Symbolic Powers and the Containment Problem

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Abstract

The symbolic powers $I^{(n)}$ of an ideal I relate to the primary decomposition of the ordinary powers I^n, and consist, under certain circumstances, of the functions that vanish up to order n in the corresponding variety. The Containment Problem of symbolic and ordinary powers of I consists of determining when $I^{(a)} \subseteq I^b$. Given a radical ideal I in a regular ring R, there is a uniform answer to this question by Ein–Lazarsfeld–Smith, Hochster–Huneke and Ma–Schwede, but that is not necessarily best possible. In particular, a question of Huneke remains open for prime ideals, and a generalization proposed by Harbourne has been shown to fail for certain radical ideals. In this dissertation, we study the Containment Problem for regular rings containing a field, presenting versions of Harbourne's Conjecture that do hold.

We show that Harbourne's Conjecture holds when R/I is F-pure, and prove stronger containments in the case when R/I is strongly F-regular, results that are joint work with Craig Huneke. We also answer Huneke's question positively for monomial curves in dimension 3, provide evidence that Harbourne's Conjecture might always hold eventually, and study the (non-)relationship between the Containment Problem and the finite generation of the symbolic Rees Algebra. Finally, we discuss algorithms for computing symbolic powers over a polynomial ring.

Chapter 0

Introduction

Hilbert's Nullstellensatz provides a dictionary between algebra and geometry: there is a bijection between subvarieties of $\mathbb{C}^n$ and radical ideals in $\mathbb{C}[x_1, \ldots, x_n]$, with each variety V corresponding to the ideal I of polynomials that vanish along V. But how do we measure that vanishing? A natural measure would be to ask which polynomials in I vanish up to order n along V. A classical result of Zariski and Nagata says that these coincide with the n-th symbolic power of I.

Historically, symbolic powers first arose from the theory of primary decomposition. In 1905, Emanuel Lasker [Las05] showed that every ideal in a polynomial or power series ring over a field has a primary decomposition, and in 1921, Emmy Noether [Noe21] extended Lasker's result to any noetherian ring. This can be thought of as a generalization of the fundamental theorem of arithmetic.

The topic of primary decomposition is closely related to the subject of associated primes. Algebraically, a primary decomposition

$$I = Q_1 \cap \cdots \cap Q_k$$

can be taken such that the radicals of the primary ideals Q_i are distinct and range over all the associated primes of I. The components corresponding to the minimal associated primes of I are unique and given by a simple yet difficult to compute formula: the preimage of I after localization at the given minimal prime. Geometrically, this says that any algebraic variety can be decomposed uniquely as a union of irreducible components. The embedded components, corresponding to non-minimal primes, are not necessarily unique, and somewhat more mysterious.

When P is a prime ideal, its powers might not be primary, but we can still find a primary decomposition for each P^n. The unique minimal primary component of P^n, with radical P itself, is the n-th symbolic power of P, $P^{(n)}$. Given a general[1] ideal I, its n-th symbolic power is given by the intersection of the minimal components of I^n.

In 1949, Zariski gave the first proof of what is now known as the Zariski–Nagata theorem [Zar49, Nag62]: the n-th symbolic power of a prime ideal P in a regular ring consists of the functions that vanish up to order n on the variety $V(P)$. This result, which has been extended by Eisenbud and Hochster [EH79], can be phrased in terms of differential operators:[2] the n-th symbolic power of a prime ideal P in a polynomial ring over a perfect field consists of the elements that are sent back inside P by any differential operator up to order $n-1$. This can be made quite general using Grothendieck's definition of the ring of differential operators. This description of symbolic powers in terms of differential operators, however, does not follow in mixed characteristic; in a sense, differential operators do not exclude enough elements from the symbolic power of a prime ideal containing a prime integer in a polynomial ring over a ring of mixed characteristic. This setback can be fixed using *more* maps. In joint work with Alessandro De Stefani and Jack Jeffries [DSGJ], which is not part of this thesis, we showed that there is a notion of mixed differential powers that does coincide with symbolic powers, by using Buium's p-derivations together with Grothendieck's differential operators.

In particular, the Zariski–Nagata Theorem says that the symbolic powers have nice geometric properties, while the ordinary algebraic powers have no precise geometric

[1] We note that, in fact, there are two natural ways of defining symbolic powers, and while those do coincide when the ideal I has no embedded primes itself, which is the case of all deals studied in this thesis, the two definitions are in fact different for a general ideal I.

[2] See, for example, [Eis95, Theorem 3.14], or [DDSG$^+$17, Theorem] for a general proof for polynomial rings over a perfect field.

meaning. On the other hand, $P^{(n)}$ can be extremely difficult to compute algebraically, while determining P^n from P is fairly simple. Comparing the ordinary and symbolic powers of an ideal becomes a natural question.

Determining for which ideals all corresponding symbolic and ordinary powers coincide is still an open question, even over a regular ring. In 1973, Hochster [Hoc73] gave criteria on a prime ideal P that imply $P^{(n)} = P^n$ for all $n \geqslant 1$, criteria generalized to primary ideals by Li and Swanson in 2006 [LS06]. However, there is still no complete characterization of which ideals I have the property that $I^{(n)} = I^n$ for all $n \geqslant 1$. Other sufficient conditions are known, and the question is completely settled in certain cases. However, the question is still open even for squarefree monomial ideals, although a question known as the Packing Problem proposes a certain combinatorial condition as the answer in this case.

In the 1980s and 1990s, much effort was devoted to studying symbolic Rees algebras, $\oplus I^{(n)} t^n$, especially for prime ideals. The main motivation was a question raised by Cowsik in the 1980s: should the symbolic Rees algebra always be noetherian, in particular for prime ideals P in a regular local ring R such that $\dim(R/P) = 1$? Cowsik's motivation was a result of his [Cow84] showing that a positive answer would imply that all such primes are set-theoretic complete intersections, that is, complete intersections up to radical. Eliahou, Huckaba, Huneke, and Vasconcelos proved various criteria that imply noetherianity. However, in 1985, Paul Roberts [Rob85] answered Cowsik's question negatively, building an example based on Nagata's [Nag65] counterexamples to Hilbert's 14th Problem. Space monomial curves, however, were known to be set-theoretic complete intersections [Bre79, Her80, Val81], and much work was devoted to studying their symbolic Rees algebras. Surprisingly, the answer to Cowsik's question is negative even for this class of primes, with the first non-noetherian example found by Morimoto and Goto [GM92]. Cutkostky [Cut91] also gave criteria to determine whether the symbolic Rees algebra of a given space monomial curve is noetherian. The symbolic powers of this class of prime ideals are thus particularly interesting, and we study them in Chapter 3.

Besides being an interesting subject in its own right, symbolic powers appear as auxiliary tools in several important results in commutative algebra, such as Krull's Principal Ideal Theorem, Chevalley's Lemma, or in giving a proof in prime characteristic for the fact that regular local rings are UFDs. Hartshorne's proof of the Hartshorne–Lichtenbaum Vanishing Theorem also makes use of symbolic powers. Explicitly, Hartshorne's proof of this local cohomology result uses the fact that certain symbolic and -adic topologies are equivalent, and thus local cohomology can be computed using symbolic powers.

In the 1970s and 1980s, there was a lot of interest in comparing the topologies determined by the ordinary and symbolic powers by Huckaba, Katz, McAdams, Ratliff, Schenzel, Verma, and others. The fact that two topologies coincide for a given ideal I is equivalent to saying that for all positive integers b, there exists a value a such that $I^{(a)} \subseteq I^b$. Schenzel asked if this should imply that there exists a constant k such that $I^{(kn)} \subseteq I^n$; in 2000, Irena Swanson [Swa00] answered this question positively.

Soon after Swanson's theorem, Ein, Lazarsfeld, and Smith showed that over an affine variety of characteristic 0, this constant k can be taken to be the height of I, if I is a prime ideal. Hochster and Huneke generalized the result for the case of a regular ring containing a field and any ideal I, where for a radical ideal I the constant k can be taken to be the big height of I, an invariant depending only on the associated primes of I. In particular, these results imply that $I^{(dn)} \subseteq I^n$ for all $n \geqslant 1$, where the constant d can be taken to be independent of the choice of ideal. Whether such a uniform result holds for prime ideals in a more general setting is still an open question. However, this has been settled in some cases, by work of Huneke, Katz, Validashti, Walker, and others [HKV09, HKV15, Wal17, Wal16a, Wal16b].

These results do not, however, settle Schenzel's question [Sch86] of determining the function f that returns the smallest value $f(b) = a$ such that $I^{(a)} \subseteq I^b$. This question has became known as the Containment Problem. In particular, a 2000 question of Huneke remains open: if P is a prime ideal of height 2 in a regular ring, must $P^{(3)} \subseteq P^2$? Very little is known about this question for the special case of prime ideals. In Chapter 3, we answer this question positively for the defining ideal of a space monomial curve parameterized by (t^a, t^b, t^c) over a field k.

In 2006, Brian Harbourne proposed a generalization of Huneke's question: that $I^{(hn-h+1)} \subseteq I^n$ should hold for all $n \geqslant 1$ and all radical ideals of big height h. In the following years, this was shown to hold for various classes of ideals; however, a counterexample was found in 2011 by Dumnicki, Szemberg, and Tutaj-Gasińska. This counterexample, and subsequent ones coming from very specific configurations of points in $\mathbb{P}^n$, fail the containment $I^{(3)} \subseteq I^2$ asked by Huneke. These have been studied by many, among them Harbourne, Huizenga, Malara, Nagel, Seceleanu, Szemberg, and Szpond. However, there is still much about the nature of these examples and what specific properties they have that lead to $I^{(3)} \nsubseteq I^2$ that is not understood. Still, these examples seem to be somewhat rare.

This thesis focuses on conditions that do imply Harbourne's Conjecture, or variations of it. We show that the conjecture holds for nice classes of ideals, and give evidence that it might always hold *eventually*. In particular, in Chapter 2 we present joint work with Craig Huneke showing that the conjecture does hold when I defines an F-pure ring. Moreover, as discussed in Chapter 4, there is evidence that the conjecture might always hold eventually, meaning for all n sufficiently large — what n might be sufficiently large likely depends on the ideal. We also study the Containment Problem for space monomial curves in Chapter 3, and for ideals whose symbolic Rees algebra is noetherian in Chapter 5.

0.1 Structure of the thesis

In Chapter 1, we introduce symbolic powers and the Containment Problem in the context of regular rings; there are no new results in this chapter. New results can be found in Chapters 2 through 7, the contents of which we will briefly describe below.

Chapter 2

The main results in this chapter are joint work with Craig Huneke, and have been published in [GH17]. In Section 2.11, we set up the necessary background, introducing in Subsection 2.1.1 some of the characteristic p techniques that we will use, recording a proof by Hochster and Huneke that has not explicitly appeared in print in Subsection 2.1.2 — namely, that Harbourne's Conjecture holds for powers of p — and introducing in Subsection 2.1.3 F-pure and strongly F-regular rings. In Section 2.2, the main results of [GH17] are proved. In particular, in Subsection 2.2.1 we show that Harbourne's Conjecture holds for all ideals defining an F-pure ring, and in Subsection 2.2.2 we show that in fact for ideals defining strongly F-regular rings, a stronger containment can be obtained. As a corollary, we show that for all height 2 prime ideals P defining a strongly F-regular ring, $P^{(n)} = P^n$ for all $n \geqslant 1$.

Chapter 3

The main achievement in Chapter 3 is answering a question of Huneke (given a height 2 prime P, is $P^{(3)} \subseteq P^2$?) positively for space monomial curves. More generally, we study the ideals I in $k[x, y, z]$ of height 2 generated by the maximal minors of a 2×3 matrix M. We follow a technique by Alexandra Seceleanu [Sec15], which we rewrite more generally — her work focused on the $I^{(3)} \subseteq I^2$ containment — to give conditions equivalent to $I^{(a)} \subseteq I^b$ in terms of linear algebra.

Section 3.1 contains the preliminaries; in Subsection 3.1.1 we discuss a homological criterion for $I^{(a)} \subseteq I^b$ that is the base of Seceleanu's work, and in Subsection 3.1.2 we set up some background on Rees algebras. The main results can be found in Section 3.2. In particular, we show in Subsection 3.2.1 that $I^{(a)} \subseteq I^b$ if and only if certain vectors are in the image of a given matrix, whose entries are determined by the entries of M. We then show that this latter condition can be simplified (from a computational point of view), and we apply these results to the containments $I^{(3)} \subseteq I^2$, $I^{(4)} \subseteq I^3$ and $I^{(5)} \subseteq I^3$, in Subsections 3.2.2, 3.2.3 and 3.2.4, respectively. We also give sufficient criteria for each of these containments to hold, depending only on the entries of M, that are easy to check. We apply these criteria to show that $P^{(3)} \subseteq P^2$ and $P^{(5)} \subseteq P^3$ for all space monomial curves.

Chapter 4

This chapter revolves around the idea that even if I is such that Harbourne's Conjecture fails for specific values of n, the containments predicted by Harbourne's Conjecture might still hold for large n. In Subsection 4.1.1, we discuss the main result in this chapter, Theorem 4.6, where we show that if $I^{(hn-h)} \subseteq I^n$ holds for some value of n, then such a containment must hold for all n large enough. We then discuss evidence that in fact $I^{(hn-C)} \subseteq I^n$ might hold for any value of C as long as we take n to be large enough, in Subsection 4.1.2; in particular, we discuss how a strict upper bound on an invariant of Bocci and Harbourne, known as the resurgence of I, would imply such a result.

Chapter 5

In Chapter 5, we study the Containment Problem for ideals I whose symbolic Rees algebra is noetherian and generated in degree up to d. Section 5.1 contains general facts about symbolic Rees algebras, introducing symbolic Rees algebras in Subsection 5.1.1 and proving in Subsection 5.1.2 that $I^{(dn-d+1)} \subseteq I^n$. Section 5.2 is dedicated to studying some of the questions raised in previous chapters in this context. In Subsection 5.2.1, we prove that Harbourne's Conjecture holds as long as the containments required are verified up to a point. In Subsection 5.2.2, we show that the stable version of Harbourne's Conjecture would be implied by a positive answer to a simple number theory question, which we use to prove the conjecture in some cases. In Subsection 5.2.3, we show that the stable version of Harbourne's Conjecture holds as long as $I^{(hn)} = \left(I^{(h)}\right)^n$ for all n, where I is a radical ideal of big height h over a regular ring containing a field.

Chapter 6

A method known as reduction to characteristic p allows for extending results in positive characteristic to the equicharacteristic zero case. In Chapter 6, we apply these methods to the results in Chapter 2, and discuss why we cannot apply reduction to characteristic p techniques to some of the positive characteristic results in Chapter 5. In Subsection 6.1.1 we discuss general background on reduction to positive characteristic, following [HH99]. In Subsection 6.1.2, we focus on how these techniques can be applied to symbolic powers, following the work of Hochster and Huneke in [HH02, Section 4]. We then apply these techniques in Section 6.2. In Subsection 6.2.1, we prove equicharacteristic 0 versions of the results in Chapter 2, results which are joint work with Craig Huneke and that have appeared in [GH17]. In Subsection 6.2.2, we discuss how one could use these techniques to prove a stable version of Harbourne's Conjecture in equicharacteristic 0, and how the work we have done in Subsection 5.2.2 cannot be applied for this purpose.

Chapter 7

In Chapter 7, we discuss the concrete problem of actually computing the symbolic powers of ideals in a polynomial ring. In Subsection 7.1.1, we briefly discuss strategies one can adopt given certain properties of the ideal, which have been used in a Macaulay2 package written in collaboration with Ben Drabkin, Alexandra Seceleanu, and Branden Stone, and to which Andrew Conner and Diana Zhong have also contributed code. In Subsection 7.1.2, we prove a formula for the symbolic powers of radical ideals in prime characteristic.

Chapter 1

Symbolic Powers

In this chapter, we introduce symbolic powers and the Containment Problem, and recall various classical commutative algebra results that we will need in the remaining chapters. In order to study symbolic powers, we must first understand primary decomposition. We recall some facts regarding primary decomposition in Subsection 1.1.1, introduce symbolic powers in Subsection 1.1.2 and discuss the Containment Problem in Section 1.2.

1.1 An introduction to Symbolic Powers

1.1.1 Primary decomposition and associated primes

Definition 1.1. *An ideal Q in a ring R is called* primary *if the following holds for all $a, b \in R$: if $ab \in Q$, then $a \in Q$ or $b^n \in Q$ for some $n \geqslant 1$.*

We can rewrite the previous definition using the radical of Q:

Definition 1.2. *Given an ideal I in a ring R, the* radical *of I is the ideal*

$$\sqrt{I} := \{a \in R : a^n \in I \text{ for some } n \geqslant 1\}.$$

Moreover, we say that an ideal I is radical *if $\sqrt{I} = I$.*

Lemma 1.3. *Let $\operatorname{Min}(I)$ denote the set of minimal prime ideals over I. Then*

$$\sqrt{I} \quad = \quad \bigcap_{P \supseteq I} P \quad = \bigcap_{\substack{P \supseteq I \\ P \in \operatorname{Min}(I)}} P.$$

Proof. See [AM69, Proposition 1.14]. □

Remark 1.4. Radical ideals over a noetherian ring are thus given by a finite intersection of prime ideals.

Remark 1.5. From the definition, it follows that the radical of a primary ideal is always a prime ideal. If the radical of a primary ideal Q is the prime ideal P, we say that Q is P-primary. Note, however, that not all ideals with a prime radical are primary, as we will see in Example 1.22. However, if the radical of an ideal I is maximal, then I is indeed primary.

Given an ideal I, we would like to decompose it as an intersection of primary ideals:

Definition 1.6 (Irredundant Primary Decomposition)**.** *A* primary decomposition *of the ideal I consists of primary ideals $Q_1, \ldots, Q_n$ such that $I = Q_1 \cap \cdots \cap Q_n$. An* irredundant *primary decomposition of I is one such that no Q_i can be omitted, and such that $\sqrt{Q_i} \neq \sqrt{Q_j}$ for all $i \neq j$.*

Remark 1.7. Any primary decomposition can be simplified to an irredundant one. This can be achieved by deleting unnecessary components and intersecting primary ideals with the same radical, since the intersection of primary ideals with the same radical P is in fact a P-primary ideal.

In our setting, primary decompositions always exist:

Theorem 1.8 (Lasker–Noether)**.** *Every ideal in a noetherian ring has a primary decomposition.*

Proof. For the original results, see [Las05, Noe21]. For a modern proof, see [Mat80, Section 8]. □

Primary decompositions are closely related to associated primes:

Definition 1.9 (Associated Prime)**.** *Let M be an R-module. A prime ideal P is an* associated prime *of M if the following equivalent conditions hold:*

(a) There exists a nonzero element $a \in M$ such that $P = \operatorname{ann}_R(a)$.

(b) There is an inclusion of R/P into M.

If I is an ideal of R, we refer to an associated prime of the R-module R/I as simply an associated prime of I. We will denote the set of associated primes of I by $\operatorname{Ass}(R/I)$.

We will mostly deal with associated primes of ideals. When the ring is noetherian, the set of associated primes of a nonzero ideal I is always non-empty and finite. Moreover, $\operatorname{Ass}(R/I) \subseteq \operatorname{Supp}(R/I)$, where $\operatorname{Supp}(M)$ denotes the support of the module M, meaning the set of primes p such that $M_p \neq 0$. In fact, the minimal primes of the support of R/I coincide with the minimal associated primes of I. In particular, all minimal primes of I are associated. For proofs of these facts and more on associated primes, see [Mat80, Section 7].

Lemma 1.10. *Let R be a noetherian ring and I an ideal in R. A prime ideal P is associated to I if and only if $\operatorname{depth}(R_P/I_P) = 0$.*

Proof. By [Mat89, Theorem 6.2], P is an associated prime of I if and only if P_P is an associated prime of I_P. It is enough to show the following claim: in a local ring $(R, \mathfrak{m})$, the maximal ideal $\mathfrak{m}$ is associated to 0 if and only if $\operatorname{depth}(R) = 0$. To prove the claim, note that $\operatorname{depth}(R) = 0$ if and only if $\operatorname{Hom}(R/\mathfrak{m}, R) \neq 0$, which is equivalent to the existence of a nonzero element $a \in R$ such that $\mathfrak{m}a = 0$. This can be rewritten as $\mathfrak{m} = \operatorname{ann}(a)$, which is equivalent to $\mathfrak{m}$ being an associated prime of 0. □

Given an ideal I, we will be interested not only in its associated primes, but also in the associated primes of its powers. Fortunately, the set of prime ideals that are associated to some power of I is finite, a result first proved by Ratliff in [Rat76] and then extended by Brodmann [Bro79].

Definition 1.11. *Let R be a noetherian domain and I a nonzero ideal in R. We define*

$$A(I) = \bigcup_{n \geqslant 1} \operatorname{Ass}\left(R/I^n\right).$$

Theorem 1.12 (Brodmann, 1979)**.** *Let R be a noetherian domain and I a nonzero ideal in R. For n sufficiently large, $\operatorname{Ass}\left(R/I^n\right)$ is independent of n. In particular, $A(I)$ is a finite set.*

The relationship between primary decomposition and associated primes is as follows:

Theorem 1.13 (Primary Decomposition)**.** *Let $I = Q_1 \cap \cdots \cap Q_n$ be an irredundant primary decomposition of I, where Q_i is a P_i-primary ideal for each i. Then*

$$\operatorname{Ass}(R/I) = \{P_1, \ldots, P_n\}.$$

Moreover, if P_i is minimal in $\operatorname{Ass}(R/I)$, then Q_i is unique, and given by

$$Q_i = I_{P_i} \cap R,$$

where $- \cap R$ denotes the pre-image in R via the natural map $R \longrightarrow R_P$. If P_i is an embedded prime of I, meaning that P_i is not minimal in $\operatorname{Ass}(R/I)$, then the corresponding primary component is not necessarily unique.

Proof. See [Mat80, Section 8]. □

Finally, we record a definition we will use repeatedly: colon ideals.

Definition 1.14 (Colon ideal)**.** *Let I and J be ideals in a ring R. The colon ideal $(J : I)$ is the ideal consisting of elements $r \in R$ such that $rI \subseteq J$.*

1.1.2 Symbolic powers: definition, basic properties and the equality problem

We will now restrict our study to ideals with no embedded primes. Later, we will restrict our study even further to radical ideals, which not only have no embedded primes but also coincide with the (finite) intersection of their minimal primes. Prime ideals will be of particular interest.

Definition 1.15 (Symbolic Powers)**.** *Let R be a noetherian ring, and I an ideal in R with no embedded primes. The n-th* symbolic power *of I is the ideal defined by*

$$I^{(n)} \quad = \bigcap_{P \in \operatorname{Ass}(R/I)} \left(I^n R_P \cap R\right).$$

Remark 1.16. The assumption that I has no embedded primes implies in particular that $\operatorname{Ass}(I) = \operatorname{Min}(I)$. However, when I has embedded primes, we do have two distinct possible definitions for symbolic powers, given by intersecting $I^n R_P \cap R$ with P ranging over $\operatorname{Ass}(I)$ or $\operatorname{Min}(I)$. We will focus on ideals with no embedded primes, so this distinction is not relevant. However, we note that there are statements in Chapter 5 with no assumption on I, and these do in fact hold for both definitions of symbolic power.

Both definitions have advantages. When we take P ranging over $\operatorname{Ass}(I)$, we get $I^{(1)} = I$, while taking P ranging over $\operatorname{Min}(I)$ means that $I^{(n)}$ coincides with the intersection of the primary components of I^n corresponding to its minimal primes.

Example 1.17. If I is a squarefree monomial ideal in $k[x_1, \ldots, x_n]$, then I is a radical ideal whose minimal primes are generated by variables. Writing an irredundant primary decomposition $I = \bigcap\limits_i Q_i$, where each Q_i is an ideal generated by variables, we have $I^{(n)} = \bigcap\limits_i Q_i^n$. For more on symbolic powers of monomial ideals, see [SMCH16].

The following lemma lists some basic properties of symbolic powers:

Lemma 1.18. *Let I be an ideal with no embedded primes in a noetherian ring R.*

(a) $I^{(1)} = I$;

(b) For all $n \geqslant 1$, $I^n \subseteq I^{(n)}$.

(c) $I^a \subseteq I^{(b)}$ *if and only if* $a \geqslant b$.

(d) If $a \geqslant b$, then $I^{(a)} \subseteq I^{(b)}$.

(e) For all $a, b \geqslant 1$, $I^{(a)} I^{(b)} \subseteq I^{(a+b)}$.

(f) $I^n = I^{(n)}$ if and only if I^n has no embedded primes.

Proof.

(a) Since all of the associated primes of I are minimal, Theorem 1.13 guarantees that

$$I = \bigcap_{P \in \operatorname{Ass}(R/I)} (IR_P \cap R) = I^{(1)}.$$

(b) For all associated primes P of I, $I^n \subseteq I^n R_P \cap R$. In other words, any set is contained in the preimage of its own image by any map.

(c) If $a \geqslant b$, then $I^a \subseteq I^b \subseteq I^{(b)}$. Conversely, if $I^a \subseteq I^{(b)}$, then given an associated prime P of I, we must have $(I_P)^a = (I^a)_P \subseteq \left(I^{(b)}\right)_P = (I_P)^b$. Write $J = I_P$. If $a < b$, it would follow that $J^a = J^b$, which by Nakayma's Lemma implies $J = 0$, and thus $I = 0$.

(d) Follows from the fact that $I^a \subseteq I^b$, and that taking preimages preserves inclusions.

(e) The containment follows from the identity $I^a I^b = I^{a+b}$. If $x \in I^{(a)}$ and $y \in I^{(b)}$, then the natural map $R \longrightarrow R_P$ takes xy into the image of I^{a+b}.

(f) Since $\sqrt{I^n} = \sqrt{I}$, the minimal primes of I^n coincide with those of I. Therefore, an irredundant primary decomposition of I^n consists of

$$I^n = I^{(n)} \cap Q_1 \cap \cdots \cap Q_k,$$

where $Q_1, \ldots, Q_k$ are primary components corresponding to embedded primes of I^n. There are no such components precisely when $I^n = I^{(n)}$.

□

As (f) suggests, powers of ideals with no embedded primes might have embedded primes, and in particular the converse containments to (b) and (e) do not hold in general:

Example 1.19. Consider a field k and let $R = k[x, y, z]$. Let I be the following radical ideal:

$$I = (xy, xz, yz) = (x, y) \cap (x, z) \cap (y, z).$$

When we localize at each of the associated prime ideals of I, which are (x, y), (x, z) and (y, z), the third variable gets inverted, so that the remaining two ideals become the whole ring. Moreover, the pre-image of $(x, y)^n R_{(x,y)}$ in R is $(x, y)^n$. Thus the symbolic powers of I are given by

$$I^{(n)} = (x, y)^n \cap (x, z)^n \cap (y, z)^n.$$

This matches the formula in Example 1.17. In particular, $xyz \in I^{(2)}$. However, all homogeneous elements in I^2 have degree at least 4, since I is a homogeneous ideal generated in degree 2. Therefore, $xyz \notin I^2$, and $I^2 \neq I^{(2)}$. In fact, the maximal ideal (x, y, z) is an associated prime of I^2, since $(x, y, z) = (I^2 : xyz)$.

Symbolic powers do coincide with ordinary powers if the ideal is generated by a regular sequence in a Cohen-Macaulay ring:

Theorem 1.20 (see [Hoc73])**.** *Let R be a noetherian ring. If I is an ideal generated by a regular sequence, then the associated primes of I^n coincide with those of I.*

Proof. Since I is generated by a regular sequence, the associated graded ring of I, $\bigoplus_{n \geqslant 1} I^n/I^{n+1}$, is isomorphic to a polynomial ring over R/I in as many variables as generators of I. In particular, I^n/I^{n+1} is a free module over R/I for each n. Therefore, the associated primes of R/I and I^n/I^{n+1} coincide. Now consider the following exact sequence:

$$0 \longrightarrow I^n/I^{n+1} \longrightarrow R/I^{n+1} \longrightarrow R/I^n \longrightarrow 0\ .$$

By [Mat80, Lemma 7.F], $\operatorname{Ass}(R/I^{n+1}) \subseteq \operatorname{Ass}(I^n/I^{n+1}) \cup \operatorname{Ass}(R/I^n)$. When $n = 1$, this implies that $\operatorname{Ass}(R/I^2) \subseteq \operatorname{Ass}(R/I)$, and since $\operatorname{Ass}(R/I) \subseteq \operatorname{Ass}(R/I^2)$ always holds, we conclude that the associated primes of I^2 coincide with those of I. Now proceeding by induction, the result follows. $\square$

As we have mentioned before, the symbolic powers of prime ideals will be of special interest to us.

