Quantum Covering Groups and Quantum Symmetric Pairs

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Abstract

A quantum covering group $\mathbf{U}_\pi$ is an algebra with parameters q and π subject to $\pi^2 = 1$ and it admits an integral form; it specializes to the usual quantum group at $\pi = 1$ and to a quantum supergroup of anisotropic type at $\pi = -1$. In this dissertation, we establish the Frobenius-Lusztig homomorphism and Lusztig-Steinberg tensor product theorem in the setting of quantum covering groups at roots of 1, recovering Lusztig's constructions for quantum groups at roots of 1 when we specialize at $\pi = 1$.

We develop a theory of quantum symmetric pair $(\mathbf{U}_\pi, \mathbf{U}_\pi^\imath)$, where $\mathbf{U}_\pi^\imath$ is a coideal subalgebra of $\mathbf{U}_\pi$. When specializing at $\pi = 1$, the pair $(\mathbf{U}_\pi, \mathbf{U}_\pi^\imath)$, reduces to a quantum symmetric pair of G. Letzter and its Kac-Moody generalization by Kolb. We give a Serre presentation for $\mathbf{U}_\pi^\imath$ of quantum symmetric pairs $(\mathbf{U}_\pi, \mathbf{U}_\pi^\imath)$ for quantum covering groups, introducing the $\imath^\pi$-*Serre relations* and $\imath^\pi$-*divided powers*. We also develop a quasi K-matrix in this setting, which leads to a construction of $\imath$canonical bases for the highest weight integrable $\mathbf{U}_\pi$-modules and their tensor products regarded as $\mathbf{U}_\pi^\imath$-modules, as well as an $\imath$canonical basis for the modified form of the $\imath$quantum group $\mathbf{U}_\pi^\imath$. Again, specializing at $\pi = 1$ we recover the Serre presentation of $\mathbf{U}^\imath$ by Chen-Lu-Wang and the canonical basis construction of Bao-Wang. The specialization at $\pi = -1$ leads to new constructions for quantum supergroups.

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Contents

Chapter 1

Introduction

In representation theory, *quantum groups* are a key object of interest. Originally used by Drinfeld and Jimbo to study integrable systems, quantum groups are deformations of universal enveloping algebras of Lie algebras through the addition of a parameter q, yielding a richer algebraic structure known as a *Hopf algebra*. Today, quantum groups have many interesting applications such as knot invariants, modular representation theory, and categorification.

Categorification is the process of taking a familiar algebraic construction and adding a layer of categorical structure. For instance, the homology groups of a manifold can be viewed as a categorification of the Euler characteristic. This process allows us to cast a familiar construction as the shadow of some higher structure, leading to deeper insights.

An important structure in the study of quantum groups and categorification are *canonical bases*, which are bases for quantum groups with certain nice properties, such as enabling one to write down a basis for any simple module (hence the word 'canonical'). Another such property is the positivity of structure constants for multiplication in symmetric type, which hints at geometry and categorification (these two concepts are often intertwined). In fact, canonical bases have a geometric origin – they arise as shadows of intersection cohomology sheaves. Canonical bases are often used in working with integral forms of quantum groups, which is a version over the ring $\mathcal{A} = \mathbb{Z}[q, q^{-1}]$ that is ubiquitous in categorification, where q becomes

the shadow of a grading shift. Two important examples of categorification are the KLR construction for quantum groups by M. Khovanov, A. Lauda and R. Rouquier using quiver Hecke algebras, and Soergel bimodules for Hecke algebras by B. Elias and G. Williamson. In this setting, canonical bases above arise as the shadow of indecomposables.

The work in this dissertation lies at the intersection of a few notions that are built on this rich foundation of quantum groups. These include the *quantum covering groups* and *quantum symmetric pairs*. In the first part, we will discuss constructions at roots of unity for quantum covering groups. The second part will feature a Serre presentation, bar involution and canonical basis for quantum symmetric pairs of quantum covering groups. A brief overview of these notions are given in the following paragraphs.

Quantum covering groups

A quantum covering group $\mathbf{U}_\pi$, introduced in [CHW13] (cf. [HW15]) is an algebra defined via a super Cartan datum I (a finite indexing set associated to Kac-Moody superalgebras with no isotropic odd roots). $\mathbf{U}_\pi$ depends on two parameters q and π, where $\pi^2 = 1$. A quantum covering group specializes at $\pi = 1$ to the quantum group above, and at $\pi = -1$ to a quantum supergroup of anisotropic type (see [BKM98]). In addition to the usual Chevalley generators, we have generators J_i for each $i \in I$. If one writes K_i as q^{h_i}, then analogously we will have $J_i = \pi^{h_i}$. The parameter π can be seen as a shadow of a parity shift functor in .D. Hill and W. Wang's ([HW15]) categorification of quantum groups by the *spin* quiver Hecke superalgebras introduced in [KKT16]. Since then, further progress has been made on the odd/spin/super categorification of quantum covering groups; see [KKO14, EL16, BE17].

Just like for quantum groups, a theory of canonical bases for integrable modules of $\mathbf{U}_\pi$ and its modified (idempotented) form $\dot{\mathbf{U}}_\pi$ has been developed, in [CHW14, Cl14].

Quantum covering groups at roots of 1

A Drinfeld-Jimbo quantum group with the quantum parameter q admits an integral $\mathbb{Z}[q, q^{-1}]$-form; its specialization at q being a root of 1 were studied by G. Lusztig in [Lu90a, Lu90b], [Lu94, Part V] and also by many other authors. In these works Lusztig developed the quantum group version of Frobenius homomorphism and Frobenius kernel (known as small quantum groups), as a quantum analogue of several classical concepts arising from algebraic groups in a prime characteristic. The quantum groups at roots of 1 and their representation theory form a substantial part of Lusztig's program on modular representation theory, and they have further impacted other areas including geometric representation theory and categorification.

The first part of this dissertation details generalizations of these constructions to quantum covering groups in joint work with T. Sale and W. Wang [CSW18]. In Theorem 4.5, we formulate a Lusztig-Steinberg tensor product theorem for ${}_R\mathbf{f}$ the half quantum group at a root of unity, and then establish the Frobenius-Lusztig homomorphism between ${}_R\mathbf{f}$ and its quasi-classical counterpart ${}_R\mathbf{f}^\diamond$:

Theorem A (Theorem 4.7, Frobenius-Lusztig homomorphism). There exists a homomorphism $\mathrm{Fr} : {}_R\mathbf{f} \longrightarrow {}_R\mathbf{f}^\diamond$ which for all $i \in I, n$ sends the generators $\theta_i^{(n)}$ to $\theta_i^{(n/\ell_i)}$ if ℓ_i divides n, and to 0 otherwise.

We then show that the homomorphism Fr can be extended to the modified quantum covering group in Theorem 4.8. We then use this to define the small quantum covering group and show that it is a finite-dimensional Hopf algebra when $\mathbf{U}_\pi$ is of finite type.

Quantum symmetric pairs

A quantum symmetric pair $(\mathbf{U}, \mathbf{U}^\imath)$ is a quantization of the symmetric pair of enveloping algebras $(\mathbf{U}(\mathfrak{g}), \mathbf{U}(\mathfrak{g}^\theta))$ where $\theta : \mathfrak{g} \to \mathfrak{g}$ is an involution of the Lie algebra $\mathfrak{g}$. Originally developed for applications in harmonic analysis for quantum group analogs of symmetric spaces, G. Letzter developed a comprehensive theory of quantum symmetric pairs for all

semisimple $\mathfrak{g}$ in [Le99]. The algebraic theory of quantum symmetric pairs was subsequently extended to the setting of Kac-Moody algebras in [Ko14]. The *ıquantum group* $\mathbf{U}^\imath$ is a subalgebra of the quantum group $\mathbf{U}$ satisfying a *coideal property*; coideal subalgebras provide important substructure for $\mathbf{U}$, since Hopf subalgebras are rare 'in nature'.

More recent developments have made it apparent that quantum symmetric pairs play an important role in representation theory at large. In a series of groundbreaking papers, H. Bao and W. Wang proposed a program of canonical bases for quantum symmetric pairs [BW18a, BW18b, BW18c]. They performed their program for the Type AIII/IV symmetric pairs $(\mathfrak{sl}_{2N}, \mathfrak{s}(\mathfrak{gl}_N \times \mathfrak{gl}_N))$ and $(\mathfrak{sl}_{2N+1}, \mathfrak{s}(\mathfrak{gl}_N \times \mathfrak{gl}_{N+1}))$ and applied it to tensor products of their $\mathbf{U}^\imath$-modules, establishing a Kazhdan-Lusztig theory and irreducible character formula for the category $\mathcal{O}$ of the ortho-symplectic Lie superalgebra $\mathfrak{osp}(2n+1\,|\,2m)$, a feat for which they were awarded the 2020 Chevalley prize in Lie Theory. Together with previously known results, these recent developments suggest that quantum symmetric pairs allow as deep a theory as quantized enveloping algebras themselves. In fact, $\mathbf{U}$ can be viewed as a special type of quantum symmetric pair, the *diagonal quantum symmetric pair* $(\mathbf{U} \otimes \mathbf{U}, \imath(\mathbf{U}))$ where $\imath = (\omega \otimes 1)\Delta : \mathbf{U} \to \mathbf{U} \otimes \mathbf{U}$. It is thus reasonable to expect that many results about quantized groups have their counterparts in the realm of quantum symmetric pairs.

A fundamental property of any quantum group $\mathbf{U}$ is the existence of a universal R-matrix, an element in the completion of a tensor product of $\mathbf{U}$ with itself which gives rise to solutions of the quantum Yang-Baxter equation for suitable representations of $\mathbf{U}$. The existence of a universal R-matrix is crucial to V. Drinfeld and M. Jimbo's investigation into the theory of quantum integrable systems [Dri87], [Jim85], and has applications to the construction of knot invariants [RT90]. The analog of the quantum Yang-Baxter equation for quantum symmetric pairs is known as the boundary quantum Yang-Baxter equation, or the (quantum) reflection equation cf. [Che84]. An an element providing solutions of the reflection equation in all representations is called a 'universal K-matrix', a term first used in E. Sklyanin's investigation of quantum integrable models with non-periodic boundary conditions [Skl88], [KS92].

For a quantum group $\mathbf{U}$ with negative part $\mathbf{U}^-$ and postive part $\mathbf{U}^+$, the quasi R-matrix for $\mathbf{U}$ is a canonical element in a completion of $\mathbf{U}^- \otimes \mathbf{U}^+$ which appears as an intertwiner of two bar involutions on $\Delta(\mathbf{U})$. The quasi R-matrix has a simpler expression than the universal R-matrix, and is used in [Lu94, Part IV] to construct canonical bases for tensor products of $\mathbf{U}$ and $\dot{\mathbf{U}}$. For quantum symmetric pairs an analogue intertwining the bar involutions on $\mathbf{U}^\imath$ and $\mathbf{U}$, the *quasi K-matrix* is constructed for special cases in [BW18a] and more generally in [BaK19]. An immediate application of the quasi K-matrix in [BW18a, BW18b, BW18c] is the construction of canonical bases for tensor products of $\mathbf{U}^\imath$-modules and $\dot{\mathbf{U}}^\imath$.

For the negative half $\mathbf{U}^-$ of the quantum group in rank one $\mathbf{U} = \mathbf{U}_q(\mathfrak{sl}_2)$, the Lusztig divided powers are monomials in a single variable F, and they form the canonical basis for $\mathbf{U}^-$. The canonical basis for $\mathbf{U}^\imath$ in rank one is formed by the *$\imath$-divided powers*, introduced in [BW18b, BW18c] and further explored in [BeW18]. Instead of being monomials, they are polynomials in a single variable B. They give bases for finite-dimensional simple $\mathfrak{sl}_2$-modules, and have two different formulas, $B^{(n)}_{\mathrm{ev}}$ and $B^{(n)}_{\overline{1}}$, depending on the parity of the corresponding highest weight, which is a non-negative integer.

The $\imath$-divided powers and their expansion formulas in [BeW18] formed a cornerstone of the construction of the Serre presentation for quasi-split $\imath$-quantum groups established in H. Chen, M. Lu and W. Wang in [CLW18]. In [BW18b, BW18c], $\imath$-divided powers for $i \in I$ with $\tau i = i$ were defined using the same formulas, and then shown to generate as an algebra the integral form ${}_{\mathcal{A}}\dot{\mathbf{U}}^\imath$ of the modified quantum group.

Serre presentation for $\imath$quantum covering groups

For quantum covering groups $\mathbf{U}_\pi$ of super Kac-Moody type and a diagram involution τ, (quasi-split) quantum symmetric pairs $(\mathbf{U}_\pi, \mathbf{U}^\imath_\pi)$ we define $\mathbf{U}^\imath_\pi$ with generators K_μ, $\widetilde{J}_i$ (products of K_i and J_i respectively which keep track of the q and π grading), and B_i satisfying the embedding

formula

$$B_i = F_i + q_i^{-1} E_{\tau i} K_i^{-1}.$$

The *ıquantum covering group* $\mathbf{U}^\imath_\pi$ is a right coideal subalgebra.

The $\imath$-divided powers above have a generalization to $\mathbf{U}^\imath_\pi$, the *$\imath^\pi$-divided powers* $B^{(m)}_{i,\bar{1}}$ and $B^{(m)}_{i,\bar{0}}$ which are given in the formulas (7.31) and (7.32) below for $i \in I$ with $\tau i = i$. The new facets π and J of quantum covering groups are incorporated into these formulas, and when we specialize at $\pi = 1$ and $\widetilde{J}_i = 1$, we obtain the $\imath$-divided powers above. The $\imath^\pi$-divided powers also satisfy a collection of expansion formulas which are instrumental in the following result:

Theorem B (Theorem 8.1, Serre presentation for $\mathbf{U}^\imath_\pi$). $\mathbf{U}^\imath_\pi$ is generated by B_i K_μ, $\widetilde{J}_i$, subject to a few standard relations and the following:

$$\text{If } \tau i = i \neq j, \quad \sum_{n=0}^{1-a_{ij}} (-1)^n \pi_i^{n+\binom{n}{2}} B^{(n)}_{i,\overline{a_{ij}}+\overline{p_i}} B_j B^{(1-a_{ij}-n)}_{i,\overline{p}_i} = 0. \tag{B1}$$

$$\text{If } \tau i \neq i, \quad \sum_{n=0} /^{1-a_{i,\tau i}} (-1)^n \pi_i^{n+\binom{n}{2}} B_i^{(n)} B_{\tau i} B_i^{(1-a_{i,\tau i}-n)} = \frac{1}{\pi_i q_i - q_i^{-1}} \tag{B2}$$

$$\cdot \left(q_i^{a_{i,\tau i}} (\pi_i q_i^{-2}; \pi_i q_i^{-2})_{-a_{i,\tau i}} B_i^{(-a_{i,\tau i})} \mathcal{Z}_i \; - (\pi_i q_i^2; \pi_i q_i^2)_{-a_{i,\tau i}} B_i^{(-a_{i,\tau i})} \mathcal{Z}_{\tau i} \right).$$

Relation (B1) is the $\imath^\pi$-Serre relation – when $\pi = 1$, this specializes to the $\imath$-Serre relation in [CLW18]. Note that here $p_i = 0$ or 1, giving a collection of alternate presentations. Similar to the proof therein (Theorem 4.8), the expansion formulas for the $\imath^\pi$-divided powers mentioned above can be used to reduce the proof of (B1) to a q-binomial identity. The second relation (B2) specializes to [BaK15, Theorem 3.6]) when $\pi = 1$.

Quasi K-matrix and canonical basis for $\imath$quantum covering groups

As foreshadowed above above, the quasi K-matrix is a natural starting point in the investigation of a theory of canonical basis for $\mathbf{U}^\imath_\pi$. For regular quantum groups, the bar involutions $\psi_\imath$ on $\mathbf{U}^\imath$ and ψ on $\mathbf{U}$ are not compatible; $\psi_\imath$ is not simply the restriction of ψ to the subalgebra

$\mathbf{U}^\imath$. Thus, one can define a quasi K-matrix Υ that 'intertwines' these two bar involutions. In the case of the diagonal quantum symmetric pair, the quasi K-matrix arises naturally from Lusztig's quasi $\mathcal{R}$-matrix. An application of the quasi K-matrix is transforming involutive based $\mathbf{U}$-modules ($\mathbf{U}$-modules with distinguished bases compatible with the bar-involution ψ on $\mathbf{U}$), into involutive based $\mathbf{U}^\imath$-modules, compatible with the bar-involution $\psi_\imath$ on $\mathbf{U}^\imath$. In the quantum covering setting, the quasi $\mathcal{R}$-matrix is constructed in [CHW13, Theorem 3.1.1]. In this dissertation, we have the following generalization for quasi-split $\mathbf{U}^\imath_\pi$:

Theorem C (Theorem 9.2). There exists a unique family of elements $\Upsilon_\mu \in (\mathbf{U}^+_\pi)_\mu$ such that $\Upsilon_0 = 1$ and $\Upsilon = \sum_\mu \Upsilon_\mu$ where $p(\mu)$ is even, satisfying the following identity in $\widehat{\mathbf{U}_\pi}$:

$$\psi_\imath(u)\Upsilon = \Upsilon\psi(u), \quad \text{for all } u \in \mathbf{U}^\imath_\pi.$$

When we specialize at $\pi = 1$, we obtain the quasi K-matrix of [BW18a] and [BaK19]. In rank one i.e. when I is a single odd root, the quasi K-matrix takes on the form

$$\Upsilon = \sum_{k\geq 0}(-\pi)^k(\pi q - q^{-1})^k q^{2k-k^2}[2k-1]^{!!}_\pi E^{(2k)},$$

where $[2k-1]^{!!}_\pi := [2k-1]_\pi \cdot [2k-3]_\pi \cdot \ldots \cdot [1]$.

The quasi K-matrix Υ is invertible, and its inverse is obtained by applying the bar involution. Crucially, Υ has the property that it preserves the integrality of the $\mathcal{A}$-forms of integrable highest weight $\mathbf{U}^\imath_\pi$-modules and their tensor products. Using this property of integrality of the action of their quasi K-matrix, Bao and Wang defined in [BW18a, BW18c] a new bar involution on based $\mathbf{U}$-modules (modules M with a distinguished basis B, and compatible involution ψ) thus enabling the construction of $\imath$-canonical bases of these modules (which are now based $\mathbf{U}^\imath$-modules) from their canonical bases. With the $\imath^\pi$-divided powers above, these constructions also lead to a theory of canonical basis for integrable based $\mathbf{U}^\imath_\pi$-modules - examples include highest weight integrable modules and their tensor products (following [BW16]).

Here the integral form is now over $\mathcal{A}^\pi := \mathbb{Z}^\pi[q, q^{-1}]$, and we have a π-*basis* - a 'signed' basis that for the half quantum group $\mathbf{f}$ specializes to the Lusztig-Kashiwara canonical basis when $\pi = 1$, and when $\pi = -1$ specializes to Lusztig's signed basis [Lu94, Chapter 14].

Theorem D (Theorem 10.2). Let (M, B) be a based $\mathbf{U}_\pi$-module whose weights are bounded above. Assume the involution $\psi_\imath := \Upsilon \circ \psi$ of M preserves the $\mathcal{A}^\pi$-submodule ${}_{\mathcal{A}}M$. The $\mathbf{U}^\imath$-module M admits a unique π-basis $B^\imath := \{b^\imath | b \in B\}$, which is $\psi_\imath$-invariant and of the form

$$b^\imath = b + \sum_{b' \in B, b' < b} t_{b;b'} b', \quad \text{for} \quad t_{b;b'} \in q^{-1}\mathbb{Z}^\pi[q^{-1}].$$

$B^\imath$ forms an $\mathcal{A}^\pi$-basis for the $\mathcal{A}^\pi$-lattice ${}_{\mathcal{A}}M$ (generated by B), and forms a $\mathbb{Z}^\pi[q^{-1}]$-basis for the $\mathbb{Z}^\pi[q^{-1}]$-lattice $\mathcal{M}$ (generated by B).

We conclude by constructing a canonical basis for the modified form $\dot{\mathbf{U}}^\imath_\pi$, generalizing [BW18b, BW18c]:

Theorem E (Theorem 10.10). Let $\zeta_\imath \in X_\imath$ and $(b_1, b_2) \in B \times B$. The set

$$\dot{\mathbb{B}}^\imath = \{b_1 \diamondsuit^\imath_{\zeta_\imath} b_2 \big| \zeta_\imath \in X_\imath, (b_1, b_2) \in B \times B\}$$

forms a $\mathbb{K}(q)$-basis of $\dot{\mathbf{U}}^\imath$ and an $\mathcal{A}^\pi$-basis of ${}_{\mathcal{A}}\dot{\mathbf{U}}^\imath$, where $b_1 \diamondsuit^\imath_{\zeta_\imath} b_2$ is $\psi_\imath$-invariant and is the unique element $b_1 \diamondsuit^\imath_{\zeta_\imath} b_2 = u \in \dot{\mathbf{U}}^\imath$ such that for all $\lambda, \mu \gg 0$ with $\overline{\lambda + \mu} = \zeta_\imath$,

$$u(\eta_\lambda \otimes \eta_\mu) = (b_1 \diamondsuit_{\zeta_\imath} b_2)^\imath_{\lambda,\mu} \in L^\imath(\lambda, \mu) = L(\lambda + \mu).$$

Organization

The first part of the dissertation is organized as follows: In Chapter 2, we cover the foundational details of quantum covering groups. In Section 3.1, we establish several basic properties of the (q, π)-binomial coefficients at roots of 1, generalizing Lusztig [Lu94, Chapter 34]. In the

same chapter, we recall the definitions of the half quantum covering group ${}_R\mathbf{f}$ and the whole (respectively, the modified) quantum covering group $\mathbf{U}$ (respectively, ${}_R\dot{\mathbf{U}}$) over some ring R^π, associated to a super Cartan datum. We give a presentation of ${}_R\dot{\mathbf{U}}$ and a presentation of the quasi-classical counterpart ${}_R\mathbf{f}^\diamond$ of ${}_R\mathbf{f}$, generalizing [Lu94, 33.2].

Our Chapter 4 is a generalization of [Lu94, Chapter 35]. We establish in Theorem 4.1 a R^π-superalgebra homomorphism $\mathrm{Fr}' : {}_R\mathbf{f}^\diamond \longrightarrow {}_R\mathbf{f}$, which sends the generators $\theta_i^{(n)}$ to $\theta_i^{(n\ell_i)}$ for all $i \in I, n$. This is followed by the Lusztig-Steinberg tensor product theorem for ${}_R\mathbf{f}$ which we prove in Theorem 4.5. Next we establish in Theorem 4.7 the Frobenius-Lusztig homomorphism $\mathrm{Fr} : {}_R\mathbf{f} \longrightarrow {}_R\mathbf{f}^\diamond$ which sends the generators $\theta_i^{(n)}$ to $\theta_i^{(n/\ell_i)}$ if ℓ_i divides n, and to 0 otherwise, for all $i \in I, n$. We further extend the homomorphism Fr to the modified quantum covering group in Theorem 4.8.

Finally in Chapter 5, we formulate the small quantum covering groups and investigate its Hopf algebra structure. In the finite type case corresponding to type $B(0, n)$, we show that the small quantum covering group is finite dimensional, and we compute its dimension.

In Part 2, we discuss key constructions for quasi-split quantum symmetric pairs $(\mathbf{U}_\pi, \mathbf{U}_\pi^\imath)$ for quantum covering groups, dropping the subscript π. In Chapter 6 we introduce the $\imath$quantum covering group $\mathbf{U}^\imath$, giving its structure and size. In the following chapter, we introduce the $\imath^\pi$-divided powers and prove a handful of their expansion formulas, which we will use to prove the validity of the $\imath^\pi$-Serre relations in Chapter 8. The rest of Chapter 8 contains a statement and proof of the Serre presentation for quasi-split $\mathbf{U}^\imath$, which uses an approach inspired by [CLW18], reducing the main result to the (q, π)-binomial identity in §8.3.

In Chapter 9, we see that the Serre presentation enables the definition of a bar involution on $\mathbf{U}^\imath$ (§9.1), and in the same chapter a quasi K-matrix Υ intertwining this bar involution is constructed, and we show that Υ preserves the integral forms of various based modules and their tensor products. Finally, in Chapter 10, a theory of canonical basis for tensor products of $\mathbf{U}^\imath$-modules and $\dot{\mathbf{U}}^\imath$ is formulated, using a quasi R-matrix $\Theta^\imath$ for $\mathbf{U}^\imath$ constructed from Υ.

Notation

As a remark: we will drop the subscript π from $\mathbf{U}_\pi$ and related notation in the following chapters, so $\mathbf{U}$ will be understood to refer to the quantum covering group. We will explicitly mention when we are referring to the usual quantum group e.g. when we specialize $\pi = 1$.

Part I

Quantum Covering Groups at Roots of 1

Chapter 2

Quantum Covering Groups

In this chapter, we will give an overview of the details of quantum covering groups. We will go over basic notation, conventions and constructions that are fundamental to the main results in the subsequent chapters.

2.1 Foundations and structure

We start by recalling the definition of a quantum covering group from [CHW13] starting with a *super Cartan datum* and a root datum.

Super Cartan data

A *Cartan datum* is a pair $(I, \cdot)$ consisting of a finite set I and a symmetric bilinear form $\nu, \nu' \mapsto \nu \cdot \nu'$ on the free abelian group $\mathbb{Z}[I]$ with values in $\mathbb{Z}$ satisfying

(a) $d_i = \frac{i \cdot i}{2} \in \mathbb{Z}_{>0}$;

(b) $2\frac{i \cdot j}{i \cdot i} \in -\mathbb{N}$ for $i \neq j$ in I.

If the datum can be decomposed as $I = I_0 \coprod I_1$ such that

(c) $I_1 \neq \emptyset$,

(d) $2\frac{i\cdot j}{i\cdot i} \in 2\mathbb{Z}$ if $i \in I_1$,

then it is called a *super Cartan datum*; cf. [CHW13].

We denote the parity $p(i) = 0$ for $i \in I_0$ and $p(i) = 1$ for $i \in I_1$. Following [CHW13], we will always assume a super Cartan datum satisfies the additional *bar-consistent* condition:

(e) $\frac{i\cdot i}{2} \equiv p(i) \mod 2, \quad$ for all $i \in I$.

This condition is always satisfied for super Cartan data of finite or affine type, with one exception.

Note that (d) and (e) imply that

(f) $i \cdot j \in 2\mathbb{Z}$ for all $i, j \in I$.

The $i \in I_{\bar{\mathbf{0}}}$ are called even, $i \in I_{\bar{\mathbf{1}}}$ are called odd. We extend the parity function $p : I \to \{0, 1\}$ to the homomorphism $p : \mathbb{Z}[I] \to \mathbb{Z}$. Then p induces a $\mathbb{Z}_2$-grading on $\mathbb{Z}[I]$ which we shall call the parity grading.

A super Cartan datum $(I, \cdot)$ is said to be of *finite* (resp. *affine*) type exactly when $(I, \cdot)$ is of finite (resp. affine) type as a Cartan datum (cf. [Lu94, §2.1.3]). In particular, the only super Cartan datum of finite type is type $B(0, n)$ for $n \geq 1$; the corrresponding the Lie superalgebras are the orthosymplectic Lie superalgebras $\mathfrak{osp}(1|2n)$.

A *root datum* associated to a super Cartan datum $(I, \cdot)$ consists of

(a) two finitely generated free abelian groups Y, X and a perfect bilinear pairing $\langle \cdot, \cdot \rangle : Y \times X \to \mathbb{Z}$;

(b) an embedding $I \subset X$ $(i \mapsto i')$ and an embedding $I \subset Y$ $(i \mapsto i)$ satisfying

(c) $\langle i, j' \rangle = \frac{2i\cdot j}{i\cdot i}$ for all $i, j \in I$.

We will always assume that the root datum is X-*regular* (respectively Y-*regular*) image of the embedding $I \subset X$ (respectively, the image of the embedding $I \subset Y$) is linearly independent in X (respectively, in Y).

We also define a partial order $\leq$ on the weight lattice X as follows: for $\lambda, \lambda' \in X$,

$$\lambda \leq \lambda' \text{ if and only if } \lambda' - \lambda \in \mathbb{N}[I]. \tag{2.1}$$

The matrix $A := (a_{ij}) := \langle i, j' \rangle$ is a *symmetrizable generalized super Cartan matrix*: if $D = \mathrm{diag}(d_i \,|\, i \in I)$, then DA is symmetric.

Let π be a parameter such that

$$\pi^2 = 1.$$

For any $i \in I$, we set

$$q_i = q^{i \cdot i/2}, \qquad \pi_i = \pi^{p(i)}.$$

Note that when the datum is consistent, $\pi_i = \pi^{\frac{i \cdot i}{2}}$; by induction, we therefore have $\pi^{p(\nu)} = \pi^{\nu \cdot \nu/2}$ for $\nu \in \mathbb{Z}[I]$. We extend this notation so that if $\nu = \sum \nu_i i \in \mathbb{Z}[I]$, then

$$q_\nu = \prod_i q_i^{\nu_i}, \qquad \pi_\nu = \prod_i \pi_i^{\nu_i}.$$

For any ring R we define a new ring $R^\pi = R[\pi]/(\pi^2 - 1)$ (with π commuting with R). Below, we will work over $\mathbb{K}(q)^\pi$ where $\mathbb{K}$ is a field of characteristic 0, and we will also consider algebras over the ring $\mathcal{A}^\pi$, where $\mathcal{A} = \mathbb{Z}[q, q^{-1}]$.

