Supporting Students' Mathematical Modeling and Informed Engineering Design in Geometry Classrooms: A Design-Based Approach

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Dissertation Defense

Paper One

Abstract

While mathematical modeling and engineering design are both seen as a means to promote deeper, more integrated, and more applied student understanding, there has been little focus on leveraging their synergistic use in K-12 classrooms. This study explores how to support mathematical modeling and engineering design coherently in mathematics classrooms. We present two cycles of a design-based research study that used a web-based learning environment to support students in an engineering design project. In the first cycle, students (n = 44) used mathematical models in their designs, and in the second cycle students (n = 66) created mathematical models from testing prototype designs. We examined the impact of the projects on student understanding of geometry, mathematical modeling, and engineering design in K-12 classrooms. We draw on pretest and posttest data, embedded assessments, learning environment log-data, student explanations and artifacts, as well as classroom observations to support our results. Results indicate significant gains for connected mathematical understanding and that technology supports for mathematical modeling and engineering design can have a positive impact on the student understanding of these practices.

Keywords: mathematical modeling, engineering design, geometry, K-12, technology

Introduction

Nationally, there have been concurrent calls to increase the use of mathematical modeling (e.g., NGA, 2010) and to increase the use of engineering design (e.g., NRC, 2012) in K-12 classrooms. Both calls cite similar goals—to promote deeper, more integrated, and more applied student understanding—but because these calls originate in different disciplinary areas (mathematics and science, respectively) there has been little focus on their synergy (NRC, 2009), and there has been little focus on the possible benefits of concretely connecting these approaches in K-12 classrooms.

Mathematical modeling is "a process that uses mathematics to represent, analyze, make predictions or otherwise provide insight into real-world phenomena" (Bliss & Libertini, 2016, p.8). Mathematical modeling is an iterative process that involves "using mathematics or statistics to describe (i.e., model) a real world situation and deduce additional information about the situation by mathematical or statistical computation and analysis" (Common Core Standards Writing Team, 2013, p. 5). These definitions make clear that while modeling is firmly grounded in a real-world context, its role is to better understand or make predictions by creating abstract mathematical representations of a real-world context (Cirillo, Pelesko, Felton-Koestler, & Rubel, 2016). Engineering design is also an iterative process that is grounded in a real-world context; however, engineering design typically takes the insights developed from mathematical modeling or elsewhere to build prototype designs, test them, and refine them not in an abstract mathematical sense, but within the real-world context itself (e.g., Burghardt & Hacker, 2004). Mathematical modeling might predict how a rocket's dimensions impact its flight path, and engineering design uses these predictions to build and test the actual rocket. While mathematical modeling and engineering design are distinct, they have much in common and complement each other (see Figure 1).

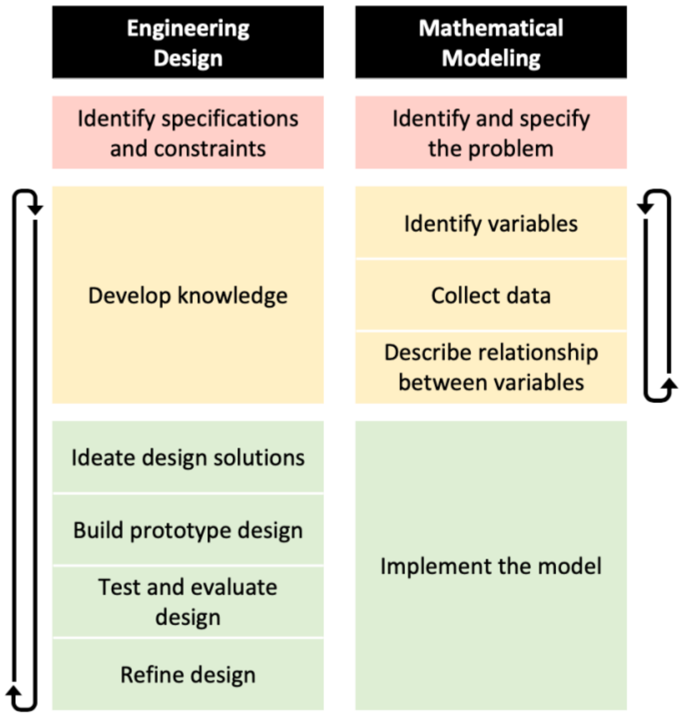

Figure 1. The similarities and synergy between engineering design and mathematical modeling. Similar colors indicate where engineering design and mathematical modeling stages align, with multiple mathematical modeling stages aligning with engineering design and vice versa. Circular arrows indicate where iteration most commonly occurs within engineering design and mathematical modeling.

This synergistic relationship between engineering design and mathematical modeling has been highlighted in studies of the practice of both professional engineers (Gainsburg, 2006) and college student engineers (Cardella, 2010). These studies noted that in practice, the iterative refining and testing of mathematical models is used to inform the iterative refining and testing of prototype designs, which can in turn inform future refinements of the mathematical models. Given that this relationship exists, several studies have investigated how to include mathematical modeling into engineering curricula at the undergraduate level (Diefes-Dux et al., 2004; Hamilton, Lesh, Lester, & Brilleslyper, 2008; Wedelin & Adawi, 2014; Carberry & Mckenna, 2014).

However, in pre-college settings, the integration of engineering design and mathematical modeling is rare (Becker & Park, 2011). A few studies have integrated mathematics and engineering, some by using mathematics content knowledge to inform design projects (e.g., Burghardt, Hecht, Russo, Lauckhardt, & Hacker, 2010; Narode, 2011), and others by using design projects to contextualize or motivate mathematics learning (e.g., Jacobson & Lehrer, 2000; Schroeder, Lee, & Mohr-Schroeder, 2015; Chou, Chen, Wu, & Carey, 2017; Kertil & Gurel, 2016). However, only a few projects incorporate the interconnected relationship between engineering design and mathematical modeling, likely due to the difficulties implementing such projects (Roehrig, Moore, Wang, & Park, 2011).

This study aims to explore how to support the synergistic use of both engineering design and mathematical modeling in pre-college geometry classrooms. Specifically, we will seek to answer these questions:

- How might interweaving mathematical modeling and informed engineering design help students develop a more connected understanding of geometry?
- What kinds of supports help students to engage in mathematical modeling in the context of engineering design?

This paper uses a design-based research approach to investigate these questions. Design-based research, or design research (Sandoval, 2014) seeks to study learning environments in the settings for which they are designed (e.g., real classrooms) and ideally contribute to fundamental research on teaching and learning (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003; Design-Based Research Collective, 2003). By testing theoretically-driven hypotheses about learning environments in authentic classroom settings (as opposed to a more clinical or contrived setting), design research seeks to articulate what works under what circumstances. In this paper we present two cycles of a design-based research study. The first cycle examines students *using* mathematical models within an engineering design project. The second cycle examines students *creating* the mathematical models within an engineering design project using data they collect during the project. We structure this paper by describing the first cycle of the project, its results, and how we used these results to inform our revisions of the project for the second cycle. We then describe how the second cycle of the project was implemented, its results, and discuss the overall implications and limitations this study.

Literature Review

Informed Engineering Design

Informed Engineering Design (IED) emphasizes engineering design practices that can support learning in K-12 classrooms (Burghardt & Hacker, 2004). IED builds upon the many models of engineering design (e.g., Crismond & Adams, 2012) with a specific focus on supporting understanding and application of relevant mathematics and science principles through alignment with design criteria. For example, a design project that asks students to build a bridge with specific materials to hold a certain amount of weight is unlikely to encourage students to consider how to mathematically model a bridge if they are able to successfully meet project criteria through trial and error. In contrast, design criteria that call for students to explain where and why their design will break can encourage students to investigate and apply their understanding of forces and material science by creating mathematical models of the prototypes.

The IED approach is based the experience of having implemented engineering design projects in precollege classroom with hundreds of students (Akins & Burghardt, 2006; Burghardt et al., 2010; Burghardt & Krowles, 2006). IED involves giving students opportunities to understand the real-world problem or design challenge, to develop relevant knowledge and new ideas that relate to the problem, to generate designs that incorporate both their new and prior ideas, to conduct tests and experiments of their designs, and to evaluate their designs and the ideas that underpinned their designs (see Figure 2).

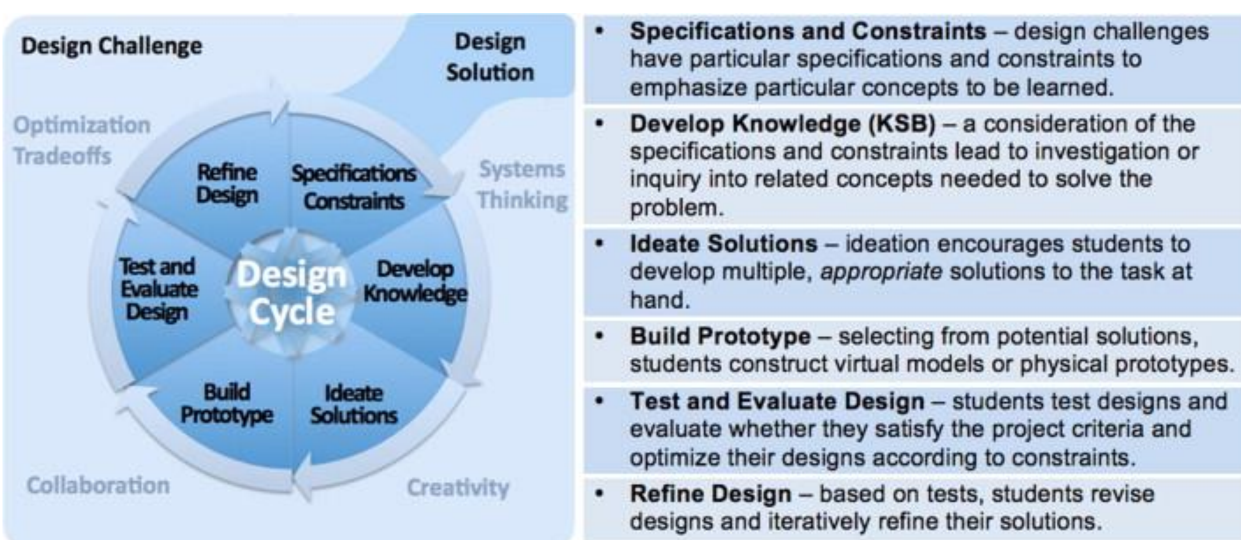

Figure 2. Stages of Informed Engineering Design (Chiu et al., 2013).

Mathematical Modeling

Mathematical modeling is a major area of focus for mathematics educators and researchers (e.g., NGA, 2010; Hirsch & Roth McDuffie, 2016). While there are a variety of definitions for mathematical modeling, there is a broad consensus, summarized by Bliss and Libertini (2016, p.8; see Figure 3), that it is "a process that uses mathematics to represent, analyze, make predictions or otherwise provide insight into real-world phenomena." This definition of mathematical modeling emphasizes that the process begins in a real-world context and involves *mathematizing* or *formulating* the context by identifying variables, making assumptions, and choosing what is important (e.g., Bliss, Fowler, & Galluzo, 2014). In addition, mathematical modeling involves *interpreting* or *testing* the mathematical model that has been created. Typically, the model is judged by its validity, or how well it is able to "construct, describe, explain, manipulate, predict or control systems that occur in the world" (Lesh, Doerr, Carmona, & Hjalmarson, 2003, p. 225). Furthermore, the processes of mathematical modeling—of mathematizing the real-world context into a mathematical model and using predictions from the model in real-world decisions—are cyclical and iterative, where the limited validity of a model motivates further cycles and further revisions to the model (Cirillo, Pelesko, Felton-Koestler, & Rubel, 2016).

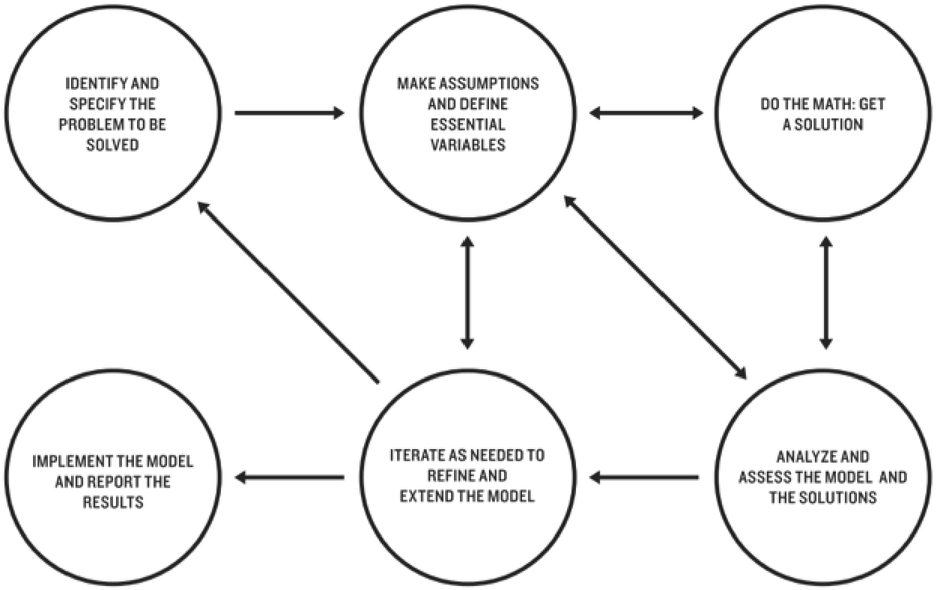

Figure 3. Mathematical Modeling Processes as outlined by Bliss and Libertini (2016).

