Quantal Response Equilibria for Models of Price Competition

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ABSTRACT

There is a growing body of data from game theory and industrial organization experiments that reveals systematic deviations from Nash equilibrium behavior. In this thesis, the perfectly rational decision-making embodied in Nash equilibrium is generalized to allow for endogenously determined decision errors. Firms choose among strategies based on their expected payoffs, but make decision errors based on a probabilistic or quantal choice model. Such errors may either be due to mistakes or to unobserved random variations in payoff functions. For a given error distribution a quantal response equilibrium is a fixed point in choice probabilities. Closed-form solutions for equilibrium price distributions with endogenous errors are derived for models of price competition. Numerical methods are used to examine more complex market models. This thesis establishes differences in the qualitative properties of Nash and quantal response equilibria in models of price competition.

This thesis consists of two parts. In the first part, chapters 3 and 4, a parametric class of quantal response functions is derived from a model of multiplicative random errors. This "power function" decision rule is used to derive the equilibrium price distribution with endogenous errors in a simple model of price competition. The power-function quantal response equilibrium is appealing since it thereby accounts for systematic deviations from the Bertrand-Nash equilibrium.

The second part of this thesis, chapters 5 and 6, applies the quantal response equilibrium to a series of increasingly complex models, with step-function demand and supply structures, of the type used in market experiments. In some of these models, the price distribution in a quantal response equilibrium is affected by changes in structural variables although the Nash equilibrium remains unaltered. It is also shown that the quantal response equilibrium stochastically dominates the Nash equilibrium in mixed-strategies in a model with market power and increasing costs. The Nash and quantal response equilibria differ in a model with market power and constant costs. In other models, however, it is shown that the Nash and quantal response equilibria are identical. This is the case in a (first-price) all-pay auction presented in chapter 6.

TABLE OF CONTENTS

LIST OF FIGURES

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CHAPTER 1

INTRODUCTION

This thesis generalizes the standard Nash equilibrium analysis of Bertrand-Nash competition to allow for endogenously determined choice errors. In this model, players make decision errors based on a probabilistic or quantal choice model, and understand other players do so as well. The first part of this thesis derives a parametric class of quantal response functions from a model with multiplicative errors. The second part of the thesis investigates the use of the quantal response equilibrium in models of price competition.

Although the Nash equilibrium is widely used in economic theory, there is some dissatisfaction with this concept. One criticism is that rationality is less restrictive than the Nash equilibrium implies. In general, there are many more strategies that may be considered rational choices, according to some beliefs, than merely those choices described as Nash equilibrium strategies (Bernheim (1984) and Pearce (1984)). An opposite criticism finds the Nash equilibrium concept to be too unrestrictive because it allows for behavior that is intuitively unreasonable. The literature on *refinements*, beginning with Selten (1975), has developed a series of proposed rules for eliminating such implausible equilibria. Articles on refinements typically begin with examples having several Nash equilibria, some of which are intuitively implausible because, for instance, they are based on strategies that can be interpreted as noncredible threats.[1] Some economists have therefore proposed that an analysis of learning and adjustment is the most useful way to proceed. The literature on the topic of evolutionary game theory consistently adopts this assumption, going back at least to Alchian's (1950) seminal paper and to Simon's (1957) work on bounded rationality.[2] There is also much recent work on *naive* (non-strategic) learning models, showing behavior that converges to a Nash equilibrium.[3]

Most theorists are uneasy about models of limited rationality, in part because of the looseness and the multiplicity of possible approaches. Yet data from laboratory games with human subjects provide empirical regularities that can guide theoretical work on learning and adjustment. As a first approximation, evidence from game experiments tends to conform to Nash equilibrium predictions (Davis and Holt, 1993, chapter 2). However, some features of the data from market experiments have been difficult to explain in this way. Systematic deviations from rational behavior have been observed

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[1] For further discussion of related issues, see Kohlberg and Mertens (1986) and Kreps and Wilson (1982).

[2] This literature is characterized by individuals who make choices based on rules of thumb or who have some very rigid method of choice.

[3] See Kalai and Lehrer (1991) and the references therein. Brandts and Holt (1992), (1993), show that adaptive behavior in laboratory games can result in equilibrium patterns that are ruled out by almost all standard refinements of the sequential Nash equilibrium.

in experiments with the Nash equilibrium at the boundary of the set of feasible decisions, e.g., in ultimatum and public goods games.[4] For instance, in a 1-dollar ultimatum bargaining game, the sender proposes a split which the receiver must either accept or reject. A rejection results in earnings of 0 for both players. For this game, a subgame perfect Nash outcome is 1 penny for the receiver and 99 cents for the sender. Yet the actual outcomes of experimental ultimatum games are not nearly so asymmetric.

The ultimatum game can be given a simple market interpretation, with a single seller proposing a price that the buyer must either accept or reject. In market games with multiple price-setting sellers, however, the Nash equilibrium may involve randomization if sellers' production capacities are limited. Experimental data seem to track the qualitative features of Nash equilibria in such games, but prices are often much higher than the equilibrium predictions (Davis and Holt, 1994).

In order to sort out the reasons for the observed departures from the Nash prediction, a useful positive theory of behavior in games could begin by qualifying the assumption that individuals are perfect maximizers of their own money payoffs. Several authors have relaxed the perfect rationality assumption in experimental games: Brown and Rosentahl (1990), Camerer and Weigelt (1988), Mckelvey and Palfrey (1992, 1993), Banks et al. (1994), Brandts and Holt (1992), and Palfrey and Rosenthal (1991, 1992).

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[4] See Davis and Holt (1993), chapters 5 and 6. One way to move the Nash equilibrium away from the boundary in these games is suggested by Palfrey and Prisbrey (1993) and Prisbrey (1994).

One way is to introduce decision error, i.e. in choosing their strategies players make mistakes. Players 'tremble' and therefore every strategy (even a dominated strategy) is played with a strictly positive probability. In the case of vanishingly small strategy errors, this approach was originally used to rule out unintuitive criteria in, especially, extensive form games (Selten, 1975).

As a first step, it is useful to distinguish two sources of deviations from the Nash equilibria as calculated from expected money payoffs. First, *systematic* deviations may be due to the importance of neglected factors, such as altruism, envy, fairness, etc. These factors are likely to be more important in bargaining and public goods games than in impersonal market situations. Second, *nonsystematic* or random "errors" can follow from mistakes in recording decisions, from time constraints as in chess games, or from random errors in evaluating small differences in expected payoffs. Experimental evidence suggests that nonsystematic errors can occur in strategic situations (McKelvey and Palfrey, 1993) and also in simpler individual decision-making tasks (Anderson, 1994).

This thesis investigates the quantal response equilibrium in models of price competition in which boundedly rational players interact. In contrast to the classical conception of rationality that is based on unlimited capacity, boundedly rational players are limited by their own computational ability. Boundedly rational players have been most commonly characterized by either the random choice or the random utility version of discrete choice theory. McKelvey and Palfrey (1993) used the discrete choice framework to analyze the strategic interaction of multiple individuals. They proposed a game theoretic equilibrium concept, the quantal response equilibrium. In this model, players make choice errors based on a quantal or discrete choice model, and assume other players do so as well. The added complexity in applying the quantal response approach to game theory - as opposed to individual decision making - is that the choice probabilities of the players have an important interactive component, since they are determined simultaneously in equilibrium.

To illustrate the effects of choice errors in a market model, consider the quantal response equilibrium for a simple Bertrand game with zero production cost and two price choices. In this game, each seller simultaneously chooses between a high price P_H, and a low price P_L. The combination of prices determines payoffs as shown in the table below, where seller 1's payoff is listed to the left in each cell. The profits from defection, π_d, exceed those from cooperation, π_c, which in turn exceed the profit π_n from the Nash equilibrium: $\pi_d > \pi_c > \pi_n > 0$. The only Nash equilibrium outcome is (π_n, π_n).

		Seller 2	
		P_H	P_L
Seller 1	P_H	π_c , π_c	0 , π_d
	P_L	π_d , 0	π_n, π_n

Next, consider the effects of decision errors in this price game. A particularly simple approach to modeling such errors is based on the random choice framework initiated by Luce (1959).[5] If the expected payoff of decision i is $u_i \geq 0$, then under the Luce model, with two decisions, the probability of choosing i is

(1) $$Pr(choose\ i) = \frac{u_i}{u_1 + u_2} \qquad i = 1, 2.$$

These choice probabilities reflect boundedly rational behavior in the sense that a player does not always choose the decision with the higher payoff. Let σ denote the probability that seller 2 chooses the cooperative decision P_H. Given this probability, seller 1's expected payoff is $u_H = \sigma\pi_c$ for decision P_H and $u_L = \sigma\pi_d + (1\text{-}\sigma)\pi_n$ for decision P_L. Using the Luce choice function (1), player 1 will choose decision P_i with probability $u_i/(u_H+u_L)$, or

[5] The Luce model is explained in detail in the next chapter.

(2) $$Pr(\,choose\ P_H\,) = \frac{\sigma\pi_c}{\sigma\pi_c + \sigma\pi_d + (1-\sigma)\pi_n}\,.$$

The equilibrium consistency requirement is that choice probabilities correspond to beliefs. In particular, the right side of equation (2) must equal σ, which provides an equation that can be solved for σ:[6]

(3) $$\sigma = \frac{\pi_c - \pi_n}{\pi_c - \pi_n + \pi_d}\,.$$

Clearly, the probability σ that a seller chooses the high "cooperative" price P_H is positively related to the gain from cooperation, $(\pi_c-\pi_n)$, and negatively related to the payoff from defection, π_d. Since P_L is a dominant strategy in the Nash game without errors as long as $\pi_d > \pi_c$, σ can be interpreted as the probability of making an error.

This thesis investigates the quantal response equilibrium in markets in which sellers post prices simultaneously. The laboratory implementation of this model is commonly called a "posted offer auction." The posted offer institution is interesting because laboratory data suggest that prices deviate from Bertrand-Nash predictions in a systematic manner. Although this thesis has implications for experimental data, it mainly focuses on differences in the qualitative properties of Nash and quantal response equilibria.

Chapter 2 first describes models of individual decision making and proceeds to introduce the quantal response equilibrium. In chapter 3, a parametric class of quantal response functions is derived from a model with multiplicative random errors. This

[6] Note that the Nash equilibrium condition, $\sigma=0$, does not satisfy (3).

functional form is a useful method of modeling decision errors in markets with posted prices. The power function derivation also provides a convenient parameterization. At one extreme, individuals choose randomly, independent of expected payoffs. At the other extreme, individuals always choose the decision with the highest expected payoff. The power function decision rule is used in chapter 4 to derive the equilibrium price distribution in a simple duopoly model.

The second part of this thesis applies the approach derived in the first part. Chapter 5 examines the consequences of market structure for equilibrium price distributions. In this chapter, the quantal response equilibrium is computed for a series of price-setting models, each type reflecting designs actually used in market experiments. In some of these models, the price distribution in a quantal response equilibrium is affected by changes in structural variables although the Nash equilibrium remains unaltered. Chapter 6 compares the Nash and the quantal response equilibrium for discrete and continuous bid choices in a (first-price) all-pay auction model. In this model, each firm submits a bid for a prize. All firms forfeit their bids, but the firm submitting the highest bid wins the prize. Two particular parametric classes of quantal response functions, the logit and the power function, are used to show how the quantal response equilibrium of the all-pay auction can be computed.

CHAPTER 2

PROBABILISTIC THEORIES OF CHOICE

2.1 INTRODUCTION

The first part of this chapter describes several models of individual decision making. Anderson, de Palma, and Thisse (1992) distinguish two interpretations of discrete choice theories. In the first interpretation the utility is constant but the decision rule is random (Luce, 1959; Tversky, 1972a). By contrast, the second interpretation assumes that utility is random while the decision rule is constant (Thurstone, 1927; McFadden, 1984). These approaches are formulated for individual decisions where the probability of making a decision is a function of the expected payoffs of all possible decisions. The quantal response equilibrium, presented at the end of this chapter, analyzes such behavior in an interactive environment. This equilibrium concept has its origins in discrete choice response models. However, the quantal response equilibrium requires the expected payoffs themselves to be functions of the discrete choice probabilities, while the decision maker's expected payoffs in standard discrete choice models are exogenous.

2.2 INDIVIDUAL DECISION THEORIES OF CHOICE

When confronted with the same alternatives, under similar conditions, individuals do not consistently make the same choice. Some explanations for such behavior include learning, changes in taste over time, and saturation. However, even when the effect of such factors is minimal, the lack of consistency persists. To account for this observed choice behavior, some theorists have proposed that individual choice is the result of a random process. Probabilistic theories of choice can be divided into two basic types: constant utility models and random utility models.

Constant utility models assume that each alternative has a scale value or fixed "utility". Therefore, the probability of choosing one alternative over an other is a function of the distance between the "utilities" derived from the two alternatives. Under this interpretation, the decision problem is viewed as a discrimination problem where the individual is trying to determine which alternative would yield higher scale value. The greater the distance between the scale values, the easier for the individual to differentiate among alternatives (Luce, 1959; Tversky, 1972a). By contrast, random utility models assume that the individual always chooses the alternative that has the highest utility, but the difference is that the utilities themselves are random variables rather than constants (Thurstone, 1927; McFadden, 1984). The actual choice mechanism is basically deterministic for the individual, but the utility of each alternative is random from the point of view of the modeler.

THE LUCE MODEL

Luce (1959) has proposed a constant utility model based on the notion that individual choice behavior can be modeled by a random process.[1] Luce assumes that choice probabilities satisfy a simple, but powerful, axiom that serves as a cornerstone of the model. To illustrate the Luce choice axiom, consider the following setup. Assume that all the choice probabilities are neither 0 nor 1. Let U be a finite set of alternatives and define J to be any subset of U that contains a given alternative i. Also, let Pr(J;U) be the probability of choosing an element of J when the set of feasible choices is U. Luce's choice axiom states that the probability of choosing a particular alternative from the entire set U, equals the probability that the alternative will be in the subset J, multiplied by the probability of choosing some alternative from J:

(1) $$Pr(i;U) = Pr(i;J)\,Pr(J;U) \qquad for\ \ i \in J\ ,\ J \subset U\ .$$

The Luce choice axiom in (1) implies that irrelevant alternatives outside of J can be deleted from any choice without affecting the ratios of choice probabilities between two alternatives in the subset J. Hence this implies independence of irrelevant alternatives. To see this, consider the equations in (2):

(2) $$Pr(1;U) = Pr(1;J)\,Pr(J;U)$$

$$Pr(2;U) = Pr(2;J)\,Pr(J;U)\ .$$

Dividing one probability from the other, one obtains the constant ratio rule:

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[1] Anderson, de Palma, and Thisse (1992) discuss in detail the Luce model.

(3) $$\frac{Pr(1;U)}{Pr(2;U)} = \frac{Pr(1;J)}{Pr(2;J)} .$$

Since J can be {1,2}, equation (3) can also be expressed as

(4) $$\frac{Pr(1;U)}{Pr(2;U)} = \frac{Pr(1;\{1,2\})}{Pr(2;\{1,2\})} .$$

The above rule corresponds to a version of the property known as independence of irrelevant alternatives.[2]

An important consequence of the Luce choice axiom is the existence of a scale value for each alternative, such that the probability of choosing that alternative equals its scale value divided by the sum of scale values for all alternatives. Such a scale value is based on the independence of irrelevant alternatives property as it is shown next. Because the summation over all j of Pr(j;U) equals 1,

(5) $$Pr(i;U) = \frac{Pr(i;U)}{\sum_j Pr(j;U)} = \frac{1}{\sum_j \frac{Pr(j;U)}{Pr(i;U)}} .$$

Alternatively, the constant ratio rule (3) can be used to express (5):

(6) $$Pr(i;U) = \frac{1}{\sum_j \frac{Pr(j;U)}{Pr(i;U)}} = \frac{1}{\sum_j \frac{Pr(j;\{i,j\})}{Pr(i;\{i,j\})}} .$$

Now let y be an arbitrary element from the set U and define the scale value of a given

[2] Luce (1959) shows that the constant ratio rule in (3) is equivalent to the choice axiom in (1).

alternative i, v_i:

(7)
$$v_i = \frac{Pr(i;\{y,i\})}{Pr(y;\{y,i\})} = \frac{Pr(i;U)}{Pr(y;U)},$$

where the final equation follows from (3). By applying the constant ratio rule again, equation (7) implies:

(8)
$$\frac{v_1}{v_2} = \frac{Pr(1;U)Pr(y;U)}{Pr(y;U)Pr(2;U)} = \frac{Pr(1;U)}{Pr(2;U)} = \frac{Pr(1;\{1,2\})}{Pr(2;\{1,2\})}.$$

From equations (7) and (8), we have the following result:

(9)
$$Pr(i;U) = \frac{1}{\sum_j \frac{Pr(j;\{i,j\})}{Pr(i;\{i,j\})}} = \frac{1}{\sum_j \frac{v_j}{v_i}} = \frac{v_i}{\sum_j v_j}.$$

Thus, if the Luce choice axiom is satisfied, the choice probabilities can expressed as proportions of scale values, as in (9). With two alternatives, (9) becomes

(10)
$$Pr(1;\{1,2\}) = \frac{v_1}{v_1 + v_2}.$$

Clearly, from (10) the probability of choosing alternative 1 increases according to the scale value associated with it, but decreases with the scale value associated with decision 2. Thus the Luce model implies boundedly rational behavior in the sense that a player does not always choose the decision with the highest scale value. Block and Marshack (1960) show that an equivalence between the Luce model and the multinomial logit model exits. To see this, let x_i=ln v_i and rewrite (10) as $(e^{x_1})/(e^{x_1} + e^{x_2})$.

Note that (7) is unique up to multiplication by a positive constant. To see this, let z be another arbitrary element from the set U. Hence, the new scale value is given as

(11) $$v_i^* = \frac{Pr(i;\{z,i\})}{Pr(z,\{z;i\})} .$$

Using equation (6) and the definition of v_i, we have

(12) $$v_i^* = \frac{Pr(i;U)}{Pr(z;U)} = \frac{Pr(y;U)Pr(i;U)}{Pr(z;U)Pr(y;U)} = \frac{Pr(y;U)}{Pr(z;U)}v_i = Av_i ,$$

where A is a constant.

One criticism of the probabilistic choice rule in (9) (and thus of independence of irrelevant alternatives) is that it may not hold true in situations where the choice is divided in some manner. To illustrate this criticism, Debreu (1960) offered the following example. Assume that the choice set contains three elements: a recording of the Debussy quartet, D; a recording of a Beethoven symphony conducted by f, B_f; and a recording of the same symphony conducted by k, B_k. Let U be the entire recording music menu and J be the subset containing the Beethoven recordings, i.e., B_f and B_k. Suppose that a subject selects B_f with probability 1/2, when presented with $\{B_k,B_f\}$, so that these alternatives have the same scale values, i.e., $v_{B_f} = v_{B_k}$. Further, when the subject is confronted with either {D, B_k} or {D, B_f}, D is selected with probability 3/5. From (10), the probability 3/5 implies $v_D = (3/2)v_{B_f} = (3/2)v_{B_k}$. According to the Luce model with these scale values, when presented with {D, B_f, B_k}, D must be chosen with probability 3/7. Thus when making a decision between D and B_f, the subject would rather have Debussy. However, when choosing between D, B_f, and B_k, while being indifferent between B_f and B_k, the subject is more likely to choose one of the Beethoven recordings. Debreu concluded that the Luce choice axiom is only appropriate when the choice sets have equally dissimilar alternatives. Another possible explanation of subjects' incorrect choices is that they are due to mistakes in recording decisions, so the addition of "irrelevant" choice alternatives can affect choice probabilities.

THE TVERSKY MODEL

Tversky (1972a) and others have argued that the restrictions imposed by the independence of irrelevant alternatives property of the Luce model are very unappealing in many applications. One alternative is the elimination-by-aspects model proposed by Tversky. This model can be interpreted as the result of a choice process in which each "alternative" choice is characterized by a finite number of "aspects". The characteristics or aspects are taken as desirable features from the point of view of a given individual. Each aspect has a positive "utility" value. The decision maker picks an aspect using a Luce-like ratio-of-utility-values rule and then restricts further attention to alternatives that possess the aspect selected. Then another aspect is selected, etc. In each stage the individual samples from the remaining aspects, eliminating alternatives that fail to have the sampled aspect. The selection of aspects is random, but the elimination of alternatives which lack the selected aspect is deterministic.

To illustrate this model, suppose there are four aspects, I through IV, with (utility) values v_I, v_{II}, v_{III} and v_{IV} which are assumed to be strictly positive.[3] The utility values can differ across aspects, since some aspects may be more important than others to the decision makers. Assume that the choice set U contains the same elements, $\{D, B_f, B_k\}$. These alternatives, along with their associated aspects are assumed to be as follows:

	I	II	III	IV
D	v_I	0	v_{III}	v_{IV}
B_f	0	v_{II}	v_{III}	v_{IV}
B_k	v_I	v_{II}	0	v_{IV}

The fourth aspect that is common to all choices does not enter in the decision process and hence is eliminated from the table above, as indicated by the grey shading. It follows from the payoff table structure that D would be chosen if aspect I is selected first (which rules out B_f) and aspect III is selected second (which rules out B_k). Similarly, D would be also chosen if aspect III were selected first and aspect I were selected second Thus the probability of choosing alternative D is calculated as follows:

[3] This is a modified example from Tversky (1972b), 281-84.

$Pr(D,\{D, B_f, B_k\})$ = Pr(I chosen as the 1st aspect)*Pr(III chosen as the 2nd aspect)

+ Pr(II chosen as the 1st aspect)*0

+ Pr(III chosen as the 1st aspect)*Pr(I chosen as the 2nd aspect).

The probability that aspect I is selected first is assumed to be a ratio of v_I to the sum of the utilities for all three options: $v_I/(v_I+v_{II}+v_{III})$. The probability that III is selected second is a ratio of v_{III} to the sum of the utilities for the remaining two options:$v_{III}/(v_{II}+v_{III})$. It follows from this logic that the probability of choosing D is

(13)

$$Pr(D;\{D,B_f,B_k\}) = \frac{v_I}{v_I+v_{II}+v_{III}} \cdot \frac{v_{III}}{v_{II}+v_{III}} + \frac{v_{II}}{v_I+v_{II}+v_{III}} \cdot 0 + \frac{v_{III}}{v_I+v_{II}+v_{III}} \cdot \frac{v_I}{v_I+v_{II}} .$$

An interesting feature of this model is that it is capable of addressing Debreu's critique. Clearly, in Debreu's example the two Beethoven recordings share more aspects in common than either shares with the Debussy recording. This is illustrated in the simple example below:

	I	II
D	v_I	0
B_f	0	v_{II}
B_k	0	v_{II}

Let $v_I = 3$ and $v_{II} = 2$. Notice that in a pairwise comparison between D and B_k. D is chosen if aspect I is selected first, with occurs with probability $v_I/(v_I + v_{II}) = 3/5$. In choosing between D, B_k, and B_f, the probability of choosing D is again 3/5 since the only way that D is to be chosen is for aspect I to be selected first, which happens with probability $v_I/(v_I + v_{II})$. In the Luce model, the probability of choosing D is the ratio of v_I to the sum of the utilities for all three options: $v_I/(v_I + v_{II} + v_{III}) = 3/7$. Hence the Tversky model predicts a higher probability of choosing the alternative D from $\{D, B_f, B_k\}$ than in the Luce model. This higher probability is more consistent with the preference for D in the choice between D and B_f or between D and B_k.

THE RANDOM UTILITY APPROACH

The Tversky model is one way of accounting for similarities among alternatives. Another approach is the multinomial probit model, in which the residuals in the random utility-model have a multivariate normal distribution. This model was first proposed by Thurstone (1927). This model was constructed in order to explain the fluctuations in the psychological evaluation of objects. Thurstone provides a theory for converting the proportion of times one alternative is judged greater than another into a measure of the subjective difference between them. The model assumes that in each comparison of alternatives, a judgment is made as to which alternative is preferred. Such comparisons are replicated independently, and the proportion of judgments between the two alternatives is determined. The model further assumes that when an individual chooses between the two alternatives -- e.g, a judgement about which one is greater -- each alternative causes a subjective experience of some degree of intensity. Thus, the individual's response reflects which alternative gives rise to the highest subjective experience. However, in independent replications, an alternative does not necessarily cause the same intensity of subjective experience. Rather there is a normal distribution of such "discriminal processes", as Thurstone calls them, reflecting their relative probability of occurrence on any one trial. To illustrate Thurstone's model, consider the case of two alternatives. Assume that a subject's utility derived from alternatives 1 and 2 can be written as

$$V_1^* = v_1 + \varepsilon_1$$

$$V_2^* = v_2 + \varepsilon_2 ,$$

where $v_1 - v_2$ is the measurable psychological distance and the ε_i are the residual random elements. Now consider the probability that the first alternative will be chosen:

$$\begin{aligned} \Pr(\text{ choose 1 }) &= \Pr(v_1 + \varepsilon_1 > v_2 + \varepsilon_1) \\ &= \Pr(v_1 - v_2 > \varepsilon_2 - \varepsilon_1) \\ &= H(v_1 - v_2) , \end{aligned}$$

where H(*) denotes a cumulative distribution. If the errors are identically, independently, and normally distributed, then the probability that an individual chooses alternative 1 can be expressed in terms of a probit decision rule. The above expression is analogous to Thurstone's law of comparative judgment.

