SYMMETRIES OF 2+1D NON-HERMITIAN CHIRAL MAJORANA SURFACE STATES IN TOPOLOGICAL SUPERCONDUCTOR

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(ABSTRACT)

This thesis investigates the emergent symmetries and classifications of non-Hermitian chiral Majorana surface states in topological superconductors, with a focus on 2+1-dimensional (2+1D) boundary modes. Using tools from conformal field theory (CFT), the research explores how 1+1D chiral Majorana edge theories can be used to analyse 2+1D surfaces with complex energy spectra, where steady-state configurations replace traditional ground states. The study develops a framework for deriving equations of motion, operator product expansions, and correlation functions on 2+1D boundaries.

Dedication

To my parents for their unconditional love and comprehensive support throughout my academic endeavors.

And, of course, to my friends and fellow Georgians, to those who for the past two years fought for my rights there where I could not and are fighting still for the most precious idea - freedom.

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Chapter 1

Introduction

In recent years, the study of topological phases of matter has provided a unifying framework for understanding complex quantum systems. These phases, instead of the local order parameters, are characterised by global topological invariants. Topological superconductors and insulators, due to their robustness to the disorder have been found to realise unique symmetry-protected topological surface states. This thesis investigates the symmetries and classifications of non-Hermitian chiral Majorana surface states in topological superconductors, to investigate emergent 2+1D symmetries. Building on the tenfold Altland-Zirnbauer symmetry classes, which classify materials based on time-reversal, particle-hole, and chiral symmetries, recent studies have also incorporated spatial symmetries such as reflection and inversion. The aforementioned classifications, discussed in Classification of Topological Quantum Matter with Symmetries [2], provide a rigorous framework for understanding the connections between symmetry and topology in both, gapped and gapless systems. This framework is essential for identifying protected surface states, such as Majorana fermions, and understanding their stability under perturbations.

A complementary perspective is studied in Topology of Crystalline Insulators and Superconductors [8], which extends the classification of topological phases to systems with crystalline symmetries. This extension emphasises the role of spatial symmetry in quasiparticle excitations, such as surface Majorana modes, which in recent years have been shown to be an integral part of realising topological quantum bits [6].

In this thesis, conformal field theory (CFT) is employed as a mathematical tool to analyse the emergent symmetries of 2D systems, with the goal of extending description to 2+1D boundaries. CFT provides a natural framework for describing chiral edge modes in topological systems, as it captures the behavior of fields under conformal transformations. This is particularly relevant for understanding the behavior of Majorana fermions, which obey conformal symmetry at low energies.

The central aim of this thesis is to investigate the symmetries of non-Hermitian systems, particularly in 2+1 dimensions, with the help of the 1+1D theory discussed in earlier chapters. Taking the insights gained from 1+1D chiral Majorana systems into consideration, the study extends these ideas to explore the dynamics and emergent properties of 2+1D surface states in non-Hermitian topological systems.

Outline of the Thesis

The thesis will be constructed in the following way: we will first review the conformal field theory and discuss how this mathematical tool can be used to investigate emergent symmetries in 2 dimensions. Then, to set up a problem for a three-dimensional topological superconductor with vertical hopping, we will dive into the discussion of a 2D ”p+ip” superconductor with a 1D surface and the emergent Majorana phases there. Once we have established the 1+1D theory, we will discuss the properties of the 3-dimensional superconductor that leads to Surface Majorana states. Then, for 2+1D surface states we will derive the equation of motion and discuss the expectation value for the product of operators and the challenges in obtaining them.

Chapter 2

Review of Literature

Symmetries have been studied from the very beginnings of civilisation and for very good reasons. For one, it is one of ”the chief forms of beauty”, according to *the philosopher* Aristotle. Whether it is the symmetry or the one that is broken, the form of beauty is a matter of aesthetics; however, there is no denying his implications that the study of reality around us often requires the inquiry in symmetries that manifest themselves in nature. Today, with natural philosophies spoken with the language of mathematics, we have gained a much deeper understanding of how symmetries truly reveal nature. One of the most useful approaches is what we know as Noether’s theorem. Considering infinitesimal symmetry transformations Noether’s theorem allows us to find conserved currents, charges, or Stress-Energy tensors, which allows us to study different systems and their symmetries in detail. However, in order to make use of Noether’s theorem, we first need to know what are the symmetry transformations of the fields and coordinates, which may not always be the case.

The Conformal Field Theory (CFT) is another mathematical theory that offers us a delicate way of studying symmetries. The CFT proves to be extremely useful, especially in two dimensions, where the transformation parameters obey Cauchy-Riemann equations, bringing complex analysis tools to our aid. Since the main skeleton of the theoretical framework required throughout the investigation of Majorana surface state symmetries are different aspects of CFT, in this chapter some detailed discussion will be provided.

The sources on which I will be heavily relying throughout this chapter are the classics of Conformal Field Theory.

[3]: Di Francesco, Philippe, Pierre Mathieu, and David Sénéchal. *Conformal Field Theory.* New York: Springer, 1997.

[4]: Ginsparg, Paul. *Applied Conformal Field Theory.* 1988. arXiv:hep-th/9108028.

