Modeling and Measuring Systemic Risk in Financial Markets

by

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Abstract

A modern financial system has a complex and dynamic nature formed by a noncentralized set of decision-makers and evolves through their interactions. Through the resulting interdependencies, financial distresses may cause or intensify system-level losses for individual stakeholders or even lead to financial crises and/or economic depression. The principal aim of this dissertation is to establish a normative framework supporting policy-level decisions related to systemic risk in financial markets. The proposed framework provides the analytical tools and optimization formulations needed in the design of regulatory policies to prevent and mitigate systemic risk, addressing the inefficiency and unintended consequences of conventionally developed regulatory rules and policies.

We use a systems theoretic approach that can capture the systemic nature and behavior of financial markets towards an alternative normative framework on systemic risk in financial markets. We apply this approach for modeling and measuring systemic risk to provide a basis for an incentive-compatible regulatory design framework.

Keywords and phrases: Systems Theory, Systemic Risk, Regulatory Systems, Banking Regulations, Financial Intermediary, Interbank Networks, Incomplete Financial Markets, Microfoundational Modeling, Financial Contagion, Financial Crisis, Financial Distress, Capital Structure, General Equilibrium of Regulated Financial Markets

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To my lovely wife, Neda.

Contents

List of Symbols

The next list describes some of the main symbols that will be later used within the body of the document

Banking and Financial Intermediation

$\mathcal{N}$	Set of Banks in the financial system
y	Per capita investment in short (liquid) assets
x	Per capita investment in long assets
y_i	Bank i's per capita investment in short (liquid) assets
x_i	Bank i's per capita investment in long assets
R	Rate of return of long assets at t_2, $(R > 1)$
r	Liquidation rate of long assets at t_1, $(r < 1)$
$\boldsymbol{S}$	Space of all possible states
Σ	All the collections of the economic scenarios
$\boldsymbol{P}$	Probability measure equipped with $\boldsymbol{S}$
w	Random Variable of aggregate proportion of early consumers in the economy
$w(s)$	State contingent aggregate proportion of early consumers in the economy
$w_i(s)$	State contingent liquidity needs of bank $i \in N$ in a given state $s \in S$

$\overline{c^1}$ Face value of the consumption allocation to early consumers (t_1)

$c^1(s)$ State contingent consumption allocation to early consumers (t_1) at state $s \in S$

c^1 Random variable of the consumption allocation to early consumers (at t_1) defined on the probability space, $(\boldsymbol{S}, \Sigma, \boldsymbol{P})$

$c^2(s)$ State contingent consumption allocation to late consumers (at t_2) at state $s \in S$

c^2 Random variable of consumption allocation to late consumers (at t_2) defined on the probability space, $(\boldsymbol{S}, \Sigma, \boldsymbol{P})$

D_{ij} Deposit of bank i in bank j at t_0

P^s_{ji} Bank i withdrawal at t_1 from her deposits in bank j

$y_i^{exc}(s)$ Excess liquidity transferred by bank i from t_1 to t_2

$\alpha(s)$ Proportion of the long assets liquidated early at t_1

$\alpha_i(s)$ Proportion of the bank i's long assets liquidated early at t_1

Banking, Financial System, and Real Economy

h Representative household

$\mathcal{N}$ Set of Banks in the financial system

$\mathcal{F}$ Set of Firms in the financial system

$U^h(.)$ Representative household's utility function

w_0^h Endowment of Representative household

x_j^h Representative household's depositing in bank j, $\forall j \in \mathcal{N}$

y_f^h Investments of the representative household in firm $f \in \mathcal{F}$'s equity

R_f	Face value return rate of loans issued to fund the potential project of firm $f \in \mathcal{F}$
$\bar{R}_f^s$	state-contingent return rate of loans issued to fund the potential project of firm $f \in \mathcal{F}, \ \forall s \in \mathcal{S}$
y_f^j	loan contract issued by bank $j \in \mathcal{N}$ for the firms, $f \in \mathcal{F}$
ρ_f^s	The proportion of the face value of firm f's obligations that is paid at t_2, due to the limited liability provision, if $s \in \mathcal{S}$ realize
eq_0^j	Equity of bank, $j \in \mathcal{N}$
Res_j	Minimum reserve requirement for bank $j \in \mathcal{N}$ to be held in safe assets
L_j	Maximum leverage allowance of bank $j \in \mathcal{N}$ to invest in risky assets
x_i^j	Interbank deposit of bank $j \in \mathcal{N}$ in bank $i \in \mathcal{N}$
x_f^j	Investment of the bank $j, \ \forall j \in \mathcal{N}$ in loan issued to fund the potential project of firm $f \in \mathcal{F}$
l_j	leverage adjustment to the existing leverage allowance of L_j and thus the new leverage allowance for each bank would be $L_j^{new} = L_j - l_j$
α_j	Alternative notation for leverage adjustment where $\alpha_j = \frac{L_i - l_i}{L_j}$

List of Tables

Chapter 1

Introduction

In the literature of systems theory, it is well understood that increasing complexity of a system eventually leads to the emergence of phenomena or behaviors of a systemic nature that are not analytically tractable with standard methodologies, which ignore system-level constructs and processes. The 2007-09 financial crisis illustrated the emergence and growing significance of systemic behaviors and phenomena in financial markets and, therefore, their growing contribution to the overall dynamics and behavior of financial markets. The 2007-09 failure of conventional regulatory and policy-making approaches, even widely accepted standards such as Basel I and Basel II, should not be surprising, given that some of their most essential (regulatory) rules and policies have no particular scientific basis (Fabozzi et al. 2010).

During the last decade, an extensive body of literature has been developed re-thinking the standard approaches to modeling and analyzing financial markets, trying to explain the 2007-09 financial crisis as a direct consequence of so-called "systemic risk". However, the literature on systemic risk in financial markets never reached a consensus on a unified definition of systemic risk or how it relates to market failure phenomena originated from system-level features and structures. The lack of a unifying and clear conceptual framework and the pressing need to, at least partially, make sense of the financial crisis gave rise to reverse definitions that represent systemic risk through its symptoms. For example, it is conventional wisdom accepted across the literature that systemic risk corresponds or even equates with extreme events like financial crisis triggered by (an) exogenous event(s). Likewise, on the normative side, the same combination of conceptual gaps and social pressure for restrictive interventions gave rise to regulatory standards and policies explicitly developed to address systemic risk that are empirically rather than theoretically motivated (Allen and Gale 2007, Gale 2018). Early empirical and theoretical results[1] on the failure of more recent regulatory standards and policies, such as Basel III, which was specifically developed in response to the 2007-09 financial crisis and to address systemic risk, supports the impression that the trial and error approach and bare reliance on empirical analysis are not a reasonable basis for developing regulatory and policy-making solutions, at least with regard to systemic risk.

On the other hand, it is well known that external interventions of policies and regulatory rules have the great potential to trigger unanticipated changes in the dynamics of the system and may result in undesired system-level outcomes, even beyond the risks that the regulatory rules and policies aim to mitigate (for example see lemma 3.2.3, and corollary 3.2.6 in section 3.2.2). These inefficiencies, as well as the potential unintended consequences of conventional regulatory rules and policies, call for a comprehensive and critical assessment of our approach to systemic risk.

The principal aim of this dissertation is to establish a normative framework supporting policy-level decisions related to systemic risk in the banking section. We argue that the proposed framework provides the analytical tools and optimization formulations needed in the design of regulatory policies to prevent and mitigate systemic risk. We use a systems theoretic approach that is capable of capturing the systemic nature and behavior of financial markets. We apply this approach for modeling and measuring systemic risk to provide a basis for an incentive-compatible regulatory design

[1] For a discussion on empirical results showing the inefficiency and failure of the banking regulations see Chinazzi and Fagiolo (2013). For example, for theoretical inconsistencies in the Basel II regulatory standard see Rajaratnam et al. (2017). For some results on theoretical inconsistencies in the Basel III regulatory standard see section 2.1.4, and also corollary 3.2.6 in section 3.2.2.

framework.

Though our approach is based in systems theory, we have avoided the use of the highly abstract language and terminology of this field. Rather, we rely on the more widely known analytical tools and frameworks of finance, economics, and operations research. In some cases, we adapt the use of these tools to support a systems theoretic approach. For example, we use general equilibrium theory as an analytical mechanism to translate communications across the system borders, contrary to its common usage in finance and economics to model the system dynamics around a steady state.

1.1 Financial Markets as Complex Adaptive Systems

A modern financial system has a complex and dynamic nature formed by a noncentralized set of decision-makers and evolves through their interactions and the real economy. Through the resulting inter-dependencies, financial distresses may propagate in the system, which in turn may cause or intensify system-level losses for individual stakeholders or even lead to financial crises and/or economic depression. The primary role of a central bank (regulator and policy-maker) is to efficiently maintain the stability of the economy through system-level risk-mitigating policies, which, to be more specific, should aim to reduce the vulnerabilities and increase the financial system's ability to absorb unexpected events. From the regulatory and policy-maker's perspective, anything detrimental to the functionality of a system for social welfare and equity is undesirable.

In this dissertation, we focus on the risk-sharing functionality of the banking system in the face of uncertainties in financial markets. Thus, any structural feature of the banking system that affects the banking system's risk-sharing functionality is considered undesirable. Likewise, the losses that may realize/emerge due to such dysfunctionalities are referred to as systemic losses. Note that here we are framing the problem from the regulator's/policy-maker's perspective. Thus, we limit our scope to social welfare and equity losses that could be avoided by a properly functioning system (not necessarily a banking system).

Through the use of a generic framework that relates systemic structures and features of the banking system to loss exposures of social welfare and equity, we can formally define what we mean in this dissertation by "systemic risk" to be any loss exposure of social welfare and equity due to system-level risk sharing dysfunctionality of the banking system. We use this definition not only in its positive capacity but also for developing a framework that can address systemic risk in a normative way.

To develop a normative framework, we need to start with a proper understanding of how the systemic structures and features of a banking system relate to social welfare and equity loss. Such understanding is the basis for considering how to design a practical set of normative actions from the regulator/policy-maker perspective that directly or indirectly address the recognized or even hidden dysfunctionalities. A proper normative approach also requires a proper measuring framework within its evaluation mechanism that relates the structure of the system to the market parameters and decision variables of different stakeholders including bankers, consumers, firms, and especially a policy-maker/regulator. Related issues are discussed in the following sections.

1.2 Toward a Normative Framework for Systemic Risk

Developing a theoretically grounded normative framework to address systemic risk requires a modeling and measuring framework that relates the structural and system-level features of the system to possible system-level loss exposures. In other words, the capacity of a normative framework to adequately capture and efficiently address systemic risk depends on how the feed-back and feed-forward signals of its intervention mechanisms relate to the structural and system-level features of the banking system as well as its interactions with the general economy to the possible emergence of undesirable system-level phenomena. It is also essential to have a proper evaluation mechanism and thus a measuring process that determines what undesirable means[2] from the perspective of a

[2] For example, in the literature of systemic risk, financial contagion is considered as an undesirable systemic phenomenon. Thus, the risk of the emergence of financial contagion and the losses associated with that are considered a systemic risk. However, in this dissertation we do not directly relate any

given stakeholder (here the regulator) and its relation to the stakeholder's intervention signals. Thus, we treat the modeling and measuring of systemic risk in financial markets as a design problem that aims to create a theoretically grounded normative capacity for regulatory and policy-making frameworks on systemic risk.

Next, we discuss our approach to define and formulate a systemic risk measure as a basis of an appropriate assessment mechanism in a prescriptive framework. It leads to a discussion on how any (systemic risk) measure is (implicitly) derived from an underlying model and thus what is the best modeling approach that could be used as a basis of our systemic risk measure[3].

1.2.1 Measuring Systemic Risk in a Normative Framework

Risk measures are formulated to quantify some uncertainty in terms of a type of loss exposure. To make sense of what a specific risk measure quantifies, we should have a clear understanding about the scope of the uncertainty it captures as well as the type of loss it is concerned with. However, the main drawback of the existing systemic risk measures is that they do not capture how uncertainty through a structural or systemic feature of the system relates[4] to a system-level loss exposure. In other words, these measures do not relate the loss exposures they quantify to the (systemic) structure behind the emergence of these losses.

Furthermore, what we aim to achieve using the systemic risk measure has a definitive role in its formulation. Therefore, let us explicitly state that the systemic risk measures developed in this dissertation are aimed to provide a basis for the evaluation

specific phenomena to be systemic. Instead, we use the notion of system-level loss that could be associated with systemic features and structures as a definitive basis of what systemic risk means and how it should be modeled, measured, and incorporated in any normative approach.

[3]As it is supposed to be used as a basis of the evaluation mechanism of a normative framework.

[4]It is not straightforward to develop a comprehensive model that captures how system-level features and structures relate uncertainty to system-level loss exposures even in aggregate. It becomes even more complicated if loss exposures are defined from the perspective of a specific system or an individual in interaction with but external to the primary system.

mechanisms used by a normative framework for systemic risk. A mechanism to evaluate the structure of a system, its components as well as the interactions of the components in terms of their contribution to the emergence of undesirable system-level phenomena. Thus, such a measure should be formulated based on a model of the system that relates system-level features and structures to system-level losses and therefore provides us with a theoretically grounded basis for an evaluation mechanism in a normative way.

Next, we discuss how we developed a modeling framework capable of providing the underlying structure needed for such a measuring process and other features required for a proper normative framework for systemic risk.

1.2.2 Modeling Systemic Risk for a Normative Framework

Following the classical steady-state approach in systems theory, to explain an observed phenomenon, we should develop a theoretical formulation of the system and use analytical tools. In the case of systemic risk in financial markets, micro-foundational modeling of a banking system would be a proper starting point. Then, general equilibrium theory can be used to explain how the structural features of the banking system relate to system-level phenomena like financial contagion in a positive way, which in turn helps us to develop normative frameworks to efficiently address the underlying dynamics of systemic risk.These are the steps that should be taken to develop and justify a regulatory and policy framework in order to keep it consistent with the core features of systems theory.

Unfortunately, as shown in chapter 3, the existing micro-foundational models are not yet capable of relating phenomena like financial contagion to any specific market failure dynamics of the banking system. A path forward following the classical steady-state approach would be to extend the existing theoretical models by including more structural features, and/or relaxing more simplifying assumptions up to the point that the analytical results provide us with an explanation of how a systemic feature of the banking system relates to the observed system-level loss expusure. We take this path in chapter 3, as we extend the latest theoretical modeling framework of the banking system with a focus on the risk-sharing functionality of the system. In particular, we relaxed the "no-aggregate uncertainty" and "pecking order" assumptions made in the existing formulations and showed that even with these extensions, we could not explain the emergence of financial contagion.

One may continue with more extensions and finally get enough of the complexities formulated in the model to capture the system-level dynamics that lead to the emergence of, say, financial contagion. However, there are three main disadvantages to theoretical models which go beyond the formulation of the main characteristics and principles of a complex system:

1. There are some dynamics and system components that are not observable and thus we can not explicitly formulate them in the theoretical models. For example, the individual-level decision-making preferences, risk profiles, and behavioral dynamics and dependencies are not observable.

2. Even if we could explicitly formulate the heterogeneous components of the system, it would be both computationally and analytically in-tractable. Indeed, high fidelity multiagent models as well as numerical simulations of stochastic dynamic general equilibrium models have shown that the nonlinearity inherent in the structure of the banking system makes it really difficult to get conclusive and analytical results in a level that is usable for policy-making and regulatory purposes.

3. The evolution of many observable and non-observable aspects of the financial system rapidly outdates any theoretical model that goes beyond the very basic features and principals of the system.

Thus, in chapter 5, we take a different approach than the conventional theoretical framework. Instead of representing a system by a theoretical model, we use a theoretical model to give a structure to an observed realization of the system. Such a "model-driven state representation" of a system only captures the dynamics of the banking system to the extent formulated in the theoretical model. Anything beyond the model's scope stays hidden but present in the parametric structure of the representation. This approach provides us with the opportunity to take advantage of some generally simple theoretical results to design interaction mechanisms with the system that controls the dynamics of the hidden parts of the system constant and thus makes it possible to rely upon the dynamics of the modeled parts of the system as an underlying structure that governs the overall system's response to those interactions.

To be specific, on the systemic risk of financial markets, we used an explicit formation of the bankruptcy law to give a structure to an observed state representation of the banking system. This representation formulates how the bankruptcy law connects the banks' balance sheets in a given state. Instead of incorporating a general equilibrium model of a heterogeneous banking system in our formulation, we reorganize the parametric structure of the formulation to capture some of the known dynamics of the financial general equilibrium model. Specifically, our state representation reformulates the observed data of the banks' obligations to each other in terms of the banks' proportional investment (decisions) in each other. Thus, the resulting bankruptcy law formulation of the banking system has a two-layer parametric setting that simultaneously captures the obligation of the banks to each other as well as their proportional investments in each other. The proportional investment, although not enough to model the system-level dynamics behind the banks' decision-making structure and equilibrium formation, makes it possible to take advantage of some analytical results we already have from theoretical models to interact with the system in the given state without neglecting the hidden part of the steady state dynamics.The analytical basis we rely on to keep the dynamics of the hidden parts of the system constant while we interact with the modeled parts of the system are derived from the following decision-theoretical result in the financial mathematics literature: if nothing other than the size of the available funds of an investor changes, then under very generic assumptions, we can expect her proportional investments to remain the same, no matter what is her decision-making structure. Using this theoretical basis, our formulation of the banking system captures the dynamics of the general equilibrium model behind the formation of its observed state if the external interactions with the system are limited to changing only the size of the funding available to the banks while the market parameters are kept constant. In other words, any regulatory and policy-making interactions that do not affect any decision-making parameters involved in the bankers' investments other than the size of the funds available to each bank could be analyzed based on our formulation and specifically in terms of the system-level risk effects of such policies and regulations.

To simplify the modeling, we initially limit the feasible regulatory interventions to a set of requirements that do not change the overall size of the supply and demand of each asset in the financial market. With this, we are implicitly assuming that the market prices of the assets in financial markets do not change if the size of their supply and demand are kept intact, and thus the proportional investments of the bankers remain the same. This assumption ignores the fact that the asset pricing in financial markets goes beyond the simple supply and demand sizes and may depend on the capital structure of the banks and the differences in their risk profiles.

To generalize the initial modeling framework that incorporates more complicated market dynamics that govern the bankers' investment decisions, we formulate a general equilibrium model of the loan and equity market in the second part of chapter 5. The equilibrium model formulates a representative consumer, a set of heterogeneous banks, and a set of heterogeneous firms. We do not directly solve or analyze this model due to the difficulties and limitations discussed earlier. Instead, we use this formulation as a proxy to relate the modeled part of our model to the hidden structure of the system. As we will show in the second part of chapter 5, if we consider the observed market state as a solution of the equilibrium model, the equilibrium model can be used to derive the conditions required to guarantee that all market features remain the same while the size of the available funds to each individual bank changes. With this, we can relax the initial assumption made in the first part of chapter 5, which provides us with the theoretical guarantee needed to reliably assume that the banks' proportional investment decisions remain the same if the regulatory interventions satisfy the conditions derived from the general equilibrium model.

Finally, we use this framework to design and optimize a regulatory and policy-making intervention mechanism that adjusts the size of each bank in a controlled way without interrupting other features of the market[5] as an example to show how a normative framework for systemic risk could be developed based on our framework.

1.3 Preliminaries

In this dissertation, we use a collection of concepts borrowed from different disciplinary areas. Some are used across multiple fields but may have different names or have defined differently. To keep the dissertation more easily comprehensible for readers with different backgrounds, we briefly review here some of the main concepts and definitions used in the following chapters.

1.3.1 State-Contingent Contracts and Arrow-Debreu Securities

A state-contingent security or contract pays one unit of consumption in one state of nature and nothing otherwise. An state-contingent security is also called Arrow-Debreu security, referring to the contributions of Arrow and Debreu (1954) who extended the general equilibrium analysis to incorporate *time* and *uncertainty* by introducing an event-tree to describe the uncertainty and a structure of markets in which date-event contingent commodities can be traded at an initial date. This model has come to be known as the Arrow-Debreu model. In the Arrow-Debreu model, a large collection of contingent contracts—one for each good for each possible date-event in the future—is traded at the initial date and thereafter no further trade occurs; agents simply deliver or receive delivery on the contractual commitments made at the initial date.

1.3.2 Complete and Incomplete Markets

Markets are said to be complete if there is a state-contingent security for every state of nature at the following date. Pareto optimal allocations of an Arrow-Debreu equilibrium

[5] As we see in chapter 5, the regulatory design framework makes sure that regulatory interventions satisfy the conditions derived from a general equilibrium model

can be achieved in complete markets by the equivalent market structure in which agents trade, at each date-event, spot contracts calling for the current delivery of each good, and Arrow securities for the delivery of income at each of the contingencies at the following date.

Contractual Incompleteness

In an ideal world, people can write a complete contingent contract that induces all parties to take the 'right' actions in every possible state of the world, which leads to a Pareto efficient outcome. Then, when contracts do not deal with all relevant contingencies, they are incomplete. The parties' ability to write such a contract might well be limited for the following reasons:

- Unforseen contingencies: "Parties cannot define the ex ante contingencies that may occur later on."
- Cost of writing contracts: "Even if one could foresee all contingencies, they might be so numerous that it would be too costly to describe them in a contract."
- Cost of enforcing contracts: "Courts must understand the terms of the contract and verify the contracted upon contingencies and actions in order to enforce the contract."

For the above reasons, the contracts could be incomplete in the sense that they leave out what to do in some contingencies.

From a theoretical perspective, incomplete markets complicate the study of financial market equilibrium, portfolio optimization, and derivative securities. Although the theory of derivative securities in complete markets is understood very well and is the subject of numerous textbook accounts, there is as yet no fully developed, sound theoretical framework for incomplete markets.In the financial economics literature, market completeness often is associated with equal inter-temporal rates of substitution among agents (see Saito 1999, Anderlini and Felli 1999) and thus a calibrated representative agent model would reflect aggregate preferences, and microeconomic data would show that households are capable of fully insuring themselves against idiosyncratic risks.

The necessary and sufficient conditions on individuals' utility functions for all Pareto optimal allocations to be achievable by holding the portfolio of all assets and borrowing or lending are discussed, as well as the relationship between these conditions and an aggregation result in different settings.

1.3.3 Pareto Efficiency

A solution of a multi-objective optimization problem is Pareto efficient if there does not exist any other solution that can strictly improve one of the objective functions without worsening at least one other objective function. This is equivalent to the Pareto efficiency concept in economics that is merely defined in a different way.

The Pareto efficiency of an economy generally means that there is no alternative (feasible) economy that improves the utility of an stakeholder without worsening another stakeholder's utility. For example, an allocation of state-contingent contracts is said to be Pareto optimal or Pareto efficient if it is feasible and if there do not exist other allocations which are feasible and can strictly increase at least one individual's utility without decreasing the utilities of others. For example, an allocation that gives a single individual all the consumption available and others nothing is a Pareto optimal allocation.

1.3.4 Social Planner as a Theoretical Device

In welfare economics, a social planner is a hypothetical decision-maker who has access to all information and makes a decision on behalf of all parties aiming for the collectively best outcome through maximizing a social welfare function[6]. Modern welfare economics focuses on how the social planning problem relates to competitive equilibrium models

[6] Social planner and social welfare function are purely conceptual and in reality there are no clearly articulated social welfare functions in any economy.

based on the first and second welfare theorems (Mankiw and Mankiw 1998).

The first welfare theorem tells us that if an allocation and a set of prices constitute a competitive equilibrium, then the allocation will be Pareto efficient.

This tells us that the competitive equilibrium is efficient; the only option we have is to redistribute endowments in order to achieve a different point on the contract curve, but there is nothing more we can do.

The second welfare theorem tells us that any social planning problem can be decentralized as a competitive equilibrium. In other words, a social planner can achieve any Pareto-optimal outcome by an appropriate redistribution of wealth by means of a competitive market. The idea of social planning encompasses not merely planning in the domain of welfare activities and social services, but also social aspects of physical, fiscal, or economic plans, and constitutes one necessary axis in comprehensive planning (Nehnevajsa and Kahn 1971).

1.3.5 Constraint Efficiency

It is tempting to conclude that a market structure has failed in case the market outcome is not Pareto-efficient; but this assumes that the planner is not subject to imperfections like transaction costs and other frictions that prevent markets from being complete. Before we decide that the market has failed and that some intervention is required, we should ask whether the central planner could do better if he were constrained to use only the trading opportunities available to the market participants. For example, it is clear that a planner can improve on the laisser-faire allocation by transferring goods from intermediaries whose depositors have a low marginal utility of consumption to intermediaries whose depositors have a high marginal utility of consumption. In doing so, the planner is performing the function of the missing markets for Arrow securities that allow intermediaries to transfer wealth across states and achieve the first best. However, if the market participants are prevented by transaction costs or other frictions from making these trades, perhaps the planner will be too. This suggests that the appropriate test for market failure is to ask whether a planner could improve on the laisser-faire allocation using the same technology available to the market participants.

1.3.6 Solvency, Insolvency, and Bankruptcy

A bank is said to be *solvent*, if it can meet the demands of every depositor who wants to withdraw (including banks in other regions) by using only its liquid assets, that is, the short asset and the deposits in other regions. The bank is said to be *insolvent* if it can meet the demands of its deposits but only by liquidating some of the long asset. Finally, the bank is said to be *bankrupt* if it cannot meet the demands of its depositors by liquidating all its assets.

1.4 Structure of the Dissertation

In chapter 2, we discuss the main ideas behind banking regulations and financial markets to provide a basis for our normative and positive discussions on what systemic risk is and thus what is a proper modeling and measuring approach to capture its main characteristics and dynamics. This includes a short critical review of the latest banking regulations and a discussion on how the lack of a unified theoretical ground in developing these regulations has unintended consequences for the banking system and the general economy. On that front, we make an example based on the Basel III framework to show how the interaction of regulatory rules, neglected in the development of this framework results in generic inconsistencies and has unintended but significant system-level consequences. In the second part of chapter 2, we critically review the literature of two modeling approaches on systemic risk with a brief discussion on what are their theoretical strengths and shortcomings.

In chapter 3, we extend the general equilibrium model of a banking system with an interbank deposit market as a financial intermediary to incorporate aggregate uncertainty and standard deposit contracts. We show that an equilibrium behavior of a competitive banking system decentralizes the social planner's allocation under the same assumption without any regulation imposed and therefore it is constraint efficient. This shows that the financial contagion is not an equilibrium feature of the decentralization functionality of the banking system even under aggregate uncertainty and standard (incomplete) deposit contracts.

In chapter 4, we address a mathematical limitation that arises in chapter 5 in the formulation of the marginal value function of the leverage allowance problem. Similar problem arises in the the analytical studies on stability, local controllability, and comparative statics of optimization and control problems, where it would be misleading to use generic formulations of the directional derivative of the value functions in nonconvex optimization problems if under degeneracy.

In chapter 5, we turn our critical reflections in previous chapters on the conventional approaches towards an alternative normative framework on systemic risk in financial markets. We initially develop a modeling formulation of banking system in correspondence with the real economy where bankruptcy law and investment decisions forming the structure of the banking system simultaneously captured in a two-layer parametric setting simultaneously. We use this modeling framework to derive a measure of systemic risk as a means to evaluate the components of the system, their interactions, and the structure of the system in terms of their contribution to the emergence of undesirable system-level phenomena. This goes beyond a measure for monitoring the state of the system and aims to be used in a model-based evaluation mechanism capable of relating potential interventions to structural changes and thus our normative aims. This leads to our discussion on how considering the proportional investment decisions of different stakeholders of the financial markets as the building blocks of the realized structure of the banking system presents a framework to design a system-level tractable adaptive capacity that could be used in a normative way and as a basis of a regulatory framework on systemic risk. The proportional investments, although not enough to model the system-level dynamics behind the banks' decision-making structure and equilibrium formation, makes it possible to take advantage of some analytical results we have from theoretical models to interact with the system in the given state without neglecting the hidden part of the steady-state dynamics. This provides us with a theoretical basis to design and take advantage of the banking system's adaptive capacity, developing a regulatory setting that systematically adjusts the banks' existing regulatory requirements as an interaction mechanism with the banking system towards our normative aims addressing systemic risk.

We initially develop and optimize a leverage allowance adjustment framework (LAA under proportionality assumption) as a regulatory setting, assuming that if the regulator changes the leverage allowance of a given bank, the banks' proportional investment decisions do not change. However, to provide some basis for this assumption, we limit the regulator's actions to ensure that the size of the aggregate funding supply does not change after the adjustments. Finally, to extend the initial LAA framework to be usable in a more generic setting, we develop a general equilibrium representation of a regulated financial market as a basis to derive the sufficient conditions that guarantee that the proportionality assumption holds in a more general setting. The sufficient conditions of the extended LAA framework (LAA under sufficient conditions) restructure and reduce the complexity of the banking system's adaptive capacity as it provides us with a market-based mechanism to control for the dynamics governed by the hidden structure of the banking system and simultaneously use the regulatory intervention to restructure the observable structure of the banking system. This reduces the complexity of the system significantly as it provides us with a basis to rely on the market-based interaction mechanisms developed based on our framework and leveraging the banking system's observable structure toward our normative aims and without neglecting the hidden structure of the system.

Chapter 2

Preliminary Analysis and Critical Literature Review

In this chapter, we discuss the main ideas behind banking regulations and financial markets to provide a basis for our normative and positive discussions on the nature of systemic risk and ways in which it can be modeled and measured. The discussion includes a short critical review of the latest banking regulations and a discussion on how the lack of a unified theoretical ground in developing these regulations has unintended consequences for the banking system and the general economy. On that front, we make an example based on the Basel III framework to show how the interactions of the regulatory rules, neglected in this framework's development, result in generic inconsistencies and have unintended but significant system-level consequences.

Next, we review the literature of different approaches and methodologies to model, measure, and manage systemic risk, categorizing them into two groups. The first group takes a structural modeling approach using contingent claims analysis of the financial institutions' liabilities. The second group takes a macro-finance modeling approach through reduced-form economic models and measures of systemic risk.

2.1 Banking Regulations and Financial Markets

It is well known in the literature of financial economics that if a complete financial market (as defined in section 1.3.2) was a practical possibility, then the market mechanisms would allocate the resources most efficiently. In that case there would be no need for any intermediary such as a banking system in financial markts . Next, we discuss how the banking system can be understood as a (partial) alternative to a complete market, followed by a brief discussion on the necessity of banking regulations.

We propose a systems theoretic perspective to categorize the potential issues that motivate policy and regulatory interventions. Finally, to show how the conventional perspective fails as a basis for developing a proper and unified regulatory framework, we discuss a fundamental inconsistency in the setting of the latest and widely accepted standards that are developed based on the conventional perspective.

2.1.1 Banking System's Role and Functionality

The main rationale behind the existence of any intermediary system in financial markets is to reduce the incompleteness of the market and increase the efficiency of the resource allocation as much as possible. The banking system is a decentralized intermediation alternative that provides a channel of interaction between households and the real economy in a practical way aiming for the (partial) realization of the complete market dynamics. The essential features of the banking system that characterize the intermediation and thus interaction dynamics are associated with the liquidity and credit transformation functionality of the banking system through the maturity mismatch resolution of financial contracts. In other words, credit transformation provides funding for riskier and long-term assets, whereas liquidity transformation produces more liquid and short-term liabilities. It is essential to understand that proper credit and liquidity transformation rely on the efficiency of the resolution of the maturity mismatch between the assets and liabilities created by the banks' interactions with their customers and the real economy considering the uncertainty and heterogeneity of the customers and real economy[1]. This aligns with the main characteristics of complete markets where the high fidelity interaction increases the short-term liquidity and long-term funding through the risk-sharing functionality of the complete contingent contracts.

2.1.2 Regulatory Rational

The decentralized nature of the banking system, and thus the possibility of market failure, especially in the face of high uncertainty and heterogeneity of the financial markets, may require regulatory and policy interventions to increase the system's efficiency and reduce its externalities to other systems. However, even with the extensive literature on the details of the highly complex [2] regulatory frameworks in place, there is no theory-based rationale to unify the different components of the banking regulations that can address the following questions:

- What market failure necessitates the imposition of regulatory requirements?
- Why can't the market be left to determine the appropriate levels of liquidity and capital?
- What is a proper role for a regulatory framework, at least from a theoretical point of view?

To analyze the necessity and functionality of a regulatory framework, we categorize the potential issues that motivate external interventions into two categories:

1. The structural characteristics of the banking system in its fundamentals may result in inefficiencies and externalities. The main question here is if the functionalities of the banking system are capable of (at least from a theoretical point

[1] If a bank fails to properly resolve the maturity mismatch by holding an optimal level of liquidity, it is exposed to a substantial liquidity shortage. Similarly, if a bank fails in the proper diversification of the credit transformation, it is exposed to capital shortage.

[2] The complexity of the current regulatory framework is, for the most part, the result of rules created ad hoc to address emerging risks that threaten to disrupt financial markets (Basel Committee on Banking Supervision 2016).

of view) increasing the efficiency of the market or even replicating the efficiency of the complete market? And if not, could we use regulatory and policy-making interventions to improve the system's performance at a systemic level?

2. Practical limitations within the banking system and in interaction with other systems may be reducing the system's efficiency and creating externalities. The main question here is whether the functionality of the banking system is impaired by practical limitations and imperfections? If that is the case, could we use regulatory and policy-making tools and interventions to support the system achieve higher levels of its intended functionality?

On the other hand, the mainstream literature categorizes the issues that motivate regulatory and policy-making interventions from a different perspective (Dewatripont and Tirole 1994a,b):

1. The failure of banks, especially if it has system-level consequences resulting in the financial market's instability or even financial crisis. Most of the literature would relate these types of issues to systemic risk as a kind of market failure; however, there is no consensus about many aspects of what systemic risk is.

2. The risk exposure of consumers motivates regulations that are aimed at protecting individuals from extensive exposure to risk. Interest rate ceilings on loans and conflict of interest rules are examples of such regulations and policies. Some more generic rules and policies are also associated with consumer protection in the literature due to their efficiency increasing functionality.

3. Broader social and economic concerns and objectives may be used as a rationale behind some policies and regulations, such as prevention of money laundering using the reporting requirements for large cash transactions have pointed to another category of rationale for justifying banking regulation.

4. Since depositors have limited capacity to monitor if the bankers' actions are aligned with the depositors' interests or not, some regulations may be required to

ensure that bankers act in the interests of their depositors.

In this thesis, we use the first categorization as a basis of our modeling and analysis of systemic risk, even though it has a higher level of abstraction compared to the second one, which represents the mainstream perspective on regulation and policy-making. The next chapters would be focused on why. Before getting to the details of why our perspective is a better alternative, as discussed in the next chapters, it is useful to briefly[3]review the structure of the most recent rules, developed based on the mainstream perspective on banking regulations and aimed to prevent systemic risk. We also discuss one of the complications that arise from the mainstream setting due to the negligence of the systemic interactions of different components of the regulatory framework.

2.1.3 Stylized Model of the Banking Regulations

Even though there is no agreement about the nature of systemic risk as a kind of market failure, it is widely accepted that a combination of policies like capital adequacy ratios, liquidity requirements, reserve requirements, deposit insurance, and asset restrictions, if used properly, could limit systemic risk. However, the development and calibration of these regulations have largely proceeded independently. There have been limited attempts incorporating the interactions of these rules in the development of a unified regulatory framework[4]. To illustrate why it is important to explicitly consider the interactions of the rules in developing a regulatory framework, in this section, we briefly discuss the development of the Basel Accords, which are the latest and widely accepted regulatory frameworks. Basel Accords have been developed based on the conventional

[3]There have been a number of good surveys and overviews of banking regulation. These include Herring and Santomero (2000), Santos (2001), Freixas and Santomero (2004), Barth et al. (2005, 2013, 2020). Therefore, this section will be focused on the latest developments in banking regulations reflected in the Basel Accords and specifically Basel III.

[4]There are some literature reviews on the interactions of different regulatory components from a normative perspective. These studies mostly rely on how an existing regulatory framework could be optimally calibrated. There is not so much work done on how should a regulatory framework develop considering the systemic interaction of its different components in an economic setting

perspective and are mainly aimed at preventing systemic risk, especially in the latest Accord of Basel III. These accords provide an example of regulations that are empirically rather than theoretically motivated and miss the theoretical basis to support that the interactions of the proposed rules are aligned with the regulatory and policy-making intentions. Despite the limited literature on the full interactions among the proposed set of rules in these accords, it is already known that the interaction dynamics are inconsistent and may result in unintended consequences and even self-defeating of the regulations. To illustrate such an inconsistency in the fundamental setting of these rules, we use a simple formulation of the most recent policies proposed by Basel III in a stylized setting and show how explicit redundancy appears in it. Later in chapter 3, we prove that such a redundancy has a destabilizing effect on the liquidity pricing, even under no aggregate uncertainty.

