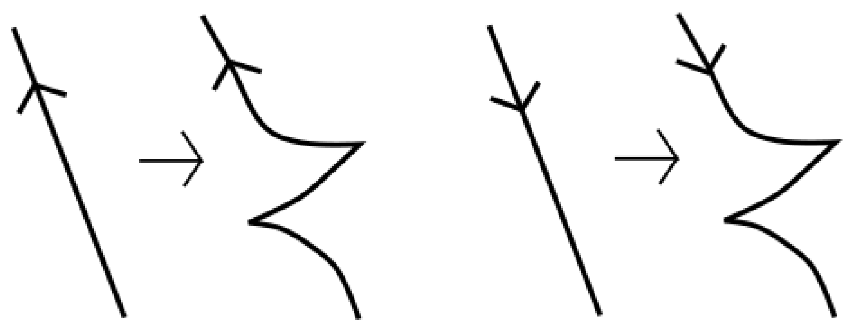

Figure 5.3: Two different ways of adding zig-zags.

Different combinations of zig-zags generally lead to different Stein structures on $\Sigma_g \times D^2$. In particular, we will be content with stating the following fact, which is a consequence of Gompf ([11], Proposition 2.3), and omit the proof: if p downward zig-zags and q upward zig-zags are added to the Legendrian knot K in Figure 5.2, where $p+q=2g-2$, then the dual of the first Chern class of the resulting Stein structure on $\Sigma_g \times D^2$ is $2(p-q)[D^2]$, where $[D^2]$ is the dual of the generator of $H^2(\Sigma_g \times D^2)$. By choosing $p = 0, 1, \ldots, 2g-2$, we have proved the main result of this section:

Lemma 5.3.3. *For all $-(g-1) \leq k \leq (g-1)$, there exists a Stein structure on $\Sigma_g \times D^2$ which restricts to a contact structure ξ_k on the boundary satisfying $\langle c_1(\xi_k), [\Sigma_g] \rangle = 2k$.*

The same result can also be found in [11], Exercises 11.2.5(a) and 11.3.2(a).

5.4 Heegaard Floer homology of $\Sigma_g \times S^1$

This section completes the agenda stated at the beginning of the previous section. To this end, we define the following $\mathbb{Z}$-graded group

$$X(g,d) \cong H_*(Sym^d(\Sigma_g)) \cong \bigoplus_{i=0}^{d} \Lambda^{2g-i} H^1(\Sigma_g) \otimes (\mathbb{Z}[U]/U^{d+1-i}), \tag{5.5}$$

where U carries degree -2. Now recall that ([21], Theorem 9.3) for all $k \neq 0$, we have an isomorphism of relatively graded groups

$$HF^+(\Sigma_g \times S^1, t_k) \cong X(g,d), \tag{5.6}$$

where $d = g-1-|k|$ and t_k is the spin^c structure of $\Sigma_g \times S^1$ satisfying $\langle c_1(t_k), [\Sigma]\rangle = 2k$. Furthermore, if $3d < 2g-1$, then Equation (5.6) is an isomorphism of $(\mathbb{Z}[U] \otimes \Lambda^* H_1(\Sigma))$-modules, where the action of $\gamma \in H_1(\Sigma)$ on $X(g,d)$ is given by

$$D_\gamma(\omega \otimes U^j) = (\iota_\gamma \omega) \otimes U^j + PD(\gamma) \wedge \omega \otimes U^{j+1}. \tag{5.7}$$

One can check from Equations 5.5 and 5.7 that $X(g,d)$ is supported in $2d+1$ adjacent gradings, the subgroup supported by the bottom grading of $X(g,d)$ is generated over $\mathbb{Z}$ by $B := U^d \otimes \omega$, where ω generates $\Lambda^{2g} H^1(\Sigma_g \times S^1)$, and that the kernel of the H_1-action is exactly the integer multiples of B. While $HF^+(\Sigma_g \times S^1, t_k)$

is only relatively $\mathbb{Z}/2k$-graded, we will say "the bottom grading of $HF^+(\Sigma_g \times S^1, t_k)$" to mean the relative grading of $HF^+(\Sigma_g \times S^1, t_k)$ supported by B', the image of B under the isomorphism (5.6). The term "top grading of $HF^+(\Sigma_g \times S^1, t_k)$ is defined analogously.

