Dyer-Lashof operations as extensions and an application to $H_*(BU)$

Brian Thomas

A Dissertation Presented to the Graduate Faculty
of the University of Virginia in Candidacy for the Degree of
Doctor of Philosophy

Department of Mathematics

University of Virginia

May, 2019

©Copyright by Brian Thomas 2019

Abstract

Algebraic topology is concerned with the algebraic structure associated to topological spaces. There are algebraic operations Q^k, called Dyer-Lashof operations, that act on the homology of highly structured spaces. We explore a connection between these operations and Ext groups between unstable modules over the Steenrod algebra. This allows us to make calculations in the stable world, which is often easier. By using a purely algebraic spectral sequence developed by Kuhn and McCarty, along with these Ext groups, one can obtain information on how the Q^k act on $H_*(\Omega^\infty X)$ for connective spectra X. The Ext groups are still not easy, but as an application of our method, we show how to calculate the Q^k when $X = \Sigma^2 ku$ which has $H_*(\Omega^\infty X) = H_*(BU)$, obtaining an action of the Dyer-Lashof algebra that was previously shown by Kochman and Priddy.

Acknowledgments

First, I want to thank my advisor, Nick Kuhn, who introduced me to the problems in this thesis, gave much of his time, and contributed several key ideas, particularly the proof of Theorem 3.3.16. I want to thank Prasit Bhattacharya for invaluable conversations about how to approach calculations. I also benefited mathematically from talks with several other students and friends, in particular John Berman, Matt Gagne, Gabe Islambouli, Jim Phillips, and Ted Williams. I owe thanks to Raven and Allison for tea time, and thanks to Julie Bergner for feedback toward improving the clarity of this thesis. I'd like to thank my family: Mom, Dad, and Megan, for their continued support throughout these six years.

Thank you, Anna, for being you.

Contents

0.1 Notation 1

1 Introduction **3**

1.1 Outline 6

2 Background **9**

2.1 The Steenrod algebra 9

2.2 Destabilization and the Dyer-Lashof algebra 12

2.3 A spectral sequence for $H_*(\Omega^\infty X)$ 15

2.4 Ext groups and Yoneda's description 15

2.5 Brown-Gitler modules 17

3 Dyer-Lashof operations on the derived functors of destabilization **19**

3.1 The derived functors Ω^∞_s 20

3.2 The Singer complex and the action of Q^k 23

3.3 Q^k_n as extensions 25

4 Calculating the derived functors for $\Sigma^{2l}\mathcal{A} /\!\!/ E(1)$ **33**

4.1 The stable category of $E(n)$-modules 33

4.2 Classification of $J(n)$ as $E(1)$-modules 36

4.3 The Ext groups of projective space modules 41

4.4 The modules $L_s(\Sigma^{2l}\mathcal{A} /\!\!/ E(1))$ 47

4.5 2-completed K-theory of Brown-Gitler spectra 53

5 Generating $L_s(\Sigma^2\mathcal{A} /\!\!/ E(1))$ **55**

5.1 Reducing Q^k to $E(1)$ 56

5.2 Verifying the algebraic and geometric conditions 58

5.3 Proof of Theorem 5.2.1 60

5.4 Consequences 74

5.5 Other Q^k calculations 75

6 Further directions **79**

A Proof of Theorem 3.3.16 **81**

A.1 Proof of Proposition A.0.5 84

A.2 Proof of Proposition A.0.7 87

0.1 Notation

We list commonly-used notation here for reference.

$H^*(-)$	Mod 2 cohomology
$H_*(-)$	Mod 2 homology
$\lvert x \rvert$	Degree of an element x
$\mathcal{A}$	The mod 2 Steenrod algebra
$\mathcal{R}$	The mod 2 Dyer-Lashof algebra
$\mathcal{A} /\!\!/ B$	A "mod mod" quotient of $\mathcal{A}$ (see Definition 2.1.7)
$M^\vee$	The linear dual of M
$E(n)$	Exterior subalgebra of $\mathcal{A}$ on generators $Q_0, Q_1, \dots, Q_n$
ξ_n	Generators of the dual Steenrod algebra
ζ_n	Anti-automorphic image of ξ_n
$\mathcal{M}$	Category of (left or right, depending on context) $\mathcal{A}$–modules
$\mathcal{U}$	Category of unstable $\mathcal{A}$–modules
$\Sigma^n M$	n-th shift operator on an $\mathcal{A}$-module M
$\sigma^n m$	An element of $\Sigma^n M$
$\Sigma^n X$	n-th suspension of a space or spectrum X
ku	The connective K-theory spectrum
$J(n)$	n-th cohomological Brown-Gitler spectrum
$\mathcal{P}^m$	Reduced cohomology of $\mathbb{R}P^n$
$\Omega^\infty X$	Infinite loop space of a spectrum
$\Omega^\infty M$	"Destabilization" of a $\mathcal{A}$-module
$\Omega^\infty_s M$	Derived functors of destabilization
$L_s(M)$	See Notation 3.1.8

Chapter 1

Introduction

In algebraic topology, one of the central goals is to distinguish two topological spaces from each other. The idea is to assign algebraic information to spaces such that, if two spaces are the same (say, up to homotopy equivalence), then the algebraic information agrees. Poincaré showed in 1895 that we can assign numbers to spaces that detect holes in the spaces, and later this generalized to assigning groups instead, called homology groups. Around this time, cohomology groups and homotopy groups were also being studied. As one ventures further into the depths of algebraic topology, one finds there is a plethora of algebraic information that is invariant under homotopy equivalence, and can often be used to detect when two spaces are different. If a space itself is highly structured, such as an "infinite loop space," often coming from a geometric source, then there are operations acting on the homology groups of that space.

Computing homotopy groups or homology groups of spaces is hard in practice. For instance, not all homotopy groups of the two-dimensional sphere are known. For this reason, homotopy theorists often work in the "stable" world of *spectra* rather than spaces, and work with stable homotopy groups, which are easier to compute. Since (co)homology is stable in nature, it carries over unchanged. One can think of stable homotopy as a topological analog of (the derived category of) abelian groups, which is convenient to work in.

This thesis is concerned with the calculational aspect of obtaining "unstable" information, like in the case of infinite loop spaces, from stable information. Given a spectrum X, its (mod 2) homology $H_*(X)$ is a right module over the Steenrod algebra, $\mathcal{A}$. However, the homology of its infinite loop space $\Omega^\infty X$ is also a left module over the Dyer-Lashof algebra, $\mathcal{R}$, where the operations are intertwined with $\mathcal{A}$ via the Nishida relations. As the passage from spaces to spectra is thought of as moving to a setting of stable phenomena, the passage from a spectrum to its associated infinite loop space can be thought of as *destabilization*.

In [17], Kuhn and McCarty study a spectral sequence which takes as input the right $\mathcal{A}$-module $H_*(X)$, where X is a connective spectrum, and outputs $H_*(\Omega^\infty X)$ with all of its structure, as a bi-commutative Hopf algebra carrying actions of both the Steenrod and Dyer-Lashof algebras. The goal is thus:

Goal: When can we determine $H_*(\Omega^\infty X)$ from $H_*(X)$?

In general, this is hard because the differentials in the spectral sequence can be complicated. However, under certain conditions (Theorem 3.1.9), the E^∞ term of this spectral sequence can be identified as an algebraic functor of $H_*(X)$, and hence can be computed via algebraic methods alone.

More precisely, a right $\mathcal{A}$-module M has a maximal *unstable* submodule, $\Omega^\infty M$, which we think of as the algebraic version of destabilization. Then the algebraic E^∞ term is a certain enveloping algebra construction applied to the bi-graded module

$$L_*(M) = \bigoplus_s \Omega\Omega_s^\infty(\Sigma^{1-s}M),$$

where Ω_s^∞ is the s-th derived functor of Ω^∞, and Ω is the "unstable desuspension" $\Omega^\infty\Sigma^{-1}$. We explain this notation in detail in Section 3.1.

The goal then becomes to compute $L_*(M)$ when $M = H_*(X)$. The module $L_*(M)$ is "almost" the bi-graded object $\bigoplus_s \mathrm{Ext}_{\mathcal{A}}^{s,s}(M^\vee, J(*))$, where s corresponds to one grading, and $*$ to another. Here, $J(n)$ is the n-th Brown-Gitler module (see Section 3.3). This object has a right action by $\mathcal{A}$ via maps

$$\mathrm{Sq}^i\colon J(n) \to J(n-i),$$

and a left action by Dyer-Lashof operations Q^k, which turn out to be given by the Yoneda product with certain extensions $Q(n,k) \in \mathrm{Ext}^1_{\mathcal{A}}(J(n), \Sigma J(n+k))$. In unpublished work, Nick Kuhn has explicitly identified these; this is stated in Theorem 3.3.16 with a proof in the Appendix.

We call the conditions above, which guarantee the algebraic E^∞ coincides with the regular E^∞, the *algebraic* and *geometric* conditions. In [17], examples where these conditions hold were of two sorts: spectra related to suspension spectra, where the spectral sequence collapses at E^1, and Eilenberg-MacLane spectra, where the derived functors end up vanishing.

This thesis studies the case when $X = \Sigma^2 ku$, where ku is connective K-theory, so that $\Omega^\infty X = BU$. It is known that $H^*(ku) = \mathcal{A} /\!\!/ E(1)$, a quotient of the Steenrod algebra, where $E(1)$ is the subalgebra of $\mathcal{A}$ generated by $Q_0 = \mathrm{Sq}^1$ and $Q_1 = \mathrm{Sq}^1\mathrm{Sq}^2 + \mathrm{Sq}^2\mathrm{Sq}^1$. In this case, it was thought that the derived functors of $H_*(ku)$ would be quite complicated. This turns out to be true, but we are still able make a full calculation (Theorem 4.4.8). Moreover, nothing was known about the Yoneda product with the Dyer-Lashof operations, which we investigate and use to show that the algebraic condition holds for X (see Chapter 5). The fact that ku is complex-oriented makes it is easy to show that the geometric condition holds as well. Therefore, the results of Kuhn and McCarty allow us to make a direct connection between the purely algebraic $L_*(H_*(X))$ and $H_*(\Omega^\infty X) = H_*(BU)$ as Hopf algebras with all of the corresponding actions by $\mathcal{A}$ and $\mathcal{R}$. This allows us to calculate the operations Q^k on this algebra using the extensions $Q(n,k)$.

We stress that the Dyer-Lashof operations on $H_*(BU)$ have already been known since the '70s due to Kochman, and in a more systematic way due to Priddy. However, the novelty of our new approach is that, one, we obtain the rich structure of $H_*(BU)$ starting only with knowledge of the easily-described $\mathcal{A}$-module $H_*(ku)$, and two, this approach is amenable to generalization to other quotients (or dually, subalgebras) of the Steenrod algebra, such as $E(n)$ which relates to truncated Brown-Peterson spectra, or $\mathcal{A}(2)$ which relates to tmf. See Chapter 6.

1.1 Outline

Our main results are

(1) A complete calculation of $\mathrm{Ext}_{\mathcal{A}}^{*,*}(H_*(ku), J(n))$ and a calculation of $L_*(M)$ for M being any suspension of $H_*(ku)$. See, for example, Corollary 4.4.5 and Theorem 4.4.8;

(2) If $M = \Sigma^2 H_*(ku)$, then $L_*(M) = \bigoplus_s \mathrm{Ext}_{\mathcal{A}}^{s,s}(M, J(*))$ is generated by $L_0(M) = \mathrm{Hom}_{\mathcal{A}}(M, J(*))$ as a module over the Dyer-Lashof algebra via Yoneda products with the extensions $Q(n,k)$ (Corollary 5.2.2). As a corollary to this, we are able to make Q^k calculations in $H_*(BU)$.

The thesis is arranged as follows.

In Chapter 2, we give an overview of necessary background information on modules over the Steenrod algebra, the Dyer-Lashof algebra, the spectral sequence of Kuhn-McCarty, and techniques from homological algebra.

In Chapter 3, we set up the main theory of the derived functors Ω^∞, denoted Ω^∞_s. We describe the conditions needed to work with the algebraic spectral sequence (Theorem 3.1.9), and introduce $L_*(M)$. We also elucidate the connection between Ω^∞_s, Dyer-Lashof operations, and Ext groups. In particular, we illustrate how Dyer-Lashof operations Q^k can be thought of as extensions of unstable modules, denoted $Q(n,k)$ for varying n, in Section 3.3.

In Chapter 4, we give a complete computation of the derived functors of (arbitrary suspensions of) $H_*(ku)$ by connecting the derived functors to certain Ext groups. Since $H_*(ku) = (\mathcal{A} /\!\!/ E(1))^\vee$, we use a change of rings so that our calculation reduces to $\mathrm{Ext}_{E(1)}^{*,*}(\mathbb{F}_2, J(n))$. From here, we need only understand the $\mathcal{A}$-modules $J(n)$ when restricted to $E(1)$. Being injective objects, the $J(n)$ are large and complicated. However, for the purposes of Ext computations over $E(1)$, we can trade them in for the cohomology of projective spaces by working in the stable category of $E(1)$-modules (Theorem 4.2.9). These modules are "skinny"; they are one-dimensional in every degree, and hence the $\mathrm{Ext}_{E(1)}$ groups are tractable (Theorem 4.3.1). This leads to (1) above.

Finally, in Chapter 5, we explicitly identify the extensions $Q(n,k)$ when restricted to $E(1)$. The $Q(n,k)$ are extensions of $\mathcal{A}$-modules, but by doing various inductive calculations over $\mathcal{A}$, we can conclude that the $Q(n,k)$ are not split over $E(1)$. Moreover, we are able to show that they induce isomorphisms when we splice with them. This is stated precisely in Theorem 5.2.1, and gives (2) above. The proof takes a bit of work and is in Section 5.3. We conclude that $L_*(M)$ gives an associated graded for $H_*(BU)$, and thus our nonzero Q^k calculations let us deduce how certain Q^k act on $H_*(BU)$, independently of any previous knowledge about $H_*(BU)$. Examples of how this is done are given in Sections 5.4 and 5.5. For example, we can identify a generating set for $H_*(BU)$ over $\mathcal{R}$ (Corollary 5.4.1).

Chapter 2

Background

We will record necessary background information, focusing on what is needed for calculations with $\mathcal{A}$-modules and Ext groups of $\mathcal{A}$-modules.

Warning 2.0.1. Here, and in the rest of this thesis, (co-)homology will be with coefficients in $\mathbb{F}_2$, and whenever a prime number is suppressed, that prime is $p = 2$.

2.1 The Steenrod algebra

Let $H = H\mathbb{F}_2$ be the Eilenberg-MacLane spectrum corresponding to mod 2 cohomology. If X is a space, or more generally, a spectrum, then its cohomology ring $H^*(X)$ has natural stable operations

$$\mathrm{Sq}^i \colon H^*(X) \to H^{*+i}(X)$$

called *Steenrod squares* that satisfy various properties, such as the Cartan formula. The *Steenrod algebra* at $p = 2$, denoted $\mathcal{A}$, is the graded $\mathbb{F}_2$-algebra generated by the Steenrod squares Sq^i, where Sq^i is in degree i, subject to the *Adem relations* and $\mathrm{Sq}^0 = 1$. The Steenrod algebra is covered in depth in many places [35, Chapter 18], [27], [28]. Given a space or spectrum X, we can rephrase the above by saying that $H^*(X)$ is a graded left $\mathcal{A}$-module. Dually, the homology $H_*(X)$ can be regarded as a right $\mathcal{A}$-module, where the operations *decrease* degree.

If X is a space, then $H^*(X)$ is an *unstable* left $\mathcal{A}$-module, and $H_*(X)$ is an unstable right $\mathcal{A}$-module, which we describe in the following definition.

Definition 2.1.1. A left $\mathcal{A}$-module M is *unstable* if $\mathrm{Sq}^n(x) = 0$ whenever $n > |x|$, for $x \in M$. Dually, a right $\mathcal{A}$-module M is *unstable* if $(x)\,\mathrm{Sq}^n = 0$ whenever $n > \frac{|x|}{2}$, for $x \in M$.

Milnor in [26] described the dual of $\mathcal{A}$, denoted $\mathcal{A}_*$, as a polynomial algebra on generators ξ_n in degrees $2^n - 1$, which are dual to $\mathrm{Sq}^{2^{n-1}}\,\mathrm{Sq}^{2^{n-2}} \cdots \mathrm{Sq}^2\,\mathrm{Sq}^1$ with respect to the "admissible basis". We will often make a passage between $\mathcal{A}$ and $\mathcal{A}_*$, and it is important to note that, under this duality, quotients become sub-objects, and vice versa. This duality makes $\mathcal{A}$ a Hopf algebra, with conjugation χ and coproduct Δ (notation we will use for $\mathcal{A}_*$ as well). Let $\zeta_n = \chi(\xi_n)$.

Lemma 2.1.2. *Let Δ denote the coproduct of $\mathcal{A}$ or $\mathcal{A}_*$, and let χ denote the canonical anti-automorphism. Then*

(a) $\Delta(Sq^k) = \sum_{0 \le i \le k} Sq^i \otimes Sq^{k-i}$,

(b) $\chi(Sq^k) = \sum_{1 \le i \le k} Sq^i \chi(Sq^{k-i})$ *and* $\chi(Sq^0) = 1$,

(c) $\Delta(\xi_n) = \sum_{0 \le i \le n} \xi_{n-i}^{2^i} \otimes \xi_i$,

(d) $\zeta_n = \chi(\xi_n) = \sum_{0 \le i \le n-1} \xi_{n-i}^{2^i} \chi(\xi_i)$,

(e) $\Delta(\zeta_n) = \sum_{0 \le i \le n} \zeta_i \otimes \zeta_{n-i}^{2^i}$.

Proof: The coproduct statements for Sq^k and ξ_n can be found in [26]. The statements about χ are formal for Hopf algebras, once the coproduct is known. The coproduct for ζ_n can then be deduced from that of ξ_n. □

Remark 2.1.3. If we swap the factors of the tensor products in (c) and (e), we do not get an equivalent coproduct, i.e., $\mathcal{A}_*$ is not co-commutative.

Lemma 2.1.4 ([26], [37]). *The element ξ_n is dual to the Milnor primitive Q_{n-1}, which is characterized by $Q_n = [Q_{n-1}, \mathrm{Sq}^{2^n}]$ and $Q_0 = \mathrm{Sq}^1$.*

The Steenrod algebra acts on itself and thus also acts on its dual, both on the left and right. Note that there is an (anti-)isomorphism

$$\mathcal{A}_* \cong \mathbb{F}_2[\xi_1, \xi_2, \dots] \cong \mathbb{F}_2[\zeta_1, \zeta_2, \dots]$$

induced by χ, which changes the action of the Steenrod algebra on $\mathcal{A}_*$. It is often more convenient for us to describe homology in terms of the ζ_n.

Lemma 2.1.5. *Let* Sq *denote the total Steenrod square. The Steenrod algebra acts on $\mathcal{A}_*$ via:*

(a) $\xi_n \cdot \mathrm{Sq} = \xi_n + \xi_{n-1}$,
(b) $\zeta_n \cdot \mathrm{Sq} = \sum_{0 \le i \le n} \zeta_{n-i}^{2^i}$,
(c) $\mathrm{Sq} \cdot \xi_n = \xi_n + \xi_{n-1}^2$,
(d) $\mathrm{Sq} \cdot \zeta_n = \sum_{0 \le i \le n} \zeta_i$,
where $\zeta_0 = \xi_0 = 1$.

Corollary 2.1.6. *Fix n. For $0 \le k \le n$, we have $\zeta_n \cdot \mathrm{Sq}^{2^k-1} = \zeta_{n-k}^{2^k}$ and $\zeta_n \cdot \mathrm{Sq}^l = 0$ for all other l. In particular, $\zeta_n \cdot \mathrm{Sq}^{2^n-1} = 1$.*

Proof: The degree of $\zeta_{n-k}^{2^k}$ is $2^k(2^{n-k} - 1) = 2^n - 2^k$. In other words, the degree of $\zeta_{n-k}^{2^k}$ is $2^k - 1$ lower than that of ζ_n. The total square in Lemma 2.1.5 shows that these are the only elements hit. □

Definition 2.1.7. Let A be an augmented algebra, and let $\widetilde{A}$ be the kernel of the augmentation. If $B \subset A$ is a sub-augmented algebra, let

$$\mathcal{A} /\!\!/ B = \mathcal{A} \otimes_B \mathbb{F}_2 = \mathcal{A}/(\widetilde{B}),$$

which is again an augmented algebra.

Remark 2.1.8. If A is a connected Hopf algebra and $B \subset A$ is a sub-Hopf algebra, then A is a free B-module.

The Steenrod algebra has many interesting sub-algebras, and the ones we list in the following definition are also sub-Hopf algebras (closed under the Hopf structure).

Definition 2.1.9. (a) Let $\mathcal{A}(n)$ denote the sub-algebra generated by $\mathrm{Sq}^1, \mathrm{Sq}^2, \ldots, \mathrm{Sq}^{2^n}$.

(b) Let $E(n) \subset \mathcal{A}(n)$ denote the exterior sub-algebra generated by $Q_0, Q_1, \ldots, Q_n$.

Example 2.1.10. (a) Let $X = \mathbb{R}P^\infty$. Then $H^*(X) \cong \mathbb{F}_2[x]$ where $|x| = 1$, and $\mathcal{A}$ acts on $H^*(X)$ via $\mathrm{Sq}^k x^n = \binom{n}{k} x^{n+k}$.

(b) Let $X = ku$, the connective K-theory spectrum. Then $\Omega^\infty ku = BU \times \mathbb{Z}$. We have $H^*(X) \cong \mathcal{A} /\!\!/ E(1)$ as left $\mathcal{A}$-modules and

$$H_*(X) \cong \mathbb{F}_2[\zeta_1^2, \zeta_2^2, \zeta_3, \zeta_4, \ldots] \subset \mathcal{A}_*.$$

(c) In general, let $X = BP\langle n\rangle$, a truncated Brown-Peterson spectrum. Then $H^*(X) \cong \mathcal{A} /\!\!/ E(n)$, and

$$H_*(BP\langle n\rangle) = \mathbb{F}_2[\zeta_1^2, \ldots, \zeta_{n+1}^2, \zeta_{n+2}, \zeta_{n+3}, \ldots] \subset \mathcal{A}_*.$$

(d) If $X = tmf$, the connective spectrum of topological modular forms, then $H^*(X) \cong \mathcal{A} /\!\!/ \mathcal{A}(2)$, and

$$H_*(X) \cong \mathbb{F}_2[\zeta_1^8, \zeta_2^4, \zeta_3^2, \zeta_4, \zeta_5, \ldots] \subset \mathcal{A}_*.$$

Remark 2.1.11. In examples (b), (c), and (d), the right action of $\mathcal{A}$ on $H_*(X)$ is induced by the one on $\mathcal{A}_*$ given in Lemma 2.1.5.

2.2 Destabilization and the Dyer-Lashof algebra

The upside of cohomology operations is that they enrich our algebraic invariants with more structure, which is especially useful in "non-existence" proofs. For example, although $\Sigma\mathbb{C}P^2$ and $S^3 \vee S^5$ have isomorphic cohomology rings, the former has a nontrivial Sq^2 whereas the latter is a trivial $\mathcal{A}$-module, showing they are not homotopy-equivalent. In this section, we review the Dyer-Lashof operations, which are homology operations that exist for certain spaces and spectra which add even more algebraic structure that is invariant under homotopy equivalence.

Given a spectrum X, one can take its infinite loop space $\Omega^\infty X$, and in general, $\Omega^\infty \Sigma^n X$ will retrieve the n-th space of (the Ω-spectrum associated to) X. Since Ω^∞ maps spectra to spaces, it can be thought of as *destabilization*.

Example 2.2.1. (a) If $X = H\mathbb{Z}$, then $\Omega^\infty \Sigma^n X = K(\mathbb{Z}, n)$, an Eilenberg-MacLane space. For example, $\Omega^\infty X = \mathbb{Z}$, $\Omega^\infty \Sigma X = S^1$, and $\Omega^\infty \Sigma^2 X = \mathbb{C}P^\infty$.

(b) If $X = H\mathbb{F}_2$, then $\Omega^\infty \Sigma X = K(\mathbb{F}_2, 1) = \mathbb{R}P^\infty$.

(c) Let $X = ku$, connective K-theory. Then, e.g., $\Omega^\infty X = BU \times \mathbb{Z}$, $\Omega^\infty \Sigma^2 X = BU$, and $\Omega^\infty \Sigma^4 X = BSU$. We have $H^*(BU) \cong \mathbb{F}_2[c_1, c_2, \dots]$, where $|c_i| = 2i$ and the c_i are the Chern classes. On the other hand, $H_*(BU)$ is also polynomial: $H_*(BU) \cong \mathbb{F}_2[a_2, a_4, \dots]$ where $|a_{2i}| = 2i$ and c_n is dual to a_2^n with respect to this basis. The co-algebra structure is given by $\Delta(a_{2n}) = \sum_i a_{2i} \otimes a_{2n-2i}$.

Spaces of the form $\Omega^\infty X$ have a lot of structure, and they have been studied extensively [2], [23]. Whereas $H_*(X)$ is a right $\mathcal{A}$-module, $H_*(\Omega^\infty X)$ has more additional structure. It is a Hopf algebra with product induced by the Pontryagin product, and it is a left module over the Dyer-Lashof algebra $\mathcal{R}$ (to be defined below), where the action is tied together with that of $\mathcal{A}$ via the Nishida relations described below. We follow [21] for the following definition.

Definition 2.2.2. Let $\mathcal{F}$ denote the free graded associative algebra with unit generated by the symbols Q^i, $i \geq 0$, where $|Q^i| = i$. For a string $I = (i_1, \dots, i_n)$, define $Q^I = Q^{i_1} \cdots Q^{i_n}$. Define the *excess* of Q^I to be $\sum_j i_j - 2i_{j+1}$, and let $\mathcal{J}$ be the ideal generated by

(i) $Q^a Q^b + \sum_i \binom{i-b-1}{2i-a} Q^{a+b-i} Q^i$ for $a > 2b$, and

(ii) Q^I with excess less than 0.

The *Dyer-Lashof algebra* is defined to be $\mathcal{R} = \mathcal{F}/\mathcal{J}$. The relations in (i) are called the *Adem relations*.

Dyer-Lashof operations exist whenever there is an H_∞–structure on a spectrum or space (see [7]), although one can relax this to get a "partial" action of the Q^i under weaker conditions. As mentioned above, the action of $\mathcal{R}$ is intertwined with that of $\mathcal{A}$ via the formulas

$$(Q^a x)\,\mathrm{Sq}^b = \sum_i \binom{a-b}{b-2i} Q^{a-b+i}(x\,\mathrm{Sq}^i),$$

which are called the *Nishida relations*. These operations were introduced by Araki and Kudo in [16] and were later studied in more detail by Dyer and Lashof in [9] as operations acting on the homology of infinite loop spaces (E_∞-spaces). In this case, there is also the condition $Q^{|x|}x = x^2$.

