Reduction and Deformation of One-point Galois Covers

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Abstract

The étale fundamental group of an algebraic curve encodes information about both its finite étale covers and the unramified extensions of its function field. Over an algebraically closed field of characteristic 0, the Riemann existence theorem provides a powerful tool to compute these fundamental groups. The situation is, however, significantly more complicated over fields of positive characteristic p. In this thesis, I extend two techniques for studying these fundamental groups. The first relies on the relationship between the fundamental group of a curve and that of its reduction and involves showing that covers of elliptic curves defined over “small" fields branched at exactly one point have good reduction to positive characteristic. This generalizes results of Raynaud and Obus. The second generalizes techniques that Pries used to show the existence of a cover of $\mathbb{P}^1$ with a single wildly ramified branch point whose conductor is as small as possible. These techniques include studying degenerations of one-point covers in positive characteristic.

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Contents

Chapter 1

Introduction

1.1 Overview

We often attempt to study an object via its covers. In algebraic topology, the theory of covering spaces provides insight into such objects as fundamental groups. In number theory, we are able to study the arithmetic of a field through its extensions, giving rise to Galois theory. In algebraic geometry, we may look at branched covers of algebraic curves. Here, we obtain not only topological information about a curve, such as the subgroup structure of its fundamental group, but also Galois-theoretic information. The field of rational functions on the covering space forms an extension of the field of rational functions on the curve, which allows us to obtain arithmetic information as well.

Given a curve X defined over a field K and a fixed branch locus B, one defines its *algebraic fundamental group* $\pi_1(X \setminus B)$ as the inverse limit of the automorphism groups $\mathrm{Aut}(Y \to X)$, where Y ranges over finite Galois covers of X branched over B. This is analogous to viewing the topological fundamental group of a space as the group of deck transformations of its universal cover. When K has characteristic zero, one can compute this explicitly with loops as one would in the complex-analytic case. Indeed, by the Riemann Existence Theorem ([Gro71], Exposé XII, Corollaire 5.2), the fundamental group here is the profinite completion of the topological fundamental group of X viewed as a Riemann surface over $\mathbb{C}$.

When k has positive characteristic p, however, the situation is more difficult. In this case, one may have wildly ramified branch points, which complicate the computation of the algebraic fundamental group. For instance, in characteristic p, P^1 has nontrivial covers branched over a single point; in other words, one has covers that do not arise from the complex-analytic topology, in which $P^1 \setminus \{\infty\}$ is simply connected. This thesis addresses two methods for studying fundamental groups in positive characteristic.

First, one may instead choose to work with the *tame fundamental group*. Let K be a complete discretely valued field of characteristic zero whose residue field k has positive characteristic $p > 0$. Here, we construct the p-tame fundamental group by taking the inverse limit of only the $\mathrm{Aut}(Y \to X)$ for which $Y \to X$ has prime-to-p ramification indices. If X is a curve with good reduction over K (§2.4), we have, due to Grothendieck, a surjection

$$sp : \pi_1(X \setminus B)^{p-tame} \to \pi_1(\overline{X} \setminus B)^{tame}.$$

Understanding $\pi_1(\overline{X} \setminus B)^{tame}$ then amounts to understanding the kernel of the map sp. We have the monodromy action of $\pi_1(X \setminus B)^{p-tame}$ on p-tame covers of X with branch locus B, and so an element of $\pi_1(X \setminus B)^{p-tame}$ will be in the kernel exactly when it acts trivially on every such cover having good reduction.

Our task, then, is to determine when such covers have good reduction.

In [Ray99], Raynaud gave a criterion for certain covers to have good reduction:

Theorem 0. *Let* G *be a finite group with Sylow* p*-subgroup of order* p*. Let* k *be an algebraically closed field of positive characteristic* p*,* $K = \mathit{Frac}(W(k))$*, and* L *be a finite extension of* K *such that the absolute ramification index of* K *is less than* $\frac{p-1}{m_G}$*. Let* $f : Y \to \mathbb{P}^1$ *be a* G*-cover defined over* L *branched above the points* $0, 1, \infty$*. Then* f *has potentially good reduction.*

In the same paper, Raynaud suggested that one may obtain a similar result for both three-point covers and *one-point covers* (that is to say, covers of elliptic curves branched over one point) whose Galois groups have arbitrarily large cyclic Sylow p-subgroups. Indeed, Raynaud also proved Theorem 0 for one-point covers and the two cases often have analogous properties: for instance, both have the same étale fundamental group. Moreover, in [Obu17], Theorem 5.2, Obus proved the analogue of Theorem 0 when one allows G to have an arbitrary cyclic Sylow p-subgroup. We complete the analogy in the following:

Theorem 1.1.1. *Let* G *be a finite group with cyclic Sylow* p*-subgroup. Let* k *be an algebraically closed field of positive characteristic* p*,* $K = \mathit{Frac}(W(k))$*, where* $W(k)$*, and* L *be a finite extension of* K *such the absolute ramification index of* K *is less than* $\frac{p-1}{m_G}$*. Let* $f : Y \to E$ *be a one-point* G*-cover defined over* K*, where* E *is an elliptic curve having good reduction and* Y *is a curve of genus* $g \geqslant 2$*. Then* f *has potentially good reduction.*

The second method concerns studying covers in characteristic p directly. Let n denote the minimal number of generators of $G/p(G)$, where $p(G)$ is the subgroup of G generated by all p-subgroups of G. Abhyankar's conjecture, since proven by Raynaud ([Ray90]) and Harbater ([Har94]), gives explicit criteria for a finite group G to be realized as the Galois group of a cover of a curve of genus g with r branch points. Specifically, G can be realized if and only if

$$n \leqslant 2g + r$$

It does not, however, provide any description of the inertia groups above these branch points.

Suppose G has a Sylow p-subgroup of the form $\mathbb{Z}/p$ and suppose K is a discretely valued field with residue field k, both of which have characteristic $p > 0$. In [Pri02a], Pries gives the minimal conductor j_{min} a G-cover of $\mathbb{P}^1$ defined over K branched over a single point with inertia group $\mathbb{Z}/p \rtimes \mathbb{Z}/m$, with m prime to p, can have. Moreover, she proves the existence of a cover of $\mathbb{P}^1$ with realizing this conductor. Her proof draws on two main results. First, given such a G-cover $f : Y \to \mathbb{P}^1$ whose conductor is not minimal, she proves the existence of a degeneration (§2.8) of f ([Pri02b], Theorem 3.3.7); that is, a family of covers over a scheme S, one member of which is f and another member of which has bad reduction. Second, she derives a vanishing cycles formula ([Pri02a], Theorem 2.8), which relates the arithmetic genus of the base curve to ramification data of the cover (which includes the conductors of branch points). The idea of the proof is as follows: if f does not have minimal conductor, we can generate a new cover of $\mathbb{P}^1$ branched over one point from its degeneration. The vanishing cycles formula allows us to show that its conductor is smaller than that of the original cover. We the repeat this process until we have a cover with minimal conductor.

I have generalized these techniques to the case when G has an arbitrary cyclic Sylow p-subgroup. For the degenerations, I have the following:

Theorem 1.1.2. *Let $f_k : Y_k \to \mathbb{P}^1$ be a G-cover branched at exactly one point, at which it has inertia group $\mathbb{Z}/p^n \rtimes \mathbb{Z}/m$ with m prime to p. There exists a degeneration of f over Ω for some proper connected variety Ω.*

As for the vanishing cycles formula, I have Theorem 4.2.1. The exact statement here is omitted due to its cumbersome notation.

1.2 Structure of the thesis

Chapter 2 provides the background needed for Chapters 3 and 4. Section 2.1 gives some common notation and results concerning extensions of discretely valued fields. Section 2.2 specifically concerns results on so-called tame-by-cyclic extensions in positive characteristic, mostly due to Obus and Pries ([OP10]). In particular, it exhibits a parameter space for such extensions of $k((t))$. Sections 2.3, 2.4, and 2.5 give information about covers of curves and their models. In particular, it introduces the stable model, which will be of key importance in the proof of Theorem 1.1.1. Section 2.6 gives some results concerning the inertia groups of the special fiber of the stable model. Section 2.7 gives an overview of the auxiliary cover attached to the cover f, originally due to Raynaud ([Ray99]) and generalized by Obus ([Obu13a]). It also plays a key role in the proof of Theorem 1.1.1, as it allows us to work with a cover with some of the same ramification data as f has but with a less complicated Galois group. Finally, Section 2.8 gives some definitions and results on families of wildly ramified covers of $\mathbb{P}^1$, mostly due to Pries ([Pri02b]).

Chapter 3 contains the proof of Theorem 1.1.1. The proof follows the idea of the proof in [Obu17]. In Section 3.1, we examine the possible torsor structures components of the stable model can have. Certain torsor structures over certain components indicate that a cover has bad reduction. In Section 3.2, we show that a cover as described in Theorem 1.1.1 cannot have these torsor structures, allowing us to conclude good reduction.

Chapter 4 contains the proofs of Theorem 1.1.2 and Theorem 4.2.1. Section 4.1 proves Theorem 1.1.2 by exhibiting a parameter space for $\mathbb{Z}/p^n \rtimes \mathbb{Z}/m$-covers of $\mathbb{P}^1$ and using it to find a family with a member having bad reduction. Section 4.2 contains the proof of Thoerem 4.2.1, which is an explicit calculation involving the comparison of the genera of the generic fiber and special fiber.

1.3 Notation

Throughout this thesis, if K is a complete discretely valued field, $\mathcal{O}_K$ will refer to its ring of integers and k will refer to its residue field. We say that K has *mixed characteristic* $(0, p)$ when K has characteristic 0 and k has positive characteristic p. We say that it has *equal characteristic* p when both K and k have positive characteristic p. If K has discrete valuation ν, we call $e(K) = \nu(p)$ its absolute ramification index.

A G-*cover* of K-curves refers to a finite map of curves $f : Y \to X$ together with an isomorphism $G \xrightarrow{\sim} \mathrm{Aut}(Y/X)$, where both X and Y are smooth, projective, and geometrically integral K-curves and the extension $K(Y)/K(X)$ is Galois with group G. If K has characteristic 0, a *one-point cover* refers to a cover of an elliptic curve branched at exactly one point; we typically assume that this point is the origin If K has characteristic $p > 0$, a one-point cover refers to a cover of

$\mathbb{P}^1$ branched at exactly one point; we typically assume this point is ∞.