Remark 1.21. In the case of a prime ideal P, its n-th symbolic power is given by

$$P^{(n)} = P^n R_P \cap R = \{a \in R : sa \in P^n \text{ for some } s \notin P\}.$$

The n-th symbolic power of P is the unique P-primary component in an irredundant primary decomposition of P^n. Moreover, $P^{(n)}$ is the smallest P-primary ideal containing P^n. Indeed, if Q is a P-primary ideal and $P^n \subseteq Q$, consider any element $a \in P^{(n)}$ and $s \notin P = \sqrt{Q}$ such that $sa \in P^n$. Since $sa \in Q$ and Q is P-primary, then $a \in Q$, so that $P^{(n)} \subseteq Q$ as claimed.

The equality $P^{(n)} = P^n$ is equivalent to P^n being a primary ideal. In particular, if $\mathfrak{m}$ is a maximal ideal, $\mathfrak{m}^n = \mathfrak{m}^{(n)}$ for all n; indeed, an embedded prime of $\mathfrak{m}^n$ would be a prime ideal strictly containing the only minimal prime, $\mathfrak{m}$ itself, and such a prime cannot exist.

In particular, the symbolic powers of a prime ideal are not, in general, trivial:

Example 1.22. Consider a field k and an integer $n > 1$ and let $A = k[x, y, z]$, $p = (x, z)$, $I = (xy - z^n)$ and $R = A/I$. Using $\overline{a}$ to denote the image of an element or ideal a in R via the natural projection map, note that $\overline{p}$ is a prime ideal in R, and that $\overline{y} \notin \overline{p}$. Since $\overline{x}\,\overline{y} = \overline{z}^n \in (\overline{p})^n$, we have $\overline{x} \in (\overline{p})^{(n)}$. However, $\overline{x} \notin (\overline{p})^n$.

Note that, in particular, $(\overline{p})^n$ is not a primary ideal, even though its radical is the prime ideal $\overline{p}$.

The equality of ordinary and symbolic powers of a prime ideal might fail even over a regular ring:

Example 1.23. Let k be a field, $R = k[x,y,z]$, and consider the map $\psi : R \longrightarrow k[t]$ given by $\psi(x) = t^3$, $\psi(y) = t^4$ and $\psi(z) = t^5$. Let P be the prime ideal

$$P = \ker\psi = \left(\underbrace{x^2y - z^2}_{f}, \underbrace{xz - y^2}_{g}, \underbrace{yz - x^3}_{h} \right).$$

We will see that $P^{(2)} \neq P^2$.

First, consider a non-standard grading on R under which ψ is a degree 0 map, and I is a homogeneous ideal: give x degree 3, y degree 4, and z degree 5. With this grading, $\deg(f) = 10$, $\deg(g) = 8$ and $\deg(h) = 9$, and the polynomial $fg - h^2$ is homogeneous of degree 18. Note that $fg - h^2 = xq$, for some q of degree $18 - 3 = 15$. Since $x \notin P$ and $fg - h^2 \in P^2$, we have $q \in P^{(2)}$. However, since all elements in P have degree at least 8, then all elements in P^2 must have degree at least 16, so that $q \notin P^2$. We conclude that $P^2 \neq P^{(2)}$.

We will study a class of ideals generalizing this example further in Chapter 3.

In general, the question of when the symbolic and ordinary powers of a given ideal coincide is open:

Question 1.24. Let R be a regular ring. For which ideals I with no embedded primes in R do we have $I^{(n)} = I^n$ for all $n \geqslant 1$? Is there an invariant d depending on the ring R or the ideal I such that $I^{(n)} = I^n$ for all $n \leqslant d$ (or for $n = d$) implies that $I^{(n)} = I^n$ for all $n \geqslant 1$?

There are some settings under which the answer to this question is known. The following is [Hun86, Corollary 2.5]:

Theorem 1.25 (Huneke, 1986)**.** *Let R be a regular local ring of dimension 3, and P a prime ideal in R of height 2. The following are equivalent:*

(a) $P^{(n)} = P^n$ *for all* $n \geqslant 1$*;*

(b) $P^{(n)} = P^n$ *for some* $n \geqslant 2$*;*

(c) P is generated by a regular sequence.

In particular, for a height 2 prime P in a regular local ring of dimension 3, we have $P^{(n)} \neq P^n$ for all $n \geqslant 2$ as long as P has at least 3 generators. This suggests a relationship between minimal number of generators and equality of ordinary and symbolic powers of ideals.

Theorem 1.26 (Cooper, Fatabbi, Guardo, Lorenzini, Migliore, Nagel, Seceleanu, Szpond, and Van Tuyl, 2016, [CFG+16])**.** *Let $R = k[x_0, \ldots, x_n]$ be a polynomial ring over a field k. Let I be a height 2 ideal in R such that R/I is Cohen-Macaulay and such that I_P is generated by a regular sequence for all primes $P \neq (x_0, \ldots, x_n)$ containing I. Then $I^{(k)} = I^k$ for all $k < n$ regardless of the minimal number of generators of I. Moreover, the following statements are equivalent:*

(a) $I^{(k)} = I^k$ for all $k \geqslant 1$;

(b) $I^{(n)} = I^n$;

(c) I is generated by at most n elements.

Remark 1.27. Notice that if P is a height 2 prime ideal in a polynomial ring in 3 variables, meaning that $n = 2$ in the statement of Theorem 1.26, then the conclusions of Theorems 1.25 and 1.26 coincide, although Theorem 1.25 also adds the equivalence with condition

(d) $I^{(k)} = I^k$ for some $k \geqslant 2$;

This suggests that Theorem 1.26 might hold if we add condition (d) to the equivalences stated.

In Chapter 2, we will discuss conditions that imply $I^{(n)} = I^n$ for all $n \geqslant 2$.

Example 1.28. In Example 1.23, we saw that the height 2 prime in $k[x, y, z]$ defined by the map to $k[t]$ that sends x to t^3, y to t^4, and z to t^5, which is minimally generated by 3 elements, has $P^{(2)} \neq P^2$. Theorem 1.25 actually guarantees that $P^{(n)} \neq P^n$ for all $n \geqslant 2$.

The problem of equality of symbolic and ordinary powers of ideals is also understood for licci prime ideals [HU89, Corollary 2.9].

To close this section, we note that symbolic powers can also be described via saturations.

Definition 1.29. *Let I, J be ideals in a ring R. The* saturation *of I with respect to J is the ideal given by*

$$(I : J^\infty) := \bigcup_{n \geqslant 1} (I : J^n) = \{r \in R : rJ^n \subseteq I \textit{ for some } n \geqslant 1\}.$$

Lemma 1.30. *Let I be an ideal in a noetherian ring R with no embedded primes. Then there exists an ideal J such that for all $n \geqslant 1$,*

$$I^{(n)} = (I^n : J^\infty).$$

Proof. Recall that we use the notation $A(I)$ to refer to set of primes that are associated to some power of I (cf. Definition 1.11), which is a finite set by Theorem 1.12. If $A(I)$ consists only of minimal primes of I, then all symbolic and ordinary powers of I coincide, so that we can take $J = R$. Otherwise, let $P_1, \ldots, P_k$ be the primes in $A(I)$ that are not minimal over I. Consider an element s not in any minimal prime of I and such that $s \in P_1 \cap \cdots \cap P_k$. For each n, write

$$I^n = I^{(n)} \cap Q_1 \cap \cdots \cap Q_t,$$

where each Q_j is a primary ideal with radical one of the P_i. Then, since $s \in \sqrt{Q_i}$ for all i, we have $(Q_i : s^\infty) = R$, and thus

$$(I^n : s^\infty) = \left(I^{(n)} : s^\infty\right) \cap (Q_1 : s^\infty) \cap \cdots \cap (Q_t : s^\infty) = (I^{(n)} : s^\infty).$$

Moreover, $I^{(n)}$ is an intersection of primary ideals, the radicals of which do not contain s. Therefore, $(I^{(n)} : s^\infty) = I^{(n)}$. □

Remark 1.31. Note that the ideal J in Lemma 1.30 can be taken to be the intersection of all the non-minimal primes in $A(I)$. Indeed, given $s \in J$ that is not contained in any minimal prime of I, the proof of Lemma 1.30 shows that $(I^n : s^\infty) = I^{(n)}$, and since $s \in J$, then

$$(I^n : J^\infty) \subseteq (I^n : s^\infty) = I^{(n)}.$$

Moreover, writing an irredundant primary decomposition

$$I^n = I^{(n)} \cap Q_1 \cap \cdots \cap Q_k,$$

we have $J \subseteq \sqrt{Q_1} \cap \cdots \cap \sqrt{Q_k}$. Therefore, there exists a power of J, say J^k, that is contained in $Q_1 \cap \cdots \cap Q_k$, so that $I^{(n)} J^k \subseteq I^n$. This proves that

$$I^{(n)} \subseteq (I^n : J^\infty).$$

Note that we may also take J to be the intersection of any finite set of primes $P \supseteq I$ that are not minimal over I, as long as this set includes all of the non-minimal primes in $A(I)$. Indeed, in the proof above we only used two facts: that J contains some element not in any minimal prime of I, and that $J \subseteq \sqrt{Q}$ whenever Q is a non-minimal but irredundant primary component of I^n for some n.

Saturations are computationally much easier to calculate than finding primary decompositions, as we will see in Chapter 7. However, despite the previous remark, finding J as in Lemma 1.30 requires some concrete knowledge of $A(I)$ – knowing an upper bound for the value n at which $\operatorname{Ass}(R/I^n)$ stabilizes would suffice. Unfortunately, the proof of Theorem 1.12 is not constructive. We will, however, make use of saturations in certain cases when we do have control over what ideal J to consider, especially in Chapter 3.

1.2 The Containment Problem

Given a radical ideal I in a regular ring, the Containment Problem for I consists of determining for which values of a and b does the containment $I^{(a)} \subseteq I^b$ hold. In

Subsection 1.2.1 we introduce the problem and discuss a beautiful answer to this question by Ein–Lazarsfeld–Smith, Hochster–Huneke, and Ma–Schwede. We then discuss Harbourne's Conjecture in Subsection 1.2.2. We will present new results related to this question in Chapters 2, 3, 4, and 5.

1.2.1 A famous containment

We saw in Lemma 1.18 that $I^a \subseteq I^{(b)}$ if and only if $a \geqslant b$. Containments of type $I^{(a)} \subseteq I^b$ are a lot more interesting, and the motivation behind this dissertation.

Question 1.32 (Containment Problem)**.** Let R be a regular ring and I be a radical ideal in R. When is $I^{(a)} \subseteq I^b$?

Example 1.33. We saw in Example 1.19 that the second symbolic power of the ideal $I = (xy, xz, yz) \subseteq k[x, y, z]$ does not coincide with its square. However, the containment $I^{(3)} \subseteq I^2$ does hold. To check this, we can explicitly write down a set of generators for $I^{(3)}$. The monomial ideal I contains a monomial $x^a y^b z^c$ if and only if $a + b \geqslant 3$, $a + c \geqslant 3$, and $b + c \geqslant 3$; these conditions are in turn equivalent to $x^a y^b z^c$ being an element of $(x, y)^3$, $(x, z)^3$ or $(y, z)^3$. Therefore,

$$I^{(3)} = \left(x^2y^2z, x^2yz^2, xy^2z^2, x^3y^3, x^3z^3, y^3z^3\right) \subseteq I^2.$$

Does Question 1.32 always make sense? That is, given b, must there exist an a such that $I^{(a)} \subseteq I^b$? If so, then the two graded families of ideals $\{I^n\}$ and $\left\{I^{(n)}\right\}$ are cofinal, and thus induce equivalent topologies. In 1985, Schenzel [Sch85] gave a characterization of when these two families are cofinal. In particular, if R is a regular ring and I is a radical ideal in R, then $\{I^n\}$ and $\left\{I^{(n)}\right\}$ are cofinal. Schenzel's characterization did not, however, provide information on the relationship between a and b.

Theorem 1.34 (Swanson, 2000, [Swa00])**.** *Let R be a noetherian ring, and I and J two ideals in R. The following are equivalent:*

(i) $\{I^n : J^\infty\}$ *is cofinal with* $\{I^n\}$*.*

(ii) There exists an integer c *such that* $(I^{cn} : J^\infty) \subseteq I^n$ *for all* $n \geqslant 1$*.*

In particular, given a radical ideal in a regular ring, there exists an integer c such that $I^{(cn)} \subseteq I^n$ for all $n \geqslant 1$. More surprisingly, this constant can be taken uniformly, meaning depending only on R.

Definition 1.35 (Big height)**.** *Let* I *be an ideal with no embedded primes. The* big height *of* I *is the maximal height of an associated prime of* I*. If the big height of* I *coincides with the height of* I*, meaning that all associated primes of* I *have the same height, we say that* I *has* pure height*.*

Theorem 1.36 (Ein-Lazarsfeld-Smith [ELS01], Hochster-Huneke [HH02], Ma-Schwede [MS17])**.** *Let* R *be a regular ring and* I *a radical ideal in* R*. If* h *is the big height of* I*, then* $I^{(hn)} \subseteq I^n$ *for all* $n \geqslant 1$*.*

Remark 1.37. In particular, $I^{(n)} \subseteq I^{\left\lfloor \frac{n}{h} \right\rfloor}$ for all $n \geqslant 1$.

Ein, Lazarsfeld, and Smith first proved this theorem in the equicharacteristic 0 geometric case, using multiplier ideals. Hochster and Huneke then used reduction to characteristic p and tight closure techniques to prove the result in the equicharacteristic case. Recently, Ma and Schwede built on ideas used in the recent proof of the Direct Summand Conjecture to define a mixed characteristic analogue of multiplier ideals, allowing them to deduce the mixed characteristic version of Theorem 1.36.

Given an ideal I and $t \geqslant 0$, the multiplier ideal $\mathcal{J}(R, I^t)$ measures the singularities of $V(I) \subseteq \operatorname{Spec}(R)$, scaled by t. We refer to [ELS01, MS17] for the definition. The proof of Theorem 1.36 in the characteristic 0 case relies on a few key properties of multiplier ideals:

- $I \subseteq \mathcal{J}(R, I)$;
- For all $n \geqslant 1$, $\mathcal{J}\left(R, \left(P^{(nh)}\right)^{\frac{1}{n}}\right) \subseteq P$ as long as P is a prime of height h;

- For all integers $n \geqslant 1$, $\mathcal{J}\left(R, I^{tn}\right) \subseteq \mathcal{J}\left(R, I^{t}\right)^{n}$

Then, given a prime ideal P of height h,

$$P^{(hn)} \subseteq \mathcal{J}\left(R, \left(P^{(nh)}\right)\right) \subseteq \mathcal{J}\left(R, \left(P^{(nh)}\right)^{\frac{1}{n}}\right)^{n} \subseteq P^{n}.$$

In characteristic p, a similar proof works, replacing multiplier ideals by test ideals. However, the proof given by Hochster and Huneke relies only on basic properties of the Frobenius map, which we shall describe in Chapter 2.

Remark 1.38. As a corollary of Theorem 1.36, we obtain a uniform constant c as in Theorem 1.34. Indeed, the big height of any ideal is at most the dimension d of the ring, so that $I^{(dn)} \subseteq I^{n}$ for all n. If we restrict to prime ideals in R, this constant can be improved to $d-1$, since the ordinary and symbolic powers of any maximal ideal coincide.

The question of whether there exists a uniform constant c such that $I^{(cn)} \subseteq I^{n}$ for all $n \geqslant 1$ over all ideals I for which the I-symbolic and I-adic topologies are equivalent is still open in the non-regular setting. However, there are some cases where it has been solved. For the case of monomial prime ideals over normal toric rings, see the work of Robert M. Walker [Wal16a, Wal17].

Theorem 1.39 (Huneke-Katz-Validashti, 2009, [HKV09])**.** *Let R be an equicharacteristic reduced local ring such that R is an isolated singularity. Assume either that R is equidimensional and essentially of finite type over a field of characteristic zero, or that R has positive characteristic and is F-finite. Then there exists $h \geqslant 1$ with the following property: for all ideals I with positive grade for which the I-symbolic and I-adic topologies are equivalent, $I^{(hn)} \subseteq I^{n}$, for all $n \geqslant 1$.*

Finally, we note that big height cannot be replaced by height in Theorem 1.36.

Example 1.40. Consider the ideal

$$I = (x, y) \cap (y, z) \cap (x, z) \cap (a) = (xya, xza, yza) \subseteq k[x, y, z, a],$$

which has height 1 and big height 2. If we replaced big height by height in Theorem 1.36, we would have $I^{(n)} = I^n$ for all $n \geqslant 1$. However, similarly to Example 1.19, $I^{(2)} \neq I^2$. Indeed, note that

$$xyza^2 \in I^{(2)} = (x,y)^2 \cap (y,z)^2 \cap (x,z)^2 \cap (a)^2,$$

whereas all elements in I^2 must have degree at least 6.

Remark 1.41. This example also provides a recipe for creating counterexamples to $I^{(cn)} \subseteq I^n$, where c is the height of I, for any given height c. Indeed, fix a value h for the big height, and a height $c < h$, and find n such that $n \geqslant \frac{h}{h-c}$, which implies that $hn - h \geqslant cn$. If we find an ideal I of height c and big height h with $I^{(hn-h)} \nsubseteq I^n$, we are done.

In Example 2.32, we will see how to build an ideal J of pure height h in a polynomial ring v variables, where v depends on h and n, such that $J^{(hn-h)} \nsubseteq J^n$. Consider such an ideal J in the variables $x_1, \ldots, x_v$, and take the expansion of J in the ring $k[x_1, \ldots, x_v, y_1, \ldots, y_c]$. Let $I = J \cap (y_1, \ldots, y_c)$, which by construction is an ideal of height c and big height h. Note that the intersection and the product of two ideals generated in distinct sets of variables coincide, so that all powers of I are given by

$$I^m = (J\,(y_1, \ldots, y_c))^m = J^m\,(y_1, \ldots, y_c)^m = J^m \cap (y_1, \ldots, y_c)^m.$$

Moreover, all symbolic powers of I are given by

$$I^{(m)} = J^{(m)} \cap (y_1, \ldots, y_c)^m = J^{(m)}\,(y_1, \ldots, y_c)^m.$$

Therefore, $I^{(a)} \subseteq I^b$ if and only if $J^{(a)} \subseteq J^b$, and in particular $I^{(hn-n)} \nsubseteq I^n$, so that $I^{(cn)} \nsubseteq I^n$.

1.2.2 Harbourne's Conjecture

The containments provided by Theorem 1.36 are not necessarily best possible.

Example 1.42. The ideal $I = (x,y)\cap(x,z)\cap(y,z)$ from Example 1.19 has big height 2, so that Theorem 1.36 implies that $I^{(2n)} \subseteq I^n$ for all $n \geqslant 1$. However, as we saw in Example 1.33, $I^{(3)} \subseteq I^2$, even though the theorem only guarantees $I^{(4)} \subseteq I^2$.

Question 1.43 (Huneke, 2000)**.** Let P be a prime ideal of height 2 in a regular ring containing a field. Does the containment $P^{(3)} \subseteq P^2$ always hold?

As we will see in Theorem 3.31, space monomial curves such as the one in Example 1.23 always verify this containment. Apart from when the equality of ordinary and symbolic powers has been shown to hold, there are virtually no other cases where the answer to Question 1.43 is known. In particular, the question remains open for general prime ideals in dimension 3.

Harbourne proposed the following generalization of Question 1.43, which can be found in [HH13, BRH+09]:

Conjecture 1.44 (Harbourne, 2006)**.** *Let I be a radical homogeneous ideal in $k[\mathbb{P}^N]$, and let h be the big height of I. Then for all $n \geqslant 1$,*

$$I^{(hn-h+1)} \subseteq I^n.$$

Remark 1.45. Equivalently, Harbourne's Conjecture asks if $I^{(n)} \subseteq I^{\lceil \frac{n}{h} \rceil}$ for all $n \geqslant 1$.

Remark 1.46. When $h = 2$, the conjecture asks that $I^{(2n-1)} \subseteq I^n$, and in particular that $I^{(3)} \subseteq I^2$.

There are various cases where this conjecture is known to hold: if I is a monomial ideal (which first appeared in [BRH+09, Example 8.4.5]), if I corresponds to a general set of points in $\mathbb{P}^2$ [BH10a] or $\mathbb{P}^3$ [Dum15], and if I corresponds to a star configuration of points [HH13]. As we will see in Chapter 2, the conjecture also holds if I defines an F-pure ring, which in particular recovers the result for monomial ideals.

Unfortunately, Conjecture 1.44 turns out to be too general; it does not hold for all homogeneous radical ideals.

Example 1.47 (Fermat configurations of points)**.** Let $n \geqslant 3$ be an integer and consider a field k of characteristic not 2 such that k contains n distinct roots of unity. Let $R = k[x, y, z]$, and consider the ideal

$$I = (x(y^n - z^n), y(z^n - x^n), z(x^n - y^n)) \, .$$

When $n = 3$, this corresponds to a configuration of 12 points in $\mathbb{P}^2$, as described in Figure 1.1. Over $\mathbb{P}^2(\mathbb{C})$, these 12 points are given by the 3 coordinates points plus the 9 points defined by the intersections of $y^3 - z^3$, $z^3 - x^3$ and $x^3 - y^3$.

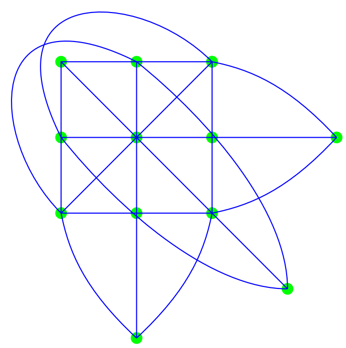

Figure 1.1: Fermat configuration of points when $n = 3$.

The ideal I is radical and has pure height 2. However, $I^{(3)} \nsubseteq I^2$, since the element $f = (y^n - z^n)(z^n - x^n)(x^n - y^n) \in I^{(3)}$ but not in I^2. This can be shown via geometric arguments, noting that f defines 9 lines, some three of which go through each of the 12 points.

This was first shown in [DSTG13] over $k = \mathbb{C}$, and then generalized in [HS15, Proposition 3.1] to any k and any n provided the condition above on the existence of enough roots of unity is satisfied.

Other configurations of points in $\mathbb{P}^2$ have been shown to produce ideals that fail the containment $I^{(3)} \subseteq I^2$, such as the Klein and Wiman configurations of points [Sec15]. Given a configuration of points in $\mathbb{P}^k$ that produces an ideal I with $I^{(hn-h+1)} \nsubseteq I^n$, one can produce other counterexamples to the same type of containment by applying flat morphisms $\mathbb{P}^k \to \mathbb{P}^k$, by work of Solomon Akesseh [Ake17].

Example 1.48. In [HS15], Harbourne and Seceleanu showed that the containment $I^{(hn-h+1)} \subseteq I^n$ can fail for arbitrarily high values of n in characteristic $p > 0$. However, their counterexamples are constructed depending on n, meaning that given n, there exists an ideal I_n of pure height 2 (corresponding, once more, to a configuration of points in $\mathbb{P}^2$) which fails $I_n^{(hn-h+1)} \subseteq I_n^n$.

There are, however, no known counterexamples to the following conjecture:

Conjecture 1.49. *Let R be a regular ring containing a field and I a radical ideal in R with big height h. Then there exists m such that the following containment holds for all $n \geqslant m$:*

$$I^{(hn-h+1)} \subseteq I^n.$$

There are also no known prime counterexamples to Harbourne's Conjecture. In particular, Question 1.43 remains open. In Chapter 3, we will answer this question positively for space monomial curves.

To close this chapter, we record a lemma that we will use repeatedly: the fact that $I \subseteq J$ is a local statement.

Lemma 1.50. *Let R be a noetherian ring and I and J ideals in R. The containment $I \subseteq J$ holds if and only if for all $P \in \operatorname{Ass}(R/J)$, $I_P \subseteq J_P$.*

Proof. One implication in obvious, since localization preserves containments. If J has no embedded primes, the converse follows from the fact that $I_P \subseteq J_P$ for all $P \in \operatorname{Ass}(R/J)$ implies that

$$I \quad \subseteq \bigcap_{P \in \operatorname{Ass}(R/J)} J_P \cap R = J^{(1)} = J.$$

For the general case, consider the complement W of the union of all associated primes of R/J, and recall that W coincides with the nonzerodivisors of R/J. Our assumption is equivalent to $I_W \subseteq J_W$. Given $a \in I$, this implies that $sa \in J$ for some $s \in W$. In other words, $a + J$ is taken to 0 in R/J by a nonzerodivisor, so we conclude that $a \in J$. □

Chapter 2

Symbolic powers in characteristic p

Characteristic p techniques are very useful for studying symbolic powers. In particular, Hochster and Huneke used such tools to prove Theorem 1.36 in [HH02], and also in their subsequent paper [HH07]. In this chapter, we discuss some of those techniques, including ideas from [HH02], introduce F-pure and strongly F-regular rings, and then prove Harbourne's Conjecture for ideals defining F-pure rings in Section 2.2. We then show that a stronger statement holds for ideals defining strongly F-regular rings. The new results in Section 2.2 are joint work with Craig Huneke, and can be found in [GH17].

All the rings in this chapter contain a field of prime characteristic p.

2.1 Preliminaries

2.1.1 A short introduction to characteristic p techniques

When dealing with rings of prime characteristic p, we gain a powerful tool:

Definition 2.1. *Let R be a ring of prime characteristic p. The* Frobenius *map is the R-homomorphism defined by $F(x) = x^p$. We denote the e-th iteration of the Frobenius map, $F^e(x) = x^{p^e}$, by F^e. Applying the e-iteration of Frobenius to an ideal I in R returns an ideal, the e-th* Frobenius power *of I, which we denote by $I^{[p^e]}$:*

$$I^{[p^e]} := \left(a^{p^e} : a \in I\right).$$

Remark 2.2. If $I = (a_1, \ldots, a_n)$, then $I^{[p^e]} = \left(a_1^{p^e}, \ldots, a_n^{p^e}\right)$.

The following simple lemma is a key idea for dealing with symbolic powers in prime characteristic:

Lemma 2.3. *Let I be an ideal generated by g elements in a ring R of prime characteristic p. Then for all $q = p^e$, we have*

$$I^{gq-g+1} \subseteq I^{[q]}.$$

Proof. Consider minimal generators $(x_1, \ldots, x_n) = I$. The ideal $I^{n(q-1)+1}$ is generated by elements of the form $x_1^{a_1} \cdots x_n^{a_n}$ with $a_1 + \cdots + a_n \geqslant n(q-1)+1$. Take one such generator; by the Pigeonhole Principle, there must be an i such that $a_i \geqslant q$, and thus $x_1^{a_1} \cdots x_n^{a_n} \in I^{[q]}$. □

A similar proof shows the following stronger version of the previous lemma:

Lemma 2.4 (see Lemma 2.4 (a) in [HH02])**.** *Let R be a ring, and $I = (u_1, \ldots, u_h)$ an ideal in R. Then for all integers $t \geqslant 1$ and $k \geqslant 0$,*

$$I^{ht+kt-h+1} \subseteq \left(u_1^t, \ldots, u_h^t\right)^{k+1}.$$

Hence, if R has prime characteristic $p > 0$ and $q = p^e$,

$$I^{hq+kq-h+1} \subseteq \left(I^{[q]}\right)^{k+1} = \left(I^{k+1}\right)^{[q]}.$$

We will be focusing on regular rings of prime characteristic. One of the main results we will need is that over a regular ring, the Frobenius map is flat. However, this is also one of the points where the assumption that we are working over a regular ring is crucial: the flatness of Frobenius *characterizes* regular rings.

Theorem 2.5 (Kunz, 1969 [Kun69])**.** *If R is a reduced local ring of prime characteristic p, R is flat over R^p if and only if R is a regular ring.*

This theorem has many important consequences.

Lemma 2.6. *Let R be a regular of characteristic p. For all ideals I and J in R and all $q = p^e$,*

$$(J : I)^{[q]} = \left(J^{[q]} : I^{[q]}\right).$$

Proof. Since $R^{[q]}$ is a flat R-module by Theorem 2.5, this is a particular case of [Mat89, Theorem 7.4 (iii)]. □

Lemma 2.7. *Over a regular ring, the Frobenius map preserves associated primes. In particular, if I is an ideal with no embedded primes in a regular ring R of prime characteristic p, then for all $q = p^e$, the ideal $I^{[q]}$ has no embedded primes.*

Proof. The Frobenius map is exact [Kun69, Theorem 2.1], and thus it takes minimal free resolutions to minimal free resolutions. In particular, if Q is a prime ideal in R, the Frobenius map takes a minimal free resolution of $(R/I)_Q$ to a minimal free resolution of $\left(R/I^{[q]}\right)_Q$. Moreover, the length of the resolution is preserved, so that the projective dimensions coincide. By the Auslander-Buchsbaum formula, this implies that the depths also coincide, so that

$$\operatorname{depth}(R/I)_Q = 0 \text{ if and only if } \operatorname{depth}\left(R/I^{[q]}\right)_Q = 0.$$

By Lemma 1.10, this completes the proof. □

2.1.2 A proof by Hochster and Huneke

In Chapter 1, we discussed the famous containment theorem of Ein-Lazarsfeld-Smith, Hochster-Huneke, and Ma-Schwede, and how the theory of multiplier ideals can be used to prove this result. We will now discuss a different approach to proving this theorem, taken by Hochster and Huneke in [HH02]. The key ideas in their proof are surprisingly simple, and understanding these is of extreme importance for dealing with symbolic powers in prime characteristic. With the goal of applying some of their techniques to address other problems involving symbolic powers, we will now discuss some of the ingredients in Hochster and Huneke's proof of Theorem 1.36.