2.1 Brief Review of Conformal Field Theory

2.1.1 Conformal Group

Conformal Group by definition is a group of coordinate transformations that leaves the metric invariant up to a scale change. That is, under coordinate transformation $x \to x'$, the new metric is proportional to one before the transformation

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = \Omega(x) g_{\mu\nu}(x) \tag{2.1.1}$$

Where we have $g_{\mu\nu}$ from the definition of the line element: $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$. To find the infinitesimal generators of the conformal group we perform transformation $x \to x' = x + \epsilon(x)$. Under this transformation, the line element becomes:

$$ds^2 \to ds^2 + (\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\nu) dx^\mu dx^\nu \tag{2.1.2}$$

If we consider the general transformation of the metric tensor for the given infinitesimal coordinate transformation, in first order we have:

$$\begin{aligned} g'_{\mu\nu} &= \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta} \\ &= g_{\mu\nu} - (\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) \end{aligned} \tag{2.1.3}$$

Comparing Eqn2.1.3 to Eqn2.1.1 we can see that $\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu$ should be proportional to $g_{\mu\nu}$. Considering a case where the transformation is a transformation of the cartesian metrics, $g_{\mu\nu} = \eta_{\mu\nu}$, with $\eta_{\mu\nu} = diag(1, ..., 1)$, then the condition 2.1.1 requires:

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \frac{2}{d}(\partial \cdot \epsilon)\eta_{\mu\nu} \tag{2.1.4}$$

2.1.2 Conformal Algebra in 2D

Now we discuss the special case of the conformal transformations and choose $d = 2$ to be the dimensions of the space. Then the Eqn2.1.4 in 2 dimensions with flat space-time yields:

$$\partial_1 \epsilon_1 = \partial_2 \epsilon_2, \quad \partial_1 \epsilon_2 = -\partial_2 \epsilon_1, \tag{2.1.5}$$

Which we can immediately recognise to be Cauchy-Riemann equations. This tempts us to define complex coordinates $z, \bar{z} = x^1 \pm ix^2$, and generators $\epsilon(z), \bar{\epsilon}(\bar{z}) = \epsilon^1 \pm i\epsilon^2$. The conformal coordinate transformation for complex coordinates is now given by analytic complex functions

$$z \to f(z), \quad \bar{z} \to \bar{f}(\bar{z}) \tag{2.1.6}$$

The infinitesimal form of the complex coordinate transformations takes the form:

$$z \to z + \epsilon_n(z), \quad \bar{z} \to \bar{z} + \bar{\epsilon}_n(\bar{z}) \tag{2.1.7}$$

With $n \in \mathbb{Z}$. Choosing the basis to be $\bar{\epsilon}_n(\bar{z}) = -z^{n+1}$, and $\bar{\epsilon}_n(\bar{z}) = -\bar{z}^{n+1}$, the infinitesimal generators become

$$l_n = -z^{n+1}\partial_z, \quad \bar{l}_n = -\bar{z}^{n+1}\partial_{\bar{z}} \tag{2.1.8}$$

These generators satisfy the algebra:

$$[l_n, l_m] = (n-m)l_{n+m}, \quad [\bar{l}_n, \bar{l}_m] = (n-m)]\bar{l}_{n+m}, \quad [l_n, \bar{l}_m] = 0 \tag{2.1.9}$$

Later we will find that there is a correction piece for the quantum mechanical case.

2.1.3 Conformal Fields in 2 Dimensions, Radial Quantisation

Following up on transformations given by Eqn2.1.6, the line element transforms as:

$$ds^2 = dzd\bar{z} \to \left(\frac{\partial f}{\partial z}\right)\left(\frac{\partial \bar{f}}{\partial \bar{z}}\right) dzd\bar{z} \tag{2.1.10}$$

We can generalise this transformation for primary fields defined on a complex plane:

$$\Phi(z, \bar{z}) \to \left(\frac{\partial f}{\partial z}\right)^h \left(\frac{\partial \bar{f}}{\partial \bar{z}}\right)^{\bar{h}} \Phi(f(z), \bar{f}(\bar{z})) \tag{2.1.11}$$

Where h-is the conformal wheight of the field ($h = \frac{1}{2}$ for fermions in 2 dimensions).

The transformations of secondary fields do not obey this expression, but we still can investigate their properties by considering their product expansions, as we will discuss it later in this chapter.

We now delve deeper into conformal theories for quantum fields. In flat space-time, light cone coordinates are constructed as: $\zeta, \bar{\zeta} = x^0 \pm ix^1$. We then compactify the spatial coordinate by parameterising it on the cylinder, $x^1 = x^1 + 2\pi$. As we will see later, conformal mapping $\zeta \rightarrow z = exp(\zeta) = exp(x^0 + ix^1)$ will be of our uttermost interest. Given conformal transformation maps cylinder onto the complex plane. The radial coordinate on the complex plane then corresponds to the time, with infinite past $x^0 = -\infty$ mapped to the origin, radial coordinate increasing with time. In other words, equal time surfaces $x^0 = const$ are mapped to circles on a complex plane. The time translation $x^0 \rightarrow x^0 + a$ on the complex plane becomes dilation $z \rightarrow ze^a$, so the dilation generator is analogous to Hamiltonian. Note that holomorphic and anti-holomorphic fields correspond to i.e. left, or right-moving fields. For the rest of the thesis, we will drop anti-holomorphic parts for brevity and simplicity since the discussion will mainly involve holomorphic fields.

Generally, symmetry generators are given by Noether's theorem. Infinitesimal symmetry variation in any field A can be calculated by a commutator of that field to the conserved charge: $\delta_\epsilon A = \epsilon[Q, A]$, where $Q = \int d^d x j_0(x)$. Where $j_\mu = T_{\mu\nu}\epsilon^\nu$ is the conserved current associated with the symmetries of the theory and $T_{\mu\nu}$ is the Energy-Momentum (EM) tensor. Integration over the equal-time surface gives us the conserved charge: $Q = \int dx j_0(x) \rightarrow \int d\theta j_r(\theta)$. Then the conserved charge expressed in complex variables becomes

$$Q = \frac{1}{2\pi i} \oint dz T(z)\epsilon(z) \tag{2.1.12}$$

(ingoring the anti-holomorphic part here "... + $\bar{T}(\bar{z})\bar{\epsilon}(\bar{z})$"). The handiness of this approach is the following. The symmetry generators require us to calculate "equal-time" commutators between the fields and conserved charges, given to us by contour integrals over the Energy-Momentum Tensor and the fields themselves

$$\delta_\epsilon \psi(w) = \frac{1}{2\pi i} \oint [dz T(z)\epsilon(z), \psi(w)] \tag{2.1.13}$$

In order to evaluate such infinitesimal symmetry transformation of any field, let us provide some further insight into the techniques of dealing with commutators on the conformal plane.