Note that if k is small enough so that $2|k| < 2d+1$, then the relative gradings supported by $HF^+(\Sigma_g \times S^1)$ "roll over", and in particular, the bottom grading of $HF^+(\Sigma_g \times S^1, t_k)$ would no longer be generated by B'. By the same token, gradings can only provide non-vacuous constraints of the relative invariants if $HF^+(\Sigma_g \times S^1, t_k)$ is not supported by all the gradings available in the relative $\mathbb{Z}/2|k|$-grading, that is, if $2d+1 < 2|k|$. In fact, since we will formulate our constraints of the relative invariants in terms of Θ-invariants which repeat itself under an increment of $|k|$ in the relative grading (see the discussion after Equation 5.2), we will assume further that $2d+1 < |k|$, i.e. $3d < g-2$, so that Lemma 5.4.2, Lemma 5.4.3 and Proposition 5.5.1 will be non-vacuous. Notice that $3d < g-2$ is stronger than $3d < 2g-1$, and therefore the isomorphism (5.6) holds as $(\mathbb{Z}[U] \otimes \Lambda^* H_1(\Sigma))$-modules.

Our next task is to identify the gradings of the Heegaard Floer groups of $\Sigma_g \times S^1$ with the plane fields ξ_k constructed in Lemma 5.3.3.

Recall that, given a contact structure ξ on a 3-manifold Y, Ozsváth and Szabó defined in [25] a contact invariant $c(\xi) \in \widehat{HF}(-Y, t_\xi)$. We will focus on the variant $c^+(\xi)$ which is the image of $c(\xi)$ in $HF^+(-Y, t_\xi)$ in the long exact sequence 2.3.

Properties 5.4.1. *$c^+(\xi)$ has the following properties:*

1. *$c^+(\xi)$ is in the kernel of the H_1-action. ([25], Remark 4.5)*

2. *If ξ is a Stein fillable contact structure on Y, then $c^+(\xi)$ is non-trivial. (See Ghiggini, [9] Theorem 2.13, for example.)*

3. *$c^+(\xi)$ is supported by the absolute plane field grading $[\xi_k]-1$. (See Hedden-Mark [13], Corollary 4.6)*

These facts together imply that the bottom grading of $HF^+(-\Sigma_g \times S^1, t_k)$ is supported by the plane field $[\xi_k] - 1$. Since the map τ (see exact sequence 2.2) decreases relative grading by 1, it follows that the bottom group of $HF^-(-\Sigma_g \times S^1, t_k)$ has plane field grading $[\xi_k] - 2$.

With this in place, we are ready to calculate the Θ-invariants of the plane field gradings (which we call Θ-*gradings* for brevity) supported by $HF^-(-\Sigma_g \times S^1, t_k)$, by utilizing the previously constructed Stein structures on $\Sigma \times D^2$. If $\mathcal{S}_k$ is the spin^c structure on $\Sigma \times D^2$ that extends t_k, then $PDc_1(\mathcal{S}_k) \in H^2(\Sigma \times D^2, \Sigma \times S^1)$ can be represented by $2k(\{*\} \times D^2)$. We will always choose the framing on the class $PDc_1(t_k) = 2k[\{*\} \times S^1]$ (see the paragraph preceding Definition 5.1.3) to be the product framing on the smooth cycle $2k(\{*\} \times S^1)$.

Lemma 5.4.2. *The bottom group of $HF^-(-\Sigma_g \times S^1, t_k)$ has Θ-grading*

$$\Theta_{-\Sigma_g \times S^1, f}([\xi_k] - 2) \equiv -4d - 8. \tag{5.8}$$

As a result, $HF^-(-\Sigma_g \times S^1, t_k)$ is supported by Θ-gradings $4m - 8 \mod 4|k|$, where $-d \leq m \leq d$.

