Parallels Between Heegaard Splittings and Trisections of 4-manifolds

Gabriel Islambouli

A Dissertation Presented to the Graduate Faculty
of the University of Virginia in Candidacy for the Degree of
Doctor of Philosophy

Department of Mathematics

University of Virginia

May, 2019

Abstract

For over a century, Heegaard splittings have been recognized as a useful way to describe a 3-manifold. In 2016, Gay and Kirby [14] introduced a new decomposition of 4-manifolds called a trisection. They showed that the theory of trisections has many parallels with the theory of Heegaard splittings, including a diagrammatic theory and a stable equivalence theorem.

This dissertation develops the analogies between the theories further in two directions. In one direction, we show that invariants of 3-manifolds defined using Heegaard splittings can be adapted in order to provide invariants of 4-manifolds. More precisely, given two smooth, oriented, closed 4-manifolds, M_1 and M_2, we adapt work of Johnson [25] to construct two invariants, $D^P(M_1, M_2)$ and $D(M_1, M_2)$, coming from distances in the pants complex and the dual curve complex, respectively. Our main results are that the invariants are independent of the choices made throughout the process, as well as interpretations of "nearby" manifolds.

In another direction, we show that tools used to distinguish Heegaard splittings of a 3-manifold can be adapted to distinguish trisections of 4-manifolds. As a result, we exhibit the first examples of inequivalent trisections. We in fact show that, for every $k \geq 2$, there are infinitely many manifolds admitting $2^k - 1$ non-diffeomorphic $(3k, k)$-trisections. Here, the manifolds are spun Seifert fiber spaces and the trisections come from Meier's spun trisections [34].

Acknowledgments

First and foremost, I would like to thank my advisor, Slava Krushkal. Without his patience and guidance, this work would not have been possible. The faculty at the University of Virginia have been incredibly welcoming. Among them, I would in particular like to thank Thomas Koberda, Sara Maloni, and Tom Mark for exposing me to many beautiful areas of mathematics. I would also like to thank Michael Willis, Jonathan Simone, Brian Thomas, and Matthew Gagne for many engaging conversations during mathematically formative years. The community of trisectors has been incredibly encouraging and energizing. In particular, I'd like to thank David Gay, Jeff Meier, and Alex Zupan for taking me under their wing when I was just learning the subject. Personally, I owe a great debt to Amanda, Marguerite, Oliver, and my parents for their unending support.

Contents

1 Decomposing manifolds via smooth functions **8**

1.1 Preliminary definitions and Conventions 8

1.2 Cerf Theory and Modifying Handle Decompositions 10

1.2.1 Homotopies of Cerf Functions 12

1.2.2 Handlebodies and Spines 15

1.3 Heegaard Splittings of 3-Manifolds 19

1.3.1 Stabilization of Heegaard Splittings 23

1.4 Trisections of 4-Manifolds 25

1.4.1 Existence and uniqueness 31

1.4.2 Relation to Handle decompositions 32

1.4.3 Stabilizations 35

2 Distinguishing Decompositions of Manifolds **40**

2.1 Nielsen equivalence 40

2.2 Nielsen equivalence in Fuchsian groups 43

2.2.1 Nielsen equivalence and automorphisms 46

2.3 Spines of handlebodies and Nielsen equivalence 47

2.4 Invariants of decompositions of manifolds 49

2.5 Seifert fiber spaces and their Heegaard splittings 52

2.6 Meier's spun trisections 57

2.7 Spinning Seifert fiber spaces 62

2.7.1 Spun small Seifert fiber spaces 63

3 Invariants Coming from the Pants Complex 66

3.1 Introduction 66

3.1.1 Simplicial Complexes Associated to Surfaces 66

3.2 Distances of handlebodies, Heegaard splittings and 3-manifolds 70

3.3 Distances of trisections 72

3.4 Some Bounds 80

3.5 Nearby Manifolds 83

3.6 Complexes of Trisections 92

3.7 Quadrisected embeddings 95

List of Figures

1-1 An example of a Cerf Graphic. Handles are labeled according to their index. Cusps correspond to handle creations/cancellations. 12

1-2 A Reidemeister 2 move on a Cerf graphic. 13

1-3 A Reidemeister 3 move on a Cerf graphic. 14

1-4 A cusp poke on a Cerf graphic. 14

1-5 A cusp merge on a Cerf graphic. 14

1-6 An eye birth on a Cerf graphic. 14

1-7 Flipping an indefinite fold of 4-dimensions. Red arrows indicate where the next move will be taking place, and black arrows indicate the direction in which the genus of the fiber decreases 16

1-8 The process of sliding an edge which represents the generator a over an edge representing the generator b produces a spine which has edges representing the generators ab and b. 18

1-9 Left: A Heegaard diagram for $S^1 \times S^2$. Right: A Heegaard diagram for S^3. 22

1-10 A 3-dimensional 2-handle sliding over another 2-handle. The attaching regions are shaded in red. 23

1-11 A schematic of a trisection. Each X_i is diffeomorphic to a 4-dimensional handlebody and each H_{ij} is diffeomorphic to a 3-dimensional handlebody. The three H_{ij} meet in a closed surface indicated by a dot in the center of the trisection. Any two of the H_{ij} form a Heegaard splitting for some connected sum of copies of $S^1 \times S^2$. 27

1-12 Reconstructing a 4-manifold from a tripod. We first thicken the whole tripod into a 4-dimensional manifold with boundary. The boundary can then be capped off in a unique manner. 28

1-13 The genus 1 trisections. Top Left: S^4. Top Right: $S^1 \times S^3$. Bottom Left: $\mathbb{C}P^2$. Bottom Right: $\overline{\mathbb{C}P^2}$ 30

1-14 A trisected Cerf function. The white blocks are smoothly just products, and while they may contain many crossings, they contain no cusps. A $(g-k)-$trisection corresponds to a function where each sector has k folds without cusps and $g-k$ folds with cusps. 33

1-15 Extracting a handle decomposition from a Cerf graphic. After projecting onto the horizontal axis, each vertical tangency becomes a critical point of the index indicated below. The red lines are the delineations of the sectors. 34

1-16 The (3,1)-trisection of S^4 used to define the stabilization operation. . 36

1-17 A 2-stabilization from the perspective of a Morse 2-function. 38

1-18 The neighbourhood of a boundary parallel arc in a 3-dimensional handlebody. 38

2-1 The graph describing the vertical Heegard Splitting $\Sigma(1, 2, 5)$. Note that the exceptional fibers f_1, f_2, and f_5 are included in this graph, but are not pictured. 55

2-2 Crossing a punctured Heegaard splitting with S^1 produces a Cerf function to the annulus. The map can be extended to the disk after capping the 4-manifold off with $S^2 \times D^2$ 58

2-3 Left: The trisected Morse 2 function for $S(\Sigma)$. Right: The portion without cusps of X_i can be identified with $H_1 \times I$ 59

2-4 Top: A schematic picture of spines of H_1 of the vertical Heegaard splittings of S(0,3). Horizontal tubes lie in a neighborhood of the base sphere while vertical tubes are neighborhoods of fibers. Bottom: A spine of a genus 3 handlebody which contains the spines of both of the vertical Heegaard splittings above. 64

3-1 Above: An S-move in the pants complex. Below: An A-move in the pants complex. 68

3-2 The distance between two trisections is the minimum distance between the sets $\hat{\phi}(H^*_{\gamma_1})$ and $H^*_{\gamma_2}$ 74

3-3 Left: Stabilizing a trisection amounts to gluing this diagram onto a punctured trisection diagram. Right: A pants decomposition for one handlebody of the stabilizing surface 75

3-4 $S^2 \times S^2$ and $\widetilde{S^2 \times S^2}$ are distance one in the pants complex 83

3-5 A schematic of the construction in Theorem 3.5.3 87

3-6 Resolving a Reidemeister 2 move of the attaching link on the Heegaard Surface 89

3-7 Resolving a Reidemeister 1 move to change the surface framing of l_1 by 1. Parallel curves such as l_2 can be sent over the stabilizing surface without twisting about α_i to preserve the surface framing 91

3-8 Dehn twisting the sliding arc about the target curve does not change the isotopy type of slid curve. 91

3-9 S^4 (shown on the left) is adjacent to $S^2 \times S^2$ (shown on the right) in $P(2,0)$. 95

3-10 A schematic of a quadrisection. We will be interested in the 3-manifold given by the Heegaard splitting $H_{12} \cup H_{34}$ 96

3-11 A (1;0,0,0,0)-quadrisection of $\mathbb{R}P^3$ in $S^2 \tilde{\times} S^2$. A Heegaard diagram for $\mathbb{R}P^3$ can be seen in the purple and green curves. 97

3-12 A (2;0,0,0,0)-quadrisection of L(7,2) in $\sharp^2 S^2 \times S^2$ 99

Introduction

The topology of smooth 4-dimensional manifolds is one of the most rich and yet poorly understood areas in low dimensional topology. The complexity here has origins in Milnor's discovery of manifolds homeomorphic, but not diffeomorphic to the 7-sphere [37], which showed how the category of smooth manifolds and the category of topological manifolds can diverge. Nowhere is this difference more pronounced than in dimension four. Here, following work of Freedman [13], simply connected topological 4-manifolds are well understood, whereas, by contrast, there is not even a clear conjectural picture for simply connected smooth 4-manifolds, with work of Donaldson [9] showing that the smooth and topological categories widely differ here. In many ways, smooth 4-manifolds are the final frontier for smooth manifold theory, as geometric techniques have lead to great insights into the structure of 2- and 3-dimensional manifolds, and, in dimensions greater than four, surgery theoretic techniques have had great success.

Due to the success of mathematics in treating 2- and 3-dimensional manifolds, as well as the inherent difficulty in visualizing objects in 4-dimensions, smooth 4-manifold theory often attempts to proceed by analogy with lower dimensional objects. Much of this dissertation will be focused on this very approach, in particular drawing analogies with dimension three. One of the most basic ways to describe a 3-dimensional object is through a Heegaard splitting. This construction is the starting point of many modern approaches in 3-manifold topology; underpinning the theories of Heegaard Floer homology [43], thin position for 3-manifolds [47], and Dehn surgery on links [52] [31].

A Heegaard splitting is a decomposition of a 3-manifold into two 3-dimensional 1-handlebodies, each of which individually contains very little topological information beyond its genus. The complexity of the 3-manifold then arises once we glue these pieces together along their boundary. This leads to a rich interplay between the theory of Heegaard splittings and the theory of mapping class groups of surfaces which far from being completely understood, even in light of the recent progress on the topology of 3-dimensional manifolds.

Recently, Gay and Kirby [14] introduced a promising new perspective on smooth 4-manifolds called a trisection, which serves as a bridge between the unknown world of 4-dimensions and the generally well understood world of 2- and 3-dimensional manifolds. Much like a Heegaard splitting, a (g, k)-trisection is a decomposition a a smooth, closed 4-manifold into three genus g 4-dimensional 1-handlebodies, where each pair meets along a genus k 3-dimensional 1-handlebody. Again the complexity is moved from the topology of these pieces into the way they are glued together. In addition, Gay and Kirby give a diagrammatic theory for trisections akin to the diagrammatic theory of a Heegaard splitting, which takes place entirely on a 2-dimensional surface, intertwining constructions on surfaces with constructions on 4-manifolds. The overall theme of this dissertation is pushing the analogies between these two theories further.

In attempting to understand a fixed manifold by its Heegaard splittings or trisections, the first task is to relate the different ways one can decompose the manifold. Herein lies one of the most powerful similarities between the theories. For both Heegaard splittings and trisections, there is a simple operation, called stabilization, which relates any two decompositions. More precisely, any two decompositions of a fixed manifold become isotopic after some number of stabilizations. This gives one a straightforward path for relating properties of the decompositions to properties of the manifolds themselves; namely, one defines some property of the decomposition and analyzes how the property changes when one stabilizes.

The stable equivalence of these decompositions gives the set of Heegaard splittings of a 3-manifold, or trisections of a 4-manifold, the structure of a connected tree, with each vertex corresponding to an isotopy class of a decomposition with two decompositions joined by an edge if they are related by a stabilization. Much work has been done on the structure of this Heegaard tree for individual manifolds. Perhaps the most famous in a large collection of such results is Waldhausen's theorem [50], which states that S^3 has a unique Heegaard splitting in each genus, up to isotopy. In a similar vein, Bonahon and Otal have shown that lens spaces have a unique splitting in each dimension [5]. The first examples of non-isotopic Heegaard splittings of the same genus came in 1970, when Engmann constructed two non-isotopic Heegaard splittings of genus 2 on the connected sum of two lens spaces [10]. By contrast, in part due to the recency of the theory, very little is known about the structure of the trisection tree. Unlike Heegaard splittings, there is no known bound on the number of stabilizations needed to make two trisections of the same genus isotopic, nor any manifolds which have their trisection tree understood.

In Chapter 2 of this dissertation, we analyze the structure of the trisection tree for certain manifolds. In particular, we construct the first examples of trisections of a fixed manifold of the same genus which are not isotopic. To do this, we will extend techniques which work for Heegaard splittings of 3-manifolds to trisections of 4-manifolds. More precisely, to each isotopy class of trisections, we associate 3 Nielsen classes of generators of the fundamental group, which we denote $\mathscr{N}(X_1)$, $\mathscr{N}(X_2)$, and $\mathscr{N}(X_3)$. This is akin to a Heegaard splitting, which is well known to admit two Nielsen classes, a fact which has been used by numerous authors in order to distinguish Heegaard splittings (see for example [3], [32] and [10]).

In applying these invariants, we look to the spun trisections of spun 4-manifolds constructed by Meier in [34]. The construction takes, as input, a Heegaard splitting of a 3-manifold, $M^3 = H_1 \cup_\Sigma H_2$, and produces a trisection of the spin of M^3, $S(M^3) = X_1 \cup X_2 \cup X_3$. It is easy to show that $\pi_1(M^3) = \pi_1(S(M^3))$. Our main application hinges on a refinement of this fact, which states that, in some sense, spinning a Heegaard splitting induces a Nielsen equivalence. More precisely we show the following result.

Theorem 2.6.3. *Let M^3 be a closed, orientable, 3-dimensional manifold, and let $H_1 \cup_\Sigma H_2$ and $H_1' \cup_{\Sigma'} H_2'$ be Heegaard splittings of M. Then if $\mathscr{N}(H_1) \neq \mathscr{N}(H_1')$, the trisections $S(\Sigma)$ and $S(\Sigma')$ are not isotopic. Moreover, if for all $\phi \in Aut(\pi_1(M))$, $\phi(\mathscr{N}(H_1)) \neq \phi(\mathscr{N}(H_1'))$ then the trisections $S(\Sigma)$ and $S(\Sigma')$ are not diffeomorphic.*

Historically, most inequivalent Heegaard splittings have been distinguished by their Nielsen classes (with a notable exception being [26]). Therefore, in some sense, the theorem above embeds much of the complexity of Heegaard splittings into the realm of trisections. This shows that the theory for 4-manifolds is likely to be much more complex than the theory for 3-manifolds, as one would expect. Focusing our attention on the Heegaard splittings of Seifert fibered spaces the work of Lustig and Moriah in [32] and Boileau, Collins, and Zieschang in [3], gives us the following corollary.

Corollary 2.7.1. *For every $n \geq 2$, there exist 4-manifolds which admit non-isotopic $(3n, n)-$trisections of minimal genus.*

In another direction, Chapter 3 further develops the analogies between the diagrammatic theories of Heegaard splittings and Trisections. Specifically, we draw analogies between the theory of trisections and the work in [25], where Johnson uses two closely related simplicial complexes associated to surfaces in order to define invariants of 3-manifolds. In particular, by passing through Heegaard splittings, Johnson defines distances between two 3-manifolds in the pants complex and the dual curve complex, which in the end is independent of the particular Heegaard splittings chosen. An interesting interpretation of the distance between 3-manifolds M_1 and M_2 in the dual curve complex is that it is equal to the minimum number of components of a link $L \subset M_1$ such that Dehn surgery along L produces M_2.

One of the key observations which allows us to extend this work into 4-dimensions is that, if we have two 4-manifolds equipped with (g, k)-trisections for the same g

and k, we may cut out a chosen X_i from each of them, and glue them together in a way which respects the structure on the boundary of X_i. This gives a way to view all relevant curves of the trisection diagrams on a single surface, and hence compare them in the chosen complex. We are then readily able to define two non-trivial distances between trisections: $D(T_1, T_2)$ and $D^P(T_1, T_2)$.

Next, we use the distances between trisections to define distances between 4-manifolds. For this task, we look towards the stable equivalence of trisections. If T is a genus h trisection, and $g = h + 3n$ for $n \in \mathbb{N}$, then stabilizing T n times produces a a genus g trisection of the same 4-manifold, which we denote T^g. The main theorem of Chapter 3 is the following.

Theorem 3.3.5. *Let M_1 and M_2 have trisections T_1 and T_2, respectively. Then the limit $\lim_{g\to\infty} D(T_1^g, T_2^g)$ is well defined and depends only on the underlying manifolds, M_1 and M_2.*

By carrying out this construction in both complexes, we obtain two natural number valued invariants of two 4-manifolds, $D(M_1, M_2)$ and $D^P(M_1, M_2)$. Much of the rest of the chapter is dedicated to exploring properties of these invariants. For example, if $\sigma(M)$ denotes the signature of M, we obtain the following inequality.

Proposition 3.4.3. $D(M_1, M_2) \geq \frac{1}{2}|\sigma(M_1) - \sigma(M_2)|$.

We also give interpretations of nearby manifolds in terms of Kirby calculus. We first show that manifolds which are close in the pants complex have very similar Kirby diagrams. More precisely, we show the following.

Theorem 3.5.3. *If $D^P(M_1, M_2) = 1$, then M_1 and M_2 have Kirby diagrams which are identical, except for the framing on some 2-handle.*

We also show that manifolds with similar Kirby diagrams are close in the pants complex, which is encompassed in the following theorem.

Theorem 3.5.5. *Let M_1 and M_2 be non-diffeomorphic 4-manifolds with the same Euler characteristic which have Kirby diagrams K_1 and K_2, respectively. If K_1 and K_2 only differ in the framing of some 2-handle, where the framing differs by 1, then $D^P(M_1, M_2) = 1$.*

Our line of inquiry in constructing these invariants leads naturally to the construction of graphs of 4-manifolds coming from subgraphs of the pants complex and the dual curve complex. We are also led to a class of embeddings of 3-manifolds in 4-manifolds which may be represented as curves on a surface.

Remark 0.0.1. Much of the work in this dissertation has appeared in the papers [23] and [24] by the author.

Chapter 1

Decomposing manifolds via smooth functions

1.1 Preliminary definitions and Conventions

As we will be dealing with parallels between manifolds of different dimensions, superscripts will be used help distinguish the dimension of a given space e.g. M^n denotes an n- dimensional manifold. Throughout this dissertation, all manifolds will be assumed to be smooth, compact, and orientable. Smooth manifolds are known admit many smooth functions to $\mathbb{R}$. Among these functions, Morse functions have proven to be immensely useful in studying the smooth topology of manifolds. Though we assume familiarity with Morse functions on the level of, say, [36] we briefly recall the definitions here for completeness.

Definition 1.1.1. Let $f : M \to \mathbb{R}$ be a smooth function, and let $p \in M$ be a point such that M has local coordinates $(x_1, ..., x_n)$ near p. Then the **Hessian** of f at p is given by $\mathrm{Hess_f(p)} = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1}(p) & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n}(p) \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1}(p) & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_n}(p) \end{pmatrix}$

Definition 1.1.2. A smooth function $M \to \mathbb{R}$ is called a **Morse function** if at every critical point takes a distinct critical value, and at every critical point p, the determinant of $Hess_f(p)$ is nonzero.

One reason Morse functions are useful is because these functions have easy to understand local models at the inverse image of any point in $\mathbb{R}$. At regular values, the inverse function theorem guarantees that the inverse image will be a codimension 1 manifold. At critical values, the famous Morse Lemma [41] provides useful local models.

Lemma 1.1.3. *Let f be a smooth function, and let p be a nondegenerate critical point. Then there exists a neighbourhood U containing p with a chart $(x_1, ..., x_n)$, such that $x_i(p) = 0$ for all i, and $f = f(p) - x_1^2 - x_2^2 - ... - x_k^2 + x_{k+1}^2 + ... + x_n^2$. The number k is independent of the choices made, and is called the **index** of p.*

Recall that an n-dimensional k-handle attachment is the process of attaching $D^k \times D^{n-k}$ along $S^{k-1} \times D^{n-k}$ to the boundary of an n-manifold. In the following theorem, we let M_a denote the inverse image $f^{-1}(-\infty, a]$ of some Morse function f.

Theorem 1.1.4. *If a Morse function f contains no critical values in the interval $[a, b]$ then M_a is diffeomorphic to M_b Moreover, if f has a critical value of index k at p, then $M_{p+\epsilon}$ is the result of attaching a k-handle to $M_{p-\epsilon}$.*

At times, it will be useful to "turn a manifold upside down" in order to analyze its structure from a different perspective. The following lemma provides a method for doing so.

Lemma 1.1.5. *If $f : M^n \to \mathbb{R}$ is a Morse function, so is $-f : M^n \to \mathbb{R}$. Furthermore, if f has a critical value at p of index k, then $-p$ is a critical value of $-f$ of index $n-k$.*

1.2 Cerf Theory and Modifying Handle Decompositions

Given two Morse functions, f_0 and f_1 on a fixed manifold, it is natural to ask how these functions are related. Of course, since $\mathbb{R}$ is contractible, these two maps are homotopic by some homotopy f_t. One may hope that we can choose f_t so that it is Morse for all values of $t \in [0, 1]$. This can quickly be shown to be too much to ask for, as one may readily construct Morse functions with different numbers of critical values. Towards a controlled homotopy where the singularities are manageable, we introduce Cerf functions.

Definition 1.2.1. A 1-parameter family of functions $f_t : M \to \mathbb{R}$ is called **Cerf** if for all but finitely many values of t, f_t is Morse. Moreover, if at some time, t_0, f_{t_0} is not Morse, then f_{t_0} only fails to be Morse at a single critical value u and either:

1. Two critical values cross at u.

2. A cancelling k-handle and $k+1$-handle are born. More precisely, there exist local coordinates $(x_1, ..., x_n, t)$ on $U \subset M \times \mathbb{R}$ such that locally, f_t is given by $f_t(x_1, ..., x_n) = -x_1^2 - ... - x_k^2 + x_{k+1}^3 - (t - t_0)x_{k+1} + x_{k+2}^2 + ... + x_n^2$.

3. A k handle and a $k+1$ handle cancel. This phenomenon can be modeled locally by reversing the t coordinate in the model for handle births.

Just as Morse functions are generic in the space of smooth functions, Cerf functions are generic in the space of one parameter families of smooth functions. That is, any smooth family of maps $f_t : M \to \mathbb{R}$ where f_0 and f_1 are Morse can be perturbed rel f_0 and f_1 to be Cerf. A useful consequence of this is the following theorem.

Theorem 1.2.2. *Any two Morse functions $f_0, f_1 : M \to \mathbb{R}$ are connected by a family of functions $f_t, t \in [0,1]$ which is Cerf.*

A convenient way to think about Cerf functions is through a Cerf graphic, as pictured in Figure 1.2. Here, we track the critical values of the Morse functions f_t as t varies. The horizontal coordinate corresponds to t whereas the vertical coordinate corresponds to the value of the function. The paths of the critical values in the Cerf graphic will be called **critical folds**, and at times it will be useful to label them with their index. In light of Lemma 1.1.5, one must also provide a co-orientation when specifying the index of a critical fold as an index k critical point with one co-orientation is an index $n-k$ critical point with the opposite co-orientation.

As a shift in perspective, one may consider the family of functions $f_t : M \to \mathbb{R}$ to be a single function $F : M \times I \to \mathbb{R}^2$ by setting $F(x) = (t, f_t(x))$. A generic smooth function from any manifold to $\mathbb{R}^2$, which locally looks like a Cerf function

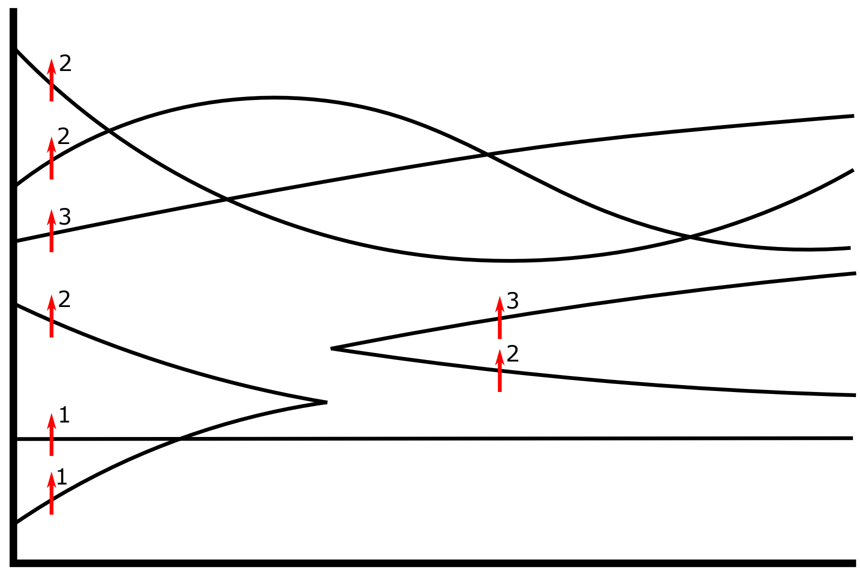

Figure 1-1: An example of a Cerf Graphic. Handles are labeled according to their index. Cusps correspond to handle creations/cancellations.

is sometimes called a **Morse 2-function** to emphasize this perspective. Here, the genericity of Cerf functions provides local models for Morse 2-functions; namely they are either locally homotopies which are Morse at every time, or otherwise will have a cusp or crossing. This observation plays a heavy role in the theory of trisections, which arise as certain Morse 2-functions on 4-manifolds. In this dissertation we will use the term Cerf function for both perspectives.