2.1.4 Commutators and Operator Product Expansion

In order to evaluate expressions involving commutators, we first need to define them. We note, that product of operators $A(z)B(w)$ is only defined if they are radially ordered (following from the requirement that in Euclidian space-time they need to be time ordered in order for the Green's function to converge), i.e. $|z| > |w|$. Radial ordering, conformal plane analogue to the time ordering in Euclidian space is defined as follows:

$$R(A(z)B(w)) = \begin{cases} A(z)B(w) & |z| > |w| \\ B(w)A(z) & |z| < |w| \end{cases} \tag{2.1.14}$$

Where if A and B are fermionic fields the latter case will pick up a minus sign. We now apply radial ordering to expressions of the sort given by Eqn2.1.13. Then the contour integral, when w is inside of the contour is evaluated first, and then the integral -

when it lies outside, effectively reducing the commutator to a single integral drawn tightly around w.

Then the "equal-time" commutator of the field operator with the contour integral of the EM tensor can be calculated by evaluating the contour integral of the radially ordered product of the field operator to the spatial integral of the EM Tensor. Using this definition, the infinitesimal symmetry variation in the field becomes:

$$\delta_\epsilon \psi(w) = \frac{1}{2\pi i}\left(\oint_{|z|>|w|} - \oint_{|z|<|w|}\right) dz\epsilon(z) R\left(T(z)\psi(w)\right) \tag{2.1.15}$$

In order for the charge 2.1.12 to induce correct infinitesimal conformal transformations, we deduce what short-distance singularities of the product $T(z)\psi(z)$ should be. From the properties of short-distance operator product expansion, we can also define the quantum EM tensor. The expansion leading to correct infinitesimal transformations takes the form:

$$T(z)\psi(w) = \frac{h}{(z-w)^2}\psi(w) + \frac{1}{z-w}\partial_w \psi(w) + \ldots \tag{2.1.16}$$

Here we drop the radial ordering symbol. This operator product expansion (OPE) defines the primary fields and encodes their conformal transformation properties.

However, there are fields that do not transform according to Eqn2.1.11 under conformal transformations. For example, derivatives of fields obey more complicated transformation properties. An example of such a field is the EM tensor, which as we will see later has a derivative of a field in it. An expansion for EM Tensor, similar to

OPE gien by Eqn2.1.16 is given by

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{z-w}\partial T(w) \tag{2.1.17}$$

Where coefficient c is known as *central charge* and is equal to 1/2 for fermionic fields. To compute Virasoro algebra, we now define Laurent's expansion of EM Tensor

$$T(z) = \sum_{n\in\mathcal{Z}} z^{-n-2}L_n \quad \text{with} \quad L_n = \frac{1}{2\pi i}\oint dz T(z) z^{n+1} \tag{2.1.18}$$

Using the OPE of the EM tensor given above, we can compute the commutators between the modes L_n and L_m

$$[L_m, L_n] = \left(\oint \frac{dz}{2\pi i}\oint \frac{dw}{2\pi i} - \oint \frac{dw}{2\pi i}\oint \frac{dz}{2\pi i}\right) z^{n+1}w^{n+1}T(z)T(w) \tag{2.1.19}$$

After computation, we get the algebra for the transformation generators:

$$[L_m, L_n] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n,-m} \tag{2.1.20}$$

we have the similar results for $\bar{L}_n$ and $\bar{L}_m$, with central charge and L_{n+m} conjugated to $\bar{c}$ and $\bar{L}_{n+m}$.

2.2 Majoranas in a Nutshell

Majorana particles have been long hypothesised to exist as fermions that are their own antiparticles [7]. As a consequence, Majorana particles can be described by single real-valued wavefunctions. Mathematically Majorana particles can be expressed as a superposition of fermionic creation and annihilation operators (or vice versa, we can define fermionic operators as a superposition of two Majorana particles):

$$\psi_{Majorana} \propto c^\dagger + c \tag{2.2.1}$$

It is evident from the definition that $\psi = \psi^\dagger$. We choose a normalisation where the continuous fields satisfy the anti-commutation relation given by

$$\{\psi(x), \psi(x')\} = \delta(x - x') \tag{2.2.2}$$

This, combined with the condition that Majoranas are their own anti-particles has an implication, that their Fourier transforms relate to each other by the following expression

$$\psi^\dagger(k) = \psi(-k). \tag{2.2.3}$$

Chapter 3

Hermitian Case

Having discussed the literature needed to investigate the emergent symmetries, we now move on to some specific examples. Here we will start by discussing the most general theory for superconductivity and then specify the parameters according to the materials of our interest. Namely, ones that support emergent Majorana modes on their surface. As an introduction to the 2+1D surface, we will first discuss the 1+1D surface case.

The discussion and formulae in the following chapter are from the sources:
[1]: Altland, Alexander, and Benjamin Simons. *Condensed Matter Field Theory*. 2nd ed. Cambridge: Cambridge University Press, 2010.
[2]: Chiu, Ching-Kai, Jeffrey C. Y. Teo, Andreas P. Schnyder, and Shinsei Ryu. *"Classification of Topological Quantum Matter with Symmetries."* Reviews of Modern Physics 88, no. 3 (2016): 035005.

