Right-Angled Artin Groups in Mapping Class Groups

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Abstract

The study of free, and more generally, right-angled Artin subgroups of mapping class groups has a long history in geometric group theory. The goal of this thesis is to present a general embedding theorem for such subgroups by combining ideas from Koberda and Clay-Leininger-Mangahas in a unified way, via the hierarchical structure of curve graphs as laid out by Masur-Minsky. The original mathematics in this dissertation is contained in the article [45].

Prof. Thomas Koberda (Advisor)

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Chapter 1

Introduction

Let S be an oriented surface of finite type, *i.e.* $\pi_1(S)$ is finitely generated. The *mapping class group* of S is the group of homotopy classes of orientation-preserving homeomorphisms of S,

$$MCG(S) := Homeo^+(S)/homotopy.$$

The study of free subgroups of $MCG(S)$, in a sense, goes back to Klein [34], who introduced the well-known Ping-Pong Lemma and showed that the matrices

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

generate a free subgroup of $SL(2, \mathbb{Z})$. We may identify $SL(2, \mathbb{Z})$ with $MCG(T^2)$, the mapping class of the torus, and these matrices correspond to the squares of the Dehn twists about the standard meridian and longitude curves. More generally (i.e. for hyperbolic surfaces), it follows from Ivanov's [29] and McCarthy's [43] proofs of the Tits alternative for $MCG(S)$ that there is in fact an abundance of free subgroups.

We wish to broaden our view to the larger class of *right-angled Artin groups* (hereafter called RAAGs). Recall that a RAAG has a presentation determined by a finite simplicial graph Γ with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$:

$$A(\Gamma) = \langle v_i \in V(\Gamma) \mid [v_i, v_j] = 1 \iff (v_i, v_j) \in E(\Gamma) \rangle.$$

Regarding subgroups of $MCG(S)$ of this form, Koberda showed that they too can be found in abundance. All of the terminology below will be defined shortly, and the author notes that the given restrictions are mild.

Theorem 1 (Koberda, [35] Theorem 1.1)**.** *Let $\{f_1, \ldots, f_m\}$ be an irredundant collection of pure mapping classes supported on connected subsurfaces $S_1, \ldots, S_m \subseteq S$. There exists some $N \neq 0$ such that for all $n \geq N$,*

$$\langle f_1^n, \ldots, f_m^n \rangle \cong A(\Gamma),$$

where Γ is the co-intersection graph of the subsurfaces $\{S_i\}$.

Theorem 1 admits two interpretations. The first, as suggested above, is that any infinite subgroup of a fixed mapping class group contains a RAAG (in fact, infinitely many). Secondly, given a RAAG, one can embed it in a mapping class group provided the corresponding surface is complicated enough to contain a collection of subsurfaces with the appropriate intersection pattern. The relationship between RAAGs and mapping class groups goes further - via the work of Kim-Koberda in [33], there are a number of analogies between (subgroups of) RAAGs and (subgroups of) mapping class groups. These are obtained by comparing the action of a mapping class group on its corresponding *curve graph* to the action of a RAAG on its corresponding *extension graph*; we will discuss the former in some detail in Chapter 3, and though the theory of the latter is quite similar, it is beyond the scope of the present discussion.

Koberda's proof of Theorem 1 goes by playing ping-pong on a space of geodesic laminations on S, and relies on certain non-constructive compactness and continuity arguments, so it is not immediately clear how the number N depends on S or on the given mapping classes. The goal of this dissertation is to effectivize and strengthen Koberda's theorem. The constant in the statement of the theorem below is explicitly computed in Chapter 5.

Theorem 2. *Let $\{f_1, \ldots, f_m\}$ be an irredundant collection of pure mapping classes supported on connected subsurfaces $S_1, \ldots, S_m \subseteq S$. There exists an explicit constant $N = N(\{f_i\})$, depending only on certain geometric data extracted from the given collection of mapping classes, such that for all $n \geq N$,*

$$H = \langle f_1^n, \ldots, f_m^n \rangle \cong A(\Gamma),$$

where Γ is the co-intersection graph of the subsurfaces $\{S_i\}$. Moreover, increasing N in a controlled way, we can guarantee that H is undistorted in $MCG(S)$.

In the "moreover" statement, we say a finitely generated subgroup H of a finitely generated group G is *undistorted* if, roughly, the intrinsic word metric on H agrees with the extrinsic word metric inherited from G, up to linearly bounded error. Computing the constant explicitly in the case that all mapping classes in question are Dehn twists, we have the following corollary.

Corollary 1. *Let $\{t_1, \ldots, t_m\}$ be a collection of Dehn twists about distinct simple closed curves $\{\beta_1, \ldots, \beta_m\}$, and let*

$$N = 18 + \max_{i,j} i(\beta_i, \beta_j),$$

where $i(\cdot, \cdot)$ denotes geometric intersecion number. Then for all $n \geq N$, we have

$$\langle t_1^n, \ldots, t_m^n \rangle \cong A(\Gamma),$$

where Γ is the subgraph of the curve graph $\mathcal{C}(S)$ spanned by $\{\beta_1, \ldots, \beta_m\}$.

A similar bound has been found by Seo using methods from hyperbolic and coarse geometry, and Bass-Serre theory. It is worth mentioning that if there are more than two mapping classes involved, the constant N *necessarily* depends on the given mapping classes, as the following example illustrates. Let β_1 and β_2 be two non-trivially intersecting simple closed curves, and consider the Dehn twists

$$t_1 = t_{\beta_1}, \;\; t_2 = t_{\beta_2}, \;\; \text{and} \;\; t_3 = t_1^{2^K} t_2 t_1^{-2^K}$$

for some $K > 0$. Then for no $1 \leq k \leq K$ is $\langle t_1^{2^k}, t_2^{2^k}, t_3^{2^k} \rangle$ isomorphic to a free group of rank 3, even though the corresponding twisting curves pairwise intersect.

Using similar methods, we are also able to determine the Nielsen-Thurston type for all elements of these subgroups.

Theorem 3. *Let H be as in Theorem 2. Then every $h \in H$ is pseudo-Anosov on its support. In particular, if the support of h is all of S, then h is pseudo-Anosov.*

Every RAAG can be embedded in some mapping class group, and moreover there is always an embedding that satisfies the hypotheses of Theorem 2 - using Crisp-Wiest's construction [16], every RAAG can be realized as a subgroup of a mapping class group satisfying the hypotheses of Theorem 2 by sending the vertex generators to the $N \geq 19$ powers of a suitable collection of Dehn twists. The control over the geometry of this embedding afforded by Theorem 2 allows one to transfer questions about RAAGs into a topological setting. For example, an immediate consequence of Theorem 3 is that RAAGs are torsion-free, which was previously known by more advanced techniques due to Charney (*cf.* [12]). It would be interesting to explore just how much of the combinatorial/geometric group theory of RAAGs can be deduced via such embeddings.

This dissertation is organized as follows: in Chapter 2 we establish the relevant terminology and some basic notions we will need from coarse geometry, geometric group theory (including a proof of a new ping-pong lemma for RAAGs), and the theory of surfaces and their mapping class groups. In Chapter 3 we recall the work of Masur-Minsky on the geometry of the curve graph, including the construction of subsurface projections and other relevant results from [42], which we use to build our ping-pong table. In Chapter 4 we briefly recall the idea of the proof of Koberda's theorem before giving a concise treatment of the Clay-Leininger Mangahas construction of admissible embeddings; also in this section is a generalization of the famous Behrstock inequality regarding projection distance bounds between geodesics in the curve graph and its curve subgraphs. In Chapter 5 we carry out the proofs of the main theorems and discuss an application to the theory of convex cocompact subgroups of mapping class groups; this final section is self-contained.

Chapter 2

Background

2.1 A Bit of Coarse Geometry

For a thorough treatment of coarse geometry, see [10]. Let (X_1, d_{X_1}) and (X_2, d_{X_2}) be metric spaces. We say a (not-necessarily-continuous) map $f : X_1 \to X_2$ is a (A, B)-*quasi-isometric embedding* if there are constants $A \geq 1$ and $B \geq 0$ such that for all $x, y \in X_1$,

$$\frac{1}{A} d_{X_1}(x, y) - B \leq d_{X_2}(f(x), f(y)) \leq A d_{X_1}(x, y) + B.$$

If there is a constant $D > 0$ such that any $x_2 \in X_2$ is within a distance D of $f(X_1)$, we further say f is a *quasi-isometry*, and that X_1 and X_2 are *quasi-isometric*. Though it might seem like such an equivalence relation on metric spaces is too rough to be useful, it turns out that many geometric properties of interest are in fact preserved under quasi-isometries (and quasi-isometric embeddings). One such property that has played a central role in geometric group theory since its inception is Cannon-Gromov-Rips' δ*-hyperbolicity*, a coarse notion of negative curvature reminiscent of classical hyperbolic geometry. We say a metric space X is δ-hyperbolic if there exists a uniform constant δ so that all geodesic triangles are "δ-slim", *i.e.* each side is contained in the union of the δ-neighborhoods of the other two; see Figure 2-1.

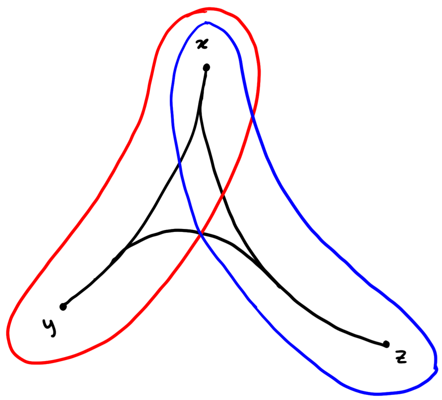

Figure 2-1: The geodesic between y and z is contained in the union of the δ-neighborhoods of those between x and y and x and z

We note that there are many other equivalent characterizations of δ-hyperbolicity, each with their own advantages. While the theory of δ-hyperbolic geometry is vast and beautiful, of particular use to us will be the constraints it puts on the behavior of nearest-point projections. We record the following two lemmas for later use. The first says that nearest-point projections of points are uniformly bounded-diameter sets.