In an early example, Madsen studied the operations on $H_*(G)$ in his thesis, where $G = \lim_n Map(S^n, S^n)$ [21]. The operations on $H_*(BU) \cong \mathbb{F}_2[a_2, a_4, a_6, \dots]$ were computed mod decomposables in [15], and later closed formulas were given in [32]. A good summary of these calculations can be found in [29]. In Chapter 5, we will show that our approach to Dyer-Lashof operations as extensions of unstable modules gives a different approach for making calculations. For example, we can identify $\{a_{2^k} \mid k \geq 1\}$ as a generating set for $H_*(BU)$ over $\mathcal{R}$ (Corollary 5.4.1), independently of these previous results.

Computing the action of $\mathcal{R}$ can be difficult, but this knowledge yields valuable information about the space or spectrum which can lead to non-existence results. For example, in (chromatic) homotopy theory and derived algebraic geometry, one would often like an object to be E_∞, a type of structure that records an infinite heirarchy of commutativity up to homotopy in a "coherent" way. An E_∞-map between E_∞ objects must respect the action of $\mathcal{R}$. For example, if $k(n)$ is the n-th connective Morava K-theory spectrum, then $k(n)$ is not an E_∞-ring spectrum for $n > 0$; otherwise the subalgebra embedding

$$H_*(k(n)) \cong \mathbb{F}_2[\zeta_1, \dots, \zeta_n, \zeta_{n+1}^2, \zeta_{n+2}, \dots] \subset \mathcal{A}_*$$

would be closed under Dyer-Lashof operations, but one can check this is not true using, e.g., calculations in [7, Chapter 3]. Another example is in [13], where Hu-Kriz-May that there cannot be an E_7-map, or therefore an E_∞-map, $BP \to MU$ at $p = 2$. In [19], Lawson applies Dyer-Lashof operations to "secondary operations" to ultimately determine that BP is not an E_∞-ring spectrum, a problem that was open for 40 years.

2.3 A spectral sequence for $H_*(\Omega^\infty X)$

In [17], Kuhn and McCarty studied a spectral sequence obtained by applying $H_*(-)$ to the Goodwillie tower associated to the co-unit $\Sigma^\infty\Omega^\infty X \to X$. This yields a spectral sequence in the category of Hopf algebras with E^1 page generated by $H_*(D_k(X))$ for $k \geq 1$, where $D_k(X)$ denotes the k-th extended power of X, and converges to $H_*(\Omega^\infty X)$, which is equipped with its Dyer-Lashof operations.

Theorem 2.3.1 ([17, Prop 1.2])**.** *Let X be a spectrum. Then there is a spectral sequence*

$$E^1_{-k,-k+*}(X) \cong H_*(D_k(X)) \Rightarrow H_*(\Omega^\infty X)$$

which converges strongly when X is connective.

The E^1 page is obtained from the homology of the extended powers, $H_*(D_nX)$, where Dyer-Lashof operations originate. Heuristically, we might think of the Dyer-Lashof operations as being "freely introduced" subject to the Nishida relations and the $\mathcal{A}$ action on $H_*(X)$. The ones that survive the spectral sequence are ones that make it to (the associated graded of) $H_*(\Omega^\infty X)$. This is stated more precisely in Theorem 1.6 of the same paper.

2.4 Ext groups and Yoneda's description

Given a ring R, the groups $\mathrm{Ext}^s_R(-,-)$ are defined as the derived functors of $\mathrm{Hom}_R(-,-)$ in the category of left (or right) R-modules. For us, R will be a graded ring. We define

$$\mathrm{Hom}_R^t(M, N) = \mathrm{Hom}_R(M, \Sigma^t N) = \mathrm{Hom}_R(\Sigma^{-t} M, N)$$

for graded R-modules M and N, where $(\Sigma N)^r = N^{r-1}$. Using this notation, $\mathrm{Ext}_R^{s,t}$ is the s-th derived functor of Hom_R^t.

Using the derived functor definition of Ext is sometimes hard to use in computations because of its abstract and choice-heavy nature. There is an alternative description due to Yoneda, as follows. An element of $\mathrm{Ext}_R^{s,t}(M, N)$ can be thought of as an s-fold extension of R-modules:

$$0 \to \Sigma^t N \to P_s \to P_{s-1} \to \cdots \to P_1 \to M \to 0,$$

with a suitable definition of equivalence [20, p. 83]. Unfortunately, it is difficult to determine if two s-fold extensions represent the same element in Ext^s for $s > 1$, but we will not need to do so here. The situation is easier for $s = 1$; for example, an extension is 0 if and only if it is split.

Yoneda's description of Ext makes it easy to define a type of module structure. There is a composition

$$\mathrm{Ext}_R^{s,t}(L, N) \otimes \mathrm{Ext}_R^{s',t'}(M, L) \to \mathrm{Ext}_R^{s+s',t+t'}(M, N),$$

called the *Yoneda splice* or the *Yoneda product*, which we first illustrate for $s, s' \geq 1$. If $\varepsilon \in \mathrm{Ext}_R^{s,t}(L, N)$ is given by

$$0 \to \Sigma^t N \to P_s \to \cdots \to P_1 \to L \to 0$$

and $\varepsilon' \in \mathrm{Ext}_R^{s',t'}(M, L)$ is given by

$$0 \to \Sigma^{t'} L \to Q_{s'} \to \cdots \to Q_1 \to M \to 0,$$

define the composite $\varepsilon \circ \varepsilon' \in \mathrm{Ext}_R^{s+s',t+t'}(M,N)$ to be the exact sequence

$$0 \to \Sigma^{t+t'} N \to \Sigma^{t'} P_s \to \dots \Sigma^{t'} P_1 \to Q_{s'} \to \dots Q_1 \to M \to 0,$$

where $\Sigma^{t'} P_1 \to Q_{s'}$ is the splice $\Sigma^{t'} P_1 \to \Sigma^{t'} L \to Q_s$. If $s = 0$ then $\varepsilon\colon L \to \Sigma^t N$, and $\varepsilon \circ \varepsilon'$ is defined using the following diagonal map:

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & \Sigma^{t'} L & \longrightarrow & Q_{s'} & \longrightarrow & Q_{s'-1} \dots & \longrightarrow & Q_1 & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & \nearrow & & & & & & & \\
0 & \longrightarrow & \Sigma^{t+t'} N & \longrightarrow & P & & & & & & & &
\end{array}$$

where the left square is a pushout. When $s' = 0$, we obtain a similar situation using a pullback instead. These cases of $s = 0$ and $s' = 0$ are just restatements of the covariance and contravariance of Ext_R, respectively.

The Yoneda product is associative and unital when defined. If R is a Hopf algebra over $\mathbb{F}_2$, such as $\mathcal{A}$ or one of its many sub-Hopf algebras, then $\mathrm{Ext}_R^{*,*}(\mathbb{F}_2, \mathbb{F}_2)$ is a graded commutative $\mathbb{F}_2$-algebra, and $\mathrm{Ext}_R^{s,t}(\mathbb{F}_2, M)$ is a right $\mathrm{Ext}_R^{*,*}(\mathbb{F}_2, \mathbb{F}_2)$-module. By flipping the order of composition, we obtain a left module as well.

We recommend [3] for a "user's guide" approach to Ext as it relates to stable homotopy theory calculations, and [4, Chapter 2] or [20, Chapter 3] for details Yoneda's Ext.

Warning 2.4.1. We will often conflate an element $\varepsilon \in \mathrm{Ext}^1(C, A)$ with the extension $0 \to A \to \varepsilon \to C \to 0$.

2.5 Brown-Gitler modules

The classical mod 2 Brown-Gitler spectra were originally given in [6]. We will be mainly concerned with their cohomology.

Definition 2.5.1. Let $T(n)$ be the nth Brown-Gitler spectrum. Let $J(n) = H^*(T(n))$ and $G(n) = H_*(T(n))$. The modules $J(n)$ are called the *nth (cohomological) Brown-Gitler modules*, and the $G(n)$ are the *nth (homological) Brown-Gitler modules*.

For an unstable left $\mathcal{A}$–module M, we have

$$\mathrm{Hom}_{\mathcal{U}}(M, J(n)) \cong \mathrm{Hom}_{\mathbb{F}_2}(M^n, \mathbb{F}_2),$$

where $\mathcal{U}$ denotes the category of unstable $\mathcal{A}$–modules. It follows that $J(n)$ is an injective unstable $\mathcal{A}$-module. In fact, it is the injective envelope of $\widetilde{H}^*(S^n) = \Sigma^n \mathbb{F}_2$ in the category of unstable left $\mathcal{A}$–modules. Similarly, $G(n)$ is the projective cover of $\widetilde{H}_*(S^n)$ in the category of unstable right modules, and $\mathrm{Hom}_{\mathcal{U}}(G(n), M) \cong M_n$ for M an unstable right $\mathcal{A}$–module. See [18, p. 47] or [25, p. 74] for more details.

Proposition 2.5.2 ([33, Proposition 2.3.3])**.** *There is an isomorphism* $J(2n+1) \cong \Sigma J(2n)$ *as* $\mathcal{A}$*-modules.*

Chapter 3

Dyer-Lashof operations on the derived functors of destabilization

Let M be a right $\mathcal{A}$-module. We will develop a theory of Dyer-Lashof operations on the doubly graded module $\Omega^\infty_* \Sigma^{-*} M$, where Ω^∞_* are the derived functors of Ω^∞ (Definition 3.1.5). More explicitly, there are natural operations

$$Q^k \colon (\Omega^\infty_s \Sigma^{-s} M)_n \to (\Omega^\infty_{s+1} \Sigma^{-(s+1)} M)_{n+k}$$

for all $s, n \geq 0$ which satisfy the usual properties, such as the Adem relations. These operations arise via maps on the chain level using the Singer complex. Using Proposition 3.3.2, these operations are equivalent to operations

$$Q^k \colon \operatorname{Ext}^{s,s}_{\mathcal{A}}(M, J(n)) \to \operatorname{Ext}^{s+1,s+1}_{\mathcal{A}}(M, J(n+k)).$$

By the Yoneda lemma, these operations are encoded as elements

$$Q^k_n \in \operatorname{Ext}^{1,1}_{\mathcal{A}}(J(n), J(n+k)),$$

which are described in detail in §4.4. This connection to Ext allows us to bypass complicated calculations in chain complexes, and we will use this to our advantage in Chapter 5.

In order to define the Singer complex, some basic theory surrounding unstable $\mathcal{A}$-modules is needed.

3.1 The derived functors Ω_s^∞

Let $\mathcal{M}^L$ denote the category of left (cohomological) $\mathcal{A}$-modules, and $\mathcal{M}^R$ denote the category of right (homological) $\mathcal{A}$-modules. Let $\mathcal{U}^L \subset \mathcal{M}^L$ denote the full subcategory of left $\mathcal{A}$-modules that are unstable, and let $\mathcal{U}^R \subset \mathcal{M}^R$ denote the right $\mathcal{A}$-modules that are unstable. We will sometimes use $\mathcal{M}$ or $\mathcal{U}$ to refer to left or right modules ambiguously when the context is clear or when it is irrelevant.

Remark 3.1.1. We assume that all of the $\mathcal{A}$-modules in this thesis are *locally finite*, which means they are finitely generated over $\mathbb{F}_2$ in every graded dimension. We will often switch between left and right modules, and this condition makes this duality run smoothly.

We commonly pass from $\mathcal{M}$ to $\mathcal{U}$, and the following functor is the basic way to do this.

Definition 3.1.2. (a) For a left $\mathcal{A}$-module M, let $\Omega^\infty(M)$ be the largest unstable quotient of M. We call this functor *cohomological destabilization.*

(b) For a a right $\mathcal{A}$-module M, let $\Omega^\infty(M)$ be the largest unstable submodule of M. We call this functor *homological destabilization.*

When the context is clear, we refer to Ω^∞ simply as *destabilization.*

Lemma 3.1.3. *(a) The inclusion* $\mathcal{U}^L \hookrightarrow \mathcal{M}^L$ *has left adjoint* $\Omega^\infty \colon \mathcal{M}^L \to \mathcal{U}^L$.

(b) The inclusion $\mathcal{U}^R \hookrightarrow \mathcal{M}^R$ *has right adjoint* $\Omega^\infty \colon \mathcal{M}^R \to \mathcal{U}^R$.

Proof: We prove (a); the proof of (b) is similar. Let $M \in \mathcal{M}^L$ and $N \in \mathcal{U}^L$. We will show $\mathrm{Hom}_{\mathcal{A}}(M, N) \cong \mathrm{Hom}_{\mathcal{U}}(\Omega^\infty M, N)$.

Let $I = \mathcal{A}\{\mathrm{Sq}^k x \mid k > |x|\}$ so that $\Omega^\infty M = M/I$. Given $f\colon M \to N$, we have $I \subset \ker(f)$: if $y \in I$, then $y = a\,\mathrm{Sq}^k x$ where $k > |x|$, hence $f(y) = a\,\mathrm{Sq}^k f(x) = 0$

because $|x| = |f(x)|$ and N is unstable. Thus f factors uniquely as $M \twoheadrightarrow \Omega^\infty M \to N$. One can check that this defines a bijection. $\square$

The functor $\Omega^\infty\colon \mathcal{M} \to \mathcal{U}$ should be thought of as the algebraic version of the usual topological destabilization functor $\Omega^\infty\colon \mathbf{Sp} \to \mathbf{Top}$. Note, however, that in general $H_*(\Omega^\infty X) \neq \Omega^\infty H_*(X)$, even as groups. The latter is simpler to compute in many cases, whereas the former has extra structure, such as natural Dyer-Lashof operations. The relationship between topological and algebraic destabilization has been studied via spectral sequences in [17] and [12], and in [24] in the other direction.

If an $\mathcal{A}$-module M is "far" from being unstable, then $\Omega^\infty M$ is accordingly small. As an example, we have the following calculation (notice that this applies to $\mathcal{A}$ itself).

Lemma 3.1.4. *If $M = \mathcal{A} /\!\!/ B$ then $\Omega^\infty(M) = \mathbb{F}_2$.*

Proof: Since $M = \mathcal{A}/(\widetilde{B})$, we see that $1 \in M$ and so any $\mathrm{Sq}^n \cdot 1$ in M with $n > 0$ is 0 in the quotient when we apply Ω^∞. $\square$

We now proceed to describe the conditions which guarantee that the E^∞ term of the spectral sequence of [17] can be computed algebraically. We need some definitions and notation first.

Definition 3.1.5. (a) The cohomological Ω^∞, being right exact, has left-derived functors $\Omega^\infty_s\colon \mathcal{M}^L \to \mathcal{U}^L$ for $s > 0$.

(b) The homological Ω^∞, being left exact, has right-derived functors $\Omega^\infty_s\colon \mathcal{M}^R \to \mathcal{U}^R$ for $s > 0$.

Note that $\Omega^\infty_0 = \Omega^\infty$. The Ω^∞_s are called the *derived functors of destabilization*. If necessary, we will distinguish these by the words *homological* or *cohomological* for right or left $\mathcal{A}$–modules, respectively.

Definition 3.1.6. Let $\Omega\colon \mathcal{U} \to \mathcal{U}$ be the left (resp. right) adjoint to the shift operator $\Sigma\colon \mathcal{U} \to \mathcal{U}$ for left (resp. right) $\mathcal{A}$–modules.

The functor Ω should be thought of as the "unstable desuspension". It can be written $\Omega^\infty\Sigma^{-1}$, for left or right $\mathcal{A}$–modules.

In [17, §5.4], Kuhn and McCarty investigate a purely algebraic spectral sequence, E_{alg}. This spectral sequence starts with a right $\mathcal{A}$–module and adds Dyer-Lashof operations as "freely as possible", subject to forced relations like the Nishida relations. To make this more precise, we need the following definitions.

Definition 3.1.7. Let $N \in \mathcal{U}^R$ with a left action by the Dyer-Lashof algebra. Define a Hopf algebra $U_Q(M)$ by

$$U_Q(M) = S^*(M)/(Q^{|x|}x - x^2),$$

where $S^*(M)$ is the free commutative algebra generated by M, and the coalgebra structure is determined by making M primitive. That is, for $m \in M$, we have $\Delta(m) = 1 \otimes m + m \otimes 1$.

In other words, $U_Q(M)$ is the "free" primitively-generated Hopf algebra on M with the squaring condition $Q^{|x|}(x) = x^2$.

Notation 3.1.8. Let $L_s(M) = \Omega\Omega^\infty_s(\Sigma^{1-s}M)$ for an $\mathcal{A}$-module M. Note that $L_0(M) = \Omega^\infty M$.

The algebraic spectral sequence has $E^1_{alg} = U_Q(\mathcal{R}_*(M))$ (described in the next section) and $E^\infty_{alg} = U_Q(L_*(M))$.

Theorem 3.1.9 ([17, Corollary 1.14])**.** *Suppose X is a spectrum such that*

(a) $\Omega^\infty H_*(X) = L_0 H_*(X)$ generates $L_*(H_*(X))$ as a left module over $\mathcal{R}$, and

(b) the evaluation map $H_*(\Omega^\infty X) \to \Omega^\infty H_*(X)$, induced by the co-unit $\Sigma^\infty\Omega^\infty \Rightarrow 1$, is onto.

Then the spectral sequence in Theorem 2.3.1 agrees with E_{alg}. Thus $U_Q(L_*(M))$ gives an associated graded for $H_*(\Omega^\infty X)$.

Remark 3.1.10. We refer to (a) as the "algebraic" condition and to (b) as the "geometric" condition, as it comes from the evaluation map of topological origin.

Thus, if one is able to check these conditions for a spectrum X, then one can compute the associated graded for $H_*(\Omega^\infty X)$. This can lead to a determination of how certain Dyer-Lashof operations act on $H_*(\Omega^\infty X)$, through knowledge of $L_*(H_*(X))$ alone, which is a purely algebraic object. In Chapter 5, we will apply this strategy to connective K-theory.

3.2 The Singer complex and the action of Q^k

The Singer complex is a way to directly compute Ω^∞_s from a chain complex. This idea originates in [34] for related functors. We follow the approach of [17, §4].

If X is a spectrum, there is a natural map $\varepsilon\colon \Sigma D_d X \to D_d \Sigma X$, where D_d is the dth extended power functor, which has known behavior in homology.

Definition 3.2.1. Let $\mathcal{R}_s H_*(X) = \operatorname{im}\{\varepsilon_*\colon H_*(\Sigma D_{2^s}(\Sigma^{-1}X)) \to H_*(D_{2^s}X)\}$.

By knowledge of ε, we can deduce

$$\mathcal{R}_s H_*(X) = \langle Q^I x \mid l(I) = s, x \in H_*(X)\rangle/(\text{unstable and Adem relations}),$$

where the unstable condition for Dyer-Lashof operations is: $Q^i x = 0$ if $i < |x|$. This description is used to extend the functor $\mathcal{R}_s$ to arbitrary (locally finite) $\mathcal{A}$-modules M. Multiplication is encoded via the map

$$\mu\colon \mathcal{R}_s(\mathcal{R}_t M) \to \mathcal{R}_{s+t}(M)$$

corresponding to the natural transformation $\mu\colon D_{2^s} D_{2^t}(X) \to D_{2^{s+t}}(X)$.

Theorem 3.2.2 ([17, Theorem 1.16])**.** *For a right $\mathcal{A}$–module M, let $d_s\colon \mathcal{R}_s(M) \to \mathcal{R}_{s+1}(\Sigma M)$ be defined by*

$$d_s(Q^I x) = \sum_{i\geq 0} Q^I Q^{i-1}(\sigma x \operatorname{Sq}^i).$$

Then d_s is a differential and the homology of the chain complex $\Sigma\mathcal{R}_*(\Sigma^{*-1}M)$ with respect to Σd_* is $\Omega^\infty_*(M)$.

Proposition 3.2.3. *There are natural Dyer-Lashof operations*

$$Q^k\colon (\Omega^\infty_s\Sigma^{-s}M)_n \to (\Omega^\infty_{s+1}\Sigma^{-(s+1)}M)_{n+k}$$

which satisfy the usual properties: Adem and Nishida relations, and instability.

Proof: The $\mathbb{F}_2$-linear map Q^k can be written as a chain map:

$$\begin{array}{ccccc} \Sigma\mathcal{R}_{s-1}(\Sigma^{-2}M) & \xrightarrow{d_{s-1}} & \Sigma\mathcal{R}_s(\Sigma^{-1}M) & \xrightarrow{d_s} & \Sigma\mathcal{R}_{s+1}(M) \\ \downarrow{\scriptstyle Q^k} & & \downarrow{\scriptstyle Q^k} & & \downarrow{\scriptstyle Q^k} \\ \Sigma\mathcal{R}_s(\Sigma^{-2}M) & \xrightarrow{d_s} & \Sigma\mathcal{R}_{s+1}(\Sigma^{-1}M) & \xrightarrow{d_{s+1}} & \Sigma\mathcal{R}_{s+2}(M) \end{array} \tag{3.2.4}$$

Here, Q^k is given by left multiplication. For example,

$$Q^k\colon \Sigma\mathcal{R}_s(\Sigma^{-1}M) \to \Sigma\mathcal{R}_{s+1}(\Sigma^{-1}M)$$

is defined by

$$\sigma Q^I\sigma^{-1}m \mapsto \sigma Q^kQ^I\sigma^{-1}m,$$

where the Adem relations are applied to Q^kQ^I. One can check directly that this commutes with the differential by an easy calculation. Applying homology to the middle vertical arrow gives the map described in the proposition. The fact that these Q^k satisfy the properties of Dyer-Lashof operations follows because of their source (see [17, Lemma 4.8(a)]). □

Remark 3.2.5. The module $\Omega^\infty_*\Sigma^{-*}M$ can be thought of as doubly graded, where (i,j) corresponds to $\Omega^\infty_i\Sigma^{-i}$ in the first component and the total internal grading coming from M in the second component. Then $\mathcal{A}$ acts as $a\colon (i,j)\mapsto(i,j-|a|)$ and $\mathcal{R}$ acts as $q\colon (i,j)\mapsto(i+1,j+|q|)$.

The operations Q^k also act on $L_*(M)$ in a similar way, using the fact that $L_s(M)$ is the image of the natural map $\varepsilon\colon \Omega^\infty_s(\Sigma^{-s}M)\to\Sigma^{-1}\Omega^\infty_s(\Sigma^{1-s}M)$; see [17, §4.7].

3.3 Q_n^k as extensions

We begin with a link between the unstable derived functors of destabilization and stable Ext groups. Theorems of this form appear in [18] or [14, Lemma 2.1], and [11, Corollary 1.9] in the dual form.

Proposition 3.3.1. *Let M and I be left $\mathcal{A}$–modules with I a $\mathcal{U}$-injective. Then there is a natural isomorphism*

$$\mathrm{Ext}_{\mathcal{A}}^{s,t}(M, I) \cong \mathrm{Hom}_{\mathcal{U}}(\Omega_s^{\infty}(\Sigma^{-t} M), I).$$

Corollary 3.3.2. *Let M be a left $\mathcal{A}$-module. Then there is an equivalence of unstable right $\mathcal{A}$-modules*

$$\mathrm{Ext}_{\mathcal{A}}^{s,t}(M, J(*)) \cong \mathrm{Hom}_{\mathcal{U}}(\Omega_s^{\infty}\Sigma^{-t} M, J(*)) \cong (\Omega_s^{\infty}\Sigma^{-t} M^{\vee})_*$$

where $M^{\vee}$ denotes the dual of M and the action on the two modules on the left is induced by the maps

$$\cdot Sq^i \colon J(n) \to J(n-i).$$

Remark 3.3.3. Let superscripts denote degrees in a left module, and subscripts degrees in a right module. If M is a left $\mathcal{A}$-module, then $M_n = (M^n)^{\vee} \cong \mathrm{Hom}_{\mathcal{U}}(M, J(n))$, and $(\Omega_s^{\infty} M)_n = (\Omega_s^{\infty} M^{\vee})_n$. Note $M^{\vee}$ is a right $\mathcal{A}$-module, so $\Omega_s^{\infty} M^{\vee}$ is an unstable right $\mathcal{A}$-module.

Because of this connection, it is easier to do calculations with $\Omega_s^{\infty}\Sigma^{1-s} M$ rather than $L_s(M)$ directly. However, the following corollary shows that in the case when $L_s(M)$ is evenly graded, $L_s(M)$ can be computed directly from Ext groups. This will be employed in Chapter 4.

Corollary 3.3.4. *Let M be a left $\mathcal{A}$–module. Since $\Sigma J(2n) \cong J(2n+1)$ is an injective, we obtain*

$$\mathrm{Ext}_{\mathcal{A}}^{s,t}(M, J(2n)) \cong \mathrm{Hom}_{\mathcal{U}}(\Omega\Omega_s^{\infty}\Sigma^{1-t} M, J(2n)).$$

Thus

$$\mathrm{Ext}_{\mathcal{A}}^{s,s}(M, J(2n)) \cong \mathrm{Hom}_{\mathcal{U}}(L_s M, J(2n)) \cong (L_s M^\vee)_{2n},$$

which holds for all $s \geq 0$*, noting that*

$$\mathrm{Hom}_{\mathcal{A}}(M, J(2n)) \cong \mathrm{Hom}_{\mathcal{U}}(\Omega^\infty M, J(2n)) \cong (\Omega^\infty M^\vee)_{2n}.$$

By Corollary 3.3.2, we see that the operations constructed in Proposition 3.2.3 are equivalent to having natural operations

$$Q_n^k\colon \mathrm{Ext}_{\mathcal{A}}^{s,s}(M, J(n)) \to \mathrm{Ext}_{\mathcal{A}}^{s+1,s+1}(M, J(n+k)).$$

Thus, by a derived Yoneda lemma, these operations are induced by elements

$$Q_n^k \in \mathrm{Ext}_{\mathcal{A}}^{1,1}(J(n), J(n+k)).$$

Note that we are conflating Q_n^k with its induced map $(Q_n^k)_*$ on $\mathrm{Ext}_{\mathcal{A}}$.

Remark 3.3.5. One can deduce the existence of Q_n^k by using a Yoneda lemma in the derived category of $\mathcal{A}$-modules where Hom is given by graded Ext. Another way of saying this is that we have a natural transformation between two δ-functors, $\mathrm{Ext}^{s,s}(-, J(n))$ and $\mathrm{Ext}^{s+1,s+1}(-, J(n+k))$, which is determined by its value when $s = 0$. See [36, Chapter 2].

In order to make computations in Chapter 5, we need to explore this link to Ext. The following proposition connects the Singer complex itself to $\mathrm{Ext}_{\mathcal{A}}$. Recall that $G(n)$ denotes the dual of $J(n)$.

Proposition 3.3.6. *There is an epimorphism*

$$(\Sigma \mathcal{R}_1 \Sigma^{-1} G(n))_{n+k} \twoheadrightarrow \mathrm{Ext}_{\mathcal{A}}^{1,1}(J(n), J(n+k))$$

which is an isomorphism if $k \geq 0$*.*

Proof: There is an isomorphism $\mathrm{Ext}_{\mathcal{A}}^{1,1}(J(n), J(n+k)) \cong (\Omega_1^\infty \Sigma^{-1} G(n))_{n+k}$ from

Proposition 3.3.2, and the latter group is a quotient of $(\Sigma\mathcal{R}_1\Sigma^{-1}G(n))_{n+k}$ which can be seen on the chain level: the differential at this step is zero, so everything in the latter group is a cycle. But one can check that the differential can only kill things in $\Sigma\mathcal{R}_1\Sigma^{-1}G(n)$ in degrees $< n$. □

Remark 3.3.7. Using a similar argument, one can replace $G(n)$ by any finite unstable module M, where we replace $J(n)$ with $M^\vee$. The surjection always holds, and the isomorphism holds in degrees above the degree of the top class of $G(n)$.