This definition of mathematical modeling emphasizes that it is a process and that students are engaged in modeling practices. As such this definition aligns with that of other authors who have distinguished mathematical modeling from models or manipulatives to help communicate mathematical ideas (Cirillo, Pelesko, Felton-Koestler, & Rubel, 2016). For example, the "area model" can be used to help students find the product 13 x 27 by considering (10 + 3)(20 + 7). But this same model can be used to help students correctly expand (x + 3)(2x + 7). The definition of mathematical modeling using in this paper is also in contrast to mathematics application questions that take a mathematical rule and ask students to use it in a context. This application approach is common in many "word-problems" or "real-world problems" found in traditional mathematics textbooks (Meyer, 2015). Mathematical modeling is also defined to be broader than mathematical problem solving, with the later tending to pose mathematical problems that require students to recognize patterns and find answers within the realm of mathematics (Pollak, 2011).

For example, when students are asked to derive the area of a triangle formula they often start and end with mathematical phenomena. The application of a mathematical rule and the recognition of patterns are both important mathematical enterprises and can form elements of the mathematical modeling process, but by themselves are not sufficient to define mathematical modeling as used in this paper.

Model-eliciting activities (MEAs) can be used to support the practice of Mathematical Modeling in the classroom. In coining the term, Lesh and colleagues (2000) argue that for students, an MEA should "reveal explicitly the development of constructs (conceptual models) that are significant from a mathematical point of view and powerful from a practical point of view" (Lesh et al, 2000, p.608). To do this they argue that an MEA should involve students creating a model that symbolically describes a meaningful situation, and also allow students to judge the value of the model themselves (Lesh et. al., 2000). Lesh and colleagues distinguish MEAs from other types of mathematical problems where "the problem solver's goal is merely to produce a brief answer to a question that was formulated by others (within a situation that was described by others, getting from givens to goals that are specified by others, and using strings of facts and rules that are restricted artificially by others)" (p.594). Lesh and colleagues also advocate MEAs because the process of mathematical modeling is able to reveal students' "ways of thinking" (p. 594) and because they "provide the conceptual foundations for deeper and higher order understandings" (p.592).

Challenges to Implementation

Although research suggests that implementing engineering design projects into mathematics classes can help students develop a more connected understanding of mathematical concepts, this also creates challenges for educators. First, mathematics teachers who implement these projects may not have engineering expertise and need support implementing informed engineering design projects in their classrooms (e.g., Purzer, Moore, Baker, & Berland, 2014), especially when these projects use new technologies and materials (e.g., Wang, Moore, Roehrig & Park, 2011). Many K-12 students and teachers have little exposure to engineering (Katehi, Pearson, & Feder, 2009) and as a result, classroom design projects may focus heavily on building products to the detriment of other design practices. In addition, engineering design projects differ from typical mathematical problems in that there is no one right answer; every student group can have a distinct, successful solution. Thus, teachers need support to provide differentiated guidance, feedback, and troubleshooting to help students develop understanding through design. Many of these challenges have also been highlighted when teachers implement mathematical modeling projects into mathematics classes (e.g., Anhalt, Cortez, & Bennett, 2018).

To address these challenges, computer-based learning environments have been used in K-12 settings. Such learning environments can help students engage in engineering design and mathematical modeling practices (e.g., White & Frederiksen, 2005) and can encourage student collaboration and knowledge building (Slotta & Linn, 2009). They can also provide "distributed scaffolding" that can support students throughout the designing and modeling process (Puntambekar & Kolodner, 2005) and they can incorporate simulations and visualizations to help students develop connections between ideas.

Supporting Engineering Design and Mathematical Modeling with WISE

The Web-based Inquiry Science Environment (WISE; Slotta & Linn, 2009), is a free, web-based, open-source learning environment that provides explicit supports for inquiry learning and engineering design projects (Chiu et al., 2013). WISE includes a diversity of tools such as drawing and simulation technologies, advanced assessment tools, collaboration tools, and reflection supports. WISE provides teacher functionality such as monitoring of student progress, automated scoring of student work, and the ability to build and customize projects. WISE also provides logging of students' interactions within the environment as part of researcher tools.

In the past, WISE has been used to support engineering design projects by using its functionality to *make practices accessible*, *make thinking visible*, and *help students learn from others,* as well as *promote reflection,* and *support multiple representations of solutions* (Chiu, Gonczi, Fu, & Burghardt, 2017). This functionality is likely to also support the implementation of mathematical modeling activities, however, this has yet to be explored.

Making practices accessible. WISE supports students and teachers with little engineering or modeling experience to engage in these practices. For example, by making informed engineering design processes (see Figure 4) explicit, and by connecting each step of the project with a corresponding design stage we are able to orientate students are during the design process (Cordray, Harris, & Klein, 2009; Cunningham, 2009; Martin, Rivale, & Diller, 2007). In addition, some steps in the project will ask students to reflect on their current stage in the design cycle. For example, one activity in the *Specifications and Constraints* design stage may ask students to articulate what the specifications and constraints are (Figure 4). Although steps are presented in a sequential manner, students are able to skip ahead or revisit steps as needed to during the project.

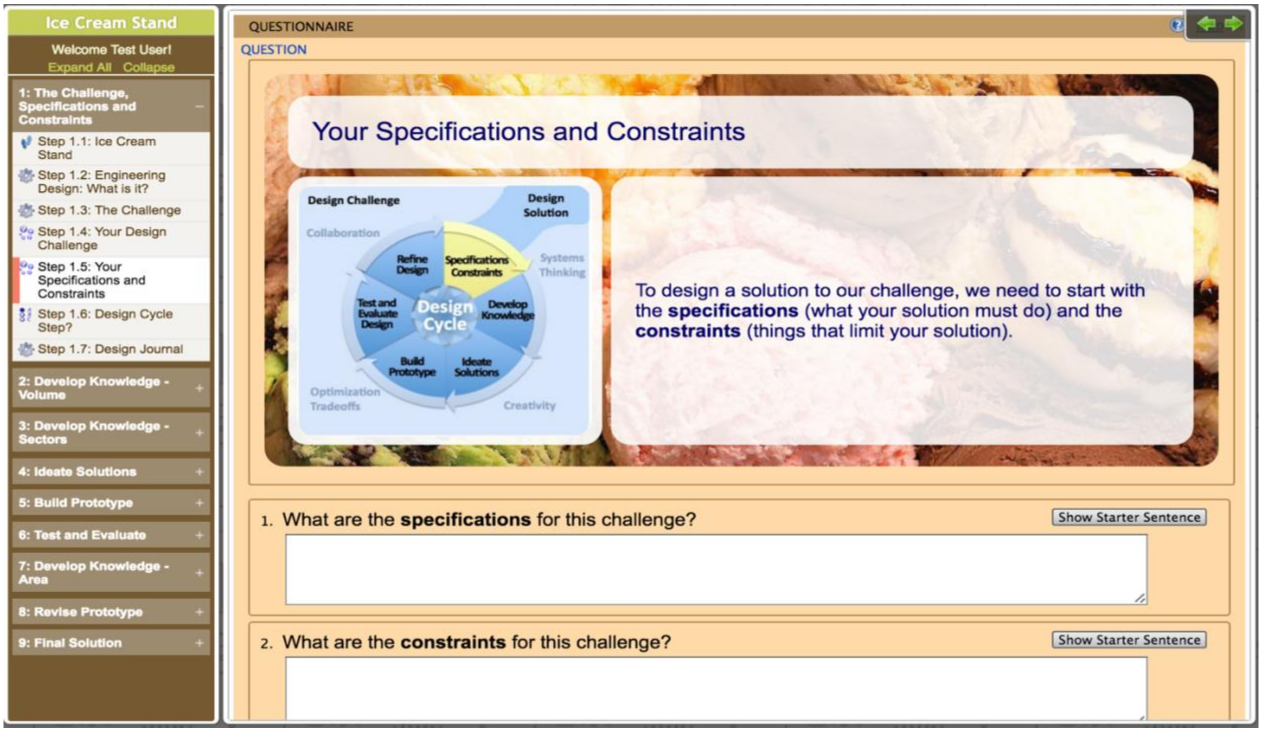

Figure 4. Students articulate specifications and constraints during a project challenge activity in WISE.

Making thinking visible. The Idea Manager (see Figure 5; Matuk et al., 2012) helps make students' thinking visible across multiple contexts by using two connected tools, the Idea Basket and the Explanation Builder. The Idea Basket allows students to document and "add ideas" over the course of a WISE project and makes those ideas visible in a persistent repository. Students add ideas to their basket by typing in a short description and specifying other attributes through flags or tags. The WISE Explanation Builder (Figure 5) provides a scaffold for students constructing evidence-based explanations and arguments by leveraging the ideas gathered in the basket. Students can drag and drop their ideas from their basket into an organizing space, supporting the distinguishing and sorting of ideas. Students then use their organized ideas to construct an explanation or argument. The tool helps students bring together multiple sources of evidence to support a coherent explanation or argument. The Idea Manager has been classroom tested in science inquiry units (e.g., McElhaney et al., 2012; Tate, Feng, & McElhaney, 2016) but not in engineering design or mathematical modeling contexts.

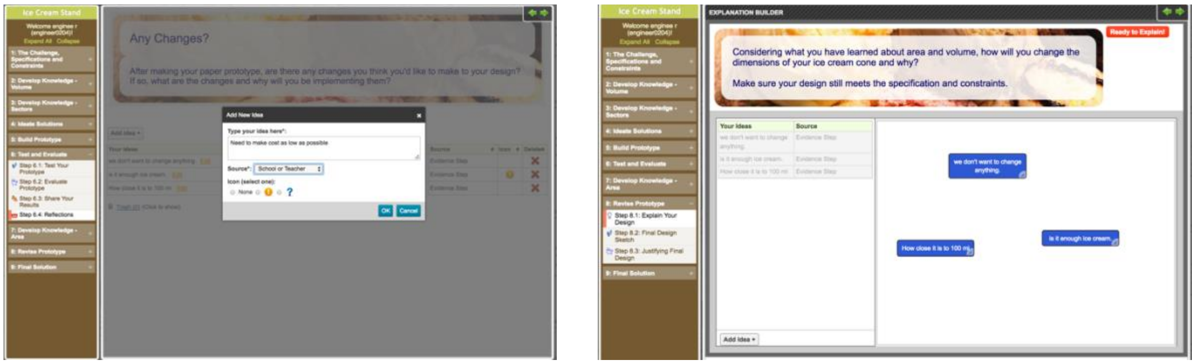

Figure 5. Students can add ideas to their Idea Basket (left) and then sort ideas to build a design explanation with the Explanation Builder (right).

Helping students learn from others. To help students create and share representations of ideas, students use a Design Wall (see Figure 5) where they can post images or text and get feedback from their peers. The Design Wall is similar to blog or social networking websites in functionality. Students can post images for inspiration in the ideation phase, share pictures of initial or revised prototypes and models and get feedback through commenting functionality from other students in or outside of their class.

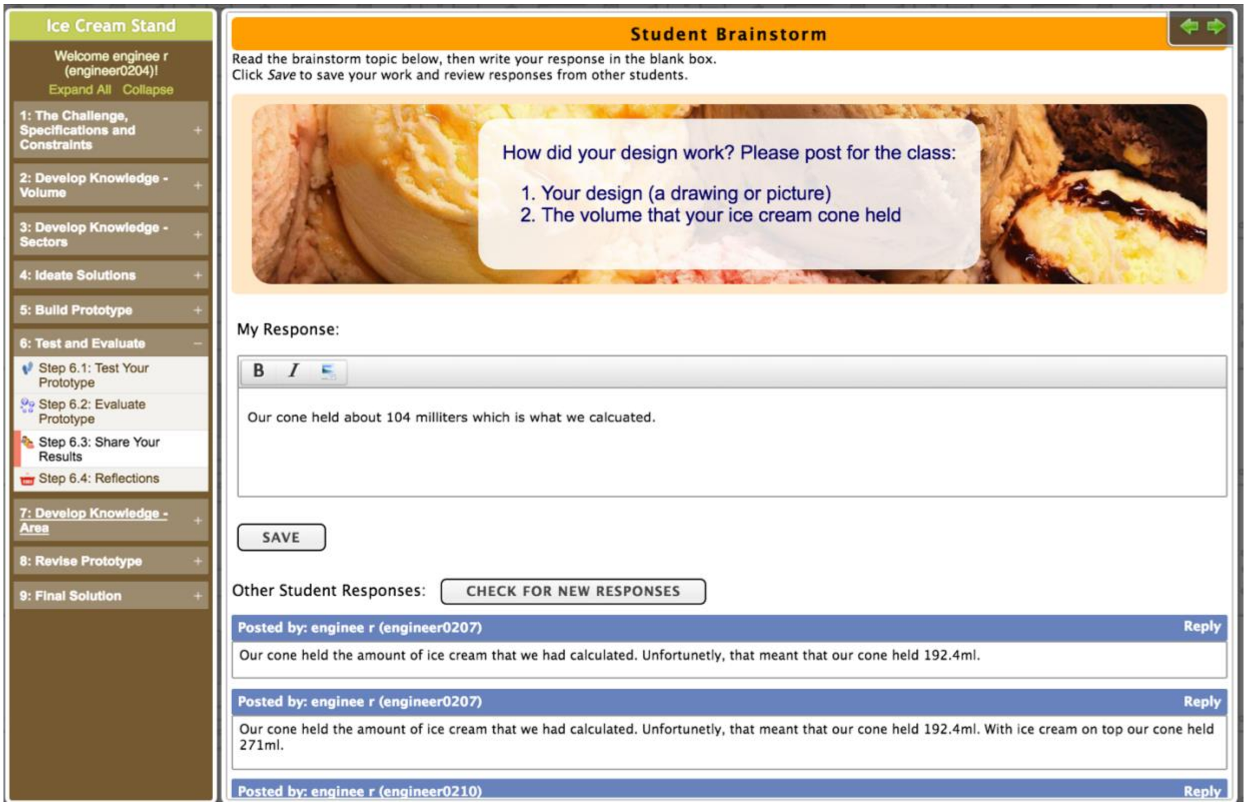

Figure 5. Students can post pictures and comments to a "wall" to share designs.