3.1 2D BdG Hamiltonian and Surface States of 1D Chiral Fermions

The Bogaliubov-de Gennes Hamiltonian arises from the "mean-field" consideration of BCS theory. The BCS formalism is one of the most basic ways to capture the most essential physics of a thin shell around the Fermi surface, where the superconducting phenomenon occurs.

$$\hat{H} = \sum_{\boldsymbol{k}\sigma} \epsilon_{\boldsymbol{k}} \hat{n}_{\boldsymbol{k}} - \frac{g}{L^d} \sum_{\boldsymbol{k},\boldsymbol{k'},q} c^{\dagger}_{\boldsymbol{k}+q\downarrow} c^{\dagger}_{-\boldsymbol{k}\downarrow} c_{\boldsymbol{k'}-q\uparrow} c_{\boldsymbol{k'}\uparrow} \tag{3.1.1}$$

Where g is a positive constant, L is the length dimension of the system and d is the dimensions of the space. Assuming that the ground state $|\Omega_s\rangle$ is characterised by the presence of a macroscopic number of Cooper pairs. That is, $c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow}$ has non-vanishing GS expectation value for below the critical temperature T_c:

$$\Delta(\mathbf{k}) = \frac{g}{L^d} \left\langle \Omega_\mathbf{s} \left| c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow} \right| \Omega_\mathbf{s} \right\rangle, \quad \bar{\Delta}(\mathbf{k}) = \frac{g}{L^d} \left\langle \Omega_\mathbf{s} \left| c^{\dagger}_{\mathbf{k}\uparrow} c^{\dagger}_{-\mathbf{k}\downarrow} \right| \Omega_\mathbf{s} \right\rangle, \tag{3.1.2}$$

Those expectation values $\Delta(\mathbf{k})$ are called order parameters of the superconducting transition. For that, we note, that two fermion states $|\mathbf{k}\uparrow, -\mathbf{k}\downarrow\rangle$ has bosonic nature, thus $c^{\dagger}_{\mathrm{k}\uparrow} c^{\dagger}_{-\mathrm{k}\downarrow}$ is to be treated as an operator that creates bosonic state. The BdG Hamiltonian, after switching to a mean-field picture takes the form:

$$\hat{H} - \mu\hat{N} \simeq \sum_{\mathbf{k}} \left[\xi_{\mathbf{k}} c^{\dagger}_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} - \left(\bar{\Delta} c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow} + \Delta c^{\dagger}_{\mathbf{k}\uparrow} c^{\dagger}_{-\mathbf{k}\downarrow} \right) \right] \tag{3.1.3}$$

Where $\xi_{\mathbf{k}} = \epsilon_{\mathbf{k}} - \mu$. We now define two-component Nambu-spinors using fermionic operators, which then allows us to write Hamiltonian in matrix form in k-space

$$\Psi^{\dagger}_{\mathbf{k}} = \left(c^{\dagger}_{\mathbf{k}\uparrow}, c_{-\mathbf{k}\downarrow} \right), \quad \Psi_{\mathbf{k}} = \begin{pmatrix} c_{\mathbf{k}\uparrow} \\ c^{\dagger}_{-\mathbf{k}\downarrow} \end{pmatrix}, \tag{3.1.4}$$

Using "Nambu" operators, we can finally express Hamiltonian in k-space. For 2-dimensional problem we have $\vec{k} \in \mathcal{R}^2, \quad k = (k_x, k_z)$, and $k^2 = k_x^2 + k_z^2$. So the mean-field Hamiltonian for chiral "p+ip" SC [9] can be written in terms of 2D "p+ip" BdG Hamiltonian

$$\hat{H}_0 - \mu\hat{N} = \sum_{\mathbf{k}} \Psi_{\mathbf{k}}^{\dagger} H_{BdG}^{p+ip}(k) \Psi_{\mathbf{k}} \tag{3.1.5}$$

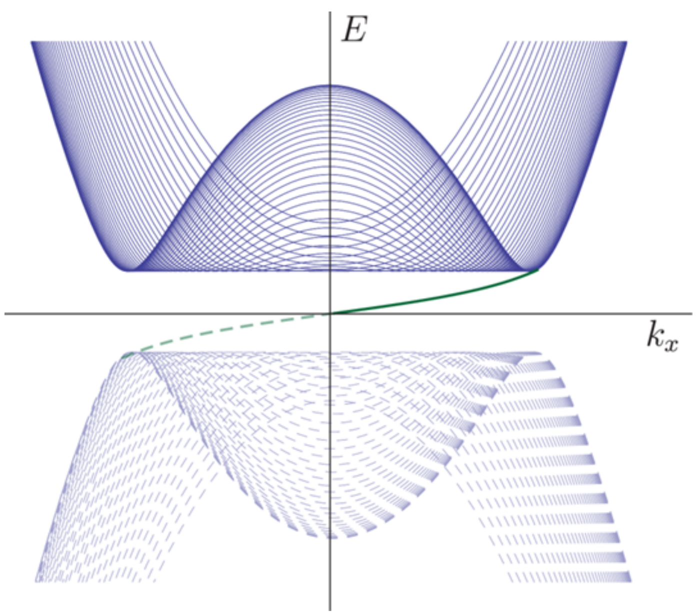
For "p+ip" type o SC we have $\Delta(\mathbf{k}) = \Delta(k_x + ik_z)$, where now Δ is just some complex parameter. Then the BdG Hamiltonian in k-space can be written:

$$H_{BdG}^{p+ip}(k) = \left(\frac{k^2}{2m} - \mu\right)\tau_z + \Delta k_x \tau_x + \bar{\Delta} k_z \tau_y \tag{3.1.6}$$

and τ_x, τ_y, τ_z are pauli matrices. The positive energy for Eqn3.1.6 is

$$E(k) = \sqrt{\left(\frac{k^2}{2m} - \mu\right)^2 + \Delta^2\left(k_x^2 + k_z^2\right)} \tag{3.1.7}$$

The parameter $|\Delta|$ can be identified as the gap between two energy branches.



q

Figure 3.1: In this figure the Energy branches are plotted. The green lines represent Majorana particles with linear dispersion at low k values.

The Hamiltonian that describes the edge modes represented by the green line in Fig. 3.1 is given by the τ_x component of k-space BdG:

$$H_{edges}(k_x) = \Delta k_x \tau_x \tag{3.1.8}$$

This edge Hamiltonian includes both left and right-moving excitations. In the next section, we will focus on only one of the chiralities.