Following Thurstone, McFadden (1984) assumed that utility is a random function. However, McFadden's interpretation of discrete choice theory is conceptually very different from the previous theories. Under his framework, the decision-makers are rational in the sense that they make choices that maximize their perceived utility. From the econometrician's point of view, "errors" result from unobserved characteristics influencing an individual's choice. For example, the v_i scale or utility discussed earlier could represent the observed parts of an individual's utility. It follows that the optimal decision may also depend on random unobserved utility elements or on latent variables such as idiosyncratic preferences or specific tastes for a given choice. The distribution of the random payoff elements determines the form of the probabilistic choice function (e.g., logit, probit).

McFadden (1984) shows that if the errors, ε_i, are identically, independently, and log Weibull distributed, then the probability that an individual chooses alternative 1 can be expressed in terms of the following logistic function. Luce and Suppes (1965) attribute this result to an unpublished paper by Holman and Marley. The appendix at the end of this chapter provides the derivation.

$$\textbf{(14)} \qquad Pr(\ choose\ 1\ ) = \frac{e^{\lambda v_1}}{e^{\lambda v_2} + e^{\lambda v_1}} \ .$$

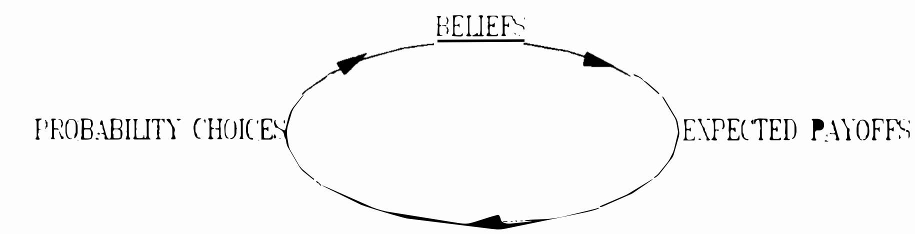
As $1/\lambda$ goes to ∞ in equation (14), it can be shown that the variance of the error terms tends to infinity. Thus the individual will choose between decisions 1 and 2 with equal probability, regardless of the expected payoffs, as can be seen from the limiting case of (14) with $\lambda=0$. The error variance, $1/\lambda$ goes to 0 as λ goes to ∞, and therefore, it follows from (14) that the probability of choosing the option with the higher expected payoff goes to 1.

In the Luce and McFadden-Thurstone models, the probability of choosing one alternative over another is expressed as an increasing function of the difference between their scale values in the Luce model and utility under the McFadden-Thurstone approach. It is also possible to interpret the random residuals as being caused by decision errors. Under the decision error interpretation, these choice probabilities reflect boundedly rational behavior, in the sense that an individual does not always choose the decision with the highest utility. These choice probabilities reflect a tendency toward utility maximization because a non-optimal choice is less likely when the difference in the underlying utilities is large. The next section incorporates the framework into an equilibrium analysis.

2.3 THE QUANTAL RESPONSE EQUILIBRIUM

The idea of a quantal response equilibrium was used in chapter 1 to model decision errors in a market game. Unlike a Nash equilibrium, where players use best responses to others' strategies, quantal responses are *stochastic* best responses. In particular, players are more likely to choose better strategies than worse strategies, but do not play a best response with probability one. The idea has its origins in statistical limited dependent variable models such as discrete choice (in economics and psychology) and stimulus/response models in biology. The added complexity in applying the quantal response approach to game theory -- in contrast to individual choice -- is that the choice probabilities of the players have an important interactive component, since they are simultaneously determined in equilibrium.

In a quantal response equilibrium, a player's beliefs about others' actions will determine the player's own expected payoffs, which in turn determine the player's choice probabilities via a quantal response function. The model is closed by requiring the choice probabilities to be consistent with the initial beliefs. The circuit may be summarized as follows:



Formally, Mckelvey and Palfrey's (1993) quantal response equilibrium is a fixed point in choice probabilities. Define π as the set of all possible combinations of the expected payoffs for all players in a finite normal form game. Let δ be the Cartesian product of the mixed strategies for all players, and let $\hat{p}$ be an element of δ, i.e., $\hat{p}$ specifies a particular mixed strategy for each player. Denote a vector of all expected payoffs as $e(\hat{p})$.

Thus, $e(\dot{p})$ maps a particular array of mixed strategies, $\dot{p}$, into a vector of players' expected payoffs, π. A discrete choice function σ maps expected payoffs into a mixed strategy for a single player. The function σ is assumed to be continuous and monotonically increasing in the payoffs. Let T^{σ} represent the resulting mapping from the set of all possible combinations of players' expected payoffs to their choice probabilities, $T^{\sigma} : \pi \rightarrow \delta$. To summarize, $e(\dot{p}) : \delta \rightarrow \pi$ maps mixed strategy probabilities to expected payoffs, and $T^{\sigma} : \pi \rightarrow \delta$ maps expected payoffs to mixed strategy probabilities.

The equilibrium is a fixed point:

DEFINITION. A Quantal Response Equilibrium is a $\dot{p}$ such that $\dot{p} = T^{\sigma}(e(\dot{p}))$.

The Brouwer fixed point theorem implies the existence of such an equilibrium, since $T^{\sigma}(e(\dot{p}))$ is a continuous function that maps a compact set δ into itself. This result is due to McKelvey and Palfrey (1993).

2.4 CONCLUSIONS

All the models presented in the first part of this chapter try to account for the randomness in observed choice behavior. The models of McFadden and Thurstone share the property that the probability of choosing a particular alternative over another is expressed as an increasing function of the difference between their utilities. To see this, note that in these models there exists a distribution function F such that $Pr(x;\{x,y\}) = F[u_x - u_y]$. This property is useful because it allows to test hypotheses about individual choice behavior when variability is apparent in the data.

A drawback of the Luce model is the independence-of-irrelevant-alternatives property. As Debreu (1960) pointed out, this property leads to inconsistencies in some choice situations. The Tversky model is one way to account for the unintuitive implications of the independence-of-irrelevant alternatives property.[4] In the Tversky model each alternative is described by a set of aspects, and at each stage of the selection process an aspect is selected from the ones included in the available alternatives, with a probability that is proportional to its value. The Tversky model has not found many applications in economics since this model is very restrictive. For example, as the choice set increases the probability of choosing an alternative becomes computationally

[4] Another approach is the nested logit. Under this framework choice is modeled in a two stage-nested process. A detailed discussion of this model is found in Anderson, de Palma, and Thisse (1992).

infeasible.[5] Another problem with the Tversky model is that the aspects are binary in nature. However, many economic problems often require the aspects to be treated as continuous random variables, e.g., quantity, price and quality. The Tversky model is a "myopic" model in the sense that it assumes that all errors are made in the aspect selection and none in the choice of alternatives. This model could be of interest in situations that require the comparison spectrum to be multidimensional.

The Luce, logit and probit models are all closely related. McFadden's logit model is a special case of the Luce model in which the v_i in equation (14) of this chapter are a transformation of the scale values. Thurstone's model is similar to McFadden's model, except that the error terms are normally rather then log Weibull distributed. The resulting probit model has been applied to situations with small numbers of alternatives because the computations involve evaluating multiple integrals. The logit model has been used as an alternative to the probit model because the logistic distribution and the cumulative normal distribution do not differ greatly and often both deliver similar results.

The last part of this chapter presented the quantal response equilibrium. This equilibrium concept is a method of modeling decision errors in an interactive decision environment which uses the basis borrowed from the work in discrete choice theory of Luce, McFadden and Thustone. In the quantal response equilibrium, a player's beliefs

[5] To counter this problem, Tversky and Sattah (1979) specialized the elimination-by-aspects model to a situation where the alternatives are given in a tree structure. Although this model involves fewer parameters than the Tversky's model, it has not found many empirical applications.

about others' actions will determine the player's own expected payoffs, which in turn determine the player's choice probabilities via a quantal response function. The model is closed by requiring the choice probabilities to be consistent with the initial beliefs. In the remainder of this thesis we use two specialized versions of the quantal response equilibrium, one based on the logit function and the other based on a "power function" to be derived on the next chapter.

2.5 APPENDIX

DERIVATION OF THE BINOMIAL LOGIT MODEL

Assume that a subject's utility derived from alternative i can be written as

(15) $$V_i^* = v_i + \varepsilon_i \ .$$

In this expression the modeler is unable to predict the subject's choice, since ε_i cannot be observed. The observable part of the utility is v_i, while ε_i captures the unmeasured aspects of utility. When there are two alternatives, the probability of choosing 1 is given by

(16) $$Pr(\ choose\ 1\ ) = Pr(\ \varepsilon_1 + v_1 - v_2 > \varepsilon_2\ ).$$

Let the cumulative distribution function of the error term, $H(\varepsilon)$, be[6]

(17) $$H(\epsilon) = e^{-e^{-\lambda\epsilon}} \qquad \varepsilon \in (-\infty, \infty)\ .$$

The corresponding density, $h(\varepsilon)$, is

(18) $$h(\epsilon) = \lambda e^{-\lambda\epsilon} e^{-e^{-\lambda\epsilon}}.$$

For a given realization ε_1 of the error term, it follows from (18) that alternative 1 will be chosen with probability $H(\varepsilon_1 + v_1 - v_2)$. Thus the probability of choosing alternative 1

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[6] The distribution function

$$H(\epsilon) = e^{-e^{-\lambda\epsilon}} \qquad \varepsilon \in (-\infty, \infty)\ .$$

is called the Type I extreme-value distribution, or log Weibull distribution, by Johnson and Kotz (1970,p.272).

is obtained by integrating $H(\varepsilon_1 + v_1 - v_2)$ over all possible values of ε_1:

(19)

$$Pr(\ choose\ 1\ ) = \int_{-\infty}^{\infty} h(\epsilon_1)\ H(\epsilon_1 + v_1 - v_2)\ d\epsilon_1$$

$$= \int_{-\infty}^{\infty} \lambda e^{-\lambda\epsilon_1} e^{-e^{-\lambda\epsilon_1}} e^{-e^{-\lambda(\epsilon_1+v_1-v_2)}} d\epsilon_1 \quad .$$

Let $\tau = e^{-\lambda\varepsilon_1}$ so $d\tau = -\lambda e^{-\lambda\varepsilon_1}$ and define $y_1 = e^{\lambda v_1}$, $y_2 = e^{\lambda v_2}$. As ε goes from $-\infty$ to ∞, τ goes from ∞ to 0, so equation (19) becomes

$$Pr(\ choose\ 1\ ) = \int_{\infty}^{0} (-d\tau) e^{-\tau} e^{-\tau(\frac{y_2}{y_1})}$$

$$= \int_{0}^{\infty} e^{-\tau} . e^{-\tau[\frac{y_2}{y_1}]} d\tau$$

$$= \int_{0}^{\infty} e^{\frac{-\tau(y_1+y_2)}{y_1}} d\tau$$

$$= -\frac{y_1}{y_1+y_2} [e^{-\tau[\frac{y_1+y_2}{y_1}]}]\Big|_{0}^{\infty}$$

$$= \frac{y_1}{y_1+y_2} \ ,$$

where the second equality follows from a reversal of the limits of integration. Notice from the definitions of y_1 and y_2 that

$$Pr(\ choose\ 1\ ) = \frac{e^{\lambda v_1}}{e^{\lambda v_1} + e^{\lambda v_2}} \quad ,$$

which is the logit formulation.

CHAPTER 3

THE POWER-FUNCTION QUANTAL RESPONSE EQUILIBRIUM

3.1 INTRODUCTION

The Luce framework, as discussed in the previous chapter, provides a rather rigid relationship between the underlying utilities and the choice probabilities of the individuals. Recall that the Luce model choice probabilities are given as ratios of expected payoffs:

$$\textbf{(1)} \qquad Pr(\ choose\ i\ ) = \frac{\pi_i}{\pi_1 + \pi_2}\ , \qquad for\ \ i = 1,2.$$

The above expression can be parameterized in a more general form that permits an arbitrary degree of bounded rationality, with, fully rational individuals at one extreme. At the other extreme, there is absolutely no connection between expected payoffs and choice probabilities. The power-function quantal response equilibrium derived in this chapter generalizes equation (1) by having each expected payoff raised to a power. This functional form turns out to be a useful way to model decision errors in models of price competition since it often leads to tractable solutions and comparative statics results. The power-function quantal response equilibrium is based on random utility maximization with multiplicative error terms. In the power-function quantal response equilibrium, each player's quantal response function will have a power parameter which, when equal to 1, yields the Luce model. The parameter, however, can take on any value between 0 and ∞. The logistic quantal response equilibrium is another specialized version of the quantal response equilibrium which will be also used in subsequent chapters. The logistic equilibrium is based on random utility maximization with additive error terms.

This chapter is organized as follows: Section 3.2 introduces the power function model. Section 3.3 compares the power-function and the logit formulations. Section 3.4 derives the power-function model from a random-utility framework. Section 3.5 compares the equilibrium properties of the power-function and the logistic quantal response equilibrium in a simple market game with two possible price choices.

3.2 THE POWER-FUNCTION MODEL

For simplicity in exposition assume that a single decision maker must choose between two alternatives, 1 and 2. The corresponding expected payoffs, π_1 and π_2, are assumed to be strictly positive. Under the power function model, the probability of choosing alternative 1 is given by:

(2) $$Pr(\ choose\ 1\ ) = \frac{\pi_1^{\lambda}}{\pi_1^{\lambda} + \pi_2^{\lambda}}\ ,$$

where the ratio of expected payoffs is raised to a power λ. In (2), λ is a nonnegative parameter that measures the degree of rationality of the individuals. As λ goes to 0, the individual chooses each decision with equal probability, regardless of expected payoffs. As λ goes to ∞, the decision with the highest expected payoff is selected with probability

1.

The model in (2) can be used to examine decisions errors in an equilibrium framework. The power-function quantal response equilibrium is a specific version of the quantal response equilibrium that uses equation (2) to determine behavior in an interactive environment. Before deriving the power function (2) in a random utility model, it is useful to compare the power function with the more standard logit response function.

3.3 A COMPARISON OF THE LOGISTIC AND THE POWER FUNCTION MODELS

This section examines some properties of two parametric versions of the quantal response equilibrium. These two versions are the logit:

(3)
$$Pr(\ choose\ i\ ) = \frac{e^{\lambda\pi_i}}{e^{\lambda\pi_1} + e^{\lambda\pi_2}}, \quad i = 1, 2,$$

with $\lambda>0$, and the power function:

(4)
$$Pr(\ choose\ i\ ) = \frac{\pi_i^{\lambda}}{\pi_1^{\lambda} + \pi_2^{\lambda}}, \quad i = 1, 2,$$

with $\lambda > 0$ and $\pi_i > 0$. The logit odds ratio is written as

(5)
$$\frac{Pr(\ choose\ 1\ )}{Pr(\ choose\ 2\ )} = e^{\lambda(\pi_1 - \pi_2)};$$

where the probability ratio for the logit is a function of the expected payoff *difference*.

Similarly, the power function odds ratio is giving by

(6)
$$\frac{Pr(\ choose\ 1\ )}{Pr(\ choose\ 2\ )} = \left[\frac{\pi_1}{\pi_2}\right]^{\lambda},$$

where the probability ratio for the power function is a function of the expected payoff *ratio*.

Notice that the logit and the power function decision rules satisfy the independence-of-irrelevant alternatives property discussed in chapter 2. That is, irrelevant alternatives other than 1 and 2 can be deleted from any choice without affecting the ratios of choice probabilities between the alternatives 1 and 2.

SENSITIVITY TO THE ERROR RATE

In what follows, it is shown that as the error rate, $1/\lambda$, decreases, the probability of choosing the best alternative increases for both specifications.[1] Taking the partial derivative of the logit odds ratio with respect to λ, we obtain:

(7)
$$\frac{\partial\left[\frac{Pr(\ choose\ 1\ )}{Pr(\ choose\ 2\ )}\right]}{\partial\lambda} = (\pi_1 - \pi_2)\ .\ e^{\lambda(\pi_1 - \pi_2)},$$

Similarly, the partial derivative of the power-function with respect to λ yields

For $\pi_1 > \pi_2$, an increase in λ increases the odds ratio in both formulations. From (8), the

[1] In a product differentiation context, Perloff, J. and Salop, S. (1986) interpret error as a price "mistake".

(8)
$$\frac{\partial \left[\frac{Pr(\ choose\ 1\ )}{Pr(\ choose\ 2\ )} \right]}{\partial \lambda} = \lambda \log \left[\frac{\pi_1}{\pi_2} \right] \cdot \left[\frac{\pi_1}{\pi_2} \right]^{\lambda} .$$

probability that an individual chooses the alternative that yields the highest expected payoff, π_1, goes to 1 as λ goes to ∞. On the other hand, if $\pi_1=\pi_2$, the logit and the power function odds ratios become 1 irrespective of λ.

Next, we examine how choice probabilities are affected by payoff transformations. This is done because in designing and evaluating experiments it is important to determine how salient payoffs affect individual decisions.

ADDITIVE CONSTANT

Consider the effect of adding a fixed constant τ to all payoffs, e.g a lump-sum subsidy or a fixed cost. Notice from equation (9) that the additive constant τ does not change the logit odds ratio:

(9)
$$\frac{Pr(\ choose\ 1\ )}{Pr(\ choose\ 2\ )} = e^{\lambda\,((\pi_1\,+\,\tau)\,-\,(\pi_2\,+\,\tau))} ,$$

A similar argument shows that τ affects the power-function choice probabilities:

(10)
$$\frac{Pr(\ choose\ 1\ )}{Pr(\ choose\ 2\ )} = \left[\frac{\pi_1 + \tau}{\pi_2 + \tau} \right]^{\lambda} .$$

Without loss of generality, assume that $\pi_1 > \pi_2$. The partial derivative of the power-function odds ratio with respect to τ is given by

(11) $$\frac{\partial \left[\dfrac{Pr(\ choose\ 1\ )}{Pr(\ choose\ 2\ )}\right]}{\partial\, d\tau} = \lambda \left[\frac{\pi_2 - \pi_1}{(\pi_2 + \tau)^2}\right] \cdot \left[\frac{\pi_1 + \tau}{\pi_2 + \tau}\right]^{\lambda - 1} .$$

It follows from (11) that an increase in τ decreases the power-function odds ratio.

To summarize:

<u>*Proposition 1*</u>

Adding a constant τ to each payoff does affect the power-function odds ratio but has no effect on the logit odds ratio. Furthermore, an increase in τ decreases the power-function odds ratio.

MULTIPLICATIVE CONSTANT

Consider the effect of multiplying all payoffs by a constant τ, e.g. τ could correspond to one minus the marginal tax rate. Notice from (12) that multiplying each payoff by a constant τ changes the logit odds ratio as follows

(12) $$\frac{Pr(\ choose\ 1\ )}{Pr(\ choose\ 2\ )} = e^{\lambda\tau(\pi_1 - \pi_2)} .$$

Given $\pi_1 > \pi_2$, an increase in τ increases the logit odds ratio as it is shown in (13):

(13) $$\frac{\partial \left[\dfrac{Pr(\ choose\ 1\ )}{Pr(\ choose\ 2\ )}\right]}{\partial\lambda} = \lambda\,(\pi_1 - \pi_2)\,.\,e^{\lambda\tau(\pi_1 - \pi_2)} .$$

However, the power-function choice probabilities remain unchanged.

To summarize:

Proposition 2

The multiplication of each payoff by a nonzero constant increases the logit odds ratio but has no effect on the power-function odds ratio. Furthermore, an increase in τ increases the logit odds ratio.

In this section, it was shown that the probability of choosing alternative 1 over alterative 2 is an increasing function of the *difference* between the expected payoff values in the logit formulation, while it is an increasing function of the *ratio* of expected payoffs in the power function formulation. Another result is that adding a constant to each payoff does not affect the logit odds ratio but it does affect the power-function odds ratio. Lastly, when each payoff is multiplied by a constant, the logit odds ratio increases but the power-function odds ratio remains unaffected. In the next section, we discuss the random-utility foundations of the power function model.

3.4 A RANDOM-UTILITY DERIVATION OF THE POWER-FUNCTION MODEL

The power function model in (2) can be derived from the random utility maximization approach with multiplicative errors. There are many ways to model the stochastic behavior of the error term in the payoff function. Previous research has focused on either normally or log Weibull distributed errors, which yield the probit and logit

To summarize:

Proposition 2

The multiplication of each payoff by a nonzero constant increases the logit odds ratio but has no effect on the power-function odds ratio. Furthermore, an increase in τ increases the logit odds ratio.

In this section, it was shown that the probability of choosing alternative 1 over alterative 2 is an increasing function of the *difference* between the expected payoff values in the logit formulation, while it is an increasing function of the *ratio* of expected payoffs in the power function formulation. Another result is that adding a constant to each payoff does not affect the logit odds ratio but it does affect the power-function odds ratio. Lastly, when each payoff is multiplied by a constant, the logit odds ratio increases but the power-function odds ratio remains unaffected. In the next section, we discuss the random-utility foundations of the power function model.

3.4 A RANDOM-UTILITY DERIVATION OF THE POWER-FUNCTION MODEL

The power function model in (2) can be derived from the random utility maximization approach with multiplicative errors. There are many ways to model the stochastic behavior of the error term in the payoff function. Previous research has focused on either normally or log Weibull distributed errors, which yield the probit and logit decision rules respectively.[2] The power function decision rule can be derived from random utility expressed as a product: $U_i=\pi_i\kappa_i$, where κ_i is an identical and independently distributed multiplicative error term known to a player, and π_i is a nonnegative expected payoff. With two alternatives, the probability that a player selects decision 1 is

(14) $$Pr(\ choose\ 1\ ) = Pr(\pi_1\kappa_1 > \pi_2\kappa_2)\ .$$

Making a logarithmic transformation, we have

(15) $$Pr(\ choose\ 1\ ) = Pr(\ \ln \pi_1 + \ln \kappa_1 > \ln \pi_2 + \ln \kappa_2\ )\ .$$

Let G(*) denote the distribution of κ, such that [3]

(16) $$G(\kappa) = e^{-(\kappa)^{-\lambda}} \qquad \kappa \in [0,\infty)\ , \quad \lambda > 0\ .$$

Define a transformation of the error term: $\varepsilon = \ln \kappa$ or $\kappa = e^{\varepsilon}$. Substitute e^{ε} for κ in (16) to obtain the distribution function:

(17) $$H(\epsilon) = e^{-e^{-\lambda\epsilon}}\ ,$$

which is a log Weibull distribution with parameter λ. When an additive random utility error, ε, is log Weibull distributed, Luce and Suppes (1967) have shown that the standard

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[2] There are a number of other papers that use explicit models of the error structure. Logit and probit specifications of the errors in the analysis of experimental data are common Palfrey and Rosenthal (1991), Palfrey and Prisbrey (1992), Stahl and Wilson (1993), Anderson (1994), and Harless and Camerer (1994). Zauner (1994) uses a Harsanyi (1973) equilibrium model with independent normal errors to explain data from a centipede game reported by McKelvey and Palfrey (1992).

[3] In the analysis that follows, the i subscript is dropped from the error terms since the errors are i.i.d.

logit decision rule is derived from (17), as shown in chapter 2. Since the logarithmic transformation of the multiplicative error in (17) is additive in the logarithm of π_i, the relevant probabilistic choice function is the logit formulation with the expected payoff, v_i, replaced by log π_i. Hence

(18) $$e^{\lambda v_i} = e^{\lambda \log \pi_i} = \pi_i^{\lambda} ,$$

and the logistic choice rule in equation (3) reduces to the power function rule in equation (4) of this chapter.

To summarize:

Proposition 3

If the payoff function is random and multiplicative, $\pi_i \kappa_i$, with the error terms identically and independently distributed as $G(\kappa_i) = e^{-(\kappa i)^{-\lambda}}$, the probabilistic choice function is the power function: Pr(choose i) = $(\pi_i)^{\lambda} / \sum (\pi_j)^{\lambda}$.