When G has a cyclic Sylow p-subgroup P, we denote by m_G the value $|N_G(P)|/Z_G(P)|$.

Let $f : Y \to X$ be an arbitrary scheme morphism and H a finite group with $H \hookrightarrow \mathrm{Aut}(Y/X)$. If G is a finite group containing H, we define $\mathrm{Ind}_H^G f : \mathrm{Ind}_H^G Y \to X$ by setting $\mathrm{Ind}_H^G Y$ equal to the disjoint union of $[G : H]$ copies of Y (indexed by the the left cosets of H in G) and applying f to each copy. The group G then acts on $\mathrm{Ind}_H^G Y$, giving rise to a G-cover $\mathrm{Ind}_H^G f$.

Chapter 2

Background

2.1 Extensions of discretely valued fields

In this section, we recall some facts about discretely-valued fields and their extensions; see, for instance, [Ser79], §IV.

Let K be a complete discretely valued field with ring of integers $\mathcal{O}_K$ and residue field k of positive characteristic $p > 0$. Let L be a finite Galois extension of K with Galois group G. The field L is complete with respect to the valuation ν_L, which extends that of K.

The group G has a filtration $G = G_0 \subseteq G_1 \subseteq \cdots$, where $G_i = \{g \in G : \nu_L(gx - x) \geqslant i + 1\}$. One calls these groups the *higher ramification groups with respect to the lower numbering*. There is a corresponding filtration $G = G^0 \subseteq G^{u_1} \subseteq G^{u_2} \subseteq \cdots$ called the *higher ramification groups with respect to the upper numbering*. There is a function φ from the lower numbering to the upper numbering with the property that $\varphi(0) = 0$ and $\varphi(i) \leqslant \varphi(j)$ whenever $i \leqslant j$. The lower numbering is invariant under taking subgroups and the upper numbering is invariant under taking quotients.

Any number i for which $G_{i+1} \subsetneq G_i$ is called a *lower jump* of the extension L/K. Similarly, any number j for which $G^{j+\epsilon} \subsetneq G^j$ for all $\epsilon > 0$ is called an *upper jump* of L/K. For sufficiently large $i \in \mathbb{Z}$, $G_i = G^i = \{0\}$. The largest number j for which $G^j \neq \{0\}$ (that is, the largest upper jump) is called the *conductor of higher ramification of* L/K.

The extension L/K with respective residue fields λ and κ is called *unramified* if $[L : K] = [\lambda : \kappa]$. It is called *tamely ramified* if λ/κ is separable and $[L : T]$ is prime to p, where T is the maximal unramified subextension of L/K. If it is not tamely ramified, it is *wildly ramified*.

2.2 Wildly ramified cyclic extensions

Let k be an algebraically closed field of positive characteristic $p > 0$. If R is a complete equal characteristic discrete valuation ring with fraction field K, uniformizer t, and residue field k, it is known (c.f. [Ser79], II, Theorem 2) that $R \cong k[\![t]\!]$ and $K \cong k((t))$. If L is a separable Galois field extension of K with group G and S is the integral closure of R in L, then S is a Galois extension of rings with group G totally ramified over the prime ideal (t).

By [Lan93], p. 331, Ex. 50, every Galois extension of K with Galois group $G \cong \mathbb{Z}/p^n$ has Witt vector equations $(y_1^p, \ldots, y_n^p) = (y_1, \ldots, y_n) + (x_1, \ldots, x_n)$, where every x_i is in K and the polynomial $z^p - z = x_i$ has no roots in K. There is a generator τ of $\mathbb{Z}/p^n$ whose action on Witt vectors is $\tau(y_1, \ldots, y_n) = (y_1, \ldots, y_n) + (1, 0, \ldots, 0)$. One says that $(x_1, \ldots, x_n)$ is in *standard form* if each x_i is in $k[t]$ and either $x_i = 0$ or no exponent of x_i is divisible by p. The upper (and therefore lower) jumps of such an extension can be determined entirely from this Witt vector equation (c.f. [OP10] Lemma 3.4):

Lemma 2.2.1. *Let L/K be a $\mathbb{Z}/p^n$-Galois extension with Witt vector $(x_1, \ldots, x_n)$ in standard form. Let $u = \max\{-p^{n-i}\nu(x_i)\}_{i=1}^n$. Then u is the conductor of L/K.*

The following lemma describes which integer sequences can occur as the sequence of upper jumps of a $\mathbb{Z}/p^n$-Galois extension of K (c.f. [OP10], Lemma 3.6):

Lemma 2.2.2. *A sequence of positive integers $w_1 \leqslant \ldots \leqslant w_n$ occurs as the set of upper jumps of a $\mathbb{Z}/p^n$-extension of K if and only if $p \nmid w_1$ and, for $1 < i \leqslant n$, either $w_i = pw_{i-1}$ or both $w_i > pw_{i-1}$ and $p \nmid w_i$.*

Now let $G = \mathbb{Z}/p^n \rtimes \mathbb{Z}/m$ with m prime to p. Let $\sigma \in G$ have order p^n and let $m' = \frac{|Z_G(\sigma)|}{p^n}$. Let L/L_0 denote the $\mathbb{Z}/p^n$-subextension. This extension L/K is determined entirely by its upper and lower jumps. The following ([OP10], Theorem 5.2) gives necessary and sufficient conditions for a sequence of rational numbers to occur as the upper jumps of a G-Galois extension of K:

Theorem 2.2.3. *Let $G = \mathbb{Z}/p^n \rtimes \mathbb{Z}/m$ with m prime to p. A sequence of rational numbers $u_1 \leqslant \ldots \leqslant u_n$ occurs as the set of jumps in the upper numbering of the ramification filtration of a G-Galois extension of $k((t))$ if and only if:*

(i) $u_i \in \frac{1}{m}\mathbb{N}$ *for* $1 \leqslant i \leqslant n$*,*

(ii) $\gcd(m, mu_1) = m'$*,*

(iii) $p \nmid mu_1$ *and, for* $1 < i \leqslant n$*, either* $u_i = pu_{i-1}$ *or both* $u_i > pu_{i-1}$ *and* $p \nmid mu_i$*,*

(iiii) and $mu_i \equiv mu_1 \mod m$ *for* $1 \leqslant i \leqslant n$*.*

We want to parameterize all such extensions. The following ([OP10], Lemma 4.4) describes the isomorphism classes of such an extension:

Lemma 2.2.4. *There is a Witt vector* $(x_1, \ldots, x_n)$ *in standard form for* L/L_0 *and it is uniquely determined up to multiplication by a root of unity* $\mu_{m/m'}$*. There are* $\frac{\phi(m)}{\phi(m/m')}$ *non-isomorphic* G*-Galois structures on the extension* L/K *such that* σ *acts on Witt vectors as* $\sigma(y_1, \ldots, y_n) = (y_1, \ldots, y_n) + (1, 0, \ldots, 0)$*.*

In fact, there is a scheme $\mathcal{M}_\eta$, where η is a ramification filtration of a G-Galois extension, parameterizing G-Galois extensions of K with ramification filtration η. That is, a scheme whose k-points are in bijection with G-Galois extensions of $k((t))$ with upper ramification filtration η. Following [OP10], we describe this scheme.

Given positive integers w and m, let $\epsilon_p(w, m) = \{e \in \mathbb{Z} \,|\, 1 \leqslant e \leqslant w, e \equiv w \mod m, p \nmid e\}$.

Given a positive integer N, the root of unity $\mu_{m/m'}$ acts on affine N-space $\mathbb{A}^N$ by multiplication on each coordinate. We denote the quotient by $\mathbb{A}^N/\mu_{m/m'}$. The following ([OP10], Theorem 5.6) describes the parameter space $\mathcal{M}_\eta$:

Theorem 2.2.5. *Let* $G = \mathbb{Z}/p^n \rtimes \mathbb{Z}/m$ *with* m *prime to* p. *Let* $u_1 \leqslant \ldots \leqslant u_n$ *be a sequence satisfying the conditions of Theorem 2.3 and let* η *denote the ramification filtration of* G *with upper jumps* $u_1 \leqslant \ldots \leqslant u_n$. *Let* $N_\eta = \sum_{i=1}^{n} |\epsilon_p(mu_i, m)|$. *Then there is an open subscheme* U_η *of* $\mathbb{A}^{N_\eta}/\mu_{m/m'}$ *and a finite étale map* $\pi : \mathcal{M}_\eta \to U_\eta$ *of degree* $\frac{\phi(m)}{\phi(m/m')}$ *such that the* k*-points of* $\mathcal{M}_\eta$ *are in natural bijection with isomorphism classes of* G*-Galois extensions of* $k((t))$ *with ramification filtration* η.

2.3 Covers of curves

Let $f : Y \to X$ be a morphism of smooth, projective, geometrically irreducible curves. One calls f a *branched cover* if f is finite, surjective, and generically étale. One calls f a *Galois cover with group* G (or simply a G-*cover*) if one also has an isomorphism of G with $\mathrm{Aut}(Y/X)$ and the extension $K(Y)/K(X)$ is Galois with group G. If f is defined over a discretely valued field K with residue characteristic $p > 0$, one calls f a *tame* cover if every ramification index is prime to p.

For a cover $f : Y \to X$, where Y and X have respective genera g_Y and g_X, the Riemann-Hurwitz formula ([Har77], IV §2) states that

$$2g_Y - 2 = (\deg(f)(2g_X - 2) + |\Delta|$$

where Δ is the ramification divisor.

If $x \in X, y \in Y$ with $f(y) = x$, one has the extension of complete discrete valuation rings $\widehat{\mathcal{O}}_{Y,y}/\widehat{\mathcal{O}}_{X,x}$. One says that f is ramified, tamely ramified, or wildly ramified at y when this extension is.