1.2.1 Homotopies of Cerf Functions

Just as any two Morse functions are connected by a Cerf function, any two Cerf functions are joined by a family of functions which are Cerf at all but finitely many times. An analysis of the singularities which occur in these functions was first carried out by Jean Cerf in [7], where he showed that, in many cases, pseudo-isotopies of

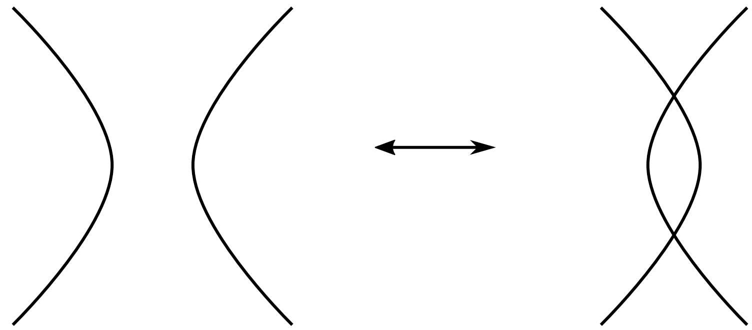

Figure 1-2: A Reidemeister 2 move on a Cerf graphic.

manifolds can be upgraded to isotopies. A modern treatment of these two parameter functions can be found in [19] and [15]. We summarize the results here.

Theorem 1.2.3. *[7] Let $f_{t,0}$ and $f_{t,1}$ be two Cerf functions on a smooth manifold M. Then $f_{t,0}$ and $f_{t,1}$ are connected by a homotopy $f_{t,s}$ of smooth functions. For all but finitely many times s, $f_{t,s}$ is a Cerf function. Moreover, if f_{t,s_0} is not Cerf, then for some small value of ϵ, the Cerf graphic for $f_{t,s_0-\epsilon}$ and the Cerf graphic for $f_{t,s_0+\epsilon}$ are related by one of the following moves or their inverses.*

1. *A Reidemeister 2 move as shown in Figure 1.2.1*
2. *A Reidemeister 3 move as shown in Figure 1.2.1*
3. *A cusp poke as shown in Figure 1.2.1*
4. *A cusp merge as shown in Figure 1.2.1*
5. *An eye birth as shown in Figure 1.2.1*

The reader should be warned that Theorem 1.2.3 concerns possible modifications of a Cerf graphic and does not permit one to arbitrarily modify a given Cerf graphic by

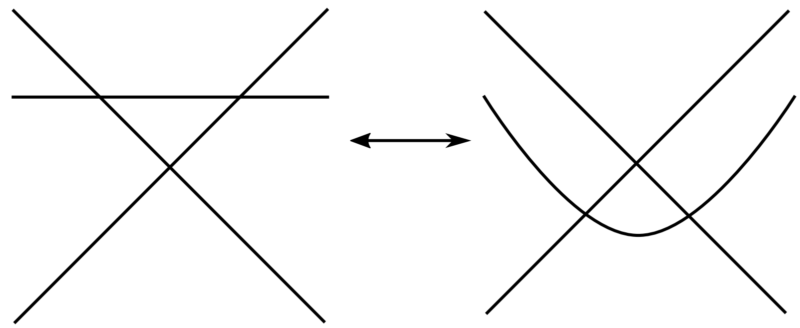

Figure 1-3: A Reidemeister 3 move on a Cerf graphic.

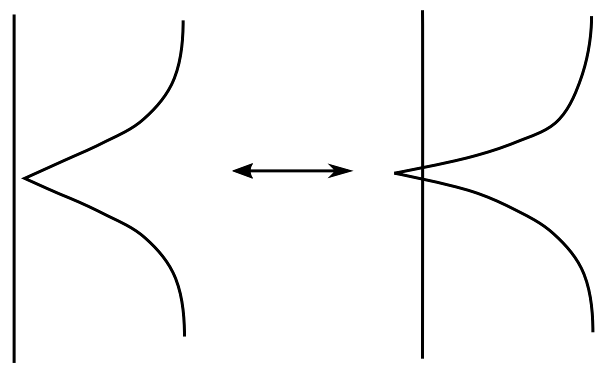

Figure 1-4: A cusp poke on a Cerf graphic.

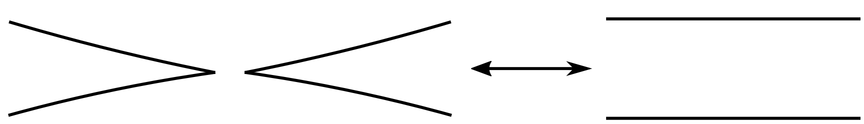

Figure 1-5: A cusp merge on a Cerf graphic.

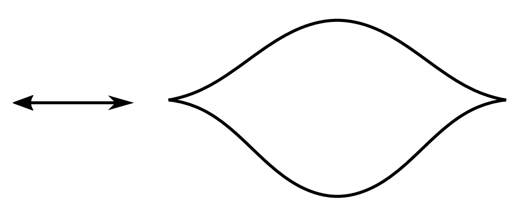

Figure 1-6: An eye birth on a Cerf graphic.

these moves. Nevertheless, much work has been done on exactly when modifications can be performed on a Cerf graphic, especially with respect to Cerf functions on 4-manifolds. On a 4-manifold, inverse images of regular values are orientable surfaces, and folds can be of index 0, 1, 2 or 3. We follow the convention that index 0 and 3 folds will be labeled numerically, whereas index 1 and 2 folds will be co-oriented towards the direction where the genus of the fiber decreases. One move which will be particularly useful to us is the process of flipping an inward pointing fold on a 4-manifold inside out. This process, originally carried out in [34], can be seen in Figure 1-7, and we refer the reader to [1] to see that these moves are indeed valid, as well as for a coherent treatment of the modification of Cerf diagrams.

1.2.2 Handlebodies and Spines

It is straightforward to show that the diffeomorphism type of attaching a k-handle to a manifold M depends only on the isotopy class of the attaching map in ∂M and a framing of the neighbourhood of the attaching region. We would like to determine the result of attaching 1-handles to $M = D^n$. Recall that the attaching region of an n-dimensional 1-handle is given by $S^0 \times D^{n-1}$ and the boundary of D^n is S^{n-1}. If $n > 2$, and $(b_1, ..., b_k)$ and $(b'_1, ..., b'_k)$ are arbitrary collections of $n-1$ balls embedded in S^{n-1}, then there exists an isotopy of S^{n-1} sending $b_i \to b'_i$. Also, there are two ways to frame a copy of S^0 in a manifold, but one of these framings will produce a non-orientable manifold. As a result, there is a unique way to attach these 1-handles and end up with an orientable manifold, and we may unambiguously make the following

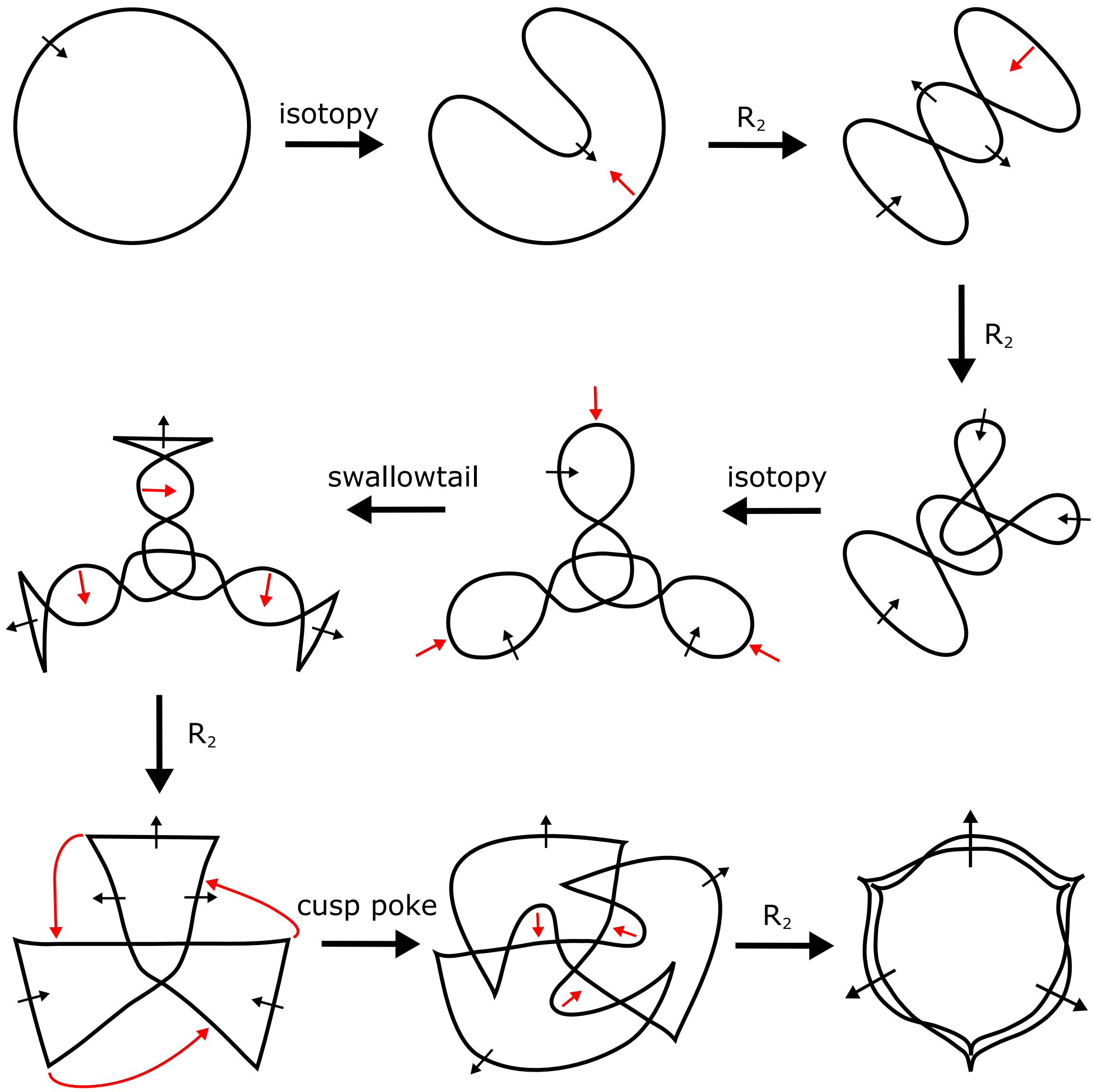

Figure 1-7: Flipping an indefinite fold of 4-dimensions. Red arrows indicate where the next move will be taking place, and black arrows indicate the direction in which the genus of the fiber decreases

definition.

Definition 1.2.4. Let $g > 0$ and $n > 3$ be integers. We define the **genus g handlebody of dimension n** to be the unique orientable manifold obtained by attaching g n-dimensional 1-handles to D^n. We will denote this manifold by $\boldsymbol{H_g^n}$.

Handlebodies deformation retract onto graphs embedded within them. We call any embedded graph which is a deformation retract of H_g^n a **spine** of H_g^n. Most naturally, one can construct a spine by connecting the cores of the 1-handles to a common point in the interior of the n-ball. Alternatively, one can define an n-dimensional handlebody to be the neighbourhood of a graph embedded in $\mathbb{R}^n$, in which case a spine appears as the graph itself.

In general, a spine may contain an arbitrary number of vertices, however, for our purposes, we will be particularly concerned with spines which have a single vertex. In addition, our spines will come with an orientation of each edge. Henceforth, all spines of handlebodies will be assumed to have one vertex and carry an orientation, unless otherwise noted. In this case, a spine of H_g^n will be a wedge of g circles with each circle carrying an orientation. We will consider spines up to isotopy in H_g^n. If $g < 2$, then a handlebody has a unique spine, but otherwise, there are infinitely many spines.

A spine of a handlebody can be altered in controlled ways to obtain a new spine of the handlebody. One may reverse the orientation of any edge, and the resulting graph is clearly still a spine. A more interesting move on one vertex spines is what is called an **edge slide**. Informally, this amounts to sliding the end of a loop over

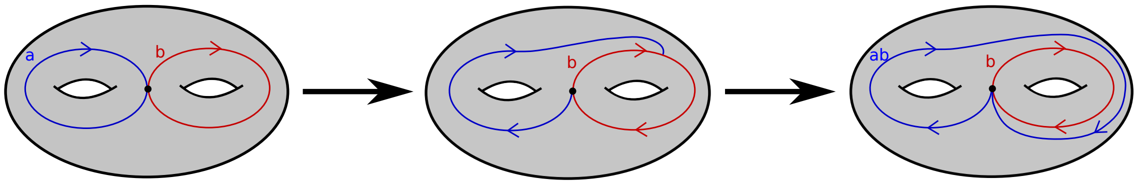

Figure 1-8: The process of sliding an edge which represents the generator a over an edge representing the generator b produces a spine which has edges representing the generators ab and b.

another loop in the direction of the second loops orientation and returning back to the base vertex. This is illustrated in Figure 1.2.2.

Though not very enlightening, we also give a formal definition of an edge slide because it will play an important role in this dissertation. Let S be a spine of H_g^n, and let l_1 and l_2 be two loops of S parameterized by $f_1, f_2 : [0,1] \to H_g^n$. Let $g : D^2 \to H_g^n$ be an embedding of a 2-dimensional disk so that the boundary S^1 is parameterized by $[0,1]$ and oriented in the direction of increasing real number values with $h = g|_{S^1}$. Suppose that $h([0,\frac{1}{3}]) = f_1([\frac{2}{3},1])$ and $h([\frac{1}{3},\frac{2}{3}]) = f_2([0,1])$ where both restricted functions are orientation preserving homeomorphisms of the interval. We may obtain a new one vertex spine of H_g^n by leaving all edges of G unchanged except for l_1 which is replaced by $(l_1 \backslash h([0,\frac{1}{3}])) \cup \overline{h([\frac{2}{3},1])}$ (where the bar indicates opposite orientation). This process is called sliding l_1 over l_2.

It is straightforward to see that $\pi_1(H_g^n)$ is a free group on g generators. Moreover, any spine of H_g^n specifies a basis for F_n. Edge slides give ways to obtain a new set of generators of the free group, which will be elaborated on in Chapter 2. In fact, we will see in Lemma 2.3.1 that any basis of the free group can be obtained by successively applying edge slides and reversals of orientations.

1.3 Heegaard Splittings of 3-Manifolds

In his 1898 thesis [21], Poul Heegaard introduced a decomposition of 3-manifolds which we now call a Heegaard splitting. This construction allowed him to understand how torsion arises in the homology groups of manifolds, leading to a counterexample to the Poincare's original statement of Poincare duality. In 1952, work of Moise [39] on the triangulability of 3-manifolds showed that every 3-manifold admits a Heegaard splitting. We begin with a definition of these decompositions.

Definition 1.3.1. A **genus g Heegaard splitting** of a closed, orientable 3-manifold M is a genus g surface, Σ_g, embedded in M such that the complement of an open regular neighborhood of Σ_g in M has two components, H_1 and H_2, each homeomorphic to H^3_g. We call such a surface Σ_g a **Heegaard surface** for M.

There are various notions of equivalence of Heegaard splittings all of which can differ in subtle ways. A coherent treatment of these equivalence relations can be found in [2]. For our purposes we will focus on the two equivalence relations defined below.

Definition 1.3.2. Two Heegaard splittings defined by $H_1 \cup_\Sigma H_2$ and $H'_1 \cup_{\Sigma'} H'_2$ are **isotopic as Heegaard splittings** if Σ is isotopic Σ' by an isotopy taking H_1 to H'_1 and H_2 to H'_2. Two Heegaard splittings, denoted as above, are **homeomorphic as Heegaard splittings** if there is a homeomorphism $f : M^3 \to M^3$ so that $f(H_1) = H'_1$ and $f(H_2) = H'_2$.

A natural way to obtain a Heegaard splitting is through the modification of a

Morse function on M. It is a standard result that handles can be attached in order of increasing index, so that 0-handles are attached before 1-handles, 1-handles are attached before 2-handles, etc. It is also a standard result that a connected n-dimensional manifold admits a Morse function with a single 0-handle and a single n-handle. The upshot for 3-manifolds is that one can modify a handle decomposition until it consists of a single 0-handle, followed by a collection of 1-handles, a collection of 2-handles, and finally, a single 3-handle. We will call such Morse functions on a 3-manifold **Heegaard-Morse** functions.

From the discussion in Section 1.2.2 we see that the 0- and 1-handles in a handle decomposition coming from a Heegaard-Morse function form a genus g handlebody. Furthermore, an application of Lemma 1.1.5 shows that the 2- and 3-handles are also handlebodies. Therefore, a level surface of such a Morse function located after the 1-handles, but before the 2-handles are attached is a Heegaard surface.

Recall that the diffeomorphism type of a manifold obtained by handle attachments depends only on the isotopy class of the attaching maps. We seek to understand Heegaard splittings visually by looking at these attaching maps. To this end, note that a 2-handle in 3-dimensions is attached to a surface along $S^1 \times D^1$. In the case that the 3 manifold is orientable, the level sets are all in fact orientable surfaces. It can be taken as the definition of an orientable surface that the tubular neighbourhood of a simple closed curve is in fact the trivial bundle over that curve. As a result, one can specify the attaching region of a 3-dimensional 2-handle uniquely through a simple closed curve drawn on a surface.

Given a Heegaard-Morse function, the 2-handle attachments can be described by a collection of simple closed curves drawn on the Heegaard surface. The duality between handle attachments and surgery implies that, since there is exactly one 3-handle which must be attached along a copy of S^2, the curves describing the 2-handles must be non separating curves, and surgery along them must produce a 2-sphere. This in turn implies that there must in fact be g of these curves, where g is the genus of the Heegaard surface. This motivates the following definition.

Definition 1.3.3. Given a genus g surface Σ_g, a **cut system** is a collection of g disjoint simple closed curves such that surgery along these curves produces S^2.

To rephrase out earlier discussion, we have shown that the attaching regions of the 2-handles coming from a Heegaard-Morse function form a cut system on the Heegaard surface. In fact, by turning the Heegaard-Morse function upside down in the sense of Lemma 1.1.5, the 1-handles become 2-handles, and so they too can be described by a cut system. One can thus encode a Heegaard splitting coming from a Heegaard-Morse function as two cut systems on the Heegaard Surface. This leads us to the following definition.

Definition 1.3.4. A **Heegaard Diagram** is a triple $(\Sigma_g, \alpha, \beta)$ where Σ_g is a surface of genus g, and $\alpha = (\alpha_1, ...\alpha_g)$ and $\beta = (\beta_1, ..., \beta_g)$ are cut systems for Σ_g.

We have seen that Heegaard-Morse functions on a 3-manifold, and therefore the 3-manifold itself, can be encoded by a Heegaard diagram. Conversely, given a Heegaard diagram, (Σ, α, β), one can reconstruct a 3-manifold. To do this, first take the product $\Sigma \times [0, 1]$. Attach 3-dimensional 2-handles to $\Sigma \times \{0\}$ as prescribed by the α curves. Since α was a cut system, the remaining boundary is a 2-sphere, which, by a standard

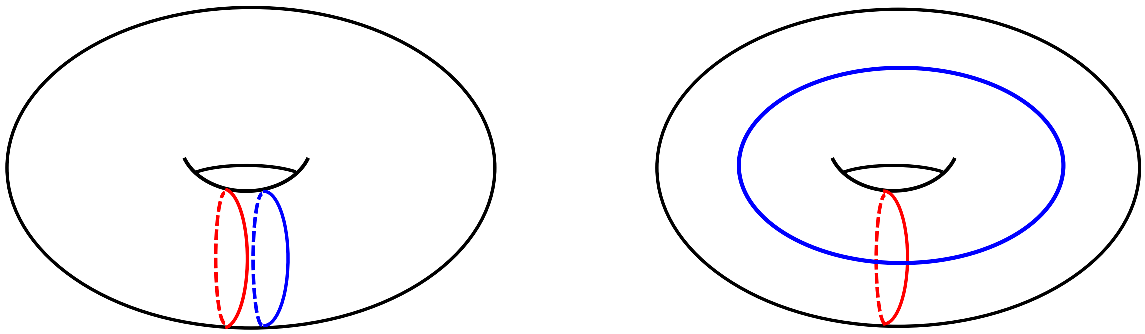

Figure 1-9: Left: A Heegaard diagram for $S^1 \times S^2$. Right: A Heegaard diagram for S^3.

coning argument, can be uniquely filled in with a 3-ball. Repeat this process on the other side, this time attaching 2-handles to $\Sigma \times \{1\}$ along the β curves and capping off with a ball in a unique manner. The result is a closed 3-manifold, unambiguously described by the cut systems.

A few Heegaard diagrams will be particularly relevant for this dissertation. In Figure 1.3 we see Heegaard Diagrams for $S^1 \times S^2$ and S^3. To see that these diagrams correspond to their respective manifolds, we first focus on the left diagram. The red and blue curves bound disks consisting of the cores of the attached 2-handles. When placed on top of each other, the red and blue curves correspond to a 2-sphere. Taking the curves around the complementary circle factor of the torus, we find an S^1 parameterizated family of the previously described 2-sphere, showing that this manifold is in fact $S^1 \times S^2$. On the right, we see two curves intersecting geometrically once. The red curve can be seen as the belt sphere for a 1-handle and the blue curve, we have seen, is the attaching region for a 2-handle. The fact that they intersect once is exactly the criteria we need in order to cancel the handles. In other words, this

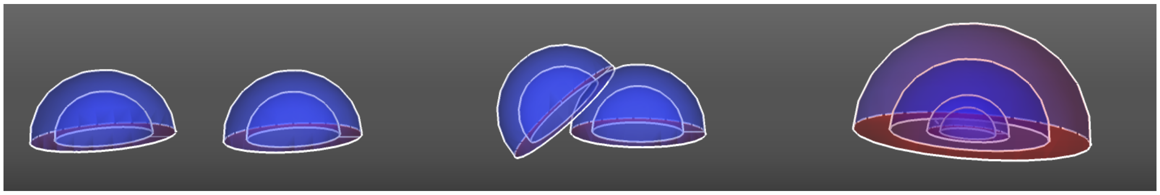

Figure 1-10: A 3-dimensional 2-handle sliding over another 2-handle. The attaching regions are shaded in red.

manifold can instead be built with only a 0-handle and a 3-handle, which implies that the manifold is homeomorphic to S^3.

1.3.1 Stabilization of Heegaard Splittings

In this section, we seek to describe a set of diagrammatic moves to determine when two diagrams represent the same 3-manifold. Our strategy will be to determine moves which take us between the corresponding Heegaard-Morse functions, and track their effect on the attaching regions of the handles. The first move we discuss is the 2-handle slide.

In an isotopy of a manifold, the 2 handles may follow paths which take them over other 2-handles, as shown in Figure 1-10. In this case, the isotopy class of the attaching maps can change, but this change is quite straightforward to see. Let α_1 and α_2 be the attaching curves for a pair of 2-handles, h_1 and h_2, respectively. We define a band sum of α_1 and α_2 to be the internal connected sum on the surface of these two curves along some chosen path, noting that the resulting isotopy class depends on this path. Sliding h_1 over h_2 amounts to leaving α_2 unchanged, and replacing α_1 with a band sum of α_1 and a push off of α_2. Given a cut system α, we refer to the process of replacing some curve in the system by a band connected sum with some other curve in the system by a path disjoint from all the curves as simply a **handle slide**.

Up to handle slides, the connected sum operation is well defined on Heegaard diagrams. Recall that S^3 admits a genus 1 Heegaard splitting, whose diagram is shown in Figure 1.3. The process of taking a connected sum with this Heegaard splitting is called **stabilization**. If $\Sigma \subset M$ is a genus g Heegaard splitting, then the stabilization will be a genus $g+1$ Heegaard splitting of the same manifold which we will denote Σ^{g+1}. Note that stabilization is unique up to isotopy, so the set of all Heegaard splittings of M up to a chosen equivalence forms a tree which we will call the **Heegaard tree** of M.

The Heegaard splitting for S^3 can be seen as a cancelling pair of 1- and 2-handles. From this perspective, it become clear that stabilization corresponds to modifying a Heegaard-Morse function by the addition of a cancelling 1- and 2-handle pair. The two Heegaard-Morse functions are connected by a Cerf graphic consisting of exactly one birth. Conversely, a stabilized Heegaard splitting is connected to its destabilizations via a Cerf Graphic with exactly one death. The following theorem of Laudenbach [29], which we specialize to the case at hand, elaborates on this idea.

Theorem 1.3.5. *(Theorem 1.3 of [29]) Let M^3 be a closed connected 3-manifold. Any two Heegaard-Morse functions f_0 and f_1 are joined by a Cerf function (f_t) such that, for all t except some finite set $\{t_1, ..., t_n, t_{n+1}, ..., t_{n+m}\}$ f_t is a Heegaard-Morse function. Moreover, $t_1, ..., t_n$ are birth singularities and lie in $[0, \frac{1}{3}]$, and $t_{n+1}, ..., t_{n+m}$ are birth singularities and lie in $[\frac{1}{3}, \frac{2}{3}]$.*

In particular, the Heegaard-Morse function $f_{\frac{1}{2}}$ describes a Heegaard splitting that is a common stabilization of those associated with f_0 and f_1. This recaptures a theorem originally due to Reidemeister and Singer.

Theorem 1.3.6. *([44], [49]) Given a fixed 3-manifold M, any two Heegaard splittings have a common stabilization. That is any two Heegaard splittings will become isotopic after sufficient stabilizations.*

This theorem gives the set of Heegaard splittings of a fixed manifold the structure of a poset where $\Sigma \leq \Sigma'$ if and only if Σ stabilizes to Σ'. The number of Heegaard splittings in a given genus, can vary quite wildly. A central part of this dissertation and, more broadly, the theory of Heegaard splittings, is based around the structure of the Heegaard tree of particular manifolds. Combining Waldhausen's work on the Heegaard structure of S^3 [50] with Haken's work on the reducibility of Heegaard splittings [18] we obtain that the Heegaard splittings of connected sums of $S^1 \times S^2$ are particularly simple. More precisely, we obtain the following theorem.

Theorem 1.3.7. *([18] [50]) The manifolds S^3 and $\#^k S^1 \times S^2$ for $k \in \mathbb{Z}^+$ have a unique Heegaard splitting in each genus. In other words, all Heegaard splittings of these manifolds are stabilizations of the unique Heegaard splitting of genus 0 or k, respectively.*

1.4 Trisections of 4-Manifolds

Trisections were introduced by Gay and Kirby in [14] as a 4-dimensional analogue of a Heegaard splitting. One would perhaps hope to generalize the definition of a

Heegaard splitting to 4-manifolds by cutting an ordered Morse function in half as we did with a Heegaard splitting. One must decide, then, which half receives the 2-handles of the decomposition, after which the other half is diffeomorphic to H^4_k. The following theorem of Laudenbach and Poenaru shows, however, that this approach is doomed.