Proposition 3.3.8. *If $k \geq 0$, the element Q^k_n corresponds to $\sigma Q^k \sigma^{-1}\iota_n \in \Sigma\mathcal{R}_1\Sigma^{-1}G(n)$.*

Proof: By Proposition 3.3.6, $(\Sigma\mathcal{R}_1\Sigma^{-1}G(n))_{n+k} \cong \mathrm{Ext}^{1,1}_{\mathcal{A}}(J(n), J(n+k))$, and we have the diagram

$$\begin{array}{ccc} (\Omega^\infty_s\Sigma^{-s}M^\vee)_n & \xrightarrow{\cong} & \mathrm{Ext}^{s,s}_{\mathcal{A}}(M, J(n)) \\ \downarrow{\scriptstyle Q^k_n} & & \downarrow{\scriptstyle Q^k_n} \\ (\Omega^\infty_{s+1}\Sigma^{-(s+1)}M^\vee)_{n+k} & \xrightarrow{\cong} & \mathrm{Ext}^{s+1,s+1}_{\mathcal{A}}(M, J(n+k)) \end{array} \tag{3.3.9}$$

Note that the left multiplication map Q^k in Proposition 3.2.3 factors as

$$\begin{array}{ccccc} & & \Sigma\mathcal{R}_1\Sigma^{-1}(\Sigma\mathcal{R}_s\Sigma^{-1}M) & & \\ & \nearrow{\scriptstyle \sigma Q^k\sigma^{-1}} & & \searrow{\scriptstyle \mu} & \\ \Sigma\mathcal{R}_s\Sigma^{-1}M & & \xrightarrow{\quad Q^k \quad} & & \Sigma\mathcal{R}_{s+1}\Sigma^{-1}M \end{array}$$

where μ "applies" the Adem relations. Since $\Sigma\mathcal{R}_1\Sigma^{-1}$ is exact, we can pass to homology to obtain

$$\begin{array}{ccccc} & & \Sigma\mathcal{R}_1\Sigma^{-1}(\Omega^\infty_s\Sigma^{-s}M) & & \\ & \nearrow{\scriptstyle \sigma Q^k\sigma^{-1}} & & \searrow{\scriptstyle \mu} & \\ \Omega^\infty_s\Sigma^{-s}M & & \xrightarrow{\quad Q^k \quad} & & \Omega^\infty_{s+1}\Sigma^{-(s+1)}M \end{array}$$

If $x_n \in (\Omega^\infty_s\Sigma^{-s}M)_n$, then x_n is represented by $x_n\colon G(n) \to \Omega^\infty_s\Sigma^{-s}M$ sending the generator $\iota_n \mapsto x_n$. Therefore, we have

$$\Sigma\mathcal{R}_1\Sigma^{-1}G(n) \to \Sigma\mathcal{R}_1\Sigma^{-1}(\Omega^\infty_s\Sigma^{-s}M) \to \Omega^\infty_{s+1}\Sigma^{-(s+1)}M$$

which maps $\sigma Q^k \sigma^{-1} \iota_n \mapsto Q^k x_n$, so that this map has the same image as the map $G(n+k) \to \Omega^\infty_{s+1} \Sigma^{-(s+1)} M$ classifying $Q^k x_n$. □

Remark 3.3.10. Another way to see Proposition 3.3.8 is consider the universal case when $M = J(n)$ (so $M^\vee = G(n)$) and $s = 0$ in the diagram (3.3.9).

Knowing that the elements Q^k_n correspond to $\sigma Q^k \sigma^{-1} \iota_n$, and hence $(Q^k_n)_*$ to Q^k, the natural question is, can we explicitly describe these extensions? If we can answer that question, then we will be able to do explicit computations via the Yoneda splice. The extensions will be constructed from pushouts and tensor products, so we first define the "Hopf algebra tensor product," or smash product.

Definition 3.3.11 ([22, p. 186])**.** Given two left $\mathcal{A}$-modules M and N, the *Hopf algebra tensor product* of M and N will be denoted $M \otimes N$, and is defined as follows. The underlying graded group is simply $M \otimes_{\mathbb{F}_2} N$, and the $\mathcal{A}$ action is given by $a(x \otimes y) = \sum_i a_i x \otimes a'_i y$ where $a \in \mathcal{A}$ and $\Delta(a) = \sum_i a_i \otimes a'_i$. This makes $M \otimes N$ a left $\mathcal{A}$-module.

The following lemma is straightforward from the definition.

Lemma 3.3.12. *(1)* $\mathbb{F}_2 \otimes M \cong M \cong M \otimes \mathbb{F}_2$

(2) $\otimes$ *is exact in either variable and commutes with colimits.*

(3) Since $\mathcal{A}$ *is co-commutative,* $\otimes$ *is commutative.*

(4) If M *and* N *are unstable, then so is* $M \otimes N$*.*

(5) If M *is a left* $\mathcal{A}$*-module, then* $\mathcal{A} \otimes M$ *is a free* $\mathcal{A}$*-module.*

For example, if we label the generators of $J(2)$ by y_1 and y_2, with $\mathrm{Sq}^1 y_1 = y_2$, then in $J(2) \otimes J(2)$, we have four classes contained in dimensions 2, 3, and 4, with the following actions:

$$\mathrm{Sq}^1(y_1 \otimes y_1) = y_2 \otimes y_1 + y_1 \otimes y_2$$
$$\mathrm{Sq}^1(y_1 \otimes y_2) = y_2 \otimes y_2$$
$$\mathrm{Sq}^1(y_2 \otimes y_1) = y_2 \otimes y_2$$
$$\mathrm{Sq}^2(y_1 \otimes y_1) = y_2 \otimes y_2$$

Notice that a nontrivial Sq^2 now exists because $\mathrm{Sq}^1 \otimes \mathrm{Sq}^1$ is a summand of $\Delta(\mathrm{Sq}^2)$.

Notation 3.3.13. Let $\mathcal{P}_0^\infty = H^*(\mathbb{R}P^\infty) = \mathbb{F}_2[t]$ and $\mathcal{P}_{-1}^\infty = \mathbb{F}_2[t]\cdot t^{-1}$, where $\mathrm{Sq}^n t^{-1} = t^{n-1} \neq 0$ for all $n \geq 0$. In other words, $\mathcal{P}_{-1}^\infty$ is the cohomology of Singer's $\mathbb{R}P_{-1}^\infty$.

The following short exact sequence will be our building block for the construction of Q_n^k as an extension.

Lemma 3.3.14. *There is a short exact sequence of $\mathcal{A}$-modules*

$$0 \longrightarrow \mathcal{P}_0^\infty \longrightarrow \mathcal{P}_{-1}^\infty \longrightarrow \Sigma^{-1}\mathbb{F}_2 \longrightarrow 0$$

arising from the cofibration sequence

$$S^{-1} \to \mathbb{R}P_{-1}^\infty \to \mathbb{R}P^\infty.$$

Let t_k denote the dual of t^k.

Definition 3.3.15. For a given positive n and k, let $e(n,k,j)$ denote the element

$$t_{k+j} \otimes (\iota_n \cdot \chi(\mathrm{Sq}^j)) \in [H_{k+j}(\mathbb{R}P^\infty) \otimes G(n)]_{n+k}.$$

Dually, this can be thought of as an $\mathcal{A}$-module map

$$e(n,k,j)\colon \mathcal{P}_0^\infty \otimes J(n) \longrightarrow J(n+k).$$

Let $q(n,k) = \sum_j e(n,k,j)$.

The following theorem is the main result allowing the explicit construction of the Q_n^k which come from the Yoneda product. We will often write $Q(n,k)$ for the extension itself, or the element in $\mathrm{Ext}^1_{\mathcal{A}}$, depending on context. The proof is delayed to the Appendix.

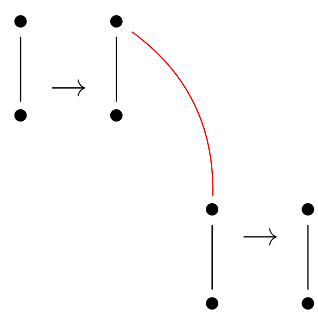
Theorem 3.3.16. *Let $Q(n,k)$ be the extension formed by the pushout*

$$\begin{array}{ccccc}
\mathcal{P}_0^\infty \otimes J(n) & \longrightarrow & \mathcal{P}_{-1}^\infty \otimes J(n) & \longrightarrow & \Sigma^{-1}J(n) \\
{\scriptstyle q(n,k)}\downarrow & & \downarrow & & \downarrow{\scriptstyle =} \\
J(n+k) & \longrightarrow & Q(n,k) & \longrightarrow & \Sigma^{-1}J(n)
\end{array}$$

Then $Q(n,k) \in \mathrm{Ext}_{\mathcal{A}}^{1,1}(J(n), J(n+k))$ *corresponds to* $\sigma Q^k \sigma^{-1}\iota_n \in \Sigma\mathcal{R}_1\Sigma^{-1}G(n)$ *under the isomorphism in Proposition 3.3.6.*

Corollary 3.3.17. *The action of* Q_n^k *on* $\mathrm{Ext}_{\mathcal{A}}^{*,*}$ *is given by the Yoneda splice with the extension* $Q(n,k)$ *in Theorem 3.3.16.*

Example 3.3.18. $Q(2,1) \in \mathrm{Ext}_{\mathcal{A}}^{1,1}(J(2), J(3))$ is the nonzero extension pictured, where the straight lines are Sq^1 and the curved lines are Sq^2:



See (4.1.2) for more details about what these diagrams represent.

Remark 3.3.19. More generally, the short exact sequence $0 \to J(2n-1) \to Q(n, n-1) \to \Sigma^{-1}J(n) \to 0$ agrees after suspension with the Mahowald sequence $0 \to \Sigma J(2n-1) \to J(2n) \to J(n) \to 0$.

The Yoneda product with $Q(n,k)$ induces a natural map

$$Q(n,k) \circ - \colon\ \mathrm{Ext}_{\mathcal{A}}^{s,s}(M, J(n)) \to \mathrm{Ext}_{\mathcal{A}}^{s+1,s+1}(M, J(n+k))$$

for all left $\mathcal{A}$–modules M, which we now know is Q_n^k. Because of Proposition 3.3.8, we know that it satisfies the usual properties of Dyer–Lashof operations.

In particular, one has stability, which we now describe. The inclusion $\varepsilon : \Sigma J(n) \to$

$J(n+1)$ induces

$$\varepsilon_* : \mathrm{Ext}_{\mathcal{A}}^{s,s}(M, J(n)) = \mathrm{Ext}_{\mathcal{A}}^{s,s}(\Sigma M, \Sigma J(n)) \to \mathrm{Ext}_{\mathcal{A}}^{s,s}(\Sigma M, J(n+1)).$$

We therefore have commutative diagrams

$$\begin{array}{ccc} \mathrm{Ext}_{\mathcal{A}}^{s,s}(M, J(n)) & \xrightarrow{\varepsilon_*} & \mathrm{Ext}_{\mathcal{A}}^{s,s}(\Sigma M, J(n+1)) \\ \big\downarrow{\scriptstyle Q_n^k \circ -} & & \big\downarrow{\scriptstyle Q_{n+1}^k \circ -} \\ \mathrm{Ext}_{\mathcal{A}}^{s+1,s+1}(M, J(n+k)) & \xrightarrow{\varepsilon_*} & \mathrm{Ext}_{\mathcal{A}}^{s+1,s+1}(\Sigma M, J(n+1+k)) \end{array}$$

and thus it makes sense to just write Q^k for Q_n^k. In other words, we have well-defined operations

$$Q^k : \mathrm{Ext}_{\mathcal{A}}^{s,s}(M, J(*)) \to \mathrm{Ext}_{\mathcal{A}}^{s+1,s+1}(M, J(*+k))$$

satisfying standard properties.

Remark 3.3.20. A careful reader will see that if $x \in \mathrm{Ext}_{\mathcal{A}}^{s,s}(M, J(n))$, then we can be sure that $Q^k x = 0$ only if $k < n-1$, as $\sigma^{-1}\iota_n$ has degree $n-1$, and not n. However, under this translation to Ext, it follows that $L_s(M)_n$ is precisely equal to the image of $\varepsilon_* : \mathrm{Ext}_{\mathcal{A}}^{s,s}(M, J(n)) \to \mathrm{Ext}_{\mathcal{A}}^{s,s}(\Sigma M, J(n+1))$. From this, we can deduce that if $x \in L_s(M)_n$, then $Q^k x = 0$ if $k < n$,

Chapter 4

Calculating the derived functors for $\Sigma^{2l}\mathcal{A} /\!\!/ E(1)$

In this chapter, we will compute the functors $L_s(M)$ when M is the dual of $\Sigma^{2l}\mathcal{A} /\!\!/ E(1)$, for any $l \geq 0$ (Theorem 4.4.8). The computation will be as a graded $\mathbb{F}_2$-vector space (for each $s \geq 0$), although the module structure over $\mathcal{A}$ can be inferred from that of M and the Nishida relations. We will do this by first computing $\Omega^\infty_s \Sigma^{-t} M$ for $s \geq 0$ and all integers t, which is in turn computed via $\mathrm{Ext}_{\mathcal{A}}$ groups, which involve the Brown-Gitler modules $J(n)$.

We begin with a discussion of stable $E(1)$-modules, and then classify Brown-Gitler modules in the stable category of $E(1)$-modules. This allows a direct $\mathrm{Ext}_{E(1)}(\mathbb{F}_2, J(n))$ computation, which gives us $\mathrm{Ext}^{*,*}_{\mathcal{A}}(\Sigma^{2l}\mathcal{A} /\!\!/ E(1), J(n))$ via a change of rings. This then leads to a calculation of L_* because of "even-ness" properties (Corollary 4.4.6).

4.1 The stable category of $E(n)$-modules

In this section, we will develop a basic theory of modules over the sub-Hopf algebra $E(n) \subset \mathcal{A}$. In the rest of this chapter, we will focus on $n = 1$, but the results in this section can be said more generally with little difficulty, so we work in the context of all $n \geq 0$. Since the calculation of Ext groups over $E(1)$ is a primary goal, we will think stably. In other words, we will focus on maps between $E(1)$ modules which induce isomorphisms on Ext^s for $s > 0$.

Recall that $E(n)$ is the exterior algebra $\Lambda(Q_0, Q_1, \ldots, Q_n)$, where the Q_n are the Milnor primitives in $\mathcal{A}$. When $n = 1$, we have

$$H^*(ku) = \mathcal{A} /\!\!/ E(1) = \mathcal{A}/\mathcal{A}\{Sq^1, Sq^2Sq^1 + Sq^3\} \cong \mathcal{A}/\mathcal{A}\{Sq^1, Sq^3\}.$$

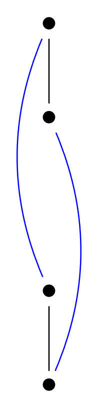
The dual is thus a subalgebra of $\mathcal{A}_*$,

$$H_*(ku) = \mathbb{F}_2[\zeta_1^2, \zeta_2^2, \zeta_3, \zeta_4, \ldots],$$

where the right $\mathcal{A}$-module structure is given by Lemma 2.1.5.

Remark 4.1.1. Since $\Delta(Q_i) = Q_i \otimes 1 + 1 \otimes Q_i$ for all i, the algebra $E(n)$ is a sub-*Hopf* algebra of $\mathcal{A}$.

We will denote modules over $\mathcal{A}$ or $E(1)$ by *cell diagrams*, where the cells (nodes) represent generators in various degrees, and the lines between them represent the actions of various operations in $\mathcal{A}$. For example, $E(1)$ can be drawn as a module over itself as:


(4.1.2)

where the cells are in degrees 0, 1, 3, and 4; the black, straight lines indicate $Q_0 = \mathrm{Sq}^1$; and the blue, curved lines indicate Q_1.

The Steenrod algebra can be thought of as a left or right module over $E(n)$ via multiplication, and its structure as an $E(n)$-module is illustrated in the following theorem.

Theorem 4.1.3. *(a) $E(n)$, being a finite connected Hopf algebra, is Frobenius. In* *particular, this means that for $E(n)$-modules, free, projective, flat, and injective are all equivalent notions.*

(b) $\mathcal{A}$ is a free right $E(n)$-module, hence also flat and injective.

Proof: See [22, p. 190-191] for (a). For (b), see for instance Proposition 1 on p. 331, and the proof of Theorem 8(a) on p. 320 in [22]. □

Definition 4.1.4. Two $E(n)$-modules M and N are said to be $E(n)$-*stably equivalent*, denoted $M \underset{E(n)}{\sim} N$, if

$$M \oplus \left(\bigoplus_j \Sigma^{k_j} E(n) \right) \cong N \oplus \left(\bigoplus_i \Sigma^{l_i} E(n) \right).$$

Lemma 4.1.5. *Suppose $M \underset{E(n)}{\sim} N$, and let L be any $E(n)$-module. Then*

(a) $\mathrm{Ext}^{s,t}_{E(n)}(M, L) \cong \mathrm{Ext}^{s,t}_{E(n)}(N, L)$ for $s \geq 1$, and

(b) $\mathrm{Ext}^{s,t}_{E(n)}(L, M) \cong \mathrm{Ext}^{s,t}_{E(n)}(L, N)$ for $s \geq 1$.

Proof: By the additivity of Ext, we need only show $\mathrm{Ext}^{s,t}_{E(n)}(E(n), L) = 0$ for (a) and $\mathrm{Ext}^{s,t}_{E(n)}(L, E(n)) = 0$ for (b). The former follows because $E(n)$ is free over itself hence projective; the latter follows because $E(n)$ is a Frobenius algebra, hence also injective over itself by Theorem 4.1.3. □

Proposition 4.1.6. *The algebra cohomology of $E(1)$ is given by $\mathrm{Ext}^{*,*}_{E(1)}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathbb{F}_2[h_0, v_1]$ where $|h_0| = (1, 1)$ and $|v_1| = (1, 3)$. Thus the even line (Adams $t - s$ grading) groups are all $\mathbb{F}_2$ and the odd line groups are all 0.*

Proof: The first statement follows from the basic theory of Koszul resolutions.

Next, since h_0 and v_1 generate $\mathrm{Ext}^{*,*}_{E(1)}(\mathbb{F}_2, \mathbb{F}_2)$ as an algebra, we obtain elements in dimensions $(m+n, m+3n)$ for all $m, n \geq 0$. Thus we only obtain nonzero elements on the $m+3n-m-n = 2n$ line, for all n. To see that each nonzero group is precisely $\mathbb{F}_2$, assume two elements $h_0^m v_1^n$ and $h_0^{m'} v_1^{n'}$ are in the same degree. Then $m+n = m'+n'$ and $m+3n = m'+3n'$, and so $m'+n'+2n = m'+3n'$ hence $n = n'$ and thus $m = m'$ as well. So the two elements are the same, and each non-zero degree must have only one $\mathbb{F}_2$-generator. □

One can similarly prove the following, although we will not need it.

Proposition 4.1.7. *The algebra cohomology of $E(n)$ is given by $Ext^{*,*}_{E(n)}(\mathbb{F}_2, \mathbb{F}_2) = \mathbb{F}_2[h_0, v_1, v_2, \ldots, v_n]$ with $|v_n| = (1, 2^{n+1} - 1)$.*

Remark 4.1.8. These algebra cohomologies are the E_2 pages for the Adams spectral sequence computing $\pi_*(BP\langle n\rangle)^\wedge_2 = \mathbb{F}_2[v_1, \ldots, v_n]$, where $|v_n| = 2^{n+1} - 2$ and h_0 becomes multiplication by 2.

4.2 Classification of $J(n)$ as $E(1)$-modules

We will now classify the Brown-Gitler modules $J(n)$ stably over $E(1)$.

Definition 4.2.1. (a) Let $\nu(n)$ denote the largest power of 2 that divides n (i.e. 2-adic valuation).

(b) Let $\alpha(n)$ denote the number of 1s in the binary expansion of n.

It is an easy exercise to make the following calculations, which will be useful later.

Lemma 4.2.2. *(a)* $\alpha(n) = \alpha(n-1) - \nu(n) + 1$

(b) $\alpha(2n) = \alpha(n)$

(c) $\alpha(2n+1) = \alpha(2n) + \alpha(1) = \alpha(n) + 1$

(d) $\alpha(2^k) = 1$

(e) $\nu(2n) = \nu(n) + 1$

In order to classify $J(n)$ over $E(1)$, we will need to compute the "Margolis homology" of $J(n)$ with respect to Q_0 and Q_1.

Definition 4.2.3. Let $e \in \mathcal{A}$ be an exterior element, so $e^2 = 0$. Then e acts as a differential on any $\mathcal{A}$-module (or simply $\Lambda(e)$-module) M. We define the *Margolis homology* of M with respect to e to be the homology with respect to e, that is,

$$H(M; e) = \frac{\ker(e)}{\mathrm{im}(e)}.$$

In order to compute $H(J(n); Q_0)$ and $H(J(n); Q_1)$, it is easier to combine all of the $J(n)$ into a bi-graded module first. The dual version of the following theorem was first proved in [25, Theorem 6.17].

Theorem 4.2.4 ([33, p. 43])**.** *The modules $J(n)$ assemble into a bi-graded $\mathcal{A}$-algebra*

$$J(*)^* \cong \mathbb{F}_2[x_0, x_1, x_2, \dots],$$

where $|x_i| = (1, 2^i)$. Here, an element in bi-degree (t, n) means it belongs to $J(n)^t$, so that $x_i \in J(2^i)^1$, and the bi-degree of a product is obtained by adding the bi-degrees of the factors. The left $\mathcal{A}$ action is determined by $\mathrm{Sq}^1(x_i) = x_{i-1}^2$ and the Cartan formula.

Remark 4.2.5. We use the convention that $x_k = 0$ if $k < 0$.

It follows from this theorem that $\mathrm{Sq}(x_i) = x_i + x_{i-1}^2$, and that $\mathrm{Sq}^k(x_0) = 0$ for any $k > 0$. Since $J(*)^*$ is an $\mathcal{A}$-algebra, one can determine the action of $\mathcal{A}$ on products of elements using Definition 3.3.11. For example, we have $x_1x_2 \in J(2+4)^2 = J(6)^2$ and $\mathrm{Sq}^1(x_1x_2) = x_0^2x_2 + x_1^3$.

Lemma 4.2.6. *(a) $H(J(*); Q_0) \cong \dfrac{\mathbb{F}_2[x_0]}{\langle x_0^2\rangle}$, and*

(b) $H(J(*); Q_1) \cong \dfrac{\mathbb{F}_2[x_0, x_1]}{\langle x_0^4, x_1^4\rangle} \otimes \Lambda(x_2^2, x_3^2, \dots)$.

Proof: It suffices to work on the generators x_i for $J(*)$, and in this setting, we have a Künneth formula. From the fact that $\mathrm{Sq}(x_i) = x_i + x_{i-1}^2$, we deduce $Q_1(x_i) = x_{i-2}^4$ (and $Q_0(x_i) = x_{i-1}^2$). Furthermore, $Q_0(x_i^2) = Q_1(x_i^2) = 0$ for any x_i since both Q_0 and Q_1 are primitive elements of $\mathcal{A}$. Since $Q_0(x_0) = 0$ and $Q_1(x_0) = Q_1(x_1) = 0$, we have

$$H(J(*); Q_0) \cong \frac{\mathbb{F}_2[x_0, x_1^2, x_2^2, \dots]}{\langle x_0^2, x_1^2, x_2^2, \dots\rangle}$$

and

$$H(J(*); Q_1) \cong \frac{\mathbb{F}_2[x_0, x_1, x_2^2, x_3^2 \dots]}{\langle x_0^4, x_1^4, x_2^4, x_3^4, \dots\rangle}$$

and the lemma follows. □

We will need the following theorem of Margolis which classifies indecomposable $E(1)$–modules.

Theorem 4.2.7. *(a) Every $E(1)$–module M can be written uniquely (up to isomorphism) as $M \cong F \oplus L$, where F is a free module, and L is the coproduct of "lightning flash modules".*

(b) If M is a finite $E(1)$–module such that $H(M;Q_0) = 0$, then L in part (a) must be some suspension of the $E(1)$–module $L(k)$ defined as follows: $L(k)$ is generated by $z_0, z_1, \ldots, z_k$ subject to the relations $Q_1 z_i = Q_0 z_{i+1}$ for all i and $Q_1 z_k = 0$.

Proof: Part (a) follows from [22, Chapter 18, Theorem 5]. Part (b) follows from [22, Chapter 18, Proposition 7], where $L(k)$ is denoted there by $L(k,1,0)$. $\square$

Notation 4.2.8. Let $\mathcal{P}^m := \widetilde{H}^*(\mathbb{R}P^m)$ for $m \geq 1$. It will often be convenient to let $\mathcal{P}^0 = \Sigma^2\mathbb{F}_2$.

Theorem 4.2.9. *Let $\mathcal{P}^m$ be as in Notation 4.2.8 for $m \geq 0$.*

(a) If n is even, $J(n) \underset{E(1)}{\sim} \Sigma^{2\alpha(n)-2}\mathcal{P}^{2\nu(n)}$.

(b) If n is odd, $J(n) \underset{E(1)}{\sim} \Sigma^{2\alpha(n)-3}\mathcal{P}^{2\nu(n-1)}$.

Remark 4.2.10. The definition of $\mathcal{P}^0$ is only relevant for $n = 0, 1$.

Proof: We will prove the theorem for even n, and the odd case follows from the equivalence $J(2n+1) \cong \Sigma J(2n)$ and Lemma 4.2.2. Furthermore, we can assume $n > 2$ since $J(2) = \mathcal{P}^2$.

Assume $n > 2$ is even. From Lemma 4.2.6, we deduce $H(J(n);Q_0) = 0$, so $J(n)$ is $E(1)$–stably equivalent to L as described in Theorem 4.2.7(b), which has k generators. We must determine k and the number of suspensions of $L(k)$. First we compute $H(J(n);Q_1)$.

Let

$$P = \frac{\mathbb{F}_2[x_0, x_1]}{\langle x_0^4, x_1^4 \rangle}$$

and $\Lambda = \Lambda(x_2^2, x_3^2, \dots)$. By Lemma 4.2.6, $H(J(*);Q_1) \cong P \otimes \Lambda$. The elements in Λ are in $J(8k)^2$ for some $k \geq 1$, and those in P range from $J(0)^0$ to $J(9)^6$. Therefore, if we let y denote a generic element in Λ, then

$$H(J(n);Q_1) = \begin{cases} \{x_0^2 y, x_1 y\} & n = 8k+2 \\ \{x_0^2 x_1 y, x_1^2 y\} & n = 8k+4 \\ \{x_0^2 x_1^2 y, x_1^3 y\} & n = 8k+6 \\ \{y, x_0^2 x_1^3 y'\} & n = 8k. \end{cases}$$

In particular, $H(J(n);Q_1)$ is two-dimensional. One can check that $L(k)$ is the same module as $\Sigma^{-1}\mathcal{P}^{2(k+1)}$ restricted to $E(1)$ (see Example 4.2.12 for $L(2)$). To determine k and the number of suspensions of $L(k)$, we will use Lemma 4.2.2 several times. We proceed by considering $n \pmod 8$.