Proof. Since $PDc_1(\mathcal{S}_k)$ can be represented by $2k(\{*\} \times D^2)$, it is easy to see that $Q_f(PDc_1(\mathcal{S}_k)) = 0$ for all k. Thus

$$\begin{aligned}\Theta_{\Sigma_g \times S^1, f}([\xi_k]) &= Q_f(PDc_1(\mathcal{S}_k)) - 2\chi(\Sigma_g \times D^2) - 3\sigma(\Sigma_g \times D^2) \\ &= -2(2-2g) \\ &\equiv 4d \mod 4|k|.\end{aligned}$$

Since Θ reverses sign when the 3-manifold reverses orientation, it follows from Equation 5.2 that

$$\Theta_{-\Sigma_g \times S^1, f}([\xi_k]) \equiv -4d, \qquad \Theta_{-\Sigma_g \times S^1, f}([\xi_k] - 2) \equiv -4d - 8.$$

So we have proved the first statement, and the second statement follows immediately from the fact that $HF^-(-\Sigma_g \times S^1, t_k)$ takes up $2d+1$ adjacent Ozsváth-Szabó relative gradings. $\square$

Similarly, we have the following lemma for cobordisms from S^3 to $\Sigma_g \times S^1$:

Lemma 5.4.3. *The top group of $HF^-(\Sigma_g \times S^1, t_k)$ has Θ-grading $4d - 8 \mod 4|k|$. As a result, $HF^-(\Sigma_g \times S^1, t_k)$ is supported by Θ-gradings $4m - 8 \mod 4|k|$, where $-d \leq m \leq d$.*

Proof. Recall that the action on the plane fields depends on the orientation of the ambient manifold. In particular, for any plane field ξ in Y, we have $[\xi] +_Y n = [\xi] -_{-Y} n$, where the subscripts are to emphasize the ambient orientation of the manifold.

Now, the bottom grading of $HF^+(-\Sigma_g \times S^1, t_k)$ is supported by $[\xi_k] -_{(-\Sigma_g \times S^1)} 1$, so by duality ([13], Equation (14)), the top grading of $HF^-(\Sigma_g \times S^1, t_k)$ is supported by $[\xi_k] -_{(-\Sigma_g \times S^1)} 1 -_{(\Sigma_g \times S^1)} 3 = [\xi_k] -_{(\Sigma_g \times S^1)} 2$. Now

$$\begin{aligned}
\Theta_{\Sigma_g \times S^1, f}([\xi_k] - 2) &= \Theta_{\Sigma_g \times S^1, f}([\xi_k]) - 8 \\
&= 0 - 2\chi(\Sigma_g \times D^2) - 3\sigma(\Sigma_g \times D^2) - 8 \\
&\equiv 4d - 8 \qquad \mod 4|k|,
\end{aligned}$$

This proves the first statement, and the second statement immediately follows. □

5.5 Degree shift criterion for fiber sums

With these lemmas in place, we are ready to prove a degree shift criterion for the non-vanishing of the Ozsváth-Szabó 4-manifold invariant $\Phi_{X,s}$.

Proposition 5.5.1. *Let $X = Z_1 \cup_Y Z_2$ be the oriented 4-manifold formed by gluing Z_1 and Z_2 along their common boundary $Y = \Sigma_g \times S^1$, so that $\partial Z_1 = -\Sigma_g \times S^1$ and $\partial Z_2 = \Sigma_g \times S^1$; assume also that $b_2^+(Z_1), b_2^+(Z_2) \geq 1$. Let s be any spin^c structure on X, the restriction of s to Z_i be s_i, the restriction of s to Y be t_k where $\langle c_1(t_k), [\Sigma_g] \rangle = 2k$, and f be the framing on $PDc_1(t_k)$ defined before Lemma 5.4.2. Assume $3d < g-2$ (see discussion preceding Properties 5.4.1). If $\Phi_{X,s} \neq 0$, then for $i = 1, 2$, the shift terms $\mathcal{S}_i = Q_f(PDc_1(s_i)) - 2\chi(Z_i) - 3\sigma(Z_i)$ must satisfy*