3.2 Chiral Majorana Fermions on 1+1D Boundary

Figure 3.2: The left and right moving Majorana modes represented by left and right arrows on the edges of SC respectively.

For 2 dimensional SC we have boundaries (or edges) specified by x-coordinate. Now we assume that those modes are not interacting. We also drop the spin indices since from now on we will be assuming the spinless/spin-polarised fermions. Then the analogue of Eqn2.2.1 for the edge-Majorna excitations expressed in terms of the fermionic operators in k-space become:

$$\psi(k_x) \propto \int dz\{\phi^\dagger(z)c^\dagger_{k_x}(z) + \phi(z)c_{-k_x}(z)\} \tag{3.2.1}$$

Where $\phi(z)$ is a rapidly decaying function with respect to increasing z from the surface. This allows us to arrive at the low-energy effective Hamiltonian operator on the edges for the Majorana excitations

$$\hat{H}_{edge} = \sum_{k_x} \Delta k_x \psi^{\dagger}_{k_x} \psi_{k_x} \tag{3.2.2}$$

Observing the linear dispersion relation on the boundaries, we write the boundary Hamiltonian density in coordinate representation:

$$\mathcal{H} = -\frac{i\hbar v}{2} \psi \partial_x \psi \tag{3.2.3}$$

We seek to find the equations of motion. For that, we first introduce the Fourier transform for the Majorana operators. Generally, Majoranas obey anti-periodic boundary conditions: $\psi(x+L) = -\psi(x)$. With anti-periodicity discretising the momentum and for $x \to x+L$ giving an overall minus sign, we carry out discrete summation over half-integers, i.e. $m \in \mathcal{Z} + 1/2$

$$\begin{gathered}\psi(x) = \frac{1}{\sqrt{L}} \sum_{m \in \mathbb{Z}+1/2} e^{\frac{2\pi i}{L} m x} \psi_m, \\ \psi_m = \frac{1}{\sqrt{L}} \int_0^L dx e^{-\frac{2\pi i}{L} m x} \psi(x)\end{gathered} \tag{3.2.4}$$

Following these definitions, anti-commutation relations in the momentum space become

$$\{\psi_m, \psi_{m'}\} = \delta_{m,-m'} \tag{3.2.5}$$

Using the Fourier transform given above, momentum-space anti-commutation relations, and discrete version of condition 2.2.3: $\psi_m^\dagger = \psi_{-m}$ we can check that the Hamiltonian density in Eqn3.2.3 gives the correct k-space BdG edge hamiltonian:

$$H = \sum_{m,n} \frac{\pi\hbar v}{L} m\psi_m^\dagger \psi_m \tag{3.2.6}$$

Note that the Hamiltonian provided here is very same as the Hamiltonian given in Eqn3.2.2, with specified momentum and gap parameter. Now, proceeding with our task of finding the equations of motion for Majorana fermions, we write the Heisenberg equation of motion to find time evolution for fermionic operators ψ_m:

$$\begin{aligned} \frac{d\psi_m}{dt} &= \frac{1}{i\hbar}[\psi_m, H] \\ &= -\frac{2\pi i v m}{L}\psi_m \end{aligned} \tag{3.2.7}$$

The solution to the above differential equation is trivial, thus for the momentum-space operators ψ_m we have the following time dependence:

$$\psi_{mn}(t) = e^{-\frac{2\pi i m}{L} vt}\psi_m \tag{3.2.8}$$

Plugging the above result in Fourier transformation we get Majorana fields as a function of coordinate and time

$$\psi(x,t) = \psi(x - vt) = \frac{1}{\sqrt{L}} \sum e^{\frac{2\pi i m}{L}(x-vt)}\psi_m \tag{3.2.9}$$

For simplicity let's choose $v = 1$. The equation of motion for such a field is

$$(\partial_x + \partial_t)\,\psi(x,t) = 0 \tag{3.2.10}$$

To investigate the symmetries of 1+1D Fermions described by the equation above, we perform a Wick rotation: $\tau = it$. Under which the above expression becomes:

$$\psi(x,t) = \frac{1}{\sqrt{L}} \sum e^{\frac{2\pi i m}{L}(x+i\tau)} \psi_m \tag{3.2.11}$$

We can now recognise the patterns that lead to the conformal transformation of the fields. Namely, transformation that will parameterise coordinates x, τ onto the cylinder and then map them to the complex plane.

First, from flat Euclidian space and time, we map our fields to the cylinder $\zeta, \bar{\zeta} = \tau \mp ix$. Then we consider conforomal mapping $\zeta \to z = exp(\frac{2\pi i}{L}\zeta)$. Under such mapping conformal field transformation is given by:

$$\psi_{cyl}(\zeta) = \left(\frac{dz}{d\zeta}\right)^h \phi_{pl}(z(\zeta)) \tag{3.2.12}$$

With conformal weight $h = \frac{1}{2}$ for fermions, dropping the subscript "pl" from now on, we can now express the field on the complex plane as:

$$\psi(z) = \frac{1}{\sqrt{L}} \sum_{m \in \mathbb{Z}+1/2} z^{m-\frac{1}{2}} \psi_m \tag{3.2.13}$$

Now we calculate the ground state correlation function between two time-ordered fields. Time ordering here would mean. For that, we define the ground state as:

$$\psi_m \left|GS\right\rangle = 0, \quad \text{for} \quad m < 0 \tag{3.2.14}$$

Time-ordered product for two conformal fields $\psi(z)$ and $\psi(w)$ is the same as radial ordering on complex plane for the variables z and w, i.e. $|z| > |w|$

$$\begin{aligned}
\langle\psi(z)\psi(w)\rangle &\equiv \langle GS|\psi(x,t),\psi(x',t')|GS\rangle \\
&= \sum_{m,m'\in\mathbb{Z}+1/2} \langle GS|z^{m+\frac{1}{2}}w^{m'-\frac{1}{2}}\psi_m\psi_{m'}|GS\rangle \\
&= \sum_{m,m'\in\mathbb{Z}+1/2} z^{m+\frac{1}{2}}w^{m'-\frac{1}{2}}\left\langle\psi_m\psi_{m'}\right\rangle \\
&= \sum_{m\in\mathbb{Z}+1/2} z^{-m-\frac{1}{2}}w^{m-\frac{1}{2}} \\
&= \frac{1}{z-w}
\end{aligned} \tag{3.2.15}$$

Where in the last line we have used the fact that variables are radially ordered, implying convergence of geometric series everywhere. Note that the summation will be carried out over positive m due to the choice of GS.