Recall also the (q, π)-*integers* and (q, π)-*binomial coefficients* in [CHW13]: we shall denote

$$[n] = \begin{bmatrix} n \\ 1 \end{bmatrix} = \frac{(\pi q)^n - q^{-n}}{\pi q - q^{-1}} \quad \text{for } n \in \mathbb{Z},$$

$$[n]^! = \prod_{s=1}^{n} [s] \quad \text{for } n \in \mathbb{N},$$

and with this notation we have

$$\begin{bmatrix} m \\ n \end{bmatrix} = \frac{[m]^!}{[n]^![m-n]^!} \quad \text{for } 0 \leq n \leq m.$$

We denote by $[n]_i, [m]_i^!$, and $\begin{bmatrix} n \\ m \end{bmatrix}_i$ the variants of $[n], [m]!$, and $\begin{bmatrix} n \\ m \end{bmatrix}$ with q replaced by q_i and π replaced by π_i, and $\begin{bmatrix} m \\ n \end{bmatrix}_{q^2}$ the variant with q replacing q^2.

For any $i \neq j$ in I, we define the following polynomial in two (noncommutative) variables x and y:

$$F_{ij}(x,y) = \sum_{n=0}^{1-a_{ij}} (-1)^n \pi_i^{np(j)+\binom{n}{2}} \begin{bmatrix} 1-a_{ij} \\ n \end{bmatrix}_i x^n y x^{1-a_{ij}-n}. \tag{2.2}$$

The quantum covering group

Let $\mathbf{U}$ denote the quantum covering group associated to the root datum $(Y, X, ...)$ introduced in [CHW13]. By [CHW13, Proposition 3.4.2], $\mathbf{U}$ is a unital $\mathbb{K}(q)^\pi$-superalgebra with generators

$$E_i \quad (i \in I), \quad F_i \quad (i \in I), \quad J_\mu \quad (\mu \in Y), \quad K_\mu \quad (\mu \in Y),$$

subject to the relations (a)-(f) below for all $i, j \in I, \mu, \mu' \in Y$:

$$E_i \quad (i \in I), \quad F_i \quad (i \in I), \quad J_\mu \quad (\mu \in Y), \quad K_\mu \quad (\mu \in Y),$$

with parity $p(E_i) = p(F_i) = p(i)$ and $p(K_\mu) = p(J_\mu) = 0$, subject to the relations (a)-(f) below for all $i, j \in I, \mu, \mu' \in Y$:

$$K_0 = 1, \quad K_\mu K_{\mu'} = K_{\mu+\mu'}, \tag{R1}$$

$$J_{2\mu} = 1, \quad J_\mu J_{\mu'} = J_{\mu+\mu'}, \tag{R2}$$

$$J_\mu K_{\mu'} = K_{\mu'} J_\mu, \tag{R3}$$

$$K_\mu E_i = q^{\langle \mu, i' \rangle} E_i K_\mu, \quad J_\mu E_i = \pi^{\langle \mu, i' \rangle} E_i J_\mu, \tag{R4}$$

$$K_\mu F_i = q^{-\langle \mu, i' \rangle} F_i K_\mu, \quad J_\mu F_i = \pi^{-\langle \mu, i' \rangle} F_i J_\mu, \tag{R5}$$

$$E_i F_j - \pi^{p(i)p(j)} F_j E_i = \delta_{i,j} \frac{\widetilde{J}_i \widetilde{K}_i - \widetilde{K}_{-i}}{\pi_i q_i - q_i^{-1}}, \tag{R6}$$

$$(q,\pi)\text{-Serre relations} \qquad F_{ij}(E_i, E_j) = 0 = F_{ij}(F_i, F_j), \text{ for all } i \neq j. \tag{R7}$$

where for any element $\nu = \sum_i \nu_i i \in \mathbb{Z}[I]$ we have set $\widetilde{K}_\nu = \prod_i K_{d_i \nu_i i}$, $\widetilde{J}_\nu = \prod_i J_{d_i \nu_i i}$. In particular, $\widetilde{K}_i = K_{d_i i}$, $\widetilde{J}_i = J_{d_i i}$. Under the bar-consistency condition (e), $\widetilde{J}_i = 1$ for $i \in I_{\overline{\mathbf{0}}}$ while $\widetilde{J}_i = J_i$ for $i \in I_{\overline{\mathbf{0}}}$. Note that by the same condition a_{ij} is always even for $i \in I_{\overline{\mathbf{1}}}$, and so J_i is central for all $i \in I$. As usual, denote by $\mathbf{U}^-$, $\mathbf{U}^+$ and $\mathbf{U}^0$ the subalgebras of $\mathbf{U}$ generated by $\{E_i \,|\, i \in I\}$, $\{F_i \,|\, i \in I\}$ and $\{J_\mu, K_\mu \,|\, \mu \in Y\}$ respectively. Also denote $\mathbf{U}^{0'} = \{J_i, K_i \,|\, i \in I\}$. We endow $\mathbf{U}$ with a $\mathbb{Z}[I]$-grading $|\cdot|$ by setting $|E_i| = i$, $\quad |F_i| = -i$, $\quad |J_\mu| = |K_\mu| = 0$. The parity on $\mathbf{U}$ is given by $p(E_i) = p(F_i) = p(i)$ and $p(K_\mu) = p(J_\mu) = 0$.

The specialization at $\pi = 1$ of the algebra $\mathbf{U}$, which we will denote by $\mathbf{U}|_{\pi=1}$, is a variant of the usual Drinfeld-Jimbo quantum group with extra central elements J_μ, with many properties specializing to that of [Lu94], cf. [CHW13]. The specialization at $\pi = -1$ of the algebra $\mathbf{U}/\mathcal{J}$ is naturally identified with a quantum group associated to the Cartan datum $(I, \cdot)$.

If we write $F_i^{(n)} = F_i^n/[n]_i^!$ and $E_i^{(n)} = E_i^n/[n]_i^!$ for $n \geq 1$ and $i \geq 1$, then the (q,π)-Serre relations (R7) can be rewritten as:

$$\sum_{n=0}^{1-a_{ij}} (-1)^n \pi_i^{np(j)+\binom{n}{2}} F_i^{(n)} F_j F_i^{(1-a_{ij}-n)} = 0 \tag{2.3}$$

and

$$\sum_{n=0}^{1-a_{ij}} (-1)^n \pi_i^{np(j)+\binom{n}{2}} E_i^{(n)} E_j E_i^{(1-a_{ij}-n)} = 0. \tag{2.4}$$

By [CHW13, Propositions 1.4.1, 3.4.1], the unital $\mathbb{Q}(q)^\pi$-superalgebra $\mathbf{f}$ is generated by θ_i

$(i \in I)$ subject to the super Serre relations

$$\sum_{n+n'=1-\langle i,j'\rangle} (-1)^{n'} \pi_i^{n'p(j)+\binom{n'}{2}} \theta_i^{(n)} \theta_j \theta_i^{(n')} = 0$$

for any $i \neq j$ in I; here a generator θ_i is even if and only if $i \in I_0$. There is an $\mathcal{A}^\pi$-form for $\mathbf{f}$, which we call ${}_{\mathcal{A}}\mathbf{f}$. It is generated by the divided powers $\theta_i^{(n)} = \theta_i^n/[n]^!_{q_i,\pi_i}$ for all $i \in I, n \geq 1$. As R^π is an $\mathcal{A}^\pi$-algebra (cf. §3.1), by a base change we define ${}_R\mathbf{f} = R^\pi \otimes_{\mathcal{A}^\pi} {}_{\mathcal{A}}\mathbf{f}$.

The algebra $\mathbf{U}$ has an $\mathcal{A}^\pi$-form ${}_{\mathcal{A}}\mathbf{U}$. By a base change, we obtain ${}_R\mathbf{U} = R^\pi \otimes_{\mathcal{A}^\pi} {}_{\mathcal{A}}\mathbf{U}$. Let ${}_R\mathbf{U}^+$ (resp. ${}_R\mathbf{U}^-$) denote the subalgebra of ${}_R\mathbf{U}$ generated by the $E_i^{(n)} = E_i^n/[n]^!_{\mathbf{q}_i,\pi_i}$ (resp. $F_i = F_i^n/[n]^!_{\mathbf{q}_i,\pi_i}$). As a R^π-algebra ${}_R\mathbf{f}$ is isomorphic to ${}_R\mathbf{U}^+$ (resp. ${}_R\mathbf{U}^-$) via the map $x \mapsto x^+$ (resp. $x \mapsto x^-$), where $(\theta_i^{(n)})^+ = E_i^{(n)}$ (resp. $(\theta_i^{(n)})^- = F_i^{(n)}$.

Denote by $X^+ = \{\lambda \in X \mid \langle i, \lambda\rangle \in \mathbb{N}, \text{ for all } i \in I\}$, the set of dominant integral weights.

For $\lambda \in X$, let $M(\lambda)$ be the Verma module of $\mathbf{U}$, and we can naturally identify $M(\lambda) = \mathbf{f}$ as $\mathbb{K}(q)^\pi$-modules. The ${}_{\mathcal{A}}\mathbf{U}$-submodule ${}_{\mathcal{A}}M(\lambda)$ can be identified with ${}_{\mathcal{A}}\mathbf{f}$ as $\mathcal{A}^\pi$-free modules. For $\lambda \in X^+$, we define the integrable $\mathbf{U}$-module $V(\lambda) = M(\lambda)/J_\lambda$, where J_λ is the left $\mathbf{f}$-module generated by $\theta_i^{\langle i,\lambda\rangle+1}$ for all $i \in I$. Let ${}_RM(\lambda) = R^\pi \otimes_{\mathcal{A}^\pi} {}_{\mathcal{A}}M(\lambda)$ for $\lambda \in X$, and ${}_RV(\lambda) = R^\pi \otimes_{\mathcal{A}^\pi} {}_{\mathcal{A}}V(\lambda)$ for $\lambda \in X^+$.

The following lemmas on the twisted derivation (defined in [CHW13, §1.5]) will be important tools for the construction of the quasi K-matrix in part 3. The first is from [CHW13, Lemma 1.5.2]) (cf. [Lu94, Lemma 1.2.15] for the quantum group version):

Lemma 2.1. *Let $x \in \mathbf{f}_\nu$ where $\nu \in \mathbb{N}[I]$ is nonzero.*

(a) If $r_i(x) = 0$ for all $i \in I$, then $x = 0$.

(b) If ${}_ir(x) = 0$ for all $i \in I$, then $x = 0$.

Just as in [BW18a], the following lemma will play a useful role (cf. [BW18a, Lemma 1.1])

Lemma 2.2. *${}_jr \circ r_i = r_i \circ {}_jr$ for all $i, j \in I$*

Proof. It suffices to show this for homogeneous $x \in {}'\mathbf{f}_\mu$, using induction on the height of μ; for $x = 1$ both sides are identically 0, and from the inductive definition, we have

$$r_j \circ {}_i r(xy) = {}_i r(x) r_j(y) + \pi^{p(y)p(j)} q^{|y|\cdot j} r_j({}_i r(x)) y + \pi^{p(x)p(i)} q^{|x|\cdot i} x r_j({}_i r(y)) \\ + \pi^{p(x)\cdot p(i) + p({}_i r(y))p(j)} q^{|x|\cdot i + |{}_i r(y)|\cdot j} r_j(x) {}_i r(y)$$

and

$${}_i r \circ r_j(xy) = {}_i r(x) r_j(y) + \pi^{p(y)p(j)} q^{|y|\cdot j} {}_i r(r_j(x)) y + \pi^{p(x)p(i)} q^{|x|\cdot i} x {}_i r(r_j(y)) \\ + \pi^{p(y)\cdot p(j) + p(r_j(x))p(i)} q^{|y|\cdot j + |r_j(x)|\cdot i} r_j(x) {}_i r(y),$$

and since $p(r_k(z)) = p(z) - p(k)$, the π powers in the last term of each of the two expressions on the right is equal to $p(x)p(i) + p(y)p(j) - p(i)p(j)$; similarly $|r_k(z)| = |z| - k$ so the q powers are $|x| \cdot i + |y| \cdot j - i \cdot j$, and so the two expressions agree by application of the inductive hypothesis. □

Here, as in [CHW13], we will use the following conventions for the comultiplication:

$$\Delta(E_i) = E_i \otimes 1 + \widetilde{J}_i \widetilde{K}_i \otimes E_i \quad (i \in I) \qquad \Delta(F_i) = F_i \otimes \widetilde{K}_{-i} + 1 \otimes F_i \quad (i \in I), \tag{2.5}$$

$$\Delta(K_\mu) = K_\mu \otimes K_\mu \quad (\mu \in Y) \qquad \Delta(J_\mu) = J_\mu \otimes J_\mu \quad (\mu \in Y). \tag{2.6}$$

2.2 The modified algebra $\dot{\mathrm{U}}$

In [Lu94, Chapter 23] a modified form of the quantum group is introduced, featuring orthogonal idempotents that behave like projections onto weight spaces. For quantum covering groups, the modified form $\dot{\mathbf{U}}$ is defined in [CFLW, Definition 4.2] to be the (non-unital) $\mathbb{K}(q)^\pi$-algebra generated by the symbols $\mathbf{1}_\lambda$ (the *orthogonal idempotents*), $E_i \mathbf{1}_\lambda$ and $F_i \mathbf{1}_\lambda$, for $\lambda \in X$

and $i \in I$, subject to the relations:

$$\mathbf{1}_\lambda \mathbf{1}_{\lambda'} = \delta_{\lambda,\lambda'} \mathbf{1}_\lambda,$$

$$(E_i \mathbf{1}_\lambda)\mathbf{1}_{\lambda'} = \delta_{\lambda,\lambda'} E_i \mathbf{1}_\lambda, \quad \mathbf{1}_{\lambda'}(E_i \mathbf{1}_\lambda) = \delta_{\lambda',\lambda+i'} E_i \mathbf{1}_\lambda,$$

$$(F_i \mathbf{1}_\lambda)\mathbf{1}_{\lambda'} = \delta_{\lambda,\lambda'} F_i \mathbf{1}_\lambda, \quad \mathbf{1}_{\lambda'}(F_i \mathbf{1}_\lambda) = \delta_{\lambda',\lambda-i'} F_i \mathbf{1}_\lambda,$$

$$(E_i F_j - \pi^{p(i)p(j)} F_j E_i)\mathbf{1}_\lambda = \delta_{ij} \left[\langle i, \lambda\rangle\right]_{v_i,\pi_i} \mathbf{1}_\lambda,$$

$$\textstyle\sum_{n+n'=1-\langle i,j'\rangle} (-1)^{n'} \pi_i^{n'p(j)+\binom{n'}{2}} E_i^{(n)} E_j E_i^{(n')} \mathbf{1}_\lambda = 0 \ \ (i \neq j),$$

$$\textstyle\sum_{n+n'=1-\langle i,j'\rangle} (-1)^{n'} \pi_i^{n'p(j)+\binom{n'}{2}} F_i^{(n)} F_j F_i^{(n')} \mathbf{1}_\lambda = 0 \ \ (i \neq j),$$

where $i, j \in I$, $\lambda, \lambda' \in X$, and we use the notation $xy\mathbf{1}_\lambda = (x\mathbf{1}_{\lambda+|y|})(y\mathbf{1}_\lambda)$ for $x, y \in \mathbf{U}$. A more in-depth treatment of $\dot{\mathbf{U}}^\imath$ can be found in [Cl14], and covers its tensor modules and canonical bases (§3.3 and §4 of *loc.cit.* respectively).

The modified quantum covering group $\dot{\mathbf{U}}$ admits an $\mathcal{A}^\pi$-form, ${}_{\mathcal{A}}\dot{\mathbf{U}}$ and so we can define ${}_R\dot{\mathbf{U}} = R^\pi \otimes_{\mathcal{A}^\pi} {}_{\mathcal{A}}\dot{\mathbf{U}}$. We will give here a presentation for ${}_R\dot{\mathbf{U}}$.

Lemma 2.3. *The modified quantum covering group ${}_R\dot{\mathbf{U}}$ is generated as an R^π-algebra by $x^+\mathbf{1}_\lambda x'^-$ or equivalently by $x^-\mathbf{1}_\lambda x'^+$, where $x \in {}_R\mathbf{f}_\mu, x' \in {}_R\mathbf{f}_\nu$ and $\lambda \in X$, subject to the following relations:*

$$\begin{aligned}
&(\theta_i^{(N)})^+ \mathbf{1}_\lambda (\theta_i^{(M)})^- \\
&\quad = \sum_{t\geq 0} \pi_i^{MN-\binom{t+1}{2}} (\theta_i^{(M-t)})^- \begin{bmatrix} M+N+\langle i,\lambda\rangle \\ t \end{bmatrix}_{\mathbf{q}_i,\pi_i} \mathbf{1}_{\lambda+(M+N-t)i'} (\theta_i^{(N-t)})^+, \\
&(\theta_i^{(N)})^- \mathbf{1}_\lambda (\theta_i^{(M)})^+ \\
&\quad = \sum_{t\geq 0} \pi_i^{MN+t\langle i,\lambda\rangle-\binom{t}{2}} (\theta_i^{(M-t)})^+ \begin{bmatrix} M+N-\langle i,\lambda\rangle \\ t \end{bmatrix}_{\mathbf{q}_i,\pi_i} \mathbf{1}_{\lambda-(M+N-t)i'} (\theta_i^{(N-t)})^-,
\end{aligned}$$

$$(\theta_i^{(N)})^+ (\theta_j^{(M)})^- \mathbf{1}_\lambda = \pi^{MNp(i)p(j)} (\theta_j^{(M)})^- (\theta_i^{(N)})^+ \mathbf{1}_\lambda, \ \textit{for } i \neq j,$$

$$x^+\mathbf{1}_\lambda = \mathbf{1}_{\lambda+\mu}x^+, \quad x^-\mathbf{1}_\lambda = \mathbf{1}_{\lambda-\mu}x^-,$$

$$(x^+\mathbf{1}_\lambda)(\mathbf{1}_{\lambda'}x'^-) = \delta_{\lambda,\lambda'}x^+\mathbf{1}_\lambda x'^-, \quad (x^-\mathbf{1}_\lambda)(\mathbf{1}_{\lambda'}x'^+) = \delta_{\lambda,\lambda'}x^-\mathbf{1}_\lambda x'^+,$$

$$(x^+\mathbf{1}_\lambda)(\mathbf{1}_{\lambda'}x'^-) = \delta_{\lambda,\lambda'}\mathbf{1}_{\lambda+\mu}x^+x'^-, \quad (x^-\mathbf{1}_\lambda)(\mathbf{1}_{\lambda'}x'^+) = \delta_{\lambda,\lambda'}\mathbf{1}_{\lambda-\mu}x^-x'^+,$$

$$(rx + r'x')^\pm\mathbf{1}_\lambda = rx^\pm\mathbf{1}_\lambda + r'x'^\pm\mathbf{1}_\lambda, \ \textit{where } r, r' \in R^\pi.$$

Proof. This is proved in the same way as [Lu94, §31.1.3]. Let A be the R^π-algebra with the above generators and relations. All of these relations are known to hold in ${}_R\dot{\mathbf{U}}$. The first three are shown to hold in ${}_R\dot{\mathbf{U}}$ by a direct application of [CHW13, Lemma 2.2.3] as in [Cl14, Lemma 4] while the remaining ones are clear. However, there was an error in the second relation of [Cl14, Lemma 4], so we derived that relation from [CHW13, Lemma 2.2.3] in [CSW18]. We have

$$\begin{aligned}
&(\theta_i^{(N)})^-\mathbf{1}_\lambda(\theta_i^{(M)})^+ \\
&= (\theta_i^{(N)})^-(\theta_i^{(M)})^+\mathbf{1}_{\lambda-Mi'} \\
&= \sum_{t\geq 0}(-1)^t\pi_i^{(M-t)(N-t)-t^2}(\theta_i^{(M-t)})^+\begin{bmatrix}\tilde{K}_i; M+N-(t+1)\\ t\end{bmatrix}_{\mathbf{q}_i,\pi_i}(\theta_i^{(N-t)})^-\mathbf{1}_{\lambda-Mi'} \\
&= \sum_{t\geq 0}(-1)^t\pi_i^{(M-t)(N-t)-t^2}(\theta_i^{(M-t)})^+\begin{bmatrix}\langle i,\lambda\rangle - M - N + t - 1\\ t\end{bmatrix}_{\mathbf{q}_i,\pi_i}\mathbf{1}_{\lambda-(M+N-t)i'}(\theta_i^{(N-t)})^- \\
&= \sum_{t\geq 0}\pi_i^{MN+t\langle i,\lambda\rangle-\binom{t}{2}}(\theta_i^{(M-t)})^+\begin{bmatrix}M+N-\langle i,\lambda\rangle\\ t\end{bmatrix}_{\mathbf{q}_i,\pi_i}\mathbf{1}_{\lambda-(M+N-t)i'}(\theta_i^{(N-t)})^-
\end{aligned}$$

where in the last step, we used [CHW13, (1.10)] with $a = M + N - \langle i, \lambda\rangle$. Hence the natural homomorphism $A \longrightarrow {}_R\dot{\mathbf{U}}$ is surjective. Let $\mathbf{S}$ be an R^π-basis of ${}_R\mathbf{f}$ consisting of weight vectors. Then $\{x^+\mathbf{1}_\lambda x'^-|x, x' \in \mathbf{S}, \lambda \in X\}$ can be seen to be an R^π-basis for A, and it is known to be one for ${}_R\dot{\mathbf{U}}$ (cf. [Cl14, Lemma 5]). Thus, the natural homomorphism is, in fact,

an isomorphism. $\square$

Let $\mathcal{A} = \mathbb{Z}^\pi[q, q^{-1}]$. There is an $\mathcal{A}$-subalgebra ${}_\mathcal{A}\dot{\mathbf{U}}$ generated by $E_i^{(n)}\mathbf{1}_\lambda, F_i^{(n)}\mathbf{1}_\lambda$ for $i \in I$ and $n \geq 0$ and $\lambda \in X$. Note that $\dot{\mathbf{U}}$ is naturally a $\mathbf{U}$-bimodule, and in particular we have

$$K_h\mathbf{1}_\lambda = \mathbf{1}_\lambda K_h = q^{\langle h,\lambda\rangle}\mathbf{1}_\lambda, \quad \text{for all } h \in Y.$$

We have the mod 2 homomorphism $\mathbb{Z} \to \mathbb{Z}_2, k \mapsto \overline{k}$, where $\mathbb{Z}_2 = \{\overline{0}, \overline{1}\}$. Let us fix an $i \in I$. Define

$$\dot{\mathbf{U}}_{i,\mathrm{ev}} := \bigoplus_{\lambda:\, \langle h_i,\lambda\rangle \in 2\mathbb{Z}} \dot{\mathbf{U}}\mathbf{1}_\lambda, \qquad \dot{\mathbf{U}}_{i,\mathrm{odd}} := \bigoplus_{\lambda:\, \langle h_i,\lambda\rangle \in 1+2\mathbb{Z}} \dot{\mathbf{U}}\mathbf{1}_\lambda. \tag{2.7}$$

Then $\dot{\mathbf{U}} = \dot{\mathbf{U}}_{i,\mathrm{ev}} \oplus \dot{\mathbf{U}}_{i,\mathrm{odd}}$. Similarly, letting ${}_\mathcal{A}\dot{\mathbf{U}}_{i,\mathrm{ev}} = \dot{\mathbf{U}}_{i,\mathrm{ev}} \cap_\mathcal{A} \dot{\mathbf{U}}$ and ${}_\mathcal{A}\dot{\mathbf{U}}_{i,\mathrm{odd}} = \dot{\mathbf{U}}_{i,\mathrm{odd}} \cap_\mathcal{A} \dot{\mathbf{U}}$, we have ${}_\mathcal{A}\dot{\mathbf{U}} = {}_\mathcal{A}\dot{\mathbf{U}}_{i,\mathrm{ev}} \oplus {}_\mathcal{A}\dot{\mathbf{U}}_{i,\mathrm{odd}}$.

For our later use, with $i \in I$ fixed once for all, we need to keep track of the precise value $\langle h_i, \lambda\rangle$ in an idempotent $\mathbf{1}_\lambda$ but do not need to know which specific weights λ are used. Thus it is convenient to introduce the following generic notation

$$\mathbf{1}_m^\star = \mathbf{1}_{i,m}^\star, \qquad \text{for } m \in \mathbb{Z}, \tag{2.8}$$

to denote an idempotent $\mathbf{1}_\lambda$ for some $\lambda \in X$ such that $m = \langle h_i, \lambda\rangle$. In this notation, the identities in [Cl14] (with a correction provided in [CSW18, Lemma 3.2]) can be written as

follows: for any $m \in \mathbb{Z}$, $a, b \in \mathbb{Z}_{\geq 0}$, and $i \neq j \in I$,

$$E_i^{(a)} \mathbf{1}_{i,m}^{\star} = \mathbf{1}_{i,m+2a}^{\star} E_i^{(a)}, \quad F_i^{(a)} \mathbf{1}_{i,m}^{\star} = \mathbf{1}_{i,m-2a}^{\star} F_i^{(a)}; \tag{2.9}$$

$$E_j \mathbf{1}_{i,m}^{\star} = \mathbf{1}_{i,m+a_{ij}}^{\star} E_j, \qquad F_j \mathbf{1}_{i,m}^{\star} = \mathbf{1}_{i,m-a_{ij}}^{\star} F_j; \tag{2.10}$$

$$F_i^{(a)} E_i^{(b)} \mathbf{1}_{i,m}^{\star} = \sum_{j=0}^{\min\{a,b\}} \pi_i^{ab+jm+\binom{j}{2}} \begin{bmatrix} a-b-m \\ j \end{bmatrix}_i E_i^{(b-j)} F_i^{(a-j)} \mathbf{1}_{i,m}^{\star}; \tag{2.11}$$

$$E_i^{(a)} F_i^{(b)} \mathbf{1}_{i,m}^{\star} = \sum_{j=0}^{\min\{a,b\}} \pi_i^{ab+\binom{j+1}{2}} \begin{bmatrix} a-b+m \\ j \end{bmatrix}_i F_i^{(b-j)} E_i^{(a-j)} \mathbf{1}_{i,m}^{\star}. \tag{2.12}$$

From now on, we shall always drop the index i to write the idempotents as $\mathbf{1}_m^{\star}$.

Remark 2.4. If $u \in \mathbf{U}$ satisfies $u\mathbf{1}_{2k-1}^{\star} = 0$ for all possible idempotents $\mathbf{1}_{2k-1}^{\star}$ with $k \in \mathbb{Z}$ (or respectively, $u\mathbf{1}_{2k}^{\star} = 0$ for all possible $\mathbf{1}_{2k}^{\star}$ with $k \in \mathbb{Z}$), then $u = 0$.

Convention

We impose a mild *bar-consistent* assumption on the super Cartan datum in this paper, following [HW15, CHW14]. This assumption ensures that the new super Cartain datum and root datum arising from considerations of roots of 1 work as smoothly as one hopes. The assumption turns out to be also most appropriate again for the existence of Frobenius-Lusztig homomorphisms for quantum covering groups.

Chapter 3

Notation and formulas

In this chapter, we will introduce notation for the rest of this part, and establish several basic formulas of the (q, π)-binomial coefficients at roots of 1. They specialize to the formulas in [Lu94, Chapter 34] at $\pi = 1$. We also describe a presentation for the quasi-classical counterpart of modified quantum covering groups.

3.1 Identities for (q, π)-binomials at roots of 1

Let π and q be formal indeterminants such that $\pi^2 = 1$. Fix $\sqrt{\pi}$ such that $\sqrt{\pi}^2 = \pi$. In contrast to earlier papers on the quantum covering groups [CHW13, CHW14, CFLW, Cl14], it is often helpful and sometimes crucial for the ground rings considered in this paper to contain $\sqrt{\pi}$, and for the sake of simplicity we choose to do so uniformly from the outset. For any ring S with 1, define the new ring

$$S^\pi = S \otimes_{\mathbb{Z}} \mathbb{Z}[\sqrt{\pi}].$$

We shall use often the following two rings:

$$\mathcal{A} = \mathbb{Z}[q, q^{-1}], \qquad \mathcal{A}^\pi = \mathbb{Z}[q, q^{-1}, \sqrt{\pi}].$$

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$. For $a \in \mathbb{Z}$ and $n \in \mathbb{N}$, we define the (q, π)-*integer*

$$[a]_{q,\pi} = \frac{(\pi q)^a - q^{-a}}{\pi q - q^{-1}} \in \mathcal{A}^\pi,$$

and then define the corresponding (q, π)-factorials and (q, π)-binomial coefficients by

$$[n]^!_{q,\pi} = \prod_{i=1}^{n} [i]_{q,\pi}, \qquad \begin{bmatrix} a \\ n \end{bmatrix}_{q,\pi} = \frac{\prod_{i=1}^{n} [a+1-i]_{q,\pi}}{[n]^!_{q,\pi}}.$$

For an indeterminant v, we denote the v-integers

$$[a]_v = \frac{v^a - v^{-a}}{v - v^{-1}}$$

and we similarly define the v-factorials $[n]^!_v$ and v-binomial coefficients $\begin{bmatrix} a \\ n \end{bmatrix}_v$. We denote by $\binom{a}{n}$ the classical binomial coefficients.