$$\mathcal{S}_1, \mathcal{S}_2 \equiv 4m \mod 4|k|, \qquad \text{where } -d \leq m \leq d. \tag{5.9}$$

and that

$$\mathcal{S}_1 + \mathcal{S}_2 \equiv 0 \qquad \text{mod } 4|k|. \tag{5.10}$$

Remark 5.5.2. The *simple type conjecture* states that if X is a simply connected closed 4-manifold with $b_2^+(X) \geq 2$, then any Ozsváth-Szabó basic class s satisfies $c_1^2(s) - 2\chi(X) - 3\sigma(X) = 0$. Since Q, χ and σ are all additive, this implies that

$$\mathcal{S}_1 + \mathcal{S}_2 \equiv 0 \quad \text{mod } 2d(s).$$

Equation (5.10) implies the above equation, and equation (5.9) asserts that the individual terms $\mathcal{S}_1$ and $\mathcal{S}_2$ must lie in certain intervals in some $\mathbb{Z}/m$, which the simple type conjecture does not assert.

Before proving Proposition 5.5.1, let's give an example of how the degree shift criterion can be useful in showing that certain spin^c structures are not Ozsváth-Szabó basic classes.

Let $X(m,n)$ be the smooth 4-manifold constructed by taking the double branched cover of $S^2 \times S^2$ branched along a surface obtained by smoothing the union of $2m$ parallel copies of $S^2 \times \{pt\}$ and $2n$ copies of $\{pt\} \times S^2$. The projections $S^2 \times S^2 \longrightarrow S^2$ to the first and second factor realize $X(m,n)$ as a genus $m-1$ fibration and a genus $n-1$ fibration respectively, both of which can be perturbed into Lefschetz fibrations. Then $X(2,n)$ is the elliptic surface $E(n)$ (cf. [11], Section 8.4), and $X(2k,n)$ is the k-fold fiber sum of $E(n)$ with respect to the genus $n-1$ Lefschetz fibration structure (where the gluing map $\Sigma_{n-1} \times S^1 \longrightarrow \Sigma_{n-1} \times S^1$ is the identity map on each fiber). Also, $X(m,n)$ is a minimal general type complex surface for $m, n \geq 3$ (see [11], Chapter 8.4 and [28], Proposition 1.3). From now on, we fix $m = 6$ and denote $X(6,n)$ by X, so that X is the 3-fold fiber sum of $E(n)$.

Ozsváth and Szabó's definition of absolute invariant in [26], Chapter 9 is slightly more general than Definition 2.8.3, in the sense that the former definition allows the possibility that the image of $\Theta^- \in HF^-(S^3)$ under $F^{mix}_{X-D^4-D^4}$ lands on gradings of $HF^+(S^3)$ other than the minimal grading 0. Thus, in Ozsváth and Szabó's definition, if s is a spin^c structure on X so that $\Phi_{X,s} \neq 0$, then $F^{mix}_{X-D^4-D^4}$ could shift the $\mathbb{Q}$-degree by any even number at least 2 (instead of just 2). Then it follows from the degree shift formula for $\mathbb{Q}$-grading (Equation 2.5) that

$$c_1^2(s) - 2\chi(X) - 3\sigma(X) \geq 0. \tag{5.11}$$

(Whereas the simple type conjecture (Remark 5.5.2) asserts that only the equality may hold.)

One can check that $2\chi(X) + 3\sigma(X) = 2(44n - 16) + 3(-24n) = 16n - 32$. The canonical spin^c structure s^* of $X(m,n)$ is characterized by $PDc_1(s^*) = (2-n)[\Sigma_{m-1}] + (2-m)[\Sigma_{n-1}]$. Since Σ_{m-1} and Σ_{n-1} intersect as two positive points, we see in our situation $m = 3$ that $c_1^2(s^*) = \Big((2-n)[\Sigma_5] - 4[\Sigma_{n-1}]\Big)^2 = 16n - 32$. Therefore s^* satisfies both the constraints given by (5.11) and the simple type conjecture.