Remark 2.8. One of the key ingredients needed to show $I^{(hn)} \subseteq I^n$ is Lemma 2.3, which uses the minimal number of generators of I. In fact, we need to know the number of generators of I when localized at each associated prime of I. If $I = Q$ is a prime ideal of height h, then the only associated prime of Q is Q itself, and Q_Q is the maximal ideal of a regular local ring of dimension h, so that it is minimally generated by h elements. For a radical ideal I of big height h, I_P is generated by at most h elements when localized at any of its associated primes P. Indeed, since I is radical, $I = P \cap J$, where J contains elements not in P, and thus $I_P = P_P$, which is generated by as many elements as the height of P. By definition, this is at most h.

The results in [HH02] cover a more general case, not assuming that I is radical. The main ideas are still the same, but the maximal value for the minimal number of generators of I_P, when P runs through the associated primes of I, is no longer necessarily h. To overcome this issue, we can substitute I_P by a minimal reduction of I_P, which is generated by as many elements as the analytic spread of I_P. We note that this number is at most the height of P. The general form of Theorem 1.36 is then as follows: $I^{(hn)} \subseteq I^n$ for all $n \geqslant 1$, where h can be taken to be

- the maximal value of the minimal number of generators of I_P, where P runs through the associated primes of I,
- the maximal height of an associated prime of I, or
- the maximal analytic spread of I_P, where P runs through the associated primes I.

We note that when I is radical, all these invariants coincide. For more on reductions and a definition of analytic spread, see [SH06, Chapter 8].

Lemma 2.9 (Hochster-Huneke [HH02])**.** *Suppose that I is a radical ideal of big height h in a regular ring R containing a field of characteristic $p > 0$. For all $q = p^e$,*

$$I^{(hq)} \subseteq I^{[q]}.$$

Proof. By Lemma 1.50, it is enough to show the containment holds once we localize at the associated primes of $I^{[q]}$. By Lemma 2.7, the associated primes of $I^{[q]}$ coincide with those of I. So let P be an associated prime of I, and recall that I_P is generated by at most h elements. Over R_Q, the containment becomes

$$I_Q^{hq} \subseteq I_Q^{[q]},$$

which follows by Lemma 2.3. □

In fact, the same proof using the full power of Lemma 2.3 gives Harbourne's Conjecture 1.44 for powers of p, a fact first noted by Craig Huneke:

Lemma 2.10. *Suppose that I is a radical ideal of big height h in a regular ring R containing a field of characteristic $p > 0$. For all $q = p^e$,*

$$I^{(hq-h+1)} \subseteq I^{[q]} \subseteq I^q.$$

As an easy corollary, we obtain an affirmative answer to Huneke's Question 1.43 in characteristic 2:

Corollary 2.11. *If I is a radical ideal of big height 2 in a regular ring R containing a field of characteristic 2, then $I^{(3)} \subseteq I^2$.*

Proof. Take $q = 2$ and $h = 2$ in Lemma 2.10. □

The following is [ELS01, Theorem 2.2] in the case of smooth complex varieties, and more generally [HH02, Theorem 2.6]:

Theorem 2.12 (Hochster-Huneke, Theorem 2.6 in [HH02])**.** *Let I be a radical ideal of a regular ring containing a field, and let h be the big height of I. Then for all $n \geqslant 1$ and all $k \geqslant 0$, $I^{(hn+kn)} \subseteq \left(I^{(k+1)}\right)^n$.*

When $k = 0$, this gives $I^{(hn)} \subseteq I^n$. Moreover, in characteristic p, techniques such as the ones used in [HH02, Theorem 2.6] can be used to obtain a generalized version of Harbourne's Conjecture for powers of p, which can also be found in [GH17, Lemma 2.6]:

Lemma 2.13. *Let I be a radical ideal in a regular ring R of characteristic $p > 0$ and h the big height of I. For all $q = p^e$,*

$$I^{(hq+kq-h+1)} \subseteq \left(I^{(k+1)}\right)^{[q]}.$$

Proof. By Lemma 1.50, it is enough the show that the statement holds once we localize at each associated prime of $\left(I^{(k+1)}\right)^{[q]}$. Since $\operatorname{Ass}\left(I^{(k+1)}\right) = \operatorname{Ass}(I)$, Lemma 2.7 shows it is enough to show that the statement holds at the associated primes of I. If P is an associated prime of I, the statement we need to show over R_P is the following:

$$P_P^{hq+kq-h+1} \subseteq \left(P_P^{[q]}\right)^{k+1}.$$

The claim now follows by Lemma 2.4. □

2.1.3 F-pure and strongly F-regular rings

We now introduce the notions of F-pure and strongly F-regular ring, which we will need for the remainder of this chapter.

Definition 2.14 (F-finite ring)**.** *Let A be a noetherian ring of characteristic $p > 0$. We say that A is* F-finite *if A is a finitely generated module over itself via the action of the Frobenius map.*

Definition 2.15. *If A is F-finite and reduced, the ring of p^e-roots of A is denoted by $F_*^e A$, and the inclusion $A \hookrightarrow F_*^e A$ can be identified with F^e. The fact that A is F-finite implies that $F_*^e A$ is a finitely generated module over A for all $q = p^e$.*

Example 2.16. If k is a perfect field, then $k[x_1, \ldots, x_n]$ is F-finite. In fact, every ring R essentially of finite type over k is F-finite.

We will study F-pure rings, which were introduced by Hochster and Roberts in [HR76].

Definition 2.17 (F-pure ring)**.** *Let A be a noetherian ring of characteristic $p > 0$. We say that A is* F-pure *if for any A-module M, $F \otimes 1 : A \otimes M \longrightarrow A \otimes M$ is injective.*

Definition 2.18 (F-split ring)**.** *Let A be a noetherian ring of characteristic $p > 0$. We say that A is* F-split *if the inclusion $R \hookrightarrow F^e_* R$ splits for every (equivalently, some) $q = p^e$.*

Lemma 2.19. *If A is F-finite, then A is F-pure if and only A is F-split.*

Proof. See [HR76, Corollary 5.3]. □

Example 2.20. Every regular ring is F-pure.

The following theorem characterizes ideals that define F-pure rings over a regular ring:

Theorem 2.21 (Fedder's Criterion for F-purity, Theorem 1.12 in [Fed83])**.** *Let $(R, \mathfrak{m})$ be a regular local ring of characteristic $p > 0$. Given an ideal I in R, R/I is F-pure if and only if for all $q = p^e \gg 0$,*

$$\left(I^{[q]} : I\right) \nsubseteq \mathfrak{m}^{[q]}.$$

Lemma 2.22. *If I be a squarefree monomial ideal in a polynomial ring over a field, then R/I is an F-pure ring.*

Proof. Assume that the ring is $R = k[x_1, \ldots, x_d]$. Any monomial ideal I is generated by elements of the form $x_{i_1} \cdots x_{i_k}$, where the i_j are distinct. Therefore, we must have $(x_1 \cdots x_d)^{q-1} \in (I^{[q]} : I)$, since $(x_1 \cdots x_d)^{q-1} \cdot x_{i_1} \cdots x_{i_k}$ is a multiple of $(x_{i_1} \cdots x_{i_k})^q$. On the other hand, $(x_1 \cdots x_d)^{q-1} \notin \mathfrak{m}^{[q]}$. By Fedder's Criterion, R/I is F-pure. □

Definition 2.23 (Strongly F-regular ring)**.** *An F-finite reduced ring A is* strongly F-regular *if given any $f \in A$, f a nonzerodivisor in A, there exists $q = p^e$ such that the inclusion $F^e_*(f)A \longrightarrow F^e_* A$ splits.*

Strongly F-regular rings, first introduced by Hochster and Huneke in [HH89], are normal and Cohen-Macaulay.

Remark 2.24. By letting $f = 1$ in the definition, we see that strongly F-regular rings are F-split, and thus F-pure, by Lemma 2.19.

Example 2.25. Let k be a field of characteristic p, and consider a generic $m \times n$ matrix M. Given any $t \leqslant m, n$, the k-algebra generated by all the t-minors of M is strongly F-regular. Moreover, if R is the polynomial ring over k generated by the entries of M, and $I = I_t(M)$, then R/I is strongly F-regular. More generally, rings of invariants of classical groups, after reduction modulo p, are strongly F-regular [SvdB97, Theorem 5.2.3].

Example 2.26. Let k be a field of characteristic p and $A = k[x_1, \ldots, x_n]$. A Veronese subring of A is a k-algebra generated by all monomials in A of a fixed degree. Veronese rings are always strongly F-regular. More generally, given a homogeneous ideal J in A, if A/J is strongly F-regular, then any Veronese subring of A/J is also strongly F-regular.

Locally acyclic cluster algebras [BMRS15] and certain ladder determinantal varieties [GS95] are also strongly F-regular. See also [BT06].

There is a criterion similar to Theorem 2.21 for strongly F-regular rings:

Theorem 2.27 (Glassbrenner's Criterion for strong F-regularity, see [Gla96])**.** *Let $(R, \mathfrak{m})$ be an F-finite regular local ring of prime characteristic p. Given a proper radical ideal I of R, R/I is strongly F-regular if and only if for each element $c \in R$ not in any minimal prime of I, $c\left(I^{[q]} : I\right) \nsubseteq \mathfrak{m}^{[q]}$ for all $q = p^e \gg 0$.*

Lemma 2.28. *Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring over a field k and I a squarefree monomial ideal in R. If R/I is strongly F-regular, then I is generated by variables.*

Proof. Strongly F-regular rings are products of domains [HH94a, Theorem 5.5 c)], and since R/I is a graded ring, it must be a domain. If $x_i x_j$ divides a minimal generator of I, then x_i and x_j are zerodivisors on R/I. Therefore, I must be generated by variables. □

2.2 Harbourne's Conjecture for F-pure rings

We will now use the characteristic p tools that we have discussed in the previous section to show that Harbourne's Conjecture does hold for ideals defining F-pure rings. For ideals defining strongly F-regular rings, we prove that a tighter containment holds. All the results in this section are joint work with Craig Huneke, and can be found in [GH17].

2.2.1 The F-pure case

Takagi-Yoshida [TY08, Theorem 3.3] and Hochster-Huneke [HH07, Theorem 3.6] independently showed that if R/I is F-pure, then Theorem 1.36 can always be improved by 1. More precisely, $I^{(hn-1)} \subseteq I^n$ for all $n \geqslant 1$, where h is the big height of I. For ideals with big height $h = 2$, this is in fact Harbourne's Conjecture. While Takagi and Yoshida used the theory of test ideals, Hochster and Huneke relied on the same type of techniques we have discussed in the previous section. Takagi and Yoshida's result, however, is stronger, and somewhat reminiscent of Theorem 2.12:

Theorem 2.29. *Let R be an excellent regular ring of characteristic $p > 0$, and let I be an ideal of positive height. Write h for the big height of I. If R/I is F-pure, then*

$$I^{(hn+kn-1)} \subseteq \left(I^{(k+1)}\right)^n.$$

We might then expect to obtain stronger containment results when R/I is F-pure. And indeed, we will show that Harbourne's Conjecture always holds for I when R/I is F-pure.

Naively, the idea of the proof is to study the colon ideal $\left(I^n : I^{(hn-h+1)}\right)$. The colon ideal $(J : I)$ measures the failure of $I \subseteq J$, and $(J : I) = R$ precisely when $I \subseteq J$. In order to show that $\left(I^n : I^{(hn-h+1)}\right) = R$, we need to show that this ideal contains some *large* ideal; Fedder's Criterion 2.21 provides the perfect candidate.

Lemma 2.30 (Lemma 3.2 in [GH17])**.** *Let R be a regular ring of characteristic $p > 0$. Let I be a radical ideal in R and h the big height of I. For all $n \geqslant 1$ and for all $q = p^e \gg 0$,*

$$\left(I^{[q]} : I\right) \subseteq \left(I^n : I^{(hn-h+1)}\right)^{[q]}.$$

Proof. Recall that

$$\left(I^n : I^{(hn-h+1)}\right)^{[q]} = \left((I^n)^{[q]} : \left(I^{(hn-h+1)}\right)^{[q]}\right),$$

by Lemma 2.6. Take $s \in \left(I^{[q]} : I\right)$. Then $sI^{(hn-h+1)} \subseteq sI \subseteq I^{[q]}$, so

$$s\left(I^{(hn-h+1)}\right)^{[q]} \subseteq \left(sI^{(hn-h+1)}\right)\left(I^{(hn-h+1)}\right)^{q-1} \subseteq I^{[q]}\left(I^{(hn-h+1)}\right)^{q-1}.$$

We will show that

$$\left(I^{(hn-h+1)}\right)^{q-1} \subseteq \left(I^{n-1}\right)^{[q]},$$

which implies that

$$s\left(I^{(hn-h+1)}\right)^{[q]} \subseteq (I^n)^{[q]},$$

completing the proof.

Notice that, by Lemma 1.18,

$$\left(I^{(hn-h+1)}\right)^{q-1} \subseteq I^{((hn-h+1)(q-1))}.$$

We claim that $\left(I^{(hn-h+1)}\right)^{q-1} \subseteq I^{((hq-1)(n-1)+h(n-1))}$, and to show that, it is enough to prove that

$$(hn-h+1)(q-1) \geqslant (hq-1)(n-1) + h(n-1).$$

This holds if $q \geqslant (2h-1)(n-1)+1$.

Applying Theorem 2.12 with $k = hq - 1$, and with $n - 1$ in place of n, we obtain the following containment:

$$I^{(h(n-1)+(hq-1)(n-1))} \subseteq \left(I^{(hq)}\right)^{n-1}.$$

By Lemma 2.9, $I^{(hq)} \subseteq I^{[q]}$. Then

$$\left(I^{(hn-h+1)}\right)^{q-1} \subseteq \left(I^{[q]}\right)^{n-1}.$$

As noted before, this concludes the proof. □

We can now show that Harbourne's conjecture holds for ideals defining F-pure rings:

Theorem 2.31 (Theorem 3.3 in [GH17])**.** *Let R be a regular ring of characteristic $p > 0$. Let I be an ideal in R with R/I F-pure, and let h be the big height of I. Then for all $n \geqslant 1$, $I^{(hn-h+1)} \subseteq I^n$.*

Proof. First, note that we can reduce to the local case, by Lemma 1.50. Indeed, the big height of an ideal does not increase under localization, and all localizations of an F-pure ring are F-pure [HR74, 6.2]. So suppose that $(R, \mathfrak{m})$ is a regular local ring, and that R/I is F-pure.

Fix $n \geqslant 1$, and consider q as in Lemma 2.30. Then

$$\left(I^{[q]} : I\right) \subseteq \left(I^n : I^{(hn-h+1)}\right)^{[q]}.$$

If $I^{(hn-h+1)} \nsubseteq I^n$, then $\left(I^n : I^{(hn-h+1)}\right)^{[q]} \subseteq \mathfrak{m}^{[q]}$, contradicting Fedder's Criterion (Theorem 2.21). □

This result is sharp over the class of ideals defining F-pure rings, as the following example shows. This example is a special case of the star configurations of points in [HH13], and has also appeared in [HH07, Example 1.2].

Example 2.32. Let $I \subseteq R := k[x_1, \dots, x_v]$ be the following ideal:

$$I = \bigcap_{i \neq j} (x_i, x_j) = (x_1 \dots \hat{x}_i \dots x_v : 1 \leqslant i \leqslant v) \, .$$

As we saw in Example 1.17, the symbolic powers of I can be written as

$$I^{(m)} = \bigcap_{i \neq j} (x_i, x_j)^m \, .$$

A monomial $x_1^{a_1} \cdots x_v^{a_v}$ is in $(x_i, x_j)^m$ if and only if $a_i + a_j \geqslant m$; thus $x_1^{a_1} \cdots x_v^{a_v} \in I^{(m)}$ if and only if $a_i + a_j \geqslant m$ for all $i \neq j$. In particular, $x_1^{n-1} \cdots x_v^{n-1} \in I^{(2n-2)}$. On the other hand, elements of I have degree at least $v-1$, so that elements of I^n have degree at least $(v-1)n = vn - n$. If $n < v$, then $vn - v < vn - n$, and since the degree of $x_1^{n-1} \cdots x_v^{n-1}$ is $v(n-1) = vn - v$, $x_1^{n-1} \cdots x_v^{n-1} \notin I^n$.

Notice that all the associated primes of I have height 2, so $h = 2$. Theorem 2.31 says that $I^{(2n-1)} \subseteq I^n$ for all n, but the previous argument shows that $I^{(2n-2)} \nsubseteq I^n$ for all $n < v$. However, we will see in Example 4.11 that in fact once we fix v, we do have $I^{(2n-2)} \subseteq I^n$ for all $n \gg 0$.

This example can be generalized to any $h \geqslant 2$, by taking $I = \bigcap_{i_1 < \dots < i_h} (x_{i_1}, \dots, x_{i_h})$. In that case, $I^{(hn-h)} \nsubseteq I^n$ for all $n < (h-1)v$.

However, we do not know of an example of an ideal I with $I^{(hn-h)} \nsubseteq I^n$ for all n, or even for infinitely many values of n. We will explore this idea further in Chapter 4.

2.2.2 The strongly F-regular case

Example 2.32 gives classes of squarefree monomial ideals for which Harbourne's Conjecture is best possible. However, if we exclude such examples, we can obtain better containments. In fact, if R/I is strongly F-regular, we can replace h by $h-1$ in Harbourne's Conjecture. To prove this fact, we will need the following lemma: x

Lemma 2.33 (Lemma 4.2 in [GH17]). *Let R be a regular ring of characteristic $p>0$, I an ideal in R, and $h \geqslant 2$ the big height of I. Then for all $d \geqslant h-1$ and for all $q=p^e$,*

$$\left(I^d : I^{(d)}\right)\left(I^{[q]} : I\right) \subseteq \left(II^{(d+1-h)} : I^{(d)}\right)^{[q]}.$$

Proof. Let $t \in \left(I^d : I^{(d)}\right)$ and $s \in \left(I^{[q]} : I\right)$. It suffices to prove that

$$st\left(I^{(d)}\right)^{[q]} \subseteq \left(II^{(d+1-h)}\right)^{[q]}.$$

First, note that

$$st\left(I^{(d)}\right)^{[q]} \subseteq s\left(tI^{(d)}\right)\left(I^{(d)}\right)^{q-1} \subseteq sI^d\left(I^{(d)}\right)^{q-1} \subseteq I^{[q]}I^{d-1}\left(I^{(d)}\right)^{q-1}.$$

Now since $d-1+d(q-1)=dq-1$, we have $I^{d-1}\left(I^{(d)}\right)^{q-1} \subseteq I^{(qd-1)}$. By Lemma 2.13, we get

$$I^{(qd-h+1)} \subseteq \left(I^{(d+1-h)}\right)^{[q]}.$$

As long as $h \geqslant 2$, we have $qd-1 \geqslant qd-h+1$, which implies that $I^{(qd-1)} \subseteq \left(I^{(d+1-h)}\right)^{[q]}$. Then

$$st\left(I^{(d)}\right)^{[q]} \subseteq I^{[q]}I^{d-1}\left(I^{(d)}\right)^{q-1} \subseteq \left(II^{(d+1-h)}\right)^{[q]},$$

as desired. □

We can now show the main result on ideals defining strongly F-regular rings:

Theorem 2.34 (Theorem 4.1 in [GH17]). *Let R be an F-finite regular ring of characteristic $p>0$, and I an ideal of big height $h \geqslant 2$ such that R/I is strongly F-regular. Then $I^{(d)} \subseteq II^{(d+1-h)}$ for all $d \geqslant h-1$. In particular,*

$$I^{((h-1)n+1)} \subseteq I^{n+1}$$

for all $n \geqslant 1$.

Proof. We first note that the second statement follows from the first, by induction. To see this, assume we have shown that $I^{(d)} \subseteq II^{(d+1-h)}$ for all $d \geqslant h-1$. When

$d = h$, this means that $I^{(h)} \subseteq II^{(1)} \subseteq I^2$, which is the statement we are trying to show for the case $n = 1$. The induction step follows from choosing $d = (h-1)(n+1)+1$.

We prove the first statement by contradiction. As before, we can reduce to the case where $(R, \mathfrak{m})$ is a regular local ring. Note that strong F-regularity is a local property, that is, R is strongly F-regular if and only if all of its localizations are strongly F-regular [HH89, 3.1 (a)].

Suppose that $\left(II^{(d+1-h)} : I^{(d)}\right) \subseteq \mathfrak{m}$. We claim that we can always find an element $t \in \left(I^d : I^{(d)}\right)$ not in any minimal prime of I. Indeed, either $I^{(d)} = I^d$, in which case $\left(I^d : I^{(d)}\right) = R$, or $I^d = I^{(d)} \cap J$, where J is an intersection of embedded components of I^d. By definition, J must contain elements not in a minimal prime of I, and any such element must be in $\left(I^d : I^{(d)}\right)$.

By Lemma 2.33,

$$t\left(I^{[q]} : I\right) \subseteq \left(II^{(d+1-h)} : I^{(d)}\right)^{[q]} \subseteq \mathfrak{m}^{[q]}.$$

By Glassbrenner's Criterion, Theorem 2.27, this contradicts the fact that R/I is strongly F-regular. □

Remark 2.35. Note that if R is a local ring and R/I is strongly F-regular, then I is a prime ideal, so that the big height of I is in fact the height of I.

For primes of height 2, Theorem 2.34 actually gives equality:

Corollary 2.36. *Let R be a regular ring of characteristic $p > 0$, and I a height 2 prime such that R/I is strongly F-regular. Then $I^{(n)} = I^n$ for all $n \geqslant 1$.*

This gives non-trivial classes of ideals with $I^{(n)} = I^n$ for all $n \geqslant 1$.

Example 2.37. Let $S = k[s^3, s^2t, st^2, t^3] \subseteq k[s,t]$, where k is a field of characteristic $p > 3$. This is a Veronese subring of $k[s,t]$, and thus strongly F-regular. We can write S as a quotient of $k[a,b,c,d]$ by a 3-generated height 2 prime ideal,

$$P = \left(b^2 - ac, c^2 - bd, bc - ad\right).$$

By Corollary 2.36, $P^{(n)} = P^n$ for all $n \geqslant 1$.

Example 2.38. Let $R = k[a, b, c, d]$, where k is a field of prime characteristic $p > 2$, n be an integer, and let I be the ideal of 2×2 minors of the 2×3 matrix

$$\begin{pmatrix} a^2 & b & d \\ c & a^2 & b^n - d \end{pmatrix}.$$

By [Sin99, Proposition 4.3], R/I is strongly F-regular. Since I is has height 2, Corollary 2.36 says that $I^{(k)} = I^k$ for all k.

We do not know if Theorem 2.34 is sharp when $h > 2$. There are, however, subclasses of ideals defining strongly F-regular rings for which we can obtain better containments. The following is Example 3 in [GH17].

Example 2.39 (Determinantal ideals)**.** Consider a generic $n \times n$ matrix X, a field K of characteristic 0 or $p > \min\{t, n-t\}$, and let I_t denote the ideal of t-minors of X in $R = K[X]$. These ideals of minors define strongly F-regular rings in characteristic p, by [HH94b, 7.14]. For which values of k and m do we have $I_t^{(k)} \subseteq I_t^m$? We claim that this holds when $\frac{n}{t(n-t+1)} k \geqslant m$.

Fix k and consider $m \leqslant \frac{nk}{t(n-t+1)}$. By [BV88, Theorem 10.4], given s_i-minors δ_i of X, we have $\delta_1 \cdots \delta_u \in I_t^{(k)}$ if and only if $\displaystyle\sum_{i=1}^{u} \max\{0, s_i - t + 1\} \geqslant k$, and moreover, $I_t^{(k)}$ is generated by such products. Notice that any factors corresponding to minors of size less than t are irrelevant to determine whether or not the given product is in $I_t^{(k)}$.

So consider $\delta_1, \ldots, \delta_u$, with δ_i an s_i-minor for each i, such that $\delta_1 \cdots \delta_u \in I_t^{(k)}$, and write $s = s_1 + \cdots + s_u$. Using the remark above, we may assume that $s_i \geqslant t$ for each i. We want to show that for all such possible choices of δ_i, $\delta_1 \cdots \delta_u \in I_t^m$. Since $\delta_1 \cdots \delta_u \in I_t^{(k)}$, then

$$\sum_{i=1}^{u} \max\{0, s_i - t + 1\} \geqslant k$$

which, since we are assuming $s_i \geqslant t$ for all i, can be rewritten as

$$(*) \qquad s \geqslant k + u(t-1).$$

Moreover, s_i is the size of a minor of an $n \times n$ matrix, and thus each $s_i \leqslant n$. In particular, this implies that $un \geqslant s$. Therefore, we must have

$$un \geqslant k + u(t-1),$$

so that

$$u \geqslant \frac{k}{n-t+1}.$$

Combining this with $(*)$, we obtain

$$s \geqslant k + \frac{k}{n-t+1}(t-1) = \frac{nk}{n-t+1}.$$

By [DEP80, 7.3], $I_t^m = I_t^{(m)} \cap I_{t-1}^{(2m)} \cap \cdots \cap I_1^{(tm)}$. Thus it suffices to show that $\delta_1 \cdots \delta_u \in I_t^m$. In particular, for each $1 \leqslant j \leqslant t$, we need to show that $\delta_1 \cdots \delta_u \in I_j^{((t-j+1)m)}$, which is equivalent to the following inequalities:

$$\begin{aligned}
&s \geqslant tm \\
&s \geqslant (t-1)m + u = tm + (u-m) \\
&s \geqslant (t-2)m + 2u = tm + 2(u-m) \\
&\quad\vdots \\
&s \geqslant m + (t-1)u = tm + (t-1)(u-m).
\end{aligned}$$

Note that all these inequalities are convex combinations of the first and the last one, so that they are verified as long as the first and last one are, that is, if and only if

$$\begin{aligned}
&s \geqslant tm \\
&s \geqslant m + (t-1)u.
\end{aligned}$$

Since we are assuming that $s \geqslant k + (t-1)u$, then $s \geqslant m + (t-1)u$ as long as $k \geqslant m$, which must be satisfied by any m with $I_t^{(k)} \subseteq I_t^m$. It remains to check that $s \geqslant tm$.

We have seen above that $s \geqslant \frac{nk}{n-t+1}$. Given this, $s \geqslant tm$ is satisfied as long as $\frac{nk}{n-t+1} \geqslant tm$, which can be rewritten as $k \geqslant \frac{(n-t+1)t}{n} m$, which was our claim.

We conclude that $I^{(k)} \subseteq I^m$ as long as $k \geqslant \frac{t(n-t+1)}{n} m$. This bound is much better than that of Theorem 2.34 when n is large and t is close to $\frac{n}{2}$. We also note that the computations above can be used to show that, given k, this is actually the best possible value of m.

Note that we can obtain the same containment results for the ideal of $t \times t$ minors of a symmetric $n \times n$ matrix or the ideal of $2t$-Pfaffians of a generic $n \times n$ matrix, using Proposition 4.3 and Theorem 4.4 in [JMnV15] for the symmetric case and Theorem 2.1 and Theorem 2.4 in [DN96] for the Pfaffians.

The key ingredient that allowed us to completely answer the Containment Problem for such ideals is the explicit description of the symbolic and ordinary powers given by [DEP80, JMnV15, DN96]. In general, there is no such nice characterization of symbolic powers.

Chapter 3

Height 2 ideals in 3 variables

If P is a prime ideal of height 2 in a regular ring, is $P^{(3)} \subseteq P^2$? This question of Huneke, which motivated Harbourne's Conjecture,[1] is still open. The simplest interesting case is when P is a 3-generated ideal in dimension 3, and among these, the defining ideals of space monomial curves $k[t^a, t^b, t^c]$. All of these space monomial curves are defined by the 2×2 minors of a 2×3 matrix. Following the work of Alexandra Seceleanu in [Sec15], we study the Containment Problem for ideals given by the 2×2 minors of a 2×3 matrix, and prove that $P^{(3)} \subseteq P^2$ does hold for space monomial curves. We then apply the same techniques to find sufficient conditions for other containments of the type $I^{(a)} \subseteq I^b$ to hold.