Lemma 1. *Let $x \in X$, a δ-hyperbolic space, and let $\alpha \subset X$ be a bi-infinite geodesic. Denote by $\pi_\alpha : X \to \alpha$ a set-valued nearest-point projection. Then*

$$diam_X\{\pi_\alpha(x)\} \leq 4\delta.$$

The second says that if the nearest-point projections of two points are far apart, then the points must have been far apart.

Lemma 2. *Let $\alpha \subset X$ be a bi-infinite geodesic and $x, y \in X$ be distinct points. For subsets $A, B \subset X$ let $d_X(A, B) = diam_X(A \cup B)$. Then*

$$d_X(\pi_\alpha(x), \pi_\alpha(y)) \leq d_X(x, y) + 24\delta.$$

The preceding lemma generalizes to a so-called “distance formula” for hyperbolic spaces, originally observed by Bestvina-Bromberg-Fujiwara [5], see also [13]. In the statement, a WPD element is a generalization of a classical hyperbolic (or loxodromic) isometry, and in particular such an element has a “quasi-axis” on which it acts by translation.

Theorem 4 (Clay-Mangahas-Margalit, [13] Proposition 7.1)**.** *Let G be a group acting by isometries on a δ-hyperbolic space X, and let $\{f_i\}$ be a finite collectin of WPD elements. Let $\mathcal{A}$ be the collection of all quasi-axes of the G-conjugates of the f_i. There exists constants A, B, and M such that for all $x, y \in X$,*

$$d_X(x,y) \geq \frac{1}{A} \sum_{\alpha \in \mathcal{A}} \{\{d_X(\pi_\alpha(x), \pi_\alpha(y))\}\}_M - B$$

where $\{\{z\}\}_M$ is equal to z if $z \geq M$ or 0 if $z < M$.

Finally, recall that to a group G with generating set Y we may associate the Cayley graph $Cay(G,Y)$, and that equipped with the graph metric $Cay(G,Y)$ is a metric space. Moreover, if G is finitely generated, then any two metrics coming from different finite generating sets Y and Y' yield quasi-isometric Cayley graphs; see Fig. 2-2 for an example of two generating sets for $\mathbb{Z}$, and observe how the roughly looks the same “from a distance". In this way, we can view groups themselves as metric spaces, and the coarse geometry of the Cayley graph has consequences for the algebra of the group. One example of this phenomenon is hyperbolicity - we say a finitely generated group G is hyperbolic if some Cayley graph for G is δ-hyperbolic. This strong geometric property provides a host of consequences for G, for example finite presentability and satisfaction of the Tits alternative.

We may now put a (left-invariant) metric d_G on G, the *word metric*, defined by

$$d_G(g,h) := d_{Cay(G,Y)}(g,h) = d_{Cay(G,Y)}(1, h^{-1}g).$$

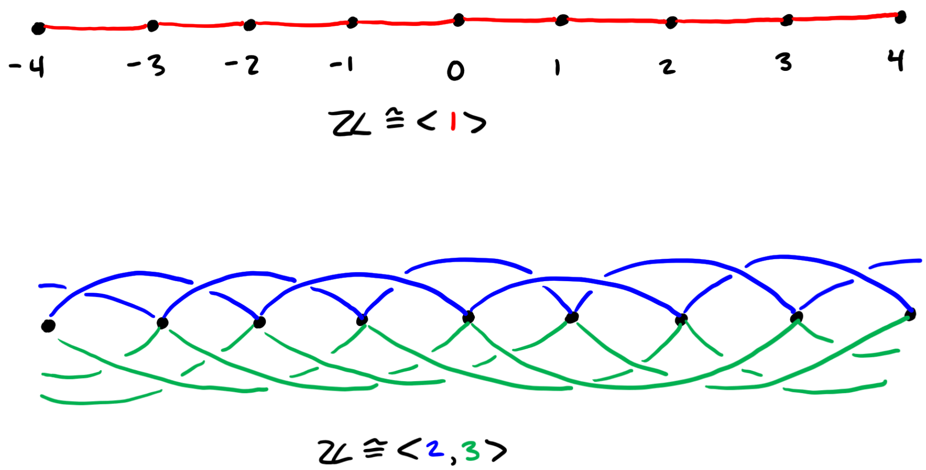

Figure 2-2: Different finite generating sets yield quasi-isometric Cayley graphs.

Regarding the statement of Theorem 2, we say that a finitely generated subgroup $H < G$ is *undistorted* if the inclusion of H into G is a quasi-isometric embedding with respect to their respective word metrics. This means that the intrinsic word metric on the subgroup agrees up to controlled error with the extrensic word metric inherited from the ambient group. This coarse-geometric property, in some cases, has algebraic consequences for the subgroup in question- for example, undistorted subgroups of hyperbolic groups are again hyperbolic. As we will see later, a certain class of undistorted subgroups of mapping class groups give rise to hyperbolic surface group extensions.

We end this section with what is sometimes called the "Fundamental Lemma of Geometric Group Theory", also known as the Milnor-Schwarz lemma.

Lemma 3 (Milnor-Schwarz, *cf.* [19] Theorem 8.2)**.** *Suppose a group G is acting by isometries on a geodesic metric space X, such that the action is properly discontinuous and cocompact. Then G is finitely generated and quasi-isometric to X.*

2.2 RAAGs

Excellent surveys on right-angled Artin groups can be found in [12] and [36], and all of the facts stated below can be found therein. Given a finite simplicial graph Γ with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$, the *right-angled Artin group on* Γ is the group with the presentation

$$A(\Gamma) := \langle v_i \in V(\Gamma) \mid [v_i, v_j] = 1 \iff (v_i, v_j) \in E(\Gamma) \rangle.$$

We call the v_i the *vertex generators* of $A(\Gamma)$. The standard examples of such groups are free groups (when Γ has no edges), free abelian groups (when Γ is a complete graph), and free and direct products of such groups (corresponding to disjoint union of graphs and join of graphs, respectively). Though most RAAGs don't admit such simple descriptions, there is enough rigidity present that one can show RAAGs satisfy many desirable group-theoretic properties. For example, Green provides a normal form (outlined in the proof of Lemma 4 below) for elements of RAAGs in terms of subgroups generated by join subgraphs, and Hermiller-Meier showed that any representative of an element in a RAAG can be put into this normal form using a finite set of moves, thus providing a solution to the so-called "word problem" for RAAGs.

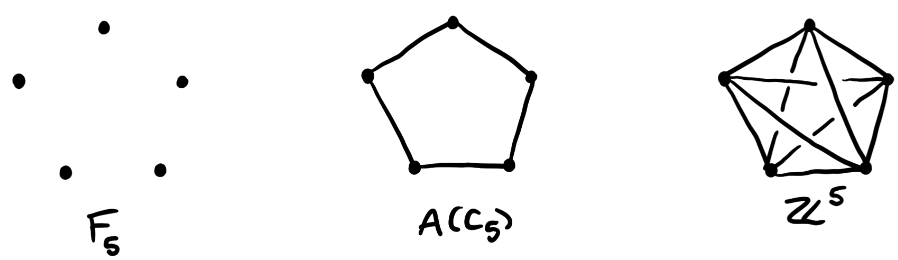

Figure 2-3: RAAGs with a fixed number of generators are on a spectrum from non-abelian free groups to free abelian groups.

In many ways, the algebra of a RAAG is tied to various properties of its defining graph. As mentioned above, free and direct products of RAAGs precisely correspond to disjoint unions and joins of graphs. Another deceptively simple fact is that two RAAGs are isomorphic if and only if their defining graphs are isomorphic. Interesting properties of graphs, for example colorability and Hamiltonicity, the decidibility of which are NP-complete problems, have group theoretic characterizations in terms of homomorphisms between RAAGs and group cohomology, respectively (*cf.* [37]). A particular example we wish to highlight is the usefulness of RAAGs in cryptography; the following example comes from [21], see also [22]. It is known that determining if two given graphs are isomorphic is computationally difficult, but it is easy enough to verify if a given map is an isomorphism. Using this with the above fact that RAAGs are isomorphic if and only if their defining graphs are, one can design a secure authentication scheme as follows. Alice wants to verify her identity to Bob in a way that can't be easily mimicked. She has a private isomorphism between two RAAGs, as well as a third RAAG isomorphic to the first two, which she sends to Bob. To verify Alice's identity, Bob sends Alice a succession of random bits. If a bit is 0, Alice sends her "public" isomorphism, and if a bit is 1, she sends her public isomorphism composed with her private one. After each retrieval, Bob verifies that he's received an isomorphism, and after enough successful trials, Bob can be confident that Alice is who she says she is.

From a coarse-geometric point of view, RAAGs also serve as an interpolation between flat and hyperbolic geometry - indeed, they are prototypical examples of CAT(0) groups. Because of this, the study of RAAGs and their subgroups is quite rich and has played a central role in many recent breakthroughs in geometric group theory, see for example [50]. Further, RAAGs and mapping class groups serve as the motivating examples of hierarchically hyperbolic groups [2].

The following is an alternate version of the ping-pong lemma for RAAGs attributed to Farb, itself a generalization of the classical ping-pong lemma for free groups. It is the main tool used in proving Theorem 2.