Theorem 1.4.1. *[30] Any diffeomorphism of $\#^k S^1 \times S^2$ extends across H^4_k. As a result, there is a unique manifold obtained by capping off a 4-manifold with boundary $\#^k S^1 \times S^2$ with H^4_k.*

By this theorem, when attempting to cut a closed 4-manifold in half, the entire manifold is determined by the half with the 2-handles, so no reduction in complexity can be achieved in this manner. With this limitation in mind, the next best hope is instead to decompose a 4-manifold into 3 handlebodies. Surprisingly, this is indeed possible. In fact, many of the most appealing features of the 3-dimensional theory, such as a diagrammatic theory and stable equivalence, are present in the 4-dimensional theory. We begin with a formal definition of a trisection; a schematic for a trisection of a smooth, orientable, closed 4-manifold can be seen in Figure 1-11, and should be used as a reference while absorbing the formal definition.

Definition 1.4.2. A **(g, k)-trisection** of a 4-manifold, M, is a decomposition $M = X_1 \cup X_2 \cup X_3$ such that:

1. $X_i \cong H^4_k$. We will call each X_i a **sector** of the trisection.

2. $X_i \cap X_j = H_{ij} \cong H^3_g$ for $i \neq j$.

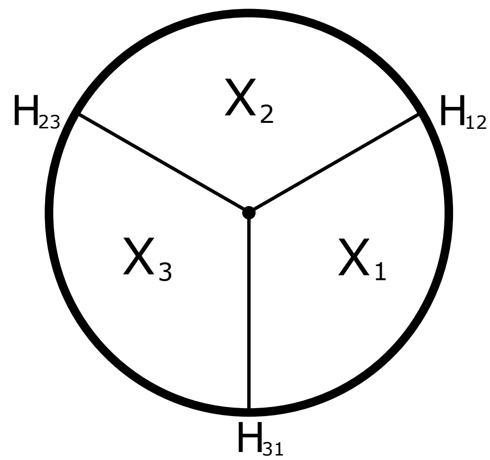

Figure 1-11: A schematic of a trisection. Each X_i is diffeomorphic to a 4-dimensional handlebody and each H_{ij} is diffeomorphic to a 3-dimensional handlebody. The three H_{ij} meet in a closed surface indicated by a dot in the center of the trisection. Any two of the H_{ij} form a Heegaard splitting for some connected sum of copies of $S^1 \times S^2$.

3. $\partial X_i = H_{ij} \cup H_{ik}$ is the genus g Heegaard splitting for $\partial X_i = \#^k S^1 \times S^2$.

4. $X_1 \cap X_2 \cap X_3$ is a genus g surface which we call the **trisection surface**.

Two trisections of a fixed 4-manifold, M, defined by $X_1 \cup X_2 \cup X_3$ and $X_1' \cup X_2' \cup X_3'$ are **homeomorphic as trisections** if there is a homemorphism of M such that $f(X_i) = X_i'$. Two trisections are **isotopic as trisections** if there is an isotopy, f_t, of M such that $f_0 = id$ and $f_1(X_i) = X_i'$.

Of particular importance in this decomposition is the union $H_{12} \cup H_{23} \cup H_{31}$, which we will call the **tripod** of the trisection. Starting with the tripod, one can uniquely reconstruct the 4-manifold which it came from, as well as the trisection structure on it. This process is illustrated in Figure 1-12 and can be referenced in the following description. To re-obtain the closed 4-manifold, one first thickens the tripod by taking the product of the trisection surface with D^2 and the product of each of the handlebodies with D^1. This leaves a manifold with three boundary components, each

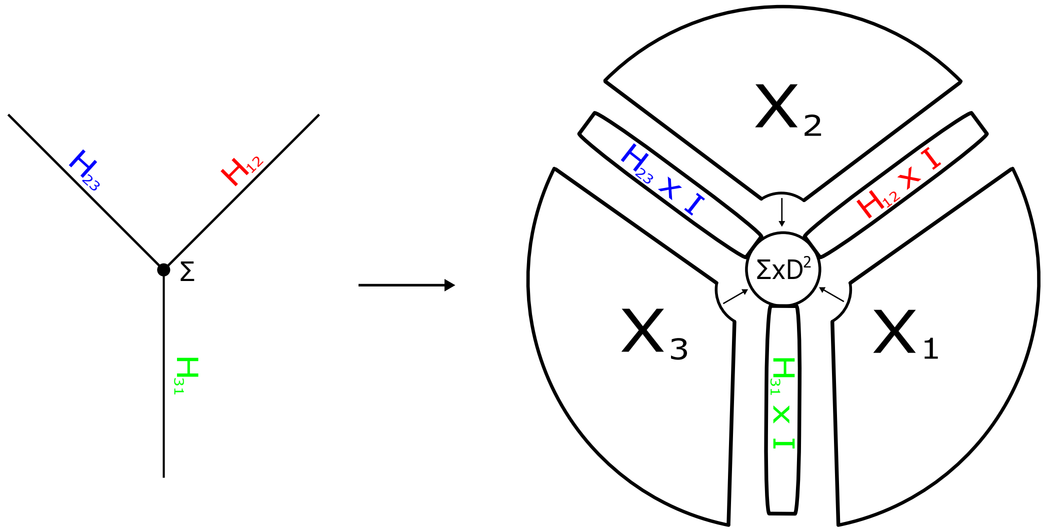

Figure 1-12: Reconstructing a 4-manifold from a tripod. We first thicken the whole tripod into a 4-dimensional manifold with boundary. The boundary can then be capped off in a unique manner.

diffeomorphic to $\#^k S^1 \times S^2$. By Theorem 1.4.1, each of these boundary components can be uniquely capped off with H_k^4. As there were no choices to be made in either the thickenings or the capping of the boundary components, we have uniquely constructed a trisected 4-manifold.

We next seek to further reduce the information of a tripod to information contained on the trisection surface. Recall that a 3-dimensional handlebody can be obtained from a genus g surface by attaching g 3-dimensional 2-handles and a 3-handle to the surface. In Section 1.3, we saw that this information can be specified by a cut system (see Definition 1.3.3). Since a tripod consists of three 3-dimensional handlebodies meeting at a surface, it can be completely encoded by three cut systems drawn on the trisection surface. These three cut systems are far from arbitrary and are severely restricted by condition 3 of the definition of a trisection (Definition 1.4.2). We record a formal definition of these restrictions below.

Definition 1.4.3. A (g,k)-**trisection diagram** is a quadruple $(\Sigma, \alpha, \beta, \gamma)$ such that

1. Σ is a genus g surface.

2. α, β, and γ are cut systems for Σ

3. The triples (Σ, α, β), (Σ, β, γ), and (Σ, γ, α) are Heegaard diagrams for $\#^k S^1 \times S^2$.

Given a trisection, the preceding discussion shows how one immediately obtains a trisection diagram by recording cut systems for the 3-dimensional handlebodies on the trisection surface. Moreover, each of the steps can be reversed. That is, if we start with a trisection diagram, we can attach 2-handles and a 3-handle as prescribed by the cut system. This gives us a trisection tripod, which, as outlined in Figure 1-12, uniquely encodes a trisection. Isotopies of the 4-manifold will usually leave the cut systems of the tripod invariant, however, at times the 2-handles will slide over each other and off of the surface. The latter case corresponds to modifications of the trisection diagram by handle slides within each cut system. Moreover, a diffeomorphism of the 4-manifold will correspond to a diffeomorphism of the trisection diagram. We therefore have a one to one correspondence between trisections of a 4-manifold up to trisected diffeomorphism and trisection diagrams up to handle slides and diffeomorphisms.

At times, it will be useful to relax the condition that all of the X_i in a trisection are diffeomorphic to the same 4-dimensional handlebody. In particular, we will allow $X_i \cong \natural^{k_i} S^1 \times D^3$ where for $i \neq j$ it is possible that $k_i \neq k_j$. In this case, we insist that $\partial X_i = H_{ij} \cup H_{ik}$ is a genus g Heegaard splitting for $\partial X_i \cong \#^{k_i} S^1 \times S^2$. We

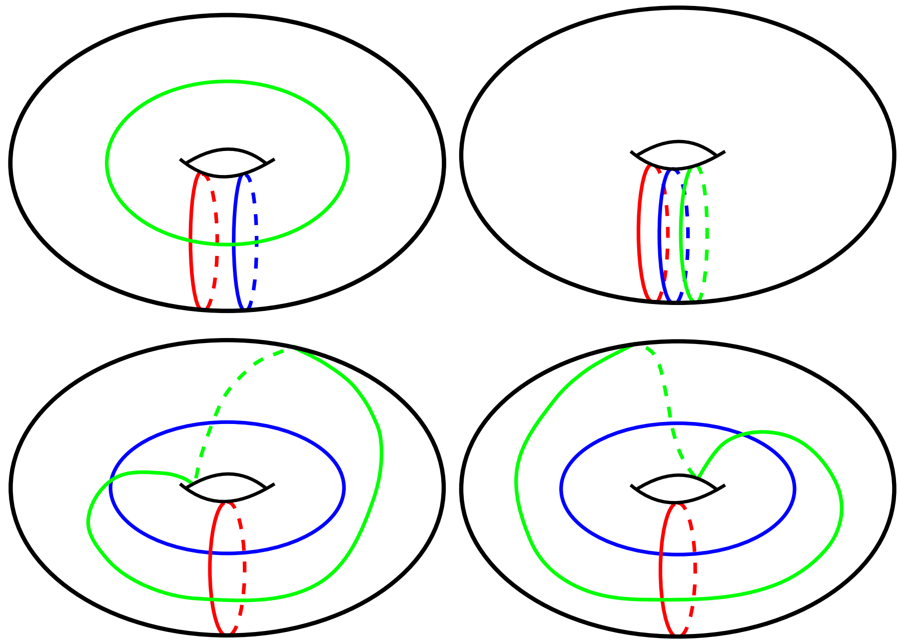

Figure 1-13: The genus 1 trisections. Top Left: S^4. Top Right: $S^1 \times S^3$. Bottom Left: $\mathbb{C}P^2$. Bottom Right: $\overline{\mathbb{C}P^2}$

will call this more general setup an **unbalanced $(g; k_1, k_2, k_3)$-trisection**. Most of the underlying theory in this more general set up is unchanged. In particular, one can trace through the previous discussion and find that it in no way depends on the trisection being balanced, so that unbalanced trisections admit a similar diagrammatic theory. See Figure 1-13 for some trisection diagrams, and observe that all except the genus 1 trisection of S^4 is balanced.

By a straightforward combinatorial argument, once can show that, up to permutations of the colors, the trisection diagrams in Figure 1-13 are actually a complete list of genus 1 trisection diagrams. By permuting the colors, we see that S^4 has unbalanced trisections with parameters $(1; 1, 0, 0)$, $(1; 0, 1, 0)$, and $(1; 0, 0, 1)$. These trisections will be useful for balancing trisections which arise "in nature." For simplicity, unless otherwise noted, all trisections will be assumed to be balanced.

1.4.1 Existence and uniqueness

An essential part of the theory of Heegaard splittings is the existence and stable equivalence of Heegaard splittings. These results give a natural way to define invariants on a 3-manifold. Namely, one gives an invariant of a Heegaard splitting and show that this invariants changes in a controlled way with respect to stabilization. We seek to set up a parallel picture for 4-manifolds. Naturally, the first problem to address is the existence of a trisection on a given 4-manifold. In their foundational paper, Gay and Kirby [14] prove the following result.

Theorem 1.4.4. *Every smooth, oriented, closed 4-manifold admits a (g, k)-trisection for some g and k*

Gay and Kirby in fact give two proofs of this theorem, one based on Kirby diagrams, and another based on Cerf functions. For now, we will discuss the proof based on the Cerf theory discussed in Section 1.2. We will proceed by analogy with the 3-dimensional case. To get a Heegaard splitting, we "bisected" a Morse function to the interval so that the surfaces corresponding to regular values monotonically grew in complexity when moving towards the center. The strategy for the 4-dimensional case is similar. To obtain a trisection, we will "trisect" a Cerf function to the disk such that, when moving towards the center, the complexity of the surfaces corresponding to regular values grow in complexity. This monotonic increase in complexity allows one to identify the pieces with standard manifolds.

Just as every Morse function on a manifold was not amenable to such a bisection (we needed that the function was ordered), an arbitrary Cerf diagram does not necessarily admit a trisection. To construct such a function, Gay and Kirby start with an ordered handle decomposition of a manifold, and construct a map from each set of handles to the disk. They then modify this so that when moving towards the center of the disk, one only passes through index 0 and index 1 folds and so that the cusps each fold can be distributed into three sectors such that no sector contains a fold with two consecutive cusps. By distributing the cusps to three sectors as described before, we obtain a trisected Cerf function whose critical image is shown in in Figure 1-14. Taking a path from the outside of the disk to the center away from the cusps, one sees a 3 manifold built out of a 0-handle and g 1-handles, that is, a 3-dimensional genus g handlebody. Sweeping this path out through one of the sectors we see that $g-k$ of these 1 handles are cancelled by 2-handles when we pass through each cusp, leaving k of the 1 handles surviving in this sector. In other words the inverse image of any sector is diffeomorphic to $H^3_k \times I = H^4_k$. The other conditions for being a trisection can also easily be checked, by which they conclude that any 4-manifold admits a trisection.

1.4.2 Relation to Handle decompositions

Just as Heegaard splittings are intimately connected to handle decompositions, so too are trisections; though the connection is a bit more subtle. Given a trisection $M = X_1 \cup X_2 \cup X_3$ one can construct a trisected Cerf function as in Figure 1-14. We can then apply a homotopy of the Cerf function until it appears as in Figure 1-15. The projection of this function onto the horizontal axis is a Morse function, and

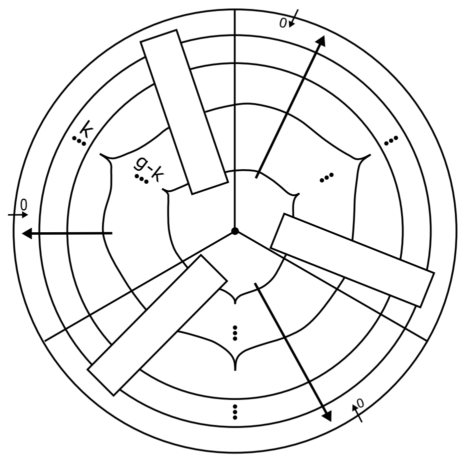

Figure 1-14: A trisected Cerf function. The white blocks are smoothly just products, and while they may contain many crossings, they contain no cusps. A $(g-k)-$trisection corresponds to a function where each sector has k folds without cusps and $g-k$ folds with cusps.

we therefore obtain a handle decomposition, with indices as indicated in the figure (one can deduce these using local models for folds). Here, we see that the 0- and 1-handles lie in one region of the trisection, the 2-handles in the other and the 3- and 4-handles appear in the final sector. One of the strengths of viewing a 4-manifold as a trisection is the symmetry of the sectors. Here this manifests itself as the following lemma, which will be instrumental in Chapter 2.

Lemma 1.4.5. *Given a trisection $M^4 = X_1 \cup X_2 \cup X_3$, we can obtain a handle decomposition of M such that, for any $i \in \{1, 2, 3\}$, X_i is the union of the 0- and 1-handles of the decomposition.*

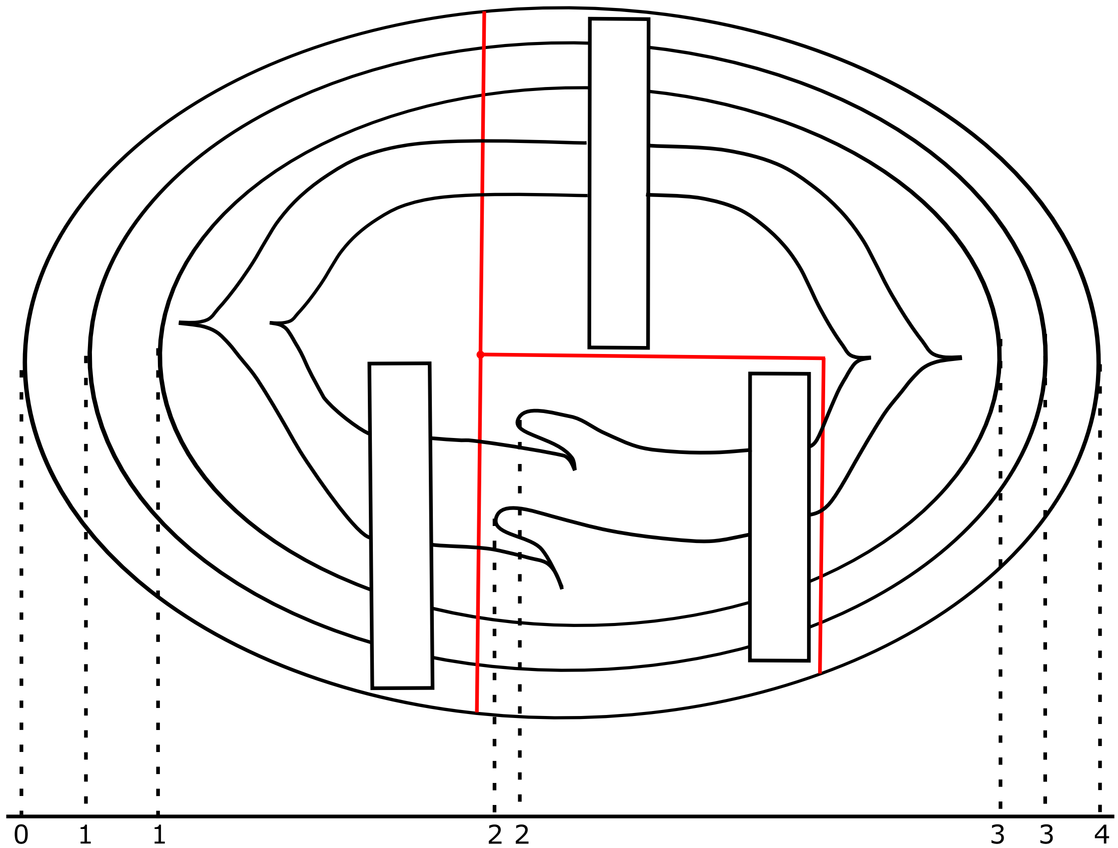

Figure 1-15: Extracting a handle decomposition from a Cerf graphic. After projecting onto the horizontal axis, each vertical tangency becomes a critical point of the index indicated below. The red lines are the delineations of the sectors.

1.4.3 Stabilizations

The stabilization operation plays a central role in this dissertation. In Chapter 2, we show that there are trisections which require this stabilization in order to become equivalent. In Chapter 3, we use this operation to upgrade invariants of trisections into invariants of manifolds. The stabilization operation for trisections can be viewed through three lenses: diagrammatically, through Cerf graphics, or intrinsically. In all cases, stabilization is a compound move comprised of three ”unbalanced" stabilizations. We first treat the diagrammatic case, as it is the most straightforward.

If T_1 is a (g_1, k_1)-trisection, and T_2 is a (g_2, k_2)-trisection, we may form their connected sum $T_1 \# T_2$, which inherits the structure of a $(g_1 + g_2, k_1 + k_2)$-trisection. On the level of diagrams, this amounts to taking the connected sum of trisection diagrams for T_1 and T_2. There seems to be a choice involved, as one must choose disks on both trisection diagrams to excise and attach a tube, and these disks may lie in different positions with respect to the cut systems. It is not too hard to see however (an explicit outline of this is given in Proposition 3.5 of [35]), that the meridian of the connected sum tube bounds a disk in all three resulting handlebodies. The choice therefore is immaterial up to handle slides.

As noted before, S^4 admits three unbalanced trisections of genus 1. Forming their connected sum gives the balanced $(3, 1)$-trisection shown in Figure 1-16. If T is a (g, k)-trisection for M^4, we may form a $(g + 3, k + 1)$-trisection for M by taking a connected sum with the aforementioned trisection for S^4. The resulting $(g+3, k+1)$-trisection is called a **stabilization** of T. We may also take the connected sum of

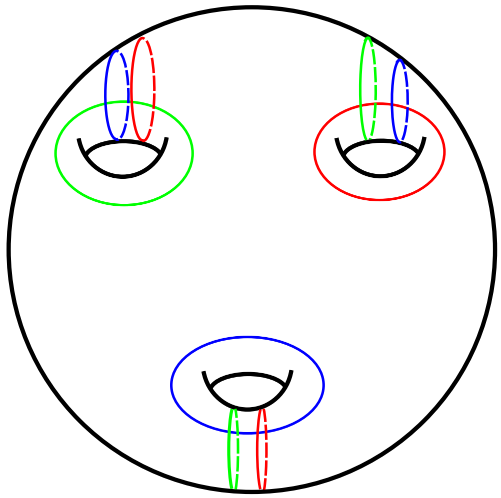

Figure 1-16: The (3,1)-trisection of S^4 used to define the stabilization operation.

T with one of the unbalanced genus 1 trisections of S^4. The resulting (possibly unbalanced) trisection is called an **i-stabilization** of T, where $i \in \{1, 2, 3\}$ indicates that we are summing with the unbalanced genus 1 trisection of S^4 where $k_i \neq 0$.

Now that we have a rigorous definition of a stabilization, we will set up some notation and state the main theorem of this section before proceeding to the alternative perspectives. Let T be a genus h trisection and let $g = h + 3n$ for some $n \in \mathbb{N}$. We denote by $\boldsymbol{T^g}$ the $(h + 3n, k + n)$-trisection obtained by stabilizing T n times to a genus g trisection. If T has spine $H_{12} \cup H_{23} \cup H_{31}$ and trisection surface Σ, we will denote the tripod and trisection surface of T^g by $H^g_{12} \cup H^g_{23} \cup H^g_{31}$ and Σ^g. The following theorem has shown to itself to be essential for extending invariants of trisections to invariants for 4-manifolds. It can be seen as the analogue of Theorem 1.3.6 for trisections.

Theorem 1.4.6. *(Theorem 11 of [14]) If T_1 and T_2 are trisections of the same manifold X, then there exists a natural number n so that T^n_1 and T^n_2 are isotopic as* *trisections. That is, if $T_1^n = X_1 \cup X_2 \cup X_3$ and $T_2^n = Y_1 \cup Y_2 \cup Y_3$, then there exists a self diffeomorphism f of X isotopic to the identity so that $f(X_i) = Y_i$.*

From the perspective of a Cerf function, the stabilization operation comes from the eye birth modification, drawn in Figure 1.2.1. Note that one can always introduce an eye into any point of a Cerf graphic without changing the underlying manifold. Given a trisected Cerf function, we may introduce an eye into the center and distribute the two cusps such that they lie in different sectors of the trisection, as illustrated in Figure 1-17. Since all folds are still of index 1 when pointing towards the center, the resulting Cerf graphic is still naturally trisected.

The sectors of the trisections receiving the cusps have the genus of the Heegaard splitting on their boundary increase, but the diffeomorphism type of the boundary is left unchanged. On the other hand, the sector which does not have a cusp picks up one more factor of $S^1 \times S^2$ through this process. We see immediately that if we started with a balanced trisection, we now have an unbalanced trisection. We may remedy this by doing this operation a total of three times, where each time we choose a different sector receive the region with no cusps. In fact, by determining the attaching regions for the added fold singularities, one can show that each eye added in this fashion is in fact an i-stabilization where X_i is the sector which does not receive a cusp.

We next turn our attention to the intrinsic perspective of the stabilization operation. We begin with the case of Heegaard splittings, and proceed by analogy. Let $M^3 = H_1 \cup_\Sigma H_2$ be a genus g Heegaard splitting, and let a be a boundary parallel

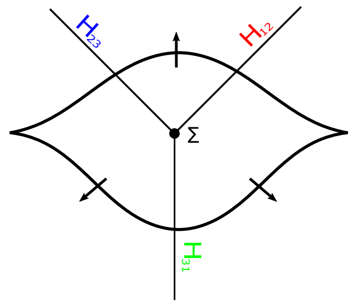

Figure 1-17: A 2-stabilization from the perspective of a Morse 2-function.

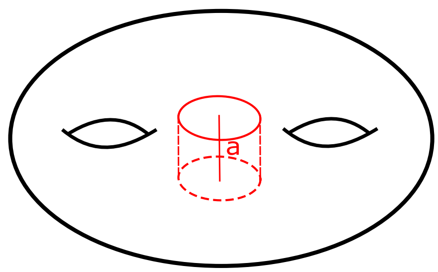

Figure 1-18: The neighbourhood of a boundary parallel arc in a 3-dimensional handlebody.

arc in H_1. Note that removing an open regular neighbourhood of a from H_1 produces a handlebody of genus $g+1$, as can be readily seen in Figure 1-18. Furthermore, the closed regular neighbourhood of a meets H_2 along two disks. It follows that if we were to redistribute a closed regular neighbourhood of a to H_2, we would be adding a 3-dimensional 1-handle to H_2, which again produces a handlebody of genus $g+1$. To summarize, if $\nu(a)$ denotes the open regular neighbourhood of a, then the decomposition $M^3 = H_1 \backslash \nu(a) \cup (H_2 \cup \overline{\nu(a)})$ is a genus $g+1$ Heegaard splitting of M.

Moving to 4-dimensions, let $M^4 = X_1 \cup X_2 \cup X_3$ be a (g,k)-trisection, and let a_{12}, a_{23}, and a_{31} be boundary parallel arcs in the handlebodies H_{12}, H_{23}, and H_{31} respectively. We claim that the decomposition $M^4 = ((X_1 \cup \overline{\nu(a_{23})}) \backslash (\nu(a_{12}) \cup \nu(a_{31}))) \cup ((X_2 \cup \overline{\nu(a_{31})}) \backslash (\nu(a_{12}) \cup \nu(a_{23}))) \cup ((X_3 \cup \overline{\nu(a_{12})}) \backslash (\nu(a_{23}) \cup \nu(a_{31})))$ is again a trisection.

By our previous discussion on the intrinsic stabilization of Heegaard splittings, it follows that the 3-dimensional handlebodies in the trisection become 3-dimensional genus $g+3$ handlebodies in this new decomposition, where a genus is added each time and arc is added or removed. Adding a 4-dimensional regular neighbourhood of an arc which meets X_i in two balls corresponds to adding a 4-dimensional 1-handle to X_i, and removing open regular neighbourhoods of arcs can be realized by an isotopy of the X_i, and does not affect the diffeomorphism type. In summary, this operation produces a $(g+3, k+1)$-trisection of the same 4-manifold, and it is straightforward to check that this operation is equivalent to the previously defined stabilizations. The following chapter shows that this operation is in fact necessary in order to make some trisections of the same genus isotopic.