Basel Accords and Bank Structures

In describing the early days of the Basel Committee, Goodhart (2011) notes that the original intent was to have a liquidity requirement to complement the capital requirement. This is aligned with our earlier discussion on the main functionalities of a banking system[5] and how regulatory should be developed assuring that these functionalities are performed as intended. The latest Accord, Basel III, includes two capital regulations[6] and two liquidity regulations[7]. We can denote the different components of an extended banking (balance sheet) structure and the related regulatory parameters as summarized in table 2.1 as defined in the latest version of Basel III (see Basel Committee on Banking Supervision 2020b, Cecchetti and Kashyap 2018). To simplify the formulation of regulatory requirements, let us define the aggregate risky assets of a bank, denoted by A combining all on and off-balance sheet risky assets. Similarly, let us define the aggregate short-term (deposits) and long-term (bonds) liabilities denoted by D and B. We can also drop the indexes of the regulatory parameters for the aggregate balance

[5]To provide liquidity and reduce the investment risk through sharing it

[6]The capital adequacy and leverage ratio

[7]The liquidity coverage ratio and the net stable funding ratio

ASSETS	LIABILITIES
Regulatory parameters	Regulatory parameters
$\boldsymbol{R}$: Safe assets (HQLA[8])	$\boldsymbol{D_s}$: Deposits (runable liabilities)
	l_s^D: LCR run-off rate of D
$\boldsymbol{A_i^{\text{On}}}$: On-balance sheet risky asset $\boldsymbol{i}$	a_s^D: rate of availability of the stable funding of D
r_i^{On}: risk weight of asset i	
f_i^{On}: stable funding rate required for risky assets	
	$\boldsymbol{B_k}$: Bonds (long-term liabilities)
$\boldsymbol{A_j^{\text{Off}}}$: Off-balance sheet risky asset $\boldsymbol{j}$	a_k^B: availability rate of stable funds for long-term liabilities
r_j^{Off}: risk weight of asset j	
l_j^{Off}: LCR run-off rate of off-balance sheet assets due to their contingent liabilities	$\boldsymbol{E}$: Equity

Table 2.1: Banking Extended Balance Sheet Structure and Regulatory Parameters

sheet items. Then the simplified formulation of a baking structure and the regulatory parameters can be summarized in table 2.2.

Note that in table 2.2, without loss of generality, we adjusted the aggregated LCR run-off rates of off-balance sheet assets[10] , l_j^{off}, to get the LCR run-off rate of aggregate risky assets denoted by l^A. We briefly discuss the four main regulatory requirements introduced in Basel III, categorized under capital and liquidity requirement rules, followed by their stylized formulation. Later, we use the stylized formulation of the rules to study the interactions of the policies so that we can examine the overall consistency of the regulatory system.

[10]The proportion of risky assets that are exposed to contingent liabilities

ASSETS	LIABILITIES
Regulatory parameters	Regulatory parameters
$\boldsymbol{R}$: Safe assets (HQLA[9])	$\boldsymbol{D}$: Deposits (runable liabilities)
	l^D: LCR run-off rate of D
$\boldsymbol{A}$: On and Off-balance sheet risky assets	a^D: rate of availability of the stable funding of D
r^{A}: risk weight of risky assets	
f^{A}: stable funding rate required for risky assets	$\boldsymbol{B}$: Bonds (long term liabilities)
	a^B: availability rate of stable funds for long-term liabilities
l^{A}: LCR run-off rate of aggregate risky assets	
	$\boldsymbol{E}$: Equity

Table 2.2: Simplified Balance Sheet Structure and Regulatory Parameters

Capital Regulations

Capital requirements generally set a minimum level of capital that a bank must maintain in relation to its assets. In the literature on capital adequacy, it is often argued that capital adequacy requirements are necessary to control the moral hazard problems generated by the existence of deposit insurance[11]. These rules may take the form of a simple fraction of the assets or a more complicated formula.

The first Basel Accord imposed uniform capital adequacy requirements on the banks of all signatory countries. The second Basel Accord (Basel Committee on Banking Supervision 2006) introduces more sophisticated methods of determining the appropriate level of capital for banks, but the idea that banks must be compelled to hold the appropriate level of capital remains a basic principle of the regulatory system[12]. Basel

[11] Deposit insurance was introduced in the 1930s to prevent bank runs or, more generally, financial instability. Because banks issue insured debt-like obligations (e.g., bank deposits), they have an incentive to engage in risk-shifting behavior.

[12] These rules, which are designed to ensure sufficient buffers should banks face losses, are the outgrowth of decades of experience dating to the original agreements in 1975(see on Banking Supervision

III (Basel Committee on Banking Supervision 2020a) includes two capital regulations, the capital adequacy, and the leverage ratio requirement focusing on high-quality capital, predominantly in the form of shares and retained earnings that can absorb losses[13]. These two rules ensure that a bank's equity is higher than a fraction of the sum of assets, weighted or unweighted by their riskiness:

1. Risk-weighted Capital Requirement

   The size of a bank's capital as a risk buffer should be proportional to the exposure of the bank's assets to potential losses. In other words, banks should hold higher equity if they have invested in riskier assets. This rule requires a bank to maintain specified minimum levels of regulatory capital[14], determined by the regulator and denoted by c, with each level set as a percentage of risk-weighted assets. Using the simplified[15] notations and definitions introduced in table 2.2 we can formulate risk-weighted capital requirement by

$$\frac{E}{r^A A} \geq c \tag{2.1}$$

   Basel III initially was more focused on how the regulatory capital,E, should be determined in the capital ratio calculation. The 2017 reform tried to restore the credibility of the risk characteristics of each type of asset[16]focusing on the risk weights in capital ratio calculation. The 2017 reforms also aimed at improving the comparability of the banks' capital ratios.

2011).

[13]The new features include specific classification criteria for the components of regulatory capital. (Basel Committee on Banking Supervision 2020c)

[14]Common Equity Tier 1, Tier 1, and total capital (see Basel Committee on Banking Supervision (2020a))

[15]The equivalent formulation, before simplifying the balance sheet structure, using the notations and definitions introduced in table 2.1 for the risk weighted capital requirement would be:

$$\frac{E}{\left[\sum_i r_i^{\text{on}} A_i^{\text{on}} + \sum_j r_j^{\text{off}} A_j^{\text{off}}\right]} \geq c$$

[16]As an indication of how risky it is for the bank to hold the asset.

2. Leverage Ratio Requirement

   Since risk-based capital ratios, even when they were set seemingly high enough, did not prevent the banks' excessive leverage built up during the 2007-09 financial crisis, the leverage ratio rule as a simple, non-risk-based "backstop" was introduced within the Basel III framework to supplement the risk-based capital requirements[17].

   Leverage ratio requirements ensure that the regulatory capital of a bank, E, proportional to the aggregate leverage of a bank (the sum of its assets regardless of their riskiness), is higher than a minimum leverage ratio. Using the notation and definitions introduced in table 2.2 we can formulate leverage ratio requirement by[18]:

$$\frac{E}{R+A} \geq \delta \tag{2.2}$$

   where δ denotes the minimum leverage ratio set by the regulator.

Remark 1 *Contrary to the risk-weighted capital requirement that forces banks that have riskier assets to hold more capital, the leverage ratio requirement ties the level of capital to the overall size of the bank (including off-balance sheet items) regardless of the risk weight of the assets.*

Liquidity Regulations

Basel Committee's key reforms in Basel III include a set of rules developed to improve the banking sector's ability to absorb liquidity shocks arising from financial and eco-

[17] The 2017 reforms made some adjustments to the implementation of this rule focusing on Global systemically important banks (G-SIBs) and required them to keep a higher leverage ratio.

[18] The equivalent formulation, before simplifying the balance sheet structure, using the notations and definitions introduced in table 2.1 for leverage ratio requirement would be:

$$\frac{E}{R+A^{\text{on}}+A^{\text{off}}} \geq \delta$$

nomic stress, whatever the source, thus reducing the risk of spillover from the financial sector to the real economy (Basel Committee on Banking Supervision 2013).

More specifically, liquidity requirements are used to avoid the inefficient liquidation of illiquid technology and ensure that banks can withstand funding reductions such as deposit withdrawals or liquidity demands arising from off-balance sheet activities. Thus, the financial failure of a bank due to early liquidation does not spill over to other banks. The liquidity requirements include two regulations designed to address two separate but complementary issues(Basel Committee on Banking Supervision 2014), the market liquidity concerns and the maturity mismatch problem[19]. The first rule, liquidity coverage ratio (LCR), requires the bank to hold sufficient high-quality liquid assets to survive a significant stress scenario lasting for 30 days. LCR has been designed to assure the short-term resilience of a bank to its liquidity risk exposure. The second rule, net stable funding ratio (NSFR), mitigates the long-term exposure of banks to funding stress by requiring banks to fund their activities with sufficiently stable sources.

1. Liquidity Coverage Ratio (LCR)

   LCR was designed to ensure that banks hold a sufficient reserve of high-quality liquid assets (HQLA) to allow them to survive a period of significant liquidity stress lasting 30 calendar days. Using the simplified[20] notations and definitions introduced in table 2.2 we can formulate LCR requirement by:

$$R \geq \ell^d D + \ell^A A \tag{2.4}$$

   where $\ell^A A$ represents the contingent liabilities due to off-balance-sheet assets of the banks. Since in the simplification of the balance sheet we aggregated the on-

[19] The liquidity coverage ratio (LCR) is concerned with the level of market liquidity. The net stable funding ratio (NSFR) aims to resolve the maturity mismatch problem.

[20] The equivalent formulation, before simplifying the balance sheet structure, using the notations and definitions introduced in table 2.1 for LCR requirement would be:

$$R \geq \sum_s \ell_s^d D_s + \sum_j \ell_j^{\text{off}} \ A_j^{\text{off}} \tag{2.3}$$

Thus, to meet the LCR requirement, a bank's reserve of liquid assets, R, should be higher than a weighted sum of its (risky) liabilities.

and off-balance sheet assets, the liquidity risk exposures due to the off-balance sheet items can be formulated as a proportion of the aggregate on and off-balance sheet items[21].

Thus, to meet the LCR requirement, a bank's reserve of liquid assets[22] , R, should be higher than a weighted sum of its (risky) liabilities. Note that in the simplified formulation.

2. Net Stable Funding Ratio (NSFR)

   NSFR is designed to ensure that banks' risky assets are sufficiently funded by the liability side of the banks' balance sheets. NSFR requirement assigns different weights to funding resources, proportional to their reliability. This means that if a funding resource (as a liability of a bank) has a longer term maturity, it is considered more reliable, and thus its weight in the NSFR formulation should be higher. NSFR also assigns different rates to the risky assets[23]based on their liquidity needs and maturity characteristics. This means that an asset that has a longer term maturity and/or has higher risk exposure needs more stable funding, and thus its weight in the NSFR formulation should be higher. Using the simplified[24] notations and definitions introduced in table 2.2 we can formulate NSFR requirement by:

$$a^B B + a^D D + E \geq fA \tag{2.6}$$

[21]Here we are implicitly assuming, without loss of generality, that each bank keeps the proportionality of the off- and on-balance sheet investments of its aggregate risky assets fixed.

[22]we use “liquid assets”, “high-quality liquid assets (HQLA)”, “reserves” and “risk-free assets” interchangeably.

[23]Including on- and off-balance-sheet assets

[24]The equivalent formulation, before simplifying the balance sheet structure, using the notations and definitions introduced in table 2.1 for NSFR requirement would be:

$$\sum_k a_k^B B_k + \sum_s a_s^D D_s + E \geq \sum_i f_i A_i \tag{2.5}$$

Note that the aggregation of the weights of similar types of assets and liability in the simplified formulation does not change how NSFR affects the interactions of different asset and liability types in the bank's structure.

where the left-hand side of eq. (2.6) represents the level of stable funding resources that are available to a bank, whereas the right-hand side of eq. (2.6) represents the required stable funding of the bank.

Then, NSFR simply says that a bank's stable funding resources should be higher than its required stable funding.

Remark 2 *The technical formulation of NSFR requires a weighted sum of liabilities to be higher than a weighted sum of assets, whereas LCR formulation requires a weighted sum of assets to be higher than a weighted sum of liabilities. This means that the NSFR requirement effectively reverses the LCR requirement.*

As we will see, the fact that the LCR and NSFR effectively reverse the inequality between assets and liabilities creates some complications.

2.1.4 Balance Sheet Interactions of the Regulatory Requirements

The regulatory framework, in general, is a system of different structural requirements aimed at restricting the structure of a bank and, in aggregate, the structure of a banking system in a way that efficiently reduces or eliminates the market failure issues (see section 2.1.2).

From a technical perspective, however, each regulation is a structural requirement that restricts the banks' balance sheet configuration. Similarly, a regulatory framework is a collection of different structural requirements forcing banks to adapt their structures in a way that the setting of their balance sheets complies with all requirements simultaneously. If and how a regulatory framework achieves its intended goal of reducing or eliminating some market failures would not be answered if each individual regulation is considered and analyzed independently, neglecting how it relates and interacts with other regulations and other system components. The interactions that emerge from the simultaneous adaption of the regulatory framework are highly complexand mostly unknown in the literature of banking and financial economics.

The most basic analysis of the interactions of the regulatory requirements would only consider the balance sheet relations of the regulations regardless of the economic power behind the formation of the balance sheet structure. With that, we may have some initial ideas about the consistency of the regulations within a framework to start with, and then we can go beyond the mechanical relations and consider a more complicated interaction in a richer model of the system.

Next, we study the balance sheet interactions of the capital and liquidity requirements defined in Basel III, using the simplified model of the banking balance sheet structure[25] We formulated in section 2.1.3.

Liquidity and Capital Requirements Interactions in Basel III

To capture the basic features of Basel III's liquidity and capital requirements and their interactions, the formulation of the regulatory system should incorporate all four rules of the risk-weighted capital ratio, the leverage ratio, liquidity coverage ratio and net stable funding ratio, formulated by Equations (2.1) to (2.6) using a unified system subject to the balance sheet identity equation

$$R + A = B + D + E \tag{2.7}$$

which relates all different components of the banking balance sheet[26]. The balance sheet identity equation provides a channeling mechanism that gives structure to the interactions of the aforementioned four regulatory rules. Using eq. (2.7) we can rewrite

[25]To capture the balance sheet interactions of different regulatory requirements, we should use the formulation of the banking structure that the regulatory framework has been designed for. That is the reason we formulated a model of the banking balance sheet structure based on Basel III's definition of different components of a banking balance sheet in section 2.1.3.

[26]The equivalent formulation, before simplifying the balance sheet structure, using the notations and definitions introduced in table 2.1 for the balance sheet identity equation would be:

$$R + \sum_i A_i^{\text{on}} + \sum_j A_j^{\text{off}} = \sum_k B_k + \sum_s D_s + E \tag{2.8}$$

eqs. (2.1), (2.2), (2.4) and (2.6) in two equivalent forms:

$$E \geq Cr^A A \qquad \text{(Risk-weighted Capital Requirement)}$$

$$E \geq \frac{\sigma(B+D)}{1-\sigma} \qquad \text{(Leverage Ratio)}$$

$$E \leq (1-l^A)A - (D+l^d)D - B \qquad \text{(Liquidity Coverage Ratio)}$$

$$E \geq f^A A - a^D D - a^B B \qquad \text{(Net stable Funding Ratio)}$$

where we have reformulated both capital and liquidity requirements as a system of capital (equity) restrictions in terms of risky assets and liabilities and

$$R \geq (Cr^A - 1)A + B + D \qquad \text{(Risk-weighted Capital Requirement)}$$

$$R \geq \frac{B+D}{1-\sigma} - A \qquad \text{(Leverage Ratio)}$$

$$R \geq l^A A + l^d D \qquad \text{(Liquidity Coverage Ratio)}$$

$$R \geq (f^A - 1)A - (1-a^B)B + (1-a^D)D \qquad \text{(Net stable Funding Ratio)}$$

where we have reformulated both capital and liquidity requirements as a system of reserve (safe assets) restrictions in terms of risky assets and liabilities. In other words, all regulations can be replaced by a set of capital requirements or reversely by a set of liquidity requirements.

It is easy to see that this system does not have a nonzero solution for all rules to be binding. This means that for any given parametric setting of the regulations, at least one of the rules would be redundant for a complying bank. This has multiple implications for individual banks' structure(see Cecchetti and Kashyap 2018)) that illustrates how interaction of the four central regulations of the latest post-crisis regulatory framework has unintended consequences. For example, as discussed earlier, the LCR and the NSFR rules are designed as complementary regulations, where the LCR provides short-term stability that safeguards an institution against a run and the NSFR buys time that could facilitate a resolution. Since for any bank in the system either the LCR and the NSFR would be binding, we should expect the banking system to be divided into two groups at any given time. The banks in the first group just have enough liquidity to cope with the short-term liquidity shocks, which is redundant for the long-term shocks. The banks in the other group, however, just have enough liquidity to cope with the long-term liquidity shocks, which is redundant for the short-term shocks. This may seem necessary to protect the banking system in the face of both short- and long-term liquidity stress. However, the theoretical model of the financial system developed in chapter 3 shows that this is not only economically efficient but also does not reduce the risk exposure of the banking system. Specifically, as shown in 3.2.5, redundancy of short-term or long-term liquidity, specifically when consumers expect to receive positive endowments in the future, would lead to aggregate social welfare losses with no reduction in the risk exposure of the banking system. In other words, the totally unregulated banking system developed in Financial Market section in chapter 3 more efficiently copes with liquidity shocks than the same system but subject to the liquidity regulations of the Basel III framework. More importantly, this setting may result in liquidity shortage or liquidity abundance, which, as shown in the proof of 3.2.5, leads to the distortion of the liquidity pricing in the market, which in turn has significant consequences for the banking system as well as the general economy. None of these involve bankruptcy or even insolvency of any bank, yet losses are unintended consequences of systemic structures and interactions. Thus, we consider these types of losses and risks to be systemic.

2.2 Literature Review of Structural Modeling Approach

Methods that model and analyze the structure and nature of the relationships between financial institutions aim to capture and/or relate the system-level dynamics to the micro-foundations of the system. Micro-foundations of a system include micro-level dynamics, processes, and structures that may have system-level implications. As follows, we review two methodological approaches of structural modeling of systemic risk: i) micro-foundational modeling and ii) stylized network modeling. Micro-foundational modeling approaches are mainly focused on liquidity distress as the main channel of distress contingency. On the other hand, stylized network modeling approaches are focused on formulating the default contingency. Both approaches are extended to capture the other main distress channel or even more complicated phenomena like fire-sales and spill-overs, but so far, there has not been a structural modeling approach that formulates both distress channels in a unified setting.

2.2.1 Micro-foundational Economic Modeling

We discuss this strain in the literature based on the model by Allen and Gale (2000) for two reasons. First, this work and its proceeding literature rely on the mainstream theory of banking and financial intermediation, developed in the 80s, (Diamond and Dybvig 1983, 1986) and capture the essence of the most rigorous theoretical frameworks in financial economics and relate them to systemic risk in banking and financial systems. Second, this provides us a microeconomic explanation on why we need financial intermediaries for providing liquidity insurance as well as how the intermediary mechanisms work in different circumstances when consumers have uncertain liquidity needs.

2.2.2 Stylized Network Modeling

Later, Eisenberg and Noe (2001a) proposed an elegant financial model for interbank lending networks, which, consistent with the bankruptcy laws, shows how the propagation of the distress originates from the missing payments in a network and leads to the contractual dependencies of the banks in their debt clearing payments. Building on the Eisenberg and Noe (2001a) work, the literature has grown around the theoretical modeling of the interconnections of a financial system. For example, Rogers and Veraart (2013), Elliott et al. (2014) and Glasserman and Young (2015) generalized the basic model by considering new parameters like bankruptcy and liquidity cost. Another example is the Capponi and Chen (2015)'s extension of the interbank clearing mechanism to a multi-period version, which helps to analyze the effect of different distress mitigation policies through time.

Another approach in structural modeling studies different network topologies to see which structural properties of an interbank network reinforce the systemic contagion of the losses (Benoit et al. 2017) in the system. Despite the fact that there is no rigorous framework that can connect the relevant economic forces with the different network structures and topologies (Bo and Capponi 2015), the emerged literature on the banking networks provides us with an insight into how and to what extent the interconnections of the components of a system can translate to the mechanisms for the aggregation and propagation of the shocks/distress through a system. The significance of the micro-level shocks have been a matter of debate in the Macro-economic literature (see Gabaix (2011) and Lucas (1977)). Accordingly it was essential to ground a network theoretical basis Hosseininia and Dadgostari (2013a), Dadgostari and Hosseininia (2013), Hosseininia and Dadgostari (2013b) for the significance of the non-Macro economic shocks in the creation of a sizable volatility and fluctuations in a multi-sector economy (Acemoglu et al. 2012, 2013) and also to study how the non-Macro shocks spread in the networks with different structures (Cifuentes et al. 2005, Gai and Kapadia 2010, Nier et al. 2007, Allen and Babus 2009, Acemoglu et al. 2015, Chen et al. 2013).

Despite the extensive progress in this strand of research, due to missing economic framework that captures the underlying market dynamics that lead to the counterparty contingencies in the first place and relates the structure of the system to economic incentives, we can not derive meaningful policy-making and regulatory implications for systemic risk management. Here we study some of these drawbacks in this strand of the literature that our proposed framework aims to address.

- Despite the theoretical work that supports the possible significance of the domino effects of contingencies working through the balance sheet mechanics of the banking system, other essential mechanisms at work in the formation of the financial network and the build-up and unfolding of distress lie in the interaction between leverage, interest rates, asset prices, and portfolio decisions. Models of system-wide counterparty contingencies arising from the balance sheet mechanics of the interbank contractual dependencies need to incorporate these underlying economic parameters and their dynamics into account to relate its measurements to the real economy's dynamics and enable the regulators and policy-making decisions to use these models in a causal sense to monitor and manage the market and systemic risk. (Regarding contagion, "it takes two to tango," meaning that the causes of contagion relate to both its transmission and its reception: for a loan to lead to a contagious default, the lender must be leveraged, and the borrower's asset portfolio must put it at risk of default.)

- The risk measures developed in this strand of research, for exogenous shocks of different types, quantify the number of the defaulted banks or levels of the banks' losses and relate those results to different levels of the systemic importance of individual banks and how it associates to different structures or topologies of the financial network without thinking about how a bank's participation in the system affects other market participants and the structure of the system. Therefore, none of these modeling frameworks and developed measures has the property that causally relates the banks' decisions and regulatory policies to the level of individual banks' contributions to overall risk exposures. (the sum over all firms equals a measure of systemic risk, and they do not explain what fraction of systemic risk is caused by each firm. That is the defining property of systemic)

- These existing structural models view the effects of exogenous and independent shocks and consequent contagion on a given structure of a financial system, fixed and do not hold banks responsible for the impact that banks decisions have on other banks and the overall system. (in a way that holds firms responsible for their contributions to contagion, it is necessary to go beyond portfolio risk attribution methods.)

- By treating the risk exposures of the external assets of the banks to be independent, the regulatory and policy-making implications of existing models overlook the endogenous correlations that arise from the overlapped investment portfolios of the banks and thus common shocks. (Examples are correlations of credit risks that arise from a common dependence of loan customers on the business cycle (loans to non-financial firms) or on real-estate prices (home mortgages). The common dependence of asset prices and bank funding conditions on market rates of interest is also overlooked. Indeed, for positions in the bank book, the risk that interest rate increases in the market might make funding more expensive is neglected altogether; this is the risk that caused a large part of the US savings and loans industry to be defacto insolvent as of 1981.)

- Moreover, it is less clear how to link the measures produced by these tools to regulatory interventions. For example, observing that a bank has become more central in the interbank market does not translate directly into a clear policy response. More structural models, linking risk estimates to well-defined policy objectives and available tools, would be useful to regulators.

2.3 Literature Review of Macro-Prudential Modeling

The other strand of the literature, which has been the main stream of research and practice (Silva et al. 2017) in this field over the past 5 years, studies systemic risk as a Macro-finance phenomenon which should be measured as an overall distress to the system.

In this approach, the contribution of each component to the aggregate systemic risk is attributed through the axiomatic connection of the tail distributions of the institutions' balance sheets and the market, based on the data obtained from the current (assumingly efficient) market prices of the securities issued by the financial institutions or the derivatives written on them. To be more specific, Macro-financial approaches develop statistical methods that investigate the impact of the distress of an institution on the market or the impact of a distressed market on an individual institution, treating the financial system as a portfolio of firms/institutions with cross-sectional investments in the real market to capture the left tail dependencies of the market returns and the firms' returns.

The Macro-finance models theoretically follow the classic economic models in the corporate finance and credit risk management literature, and most significantly dissolve most of the systemic structure of the market through the aggregation of the banks' balance sheets by simple summations, like formulating the overall banking system's return as a value-weighted average of all banks' returns, and similarly, the overall banking system's loss as the sum of the banks' losses and overall banking system's capital-shortfall as the sum of the banks' capital-shortfalls. On this basis value-at-risk of the financial system is formulated and through different statistical approaches estimated. Adrian and Brunnermeier (2016), measures the financial sector's VaR given that a bank has had a VaR loss, which they denote CoVaR. Then they define a measure of the contribution of a bank to the overall systemic risk, through the change in the financial system's VaR, conditional on the bank to be under stress relative to not being under stress, denoted as ΔCoVar. As a statistical tail dependency measure, ΔCoVar captures the component of the loss that comoves with the distress of a particular bank and best viewed as a useful reduced-form analytical tool capturing (tail) comovements. Contrary to the way that ΔCoVar and its extensions are formulated, capital-shortfall models (see (Acharya et al. 2017, 2012, Brownlees and Engle 2016)), reverse the conditioning to shift the focus to the question of how much is a bank's capital shortfall (as a proxy or its risk exposure) given that the financial system's capital shortfall is above some given threshold (as a proxy of financial crisis). For example, Acharya et al. (2010), builds an economic model based on an analogical comparison of how banks break down a firm-wide loss in to the separate losses of the internal groups of the firm with how the overall loss of the banking system can be broken down in to the separate loses of the individual financial institutions, constituting the banking system. To attribute systemic risk to the individual institutions, a measure of "tail" losses of financial products has been defined as *Expected Shortfall (ES)*, which in turn is used to extend the measure of risk for the whole financial system. Later, (Acharya et al. 2017) extends this approach to support the theoretical ground for SRISK[27], the systemic risk Measure first introduced at 2012

[27]In Acharya et al. (2017)'s paper, SRISK is refereed by the generic name of "Systemic Expected Shortfall" or SES

by Acharya et al. and is implemented in VLab[28] to measure systemic risk in real time for different countries and financial institutions.

Despite the extensive progress in this strand of research, due to big theoretical gaps, some of the fundamental properties that reflect the systemic nature of financial markets are missed in the existing modeling frameworks, which would be essential to develop appropriate regulatory and policy-making frameworks. Here we study some of these drawbacks in the main strands of the literature that our proposed framework aims to address.

- Aggregation of the banks' balance sheets through simple summation not only dissolves the systemic interactions and dependencies of the banks in the interbank loan market, and consequently misses the systemic phenomena like contagion, but also may have misleading consequences. For example, the appearance of some assets of banks in other banks' liabilities would virtually inflate the size of the aggregate financial market balance sheet due to multiple accounts of the transmitted obligations.

- To separate systemic losses from non-systemic losses, capital shortfall models assume that a systemic event, systemic crisis, financial crisis, and severe market decline are the same phenomenon and correspond to scenarios where the market declines below a given threshold. To do so and following the aforementioned modeling approach of aggregating banks' balance sheets, Acharya et al. (2017) defines a financial crisis as a market status where the aggregate capital (sum of all banks' capitals) is below some given percentage of the aggregate assets (sum of the banks assets). Similarly, Brownlees and Engle (2016) defines a financial crisis as a market status where the overall market return as a value-weighted average of all banks returns is below some given threshold. Finally, systemic risk is defined as an expected loss or capital shortfall of the financial market, conditional on a financial crisis, which in turn is the sum of the banks' losses or capital shortfalls

[28] The Volatility Laboratory of NYU Stern (`https://vlab.stern.nyu.edu`)

conditional on a financial crisis. Equivalence of a systemic event and financial crisis is a misleading assumption since the systemic interactions of the banks and its implications to the market dynamics, does not necessarily correspond to or is not conditioned on a crisis. In other words, systemic nature of the financial market would expose some banks to risks that may not necessarily result in a financial crisis and vice versa.

- As statistical tail-dependency measures, ΔCoVar and capital shortfall measures are reduced-form measures which may be useful to predict when an institution or the financial market is under stress but not on why it is under stress and what the causes of the stress are and consequently how much an institution causally contributes to risk exposures. Therefore, the measured risk exposures are not equivalent to the level of risk contributions in a causal sense. This is the fundamental drawback in this strand of literature that impedes fair policy-making solutions to internalize systemic risk.

- Moreover, since these measures do not causally recognize the mechanisms of risk exposure, not only can not allocate the sources of externalities to different financial institutions, but also it is unclear how to break down the measured risk exposures to systemic and non-systemic. Therefore, in our view, it is misleading to interpret the risk exposures identified and measured through the left tail dependency of the banks and the financial market as systemic risk. Companies responsible for the stress may not be the ones that suffer most from the system being under stress. In addition, a company may indeed be responsible for creating systemic risk without being under stress when the system is under stress. (example)

2.3.1 Economic Framework of Macro-Prudential Modeling

Here we use the economic framework developed by Acharya et al. (2012) as a basis to analysye the basis of the Macro-Prudential Modeling approach Acharya et al. (2012)

aims to build a theoretical ground for SRISK[29], the systemic risk Measure first introduced at 2012 by Acharya et al. and is implemented in VLab [30] to measure systemic risk in real time for different countries and financial institutions. The employed empirical analysis to see how much this measure is reliable to predict systemic risk in the 2007-2009 financial crisis, supplemented with a discussion on its consistency with the results of the stress-test performed by bank regulators during the spring of 2009, supports the practicality of the application of this measure for the use of regulators too. To formally develop a model-based measure for systemic risk, bringing together the theoretical foundations and practical needs and constraints (mostly data access constraints) of regulators, they have built an economic model capturing the systemic risk externalities of the banking system for the rest of the economy, where a regulator tries to control the systemic risk exposure of the economy through an optimal taxation policy. In fact, the taxation policy aims to optimally force each financial institution to internalize the costs of their contribution to the overall systemic risk. More precisely, the economic framework models the financial system as a set of institutions ("Banks") that take their investment decisions considering the taxation policy, which charges banks proportional to their contribution to systemic risk.

To build an appropriate economic model, the first two standards (firm-level) risk measures[31] are compared, and due to the capability of *Expected Shortfall (ES)* to measure "tail" losses of financial products properly; it is selected to be extended to a measure of risk for the whole financial system. To make such an extension, use an analogical comparison of how banks break down a firm-wide loss into separate losses of internal groups of the bank with how the overall loss of the financial system can be broken down in to separate losses of the single financial institutions, constituting the banking system.

Formally, if R denotes the total return of a bank and r_i represents the return associated with a group i in the bank, then $R = \sum_i y_i r_i$ should hold where y_i stands for the weight of the group i in the bank's portfolio. Accordingly, the expected loss of

[29] SRISK is refereed to as as a generic name of "Systemic Expected Shortfall" or *SES*

[30] The Volatility Laboratory of NYU Stern `https://vlab.stern.nyu.edu`

[31] VaR and Expected Shortfall

the bank is formulated as:

$$ES_\alpha = -\sum_i y_i E[r_i | R \leq -VaR_\alpha] \tag{2.9}$$

where VaR_α is the most that the bank loses with a confidence level of $1-\alpha$, and accordingly, the marginal expected shortfall for each group i is:

$$MES^i_\alpha = \frac{\partial ES_\alpha}{\partial y_i} = -E[r_i | R \leq -VaR_\alpha] \tag{2.10}$$

Here, the marginal expected shortfall of group i in the bank, denoted by MES^i_α measures how the group i's expected loss contributes to the overall loss of the bank. This approach suggests that the financial loss of the banking system can be modeled in the same way, where we add all expected shortfalls of the banks to get the total loss of the overall banking system under the systemic risk. Then each bank's contribution to systemic risk can be modeled using the same formulation of marginal expected shortfall or MES^i used for the calculation of expected loss contribution of groups within a bank.

Building on top of that assumption, the next step is to develop an equilibrium model of the banking system, which provides us with a way to calculate the marginal expected shortfall of each bank, used ultimately as a measure to estimate the expected contribution of each financial institution to the overall systemic risk. We would not go into a detailed explanation of how the banking equilibrium is formulated, but in short, it is assumed that each bank invests a portion of its equity ($\bar{w}^i_0$), denoted by w^i_0 and the raised debt of b^i in the economy at $t = 0$ (pre-stress), forming the budget constraint $w^i_0 + b^i = a^i$ where a^i denotes the total assets of the bank i. Accordingly, the bank i's pre-stress income (denoted by $\hat{y}^i$) is determined by the aggregate return of its assets. It should be mentioned that the asset return rates are bank-specific, which means that the model does not capture the correlated asset returns of banks. Then the cost of distress for each bank is formulated by $(\Phi(\hat{y}^i, f^i))$ that depend on the (pre-stress) market value of the bank's assets, $\hat{y}^i$ and the face value of the bank's assets, f^i. Consequently, the net value of the bank i at time 1 (after distress) is determined as:

$$w^i_1 = \hat{y}^i - \Phi(\hat{y}^i, f^i) - f^i \tag{2.11}$$

Then, the expected return of the bank, considering the limited liability constraint, would be determined as $1_{[w_1^i>0]} \times w_1^i$ and accordingly each bank solves the following problem:

$$\max_{x} \quad c[\bar{w}^i - w^i - \tau^i] + E[u(1_{[w_1^i>0]} \times w_1^i)] \tag{2.12}$$

where x denotes the investment decision space of bank i, that is constrained by equations representing the bank i's balance sheet including (2.11) and the tax constraint coming from the planner's optimization problem (the regulator's taxation policy). In fact, the optimization problems of the banks in the financial system link to each other, forming an equilibrium model by taxing variables, which is the regulator's policy-making variable, determined by the planner's (regulator's) optimization problem.

Planner's problem maximizes a welfare function $P^1 + P^2 + P^3$ for the regulator, where P^1 is the sum of utilities of the bank owners, P^2 is the expected cost of debt insurance fund managed by the regulator (cost is considered negative to be minimized by the maximization) and P^3 formulates the externality of a financial crisis. In the planner's problem, P^3 plays an essential role in how the regulator models the systemic risk which eventually shapes the equilibrium of the banking system and how each bank internalizes its associated systemic risk tax[32]. Therefore, we explain the formulation of P^3 in more detail to represent how the cost of systemic risk is modeled. It is assumed that a financial crisis happens when the aggregated banking capital after distress ($t = 1$), formulated as $W_1 = \sum_i w_1^i$ falls below a fraction z[33] of the aggregate assets of banks, formulated as $A = \sum_i a^i$. In other words, when $W_1 \geq zA$ holds, there is no financial crisis and no systemic risk externality associated with that, but as soon as $W_1 < zA$, the system is under-capitalized as much as $zA - W_1$ and this would cause financial failure with the externality cost that linearly grows with the level system is under-capitalized. If e measures the severity of the linear growth of externality by the level of being under-capitalized, the expected cost of the systemic risk for the economy would be measured

[32] which is supposed to (proportional to) the bank's contribution to the overall systemic risk

[33] z is usually set in the range of %8 to %12 if all assets have risk-weighting of close to %100 under Basel I capital requirements.

as:

$$P^3 = E[e \times 1_{[W_1 < zA]}(zA - W_1)] \tag{2.13}$$

In fact, the above formulation sets a predetermined threshold on the adequacy of aggregate capital of the financial system, to avoid systemic risk externalities that are assumed to be early fire-sales and restricted credit supply.

According to this definition of systemic risk, we can use the same conceptual framework that individual banks use to measure the share of financial shortfall caused by a group within the bank, to measure the share of under-capitalization caused by a bank within a financial system, in the case of the financial crisis. Formally, similar to the definition of the marginal expected shortfall of group i in the bank, MES_α^i, the "Systemic Expected Shortfall" of the bank i within the financial system, is defined as:

$$SES^i = E[za^i - w_1^i | W_1 < zA], \tag{2.14}$$

where it is trivial that $\sum_i SES^i = zA - W_1$.

It turns out that at the equilibrium, the optimal tax policy is the sum of two components; $\tau^{i*} = \tau^{i*}_{\text{Ins Risk}} + \tau^{i*}_{\text{Sys Risk}}$, where $\tau^{i*}_{\text{Ins Risk}}$ determines the tax each bank should pay for the institutional-risk exposure of its guaranteed liabilities and $\tau^{i*}_{\text{Sys Risk}}$ addressing the systemic risk externalities of the banks, which taxes each bank, proportional to its expected contribution to under-capitalization of the system, given the financial crisis or in formal terms:

$$\tau^{i}_{\text{Sys Risk}} \propto Pr(W_1 < zA) \times SES^i \tag{2.15}$$

that is as we expect, consistent with the definition made earlier in this section for systemic risk and the contributed share of each bank in the overall systemic risk.

2.3.2 Expected Shortfall as a Measure of Systemic Risk

To implement an optimal tax policy, the regulator has to determine the expected contribution of each bank in the overall systemic risk according to (2.15). To do so, the regulator needs to calculate the probability of a financial crisis, $Pr(W_1 < zA)$ and the

"Systemic Expected Shortfall" of each bank[34], SES^i according to (2.14). In practice, these parameters may not be measurable, and accordingly, regulators should use any variable predictive of these parameters. Then, similar to (2.10), a marginal expected shortfall measure is defined for net equity returns ($\frac{w_1^i}{w_0^i} - 1$) in the worst 5% market outcomes in "moderately bad days", represented by $I_{5\%}$, which leads to:

$$MES_{5\%}^i = -E[\frac{w_1^i}{w_0^i} - 1|I_{5\%}] \tag{2.16}$$

Then, Extreme Value Theory is used for the estimation of return distribution given a real financial crisis.

Next, a proposition is proved that indicates that the defined marginal expected shortfall measure for the net equity returns of each bank ($MES_{5\%}^i$) has a linear relationship with the Systemic Expected shortfall (SES^i) of the banks, which we intend to determine as a measure of systemic risk (contribution of each bank).

$$SES^i = za^i - w_0^i + kMES_{5\%}^i + cte \tag{2.17}$$

Accordingly, $MES_{5\%}^i$ along with the the leverage ($\frac{(a_i)}{w_0^i}$) can be used as a predictive variable for SES^i, and then the regulator can use the resulting estimation of SES^i, to calculate the optimal tax policy and control systemic risk.