3.3 EM Tensor

The EM tensor, generally, can be calculated from Noether's theorem by assuming the invariance of the action with respect to space-time translations. To set up the discussion, we will briefly write out the result of Noether's theorem and then show what the EM tensor looks like in complex coordinates.

Introducing symmetry transformations for the coordinates and the fields:

$$\begin{aligned} x'^{\mu} &= x^{\mu} + \omega_a \frac{\delta x^{\mu}}{\delta \omega_a} \\ \Phi'\left(\boldsymbol{x}'\right) &= \Phi(\boldsymbol{x}) + \omega_a \frac{\delta \mathcal{F}}{\delta \omega_a}(\boldsymbol{x}) \end{aligned} \tag{3.3.1}$$

Where ω_a are the infinitesimal symmetry generators, we can write out the conserved current for a given Lagrangian:

$$j_a^{\mu} = \left\{ \frac{\partial \mathcal{L}}{\partial\left(\partial_{\mu}\Phi\right)} \partial_{\nu}\Phi - \delta_{\nu}^{\mu}\mathcal{L} \right\} \frac{\delta x^{\nu}}{\delta \omega_a} - \frac{\partial \mathcal{L}}{\partial\left(\partial_{\mu}\Phi\right)} \frac{\delta \mathcal{F}}{\delta \omega_a} \tag{3.3.2}$$

If we assume simple space-time translations, that is: $x \rightarrow x^{\mu} + \epsilon^{\mu}$, or for Eqn3.3.1, that would be: $\frac{\delta x^{\mu}}{\delta \omega_a} = \delta_{\nu}^{\mu}$, we gat what is known as EM tensor.

We now write down the action with both chiralities:

$$S = \frac{1}{8\pi} \int (\psi \bar{\partial} \psi + \bar{\psi} \partial \bar{\psi}) \tag{3.3.3}$$

We can calculate EM tensor by taking $\mu = 0, 1$ for z and $\bar{z}$ in Noethers theorem to compute

$$T(z) = -\frac{1}{2} \psi \partial_z \psi \tag{3.3.4}$$

Or it's anti-holomorphic counterpart.

Chapter 4

2+1D non-Hermitian Case

We now move on to the discussion of 3D superconductors with 2+1D surfaces. This type of superconductor exhibits different types of symmetries, however, as we will see later they still support Majorana edge modes. The topological classification for non-Hermitian band Hamiltonians had been previously studied by Gong et al. [5].

4.1 Combined Symmetries in non-Hermitian Systems

The BdG Hamiltonian in its most general form written in terms of fermionic operators is given by

$$\hat{H} = \int \frac{d^d k}{(2\pi)^d} \begin{pmatrix} c^\dagger_{\mathbf{k}} & c_{-\mathbf{k}} \end{pmatrix} H(k) \begin{pmatrix} c_{\mathbf{k}} \\ c^\dagger_{-\mathbf{k}} \end{pmatrix} \tag{4.1.1}$$

with,

$$H(k) = \begin{pmatrix} h_{11}(\mathbf{k}) & \Delta_{12}(\mathbf{k}) \\ \Delta_{21}(\mathbf{k}) & h_{22}(\mathbf{k}) \end{pmatrix} \tag{4.1.2}$$

For non-hermitian systems, we have the ”Nambu” particle-hole symmetry so that the

BdG Hamiltonian obeys

$$H(k) = -U_c H(-k)^T U_c^{-1}, \quad \text{with} \quad U_C = \tau_x = \begin{bmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{bmatrix}. \tag{4.1.3}$$

This identity is equivalent to the anti-commutation relations satisfied by the fermion operators.

4.2 2+1D Boundary non-Hermitian BdG Hamiltonian

For this section, we now consider three-dimensional topological superconductor. In Fig.4.1 we can see what the geometry of 3D "p+ip" SC looks like. We have stacks of 2D "p+ip" SC-s, presented by layers in $x-z$ plane ($\mathbf{k} = (k_x, k_z)$). We have a driven system, that supports hopping along the y axis but not in the opposite direction.

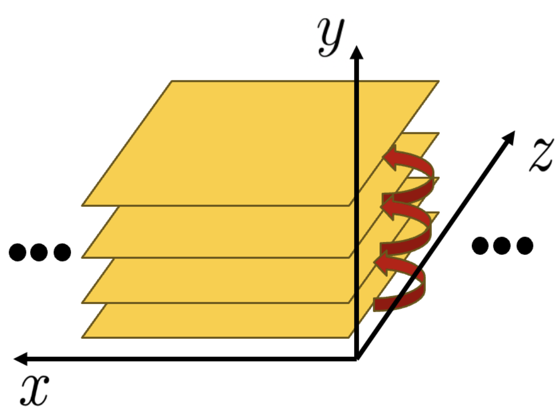


Figure 4.1: Depiction of 3-dimensional "p+ip" SC. Each layer represents 2-dimensional "p+ip" SC. We allow electrons to hop from one layer to another, however, we restrict the motion alongside the y-axis, i.e. electrons can only move upwards.