Promote reflection. WISE uses a *Notebook* to help students record and reflect upon design processes. All student work within WISE is recorded in the *Notebook*, including drawings, open responses, pictures, and critiques of others' work. From the *Notebook*, students can select and annotate specific artifacts to include in their *Portfolio*, which is used to share with teachers or their peers. Both the *Notebook* and *Portfolio* support practices of communication as well as promoting reflection.

Support multiple representations of solutions. Building upon research demonstrating the benefit of providing multiple representations for learning (e.g., Ainsworth, 2006), WISE projects support both virtual and physical representations of solutions. As WISE is computer-based, it can scaffold the use of rich simulations, visualizations, or computer-aided design technologies to represent solutions. Studies demonstrate that manipulating 3-d visualizations of objects can positively benefit student understanding of geometry concepts (Sung, Shih, & Chang, 2015; Schroeder, Lee, & Mohr-Schroeder, 2015; Chou, Chen, Wu, & Carey, 2017). WISE also enables data tables and graphing tools to be embedded into the project to encourage modeling practices. However, because physical representations also provide unique affordances such as dealing with tolerances or error and offer comparisons for virtual solutions (e.g., Blikstein et al., 2012), we aimed for the design project in this study to leverage both virtual and physical representations.

This study used WISE technologies to support students' synergistic incorporation of mathematical modeling into informed engineering design projects in geometry classrooms. The next section presents the first iteration of this project and discusses how the results informed the subsequent project revision. The following section describes the second cycle of the project, its results, and a discussion of the overall implications and limitations for this study.

Project Cycle 1: Using Models

Participants and Context

Participants (*n* = 44) were students from two classes of 8th grade geometry taught by the same teacher. School demographics consisted of 19.1% Black, 20.2% Hispanic, and 46.8% White students with 44.9% students receiving free or reduced lunch and 21.6% of students classified with Limited English Proficiency. The teacher had over 5 years of experience and an advanced degree in mathematics.

The Ice Cream Cone Project

The project was co-developed with the participating teacher. After an initial meeting with researchers, the teacher provided concepts that she felt needed connection to each other and to authentic contexts. Together the team devised the ice cream cone project to address circle geometry concepts. The project challenged students to create a waffle cone to hold a given volume of ice cream using the least area of waffle (see Figure 6).

Figure 6. The waffle maker (left) and a student made waffle cone (right).

Connecting mathematical ideas. The mathematical ideas that this project aimed to help students to connect were the dimensions, area, and volume of circles, sectors, and cones. Table 1 shows the Common Core Standards for Mathematics Content addressed by the project. The project addresses standards for both 8th grade geometry and high school geometry.

Table 1

Common Core State Standards for Mathematics Content targeted in the Ice Cream Cone Project

CCSS Standard	Description	Project Context
CCSS.MATH. CONTENT.HSG. GMD.A.3	Use volume formulas for cylinders, pyramids, cones, and spheres to solve problems.*	Calculate maximum volume of ice cream held by various prototypes of cones.

and CCSS.MATH. CONTENT.8.G.C.9	Know the formulas for the volumes of cones, cylinders, and spheres and use them to solve real-world and mathematical problems.	
CCSS.MATH. CONTENT.8.G.B.7	Apply the Pythagorean Theorem to determine unknown side lengths in right triangles in real-world and mathematical problems in two and three dimensions.	Translate a two-dimensional sector into a three-dimensional cone by applying the Pythagorean theorem.
CCSS.MATH. CONTENT.HSG.C.B.5	Find arc lengths and areas of sectors of circles	Calculate arc length of sector and connect to circumference of cone, calculate area for cost.
CCSS.MATH. CONTENT.HSG.GMD .B.4	Visualize relationships between two-dimensional and three-dimensional objects	Connect two-dimensional waffle sector with a three-dimensional ice cream cone.
CCSS.MATH. CONTENT.HSG.MG. A.3	Apply geometric methods to solve design problems (e.g., designing an object or structure to satisfy physical constraints or minimize cost).*	Design an ice cream cone to maximize volume and minimize cost within constraints of waffle maker.

*Note: Asterisks on CCSS Content standards represent places of intersections with modeling practices.

This project aimed to help students connect the properties of sectors and cones. Typically, students learn about cones alongside other three-dimensional shapes such as prisms and pyramids. Students often study sectors within two-dimensional circle topics. However, because the net of a cone (excluding the circular base) is a sector (see Figure 7) the area of a sector can be connected with the curved surface area of the cone, the radius of the sector can be connected to the slant height of the cone, and the arc length of the sector can be connected to the circumference of the base of the cone. In addition, since the volume of a cone is usually expressed in terms of the vertical height of the cone rather than the slant height of the cone, we use the Pythagorean theorem to help students distinguish between slant and vertical height, and to understand how the slant height, vertical height and cone base radius relate to each other. Figure 8 shows how we intended students to connect the sector properties of arc length and radius with cone properties of base radius and slant height.

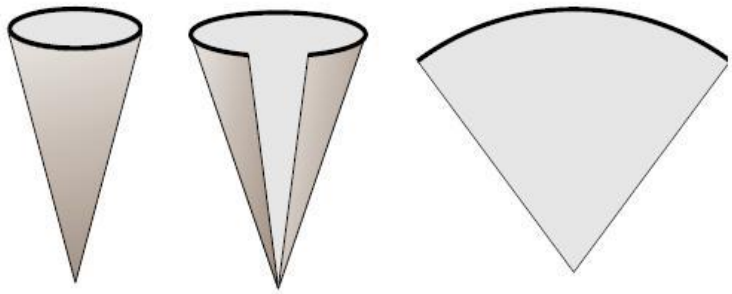

Figure 7. The net of a cone (excluding the circular base) is a sector.

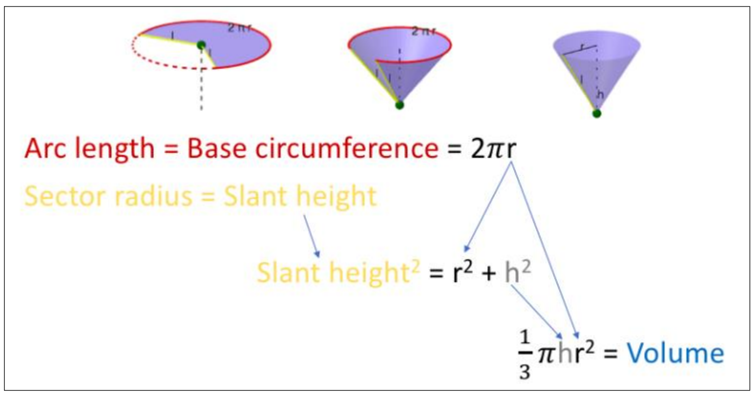

Figure 8. The connections between sectors and cones that this project intended to give opportunities for students to make.

Supporting engineering design and mathematical modeling. The project supported students to engage in engineering design by creating activities that aligned with the informed engineering design stages (see Figure 2). A detailed list of the steps in the project and how they align to each of these stages can be found in Appendix A. Some of the steps that were aligned with the "develop knowledge" engineering design stage also involved students engaging with mathematical modeling. For example, in order for students to learn about how the properties of cones and sectors were connected, the students were able to manipulate Geogebra visualizations (e.g., Hohenwarter & Hohenwarter, 2009) that we created for this project to investigate how the variables such as slant height, vertical height and cone base radius relate to each other (Figure 9).

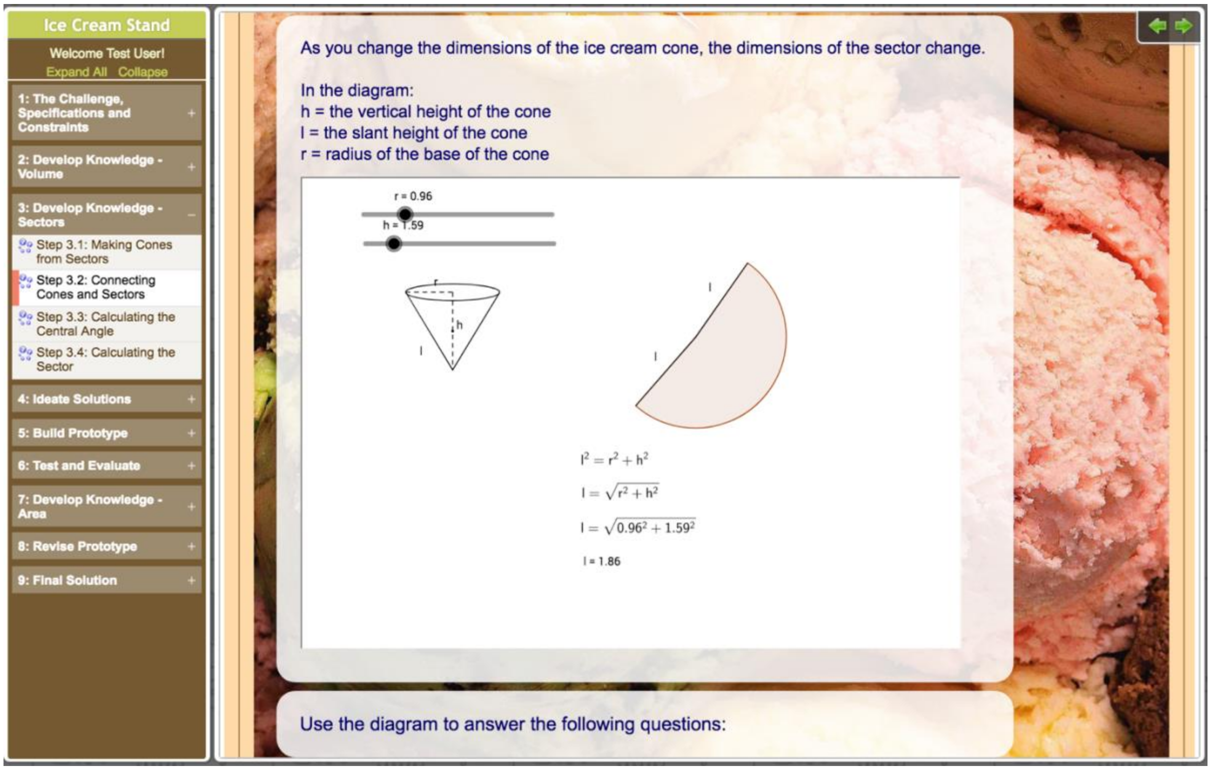

Figure 9. Students work with interactive GeoGebra visualization within WISE.

Activities in the project that were connected with the "ideate solutions" engineering design stage encouraged students to create multiple solutions informed by mathematical models. Students were encouraged to draw plans for their cones and use the mathematical models to calculate the necessary dimensions of the sector for their design. Students were prompted to justify their designs by asking how their designs meet the requirements of the challenge, and to reflect on potential problems with their design. The WISE project steps guided reflection by asking students to add the ideas that they considered important while making their initial designs into their Idea Baskets.

Students used their plans to build physical prototypes of their ice cream cone designs with paper (see Figure 10). To test their designs, students filled their paper cones with dried rice and measured the volume of rice with a graduated cylinder. They could then compare their results with their calculations to see how well their prototype designs performed. Students posted pictures or drawings of their prototypes, annotated with the volume they held, on the Design Wall to share with the class. After looking at other designs both in person and within WISE, students reflected on their designs using the Idea Basket. Students were prompted to explain and justify future changes that they would like to make to their designs. After making necessary revisions to their designs, the project guided students to another iteration of "develop knowledge" that focused on how to calculate the area of a sector. Students were again prompted to reflect and add any ideas about their designs into their Idea Basket. Students were able to make multiple iterations on their paper prototypes if needed. Students then built their final prototype out of sheet metal and used it to shape an edible flat waffle into an ice cream cone shape, which they filled with ice cream. The teacher created a chart on the whiteboard in front of the class for students to put in their final values for the dimensions, cost, and volume of ice cream for each group's design. Consequently, throughout the engineering design stages of ideating, building, testing and refining stages, there was a strong interconnectedness with the mathematical models of the properties of sectors and cones.

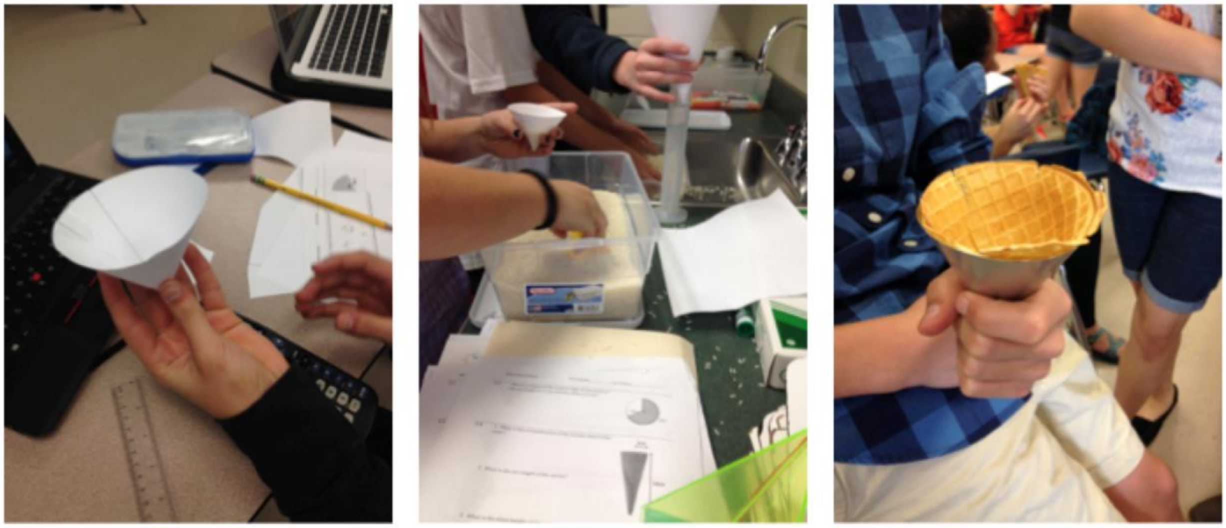

Figure 10. In the Ice Cream project, students created paper prototypes of cones, tested calculations by filling the cone with rice and measuring volume with graduated cylinders, and then created a metal prototype used to make an actual waffle cone.