2.4 Conclusion

Most of the literature on systemic risk is focused on how the economic externality caused by systemic risk should be internalized by the system through the deployment of appropriate monetary policies. We discussed the existing regulatory frameworks, their generic formulations, and the rationale behind their definitions. We also reviewed the stylized modeling and macro-prudential modeling approaches to systemic risk how each of these modeling frameworks addresses financial institutions' contribution to systemic risk. As follows, we discuss how the misconceptions in the definition of a system[35], re-

[34] conditional loss of a bank if a crisis occurs

[35] In defining banking system, financial system and systemic risk

sults in major technical issues.These are conceptually connected issues but are discussed separately to keep it simple.

1. The conventional economic models of financial systems formalize the risk exposure of the system, in analogy to how an individual firm is exposed to financial risks. In the case of a firm that manages its internal groups and particularly their exposure to risk, it is natural to assume that their aggregate returns are consolidated in the same fund, facing an aggregate exposure to risk. However, the financial system does not work in that way, and the role of the regulator is to keep interaction of the components of the system as much as possible competitive. Within a firm, if the balance of losses and gains cover each other enough, the aggregate risk exposure of the firm will be zero. No matter how severe the losses are, as far as that the gains are high enough, the firm will be financially stable. On the other hand, this is not the case for the financial system. In the financial system, if some banks lose so much and default, no matter if the aggregate return of the system is zero, the banks which have had positive returns would not cover the losses of the defaulted banks, and accordingly, the risk exposures are not canceled out. This comes from the fundamental difference of the nature of how the financial system of an individual firm is organized and the banking system constituted of competitive banks in a free financial market. Building a risk measure based on this assumption, not only will result in unfair regulations but also does not capture any notion of the systemic nature of the risk, which exactly depends on how banks interact with each other and how the structure of the financial system is shaped.

2. Modeling the prestress income of the banks, where the investment of the bank i in asset j is denoted by x^i_j, the return parameters associated to assets are bank-specific and denoted by r^i_j, which represents the per-dollar return of asset j for the bank i. Therefore, even if two banks have invested in the same asset, they may have completely independent return variables, and accordingly, the connection of banks through the correlation of their returns is not captured in the model. This is another systemic component removed from the financial system's model, reduc-

ing the banking system from a dynamically correlated system, integrated through the market, to a statistically independent set of agents, with no integrating mechanism, other than the regulators taxing policy.

3. The cost of distress for each bank is formulated by $(\Phi(\hat{y}^i, f^i))$. Consequently, the net value of the bank i at time 1 (after distress) is determined as:

$$w_1^i = \hat{y}^i - \Phi(\hat{y}^i, f^i) - f^i$$

Since the formulation of "cost of distress", only depend on the (pre-stress) market value of the bank's assets, $\hat{y}^i$ and the face value of the bank's assets, f^i, it is not measuring how the bank i's investments in the market may perform under stress and accordingly, w_1^i representing the net value of the bank i after the crisis, cannot really depend on the performance of the investments under stress. Then if the net value variable w_1^i, does not contain that information, the measure of under-capitalization of bank i, given the financial crisis, will be a generic approximation of capital shortfall of the bank i, given the composition of the investment portfolio of the bank i.

4. As explained earlier, from this perspective, a financial crisis happens when the aggregated banking capital falls below a fraction of the aggregate assets of the banks. In other words, when $W_1 \geq zA$ holds, there is no financial crisis and no systemic risk externality associated with that, but as soon as $W_1 < zA$, the system is under-capitalized as much as $zA - W_1$, and this would cause the financial crisis and systemic risk. Accordingly, the systemic risk contribution of banks is formulated as:

$$SES^i = E[za^i - w_1^i | W_1 < zA]$$

only is measured when the financial crisis has already happened. In fact, the developed systemic risk measure, SES^i examines each bank's stress conditional on the occurrence of a financial crisis. There are two problems with the above assumptions and definitions. First is the definition of the financial crisis. It is defined according to if, in aggregate, the system is under-capitalized or not. Based on this

> definition, no matter how the banks' capitalization is distributed in the system and how the structure of the liabilities of banks is shaped, it should be considered under crisis if the aggregate capitalization is below a predefined threshold. The second problem is that the financial crisis is considered a prerequisite for systemic risk exposure, which is not the case. It does not require a financial crisis to happen for the system to be exposed to systemic risk. Even if a couple of banks default and it costs the regulator and the system some loss, it does not necessarily mean that a financial crisis has accrued. In fact, there is a well-developed literature (Acemoglu et al. 2012, 2013) on how a set of small economic shocks may cause some systemic risk events, which may result in a financial crisis if the system is fragile or may not result in a financial crisis if the system can absorb those shocks efficiently, and prevent extensive propagation of defaults and economic spillovers.

The above issues result in an equilibrium model in which there is no equation or variable to represent the banks' connection, and accordingly, this equilibrium model does not model a system. In fact, the equilibrium model tries to design a system by introducing tax variables, which are the only variables connecting the bank optimization problems. The proposed tax variables come from an optimization problem solved by the regulator to determine the tax associated with each bank. Accordingly, the equilibrium model assumes that the financial system is a set of agents constituting a market in which they invest in separate and independent markets. Their decisions do not systematically harm or help each other, competing over building portfolios that maximize their revenues after the assigned taxes. The regulator hopes that by imposing a tax on each bank, proportional to the level of expected capital shortfall of that bank, the system balances itself around an optimal equilibrium in which each bank has internalized the cost of its contribution to the systemic risk and, accordingly, the aggregate systemic risk is minimized. It is set similar to how the externalities of CO_2 emissions of industrial facilities are modeled in the economic equilibrium models where a tax associated with the level of CO_2 emissions of each facility internalizes the cost of the pollution and balances the equilibrium around an optimal point. The difference is that the size and position of the CO_2 emission of a factory are not systematically related to the CO_2 emission of other facilities, and accordingly, taxing CO_2 emissions is an efficient way to control those externalities. In the case of systemic risk contribution (externality) of each bank, we cannot make such assumptions. For example, a bank defaulting with the same size of capital shortfall may cause some other banks to default or not, according to the structure of the financial system. Therefore, the size of a bank's contribution to the systemic risk may be different depending on where and how the bank is located in the system and, in general, how the financial system is shaped. To capture those properties in the taxation optimization problem, the connections of banks (collateral obligations and other components of the system) should be considered in the optimization problems of banks. With those conditions held in the formulation of the equilibrium, the taxing policy, resulting from the solution of the equilibrium model, will determine the market game rules in a way, which is fair and minimizes the overall systemic risk exposure of the system.

Finally, it seems that there is a misconception are organized around the definition of systemic risk as an aggregate under-capitalization of the financial system, which finally leads to regulations and policies which penalize individual banks according to how much a bank's expected capital shortfall is, if it defaults, with no connection with other banks and the system as a whole. Empirical results do well approximate the capital shortfalls under crisis and the stress-test results, but assuming those approximations to be the systemic risk contribution of the banks is an arguable claim according to the above discussions.

Chapter 3

Financial Intermediation, Banking, and Decentralization

In this chapter, we extend the general equilibrium model of a banking system with an interbank deposit market as a financial intermediary to incorporates aggregate uncertainty and standard deposit contracts. We show that an equilibrium behavior of a competitive banking system decentralizes the social planner's allocation under the same assumption without any regulation imposed and therefore it is constraint efficient. This shows that the financial contagion is not an equilibrium feature of the decentralization functionality of the banking system even under aggregate uncertainty and standard (incomplete) deposit contracts(as defined in sections 1.3.1 and 1.3.2).

3.1 Introduction

Our understanding of the banking system's role as a financial intermediary and its mechanisms and functionalities to provide liquidity insurance has its roots in the theory of banking mainly developed after the 80s by the seminal contributions of Diamond and Dybvig (1983, 1986), Diamond (1991), Allen and Gale (2000).The theory of banking and financial intermediation was mainly motivated by instabilities that emerged after the financial market's deregulation in the 70s, aiming to go beyond empirical studies' limitations to explain the advantages of a competitive banking system compared to other intermediary alternatives. It was also essential to have a theoretical framework to study how a banking system relates to efficiency and stability of financial markets and the general economy. The widely spread and disruptive financial crisis of 2007-08 drew attention to phenomena like financial contagion and systemic risk and how the banking system may contribute directly or indirectly to such externalities.

The call for more restrictive regulation of the banking system aimed at preventing systemic risk and financial contagion requires in depth understanding of the how a banking system contributes to systemic risk. Such a system theoretic understanding of the economic powers behind such externalities are essential in the efficiency of any regulatory and policymaking intervention designed to address such market failures. In this chapter we focus on modeling and analysis of structural characteristics of the banking system in its fundamentals aiming to characterize the systemic behavior of the system under standard banking assumptions. The main question here is if the structure and functionalities of the banking system in its fundamentals are capable of replicating (at least from theoretical point of view) the risk sharing efficiency of the complete markets under standard[1] banking assumptions? We address the issues beyond the standard banking assumption in the next chapters. This includes the practical limitations and imperfections within the banking system and also in its interactions with other systems that have systemic consequences reducing the system's efficiency and creating negative externalities. The aim of this chapter is to challenge the conventional view on systemic risk showing that the emergence of the banking structures and dynamics that are known in the literature to be susceptible to financial contagion (Acemoglu et al. 2010, 2013, Allen and Gale 2017) do not relate to the systemic fundamentals and functionalities of banking systems under standard assumptions of the financial intermediation and banking theory.

More specifically we show that under aggregate uncertainty and standard deposit contracts the financial system formed in equilibrium is indeed a decentralization of

[1]These include incomplete deposit contracts and aggregate uncertainty

the social planner's allocation. Thus, it is constrained efficient[2] and occurrence of the financial contagion in such economy can not be explained by the banking system's dynamics or functionalities.

In other words to understand any risk-sharing inefficiency of the banking system we should go beyond the decentralization functionality of the banking system. Hence any regulation that only targets the banking decentralization functionality can not reduce such inefficiencies and may even have unintended consequences. Of course, we are not saying that systemic nature and functionalities of the banking system do not matter in modeling and preventing systemic risk, only that there is no theoretical basis supporting the market failure of the banking system in its nature.

These results have significant regulatory and policymaking implications. The general procedure in developing a regulatory or policymaking framework requires a clear identification (or at least evidence) of the mechanisms by which a market failure is created to justify and develop efficient interventions that address the economic powers causing the failure (Allen and Gale 2007).

This is well-known in the literature that if there is no aggregate uncertainty, decentralization of the banking system is efficient (Allen and Gale 2000, Gale et al. 2017, Freixas and Jorge 2008). It is also constrained efficient with aggregate uncertainty if banks are connected in an appropriate sense" (Diamond and Rajan 2005, Castiglionesi 2013, Allen and Gale 2017, Allen et al. 2020)

In particular this paper extends the modeling framework of the central planner's economy to incorporates both aggregate uncertainty and standard deposits contracts (incomplete market) in her decision problem. This modeling framework provides us with a proper benchmark to compare different market settings with the best theoretical

[2]If an allocation is constrained efficient, the risk-sharing is not as efficient as it would be in a complete market. Therefore, since a social planner (as a benchmark) subject to the same constraints of the banking system could not do any better if replaced by the banking system; the inefficiency is not caused by the banking decentralization. Hence, any regulation that affects the banking's decentralization system can not reduce the inefficiencies.

alternative under the same economic assumptions. As discussed earlier it turns out that the financial equilibrium of the banking system reduces to the social planner's allocation and therefore banking system should be constraint efficient. This means that the illustrative examples of "financial contagion" in the literature[3], if possible, are formed by economic powers other than the decentralization functionality of the banking system.

Therefore, to address externalities like financial contagion and systemic risk, we need to extend the existing theoretical frameworks to the extent that they can identify and use the microeconomic foundation of such phenomena for guiding the regulatory and policymaking intervention capable of internalizing the financial contagion.

To build the ground for a brief review of the basic theory of banking and financial intermediation, we begin with section 3.2.1, formulating a simple situation in which consumers have no interactions (either through markets or financial intermediary) and therefore optimize their consumption bundle in isolation. Next, in section 3.2.2, we allow interaction between consumers through a liquidity market at t_1 and show how the Market allocation economy increases the consumers' welfare compared to Autarky, but not perfectly efficient due to the missing market at t_0. We also generalize the market allocation allowing for non-zero endowments at t_1, and show how the liquidity pricing is easily distorted in Market allocation, even without aggregate uncertainty. Finally, we show that a liquidity market at t_1 increases the expected utility of every consumer, both for classic market allocation economy compared to Autarky economy and generalized market allocation economy compared with generalized Autarky economy. In section 3.3, we replace the market at t_1 with a central social planner and show how it may optimally share risk through consumers with and without aggregate uncertainty. In this section, the social planner with aggregate uncertainty is limited to standard type deposit contracts with consumers to formalize an appropriate benchmark for evaluating

[3]financial contagion only has been investigated as a financial possibility through illustrative examples of some forms of the banking networks (e.g., ring shape networks) which are fragile under small shocks like excess liquidity demand (Allen and Gale 2007) or asset price shocks Acemoglu et al. (2010) in the literature

the banking system's efficiency. section 3.4 formalizes a banking system with standard deposit contracts and an interbank deposit market at t_0. We develop general equilibrium models of the system with and without uncertainty and show that in both cases, the banking system is capable of decentralizing the allocation and risk-sharing solution of the social planner under similar conditions.

3.2 Liquidity, Uncertainty and Financial Market

3.2.1 Autarky Economy

Consider an economy with three dates, $t_0 < t_1 < t_2$, and the numeraire as a single consumption good. Assume that consumers are identical agents with an increasing, strictly concave, and twice continuously differentiable function $u(.)$ and do not interact with each other at any date.

At t_0 consumers have access to a market with two types of assets, liquid assets and illiquid assets. Liquid assets retain their value through time and therefore if sold at either t_1 or t_2, pay equal to their original value at t_0. On the other hand, illiquid assets return $R > 1$ at t_2, but if sold (liquidated) prematurely at t_1 return $r < 1$.There is no investment or trading opportunity at t_1 and therefore there is no difference if an agent consumes liquid assets at t_1 or t_2.

We assume that each consumer is limited to 1 unit of endowment.

Therefore, a consumer's investment decision at t_0 can be summarized by (x, y) as y and x stand for investment on liquid assets and illiquid assets respectively where $y + x = 1$.

Let us introduce a notion of uncertainty regarding the liquidity needs of consumers at t_1. Suppose that the proportion of consumers that face urgent liquidity needs at t_1 is known in advance and denoted by $0 \leq w \leq 1$. Therefore, there is no uncertainty about the aggregate liquidity needs in this setting. However, suppose that each individual consumer does not know in advance if she will be within the proportion of consumers with urgent liquidity need at t_1 or not. Therefore, each individual consumers have uncertainty about her liquidity need at t_1, which can be formulated by setting her probability of facing urgent liquidity need at t_1 to w.

At t_1 uncertainty resolves and each consumer learns if she has urgent liquidity needs at the time or not. From know on we differentiate between consumers with liquidity needs at t_1 referring to them as *early consumers* whereas consumers who have no urgent liquidity needs at t_1 are referred as *late consumers*.

This means that in different realizations of the economy at t_1, an individual consumer may have different realizations about being early or late consumer, but the proportion of early consumers do not depend on the realization of the economy and is known in advance. Later in Section 3.3.3, we make a broader definition of uncertainty to incorporate the possibility of aggregate uncertainty in our models, but for know this definition is enough.

Early consumers sell all their assets at t_1 and each receives $y+r(1-x)$, whereas late consumers wait for the higher rate of return, R, on illiquid assets at t_2 and each receives $y + Rx$. Notice that, since consumers have no interactions at any date, they cannot trade liquidity at t_1. Therefore, late consumers cannot lend or trade their unused liquid assets at t_1 for illiquid assets of early consumers.

Therefore, each consumer's expected consumption utility is $U^A(w) = wU(y+rx)+(1-w)U(y+Rx)$.

With this setting we can formalize each consumer's decision problem as she chooses an investment portfolio, (x, y), at t_0 maximizing her expected consumption utility at t_2:

$$
\begin{aligned}
P^A: \quad & \underset{x,y \in \mathbb{R}^+}{\text{maximize}} \quad && wU(y+rx) + (1-w)U(y+Rx) \\
& \text{subject to} && y + x = 1
\end{aligned}
\tag{3.1}
$$

Proposition 3.2.1 (optimal portfolios in Autarky) *In an Autarky economy, the optimal investment portfolio of a consumer, (x^*, y^*), with a strictly increasing and concave utility function, u(.), satisfies the following:*

(i) $0 < x^* \leq 1$ *and* $0 \leq y^* \leq 1 \iff wr + (1-w)R > 1$

(ii) $x^* = 0,\ y^* = 1 \iff wr + (1-w)R \leq 1$

Proof: First, note that there exist an optimal solution for P^A, since the problem's objective function, $u(.)$, is continues over its compact feasible space. Also note that the left hand sides of (i) and (ii) represent mutually exclusive and collectively exhaustive subsets of the set of optimal solutions.

Let us form the Largangean function of P^A as:

$$\mathcal{L}(x, y, \mu) = wU(y + rx) + (1-w)U(y + Rx) + \mu(y + x - 1)$$

where μ is the Lagrange multiplier associated with (3.1) constraint.
For the forward direction of (i), we have $x^* > 0$ and $y^* \geq 0$ and therefore, KKT conditions require:

$$\frac{\partial \mathcal{L}}{\partial y} = wrU'(y + rx) + (1-w)RU'(y + Rx) + \mu \leq 0 \tag{3.2a}$$

$$\frac{\partial \mathcal{L}}{\partial x} = wU'(y + rx) + (1-w)U'(y + Rx) + \mu = 0 \tag{3.2b}$$

if we subtract (3.2a) form (3.2b), we can derive:

$$\frac{U'(y + rx)}{U'(y + Rx)} \leq \frac{(1-w)(R-1)}{w(1-r)} \tag{3.3a}$$

We have $R > r$ and thus for $x^* > 0$ and $y^* \geq 0$ we have $y + rx > y + Rx$. Thus, we have we have $\frac{U'(y+rx)}{U'(y+Rx)} > 1$. Since $u(.)$ is strictly concave by definition. So, the right-hand side of (3.3a) is greater than 1 and therefore we have:

$$(1-w)R + wr > 1 \tag{3.4}$$

For the forward direction of (ii), we have $x^* = 0$ and $y^* = 1$, and therefore, KKT conditions require:

$$\frac{\partial \mathcal{L}}{\partial y} = wrU'(1) + (1-w)RU'(1) + \mu = 0 \tag{3.5a}$$

$$\frac{\partial \mathcal{L}}{\partial x} = wU'(1) + (1-w)U'(1) + \mu \leq 0 \tag{3.5b}$$

and therefore:

$$(1-w)R + wr \leq 1 \tag{3.6}$$

For the reverse direction of (i), suppose not. Then, since $x, y \in [0, 1]$, the only remaining solution in the feasible space to be optimal is $x = 0$ and $y = 1$. This contradicts with the forward direction of (ii).

For the reverse direction of (ii), suppose not. Then, since $y + x = 1$, we should have $0 < x^* \leq 1$ and $0 \leq y^* \leq 1$. This contradicts with the forward direction of (ii). With this we have concluded the proofs of both directions of (i) and (ii). ■

Corollary 3.2.2 *In an Autarky economy, a risk averse consumer do not invest in illiquid assets unless* $wr + (1-w)R > 1$.

As follows I discuss the optimal investment decisions of a consumer in an Autarky economy in different parametric settings to compare the outcome of Autarky economy with its alternatives discussed in later sections. We can show that a consumer only starts investing in illiquid assets if the expected return of illiquid asset outperform investment in liquid asset. This is consistent with our intuition about how a risk averse investor decides about her portfolio. But later we show that there are alternative economies that can provide risk sharing strategies that reduce the risk exposures of consumers and therefore investment portfolios that violate eq x could be optimal and therefore, consumers can gain higher welfare through higher expected returns on their investments in alternative economies.

These results are the extension of Allen and Gale (2001, 2004, 2000, 2007), Diamond and Dybvig (1983), Allen and Gale (2017), but extends them to non-interior solutions of consumers and therefore covers all possible scenarios in an Autarky economy.

3.2.2 Financial Market

Similar to Autarky economy, we have three dates, $t_0 < t_1 < t_2$, and the numeraire as a single consumption good. Consumers are identical agents with an increasing, strictly concave, and twice continuously differentiable utility function $u(.)$. Again, at t_0 each consumer is limited to 1 unit of endowment and have access to a market with two type of assets, liquid assets and illiquid assets. Liquid assets retain their value through time whereas illiquid assets return $r < 1$ if sold (liquidated) prematurely at t_1 or $R > 1$ if sold at t_2. Therefore, a consumer's investment decision at t_0 can be summarized by (x, y) as y and x stand for investment on liquid assets and illiquid assets respectively where $y + x = 1$.

Contrary to Autarky economy where consumers had no interactions at t_1, here consumers have access to a financial market where they can trade illiquid assets for liquidity at t_1. Notice that we yet do not have any financial intermediary like a banking system. Also consumers do not have access to any financial market at t_0 where they could trade Arrow-like securities (contingent claims; see section 1.3.1). Therefore, this is an incomplete market and therefore the risk sharing equilibrium, is not efficient.

Similar to Autarky economy the only source of uncertainty comes from the liquidity needs of consumers at t_1. With a certain identical probability of $0 \leq w \leq 1$, each consumer face liquidity needs at t_1 and has to liquidate all her assets for consumption. Otherwise, and thus with the probability of $1 - w$ has no liquidity needs at t_1, and can wait until t_2 for higher rate of return on her illiquid assets. Therefore, individual consumers have uncertainty about their consumption needs but since for each consumer the probability of facing liquidity needs at t_1 is identical and known in advance there is no aggregate uncertainty.

As uncertainty resolves at t_1, each consumer learns if she is an early consumer or a late consumer. Late consumers do not need their liquid assets at the time whereas early consumers cannot wait until t_2 to sell their illiquid assets. Therefore, there exist both demand and supply for liquidity. Early consumers trade their illiquid assets for liquidity if priced higher than the liquidation rate of r. On the other hand, late consumers trade their liquidity for illiquid assets if it is priced high enough to outperform the unit rate of return of liquid assets. More formally, if we denote the equilibrium price of liquidity by p, early consumers trade their illiquid assets for liquidity if $p > r$, whereas late consumers trade their liquidity for illiquid assets if $\frac{R}{p} > 1$. In summery equilibrium price satisfies $r \leq p \leq R$. Note that, the equilibrium price of p for liquidity is equivalent to the equilibrium price of $\frac{1}{p}$ for illiquid assets.

Therefore, early consumers trade all of their illiquid assets for liquidity and each receives $c^1 = y + px$, whereas late consumers trade all of their liquidity for illiquid assets and each receives $c^2 = R(\frac{y}{p} + x)$

Therefore, each consumer's expected consumption utility is $U^M(w) = wU(c^1) + (1-w)U(c^2) = wU(y+px) + (1-w)U(R(\frac{y}{p}+x))$.

With this setting we can formalize the competitive equilibrium problem where each consumer chooses an investment portfolio, (x, y), at t_0 maximizing her expected consumption utility at t_2:

$$P^M : \quad \underset{x, y \in \mathbb{R}^+}{\text{maximize}} \quad wU(c^1) + (1-w)U(c^2) \tag{3.7a}$$

$$\text{subject to} \quad y + x = 1 \tag{3.7b}$$

$$c^1 = y + px \tag{3.7c}$$

$$c^2 = R(\frac{y}{p} + x) \tag{3.7d}$$

with the market clearing equation as:

$$(1-w)y = pwx \tag{3.8}$$

This formulation was first introduced by Allen and Gale (2000) based on the theoretical work of Diamond and Dybvig (1983). From now on, we refer to this equilibrium problem as the "classic market allocation equilibrium". Note that in the classic "market allocation equilibrium", investment portfolio of each consumer, (x^*, y^*) at t_0, and the equilibrium price of long assets at t_1 that solves (3.7a), satisfy the following:

(i) $x^* = 1 - w$ and $y^* = w$

(ii) $0 < w < 1 \Longrightarrow p^* = 1$

Recall that "Market allocation equilibrium" models, Allen and Gale (2000, 2007, 2017) assume that consumers are limited to unit size of endowment at t_0 and thus they do not endowed at t_1 and t_2. This results in equilibrium solutions at borders ($w = 0$ and $w = 1$) where there exist no excess short or long assets at t_1. For example, when we have $w = 1$, meaning that all consumers are early consumers, they only invest in short assets and therefore at t_1 no one holds any long assets to trade and therefore the pricing is impossible. In other words, due to this assumption, we expect the market to vanish at borders ($w = 0$ and $w = 1$) since there is no supply or demand in such circumstances. With that the equilibrium pricing losses its meaning at borders. Later in this section, we relax the assumption that consumers are restricted to a unit of endowment at t_0 and receive no endowment at t_1. This not only makes it possible to understand the pricing and market dynamics in borders, but also has implications for non-border circumstances. Therefore, let us generalize the classic market allocation economy of Allen and Gale (2000) by introducing some deterministic amount of endowments received by each consumer at t_1. Formally, let us assume that each consumer is endowed with an amount $\epsilon \geq 0$ endowment of short assets as well as long assets at t_1. Therefore, each consumer at t_1 owns $y + \epsilon$ of short assets and $x + \epsilon$ of long assets. The size of ϵ is known in advance and prior to consumers investment decisions at t_0. Also, for the sake of simplicity and symmetry, the size of long and short endowments are assumed to be equal for each consumer. With extra endowments received by each consumer at t_1, the early consumers have some extra long assets to sell.

Similarly, the late consumers have extra liquidity to buy long assets. This creates a trading opportunity at t_1 even if all consumers have had invested their initial endowment in the same type of asset at t_0. Here we explicitly assume that there exists an outside market for trading long assets for short assets where a consumer can sell a long asset at the liquidation rate of r or buy a long asset at the return rate of R. This assumption is implicit in the classic model of Allen and Gale (2000), as they assume $r \leq p \leq R$ should hold in their formulations of the market allocation. Therefore, we are not changing the market assumption in that regard. Similar to the initial market allocation model, when uncertainty resolves at t_1 early consumers trade all their long assets, $x + \epsilon$ for liquidity. We denote the amount of an early consumers long assets traded in the outside market with x^{ext}. Therefore, early consumer's earnings, $c^1 = y + \epsilon + (x + \epsilon - x^{ext})p + x^{ext}r$, is composed of two parts. The first part, is her earnings from trading a portion of her assets with late consumers at the equilibrium price, p, and the second part is her earnings from liquidating the remaining portion, x^{ext}, in the outside market.

On the other hand, the amount of a late consumer's short assets traded in the outside market is denoted by y^{ext}. So the late consumer's earnings, $c^2 = (\frac{y+\epsilon-y^{ext}}{p})R + (x+\epsilon)R + y^{ext}$, is composed of two parts. The first part is her earnings from trading a portion of her short assets with early consumers at the equilibrium price, $\frac{R}{p}$, and the second part is her earnings from liquidating the remaining portion, x^{ext} in the outside market.

It is easy to see that if in the equilibrium long asset price is higher than r, then an early consumer would not liquidate any portion of her long assets at the outside market. As we will show later, the equilibrium price would not go lower than the liquidation rate and therefore it may seem unnecessary to have an outside market. However, since we have introduced new endowments at t_1, there are some equilibrium circumstances where all consumers are early consumers, $w = 1$, and have only invested in short assets, $x = 1$, and therefore at t_1, there is no demand for their new endowments in long assets. With this, if we do not have an outside market formulated in our model, the equilibrium price would be zero for long assets which is unrealistic. Similarly, there are some equilibrium circumstances where all consumers are late consumers, $w = 0$, and have only invested in short assets, $y = 1$, and therefore at t_1, there is no demand for their new endowments in short assets. In this case, without an outside market the equilibrium price would be zero for short assets which in unrealistic too.

Recall that in the classic models of Allen and Gale (2000) the type of equilibrium solutions where there exist excess short or long assets at t_1 are neglected due to the assumption of zero endowment at t_1. With this assumption, at borders ($w = 0$ and $w = 1$), the market vanishes at t_1 since there is no supply and demand and therefore, the equilibrium pricing losses its meaning. For example, when we have $w = 1$, meaning that all consumers are early consumers, they only invest in short assets and therefore at t_1 no one holds any long assets to trade and therefore the pricing is impossible. This setting generalizes the classic market allocation economy and classic market equilibrium model of Allen and Gale (2000), as each consumer receives equal endowments of short and long assets at t_0. We denote the size of both endowments for each consumer by $\epsilon \geq 0$ and having that known in advance, each consumer chooses an investment portfolio, (x, y), at t_0 maximizing her expected consumption utility at t_2:

$$P_1^{GM}: \quad \underset{x,y \in \mathbb{R}^+}{\text{maximize}} \quad wU(c^1) + (1-w)U(c^2) \tag{3.9a}$$

$$\text{subject to} \quad y + x = 1 \tag{3.9b}$$

$$c^1 = y + \epsilon + (x + \epsilon - x^{ext})p + x^{ext}r \tag{3.9c}$$

$$c^2 = (\frac{y + \epsilon - y^{ext}}{p})R + (x + \epsilon)R + y^{ext} \tag{3.9d}$$

with the market clearing equation as:

$$(1 - w)(y + \epsilon - y^{ext}) = pw(x + \epsilon - x^{ext}) \tag{3.10}$$

This formulation yet does not introduce any aggregate uncertainty to the system. From now on we refer to this economy set that allows equal endowments of long and short assets at t_1 and its equilibrium problem as "generalized market allocation economy" and "generalized market allocation equilibrium" respectively.

Lemma 3.2.3 *In generalized market allocation economy, in equilibrium if $x^{ext} > 0$ and $y^{ext} > 0$ either $w = 0, p = R$ or $w = 1, p = r$.*

Proof: Let us substitute c^1 and c^2 in the (3.9a) with (3.9c), and (3.9d), we can rewrite P_1^{GM} as:

$$
\begin{aligned}
P_2^{GM}: \quad & \underset{x,y\in\mathbb{R}^+}{\text{maximize}} \quad && wU(y+\epsilon+(x+\epsilon-x^{ext})p+x^{ext}r) \\
& && +(1-w)U((\frac{y+\epsilon-y^{ext}}{p})R+(x+\epsilon)R+y^{ext}) \\
& \text{subject to} && y+x=1
\end{aligned} \tag{3.11}
$$

Forming the Lagrangian function of P_2^{GM} as:

$$
\mathcal{L}(x,y,\mu) = wU(y+px)+(1-w)U(R(\frac{y}{p}+x))+\mu(y+x-1) \tag{3.12}
$$

where μ is the Lagrange multiplier associated with (3.11) constraint we have the following first-order conditions of the equilibrium problem:

$$
\begin{aligned}
\frac{\partial\mathcal{L}}{\partial x} &= wpU'\big(y+\epsilon+(x+\epsilon-x^{ext})p+x^{ext}r\big) \\
&\quad +(1-w)RU'\big((\frac{y+\epsilon-y^{ext}}{p})R+(x+\epsilon)R+y^{ext}\big)\begin{cases}\leq 0 & \text{if } x>0\\ =0 & \text{if } x=0\end{cases}
\end{aligned} \tag{3.13a}
$$

$$
\begin{aligned}
\frac{\partial\mathcal{L}}{\partial y} &= wU'\big(y+\epsilon+(x+\epsilon-x^{ext})p+x^{ext}r\big) \\
&\quad +(1-w)(\frac{R}{p})U'\big((\frac{y+\epsilon-y^{ext}}{p})R+(x+\epsilon)R+y^{ext}\big)\begin{cases}\leq 0 & \text{if } x>0\\ =0 & \text{if } x=0\end{cases}
\end{aligned} \tag{3.13b}
$$

$$
\frac{\partial\mathcal{L}}{\partial x^{ext}} = w(r-p)U'\big(y+\epsilon+(x+\epsilon-x^{ext})p+x^{ext}r\big)\begin{cases}\leq 0 & \text{if } x^{ext}>0\\ =0 & \text{if } x^{ext}=0\end{cases} \tag{3.13c}
$$

$$
\frac{\partial\mathcal{L}}{\partial y^{ext}} = (1-w)(1-\frac{R}{p})U'\big((\frac{y+\epsilon-y^{ext}}{p})R+(x+\epsilon)R+y^{ext}\big)\begin{cases}\leq 0 & \text{if } y^{ext}>0\\ =0 & \text{if } y^{ext}=0\end{cases} \tag{3.13d}
$$

subtracting 3.13b from 3.13a we get:

$$w(p-1)U'\big(y+\epsilon+(x+\epsilon-x^{ext})p+x^{ext}r\big)$$
$$+(1-w)(R-\frac{R}{p})U'\big((\frac{y+\epsilon-y^{ext}}{p})R+(x+\epsilon)R+y^{ext}\big)\begin{cases}=0 & \text{if } x>0, y>0\\ \geq 0 & \text{if } x=1, y=0\\ \leq 0 & \text{if } x=0, y=1\end{cases} \tag{3.14}$$

Therfore, if $x^{ext} > 0$, $y^{ext} > 0$, since $U'(.) > 0$ by definition, (3.13c) and (3.13d) imply:

$$w(r-p)=0, \qquad and \qquad (1-w)(1-\frac{R}{p})=0$$

Since $r < R$ by assumption, $r-p$ and $1-\frac{R}{p}$ cannot be zero in the same time. Therefore, either $w=0, p=R$ or $w=1, p=r$, which concludes the proof. ■

Intuitively, this lemma means that it is impossible to have both non-zero excess short assets and excess long assets at t_1, unless there is no short assets demand $(w=0)$ or the opposite, that there is long asset demand$(w=1)$ at t_1.

Corollary 3.2.4 *If $0 < w < 1$, then $x^{ext}=0$ and/or $y^{ext}=0$.*

Proof: It is an immediate result of 3.2.3 ■

Proposition 3.2.5 (Generalized market allocation) *In generalized market allocation economy, equilibrium investment portfolio of each consumer, (x^*, y^*) at t_0 is given by:*

$$x^* = f((1-w)+\epsilon(1-2w)) \tag{3.15a}$$

$$y^* = f(w-\epsilon(1-2w)) \tag{3.15b}$$

$$\text{where } f(z)=\begin{cases} z & \text{if } 0\leq z\leq 1\\ 1 & \text{if } z>1\\ 0 & \text{if } z<0\end{cases}.$$

Proof: For $\epsilon = 0$ the problem is reduced to the classic market allocation problem and thus we refer the reader to Allen and Gale (2000) for the proof[4]. Therefore, here we only consider the cases where $\epsilon > 0$. Let us break up the proof to two parts. In part 1, we investigate the equilibrium solutions at borders ($w = 0$ and $w = 1$) whereas in part 2 we verify the same results for $0 < w < 1$.

part 1 : **for** $\boldsymbol{w}$**= 0**, using (3.13d), we get:

$$(1 - \frac{R}{p})U'\big((\frac{y + \epsilon - y^{ext}}{p})R + (x + \epsilon)R + y^{ext}\big) = 0$$

and therefore, since $U'(.) > 0$, we have $p^* = R$. With that and since $R > 1$, the left hand side of (3.14) would be $(R-1)U'(.)$ which is strictly positive and implies $x^* = 1, y^* = 0$.

This proves the validity of (3.15) for $w = 0$ and $\epsilon > 0$ as $x^* = f(1 + \epsilon) = 1, y^* = f(-\epsilon) = 0$

for $\boldsymbol{w}$**= 1**, using (3.13c), we get:

$$(r - p)U'\big(y + \epsilon + (x + \epsilon - x^{ext})p + x^{ext}r\big) = 0$$

and therefore, since $U'(.) > 0$, we have $p^* = r$. With that and since $r < 1$, the left hand side of (3.14) would be $(r-1)U'(.)$ which is strictly negative and implies $x^* = 0, y^* = 1$.

This proves the validity of (3.15) for $w = 1$ and $\epsilon > 0$ as $x^* = f(-\epsilon) = 0, y^* = f(1 + \epsilon) = 1$

part 2 : **for 0 <**$\boldsymbol{w}$**< 1**, recall from 3.2.4 that have $x^{ext} = 0$ and/or $y^{ext} = 0$. therefore, if $0 < w < 1$, there exist only 3 possible solution sets for the market allocation[5]. Accordingly, we separate this part to 3 subparts and show that the market allocation is given by eq for all of these solution sets.

2.1 : **for** $\boldsymbol{x^{ext} = 0, y^{ext} > 0}$ using (3.13d), we get:

$$(1 - \frac{R}{p})U'\big((\frac{y + \epsilon - y^{ext}}{p})R + (x + \epsilon)R + y^{ext}\big) = 0$$

[4]The investment decision of each consumer is given by $x^* = 1 - w$ and $y^* = w$ if $\epsilon = 0$.

[5]$x^{ext} = 0, y^{ext} > 0$ or $x^{ext} > 0, y^{ext} = 0$ or $x^{ext} = 0, y^{ext} = 0$.

and therefore, since $U'(.) > 0$, we have $p^* = R$. With that and since $R > 1$, the left hand side of (3.14) would be $(R-1)U'(.)$ which is strictly positive and implies $x^* = 1, y^* = 0$. Also, note that for $x^* = 1, y^* = 0$ and $p^* = R$, (3.10) leads to

$$p^* = \frac{(1-w)(\epsilon - y^{ext})}{w(1+\epsilon)} = R$$

since $R > 1$, such equilibrium solutions require the parametric setting of the market to satisfy:

(A) $w < \frac{1}{1+R}$ and $\epsilon > \frac{w}{1-2w}$

This proves the validity of (3.15) since by (A) we have $(1-w)+\epsilon(1-2w) \geq 1$ and $w - \epsilon(1-2w) \leq 0$ which leads to $x^* = f((1-w)+\epsilon(1-2w)) = 1, y^* = f(w - \epsilon(1-2w)) = 0$.

