Topology, combinatorics, and categories: from Δ to Θ_2

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Abstract

The predominant purpose of this paper is to provide a set of generating maps for the category Θ_2 and the relations amongst them. To understand Θ_2 we examine the category Δ, of which Θ_2 is a generalization. We describe a natural way that Δ arises as one tries to model topological spaces combinatorially by means of simplices, leading to the concept of a simplicial set. We show that the generating maps of Δ can be utilized to define generating maps for Θ_2, with many of the relations amongst these generating maps for Θ_2 coming from the relations of the generating maps of Δ.

1 Introduction

Often in mathematics a question can be difficult to answer in the setting in which it is posed, whereas taking a different perspective on the question by using tools of another area can alleviate that difficulty. For example, sometimes we can determine when two topological spaces are not homeomorphic by comparing their fundamental groups: if the fundamental groups of two spaces are not isomorphic, then the two spaces are not homeomorphic.

We can look to translate certain topological spaces into a combinatorial setting, that is, a setting consisting of a set of discrete points and subsets thereof. For example, we might represent a triangle in $\mathbb{R}^2$ by three points $\{x_0, x_1, x_2\}$; since any two nondegenerate triangles (ones with positive area) in $\mathbb{R}^2$ are homeomorphic, we might expect that the information that these three vertices communicate is sufficient for working with the triangle in a topological context.

The virtue of the fundamental group in the example comes from our ability to compute the group itself. Defining this group would be of little use if we could not understand it. Thus, when translating from a topological setting into a combinatorial setting, we look to understand the structure of the combinatorics in order to reap the benefits of this shift in perspective.

It turns out that we can model sufficiently well-behaved topological spaces combinatorially using the category Δ, that of finite ordinals and order preserving maps between them. We can understand the structure of these models, called simplicial sets, by understanding the functions between objects of Δ. One way to understand these functions is by a set of generating maps, the composites of which give every possible map between two objects of Δ, and the relations amongst these generating maps.

The category Δ can also be used to study categories. We can consider taking the collection of n composable morphisms in a category for varying n; Δ serves as a useful tool for organizing this information. Just as in our original view of the category, we can refer to its objects and morphisms, but now, we can also refer to the composable pairs of morphisms, and the composable triples of morphisms, and so on. Understanding the combinatorics of Δ through generating maps can then help us understand the combinatorics of categories. The category Θ_2, a generalization of Δ, can serve a similar role for something called a 2-category. The primary goal of this paper is to provide a set of generating maps for Θ_2 and the relations amongst these maps.

Category theory provides the language in which we communicate some of the concepts we consider in this paper, particularly those concepts in the latter half, so we begin with some basic category theory background in Section 2. In Section 3 we discuss simplicial complexes as a precursor to simplicial sets; Δ first arises here within the discussion of simplicial sets. As we show in Section 4, a set of generating maps for Θ_2 can be defined in terms of the generating maps of Δ.

2 Category theory background

Here we present the category theory that will come up later in the paper. The reader who is already familiar with the basic concepts of category theory (categories, functors, natural transformations) may wish to skip to Section 3.

Often in math courses we study a particular object and deduce properties of the object, e.g., $\mathbb{R}^n$ is contractible. The idea of a category is to shift our perspective from these particular instances of an object, take a step back, and consider the collection of all objects of a given sort along with a specified type of morphism between them. For example, in the context of topology, the relevant type of map between spaces is a continuous function, and the collection of topological spaces and the continuous functions between them form a category.

As mentioned in the introduction, the collection of all finite ordinals and the order-preserving maps between them form a category Δ, which occupies a vital role in this paper. Part of this role is serving as the principal component in the definition of the category Θ_2, which is the central focus of Section 4. Furthermore, understanding the notion of a category is essential in understanding other categorical concepts that are used in this paper.

Definition 2.1. A *category* consists of a collection of *objects* and a collection of *morphisms* (also called *arrows*) such that:

- each morphism has a specified *domain* and *codomain*; writing $f : X \to Y$ signifies that f is a morphism with domain X and codomain Y;
- for each object X, there is a unique *identity morphism* $1_X : X \to X$; and
- given two morphisms $f : X \to Y$ and $g : Y \to Z$, there is a *composite morphism* $gf : X \to Z$.

This data is subjected to the following two conditions.

- For any $f : X \to Y$, $f1_X$ and $1_Y f$ are both equal to f.
- For any composable triple of morphisms f, g, h, the composites $f(gh)$ and $(fg)h$ are equal; i.e., composition of morphisms is associative.

Example 2.2. Here are some examples of categories consisting of common mathematical objects.

(i) The category **Set** has sets as its objects, and functions between two specified sets as morphisms.

(ii) **Top** is the category whose objects are topological spaces and whose morphisms are continuous maps.

(iii) The category **Group** has groups as its objects and group homomorphisms as its morphisms.

Note that, in order for these examples to form categories, it is essential that composites of the relevant type of map result in another map of the same type: the composition of two continuous maps is again continuous; the composition of two group homomorphism is again a group homomorphism. The morphisms of a category that are invertible are called *isomorphisms*. In **Set**, the isomorphisms are bijective functions between sets; in **Top**, the isomorphisms are homeomorphisms, i.e., continuous maps with continuous inverses; in **Group**, the isomorphisms are invertible group homomorphisms. In each of the above examples, the morphisms are functions between underlying sets of the objects; however, this need not be the case.

Example 2.3. The morphisms in each of the following categories are not morphisms of sets.

(i) A group G may be regarded as a category $\mathbf{B}G$ with a single object. The morphisms of $\mathbf{B}G$ are the elements of the group, with composition given by group multiplication and thus the identity element e serving as the identity morphism.

(ii) The homotopy category **Htpy** has topological spaces as its objects, like **Top**, but its morphisms are homotopy classes of continuous maps.

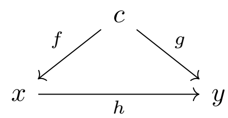
(iii) For a unital ring R, the category $\mathbf{Mat}_R$ has as objects positive integers and a morphism from n to m is given by an $m \times n$ matrix with entries in R. Composition of a morphisms is given by matrix multiplication.

Definition 2.4. Given a category $\mathcal{C}$, the *opposite category* $\mathcal{C}^{op}$ is obtained by taking the same objects as $\mathcal{C}$ while reversing the direction of the morphisms. That is, $\mathcal{C}^{op}$ has the same objects as $\mathcal{C}$, but there is a morphism $f^{op} : Y \to X$ in $\mathcal{C}^{op}$ if and only if there is a morphism $f : X \to Y$ in $\mathcal{C}$.

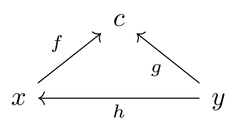
- For each object X, the arrow 1_X^{op} is the identity morphism.
- A pair f^{op}, g^{op} is composable as $f^{op} \cdot g^{op}$ in $\mathcal{C}^{op}$ precisely when the composite $g \cdot f$ is defined in C, in which case, $f^{op} \cdot g^{op} = (g \cdot f)^{op}$.

The similarity in structure of a category to its opposite category manifests in the concept of *duality*. Any theorem of a category $\mathcal{C}$ has a *dual theorem* for the category $\mathcal{C}^{op}$, proven by the dual of the original proof, more or less obtained by "reversing all the arrows." Since $(\mathcal{C}^{op})^{op} = \mathcal{C}$, a theorem holds in $\mathcal{C}$ if and only if its dual theorem holds in $\mathcal{C}^{op}$, where the 'dual theorem' is obtained by interchanging the domain and codomain of each morphism as well as the order of composing two morphisms.

Example 2.5. For an example application of the concept of duality, consider the following fact: For any category $\mathcal{C}$ and any object c in $\mathcal{C}$, there exists a category $c/\mathcal{C}$ whose objects are morphisms $f : c \to x$ and a morphism from $f : c \to x$ to $g : c \to y$ is a morphism $h : x \to y$ so that the diagram



commutes, i.e., $g = hf$. Since this fact holds for the general category $\mathcal{C}$, it too holds for $\mathcal{C}^{op}$. But, in C^{op}, the diagram above has arrows running in the opposite directions. Thus we can immediately deduce the following fact by duality: For any category $\mathcal{C}$ and any object c in $\mathcal{C}$, there exists a category $\mathcal{C}/c$ whose objects are morphisms $f : x \to c$ and a morphism from $f : x \to c$ to $g : y \to c$ is a morphism $h : y \to x$ so that the diagram



commutes, i.e., $g = fh$.