Chapter 2

Distinguishing Decompositions of Manifolds

2.1 Nielsen equivalence

In this chapter, we will show how to construct inequivalent decompositions of the same genus on fixed 3- and 4-manifolds. The work on 3-manifolds is not novel, however, the adaptation and application to 4-dimensional decompositions is. Our strategy will be to develop an algebraic invariant which is sensitive to the different ways we can decompose a manifold. We start by defining an equivalence relation between generating sets of the same size of a fixed group.

Definition 2.1.1. Let $F_n = F[x_1, ..., x_n]$ be the free group of rank n with basis $(x_1, ..., x_n)$. Let G be a finitely generated group and $A = (a_1, ..., a_n)$ and $B = (b_1, ...b_n)$ be generating sets of size n for G. A and B are called **Nielsen equivalent**

if there exists a basis $(y_1, y_2, ..., y_n)$ for $F[x_1, ...x_n]$ and a homomorphism $\phi : F_n \to G$ so that $\phi(x_i) = a_i$ and $\phi(y_i) = b_i$.

Given a generating set A, we will denote its Nielsen class by $\mathscr{N}(\boldsymbol{A})$. It is a classical result [42] that the automorphism group of the free group is generated by the elementary Nielsen transformations. Given the free group of rank n with ordered basis $(x_1, x_2, ..., x_n)$, the Nielsen transformations are the following:

1. Swap x_1 and x_2.

2. Cyclically permute $(x_1, x_2, ..., x_n)$ to $(x_2, x_3, ..., x_n, x_1)$.

3. Replace x_1 with x_1^{-1}.

4. Replace x_1 with x_1x_2.

In light of this result, we obtain an alternative characterization of Nielsen equivalence. Given two generating sets of a group, $A = (a_1, ..., a_n)$ and $B = (b_1, ...b_n)$, write the b_i as words, w_i, in the generators $(a_1, a_2, ..., a_n)$ to obtain an ordered list of words, $(w_1, w_2, ..., w_n)$. A and B are Nielsen equivalent if and only if we can successively apply the automorphisms 1-4 above to get from the ordered set $(a_1, a_2, ..., a_n)$ to $(w_1, w_2, ..., w_n)$. Here, we are allowed to simplify words using any applicable relations in the group.

Despite the simplicity of the moves, it is usually difficult to tell if two generating sets are fact Nielsen equivalent. To this end, we seek to develop invariants of Nielsen classes in order to more readily distinguish them. It often happens that there are no obvious computable invariants of Nielsen classes in a given group, but when passing to a quotient, geometric or algebraic properties allow one to construct powerful invariants. It is therefore helpful to develop a relation between the Nielsen classes of a group and the Nielsen classes of the induced generators in a quotient group. The following lemma shows that we can pass to quotients in order to distinguish generators.

Lemma 2.1.2. *Let G be a group, and $A = (a_1, ..., a_n)$ and $B = (b_1, ...b_n)$ be generating sets of size n for G. Let H be a quotient of G, with quotient map $\phi : G \to H$ and let $\phi(A) = (\phi(a_1), ..., \phi(a_n))$ and $\phi(B) = (\phi(b_1), ...\phi(b_n))$ be the induced generating set of H under the quotient map. Then if $\phi(A)$ and $\phi(B)$ are not Nielsen equivalent, A and B are not Nielsen equivalent.*

Proof. We will prove the contrapositive. Suppose A and B are Nielsen equivalent. Then there is a sequence of elementary Nielsen transformations taking the generators in A and to the generators in B. Using the fact that ϕ is a group homomorphism, one can easily check that the elementary Nielsen transformations on the a_i descend to elementary Nielsen transformations on the $\phi(a_i)$. Therefore, the sequence of transformations taking A to B gives a sequence of transformations taking $\phi(A)$ to $\phi(B)$. $\square$

We are now in a good position to give a relatively simple example which led Engmann [10] and Birman [2] to construct the first inequivalent Heegaard splittings. Consider the group $\mathbb{Z}_7 * \mathbb{Z}_7$ given by the presentation $\langle a, b | a^7, b^7 \rangle$. Notably, this group is the fundamental group of the 3-manifold $L(7, 2)\#L(7, 2)$. Consider the generating sets (a, b) and (a, b^5). Under the abelianization map, these map to $((1, 0), (0, 1))$ and

$((1,0),(0,5))$ in $\mathbb{Z}_7 \times \mathbb{Z}_7$.

We will to show that these generating sets are inequivalent in $\mathbb{Z}_7 \times \mathbb{Z}_7$. To see this, we will first arrange the generators in a matrix. One can then check that the elementary Nielsen transformations 1-3 change the mod 7 determinant by a factor of -1 and that transformation 4 preserves the determinant of this matrix. Therefore the determinant, up to sign, is an invariant of the Nielsen class. Since $1 \neq \pm 5 (mod 7)$ we conclude that these generators are not Nielsen equivalent. By Lemma 2.1.2 we learn that (a, b) and (a, b^5) are inequivalent generating sets of $\mathbb{Z}_7 * \mathbb{Z}_7$.

2.2 Nielsen equivalence in Fuchsian groups

A particularly interesting case of Nielsen equivalence occurs in abstract Fuchsian groups. Recall that a Fuchsian group is a discrete subgroup of $PSL(2, \mathbb{R})$. An abstract Fuchsian group is a group which embeds as a discrete subgroup of $PSL(2, \mathbb{R})$. For our applications, we will primarily be concerned with abstract Fuchsian groups whose finite order elements have odd and pairwise co-prime orders. If $m \geq 2$ is an integer and $\{\alpha_1, ..., \alpha_m\}$ is a set of m pairwise distinct integers, these groups have a presentation given by

$$\langle s_1, ..., s_n, a_1, b_1, ..., a_g, b_g | s_i^{\alpha_i}, s_1 s_2 ... s_n \Pi[a_i, b_i] \rangle.$$

Viewing $PSL(2, \mathbb{R})$ as the group of isometries of the hyperbolic plane gives these groups some geometric meaning. The quotient of $\mathbb{H}^2$ by a Fuchsian group inherits the structure of a 2-dimensional orbifold, and for this reason, abstract Fuchsian groups naturally come up when studying Seifert fibered spaces, which may be regarded as circle bundles over orbifolds. A theorem of Rosenberger considerably narrows the possible Nielsen classes of such a group to a tangible set.

Theorem 2.2.1. *([45], Satz 2.2) Let $m \geq 2$ be an integer and let $\{\alpha_1, ..., \alpha_m\}$ be a set of m pairwise distinct integers. Let G be the Fuchsian group given by the presentation*

$$\langle s_1, ..., s_n, a_1, b_1, ..., a_g, b_g | s_i^{\alpha_i}, s_1 s_2 ... s_n \Pi[a_i, b_i] \rangle.$$

Then every generating system of size $2g+m-1$ is Nielsen equivalent to one of the form $\{s_1^{\beta_1}, ..., s_{k-1}^{\beta_{k-1}}, s_{k+1}^{\beta_{k+1}}, ..., s_n^{\beta_n}, a_1, b_1, ...a_g, b_g\}$ where $gcd(\alpha_i, \beta_i) = 1$ and $1 \leq \beta_i \leq \alpha_i/2$.

In order to develop invariants of such generators, we turn towards the Fox derivative, an important derivation in the free group developed by Fox in a series of five papers starting with [11]. Given a free group $F_n = F[x_1, ...x_n]$, the ith Fox derivative $\partial/\partial x_i$ is the unique $\mathbb{Z}$ linear function $\mathbb{Z}F_n \to \mathbb{Z}F_n$ satisfying $\partial x_j/\partial x_i = \delta_{i,j}$ and $\partial ab/\partial x_i = \partial a/\partial x_i + a\partial b/\partial x_i$. Given another basis of F_n, $\{y_1, ..., y_n\}$, we can form the Jacobian matrix $(\partial y_i/\partial x_j)_{i,j}$.

Given two generating sets of a Fuchsian group G, one may hope to lift both sets to the free group, and compute the Jacobian matrix of this change of basis. This approach immediately runs into trouble, as the lifts of generators to the free group are highly non-unique. The indeterminacy of these lifts is exactly the relations of the group. The solution of Lustig and Moriah [32] to this issue is to find some representation in which the relations contribute nothing to the determinant of this Jacobian matrix.

Fuchsian groups, by their definition, admit faithful representations into $PSL(2,\mathbb{R})$, which we may regard as a faithful representation into $PSL(2,\mathbb{C})$. It is known that a Fuchsian group admits a faithful representation into $SL(2,\mathbb{C})$ if and only if the orders of the parabolic elements are odd [8][28]. Let G be a Fuchsian group whose parabolic elements have odd order, and let $\rho : G \to SL(2,\mathbb{C})$ be a faithful representation. Given a surjective homomorphism $f : F_n \to G$ and two n-element generating sets U and V of G, we may lift these generators arbitrarily to the free group, and compute the Jacobian matrix, which we denote by $\partial\mathscr{U}/\partial\mathscr{V}$. Then by applying $\rho \circ f$ elementwise to this matrix and "unbracketing," we obtain a matrix of complex numbers. We denote this determinant by $det(\partial\mathscr{U}/\partial\mathscr{V})$. Surprisingly, Lustig and Moriah obtain the following results.

Proposition 2.2.2. *(Corollary 1.8 and 1.10 of [32]) The complex number $det(\partial\mathscr{U}/\partial\mathscr{V})$ does not depend on the choice of faithful representation. Furthermore, $det(\partial\mathscr{U}/\partial\mathscr{V})$ depends only on the Nielsen classes of U and V.*

It turns out that this invariant is enough to completely classify Nielsen classes of these Fuchsian groups. More precisely, we have the following theorem.

Theorem 2.2.3. *Let G be the Fuchsian group given by the presentation*

$$\langle s_1, ..., s_n, a_1, b_1, ..., a_g, b_g | s_i^{\alpha_i}, s_1 s_2 ... s_n \Pi[a_i, b_i]\rangle$$

with $m \geq 3$ or $g \geq 1$ such that the α_i are all odd and pairwise relatively prime. Two generating sets of the form $\{s_1^{\beta_1}, ..., s_{k-1}^{\beta_{k-1}}, s_{k+1}^{\beta_{k+1}}, ..., s_n^{\beta_n}, a_1, b_1, ...a_g, b_g\}$ and $\{s_1^{\beta'_1}, ..., s_{j-1}^{\beta'_{j-1}},$

$s_{j+1}^{\beta'_{j+1}}, ..., s_n^{\beta'_n}, a_1, b_1, ...a_g, b_g\}$ *where* $gcd(\alpha_i, \beta_i) = gcd(\alpha_i, \beta'_i) = 1$, $1 \leq \beta_i, \beta'_i \leq \alpha_i/2$ *are Nielsen equivalent if and only if either:*

1. $j = k$ *and* $\beta_i = \beta'_i$ *or*

2. $j \neq k$, $\beta_j = \beta'_k = 1$, *and* $\beta_i = \beta'_i$ *for* $i \neq j, k$

It should be noted that other authors have distinguished Nielsen classes of generators in other classes of Fuchsian groups as well. Notably Boileau, Collins and Zeischang [3] completely determine the Nielsen classes in the case where $m = 3$ and $g = 0$. Much of the work on this topic is done with the goal of distinguishing Heegaard splittings, and it has proven to be a very effective tool.

2.2.1 Nielsen equivalence and automorphisms

Automorphisms of a group act on the Nielsen classes of a group by simply applying the automorphism to each element of the generating set. In general, the Nielsen class of A may not be equal to the Nielsen class of $\phi(A)$. This, however, turns out not to be the case for the groups we are considering. The following theorem of Lustig, Moriah, and Rosenberger in [33] is pertinent.

Theorem 2.2.4. *Let* $m \geq 2$ *be an integer and let* α_i *be a set of* m *pairwise distinct integers. Let* G *be the Fuchsian group given by the presentation*

$$\langle s_1, ..., s_n, a_1, b_1, ..., a_g, b_g | s_i^{\alpha_i}, s_1 s_2 ... s_n \Pi[a_i, b_i] \rangle.$$

Consider the generating system given by $A = \{x_1 = s_1^{\beta_1}, ..., x_{k-1} = s_{k-1}^{\beta_{k-1}}, x_{k+1} =$

$s_{k+1}^{\beta_{k+1}}, ..., x_n = s_n^{\beta_n}, a_1, b_1, ...a_g, b_g\}$ *where* $(\alpha_i, \beta_i) = 1$. *Then any automorphism* $h : G \to G$ *is induced by some automorphism of the free group* $F_{2g+m-1} = F[X_1, ..., X_{k-1}, X_{k-1}, ... X_n, A_1, B_1, ..., A_g, B_g]$ *with respect to the surjection* $F_{2g+m-1} \to G$ *given by* $X_i \mapsto x_i$, $A_i \mapsto a_i$, $B_i \mapsto b_i$.

Corollary 2.2.5. *Let* G *be a Fuchsian group, and* A *be a generating set of* G *satisfying the hypothesis of Theorem 2.2.4. Let* ϕ *be any automorphism of* G. *Then* $\mathscr{N}(A) = \mathscr{N}(\phi(A))$

Proof. Let $f : F[X_1, ..., X_{k-1}, X_{k-1}, ..., X_n, A_1, B_1, ..., A_g, B_g] \to G$ be the surjection given in Theorem 2.2.4 and let ψ be the automorphism of the free group which induces ϕ, as guaranteed by the same theorem. Now $\psi(X_1), ..., \psi(X_{k-1}), \psi(X_{k-1})$,...,$\psi(X_n), \psi(A_1), \psi(B_1), ..., \psi(A_g), \psi(B_g)$ is again some generating set for the free group. Moreover, $f(\psi(X_i)) = \phi(x_i), f(\psi(A_i)) = \phi(a_i)$, and $f(\psi(B_i)) = \phi(b_i)$. In other words, f sends the original generating set to A and ψ of the generating set to $\phi(A)$, which is precicely the definition of Nielsen equivalence. □

2.3 Spines of handlebodies and Nielsen equivalence

In this section, we seek to relate the algebra and topology which have been discussed. In particular, we seek to tie Nielsen equivalent generating sets to the generating sets of handlebodies coming from spines discussed in Section 1.2.2. The following lemma relates the moves on spines to the Nielsen transformations discussed in Section 2.1 and is straightforward to prove.

Lemma 2.3.1. *Let S be a spine of H_g^n consisting of loops labeled $x_1, ..., x_g$. Let $\pi_1(H_g^n) = F_g$ have the ordered basis consisting of the labels $(x_1, ..., x_n)$. Then Nielsen transformations on F_g of type 1 and 2 correspond to relabeling the edges of S. Type 3 transformations correspond to reversing the orientation of an edge and type 4 transformations correspond to edge slides. Therefore, all automorphisms of $\pi_1(H_g^n)$ are realized by permutations of labels, edge slides, and reversals of orientations on spines.*

Since $Aut(F_n)$ is generated by the Nielsen transformations, the previous lemma shows that any ordered basis of the free group can be realized as a spine of a given handlebody. The next logical question is the uniqueness of this realization. This matter is especially delicate in dimension 3, since there are always many isotopy classes of simple closed curves in a given homotopy class. Surprisingly, insisting that these loops form a spine is enough to narrow the large number isotopy classes to a single one. The approach here is to show that any two spines are related by the moves given above and to note that each of these moves acts non-trivially on the homotopy class of an edge in a spine.

Lemma 2.3.2. *Any two spines of H_g^n are related by changing orientations and edge slides.*

Proof. In dimension 3, this is well known, but we sketch a proof for completeness. In a 3-dimensional handlebody, a spine gives rise to a unique set of disks dual to the spine. In our situation, where the spine has a single vertex, we get a minimal disk system, that is, a collection of disks which cuts the handlebody into a single 3-ball. Conversely, a minimal disk system also gives rise to a unique spine. It follows from the work of Reidemeister and Singer in [44] and [49] that any two minimal disk systems for a handlebody are related by a sequence of disk slides. If D_1 and D_2 are dual disks for l_1 and l_2, respectively, then the disk system obtained by disk sliding D_1 over D_2 is dual to the spine obtained by edge sliding l_2 over l_1. Thus any spine can be obtained by converting to disk systems and performing the dual edge slides prescribed by the disk slides between the disk systems.

In dimensions $n \geq 4$, the situation is simpler. Let S and S' be spines consisting of loops $s_1, ..., s_g$ and $s'_1, ..., s'_g$ respectively. The homotopy classes of these loops specify 2 bases for F_g. These bases are related by Nielsen transformations which, by Lemma 2.3.1, can be realized geometrically as moves on spines. Apply these moves to S until we obtain loops in the same homotopy class as the loops in S'. Now since $n \geq 4$ homotopic loops are in fact isotopic. Then each loop of S can be isotoped to the corresponding loop of L' homotopic to it by an isotopy which, by general position, misses the other loops. □

2.4 Invariants of decompositions of manifolds

In this section, we will briefly review some relevant properties of the decompositions of 3- and 4-manifolds into handlebodies, and find some invariants of these decompositions. We begin with dimension 3. Recall that a Heegaard splitting can naturally be seen as the middle level of a Heegaard-Morse function. From this point of view, H_1 is the union of the 0- and 1-handles, so that the inclusion map $i : H_1 \hookrightarrow M^3$ induces a surjection of the fundamental groups. The fundamental group of H_1 is likewise surjected onto by the fundamental group of its spine under the inclusion map. By inverting the Morse function, H_2 becomes the union of the 0- and 1-handles, so the fundamental group of the spine of H_2 also surjects onto $\pi_1(M^3)$. Thus, a Heegaard splitting determines two sets of generators for $\pi_1(M^3)$. In light of Lemmas 2.3.1 and 2.3.2, the spines of each handlebody give well defined Nielsen equivalence classes of $\pi_1(M^3)$. This motivates the following definition, which will also be used in decompositions in other dimensions.

Definition 2.4.1. Let $i : H_g^n \hookrightarrow M^n$ be an embedding of an n-dimensional genus g handlebody which induces a surjection of fundamental groups, and let H denote the image of H_g^n in M^n. We will denote by $\boldsymbol{\mathcal{N}(H)}$ the Nielsen class of the generators of $\pi_1(M^n)$ obtained from spines of H. If f is a self homeomorphism of M^n we will denote by $\boldsymbol{f(\mathcal{N}(H))}$ the Nielsen class obtained by applying the induced map on fundamental groups to each loop of some spine of H.

The following proposition has been used extensively to distinguish Heegaard splittings (see [3] and [32] for particular applications).

Proposition 2.4.2. *Let $H_1 \cup_\Sigma H_2$ and $H_1' \cup_{\Sigma'} H_2'$ be two genus g Heegaard splittings of M^3. If these Heegaard splittings are isotopic, then $\mathcal{N}(H_1) = \mathcal{N}(H_1')$, and $\mathcal{N}(H_2) = \mathcal{N}(H_2')$. If the Heegaard splittings are homeomorphic by some homeomorphism f, then $f(\mathcal{N}(H_1)) = \mathcal{N}(H_1')$ and $f(\mathcal{N}(H_2)) = \mathcal{N}(H_2')$.*

Proof. If $H_1 \cup_\Sigma H_2$ and $H_1' \cup_{\Sigma'} H_2'$ are isotopic, then the isotopy takes H_1 to H_1'. In particular, a spine for H_1 is taken to a spine for H_1'. By Lemma 2.3.2, these spines can be made equivalent by a series of edge slides and reversals of orientations. By

Lemma 2.3.1, this implies that the generators of $\pi_1(M^3)$ coming from the spines are Nielsen equivalent. An identical argument shows that $\mathscr{N}(H_2) = \mathscr{N}(H_2')$. If the splittings are homeomorphic, a similar argument applies after the application of the homeomorphism. $\square$

We next turn our attention to 4-dimensional decompositions. Recall that, by Lemma 1.4.5, we may obtain a handle decomposition of a 4-manifold, M, from a trisection, $M = X_1 \cup X_2 \cup X_3$, so that any X_i is the union of the 0- and 1-handles. From this point of view, it is clear that X_i generates $\pi_1(M)$. Using Lemmas 2.3.2 and 2.3.1 we see that we in fact get 3 Nielsen equivalence classes of generators. The following proposition has a proof which is nearly identical to that of Proposition 2.4.2.

Proposition 2.4.3. *Let $X_1 \cup X_2 \cup X_3$ and $X_1' \cup X_2' \cup X_3'$ be two (g,k)-trisections of M^4. If these trisections are isotopic, then $\mathscr{N}(X_i) = \mathscr{N}(X_i')$. If the trisections are diffeomorphic by some diffeomorphism f, then $f(\mathscr{N}(X_i)) = \mathscr{N}(X_i')$.*

Proof. If $X_1 \cup X_2 \cup X_3$ and $X_1' \cup X_2' \cup X_3'$ are isotopic, then in particular a spine for X_i is taken to a spine for X_i'. By Lemma 2.3.2 these spines are related by a series of edge slides and reversals of orientations. By Lemma 2.3.1 this implies that the generators of $\pi_1(M^4)$ coming from the spines are Nielsen equivalent. If the splittings are homeomorphic, a similar argument applies after the application of the homeomorphism. $\square$

2.5 Seifert fiber spaces and their Heegaard splittings

Seifert fiber spaces provide a rich set of examples of Heegaard splittings which may be distinguished using Proposition 2.4.2. These spaces were classified in 1933 by Seifert [48] and have since provided important examples of 3 dimensional manifolds. They play an especially important role in the geometrization of 3-manifolds, since 3-manifolds admitting six of the possible eight geometries admit Seifert fiberings. In what follows, we will primarily be interested in fully orientable Seifert fibrations; that is, fibrations whose total space and base orbifold are orientable. All of these manifolds can be obtained by starting with a surface bundle over a circle, and performing surgery along the circles $S^1 \times \{x_0\}, S^1 \times \{x_1\}, ... S^1 \times \{x_r\}$ for $\{x_1, ..., x_r\} \subset \Sigma_g$. We will denote by $S(g, e; (\alpha_1, \beta_1), (\alpha_2, \beta_2), ..., (\alpha_r, \beta_r))$ the unique fully orientable Seifert fiber space with a genus g base surface, Euler class e (as an S^1 bundle), and r exceptional fibers of type $\frac{\beta_i}{\alpha_i}$.

From an abstract point of view, applying the long exact sequence of a fibration shows that if S is a Seifert fibered manifold with base surface B, then the fundamental group fits into the exact sequence $\pi_1(S^1) \to \pi_1(S) \to \pi_1(B)$, where by $\pi_1(B)$ we mean the orbifold fundamental group. By their definition, orbifold fundamental groups fit into our class of Fuchsian groups. We therefore see that $\pi_1(S)/\mathbb{Z}$ is some Fuchsian group, which gives us hope for applying the results of Section 2.2. Before being able to apply these results however, we need a more concrete description of the generators of the fundamental group.

By viewing a fully orientable Seifert fiber space, S, as surgery on the product of a surface and a circle, we may use Van Kampen's theorem in order to obtain a presentation of $\pi_1(S)$. More precisely, after cutting out the solid tori $S^1 \times \{x_0\}, S^1 \times \{x_1\}, ...S^1 \times \{x_r\}$ for $\{x_1, ..., x_r\} \subset \Sigma_g$, we are left with a manifold with fundamental group generated by the standard generators on a the surface, $a_1, b_1, ..., a_g, b_g$, the regular fiber, f, and the meridians of the solid tori q_i. Doing $\frac{\beta_i}{\alpha_i}$ surgery introduces relations for the form $q^{\alpha_i} f^{\beta_i}$. Combining the relations coming from the surface to the relations coming from the surgeries and, noting that the fiber still lies in the center, we obtain the presentation:

$$\pi_1(S) = \langle a_1, b_1, ...a_g, b_g, q_1, ...q_r, f | [f, q_i], [f, a_i], [f, b_i], q^{\alpha_i} f^{\beta_i}, q_1...q_r[a_1, b_1]...[a_g, b_g]f^e \rangle.$$

From this perspective, it is evident that the quotient of this group by the cyclic group $\langle f \rangle$ is a Fuchsian group. Our next goal is to construct Heegaard splittings which see these group elements which survive in this quotient. In [40], it is shown that irreducible Heegaard splittings of Seifert fiber spaces are all constructed by one of two methods called horizontal or vertical. Vertical Heegaard splittings are well distinguished by the Neilsen classes they induce. On the other hand, it was shown in [26] that there are Seifert fiber spaces which admit infinitely non-isotopic horizontal Heegaard splittings which nevertheless induce Nielsen equivalent generating sets. For this reason we focus on the vertical splittings.

To construct a vertical Heegaard splitting, we start by describing a graph in $S(g, e; (\alpha_1, \beta_1), ..., (\alpha_r, \beta_r))$. Let Σ be the base surface and $e_1, ..., e_r$ be the images of the exceptional fibers, $f_1, ..., f_r$ on Σ. Choose a basepoint, p, on Σ which is the image of a regular fiber. Choose some non-empty, proper subset of indices $\{i_1, ..., i_j\} \subset \{1, ..., r\}$ and let σ_{i_k} be an arc based at p joining p to e_{i_k}. Let $\{m_1, ..., m_{r-j}\}$ be the complementary set $\{1, ..., r\} \backslash \{i_1, ..., i_j\}$. Let q_{m_k} be a loop based at p which winds around e_{m_k} once. Finally, let $a_1, b_1, ..., a_g, b_g$ be the usual collection of curves based at p which cut Σ into a disk. Choose all curves so that they are disjoint, except for at p. Let $\Gamma(i_1, ..., i_j)$ be the graph consisting of $a_1, b_1, ..., a_g, b_g, \sigma_{i_1}, f_{i_1}, ..., \sigma_{i_j}, f_{i_j}$, $q_{m_2}, ..., q_{m_{r-j}}$. Note that q_{m_1} was excluded, this was arbitrarily chosen and any q_{m_k} may be excluded.