2.2 **: for $\boldsymbol{x^{ext} > 0, y^{ext} = 0}$**, using (3.13c), we get:

$$(r-p)U'\big(y + \epsilon + (x + \epsilon - x^{ext})p + x^{ext}r\big) = 0$$

and therefore, since $U'(.) > 0$, we have $p^* = r$. With that and since $r < 1$, the left hand side of (3.14) would be $(r-1)U'(.)$ which is strictly negative and implies $x^* = 0, y^* = 1$.

Moreover, note that for $x^* = 0, y^* = 1$ and $p^* = r$, (3.10) leads to

$$p^* = \frac{(1-w)(1+\epsilon)}{w(\epsilon - x^{ext})} = r$$

since $r < 1$, such equilibrium solutions require the parametric setting of the market to satisfy:

(B) $w > \frac{1}{1+r}$ and $\epsilon > \frac{1-w}{2w-1}$

This proves the validity of (3.15) since by (B) we have $(1-w)+\epsilon(1-2w) \leq 0$ and $w - \epsilon(1-2w) \geq 1$ which leads to $x^* = f((1-w)+\epsilon(1-2w)) = 0, y^* = f(w - \epsilon(1-2w)) = 1$.

2.3 : for $x^{ext} = 0, y^{ext} = 0$:

if $p^* = 1$, then (3.10) reduces to

$$(1 - w)(y + \epsilon) = w(x + \epsilon)$$

and leads to:

$$x^* = (1 - w) + \epsilon(1 - 2w)$$
$$y^* = w - \epsilon(1 - 2w)$$

however, note that since $y+x = 1$ and $x, y \geq 0$, such equilibrium solutions require the parametric setting of the market to satisfy one of the following:

(C1) $w < \frac{1}{2}$ and $0 \leq \epsilon \leq \frac{w}{1-2w}$

(C2) $w = \frac{1}{2}$ and $0 \leq \epsilon$

(C3) $w > \frac{1}{2}$ and $0 \leq \epsilon \leq \frac{1-w}{2w-1}$

This proves the validity of (3.15) since by (C1),(C2) and (C2) we have $0 \leq (1 - w) + \epsilon(1 - 2w) \leq 1$ and $0 \leq w - \epsilon(1 - 2w) \leq 1$ which leads to $x^* = f((1-w)+\epsilon(1-2w)) = (1-w)+\epsilon(1-2w), y^* = f(w-\epsilon(1-2w)) = w - \epsilon(1 - 2w)$.

if $p^* < 1$, then the left hand side of the (3.14) is strictly negative and therefore $x^* = 0, y^* = 1$. Note that for $p^* < 1$ with $x^* = 0, y^* = 1$ to be an equilibrium solution (3.10) requires

$$p^* = \frac{(1 - w)(1 + \epsilon)}{w\epsilon} \leq 1$$

and thus the the parametric setting of the market should satisfy:

(D) $w > \frac{1}{2}$ *and* $\epsilon \geq \frac{w-1}{1-2w}$

This proves the validity of (3.15) since by (D) we have $0 \leq (1 - w) + \epsilon(1 - 2w) \leq 1$ and $0 \leq w - \epsilon(1 - 2w) \leq 1$ which leads to $x^* = f((1 -$

$w)+\epsilon(1-2w)) = (1-w)+\epsilon(1-2w), y^* = f(w-\epsilon(1-2w)) = w-\epsilon(1-2w)$.

if $p^* > 1$, then the left hand side of the (3.14) is strictly positive and therefore $x^* = 1, y^* = 0$. Note that for $p^* > 1$ with $x^* = 1, y^* = 0$ to be an equilibrium solution (3.10) requires

$$p^* = \frac{(1-w)\epsilon}{w(1+\epsilon)} \geq 1$$

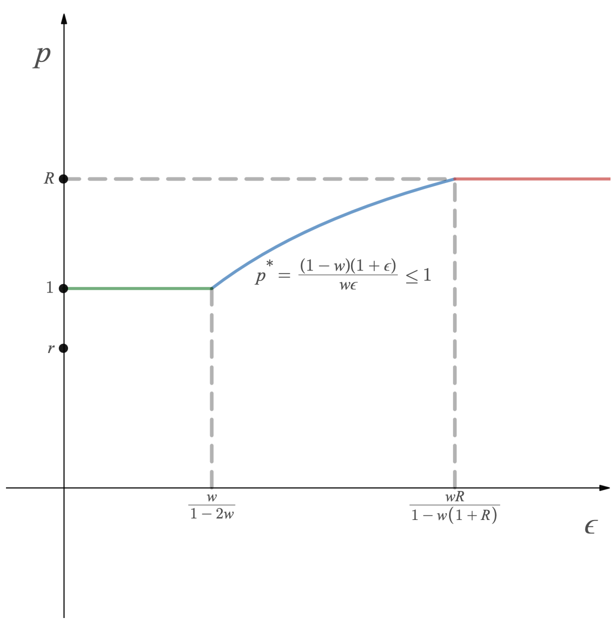
and thus the the parametric setting of the market should satisfy:

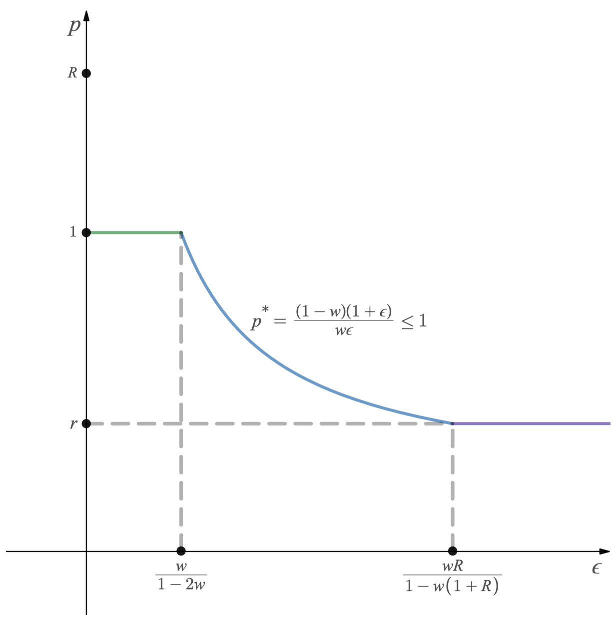
(E) $\quad w > \frac{1}{2} \quad and \quad \epsilon \geq \frac{w-1}{1-2w}$

This proves the validity of (3.15) since by (E) we have $0 \leq (1-w)+\epsilon(1-2w) \leq 1$ and $0 \leq w-\epsilon(1-2w) \leq 1$ which leads to $x^* = f((1-w)+\epsilon(1-2w)) = (1-w)+\epsilon(1-2w), y^* = f(w-\epsilon(1-2w)) = w-\epsilon(1-2w)$.

with this, we have concluded the proof of proposition 3.2.5. ■

The following figures summarize the proposition 3.2.5:





Corollary 3.2.6 (Generalized market allocation liquidity pricing) *In the generalized market allocation economy, the equilibrium liquidity pricing is given by:*

$$p^* = \begin{cases} min\{1, \frac{(1-w)\epsilon}{(1+\epsilon)w}\} & if\ 0 \leq w \leq \frac{1}{2} \\ max\{1, \frac{(1-w)(1+\epsilon)}{\epsilon w}\} & if\ \frac{1}{2} \leq w \leq 1 \end{cases} \tag{3.16}$$

for $\epsilon > 0$.

Proof: For $0 \leq w \leq \frac{1}{2}$ and $\epsilon \leq \frac{w}{1-2w}$ we have $\frac{(1-w)\epsilon}{(1+\epsilon)w} \geq 1$ and therefore by (3.16) we get $p^* = 1$. This result has been proven in proposition 3.2.5 (see **part 1** and **2.3** in **part 2**).

For $0 \leq w \leq \frac{1}{2}$ and $\epsilon \geq \frac{w}{1-2w}$ we have $0 \leq \frac{(1-w)\epsilon}{(1+\epsilon)w} \leq 1$ and therefore by (3.16), we should have $p^* = \frac{(1-w)\epsilon}{(1+\epsilon)w}$. This result has been proven in proposition 3.2.5 (see **part 1** and **2.1**, **2.3** in **part 2**).

For $\frac{1}{2} \leq w \leq 1$ and $\epsilon \geq \frac{w-1}{1-2w}$ we have $\frac{(1-w)(1+\epsilon)}{\epsilon w} \geq 1$ and therefore by (3.16), we should have $p^* = 1$. This result has been proven in proposition 3.2.5 (see **part 1** and **2.3** in **part 2**).

For $\frac{1}{2} \leq w \leq 1$ and $\epsilon \leq \frac{w-1}{1-2w}$ we have $\frac{(1-w)(1+\epsilon)}{\epsilon w} \leq 1$ and therefore by (3.16), as proven in proposition 3.2.5 (see **part 1** and **2.2**, **2.3** in **part 2**) we should have $p^* = \frac{(1-w)(1+\epsilon)}{\epsilon w} \leq 1$. ■

This is well known in the literature that the classic market allocation economy is more efficient than Autarky economy. Recall that for each consumer, the expected consumption utility when there is no endowment at t_1 in Autarky economy is $U^A(w) = wU(y + rx) + (1 - w)U(y + Rx)$, whereas in classic market allocation it is $U^M(w) = wU(1) + (1 - w)U(R)$. It is easy to see that since $y + x = 1$ and $r < 1 < R$, we always have $y + rx \leq 1$ and $y + Rx \leq R$ and therefore $U^A(w) \leq U^M(w)$ with strict inequality if $0 < w < 1$ and therefore the Market allocation economy is more efficient than Autarky economy.

The same result holds for the generalized case introduced here where consumers may receive non-zero endowments of the short and long assets at t_1 or in other words $\epsilon > 0$. Recall that the expected consumption utility of $U^{GM}(w) = wU(y+\epsilon+(x+\epsilon-x^{ext})p+x^{ext}r)+(1-w)U((\frac{y+\epsilon-y^{ext}}{p})R+(x+\epsilon)R+y^{ext}$ in Generalized market allocation economy, whereas in Autarky economy with non-zero endowments, expected consumption utility of each consumer is $U^{GA}(w) = wU(y + \epsilon + (x + \epsilon)r) + (1 - w)U(y + \epsilon + (x + \epsilon)R)$. Here again, Generalized market allocation is more efficient that Autarky allocation since $r < 1 < R$ and $y^{ext} \geq 0$ and $y^{ext} \geq 0$ leads to $y + \epsilon \leq (\frac{y+\epsilon-y^{ext}}{p})R + y^{ext}$ and $(x + \epsilon)r \leq x + \epsilon - x^{ext})p + x^{ext}r$ and therefore $U^{GM}(w) \geq U^{GA}(w)$ for any $w \in W$.

3.3 Social Planner and Optimal Risk Sharing

Suppose that consumers do not have access to a liquidity market and instead a social planner plays the intermediary role of collecting the endowments, making investment decisions, and finally allocating the consumption profiles to early and late consumers at t_1 and t_2 respectively, maximizing the aggregate social welfare. Such an economy is used as a theoretical benchmark to compare the performance of more realistic alternative economies in terms of the aggregate social welfare. In this section, we first briefly review Allen and Gale (2000)'s model of social planner's allocation where it is assumed that no aggregate uncertainty exists. Allen and Gale (2000)'s model is widely accepted in the literature as a main benchmark to compare the performance of different market settings.In the second part of this section, we introduce a generalized formulation of the planner's allocation, that allows aggregate uncertainty and formulates it in the social planner's problem. We argue that the generalized formulation is a better benchmark if comparing alternative market settings where there exists aggregate uncertainty in the liquidity market. It is more informative as it controls the effect of aggregate uncertainty and helps us to understand how well an alternative market setting performs compared to a benchmark developed under the same assumption (and here, under aggregate uncertainty.) Later in the section 3.4, we formulate the decentralized alternatives of both benchmark economies introducing a banking system with an interbank deposit market and compare their optimal allocations of consumption profiles to early and late consumers.

3.3.1 Aggregate Uncertainty

The possibility of aggregate uncertainty means that not only we have uncertainty about the liquidity need of individual consumers, but also there exists uncertainty about the aggregate liquidity need of consumers too.

To be able to incorporate the possibility of aggregate uncertainty in our formulations properly, we need to be more specific about the formal definition of uncertainty in the economy. Let us define the aggregate proportion of early consumers as a state contingent variable, $0 \leq w(s) \leq 1$, for any $s \in \boldsymbol{S}$, where $\boldsymbol{S}$ represents the space of all possible states. This is equivalent to defining $0 \leq w \leq 1$, as a random variable on the probability space, $(\boldsymbol{S}, \Sigma, \boldsymbol{P})$, where Σ denotes all the collections of the economic scenarios and $\boldsymbol{P}$ is the probability measure equipped with $\boldsymbol{S}$. We use state contingent and random variable definitions of uncertainty interchangeably. More specifically, the notation of w is used referring to the random variable definition whereas the notation $w(s)$; $\forall s \in S$ is used referring to the equivalent state contingent definition.

In this setting, the economies that have no aggregate uncertainty can be represented as if $w(s) = w$ for all $s \in S$ as a certain scalar known at t_0. In addition, as defined in 3.2.1 in an economy without aggregate uncertainty, we can use the same notation of w to denote the probability of a consumer facing liquidity need at t_1 (being an early consumer). This is equivalent to assuming that the binary random variables of individual consumers being an early consumer at t_1 are equal in distribution (see the definition of equality in distribution in Klenke (2008), Jacod and Protter (2000)). Therefore, individual consumers have symmetric expectations of being an early consumer at t_1, even if different consumers have different probabilities of being an early consumer at the same states[6].

On the other hand, an economy with aggregate uncertainty can be represented as if there exist $s_a, s_b \in \boldsymbol{S}$ where $w(s_a) \neq w(s_b)$, or equivalently if w is a random variable defined on the probability space, $(\boldsymbol{S}, \Sigma, \boldsymbol{P})$.

Therefore, under aggregate uncertainty, the planner's investment in short and long assets at t_0 leads to a state-contingent/stochastic consumption bundle offered by the social planner at t_1 and t_2.

3.3.2 Planner's Allocation without Aggregate Uncertainty

The economy has three times, $t_0 < t_1 < t_2$, and the numeraire as a single consumption good. Moreover, assume that consumers are identical agents who are limited to 1 unit of endowment at t_0 and have a strictly increasing, strictly concave, and twice continuously differentiable utility function $u(.)$. However, contrary to the Autarky and classic market allocation economies, consumers do not have direct access to a market of short and long assets. Instead, a central social planner makes per capita investment decisions in the long and short assets on behalf of consumers. The symmetric setting of the economy lets us use the same utility function of individual consumers to formulate the aggregate social welfare of the consumers as the social planner's utility function. At t_0 social

[6]Under aggregate uncertainty, probability of a consumer to be an early consumer is a state contingent/random variable and may be different from the state contingent proportion of early consumers at the same state.

planner makes per capita investment decisions in long and short assets, denoted by (x, y) as y and x stand for investment on short assets and long assets respectively where $y + x = 1$. We also assume certain rates of return of short and long assets, denoted by $0 \leq r < 1$ and $R > 1$ respectively. Similar to the Autarky and classic market allocation economies, the only source of uncertainty comes from the liquidity needs of consumers at t_1. With a certain identical probability of $0 \leq w \leq 1$, each consumer face liquidity needs at t_1. Therefore, with the probability of $1 - w$ each consumer has no liquidity needs at t_1, and can wait until t_2. Therefore, individual consumers have uncertainty about their consumption needs, but since for each consumer, the probability of facing liquidity needs at t_1 is identical and known in advance, there is no aggregate uncertainty.

As the uncertainty resolves at t_1 each consumer learns if she is an early consumer or a late consumer. Social planner allocates c^1 unites of consumption to early consumers at t_1. On the other hand, since late consumers do not need liquidity at t_2, social planner wait until t_2 to use the higher rate of returns of the long assets $(R > 1)$ to be able to allocate a higher level of consumption to late consumers, denoted by c^2 at t_2. With that, we can formalize the social planner's aggregate welfare utility function as $U^{SP} = wU(c^1) + (1 - w)U(c^2)$ and thus the social planner's decision problem as:

$$P_1^{SP}: \quad \underset{x, y \in \mathbb{R}^+}{\text{maximize}} \quad wU(c^1) + (1 - w)U(c^2) \tag{3.17a}$$

$$\text{subject to} \quad y + x = 1 \tag{3.17b}$$

$$y = wc^1 \tag{3.17c}$$

$$Rx = (1 - w)c^2 \tag{3.17d}$$

Note that (3.17b) represents the social planner's per capita investment in short and long assets at t_0. Since there is no aggregate uncertainty, then the optimal investment in short (liquid) assets would be equal to the consumption allocated to early consumers, as denoted by (3.17c). Similarly, as denoted by (3.17d), the optimal investment in long assets would be equal to the consumption allocated to late consumers discounted by R.

It is easy to see that the equilibrium solution of the classic market allocation is a feasible solution for the social planner too, however it may not be the social planner's optimal solution. To see that, let us substitute $x = \frac{(1-w)}{R}c^2$ and $y = wc^1$ and rewrite P_1^{SP} as:

$$P_2^{SP}(w): \quad \underset{c^1, c^2 \in \mathbb{R}^+}{\text{maximize}} \quad wU(c^1) + (1-w)U(c^2) \tag{3.18a}$$

$$\text{subject to} \quad \frac{(1-w)}{R}c^2 + wc^1 = 1 \tag{3.18b}$$

and form the Largangean function of $P_2^{SP}(w)$ as:

$$\mathcal{L}(c^1, c^2, \lambda) = wU(c^1) + (1-w)U(c^2) - \lambda(\frac{(1-w)}{R}c^2 + wc^1 - 1)$$

where λ is the Lagrange multiplier associated with (3.18b) constraint.

Then first-order optimality conditions require

$$\frac{\partial \mathcal{L}}{\partial c^1} = \quad w\ (U'(c^1) - \lambda) \quad \begin{cases} \leq 0 & \text{if } c^1 = 0 \\ = 0 & \text{if } c^1 > 0 \end{cases} \tag{3.19a}$$

$$\frac{\partial \mathcal{L}}{\partial c^2} = (1-w)(U'(c^2) - \frac{\lambda}{R}) \begin{cases} \leq 0 & \text{if } c^2 = 0 \\ = 0 & \text{if } c^2 > 0 \end{cases} \tag{3.19b}$$

to hold for $c^1 > 0$ and $c^2 > 0$ and thus $\frac{\partial \mathcal{L}}{\partial c^2} = 0$ and $\frac{\partial \mathcal{L}}{\partial c^1} = 0$. This leads to $U'(c^1) = \lambda$ and $U'(c^2) = \frac{\lambda}{R}$ and therefore we expect

$$\frac{U'(c^1)}{U'(c^2)} = R > 1 \tag{3.20}$$

to hold at the optimal solution of the social planner's allocation. Now we can see that the optimal solution to classic market allocation, $c^1 = 1$ and $c^2 = R$ does not satisfy (3.20) for some utility functions like $U(c) = x - (1-x)^2$ and therefore is not an optimal solution of the social planner's allocation problem.

3.3.3 Planner's Allocation with Aggregate Uncertainty

So far, we have assumed that there is no aggregate uncertainty in the economy and compared the efficiency of different allocations, including the benchmark (social planner's) allocation under that assumption. In this section, we extend the benchmark (social planner's) allocation allowing for aggregate uncertainty, to be the basis of the comparison of alternative allocation settings under aggregate uncertainty. More specifically, in the following sections, we formulate the banking system's allocation with aggregate uncertainty and show that it can perfectly decentralize the benchmark's allocation. This is contrary to the widely accepted idea in the literature that the financial market's fragility and possibility of contagion is an externality of decentralization of the financial market under aggregate uncertainty.

Suppose that instead of a fully contingent consumption bundle, the social planner commits to a certain per capita consumption of $\overline{c^1}$ to early consumers and a state-contingent payoff of $c^2(s)$ to late consumers[7] subject to $\overline{c^1} \leq c^2(s)$ for any $s \in \boldsymbol{S}$. Of course, with this setting, social planner may default on her commitment to allocate the fixed consumption of $\overline{c^1}$ to early consumers. Therefore, the actual consumption allocated to early consumers is indeed state contingent.

Planner is solvent in a given state if she can fully payoff the committed $\overline{c^1}$ of consumption to early consumers and also a state-contingent payoff of $c^2(s)$ to late consumers where $c^2(s) > \overline{c^1}$ holds. We assume that late consumers would behave like early consumers (in a sense similar to the bank-run phenomena) as soon as the allocated consumption to late consumers equates or falls bellow the consumption of the early consumers[8]. Therefore, the planner defaults in a given state if she can not fully payoff the committed $\overline{c^1}$ to early consumers as well as $c^2(s) > \overline{c^1}$ to late consumers. In such circumstances, due to the bank-run-like behavior of consumers, the actual consumption allocated to both early and late consumers is the per capita liquidation value of the planner's investment at t_1 or formally $y + rx$.

The planner's problem is to maximize the expected aggregate social welfare of the consumers and therefore her investment decision may efficiently lead to her bankruptcy

[7] To simplify the notation we use $c^1(s)$ and $c^1(s)$ interchangeably referring to state contingent consumption at t_1 and similarly $c_{t_2}(s)$ and $c^2(s)$ referring to state contingent consumption at t_2.

[8] Thus insolvency and bankruptcy are the same in this setting as planner in both cases has to liquidate all of her assets at t_1 to payoff the consumers

in certain states, especially if the probability of the realization of those states are low enough. Therefore, despite the fact that the consumption allocated by the planner to early consumers is supposed to be risk free and independent from the state of the world at t_1, the existence of aggregate uncertainty may lead the optimal investment decision of the planner to be her bankruptcy in some states and therefore the risk exposure of early consumers.

Under aggregate uncertainty, the planner's committed early consumption profile, denoted by $\overline{c^1}$ only realizes in circumstances that planner is not bankrupt. To capture the bankruptcy circumstances we need to introduce a state contingent variable, denoted by $c^1(s)$, that represents the actual consumption profile allocated to early consumers including the planner's bankruptcy circumstances too. Note that $c^1(s) = y + rx$ if planner is bankrupt in state s and otherwise $c^1(s) = \overline{c^1}$. This helps to internalize the risk exposures of the social planner and thus consumers through formulating the possible bankruptcy scenarios of the planner into her decision problem: Accordingly, the social planner's decision problem solves[9]:

$$\hat{P}_2^{SP}: \max_{x,y\in\mathbb{R}^+} \quad \sum_{s\in S} \boldsymbol{P}(s).\big[wU(c^1(s) + (1-w)U(c^2(s)\big] \tag{3.21a}$$

$$\text{s.t.} \quad y + x = 1 \tag{3.21b}$$

$$y + \alpha^s xr = w^s c^1(s) + y^{exc}(s) \qquad \forall s \in S \tag{3.21c}$$

$$y^{exc}(s) + R(1-\alpha^s)x = (1-w^s)c^2(s) \qquad \forall s \in S \tag{3.21d}$$

$$y + rx \le c^1(s) \le \overline{c^1} \qquad \forall s \in S \tag{3.21e}$$

$$c^1(s) \le c^2(s) \le y + rx + M(c^2(s) - c^1(s)) \qquad \forall s \in S \tag{3.21f}$$

$$0 \le \alpha^s \le 1, \qquad \forall s \in S \tag{3.21g}$$

where for a given state, $s \in S$, α^s is a decision variable that denotes the proportion of the long assets liquidated early at t_1 and y^{exc} is a decision variable that denotes the

[9] If we use the notation of c^1 and c^2 to refer to the random variable definition of the equivalent state contingent variables, $c^1(s)$ and $c^2(s)$; $\forall s \in S$, respectively, the objective function can alternatively be formulated as: $E_{\boldsymbol{P}}\big[wU(c^1) + (1-w)U(c^2)\big] = \sum_{s\in S} \boldsymbol{P}(s).\big[wU(c^1(s) + (1-w)U(c^2(s)\big]$

excess liquidity transferred from t_1 to t_2. (3.21b) is the planner's budget constraint at t_0 and (3.21c) and (3.21d) are the budget constraint at t_1 and t_2 respectively.

The model formulates both types of the state-contingent (forward) decision problems of the planner, either the planner investment profile of (x, y) at t_0 in a given state, $s \in S$ at t_1 leads to her being solvent or bankrupt.

If the planner is not bankrupt ($c^2(s) > c^1(s)$), then α^s in (3.21c) is used to formulate the circumstances where the investments in short assets, y is less than the liquidity needs of the early consumers at state s and therefore the planner liquidates a portion ($0 \leq \alpha^s \leq 1$) of his long assets at the rate of r to payoff the early consumers. $y^{exc}(s)$ is used in (3.21c) to capture the circumstances where the investments in short assets, y, is more than the liquidity needs of the early consumers at state s. Accordingly, the planner transfers the excess of the short assets, $y^{exc}(s)$ to the budget constraint at t_2 formulated by (3.21d) to determine the consumption profile of the late consumers.

It is easy to see that if we have a shortage of liquidity at t_1 then $y^{exc}(s) = 0$ and $\alpha^s > 0$, where as If there is an excess liquidity at t_1 then $y^{exc}(s) > 0$ and $\alpha^s = 0$ and finally if the liquidity demand and supply match exactly in a given state, then $y^{exc}(s) = 0$ and $\alpha^s = 0$.

Moreover (3.21e) is used to determine the optimal and unique early consumption profile of $\overline{c^1}$ across all the states that the planner is solvent. Note that (3.21e) and the strict concavity of the objective function,(3.21a), assure that if planner is not bankrupt, then $c^1(s) = \overline{c^1}$ holds for the optimal solution of P^3 [10]. If the planner is bankrupt ($c^2(s) = c^1(s)$), then (3.21f) and (3.21e), where M is a arbitrary large number, assure that the consumption profile allocated to all consumers is the per capita liquidation value of the planner's investments at t_1 or formally $c^1(s) = c^2(s) = y + rx$.

This is also consistent with the planners budget constraints at t_1 and t_2, since if $c^1(s) = c^2(s) = y + rx$ holds then (3.21c) and (3.21d) lead to $\alpha^s = 1$ which captures the bank-run-like behavior of the consumers under the planners bankruptcy.

[10] social planner pays off the committed consumption profile allocated to early consumers, $\overline{c^1}$, if not bankrupt

3.4 Banking System and Decentralization

In this section, we consider the intermediary role of a banking system as a main provider of liquidity insurance to consumers. We first briefly review the main modeling approach that adopts the theory of banking developed by Bryant (1980) and Diamond and Dybvig (1983) for the analysis of liquidity shocks and their implications for financial crises. As shown by Allen and Gale (2017), within that framework, the optimal risk sharing of the social planner under no aggregate risk can be perfectly decentralized by a banking system. However, as argued in Castiglionesi (2013) and Brusco and Castiglionesi (2007), when one of the assumptions in the benchmark model is removed and therefore the market has some imperfections, the banking system fails to achieve the social optimum which justifies imposing some liquidity regulations.

Here we argue that liquidity regulations on the banking system could not be well justified (at least from the theoretical point of view) unless their efficiency is compared with a benchmark under the same assumptions. This is essential to have a clear microeconomic foundation of externality in liquidity provision to justify and develop a regulatory framework. Therefore, we will formalize a decentralized banking system in the second part of this section to compare it with a benchmark of the social planner that we formalized in the previous section, whereas both have to deal with the same market imperfection of aggregate uncertainty. As showed in section 3.3.3, under aggregate uncertainty, it is quite possible to have an optimal social planner's allocation that efficiently results in her bankruptcy in some states and therefore exposure of the consumers to liquidity risk. In the second part of this section, we show (see theorem 3.4.4) that, under aggregate uncertainty, a competitive banking system with access to interbank deposit contracts would decentralize the same equilibrium solution, and therefore do not expose the consumers to any risk more than what a benchmark social planner's solution would do.

3.4.1 Banking System's Allocation Without Aggregate Risk

Suppose that a competitive banking system with access to interbank deposit contracts (or more generally, the money market) plays an intermediary role in the liquidity market, as banks collect the deposits of the consumers and invest them in short and long assets with certain rates of return. We assume that there is no aggregate uncertainty about the liquidity needs of the consumers at t_1, but individual consumers are exposed to different liquidity uncertainties and therefore the banks are supposed to provide liquidity insurance for individual consumers. Contrary to the social planner, individual banks may be exposed to liquidity uncertainties, if there is no interaction between them. We see later that the set of interbank deposit contracts is indeed an efficient integration mechanism that helps banks to perfectly share their exposures to uncertainty and risk and reproduce the same consumption allocations of the social planner for their customers.

To formalize the economic setting, suppose that at t_0 each consumer invests her single unit of endowment in a deposit contract with one of the banks in the banking system. Deposit contract of a bank at t_0 is defined as a per capita payoff of c^1 if the depositor withdraws at t_1 or a per capita payoff of c^2 if the depositor waits until t_2 for withdrawal. We assume that, at t_0, each bank makes decision about the per capita investments of the collected endowments in short and long assets, denoted by (x_i, y_i) as y_i and x_i stand for investments of bank $i \in N$, for short assets and long assets respectively. We also assume certain rates of return of short and long assets, denoted by $0 \leq r < 1$ and $R > 1$ respectively. maximizes the aggregate expected welfare of its consumers at t_1 and t_2 as it decides about (allocates) the contractual payoffs (consumption profiles) to early and late consumers at t_0 and t_1, respectively. Note that due to the symmetric setting of the economy at t_0, the payoff contracts of the banks in equilibrium are identical and due to the strict concavity of the utility function we have $c^2 > c^1$.

Even though there is no aggregate uncertainty in this economy, since each bank's collection of customers may have different liquidity needs in different states at t_1, the liquidity needs of individual banks are uncertain too. Recall the equivalent notations introduced to capture the uncertainty of the liquidity needs of consumers in section 3.3.3. To keep the notation simple and also illustrate how individual banks in this framework are treated as an aggregation of their customers (consumers), we use similar notational definitions for individual banks' here. With an abuse of notation, let us denote the proportion of bank i's consumers that are early consumer at a given state $s \in S$ by $w_i(s)$ where:

$$\frac{1}{n}\sum_{i=1}^{N} w_i(s) = w(s) \tag{3.22}$$

holds. Equivalently, we denote the state contingent liquidity needs of bank $i \in N$ in a given state $s \in S$ by $w_i(s)$. Note that $w(s)$ in (3.22) is the state contingent aggregate proportion of early consumers in the economy.

At t_1 uncertainties resolve and early consumers withdraw the payoff of c^1 from their banks whereas the late consumers wait until t_2 to receive the higher payoff of c^2. Allen and Gale (2000) showed that if banks have access to an interbank deposit market, they can resolve these uncertainties using bilateral deposit contracts and perfectly share their exposure to risk and indeed reproduce the same consumption allocation of the social planner for their customers.

To illustrate the risk sharing mechanism of the bilateral deposit contacts and the functionality of the interbank market in this economy, let us borrow an example from Allen and Gale (2000).

Consider a banking system comprised of four banks indexed by $N = \{1, 2, 3, 4\}$. There are two possible states, denoted by s_a, s_b, with equal probabilities of occurrence at t_1 regarding the liquidity needs of the individual banks' customers (consumers) as summarized in Table 3.1. Let us fix the state contingent liquidity needs of the banks to be either $w_i(s) = \lambda_H$ or $w_i(s) = \lambda_L$, where $\lambda_H > \lambda_L$. For example, if state s_a realizes, $w_1^{s_a} = \lambda_H$ proportion of bank 1 customers face liquidity need and withdraw their deposits at t_1. Suppose that at t_0 banks make equal bilateral deposits in other

Table 3.1: Banking system in an economy without aggregate uncertainty

state	prob	Bank1	Bank 2	Bank 3	Bank 4	Aggregate
s_a	$\frac{1}{2}$	$w_1(s_a) = \lambda_H$	$w_2(s_a) = \lambda_L$	$w_3(s_a) = \lambda_H$	$w_4(s_a) = \lambda_L$	$\frac{\lambda_H+\lambda_L}{2}$
s_b	$\frac{1}{2}$	$w_1(s_b) = \lambda_L$	$w_2(s_b) = \lambda_H$	$w_3(s_b) = \lambda_L$	$w_4(s_b) = \lambda_H$	$\frac{\lambda_H+\lambda_L}{2}$

banks and each bank's aggregate deposit is $\frac{\lambda_H+\lambda_L}{2}$. Note that due to the symmetry of deposits, they do not not violate[11] the budget constraints of the banks at t_0. At t_1 each bank with high liquidity need of λ_H withdraws d^1 of her deposits to be able to payoff her early consumers at the rate of c^1. This liquidity is supplied using the excess liquidity of the banks with low liquidity need of λ_L at t_1 [12]. Note that this transfer of liquidity at t_1 creates a difference in the bilateral deposit accounts of the banks and requires the banks that received d^1 of liquidity from other banks at t_1 to payback them as much as d^2 at t_2. Note that since banks have identical endowments of one unit as well as equal deposits in other banks with symmetric exposures to liquidity needs at t_1, they should offer identical consumption bundle at t_0 in equilibrium. With that setting, we can formalize the problem of each bank as:

$$\max_{x,y\in\mathbb{R}^+} \quad \tfrac{1}{2}(\lambda_H U(c^1) + (1-\lambda_H)U(c^2)) + \tfrac{1}{2}(\lambda_L U(c^1) + (1-\lambda_L)U(c^2)) \tag{3.23a}$$

$$\text{s.t.} \quad y + x = 1 \tag{3.23b}$$

$$y - d^1 = \lambda_L c^1 \tag{3.23c}$$

$$Rx + d^2 = (1-\lambda_L)c^2 \tag{3.23d}$$

$$y + d^1 = \lambda_H c^1 \tag{3.23e}$$

$$Rx - d^2 = (1-\lambda_H)c^2 \tag{3.23f}$$

where (3.23c) and (3.23e) formulate d^1 of the aggregate deposit payoff at the rate of c^1 withdrawn by the banks with high liquidity need from the banks with low liquidity need

[11] since each bank deposits the same size of what it receive from other banks and therefore they cancel out each other.

[12] a banks with low liquidity need after the payoff of to her early consumers at t_1, has excess liquidity of $\boldsymbol{y} - \lambda_l$ that can be used by the banks with high liquidity needs.

at t_1 in to the budget constraints of the banks at t_1. On the other hand, (3.23e) and (3.23f) formulate d^2 of the aggregate deposit payoff at the rate of c^2 paid by the banks with high liquidity need of λ_H to other banks at t_2 into the budget constraints of the banks at t_2. Note that since at t_0 the interbank deposits are symmetric they cancel out each other and therefore there is no need to include them in the formulation of budget constraint at t_0 as represented by (3.23b). Note that if we subtract (3.23c) from (3.23e), and similarly subtract (3.23d) from (3.23f), solving for d^1 and d^2 respectively we can see how the liquidity transfers of $d^1 = (\frac{\lambda_H - \lambda_L}{2})c^1$ at t_1 and their equivalent paybacks of $d^2 = (\frac{\lambda_H - \lambda_L}{2})c^2$ at t_2 are using the bilateral deposit contracts of banks at t_0 help banks to make sure that they do have access to the liquidity they need at t_1, no matter how much is the aggregate liquidity need of their consumers. Therefore, the banking system is capable of perfectly sharing their exposure to liquidity shocks and reproducing the consumption allocation of the social planner. To see that, let us replace the four banks of the above example with a social planner and see how the consumption bundle is allocated. Since the aggregate liquidity needed at t_1 is $\frac{\lambda_H + \lambda_L}{2}$ the social planners problem, P_1^{SP}, would be:

$$\max_{x,y \in \mathbb{R}^+} \quad (\tfrac{\lambda_H + \lambda_L}{2})U(c^1) + (1 - \tfrac{\lambda_H + \lambda_L}{2})U(c^2) \tag{3.24a}$$

$$\text{s.t.} \quad y + x = 1 \tag{3.24b}$$

$$y = (\tfrac{\lambda_H + \lambda_L}{2})c^1 \tag{3.24c}$$

$$Rx = (\tfrac{\lambda_H + \lambda_L}{2})c^2 \tag{3.24d}$$

which is equivalent to the decision problem of the individual banks[13] and therefore has the same optimal solution of the banking system's equilibrium. For the general formu-

[13]Objective function of the banking system, (3.23) and the objective function of the social planner, (3.24a), are identical. Moreover, the budget constraint of the banking system, (3.23b), is identical to the budget constraint of the social planner, (3.24b). Adding the budget constraints of a bank at t_1, (3.23c) and (3.23e), reduces to in the budget constraint of the social planner at t_1, (3.24c). Similarly adding the budget constraints of banking system at t_2, (3.23c) and (3.23e), results in the budget constraint of the social planner at t_2, (3.24c). Therefore, the banking system's problem reduces to the social planner's allocation problem

lation of the banking system under no aggregate uncertainty and a detailed discussion on how the banking system efficiently decentralizes the social planners' allocation, we refer the reader to Allen and Gale (2017, 2004, 2007).