Given a category, it can be natural to consider some subcollection of its objects that share a common feature. For instance, in **Top**, we might wish to restrict our attention to a certain class of nice spaces, such as compactly generated Hausdorff spaces or, even more restrictively, CW complexes.

Definition 2.6. A *subcategory* $\mathcal{S}$ of a category $\mathcal{C}$ consists of a subcollection of objects of $\mathcal{C}$ along with a subcollection of morphisms of $\mathcal{C}$ such that:

- the domain and codomain of every morphism in $\mathcal{S}$ are objects of $\mathcal{S}$;
- for every object X in $\mathcal{S}$, the identity morphism 1_X from $\mathcal{C}$ is in $\mathcal{S}$; and
- for any composable pair of morphisms f and g in $\mathcal{S}$, the composite $g \circ f$ in $\mathcal{C}$ is also in $\mathcal{S}$.

Note that the definition of a subcategory $\mathcal{S}$ guarantees that $\mathcal{S}$ is a category itself.

Sets, groups, topological spaces—they are all a type of mathematical object, and with them we considered specific sorts of morphisms. Categories themselves are a type of mathematical object, which brings us to the question, what is a morphism of categories?

Definition 2.7. A *functor* $F : \mathcal{C} \to \mathcal{D}$ between categories $\mathcal{C}$ and $\mathcal{D}$ consists of the following data:

- for each object c in $\mathcal{C}$, an associated object Fc in $\mathcal{D}$; and
- for each morphism $f : c \to c'$ in $\mathcal{C}$, an associated morphism $Ff : Fc \to Fc'$ in $\mathcal{D}$.

These assignments are subject to the two following functorality axioms:

- for each object c in $\mathcal{C}$, $F(1_c) = 1_{Fc}$, where 1_{Fc} is the identity morphism of the object Fc in $\mathcal{D}$; and

- for any composable pair f, g in $\mathcal{C}$, $F(g \cdot f) = Fg \cdot Ff$.

Example 2.8. There is a functor $P : \mathbf{Set} \to \mathbf{Set}$ that sends a set A to its power set $PA = \{A' \subseteq A\}$ and a function $f : A \to B$ to the direct-image function $f_* : PA \to PB$ that maps $A' \subseteq A$ to $f(A') \subseteq B$.

Example 2.9. Those familiar with a little algebraic topology will know of the *fundamental group*, which defines a functor $\pi_1 : \mathbf{Top}_* \to \mathbf{Group}$ sending a pointed topological space (X, x_0) to its fundamental group $\pi_1(X, x_0)$. A continuous map $f : (X, x_0) \to (Y, y_0)$ of based spaces induces a group homomorphism $f_* : \pi_1(X, x_0) \to (Y, y_0)$.

Knowing that an assignment between two categories of mathematical objects is functorial can be of great use. For example, using functoriality one can reduce the Brouwer Fixed Point Theorem (that every continuous endomorphism of a 2-dimensional disk has a fixed point) to the algebraic fact that $0 \neq 1$; see Theorem 1.3.3 in [1].

We do not have to stop at the concept of a functor, a morphism between categories; we can take another step up and consider a morphism between functors, something called a *natural transformation.* In fact, Saunders Mac Lane, one of the founders of category theory, said that the desire to define a natural transformation is what lead to the definition of a functor, which in turn lead to the definition of a category. In this way, our presentation here of basic categorical concepts runs in reverse as compared to the initial motivation of the definitions themselves [2].

Definition 2.10. Let $\mathcal{C}$ and $\mathcal{D}$ be categories and $F, G : \mathcal{C} \to \mathcal{D}$ be two functors between them. A *natural transformation* $\alpha : F \Rightarrow G$ consists of a morphism $\alpha_c : Fc \to Gc$ for each object c in $\mathcal{C}$ so that for any morphism $f : c \to c'$ in $\mathcal{C}$, the following square of morphisms in $\mathcal{D}$ commutes:

$$\begin{array}{ccc} Fc & \xrightarrow{\alpha_c} & Gc \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle Gf} \\ Fc' & \xrightarrow[\alpha_{c'}]{} & Gc' \end{array}$$

The collection of all α_c define the *components* of the natural transformation. A natural transformation in which every component α_c is an isomorphism is called a *natural isomorphism* and is depicted $\alpha : F \cong G$.

We depict such a natural transformation $\alpha : F \Rightarrow G$ between two functors $F, G : \mathcal{C} \to \mathcal{D}$ by the diagram

$$\mathcal{C} \underset{G}{\overset{F}{\underset{\Downarrow \alpha}{\rightrightarrows}}} \mathcal{D}.$$

Example 2.11. There is a natural transformation $\eta : 1_{\mathbf{Set}} \Rightarrow P$, from the identity functor on **Set** to the power set functor, whose components $\eta_A : A \to PA$ are the functions that carry $a \in A$ to the singleton subset $\{a\} \in PA$. Naturality here says that for any $f : A \to B$ in **Set**, taking $a \in A$ to $\{a\} \in PA$, then to $f_*(\{a\}) = \{f(a)\}$ results in the same set as first taking a to $f(a)$, then $f(a)$ to $\{f(a)\}$, which is indeed the case.

Between any two categories we have a collection of functors, and between any two of these functors we have a collection of natural transformations. For two categories $\mathcal{C}$ and $\mathcal{D}$, we can consider the functors $\mathcal{C} \to \mathcal{D}$ as the objects of a category itself, with natural transformations as the morphisms between these functors. A *2-category* is a category such that the collection of 1-morphisms between any two objects are the objects of a category, whose morphisms are some type of 2-morphism between the 1-morphisms. One way to view a 2-category is that the 1-morphisms resemble functors, and the 2-morphisms resemble natural transformations.

The category theory presented here will be useful as a language for discussing simplicial sets and further the category Θ_2, which will be discussed in Section 4. Next, however, before employing these categorical tools, we look at the concept of *simplices* and assembling more complicated structures out of these.

3 Simplicial complexes and simplicial sets

3.1 Simplicial complexes and simplicial maps

Simplicial complexes are built out of basic blocks, called *n-simplices*: a *geometric n-simplex* is the convex hull of $n+1$ geometrically independent points $v_0, v_1, \ldots, v_n$. By 'geometrically independent,' we mean that the n vectors $v_1 - v_n, \ldots, v_n - v_0$ are linearly independent. For example, a 0-simplex is a point, a 1-simplex an interval, a 2-simplex a triangle, a 3-simplex a tetrahedron, and so on. We call the points (0-simplices) *vertices*, and a *face* of a simplex is the convex hull of some subset of the set of vertices. Note that, by definition, the face of a simplex is itself a simplex; for example, a face of a 2-simplex, a triangle, one of its three sides (1-simplices), or simply one of its vertices (0-simplices).

Ultimately, we want to use simplices to model topological spaces; however, the n-simplex is homeomorphic to a closed n-dimensional ball, so simplices alone only model a rather small collection of spaces. We can assemble more complicated spaces out of these simplices, in a way that can be informally thought of as gluing together n-simplices (for various n) along faces.

Definition 3.1. A *geometric simplicial complex* X in $\mathbb{R}^N$ consists of a collection of simplices in $\mathbb{R}^N$ such that

(i) every face of a simplex of X is in X, and

(ii) the intersection of any two simplices of X is a face of each of them, or empty.

We label a simplicial complex by first labeling its vertices, then considering a collection of $k+1$ of these vertices as constituting a k-simplex of the complex. If the vertices $\{v_{i_0}, \ldots, v_{i_k}\}$ are the vertices of a simplex, we label that simplex $[v_{i_0}, \ldots, v_{i_k}]$.

Example 3.2. We may think of any k-simplex $[v_0, \ldots, v_k]$ as a simplicial complex itself, consisting of a single k-simplex along with all of its faces; i.e., every $[v_{i_0}, \ldots, v_{i_l}]$ for $\{v_{i_0}, \ldots, v_{i_l}\} \subseteq \{v_0, \ldots, v_k\}$.

As a model of topological spaces, we are often concerned with the homeomorphism type and combinatorial information of a geometric simplicial complex, in which case we can ignore its precise embedding in Euclidean space. The combinatorial information of the geometric simplicial complex, i.e., the set of vertices and the subsets of vertices that constitute simplices, entails all the topological information that we need.

Definition 3.3. An *abstract simplicial complex* consists of a set X^0 of "vertices" and, for each k, a set X^k composed of $k+1$ element subsets of X^0. These are subjected to the condition that any $j+1$ element subset of a set in X^k is an element of X^j.

The elements of X^k are the abstract k-simplices, and the subset condition can be understood as requiring the face of any k-simplex to again be a simplex in the abstract simplicial complex.