We now include the hopping term in the Hamiltonian and follow the similar procedure of first expressing it in k-space and then writing it in terms of Nambu spinors:

$$\begin{aligned}\hat{H}-\mu\hat{N} &= \sum_y \left(\hat{H}_0-\mu\hat{N}+t\sum_{\mathbf{k}} c^\dagger_{\mathbf{k},y+1}c_{\mathbf{k},y}\right)\\ &= \sum_{\mathbf{k}} \Psi^\dagger_{\mathbf{k},k_y}\left[H^{p+ip}_{BdG}(\mathbf{k})+t\cos(k_y)\tau_z+it\sin(k_y)\mathbb{I}\right]\Psi_{\mathbf{k},k_y}\end{aligned} \tag{4.2.1}$$

Finally, we can write the new BdG Hamiltonian with the hopping:

$$H^{NH}(\mathbf{k},k_y)=H^{p+ip}_{BdG}(\mathbf{k})+t\cos(k_y)\tau_z+it\sin(k_y)\mathbb{I}, \tag{4.2.2}$$

where H^{p+ip}_{BdG} is the BdG Hamiltonian for the 2D hermitian $p+ip$ superconductor presented in Eqn3.1.6, and $\Psi^\dagger_{\mathbf{k},k_y}=(c^\dagger_{\mathbf{k},k_y},c_{-\mathbf{k},-k_y})$ is the Nambu fermion vector.

We know that the first term gives us the linear disperssion 3.2.2 and the corresponding boundary Hamitonian discussed in section 3.2. We now consider the hopping term in better detail

$$t\sum_{k,y}c^\dagger_{k,y+1}c_{k,y}=\frac{1}{2}t\sum_{k,y}c^\dagger_{k,y+1}c_{k,y}-c_{k,y}c^\dagger_{k,y+1} \tag{4.2.3}$$

Introducing a very similar cutoff as in Eqn3.2.1, however, we now do it for each stack

$$\psi^y(k_x)\propto\int dz\{\phi^\dagger(z)c^{y\dagger}_{k_x}(z)+\phi(z)c^y_{-k_x}(z)\} \tag{4.2.4}$$

So the hopping part of the Hamiltonian becomes proportional to

$$\tilde{t}\sum_{k_x,y}\psi^{y+1\dagger}_{k_x}\psi^y_{k_x} \tag{4.2.5}$$

resulting in the non-hermitian chiral Majorana fermion on the surface boundary described the non-hermitian Hamiltonian density near $k_y = 0$

$$\mathcal{H} = -\frac{i\hbar v}{2}\psi(\partial_x + i\partial_y)\psi \tag{4.2.6}$$

There is another chiral Majorana fermion with the opposite handedness $(\partial_x - i\partial_y)$ near $k_y = \pi$.

4.3 Calculations for 2+1D Product of Operators

Now for the 2+1D boundary, the Majorana particles are parameterised on a torus:

$$\psi = \psi(x,y), \quad (x,y) \in [0,L]^2 \qquad \text{and} \qquad \psi(x+L,y) = -\psi(x,y) = \psi(x,y+L) \tag{4.3.1}$$

They satisfy the anti-commutation relations:

$$\{\psi(x,y), \psi(x',y')\} = \delta^{(2)}(x-x', y-y') \tag{4.3.2}$$

And their Fourier transform could be of the form:

$$\psi(x,y) = \frac{1}{L}\sum_{m,n\in\mathbb{Z}+1/2} e^{\frac{2\pi i}{L}(mx+ny)}\psi_{mn},$$

$$\tag{4.3.3}$$

$$\psi_{mn} = \int_0^L dxdy e^{-\frac{2\pi i}{L}(mx+ny)}\psi(x,y)$$

Anti-commutation relation becomes:

$$\{\psi_{mn}, \psi_{m'n'}\} = \delta_{m,-m'}\delta_{n,-n'} \tag{4.3.4}$$

Followin the very same procedure as in sec3.2, we find the Hamiltonian in k-space:

$$H = \sum_{m,n} -\frac{\pi\hbar v}{L}(m + in)\gamma^{\dagger}_{mn}\gamma_{mn} \tag{4.3.5}$$

However, note that the Energy here takes complex values. For our discussion, it is of peculiar interest since the complex energy values no longer allow us to explicitly define the GS. In fact, the GS no longer exists since we can no longer have the minimum energy of the system. What we have now are families of steady states.

Following up on a procedure, we determine the time evolution of the fermionic operators ψ_{mn}:

$$\begin{aligned}\frac{d\psi_{mn}}{dt} &= \frac{1}{i}[\psi_{mn}, H] \\ &= -\frac{2\pi i v(m + in)}{L}\psi_{mn}\end{aligned} \tag{4.3.6}$$

Again, we set $v = 1$, and after solving the differential equation given above, we arrive at the fields as a function of time:

$$\psi(x, y, t) = \frac{1}{L}\sum e^{\frac{2\pi i}{L}(m(x-t)+n(y-it))}\psi_{mn} \tag{4.3.7}$$

The equation of motion that Field in Eqn4.3.5 satisfies is:

$$\left(\partial_x + i\partial_y + \partial_t\right)\psi(x, y, t) = 0 \tag{4.3.8}$$

Here we see that conformal symmetry naturally arises for the y and t. However, for a pair of variables of x and t Wick's rotation is still necessary to map the fields onto the cylinder. If we were to complexify time, we see that it would mess up the y and t pair. So now we choose to complexify x direction and introduce: $\mathcal{X} = ix$. That would allow us to define a pair of variables:

$$\eta = iy + t \quad and \quad \zeta = \mathcal{X} - it \tag{4.3.9}$$

The field expansion on the complex plane is then proportional to:

$$\sum_{m,n \in \mathbb{Z}+1/2} z^{m-\frac{1}{2}} w^{n-\frac{1}{2}} \tag{4.3.10}$$

with $z \equiv \exp(\frac{2\pi}{L}\zeta)$ and $w \equiv \exp(\frac{2\pi}{L}\eta)$