Lemma 4 (Ping-Pong). *Let $A(\Gamma)$ be a right-angled Artin group acting on a set X*

such that there exist non-empty subsets $X_i' \subseteq X_i \subset X$ *for each vertex generator* v_i *satsifying*

1. *For* $i \neq j$, *if* $X_i \cap X_j \neq \varnothing$, *then there exists* $x_i \in X_i$ *which does not belong to* X_j, *and vice versa*

2. *If* u *is a word in the vertex generators not containing a power of* v_j, *wherein every vertex generator commutes with* v_j, *then* $u(X_j') \subseteq X_j$

3. *If* v_i *and* v_j *do not commute, then* X_i *and* X_j *are disjoint and* $v_i^r(X_j) \subset X_i'$ *for all* $r \neq 0$, *and vice versa*

Then the $A(\Gamma)$ *action on* X *is faithful.*

Proof. If Γ decomposes as a join, then $A(\Gamma)$ splits as a direct product, and we can play ping-pong on each factor. Hence, we will assume that Γ is not a join, so that, in particular, for each vertex generator v_i there is at least one other vertex generator v_j which does not commute with it. Let $w \neq 1 \in A(\Gamma)$ be a word in the vertex generators. We begin by putting w into a normal form called *central form*, due to Green [24]. Given a representative of w written in the vertex generators, we can perform two operations which do not change the equivalence class of w: a shuffle, where we replace a subword $v_i^r v_j^s$ with $v_j^s v_i^r$ if v_i and v_j commute, and a deletion, where we remove subwords $v_i^r v_i^{-r}$. Starting with any representative of w (in the vertex generators), we can perform these two operations until w may be written as

$$w = u_k v_{i_k}^{r_k} u_{k-1} v_{i_{k-1}}^{r_{k-1}} \cdots u_1 v_{i_1}^{r_1}$$

where

- each u_j is a word in the vertex generators of $A(\Gamma)$, such that each generator appearing in u_j commutes with each other generator appearing in u_j

- v_{i_j} commutes with each generator appearing in u_j

- v_{i_j} does not commute with $v_{i_{j+1}}$ for all $1 \leq j < k$.

We call k the *central-word length* of w.

We now show that w acts non-trivially on X. First suppose that $k = 1$, so that $w = u_1 v_{i_1}^{r_1}$. By assumption there is some generator v_j which does not commute with v_{i_1}, and we choose $x_j \in X_j$. Applying (3), we have $v_{i_1}^{r_1} x_j \in X'_{i_1}$, then applying (2) we have $u_1 v_{i_1}^{r_1} x_j \in X_{i_1}$. Again by (3), since $X_{i_1} \cap X_j = \varnothing$, we see that $w x_j \neq x_j$ and we are done. Now suppose $k \geq 2$ and that w is written in central form. If v_{i_2} and v_{i_k} are distinct, then either by (1) or (3) we can choose $x_{i_2} \in X_{i_2}$ which does not belong to X_{i_k}; note that since v_{i_2} and v_{i_1} don't commute by assumption, x_{i_2} also does not belong to X_{i_1}. Repeatedly applying the argument above to this word, we have that $w x_{i_2} \in X_{i_k}$, so in particular $w x_{i_2} \neq x_{i_2}$. Finally, if $v_{i_2} = v_{i_k}$, then we can conjugate w by $v_{i_2}^{r_k}$, choose $x_{i_1} \in X_{i_1}$, and apply the same process to

$$v_{i_2}^{r_k} w v_{i_2}^{-r_k} = u_k v_{i_2}^{2r_k} u_{k-1} v_{i_{k-1}}^{r_{k-1}} \cdots u_1 v_{i_1}^{r_1} v_{i_2}^{-r_k},$$

which is indeed in central form. □

2.3 Surfaces and their Mapping Class Groups

The standard resource for the theory of mapping class groups is [19]. Let S be a connected, oriented finite-type surface, possibly with punctures; most of the surfaces we consider will satisfy $\chi(S) < 0$, and thus will admit a hyperbolic metric, though all of the following definitions work just as well in the remaining cases. The *mapping class group* of S, denoted $MCG(S)$, is the group of homotopy classes of orientation-preserving homeomorphisms of S, and we call elements of $MCG(S)$ *mapping classes*. Mapping class groups play an important role in low-dimensional geometry and topology; not only are they the symmetry groups of surfaces, but they are also important in the construction and general theory of 3- and 4-manifolds. They are also of interest purely as groups. While $Homeo^+(S)$ is uncountable and difficult to study, $MCG(S)$ is, incredibly, fintely presented, and has a solvable word problem. Mapping class groups also contain many interesting groups: surface and 3-manifold groups (*i.e.* $\pi_1(M)$ where M is a surface or 3-manifold), RAAGs, and mapping class groups of other surfaces to name a few.

We frequently study mapping class groups of surfaces via their action on simple closed curves and subsurfaces. An *essential simple closed curve* (hereafter just "simple closed curve") is the free homotopy class of a non-nullhomotopic and non-peripheral simple closed curve on S, and an *essential subsurface* (hereafter just "subsurface") $S' \subseteq S$ is either a regular neighborhood of an essential simple closed curve (*i.e.* an annulus), or a component of the complement of a collection of pairwise disjoint essential simple closed curves (*i.e.* the complement of a multi-curve). For both simple closed curves and subsurfaces, we will not distinguish between a representative and its homotopy class. With respect to the natural action of $MCG(S)$ on simple closed curves, there is a trichotomy, due to work of Nielsen and Thurston (*cf.* [11]): given $f \in MCG(S)$, f is either

1. finite order,
2. reducible, *i.e.* infinite order and preserves a non-empty multi-curve C, or
3. pseudo-Anosov, *i.e.* infinite order and no power of f preserves any multi-curve

For a reducible mapping class f, it follows from Birman-Lubotzky-McCarthy that some power f fixes a multi-curve C point-wise, and restricts to a pseudo-Anosov mapping class or the identity on each component of $S \backslash C$. We call such "partial pseudo-Anosov" mapping classes, as well as pseudo-Anosov mapping classes, *pure*. The *support* of a pure mapping class f is all of S if f is pseudo-Anosov, an annulus about the twisting curve if f is a Dehn twist, or the components of $S \backslash C$ where the action of f is non-trivial if f is a partial pseudo-Anosov mapping class. A partial pseudo-Anosov mapping class f could also multi-twist about its fixed multicurve - in this case we define the support to be the components of $S \backslash C$ where the action of f is non-trivial together with the annular neighborhoods of those curves in C where f is twisting. In particular, if a partial pseudo-Anosov mapping class f exhibits such “boundary twisting", its support is disconnected by definition. Given an element of the mapping class group written as a product of pure mapping classes, we define its support as follows: from the list of pure mapping classes appearing in a cyclically reduced conjugate of the given representative, we extract a list of the supports of the pure mapping classes that appear, and take their union. We then translate this subsurface by the word which conjugated the given element to its cyclically reduced representative.

Chapter 3

The Foundation

3.1 Curve Graphs and Subsurface Projection

The *curve graph* of S, denoted $\mathcal{C}(S)$, is the graph whose vertices are simple closed curves, and whose edges are spanned by vertices corresponding to pairs of simple closed curves which can be realized disjointly; see Figure 3-1 for small portion of the curve graph of a genus 2 surface. We note that almost all curve graphs are infinite-diameter and locally infinite.

We equip $\mathcal{C}(S)$ with the graph metric. A celebrated theorem of Masur-Minsky says that with this metric, $\mathcal{C}(S)$ is δ-hyperbolic. Moreover, Aougab [1], Bowditch

Figure 3-1: A collection of simple closed curves on a surface and the subgraph they span in the curve graph.

[8], and Clay-Rafi-Schleimer [14] have shown that the hyperbolicity constant δ can be made independent of S, and Hensel-Przytycki-Webb [27] have shown that $\delta = 17$ suffices. In the sequel, we will use the notation δ instead of its explicit value to make clear the dependence on the hyperbolic geometry of $\mathcal{C}(S)$.

Our ping-pong sets will be given in terms of Ivanov-Masur-Minsky's *subsurface projections* of simple closed curves to subsurfaces of S. For now, we assume $\chi(S) < 0$, fix a hyperbolic metric on S, and for each simple closed curve we take its unique geodesic representative. Given a non-annular subsurface $S' \subset S$, we define a coarse "projection" map $\pi_{S'} : \mathcal{C}(S) \to \mathcal{C}(S')$ as follows. Let γ be a simple closed curve on S. If γ is disjoint from S' entirely, then $\pi_{S'}(\gamma) = \varnothing$, and if γ is properly contained is S' then $\pi_{S'}(\gamma) = \gamma$. Otherwise, γ non-trivially intersects $\partial S'$, and we define $\pi_{S'}(\gamma)$ to be the set of simple closed curves obtained by considering each arc α of $\gamma \cap S'$ and taking the boundary of a regular neighborhood of $\alpha \cup \partial S'$, see Figure 3-2. Note that

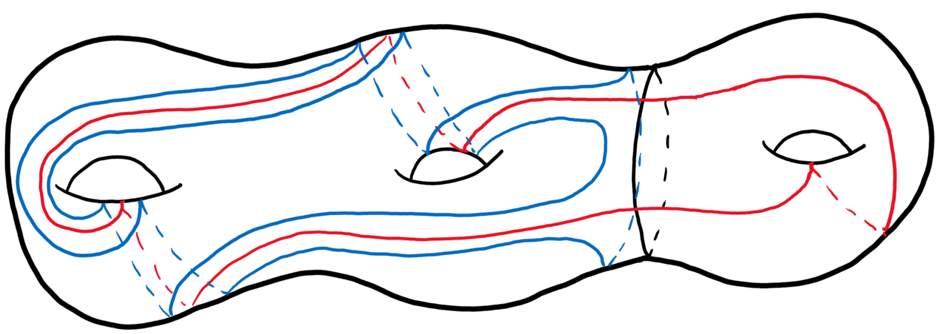

Figure 3-2: The projection of the red curve to the left genus two subsurface consists of the blue curves.

geodesic simple closed curves are necessarily in minimal position, so that no such arc can be homotoped out of S', and thus each simple closed curve obtained this way is essential.