The appropriate type of morphism between simplicial complexes is a *simplicial map*. If X and Y are simplicial complexes, then a simplicial map $f : X \to Y$ is determined by sending the vertices $\{v_i\}$ of X to vertices $\{f(v_i)\}$ of Y such that, if $[v_{i_0}, \dots, v_{i_k}]$ is a simplex of X, then $[f(v_{i_0}), \dots, f(v_{i_k})]$ is a simplex of Y (the $f(v_{i_j})$ need not be distinct). A function on the vertices $X^0 \to Y^0$ determines the entire function $f : X \to Y$. For abstract simplicial complexes, a map on the vertices determines the entire function because the simplices are represented by sets of vertices. For geometric simplicial complexes, a map on the vertices determines the entire function between complexes by by linear interpolation on each simplex with respect to barycentric coordinates.

Example 3.4. Let $[v_{i_0}, \dots, v_{i_k}]$ be a k-simplex in a simplicial complex X. Then $K = [v_{i_0}, \dots, v_{i_k}]$ is itself a simplicial complex, so the map $K \to X$ taking each v_{i_j} to itself is a simplicial map taking K identically to itself in X. This simplicial map is the inclusion $K \hookrightarrow X$ of a face K into a simplicial complex X. Figure 1 depicts such an inclusion.

Example 3.5. Another important example of a simplicial map is one that collapses a simplex. Consider, for example, the 4-simplex $[v_0, v_1, v_2, v_3, v_4]$, one of whose faces is $[v_0, v_1, v_2, v_3]$. Now let $f : [v_0, v_1, v_2, v_3, v_4] \to [v_0, v_1, v_2, v_3]$ be the map $v_0 \mapsto v_0$, $v_1 \mapsto v_1$, $v_2 \mapsto v_1$, $v_3 \mapsto v_2$, and $v_4 \mapsto v_3$. This map f collapses the 4-simplex $[v_0, v_1, v_2, v_3, v_4]$ down to its face the 3-simplex $[v_0, v_1, v_2, v_3]$, so f sends a 4-simplex to 3-simplex. One of the virtues of working with simplicial sets instead of simplicial complexes is the ability simplicial set theory lends us to still see the image of the 4-simplex sitting inside of the 3-simplex as a "degenerate" simplex.

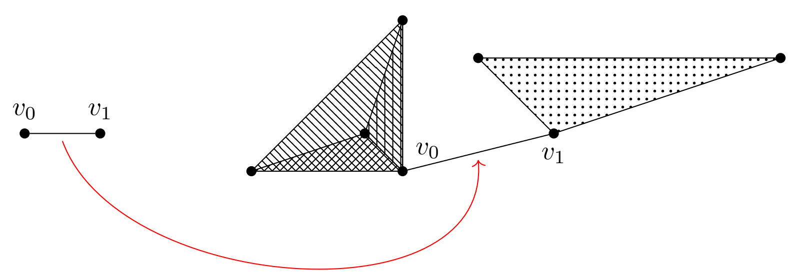


Figure 1: The inclusion of a 1-simplex $[v_0, v_1]$ into a simplicial complex of which it is a face.

We denote an (abstract) n-simplex $[v_0, \dots, v_n]$ by Δ^n. One may define a *geometric realization* function that takes this abstract simplex Δ^n to a geometric simplex situated in Euclidean space $|\Delta^n|$, by assigning certain vectors in Euclidean space to the vertices v_i and then taking the convex hull of this set of vectors.

Simplicial maps of geometric simplicial complexes determine simplicial maps of abstract simplicial complexes by recording where each vertex of the domain goes. Conversely, note that a simplicial map is described entirely in terms of abstract simplicial complex information, by picking a vertex in the image for each vertex in the domain to be sent to. Moreover, with the notion of simplicial homeomorphism we are able to identify an abstract simplicial complex with all of the geometric simplicial complexes that have the same combinatorial data, up to simplicial homeomorphism. This identification justifies our abandoning of "geometric" or "abstract" and simply saying "simplicial complex."

Unfortunately, simplicial complexes do not behave as well as we would like them in order for them to model topological spaces by combinatorial information. This may be seen through the behavior of the geometric realization function with respect to taking products.

Definition 3.6. Let X and Y be two simplicial complexes. Their product $X \times Y$ is defined by

$$(X \times Y)^0 = X^0 \times Y^0$$

where $\sigma \subseteq (X \times Y)^0$ is a simplex of $X \times Y$ if and only if $p_X(\sigma)$ is a simplex of X and $p_Y(\sigma)$ is a simplex of Y, for projection maps $p_X : X^0 \times Y^0 \to X^0$ and $p_Y : X^0 \times Y^0 \to Y^0$.

Example 3.7. Consider the geometric realization of a 1-simplex, $|\Delta^1|$. This space is just an interval I, so taking the product $|\Delta^1| \times |\Delta^1|$ yields a square. On the other hand, consider the product $\Delta^1 \times \Delta^1$ of simplicial complexes. We get a vertex set

$$V^0 = \{(v_0, v_0), (v_0, v_1), (v_1, v_0), (v_1, v_1)\}$$

and a set of simplices consisting of all subsets of V^0. Thus, the geometric realization $|\Delta^1 \times \Delta^1|$ is a tetrahedron. So, in short, $|\Delta^1| \times |\Delta^1|$ is not equal to, or even homeomorphic to, $|\Delta^1 \times \Delta^1|$.

This disagreement is a problem, particularly when we are introducing these combinatorial objections to study topological spaces. To resolve this issue, we impose a partial ordering on the vertices of simplicial complexes.

Definition 3.8. An *oriented simplicial complex* is a simplicial complex with a partially ordered vertex set.

We can turn any abstract simplicial complex into an oriented simplicial complex by imposing a partial ordering on its vertices. Then each simplex has an orientation of its faces; hence we may be think of a 1-simplex as

$$v_0 \bullet \longrightarrow \bullet\, v_1$$

if $v_0 < v_1$ in the partial ordering. We denote an oriented n-simplex by $\Delta[n]$. We can realize geometrically these oriented simplices just as before, ignoring orientations.

Example 3.9. Back to the previous example involving $\Delta[1] \times \Delta[1]$, where we write $\Delta[1] = \{v_0, v_1\}$ with $v_0 < v_1$. Now the set of simplices is not all of $\{v_0, v_1\}^2$ since the vertices (v_0, v_1) and (v_1, v_0) are incomparable. As a result, we get an ordered simplicial complex whose realization is a square, as we would expect.

Ordering the vertices helps resolve the problem with taking products and geometric realizations. However, even ordered simplicial complexes present issues with an important construction involving spaces: quotients.

Example 3.10. We can take the quotient of an interval by identifying its two boundary points, resulting in a circle. If we try to do this on the ordered simplicial complex $\Delta[1]$, we identify the two vertices as one, but then we must end up with a 0-simplex, since simplices are completely determined by their vertices.

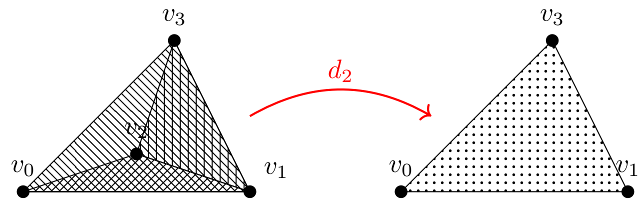


Figure 2: The face map $d_2 : [v_0, v_1, v_2, v_3] \to [v_0, v_1, v_3]$ taking the tetrahedron to its 2-simplex face obtained by removing the vertex 2.

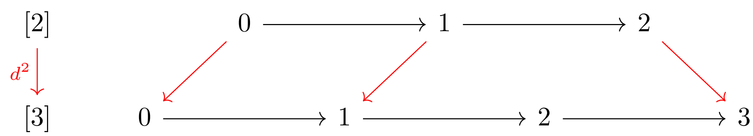


Figure 3: The map d^2 taking $[0, 1, 2]$ to $[0, 1, 3]$ in $[0, 1, 2, 3]$.

Because of this discordance between ordered simplicial complexes and quotient spaces, a ubiquitous construction in topology, we look to abandon the condition that simplices be determined by their vertices. Once we allow for this repetition of vertices in a simplex, we start to look at simplices that look rather different from the simplices like an interval, a triangle, a tetrahedron, etc., with which we are familiar. The type of combinatorial object capturing this idea with which we will be concerned are *simplicial sets*.