First suppose $n = 8k+2$, so that $H(J(n);Q_1)$ has generators $x_0^2 y$ and $x_1 y$. Note that $\mathrm{Sq}^1(x_0^2 y) = 0$ and $\mathrm{Sq}^1(x_1 y) = x_0^2 y$, so these generators are connected by Sq^1. This implies $m = 0$, that is, L is some shift of $\mathcal{P}^2$. The bottom class of this $\mathcal{P}^2$ is $x_1 y$ where y is a product of x_i^2 for $i \geq 2$. The degree of y is given by twice the number of its x_i^2 factors. On the other hand, the number of x_i^2 factors corresponds to writing $8k$ as a sum of the numbers $8, 16, 32, 64, \ldots$, or writing k 2-adically. Thus y is in degree $2\alpha(k)$, so that the bottom class of $\mathcal{P}^2$ is in dimension $1 + 2\alpha(k)$. But,

$$\alpha(n) = \alpha(4k+1) = \alpha(4k) - \nu(4k+1) + 1 = \alpha(k) + 1.$$

Thus $\alpha(k) = \alpha(n) - 1$, so the bottom class of $\mathcal{P}^2$ is in degree $2\alpha(n) - 1$, and it follows that $L = \Sigma^{2\alpha(n)-2}\mathcal{P}^2$.

Next, suppose $n = 8k+4$ and proceed similarly. The Margolis homology generators are $x_0^2 x_1 y$ and $x_1^2 y$. In this case $\mathrm{Sq}^1(x_0^2 x_1 y) = x_0^4 y$ and $\mathrm{Sq}^1(x_1^2 y) = 0$. This implies $m = 1$ and L is a shift of $\mathcal{P}^4$. As above, y is in degree $2\alpha(8k) = 2\alpha(k)$. Thus $|x_1^2 y| = 2\alpha(k) + 2$ but the bottom class in $\mathcal{P}^4$ is in degree $2\alpha(k) + 1$. One can check $\alpha(k) = \alpha(n) - 1$ again, and so $L = \Sigma^{2\alpha(n)-2}\mathcal{P}^4 = \Sigma^{2\alpha(n)-2}\mathcal{P}^{2\nu(n)}$ (as an $E(1)$-module).

Next, suppose $n = 8k+6$. The Margolis homology generators are $x_0^2x_1^2y$ and x_1^3y, and $\mathrm{Sq}^1(x_1^3y) = x_0^2x_1^2y$, so we must have $m = 0$, that is, L is a shift of $\mathcal{P}^2$. The bottom class of this $\mathcal{P}^2$ is x_1^3y. Reasoning as above, we see that the degree of this class is $3 + 2\alpha(k)$. Note that

$$\alpha(n) = \alpha(4k+3) = \alpha(4k+2) - \nu(4k+3) + 1 = \alpha(2k+1) + 1$$

and

$$\alpha(2k+1) = \alpha(2k) - \nu(2k+1) + 1 = \alpha(k) + 1.$$

Putting these together gives $\alpha(k) = \alpha(n) - 2$, and thus the bottom class of $\mathcal{P}^2$ is in degree $2\alpha(n) - 1$, and so $L = \Sigma^{2\alpha(n)-2}\mathcal{P}^2$.

Finally, suppose $n = 8k$. The Margolis homology generators are y and $x_0^2x_1^3y'$ where y' is a (possibly empty, in the case $k = 1$) product of x_i^2 for $i \geq 2$ as well. Reasoning as above, we know $|y| = 2\alpha(k) = 2\alpha(n)$, whereas $y' \in J(8(k-1))$ is in degree $2\alpha(k-1)$, so $|x_0^2x_1^3y'| = 5 + 2\alpha(k-1)$. Since these Q_1-Margolis homology generators are not connected by Sq^1, we know $m > 0$. This implies that the bottom class of L is in degree $2\alpha(k)-1$ and the top class is in $6+2\alpha(k-1) = 4+2\alpha(k)+2\nu(k)$. Note that $\nu(k) = \nu(n) - 3$ and the difference in degree between the top and bottom classes of L is $5 + 2\nu(k)$, whence

$$L = \Sigma^{2\alpha(n)-2}\mathcal{P}^{6+2\nu(k)} = \Sigma^{2\alpha(n)-2}\mathcal{P}^{2\nu(n)}$$

as $E(1)$-modules. □

Corollary 4.2.11. *If $s \geq 1$, then*

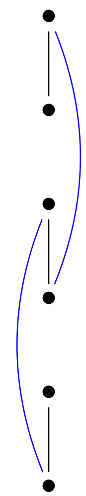
$$\mathrm{Ext}_{E(1)}^{s,t}(\mathbb{F}_2, J(n)) \cong \mathrm{Ext}_{E(1)}^{s,t+2\alpha(n)-2}(\mathbb{F}_2, \mathcal{P}^{2\nu(n)})$$

for even n, and

$$\mathrm{Ext}_{E(1)}^{s,t}(\mathbb{F}_2, J(n)) \cong \mathrm{Ext}_{E(1)}^{s,t+2\alpha(n)-3}(\mathbb{F}_2, \mathcal{P}^{2\nu(n-1)})$$

for odd n.

Example 4.2.12. Here is $\mathcal{P}^6$ restricted to $E(1)$. Note $J(8) \underset{E(1)}{\sim} \mathcal{P}^6$. The bottom class is in degree 1.



4.3 The Ext groups of projective space modules

Motivated by Corollary 4.2.11, we now compute $\mathrm{Ext}^{*,*}_{E(1)}(\mathbb{F}_2, \mathcal{P}^{2m})$ for all $m \geq 1$, as right modules over $\mathrm{Ext}^{*,*}_{E(1)}(\mathbb{F}_2, \mathbb{F}_2)$. The groups $\mathrm{Ext}^{*,*}_{\mathcal{A}(1)}(\mathcal{P}^{\infty}_m, \mathbb{F}_2)$ were computed by Gitler, Mahowald, and Milgram in [10], and later by Davis in [8] for $\mathcal{P}^{2m}$, in their study of non-immersions of projective spaces. In the former, they only calculate the groups, and in the latter, the module structure is with respect to a different group, that for instance does not include a v_1 action. It may be possible to deduce some of our results here from a combination of their results, duality, and restriction of the module structure, but the author is not sure if that is feasible. In any case, our proof is straightforward, with the module structure being deduced from an explicit equivalence of extensions.

Theorem 4.3.1. *Let $E = \mathrm{Ext}^{*,*}_{E(1)}(\mathbb{F}_2, \mathbb{F}_2) = \mathbb{F}_2[h_0, v_1]$. As a right E-module,*

$$\mathrm{Ext}^{*,*}_{E(1)}(\mathbb{F}_2, \mathcal{P}^{2m}) \cong \frac{\{u_{-2}, u_{-4}, \ldots, u_{-2m}\} \cdot E}{\langle u_{-2k+2}h_0 + u_{-2k}v_1 \mid 2 \leq k \leq m\rangle},$$

where $|u_{-2k}| = (0, -2k)$.

Remark 4.3.2. The theorem implies that as a module over $\mathbb{F}_2[v_1]$,

$$\mathrm{Ext}^{*,*}_{E(1)}(\mathbb{F}_2, \mathcal{P}^{2m}) \cong \bigoplus_{1 \le k \le m} \Sigma^{0,-2k}\, \mathbb{F}_2[v_1].$$

In other words, the doubly graded $\mathrm{Ext}^{*,*}_{E(1)}(\mathbb{F}_2, \mathcal{P}^{2m})$ consists of m independent v_1-towers on the u_{-2k}. For degree reasons, this means that each bigraded degree consists of a single $\mathbb{F}_2$ or is zero.

Corollary 4.3.3. $\mathrm{Ext}^{s,t}_{E(1)}(\mathbb{F}_2, \mathcal{P}^{2m}) = \mathbb{F}_2$ *if and only if* $(s,t) = (s, 3s-2k)$ *for* $s \ge 0$ *and some* $1 \le k \le m$*. In all other bi-degrees, the group is zero.*

Proof: The (s,t) Ext group is nonzero if and only if it is generated by $v_1^r u_{-2k}$ for some $r \ge 0$ and $1 \le k \le m$. The degree of this is $(r, 3r) + (0, -2k) = (r, 3r-2k)$. So $(s,t) = (r, 3r-2k) = (s, 3s-2k)$. Also because of degree, there can only be a single generator in each bi-degree. □

The proof of Theorem 4.3.1 relies on multiple long exact sequences of Ext groups associated to various short exact sequences of projective space modules. We first analyze the types of $E(1)$-module maps between such modules. The following lemma is straightforward.

Lemma 4.3.4. *(a) There is a short exact sequence*

$$\Sigma^2\mathbb{F}_2 \to \mathcal{P}^2 \to \Sigma\mathbb{F}_2,$$

which represents $h_0 \in \mathrm{Ext}^{1,1}_{E(1)}(\mathbb{F}_2, \mathbb{F}_2)$*.*

(b) For $m \ge 2$*, there is a short exact sequence*

$$0 \to \Sigma^2\mathcal{P}^{2m-2} \to \mathcal{P}^{2m} \to \mathcal{P}^2 \to 0,$$

which is nontrivial in $\mathrm{Ext}^{1,2}_{E(1)}(\mathcal{P}^2, \mathcal{P}^{2m-2})$*.*

Proposition 4.3.5. *For $s \geq 0$, there are short exact sequences*

$$\mathrm{Ext}_{E(1)}^{s-1,t+1}(\mathbb{F}_2, \mathbb{F}_2) \xrightarrow{h_0} \mathrm{Ext}_{E(1)}^{s,t+2}(\mathbb{F}_2, \mathbb{F}_2) \longrightarrow \mathrm{Ext}_{E(1)}^{s,t}(\mathbb{F}_2, \mathcal{P}^2),$$

where $\mathrm{Ext}_{E(1)}^{<0,t}(-,-) = 0$ *and* $\mathrm{Ext}_{E(1)}^{0,t} = \mathrm{Hom}_{E(1)}^{t}$. *Thus*

$$\mathrm{Ext}_{E(1)}^{s,t}(\mathbb{F}_2, \mathcal{P}^2) \cong \mathrm{Ext}_{E(1)}^{s,t+2}(\mathbb{F}_2, \mathbb{F}_2)/h_0(\mathrm{Ext}_{E(1)}^{s-1,t+1}(\mathbb{F}_2, \mathbb{F}_2)).$$

Proof: The short exact sequence in Lemma 4.3.4(a) gives rise to a long exact sequence associated to the functor $\mathrm{Hom}_{E(1)}^{t}$:

$$\cdots \longrightarrow \mathrm{Ext}_{E(1)}^{s-1,t+1}(\mathbb{F}_2, \mathbb{F}_2) \xrightarrow{\delta} \mathrm{Ext}_{E(1)}^{s,t+2}(\mathbb{F}_2, \mathbb{F}_2) \longrightarrow \mathrm{Ext}_{E(1)}^{s,t}(\mathbb{F}_2, \mathcal{P}^2) \longrightarrow \mathrm{Ext}_{E(1)}^{s,t+1}(\mathbb{F}_2, \mathbb{F}_2) \xrightarrow{\delta} \mathrm{Ext}_{E(1)}^{s+1,t+2}(\mathbb{F}_2, \mathbb{F}_2) \longrightarrow \cdots$$

The connecting homomorphism δ is given by multiplication by $h_0 \in \mathrm{Ext}_{E(1)}^{1,1}(\mathbb{F}_2, \mathbb{F}_2)$ which implies that δ is injective since multiplication by h_0 has no kernel in $\mathrm{Ext}_{E(1)}^{*,*}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathbb{F}_2[h_0, v_1]$. Thus the long exact sequence splits into short exact ones, which proves the proposition for $s \geq 1$. When $s = 0$, the statement reduces to the claim that $\mathrm{Hom}^{t+2}(\mathbb{F}_2, \mathbb{F}_2) \to \mathrm{Hom}^{t}(\mathbb{F}_2, \mathcal{P}^2)$ is an isomorphism for all t, which is clear since both groups are 0 unless $t = -2$. $\square$

Corollary 4.3.6. *Let u_{-2} be the unique nonzero $E(1)$-module map $\Sigma^2\mathbb{F}_2 \to \mathcal{P}^2$. Then $u_{-2} \in \mathrm{Ext}_{E(1)}^{0,-2}(\mathbb{F}_2, \mathcal{P}^2)$ and*

$$\mathrm{Ext}_{E(1)}^{*,*}(\mathbb{F}_2, \mathcal{P}^2) \cong \{u_{-2}\} \cdot \mathbb{F}_2[v_1].$$

Proof: The isomorphism in Proposition 4.3.5 says that $\mathrm{Ext}_{E(1)}^{*,*}(\mathbb{F}_2, \mathcal{P}^2)$ is a quotient of $\mathrm{Ext}_{E(1)}^{*,*}(\mathbb{F}_2, \mathbb{F}_2)$, but shifted down by two. In Proposition 4.1.6, we see that there is exactly one generator $h_0^a v_1^b$ in $\mathrm{Ext}_{E(1)}^{a+b,a+3b}(\mathbb{F}_2, \mathbb{F}_2)$, whenever this group is nonzero. Thus when we quotient out by $h_0(\mathrm{Ext}_{E(1)}^{*-1,*-1}(\mathbb{F}_2, \mathbb{F}_2))$ we are left with $\mathbb{F}_2[v_1]$.

Now, under the isomorphism $\mathrm{Hom}_{E(1)}^{t}(\mathbb{F}_2, \mathcal{P}^2) \cong \mathrm{Hom}_{E(1)}^{t+2}(\mathbb{F}_2, \mathbb{F}_2)$ when $s = 0$ in Proposition 4.3.5, the identity $\mathbb{F}_2 \to \mathbb{F}_2$ corresponds to u_{-2}, and so v_1^a corresponds to $u_{-2}v_1^a$, and this defines the action of v_1 on u_{-2}. □

This corollary proves Theorem 4.3.1 in the case of $\mathcal{P}^2$. We now generalize this result.

Proof of Theorem 4.3.1. We induct on m. By Corollary 4.3.6, the claim holds for $m = 1$. Now let $m \geq 2$ and assume the claim for all $m' < m$ so that in particular,

$$\mathrm{Ext}_{E(1)}^{*,*}(\mathbb{F}_2, \mathcal{P}^{2m-2}) \cong \frac{\{u_{-2}, u_{-4}, \ldots, u_{-2m+2}\} \cdot E}{\langle u_{-2k+2}h_0 + u_{-2k}v_1 \mid 2 \leq k \leq m-1 \rangle}.$$

Let u_{-2k} be the unique nonzero $E(1)$-module map $\Sigma^{2k}\mathbb{F}_2 \to \mathcal{P}^{2m}$ for $1 \leq k \leq m$, so that $u_{-2k} \in \mathrm{Ext}_{E(1)}^{0,-2k}(\mathbb{F}_2, \mathcal{P}^{2m})$. Associated to the short exact sequence in Lemma 4.3.4(b), there is a long exact sequence corresponding to the functor $\mathrm{Hom}_{E(1)}^{t}(\mathbb{F}_2, -)$. The boundary map is given by

$$\delta\colon\ \mathrm{Ext}_{E(1)}^{s,t}(\mathbb{F}_2, \mathcal{P}^2) \longrightarrow \mathrm{Ext}_{E(1)}^{s+1,t+2}(\mathbb{F}_2, \mathcal{P}^{2m-2}).$$

We claim this map is always zero for $s \geq 0$. To see this, note that the domain is nonzero if and only if $(s,t) = (s, 3s-2)$ by Corollary 4.3.6. This is true if and only if $(s+1, t+2) = (s+1, 3s)$. But the target of δ is nonzero if and only if $(s+1, t+2) = (s+1, 3(s+1) - 2k)$ for some $1 \leq k \leq m-1$, by our induction hypothesis. Since $3s \neq 3(s+1) - 2k$ for any k, this means δ must be zero in this case. Otherwise, the domain is 0, so δ is obviously zero.

Therefore, for $s \geq 0$, there are short exact sequences

$$\mathrm{Ext}_{E(1)}^{s,t+2}(\mathbb{F}_2, \mathcal{P}^{2m-2}) \longrightarrow \mathrm{Ext}_{E(1)}^{s,t}(\mathbb{F}_2, \mathcal{P}^{2m}) \longrightarrow \mathrm{Ext}_{E(1)}^{s,t}(\mathbb{F}_2, \mathcal{P}^{2}).$$

Now, if $(s,t) = (s, 3s-2)$, then the first group is 0 by the induction hypothesis, and so

$$\mathrm{Ext}_{E(1)}^{s,3s-2}(\mathbb{F}_2, \mathcal{P}^{2m}) \xrightarrow{\cong} \mathrm{Ext}_{E(1)}^{s,3s-2}(\mathbb{F}_2, \mathcal{P}^{2}) \cong \mathbb{F}\{u_{-2}v_1^s\}.$$

We use the same label $u_{-2}v_1^s$ for this corresponding element in the domain. For any other bi-degree (s,t), $\mathrm{Ext}_{E(1)}^{s,t}(\mathbb{F}_2, \mathcal{P}^2) = 0$, so

$$\mathrm{Ext}_{E(1)}^{s,t+2}(\mathbb{F}_2, \mathcal{P}^{2m-2}) \xrightarrow{\cong} \mathrm{Ext}_{E(1)}^{s,t}(\mathbb{F}_2, \mathcal{P}^{2m}).$$

Under this isomorphism, we let $u_{-2(k+1)}v_1^s$ denote the image of $u_{-2k}v_1^s$ for $1 \leq k \leq m-1$, noting that we move from bi-degree $(s, 3s-2k)$ to $(s, 3s-2k-2) = (s, 3s-2(k+1))$.

This defines the generators $\{u_{-2}, \ldots, u_{-2m}\}$ in $\mathrm{Ext}_{E(1)}^{*,*}(\mathbb{F}_2, \mathcal{P}^{2m})$, and it remains to prove the relations $u_{-2k+2} \circ h_0 = u_{-2k} \circ v_1$ for $2 \leq k \leq m-1$. First we note that these are the only possible relations on the generators (referring to both u_{-2k}, and h_0 and v_1, but not their powers), for degree reasons, which can be deduced using Corollary 4.3.3 applied to the inductive hypothesis along with case analysis for m. In particular, $u_{-2m} \circ h_0 = 0$.

To prove $u_{-2k+2} \circ h_0 = u_{-2k} \circ v_1$, we use the definition of the Yoneda product. First, $u_{-2k+2} \colon \Sigma^{2k-2}\mathbb{F}_2 \to \mathcal{P}^{2m}$ and $h_0 \colon \Sigma\mathbb{F}_2 \hookrightarrow H \twoheadrightarrow \mathbb{F}_2$ gives $u_{-2k+2} \circ h_0$ as the pushout, $\mathcal{E}$:

$$\begin{array}{ccccc}
\Sigma^{2k-2}\mathbb{F}_2 & \longrightarrow & \Sigma^{2k-3}H & \longrightarrow & \Sigma^{2k-3}\mathbb{F}_2 \\
\downarrow{\scriptstyle u_{-2k+2}} & & \downarrow & & \downarrow{\scriptstyle =} \\
\mathcal{P}^{2m} & \longrightarrow & \mathcal{E} & \longrightarrow & \Sigma^{2k-3}\mathbb{F}_2
\end{array}$$

Here, H is $\Sigma^{-1}\mathcal{P}^2$, and $\mathcal{E}$ is obtained by identifying the top cell of H with the cell in $\mathcal{P}^{2m}$ in degree $2k-2$. On the other hand, $u_{-2k} \colon \Sigma^{2k}\mathbb{F}_2 \to \mathcal{P}^{2m}$ and $v_1 \colon \Sigma^3\mathbb{F}_2 \hookrightarrow V \twoheadrightarrow \mathbb{F}_2$, where V is the module

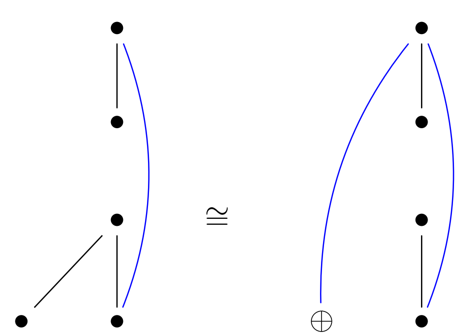
where the blue line is a Q_1 attaching a degree 0 cell to a degree 3 cell. Then $u_{-2k} \circ v_1$ is the pushout

$$\begin{array}{ccccc} \Sigma^{2k}\mathbb{F}_2 & \longrightarrow & \Sigma^{2k-3}V & \longrightarrow & \Sigma^{2k-3}\mathbb{F}_2 \\ {\scriptstyle u_{-2k}}\downarrow & & \downarrow & & \downarrow{\scriptstyle =} \\ \mathcal{P}^{2m} & \longrightarrow & \mathcal{E}' & \longrightarrow & \Sigma^{2k-3}\mathbb{F}_2 \end{array}$$

In other words, $\mathcal{E}'$ is obtained by gluing the top cell of V to the degree $2k$ cell in $\mathcal{P}^{2m}$. Therefore both $\mathcal{E}$ and $\mathcal{E}'$ have one cell in every dimension except $2k-3$, and a change of basis sending the bottom cell of H in $\mathcal{E}$ to the sum of the two elements in degree $2k-3$ yields a module isomorphic to $\mathcal{E}'$ since the sum supports a Q_1 and no Sq^1. (See Example 4.3.7).

This completes the proof. □

Example 4.3.7. The change of basis showing $u_{-2} \circ h_0 = u_{-4} \circ v_1$ in $\mathcal{P}^4$.



□

Remark 4.3.8. Note that the generator u_{-2} is always pulled back from $\mathrm{Ext}_{E(1)}(\mathbb{F}_2, \mathcal{P}^2)$ whereas $u_{-4}, \ldots, u_{-2k}$ are pushed forward from a twice suspended $\mathrm{Ext}_{E(1)}(\mathbb{F}_2, \mathcal{P}^{2m-2})$, so this is where the shift comes from. For example, in $\mathrm{Ext}_{E(1)}(\mathbb{F}_2, \mathcal{P}^4)$, u_{-4} comes from $\mathrm{Ext}^{0,-4}_{E(1)}(\mathbb{F}_2, \Sigma^2\mathcal{P}^2) = \mathrm{Ext}^{0,-2}_{E(1)}(\mathbb{F}_2, \mathcal{P}^2)$. This is not surprising given the short exact sequence in Lemma 4.3.4.

4.4 The modules $L_s(\Sigma^{2l}\mathcal{A} /\!\!/ E(1))$

Let M be the dual of $\Sigma^{2l}\mathcal{A} /\!\!/ E(1)$ and recall that $L_s M = \Omega\Omega_s^{\infty}\Sigma^{1-s}M$ by definition. By Proposition 3.3.2, we can instead calculate $\operatorname{Ext}_{\mathcal{A}}^{s,t}(\Sigma^{2l}\mathcal{A} /\!\!/ E(1), J(n))$ as a first approximation. By a change of rings and also Corollary 4.2.11, this requires understanding the Ext groups of $\mathcal{P}^m$ over $E(1)$ for $s \geq 1$ which was accomplished in the previous section (the case $s = 0$ is covered by Lemma 3.1.4). The objective of this section will be to put all of this together in the appropriate degrees with the appropriate suspensions and obtain a complete calculation of $L_*(M)$ as a graded $\mathbb{F}_2$-vector space. See Example 4.4.10 for $l = 0$.

Remark 4.4.1. Although the calculations in this section apply to $\Sigma^{2l}\mathcal{A} /\!\!/ E(1)$, we keep $\Sigma^2\mathcal{A} /\!\!/ E(1)$ in mind as the motivating case, which we will return to in Chapter 5. In this case, there is a fibration sequence

$$\Sigma^2 ku \to ku \to H\mathbb{Z}$$

realizing $\Omega^\infty\Sigma^2 ku = BU$ on the level of infinite loop spaces. Thus, information about the derived functors of $H_*(\Sigma^2 ku) = \Sigma^2\mathcal{A} /\!\!/ E(1)$ yields information about $H_*(BU) = \mathbb{F}_2[a_2, a_4, \ldots]$, whereas $H_*(ku)$ yields information about $H_*(BU \times \mathbb{Z}) = \mathbb{F}_2[a_0^{\pm 1}, a_2, a_4, \ldots]$. However, $\Sigma^2 ku$ has the advantage of being connective, easing convergence issues for the algebraic spectral sequence.

We know that $H_*(\Omega^\infty\Sigma^2 ku) \cong H_*(BU)$, but we can also determine the algebraic destabilization $\Omega^\infty H_*(\Sigma^2 ku)$.

Lemma 4.4.2. *We have $\Omega^\infty(\Sigma^2\mathcal{A} /\!\!/ E(1)) \cong \Phi F(1)$, the double of the free $\mathcal{U}$–module on one generator.*

Proof: We work dually. By definition, $(\Omega^\infty(\Sigma^2\mathcal{A} /\!\!/ E(1)))^\vee$ is the largest stable submodule of $(\Sigma^2\mathcal{A} /\!\!/ E(1))^\vee$. Note that $\xi_k^2 = \chi(\zeta_k^2) \in (\mathcal{A} /\!\!/ E(1))^\vee$ so $\sigma^2\xi_k^2 \in (\Sigma^2\mathcal{A} /\!\!/ E(1))^\vee$. This element has degree 2^{k+1} for $k \geq 0$ where $\xi_0 = 1$. One has $(\sigma^2\xi_k^2)\operatorname{Sq} = \sigma^2(\xi_k^2 + \xi_{k-1}^2)$ and thus $(\sigma^2\xi_k^2)\operatorname{Sq}^{2^k} = \sigma^2\xi_{k-1}^2$. This operation is the largest that ξ_k^2 supports and it passes the unstable condition. These operations are squares because of degree so this gives the algebra structure. Moreover, one can check that these are the only elements that "survive" after applying Ω^∞, because of degree and the action of Sq on ζ_k.

Alternatively, one can work cohomologically, noting the proof of Lemma 3.1.4, except now $\mathrm{Sq}^2(\sigma^2 1) = \sigma^2 \mathrm{Sq}^2$ is nonzero in the quotient, and this is the only nonzero operation on $\sigma^2 1$ since Sq^1 vanishes in $\mathcal{A} /\!\!/ E(1)$. Then one can check that we only get Sq^4 on Sq^2, and in general, we only have powers of two. □

We now proceed toward calculating $L_s(\Sigma^{2l}\mathcal{A} /\!\!/ E(1))$, and we begin by recording a standard change-of-rings theorem.

Theorem 4.4.3. *(Shapiro's lemma) Let N be an $\mathcal{A}$-module. Then there is a natural isomorphism*

$$\mathrm{Ext}^n_{\mathcal{A}}(\mathcal{A} /\!\!/ E(1), N) \to \mathrm{Ext}^n_{E(1)}(\mathbb{F}_2, N)$$

for all $n \geq 0$, where the N on the right is restricted to $E(1)$.