3.1 Preliminaries

3.1.1 A Homological Criterion

As we saw in Lemma 1.30, the symbolic powers of an ideal I can be computed by taking the saturations of the ordinary powers I^n with respect to some fixed ideal J. However, determining which ideal J to consider for a given I can be fairly difficult. Fortunately, if I has height $\dim R - 1$, we do have control over which J to consider.

Lemma 3.1. *Let I be an ideal of pure height $d - 1$ with no embedded primes in a local ring of dimension d with maximal ideal $\mathfrak{m}$. Then the symbolic powers of I can*

[1] See Question 1.43 and Conjecture 1.44.

be computed by taking saturations with the maximal ideal, that is, for all $n \geqslant 1$.

$$I^{(n)} = (I^n : \mathfrak{m}^\infty)\,.$$

Proof. The only potential embedded prime of any power of I is $\mathfrak{m}$. Therefore, the result follows by 1.31. ☐

In order to describe saturations, local cohomology comes in handy:

Definition 3.2 (Local Cohomology). *Let R be a noetherian ring and I an ideal in R. Consider the functor $\Gamma_I(-)$ defined on the category of R-modules by taking a module M to the its submodule*

$$\Gamma_I(M) = \{m \in M \mid \text{ there exists } s \geqslant 1 : I^s m = 0\} = (0 :_M I^\infty)$$

and defined naturally on maps by pre-composing with the restriction map. This is a left exact functor, and its right derived functors are denoted by $\mathrm{H}^i_I(-)$. The module $\mathrm{H}^i_I(M)$ is called the i-th local cohomology *module of M with respect to I.*

Alternatively, $\mathrm{H}^i_I(M)$ can be defined as the i-th homology module of the Čech Complex $\check{C}(\underline{x}; M)$. The Čech Complex $\check{C}(\underline{x}; M)$ is defined by taking $\check{C}(\underline{x}; M) = \check{C}(\underline{x}) \otimes M$, where $I = (x_1, \ldots, x_n)$ and $\check{C}(\underline{x})$ is defined as follows.

For each $x \in R$, denote by R_x and M_x the localizations by the multiplicative set of powers of x. Consider $x_1, \ldots, x_n \in R$, $\underline{x} := x_1, \ldots, x_n$ and the complexes

$$\check{C}(x_i):\ 0 \longrightarrow R \longrightarrow R_{x_i} \longrightarrow 0\,. \qquad r \longmapsto \tfrac{r}{1}$$

The Čech complex *on $\underline{x}$ is the chain complex*

$$\check{C}(\underline{x}) := \check{C}(x_1, \ldots, x_n) = \bigotimes_{i=1}^{n} (\, 0 \longrightarrow R \longrightarrow R_{x_i} \longrightarrow 0\,) = \bigotimes_{i=1}^{n} \check{C}(x_i),$$

that is,

$$0 \longrightarrow R \longrightarrow \oplus_{i=1}^n R_{x_i} \longrightarrow \oplus_{i<j}^n R_{x_i x_j} \longrightarrow \ldots \longrightarrow R_{x_1 \ldots x_n} \longrightarrow 0\,.$$

Therefore, $\mathrm{H}^i_I(M)$ *is given by the* i*-th homology module of the complex*

$$0 \longrightarrow M \longrightarrow \bigoplus_{i=1}^{n} M_{x_i} \longrightarrow \bigoplus_{i<j}^{n} M_{x_i x_j} \longrightarrow \dots \longrightarrow M_{x_1 \dots x_n} \longrightarrow 0\,.$$

For the equivalence of these two definitions, see [BS98, §5.1].

Local cohomology modules measure depth and dimension, and have many other important properties. For more on local cohomology, see [ILL+07] or [BS98].

Lemma 3.3. *Let* I *be an ideal of height* $d-1$ *with no embedded primes in a local ring of dimension* d *with maximal ideal* $\mathfrak{m}$*. Then for each* $n \geqslant 1$*,*

$$\mathrm{H}^0_{\mathfrak{m}}\left(R/I^n\right) = I^{(n)}/I^n.$$

Proof. By definition,

$$\mathrm{H}^0_{\mathfrak{m}}\left(R/I^n\right) = 0 :_{R/I^n} \mathfrak{m}^\infty = (I^n : \mathfrak{m}^\infty)/I^n.$$

The statement follows from Lemma 3.1. □

Lemma 3.4. *Let* I *be an ideal of height* $d-1$ *with no embedded primes in a local ring of dimension* d *with maximal ideal* $\mathfrak{m}$*, and consider integers* $a \geqslant b$*. The containment* $I^{(a)} \subseteq I^b$ *holds if and only if the map*

$$\mathrm{H}^0_{\mathfrak{m}}\left(R/I^a\right) \longrightarrow \mathrm{H}^0_{\mathfrak{m}}\left(R/I^b\right)$$

induced by the surjection $R/I^a \to R/I^b$ *is the zero map.*

Proof. By Lemma 3.3, the map in question is the map

$$I^{(a)}/I^a \longrightarrow I^{(b)}/I^b,$$

which has image $I^{(a)}/I^b$. This is the zero map if and only if $I^{(a)} \subseteq I^b$. □

We can now replace the question of whether $I^{(a)} \subseteq I^b$ by a question that involves only ordinary powers of I. The following theorem is [Sec15, Proposition 3.1] rewritten for any dimension:

Lemma 3.5. *Let I be an ideal of height $d-1$ with no embedded primes in a regular local ring of dimension d with maximal ideal $\mathfrak{m}$, and consider integers $a \geqslant b$. Let π be the natural projection map $R/I^a \longrightarrow R/I^b$, and ι be the natural inclusion map $I^a \subseteq I^b$. The following are equivalent:*

(a) $I^{(a)} \subseteq I^b$;

(b) The map $\operatorname{Ext}^d_R(\pi) : \operatorname{Ext}^d_R\left(R/I^b, R\right) \longrightarrow \operatorname{Ext}^d_R\left(R/I^a, R\right)$ *is the zero map;*

(c) The map $\operatorname{Ext}^{d-1}_R(\iota) : \operatorname{Ext}^{d-1}_R\left(I^b, R\right) \longrightarrow \operatorname{Ext}^{d-1}_R\left(I^a, R\right)$ *is the zero map.*

Proof. The proof follows by Local Duality (cf. [BS98, Theorem 11.2.6]) and Ext-shifting. More precisely: under our assumptions, Local Duality implies that the map

$$\mathrm{H}^0_{\mathfrak{m}}\left(R/I^a\right) \longrightarrow \mathrm{H}^0_{\mathfrak{m}}\left(R/I^b\right)$$

is the zero map if and only if the map $\operatorname{Ext}^d_R(\pi) : \operatorname{Ext}^d_R\left(R/I^b, R\right) \longrightarrow \operatorname{Ext}^d_R\left(R/I^a, R\right)$ is the zero map, so that the equivalence between (a) and (b) follows by Lemma 3.4. The equivalence between (b) and (c) follows from looking at the maps between the long exact sequence in local cohomology induced by applying the functor $\Gamma_I(-)$ to the following diagram, where the rows are short exact sequences:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I^a & \longrightarrow & R & \longrightarrow & R/I^a & \longrightarrow & 0\,. \\
 & & \big\downarrow{\scriptstyle \iota} & & \big\| & & \big\downarrow{\scriptstyle \pi} & & \\
0 & \longrightarrow & I^b & \longrightarrow & R & \longrightarrow & R/I^b & \longrightarrow & 0
\end{array}$$

□

Remark 3.6. The assumption that R is regular is not necessary in Lemma 3.5, the same proof works if R is any Gorenstein local ring. However, we will only use this result in the regular setting.

We have now converted the Containment Problem into a purely homological question, one which we can answer if we can find free resolutions $F_*(n)$ for all the powers of I together with compatible lifts of the inclusion maps $I^a \subseteq I^b$ for each $a \geqslant b$:

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & F_{d-1}(a) & \longrightarrow & \cdots & \longrightarrow & F_0(a) & \longrightarrow & I^a & \longrightarrow & 0\,. \\
 & & \Big\downarrow{\scriptstyle g} & & & & \Big\downarrow & & \Big\downarrow & & \\
\cdots & \longrightarrow & F_{d-1}(b) & \longrightarrow & \cdots & \longrightarrow & F_0(b) & \longrightarrow & I^b & \longrightarrow & 0
\end{array}$$

Given Lemma 3.4, determining if $I^{(a)} \subseteq I^b$ is equivalent to determining if g induces the zero map on $\operatorname{Ext}_R^{d-1}(-, R)$. In general, this question can be very difficult to solve. However, we will study cases where we can both find resolutions for all powers of I and compatible lifts of all inclusions $I^{(a)} \subseteq I^b$.

Remark 3.7. All the results above also hold when we take R to be a polynomial ring over a field k and I to be a homogeneous ideal, where $\mathfrak{m}$ represents the unique homogeneous maximal ideal.

3.1.2 Rees Algebra preliminaries

To study *all* the powers of I, we will use the Rees algebra of I:

Definition 3.8 (Rees Algebras)**.** *Let I be an ideal in a noetherian ring R. The* Rees algebra $\mathcal{R}(I)$ *of I is the graded algebra given by*

$$\mathcal{R}(I) = \bigoplus_{n \geqslant 0} I^n t^n \subseteq R[t],$$

whose degree-n piece is isomorphic to I^n.

Remark 3.9. We can view the Rees algebra of I as a quotient of a polynomial ring in R. Indeed, suppose that I is generated by elements $f_1, \ldots, f_n \in R$. Consider the polynomial ring $S = R[T_1, \ldots, T_n]$, where elements in R have degree 0 and each T_i has degree 1, and the graded map $S \longrightarrow \mathcal{R}(I)$ given by $T_i \mapsto f_i t$. This map determines $\mathcal{R}(I)$ as a quotient of S. If we can resolve $\mathcal{R}(I)$ over S, we can use that resolution to find resolutions of each power of I over R by taking the strands in each degree. Via this map, we can write $\mathcal{R}(I) \cong S/L$, where

$$L = \big(F(T_1, \ldots, T_n) \in R\,[T_1, \ldots, T_n] \;\big|\; F(f_1, \ldots, f_n) = 0\big)\,.$$

Recall that the symmetric algebra of I is the quotient of the tensor algebra $\bigoplus_{n\geqslant 0} I^{\otimes n}$ by the ideal generated by the simple tensors of the form $u \otimes v - v \otimes u$. The symmetric algebra of I can also be written as a quotient of S, as $\mathrm{Sym}(I) \cong S/L_1$ with

$$L_1 = \left(\sum_{i=1}^{n} a_i T_i \;\middle|\; \sum_{i=1}^{n} a_i f_i = 0 \right).$$

Notice that L_1 coincides with the ideal generated by the degree 1 part of L above.

Moreover, the multiplication map on $I \otimes I$ induces a surjective graded map $\mathrm{Sym}(I) \to \mathcal{R}(I)$.

Definition 3.10. *We say that an ideal I is* of linear type *if the map* $\mathrm{Sym}(I) \to \mathcal{R}(I)$ *is an isomorphism.*

Remark 3.11. Equivalently, I is of linear type if $L = L_1$ in Remark 3.9, which is equivalent to L being generated in degree 1.

The ideals of linear type are those with the simplest possible Rees algebra. In general, Rees algebras are very difficult to resolve over S, but not if I is of linear type. This is the property that we will make use of in the next subsection.

3.2 Criteria in terms of matrices

We will now restrict our attention to the polynomial ring $R = k[x, y, z]$ and to ideals of the form $I = I_2(M)$, where M is the matrix

$$M = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}$$

and where $I_2(M)$ denotes the ideal of 2×2 minors of M. So $I = (f_1, f_2, f_3)$, where

$$f_1 = a_2 b_3 - a_3 b_2, \quad f_2 = a_3 b_1 - a_1 b_3 \quad \text{and} \quad f_3 = a_1 b_2 - a_2 b_1.$$

We will furthermore assume that I is a local complete intersection, which means that when localizing at a minimal prime of I, I is generated by 2 elements.

Whenever we write S, we mean the algebra $S = R[T_1, T_2, T_3]$, and we will write $F := a_1T_1 + a_2T_2 + a_3T_3$ and $G := b_1T_1 + b_2T_2 + b_3T_3$.

3.2.1 General criteria

Lemma 3.12. *Let I be as above. Then I is of linear type.*

Proof. By [Hun80], ideals generated by d-sequences are of linear type, and I is generated by a d-sequence. □

In particular, we know how to resolve $\mathcal{R}(I)$ over S. Recall that we write $S(-n)$ to indicate the graded S-module with the same module structure as S, but with a graded structure given by shifting the degrees in S, via $S(-n)_m = S_{m-n}$.

Lemma 3.13. *The Rees algebra of I is a complete intersection over S, with minimal free resolution given by*

$$0 \longrightarrow S(-2) \xrightarrow{\varphi} S(-1) \oplus S(-1) \xrightarrow{\psi} S \longrightarrow \mathcal{R}(I) \longrightarrow 0\,,$$

where $\varphi = \begin{bmatrix} F \\ G \end{bmatrix}$ *and* $\psi = \begin{bmatrix} G & -F \end{bmatrix}$.

Proof. Our assumption that I has height 2 implies in particular that the two rows of M are linearly independent, so that F, G is a regular sequence in S. All we need to see is that the kernel $L = L_1$ of $T_i \mapsto f_i t$ is the ideal (F, G), since the resolution given is simply the Koszul complex on the regular sequence F, G. A minimal free resolution of I is given by

$$0 \longrightarrow R^2 \xrightarrow{M^T} R^3 \xrightarrow{[f_1\ f_2\ f_3]} I \longrightarrow 0\,.$$

In particular, any relation between the generators of I is an R-linear combination of the relations (a_1, a_2, a_3) and (b_1, b_2, b_3), so that F and G minimally generate L. □

Discussion 3.14. Since $\mathcal{R}(I)_n = I^n t^n$, we can now extract resolutions for all powers of I by taking the degree n strand of the resolution for $\mathcal{R}(I)$ over S. For that, note that S_n is the free R-module generated by all monomials of degree n in T_1, T_2, and T_3. There are $\binom{n+2}{2}$ monomials of degree n in 3 generators, so we will identify S_n with $R^{\binom{n+2}{2}}$. Thus I^n has a free resolution over R given by

$$0 \longrightarrow R^{\binom{n}{2}} \xrightarrow{\varphi(n)} R^{\binom{n+1}{2}} \oplus R^{\binom{n+1}{2}} \xrightarrow{\psi(n)} R^{\binom{n+2}{2}} \longrightarrow I^n \longrightarrow 0$$

where $\varphi(n)$ is the map induced by multiplication by F on the first copy of $R^{\binom{n+1}{2}}$ and by multiplication by G on the second copy of $R^{\binom{n+1}{2}}$.

More precisely, let α and β be generators of the free S-module in homological degree 1 in the minimal free resolution

$$0 \longrightarrow S(-2) \xrightarrow{\varphi} S(-1) \oplus S(-1) \xrightarrow{\psi} S \longrightarrow \mathcal{R}(I) \longrightarrow 0\ ,$$

and let γ be a generator of the free S-module in homological degree 2. With this notation, $T_1^i T_2^j T_3^k \gamma$, where $i+j+k = n-2$, will denote a generator in homological degree 2 in

$$0 \longrightarrow R^{\binom{n}{2}} \xrightarrow{\varphi(n)} R^{\binom{n+1}{2}} \oplus R^{\binom{n+1}{2}} \xrightarrow{\psi(n)} R^{\binom{n+2}{2}} \longrightarrow I^n \longrightarrow 0\ ,$$

while $T_1^i T_2^j T_3^k \alpha$ and $T_1^i T_2^j T_3^k \beta$, where $i+j+k = n-1$, denote generators in homological degree 1. With this notation, given $i+j+k = n-2$, the map $\varphi(n)$ takes $T_1^i T_2^j T_3^k \gamma$ to

$$\left(a_1 T_1^{i+1} T_2^j T_3^k + a_2 T_1^i T_2^{j+1} T_3^k + a_3 T_1^i T_2^j T_3^{k+1}\right)\alpha + \left(b_1 T_1^{i+1} T_2^j T_3^k + b_2 T_1^i T_2^{j+1} T_3^k + b_3 T_1^i T_2^j T_3^{k+1}\right)\beta.$$

If we represent $\varphi(n)$ as a matrix with $n-2$ columns and $2(n-1)$ rows, the entry in the column corresponding to $T_1^i T_2^j T_3^k \gamma$, where $i+j+k = n-2$, and the row corresponding to $T_1^u T_2^w T_3^v \delta$, where $u+w+v = n-1$ and $\delta = \alpha$ or $\delta = \beta$, has

- a_1 if $\delta = \alpha$, $u = i+1$, $w = j$ and $v = k$;
- a_2 if $\delta = \alpha$, $u = i$, $w = j+1$ and $v = k$;

- a_3 if $\delta = \alpha$, $u = i$, $w = j$ and $v = k + 1$;
- b_1 if $\delta = \alpha$, $u = i + 1$, $w = j$ and $v = k$;
- b_2 if $\delta = \alpha$, $u = i$, $w = j + 1$ and $v = k$;
- b_3 if $\delta = \alpha$, $u = i$, $w = j$ and $v = k + 1$.

Fix the lexicographical order on the monomials in T_1, T_2, and T_3 and write each of the maps we are considering in the ordered basis given by this order.

Example 3.15. When $n = 2$, $\varphi(2)$ can be represented by the following 6×1 matrix:

$$\begin{bmatrix} a_1 & a_2 & a_3 & b_1 & b_2 & b_3 \end{bmatrix}^T.$$

Example 3.16. When $n = 3$, $\varphi(3)$ can be represented by a 12×3 matrix, whose transpose we record below, indicating which basis elements each row and column corresponds to:

	$T_1^2\alpha$	$T_1T_2\alpha$	$T_1T_3\alpha$	$T_2^2\alpha$	$T_2T_3\alpha$	$T_3^2\alpha$	$T_1^2\beta$	$T_1T_2\beta$	$T_1T_3\beta$	$T_2^2\beta$	$T_2T_3\beta$	$T_3^2\beta$
$T_1\gamma$	a_1	a_2	a_3	0	0	0	b_1	b_2	b_3	0	0	0
$T_2\gamma$	0	a_1	0	a_2	a_3	0	0	b_1	0	b_2	b_3	0
$T_3\gamma$	0	0	a_1	0	a_2	a_3	0	0	b_1	0	b_2	b_3

Now that we have found resolutions for all powers of I, it remains to find lifts for all inclusion maps $I^n \subseteq I^{n-1}$ that are compatible with these resolutions.

Definition 3.17. *The* Euler operator *is the differential* $D = f_1\frac{\partial}{\partial T_1} + f_2\frac{\partial}{\partial T_2} + f_3\frac{\partial}{\partial T_3}$ *on* S.

Lemma 3.18. *The Euler operator* $D = f_1\frac{\partial}{\partial T_1} + f_2\frac{\partial}{\partial T_2} + f_3\frac{\partial}{\partial T_3}$ *on* S *induces the degree* -1 *map on* $\mathcal{R}(I)$ *that takes an homogeneous element* $gt^n \in I^n t^n$ *to* $ngt^{n-1} \in I^{n-1}t^{n-1}$.

Proof. Note that $D(F) = D(G) = 0$, since

$$D(F) = a_1f_1 + a_2f_2 + a_3f_3 = 0$$

and

$$D(G) = b_1 f_1 + b_2 f_2 + b_3 f_3 = 0.$$

In particular, D induces a map on $S/(F, G) \cong \mathcal{R}(I)$. Now to check that the induced map is indeed as claimed, it is enough show that the claim holds for elements of the form $gt^n = f_1^i f_2^j f_3^k t^n$, where $i + j + k = n$. Note that such an element is the image of $T_1^i T_2^j T_3^k$ via the surjection $S \longrightarrow \mathcal{R}(I)$, and that

$$D\left(T_1^i T_2^j T_3^k\right) = i f_1 T_1^{i-1} T_2^j T_3^k + j f_2 T_1^i T_2^{j-1} T_3^k + k f_3 T_1^i T_2^j T_3^{k-1}.$$

Now the map $S \longrightarrow \mathcal{R}(I)$ takes this element to

$$\left(i f_1 f_1^{i-1} f_2^j f_3^k + j f_2 f_1^i f_2^{j-1} f_3^k + k f_3 f_1^i f_2^j f_3^{k-1}\right) t^{n-1} = ngt^{n-1}.$$

$\square$

In what follows, we will write D_n to represent the map $S_n \longrightarrow S_{n-1}$ induced by the Euler operator D.

Corollary 3.19. *The following is a commutative diagram with exact rows:*

$$\begin{CD}
0 @>>> R^{\binom{n-1}{2}} @>{\varphi(n-1)}>> R^{\binom{n}{2}} \oplus R^{\binom{n}{2}} @>{\psi(n-1)}>> R^{\binom{n+1}{2}} @>>> I^{n-1} @>>> 0 \\
@. @A{D_{n-2}}AA @AA{D_{n-1}}A @AA{D_n}A @AA{n\,\iota}A \\
0 @>>> R^{\binom{n}{2}} @>{\varphi(n)}>> R^{\binom{n+1}{2}} \oplus R^{\binom{n+1}{2}} @>{\psi(n)}>> R^{\binom{n+2}{2}} @>>> I^{n} @>>> 0
\end{CD}$$

Proof. The only thing we need to check is that D commutes with the maps $[F\,G]$ and $[G\ -F]^T$ that appear in the minimal free resolution of $\mathcal{R}(I)$. In other words, the statement follows once we show that $DF = FD$ and $DG = GD$. By the Leibniz rule, $D(Fs) = F(D)s + FD(s)$ for each s. Note that $D(F) = a_1 f_1 + a_2 f_2 + a_3 f_3 = 0$, so that $D(Fs) = FD(s)$. Similarly for G. $\square$

Corollary 3.20. *Let $n \geqslant m \geqslant 1$ be integers. Write $D_{n,m} := D_{n-2} D_{n-3} \cdots D_m D_{m-1}$ and let ι denote the inclusion map $I^n \subseteq I^m$. The following is a commutative diagram* *with exact rows:*

$$\begin{array}{ccccccccc}
0 \longrightarrow & R^{\binom{m}{2}} & \xrightarrow{\varphi(m)} & R^{\binom{m+1}{2}} \oplus R^{\binom{m+1}{2}} & \xrightarrow{\psi(m)} & R^{\binom{m+2}{2}} & \longrightarrow & I^m & \longrightarrow 0 \\
 & \uparrow {\scriptstyle D_{n,m}} & & \uparrow & & \uparrow & & \uparrow {\scriptstyle \frac{n!}{m!}\iota} & \\
0 \longrightarrow & R^{\binom{n}{2}} & \xrightarrow[\varphi(n)]{} & R^{\binom{n+1}{2}} \oplus R^{\binom{n+1}{2}} & \xrightarrow[\psi(n)]{} & R^{\binom{n+2}{2}} & \longrightarrow & I^n & \longrightarrow 0
\end{array}$$

Proof. Compose successive commutative diagrams as in Corollary 3.19. □

We write A^T for the transpose matrix of A.

Corollary 3.21. *Suppose that* $\operatorname{char} k$ *does not divide* $\frac{n!}{m!}$. *Then* $I^{(n)} \subseteq I^m$ *if and only if the image of* $(D_{n,m})^T$ *is contained in the image of* $(\varphi(n))^T$.

Proof. By Lemma 3.5, $I^{(n)} \subseteq I^m$ if and only if $\operatorname{Ext}^2_R(\iota) = 0$. Under the assumption that $\operatorname{char} k \nmid \frac{n!}{m!}$, $\operatorname{Ext}^2_R(\iota) = 0$ if and only if $\frac{n!}{m!}\operatorname{Ext}^2_R(\iota) = \operatorname{Ext}^2_R\left(\frac{n!}{m!}\iota\right) = 0$. Finally, $\operatorname{Ext}^2_R(\iota)$ can be computed by applying $\operatorname{Hom}_R(-, R)$ to $D_{n,m}$. Given a matrix representing the map $D_{n,m}$, applying $\operatorname{Hom}_R(-, R)$ corresponds to matrix transposition. Therefore, $\operatorname{Ext}^2_R(I^n, R)$ can be computed by taking the quotient of $R^{\binom{n}{2}}$ by the image of the transpose of $\varphi(n)$. The result follows. □

It is now convenient to rewrite the previous statement:

Lemma 3.22. *Let* $n \geqslant m \geqslant 1$. *Suppose that* $\operatorname{char} k$ *does not divide* $\frac{n!}{m!}$. *Then* $I^{(n)} \subseteq I^m$ *if and only if for each* $i + j + k = m - 2$,

$$(n-m)! \sum_{u+w+v=n-m} \binom{i+u}{u}\binom{j+v}{v}\binom{k+w}{w} f_1^u f_2^v f_3^w T_1^{i+u} T_2^{j+v} T_3^{k+w}$$

is an R*-linear combination of*

$$\left\{ \begin{array}{c} a_1 T_1^{d-1} T_2^e T_3^f + a_2 T_1^d T_2^{e-1} T_3^f + a_3 T_1^d T_2^e T_3^{f-1}, \\ b_1 T_1^{d-1} T_2^e T_3^f + b_2 T_1^d T_2^{e-1} T_3^f + b_3 T_1^d T_2^e T_3^{f-1} \end{array} \right\}_{d+e+f=n-1}.$$

where $T_i^c = 0$ *if* $c < 0$.

Proof. Given $i + j + k = m - 2$,

$$(n-m)! \sum_{u+w+v=n-m} \binom{i+u}{u}\binom{j+v}{v}\binom{k+w}{w} f_1^u f_2^v f_3^w T_1^{i+u} T_2^{j+v} T_3^{k+w}$$

is the image under $(D_{n,m})^T$ of $T_1^i T_2^j T_3^k$. In other words, the coefficients of the term in $T_1^{i+u} T_2^{j+v} T_3^{k+w}$ above corresponds to the entry of the matrix for $(D_{n,m})^T$ in the column corresponding to the basis element $T_1^u T_2^v T_3^k$ and the row corresponding to the basis element $T_1^{i+u} T_2^{j+v} T_3^{k+w}$. To see this, note first that for each choice of $u+v+w = n-m$, the image of $T_1^{i+u} T_2^{j+v} T_3^{k+w}$ under $D_{n,m}$ is $T_1^i T_2^j T_3^k$, up to multiplying by an element in R. To check that the coefficients are as claimed, note that these can be computed by counting how many different ways we can compose the differential operators $f_1 \frac{\partial}{\partial T_1}$ (u times), $f_2 \frac{\partial}{\partial T_2}$ (v times), and $f_3 \frac{\partial}{\partial T_3}$ (w times), which gives

$$\frac{(u+v+w)!}{u!v!w!},$$

and multiplying by the coefficients that appear when applying $\frac{\partial^u}{\partial (T_1)^u}$ to T^{i+u} to obtain T^i, and similarly for T_2 and T_3, which gives

$$\frac{(i+u)!}{i!} \frac{(j+v)!}{j!} \frac{(k+w)!}{k!}.$$

Multiplying these together and simplifying, we obtain

$$(u+v+w)! \frac{(u+i)!}{u!i!} \frac{(j+v)!}{j!v!} \frac{(k+w)!}{k!w!},$$

which can be rewritten as claimed.

On the other hand, the elements in

$$\left\{ \begin{array}{c} a_1 T_1^{d-1} T_2^e T_3^f + a_2 T_1^d T_2^{e-1} T_3^f + a_3 T_1^d T_2^e T_3^{f-1}, \\ b_1 T_1^{d-1} T_2^e T_3^f + b_2 T_1^d T_2^{e-1} T_3^f + b_3 T_1^d T_2^e T_3^{f-1} \end{array} \right\}_{d+e+f=n-1}$$

generate the image of $(\varphi(n))^T$, and in fact each of these elements corresponds to a column of the matrix for $(\varphi(n))^T$ in the basis we are considering. Therefore, the statement follows from 3.21. □

Lemma 3.23. *Let $P(T)$ be a monomial in T_1, T_2, and T_3 of degree $n-3$. Given $i \neq j$, $i, j \in \{1,2,3\}$,*

$$(f_iT_i - f_jT_j)\, P(T)\gamma$$

is in the image of $(\varphi(n))^T$.