Now for $|z| > |z'|$ and $|w| > |w'|$, we calculate expectation value for the product of operators for the steady state ($\langle SS|...|SS\rangle$):

$$\begin{aligned}
\langle \psi(z,w)\psi(z',w')\rangle \equiv{}& \langle SS|\psi(x,y,t), \psi(x',y',t')|SS\rangle \\
={}& \sum_{m,m',n,n' \in \mathbb{Z}+1/2} \langle SS| z^{m+\frac{1}{2}} w^{n-\frac{1}{2}} \tilde{z}^{m'+\frac{1}{2}} \tilde{w}^{n'-\frac{1}{2}} \psi_{mn}\psi_{m'n'}|SS\rangle \\
={}& \sum_{m,m',n,n' \in \mathbb{Z}+1/2} z^{m+\frac{1}{2}} w^{m'-\frac{1}{2}} \tilde{z}^{m'+\frac{1}{2}} \tilde{w}^{n'-\frac{1}{2}} \langle \psi_m \psi_{m'}\rangle \\
={}& \sum_{m,n \in \mathbb{Z}+1/2} z^{-m-\frac{1}{2}} w^{m-\frac{1}{2}} \tilde{z}^{m+\frac{1}{2}} \tilde{w}^{n-\frac{1}{2}} \\
={}& \sum_{m,n \in \mathbb{Z}+1/2} \frac{1}{\sqrt{z\tilde{z}}} \frac{1}{\sqrt{w\tilde{w}}} \left(\frac{\tilde{z}}{z}\right)^m \left(\frac{\tilde{w}}{w}\right)^n
\end{aligned} \tag{4.3.11}$$

Again, we note that the $|SS\rangle$ does not correspond to the lowest energy state, but to the family of steady states. If we examine the sum in Eqn4.3.11, we will see that for any choice of states that are annihilated by ψ_{mn}, the sum will diverge. That is because the family of such states should be exactly the half of the states, so no matter how we choose the half of the k-space, either, sum over n, or sum over m will give us terms that diverge.

However, formally we can recognise the same pattern and not yet specify the range of values m and n take, the product of operators should yield:

$$\psi(z,w)\psi(z',w') = \frac{1}{z-\tilde{z}}\frac{1}{w-\tilde{w}} \tag{4.3.12}$$

This can also be verified by dimension analysis. The field has a scaling dimension of $1/length$ in 2+1D. From dimension analysis we can predict that a more general form of the expectation value could be of the form:

$$\psi(z,w)\psi(z',w') = \frac{(w-\tilde{w})^{(n-1)}}{(z-\tilde{z})^{(n+1)}} + \frac{(z-\tilde{z})^{(n-1)}}{(w-\tilde{w})^{(n+1)}} \tag{4.3.13}$$

We note that the expression is symmetric under the exchange of variables $z, \tilde{z}$ and $w, \tilde{w}$.

Chapter 5

Conclusions

In this thesis we first discussed some insights from the conformal field theory in 2-dimensions, to establish a mathematical basis for investigating the symmetries of emergent Majorana particles on the 1+1D edges of 2-dimensional "p+ip" superconductor. Later, after discussing the theory of superconductivity in its most general form, we showed how those edge modes can be manifested in certain types of SC-s. Then, having 1+1D edge modes investigated, using similar procedures and techniques we have proposed calculations for 2+1D steady state expectation value.

Bibliography

[1] Alexander Altland and Benjamin Simons. *Condensed Matter Field Theory*. 2nd. Cambridge: Cambridge University Press, 2010. ISBN: 978-0521769754.

[2] Ching-Kai Chiu et al. "Classification of topological quantum matter with symmetries". In: *Rev. Mod. Phys.* 88 (3 Aug. 2016), p. 035005. DOI: 10.1103/RevModPhys.88.035005. URL: https://link.aps.org/doi/10.1103/RevModPhys.88.035005.

[3] Philippe Di Francesco, Pierre Mathieu, and David Sénéchal. *Conformal Field Theory*. New York: Springer, 1997. ISBN: 978-0387947853.

[4] Paul Ginsparg. *Applied Conformal Field Theory*. 1988. arXiv: hep-th/9108028 [hep-th]. URL: https://arxiv.org/abs/hep-th/9108028.

[5] Zongping Gong et al. "Topological Phases of Non-Hermitian Systems". In: *Physical Review X* 8.3 (Sept. 2018). ISSN: 2160-3308. DOI: 10.1103/physrevx.8.031079. URL: http://dx.doi.org/10.1103/PhysRevX.8.031079.

[6] A.Yu. Kitaev. "Fault-tolerant quantum computation by anyons". In: *Annals of Physics* 303.1 (2003), pp. 2–30. ISSN: 0003-4916. DOI: https://doi.org/10.1016/S0003-4916(02)00018-0. URL: https://www.sciencedirect.com/science/article/pii/S0003491602000180.

[7] Ettore Majorana and Luciano Maiani. "A symmetric theory of electrons and positrons". In: *Ettore Majorana Scientific Papers: On occasion of the centenary of his birth*. Ed. by Giuseppe Franco Bassani. Berlin, Heidelberg: Springer Berlin

Heidelberg, 2006, pp. 201–233. ISBN: 978-3-540-48095-2. DOI: 10.1007/978-3-540-48095-2_10. URL: https://doi.org/10.1007/978-3-540-48095-2_10.

[8] Ken Shiozaki and Masatoshi Sato. “Topology of crystalline insulators and superconductors”. In: *Physical Review B* 90.16 (Oct. 2014). ISSN: 1550-235X. DOI: 10.1103/physrevb.90.165114. URL: http://dx.doi.org/10.1103/PhysRevB.90.165114.

[9] Grigory E. Volovik. *The Universe in a Helium Droplet*. Oxford University Press, Feb. 2009. ISBN: 9780199564842. DOI: 10.1093/acprof:oso/9780199564842.001.0001. URL: https://doi.org/10.1093/acprof:oso/9780199564842.001.0001.