Data Sources

Pretest and posttest. The pretest and posttest assessments were identical and each consisted of six open-response questions (see Appendix B). The first three items were adapted from a set of released state mandated assessment questions and required students to use discrete pieces of mathematical knowledge. The first test item assessed knowledge of arc length of a sector, the second assessed knowledge of area of a sector and the third assessed volume of a cone from direct measurements. Students could answer these questions correctly by identifying and using an appropriate formula. For clarity, we will refer to these items that assess discrete mathematics knowledge as DM1, DM2, and DM3.

The next three items were created by the researchers to capture integrated mathematical knowledge of sectors, cones and/or the Pythagorean theorem. The fourth test item assessed ability to distinguish between slant height and vertical height and use the Pythagorean relationship between them and the cone's base radius to calculate cone volume. The fifth test item assessed connections among properties of a cone related to the properties of a sector, for example, that the circumference of the cone base equals the arc length of the sector, or the slant height of the cone equals the radius of a sector. The sixth and final test item assessed the students' ability to explain how to calculate the volume of a cone given the dimensions of a sector that is its net. Correctly answering this question required multiple steps and ideas to be connected with each other. For example, to correctly explain the solution students need to articulate how to calculate the arc length of the sector, equate it to the cone base circumference and use the circumference to find the cone base radius. In addition, students need to explain that the sector radius must be equated to the cone slant height and combined with the cone base radius, using the Pythagorean theorem, to find the vertical cone height, which can be used in the standard cone volume formula. For clarity, we will refer to these items that assess connected mathematics knowledge as CM1, CM2, and CM3.

Embedded assessments. Four embedded assessment items specifically targeted students' understanding of the relationship between slant height and vertical height. In order for students to integrate ideas about sectors and cones, students need to understand that when a sector is formed into a cone, the sector radius becomes the slant height. Combined with the cone base radius, students can use the Pythagorean theorem to find the vertical cone height. Two embedded items targeted students' conceptual understanding by explaining relationships. For example, after working with interactive visualizations that allowed students to change the dimensions of the cone (Figure 10), students were asked "Why is the slant height always bigger than the vertical height?" (EA1). Students were also asked to explain calculations of their cone designs. For example, prompts asked students, "How did you calculate the radius of the sector from your cone dimensions r and h?" (EA3). Two embedded items targeted students accompanying procedural understanding of calculating the values, asking students to calculate the radius and central angle of the sector needed to make a specific cone (EA2) and the area and cost of a sector (EA4).

WISE log data. WISE recorded each student's progress throughout the project. Log data included how long and when students clicked on specific steps, how long they were on that step, any student responses to questions asked during that step, how many times the step had been visited, and how many revisions to responses the student had made, along with the content of those revisions.

Classroom observations/video. Two student groups were videotaped during the study to triangulate log data. Researchers also took written observations during project implementation.

Paper packet. To capture students' handwritten mathematical calculations, a paper packet accompanied the project. This packet contained the same questions from the WISE project where students needed to perform mathematical calculations and provided space for students to sketch their cone designs.

Procedures

About a week prior to the start of the project, the paper pretest was administered individually to students. Students were given access to a formula sheet that was the same as that used by the students during their state mandated assessments. The posttest was administered in an identical manor a few days after the project was completed.

The project lasted for three 85-minute blocks on consecutive days during their normal scheduled mathematics instruction, in their normal mathematics classroom. All students had a personal laptop provided by the school district that they were familiar with using. Students sat in pairs and were allowed to discuss their ideas with each other, to ask each other questions and seek assistance from the classroom teacher or researchers if needed. But students were expected to answer the questions within WISE and the accompanying paper packet independently. Both the teacher and the researchers acted as additional resources for students who needed additional support. This assistance usually took the form of referring students back to the information or instructions within the WISE project, but also included asking students the same or similar questions verbally, listening to the verbal response and encouraging the student to then write their response.

Data Analysis

The test items that assessed discrete mathematics knowledge (DM1-3) were scored using a 5-point rubric developed for each question (see Appendix D). Higher scores were assigned to responses that accurately identified and applied the knowledge necessary to answer the given question. For example, on item DM1 students received a 0 for a blank or irrelevant comment, a 1 for using formulas for the circumference of a circle or indicating some portion of the circle is needed. Students scored a 2 for writing an expression for the arc length as a proportion of the circumference. Students scored a 3 for correctly calculating the numerical value for the arc length with appropriate justification, and a 4 for correctly calculating the numerical value for the arc length with appropriate justification and correct units.

The test items that assessed connected mathematics knowledge (CM1-3) were scored using an adapted 5-point Knowledge Integration (KI) rubric developed for each question (e.g., Liu, Lee, & Linn, 2011). Higher scores were assigned to responses that accurately identified and integrated relevant ideas. Adapting a KI rubric involved coding for irrelevant, alternative, partial, and normative ideas. For example, on item CM1 students received a 0 for a blank or irrelevant idea. Students scored a 1 for alternative ideas, such as incorrectly identifying the vertical height.

Students scored a 2 for a partial link, writing an expression for the arc length as a proportion of the circumference (partial link). Students scored a 3 for correctly calculating the numerical value for the arc length with appropriate justification (normative link). Students scored a 4 for correctly calculating the numerical value for the arc length with appropriate justification and correct units (multiple links). Twenty percent of the test items were randomly selected and independently scored by two graders until ninety percent agreement was reached, after which the entire set of responses were scored by one researcher.

The four embedded assessment items were coded using a 4-point rubric specifically designed to capture if students were able to distinguish the relationship between the slant and vertical height. The rubric was used for both open response explanations and procedural questions. Twenty percent of the student responses to these questions were randomly selected and coded independently by two researchers. One hundred percent agreement was obtained, after which the entire set of responses were scored by one researcher.

The log data were initially cleaned as some students left browser tabs open after leaving class. To correct for this problem, log records with times that extended outside of the class time were shortened to end at the end of class. Records were categorized into engineering design stages based on the project step. The total duration for how long students spent in each stage were calculated from time stamps in the log data. The average time per step was calculated as various activities had different numbers of steps. The number of visits to each step and revisions to embedded assessments and returns to steps were also calculated.

Results

Research Question 1: How might interweaving mathematical modeling and informed engineering design help students develop a more connected understanding of geometry?

Pretest and posttests. After excluding missing data or those missing consent forms, data from 40 students were analyzed. Overall, average pretest and posttest scores significantly improved from pretest to posttest with a large effect size (see Table 2). Breaking down the results by test item, no significant gains were found for questions DM1 and DM2, but significant gains were found for items DM3, and CM1, 2, and 3 with large effect sizes.

Table 2

Pretest and posttest means and their differences by item

Item	Mean Pretest	Mean Posttest	Mean Difference (Post Test - Pre Test)	t	df	Sig. (2-tailed)	Cohen's d effect size
DM1	3.025	2.950	-0.075	-0.301	39	0.765	-0.048
DM2	3.450	3.425	-0.025	-0.105	39	0.917	-0.017
DM3	3.375	3.825	0.450	5.152	39	0.000	0.815
CM1	1.125	2.775	1.650	7.840	39	0.000	1.240
CM2	2.600	3.800	1.200	6.000	39	0.000	0.949
CM3	1.650	2.925	1.275	5.855	39	0.000	0.926
Total	15.225	19.700	4.475	9.643	39	0.000	1.525

Embedded assessments. Means and standard deviations for embedded assessments, along with the accompanying pretest and posttest item score are presented in Table 3. Scores represent students making progress distinguishing between slant height and vertical height over time, with all students answering correctly for the second embedded assessment question (EA2).

Table 3

Mean and standard deviation for student embedded assessment scores

Embedded assessment item	Mean	Standard Deviation
EA1	2.11	0.83
EA2	3.00	0.00
EA3	2.77	0.65
EA4	2.83	0.71

Research Question 2: What kinds of supports help students to engage in mathematical modeling in the context of engineering design?

Duration of steps and practices. Figure 11 displays the average time per step within each design practice and the total time for each design practice. The average time a student spent on the project was 185 minutes (see Table 4). Log data show students spent the most total time in Develop Knowledge and Ideate Solutions phases. Students generally spent almost half the time of the project on steps that involved developing mathematical knowledge, about a quarter of the time ideating design solutions, and a quarter of the time building, testing and refining their design solutions.

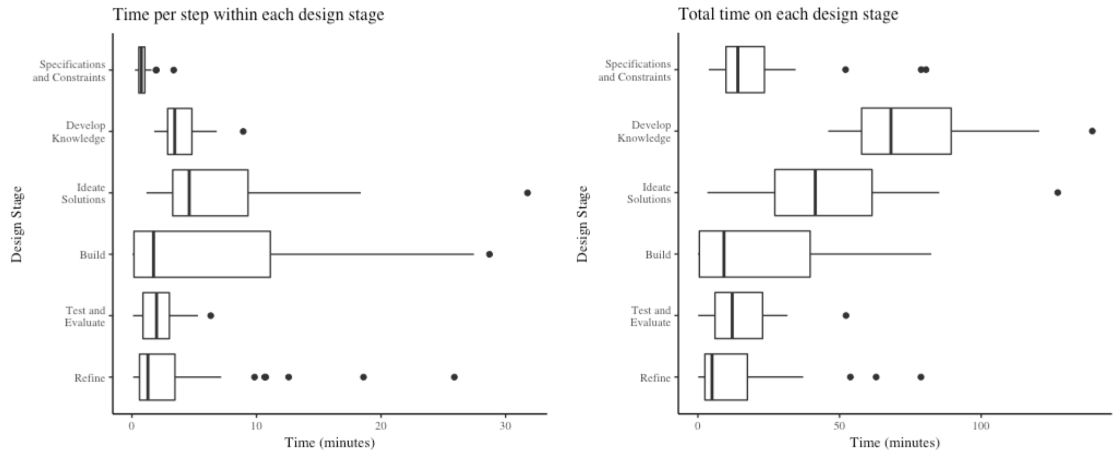

Figure 11. The time per step visit and the total time that students were logged, for each design stage.

Table 4

Mean and standard deviation for the time per step visit and the total time that students were logged, for each design stage (minutes)

Design Stage	Time per step (minutes)		Total time (minutes)	
	Mean	Standard Deviation	Mean	Standard Deviation
Specifications and Constraints	0.91	0.56	19.68	16.82
Develop Knowledge	3.94	1.56	75.56	23.83
Ideate Solutions	6.88	6.19	45.72	25.11
Build Prototype	5.76	7.62	19.81	22.70
Test and Evaluate Solutions	2.13	1.51	14.93	11.50
Refine Solution	3.68	5.73	13.93	19.18

Visiting Steps. Figure 12 displays the total number of visits made to each step. As students were able to return and review prior steps within WISE, students visited some steps more frequently than other steps. With an n = 40, 40 steps would roughly correspond to students progressing through the project without returning to or revisiting steps. Students visited steps 3,4 and 5 most often. These steps described the details of the design challenge (steps 3 and 4) including the project specifications and constraints (step 5). The next most frequent steps visited were Develop Knowledge steps (steps 8 through 14).

Looking specifically into the Develop Knowledge steps, log data showed some students (*n* = 17) revisiting the interactive visualizations (steps 11 and 12) and around half (*n* = 23) of students proceeding without revisiting the visualizations (see Figure 13 for examples). The connected mathematics pretest scores (items CM1-3) for students who revisited the visualization steps were significantly less than those that did not revisit the visualization steps (Revisit: M(SD) = 4.65(1.50); No Revisit: M(SD) = 5.91(1.98); $t = 2.31$, $p = 0.026$) suggesting that students with higher prior connected mathematics knowledge were less inclined to revisit the visualizations. However, the connected mathematics scores for these same groups was not significantly different (Revisit: M(SD) = 9.41(1.62); No Revisit: M(SD) = 9.57(2.37); $t = 0.24$, $p = 0.81$), indicating that the group that did revisit the visualizations were able to make increased integrated knowledge gains.

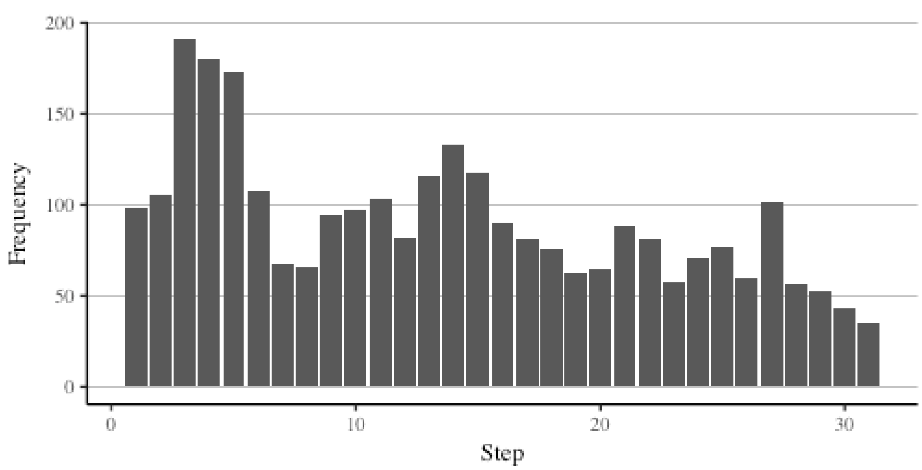

Figure 12. The total number of visits made to each step by all students.