Suppose the Galois group of $\widehat{\mathcal{O}}_{Y,y}/\widehat{\mathcal{O}}_{X,x}$ is $\mathbb{Z}/p^n \rtimes \mathbb{Z}/m$. Let $\{j_1, \ldots, j_n\}$ and $\{u_1, \ldots u_n\}$ be the lower and upper jumps, respectively, of this extension. Let $|\Delta_y|$ denote the degree of Δ at y. Then we have the following (c.f. [Obu12], Lemma 2.7):

Lemma 2.3.1. *In terms of the lower jumps, one has*

$$|\Delta_y| = p^n m - 1 + \sum_{i=1}^{n} j_i p^{n-i}(p-1) = p^n m - 1 + \sum_{i=1}^{n} (p^{n-i+1} - 1)(j_i - j_{i-1}).$$

In terms of the upper jumps, one has

$$|\Delta_y| = p^n m - 1 + \sum_{i=1}^{n} m p^{i-1}(p^{n-i+1} - 1)(u_i - u_{i-1}).$$

2.4 Models of curves

Let X be a curve defined over the complete, discretely valued field K. A *model* of X over $\mathcal{O}_K$ is an integral, normal scheme $\mathcal{X}$, projective and flat over $\mathrm{Spec}(\mathcal{O}_K)$, whose generic fiber $\mathcal{X}_K$ is isomorphic to X. One denotes by $\overline{X}$ the special fiber $\mathcal{X} \times_{\mathcal{O}_K} k$, called the *reduction* of X.

In general, the model $\mathcal{X}$ is not unique. If one can choose a model $\mathcal{X}$ in which $\overline{X}$ is smooth, one says that X has *good reduction* and calls $\mathcal{X}$ a *good model* of X. Otherwise, one says that X has *bad reduction*. If one can choose a good model over a finite extension L of K, one says that X has *potentially good reduction*.

In the language of varieties, if one has a smooth variety X defined by a choice of polynomials $\{f_1, \dots, f_n\}$ over K, one can reduce these polynomials modulo the maximal ideal π of $\mathcal{O}_K$ (by clearing denominators, one can always define the polynomials over $\mathcal{O}_K$). The polynomials $\{f_1 \mod \pi, \dots, f_n \mod \pi\}$ define a variety $\overline{X}$ over κ. If $\overline{X}$ is smooth, X would have good reduction. If $\overline{X}$ is singular, one may be able to perform a change of variables over an extension L of K that result in a smooth curve over κ; in this case, X would have potentially good reduction.

If X has bad reduction, by the Semistable Reduction Theorem ([DM69], Corollary 2.7) one can choose a model (after perhaps taking a finite extension L of K) in which $\overline{X}$ satisfies the following:

1. All singularities of $\overline{X}$ are ordinary double points.

2. Any irreducible genus 0 component of $\overline{X}$ contains at least three marked points.

Here, “marked points" describes points of intersection with the rest of $\overline{X}$. One calls such a model a *semistable model* of X. In general, any curve satisfying conditions (1) and (2) is called *semistable*. In some sense, $\overline{X}$ is “as close to smooth as possible."

2.5 Models of covers

Let $f : Y \to X$ be a cover of curves. A *model* of f is a cover of relative curves $\mathcal{Y} \to \mathcal{X}$ over $\mathcal{O}_K$ whose generic fiber is isomorphic to f.

Suppose X has good reduction over $\mathcal{O}_K$. One says that f has *good reduction* if one can choose a model of f in which $\overline{Y}$ is smooth and the extension $K(\overline{Y})/K(\overline{X})$ is separable. If one can only do this over a finite extension L of K, one says that f has *potentially good reduction*. If one can do neither, one says that f has *bad reduction*.

As with models of curves, if f has bad reduction, one can construct a model of f which is “as close to smooth as possible." After perhaps a finite extension L of K, one obtains a semistable model Y^{st} of Y (in which “marked points" now include both points of intersection of a component with the rest of $\overline{Y}$ or specializations of ramification points). The action of G on Y^{st} induces an action on $\overline{Y}$. By [Ray99], Corollary 2.3.3i, the quotient Y^{st}/G is a semistable model of X, denoted X^{st}. One then has $f^{st} : Y^{st} \to X^{st}$, the *stable model* of f, and $\overline{f} : \overline{Y} \to \overline{X}$, the *stable reduction* of f. If $\overline{Y}$ is smooth in the stable model and the extension $K(Y)/K(X)$ is separable, one can say that f has good reduction.

The stable model of f can be achieved over a minimal finite extension K^{st} of K. By [Ray99], Théorème 2.2.2, the field K^{st} is the fixed field of the subgroup of $G_K = \mathrm{Gal}(\overline{K}/K)$ generated by all $\sigma \in G_K$ that act trivially on $\overline{Y}$. This extension is Galois over K; one denotes its Galois group by Γ_K.

Since X has good reduction, the special fiber $\overline{X}^{st}$ is a blow-up of a fixed good model of X. One calls the strict transform of its special fiber the *original* *component*, denoted $\overline{X}_0$.

2.6 Ramification on the special fiber

The following is [Ray99], Proposition 2.4.11:

Proposition 2.6.1. *The inertia groups of* $\overline{f} : \overline{Y} \to \overline{X}$ *at the points of* $\overline{Y}$ *are as follows:*

1. *At the generic points of irreducible components, the inertia groups are* p*-groups.*

2. *At each node at the intersection of irreducible components* $\overline{V}$ *and* $\overline{V}'$*, the inertia group is the extension of a cyclic, prime-to-*p *order group by a* p*-group generated by the inertia groups of the generic points of* $\overline{V}$ *and* $\overline{V}'$*.*

3. *At the specialization of a ramification point* y *of* Y *that specializes to a smooth point of* $\overline{V} \subset \overline{Y}$*, the inertia group is an extension of the prime-to-*p *part of the inertia group at* y *by the inertia group of the generic point of* $\overline{V}$*.*

If $\overline{V}$ is an irreducible component of $\overline{Y}$, we will denote by $I_{\overline{V}}$ and $D_{\overline{V}}$ the inertia and decomposition groups, respectively, at the generic point of $\overline{V}$.

For an irreducible component $\overline{U}$ of $\overline{X}$, the inertia groups of the components of $\overline{Y}$ above $\overline{U}$ are conjugate p-groups. If they have order p^i, we call $\overline{U}$ a p^i *component*. We say that $\overline{U}$ is *étale* when $i = 0$ and *inseparable* otherwise. This comes from the fact that, since $\overline{Y}$ is reduced, the inertia arises from an inseparable extension of residue fields at this generic point.

For the remainder of this section, assume G has a cyclic Sylow p-subgroup. The following is [Obu12], Corollary 2.11, which gives the relationship between the inertia groups of intersecting components:

Corollary 2.6.2. *If $\overline{U}$ and $\overline{U}'$ are two intersecting components of $\overline{X}$, then either $I_{\overline{U}} \subseteq I_{\overline{U}'}$ or $I_{\overline{U}'} \subseteq I_{\overline{U}}$.*

Moreover, the ramification index of a point $x \in X$ places restrictions on to what components this point can specialize (c.f. [Obu12]):

Proposition 2.6.3. *If $x \in X$ is branched with index $p^a s$, where $(p, s) = 1$, then x specializes to a p^a-component.*

We call an irreducible component $\overline{U}$ a *tail* if it intersects the rest of $\overline{X}$ at exactly one point; otherwise, it is an *interior component*. If, in addition, a tail contains the specialization of a branch point, we call this tail *primitive*; otherwise, we say that it is *new*. This follows the convention established, for example, in [Ray99], Definition 3.3.1. When the genus of X is greater than 0, we assume that each tail contains the specialization of at most one branch point. We index the étale tails by the set B_{et}, which we divide into the new étale tails and primitive étale tails by the sets B_{new} and B_{prim}, respectively.

Now let x be a point of intersection of two components $\overline{U}$ and $\overline{U}'$ with $I_{\overline{U}'} \subseteq I_{\overline{U}}$. Let y be a point lying above x; the point y is the intersection point of two components $\overline{V}$ and $\overline{V}'$ lying above $\overline{U}$ and $\overline{U}'$, respectively. Following [Obu12], Definition 2.1.8, we call the conductor of higher ramification (defined in [Ser79], IV §3, for instance) of the extension of complete discrete valuation rings $\hat{\mathcal{O}}_{\overline{U}',x} \hookrightarrow \hat{\mathcal{O}}_{\overline{V}',y}$ the *effective ramification invariant* at x, denoted σ_x. If x is on a tail $\overline{X}_b$ we will often simply write σ_b for σ_x.

We recall the following, which places some bounds on these invariants; it is originally found in [Obu12] in the form of Lemmas 2.20 and 4.2:

Lemma 2.6.4. *The effective ramification invariants σ_b are positive and lie in $\frac{1}{m_G}\mathbb{Z}$. If σ_b is a new tail, we have $\sigma_b \geqslant 1 + \frac{1}{m_G}$.*

Let E be an elliptic curve with good reduction over R and let $\langle \sigma_b \rangle$ denote the fractional part of σ_b. Suppose $G = \mathbb{Z}/p^s \rtimes \mathbb{Z}/m$. If $f : Y \to E$ is a G-cover branched at the points $P_1, \dots, P_r$, $r > 1$ (note that we now require $r > 1$ since any one-point G-cover with G abelian, as g would be when $r = 1$, is merely an isogeny of elliptic curves; see Lemma 3.2.1). In this situation, we can describe $\langle \sigma_b \rangle$ more precisely. We can decompose f into an étale $\mathbb{Z}/p^n$-cover $Y \to Z$ and a $\mathbb{Z}/m$-cover $g : Z \to E$. The latter is given, birationally, by $z^m = f_r \cdot f_u$, where f_r and f_u are rational functions on E corresponding to, respectively, a divisor in the form $\sum_{i=1}^{r} a_i P_i$, $0 < a_i < m$, and an m-divisible divisor on E. We say that $f : Y \to E$ is of *multiplicative type* when $\sum_{i=1}^{r} a_i = m$. The meaning of multiplicative type will become clearer later (Proposition 3.1.2). The following is a generalization of [Wew03], Proposition 1.8, originally found in [Obu17], Proposition 3.6. It was originally proven in the case in which the base curve is $\mathbb{P}^1$, but can be proven identically in the case in which the base curve is an elliptic curve E.