Let $H_1(i_1, ..., i_j)$ be a tubular neighborhood of $\Gamma(i_1, ..., i_j)$ in $S(g, e; (\alpha_1, \beta_1), ..., (\alpha_r, \beta_r))$. This is clearly a handlebody of genus $2g + m - 1$. Moreover, the complement of this handlebody, $H_2(i_1, ..., i_j) = S(g, e; (\alpha_1, \beta_1), ..., (\alpha_r, \beta_r)) \backslash int(H_1(i_1, ..., i_j))$ is also a handlebody. We refer the reader to Section 2 of [32] for a proof of this fact. Moreover, in Remarks 2.1, 2.2, and 2.3 of [32] it is argued that the isotopy class of this Heegaard splitting only depends on the choices of $\{i_1, ..., i_j\}$ and that complementary choices of sets give the same Heegaard splitting. We may therefore denote by $\Sigma(i_1, ...i_j)$ the unique Heegaard splitting described above. Any Heegaard splitting obtained by the above process is called a **vertical Heegaard splitting**. A straightforward counting argument shows that $S(g, e; (\alpha_1, \beta_1), ..., (\alpha_r, \beta_r))$ has at most $2^{r-1} - 1$ distinct vertical Heegaard splittings of genus $2g + m - 1$.

In order to distinguish these splittings, we seek to understand the spines of these handlebodies. We first fix some notation for some of the relevant curves on the surface. Given a Seifert fiber space Σ with r exceptional points $x_1, ...x_r$, we take X_i to corresponding exceptional fibers. Let σ_i be a path from b to x_i and let Y_i be the

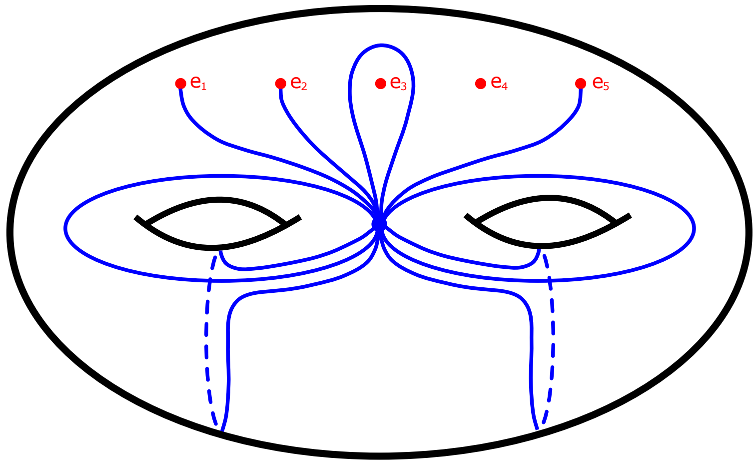

Figure 2-1: The graph describing the vertical Heegard Splitting $\Sigma(1,2,5)$. Note that the exceptional fibers f_1, f_2, and f_5 are included in this graph, but are not pictured.

path $\sigma_i X_i \sigma_{i^{-1}}$. Then tracing through the spines of the construction above we obtain the following:

Lemma 2.5.1. *(Lemma 2.6 of [32])*

Let $S(g,e;(\alpha_1,\beta_1),...,(\alpha_r,\beta_r))$ be a Seifert fiber space, $\Sigma(i_1,...i_j)$ be a vertical Heegaard splitting, and $\{m_1,...,m_{r-j}\}$ be the complementary set $\{1,...,r\}\backslash\{i_1,...,i_j\}$. Then the spine of H_1 corresponds to the generators $(a_1,b_1,...,a_g,b_g,Y_{i_1},...Y_{i_j},q_{l_1},...q_{l_{m-j-2}})$ for any $m-j-2$ elements in $\{1,...,m\}\backslash\{i_1,...,i_j\}$ and the spine of H_2 corresponds to the generators $(a_1,b_1,...,a_g,b_g,\ Y_{l_0},...Y_{l_j},q_{i_1},...q_{i_j})$.

Our next goal is to determine the image of these generators in the quotient group. We would like to describe these in terms of meridian and longitude pairs for the neighbourhood of an exceptional fiber. Let μ_i and λ_i be a meridian longitude pair for the boundary of a regular neighbourhood of X_i, and let f be the homotopy class of a regular fiber on the boundary torus. Recall that the exceptional fiber was obtained by doing $\frac{\beta_i}{\alpha_i}$ surgery on a regular fiber. If we let γ_i and δ_i be integers such that $\beta_i\gamma_i - \alpha_i\delta_i = 1$, then we see that $f = \mu^{\gamma_i}\lambda_i^{\alpha_i}$ and $\lambda_i = q_i^{-\gamma_i}h^{-\delta_i}$. We then see that in $\pi_1(S)/\langle f\rangle$, $X_i = \lambda_i = q_i^{-\gamma_i}$ and that $X_i^{\alpha_i} = q_i^{\alpha_i(-\gamma_i)} = \mu_i = 1$. In summary, we obtain the following:

Corollary 2.5.2. *(Lemma 2.7 of [32]) In the quotient $\pi_1(M)/\langle f\rangle$ the homotopy class of the spines of H_1 in a vertical Heegaard splitting map to the generators $\{q_{i_1}^{-\gamma_{i_0}}, ..., q_{i_j}^{-\gamma_{i_j}}, a_1, b_1, ...a_g, b_g\}$, and the spine for H_2 maps to the generators $\{q_{l_1}^{-\gamma_{l_0}}, ..., q_{m-j-2}^{-\gamma_{l_{m-j-2}}}, a_1, b_1, ...a_g, b_g\}$*

By combining the previous corollary with Theorem 2.2.3 we obtain the following theorem:

Theorem 2.5.3. *Let M be a Seifert fiber space with Seifert invariants $S(g, e; (\alpha_1, \beta_1), ..., (\alpha_r, \beta_r))$ satisfying the following conditions:*

- *$g > 0$ and $r > 0$, or $r \geq 3$*
- *$\beta_i \not\equiv \pm 1 (mod\, \alpha_i)$*
- *All of the α_i are odd and pairwise relatively prime.*

Then for $i \in \{1, 2\}$, $\mathcal{N}(H_i(i_1, ..., i_j)) = \mathcal{N}(H_i(k_1, ..., k_j))$ if and only if the sets $\{i_1, ..., i_j\}$ and $\{k_1, ..., k_j\}$ are equal or complementary to each other in $\{1, 2, ..., m\}$

Corollary 2.5.4. *A Seifert fiber space satisfying the criteria of Theorem 2.5.3 admits $2^{r-1} - 1$ non isotopic vertical Heegaard splittings of genus $2g + r - 1$, all of which are distinguished by their Nielsen classes.*

2.6 Meier's spun trisections

Having constructed non-isotopic Heegaard splittings on some fixed 3-manifolds we next set our sights on constructing inequivalent trisections on some 4-manifolds. Our approach here will be to leverage the previous work done on Heegaard splittings. For this task, we would like to construct 4-manifolds which are, in some sense, very similar to 3-manifolds. Among a few candidates, it turns out that spun manifolds are most suited for this task.

To construct a spun 4-manifold, we start with a closed 3-manifold M, and denote by $M_\circ$ the punctured manifold, obtained by removing a 3-ball from M. Now $M_\circ \times S^1$ has boundary $S^2 \times S^1$, which we may fill in with $S^2 \times D^2$ in two ways. Let $S(M) = M_\circ \cup_{id} S^2 \times D^2$ be the result of capping off $M_\circ$ with $S^2 \times D^2$ via the identity map and let $S^*(M) = M_\circ \cup_\tau S^2 \times D^2$ be the result of capping off $M_\circ$ with $S^2 \times D^2$ via the unique map of $S^2 \times S^1$ which does not extend across $S^2 \times D^2$. In other words, $S(M)$ and $S^*(M)$ differ by a Gluck twist about the copy of S^2 being attached. It is straightforward to see that $\pi_1(S(M)) = \pi_1(S^*(M)) = \pi_1(M)$.

In [34], Meier gives a construction which, given a genus g Heegaard splitting, $M = H_1 \cup_\Sigma H_2$, produces $(3g, g)$-trisections, $S(\Sigma)$ and $S^*(\Sigma)$, of $S(M)$ and $S^*(M)$ respectively. Shortly thereafter, Hayano [20] showed that $S(\Sigma)$ and $S^*(\Sigma)$ are simplified trisections, as defined by Baykur and Saeki in [1]. The construction begins with a genus g Heegaard splitting, $M = H_1 \cup_\Sigma H_2$. This splitting can be realized by a Heegaard-Morse function, $f : M \to [0, 2]$, with a unique index 0 critical point with critical value 0, g index 1 critical points which take distinct values in $(0, 1)$, g index 2

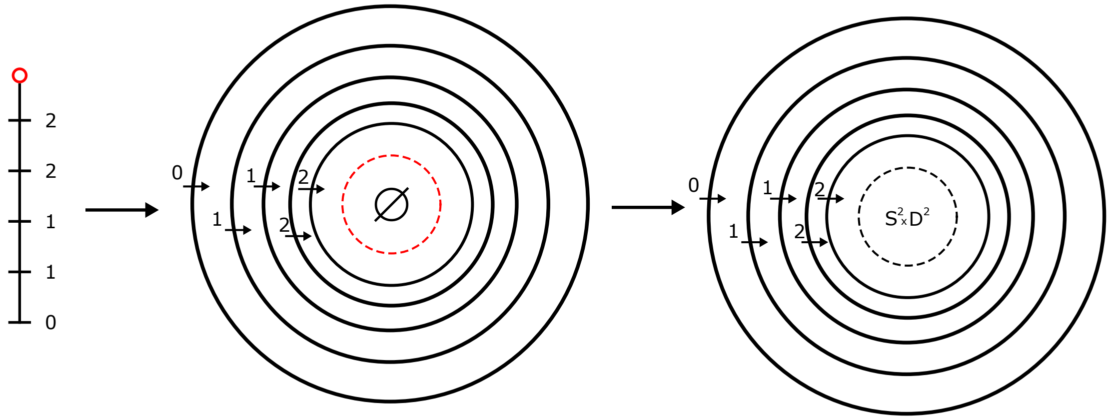

Figure 2-2: Crossing a punctured Heegaard splitting with S^1 produces a Cerf function to the annulus. The map can be extended to the disk after capping the 4-manifold off with $S^2 \times D^2$

.

critical points which take distinct values in $(1, 2)$, and a unique index 3 critical point with critical value 2.

Next, remove the 3-handle corresponding to the index 3 critical point in order to puncture M in H_2, and cross with S^1 to obtain $M_\circ \times S^1$. If we parametrize D^2 using polar coordinates, we obtain a generic smooth function $\tilde{f} : M_\circ \times S^1 \to D^2$ defined by $\tilde{f}(x, \theta) = (f(x), \theta)$. One can then fill in the boundary component of $M_\circ \times S^1$ with $S^2 \times D^2$ and extend $\tilde{f}$ across the inner disk of D^2 in the obvious way. This process is illustrated in Figure 2-2.

This Cerf function is not yet trisected, as there are critical folds which are of index 2 when co-oriented towards the center of the disk. In order to deal with these, we apply the compound move shown in Figure 1-7 in order to "flip" them so that all folds are of index 1 when moving towards the center of the disk. After these moves, we have a trisected Cerf function describing a $(3g, g)$-trisection of $S(M)$. An important

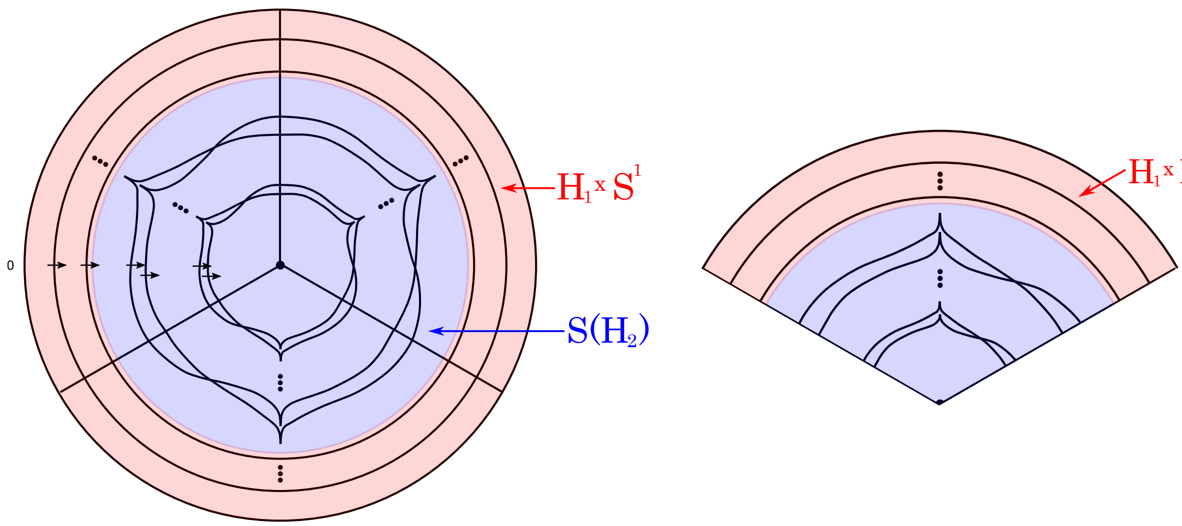

Figure 2-3: Left: The trisected Morse 2 function for $S(\Sigma)$. Right: The portion without cusps of X_i can be identified with $H_1 \times I$

observation made in [34] is that, throughout this process, H_1 is left unaltered. It is neither punctured, nor are there any modifications done to the Morse 2-function in the region corresponding to $H_1 \times S^1$. Therefore, the Morse 2-function for $S(\Sigma)$ can be decomposed into two pieces; one of which is $H_1 \times S^1$, and the other of which is the result of applying the spin construction to H_2. These two pieces meet along an embedded copy of $\Sigma \times S^1$. This decomposition is shown in Figure 2-3. This leads us to the next lemma.

Lemma 2.6.1. *Let $H_1 \cup_\Sigma H_2$ be a genus g Heegaard splitting of a closed 3-manifold M, and let $S(\Sigma) = X_1 \cup X_2 \cup X_3$ be the $(3g, g)$-trisection constructed in Section 2.6. Then if S is a spine for H_1, $f_{\frac{1}{6}}(S)$ is a spine for X_1.*

Proof. Note that X_1 consists of two pieces, one of the pieces is one third of $H_1 \times S^1$, which we will call $\tilde{X}_1$, and the other of which is one third of $S(H_2)$ (see the right side of Figure 2-3). The portion consisting of one third of $S(H_2)$ consists of only cusped folds and so it is a collar on $\tilde{X}_1$. Therefore, there is a deformation retraction of X_1, as a subset of $S(M)$, onto $\tilde{X}_1$. $\tilde{X}_1$, in turn, can be viewed as $f_{[0,\frac{1}{3}]}$ and so it deformation retracts onto $f_{\frac{1}{6}}$. This is an embedded copy of H_1 and so it retracts onto $f_{\frac{1}{6}}(S)$. After composing all of these retractions, we obtain that X_1 retracts onto $f_{\frac{1}{6}}(S)$. $\square$

Lemma 2.6.2. *Let $H_1 \cup_\Sigma H_2$ be a genus g Heegaard splitting of a closed 3-manifold M and let S be a spine for H_1. Let P the presentation for $\pi_1(M)$ consisting of $\langle s_1, s_2, ..., s_g | R\rangle$ where the s_i are the homotopy classes of the loops in S, and R is the set of relations induced by H_2. Then for any $\theta \in [0, 1)$, the set of generators given by the loops of $f_\theta(S)$ induce the same presentation for $\pi_1(S(M))$.*

Proof. We will break down each step of the construction of $S(M)$ and track presentations of the fundamental groups along the way. The first step is to puncture M to obtain $M_\circ$. There is a natural inclusion $i : M_\circ \hookrightarrow M$. This induces an isomorphism, i_*, on fundamental groups, and so there is an inverse $i_*^{-1} : \pi_1(M) \to \pi_1(M_\circ)$. Since S is left unchanged, we may identify $\pi_1(M_\circ)$ with P so that $i_*^{-1}(s_i) = s_i$. Also, since H_1 is unaltered in this process, $i|_{H_1}$ is a bijection, so it has an inverse, $i|_{H_1}^{-1}$.

Next, for each θ there is an inclusion given by $x \mapsto (x, \theta)$. Fix an angle θ and denote by $j : M_\circ \hookrightarrow M_\circ \times S^1$ the corresponding inclusion. We may identify $\pi_1(M_\circ \times S^1)$ with the presentation $\langle s_1, s_2, ..., s_g, z | R, zs_i = s_i z\rangle$ of $\pi_1(M) \times \mathbb{Z}$ with $j_*(s_i) = s_i$. Finally, we have an inclusion map $k : M_\circ \times S^1 \hookrightarrow S(M)$. k_* induces the projection map $\pi_1(M) \times \mathbb{Z} \to \pi_1(M)$. We may identify $\pi_1(S(M))$ with P so that $k_*(s_i, z) = k_*(s_i, 0) = s_i$.

Now note that $k \circ j \circ i|_{H_1}^{-1} = f_\theta$. Moreover we have arranged matters so that if we identify both $\pi_1(M)$ and $\pi_1(S(M))$ with the presentation P, then $f_*(s_i) = s_i$. This completes the proof.

□

Combining the previous two lemmas, we can see how to construct a spine for $S(M)$ which generates the fundamental group in the same way that one of the handlebodies of a Heegaard splitting does. This is the main ingredient in our proof of the main theorem.

Theorem 2.6.3. *Let M^3 be a closed, orientable, 3-dimensional manifold, and let $H_1 \cup_\Sigma H_2$ and $H_1' \cup_{\Sigma'} H_2'$ be Heegaard splittings of M. Then if $\mathscr{N}(H_1) \neq \mathscr{N}(H_1')$, the trisections $S(\Sigma)$ and $S(\Sigma')$ are not isotopic. Moreover, if for all $\phi \in Aut(\pi_1(M))$, $\phi(\mathscr{N}(H_1)) \neq \phi(\mathscr{N}(H_1'))$ then the trisections $S(\Sigma)$ and $S(\Sigma')$ are not diffeomorphic.*

Proof. By an isotopy of Σ, we may arrange so that $H_2 \cap H_2' \neq \emptyset$. We may then use the spin construction to obtain trisections $S(\Sigma) = X_1 \cup X_2 \cup X_3$ and $S(\Sigma') = X_1' \cup X_2' \cup X_3'$. Let S and S' be spines of H_1 and H_1' respectively. Let f_θ and g_θ be the S^1 parameterized families of embeddings of H_1 and H_1' respectively into $S(M)$. By Lemma 2.6.1, $f_{\frac{1}{6}}(S)$ is a spine for X_1 and $g_{\frac{1}{6}}(S')$ is a spine for X_1'. By Lemma 2.6.2, we may identify the fundamental groups of M and $S(M)$ so that these spines for X_1 and X_1' induce the same sets of generators as the spines for H_1 and H_1'. Therefore, $\mathscr{N}(H_1) = \mathscr{N}(X_1)$ and $\mathscr{N}(H_1') = \mathscr{N}(X_1')$. By assumption, $\mathscr{N}(H_1) \neq \mathscr{N}(H_1')$ so that $\mathscr{N}(X_1) \neq \mathscr{N}(X_1')$. By Proposition 2.4.3, $S(\Sigma)$ and $S(\Sigma')$ are not isotopic.

Similarly, let us suppose towards a contradiction that $S(\Sigma)$ and $S(\Sigma')$ are diffeomorphic by some diffeomorphism h. Then by the second part of Proposition 2.4.3, $h(\mathscr{N}(X_1)) = \mathscr{N}(X_1')$. But since $\mathscr{N}(H_1) = \mathscr{N}(X_1)$ and $\mathscr{N}(H_1') = \mathscr{N}(X_1')$ this implies that $h(\mathscr{N}(H_1)) = \mathscr{N}(H_1')$. This contradicts the assumption that no such h exists.

$\square$

2.7 Spinning Seifert fiber spaces

In this section, we combine with work done in the previous two sections to construct trisections which can be distinguished using Theorem 2.6.3. Recall that, by Corollary 2.5.4 the Seifert fiber space $S(g, e; (\alpha_1, \beta_1), ..., (\alpha_m, \beta_m))$ has exactly $2^{m-1}-1$ distinct vertical Heegaard splittings all of which are distinguished by their Nielsen classes. Moreover, case (b) of Theorem 1.1 of [4] states that the Heegaard genus of these manifolds is the same as the rank of their fundamental group, and both are equal to $2g+m-1$. Applying Meier's spin construction to the Heegaard splitting $\Sigma(i_1, ...i_j)$ therefore produces a $(3(2g+m-1), (2g+m-1))$-trisection of $S(\Sigma(i_1, ...i_j))$. Since the rank of the fundamental group of the spun manifold is equal to the $2g+m-1$, we see that these are in fact minimal genus trisections of $S(\Sigma(i_1, ...i_j))$. We distill the results stated above in the following corollary.

Corollary 2.7.1. *For every $n \geq 2$, there exist 4-manifolds which admit non-isotopic $(3n, n)-$trisections of minimal genus.*

We next turn our attention to diffeomorphism classes of trisections. In this case, if we seek to distinguish two trisections by their generating sets, Proposition 2.4.3 shows that we must also consider the effect of applying an automorphism of the group on a set of generators. The groups in question are fundamental groups of Seifert fiber spaces. As noted before, the fundamental group of a Seifert fiber space modulo its center is a Fuchsian group. Any diffeomorphism of a trisection will induce a map on fundamental groups which sends the center to itself. This map therefore descends to the quotient by the center, so we can apply Theorem 2.2.4 to see that the Nielsen classes previously distinguished do not become equivalent after an automorphism of a group. This leads to the following corollary.

Corollary 2.7.2. *For every $n \geq 2$, there exist 4-manifolds which admit non-diffeomorphic $(3n, n)$- trisections of minimal genus.*

2.7.1 Spun small Seifert fiber spaces

Seifert fiber spaces whose base space is a sphere with 3 exceptional fibers are called **small Seifert fiber spaces**. These manifolds admit at most three vertical Heegaard splittings of genus 2. In [3], the authors show that if for every singular fiber $\beta_i \not\equiv \pm 1 (mod\, \alpha_i)$ then $S(0, e; (\alpha_1, \beta_1), (\alpha_2, \beta_2), (\alpha_3, \beta_3))$ admits exactly three Heegaard splittings up to isotopy, all of which are vertical and distinguished by their Nielsen classes. These manifolds also admit a genus 3 Heegaard splitting obtained as follows: Take a path on the base space from a base point to all three critical points and connect each path to the respective fiber to form a wedge of 3 circles. Let H_1 to be a regular neighborhood of this graph and let H_2 be the complement of the interior of H_1. Note that the graph we constructed naturally forms the spine for H_1.

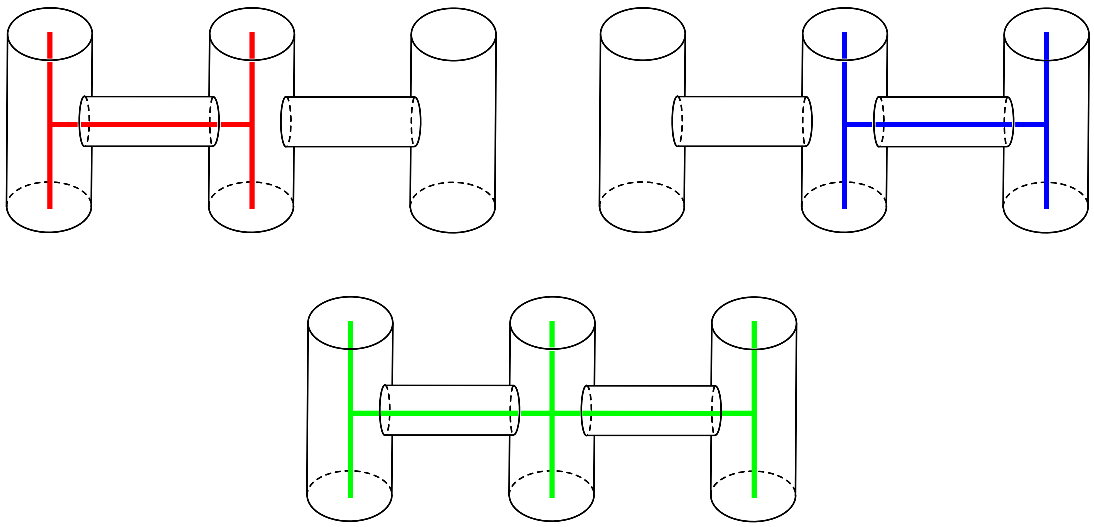

Figure 2-4: Top: A schematic picture of spines of H_1 of the vertical Heegaard splittings of S(0,3). Horizontal tubes lie in a neighborhood of the base sphere while vertical tubes are neighborhoods of fibers. Bottom: A spine of a genus 3 handlebody which contains the spines of both of the vertical Heegaard splittings above.

This construction closely resembles the construction of the vertical Heegaard splittings and one readily sees that the spines of H_1 of each of the vertical Heegaard splittings are subgraphs of the spine of H_1 of the genus 3 splitting we constructed above. Two of these Heegaard splittings together with the genus 3 splitting are depicted in Figure 2.7.1. The following corollary of the classification of Heegaard splittings of handlebodies in [47] tells us that these splittings are closely related.

Corollary 2.7.3. *Let $H_1 \cup_\Sigma H_2$ and $H_1' \cup_{\Sigma'} H_2'$ be two Heegaard splittings of M^3 and let S and S' be spines of H_1 and H_1', respectively. Then if S is a subgraph of S' then $H_1' \cup_{\Sigma'} H_2'$ is a stabilization of $H_1 \cup_\Sigma H_2$.*

We therefore conclude that all three of the vertical Heegaard splittings of $S(0, e; (\alpha_1, \beta_1), (\alpha_2, \beta_2), (\alpha_3, \beta_3))$ become isotopic after a single stabilization. An easy analysis of the local diagrammatic modifications in [34] shows that the spin of a stabilized Heegaard splitting is a stabilized trisection. This implies that $S(\Sigma(1,2))$, $S(\Sigma(2,3))$, and $S(\Sigma(1,3))$ are all non-isotopic trisections which become pairwise isotopic after a single balanced stabilization. The following proposition shows that a balanced stabilization as opposed to an unbalanced stabilization is required in a very strong sense.