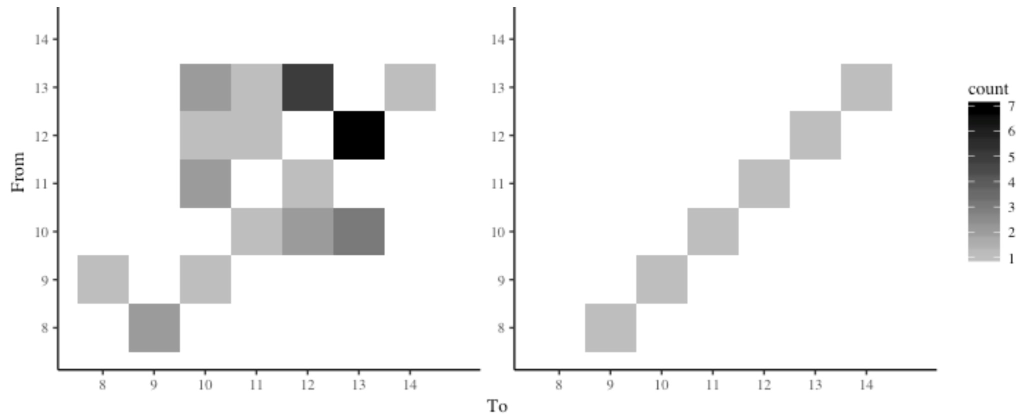

Figure 13. The transitions between Develop Knowledge steps for two example students. The student on the left revisits the visualization steps (steps 11 and 12) many times. The student on the right does not revisit any steps and proceeds through the steps incrementally.

Discussion

The first cycle of this study explored how a project to support a synergistic use of engineering design and mathematical modeling to help students develop a connected understanding of circles, sectors, and cones. Overall, student scores from pretest to posttest improved significantly. Embedded assessments captured how students were able to use WISE to support engineering design and mathematical modeling practices to connect and apply mathematical knowledge. Students engaged in the engineering design and mathematical modeling practices synergistically, with log data reflecting that the students spend most of their time on those steps where designing and modeling informed each other, that is, the Developing Knowledge and Ideating Solutions activities.

Developing connected mathematical understanding. From pretest to posttest, most of the total learning gain was from the connected mathematics items (CM1-3) rather than the discrete mathematics items (DM1-3). The lack of gain on the discrete mathematics items could potentially be due to a ceiling effect, as scores started relatively high. Students began the project able to perform well on items that assessed standard, discrete, well-defined knowledge. However, the pretest scores for questions that required student to connect this knowledge began low, reflecting the isolated nature of students' ideas. By the posttest, students made significant progress connecting ideas about cones and sectors. For example, although many students were able to find the volume of a cone by directly applying a formula (DM3), when given a slant height instead of a vertical height (CM1) many students did not make the correct connections among ideas on the pretest, but were able to connect ideas about right triangles and cones and solve for the correct volume by the posttest. These results point to the utility of using our informed engineering design and mathematical modeling project to help students develop a connected mathematical understanding.

The use of mathematical modeling with the engineering design project could have contributed to these gains. The two interactive geometry visualizations that emphasized the relationships between sectors and cones were specifically designed for this project as models for the students to use. Students who revisited the visualizations began with significantly lower pretest KI scores but ended with no difference on posttest scores. These results suggest that students who might have needed extra support were able to use the visualizations to develop connections among targeted concepts. These results align with other research showing the benefit of interactive geometry visualizations on student understanding (e.g., Hollebrands, 2007; Zengin, Furkan, & Kutluca, 2012).

Engaging in engineering design and mathematical modeling practices. Although students had no prior experience conducting engineering design projects in their geometry class, log data and video observations indicated that through the WISE supports, students were able to engage in the engineering design practices of defining project criteria, developing knowledge, ideating solutions, building, testing, and evaluating prototypes, and refining solutions, as well as the mathematical modeling practices of defining variables and finding solutions. That not all the mathematical modeling practices were observed is due to the fact that the project focused on students using models rather than creating models. These observations informed our revisions of the project in Cycle 2.

Research demonstrates that implementing engineering design projects in mathematics classrooms can often focus on trial and error approaches and lack deep connections to developing content understanding. This was not observed. In addition to the pre/posttest results, the log data suggests that students in the Ice Cream project spent nearly half of their time in the Develop Knowledge activities that involved using mathematical models, representing a large emphasis on developing integrated understanding of geometry ideas. As many teachers have time constraints and engineering design or mathematical modeling projects can be seen as something extra, these results align with other research indicating that this "something extra" can have a powerful impact on students learning content knowledge (e.g., Jacobson & Lehrer, 2000; Puntambekar & Kolodner, 2005).

Although the participating geometry teacher had very little experience with engineering design or mathematical modeling activities, she was able to facilitate the Ice Cream project in her classroom. The teacher had little difficulty helping students troubleshoot when needed or managing the materials and chaos of students building waffle cones and eating ice cream in her class. The teacher was also able to adapt and build upon the WISE project as she created her own table on the whiteboard for her students to share their final results with each other at the end of the project. Results point to the potential of WISE to help mathematics teachers implement student-centered, project-based projects in their classrooms.

Project Cycle 2: Creating Models

Participants and Context

Participants (n = 66) were students from five classes of high school geometry all of which were taught by the same teacher. School demographics consisted of 34% Black, 9% Hispanic, and 45% White students with 45% of students receiving free or reduced lunch. The teacher had one year of prior teaching experience, and this was his first year teaching geometry.

Revisions to the Ice Cream Cone Project

The project was revised so that the students were asked to *create* mathematical models from data that they collected. This was a change from the first cycle in which students *used* premade mathematical models to inform their designs. This revision reflects the findings from Cycle 1 that not all mathematical modeling practices had been supported. These revisions were intended to give better insight into how to support mathematical modeling practices such as "identifying variables". This change in emphasis meant that the project included building and measuring steps within the modeling activities so that the data collected informed the mathematical models that the students created.

To collect the data, students first made paper cones, using instructions within WISE to draw out a 18cm diameter circle, cut out a sector from the circle and tape the radii of the sector together to form a cone (see Figure 14). Next, the project asked students to use string to measure the arc length of the sector, a protractor to measure the central angle of the sector, a ruler to measure the radius of the sector as well as the slant height of the cone and the radius of the base of the cone, and finally by filling their cones with dry rice and pouring the rice into a graduated cylinder they measured the volume of their cone.

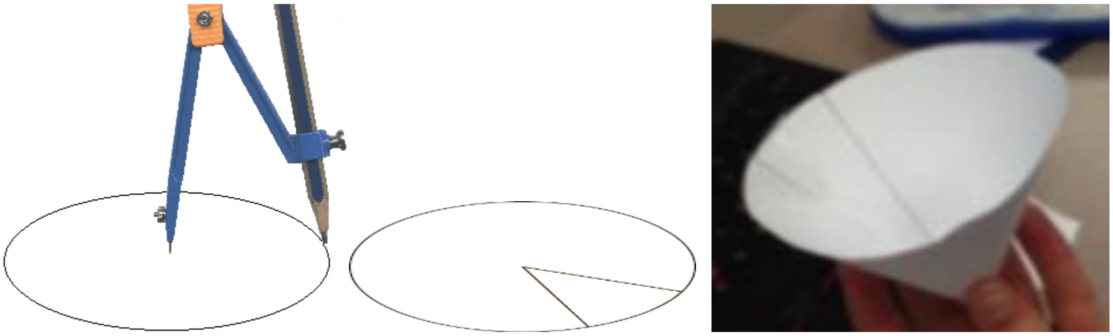

Figure 14. The revised cycle two WISE project instructed students how to make paper cone prototypes that they tested to generate data that could be modelled.

Students were given approximately 60 minutes to make additional sectors and cones, take measurements (see Figure 15) and record their measurements in a data table (see Figure 16).

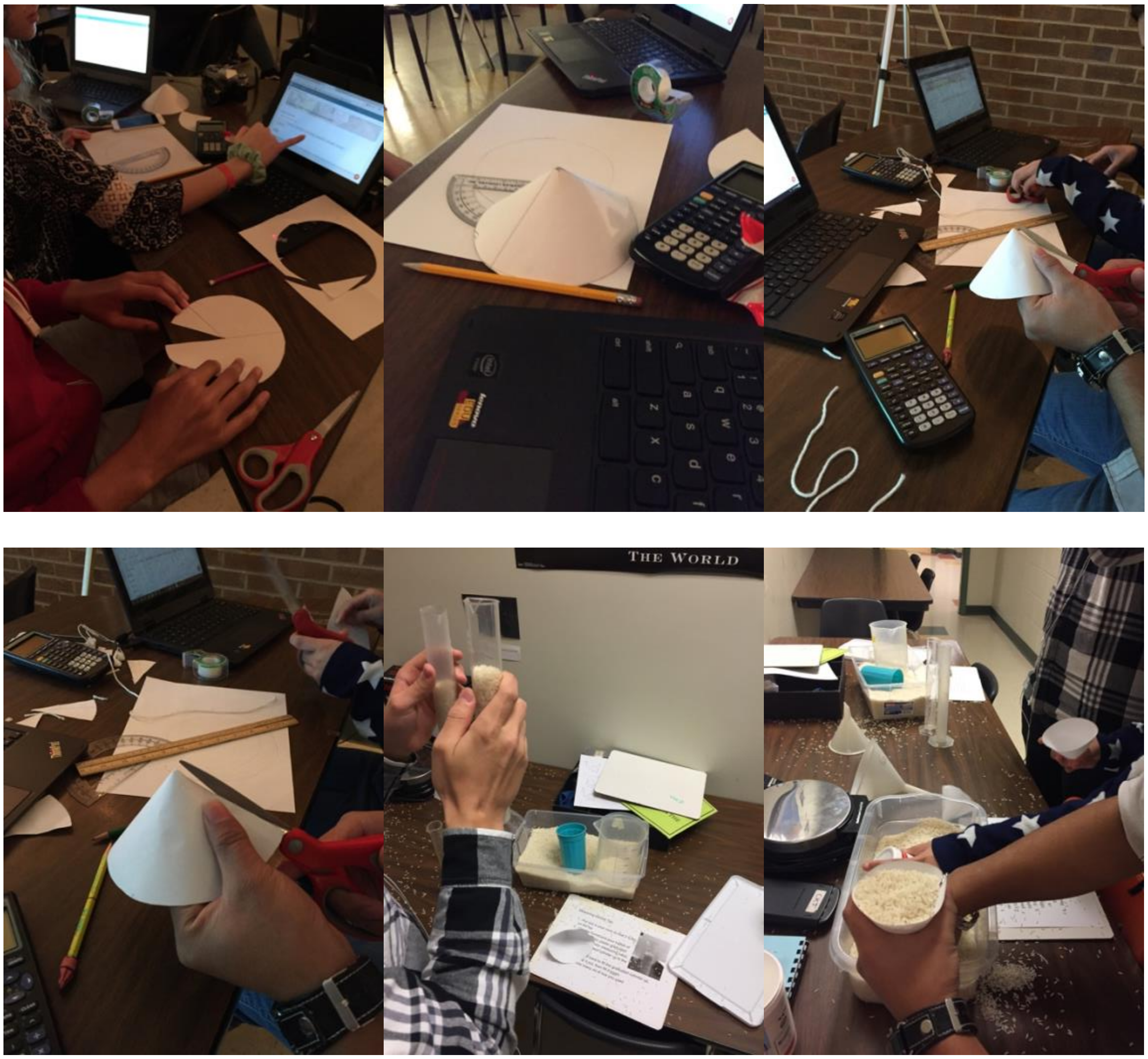

Figure 15. Students building and measuring prototype cones

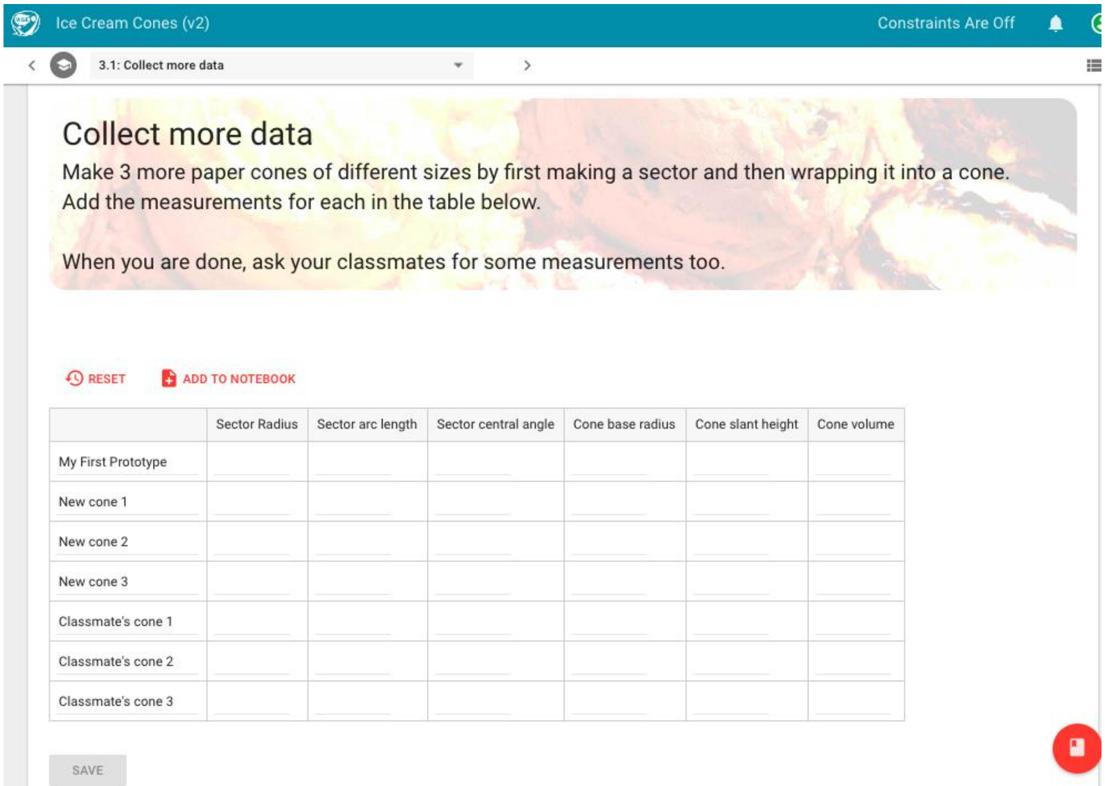

Figure 16. Students were encouraged by the project to test their prototype paper cones and record their measurements to use for modeling in a later activity.