Proposition 2.6.5. *Let $f : Y \to E$ be a $G = \mathbb{Z}/p^s \rtimes \mathbb{Z}/m$-cover branched in $r \geqslant 1$ points and let f^{ss} be a fixed semistable model for f. Suppose $\overline{X}_b$ is an étale tail of $\overline{E}^{ss}$*

containing the specialization of a unique branch point P_i *with corresponding* a_i. *Then* $\langle\sigma_b\rangle = \frac{a_i}{m}$.

In [Ray99], Corollary 3.4.4, we find the *vanishing cycles formula*, which relates the invariants σ_b and the genus of X in the case in which G has a Sylow p-subgroup of the form $\mathbb{Z}/p$. This formula is generalized to the case in which G has a *cyclic* Sylow p-subgroup in [Obu12], Theorem 3.14; we recall this version below.

Theorem 2.6.6. *Let* $f : Y \to X$ *be a* G*-Galois cover branched at* $r \geqslant 1$ *points having bad reduction, where* G *has a cyclic Sylow* p*-subgroup; let* g *be the genus of* X*; for each* $b \in B_{et}$, *let* σ_b *be the corresponding effective ramification invariant. Then we have*

$$2g - 2 + r = \sum_{b \in B_{et}} (\sigma_b - 1)$$

In particular, when X is an elliptic curve, we have

$$r = \sum_{b \in B_{new}} (\sigma_b - 1) + \sum_{b \in B_{prim}} \sigma_b$$

2.7 The auxiliary cover

If $f : Y \to X$ is a G-cover with bad reduction (in particular, $\overline{X}$ has a component other than the original component $\overline{X}_0$), one can construct (as originally in [Ray99] and adapted for arbitrary G in [Obu13b]) the *auxiliary cover* over a finite extension K' of K. This cover $f^{aux} : Y^{aux} \to X^{aux}$ is Galois with group

$G^{aux} \leqslant G$ and has the following properties:

(i) $(X^{aux})^{ss} = X^{st}$ and $\overline{X}^{aux} = \overline{X}$.

(ii) There is an étale neighborhood Z of the union of the inseparable components of $\overline{X}$ with the property that, as covers, $f^{st} \times_{X^{st}} Z \cong \left(\mathrm{Ind}_{G^{aux}}^{G}(f^{aux})^{st}\right) \times_{X^{st}} Z$.

(iii) The branch locus of f^{aux} consists of a branch point x_b of index m_b for each étale tail $\overline{X}_b$ of $\overline{X}$ for which $m_b > 1$. If $\overline{X}_b$ is primitive, so that it contains a branch point of f, x_b is this corresponding branch point. If $\overline{X}_b$ is new, x_b specializes to a smooth point of $\overline{X}$ on $\overline{X}_b$.

(iv) Given an étale tail $\overline{X}_b$ of $\overline{X}$ and an irreducible component $\overline{V}_b$ of $\overline{Y}^{aux}$ above $\overline{X}_b$ with effective ramification invariant σ_b^{aux}, we have $\sigma_b^{aux} = \sigma_b$.

(v) If N is the maximal prime-to-p normal subgroup of G^{aux}, then $G^{aux}/N \cong \mathbb{Z}/p^s \rtimes \mathbb{Z}/m_{G^{aux}}$ with $s \geqslant 1$.

The following is a recollection of its construction: one removes from $\overline{Y}$ every component above the étale tails of $\overline{X}$. What remains is possibly disconnected; fix one connected component and denote it $\overline{V}$.

For each $b \in B_{prim}$, let α_b be the branch point of f specializing to $\overline{X}_b$, let $\overline{x}_b$ be the point of intersection of of $\overline{X}_b$ with the rest of $\overline{X}$, and $p^r m_b$, with $(p, m_b) = 1$, be the ramification index above $\overline{x}_b$. By Proposition 2.2, $\overline{X}_b$ intersects a p^r-component. At each point $\overline{v}_b \in \overline{V}$ above $\overline{x}_b$, one attaches a Katz-Gabber cover (as in [Ray99], Théorème 3.2.1). The branch locus of this cover comprises a tamely ramified branch point at the specialization $\overline{\alpha}_b$ of α_b of order m_b and a wildly ramified branch point at $\overline{x}_b$ of order $p^r m_b$. One can write this cover as the composition of a tame cover branched of order m_b at $\overline{\alpha}_b$ and $\overline{x}_b$ and a wild cover branched of order p^r at $\overline{x}_b$.

For each $b \in B_{new}$, one does the same, but introduces an arbitrary branch point $\overline{\alpha}_b$ distinct from $\overline{x}_b$ of order m_b.

In particular, the auxiliary cover has a simpler Galois group but at the expense of possibly introducing more branch points.

Now suppose G has a cyclic Sylow p-subgroup. From [Obu13b], Proposition 7.3, one has the following:

Proposition 2.7.1. *If G has a cyclic Sylow p-subgroup, then G^{aux} has a normal subgroup of order p.*

This allows one to make a further simplification: G^{aux} then has a maximal normal prime-to-p subgroup N with the property that $G^{aux}/N = G^{str} \cong \mathbb{Z}/p^n \rtimes \mathbb{Z}/m_{G^{aux}}$, where $m_{G^{aux}} | m_G$. Then $Y^{str} = Y^{aux}/N$ is a G^{str}-Galois cover of X, called the *strong auxiliary cover*.

Finally, passing to the auxiliary cover does not change the ramification invariants (§2.7) (c.f. [Obu17], Proposition 4.1):

Proposition 2.7.2. *Given an étale tail $\overline{X}_b$ of $\overline{X}$ and an irreducible component $\overline{V}_b$ of $\overline{Y}^{aux}$ above $\overline{X}_b$ with effective ramification invariant σ_b^{aux}, we have $\sigma_b^{aux} = \sigma_b$.*

Moreover, since taking quotients by subgroups of prime-to-p order does not affect the ramification invariants, they also remain the same when using the strong auxiliary cover.

2.8 Families of covers in positive characteristic

Let K be a discretely valued field with valuation ring $\mathcal{O}_K$ and algebraically closed residue field k. We now have $\mathrm{char}(K) = \mathrm{char}(k) = p > 0$. Let G be a finite group. In this section, we will outline some facts involving families of covers defined over k. See also [Pri02b].

Let S be an irreducible k-scheme. A *family of curves* over S is a flat morphism $X \to S$ of relative dimension 1. That is, each fiber of this morphism is a k-curve. A *family of* G*-covers* is a flat S-morphism $f_S : Y_S \to X_S$, where X_S and Y_S are families of curves over S, the action of G on Y_S is compatible with the map f_S, and the fibers of f_S are G-covers.

Let $f_S : Y_S \to X_S$ and $f'_S : Y'_S \to X_S$ be two families of G-covers.

- The covers f_S and f'_S are *weakly isomorphic* if there is a G-equivariant S-isomorphism $\phi_Y : Y'_S \to Y_S$ and S-isomorphism $\phi_X : X_S \to X_S$ which satisfy $\phi_X \circ f'_S = f_S \circ \phi$.
- They are *isomorphic* if ϕ_X can be taken to be the identity isomorphism.
- We say that f_S is *weakly constant* if there is a G-cover $f : Y_k \to X_k$ for which f_S is weakly isomorphic to $f \times_k S : Y_k \times_k S \to X_k \times_k S$.
- We say that it is *constant* if it is isomorphic to $f \times_k S : Y_k \times_k S \to X_k \times_k S$.

Let g be the genus of the fibers of Y_S. We say that Y_S is *isotrivial* if the induced morphism $S \to \mathcal{M}_g$, where $\mathcal{M}_g$ is the moduli space of genus g curves, is constant.

Given a cover $f : Y \to X$, the *germ* $\widehat{X}_x$ is the spectrum of the complete local ring at $x \in X$. Let $y \in Y$ be a closed point with $f(y) = x$. The germ $\hat{f}_y$ of f at y is the cover of spectra of complete local rings $\hat{f}_y : \widehat{Y}_y \to \widehat{X}_x$. The following (c.f. [Pri02b], Proposition 2.1.3) gives a local criterion for families of branched covers to be constant:

Proposition 2.8.1. *Let X_k be a smooth connected curve and let $B_k = x_1, \dots, x_m$ be a finite set of points of X_k, let S be a connected k-scheme, and let $f_S : Y_S \to X_k \times_k S$ be a family of G-covers of smooth connected curves over S branched only at $x_i \times_k S$, $1 \leqslant i \leqslant m$. Then f_S is constant if and only if $\hat{f}_{x_i \times_k S}$ is constant for $1 \leqslant i \leqslant m$.*

Now we consider the case of G-covers of $\mathbb{P}^1_S$ branched only at ∞. Let A_S be the group of *affine S-linear transformations* of $\mathbb{P}^1_S$; that is the subgroup of the automorphism group of $\mathbb{P}^1_S$ that fixes ∞. We note that a family of G-covers f_S is weakly constant if and only if $T \circ f_S$ is constant for some $T \in A_S$. The following (c.f. [Pri02b], Proposition 2.3.2) gives a local criterion for a family of covers to be constant:

Proposition 2.8.2. *Let S be a connected K-scheme, let $f_S : Y_S \to \mathbb{P}^1_S$ be a G-cover branched only at ∞, and let $\hat{f}_\infty$ be the germ at a point above ∞. Then f_S is weakly constant if and only if there exists an affine linear transformation $T \in A_S$ such that the cover $T \circ \hat{f}_\infty$ of germs is constant.*

The proof follows from Proposition 2.8.1.

Now let $f : Y \to \mathbb{P}^1_k$ be a G-cover of smooth connected curves branched only at ∞ at which it has inertia group I. A *deformation* of f over $\operatorname{Spec}(\mathcal{O}_K)$ is a G-cover $f_{\mathcal{O}_k} : Y_{\mathcal{O}_K} \to \mathbb{P}^1_{\mathcal{O}_K}$ with the closed fiber $f_{\mathcal{O}_K} \times_K k$ isomorphic to f, branched at a unique $\mathcal{O}_K$-point which specializes to ∞ with inertia group I at its generic point. We say that f is *smooth* if each fiber of $Y_{\mathcal{O}_K}$ is smooth.