Proof: This is the derived version of the induction/restriction adjunction. By Theorem 4.1.3, we know that $\mathcal{A}$ is a flat (right) $E(1)$-module. Thus we can turn a free resolution of $\mathbb{F}_2$ by $E(1)$-modules into a free resolution of $\mathcal{A} /\!\!/ E(1) = \mathcal{A} \otimes_{E(1)} \mathbb{F}_2$ by $\mathcal{A}$-modules via $\mathcal{A}\otimes_{E(1)}$. There is a canonical map of resolutions $F_* \to \mathcal{A} \otimes_{E(1)} F_*$ which yields a map of complexes

$$\mathrm{Hom}_{\mathcal{A}}(\mathcal{A} \otimes_{E(1)} F_*, N) \to \mathrm{Hom}_{E(1)}(\mathcal{A} \otimes_{E(1)} F_*, N) \to \mathrm{Hom}_{E(1)}(F_*, N).$$

This composite is precisely the map in the induction/restriction adjunction, hence it is a natural isomorphism, and we therefore get an isomorphism on homology. □

Corollary 4.4.4. *Let $M = \Sigma^{2l}\mathcal{A} /\!\!/ E(1)$. Then by base change,*

$$(\Omega^\infty_s \Sigma^{-t} M^\vee)_* \cong \mathrm{Ext}^{s,t-2l}_{E(1)}(\mathbb{F}_2, J(*)),$$

as right $\mathcal{A}$-modules.

Proof: This follows from Proposition 3.3.2 and Theorem 4.4.3. □

The calculation of the $\mathrm{Ext}_{E(1)}$ groups of $\mathcal{P}^{2m}$ let us conclude the following.

Proposition 4.4.5. *Let $n \geq 2$ and $s \geq 1$.*

(a) If n is even, then $\mathrm{Ext}^{s,t}_{E(1)}(\mathbb{F}_2, J(n)) = \mathbb{F}_2$ if and only if $t = 3s - 2k - 2\alpha(n) + 2$ for some $1 \leq k \leq \nu(n)$.

(b) If n is odd, then $\mathrm{Ext}^{s,t}_{E(1)}(\mathbb{F}_2, J(n)) = \mathbb{F}_2$ if and only if $t = 3s - 2k - 2\alpha(n) + 3$ for some $1 \leq k \leq \nu(n-1)$ (note that $\nu(n-1) \geq 1$).

In all other cases, the group is zero.

Proof: This follows from Corollaries 4.2.11 and 4.3.3. □

Corollary 4.4.6. *Let M be the dual of $\Sigma^{2l}\mathcal{A}/\!\!/E(1)$, where $l \geq 0$. Then the $\mathcal{A}$-module $L_s(M)$ is evenly graded for $s \geq 1$. If $l = 0$ or $l = 1$, then $L_0(M)$ is evenly graded as well.*

Proof: Let $s \geq 1$. We will show that $\Omega^\infty_s \Sigma^{1-s}(M)$ is oddly graded, which implies that $L_s(M)$ is evenly graded since it is obtained by applying $\Omega = \Omega^\infty\Sigma^{-1}$.

To this end, first note

$$\Omega^\infty_s \Sigma^{1-s}(M)_n \cong \mathrm{Ext}^{s,s-1-2l}_{E(1)}(\mathbb{F}_2, J(n))$$

by Corollary 4.4.4. Now, apply the previous proposition with $t = s-1-2l$. If $n \geq 2$ is even, then the above Ext group is nonzero if and only if $-1-2l = 2s-2k-2\alpha(n)+2$, which cannot hold. If $n = 0$, the group is 0 for degree reasons.

The claims for $s = 0$ follow from Lemma 3.1.4 and Lemma 4.4.2 for $l = 0$ and $l = 1$, respectively. □

Corollary 4.4.7. *Let M denote the dual of $\Sigma^{2l}\mathcal{A}/\!\!/E(1)$. Then for $s \geq 1$,*

$$L_s(M)_* \cong \mathrm{Ext}^{s,s-2l}_{E(1)}(\mathbb{F}_2, J(*)),$$

as right $\mathcal{A}$-modules. If $l = 0$ or $l = 1$, then this holds for $s = 0$ as well.

Proof: By Corollary 4.4.6, $L_s(M)$ is evenly graded for the stated values of s, so the claim then follows from Proposition 3.3.4 and Theorem 4.4.3. $\square$

We can now state the main result of this chapter.

Theorem 4.4.8. *Let* $M = (\Sigma^{2l}\mathcal{A} /\!/ E(1))^\vee$.

(a) Let $l = 0$ *and fix* $s \geq 0$. *Then*

$$L_s(M)_n = \begin{cases} \mathbb{F}_2 & n = 0 \\ \mathbb{F}_2 & n \geq 2 \text{ is even and } s - \nu\left(\frac{n}{2}\right) \leq \alpha(n) \leq s \\ 0 & \text{otherwise.} \end{cases}$$

(b) Let $l = 1$ *and fix* $s \geq 0$. *Then*

$$L_s(M)_n = \begin{cases} \mathbb{F}_2 & s = 0 \text{ and } n = 2^k \text{ for } k \geq 1 \\ \mathbb{F}_2 & s \geq 1, n \geq 2 \text{ is even, and } (s+1) - \nu\left(\frac{n}{2}\right) \leq \alpha(n) \leq s+1 \\ 0 & \text{otherwise.} \end{cases}$$

(c) Let $l > 1$ *and fix* $s \geq 1$. *Then*

$$L_s(M)_n = \begin{cases} \mathbb{F}_2 & n \geq 2 \text{ is even and } (s+l) - \nu\left(\frac{n}{2}\right) \leq \alpha(n) \leq s+l \\ 0 & \text{otherwise.} \end{cases}$$

Proof: By Corollary 4.4.7, $L_s(M)_n \cong \mathrm{Ext}_{E(1)}^{s,s-2l}(\mathbb{F}_2, J(n))$ whenever $s \geq 1$. Furthermore, we only need to consider even dimensions. The theorem then ultimately follows from Proposition 4.4.5, although there are some edge cases to consider.

First let $l = 0$, so $M = \mathcal{A} /\!/ E(1)$. If $s = 0$, then $L_0(M) = \mathbb{F}_2$ concentrated in degree 0 (Lemma 3.1.4). Let $s \geq 1$. When $n = 0$, we have $\mathrm{Ext}_{E(1)}^{s,s}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathbb{F}_2$ by Proposition 4.1.6. For even $n \geq 2$, the result follows from Proposition 4.4.5, as the condition

$$s = 3s - 2k - \alpha(n) + 2, \quad \text{for some } 1 \leq k \leq \nu(n)$$

can be rewritten as

$$s - \nu\left(\frac{n}{2}\right) \leq \alpha(n) \leq s,$$

using the fact that $\nu\left(\frac{n}{2}\right) = \nu(n) - 1$ if n is even.

If $l = 1$, then $L_0(M) \cong \Phi F(1)$ (Lemma 4.4.2). Let $s \geq 1$. If $n = 0$, then $\mathrm{Ext}^{s,s-2}_{E(1)}(\mathbb{F}_2, \mathbb{F}_2) = 0$. If $n \geq 2$ is even, then the condition

$$s - 2 = 3s - 2k - 2\alpha(n) + 2, \quad \text{for some } 1 \leq k \leq \nu(n)$$

can be rewritten as

$$(s+1) - \nu\left(\frac{n}{2}\right) \leq \alpha(n) \leq s+1,$$

similarly to part (a).

Part (c) is similar to (b), except that we make no claim about $L_0(M)$. □

Note that $s \geq 1$, then the above theorem implies the relationship

$$L_s(\Sigma^{2l}\mathcal{A} /\!\!/ E(1))_* \cong L_{s+l}(\mathcal{A} /\!\!/ E(1))_*$$

for $* > 0$. This can be seen as being induced by the element $v_1 \in \mathrm{Ext}^{1,3}_{E(1)}(\mathbb{F}_2, \mathbb{F}_2)$.

We will now obtain a handle on which degrees are nonzero in terms of $\alpha(n)$ and $\nu(n)$, which we may sometimes refer to as the α/ν filtration. This is encapsulated in the following remark.

Remark 4.4.9. The condition on $\alpha(n), \nu(n)$, and s can be rephrased in several ways. One way is

$$(s+l) - (\nu(n) - 1) \leq \alpha(n) \leq (s+l),$$

for all l, and whenever $s \geq 1$. This can be rephrased again by saying $\alpha(n) = s+1-k+l$ for some $1 \leq k \leq \nu(n)$ (where n is even). Another is

$$\alpha(2n) \leq s \leq \nu(2n) + \alpha(2n) - 1.$$

Often we think of s as fixed, and we try to phrase the condition in terms of $\alpha(n)$, as this parameter is easiest to control.

Example 4.4.10. Following the previous remark, we illustrate $L_s(\mathcal{A}/\!\!/E(1))$ for some s.

(1) When $s = 0$, we have seen $L_0(\mathcal{A} /\!\!/ E(1))_* = \mathbb{F}_2$, concentrated in degree 0.

(2) When $s = 1$, we see that $L_1(\mathcal{A} /\!\!/ E(1))_{2n} = \mathbb{F}_2$ if and only if $n = 0$ or

$$2 - \nu(2n) \leq \alpha(2n) \leq 1.$$

This translates to $\alpha(2n) = 1$ and $\nu(2n) \geq 1$. Note that $\nu(2n) \geq 1$ is a non-condition for even numbers larger than 0. So $L_1(\mathcal{A} /\!\!/ E(1)) = \mathbb{F}_2$ for $2n = 0, 2, 4, 8, 16, \ldots, 2^k, \ldots$.

(3) When $s = 2$, we see that $L_2(\mathcal{A} /\!\!/ E(1))_{2n} = \mathbb{F}_2$ if and only if $n = 0$ or

$$3 - \nu(2n) \leq \alpha(2n) \leq 2.$$

This translates to (1) $\alpha(2n) = 1$ and $\nu(2n) \geq 2$, or (2) $\alpha(2n) = 2$ and $\nu(2n) \geq 1$. The first group consists of $2n = 4, 8, 16, \ldots$ and the second group consists of all even numbers with $\alpha = 2 : 6, 10, 12, 18, 20, \ldots$. Together with $n = 0$, these groups exhaust all non-trivial dimensions for $L_2(\mathcal{A} /\!\!/ E(1))$.

(4) In general for $L_s(\mathcal{A} /\!\!/ E(1))$, the condition

$$s - (\nu(2n) - 1) \leq \alpha(2n) \leq s$$

stratifies $2n$ into the following groups: $\alpha(2n) = 1$ and $\nu(2n) \geq s$, $\alpha(2n) = 2$ and $\nu(2n) \geq s - 1$, $\ldots$, $\alpha(2n) = s$ and $\nu(2n) \geq 1$, where we take note that $\nu(2n) \geq 1$ is the same as no condition for $n \neq 0$. When we include the $\mathbb{F}_2$ in dimension 0, this exhausts all non-zero dimensions for $L_s(\mathcal{A} /\!\!/ E(1))$.

Example 4.4.11. When $M = \Sigma^2\mathcal{A} /\!\!/ E(1)$, the conditions above all shift because $L_s(M) \cong L_{s+1}(\mathcal{A} /\!\!/ E(1))$. In this particular case, this is even true for $s = 0$ (except for in degree 0). For example, when $s = 1$, we have that $L_1(M)_{2n} = \mathbb{F}_2$ whenever $\alpha(2n) = 1$ and $\nu(2n) \geq 2$, or whenever $\alpha(2n) = 2$ and $\nu(2n) \geq 1$.

In general, fix $s \geq 0$. Then when $\alpha(n) = 1$, $L_s(M)_n$ will be nonzero if and only if $\nu\left(\frac{n}{2}\right) \geq s$ so $\nu(n) \geq s+1$. In this case, $L_s(M)_n \cong \mathbb{F}_2\{v_1^s u_{-2(s+1)}\}$. If $\alpha(n) = 2$, then $L_s(M)_n$ is nonzero if and only if $\nu(n) \geq s$. If $\alpha(n) = s+1$, then $\nu\left(\frac{n}{2}\right) \geq 0$ so $L_s(M)_n$ is nonzero if and only if $\nu(n) \geq 1$, which is actually not a constraint.

4.5 2-completed K-theory of Brown-Gitler spectra

It is worth noting that, as a corollary to our $L_s(M)$ computation for $M = \mathcal{A} /\!\!/ E(1)$, we obtain information about the 2-completed complex K-theory of Brown-Gitler spectra. This follows from the calculation in Proposition 4.4.5, where the E_2 term of the Adams spectral sequence

$$\mathrm{Ext}_A^{s,t}(\mathcal{A} /\!\!/ E(1), J(n)) \Rightarrow [T(n), \mathrm{ku}]_{t-s} = \mathrm{ku}^{s-t}(T(n))$$

is computed for $s \geq 1$. Moreover, the E_2 term is evenly graded. If n is even,

$$\mathrm{Ext}_{E(1)}^{s,t}(\mathbb{F}_2, J(n)) \cong \mathrm{Ext}_{E(1)}^{s,t+2\alpha(n)-2}(\mathbb{F}_2, \mathcal{P}^{2\nu(n)}),$$

which is only nonzero when $t = 3s - 2k - 2\alpha(n) + 2$ for some $1 \leq k \leq \nu(n)$, so only the $2(s-k)$ line is nonzero. This is similar for n odd noting the usual suspension isomorphism.

A calculation of the real K-theory of Brown-Gitler spectra was done in [30] using Hopf ring methods, and a more general result is proved in [31].

Chapter 5

Generating $L_s(\Sigma^2 \mathcal{A} /\!\!/ E(1))$

Let $M = (\Sigma^2 \mathcal{A} /\!\!/ E(1))^\vee$. In this chapter, we will show that $L_0(M) = \Omega^\infty(M) = \Phi F(1)$ generates $L_*(M)$ as a left module over $\mathcal{R}$ (Corollary 5.2.2), using an "α-filtration". In other words, the "algebraic condition" in Theorem 3.1.9 holds. Along with the "geometric condition" (Proposition 5.2.4), this shows that $U_Q(L_*M)$ (recall Definition 3.1.7) is an associated graded for $H_*(\Omega^\infty M) = H_*(BU)$, as an algebra over $\mathcal{R}$. The action by $\mathcal{R}$ was computed in [15] and [32]. Our methods retrieve some of these results, independently of their work.

The plan will be to combine the work of the previous two sections. We start with the general theory of Dyer-Lashof operations as extensions (Theorem 3.3.16). Then, using our computation of $L_*(M)$, we are able to show these extensions, when restricted to $E(1)$, are not split. Moreover, the Yoneda product with respect to these extensions yields an isomorphism, which we determine by explicitly constructing extensions. This is encapsulated in Theorem 5.2.1, and the proof in Section 5.3. In Sections 5.4 and 5.5, we give examples of how to use this technique to make explicit calculations with the Q^k.

5.1 Reducing Q^k to $E(1)$

Let $M = \Sigma^{2n}\mathcal{A} /\!\!/ E(1)$. In Chapter 3, we saw that the Dyer-Lashof operations on $L_*(M)$ can be interpreted as natural maps

$$\mathrm{Ext}_{\mathcal{A}}^{s,s-2n}(\mathcal{A}\otimes_{E(1)}\mathbb{F}_2, J(n)) \xrightarrow{Q_n^k} \mathrm{Ext}_{\mathcal{A}}^{s+1,s+1-2n}(\mathcal{A}\otimes_{E(1)}\mathbb{F}_2, J(n+k)),$$

via the Yoneda splice with $Q_n^k \in \mathrm{Ext}_{\mathcal{A}}^{1,1}(J(n), J(n+k))$. After a change of rings (Theorem 4.4.3), we want to instead think of these operations as maps

$$\mathrm{Ext}_{E(1)}^{s,s-2n}(\mathbb{F}_2, J(n)) \xrightarrow{Q_n^k} \mathrm{Ext}_{E(1)}^{s+1,s+1-2n}(\mathbb{F}_2, J(n+k))$$

induced by the restriction of Q_n^k to $E(1)$. The next proposition shows that this is valid.

Proposition 5.1.1. *The following diagram commutes:*

$$\begin{array}{ccc} \mathrm{Ext}_{\mathcal{A}}^{s,s}(\mathcal{A}\otimes_{E(1)}\mathbb{F}_2, J(n)) & \xrightarrow{Q_n^k} & \mathrm{Ext}_{\mathcal{A}}^{s+1,s+1}(\mathcal{A}\otimes_{E(1)}\mathbb{F}_2, J(n+k)) \\ \downarrow \cong & & \downarrow \cong \\ \mathrm{Ext}_{E(1)}^{s,s}(\mathbb{F}_2, J(n)) & \xrightarrow{Q_n^k|_{E(1)}} & \mathrm{Ext}_{E(1)}^{s+1,s+1}(\mathbb{F}_2, J(n+k)) \end{array}$$

where the modules $J(n)$ and $J(n+k)$ in the bottom row are restricted to $E(1)$.

Proof: The vertical isomorphisms are those in Theorem 4.4.3, and each factors as

$$\mathrm{Ext}_{\mathcal{A}}^{s,s}(\mathcal{A}\otimes_{E(1)}\mathbb{F}_2, -) \xrightarrow{\mathrm{res}} \mathrm{Ext}_{E(1)}^{s,s}(\mathcal{A}\otimes_{E(1)}\mathbb{F}_2, -) \xrightarrow{i^*} \mathrm{Ext}_{E(1)}^{s,s}(\mathbb{F}_2, -),$$

where $i\colon \mathbb{F}_2 \to (\mathcal{A}\otimes_{E(1)}\mathbb{F}_2)\big|_{E(1)}$ is the canonical $E(1)$-map sending $x \to 1\otimes x$ (see [4, p. 47]).

To show the diagram commutes, we first trace down then right. Suppose we have a length s extension of $\mathcal{A}$-modules, which we denote x_n:

$$\Sigma^s J(n) \longrightarrow M_s \longrightarrow \cdots \longrightarrow M_1 \longrightarrow \mathcal{A}\otimes_{E(1)}\mathbb{F}_2.$$

Let $\overline{x_n}$ denote the image of x_n under the left vertical isomorphism. Let $Z = (\mathcal{A} \otimes_{E(1)} \mathbb{F}_2)\big|_{E(1)}$ and first form the pullback

$$\begin{array}{ccc} M_1\big|_{E(1)} \underset{Z}{\times} \mathbb{F}_2 & \longrightarrow & \mathbb{F}_2 \\ \downarrow & & \downarrow i \\ M_1\big|_{E(1)} & \longrightarrow & Z \end{array}$$

Then $\overline{x_n}$ is given by

$$\Sigma^s J(n)\big|_{E(1)} \to M_s\big|_{E(1)} \to \cdots \to M_2\big|_{E(1)} \to M_1\big|_{E(1)} \underset{Z}{\times} \mathbb{F}_2 \to \mathbb{F}_2.$$

Now we apply the Yoneda splice with respect to $Q_n^k\big|_{E(1)}$, which is given by

$$\Sigma J(n+k)\big|_{E(1)} \longrightarrow Q(n,k)\big|_{E(1)} \longrightarrow J(n)\big|_{E(1)}.$$

We obtain $Q_n^k\big|_{E(1)} \circ \overline{x_n}$:

$$\Sigma^{s+1} J(n+k)\big|_{E(1)} \to \Sigma^s Q(n,k)\big|_{E(1)} \to M_s\big|_{E(1)} \to \cdots \to M_1\big|_{E(1)} \underset{Z}{\times} \mathbb{F}_2 \to \mathbb{F}_2.$$

On the other hand, we can trace right then down. First we get $Q_n^k \circ x_n$:

$$\Sigma^{s+1} J(n+k) \longrightarrow \Sigma^s Q(n,k) \longrightarrow M_s \longrightarrow \cdots \longrightarrow M_1 \longrightarrow \mathcal{A} \otimes_{E(1)} \mathbb{F}_2.$$

Then before applying the vertical isomorphism, we restrict to $E(1)$ and take the pullback at the end of the sequence with respect to i. This gives $M_1 \underset{Z}{\times} \mathbb{F}_2$ again, so that $\overline{Q_n^k \circ x_n}$ is given by:

$$\Sigma^{s+1} J(n+k)\big|_{E(1)} \to \Sigma^s Q(n,k)\big|_{E(1)} \to M_s\big|_{E(1)} \to \cdots \to M_1\big|_{E(1)} \underset{Z}{\times} \mathbb{F}_2 \to \mathbb{F}_2.$$

Since the splicing maps are the same, this is the same sequence as above, proving the lemma.

Note that this argument is for $s \geq 1$, and $s = 0$ can be shown similarly using the adjunction at the level of Hom. $\square$

Remark 5.1.2. In general, we will drop the $\big|_{E(1)}$ symbol if the context is clear.

Proposition 5.1.1 says that when we restrict to $\mathrm{Ext}_{E(1)}$ in studying $L_*(M)$ via change of rings, we may likewise restrict Q_n^k to an element of $\mathrm{Ext}^{1,1}_{E(1)}(J(n), J(n+k))$. To this end, the basic strategy for showing that $L_0(M)$ generates $L_*(M)$ is as follows. First, we will classify the groups $\mathrm{Ext}^{1,1}_{E(1)}(J(n), J(n+k))$ for the relevant n and k. Then, we will carefully study the extensions $Q(n,k)$ as extensions over $\mathcal{A}$. Using this, we will show that when we restrict these to $E(1)$, we (stably) end up with the extensions that we've classified over $E(1)$. These extensions will be explicitly given and we can determine their effect on $\mathrm{Ext}_{E(1)}$ groups.

Lemma 5.1.3. *Restriction to $E(1)$, being a left and right adjoint, commutes with pushouts, pullbacks, and tensor products.*

Remark 5.1.4. What one cannot do, however, is make a correspondence between $\mathrm{Hom}_{\mathcal{A}}$ and $\mathrm{Hom}_{E(1)}$. In particular, $\mathrm{Hom}_{\mathcal{A}}(M, J(n))$ identifies as the dual of M^n, for M unstable (or take $\Omega^\infty M$ for general M). But there can be more maps in $\mathrm{Hom}_{E(1)}(M, J(n))$ after restricting.

5.2 Verifying the algebraic and geometric conditions

In order to show that $L_*(M)$ gives an associated graded for $H_*(BU)$, we need to show the algebraic and geometric conditions hold in Theorem 3.1.9. The algebraic condition follows from the following theorem, whose proof is delayed until the next section.

Theorem 5.2.1. *Let n be even and let $2^l \geq n$. Then the operations $Q_n^{2^l}\big|_{E(1)}$ are nonzero, and moreover, each of these operations induces an isomorphism*

$$\mathrm{Ext}^{s,s-2}_{E(1)}(\mathbb{F}_2, J(n)) \longrightarrow \mathrm{Ext}^{s+1,(s+1)-2}_{E(1)}(\mathbb{F}_2, J(n+2^l))$$

for all $s \geq 0$ via the Yoneda splice.

Corollary 5.2.2. *If $M = \Sigma^2 \mathcal{A} /\!/ E(1)$, then $L_0(M)$ generates $L_*(M)$ as a left module over the Dyer-Lashof algebra.*

Proof: First note that, by Corollary 4.4.7, we can rephrase Theorem 5.2.1 by saying

$$L_s(M)_n \xrightarrow{Q_n^{2^l}} L_{s+1}(M)_{n+2^l}$$

is an isomorphism whenever $2^l \geq n$ and n is even. We will let x_d denote an element of degree d, which must be unique in this setting. Let $x_n \in L_s(M)_n$ be nonzero for a given $s \geq 1$. We will show that $x_n = Q^I y_m$ where $y_m \in L_0(M)$, by induction on s.

If $\alpha(n) = 1$, write $n = 2^k$. Then

$$L_{s-1}(M)_{2^{k-1}} \xrightarrow{Q_{2^{k-1}}^{2^{k-1}}} L_s(M)_{2^k}$$

is an isomorphism, and in particular, $L_{s-1}(M)_{2^{k-1}} \neq 0$. Thus $x_n = Q^{2^{k-1}} x_{2^{k-1}}$.

If $\alpha(n) > 1$, write $n = 2^{k_1} + \cdots + 2^{k_\alpha}$ where $k_1 < k_2 < \cdots < k_\alpha$. Then

$$L_{s-1}(M)_{n-2^{k_\alpha}} \xrightarrow{Q_{n-2^{k_\alpha}}^{2^{k_\alpha}}} L_s(M)_n$$

is an isomorphism, and in particular, $L_{s-1}(M)_{n-2^{k_\alpha}} \neq 0$. Thus $x_n = Q^{2^{k_\alpha}} x_{n-2^{k_\alpha}}$.

The corollary now follows by induction. □

Remark 5.2.3. As soon as we say that $x_n \in L_s(M)_n$ is nonzero, many conditions are automatically fixed. For instance, if $x_{2^k} \neq 0$ in $L_s(M)$, when we claim $Q^{2^{k-1}} x_{2^{k-1}} = x_{2^k}$, so we are implicitly claiming that $L_{s-1}(M)_{2^{k-1}} \neq 0$. This has to be true because, in this case, we must have $k \geq s+1$, and thus $k-1 \geq s$ which implies our claim. All of this is included in the statement of Theorem 5.2.1, and we will see it in detail during the proof of the theorem.

A question that still remains is whether $Q_n^n \neq 0$, and we conjecture that $Q(n,n)$ are not split over $E(1)$ as well. This would yield algebraic information in the associated graded, namely, that $Q^{|x|}x = x^2$.

Next, we prove the geometric condition, which is dual to the version in Theorem

3.1.9.

Proposition 5.2.4. *Let $X = \Sigma^2 ku$. Then the evaluation map $\Sigma^\infty \Omega^\infty X \to X$ induces a monomorphism*

$$\Omega^\infty H^*(X) \hookrightarrow H^*(\Omega^\infty X).$$

Proof: There is a cofibration sequence [1, p. 336] $\Sigma^2 ku \to ku \to H\mathbb{Z}$, which induces the diagram

$$\begin{array}{ccc} \Sigma^\infty BU & \longrightarrow & \Sigma^2 ku \\ \downarrow & & \downarrow \\ \Sigma^\infty \mathbb{C}P^\infty & \longrightarrow & \Sigma^2 H\mathbb{Z} \end{array}$$

since the co-unit $\Sigma^\infty \Omega^\infty \Rightarrow 1$ is a natural transformation, $\Omega^\infty \Sigma^2 ku = BU$, and $\Omega^\infty \Sigma^2 H\mathbb{Z} = \mathbb{C}P^\infty$. Note that $H^*(ku) \cong \mathcal{A} /\!/ E(1)$ and $H^*(H\mathbb{Z}) \cong \mathcal{A} /\!/ \mathrm{E}(0)$. Furthermore, the map $\Sigma^2 \mathcal{A} /\!/ \mathrm{E}(0) \to \Sigma^2 \mathcal{A} /\!/ E(1)$ is a surjection because of the above cofibration sequence, and since Ω^∞ is right exact as a cohomological functor, we obtain a surjection $\Omega^\infty \Sigma^2 \mathcal{A} /\!/ \mathrm{E}(0) \to \Omega^\infty \Sigma^2 \mathcal{A} /\!/ E(1)$. Therefore we have the following diagram after applying $H^*(-)$ to the previous:

$$\begin{array}{ccc} H^*(BU) & \longleftarrow & \Omega^\infty \Sigma^2 \mathcal{A} /\!/ E(1) \\ \uparrow & & \uparrow \\ H^*(\mathbb{C}P^\infty) & \longleftarrow & \Omega^\infty \Sigma^2 \mathcal{A} /\!/ \mathrm{E}(0) \end{array}$$

where the right map is a surjection. The left map is an injection because $BU \simeq \mathbb{C}P^\infty \times BSU$ so $\mathbb{C}P^\infty$ splits off BU because of the complex orientation. Finally, it is classical that the bottom map is an injection. It follows that the top map must be injective as well. □

Remark 5.2.5. Recalling that $ku = BP\langle 1 \rangle$, a similar argument shows that the geometric condition holds for $\Sigma^2 BP\langle n \rangle$.