3.5 A BERTRAND DUOPOLY EXAMPLE

In this section we examine the properties of the power-function and logit approaches in a simple market game with two possible price choices. Consider the symmetric Bertrand game in the table below. In this game, each seller simultaneously chooses between a high price P_H, and a low price P_L. The combination of prices determines payoffs as shown in the table below, where seller 1's payoff is listed to the left in each cell. The profits from defection, π_d, exceed those from cooperation, π_c, which in turn exceed those from the Nash equilibrium: $\pi_d > \pi_c > \pi_n > 0$. The Nash equilibrium is the outcome (π_n, π_n).

		Seller 2	
		P_H	P_L
Seller 1	P_H	π_c , π_c	0 , π_d
	P_L	π_d , 0	π_n, π_n

Denote by σ denote the probability that seller 2 chooses the cooperative decision P_H. Given this probability, seller 1's expected payoff is $u_H = \sigma\pi_c$ for decision P_H and $u_H = \sigma\pi_d + (1-\sigma)\pi_n$ for decision P_L. Then in a logit quantal response equilibrium we have

$$\textbf{(19)} \qquad \sigma = \frac{e^{\lambda(\sigma\pi_c)}}{e^{\lambda\sigma\pi_c} + e^{\lambda(\sigma\pi_d + (1-\sigma)\pi_n)}} \quad ,$$

where (19) is a one's seller's stochastic "best reply" to the other seller's price strategy.

The probability of choosing P_L is

(20)
$$1 - \sigma = \frac{e^{\lambda(\sigma(\pi_d) + (1-\sigma)\pi_n)}}{e^{\lambda\sigma\pi_c} + e^{\lambda(\sigma\pi_d + (1-\sigma)\pi_n)}} .$$

It follows from (19) and (20) that the odds ratio is

(21)
$$\frac{\sigma}{(1-\sigma)} = e^{\lambda(\sigma\pi_c - \sigma\pi_d - (1-\sigma)\pi_n)} .$$

Note that the Nash equilibrium condition, σ=0, does not satisfy (21). In order to determine the effect of λ in the endogenous probability σ, take the logarithm of both sides of (21) to obtain:

(22)
$$\log\sigma - \log(1-\sigma) = \lambda(\sigma\pi_c - \sigma\pi_d - (1-\sigma)\pi_n) .$$

Then take the total derivative of (22). After arranging some terms we have

(23)
$$d\sigma\left[\frac{1}{\sigma} + \frac{1}{1-\sigma} - \lambda\pi_c + \lambda\pi_d + \lambda\pi_n\right] = d\lambda\left[\sigma\pi_c - \sigma\pi_d - (1-\sigma)\pi_n\right] .$$

Dividing through by the differential dλ and rearranging, one obtains the derivative

(24)
$$\frac{d\sigma}{d\lambda} = \frac{\sigma(\pi_c - \pi_d) - (1-\sigma)\pi_n}{\left[\frac{1}{\sigma} + \frac{1}{1-\sigma} + \lambda(\pi_d + \pi_n - \pi_c)\right]} .$$

Since $\pi_d > \pi_c - \pi_n$, the sign of (24) is positive. Then as λ increases, errors decrease and people play more like conventional Nash players.

The probability of choosing P_L is

(20) $$1 - \sigma = \frac{e^{\lambda(\sigma(\pi_d) + (1-\sigma)\pi_n)}}{e^{\lambda\sigma\pi_c} + e^{\lambda(\sigma\pi_d + (1-\sigma)\pi_n)}} .$$

It follows from (19) and (20) that the odds ratio is

(21) $$\frac{\sigma}{(1-\sigma)} = e^{\lambda(\sigma\pi_c - \sigma\pi_d - (1-\sigma)\pi_n)} .$$

Note that the Nash equilibrium condition, σ=0, does not satisfy (21). In order to determine the effect of λ in the endogenous probability σ, take the logarithm of both sides of (21) to obtain:

(22) $$\log\sigma - \log(1-\sigma) = \lambda(\sigma\pi_c - \sigma\pi_d - (1-\sigma)\pi_n) .$$

Then take the total derivative of (22). After arranging some terms we have

(23) $$d\sigma\left[\frac{1}{\sigma} + \frac{1}{1-\sigma} - \lambda\pi_c + \lambda\pi_d + \lambda\pi_n\right] = d\lambda\left[\sigma\pi_c - \sigma\pi_d - (1-\sigma)\pi_n\right] .$$

Dividing through by the differential $d\lambda$ and rearranging, one obtains the derivative

(24) $$\frac{d\sigma}{d\lambda} = \frac{\sigma(\pi_c - \pi_d) - (1-\sigma)\pi_n}{\left[\frac{1}{\sigma} + \frac{1}{1-\sigma} + \lambda(\pi_d + \pi_n - \pi_c)\right]} .$$

Since $\pi_d > \pi_c - \pi_n$, the sign of (24) is positive. Then as λ increases, errors decrease and people play more like conventional Nash players.

Next we examine the equilibrium power-function odds ratio. Notice from (25) that the Nash equilibrium (σ=0) satisfies the power-function quantal response equilibrium. Assuming $\sigma \neq 0$, we have

$$\text{(25)} \qquad \frac{\sigma}{(1-\sigma)} = \left(\frac{\sigma \pi_c}{\sigma \pi_d + (1-\sigma)\pi_n} \right)^{\lambda} .$$

Taking the logarithm of both sides of (25) results in the following equation

$$\text{(26)} \qquad \log \sigma - \log(1-\sigma) = \lambda \left[\log(\sigma \pi_c) - \log(\sigma \pi_d + (1-\sigma)\pi_n) \right].$$

The implicit derivative of σ with respect to λ in equation (26) yields

$$\text{(27)} \qquad \frac{d\sigma}{d\lambda} = \frac{\log(\sigma \pi_c) - \log(\sigma \pi_d + (1-\sigma)\pi_n)}{\left[\frac{1}{\sigma} + \frac{1}{1-\sigma} - \frac{\lambda}{\sigma} + \frac{\lambda(\pi_d - \pi_n)}{\sigma \pi_d + (1-\sigma)\pi_n} \right]}.$$

Notice that the numerator of (27) is negative since $\log(\sigma\pi_c) < \log(\sigma\pi_d + (1-\sigma)\pi_n)$, or equivalently, $\sigma(\pi_d-\pi_c) > -(1-\sigma)\pi_n$. A sufficient condition for the sign of equation (27) to be negative is that $\lambda < 1$. In this case, the probability of cooperation again decreases.

This example shows that the power function and the logit can yield different comparative statics results. For example, the Nash equilibrium (σ=0) is a quantal response equilibrium for the power function but not for the logit. By comparing the Nash and quantal response equilibrium, it is interesting to note that the quantal response equilibrium always assigns positive probability to both pure price strategies. Furthermore, as λ increases the probability of choosing the cooperative, high-price decision also increases in a logistic quantal response equilibrium. Accordingly, the model is able to account for systematic deviations from Bertrand-Nash equilibrium. The analysis of the Bertrand model also suggests that a sufficient amount of decision error can actually make individuals better off since it increases the probability of choosing the cooperative, high-price decision.

In the market model to be examined in the next chapter, the quantal response equilibrium is much more difficult to compute since the range of feasible price choice decisions is assumed to be continuous and the number of firms is allowed to exceed 2. However, the intuition gained from the Bertrand model with two price choices will be useful in analyzing a more complex market structure.

CHAPTER 4

THE QUANTAL RESPONSE EQUILIBRIUM IN A DUOPOLY MODEL OF PRICE COMPETITION

This chapter applies the quantal response equilibrium to a Bertrand duopoly game with a continuum of price choices. A closed-form solution for equilibrium price distribution with endogenous errors is derived using the power function choice model. The chapter concludes with a summary of the methodology to be used in subsequent chapters.

4.1 THE MODEL

Assume a homogeneous-product duopoly with zero cost. Each seller supplies one unit to the market, and the buyer demands one unit inelastically for all prices less or equal to one. The Nash equilibrium for this game is for both sellers to charge the competitive price of zero. This price is a unique Nash equilibrium since a unilateral price increase results in no sale, and therefore, does not yield higher earnings for either seller.

Now, consider the quantal response equilibrium for the Bertrand model. To avoid a cumbersome analysis of demand division when prices are equal, the price p will be treated as a continuous variable. Let F(p) denote the continuous distribution of the other seller's price, i.e, F(p) is the probability that the other seller's price is less than p. A seller who chooses price p sells one unit with probability 1 - F(p), and earns p on that unit. Thus the profit to a seller as a function of p is given by

(1)
$$\pi(p) = p\left[\,1 - F(p)\,\right].$$

4.2 THE EQUILIBRIUM PRICE DISTRIBUTION

The price choices were discrete in the market game analyzed in the previous chapter. In that analysis, the probabilities were proportional to expected payoffs raised to a power. In the present market context, the continuous power function rule implies that the choice probability density must satisfy (2), where the density is proportional to expected profit raised to a power λ:

(2)
$$f(p) = \frac{\left(p\left[\,1 - F(p)\,\right]\right)^{\lambda}}{\int_0^1 \left(x\left[\,1 - F(x)\,\right]\right)^{\lambda} dx}.$$

Let μ denote the denominator of the right hand side of (2), which is a constant independent of p. Thus

(3)
$$f(p) = \frac{p^{\lambda}\left[\,1 - F(p)\,\right]^{\lambda}}{\mu}.$$

Equation (3) is a nonlinear differential equation that can be used to determine the distribution, F(p). The main result of this chapter is given in the next proposition:

Proposition 1

The price density

(4) $$f(p) = \frac{\lambda+1}{1-\lambda} p^{\lambda}\left[1-p^{\lambda+1}\right]^{\frac{\lambda}{1-\lambda}} .$$

For $\lambda \epsilon [0,1)$, equation (4) is a power-function quantal response equilibrium.

Proof

Notice that equation (3) can be expressed as

(5) $$\frac{f(p)}{\left[1-F(p)\right]^{\lambda}} = \frac{p^{\lambda}}{\mu} .$$

Integrating both sides of equation (5) from a nonnegative p_a to some $p^* > p_a$, we have

(6) $$\int_{p_a}^{p^*} \frac{f(p)}{[1-F(p)]^{\lambda}} dp = \int_{p_a}^{p^*} \frac{p^{\lambda}}{\mu} dp .$$

Making a change of variables on the left hand side of the above integral and defining c = F(p) and dc = f(p)dp, this becomes

(7) $$\int_{F(p_a)}^{F(p^*)} \frac{dc}{(1-c)^{\lambda}} = \int_{p_a}^{p^*} \frac{p^{\lambda}}{\mu} dp .$$

Note that the lower bound of the power-function price distribution must be zero because negative prices produce no profits, which contradicts the power function quantal response equilibrium in (2). Thus if p = 0, c = F(0) = 0. As p goes from 0 to p^*, c goes from 0 to $F(p^*)$. Assuming $\lambda \neq 1$, equation (7) can be integrated:

(8) $$-\frac{[1-c]^{1-\lambda}}{1-\lambda}\Bigg|_{0}^{F(p*)} = \frac{p^{\lambda+1}}{\mu(\lambda+1)}\Bigg|_{0}^{p*},$$

equation (8) also yields

(9) $$-\frac{[1-F(p)]^{1-\lambda}}{1-\lambda}+\frac{1}{1-\lambda} = \frac{p^{\lambda+1}}{\mu(\lambda+1)} - 0 .$$

Simplifying the notation, we obtain

(10) $$[1-F(p)]^{1-\lambda} = 1-\frac{(1-\lambda)}{\mu(1+\lambda)}p^{\lambda+1},$$

or equivalently.

(11) $$1-F(p) = \left[1-\frac{(1-\lambda)}{\mu(1+\lambda)}p^{\lambda+1}\right]^{\frac{1}{1-\lambda}}.$$

The price probability function is given by

(12) $$F(p) = 1-\left[1-\frac{(1-\lambda)}{\mu(1+\lambda)}p^{\lambda+1}\right]^{\frac{1}{1-\lambda}}.$$

where μ is a constant to be determined. Note that the lower boundary condition, F(0)=0, is satisfied. The other boundary condition, F(p)=1, in turn implies that $\mu = (1-\lambda)/(1+\lambda) > 0$. Thus replacing μ in (12), one obtains the power function cumulative probability and the corresponding density:

(13) $$F(p) = 1-\left[1-p^{\lambda+1}\right]^{\frac{1}{1-\lambda}}.$$

The corresponding price choice probability is

(14) $$f(p) = \frac{\lambda+1}{1-\lambda} p^{\lambda}\left[1-p^{\lambda+1}\right]^{\frac{\lambda}{1-\lambda}} .$$

It follows from (13) and from the definition of μ that the density in (14) satisfies the equilibrium condition in (2). As $\lambda \rightarrow 0$, $f(p) \rightarrow 1$, and the distribution of prices becomes the uniform distribution. Recall that the Nash equilibrium price is zero in this Bertrand model. The quantal response equilibrium in (13) therefore, produces systematic departures from the Nash equilibrium with $p>0$, even though the expected value of the error term is 0. The proposition below shows that the power function equilibrium price distribution, f(p), captures the extent to which a player's behavior deviates from the Nash equilibrium zero price outcome. To summarize:

Proposition 2

As the error rate decreases, $\lambda \rightarrow 1$, the power function cumulative probability converges to the Nash equilibrium: $F(p) \rightarrow 1$ for all $p > 0$.

Proof

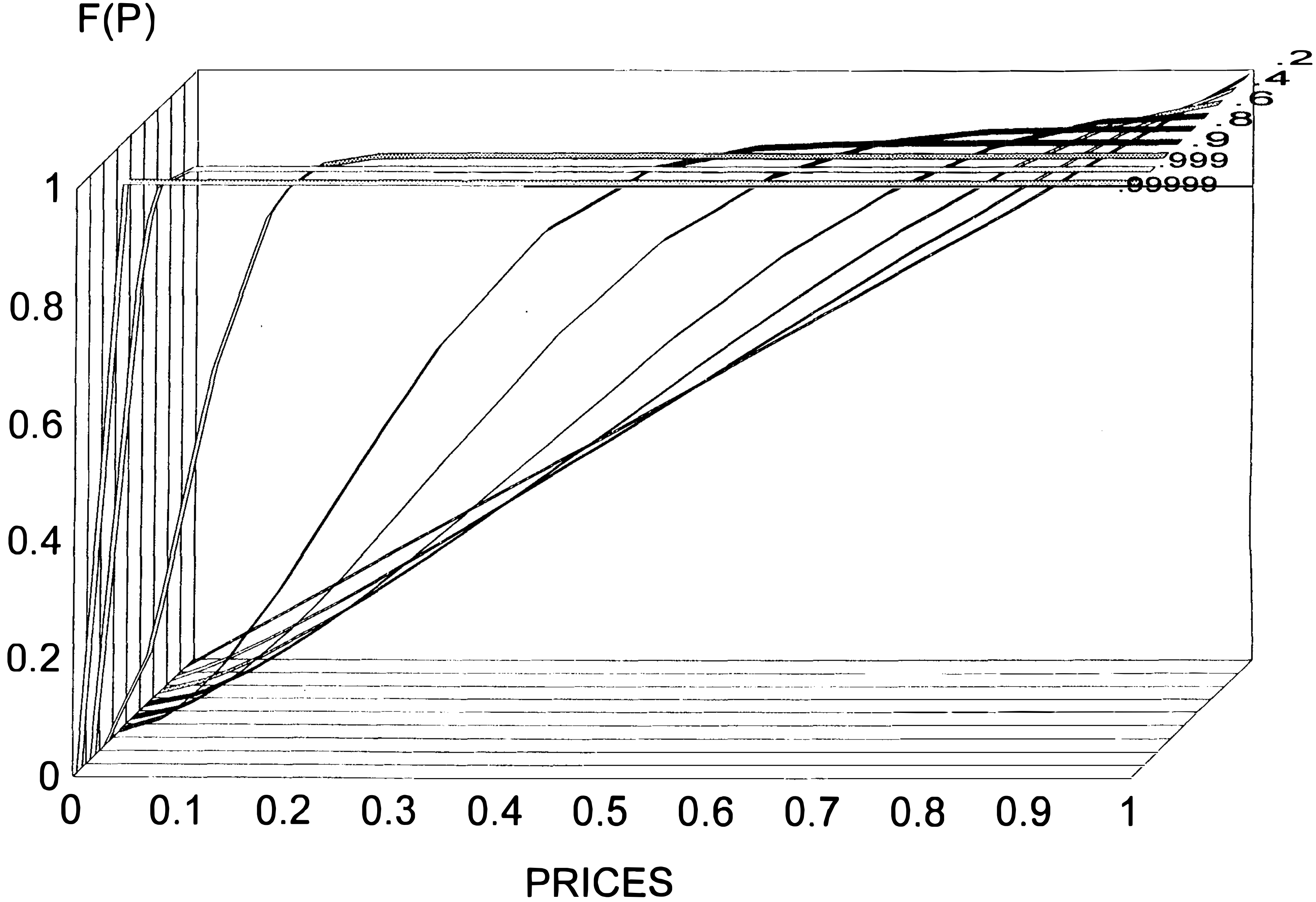
We need to show that, for any p value, $[\ 1 - p^{(1+\lambda)}\ ]^{1/1-\lambda}$ in equation (13) vanishes as $\lambda \rightarrow 1$. Consider $\lambda < 1$ and notice that $p^{(1+\lambda)} > p^2$ for all $p \varepsilon (0,1)$, so $1 - p^{(1+\lambda)} < 1 - p^2 < 1$. Hence $[1-p^{(1+\lambda)}]^{1/1-\lambda} < [\ 1 - p^2\ ]^{1/1-\lambda}$. Since $1 - p^2 < 1$ and the exponent, $1/1-\lambda$, goes to ∞ as $\lambda \rightarrow 1$, it follows that $[\ 1 - p^{(1+\lambda)}\ ]^{1/1-\lambda}$ converges to 0. Therefore $F(p) \rightarrow 1$.

Now consider $\lambda \geq 1$. The power-function quantal response equilibrium condition in equation (2) implies that a quantal response equilibrium is the degenerate distribution $F(p) = 1$ for $p \geq 0$.

For purpose of comparison with the Nash equilibrium, it is useful to illustrate proposition 2 in terms of a graph showing the relationship between the power-function quantal response equilibrium predictions and prices. In Figure 1 the power-function quantal response equilibrium distribution is measured along the vertical axis while the horizontal axis represent prices. The figure shows that as the error rate, $1/\lambda$, goes to 0 the mass of probability is concentrated in the range of prices between 0 and .1. The density is close to zero for all prices above .1.

The baseline Bertrand pricing game shown above is appealing because it delivers an explicit quantal response solution. The next chapters, however, show that the assumptions of a continuous price distribution and a power-function are not essential for a solution. Numeric methods are used in subsequent chapters for more complex examples, which involve different cost and demand structures.

FIGURE 4.1



Key:

4.3 CONCLUSIONS

This chapter used the power-function quantal response equilibria to model decision errors in a Bertrand duopoly model with continuous price choices. It is shown that a sufficient degree of errors is needed in order to break away from the Nash equilibrium. The calculation of the quantal response equilibria for this Bertrand game is more involved than the one presented for the model in the previous chapter. First, an explicit solution for the nonlinear differential equation in the price distribution in equation (2) must be derived. Given this solution, one uses the appropriate boundary conditions to obtain the support of the equilibrium price distribution. The power-function equilibrium price distribution derived in this chapter is appealing since it leads to a comparative statics result for the error-rate parameter. In chapter 5 this approach is used to examine the effects of market structure on the endogenous equilibrium price distributions. In addition, the methodoly derived in this chapter permits analysis of other models like the all-pay auction to be presented in chapter 6.

CHAPTER 5

QUANTAL RESPONSE EQUILIBRIA FOR POSTED-OFFER AUCTION MARKETS

5.1 INTRODUCTION

The Bertrand model describes competition among a group of price-setting sellers. The laboratory implementation of the Bertrand model is a posted-offer auction. In this institution, sellers submit prices simultaneously and then randomly designated buyers purchase at the posted prices. The Bertrand-Nash equilibrium will differ from the competitive equilibrium when sellers set prices above the competitive level. In such situations, experimental evidence indicates that competitive equilibrium pricing, Edgeworth cycles in prices and mixed-strategy Nash equilibrium are not completely consistent with experimental data (Brown-Kruse et al.,1993). Brown-Kruse et al. report (1993) that average seller price decreases over time for the first 20 periods of the experiment. However, with the exception of two experiments, prices do not converge to the competitive equilibrium over time. Observed pricing does not conform to Edgeworth cycle theory although experiments exhibit upward and downward price swings of the sort predicted by the Edgeworth cycle theory. Also observed pricing is not consistent with the mixed-strategy Nash equilibrium distribution. Average prices tend to exceed predicted mixed-strategy Nash equilibrium prices although price dispersion for aggregate data is similar to the dispersion predicted by the mixed-strategy Nash equilibrium (Holt and Davis, 1994 and Brown-Kruse et al., 1993). Another empirical feature of experimental models of price competition is that market models that share identical Nash equilibrium often exhibit different average prices. In particular, market models with an increasing costs exhibit higher average prices than market models with constant costs (Holt and Davis, 1990). Certain factors have been associated with systematic price deviations from Bertrand-Nash equilibrium in posted offer markets: cost structure, low excess supply at prices above the competitive price, small numbers of sellers, and market power (Davis and Williams, 1990, Wellford et al., 1990, and Davis and Holt, 1994, Brown-Kruse et al.,1993).

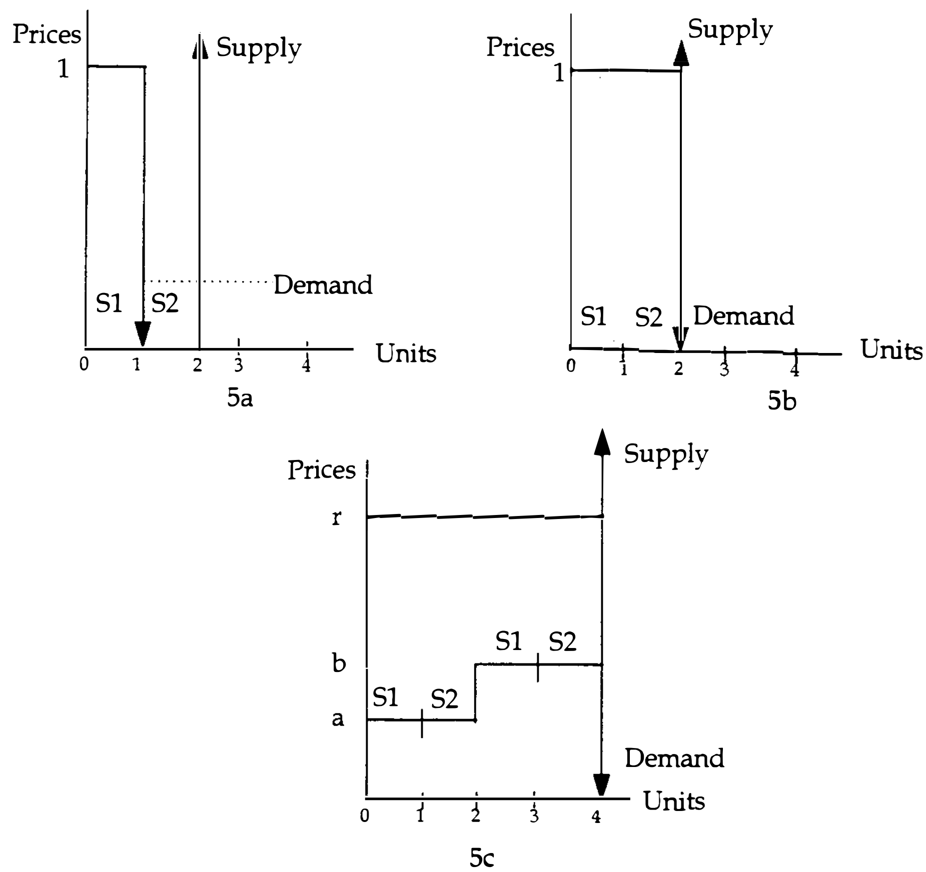
This chapter uses the quantal response equilibrium to model behavior in posted offer markets. The objective is to derive testable propositions about the effects of changes in market structure such as cost structure, market power and seller concentration on equilibrium price distributions. In the rest of this chapter, we study two particular parametric classes of quantal response functions: the logit and the power function. As shown in chapter 3, these functional forms differ in the error structure. In contrast to the logit, the power function turns out to be computationally convenient for a wide class of posted-offer markets. Consequently, the logit equilibrium is not computed for all the market designs to be presented in this chapter.