3.2 Simplicial sets

Let $[n]$ denote the ordered set $\{0 < 1 < \cdots < n\}$. We can consider order-preserving maps $[n] \to [m]$. For example, $0 \mapsto 0$ and $1 \mapsto 2$ is an order preserving map $[1] \to [2]$. We might also consider the map $[1] \to [2]$ sending both 0 and 1 to 1. Such a map can be obtained as a composite $[1] \to [0] \to [2]$, and we are interested in generating maps between these ordered sets.

All order-preserving maps $[n] \to [m]$ can be written as a finite composition of maps $[k-1] \to [k]$ and $[k+1] \to [k]$ of the following form. We have $n+1$ maps $d^i : [n-1] \to [n]$ for $0 \le i \le n$, each specified by picking an element i of $[n]$ to leave out of the image, and $n+1$ maps $s^i : [n+1] \to [n]$ for $0 \le i \le n$, specified by picking an element of $[n]$ to which two consecutive elements of $[n+1]$ map. More explicitly, $d^i(j) = j$ for $j < i$ and $d^i(j) = j+1$ for $j \ge i$; $s^i(j) = j$ for $j \le i$ and $s^i(j) = j - 1$ for $j > i$. Since the i for which the maps d^i and s^i are defined depends on the domain object $[n-1]$ or $[n+1]$, respectively, to be precise we should write d^i_n and s^i_n, but for simplicity we omit the script n; whenever we come across $d^i : [n-1] \to [n]$ or $s^i : [n+1] \to [n]$, we must remember that these two types of maps are only defined for $0 \le i \le n$.

In order to consider specific subsets of $[n]$, let $[0, \ldots, n]$ also stand for $[n]$. This notation for $[n]$ is suggestive as it resembles our notation $[v_0, \ldots, v_n]$ for an n-simplex previously. Under this convention, for example, we can denote the subset $\{0, 1, 3\}$ of $[3] = \{0, 1, 2, 3\}$ by $[0, 1, 3]$. If we think of $[n]$ as an n-simplex, although it is really just the ordered set $\{0, \ldots, n\}$, then the maps d^i *look* like the inclusion of a face into a simplex. For example, the map d^2 takes the 2-simplex $[v_0, v_1, v_2]$ to $[v_0, v_1, v_2, v_3]$ by $d^2(v_i) = v_i$ for $i < 2$, and

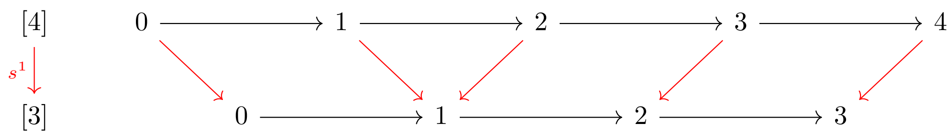


Figure 4: The map s^1 taking $[0, 1, 2, 3, 4]$ to $[0, 1, 1, 2, 3]$.

$d^2(v_i) = v_{i+1}$ for $i \geq 2$; hence the image of the 2-simplex $[v_0, v_1, v_2]$ under d^2 is $[v_0, v_1, v_3]$, the face of a 3-simplex $[v_0, v_1, v_2, v_3]$ obtained by removing the vertex v_2; see Figure 2. Similarly, $d^2 : [2] \to [3]$ sends $[0, 1, 2]$ to $[0, 1, 3]$; see Figure 3. Hence, the opposite maps $(d^i)^{op} =: d_i : [n] \to [n-1]$ take $[n]$ to the subset $[0, \ldots, \widehat{i}, \ldots, n]$, where $\widehat{i}$ means the element i is excluded. These maps d_i, when applied to a simplex, return the face of the simplex obtained by removing the i-th vertex; for this reason, such a map d_i is called a *face map*. It is important to note that each d_i simply assigns to an n-simplex one of its faces, with no point-set topological or simplicial map meant.

On the other hand, for example, the map s^1 takes $[v_0, v_1, v_2, v_3, v_4]$ to $[v_0, v_1, v_1, v_2, v_3]$ by $s^1(v_i) = v_i$ for $i \leq 1$, and $s^1(v_i) = v_{i-1}$ for $i > 1$. The simplex $[v_0, v_1, v_1, v_2, v_3]$ is a *degenerate* simplex: a simplex is degenerate it if not all of its vertices are unique. Similarly, the map $s^1 : [4] \to [3]$ sends $[0, 1, 2, 3, 4]$ to $[0, 1, 1, 2, 3]$; see Figure 4. As with the maps d^i, here we can consider the map $s_1 := (s^1)^{op}$ to be defined so that $s_1[0, 1, 2, 3]$ is $[0, 1, 1, 2, 3]$, and more generally, $s_i[0, \ldots, n] = [0, \ldots, i, i, \ldots, n]$. The maps s_i may not seem to have such a natural interpretation as our face maps d_i. The way to think of these is to first recall Example 3.5. There we considered a map collapsing a 2-simplex to one of its faces. This was in fact an example of the map s_1 depicted above applied to general simplices, where $s_1[v_0, v_1, v_2, v_3, v_4] = [v_0, v_1, v_1, v_2, v_3]$. We consider the map s_i to take an n-simplex, and return the *degenerate* $(n+1)$-simplex sitting inside the n-simplex by repeating the i-th vertex. Hence, we refer to s_i as *degeneracy maps*. We have violated our earlier rule that a simplex $[v_0, \ldots, v_n]$ must have distinct vertices satisfying $v_i < v_j$ if $i < j$. Now, we allow $[v_0, \ldots, v_n]$ to stand for a simplex if $v_i \leq v_j$ if $i < j$.

In summary, we have $n + 1$ face maps $d_i : [n] \to [n-1]$ and $n + 1$ degeneracy maps $s_i : [n] \to [n+1]$, for $0 \leq i \leq n$, that allow us to refer to the various faces and degeneracies present in any given simplex. We may see that the maps d_i and s_i satisfy certain relations between one another. For example, if $i < j$, then $d_i d_j = d_{j-1} d_i$; indeed,

$$\begin{aligned} d_i d_j[0, \ldots, n] &= d_i[0, \ldots, \widehat{j}, \ldots, n] \\ &= [0, \ldots, \widehat{i}, \ldots, \widehat{j}, \ldots, n] \\ &= d_{j-1}[0, \ldots, \widehat{i}, \ldots, n] \\ &= d_{j-1} d_i[0, \ldots, n]. \end{aligned}$$

Composing d_i and s_j maps yields various other relations that can be checked similarly to above. These relations lead to the following definition of a simplicial set.

Definition 3.11 (Combinatorial definition of simplicial set)**.** A *simplicial set* consists of a sequence of sets $X_0, X_1, \ldots$ and, for each $n \geq 0$, maps $d_i : X_n \to X_{n-1}$ and $s_i : X_n \to X_{n+1}$

for each i with $0 \leq i \leq n$ such that

$$\begin{aligned} d_i d_j &= d_{j-1} d_i \quad \text{if } i < j, \\ d_i s_j &= s_{j-1} d_i \quad \text{if } i < j, \\ d_j s_j &= d_{j+1} s_j = \text{id}, \\ d_i s_j &= s_j d_{i-1} \quad \text{if } i > j+1, \\ s_i s_j &= s_{j+1} s_i \quad \text{if } i \leq j. \end{aligned} \tag{3.12}$$

Example 3.13. Any ordered simplicial complex can be made into a simplicial set by adjoining all degenerate simplices. Let X be an ordered simplicial complex. Then we obtain a simplicial set $\overline{X}$ where $\overline{X}_n$ consists of all simplices of the form $[v_{i_0}, \ldots, v_{i_n}]$ for $v_{i_j} \leq v_{i_k}$ when $j < k$, where the *set* of vertices $\{v_{i_0}, \ldots, v_{i_n}\}$ constitutes a simplex in X, so that the vertices need not be unique.

Example 3.14. The standard 0-simplex $X = \Delta[0]$ thought of as a simplicial set is the unique simplicial set with a single simplex in each dimension, i.e., one element $[0, \ldots, 0]$ ($n+1$ zeros) in each X_n, for $n \geq 0$.

Example 3.15. We can form a simplicial set from a topological space by forming the singular chain complex of the space. For a topological space X, let $\mathcal{S}(X)_n$ be the set of all continuous maps from $|\Delta^n|$ to X. To form a simplicial set, we must define the face and degeneracy maps. Given a singular simplex $\sigma : |\Delta^n| \to X$ in $\mathcal{S}(X)_n$, define the singular simplex $d_i \sigma$ by precomposing σ with the simplicial inclusion $[0, \ldots, \widehat{i}, \ldots, n] \to [0, \ldots, n]$. Similarly for the degeneracy maps, s_i takes σ to the singular simplex defined by precomposing σ with the geometric collapse $[0, \ldots, n+1] \to [0, \ldots, \widehat{i}, \widehat{i}, \ldots, n]$.