The following (c.f. [Pri02a], Proposition 3.1.8), gives a criterion for a smooth deformation to be isotrivial:

Proposition 2.8.3. *Let* $f : Y \to \mathbb{P}^1_k$ *be a* G*-cover of smooth connected curves branched only at* ∞ *and suppose the genus of* Y *is at least* 2*. A smooth deformation* $f_{\mathcal{O}_K}$ *of* f *is isotrivial if and only if there is an étale cover* $S \to Spec(\mathcal{O}_K)$ *such that the pullback* f_S *is in the orbit of* f *under* A_S*. Moreover,* f_S *is weakly constant if and only if there is an étale cover* $S \to Spec(\mathcal{O}_K)$ *such that* $T \circ \hat{f}_\infty$ *is constant for some* $T \in A_S$*.*

Finally, we address *degenerations*. Let $f_k : Y_k \to \mathbb{P}^1_k$ be a G-cover of smooth connected curves and let S be a connected scheme of finite type over k. A *degeneration* of f_k over S is a deformation $f_S : Y_S \to \mathbb{P}^1_S$ over S with closed k-points $s_0, s_1 \in S$ satisfying the following:

(i) the curves Y_S and $\mathbb{P}^1_S$ are flat proper S-curves

(ii) the cover f_S is a G-cover of semistable curves with f_k and f_{s_0} isomorphic as G-covers

(iii) the pullback of f_S to $\widehat{S}_{s_0}$ is a smooth deformation of f_{s_0}

(iv) the cover f_{s_1} has bad reduction.

Chapter 3

Good reduction of covers of elliptic curves

In this section, we will prove Theorem 1.1.1:

Theorem 3.0.1. *Let* G *be a finite group with cyclic Sylow* p*-subgroup. Let* $K_0 = \mathrm{Frac}(W(k))$*, where* k *is an algebraically closed field of characteristic* $p > 0$*. Let* K *be a finite extension of* K *such that* $e(K) < \frac{p-1}{m_G}$*. Let* $f : Y \to E$ *be a one-point* G*-cover defined over* K*, where* Y *is a curve of genus* $g \geqslant 2$*. Then* f *has potentially good reduction.*

The idea of the proof is as follows: above each inseparable component $\overline{U}$ of the special fiber $\overline{X}$ of the stable model, we look at the quotient of a component of $\overline{Y}$ by a certain subgroup of its decomposition group $D_{\overline{V}}$, where $\overline{V}$ is a component of $\overline{Y}$ above $\overline{U}$. The corresponding extension of complete local rings at their generic points has the structure of a torsor under a group scheme of p-power rank. Certain types of torsors, specifically those under group schemes μ_{p^i} of roots of unity under multiplication, allow us to detect when f has bad reduction. By studying the possible torsor structures, we are able to rule out these types of torsors when the cover is defined over such a field as in the theorem, which allow us to conclude that f has good reduction.

3.1 Reduction types

Supposing that f has bad reduction, we fix a semistable model $f^{ss} : Y^{ss} \to E^{ss}$. Let $\overline{U}$ be an inseparable component of $\overline{E}^{ss}$ and let $\overline{V}$ be a component of $\overline{Y}^{ss}$ lying above $\overline{U}$ having decomposition and inertia group $D_{\overline{V}}$ and $I_{\overline{V}}$, respectively. The group $D_{\overline{V}}$ has a normal subgroup of order p, since $I_{\overline{V}}$ is normal in $D_{\overline{V}}$ and $I_{\overline{V}}$ is cyclic. Then [Obu12], Corollary 2.4, implies that $D_{\overline{V}}$ has a maximal normal prime-to-p subgroup N with $D_{\overline{V}}/N \cong \mathbb{Z}/p^j \rtimes \mathbb{Z}/m_{D_{\overline{V}}}$ for some $j > 0$; we denote by P the Sylow p-subgroup of $D_{\overline{V}}/N$.

Let η be the generic point of $\overline{V}/N$ in the quotient curve $\overline{Y}/N$. The group P acts on $\hat{\mathcal{O}}_{Y_R/N,\eta}$; we denote this ring by B and its subring fixed by P by A. The action of P on the residue field of B is trivial, so that B is a totally ramified extension of A. We say that f has *multiplicative reduction* when B/A has the structure of a μ_{p^j}-torsor.

Following [Obu17], we have an equivalent formulation which will be of use to us. We say that B/A, as above, is *of* μ_{p^j}*-type* if $\mathrm{Frac}(B)$ is a Kummer extension of $\mathrm{Frac}(A)$ given by the adjunction of a $p^{j^{th}}$ root of a unit in A that does not reduce to a p^{th} power in the residue field of A. Similarly, we say that B/A is *potentially of* μ_{p^j}*-type* if the base change by some finite extension K'/K is of μ_{p^j}-type. The former is equivalent to B/A having the structure of a μ_{p^j}-torsor.

Now we assume that G is of the form $\mathbb{Z}/p^s \rtimes \mathbb{Z}/m$ and is branched at the points $P_1, \ldots, P_r$, $r > 1$. As we saw in §2.6, we can decompose f into an étale $\mathbb{Z}/p^s$-cover $Y \to Z$ and a branched $\mathbb{Z}/m$-cover, $Z \to E$: the latter is given, birationally, by $z^m = f_r \cdot f_u$, where f_r and f_u are rational functions on E corresponding to, respectively, a divisor in the form $\sum_{i=1}^{r} a_i P_i$, $0 < a_i < m$, and an m-divisible divisor on E. Also recall that we say that f is of multiplicative type if $\sum a_i = m$.

The quotient of G by its subgroup of index pm gives a quotient cover of E with Galois group $\mathbb{Z}/p \rtimes \mathbb{Z}/m$. This corresponds to the reduction $\overline{\chi} : \mathbb{Z}/m \to \mathbb{F}_p^{\times}$ of a character $\chi : \mathbb{Z}/m \to \mu_m(K)$ mod ν_K, where ν_K is the valuation on K. The corresponding p-cyclic cover above Z is étale, so it gives to a nontrivial class in $H^1_{et}(Z, \mathbb{Z}/p)_\chi$. Letting J_Z denote the Jacobian of Z, we have a canonical isomorphism $H^1_{et}(Z, \mathbb{Z}/p)_\chi \cong J_Z[p]_\chi(-1) = J_Z[p]_\chi(-1) = \mathrm{Hom}_{\mathbb{F}_p}(\mu_p(\overline{K}), J_Z[p]_\chi)$ (see [Mil80], III §4). Choosing a p^{th} root of unity in a fixed algebraic closure $\overline{K}$ of K, we identify $J_Z[p]_\chi(-1)$ with $J_Z[p]_\chi$. We have the following result concerning the structure of the Jacobian $\mathrm{Jac}(\overline{Z})[p]_{\overline{\chi}}$ of the special fiber $\overline{Z}$, which we denote $\overline{J}[p]_{\overline{\chi}}$:

Proposition 3.1.1. *Let* $\overline{g} : \overline{Z} \to \overline{E}$ *be an* m*-cyclic cover branched at* $r \geqslant 2$ *points of multiplicative type. Then* $\overline{J}[p]_{\overline{\chi}} \cong \left(\mathbb{Z}/p\right)^{r-1} \times \mu_p$.

Proof. The cover f is given, birationally, by the equation $z^m = f_r \cdot f_u$, where f_r

and f_u are rational functions on E corresponding to, respectively, a divisor in the form $\sum_{i=1}^{r} a_i P_i$ and an m-divisible divisor on E, $\sum_{j=1}^{\rho} m b_j Q_j$. The branch locus of f comprises $\{P_1, \ldots, P_r\}$.

The action of $\mathbb{Z}/m$ on $\overline{g}_* \mathcal{O}_{\overline{Z}}$ arising from that on $\overline{Z}$ gives us the decomposition

$$\overline{g}_* \mathcal{O}_{\overline{Z}} = \bigoplus \mathcal{L}_{\overline{\psi}},$$

where $\overline{\psi}$ runs through all characters $\mathbb{Z}/m \to k^\times$, into isotypical line bundles.

We view z as a rational section of the sheaf $\mathcal{L}_{\overline{\chi}}$. We compute that the order of vanishing of z is $0 < \frac{a_i}{m} < 1$ at each P_i and $\frac{m b_j}{m} = b_j$ at each Q_j. Moreover, since z has a pole of order $\frac{1}{m}\big(\sum a_i + m \sum b_j\big) = 1 + \sum b_j$ at ∞, we compute that $\deg(\mathcal{L}_{\overline{\chi}}) = \sum b_j - (1 + \sum b_j) = -1$.

We perform a similar calculation for the rational section z^{m-1} of $\mathcal{L}_{\overline{\chi}^{-1}}$. It has order of vanishing $a_i - 1$ at each $\overline{P}_i$, since $\frac{a_i}{m}(m-1) = (a_i - 1) + (1 - \frac{a_i}{m})$ and $0 < \frac{a_i}{m} < 1$. Similarly, z^{m-1} has order of vanishing $b_j(m-1)$ at each Q_j and a pole of order $(1 + \sum b_j)(m-1)$ at ∞, so that $\deg(\mathcal{L}_{\overline{\chi}^{-1}}) = (m - r) + (m-1)\sum b_j - (1 + \sum b_j)(m-1) = 1 - r$. So $H^1(\overline{Z}, \mathcal{O}_{\overline{Z}})_{\overline{\chi}}$ and $H^1(\overline{Z}, \mathcal{O}_{\overline{Z}})_{\overline{\chi}^{-1}}$ have respective dimensions 1 and $r - 1$.

Combining [Oda69], Corollary 5.11, and [Wed08], Theorem 2.1, we may compute the rank of $\overline{J}[p]_{\overline{\chi}}$ as a group scheme as

$$\dim_k(H^1_{dR}(\overline{Z})_{\overline{\chi}}) = \dim_k(H^1(\overline{Z}, \mathcal{O}_{\overline{Z}})_{\overline{\chi}}) + \dim_k(H^0(\overline{Z}, \Omega_{\overline{Z}/k})_{\overline{\chi}}) = r$$

since $H^0(\overline{Z}, \Omega_{\overline{Z}/k})_{\overline{\chi}}$ is dual to $H^1(\overline{Z}, \mathcal{O}_{\overline{Z}})_{\overline{\chi}^{-1}}$.