The chapter is structured as follows: In sections 5.1 and 5.2, the quantal response equilibrium is calculated for posted offer markets with severe capacity constraint that have one or more cost steps. In these designs, the competitive equilibrium price is the Nash equilibrium. An interesting feature of some of these models is that the quantal response equilibrium proves to be sensitive to changes in the cost and demand parameters that do not affect the Bertrand-Nash equilibrium. Sections 5.3 and 5.4 use the quantal response equilibrium to investigate the effects of market power on equilibrium prices. The Nash equilibrium in these markets involves mixed strategies. Specifically, section 5.3 examines a model with market power and constant marginal cost. In this model, we show that the Nash equilibrium in mixed-strategies and quantal response equilibria differ. However, this is not true in general. as it will be shown in the next chapter. Section 5.4 analyses the Nash and quantal response equilibria for continuous and for discrete price choices in a market power model with increasing costs. Section 5.5 investigates the effects of seller concentration on the quantal response equilibrium price distribution. This chapter ends with a summary of the main conclusions.

A BASIC MODEL WITH NO MARKET POWER

We begin with a review of the duopoly model from chapter 4, shown in figure 5a. Figures 5b and 5c are used in later games. Sellers' units are indicated on the market supply curve by designations. S1 and S2. for sellers 1 and 2 respectively. It is assumed that sellers choose prices simultaneously and share demand in the event of a tie. A well-

FIGURE 5.1


Prices
1
Supply
Demand
S1
S2
Units
0
1
2
3
4
5a
Prices
1
Supply
S1
S2
Demand
Units
0
1
2
3
4
5b
Prices
Supply
r
b
a
S1
S2
S1
S2
Demand
Units
0
1
2
3
4
5c

known result is that the Bertrand-Nash equilibrium is for both sellers to charge the competitive price. In this sense, sellers have no market power in this design.

The quantal response equilibrium for the model in figure 5a is characterized by a price distribution for each seller, F(p). Thus, F(p) is the probability that p is the highest price posted. A seller who chooses price p sells the unit with probability 1-F(p). The expected profit to a seller as a function of p is

(1) $$\pi(p) = p\,[1-F(p)]\ .$$

In the present market context, the power function decision rule with $\lambda > 0$ results in the following condition for a quantal response equilibrium:

(2) $$f(p) = \frac{(p\,[1-F(p)])^{\lambda}}{\mu}\ ,$$

$$\mu = \int_0^1 (x\,[1-F(x)])^{\lambda}\, dx\ ,$$

where μ is a constant, independent of p. The above equation parameterizes the set of possible equilibrium response functions f(p) with the parameter λ, which is inversely related to the level of error. For $\lambda < 1$, the equilibrium price distribution is

(3) $$F(p) = 1 - \left[1-p^{\lambda+1}\right]^{\frac{1}{1-\lambda}}\ ,$$

with the corresponding equilibrium price density:[1]

[1] It can be verified that equations (3) and (4) satisfy (2) with $\mu=(1-\lambda)/(1+\lambda)$.

(4) $$f(p) = \frac{\lambda+1}{1-\lambda} p^{\lambda} \left[1 - p^{\lambda+1}\right]^{\frac{\lambda}{1-\lambda}} .$$

It was shown in chapter 4 that, as λ goes to 1, all the mass of probability is concentrated on the set of prices near 0. This result is appealing, since the model thereby accounts for systematic deviations from the Bertrand-Nash equilibrium. [2]

5.2 A MODEL WITH SEVERE CAPACITY CONSTRAINTS AND CONSTANT COSTS

In the next market design, figure 5b, each seller's supply remains constant but demand is increased from 1 to 2 units at prices below 1. The range of competitive prices is from 0 to 1, since total capacity is equal to the total demand. The pure-strategy Nash equilibrium is for both sellers to charge a price of 1, the maximum competitive price. Thus no seller has an incentive to increase the price from the common maximum competitive price, since nothing can be sold at higher prices. At any price below 1, a unilateral price reduction lowers earnings from 1 to p.

Now assume that the sellers' best responses are probabilistic rather than deterministic. When a seller chooses a price of p, the seller has the higher price with probability F(p) and the lower price with probability 1-F(p). Since a seller's unit always sells when $p < 1$, the earnings are

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[2] As noted in chapter 4, when $\lambda \geq 1$, one quantal response equilibrium is the degenerate distribution $F(p) = 1$ for $p \geq 0$.

(5) $$\pi(p) = p\ , \qquad\qquad for\ p < 1\ .$$

For any given $\lambda > 0$, the power function response probabilities are:

(6) $$f(p) = \frac{p^{\lambda}}{\mu}\ , \qquad\qquad for\ p < 1\ ,$$

$$\mu = \int_0^1 x^{\lambda}\, dx.$$

Integrating equation (6), one obtains the price distribution function:

(7) $$F(p) = \frac{p^{\lambda+1}}{(\lambda+1)\mu}\ , \qquad\qquad for\ p < 1\ .$$

The next task is to determine the constant μ from the analysis of boundary conditions. Let $\underline{p}$ and $\overline{p}$ denote the bounds of the support of the price distribution. To determine the upper bound, note that the seller's profits are 0 for $\overline{p} > 1$. This result in turn implies that the equilibrium density in (6) is 0 for $p > 1$. On the other hand, $\overline{p} < 1$ implies that the seller's expected payoff is strictly greater than 0 for $p > \overline{p}$. Since $f(p)$ is strictly positive by (6), this result contradicts the definition of $\overline{p}$. Therefore $F(1)=1$ by definition. The boundary conditions, $F(0)=0$ and $F(1)=1$, imply that $\mu = 1/(\lambda+1) > 0$. Substituting this result back in (7), one obtains the equilibrium price distribution:

(8) $$F(p) = p^{\lambda+1}\ , \qquad\qquad for\ p < 1\ .$$

In a quantal response equilibrium, sellers 1 and 2 post prices according to (8). As λ goes to ∞, the probability of choosing p goes to 0 for $p \epsilon (0,1)$ and all probability mass is concentrated on the Nash equilibrium price of 1. For finite λ, the result in (8) is appealing since the model predicts systematic deviations from the Nash equilibrium, despite the fact that the error structure is assumed to be unbiased.

By comparing this result with the one obtained for the Bertrand model with two price choices, it is interesting to note that in figure 5b a sufficient amount of decision error make individuals worse off. This is because errors lower the probability of choosing the price of 1.

In order to contrast the above result with an alternative quantal response function, we next analyze the model in figure 5b using the logistic formulation. If each seller uses the logistic quantal response function, then we have

(9)
$$f(p) = \frac{e^{\lambda p}}{\mu}, \qquad \textit{for } p < 1,$$
$$\mu = \int_0^1 e^{\lambda x}\, dx,$$

where μ is the integral over all prices. The corresponding price distribution is

(10)
$$F(p) = \frac{e^{\lambda p}}{\lambda\mu} + k, \qquad \textit{for } p < 1,$$

where k is a constant of integration. In order to obtain the equilibrium price distribution, μ and k must be determined. The notation used here is identical to the notation introduced previously. The boundary condition $F(0)=0$ in turn implies that $k = -1/\lambda\mu$. Consider the upper bound $\bar{p}$. From (9), the probability density is positive for $\bar{p} > 1$.

On the other hand, $\bar{p} < 1$ implies that the seller's expected payoff is strictly greater than 0 for $p > \bar{p}$. Since f(p) is strictly positive by (9), this result contradicts the definition of $\bar{p}$. Therefore $F(\bar{p})=1$. Then, it follows from (10) with $F(1)=1$ and $k=-1/\lambda\mu$ that $\mu = (e^{\lambda}-1)/\lambda$. Substituting the solutions for μ and k back into equation (10), one obtains the quantal response equilibrium distribution function:

(11) $$F(p) = \frac{e^{\lambda p} - 1}{e^{\lambda} - 1}, \qquad for\ p < 1 .$$

In order to evaluate (11) as λ goes to ∞, we apply L'Hopital's rule. Differentiating both parts of the fraction with respect to λ and taking the limit, one obtains

(12) $$\lim_{\lambda\to\infty} F(p) = \lim_{\lambda\to\infty} \frac{pe^{\lambda p}}{e^{\lambda}} = \lim_{\lambda\to\infty} p\, e^{\lambda[p-1]} = 0 ,$$

since $p < 1$. This result implies that as the error rate, $1/\lambda$, goes to 0, sellers behave as conventional Nash players. Similarly, L'Hospital's rule can be used to show that as $1/\lambda$ goes to ∞, F(p) goes to p and prices are uniformly distributed.

In summary, in the model in figure 5b, the power function and the logit equilibrium predict systematic departures from the Bertrand-Nash equilibrium for finite error parameters, and convergence to the Nash equilibrium as the errors vanish.

5.3 A MODEL WITH SEVERE CAPACITY CONSTRAINTS AND INCREASING COSTS

Consider a more complex market model, figure 5c, where each seller has 1 unit with a low cost denoted by a and 1 unit with a high cost denoted by b, with a < b < r. Here r is the reservation price. The market demand is rectangular with 4 units demanded for prices below r. The intersection of the high marginal cost with demand determines the range of competitive prices, [b,r]. The Nash equilibrium is the highest competitive price, r. Clearly, a seller posting a price above r earns 0 profits, while a unilateral price reduction does not increase sales. When demand is divided equally at the Nash price, each seller sells 2 units.

Although the market designs 5b and 5c may share identical Nash equilibrium, in experiments different average prices are observed. The obvious question is why the presence of the two-cost step structure should have any effect on equilibrium price. In light of this experimental result, we next compute the quantal response equilibrium for figure 5c. The calculation involves two parts, distinguished by the relation of price to the high cost, b. Note that for any p below the high cost step, b, both sellers sell 1 unit with probability 1, so

(13) $$\pi(p) = (p - a), \qquad \textit{for } p \in [a, b) \; .$$

Similarly, both sell 2 units for prices above the high cost step b:

(14) $$\pi(p) = (2p - b - a), \qquad for\ p \in [b, r].$$

POWER FUNCTION RESPONSES

In this section the power function and logistic responses will be compared. First consider the power function formulation. For any given $\lambda > 0$, the power-function conditions for a quantal response equilibrium are:

(15)
$$f(p) = \frac{(p-a)^{\lambda}}{\mu}, \qquad for\ p \in [a,b),$$
$$f(p) = \frac{(2p-b-a)^{\lambda}}{\mu}, \qquad for\ p \in [b,r].$$

Notice that the densities in (15) must integrate to 1. Hence μ is written as

(16) $$\mu = \int_a^b (x-a)^{\lambda}\, dx + \int_b^r (2x-b-a)^{\lambda}\, dx .$$

Integrating the densities in (15), we have

(17)
$$F(p) = \frac{(p-a)^{\lambda+1}}{\mu(\lambda+1)} + k_1, \qquad for\ p \in [a, b),$$
$$F(p) = \frac{(2p-b-a)^{\lambda+1}}{2\mu(\lambda+1)} + k_2, \qquad for\ p \in [b, r],$$

where k_1 and k_2 are the constants of integration. Note that $F(a)=0$ implies $k_1=0$. The constant k_2 is chosen so that $F(b)^+=F(b)^-$. It follows from (17) that $k_2=(b-a)^{\lambda+1}/2\mu(\lambda+1)$. The constant μ is next determined. First, consider the upper bound $\overline{p}$. Equation (15) implies that the probability density is 0 for $p > r$ since the expected payoff is 0 for prices in this range. Therefore it must be the case that F(r) = 1. Substituting this result back into (17) evaluated at p=r and using the formula for k_2, it can be shown that $\mu = [(2r-b-a)^{\lambda+1} + (b-a)^{\lambda+1}]/2(\lambda+1)$. Substituting the formula for μ back into (17), it follows that the equilibrium probability functions are written as:

(18)

$$F(p) = \frac{2(p-a)^{\lambda+1}}{(2r-a-b)^{\lambda+1} + (b-a)^{\lambda+1}}, \qquad for\ p \in [a, b)\ ,$$

$$F(p) = \frac{(2p-b-a)^{\lambda+1} + (b-a)^{\lambda+1}}{(2r-a-b)^{\lambda+1} + (b-a)^{\lambda+1}}, \qquad for\ p \in [b, r]\ ,$$

with the corresponding equilibrium probability densities:

(19)

$$f(p) = \frac{2(p-a)^{\lambda}(\lambda+1)}{(2r-b-a)^{\lambda+1} + (b-a)^{\lambda+1}}, \qquad for\ p \in [a,b)\ ,$$

$$f(p) = \frac{2(2p-b-a)^{\lambda}(\lambda+1)}{(2r-b-a)^{\lambda+1} + (b-a)^{\lambda+1}}, \qquad for\ p \in [b,r]\ .$$

It follows from equation (18) and from the definition of μ that equations (19) satisfy the quantal response equilibrium conditions in (15). As λ goes to 0, F(p) goes to (p-a)/(r-a), which is a uniform distribution resulting from maximal decision error. Next we apply L'Hospital's rule to evaluate (18) as λ goes to ∞. Differentiating both parts of the fraction in (18) with respect to λ, for $p \epsilon [a,b)$, we have

(20) $$\lim_{\lambda \to \infty} F(p) = \lim_{\lambda \to \infty} \frac{\ln[2(p-a)] e^{[\lambda+1]\ln[2(p-a)]}}{\ln(2r-b-a) e^{[\lambda+1]\ln(2r-b-a)} + \ln(b-a) e^{[\lambda+1]\ln(b-a)}}, \qquad for\ p \in [a,b)\,.$$

Dividing all the terms of the above equation by $e^{[\lambda+1]\ln(2r-b-a)}$ yields

$$\text{(21)} \qquad \lim_{\lambda \to \infty} F(p) = \lim_{\lambda \to \infty} \frac{\ln[2(p-a)]e^{[\lambda+1][\ln[2(p-a)] - \ln(2r-b-a)]}}{\ln(2r-b-a) + \ln(b-a)e^{[\lambda+1][\ln(b-a) - \ln(2r-b-a)]}}, \qquad for\; p \in [a,b).$$

If the power in the exponent function in the numerator in (21) is negative then as λ goes to ∞, F(p) goes to 0. The power in the exponent is negative when ln[2(p-a)] - ln (2r-b-a) < 0, or equivalently, 2(r-p) $>$ b-a. Given pϵ[a,b), a sufficient condition for the numerator in (21) to approach 0 as λ goes to ∞ is that (r-a) $>$ (b-a)/2, which is clearly true. Therefore, as λ goes to ∞, F(p) goes to 0 and all of the probability is on the upper price range, where the Nash equilibrium is located.

Applying L'Hopital's rule and hence differentiating both parts of the fraction in (18) with respect to λ for pϵ[b,r], we have

$$\text{(22)} \qquad \lim_{\lambda \to \infty} F(p) = \lim_{\lambda \to \infty} \frac{\ln(2p-b-a)e^{[\lambda+1]\ln(2p-b-a)} + \ln(b-a)e^{[\lambda+1]\ln(b-a)}}{\ln(2r-b-a)e^{[\lambda+1]\ln(2r-b-a)} + \ln(b-a)e^{[\lambda+1]\ln(b-a)}}, \qquad for\; p \in [b, r].$$

Dividing all the terms in the above equation by $e^{[\lambda+1]\ln(2r-b-a)}$ yields

$$\text{(23)} \quad \lim_{\lambda \to \infty} F(p) = \lim_{\lambda \to \infty} \frac{\ln(2p-b-a)e^{[\lambda+1][\ln(2p-b-a) - \ln(2r-b-a)]} + \ln(b-a)e^{[\lambda+1][\ln(b-a) - \ln(2r-b-a)]}}{\ln(2r-b-a) + \ln(b-a)e^{[\lambda+1][\ln(b-a) - \ln(2r-b-a)]}}, \qquad for\; p \in [b, r].$$

If the powers in the exponent functions in the numerator in (23) are negative, then as λ goes to ∞, F(p) goes to 0. The powers are negative for when ln(2p-b-a) -ln(2r-b-a) $<$ 0 and ln(b-a) - ln(2r-b-a) < 0. The first inequality is true since r $>$ p. The second inequality is also true since r $>$ b. Therefore, F(p) goes to 0 as λ goes to ∞, and the price distribution converges to the Nash equilibrium price of r.

Another interesting property of this model is that a change in the cost structure does not alter the Nash equilibrium as long as b remains below r. However, a change in the cost parameters may alter the price distribution in a quantal response equilibrium. Next, we examine how changes in the cost parameters affect the quantal response equilibrium. As before, each price range must be considered separately. For $p \epsilon [a,b)$, the first partial derivatives of F(p) in the top part of (18) with respect to a and b are:

$$\text{(24)} \quad i)\ \frac{\partial F(p)}{\partial a} = \frac{2(\lambda+1)(p-a)^{\lambda}[(2r-b-a)^{\lambda}(p-2r+b)+(p-b)(b-a)^{\lambda}]}{[(2r-a-b)^{\lambda+1}+(b-a)^{\lambda+1}]^2} < 0, \qquad for\ p \in [a,b),$$

$$ii)\ \frac{\partial F(p)}{\partial b} = -\frac{2(\lambda+1)(p-a)^{\lambda+1}[(b-a)^{\lambda}-(2r-b-a)^{\lambda}]}{[(2r-a-b)^{\lambda+1}+(b-a)^{\lambda+1}]^2} > 0 \qquad for\ p \in [a,b).$$

where the inequality claims are next verified. The sign of the equation 24(i) is negative if p-2r+b < 0 and p-b < 0, which is true since p < b and b < r. Thus, an increase in a *decreases* F(p). The sign of the equation 24(ii) is positive since (2r-b-a) > b-a, or equivalently, r > b. Hence, an increase in b *increases* F(p) on [a,b).

For $p \epsilon [b,r]$, the first partial derivative of F(p) with respect to a is

$$\text{(25)} \quad \frac{\partial F(p)}{\partial a} = \frac{2(\lambda+1)[(b-a)^{\lambda}[(2r-a-b)^{\lambda}(b-r)+(2p-b-a)^{\lambda}(p-b)]+(2r-b-a)^{\lambda}(2p-b-a)^{\lambda}(p-r)]}{\omega} < 0, \qquad for\ p \in [b,r),$$

where the denominator of (25) is given by $w=[(2r-a-b)^{\lambda+1} + (b-a)^{\lambda+1}]^2$. The sign in equation (25) is negative if the following is true

$$(26) \qquad (2r-a-b)^{\lambda}(b-r) > (2p-a-b)^{\lambda}(p-b)\ , \qquad for\ p \in [b,r)\ ,$$
$$\left(\frac{2r-a-b}{2p-a-b}\right)^{\lambda} > \left(\frac{p-b}{b-r}\right).$$

Notice that (2r-a-b)/(2p-a-b) > 1 in (26) since r > p. On the other hand, the term (p-b)/(b-r) is always negative since p > b and r > b. Therefore, an increases in a *decreases* F(p) on [b,r). The first partial derivative of F(p) with respect to the high cost unit b yields

$$(27) \quad \frac{\partial F(p)}{\partial b} = \frac{2(\lambda+1)\left[(b-a)^{\lambda}\left[(2r-a-b)^{\lambda}(r-a) + (2p-b-a)^{\lambda}(a-p)\right] + (2r-b-a)^{\lambda}(2p-b-a)^{\lambda}(p-r)\right]}{\omega} > 0 \qquad for\ p \in [b,r)\ .$$

The sign in equation (27) is positive if $(2r-b-a)^{\lambda}/(2p-a-b)^{\lambda}$ > (a-p)/(r-a). Since r > p and p > a, it follows that the left hand side of the inequality is positive, and the right hand side is negative. Therefore, an increase in b increases F(p) in (27). In order to contrast our findings with an alternative quantal response specification, we next consider the logistic quantal response equilibrium.

LOGISTIC RESPONSES

The logistic equilibrium for the model in figure 5c is next examined. Recall that the calculation of the quantal response equilibrium for this model involves two parts. For any $\lambda > 0$, the conditions for a quantal response equilibrium are given by

where μ is the integral over all prices.

$$(29) \qquad \mu = \int_a^b e^{\lambda(x-a)}\, dx + \int_b^r e^{\lambda(2x-b-a)}\, dx\ .$$

$$(28)\qquad f(p) = \frac{e^{\lambda(p-a)}}{\mu}, \qquad for\ p \in [a,b),$$
$$f(p) = \frac{e^{\lambda(2p-b-a)}}{\mu}, \qquad for\ p \in [b,r],$$

It is readily verified from (28) that the price distributions are

$$(30)\qquad F(p) = \frac{e^{\lambda(p-a)}}{\mu\lambda} + k_1, \qquad for\ p \in [a,b),$$
$$F(p) = \frac{e^{\lambda(2p-b-a)}}{2\mu\lambda} + k_2, \qquad for\ p \in [b,r].$$

The boundary condition F(a)=0 implies $k_1 = -1/\mu\lambda$. As before, k_2 is chosen so that $F(b)^+ = F(b)^-$. It follows from this condition that $k_2 = (e^{\lambda(b-a)}-2) / 2\mu\lambda$. The boundary condition F(r)=1 in turn implies that $\mu = [e^{\lambda(2r-b-a)} + e^{\lambda(b-a)}-2]/2\lambda$. Substituting these results back into (30) we have

$$(31)\qquad i)\ F(p) = \frac{2[e^{\lambda(p-a)} - 1]}{e^{\lambda(2r-b-a)}+e^{\lambda(b-a)}-2}, \qquad for\ p \in [a,b),$$
$$ii)\ F(p) = \frac{e^{\lambda(2p-b-a)}+e^{\lambda(b-a)}-2}{e^{\lambda(2r-b-a)} + e^{\lambda(b-a)} - 2}, \qquad for\ p \in [b,r].$$

In order to determine the limit of F(p) as λ goes to 0, we next take the derivative with respect to λ of both parts of the fraction in equation 31(i):

(32) $$\lim_{\lambda\to 0} F(p) = \lim_{\lambda\to 0} \frac{2(p-a)e^{\lambda(p-a)}}{(2r-b-a)e^{\lambda(2r-b-a)} + (b-a)e^{\lambda(b-a)}}, \qquad for\ p \in [a,b).$$

From (32), as λ goes to 0, F(p) goes to (p-a)/(r-a), and prices are uniformly distributed. To obtain the limit of F(p) as λ goes to ∞, we divide both parts of the fraction in (32) by $e^{\lambda(2r-b-a)}$ to obtain:

(33) $$\lim_{\lambda\to\infty} F(p) = \lim_{\lambda\to\infty} \frac{2(p-a)e^{\lambda(p-2r+b)}}{(2r-b-a)+(b-a)e^{2\lambda(b-r)}}, \qquad for\ p \in [a,b).$$

The power of the exponent in the numerator in (33) is negative if 2r > p+b, which is true since r > p and r > b. Then, as λ goes to ∞, F(p) goes to 0 on [a,b), which puts all mass in the upper range where the Nash price is located. Similarly, applying l'Hopital's rule to the equation 31(ii) yields

(34) $$\lim_{\lambda\to 0} F(p) = \lim_{\lambda\to 0} \frac{(2p-b-a)e^{\lambda(2p-b-a)}+(b-a)e^{\lambda(b-a)}}{(2r-b-a)e^{\lambda(2r-b-a)}+(b-a)e^{\lambda(b-a)}}, \qquad for\ p \in [b,r].$$

It follows from (34) that as λ goes to 0, F(p) goes to the uniform distribution: (p-a)/(r-a). Next, we evaluate (34) as λ goes to ∞. By dividing both the numerator and denominator of (34) by $e^{\lambda(2r-b-a)}$, one obtains

(35) $$\lim_{\lambda\to\infty} F(p) = \lim_{\lambda\to\infty} \frac{(2p-b-a)e^{\lambda 2(p-r)}+(b-a)e^{-\lambda 2r}}{(2r-b-a)+(b-a)e^{-\lambda 2r}}, \qquad for\ p \in [b,r].$$

Notice that the power of the first exponent in the numerator in (35) is negative since r > p. Hence, as λ goes to ∞, the numerator in (35) goes to 0. Therefore F(p) goes to

0 on the upper range, and the distribution converges to one that puts all mass at the Nash price of r.