In the rest of this chapter, the notation v is auxiliary, and we will identify

$$v := \sqrt{\pi} q,$$

and hence, for $n, t \in \mathbb{N}$,

$$[n]_{q,\pi} = \sqrt{\pi}^{n-1} [n]_v, \qquad [n]^!_{q,\pi} = \sqrt{\pi}^{n(n-1)/2} [n]^!_v,$$

$$\begin{bmatrix} n \\ t \end{bmatrix}_{q,\pi} = \sqrt{\pi}^{(n-t)t} \begin{bmatrix} n \\ t \end{bmatrix}_v. \tag{3.1}$$

Fix $\ell \in \mathbb{Z}_{>0}$ and let $\ell' = \ell$ or 2ℓ if ℓ is odd and let $\ell' = 2\ell$ if ℓ is even. Let

$$\mathcal{A}' = \mathcal{A}/\langle f(q)\rangle,$$

where $\mathcal{A}/\langle f(q)\rangle$ denotes the ideal generated by the ℓ'-th cyclotomic polynomial $f(q)$; we denote by $\varepsilon \in \mathcal{A}'$ the image of $q \in \mathcal{A}$. Take R to be an $\mathcal{A}'$-algebra with 1 (and so also an $\mathcal{A}$-algebra). Introduce the following root of 1 in R^π:

$$\mathbf{q} = \sqrt{\pi}\varepsilon \in R^\pi. \tag{3.2}$$

Then the element

$$\mathbf{v} := \sqrt{\pi}\mathbf{q} \in R^\pi$$

satisfies that

$$\mathbf{v}^{2\ell} = 1, \qquad \mathbf{v}^{2t} \neq 1 \quad (\text{ for all } t \in \mathbb{Z}, \ell > t > 0). \tag{3.3}$$

Consider the specialization homomorphism $\phi : \mathcal{A}^\pi \to R^\pi$ which sends q to $\mathbf{q}$ and $\sqrt{\pi}$ to $\sqrt{\pi}$. We shall denote by $[n]_{\mathbf{q},\pi}$ and $\begin{bmatrix} n \\ t \end{bmatrix}_{\mathbf{q},\pi}$ the images of $[n]_{q,\pi}$ and $\begin{bmatrix} n \\ t \end{bmatrix}_{q,\pi}$ under ϕ respectively, and so on.

The following lemma is an analogue of [Lu94, Lemma 34.1.2], which can be in turn recovered by setting $\pi = 1$ below.

Lemma 3.1. *(a) If $t \in \mathbb{Z}_{>0}$ is not divisible by ℓ and $n \in \mathbb{Z}$ is divisible by ℓ, then*

$$\begin{bmatrix} n \\ t \end{bmatrix}_{\mathbf{q},\pi} = 0.$$

(b) If $n_1 \in \mathbb{Z}$ and $t_1 \in \mathbb{N}$, then we have

$$\begin{bmatrix} \ell n_1 \\ \ell t_1 \end{bmatrix}_{\mathbf{q},\pi} = \pi^{\ell^2 t_1 (n_1 - (t_1-1)/2)} \mathbf{q}^{\ell^2 t_1 (n_1+1)} \binom{n_1}{t_1}.$$

(c) Let $n \in \mathbb{Z}$ and $t \in \mathbb{N}$. Write $n = n_0 + \ell n_1$ with $n_0, n_1 \in \mathbb{Z}$ such that $0 \le n_0 \le \ell - 1$ and write $t = t_0 + \ell t_1$ with $t_0, t_1 \in \mathbb{N}$ such that $0 \le t_0 \le \ell - 1$. Then we have

$$\begin{bmatrix} n \\ t \end{bmatrix}_{\mathbf{q},\pi} = \pi^{\ell(n_0-t_0)t_1 + \ell^2(n_1-(t_1-1)/2)t_1} \mathbf{q}^{\ell(n_0 t_1 - n_1 t_0) + \ell^2(n_1+1)t_1} \begin{bmatrix} n_0 \\ t_0 \end{bmatrix}_{\mathbf{q},\pi} \binom{n_1}{t_1}.$$

Proof. One proof would be by imitating the arguments for [Lu94, Lemma 34.1.2]. Below we shall use an alternative and quicker approach, which is to convert [Lu94, Lemma 34.1.2] into our current statements using (3.1) via the substitution $\mathbf{v} = \sqrt{\pi}\mathbf{q}$. Part (a) immediately follows from [Lu94, Lemma 34.1.2(a)].

(b) By applying [Lu94, Lemma 34.1.2(b)] to $\begin{bmatrix} \ell n_1 \\ \ell t_1 \end{bmatrix}_{\mathbf{v}}$ and using (3.1), we have

$$\begin{bmatrix} \ell n_1 \\ \ell t_1 \end{bmatrix}_{\mathbf{q},\pi} = \sqrt{\pi}^{\ell t_1(\ell n_1 - \ell t_1)} \begin{bmatrix} \ell n_1 \\ \ell t_1 \end{bmatrix}_{\mathbf{v}} = \sqrt{\pi}^{\ell^2 t_1 (n_1 - t_1)} \mathbf{v}^{\ell^2 t_1 (n_1+1)} \binom{n_1}{t_1},$$

which can be easily shown to be equal to the formula as stated in the lemma.

(c) Note that

$$\sqrt{\pi}^{(n-t)t} = \sqrt{\pi}^{\ell((n_0-t_0)t_1 + (n_1-t_1)t_0)} \sqrt{\pi}^{\ell^2(n_1-t_1)t_1} \sqrt{\pi}^{(n_0-t_0)t_0}. \tag{3.4}$$

By applying [Lu94, Lemma 34.1.2(c)] to $\begin{bmatrix} n \\ t \end{bmatrix}_{\mathbf{v}}$ and using (3.1)-(3.4), we have

$$\begin{aligned}
\begin{bmatrix} n \\ t \end{bmatrix}_{\mathbf{q},\pi} &= \sqrt{\pi}^{(n-t)t} \begin{bmatrix} n \\ t \end{bmatrix}_{\mathbf{v}} \\
&= \sqrt{\pi}^{(n-t)t} \mathbf{v}^{\ell(n_0t_1-n_1t_0)+\ell^2(n_1+1)t_1} \begin{bmatrix} n_0 \\ t_0 \end{bmatrix}_{\mathbf{v}} \binom{n_1}{t_1} \\
&= \sqrt{\pi}^{\ell((n_0-t_0)t_1+(n_1-t_1)t_0)} \sqrt{\pi}^{\ell^2(n_1-t_1)t_1} \sqrt{\pi}^{\ell(n_0t_1-n_1t_0)+\ell^2(n_1+1)t_1} \\
&\qquad \times \mathbf{q}^{\ell(n_0t_1-n_1t_0)+\ell^2(n_1+1)t_1} \left(\sqrt{\pi}^{(n_0-t_0)t_0} \begin{bmatrix} n_0 \\ t_0 \end{bmatrix}_{\mathbf{v}} \right) \binom{n_1}{t_1} \\
&= \pi^{\ell(n_0-t_0)t_1+\ell^2(n_1-(t_1-1)/2)t_1} \mathbf{q}^{\ell(n_0t_1-n_1t_0)+\ell^2(n_1+1)t_1} \begin{bmatrix} n_0 \\ t_0 \end{bmatrix}_{\mathbf{q},\pi} \binom{n_1}{t_1}.
\end{aligned}$$

The lemma is proved. □

Note that, due to our choice of $\mathbf{q} = \sqrt{\pi}\varepsilon$, we also have an analogue of equation (e) in the proof of [Lu94, Lemma 34.1.2]:

$$\mathbf{v}^{\ell^2+\ell} = \pi^{(\ell+1)\ell/2}\mathbf{q}^{\ell^2+\ell} = (-1)^{\ell+1}. \tag{3.5}$$

The following is an analogue of [Lu94, §34.1.3(a)].

Lemma 3.2. *Let $b \geq 0$. Then*

$$\frac{[\ell b]^!_{\mathbf{q},\pi}}{([\ell]^!_{\mathbf{q},\pi})^b} = b!(\pi\mathbf{q})^{\ell^2 b(b-1)/2}.$$

Proof. Recall $\mathbf{v} = \sqrt{\pi}\mathbf{q}$. Using (3.1) and [Lu94, §34.1.3(a)], we have

$$\begin{aligned}[\ell b]^!_{\mathbf{q},\pi}/([\ell]^!_{\mathbf{q},\pi})^b &= \sqrt{\pi}^{\ell b(\ell b-1)/2-b\ell(\ell-1)/2}[\ell b]^!_{\mathbf{v}}/([\ell]^!_{\mathbf{v}})^b \\ &= \sqrt{\pi}^{\ell^2 b(b-1)/2} b!\mathbf{v}^{\ell^2 b(b-1)/2} = b!(\pi\mathbf{q})^{\ell^2 b(b-1)/2}.\end{aligned}$$

The lemma is proved. □

Below is a π-analogue of [Lu94, Lemma 34.1.4].

Lemma 3.3. *Suppose that* $0 \le r \le a < \ell$. *Then,*

$$\sum_{s=0}^{\ell-a-1} (-1)^{\ell-r+1+s}\pi^{\binom{s+1}{2}+s(r-\ell)}\mathbf{q}^{-(\ell-r)(a-\ell+1+s)+s}\begin{bmatrix}\ell-r\\ s\end{bmatrix}_{\mathbf{q},\pi} = \pi^{\binom{r}{2}-\binom{l}{2}-a(r-l)}\mathbf{q}^{\ell(a-r)}\begin{bmatrix}a\\ r\end{bmatrix}_{\mathbf{q},\pi}.$$

Proof. Plugging $\mathbf{v} = \sqrt{\pi}\mathbf{q}$ into [Lu94, Lemma 34.1.4] and using (3.1), we obtain

$$\begin{aligned}\sum_{s=0}^{\ell-a-1} (-1)^{\ell-r+1+s}\sqrt{\pi}^{-(\ell-r)(a-\ell+1+s)+s+s(s-\ell+r)}\mathbf{q}^{-(\ell-r)(a-\ell+1+s)+s}\begin{bmatrix}\ell-r\\ s\end{bmatrix}_{\mathbf{q},\pi} \\ = \sqrt{\pi}^{\ell(a-r)+r(r-a)}\mathbf{q}^{\ell(a-r)}\begin{bmatrix}a\\ r\end{bmatrix}_{\mathbf{q},\pi}.\end{aligned}$$

Rearranging the $\sqrt{\pi}$ terms, we have

$$\begin{aligned}\sum_{s=0}^{\ell-a-1} (-1)^{\ell-r+1+s}\sqrt{\pi}^{s(s+1)+2s(r-\ell)}\mathbf{q}^{-(\ell-r)(a-\ell+1+s)+s}\begin{bmatrix}\ell-r\\ s\end{bmatrix}_{\mathbf{q},\pi} \\ = \sqrt{\pi}^{r(r-1)-\ell(\ell-1)-2a(r-l)}\mathbf{q}^{\ell(a-r)}\begin{bmatrix}a\\ r\end{bmatrix}_{\mathbf{q},\pi}.\end{aligned}$$

from which the desired formula is immediate. □

3.2 The quasi-classical algebras

Definitions and lemmas

Define

$$\ell_i = \min\{r \in \mathbb{Z}_{>0} \mid r(i \cdot i)/2 \in \ell\mathbb{Z}\}.$$

The next lemma follows by the definition of ℓ_i and the bar-consistency condition of I.

Lemma 3.4. *For each $i \in I_1$, ℓ_i has the same parity as ℓ.*

Then $(I, \diamond)$ is a new root datum by [Lu94, 2.2.4], where we let

$$i \diamond j = (i \cdot j)\ell_i\ell_j, \qquad \text{for all } i, j \in I.$$

Note that if ℓ is odd, then $(I, \diamond)$ is a super Cartan datum with the same parity decomposition $I = I_0 \cup I_1$ as for $(I, \cdot)$ by Lemma 3.4; if ℓ is even, then $(I, \diamond)$ is a (non-super) Cartan datum with $I_1 = \emptyset$.

We shall write $Y^\diamond, X^\diamond$ in this paper what Lusztig [Lu94, 2.2.5] denoted by Y^*, X^* respectively, and we will use superscript $^\diamond$ in related notation associated to $(Y^\diamond, X^\diamond, I, \diamond)$ below. More explicitly, we set $X^\diamond = \{\zeta \in X | \langle i, \zeta\rangle \in \ell_i\mathbb{Z}, \text{ for all } i \in I\}$ and $Y^\diamond = \mathrm{Hom}_{\mathbb{Z}}(X^\diamond, \mathbb{Z})$ with the obvious pairing. The embedding $I \hookrightarrow X^\diamond$ is given by $i \mapsto i'^\diamond = \ell_i i' \in X$, while embedding $I \hookrightarrow Y^\diamond$ is given by $i \mapsto i^\diamond \in Y^\diamond$ whose value at any $\zeta \in X^\diamond$ is $\langle i, \zeta\rangle/\ell_i$. It follows that $\langle i^\diamond, j'^\diamond\rangle = 2i \diamond j/i \diamond i$.

If ℓ is odd, then $(Y^\diamond, X^\diamond, \cdots)$ is a new super root datum satisfying (a)-(d) above and in addition the bar-consistency condition (e). Indeed, we have $2\frac{i\diamond j}{i\diamond i} = 2\frac{i\cdot j}{i\cdot i}\frac{\ell_j}{\ell_i} \in 2\mathbb{Z}$ by Lemma 3.4, whence (d), and $\frac{i\diamond i}{2} = \frac{i\cdot i}{2}\ell_i^2 \equiv p(i) \mod 2$ by Lemma 3.4, whence (e). If ℓ is even, then $(Y^\diamond, X^\diamond, \cdots)$ is a new (non-super) root datum just as in [Lu94, 2.2.5].

The algebras ${}'\mathbf{f}^\diamond$, $\mathbf{f}^\diamond$ and ${}_R\mathbf{f}^\diamond$ are defined in the same way as $\mathbf{f}$ using the Cartan datum $(I, \diamond)$, and the algebra $\mathbf{U}^\diamond$ is defined in the same way as $\mathbf{U}$ based on the root datum $(Y^\diamond, X^\diamond, ...)$.

The algebra $\dot{\mathbf{U}}^\diamond$

The algebra $\dot{\mathbf{U}}^\diamond$ is defined in the same way as $\dot{\mathbf{U}}$ using $\mathbf{U}^\diamond$ and $(Y^\diamond, X^\diamond, ...)$, and so it also has an $\mathcal{A}^\pi$-form ${}_{\mathcal{A}}\dot{\mathbf{U}}^\diamond$ and we can define ${}_R\dot{\mathbf{U}}^\diamond = R^\pi \otimes_{\mathcal{A}^\pi} {}_{\mathcal{A}}\dot{\mathbf{U}}^\diamond$.

Remark 3.5. If ℓ is even, then ${}_R\mathbf{f}^\diamond$ is a (non-super) algebra; if ℓ is odd, then the θ_i in ${}_R\mathbf{f}^\diamond$ and ${}_R\mathbf{f}$ for any given i have the same parity.

For $i \in I$, we denote

$$q_i^\diamond = q^{i\diamond i/2} = q_i^{\ell_i^2}, \qquad \mathbf{q}_i^\diamond = \mathbf{q}^{i\diamond i/2} = \mathbf{q}_i^{\ell_i^2}, \qquad \pi_i^\diamond = \pi^{i\diamond i/2} = \pi_i^{\ell_i^2}. \tag{3.6}$$

Lemma 3.6. *Let $i \in I_1$.*

(a) If ℓ is odd, then $\pi_i^\diamond = \pi_i$.

(b) If ℓ is even, then $\pi_i^\diamond = 1$.

Proof. Recall from Lemma 3.4 that ℓ_i must have the same parity as ℓ. The claim on $\pi_i^\diamond$ follows now from (3.6). □

For each $i \in I$, we have

$$\pi_i^\diamond \mathbf{q}_i^{\diamond 2} = (\pi_i \mathbf{q}_i^2)^{\ell_i^2} = 1. \tag{3.7}$$

Following Lusztig [Lu94], we will refer to the quantum supergroup ${}_R\mathbf{f}^\diamond$ associated to $(Y^\diamond, X^\diamond, \cdots)$ as *quasi-classical*; cf. (3.7).

Proposition 3.7. *Let R be the fraction field of $\mathcal{A}'$. The quasi-classical algebra ${}_R\mathbf{f}^\diamond$ is isomorphic to ${}_R\tilde{\mathbf{f}}^\diamond$, the R^π-algebra generated by θ_i, $i \in I$, subject to the super Serre relations:*

$$\sum_{n+n'=1-\langle i,j'\rangle^\diamond} (-1)^{n'} (\pi_i^\diamond)^{np(j)+\binom{n}{2}} \theta_i^{(n)} \theta_j \theta_i^{(n')} = 0 \qquad (i \neq j \in I).$$

Proof. When $\pi_i = 1$ or ℓ is even, $\pi_i^\diamond = 1$ and $\mathbf{q}_i^\diamond = \pm 1$ for each $i \in I$. Hence, in this case the lemma reduces to [Lu94, §33.2].

Now let ℓ be odd and $\pi = -1$. We make use of the *weight-preserving* automorphism $\dot{\Psi}$ of ${}_R\dot{\mathbf{U}}^\diamond$ (called a twistor) given in [CFLW, Theorem 4.3] when the base ring contains $\sqrt{-1}$. We will only recall the basic property of $\dot{\Psi}$ which we need, and refer to [CFLW] for details. Note that for all $i \in I$, $\mathbf{q}_i^\diamond$ is a power of $\sqrt{-1}$ with at least one of the $\mathbf{q}_i^\diamond = \pm\sqrt{-1}$. Thus, $\pm\sqrt{-1}$ will play the role played by the v in [CFLW, Theorem 4.3], which we will denote by $\tilde{v}$ in this proof so as not to confuse it with the v defined in this paper. Recall $\dot{\Psi}$ takes π to $-\pi$ and $\tilde{v}$ to $\sqrt{-1}\tilde{v}$. When we specialize $\pi = -1$ and $\tilde{v} = \pm\sqrt{-1}$, we obtain an R-linear isomorphism of that specialization of ${}_R\dot{\mathbf{U}}^\diamond$, denoted by ${}_R\dot{\mathbf{U}}^\diamond|_{-1}$, with the (quasi-classical) modified quantum group corresponding to the specialization $\pi = 1$ and $\mathbf{q}_j^\diamond = \pm 1$, denoted by ${}_R\dot{\mathbf{U}}^\diamond|_1$.

Write

▷ ${}_{R_{-1}}\mathbf{f}$ for the half quantum (super)group over R corresponding to the former (i.e., $\pi = -1$);

▷ ${}_{R_1}\mathbf{f}^\diamond$ for the half (quasi-classical) quantum group over R corresponding to the latter (i.e., $\pi = 1$); cf. [Lu94, 33.2].

Recall that ${}_R\mathbf{f}^\diamond$ is a direct sum of finite-dimensional weight spaces ${}_R\mathbf{f}_\nu^\diamond$, where $\nu \in \mathbb{Z}_{\geq 0}[I]$. The weight-preserving isomorphism $\dot{\Psi}$ above implies that

$$\dim_{R^\pi}({}_R\mathbf{f}_\nu^\diamond) = \dim_R({}_{R_{-1}}\mathbf{f}_\nu^\diamond) = \dim_R {}_{R_1}\mathbf{f}_\nu^\diamond, \qquad \text{for all } \nu.$$

As ${}_{R_1}\mathbf{f}^\diamond$ is quasi-classical in the sense of [Lu94, 33.2], we have $\dim_R {}_{R_1}\mathbf{f}_\nu^\diamond = \dim_R {}_{R_1}\mathbf{f}_\nu$ for all ν, by [Lu94, 33.2.2], where ${}_{R_1}\mathbf{f}$ is the enveloping algebra of the half KM algebra over R. Hence we have

$$\dim_{R^\pi}({}_R\mathbf{f}_\nu^\diamond) = \dim_R({}_{R_1}\mathbf{f}_\nu), \qquad \text{for all } \nu. \tag{3.8}$$

Since the super Serre relations hold in ${}_R\mathbf{f}^\diamond$ (cf. [CHW13, Proposition 1.7.3]) we have a surjective algebra homomorphism $\varphi : {}_R\tilde{\mathbf{f}}^\diamond \longrightarrow {}_R\mathbf{f}^\diamond$ mapping $\theta_i \mapsto \theta_i$ for all i. Then φ maps each weight space ${}_R\tilde{\mathbf{f}}_\nu^\diamond$ onto the corresponding weight space ${}_R\mathbf{f}_\nu^\diamond$. As ${}_R\tilde{\mathbf{f}}^\diamond$ has a Serre-type

presentation by definition, it follows by [KKO14, CHW14] that $\dim_{R^\pi}({}_R\tilde{\mathbf{f}}_\nu) = \dim_R({}_{R_1}\mathbf{f}_\nu)$ for each ν. This together with (3.8) implies that $\dim_{R^\pi}({}_R\tilde{\mathbf{f}}_\nu) = \dim_{R^\pi}({}_R\mathbf{f}^\diamond_\nu)$. Therefore φ is a linear isomorphism on each weight space and thus an isomorphism. $\square$

An analogue of Lusztig's Lemma 35.1.5

Below we provide an analogue of [Lu94, 35.1.5], which is a relation for (q, π)-binomial terms when the arguments are divisible by ℓ_i.

Lemma 3.8. *Assume that both $n \in \mathbb{Z}$ and $t \in \mathbb{N}$ are divisible by ℓ_i. Then*

$$\begin{bmatrix} n \\ t \end{bmatrix}_{\mathbf{q}_i, \pi_i} = \begin{bmatrix} n/\ell_i \\ t/\ell_i \end{bmatrix}_{\mathbf{q}_i^\diamond, \pi_i^\diamond}.$$

(Setting $\pi = 1$ in the above formula recovers [Lu94, 35.1.5].)

Proof. By Lemma 3.1(b), we have

$$\begin{bmatrix} n \\ t \end{bmatrix}_{\mathbf{q}_i, \pi_i} = \pi_i^{t(n-(t-\ell_i)/2)} \mathbf{q}_i^{t(n+\ell_i)} \binom{n/\ell_i}{t/\ell_i}.$$

Note that $\pi_i^\diamond \mathbf{q}_i^{\diamond 2} = (\pi \mathbf{q}^2)^{\frac{i \cdot i}{2}\ell_i^2}$. Since $(\pi \mathbf{q}^2)^{2\ell} = 1$ and ℓ divides $\frac{i \cdot i}{2}\ell_i^2$ by the definition of ℓ_i, we have $(\pi_i^\diamond \mathbf{q}_i^{\diamond 2})^2 = 1$. Hence by (3.6) and Lemma 3.1(b) with $\ell = 1$ we have

$$\begin{bmatrix} n/\ell_i \\ t/\ell_i \end{bmatrix}_{\mathbf{q}_i^\diamond, \pi_i^\diamond} = \pi_i^{t(n-(t-\ell_i)/2)} \mathbf{q}_i^{t(n+\ell_i)} \binom{n/\ell_i}{t/\ell_i}.$$

The lemma follows. $\square$

Chapter 4

A Frobenius-Lusztig homomorphism

We now establish a Frobenius-Lusztig homomorphism between the quasi-classical covering group and the quantum covering group at roots of 1, extending results in [Lu94, §35]. We also formulate a Lusztig-Steinberg tensor product theorem in this setting.

Assumptions

Following [Lu94, 35.1.2], in this and following sections we shall impose the following assumptions:

(a) for any $i \neq j \in I$ with $\ell_j \geq 2$, we have $\ell_i \geq -\langle i, j' \rangle + 1$.

(b) $(I, \cdot)$ has no odd cycles.

4.1 A generating theorem for the R^π-superalgebra ${}_R\mathbf{f}$

We will prove below a generalization of [Lu94, Theorem 35.1.8].

Theorem 4.1. *There is a unique R^π-superalgebra homomorphism*

$$\mathrm{Fr}' : {}_R\mathbf{f}^\diamond \longrightarrow {}_R\mathbf{f}, \qquad \mathrm{Fr}'(\theta_i^{(n)}) = \theta_i^{(n\ell_i)} \quad (\textit{ for all } i \in I, n \in \mathbb{Z}_{>0}).$$

(Be aware that the two θ_i's above belong to different algebras and hence are different. Theorem 4.1 is consistent with Remark 3.5.)

The rest of the section is devoted to a proof of Theorem 4.1. The same remark as in [Lu94, 35.1.11] allows us to reduce the proof to the case when R is the quotient field of $\mathcal{A}'$, which we will assume in the remainder of this and the next section.

Recall from (3.3) that $\pi^\ell \mathbf{q}^{2\ell} = 1$ and $\pi^t \mathbf{q}^{2t} \neq 1$ for $0 < t < \ell$. By the definition of ℓ_i, we have $\pi_i^\ell \mathbf{q}_i^{2\ell} = 1$ and $\pi_i^t \mathbf{q}_i^{2t} \neq 1$ for $0 < t < \ell_i$. Then $[t]^\pi_{\mathbf{q}_i}!$ is invertible in R^π, for $0 < t < \ell_i$.

The following is an analogue of [Lu94, Lemma 35.2.2] and the proof uses now Lemmas 3.1 and 3.2.

Lemma 4.2. *The R^π-superalgebra ${}_R\mathbf{f}$ is generated by the elements $\theta_i^{(\ell_i)}$ for all $i \in I$ and the elements θ_i for $i \in I$ with $\ell_i \geq 2$.*

Proof. By definition the algebra ${}_R\mathbf{f}$ is generated by $\theta_i^{(n)}$ for all $i \in I$ and $n \geq 0$. We can write $n = a + \ell_i b$, for $0 \leq a < \ell_i$ and $b \in \mathbb{N}$. We note the following three identities in ${}_R\mathbf{f}$:

$$\theta_i^{(a+\ell_i b)} = \mathbf{q}_i^{\ell_i ab} \theta_i^{(a)} \theta_i^{(\ell_i b)}, \tag{4.1}$$

$$\theta_i^{(a)} = [a]^{-1}_{\mathbf{q}_i, \pi_i} \theta_i^a, \tag{4.2}$$

$$\theta_i^{(\ell_i b)} = (b!)^{-1} (\pi_i \mathbf{q}_i)^{-\ell_i^2 \binom{b}{2}} (\theta_i^{(\ell_i)})^b, \tag{4.3}$$

where (4.1) follows by Lemma 3.1 and (4.3) follows by Lemma 3.2, respectively. (Note that a sign in the power of $\mathbf{v}_i$ in the identity (b) in [Lu94, proof of Lemma 35.2.2] is optional, but the sign cannot be dropped from the power of $\mathbf{q}_i$ in (4.3).) The lemma follows. □

Proof of Theorem 4.1

The uniqueness part is clear.