More generally, consider spin^c structures s on X where $PDc_1(s)$ is of the form $k[\Sigma_5] + p[\Sigma_{n-1}]$ for some integers k and p. Then $c_1^2(s) = 4pk$. Therefore, if $\Phi_{X,s} \neq 0$, then (5.11) implies that $pk \geq 4n-8$. We shall apply Proposition 5.5.1 to show that there exist values of p, k that satisfy $pk \geq 4n-8$ where $\Phi_{X,s}$ still vanishes. To this end, decompose X into the fiber sum of $E(n)\#E(n)$ and $E(n)$, and let $Z = E(n)\#E(n) - \Sigma_{n-1} \times D^2$. Notice that $PDc_1(s|_Z)$ can be represented by a subset of Σ_5 which is a genus 3 surface with two boundary components, and $PDc_1(s|_{\Sigma_{n-1}\times S^1}) = 2k[S^1]$. With our choice of framing before Lemma 5.4.2, it is clear that $Q_f(PDc_1(s|_Z)) = 0 \in \mathbb{Z}/4|k|$. Then one can calculate the shift term $\mathcal{S}$ for Z:

$$\begin{aligned}
\mathcal{S} &= Q_f(PDc_1(s|_Z)) - 2\chi(Z) - 3\sigma(Z) \\
&= 0 - 2(30n-12) - 3(-16n) \\
&= -12(n-2) \\
&= -12(d+k) \qquad \text{(Recall } d = g-1-k\text{, here with } g = n-1\text{)} \\
&= -12d \mod 4|k|.
\end{aligned}$$

If n and k are sufficiently large and d is sufficiently small compared to k (more precisely, when $4d < |k|$), then $\mathcal{S}$ would not satisfy $-4d \leq \mathcal{S} \leq 4d$ imposed by (5.9), implying that $\Psi_{Z,s|_Z} = 0$, hence $\Phi_{X,s} = 0$.

Proof of Proposition 5.5.1. The bottom generator p of $HF^+(S^3)$ is supported by the standard contact structure of S^3 which is Stein fillable by D^4. So if we choose f' to be the 0-framing of the unknot in S^3, then p has Θ-grading

$$Q_{f'}(PDc_1(D^4)) - 2\chi(D^4) - 3\sigma(D^4) = -2. \tag{5.12}$$

It follows that the top generator q of $HF^-(S^3)$ has Θ-grading -10 (with the same framing f'). Therefore, by Proposition 5.2.1 and Lemma 5.4.2, we have

$$-10+Q_{f',f}(PDc_1(s_1))-2\chi(Z_1-D^4)-3\sigma(Z_1-D^4) \equiv -8+4m \mod 4|k| \quad (-d \leq m \leq d). \tag{5.13}$$

Thus our assertion on $\mathcal{S}_1$ in Equation (5.9) follows from adding Equations (5.12) and (5.13); the case for $\mathcal{S}_2$ can be proved similarly.

To see why Equation (5.10) is true, note that if $\Phi_{X,s} \neq 0$, then the relative invariants Ψ_{Z_1,s_1} and Ψ_{Z_2,s_2} have to pair non-trivially in the pairing mentioned in Theorem 2.8.4:

$$\underline{HF}^-(-\Sigma_g \times S^1, t_k; M_{Z_1}) \otimes \underline{HF}^-(\Sigma_g \times S^1, t_k; M_{Z_2}) \longrightarrow M_{X,\Sigma_g \times S^1}.$$

By definition of the pairing (see [16], Section 10.3 or [26], Section 5.1), the top i-th group of $\underline{HF}^-(-\Sigma_g \times S^1, t_k; M_{Z_1})$ only pairs non-trivially with the bottom i-th group of $\underline{HF}^-(\Sigma_g \times S^1, t_k; M_{Z_2})$. Since the i-th highest possible value of $\mathcal{S}_1$ is $4(d-i+1) \mod 4|k|$ and the i-th lowest possible value of $\mathcal{S}_2$ is $4(-d+i-1) \mod 4|k|$, we always have $\mathcal{S}_1 + \mathcal{S}_2 \equiv 0 \mod 4|k|$ whenever the pairing is non-trivial. $\square$

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