As before, we examine a change in the cost parameters on F(p). For $p \epsilon [a,b)$, the first partial derivatives of F(p) with respect to a and b are

(36)
$$\frac{\partial F(p)}{\partial a} = \frac{4\lambda e^{\lambda(p-a)} - 2\lambda(e^{\lambda(2r-b-a)} + e^{\lambda(b-a)})}{[e^{\lambda(2r-a-b)} + e^{\lambda(b-a)} - 2]^2} < 0, \qquad for\, p \in [a,b),$$
$$\frac{\partial F(p)}{\partial b} = \frac{-2\lambda[e^{\lambda(p-a)} - 1](e^{\lambda(b-a)} - e^{\lambda(2r-b-a)})}{[e^{\lambda(2r-a-b)} + e^{\lambda(b-a)} - 2]^2} > 0, \qquad for\, p \in [a,b).$$

where the inequality claims in (36) are verified next. The numerator of the first equation in (36) is negative if $4\lambda e^{\lambda(p-a)} < 2\lambda(e^{\lambda(2r-a-b)} + e^{\lambda(b-a)})$. Dividing both sides of the inequality by $2\lambda e^{\lambda(p-a)}$ yields $2 < e^{\lambda(2r-p-b)} + e^{\lambda(b-p)}$. The powers in the exponents are positive since $r > b > p$ and $\lambda > 0$. It follows from this result that $\partial F(p)/\partial a < 0$ on [a,b). The sign of the right side of the bottom equation in (36) is positive if $e^{\lambda(p-a)} > 1$ and $e^{\lambda(b-a)} < e^{\lambda(2r-b-a)}$, which is true since $p > a$ and $r > b$.

For $p \epsilon [b,r]$, the first partial derivatives of F(p) with respect to a and b are

(37)
$$\frac{\partial F(p)}{\partial a} = \frac{2\lambda[e^{\lambda(2p-b-a)} - e^{\lambda(2r-b-a)}]}{[e^{\lambda(2r-b-a)} + e^{\lambda(b-a)} - 2]^2} < 0, \qquad for\, p \in [b,r),$$
$$\frac{\partial F(p)}{\partial b} = \frac{2\lambda[e^{\lambda(b-a)} - 1](e^{\lambda(2r-b-a)} - e^{\lambda(2p-b-a)})}{[e^{\lambda(2r-a-b)} + e^{\lambda(b-a)} - 2]^2} > 0, \qquad for\, p \in [b,r),$$

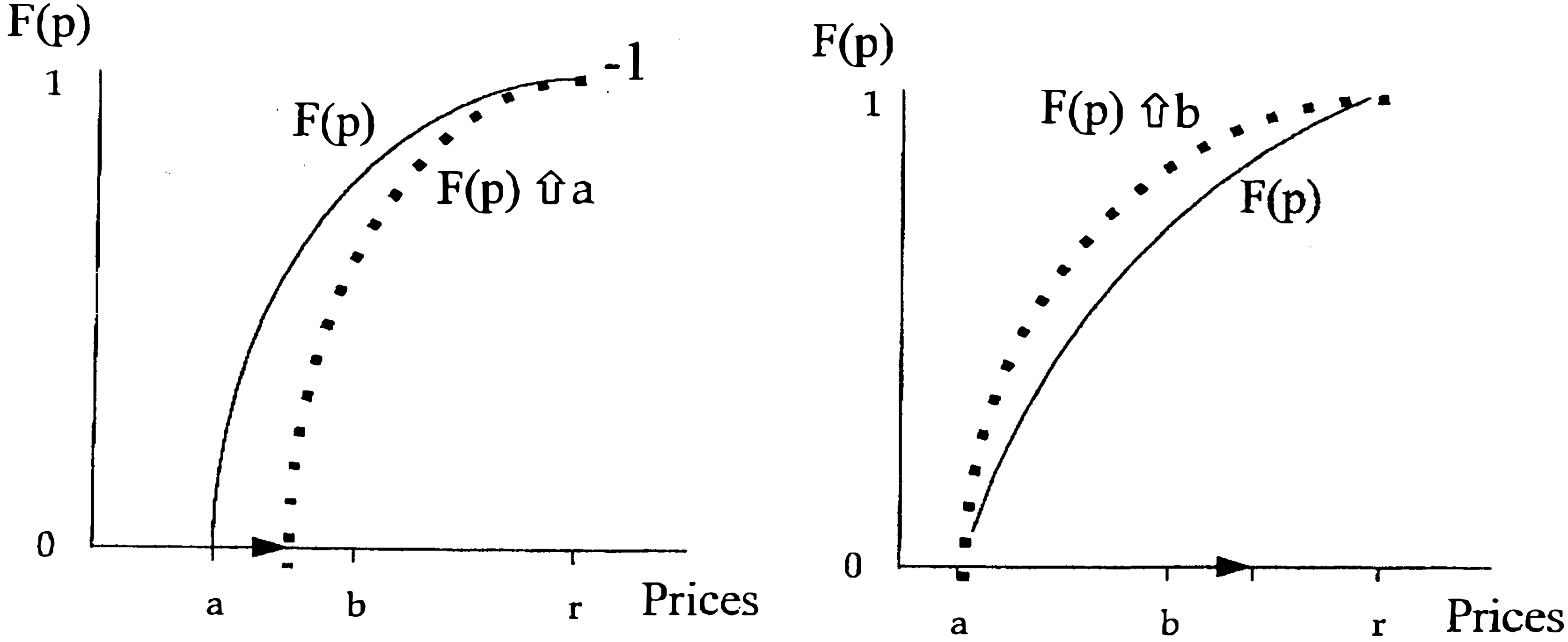
where the inequality claims in (37) are verified next. The sign of the right side of the first equation in (37) is negative since $e^{\lambda(2p-b-a)} < e^{\lambda(2r-b-a)}$, or equivalently, $r > p$. The bottom equation in (37) is positive if $e^{\lambda(b-a)} > 1$ and $e^{\lambda(2r-b-a)} > e^{\lambda(2p-b-a)}$, which is true since $b > a$ and $r > p$.

The comparative statics results are summarized in the table below:

	$p\epsilon[a,b)$		$p\epsilon[b,r)$	
	$\partial F(p)/\partial a$	$\partial F(p)/\partial b$	$\partial F(p)/\partial a$	$\partial F(p)/\partial b$
power function	−	+	−	+
logit	−	+	−	+

In summary, the model presented in this section has the feature that the Nash equilibrium is unaffected by changes in the cost structure as long as b remains below r. For the quantal response equilibria, the comparative statics results in the table above are depicted in Figure 5.2. An increase in the low-cost step stochastically raises prices in the whole range of prices. Thus, sellers in a quantal response equilibrium post stochastically higher prices when they face an increase in the low-cost step. By contrast, an increase in the high-cost step raises the distribution function for the whole range of prices. Hence, sellers post stochastically *lower* prices given an increase in the high-cost step. The intuition behind this last result is that an increase in b reduces profits for the second unit that is only sold at prices above b, which causes sellers to post stochastically lower prices in a quantal response equilibrium.

Figure 5.2



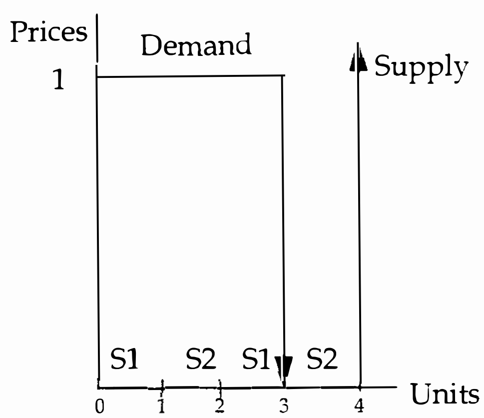
5.4 MARKET POWER AND CONSTANT COSTS

The market model in figure 5.3 illustrates the situation when there is excess supply and the Nash equilibrium involves randomization. To understand how randomization may arise, consider the duopoly model in figure 5d (Tirole, 1988). Each of the 2 sellers has the capacity to supply 2 units at 0 cost. The market quantity demanded is 3 units at any price less than or equal to 1, and 0 at any price above 1. Assume that sellers split the market in the case of ties. Further, suppose that only two prices can be posted. For example, if seller 1 offers 2 units at a price of 0 and seller 2 posts a price equal to 1/2, buyers would like to buy from seller 1. This seller will sell two units, netting a profit of 0. Seller 2 will face a residual demand of 1 unit and will net 1/2. Hence each seller has a unilateral incentive to raise price above a common competitive price of 0.

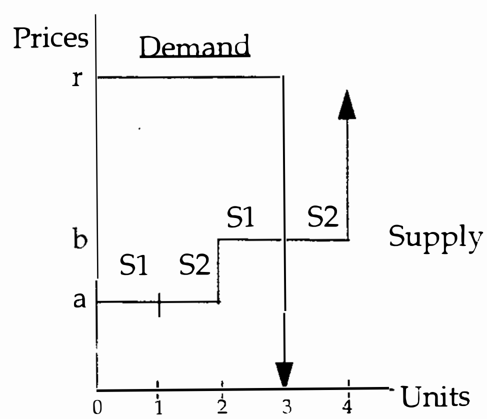
The Nash equilibrium will involve randomization.[3] The calculations are straightforward (see Holt and Solis-Soberon (1992) and the references therein). Given seller 2's capacity constraint, seller 1 can always obtain a safe payoff of 1 by charging the price of 1 and selling to the residual demand. For seller 1 to be indifferent between posting some arbitrary price p and the limit price 1, it must be the case that seller 2

[3] As can be verified, there is no equilibrium in pure strategies. There is randomization over the set of prices (1/2,1). For instance, if seller 1 posts a price of 1, seller 2's best response is to slightly cut this price and sell the 2 units. Then, seller 1's best response is to cut this price. This Edgeworth cycle of best responses continues until the price falls to 1/2. At this price, the expected payoff from selling one unit equals the expected payoff from selling 2 units at the price of 1.

FIGURE 5.3



5e



5f

prices according to a distribution F(p) that makes seller 1's expected earnings at p equal to the certain earnings. When seller 1 chooses a price of p, seller 1 has the highest price with probability F(p) and the lowest price with probability 1-F(p). Therefore, seller 1 sells 1 unit with probability F(p) and 2 units with probability 1-F(p). Hence the expected profit function for a seller as a function of p is

(38) $$\pi(p) = pF(p) + 2p[1 - F(p)].$$

In a mixed strategy Nash equilibrium, seller 1 must be indifferent among all prices over which randomization occurs. Hence, the distribution F(p) must equate the expected profit at each p in the support $[\underline{p},\bar{p}]$, to the certain profit of 1. The resulting equation yields

(39) $$F(p) = 2 - \frac{1}{p}, \qquad \textit{mixed Nash equilibrium},$$

with the corresponding probability density

(40) $$f(p) = \frac{1}{p^2}.$$

Next, we determine the upper and the lower bound of the price distribution. Notice that no price above the reservation price will be charged since the payoff to a seller is zero. From (39), the boundary condition $F(\underline{p})=0$ implies that $\underline{p}=1/2$.

In the analysis that follows, the Nash equilibrium is generalized to incorporate decision errors. First note that the expected profit in (38) can be expressed:

$\pi(p) = p[2-F(p)]$. Therefore, the power-function quantal response equilibrium condition is:

(41)
$$f(p) = \frac{\left[p(2 - F(p))\right]^{\lambda}}{\mu},$$
$$\mu = \int_{0}^{r} \left[x(2 - F(x))\right]^{\lambda} dx .$$

As before, μ is a constant independent of p. Before deriving the quantal response equilibrium price distribution, it is worth pointing out one interesting property of (41). By substituting (39) into the right side of (41), it follows that a quantal best response to the other seller's Nash equilibrium in mixed-strategies is the uniform distribution, $1/\mu$. This is because the expected profits are equal at all prices in a mixed-strategy Nash equilibrium. Hence, if the rival is using his Nash equilibrium, the seller's best response is to spread price decisions uniformly. This result shows why the quantal response equilibrium and Nash distribution cannot be the same when the Nash mixed distribution is not uniform to begin with.

To derive F(p) in (41), we integrate from 0 to some p^* to obtain

(42) $$\int_0^{p^*} \frac{f(p)}{[2-F(p)]^{\lambda}}\,dp = \int_0^{p^*} \frac{p^{\lambda}}{\mu}\,dp\ .$$

Let c = F(p) and dc = f(p)dp. As p goes from 0 to p^*, c goes from F(0)=0 to F(p^*). Assuming $\lambda \neq 1$,[4]

(43) $$-\ \frac{[2-c]^{1-\lambda}}{1-\lambda}\Bigg|_0^{F(p^*)} = \frac{p^{\lambda+1}}{\mu(\lambda+1)}\Bigg|_0^{p^*}\ .$$

Equation (43) is reexpressed as

(44) $$-\ \frac{[2-F(p^*)]^{1-\lambda}}{1-\lambda} + \frac{2^{1-\lambda}}{1-\lambda} = \frac{(p^*)^{\lambda+1}}{\mu(\lambda+1)} - 0\ .$$

To simplify the notation let p denote p^* and rearrange:

(45) $$[2-F(p)]^{1-\lambda} = 2^{1-\lambda} - \frac{(1-\lambda)}{\mu(1+\lambda)}p^{\lambda+1}\ .$$

It follows from (45) that the price distribution is

(46) $$F(p) = 2 - \left[2^{1-\lambda} - \frac{(1-\lambda)}{\mu(1+\lambda)}p^{\lambda+1}\right]^{\frac{1}{1-\lambda}}\ .$$

The next task is to determine μ. The boundary condition, F(1)=1, implies that the term in square brackets in (46) is equal to 1. Hence, $\mu=(1-\lambda)/(1+\lambda)(2^{1-\lambda}-1)$. Thus, replacing μ in (46) yields the power-function equilibrium distribution:

---

[4] If $\lambda=1$, the same method provides an explicit solution. The derivation of the price distribution is provided in the appendix at the end of this chapter.

(47) $$F(p) = 2 - \left[2^{1-\lambda} - (2^{1-\lambda} - 1)p^{\lambda+1}\right]^{\frac{1}{1-\lambda}} . \qquad Q.R.E.$$

The corresponding equilibrium density is

(48) $$f(p) = \frac{(2^{1-\lambda} - 1)(\lambda+1)p^{\lambda}}{(1-\lambda)}[2^{1-\lambda}-(2^{1-\lambda}-1)p^{\lambda+1}]^{\frac{\lambda}{1-\lambda}} .$$

It can be shown using (47) and the definition of μ that the density in (48) satisfies the equilibrium condition in (41). As λ goes to 0, F(p) goes to p and prices are uniformly distributed.

Experimental evidence indicates that observed pricing does not conform to the Edgeworth cycle theory. However, by comparing equations (54) and (62) it is readily verified that the Nash equilibrium distribution at the lower bound of the price support, $F_N(1/2)=0$, is less than the quantal response equilibrium distribution, $F_Q(1/2)$. Therefore, consider the conjecture that the Nash equilibrium distribution, $F_N(p)$, stochastically dominates (in terms of first degree dominance) the quantal response equilibrium, $F_Q(p)$, or equivalently, $F_N(p) < F_Q(p)$. From equations (54) and (62), $F_N(p) < F_Q(p)$ implies $1/p > [2^{1-\lambda}(1-p^{\lambda+1}) + p^{\lambda+1}]^{1/1-\lambda}$. Raising both sides of the inequality to the power $1-\lambda$ yields $p^{\lambda-1} > [2^{1-\lambda}(1-p^{\lambda+1}) + p^{\lambda+1}]$. Dividing by $p^{\lambda-1}$ and arranging terms we have $1 > p^{1-\lambda}2^{1-\lambda}(1-p^{1+\lambda})+p^2$. As λ goes to 1, the right hand side of the inequality goes to $2(1-p^2) + p^2 = 2-p^2$, which is greater than 1 for $p\epsilon(0,1)$. This result in turn contradicts the conjecture that the Nash equilibrium stochastically dominates the quantal response equilibrium.

FIGURE 5.4

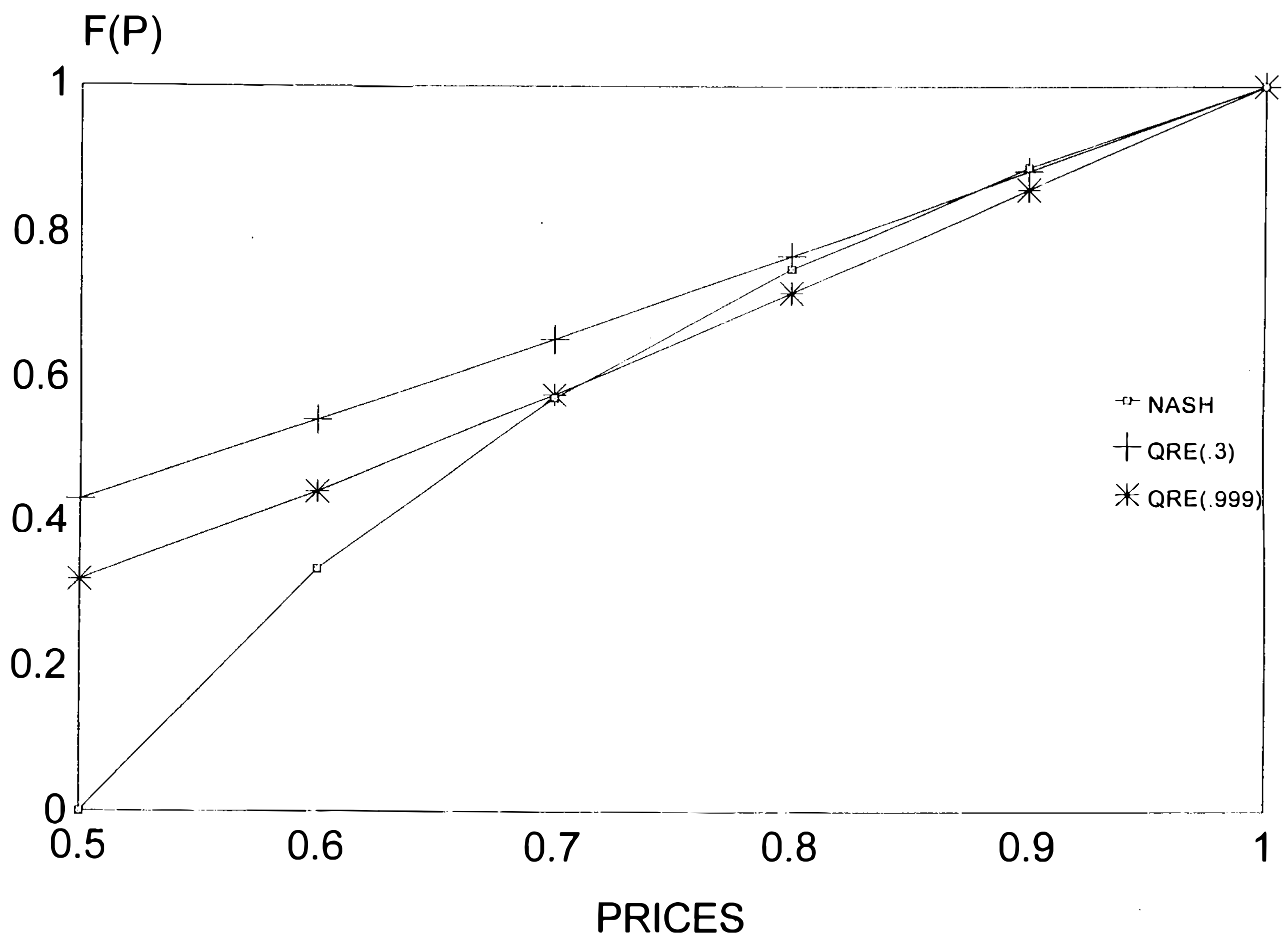


Figure 5.4 illustrates the relationship between the Nash and the quantal response equilibria predictions. In this figure the distribution functions are measured along the vertical axis, while the horizontal axis represent prices. The figure shows that as the error rate, $1/\lambda$, goes to 1, $F_Q(p)$ differs from $F_N(p)$. This figure illustrates the fact that in this model the Nash and quantal response equilibria differ. The best quantal response to the Nash equilibrium with equilibrium expected payoffs is to spread the probability uniformly across all price decisions. In some other models, however, the Nash and quantal response equilibria are identical. This is the case of the all-pay auction presented in the next chapter.

5.5 MARKET POWER AND INCREASING MARGINAL COSTS

Consider figure 5e. In the absence of the low-cost step a (if a is equal to b), the Nash equilibrium analysis for figure 5d would apply and sellers would randomize over an interval of prices that exceed b. The obvious question is whether the two-cost step structure can explain observed supracompetitive pricing (Holt and Davis, 1994). In light of some experimental results, we next examine the quantal response equilibria for the market power design in figure 5e.

In what follows the random variable p will be treated first as a continuous and then as a discrete variable. This is done because it is not possible to obtain a closed-form solution for the quantal response equilibrium with continuous price choices. However, the intuition gained from dealing with the continuous case is useful in dealing with the discrete set-up.[5] Numerical methods are used in the discrete case in order to assess the effect of market power on the equilibrium price distribution.

In figure 5e, each of the two sellers, S1 and S2, may offer a total of 2 units: 1 unit at a cost of a, and 1 unit at a per unit cost of b. A total of 3 units are demanded by a buyer at any price below the reservation value r. As in the previous example, neither seller has the capacity to meet the market demand, but there is excess industry capacity. The existence of a pure-strategy Nash equilibrium in this market design is precluded by incentives for both sellers to undercut from any common price. As before, the calculation of the mixed equilibrium involves equating expected payoffs to a constant. Let F(p) denote the continuous distribution of the other seller's price, i.e, F(p) is the probability that the other's price is less than p. A seller with a price p that is between b and r will earn a profit of p-a if the rival prices below p, and the seller will earn 2p-a-b otherwise. Then the expected profit function for p > b is written as

$$\text{(49)} \qquad \begin{aligned} \pi(p) &= F(p)(p-a)+[1-F(p)](2p-a-b) \\ &= 2p-a-b+F(p)(b-p) \ . \end{aligned}$$

For seller 1 to be willing to randomize over a range of prices, it must be the case that all prices in that range offer the same expected profit. At a price of r, the seller sells 1 unit and earns r-a for sure. Seller 1 will be indifferent between posting p and posting

[5] Holt and Solis-Soberon (1992) contain a detailed discussion of the calculation of the Nash equilibrium in posted-offer markets.

r if [1-F(p)](2p-a-b) + F(p)(p-a) = r-a. Solving for F(p), it follows that seller 2 must price so that

$$\textbf{(50)} \qquad F(p) = \frac{b+r-2p}{b-p}, \qquad for\, p \in (\frac{b+r}{2}, r) .$$

The above equation specifies the mixed distribution that seller 2 must use in order for seller 1 to be willing to choose randomly in a range of prices that yield equal expected profits. In equilibrium, both sellers randomize according to (50). The last step is to verify the lower bound of the price distribution in (50). At the lower bound, F(p)=0, so $\underline{p}=(b+r)/2$. The equilibrium distribution is bounded above by r.[6]

In laboratory experiments the set of allowable price decisions often is finite (e.g., pennies). In what follows, the mixed-strategy Nash equilibrium when prices are integer-valued is calculated. The equilibrium expected payoff, S, must satisfy (51) below. This equation is similar to the one for the continuous case. However, this equation also accounts for the payoff function that determines earnings when a seller's price matches the other's price. At this price, demand is divided equally, so each seller earns the average profit: [(2p-a-b) + (p-a)]/2. The density, $f(p_i)$, is the equilibrium probability that a price selected is p_i, $f(p_i) \geq 0$ for $p_i=p_1,..,r$:

[6] By equating the distribution in (50) to .5, one obtains the median of the mixed distribution, p = b + (2r)/3.

$$\text{(51)} \qquad S = \left[\sum_{p_i=p_1}^{p_{k-1}} f(p_i)\right](p_k - a) + f(p_k)\frac{[(2p_k - a - b) + (p_k - a)]}{2} + \left[1 - \sum_{p_i=p_1}^{p_k} f(p_i)\right](2p_k - a - b) \ ,$$

where $p_k = p_1,..,r$. Equation (51) can also be expressed as

$$\text{(52)} \qquad S = \left[\sum_{p_i=p_1}^{p_{k-1}} f(p_i) + \frac{f(p_k)}{2}\right](p_k - a) + \left[1 - \left[\sum_{p_i=p_1}^{p_{k-1}} f(p_i) + \frac{f(p_k)}{2}\right]\right](2p_k - a - b) \ .$$

The $G(p_k)$ in equation (52) is a *modified* "distribution function", that allows for the event of ties:

$$\text{(53)} \qquad G(p_k) = \sum_{p_i=p_1}^{p_{k-1}} f(p_i) + \frac{f(p_k)}{2} = \frac{(2p_k - a - b) - S}{p_k - b} \ ,$$

where the final equation follows from (52). In order to obtain the support of the equilibrium mixed-strategy Nash equilibrium, consider a set of consecutive integer-valued prices: $[p_1,\dots,r]$, where r is the largest price. Define p_L and p_H as the lowest and highest prices respectively that are selected with strictly positive probability, where $p_1 \leq p_L < p_H \leq r$. By evaluating (51) at p_H and using the fact that the sum of the densities up to $f(p_H)$ equals one, one obtains

$$\text{(54)} \qquad \begin{aligned} S &= [1 - f(p_H)]\,(p_H - a) + f(p_H)\left[\frac{(2p_H - a - b) + (p_H - a)}{2}\right] \\ &= (p_H - a) + \frac{f(p_H)}{2}[p_H - b] \ . \end{aligned}$$

Since $f(p_H) > 0$, $r \geq p_H$ and $r > b$, it follows from (54) that $S > p_H - a$. Now, we calculate the mixed equilibrium probabilities for this model. For example, suppose that $a=0$, $b=4$ and $r=9$. Conjecture that $f(p_k) = (r-b)/(b-p_k)^2$, with the upper bound $p_H = 9$ and the lower bound $p_L=7$, is a Nash equilibrium in mixed-strategies. In equilibrium, the seller must be indifferent between the prices 7, 8 and 9. Next, we verify that the seller has no incentive to deviate by choosing an outside price with positive probability. Using equation (52) and the conjecture, $f(p_k)=(r-b)/(b-p_k)^2$, one can show that $S_7 = S_8 = S_9 \approx 9.16$.[7] The equilibrium probabilities are: f(7)= 5/9, f(8)= 5/16 and f(9)= 5/25. The equilibrium distribution function that results is G(7)= 5/9, G(8)=125/144 and G(9)=1. The mixed-strategy Nash distribution function is depicted in Figure 5.5. In this figure the horizontal axis represents prices, while the distribution function is labeled on the vertical axis.