By Lemma 3.2 (with $\ell = 1$), we have

$$[n]^!_{\mathbf{q}_i^\diamond, \pi_i^\diamond} = (\pi_i \mathbf{q}_i)^{\ell_i^2 \binom{n}{2}} n!. \tag{4.4}$$

We first observe that the existence of a homomorphism Fr' such that $\mathrm{Fr}'(\theta_i) = \theta_i^{(\ell_i)}$ implies that $\mathrm{Fr}'(\theta_i^{(n)}) = \theta_i^{(n\ell_i)}$ for all $n \geq 0$. Indeed, using (4.3)-(4.4) we have

$$\mathrm{Fr}'(\theta_i^{(n)}) = ([n]_{\mathbf{q}_i^\diamond, \pi_i^\diamond}!)^{-1} \mathrm{Fr}'(\theta_i)^n = \big((\pi_i \mathbf{q}_i)^{\ell_i^2 n(n-1)/2} n!\big)^{-1} \mathrm{Fr}'(\theta_i)^n = \theta_i^{(n\ell_i)}.$$

Hence it remains to show that there exists an algebra homomorphism $\mathrm{Fr}' : {}_R\mathbf{f}^\diamond \to {}_R\mathbf{f}$ such that $\theta_i \to \theta_i^{(\ell_i)}$, for all $i \in I$. By Proposition 3.7 (also cf. [CHW13]), the algebra ${}_R\mathbf{f}^\diamond$ has the following defining relations:

$$\sum_{n+n'=1-\langle i,j'\rangle^\diamond} (-1)^{n'} (\pi_i^\diamond)^{np(j)+\binom{n}{2}} \theta_i^{(n)} \theta_j \theta_i^{(n')} = 0 \qquad (i \neq j \in I).$$

By (4.4) it suffices to check the following identity in ${}_R\mathbf{f}$: for $i \neq j \in I$,

$$\sum_{n+n'=1-\langle i,j'\rangle \ell_j/\ell_i} (-1)^{n'} \pi_i^{\ell_i^2(np(j)+n(n-1)/2)} (\pi_i \mathbf{q}_i)^{-\ell_i^2 \binom{n}{2}} (\pi_i \mathbf{q}_i)^{-\ell_i^2 \binom{n'}{2}} \frac{(\theta_i^{(\ell_i)})^n}{n!} \theta_j^{(\ell_j)} \frac{(\theta_i^{\ell_i})^{n'}}{n'!} = 0,$$

which, by the identity (4.3), is equivalent to checking the following identity in ${}_R\mathbf{f}$:

$$\sum_{n+n'=1-\langle i,j'\rangle \ell_j/\ell_i} (-1)^{n'} \pi_i^{\ell_i^2(np(j)+n(n-1)/2)} \theta_i^{(\ell_i n)} \theta_j^{(\ell_j)} \theta_i^{(\ell_i n')} = 0. \tag{4.5}$$

It remains to prove (4.5). Set $\alpha = -\langle i, j'\rangle$. For any $0 \leq t \leq \ell_i - 1$, we set

$$g_t = \sum_{\substack{r,s \\ r+s=\ell_j\alpha+\ell_i-t}} (-1)^r \pi_i^{\ell_j rp(j)+r(r-1)/2} q_i^{r(\ell_i-1-t)} \theta_i^{(r)} \theta_j^{(\ell_j)} \theta_i^{(s)} \in {}_{\mathcal{A}}\mathbf{f}.$$

This is basically $f'_{i,j;\ell_j,\ell_j\alpha+\ell_i-t}$ in [CHW13, 4.1.1(d)] in the notation of θ's. By the higher super

Serre relations (see [CHW13, Proposition 4.2.4] and [CHW13, 4.1.1(e)]), we have $g_t = 0$ for all $0 \le t \le \ell_i - 1$. Set

$$g = \sum_{t=0}^{\ell_i-1} (-1)^t \pi_i^{t(t-1)/2} \mathbf{q}_i^{\ell_j \alpha t + \ell_i t - t} g_t \theta_i^{(t)},$$

which must be 0. On the other hand, setting $s' = s + t$, we have

$$(0 =)\ g = \sum_{\substack{r,s' \\ r+s' = \ell_j\alpha + \ell_i}} c_{r,s'} \theta_i^{(r)} \theta_j^{(\ell_j)} \theta_i^{(s')}, \tag{4.6}$$

where

$$c_{r,s'} = \sum_{t=0}^{\ell_i-1} (-1)^{r+t} \pi_i^{\ell_j r p(j) + r(r-1)/2 + t(t-1)/2} q_i^{r(\ell_i - 1 - t) + \ell_j \alpha t + \ell_i t - t} \begin{bmatrix} s' \\ t \end{bmatrix}_{q_i, \pi_i}.$$

Taking the image of the identity (4.6) under the map ${}_{\mathcal{A}}\mathbf{f} \to {}_R\mathbf{f}$, we have

$$\sum_{\substack{r,s' \\ r+s' = \ell_j\alpha + \ell_i}} \phi(c_{r,s'}) \theta_i^{(r)} \theta_j^{(\ell_j)} \theta_i^{(s')} = 0 \ \in {}_R\mathbf{f}.$$

For a fixed s', we write $s' = a + \ell_i n$, where $a, n \in \mathbb{Z}$ and $0 \le a \le \ell_i - 1$. Note by Lemma 3.1(c) that $\begin{bmatrix} s' \\ t \end{bmatrix}_{\mathbf{q}_i, \pi_i} = \mathbf{q}_i^{-\ell_i n t} \begin{bmatrix} a \\ t \end{bmatrix}_{\mathbf{q}_i, \pi_i}$. Now using $r + s' = \ell_j \alpha + \ell_i$ we compute

$$\begin{aligned}
\phi(c_{r,s'}) &= (-1)^r \mathbf{q}_i^{r(\ell_i - 1)} \sum_{t=0}^{\ell_i-1} (-1)^t \pi_i^{\ell_j r p(j) + r(r-1)/2 + t(t-1)/2} \mathbf{q}_i^{t(s'-1) - \ell_i n t} \begin{bmatrix} a \\ t \end{bmatrix}_{\mathbf{q}_i, \pi_i} \\
&= (-1)^r \mathbf{q}_i^{r(\ell_i - 1)} \sum_{t=0}^{a} (-1)^t \pi_i^{\ell_j r p(j) + r(r-1)/2 + t(t-1)/2} \mathbf{q}_i^{t(a-1)} \begin{bmatrix} a \\ t \end{bmatrix}_{\mathbf{q}_i, \pi_i} \\
&\overset{(a)}{=} \delta_{a,0} (-1)^{\ell_j \alpha + \ell_i - \ell_i n} \pi_i^{\ell_j r p(j) + r(r-1)/2} \mathbf{q}_i^{(\ell_i - 1)(\ell_j \alpha + \ell_i - \ell_i n)} \\
&\overset{(b)}{=} \delta_{a,0} (-1)^{\alpha \ell_j / \ell_i + 1 - n} \pi_i^{\ell_j r p(j) + r(r-1)/2 - r(\ell_i - 1)/2}.
\end{aligned} \tag{4.7}$$

The identity (a) above follows by the identity $\sum_{t=0}^{a}(-1)^t\pi_i^{t(t-1)/2}\mathbf{q}_i^{t(a-1)}\begin{bmatrix} a \\ t \end{bmatrix}_{\mathbf{q}_i,\pi_i} = \delta_{a,0}$ (see [CHW13, 1.4.4]), and (b) follows by the identity $\pi_i^{(\ell_i-1)\ell_i/2}\mathbf{q}_i^{\ell_i^2-\ell_i} = (-1)^{\ell_i+1}$ (which is an i-version of (3.5) with the help of $\pi_i^{\ell_i}\mathbf{q}_i^{2\ell_i} = 1$).

Inserting (4.7) into (4.6) and comparing with (4.5), we reduce the proof of (4.5) to verifying that $\pi_i^{\ell_i^2(np(j)+n(n-1)/2)} = \pi_i^{\ell_j\ell_i np(j)+\ell_i n(\ell_i n-1)/2-\ell_i n(\ell_i-1)/2}$, which is equivalent to verifying $\pi_i^{\ell_i^2 np(j)} = \pi_i^{\ell_j\ell_i np(j)}$. The latter identity is trivial unless both i and j are in I_1; when both i and j are in I_1, the identity follows from Lemma 3.4. Therefore, we have proved (4.5) and hence Theorem 4.1. □

4.2 A Lusztig-Steinberg tensor product theorem

First, as set-up, we develop in this subsection the analogue of [Lu94, 35.3]; recall we are still working under the assumption that R is the quotient field of $\mathcal{A}'$.

Proposition 4.3. *Let $\lambda \in X^\diamond$, i.e., $\langle i, \lambda\rangle \in \ell_i\mathbb{Z}$ for all $i \in I$. Let M denote the simple highest weight module with highest weight λ in the category of R^π-free weight $\mathbf{U}$-modules, and let η be a highest weight vector of M^λ.*

(a) If $\zeta \in X$ satisfies $M^\zeta \neq 0$, then $\zeta = \lambda - \sum_i \ell_i n_i i'$, where $n_i \in \mathbb{N}$. In particular, $\langle i, \zeta\rangle \in \ell_i\mathbb{Z}$ for all $i \in I$.

(b) If $i \in I$ is such that $\ell_i \geq 2$, then E_i, F_i act as zero on M.

(c) For any $r \geq 0$, let M'_r be the subspace of M spanned by $F_{i_1}^{(\ell_{i_1})}F_{i_2}^{(\ell_{i_2})}\dots F_{i_r}^{(\ell_{i_r})}\eta$ for various sequences $i_1, i_2, \dots, i_r$ in I. Let $M' = \sum_r M'_r$. Then $M' = M$.

Proof. The proof is completely analogous to [Lu94]. All computations are similar except that we are now working over R^π instead of R; and the results follow from Lemma 3.1, [CHW13, (4.1) and Proposition 4.2.4], and Lemma 4.2.

First, we show that

(d) $E_i M'_r = 0$, $F_i M'_r = 0$ for any $i \in I$ such that $\ell_i \geq 2$,

which is similarly proved by induction on $r \geq 0$. The base case $r = 0$ follows from the fact that $\begin{bmatrix} \langle i, \lambda \rangle \\ t \end{bmatrix}_{\mathbf{q}_i, \pi_i} = 0$ since $\lambda \in X^\diamond$ (using Lemma 3.1) and the fact that $E_j^{(n)} F_i \eta$ is an R^π-linear combination of $F_i E_j^{(n)}$ and $E_j^{(n-1)}$. For the inductive step, we want to show that $E_i F_j^{(\ell_j)} m = 0$ and $F_i F_j^{(\ell_j)} m = 0$ for any $i, j \in I$ such that $\ell_i \geq 2$ and any $m \in M'_{r-1}\zeta$. For the first one we use the fact that $E_i F_j^{(\ell_j)} m$ is an R^π-linear combination of $F_j^{(\ell_j)} E_i m$ and $F_j^{\ell_j - 1}$ in the case $\ell_j \geq 2$, and for $\ell_j = 1$ we again use $\begin{bmatrix} \langle i, \lambda \rangle \\ t \end{bmatrix}_{\mathbf{q}_i, \pi_i} = 0$ from Lemma 3.1. For the second one, we may use [CHW13, (4.1) and Proposition 4.2.4] to write $F_i F_j^{(\ell_j)} m$ as a R^π-linear combination of $F_j^{(\ell_j - r)} F_i F_j^{(r)} m$ for various r with $0 \leq r < \ell_j$, and for such r we have $F_i F_j^{(r)} m = 0$ by the induction hypothesis.

Next, we may show by induction on $r \geq 0$ that

(e) $E_i^{(l_i)} M'_r \subset M'_{r-1}$ for any $i \in I$,

(by convention $M'_{-1} = 0$); again for $m' \in M'_{r-1}$ we can use the fact that $E_i^{(l_i)} F_j^{(\ell_j)} m'$ is an R^π-linear combination of $F_j^{(\ell_j)} E_i^{(\ell_i)} m'$ (which is in M'_{r-1} by the induction hypothesis), and elements of the form $F_j^{(\ell_j - t)} E_i^{(\ell_i - t)} m'$ with $t > 0$ and $t \leq \ell_i, t \leq \ell_j$ (which as before are zero if $t < \ell_i$ or if $t = \ell_i$ and $t < \ell_j$, by (d), and are in M'_{r-1} if $t = \ell_i = \ell_j$).

The statements (d), (e) together with Lemma 4.2 show that $\sum_r M'_r$ is an ${}_R\dot{\mathbf{U}}$-submodules of M, and by simplicity of M it follows that $M = \sum_r M'_r$, from which (a) and (b) also follow. □

Corollary 4.4. *There is a unique weight ${}_R\dot{\mathbf{U}}^\diamond$-module structure on M (as in Proposition 4.3) in which the ζ-weight space is the same as that in the ${}_R\dot{\mathbf{U}}^\diamond$-modules M, for any $\zeta \in X^\diamond \subset X$, and such that $E_i, F_i \in {}_R\mathbf{f}^\diamond$ act as $E_i^{(\ell_i)}, F_i^{(\ell_i)} \in {}_R\mathbf{f}$. Moreover, this is a simple (R^π-free) highest weight module for ${}_R\dot{\mathbf{U}}^\diamond$ with highest weight $\lambda \in X^\diamond$.*

Proof. We define operators $e_i, f_i : M \to M$ for $i \in I$ by $e_i = E_i^{(\ell_i)}$, $f_i = F_i^{(\ell_i)}$. Using Theorem 4.1 we see that e_i and f_i satisfy the Serre-type relations of ${}_R\mathbf{f}^\diamond$.

If $\zeta \in X \setminus X^\diamond$ we have $M^\zeta = 0$ by Proposition 4.3(a) above. If $\zeta \in X^\diamond$ and $m \in M^\zeta$, then we have that $(e_i f_j - f_j e_i)(m)$ is equal to $\delta_{i,j} \begin{bmatrix} \langle i, \lambda \rangle \\ \ell_i \end{bmatrix}_{\mathbf{q}_i, \pi_i} \cdot m$ plus an R^π-linear combination of elements of the form $F_i^{\ell_i - t} E_i^{\ell_i - t}(m)$ with $0 < t < \ell_i$ (this follows by [Cl14, Lemma 4]) which are zero by Proposition 4.3(b). Since $\langle i, \zeta \rangle \in \ell_i \mathbb{Z}$, we see from Lemma 3.8 that

$$\begin{bmatrix} \langle i, \lambda \rangle \\ \ell_i \end{bmatrix}_{\mathbf{q}_i, \pi_i} = \begin{bmatrix} \langle i, \lambda \rangle / \ell_i \\ 1 \end{bmatrix}_{\mathbf{q}_i^\diamond, \pi_i^\diamond}$$

and so $(e_i f_j - f_j e_i)m = \delta_{i,j}[\langle i, \lambda \rangle / \ell_i]_{\mathbf{q}_i^\diamond, \pi_i^\diamond} \cdot m$. We also have that $e_i(M^\zeta) \subset M^{\zeta + \ell_i i'}$ and $f_i(M^\zeta) \subset M^{\zeta - \ell_i i'}$. Thus, we have a unital ${}_R\dot{\mathbf{U}}^\diamond$-module structure on M, and by Proposition 4.3(c) this is a highest weight module of ${}_R\dot{\mathbf{U}}^\diamond$ with highest weight λ and simplicity also follows using Lemma 4.2 in the same argument as in [Lu94]. □

Now we are ready to state our analogue of the main result of [Lu94, 35.4] on a tensor product decomposition. Let $\mathfrak{f}$ be the R-subalgebra of ${}_R\mathbf{f}$ generated by the elements θ_i for various i such that $\ell_i \geq 2$. We have $\mathfrak{f} = \oplus_\nu \mathfrak{f}_\nu$ where $\mathfrak{f} = {}_R\mathbf{f}_\nu \cap \mathfrak{f}$.

Theorem 4.5 (Lusztig-Steinberg tensor product theorem)**.** *The R^π-linear map*

$$\chi : {}_R\mathbf{f}^\diamond \otimes_R \mathfrak{f} \to {}_R\mathbf{f}, \qquad x \otimes y \mapsto \mathrm{Fr}'(x) y$$

is an isomorphism of R^π-modules.

Proof. First, we make the following statement which is similar to (but slightly less precise than) [Lu94, 35.4.2(a)].

Claim. For any $i \in I$ and $y \in \mathfrak{f}_\nu$, there exists some $a(y), b(y) \in \mathbb{Z}$ such that the difference $\theta_i^{(\ell_i)} y - \pi_i^{a(y)} \mathbf{q}_i^{b(y)} y \theta_i^{(\ell_i)}$ belongs to $\mathfrak{f}$.

For $y = y'y''$ one easily reduces the Claim to the same type of claim for y' and y''. Hence it suffices to show this Claim when y is a generator of $\mathfrak{f}$ i.e. $y = \theta_j$ where $\ell_j \geq 2$. Recall our assumption (a) in §4 that $\ell_i \geq -\langle i, j' \rangle + 1$. Hence, we may use the higher Serre relation in [CHW13, (4.1) and Proposition 4.2.4] (but with θ_i's instead of F_i's) to show that for some $a(j), b(j)$, the difference $\theta_i^{(\ell_i)}\theta_j - \pi_i^{a(j)}\mathbf{q}_i^{b(j)}\theta_j\theta_i^{(\ell_i)}$ is an R^π-linear combination of products of the form $\theta_i^{(r)}\theta_j\theta_i^{(\ell_i - r)}$ with $0 < r < \ell_i$, which are contained in $\mathfrak{f}$ by definition. The Claim is proved.

By Lemma 4.2, ${}_R\mathbf{f}$ is generated by $\theta_i^{(\ell_i)}$ and θ_j with $\ell_j \geq 2$. The surjectivity of χ follows as the Claim allows us to move factors θ_j to the right which produces lower terms in $\mathfrak{f}$.

The injectivity is proved by exactly the same argument as in [Lu94, 35.4.2] using now Proposition 4.3 and Corollary 4.4; the details will be skipped. □

The following is an analogue of [Lu94, Proposition 35.4.4], which follows by the same argument now using the anti-involution σ of ${}_R\mathbf{f}$ which fixes each θ_i (cf. [CHW13, §1.4]). We omit the detail to avoid much repetition.

Proposition 4.6. *Assume that the root datum is simply connected. Then, there is a unique $\lambda \in X^+$ such that $\langle i, \lambda \rangle = \ell_i - 1$ for all i. Let η be the canonical generator of ${}_RV(\lambda)$. The map $x \mapsto x^-\eta$ is an R^π-linear isomorphism $\mathfrak{f} \longrightarrow {}_RV(\lambda)$.*

4.3 The Frobenius-Lusztig homomorphism

The following is a generalization of [Lu94, Theorem 35.1.7]. As with Theorem 4.1, we may reduce the proof to the case when R is the quotient field of $\mathcal{A}'$ (cf. [Lu94, 35.1.11]).

Theorem 4.7. *There is a unique R^π-superalgebra homomorphism* $\mathrm{Fr} : {}_R\mathbf{f} \longrightarrow {}_R\mathbf{f}^\diamond$ *such that, for all $i \in I, n \in \mathbb{N}$,*

$$\mathrm{Fr}(\theta_i^{(n)}) = \begin{cases} \theta_i^{(n/\ell_i)}, & \text{if } \ell_i \text{ divides } n, \\ 0, & \text{otherwise.} \end{cases}$$

(We call Fr the Frobenius-Lustig homomorphism.)

Proof. The proof proceeds essentially like that of [Lu94, Theorem 35.1.7]. Uniqueness is clear; we need only prove the existence. By Theorem 4.5, there is an R^π-linear map $P : {}_R\mathbf{f} \longrightarrow {}_R\mathbf{f}^\diamond$, such that for all $i_k \in I$ and for $j_p \in I$ where $\ell_{j_p} \geq 2$

$$P(\theta_{i_1}^{(\ell_{i_1})} ... \theta_{i_n}^{(\ell_{i_n})} \theta_{j_1} ... \theta_{j_r}) = \begin{cases} \theta_{i_1} ... \theta_{i_n}, & \text{if } r = 0, \\ 0, & \text{otherwise.} \end{cases}$$

We now check that P is a homomorphism of R^π-algebras. Because ${}_R\mathbf{f}$ is generated as an R^π-module by elements of the form $x = \theta_{i_1}^{(\ell_{i_1})} ... \theta_{i_n}^{(\ell_{i_n})} \theta_{j_1} ... \theta_{j_r}$, we need to check that for any such x,

$$P(x\theta_j) = P(x)P(\theta_j) \tag{4.8}$$

for $j \in I$ such that $\ell_j \geq 2$ and

$$P(x\theta_i^{(\ell_i)}) = P(x)P(\theta_i^{(\ell_i)}) \tag{4.9}$$

for all $i \in I$. As (4.8) is obvious, we will concern ourselves with (4.9). Note that (4.9) is clear when $r = 0$. Assume now $r > 0$. Let us write $x' = \theta_{i_1}^{(\ell_{i_1})} ... \theta_{i_n}^{(\ell_{i_n})} \theta_{j_1} ... \theta_{j_{r-1}}$ and $\theta_j = \theta_{j_r}$ so that $x = x'\theta_j$. For $i = j$, we have $P(x)P(\theta_i^{(\ell_i)}) = 0$ and

$$P(x\theta_i^{(\ell_i)}) = P(x'\theta_i\theta_i^{(\ell_i)}) = P(x'\theta_i^{(\ell_i)}\theta_i) = P(x'\theta_i^{(\ell_i)})P(\theta_i) = 0,$$

where the third equality is due to (4.8). Now suppose that $i \neq j$. As $\ell_i > -\langle i, j' \rangle$, we may use the higher order Serre relations for quantum covering groups (cf. [CHW13, (4.1) and Proposition 4.2.4]) to write $\theta_j\theta_i^{(\ell_i)}$ as a linear combination of terms of the form $\theta_i^{(m)}\theta_j\theta_i^{(n)}$ where $m + n = \ell_i$ and $m \geq 1$. Because of (4.2) and (4.8), $P(x'\theta_i^{(m)}\theta_j\theta_i^{(n)}) = 0$ for $1 \leq m < \ell_i$, and $P(x'\theta_i^{(\ell_i)}\theta_j) = 0$.

Now that we know that P is an R^π-algebra homomorphism, it remains to compute $P(\theta_i^{(n)})$ for all $n \in \mathbb{Z}_{\geq 0}$. Write $n = b\ell_i + a$, where $0 \leq a < \ell_i$ and $b \in \mathbb{Z}_{\geq 0}$. Using (4.1), (4.2) and (4.3), for $a > 0$ we have

$$P(\theta^{(b\ell_i+a)}) = \mathbf{q}_i^{\ell_i ab} P(\theta_i^{(a)}) P(\theta_i^{(b\ell_i)}) = \mathbf{q}_i^{\ell_i ab} ([a]^!_{\mathbf{q}_i,\pi_i})^{-1} P(\theta_i^a) P(\theta_i^{(b\ell_i)}) = 0.$$

Similarly, for $a = 0$ we have

$$\begin{aligned} P(\theta_i^{(b\ell_i)}) &= (b!)^{-1} (\pi_i \mathbf{q}_i)^{-\ell_i^2 \binom{b}{2}} P(\theta_i^{(\ell_i)})^b \\ &= (b!)^{-1} (\pi_i^\diamond \mathbf{q}_i^\diamond)^{-\binom{b}{2}} \theta_i^b = ([b]^!_{\mathbf{q}_i^\diamond,\pi_i^\diamond})^{-1} \theta_i^b = \theta_i^{(b)}, \end{aligned}$$

where, in the third equality we used Lemma 3.2, with $\ell = 1$. Hence, P is the desired homomorphism Fr. $\square$

A Frobenius-Lusztig homomorphism for ${}_R\dot{\mathbf{U}}$

We extend the Frobenius-Lusztig homomorphism $\mathrm{Fr} : {}_R\mathbf{f} \longrightarrow {}_R\mathbf{f}^\diamond$ in Theorem 4.7 to ${}_R\dot{\mathbf{U}}$. In contrast to the quantum group setting, we have to twist Fr slightly on one half of the quantum covering group.

Theorem 4.8. *There is a unique R^π-superalgebra homomorphism* $\mathrm{Fr} : {}_R\dot{\mathbf{U}} \longrightarrow {}_R\dot{\mathbf{U}}^\diamond$ *such that for all $i \in I, n \in \mathbb{Z}, \lambda \in X$,*

$$\mathrm{Fr}(E_i^{(n)} \mathbf{1}_\lambda) = \begin{cases} \pi_i^{\binom{\ell_i}{2} n/\ell_i} E_i^{(n/\ell_i)} \mathbf{1}_\lambda, & \text{if } \ell_i \text{ divides } n \text{ and } \lambda \in X^\diamond, \\ 0, & \text{otherwise} \end{cases} \tag{4.10}$$

and

$$\mathrm{Fr}(F_i^{(n)}\mathbf{1}_\lambda) = \begin{cases} F_i^{(n/\ell_i)}\mathbf{1}_\lambda, & \textit{if } \ell_i \textit{ divides } n \textit{ and } \lambda \in X^\diamond, \\ 0, & \textit{otherwise.} \end{cases}$$

(We also call Fr in this theorem the Frobenius-Lustig homomorphism.)

Proof. Let $\mathrm{Fr} : {}_R\mathbf{f} \longrightarrow {}_R\mathbf{f}^\diamond$ be the homomorphism from Theorem 4.7. Consider the homomorphism $\tilde{\mathrm{Fr}} = \psi \circ \mathrm{Fr}$, where $\psi : {}_R\mathbf{f}^\diamond \longrightarrow {}_R\mathbf{f}^\diamond$ is the algebra automorphism such that $\theta_i^{(n)} \mapsto \pi_i^n \theta_i^{(n)}$. The proof, much like that of [Lu94, Theorem 35.1.9], amounts to checking that for $x, x' \in {}_R\mathbf{f}$ the assignment

$$x^+\mathbf{1}_\lambda x'^- \mapsto \tilde{\mathrm{Fr}}(x^+)\mathbf{1}_\lambda Fr(x'^-), \quad x^-\mathbf{1}_\lambda x'^+ \mapsto \mathrm{Fr}(x^-)\mathbf{1}_\lambda \tilde{\mathrm{Fr}}(x'^+),$$

for $\lambda \in X^\diamond$, and

$$x^+\mathbf{1}_\lambda x'^- \mapsto 0, \quad x^-\mathbf{1}_\lambda x'^+ \mapsto 0,$$

for $\lambda \in X \backslash X^\diamond$ satisfies the the appropriate relations. These are the relations of Lemma 2.3 for ${}_R\dot{\mathbf{U}}$ and for ${}_R\dot{\mathbf{U}}^\diamond$, using Lemma 3.8 to deal with the $(\mathbf{q}, \pi)$-binomial coefficients. The use of the homomorphism $\tilde{\mathrm{Fr}}$ (in place of Fr) on $\mathbf{U}^+$ is necessitated by the first and second relations in Lemma 2.3. Both sides of the first relation are mapped to zero by Fr unless $N, M \in \ell_i\mathbb{Z}$

and $\lambda \in X^\diamond$, so we focus on this case. Recalling $\mathbf{q}_i^\diamond, \pi_i^\diamond$ from (3.6), we have

$$
\begin{aligned}
&\mathrm{Fr}\left(\sum_{t\geq 0} \pi_i^{MN-\binom{t+1}{2}} F_i^{(M-t)} \begin{bmatrix} M+N+\langle i,\lambda\rangle \\ t \end{bmatrix}_{\mathbf{q}_i,\pi_i} \mathbf{1}_{\lambda+(M+N-t)i'} E_i^{(N-t)}\right) \\
&= \sum_{t\geq 0} \pi_i^{MN-\binom{t+1}{2}} \mathrm{Fr}(F_i^{(M-t)}) \begin{bmatrix} M+N+\langle i,\lambda\rangle \\ t \end{bmatrix}_{\mathbf{q}_i,\pi_i} \mathbf{1}_{\lambda+(M+N-t)i'} Fr(E_i^{(N-t)}) \\
&= \sum_{t\geq 0, t\in \ell_i\mathbb{Z}} (\pi_i^\diamond)^{(M/\ell_i)(N/\ell_i)-\binom{t/\ell_i+1}{2}} \pi_i^{t/\ell_i\binom{\ell_i}{2}} F_i^{((M-t)/\ell_i)} \begin{bmatrix} (M+N+\langle i,\lambda\rangle)/\ell_i \\ t/\ell_i \end{bmatrix}_{\mathbf{q}_i^\diamond,\pi_i^\diamond} \\
&\qquad\qquad \cdot \mathbf{1}_{\lambda+(M+N-t)i'} \pi_i^{(N-t)/\ell_i\binom{\ell_i}{2}} E_i^{((N-t)/\ell_i)} \\
&= \pi_i^{N/\ell_i\binom{\ell_i}{2}} \sum_{t\geq 0, t\in \ell_i\mathbb{Z}} (\pi_i^\diamond)^{(M/\ell_i)(N/\ell_i)-\binom{t/\ell_i+1}{2}} F_i^{((M-t)/\ell_i)} \begin{bmatrix} (M+N+\langle i,\lambda\rangle)/\ell_i \\ t/\ell_i \end{bmatrix}_{\mathbf{q}_i^\diamond,\pi_i^\diamond} \\
&\qquad\qquad \cdot \mathbf{1}_{\lambda+(M+N-t)i'} E_i^{((N-t)/\ell_i)} \\
&= \pi_i^{N/\ell_i\binom{\ell_i}{2}} E_i^{(N/\ell_i)} \mathbf{1}_\lambda F_i^{(M/\ell_i)} \\
&= \mathrm{Fr}\big(E_i^{(N)} \mathbf{1}_\lambda F_i^{(M)}\big),
\end{aligned}
$$

where we have used $\pi_i^{-\binom{t+1}{2}} = (\pi_i^\diamond)^{-\binom{t/\ell_i+1}{2}} \pi_i^{t/\ell_i\binom{\ell_i}{2}}$ and Lemma 3.8 in the second equality above.

The verification of the second relation of Lemma 2.3 is entirely similar, and the other relations therein are straightforward. $\square$

Chapter 5

Small quantum covering groups

In this chapter, we construct the small quantum covering group ${}_R\mathfrak{u}$ and study its structure. We end this part by giving a dimension formula for small quantum covering groups of finite type. Here, we take $R^\pi = \mathbb{Q}(\mathbf{q})^\pi$, where $\mathbf{q}$ is as in (3.2).

5.1 Definition and structure

Let ${}_R\dot{\mathfrak{u}}$ be the subalgebra of ${}_R\dot{\mathbf{U}}$ generated by $E_i\mathbf{1}_\lambda$ and $F_i\mathbf{1}_\lambda$ for all $i \in I$ with $\ell_i \geq 2$ and $\lambda \in X$. It is clear then, that ${}_R\dot{\mathfrak{u}}$ is spanned by terms of the form $x^+\mathbf{1}_\lambda x'^-$ where $x, x' \in \mathfrak{f}$. We follow the construction of [Lu94, §36.2.3] in extending ${}_R\dot{\mathbf{U}}$ to a new algebra ${}_R\hat{\mathbf{U}}$. Any element of ${}_R\dot{\mathbf{U}}$ can be written as a sum of the form $\sum_{\lambda,\mu\in X} x_{\lambda,\mu}$ where $x_{\lambda,\mu} \in \mathbf{1}_{\lambda R}\dot{\mathbf{U}}\mathbf{1}_\mu$ is zero for all but finitely many pairs λ, μ. We relax this condition in ${}_R\hat{\mathbf{U}}$ by allowing such sums to have infinitely many nonzero terms provided that the corresponding $\lambda - \mu$ are contained in a finite subset of X. The algebra structure extends in the obvious way. We define ${}_R\hat{\mathfrak{u}}$ to be the subalgebra of ${}_R\hat{\mathbf{U}}$ with $x_{\lambda,\mu} \in \mathbf{1}_{\lambda R}\dot{\mathfrak{u}}\mathbf{1}_\mu$.