Then $\mathcal{S}(X)$, consisting of sets $\mathcal{S}(X)_n$ for all $n \geq 0$, forms a simplicial set with the face and degeneracy maps as defined above. One must check that these face and degeneracy maps satisfy the relations of (3.12); the fact that these relations hold for singular simplices in $\mathcal{S}(X)$ follows from the relations holding for a standard simplex.

Let us employ the category theory provided at the beginning in order to provide another view of simplicial sets. The basic properties of a simplicial set derive from those of the standard n-simplex; we derived the relations among face and degeneracy maps from their behavior on the standard simplices. As such, one might find it natural to define a simplicial set as the functorial image of a category built from the standard simplices.

Definition 3.16. The category Δ has as objects the finite ordered sets $[n] = \{0, \ldots, n\}$. The morphisms in Δ are order-preserving functions $[n] \to [m]$.

As we noted at the beginning of this subsection, although not in reference to a category, the maps of this category are generated by certain maps between sets of neighboring cardinalities $d^i : [n-1] \to n$ and $s^i : [n+1] \to [n]$, $0 \leq i \leq n$. As we did before to obtain our face and degeneracy maps, we consider the opposite maps $d_i = (d^i)^{op}$ and $s_i = (s^i)^{op}$; such a move in this context is what we defined before as switching to the opposite category, Δ^{op}.

Definition 3.17 (Categorical definition of simplicial set)**.** A *simplicial set* is a functor $\Delta^{op} \to \textbf{Set}$.

A functor $X : \Delta^{op} \to \textbf{Set}$ assigns to $[n]$ a set X_n of n-simplices. Moreover, X takes the generating maps $d_i : [n] \to [n-1]$ and $s_i : [n] \to [n+1]$ from Δ^{op}, and assign them to maps

$Xd_i : X_n \to X_{n-1}$ and $Xs_i : X_n \to X_{n+1}$ that we interpret as face and degeneracy maps, respectively. Since the maps d_i and s_i in Δ^{op} satisfy those relations in the combinatorial definition of simplicial set, their images Xd_i and Xs_i satisfy the relations by functoriality.

Example 3.18. We can reconsider the example of the singular set $\mathcal{S}(Y)$ of a topological space Y from the categorical point of view. The singular set $\mathcal{S}(Y)$ is a functor $\Delta^{op} \to \mathbf{Set}$ that assigns to $[n]$ the set $\mathrm{Hom}_{\mathbf{Top}}(\Delta^n, Y)$, the set of all continuous maps $\Delta^n \to Y$. To the face and degeneracy maps d_i and s_i of Δ^{op}, the functor assigns the face and degeneracy maps as defined in Example 3.9, by precomposition. The correspondences may be depicted as

$$\begin{array}{ccc} [n] & & \mathrm{Hom}_{\mathbf{Top}}(\Delta^n, Y) \\ \Big\downarrow d_i & \Longrightarrow & \Big\downarrow \mathcal{S}(Y)(d_i) \\ [n-1] & & \mathrm{Hom}_{\mathbf{Top}}(\Delta^{n-1}, Y) \end{array} \qquad \begin{array}{ccc} [n] & & \mathrm{Hom}_{\mathbf{Top}}(\Delta^n, Y) \\ \Big\downarrow s_i & \Longrightarrow & \Big\downarrow \mathcal{S}(Y)(s_i) \\ [n+1] & & \mathrm{Hom}_{\mathbf{Top}}(\Delta^{n+1}, Y) \end{array}$$

Before we had the notion of a simplicial map between simplicial complexes. Now we consider a simplicial morphism between simplicial sets.

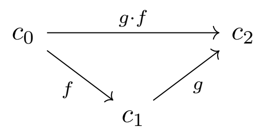
Definition 3.19. If $X, Y : \Delta^{op} \to \mathbf{Set}$ are simplicial sets, then a *simplicial morphism* $f : X \to Y$ is a natural transformation $X \Rightarrow Y$.

Put more concretely, a simplicial morphism consists of set maps $f_n : X_n \to Y_n$ for each n that commute with the face and degeneracy maps.

Example 3.20. In Example 3.3, we considered a map $f : [v_0, v_1, v_2, v_3, v_4] \to [v_0, v_1, v_2, v_3]$ that collapsed a 4-simplex into a 3-simplex. Consider the standard simplices $\Delta[4]$ and $\Delta[3]$ as simplicial sets (by adjoining all degenerate simplices) and define the map f in the analogous way: $f(0) = 0)$, $f(1) = f(2) = 1$, $f(3) = 2$, $f(4) = 3$. Then, as before, $\Delta^4 = [0, 1, 2, 3, 4]$ is taken to the degenerate simplex $[0, 1, 1, 2, 3] = s_1[0, 1, 2, 3]$. At the same time, the morphism f is doing an infinite number of other things. For instance, f takes the 0-simplices $[1], [2] \in \Delta^4$ to $[1] \in \Delta^2$. Additionally, f takes the degenerate simplex $[0, 1, 1, 2, 3, 3, 3, 4] = s_5 s_4 s_1[0, 1, 2, 3, 4] \in \Delta^4$ to the degenerate simplex $[0, 1, 1, 1, 2, 2, 2, 3] = s_5 s_4 s_2 s_1 \in \Delta^3$. These are just a few of the infinitely many things the map f is doing; however, because the map f commutes with the degeneracy maps, f is determined by what it does to the nondegenerate simplices: if $y = s_{i_1} \dots s_{i_k} x$ is some degenerate simplex for x nondegenerate, then $f(y) = f(s_{i_1} \dots s_{i_k} x) = s_{i_1} \dots s_{i_k} f(x)$.

Note that every object $[n]$ in Δ can be regarded as a category itself, with $n + 1$ objects $0, 1, \dots, n$ and an arrow $i \to j$ if and only if $i \leq j$. For any sequence of n composable maps in a category $\mathcal{C}$, i.e., something of the form $c_0 \to c_1 \to \cdots \to c_n$, there is a functor $[n] \to \mathcal{C}$ that picks out this sequence by assigning $i \to c_i$ and the maps $i \to j$ to $c_i \to c_j$. Considering the subcategory $\mathcal{S}$ of $\mathcal{C}$ as the category $c_0 \to c_1 \to \cdots \to c_n$ along with all possible composite morphisms, this functor realizes an isomorphism of categories $[n] \cong S$. In light of these isomorphisms, the objects of Δ can be thought of as representing all possible finite compositions that can take place in an ordinary category. The following is a construction of a simplicial set from a category that utilizes this point of view.

Example 3.21. Any category whose morphisms constitute a set (rather than a proper class) can be turned into a simplicial set. Given such a category $\mathcal{C}$, we define its *nerve* $N\mathcal{C}$, a simplicial set where its set of n-simplices $N\mathcal{C}_n$ is given by the set of functors $[n] \to \mathcal{C}$. The 0-simplices are then the objects of $\mathcal{C}$, the 1-simplices are the morphisms in $\mathcal{C}$, the 2-simplices are composable pairs of morphisms, and so on, where the n-simplices are given by a string of n-composable morphisms in $\mathcal{C}$. A 2-simplex in $N\mathcal{C}$ can be depicted by the following diagram in $\mathcal{C}$:



The face map d_i on an n-simplex $[n] \to C$ in $N\mathcal{C}$ is given by the composite

$$[0, \ldots, \widehat{i}, \ldots, n] \hookrightarrow [n] \to \mathcal{C},$$

and the degeneracy map s_i is defined by the composite

$$[0, \ldots, i, i, \ldots, n] \twoheadrightarrow [n] \to \mathcal{C}.$$

Remark 3.22. A category whose morphisms form a set is called a *small category.*

The nerve construction allows us to consider any category as a certain type of simplicial set. We can then understand the structure of the category by understanding the structure of this simplicial set. Put another way, our understanding of the combinatorics of the category Δ allows us to understand the combinatorics of a general category. The idea behind Θ_2, a generalization of Δ, is to be able to understand 2-categories in a similar way. As the objects of Δ represent the finite compositions that can occur in a regular category, the objects of Θ_2 represent the finite compositions that can occur in a 2-category.

As we have seen, we can give two equivalent definitions of a simplicial set. One definition we can give neatly in a single line through the language of category theory. The other definition we can give explicitly, by sequences of sets with generating maps and relations. The advantage of the second, combinatorial approach is that it gives explicit and concrete conditions under which a sequence of sets forms a simplicial set. The goal of the next section is to introduce Θ_2 and present a set of generating maps and relations that provide a combinatorial description of the structure of the category.