Now we examine sellers' best responses using the quantal response equilibrium. Let F(p) be the probability that a seller posts the highest price p. The calculation of the quantal response equilibrium for the model in figure 5e involves two parts. For a p between the low and the high cost, a seller only offers 1 unit, which always sells regardless of the other seller's price, so:

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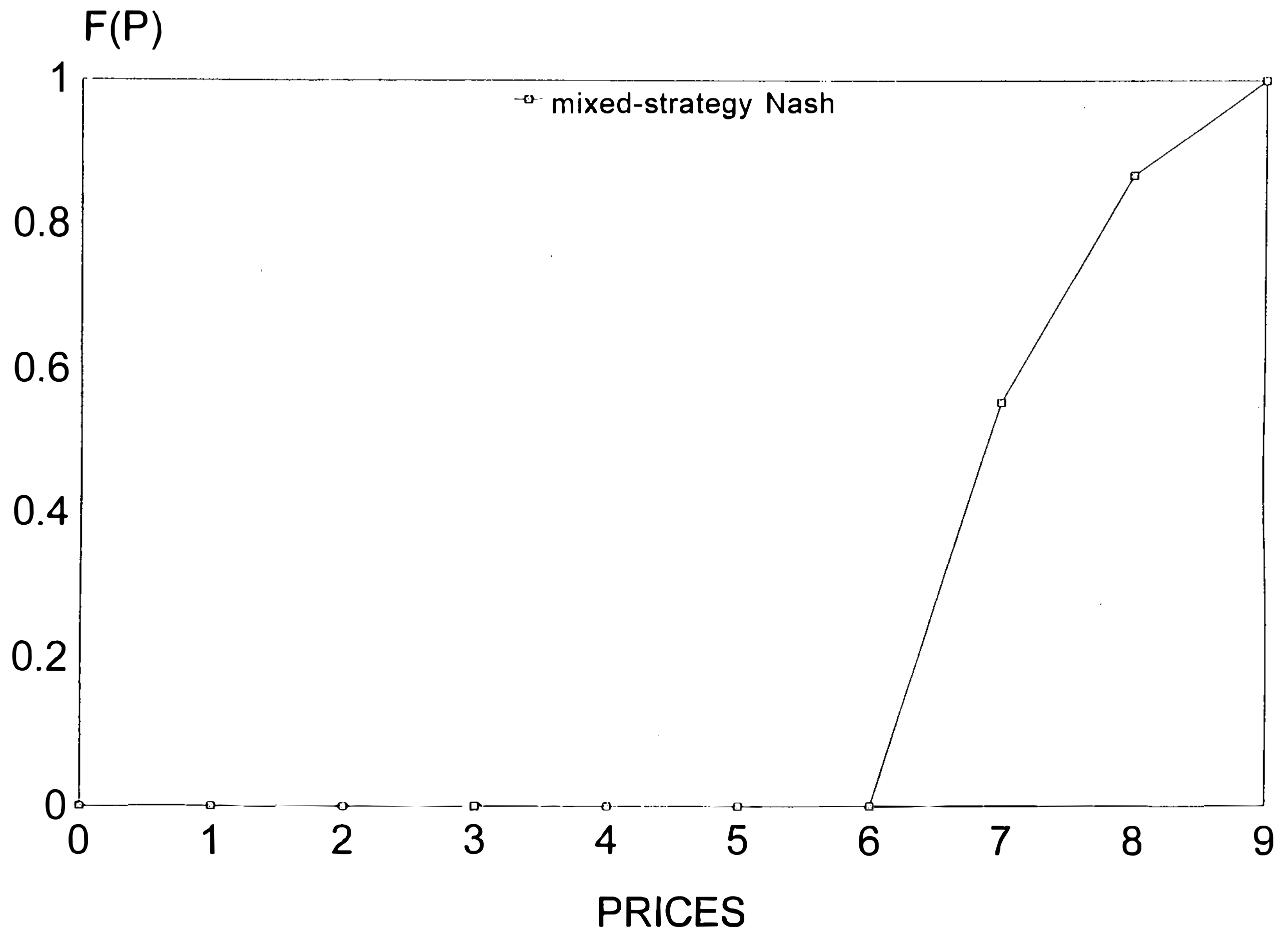
[7] These three calculations are:

$$S_7 = \frac{f(7)}{2}[(2*7-4-0)+(7-0)]+[1-f(7)](2*7-4-0),$$

$$S_8 = f(7)[8-0]+\frac{f(8)}{2}[(2*8-4-0)+(8-0)]+[1-f(7)-f(8)](2*8-4-0),$$

$$S_9 = [f(7)+f(8)+\frac{f(9)}{2}]((9-0)-(2*9-4)) + (2*9-4).$$

FIGURE 5.5



(55) $$\pi(p) = p - a, \qquad \textit{for } p \in [a,b) \ .$$

For a p between the high cost step b and the reservation price r, the seller always sells 1 unit at cost of a. When the seller charges p it may be that p is the smallest price posted. This happens if the seller's rival charges a price higher than p, an event which has probability [1- F(p)]. However, the seller also sells two units if the rival prices below the high-cost step b, this event has probability F(b). Hence the expected profit function for a seller as a function of p becomes

(56) $$\begin{aligned}\pi(p) &= (p-a) + [1 - F(p) + F(b)](2p-a-b), \qquad \textit{for } p \in [b,r]\,, \\ &= (p-a) + [z - F(p)]\,(2p-a-b)\,,\end{aligned}$$

where z=1-F(b) is a constant independent of p. The power-function quantal response equilibrium conditions are given by

(57) $$\begin{aligned}f(p) &= \frac{(p-a)^{\lambda}}{\mu}\,, \qquad \textit{for } p \in [a,b), \\ f(p) &= \frac{\big((p-a) + [z - F(p)](2p-a-b)\big)^{\lambda}}{\mu}\,, \qquad \textit{for } p \in [b,r]\ .\end{aligned}$$

Since the densities have to integrate to one, the constant μ is written as

(58) $$\mu = \int_a^b (x-a)^{\lambda}\,dx + \int_b^r \big((x-a) + [z - F(x)](2p-a-b) \big)^{\lambda}\,dx.$$

Notice that the calculation of F(p) for pϵ[a,b) parallels previous examples. However, for prices above b, an analytical solution cannot be found.

In order to determine the effects of market power on the price distribution, we next proceed to calculate the quantal response equilibrium when the price choices are discrete, and it is necessary to account for the event of ties. The notation used here is identical to the notation introduced previously. For any price between the low and the high-cost step, the seller's expected profit function is

$$\textbf{(59)} \qquad \pi(p_k) = (p_k - a)\ , \qquad \textit{for } p_k \in \{p_a, \ldots\ldots, p_{b-1}\}\ ,$$

where p_a is the price at the low-cost step, a. Similarly, p_b is the price at the high-cost step, b. Hence the expected profit function for a seller as a function of p_k is

$$\pi(p_k) = \left[\sum_{p_i = p_b}^{p_{k-1}} f(p_i) + \frac{f(p_k)}{2}\right](p_k - a) + \left[1 - \left[\sum_{p_i = p_b}^{p_{k-1}} f(p_i) + \frac{f(p_i)}{2}\right](2p_k - a - b)\right] + \sum_{p_i = p_a}^{p_{b-1}} f(p_i)\,(2p_k - a - b) \quad \textit{for } p_k \in \{p_b, .., r\}$$

(60)

The first term in equation (60) is the expected profit of a seller when his price is higher. The last term is the profit of a seller when the rival prices below b. The second term in (60) is the expected profit in the event of ties. The logit quantal response equilibrium conditions are shown in (61):

$$f(p_k) = \frac{e^{\lambda(p_k - a)}}{\mu}, \qquad \textit{for } p_k \in \{p_a, \ldots\ldots, p_{b-1}\}\ ,$$

(61)

$$f(p_k) = \frac{e^{\lambda(G(p_k)(p_k - a) + (1 - G(p_k))(2p_k - a - b) + F(p_{b-1})(2p_k - a - b))}}{\mu}, \qquad \textit{for } p_k \in \{p_b, \ldots\ldots, r\}\ ,$$

where $G(p_k)$ is the *modified* "distribution function" that allows for ties. Since the densities in (61) have to sum to one, the constant μ is written as

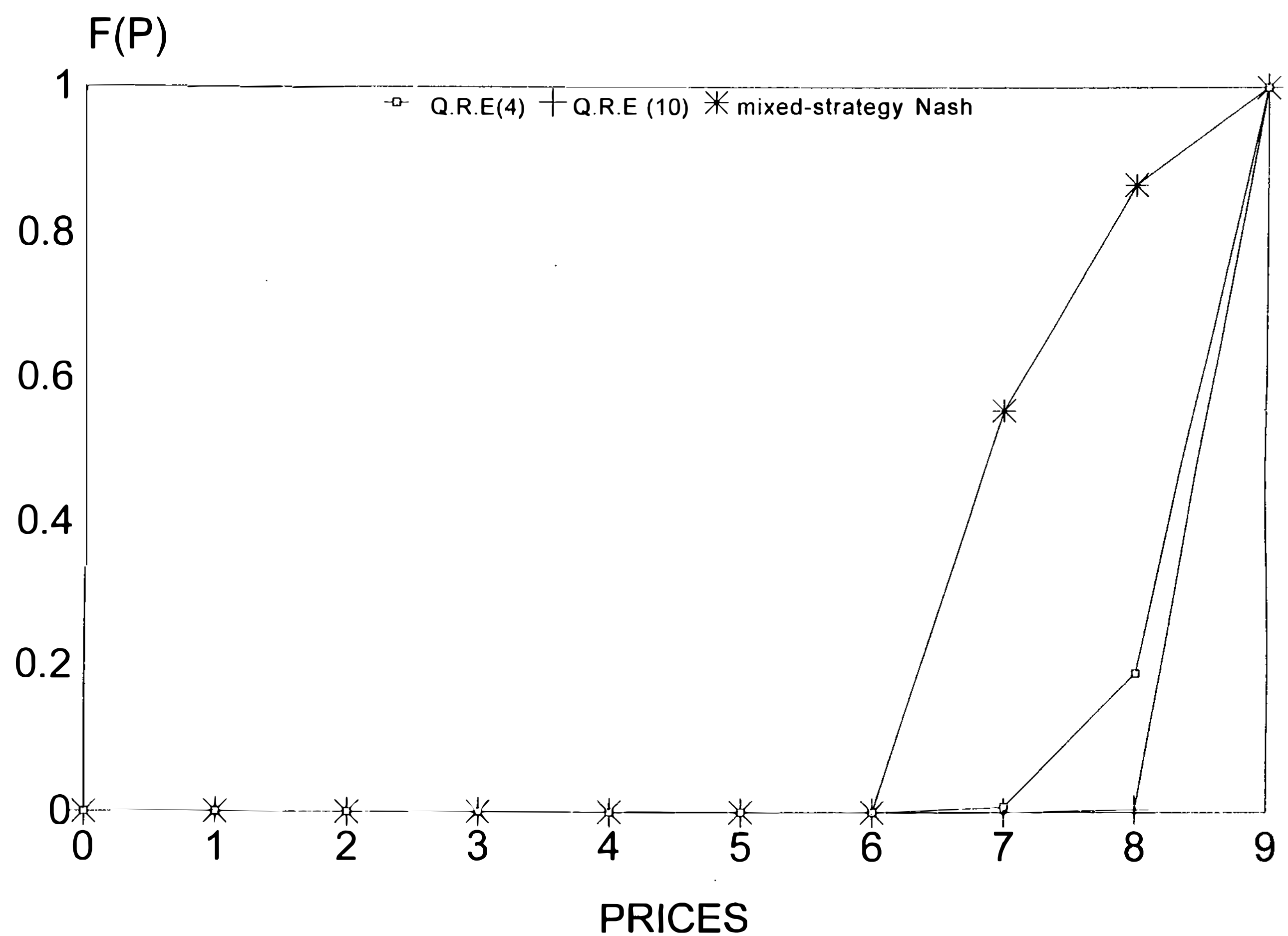
(62)

$$\mu = \sum_{x=a}^{b-1} e^{\lambda(x-a)} + \sum_{x=b}^{r} e^{\lambda[G(x)(x-a)+[1-G(x)](2x-a-b)+F(p_{b-1})(2x-a-b)]}.$$

Numerical methods are used to obtain the equilibrium price density, $f(p_k)$. The parametric values are the following: $p_k \epsilon \{0,1,2,..,9\}$, a= 0, b=4 and r=9. The quantal response function employed is the logit. The procedure for finding the equilibrium price densities is as follows. First, an initial uniform distribution of prices is provided. Using this distribution, values for the expected payoffs and the logistic probabilities are computed. Given the updated values of the function probability distribution, new values of the expected payoffs are computed and the procedure is repeated until the probability distribution converges.

The Nash and quantal response equilibrium price distributions are illustrated in Figure 5.6. In this figure the distribution functions are indicated in the vertical axis, while prices are represented in the horizontal axis. The quantal response equilibrium is plotted for different error rates. The upper and lower bound of the mixed-strategy Nash equilibrium is 9 and 7 respectively. The figure shows that as the error rate, $1/\lambda$, decreases more of the mass of probability in a quantal response equilibrium is concentrated in the reservation price of 9. In the Nash equilibrium, however, the mass of probability is concentrated near the lower bound of the mixed-strategy Nash equilibrium. With respect to the equilibrium strategies, it is interesting to note that $F_Q(p)$ dominates stochastically $F_N(p)$, in terms of first degree dominance, or equivalently, $F_Q(p)$

FIGURE 5.6


F(P)
1
0.8
0.6
0.4
0.2
0
Q.R.E(4)
Q.R.E (10)
mixed-strategy Nash
0 1 2 3 4 5 6 7 8 9
PRICES

$< F_N(p)$. This means that sellers are posting stochastically higher prices in a quantal response equilibrium than in the mixed-strategy Nash equilibrium.

5.6 SELLER CONCENTRATION

In this section we examine the quantal response equilibrium in the presence of a change in seller concentration. This structural variable is another factor that has been associated to systematic deviation from the Bertrand-Nash equilibrium in posted-offer markets (Holt and Davis, 1984).

Consider a generalization of the baseline model introduced at the beginning of this chapter to the case of N sellers. As before, each seller has 1 unit to sell at a zero cost. The quantity demanded is 1 unit for all prices less than or equal 1. A well known result is that for $N \geq 2$, where N is the number of sellers, the Bertrand-Nash equilibrium is to set price equal to marginal cost.

Now consider the calculation of the quantal response equilibrium. When a seller charges p it may be that p is the smallest price being posted. This happens only if the other sellers charge prices higher than p, an event which has probability $[1- F(p)]^{N-1}$. Therefore, the expected profit of the seller is

(63) $$\pi(p) = [1 - F(p)]^{N-1} p.$$

In the present market context, the power-function decision rule implies that the choice probabilities must satisfy:

$$f(p) = \frac{\left(p\,[1-F(p)]^{N-1}\right)^{\lambda}}{\mu} ,$$

(64)

$$\mu = \int_0^1 \left(x\,[1-F(x)]^{N-1} \right)^{\lambda} dx .$$

Equation (64) can be expressed as

(65)

$$\frac{f(p)}{[\,1-F(p)\,]^{(N-1)\lambda}} = \frac{p^{\lambda}}{\mu} .$$

Integrating both sides of equation (65) from 0 to some $p^* > 0$, we have

(66)

$$\int_0^{p^*} \frac{f(p)}{[1-F(p)]^{(N-1)\lambda}}\, dp = \int_0^{p^*} \frac{p^{\lambda}}{\mu}\, dp .$$

Let c = F(p) and dc = f(p)dp. As p goes from 0 to p^*, c goes from F(0) to F(p^*). Assuming $\lambda \neq 1$

(67)

$$\int_{F(0)}^{F(p^*)} \frac{dc}{(1-c)^{(N-1)\lambda}} = \int_0^{p^*} \frac{p^{\lambda}}{\mu}\, dp .$$

(68)

$$- \frac{[1-c\,]^{1-(N-1)\lambda}}{1-(N-1)\lambda} \Bigg|_{F(0)}^{F(p^*)} = \frac{p^{\lambda+1}}{\mu(\lambda+1)} \Bigg|_0^{p^*} .$$

Using the boundary condition F(0)=0, equation (68) is also written as

(69)

$$- \frac{[1-F(p^*)]^{1-(N-1)\lambda}}{1-(N-1)\lambda} + \frac{1}{1-(N-1)\lambda} = \frac{p^{*\,\lambda+1}}{\mu(\lambda+1)} .$$

Multiplying both sides of (69) by $1-(N-1)\lambda$ yields

Simplifying the notation, it follows from (70) that the probability distribution is

(70)
$$-[1-F(p^*)]^{1+\lambda-\lambda N} = [\frac{(1+\lambda-\lambda N)(p^*)^{\lambda+1}}{\mu(\lambda+1)} - 1] .$$

(71)
$$F(p) = 1- \left[1 - \frac{(1+\lambda-N\lambda)}{\mu(1+\lambda)} p^{\lambda+1}\right]^{\frac{1}{1+\lambda-N\lambda}} ,$$

where μ is a constant to be determined. The boundary condition, F(1)=1, implies that $\mu=(1+\lambda-\lambda N)/(\lambda+1)$. The quantal response equilibrium price distribution is [8]

(72)
$$F(p) = 1- \left[1 - p^{\lambda+1}\right]^{\frac{1}{1+\lambda-N\lambda}} ,$$

with the corresponding equilibrium price density:

(73)
$$f(p) = \left(\frac{\lambda+1}{1+\lambda-\lambda N}\right) p^{\lambda}\left[1 - p^{\lambda+1}\right]^{\frac{-\lambda+\lambda N}{1+\lambda-\lambda N}} .$$

Next we examine the effect of the number of sellers, on the endogenous equilibrium price distribution. The partial derivative of (73) with respect to N is

(74)
$$\frac{\partial F}{\partial N} = - \frac{\partial}{\partial N}\left(e^{\frac{1}{1+\lambda-N\lambda}\ln(1-p^{\lambda+1})}\right) .$$

Equation (74) is expressed as follows

(75)
$$\frac{\partial F}{\partial N} = -e^{\frac{1}{1+\lambda-N\lambda}\ln(1-p^{\lambda+1})} \frac{\partial}{\partial N}\left[\frac{1}{1+\lambda-N\lambda}\ln(1-p^{\lambda+1})\right] ,$$

The partial derivative of F(p) with respect to N is

[8] From (64), a quantal response equilibrium for $(1/N) - 1 \geq \lambda$ is the degenerate distribution $F(p)=1$ for $p\geq 0$.

(76) $$\frac{\partial F}{\partial N} = -[1-p^{\frac{1}{1+\lambda-N\lambda}}]\ \frac{\ln(1-p^{\lambda+1})\,\lambda}{(1+\lambda-N\lambda)^2}\ .$$

The logarithm in (76) is negative since $p \in (0,1)$. Therefore, as N increases the price distribution F(p) increases, and price declines stochastically.

To summarize:

As the number of sellers, N, increases, the power-function price equilibrium increases. Thus, given an increase in N, sellers post stochastically lower prices.

5.7 CONCLUSIONS

This chapter examined the effects of structural variables on equilibrium price distributions; cost structure, market power and seller concentration. The quantal response equilibrium was calculated for a series of variations of the simple duopoly model presented in chapter 4. The variations were introduced sequentially in order to develop the intuition needed to compute the quantal response equilibria in these models. The computations of the quantal response equilibria are not straightforward since it often involves determining expected payoffs for different ranges of prices, and it is not always possible to obtain closed-form solutions. Numerical methods were used in more complex market designs to assess the effects of structural variables on the equilibrium price distribution. The models considered in this chapter had 1 or 2 cost steps, severe capacity constraints and market power. Specific conclusions for the models analyzed in this chapter include:

1. With severe capacity constraints, the power-function and the logit quantal response equilibrium predict systematic departures from the Bertrand-Nash equilibrium for finite error parameters, and convergence to the Nash equilibrium as the errors vanish. Accordingly, the model is able to account for systematic price deviations in past experiments.

2. With severe capacity constraints and increasing costs, it is shown that sellers post stochastically higher prices when they face an increase in the low cost parameter. An increase in the high-cost parameter increases

prices above the high cost step, which reduces profits for the second unit that only sells at higher prices. This causes sellers to post stochastically lower prices in a quantal response equilibrium. By contrast, the Nash equilibrium is unaffected by changes in the cost parameters as long as the high cost parameter is below the reservation price.

4. With market power and constant costs, it is shown that the Nash equilibrium in mixed-strategies and the quantal response equilibrium differ.

5. With market power and increasing costs, the quantal response equilibrium stochastically dominates the Nash equilibrium (in terms of first degree dominance). In a quantal response equilibrium, as the error rate decreases, the mass of probability is concentrated in the reservation price. In the mixed-strategy Nash equilibrium, however, the mass of probability is concentrated near the lower bound of the price distribution.

6. In the N-firm model, a decrease in the number of sellers generates a stochastic increase in prices.

In this chapter, a comparison was made between the predictions of the quantal response approach and empirical features of posted-offer market experiments. The main finding is that the quantal response approach is typically consistent with supracompetitive pricing observed in posted-offer markets, including seller concentration, market power with increasing costs and cost structure.

5.8 APPENDIX II

For $\lambda=1$, the power function quantal response equilibrium condition becomes

(77) $$f(p) = \frac{p(2-F(p))}{\mu} .$$

This equation can be also arranged as follows

(78) $$-\frac{f(p)}{(2-F(p))} = -\frac{p}{\mu} .$$

The above equation is reexpressed as

(79) $$\partial \ln(2-F(p)) = -\frac{p}{\mu} .$$

Integrating from 0 to some p^*, we have

(80) $$\int_0^{p^*} \partial \ln(2-F(p)) = \int_0^{p^*} -\frac{p}{\mu} .$$

Equation (80) yields the following result:

(81) $$\ln(2-F(p^*)) = -\frac{p^{*2}}{2\mu}$$

To simplify the notation let p denote p^*. It follows from the above equation that the price distribution is

(82)
$$F(p) = 2[\, 1 - e^{-\frac{p^2}{2\mu}} \,] \;.$$

The next task is to determine μ. The boundary condition, F(1)=1, implies that μ = -1/(2*ln(1/2)).

CHAPTER 6

FIRST-PRICE, ALL-PAY AUCTIONS

6.1 INTRODUCTION

Auctions are one of the basic mechanisms for determining the prices of goods to be exchanged. In auctions, prices are determined by competition among potential buyers. Since the price in an auction is determined when the object is sold, it reflects all the available information and the preferences of the potential buyers who are bidding. Auctions may take one of two basic forms, oral or sealed-bid. In oral auctions, bidders hear one another's bids as they each made. In sealed-bid auctions, bidders simultaneously submit one or more bids to the seller without revealing their bids to one another.[1] A widely used sealed-bid auction is the first-price auction. In this auction, the highest bidder wins the item and pays the price submitted; the other bidders get and pay nothing.

The all-pay auction is similar to the first-price auction, except that losers must also pay their submitted bids. Baye et al. (1995) fully characterize the set of Nash equilibria in the first-price all-pay auction with complete information. In contrast to previous research, they show that the set of equilibria is much larger than the set of

[1] For a further discussion of auctions, see, for example, McAfee and McMillan (1987), Milgrom and Weber (1982) and Myerson (1991).

symmetric equilibria. They show also that the equilibria are not revenue equivalent in general.