3.4.2 Banking System's Allocation Under Aggregate Risk

Here we extend the banking system's allocation problem allowing for the aggregate uncertainty in the economy. Suppose that a competitive banking system have access to a similar interbank deposit market of the section 3.4.1 and plays the same intermediary role collecting consumers' endowment through deposit contracts and invest them in short and long assets. However, as we allow aggregate uncertainty of the liquidity need of individual consumers, the dynamics of the interbank deposit market may require banks in some states to liquidate some or all of their long assets early at t_0 to satisfy their budget constraints. Moreover, the possibility of bankruptcy at some states should be formulated in the bank's decision problem in the general equilibrium formulation of the banking system.

Suppose that at t_0 bank i for all $i \in N$, makes decision about the per capita investment of the collected endowments of her customers (consumers) in long and short assets, denoted by (x_i, y_i), as well as how much it deposits in other banks, denoted by $D_{ij}, \forall j \in N$. Without loss of generality, we assume that $\sum_j D_{ij} = \sum_j D_{ji}$ and therefore the budget constraint at t_0 is:

$$x_i + y_i = 1, \qquad \forall i \in N \tag{3.25}$$

Moreover, bank i commits to a certain per capita consumption of $\overline{c_i^1}$ to her early consumers and a state-contingent payoff of $c_i^2(s)$ to her late consumers. However, since under aggregate uncertainty there is a possibility of bank i's bankruptcy in certain states, the face value of the deposit contract for bank i's early customers, $\overline{c_i^1}$, only realizes if bank i is solvent. Therefore, the actual allocated consumption for early consumers of bank i is not independent from the state of the world at t_1. To capture this in our formulation that incorporates the possibility of insolvency in each banks decision problem, we define $c_i^1(s)$ as a state contingent contract[14] for the early consumers of the bank i. Later we discuss how the state contingent rate of $c_i^1(s)$ relates to the fixed rate of $\overline{c_i^1}$.

Here we use the same definition and notation of previous sections for the rates of return of short assets, $0 \leq r < 1$, and long assets, $R > 1$.

At t_1, when the uncertainty resolves, bank i learns about the proportion of her early customers as well as how much other banks withdraw from their deposits in bank i and therefore how much liquidity in aggregate she needs to payoff her obligations at t_1. .

On the other hand, her resources of funding for the payoff at t_1 are **(a)** bank i's short assets y_i, **(b)** the deposits she has in other banks D_{ij} for all $j \neq i \in N$ and finally **(c)** her long assets, x_i to be liquidated at the low rate of $r < 1$, if there is no other available source of liquidity. Then each bank's decision on how these resources should be used to pay off her early consumers and other banks at t_1 can be summarized as:

- How much of the bank's short (liquid) assets should be used and if any excess liquidity remains to be transferred to t_2. Thus, we can formulate the liquid assets that bank i uses to pay off her early consumers by $y_i - y_i^{exc}$, where $y_i^{exc} \geq 0$ denotes the excess liquidity transferred to t_2.

- How much of the bank's deposits should be withdrawn from other banks. Thus we can formulate the aggregate of withdrawn deposits of bank i from other banks by $\sum_{j \neq i} P_{ji}^s c_j^2(s)$, where P_{ji} denotes how much bank i withdraws from her deposits in bank j, at the rate of $c_j^2(s)$ for all $j \in N, j \neq i$. Therefore,

$$P_{ji}(s) \leq D_{ij} \tag{3.26}$$

  should hold.

- How much of the bank i's long assets should be liquidated at the rate of $r < 1$. If

[14] This is not an Arrow-like state contingent contract and do not imply complete market allocation.

we denote the proportion of bank i's long assets liquidated at t_1 by $0 \leq \alpha_i \leq 1$, then $\alpha_i^s x_i r$ is how much of the bank i's payment at t_1 is made using the long asset liquidations.

Note that if the excess liquidity, y_i^{exc} is non-zero, then there should be no need to use expensive liquidation of long assets to payoff early consumers and therefore we have $\alpha = 0$. On the other hand, if the bank liquidates some of her long assets (meaning that $\alpha > 0$ at the low rate of $r < 1$, then there should have remained no liquid assets to be used (meaning that $\alpha = 0$.)

Then if at the given state of $s \in S$ at t_1, she has enough resources to payoff all of her obligations at t_1 at the fix rate of $\overline{c_i^1}$ and also to payoff her late consumers and the remaining deposits of the other banks at the state contingent rate of $c_i^2(s)$, where $c_i^2(s) > \overline{c_i^1}$, then bank i is solvent. Let us denote the states that bank i is solvent at as $\bar{S}_i \subseteq S$.

Then, for all state $s \in \bar{S}_i$, we can represent bank i's budget constraints at t_1 as:

$$y_i - y_i^{exc}(s) + \alpha_i^s x_i r + \sum_{j \neq i} P_{ji}^s c_j^1(s) \tag{3.27}$$

$$= w_i^s \overline{c_i^1} + \sum_{j \neq i} P_{ij}^s \overline{c_i^1}$$

With that, it is easy to see how the budget constraints of bank i at t_2 at the states that she is solvent:

$$y_i^{exc}(s) + R(1 - \alpha_i^s) x_i + \sum_{j \neq i} (D_{ij} - P_{ji}^s) c_j^2(s) \tag{3.28}$$

$$= (1 - w_i^s) c_i^2(s) + \sum_{j \neq i} (D_{ji} - P_{ij}^s) c_i^2(s)$$

On the other hand, the bank i defaults[15] in a state $s \in S/\bar{S}_i$ if she can not fully payoff the committed $\overline{c_i^1}$ to her early depositors as well as $c_i^2(s) > \overline{c_i^1}$ to the late depositors. In

[15] since late consumers would behave like early consumers (in a sense similar to the bank-run phenomena) as soon as the allocated consumption to late consumers equates or falls bellow the allocated consumption of the early consumers, thus insolvency and bankruptcy are the same in this setting since in both cases all of the bank's assets are liquidated at t_1 to payoff the consumers

such circumstances, due to the bank-run-like behavior of consumers and following the limited liability rule, the actual consumption allocated to both early and late consumers by bank i is the per capita liquidation value of the bank's investments at t_1 or formally $y_i + rx_i$.

This shows how the actual early consumption profile offered by a bank depends on the realized state (then early consumption is state contingent, but not like Arrow securities (see section 1.3.1) as:

$$c_i^1(s) = \begin{cases} y + rx & \text{bank } i \text{ is bankrupt} \\ \overline{c_i^1} & \text{otherwise} \end{cases} \qquad \forall i \in N, \forall s \in S \tag{3.29}$$

To assure that the state contingency of (3.29) is incorporated in the budget constraints and therefore decision problems of the banks, the following, as we refer to them *bankruptcy constraints*

$$c_i^1(s) \leq c_i^2(s) \leq y + rx + (c_i^2(s) - c_i^1(s))M \tag{3.30}$$

$$y + rx \leq c_i^1(s) \leq \overline{c_i^1} \tag{3.31}$$

should hold, where M is an arbitrary large number. Bankruptcy constraints assure that if bank i is not bankrupt and thus $c_i^2(s) - c_i^1(s) > 0$, then $c_i^2(s) \leq y + rx + (c_i^2(s) - c_i^1(s))M$ and since the utility functions are strictly increasing it yields to $c_i^1(s) \leq \overline{c_i^1}$. Otherwise, 3.30 reduces to $c_i^1(s) = c_i^2(s) = y + rx$ which captures the bank-run-like behavior of the consumers under bankruptcy circumstances. With this, we can formulate the budget constraints of the banks for all possible states in s at t_1 replacing $c_i^1(s)$ with $\bar{c_i^1}$ in 3.27 which yields to:

$$y_i - y_i^{exc}(s) + \alpha_i^s x_i r + \sum_{j \neq i} P_{ji}^s c_j^1(s) \tag{3.32}$$

$$= w_i^s c_i^1(s) + \sum_{j \neq i} P_{ij}^s c_i^1(s)$$

With that, each bank's decision problem can be formulated as maximizing the expected aggregate social welfare[16]of her customers or formally

$$\underset{x_i, y_i \in \mathbb{R}^+}{\text{maximize}} \quad \sum_{s \in S} \boldsymbol{P}(s).\big[w_i^s U(c_i^1(s)) + (1 - w_i^s) U(c_i^2(s))\big] \tag{3.33a}$$

subject to the bank's budget constraints at t_0, t_1 and t_2, formulated by (3.25),3.32 and (3.28) respectively, deposit withdrawal constraints, formulated by (3.26) and finally the bankruptcy constraints, formulated by (3.30) where for any given state, $s \in S$, α_i^s is a decision variable that denotes the proportion of the long assets liquidated early at t_1 and $y_i^{exc}(s)$ is a decision variable that denotes the excess liquidity transferred by bank i from t_1 to t_2.

Note that as mentioned earlier, since the bank's utility functions are strictly increasing by assumption, then if a bank has a liquidity shortage at t_1 then $y_i^{exc}(s) = 0$ and $\alpha_i^s > 0$, where as if there is an excess liquidity at t_1 then $y_i^{exc}(s) > 0$ and $\alpha_i^s = 0$. Finally, if the liquidity demand and supply match exactly in a given state, then $y_i^{exc}(s) = 0$ and $\alpha_i^s = 0$.

Remark 3 *Since all banks have distributionally equal exposure to liquidity demand ex ante at t_0, the symmetry of the economy suggests that the bank's per capita investment decisions as well as their cross-holding interbank deposits are equal. In other words, $x_i = x_j, y_i = y_j$ and $D_{ij} = D_{ji}$ for all $i, j \in N, i \neq j$. Therefore, from know on, we set $x_i = x, y_i = y$ and $D_{ij} = D$.*

Next, we show that with this setting, in any given state $s \in S$ realized at t_1 either all banks are solvent, or all are bankrupt.

Lemma 3.4.1 (All or No Bankruptcy) *In the banking system's equilibrium, in any given state $s \in S$ realized at t_1 either all banks are solvent, or all are bankrupt.*

[16]If the notation of the random variable definition of the equivalent state contingent variables is used, the expected social welfare can be formulated in a more concise setting as $E_{\boldsymbol{P}}\big[w_i U(c_i^1) + (1 - w_i) U(c_i^2)\big]$

Proof: Let us denote the set of banks that go bankrupt in the given state of $s \in S$ by $N^{BKT} \subseteq N$.

We need to show that if there is a bankrupt bank at a given state $s \in S$, then all other banks are bankrupt too.

Suppose it is not, then if there exists a bank $i \in N$ that is bankrupt at state $s \in S$, then there should exist a bank $j' \in N; j' \neq i$ that is not bankrupt. Therefore, for bank i we have $c_i^1(s) < c^1$ whereas for bank j' we have $c_{j'}^1(s) = c^1$ and thus:

$$c_i^1(s) < c_{j'}^1(s) \tag{3.34}$$

Since bank i is bankrupt, then due to the lack of liquidity, it would liquidate all of its long assets and withdraw all of its deposits from other banks. On the other hand, the bankruptcy of bank i makes all of her depositors (all early and late consumers and other banks) to run and taking some loss withdraw their proportion of the bank i's liquidated assets at t_1.

Therefore, the bankruptcy of bank i $s \in S$, implies that $P_{ij}^s = P_{ji}^s = D$ for all $j \in N$ and $\alpha_i^s = 1$ should hold and then bank i's budget constraints at t_1 and t_2 yield to

$$y + xr + \sum_{j \neq i} Dc_j^1(s) = (1 + D)c_i^1(s) \tag{3.35}$$

and having $c_i^1(s) < c^1$ we have

$$y + xr + \sum_{j \neq i} Dc_j^1(s) < (1 + D)c^1 \tag{3.36}$$

Note that, since bank i is bankrupt we have $y_i^{exc}(s) = 0, \alpha_i(s) = 1$. On the other hand, bank j' is not bankrupt, but yet may be be solvent, (meaning $y_{j'}^{exc} > 0$ and $\alpha_i(s) = 0$, or insolvent, (meaning $y_{j'}^{exc} = 0$ and $0 < \alpha_i(s) < 1$), at the given state s. Thus, for bank j' we have $0 \leq \alpha_i(s) < 1$. Thus, the following first-order condition should hold:

$$\frac{\partial \mathcal{L}_{j'}^s}{\partial \alpha_{j'}^s} = \mu_{j'}^1(s)rx - \mu_{j'}^2(s)Rx \geq 0 \tag{3.37}$$

and therefore, we have:

$$\frac{\mu_{j'}^2(s)}{\mu_{j'}^1(s)} \leq \frac{r}{R} \tag{3.38}$$

where $\mathcal{L}^s_{j'}$ is the Lagrangian function of bank j''s problem at the given state of s and $\mu^1_{j'}(s)$ and $\mu^2_{j'}(s)$ are the Lagrangian multipliers associated with the bank's budget constraints at t_1 and t_2 respectively.

In addition, since we have $P^s_{ij'} = D > 0$ the following first-order conditions should hold:

$$\frac{\partial \mathcal{L}^s_{j'}}{\partial P^s_{ij'}} = \mu^1_{j'}(s)c^1_i(s) - \mu^2_{j'}(s)c^2_i(s) + \mu^P_{j'i}(s) = 0 \tag{3.39}$$

where $\mu^P_{j'i}(s) \geq 0$ is the Lagrangian multipliers associated with the deposit withdrawal constraints, (3.26).

Also given that bank i is bankrupt we have $c^1_i(s) = c^2_i(s) < c^1$. Then if we substitute $c^1_i(s)$ with $c^2_i(s)$ in (3.39) we obtain $\mu^1_{j'}(s) < \mu^2_{j'}(s)$ which contradicts[17] with (3.37), which concludes our proof. ■

Corollary 3.4.2 (Early Consumption Allocation in Equilibrium) *In the banking system's equilibrium, all bank's state contingent early consumption allocations are equal or formally:*

$$c^1_i(s) = c^1_j(s) \qquad \forall i, j \in N, s \in S \tag{3.40}$$

Proof: By symmetry of the economy at t_0, it is easy to see the equality of early consumption allocation of bank in their solvency states, $\bar{c^1_i} = \bar{c^1_j} \forall i, j \in N$. Thus we set $\bar{c^1} = \bar{c^1_i}, \forall i \in N$ to denote the general per capita early consumption profile offered by a solvent bank at t_1 in the banking system. On the other hand, as shown by lemma 3.4.1, if there is a bankrupt bank in a given state then all other banks are also bankrupt and therefore we have $c^1_i(s) = c^2_i(s)$ for all $i \in N$, and $P_{ij} = D$, for all $i, j \in N$. Then if for

[17] Recall that $r < 1 < R$

each bank we add the budget constraint at t_1 and t_2, we get:

$$\begin{aligned} y + xr &= c_i^1(s) + D\sum_{j\neq i}(c_i^1(s) - c_j^1(s)); \\ &= c_i^1(s) + D(n-1)c_i^1(s) - D\sum_{j\neq i} c_j^1(s); \\ &= (nD+1)c_i^1(s) - D\sum_{j} c_j^1(s); \qquad \forall i \in N \end{aligned} \tag{3.41}$$

Then if we sum both sides of (3.41) over all $i \in N$ and solve for $y + xr$ we obtain:

$$y + xr = \frac{\sum_i c_i^1(s)}{n} \tag{3.42}$$

Then substituting $\sum_j c_j^1(s)$ with $n(y + xr)$ in (3.41) we get:

$$y + xr + Dn(y + xr) = c_i^1(s)(nD+1) \qquad \forall i \in N \tag{3.43}$$

and solving for $c_i^1(s)$ we get:

$$c_i^1(s) = c_j^1(s) = y + rx, \qquad \forall i, j \in N \tag{3.44}$$

which concludes the proof. ∎

Corollary 3.4.3 (Late Consumption Allocation in Equilibrium) *In the banking system's equilibrium, all bank's state contingent late consumption allocations are equal or formally:*

$$c_i^2(s) = c_j^2(s) \qquad \forall i, j \in N, s \in S \tag{3.45}$$

Proof: By lemma 3.4.1, if there is a bankrupt bank in a given state then all other banks are also bankrupt and therefore we have $c_i^1(s) = c_i^2(s)$ for all $i \in N$. Then, using corollary 3.4.2, we have $c_i^2(s) = c_j^2(s)$, $\forall i, j \in N$, $s \in S^{BNKT}$.

Then we just need to show that corollary 3.4.3, also holds for the states that all banks are not bankrupt. Since there is no bankrupt bank in a given state $s \in S^{BNKT}$, we have $P_{ij} < D$ and thus $\mu_{ij}^P = 0$ for all $i, j \in N$, where μ_{ij}^P are the Lagrangian multipliers associated with the deposit withdrawal constraints. Thus, without loss of generality, assuming[18], that $P_{ij} > 0$, for all $i, j \in N$, the following first-order conditions

$$\frac{\partial \mathcal{L}_j^s}{\partial P_{ij}^s} = \mu_j^1(s)c^1(s) - \mu_j^2(s)c_i^2(s) = 0 \quad \forall i, j \in N, \quad s \in S^{BNKT} \tag{3.46}$$

should hold. Then for all $\forall i, j, k \in N; i \neq j \neq k$ for all $s \in S^{BNKT}$

$$\mu_i^1(s)c^1(s) = \mu_i^2(s)c_j^2(s) \tag{3.47}$$

$$\mu_j^1(s)c^1(s) = \mu_j^2(s)c_i^2(s) \tag{3.48}$$

$$\mu_k^1(s)c^1(s) = \mu_k^2(s)c_i^2(s) \tag{3.49}$$

thus

$$\frac{\mu_i^1}{\mu_i^2}(s) = \frac{\mu_j^1}{\mu_j^2}(s) = \frac{\mu_k^1}{\mu_k^2}(s) \tag{3.50}$$

and finally

$$c_i^2(s) = c_j^2(s) = c_k^2(s) \tag{3.51}$$

holds which concludes the proof. ■

Then we can rewrite the general equilibrium problem of the banking system as:

$$P^B: \max_{x,y \in \mathbb{R}^+} \; E_{\boldsymbol{P}}\big[w_i U(c_i^1) + (1 - w_i)U(c_i^2)\big], \tag{3.52a}$$

$$\text{s.t.} \quad y + \alpha_i^s x r = (w_i^s + \sum_{j \neq i}(P_{ij}^s - P_{ji}^s))c^1(s) + y_i^{exc}(s) \tag{3.52b}$$

$$y_i^{exc}(s) + R(1 - \alpha_i^s)x = (1 - w_i^s)c^2(s) + \sum_{j \neq i}(P_{ji}^s - P_{ij}^s)c^2(s) \tag{3.52c}$$

Theorem 3.4.4 (Decentralization of the Social Optimum Allocation) *Banking system's consumption allocations under standard deposit contracts and aggregate uncertainty decentralize the social optimum allocation under the same assumptions.*

Proof: Due to the symmetry of the general equilibrium problem of the banking system, the competitive equilibrium solution is given by the equivalent problem, where

[18] assume not, then we can replace the equilibrium solution by an equivalent solution by only changing D to $D + \epsilon$ and $P_{ij}, \forall i, j \in N$ to $P_{ij} + \epsilon, \forall i, j \in N$, where $\epsilon > 0$ is an arbitrary positive constant. It is easy to see that the new solution is equivalent to the initial one assuring that all deposit withdrawals at t_1 are strictly positive.

the Bergson-Samuelson function of the equally weighted average of an individual bank's utility function[19]

$$\sum_{i\in N} E_{\boldsymbol{P}}\big[w_i U(c^1) + (1-w_i)U(c^2)\big] \tag{3.54}$$

is maximized over the banking system constraints and the clearance functions. Since $\frac{1}{n}\sum_i w_i^s = w^s, \forall s \in S$ and therefore $\frac{1}{n}\sum_i w_i = w$ (see (3.22)), we can rewrite (3.53) as

$$\sum_{i\in N} E_{\boldsymbol{P}}\big[w_i U(c^1) + (1-w_i)U(c^2)\big] = E_{\boldsymbol{P}}\big[(\sum_i w_i)U(c^1) + (n - (\sum_i w_i))U(c^2)\big] \tag{3.55}$$

$$= nE_{\boldsymbol{P}}\big[(wU(c^1) + (1-w)U(c^2)\big] \tag{3.56}$$

Then, since n is a constant scalar, we can drop it from the objective function's formulation and formulate the general equilibrium problem of the banking system by:

$$\hat{P}^B: \max_{x,y\in\mathbb{R}^+} \quad \sum_{i\in N} E_{\boldsymbol{P}}\big[(wU(c^1) + (1-w)U(c^2)\big], \tag{3.57a}$$

s.t.

$$x + y = 1; \tag{3.57b}$$

$$y + \alpha_i^s x r = [w_i^s + \sum_{j\neq i}(P_{ij}^s - P_{ji}^s)]c^1(s) + y_i^{exc}(s); \qquad \forall i \in N \tag{3.57c}$$

$$y_i^{exc}(s) + (1-\alpha_i^s)xR = [1 - w_i^s + \sum_{j\neq i}(P_{ji}^s - P_{ij}^s)]c^2(s); \qquad \forall i \in N \tag{3.57d}$$

Now suppose that we add a new set of constraints, $w_i^s + \sum_{j\neq i}(P_{ij}^s - P_{ji}^s) = w^s$; and $y_i^{exc}(s) = y^{exc}(s)$ and $\alpha_i^s = \alpha(s)$ for all $i \in N$, to $\hat{P}^B$. It is easy to see that with the newly added constraints holding, then we can drop the indexing of the banks and $\hat{P}^B$ reduces to:

$$\hat{P}^B_{cons}: \max_{x,y\in\mathbb{R}^+} \quad \sum_{i\in N} E_{\boldsymbol{P}}\big[(wU(c^1) + (1-w)U(c^2)\big], \tag{3.58a}$$

[19] From here on to simplify the notation we switch to the random variable definition of the equivalent state contingent variables in our formulations. Then for (3.53) we have:

$$\sum_{i\in N} E_{\boldsymbol{P}}[w_i U(c^1) + (1-w_i)U(c^2)] = \sum_{i\in N}\sum_{s\in S} \boldsymbol{P}(s).\big[w_i^s U(c_i^1(s)) + (1-w_i^s)U(c_i^2(s))\big] \tag{3.53}$$

s.t.

$$x + y = 1; \tag{3.58b}$$

$$y + \alpha^s x r = w^s c^1(s) + y_i^{exc}(s); \tag{3.58c}$$

$$y^{exc}(s) + (1 - \alpha_i^s) x R = [1 - w^s] c^2(s); \tag{3.58d}$$

which is indeed equivalent to the social planner's optimization problem, P_1^{SP}. On the other hand, since $\hat{P}^B_{cons}$ is generated by adding new constraints to $\hat{P}^B$, then its optimal solution can not be better than the optimal solution of the banking systems problem, $\hat{P}^B$. Then, the social planner's optimal solution not only is a feasible solution for the banking system's equilibrium, but indeed is the (pareto optimal) walrasian equilibrium of the banking system, which concludes our proof. ■

3.5 Discussion and Conclusion

In this chapter, we extended the latest general equilibrium models that aim to explain system-level phenomena like financial contagion. We briefly reviewed of the basic theory of banking and financial intermediation, with section 3.2.1, formulating a simple situation in which consumers have no interaction (either through markets or financial intermediaries). Next we reviewed the classic extention of the Autorky economy in section 3.2.2, allowing interaction between consumers through a liquidity market which creates a market allocation economy and increases the consumers' welfare compared to Autarky. We generalized the classic market allocation framework allowing for non-zero endowments at t_1, and showed how the liquidity pricing is easily distorted in market allocation, even without aggregate uncertainty. Finally, we showed that liquidity markets increase the expected utility of every consumer, both for classic market allocation economy compared to Autarky economy and generalized market allocation economy compared with generalized Autarky economy.

In section 3.3, to formalize an appropriate benchmark for evaluating the banking system's efficiency as a financial intermediary, replace the liquidity market with a central social planner and showed how it may optimally share risk through consumers with and with and without aggregate uncertainty. We finally developed general equilibrium models of the banking system with and without uncertainty and showed that in both cases, the banking system is capable of decentralizing the allocation and risk-sharing solution of the social planner under similar conditions.

In conclusion, we showed that the equilibrium behavior of a competitive banking system decentralizes the social planner's allocation under the same assumption without any regulation imposed, and therefore it is constraint efficient. This means that the financial contagion is not an equilibrium feature of the decentralization functionality of the banking system even under aggregate uncertainty and standard (incomplete) deposit contracts.

Chapter 4

Marginal Value Function Under Degeneracy

4.1 Introduction

An essential part of mathematical programming is concerned with information on sensitivity, stability and comparative statics of optimization and control problems with respect to parameter perturbations and their local controllability. Directional derivatives of the optimal value functions are the main analytical tool in operations research to obtain such information Mordukhovich et al. (2009), Stechlinski et al. (2018), Aravkin et al. (2013) and have been the subject of extensive studies in the literature, with multiple formulations having been proposed under different assumptions and for different classes of non-convex optimization problems. The well-understood complications with degeneracy of optimal solutions and the implications of degeneracy for differentiability properties, especially in non-convex problems, engender theoretical and computational difficulties (see Shapiro (2005, 1988), Rockafellar and Wets (2009), Dempe (2002), Gauvin and Dubeau (1982), Fletcher (1998) and references therein). Degeneracy is not just a theoretical possibility; indeed, it systematically occurs in many important optimization and control problems Bomze (2002) such as optimal design Byeon et al. (2020), optimal control Arutyunov and Aseev (1997), maximum-clique problem Bomze et al. (1997), and scheduling, especially in large-scale Anitescu (2000) decentralized systems with game theoretic natures. For example degeneracy in robust multistage decision models is by no means is an exception, but is rather a general fact intrinsic in any robust multistage decision model, as shown in Delage and Iancu (2015) (see, also, Bertsimas et al. (2010), Iancu et al. (2013)).

Significant degeneracy issues also occur in economics and finance problems, especially where a nonconvex cost function is formulated in an equilibrium model Bomze (2002) or the parametric setting of an equilibrium formulation is stochastic Ehrenmann and Smeers (2011). Another well-known example is the Markowitz's mean-variance portfolio selection problem, as degeneracy may occur whenever the (estimated) variance-covariance matrix is singular Bomze and De Klerk (2002), which also leads to degeneracy of the capital market equilibrium solution and the asset pricing formulations in financial mathematics.

Extensive research has been done to establish solution strategies that have the convergence properties required for degeneracy and ill-conditioning of the constraints (see Coulibaly and Orban (2012), Friedlander and Tseng (2008), Ben Amor et al. (2006), Wright (2003, 2005). On the other hand, the differential properties and directional derivatives of the optimal value function in non-convex programming in most of the existing literature (see Huy et al. (2012), Nemirovski (2019), Ben-Tal et al. (2009) and references therein) assumes non-degeneracy of the optimal solution. Such assumptions would result in practical complications, as one cannot easily guarantee or check non-degeneracy of the large scale problems. Also, this leads to a significant limitation in analytical studies on stability, local controllabality and comparative statics of optimization and control problems that deal with generic formulations of systems as it would be misleading if under degeneracy one relies on regular formulations to derive control policies or does the comparative statistic analysis.

The contribution of this paper is to address cases in which the directional derivative of the optimal value function has a guarantee of local uniqueness but no guarantee against degeneracy.

4.1.1 Notation and Definitions

For a parametric vector $\theta = (\theta_1, \ldots, \theta_p)$ ranging over an open set $\Theta \subset \mathbb{R}^p$, we consider the nonlinear programming problem (omitting equality constraints):

$$\begin{aligned} \boldsymbol{P}(\theta): \quad & \underset{x \in \mathbb{R}^n}{\text{maximize}} && f(x) \\ & \text{subject to} && g_j(x,\theta) \le c_j(\theta), \quad j = 1, \ldots, m \end{aligned} \tag{4.1}$$

where $f(x) : \mathbb{R}^n \to \mathbb{R}$, $g_j(x, \theta) : \mathbb{R}^n \times \Theta \to \mathbb{R}$ and $c_j(\theta) : \Theta \to \mathbb{R}$ are continuously differentiable on $\mathbb{R}^n$, $\Theta \times \mathbb{R}^p$ and Θ, respectively, for all $j = 1, \ldots, m$. The gradient of a function with respect to x is denoted by ∇_x as a row vector. For each value of $\theta \in \Theta$, we define the set of feasible solutions as

$$F(\theta) = \{x | g_j(x, \theta) \le c_j(\theta), \quad j = 0, \ldots, m\},$$

the optimal value function of the optimization problem as

$$v(\theta) = sup\{f(x); x \in F(\theta)\},$$

and the set of local optimal solutions as

$$X(\theta) = \{x \in F(\theta) \,|\, \exists\, \epsilon > 0; \forall y \in N_\epsilon(x); f(x) \ge f(y)\},$$

where $N_\epsilon(x) = \{y \in F(\theta) \,|\, \|y - x\| \le \epsilon\}$ is an ϵ-neighborhood around $x \in F(\theta)$. A local optimal solution $\bar{x} \in X(\theta)$ is said to be strict or locally unique if there exists $\epsilon \in \mathbb{R}_{++}$ such that $N_\epsilon(\bar{x}) \cap X(\theta) = \{x\}$.

Define the Lagrangian function associated with $\boldsymbol{P}(\theta)$ as:

$$\mathcal{L}(x, \lambda, \theta) = f(x) + \sum_{j=1}^{m} \lambda_j g_j(x, \theta),$$

where for $j \in \{1, \ldots m\}$, λ_j is a Lagrangian multiplier associated with $g_j(x, \theta)$ at x and thus $\lambda = (\lambda_1, \ldots, \lambda_m)$ is a Lagrangian multiplier vector associated with $\boldsymbol{P}(\theta)$ at x.

Therefore, the Karush-Kuhn-Tucker (KKT) systems stationary conditions are:

$$\nabla_x \mathcal{L}(x, \theta, \lambda) = \nabla_x f(x) + \sum_{j=1}^{m} \lambda_j \nabla_x g_j(x, \theta) = 0 \tag{4.2a}$$

$$\nabla_{\lambda_j} \mathcal{L}(x, \theta, \lambda) = g_j(x, \theta) - c_j(\theta) \leq 0 \qquad j = 0, \ldots, m \tag{4.2b}$$

$$\lambda_j \geq 0 \qquad j = 0, \ldots, m \tag{4.2c}$$

$$\lambda_j(g_j(x, \theta) - c_j(\theta)) = 0 \qquad j = 0, \ldots, m. \tag{4.2d}$$

For a given parametric setting $\bar{\theta} \in \Theta$ and its associated local optimum at $\bar{x} \in X(\bar{\theta})$, define the set of Lagrangian multipliers vectors of $\boldsymbol{P}(\bar{\theta})$ at $\bar{x}$ by

$$M(\bar{x}, \bar{\theta}) = \{\lambda \in \mathbb{R}^m_+ \setminus \{0\} \mid \nabla_x \mathcal{L}(\bar{x}, \bar{\theta}, \lambda) = 0, \lambda_j(g_j(\bar{x}, \bar{\theta}) - c_j(\bar{x}, \bar{\theta})) = 0, j = 0, \ldots, m\}, \tag{4.3}$$

where $J_0(\bar{x}, \bar{\theta})$ denotes the index set of active constraints of $\boldsymbol{P}(\bar{\theta})$ at $\bar{x} \in F(\bar{\theta})$; specifically,

$$J_0(\bar{x}, \bar{\theta}) = \{j \in \{1, \ldots, m\} | g_j(\bar{x}, \bar{\theta}) = c_j(\bar{\theta})\},$$

Define the index set of strongly active constraints of $\boldsymbol{P}(\bar{\theta})$ at $\bar{x} \in X(\bar{\theta})$ associated with the Lagrangian multiplier vector, $\bar{\lambda} \in M(\bar{x}, \bar{\theta})$, as

$$J_0^+(\bar{x}, \bar{\theta}, \bar{\lambda}) = \{j \in J_0(\bar{x}, \bar{\theta}) \mid \lambda_j > 0\}.$$

It is well known in the literature that for a given parametric setting, $\bar{\theta} \in \Theta$ and a locally optimal solution, $\bar{x} \in X(\bar{\theta})$, and under some constraints qualifications, $M(\bar{x}, \bar{\theta})$ is a nonempty, compact and convex polyhedron. Therefore, $M(\bar{x}, \bar{\theta})$ can be represented as a convex hull of its extreme points Bonnans and Shapiro (2000). $\lambda \in M(\bar{x}, \bar{\theta})$ would be an extreme point of $M(\bar{x}, \bar{\theta})$ if and only if the gradient vectors of strongly active constraints, indexed by $J_0^+(\bar{x}, \theta, \lambda)$ are linearly independent. Then, denote the extreme points of $M(\bar{x}, \bar{\theta})$ as

$$E_M(\bar{x}, \bar{\theta}) = \{\lambda \in M(\bar{x}, \bar{\theta}) \mid det(G) \neq 0\}, \tag{4.4}$$

where $G = \{\nabla_x g_j(\bar{x}, \bar{\theta})\}_{j \in J_0^+(\bar{x}, \bar{\theta}, \lambda)}$.

Definition 4.1.1 (Directional derivative of the objective value function:) *The directional derivative of a value function $v : \Theta \to \mathbb{R}$ at $\theta \in \Theta$ in the direction of $d \in \mathbb{R}^m$ is defined as*

$$\nabla v(\theta, d) = \lim_{t \downarrow 0} \frac{v(\theta + td) - v(\theta)}{t} \tag{4.5}$$

if the limit exists. In the case that the limit does not exist, we use the upper and lower Dini directional derivatives, which always exist and provide us with the upper and lower bounds of the rates of change in the optimal value:

$$\nabla^+ v(\theta, d) = \limsup_{t \downarrow 0} \frac{v(\theta + td) - v(\theta)}{t} \tag{4.6a}$$

$$\nabla^- v(\theta, d) = \liminf_{t \downarrow 0} \frac{v(\theta + td) - v(\theta)}{t}. \tag{4.6b}$$

Clearly, the directional derivative at θ in the direction of d exists if and only if the lower and upper Dini directional derivatives coincide at θ in the direction of d.

4.2 Constraint Qualifications and Degeneracy

If $\boldsymbol{P}(\theta)$ is a convex optimization problem, the KKT stationary conditions hold at a point if and only if that point is an optimum solution. In non-convex programming, by contrast, we may deal with optimal solutions that do not satisfy the first-order (KKT type) and/or second-order stationary conditions and vice versa. There exists an extensive literature on different regulatory conditions and constraint qualifications (CQ) that guarantee the validity of different types of necessary and/or sufficient first and second-order conditions on (locally) optimal solutions. Below, we review some of these qualifications and conditions and their implications for the degeneracy of (locally) optimal solutions. These qualifications and constraints are used later as we develop directional derivatives of the optimal value function for $\boldsymbol{P}(\theta)$.

Definition 4.2.1 (LICQ) *Linear independence constraint qualification (LICQ) is satisfied at $x \in F(\theta)$ if the gradients of all active constraints, $\{\nabla_x g_j(x, \theta\}_{j \in J_0(x,\theta)}$, are linearly independent.*

If $\bar{x} \in F(\bar{\theta})$ is a local optimal solution of $\boldsymbol{P}(\bar{\theta})$ and satisfies LICQ, then $\bar{x}$ is also a KKT point (stationary solution) and therefore its associated Lagrange multiplier vector $\bar{\lambda} \in \mathbb{R}^m$ solves the KKT conditions uniquely. However, other alternative KKT points may exist that are also locally optimal and have the same unique Lagrange multiplier vector as $\bar{x}$. Therefore, LICQ helps us to determine if the KKT multipliers of a local solution are unique or not, but other conditions are needed to resolve the uniqueness of the KKT points. The following definitions and conditions help us to determine whether or not $\bar{x}$ is a unique KKT point and thus a locally strict optimizer of $\boldsymbol{P}(\theta)$.

Definition 4.2.2 (SCS) *Strict Complementarity Slackness (SCS) condition is satisfied for a Lagrange multiplier vector,* $\lambda \in M(x, \theta)$*, if* $\lambda_j > 0$*, for all* $j \in J_0(x, \theta, \lambda)$*.*

Let us use the term *full-ranked SCS* condition to refer to the case that $\bar{x}$ satisfies SCS condition with at least n independent constraints being active. Under LICQ and full-ranked SCS condition, not only is the associated Lagrange multiplier vector $\bar{\lambda}$ unique, but also $\bar{x}$ is a locally unique KKT point and thus a locally unique optimizer of $\boldsymbol{P}(\theta)$. It is sufficient for a KKT point to be a strict local optimizer of $\boldsymbol{P}(\theta)$ if it has n active constraints and satisfies both LICQ and SCS conditions. On the other hand, if the full-ranked SCS condition is not satisfied, then even though LICQ implies the uniqueness of the Lagrange multiplier vector, $\boldsymbol{P}(\theta)$ may have multiple KKT points which are locally alternative optimizers of $\boldsymbol{P}(\theta)$ with the same unique Lagrange multiplier vector and thus same optimal objective value.

Definition 4.2.3 (MFCQ) *The Mangasarian-Fromowitz constraint qualification (MFCQ) is satisfied for* $\boldsymbol{P}(\theta)$ *at* $x \in F(\theta)$ *if the cone of interior directions of* $F(\theta)$ *at* x*, defined as* $G_0 = \{d \in \mathbb{R}^m \mid \nabla_x g_j(x, \theta) d < 0, j \in J_0(x, \theta)\}$*, is non-empty. Global MFCQ holds if for all* $x \in X(\theta)$ *we have MFCQ satisfied for* $\boldsymbol{P}(\theta)$*.*

Similar to the case with LICQ, if MFCQ holds at a local optimum solution, $\bar{x} \in X(\bar{\theta})$, then the KKT stationary conditions hold at the local optimizer. Also with MFCQ holding at $\bar{x} \in X(\bar{\theta})$, the set of associated Lagrange multiplier vectors, $M(\bar{x}, \bar{\theta})$, is a nonempty and bounded convex polyhedron. This means that under MFCQ, not only may $\boldsymbol{P}(\bar{\theta})$ have alternative local solutions (KKT points), but it may be that the Lagrange multiplier vector is also not unique. With this, we are ready to define two types of degeneracy, as follows.