Proof. The statement is equivalent to the fact that $(f_iT_i - f_jT_j)\, P(T)$ is an R-linear combination of elements in

$$\left\{ \begin{array}{c} a_1T_1^{d-1}T_2^eT_3^f + a_2T_1^dT_2^{e-1}T_3^f + a_3T_1^dT_2^eT_3^{f-1}, \\ b_1T_1^{d-1}T_2^eT_3^f + b_2T_1^dT_2^{e-1}T_3^f + b_3T_1^dT_2^eT_3^{f-1} \end{array} \right\}_{d+e+f=n-1}.$$

We will prove the statement when $i = 1$ and $j = 2$. The remaining possibilities can be obtained in a similar way. If $P(T) = T_1^iT_2^jT_3^k$, then

$$\begin{aligned} & b_3\left(a_1T_1^iT_2^{k+1}T_3^j + a_2T_1^{i+1}T_2^kT_3^j + a_3T_1^{i+1}T_2^{k+1}T_3^{j-1}\right) \\ & -a_3\left(b_1T_1^iT_2^{k+1}T_3^j + b_2T_1^{i+1}T_2^kT_3^j + b_3T_1^{i+1}T_2^{k+1}T_3^{j-1}\right). \\ & = -f_2T_1^iT_2^{k+1}T_3^j + f_1T_1^{i+1}T_2^kT_3^j + 0\,T_1^{i+1}T_2^{k+1}T_3^{j-1} \\ & = (-f_2T_2 + f_1T_1)\, P(T). \end{aligned}$$

□

Remark 3.24. Note that the elements in Lemma 3.23 correspond to Koszul syzygies.

Theorem 3.25. *Let $n \geqslant m \geqslant 1$, and suppose that $\operatorname{char} k$ does not divide $\frac{n!}{m!}$. Then $I^{(n)} \subseteq I^m$ if and only if for each i, j, and k such that $i+j+k = m-2$, there exist a choice of u, v and w such that $u+v+w = n-m$ and*

$$\frac{n!}{m!}\, f_1^u f_2^v f_3^w T_1^{i+u}T_2^{j+v}T_3^{k+w}$$

is an R-linear combination of

$$\left\{ \begin{array}{c} a_1T_1^{d-1}T_2^eT_3^f + a_2T_1^dT_2^{e-1}T_3^f + a_3T_1^dT_2^eT_3^{f-1}, \\ b_1T_1^{d-1}T_2^eT_3^f + b_2T_1^dT_2^{e-1}T_3^f + b_3T_1^dT_2^eT_3^{f-1} \end{array} \right\}_{d+e+f=n-1}.$$

Remark 3.26. To show that $I^{(n)} \subseteq I^m$, we need to prove that each column of $(D_{n,m})^T$ is in the image of $\varphi(n)^T$. However, Theorem 3.25 actually says that for each of those columns, it is enough to show that the vector obtained by substituting all of the nonzero entries but one by 0 is in the image of $\varphi(n)^T$. This simplifies our calculations considerably if we want to prove that specific containments hold. All our results of this kind will follow as corollaries of this theorem.

Proof. First, note that

$$(n-m)! \sum_{u+w+v=n-m} \binom{i+u}{u}\binom{j+v}{v}\binom{k+w}{w} = (n-m)!\binom{n}{n-m} = \frac{n!}{m!}.$$

Given two choices of $u+v+w = u'+v'+w' = n-m$, we claim that

$$f_1^u f_2^v f_3^w T_1^{i+u} T_2^{j+v} T_3^{k+w} - f_1^{u'} f_2^{v'} f_3^{w'} T_1^{i+u'} T_2^{j+v'} T_3^{k+w'} \in \operatorname{im}\left(\varphi(n)\right)^T.$$

If the claim holds, then for each choice of $u+v+w = n-m$,

$$\frac{n!}{m!} f_1^u f_2^v f_3^w T_1^{i+u} T_2^{j+v} T_3^{k+w}$$

differs from

$$(n-m)! \sum_{u'+w'+v'=n-m} \underbrace{\binom{i+u'}{u'}\binom{j+v'}{v'}\binom{k+w'}{w'}}_{C(u',v',w')} f_1^{u'} f_2^{v'} f_3^{w'} T_1^{i+u'} T_2^{j+v'} T_3^{k+w'}$$

by

$$\sum_{\substack{u'+v'+w'=m-n \\ (u',v',w')\neq(u,v,w)}} C(u',v',w')\left(f_1^u f_2^v f_3^w T_1^{i+u} T_2^{j+v} T_3^{k+w} - f_1^{u'} f_2^{v'} f_3^{w'} T_1^{i+u'} T_2^{j+v'} T_3^{k+w'}\right),$$

which is in the image of $\left(\varphi(n)\right)^T$. By 3.22, the result follows.

Let us prove the claim holds when $u' = u+1$, $v' = v-1$, and $w' = w$. We want to show that

$$f_1^u f_2^v f_3^w T_1^{i+u} T_2^{j+v} T_3^{k+w} - f_1^{u+1} f_2^{v-1} f_3^w T_1^{i+u+1} T_2^{j+v-1} T_3^{k+w} \in \operatorname{im}\left(\varphi(n)\right)^T.$$

And indeed, this can be rewritten as

$$f_1^u f_2^{v-1} f_3^w \left(f_2 T_1^{i+u} T_2^{j+v} T_3^{k+w} - f_1 T_1^{i+u+1} T_2^{j+v-1} T_3^{k+w} \right)$$

is in the image of $(\varphi(n))^T$ by Lemma 3.23. The same result follows in the same manner we switch the roles of u, v and w. Now an inductive argument shows the claim holds when $|u-u'|>1$, by simply taking successive sums of differences of this sort. $\square$

We will now apply this result to specific values of n and m.

3.2.2 When is $I^{(3)} \subseteq I^2$?

Recall that

$$(\varphi(3))^T = \begin{pmatrix} a_1 & a_2 & a_3 & 0 & 0 & 0 & b_1 & b_2 & b_3 & 0 & 0 & 0 \\ 0 & a_1 & 0 & a_2 & a_3 & 0 & 0 & b_1 & 0 & b_2 & b_3 & 0 \\ 0 & 0 & a_1 & 0 & a_2 & a_3 & 0 & 0 & b_1 & 0 & b_2 & b_3 \end{pmatrix}.$$

We record here what Theorem 3.25 says in the case of $n=3$ and $m=2$:

Corollary 3.27. *Suppose that* $\operatorname{char} k \neq 2,3$. *The containment* $I^{(3)} \subseteq I^2$ *holds if and only if one (equivalently, all) of the following vectors are in the image of* $(\varphi(3))^T$*:*

$$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}, \quad \begin{bmatrix} f_1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ f_2 \\ 0 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 0 \\ 0 \\ f_3 \end{bmatrix}.$$

Theorem 3.28. *Suppose that* $\operatorname{char} k \neq 2,3$. *If* $a_1 | b_2 a_3$, *then* $I^{(3)} \subseteq I^2$.

Proof. Suppose that $b_2 a_3 = c a_1$.

By Corollary 3.27, it is enough to show that $f_1 T_1$ can be written as an R-linear combination of the columns in $(\varphi(3))^T$. Since $f_1 = a_2 b_3 - a_3 b_2$, then

$$f_1 T_1 = a_2 \left(b_3 T_1 + b_1 T_3 \right) - b_1 \left(a_3 T_2 + a_2 T_3 \right) + a_3 \left(b_2 T_1 + b_1 T_2 \right) - 2c \left(a_1 T_1 \right).$$

This can also be rewritten as

$$\begin{bmatrix} a_2b_3 - a_3b_2 \\ 0 \\ 0 \end{bmatrix} = a_2 \begin{bmatrix} T_1T_3 \\ \hline b_3 \\ 0 \\ b_1 \end{bmatrix} - b_1 \begin{bmatrix} T_2T_3 \\ \hline 0 \\ a_3 \\ a_2 \end{bmatrix} + a_3 \begin{bmatrix} T_1T_2 \\ \hline b_2 \\ b_1 \\ 0 \end{bmatrix} - 2c \begin{bmatrix} T_1^2 \\ \hline a_1 \\ 0 \\ 0 \end{bmatrix}.$$

□

Remark 3.29. The condition that $a_1|b_2a_3$ is not equivalent to $I^{(3)} \subseteq I^2$.

We will now apply this result to space monomial curves such as in Example 1.23.

Example 3.30 (Space Monomial Curves)**.** Let a, b, c be integers and consider the map $\varphi : R \longrightarrow k[t]$ given by $x \mapsto t^a$, $y \mapsto t^b$ and $z \mapsto t^c$. By [Her70, Proposition 3.3], the kernel P of this map is a prime ideal of height 2, which is the ideal of 2×2 minors of the matrix

$$M = \begin{pmatrix} x^{\alpha_3} & y^{\beta_1} & z^{\gamma_2} \\ z^{\gamma_1} & x^{\alpha_2} & y^{\beta_3} \end{pmatrix}$$

for some integers α_2, α_3, β_1, β_3, γ_1 and γ_2. These ideals come in two flavors: either they are complete intersections, meaning that they are two generated, in which case all symbolic and ordinary powers coincide, or they are minimally generated by 3 elements, in which case $P^{(n)} \neq P^n$ for all $n \geqslant 2$, by Theorem 1.25.

Theorem 3.31. *Let P be the defining ideal of a space monomial curve, such as in Example 3.30. Then $P^{(3)} \subseteq P^2$.*

Proof. If $\alpha_3 \leqslant \alpha_2$, then $a_1|b_2$. If $\alpha_3 < \alpha_2$, then row and column operations lead us to the matrix

$$\begin{pmatrix} x^{\alpha_2} & z^{\beta_1} & y^{\beta_3} \\ y^{\beta_1} & x^{\alpha_3} & z^{\beta_2} \end{pmatrix}$$

whose ideal of 2×2 minors is still P, and this matrix verifies the condition $a_1|b_2$. Corollary 3.27 completes the proof. □

Remark 3.32. Question 1.43 has an affirmative answer for these primes.

3.2.3 When is $I^{(4)} \subseteq I^3$?

Since the containment $I^{(4)} \subseteq I^2$ always holds by Theorem 1.36, the next interesting containment is $I^{(4)} \subseteq I^3$. The matrix for $(\varphi(4))^T$ is the following 6×20 matrix:

$$\begin{pmatrix}
a_1 & a_2 & a_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b_1 & b_2 & b_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & a_1 & 0 & a_2 & a_3 & 0 & 0 & 0 & 0 & 0 & 0 & b_1 & 0 & b_2 & b_3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & a_1 & 0 & a_2 & a_3 & 0 & 0 & 0 & 0 & 0 & 0 & b_1 & 0 & b_2 & b_3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & a_1 & 0 & 0 & a_2 & a_3 & 0 & 0 & 0 & 0 & 0 & b_1 & 0 & 0 & b_2 & b_3 & 0 & 0 \\
0 & 0 & 0 & 0 & a_1 & 0 & 0 & a_2 & a_3 & 0 & 0 & 0 & 0 & 0 & b_1 & 0 & 0 & b_2 & b_3 & 0 \\
0 & 0 & 0 & 0 & 0 & a_1 & 0 & 0 & a_2 & a_3 & 0 & 0 & 0 & 0 & 0 & b_1 & 0 & 0 & b_2 & b_3
\end{pmatrix}.$$

In other words, the image of $(\varphi(4))^T$ is the R-linear span of

$$\left\{ \begin{array}{c} a_1 T_1^{d-1} T_2^e T_3^f + a_2 T_1^d T_2^{e-1} T_3^f + a_3 T_1^d T_2^e T_3^{f-1}, \\ b_1 T_1^{d-1} T_2^e T_3^f + b_2 T_1^d T_2^{e-1} T_3^f + b_3 T_1^d T_2^e T_3^{f-1} \end{array} \right\}_{d+e+f=3}.$$

The characteristic condition we need now is that $\operatorname{char} k$ does not divide $\frac{4!}{3!} = 4$. In particular, 4 is invertible in k. We record the statement of Theorem 3.25 in this specific case below.

Corollary 3.33. *Suppose that* $\operatorname{char} k \neq 2$. *The containment* $I^{(4)} \subseteq I^3$ *if and only if all three of the conditions following hold:*

- *One (equivalently, all) of the following vectors are in the image of* $(\varphi(4))^T$*:*

$$\begin{bmatrix} 2f_1 \\ f_2 \\ f_3 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ f_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ f_3 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} f_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

  Note that this condition corresponds to the image of T_1 *under* $(D_{4,3})^T$*.*

- *One (equivalently, all) of the following vectors are in the image of* $(\varphi(4))^T$*:*

$$\begin{bmatrix}0\\f_1\\0\\2f_2\\f_3\\0\end{bmatrix},\quad \begin{bmatrix}0\\0\\0\\0\\f_3\\0\end{bmatrix},\quad \begin{bmatrix}0\\f_1\\0\\0\\0\\0\end{bmatrix},\quad \text{and}\quad \begin{bmatrix}0\\0\\0\\f_2\\0\\0\end{bmatrix}.$$

Note that this condition corresponds to the image of T_2 *under* $(D_{4,3})^T$*.*

- *One (equivalently, all) of the following vectors are in the image of* $(\varphi(4))^T$*:*

$$\begin{bmatrix}0\\0\\f_1\\0\\f_2\\2f_3\end{bmatrix},\quad \begin{bmatrix}0\\0\\f_1\\0\\0\\0\end{bmatrix},\quad \begin{bmatrix}0\\0\\0\\0\\0\\f_3\end{bmatrix},\quad \text{and}\quad \begin{bmatrix}0\\0\\0\\0\\0\\f_3\end{bmatrix}.$$

Note that this condition corresponds to the image of T_3 *under* $(D_{4,3})^T$*.*

We will now find some sufficient conditions for $I^{(4)} \subseteq I^3$.

Theorem 3.34. *Suppose that* $\operatorname{char} k \neq 2$*. If* $a_1|b_2|a_1^2$*,* $a_2|b_3$*, and* $b_1|a_3$*, then* $I^{(4)} \subseteq I^3$*.*

Proof. More generally, $I^{(4)} \subseteq I^3$ holds as long as

$$a_2|a_1b_3,\ a_2b_2|a_1^2b_3,\ a_1|a_3b_2,\ b_1|a_3b_3,\ a_1b_1|a_2a_3b_2,\ a_1b_1|a_3b_2b_3.$$

We will show that $f_1T_1^2$, $f_2T_2^2$, and $f_3T_3^2$ are all in the image of $(\varphi(4))^T$, which by Theorem 3.33 is enough to prove the claim.

The assumptions imply that $a_3b_2T_1^2$ is a multiple of $a_1T_1^2 \in \operatorname{im}(\varphi(4))^T$. Therefore, in order to prove that $f_1T_1^2$ is in the image of $(\varphi(4))^T$, it is enough to show that $a_2b_3T_1^2 \in \operatorname{im}(\varphi(4))^T$, which follows from

$$\begin{bmatrix}a_2b_3\\0\\0\\0\\0\\0\end{bmatrix} = b_3\begin{bmatrix}T_1^2T_2\\ \hline a_2\\a_1\\0\\0\\0\\0\end{bmatrix} - a_1\frac{b_3}{a_2}\begin{bmatrix}T_1T_2^2\\ \hline 0\\a_2\\0\\a_1\\0\\0\end{bmatrix} + \frac{a_1^2}{b_2}\frac{b_3}{a_2}\begin{bmatrix}T_2^3\\ \hline 0\\0\\0\\b_2\\0\\0\end{bmatrix}.$$

Similarly, the assumptions imply that $a_1b_3T_2^2$ is a multiple of $a_2T_2^2 \in \operatorname{im}(\varphi(4))^T$. Therefore, in order to prove that $f_2T_2^2$ is in the image of $(\varphi(4))^T$, it is enough to show that $a_3b_1T_2^2 \in \operatorname{im}(\varphi(4))^T$, which follows from

$$\begin{bmatrix} 0\\0\\0\\a_3b_1\\0\\0 \end{bmatrix} = a_3 \begin{bmatrix} T_1T_2^2 \\ \hline 0\\b_2\\0\\b_1\\0\\0 \end{bmatrix} - a_3\frac{b_2}{a_1} \begin{bmatrix} T_1^2T_2 \\ \hline a_2\\a_1\\0\\0\\0\\0 \end{bmatrix} + a_2\frac{a_3}{b_1}\frac{b_2}{a_1} \begin{bmatrix} T_1^3 \\ \hline b_1\\0\\0\\0\\0\\0 \end{bmatrix}.$$

Finally, if $b_1|a_3$, $f_3T_3^2 \in \operatorname{im}(\varphi(4))^T$ can be obtained by

$$\begin{bmatrix} 0\\0\\0\\0\\0\\a_1b_2-a_2b_1 \end{bmatrix} = a_1 \begin{bmatrix} T_2T_3^2 \\ \hline 0\\0\\0\\0\\b_3\\b_2 \end{bmatrix} - a_2 \begin{bmatrix} T_1T_3^2 \\ \hline 0\\0\\b_3\\0\\0\\b_1 \end{bmatrix} + 2a_1\frac{b_3}{a_2} \begin{bmatrix} T_2^2T_3 \\ \hline 0\\0\\0\\a_3\\a_2\\0 \end{bmatrix}$$

$$-b_3 \begin{bmatrix} T_1T_2T_3 \\ \hline 0\\a_3\\a_2\\0\\a_1\\0 \end{bmatrix} + 2a_1\frac{a_3}{b_1}\frac{b_3}{a_2} \begin{bmatrix} T_1t_2^2 \\ \hline 0\\b_2\\0\\b_1\\0\\0 \end{bmatrix} - b_3\frac{a_3}{b_1} \begin{bmatrix} T_1^2T_2 \\ \hline b_2\\b_1\\0\\0\\0\\0 \end{bmatrix} + b_3\frac{a_3}{b_1}\frac{b_2}{a_1} \begin{bmatrix} T_1^3 \\ \hline a_1\\0\\0\\0\\0\\0 \end{bmatrix}.$$

□

Theorem 3.35. *Suppose that* $\operatorname{char} k \neq 2$. *If* $a_1|b_2|a_1^2$, $a_2|b_3$, *and* $a_3|b_1$, *then* $I^{(4)} \subseteq I^3$.

Proof. First, note that part of the proof of Theorem 3.34 still applies; in particular, we can obtain $f_1T_1^2$ and $f_2T_2^2$ in the same way. To prove the statement, it is enough to show that $f_3T_3^2 \in \operatorname{im}(\varphi(4))^T$. Recall that $f_3 = a_1b_2 - a_2b_1$.

$$\begin{bmatrix} 0\\0\\0\\0\\0\\2f_3 \end{bmatrix} = 2b_2 \begin{bmatrix} T_1T_3^2 \\ \hline 0\\0\\a_3\\0\\0\\a_1 \end{bmatrix} - 3a_2\frac{b_1}{a_3} \begin{bmatrix} T_3^3 \\ \hline 0\\0\\0\\0\\0\\a_3 \end{bmatrix} - 2a_3 \begin{bmatrix} T_1T_2T_3 \\ \hline 0\\b_3\\b_2\\0\\b_1\\0 \end{bmatrix} + a_3\frac{b_3}{a_2} \begin{bmatrix} T_1T_2^2 \\ \hline 0\\a_2\\0\\a_1\\0\\0 \end{bmatrix}$$

$$+b_3 \begin{bmatrix} T_1T_2T_3 \\ \hline 0 \\ a_3 \\ a_2 \\ 0 \\ a_1 \\ 0 \end{bmatrix} - a_1 \frac{b_3}{a_2} \begin{bmatrix} T_2^2T_3 \\ \hline 0 \\ 0 \\ 0 \\ a_3 \\ a_2 \\ 0 \end{bmatrix} + 2b_1 \begin{bmatrix} T_2T_3^2 \\ \hline 0 \\ 0 \\ 0 \\ 0 \\ a_3 \\ a_2 \end{bmatrix} - a_2 \begin{bmatrix} T_1T_3^2 \\ \hline 0 \\ 0 \\ b_3 \\ 0 \\ 0 \\ b_1 \end{bmatrix}.$$

Since 2 is invertible, we are done. □

Corollary 3.36. *Suppose that k is a field with $\operatorname{char} k \neq 2$, and let $P \subseteq k[x, y, z]$ be the ideal generated by the maximal minors of*

$$\begin{pmatrix} x^{\alpha_3} & y^{\beta_1} & z^{\gamma_2} \\ z^{\gamma_1} & x^{\alpha_2} & y^{\beta_3} \end{pmatrix}$$

If $\alpha_3 \leqslant \alpha_2 \leqslant 2\alpha_3$ and $\beta_1 \leqslant \beta_3$, then $P^{(4)} \subseteq P^3$.

Remark 3.37. Note that these conditions can be stated in different ways by switching the roles of the α_i, β_i and γ_i. However, as any matrix obtained by permuting the rows and columns of our given matrix still defines the same ideal, it is enough to state these conditions as above.

Proof. The condition $a_1|b_2|a_1^2$, follows from $\alpha_3 \leqslant \alpha_2 \leqslant 2\alpha_3$, while $\beta_1 \leqslant \beta_3$ implies that $a_2|b_3$. We always have $a_3|b_1$ or $b_1|a_3$. Theorems 3.34 and 3.35 complete the proof. □

Remark 3.38. The conditions in Corollary 3.36 are not necessary. In fact, the matrix

$$M = \begin{pmatrix} x & y & z^3 \\ z & x^3 & y^3 \end{pmatrix}$$

does not verify the conditions of Theorem 3.36, and yet $I^{(4)} \subseteq I^3$ does hold for $k = \mathbb{Q}, \mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/5\mathbb{Z}$, and $\mathbb{Z}/101\mathbb{Z}$, according to Macaulay2 [GS] computations.

Theorem 3.39. *Let k be a field of characteristic not 2, and let P be the defining ideal in $k[x, y, z]$ of the curve $k[t^a, t^b, t^c]$, where $a < b < c$. If $a = 3$ or $a = 4$, then $P^{(4)} \subseteq P^3$.*

Proof. If P is a complete intersection, the theorem is trivially true, since $P^{(n)} = P^n$ for all $n \geqslant 1$, by Theorem 1.20, so suppose that P is not generated by a regular sequence.

In the proof of [Hun87, Theorem 3.14], Huneke shows that if $a = 4$ and P is not a complete intersection, then P is minimally generated by the maximal minors of

$$\begin{pmatrix} y & z & x^p \\ x^q & y^2 & z \end{pmatrix},$$

where $b = p + 2q$ and $c = 2q + 3p$. By Corollary 3.36, $P^{(4)} \subseteq P^3$.

A similar technique as the one used in the proof of [Hun87, Theorem 3.14] can be used to determine the form of P for different values of a. First, we note that (P, x) must be of the form $(P, x) = (y^{\beta_1} z^{\gamma_1}, y^{\beta_2}, z^{\gamma_3})$, and that $R/(P, x)$ has multiplicity a. Following the classification in [Poo08], we can then determine all possibilities for (P, x) when $a = 3$, and it turns out that the only possibility is $(P, x) = (x^2, y^2, xy)$. In particular, $P = I_2(M)$, where

$$M = \begin{pmatrix} x^{\alpha_3} & y & z \\ z & x^{\alpha_2} & y \end{pmatrix}.$$

Now any matrix of this form verifies the conditions in Corollary 3.36, so $P^{(4)} \subseteq P^3$. □

Remark 3.40. The same technique might be used to determine the Hilbert-Burch matrix of P for other values of a. When $a = 5$, (P, x) must be (y^2, yz, z^4), (y^3, yz, z^3), or (y^2, yz^2, z^3), via the classification in [Poo08].

- If $(P, x) = (y^2, yz, z^4)$, P must be $I_2(M)$, where

  $$M = \begin{pmatrix} x^{\alpha_3} & y & z^3 \\ z & x^{\alpha_2} & y \end{pmatrix}.$$

  This matrix verifies the conditions in Corollary 3.36 whenever $\alpha_3 \leqslant \alpha_2 \leqslant 2\alpha_3$, or $\alpha_2 \leqslant \alpha_3$.

  For example, taking $a = 5$, $b = 7$ and $c = 13$, we get $\alpha_3 = 1$ and $\alpha_2 = 3$,

  $$M = \begin{pmatrix} x^3 & y & z^3 \\ z & x & y \end{pmatrix}.$$

Note that $\alpha_2 \leqslant \alpha_3$ is verified, while the other two possible conditions are not. Thus $P^{(4)} \subseteq P^3$.

For example, taking $a = 5$, $b = 23$ and $c = 12$, we get $\alpha_3 = 5$ and $\alpha_2 = 1$,

$$M = \begin{pmatrix} x^5 & y & z^3 \\ z & x^2 & y \end{pmatrix}.$$

Note that $\alpha_2 \leqslant \alpha_3$ is verified, while the other two possible conditions are not. Thus $P^{(4)} \subseteq P^3$.

- If $(P, x) = (y^3, yz, z^3)$, P must be $I_2(M)$, where

$$M = \begin{pmatrix} x^{\alpha_3} & y & z^2 \\ z & x^{\alpha_2} & y^2 \end{pmatrix}.$$

  This verifies the conditions in Corollary 3.36 as long as $\alpha_3 \leqslant \alpha_2 \leqslant 2\alpha_3$ or $\alpha_2 \leqslant \alpha_3 \leqslant 2\alpha_2$.

- If $(P, x) = (y^2, yz^2, z^3)$, P must be $I_2(M)$, where

$$M = \begin{pmatrix} y & z & x^{\alpha_3} \\ x^{\alpha_2} & y & z^2 \end{pmatrix}.$$

  This matrix always verifies the conditions in Corollary 3.36.

In all the following examples, suppose that k is a field of characteristic not 2.

Example 3.41. Consider the matrix

$$\begin{pmatrix} x & z & y \\ y & x^2 & z \end{pmatrix},$$

whose ideal of 2×2 minors defines the space monomial curve $k[t^3, t^4, t^5]$. Corollary 3.36 gives $I^{(4)} \subseteq I^3$.

Example 3.42. Let $n \geqslant 4$ be an integer not divisible by 3. By [GNW94], the ideal of 2×2 minors of the matrix

$$\begin{pmatrix} y & x^n & x^{2n-1} \\ x^n & y^2 & z^{2n-1} \end{pmatrix},$$

define the space monomial curve $k[t^a, t^b, t^c]$ with $a = 7n-3$, $b = (5n-2)n$, and $c = 8n-3$. By Theorem 3.34, $I^{(4)} \subseteq I^3$. These examples are especially interesting because their symbolic Rees algebra, which is a topic we will discuss in Chapter 5, is not noetherian in characteristic 0, as shown in [GNW94].

Example 3.43. Let $m = 3n + q \geqslant 1$ be an integer, where $0 \leqslant q < 3$, $a = m$, $b = m+1$, and $c = m+3$. Consider the defining ideal P of $k[t^a, t^b, t^c]$ in $k[x, y, z]$. By [GNS91a, Corollary 4.6], P has one of three types:

(q = 0) P is generated by a regular sequence, and $P^{(k)} = P^k$ for all k.

(q = 1) P is generated by the maximal minors of the matrix

$$\begin{pmatrix} y & z & x^n \\ x^2 & y^2 & z^n \end{pmatrix},$$

which verifies the conditions in Theorem 3.34 as long as $n \geqslant 2$. When $n = 1$, the matrix is

$$\begin{pmatrix} y & z & x \\ x^2 & y^2 & z \end{pmatrix},$$

which verifies the conditions in Theorem 3.35. For all values of $n \geqslant 1$, we have $P^{(4)} \subseteq P^3$.

(q = 2) P is generated by the maximal minors of the matrix

$$\begin{pmatrix} y & x^2 & z^n \\ z & y^2 & x^{n+1} \end{pmatrix},$$

which verifies the conditions in Theorem 3.34 for all $n \geqslant 1$, so that $P^{(4)} \subseteq P^3$.

Example 3.44. The space monomial curve $k[t^9, t^{11}, t^{14}]$ is defined by the ideal I of maximal minors of the matrix

$$\begin{pmatrix} z & y^3 & x^3 \\ x & z^2 & y^2 \end{pmatrix}.$$

Notice that this fails the conditions of both Theorem 3.34 and Theorem 3.35. Macaulay2 [GS] computations show that $I^{(4)} \not\subseteq I^3$, and in fact, this is the smallest such example, in the sense that if a, b, c are such that $a \leqslant 7$, $b \leqslant 11$, and $c \leqslant 14$, and $(a, b, c) \neq (9, 11, 14)$ then the defining ideal of the space monomial curve $k[t^a, t^b, t^c]$ verifies $I^{(4)} \subseteq I^3$.

3.2.4 When is $I^{(5)} \subseteq I^3$?

We now find a sufficient condition for $I^{(5)} \subseteq I^3$, and show that it implies that $I^{(5)} \subseteq I^3$ for all prime ideals defining $k[t^a, t^b, t^c]$.

Theorem 3.45. *Suppose that* $\operatorname{char} k \neq 2, 3, 5$. *If* $a_1|b_2$ *and* $a_2|b_3$, *then* $I^{(5)} \subseteq I^3$.

Proof. To apply Theorem 3.25, we need that $\operatorname{char} k$ does not divide $\frac{5!}{3!} = 20$, which excludes $\operatorname{char} k = 2$ and $\operatorname{char} k = 5$. Theorem 3.25 guarantees that $I^{(5)} \subseteq I^3$ as long as $f_1^2T_1^3$, $f_2^2T_2^3$ and $f_3^2T_3^3$ are all in the image of $\varphi(5)$. We will show this holds as long 3 is also invertible.