Proposition 2.7.4. *The trisections $S(\Sigma(1,2))$, $S(\Sigma(2,3))$, and $S(\Sigma(1,3))$ become pairwise isotopic after a single balanced stabilization, however, for any two $\{k,l\} \subset \{1,2,3\}$ these trisections remain pairwise non-isotopic after any sequence of $k-$ and $l-$stabilizations.*

Proof. Let $H_1^{i,j}$ be H_1 of the Heegaard splitting $\Sigma(i,j)$ and let $X_n^{i,j}$ be X_n of $S(\Sigma(1,2))$. By Lemma 2.6.2 we have that for all $n \in \{1,2,3\}$, $\mathscr{N}(X_n^{i,j}) = H_1^{i,j}$. Choose any $\{k,l\} \subset \{1,2,3\}$ and let m be the remaining index. Then under any sequence of $k-$ and $l-$stabilizations, $X_m^{i,j}$ is unchanged. In particular, if we take $\{i,j\} \neq \{i',j'\}$ then under any sequence of $k-$ and $l-$ stabilizations $\mathscr{N}(X_m^{i,j}) \neq \mathscr{N}(X_m^{i',j'})$. By Proposition 2.4.3 this implies that the trisections remain non-isotopic. $\square$

Chapter 3

Invariants Coming from the Pants Complex

3.1 Introduction

One of the most useful features of trisections is their ability to encode the information of a 4-manifold conveniently on a surface. This allows one to translate questions about 4-manifolds to questions about surfaces. One of the most fruitful ways to answer such questions is to translate them into questions about complexes associated to the simple closed curves which lie on the surfaces. To begin, we give a brief overview of such complexes.

3.1.1 Simplicial Complexes Associated to Surfaces

The most commonly used complex associated to a surface is the curve complex. It has proven to be a useful tool in investigating a wide variety of structures, especially in illuminating the structure of the mapping class group of an orientable surface. We recall the definition here.

Definition 3.1.1. Given a closed, orientable surface, Σ, of genus $g \geq 2$, the **curve complex** of Σ, denoted $C(\Sigma)$, is a simplicial complex built out of simple closed curves on Σ. Each isotopy class of essential simple closed curves corresponds to a vertex. A collection of n vertices spans an $(n-1)$-simplex if the corresponding curves can be isotoped to be pairwise disjoint.

In his seminal work in [22], Hempel used the curve complex to give an invariant of Heegaard splittings generalizing the notions of reducibility, weak reducibility, and the disjoint curve property. While Hempel's distance is an indispensable tool for investigating the structure of Heegaard splittings of a 3-manifold, it is unlikely to be useful for constructing invariants of manifolds. This is due to the fact that the invariant completely collapses when a Heegaard splitting is stabilized. Our set up for trisections will have similar problems, so we consider the dual of the curve complex.

Definition 3.1.2. Given a closed, orientable surface of genus $g \geq 2$, the **dual curve complex** of Σ, denoted $C^*(\Sigma)$, is the simplicial complex whose vertices correspond to maximal dimensional simplices of $C(\Sigma)$. Two vertices in $C^*(\Sigma)$ have an edge between them if the corresponding maximal dimensional simplices of $C(\Sigma)$ share a codimension 1 face.

For a closed, orientable surface of genus $g \geq 2$, maximal dimensional simplices in $C(\Sigma)$ are of dimension $3g-4$ and correspond to a set $3g-3$ simple closed curves, whose union separates the surface into pairs of pants. An edge in the dual curve

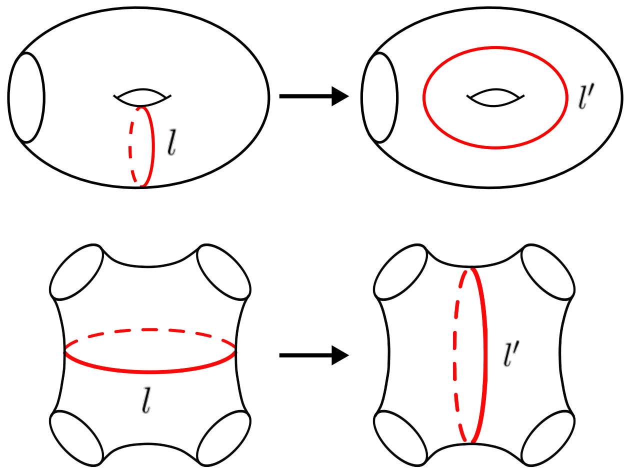

Figure 3-1: Above: An S-move in the pants complex. Below: An A-move in the pants complex.

complex therefore corresponds to starting with one pants decomposition of a surface and replacing one curve in order to obtain another pants decomposition of the surface. If instead of allowing arbitrary curve replacements, we insist that curves are replaced in the simplest way possible, we obtain the pants complex.

Definition 3.1.3. Given a surface Σ, the **pants complex** of Σ, denoted $P(\Sigma)$, is the simplicial complex whose vertices correspond to isotopy classes of pants decompositions of Σ. Two vertices v and v' in $P(\Sigma)$ are connected by an edge if the corresponding pants decompositions only differ in one curve, and the two different curves intersect minimally. That is, if $l \in v$ and $l' \in v'$ with $l \neq l'$, then either l and l' lie on a punctured torus with $|l \cap l'| = 1$ or l and l' lie on a four punctured sphere with $|l \cap l'| = 2$. In the case that l and l' lie on a punctured torus, we say that v and v' are related by an **S-move**. If l and l' lie on a four punctured sphere we say that v and v' are related by an **A-move**. See Figure 3-1 for an illustration of these moves.

The pants complex is an ubiquitous object in low dimensional topology. It captures the complex structures of the underlying surface quite well, in the sense that it is quasi-isometric to the Teichmuller space of the surface endowed with the Weil Peterson metric [6]. It also serves as a discrete analogue to the space of Morse functions on a surface. To elaborate on this, a Morse function on a surface naturally gives rise to a pants decomposition by taking the curves to be level sets of the morse function, discarding any curves that bound disks. Conversely given a pants decomposition on the surface, one may define a smooth function whose level sets are the given curves, and perturb it to a Morse function.

The duality between Morse functions and pants decompositions is used to show that the pants complex is connected. Namely, one converts the pants decompositions to Morse functions, f_0 and f_1, and uses the fact that any two Morse functions on a surface are connected by a Cerf function, f_t. The pants decompositions associated to each Morse function can only change at times t_i when f_{t_i} is not Morse. One can then analyze the level sets at the non-Morse times at deduce that A-moves and S-moves are sufficient to pass between the pants decompositions induced by $f_{t_i-\epsilon}$ and $f_{t_i+\epsilon}$.

We can leverage this to show the dual curve complex is connected as well. Note that the vertices of the pants complex and the dual curve complex are identical, and that edges in the pants complex are also edges in the dual curve complex. In other words, there is a natural map from the pants complex into the dual curve complex which is bijective on vertices and injective on edges. Since the pants complex is connected, we immediately see that the dual curve complex is also connected. We therefore get naturally defined metrics on the 1-skeletons of these complexes.

Definition 3.1.4. Let v_1 and v_2 be two vertices in $C^*(\Sigma)$. The **dual distance**, $D(v_1, v_2)$ is the length of the minimal path between v_1 and v_2 in the dual curve complex. Similarly if v_1 and v_2 are two vertices in $P(\Sigma)$, the **pants distance** $D^P(v_1, v_2)$ is the length of the minimal path between v_1 and v_2 in the pants complex.

Since the pants complex appears as a subcomplex of the dual curve complex, we get the inequality $D^P(v_1, v_2) \geq D(v_1, v_2)$. This inequality should be kept in mind when bounds are discussed later in the chapter.

3.2 Distances of handlebodies, Heegaard splittings and 3-manifolds

We say a vertex $v \in C^*(\Sigma)$ (or $P(\Sigma)$) **defines a handlebody**, H, if all of the curves in the pants decomposition corresponding to v bound disks in H. Equivalently, attaching 3-dimensional 2-handles to Σ along the curves of v and filling in the resulting 2-sphere boundary components with 3-balls produces H. Conversely, a handlebody H defines a subset of these complexes which we will denote by H^*, with the relevant complex usually being clear from the context. Given a handlebody H, and a vertex $v \in H^*$ we may continually apply a pseudo-Anosov map of the surface which extends over H to see that H^* has infinite diameter. This should be kept in mind when considering the computability of the following definition.

Definition 3.2.1. Given two handlebodies H_1 and H_2 with boundary Σ. We define the **dual distance** $D(H_1, H_2)$ of H_1 and H_2 to be the non-negative integer

$$min\{D(v_1, v_2) | v_1 \in H_1^*, v_2 \in H_2^*\}.$$

In what follows, we briefly review the work done in [25], which serves as a guideline for our generalization into 4-dimensions. We would like to compare two genus g Heegaard splittings, $H_1 \cup_\Sigma H_2$ and $H_1' \cup_{\Sigma'} H_2'$ in some chosen complex associated to a surface. As it stands, the handlebodies live in different complexes (those associated to Σ and those associated to Σ'). To fix this we will need to choose some identifications. Note that any two handlebodies of a given genus are homeomorphic. Such a homeomorphism $\phi : H_1 \to H_1'$, by restriction to Σ, induces isometries $\hat{\phi} : C^*(\Sigma) \to C^*(\Sigma')$ and $\hat{\phi} : P(\Sigma) \to P(\Sigma')$. These isometries take the handlebody subset H_1^* to $H_1'^*$ and the handlebody subset H_2^* to some subset of a complex associated to Σ' which we will denote by $\hat{\phi}(H_2^*)$. After an identification, the set of all such maps of H_1, up to isotopy forms the mapping class group of the handlebody, which we will denote by $\boldsymbol{Mod(H_1)}$. After we have identified the handlebodies using a given mapping class, the "difference" between these two Heegaard splittings is contained in the difference between the remaining handlebodies, which we quantify in the following definitions.

Definition 3.2.2. Given two genus g Heegaard splittings, $H_1 \cup_\Sigma H_2$ and $H_1' \cup_{\Sigma'} H_2'$ the **dual distance** between Σ and Σ', $D(\Sigma, \Sigma')$, is

$$min_{\phi \in Mod(H_1)}\{D(\hat{\phi}(H_2^*), H_2'^*)\}.$$

Similarly, the **pants distance**, $D^P(\Sigma, \Sigma')$, is

$$min_{\phi \in Mod(H_1)}\{D^P(\hat{\phi}(H_2^*), H_2'^*)|\}.$$

In fact, these distances stabilize to give two well defined notions of distance between two 3 manifolds, which we will denote by $D(M_1^3, M_2^3)$ and $D^P(M_1^3, M_2^3)$. Among the most elusive measures of complexity of a 3-manifold is the minimum number of components of a link $L \subset S^3$ such that surgery along L produces the given manifold. The following theorem says that the dual distance actually sees this complexity, as well as a host of related measures of complexity.

Theorem 3.2.3. *(Theorem 30 of [25]) $D(M_1^3, M_2^3)$ is equal to the minimum number of components of a link $L \subset M_1$ such that surgery along L produces M_2.*

3.3 Distances of trisections

We next seek to define analogous distance between 4-manifolds. In order to do this we begin with a distance between two trisections. Let (T_1, Σ_1) and (T_2, Σ_2) be two $(g, k)-$trisections with corresponding tripods $H_{\alpha_1} \cup H_{\beta_1} \cup H_{\gamma_1}$ and $H_{\alpha_2} \cup H_{\beta_2} \cup H_{\gamma_2}$. Both $H_{\alpha_1} \cup H_{\beta_1}$ and $H_{\alpha_2} \cup H_{\beta_2}$ are genus g Heegaard splittings for $\#^k S^1 \times S^2$. Waldhausen's theorem (Theorem 1.3.7) therefore asserts that both of these are in fact the unique genus g splitting of $\#^k S^1 \times S^2$. Therefore, there exists a map $\phi : H_{\alpha_1} \cup H_{\beta_1} \to H_{\alpha_2} \cup H_{\beta_2}$ so that $\phi(H_{\alpha_1}) = H_{\alpha_2}$ and $\phi(H_{\beta_1}) = H_{\beta_2}$. Such a map induces isometries on the various complexes associated to Σ_1 and Σ_2, and we will denote the induced isometry by $\hat{\phi}$.

If we fix an identification of both $H_{\alpha_1} \cup H_{\beta_1}$ and $H_{\alpha_2} \cup H_{\beta_2}$ with $\#^k S^1 \times S^2$, all such maps (up to isotopy with $\phi_t(\Sigma_1) = \Sigma_2$) can be identified with the mapping class group of the Heegaard splitting, which we denote $\boldsymbol{Mod(\#^k S^1 \times S^2, \Sigma_g)}$. Mapping class groups of Heegaard splittings have been studied extensively and can be quite complicated. For a hint at their complexity, when $g > k$, the group $Mod(\#^k S^1 \times S^2, \Sigma_g)$ will always have pseudo-Anosov elements [27]. Perhaps a stronger testament to their intricate structure is that the problem of determining generators of this group for splittings of S^3 has sustained nearly a century of inquiry [16] [12].

Definition 3.3.1. Let M_1 and M_2 be two 4-manifolds equipped with $(g, k)-$trisections T_1 and T_2. The **dual distance** between T_1 and T_2, $D(T_1, T_2)$, is

$$min_\phi\{D(\hat{\phi}(H^*_{\gamma_1}), H^*_{\gamma_2})\}.$$

Similarly, the **pants distance**, $D^P(T_1, T_2)$, is

$$min_\phi\{D(\hat{\phi}(H^*_{\gamma_1}), H^*_{\gamma_2})\}.$$

Where in both definitions, the the minimum is taken over all orientation preserving maps $\phi : H_{\alpha_1} \cup H_{\beta_1} \to H_{\alpha_2} \cup H_{\beta_2}$ so that $\phi(H_{\alpha_1}) = H_{\alpha_2}$ and $\phi(H_{\beta_1}) = H_{\beta_2}$

See Figure 3-2 for an illustration of the definition. Since these distances are natural number valued, they give well defined invariants of two (g, k)-trisections. Furthermore, if either distance is 0, then the distance minimizing map extends to

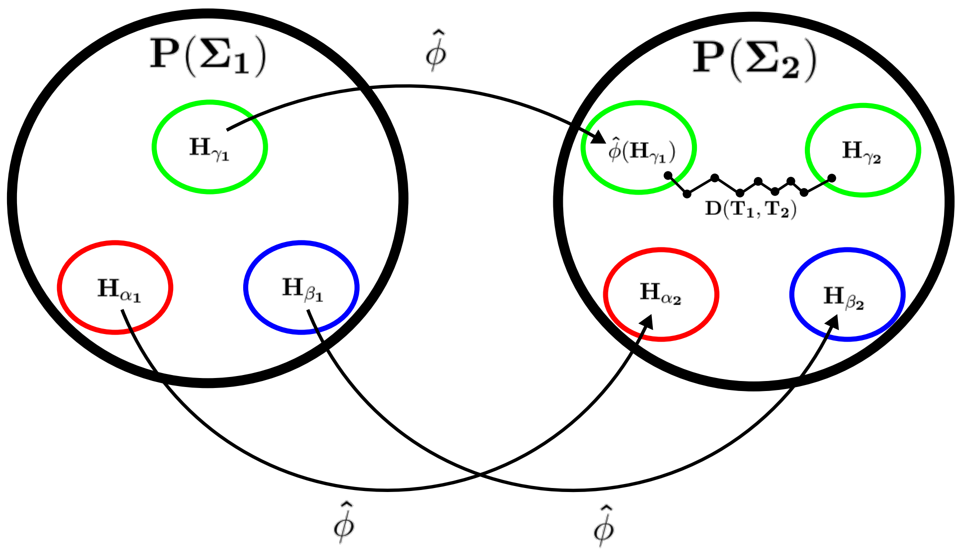

Figure 3-2: The distance between two trisections is the minimum distance between the sets $\hat{\phi}(H^*_{\gamma_1})$ and $H^*_{\gamma_2}$

a homeomorphism of tripods, which means that T_1 and T_2 are in fact diffeomorphic trisections as. Since there are many manifolds admitting a (g, k)-trisection for a given g and k with $g \neq k$ (see, for example, ([1] and [34]), this distance is nontrivial. In addition, the results of Chapter 2 show that there are in fact trisections of a fixed 4-manifold which are a nontrivial distance from each other.

We now seek to extend these distances of particular trisections to well defined distances of 4-manifolds. To do this, we need to understand how the distance behaves under stabilization. If T is a trisection with trisection surface Σ, we may stabilize T by puncturing Σ in a disk and gluing on the stabilizing surface shown in Figure 3-3. From this point of view, it is clear that we should begin by understanding paths in the complexes associated to $\Sigma \backslash D^2$. The following key lemma treating this case is contained in Lemma 15 of [25].

Lemma 3.3.2. *Let $v_1, v_2, ..., v_n$ be a minimal path in $C^*(\Sigma)$ (respectively, $P(\Sigma)$)*

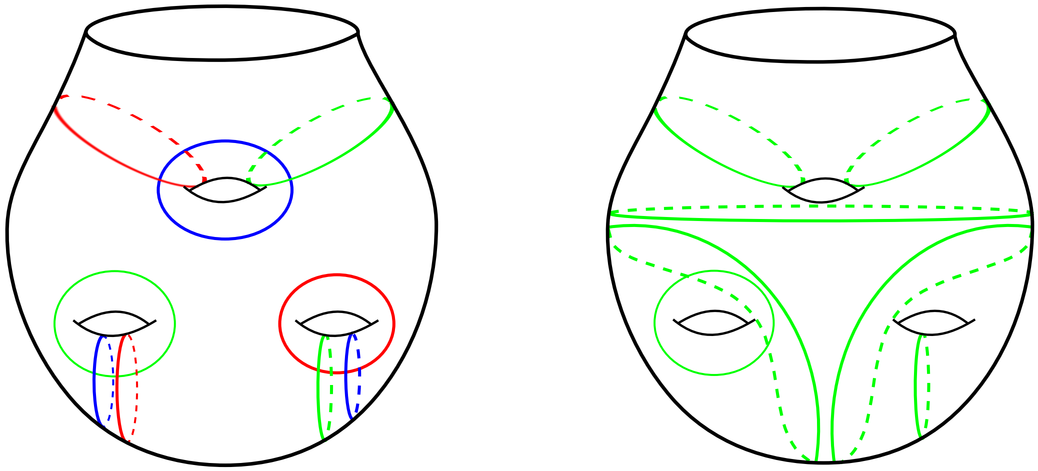

Figure 3-3: Left: Stabilizing a trisection amounts to gluing this diagram onto a punctured trisection diagram. Right: A pants decomposition for one handlebody of the stabilizing surface

between two handlebodies H_1 and H_2. Then there exists a disk $D \subset \Sigma$ and a path, $v_1', v_2', ..., v_m'$ in $C^*(\Sigma \backslash D)$ (respectively, $P(\Sigma \backslash D)$) with $m \leq 2n$ so that after capping off $\Sigma \backslash D$ with a disk, every loop in v_1' bounds a disk in H_1, and every loop in v_m' bounds a disk in H_2. Moreover, if there is some loop which is never moved in the path from v_1 to v_n, then there exists a disk D and a path $v_1', v_2', ..., v_m'$ satisfying the previous conclusions with $m = n$.

In what follows, we will adapt the work of [25] to prove that the invariants of trisections behave well under stabilization. In order to aid exposition, we will only explicitly treat the case of the distance in the dual curve complex. It should be clear after the fact that by using the part of Lemma 3.3.2 pertaining to the pants complex, all of the arguments go through virtually unchanged.

Lemma 3.3.3. *Let (T_1, Σ_1) and (T_2, Σ_2) be (g, k)-trisections, and let T_1^h and T_2^h be their genus h stabilizations. Then $D(T_1^h, T_2^h) \leq 2D(T_1, T_2)$.*

Proof. If $D(T_1, T_2) = n$ then there is a map $\phi : H_{\alpha_1} \cup H_{\beta_1} \to H_{\alpha_2} \cup H_{\beta_2}$ and a path $v_1, v_2, ..., v_n$ in $C^*(\Sigma_2)$ so that v_1 defines $\phi(H_{\gamma_1})$ and v_n defines H_{γ_2}. Let $l_i^1, ..., l_i^m$ be the loops corresponding to the pants decomposition given by v_i. Let D be a disk in the annulus formed by two parallel copies of l_1^1. Consider the pants decomposition for $\Sigma_2 \backslash D$ consisting of $l_1^1, ..., l_1^m, l_1^{m+1}$ where l_1^{m+1} is the parallel copy of l_1^1 on Σ_2 lying on the other side of D on $\Sigma_2 \backslash D$. Let v_1' be the corresponding vertex of $C^*(\Sigma_2 \backslash D)$

By Lemma 3.3.2, there is a path from v_1' to a vertex w such that if $l \in w$, then after capping off $\Sigma_2 \backslash D$ with a disk, l is isotopic to some loop in v_n. Moreover, this path is of length at most $2n$.

We first treat the case of a single stabilization. Consider the stabilization of Σ_1 produced by cutting out the disk $\phi^{-1}(D)$ gluing on a stabilizing surface to the resulting boundary component. Then $H_{\gamma_1}^{g+3}$ has a pants decomposition given by the curves in $\phi^{-1}(v_1')$ along with $\phi^{-1}(\partial D)$ and the pants decomposition for the stabilizing surface shown in Figure 3-3. By gluing on a stabilizing surface to ∂D, we can extend ϕ to a map $\phi^{g+3} : \Sigma_1^{g+3} \to \Sigma_2^{g+3}$ such that $\phi^{g+3}(w) \cup \partial D \cup \phi^{g+3}($the curves shown in Figure 3-3$)$ is a pants decomposition for $H_{\gamma 2}^{g+3}$. Since the path in $C^*(\Sigma_1 \backslash D)$ from v_1' to w takes place away from the stabilizing surface it corresponds to a path in $C^*(\Sigma_2^{g+3})$ so that $D(T_1^{g+3}, T_2^{g+3}) \leq 2D(T_1, T_2)$. To achieve the more general result, simply connect sum multiple stabilizing surfaces first before connect summing with the given trisection surfaces. $\square$

Lemma 3.3.4. *For sufficiently large g, $D(T_1^h, T_2^h) \leq D(T_1^g, T_2^g)$ when $h \geq g$.*

Proof. By the previous lemma, $D(T_1^g, T_2^g) \leq 2D(T_1, T_2)$. Choose g so that $3g - 3 >$

$2D(T_1, T_2)$. Since a pants decomposition of Σ_2^g consists of $3g-3$ loops, it follows that some loop is never moved in the path on Σ^g. In this case, we conclude by Lemma 3.3.2 that paths on Σ^g from $\hat{\phi}^g(H^g_{\gamma_1})$ to $H^g_{\gamma_2}$ lift to paths of the same length on Σ^h from $\hat{\phi}^h(H^h_{\gamma_1})$ to $H^h_{\gamma_2}$. $\square$

Theorem 3.3.5. *Let M_1 and M_2 have trisections T_1 and T_2, respectively. Then the limit $\lim_{g\to\infty} D(T_1^g, T_2^g)$ is well defined and depends only on the underlying manifolds, M_1 and M_2.*

Proof. Since the sequence $D(T_1^g, T_2^g)$ is natural number valued and non-increasing for sufficiently large g, it converges. Furthermore, by Theorem 1.4.6 any two trisections of the same manifold have a common stabilization fixing the labels of the handlebodies. Therefore, if T_1 and T_3 are distinct trisections of M_1, and T_2 and T_4 are distinct trisections of M_2 then there exists an h so that T_1^h is isotopic to T_3^h and T_2^h is isotopic to T_4^h. Then for $g > h$ we have that $D(T_1^g, T_2^g) = D(T_3^g, T_4^g)$ so that $\lim_{g\to\infty} D(T_1^g, T_2^g) = \lim_{g\to\infty} D(T_3^g, T_4^g)$. $\square$

Remark 3.3.6. The reader may be concerned that the definitions of D and D^P seem to distinguish between which third of the trisection is labeled X_1, whereas we have seemingly defined an invariant of a 4-manifold which is not sensitive to this information. However, Theorem 3.3.5 does actually encompass this case. Suppose M_1 has two trisections of the form $T_1 = (X_1, X_2, X_3)$ and $T_2 = (Y_1, Y_2, Y_3)$ such that $Y_i = X_{i-1}$ with indices taken mod 3. In [14], it is shown that any two trisections of the same manifold have a common stabilization fixing the labels of the sectors. We therefore have a map $f : M^4 \to M^4$ isotopic to the identity so that $f(Y_i^h) = X_i^h$.

We are now justified in making the following definitions.

Definition 3.3.7. Let M_1 and M_2 be two 4-manifolds which have (g, k)-trisections for the same g and k. The **dual distance** between M_1 and M_2 is $\lim_{g\to\infty} D(T_1^g, T_2^g)$ where T_1 is any trisection of M_1 and T_2 is any trisection of M_2 with the same parameters as T_1. Similarly, the **pants distance** between two 4-manifolds is $\lim_{g\to\infty} D^P(T_1^g, T_2^g)$.

Remark 3.3.8. It should be noted that in the 3-manifold case, any two pants decompositions will determine a 3-manifold, so that minimal paths between two 3-manifolds pass through intermediary 3-manifolds. This nice property simplifies many of the arguments in [25]. In our set up, we can not guarantee that H_{α_2}, H_{β_2}, and the handlebody determined by an intermediary vertex in a minimal path between $\hat{\phi}(H_{\gamma_1})$ and H_{γ_2} will still pairwise form Heegaard splittings for $\#^k S^1 \times S^2$. Therefore, these three handlebodies may not form the tripod of a trisection, and there may be no way to uniquely obtain a closed 4-manifold from this information. This leads to two natural questions:

1. Can we pass between trisections though paths whose intermediary vertices form trisections?

2. Is there any significance to the 3 handlebodies which occur in paths between two trisections?

It is clear from the definitions that trisections, T_1 and T_2, can only be compared when $(g_1, k_1) = (g_2, k_2)$. However, it is not immediately obvious when two manifolds can be compared. A necessary and sufficient condition for comparing M_1 and M_2 is that both manifolds have a (g, k)-trisection for some g and k. If a 4-manifold has a (g, k)-trisection, the Euler characteristic is given by $\chi(M) = 2 + g - 3k$, so it is necessary that $\chi(M_1) = \chi(M_2)$. The following straightforward lemma shows that this is also a sufficient condition.