Following the data collection phase, the student began creating models by looking for patterns in their data and describing any relationships they found between the following variables:

- Sector arc length and sector central angle (Relationship 1).
- Cone slant height and sector radius (Relationship 2).
- Cone base radius and sector arc length (Relationship 3).
- Cone volume, cone base radius, and cone slant height (Relationship 4).

These relationships were the same as those from cycle one of this study. However, in Cycle 2 we encouraged students to identify them from their data rather than from visualizations.

To support this, students were encouraged to form small groups and share ideas their ideas, to use a graphing tool to help them visualize their data (see Figure 17), and to access the scaffolded visualizations if needed. The teacher and researchers supported this discussion when needed.

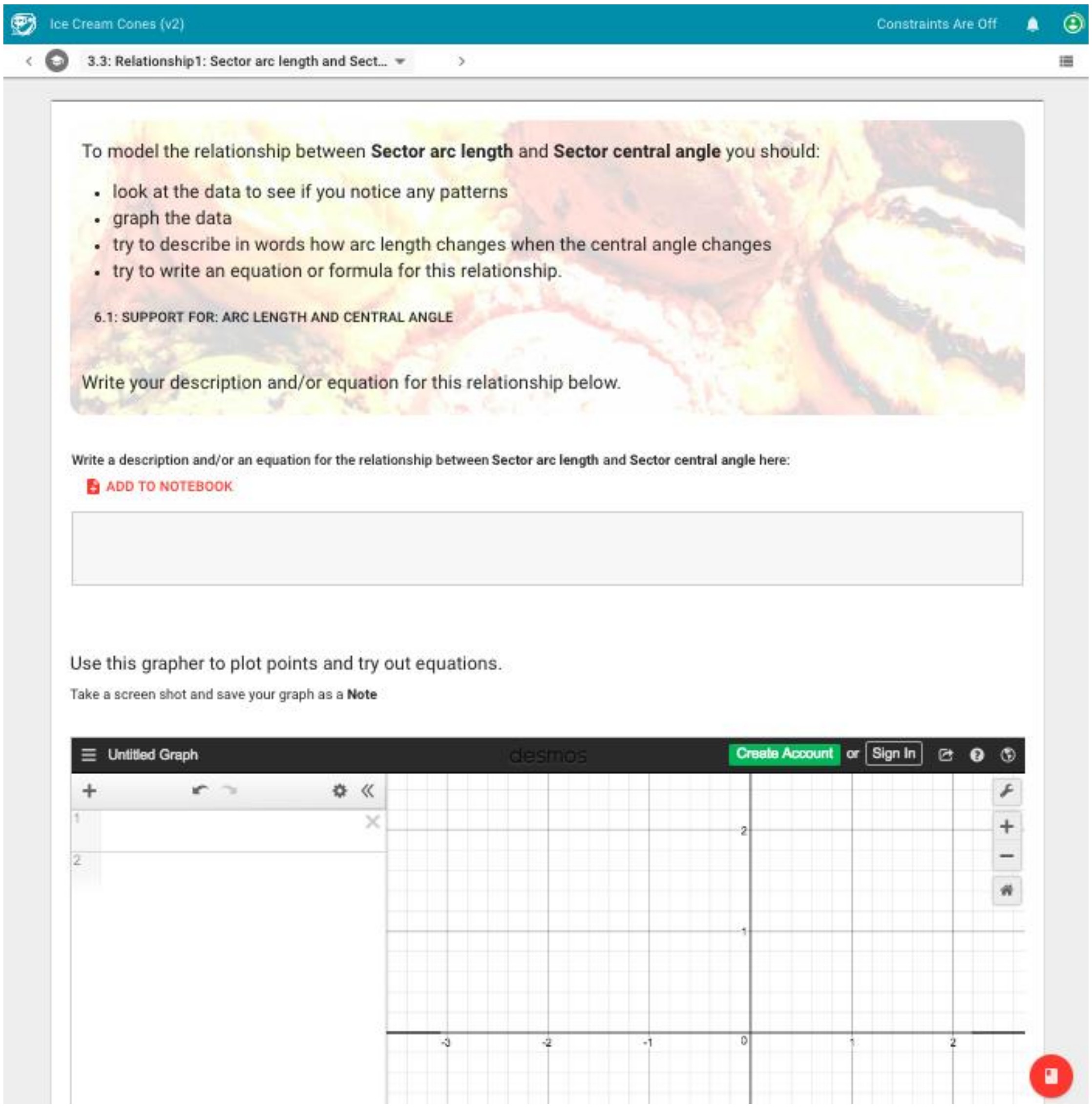

Figure 17. Describe that we are now asking student to find the relationship from the data they collected (see Figure 16). Students were given a graphing tool to use and the "SUPPORT FOR" link took student to the visualizations and models that were the focus of Cycle 1.

In addition, the challenge that the project posed to the students in Cycle 2 was changed so that students had to maximizing the amount of ice cream that a waffle cone could hold, which was considered to be more fun than the Cycle 1 challenge of minimizing the amount of waffle used to hold a fixed volume of ice cream. The full Cycle 2 project can be viewed at https://wise.berkeley.edu/project/23681.

Revisions to the Data Sources

Pretest and posttest. Because our revised project was revised to emphasize mathematical modeling and student understanding of the relationship between variables, our pretest and posttest were revised by dropping the items that assessed discrete mathematics knowledge (DM1-3) and adding items that assessed understanding of mathematical modeling and engineering design (see Appendix C). Part (a) of the first two questions (MM1 and MM2) asked students to find the relationship between two variables using data provided in a data table. Part (b) of each of these questions asked students to describe how they found this relationship. This question was intended to assess students' ability to create mathematical models from data. The second new question (ED1) asked students what the next engineering design steps were that should be taken in a given scenario. This question was intended to assess students' ability to apply the engineering design practices to new context. The connected mathematics items (CM1-3) were kept unchanged.

Embedded assessments. Given the revisions to the project the embedded assessments were also changed. In the first cycle we asked student to use the interactive visualizations and models we provided and to describe the patterns or relationships that they noticed between the variables. In this cycle we asked similar questions about these same relationships, but rather than asking student to notice these patterns by using the models we had provided for the project, we asked student to notice these patterns from the data they had collected in an earlier activity (see Figure 16). We continued to make the visualizations from the first cycle available to students as optional scaffolded materials that students could access by clicking a link (see Figure 17). Other data sources such as the WISE Log Data, Classroom Observations/Video were unchanged.

Procedures

The pretest was administered immediately prior to the start of the project, and like in the first cycle, the students answered the questions individually and had access to the formula sheet used by the students during their state mandated assessments. The posttest was administered in an identical manor immediately after the project was completed. As with the first cycle, the project occurred during three class blocks (totaling approximately 230 minutes) on consecutive days during their normal scheduled mathematics instruction, in their normal mathematics classroom. Other procedures from the first cycle were the same: All students had a personal laptop provided by the school district that they were familiar with using; students sat in pairs and were allowed to discuss their ideas with each other, to ask each other questions and seek assistance from the classroom teacher or researchers if needed.

Data Analysis

We analyzed the WISE log data, the embedded assessments, and the pretest and posttest data from cycle two using a similar approach as that used in cycle one. Items CM1-3 scored using the same 5-point Knowledge Integration (KI) rubric used in Cycle 1 (see Appendix D). Higher scores were assigned to responses that accurately identified and integrated relevant ideas. The new items MM1, MM2, and ED1 were scored using 5-point rubrics that were developed to reward more detailed articulation of models found from datasets, or of future design stages (see Appendix E). Twenty percent of each of these test items were randomly selected and independently scored by two graders until at least ninety percent agreement was reached, after which the remaining set of responses were divided into two groups, each group was scored by one researcher, and the scores combined for analysis.

The four embedded assessment items for Cycle 2 were coded using a 5-point rubric designed reward more detailed articulation of the relationship between variables (see Appendix F), and to align to scores given for similar pretest and posttest questions. All of these responses were coded independently by two researchers and any discrepancies were discussed until consensus was reached.

The log data records were categorized into engineering design stages and mathematical modeling stages (see Table 5). The total duration for how long students spent in each of these stages was calculated from time stamps in the log data. The number of visits to each step and revisions to embedded assessments and returns to steps were also calculated.

Table 5

How each step in Cycle 2 maps on to engineering design and mathematical modeling stages.

Step	Step Name	Engineering Design Stage	Mathematical Modeling Stage
1	1.1: Ice Cream Stand Project	1. Specifications and Constraints	1. Identify and Specify Problem
2	1.2: Your Design Challenge		
3	1.3: Design Notebook		
4	2.1: Sectors	2. Develop Knowledge	2. Identify variables
5	2.2: Building a paper prototype		
6	2.3: Measure your paper prototype		3. Collect data
7	3.1: Collect more data		
8	3.2: Finding relationships between variables.		4. Describe relationship between variables
9	3.3: Relationship 1		
10	3.4: Relationship 2		
11	3.5: Relationship 3		
12	3.6: Relationship 4		
13	3.7: Cone volume from Sector central angle		
14	4.1: Design your cone	3. Ideate Solutions	5. Implement the model
15	4.2: Design Calculations		
16	4.3: Build Final Solution	4. Build Prototype	
17	4.4: Test Your Design	5. Test and Evaluate	
18	5.1: Refine Design	6. Refine Design	

Results

Pretest and posttests. Overall, average pretest and posttest scores for questions that focused on connected understanding of geometry (CM1-3) significantly improved with a medium-large effect size (see Table 6). Breaking down the results by item, no significant gains were found for item CM2, but significant gains were found for items CM1 and CM3.

Table 6

Mean and standard deviations for the pretest and posttest scores for questions that focused on connected mathematical understanding, and their paired t-test results.

Item	Mean (Standard Deviation)			df	t	p	Cohen's d effect size
	Pretest	Posttest	Difference (Post-Pre)				
CM1	0.83 (0.38)	1.41 (0.80)	0.59 (0.88)	45	4.504	0.0000	0.66
CM2	0.70 (1.19)	1.07 (1.20)	0.37 (1.36)	45	1.849	0.0711	0.27
CM3	0.39 (0.49)	0.70 (0.70)	0.30 (0.79)	45	2.629	0.0117	0.39
Overall	1.91 (1.40)	3.17 (1.89)	1.26 (1.88)	45	4.551	0.0000	0.67

Average pretest and posttest scores for questions that focused on mathematical modeling and engineering design (MM1, MM2, and ED1) also significantly improved with a medium-large effect size (see Table 7).

Table 7

Mean and standard deviations for the pretest and posttest scores for questions that focused on mathematical modeling and engineering design, and their paired t-test results.

Item	Mean (Standard Deviation)			df	t	p	Cohen's d effect size
	Pretest	Posttest	Difference (Post-Pre)				
MM1(a)	1.37 (1.57)	2.24 (1.39)	0.87 (1.85)	45	3.196	0.0025	0.47
MM1(b)	0.76 (1.06)	0.76 (0.82)	0.00 (1.12)	45	0.000	1.0000	0.00
MM2(a)	0.50 (1.09)	1.33 (1.45)	0.83 (1.45)	45	3.864	0.0004	0.57
MM2(b)	0.35 (0.74)	0.61 (0.83)	0.26 (0.80)	45	2.209	0.0323	0.33
ED1	0.17 (0.38)	0.72 (0.83)	0.54 (0.84)	45	4.412	0.0001	0.65
Overall	3.15 (3.23)	5.65 (3.63)	2.50 (3.85)	45	4.409	0.0001	0.65

Visits to steps. Figure 18 shows the number of visits to each of the project steps, along with the number of students that progresses to each step. The number of students who progress beyond the data collection steadily decreases with very few students completing the project. Despite this, there is a large number of visits to the steps that correspond to the engineering design "develop knowledge" stage, or the mathematical modeling "collect data" and "describing relationships between variables" stages. Table 8 shows the average time that each student spent at each engineering design and mathematical modeling stage. This shows that students are both visiting these stages most frequently and spending the bulk of their time at these stages. Furthermore, the transitions between the mathematical modeling stages show that students tended to transition back from the describe relationships steps to the data collection steps much more than between other steps (see Figure 19).