Suppose that $b_2 = a_1c$ and $b_3 = a_2d$.

(i) $f_1^2 = (a_2b_3 - a_3b_2)^2 = a_2^2b_3^2 - 2a_2a_3b_2b_3 + a_3^2b_2^2 = a_2^4d^2 - 2a_1a_2^2a_3cd + a_1^2a_3^2c^2$.

Then $\begin{bmatrix} f_1^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T$ is given by

$$\begin{pmatrix} a_1a_3^2c^2 \\ -2a_2^2a_3cd \end{pmatrix} \begin{bmatrix} T_1^4 \\ \hline a_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + a_2^3d^2 \begin{bmatrix} T_1^3T_2 \\ \hline a_2 \\ a_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} - a_1a_2^2d^2 \begin{bmatrix} T_1^2T_2^2 \\ \hline 0 \\ a_2 \\ 0 \\ a_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + a_1^2a_2d^2 \begin{bmatrix} T_1T_2^3 \\ \hline 0 \\ 0 \\ 0 \\ a_2 \\ 0 \\ 0 \\ a_1 \\ 0 \\ 0 \\ 0 \end{bmatrix} - a_1^3d^2 \begin{bmatrix} T_2^4 \\ \hline 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ a_2 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Now in order to show that $f_2^2T_2^3$ and $f_3^2T_3^3$ are in the image of $\varphi(5)$, we will require that 3 be invertible, and show that $3f_2^2T_2^3$ and $12f_3^2T_3^3$ are in the image of $\varphi(5)$.

Note that

(ii) $f_2^2 = (a_3b_1 - a_1b_3)^2 = a_3^2b_1^2 - 2a_1a_3b_1b_3 + a_1^2b_3^2 = a_1^2a_3^2c^2 - 2a_1^2a_2a_3cd + a_1^2a_2^2d^2$,

and

(iii) $f_3^2 = (a_1b_2 - a_2b_1)^2 = a_1^2b_2^2 - 2a_1a_2b_1b_2 + a_2^2b_1^2 = a_1^4c^2 - 2a_1^2a_2b_1c + a_2^2b_1^2.$

Then:

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 3f_2^2 \\ 0 \\ 0 \\ 0 \end{bmatrix} = -a_3^2b_1c \begin{bmatrix} T_1^2T_2^2 \\ \hline 0 \\ a_2 \\ 0 \\ a_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + a_2a_3b_1c \begin{bmatrix} T_1^2T_2T_3 \\ \hline 0 \\ a_3 \\ a_2 \\ 0 \\ a_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} - a_2^2b_1c \begin{bmatrix} T_1^2T_3 \\ \hline 0 \\ 0 \\ a_3 \\ 0 \\ 0 \\ a_1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + 2a_2a_3b_1d \begin{bmatrix} T_1T_2^2 \\ \hline 0 \\ 0 \\ 0 \\ a_2 \\ 0 \\ 0 \\ a_1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-a_2^2b_1d \begin{bmatrix} T_1T_2^2T_3 \\ \hline 0 \\ 0 \\ 0 \\ a_3 \\ a_2 \\ 0 \\ 0 \\ a_1 \\ 0 \\ 0 \end{bmatrix} + \begin{pmatrix} 3a_1^2a_2d^2 \\ -9a_1a_3b_1d \end{pmatrix} \begin{bmatrix} T_2^3 \\ \hline 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ a_2 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{pmatrix} a_1a_2b_1d \\ +2a_3b_1^2 \end{pmatrix} \begin{bmatrix} T_2^3T_3 \\ \hline 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ a_3 \\ a_2 \\ 0 \\ 0 \end{bmatrix} - a_2b_1^2 \begin{bmatrix} T_2^2T_3^2 \\ \hline 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ a_3 \\ a_2 \\ 0 \end{bmatrix}$$

$$+a_3^2b_1 \begin{bmatrix} T_1T_2^3 \\ \hline 0 \\ 0 \\ 0 \\ a_1c \\ 0 \\ 0 \\ b_1 \\ 0 \\ 0 \\ 0 \end{bmatrix} - a_2a_3b_1 \begin{bmatrix} T_1T_2^2T_3 \\ \hline 0 \\ 0 \\ 0 \\ a_2d \\ a_1c \\ 0 \\ 0 \\ b_1 \\ 0 \\ 0 \end{bmatrix} + a_2^2b_1 \begin{bmatrix} T_2^2T_3^2 \\ \hline 0 \\ 0 \\ 0 \\ 0 \\ a_2d \\ a_1c \\ 0 \\ 0 \\ b_1 \\ 0 \end{bmatrix}$$

$$
\begin{bmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\0\\12f_3^2 \end{bmatrix}
= 18a_2a_3^2cd \begin{bmatrix} T_1^3T_2 \\ \hline a_2\\a_1\\0\\0\\0\\0\\0\\0\\0\\0 \end{bmatrix}
+ 9a_2^2a_3cd \begin{bmatrix} T_1^3T_3 \\ \hline a_3\\0\\a_1\\0\\0\\0\\0\\0\\0\\0 \end{bmatrix}
+ \begin{pmatrix} -8a_1a_3^2cd \\ 6a_2^2a_3d^2 \end{pmatrix} \begin{bmatrix} T_1^2T_2^2 \\ \hline 0\\a_2\\0\\a_1\\0\\0\\0\\0\\0\\0 \end{bmatrix}
$$

$$
+ \begin{pmatrix} 8a_1a_2a_3cd \\ +3a_2^3d^2 \end{pmatrix} \begin{bmatrix} T_1^2T_2T_3 \\ \hline 0\\a_3\\a_2\\0\\a_1\\0\\0\\0\\0\\0 \end{bmatrix}
+ \begin{pmatrix} -8a_1a_2^2cd \\ +27a_2a_3b_1c \end{pmatrix} \begin{bmatrix} T_1^2T_3^2 \\ \hline 0\\0\\a_3\\0\\0\\a_1\\0\\0\\0\\0 \end{bmatrix}
+ 13a_1a_2a_3d^2 \begin{bmatrix} T_1T_2^3 \\ \hline 0\\0\\0\\a_2\\0\\0\\a_1\\0\\0\\0 \end{bmatrix}
$$

$$
\begin{pmatrix} -11a_1a_2^2d^2 \\ +18a_2a_3b_1d \end{pmatrix} \begin{bmatrix} T_1T_2^2T_3 \\ \hline 0\\0\\0\\a_3\\a_2\\0\\0\\a_1\\0\\0 \end{bmatrix}
+18a_2^2b_1d \begin{bmatrix} T_1T_2T_3^2 \\ \hline 0\\0\\0\\0\\a_3\\a_2\\0\\0\\a_1\\0 \end{bmatrix}
-36a_1^2a_3d^2 \begin{bmatrix} T_2^4 \\ \hline 0\\0\\0\\0\\0\\0\\a_2\\0\\0\\0 \end{bmatrix}
+ \begin{pmatrix} 11a_1^2a_2d^2 \\ -8a_1a_3b_1d \end{pmatrix} \begin{bmatrix} T_2^3T_3 \\ \hline 0\\0\\0\\0\\0\\0\\a_3\\a_2\\0\\0 \end{bmatrix}
$$

$$
+ \begin{pmatrix} -12a_1^3cd \\ -2a_1a_2b_1d \end{pmatrix} \begin{bmatrix} T_2^2T_3^2 \\ \hline 0\\0\\0\\0\\0\\0\\0\\a_3\\a_2\\0 \end{bmatrix}
+ 27a_2b_1^2 \begin{bmatrix} T_2^2T_3^2 \\ \hline 0\\0\\0\\0\\0\\0\\0\\0\\a_3\\a_2 \end{bmatrix}
- 27a_2a_3^2c \begin{bmatrix} T_1^3T_3 \\ \hline a_2d\\0\\b_1\\0\\0\\0\\0\\0\\0\\0 \end{bmatrix}
- 18a_2a_3^2d \begin{bmatrix} T_1^2T_2^2 \\ \hline 0\\a_1c\\0\\b_1\\0\\0\\0\\0\\0\\0 \end{bmatrix}
$$

$$-9a_2^2a_3d\begin{bmatrix} T_1^2T_2T_3 \\ \hline 0 \\ a_2d \\ a_1c \\ 0 \\ b_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} - 3a_2^3d\begin{bmatrix} T_1^2T_3^2 \\ \hline 0 \\ 0 \\ a_2d \\ 0 \\ 0 \\ b_1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + 8a_1a_3^2d\begin{bmatrix} T_1T_2^3 \\ \hline 0 \\ 0 \\ 0 \\ a_1c \\ 0 \\ 0 \\ b_1 \\ 0 \\ 0 \\ 0 \end{bmatrix} - 8a_1a_2a_3d\begin{bmatrix} T_1T_2^2T_3 \\ \hline 0 \\ 0 \\ 0 \\ a_2d \\ a_1c \\ 0 \\ 0 \\ b_1 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{pmatrix} 8a_1a_2^2d \\ -27a_2a_3b_1 \end{pmatrix}\begin{bmatrix} T_1T_2T_3^2 \\ \hline 0 \\ 0 \\ 0 \\ 0 \\ a_2d \\ a_1c \\ 0 \\ 0 \\ b_1 \\ 0 \end{bmatrix} -15a_2^2b_1\begin{bmatrix} T_1T_3^3 \\ \hline 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ a_2d \\ 0 \\ 0 \\ 0 \\ b_1 \end{bmatrix} +12a_1^2a_3d\begin{bmatrix} T_2^3T_3 \\ \hline 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ a_2d \\ a_1c \\ 0 \\ 0 \end{bmatrix} + \begin{pmatrix} 12a_1^3c \\ -24a_1a_2b_1 \end{pmatrix}\begin{bmatrix} T_2T_3^3 \\ \hline 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ a_2d \\ a_1c \end{bmatrix}$$

□

Theorem 3.46. *Let k be a field of characteristic not 2, 3 or 5, and consider positive integers a, b, c. Suppose that P is the prime ideal defining $k[t^a, t^b, t^c]$. Then $P^{(5)} \subseteq P^3$.*

Proof. Recall from Example 3.30 that P is the the ideal of 2×2 minors of a matrix of the form

$$M = \begin{pmatrix} x^{\alpha_3} & y^{\beta_1} & z^{\gamma_2} \\ z^{\gamma_1} & x^{\alpha_2} & y^{\beta_3} \end{pmatrix}$$

for some integers $\alpha_2, \alpha_3, \beta_1, \beta_3, \gamma_1, \gamma_2$. If $\alpha_3 \leqslant \alpha_2$ and $\beta_1 \leqslant \beta_3$, we are done, by Theorem 3.45. If not, we can always replace M by a matrix verifying $a_1|b_2$ and $a_2|b_3$. Notice first that if M' is a matrix obtained from M by permuting rows or columns, then $I_2(M') = I_2(M) = P$. Moreover, for each of the pairs x^{α_2} and x^{α_3}, y^{β_1} and y^{β_3}, and z^{γ_1} and z^{γ_2}, one of the elements always divides the other. Cyclic permutations of the columns of M always yield a matrix where a_1 and b_2 are powers of the same variable, and a_2 and b_3 are both powers of another variable. If two of the entries in the top row of M divide the corresponding entries in the bottom row, we are done. If not, then two of the entries in the bottom row must divide the corresponding entries in the bottom row. By switching the rows and then permuting the first two columns, we obtain

$$M' = \begin{pmatrix} x^{\alpha_2} & z^{\gamma_1} & y^{\beta_3} \\ y^{\beta_1} & x^{\alpha_3} & z^{\gamma_2} \end{pmatrix}.$$

Now two entries in the top row must divide the corresponding entries in the bottom row, so we are done. □

Chapter 4

A stable version of Harbourne's Conjecture

Even though Harbourne's Conjecture does not hold for all radical ideals, there is evidence that it might always hold *eventually*, meaning that for an ideal I of big height h, $I^{(hn-h+1)} \subseteq I^n$ for all $n \gg 0$, as in Conjecture 1.49. In this chapter, we study the asymptotic behavior of symbolic powers, and suggest two different approaches to this conjecture. In Subsection 4.1.1, we prove that if the containment $I^{(hm-h)} \subseteq I^m$ holds for some m, then $I^{(hn-h)} \subseteq I^n$ holds for all $n \gg 0$. In Subsection 4.1.2, we present various classes of ideals that verify this stronger eventual containment, introduce Bocci and Harbourne's resurgence of ideals, and discuss how a bound on the resurgence $\rho(I)$ of I would prove the stable version of Harbourne's Conjecture. In fact, while $\rho(I) \leqslant h$ for all ideals I of big height h, $\rho(I) < h$ would imply that given any integer $C > 0$, $I^{(hn-C)} \subseteq I^n$ for all $n \gg 0$.

4.1 The containment problem for large values

4.1.1 A tighter containment

We recall Conjecture 1.49, for which there is no known counterexample:

Conjecture 4.1. *Let R be a regular ring and I a radical ideal in R with big height* *h. There exists m such that the following containment holds for all $n \geqslant m$:*

$$I^{(hn-h+1)} \subseteq I^n.$$

As we will see in Example 4.18, this conjecture holds for the Fermat configurations of points and other well-known counterexamples to $I^{(3)} \subseteq I^2$. This conjecture remains open in general, and there is no known counterexample. One might try to attack this conjecture assuming I has nice properties. In Subsection 5.2.2 we do just that, by discussing a characteristic p strategy for the case when the symbolic Rees algebra of I is noetherian.

As a first approach to Conjecture 1.49, we could try to prove the following:

Conjecture 4.2. *Let R be a regular ring and I a radical ideal in R with big height h. Suppose that $I^{(hm-h+1)} \subseteq I^m$ holds for some value of m. Then for all $n \gg 0$, we have*

$$I^{(hn-h+1)} \subseteq I^n.$$

We might also wonder if $n \gg 0$ can be replaced by $n \geqslant m$. If that were true, then in particular the containment $I^{(3)} \subseteq I^2$ for an ideal I of big height 2 would imply Harbourne's Conjecture 1.44. Such a result would also imply Harbourne's Conjecture for space monomial curves, since we have already shown that $I^{(3)} \subseteq I^2$ holds for such ideals (cf. Theorem 3.31).

Remark 4.3. In prime characteristic p, Theorem 2.10 guarantees that $I^{(hn-h+1)} \subseteq I^n$ always holds for some value of n – in particular, this holds when n is a power of p. Therefore, a proof of Conjecture 4.2 would imply that Conjecture 1.49 holds in positive characteristic.

While we do not know whether Conjecture 4.2 holds, there is a statement of that flavor that we can indeed prove. For that, we will need the following generalization of Theorem 2.12:

Theorem 4.4 (Mark Johnson, 2014, Theorem 4.3 (1) in [Joh14])**.** *Let R be a regular ring containing a field, and let I be a radical ideal in R with big height h. For all integers $n \geqslant 1$ and $a_1, \ldots, a_n \geqslant 0$, the following containment holds:*

$$I^{(hn+a_1+\cdots+a_n)} \subseteq I^{(a_1+1)} \cdots I^{(a_n+1)}.$$

We record a simple corollary of this theorem:

Corollary 4.5. *Let R be a regular ring containing a field, and let I be a radical ideal in R with big height h. For all integers $b, c, d \geqslant 1$,*

$$I^{(h(b+c)+cd)} \subseteq \left(I^{(d+1)}\right)^c I^b.$$

Proof. The statement follows by Theorem 4.4 with $n = b + c$, $a_1 = \cdots = a_c = d$ and $a_{c+1} = \cdots = a_{c+b} = 0$. □

Using this corollary, we can now prove the main result of this chapter:

Theorem 4.6. *Let R be a regular ring containing a field, and let I be a radical ideal in R with big height h. If*

$$I^{(hm-h)} \subseteq I^m$$

for some $m \geqslant 2$, then

$$I^{(hk-h)} \subseteq I^k$$

for all $k \gg 0$. Specifically, $I^{(hk-h)} \subseteq I^k$ holds for all $k \geqslant hm$.

Proof. If $k \geqslant hm$, write $k = hm + t$, for some $t \geqslant 0$. Apply Corollary 4.5 with $d = hm - h - 1$, $b = t$ and $c = h$. Since

$$h(b + c) + cd = h(t + h) + h(hm - h - 1) = h(hm + t - 1) = hk - h.$$

Corollary 4.5 gives

$$I^{(hk-h)} = I^{(h(b+c)+cd)} \subseteq \left(I^{(mh-h)}\right)^h I^t.$$

By assumption, $I^{(mh-h)} \subseteq I^m$. Then

$$I^{(hk-h)} \subseteq \left(I^{(mh-h)}\right)^h I^t \subseteq (I^m)^h I^t = I^{mh+t} = I^k.$$

□

Corollary 4.7. *Let R be a regular ring containing a field, and let I be a radical ideal in R with big height h. If*

$$I^{(hm-h)} \subseteq I^m$$

for some $m \geqslant 2$, then Conjecture 1.49 holds, that is, $I^{(hn-h+1)} \subseteq I^n$ holds for all $n \gg 0$.

Remark 4.8. We note that the assumption that R is a regular ring containing a field may be relaxed. Indeed, all we used the in the proof was the fact that

$$I^{(hn+a_1+\cdots+a_n)} \subseteq I^{(a_1+1)} \cdots I^{(a_n+1)}$$

for all $a_1, \ldots, a_n \geqslant 0$ and $n \geqslant 1$.

4.1.2 Evidence for a stable version of Harbourne's Conjecture

It turns out that many classes of ideals verify containments such as in Theorem 4.6. In fact, there are no known examples of ideals I that fail to verify $I^{(hn-h)} \subseteq I^n$ for some n.

Example 4.9. Suppose that I has big height 2 and $I^{(4)} \subseteq I^3$. By Theorem 4.6, this implies that $I^{(2n-2)} \subseteq I^n$ for all $n \geqslant 2 \cdot 3 = 6$.

In Theorem 3.39, we showed that the defining ideal I of the space monomial curve $k[t^a, t^b, t^c]$ verifies $I^{(4)} \subseteq I^3$ whenever $a = 3$ or $a = 4$ and $a < b < c$. Theorem 4.2 implies that in fact $I^{(2n-2)} \subseteq I^n$ for all $n \geqslant 6$ for all such ideals.

Example 4.10. In Example 3.42, we showed that the defining ideal I of the space monomial curve $k[t^a, t^b, t^c]$ given by $a = 7n-3$, $b = (5n-2)n$, and $c = 8n-3$ verifies $I^{(4)} \subseteq I^3$ as long as k is a field of $\operatorname{char} k \neq 2$. By Theorem 4.6, we now have $I^{(2n-2)} \subseteq I^n$ for all $n \geqslant 6$. This example is especially interesting because the Symbolic Rees Algebra of I is not noetherian in characteristic 0, as shown in [GNW94, Corollary 1.2].

Example 4.11. In Example 2.32, we showed that the monomial ideal I of pure height 2 in v variables given by

$$I = \bigcap_{i \neq j} (x_i, x_j) = \left(x_1 \cdots \hat{x}_i \cdots x_v \;\middle|\; 1 \leqslant i \leqslant v\right)$$

is such that $I^{(2n-2)} \nsubseteq I^n$ for all $n < v$. However, we claim that $I^{(2v-2)} \subseteq I^v$.

First, recall that $I^{(2v-2)}$ is generated by all the monomials $x_1^{a_1} \ldots x_v^{a_v}$ such that $a_i + a_j = 2v - 2$ for all $i \neq j$. It is enough to show that all such monomials are in I^v. One of those elements is $(x_1 \cdots x_v)^{v-1} \in I^{(2v-2)}$, which we can rewrite as

$$(x_1 \cdots x_v)^{v-1} = \prod_{i=1}^{v} (x_1 \cdots \hat{x}_i \cdots x_v)\,.$$

This is in fact a product of v elements in I. Given a monomial $x_1^{a_1} \cdots x_v^{a_v} \in I^{(2v-2)}$, we may now assume that $a_i < v-1$ for some i. Fix such i. For all $j \neq i$, we now have $a_j \geqslant 2v - 2 - a_i > v - 1$. Therefore, our given monomial is a multiple of $(x_1 \cdots \hat{x}_i \cdots x_v)^v$, which is in I^v.

Now the conditions of Theorem 4.6 apply, and $I^{(2n-2)} \subseteq I^n$ for all $n \gg 0$. In particular, $I^{(2n-2)} \subseteq I^n$ for all $n \geqslant 2v$.

Example 4.12 (Moh's curve)**.** It it not known whether the prime ideal P defining $k[t^6+t^{31}, t^8, t^{10}]$ in $k[x,y,z]$ is set-theoretically a complete intersection — that is, generated by 2 elements up to radical. This is the smallest example (P_3, in his notation) studied by Moh in [Moh74]. As we will discuss briefly in Chapter 5, determining whether a given ideal is set-theoretically a complete intersection is an old difficult question. See, for example, Lyubeznik's survey on this problem [Lyu89]. However, Macaulay2 [GS] computations show that $P^{(4)} \subseteq P^3$ for $k = \mathbb{Q}$. Theorem 4.2 then implies $I^{(2n-2)} \subseteq I^n$ for all $n \geqslant 6$.

Another way to study the relationship between the values of a and b such that $I^{(a)} \nsubseteq I^b$ is via the resurgence. This is an invariant defined by Bocci and Harbourne in [BH10a]:

Definition 4.13 (Resurgence)**.** *Let I be an ideal with no embedded primes in a regular ring containing a field. The* resurgence *of I is given by*

$$\rho(I) := \sup \left\{ \frac{a}{b} \mid I^{(a)} \nsubseteq I^b \right\}.$$

Remark 4.14. If I has big height h, Theorem 1.36 implies that $\rho(I) \leqslant h$. Moreover, $I^{(a)} \nsubseteq I^b$ whenever $a < b$, and thus $\rho(I) \geqslant 1$. Thus

$$1 \leqslant \rho(I) \leqslant h.$$

If $I^{(n)} = I^n$ for all $n \geqslant 1$, then $\rho(I) = 1$. However, as noted in [BH10b, Question 5.2], it is not known whether $\rho(I) = 1$ implies $I^{(n)} = I^n$ for all $n \geqslant 1$.

Remark 4.15. By definition, if $\frac{a}{b} > \rho(I)$, then $I^{(a)} \subseteq I^b$.

The resurgence can sometimes be bounded by other invariants that do not rely on computing symbolic powers, which is in fact how it has been computed in the cases where it is known. Determining the value of the resurgence can then in turn be used to find that certain containments do hold.

Lemma 4.16. *Let I be an ideal with no embedded primes in a regular ring containing a field. Let h be the big height of I. If $\rho(I) < h$, then for every integer $C > 0$, there exists N such that for all $n \geqslant N$,*

$$I^{(hn-C)} \subseteq I^n.$$

Proof. For all $n > \dfrac{C}{h-\rho(I)}$, we have

$$\frac{hn-C}{n} > \rho(I),$$

so that $I^{(hn-C)} \subseteq I^n$, by Remark 4.15. □

Remark 4.17. In particular, if $\rho(I) < h$, then Harbourne's Conjecture *eventually* holds, and moreover $I^{(hn-h)} \subseteq I^n$ for all $n \gg 0$.

Example 4.18. The Fermat [DHN+15, Theorem 2.1], Klein and Wiman [BDRH+16, Theorem 1.4] configurations all have resurgence $\frac{3}{2}$. Therefore, all of these verify $I^{(2n-2)} \subseteq I^n$ for all $n > \dfrac{2}{2-\frac{3}{2}} = 4$.

In particular, these ideals also verify Conjecture 1.49; in fact, $I^{(2n-1)} \subseteq I^n$ holds for all $n > \dfrac{1}{2-\frac{3}{2}} = 2$.

Question 4.19. Is there a radical ideal I with big height h in a regular ring such that $\rho(I) = h$?

There are no known examples with this property. If the answer is no, then in fact $I^{(hn-C)} \subseteq I^n$ for all n large enough and all radical ideals I, where large enough likely depends on the choice of I, and certainly on the constant C.

Remark 4.20. Notice that an ideal I might have $\rho(I) = h$ and still verify $I^{(hn-h)} \subseteq I^n$ for all n large. For example, if $I^{(hn-h-1)} \nsubseteq I^n$ for infinitely many values of $n \geqslant 1$, then $\rho(I) = h$.

Remark 4.21. In particular, if

$$\frac{hm-h+1}{m} > \rho(I),$$

which can be rewritten as

$$m > \frac{h-1}{h-\rho(I)},$$

then for all $n \geqslant m$,

$$I^{(hn-h+1)} \subseteq I^n.$$

The original version of Harbourne's Conjecture would then hold as long as

$$2 > \frac{h-1}{h-\rho(I)},$$

which is equivalent to

$$\frac{h+1}{2} > \rho(I).$$

When $h = 2$, we note that $\frac{3}{2}$ is the limiting value. As we saw in Example 4.18, the Fermat configurations have resurgence $\frac{3}{2}$, and so do other well-known counterexamples to Harbourne's Conjecture when $h = 2$.

This can also be applied to the case of space monomial curves. We have seen that if the characteristic of k is not 2, 3 or 5, then $P^{(3)} \subseteq P^2$ and $P^{(5)} \subseteq P^3$. Harbourne's Conjecture would follow if

$$4 > \frac{1}{2-\rho(P)},$$

or equivalently, $\rho(P) < \dfrac{7}{4}$.

Chapter 5

Symbolic Rees Algebras

In Chapter 3, we used the Rees algebra of I to study all of the powers of I at once; similarly, we can build a graded object out of the symbolic powers of I, the symbolic Rees algebra. The symbolic Rees algebra of an ideal can be fairly difficult to deal with, in particular because it is not necessarily finitely generated. In this chapter, we focus on ideals whose symbolic Rees algebra is in fact finitely generated, and try to answer the Containment Problem in that context.

In Subsection 5.1.1, we introduce the symbolic Rees algebra of I and recall some well-known results. We then assume that the symbolic Rees algebra of I is generated in degree up to d, and study the Containment Problem for such ideals I. In Subsection 5.1.2, we prove that $I^{(dn-d+1)} \subseteq I^d$ for all n. In Subsection 5.2.1, we discuss some of the questions and conjectures from previous chapters in this setting, and show that Harbourne's Conjecture holds as long as the required containment is verified up to d. Finally, we discuss a characteristic p approach to showing that Harbourne's Conjecture holds eventually when the symbolic Rees algebra of I is noetherian.

5.1 Noetherian symbolic Rees algebras

5.1.1 Preliminaries

When studying symbolic powers of ideals, it is useful to study the following graded object:

Definition 5.1. *Let R be a ring and I an ideal in R. The* symbolic Rees algebra *of I is the graded algebra*

$$\mathcal{R}_s(I) := \bigoplus_{n \geqslant 0} I^{(n)} t^n \subseteq R[t].$$

It is natural to ask when the symbolic Rees algebra is finitely generated. We start by discussing some well-known equivalent conditions.

Lemma 5.2. *Let R be a noetherian ring and I an ideal in R. The following are equivalent:*

(1) $\mathcal{R}_s(I)$ is a finitely generated R-algebra.

(2) $\mathcal{R}_s(I)$ is a noetherian ring.

(3) There exists d such that for all $n \geqslant 1$,

$$I^{(n)} = \sum_{a_1+2a_2+\cdots+da_d=n} I^{a_1} \left(I^{(2)}\right)^{a_2} \cdots \left(I^{(d)}\right)^{a_d}.$$

Furthermore, when these equivalent conditions hold, then

(4) There exists k such that $I^{(kn)} = \left(I^{(k)}\right)^n$ for all $n \geqslant 1$.

Conditions $(1)-(4)$ *are equivalent whenever R is an excellent ring.*

Proof. The fact that (1) implies (2) is a consequence of Hilbert's Basis Theorem. Moreover, since $\mathcal{R}_s(I)$ is an $\mathbb{N}$-graded algebra and $\mathcal{R}_s(I)_0 = R$ is a noetherian ring, the equivalence between (1) and (2) is a general fact about graded R-algebras; see for example [BH93, Proposition 1.5.4] for a proof.

Statement (3) says that $\mathcal{R}_s(I)$ is generated in degree up to d as an R-algebra, and thus is equivalent to (1).

To show that (3) implies (4), we follow [Ree58, Lemma 2], where in fact a stronger statement is proved. We will show that k can in fact taken to be $k = d \cdot d!$.

First, suppose that $n \geqslant k$. Then for each choice of $a_1+2a_2+\cdots+da_d = n \geqslant d \cdot d!$, we must have $ia_i \geqslant d!$ for some i, by the Pigeonhole Principle. Moreover, $q := \frac{d!}{i}$ is an integer, and thus

$$I^{a_1}\left(I^{(2)}\right)^{a_2}\cdots\left(I^{(d)}\right)^{a_d} = \left(I^{(i)}\right)^{q} I^{a_1}\left(I^{(2)}\right)^{a_2}\cdots\left(I^{(i)}\right)^{a_i-q}\cdots\left(I^{(d)}\right)^{a_d} \subseteq I^{(d!)}I^{(n-d!)}.$$

In particular, $I^{(n)} \subseteq I^{(d!)}I^{(n-d!)}$ for all $n \geqslant d \cdot d!$, which by Lemma 1.18 in fact implies that $I^{(n)} = I^{(d!)}I^{(n-d!)}$.