Let $2\tilde{\ell}$ be the smallest positive integer such that $\mathbf{q}^{2\tilde{\ell}} = 1$. Hence, $\tilde{\ell} = 2\ell$ for ℓ odd and $\tilde{\ell} = \ell$

for ℓ even. We define the cosets

$$\mathbf{c_a} = \{\lambda \in X \mid \langle i, \lambda \rangle \equiv a_i \pmod{2\tilde{\ell}}, \quad \text{for all } i \in I\}, \tag{5.1}$$

for $\mathbf{a} = (a_i | i \in I)$ with $0 \leq a_i \leq 2\tilde{\ell} - 1$. Note that there are at most $(2\tilde{\ell})^{|I|}$ such cosets and they partition X. Moreover, for each coset $\mathbf{c}$, $\mathbf{1_c} := \sum_{\lambda \in \mathbf{c}} \mathbf{1}_\lambda$ is an element of ${}_R\hat{\mathfrak{u}}$.

Let ${}_R\mathfrak{u}$ (resp. ${}_R\mathfrak{u}'$) be the R^π-submodule of ${}_R\hat{\mathfrak{u}}$ generated by the elements $x^+\mathbf{1_c}x'^-$ (resp. $x^-\mathbf{1_c}x'^+$) where $x, x' \in \mathfrak{f}$. The following is an analogue of [Lu94, Lemma 36.2.4].

Lemma 5.1. *1. For any $u \in {}_R\mathfrak{u}$ and $0 \leq M \leq \ell_i - 1$, $F_i^{(M)}u$ lies in ${}_R\mathfrak{u}$.*

2. We have ${}_R\mathfrak{u} = {}_R\mathfrak{u}'$, and ${}_R\mathfrak{u}$ is a subalgebra of ${}_R\hat{\mathfrak{u}}$.

The algebra ${}_R\mathfrak{u}$ is called the *small quantum covering group.*

Proof. We follow the proof in [Lu94]. We prove the first statement by induction on p, where our $u = E_{i_1}^{(n_1)} ... E_{i_p}^{(n_p)} x'^-$. The result is obvious for $p = 0$, so we now consider $p \geq 1$ and rewrite u as

$$u = \mathbf{1}_{\mathbf{c}'} E_{i_1}^{(n_1)} x_1^+ x'^-$$

where $x_1 = \theta_{i_2}^{(n_2)} ... \theta_{i_p}^{(n_p)}$. When $i \neq i_1$, the result is immediate, so we consider $i = i_1$. In that case, using the relations of Lemma 2.3, we have

$$F_i^{(M)} u = \sum_{\lambda \in \mathbf{c}'} \sum_{t \leq n_1, t \leq M} \pi_i^{MN + t\langle i, \lambda \rangle - \binom{t}{2}} \begin{bmatrix} n_1 + M - \langle i, \lambda \rangle \\ t \end{bmatrix}_{\mathbf{q}_i, \pi_i} \cdot E_i^{(a_1 - t)} \mathbf{1}_{\lambda - (n_1 + M - t) i'} F_i^{(M-t)} x_1^+ x'^-.$$

Fix $\mu \in \mathbf{c}'$. Then for any $\lambda \in \mathbf{c}'$, $n_1 + M - \langle i, \lambda \rangle \equiv n_1 + M - \langle i, \mu \rangle \bmod(\ell_i)$. Using Lemma

3.1 and noting that $t < \ell_i$, we have that

$$\begin{aligned}\begin{bmatrix} n_1 + M - \langle i, \lambda \rangle \\ t \end{bmatrix}_{\mathbf{q}_i, \pi_i} &= \mathbf{q}_i^{-\ell_i t(\langle i,\lambda\rangle - \langle i,\mu\rangle)} \begin{bmatrix} n_1 + M - \langle i, \mu \rangle \\ t \end{bmatrix}_{\mathbf{q}_i, \pi_i} \\ &= \begin{bmatrix} n_1 + M - \langle i, \mu \rangle \\ t \end{bmatrix}_{\mathbf{q}_i, \pi_i},\end{aligned}$$

where we used in the second equality the condition that $\langle i, \lambda \rangle - \langle i, \mu \rangle \equiv 0 \bmod (2\tilde{\ell})$. Hence, $F_i^{(M)} u$ is equal to

$$\begin{aligned}&\sum_{t \le n_1, t \le M} \pi_i^{MN + t\langle i,\mu\rangle - \binom{t}{2}} \begin{bmatrix} n_1 + M - \langle i, \mu \rangle \\ t \end{bmatrix}_{\mathbf{q}_i, \pi_i} E_i^{(a_1 - t)} (\sum_{\lambda \in \mathbf{c}'} \mathbf{1}_{\lambda - (n_1 + M - t) i'}) F_i^{(M-t)} x_1^+ x'^- \\ &= \sum_{t \le n_1, t \le M} \pi_i^{MN + t\langle i,\mu\rangle - \binom{t}{2}} \begin{bmatrix} n_1 + M - \langle i, \mu \rangle \\ t \end{bmatrix}_{\mathbf{q}_i, \pi_i} E_i^{(a_1 - t)} \mathbf{1}_{\mathbf{c}''} F_i^{(M-t)} x_1^+ x'^-,\end{aligned}$$

for some other $\mathbf{c}''$. Hence, $F_i^{(M)} u \in {}_R\mathfrak{u}$ by induction. Finally, the second statement is shown by repeated application of this result as in [Lu94, Lemma 36.2.4]. □

Hopf structure

Recall there are a comultiplication Δ and an antipode S on $\mathbf{U}$ as defined in [CHW13, Lemmas 2.2.1, 2.4.1]. Write ${}_\lambda\mathbf{U}_\mu$ for the subspace of ${}_R\dot{\mathbf{U}}$ spanned by elements of the form $\mathbf{1}_\lambda x \mathbf{1}_\mu$, where $x \in {}_R\mathbf{U}$ and write $p_{\lambda,\mu}$ for the canonical projection ${}_R\mathbf{U} \to {}_\lambda\mathbf{U}_\mu$. As in [Lu94, 23.1.5, 23.1.6], Δ and S induce R^π-linear maps

$$\Delta_{\lambda,\mu,\lambda',\mu'} : {}_{\lambda+\lambda'}\mathbf{U}_{\mu+\mu'} \longrightarrow {}_\lambda\mathbf{U}_\mu \otimes {}_{\lambda'}\mathbf{U}_{\mu'}$$

given by $\Delta_{\lambda,\mu,\lambda',\mu'}(p_{\lambda+\lambda',\mu+\mu'}(x)) = (p_{\lambda,\mu} \otimes p_{\lambda',\mu'})(\Delta(x))$, for $\lambda, \mu, \lambda', \mu' \in X$, and

$$\dot{S} : {}_R\dot{\mathbf{U}} \longrightarrow {}_R\dot{\mathbf{U}}$$

defined by $\dot{S}(\mathbf{1}_\lambda x \mathbf{1}_\mu) = \mathbf{1}_{-\mu} S(x) \mathbf{1}_{-\lambda}$ for $x \in {}_R\mathbf{U}$. For example, $\Delta(E_i) = E_i \otimes 1 + \tilde{J}_i \tilde{K}_i \otimes E_i$ in ${}_R\mathbf{U}$, and hence we obtain

$$\Delta_{\lambda-\nu+i',\lambda-\nu,\nu,\nu}(E_i \mathbf{1}_\lambda) = p_{\lambda-\nu+i',\lambda-\nu} \otimes p_{\nu,\nu}(E_i \otimes 1 + \tilde{J}_i \tilde{K}_i \otimes E_i) = E_i \mathbf{1}_{\lambda-\nu} \otimes \mathbf{1}_\nu.$$

This collection of maps is called the comultiplication on ${}_R\dot{\mathbf{U}}$, and it can be formally regarded as a single linear map

$$\dot{\Delta} = \prod_{\lambda,\mu,\lambda',\mu' \in X} \hat{\Delta}_{\lambda,\mu,\lambda',\mu'} : {}_R\dot{\mathbf{U}} \longrightarrow \prod_{\lambda,\mu,\lambda',\mu' \in X} {}_\lambda\mathbf{U}_\mu \otimes {}_{\lambda'}\mathbf{U}_{\mu'}.$$

A comultiplication $\dot{\Delta}^\diamond$ on ${}_R\dot{\mathbf{U}}^\diamond$ can be defined in the same way.

Proposition 5.2. *The Frobenius-Lusztig homomorphism* Fr *is compatible with the comultiplications on* ${}_R\dot{\mathbf{U}}$ *and* ${}_R\dot{\mathbf{U}}^\diamond$*, i.e.,* $\dot{\Delta}^\diamond \circ \mathrm{Fr} = (\mathrm{Fr} \otimes \mathrm{Fr}) \circ \dot{\Delta}$.

(In the usual quantum group setting this was noted by [Lu94, 35.1.10].)

Proof. It suffices to check on the generators $E_i^{(n)}\mathbf{1}_\lambda$ and $F_i^{(n)}\mathbf{1}_\lambda$. Let $n = m\ell_i \in \ell_i\mathbb{Z}$, and recall that $\mathrm{Fr}(E_i^{(m\ell_i)}\mathbf{1}_\lambda) = \pi_i^{\binom{\ell_i}{2}m} E_i^{(m)}\mathbf{1}_\lambda$ in ${}_R\dot{\mathbf{U}}^\diamond$. Using the formula (above [CHW13, Proposition 2.2.2])

$$\Delta(E_i^{(m)}) = \sum_{p+r=m} (\pi_i q_i)^{pr} E_i^{(p)} (\tilde{J}_i \tilde{K}_i)^r \otimes E_i^{(r)}$$

we see that the nonzero parts in $\dot{\Delta}^\diamond(\mathrm{Fr}(E_i^{(m\ell_i)}\mathbf{1}_\lambda))$ computed via (4.10) are of the form

$$\pi_i^{\binom{\ell_i}{2}m} (\pi_i^\diamond q_i^\diamond)^{(p+\langle i,\nu\rangle^\diamond)r} E_i^{(p)}\mathbf{1}_\nu \otimes E_i^{(r)}\mathbf{1}_{\lambda-\nu}, \qquad p + r = m$$

for various $\nu \in X^\diamond$, which coincides with $\mathrm{Fr} \otimes \mathrm{Fr}$ applied to terms in $\dot{\Delta}(E_i^{(m\ell_i)}\mathbf{1}_\lambda))$ of the form

$$(\pi_i q_i)^{(p\ell_i + \langle i,\nu\rangle)(r\ell_i)} E_i^{(p\ell_i)}\mathbf{1}_\nu \otimes E_i^{(r\ell_i)}\mathbf{1}_{\lambda-\nu}, \qquad p + r = m,$$

where we note there is a factor contributing from (4.10) which matches up with the previous part thanks to $\pi_i^{\binom{\ell_i}{2}p + \binom{\ell_i}{2}r} = \pi_i^{\binom{\ell_i}{2}m}$; the remaining terms are zero under $\mathrm{Fr} \otimes \mathrm{Fr}$ since at least one of the divided powers of E_i appearing in either tensor factor must be not divisible by ℓ_i.

On the other hand, if n is not divisible by ℓ_i, then the right hand side will also be zero, since all the non-zero parts of $\dot{\Delta}(E_i^{(n)}\mathbf{1}_\lambda))$ will have a tensor factor containing some divided power of E_i not divisible by ℓ_i.

$F_i^{(n)}\mathbf{1}_\lambda$ can be verified similarly. □

The maps $\dot{\Delta}$ and $\dot{S}$ restrict to maps on ${}_R\dot{\mathfrak{u}}$, which extend to R^π-linear maps $\hat{\Delta}$ and $\hat{S}$ on ${}_R\hat{\mathfrak{u}}$ in the obvious way. Henceforth, when we refer to $\hat{\Delta}$ and $\hat{S}$ we mean the restrictions to ${}_R\mathfrak{u}$.

Additionally, for any basis $\mathbf{B}$ of $\mathfrak{f}$ consisting of weight vectors, with unique zero weight element equal to 1, we define an R^π-linear map $\hat{e} : {}_R\mathfrak{u} \to R^\pi$ by:

$$\hat{e}(rb^+b'^-\mathbf{1}_{\mathbf{c_a}}) = \begin{cases} r, & \text{if } b, b' = 1 \text{ and } \mathbf{a} = \mathbf{0}, \\ 0, & \text{otherwise.} \end{cases}$$

where $b, b' \in \mathbf{B}$, $r \in R^\pi$, and $\mathbf{c_a}$ in (5.1).

Define the following elements:

$$K_i = \sum_{\lambda \in X} \mathbf{q}^{\langle i,\lambda\rangle}\mathbf{1}_\lambda, \quad J_i = \sum_{\lambda \in X} \pi^{\langle i,\lambda\rangle}\mathbf{1}_\lambda, \quad 1 = \sum_{\lambda \in X} \mathbf{1}_\lambda. \tag{5.2}$$

Proposition 5.3.

1. *The R^π-algebra ${}_R\mathfrak{u}$ has a generating set $\{E_i, F_i$ (for all i with $\ell_i \geq 2), K_i, J_i$ (for all $i \in$*

$I)\}$.

2. $({}_R\mathfrak{u}, \hat{\Delta}, \hat{e}, \hat{S})$ *forms a Hopf superalgebra.*

Proof. The elements in (5.2) can be written as

$$K_i = \sum_{\mathbf{c}} \mathbf{q}_{\mathbf{c},i} \mathbf{1}_{\mathbf{c}}, \quad J_i = \sum_{\mathbf{c}} \pi_{\mathbf{c},i} \mathbf{1}_{\mathbf{c}}, \quad 1 = \sum_{\mathbf{c}} \mathbf{1}_{\mathbf{c}},$$

where we have defined $\mathbf{q}_{\mathbf{c},i} = \mathbf{q}^{\langle i,\lambda\rangle}$ and $\pi_{\mathbf{c},i} = \pi^{\langle i,\lambda\rangle}$ for any $\lambda \in \mathbf{c}$. This implies that these elements are also in ${}_R\mathfrak{u}$. Moreover, we have

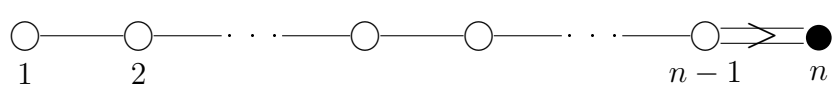
$$\mathbf{1}_{\mathbf{c}} = \prod_{i\in I} (2\tilde{\ell})^{-1}(1 + \pi_{\mathbf{c},i} J_i)(1 + \mathbf{q}_{\mathbf{c},i}^{-1} K_i + \mathbf{q}_{\mathbf{c},i}^{-2} K_i^2 + ... + \mathbf{q}_{\mathbf{c},i}^{1-\tilde{\ell}} K_i^{\tilde{\ell}-1}).$$

This proves (1).

A direct computation using these generators shows that $\hat{\Delta}$, $\hat{e}$ and $\hat{S}$ are given by the same formulas as Δ, e and S, the former maps inherit the following properties of the latter: $\hat{\Delta}$ is a homomorphism which satisfies the coassociativity (cf. [CHW13, Lemmas 2.2.1 and 2.2.3]), $\hat{e}$ is a homomorphism (cf. [CHW13, Lemma 2.2.3]), and $\hat{S}(xy) = \pi^{p(x)p(y)}\hat{S}(y)\hat{S}(x)$ (cf. [CHW13, Lemma 2.4.1]). Moreover, the image of $\hat{\Delta}$ (respectively, $\hat{S}$) lies in ${}_R\mathfrak{u} \otimes {}_R\mathfrak{u}$ (respectively, ${}_R\mathfrak{u}$). Hence (2) holds. □

5.2 Dimension formulas for finite type

We consider the Cartan datum associated to the Lie superalgebra $\mathfrak{osp}(1|2n)$, where $n = |I|$, with the Dynkin diagram



where the black node denotes the (only) odd simple root. We set

$$i \cdot i = \begin{cases} 2, & \text{if } i \text{ is odd,} \\ 4, & \text{if } i \text{ is even.} \end{cases}$$

The above Cartan datum on I is a super Cartan datum satisfying the bar-consistent condition in the sense of §2.1.

Proposition 5.4. *The small quantum covering group ${}_R\mathfrak{u}$ of type $\mathfrak{osp}(1|2n)$ is a finite dimensional R^π-module. In particular,*

$$\dim_{R^\pi}({}_R\mathfrak{u}) = \frac{\ell^{2n^2}}{\gcd(2,\ell)^{2n^2-2n}}(2\tilde{\ell})^n = \begin{cases} \ell^{2n^2}(4\ell)^n, & \textit{for } \ell \textit{ odd,} \\ \dfrac{\ell^{2n^2}}{2^{2n^2-2n}}(2\ell)^n, & \textit{for } \ell \textit{ even,} \end{cases}$$

when X is the weight lattice, and similarly,

$$\dim_{R^\pi}({}_R\mathfrak{u}) = \frac{\ell^{2n^2}}{\gcd(2,\ell)^{2n^2-2n}}2^{n-1}\tilde{\ell}^n = \begin{cases} \ell^{2n^2}2^{2n-1}\ell^n, & \textit{for } \ell \textit{ odd,} \\ \dfrac{\ell^{2n^2}}{2^{2n^2-2n}}2^{n-1}\ell^n, & \textit{for } \ell \textit{ even,} \end{cases}$$

when X is the root lattice.

Proof. Note that ${}_R\mathfrak{u}$ is a $\mathfrak{f} \otimes \mathfrak{f}^{\mathrm{opp}}$ module with basis given by the $\mathbf{1_c}$ defined above. This basis has at most $(2\tilde{\ell})^n$ elements for any X. In particular, it has $(2\tilde{\ell})^n$ elements when X is the weight lattice, and $2^{n-1}\tilde{\ell}^n$ elements when X is the root lattice, as the root lattice is index 2 in the weight lattice. Moreover, by Proposition 4.6, we have that $\dim_{R^\pi}(\mathfrak{f}^\pm) = \dim_{R^\pi}({}_RV(\lambda))$, where λ is the unique weight such that $\langle i, \lambda\rangle = \ell_i - 1$ for each $i \in I$. Let $V(\lambda)_1$ (respectively, $V(\lambda)_{-1}$) be the quotient of the Verma module of highest weight λ by its maximal ideal for the quantum group (resp. quantum supergroup) to which the quantum covering group specializes at $\pi = 1$ (respectively, $\pi = -1$) with base field $R = \mathbb{Q}(\varepsilon)$ (recall from §3.1 that ε is an ℓ'-th

root of unity). Because

$$_RV(\lambda) = (\pi+1)_RV(\lambda) \oplus (\pi-1)_RV(\lambda) \cong V(\lambda)_1 \oplus V(\lambda)_{-1}$$

and the characters of $V(\lambda)_1$ and $V(\lambda)_{-1}$ coincide for dominant weights (cf. [KKO14], [CHW14, Remark 2.5]), we have

$$\dim_{R^\pi} \mathfrak{f}^\pm = \dim_{R^\pi} \ _RV(\lambda) = \dim_R V(\lambda)_1 = \dim_R \mathfrak{f}_1^\pm = \frac{\ell^{n^2}}{\gcd(2,\ell)^{n^2-n}}$$

where $\mathfrak{f}_1$ is the (non-super) half small quantum group, i.e., $\mathfrak{f}$ specialized at $\pi = 1$. The last equality is due to [Lu90b, Theorem 8.3(iv)].

□

Part II

Quantum Symmetric Pairs for Quantum Covering Groups

Chapter 6

$\imath$Quantum Covering Groups

We will open this chapter with a short overview of the role that $\imath$-divided powers play for the classical $\imath$quantum groups. Then, we will define the $\imath$quantum covering groups and describe their size and structure. We end by giving a change of parameters isomorphism, which is the π-analogue of a construction in [CLW18] that was instrumental in streamlining the proof of main result there.

6.1 $\imath$Quantum groups and $\imath$-divided powers

For a Drinfeld-Jimbo quantum group $\mathbf{U}$ with Chevalley generators $E_i, F_i, K_i^{\pm 1}$, $i \in I$, we have a familiar presentation, its *Serre presentation*, featuring the q-Serre relations among the E_i's and F_i's. In terms of divided powers $F_i^{(n)} = F_i^n/[n]_{q_i}^!$ (where $[n]_{q_i}^!$ are the quantum factorials, which can be obtained from §3.1 by setting $\pi = 1$, cf. [Lu94]), the q-Serre relations among the F_i's has a compact form: for $i \neq j \in I$,

$$\sum_{n=0}^{1-a_{ij}} (-1)^n F_i^{(n)} F_j F_i^{(1-a_{ij}-n)} = 0. \tag{6.1}$$

Let $\Delta : \mathbf{U} \to \mathbf{U} \otimes \mathbf{U}$ comultiplication for the quantum group $\mathbf{U}$.

Quantum symmetric pairs $(\mathbf{U}, \mathbf{U}^\imath)$, are deformations of classical symmetric pairs which are defined using *Satake diagrams*, Dynkin diagrams with some nodes blackened and other nodes connected in pairs by a diagram involution. The theory of quantum symmetric pairs was systematically studied by Letzter for $\mathbf{U}$ of finite type (cf. [Le99, Le02]) and in Kac-Moody type the theory was further developed by Kolb [Ko14]. The *ıquantum group* $\mathbf{U}^\imath$ is a (right) coideal subalgebra of $\mathbf{U}$: it satisfies the property that $\Delta : \mathbf{U}^\imath \to \mathbf{U}^\imath \otimes \mathbf{U}$ (the *coideal property*). Main generators of $\mathbf{U}^\imath$ are defined in terms of generators of $\mathbf{U}$ using an embedding formula cf. (6.6):

$$B_i = F_i + \varsigma_i E_{\tau i} \widetilde{K}_i^{-1}, \qquad \text{for } i \in I, \tag{6.2}$$

where $\varsigma = (\varsigma_i)_{i \in I}$, are parameters.

A ıquantum group is called *quasi-split* (and respectively, *split*) if the underlying Satake diagram contains no black node (respectively, is equipped with the trivial involution on the Satake diagram). The origins of this terminology lie in the classical theory of real simple Lie algebras. A quasi-split ıquantum group takes only the generalized Cartan matrix and a diagram involution τ as its inputs. Examples of the split ıquantum groups were appeared previously in the literature (cf., e.g., [T93, BasK05]) and are sometimes referred to as generalized q-Onsager algebras, cf. [BaB10]. We refer to [Ko14, Introduction, (1)] for more detailed historical remarks.

In [CLW18], a Serre presentation with uniform relations for the quasi-split ıquantum groups of Kac-Moody type with general parameters is formulated precisely, generalizing the work of Letzter in finite type and Kolb in Kac-Moody type for $|a_{ij}| \leq 3$, cf. [Le02, Le03, Ko14]. A centerpiece of the Serre presentation for $\mathbf{U}^\imath$ is the *ı-Serre relations* between B_i and B_j for $\tau i = i \neq j$. These relations can be expressed with striking resemblance to the q-Serre relation (6.1): for any fixed $\overline{p} \in \mathbb{Z}_2 = \{\overline{0}, \overline{1}\}$,

$$\sum_{n=0}^{1-a_{ij}} (-1)^n B^{(n)}_{i,\overline{a_{ij}}+\overline{p}} B_j B^{(1-a_{ij}-n)}_{i,\overline{p}} = 0, \tag{6.3}$$

where the *ı-divided powers* $B_{i,\overline{p}}^{(m)}$ are polynomials (compare Lusztig's divided powers, which are monomials) in B_i which depend on a parity $\overline{p}$ arising from the parities of the highest weights of highest weight $\mathbf{U}$-modules when evaluated at the coroot h_i. The $\imath$-divided powers were introduced in [BW18a] and studied further in [BeW18], and are canonical basis elements for (the modified form of) $\mathbf{U}^\imath$ in the sense of [BW18b]. Writing the $\imath$-Serre relations (6.3) in terms of $\imath$-divided powers provided a uniform reformulation of complicated case-by-case relations for the cases $|a_{ij}| \leq 3$ in [Ko14, BaK19], which enabled the method of proof in [CLW18, §4].

A precise formulation of the Serre presentation enabled the formulation of a bar involution on a general $\imath$quantum group $\mathbf{U}^\imath$, which was predicted in [BW18a]; it allows one to write down the constraints that the parameters should satisfy, cf. [BaK15]. The bar involution on $\mathbf{U}^\imath$ is a basic ingredient for the canonical basis for $\mathbf{U}^\imath$ [BW18b, BW18c]. The $\imath$divided powers are also a key component in constructing the Frobenius-Lusztig homomorphism for $\imath$quantum groups at roots of unity in [BaS19].

6.2 Definition and basic structure

Let $(Y, X, \langle\cdot,\cdot\rangle, \cdots)$ be a root datum of (super) type $(I, \cdot)$. We call a permutation τ of the set I an *involution* of the Cartan datum $(I, \cdot)$ if $\tau^2 = \mathrm{id}$ and $\tau i \cdot \tau j = i \cdot j$ for i, $j \in I$. Note we allow $\tau = \mathrm{id}$. We will always assume that τ extends to an involution on X and an involution on Y (also denoted by τ), respectively, such that the perfect bilinear pairing is invariant under the involution τ. The permutation τ of I induces an $\mathbb{K}(q)$-algebra automorphism of $\mathbf{U}$, defined by

$$\tau : E_i \mapsto E_{\tau i}, \quad F_i \mapsto F_{\tau i}, \quad K_h \mapsto K_{\tau h}, \qquad \text{for all } i \in I, h \in Y. \tag{6.4}$$

Define

$$Y^\imath = \{h \in Y \mid \tau(h) = -h\}. \tag{6.5}$$

Just as in [CLW18], in the remainder of this dissertation we will only consider the quasi-split case (corresponding to Satake diagrams without black nodes).

Definition 6.1. The *quasi-split ıquantum group*, denoted by $\mathbf{U}^\imath_\varsigma$ or $\mathbf{U}^\imath$, is the $\mathbb{K}(q)$-subalgebra of $\mathbf{U}$ generated by

$$B_i := F_i + \varsigma_i E_{\tau i} \widetilde{K}_i^{-1}, \qquad \widetilde{J}_i \ (i \in I), \qquad K_\mu \ (\mu \in Y^\imath). \tag{6.6}$$

Here the parameters

$$\varsigma = (\varsigma_i)_{i \in I} \in (\mathbb{K}(q)^\times)^I, \tag{6.7}$$

are assumed to satisfy Conditions (6.8)–(6.10) below:

$$\overline{\varsigma_i q_i} = \varsigma_i q_i \text{ if } \tau i = i \text{ and } a_{ij} \neq 0 \text{ for some } j \in I \setminus \{i\}; \tag{6.8}$$

$$\varsigma_i = \varsigma_{\tau i} \text{ if } a_{i,\tau i} = 0; \tag{6.9}$$

$$\varsigma_{\tau i} = \pi_i q_i^{-a_{i,\tau i}} \overline{\varsigma_i} \text{ if } a_{i,\tau i} \neq 0. \tag{6.10}$$

The conditions on the parameters ensure that $\mathbf{U}^\imath$ admits a suitable bar involution (discussed in detail in Chapter 9). With the convention for the comultiplication as above, $\mathbf{U}^\imath$ is a right coideal subalgebra of $\mathbf{U}$, i.e., $\Delta : \mathbf{U}^\imath \longrightarrow \mathbf{U}^\imath \otimes \mathbf{U}$. We also note here that in [Ko14] and [CLW18] an additional set of parameters κ_i is considered; in the setting of quantum covering groups the only interesting case ($\kappa_i \neq 0$ for some $i \in I$) exists in rank 2 ($\mathfrak{osp}(1|4)$), and its Serre presentation is a straightforward generalization of the non-covering case. Thus, we can omit any discussion of κ_i from our considerations in the following chapters.

6.2.1 Structure and size of $\mathbf{U}^\imath$

A few of the results on the structure and size of $\mathbf{U}^\imath$ are collected here (cf. the non-super case in [Ko14, §5–6]). First, we define the projections P_λ and $\pi_{\alpha,\beta}$ similarly to [Ko14, §5.2]: by the triangular decomposition [CHW13, Corollary 2.3.3],

$$\mathbf{U} = \bigoplus_{\lambda \in Y} \mathbf{U}^+ \mathbf{U}_J K_\lambda S(\mathbf{U}^-),$$

where $\mathbf{U}_J = \langle J_\mu \,|\, \mu \in Y \rangle$ and S denotes the antipode of $\mathbf{U}$. For any $\lambda \in Y$ let

$$P_\lambda : \mathbf{U} \to \mathbf{U}^+ \mathbf{U}_J K_\lambda S(\mathbf{U}^-) \tag{6.11}$$

denote the projection with respect to this decomposition.