Given two simple closed curves β and γ with non-empty projection to S', we define their *projection distance* $d_{S'}(\beta, \gamma)$ to be

$$d_{S'}(\beta, \gamma) := diam_{\mathcal{C}(S')}\{\pi_{S'}(\beta) \cup \pi_{S'}(\gamma)\}$$

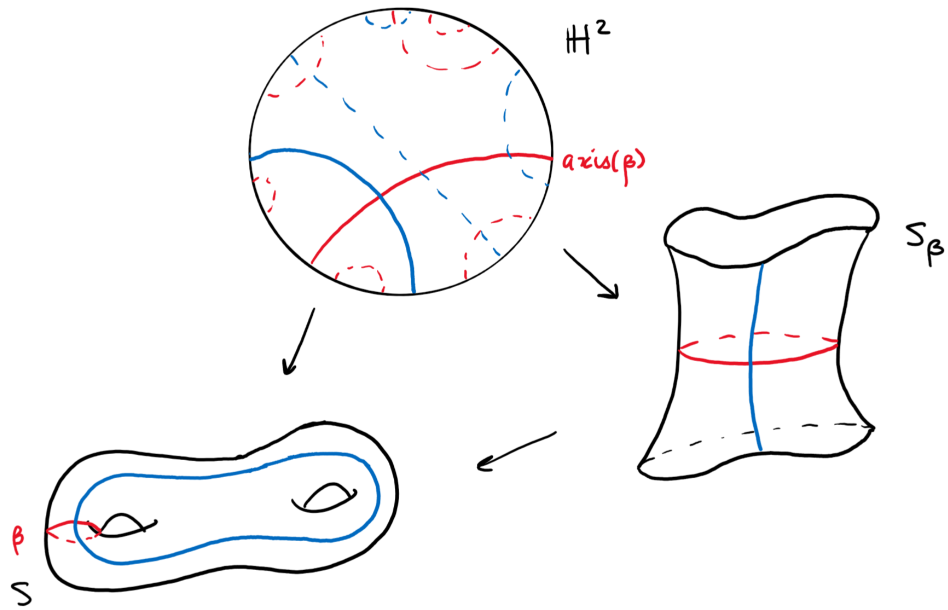

Figure 3-3: Projecting a simple closed curve to an annulus involves lifting to the infinite-sheeted cover corresponding to the core curve of the annulus.

It is a fact that these projections are coarsely Lipschitz, so that in particular the projection distance between any two disjoint simple closed curves to any subsurface is uniformly bounded (by, say, 4). Hence, it also makes sense to project a subsurface S' to another subsurface S'' when they intersect non-trivially, and the result is a bounded diameter set containing $\partial S'$.

The projection of simple closed curves to annuli is (necessarily) defined differently: again fix a hyperbolic metric on S, and let β and γ be (the unique geodesic representatives of) intersecting simple closed curves. Consider the (compactified) cover S_β of S corresponding to β. We define the projection $\pi_\beta(\gamma)$ to be the collection of lifts c of γ to the cover S_β which connect the two boundary components, see Figure 3-3. We can assemble the set of all homotopy (rel. boundary) classes of such arcs in S_β into a graph $\mathcal{A}(\beta)$, the *arc complex* of β, with edges representing pairs of vertices corresponding to homotopy (rel. boundary) classes of arcs which can be realized with disjoint interiors, and equipped with the graph metric. Given another simple closed curve δ which intersects β, we define the projection distance $d_\beta(\gamma, \delta)$ exactly as above;

roughly, this distance measures how much γ "twists" around δ relative to β. Though we chose a hyperbolic metric, it is not hard to see that $d_\beta(\delta, \gamma) \leq i(\delta, \gamma) + 1$.

Another useful fact about these projections is that they are coarsely invariant under mapping classes supported away from the subsurface we're projecting to.

Lemma 5 (Mousley, Lemma 3.1)**.** *Let $f \in MCG(S)$ be a pure mapping class supported on a subsurface $S_j \subset S$ which is disjoint from a subsurface S_i. If S_i is an annulus about a curve β, we also require that ∂S_j does not contain β. Let γ and δ be simple closed curves on S. Then*

$$|d_{S_i}(\gamma, \delta) - d_{S_i}(\gamma, f(\delta))| \leq 4.$$

In order to show that the RAAGs we build are undistorted in $MCG(S)$, we will need a way to relate word length in $MCG(S)$ to the only other available data we will have, namely projection distances. This relationship is captured by the following "distance formula" of Masur-Minsky [42], analogous to the distance formula in Theorem 4. Before stating it, we establish notation. A (complete clean) *marking* μ on S consists of a pants decomposition $\{\beta_i\}$, called the *base* of μ, together with a *transversal* for each β_i satisfying certain properties which are unnecessary for the discussion at hand. Masur-Minsky build a graph $\widetilde{\mathcal{M}}(S)$, called the *marking graph* of S, whose vertices correspond to markings and whose edges are spanned by vertices corresponding to markings related by certain *elementary moves*. Equipped with the graph metric, the graph $\widetilde{\mathcal{M}}(S)$ is locally finite and admits a cocompact action of $MCG(S)$ by isometries, so that $MCG(S)$ and $\widetilde{\mathcal{M}}(S)$ are quasi-isometric by the Milnor-Schwarz lemma. We define the projection of a marking μ to a non-annular subsurface $S' \subseteq S$ to be $\pi_{S'}(base(\mu))$ and we define the projection of μ to an annulus to be either $\pi_{S'}(base(\mu))$ if the core curve of the annulus is not in base(μ), and the projection of the corresponding transversal otherwise.

Theorem 5 (Masur-Minsky, Theorem 6.10 and Theorem 7.1)**.** *There exists* $K_0 = K_0(S) > 0$ *with the following property: for all* $K \geq K_0$*, there exist constants* $A \geq 1$ *and* $B \geq 0$ *such that for all pairs of markings* $\mu, \mu' \in \widetilde{\mathcal{M}}(S)$ *we have*

$$\frac{1}{A} \sum_{S' \subseteq S} \{\{d_{S'}(\mu, \mu')\}\}_K - B \leq d_{\widetilde{\mathcal{M}}(S)}(\mu, \mu') \leq A \sum_{S' \subseteq S} \{\{d_{S'}(\mu, \mu')\}\}_K + B,$$

where the sums are taken over all subsurfaces (including S *itself), and* $\{\{\}\}_K$ *is as in Theorem 4.*

In particular, we can approximate the $MCG(S)-$word length of a mapping class f by looking at the subsurface projections distances between μ and $f(\mu)$.

3.2 The Action on the Curve Graph

We now record a set of results of Masur-Minsky from [41] and [42] concerning the action on the curve graph of pseudo-Anosov mapping classes. The first tells us that they act on $\mathcal{C}(S)$ like hyperbolic isometries.

Proposition 1 ([41], Prop. 3.6)**.** *There exists a constant* $c = c(S) > 0$ *such that, for any pseudo-Anosov mapping class* $f \in MCG(S)$*, any simple closed curve* γ*, and any* $n \in \mathbb{Z}\backslash\{0\}$*, we have*

$$d_S(f^n(\gamma), \gamma) \geq c|n|.$$

Masur-Minsky proved the above for the so-called “non-sporadic" surfaces. For the sporadic cases, namely the once-punctured torus and four-punctured sphere, we redefine the curve graph in such a way that we obtain the Farey graph, where it is noted by Mangahas [39] that the same result follows by considering the action of hyperbolic isometries on the Farey graph embedded in $\mathbb{H}^2$. It is easy to show that Proposition 1 implies that for any simple closed curve γ and any pseudo-Anosov mapping class f, the bi-infinite sequence of curves $\{f^n(\gamma)|n \in \mathbb{Z}\}$ is an f-invariant quasi-geodesic. By restricting a pure mapping class to a pseudo-Anosov component $S' \subset S$, we obtain such a lower bound for the action of f on $\mathcal{C}(S')$, and for a power of a Dehn twist acting on its corresponding arc complex, the quantity c can be taken to be 1.

As noted above, any pseudo-Anosov mapping class f preserves many *quasi-geodesics* in $\mathcal{C}(S)$. However, later we will want to consider nearest-point projections to honest *geodesics*. In order to do so, we will use the following proposition, which says that pseudo-Anosov mapping classes are the WPD elements for the $MCG(S)$ action on $\mathcal{C}(S)$.

Proposition 2 ([42], Prop. 7.6)**.** *Let $f \in MCG(S)$ be pseudo-Anosov. There exists a bi-infinite geodesic α in $\mathcal{C}(S)$ such that for all j, α and $f^j(\alpha)$ are 2δ fellow travelers, i.e. for all $x \in \alpha$ there is some $y \in f^j(\alpha)$ such that $d_{\mathcal{C}(S)}(x, y) \leq 2\delta$, and vice-versa.*

The geodesic α and its f-translates are referred to as a *quasi-axis* for f. A straightforward computation shows that the nearest point projections of any vertex x in $\mathcal{C}(S)$ to any two geodesics in a quasi-axis are at most 10δ apart. Applying Proposition 2 to the action of f on its quasi-axis, we have:

Lemma 6 ([42], Lemma 7.7)**.** *Given $A > 0$, let N be the smallest integer such that $c(S)N > A + 10\delta$, where $c(S)$ is the constant from Proposition 2. Then for all $n \geq N$,*

$$d_{\mathcal{C}(S)}(\pi(x), \pi(f^n(x))) \geq A,$$

where π denotes a coarse nearest-point projection to the quasi-axis of f.