Many economic problems can be modeled with the all-pay auction. In situations such as lobbying for rents in regulated or protected industries, technological competition and political campaigns, the participant showing the greatest effort or expenditure wins the prize, while the others are penalized. For example, Dasgupta (1986) uses the all-pay auction to model patent races in which the bids are research and development expenditures and the prize is a patent with known value. The firm spending the most on research and development obtains the patent, while the other firms make loses since they do not recover their expenditures.[2] A characteristic of this model is that the reward structure is such that ex-post payoffs are discontinuous. This property precludes the existence of Nash equilibrium in pure strategies.[3] The Nash equilibrium of the all-pay auction with complete information typically involves the use of randomized strategies, which protect bidders from being overbid by a small amount. However, bidders can make mistakes in calculating small differences in expected payoffs. This chapter uses the quantal response approach to model behavior in the all-pay auction. Two parametric

[2] Moulin (1986) also examines this symmetric equilibrium, but interprets it as a lobbying game.

[3] Dasgupta and Maskin (1982) have shown that discontinuous games do possess mixed-strategy Nash equilibrium under certain restrictions. A sufficient set of conditions is that the firm's profit function is everywhere left lower semi-continuous in its price, (and hence weakly lower semicontinuous), the profit function is bounded, and the sum of the two firms' profit functions is continuous.

classes of quantal response functions, the power function and the logit are used to calculate the quantal response equilibrium of the all-pay auction. As shown in chapter 3, these functional forms arise from different models of the error structure. An interesting result obtained in this chapter is that the Nash equilibrium in mixed-strategies and quantal response equilibrium are identical in the all-pay auction.

This chapter consists of two parts. In the first part, the Nash equilibrium of the (first-price) all-pay auction is analyzed. Section 6.2 contains the model. Following Dasgupta (1986), the Nash equilibrium in mixed strategies for continuous bid choices is analyzed in section 6.3. In laboratory experiments the set of feasible bid decisions often is finite; the calculation of the Nash equilibrium for the all-pay auction when the allowable bids are integer-valued is the topic of section 6.4. The second part of this chapter examines the quantal response equilibrium of the all-pay auction. Specifically, sections 6.5 and 6.6 provide a general statement of the conditions under which the Nash equilibrium in mixed-strategies and quantal response equilibria are identical. Section 6.7 illustrates the calculation of the quantal response equilibrium for discrete and for continuous bid choices using two different models of the error structure. Common patterns of the Nash and quantal response equilibrium price distributions are discussed at the end of this chapter.

6.2 THE MODEL

Assume that there are 2 identical bidders (which are henceforth referred to as firms). The "bids", which can be interpreted as competitive expenditures, are set simultaneously. The firm spending the most obtains the prize. Each of these firms has an identical known valuation, v. For instance, in a research and development (R&D) race, the bids are R&D expenditures and the prize is a patent with corresponding monopoly profit. In political contests, the bids are lobbying expenditures and the prize is a political favor. The value is split in case of a tie. The payoff to firm 1 is given by

$$\Pi_1(p_1) = \begin{cases} v - p_1 & \text{for } p_1 > p_2 \\ \frac{v}{2} - p_1 & \text{for } p_1 = p_2 \\ -p_1 & \text{for } p_1 < p_2 . \end{cases} \tag{1}$$

Notice from (1) that the bid p_i is paid whether or not the prize is won. Competition in an all-pay auction may be risky because it can generate negative profits. For such a bidding contest to take place, its outcome cannot be deterministic. Each player must have at least some chance of winning in order to be willing to participate.

6.3 THE NASH EQUILIBRIUM WITH CONTINUOUS BID CHOICES

Let p_i denote the bid posted by firm i, i=1,2. Notice from (1) that at a bid p_i above v, the firm with the higher bid makes negative profits. Also no firm is allowed to set a bid p_i less than 0. The calculation of the Nash equilibrium for the all-pay auction typically involves mixed strategies. To see this, suppose there is a pure strategy equilibrium with firm 1 bidding p_1 and firm 2 bidding p_2:

$$p_1 > p_2 \geq 0.$$

Now, consider the bid p_1':

$$p_1' = (p_1+p_2)/2.$$

Since $v-(p_1+p_2)/2 > v-p_1$, firm 1 wins the prize and earns higher profits from bidding p_1' than p_1. Suppose $p_1=p_2 \geq 0$. Then, a firm by raising slightly its bid wins the prize for sure. Using this argument, it follows that there is not a pure-strategy Nash equilibrium in the all-pay auction.

In what follows, the symmetric mixed-strategy distribution with support $[\underline{p},\bar{p}]$ is constructed. There can be no mass points in the interval $[\underline{p},\bar{p}]$. The reason is that if there were mass points in this interval, it would pay for a rival to concentrate just below such a mass point, to increase its payoff.

Since there are no mass points in the equilibrium density, f(p), the equilibrium cumulative distribution function, F(p), will be a continuous function on $[\underline{p},\bar{p}]$. The possibility of ties is not considered until the next section, where bids are integer-valued. Notice that when a firm bids p, it may be that p is the highest bid being posted, in which case, the firm's profit is v-p. This happens only if the other firm bids lower than p, an event which has probability F(p). Thus the firm's expected profit function is vF(p)-p.

For firm 1 to be indifferent between bidding some arbitrary bid p and 0, it must be the case that firm 2 bids according to a distribution $F_2(p)$ that makes firm 1's expected earnings at p equal to a security expected profit, S_1. Otherwise, it would pay a firm to increase the frequency for the bid with the higher expected payoff. The equilibrium expected payoff S_1, must satisfy:

(2) $$S_1 = v F_2(p) - p \qquad for\ p \in (\underline{p}, \bar{p}) \ .$$

In a symmetric equilibrium, $F_1(p)=F_2(p)$, and in this case, (2) yields:

(3) $$F(p) = \frac{p\ +\ S}{v} \ .$$

Equation (3) determines the equilibrium distribution function, once the S constant is found from an analysis of boundary conditions, which is the next task. Recall that there are no mass points in this equilibrium. Given $F(\underline{p})=0$, it follows that $S = -\underline{p}$ in equation (2). Since bidding zero is a permissible strategy in this model, it must be the case that $\underline{p}=0$ and hence S=0. Given $F(\bar{p})=1$, equation (2) can be used to show that $v-\bar{p}=0$ so $v=\bar{p}$. Using the boundary conditions, F(0)=0 and F(v)=1, yields the equilibrium probability distribution

(4) $$F(p) = \frac{p}{v} .$$

The corresponding price density is written as

(5) $$f(p) = \frac{1}{v} .$$

In the symmetric mixed-strategy Nash equilibrium, bidders randomize according to (4). Baye, et al. (1995) have shown that with more than two players the symmetric all pay auction delivers a continuum of asymmetric equilibria. Also, in any equilibrium, the expected payoff to each player is 0.

To summarize:

Proposition 1. (Dasgupta 1986. p. 536)

In the symmetric-mixed strategy Nash equilibrium, each firm bids randomly with probability 1/v in the support [0,v]. Furthermore, expected profits are zero in equilibrium.

6.4 THE NASH EQUILIBRIUM WITH DISCRETE BID CHOICES

In laboratory experiments, the set of feasible decisions is almost always finite.[4] The calculation of the mixed-strategy Nash equilibrium when bids are restricted to be integer-valued is similar to the one for the continuous case. The equilibrium expected payoff S is given in equation (6). This equation is comparable to equation (2) but equation (6) also includes the payoff function that determines earnings when a firm's bid matches the other's bid. The density, $f(p_i)$, denotes the equilibrium probability that a price selected is p_i, where $f(p_i) \geq 0$ for $p_i = p_1, .., v$:

$$\text{(6)} \qquad S = \left[\sum_{p_i=p_1}^{p_{k-1}} f(p_i)\right](v - p_k) + f(p_k)\left(\frac{v}{2} - p_k\right) + \left[1 - \sum_{p_i=p_1}^{v} f(p_i)\right](-p_k) \ ,$$

where $p_k = p_1, .., v$. The first term in (6) is the expected profit from being the higher bidder. The second term is the expected profit of a tie at p_k. At this bid, the prize is divided equally. The last term corresponds to the expected payoff from being outbid. Equation (6) can also be expressed as

$$\text{(7)} \qquad S = \left[\sum_{p_i=p_1}^{p_{k-1}} f(p_i) + \frac{f(p_k)}{2}\right](v - p_k) + \left[1 - \left[\sum_{p_i=p_1}^{p_{k-1}} f(p_i) + \frac{f(p_k)}{2}\right]\right](-p_k) \ .$$

The $G(p_k)$ in equation (8) is a *modified* " distribution function" that allows for the event of ties:

---

[4] The analysis of mixed-equilibria for simple normal form games is discussed in Moulin (1981). Holt and Solis-Soberon (1992) contains a discussion of the calculation of the mixed-strategy equilibria in posted-offer markets.

(8)
$$G(p_k) = \sum_{p_i = p_1}^{p_{k-1}} f(p_i) + \frac{f(p_k)}{2} = \frac{p_k + S}{v},$$

where the final equality follows from (7). In order to obtain the support of the equilibrium mixed-strategy Nash equilibrium, consider a set of consecutive integer-valued bids: $[p_1, p_2, \ldots, v]$, where v is the largest integer bid. Define p_L and p_H as the lowest and highest bids respectively that are selected with strictly positive probability, where $p_1 \leq p_L < p_H \leq v$. By evaluating (6) at p_H and using the fact that the sum of the densities up to $f(p_H)$ equals one, one obtains

(9)
$$\begin{aligned} S &= [1 - f(p_H)](v - p_H) + f(p_H)(\frac{v}{2} - p_H) \\ &= v - p_H - f(p_H)\frac{v}{2} . \end{aligned}$$

Since $f(p_H) > 0$, it follows from (9) that $S < v - p_H$. Now, we calculate the mixed strategy equilibrium for this model. Conjecture that $f(p_k) = 1/v$ with the upper bound $p_H = v-1$ and the lower bound $p_1 = p_L = 0$, is a Nash equilibrium in mixed-strategies. Next, we verify that a seller is indifferent between the bids 0,1,..,v-1. By evaluating (7) at $p_H = v-1$, one obtains

(10)
$$\begin{aligned} S &= \left[F(v-2) + \frac{f(v-1)}{2}\right]v - (v-1) \\ &= \left[\frac{v-1}{v} + \frac{1}{2v}\right]v - (v-1) . \end{aligned}$$

It is straightforward to verify from (10) that $S=1/2$. An analogous argument shows that at the lower bound, $p_L=0$, $S=1/2$, and similarly for intermediate prices. In contrast to the Nash equilibrium with continuous bid choices, rents are not dissipated in equilibrium in the discrete case. A possible reason is that in the discrete case a firm has to bid higher money amounts to outbid a rival.

To summarize:

Proposition 2

The probability 1/v over the set of consecutive integer-valued bids: [0,1,...,v-1] is a mixed-strategy Nash equilibrium. Further, in equilibrium expected profits are 1/2.

In this model, bidders randomize to protect themselves from being overbid by a small amount. However, bidders can make mistakes in calculating small differences in expected payoffs. Next, we qualify the assumption that firms are perfect maximizers of their own money payoffs. This is done by introducing decision error, i.e., in choosing their bid strategies firms make mistakes. Bidders 'tremble' and therefore every bid strategy is played with a strictly positive probability.

6.5 EQUIVALENCE OF EQUILIBRIA WITH DISCRETE BID CHOICES

POWER-FUNCTION QUANTAL RESPONSES (MULTIPLICATIVE ERRORS)

For any structure of the error term (i.e multiplicative or additive), the next propositions illustrate a general statement of the equivalence between the Nash equilibrium in mixed-strategies and quantal response equilibria. Such equivalence arises because the expected profits in the mixed-strategy Nash equilibrium with discrete bid choices are equal at all bids in the support [0,1,..,v-1] (continuous case, [0,v]). Hence, if the rival is using his Nash equilibrium, the player's best quantal response is to spread bid decisions uniformly in the support [0,1...,v-1] or [0,v] respectively.

Proposition 3

Consider a game in which the strategy space, S_i, of player i is a discrete set of actions, $s_i \in S_i$, $i=1....n$. Suppose there exists a Nash equilibrium to the game at which player i plays each action in $\hat{S}_i \subset S_i$ with equal probability (i.e $S^*_i = 1/k \;\forall\; S^*_i \in \hat{S}_i$, where $k=$ number of elements in $\hat{S}_i$). Let $\pi^*_i\,(s_i)$ denote i's profit given all other players play their equilibrium mixed strategies, and suppose furthermore that $\pi^*_i\,(s_i) > 0 \;\;\forall\;\; s_i \in \hat{S}_i$ while $\pi^*_i \leq 0 \;\;\forall\;\; s_i \notin \hat{S}_i$. Then there exists a multiplicative-error quantal response equilibrium when the error structure satisfies $\varepsilon_i \geq 0$ ($F(0)=0$, no mass points at zero) and the errors are identically independently distributed. Furthermore, this multiplicative-error quantal response equilibrium is identical to the Nash equilibrium described above.

Proof

It suffices to show that player i will choose each strategy in $\hat{S}_i$ with equal probability, given the other players choose strategies in the manner described in the proposition. Clearly, since $u_j = \pi^*_i (s_j)\varepsilon_j$, no strategy outside $\hat{S}_i$ will be chosen (to do so would yield nonpositive payoff for all realizations of s_i, but nonnegative payoffs are guaranteed for $s_i \in \hat{S}_i$, the probability of "ties" at zero is zero since F(0)=0). Finally, since $\pi^*_i(s_j)=\pi^*_i (s_l)= \bar{\pi}_i > 0$ for all $s_j,s_l \in \hat{S}_i$, it remains to be shown that

$$\textbf{(11)} \qquad Pr\ (\ \bar{\pi}\,\varepsilon_j = \max_{j\ is\ \max} \bar{\pi}\,\varepsilon_l\ )\ =\ \frac{1}{k_i}\ .$$

This is true since

$$\textbf{(12)} \qquad Pr\ (\ \bar{\pi}\,\varepsilon_j = \max_{j\ is\ \max} \bar{\pi}\,\varepsilon_l\ )\ =\ \int_0^\infty F^{\,k_i-1}(\varepsilon) f(\varepsilon)\, d\,\varepsilon = \frac{F^{k_i}}{k_i}\Big|_0^\infty = \frac{1}{k_i}\ .$$

LOGISTIC QUANTAL RESPONSES (ADDITIVE ERRORS)

Proposition 4

Consider a game in which the strategy space , S_i, of player i is a discrete set of actions, $s_i \in S_i$, $i=1....n$, where $s_n < v$. Suppose there exists a Nash equilibrium to the game at which player i plays each action in $\hat{S}_i \subset S_i$ with equal probability (i.e $S^*_i = 1/k\ \forall\ S^*_i \in \hat{S}_i$, where k = number of elements in $\hat{S}i$). Let $\pi^*_i (s_i)$ denote i's profit given all other players play their equilibrium mixed strategies, and suppose furthermore that $\pi^*_i (s_i) >$

0 for all $s_i \in \hat{S}_i$ while $\pi^*_i \leq 0 \;\; \forall \;\; s_i \notin \hat{S}_i$. Then there exists an additive-error quantal response equilibrium when the error structure satisfies $\varepsilon_i \geq 0$ ($F(0)=0$, no mass points at zero) and the errors are identically independently distributed. Furthermore, this additive-error quantal response equilibrium is identical to the Nash equilibrium described above.

Proof

It suffices to show that player i will choose each strategy in $\hat{S}_i$ with equal probability, given the other players choose strategies in the manner described in the proposition. Clearly, since $u_l = \pi^*_i(s_l) + \varepsilon_l$, no strategy outside $\hat{S}_i$ will be chosen (to do so would yield nonpositive payoff $\forall$ realizations of s_i, but nonnegative payoffs are guaranteed for $s_i \in \hat{S}_i$, the probability of "ties" at zero is zero since $F(0)=0$). Finally, since $\pi^*_i(s_j) = \pi^*_i(s_l) = \bar{\pi}_i > 0 \;\; \forall \;\; s_j, s_l \in \bar{S}_i$, it remains to be shown that

(13)
$$Pr\left(\bar{\pi} + \varepsilon_j = \max_{j \text{ is max}} \bar{\pi} + \varepsilon_l\right) = \frac{1}{k_i}.$$

This is true since

(14)
$$Pr\left(\bar{\pi} + \varepsilon_j = \max_{j \text{ is max}} \bar{\pi} + \varepsilon_l\right) = \int_0^\infty F^{k_i - 1}(\varepsilon) f(\varepsilon)\, d\varepsilon = \frac{F^{k_i}}{k_i}\Bigg|_0^\infty = \frac{1}{k_i}.$$

6.6 EQUIVALENCE OF EQUILIBRIA WITH CONTINUOUS BID CHOICES

LOGISTIC QUANTAL RESPONSES (ADDITIVE ERRORS)

Proposition 5

Consider a game in which the strategy space, S_i, of player i is an interval of actions, $s \epsilon [0,v]$. Suppose there exists a Nash equilibrium to the game at which player i plays each action with probability $f(s)=1/v$. Let $\pi^*_i(s)$ denote i's profit given all other players play their equilibrium mixed strategies, and suppose furthermore that $\pi^*_i(s) \geq 0$ for all $s \epsilon [0,v]$. Then there exists an additive-error quantal response equilibrium when the error structure satisfies $\varepsilon_i \geq 0$ ($F(0)=0$, no mass points at zero) and the errors are identically independently distributed. Furthermore, this additive-error quantal response equilibrium is identical to the Nash equilibrium described above.

Proof

It suffices to show that player i will choose strategy s with equal probability, given the other players choose strategies in the manner described in the proposition. Since $\pi^*_i(s)=\bar{\pi}_i>0$ for all $s\epsilon[0,v]$, it needs to be shown that

(15) $$Pr\left(\bar{\pi} + \varepsilon_j = \max_{j \text{ is max}} \bar{\pi} + \varepsilon_i \right) = \frac{1}{n} .$$

This is true since

(16) $$Pr\left(\bar{\pi} + \varepsilon_j = \max_{j \text{ is max}} \bar{\pi} + \varepsilon_i \right) = \int_0^\infty F^{n-1}(\varepsilon) f(\varepsilon)\, d\varepsilon = \frac{F^n}{n} \Big|_0^\infty = \frac{1}{n} .$$

POWER-FUNCTION QUANTAL RESPONSES (MULTIPLICATIVE ERRORS)

Proposition 7

Consider a game in which the strategy space, S_i, of player i is an interval of actions, s∊[0,v]. Suppose there exists a Nash equilibrium to the game at which player i plays each action with probability f(s)=1/v. Let π^*_i (s) denote i's profit given all other players play their equilibrium mixed strategies, and suppose furthermore that π^*_i (s) $\geq$ 0 for all s∊[0,v]. Then there exists a multiplicative-error quantal response equilibrium when the error structure satisfies $\varepsilon_i \geq 0$ (F(0)=0, no mass points at zero) and the errors are identically independently distributed. Furthermore, this multiplicative-error quantal response equilibrium is identical to the Nash equilibrium described above.

Proof

It suffices to show that player i will choose strategy s with equal probability, given the other players choose strategies in the manner described in the proposition. For all s∊[0,v], the power-function quantal response equilibrium selects all options with equal $\bar{\pi} \geq 0$:

(17) $$\frac{\pi^{\lambda}}{n\pi^{\lambda}} = \frac{1}{n} ,$$

even when $\pi=0$ since lim $\pi \to 0^{+}$.

The set of propositions stated above shows why the quantal response equilibrium and Nash distribution for the all-pay auction are the same when the Nash mixed distribution is uniform in the support [0,1,..,v-1] for the discrete case and [0,v] for the continuous case. Consequently, adding decision error into the all-pay auction with complete information does not affect the firms' probability choices.

6.7 EXAMPLES OF THE CALCULATION OF THE QUANTAL RESPONSE EQUILIBRIUM

DISCRETE BID CHOICES AND MULTIPLICATIVE ERRORS

In this section we illustrate how the quantal response equilibrium of the all-pay auction can be computed. Recall that the the power-function quantal response equilibrium is derived from random utility with multiplicative errors. Then the power function quantal response equilibrium condition is given by

$$\text{(18)} \qquad f(p_k) = \frac{\left(\left[F(p_{k-1}) + \frac{f(p_k)}{2} \right] v - p_k \right)^{\lambda}}{\mu} \ ,$$

$$\mu = \sum_{x=0}^{v} \left(\left[F(x-1) + \frac{f(x)}{2} \right] v - x \right)^{\lambda} \ ,$$

where μ is a constant independent of prices. The probability density in (18) is obtained by solving recursively the first equation in (18), beginning with the lowest bid and working upward. For simplicity, let $\lambda=1$. Since ties are possible with integer-valued bids, it follows that $f(0)/2 > 0$. By evaluating (18) at the lowest bid $p_k=0$, it follows that $\mu = v/2$. The substitution of μ back into (18) results in the following expression

$$\text{(19)} \qquad F(p_{k-1})\, v = p_k \ .$$

Let $p_k = z$. Then, the above equation is written as

$$\text{(20)} \qquad \sum_{i=0}^{z-1} f(i)\, v = z \ ,$$

for $z = 1,\ldots,v$. Consider $z = 1$. Then, it follows from (20) that $f(0)=1/v$. Assume $z = 2$. Then, equation (20) yields

$$\text{(21)} \qquad f(0) + f(1) = \frac{2}{v}$$

Since $f(0)=1/v$, it must be the case that $f(1)=1/v$ in (21). A similar argument shows that $z=v$ yields

(22)
$$\frac{\nu - 1}{\nu} + f(\nu - 1) = \frac{\nu}{\nu} .$$

It is readily verified from (22) that f(v-1)=1/v. Now consider $\lambda > 1$. Notice that f(0) > 0 implies that $\mu = (f(0))^{\lambda-1}(v/2)^{\lambda}$. Conjecture that the equilibrium probability is 1/v. From (18), it follows that $\mu = v(1/2)^{\lambda}$. By evaluating (18) at $p_k = v-1$ and using the solution for μ, one obtains

(23)
$$f(\nu-1) = \frac{2^{\lambda}\left(\left[F(\nu - 2) + \frac{f(\nu - 1)}{2}\right]\nu - (\nu-1)\right)^{\lambda}}{\nu} .$$

Substituting the conjecture 1/v in both sides of (23), it follows that 1/v satisfies (23). A similar argument shows that the uniform distribution, 1/v, satisfies intermediate bid values.

CONTINUOUS BID CHOICES AND ADDITIVE ERRORS

For any given $\lambda > 0$, the logistic quantal response equilibrium for continuous bid choices is given by

$$f(p) = \frac{e^{\lambda[\nu F(p) - p]}}{\mu} ,$$

(24)
$$\mu = \int_0^{\nu} e^{\lambda[\nu F(x) - x]} dx .$$

where μ is a constant independent of p. The first equation in (24) is a nonlinear differential equation in the price distribution F(p). In order to obtain F(p) we first multiply both sides of the top equation in (24) by $-\lambda v e^{-\lambda v F(p)}$, which yields

(25) $$-\lambda v f(p) e^{-\lambda v F(p)} = \frac{-\lambda v e^{-\lambda p}}{\mu} .$$

Integrating over all values of p, i.e from p_a to p^*, we have

(26) $$\int_{p_a}^{p^*} -\lambda v f(p) e^{-\lambda v F(p)} dp = \int_{p_a}^{p^*} \frac{-\lambda v e^{-\lambda p}}{\mu} dp .$$

The resulting equation is written as

(27) $$e^{-\lambda v F(p)} \Big|_{F(p_a)}^{F(p^*)} = \frac{v e^{-\lambda p}}{\mu} \Big|_{p_a}^{p^*} .$$

The next task is to determine μ from an analysis of boundary conditions. Since negative prices produce no profits, conjecture that $F(p_a)=0$, with $p_a=0$. Let $\bar{p}$ denote the upper bound of the bid distribution. Using the boundary condition, $F(0)=0$, equation (27) becomes

(28) $$e^{-\lambda v F(\bar{p})} - 1 = \frac{v}{\mu}[e^{-\lambda \bar{p}} - 1] .$$

Now consider the upper bound $\bar{p}$. From (24), $\bar{p} > v$ implies that $f(p) > 0$. Since bids above v produce 0 profits, conjecture that $F(\bar{p})=1$, with $\bar{p}=v$. This conjecture in turn implies that $\mu=v$. Substituting μ back into (24), we have

(29) $$e^{-\lambda v F(p)} = e^{-\lambda p} .$$

It is readily verified from (29) that the equilibrium probability function is

$$F(p) = \frac{p}{v}, \tag{30}$$

which also satisfies the working assumption used above: $F(0)=0$ and $F(v)=1$.