5.3 Proof of Theorem 5.2.1

The strategy for proving the theorem is as follows. First we show that when we restrict the extensions $Q_n^{2^l}$ to $E(1)$, we obtain a non-split extension, i.e., a nonzero element in $\mathrm{Ext}_{E(1)}$. Then we show that the groups $\mathrm{Ext}^{1,1}_{E(1)}(J(n), J(n+2^l))$ have a single generator, and hence it must be equivalent $Q^{2^l}_n\big|_{E(1)}$. Lastly, by giving an explicit representation of this extension stably over $E(1)$, we show that the Yoneda product induced by it yields the desired isomorphism on $\mathrm{Ext}_{E(1)}$.

Recall from Theorem 3.3.16 that we use the notation

$$J(n+k) \longrightarrow Q(n,k) \longrightarrow \Sigma^{-1}J(n)$$

for the extension Q^k_n, which is obtained via the pushout:

$$\begin{array}{ccccc} \mathcal{P}^\infty_0 \otimes J(n) & \longrightarrow & \mathcal{P}^\infty_{-1} \otimes J(n) & \longrightarrow & \Sigma^{-1}J(n) \\ \downarrow{\scriptstyle q(n,k)} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle =} \\ J(n+k) & \longrightarrow & Q(n,k) & \xrightarrow{g} & \Sigma^{-1}J(n). \end{array} \tag{5.3.1}$$

We will sometimes abuse notation and call this extension $Q(n,k)$. This extension is always non-split over $\mathcal{A}$ as long as $k \geq n-1$.

As we need to make several $\mathcal{A}$-module calculations, we recall some facts about the algebraic properties of the Brown-Gitler modules $J(n)$. In particular, recall Theorem 4.2.4.

It follows from this theorem that $\mathrm{Sq}(x_i) = x_i + x^2_{i-1}$, and that $\mathrm{Sq}^k(x^m_0) = 0$ for any $m, k > 0$. Since $J(*)^*$ is an $\mathcal{A}$-algebra, one can determine the action of $\mathcal{A}$ on products of elements using Definition 3.3.11. Recall that $\mathcal{P}^\infty_0 = H^*(\mathbb{R}P^\infty)$. We let t^k denote the generator in dimension k for $k \geq 0$. We extend this denotation to $\mathcal{P}^\infty_{-1}$ by letting t^{-1} represent the bottom class. It is well-known [27] that $\mathrm{Sq}^k(t^n) = \binom{n}{k}t^{n+k}$. From this and the above theorem, and the fact that the $J(n)$ are unstable modules, one can quickly make the following calculations.

Lemma 5.3.2. *(a) $\mathrm{Sq}^{2^k}(x^{2^k}_i) = x^{2^{k+1}}_{i-1}$, and moreover, this is the only non-identity operation that $x^{2^k}_i$ supports (when $i > 0$).*

(b) $\mathrm{Sq}^{2^k}(t^{2^k}) = t^{2^{k+1}}$, and moreover, this is the only non-identity operation that t^{2^k} supports.

(c) $Q_1(x_i) = x_{i-2}^4$

(d) $Q_1(t^{2k}) = 0$

(e) $Q_1(t^{2k+1}) = t^{2k+4}$

We need a definition and some notation before proving the next lemma.

Definition 5.3.3. (a) We let $\mathrm{Sq}_0(x) = \mathrm{Sq}^{|x|}(x)$ and call this operation *squaring*.

(b) Given $t^k \otimes x \in \mathcal{P}_{-1}^{\infty} \otimes J(n)$, write $x = x_0^m y$ where y contains no factors of x_0. Then we let $\widehat{\mathrm{Sq}_0}(t^k \otimes x) = \mathrm{Sq}^{k+|y|}(t^k \otimes x)$, and we call this operation *reduced squaring*.

It is worth pointing out that the operation obtained via reduced squaring depends on the element it is being applied to, even if the degree is the same. Reduced squaring ignores copies of x_0^m, which don't support any Steenrod square, while squaring the remaining terms. Recall that

$$\Delta \mathrm{Sq}^k = \sum_{i+j=k} \mathrm{Sq}^i \mathrm{Sq}^j,$$

and Sq^k acts on $\mathcal{P}_{-1}^{\infty} \otimes J(n)$ as described in Definition 3.3.11. Then, for example, $\widehat{\mathrm{Sq}_0}(t^2 \otimes x_1^2 x_2^2) = \mathrm{Sq}^6(t^2 \otimes x_1^2 x_2^2) = t^4 \otimes x_0^4 x_1^4$ and $\widehat{\mathrm{Sq}_0}(t^2 \otimes x_0^2 x_2^2) = \mathrm{Sq}^4(t^2 \otimes x_0^2 x_2^2) = t^4 \otimes x_0^2 x_1^4$.

We will need the following technical lemma.

Lemma 5.3.4. *Consider the element $u = t^{2^k} \otimes x_0^j x_{i_1}^{2^k} \dots x_{i_m}^{2^k}$, where $1 \leq i_1 < \cdots < i_m$. If we let θ denote $\widehat{\mathrm{Sq}_0}$ iterated i_m times, then $\theta u = t^{2^{k+i_m}} \otimes x_0^{j+2^k(2^{i_1}+\cdots+2^{i_m})}$.*

Proof: We proceed by induction on m. When $m = 1$, $u = t^{2^k} \otimes x_0^j x_i^{2^k}$ where $i \geq 1$. Note that $\widehat{\mathrm{Sq}_0}(u) = t^{2^{k+1}} \otimes x_0^j x_{i-1}^{2^{k+1}}$ by Lemma 5.3.2, and the fact that all elements belong to *unstable* modules. In general, one can check that $(\mathrm{Sq}_0)^i(x_i^{2^k}) = x_0^{2^{k+i}}$, where $(\mathrm{Sq}_0)^i$ denotes i composites of Sq_0. It follows that

$$(\widehat{\mathrm{Sq}_0})^i(u) = (\mathrm{Sq}_0)^i(t^{2^k}) \otimes x_0^j (\mathrm{Sq}_0)^i(x_i^{2^k}) = t^{2^{k+i}} \otimes x_0^{j+2^{k+i}}$$

which proves the base case.

Now let $m \geq 2$ be given and assume the claim is true for $m' < m$. Reduced squaring yields

$$\widehat{\mathrm{Sq}_0}(u) = t^{2^{k+1}} \otimes x_0^j x_{i_1-1}^{2^{k+1}} \dots x_{i_m-1}^{2^{k+1}}$$

and similarly as above, we have $(\mathrm{Sq}_0)^{i_1}(x_{i_1}^{2^k}) = x_0^{2^{k+i_1}}$. It follows that

$$(\widehat{\mathrm{Sq}_0})^{i_1}(u) = (\mathrm{Sq}_0)^{i_1}(t^{2^k}) \otimes x_0^j (\mathrm{Sq}_0)^{i_1}(x_{i_1}^{2^k} \dots x_{i_m}^{2^k}) = t^{2^{k+i_1}} \otimes x_0^{j+2^{k+i_1}} x_{i_2-i_1}^{2^{k+i_1}} \dots x_{i_m-i_1}^{2^{k+i_1}}.$$

Using the induction hypothesis, we then have

$$\begin{aligned}(\widehat{\mathrm{Sq}_0})^{i_m-i_1}((\widehat{\mathrm{Sq}_0})^{i_1}(u)) &= t^{2^{k+i_1+(i_m-i_1)}} \otimes x_0^{(j+2^{k+i_1})+2^{k+i_1}(2^{i_2-i_1}+\dots+2^{i_m-i_1})} \\ &= t^{2^{k+i_m}} \otimes x_0^{j+2^k(2^{i_1}+\dots+2^{i_m})},\end{aligned}$$

as desired. □

Lemma 5.3.5. *Let $n = 2^{k_1} + \dots + 2^{k_\alpha}$ where $1 \leq k_1 < \dots < k_\alpha$, and let $2^l > n$. Then the extension $Q(n, 2^l)$, restricted to $E(1)$, is not split.*

Proof: Suppose there is an $E(1)$-map $f \colon \Sigma^{-1}J(n) \hookrightarrow Q(n, 2^l)$ such that $gf = 1$ in the diagram 5.3.1. Since $\mathcal{P}_{-1}^{\infty} \otimes J(n) \to \Sigma^{-1}J(n)$ maps $t^{-1} \otimes x \mapsto \sigma^{-1}x$, it is also true that $g(p(t^{-1} \otimes x)) = \sigma^{-1}x$, for any $x \in J(n)$.

Let $y = x_{k_1-1}^2 x_{k_2-1}^2 \dots x_{k_\alpha-1}^2$. This element is in the Margolis homology of $J(n)$ with respect to Q_1, and as such, $Q_1 y = 0$. Since f is an $E(1)$-map, we must have $Q_1 f(\sigma^{-1}y) = f(Q_1\sigma^{-1}y) = 0$. We will show that $Q_1 f(\sigma^{-1}y) \neq 0$, a contradiction. Since $gf = 1$, we deduce that $f(\sigma^{-1}y) = p(t^{-1} \otimes y) + z$ for some $z \in Q(n, 2^l)$. Here, z can be thought of an element of $\ker(g) \cong J(n + 2^l)$. Therefore $Q_1(t^{-1} \otimes y) = t^2 \otimes y$ and thus $Q_1 p(t^{-1} \otimes y) = p(t^2 \otimes y)$.

We claim that $p(t^2 \otimes y) \neq 0$. The idea is to "square our way to the top class," which must be nonzero. To this end, let θ denote the composite of $k_\alpha - 1$ iterations of $\widehat{\mathrm{Sq}_0}$. We claim $\theta(t^2 \otimes y) = t^{2^{k_\alpha}} \otimes x_0^n$. If $k_\alpha > 1$ then this follows from Lemma 5.3.4

for $\alpha(n) \geq 1$ since

$$\theta(t^2 \otimes y) = t^{2^{k_\alpha}} \otimes x_0^{2^{k_1}+\cdots+2^{k_\alpha}} = t^{2^{k_\alpha}} \otimes x_0^n,$$

which is true even if $k_1 = 1$. If we interpret 0 iterations of $\widehat{\mathrm{Sq}_0}$ as being $\mathrm{Sq}^0 = 1$, then this statement is true when $k_\alpha = 1$ as well, because in this case $n = 2$, so $t^2 \otimes y = t^2 \otimes x_0^2$. Therefore, if we let $\Theta = (\widehat{\mathrm{Sq}_0})^{l-k_\alpha} \circ \theta$, then $\Theta(t^2 \otimes y) = t^{2^l} \otimes x_0^n$. Next, by definition we have $q(n, 2^l)(t^{2^l} \otimes x_0^n) = x_0^{n+2^l}$ since $e(n, 2^l, 0)(t^{2^l} \otimes x_0^n) = x_0^{n+2^l}$ (see Definition 3.3.15). This means that in the pushout $Q(n, 2^l)$, we must have $p(t^{2^l} \otimes x_0^n) = x_0^{n+2^l} \neq 0$. But this implies $\Theta p(t^2 \otimes y) \neq 0$ because p is an $\mathcal{A}$-map, and thus $p(t^2 \otimes y) \neq 0$ as well.

Finally, we claim $\Theta(Q_1(z)) = 0$, where $z \in J(n+2^l)$ is the element above in degree $|\sigma^{-1}y| = 2\alpha(n) - 1$. If $Q_1(z) \neq 0$, then it is a sum of elements of the form $x_{k-2}^4 z'$ where $2 \leq k \leq l$. This follows from the action of Q_1 on $J(n)$ (Lemma 5.3.2) along with the fact that Q_1 is a derivation. We will show $\Theta(x_{k-2}^4 z') = 0$ for any such k, by first proving the following lemma for $k = l$.

Lemma 5.3.6. *Let Θ be the operation obtained by $l-1$ iterations of $\widehat{\mathrm{Sq}_0}$ applied to $t^2 \otimes y$. Then $\Theta(x_{l-2}^4 z') = 0$ if $|z'| = 2\alpha(n) - 2$.*

Proof: We first point out a possible piece of confusion, that Θ is obtained by $\widehat{\mathrm{Sq}_0}$ applied to $t^2 \otimes y$, which will generally *not* yield the same operation as $\widehat{\mathrm{Sq}_0}$ applied to $x_{l-2}^4 z'$. Recall that the only operations $x_{l-2}^{2^k}$ support are Sq^{2^k} (squaring) and Sq^0 (identity), and the same is true for t^{2^k}. Set $\alpha = \alpha(n)$.

First suppose $k_1 > 1$. Then the first operation in Θ is $\mathrm{Sq}_0 = \mathrm{Sq}^{(\alpha+1)(2)}$ since there are no copies of x_0^2 in y. Then $\mathrm{Sq}^{(\alpha+1)(2)}(x_{l-2}^4 z') = x_{l-3}^8(\mathrm{Sq}_0(z'))$ because we are working with unstable modules, so e.g. $\mathrm{Sq}^{(\alpha+1)(2)}(z') = 0$. Thus the only way to "distribute" $\mathrm{Sq}^{(\alpha+1)(2)}$ to yield a (possibly) nonzero element is in this way, i.e., as $\mathrm{Sq}^4 \otimes \mathrm{Sq}^{(\alpha-1)(2)}$.

Now, one can see that Θ begins

$$\Theta_{k_1} := \mathrm{Sq}^{(\alpha+1)(2^{k_1-1})} \ldots \mathrm{Sq}^{(\alpha+1)(4)} \mathrm{Sq}^{(\alpha+1)(2)}$$

at which point $\Theta_{k_1}(t^2 \otimes y) = t^{2^{k_1}} \otimes x_0^{2^{k_1}} x_{k_2-k_1}^{2^{k_1}} \dots x_{k_\alpha - k_1}^{2^{k_1}}$, and therefore the next operation of Θ is $\mathrm{Sq}^{\alpha 2^{k_1}}$. One can check that, for degree reasons again,

$$\Theta_{k_1}(x_{l-2}^4 z') = x_{l-(k_1+1)}^{2^{k_1+1}}(\delta z')$$

for some operation $\delta \in \mathcal{A}$. Furthermore,

$$|\Theta_{k_1}| = (2 + 4 + \dots + 2^{k_1 - 1})(\alpha + 1) = (2^{k_1} - 2)(\alpha + 1)$$

so $|\Theta_{k_1}(x_{l-2}^4 z')| = 2\alpha + 2 + (2^{k_1} - 2)(\alpha + 1)$ whence $|\delta z'| = 2^{k_1}(\alpha - 1)$. Therefore, when we apply $\mathrm{Sq}^{\alpha 2^{k_1}}$, the only decomposition that can yield a nonzero element is $\mathrm{Sq}^{2^{k_1+1}} \otimes \mathrm{Sq}^{(\alpha-2)(2^{k_1})}$.

This pattern continues. Because of degree and instability, every operation of Θ must square our leading factor while applying an unknown but irrelevant operation to the remaining factors. After iterating this, we come to

$$\mathrm{Sq}^{(2)(2^{k_\alpha - 1})}(x_{l-k_\alpha}^{2^{k_\alpha}}(...)) = x_{l-(k_\alpha+1)}^{2^{k_\alpha+1}}(...)$$

and the next operation in Θ is $\mathrm{Sq}^{2^{k_\alpha}}$. At this point, this operation is too small for $x_{l-(k_\alpha+1)}^{2^{k_\alpha+1}}$ to support, yet too big for the other factors (...) to support, hence it must be 0.

If $k_1 = 1$, the argument changes slightly. The first operation is $\mathrm{Sq}^{(\alpha)(2)}$ rather than $\mathrm{Sq}^{(\alpha+1)(2)}$ so the only decomposition that can yield a nonzero element is $\mathrm{Sq}^4 \otimes \mathrm{Sq}^{(\alpha-2)(2)}$. Since the degree of the non-leading factor did not matter, the rest of the arguments carry out similarly. □

We claimed that $\Theta(x_{k-2}^4 z') = 0$. The proof of the lemma shows that if $k \leq l$, then there would be a step where we reach $x_{l-k}^{2^k}(z'')$, where the next operation annihilates z'' and squares $x_{l-k}^{2^k}$. But if we follows this argument with k instead of l, then we reach $x_0^{2^k}(z'')$, and the next operation must annihilate the entire term.

Recall that $f(\sigma^{-1}y) = p(t^{-1} \otimes y) + z$, so $Q_1(f(\sigma^{-1}y)) = p(t^2 \otimes y) + Q_1 z$. We have therefore shown $\Theta(Q_1(f(\sigma^{-1}y))) \neq 0$ in $Q(n, 2^l)$, and hence $Q_1(f(\sigma^{-1}y)) \neq 0$, so f

cannot be $E(1)$-linear. Thus this extension cannot be split. $\square$

Lemma 5.3.7. *The extension $Q(2^k, 2^k)$, restricted to $E(1)$, is not split.*

Proof: Suppose there is an $E(1)$-map $f\colon \Sigma^{-1}J(2^k) \hookrightarrow Q(2^k, 2^k)$ such that $gf = 1$ in the diagram 5.3.1. Note that x_{k-1}^2 is the bottom Margolis homology generator for $J(2^k)$ with respect to Q_1. Because of degree and the commutativity of the diagram, we must have either $f(\sigma^{-1}x_{k-1}^2) = p(t^{-1} \otimes x_{k-1}^2)$ or $f(\sigma^{-1}x_{k-1}^2) = p(t^{-1} \otimes x_{k-1}^2) + x_{k+1}$. We claim the latter case cannot happen. If so, then

$$0 = f(\mathrm{Sq}^1\,\sigma^{-1}x_{k-1}^2) = \mathrm{Sq}^1\,f(\sigma^{-1}x_{k-1}^2) = p(t^0 \otimes x_{k-1}^2) + x_k^2$$

so $p(t^0 \otimes x_{k-1}^2) = x_k^2$. But then

$$\mathrm{Sq}^{2^k} \cdots \mathrm{Sq}^4\,\mathrm{Sq}^2(p(t^0 \otimes x_{k-1}^2)) = 0,$$

whereas

$$\mathrm{Sq}^{2^k} \cdots \mathrm{Sq}^4\,\mathrm{Sq}^2(x_k^2) = x_0^{2^{k+1}}$$

which is nonzero in $Q(2^k, 2^k)$ as $J(2^{k+1}) \hookrightarrow Q(2^k, 2^k)$, a contradiction.

Therefore, $f(\sigma^{-1}x_{k-1}^2) = p(t^{-1} \otimes x_{k-1}^2)$ so $Q_1 f(\sigma^{-1}x_{k-1}^2) = p(t^2 \otimes x_{k-1}^2)$. If we let Θ denote $k-1$ iterations of $\widehat{\mathrm{Sq}_0}$, then $\Theta(t^2 \otimes x_{k-1}^2) = t^{2^k} \otimes x_0^{2^k}$ by Lemma 5.3.4. Furthermore, $p(t^{2^k} \otimes x_0^{2^k}) \neq 0$ because $q(2^k, 2^k)(t^{2^k} \otimes x_0^{2^k}) \neq 0$, and thus $p(t^2 \otimes x_{k-1}^2) \neq 0$. But $Q_1(\sigma^{-1}x_{k-1}^2) = 0$ so f cannot be $E(1)$-linear, and so no splitting of $Q(2^k, 2^k)$ over $E(1)$ can exist. $\square$

The next step is to show that $\mathrm{Ext}_{E(1)}^{1,1}(J(n), J(n+2^l))$ is one-dimensional. This will follow from the next two lemmas.

Lemma 5.3.8. *Let $m \geq 1$. Then*

$$\mathrm{Ext}_{E(1)}^{1,3-2k}(\mathcal{P}^2, \mathcal{P}^{2m}) = \begin{cases} 0 & 0 \leq k \leq m-2 \\ \mathbb{F}_2 & k = m-1 \end{cases}$$

Proof: Applying $\mathrm{Hom}^{3-2k}(-, \mathcal{P}^{2m})$ to the extension h_0 (4.3.4) and using Theorem 4.3.1, we obtain an exact sequence

$$\mathrm{Ext}_{E(1)}^{1,3-2k}(\mathcal{P}^2, \mathcal{P}^{2m}) \overset{p^*}{\hookrightarrow} \mathrm{Ext}_{E(1)}^{1,3-2(k+1)}(\mathbb{F}_2, \mathcal{P}^{2m}) \xrightarrow{h_0} \mathrm{Ext}_{E(1)}^{2,6-2(k+2)}(\mathbb{F}_2, \mathcal{P}^{2m}).$$

If the last group is nonzero, it is generated by $v_1^2 u_{-2(k+2)}$, in which case $1 \leq k+2 \leq m$ so $-1 \leq k \leq m-2$. If the middle group is nonzero, it is generated by $v_1 u_{-2(k+1)}$, in which case $1 \leq k+1 \leq m$ so $0 \leq k \leq m-1$. If both of these conditions are met, then h_0 is an isomorphism because $v_1 h_0 u_{-2(k+1)} = v_1^2 u_{-2(k+2)}$. Thus $p^* = 0$ if $0 \leq k \leq m-2$.

If $k+1 = m$, then the middle group is $\mathbb{F}_2\{v_1 u_{-2m}\}$ and so h_0 is 0, meaning p^* is an isomorphism, which proves the lemma. □

Lemma 5.3.9. *Let* $m \geq 1$. *Then* $\mathrm{Ext}_{E(1)}^{1,3}(\Sigma^{2k}\mathcal{P}^{2m-2k}, \mathcal{P}^{2m}) \cong \mathbb{F}_2$ *for* $0 \leq k \leq m-1$.

Proof: We will first show these groups are at most one-dimensional as $\mathbb{F}_2$-vector spaces. To this end, apply $\mathrm{Hom}^{3-2k}(-, \mathcal{P}^{2m})$ to the sequence in Lemma 4.3.4(b), with $\mathcal{P}^{2m-2k}$ as the middle term, to obtain an exact sequence

$$\mathrm{Ext}_{E(1)}^{1,3-2k}(\mathcal{P}^2, \mathcal{P}^{2m}) \to \mathrm{Ext}_{E(1)}^{1,3-2k}(\mathcal{P}^{2m-2k}, \mathcal{P}^{2m}) \to \mathrm{Ext}_{E(1)}^{1,3-2(k+1)}(\mathcal{P}^{2m-2(k+1)}, \mathcal{P}^{2m}).$$

By Lemma 5.3.8, the first group is 0 if $0 \leq k \leq m-2$, in which case the second map is an injection. When $k = m-2$, the injection is

$$\mathrm{Ext}_{E(1)}^{1,3-2(m-2)}(\mathcal{P}^4, \mathcal{P}^{2m}) \hookrightarrow \mathrm{Ext}_{E(1)}^{1,3-2(m-1)}(\mathcal{P}^2, \mathcal{P}^{2m}) \cong \mathbb{F}_2,$$

where the isomorphism is from Lemma 5.3.8. Phrased differently, we see

$$\begin{aligned}
\mathrm{Ext}^{1,3}_{E(1)}(\mathcal{P}^{2m},\mathcal{P}^{2m}) &\hookrightarrow \mathrm{Ext}^{1,1}_{E(1)}(\mathcal{P}^{2m-2},\mathcal{P}^{2m})\\
&\hookrightarrow \mathrm{Ext}^{1,-1}_{E(1)}(\mathcal{P}^{2m-4},\mathcal{P}^{2m})\\
&\hookrightarrow \dots\\
&\hookrightarrow \mathrm{Ext}^{1,3-2(m-1)}_{E(1)}(\mathcal{P}^{2},\mathcal{P}^{2m})\\
&\cong \mathbb{F}_2,
\end{aligned}$$

which shows that the the groups in the statement of the lemma are each at most one-dimensional.

To show that each group is exactly $\mathbb{F}_2$, we show $\mathrm{Ext}^{1,3}_{E(1)}(\mathcal{P}^{2m},\mathcal{P}^{2m}) \cong \mathbb{F}_2$ by constructing a non-split extension; namely, there is an extension:

$$\Sigma^3\mathcal{P}^{2m} \xrightarrow{f} \bigoplus_{i=1}^{m} \Sigma^{2i-1}E(1) \xrightarrow{g} \mathcal{P}^{2m}$$

defined as follows. Let $\{y_4, y_5, \dots, y_{2m+3}\}$ denote the generators of $\Sigma^3\mathcal{P}^{2m}$, let

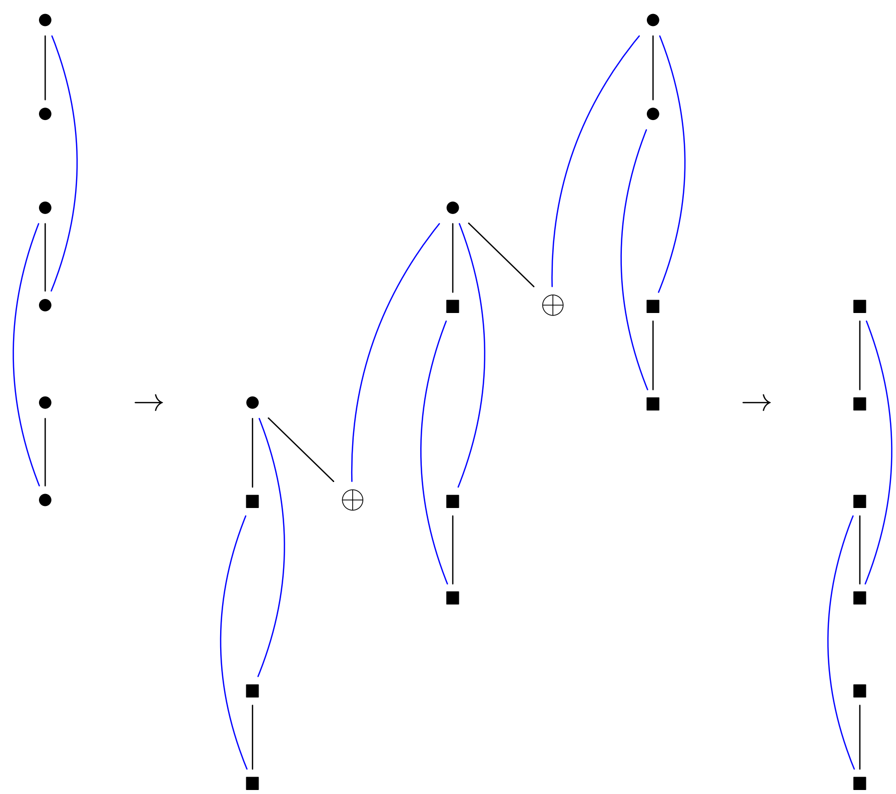
$$\{\sigma^{2i-1}x_0,\ \sigma^{2i-1}Q_0x_0,\ \sigma^{2i-1}Q_1x_0,\ \sigma^{2i-1}Q_0Q_1x_0 \mid 1 \le i \le m\}$$

denote the generators of $\bigoplus_{i=1}^{m}\Sigma^{2i-1}E(1)$, and let $\{z_1, z_2, \dots, z_{2m}\}$ denote the generators of $\mathcal{P}^{2m}$. Then we define f and g as

$$f(y_{2i}) = \begin{cases} \sigma^{2i-3}Q_1x_0 + \sigma^{2i-1}Q_0x_0 & \text{if } i \neq m+1 \\ \sigma^{2m-1}Q_1x_0 & \text{if } i = m+1 \end{cases}$$

and $g(\sigma^{2i-1}x_0) = z_{2i-1}$. The rest f and g are determined by uniquely extending these to $\mathcal{A}$-maps, and this extension is easily seen to be non-split as, e.g., the top class z_{2m} must map back to something that supports some non-trivial action. See Example 5.3.10 for $m = 3$. □

Example 5.3.10 (The extension in Lemma 5.3.9 when $m = 3$).