Now consider any $n \geqslant 1$. Since $nk \geqslant k = d \cdot d!$, then

$$I^{(kn)} = I^{(d!)}I^{(kn-d!)} = \left(I^{(d!)}\right)^2 I^{(kn-2d!)} = \cdots = \left(I^{(d!)}\right)^d I^{(kn-d\cdot d!)} \subseteq I^{(d\cdot d!)}I^{(kn-d\cdot d!)},$$

so that

$$I^{(kn)} = I^{(d\cdot d!)}I^{(kn-d\cdot d!)} = I^{(k)}I^{(k(n-1))}.$$

By induction, the statement follows.

On the other hand, if (4) holds, then the algebra

$$A := \bigoplus_{n\geqslant 0} I^{(kn)}t^{kn} = \bigoplus_{n\geqslant 0}\left(I^{(k)}t^k\right)^n \subseteq \mathcal{R}_s(I) \subseteq R[t]$$

is finitely generated. The fact that (4) implies the remaining equivalent statements will follow once we show that $\mathcal{R}_s(I)$ is a finitely generated algebra over A. To do that, we follow the argument in [Sch88, (2.2)].

Let B denote the integral closure of A inside $R[t]$. Recall[1] that B is the subring of $R[t]$ given as follows:

$$B = \left\{f \in R[t] : f^d + a_{d-1}f^{d-1} + \cdots + a_1 f + a_0 = 0 \text{ for some } f_i \in A\right\}.$$

We claim that $\mathcal{R}_s(I) = \bigoplus I^{(n)}t^n \subseteq B$. To show that, consider $u \in I^{(i)}t^i$. Then

$$u^k \in \left(I^{(i)}\right)^k t^{ik} \subseteq I^{(ki)}t^{ki} = \left(I^{(k)}t^k\right)^i,$$

so that u is a root of $T^k - \underbrace{u^k}_{\in A}$. Therefore, u is integral over A, so that $u \in B$. Since $\mathcal{R}_s(I)$ is generated by such elements, we conclude that $\mathcal{R}_s(I) \subseteq B$. Moreover, B

[1]The book [SH06] is a comprehensive reference on the subject of integral closure.

is a finitely generated module over A, by [SH06, Remark 12.3.11 or Theorem 9.2.2]. Therefore, $\mathcal{R}_s(I)$ must be finitely generated over A. $\square$

Remark 5.3. The proof of Lemma 5.2 shows that when the symbolic Rees algebra is noetherian and, say, generated in degree up to d, then for $k = d \cdot d!$, we do have $I^{(kn)} = \left(I^{(k)}\right)^n$ for all $n \geqslant 1$. In fact, it is shown in [Ree58, Lemma 2] that if the symbolic Rees algebra is generated in degrees $a_1, \ldots, a_d$, and r is the minimum common multiple of $a_1, \ldots, a_s$, then we can take $k = sr$.

Under mild assumptions, (3) above might be rewritten, as follows:

Lemma 5.4. *Let R be an excellent ring, and I an ideal in R. Suppose that k is such that $I^{(kn)} = \left(I^{(k)}\right)^n$ for all $n \geqslant 1$. Then there exists $A \geqslant 1$ such that for all $n \geqslant 1$, if $n = qk + r$, with $0 \leqslant r < k$, then*

$$I^{(n)} = \sum_{a=0}^{A} \left(I^{(k)}\right)^{q-a} I^{(ak+r)}.$$

Proof. As before, note that the R-algebra

$$B := \bigoplus_{n \geqslant 0} I^{(kn)} t^{kn} = \bigoplus_{n \geqslant 0} \left(I^{(k)} t^k\right)^n \subseteq R[t]$$

is finitely generated, and that $\mathcal{R}_s(I)$ is finitely generated over B.

Suppose that $\mathcal{R}_s(I)$ is generated over B in degrees $a_1, \ldots, a_d$. Then

$$\bigoplus_{n \geqslant 0} I^{(n)} t^n = \mathcal{R}_s(I) = I^{(a_1)} t^{a_1} B \oplus \cdots \oplus I^{(a_d)} t^{a_d} B = \bigoplus_{m \geqslant 1} I^{(a_i)} I^{(km)} t^{a_i + km}.$$

Finally, the theorem follows once we collect the pieces in degree n. $\square$

Remark 5.5. For both Lemma 5.2 (4) $\Rightarrow$ (3) and Lemma 5.4, we needed that the integral closure of a finitely generated R-algebra B in a finite extension is a finitely generated algebra over B. For that, it is enough to assume that R is excellent or analytically unramified. Rings with this property are called Nagata rings (see [Mat80, Chapter 13]). In particular, every polynomial or power series ring over a field has this property.

Given this, which ideals do have a noetherian symbolic Rees algebra? For example, the symbolic Rees algebra of a monomial ideal is noetherian [Lyu88, Proposition 1].

Question 5.6 (Cowsik)**.** Let P be a prime ideal in a regular ring R. Is the symbolic Rees algebra of P always a noetherian ring, or equivalently, a finitely generated R-algebra?

The answer to this question of Cowsik [Cow84] is no. The first counterexample was found by Roberts in [Rob85], and in fact there are space monomial curves such as the ones we studied in Chapter 3 whose symbolic Rees algebras are not noetherian [GM92, GNW94].[2] However, as we saw in Example 3.42, some of these counterexamples still verify nice containment conditions.

For height 2 prime ideals P in a dimension 3 regular local ring with infinite residue field, Huneke [Hun87, Theorem 3.1] gave a criterion to determine whether or not $\mathcal{R}_s(P)$ is noetherian. Morales [Mor91] generalized this criterion, and gave numerical criteria for values k such that $P^{(kn)} = \left(P^{(k)}\right)^n$. In [Cut91], Cutkosky shows that the symbolic Rees algebra of $k[t^a, t^b, t^c]$ is noetherian as long as $(a+b+c)^2 > abc$.

In the remainder of this chapter, we will study the Containment Problem for ideals whose symbolic Rees algebra is noetherian. For that, we will repeatedly use the following simple fact:

Remark 5.7. Suppose that $\mathcal{R}_s(I)$ is generated in degree up to d. A statement of the form $I^{(n)} \subseteq I^m$ is then equivalent to

$$\sum_{a_1+2a_2+\cdots+da_d=n} I^{a_1}\left(I^{(2)}\right)^{a_2}\cdots\left(I^{(d)}\right)^{a_d} \subseteq I^m.$$

Since $I+J \subseteq K$ if and only if $I \subseteq K$ and $J \subseteq K$, this is equivalent to

$$I^{a_1}\left(I^{(2)}\right)^{a_2}\cdots\left(I^{(d)}\right)^{a_d} \subseteq I^m$$

for all $a_1+2a_2+\cdots+da_d=n$.

[2] The result in [GNW94] has been reworked in [CT15] to show that a certain moduli space, $\overline{M}_{0,n}$ is not a Mori dream space, which means that its Cox ring is not a finitely generated k-algebra.

5.1.2 A degree version of Harbourne's Conjecture

When the symbolic Rees algebra of I is noetherian, we can obtain the following simple containment result:

Theorem 5.8. *Let R be a noetherian ring and I an ideal in R such that the symbolic Rees algebra of I is generated in degree up to d. Then for all $n \geqslant 1$,*

$$I^{(dn-d+1)} \subseteq I^n.$$

Proof. By Lemma 5.2, it is enough to show that for all choices of $a_1, \ldots, a_n \geqslant 0$ such that $a_1 + 2a_2 + 3a_3 + \cdots + da_d = dn - d + 1$,

$$I^{a_1} \left(I^{(2)}\right)^{a_2} \cdots \left(I^{(d)}\right)^{a_d} \subseteq I^{dn-d+1}.$$

To see this holds, note that $\left(I^{(i)}\right)^{a_i} \subseteq I^{a_i}$ for each i, so that

$$I^{a_1} \left(I^{(2)}\right)^{a_2} \cdots \left(I^{(d)}\right)^{a_d} \subseteq I^{a_1+a_2+\cdots+a_d}.$$

For each such choice of $a_1, \ldots, a_d$,

$$d\left(a_1 + \cdots + a_d\right) \geqslant a_1 + 2a_2 + \cdots + da_d = dn - d + 1,$$

so that

$$a_1 + \cdots + a_d \geqslant \frac{d(n-1)+1}{d}.$$

Since $a_1 + \cdots + a_d$ is an integer, we conclude that

$$a_1 + \cdots + a_d \geqslant (n-1) + 1 = n.$$ □

Given a radical ideal I with big height h whose symbolic Rees algebra is generated in degree up to d, if $d \leqslant h$, Theorem 5.8 implies that Harbourne's Conjecture 1.44 holds, and leads to stronger containment results when $d < h$. One might then ask if there is any relationship between the values of h and d. The answer is likely no. For space monomial curves, which we discussed in Chapter 3, there are well-understood examples with $d = 2, 3, 4, 5$ [GNS91b, Ree05, Ree09], whereas $h = 2$. On the other hand, Herzog and Ulrich, and independently Tom Marley, studied which space monomial curves have the property that the symbolic Rees algebra is generated in degree 2.

Corollary 5.9. *Let k be a field, $R = k[\![x, y, z]\!]$ and P be the height 2 prime defining $k[\![t^a, t^b, t^c]\!]$. Assume that P is not a complete intersection, so that it is generated by the maximal minors of*

$$M = \begin{pmatrix} x^{\alpha_3} & y^{\beta_1} & z^{\gamma_2} \\ z^{\gamma_1} & x^{\alpha_2} & y^{\beta_3} \end{pmatrix}.$$

Suppose one of the following conditions hold:

(a) $\alpha_3 = \alpha_2$ and $\beta_1 \leqslant \beta_3$ and $\gamma_2 \geqslant \gamma_1$, or

(b) $\alpha_3 \geqslant \alpha_2$ and $\beta_1 = \beta_3$ and $\gamma_2 \leqslant \gamma_1$, or

(c) $\alpha_3 \leqslant \alpha_2$ and $\beta_1 \geqslant \beta_3$ and $\gamma_2 = \gamma_1$.

Then $P^{(2n-1)} \subseteq P^n$ for all $n \geqslant 1$.

Proof. Herzog and Ulrich [HU90] and Marley have independently shown that $\mathcal{R}_s(P)$ is generated in degree up to 2 if and only if one of the conditions (a), (b) or (c) is verified. Theorem 5.8 completes the proof. □

5.2 Noetherianness and Harbourne's Conjecture (eventually)

When the symbolic Rees algebra of I is noetherian, we might expect the symbolic powers of I to have other nice properties. It is then natural to try to answer some of the questions we have asked in the case where the symbolic Rees algebra is noetherian.

5.2.1 Many open questions

Question 5.10. Let R be a regular ring and I a radical ideal in R. If the symbolic Rees algebra of I is noetherian, must I verify Harbourne's Conjecture?

The answer to this question is negative, as shown in [NS16].

Example 5.11 (Nagel–Seceleanu)**.** By [NS16, Proposition 4.1], the symbolic Rees algebra of the Fermat configuration $I = (x(y^n - z^n), y(x^n - z^n), z(x^n - y^n))$ in $K[x, y, z]$ is noetherian, and in fact $I^{(tn)} = \left(I^{(n)}\right)^t$ for all $t \geqslant 1$. However, as we saw in Example 4.18, $I^{(2m-1)} \subseteq I^m$ for all $m \geqslant 3$.

This still leaves open the question of whether all ideals with a noetherian symbolic Rees algebra verify the stable version of Harbourne's Conjecture, Conjecture 1.49.

Question 5.12. Let R be a regular ring and I a radical ideal in R with big height h. If the symbolic Rees algebra of I is noetherian, must $I^{(hn-h+1)} \subseteq I^n$ hold for all $n \gg 0$?

Similarly to Conjecture 4.2, we may ask the following weaker version:

Question 5.13. Let R be a regular ring and I a radical ideal in R with big height h. Suppose the symbolic Rees algebra of I is noetherian. Does the existence of m such that $I^{(hm-h+1)} \subseteq I^m$ imply that $I^{(hn-h+1)} \subseteq I^n$ hold for all $n \gg 0$?

And in the spirit of Lemma 4.16, we might ask the following question:

Question 5.14. Let R be a regular ring and I a radical ideal in R with big height h. Does the symbolic Rees algebra of I being noetherian imply that $\rho(I) < h$?

Unsurprisingly, if the symbolic Rees algebra of I is noetherian, it is enough to check that Harbourne's Conjecture holds for a finite number of values.

Theorem 5.15. *Let R be a noetherian ring and I an ideal in R such that the symbolic Rees algebra of I is generated in degree up to d. If h is an integer such that*

$$I^{(i)} \subseteq I^{\lceil \frac{i}{h} \rceil}$$

for all $i \leqslant d$, then for all $n \geqslant 1$,

$$I^{(hn-h+1)} \subseteq I^n.$$

Proof. Given $n \geqslant 1$,

$$I^{(hn-h+1)} \quad = \sum_{a_1+2a_2+\cdots+da_d=hn-h+1} I^{a_1}\left(I^{(2)}\right)^{a_2}\cdots\left(I^{(d)}\right)^{a_d}.$$

It is enough to show that for all $a_1,\dots,a_d \geqslant 0$ such that $a_1+2a_2+\cdots+da_d = hn-h+1$, the ideal

$$J := I^{a_1}\left(I^{(2)}\right)^{a_2}\cdots\left(I^{(d)}\right)^{a_d}$$

is contained in I^n. By assumption, $I^{(i)} \subseteq I^{\lceil \frac{i}{h} \rceil}$ for each i. Therefore, $J \subseteq I^N$, where

$$N \geqslant \sum_{i=1}^{d} a_i \left\lceil \frac{i}{h} \right\rceil \geqslant \sum_{i=1}^{d} \frac{ia_i}{h} = \frac{hn-h+1}{h}.$$

Since N is an integer, we must have

$$N \geqslant \left\lceil \frac{hn-h+1}{h} \right\rceil = n.$$

□

In characteristic 2, $I^{(3)} \subseteq I^2$ always holds by Corollary 2.11. As a consequence, we obtain Harbourne's Conjecture as long as $\mathcal{R}_s(I)$ is generated in degree up to 4.

Corollary 5.16. *Let R be a regular ring of characteristic 2. If I is a radical ideal of big height 2 such that $\mathcal{R}_s(I)$ is generated in degree up to 4, then $I^{(hn-h+1)} \subseteq I^n$ for all $n \geqslant 1$.*

For certain space monomial curves, this is enough to show Harbourne's Conjecture.

Example 5.17. For the case of the defining ideal P of a space monomial curve such as in Example 3.30, $P^{(3)} \subseteq P^2$ always holds for such ideals, by Theorem 3.31, and $P^{(5)} \subseteq P^3$ also holds by Theorem 3.46. Together with Theorem 5.15, this implies that Harbourne's Conjecture holds for such primes P whenever the symbolic Rees algebra of P is generated in degree up to 6.

In [GNS91b], Goto, Nishida, and Shimoda give explicit conditions on the Hilbert-Burch matrix of P that are equivalent to the symbolic Rees algebra being generated in degree up to 3. Similarly, in [Ree05, Ree09], Michael Reed gives equivalent conditions on the Hilbert-Burch matrix of P that are equivalent to the symbolic Rees algebra being generated in degree up to 4.

5.2.2 A prime characteristic approach

We will now discuss a characteristic p approach to Question 5.12: if the symbolic Rees algebra of I is noetherian, must I verify $I^{(hn-h+1)} \subseteq I^n$ for $n \gg 0$?

Lemma 5.18. *Suppose that R is a regular ring of prime characteristic p. If for all $a_1 + 2a_2 + \cdots + da_d = hn - h + 1$, there exists a choice of $b_i \leqslant a_i$ and $q = p^e$ such that*

$$b_1 + 2b_2 + \cdots + db_d = hq - h + 1,$$

then $I^{(hn-h+1)} \subseteq I^n$.

Proof. If such a choice of b_i exists, then $I^{a_1} \left(I^{(2)}\right)^{a_2} \cdots \left(I^{(d)}\right)^{a_d}$ can be rewritten as

$$\left(I^{b_1} \left(I^{(2)}\right)^{b_2} \cdots \left(I^{(d)}\right)^{b_d}\right) \left(I^{a_1-b_1} \left(I^{(2)}\right)^{a_2-b_2} \cdots \left(I^{(d)}\right)^{a_d-b_d}\right).$$

Note that

$$I^{b_1} \left(I^{(2)}\right)^{b_2} \cdots \left(I^{(d)}\right)^{b_d} \subseteq I^{(b_1+2b_2+\cdots+db_d)} = I^{(hq-h+1)}.$$

By Lemma 2.10, $I^{(hq-h+1)} \subseteq I^q$. Similarly,

$$I^{a_1-b_1} \left(I^{(2)}\right)^{a_2-b_2} \cdots \left(I^{(d)}\right)^{a_d-b_d} \subseteq I^{(h(n-q))} \subseteq I^{n-q}.$$

Finally, $I^q I^{n-q} = I^n$. Thus, $I^{a_1} \left(I^{(2)}\right)^{a_2} \cdots \left(I^{(d)}\right)^{a_d} \subseteq I^n$. Since the assumption was that this could be achieved for all $a_1 + 2a_2 + \cdots + da_d = hn - h + 1$, we conclude that $I^{(hn-h+1)} \subseteq I^n$. ☐

This Lemma has the advantage of converting our problem into a simple number theory question. Unfortunately, this approach only applies under certain circumstances, which depend, among other things, on the characteristic.

Theorem 5.19. *Let R be a regular ring of characteristic $p \equiv 2 \pmod 3$. If I is a radical ideal of big height 2 such that $\mathcal{R}_s(I)$ is generated in degree up to 3, then $I^{(hn-h+1)} \subseteq I^n$ for all $n \geqslant p^2 + p - 2$.*

Proof. We will apply Lemma 5.18. Since $I^k \subseteq I^{(k)}$ for all k, and $a_1+2a_2+3a_3 = 2n-1$, we might as well assume that $a_1 \leqslant 1$. Indeed, when $a_1 > 1$, we have $I^{a_1} \subseteq \left(I^{(2)}\right)^k$ or $I^{a_1} \subseteq I\left(I^{(2)}\right)^k$ for $k = \left\lfloor \frac{a_1}{2} \right\rfloor$, in which case the containment

$$I^{a_1}\left(I^{(2)}\right)^{a_2}\left(I^{(3)}\right)^{a_3} \subseteq I^n$$

will follow from

$$I^{a_1'}\left(I^{(2)}\right)^{a_2'}\left(I^{(3)}\right)^{a_3} \subseteq I^n$$

for $a_1' = a_1 - 2\left\lfloor \frac{a_1}{2} \right\rfloor$, $a_2' = a_2 + \left\lfloor \frac{a_1}{2} \right\rfloor$. If $a_1 = 1$, then $2a_2 + 3a_3 = 2n-2$. Take $n \geqslant p^2+p-2$, so that

$$2a_2 + 3a_3 = 2n - 2 \geqslant 2(p^2+p) - 4 = 2(p-1) + (2p^2-2).$$

In particular, either $a_2 \geqslant p-1$ or $a_3 \geqslant \frac{2p^2-2}{3}$. To apply Lemma 5.18 in the first case, we can choose $b_1 = 1$, $b_2 = p-1$ and $b_3 = 0$, and obtain

$$b_1 + 2b_2 + 3b_3 = 2p - 1.$$

In the second case, take $b_1 = 1$, $b_2 = 0$ and $b_3 = \frac{2p^2-2}{3}$, which is indeed an integer. Then

$$b_1 + 2b_2 + 3b_3 = 2p^2 - 1.$$

If $a_1 = 0$, then $2a_2 + 3a_3 = 2n-1$, and simple parity reasons imply that $a_3 \geqslant 1$. Equivalently, $2a_2 + 3(a_3-1) = 2n-4$. If $n \geqslant 2p-2$, then

$$2a_2 + 3(a_3 - 1) = 2n - 4 \geqslant 4p - 8 = 2(p-2) + (2p-4).$$

Then either $a_2 \geqslant p-2$ or $a_3 - 1 \geqslant \frac{2p-4}{3}$. In the first case we can chose $b_1 = 0$, $b_2 = p-2$ and $b_3 = 1$, yielding

$$b_1 + 2b_2 + 3b_3 = 2p - 1.$$

In the second case, we can chose $b_1 = 0$, $b_2 = 0$ and $b_3 = \frac{2p-1}{3}$, so that $b_1+2b_2+3b_3 = 2p-1$.

We conclude that if $p \equiv 2 \pmod 3$, then $I^{(2n-1)} \subseteq I^n$ for all

$$n \geqslant \max\left\{p^2+p-2, 2p-2\right\} = p^2+p-2. \qquad \square$$

The same proof does not follow without the condition on p.

Example 5.20. When $p \equiv 1 \pmod 3$, we must have $2p^e - 1 \equiv 1 \pmod 3$ for all e, which is a fatal difference. In fact, the technique we used in Theorem 5.19 cannot be applied to show $I^{(2n-1)} \subseteq I^n$ whenever $n \equiv 2 \pmod 3$. Indeed, when $n = 3k+2$, note that $a_1 = a_2 = 0$ and $a_3 = 2k+1$ is a solution to

$$a_1 + 2a_2 + 3a_3 = 6k + 3 = 2n - 1,$$

but there is no b such that $3b = 2p^e - 1$. Therefore, we cannot apply Lemma 5.18.

Unfortunately, Lemma 5.18 also cannot be applied when $h = 2$ and $d \geqslant 5$ in any characteristic.

Example 5.21. Let R be a regular ring of characteristic p, I a radical ideal of big height 2, and assume that $\mathcal{R}_s(I)$ is generated in degree up to d, where $d \geqslant 5$. The technique we used in Theorem 5.19 does not show that $I^{(2n-1)} \subseteq I^n$ for all n even. Indeed, suppose that $n = 2k$, and note that $a_3 = 1$, $a_4 = k-1$ and $a_i = 0$ otherwise is a solution to

$$a_1 + \cdots + da_d = 3 + 4(k-1) = 4k - 1 = 2n - 1.$$

On the other hand, if $b \leqslant 1$ and $c \leqslant k-1$ are such that

$$3b + 4c = 2p^e - 1,$$

then by parity reasons we must have $b = 1$. As a consequence, $2p^e - 1 \equiv 3 \pmod 4$, so that $2p^e$ must be a multiple of 4. Since p is prime, we must have $p = 2$. If $p \neq 2$, then in fact the same argument shows that our technique fails for all $d \geqslant 4$.

When $p = 2$, we run into a similar issue as long as $d \geqslant 5$. More specifically, this technique fails to show that $I^{(2n-1)} \subseteq I^n$ for all $n = 2k+2$. Indeed, note that $a_4 = k-1$, $a_5 = 1$ and $a_i = 0$ otherwise is a solution to

$$a_1 + \cdots + da_d = 4(k-1) + 5 = 4k + 1 = 2n - 1.$$

If $b \leqslant k-1$ and $c \leqslant 1$ are such that

$$4b + 5c = 2^{e+1} - 1,$$

then in fact we must have $c = 1$ for parity reasons, so that $e \geqslant 1$. Then 2^{e+1} is a multiple of 4, and $2^{e+1} - 1 \equiv 3 \pmod{4}$. On the other hand, $4b + 5c \equiv 1 \pmod{4}$. It is then impossible that $4b + 5c = 2^{e+1} - 1$.

When $p = 2$ and $d = 4$, Lemma 5.18 does apply. However, we already have $I^{(2n-1)} \subseteq I^n$ for all $n \geqslant 1$ in this case, by Corollary 5.16.

Unfortunately, the technique used in Theorem 5.19 also cannot be applied whenever $h \geqslant 3$.

Remark 5.22. Let R be a regular ring of characteristic p, I a radical ideal of big height $h > 2$, and assume that $\mathcal{R}_s(I)$ is generated in degree up to d. Fix n and $a_1 + 2a_2 + \cdots + da_d = hn - h + 1$. As before, we might assume $a_1 \leqslant 1$, since $I^k \subseteq I^{(k)}$ for all $k \geqslant 1$. As we will see, the hypothesis of Lemma 5.18 fail for n arbitrarily large and certain choices of $a_1 + 2a_2 + \cdots + da_d = hn - h + 1$ with $a_1 = 0$.

If $a_1 = 1$, we can in fact find b_i such as in Lemma 5.18: we claim that for each $2 \leqslant i \leqslant d$, there exist $c_i, e_i \leqslant 1$ such that $1 + ic_i = h(p^{e_i} - 1)$. Given such c_i, we can find b_i such as in Lemma 5.18 as long as

$$n \geqslant 1 + 2c_2 + \cdots + dc_d.$$

To show the claim, first note that $2a_2 + \cdots + da_d = h(n-1)$. Fix $2 \leqslant i \leqslant d$, so we can find a c_i that works as desired. Write $i = rs$ and $h = ts$, where r and t are coprime, and s is the minimum common multiple of i and h. If $p > d$, p must be invertible modulo i. In particular, there exists a solution e to $p^e \equiv 1 \pmod{i}$. Pick k such that $p^e - 1 = kr$, and let $c_i := kt$. Then

$$ic_i = (rs)(kt) = (st)(kr) = h(p^e - 1).$$

When $a_1 = 0$, we actually have $2a_2 + \cdots + da_d = h(n-1) + 1$. This is where problems arise. Let us try to find c_i such that $ic_i = h(p^{e_i} - 1) + 1$ for some c_i. For this to be possible, i must in particular be prime to h. Equivalently, we are asking that $p^e - 1 \equiv -h^{-1} \pmod{i}$. If $h \equiv 1 \pmod{i}$, this is impossible unless $i = p$. On the other hand, whenever $n \equiv 0 \pmod{i}$, we do have $h(n-1)+1 \equiv 0 \pmod{i}$. As long as $d \geqslant h-1$, there are in fact such values of i to worry about. Note, however, that this problem does not arise when $h = 2$, since in that case $h \not\equiv 1 \pmod{i}$ for all $i \geqslant 2$.

As an example, take any $k \geqslant 1$, $n = k(h-1)$, and $d \geqslant h-1$. Note that $a_{h-1} = hk-1$ and $a_i = 0$ otherwise is a solution to

$$a_1 + 2a_2 + \cdots + da_d = (h-1)(hk-1) = h(k(h-1)-1)+1 = hn-h+1.$$

In other words,

$$I^{(hn-h+1)} = \left(I^{(h-1)}\right)^{hk-1} + \sum_{a_1+2a_2+\cdots+da_d=hn-h+1} I^{a_1}\left(I^{(2)}\right)^{a_2}\cdots\left(I^{(d)}\right)^{a_d}.$$

To show that $I^{(hn-h+1)} \subseteq I^n$, we need to prove in particular that

$$\left(I^{(h-1)}\right)^{hk-1} \subseteq I^n.$$

In order to show this via Lemma 5.18, we would have to show that there exists $b \leqslant hk-1$ such that

$$(h-1)b = hp^e - h + 1.$$

But since $hp^e - h + 1 \equiv p^e \pmod{h-1}$ and p is prime, such b only exists when $h-1$ is a power of p.

We could avoid this issue by requiring that $h > d$, but Theorem 5.8 already guarantees nice containment results in that case. As a consequence, the result we are trying to show would only be interesting for $h < d$.

However, when $\mathcal{R}_s(I)$ is noetherian, we might instead use the value k for which $I^{(kn)} = \left(I^{(k)}\right)^n$.

Theorem 5.23. *Suppose that h and k are integers such that h and $h-1$ are both coprime with k. There is an infinite set of prime ideals p with the following property:*

Given a regular ring R of characteristic p, if I a radical ideal of big height h and such that $I^{(kn)} = \left(I^{(k)}\right)^n$ for all $n \geqslant 1$, then

$$I^{(hn-h+1)} \subseteq I^n$$

for all $n \geqslant N$, where N only depends on h, k, and p.

Proof. Our assumptions imply that h and $h-1$ are invertible modulo k. If $c > 1$ is an inverse to h modulo k, then $c(h-1)$ is also prime with k. By Dirichlet's Theorem, there are infinite primes p that are congruent to $c(h-1)$ modulo k. Given such a prime p,

$$h(p-1)+1 \equiv hc(h-1)-h+1 \equiv h-1-h+1 \equiv 0 \pmod{k},$$

and thus there exists b such that $hp-h+1 = bk$.