We can now combine Theorem 4 and Theorem 5 to give (half of) a more general distance formula.

Theorem 6. *Let $G < MCG(S)$ and let $\{f_i\}$ be a finite collection of pure mapping classes supported on subsurfaces $\{S_i\}$. Let $\mathcal{A}$ be the collection of all quasi-axes of the G-conjugates of the f_i (if a given f_i is reducible, we take its quasi-axis in $\mathcal{C}(S_i)$). There exist constants A', B', and M' such that for all pairs of markings $\mu, \mu' \in \widetilde{\mathcal{M}}(S)$, we have*

$$d_{\widetilde{\mathcal{M}}(S)}(\mu, \mu') \geq \frac{1}{A'} \sum_{S' \subseteq S} \sum_{\alpha \in \mathcal{A} \cap \mathcal{C}(S')} \{\{d_\alpha(\mu, \mu')\}\}_{M'} - B'$$

Chapter 4

The Inspiration

4.1 Koberda's Theorem

Recall Koberda's Theorem from the introduction.

Theorem 7. *Let $\{f_1, \ldots, f_m\}$ be an irredundant collection of pure mapping classes supported on connected subsurfaces $S_1, \ldots, S_m \subseteq S$. There exists some $N \neq 0$ such that for all $n \geq N$,*

$$\langle f_1^n, \ldots, f_m^n \rangle \cong A(\Gamma),$$

where Γ is the co-intersection graph of the subsurfaces $\{S_i\}$.

Here, *irredundancy* means that no two given mapping classes generate a virtually cyclic group, *i.e.* they don't share a common power, and the *co-intersection graph* has as vertices the subsurfaces S_i (with multiplicity) and edges between those which can be realized disjointly. The proof of this theorem is a ping-pong argument on $\mathcal{PML}(S)$, the space of *projective measured laminations* (a certain completion of the set of simple closed curves) on S, and goes roughly as follows. Each pure mapping class f_i has a pair of preserved laminations $\lambda_i^{\pm}$, the *stable* and *unstable* laminations of f_i, supported on S_i. It is a classical result of Ivanov that for each pair $\lambda_i^{\pm}$ there are open sets $U_i^{\pm}$ so that large positive powers of f_i take any lamination into U_i^+ and large negative powers of f_i take any lamination into U_i^-. One can take these sets to be disjoint and play classical ping-pong in the obvious way. However, there are several technical difficulties that arise when considering reducible mapping classes, which Koberda overcomes by lifting to the universal cover $\mathbb{H}^2$ of S and studying angles of intersection between lifts of leaves of the laminations. Both Koberda's and Ivanov's proofs rely in some way on compactness or continuity arguments, and so the given constant isn't computed explicitly (though it would be interesting to try to effectivize both arguments).

4.2 The Clay-Leininger-Mangahas Construction

We now give a brief review of the Clay-Leininger-Mangahas [15] construction of so-called "admissible" embeddings of RAAGs into mapping class groups. Our starting point is the notion of a "nice realization" of a graph (see Figure 4-1 for an example): we say a pair $(S, \{S_i\})$, consisting of a surface S and a collection of connected subsurfaces $\{S_i\}$, *nicely realizes* a graph Γ if

1. Γ is the co-intersection graph of the collection $\{S_i\}$, *i.e.* two vertices in Γ span an edge if and only if the corresponding subsurfaces are disjoint

2. each S_i is a proper, non-annular subsurface

3. if $S_i \cap S_j \neq \varnothing$, then $\partial S_i \cap \partial S_j \neq \varnothing$

Note that the conditions for a nice realization are very restricted compared to the generality of Theorem 1, in particular they exclude annuli, the whole surface, and nested subsurfaces.

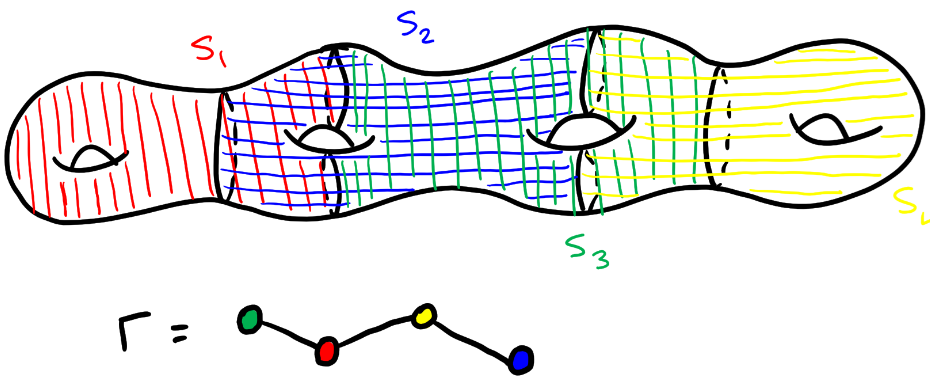

Figure 4-1: A nice realization of the path on four vertices.

Regardless, the authors are able to draw much stronger geometric conclusions about the associated RAAG subgroup of $MCG(S)$.

Theorem 8 (Clay-Leininger-Mangahas)**.** *Suppose* $(S, \{S_i\})$ *nicely realizes* Γ*, choose* $f_i \in MCG(S_i)$ *for each* i*, and let*

$$\phi : A(\Gamma) \to MCG(S)$$

be the injective homomorphism defined by $v_i \to f_i^N$*, where* N *is as in Theorem 1. There exists a constant* R *so that if the translation length of each* f_i^N *on* $\mathcal{C}(S_i)$ *is at least* R*, then* ϕ *is a quasi-isometric embedding. Moreover, each element in the image of* ϕ *is pseudo-Anosov on its support.*

We remark that although the authors assume that the map ϕ is injective as a hypothesis, this can be derived almost directly from their version Theorem 9 below, whose proof we sketch as a convenience to the reader. Though we will use projections to geodesics in curve subgraphs instead of just subsurface projections, the ideas are identical, and we highlight the necessary modifications that need to be made. A key ingredient is the following inequality due to Behrstock [4]; the version below is attributed to Leininger [39].

Lemma 7 (Behrstock Inequality)**.** *Let S_i, S_j, and S_k be three pairwise intersecting essential subsurfaces or simple closed curves. Then*

$$d_{S_i}(\partial S_j, \partial S_k) \geq 10 \implies d_{S_j}(\partial S_i, \partial S_k) \leq 4.$$

If S_i (or S_j or S_k) is an annulus, we replace ∂S_i with the core curve β_i. If all three are annuli, we may further replace 4 with 3.

In the next section we prove a generalization of this lemma (the Multi-scale Behrstock Inequality) to account for not just subsurface projections, but nearest-point projections to geodesics in curve subgraphs of $\mathcal{C}(S)$. Again, we remark that the theorem below is a modified version of the cited theorem, suited to our needs.

Theorem 9 ([15], Theorem 5.2)**.** *Let H be as above, $\mu \in \widetilde{\mathcal{M}}(S)$ be a marking on S, and let*

$$N = \frac{5K_{BGIT} + K_0 + 200\delta + 2M_3 + M_2 + M_1 + 4}{\min\limits_{1 \leq i \leq m} c(S_i)},$$

where the constants M_i are defined in Chapter 5 and are derived from projection data of the quasi-axes of the generators of H, and where K_0 is as in the Masur-Minsky distance formula. Let $w = g_1 \cdots g_k \in H$, where $g_i = (f_j^n)^{e_i}$ for $n \geq N$. Then

$$d_{g_1 \cdots g_{i-1}\alpha_j}(\mu, w\mu) \geq (K_0 + K_{BGIT} + 48\delta)|e_i|.$$

Proof. (sketch) First, we remark that while the set of mapping classes considered in [15] explicitly excludes Dehn twists, pseudo-Anosovs, and mapping classes with the same or nested supports, the Multi-scale Behrstock Inequality in the next section allows *us* to consider them. The proof goes by induction on k, the (minimal) number of “syllables" of w (note that we are *not* using central form). The base case is simply the claim that

$$\begin{aligned} d_{\alpha_j}(\mu, g_1(\mu)) &= d_{\alpha_j}(\mu, (f_j^n)^{e_1}(\mu)) \\ &\geq (K_0 + K_{BGIT} + 48\delta)|e_i|, \end{aligned}$$

which is true by construction of N. For the induction, we break w into subwords:

$$\begin{aligned} w &= (g_1 \cdots g_\ell)(g_{\ell+1} \cdots g_{i-1})g_i(g_{i+1} \cdots g_k) \\ &= abg_ic. \end{aligned}$$

Via repeated applications of the triangle inequality, using Lemma 4 where necessary, the claim reduces to the statement that the distances

$$d_{\alpha_j}(a^{-1}(\mu), \mu) \ , \ d_{\alpha_j}(c(\mu), \mu)$$

are both bounded in terms of the constants appearing in the numerator of N. This is also shown via the triangle inequality, using Lemma 4 where necessary, as well as the Multi-scale Behrstock Inequality. □

4.3 A Useful Tool

Below is a generalization of the above Behrstock inequality. The modification we will make will allow us to not only consider subsurface projections, but also nearest-point projections to geodesics in $\mathcal{C}(S)$ and its subgraphs $\mathcal{C}(S')$ for subsurfaces $S' \subset S$. For the proof, we will need the following result from [42] concerning distance bounds in the curve graph, known as the Bounded Geodesic Image Theorem. The uniform statement below is originally due to Webb [49], and the constant was recently shown by Jin [30] to be bounded above by 44.