Here, cells are represented by circles or squares. The circles get sent to circles (including the $\oplus$ symbol denoting the sum of the generators), and squares get sent to squares.

Corollary 5.3.11. *Let $n \geq 2$ be even and $2^l \geq n$. Then $\mathrm{Ext}^{1,1}_{E(1)}(J(n), J(n+2^l)) \cong \mathbb{F}_2$.*

Proof: By Theorem 4.2.9, we have $J(n) \underset{E(1)}{\sim} \Sigma^{2\alpha(n)-2}\mathcal{P}^{2\nu(n)}$.

If $2^l > n$, then $\alpha(n + 2^l) = \alpha(n) + 1$ and $\nu(n + 2^l) = \nu(n)$, whence

$$J(n + 2^l) \underset{E(1)}{\sim} \Sigma^{2\alpha(n)}\mathcal{P}^{2\nu(n)}.$$

Therefore

$$\mathrm{Ext}^{1,1}_{E(1)}(J(n), J(n + 2^l)) \cong \mathrm{Ext}^{1,3}_{E(1)}(\mathcal{P}^{2\nu(n)}, \mathcal{P}^{2\nu(n)})$$

which is $\mathbb{F}_2$ by Lemma 5.3.9.

If $2^l = n$, then $\alpha(n+2^l) = \alpha(n) = 1$ and $\nu(n+2^l) = \nu(n)+1$. Thus $J(n) \underset{E(1)}{\sim} \mathcal{P}^{2l}$

and $J(n+2^l) \underset{E(1)}{\sim} \mathcal{P}^{2l+2}$. Therefore

$$\mathrm{Ext}^{1,1}_{E(1)}(J(n), J(n+2^l)) \cong \mathrm{Ext}^{1,1}_{E(1)}(\mathcal{P}^{2l}, \mathcal{P}^{2l+2})$$

which is $\mathbb{F}_2$ by the same lemma (note that this is true for all $l \geq 1$ due to a shift). $\square$

To finish the proof of the theorem, we exhibit representatives for $\mathrm{Ext}^{1,1}_{E(1)}(J(n), J(n+2^l))$, which depend on whether $2^l > n$ or $2^l = n$.

Case 1. Suppose $2^l > n$.

If $s \geq 1$, we have

$$\begin{array}{ccc} \mathrm{Ext}^{s,s-2}_{E(1)}(\mathbb{F}_2, J(n)) & \cong & \mathrm{Ext}^{s,s-2+(2\alpha(n)-2)}_{E(1)}(\mathbb{F}_2, \mathcal{P}^{2\nu(n)}) \\ \downarrow Q_n^{2^l} & & \downarrow Q_n^{2^l} \\ \mathrm{Ext}^{s+1,(s+1)-2}_{E(1)}(\mathbb{F}_2, J(n+2^l)) & \cong & \mathrm{Ext}^{s+1,(s+1)-2+2\alpha(n)}_{E(1)}(\mathbb{F}_2, \mathcal{P}^{2\nu(n)}) \end{array} \tag{5.3.12}$$

where the vertical maps are $Q_n^{2^l}$ restricted to $E(1)$. A representative for this extension is given in Lemma 5.3.9:

$$\Sigma^3\mathcal{P}^{2m} \longrightarrow \bigoplus_{i=1}^{m} \Sigma^{2i-1}E(1) \longrightarrow \mathcal{P}^{2m}.$$

The middle term is free over $E(1)$, hence injective (Theorem 4.1.3), so the long exact sequence associated to $\mathrm{Hom}^{-2+(2\alpha-2)}_{E(1)}(\mathbb{F}_2, -)$ implies that the right vertical map in 5.3.12 is an isomorphism for $s \geq 1$.

When $s = 0$, we are in the case of $\alpha(n) = 1$ (Theorem 4.4.8), so $n = 2^k$. The following lemma states that we can trade in $J(2^k)$ for $\mathcal{P}^{2k}$ even at the "non-stable" $\mathrm{Ext}^0 = \mathrm{Hom}$ level.

Lemma 5.3.13. *There is an isomorphism*

$$\mathrm{Hom}_{E(1)}(\Sigma^2\mathbb{F}_2, J(2^k)) \xleftarrow{\cong} \mathrm{Hom}_{E(1)}(\Sigma^2\mathbb{F}_2, \mathcal{P}^{2k})$$

induced by the inclusion $\mathcal{P}^{2k} \hookrightarrow J(2^k)$.

Proof: Recall that $\mathcal{P}^{2k} \hookrightarrow J(2^k)$ is a stable $E(1)$-equivalence where the bottom class of $\mathcal{P}^{2k}$ is identified the bottom class of $J(2^k)$. There is a nonzero map $\Sigma^2\mathbb{F}_2 \to J(2^k)$ hitting $x_{k-1}^2 \in J(2^k)$ because x_{k-1}^2 supports no non-trivial operations, and moreover, this is the only such nonzero map. The same is true for $\mathcal{P}^{2k}$ and the lemma follows. $\square$

Thus (5.3.12) is still valid for $s = 0$. Returning the long exact sequence above, we have

$$\mathrm{Hom}^{2\alpha-4}_{E(1)}(\mathbb{F}_2, \bigoplus_{i=1}^{m} \Sigma^{2i-1}E(1)) = 0$$

because a nonzero $E(1)$-map $f\colon \Sigma^{4-2\alpha}\mathbb{F}_2 \to \Sigma^{2i-1}E(1)$ must hit an element in even degree which supports no non-trivial operation. One can quickly check there are no such elements. Therefore, the same long exact sequence shows that we have an isomorphism for $s = 0$ as well.

Case 2. Suppose $n = 2^k = 2^l$.

We have

$$\begin{array}{ccc} \mathrm{Ext}^{s,s-2}_{E(1)}(\mathbb{F}_2, J(2^k)) & \cong & \mathrm{Ext}^{s,s-2}_{E(1)}(\mathbb{F}_2, \mathcal{P}^{2k}) \\ \downarrow Q^{2^k}_{2^k} & & \downarrow Q^{2^k}_{2^k} \\ \mathrm{Ext}^{s+1,(s+1)-2}_{E(1)}(\mathbb{F}_2, J(2^{k+1})) & \cong & \mathrm{Ext}^{s+1,(s+1)-2}_{E(1)}(\mathbb{F}_2, \mathcal{P}^{2k+2}) \end{array} \tag{5.3.14}$$

which is valid for $s \geq 0$ as discussed above. The right vertical map is induced by the following extension.

Construction 5.3.15. Let $m \geq 1$. Let y_i denote the element of degree i in $\Sigma\mathcal{P}^{2m+2}$, and let z_i denote the element of degree i in $\mathcal{P}^{2m}$. Define $P(2m, 2m+2)$ to be the module with generators $\{\overline{y}_2, \ldots, \overline{y}_{2m+3}, \overline{z}_1, \ldots, \overline{z}_{2m}\}$ with the following $E(1)$ action. Let the action on the $\overline{y}_i$ be the same as that on y_i in $\Sigma\mathcal{P}^{2m+2}$, and let the action on the $\overline{z}_i$ be:

$$Q_1\overline{z}_{2k+1} = \overline{y}_{2k+4} + \overline{z}_{2k+4} \quad (2k+1 \neq 2m-1)$$
$$Q_1\overline{z}_{2m-1} = \overline{y}_{2m+2}$$
$$Q_1\overline{z}_{2k} = \overline{y}_{2k+3},$$

and Sq^1 acts as it does on z_i in $\mathcal{P}^{2m}$.

Lemma 5.3.16. *The extension*

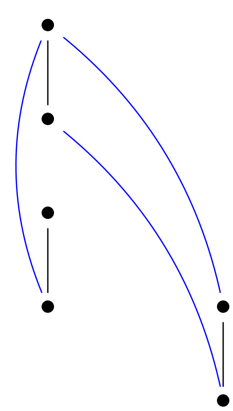
$$\Sigma\mathcal{P}^{2m+2} \to P(2m, 2m+2) \to \mathcal{P}^{2m},$$

where $y_i \mapsto \overline{y}_i$ and $\overline{z}_i \mapsto z_i$, is not split over $E(1)$.

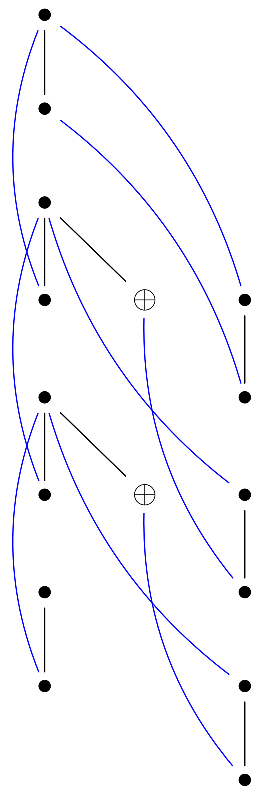
Proof: There cannot be an injective $E(1)$-map $\mathcal{P}^{2m} \to P(2m, 2m+2)$ that splits the extension because the top class $z_{2m} \in \mathcal{P}^{2m}$ must map to something in degree $2m$, but everything in degree $2m$ in $P(2m, 2m+2)$ supports a non-trivial operation, whereas z_{2m} does not. □

We illustrate this module in two cases. The blue, curved lines represent Q_1 and the black, straight lines represent $Q_0 = \mathrm{Sq}^1$.

Example 5.3.17 ($P(2,4) \sim Q_2^2$).



Example 5.3.18 ($P(6,8) \sim Q_8^8$).



We must show the extension $Q(2^k, 2^k) \sim P(2k, 2k+2)$ induces an isomorphism in 5.3.14. We have

$$\mathrm{Ext}_{E(1)}^{s,s-2}(\mathbb{F}_2, \mathcal{P}^{2k}) \neq 0 \quad \Leftrightarrow \quad \mathrm{Ext}_{E(1)}^{s+1,(s+1)-2}(\mathbb{F}_2, \mathcal{P}^{2k+2}) \neq 0$$

because in the case $\alpha = 1$, this statement is equivalent to

$$\nu(n) = k \geq s+1 \quad \Leftrightarrow \quad \nu(2n) = k+1 \geq s+2$$

which is obviously true (see Example 4.4.11). When $1 \leq s+1 \leq k$, then

$$\mathrm{Ext}_{E(1)}^{s,s-2}(\mathbb{F}_2, \mathcal{P}^{2k}) \cong \mathbb{F}_2\{u_{-2(s+1)} v_1^s\}$$

and

$$\mathrm{Ext}_{E(1)}^{s+1,(s+1)-2}(\mathbb{F}_2, \mathcal{P}^{2k+2}) \cong \mathbb{F}_2\{u_{-2(s+2)}v_1^{s+1}\}.$$

Here we are abusing notation, as the elements $u_{-2(s+1)}$ and the $u_{-2(s+2)}$ are generators in different modules. To see that $P(2k, 2k+2)$ induces an isomorphism, we'll first show $Q_{2^k}^{2^k} \circ u_{-2(s+1)} = u_{-2(s+2)} \circ v_1$. By definition, $u_{-2(s+1)} \colon \Sigma^{2(s+1)}\mathbb{F}_2 \to \mathcal{P}^{2k}$, and so the Yoneda product is given by the pullback

$$\begin{array}{ccccc}
\Sigma\mathcal{P}^{2k+2} & \longrightarrow & \mathcal{E} & \longrightarrow & \Sigma^{2(s+1)}\mathbb{F}_2 \\
\big\downarrow{=} & & \big\downarrow & & \big\downarrow{u_{-2(s+1)}} \\
\Sigma\mathcal{P}^{2k+2} & \longrightarrow & P(2k, 2k+2) & \longrightarrow & \mathcal{P}^{2k}.
\end{array}$$

In this pullback $\mathcal{E}$, everything in $P(2k, 2k+2)$ mapping to $\mathcal{P}^{2k}$ is killed except the element $\overline{z}_{2(s+1)}$ in degree $2(s+1)$ that is hit by $\Sigma^{2(s+1)}\mathbb{F}_2$. Furthermore, the copy of $\Sigma\mathcal{P}^{2k+2}$ remains, and $Q_1(\overline{z}_{2(s+1)}) = \overline{y}_{2(s+2)+1}$ by the definition of $P(2k, 2k+2)$. Since $\mathcal{P}^{2k+2}$ is shifted up by one, this gives us $u_{-2(s+2)} \circ v_1$, where $u_{-2(s+2)} \colon \Sigma^{2(s+2)}\mathbb{F}_2 \to \mathcal{P}^{2k+2}$. See Example 4.3.7 which is shifted down by one from this presentation.

Now, one can easily verify that $Q_{2^k}^{2^k}$ is an $\mathrm{Ext}_{E(1)}^{*,*}(\mathbb{F}_2, \mathbb{F}_2)$-module map, from which it follows that $Q_{2^k}^{2^k}(u_{-2(s+1)} \circ v_1^s) = u_{-2(s+2)} \circ v_1^{s+1}$, so $Q_{2^k}^{2^k}$ is an isomorphism, thus proving the case $n = 2^k$.

Remark 5.3.19. Because $u_{-2(s+1)} \circ h_0 = u_{-2(s+2)} \circ v_1$ (Theorem 4.3.1), one can rephrase this by saying $Q_{2^k}^{2^k}$ induces multiplication by h_0.

This concludes the proof of Theorem 5.2.1.

5.4 Consequences

Since the algebraic and geometric conditions in Theorem 3.1.9 hold, we deduce that $U_Q(L_*M)$ is an associated graded for $H_*(BU) \cong \mathbb{F}_2[a_2, a_4, \ldots]$. This means $L_s(M)_n \subset E^{\infty}_{-2^s, 2^s+n}$ and $y_n \in L_s(M)_n$ corresponds to an element in $H_n(BU)$. Furthermore, by [17, Theorem 1.6], the Q^k action on $L_s(M)$ corresponds to the action on the representative in $H_*(BU)$, so we can use Theorem 5.2.1 to deduce this information. For example, since Q^{2^k} is nontrivial on the element $y_2 \in L_0(M)_2$, we deduce $Q^{2^k} a_2 \neq 0$ for all k.

To make this clearer, we label elements of our associated graded. Recall that there is a sort of α/ν-based filtration (see the comments and examples following Theorem 4.4.8). We label the nonzero element of $L_s(M)_n$ in the following way.

$$\alpha(n) = s+1 \leftrightarrow x_n(0)$$
$$\alpha(n) = s \leftrightarrow x_{\frac{n}{2}}(1)$$
$$\alpha(n) = s-1 \leftrightarrow x_{\frac{n}{4}}(2)$$
$$\cdots$$
$$\alpha(n) = 1 \leftrightarrow x_{\frac{n}{2^s}}(s)$$

In general, the nonzero element of $L_s(M)_n$ is labeled as $x_{\frac{n}{2^{s+1-\alpha}}}(s+1-\alpha)$. Note that this notation uniquely defines elements in all of $L_*(M)$ because of the α/ν conditions. The notation is meant to suggest that $x_n(k)$ corresponds to $b_n^{2^k}$, where b_n make the simple system of generators of the associated graded.

Corollary 5.2.2 says that $Q^{k_{\alpha(n)}} \cdots Q^{k_2} x_{k_1}(0) \neq 0$ whenever $k_{\alpha(n)} \geq \cdots \geq k_2 \geq k_1$.

We note that $x_{2^k}(0)$ corresponds to a_{2^k} because these lie in $E^\infty_{-1,1+2^k}$ and are the only elements not hit by a Dyer-Lashof operation. We deduce the following.

Proposition 5.4.1. *$\{a_2, a_4, a_8, a_{16}, \dots\}$ is a generating set for $H_*(BU)$ over the Dyer-Lashof algebra.*

Additionally, Theorem 5.2.1 shows that e.g. $Q^{2^l} a_{2n} \neq 0$ if $2^l \geq 2n$ even without knowing which element in the associated graded corresponds to a_{2n}.

5.5 Other Q^k calculations

As a side note, we illustrate how to make computations of various Dyer-Lashof operations in $H_*(BU)$. We first show how to make direct calculations in the Singer complex. These computations are useful in guiding one toward the right answer, and may be useful if one tries to generalizes our results to $E(n)$ for $n \geq 2$.

Suppose we want to compute an operation $Q^k\colon \Omega^\infty M \to \Omega_1^\infty \Sigma^{-1} M$. Consider the diagram (recall 3.2.4):

$$\begin{array}{ccccc} & & \Sigma\mathcal{R}_0\Sigma^{-1}M & \xrightarrow{d_0} & \Sigma\mathcal{R}_1(M) \\ & & \downarrow{\scriptstyle Q^k} & & \downarrow{\scriptstyle Q^k} \\ \Sigma\mathcal{R}_0\Sigma^{-2}M & \xrightarrow{d_0} & \Sigma\mathcal{R}_1(\Sigma^{-1}M) & \xrightarrow{d_1} & \Sigma\mathcal{R}_2(M) \end{array}$$

If $m \in \Omega^\infty M$, then $d_0(m) = 0$ so $d_1(Q^k m) = 0$. Thus $Q^k m \neq 0$ in $\Omega_1^\infty \Sigma^{-1} M$ if and only if it is not hit by d_0. Let $c_k(-) = \sigma Q^k \sigma^{-1}(-)$ and $c_I = \sigma Q^I \sigma^{-1}(-)$. Then

$$\sigma d_0(\sigma^{-2} m) = \sum_{i \geq -1} c_i(m \operatorname{Sq}^i).$$

Lemma 5.5.1. *Let $m \in \Omega^\infty M$. Then $c_k m$ is killed by d_0 if and only if there exists $y \in M$ (possibly polynomial) such that $y \cdot \operatorname{Sq}^{k+1} = m$. Note $|y| = |m| + k + 1$.*

Lemma 5.5.2. $|c_i(y \operatorname{Sq}^{i+1})| = 0$ *if* $i < |\sigma^{-1} y \operatorname{Sq}^{i+1}| \Leftrightarrow i < \frac{|y|-2}{2} \Leftrightarrow i \leq \lceil |y|/2 \rceil - 2$.

Corollary 5.5.3. *The first possible nonzero term in $\sigma d(\sigma^{-2} y)$ is $c_{\lceil \frac{|y|}{2} \rceil - 1} y \operatorname{Sq}^{\lceil \frac{|y|}{2} \rceil}$.*

For example, take $m = \sigma^2 \iota \in M$ where ι is the zero dimensional generator of $\Sigma^{-2} M = H_*(ku) \cong \mathbb{F}_2[\zeta_1^2, \zeta_2^2, \zeta_3, \ldots]$. Letting $y = \sigma^2 \zeta_k$, we see the first nonzero term of $\sigma d_0(y)$ is $c_{2^{k-1}+1} y \operatorname{Sq}^{2^{k-1}+1}$. But by Corollary 2.1.6, this term vanishes, and furthermore, there are no squares on ζ_k between $\operatorname{Sq}^{2^{k-1}-1}$ and $\operatorname{Sq}^{2^k-1}$. Therefore $\sigma d_0(y) = c_{2^k-2} y \operatorname{Sq}^{2^k-1} = \sigma Q^{2^k-2} \sigma^{-1} m$. In other words,

Proposition 5.5.4. $Q^{2^k-2}(\sigma^2 \iota) = 0$ *in* $\Omega_1^\infty \Sigma^{-1} M$.

Additionally, we can make calculations using the machinery from Section 5.3, which gives us stronger results.

Proposition 5.5.5. *Let n be even, and note $J(n) \simeq \Sigma^{2\alpha(n)-2} \mathcal{P}^{2\nu(n)}$. Let $P = \mathcal{P}^{2\nu(n)}$. Then*

$$\operatorname{Ext}^{1,1}_{E(1)}(J(2), J(n)) \cong \operatorname{Ext}^{1,2\alpha-1}_{E(1)}(\mathcal{P}^2, P) \cong \mathbb{F}_2$$

if $1 \leq 3-\alpha(n) \leq \nu(n)$ *but* $1 \leq 4-\alpha(n) \leq \nu(n)$ *does not hold. The group is 0 otherwise.*

Proof: We have a long exact sequence

$$\mathrm{Ext}_{E(1)}^{1,2\alpha-2}(\mathbb{F}_2, P) \to \mathrm{Ext}_{E(1)}^{1,2\alpha-1}(\mathcal{P}^2, P) \to \mathrm{Ext}_{E(1)}^{1,2\alpha-3}(\mathbb{F}_2, P) \to \mathrm{Ext}_{E(1)}^{2,2\alpha-2}(\mathbb{F}_2, P) \to \dots$$

where the boundary map is given by multiplication by h_0. The first group is 0 so this reduces to

$$\mathrm{Ext}_{E(1)}^{1,2\alpha-1}(\mathcal{P}^2, P) \hookrightarrow \mathrm{Ext}_{E(1)}^{1,2\alpha-3}(\mathbb{F}_2, P) \to \mathrm{Ext}_{E(1)}^{2,2\alpha-2}(\mathbb{F}_2, P) \to \dots$$

Note the middle group is generated by $v_1 u_{-2(3-\alpha)}$ if this element exists and is 0 if $3-\alpha \notin \{1, \dots, \nu(n)\}$, hence the left group is also 0 in this case.

Otherwise, we can test what h_0 does by noting $h_0 v_1 u_{-2(3-\alpha)} = v_1 h_0 u_{-2(3-\alpha)}$ and the latter is equal to $v_1^2 u_{-2(4-\alpha)}$ if this element exists in $\mathrm{Ext}_{E(1)}^{*,*}(\mathbb{F}_2, P)$. In this case, h_0 is nonzero, hence an isomorphism, so that $\mathrm{Ext}_{E(1)}^{1,2\alpha-1}(\mathcal{P}^2, P) = 0$. If there is no $u_{-2(4-\alpha)}$ in $\mathrm{Ext}_{E(1)}^{*,*}(\mathbb{F}_2, P)$, then $v_1 h_0 u_{-2(3-\alpha)} = 0$ for degree reasons. This scenario equivalent to $4-\alpha \notin \{1, \dots, \nu(n)\}$. In this case, $h_0 = 0$ so that

$$\mathrm{Ext}_{E(1)}^{1,2\alpha-1}(\mathcal{P}^2, P) \cong \mathrm{Ext}_{E(1)}^{1,2\alpha-3}(\mathbb{F}_2, P) \cong \mathbb{F}_2\{v_1 u_{-2(3-\alpha)}\}.$$

This completes the proof. □

Corollary 5.5.6. *If* $n \geq 1$, *then* $\mathrm{Ext}_{E(1)}^{1,1}(J(2), J(2+(4n+2))) = 0$, *so that* $Q_2^{4n+2} = 0$ *over* $E(1)$.

Chapter 6

Further directions

Many of the techniques in this thesis would apply similarly to other subalgebras of $\mathcal{A}$, such as $E(n)$, and one can view much of this work as prototype in this regard. The geometric condition holds for $E(n)$, even when considering $E = \text{colim} E(n)$, where $A /\!\!/ E \cong H^*(BP)$. Furthermore, detecting Margolis homology generators for the Q_n is easy, although systematically tracking which $J(n)$ they lie in takes a bit of book-keeping. At key points, we used the classification of $J(n)$ as modules over $E(1)$, and this used both Margolis homology and a classification theorem of $E(1)$-modules. We guess that generalizing this classification theorem, at least to $E(2)$, is tractable, given that infinite modules can be ignored in our context.

We conjecture that $L_s(M)$ is always evenly graded for $M = A /\!\!/ E(n)$. Furthermore, at a key step in Theorem 5.2.1, we needed to show that restricting $Q(n, 2^l)$ to $E(1)$ was not split. We believe a similar technique can be used to show that this extension is not split over $E(n)$, given that the chosen Margolis homology generator y is also in the Margolis homology of Q_n if the degree is big enough.

The general strategy for sub-Hopf algebras $B \subset \mathcal{A}$ would be that Margolis homology information for $J(n)$ with respect to $P_t^s \in B$ yields information about $\text{Ext}_B(J(n))$, and this in turn yields information about the derived functors of destabilization of $\mathcal{A} /\!\!/ B$. When $s = 0$, these are the exterior generators Q_{t-1}, and the Margolis homology is easy as mentioned above. For instance, just knowing $H(J(n); Q_n)$ would yield information about Dyer-Lashof operations on $H_*(\Omega^\infty k(n); \mathbb{F}_2)$. For $\mathcal{A}(1)$, one needs $H(J(n); Q_i)$ for $i = 0, 1, 2$, which would lead to operations on $H_*(BO)$ (which is known by Kochman and Priddy). Generally, however, the Margolis homology with respect to P_t^s is quite hard if $s > 0$ because this element is not primitive, hence does not act as a derivation on $\mathcal{A}$-algebras. However, in the case of $B = \mathcal{A}(2)$ and $P_t^s = P_2^1$, the Margolis homology can be systemically determined; see [5] for details. Here, the Margolis homology of $(\mathcal{A}//\mathcal{A}(2))_*$ is computed, and this is a subalgebra of $\mathcal{A}_*$. Using the fact that $J(n)$ can be embedded in $\mathcal{A}_*$, the techniques in that paper also apply to $J(n)$ after re-indexing. We hope that this can be used to yield some information about Dyer-Lashof operations on $H_*(\Omega^\infty tmf; \mathbb{F}_2)$, even if the topological and algebraic spectral sequences do not coincide.

Appendix A

Proof of Theorem 3.3.16

Recall from Proposition 3.3.6 and the remark that follows it that if M is an unstable right $\mathcal{A}$–module, there is a natural epimorphism

$$[\Sigma\mathcal{R}_1(\Sigma^{-1}M)]_m \twoheadrightarrow \mathrm{Ext}_A^{1,1}(M^\vee, J(m)).$$

Given $x \in M_n$, we have $\sigma Q^k(\sigma^{-1}x) \in \Sigma\mathcal{R}_1(\Sigma^{-1}M)_{n+k}$. Let $Q(x,k) \in \mathrm{Ext}_A^{1,1}(M^\vee, J(n+k))$ be the corresponding extension under the above surjection, which we will write as a short exact sequence

$$0 \to J(n+k) \to Q(x,k) \to \Sigma^{-1}M^\vee \to 0.$$

Note we are slightly abusing notation.