Definition 4.2.4 (Degeneracy) *$\bar{x} \in X(\theta)$ is a degenerate local optimizer of $\boldsymbol{P}(\theta)$ if it satisfies KKT conditions but not LICQ. Moreover, a Lagrange multiplier vector, $\bar{\lambda} \in M(\bar{x}, \bar{\theta})$, is a degenerate KKT solution if the rank of the matrix formed by the gradient of the strongly active constraints, $\{\nabla_x g_j(\bar{x}, \bar{\theta}\}_{j \in J_0^+(\bar{x},\bar{\theta},\bar{\lambda})}$, is equal to n. Then we refer to **type-I-degeneracy** of $\boldsymbol{P}(\theta)$ at $\bar{x}$ if $\bar{x}$ is a degenerate local optimizer, and **type-II-degeneracy** of $\boldsymbol{P}(\theta)$ at $\bar{x}$ if all of the Lagrange multiplier vectors in $M(\bar{\theta}, \bar{x})$ are degenerate.*

Note that under this definition a locally optimal solution under LICQ is non-degenerate if and only if it satisfies full-Ranked CSC. As mentioned earlier, such a point is a unique local optimizer. Without full-ranked SCS, however, $\boldsymbol{P}(\theta)$ is type-II-degenerate at $\bar{x}$. On the other hand, if MFCQ holds at $\bar{x}$ but not LICQ, then since the set of associated Lagrange multiplier vectors is not a singleton, $\boldsymbol{P}(\theta)$ is type-I-degenerate at $\bar{x}$. Furthermore, if any $\lambda \in M(\bar{\theta}, \bar{x})$ is a degenerate multiplier vector, then $\boldsymbol{P}(\theta)$ is type-II-degenerate at $\bar{x}$ too. Therefore, under MFCQ, $\boldsymbol{P}(\theta)$ can have both types of degeneracy, which requires $\bar{x}$ to have an alternative KKT point, similar to case discussed for definition 4.2.2.

This paper's main interest is the cases that $\boldsymbol{P}(\theta)$ is type-I-degenerate and not type-II-degenerate. Therefore, we limit our study to the cases that MFCQ holds at a locally unique optimal solution, $x \in X(\theta)$ for $\boldsymbol{P}(\theta)$ but not LICQ. Failure to satisfy LICQ leads to type-I-degeneracy, and on the other hand, the local uniqueness avoids type-II-degeneracy.

Definition 4.2.5 (SSOC) *The strong second-order sufficient constraint qualification*

(SSOC) is satisfied for $\boldsymbol{P}(\theta)$ at x if

$$\nabla_x g_j(x,\theta)d = 0, \quad \forall j \in J_0^+(\bar{x},\bar{\theta}) \tag{4.7a}$$

$$d^t \nabla^2_{xx}\mathcal{L}(x,\lambda,\theta) > 0 \tag{4.7b}$$

holds for a $d \neq 0$, where ∇^2_{xx} is the Hessian matrix of second order partial derivatives (of the Lagrangian function) with respect to x.

If $\bar{x}$ is a KKT point under LICQ, then full-ranked SCS suffices for $\bar{x}$ to be a locally unique optimal solution. On the other hand, if $\bar{x}$ is a KKT point under MFCQ but not SSOC, it is necessary and sufficient that SSOC holds for $\bar{x}$ to be a locally unique optimal solution.

Definition 4.2.6 (Strongly stable optimal solution Klatte and Kummer (1999), Dempe (2002)) *A local optimal solution, $\bar{x} \in F(\bar{\theta})$ is strongly stable (strongly Lipschitz stable) if there exist neighborhoods $N_\epsilon(\bar{\theta}), N_\sigma(\bar{x}), \epsilon, \sigma > 0$ such that for all $\theta \in N_\epsilon(\bar{\theta})$ there exist a continuous function[1], $\boldsymbol{x} : \Theta \to \mathbb{R}^n$, such that $\boldsymbol{x}(\theta)$ is a unique KKT point and thus the unique local optimizer of $\boldsymbol{P}(\theta)$ in $N_\sigma(\bar{x})$ for all $\theta \in N_\epsilon(\bar{\theta})$.*

In other words, a locally optimal solution $\bar{x} \in F(\bar{\theta})$ of $\boldsymbol{P}(\bar{\theta})$ is strongly stable in the sense of Kojima Kojima (1980), Robinson and Kojima (1980) if for any "sufficiently small" perturbation of $\bar{\theta}$ there exist a unique isolated local optimal solution of the perturbed problem in a neighborhood of $\bar{x}$. It is well known in the literature (see Kojima (1980), Levy et al. (2000)) that if MFCQ holds at $\bar{x}$ but not LICQ, then $\bar{x}$ is strongly stable if and only if SSOC holds and thus $\bar{x}$ is an isolated local optimum solution of $\boldsymbol{P}(\theta)$. This also implies that if we have both MFCQ and SSOC satisfied at a KKT point, then there exist neighborhoods, $N_\epsilon(\bar{\theta}), N_\sigma(\bar{x}), \epsilon, \sigma > 0$ such that for all $\theta \in N_\epsilon(\bar{\theta})$ there exist a unique KKT point, $x(\theta)$ of $\boldsymbol{P}(\theta)$ in $N_\sigma(\bar{x})$ and this point $x(\theta)$ is thus an isolated local optimizer of $\boldsymbol{P}(\bar{\theta})$.

[1] In general $x : \Theta \to \mathbb{R}^n$ is a uniquely determined set-valued mapping, but since here it is always single-valued we formulate it as a *function* to avoid notation complications of $x(\theta)$ being a singleton set instead of the single element it contains. For more details see Rockafellar and Wets (2009).

If $\bar{x} \in F(\bar{\theta})$ is a locally unique optimal solution of $\boldsymbol{P}(\bar{\theta})$, then either LICQ and a weak version of SSOC (see Shapiro (1990)) or MFCQ and SSOC should hold for $\bar{x}$ to be the strongly stable optimal solution. In this paper, we focus on the cases in which MFCQ holds but LICQ does not, and thus $\boldsymbol{P}(\bar{\theta})$ has type-I-degeneracy.

Note that, if MFCQ is satisfied at a feasible solution, after a small perturbation, the neighborhood around the feasible point remains feasible for the post-perturbation problem. This means that under slight perturbations, feasible space does not vanish. On the other hand, even under LICQ, a slight perturbation of a problem with a compact feasible space may result in a perturbed problem with a noncompact feasible space. Therefore, we need the following compactness assumption to make sure that the optimal value remains finite under small perturbations:

Definition 4.2.7 (Local Compactness) $\boldsymbol{P}(\bar{\theta})$ *is locally compact at* $\bar{\theta} \in \Theta$ *if exists* $\sigma > 0$ *such that for any* $\theta \in N_\sigma(\bar{\theta})$, $F(\theta)$ *is a compact set.*

4.3 Initial Propositions

Lemma 4.3.1 *If* $\bar{x} \in F(\bar{\theta})$ *is a locally unique optimal solution for* $\boldsymbol{P}(\bar{\theta})$ *under MFCQ, then*

$$G^a(\bar{x}, \bar{\theta}) = \{d \in \mathbb{R}^m \setminus \{0\} \mid \nabla_x g_j(\bar{x}, \bar{\theta})d = 0; \forall j \in \tilde{J}_0^+(\bar{x}, \bar{\theta})\} \tag{4.8}$$

is an empty set, where $\tilde{J}_0^+(\bar{x}, \bar{\theta}) = \bigcup_{\lambda \in M(\bar{x}, \bar{\theta})} J_0^+(\bar{x}, \bar{\theta}, \lambda)$[2].

Proof: Note that any $\bar{d} \in G^a(\bar{x}, \bar{\theta})$ under constraint qualifications (here MFCQ) belongs to the cone of tangents of $F(\bar{\theta})$ at $\bar{x}$, defined as

$$T(\bar{x}, \bar{\theta}) = \{d : d = \lim_{k \to \infty} \lambda_k (x_k - \bar{x}), \lambda_k > 0, x_k \in F(\bar{\theta}), \text{ and } x_k \to \bar{x}\} \tag{4.9}$$

and therefore there exist a feasible sequence of $\{x_k\}$ converging to $\bar{x}$ such that directions of the cords $x_k - \bar{x}$ converge to $\bar{d}$ and, due to continuous differentiability of the constraints at $\bar{x}$, there exists $\bar{k}$ where, for $k \geq \bar{k}$, we have $\bar{d} = x_{\bar{k}} - \bar{x}$ Bazaraa et al.

[2] $\tilde{J}_0^+(\bar{x}, \bar{\theta})$ is the index set of all constraints that are strongly active in association to any $\lambda \in M(\bar{x}, \bar{\theta})$

(2006).

Now, suppose $G^a(\bar{x},\bar{\theta})$ is not an empty set. Then for any $\bar{\lambda} \in M(\bar{x},\bar{\theta})$, $\exists \bar{d} \neq 0$ where $\nabla_x g_j(\bar{x},\bar{\theta})\bar{d} = 0; \forall j \in J_0^+(\bar{x},\bar{\theta})$. As we just showed, for any $\bar{d} \in G^a(\bar{x},\bar{\theta})$, there exists $x_k \in F(\bar{\theta})$, where $\bar{d} = x_k - \bar{x}$. Since $\bar{x}$ is a locally unique optimal solution we expect $f(\bar{x}) > f(x_k)$ and therefore $f(x_k) - f(\bar{x}) = \nabla_x f(\bar{x}).(x_k - \bar{x}) = \nabla_x f(\bar{x})\bar{d} > 0$. Moreover, since $\bar{x}$ is an optimal solution of $\boldsymbol{P}(\bar{\theta})$ under MFCQ and thus the first-order optimality conditions hold we have $-\nabla_x f(\bar{x}) = \sum_{j\in J_0(\bar{x},\bar{\theta})} \lambda_j \nabla_x g_j(\bar{x},\bar{\theta})$. Since $\nabla_x g_j(\bar{x},\bar{\theta})\bar{d} = 0 \forall j \in J_0^+(\bar{x},\bar{\theta})$, multiplication of both sides by $\bar{d}$ gives $\nabla_x f(\bar{x})\bar{d} = 0$, which contradicts $\nabla_x f(\bar{x})\bar{d} > 0$ and concludes the proof. ■

Lemma 4.3.2 *If $\bar{x} \in F(\bar{\theta})$ is a locally unique optimal solution for $\boldsymbol{P}(\bar{\theta})$ under MFCQ, then for all $\lambda' \in int(M(\bar{x},\bar{\theta}))$,*

$$G^{sa}(\bar{x},\bar{\theta},\lambda') = \{d \in \mathbb{R}^m \setminus \{0\} \mid \nabla_x g_j(\bar{x},\bar{\theta})d = 0; \forall j \in J_0^+(\bar{x},\bar{\theta},\lambda')\} \tag{4.10}$$

is an empty set where $int(M(\bar{x},\bar{\theta}))$ denotes the interior of $M(\bar{x},\bar{\theta})$[3].

Proof: Recall that under MFCQ any $\lambda' \in int(M(\bar{x},\bar{\theta}))$ can be represented as a strict convex combination of extreme points of $M(\bar{x},\bar{\theta})$, denoted by $E_M(\bar{x},\bar{\theta})$. Therefore, the index set of strongly active constraints at $\bar{x}$ associated with the interior multiplier vector, λ', is given by the union of the corresponding index sets of those extreme points: $J_0^+(\bar{x},\bar{\theta},\lambda') = \bigcup_{\lambda\in E_M(\bar{x},\bar{\theta})} J_0^+(\bar{x},\bar{\theta},\lambda)$.

In addition, note that any $\lambda \in M(\bar{x},\bar{\theta})$ can be written as a strict convex combination of some (not necessarily all) of the extreme points of $M(\bar{x},\bar{\theta})$.

Therefore, the index set of the strongly active constraints at $\bar{x}$ associated with $\lambda \in M(\bar{x},\bar{\theta})$ can be obtained by the union of the corresponding index sets of the extreme points in $E_M(\bar{x},\bar{\theta})$ used in the convex combination. With that, and to obtain the index set of all constraints that are strongly active in association with any $\lambda \in M(\bar{x},\bar{\theta})$, we

[3]we denote the interior of a set, S, by $int(S)$ defined as: $int(S) = \{x \in S \mid \exists N_\epsilon(x); N_\epsilon(x) \subset S\}$

have: $\bigcup_{\lambda\in M(\bar{x},\bar{\theta})} J_0^+(\bar{x},\bar{\theta},\lambda) = \bigcup_{\lambda\in E_M(\bar{x},\bar{\theta})} J_0^+(\bar{x},\bar{\theta},\lambda)$. Therefore,

$$J_0^+(\bar{x},\bar{\theta},\lambda') = \bigcup_{\lambda\in E_M(\bar{x},\bar{\theta})} J_0^+(\bar{x},\bar{\theta},\lambda) = \bigcup_{\lambda\in M(\bar{x},\bar{\theta})} J_0^+(\bar{x},\bar{\theta},\lambda) = \tilde{J}_0^+(\bar{x},\bar{\theta}), \quad \forall\lambda' \in int(M(\bar{x},\bar{\theta}))$$

holds. This means that $G^{sa}(\bar{x},\bar{\theta},\lambda') = G^a(\bar{x},\bar{\theta})$ for any $\lambda' \in int(M(\bar{x},\bar{\theta}))$. Recalling the result from lemma 4.3.1 that $G^a(\bar{x},\bar{\theta}) = \emptyset$ means that $G^{sa}(\bar{x},\bar{\theta},\lambda') = \emptyset$ too, for any $\lambda' \in int(M(\bar{x},\bar{\theta}))$, which concludes the proof. ■

Proposition 4.3.3 *Suppose that $\bar{x}$ is a locally unique optimum solution for $\boldsymbol{P}(\bar{\theta})$ under MFCQ. For any Lagrangian multiplier vector, $\lambda' \in int(M(\bar{x},\bar{\theta}))$, there exist an expanded problem:*

$$\begin{aligned} P^{\lambda'}(\theta): \qquad & \underset{x\in\mathbb{R}^n}{\text{maximize}} \qquad f(x) \\ & \text{subject to} \qquad g_j(x,\theta) \le c_j(\theta), \quad j \in J_0^+(\bar{x},\bar{\theta},\lambda') \end{aligned} \tag{4.11}$$

where $\bar{x}$ is an strongly stable and locally unique optimal solution of $\boldsymbol{P}^{\lambda'}(\bar{\theta})$.

Proof: Since $\bar{x}$ is a local optimum solution for $\boldsymbol{P}(\bar{\theta})$ under MFCQ, it satisfies KKT conditions for $\boldsymbol{P}(\bar{\theta})$. Since $\boldsymbol{P}^{\lambda'}(\bar{\theta})$ has the same objective function an a subset of the $\boldsymbol{P}(\bar{\theta})$'s constraints KKT conditions hold for $\boldsymbol{P}^{\lambda'}(\bar{\theta})$ at $\bar{x}$ too. Similarly, CSC conditions and MFCQ hold at $\bar{x}$ for $\boldsymbol{P}^{\lambda'}(\bar{\theta})$. Moreover as shown in lemma 4.3.2 there exist no nonzero d that satisfies $\nabla_x g_j(\bar{x},\bar{\theta})d = 0$ for all $j \in J_0^+(\bar{x},\bar{\theta},\lambda')$ and therefore SSOC also holds at $\bar{x}$ for $\boldsymbol{P}^{\lambda'}(\bar{\theta})$.

As we discussed earlier, a KKT point under MFCQ and SSOC is a locally unique optimal solution as well as a strongly stable stationary solution, which concludes our proof. ■

Throughout this paper, we refer to the $\boldsymbol{P}(\theta)$ problem, defined in (4.1) as the *initial problem*. We assume $\boldsymbol{P}(\theta)$ in a given parametric setting denoted by $\bar{\theta} \in \Theta$, has a locally unique optimal solution, denoted by $\bar{x}$ unless otherwise stated. We refer to the set of problems formulated in proposition (4.3.3) as the expanded problems of the initial problem, or *expanded problems* in short. Since each expanded problem is associated with an interior element of the Lagrangian multiplier vectors of the initial problem at $\bar{x}$, we draw a distinction in notation between λ' and λ, as we use λ' only to refer to an interior element of $M(\bar{x}, \bar{\theta})$ whereas λ is used to refer to any element of $M(\bar{x}, \bar{\theta})$. We use similar approach in all definitions and notation related to expanded problems. For example we denote the optimal value function of the initial problem by $v(.)$, whereas the optimal value function of a expanded problem, associated with the Lagrangian multiplier vectors $\lambda' \in int(M(\bar{x}, \bar{\theta}))$ by $v^{\lambda'}(.)$.

Remark 4 *Due to strong stability of the expanded problems at $\bar{x}$, for every $\lambda' \in int(M(\bar{x}, \bar{\theta}))$, there exists a stable perturbation neighborhood denoted by $N_{\epsilon}^{\lambda'}(\bar{\theta}), \epsilon > 0$ and a stable solution neighborhood, denoted by $N_{\sigma}^{\lambda'}(\bar{x}), \sigma > 0$, such that, for all $\theta \in N_{\epsilon}^{\lambda'}(\bar{\theta})$ a Lipschitz continuous function, $\boldsymbol{x}^{\lambda'} : \Theta \to \mathbb{R}^n$, exists such that $\boldsymbol{x}^{\lambda'}(\theta)$ is a unique KKT point of $\boldsymbol{P}^{\lambda'}(\theta)$ in $N_{\sigma}^{\lambda'}(\bar{x})$.*

Thus $\boldsymbol{x}^{\lambda'}(\theta)$ is a unique local optimizer of $\boldsymbol{P}^{\lambda'}(\theta)$ in $N_{\sigma}^{\lambda'}(\bar{x})$ for all $\theta \in N_{\epsilon}^{\lambda'}(\bar{\theta})$. This means that if a perturbation of an expanded problem is contained in a stable perturbation neighborhood, $\theta \in N_{\epsilon}^{\lambda'}(\bar{\theta})$, then there exists a locally unique optimal solution of the perturbed problem in a stable solution neighborhood, $N_{\sigma}^{\lambda'}(\bar{x})$, where MFCQ and thus KKT conditions hold.

From now on, by perturbation we mean only those perturbations that are contained in a stable perturbation neighborhood ($\theta \in N_{\epsilon}^{\lambda'}(\bar{\theta})$). Also, the locally unique optimal solution of a perturbed problem refers to the locally unique optimal solution contained in the stable solution neighborhood ($N_{\sigma}^{\lambda'}(\bar{x})$). Note that a perturbed problem, even if it is contained in the stable perturbation neighborhood, may have locally unique optimal solutions outside the stable solution neighborhood.

Remark 5 *The continuity property of $\boldsymbol{x}^{\lambda'}(\theta)$ implies that the local optimal solution of a perturbed expanded problem, $\boldsymbol{x}^{\lambda'}(\bar{\theta} + td) \to \bar{x}$ as $t \to 0$. Also MFCQ holds at $\boldsymbol{x}^{\lambda'}(\theta)$, as far as $\theta \in N_{\epsilon}^{\lambda'}(\bar{\theta})$. Therefore, we can formulate the set of Lagrangian multipliers of*

$\boldsymbol{P}^{\lambda'}(\theta)$ at $x(\theta)$ as a nonempty, convex and compact set-valued mapping [4]

$$M^{\lambda'}(x(\theta), \theta) : N_{\epsilon}^{\lambda'}(\bar{\theta}) \rightrightarrows \mathbb{R}_{+}^{m}; \qquad \forall \lambda' \in int(M(\bar{x}, \bar{\theta})) \tag{4.12}$$

Remark 6 *The convexity and compactness of the nonempty set-valued mapping of the Lagrangian multipliers, as defined in* (4.12)*, suffices to have $M^{\lambda'}(x(\theta), \theta)) \to M^{\lambda'}(\bar{x}, \bar{\theta})$ as $\theta \to \bar{\theta}$ (see Rockafellar and Wets (2009), THM 4.32(b)). In other words, for any $\lambda' \in int(M(\bar{x}, \bar{\theta}))$ and any $\tilde{\lambda} \in M^{\lambda'}(\bar{x}, \bar{\theta})$ exists $\tilde{\lambda}(\theta) \in M^{\lambda'}(x(\theta), \theta)$, where $\tilde{\lambda}(\theta) \to \tilde{\lambda}$ as $\theta \to \bar{\theta}$. Note that $\lambda' \in M^{\lambda'}(\bar{x}, \bar{\theta})$ too and therefore exists $\lambda'(\theta) \in M^{\lambda'}(x(\theta), \theta)$, where $\lambda'(\theta) \to \lambda'$ as $\theta \to \bar{\theta}$. We do not have the same properties for the parametric perturbation of the initial problem since SSOC does not hold at $\bar{x}$ and therefore we may not have the same stability and Lipschitz continuity for optimal solution function.*

4.4 Upper Bound on the Directional Derivative

Here we use the strong stability property of the expanded problems at the locally unique optimal solution of the initial problem to derive an upper bound for the directional derivative of the initial problem's optimal value function. As shown in proposition 4.3.3 the initial problem and expanded problems have the same locally unique optimal solution and optimal value at $\bar{\theta} \in \Theta$. Therefore, if we perturb their parametric setting in the same direction, say $d \in \mathbb{R}^{P}, \|d\| = 1$ for an arbitrary $t \in \mathbb{R}_{+} \setminus \{0\}$, we expect the expanded problems to attain optimal values not worse than the initial problem after the perturbation. Then, the change of the optimal values of the expanded problems would be a natural upper bound of the change of the optimal value of the initial problem due to the perturbation.

The strong stability of the expanded problems is the main property that helps us obtain the explicit formulation of the change of their optimal value due to the perturba-

[4]a simpler definition of the set-valued mapping for the Lagrangian multipliers of the expanded problems would be $M^{\lambda'}(x(\theta), \theta) : N_{\epsilon}^{\lambda'}(\bar{\theta}) \rightrightarrows \mathbb{R}_{+}^{|J_0^{+}(\bar{x}, \bar{\theta}, \lambda')|}; \forall \lambda' \in int(M(\bar{x}, \bar{\theta}))$. However, to keep the size of the Lagrangian multiplier vectors the same across the initial and expanded problems we decided to represent the vectors in a m dimensional space.

tion and thus the upper bounds. Finally, we aggregate the upper bounds into a single upper bound as the following proposition's main result:

Proposition 4.4.1 *Let $\bar{x}$ to be a locally unique optimal solution of* (4.1) *at* $\bar{\theta} \in \Theta$. *Then*

$$\nabla v(\bar{\theta}, d) \leq \min_{\lambda' \in int(M(\bar{x},\bar{\theta}))} \sum_{j \in J_0^+(\bar{x},\bar{\theta},\lambda')} \lambda_j' \left(\nabla_\theta g_j(\bar{x},\bar{\theta}) - \nabla_\theta c_j(\bar{\theta})\right) d \tag{4.13}$$

holds if MFCQ holds at $\bar{x}$ *for* $\boldsymbol{P}(\bar{\theta})$.

Proof: Since MFCQ holds for the locally unique maximum $\bar{x}$ of (4.1) at $\bar{\theta}$, we have the necessary and sufficient conditions to have a nonempty and compact $M(\bar{x}, \bar{\theta})$ for $\boldsymbol{P}(\bar{\theta})$.

Therefore, as discussed in proposition 4.3.3, for any $\lambda' \in int(M(\bar{x},\bar{\theta}))$ there exist an expanded problem, $\boldsymbol{P}^{\lambda'}(\bar{\theta})$, with the same locally unique maximizer, $\bar{x}$. Note that since both problems have the same objective function, they have the same optimal value at $\bar{x}$. Suppose we perturb the parametric setting of $\boldsymbol{P}(.)$ and its expanded problems in the same direction $d \in \mathbb{R}^P, \|d\| = 1$, for an arbitrary $t \in \mathbb{R}_+ \setminus \{0\}$,and therefore move from $\bar{\theta}$ to $\bar{\theta} + td$. Then, the perturbed problem may have different solutions, but since $J_0^+ \subset J_0$, the optimal value of the expanded problems, $\boldsymbol{P}^{\lambda'}(\bar{\theta} + td)$, cannot be worse than the optimal value of $\boldsymbol{P}(\bar{\theta} + td)$; formally, $v(\bar{\theta} + td) \leq v^{\lambda'}(\bar{\theta} + td)$. As mentioned before, the pre-perturbation optimal value of both problems are equal, $v(\bar{\theta}) = v^{\lambda'}(\bar{\theta})$ and with that we have:

$$v(\bar{\theta} + td) - v(\bar{\theta}) \leq v^{\lambda'}(\bar{\theta} + td) - v^{\lambda'}(\bar{\theta}), \tag{4.14}$$

for all $\lambda' \in int(M(\bar{x}, \bar{\theta}))$.

If the size of the perturbation is small enough to be contained in the expanded problem's stable perturbation neighborhood (specifically, if $\bar{\theta} + td \in N_\epsilon^{\lambda'}(\bar{\theta})$ for all $\lambda' \in int(M(\bar{x}, \bar{\theta}))$, then there exist a continuous function, $\boldsymbol{x}^{\lambda'} : \Theta \to \mathbb{R}^n$, such that $\boldsymbol{x}^{\lambda'}(\theta)$ is a unique KKT point and thus the unique local optimizer of $\boldsymbol{P}^{\lambda'}(\theta)$. Then we can rewrite the right hand side of (4.14) as the difference of their corresponding objective functions at the respective optimal solutions:

$$v^{\lambda'}(\bar{\theta}+td) - v^{\lambda'}(\bar{\theta}) = f\big(\boldsymbol{x}^{\lambda'}(\bar{\theta}+td)\big) - f(\bar{x}) \tag{4.15}$$

$$= \big(\boldsymbol{x}^{\lambda'}(\bar{\theta}+td) - \bar{x}\big)\ \nabla_x f(\bar{x}) \tag{4.16}$$

$$= -\big(\boldsymbol{x}^{\lambda'}(\bar{\theta}+td) - \bar{x}\big) \sum_{j \in J_0^+(\bar{x},\bar{\theta},\lambda')} \lambda'_j \nabla_x g_j(\bar{x},\bar{\theta}), \tag{4.17}$$

for all $\lambda' \in int(M(\bar{x},\bar{\theta}))$.

Note that we used a Taylor approximation to get (4.16) and then substituted $\nabla_x f(\bar{x})$ using (4.2a) conditions[5] to obtain (4.17). We also have complimentary slackness conditions, (4.2d), that hold at the optimal solutions of the expanded problem before the perturbation:

$$\lambda'_j[g_j(\bar{x},\bar{\theta}) - c_j(\bar{\theta})] = 0, \quad \forall j \in J_0^+(\bar{x}), \tag{4.18}$$

and also after any perturbation in the stable perturbation neighbourhoods:

$$\lambda'_j(\bar{\theta}+td)\big[g_j(\boldsymbol{x}^{\lambda'}(\bar{\theta}+td), \bar{\theta}+td) - c_j(\bar{\theta}+td))\big] = 0, \quad \forall j \in J_0^+(\bar{x}), \tag{4.19}$$

for all $\lambda' \in int(M(\bar{x},\bar{\theta}))$.

Let us substitute $\lambda'_j(\bar{\theta}+td)$ in (4.19) with $\lambda'_j(\bar{\theta}+td) + \lambda'_j - \lambda'_j$ and then subtract it from (4.18) to obtain:

$$\lambda'_j\Big[\big[g_j(\boldsymbol{x}^{\lambda'}(\bar{\theta}+td), \bar{\theta}+td) - g_j(\bar{x},\bar{\theta})\big] - \big[c_j(\bar{\theta}+td) - c_j(\bar{\theta})\big]\Big] = \tag{4.20}$$
$$(\lambda'_j(\bar{\theta}+td) - \lambda'_j)\Big[c_j(\bar{\theta}+td) - g_j(\boldsymbol{x}^{\lambda'}(\bar{\theta}+td), \bar{\theta}+td)\Big] \quad \forall j \in J_0^+(\bar{x},\bar{\theta},\lambda'),$$

for all $\lambda' \in M(\bar{x},\bar{\theta})$. Then, with the 1st order Taylor approximation, we obtain

$$\begin{aligned}\lambda'_j\Big[\big[\boldsymbol{x}^{\lambda'}(\bar{\theta}+td) - \bar{x}\big]\nabla_x g_j(\bar{x},\bar{\theta})\Big] + \lambda'_j\Big[\big[\nabla_\theta g_j(\bar{x},\bar{\theta}) - \nabla_\theta c_j(\bar{\theta})\big]td\Big] \\ = (\lambda'_j(\bar{\theta}+td) - \lambda'_j)\Big[\big[\nabla_\theta c_j(\bar{\theta}) - \nabla_\theta g_j(\boldsymbol{x}^{\lambda'}(\bar{\theta}+td), \bar{\theta})\big]td \\ - \big[\boldsymbol{x}^{\lambda'}(\bar{\theta}+td) - \bar{x}\big]\nabla_x g_j(\bar{x},\bar{\theta})\Big]; \quad \forall j \in J_0^+(\bar{x},\bar{\theta},\lambda').\end{aligned} \tag{4.21}$$

[5] since $\bar{x}$ is the locally unique optimal solution of the expanded problems (see proposition 4.3.3), then (4.2a) conditions hold for the expanded problems at $\bar{x}$ and we have: $-\nabla_x f(\bar{x}) = \sum_{j \in J_0^+(\bar{x},\bar{\theta},\lambda')} \lambda'_j \nabla_x g_j(\bar{x},\bar{\theta})$.

To simplify the notation in what follows, denote the right hand side of the (4.21) by $\varphi_j(\bar{x}, \bar{\theta}, t, d)$. Then,

$$-\lambda'_j\Big[\big[\boldsymbol{x}^{\lambda'}(\bar{\theta}+td)-\bar{x}\big]\nabla_x g_j(\bar{x},\bar{\theta})\Big] = \lambda'_j\Big[\big[\nabla_\theta g_j(\bar{x},\bar{\theta})-\nabla_\theta c_j(\bar{\theta})\big]td\Big]-\varphi_j(\bar{x},\bar{\theta},t,d), \quad (4.22)$$

for all $\lambda' \in M(\bar{x}, \bar{\theta})$.

With this, we can rewrite the right-hand side of (4.17) and thus (4.14) in terms of parametric gradients and obtain the following upper bound for the optimal value changed due to the perturbation:

$$v(\bar{\theta}+td)-v(\bar{\theta}) \leq \sum_{j\in J_0^+(\bar{x},\bar{\theta},\lambda')} \Big(\lambda'_j\Big[\big[\nabla_\theta g_j(\bar{x},\bar{\theta})-\nabla_\theta c_j(\bar{\theta})\big]td\Big]-\varphi_j(\bar{x},\bar{\theta},t,d),\Big) \quad (4.23)$$

for all $\lambda' \in int(M(\bar{x}, \bar{\theta}))$. Therefore,

$$v(\bar{\theta}+td)-v(\bar{\theta}) \leq \quad (4.24)$$
$$\min_{\lambda'\in int(M(\bar{x},\bar{\theta}))} \sum_{j\in J_0^+(\bar{x},\bar{\theta},\lambda')} \Big(\lambda'_j\Big[\big[\nabla_\theta g_j(\bar{x},\bar{\theta})-\nabla_\theta c_j(\bar{\theta})\big]td\Big]-\varphi_j(\bar{x},\bar{\theta},t,d).\Big)$$

Then, the lower Dini directional derivative as defined in (4.6a) has the following upper bound:

$$\nabla^+ v(\bar{\theta},d) = \limsup_{t\downarrow 0}\frac{v(\bar{\theta}+td)-v(\bar{\theta})}{t}$$
$$\leq \min_{\lambda'\in int(M(\bar{x},\bar{\theta}))}\Big(\sum_{j\in J_0^+(\bar{x},\bar{\theta},\lambda')}\lambda'_j\Big[\big[\nabla_\theta g_j(\bar{x},\bar{\theta})-\nabla_\theta c_j(\bar{\theta})\big]d\Big]-\limsup_{t\downarrow 0}(\frac{\varphi_j(\bar{x},\bar{\theta},t,d)}{t})\Big). \quad (4.25)$$

Recall from remark 5 that $\boldsymbol{x}^{\lambda'}(.)$ is a Lipschitz continuous function for the expanded problems in the stable perturbation neighbourhood ($N_\sigma^{\lambda'}(\bar{x})$) and, therefore, $\lim_{t\downarrow 0}(\frac{x^{\lambda'}(\bar{\theta}+td)-\bar{x}}{t})$ is finite. Also, as discussed in remark 6, $\lambda'(.)$ is a continuous function for the expanded problems in the stable perturbation neighbourhood ($N_\sigma^{\lambda'}(\bar{x})$). Therefore, as $t \to 0$, we have $(\lambda'_j(\bar{\theta}+td) \to \lambda'_j)$ and with that:

$$\limsup_{t\downarrow 0}(\frac{\varphi_j(\bar{x},\bar{\theta},t,d)}{t}) = \limsup_{t\downarrow 0}(\lambda'_j(\bar{\theta}+td)-\lambda'_j)\big[\nabla_\theta c_j(\bar{\theta}) - \nabla_\theta g_j(\boldsymbol{x}^{\lambda'}(\bar{\theta}+td),\bar{\theta})\big]d \tag{4.26}$$

$$+\limsup_{t\downarrow 0}(\lambda'_j(\bar{\theta}+td)-\lambda'_j)(\frac{\boldsymbol{x}^{\lambda'}(\bar{\theta}+td)-\bar{x}}{t}) = 0. \tag{4.27}$$

Thus,

$$\nabla^+ v(\bar{\theta},d) \le \min_{\lambda'\in int(M(\bar{x},\bar{\theta}))} \sum_{j\in J_0^+(\bar{x},\bar{\theta},\lambda')} \lambda'_j\big[\nabla_\theta g_j(\bar{x},\bar{\theta}) - \nabla_\theta c_j(\bar{\theta})\big]d, \tag{4.28}$$

which concludes the proof. ■

4.5 Lower Bound on the Directional Derivative

We can derive this bound directly from the first-order Taylor approximation of the change of the optimal value of $\boldsymbol{P}(\bar{\theta})$ for the perturbation from $\bar{\theta}$ to $\bar{\theta}+td$ and therefore from its locally optimal solution to a post-perturbation optimal solution, say $\hat{x} \in X(\bar{\theta}+td)$. Here we focus on the cases where $\bar{x} \in X(\bar{\theta})$ is a locally unique optimal solution of $\boldsymbol{P}(\bar{\theta})$ under MFCQ. Under these conditions, $\boldsymbol{P}(\bar{\theta})$ at $\bar{x}$ is type-I degenerate and $\bar{x}$ may not be a strongly stable solution.

Consider perturbing the parametric setting of P from $\bar{\theta} \to \bar{\theta}+td$. Since $\boldsymbol{P}(\bar{\theta})$ satisfies MFCQ at $\bar{x}$, as discussed earlier, then $F(\theta+td)$ remains nonempty under slight perturbations. If the local compactness conditions hold at $\bar{x}$ too, then $F(\theta+td)$ remains compact and therefore there should exist an optimal solution after perturbation. However, such an optimal solution of $\boldsymbol{P}(\bar{\theta}+td)$ may not be close to $\bar{x}$ or even satisfy MFCQ conditions.

Proposition 4.5.1 *Let $\bar{x}$ to be a locally unique optimal solution of* (4.1) *at $\bar{\theta} \in \Theta$. Then,*

$$\nabla^- v(\bar{\theta},d) \ge \max_{\lambda\in int(M(\bar{x},\bar{\theta})} \sum_{j\in J_0^+(\bar{x},\bar{\theta},\lambda)} \lambda_j\left(\nabla_\theta g_j(\bar{x},\bar{\theta}) - \nabla_\theta c_j(\bar{\theta})\right)d \tag{4.29}$$

holds if MFCQ and local compactness are satisfied at $\bar{x}$ for $\boldsymbol{P}(\bar{\theta})$.

Proof: $\boldsymbol{P}(\bar{\theta})$ satisfies MFCQ at $\bar{x}$ which is its strict local optimum, then KKT conditions hold at $\bar{x}$ and therefore (4.2a) and (4.2d) hold as:

$$-\nabla_x f(\bar{x}) = \sum_{j \in J_0^+(\bar{x},\bar{\theta},\bar{\lambda})} \bar{\lambda}_j \nabla_x g_j(\bar{x}, \bar{\theta}), \tag{4.30}$$

$$\bar{\lambda}_j [g_j(\bar{x}, \bar{\theta}) - c_j(\bar{\theta})] = 0, \qquad \forall j \in J_0^+(\bar{x}, \bar{\theta}, \bar{\lambda}), \tag{4.31}$$

for all $\bar{\lambda} \in M(\bar{x}, \bar{\theta})$.