Theorem 10 (Bounded Geodesic Image Theorem)**.** *There exists a constant K_{BGIT} with the following property: if $S' \subset S$ is a subsurface and α is a geodesic in $\mathcal{C}(S)$ with the property that $\pi_{S'}(z) \neq \varnothing$ for all $z \in \alpha$, then*

$$diam_{\mathcal{C}(S')}\{\pi_{S'}(\alpha)\} \leq K_{BGIT}.$$

Lemma 8 (Multi-Scale Behrstock Inequality)**.** *Let β be a simple closed curve on S, and let α_1 and α_2 be either simple closed curves or geodesics in $\mathcal{C}(S_1) \subseteq \mathcal{C}(S)$ and $\mathcal{C}(S_2) \subseteq \mathcal{C}(S)$ respectively, where S_1 and S_2 are either proper subsurfaces of S or S itself. Then*

$$\min\{d_{\alpha_1}(b_1, a_2)\} \geq K_{BGIT} + 48\delta \implies \min\{d_{\alpha_2}(b_2, a_1)\} < K_{BGIT} + 48\delta$$

The minima are taken over $b_i \in \pi_{\alpha_i}(\beta)$ and $a_i \in \pi_{\alpha_j}(\alpha_i)$, where by π_{α_i} we mean either the previously defined projection to annuli if α_i is a simple closed curve, or the composition of subsurface projection to S_i followed by nearest-point projection to α_i if α_i is a geodesic.

Proof. We break the proof into cases depending on the type of each α_i and the configuration of the S_i within S. The game will be to show that if one of the quantities is suitably large, the other is bounded. In the arguments below there is repeated implicit use of Lemma 1 and Lemma 2.

Case 1: α_1 and α_2 are both simple closed curves.

In this case, we can use the original Behrstock Inequality.

Case 2: α_1 is a simple closed curve and α_2 is a geodesic in $\mathcal{C}(S_2)$.

We first consider the case that $S_2 = S$. If $\min d_{\alpha_1}(b_1, a_2) \geq K_{BGIT}$, then by the contrapositive of the Bounded Geodesic Image Theorem, a geodesic connecting β to $\pi_{\alpha_2}(\beta)$ passes through the 1-neighborhood of α_1. If z is the vertex on this geodesic which is adjacent to α_1, then we have

$$\begin{aligned} \min d_{\alpha_2}(b_2, a_1) &\leq d_{\alpha_2}(\beta, \alpha_1) \\ &\leq d_{\alpha_2}(\beta, z) + d_{\alpha_2}(z, \alpha_1). \end{aligned}$$

By construction, the nearest-point projections of β and z to α_2 overlap. Also, by Proposition 1 either $d_{\alpha_2}(z, \alpha_1) < 8\delta + 2$ or else $d_{\alpha_2}(z, \alpha_1) \leq 1 + 24$. Hence,

$$\begin{aligned} \min d_{\alpha_2}(b_2, a_1) &\leq 8\delta + (1 + 24\delta) \\ &= 1 + 32\delta. \end{aligned}$$

We now suppose that S_2 is a proper essential subsurface of S and that $\min d_{\alpha_1}(b_1, a_2) \geq 11$. Since each vertex in α_2 represents a curve which is disjoint from ∂S_2, we then have $d_{\alpha_1}(\beta, \partial S_2) \geq 10$. Applying the Behrstock inequality yields $d_{S_2}(\beta, \alpha_1) \leq 4$, and so

$$\begin{aligned} \min d_{\alpha_2}(b_2, a_1) &\leq d_{\alpha_2}(\beta, \alpha_1) \\ &\leq 4 + 24\delta. \end{aligned}$$

Case 3: α_1 and α_2 are both geodesics in their respective curve complexes.

We first consider the case that $S_1 = S_2$. Assume $\min d_{\alpha_2}(b_2, a_1) \geq 8\delta + 2$, so that in particular

$$d_{\alpha_2}(\beta, \pi_{\alpha_2}(\pi_{\alpha_1}(\beta))) \geq 8\delta + 2$$

By hyperbolicity ($cf.$ [42], Lemma 7.5), a geodesic in S_2 between β and $\pi_{\alpha_1}(\beta)$ passes within 2δ of the geodesic subsegment of α_2 connecting their projections. Let z be a point on the geodesic segment between β and $\pi_{\alpha_1}(\beta)$ that is at most a distance 2δ from a point y on α_2. Then we have

$$\begin{aligned}\min d_{\alpha_1}(b_1, a_2) &\leq d_{\alpha_1}(\beta, y)\\ &\leq d_{\alpha_1}(\beta, z) + d_{\alpha_1}(z, y)\\ &\leq 8\delta + (2\delta + 24\delta)\\ &= 34\delta.\end{aligned}$$

Next, if S_1 is nested in S_2 and we assume $\min d_{\alpha_2}(b_2, a_1) \geq 34\delta$, then each vertex on the geodesic between β and $\pi_{\alpha_2}(\beta)$ has distance at least 2 from ∂S_1. Hence, $d_{S_1}(\gamma, \pi_{\alpha_2}(\gamma)) \leq K_{BGIT}$, and thus

$$\begin{aligned}\min d_{\alpha_1}(b_1, a_2) &\leq d_{\alpha_1}(\beta, \pi_{\alpha_2}(\beta))\\ &\leq K_{BGIT} + 24\delta.\end{aligned}$$

Finally, we consider the case that ∂S_1 and ∂S_2 intersect. Suppose that $\min d_{\alpha_1}(b_1, a_2) \geq 11 + 48\delta$, and let z be any vertex on α_2. Then

$$\begin{aligned}d_{S_1}(\beta, \partial S_2) &\geq d_{\alpha_1}(\beta, \partial S_2) - 24\delta\\ &\geq d_{\alpha_1}(\beta, z) - d_{\alpha_1}(z, \partial S_2) - 24\delta\\ &\geq (11 + 48\delta) - (1 + 24\delta) - 24\delta\\ &= 10.\end{aligned}$$

Hence, by the Behrstock inequality, we have $d_{S_2}(\beta, \partial S_1) \leq 4$, and so, choosing y on α_1,

$$\begin{aligned}
\min d_{\alpha_2}(b_2, a_1) &\leq d_{\alpha_2}(\beta, y) \\
&\leq d_{\alpha_2}(\beta, \partial S_1) + d_{\alpha_2}(\partial S_1, y) \\
&\leq (d_{S_2}(\beta, \partial S_1) + 24\delta) + (1 + 24\delta) \\
&= 5 + 48\delta.
\end{aligned}$$

Thus $K_{BGIT} + 48\delta$ suffices for all cases.

$\square$

Chapter 5

Putting it All Together

5.1 Generation

We first show that the group generated by $\{f_1^n, \ldots, f_m^n\}$ is the expected RAAG.

Proof of generation. Let $\{f_1, \ldots, f_m\} \in$ MCG(S) be an irredundant collection of pure mapping classes with connected supporting subsurfaces $\{S_1, \ldots, S_m\}$. For each $1 \leq i \leq m$, let α_i be a geodesic in the quasi-axis for f_i in $\mathcal{C}(S_i) \subseteq \mathcal{C}(S)$ or the core curve of S_i if S_i is an essential annulus. As in the proof of Lemma 4 (ping-pong), we assume that the co-intersection graph of the collection $\{S_i\}$ is not a non-trivial join, so that for each f_i there is some f_j which does not commute with it. We will explicitly construct a constant N and a group action so that for all $n \geq N$, $\{f_1^n, \ldots, f_m^n\}$ satisfy the criteria for ping-pong. To this end, set

$$X = \{\beta \mid \beta \text{ an essential simple closed curve in } S\},$$

and for each $1 \leq i \leq m$, set

$$X_i = \{\beta \mid \min d_{\alpha_i}(b_i, a_j) > K_{BGIT} + 48\delta \quad \text{for all } j \text{ such that } S_j \cap S_i \neq \emptyset\},$$

$$X_i' = \{\beta \mid \min d_{\alpha_i}(b_i, a_j) > K_{BGIT} + 48\delta + 4 \quad \text{for all } j \text{ such that } S_j \cap S_i \neq \emptyset\},$$

where the minima are taken over $b_i \in \pi_{\alpha_i}(\beta)$ and $a_j \in \pi_{\alpha_i}(\alpha_j)$. Observe that if S_i and S_j intersect, then by the Multi-scale Behrstock Inequality their corresponding sets X_i and X_j are disjoint. Moreover, since we assumed the mapping classes were irredundant, *i.e.* no two have a common power, no two preserve the same ending lamination in the Gromov boundary $\partial\mathcal{C}(S)$. Hence, no chosen α_i fellow travels another chosen α_j, and so these geodesics have bounded diameter projections to one another..

Let w be a word in the abstract RAAG generated by $\{f_1^n, \ldots, f_m^n\}$. We begin by putting w into central form as in the proof of the ping-pong lemma: using only shuffles and deletions, we may write w as

$$w = u_k g_k u_{k-1} g_{k-1} \cdots u_1 g_1,$$

where each g_j represents some power of some f_i^n, and each u_j is a word in the generators satisfying the necessary properties of the central form. We possibly make one further modification to this representative. For each g_j which is a power of a Dehn twist, if a power of a generator appearing in the corresponding u_j is supported on a subsurface containing the twisting curve as a boundary component, we may shuffle $u_j g_j$ to $u_j' g_j'$, where g_j' is the aforementioned power of a generator and u_j' contains the original g_j instead. To see that this modification does not violate the central form, note that since g_{j-1} and g_{j+1} don't commute with g_j, their supports intersect the twisting curve of g_j, which is the boundary of the support of g_j'. Hence the supports of g_{j-1} and g_{j+1} both intersect that of g_j'.