Recall that $t_k \in H_k(P_0^\infty)$ denotes the element dual to t^k.

Definition A.0.1. Given $x \in M_n$ and k, let

$$q(x,k) = \sum_j t_{k+j} \otimes x\chi(Sq^j) \in [H_*(P_0^\infty) \otimes M]_{n+k}.$$

We can regard $q(x,k)$ as a map of unstable left A–modules

$$q(x,k) : H^*(P_0^\infty) \otimes M^\vee \to J(n+k).$$

We will prove the following theorem, which appears stronger than Theorem 3.3.16, but is actually equivalent, as $M = G(n)$ is the universal case.

Theorem A.0.2. *The extension $Q(x,k)$ is obtained as the pushout:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^*(P_0^\infty)\otimes M^\vee & \longrightarrow & H^*(P_{-1}^\infty)\otimes M^\vee & \longrightarrow & \Sigma^{-1}M^\vee & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle q(x,k)} & & \downarrow & & \| & & \\ 0 & \longrightarrow & J(n+k) & \longrightarrow & Q(x,k) & \longrightarrow & \Sigma^{-1}M^\vee & \longrightarrow & 0 \end{array}$$

When $M = G(n)$ and $x = \iota_n \in G(n)$, we write $q(n,k)$ for $q(\iota,k)$, and $Q(n,k)$ for $Q(\iota_n,k)$. The following corollary is immediate.

Corollary A.0.3 (Theorem 3.3.16)**.** *The extension $Q(n,k)$ is obtained as the pushout:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^*(P_0^\infty)\otimes J(n) & \longrightarrow & H^*(P_{-1}^\infty)\otimes J(n) & \longrightarrow & \Sigma^{-1}J(n) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle q(n,k)} & & \downarrow & & \| & & \\ 0 & \longrightarrow & J(n+k) & \longrightarrow & Q(n,k) & \longrightarrow & \Sigma^{-1}J(n) & \longrightarrow & 0 \end{array}$$

The theorem follows immediately from the two propositions below. Before stating these, it is useful to first recall a couple of things.

Let $H = H\mathbb{Z}/2$ and then let $\bar{H}$ be the cofiber of $S \to H$ with mod 2 cohomology $\bar{\mathcal{A}}$. Then we have a pullback of cofibration sequences

$$\begin{array}{ccccc} S^{-1} & \longrightarrow & P_{-1}^\infty & \longrightarrow & P_0^\infty \\ \| & & \downarrow & & \downarrow \\ S^{-1} & \longrightarrow & \Sigma^{-1}H & \longrightarrow & \Sigma^{-1}\bar{H} \end{array}$$

realizing the pushout diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Sigma^{-1}\bar{\mathcal{A}} & \longrightarrow & \Sigma^{-1}\mathcal{A} & \longrightarrow & \Sigma^{-1}\mathbb{Z}/2 & \longrightarrow & 0 \\ & & \twoheaddownarrow & & \twoheaddownarrow & & \| & & \\ 0 & \longrightarrow & H^*(P_0^\infty) & \longrightarrow & H^*(P_{-1}^\infty) & \longrightarrow & \Sigma^{-1}\mathbb{Z}/2 & \longrightarrow & 0 \end{array} \tag{A.0.4}$$

in cohomology.

Proposition A.0.5. *Let M be an unstable right $\mathcal{A}$–module.*

(a) The composite $H_*(P^\infty_{-1}) \otimes M \hookrightarrow \Sigma^{-1}\mathcal{A}_* \otimes M \xrightarrow{d_0} \Sigma\mathcal{R}_1(\Sigma^{-1}\mathcal{A}_* \otimes M)$ factors through the inclusion $\Sigma\mathcal{R}_1(\Sigma^{-1}M) \hookrightarrow \Sigma\mathcal{R}_1(\Sigma^{-1}\mathcal{A}_* \otimes M)$, defining a natural transformation β fitting into a commutative diagram

$$\begin{array}{ccc} H_*(P^\infty_{-1}) \otimes M & \xrightarrow{\beta} & \Sigma\mathcal{R}_1(\Sigma^{-1}M) \\ \downarrow & & \downarrow \\ \Sigma^{-1}\mathcal{A}_* \otimes M & \xrightarrow{d_0} & \Sigma\mathcal{R}_1(\Sigma^{-1}\mathcal{A}_* \otimes M). \end{array}$$

(b) Given $q \in [H_*(P^\infty_{-1}) \otimes M]_m$, let $\bar{q} \in [H_*(P^\infty_0) \otimes M]_m$ be its projection, let $z = \beta(q) \in [\Sigma\mathcal{R}_1(\Sigma^{-1}M)]_m$, and let $Q(z) \in Ext^{1,1}(M^\vee, J(m))$ be the associated extension. Then $Q(z)$ is obtained as the pushout:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^*(P^\infty_0) \otimes M^\vee & \longrightarrow & H^*(P^\infty_{-1}) \otimes M^\vee & \longrightarrow & \Sigma^{-1}M^\vee & \longrightarrow & 0 \\ & & \downarrow \bar{q} & & \downarrow & & \| & & \\ 0 & \longrightarrow & J(m) & \longrightarrow & Q(z) & \longrightarrow & \Sigma^{-1}M^\vee & \longrightarrow & 0. \end{array}$$

Remark A.0.6. As $H^*(P^\infty_{-1}) = \Sigma\mathcal{R}_1(\Sigma^{-1}\mathbb{Z}/2)$, β has the form $\beta : \Sigma\mathcal{R}_1(\Sigma^{-1}\mathbb{Z}/2) \otimes M \to \Sigma\mathcal{R}_1(\Sigma^{-1}M)$.

Proposition A.0.7. *(a) $\beta(t_r \otimes y) = \sum_i \sigma Q^{r+i}(\sigma^{-1} y Sq^i)$*

(b) Given $x \in M_n$ and k, let $q(x,k) = \sum_j t_{k+j} \otimes x\chi(Sq^j)$. Then $\beta(q(x,k)) = \sigma Q^k \sigma^{-1} x$.

Remark A.0.8. Proposition A.0.7 implies that β is a surjection.

A.1 Proof of Proposition A.0.5

Recall from Theorem 3.2.2 that given a right $\mathcal{A}$–module M, there is a natural chain complex

$$M \xrightarrow{d_0} \Sigma\mathcal{R}_1(M) \xrightarrow{d_1} \Sigma\mathcal{R}_2(\Sigma M) \to \dots$$

such that $\Omega^\infty(M) = \ker\{M \xrightarrow{d_0} \Sigma\mathcal{R}_1(M)\}$, and $\Omega_1^\infty(\Sigma M)$ is the first homology of this complex.

It is sometimes useful to work dually with cohomology instead, and we use a tilde to denote this. Thus, we let $\widetilde{\Omega}_*^\infty$ and $\widetilde{\mathcal{R}}_*$ be functors on left $\mathcal{A}$–modules dual to Ω_*^∞ and $\mathcal{R}_*$. Furthermore we let $\widetilde{Q}^k$ be a dual Dyer–Lashof operation, as in [17, §2.5].

We need to better explain the epimorphism

$$[\Sigma\mathcal{R}_1(\Sigma^{-1}M)]_m \twoheadrightarrow \mathrm{Ext}_{\mathcal{A}}^{1,1}(M^\vee, J(m)).$$

This factors as the composite

$$\begin{array}{ccccc} [\Sigma\mathcal{R}_1(\Sigma^{-1}M)]_m & \twoheadrightarrow & [\Omega_1^\infty(\Sigma^{-1}M)]_m & = & \mathrm{Hom}_{\mathcal{A}}(\widetilde{\Omega}_1^\infty(\Sigma^{-1}M^\vee), J(m)) \\ & \searrow\!\!\!\!\to & & & \downarrow \cong \\ & & & & \mathrm{Ext}_{\mathcal{A}}^{1,1}(M^\vee, J(m)) \end{array}$$

where the first epimorphism is explained in Proposition 3.3.2. The vertical isomorphism is from Proposition 3.3.2, and we now elaborate on that connection in the case of $\mathrm{Ext}_{\mathcal{A}}^{1,1}$.

Let I be an injective in $\mathcal{U}$, the category of unstable left $\mathcal{A}$–modules. Then there there is a natural isomorphism, for all left $\mathcal{A}$–modules N,

$$\mathrm{Hom}_{\mathcal{A}}(\widetilde{\Omega}_1^\infty(N), I) \xrightarrow{\cong} \mathrm{Ext}_{\mathcal{A}}^1(N, I),$$

with the construction as follows.

Given $\alpha : \widetilde{\Omega}_1^\infty(N) \to I$, we are looking to construct an extension

$$0 \to I \to Q(\alpha) \to N \to 0.$$

To this end, start with a short exact sequence $0 \to K \to P \to N \to 0$ with P a projective $\mathcal{A}$–module. This induces a diagram with exact rows

$$\begin{array}{ccccccccccc}
 & & 0 & \longrightarrow & K & \longrightarrow & P & \longrightarrow & N & \longrightarrow & 0 \\
 & & & & \downarrow\!\!\downarrow \scriptstyle{\pi} & & \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow & & \\
0 & \longrightarrow & \widetilde{\Omega}_1^\infty(N) & \xrightarrow{\delta} & \widetilde{\Omega}^\infty(K) & \longrightarrow & \widetilde{\Omega}^\infty(P) & \longrightarrow & \widetilde{\Omega}^\infty(N) & \longrightarrow & 0 \\
 & & & \searrow^{\alpha} & \vdots\, \scriptstyle{\widetilde{\alpha}} & & & & & & \\
 & & & & I & & & & & &
\end{array}$$

where δ is the standard connecting map, and the map $\widetilde{\alpha}$ exists because I is a $\mathcal{U}$–injective. Then the extension $Q(\alpha)$ is obtained as the pushout

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K & \longrightarrow & P & \longrightarrow & N & \longrightarrow & 0 \\
 & & \downarrow \scriptstyle{\widetilde{\alpha}\circ\pi} & & \downarrow & & \| & & \\
0 & \longrightarrow & I & \longrightarrow & Q(\alpha) & \longrightarrow & N & \longrightarrow & 0.
\end{array}$$

It is standard that the extension is independent of all choices.

Now we apply this to the situation in Proposition A.0.5(b), where $N = \Sigma^{-1}M^\vee$ with M an unstable right $\mathcal{A}$–module.

Consider the pushout diagram A.0.4 tensored with $M^\vee$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Sigma^{-1}\bar{\mathcal{A}} \otimes M^\vee & \longrightarrow & \Sigma^{-1}\mathcal{A} \otimes M^\vee & \longrightarrow & \Sigma^{-1}M^\vee & \longrightarrow & 0 \\
 & & \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow & & \| & & \\
0 & \longrightarrow & H^*(P_0^\infty) \otimes M^\vee & \longrightarrow & H^*(P_{-1}^\infty) \otimes M^\vee & \longrightarrow & \Sigma^{-1}M^\vee & \longrightarrow & 0.
\end{array} \tag{A.1.1}$$

Note that $\Sigma^{-1}\mathcal{A} \otimes M^\vee$ is a free A–module, and that $H^*(P_0^\infty) \otimes M^\vee$ is unstable, thus

$$\Sigma^{-1}\bar{\mathcal{A}} \otimes M^\vee \twoheadrightarrow H^*(P_0^\infty) \otimes M^\vee$$

factors as

$$\Sigma^{-1}\bar{\mathcal{A}} \otimes M^\vee \twoheadrightarrow \tilde{\Omega}^\infty(\Sigma^{-1}\bar{\mathcal{A}} \otimes M^\vee) \twoheadrightarrow H^*(P_0^\infty) \otimes M^\vee.$$

Now assume that β exists as in Proposition A.0.5(a), and consider the set-up of part (b). One is given $q \in [H_*(P_{-1}^\infty) \otimes M]_m$, and we let $z = \beta(q) \in [\Sigma\mathcal{R}_1(\Sigma^{-1}M)]_m$. The element q induces $\bar{q} : H^*(P_0^\infty) \otimes M^\vee \to J(m)$ and the element z determines $\bar{z} : \tilde{\Omega}_1^\infty(\Sigma^{-1}M^\vee) \to J(m)$ and thus $Q(\bar{z}) \in \mathrm{Ext}_{\mathcal{A}}^{1,1}(M^\vee, J(m))$.

Using the construction of $Q(\alpha)$ above, Proposition A.0.5(b) will be proved if we can show that the triangle in the following diagram commutes:

$$\begin{array}{ccccccc}
 & & \Sigma^{-1}\bar{\mathcal{A}} \otimes M^\vee & \longrightarrow & \Sigma^{-1}\mathcal{A} \otimes M^\vee & \longrightarrow & \Sigma^{-1}M^\vee \\
 & & \downarrow\!\!\downarrow & & \Big\downarrow & & \Big\| \\
\tilde{\Omega}_1^\infty(\Sigma^{-1}M^\vee) & \overset{\delta}{\rightarrowtail} & \tilde{\Omega}^\infty(\Sigma^{-1}\bar{\mathcal{A}} \otimes M^\vee) & & \Big\downarrow & & \Big\| \\
 & \searrow^{\bar{z}} & \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow & & \Big\| \\
 & & H^*(P_0^\infty) \otimes M^\vee & \longrightarrow & H^*(P_{-1}^\infty) \otimes M^\vee & \longrightarrow & \Sigma^{-1}M^\vee \\
 & & \Big\downarrow{\scriptstyle \bar{q}} & & & & \\
 & & J(m). & & & &
\end{array}$$

Dualizing, we need to show the following.

Lemma A.1.2. *With β as in Proposition A.0.5(a), there is a commutative diagram:*

$$\begin{array}{ccccc}
H_*(P_{-1}^\infty) \otimes M & & \xrightarrow{\quad\quad\beta\quad\quad} & & \Sigma\mathcal{R}_1(\Sigma^{-1}M) \\
\downarrow\!\!\downarrow & & & & \downarrow\!\!\downarrow \\
H_*(P_0^\infty) \otimes M & \rightarrowtail & \Omega^\infty(\Sigma^{-1}\bar{\mathcal{A}}_* \otimes M) & \overset{\delta}{\twoheadrightarrow} & \Omega_1^\infty(\Sigma^{-1}M).
\end{array}$$

We simultaneously check Proposition A.0.5(a) and Lemma A.1.2. Consider the following diagram:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \Sigma^{-1}M & \longrightarrow & H_*(P^{\infty}_{-1}) \otimes M & \longrightarrow & H_*(P^{\infty}_{0}) \otimes M & \longrightarrow & 0 \\
 & & \| & & \big\downarrow i & & \big\downarrow & & \\
 & & \| & & & & \Omega^{\infty}(\Sigma^{-1}\bar{\mathcal{A}}_* \otimes M) & & \\
 & & \| & & & & \big\downarrow j & & \\
0 & \longrightarrow & \Sigma^{-1}M & \longrightarrow & \Sigma^{-1}\mathcal{A}_* \otimes M & \xrightarrow{p} & \Sigma^{-1}\bar{\mathcal{A}}_* \otimes M & \longrightarrow & 0 \\
 & & \big\downarrow d_0 & & \big\downarrow d_0 & & \big\downarrow d_0 & & \\
0 & \longrightarrow & \Sigma\mathcal{R}_1(\Sigma^{-1}M) & \xrightarrow{k} & \Sigma\mathcal{R}_1(\Sigma^{-1}\mathcal{A}_* \otimes M) & \xrightarrow{\Sigma\mathcal{R}_1(p)} & \Sigma\mathcal{R}_1(\Sigma^{-1}\bar{\mathcal{A}}_* \otimes M) & \longrightarrow & 0 \\
 & & \big\downarrow & & & & & & \\
 & & \Omega^{\infty}_1(\Sigma^{-1}M). & & & & & &
\end{array}
$$

Here the top part of the diagram is just the diagram A.1.1 dualized.

Since $d_0 \circ j = 0$, we see that $\Sigma\mathcal{R}_1(p) \circ d_0 \circ i = 0$, and thus $d_0 \circ i$ lifts through k, which defines β and proves Proposition A.0.5(a). Now recall the construction of the connecting map δ. One lifts the image of j through p, and then the image of this under d_0 lifts through k, and one projects onto $\Omega^{\infty}_1(\Sigma^{-1}M)$. It is now clear that β and δ are compatible as stated in Lemma 2.

This completes the proof of Proposition A.0.5.

A.2 Proof of Proposition A.0.7

It is easier to first find the formula for the dual of β,

$$\widetilde{\beta} : \Sigma\widetilde{\mathcal{R}}_1(\Sigma^{-1}M^{\vee}) \to H^*(P^{\infty}_{-1}) \otimes M.$$

As constructed in the last section, this fits into a commutative diagram

$$
\begin{array}{ccccc}
\Sigma\widetilde{\mathcal{R}}_1(\Sigma^{-1}\mathcal{A} \otimes M^{\vee}) & \xrightarrow{\widetilde{d}_0} & \Sigma^{-1}\mathcal{A} \otimes M^{\vee} & \xrightarrow{p\otimes 1} & H^*(P^{\infty}_{-1}) \otimes M^{\vee} \\
\big\downarrow \pi & & & \nearrow \widetilde{\beta} & \\
\Sigma\widetilde{\mathcal{R}}_1(\Sigma^{-1}M^{\vee}). & & & &
\end{array}
$$

On one hand, we have $\pi(\sigma\widetilde{Q}^{k-1}(\sigma^{-1}\iota\otimes x)) = \sigma\widetilde{Q}^{k-1}(\sigma^{-1}x)$. Let $z \in M^\vee$. Then on the other hand, we have $\widetilde{d}_0(\sigma\widetilde{Q}^{k-1}(\sigma^{-1}\iota\otimes z)) = \mathrm{Sq}^k(\sigma^{-1}\iota\otimes z) = \sum_{i+j=k}\sigma^{-1}\mathrm{Sq}^i\iota\otimes\mathrm{Sq}^j z$, so that $((p\otimes 1)\circ\widetilde{d}_0)(\sigma\widetilde{Q}^{k-1}(\sigma^{-1}\iota\otimes z)) = \sum_{i+j=k} t^{i-1}\iota\otimes\mathrm{Sq}^j z$. We conclude that, for all $z\in M^\vee$ and k,

$$\widetilde{\beta}(\sigma\widetilde{Q}^k(\sigma^{-1}z)) = \sum_i t^i\,\mathrm{Sq}^{k-i} z.$$

Now suppose given $y \in M$ and $r \geq -1$. Then for all $z \in M^\vee$ and k, we have

$$\begin{aligned}
\langle\sigma\widetilde{Q}^k\sigma^{-1}z, \beta(t_r\otimes y)\rangle &= \langle\widetilde{\beta}(\sigma\widetilde{Q}^k\sigma^{-1}z), t_r\otimes y\rangle \\
&= \sum_i\langle t^i\,\mathrm{Sq}^{k-i} z, t_r\otimes y\rangle \\
&= \langle\mathrm{Sq}^{k-r} z, y\rangle \\
&= \langle z, y\,\mathrm{Sq}^{k-r}\rangle \\
&= \langle\sigma\widetilde{Q}^k\sigma^{-1}z, \sigma Q^k(\sigma^{-1}y\,\mathrm{Sq}^{k-r})\rangle.
\end{aligned}$$

We conclude that $\beta(t_r\otimes y) = \sum_j \sigma Q^j(\sigma^{-1}y\,\mathrm{Sq}^{j-r}) = \sum_i \sigma Q^{r+i}(\sigma^{-1}y\,\mathrm{Sq}^i)$, proving part (a).

Part (b) is now a straightforward calculation. Given $x \in M_n$ and k, let $q(x,k) = \sum_j t_{k+j}\otimes x\chi(\mathrm{Sq}^j)$. We compute:

$$\begin{aligned}
\beta(q(x,k)) &= \sum_j \beta(t_{k+j}\otimes x\chi(\mathrm{Sq}^j)) \\
&= \sum_j\sum_i \sigma Q^{k+j+i}(\sigma^{-1}x\chi(\mathrm{Sq}^j)\,\mathrm{Sq}^i) \\
&= \sum_c \sigma Q^{k+c}\sigma^{-1}\left(\sum_{i+j=c} x\chi(Sq^j)\,\mathrm{Sq}^i\right) \\
&= \sigma Q^k\sigma^{-1}x,
\end{aligned}$$

since $\sum_{i+j=c} x\chi(\mathrm{Sq}^j)\,\mathrm{Sq}^i = \begin{cases} x & \text{if } c = 0 \\ 0 & \text{otherwise.}\end{cases}$

This concludes the proof of Proposition A.0.7 and also Theorem 3.3.16.

Bibliography

[1] J. F. Adams. *Stable homotopy and generalised homology.* University of Chicago Press, Chicago, Ill.-London, 1974. Chicago Lectures in Mathematics.

[2] John Frank Adams. *Infinite loop spaces*, volume 90 of *Annals of Mathematics Studies.* Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1978.

[3] Agnès Beaudry and Jonathan A. Campbell. A guide for computing stable homotopy groups. In *Topology and quantum theory in interaction*, volume 718 of *Contemp. Math.*, pages 89–136. Amer. Math. Soc., Providence, RI, 2018.

[4] D. J. Benson. *Representations and cohomology. I*, volume 30 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, 1991. Basic representation theory of finite groups and associative algebras.

[5] Prasit Bhattacharya, Irina Bobkova, and Brian Thomas. The P_2^1 Margolis homology of connective topological modular forms. *arXiv e-prints*, page arXiv:1810.05622, Feb 2019.

[6] Edgar H. Brown, Jr. and Samuel Gitler. A spectrum whose cohomology is a certain cyclic module over the Steenrod algebra. *Topology*, 12:283–295, 1973.

[7] R. R. Bruner, J. P. May, J. E. McClure, and M. Steinberger. *H_∞ ring spectra and their applications*, volume 1176 of *Lecture Notes in Mathematics.* Springer-Verlag, Berlin, 1986.

[8] Donald Davis. Generalized homology and the generalized vector field problem. *Quart. J. Math. Oxford Ser. (2)*, 25:169–193, 1974.

[9] Eldon Dyer and R. K. Lashof. Homology of iterated loop spaces. *Amer. J. Math.*, 84:35–88, 1962.

[10] S. Gitler, M. Mahowald, and R. James Milgram. The nonimmersion problem for RP^n and higher-order cohomology operations. *Proc. Nat. Acad. Sci. U.S.A.*, 60:432–437, 1968.

[11] Paul G. Goerss. Unstable projectives and stable Ext: with applications. *Proc. London Math. Soc. (3)*, 53(3):539–561, 1986.

[12] Rune Haugseng and Haynes Miller. On a spectral sequence for the cohomology of infinite loop spaces. *Algebr. Geom. Topol.*, 16(5):2911–2947, 2016.

[13] P. Hu, I. Kriz, and J. P. May. Cores of spaces, spectra, and E_∞ ring spectra. *Homology Homotopy Appl.*, 3(2):341–354, 2001. Equivariant stable homotopy theory and related areas (Stanford, CA, 2000).

[14] David J. Hunter and Nicholas J. Kuhn. Characterizations of spectra with $\mathcal{U}$-injective cohomology which satisfy the Brown-Gitler property. *Trans. Amer. Math. Soc.*, 352(3):1171–1190, 2000.

[15] Stanley O. Kochman. Homology of the classical groups over the Dyer-Lashof algebra. *Trans. Amer. Math. Soc.*, 185:83–136, 1973.

[16] Tatsuji Kudo and Shôrô Araki. Topology of H_n-spaces and H-squaring operations. *Mem. Fac. Sci. Kyūsyū Univ. Ser. A.*, 10:85–120, 1956.

[17] Nicholas Kuhn and Jason McCarty. The mod 2 homology of infinite loopspaces. *Algebr. Geom. Topol.*, 13(2):687–745, 2013.

[18] Jean Lannes and Saïd Zarati. Sur les foncteurs dérivés de la déstabilisation. *Math. Z.*, 194(1):25–59, 1987.

[19] Tyler Lawson. Secondary power operations and the Brown-Peterson spectrum at the prime 2. *Ann. of Math. (2)*, 188(2):513–576, 2018.

[20] Saunders Mac Lane. *Homology.* Die Grundlehren der mathematischen Wissenschaften, Bd. 114. Academic Press, Inc., Publishers, New York; Springer-Verlag, Berlin-Göttingen-Heidelberg, 1963.

[21] Ib Madsen. On the action of the Dyer-Lashof algebra in $H_*(G)$. *Pacific J. Math.*, 60(1):235–275, 1975.

[22] H. R. Margolis. *Spectra and the Steenrod algebra*, volume 29 of *North-Holland Mathematical Library.* North-Holland Publishing Co., Amsterdam, 1983. Modules over the Steenrod algebra and the stable homotopy category.

[23] J. P. May. Infinite loop space theory. *Bull. Amer. Math. Soc.*, 83(4):456–494, 1977.

[24] Haynes Miller. A spectral sequence for the homology of an infinite delooping. *Pacific J. Math.*, 79(1):139–155, 1978.

[25] Haynes Miller. The Sullivan conjecture on maps from classifying spaces. *Ann. of Math. (2)*, 120(1):39–87, 1984.

[26] John Milnor. The Steenrod algebra and its dual. *Ann. of Math. (2)*, 67:150–171, 1958.

[27] John W. Milnor and James D. Stasheff. *Characteristic classes*. Princeton University Press, Princeton, N. J.; University of Tokyo Press, Tokyo, 1974. Annals of Mathematics Studies, No. 76.

[28] Robert E. Mosher and Martin C. Tangora. *Cohomology operations and applications in homotopy theory*. Harper & Row, Publishers, New York-London, 1968.

[29] Masatsugu Nagata. On the uniqueness of Dyer-Lashof operations on the Bott periodicity spaces. *Publ. Res. Inst. Math. Sci.*, 16(2):499–511, 1980.

[30] Paul Thomas Pearson. The connective real K-theory of Brown-Gitler spectra. *Algebr. Geom. Topol.*, 14(1):597–625, 2014.

[31] Geoffrey Powell. Truncated projective spaces, Brown-Gitler spectra and indecomposable $\mathcal{A}(1)$-modules. *Topology Appl.*, 183:45–85, 2015.

[32] Stewart Priddy. Dyer-Lashof operations for the classifying spaces of certain matrix groups. *Quart. J. Math. Oxford Ser. (2)*, 26(102):179–193, 1975.

[33] Lionel Schwartz. *Unstable modules over the Steenrod algebra and Sullivan's fixed point set conjecture*. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1994.

[34] William M. Singer. Iterated loop functors and the homology of the Steenrod algebra. II. A chain complex for $\Omega_s^k M$. *J. Pure Appl. Algebra*, 16(1):85–97, 1980.

[35] Robert M. Switzer. *Algebraic topology—homotopy and homology*. Springer-Verlag, New York-Heidelberg, 1975. Die Grundlehren der mathematischen Wissenschaften, Band 212.

[36] Charles A. Weibel. *An introduction to homological algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1994.

[37] R. M. W. Wood. Problems in the Steenrod algebra. *Bull. London Math. Soc.*, 30(5):449–517, 1998.