Suppose that the parametric setting of (4.1) is perturbed from $\bar{\theta} \to \bar{\theta} + td$ and, therefore, the optimal solution of $\boldsymbol{P}(.)$ changes from $\bar{x} \in X(\bar{\theta})$ to $\hat{x} \in X(\bar{\theta} + td)$. Note that there exists $\hat{x} \in X(\bar{\theta} + td)$ that is a local optimum of the perturbed problem[6]. Therefore, the perturbation results in a finite change of the optimal value as:

$$\begin{aligned} v(\bar{\theta} + td) - v(\bar{\theta}) &= f(\hat{x}) - f(\bar{x}) \\ &= (\hat{x} - \bar{x}) \quad \nabla_x f(\bar{x}) && (4.32) \\ &= -(\hat{x} - \bar{x}) \sum_{j \in J_0^+(\bar{x},\bar{\theta},\bar{\lambda})} \bar{\lambda}_j \nabla_x g_j(\bar{x}, \bar{\theta}); \qquad \forall \bar{\lambda} \in M(\bar{x}, \bar{\theta}) && (4.33) \end{aligned}$$

Here we used a Taylor approximation to get (4.32) and then substituted $\nabla_x f(\bar{x})$ from (4.30) to obtain (4.33). By subtracting $g_j(\tilde{x}, \bar{\theta} + td) - c_j(\bar{\theta} + td)$ from both sides of (4.31), we can rewrite it as:

$$\bar{\lambda}_j \Big[[g_j(\hat{x}, \bar{\theta} + td) - g_j(\bar{x}, \bar{\theta})] - [c_j(\bar{\theta} + td) - c_j(\bar{\theta})] \Big] = \bar{\lambda}_j \big[g_j(\hat{x}, \bar{\theta} + td) - c_j(\bar{\theta} + td) \big], \tag{4.34}$$

for all $j \in J_0^+(\bar{x}, \bar{\theta}, \bar{\lambda})$ and for all $\bar{\lambda} \in M(\bar{x}, \bar{\theta})$. Since the second part of the right-hand side of (4.34), $g_j(\hat{x}, \bar{\theta} + td) - c_j(\bar{\theta} + td)$, is a feasibility constraint of the perturbed problem at the optimal solution, $\hat{x} \in X(\theta + td)$, it is non-positive. Moreover, $\bar{\lambda}_j$ is a Lagrangian multiplier associated with the initial problem at its optimal solution and

[6] Due to the validity of MFCQ and LC at the optimal solution of the initial problem, the feasible space of the perturbed problem remains nonempty and compact and therefore there exist a feasible point in $\boldsymbol{F}(\theta + td)$ that optimizes the perturbed problem

thus is non-negative. Therefore, the right-hand side of (4.34) is non-positive for all $j \in J_0^+(\bar{x}, \bar{\theta}, \bar{\lambda})$ and for all $\bar{\lambda} \in M(\bar{x}, \bar{\theta})$. This leads to

$$\bar{\lambda}_j \Big[[g_j(\hat{x}, \bar{\theta} + td) - g_j(\bar{x}, \bar{\theta})] - [c_j(\bar{\theta} + td) - c_j(\bar{\theta})] \Big] \leq 0; \quad \forall j \in J_0^+(\bar{x}, \bar{\theta}, \bar{\lambda}), \forall \bar{\lambda} \in M(\bar{x}, \bar{\theta}). \tag{4.35}$$

With a Taylor approximation, we obtain

$$\bar{\lambda}_j \big[(\hat{x} - \bar{x}) \nabla_x g_j(\bar{x}, \bar{\theta}) + td \nabla_\theta g_j(\bar{x}, \bar{\theta}) \big] \leq \bar{\lambda}_j td \nabla_\theta c_j(\bar{\theta}) \quad \forall j \in J_0^+(\bar{x}, \bar{\theta}, \bar{\lambda}), \tag{4.36}$$

for all $\bar{\lambda} \in M(\bar{x}, \bar{\theta})$. Then,

$$-\bar{\lambda}_j (\hat{x} - \bar{x}) \nabla_x g_j(\bar{x}, \bar{\theta}) \geq t\, d\, \bar{\lambda}_j \big[\nabla_\theta g_j(\bar{x}, \bar{\theta}) - \nabla_\theta c_j(\bar{\theta}) \big] \quad \forall j \in J_0^+(\bar{x}, \bar{\theta}, \bar{\lambda}), . \tag{4.37}$$

for all $\bar{\lambda} \in M(\bar{x}, \bar{\theta})$. Summing over all $j \in \{0, \ldots, m\}$ gives:

$$\begin{aligned} -(\hat{x} - \bar{x}) \sum_{j \in \{0,\ldots,m\}} \bar{\lambda}_j \nabla_x g_j(\bar{x}, \bar{\theta}) &\geq t \sum_{j \in \{0,\ldots,m\}} \bar{\lambda}_j \big[\nabla_\theta g_j(\bar{x}, \bar{\theta}) - \nabla_\theta c_j(\bar{\theta}) \big] d \\ &= t \sum_{j \in J_0^+(\bar{x}, \bar{\theta}, \bar{\lambda})} \bar{\lambda}_j \big[\nabla_\theta g_j(\bar{x}, \bar{\theta}) - \nabla_\theta c_j(\bar{\theta}) \big] d, \end{aligned}$$

for all $\bar{\lambda} \in M(\bar{x}, \bar{\theta})$. Since this equation holds for all $\bar{\lambda} \in M(\bar{x}, \bar{\theta})$, we have

$$\begin{aligned} -(\hat{x} - \bar{x}) \sum_{j \in \{0,\ldots,m\}} \bar{\lambda}_j \nabla_x g_j(\bar{x}, \bar{\theta}) &\geq \max_{\lambda \in int(M(\bar{x}, \bar{\theta}))} \sum_{j \in J_0^+(\bar{x}, \bar{\theta}, \lambda)} \lambda_j \left(\nabla_\theta g_j(\bar{x}, \bar{\theta}) - \nabla_\theta c_j(\bar{\theta}) \right) td \\ &\geq \max_{\lambda \in M(\bar{x}, \bar{\theta})} \sum_{j \in J_0^+(\bar{x}, \bar{\theta}, \lambda)} \lambda_j \left(\nabla_\theta g_j(\bar{x}, \bar{\theta}) - \nabla_\theta c_j(\bar{\theta}) \right) td \end{aligned}$$

With this, we can write a lower bound for the right-hand side of (4.33) and thus (4.32) in terms of parametric gradients. This leads to the following Dini lower bound for the change of the optimal value due to the perturbation:

$$\begin{aligned} \nabla^- v(\theta, d) &= \liminf_{t \downarrow 0} \frac{v(\bar{\theta} + td) - v(\bar{\theta})}{t} \\ &\geq \max_{\lambda \in int(M(\bar{x}, \bar{\theta}))} \sum_{j \in J_0^+(\bar{x}, \bar{\theta}, \lambda)} \lambda_j \left(\nabla_\theta g_j(\bar{x}, \bar{\theta}) - \nabla_\theta c_j(\bar{\theta}) \right) d, \end{aligned}$$

which concludes the proof. ■

Proposition 4.5.2 *If MFCQ and local compactness conditions hold for a local maximum* $\bar{x}$ *of* (4.1) *at* $\bar{\theta} \in \Theta$*, then*

$$\nabla v(\bar{\theta}, d) = \sum_{j \in J_0^+(\bar{x}, \bar{\theta}, \bar{\lambda})} \bar{\lambda}_j (\nabla_\theta g_j(\bar{x}, \bar{\theta}) - \nabla_\theta c_j(\bar{\theta})); \quad \forall \bar{\lambda} \in int(M(\bar{x}, \bar{\theta})) \tag{4.38}$$

represents the directional derivative of the value function at $\bar{\theta}$ *into direction* $d \in \mathbb{R}^p$*, where* $J_0(\bar{x}, \bar{\theta}) = \{j \in J \mid g_j(\bar{x}, \bar{\theta}) = 0\}$ *and* $M(\bar{x}, \bar{\theta}) = \{\lambda = (\lambda_j; j \in J_0(\bar{x}, \bar{\theta})) | \lambda_j \geq 0, j \in J_0(\bar{x}, \bar{\theta})\}$ *denote the set of Lagrangian multipliers for* (4.1) *at* $(\bar{x}, \bar{\theta})$.

Proof: By proposition 4.4.1 and proposition 4.5.1, both of the following inequalities should hold for a locally unique optimal solution, $\bar{x}$, of the initial problem,(4.1), at $\bar{\theta} \in \Theta$:

$$\nabla^+ v(\bar{\theta}, d) \leq \min_{\lambda' \in int(M(\bar{x}, \bar{\theta}))} \sum_{j \in J_0^+(\bar{x}, \bar{\theta}, \lambda')} \lambda'_j \big[\nabla_\theta g_j(\bar{x}, \bar{\theta}) - \nabla_\theta c_j(\bar{\theta})\big] d$$

and

$$\nabla^- v(\theta, d) \geq \max_{\lambda \in int(M(\bar{x}, \bar{\theta}))} \sum_{j \in J_0^+(\bar{x}, \bar{\theta}, \lambda)} \lambda_j \left(\nabla_\theta g_j(\bar{x}, \bar{\theta}) - \nabla_\theta c_j(\bar{\theta})\right) d.$$

This leads to

$$\min_{\lambda \in int(M(\bar{x}, \bar{\theta}))} \sum_{j \in J_0^+(\bar{x}, \bar{\theta}, \lambda)} \lambda_j \left(\nabla_\theta g_j(\bar{x}, \bar{\theta}) - \nabla_\theta c_j(\bar{\theta})\right) d$$
$$\geq \max_{\lambda \in int(M(\bar{x}, \bar{\theta}))} \sum_{j \in J_0^+(\bar{x}, \bar{\theta}, \lambda)} \lambda_j \left(\nabla_\theta g_j(\bar{x}, \bar{\theta}) - \nabla_\theta c_j(\bar{\theta})\right) d,$$

and therefore we have

$$\nabla v(\bar{\theta}, d) = \sum_{j \in J_0^+(\bar{x}, \bar{\theta}, \lambda)} \lambda_j (\nabla_\theta g_j(\bar{x}, \bar{\theta}) - \nabla_\theta c_j(\bar{\theta})) d; \quad \forall \lambda \in int(M(\bar{x}, \bar{\theta})),$$

which concludes the proof. ■

Chapter 5

Modeling and Measuring Systemic Risk

In this chapter, we develop a mathematical formulation of financial systems which models an interbank network in the context of the real economy, measuring and optimizing systemic risk through a dynamic parameter setting mechanism. The initial mathematical model is built as a basis for the design of the regulatory framework. Then we propose to extend the initial model to a(n) decision-making/optimization model which directly sets the systemic parameters like leverage limits/reserve ratios as the control variables of the systemic risk. The proposed optimization model exploits the systemic structure of the contractual dependencies of the financial institutions, embedded in the initial model along with the banks' interest to maximize their shares of the financial market/leverage limits. Based on some theorems in convex analysis, we will show that such a model always has a unique solution.

5.1 Economic Framework and Regulatory System

We start with setting a baseline economy that contains the banking system as a subsystem of the real economy. Consider an economy with two dates $t_0 < t_1$ and three sets of agents, a representative household, a heterogeneous set of commercial banks/financial institutions, and a central bank[1].

Assume that at date t_0, there are a heterogeneous set of potential projects with state-contingent return rates at date t_1. There is a market at date t_0, in which the representative household and banks make investments in the potential projects maximizing their expected wealth at date t_1.

We limit the representative household's use of its endowment to investments in the (equity) of the potential projects and/or depositing in the banking system. On the other hand, banks are limited to using their endowments and the deposits they receive from the representative household and/or other banks to issue debt contracts for other banks or/and issue loans funding the potential projects. In other words, households only may deposit their endowments in the banking system and/or invest in the equity of the potential projects. Banks, however, may invest in the interbank deposit market and/or in the loans issued for funding the potential projects. We refer to loans issued for funding the potential projects as consumer loans. Both banks and consumer loan holders are protected under the limited liability provision of the bankruptcy law and thus are subject to the regulatory requirements set by the central bank.

Given the state-contingent return rates of potential projects (consumer loans), exogenous endowments, and the regulatory and legal requirements and limits, all transactions, including depositions, equity investments, and debt contracts, occur simultaneously at date t_0 and before the uncertainty is resolved at t_1. Thus, the equilibrium allocations at t_0, endogenously determine not only the size of the transactions (depositions, equity investments, and debt contracts) but also the nominal return rates of the debt contracts/loans, which create a state-contingent system of obligations at t_0.

The aforementioned economy is the basis of the parametric dynamics that appear in the financial system we define in section5.3. In other words, we assume that the contractual linkages in the financial system are equilibrium objects that are formed under uncertainty and based on the market participants' profit maximization and risk aversion profiles, which in turn govern their borrowing, lending, and investment incentives.

[1]We refer to the central bank, the regulator, and the policy-maker interchangeably).

5.2 System-level Adaptive Capacity of the Banking System

Individual banks' decision-making process, reflected in the proportional investment of each bank's available funding in different assets, forms the asset side of the individual bank's balance sheet. On the other hand, the liability side of each bank's balance sheet is formed by the proportional investments of other banks as well as the deposits of the households.[2]

Thus, the structure of the banking system is formed by a system of interconnected investment decisions realized in the form of interconnected balance sheets of the banking system. Presenting the banking system's observed structure as a realization of a system of interactive decisions provides a basis to create a tractable adaptive capacity in the banking system that could be used as a normative interaction channel with the banking system. In other words, considering the investment decisions of individual banks as the building blocks of the realized structure of the banking system presents a framework to design a system-level tractable adaptive capacity that could be used in a normative way and as a basis of a regulatory framework on systemic risk. The challenge, although, is to design an interaction mechanism that creates an analytically tractable adaptive capacity for such a complex decision system without explicitly modeling the hidden parts of the system.

In the following sections, we discuss our approach to develop such a mechanism and thus the normative capacity we need for a reliable system-level regulatory framework. Before that, we briefly discuss why the structure of the existing regulatory frameworks are incapable of creating a system-level adaptive capacity in the banking system that could for normative interactions with the banking system. Then we define two regulatory rules, leverage allowance and reserve requirements, to be used as the main

[2] In other words, the balance sheet of each individual bank is formed by an asset side: the bank's investments in the real economy and the banking system, and on the other hand, the liability side: investments of households and the banking system in the bank.

components of the interaction mechanism of our proposed regulatory system. We compare the formulation of these rules with the traditional rules and briefly discuss why they provide a reliable basis for creating the adaptive capacity we need to design a normative framework. In particular, we explain how the idea behind their definition takes advantage of the structure of the banking balance sheets as a system of decisions to create a system-level adaptive capacity.

Regulatory Requirements as a Normative Interaction Mechanism

As discussed earlier in chapter 2, capital and leverage requirements are the main regulatory tools that are used by central banks to address systemic risk in financial markets. Capital requirements, however, initially were meant to mitigate the default risk of the individual banks due to credit risk exposures. Leverage requirements were developed more recently aimed to complement the capital requirements, protecting individual banks against liquidity risk exposures. Although the leverage requirements formulations follow the same rationale of the capital requirements and thus aim to mitigate the default risk of the individual banks. The difference is that the leverage requirements are aimed at liquidity shock rather than credit defaults.

In the latest regulatory standards, like Basel III, it is assumed that these individual-level protections on larger scales lead to more stability of the banking system and thus prevent systemic risk and collapse of the financial system. These frameworks do not go any further on how the individual level protections escalate in a way that captures the system-level dynamics[3]. Formulations for capital and leverage requirements vary in different frameworks; however, their generic structure aims to ensure that the size of safe assets (liabilities) proportional to the size of risky liabilities (risky assets) is larger than a threshold set by the regulator. The difference lies in how riskiness is weighted.

Despite the intuitive appeal, as we showed in chapter 2 , the interactions of these regulations result in confliction of their aimed purposes and, as shown in chapter 3,

[3]For more details on the formal definitions of capital and leverage requirements based on the Basel III framework, see chapter 2, Section 1.

this may even cause system-level instability in financial markets with undesirable consequences for all market stakeholders. Therefore, in this chapter, we develop a modeling approach that can be used to leverage the adaptive characteristics of individual banks to interact with the system-level dynamics of the system in a normative way. Before getting to the details of our framework, let us discuss the idea behind its setting.

Let us define the leverage allowance of a bank as a regulatory requirement that limits the maximum size of a bank's investment in risky assets to a level determined by the regulator. Any other type of assets that are considered risk-free would be referred to the bank's reserve. Furthermore, let us define the reserve requirement of a bank as a regulatory measure that requires a bank to hold a minimum size of reserves (safe assets), determined by the regulator. Note that the formulation of the leverage allowance and reserve requirements are not in the ratio format; however, we can transform these requirements for each individual bank to their equivalent definitions in traditional regulatory rules[4].

Note that leverage allowance and reserve requirements only regulate the asset side of the banks' balance sheet. We do not define any requirements that directly regulate the liability side of the balance sheet, however, as we see in the next sections, the liability side of a given bank's balance sheet is formed under the collective regulation of the asset sides of other banks investments in the given bank.

5.3 Parametric Modeling of a Financial System

Consider the set of n commercial banks regulated by the central bank in the economy defined in section 5.1. We refer to this set of banks, the deposits they receive, the

[4]Leverage allowance and reserve requirements as defined here could be transformed to their ratio equivalents as defined in traditional regulatory frameworks; however, since our regulatory framework assigns specific requirements for each individual bank, we can not use the ratio format which could be only applied to assign the same requirement for all the banks in the banking system.

reserves[5] they hold, their obligations to each other[6] , and finally their investments in the consumer loan market[7], a *"state-contingent financial system"*. We assume that an state-contingent financial system is a part of the market equilibrium, formed at the initial state of the economy at t_0.

At t_0, each bank is limited by the central bank to a certain nonnegative level investment in risky assets (leverage allowance), represented for the bank i by L_i, and for the financial system by the vector $\boldsymbol{L}$. Each bank uses its assigned leverage allowance to fund the bank's both interbank loans portfolio and consumer loans portfolio, which are refereed here respectively as *"inter-network "* investments and *"external"* or out of network investments of the bank i. To simplify the notation, all other risky assets funded by the bank i, out of the banking network, are limited to be debt contracts[8] and are refereed as consumer loans, issued by the bank i. It is the decision of the bank i to determine how it assigns its leverage allowance to internal and external loans, where the bank's aggregate external investments is represented by L_i^{ex} and its aggregate inter-network investments is represented by L_i^{in}, however it is assumed that all leverage allowance is used by the bank, and accordingly it holds:

$$L_i^{ex} + L_i^{in} = L_i \qquad ; \forall i \in \mathcal{N}$$

The reserves in the central bank are represented by the *reserve vector* $\boldsymbol{R}$, where R_i is the reserve of the bank i held by the bank i itself or at the central bank.

The obligations of the banks to each other are represented by $n \times n$ *obligation matrix* $\boldsymbol{O}$, where O_{ij} is the obligation of the bank i to the bank j, $\forall i, j \in \mathcal{N}$. It is assumed that $O_{ij} \geq 0$ and $O_{ii} = 0$ which means that the obligations in the system are nonnegative and no bank makes an obligation to itself, respectively. Then we have:

[5]To simplify the notation, all risk-free assets of each bank are considered cash reserves held by the bank itself or at a Federal Reserve bank

[6]The loans/debt contracts they have issued for each other

[7]Loans/debt contracts they have issued for each other and/or for consumers/projects

[8]Recall that the banking system investments in potential projects are limited to debt contacts (consumer loans) governed under bankruptcy law provision rules.

$$\sum_{i=1}^{n} Oij = L_j^{in} \qquad ; \forall j \in \mathcal{N}$$

Recall that L_j^{in} represents that how much the bank j has invested inside the financial network or in other words, the internal leverage level for the bank j in the financial system. Let us define a $n \times n$ relative obligation matrix, $\mathbf{\Phi}$ to be the normalization of the obligation matrix with respect to the internal leverage, as follows:

$$\Phi_{ij} = \begin{cases} \frac{O_{ij}}{L_j^{in}} & \text{if } L_j^{in} \neq 0 \\ 0 & \text{if } L_j^{in} = 0 \end{cases}$$

where Φ_{ij} is the proportion of j's leverage allowance constituted by the obligation of i to j. In other words, Φ_{ij} represents the proportion of the leverage allowance, bank j has assigned to bank i. Accordingly, the total obligation of the bank i to the system is $\sum_{j=1}^{n} \Phi_{ij} L_j^{in} = \sum_{j=1}^{n} O_{ij}$ and the sum of the proportional obligations of the system to each bank, say i is $\sum_{i=1}^{n} \Phi_{ij} = 1$.

Building upon the definitions already made, we can define an *state-contingent financial system* formally given the set of the banks $\mathcal{N}$, their obligations in the system ($\boldsymbol{O}$ matrix), reserves in the central bank ($\boldsymbol{R}$ vector) and their external investments vector ($\boldsymbol{L}^{ex}$ vector) as corresponding to the tuple $(\mathcal{N}, \boldsymbol{O}, \boldsymbol{R}, \boldsymbol{L}^{ex})$. Furthermore, since for a given obligation matrix we can construct the matrix of proportional obligations ($\mathbf{\Phi}$) and internal leverage allowances ($\boldsymbol{L}^{in}$) and vice versa, then the financial system can be also represented as $(\mathcal{N}, \mathbf{\Phi}, \boldsymbol{L}^{in}, \boldsymbol{R}, \boldsymbol{L}^{ex})$ equivalently. We use the later formulation, since it leads to a parametric formulation of financial systems that incorporates regulatory variables, like leverage allowances in addition to reserve requirements in the earlier formulation.

Each bank funds some consumer loans in aggregate as much as L_i^{ex}. When period 1 arrives and the state of the world is revealed, those debt contracts/loans are executed and accordingly, the income of each bank from its consumer loan contracts are realized. Normally, banks expect to get back how much the face values of the loans are. However, some of the consumer loans may default and accordingly, the banks receive back less than the face value of loans. Assuming that an exhaustive set $\mathcal{S}$ of states of the world[9] at t_1 is finite, we represent the realized consumer loans portfolio value of the bank i at state s by x_i^s, which is nonnegative and less than or equal to L_i^{ex}. In other words, x_i^s represents the amount of the cash that bank i can expect to get back in aggregate from its external debt contracts/loans in a given economic scenario, $s \in \mathcal{S}$. Formally, realized portfolio values of the banks in the financial network form the external cash flow vector of the financial system, is presented by $\boldsymbol{x}^s = [x_i^s]_{i\in\mathcal{N}}$; where $0 \leq x_i^s \leq L_i^{ex}; \quad \forall i \in \mathcal{N}, \forall s \in \mathcal{S}$.

In turn, the external cash flow to each bank in a given state of the world at t_1 determines whether a bank remains solvent and capable of paying back the face value of its obligations to its creditors or not. In the case of some insolvencies, some obligations cannot be fulfilled to the level of their face value. and it is essential to figure out how much each bank should pay its creditors at t_1, to be legally clear. To formulate a representation of the realized system, let us define a payment vector (or clearance vector) as $\boldsymbol{P} = [P_i]_{i\in\mathcal{N}}$, where P_i shows how much the bank i should pay in sum to the system. Moreover, we need to define (for more detail) a payment matrix $\boldsymbol{p} = [p_{ij}]_{i,j\in\mathcal{N}}$, where p_{ij} shows how much the bank i should pay the bank j and accordingly it is trivial that $P_i = \sum_{j=1}^{n} p_{ij}$. Then if the bank i can pay all of its obligations in a given state, we have:

$$P_i = \sum_{j=1}^{n} Oij = \sum_{j=1}^{n} \Phi_{ij} L_j^{in} \qquad \forall i \in \mathcal{N} \tag{5.1}$$

and otherwise, when the bank i is defaulted:

$$P_i < \sum_{j=1}^{n} Oij = \sum_{j=1}^{n} \Phi_{ij} L_j^{in} \qquad \forall i \in \mathcal{N} \tag{5.2}$$

The latter case means that how much the bank i can pay is less than the sum of its obligations to the system and accordingly to determine the payments of i to its creditors ($p_{ij}, \forall j \in \mathcal{N}$) we need to (consistent with bankruptcy law), divide P_i between the creditors proportional to the size of nominal claim of each bank on i's assets which is $\frac{O_{ij}}{\sum_{k=1}^{n} O_{ik}}$ $(= \frac{\Phi_{ij} L_j^{in}}{\sum_{k=1}^{n} \Phi_{ik} L_k^{in}})$ and accordingly the bank i's payment to other banks in the

[9] As a complete description of a possible outcome of uncertainty

system is determined as:

$$p_{ij} = (\frac{\Phi_{ij} L_j^{in}}{\sum_{k=1}^{n} \Phi_{ik} L_k^{in}}) P_i \, ; \qquad \forall i, j \in \mathcal{N} \tag{5.3}$$

It is trivial that even if the bank i does not default, above equation holds and accordingly for the bank i, defaulted or not, the internal cash flow or in other words, aggregate payments of the other banks in the system to the bank i is determined as follows:

$$\text{Internal cash flow to } i = \sum_{j=1}^{n} p_{ji} = \sum_{j=1}^{n} (\frac{\Phi_{ji} L_i^{in}}{\sum_{k=1}^{n} \Phi_{jk} L_k^{in}}) P_j$$

On the other hand, since x_i is the external cash flow to i and R_i is the bank i's reserve, then it is easy to see that the total cash available to i or the aggregate value of the items in the asset side of the i's balance sheet, is determined as:

$$\text{Total cash-available to } i = x_i + R_i + \sum_{j=1}^{n} (\frac{\Phi_{ji} L_i^{in}}{\sum_{k=1}^{n} \Phi_{jk} L_k^{in}}) P_j \tag{5.4}$$

Then, due to the absolute priority provision of the bankruptcy law, bank i pays $\max\limits_{P_i \in \mathbb{R}_+} P_i$, subject to the limited liability[10] provision:

$$U_i : P_i \leq x_i + R_i + \sum_{j=1}^{n} (\frac{\Phi_{ji} L_i^{in}}{\sum_{k=1}^{n} \Phi_{jk} L_k^{in}}) P_j \qquad \text{for all } i \in \mathcal{N} \tag{5.5}$$

and its aggregate obligations:

$$V_i : P_i \leq \sum_{j=1}^{n} \Phi_{ij} L_j^{in} \qquad \text{for all } i \in \mathcal{N} \tag{5.6}$$

Thus, if the total incoming cash flow to i is more than it's aggregate obligations to the system ($P_i = \sum_{j=1}^{n} O_{ij}$), i can pay off its creditors in the full capacity ($P_i = \sum_{j=1}^{n} \Phi_{ij} L_j$) and otherwise i should pay all of its total incoming cash flow to the system, in the form, described above. In other words:

$$P_i = min\{x_i + R_i + \sum_{j=1}^{n} (\frac{\Phi_{ji} L_i^{in}}{\sum_{k=1}^{n} \Phi_{jk} L_k^{in}}) P_j, \sum_{j=1}^{n} \Phi_{ij} L_j^{in}\} \qquad \forall i \in \mathcal{N} \tag{5.7}$$

[10] Bank i pays nothing more than its total cash available

The solution determines how much each bank should pay any other bank to clear legally, which forms the clearance vector (P), associated with the realized financial system. This results in an equivalent formulation of the banking system developed by Eisenberg and Noe (2001b). However, Eisenberg and Noe (2001b)'s framework does not incorporate the decision parameters of the involved stakeholders in its formulation of the banking system, whereas (5.7) includes the regulator's decision parameters as well as the investment decisions of the banks (and the representative household) in its parametric setting. In other words, contrary to Eisenberg and Noe (2001b)'s framework that does not go beyond[11] capturing the absolute priority provision of the bankruptcy law, (5.7) combines the structure of the bankruptcy law and the representation of the stakeholders' decisions in a simultaneous formulation of the proportional distribution of each bank's obligations/debts and assets/credits in its parametric setting.

With this representation of the realized financial system, we can assume that the network of obligations are determined endogenously and seek to determine how a change in an economic/regulatory parameter relates to the change within the network of obligations and eventually to the change of the clearing payments[12]. In other words, if we formulate how the change of an economic/regulatory parameter changes the market equilibrium, since the (nominal) obligations of the banks are endogenous to that equilibrium, we can formulate how that economic/regulatory change, changes the (nominal) obligations of the banks to each other and therefore how it changes the clearing payments at any given state.

Other than the different parametric setting of our model, formulation of the problem as a system of n optimization problems, makes it possible to guarantee the existence

[11] Eisenberg and Noe (2001b) assume that the network of obligations are exogenous and aim to figure out how much banks owe to each other. Thus, the Eisenberg and Noe (2001b)'s framework only formulates the absolute priority provision of the bankruptcy law. To do so, they just need to normalize each bank's obligations to the bank's aggregate obligations to other banks.

[12] If we assume that the interbank network of obligations is endogenous to the equilibrium model developed in section 5.4.3, we can conclude that the clearing payment of the banks at t_1 is also endogenous to the same equilibrium.

and uniqueness of the clearing vector in a generic setting.

As shown in the following proposition, we can guarantee the unique solution for the clearance network, even if there exist banks which have no external cash flow at the given state, contrary to the Eisenberg and Noe (2001b) approach that requires all banks to have a strictly positive external cash flow.

Proposition 5.3.1 (Existence and uniqueness of the clearance vector) *For any financial system,* $(\mathcal{N}, \Phi, L^{in}, R, e)$*, clearance vector,* P^* *exists and it is unique.*

Proof: The clearance vector P^* simultaneously solves for $\forall i \in \mathcal{N}$ the optimization problem:

$$\begin{aligned} \max_{P_i \in \mathbb{R}_+} \quad & P_i \\ \text{s.t.} \quad & U_i : P_i \leq x_i + R_i + \sum_{j=1}^{n} (\frac{\Phi_{ji} L_i^{in}}{\sum_{k=1}^{n} \Phi_{jk} L_k^{in}}) P_j \\ & V_i : P_i \leq \sum_{j=1}^{n} \Phi_{ij} L_j^{in} \end{aligned} \tag{5.8}$$

This system of optimization problems can be reformulated in a single optimization problem which maximizes the clearance vector $P = [P_i]_{i \in \mathcal{N}}$ satisfying all the constrained of (5.8) models, $U = [U_i]_{i \in \mathcal{N}}$, $D = [D_i]_{i \in \mathcal{N}}$ (for simplicity consider $S = \{P \in \mathbb{R}_+^n : U, V\}$):

$$\begin{aligned} \max \quad & [P_1, P_2, ...P_n] \\ \text{s.t.} \quad & P \in S \end{aligned} \tag{5.9}$$

Existence: A trivial feasible solution for (5.8) is $P = \mathbf{0}$, also P is bounded as:

$$0 \leq P_i \leq \sum_{j=1}^{n} \Phi_{ij} L_j \qquad \text{for all } i \in \mathcal{N}$$

and therefore guarantees the existence of the solution for (5.8).

Uniqueness : Considering P^* to be a pareto optimal solution, by definition means that $\nexists P \in S$; $P^* \leq P$. This means that $\forall P \in S$; $P^* \nleq P$. This guarantees that at least for one component in P, say i we have $p_i^* > p_i$, however we may or may not have other components where $p_i^* = p_i$ or $p_i^* \leq p_i$. Let us rearrange the vector components as $P = (P_a, P_b)$ where P_a is a (non-zero length) sub-vector of P in which for each component $p_{i\in a}$, we have $p^*_{i\in a} \geqslant p_{i\in a}$. On the other hand, P_b is a sub-vector of P in which for each component $p_{i\in b}$, we have $p^*_{i\in b} < p_{i\in b}$. Thus P_a is a non-zero length vector but so far P_b can be a vector of zero length or not.

Assume that for a $P \in S$, we have non-zero length P_b. Then let us move $P = (P^*_a, P^*_b)$ toward $P^{new} = (P^*_a, P_b)$ and see if it violates the feasibility constraints of (5.8), Both sets of constraints $U = [U_i]_{i\in\mathcal{N}}$, $V = [V_i]_{i\in\mathcal{N}}$ should be checked to see if P^{new} is feasible or not.

Regarding the first set of constraints; $U = [U_i : P_i \leq x_i + R_i + \sum_{j=1}^{n}(\frac{\Phi_{ji}L_i}{\sum_{k=1}^{n}\Phi_{jk}L_k})P_j]_{i\in\mathcal{N}}$ in (5.8), $(\frac{\Phi_{ji}L_i}{\sum_{k=1}^{n}\Phi_{jk}L_k})$; $\forall i, j \in N$ is always positive and it does not change with the change of the solution for P, then $\forall i \in a$, and since $0 \leq p^*_i \leq p_i$; we have:

$$\begin{aligned}
p^*_i &\leq x_i + R_i + \sum_{j=1}^{n}(\frac{\Phi_{ji}L_i}{\sum_{k=1}^{n}\Phi_{jk}L_k})p^*_j \\
&\leq x_i + R_i + \sum_{j\in a}(\frac{\Phi_{ji}L_i}{\sum_{k=1}^{n}\Phi_{jk}L_k})p^*_j + \sum_{j\in b}(\frac{\Phi_{ji}L_i}{\sum_{k=1}^{n}\Phi_{jk}L_k})p_j \qquad \forall i \in a
\end{aligned}$$

Likewise, $\forall i \in b$ since $0 \leq p_i < p^*_i$; $\forall i \in a$ we have:

$$\begin{aligned}
p_i &\leq x_i + R_i + \sum_{j=1}^{n}(\frac{\Phi_{ji}L_i}{\sum_{k=1}^{n}\Phi_{jk}L_k})p_j \\
&\leq x_i + R_i + \sum_{j\in a}(\frac{\Phi_{ji}L_i}{\sum_{k=1}^{n}\Phi_{jk}L_k})p^*_j + \sum_{j\in b}(\frac{\Phi_{ji}L_i}{\sum_{k=1}^{n}\Phi_{jk}L_k})p_j \qquad \forall i \in b
\end{aligned}$$

and thus $P^{new} = (P^*_a, P_b)$ does not violate the first set of constraints.

Considering the second set of constraints; $V = [V_i : P_i \leq \sum_{j=1}^{n}\Phi_{ij}L_j]_{i\in\mathcal{N}}$ in (5.8), since the right hand side of each equation, $\sum_{j=1}^{n}\Phi_{ij}L_j$; $\forall i \in \mathcal{N}$ does not change with the change of the solution for P and considering the feasibility assumption of the P^* and P, it is immediate that:

$$p^*_i \leq \sum_{j=1}^{n}\Phi_{ij}L_j \ ; \ \forall i \in a \qquad \text{and} \qquad p_i \leq \sum_{j=1}^{n}\Phi_{ij}L_j \ ; \ \forall i \in b$$

which means $P^{new} = (P^*_a, P_b)$ does not violate any constraint in the second set of constraints of (5.8) too. Accordingly, P^{new} is a feasible point for (5.8), which since

$P^* \leq P^{new}$, contradicts the Pareto optimality assumption we made for P^* as: $\forall P \in S\ ;\ P^* \nleq P$. Then we can conclude that for a $P \in S$, we can not have non-zero length P_b and therefore:

$$\forall P \in S\ ;\ P^* \geq P$$

which means that $P*$ is an ideal maximal (pareto) point and accordingly is unique. CFinaly since $\mathbf{1}^T P = \sum_{i=1}^{n} P_i$ is a strictly increasing function, P^* maximizes $\mathbf{1}^T P$ on S which concludes the proof. ■

5.3.1 Measuring Systemic Risk

As we discussed in chapter 1, the systemic risk measures developed in this dissertation are defined from the regulatory and policy-maker's perspective and thus are concerned with the social welfare and equity loss exposures and how it relates to structural and system-level features. The most basic type of social welfare and equity loss could be defined as how much money a regulator (the central bank) loses on behalf of the taxpayers due to her commitments as an insurer of the deposit accounts in the banking system. We can formulate the aggregate loss of the regulator/central bank given a clearance vector that corresponds to a realized economic scenario/state by $\sum_{i=0}^{n}(O_{i0} - \Phi_{i0} P_i^*)$. With this, we can formally define the overall systemic risk exposure of the central bank to be the expected shortfall of the banking system to pay back the deposits of the households in the banking system:

$$ESR = E_x[\sum_{i=0}^{n}(O_{i0} - \Phi_{i0} P_i^*)] \tag{5.10}$$

where $\boldsymbol{x}$ is the random vector of the external investment payoffs of the banks, defined on the probability space, $(\mathcal{S}, \Sigma, \mu)$. As shown in the following lemma, we can abuse the structure of the clearing vector problem (5.8) to determine the realized loss of the regulator in correspondence with its external market.

Lemma 5.3.2 *$P \in \mathbb{R}_+^n$ solves the clearing vector problem for a given state $s \in S$ if and* *only if $P \in \mathbb{R}^n_+$ solves:*

$$
\begin{aligned}
SR_s = \quad & \min_{P \in \mathbb{R}^n_+} \sum_{i=0}^{n} (O_{i0} - \Phi_{i0} P_i) \\
s.t. \quad & U_i : P_i \leq x_i^s + R_i + \sum_{j=1}^{n} \left(\frac{\Phi_{ji} L_i^{in}}{\sum_{k=1}^{n} \Phi_{jk} L_k^{in}} \right) P_j \qquad \text{for all } i \in \mathcal{N} \\
& V_i : P_i \leq \sum_{j=1}^{n} \Phi_{ij} L_j^{in} \qquad \text{for all } i \in \mathcal{N}
\end{aligned}
\tag{5.11}
$$

where x_i^s represents the realized consumer loans portfolio value of the bank i at state $s \in \mathcal{S}$.

Proof: Suppose that P is the optimal solution of (5.11). Since $\sum_{i=0}^{n} O_{i0}$ is constant, P^* also maximizes $\sum_{i=1}^{n} \Phi_{i0} P_i$ on U and V. Also, $[\Phi_{i0}]_{i \in \mathcal{N}}$ is a strictly positive vector which is a scalarizing vector for (5.8), and thus P^* is a pareto optimal solution for (5.8) (see Luc 2015 - Theorem 4.3.1). ■

It is easy to see that (5.10) is the expected value of the (5.11)'s objective value function. Therefore, the same result holds for the expected value maximization problem that determines the value of ESR in (5.11).