4 The category Θ_2

The following category Θ_2 is a generalization of Δ. Understanding the combinatorics of the category Δ helps to understand the combinatorics of ordinary categories, and Θ_2 can serve a similar role to understanding 2-categories. In this section, we present a set of generating maps for this category and the relations amongst them, analogous to the generating maps d^i and s^i of Δ.

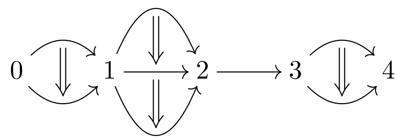
Definition 4.1. The category Θ_2 has as objects $[n]\big([k_1], \ldots, [k_n]\big)$, where $[n]$ and each $[k_i]$ are objects of Δ, and a morphism $[n]\big([k_1], \ldots, [k_n]\big) \to [m]\big([l_1], \ldots, [l_m]\big)$ is given by a function $\rho : [n] \to [m]$ in Δ, and functions $[k_i] \to [l_j]$ defined whenever $\rho(i-1) < j \leq \rho(i)$.

We refer to the objects $[n]$ and $[m]$ in the above definition as *outer objects* of Θ_2, while the $[k_i]$ and $[l_i]$ we refer to as *inner objects* of Θ_2.

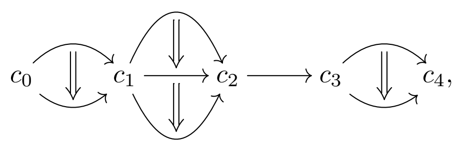
Example 4.2. The object $[4]\big([1],[2],[0],[1]\big)$ of Θ_2 can be depicted as

$$0 \xrightarrow{[1]} 1 \xrightarrow{[2]} 2 \xrightarrow{[0]} 3 \xrightarrow{[1]} 4.$$

Because the objects of Δ that label each arrow can themselves be considered as strings of arrows, we get a diagram such as



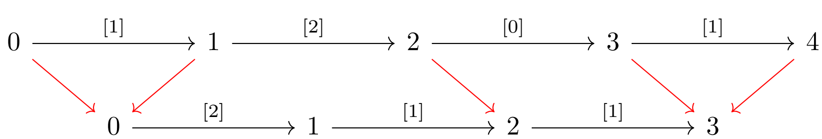
Before we described the way in which the objects of Δ represent the finite compositions of morphisms in an ordinary category and alluded to the way in which the objects of Θ_2 would do the same for a 2-category. To do so, we considered each object of Δ as a category in itself; now, we consider each object of Θ_2 as a 2-category itself. Then an example of a diagram in a general 2-category could be



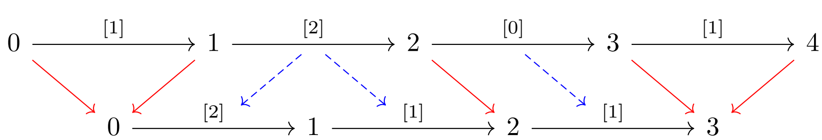
which, considered as a 2-category itself, is isomorphic to the above object $[4]\big([1],[2],[0],[1]\big)$ of Θ_2. In general, for some finite composition of morphisms in a 2-category, which constitute a subcategory $\mathcal{U}$, there is an isomorphism of categories from the appropriate object of Θ_2 to $\mathcal{U}$.

Denote a morphism $[n]\big([k_1],\ldots,[k_n]\big) \to [m]\big([l_1],\ldots,[l_m]\big)$ in Θ_2 by $\rho(\gamma_1,\ldots,\gamma_h)$ where ρ is a map $[n] \to [m]$ and γ_i is defined wherever possible (so both h and the co/domain of each γ_i depends on ρ).

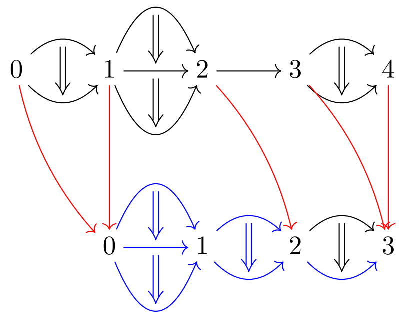
Let us consider a morphism $[4]\big([1],[2],[0],[1]\big) \to [3]\big([2],[1],[1]\big)$ in Θ_2. The first thing we need to specify in order to define such a morphism is a map $\rho : [4] \to [3]$ on the outer objects. For example:



We then have maps between the inner objects whenever it is reasonable to do so. With the above example, we have to choose a map $[2] \to [2]$, $[2] \to [1]$, and $[0] \to [1]$. We think of the three arrows $1 \to 2$ in the top object being sent to the composite of their images as arrows $0 \to 1$ and $1 \to 2$ specified by the two chosen maps $[2] \to [2]$ and $[2] \to [1]$. Which maps are specified on the inner objects can be depicted by the blue arrows below.



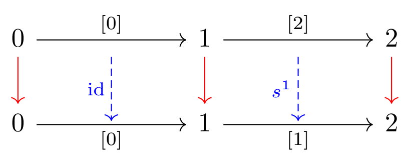
More pictorially, we can depict one choice of such a morphism in Θ_2 by



where the images of the chosen maps on the inner objects are in blue, and the map ρ is still depicted in red. For such a picture, there are still two choices for where to send the middle arrow $1 \to 2$ in the chosen map $[2] \to [1]$ in blue.

4.1 Generators and relations for Θ_2

Given any map between two objects of Θ_2, we want to be able to write it as a finite composite of maps from a certain set, a set of generating maps. A necessary condition on these generating maps is that, between any two objects of Θ_2, there must be a composite of generating maps taking us from one object to the other without collapsing any of the Δ-objects to a Δ-object smaller than its image(s) under the map in Θ_2. For instance, consider the objects $[2]\big([0],[2]\big)$ and $[2]\big([0],[1]\big)$, with a morphism in Θ_2 depicted by



that applies the map s^1 from Δ to $[2]$ and the identity elsewhere. This map cannot factor through, for example, the object $[2]\big([0],[0]\big)$ because that would require collapsing $[2]$ to a point $[0]$, then mapping that point into $[1]$: such a map picks out *one* of the two elements of $[1]$, whereas $s^1 : [2] \to [1]$ is surjective. The inability of the composite $[2] \to [0] \to [1]$ to be surjective demonstrates the issue of passing Δ-objects through smaller objects than their images under the map in question. Thus, for any map between two objects in Θ_2, we need a composite of generating maps that takes us from one object of Θ_2 to the other without collapsing Δ-objects below their images under the map in Θ_2. To avoid collapsing Δ-objects unnecessarily, we define generating maps that apply the identity on everything except for one Δ-object, where we apply a generating map of Δ to that one specified Δ-object. (The map depicted above is an example of a map of this type.)

The generators for Θ_2 presented here are of two types, ones applying a generator of Δ on an object in the 'outer' copy of Δ present in Θ_2, and ones applying a generator of Δ on an object in the 'inner' copy of Δ. For each $0 \leq i \leq n$, we have the two outer generators

$$\begin{aligned} d^i_{\mathrm{out}} &= d^i(\mathrm{id},\dots,\mathrm{id}) : [n]\big([c_1],\dots,[c_n]\big) \to [n+1]\big([c_1],\dots,[c_i],[c_i],\dots,[c_n]\big), \\ s^i_{\mathrm{out}} &= s^i(\mathrm{id},\dots,\mathrm{id}) : [n+1]\big([c_1],\dots,[c_{n+1}]\big) \to [n]\big([c_1],\dots,[c_i],[c_{i+2}],\dots,[c_n]\big), \end{aligned} \tag{4.3}$$

that apply the generating map d^i or s^i from Δ to the outer object $[n]$ and apply the identity on each of the inner objects $[c_i]$. When the outer map is s^i_{out}, the object $[c_i]$ is the only object not present in the image; whereas, when the outer map is d^i_{out}, the image has an extra copy of $[c_i]$.

For each $0 \leq k \leq n$ and $0 \leq i \leq c_k$, we have the two inner generators

$$d^j_{\text{in(k)}} = \text{id}(\text{id}, \dots, d^i, \dots, \text{id}) : [n]\big([c_1], \dots, [c_k], \dots [c_n]\big) \to [n]\big([c_1], \dots, [c_k + 1], \dots, [c_n]\big),$$
$$s^j_{\text{in(k)}} = \text{id}(\text{id}, \dots, d^i, \dots, \text{id}) : [n]\big([c_1], \dots, [c_k + 1], \dots [c_n]\big) \to [n]\big([c_1], \dots, [c_k], \dots, [c_n]\big),$$

where index k signifies the inner object $[c_k]$ to which the map d^i or s^i is to be applied, with identity maps everywhere else.