Step	Engineering Design Stage	Mathematical Modeling Stage
1 2 3	1. Specifications and Constraints	1. Identify and Specify Problem
4 5	2. Develop Knowledge	2. Identify variables
6 7		3. Collect data
8 9 10 11 12 13		4. Describe relationship between variables
14 15	3. Ideate Solutions	5. Implement the model
16	4. Build Prototype	
17	5. Test and Evaluate	
18	6. Refine Design	

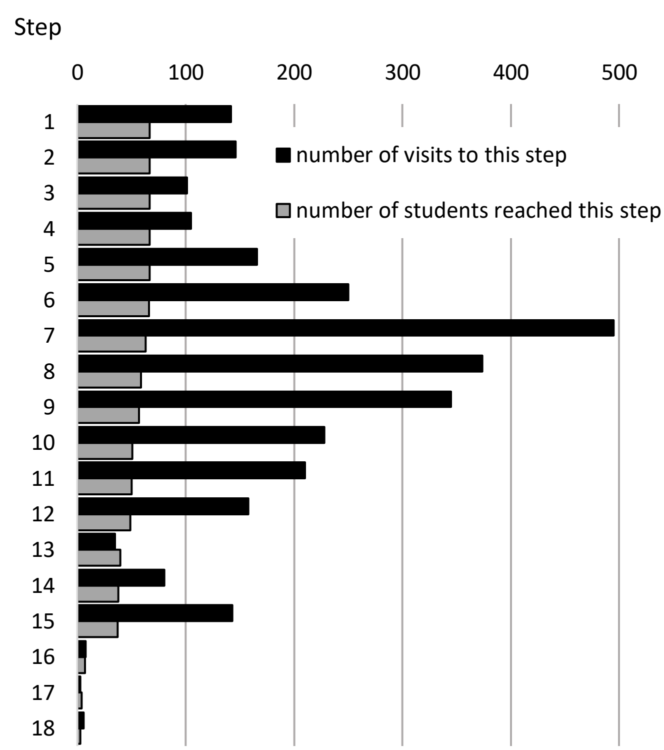

Figure 18. The total number of number of visits that all students made to each step of the project and the number of students who progressed through the project to each step.

Table 8

Mean and standard deviation time (minutes) that students were logged at each design or modeling stage, and the time (minutes) that students spent per visit to a step within that stage

Step	Engineering Design Stage		Mathematical Modeling Stage	
	Name	Mean (Standard deviation)	Name	Mean (Standard deviation)
1	1. Specifications and Constraints	6.42 (6.54)	1. Identify and Specify Problem	6.42 (6.54)
2				
3				
4	2. Develop Knowledge	95.76 (32.42)	2. Identify variables	11.36 (6.86)
5				
6			3. Collect data	69.97 (25.41)
7				
8			4. Describe relationship between variables	14.43 (14.35)
9				
10				
11				
12				
13				
14	3. Ideate Solutions	6.33 (9.81)	5. Implement the model	6.81 (10.37)
15				
16	4. Build Prototype	0.45 (1.93)		
17	5. Test and Evaluate	0.01 (0.11)		
18	6. Refine Design	0.01 (0.04)		

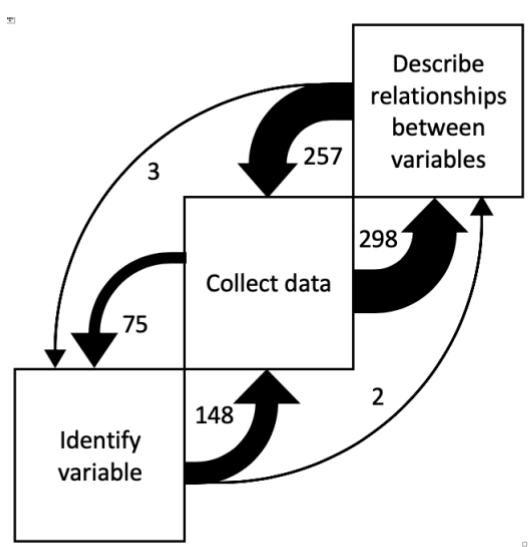

Figure 19. Transition frequencies between the mathematical modeling stages are indicated by the size of arrows.

Embedded assessments. Means and standard deviations for embedded assessments are presented in Table 9. Results show that for the question that asked students the equality between two variables (relationship 2), the students scored higher than for linear relationships between two variables (relationships 1 and 3), with students scoring lowest for a non-linear, multi-variable relationship (relationship 4).

Table 9

Mean, standard deviations, and count of embedded assessment scores

Relationship	Mean (Standard Deviation)	N
Relationship 1: Sector arc length and Sector central angle	1.02 (1.17)	41
Relationship 2: Cone slant height and Sector radius	2.07 (1.36)	30
Relationship 3: Cone base radius and Sector arc length	1.43 (1.52)	30
Relationship 4: Cone volume, Cone base radius and Cone slant height	0.79 (1.08)	29

Discussion

The second cycle of this study continued to explore how to support a synergistic use of engineering design and mathematical modeling to help students develop a connected understanding of circles, sectors, and cones. In the second cycle we placed a greater emphasis on mathematical modeling rather than engineering design by asking students to describe the relationships between variables from data that they collected rather than from their observations of the models with which they were provided.

Developing connected mathematical understanding. Overall, in Cycle 2, student scores on items that assessed the connections between cones and sectors significantly improved.

These results reaffirm the utility of using an informed engineering design and mathematical modeling project to help students develop these connections.

Rather than emphasizing that students make these connections by referring to the interactive geometry visualizations as we did in Cycle 1, this cycle emphasized students collecting data about different size cones that they make from different size sectors. The continued improvement in connected mathematical understanding is likely due in part to the data collection process which involved lots of hands-on manipulation of sectors and cones that would make students familiar with the variables that they measured.

In addition, results show that the gains in Cycle 1 are greater than in Cycle 2 and also that the pretest scores in Cycle 1 are higher than the posttest scores for Cycle 2. These differences are likely to be due in part to the different student populations in which the two cycles were implemented, but it may also indicate that for building connections between mathematical ideas, the Cycle 2 emphasis on data collection and creating mathematical models was less effective than the Cycle 1 approach of providing students with models.

Engaging in engineering design and mathematical modeling practices. As with Cycle 1, the students and the teacher in the second cycle had no prior experience conducting engineering design projects in their geometry class, and the log data results indicate that through the WISE supports, students were able to engage in the engineering design practices of finding specifications and constraints, developing knowledge, and ideating solutions, as well as the mathematical modeling stages of identifying and specifying problems, identifying variables, collecting data, describing relationships between variables, and implementing the model. That not all the engineering design stages were observed is due to the fact that the project focused on students creating mathematical models rather than repeating the engineering design cycles.

In addition, pretest to posttest results showed statistically significant gains for items that assessed the students' ability to find mathematical models and their understanding of the engineering design cycle stages. Despite this, students scored low and didn't show gains for items that asked them to articulate how they found the mathematical relationships they described, perhaps because explaining their understanding was not emphasized during the project.

Students spent a significant amount of time collecting data during Cycle 2. The data collection steps of the project involved extensive hands-on creation of different size cones from different size sectors while measuring the values of properties of each such as sector radius and arc length, and cone slant height and volume. While not directly part of creating the mathematical models, this measuring and data collection process seemed to be valuable for students to understand the variables that they would be asked to relate from classroom observations. For example, it was common for students to ask "what's the arc length?" when asked to measure arc length, indicating that the process of data collection was not just a means to obtain a dataset to analysis but also as a means for students to be able to develop an understanding of the variables. In fact, without this knowledge, common means for communicating mathematical relationships such as via formulae are likely to be less effective.

Implications

This study highlights that for mathematics students who have an emerging understanding of the mathematics they might use to model a given situation, identifying variables and the relationships between those variables is a challenging component of mathematical modeling. Common descriptions of mathematical modeling (e.g., Bliss & Libertini, 2016) often make assumptions that students have the underlying mathematical knowledge that they can use during the modeling process. From these perspectives, "identifying relationships between variables" or "doing the math" involve students selecting the most appropriate mathematical relationship from what they currently know rather than learning new mathematical relationships. This study shows that for modeling activities that are intended to help students learn about new mathematical relationships, it is important to support students to identify the relationships between variables. In Cycle 2 we were able to support students to identify relationships between variables through data collection. Data collection served to both support student understanding of the variables they were measuring, and served to provide empirical data upon which to base descriptions of the relationships between variables. Yet the use of data collection for this purpose is not commonly emphasized in descriptions of mathematical modeling. One implication of this study is that for contexts where mathematical modeling is intended to be used for students to learn about mathematical relationships, collecting physical data can be an important component of the mathematical modeling process.

This study also described two examples of the interweaving of engineering design and mathematical modeling. While distinct in many ways, their commonalties allow projects that draw upon both approaches to move fairly seamlessly from emphasizing one approach to emphasizing the other. For example, when it was possible to say that students were at some particular stage of mathematical modeling, it was also possible to identify a corresponding stage of engineering design. Alternatively, it never seemed that students switched from, say, doing engineering design to doing mathematical modeling. Rather, they moved from activities that are described well by engineering design into activities articulated well by mathematical modeling. For example, the developing knowledge process in engineering design can be elaborated upon by iterative mathematical modeling cycles, and the implementing the model stage in mathematical modeling can be elaborated on by iterative engineering design processes (see Figure 1). To some extent, this is similar to frameworks that align engineering design and science learning such as Learning by Design (Kolodner, Gray, & Fasse, 2003). In addition, while research demonstrates that implementing engineering design projects in mathematics classrooms can often focus on trial and error approaches and lack deep connections to developing mathematical content understanding, this study implies that implementing engineering design with mathematical modeling in mathematics classrooms can support connected mathematics understanding.

The study highlights that the tools within WISE that have been used previously to support engineering design and science inquiry can also be used to support engineering design and mathematical modeling. In Cycle 2 we added data collection and graphing tools for students to use in addition to the visualizations, reflection and self-explanation prompts, and sharing features from Cycle 1. Results from both cycles point to the potential of WISE to help mathematics teachers implement student-centered, project-based projects in their classrooms. The broad array of tools within WISE and their role in supporting mathematical modeling practices illustrates how technology can be used to support generative learning opportunities (Fiorella, Mayer, & Mayer, 2015) within mathematics. This study also contributes to the growing body of research investigating how to support engineering design processes in computer-based environments (Chao et al., 2017; Goldstein et al., 2015; Purzer et al., 2015).

Limitations

This study consisted of two small exploratory cycles with few student participants in each cycle, and therefore, our results may not be generalizable to other populations. Additionally, the study did not have a control group, and the learning gains we found may not be solely attributed to the ice cream cones project. Future studies involving comparison groups and isolation of specific components could provide more detailed understanding of our research questions.

Another potential limitation of this study is the way the project scaffolds engineering design and mathematical modeling. As there are specific steps through which students can linearly progress and complete, this could potentially reinforce a linear instead of iterative view of design and modeling (despite the cyclical representation and iteration within activities). On one hand, having discrete steps offers a way for teachers and students unfamiliar with engineering design and mathematical modeling to implement and engage in these practices. On the other hand, the stepwise fashion could potentially promote less informed views about these practices. Future research can explore how other kinds of environments (e.g., Easterday, Lewis, & Gerber, 2013) could support precollege students and teachers to engage in authentic engineering design practices.

Future research might consider the impact of such projects on the teachers involved in the study. For both teachers in this study, this approach was novel and their involvement in the development and implementation of the project may have impacted their instructional practices in ways that we did not record in this study. In addition, future cycles of investigation might also consider the intersecting roles of the digital tools and the physical hands-on manipulatives. Results from the two cycles in this study imply that both are important, but greater clarify on how they work together to support engineering design and mathematical modeling is needed. In addition, revisions to the project challenge itself could be made to allow increased authentic student involvement. For example, we could add other factors into the project (such as including a dome of ice cream on top of the cone) and let student pick which variables they were most interested in to optimize. Results could then be shared with classmates and combined to find solutions to a larger design challenge.

Conclusion

This study contributes to understanding how to support students' mathematical modeling and engineering design to help students develop integrated mathematics understanding. Results from the study illustrate how technology-enhanced supports can help students engage in engineering design and mathematical modeling practices. As such, the findings will be of interest to both mathematics education researchers as well as engineering education researchers.

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Appendix A

Overview of instructional steps in ice cream cones project

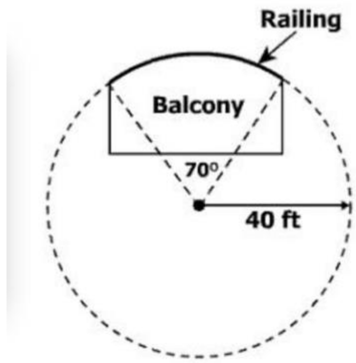
Engineering Design Cycle Stage	Step Name	Student Activity: Information	Interactive visualization	Written response	MC response	Arithmetic response	Hands-on
Specifications and Constraints	1 Introduction	✓					
	2 Engineering Design	✓					
	3 The Challenge	✓					
	4 Your Design Challenge			✓			
	5 Specs and Constraints			✓			
	6 Design Cycle				✓		
	7 Design Journal			✓			
Develop Knowledge	8 Cone Volume	✓			✓		
	9 Extra Volume	✓			✓		
	10 Total Cost					✓	
	11 Sectors as nets of Cones	✓	✓		✓		
	12 Cones to/from Sectors *	✓	✓	✓			
	13 Central Angle				✓	✓	
	14 Sector Dimensions *					✓	
Ideate Solutions	15 Design Sketch						✓
	16 Design Justification			✓			
	17 Design Calculations *			✓	✓		
	18 Your Design Ideas			✓			
Build	19 Build Prototype - Paper						✓
Test and Evaluate	20 Test Your Prototype						✓
	21 Evaluate Prototype			✓			
	22 Share Your Results			✓			
	23 Reflections			✓			
Develop Knowledge	24 Area of a Sector	✓			✓		
	25 Cone Cost *					✓	
	26 Reflections			✓			
Refine	27 Explain Your Design			✓			
	28 Final Design Sketch						✓
	29 Justifying Final Design			✓			
Build	30 Build Final Solution						✓
	31 Present Final Solution			✓			

**Denotes embedded assessment item*

Appendix B

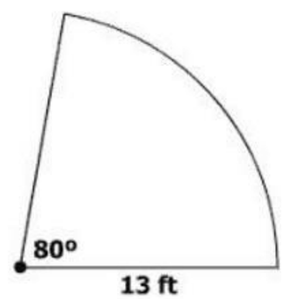
Pretest and posttest assessment items for Cycle 1

Please answer the following questions as best as you can.