Similarly, let

$$\pi_{\alpha,\beta} : \mathbf{U} \to \mathbf{U}^+_\alpha \mathbf{U}^0 \mathbf{U}^-_{-\beta} \tag{6.12}$$

denote the projection with respect to the decomposition

$$\mathbf{U} = \bigoplus_{\alpha,\beta \in Y^+} \mathbf{U}^+_\alpha \mathbf{U}^0 \mathbf{U}^-_{-\beta}.$$

Because the embedding formulas for the $\imath$quantum covering groups follow the same form as in [Ko14, (5.1)] (with $X = \emptyset$ and $s_i = 0$), we have the following technical lemma, proved in the same way as in *loc. cit.*:

Lemma 6.2. *Let $\alpha, \beta \in Q^+$. If $\pi_{\alpha,\beta}(F_{ij}(B_i, B_j)) \neq 0$ then $\lambda_{ij} - \alpha \in Q^\Theta$ and $\lambda_{ij} - \beta \in Q^\Theta$.*

Using this, we also have the following results about $\mathbf{U}^\imath$:

Proposition 6.3. *In $\mathbf{U}$, we have $P_{\lambda_{ij}}(F_{ij}(B_i, B_j)) = 0$ for all $i, j \in I$.*

Proposition 6.4. *In* $\mathbf{U}^\imath$*, we have the relation*

$$F_{ij}(B_i, B_j) \in \sum_{\{J\in\mathcal{J} \,|\, \mathrm{wt}(J)<\lambda_{ij}\}} \mathbf{U}^{0'}_{\Theta} B_J \text{ for all } i, j \in I. \tag{6.13}$$

We now show that $\mathbf{U}^\imath$ has the same size as $\mathbf{U}^-$, cf [Ko14, §6.1–2]. For any multi-index $J = (j_1, \ldots, j_n)$, define $\mathrm{wt}(J) = \sum_{i=1}^n \alpha_j$, and $F_J = F_{j_1} \ldots F_{j_n}$ and $B_J = B_{j_1} \ldots B_{j_n}$, and define $|J| = n$. Let $\mathcal{J}$ be a fixed subset of $\bigcup_{n\in\mathbb{N}_0} I^n$ such that $\{F_J \,|\, J \in \mathcal{J}\}$ is a basis of $\mathbf{U}^-$, and hence a basis of $\mathbf{U}'$ as a left $\mathbf{U}^+\mathbf{U}^{0'}$-module. Define a filtration $\mathcal{F}^*$ of $\mathbf{U}^-$ by $\mathcal{F}^n(\mathbf{U}^-) = \mathrm{span}\{F_J \,|\, J \in I^m, m \leq n\}$ for all $n \in \mathbb{N}_0$. By the homogeneity of the (q, π)-Serre relations (2.3), the set $\mathrm{span}\{F_J \,|\, J \in \mathcal{J}, |J| = n\}$ forms a basis of $\mathcal{F}^n(\mathbf{U}^-)$. Then, we have the following proposition, cf. [Ko14, Proposition 6.2]:

Proposition 6.5. *The set* $\{B_J \,|\, J \in \mathcal{J}\}$ *is a basis of the left (or right)* $\mathbf{U}^+\mathbf{U}^{0'}$*-module* $\mathbf{U}^\imath$*.*

Proof. The argument is the same as the one in [Ko14, Proposition 6.2], which is much simpler for $X = \emptyset$: for $L \in I^n$, one can obtain $B_L \in \sum_{J\in\mathcal{J}} \mathbf{U}^{0'}_{\Theta} B_J$ by an induction on $n = \mathrm{wt}(L)$ and using the (q, π)-Serre relations. We thus have that $\{B_J \,|\, J \in \mathcal{J}\}$ spans $\mathbf{U}^\imath$. The fact that $\{B_J \,|\, J \in \mathcal{J}\}$ is linearly independent follows from the specific form of the generators B_i having 'leading term' F_i and the triangular decomposition. □

6.3 Change of parameters

In [CLW18] (also cf. [Ko14, Theorem 7.1]), a change-of-parameters isomorphism is used to give a presentation of the $\imath$quantum group $\mathbf{U}^\imath_{\varsigma,\kappa}$. In particular, it is shown that the $\mathbb{K}(q)$-algebra $\mathbf{U}^\imath_{\varsigma,\kappa}$ (up to some field extension) is isomorphic to $\mathbf{U}^\imath_{\varsigma^\diamond,\mathbf{0}}$ for some distinguished parameters $\varsigma^\diamond$, i.e., $\varsigma^\diamond = q_i^{-1}$ for all $i \in I$ such that $\tau i = i$ (cf. [Le02], [Ko14, Proposition 9.2]). The same argument carries over to the quantum covering setting:

For given parameters ς satisfying (6.8)–(6.10), let $\varsigma^\diamond$ be the associated distinguished pa-

rameters such that $\varsigma_i^\diamond = \varsigma_i$ if $\tau i \neq i$, and

$$\varsigma_i^\diamond = q_i^{-1}, \text{ if } \tau i = i. \tag{6.14}$$

Let $\mathbf{U}^\imath_{\varsigma^\diamond}$ be the $\imath$quantum covering group with the parameters $\varsigma^\diamond =$ for all $i \in I$. Let $\mathbb{F} = \mathbb{K}(q)(a_i \mid i \in I \text{ such that } \tau i = i)$ be a field extension of $\mathbb{K}(q)$, where

$$a_i = \sqrt{q_i \varsigma_i}, \qquad \text{for all } i \in I \text{ such that } \tau i = i. \tag{6.15}$$

Denote by ${}_{\mathbb{F}}\mathbf{U}^\imath_\varsigma = \mathbb{F} \otimes_{\mathbb{K}(q)} \mathbf{U}^\imath_\varsigma$ the $\mathbb{F}$-algebra obtained by a base change.

Proposition 6.6. *There exists an isomorphism of $\mathbb{F}$-algebras*

$$\phi_\imath : {}_{\mathbb{F}}\mathbf{U}^\imath_{\varsigma^\diamond} \longrightarrow {}_{\mathbb{F}}\mathbf{U}^\imath_\varsigma,$$

$$B_i \mapsto \begin{cases} B_i, & \text{if } \tau i \neq i, \\ a_i^{-1} B_i, & \text{if } \tau i = i; \end{cases} \qquad K_\mu \mapsto K_\mu, \quad (\text{ for all } i \in I, \mu \in Y^\imath),$$

In particular, this enables us to use the formulas for $\imath^\pi$-divided powers in the next section free of unwieldy coefficients.

Chapter 7

$\imath^\pi$-divided powers and expansion formulas

In this chapter we will define the $\imath^\pi$-divided powers, which are generalizations of the $\imath$-divided powers developed in [BeW18] to the quantum covering group setting. The $\imath^\pi$-divided powers can be thought of as a canonical basis for $\mathbf{U}^\imath$ in rank one, which is just a polynomial ring over $\mathbb{K}(q)^\pi$. They can be written down explicitly in terms of the Chevalley generators for $\mathbf{U}$; these expansion formulas will turn out to be crucial in the following sections.

7.1 The algebras $\mathbf{U}^\imath$ and $\dot{\mathbf{U}}$ in rank one

Recall from [CHW13, 2.1] that the rank one quantum covering group $\mathbf{U}$ with a single odd root i.e. type $I = I_{\bar{1}} = \{1\}$ is the $\mathbb{K}(q)^\pi$-algebra generated by $E, F, K^{\pm 1}, J$, subject to the relations: $KK^{-1} = K^{-1}K = 1$, and

$$\begin{gathered} JK = KJ, \quad JE = EJ, \quad JF = FJ, \quad J^2 = 1, \\ KEK^{-1} = q^2 EK, \quad KFK^{-1} = q^{-2} FK, \\ EF - \pi FE = \frac{JK - K^{-1}}{\pi q - q^{-1}}. \end{gathered} \tag{7.1}$$

The rank one $\imath$quantum covering group $\mathbf{U}^\imath$ is generated as a $\mathbb{K}(q)^\pi$-algebra by a single generator

$$B = F + q^{-1}EK^{-1}.$$

Lemma 7.1. *There is an anti-involution ς of the $\mathbb{K}$-algebra $\mathbf{U}$ fixing the generators $E, F, K^{\pm 1}, J$ and sending $q \mapsto q^{-1}$.*

Proof. We have

$$\varsigma(KEK^{-1}) = K^{-1}EK = q^{-2}E = \varsigma(q^2E), \quad \varsigma(KFK^{-1}) = K^{-1}FK = q^2F = \varsigma(q^{-2}F).$$

We also have

$$\varsigma(EF - \pi FE) = FE - \pi EF = \frac{JK - K^{-1}}{\pi q^{-1} - q} = \varsigma\left(\frac{JK - K^{-1}}{\pi q - q^{-1}},\right)$$

and so ς preserves all the relations in (7.1) (since J is central). $\square$

Note that $\varsigma([n]) = \pi^{n-1}[n]$, and so $\varsigma[n]^! = \pi^{\binom{n}{2}}[n]$.

The algebra $\dot{\mathbf{U}}$ in rank one

Denote by $\dot{\mathbf{U}}$ the modified quantum group of $\mathfrak{osp}(1|2)$, as the odd rank one case of §2.2.

Let ${}_{\mathcal{A}}\dot{\mathbf{U}}$ be the $\mathcal{A}$-subalgebra of $\dot{\mathbf{U}}$ generated by $E^{(n)}\mathbf{1}_\lambda, F^{(n)}\mathbf{1}_\lambda, \mathbf{1}_\lambda$, for all $n \geq 0$ and $\lambda \in \mathbb{Z}$. There is a natural left action of $\mathbf{U}$ on $\dot{\mathbf{U}}$ such that $K\mathbf{1}_\lambda = q^\lambda \mathbf{1}_\lambda$ and $J\mathbf{1}_\lambda = \pi^\lambda \mathbf{1}_\lambda$. Denote by

$${}_{\mathcal{A}}\dot{\mathbf{U}}_{\mathrm{ev}} = \bigoplus_{\lambda \in \mathbb{Z}} {}_{\mathcal{A}}\dot{\mathbf{U}}\mathbf{1}_{2\lambda}, \qquad {}_{\mathcal{A}}\dot{\mathbf{U}}_{\mathrm{odd}} = \bigoplus_{\lambda \in \mathbb{Z}} {}_{\mathcal{A}}\dot{\mathbf{U}}\mathbf{1}_{2\lambda-1}.$$

We have ${}_{\mathcal{A}}\dot{\mathbf{U}} = {}_{\mathcal{A}}\dot{\mathbf{U}}_{\mathrm{ev}} \oplus {}_{\mathcal{A}}\dot{\mathbf{U}}_{\mathrm{odd}}$. By a base change we define $\dot{\mathbf{U}}_{\mathrm{ev}}$ and $\dot{\mathbf{U}}_{\mathrm{odd}}$ accordingly so that $\dot{\mathbf{U}} = \dot{\mathbf{U}}_{\mathrm{ev}} \oplus \dot{\mathbf{U}}_{\mathrm{odd}}$.

7.2 Recursive definition and closed form formulas

We have the following generalizations of the formulas for $\imath$divided powers developed in [BeW18], with the new additions of π and J highlighted in blue: the even $\imath^{\pi}$-divided powers $B_{\bar{0}}^{(n)}$ satisfy and are in turn determined by the following recursive relations:

$$\begin{aligned} B \cdot B_{\bar{0}}^{(2a-1)} &= [2a] B_{\bar{0}}^{(2a)}, \\ B \cdot B_{\bar{0}}^{(2a)} &= [2a+1] B_{\bar{0}}^{(2a+1)} + [2a] J B_{\bar{0}}^{(2a-1)}, \qquad \text{for } a \geq 1. \end{aligned} \tag{7.2}$$

where $[n] := [n]_{q,\pi}$ here denotes the (q,π)-integer; for the remainder of this section these subscripts will be suppressed.

Analogously, the odd $\imath$divided powers $B_{\bar{1}}^{(n)}$ satisfy (and are determined by) the following recursive relations:

$$\begin{aligned} B \cdot B_{\bar{1}}^{(2a)} &= [2a+1] B_{\bar{1}}^{(2a+1)}, \\ B \cdot B_{\bar{1}}^{(2a+1)} &= [2a+2] B_{\bar{1}}^{(2a+2)} + [2a+1] \pi J B_{\bar{1}}^{(2a)}, \qquad \text{for } a \geq 0. \end{aligned} \tag{7.3}$$

Solving these recursive formulas, we arrive at the following closed form formulas:

$$\begin{aligned} B_{\bar{0}}^{(2a)} &= \frac{B^2(B^2 - [2]^2 J) \cdots (B^2 - [2a-4]^2 J)(B - [2a-2]^2 J)}{[2a]!}, \\ B_{\bar{0}}^{(2a+1)} &= \frac{B^2(B^2 - [2]^2 J) \cdots (B^2 - [2a-2]^2 J)(B - [2a]^2 J)}{[2a+1]!}, \qquad \text{for } a \geq 0, \end{aligned} \tag{7.4}$$

and

$$\begin{aligned} B_{\bar{1}}^{(2a)} &= \frac{(B^2 - \pi J)(B^2 - \pi [3]^2 J) \cdots (B - \pi [2a-1]^2 J)}{[2a]!}, \\ B_{\bar{1}}^{(2a+1)} &= \frac{B(B^2 - \pi J)(B^2 - \pi [3]^2 J) \cdots (B - \pi [2a-1]^2 J)}{[2a+1]!}, \qquad \text{for } a \geq 0. \end{aligned} \tag{7.5}$$

For example, $B_{\bar{0}}^{(0)} = 1$, $B_{\bar{0}}^{(1)} = B$, $B_{\bar{0}}^{(2)} = B^2/[2]$, and $B_{\bar{0}}^{(3)} = B(B^2 - J[2]^2)/[3]!$, and $B_{\bar{1}}^{(0)} = 1, B_{\bar{1}}^{(1)} = B, B_{\bar{1}}^{(2)} = (B^2 - \pi J)/[2]$ and $B_{\bar{1}}^{(3)} = B(B^2 - \pi J)/[3]!$.

7.3 Expansion formulas

In this subsection we will formulate a number of useful expansion formulas for $B_{\bar{0}}^{(n)}$ and $B_{\bar{1}}^{(n)}$ in terms of the Chevalley generators for $\mathbf{U}$, cf. [BeW18]. We set

$$\check{E} := q^{-1}EK^{-1}, \qquad h := \frac{K^{-2} - J}{q^2 - \pi}, \qquad B := \check{E} + F. \tag{7.6}$$

Define, for $a \in \mathbb{Z}, n \geq 0$,

$$\begin{bmatrix} h; a \\ n \end{bmatrix} = \prod_{i=1}^{n} \frac{q^{4a+4i-4}K^{-2} - J}{q^{4i} - 1}, \qquad [h; a] = \begin{bmatrix} h; a \\ 1 \end{bmatrix}. \tag{7.7}$$

Note that $h = q[2]\,[h; 0]$.

It follows from (7.1) that, for $a \in \mathbb{Z}$ and $n \geq 0$,

$$F\check{E} = h + \pi q^{-2}\check{E}F, \qquad \begin{bmatrix} h; a \\ n \end{bmatrix} F = F \begin{bmatrix} h; a+1 \\ n \end{bmatrix}, \qquad \begin{bmatrix} h; a \\ n \end{bmatrix} \check{E} = \check{E} \begin{bmatrix} h; a-1 \\ n \end{bmatrix}. \tag{7.8}$$

Also define for $a \in \mathbb{Z}, n \geq 1$,

$$\left[\!\!\left[\begin{matrix} h; a \\ 0 \end{matrix} \right]\!\!\right] = 1, \qquad \left[\!\!\left[\begin{matrix} h; a \\ n \end{matrix} \right]\!\!\right] = \prod_{i=1}^{n} \frac{q^{4a+4i-4}K^{-2} - \pi q^2 J}{q^{4i} - 1}, \qquad [\![h; a]\!] = \left[\!\!\left[\begin{matrix} h; a \\ 1 \end{matrix} \right]\!\!\right]. \tag{7.9}$$

Note $h = q[2][\![h; 0]\!] + 1$. It follows from (7.1) and (7.9) that, for $n \geq 0$ and $a \in \mathbb{Z}$,

$$\left[\!\!\left[\begin{matrix} h; a \\ n \end{matrix} \right]\!\!\right] F = F \left[\!\!\left[\begin{matrix} h; a+1 \\ n \end{matrix} \right]\!\!\right], \qquad \left[\!\!\left[\begin{matrix} h; a \\ n \end{matrix} \right]\!\!\right] \check{E} = \check{E} \left[\!\!\left[\begin{matrix} h; a-1 \\ n \end{matrix} \right]\!\!\right]. \tag{7.10}$$

Just as in the even case, we also have

$$\left[\!\!\left[\begin{matrix} h; a \\ n \end{matrix}\right]\!\!\right] \mathbf{1}_{2\lambda-1} = q^{2n(a-\lambda)} \begin{bmatrix} a-\lambda-1+n \\ n \end{bmatrix}_{q^2} \mathbf{1}_{2\lambda-1} \in {}_{\mathcal{A}}\dot{\mathbf{U}}_{\mathrm{odd}}. \tag{7.11}$$

Lemma 7.2. *For $n \in \mathbb{N}$, we have*

$$\check{E}^{(n)} = q^{-n^2} E^{(n)} K^{-n}.$$

Proof. Follows by induction on n, using (7.1) and (7.6). □

Lemma 7.3. *The following formula holds for $n \geq 0$:*

$$F\check{E}^{(n)} = (\pi q^{-2})^n \check{E}^{(n)} F + \check{E}^{(n-1)} \frac{q^{3-3n}K^{-2} - (\pi q)^{1-n} J}{q^2 - \pi}. \tag{7.12}$$

Proof. We shall prove the following equivalent formula by induction on n:

$$F\check{E}^n = (\pi q^{-2})^n \check{E}^n F + (q^2-\pi)^{-1}[n]\check{E}^{n-1}\big(q^{3-3n}K^{-2} - (\pi q)^{1-n}J\big).$$

The base case when $n = 1$ is covered by (7.8). Assume the formula is proved for $F\check{E}^n$. Then by inductive assumption we have

$$\begin{aligned} F\check{E}^{n+1} &= (\pi q^{-2})^n \check{E}^n F\check{E} + (q^2-\pi)^{-1}[n]\check{E}^{n-1}\big(q^{3-3n}K^{-2} - (\pi q)^{1-n}J\big)\check{E} \\ &= (\pi q^{-2})^n \check{E}^n(\pi q^{-2}\check{E}F + (q^2-\pi)^{-1}\big(K^{-2} - J\big)) + (q^2-\pi)^{-1}[n]\check{E}^n\big(q^{-1-3n}K^{-2} - (\pi q)^{1-n}J\big) \\ &= (\pi q^{-2})^{n+1}\check{E}^{n+1}F + (q^2-\pi)^{-1}[n+1]\check{E}^n\big(q^{-3n}K^{-2} - (\pi q)^{-n}J\big), \end{aligned}$$

since $[n+1] = (\pi q)^n + q^{-1}[n] = \pi q[n] + q^{-n}$. The lemma is proved. □

For $n \in \mathbb{N}$, we denote

$$b_\pi^{(n)} = \sum_{a=0}^{n} (\pi q)^{-a(n-a)} \check{E}^{(a)} F^{(n-a)}. \tag{7.13}$$

The $\check{E}hF$-formula for $B_{\bar{0}}^{(n)}$

Recall $\begin{bmatrix} h; a \\ n \end{bmatrix}$ from (7.7).

Example 7.4. *We computed the following examples of $B_{\bar{0}}^{(n)}$, for $2 \leq n \leq 4$:*

$$B_{\bar{0}}^{(2)} = \frac{B^2}{[2]} = b_\pi^{(2)} + \pi q[h;0],$$
$$B_{\bar{0}}^{(3)} = \frac{B^3 - J[2]^2 B}{[3]^!} = b_\pi^{(3)} + \pi q^3 [h;-1]F + \pi q^3 \check{E}[h;-1],$$
$$B_{\bar{0}}^{(4)} = \frac{B^4 - J[2]^2 B^2}{[4]^!} = b_\pi^{(4)} + \pi q \check{E}^{(2)}[h;-1] + \pi q[h;-1]F^{(2)} + \check{E}[h;-1]F + q^6 \begin{bmatrix} h;-1 \\ 2 \end{bmatrix}.$$

Theorem 7.5. *For $m \geq 1$, we have*

$$B_{\bar{0}}^{(2m)} = \sum_{c=0}^{m} \sum_{a=0}^{2m-2c} (\pi q)^{\binom{2c}{2} - a(2m-2c-a)} \check{E}^{(a)} \begin{bmatrix} h; 1-m \\ c \end{bmatrix} F^{(2m-2c-a)}, \tag{7.14}$$

$$B_{\bar{0}}^{(2m-1)} = \sum_{c=0}^{m-1} \sum_{a=0}^{2m-1-2c} (\pi q)^{\binom{2c+1}{2} - a(2m-1-2c-a)} \check{E}^{(a)} \begin{bmatrix} h; 1-m \\ c \end{bmatrix} F^{(2m-1-2c-a)}. \tag{7.15}$$

Proof. We prove the formulae for $B_{\bar{0}}^{(n)}$ by using the recursive relations (7.2) and induction on n. The base cases for $n = 1, 2$ are clear. The induction is carried out in 2 steps.

(1) First by assuming the formula for $B_{\bar{0}}^{(2m-1)}$ in (7.15), we shall establish the formula (7.14) for $B_{\bar{0}}^{(2m)}$, via the identity $[2m]B_{\bar{0}}^{(2m)} = B \cdot B_{\bar{0}}^{(2m-1)}$ in (7.2).

Recall the formula (7.15) for $B_{\bar{0}}^{(2m-1)}$. Using $B = \check{E} + F$ and applying (7.12) to $F\check{E}^{(a)}$ we

have

$$B \cdot B_{\bar{0}}^{(2m-1)} = \sum_{c=0}^{m-1} \sum_{a=0}^{2m-1-2c} (\pi q)^{\binom{2c+1}{2}-a(2m-1-2c-a)} B \check{E}^{(a)} \begin{bmatrix} h; 1-m \\ c \end{bmatrix} F^{(2m-1-2c-a)} \tag{7.16}$$

$$= \sum_{c=0}^{m-1} \sum_{a=0}^{2m-1-2c} (\pi q)^{\binom{2c+1}{2}-a(2m-1-2c-a)} \cdot$$

$$\left(\check{E}\check{E}^{(a)} + (\pi q^{-2})^a \check{E}^{(a)} F + \check{E}^{(a-1)} \frac{q^{3-3a}K^{-2} - (\pi q)^{1-a} J}{q^2 - \pi} \right) \begin{bmatrix} h; 1-m \\ c \end{bmatrix} F^{(2m-1-2c-a)}$$

$$= \sum_{c=0}^{m-1} \sum_{a=0}^{2m-1-2c} (\pi q)^{\binom{2c+1}{2}-a(2m-1-2c-a)} \cdot$$

$$\left([a+1] \check{E}^{(a+1)} \begin{bmatrix} h; 1-m \\ c \end{bmatrix} F^{(2m-1-2c-a)} + (\pi q^{-2})^a [2m-2c-a] \check{E}^{(a)} \begin{bmatrix} h; -m \\ c \end{bmatrix} F^{(2m-2c-a)} \right.$$

$$\left. + \check{E}^{(a-1)} \frac{q^{3-3a}K^{-2} - (\pi q)^{1-a} J}{q^2 - \pi} \begin{bmatrix} h; 1-m \\ c \end{bmatrix} F^{(2m-1-2c-a)} \right).$$

We reorganize the formula (7.16) in the following form

$$[2m] \cdot B_{\bar{0}}^{(2m)} = B \cdot B_{\bar{0}}^{(2m-1)} = \sum_{c=0}^{m} \sum_{a=0}^{2m-2c} \check{E}^{(a)} f_{a,c}(h) F^{(2m-2c-a)},$$

where

$$f_{a,c}^{\pi}(h) = (\pi q)^{\binom{2c+1}{2}-(a-1)(2m-2c-a)} [a] \begin{bmatrix} h; 1-m \\ c \end{bmatrix}$$

$$+ \left(\pi^a (\pi q)^{\binom{2c+1}{2}-a(2m-1-2c-a)-2a} [2m-2c-a] \begin{bmatrix} h; -m \\ c \end{bmatrix} \right.$$

$$\left. + q^{\binom{2c-1}{2}-(a+1)(2m-2c-a)} \frac{q^{-3a}K^{-2} - (\pi q)^{-a} J}{q^2 - \pi} \begin{bmatrix} h; 1-m \\ c-1 \end{bmatrix} \right).$$

A direct computation gives us

$$
\begin{aligned}
f^{\pi}_{a,c}(h) &= (\pi q)^{\binom{2c}{2}-a(2m-2c-a)}(\pi q)^{2m-a}[a]\begin{bmatrix} h;1-m \\ c \end{bmatrix} + (\pi q)^{\binom{2c}{2}-a(2m-2c-a)}\cdot \\
&\quad \cdot\left(\pi^a(\pi q)^{2c-a}[2m-2c-a]\frac{q^{-4m}K^{-2}-J}{q^{4c}-1} + (\pi q)^{1+a-2m}\frac{q^{-3a}K^{-2}-(\pi q)^{-a}}{q^2-\pi}\right)\begin{bmatrix} h;1-m \\ c-1 \end{bmatrix} \\
&= (\pi q)^{\binom{2c}{2}-a(2m-2c-a)}(\pi q)^{2m-a}[a]\begin{bmatrix} h;1-m \\ c \end{bmatrix} + (\pi q)^{\binom{2c}{2}-a(2m-2c-a)}\cdot \\
&\quad \cdot\left(\pi^a(\pi q)^{2c-a}[2m-2c-a]\frac{q^{-4m}K^{-2}-J}{q^{4c}-1} + (\pi q)^{2c+a-2m}[2c]\frac{q^{-3a}K^{-2}-(\pi q)^{-a}}{q^{4c}-1}\right)\begin{bmatrix} h;1-m \\ c-1 \end{bmatrix} \\
&= (\pi q)^{\binom{2c}{2}-a(2m-2c-a)}(\pi q)^{2m-a}[a]\begin{bmatrix} h;1-m \\ c \end{bmatrix} + (\pi q)^{\binom{2c}{2}-a(2m-2c-a)}q^{-a}[2m-a]\begin{bmatrix} h;1-m \\ c \end{bmatrix} \\
&= (\pi q)^{\binom{2c}{2}-a(2m-2c-a)}((\pi q)^{2m-a}[a]+q^{-a}[2m-a])\begin{bmatrix} h;1-m \\ c \end{bmatrix} \\
&= (\pi q)^{\binom{2c}{2}-a(2m-2c-a)}[2m]\begin{bmatrix} h;1-m \\ c \end{bmatrix}.
\end{aligned}
$$

Hence we have obtained the formula (7.14) for $B^{(2m)}_{\overline{0}}$.