5.4 Leverage Alowance Adjustment Framework (LAA)

In this section, we study how by adjusting the leverage allowance of individual banks in a systematic way, we can change the structure of the banking system in a controlled way without interrupting other features of the market that govern the dynamics of the hidden parts of the system. Such a solution enables us to design and optimize regulatory and policy-making intervention mechanisms (not limited to normative interventions addressing systemic risk) using the observable structure of the banking system without neglecting its hidden parts.

To simplify the modeling, in the first part of this section, we assume that if the leverage allowance of a given bank is changed by the regulator, the bank adjusts the sizes of its investments to meet the new requirement, keeping the proportionality of her leverage allowance distribution. However, to provide some basis for this assumption, we limit the the feasible regulatory interventions to a set of requirements that do not change the overall size of the supply and demand of each asset in the financial market. Using this setting, we develop a model to optimize the resulting intervention of the regulator addressing systemic risk in financial markets.

In the second part of this section, we extend the initial LAA framework by incorporating the sufficient conditions derived from an equilibrium formulation of the market to ensure that proportionality conditions hold under feasible regulatory interventions. We use the resulting sufficient conditions in formulating the regulator's optimization problem, which provide a theoretical guarantee that the regulatory interventions derived from our proposed framework adjust the size of each bank in a controlled way.

5.4.1 LAA Under the Proportionality Assumption

Suppose that all leverage allowance of a bank (determined by the regulator/central bank) is used to invest in risky assets based on the banks' decision making process (hidden from the regulator). Additionally, assume that if the leverage allowance of a given bank is changed by the regulator, the bank adjusts the sizes of its investments to meet the new requirement, however, it keeps the proportionality of the leverage allowance distribution the same.It means that considering the new leverage allowance assigned to the bank i, it is assumed that the bank's decision makers will modify their investment portfolio in a way that each individual investment of the portfolio, keeps its proportional size to the size of the whole portfolio. This assumption implies that in the short term, no bank changes its investment strategy[13] by the change of its aggregate investment size. Formally, let us introduce a new vector variable $\boldsymbol{l}$, where l_i represents the adjustment, determined by the central bank, for the leverage allowance of bank i. Then

[13]By investment strategy we mean how each bank forms its investment portfolio by proportional assignment of funding to different assets. For example, if the leverage allowance of a bank is reduced to half by the central bank, every investment of that bank is reduced to half, following the same investment strategy was adopted before the allowance reduction.

the new leverage allowance for each bank changes to $L_i^{new} = L_i - l_i$ constituting new leverage allowance vector as $\boldsymbol{L}^{new}$. We differentiate between the leverage allowance used for investments within the banking system, denoted by L_i^{ex} and the leverage allowance used for investments in the real economy, denoted by L_i^{ex}. Therefore, to to comply with the new allowances we should have:

$$L_i^{new} = L_i - l_i = L_i^{ex} - l_i^{ex} + L_i^{in} - l_i^{in}$$

where l_i^{ex} is the adjustment to the leverage assignment of the bank i to the external market and l_i^{in} is the adjustment to the leverage assignment of the bank i to the internal market. It is the bank i's decision to determine how its investment portfolio should be re-structured to meet the new leverage allowance, but since we assumed that in short term, the bank i keeps its portfolio's structure the same as before, the proportional distribution of her leverage allowance remains the same as before and therefore the limit adjustments by central bank and accordingly proportional funding of external and internal markets should remain the same too or formally;

$$\frac{L_i^{ex} - l_i^{ex}}{L_i^{new}} = \frac{L_i^{ex}}{L_i} \qquad \text{and} \qquad \frac{L_i^{in} - l_i^{in}}{L_i^{new}} = \frac{L_i^{in}}{L_i}$$

should hold. As the new leverage allowances are determined, the bank i which receives a leverage allowance reduction, should gradually liquidate some portions of the debts and loans that the bank owns and use that liquidity to pay back some portion of its obligations to other banks and maybe to increase its cash reserve. Bank i receives $l_i = l_i^{ex} + l_i^{in}$ from liquidation of some portions of its assets, and updates its obligations to O_{ij}^{new}, where $O_{ij} - O_{ij}^{new}$ is the amount of the payback by the bank i to the bank j, and accordingly $\sum_{j=1}^{n}(O_{ij} - O_{ij}^{new})$ is the sum of the paybacks by the bank i to the financial system. Then, since the sum of the incoming liquidity to the bank i should be equal to the sum of the new cash reserve and aggregate outgoing liquidity as paybacks, we have:

$$\sum_{j=1}^{n}(O_{ij} - O_{ij}^{new}) + R_i^{new} = l_i^{ex} + l_i^{in}; \qquad \forall i \in \mathcal{N} \tag{5.12}$$

Recal that $\Phi_{ij}, \forall i, j$ represents proportional size of the investment of i in j to the size of the aggregate investment of i in the whole financial system. Thus, we can use (5.1) to separate the parameter Φ_{ij} which refers to the preserved proportional size of investments from the leverage allowance, which is supposed to be variable (decision variable of the regulator) Then we use $O_{ij}^{new} = \Phi_{ij}(L_j^{in} - l_j^{in})$ to follow the proportionality assumption and accordingly:

$$O_{ij} - O_{ij}^{new} = \Phi_{ij}L_j^{in} - \Phi_{ij}(L_j^{in} - l_j^{in}) = \Phi_{ij}l_j^{in}$$

then (5.12) can be rewritten as:

$$\sum_{j=1}^{n} \Phi_{ij}l_j^{ex} + R_i^{new} = l_i^{ex} + l_i^{in}$$

The same arguments hold for the banks that receive a leverage increase, where l_i should be negative.

Suppose the regulator decides to keep the economy's size the same as before the leverage adjustments[14]and does not intend to inject liquidity in the market. In that case,

$$\sum_{i=1}^{n}(l_i^{ex} + l_i^{in}) = 0$$

holds, which means that the aggregate leverage increases are canceled out by the aggregate leverage decreases, and accordingly, the financial system keeps its initial size.

[14]If the central bank intends to increase or decrease the size of the aggregate leverage of the financial system with no liquidity infusion, to formulate the expansion and the balance of leverages and reserves adjustments, we introduce E to refer to the expansion size. If the expansion size is exogenously determined by the central bank based on the central bank's monetary policy. Moreover, to keep the balance of the adjustments in the network considering the expansion policy;

$$\sum_{i=1}^{n}(l_i^{ex} + l_i^{in}) = -E$$

should hold which results in the money multiplier effect of $MME = \frac{E}{\sum_{i=1}^{n} R_i^{new}}$ due to the adjustments. It should be mentioned that when E is determined as a positive parameter, it models an expansion policy and when E is determined as a negative parameter, it models an extraction policy.

Considering the leverage allowance to be adjustable subject to these constraints [15] it provides us with the basis to develop a decision making model for the regulator/central bank.Next, we formulate a decision model of a regulator using leverage allowance adjustments as a normative decision variable that aims to minimize its overall systemic risk exposure, ESR.

5.4.2 Optimizing LAA Under the Proportionality Assumption

Suppose that the regulator/central bank adjusts the leverage allowance of each bank separately as a regulatory requirement. Given the financial system, $(\mathcal{N}, \boldsymbol{\Phi}, \boldsymbol{L}^{in}, \boldsymbol{R}, \boldsymbol{L}^{ex})$, the leverage allowance adjustments of l resulting in the new leverage allowances of $\boldsymbol{L}^{new} = \boldsymbol{L} - \boldsymbol{l}$, transforms the financial system to $(\mathcal{N}, \boldsymbol{\Phi}, \boldsymbol{L}^{in} - \boldsymbol{l}^{in}, \boldsymbol{R} + \boldsymbol{R}^{new}, \boldsymbol{L}^{ex} - \boldsymbol{l}^{ex})$ and accordingly the realized financial system transforms to $(\mathcal{N}, \boldsymbol{\Phi}, \boldsymbol{L}^{in} - \boldsymbol{l}^{in}, \boldsymbol{R} + \boldsymbol{R}^{new}, (\frac{\boldsymbol{L}^{ex} - \boldsymbol{l}^{ex}}{L_i^{ex}})x_i)$ for any given economic scenario where $0 \leq x_i \leq L_i^{ex}; \quad \forall i \in \mathcal{N}$.

Then the regulator's decision problem to minimize its financial loss or equivalently, to maximize the clearance vector, for a given economic scenario, $s \in \mathcal{S}$, solves

$$
\begin{aligned}
\max_{P,l \geq 0} \quad & \sum_{i=1}^{n} \left(P_i - \sum_{k=1}^{n} \Phi_{ik}(L_k - l_k)) \right) \\
\text{s.t.} \quad & P_i \ \leq \ x_i \left(\frac{L_i^{ex} - l_i^{ex}}{L_i^{ex}} \right) + R_i l_i^{ex} + l_i^{in} - \sum_k l_k^{in} \Phi_{ik} + \\
& \qquad \sum_{j=1}^{n} \left(\frac{\Phi_{ji} \left(L_i^{in} - l_i^{in} \right)}{\sum_{k=1}^{n} \Phi_{jk} \left(L_k^{in} - l_k^{in} \right)} \right) P_j; \qquad \forall i \in \mathcal{N} \\
& P_i \ \leq \ \sum_{j=1}^{n} \Phi_{ij}(L_j - l_j); \qquad \forall i \in \mathcal{N} \\
& l_i \ = \ l_i^{in} + l_i^{ex}; \qquad \forall i \in \mathcal{N} \\
& \frac{l_i^{in}}{l_i} \ = \ \frac{L_i^{in}}{L_i} \qquad \forall i \in \mathcal{N} \\
& \sum_{i=1}^{n} l_i = 0;
\end{aligned}
\tag{5.13}
$$

[15] These constraints make sure that the supply and demand of different assets as well as money do not change and thus the leverage adjustments do not lead to inbalancing overflows or iliquidity of the market

where $0 \leq x_i^s \leq L_i^{ex}$; $\forall i \in \mathcal{N}$.

It is easy to see that 5.13 is a fractional nonconvex optimization problem. Despite the nonconvexity of 5.13 formulation, we show in the following proposition that 5.13 always has a unique optimal solution. Because of this property, we can formulate the 5.13's marginal value function using the results we have from chapter 4.

Proposition 5.4.1 (Unique optimal solution for the leverage allowance problem) *The regulator's leverage allowance problem formulated in 5.13 always has a unique optimal solution*

Proof: Let us define $\Gamma = [\Phi_{ji}\left(L_i^{in} - l_i^{in}\right)]_{i,j \in \mathcal{N}}$ and $\theta_{ji} = \frac{\Gamma_{ij}}{\Gamma_{i.}}$ where $[\Gamma_{i.}]_{i \in \mathcal{N}}$ represents the vector of proportional obligations to i (proportional to the sum of obligations of each bank to the system). Clearly all components of Γ are nonnegative and since there is no self obligation, we have $\Gamma_{ii} = 0$. Also it is trivial that $\sum_{i=1}^{n} \Gamma_{ij} = 1$.

Now we can rewrite the (5.13) as:

$$
\begin{aligned}
P: \max_{P,l,\Gamma \geq 0} \quad & \sum_{i=1}^{n}(P_i - \sum_{k=1}^{n} \Phi_{ik}(L_k - l_k))) && && (5.14a)\\
\text{s.t.} \quad & P_i \leq x_i \left(\frac{L_i^{ex} - l_i^{ex}}{L_i^{ex}}\right) + l_i^{ex} + \left(\frac{L_i^{in}}{L_i^{ex}}\right) e_i^{ex} && && \\
& \qquad - \sum_{k} \left(\frac{L_k^{in}}{L_k^{ex}}\right) \phi_{ik} e_k^{ex} + R_i + \sum_{j=1}^{n} \theta_{ji} && \forall i \in \mathcal{N} && (5.14b)\\
& \Phi_{ji} L_i^{in} - \Phi_{ji} \frac{L_i^{in}}{L_i^{ex}} l_i^{ex} = \Gamma_{ji} && \forall i \in \mathcal{N} && (5.14c)\\
& \Gamma_{ji} \cdot \Gamma_j^{-1} \cdot P_j = \theta_{ji} && \forall i, j \in \mathcal{N} && (5.14d)\\
& \sum_{i} \Gamma_{ji} \leq \Gamma_{j.} && \forall j \in \mathcal{N} && (5.14e)\\
& P_i \leq \Gamma_{i.}; && \forall i \in \mathcal{N} && (5.14f)\\
& \sum_{i=1}^{n} \left(\frac{L_i}{L_i^{ex}}\right) l_i^{ex} = 0 && \forall i \in \mathcal{N} && (5.14g)
\end{aligned}
$$

With this reformulation, all constraints are linear but 5.14d which can be represented as a standard "difference of convex function" (dc functions; see Strekalovsky (2020a), Kononov et al. (2020)). Due to the theoretical foundation developed in Strekalovsky (2020b, 2019), by means of the exact penalization techniques, we can reduce 5.14d constraints to a family of convex problems. This is enough to prove that (5.13) has a unique optimal solution. ■

The model developed here allows us to optimally use the leverage allowance requirements of the banks to restructure the observed banking system addressing systemic risk.

5.4.3 LAA Under Sufficient Conditions

To simplify the initial modeling in the previous sections, we assumed that if the leverage allowance of a given bank is changed by the regulator, the bank adjusts the sizes of its investments to meet the new requirement, keeping the proportionality of the leverage allowance distribution. With this, we implicitly assumed that the regulatory interventions do not have any effect on the market prices of the assets. Therefore, each bank will form the same investment portfolio in terms of her proportional assignment of funding to different assets. To provide some basis for this assumption, we limited the feasible regulatory interventions to a set of leverage allowance adjustments that do not change the overall size of the supply and demand of any asset in the financial market. However, since asset pricing in financial markets goes beyond simple supply and demand sizes and may depend on many other features[16], in this section, we extend the initial framework by developing an equilibrium formulation of regulated financial markets as a basis to derive a set of market-based conditions required to guarantee that the proportionality assumption holds. Our general equilibrium model of regulated financial markets under uncertainty endogenously incorporates more complicated features of the financial markets (that govern the banking system's dynamics), including the bankers' and household's investment decisions, asset pricing, bankruptcy law, and regulatory requirements. In particular, the extended formulation captures the interactions of a heterogeneous collection of banks, a heterogeneous collection of firms, and a representative household of the economy discussed in section 5.1[17].

[16]Including the capital structure of banks, structure of the interbank deposit market, the differences in risk profiles, and even the regulatory requirements

[17]The equilibrium model formulates an equity and debt market as well as an interbak loan market, where banks and consumer loan holders are protected under the limited liability provision of the

Considering the observed market state as a solution of the equilibrium model, the equilibrium formulation can be used to derive the sufficient conditions to guarantee that all market features remain the same while the size of the available funds to each individual bank changes. In other words, we use the equilibrium formulation as a basis to derive the sufficient conditions to make sure that the proportionality assumption holds in a more general setting compared to our initial setting in this chapter.

Therefore, we can incorporate the resulting sufficient conditions in developing our regulatory framework to ensure that the intervention mechanisms keep the dynamics of the hidden parts of the banking system in control in a way that we can reliably leverage the observable structure of the banking system toward our normative aims and without neglecting the hidden structure of the system.

In other words, regulatory frameworks that are designed constrained to the conditions derived from the equilibrium model restructure the adaptive capacity of the banking system to be reliably responsive to the normative interventions of the regulatory system.

This reduces the nonlinearity and thus the complexity of the regulatory system's interaction with the banking system, and enables the regulator to take advantage of a more tractable adaptive capacity of the banking system compared to the conventional approaches.

Equilibrium Representation of Regulated Financial Markets Under Uncertainty

Let us develop a general equilibrium formulation of the baseline economy we discussed in section 5.1. Suppose that at t_0, each firm, denoted by $f \in \mathcal{F}$, raises funding through investments of the households in its equity, y_f^h as well as loans from the banking system,

bankruptcy law and also are subject to the regulatory requirements set by the central bank. In particular, the representative household is limited to investments in the (equity) of the firms and/or depositing in the banking system. Banks are limited to using their endowments and the deposits from the household and/or other banks to issue debt contracts for other banks or/and issue loans for funding firms.

$j_{j\in\mathcal{N}}$, as much $\sum_j y_f^j$ with face value return rate of R_f at date t_1. The firm invests all of the raised funding in a potential project with the state-contingent return rate of $\bar{R}_f^s$, $\forall s \in \mathcal{S}$. Since the return rates of investments in potential projects are state-contingent, the realized return on the firm's investment in some states may not be enough to payback the face value of the firm's obligation on the loans received from the banking system. Therefore, at t_1, given the realized return rate of the potential projects (realized state), each firm, say $\boldsymbol{f} \in \mathcal{F}$, solves the following accounting problem

$$P^f(R, y_{.f}): \max_{\rho_f \in [0,1]} \quad \rho_f^s \tag{5.15a}$$

$$\text{s.t.} \quad \rho_f^s \tilde{R}_f \sum_j y_f^j \;\le\; R_f^s\Big(\sum_j y_f^j + y_f^h\Big) \tag{5.15b}$$

$$\rho_f^s \;\le\; 1 \tag{5.15c}$$

to determine if she can pay the face value of its obligations ($\rho_f^s = 1$) or is bankrupt ($\rho_f^s < 1$), where $0 \le \rho_f^s \le 1 \ \forall f \in \mathcal{F}, \forall s \in \mathcal{S}$ represents the proportion of the firm f's obligations, that should be payed by firm f to the banking system (given the realized state s at t_1) due to the limited liability provision of the bankruptcy law.

The representative household, h, decides at t_0, how to use her endowment of w_0^h, investing in the (equity) of firms, y_f^h, $\forall f \in \mathcal{F}$, and depositing in the banking system, X_j^h, $\forall j \in \mathcal{N}$, maximizing the expected utility $U^h(.)$ of her whealth at t_1. Note that the state contingent value of the representative household investments in the equity of each firm at t_1 is

$$R_f^s(\sum_j y_j^h + y_f^h) - \rho_f^s \tilde{R}_f \sum_j y_f^j$$

where $R_f^s(\sum_j y_j^h + y_f^h)$ is the state contingent value of the firm f's assets at t_1 and $\rho_f^s \tilde{R}_f \sum_j y_f^j$ is what should be paid to the firm's debt holders (banking system) in the given state of $s \in \mathcal{S}$ at t_1. Therefore, the representative household solves

$$P^h(\tilde{R}, w_0^h, y_j^h, R^s):$$

$$\max_{\substack{x_j^h, y_f^h \in \mathbb{R}_+ \\ \forall j \in \mathcal{N}, \forall f \in \mathcal{F}}} E_{R_f}\Big[U_h\big(\sum_f \Big(R_f^s(\sum_j y_j^h + y_f^h) - \rho_f^s \tilde{R}_f \sum_j y_f^j\Big) + \sum_j x_j^h\big)\Big] \tag{5.16a}$$

$$\text{s.t.} \qquad \sum_f y_f^h + \sum_j x_j^h = w_0^h \tag{5.16b}$$

$$\Big(R_f^s(\sum_j y_j^h + y_f^h) - \rho_f^s \tilde{R}_f \sum_j y_f^j\Big)$$

where, (5.16b) represents its budget constraint.

On the other hand, each bank, $j \in \mathcal{N}$, decides how to use its equity of eq_0^j at t_0 and the deposits she receives from the households, x_j^h, and other banks, $\sum_{i \neq j} x_j^i$, to invest in the loan contracts issued for the firms, y_f^j, $\forall f \in \mathcal{F}$, as well as the interbank deposit market, x_i^j, $\forall i \in \mathcal{N}$, constrained to the regulatory requirements of

$$\text{Res}_j \geq (eq_j^0 + x_j^h + \sum_{i \neq j} x_j^i) - (\sum_{i \neq j} x_i^j + \sum_f y_f^j) \tag{5.17}$$

$$\sum_{i \neq j} \tilde{R}_i x_i^j + \sum_f \tilde{R}_f y_f^j \leq \; L^j \tag{5.18}$$

(5.17) formulates a reserve requirement for bank j to hold the minimum of Res_j in safe assets. (5.18) formulate the leverage allowance of bank j to invest the maximum of L_j in risky assets. Therefore, each bank, say $j \in \mathcal{N}$ solves

$$P^{B_j}(\tilde{R}, \rho, eq_j^0, L_j, \text{Res}_j):$$

$$\max_{\substack{x_i^j, y_i^j \in \mathbb{R}_+ \\ \forall i \in \mathcal{N}}} \text{E}_\rho \Big[U_j\Big(\sum_{i \neq j} \rho_i \tilde{R}_i x_i^j + \sum_f p_f \tilde{R}_f y_f^j - \rho_j(\sum_{i \neq j} \tilde{R}_j x_j^i + x_j^h)\Big)\Big] \tag{5.19a}$$

s.t.

$$\sum_{i \neq j} \tilde{R}_i x_i^j + \sum_f \tilde{R}_f y_f^j \; \leq \; L^j \tag{5.19b}$$

$$\text{Res}_j + \sum_{i \neq j} x_i^j + \sum_f y_f^j \; = \; eq_j^0 + x_j^h + \sum_{i \neq j} x_j^i \tag{5.19c}$$

$$\rho_j^s \Big[\Big(\tilde{R}_j \sum_{i \neq j} x_j^i\Big) + x_j^h\Big] + eq_j^{1s} \; =$$

$$\sum_{i \neq j} \rho_i^s \tilde{R}_i x_i^j + \rho_i^s \tilde{R}_i x_i^j + \sum_f \rho_f^s \tilde{R}_f y_f^j + FR^s(\alpha_j)\,\text{Res}\, j \tag{5.19d}$$

$$\rho_j^s \; \leq 1 \tag{5.19e}$$

Given the state-contingent return rates of potential projects (consumer loans), exogenous endowments, and the regulatory and legal requirements and limits, all transactions, including depositions, equity investments, and debt contracts, occur simultaneously at date t_0 and before the uncertainty is resolved at t_1. Thus, the equilibrium allocations at t_0, endogenously determine not only the size of the transactions (depositions, equity investments, and debt contracts) but also the nominal return rates of the debt contracts/loans, which create a state-contingent system of obligations at t_0.

Sufficient Conditions for Regulatory Adjustments

As discussed in the previous section, the banking system of the baseline economy is only regulated by the maximum leverage allowance requirement formulated in (5.18) and the minimum reserve requirement formulated in (5.17). Here we define the basic setting of a regulatory intervention as a system of adjustments to the existing regulatory requirements of the banking system. Then we make an example of an intervention mechanism that is developed based on the intervention setting to interact with the observed structure of the banking system without neglecting the hidden parts of the banking system. Finally, we develop a model to optimize the designed mechanism for addressing systemic risk.

Regulatory Setting: Suppose that the regulator by setting the adjustment parameters of $\alpha_j \in \mathbb{R}_+;\ \forall j \in \mathcal{N}$, changes the banks existing leverage allowances and reserve requirements from L_j to $L_j^{new} = \alpha_j L_j$, and from Res^j to $\text{Res}^j + (1-\alpha^j)(\sum_{i \neq j} x_j^i + \sum_f y_f^j)$ respectively[18]. Therefore, new regulatory requirements could be formalized as an adjustment of the old regulations, denoted by $\boldsymbol{\alpha} = [\alpha_i]_{i \in \mathcal{N}}$.

[18] We used a different notation to denote leverage adjustment in section 5.4, where l_j represented the adjustment to the existing leverage allowance of L_j and thus the new leverage allowance for each bank would be $L_j^{new} = L_j - l_j$. To simplify the notation of this section, we use an equivalent notation as $\alpha_j = \frac{L_i - l_i}{L_j}$.

Regulatory Interaction: Here, as an example of how our regulatory setting could be used as a basis to design unified normative interventions, we develop a set of regulatory interventions that, are capable of restructuring the observable parts of the banking system without neglecting the hidden dynamics of the banking system. Later we show how we can optimize the restructuring toward our normative aims, which is to address systemic risk in financial markets. To do so, we define the following sets of constraints on regulatory interventions ($\boldsymbol{\alpha}$) that as shown by proposition 5.4.2 are sufficient to guarantee that the interventions that satisfy these constraints restructure the banking system without changing the essential features of the market that govern the hidden dynamics of the system.

First set of constraints ensures that the size of the aggregate funding supply for each individual firm does not change after the leverage allowance adjustments, we require

$$\sum_j (1-\alpha^j) y_f^j = 0; \qquad \forall f \in \mathcal{F} \tag{5.20}$$

Therefore, the firms accounting problems do not change after the regulatory intervention.

Second, to ensure that the adjusted obligations ($\alpha^i x_j^i$) on the liability side of the banks clear, we require

$$\sum_j \sum_{i \neq j} (1-\alpha^i) x_j^i = 0 \tag{5.21}$$

where $\alpha^i \in \mathbb{R}_+; \forall i \in \mathcal{N}$. This constraint ensures that the effect of the intervention on the liability side of the banks only changes the distribution of the representative household's deposits in the banking system, but not the aggregate size of its deposits.

5.4.4 Optimizing LAA Under Sufficient Conditions

In proposition 5.4.2 we show that the regulatory adjustments constrained to (5.20) and (5.21), restructure the optimization problems of different stakeholders of the financial markets in a way that new equilibrium is an explicit function of the initial equilibrium soultion and the intervention vector ($\boldsymbol{\alpha}$).

In other words, we do not need to solve the equilibrium problem (and thus we do not need to observe the structure of the hidden parts of the system that are involved in the equilibrium formulation) to determine what will be the new equilibrium after any intervention that satisfies the sufficient conditions of proposition 5.4.2.

Equilibrium Representation After Regulatory Adjustments

Here we show that the regulatory adjustments satisfying (5.20) and (5.21), restructure the banking system, but do not change the pricing and proportionality of banks investment.

Proposition 5.4.2 (Equilibrium After Regulatory Adjustments) *Suppose that* $e = (x_i^j, y_f^j, x_j^h, y_f^h, \rho_j, \tilde{R}_j)$ *is an equilibrium solution of the economy formulated in section 5.4.3. Then if an intervention vector,* $\boldsymbol{\alpha}$*, satisfies* (5.20) *and* (5.21)*, then* $e^{new} = (\alpha^j x_i^j, \alpha^j y_f^j, x_j^h + \sum_{i \neq j}(1-\alpha^i)x_i^j, y_f^h, \rho_j, \tilde{R}_j)$ *is an equilibrium solution of the economy after the regulatory intervention.*

Proof: Due to (5.20), we have $\sum_j \alpha^j y_f^j = \sum_j y_f^j$ and therefore, for the firm's accounting problem for any given state, $s \in \mathcal{S}$, , e and e^{new} with the change of variable solve the same optimization problem.

Since (5.21) holds, replacing the decision variables of the representative household problem, , with $x_j^h + \sum_{i \neq j}(1 - \alpha^i)x_i^j, y_f^h$, reduces the budget constraint of the new problem to the budget constraint of the initial problem, (5.16b). Likewise, due to (5.20) and (5.21), the objective function of the new problem is also equivalent to the objective function of the initial problem, (5.16a). therefore the change of variable results in the no change in the representative household's optimization problem and thus, e^{new} optimizes the household problem after the intervention.

Similarly, for the bank j's optimization problem, , with the change of variable of e to e^{new} the leverage allowance constraint of the new problem

$$\alpha^j \left(\sum_{i \neq j} \tilde{R}_i x_i^j + \sum_f \tilde{R}_f y_f^j \right) \leq \alpha^j L^j$$

is equivqlent to the initial equilibriums leverage allowance constraint for bank j, (5.23b). The reserve requirement constraint of the new problem

$$\mathrm{Res}\, j + (1-\alpha^j)\left(\sum_{i\neq j} x_i^j + \sum_f y_f^j\right) + \sum_{i\neq j} \alpha^j x_i^j + \sum_f \alpha^j y_f^j = eq_j^0 + \sum_{i\neq j} \alpha^i x_j^i + x_j^h + \sum_{i\neq j} \left(1-\alpha^i\right) x_j^i j$$

also reduces to (5.19c); $\mathrm{Res}_j + \sum_{i\neq j} x_i^j + \sum_f y_f^j = eq_j^0 + x_j^h + \sum_{i\neq j} x_j^i$ due to (5.20) and (5.21). Also the bank j's return constraint at t_1

$$\rho_j^s\Big[\Big(\tilde{R}_j \sum_{i\neq j} \alpha^i x_j^i\Big) + x_j^j + \sum_{i\neq j} \left(1-\alpha^i\right) x_j^i\Big] + eq_j^{new} = \tag{5.22}$$

$$\alpha^j\Big[\sum_{i\neq j} \rho_i^s \tilde{R}_i x_i^j + \sum_f \rho_f^s \tilde{R}_f y_f^j\Big] + FR^s\left(\alpha^j\right)\Big[\mathrm{Res}^j + \left(1-\alpha^j\right)\Big(\sum_{i+j}^{j} x_i^j + \sum_F y_i^j\Big)\Big]$$

reduces to (5.19d), since

$$FR^s\Big(\alpha^j\Big)\Big[\mathrm{Res}^j + \Big(1-\alpha^j\Big)\Big(\sum_{i+j}^{j} x_i^j + \sum_F y_i^j\Big)\Big] =$$

$$\mathrm{Res}^j + \Big(1-\alpha^j\Big)\Big(\sum_{i+j} x_i^j + \sum_f y_f^j\Big) \qquad \text{(Reserve requirement after adjustment)}$$

$$+\,\rho_j^s\Big(\tilde{R}_j \sum_{i\neq j} \alpha^i x_j^i\Big) - \sum_{i+j} x_j^i \qquad \text{(Capitalization cost using interbank loans after adjustment)}$$

$$+\,\rho_j^s\Big(x_j^h + \sum_{i\neq j}\Big(1-\alpha^i\Big)x_j^i\Big) - x_j^h \qquad \text{(Capitalization cost using deposit market after adjustment)}$$

$$-\,\alpha^j\Big(\rho_j^s\Big[\tilde{R}_j \sum_{i\neq j} x_j^i\Big] - \sum_{i+j} x_j^i\Big) \qquad \text{(Capitalization cost using interbank loans before adjustment))}$$

$$-\,\alpha^j\Big(\rho_j^s x_j^h - x_j^h\Big) \qquad \text{(Capitalization cost using deposit market before adjustment)}$$

Finally, $\rho_j^s \leq 1$, (5.19e) should hold identically for both equilibrium systems. With that, it is clear that the objective function of the equilibrium problem before and after the intervention is optimized with their equivalent solutions, which concludes our proof. ■

Using the result of proposition 5.4.2, we can rewrite LAA optimization problem as:

$$P_2^{LAA}(\Phi, L, x, y, \tilde{R}):$$

$$\max_{P,\, l} \quad \sum_{i=1}^{n} \Big(P_i - \sum_{k=1}^{n} \Phi_{ik}(L_k - l_k)) \Big) \tag{5.23a}$$

$$\text{s.t.} \quad P_j \leq \big(\frac{\tilde{R}_j - 1}{\tilde{R}_j}\big) \sum_{i=1}^{n} \phi_{ji} \big[L_i^{in} - e_i^{in} - \frac{L_i^{in}}{L_j^{in}} (L_j^{in} - e_j^{in}) \big] + \big(L_j^{ex} - l_j^{ex}\big) + \text{Res}_j + \big(\frac{l_j}{L_j}\big) \big[\sum_{i+j} x_i^j + \sum_i y_i^j \big] + \sum_{i=1}^{n} \big(\frac{\phi_{ij}\big(L_j^{in} - l_j^{in}\big)}{\sum_{k=1}^{n} \phi_{ik}\big(L_k^{in} - l_k^{in}\big) + \bar{x}_i^m} \big) P_i \tag{5.23b}$$

$$P_j \leq \sum_{i=1}^{n} \phi_{ji} \Big(L_i^{in} - l_i^{in} \Big) + x_j^d + \sum_{i \neq j} \Big(\frac{\ell_i^{in}}{L_i^{ir}} \Big) x_j^i \tag{5.23c}$$

Similar to 5.13, the LAA optimization problem under sufficient condition, $P_2^{LAA}(\Phi, L, x, y, \tilde{R})$, is a fractional nonconvex optimization problem, however, since the result of proposition 5.4.1 holds, despite the nonconvexity of the its formulation, $P_2^{LAA}(\Phi, L, x, y, \tilde{R})$ always has a unique optimal solution.

5.5 Conclusion

In this chapter, we developed a framework to design a regulatory and policy-making intervention mechanism that captures how the bankruptcy law and the interactive decision-making dynamics give structure to the banking system and create system-level risk exposures for the banking system in particular and the whole economy in general. Contrary to the conventional approach, we used general equilibrium theory as an organizing structure and rational behind any observed state of the banking system rather than a model of the system. We develop a model-based measure of systemic loss, which is the basis for the regulatory and policy-making intervention evaluation mechanism to adjust the size of the banks in a controlled way without interrupting other features of the banking system and financial market. This provides us with a theoretical basis to design and take advantage of the banking system's adaptive capacity, developing a regulatory setting that systematically adjusts the banks' existing regulatory requirements as an interaction mechanism with the banking system towards our normative aims addressing systemic risk.

We initially develop and optimize a leverage allowance adjustment framework (LAA under proportionality assumption) as a regulatory setting, assuming that if the regulator changes the leverage allowance of a given bank, the banks' proportional investment decisions do not change. However, to provide some basis for this assumption, we limit the regulator's actions to ensure that the size of the aggregate funding supply does not change after the adjustments. Finally, to extend the initial LAA framework to be usable in a more generic setting, we develop a general equilibrium representation of a regulated financial market as a basis to derive the sufficient conditions that guarantee that the proportionality assumption holds in a more general setting. The sufficient conditions of the extended LAA framework (LAA under sufficient conditions) restructure and reduce the complexity of the banking system's adaptive capacity as it provides us with a market-based mechanism to control for the dynamics governed by the hidden structure of the banking system and simultaneously use the regulatory intervention to restructure the observable structure of the banking system. This reduces the complexity of the system significantly as it provides us with a basis to rely on the market-based interaction mechanisms developed based on our framework and leveraging the banking system's observable structure toward our normative aims and without neglecting the hidden structure of the system.

Chapter 6

Discussion and Conclusion

In modern economies, the stability and efficiency of the financial system as a complex interconnected set of institutions are of fundamental importance. The complexity of the financial system and the importance of its stability and efficiency are due to numerous inter-dependencies within the financial system and many interaction channels between the financial institutions/system and the other sectors of the economy.

The failure of conventional approaches, given the empirical results on the performance of the banking regulations and further various theoretical inconsistencies in the newly developed and widely accepted regulatory standards was discussed in chapter 1 and chapter 2.

We also discussed in chapter 3 why conventional approaches in the literature of systemic risk fail to explain the phenomena they associate with systemic risk based on a coherent theoretical or emperical framework. We also extended the latest micro-foundational modeling of the banking system and showed that even more complicated equilibrium modeling is not capable of relating any specific market failure to the structural features of the banking system. of the banking system. In particular, we showed that the financial contagion phenomenon[1] is not an equilibrium feature of the decen-

[1] We relaxed the "no-aggregate uncertainty" and "pecking order" assumptions made in the existing formulations and showed that even with these extensions, we could not explain the emergence of financial contagion.

tralization functionality of the banking system even under aggregate uncertainty and standard (incomplete) deposit contracts.

In chapter 5, we turn our critical reflections in previous chapters towards an alternative normative framework on systemic risk in financial markets. We initially developed a new formulation of the banking system in correspondence with the real economy, where bankruptcy law and investment decisions forming the structure of the banking system are simultaneously captured in a two-layer parametric setting. We used the structure of two-layer parametric as a basis to develop the analytical tools and optimization formulations needed to design a normative framework to prevent and mitigate systemic risk in financial markets, addressing the inefficiency and unintended consequences of conventionally developed regulatory rules and policies. We developed a mathematical framework to model a financial system in the context of the real economy, measuring and optimizing systemic risk through a dynamic parameter setting mechanism. In particular, we developed a model-based measure of systemic loss that captures how the bankruptcy law and the interactive decision-making dynamics give structure to the dynamics of the banking system and create system-level risk exposures. However, integration of the bankruptcy law and fully-fledged decision-making dynamics in a unified setting results, if done in conventional ways, in highly non-tractable nonlinear general equilibrium formulations, thus practically not usable for developing a reliable regulatory framework. Therefore, contrary to the conventional approach, we used general equilibrium theory as an organizing structure and rationale behind any observed state of the banking system rather than a mathematical framework to develop a fully-fledged model of the system. This approach enabled us to capture the interactions and thus the banking system's decision-theoretic dynamics, using a general equilibrium formulation of the interactions rather than a comprehensive model of the system. Thus, we leveraged the observable dynamics of the system to interact with the observable part of the system's dynamics while reducing the inefficiencies caused by the hidden dynamics of the system through adaptive control of the observable dynamics. We used this interaction mechanism as a basis to develop two regulatory frameworks that allow us to indirectly optimize the aggregate systemic risk exposure of the economy by setting leverage allowance for each institution, proportional to its contribution to the aggregate systemic risk, keeping the size of the economy the same as before. This mechanism exploits the adaptive capacity of the system developed based on the structure of the contractual dependencies of the financial institutions along with their interest to maximize their shares of the financial market/leverage limits. This provides us with the theoretical guarantee that any regulatory and policy-making interactions developed based on this framework capture the dynamics of the general equilibrium model behind the formation of its observed state, ensuring that the interventions do not affect any decision-making parameters involved in the bankers' investments other than the size of the funds available to each bank. Finally, this framework enables us to leverage the observable dynamics of the system to interact with the observable part of the system and thus reduce the complexity of the dynamics of the hidden parts. This reduces the complexity of the system significantly as it provides us with a basis to rely on the market-based interaction mechanisms developed based on our framework and leveraging the banking system's observable structure toward our normative aims and without neglecting the hidden structure of the system.

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