We may now play ping-pong. Up to relabelling, we assume $g_1 = f_1^{nr_1}$, $g_2 = f_2^{nr_2}$, and $g_k = f_j^{nr_j}$ for some j. Choose $\beta \in X_2 \backslash (X_2 \cap X_j)$; either g_2 and g_k don't commute, so their corresponding sets X_2 and X_j are disjoint, or they commute and their supports are disjoint, and we can choose a β which intersects S_2 but not S_j. If g_k is also a power of f_2^n, conjugate w by g_k, choose $\beta \in X_1$, and run the same argument below. Since g_1 and g_2 don't commute, their corresponding sets X_1 and X_2

are disjoint. In particular, since $\beta \in X_2$, it satisfies

$$\min d_{\alpha_1}(b_1, a_2) \leq K_{BGIT} + 48\delta.$$

For each ℓ such that $S_\ell \cap S_1 \neq \varnothing$, we have

$$\begin{aligned}
\min d_{\alpha_1}(b_1, a_\ell) &\leq d_{\alpha_1}(\beta, \alpha_\ell) \\
&\leq d_{\alpha_1}(\beta, \alpha_2) + d_{\alpha_1}(\alpha_2, \alpha_\ell) \\
&\leq (K_{BGIT} + 48\delta + diam\{\pi_{\alpha_1}(\beta)\} + diam\{\pi_{\alpha_1}(\alpha_2)\}) + M_1 \\
&= K_{BGIT} + 48\delta + 4\delta + M_2 + M_1 \\
&= K_{BGIT} + 52\delta + M_2 + M_1,
\end{aligned}$$

where

$$\begin{aligned}
M_1 &= \max_{1 \leq i, \ell, s \leq m} d_{\alpha_i}(\alpha_\ell, \alpha_s), \\
M_2 &= \max_{1 \leq i, j \leq m} diam\{\pi_{\alpha_i}(\alpha_j)\}.
\end{aligned}$$

Choosing $b' \in \pi_{\alpha_1}(\beta)$ and $a'_\ell \in \pi_{\alpha_1}(\alpha_\ell)$ which realize $\min d_{\alpha_1}(b_1, a_\ell)$, we have

$$\begin{aligned}
d_{\alpha_1}(f_1^N(b'), a_{\ell'}) &\geq d_{\alpha_1}(f_1^N(b'), b') - d_{\alpha_1}(b', a'_\ell) \\
&\geq d_{\alpha_1}(f_1^N(b'), b') - (K_{BGIT} + 52\delta + M_2 + M_1).
\end{aligned}$$

Hence, if

$$\begin{aligned}
d_{\alpha_1}(f_1^N(b'), b') &\geq 2K_{BGIT} + 110\delta + M_2 + M_1 + 4 \\
&\quad + diam\{\pi_{\alpha_1}(f_1^N(b'))\} + diam\{\pi_{\alpha_1}(b')\} \\
&= 2K_{BGIT} + 118\delta + M_2 + M_1 + 4,
\end{aligned}$$

we will have $f_1^N(b') \in X_1'$. Invoking Lemma 6, we set

$$N = \frac{5K_{BGIT} + 200\delta + M_2 + M_1 + 4}{\min\limits_{1 \le i \le m} c(S_i)},$$

which is in fact much larger than we need here, but will be useful later. Thus, $g_1(\beta) \in X_1$, and by Lemma 2, $u_1 g_1(\beta) \in X_1$. Running this process until it terminates after the application of $u_k g_k$, we see that $w(\beta) \in X_j$, and we are done.

□

If we restrict to the case where all the f_i are Dehn twists, the constant N simplifies quite a bit.

Corollary 2. *Let $\{t_1, \ldots, t_m\}$ be a collection of Dehn twists about distinct essential simple closed curves $\{\beta_1, \ldots, \beta_m\}$ on S, and let*

$$N = 18 + \max_{i,j} i(\beta_i, \beta_j).$$

Then for all $n \ge N$, we have

$$\langle t_1^n, \ldots, t_m^n \rangle \cong A(\Gamma),$$

where Γ is the subgraph of $\mathcal{C}(S)$ spanned by the curves $\{\beta_i\}$.

Proof. As we are dealing only with essential annuli, we don't need to account for the constant $c(S_i)$ from Proposition 2 (since for Dehn twists, $c = 1$), and we can use the original Behrstock inequality. Following the proof of Theorem 2, for each $1 \le i \le m$ we set

$$X_i = \{\gamma \mid d_{\beta_i}(\gamma, \beta_j) \ge 10 \quad \text{for all } j \text{ such that } \beta_j \cap \beta_i \ne \emptyset\},$$
$$X_i' = \{\gamma \mid d_{\beta_i}(\gamma, \beta_j) \ge 14 \quad \text{for all } j \text{ such that } \beta_j \cap \beta_i \ne \emptyset\},$$

and we write $w = u_k g_k \cdots u_1 g_1$, where each g_j is a power of some t_i^n, in central form; relabelling, we assume $g_1 = t_1^{nr_i}$, $g_2 = t_2^{nr_2}$, and $g_k = t_j^{nr_k}$ for some j. Choose $\beta \in X_2 \backslash (X_2 \cap X_j)$, or in the case that g_k is also a power of t_2^n, conjugate w by g_k, choose $\gamma \in X_1$, and run the same argument below.

Since $\gamma \in X_2$, we have $d_{\beta_1}(\gamma, \beta_2) \leq 3$. For any ℓ such that $i(\beta_1, \beta_\ell) \neq 0$, we then have

$$\begin{aligned} d_{\beta_1}(\gamma, \beta_\ell) &\leq d_{\beta_1}(\gamma, \beta_2) + d_{\beta_1}(\beta_2, \beta_\ell) \\ &\leq 3 + M_1. \end{aligned}$$

Then

$$\begin{aligned} d_{\beta_1}(t_1^N(\gamma), \beta_\ell) &\geq d_{\beta_1}(t_1^N(\gamma), \gamma) - d_{\beta_1}(\gamma, \beta_\ell) \\ &\geq N - 3 - M_1. \end{aligned}$$

Hence, setting $N = 17 + M_1$ suffices to finish the proof. But we previously noted that

$$\begin{aligned} M_1 &= \max_{1 \leq i, \ell, s \leq m} d_{\beta_i}(\beta_\ell, \beta_s) \\ &\leq \max_{1 \leq \ell, s \leq m} i(\beta_\ell, \beta_s) + 1, \end{aligned}$$

so we set $N = 18 + \max\limits_{1 \leq \ell, s \leq m} i(\beta_\ell, \beta_s)$ so that the constant is independent of any choice of hyperbolic metric. $\square$

This should be compared to the main theorem of [46], where a similar (quadratic) bound was computed. As an easy application, we state the following.

Corollary 3. *Let $\{\beta_1, \ldots, \beta_m\}$ be a collection of essential simple closed curves such that no three curves pairwise intersect. Then the 19^{th} powers of the corresponding Dehn twists generate a RAAG.*

Proof. Since no three curve pairwise intersect, the projection distances $d_{\beta_i}(\beta_j, \beta_k)$ are uniformly bounded above by 1. □

5.2 Undistortion

We now show that the subgroups generated in the previous section are undistorted in $MCG(S)$, after increasing the power N by a controlled amount. The proof of undistortion below is nearly identical to that of Clay-Leininger-Mangahas.

Proof of undistortion. Via the quasi-isometry between $MCG(S)$ and the marking graph $\widetilde{\mathcal{M}}(S)$, it suffices to show that there are constants $A \geq 1$ and $B \geq 0$ such that for all $w \in H$

$$\frac{1}{A} d_{\widetilde{\mathcal{M}}(S)}(\mu, w\mu) - B \leq d_H(1, w) \leq A d_{\widetilde{\mathcal{M}}(S)}(\mu, w\mu) + B.$$

For any group G acting by isometries on a metric space (X, d_X), we always have

$$d_X(x, gx) \leq A d_G(1, g),$$

where $A \geq \max d_X(x, s_i x)$, and s_i is a generator for G. Hence, we need only to find A and B so that for all $w \in H$

$$d_H(1, w) \leq A d_{\widetilde{\mathcal{M}}(S)}(\mu, w\mu) + B.$$

Let H be as above and let N be as in Theorem 5. Let $w = (f_{\ell_k}^n)^{e_k} \cdots (f_{\ell_1}^n)^{e_1}$, $n \geq N$

and set $g_j = (f^n_{\ell_j})^{e_i}$. Then

$$\begin{aligned} d_H(1,w) &= \sum_{i=1}^{k} e_i \\ &\leq \sum_{i=1}^{k} K_0 e_i \\ &\leq \sum_{i=1}^{k} d_{g_1 \cdots g_{i-1}\alpha_{\ell_i}}(\mu, w\mu). \end{aligned}$$

Grouping together axes which are in the same subsurface, we can apply the combination distance formula of Theorem 6 to get

$$\begin{aligned} \sum_{i=1}^{k} d_{g_1 \cdots g_{i-1}\alpha_{\ell_i}}(\mu, w\mu) &\leq \sum_{i=1}^{k} A'(d_{g_1 \cdots g_{i-1} S_{\ell_i}}(\mu, w\mu)) + B' \\ &\leq A' \sum_{S' \subseteq S} \{\{d_{S'}(\mu, w\mu)\}\}_K \\ &\leq A'' d_{\widetilde{\mathcal{M}(S)}}(\mu, w\mu) + B'', \end{aligned}$$

where $K \geq K_0$, and the last inequality follows from the Masur-Minsky distance formula (adjusting the coarse constants as necessary). □

5.3 Classification

Finally, we show that each $w \in H$ is pseudo-Anosov on its support. This will follow from showing that for any w, we can find an essential simple closed curve whose orbit goes off to infinity in $\mathcal{C}(S)$. We begin by stating a lemma of Bestvina-Bromberg-Fujiwara [5].