6.8 CONCLUSIONS

In this chapter, we examine the quantal response equilibrium for discrete and continuous bid choices of a (first-price) all-pay auction model. The main result derived in this chapter is that for any structure of the error terms the Nash in mixed-strategies and quantal response equilibrium are identical under certain restrictions: The Nash equilibrium price distribution is uniform in the support and the error terms are independently distributed. This is the case for the all-pay auction since the expected profits in the mixed-strategy Nash equilibrium are equal at all bids in the support [0,v] in the continuous case, and [0,1,..v-1] in the discrete case. Hence, if the rival is using his Nash equilibrium, 1/v, the seller's best response is to spread bid decisions uniformly in the corresponding support. Another interesting result derived in this chapter is that the Nash equilibrium in mixed-strategies with discrete bid choices is for each firm to choose the probability 1/v over the set of consecutive integer value bids: 0,1,....v-1 for an expected payoff of 1/2. In contrast with the continuous case, rents are not dissipated in equilibrium in the discrete case.

CHAPTER 7

CONCLUSIONS AND EXTENSIONS

This thesis consists of several essays on quantal response equilibria for models of price competition. The first part pertains to the derivation of the "power function" quantal response equilibrium from a model of multiplicative random errors. The second part uses the quantal response behavior to model equilibrium in models of price competition.

A large experimental literature documents systematic deviations from the Nash equilibrium in game theory and industrial organization experiments. This thesis examined the equilibrium properties of models of price competition in which decision error may arise. The approach used in this thesis is the quantal response equilibrium. Capturing decision error in a way that is clearly spelled out and not ad hoc is a difficult task, the quantal response equilibrium does this by using the basis borrowed from discrete choice theory of Luce (1959), McFadden (1984) and Thurstone (1927). As discussed in chapter 2, the added complexity in applying the quantal response equilibrium to game theory -- in contrast to individual choice -- is that the choice probabilities of the players have an important interactive component, since they are simultaneously determined in equilibrium. In a quantal response equilibrium, a player's beliefs about others' actions will determine the player's own expected payoffs, which in turn determine the player's choice probabilities via a quantal response function. The model is closed by requiring the choice probabilities to be consistent with the initial beliefs.

Chapter 3 derives the power-function quantal response equilibrium. This functional form is a useful way to model decision errors in models of price competition since it often leads to tractable solutions and comparative statics results. The power-function quantal response equilibrium is based on random utility maximization with multiplicative error terms.

The power function approach can be thought of as a generalization of the Luce model of choice probability ratios that earn properties to utility ratios. In the power function model, the choice probability ratios are expected payoff ratios raised to some power. The power parameter governs the extent to which a player's deviates from being conventional expected utility maximizer. At one extreme, individuals choose randomly, independent of expected payoffs. At the other extreme, individuals always choose the decision with the highest expected payoff.

The last part of chapter 3 established differences in the qualitative properties of Nash and quantal response equilibria in a standard prisoners' dilemma game with two possible price choices. In this market game, the Nash equilibrium is also a power-function quantal response equilibrium, but this is not true for the logit model. Another interesting consequence of this model is that, as the error rate, $1/\lambda$, increases, the probability of choosing the cooperative, high-price decision also increases. The analysis of the prisoners' dilemma game with two price choices suggests that a sufficient amount of decision error can actually make individuals better off since it increases the probability of choosing the cooperative, high-price decision. This result provides a natural null hypothesis for experimental analysis.

In chapter 4, the power-function equilibrium price distribution is derived for a simple Bertrand duopoly game. The power-function price distribution allow us to derive intuitive comparative results for the error-rate parameter, $1/\lambda$. Accordingly, the model is able to account for systematic price deviations from the Bertrand-Nash equilibrium.

Chapter 5 examined the consequences of cost structure, market power and seller concentration on the endogenous equilibrium price distributions. This is done because such structural variables have been associated with systematic price deviations in models of price competition (Holt and Davis, 1990, Davis and Williams, 1990, Wellford et al., 1990 Brown-Kruse et al., 1993 and Davis and Holt, 1994). We first analyzed a model with severe capacity constraints. It is shown that, the quantal response equilibrium predicts systematic departures from the Bertrand-Nash equilibrium for finite error parameters, and convergence to the Nash equilibrium as the errors vanish. Experimental evidence indicates that although market models may share identical Nash equilibrium different average prices are observed (Holt and Davis, 1990). In light of this experimental result, we introduce increasing costs to the baseline model with severe capacity constraints. We found that, an increase in the low cost parameter stochastically raises prices. Thus, sellers in a quantal response equilibrium post higher average prices when they face an increase in the low cost parameter. However, the Nash equilibrium in this model is unaffected by changes in the cost structure. Thus, the model has the potential of explaining qualitative features of some experimental results (Holt and Davis, 1990). A change in the number of sellers is another factor that has been associated to systematic price deviations from the Bertrand-Nash equilibrium in posted-offer markets (Holt and Davis, 1994). It was found that a decrease in the number of sellers generates a stochastic increase in prices. By contrast the Nash equilibrium is unaltered.

The second part of chapter 5 examined competition between two price setting sellers, each of whom faces a production capacity constraint. Experimental evidence indicates that competitive equilibrium pricing, Edgeworth cycles in prices and mixed strategy Nash equilibrium in prices are not completely consistent with the experimental data. As the error decreases, the quantal response equilibrium does not select the competitive outcome, which is consistent with experimental data.

Another empirical feature of experimental models of price competition is that prices do not conform to Edgeworth cycle theory although experiments exhibit upward and downward swings of the sort predicted by the Edgeworth cycle theory. Because of this we examine the quantal response equilibrium of the baseline model with production capacity constraints. The conjecture that the quantal response equilibrium stochastically dominates the Nash equilibrium cannot be rejected. Furthermore, it is shown that if a rival is using his Nash equilibrium strategy, the seller's best quantal response equilibrium is to spread price decisions uniformly. This is because the expected profits are equal at all prices in a mixed-strategy Nash equilibrium. A counterexample of this result is the all-pay auction model analyzed in chapter 6.

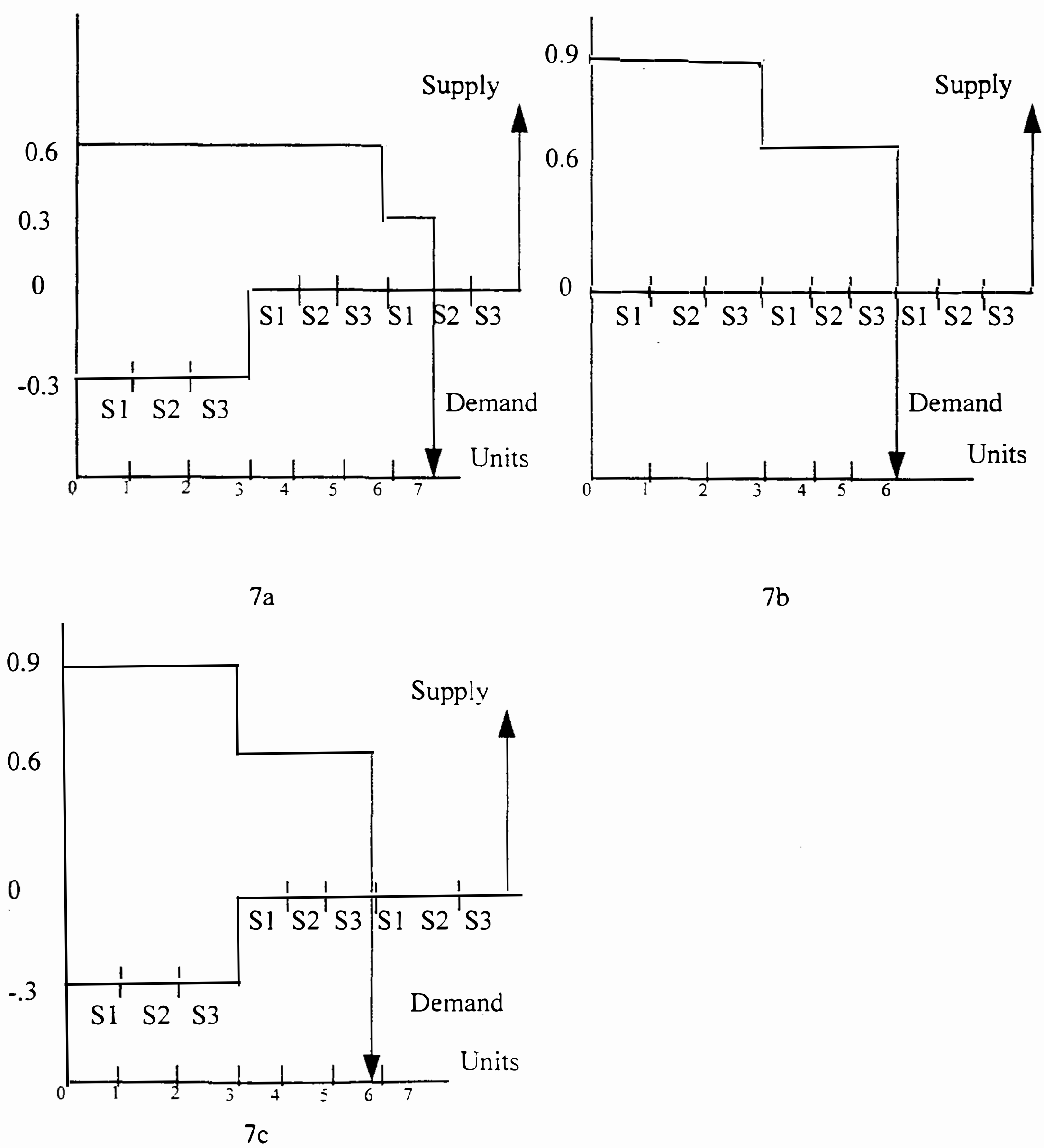
The last part of chapter 5 examined a variation of the baseline model with capacity constraints. Using numerical methods it was shown that, as the error rate decreases, the probability mass is concentrated at the reservation price in a quantal response equilibrium. In the mixed-strategy Nash equilibrium, however, the probability mass is concentrated near the lower bound of the price distribution. This result is consistent with qualitative features of the experimental posted offer markets results of Holt and Davis, 1994.

Chapter 6 analyzed the all-pay auction model. The Nash equilibria of the all-pay auction typically involves randomization, which protect bidders from being overbid by a small amount. However, bidders can make mistakes in calculating small differences in expected payoffs. The Nash equilibrium for the all-pay auction is for each firm to choose bids with a uniform probability in the price support. For any specification of the error structure, it is shown that the Nash and quantal response equilibria for the all-pay auction are identical. Moreover, if a game has the following properties, then the Nash equilibrium will be a quantal response equilibrium: The expected payoff from choosing a particular bid strategy is constant over the bid support, each bid strategy in the support has the same uniform probability of being chosen and finally, the error terms must be independently distributed.

The theoretical results obtained in this thesis are motivated by stylized patterns in experimental data and will be used to suggest designs for further experiments. Even though one important feature of the approach derived in this thesis was its simplicity we would like to outline some extensions of the approach and give directions for further research.

One extension is to apply the quantal response equilibrium to posted-offer experimental data. The posted-offer triopolies conducted by Holt and Davis (1990) are especially interesting since the observed median prices for the first 15 market periods reveal systematic deviations from the Bertrand-Nash prediction. Figure 7.1 illustrates these market designs. The median prices are labeled on the vertical axis while the horizontal axis represents output or units. Figures 7.2, 7.3 and 7.4 show the corresponding distribution function of prices for each market design. As can be seen from the figures, observed prices were above the pure-strategy Nash equilibrium in the two non-power market design. As a first step, the market models in figure 7.1 were examined using numerical methods. In these simulations a logistic error rate of 1/8 seems to track the qualitative features of the data in the non-power market designs. The next step in this research will be a statistical analysis of the data, using standard maximum likelihood techniques in a structural model that is directly implied by the quantal response equilibrium.

FIGURE 7.1


0.6
0.3
0
-0.3
Supply
S1 S2 S3 S1 S2 S3
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0 1 2 3 4 5 6 7
7a
0.9
0.6
0
Supply
S1 S2 S3 S1 S2 S3 S1 S2 S3
Demand
Units
0 1 2 3 4 5 6
7b
0.9
0.6
0
-.3
Supply
S1 S2 S3 S1 S2 S3
S1 S2 S3
Demand
Units
0 1 2 3 4 5 6 7
7c

124

FIGURE 7.2
MARKET POWER AND INCREASING COSTS

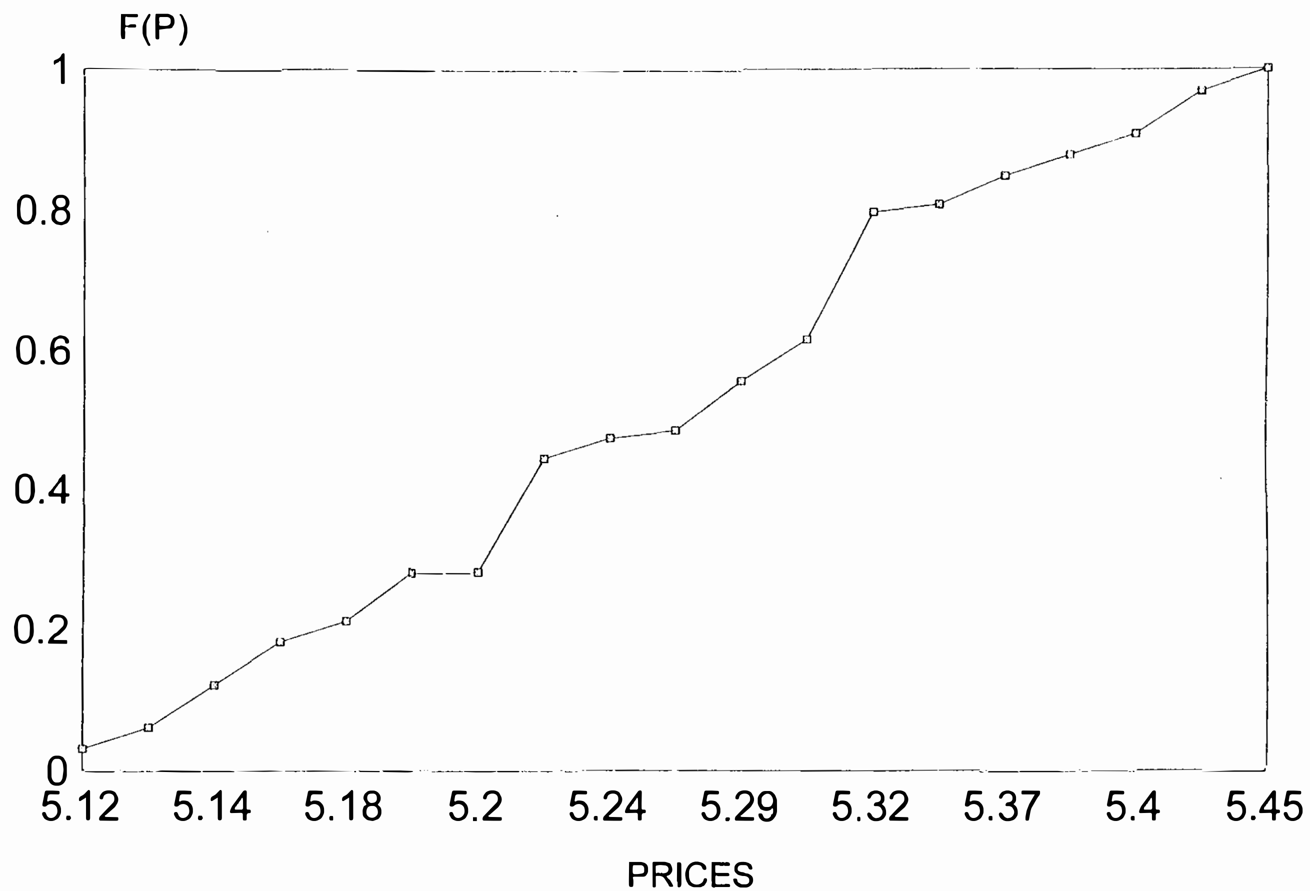


FIGURE 7.3
NO MARKET POWER AND CONSTANT COSTS

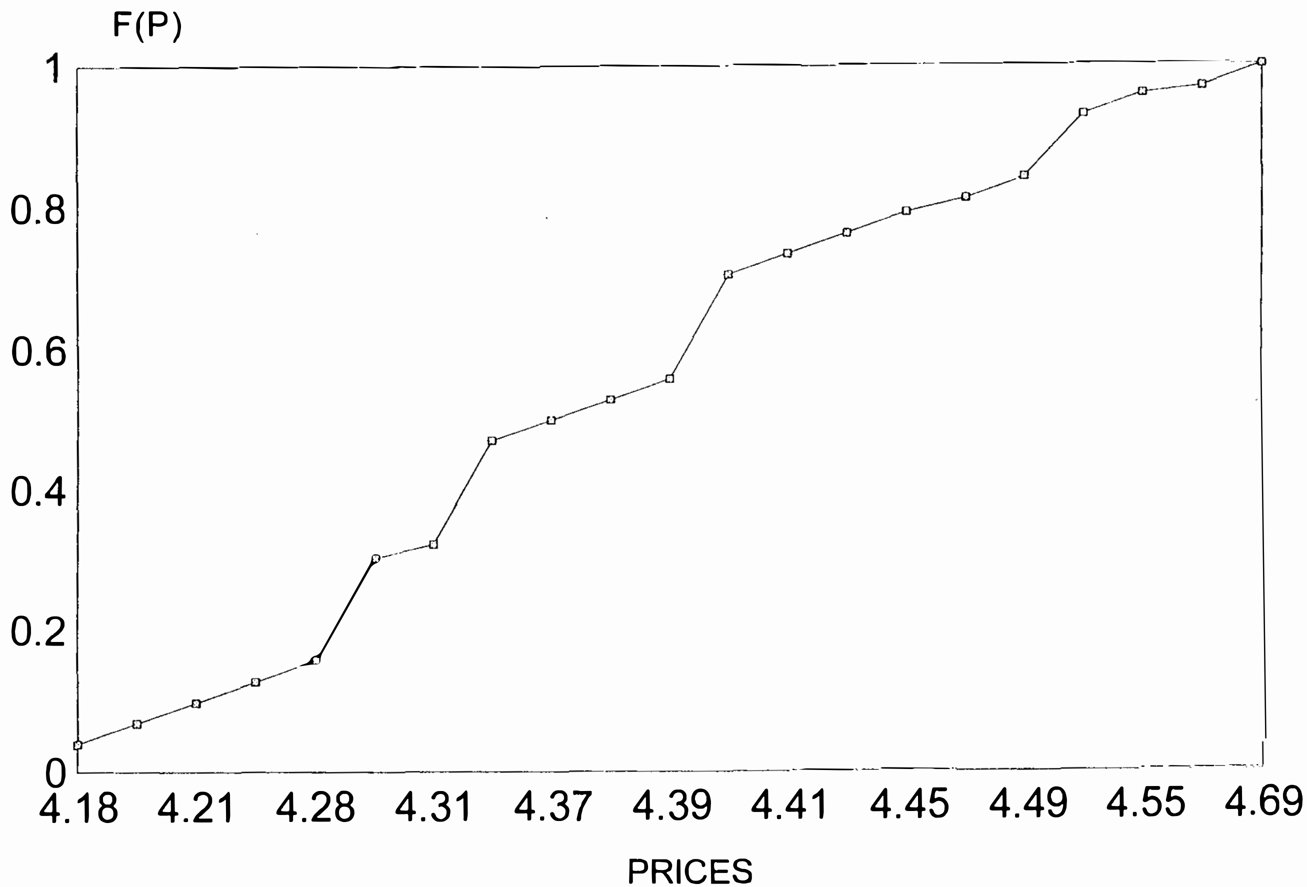
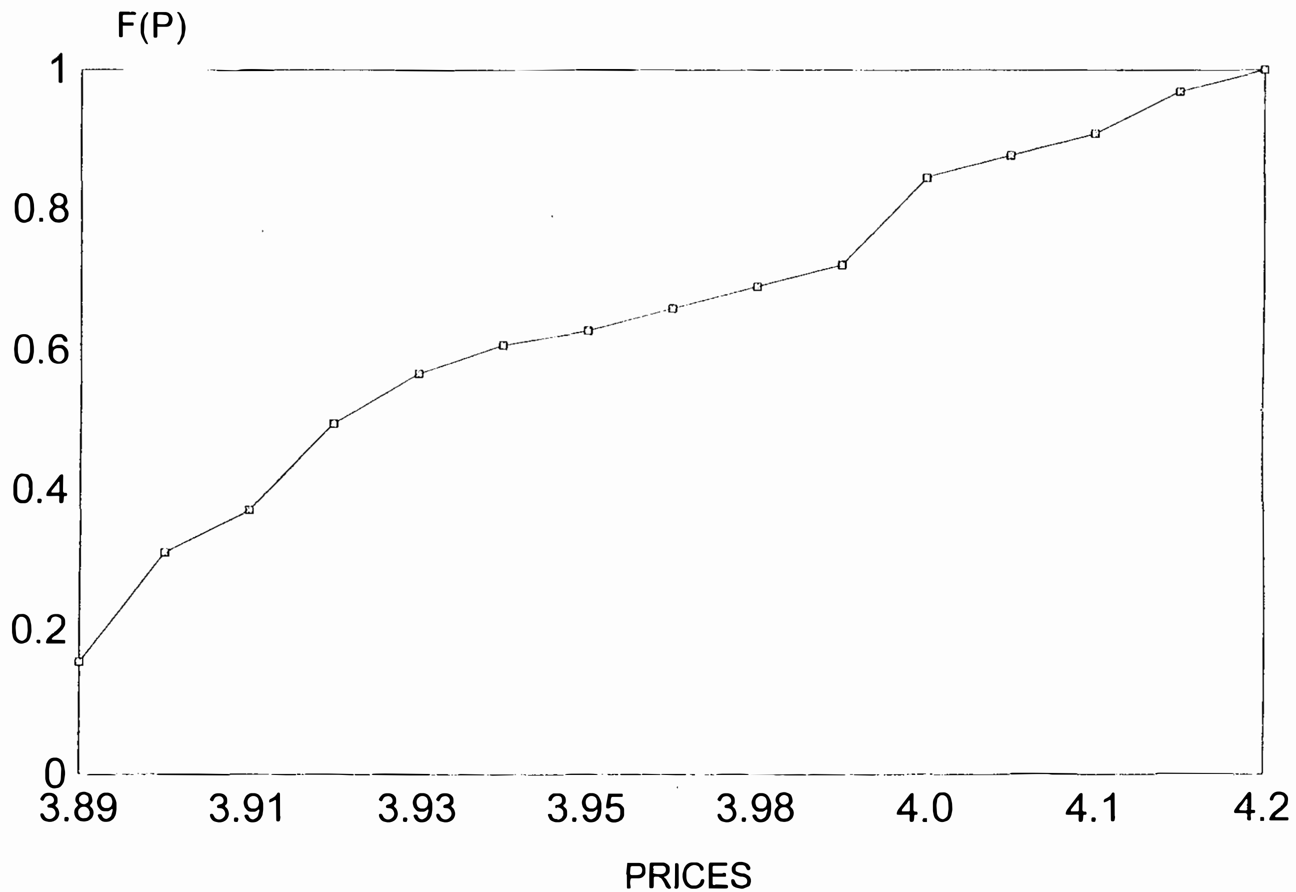


FIGURE 7.4
NO MARKET POWER AND INCREASING COSTS



Another extension is to examine ultimatum bargaining games in which there are decision errors in the buyer's purchase choices. There is a large experimental literature that documents systematic deviations from the Nash equilibrium in bargaining games. Systematic deviations in these games have been attributed to perceptions of fairness, focalness, and to random "errors" (Prisbrey, 1994). In principle, it is not difficult to extend the model to allow for buyer's decision errors. However, further work is needed in order to determine whether the model can predict systematic price deviations in ultimatum experiments. Another promising direction is to account for price choice decisions under horizontal product differentiation, in which the "error" rate is a measure of location or differentiation. Experimental evidence shows that in a Hotelling duopoly model, sellers' prices did not converge to the Nash prediction. In fact prices seem to be higher with greater distance between firms, even when different locations have identical Nash equilibrium (Brown-Kruse, 1989). The quantal response equilibrium is a well suited model to explain such deviations since structural variables affect the equilibrium price distribution but typically do not affect the Nash equilibrium. Under product differentiation, however, the quantal response equilibrium condition becomes a complex second-order nonlinear differential equation in the price distribution. Therefore performance of the model may be based on simulations.

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