There is a canonical k-linear isomorphism $\mathrm{Lie}(J[p]) \cong H^1(\overline{Z}, \mathcal{O}_{\overline{Z}})$; it is compatible with the $\mathbb{Z}/m$ action, so we also have an isomorphism of $\overline{\chi}$-eigenspaces. So $\mathrm{Lie}(J[p]_{\overline{\chi}})$ has dimension 1 and $J[p]_{\overline{\chi}}$ is isomorphic to $\left(\mathbb{Z}/p\right)^{r-1} \times G$, where G is either μ_p or α_p. Since $\overline{J}[p]_{\overline{\chi}^{-1}}$ is dual to $\overline{J}[p]_{\overline{\chi}}$, we also have that $\overline{J}[p]_{\overline{\chi}^{-1}}$ is isomorphic to $\left(\mu_p\right)^{r-1} \times G'$, where G' will be dual to G. Since $\mathrm{Lie}(J[p])_{\overline{\chi}^{-1}}$ has dimension $r-1$, we must have $G' = \mathbb{Z}/p$, and so $\overline{J}[p]_{\overline{\chi}} = \left(\mathbb{Z}/p\right)^{r-1} \times \mu_p$, as we wanted.

□

The following is analogous to [Obu17], Proposition 3.7 and gives the reasoning behind the terminology *multiplicative type*:

Proposition 3.1.2. *Let* $f : Y \to X = E$ *be a* $G = \mathbb{Z}/p^s \rtimes \mathbb{Z}/m$*-Galois cover of multiplicative type. Then any semistable model* $f^{ss} : Y^{ss} \to X^{ss}$ *of* f *has multiplicative reduction above every inseparable component of* $\overline{X}^{ss}$*.*

Proof. If $s = 1$, then by Proposition 3.1.1 and [Ray90], p. 190, the p-cyclic cover $Y \to Z$ reduces to a μ_p-cover above the generic point of each component, and so will have multiplicative reduction above every component.

If $s > 1$, then f has a quotient $\mathbb{Z}/p \rtimes \mathbb{Z}/m$-cover, h, which will also be of multiplicative type, so that any semistable model of h will have multiplicative reduction over every component of $\overline{X}^{ss}$. In particular, every component of $\overline{X}^{ss}$ will be a p^s-component for f. Let K^{ss} be a field of definition for f^{ss}, a semistable model for f; since K has algebraically closed residue field, [Obu17], Lemma 2.2 implies that f^{ss} has multiplicative reduction over all of $\overline{X}$, as we wanted.

□

Recall that the strong auxiliary cover of a G-cover, where G has a cyclic Sylow p-subgroup, is a quotient of the auxiliary cover with the property that its Galois group G is isomorphic to $\mathbb{Z}/p^s \rtimes \mathbb{Z}/m_{aux}$. We also recall that the upper-numbered filtration for the higher ramification groups is unaffected by taking quotients, so both the auxiliary cover and the strong auxiliary cover will have the same effective ramification invariants as the original cover.

Strong auxiliary covers arising from one-point covers have the following property:

Proposition 3.1.3. *If the strong auxiliary cover of a one-point cover with prime-to-p branching ramifies (that is, is not simply everywhere étale), it is of multiplicative type.*

Proof. Let $\overline{X}_b$ be an étale tail of $\overline{X}^{str}$ with effective ramification invariant σ_b. We have seen that passing to the strong auxiliary cover preserves effective ramification invariants, so by Proposition 2.6.5 and Theorem 2.6.6, $\sum\langle\sigma_b\rangle = 1$. If there were only one étale tail, this would mean that $\sigma_b = 1$ and f^{str} would be unramified, a contradiction. So we assume there are at least two étale tails. Then by Lemma 2.6.4, each σ_b is a non-integer, so that each étale tail has $m_b > 1$. So, applying Proposition 2.6.5, we see that $\sum\langle\sigma_b\rangle = \sum \frac{a_i}{m} = 1$, so that $\sum a_1 = m$ and f^{str} is of multiplicative type.

□

Proposition 3.1.3 allows us to deduce multiplicative reduction over certain components of the stable model of our cover:

Corollary 3.1.4. *Let* $f : Y \to E$ *be a branched one-point cover with bad reduction and prime-to-*p *branching indices. Then the stable model of* f *has multiplicative reduction over the original component.*

Proof. By Propositions 3.1.2 and 3.1.3, any semistable model of the strong auxiliary cover, f^{str}, has multiplicative reduction over the original component. Since f^{str} is a prime-to-p quotient of the auxiliary cover f^{aux}, any semistable model of f^{aux} will also have multiplicative reduction over the original component. The cover, f, will be isomorphic to the disjoint union of copies of f^{aux} over an étale neighborhood of the original component, so that the stable model of f will also have multiplicative reduction over the original component, as we wanted.

□

3.2 Potentially Good Reduction

Now let $f : Y \to X$ be a cover as in §2 with the additional requirement that $X \cong E$, an elliptic curve having good reduction over K.

Lemma 3.2.1. *Let* $f : X \to E$ *be a* G*-cover with* G *abelian branched at at most 1 point. Then* f *is, indeed, an étale cover* $f : X \to E$*.*

Proof. The étale fundamental group $\pi_1(E \setminus \{O\})$ of $E \setminus \{O\}$ has the presentation $\{a, b, c \,|\, aba^{-1}b^{-1}c = 1\}$. The map $\pi_1(E \setminus \{O\}) \to G$ factors via the abelianization $\pi_1(E \setminus \{O\}) \to \pi_1(E \setminus \{O\})^{ab} = \{ab \,|\, aba^{-1}b^{-1} = 1\} = \pi_1(E)$, so that f corresponds to a finite index open subgroup of $\pi_1(E)$. So the map f is, indeed, étale, as we wanted.

□

The following is analogous to [Obu17], Lemma 4.5:

Lemma 3.2.2. *Let* $f : Y \to E$ *be a tamely ramified cover, branched at* $r \geqslant 1$ *points; let* $\overline{V}$ *be an irreducible component above the original component with decomposition group* $D_{\overline{V}}$*. Then* $m_{D_{\overline{V}}} > 1$*.*

Proof. It suffices to work with the strong auxiliary cover f^{str}, which, in light of Lemma 3.2.1, we assume has more than one branch point. It will be enough to show that the decomposition group of a component above the original component is not a p-group.

The cover f^{str} has a $\mathbb{Z}/m_{G_{str}}$-quotient cover given (as, for instance, in §2) birationally by $z^m = f_r \cdot f_u$, where the rational function f_r corresponds to the divisor $\sum_1^\rho a_i P_i$ on E, with $0 < a_i < m$. It then suffices to show that the reduction of f_r is not an m^{th} power in $k(X)$.

The branch points of f specialize to distinct points on the special fiber, so we have at least r different residue classes among them. By Proposition 2.6.5 and Theorem 2.6.6, $\sum_{i=1}^\rho a_i = m \cdot r$, so that some subset of the a_i satisfy $0 < \sum a_i < m$. So the factor f_r does not reduce to an m^{th} power, as we wanted.

□

Now let G be a finite group with nontrivial cyclic Sylow p-subgroup. Let k be an algebraically closed field of characteristic p; let $K_0 = \mathrm{Frac}(W(k))$ and let K be a finite extension with $e(K) < \frac{p-1}{m_G}$, where $e(K)$ is the absolute ramification index of K and $m_G = |N_G(P)|/Z_G(P)|$. Let $f : Y \to E$ be a G-cover defined over

K, where E is an elliptic curve having good reduction over the valuation ring R of K, branched at the distinct K-points $\{x_1, \ldots, x_r\}$, with $r \geqslant 1$; suppose that the branch points of f specialize to distinct points on the special fiber of the good model of E over R.

The following is the key step in the proof of good reduction and is analogous to [Obu17], Proposition 5.1. We recall the statement; the proof is identical in our situation, with Lemma 3.2.2 assuming the role of [Obu17], Lemma 4.5.

Proposition 3.2.3. *If f has bad reduction, then the stable model of f does not have multiplicative reduction over the original component.*

We then obtain the following by applying this to the $r = 1$ case.

Theorem 3.2.4. *Let G be a finite group with cyclic Sylow p-subgroup. Let $K_0 = \mathrm{Frac}(W(k))$, where k is an algebraically closed field of characteristic $p > 0$. Let K be a finite extension of K such that $e(K) < \frac{p-1}{m_G}$. Let $f : Y \to E$ be a one-point G-cover defined over K, where Y is a curve of genus $g \geqslant 2$. Then f has potentially good reduction.*

Proof. By [Ray99], Lemme 4.2.13, the branch index of f is not divisible by p. If f were to have bad reduction, Corollary 3.1.4 would imply that f has multiplicative reduction over the original component, contrary to Proposition 3.2.3. So f has potentially good reduction.

□

Chapter 4

Degenerations of wildly ramified covers of $\mathbb{P}^1$

Let K be a complete equal characteristic $p > 0$ discretely valued field with valuation ring $\mathcal{O}_K$, prime ideal π, and algebraically closed residue field k.

4.1 The existence of a degeneration

Theorem 2.2.5 provides a parameter space for G-extensions of $k((t))$ with ramification filtration η. We would want this to yield a corresponding parameterization of G-covers $f : Y \to \mathbb{P}^1$ branched only at ∞ at which it has inertia group $\mathbb{Z}/p^r \rtimes \mathbb{Z}/m$ with m prime to p and ramification filtration η. For this, we need the language of *thickening problems* (c.f. [HS99], §3).

Let X be a projective connected smooth reduced k-curve and let B be a finite closed subset of X. A *thickening problem* of covers for X equipped with B

comprises the following data:

(i) A cover $f : Y \to X$ of geometrically connected reduced projective k-curves.