Lemma 9 ([5], Lemma 4.20)**.** *Let $\{\beta_i\}_{i=0}^{k}$ be a sequence of essential simple closed* *curves in $\mathcal{C}(S)$ such that each consecutive triple of curves satisfies*

$$d_{S_i}(\beta_{i-1}, \beta_{i+1}) \geq 3K_{BGIT},$$

where S_i is an essential subsurface with $\beta_i \in \partial S_i$. Then

$$d_{\mathcal{C}(S)}(\beta_0, \beta_k) = \sum_{i=1}^{k} d_{\mathcal{C}(S)}(\beta_{i-1}, \beta_i) - 2k$$

We will construct such a sequence so that consecutive curves are distance at least 3 apart, which by the above lemma must go off to infinity.

Proof of Theorem 3. Let H be a subgroup as constructed in the previous section. Without loss of generality, we assume that the support of $w \in H$ is all of S (the same argument holds restricting to the curve graph of the support in the case that the support is a proper subsurface). Write $w = u_1 g_1 \cdots u_k g_k$ in central form, where each g_i is a power of some generator f_j^n, $n \geq N$, of H.

If each g_i is a pseudo-Anosov mapping class, then by Theorem 9, there is a generator such that the appropriate translate of its axis "witnesses" a large distance between any essential simple closed curve β and its image $w\beta$, *i.e.* for some j,

$$d_{u_1 g_1 \cdots u_{j-1} g_{j-1} \alpha_j}(\beta, w\beta) \geq K_0 + K_{BGIT} + 48\delta.$$

In this case, it follows from Theorem 4 that w takes β "off to infinity", *i.e.* w is pseudo-Anosov.

Now assume that at least one g_i is reducible with support S'; up to conjugation, we may assume that g_1 is a power of this reducible. We first claim that $\beta \in \partial S'$ and $w\beta$ fill S, *i.e.* have distance at least 3 in $\mathcal{C}(S)$. As is noted in ([15], Lemma 6.2), the subsurfaces supporting the g_i fill S if and only if the subsurfaces $u_1 g_1 \cdots u_{j-1} g_{j-1} S_j$, where $1 \leq j \leq k$ and S_j is the support of g_j, also fill S. This implies that β and $w\beta$ fill S. Indeed, suppose γ is another essential simple closed curve. As the subsurfaces

$u_1 g_1 \cdots u_{j-1} g_{j-1} S_j$ fill S, γ has non-trivial projection to at least one of them. But in this subsurface, β and $w\beta$ have large projection, so γ cannot be disjoint from both simultaneously. Hence, β and $w\beta$ fill S, *i.e.* $d_{\mathcal{C}(S)}(\beta, w\beta) \geq 3$, and the same is true of $w^\ell \beta$ and $w^{\ell+1}\beta$ for all $\ell \in \mathbb{Z}$. It remains to show that the sequence $\{w^\ell \beta\}$ satisfies

$$d_{w^\ell S'}(w^{\ell-1}\beta, w^{\ell+1}\beta) \geq 3K_{BGIT}$$

which by equivariance of projections is equivalent to

$$d_{S'}(w^{-1}\beta, w\beta) \geq 3K_{BGIT}$$

Using the given expression for w and the triangle inequality, we have

$$\begin{aligned} d_{S'}(u_1 g_1 \cdots u_k g_k \beta, u_k^{-1} g_k^{-1} \cdots u_1^{-1} g_1^{-1} \beta) \geq d_{S'}(u_1 g_1 \cdots u_k g_k \beta, u_2 g_2 \cdots u_k g_k \beta) \\ - d_{S'}(u_2 g_2 \cdots u_k g_k \beta, u_k^{-1} g_k^{-1} \cdots u_2^{-1} g_2^{-1} \beta). \end{aligned}$$

The subtracted term on the right-hand side satisfies

$$\begin{aligned} d_{S'}(u_2 g_2 \cdots u_k g_k \beta, u_k^{-1} g_k^{-1} \cdots u_2^{-1} g_2^{-1} \beta) \leq d_{S'}(u_2 g_2 \cdots u_k g_k \beta, \alpha_j) + d_{S'}(\alpha_j, \alpha_i) \\ + d_{S'}(\alpha_i, u_k^{-1} g_k^{-1} \cdots u_2^{-1} g_2^{-1} \beta), \end{aligned}$$

where $g_2 = (f_j^n)^{e_2}$ and $g_k = (f_i^n)^{e_k}$. Setting

$$R = d_{S'}(u_1 g_1 \cdots u_k g_k \beta, u_2 g_2 \cdots u_k g_k \beta),$$

what we are trying to show reduces to

$$R \geq d_{S'}(u_2 g_2 \cdots u_k g_k \beta, \alpha_j) + M_1 + d_{S'}(\alpha_i, u_k^{-1} g_k^{-1} \cdots u_2^{-1} g_2^{-1} \beta) + 3K_{BGIT}$$

By the construction of N, R is at least $\{\text{numerator of N}\} - 4$. Moreover, the first and

third terms on the right-hand side are both bounded above by
$K_{BGIT}+48\delta$ (the bound from the Multi-scale Behrstock Inequality) - indeed, we have by Theorem 9 that

$$d_{\alpha_j}(u_2g_2\cdots u_kg_k\beta,\beta) \geq K_{BGIT}+48\delta$$

and so

$$d_{S'}(u_2g_2\cdots u_kg_k\beta,\alpha_j) < K_{BGIT}+48\delta$$

by the Multi-scale Behrstock Inequality; the same argument holds for the other term. Thus the inequality we are trying to show is

$$\{\text{numerator of N}\} - 4 \geq 5K_{BGIT}+96\delta+M_1$$

which is true by construction. □

5.4 Application

The mapping class group plays an important role in the study of surface group extensions, *i.e.* short exact sequences of groups of the form

$$1 \to \pi_1(S) \to H \to G \to 1,$$

where $\pi_1(S)$ denotes the fundamental group of a closed surface. Indeed, by general principles such a short exact sequence gives rise to a homomorphism $G \to Out(\pi_1(S))$, the outer automorphism group of $\pi_1(S)$, and a classical theorem of Dehn-Nielsen-Baer (*cf* [19]) says that $Out(\pi_1(S))$ contains $MCG(S)$ with finite index. A natural question to ask is: under what conditions on G can we guarantee that the extension H is hyperbolic? The study of this question was initiated by Farb and Mosher in [20], where they introduced the notion of *convex cocompactness* for subgroups of mapping class groups in analogy with the property of the same name for Kleinian groups. Their original definition concerned the action of G on the Teichmüller space of S, but Kent-Leininger [32] and independently Hamenstädt [25] proved the following is equivalent: a subgroup $G < MCG(S)$ is called convex cocompact if the orbit map $(g \to g \cdot x)$ into the curve graph $\mathcal{C}(S)$ is a quasi-isometric embedding. It follows from general considerations that G is then hyperbolic and *purely pseudo-Anosov*, *i.e.* every infinite-order element is a pseudo-Anosov mapping class. Combining work of Farb-Mosher and Hamenstädt, this condition is equivalent to hyperbolicity of the extension H. It was later shown by Bestvina-Bromberg-Kent-Leininger [6] that convex cocompact subgroups are precisely those which are undistorted and purely pseudo-Anosov. Using this last characterization, we have an immediate corollary to Theorems 2 and 3.

Corollary 4. *If every mapping class in the generating set for H is pseudo-Anosov, then H is an undistorted, purely pseudo-Anosov free subgroup of $MCG(S)$, i.e. H is a convex cocompact free group.*

Such groups have been dubbed "Schottky subgroups" by Farb-Mosher in analogy with the classical Schottky construction in hyperbolic space. Indeed, that *some* power of a collection of pseudo-Anosov mapping classes generate a convex cocompact subgroup was known to Farb-Mosher, though their argument is not effective.

The Bestvina-Bromberg-Kent-Leininger characterization of convex cocompactness begs a more general question: what can we say about undistorted subgroups of other geometrically interesting groups in which every element is loxodromic with respect to some action on a nice space? This was answered by Koberda-Mangahas-Taylor [38] in the context of a RAAG acting on its *extension graph*, an analog of the curve graph in that setting. In particular, they showed that *purely loxodromic* subgroups of a RAAG are precisely those which are undistorted and whose orbit into the extension graph is a quasi-isometric embedding, and thus are the analogs of convex cocompact subgroups of mapping class groups in the RAAG setting. Because the embeddings we've constructed are quasi-isometric embeddings, if we can guarantee that loxodromic elements of the RAAGs are sent to pseudo-Anosov mapping classes, we'll have an (even greater) abundance of convex cocompact subgroups of mapping class groups. For a general RAAG, it is unknown if such embeddings exist, but in forthcoming work we construct such embeddings for a few infinite families of graphs. In the statement below, a graph is of type "anti-P" for a property P if the opposite graph Γ^{op} satisfies P.

Theorem 11. *If Γ is a tree of diameter at least 3, an anti-tree, or an anti-cycle, then there exists a surface S and a quasi-isometric embedding $A(\Gamma) \hookrightarrow MCG(S)$ as above with the property that for each $\lambda \in A(\Gamma)$ loxodromic, its image in $MCG(S)$ is pseudo-Anosov. In particular, every purely loxodromic subgroup of $A(\Gamma)$ is sent to a convex cocompact subgroup of $MCG(S)$.*

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