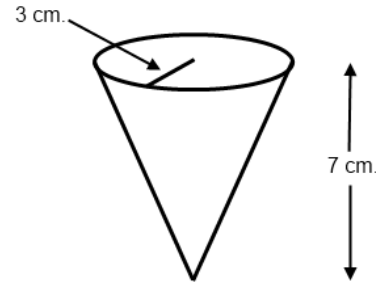


1. An architect used this diagram to design a curved balcony. She drew a circle with a radius of 40 feet and a central angle of 70° to determine the length of railing needed for the balcony.

What is the length of the railing needed for the curved section of the balcony? Show your work.



2. Flowers were planted in a section of a circular garden as shown. What is the area of this section of the garden? Show your work.

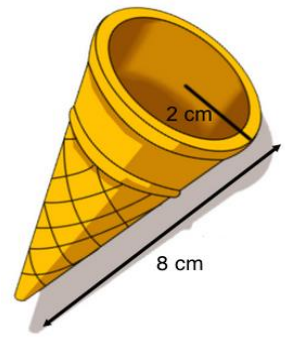


3. The volume of a cone is given by the formula $V = \frac{1}{3} \pi r^2 h$.

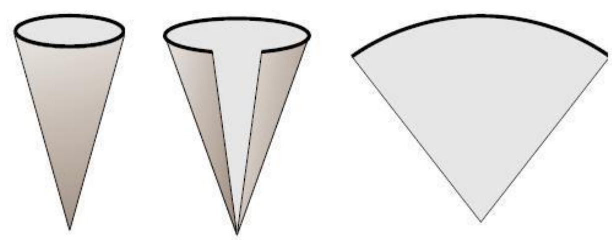
What is the volume of this cone? Show your work.

4. The volume of a cone is given by the formula $V = \frac{1}{3} \pi r^2 h$.

What is the volume of this cone? Show your work.

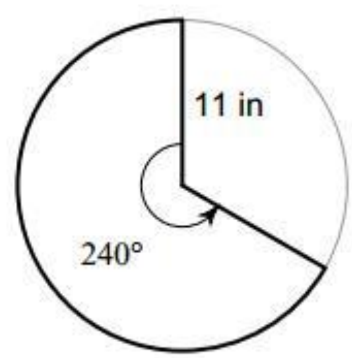


5. The diagram on the right shows how a cone can be cut and flattened into a sector.



Describe which properties of the cone are equal in size to properties of the sector.

6. The following sector will be rolled up into a cone. How would you use the dimensions of this sector to find the volume of the cone? Describe the steps.



Appendix C

Pretest and posttest assessment items added for Cycle 2

4. Renita counted the number of branches on different oak trees and birch trees. She also recorded the thickness of each tree in inches. Her measurements were:

Oak Trees

Thickness (inches)	Branches
3	34
4	46
5	59
6	70
7	82

Birch Trees

Thickness (inches)	Branches
5	64
6	127
2	8
4	31
3	16

Using the measurements for Oak Trees:
a) What is the relationship between tree thickness and the number of branches?

b) What steps did you take to find this relationship?

Using the measurements for Birch Trees:
a) What is the relationship between tree thickness and the number of branches?

b) What steps did you take to find this relationship?

5. Rafael's class is working on creating a design for an ice cream cooler. Rafael built a model with cardboard, Styrofoam, and duct tape. He is planning to turn the model into the teacher.

What additional steps should Rafael take to improve his design? Explain your reasoning.

Appendix D

Pretest and posttest assessment item rubrics.

For DM1:

Score	Descriptor	Example
0	Blank or frivolous comment. OR incorrect formula for circumference alone	idk
1	Clear indication that the circumference of the circle (words or formula, $2\pi r$) OR some portion (ratio or proportion) of the circle is needed	2 pi r OR 70/360
2	Writes an expression for the arc length as a portion (ratio or fraction) of the circumference	of the form: x /360 * 2 pi r
3	Correctly calculates the numerical value for the arc length (48.87 or $140\pi/9$) with appropriate justification (e.g. an expression described in #2 above)	
4	Correctly calculates the numerical value for the arc length (48.87 or $140\pi/9$) with correct units (ft or feet) and with appropriate justification (e.g. an expression described in #2 above)	

For DM2:

Score	Descriptor	Example
0	Blank or frivolous comment.	idk
1	Clear indication that the area of the circle (words or formula, πr^2) OR some portion of the circle is needed	pr r^2 OR 80/360
2	Writes an expression for the sector's area as a portion of the circle's area (ratio or fraction)	or the form: x /360 * pi r^2
3	Correctly calculates the numerical value for the sector's area (118 or $338\pi/9$) with appropriate justification (e.g. an expression described in #2 above)	
4	Correctly calculates the numerical value for the sector's area (118 or $338\pi/9$) with correct units (ft^2 or square feet) and with appropriate justification (e.g. an expression described in #2 above)	

For DM3:

Score	Descriptor	Example
0	Blank or frivolous comment.	idk
1	Makes an effort to use the information in the diagram but misidentifies the information.	
2	Identifies the radius = r = 3 and the height = h = 7 and/or substitutes these values into the given formula.	
3	Correctly calculates the numerical value for the cone's volume (65.97 or 21π) with appropriate justification (e.g. an expression described in #2 above)	
4	Correctly calculates the numerical value for the cone's volume (65.97 or 21π) with correct units (cm^3 or cubic centimeters or cc) and with appropriate justification (e.g. an expression described in #2 above)	

For CM1:

Score	Descriptor	Example
0	Blank or frivolous comment.	idk
1	Makes an effort to use the information in the diagram but misidentifies the information (such as h = 8 cm).	
2	Identifies the radius = r = 2cm and attempts to calculate the vertical height, h, from the given slant height (= 8cm) and r (with correctly or incorrectly) and/or substitutes the correct values for r and h into the given formula.	$h = \sqrt{(8^2 - 2^2)} = \sqrt{60} = 7.746$
3	Correctly calculates the numerical value for the cone's volume (32.45, NOT 33.5) with appropriate justification (e.g. an expression described in #2 above)	
4	Correctly calculates the numerical value for the cone's volume (32.45, NOT 33.5) with correct units (cm^3 or cubic centimeters of cc) and with appropriate justification (e.g. an expression described in #2 above)	

For CM2:

Score	Descriptor	Example
0	Blank or frivolous comment.	idk
1	Properties of the cone and/or the sector are specified without any connections OR at least **one** alternative connection that indicates equality is made.	
2	At least **one** partial connection that indicates equality is made. AND **zero** normative connections that indicate equality are made	
3	**One** normative connection that indicates equality is made. (Additional partial connections may be made if they don't contradict the normative connection) (Zero alternative connections may be made)	
4	**Two or more** normative connections that indicate equality are made. (Additional partial connections may be made if they don't contradict the normative connection) (Zero alternative connections may be made)	

Properties of Cones and Sectors that are equal:

Normative:
- Circumference of base (cone) = Arc length (sector)
- Slant height (cone) = radius (sector)
- Lateral surface area (cone) = area (sector)
- ~~Lateral~~ surface area (cone) = area (sector)

Partial:
- Circumference of base (cone) = ~~Arc~~ length (sector)
- Height (cone) = radius (sector)
- Vertical height (cone) = radius (sector)
- ~~Lateral surface~~ area (cone) = area (sector)

Alternative:
- Area of base (cone) = area (sector)
- Radius of base (cone) = radius (sector)
- Lengths are the same (unspecified properties of cone or sector)
- Heights are the same (unspecified properties of cone or sector)

For CM3:

Score	Descriptor	Example
0	Blank or frivolous comment.	idk
1	One step is stated	
2	Two or three steps are stated in a normative order.	
3	Four or five normative steps are stated in a normative order.	a), b), c), f). [two missing steps] a), c), b), d), e), f) [one out of order]
4	All six normative steps are stated in a normative order.	

Normative responses involve 6 steps in this order [or with steps c) and d) switched]

a) Find arc length (of sector) from given radius and central angle (of sector) using proportion of $2\pi r$.
   $2\pi\ (11)/\text{Arc} = 360/240 \Rightarrow \text{Arc} = 2\pi\ (11)(240)/360 = 44\ \pi\ /3 = 46.08$ inch
b) Find circumference of base (of cone) from arc length (of sector) using equality property.
   Circumference = $44\ \pi\ /3 = 46.08$ inch
c) Find radius of base (of cone) from circumference of base (of cone) using $C/2\pi$.
   $r = (44\ \pi\ /3\ )/\ 2\pi = 22/3 = 7.33$ inch
d) Find slant height (of cone) from radius (of sector) using equality property.
   11 inch
e) Find vertical height (of cone) from slant height (of cone) and radius of base (of cone) using the Pythagorean theorem, $h = \sqrt{(s^2 - r^2)}$.
   $\sqrt{(11^2 – 7.33^2)} = 8.20$ inch
f) Find volume of cone) from vertical height (of cone) and radius of base (of cone) using the volume formula,
   $^1/_3\ \pi\ r^2\ h = 462$ inch^3

Appendix E

Pretest and posttest rubrics for Cycle 2.

For MM1(a):

0	Blank, idk, etc.
1	Don't notice a relationship; "there is no pattern" OR Notices an alternative relationship
2	Notice that the number of branches increase (without saying thickness increases)
3	Notice that the number of branches increase as thickness increases. OR Positive relationship between thickness and branches. OR "Linear" increase.
4	Notice that the number of branches increase By 12 or [11, 13] for every one increase in thickness. OR that number is about 11 times larger than thickness. OR Normative mathematical relationship: Thickness = [11-13]*branches + [-2, +2]

For MM1(b):

0	Blank, idk, etc.
1	Looked at the table (vague). Said a mathematical operation without specific details (e.g. subtraction) Said they compared something (vague) Restated part a) pattern
2	Evidence of an attempt to find a model or pattern but didn't explain the step(s)
3	Found differences between table values (either said they did or they show differences) OR Found "multiply by 11" pattern OR Made a graph (or said they made a graph). (NOT, "looked at the graph" without saying that they made the graph)
4	Three or more clearly explained steps.

For MM2(a):

0	Blank, idk, etc.
1	Don't notice a relationship: "there is no pattern" "It goes up and down" OR Notices an alternative relationship
2	Notice that the number of branches increase (without saying thickness increases)
3	Notice that the number of branches increase as thickness increases. OR Positive relationship between thickness and branches.
4	Notice that the number of branches increase more (or doubles) for each additional one increase in thickness (ie. non-linear, quadratic, exponential) OR Normative mathematical relationship: Thickness = $2*2^{(thickness)}$

For MM2(b):

0	Blank, idk, etc.
1	Looked at the table (vague). Said a mathematical operation without specific details (e.g. subtraction)
2	Evidence of an attempt to find a model or pattern but didn't explain the step(s) OR Evidence of an attempt to reorder the table to make thickness in ascending order but didn't explain the step(s)
3	Reordered the table to make thickness in ascending order and attempt to find mathematical relationship OR Found multiply by 2 pattern OR Made a graph (or said they made a graph). (NOT, "looked at the graph" without saying that they made the graph)
4	Three or more clearly explained steps.

For ED1:
Score = Number of design steps mentioned based on Informed Engineering Design cycle.

Appendix F

Embedded Assessment questions from Cycle 2.

Step 3.3. Relationship 1: Sector arc length and Sector central angle

0	Blank, idk, frivolous, etc. OR How they measured the variables (string, or protractors) OR Math facts that involve one of the variables only
1	Alternative relationship (e.g. "one increases the other decreases") OR "They are not related"
2	Partial relationship identified. E.g.: string length and angle
3	As one increases the other increases (positive relationship) OR Linear relationship
4	Arc length = central angle * (2*pi*r)/360 OR Central angle = arc length * (360/2pi*r)

Step 3.4. Relationship 2: Cone slant height and Sector radius

0	Blank, idk, frivolous, etc. OR How they measured the variables (string, or protractors) OR Math facts that involve one of the variables only
1	Alternative relationship (e.g. "one increases the other decreases") OR "They are not related"
2	Partial relationship identified. E.g.: string length and ruler
3	They are (almost or exactly) the same/equal.

Step 3.5. Relationship 3: Cone base radius and Sector arc length

0	Blank, idk, frivolous, etc. OR How they measured the variables (string, or protractors) OR Math facts that involve one of the variables only
1	Alternative relationship (e.g. "one increases the other decreases") OR "They are not related"
2	Partial relationship identified. E.g.: string length and ruler
3	As one increases the other increases (positive relationship) OR Linear relationship
4	Arc length = 2*pi*radius OR Radius = Arc length / (2*pi)

Step 3.6. Relationship 4: Cone volume, Cone base radius and Cone slant height

0	Blank, idk, frivolous, etc. OR How they measured the variables (string, or protractors) OR Math facts that involve one of the variables only
1	Alternative relationship (e.g "one increases the other decreases") OR "They are not related"
2	Partial relationship identified (between only 2 of the 3 variables) Describe a relationship between radius and volume or slant height and volume
3	Describe a relationship between radius and volume and also slant height and volume
4	Combined: Volume = ⅓ pi r^2 sqrt(s^2 - r^2)