Lemma 3.3.9. *$D(M_1, M_2)$ and $D^P(M_1, M_2)$ are well defined whenever $\chi(M_1) = \chi(M_2)$.*

Proof. Let M_1 have a (g_1, k_1)-trisection, T_1, and let M_2 have a (g_2, k_2)-trisection, T_2. Now since $\chi(M_1) = \chi(M_2)$, $2 + g_1 - 3k_1 = 2 + g_2 - 3k_2$. Without loss of generality, assume $k_1 > k_2$. Then by stabilizing T_2 $(k_1 - k_2)$ times we get a new trisection of M_2, T_2' with $k_2' = k_1$ and $g_2' = g_2 + 3(k_1 - k_2) = g_2 + (g_1 - g_2) = g_1$, hence these two trisections can be compared. $\square$

Remark 3.3.10. If we don't insist on the trisections being balanced, we may compare any two 4-manifolds, regardless of their Euler characteristics. To do this, let M_1 and M_2 be 4-manifolds with corresponding trisections, T_1 and T_2. We may perform 1-stabilizations until they have the same k_1. Next, perform 2-stabilizations until both trisections have the same genus. We now have two trisections, $T_1' = X_1 \cup X_2 \cup X_3$ and $T_2' = Y_1 \cup Y_2 \cup Y_3$ so that both ∂X_1 and ∂Y_1 are diffeomorphic to $\#^{k_1} S^1 \times S^2$. Moreover, both ∂X_1 and ∂Y_1 have the structure of a genus g Heegaard splitting, so as before, there are diffeomorphisms respecting the structure of the Heegaard splittings. This allows us to carry through with the definition of the distance between trisections virtually unchanged.

It is a quick corollary of Theorem 1.4.6 that unbalanced trisections of the same

4-manifold with the same parameters (i.e. (g, k_1, k_2, k_3)) become isotopic after some number of balanced stabilizations. This allows us to carry through with Theorem 3.3.5 and define a distance between any two manifolds which stabilizes as the initial trisections are stabilized. A slight caveat is that this will, in general, depend on the values of the k_i. Nevertheless, we can still get a well defined distance, depending only on the underlying manifolds, by further minimizing over all choices of k_i.

It is natural to ask whether these distances induce a metric on the set of 4-manifolds with the same Euler characteristic. It follows quickly from the definition that these distances are 0 if and only if the manifolds are diffeomorphic. These distances are also symmetric, since if ϕ is the distance minimizing map for $D(M_1, M_2)$ (or $D^P(M_1, M_2)$), then ϕ^{-1} minimizes the distance $D(M_2, M_1)$ (respectively $D^P(M_2, M_1)$). The triangle inequality, however, is likely false. This is due to the fact that we are minimizing over handlebody sets of infinite diameter in the complexes. Given three handlebodies H_1, H_2, and H_3, the representative of H_2 closest to H_1 may be far away from the representative of H_2 closest to H_3.

3.4 Some Bounds

Lemma 3.4.1. *If $D(M_1, M_2) = n$ and $D(M_3, M_4) = m$, then $D(M_1\#M_3, M_2\#M_4) \leq n + m$.*

Proof. Stabilize trisections of M_1 and M_2 to genus g trisections, (T_1^g, Σ_1^g) and (T_2^g, Σ_2^g), with $3g-3 > n$. Also, stabilize trisections of M_3 and M_4 to genus h trisections,(T_3^h, Σ_3^h) and (T_2^h, Σ_2^h), with $3h-3 > m$. Since a pants decomposition for Σ_2^g has $3g-3$ loops, it follows that some loop in the path from $\hat{\phi}^g(H^g_{\gamma_1})$ to $H^g_{\gamma_2}$ is never moved, where $\hat{\phi}$ is the distance minimizing map. Let $v'_1, v'_2, ..., v'_n$ be the path in the dual curve complex (or the pants complex) guaranteed by Lemma 3.3.2 on $C^*(\Sigma^g_2 \backslash D)$, and let $w'_1, w'_2, ..., w'_n$ be the path guaranteed by the same lemma on $C^*(\Sigma^h_4 \backslash D')$.

Form the connect sum, $\Sigma^g_2 \# \Sigma^h_4$ along the disks D and D'. Let $v'_i \cup w'_j$ be the pants decomposition of $\Sigma^g_2 \# \Sigma^h_4$ consisting of the pants decomposition for Σ^g_2 induced by v'_i, the pants decomposition for Σ^h_4 induced by w'_j, along with the additional curve $\partial D = \partial D'$. Then the path $v'_1 \cup w'_1, v'_2 \cup w'_1, ..., v'_n \cup w'_1, v'_n \cup w'_2, ..., v'_n \cup w'_m$ is a path of length $n + m$ from $M_1 \# M_3$ to $M_2 \# M_4$. □

Corollary 3.4.2. *For any N with $\chi(N) = 2$, $D(M_1 \# N, M_2) \leq D(M_1, M_2) + D(N, S^4)$*

The question of whether $D(M_1 \# M_3, M_2 \# M_4) = n + m$ is quite easily shown to be false. For example, if M_1 and M_2 are homeomorphic, but not diffeomorphic, 4-manifolds that become diffeomorphic after a single connected sum with $S^2 \times S^2$, then $D(M_1, M_2) \neq 0$ whereas $D(M_1 \# S^2 \times S^2, M_2 \# S^2 \times S^2) = 0$.

We next seek to prove a lower bound on the distance between two manifolds based on the difference of their signatures. To this end, we briefly discuss how this information can be recovered from a trisection. Given a genus g surface Σ, choose a symplectic basis for $H_1(\Sigma_g, \mathbb{R})$. That is, a basis $\{a_1, b_1, ..., a_g, b_g\}$ so that for all i and j, $|a_i \cap a_j| = |b_i \cap b_j| = 0$ and $|a_i \cap b_j| = \delta_{ij}$. Let ω be the associated symplectic form on $\mathbb{R}^{2g}$.

Given a trisection with tripod $H_1 \cup H_2 \cup H_3$, we get 3 Lagrangian subspaces of $H_1(\Sigma_g, \mathbb{R})$ given by $L_i = ker(i_* : H_1(\Sigma_g, \mathbb{R}) \to H_1(H_i, \mathbb{R})$. We may define a symmetric bilinear form, q, on $L_1 \oplus L_2 \oplus L_3$ by $q((x_1, x_2, x_3), (y_1, y_2, y_3)) = \omega(x_1, y_2) + \omega(y_1, x_2) + \omega(x_2, y_3) + \omega(y_2, y_3) + \omega(x_3, y_1) + \omega(y_3, x_1) + \omega(x_3, y_1)$. In [14], it is observed that the signature of the matrix associated to this bilinear form is the signature of the original 4-manifold. While intermediary vertices in minimal paths between handlebodies will not always define trisections, the signature of this matrix will always be well defined. As a result, we obtain the following proposition.

Proposition 3.4.3. $D(M_1, M_2) \geq \frac{1}{2}|\sigma(M_1) - \sigma(M_2)|$.

Proof. Suppose $D(M_1, M_2) = n$ with a path $v_1, ..., v_n$, so that v_1 defines $\phi(H_{\gamma_1})$ and v_n defines H_{γ_2}. At each vertex, we have a triple of handlebodies determined by H_{α_2}, H_{β_2}, and the handlebody determined by v_i. These in turn determine three Lagrangian subspaces of $H_1(\Sigma_2, \mathbb{R})$: L_1, L_2, and L_{v_i}. Going from v_i to v_{i+1} involves changing a single curve in the pants decomposition so that L_{v_i} and $L_{v_{i+1}}$ have bases in $H_1(\Sigma_2, \mathbb{R})$ which are the same except for possibly one vector.

Let M_i be the matrix corresponding to the symmetric bilinear form q_i on $L_1 \oplus L_2 \oplus L_{v_i}$. Then M_i has real eigenvalues $\lambda_1 \leq \lambda_2 ... \leq \lambda_{3g}$. Let $a_1, ..., a_g$ be a basis for L_{v_i}. In going from L_{v_i} to $L_{v_{i+1}}$, it is possible that none of the basis vectors are changed, in which case the signature of the matrix is obviously unchanged. It is also possible that one vector, say a_j is changed. Let M_i' be the matrix obtained by deleting the row and column corresponding to a_j, and let $\lambda_1' \leq \lambda_2' ... \leq \lambda_{3g-1}'$ be its eigenvalues. By the Cauchy interlacing theorem, $\lambda_1 \leq \lambda_1' \leq \lambda_2 \leq \lambda_2' ... \leq \lambda_{3g-1}' \leq \lambda_{3g}$ so that $|\sigma(M_i) - \sigma(M_i')| \leq 1$. Similarly, we may obtain M_i' by deleting a row and column of M_{i+1} so that $|\sigma(M_{i+1}) - \sigma(M_i')| \leq 1$. Therefore, $|\sigma(M_{i+1}) - \sigma(M_i)| \leq 2$. The result

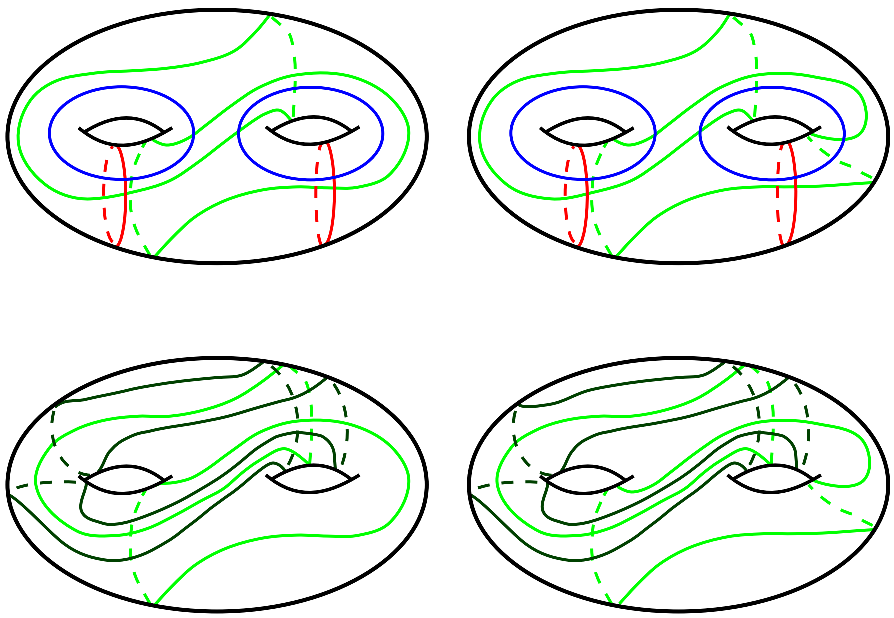

Figure 3-4: $S^2 \times S^2$ and $S^2 \widetilde{\times} S^2$ are distance one in the pants complex

immediately follows. $\square$

Comparing the standard $(g, 0)-$trisections of $\#^g \mathbb{C}P^2$ and $\#^g \overline{\mathbb{C}P^2}$ one can see that this bound is sharp. Moreover, we may conclude that $\lim_{g \to \infty} D(\#^g \mathbb{C}P^2, \#^g \overline{\mathbb{C}P^2}) = \infty$.

3.5 Nearby Manifolds

We next seek to build some intuition as to what it means when 4-manifolds are close to each other with respect to our distances. We first consider an illustrative example. The top of Figure 3-4 shows trisection diagrams T_1 for $S^2 \times S^2$ and T_2 for $S^2 \widetilde{\times} S^2$ where the α and β handlebodies are identical. Below that are pants decompositions for the γ handlebodies which are identical except for in one curve. The curves which are different intersect exactly once, showing that $D^P(T_1, T_2) = 1$. Moreover, we have 2 curves in the pants decomposition which never move, so by

Lemma 3.3.2, the path lifts to new paths of distance one on all stabilizations. Since these manifolds are non-diffeomorphic, we may therefore conclude that $D^P(S^2 \times S^2, S^2 \widetilde{\times} S^2) = 1$.

In [14], it is shown how to obtain a Kirby diagram from a trisection diagram, and these particular trisection diagrams give rise to Kirby diagrams which are identical except for in the framing of a 2-handle. We seek to show that this is in fact the case in general. That is, if $D^P(M_1, M_2) = 1$ then M_1 and M_2 have Kirby diagrams which are identical except for in the framing of some 2-handle. To do this, we first consider what it means for two handlebodies to be distance one apart in the pants complex.

Lemma 3.5.1. *Let H_1 and H_2 be two genus g handlebodies with boundary Σ. If $D^P(H_1, H_2) = 1$ then the manifold $H_1 \cup_\Sigma H_2 \cong \#^{g-1} S^1 \times S^2$.*

Proof. Let $v_1, v_2 \in P(\Sigma)$ define H_1 and H_2 respectively with $D^P(v_1, v_2) = 1$. The pants decompositions corresponding to these vertices are exactly the same except for some loops $l_1 \in v_1$ and $l_2 \in v_2$. Moreover, since A-moves do not change the handlebody and $H_1 \neq H_2$ we know that l_1 and l_2 lie in a punctured torus with $|l_1 \cap l_2| = 1$. We may therefore build a Heegaard diagram for $H_1 \cup_\Sigma H_2$ consisting of $g-1$ identical loops in both v_1 and v_2, along with l_1 and l_2. It is easy to see that this is a once stabilized splitting for $\#^{g-1} S^1 \times S^2$. $\square$

Genus g Heegaard splittings of $\#^{g-1} S^1 \times S^2$ are in some sense the second most simple Heegaard splittings in a given genus after $\#^g S^1 \times S^2$. Genus g trisections where two of the handlebodies form $\#^g S^1 \times S^2$ are easily shown to be diffeomorphic to $\#^g S^1 \times S^3$. Given these facts, it would be reasonable to assume that genus g

trisections where two of the handlebodies form $\#^{g-1}S^1 \times S^2$ are also relatively simple. The following theorem of [35] pertaining to unbalanced trisections makes this precise.

Theorem 3.5.2. *(Theorem 1.2 of [35]) Suppose that M admits a $(g; g-1, k_2, k_3)$-trisection T, and let $k' = max\{k_2, k_3\}$. Then M is diffeomorphic to either $\#^{k'}S^1 \times S^3$ or to the connect sum of $\#^{k'}S^1 \times S^3$ with one of either $\mathbb{C}P^2$ or $\overline{\mathbb{C}P^2}$, and T is a connect sum of genus 1 trisections.*

Theorem 3.5.3. *If $D^P(M_1, M_2) = 1$, then M_1 and M_2 have Kirby diagrams which are identical, except for the framing on some 2-handle.*

Proof. Suppose the distance has stabilized in T_1 for M_1 and T_2 for M_2, where T_1 and T_2 are (g, k)-trisections. We will construct 2 new manifolds from these trisections following the schematic found in Figure 3-5. Consider the manifold obtained by removing X_1 from T_1 and X_1' from T_2, and gluing the resulting two manifolds by the distance minimizing map. Since $D^P(M_1, M_2) = 1$, Lemma 3.5.1 implies that the 3-manifold $H_{\gamma_1} \cup H_{\gamma_2}$ is diffeomorphic to $\#^{g-1}S^1 \times S^2$. We may cut the resulting 4-manifold along $H_{\gamma_1} \cup H_{\gamma_2}$ to obtain two 4-manifolds each with boundary $\#^{g-1}S^1 \times S^2$. We may fill in each of the resulting manifolds with boundary with $\natural^{g-1}S^1 \times D^3$ in order to obtain two trisected, closed 4-manifolds.

We will focus on the manifold with trisection tripod $H_\alpha \cup H_{\gamma_1} \cup H_{\gamma_2}$. This closed 4-manifold inherits the structure of a $(g; g-1, g-k, g-k)$-trisection. By Theorem 3.5.2, this trisection is a connect sum of genus 1 trisections. In particular, there are curves $l_1, ..., l_g$, all bounding disks in H_α, H_{γ_1}, and H_{γ_2}, which cut Σ into g once punctured tori and a sphere with g holes. We may also ensure that each of these tori contain one α, γ_1, and γ_2 curve so that in all but one particular torus, the γ_1 and γ_2 curves are identical.

Let α^0, γ_1^0, and γ_2^0 be the three curves on the same punctured torus with $\gamma_1^0 \neq \gamma_2^0$. By virtue of the classification of genus 1 trisections, as well as the fact that $\gamma_1^0 \neq \gamma_2^0$, these three curves either form a diagram for $\mathbb{C}P^2$ or for S^4. However, if both T_1 and T_2 are balanced trisections, the three curves must form $\mathbb{C}P^2$; for otherwise, $H_\alpha \cup H_\beta$ is $\#^k S^1 \times S^2$ whereas $H_\alpha \cup H_{\gamma'}$ is $\#^{k\pm1} S^1 \times S^2$. After a diffeomorphism, α^0, γ_1^0 and γ_2^0 form the trisection diagram for $\mathbb{C}P^2$ shown in Figure 1-13. From here, it can be seen that γ_1^0 and γ_2^0 are related by a Dehn twist about a curve bounding a disk in H_α so that after pushing them into H_α they become isotopic.

We may now take a diffeomorphism of the surface and perform handle slides of the α and β curves so that the α and β curves form the standard Heegaard diagram for $\#^k S^1 \times S^2$. By pushing the $g-k$ γ_1 and γ_2 curves dual to α curves into H_α, and giving them the surface framing, we obtain framed links L_1 and L_2 in $H_\alpha \cup H_\beta$. On page 3104 of [14], it is observed that the L_i the are the framed attaching link for the 2-handles in a handle decomposition for M_i. Note that $g-k-1$ of these curves are identical, and the final curves have been shown to be isotopic in H_α, which completes the argument.

$\square$

Remark 3.5.4. Note that the construction of $D^P(T_1, T_2)$ can be generalized to encompass unbalanced trisections where one of the k_i agree on each trisection. We may then mimic the proof of Theorem 3.5.3 to study adjacent manifolds represented by

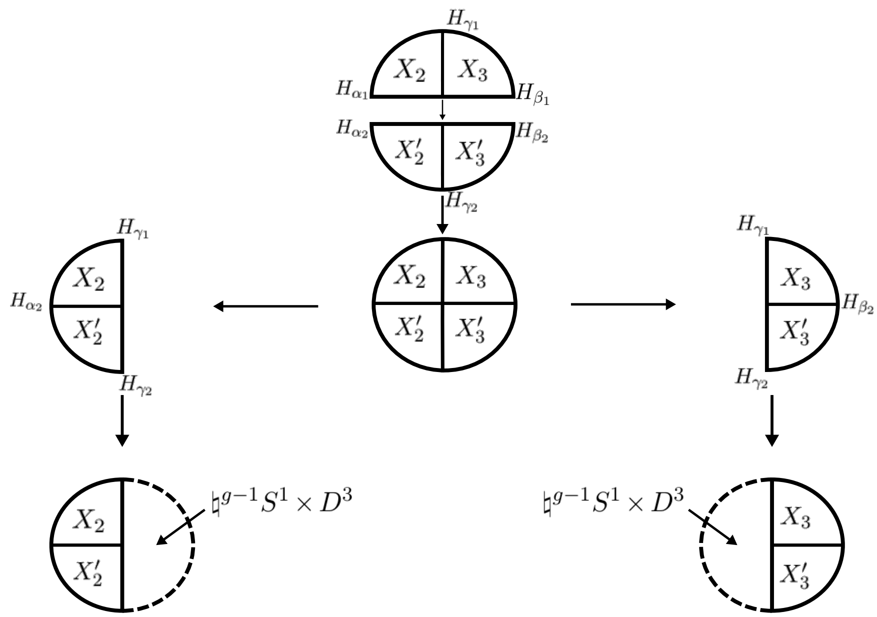

Figure 3-5: A schematic of the construction in Theorem 3.5.3

unbalanced trisections. The proof goes through unchanged except that we must also consider the possibility that α^0, γ_1^0 and γ_2^0 form the unbalanced trisection diagram for S^4 shown in Figure 1-13. In this case, the γ curve parallel to the α curve does not play a role in the induced Kirby diagram whereas the γ curve dual to the α curve manifests itself as a 2-handle. We may therefore conclude that distance 1 in this more general case corresponds to either changing a handle framing by 1 (the balanced case) or adding or removing a 2-handle.

We now seek to prove a partial converse to Theorem 3.5.3. We begin by understanding how to obtain a trisection diagram from a Kirby diagram. To do this, we follow the proof for the existence of trisections given in [14], while taking a little extra care to the particular diagram constructed. Take a Kirby diagram for M with k_1 1-handles and k_2 2-handles. The 0- and 1-handles form $\natural^{k_1} S^1 \times D^3$ and have boundary $\#^{k_1} S^1 \times S^2$. We may take a genus k_1 Heegaard splitting for this boundary and draw k_1

parallel α and β curves on the surface which bound disks in both handlebodies. Now the framed attaching link for the 2-handles projects onto the Heegaard surface, with perhaps a few crossings. Do Reidemeister 1 moves on the link on the surface to make the surface framing match the handle framing, and do a Reidemeister 2 move on each component to make sure it has at least 1 self crossing. Stabilize the Heegaard surface at each of the crossings to resolve them, resolving the self crossings as in Figures 3-6 and 3-7. By construction, for each component of the link, L_i, we may choose a dual α curve, α_i, that no other link component component intersects. Then we may slide any other α curve, α_j, along L_i over α_i to eliminate any intersections between L_i and α_j.

Embedded on the Heegaard surface, we now see g α curves, g β curves, and k_2 curves coming from the attaching link which are dual to k_2 α curves and disjoint from the rest of the α curves. We complete L to the set of γ curves by adding in $g - k_2$ curves parallel to each α curve which does not intersect any component of L. It is clear that the pairs of curves (α, β) and (α, γ) are Heegaard diagrams for the connect sum of some number of copies of $S^1 \times S^2$. What is left to check is that the same holds for the pair (β, γ).

The γ curves define a handlebody, H_γ. Note that this handlebody is the result of pushing the γ curves dual to α curves into H_α and performing surface framed Dehn surgery on them. But these dual curves come from the attaching link for the 2-handles of a closed 4-manifold. After attaching 2-handles along these curves pushed into H_α, H_α becomes H_γ, but H_β remains unchanged. Now H_γ and H_β form a Heegaard splitting for the boundary of the 3- and the 4-handles so that the pair (β, γ) is indeed

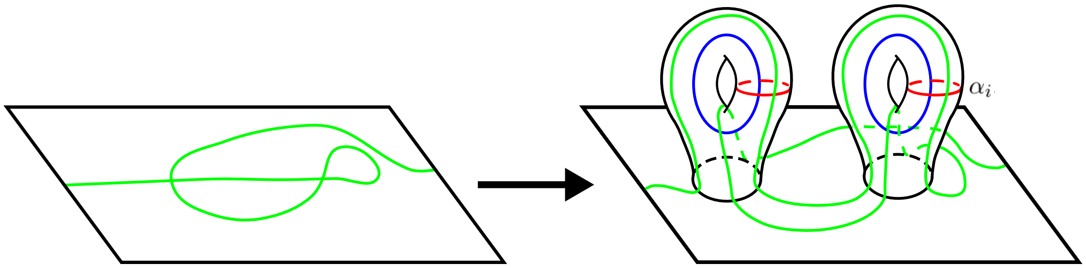

Figure 3-6: Resolving a Reidemeister 2 move of the attaching link on the Heegaard Surface

a Heegaard diagram for some number of copies of $S^1 \times S^2$. We now have a possibly unbalanced trisection diagram for M, which we may balance by connect summing with the genus 1 unbalanced trisection diagrams for S^4.

Theorem 3.5.5. *Let M_1 and M_2 be non-diffeomorphic 4-manifolds with the same Euler characteristic which have Kirby diagrams K_1 and K_2, respectively. If K_1 and K_2 only differ in the framing of some 2-handle, where the framing differs by 1, then $D^P(M_1, M_2) = 1$.*

Proof. Let L_i be the framed attaching links for K_i and let l_i be the component of the L_i in which the framing differs. Without loss of generality, suppose that $|fr(l_1)| > |fr(l_2)|$ where $fr(l_i)$ is the framing of l_i. Since K_1 and K_2 have the same 0- and 1-handles, we may project both attaching links onto the Heegaard surface for the boundary of the union of the 0- and 1-handles. Introduce self intersections as previously described to obtain dual α curves, and to make the surface framing match the handle framing. This results in almost the same immersed curves on the Heegaard surface, except that l_1 has one more kink in it than l_2. Stabilize the Heegaard surface at all crossings, and in the extra kink, send l_2 over the stabilizing genus without twisting so as not to change the framing. See Figure 3-7 for an illustration of this process.

We must now choose dual α curves for each component of the link in order to eliminate intersections. Let α_i be the α curve in the stabilization where l_1 and l_2 differ. Choose α_i to be the α curve dual to both l_1 and l_2 and choose arbitrary dual α curves for the rest of the components of the L_i. We now claim that eliminating the extra α intersections with L_i by sliding curves off over the dual α curves along arcs parallel to the link components results in identical α curves. Sliding any curve along a link component which is not l_1 or l_2 obviously results in the same curve since we have constructed these curves to be identical. Moreover, sliding an α curve over α_i along l_1 is isotopic to sliding the α curve over α_i along l_2 as can be seen in Figure 3-8.

We now have possibly unbalanced trisection diagrams for M_1 and M_2 with identical α and β curves. We seek to show that the k_i for both of these manifolds are equal so that we may connect sum with the same unbalanced trisections of S^4 in order to balance them. It is straightforward to show that a $(g; k_1, k_2, k_3)$-trisection has Euler characteristic $2 + g - k_1 - k_2 - k_3$. First note that both of these trisections have the same genus. Furthermore, k_1 is the number of copies of $S^1 \times S^2$ formed by the α and β curves, which is clearly the same for both trisections. In addition, k_3 comes from the α and γ curves, which we have constructed to be the same in both cases. Finally, the assumption that M_1 and M_2 have the same Euler characteristic ensures that these manifolds have equal k_3 so that we may balance these trisections in an identical manner.