(ii) For every $x \in B$, a Noetherian normal complete local domain R_x containing $\mathcal{O}_K$ whose maximal ideal contains π together with a finite generically separable R_x-algebra A_x.

(iii) For every $x \in B$, a pair of k-algebra isomorphisms $F_x : R_x/\pi R_x \to \widehat{\mathcal{O}}_{X,x}$ and $E_x : A_x/\pi A_x \to \widehat{\mathcal{O}}_{Y,x}$ which are compatible with the inclusions $R_x \hookrightarrow A_x$ and $\widehat{\mathcal{O}}_{X,x} \hookrightarrow \widehat{\mathcal{O}}_{Y,x}$.

A *relative* thickening problem is a thickening problem with the following fourth condition:

(iv) A projective normal $\mathcal{O}_K$-curve X' which is a trivial deformation away from B such that $X' \times_{\mathcal{O}_K} k \cong X$ and that the pullback of X' to the complete local ring at the point $x \in B$ is isomorphic to R_x.

We say that a thickening problem is G-*Galois* (or just a G-*thickening problem*) if every $R_x \hookrightarrow A_x$ is a G-extension (that is, a flat morphism on the level of rings whose corresponding extension of fraction fields is G-Galois) and every isomorphism E_x is compatible with the action of G.

A *solution* to a thickening problem is a cover $f' : Y' \to X'$ of projective normal $\mathcal{O}_K$-curves satisfying the following:

(i) The closed fiber of f' is isomorphic to f.

(ii) The pullback of f' to the formal completion of X' along $X \setminus B$ is a trivial deformation of the restriction of f to $X \setminus B$.

(iii) The pullback of f' to the complete local ring at each $x \in B$ is isomorphic to the induced map on spectra of $R_x \hookrightarrow A_x$.

It is a solution to a relative thickening problem if f' is compatible with all of the above isomorphisms. We have the following on solutions to thickening problems for covers (c.f. [HS99], Theorem 4):

Theorem 4.1.1. *(Harbater, Stevenson.) Every thickening problem for covers has a solution. The solution is unique for relative thickening problems. The solution is Galois if the thickening problem is.*

We want to use this idea to build a smooth deformation $Y_{\mathcal{O}_K} \to \mathbb{P}^1$ from the extension given by an $\mathcal{O}_K$-point of U_η. We first need to recall a lemma ([Pri02b], Lemma 2.1.4) which helps us to work with covers for which the decomposition group of a generic ramification point may be different from the inertia group at that ramification point:

Lemma 4.1.2. *Let S be an irreducible k-scheme and let $f_S : Y_S \to X_S$ be a G-cover of smooth connected curves over S branched at a finite set of S-sections with disjoint support. There exists an irreducible scheme S' and a finite surjection $i : S' \to S$ for which the generic point x of every ramification point of the pullback $i^* f_S$ has a decomposition group isomorphic to the inertia group.*

We are then able to prove the following proposition, giving a parameter space for G-covers with specified inertia:

Proposition 4.1.3. *Let* $f : Y_k \to \mathbb{P}_k$ *be a one-point* G*-cover of smooth connected curves branched only at* ∞ *at which it has inertia* $I = \mathbb{Z}/p^n \rtimes \mathbb{Z}/m$*, with* m *prime to* p*, and ramification filtration* η*. Let* $\alpha \in U_\eta$ *(in Theorem 2.2.5) correspond to the* $\mathbb{Z}/p^n \rtimes \mathbb{Z}/m$*-cover* $\hat{f} : \widehat{Y}_k \to Spec(k[[t]])$ *at the point* ∞*. Let* $S = Spec(\mathcal{O}_K)$*. Then smooth deformations of* f *over* S *are parameterized by the* S*-points of* U_η *whose closed point is* α*.*

Proof. By Theorem 2.2.5, any S-point of U_η whose closed point is α corresponds to an I-cover $\hat{f}_S$ of $\mathrm{Spec}(\mathcal{O}(S)[[t]])$ with filtration η whose closed fiber is isomorphic to $\hat{f}$. Let $X' = \mathbb{P}^1_S$. The covers f and $\hat{f}_S$, together with the isomorphism between their closed fibers, constitute a relative G-thickening problem. The unique solution to this problem guaranteed by Theorem 4.1.1 gives the smooth deformation f_S.

Now let $f_S : Y_S \to \mathbb{P}^1_S$ be any such smooth deformation. We choose our coordinates on $\mathbb{P}^1_S$ such that f_S is branched at ∞. By lemma 4.1.2, there is an irreducible scheme S' and a finite cover $S' \to S$ with the property that the decomposition group D_y of any generic ramification point y of the pullback $f_{S'}$ is equal to the inertia group I_y. By Theorem 2.2.5, the cover $\hat{f}_{S'}$ at ∞ corresponds to an S'-point of U_η whose closed point is α.

These two directions are clearly mutually inverse.

□

The following allows us to use the space U_η to classify non-isotrivial deformations:

Theorem 4.1.4. *With the hypotheses of Proposition 4.1.3, also suppose that the genus of $Y_k \geqslant 2$. Then the smooth non-isotrivial deformations of f are parameterized by the S-points of the space U_η whose closed point is α and which are not contained in the orbit of α under the action of $A_{S'}$ for any finite étale cover $S' \to S$. In particular, if $n \geqslant 3$, there exist smooth non-isotrivial deformations of f over S.*

Proof. From Proposition 4.1.3, we know the smooth deformations of f over S are parameterized by S-points of the space U_η. By Propisition 2.8.3, a deformation is not istrivial if and only if α is not contained in the orbit of α under the action of $A_{S'}$ for any finite étale cover $S' \to S$.

Also by Proposition 2.8.3, isotriviality is equivalent to there existing an étale cover $S' \to S$ and some $T \in A_{S'}$ for which $T\hat{f}_{S'}$ is constant. There is a two-dimensional orbit space under the action of any $A_{S'}$, so there are S-points not contained in the orbit if and only if the dimension of U_η is at least 3. By Theorem 2.2.5, the dimension of U_η is given by $N_\eta = \sum_{i=1}^{n} |\epsilon_p(mu_i, m)|$. By Theorem 2.2.3, $p \nmid mu_1$, $mu_1 \leqslant mu_i$, and $mu_1 \equiv mu_i \mod m$ for $1 \leqslant i \leqslant n$. In particular, $mu_1 \in \epsilon_p(mu_i, m)$ for every i. So $N_\eta \geqslant n \geqslant 3$.

□

Next, we want to address degenerations. The main theorem is the following:

Theorem 4.1.5. *Let $f_k : Y_k \to \mathbb{P}^1$ be a G-cover branched at exactly one point, at which it has inertia group $\mathbb{Z}/p^n \rtimes \mathbb{Z}/m$ with m prime to p. There exists a degeneration of f over Ω for some proper connected variety Ω.*

The proof involves the use of ruled schemes. Given a smooth connected scheme Ω, a *ruled scheme* P_Ω over Ω is a scheme equipped with a flat morphism $P_\Omega \to \Omega$ whose fibers are all isomorphic to projective lines and for which there exists a section $\Omega \to P_\Omega$. The following result is due to Pries ([Pri02b], Theorem 3.3.4) concerning isotriviality of families of covers of ruled schemes:

Theorem 4.1.6. *Let Ω be a proper irreducible k-scheme and let P_Ω be a ruled scheme over Ω. Let f_Ω be a family of G-covers of smooth connected curves branched over exactly one section. Then f_Ω is isotrivial.*

We are then ready to prove Theorem 4.1.5.

Proof. (of Theorem 4.1.5) By Theorem 4.1.4, there is a non-isotrivial smooth deformation $f_{\mathcal{O}_K}$ over $\mathcal{O}_K$ of f_k. By the Artin Approximation Theorem, the cover $f_{\mathcal{O}_K}$ descends to a cover $f_{\Omega_\circ} : Y_{\Omega_\circ} \to \mathbb{P}^1_{\Omega_\circ}$ branched over ∞ over some variety $\Omega_\circ$ of finite type over k. Let Ω be the smooth compactification of $\Omega_\circ$ and let Y_Ω be the normalization of $\mathbb{P}^1_\Omega$ in $Y_{\Omega_\circ}$. The family of curves $Y_\Omega \to \mathbb{P}^1_\Omega$ is non-isotrivial since $f_{\mathcal{O}_K}$ is. By Theorem 4.1.6, there must be at least one cover in the family whose special fiber is not smooth; in other words, there is a fiber at which f_Ω has bad reduction.

□

4.2 The vanishing cycles formula

Let $f : Y \to X$ be a branched G-cover of curves defined over $\mathcal{O}_K$. Let g_X and g_Y denote the arithmetic genus of the curves X and Y, respectively. Let

$\mathcal{W}$ and $\mathcal{B}$ denote the set of wildly ramified branch points of f and the set of terminal components of $\overline{X}$, respectively. Recall (§2.6) the effective ramification invariant $\sigma_x = \frac{h_x}{m_x}$ attached to the branch point x. Also recall that the point at which each terminal component $\overline{X}_b$ on the special fiber intersects the rest of $\overline{X}$ is a wildly ramified branch point of the special fiber. We denote the invariants attached to these branch points by $\sigma_b = \frac{h_b}{m_b}$. Let $u_{i,x}$ (resp. $u_{i,b}$) denote the i^{th} upper ramification jump at the wildly ramified branch point $x \in \mathcal{W}$ (resp. at the intersection point of the tail $\overline{X}_b$ with the rest of $\overline{X}$). Suppose $G \cong \mathbb{Z}/p^n \rtimes \mathbb{Z}/m$. At each branch point x, the inertia group has the structure $\mathbb{Z}/p^{r_x} \rtimes \mathbb{Z}/m_x$. Following [Obu12], Theorem 3.14, [Pri02a], Theorem 2.8, etc., we have the following vanishing cycles formula:

Theorem 4.2.1. $2g_X - 2 + \sum_{x \in \mathcal{W}} \left(\frac{\sigma_x}{p^{r_x - 1}} + 1\right) + \sum_{x \in \mathcal{W}} \sum_{i=1}^{r_x - 1} \left(1 - \frac{1}{p^{r_x - i}}\right)(u_{i,x} - u_{i-1,x}) = \sum_{b \in \mathcal{B}} \left(\frac{\sigma_b}{p^{r_b - 1}} - 1\right) + \sum_{b \in \mathcal{B}} \sum_{i=1}^{r_b - 1} \left(1 - \frac{1}{p^{r_b - i}}\right)(u_{i,b} - u_{i-1,b}).$

Proof. Since the strong auxiliary cover has the same ramification data as the original cover, we may replace f with f^{aux}.