(2) Now by assuming the formula for $B^{(2m)}_{\overline{0}}$ in (7.14), we shall establish the following formula (with m in (7.15) replaced by $m+1$)

$$
B^{(2m+1)}_{\overline{0}} = \sum_{c=0}^{m}\sum_{a=0}^{2m+1-2c}(\pi q)^{\binom{2c+1}{2}-a(2m+1-2c-a)}\check{E}^{(a)}\begin{bmatrix} h;-m \\ c \end{bmatrix}F^{(2m+1-2c-a)}. \tag{7.17}
$$

Recall the formula for $B^{(2m)}_{\overline{0}}$ in (7.14). Using $B=\check{E}+F$ and applying (7.12) to $F\check{E}^{(a)}$ we

have

$$B \cdot B_{\overline{0}}^{(2m)} = \sum_{c=0}^{m} \sum_{a=0}^{2m-2c} (\pi q)^{\binom{2c}{2} - a(2m-2c-a)} B \check{E}^{(a)} \begin{bmatrix} h; 1-m \\ c \end{bmatrix} F^{(2m-2c-a)}$$

$$= \sum_{c=0}^{m} \sum_{a=0}^{2m-2c} (\pi q)^{\binom{2c}{2} - a(2m-2c-a)} \cdot$$

$$\cdot \left(\check{E}\check{E}^{(a)} + (\pi q^{-2})^a \check{E}^{(a)} F + \check{E}^{(a-1)} \frac{q^{3-3a} K^{-2} - (\pi q)^{1-a} J}{q^2 - \pi} \right) \begin{bmatrix} h; 1-m \\ c \end{bmatrix} F^{(2m-2c-a)}.$$

We rewrite this as

$$B \cdot B_{\overline{0}}^{(2m)} = \sum_{c=0}^{m} \sum_{a=0}^{2m-2c} (\pi q)^{\binom{2c}{2} - a(2m-2c-a)} \cdot \left([a+1] \check{E}^{(a+1)} \begin{bmatrix} h; 1-m \\ c \end{bmatrix} F^{(2m-2c-a)} \right. \tag{7.18}$$

$$+ (\pi q^{-2})^a [2m+1-2c-a] \check{E}^{(a)} \begin{bmatrix} h; -m \\ c \end{bmatrix} F^{(2m+1-2c-a)}$$

$$\left. + \check{E}^{(a-1)} \frac{q^{3-3a} K^{-2} - (\pi q)^{1-a} J}{q^2 - \pi} \begin{bmatrix} h; 1-m \\ c \end{bmatrix} F^{(2m-2c-a)} \right).$$

We shall use (7.2), (7.18) and (7.15) to obtain a formula of the form

$$[2m+1] B_{\overline{0}}^{(2m+1)} = B \cdot B_{\overline{0}}^{(2m)} - [2m] J B_{\overline{0}}^{(2m-1)} = \sum_{c=0}^{m} \sum_{a=0}^{2m+1-2c} \check{E}^{(a)} g_{a,c}^{\pi}(h) F^{(2m+1-2c-a)}, \tag{7.19}$$

for some suitable $g^{\pi}_{a,c}(h)$. Then we have

$$
\begin{aligned}
g^{\pi}_{a,c}(h) &= (\pi q)^{\binom{2c}{2}-(a-1)(2m+1-2c-a)}[a] \begin{bmatrix} h; 1-m \\ c \end{bmatrix} \\
&\quad + \pi^a (\pi q)^{\binom{2c}{2}-a(2m-2c-a)-2a}[2m+1-2c-a] \begin{bmatrix} h; -m \\ c \end{bmatrix} \\
&\quad + (\pi q)^{\binom{2c-2}{2}-(a+1)(2m+1-2c-a)} \frac{q^{-3a}K^{-2} - (\pi q)^{-a} J}{q^2 - \pi} \begin{bmatrix} h; 1-m \\ c-1 \end{bmatrix} \\
&\quad - (\pi q)^{\binom{2c-1}{2}-a(2m+1-2c-a)}[2m] \begin{bmatrix} h; 1-m \\ c-1 \end{bmatrix} \\
&= \pi^a (\pi q)^{\binom{2c+1}{2}-a(2m+1-2c-a)} (\pi q)^{-2c-a}[2m+1-2c-a] \begin{bmatrix} h; -m \\ c \end{bmatrix} + (\pi q)^{\binom{2c+1}{2}-a(2m+1-2c-a)} X,
\end{aligned}
$$

where

$$
\begin{aligned}
X &= (\pi q)^{2m+1-4c-a}[a] \begin{bmatrix} h; 1-m \\ c \end{bmatrix} \\
&\quad + (\pi q)^{-2m+a-4c+2} \frac{q^{-3a}K^{-2} - (\pi q)^{-a} J}{q^2 - \pi} \begin{bmatrix} h; 1-m \\ c-1 \end{bmatrix} - (\pi q)^{1-4c}[2m] J \begin{bmatrix} h; 1-m \\ c-1 \end{bmatrix}.
\end{aligned}
$$

A direct computation allows us to simplify the expression for X as follows:

$$\begin{aligned}
X &= \Big((\pi q)^{2m+1-4c-a}[a]\frac{q^{4c-4m}K^{-2}-J}{q^{4c}-1} \\
&\qquad + (\pi q)^{-2m+a-4c+2}\frac{q^{-3a}K^{-2}-(\pi q)^{-a}J}{q^2-\pi} - (\pi q)^{1-4c}[2m]\Big)\begin{bmatrix} h; 1-m \\ c-1 \end{bmatrix} \\
&= (\pi q)^{2m-2c-a+1}[2c+a]\frac{q^{-4m}K^{-2}-J}{q^2-1}\begin{bmatrix} h; 1-m \\ c-1 \end{bmatrix} \\
&= (\pi q)^{2m-2c-a+1}[2c+a]\begin{bmatrix} h; -m \\ c \end{bmatrix}.
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
g^{\pi}_{a,c}(h) &= \pi^a(\pi q)^{\binom{2c+1}{2}-a(2m+1-2c-a)}(\pi q)^{-2c-a}[2m+1-2c-a]\begin{bmatrix} h; -m \\ c \end{bmatrix} \\
&\qquad + (\pi q)^{\binom{2c+1}{2}-a(2m+1-2c-a)}(\pi q)^{2m-2c-a+1}[2c+a]\begin{bmatrix} h; -m \\ c \end{bmatrix} \\
&= (\pi q)^{\binom{2c+1}{2}-a(2m+1-2c-a)}[2m+1]\begin{bmatrix} h; -m \\ c \end{bmatrix}.
\end{aligned}$$

Recalling the identity (7.19), we have thus proved the formula (7.17) for $B^{(2m+1)}_{\overline{0}}$, and hence completed the proof of Theorem 7.5. □

Reformulations of the expansion formulas for $B^{(n)}_{\overline{0}}$

We can apply the anti-involution ς in Lemma 7.1 to the formulas in Theorem 7.5 to obtain the following $Fh\check{E}$-expansion formulas (cf. [BeW18, Proposition 2.7]):

Proposition 7.6. *For $m \geq 1$, we have*

$$B_{\bar{0}}^{(2m)} = \sum_{c=0}^{m} \sum_{a=0}^{2m-2c} (-1)^c q^{3c+a(2m-2c-a)} F^{(a)} \begin{bmatrix} h; m-c \\ c \end{bmatrix} \check{E}^{(2m-2c-a)},$$

$$B_{\bar{0}}^{(2m-1)} = \sum_{c=0}^{m-1} \sum_{a=0}^{2m-1-2c} (-1)^c q^{c+a(2m-1-2c-a)} F^{(a)} \begin{bmatrix} h; m-c \\ c \end{bmatrix} \check{E}^{(2m-1-2c-a)}.$$

Proof. The involution ς in Lemma 7.1 fixes $F, \check{E}, J, K^{-1}$ and sends

$$B_{\bar{0}}^{(n)} \mapsto \pi^{\binom{n}{2}} B_{\bar{0}}^{(n)}, \qquad \begin{bmatrix} h; a \\ n \end{bmatrix} \mapsto (-1)^n q^{2n(n+1)} \begin{bmatrix} h; 1-a-n \\ n \end{bmatrix}, \qquad \text{for all } a \in \mathbb{Z}, n \in \mathbb{N}.$$

Applying ς to (7.14), we end up with $\pi^{\binom{2m}{2}}$ on the LHS and $\pi^{\binom{a}{2}+\binom{2m-2c-a}{2}}$ on the RHS. Dividing through by $\pi^{\binom{2m}{2}}$, we see that the powers of π inside the double sum work out to

$$\pi^{\binom{2m-2c-a}{2}+\binom{a}{2}-\binom{2m}{2}} \pi^{\binom{2c}{2}+a} = \pi^{a+c}\pi^{c+a} = 1.$$

Similarly for the odd power case (7.15), the powers of π in the double sum work out to $\pi^{c+a-a}\pi^c = 1$. Thus, both formulas are identical to the non-super case in [BeW18, Proposition 2.7]. $\square$

For $\lambda \in \mathbb{Z}$,

$$\begin{bmatrix} h; a \\ n \end{bmatrix} \mathbf{1}_{2\lambda} = q^{2n(a-1-\lambda)} \begin{bmatrix} a-1-\lambda+n \\ n \end{bmatrix}_{q^2} \mathbf{1}_{2\lambda} \in {}_{\mathcal{A}}\dot{\mathbf{U}}_{\mathrm{ev}}, \tag{7.20}$$

even though $\begin{bmatrix} h; a \\ n \end{bmatrix}$ does not lie in ${}_{\mathcal{A}}\mathbf{U}$ in general (cf. [BeW18]).

Thus, by the same argument as [BeW18, Proposition 2.8], we have the following reformulation of Theorem 7.5; the only difference here is the factor of π^a, which comes from

Lemma 7.2):

Proposition 7.7. *For $m \geq 1$ and $\lambda \in \mathbb{Z}$, we have*

$$B_{\bar{0}}^{(2m)}\mathbf{1}_{2\lambda} = \sum_{c=0}^{m}\sum_{a=0}^{2m-2c} \pi^a(\pi q)^{2(a+c)(m-a-\lambda)-2ac-\binom{2c+1}{2}} \begin{bmatrix} m-c-a-\lambda \\ c \end{bmatrix}_{q^2} E^{(a)}F^{(2m-2c-a)}\mathbf{1}_{2\lambda}, \tag{7.21}$$

$$B_{\bar{0}}^{(2m-1)}\mathbf{1}_{2\lambda} = \sum_{c=0}^{m-1}\sum_{a=0}^{2m-1-2c} \tag{7.22}$$

$$\pi^a(\pi q)^{2(a+c)(m-a-\lambda)-2ac-a-\binom{2c+1}{2}} \begin{bmatrix} m-c-a-\lambda-1 \\ c \end{bmatrix}_{q^2} E^{(a)}F^{(2m-1-2c-a)}\mathbf{1}_{2\lambda}.$$

In particular, we have $B_{\bar{0}}^{(n)}\mathbf{1}_{2\lambda} \in {}_{\mathcal{A}}\dot{\mathbf{U}}_{\mathrm{ev}}$, for all $n \in \mathbb{N}$.

The $\check{E}hF$-formula for $B_{\bar{1}}^{(n)}$

Recall that $[\![h;0]\!] = \left[\!\!\left[\begin{matrix} h;0 \\ 1 \end{matrix}\right]\!\!\right]$.

Example 7.8. *We have the following examples of $B_{\bar{1}}^{(n)}$, for $2 \leq n \leq 4$:*

$$B_{\bar{1}}^{(2)} = \frac{B^2 - \pi J}{[2]!} = b_{\pi}^{(2)} + \pi q[\![h;0]\!],$$

$$B_{\bar{1}}^{(3)} = \frac{B^3 - \pi JB}{[3]!} = b_{\pi}^{(3)} + \pi q^{-1}[\![h;0]\!]F + \pi q^{-1}\check{E}[\![h;0]\!],$$

$$B_{\bar{1}}^{(4)} = \frac{(B^2 - \pi J[3]^2)(B^2 - \pi J)}{[4]!} = b_{\pi}^{(4)} + \pi q\check{E}^{(2)}[\![h;-1]\!] + \pi q[\![h;-1]\!]F^{(2)} + \check{E}[\![h;-1]\!]F + q^6\left[\!\!\left[\begin{matrix} h;-1 \\ 2 \end{matrix}\right]\!\!\right].$$

Theorem 7.9. *For $m \geq 0$, we have*

$$B_{\bar{1}}^{(2m)} = \sum_{c=0}^{m} \sum_{a=0}^{2m-2c} (\pi q)^{\binom{2c}{2}-a(2m-2c-a)} \check{E}^{(a)} \left[\!\!\left[\begin{matrix} h; 1-m \\ c \end{matrix} \right]\!\!\right] F^{(2m-2c-a)}, \tag{7.23}$$

$$B_{\bar{1}}^{(2m+1)} = \sum_{c=0}^{m} \sum_{a=0}^{2m+1-2c} (\pi q)^{\binom{2c-1}{2}-1-a(2m+1-2c-a)} \check{E}^{(a)} \left[\!\!\left[\begin{matrix} h; 1-m \\ c \end{matrix} \right]\!\!\right] F^{(2m+1-2c-a)}. \tag{7.24}$$

Proof. As in [BeW18], we prove the formulae for $B_{\bar{1}}^{(n)}$ by induction on n. The base case for $n = 1$ is clear. The induction is carried out in 2 steps.

(1) First by assuming the formula for $B_{\bar{1}}^{(2m)}$ in (7.23), we shall establish the formula (7.24) for $B_{\bar{1}}^{(2m+1)}$, via the identity $[2m+1]B_{\bar{1}}^{(2m+1)} = B \cdot B_{\bar{1}}^{(2m)}$ in (7.3).

Recall the formula (7.23) for $B_{\bar{1}}^{(2m)}$. Using $B = \check{E} + F$ and applying (7.12) to $F\check{E}^{(a)}$ we have

$$\begin{aligned} B \cdot B_{\bar{1}}^{(2m)} &= \sum_{c=0}^{m} \sum_{a=0}^{2m-2c} (\pi q)^{\binom{2c}{2}-a(2m-2c-a)} B\check{E}^{(a)} \left[\!\!\left[\begin{matrix} h; 1-m \\ c \end{matrix} \right]\!\!\right] F^{(2m-2c-a)} \qquad (7.25) \\ &= \sum_{c=0}^{m} \sum_{a=0}^{2m-2c} (\pi q)^{\binom{2c}{2}-a(2m-2c-a)} \cdot \\ &\qquad \left(\check{E}\check{E}^{(a)} + (\pi q^{-2})^a \check{E}^{(a)} F + \check{E}^{(a-1)} \frac{q^{3-3a}K^{-2} - (\pi q)^{1-a} J}{q^2 - \pi} \right) \left[\!\!\left[\begin{matrix} h; 1-m \\ c \end{matrix} \right]\!\!\right] F^{(2m-2c-a)} \\ &= \sum_{c=0}^{m} \sum_{a=0}^{2m-2c} (\pi q)^{\binom{2c}{2}-a(2m-2c-a)} \cdot \\ &\left([a+1]\check{E}^{(a+1)} \left[\!\!\left[\begin{matrix} h; 1-m \\ c \end{matrix} \right]\!\!\right] F^{(2m-2c-a)} + (\pi q^{-2})^a [2m+1-2c-a] \check{E}^{(a)} \left[\!\!\left[\begin{matrix} h; -m \\ c \end{matrix} \right]\!\!\right] F^{(2m+1-2c-a)} \right. \\ &\qquad \left. + \check{E}^{(a-1)} \frac{q^{3-3a}K^{-2} - (\pi q)^{1-a} J}{q^2 - \pi} \left[\!\!\left[\begin{matrix} h; 1-m \\ c \end{matrix} \right]\!\!\right] F^{(2m-2c-a)} \right). \end{aligned}$$

We reorganize the formula (7.25) in the following form

$$[2m+1]B_{\bar{1}}^{(2m+1)} = B \cdot B_{\bar{1}}^{(2m)} = \sum_{c=0}^{m} \sum_{a=0}^{2m+1-2c} \check{E}^{(a)} \mathtt{f}_{a,c}^{\pi}(h) F^{(2m+1-2c-a)},$$

where

$$\begin{aligned}
\mathtt{f}_{a,c}^{\pi}(h) = (\pi q)^{\binom{2c}{2}-(a-1)(2m+1-2c-a)}[a] \left[\!\!\left[\begin{matrix} h; 1-m \\ c \end{matrix} \right]\!\!\right] \\
+ \Bigg(\pi^a (\pi q)^{\binom{2c}{2}-a(2m-2c-a)-2a}[2m+1-2c-a] \left[\!\!\left[\begin{matrix} h; -m \\ c \end{matrix} \right]\!\!\right] \\
+ (\pi q)^{\binom{2c-2}{2}-(a+1)(2m+1-2c-a)} \frac{q^{-3a}K^{-2} - (\pi q)^{-a} J}{q^2 - \pi} \left[\!\!\left[\begin{matrix} h; 1-m \\ c-1 \end{matrix} \right]\!\!\right] \Bigg).
\end{aligned}$$

A direct computation gives us

$$\begin{aligned}
\mathtt{f}_{a,c}^{\pi}(h) &= (\pi q)^{\binom{2c-1}{2}-1-a(2m+1-2c-a)} (\pi q)^{2m+1-a}[a] \left[\!\!\left[\begin{matrix} h; 1-m \\ c \end{matrix} \right]\!\!\right] + (\pi q)^{\binom{2c-1}{2}-1-a(2m+1-2c-a)} \cdot \\
&\quad \cdot \Bigg(\pi^a (\pi q)^{2c-a}[2m+1-2c-a] \frac{q^{-4m}K^{-2} - \pi q^2 J}{q^{4c}-1} \\
&\quad + (\pi q)^{2+a-2m} \frac{q^{-3a}K^{-2} - (\pi q)^{-a}}{q^2 - \pi} \Bigg) \left[\!\!\left[\begin{matrix} h; 1-m \\ c-1 \end{matrix} \right]\!\!\right] \\
&= (\pi q)^{\binom{2c-1}{2}-1-a(2m+1-2c-a)} (\pi q)^{2m+1-a}[a] \left[\!\!\left[\begin{matrix} h; 1-m \\ c \end{matrix} \right]\!\!\right] \\
&\quad + (\pi q)^{\binom{2c-1}{2}-1-a(2m+1-2c-a)} q^{-a}[2m+1-a] \left[\!\!\left[\begin{matrix} h; 1-m \\ c \end{matrix} \right]\!\!\right] \\
&= (\pi q)^{\binom{2c-1}{2}-1-a(2m+1-2c-a)} [2m+1] \left[\!\!\left[\begin{matrix} h; 1-m \\ c \end{matrix} \right]\!\!\right].
\end{aligned}$$

Hence we have obtained the formula (7.24) for $B_{\bar{1}}^{(2m+1)}$.

(2) Now by assuming the formula for $B_{\bar{1}}^{(2m+1)}$ in (7.24), we shall establish the following formula (with m in (7.23) replaced by $m+1$)

$$B_{\bar{1}}^{(2m+2)} = \sum_{c=0}^{m+1} \sum_{a=0}^{2m+2-2c} (\pi q)^{\binom{2c}{2}-a(2m+2-2c-a)} \check{E}^{(a)} \left[\!\!\left[\begin{matrix} h; -m \\ c \end{matrix} \right]\!\!\right] F^{(2m+2-2c-a)}. \tag{7.26}$$

Recall the formula (7.24) for $B_{\bar{1}}^{(2m+1)}$. Using $B = \check{E} + F$ and applying (7.12) to $F\check{E}^{(a)}$ we have

$$\begin{aligned} B \cdot B_{\bar{1}}^{(2m+1)} &= \sum_{c=0}^{m} \sum_{a=0}^{2m+1-2c} (\pi q)^{\binom{2c-1}{2}-1-a(2m+1-2c-a)} B\check{E}^{(a)} \left[\!\!\left[\begin{matrix} h; 1-m \\ c \end{matrix} \right]\!\!\right] F^{(2m+1-2c-a)} \\ &= \sum_{c=0}^{m} \sum_{a=0}^{2m+1-2c} (\pi q)^{\binom{2c-1}{2}-1-a(2m+1-2c-a)}. \\ &\qquad \cdot \left(\check{E}\check{E}^{(a)} + (\pi q^{-2})^a \check{E}^{(a)} F + \check{E}^{(a-1)} \frac{q^{3-3a}K^{-2} - (\pi q)^{1-a} J}{q^2 - \pi} \right) \left[\!\!\left[\begin{matrix} h; 1-m \\ c \end{matrix} \right]\!\!\right] F^{(2m+1-2c-a)}. \end{aligned}$$

We rewrite this as

$$\begin{aligned} B \cdot B_{\bar{1}}^{(2m+1)} = \sum_{c=0}^{m} \sum_{a=0}^{2m+1-2c} (\pi q)^{\binom{2c-1}{2}-1-a(2m+1-2c-a)} \cdot \Bigg(& [a+1]\check{E}^{(a+1)} \left[\!\!\left[\begin{matrix} h; 1-m \\ c \end{matrix} \right]\!\!\right] F^{(2m+1-2c-a)} \\ & + (\pi q^{-2})^a [2m+2-2c-a] \check{E}^{(a)} \left[\!\!\left[\begin{matrix} h; -m \\ c \end{matrix} \right]\!\!\right] F^{(2m+2-2c-a)} \\ & + \check{E}^{(a-1)} \frac{q^{3-3a}K^{-2} - (\pi q)^{1-a} J}{q^2 - \pi} \left[\!\!\left[\begin{matrix} h; 1-m \\ c \end{matrix} \right]\!\!\right] F^{(2m+1-2c-a)} \Bigg). \end{aligned} \tag{7.27}$$

We shall use (7.3), (7.27) and (7.23) to obtain a formula of the form

$$[2m+2]B_{\overline{1}}^{(2m+1)} = B\cdot B_{\overline{1}}^{(2m+1)} - \pi[2m+1]JB_{\overline{1}}^{(2m)} = \sum_{c=0}^{m+1}\sum_{a=0}^{2m+2-2c} \check{E}^{(a)}\mathtt{g}_{a,c}^{\pi}(h)F^{(2m+2-2c-a)}, \quad (7.28)$$

for some suitable $\mathtt{g}_{a,c}^{\pi}(h)$. Then we have

$$\begin{aligned}
\mathtt{g}_{a,c}^{\pi}(h) &= (\pi q)^{\binom{2c-1}{2}-1-(a-1)(2m+2-2c-a)}[a] \left[\!\!\left[\begin{matrix} h; 1-m \\ c \end{matrix}\right]\!\!\right] \\
&\quad + \pi^a (\pi q)^{\binom{2c-1}{2}-1-a(2m+1-2c-a)-2a}[2m+2-2c-a] \left[\!\!\left[\begin{matrix} h; -m \\ c \end{matrix}\right]\!\!\right] \\
&\quad + (\pi q)^{\binom{2c-3}{2}-1-(a+1)(2m+2-2c-a)} \frac{q^{-3a}K^{-2} - (\pi q)^{-a}J}{q^2-\pi} \left[\!\!\left[\begin{matrix} h; 1-m \\ c-1 \end{matrix}\right]\!\!\right] \\
&\quad - (\pi q)^{\binom{2c-2}{2}-a(2m+2-2c-a)}[2m+1] \left[\!\!\left[\begin{matrix} h; 1-m \\ c-1 \end{matrix}\right]\!\!\right] \\
&= \pi^a (\pi q)^{\binom{2c}{2}-a(2m+2-2c-a)}(\pi q)^{-2c-a}[2m+2-2c-a] \left[\!\!\left[\begin{matrix} h; -m \\ c \end{matrix}\right]\!\!\right] + (\pi q)^{\binom{2c}{2}-a(2m+2-2c-a)}\mathtt{X}^{\pi},
\end{aligned}$$

where

$$\begin{aligned}
\mathtt{X}^{\pi} &= (\pi q)^{2m+2-4c-a}[a] \left[\!\!\left[\begin{matrix} h; 1-m \\ c \end{matrix}\right]\!\!\right] \\
&\quad + (\pi q)^{-2m+3-4c+a} \frac{q^{-3a}K^{-2} - (\pi q)^{-a}J}{q^2-\pi} \left[\!\!\left[\begin{matrix} h; 1-m \\ c-1 \end{matrix}\right]\!\!\right] - (\pi q)^{3-4c}[2m+1] \left[\!\!\left[\begin{matrix} h; 1-m \\ c-1 \end{matrix}\right]\!\!\right].
\end{aligned}$$

A direct computation allows us to simplify the expression for $\mathtt{X}^{\pi}$ as follows:

$$\begin{aligned}
\mathtt{X}^{\pi} &= \Bigg((\pi q)^{2m+2-4c-a}[a]\frac{q^{4c-4m}K^{-2}-\pi q^2 J}{q^{4c}-1} \\
&\qquad + (\pi q)^{-2m+3-4c+a}\frac{q^{-3a}K^{-2}-(\pi q)^{-a}J}{q^2-\pi} - (\pi q)^{3-4c}[2m+1]\Bigg) \left[\!\!\left[\begin{matrix} h; 1-m \\ c-1 \end{matrix} \right]\!\!\right] \\
&= (\pi q)^{2m+2-2c-a}[2c+a]\frac{q^{-4m}K^{-2}-\pi q^2 J}{q^{4c}-1} \left[\!\!\left[\begin{matrix} h; 1-m \\ c-1 \end{matrix} \right]\!\!\right] \\
&= (\pi q)^{2m+2-2c-a}[2c+a] \left[\!\!\left[\begin{matrix} h; -m \\ c \end{matrix} \right]\!\!\right].
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
\mathtt{g}^{\pi}_{a,c}(h) &= (\pi q)^{\binom{2c}{2}-a(2m+2-2c-a)} q^{-2c-a}[2m+2-2c-a] \left[\!\!\left[\begin{matrix} h; -m \\ c \end{matrix} \right]\!\!\right] \\
&\qquad + (\pi q)^{\binom{2c}{2}-a(2m+2-2c-a)} (\pi q)^{2m+2-2c-a}[2c+a] \left[\!\!\left[\begin{matrix} h; -m \\ c \end{matrix} \right]\!\!\right] \\
&= (\pi q)^{\binom{2c}{2}-a(2m+2-2c-a)}[2m+2] \left[\!\!\left[\begin{matrix} h; -m \\ c \end{matrix} \right]\!\!\right],
\end{aligned}$$

where the last equality uses the general identity $q^{-l}[k-1]+(\pi q)^{k-1}[l]=[k]$. Recalling the identity (7.28), we have proved the formula (7.26) for $B_{\bar{1}}^{(2m+2)}$, and hence completed the proof of Theorem 7.9. □

Reformulation of the expansion formulas for $B_{\bar{1}}^{(n)}$

Just as with the even parity case, we can apply the anti-involution ς in Lemma 7.1 to the formulas in Theorem 7.5 to obtain the following $Fh\check{E}$-expansion formulas:

Proposition 7.10. *For $m \geq 0$, we have*

$$B_{\bar{1}}^{(2m)} = \sum_{c=0}^{m} \sum_{a=0}^{2m-2c} (-1)^c q^{-c+a(2m-2c-a)} F^{(a)} \left[\!\!\left[\begin{matrix} h; 1+m-c \\ c \end{matrix} \right]\!\!\right] \check{E}^{(2m-2c-a)},$$

$$B_{\bar{1}}^{(2m+1)} = \sum_{c=0}^{m} \sum_{a=0}^{2m+1-2c} (-1)^c q^{c+a(2m+1-2c-a)} F^{(a)} \left[\!\!\left[\begin{matrix} h; 1+m-c \\ c \end{matrix} \right]\!\!\right] \check{E}^{(2m+1-2c-a)}.$$

Proof. This time ς fixes $F, \check{E}, J, K^{-1}$ and sends

$$B_{\bar{1}}^{(n)} \mapsto B_{\bar{1}}^{(n)}, \quad \left[\!\!\left[\begin{matrix} h; a \\ n \end{matrix} \right]\!\!\right] \mapsto (-1)^n q^{2n(n-1)} \left[\!\!\left[\begin{matrix} h; 2-a-n \\ n \end{matrix} \right]\!\!\right], \quad \text{for all } a \in \mathbb{Z},\ n \in \mathbb{N}.$$

The rest of the calculation is very similar to the even case above, and we obtain as before formulas that are formally the same as the non-super case, though there are factors of π and J contained in $\left[\!\!\left[\begin{matrix} h; a+1 \\ n \end{matrix} \right]\!\!\right]$. □

For $\lambda \in \mathbb{Z}$, recall from 7.11 that we have

$$\left[\!\!\left[\begin{matrix} h; a \\ n \end{matrix} \right]\!\!\right] \mathbf{1}_{2\lambda-1} = q^{2n(a-\lambda)} \left[\begin{matrix} a-\lambda-1+n \\ n \end{matrix} \right]_{q^2} \mathbf{1}_{2\lambda-1} \in {}_{\mathcal{A}}\dot{\mathbf{U}}_{\mathrm{odd}}. \tag{7.29}$$

Hence, by a similar argument to the even parity case, we have the following reformulation of Theorem 7.9 (the extra factor of π^a comes from Lemma 7.2):

Proposition 7.11. *For $m \geq 0$ and $\lambda \in \mathbb{Z}$, we have*

$$B_{\bar{1}}^{(2m)}\mathbf{1}_{2\lambda-1} = \sum_{c=0}^{m}\sum_{a=0}^{2m-2c} \pi^a(\pi q)^{2(a+c)(m-a-\lambda)-2ac+a-\binom{2c}{2}} \begin{bmatrix} m-c-a-\lambda \\ c \end{bmatrix}_{q^2} E^{(a)}F^{(2m-2c-a)}\mathbf{1}_{2\lambda-1},$$

$$B_{\bar{1}}^{(2m+1)}\mathbf{1}_{2\lambda-1} = \sum_{c=0}^{m}\sum_{a=0}^{2m+1-2c} \pi^a(\pi q)^{2(a+c)(m-a-\lambda)-2ac+2a-\binom{2c}{2}} \begin{bmatrix} m-c-a-\lambda+1 \\ c \end{bmatrix}_{q^2} E^{(a)}F^{(2m+1-2c-a)}\mathbf{1}_{2\lambda-1}.$$

In particular, we have $B_{\bar{1}}^{(n)}\mathbf{1}_{2\lambda-1} \in {}_{\mathcal{A}}\dot{\mathbf{U}}_{\mathrm{odd}}$, for all $n \in \mathbb{N}$.

7.4 Definition for arbitrary $\mathbf{U}^\imath$

Let $\mathbf{U}^\imath = \mathbf{U}^\imath_\varsigma$ be an $\imath$quantum group with parameter ς, for a given root datum $(Y, X, \langle\cdot,\cdot\rangle, \ldots)$.

Definition 7.12. For $i \in I$ with $\tau i \neq i$, imitating Lusztig's divided powers, we define the *divided power* of B_i to be

$$B_i^{(m)} := B_i^m/[m]_i^!, \qquad \text{for all } m \geq 0, \qquad \text{when } i \neq \tau i. \tag{7.30}$$

For $i \in I$ with $\tau i = i$, the $\imath^\pi$*-divided powers* are defined to be

$$B_{i,\bar{1}}^{(m)} = \frac{1}{[m]_i^!}\begin{cases} B_i\prod_{j=1}^k(B_i^2 - \varsigma_i q_i[2j-1]_i^2\widetilde{J}_i) & \text{if } m = 2k+1, \\ \prod_{j=1}^k(B_i^2 - \varsigma_i q_i[2j-1]_i^2\widetilde{J}_i) & \text{if } m = 2k; \end{cases} \tag{7.31}$$

$$B_{i,\bar{0}}^{(m)} = \frac{1}{[m]_i^!}\begin{cases} B_i\prod_{j=1}^k(B_i^2 - \varsigma_i\pi_i q_i[2j]_i^2\widetilde{J}_i) & \text{if } m = 2k+1, \\ \prod_{j=1}^k(B_i^2 - \varsigma_i\pi_i q_i[2j-2]_i^2\widetilde{J}_i) & \text{if } m = 2k. \end{cases} \tag{7.32}$$

When we specialize $\pi_i = 1$ and $\widetilde{J}_i = 1$, we obtain the $\imath$-divided powers in [CLW18] from the formulas above. In the case when the parameter $\varsigma_i = q_i^{-1}$ for $\tau i = i$, this is the rank one case

described above, and all formulas and results there hold for $B^{(n)}_{i,\overline{p}}$. Using 6.3, we note that we can obtain $\mathbf{U}^i$ with general parameters ς_i from this special case by the rescaling isomorphism therein.