Finally, we complete both sets of γ curves to pants decompositions of the handlebodies to finish the argument. To this end, note that l_1 and l_2 intersect transversely in

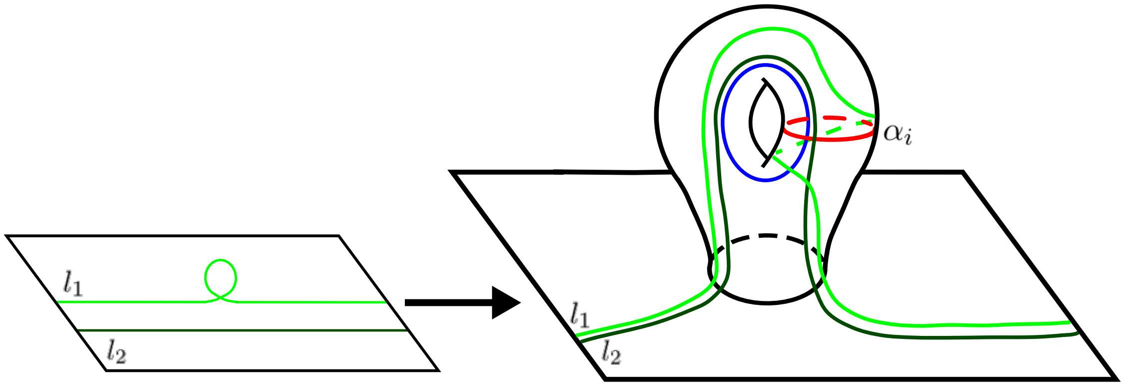

Figure 3-7: Resolving a Reidemeister 1 move to change the surface framing of l_1 by 1. Parallel curves such as l_2 can be sent over the stabilizing surface without twisting about α_i to preserve the surface framing

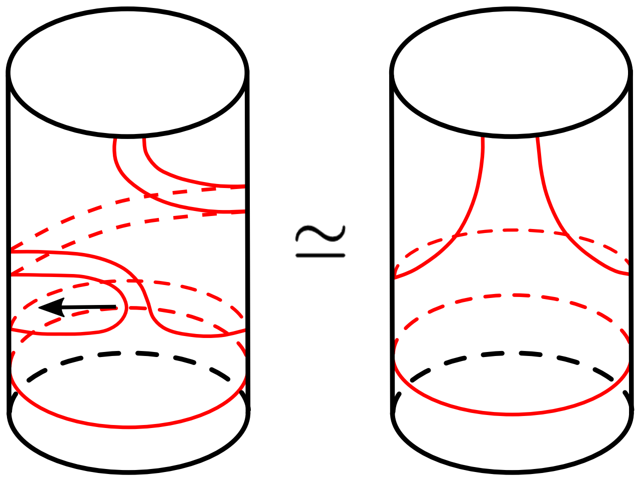

Figure 3-8: Dehn twisting the sliding arc about the target curve does not change the isotopy type of slid curve.

one point so that the boundary of a regular neighborhood of the curves bounds a disk in both handlebodies. This cuts off a punctured torus containing l_1 and l_2. Outside of this punctured torus the γ handlebodies are identical and so we may complete them to an arbitrary pants decomposition. The resulting pants decompositions are easily seen to be one apart in the pants complex. □

We may also alter the framings of 2-handles by Dehn twisting about a chosen dual α curve which intersects no other component of the attaching link (recall that such curves may always be created by the introduction of self crossings in the link component). The result of repeatedly Dehn twisting a link component about the given α curve may intersect our original link component many times, however both curves lie in the punctured torus filled by the α curve and the original link component. In addition, adding more Dehn twists to the sliding arc in Figure 3-8 does not change the isotopy type of the resulting curve, so that we may again eliminate intersections via isotopic handle slides of the α curves. These are all the essential ingredients to the following theorem, whose details we leave to the reader.

Theorem 3.5.6. *Let M_1 and M_2 be non-diffeomorphic 4-manifolds with the same Euler characteristic which have Kirby diagrams, K_1 and K_2, respectively. If K_1 and K_2 only differ in the framing of some 2-handle, then $D(M_1, M_2) = 1$.*

3.6 Complexes of Trisections

We next seek to define a collection of graphs associated to trisections. Here, it is useful to consider the more general case of unbalanced trisections. Fix a surface, Σ, and two handlebodies, H_α and H_β, with boundary Σ, so that $H_\alpha \cup_\Sigma H_\beta \cong \#^{k_1} S^1 \times S^2$. We may identify the first two handlebodies in a $(g; k_1, k_2, k_3)$-trisection with $H_\alpha \cup_\Sigma H_\beta$. The third handlebody then gives rise to some handlebody subset of $P(\Sigma)$. We therefore have a subcomplex of the pants complex associated to any (possibly unbalanced) trisection with parameters $(g; k_1, -, -)$. This motivates the following definition.

Definition 3.6.1. Fix a genus g surface surface Σ and two handlebodies H_α and

H_β so that $H_\alpha \cup_\Sigma H_\beta \cong \#^{k_1} S^1 \times S^2$. Define the $(g; k_1, -, -)$ complex of trisections, $\boldsymbol{P(g, k_1)}$, to be the full subgraph of the pants complex whose vertices are $\{\gamma \in P(\Sigma) |$ γ defines H_γ, $H_\alpha \cup_\Sigma H_\gamma \cong \#^{k_2} S^1 \times S^2$, and $H_\beta \cup_\Sigma H_\gamma \cong \#^{k_3} S^1 \times S^2\}$.

Definition 3.6.2. $\gamma \in P(g, k_1)$ is a **representative** for a trisection T if γ defines H_γ and $H_\alpha \cup H_\beta \cup H_\gamma$ is a tripod for T. We say T_1 and T_2 are **adjacent** in $P(g, k_1)$ if they have representatives which are adjacent.

Note that a trisection has many representatives in $P(g, k_1)$. Not only are there infinitely many vertices in the pants complex defining the same handlebody, but multiple different handlebodies may represent the same trisection. For example, if $k_1 > 0$ there is some nonseparating curve which bounds disks in both H_α and H_β. A Dehn twist about this curve will usually change H_γ, but will give rise to a diffeomorphic trisection. More generally, we could take any element of the mapping class group $Mod(H_\alpha \cup H_\beta, \Sigma)$ which does not extend across H_γ to produce similar results.

Lemma 3.6.3. *Let T be a stabilized trisection of M^4. Then there exists a trisection T' for $M \# \mathbb{C}P^2$ so that T and T' are adjacent in $P(g, k_1)$.*

The proof of the previous lemma is straightforward. We may in fact weaken the hypothesis that T is stabilized to the condition that T is 2- or 3-stabilized, but the lemma as stated will be sufficient for our needs. This lemma is useful to us because 4-manifolds can change drastically under connect sums with $\mathbb{C}P^2$ and $\overline{\mathbb{C}P^2}$. The following corollary of Wall's theorem in [51] makes this precise.

Proposition 3.6.4. *(Corollary 9.1.14 of [17]) Let M_1 and M_2 be simply connected 4-manifolds. Then there exist natural numbers l_1, l_2, m_1, m_2 so that $M_1 \#^{l_1} \mathbb{C}P^2 \#^{m_1} \overline{\mathbb{C}P^2}$*

is diffeomorphic to $M_2 \#^{l_2} \mathbb{C}P^2 \#^{m_2} \overline{\mathbb{C}P^2}$

We are now well equipped to prove the main proposition of this section.

Proposition 3.6.5. *Let* M_1 *and* M_2 *be simply connected, smooth, closed 4-manifolds. Then there exist* $(g, k, -, -)$*-trisections,* T_1 *and* T_2*, for* M_1 *and* M_2 *respectively, so that* T_1 *and* T_2 *are in the same connected component of* $P(g, k)$*.*

Proof. Take arbitrary trisections, T_1 of M_1, and T_2 of M_2. Now 1- and 2-stabilize them so that they have the same genus, g, and the same k_1. We will first calculate the number of stabilizations needed for the construction. Let l_1, l_2, m_1, m_2 be as in Proposition 3.6.4. Let $a = max\{l_1 + m_1, l_2 + m_2\}$. After 2-stabilizing T_1 and T_2 a times, we may change each 2-stabilization into a summand of $\mathbb{C}P^2$ or $\overline{\mathbb{C}P^2}$ to obtain two (possibly different) trisections for the same 4-manifold. By Theorem 1.4.6, we may perform some number of balanced stabilizations on the resulting trisections until they are isotopic. Let b be the number of stabilizations needed to make the resulting trisections isotopic.

We claim that T_1^{g+a+3b} and T_2^{g+a+3b} can be connected in $P(g + a + 3b, k_1 + b)$. To see this, observe that by Lemma 3.6.3, each 2-stabilization can be changed into an extra factor of $\mathbb{C}P^2$ or $\overline{\mathbb{C}P^2}$ adjacent to T_1^{g+a+3b} or T_2^{g+a+3b}. Changing each 2-stabilization in T_1^{g+a+3b} to the appropriate $\mathbb{C}P^2$ or $\overline{\mathbb{C}P^2}$ summand successively leads to a path to a trisection of $M_1 \#^{l_1} \mathbb{C}P^2 \#^{m_1} \overline{\mathbb{C}P^2}$ which we know to be diffeomorphic to $M_2 \#^{l_2} \mathbb{C}P^2 \#^{m_2} \overline{\mathbb{C}P^2}$. Moreover, the constructed trisections have been stabilized enough to become isotopic. □

It is especially interesting to know which manifolds are adjacent to S^4, for if N

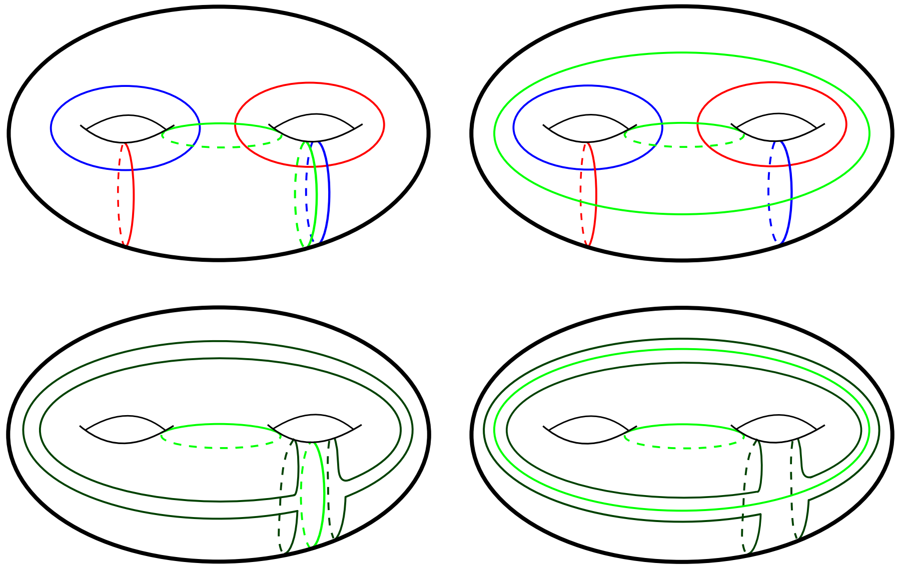

Figure 3-9: S^4 (shown on the left) is adjacent to $S^2 \times S^2$ (shown on the right) in $P(2,0)$.

is adjacent to S^4, then for any M, we may stabilize a trisection to get an adjacent trisection for $M\#N$. It is straightforward to see that S^4 is adjacent to $\mathbb{C}P^2$, $\overline{\mathbb{C}P^2}$ and $S^1 \times S^3$. Furthermore, Figure 3-9 shows that S^4 is also adjacent to $S^2 \times S^2$. It is tempting to believe that this is a complete list of manifolds. In light of Remark 3.5.4, manifolds adjacent to S^4 correspond to starting with some (perhaps very complicated) Kirby diagram for S^4 and then changing the framing of some 2-handle, or adding/removing a 2-handle. We conclude this section with a question.

Question 3.6.6. Which 4-manifolds are adjacent to S^4?

3.7 Quadrisected embeddings

In considering the gluings of pieces of trisections required to compute our various distances, we are led naturally to 4-manifolds with visibly embedded 3-manifolds.

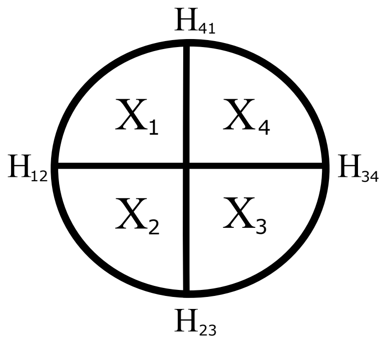

Figure 3-10: A schematic of a quadrisection. We will be interested in the 3-manifold given by the Heegaard splitting $H_{12} \cup H_{34}$

.

We begin with a definition of these embeddings. The definition is accompanied by a schematic shown in Figure 3-10 which should be used as a reference while parsing the formal definition.

Definition 3.7.1. A **Quadrisection** of a 4-manifold is a decomposition $M = X_1 \cup X_2 \cup X_3 \cup X_4$ such that

- $X_i \cong H^4_k$
- $X_i \cap X_{i+1} = H_{i(i+1)} \cong H^3_g$ for $i \neq j$
- $\partial X_i = H_{i(i+1)} \cup H_{(i-1)i}$ is a genus g Heegaard splitting for $\partial X_i = \sharp^k S^1 \times S^2$

with all indices taken mod 4.

As in the case of trisections, the fact that $\partial X_i = \sharp^k S^1 \times S^2$ implies that a quadrisection is completely determined by the union $H_{12} \cup H_{23} \cup H_{34} \cup H_{41}$ which we call the **quadrapod** of the quadrisection. That is, given a quadrapod, we may carry out an operation similar to that illustrated in Figure 1-12, using the fact that the boundaries

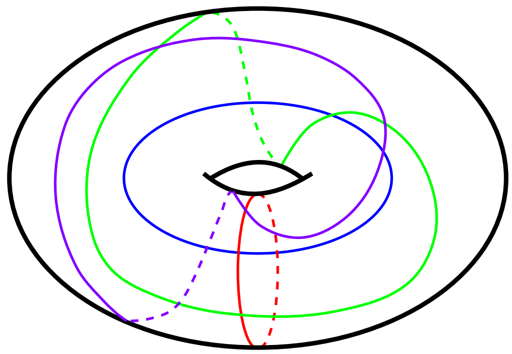

Figure 3-11: A (1;0,0,0,0)-quadrisection of $\mathbb{R}P^3$ in $S^2\tilde{\times}S^2$. A Heegaard diagram for $\mathbb{R}P^3$ can be seen in the purple and green curves.

form copies of $\sharp^k S^1 \times S^2$ which may be uniquely filled in with H^4_k. This again leads us to a natural diagrammatic theory of such objects, whose diagrams we describe below.

Definition 3.7.2. A **quadrisection diagram** is a surface, Σ, together with 4 cut systems $\alpha, \beta, \gamma, \delta$ such that the four pairs (α, β), (β, γ), (γ, δ), and (δ, α), are Heegaard diagrams for a connected sum of $S^1 \times S^2$.

By adapting the ideas used to construct trisection diagrams in the lead up to the proof of Theorem 3.5.5, we may make any surgery link for a 3-manifold lie in a nice position with respect to some Heegaard splitting for S^3. This is the starting point for the proof of the main proposition of this section.

Proposition 3.7.3. *Every 3-manifold admits a quadrisected embedding into $\sharp^n S^2 \times S^2$ for some integer n.*

Proof. Recall that every 3-Manifold admits a surgery diagram such that all components are even integer framed. This fact was originally shown in [38], and a modern proof can be found in Theorem 4.1 of [46]. Given an arbitrary 3-manifold M, let

$L = \cup_i L_i$ be an even integer framed link such that surgery on L produces M. Take an abritrary Heegaard surface for S^3, and project L onto it. In general the link will intersect itself many times after this projection, but this may be dealt with by stabilizing the Heegaard surface. Further stabilize the surface, sending the link over the stabilizing surface until we arrive at a Heegaard splitting $H_\alpha \cup H_\beta$ in which there exists a collection of properly embedded disks α_i in H_α so that $|\alpha_i \cap L_j| = \delta_{ij}$. Do right or left handed Dehn twists of L_i about α_i until the surface framing matches the prescribed even surgery coefficient for L_i. This condition implies that, after pushing L into H_α and doing surface framed surgery, the homeomorphism type is still a handlebody of genus g, which we call H_γ.

We also see that $H_\gamma \cup H_\beta$ is a Heegaard splitting for the given 3-manifold M, and since the γ curves were dual to α curves we see that $H_{\alpha|} \cup H_\gamma$ is $\sharp^n S^1 \times S^2$ for some k. We may fill in the S^3 boundary component with a ball, and the $\sharp^n S^1 \times S^2$ component with H^4_k in order to obtain half of a quadrisection diagram whose boundary is our given manifold. Doubling this entire construction, we get a $(0, k, k, 0)$-quadrisection with a Heegaard splitting $H_\gamma \cup H_\beta$ of M clearly appearing in the middle. Moreover, on the level of 4-manifolds, since all of the curves for this link were integer framed, we have simply constructed the double of a 2-handlebody. The fact that all of the curves were additionally even framed implies that this double is in fact $\sharp^n S^2 \times S^2$ (see Corollary 5.1.2 of [17] for reference). $\square$

Remark 3.7.4. The previous construction follows a classical proof which shows that every 3-manifold embeds in $\sharp^n S^2 \times S^2$ for some n where one simply doubles an even

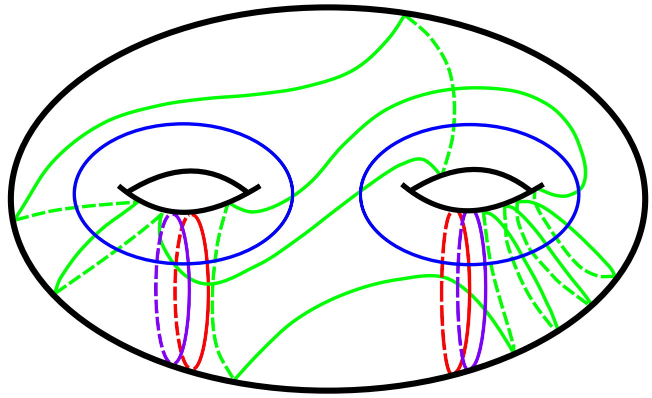

Figure 3-12: A (2;0,0,0,0)-quadrisection of L(7,2) in $\sharp^2 S^2 \times S^2$

framed surgery link. In comparing the two arguments one should note that no new factors of $S^2 \times S^2$ were introduced in order to have the 3-manifold lie in quadrisected position.

The proof of Proposition 3.7.3 actually gives us an algorithm for realizing a Quadrisection diagram of a given 3-manifold in $\sharp^n S^2 \times S^2$. For illustration, we carry the algorithm out for $L(7,2)$. We begin with the description of $L(7,2)$ as surgery on the Hopf link with framings 2 and 3. Projecting this link onto a Heegaard surface for S^3, and making the surface framing match the surgery coefficients gives us the green curves in Figure 3-12. Doubling the α curves in the Heegaard decomposition gives us the full quadrisection diagram for an embedding of $L(7,2) \subset \sharp^2 S^2 \times S^2$.

Bibliography

[1] R. I. Baykur and O. Saeki. Simplifying indefinite fibrations on 4-manifolds. *ArXiv e-prints*, May 2017.

[2] Joan S. Birman. *On the equivalence of Heegaard splittings of closed, orientable 3-manifolds*. Princeton Univ. Press, Princeton, N.J., 1975.

[3] M. Boileau, D. J. Collins, and H. Zieschang. Genus 2 Heegaard decompositions of small Seifert manifolds. *Ann. Inst. Fourier (Grenoble)*, 41(4):1005–1024, 1991.

[4] M. Boileau and H. Zieschang. Heegaard genus of closed orientable Seifert 3-manifolds. *Invent. Math.*, 76(3):455–468, 1984.

[5] Francis Bonahon and Jean-Pierre Otal. Scindements de Heegaard des espaces lenticulaires. *Ann. Sci. École Norm. Sup. (4)*, 16(3):451–466 (1984), 1983.

[6] Jeffrey F. Brock. The Weil-Petersson metric and volumes of 3-dimensional hyperbolic convex cores. *J. Amer. Math. Soc.*, 16(3):495–535, 2003.

[7] Jean Cerf. La stratification naturelle des espaces de fonctions différentiables réelles et le théorème de la pseudo-isotopie. *Inst. Hautes Études Sci. Publ. Math.*, (39):5–173, 1970.

[8] Marc Culler. Lifting representations to covering groups. *Adv. in Math.*, 59(1):64–70, 1986.

[9] S. K. Donaldson. An application of gauge theory to four-dimensional topology. *J. Differential Geom.*, 18(2):279–315, 1983.

[10] Renate Engmann. Nicht-homöomorphe Heegaard-Zerlegungen vom Geschlecht 2 der zusammenhängenden Summe zweier Linsenräume. *Abh. Math. Sem. Univ. Hamburg*, 35:33–38, 1970.

[11] Ralph H. Fox. Free differential calculus. I. Derivation in the free group ring. *Ann. of Math. (2)*, 57:547–560, 1953.

[12] M. Freedman and M. Scharlemann. Powell moves and the Goeritz group. *ArXiv e-prints*, April 2018.

[13] Michael Hartley Freedman. The topology of four-dimensional manifolds. *J. Differential Geom.*, 17(3):357–453, 1982.

[14] David Gay and Robion Kirby. Trisecting 4–manifolds. *Geom. Topol.*, 20(6):3097–3132, 2016.

[15] David T. Gay and Robion Kirby. Indefinite Morse 2-functions: broken fibrations and generalizations. *Geom. Topol.*, 19(5):2465–2534, 2015.

[16] Lebrecht Goeritz. Die abbildungen der brezelfläche und der vollbrezel vom geschlecht 2. *Abh. Math. Sem. Univ. Hamburg*, 9(1):244–259, 1933.

[17] Robert E. Gompf and András I. Stipsicz. 4-*manifolds and Kirby calculus*, volume 20 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1999.

[18] Wolfgang Haken. Some results on surfaces in 3-manifolds. In *Studies in Modern Topology*, pages 39–98. Math. Assoc. Amer. (distributed by Prentice-Hall, Englewood Cliffs, N.J.), 1968.

[19] Allen Hatcher and John Wagoner. *Pseudo-isotopies of compact manifolds*. Société Mathématique de France, Paris, 1973. With English and French prefaces, Astérisque, No. 6.

[20] K. Hayano. On diagrams of simplified trisections and mapping class groups. *ArXiv e-prints*, November 2017.

[21] P. Heegaard. Forstudier til en topologisk Teori for de algebraiske Fladers Sammenhang. Kjöbenhavn. 104 S. 8° (1898)., 1898.

[22] John Hempel. 3-manifolds as viewed from the curve complex. *Topology*, 40(3):631–657, 2001.

[23] Gabriel Islambouli. Comparing 4-manifolds in the pants complex via trisections. *Algebr. Geom. Topol.*, 18(3):1799–1822, 2018.

[24] Gabriel Islambouli. Nielsen equivalence and trisections of 4-manifolds. *ArXiv e-prints*, April 2018.

[25] Jesse Johnson. Heegaard splittings and the pants complex. *Algebr. Geom. Topol.*, 6:853–874, 2006.

[26] Jesse Johnson. Horizontal Heegaard splittings of Seifert fibered spaces. *Pacific J. Math.*, 245(1):151–165, 2010.

[27] Jesse Johnson and Hyam Rubinstein. Mapping class groups of Heegaard splittings. *J. Knot Theory Ramifications*, 22(5):1350018, 20, 2013.

[28] Irwin Kra. On lifting Kleinian groups to $\mathrm{SL}(2, \mathbf{C})$. In *Differential geometry and complex analysis*, pages 181–193. Springer, Berlin, 1985.

[29] François Laudenbach. A proof of Reidemeister-Singer's theorem by Cerf's methods. *Ann. Fac. Sci. Toulouse Math. (6)*, 23(1):197–221, 2014.

[30] François Laudenbach and Valentin Poénaru. A note on 4-dimensional handlebodies. *Bull. Soc. Math. France*, 100:337–344, 1972.

[31] W. B. R. Lickorish. A representation of orientable combinatorial 3-manifolds. *Ann. of Math. (2)*, 76:531–540, 1962.

[32] Martin Lustig and Yoav Moriah. Nielsen equivalence in Fuchsian groups and Seifert fibered spaces. *Topology*, 30(2):191–204, 1991.

[33] Martin Lustig, Yoav Moriah, and Gerhard Rosenberger. Automorphisms of Fuchsian groups and their lifts to free groups. *Canad. J. Math.*, 41(1):123–131, 1989.

[34] Jeffrey Meier. Trisections and spun 4-manifolds. *ArXiv e-prints*, August 2017.

[35] Jeffrey Meier, Trent Schirmer, and Alexander Zupan. Classification of trisections and the generalized property R conjecture. *Proc. Amer. Math. Soc.*, 144(11):4983–4997, 2016.

[36] J. Milnor. *Morse theory.* Based on lecture notes by M. Spivak and R. Wells. Annals of Mathematics Studies, No. 51. Princeton University Press, Princeton, N.J., 1963.

[37] John Milnor. On manifolds homeomorphic to the 7-sphere. *Ann. of Math. (2)*, 64:399–405, 1956.

[38] J.W. Milnor. *Differentiable manifolds which are homotopy spheres.* 1959.

[39] Edwin E. Moise. Affine structures in 3-manifolds. V. The triangulation theorem and Hauptvermutung. *Ann. of Math. (2)*, 56:96–114, 1952.

[40] Yoav Moriah and Jennifer Schultens. Irreducible Heegaard splittings of Seifert fibered spaces are either vertical or horizontal. *Topology*, 37(5):1089–1112, 1998.

[41] Marston Morse. *The calculus of variations in the large*, volume 18 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 1996. Reprint of the 1932 original.

[42] Jakob Nielsen. Die Isomorphismengruppe der freien Gruppen. *Math. Ann.*, 91(3-4):169–209, 1924.

[43] Peter Ozsváth and Zoltán Szabó. Holomorphic disks and three-manifold invariants: properties and applications. *Ann. of Math. (2)*, 159(3):1159–1245, 2004.

[44] Kurt Reidemeister. Zur dreidimensionalen Topologie. *Abh. Math. Sem. Univ. Hamburg*, 9(1):189–194, 1933.

[45] Gerhard Rosenberger. Automorphismen ebener diskontinuierlicher Gruppen. In *Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference (State Univ. New York, Stony Brook, N.Y., 1978)*, volume 97 of *Ann. of Math. Stud.*, pages 439–455. Princeton Univ. Press, Princeton, N.J., 1981.

[46] Nikolai Saveliev. *Lectures on the topology of 3-manifolds*. De Gruyter Textbook. Walter de Gruyter & Co., Berlin, revised edition, 2012. An introduction to the Casson invariant.

[47] Martin Scharlemann and Abigail Thompson. Heegaard splittings of (surface) $\times I$ are standard. *Math. Ann.*, 295(3):549–564, 1993.

[48] H. Seifert. Topologie Dreidimensionaler Gefaserter Räume. *Acta Math.*, 60(1):147–238, 1933.

[49] James Singer. Three-dimensional manifolds and their Heegaard diagrams. *Trans. Amer. Math. Soc.*, 35(1):88–111, 1933.

[50] Friedhelm Waldhausen. Heegaard-Zerlegungen der 3-Sphäre. *Topology*, 7:195–203, 1968.

[51] C. T. C. Wall. On simply-connected 4-manifolds. *J. London Math. Soc.*, 39:141–149, 1964.

[52] Andrew H. Wallace. Modifications and cobounding manifolds. IV. *J. Math. Mech.*, 12:445–484, 1963.