On the other hand, by Lemma 5.4 there exists A such that for all $n \geqslant 1$,

$$I^{(n)} = \sum_{a=0}^{A} \left(I^{(k)}\right)^{s-a} I^{(ak+r)}.$$

where $n = sk+r$ and $0 \leqslant r < k$. We claim that as long as $hn-h+1 \geqslant (A+b)k$, then $I^{(hn-h+1)} \subseteq I^n$. Rewrite this in the form $n \geqslant N$.

To show the claim, fix such n and write $hn-h+1 = sk+r$ with $0 \leqslant r < k$. Then

$$I^{(hn-h+1)} = \sum_{a=0}^{A} \left(I^{(k)}\right)^{s-a} I^{(ak+r)}.$$

Note that

$$s-a \geqslant s-A = \left\lfloor \frac{hn-h+1}{k} \right\rfloor - A \geqslant (b+A)-A = b,$$

and thus $k(s-a) \geqslant bk$. In particular,

$$I^{(hn-h+1)} = \left(I^{(k)}\right)^{b} \sum_{a=0}^{A} \left(I^{(k)}\right)^{s-a-b} I^{(ak+r)}.$$

The result now follows by Lemma 5.18. We note, however, that N depends on p. □

Remark 5.24. We note that finding an appropriate value of k to apply Theorem 5.23 might be a difficult task, even if we know that $\mathcal{R}_s(I)$ is generated in degree up to d. By Remark 5.3, we might take $k = d \cdot d!$, but that only makes the conditions in Theorem 5.23 harder to meet. We recall that by Theorem 5.8, $I^{(hn-h+1)} \subseteq I^n$ for $n \gg 0$ already holds whenever $d \leqslant h$. Therefore, as we have noted before, we are only interested in the case when $d > h$. But taking $k = d \cdot d!$ will then never produce values meeting the conditions of Theorem 5.8.

However, in some circumstances we may indeed find values of k such as in Theorem 5.23. When $h = 2$, Theorem 5.23 applies whenever we can take k to be an odd number.

Example 5.25 (Fermat configurations yet again)**.** As noted in Example 5.11, we can take $k = n$ for the Fermat configuration $I = (x(y^n - z^n), y(x^n - z^n), z(x^n - y^n))$, by [NS16, Proposition 4.1]. If n is odd, then I verifies Conjecture 1.49. On the other hand, we have already seen in Example 4.18 that I does verify Conjecture 1.49, since $\rho(I) < 2$.

5.2.3 The case $I^{(hn)} = \left(I^{(h)}\right)^n$

When $I^{(hn)} = \left(I^{(h)}\right)^n$, we can improve on the ideas from the previous subsection, together with the following corollary [DDSG+17, Corollary 2.36] of a result of Hübl [Hüb05, Theorem 1.2].

Lemma 5.26 (Hübl)**.** *Let k be a field of characteristic zero, and let S be a smooth algebra, essentially of finite type over k. Given a radical ideal I in S, there exists $N > 0$ such that $\left(I^{(2)}\right)^n \subseteq I^{n+1}$ for all $n \geqslant N$.*

We can apply this lemma to obtain eventual results in the flavor of Chapter 4.

Corollary 5.27. *Let k be a field of characteristic zero, and let S be a smooth algebra, essentially of finite type over k. If I is a radical ideal in S such that $I^{(km)} = \left(I^{(k)}\right)^m$ for all $m \geqslant 1$, then for every integer C, $I^{(kn)} \subseteq I^{n+C}$ for all $n \gg 0$.*

Proof. First note that successive applications of Lemma 5.26 lead to $\left(I^{(2)}\right)^n \subseteq I^{n+C}$ for all $n \gg 0$. Fixing N such that $\left(I^{(2)}\right)^N \subseteq I^{N+C}$, then for all $n \geqslant N$ we have

$$I^{(kn)} = \left(I^{(k)}\right)^n = \left(I^{(k)}\right)^{n-N} \left(I^{(k)}\right)^N \subseteq I^{n-N} I^{N+C} = I^{n+C}. \qquad \square$$

Theorem 5.28. *Let I be a radical ideal of big height h in a regular ring R containing a field. If I is such that $I^{(hm)} = \left(I^{(h)}\right)^m$ for all $m \geqslant 1$, then $I^{(hn-h+1)} \subseteq I^n$ for all $n \gg 0$.*

Proof. By Lemma 5.4, there exists an integer $A \geqslant 1$ such that for all $n \geqslant 1$,

$$I^{(hn-h+1)} = \sum_{a=0}^{A} \left(I^{(h)}\right)^{n-1-a} I^{(ha+1)}.$$

In prime characteristic p, consider e such that $q := p^e \geqslant A+1$. Whenever $n \geqslant q$, we have $hn - h + 1 \geqslant hq - h + 1 \geqslant hA + 1$, and thus

$$I^{(hn-h+1)} = \sum_{a=0}^{A} \left(I^{(h)}\right)^{n-1-a} I^{(ha+1)} = \sum_{a=0}^{A} \left(I^{(h)}\right)^{n-q} \left(I^{(h)}\right)^{q-1-a} I^{(ha+1)} \subseteq I^{(h(n-q))} I^{(hq-h+1)}.$$

As in the proof of Lemma 5.18, $I^{(hq-h+1)} \subseteq I^q$ by Lemma 2.10 and $I^{(h(n-q))} \subseteq I^{n-q}$ by Theorem 1.36. Therefore,

$$I^{(hn-h+1)} \subseteq I^{(hq-h+1)} I^{(h(n-q))} \subseteq I^q I^{n-q} = I^n.$$

In characteristic 0, consider N such as in Lemma 5.26, that is, such that $\left(I^{(2)}\right)^n \subseteq I^{n+1}$ for all $n \geqslant N$. Let $n \geqslant N + A + 1$. Then

$$I^{(hn-h+1)} = \sum_{a=0}^{A} \left(I^{(h)}\right)^{n-1-a} I^{(ha+1)} = \sum_{a=0}^{A} \left(I^{(h)}\right)^{N} \left(I^{(h)}\right)^{n-1-N-a} I^{(ha+1)}.$$

By Theorem 1.36, $I^{(ha+1)} \subseteq I^a$. Moreover, $\left(I^{(h)}\right)^{n-1-N-a} \subseteq I^{n-1-N-a}$ since $I^{(h)} \subseteq I$. By choice of N, $\left(I^{(h)}\right)^N \subseteq \left(I^{(2)}\right)^N \subseteq I^{N+1}$. Therefore,

$$I^{(hn-h+1)} = \sum_{a=0}^{A} \left(I^{(h)}\right)^{N} \left(I^{(h)}\right)^{n-1-N-a} I^{(ha+1)} \subseteq \sum_{a=0}^{A} I^{N+1} I^{n-1-N-a} I^a = I^n. \qquad \square$$

Morales [Mor91] found classes of space monomial curves verifying $I^{(2n)} = \left(I^{(2)}\right)^n$ for all $n \geqslant 1$, to which can now apply Theorem 5.28.

Corollary 5.29. *Consider coprime integers a and b. Let k be a field and P be the defining ideal in $k[x,y,z]$ of the space monomial curve $k[t^a, t^{a+b}, t^{a+2b}]$. For all $n \gg 0$, $P^{(2n-1)} \subseteq P^n$.*

Proof. By [Mor91, Corollary 4.6], $P^{(2n)} = \left(P^{(2)}\right)^n$ for all $n \geqslant 1$, so we can apply Theorem 5.28. □

Chapter 6

Reduction to characteristic p

There are characteristic free questions that are easier to attack using positive characteristic techniques. Hochster and Huneke's proof that $I^{(hn)} \subseteq I^n$, or the results we discussed in Chapter 2, are examples of this. More surprisingly, a characteristic p solution to a question can sometimes be enough to solve the equicharacteristic 0 case, via a method known as reduction to positive characteristic.

In this Chapter, we use reduction to positive characteristic to deduce equicharacteristic 0 versions of the results in Chapter 2, and discuss why the same techniques cannot be applied to the results in Subsection 5.2.2.

6.1 Background

In this Section, we survey some background on reduction to characteristic p, following [HH99], and discuss how these techniques can be applied to the containment problem, as in [HH02]. For a thorough treatment of reduction to characteristic p techniques, see [HH99].

6.1.1 Reduction to characteristic p techniques

Say we want to study a finitely generated algebra R over a field k of characteristic 0, or that R is essentially of finite type over k, meaning that R is the localization of a quotient of $k[x_1, \ldots, x_n]$. The idea of descent theory is to replace k by a finitely generated $\mathbb{Z}$-subalgebra A of k, to replace finitely generated k-algebras by finitely generated A-algebras, and modules over finitely generated modules over those k-algebras by finitely generated modules over those A-algebras. We also want these replacement modules to be free over A, as modules over k are always free.

Say $R = k[x_1, \dots, x_n]/(G_1, \dots, G_s)$, where the G_i are polynomials over k in the indeterminates x_i. We start by collecting all the (finitely many) coefficients appearing in each G_i, and take A to be the free $\mathbb{Z}$-module generated by those coefficients, replacing R by $R_A = A[x_1, \dots, x_n]/(G_1, \dots, G_s)$. If we want to make assertions about an ideal I in R, we may extend A to include all the (finitely many) coefficients of the generators of I, so that there is a natural corresponding ideal I_A in R_A. If $I \subseteq J$ are ideals in R, then we can extend A even further to include both all the coefficients of the generators g_i of I and f_j of J, but also the coefficients of all the $a_{i,j} \in R$ such that $g_i = \sum_j a_{i,j} f_j$. We then get $I_A \subseteq J_A$. Note also that if A contains all the coefficients of u such that $u \in I$ and $u \notin J$, then $u_A = u$ verifies $u_A \in I_A$ and $u_A \notin J_A$.

More generally, given an R-module M, we can choose a model M_A for M such that $M = M_A \otimes_A k$. In particular, given $M \neq 0$, a model M_A for M must still be nonzero. The process we described above does not necessarily lead to free A-modules; but we can indeed chose such models M_A to be free.

Lemma 6.1 (Generic freeness)**.** *Let A be a Noetherian domain, R a finitely generated A-algebra, S a finitely generated R-algebra, W a finitely generated S-module, M a finitely generated R-submodule of W and N a finitely generated A-submodule of W. Let $V = W/(M + N)$. Then there exists a nonzero $a \in A$ such that V_a is free over A_a.*

Proof. This is a result originally due to Grothendieck [Gro67, Théorèm 6.9.1]. For a proof of this strong version, see [HH88, Lemma 8.1]. □

As a consequence, we can chose an element f in our original choice of A such that

M_{A_f} is free over A_f for a given R-module M; we then take such A_f as our finitely generated $\mathbb{Z}$-algebra replacing k, and write it as A. Again, M can now be recovered via localization. In particular, this allows us to choose R_A free over A.

After this descent to $A[x_1, \ldots, x_n]/(G_1, \ldots, G_s)$, we can now pass to prime characteristic. First, take any maximal ideal $\mathfrak{m}$ in A, and note that $\mathfrak{m} \cap \mathbb{Z}$ is a maximal ideal in $\mathbb{Z}$, by the Nullstellensatz for Jacobson rings (see for example [Eis95, Theorem 4.19]). In other words, $\mathfrak{m} \cap \mathbb{Z} = p\mathbb{Z}$ for some prime p, and thus $\mathbb{Z}/p\mathbb{Z} \subseteq A/\mathfrak{m}$. We conclude that $A/\mathfrak{m}$ has prime characteristic; in fact, $A/\mathfrak{m}$ is a field and a finitely generated algebra over $\mathbb{Z}/p\mathbb{Z}$, and thus algebraic over $\mathbb{Z}/p\mathbb{Z}$, by [Mat89, Theorem 5.2]. Finitely generated algebraic extensions are also module finite, and thus $A/\mathfrak{m}$ must be a finitely generated module over $\mathbb{Z}/p\mathbb{Z}$, and thus a finite field. Finally, we replace A by each $A/\mathfrak{m}$.

However, when $R = k[\![x_1, \ldots, x_n]\!]/(G_1, \ldots, G_s)$, the approach we described above to find a module R_A for R does not quite work, since each G_i might have an infinite number of nonzero coefficients, requiring us to add an infinite number of generators to A. Instead, we can reduce to the case of algebras of finite type over k using Artin Approximation [Art69, AR88].

Theorem 6.2 (Artin Approximation)**.** *Let k be a field and $T = k[\![x_1, \ldots, x_d]\!]$. Every k-algebra homomorphism of a finitely generated k-algebra T to R factors through some S, meaning $T \to S \to R$ for some k-algebra homomorphisms, where S is regular and of finite type over $k[x_1, \ldots, x_d]$.*

6.1.2 Reduction to characteristic p and symbolic powers

The following theorem by Hochster and Huneke allows us to apply reduction to characteristic p to prove statements about symbolic powers. This might be applied to prove containments *for all* ideals I, such as in [HH02, Theorem 4.4], but it may also be applied to specific classes of ideals — for example, if I has a property that passes to the reduction of I to characteristic p. The following is shown in [HH02, Theorem 4.4], using the reduction to positive characteristic techniques described in [HH99].

Theorem 6.3 (Hochster–Huneke)**.** *Let R be a regular ring essentially of finite type over a field k of characteristic 0. Fix positive integers a and b, and let I be an ideal in R. Consider a model A as described in the previous section, and assume that for almost all fibers, the containment $I_A^{(a)} \subseteq I_A^b$ holds for the map $\mathbb{Z} \longrightarrow R_A$ after passing to fibers over closed points of $\operatorname{Spec} \mathbb{Z}$. Then $I^{(a)} \subseteq I^b$.*

Remark 6.4 (Uniqueness)**.** We note that in principle the statement of Theorem 6.3 is not well-defined, simply because the construction of A we described is not uniquely defined. However, as noted in [HH99, (2.1.12)], given two choices of $\mathbb{Z}$-algebras A and B, all our choices of R_A, I_A, and so on become the same for a sufficiently large finitely generated $\mathbb{Z}$-algebras C of k with $A \subseteq C$ and $B \subseteq C$. Therefore, we will assume that we have enlarged our ring R_A enough that there is no ambiguity.

Remark 6.5. Hochster and Huneke's result is true more generally, without the essentially of finite type assumption. For a regular local ring R, equidimensional over a field k, but not necessarily of finite type over k, one may reduce to the finite type case using Artin Approximation, as in [HH02, Theorem 4.3].

6.2 Applications to the containment problem

In this Section, we apply reduction to positive characteristic techniques to the containment problem: in Subsection 6.2.1, we give equal characteristic 0 versions of Theorems 2.31 and 2.34, which can also be found in [GH17], and in Subsection 6.2.2 we discuss why these techniques cannot be used to extend some of our characteristic p results from Chapter 5.

6.2.1 An equicharacteristic 0 case of Harbourne's Conjecture

We have shown in Chapter 2 that if R/I is F-pure, then I verifies Harbourne's Conjecture. If I is an ideal in a regular ring essentially of finitely type over a field of characteristic 0, we can now prove an analogous result when I reduces modulo p to an ideal defining an F-pure ring at least for infinitely many values of p.

Definition 6.6 (dense F-type)**.** *Let R be a ring essentially of finite type over a field k of characteristic zero. Suppose we are given a model R_A of R over a finitely generated $\mathbb{Z}$-subalgebra A of k. We say that R is of* dense F-pure type *(respectively* dense strongly F-regular type*) if there exists a dense subset of closed points $S \subseteq \operatorname{Spec} A$ such that R_μ is F-pure (respectively strongly F-regular) for all $\mu \in S$.*

Remark 6.7. Note that the definition above is independent of the choice of R_A, since these types of singularities are stable under base change.

Theorem 6.8. *Let R be a regular ring, essentially of finite type over a field of characteristic 0. Let I be an ideal in R such that R/I is of dense F-pure type, and let h be the big height of I. Then, for all integers $n \geqslant 1$,*

$$I^{(hn-h+1)} \subseteq I^n.$$

Proof. After reducing to characteristic $p > 0$, the statement follows from Theorem 2.31 and Theorem 6.3. □

Similarly to Theorem 6.8, we obtain a characteristic 0 version of Theorem 2.34.

Theorem 6.9. *Let R be a regular ring, essentially of finite type over a field of characteristic 0. Let I be an ideal in R such that R/I is of dense strongly F-regular type, and let h be the largest height of any minimal prime of I. Then, for all integers $n \geqslant 1$,*

$$I^{((h-1)n+1)} \subseteq I^{n+1}.$$

Proof. Our assumption means that given models R_A and $(R/I)_A$ of R and R/I over a finitely generated $\mathbb{Z}$-subalgebra A of k, there exists a dense subset of closed points $S \subseteq \operatorname{Spec} A$ such that $(R/I)_\mu$ is strongly F-regular for all $\mu \in S$.

As in the proof of Theorem 6.8, we use the standard descent theory of [HH99, Chapter 2], reducing the problem to the positive characteristic case, following the same steps as in [HH02, Theorem 4.2]. In other words, we use Theorem 6.3. After reducing to characteristic $p > 0$, the statement follows from Theorem 2.34. □

6.2.2 Stable (non-)results

One might hope that showing $I^{(hn-h+1)} \subseteq I^n$ holds for all $n \gg 0$ in characteristic p for at least an infinite number of values of p might imply $I^{(hn-h+1)} \subseteq I^n$ holds for all $n \gg 0$ in characteristic 0, using reduction to characteristic p techniques. However, that would require the n large enough to guarantee $I^{(hn-h+1)} \subseteq I^n$ to not depend on p.

Indeed, suppose there exists N, not depending on p, such that for all $n \geqslant N$, $I_p^{(hn-h+1)} \subseteq I_p^n$ for p in some infinite set. Then by Theorem 6.3 this implies that $I^{(hn-h+1)} \subseteq I^n$ for all $n \geqslant N$. On the other hand, suppose we can only show that there exists an infinite set of primes p and corresponding integers $N(p)$ such that $I_p^{(hn-h+1)} \subseteq I_p^n$ for all $n \geqslant N(p)$, and that $\lim\limits_{p \to \infty} N(p) = \infty$. Then this does not show that $I^{(hn-h+1)} \subseteq I^n$ for all $n \gg 0$. In fact, given a fixed n, $I_p^{(hn-h+1)} \subseteq I_p^n$ must hold for all p such that $n \geqslant N(p)$, but such p form a finite set.

In particular, reduction to characteristic p does not directly lead to a characteristic 0 version of Theorem 5.23.

Chapter 7

Algorithms for computing symbolic powers

7.1 Macaulay2 Algorithms

Given an ideal in a polynomial ring, how would we go about actually computing its symbolic powers? In particular, we would like to know how to do this with the help of a computer, perhaps using *Macaulay2* [GS]. Macaulay2 is a software system devoted to supporting research in algebraic geometry and commutative algebra, and in particular it supports computations over polynomial rings.

The Macaulay2 package `SymbolicPowers` allows the user to compute symbolic powers in a polynomial ring. This package was written in collaboration with Ben Drabkin, Alexandra Seceleanu, and Branden Stone, and Andrew Conner and Diana Zhong have also contributed code. The package also includes other features, as described in [DGSS17], and can be found at `https://github.com/eloisagrifo/SymbolicPowers`.

7.1.1 General approach

We will now describe some strategies for computing symbolic powers of ideals with no embedded primes in a polynomial ring. These are the basis of the `SymbolicPowers` package.

Let k be a field, $R = k[x_1, \dots, x_d]$, and consider an ideal I in R with no embedded primes. Say we want to compute the n-th symbolic power of I. Macaulay2 allows us to find a primary decomposition for I and I^n, but the process can be fairly slow.

There are many algorithms for computing primary decompositions of ideals, but as noted in [DGP99], "providing efficient algorithms for primary decomposition of an ideal [...] is [...] still one of the big challenges for computational algebra and computational algebraic geometry." In [DGP99], Decker, Gert-Martin, and Gerhard compare four different algorithms [Rit50, Wu84, GTZ88, EHV92, SY96] for computing primary decomposition, using the computer algebra system SINGULAR [DGPS16]. Note that even for the case of monomial ideals, finding a primary decompositions is an NP-complete problem [HsS02].

It is then convenient to find algorithms for computing $I^{(n)}$ that avoid explicitly determining a primary decomposition for I^n, and ideally also I.

Suppose that I is a radical **homogeneous** ideal of pure height $d-1$. As noted in Lemma 3.1, we can explicitly compute the symbolic powers of I by taking $I^{(n)} = I^n : (x_1, \dots, x_d)^\infty$. In particular, this is a fairly efficient way to compute the symbolic powers of the ideals such as the ones we studied in Chapter 3.

If I is a squarefree monomial ideal, we saw in Example 1.17 that an explicit primary decomposition of I, say $I = \bigcap\limits_i Q_i$, is such that each Q_i is an ideal generated by variables, and $I^{(n)} = \bigcap\limits_i Q_i^n$. While computing primary decompositions is usually a fairly lengthy process, Macaulay2 uses a more efficient method when I is a monomial ideal.

Unfortunately, when I does not fall under one of the previous conditions, computing $I^{(n)}$ might in fact require finding primary decompositions for both I and I^n. If I is prime, or even primary, it is enough to scan the a list of primary components of I^n and find the component with radical $\sqrt{I}$.

There is also an algorithm by Seth Sullivant that follows from [Sul08, Proposition 2.8]; this can also be found in the symbolic powers package. However, this method requires dealing with ideals in $2d$ variables.

In Subsection 7.1.3, we will compare these different algorithms.

7.1.2 An algorithm in prime characteristic

If k has prime characteristic and the ideal I is prime, we can compute symbolic powers in yet another way.

Theorem 7.1. *Let k be a field of characteristic $p > 0$ and $R = k[x_1, \dots, x_n]$. If I is a radical ideal of pure height h in R, then for all $a \geqslant 1$,*

$$I^{(a)} = \left(I^{[q]} : I^{h(q-1)+1-a}\right),$$

where $q = p^e$ is such that $a \leqslant q$.

Proof. To show $(\subseteq)$, note that

$$I^{(a)} I^{h(q-1)+1-a} \subseteq I^{(h(q-1)+1)} \subseteq I^{[q]},$$

by Lemma 2.10. To prove $(\supseteq)$, we only need to show the containment holds after localizing at each associated prime of I, by Lemma 1.50 and Lemma 2.7. Since I is radical, I_P becomes the maximal ideal for each associated prime P. The assumption that I has pure height h guarantees that R_P has dimension $d = h$ for all P. Thus, it is enough to prove that given a regular local ring of characteristic $p > 0$ and dimension d with maximal ideal $\mathfrak{m}$, and a such that $a \leqslant q$, the following holds:

$$\left(\mathfrak{m}^{[q]} : \mathfrak{m}^{d(q-1)+1-a}\right) \subseteq \mathfrak{m}^a.$$

By [BS15, Lemma 3.2], for all $0 \leqslant k \leqslant d(q-1)$, we have

$$\left(\mathfrak{m}^{[q]} : \mathfrak{m}^k\right) = \mathfrak{m}^{[q]} + \mathfrak{m}^{d(q-1)-1-k}.$$

When $k = d(q-1) + 1 - a$, we have $d(q-1) - 1 - k = a$, and

$$\left(\mathfrak{m}^{[q]} : \mathfrak{m}^{d(q-1)+1-a}\right) = \mathfrak{m}^{[q]} + \mathfrak{m}^a.$$

On the other hand, since $a \leqslant q$, then $\mathfrak{m}^{[q]} \subseteq \mathfrak{m}^q \subseteq \mathfrak{m}^a$, so that in fact

$$\left(\mathfrak{m}^{[q]} : \mathfrak{m}^{d(q-1)+1-a}\right) = \mathfrak{m}^a. \qquad \square$$

Remark 7.2. Computing ordinary powers of an ideal can be fairly slow. Given a, one might take p such that $p^{e-1} < a \leqslant q$ to use the formula in Theorem 7.1.

Remark 7.3 (Characteristic 0)**.** Suppose that I is a radical ideal in $\mathbb{Q}[x_1, \ldots, x_d]$, whose generators are polynomials with integer coefficients, say $I = (f_1, \ldots, f_n)$. For each prime integer p, write I_p for the ideal generated over $\mathbb{F}_p$ by the reduction modulo p of the f_i. Fix n, and consider the symbolic powers $I_p^{(n)}$ for each p. If $p \gg 0$, a set of generators for $I_p^{(n)}$ may be lifted to $\mathbb{Z}[x_1, \ldots, x_d]$ to obtain generators for $I^{(n)}$. Theorem 7.1 can then be used to compute the symbolic powers I_p.

The issue, however, is determining what values of p to take.

7.1.3 Comparing algorithms

How do these algorithms compare? We will compare approximate times for each of these algorithms applied to the same ideals using Macaulay2 [GS] on the same machine. The algorithms we are comparing are based on the code in the `SymbolicPowers` package.

`pdec`: find the associated primes of I and a primary decomposition for I^n, and then intersected the components of I^n whose radicals agree with the minimal primes of I

```
pdec(Ideal,ZZ) := Ideal => (I,n) -> (assI := associatedPrimes(I);
      decomp := primaryDecomposition power(I,n);
      intersect select(decomp, a -> any(assI, i -> radical a==i)))
```

`sat`: take the saturation of I^n with respect to the irredundant maximal ideal, using the Macaulay2 function `saturate`.

`mon`: the algorithm for symbolic powers of squarefree monomial ideals given by Example 1.17.

```
mon(Ideal,ZZ) := Ideal => (I,n) -> (assP := associatedPrimes(I);
     intersect apply(assP, i -> i^n))
```

`charp`: the algorithm in prime characteristic given by Theorem 7.1.

```
charp(Ideal,ZZ) := Ideal => (I,n) -> (R := ring I; p := char R;
h := bigHeight I;
(e := ceiling(log_p n);
q := p^e; c := q-1; d := h*c-n+1; J:= I^d;
(frobeniusPower(I,q)):J))
```

The following tables contain an approximation of the time in seconds it takes to run each of these algorithms with the given ideals over the given fields. Fermat denotes the ideal $(x(y^3-z^3), y(z^3-x^3), z(x^3-y^3)) \in k[x, y, z]$, where k is the indicated field. The defining ideal of $k[t^a, t^b, t^c]$ in $k[x, y, z]$ is represented by (t^a, t^b, t^c). Finally, (xy, xz, yz) is simply the monomial ideal (xy, xz, yz) in $k[x, y, z]$, and in the monomial ideals table, the second ideal is generated by all squarefree monomials of degree 4 in 5 variables.

Note that the computations run faster for squarefree monomial ideals, in particular because Macaulay2 [GS] runs a more efficient algorithm to find their primary decomposition.

The `charp` algorithm can be extremely slow even for small powers if the characteristic is much larger than the power we are considering. For example, to compute $I^{(2)}$ over characteristic 101, we must find I^{98} and $I^{[101]}$. However, `charp` is faster when taking large powers that are close to the characteristic.

Ideal	Field	$I^{(2)}$			$I^{(3)}$			$I^{(5)}$		
		`pdec`	`sat`	`charp`	`pdec`	`sat`	`charp`	`pdec`	`sat`	`charp`
Fermat	$\mathbb{Q}$	.69	.003		1.37	.015		13.26	.067	
Fermat	$\mathbb{F}_7$	.89	.003	.095	1.25	.01	.12	3.81	.041	.12
Fermat	$\mathbb{F}_{101}$	.52	.003	>5400	.89	.044		3.98	.064	
(t^3, t^4, t^5)	$\mathbb{Q}$	.1	.004		.094	.004		.18	.045	
(t^3, t^4, t^5)	$\mathbb{F}_7$	.09	.0076	.088	.099	.006	.087	.12	.039	.13
(t^3, t^4, t^5)	$\mathbb{F}_{101}$	.08	.005		.087	.0058		.12	.043	>3600

Figure 7.1: Comparison of `pdec`, `sat` and `charp`.

Ideal	Field	$I^{(2)}$			$I^{(3)}$			$I^{(20)}$		
		`pdec`	`sat`	`mon`	`pdec`	`sat`	`mon`	`pdec`	`sat`	`mon`
(xy, xz, yz)	$\mathbb{Q}$	.068	.002	.016	.041	.003	.016	.14	.11	.045
	$\mathbb{F}_{101}$	.015	.0015	.012	.016	.002	.03	.07	.087	.013
all degree 2 in 5 variables	$\mathbb{Q}$	.037	.004	.023	.087	.017	.024	>5400		14.9
	$\mathbb{F}_{101}$	.065	.005	.019	.047	.046	.019		>5400	17.02

Figure 7.2: Comparing algorithms for monomial ideals.

		$p = 17$			$p = 19$			$p = 23$		
		`charp`	`sat`	`mon`	`charp`	`sat`	`mon`	`charp`	`sat`	`mon`
Ideal	Fermat	19	21		37	42		134	165	
	(t^3, t^4, t^5)	32	1		72	1.8		335	5.6	
	(xy, xz, yz)	18.6	.08	.004	38.6	.1	.004	129.6	.18	.004

Figure 7.3: Approximate times in seconds for computing $I^{(p)}$ in characteristic p.

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