We may decompose f into a $\mathbb{Z}/p$-cover $Y \to Z$ and a $\mathbb{Z}/p^{n-1} \rtimes \mathbb{Z}/m$-cover $Z \to X$, whose degree we denote by d. Let g_Z denote the arithmetic genus of Z. By the Hurwitz formula, we have on the generic fiber of $Z \to X$

$$2g_Z - 2 = d(2g_X - 2) + \sum_{b \in \mathcal{B}} \frac{d}{m_b}(m_b - 1) \qquad (\star)$$

$$+\sum_{x\in\mathcal{W}}\frac{d}{m_x p^{r_x-1}}\Big(p^{r_x-1}m_x-1+\sum_{i=1}^{r_x-1}\big(m_x p^{i-1}(p^{r_x-i}-1)(u_{i,x}-u_{i-1,x})\big)\Big)$$

and on the generic fiber of $Y\to Z$, we have

$$2g_Y-2=p(2g_Z-2)+\sum_{x\in\mathcal{W}}\Big(\frac{d}{p^{r_x-1}m_x}(h_x+1)(p-1)\Big).$$

Combining the two, we have

$$2g_Y-2=pd(2g_X-2)+p\sum_{b\in\mathcal{B}}\frac{d}{m_b}(m_b-1) \qquad (\star\star)$$

$$+p\sum_{x\in\mathcal{W}}\frac{d}{m_x p^{r_x-1}}\Big(p^{r_x-1}m_x-1+\sum_{i=1}^{r_x-1}\big(m_x p^{i-1}(p^{r_x-i}-1)(u_{i,x}-u_{i-1,x})\big)\Big)$$

$$+\sum_{x\in\mathcal{W}}\Big(\frac{d}{p^{r_x-1}m_x}(h_x+1)(p-1)\Big).$$

Now we perform a similar computation on the special fiber. Let $\overline{X}^{\circ}$ be the closure of the union of the irreducible components of $\overline{X}$ that are not tails and let $\overline{Z}^{\circ}$ and $\overline{Y}^{\circ}$ be the corresponding inverse images. The curves $\overline{Z}^{\circ}$ and $\overline{Y}^{\circ}$ have the same genus. It follows that

$$g_Y=g_Z+\sum_{b\in\mathcal{B}}\frac{d}{p^{r_b-1}m_b}g_{\overline{Y}_b}-\sum_{b\in\mathcal{B}}\frac{d}{p^{r_b-1}m_b}g_{\overline{Z}_b}$$

We know from the Hurwitz formula:

$$2g_{\overline{Y}_b}-2=p(2g_{\overline{Z}_b}-2)+(h_b+1)(p-1),$$

which means

$$g_Y = g_Z + \sum_{b\in\mathcal{B}} \frac{d}{p^{r_b-1}m_b}\frac{(h_b-1)(p-1)}{2} + (p-1)\sum_{b\in\mathcal{B}} \frac{d}{p^{r_b-1}m_b} g_{\overline{Z}_b}.$$

Another calculation gives

$$2g_{\overline{Z}_b} - 2 = -2p^{r_b-1}m_b + p^{r_b-1}(m_b-1)$$

$$+\Big(p^{r_b-1}m_b - 1 + \sum_{i=1}^{r_b-1}\big(m_b p^{i-1}(p^{r_b-i}-1)(u_{i,b}-u_{i-1,b})\big)\Big).$$

Solving for $g_{\overline{Z}_b}$, substituting into the previous equation, and simplifying, we get

$$g_Y = g_Z + \sum_{b\in\mathcal{B}} \frac{d}{p^{r_b-1}m_b}\frac{(h_b-1)(p-1)}{2} + (p-1)\sum_{b\in\mathcal{B}} \frac{d}{p^{r_b-1}m_b} + (p-1)\sum_{b\in\mathcal{B}} d$$

$$+(p-1)\sum_{b\in\mathcal{B}} \frac{d}{2m_b}(m_b-1)$$

$$+(p-1)\sum_{b\in\mathcal{B}} \frac{d}{2p^{r_b-1}m_b}\Big(p^{r_b-1}m_b - 1 + \sum_{i=1}^{r_b-1}\big(m_b p^{i-1}(p^{r_b-i}-1)(u_{i,b}-u_{i-1,b})\big)\Big).$$

From this we get

$$2(g_Y-2)=2(g_Z-2)+\sum_{b\in\mathcal{B}}\frac{d}{p^{r_b-1}m_b}(h_b+1)(p-1)+(p-1)\sum_{b\in\mathcal{B}}2d+(p-1)\sum_{b\in\mathcal{B}}\frac{d}{m_b}(m_b-1)$$

$$+(p-1)\sum_{b\in\mathcal{B}}\frac{d}{p^{r_b-1}m_b}\Big(p^{r_b-1}m_b-1+\sum_{i=1}^{r_b-1}\big(m_bp^{i-1}(p^{r_b-i}-1)(u_{i,b}-u_{i-1,b})\big)\Big).$$

Combining this with $\star$, we get

$$2g_Y-2=d(2g_X-2)+\sum_{b\in\mathcal{B}}\frac{d}{m_b}(m_b-1) \qquad (\star\star\star)$$

$$+\sum_{x\in\mathcal{W}}\frac{d}{m_xp^{r_x-1}}\Big(p^{r_x-1}m_x-1+\sum_{i=1}^{r_x-1}\big(m_xp^{i-1}(p^{r_x-i}-1)(u_{i,x}-u_{i-1,x})\big)\Big)$$

$$+\sum_{b\in\mathcal{B}}\frac{d}{p^{r_b-1}m_b}(h_b+1)(p-1)+(p-1)\sum_{b\in\mathcal{B}}2d+(p-1)\sum_{b\in\mathcal{B}}\frac{d}{m_b}(m_b-1)$$

$$+(p-1)\sum_{b\in\mathcal{B}}\frac{d}{p^{r_b-1}m_b}\Big(p^{r_b-1}m_b-1+\sum_{i=1}^{r_b-1}\big(m_bp^{i-1}(p^{r_b-i}-1)(u_{i,b}-u_{i-1,b})\big)\Big).$$

Equating $\star\star$ and $\star\star\star$ and dividing through by d, we get

$$p(2g_X-2)+p\sum_{b\in\mathcal{B}}\frac{1}{m_b}(m_b-1)$$

$$+p\sum_{x\in\mathcal{W}}\frac{1}{m_xp^{r_x-1}}\Big(p^{r_x-1}m_x-1+\sum_{i=1}^{r_x-1}\big(m_xp^{i-1}(p^{r_x-i}-1)(u_{i,x}-u_{i-1,x})\big)\Big)$$

$$+\sum_{x\in\mathcal{W}}\Big(\frac{1}{p^{r_x-1}m_x}(h_x+1)(p-1)\Big).$$

$$= (2g_X - 2) + \sum_{b \in \mathcal{B}} \frac{1}{m_b}(m_b - 1) \qquad (\star\star\star)$$

$$+ \sum_{x \in \mathcal{W}} \frac{1}{m_x p^{r_x - 1}} \Big(p^{r_x - 1} m_x - 1 + \sum_{i=1}^{r_x - 1} \left(m_x p^{i-1} (p^{r_x - i} - 1)(u_{i,x} - u_{i-1,x}) \right) \Big)$$

$$+ \sum_{b \in \mathcal{B}} \frac{1}{p^{r_b - 1} m_b} (h_b + 1)(p - 1) + (p - 1) \sum_{b \in \mathcal{B}} 2 + (p - 1) \sum_{b \in \mathcal{B}} \frac{1}{m_b}(m_b - 1)$$

$$+(p - 1) \sum_{b \in \mathcal{B}} \frac{1}{p^{r_b - 1} m_b} \Big(p^{r_b - 1} m_b - 1 + \sum_{i=1}^{r_b - 1} \left(m_b p^{i-1} (p^{r_b - i} - 1)(u_{i,b} - u_{i-1,b}) \right) \Big).$$

By now the reader will have either gotten the idea or stopped reading altogether. Combining like terms and simplifying, we get

$$(p-1)(2g_X - 2) + (p-1) \sum_{x \in \mathcal{W}} \left(\frac{\sigma_x}{p^{r_x - 1}} + 1 \right) + (p-1) \sum_{x \in \mathcal{W}} \sum_{i=1}^{r_x - 1} \left(1 - \frac{1}{p^{r_x - i}} \right) (u_{i,x} - u_{i-1,x})$$

$$= (p - 1) \sum_{b \in \mathcal{B}} \left(\frac{\sigma_b}{p^{r_b - 1}} - 1 \right) + (p - 1) \sum_{b \in \mathcal{B}} \sum_{i=1}^{r_b - 1} \left(1 - \frac{1}{p^{r_b - i}} \right) (u_{i,b} - u_{i-1,b})$$

Dividing by $p - 1$, we get

$$2g_X - 2 + \sum_{x \in \mathcal{W}} \left(\frac{\sigma_x}{p^{r_x - 1}} + 1 \right) + \sum_{x \in \mathcal{W}} \sum_{i=1}^{r_x - 1} \left(1 - \frac{1}{p^{r_x - i}} \right) (u_{i,x} - u_{i-1,x})$$

$$= \sum_{b \in \mathcal{B}} \left(\frac{\sigma_b}{p^{r_b - 1}} - 1 \right) + \sum_{b \in \mathcal{B}} \sum_{i=1}^{r_b - 1} \left(1 - \frac{1}{p^{r_b - i}} \right) (u_{i,b} - u_{i-1,b})$$

as we wanted.

□

We note that, when $X = \mathbb{P}^1$, $|\mathcal{W}| = 1$, and $r = 1$, we recover the vanishing cycles formula of [Pri02a].

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