Thus the four types of generators for Θ_2 are

$$\begin{matrix} d^i_{\text{out}} & d^i_{\text{in(k)}} \\ s^i_{\text{out}} & s^i_{\text{in(k)}} \end{matrix}$$

To get a visual sense for what these generators look like, we depict each of the four types of generators when $n = 3$: Figure 5 portrays the map d^2_{out}; Figure 6 portrays the map s^1_{out}; Figure 7 portrays the map $d^i_{\text{in(2)}}$; Figure 8 portrays the map $s^i_{\text{in(3)}}$.

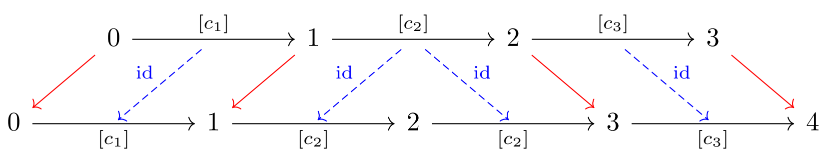


Figure 5: The outer generator d^2_{out} taking $[3]\big([c_1], [c_2], [c_3]\big)$ to $[4]\big([c_1], [c_2], [c_2], [c_3]\big)$.

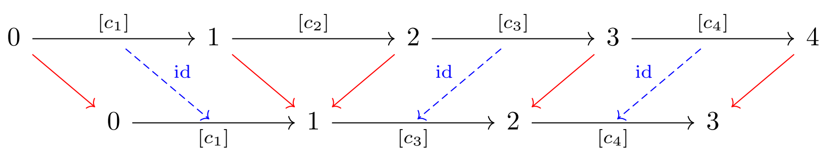


Figure 6: The outer generator s^1_{out} taking $[4]\big([c_1], [c_2], [c_3], [c_4]\big)$ to $[3]\big([c_1], [c_3], [c_4]\big)$.

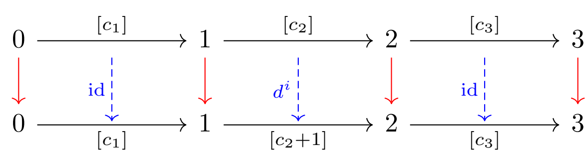


Figure 7: The inner generator $d^i_{\text{in(2)}}$ taking $[3]\big([c_1], [c_2], [c_3]\big)$ to $[3]\big([c_1], [c_2 + 1], [c_3]\big)$.

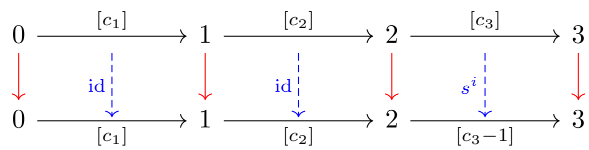


Figure 8: The inner generator $s^i_{\mathrm{in}(3)}$ taking $[3]\big([c_1],[c_2],[c_3]\big)$ to $[3]\big([c_1],[c_2],[c_3-1]\big)$.

Example 4.4. To get a further sense for how these generators behave in action, let us consider two types of copies of Δ that sit inside Θ_2; in other words, two subcategories of Θ_2 that are isomorphic to Δ.

(i) The subcategory consisting of all objects of the form $[1]\big([n]\big)$ for any object $[n]$ in Δ along with maps $d^i_{\mathrm{in}(1)}$ and $s^i_{\mathrm{in}(1)}$ is isomorphic to Δ. The functor demonstrating this isomorphism may be defined by sending the object $[1]\big([n]\big)$ to $[n]$, and sending the map $d^i_{\mathrm{in}(1)}$ to d^i and $s^i_{\mathrm{in}(1)}$ to s^i, with its inverse defined by reversing these assignments.

This example generalizes to the subcategory of Δ consisting of all objects of the form $[m]\big([c_1],\dots,[c_{k-1}],[n],[c_{k+1}],\dots,[c_m]\big)$ for all c_j fixed as well as n fixed, with maps $d^i_{\mathrm{in(k)}}$ and $s^i_{\mathrm{in(k)}}$. It is worth noting that when we consider this more general case, the subcategory is not *full*, meaning it does not include all maps between these objects in Θ_2.

(ii) The subcategory composed of objects of the form $[n]\big([0],\dots,[0]\big)$ and maps d^i_{out} and s^i_{out}. The functor sending $[n]\big([0],\dots,[0]\big)$ to $[n]$, d^i_{out} to d^i and s^i_{out} to s^i realizes this isomorphism.

Like in (i), this example generalizes. The subcategory of all objects of the form $[n]\big([m],\dots,[m]\big)$, for m fixed, and maps d^i_{out} and s^i_{out} is isomorphic to Δ as well, with the functor demonstrating this isomorphism defined similarly to before. However, if $m>0$, then this subcategory is not full.

Before we discuss the relations between these generators, let us show that these maps really do generate all maps in Θ_2.

Proposition 4.5. Any map

$$\varphi=\rho(\gamma_1,\dots,\gamma_h):[n]\big([k_1],\dots,[k_n]\big)\to[m]\big([l_1],\dots,[l_m]\big)$$

in Θ_2 can be written as a finite composite of d^i_{out}, s^i_{out}, $d^i_{\mathrm{in(k)}}$, and $s^i_{\mathrm{in(k)}}$.

Proof. Since $\rho:[n]\to[m]$ is a map in Δ, we may write it as a finite composite of maps d^i and s^i. Hence, in Θ_2, applying this composite map with each d^i and s^i replaced by d^i_{out} and s^i_{out}, respectively, sends $[n]\big([k_1],\dots,[k_n]\big)$ to $[m]\big([k_{i_1}],\dots,[k_{i_h}]\big)$, where the k_{i_j} may repeat. The objects $[k_{i_j}]$ are all of the objects on which the inner maps γ_h are defined.

Each γ_j is a map in Δ taking $[k_{i_j}]$ to $[l_j]$, so we may write this as a finite composition of maps d^i and s^i. Then we may apply this sequence of maps, with d^i and s^i replaced by $d^i_{\mathrm{in(i_j)}}$ and $s^i_{\mathrm{in(i_j)}}$, doing so in turn for each $1\le j\le m$, resulting in $[m]\big([l_1],\dots,[l_m]\big)$. □

Now we want to consider the relations amongst these generators. For the outer generators, we may see that the relations between these are given precisely by the relations (3.12) in the combinatorial definition of simplicial set, where we switch the order of composition since $d_i = (d^i)^{op}$ and $s_i = (s^i)^{op}$. Explicitly,

$$\begin{aligned} d^j_{\text{out}} d^i_{\text{out}} &= d^i_{\text{out}} d^{j-1}_{\text{out}} && \text{if } i < j, \\ s^j_{\text{out}} d^i_{\text{out}} &= d^i_{\text{out}} s^{j-1}_{\text{out}} && \text{if } i < j, \\ s^j_{\text{out}} d^j_{\text{out}} &= s^j_{\text{out}} d^{j+1}_{\text{out}} = \text{id}, \\ s^j_{\text{out}} d^i_{\text{out}} &= d^{j-1}_{\text{out}} s^j_{\text{out}} && \text{if } i > j+1, \\ s^j_{\text{out}} s^i_{\text{out}} &= s^i_{\text{out}} s^{j+1}_{\text{out}} && \text{if } i \leq j. \end{aligned} \tag{4.6}$$

For the inner generators, when the indices of two of these maps are distinct, the two maps commute:

$$\begin{aligned} d^i_{\text{in(k)}} d^j_{\text{in(h)}} &= d^j_{\text{in(h)}} d^i_{\text{in(k)}} && \text{if } k \neq h, \\ d^i_{\text{in(k)}} s^j_{\text{in(h)}} &= s^j_{\text{in(h)}} d^i_{\text{in(k)}} && \text{if } k \neq h, \\ s^i_{\text{in(k)}} s^j_{\text{in(h)}} &= s^j_{\text{in(h)}} s^i_{\text{in(k)}} && \text{if } k \neq h. \end{aligned} \tag{4.7}$$

When the indices agree on two inner generators, i.e. $k = h$ above, the relations again come from the relations (3.12) in the combinatorial definition of simplicial set, where we switch the order of composition:

$$\begin{aligned} d^j_{\text{in(k)}} d^i_{\text{in(k)}} &= d^i_{\text{in(k)}} d^{j-1}_{\text{in(k)}} && \text{if } i < j, \\ s^j_{\text{in(k)}} d^i_{\text{in(k)}} &= d^i_{\text{in(k)}} s^{j-1}_{\text{in(k)}} && \text{if } i < j, \\ s^j_{\text{in(k)}} d^j_{\text{in(k)}} &= s^j_{\text{in(k)}} d^{j+1}_{\text{in(k)}} = \text{id}, \\ s^j_{\text{in(k)}} d^i_{\text{in(k)}} &= d^{j-1}_{\text{in(k)}} s^j_{\text{in(k)}} && \text{if } i > j+1, \\ s^j_{\text{in(k)}} s^i_{\text{in(k)}} &= s^i_{\text{in(k)}} s^{j+1}_{\text{in(k)}} && \text{if } i \leq j. \end{aligned} \tag{4.8}$$

We have seen the relations between the outer maps, and we have seen the relations between the inner maps, so now we look to the relations between the outer maps and the inner maps of Θ_2.

Composing inner and outer d-maps we have

$$\begin{aligned} d^i_{\text{in(k)}} d^j_{\text{out}} &= d^j_{\text{out}} d^i_{\text{in(k)}} && \text{if } k < j, \\ d^i_{\text{in(k)}} d^j_{\text{out}} &= d^j_{\text{out}} d^i_{\text{in(k−1)}} && \text{if } k > j+1, \\ d^i_{\text{in(j)}} d^j_{\text{out}} &= s^{i-1}_{\text{in(j+1)}} d^j_{\text{out}} d^i_{\text{in(j)}} = s^i_{\text{in(j+1)}} d^j_{\text{out}} d^i_{\text{in(j)}}, \\ d^i_{\text{in(j+1)}} d^j_{\text{out}} &= s^{i-1}_{\text{in(j)}} d^j_{\text{out}} d^i_{\text{in(j)}} = s^i_{\text{in(j)}} d^j_{\text{out}} d^i_{\text{in(j)}}. \end{aligned} \tag{4.9}$$

The reason the inner s-maps appear in the case where $k = j$ and $k = j+1$ is due to the way the outer maps d^j_{out} are defined in (2). When applying d^j_{out} to an object, the j-th object $[c_j]$ is duplicated in the image, so then applying an inner map $d^i_{\text{in(j)}}$ to one of these duplicates leaves one $[c_j]$ and one $[c_j + 1]$ in the image. When switching the order of composition, if we apply an inner map to $[c_j]$ first, giving $[c_j + 1]$, this results in $[c_j + 1]$ being duplicated in the image after applying d^j_{out}, hence we must apply $s^i_{\text{in(j)}}$ or $s^i_{\text{in(j+1)}}$ (depending on if $k = j$

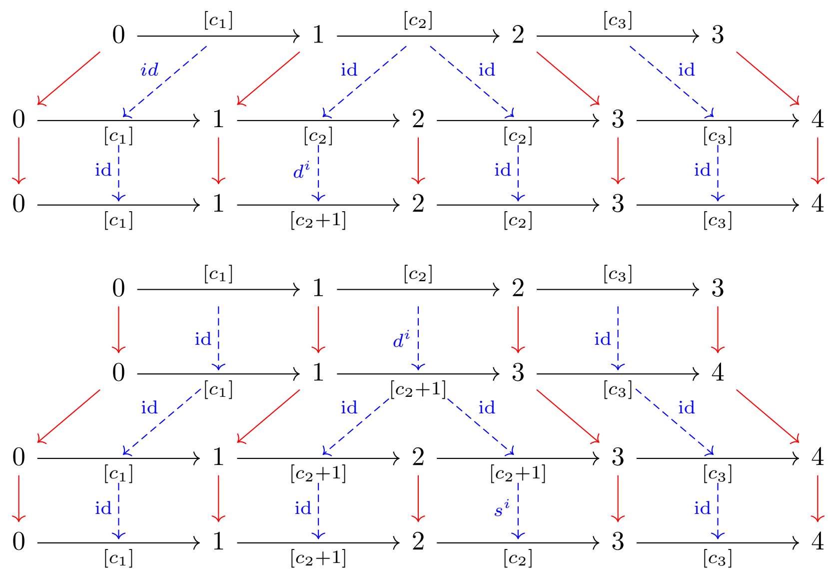


Figure 9: Top diagram: $d^i_{\text{in}(2)}d^2_{\text{out}}$. Bottom diagram: $s^i_{\text{in}(3)}d^2_{\text{out}}d^i_{\text{in}(2)}$.

or $k = j+1$) in order to result in one $[c_j]$ and one $[c_j+1]$ in the image. Pictorially this may be seen in Figure 9, where the relation

$$d^i_{\text{in}(2)}d^2_{\text{out}} = s^i_{\text{in}(3)}d^2_{\text{out}}d^i_{\text{in}(2)}$$

is displayed. Note that the s^i in the bottom diagram can be replaced by s^{i-1} (as expressed by the third relation in (4.9)). Moreover, the reason that this works is from the relation $s^i d^i = s^{i-1}d^i = \text{id}$ in Δ.

Composing inner and outer s-maps we have

$$\begin{aligned} s^i_{\text{in(k)}}s^j_{\text{out}} &= s^j_{\text{out}}s^i_{\text{in(k)}} && \text{if } k < j, \\ s^i_{\text{in(k)}}s^j_{\text{out}} &= s^j_{\text{out}}s^i_{\text{in(k+1)}} && \text{if } k \geq j, \\ s^j_{\text{out}}s^i_{\text{in(j)}} &= s^j_{\text{out}}. \end{aligned} \tag{4.10}$$

Composing one inner s-map and one outer d-map we have

$$\begin{aligned} s^i_{\text{in(k)}}d^j_{\text{out}} &= d^j_{\text{out}}s^i_{\text{in(k)}} && \text{if } k < j, \\ s^i_{\text{in(k)}}d^j_{\text{out}} &= d^j_{\text{out}}s^i_{\text{in(k}-1)} && \text{if } k > j+1, \end{aligned} \tag{4.11}$$

where the case for $k = j$ and $k = j+1$ is handled by the third and fourth relations of (6).

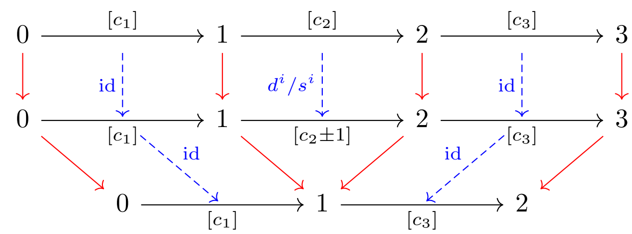


Figure 10: Diagram displaying the relations $s^j_{\text{out}} d^i_{\text{in(j)}} = s^j_{\text{out}}$ and $s^j_{\text{out}} s^i_{\text{in(j)}} = s^j_{\text{out}}$. In either case, $d^i_{\text{in(j)}}$ and $s^i_{\text{in(j)}}$ have no affect on the image because their inner index j agrees with that of the outer map s^j_{out}.

Composing one inner d-map and one outer s-map we have

$$\begin{aligned} d^i_{\text{in(k)}} s^j_{\text{out}} &= s^j_{\text{out}} d^i_{\text{in(k)}} && \text{if } k < j, \\ d^i_{\text{in(k)}} s^j_{\text{out}} &= s^j_{\text{out}} d^i_{\text{in(k+1)}} && \text{if } k \geq j, \\ s^j_{\text{out}} d^i_{\text{in(j)}} &= s^j_{\text{out}}. \end{aligned} \tag{4.12}$$

From the third relation in (4.10) and the third relation in (4.12), we see that the inner s-map and inner d-map, respectively, are in a way 'canceled out' by the outer s-map, due to the way the index of the inner maps agrees with the index of the outer map; both can be seen in Figure 10.

In the case of a simplicial set, i.e., a functor $\Delta^{op} \to \mathbf{Set}$, the generating maps of Δ along with their relations ultimately provided the sufficient information to check that a sequence of sets and maps satisfy in order to constitute a simplicial set. In this way, those interested in considering functors $\Theta_2 \to \mathbf{Set}$ can use the relations above as explicit conditions under which something constitutes a Θ_2-set.

Remark 4.13. We would have liked to have proved that the relations above are all of the relations on these generators for Θ_2 in the sense that, if we are given some other relation on the generators, we can express that given relation in terms of the ones above. We believe that the relations listed above are in fact all of them, and proving that we can write any relation on the generators of Θ_2 in terms of those relations above, or if necessary, augmenting this list of relations until it is sufficient for expressing any general relation, is the next step in the project.

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