Chapter 8

A Serre presentation for $\mathbf{U}^\imath$

We are now ready to state and prove one of the main results in this dissertation, a Serre presentation for $\mathbf{U}^\imath$, which parallels the main result in [CLW18], Theorem 3.1. In addition to a handful of standard relations, this presentation also features two novel relations: for $\tau i \neq i$, a Serre-type relation between B_i and $B_{\tau i}$ with a 'correction term' ((8.6) below) and for $\tau i = i \neq j$, a Serre-type relation, the $\imath^\pi$-*Serre relations* between B_i and B_j, which can be neatly written in terms of the $\imath^\pi$-divided powers ((8.7) below).

8.1 Statement of the result

Denote

$$(a;x)_0 = 1, \qquad (a;x)_n = (1-a)(1-ax)\cdots(1-ax^{n-1}), \qquad \text{for all } n \geq 1.$$

Theorem 8.1. *Fix $\overline{p}_i \in \mathbb{Z}_2$ for each $i \in I$. The $\mathbb{K}(q)^\pi$-algebra $\mathbf{U}^\imath$ has a presentation with generators B_i, $\widetilde{J}_i$ $(i \in I)$, K_μ $(\mu \in Y^\imath)$ and the following relations* (8.1)–(8.7)*: for $\mu, \mu' \in Y^\imath$*

and $i \neq j \in I$,

$$\widetilde{J}_i \text{ is central,} \tag{8.1}$$

$$K_\mu K_{-\mu} = 1, \quad K_\mu K_{\mu'} = K_{\mu+\mu'}, \tag{8.2}$$

$$K_\mu B_i - q_i^{-\langle \mu, \alpha_i \rangle} B_i K_\mu = 0, \tag{8.3}$$

$$[B_i, B_j] = 0, \qquad \text{if } a_{ij} = 0 \text{ and } \tau i \neq j, \tag{8.4}$$

$$\sum_{n=0}^{1-a_{ij}} (-1)^n \pi_i^{np(j)+\binom{n}{2}} B_i^{(n)} B_j B_i^{(1-a_{ij}-n)} = 0, \qquad \text{if } j \neq \tau i \neq i, \tag{8.5}$$

$$\sum_{n=0}^{1-a_{i,\tau i}} (-1)^n \pi_i^{n+\binom{n}{2}} B_i^{(n)} B_{\tau i} B_i^{(1-a_{i,\tau i}-n)} = \frac{1}{\pi_i q_i - q_i^{-1}} \tag{8.6}$$

$$\cdot \left(q_i^{a_{i,\tau i}} (\pi_i q_i^{-2}; \pi_i q_i^{-2})_{-a_{i,\tau i}} B_i^{(-a_{i,\tau i})} \widetilde{J}_i \widetilde{K}_i \widetilde{K}_{\tau i}^{-1} - (\pi_i q_i^2; \pi_i q_i^2)_{-a_{i,\tau i}} B_i^{(-a_{i,\tau i})} \widetilde{J}_{\tau i} \widetilde{K}_{\tau i} \widetilde{K}_i^{-1} \right), \text{ if } \tau i \neq i,$$

$$\sum_{n=0}^{1-a_{ij}} (-1)^n \pi_i^{n+\binom{n}{2}} B_{i,\overline{a_{ij}}+\overline{p}_i}^{(n)} B_j B_{i,\overline{p}_i}^{(1-a_{ij}-n)} = 0, \qquad \text{if } \tau i = i \neq j. \tag{8.7}$$

Proof. Granting first that (8.6) and (8.7) both hold in $\mathbf{U}^\imath$, the same argument used in [CLW18] is also applicable here in this setting. The main ingredients are the results in §6.2.1 above; we have a generalization of [Ko14, Theorem 7.1] when X (corresponding to the black nodes) is empty.

Thus, the remaining work lies in showing that both (8.6) and (8.7) holds in $\mathbf{U}^\imath$. We will do so in the subsequent sections, in Proposition 8.3 of §8.2 and Theorem 8.6 of §8.4 respectively. □

Before moving on, we will display here the Serre presentation for split $\mathbf{U}^\imath$, which takes on a particularly simple form (recall that a quasi-split $\imath$quantum group $\mathbf{U}^\imath$ is split if $\tau = \mathrm{id}$):

Theorem 8.2. *Fix* $\overline{p}_i \in \mathbb{Z}_2$, *for each* $i \in I$. *Then the split* $\imath$*quantum group* $\mathbf{U}^\imath$ *has a Serre*

presentation with generators B_i $(i \in I)$ and relations

$$\sum_{n=0}^{1-a_{ij}} (-1)^n \pi_i^{n+\binom{n}{2}} B^{(n)}_{i,\overline{a_{ij}}+\overline{p}_i} B_j B^{(1-a_{ij}-n)}_{i,\overline{p}_i} = 0.$$

Moreover, $\mathbf{U}^\imath$ admits a $\mathbb{K}(q)$-algebra anti-involution σ which sends $B_i \mapsto B_i$ for all i.

Proof. This follows from Theorem 8.1 by noting that $Y^\imath = \emptyset$ and $\tau i = i$ for all $i \in I$. □

8.2 Serre relation when $\tau i \neq i$

In this section we will show that (8.6) holds, following [BaK15, §3.5]. Recall the projections P_λ and $\pi_{0,0}$ defined above, which are also in [BaK15].

Proposition 8.3. *If $\tau i \neq i$, the following relation holds in $\mathbf{U}^\imath_\varsigma$:*

$$\sum_{n=0}^{1-a_{i,\tau i}} (-1)^n \pi_i^{n+\binom{n}{2}} B_i^{(n)} B_{\tau i} B_i^{(1-a_{i,\tau i}-n)} = \frac{1}{\pi_i q_i - q_i^{-1}}$$
$$\cdot \left(q_i^{a_{i,\tau i}} (\pi_i q_i^{-2}; \pi_i q_i^{-2})_{-a_{i,\tau i}} B_i^{(-a_{i,\tau i})} \widetilde{J}_i \widetilde{K}_i \widetilde{K}_{\tau i}^{-1} - (\pi_i q_i^2; \pi_i q_i^2)_{-a_{i,\tau i}} B_i^{(-a_{i,\tau i})} \widetilde{J}_{\tau i} \widetilde{K}_{\tau i} \widetilde{K}_i^{-1} \right).$$

Proof. Recall now that i and $j = \tau(i) \neq i$ must have the same parity, and if both i and j are even roots there is nothing to prove. Thus, we may assume that i and j are odd roots, and so by the bar-consistency condition $m = 1 - a_{ij}$ is odd. Also set $\lambda_{ij} = m \cdot i + j$ and with the notation above set $Q_{-\lambda_{ij}} = \mathrm{id} \otimes (P_{-\lambda_{ij}} \circ \pi_{0,0})$ as the vector space endomorphism of $\mathbf{U} \otimes \mathbf{U}$.

By a construction parallel to [Ko14, (7.8)], for $Y = F_{ij}(B_i, B_j)$ we have the relation

$$C_{ij}(\mathbf{c}) = -(\mathrm{id} \otimes \varepsilon) \circ Q_{-\lambda_{ij}}(\Delta(Y) - Y \otimes K_{-\lambda_{ij}}). \tag{8.8}$$

Just as in *loc. cit.*, we can compute $\Delta(Y)$ from the formulas

$$\Delta(B_i) = B_i \otimes K_i^{-1} + 1 \otimes F_i + \varsigma_i Z_i \otimes E_j K_i^{-1}$$
$$\Delta(B_j) = B_j \otimes K_j^{-1} + 1 \otimes F_j + \varsigma_j Z_j \otimes E_i K_j^{-1}$$

where $Z_k = J_{\tau(k)} K_{\tau(k)} K_k^{-1}$ for $k = i, j$, and so we have that

$$Q_{-\lambda_{ij}}(\Delta(Y) - Y \otimes K_{\lambda_{ij}}) = (a_j B_i^{m-1} \varsigma_j Z_j + a_i B_i^{m-1} \varsigma_i Z_i) \otimes K_{-\lambda_{ij}} \tag{8.9}$$

where a_i and a_j can be determined explicitly using the commutation relations

$$Z_j B_i = q_i^{-(m+1)} B_i Z_j, \qquad Z_i B_i = q_i^{m+1} B_i Z_i.$$

For instance,

$$\begin{aligned} a_j B_i^{m-1} \varsigma_j Z_j \otimes K_{-\lambda_{ij}} &= Q_{-\lambda_{ij}} \Bigg(\sum_{k=0}^{m} (-1)^k \pi_i^{\binom{k}{2}+k} \begin{bmatrix} m \\ k \end{bmatrix}_i \\ &\quad \cdot \sum_{l=0}^{m-k-1} (B_i^l \otimes K_i^{-l})(1 \otimes F_i)(B_i^{m-1-k-l} \otimes K_i^{-(m-1-k-l)})(\varsigma_j Z_j \otimes E_i K_j^{-1})(B_i^k \otimes K_i^{-k}) \Bigg) \\ &= \sum_{k=0}^{m} \frac{(-1)^k \pi_i^{\binom{k}{2}+k} \pi_i}{(\pi_i q_i - q_i^{-1})} \begin{bmatrix} m \\ k \end{bmatrix}_i \sum_{l=0}^{m-k-1} \pi_i^{m-1-l} \cdot \pi_i^k q_i^{-(m+1)k-2(m-k-l-1)} B_i^{m-1} \varsigma_j Z_j \otimes K_{-\lambda_{ij}}, \end{aligned}$$

where the extra factors of π_i come from multiplying out $1 \otimes F_i$ and $B_i^{m-1-k-l} \otimes K_i^{m-1-k-l}$ and $B_i^k \otimes K_i^k$, and $\varsigma_j Z_j \otimes E_i K_j^{-1}$ and $B_i^k \otimes K_i^k$ respectively since multiplication in $\mathbf{U} \otimes \mathbf{U}$ is defined according to the rule $(a \otimes b)(c \otimes d) = \pi^{p(b)p(c)} ac \otimes bd$. A further factor of π_i comes from the

following:

$$Q_{-\lambda_{ij}}(K_i^{-(m-k-1)}F_iE_iK_j^{-1}K_i^{-k}) = Q_{-\lambda_{ij}}(K_i^{-(m-k-1)}\left(\pi_iE_iF_i - \pi_i\frac{J_iK_i - K_i^{-1}}{\pi_iq_i - q_i^{-1}}\right)K_j^{-1}K_i^{-k})$$
$$= \frac{\pi_i}{\pi_iq_i - q_i^{-1}}K_i^{-m}K_j^{-1}.$$

Note that $m-1 = -a_{ij}$ is always even (by bar-consistency), and so $\pi_i^{m-1} = 1$. Thus,

$$a_j = \sum_{k=0}^{m}\frac{(-1)^k\pi_i^{\binom{k}{2}}}{(\pi_iq_i - q_i^{-1})}\begin{bmatrix} m \\ k \end{bmatrix}_i \sum_{l=0}^{m-k-1} q_i^{-(m-1)k-2(m-1)}\pi_i^l q_i^{2l}$$
$$= \sum_{k=0}^{m}\frac{(-1)^k\pi_i^{\binom{k}{2}}}{(\pi_iq_i - q_i^{-1})}\begin{bmatrix} m \\ k \end{bmatrix}_i q_i^{-(m-1)k-2(m-1)}\frac{(\pi_iq_i^2)^{m-k} - 1}{\pi_iq_i^2 - 1}.$$

This time, we may use [CHW13, (1.12)], which after applying the bar involution yields

$$\sum_{k=0}^{m}\pi_i^{\binom{k}{2}}q_i^{-k(m-1)}\begin{bmatrix} m \\ k \end{bmatrix}_i z^k = \prod_{j=0}^{m-1}(1 + (\pi_iq_i^{-2})^jz); \tag{8.10}$$

in particular,

$$\sum_{k=0}^{m}\pi_i^{\binom{k}{2}}q_i^{-k(m-1)}\begin{bmatrix} m \\ k \end{bmatrix}_i (-1)^k = 0;$$

and

$$\sum_{k=0}^{m}\pi_i^{\binom{k}{2}}q_i^{-k(m-1)}\begin{bmatrix} m \\ k \end{bmatrix}_i (-\pi_iq_i^{-2})^k = \prod_{j=0}^{m-1}(1 - (\pi_iq_i^{-2})^{j+1}) = (\pi_iq_i^{-2};\pi_iq_i^{-2})_m,$$

(Recall that $(x;x)_m := \prod_{j=1}^{m}(1-x^j)$) and so (remembering that $\pi_i^m = \pi_i$ since m is odd) we

have

$$a_j = \frac{\pi_i q_i^{-2(m-1)}(\pi_i q_i^2)^m}{q_i(\pi_i q_i - q_i^{-1})^2}(\pi_i q_i^{-2};\pi_i q_i^{-2})_m = \frac{q_i}{(\pi_i q_i - q_i^{-1})^2}(\pi_i q_i^{-2};\pi_i q_i^{-2})_m. \tag{8.11}$$

Similarly, for a_i we have additional factors of $\pi_i^{\binom{k}{2}+k}$ from the super-Serre relations and π_i^l from the tensor product multiplication:

$$\begin{aligned}
a_i &= \frac{\pi_i}{\pi_i q_i - q_i^{-1}} \sum_{k=0}^{m} (-1)^k \pi_i^{\binom{k}{2}+k} \begin{bmatrix} m \\ k \end{bmatrix}_i \sum_{l=0}^{k-1} q_i^{(k-1)(m+1)} \pi_i^l q_i^{-2l} \\
&= \frac{\pi_i}{\pi_i q_i - q_i^{-1}} \sum_{k=0}^{m} (-1)^k \pi_i^{\binom{k}{2}+k} \begin{bmatrix} m \\ k \end{bmatrix}_i q_i^{(k-1)(m+1)} \frac{1-(\pi_i q_i^{-2})^k}{1-\pi_i q_i^{-2}} \\
&= \frac{\pi_i(\pi_i q_i)}{(\pi_i q_i - q_i^{-1})^2} q_i^{-(m+1)} \sum_{k=0}^{m} (-1)^k \pi_i^{\binom{k}{2}} \pi_i^k q_i^{k(m+1)} \begin{bmatrix} m \\ k \end{bmatrix}_i (1-(\pi_i q_i^{-2})^k) \\
&= \frac{q_i}{(\pi_i q_i - q_i^{-1})^2} q_i^{-(m+1)} \sum_{k=0}^{m} (-1)^k \pi_i^{\binom{k}{2}} q_i^{k(m-1)} \begin{bmatrix} m \\ k \end{bmatrix}_i ((\pi_i q_i^2)^k - 1) \\
&= \frac{q_i^{-m}}{(\pi_i q_i - q_i^{-1})^2}\Big((\pi_i q_i^2;\pi_i q_i^2)_m - 0\Big) = \frac{q_i^{-m}}{(\pi_i q_i - q_i^{-1})^2}(\pi_i q_i^2;\pi_i q_i^2)_m,
\end{aligned}$$

this time using [CHW13, (1.12)] directly (without the need for applying the bar involution). Putting this together with 8.9 and applying $-\mathrm{id} \otimes \varepsilon$, we obtain

$$C_{ij}(\mathbf{c}) = \frac{-1}{(\pi_i q_i - q_i^{-1})^2}(q_i^{-m}(\pi_i q_i^2;\pi_i q_i^2)_m B_i^{m-1}\varsigma_i Z_i + q_i(\pi_i q_i^{-2};\pi_i q_i^{-2})_m B_i^{m-1}\varsigma_j Z_j). \tag{8.12}$$

Dividing through by $[m]_i^!$ and simplifying yields the version with divided powers presented in Theorem 8.1. □

8.3 A (q,π)-binomial identity

We state and prove here a (q,π)-binomial identity that will be crucial to the proof of Proposition 8.7 in the next section: for

$$w \in \mathbb{Z}, \quad u, \ell \in \mathbb{Z}_{\geq 0}, \text{ with } u, \ell \text{ not both } 0, \tag{8.13}$$

we define

$$\begin{aligned} &T(w,u,\ell)_{q,\pi} \qquad (8.14)\\ &= \sum_{\substack{c,e,r\geq 0\\ c+e+r=u}} \sum_{\substack{t=0\\ 2|(t+w-r)}}^{\ell} \\ &\quad \pi^{lt+r+e+\binom{t}{2}} (\pi q)^{-t(\ell+u-1)+(\ell+u)(c-e)} \begin{bmatrix} \ell \\ t \end{bmatrix} \begin{bmatrix} w+t-\ell \\ r \end{bmatrix} \begin{bmatrix} u-1+\frac{w+t-r}{2} \\ c \end{bmatrix}_{q^2} \begin{bmatrix} \frac{w+t-r}{2}-\ell \\ e \end{bmatrix}_{q^2} \\ &- \sum_{\substack{c,e,r\geq 0\\ c+e+r=u}} \sum_{\substack{t=0\\ 2\nmid(t+w-r)}}^{\ell} \\ &\quad \pi^{lt+c+\binom{t}{2}} (\pi q)^{-t(\ell+u-1)+(\ell+u-1)(c-e)} \begin{bmatrix} \ell \\ t \end{bmatrix} \begin{bmatrix} w+t-\ell \\ r \end{bmatrix} \begin{bmatrix} u+\frac{w+t-r-1}{2} \\ c \end{bmatrix}_{q^2} \begin{bmatrix} \frac{w+t-r-1}{2}-\ell \\ e \end{bmatrix}_{q^2}. \end{aligned}$$

When we specialize at $\pi = 1$, we have $T(w,u,\ell)_{q,1} = T(w,u,\ell)$ as defined in [CLW18, (3.18)].

Proposition 8.4 ([CLW18], Theorem 3.6)**.** *The identity $T(w,u,\ell) = 0$ holds, for all integers w, u, ℓ as in* (8.13)*.*

As pointed out in [CLW18], a direct proof of this proposition proved challenging. Instead, the authors approached this by first introducing a more general q-binomial identity in several more parameters. This general identity specialized to the one above and satisfied certain recurrence relations, thus completing the proof with an inductive argument (details in §5 of

[CLW18]). Fortunately for us, we can sidestep the complicated process above for the analogous result here in our setting by making a deft substitution and leveraging the earlier result:

Proposition 8.5. *The identity* $T(w,u,\ell)_{q,\pi}=0$ *holds, for all integers* w,u,ℓ *as in* (8.13).

Proof. By a substitution of $q\mapsto\sqrt{\pi}q$ in $T(w,u,l)$, we obtain

$$T(w,u,l)|_{q\mapsto\sqrt{\pi}q}=\sqrt{\pi}^{u^2-lu-uw}T(w,u,\ell)_{q,\pi},$$

and so the result follows from Proposition 8.4. □

8.4 Proof of the $\imath^\pi$-Serre relations

This section is devoted to a proof of the following theorem:

Theorem 8.6. *The* $\imath^\pi$*-Serre relations* (8.7),

$$\sum_{n=0}^{1-a_{ij}}(-1)^n\pi_i^{n+\binom{n}{2}}B^{(n)}_{i,\overline{a_{ij}}+\overline{p}_i}B_jB^{(1-a_{ij}-n)}_{i,\overline{p}_i}=0,\qquad \text{if } \tau i=i\neq j.$$

hold in the $\imath$*quantum covering group* $\mathbf{U}^\imath$.

The general strategy will rely on applying a few reductions to reduce (8.7) to the (q,π)-binomial above, which vanishes as we saw in Proposition 8.5. Using the isomorphism ϕ in Proposition 6.6, the $\imath$Serre relations for $\mathbf{U}^\imath_{q_i^{-1}}$ is transformed into the $\imath$Serre relations (8.7) for $\mathbf{U}^\imath_\varsigma$ with general parameters. Hence just as in [CLW18], we will work with the $\imath$quantum groups with distinguished parameters, $\mathbf{U}^\imath=\mathbf{U}_{q_i^{-1}}$, as a first reduction of the $\imath$Serre relations. A subsequent 'reduction by equivalence' as in §4.1 of [CLW18] can be applied, further reducing (8.7) to

$$\sum_{n=0}^{1-a_{ij}}(-1)^nB^{(n)}_{i,\overline{a_{ij}}+\overline{p}}F_jB^{(1-a_{ij}-n)}_{i,\overline{p}}=0 \tag{8.15}$$

for each $\overline{p}\in\mathbb{Z}_2$, where $i\in I$ such that $\tau i=i$, $j\neq i$.

Now fix $i = 1$ and $j = 2$. Note that when $p(1)$ is even, there are no additional formulas to prove since $\pi_1 = 1$. Thus, we may assume that $p(1)$ is odd, and so due to the bar-consistency condition ([CHW13, 1.1(d)]) we must have $a_{12} \in -2\mathbb{N}$. Hence, it is sufficient to prove that:

Proposition 8.7. *Suppose that $a_{12} = -2m \in -2\mathbb{N}$. Then,*

$$\sum_{n=0}^{2m+1} (-1)^n \pi_1^{np(2)+\binom{n}{2}} B_{1,\bar{0}}^{(n)} F_2 B_{1,\bar{0}}^{(2m+1-n)} = 0, \text{ and} \tag{8.16}$$

$$\sum_{n=0}^{2m+1} (-1)^n \pi_1^{np(2)+\binom{n}{2}} B_{1,\bar{1}}^{(n)} F_2 B_{1,\bar{1}}^{(2m+1-n)} = 0. \tag{8.17}$$

Proof. Just as in [CLW18, §4], we will show that (8.16) holds by showing that

$$\sum_{n=0}^{2m+1} (-1)^n \pi_1^{np(2)+\binom{n}{2}} B_{1,\bar{0}}^{(n)} F_2 B_{1,\bar{0}}^{(2m+1-n)} \mathbf{1}_{2\lambda}^{\star} = 0. \tag{8.18}$$

for all λ, using Remark 2.4.

Using Proposition 7.7 to expand $B_{1,\bar{0}}^{(n)}$ and $B_{1,\bar{0}}^{(2m+1-n)}$ and (2.11) to collect the factors of E_1, we have (cf. [CLW18, (4.15)])

$$\sum_{n=0}^{2m+1}(-1)^n\pi_1^{np(2)+\binom{n}{2}}B_{1,\bar{0}}^{(n)}F_2B_{1,\bar{0}}^{(2m+1-n)}\mathbf{1}_{2\lambda}^{\star}= \tag{8.19}$$

$$\sum_{n=0,2|n}^{2m}\sum_{c=0}^{m-\frac{n}{2}}\sum_{e=0}^{\frac{n}{2}}\sum_{a=0}^{2m+1-n-2c}\sum_{d=0}^{n-2e}\sum_{r=0}^{\min\{a,n-2e-d\}}$$

$$\cdot\,\pi_1^{a+d+ap(2)+ad+\binom{r}{2}+\binom{n}{2}}(\pi_1q_1)^{(a+c+d+e)(2m+1-n-2\lambda-2a-2c-2d-2e)+d}$$

$$\cdot\begin{bmatrix}a+d-r\\ d\end{bmatrix}_{q_1}\begin{bmatrix}2m+2-n-2\lambda-2e-d-3a-4c\\ r\end{bmatrix}_{q_1}\begin{bmatrix}m-\frac{n}{2}-\lambda-c-a\\ c\end{bmatrix}_{q_1^2}$$

$$\cdot\begin{bmatrix}m+1-\frac{n}{2}-\lambda-e-d-2a-2c\\ e\end{bmatrix}_{q_1^2}E_1^{(a+d-r)}F_1^{(n-2e-d-r)}F_2F_1^{(2m+1-n-2c-a)}\mathbf{1}_{2\lambda}^{\star}$$

$$-\sum_{n=1,2\nmid n}^{2m+1}\sum_{c=0}^{m+\frac{1-n}{2}}\sum_{e=0}^{\frac{n-1}{2}}\sum_{a=0}^{2m+1-n-2c}\sum_{d=0}^{n-2e}\sum_{r=0}^{\min\{a,n-2e-d\}}$$

$$\cdot\,\pi_1^{a+d+(a+1)p(2)+a+ad+\binom{r}{2}+\binom{n}{2}}(\pi_1q_1)^{(a+c+d+e)(2m+1-n-2\lambda-2a-2c-2d-2e)-a-2c}$$

$$\cdot\begin{bmatrix}a+d-r\\ d\end{bmatrix}_{q_1}\begin{bmatrix}2m+2-n-2\lambda-2e-d-3a-4c\\ r\end{bmatrix}_{q_1}\begin{bmatrix}m+\frac{1-n}{2}-\lambda-c-a\\ c\end{bmatrix}_{q_1^2}$$

$$\cdot\begin{bmatrix}m+\frac{1-n}{2}-\lambda-e-d-2a-2c\\ e\end{bmatrix}_{q_1^2}E_1^{(a+d-r)}F_1^{(n-2e-d-r)}F_2F_1^{(2m+1-n-2c-a)}\mathbf{1}_{2\lambda}^{\star}.$$

By the same series of substitutions as detailed in [CLW18], we may collect the q- and q^2-binomial factors and the π_1 factors into a sum $S(y,u,\ell,\lambda)_\pi$ (the rest can be factored out) to obtain

$$\sum_{n=0}^{2m+1}(-1)^n\pi_1^{np(2)+\binom{n}{2}}B_{1,\bar{0}}^{(n)}F_2B_{1,\bar{0}}^{(2m+1-n)}\mathbf{1}_{2\lambda}^{\star}=\sum_{\substack{\ell,y,u\geq 0;u+\ell>0\\ \ell+y+2u\leq 2m+1}} \tag{8.20}$$

$$\pi_1^{(l+y)p(2)+l+\binom{y}{2}}(\pi_1q_1)^{(\ell+u)(2m+1-2\lambda-2\ell-3u-y)}S(y,u,\ell,\lambda)_\pi E_1^{(\ell)}F_1^{(y)}F_2F_1^{(2m+1-\ell-y-2u)}\mathbf{1}_{2\lambda}^{\star},$$

where $S(y,u,\ell,\lambda)_\pi$ is a sum over n (with a difference when $2|n$ and $2\nmid n$ as above) and over

$c, e, r \geq 0, c + e + r = u$ cf. [CLW18, 4.16].

Then, using the new variables $t := -u - y - e + c + n$ and $w := 2m + 2 - 2\lambda - 2l - 4u - y$ in §4.4 of [CLW18], we have that $S(y, u, \ell, \lambda)_\pi = T(w, u, \ell)_{q,\pi}$. Thus, the right-hand side vanishes by Theorem 8.5 and so (8.16) holds.

Just as in [CLW18], a similar argument shows that (8.17) holds. $\square$

Chapter 9

Bar involution and quasi K-matrix

The Serre presentation for $\mathbf{U}^\imath$ enables the definition of a bar involution $\psi_\imath$ on $\mathbf{U}^\imath$, which is not simply the restriction of the bar involution ψ on $\mathbf{U}$. For instance, B_i has the image $F_i + c_i E_{\tau i} K_i^{-1}$ under the embedding $\mathbf{U}^\imath \hookrightarrow \mathbf{U}$, and $\psi_\imath$ fixes B_i, but $\psi(F_i + c_i E_i K_i^{-1}) = F_i + \overline{c_i} E_{\tau i} J_i K_i$.

In this section, we will construct a quasi K-matrix Υ that 'intertwines' these two bar involutions (the quasi K-matrix goes by the name 'intertwiner' in [BW18a, Chapter 2]). The quasi K-matrix has the property that its action is integral, in the sense that it preserves the $\mathcal{A}^\pi$-forms (i.e. integral forms) of certain integrable highest weight $\mathbf{U}$-modules and their tensor products. This property will be used in the development of a theory of canonical bases for $\mathbf{U}^\imath$ in the next section, cf. [BW18b, BW18c].

9.1 Bar involution on $\mathbf{U}^\imath$

Recall the three conditions (6.8)–(6.10) on ς_i in Definition 6.1. We may now conclude the existence of the bar involution for the quasi-split $\imath$quantum group $\mathbf{U}^\imath := \mathbf{U}^\imath_\varsigma$, granting that these conditions on ς_i are satisfied:

Proposition 9.1. *Assume that the parameters ς_i, for $i \in I$, satisfy the conditions* (6.8)–(6.10),

which we recall here:

(6.8) $\overline{\varsigma_i q_i} = \varsigma_i q_i$, *if* $\tau i = i$ *and* $a_{ij} \neq 0$ *for some* $j \in I \setminus \{i\}$;

(6.9) $\overline{\varsigma_i} = \varsigma_i = \varsigma_{\tau i}$, *if* $\tau i \neq i$ *and* $a_{i,\tau i} = 0$;

(6.10) $\varsigma_{\tau i} = \pi_i q_i^{-a_{i,\tau i}} \overline{\varsigma_i}$, *if* $\tau i \neq i$ *and* $a_{i,\tau i} \neq 0$.

Then there exists a $\mathbb{K}$*-algebra automorphism* $^{-} : \mathbf{U}^\imath \to \mathbf{U}^\imath$ *(called a bar involution) such that*

$$\overline{q} = \pi q^{-1}, \quad \overline{K_\mu} = J_\mu K_\mu^{-1}, \quad \overline{B_i} = B_i, \qquad \textit{for all } \mu \in Y^\imath, i \in I.$$

Proof. Under the assumptions, the $\imath$-divided powers $B_i^{(n)}$ in (7.30) and $B_{i,\overline{p}}^{(n)}$, for $\overline{p} \in \mathbb{Z}_2$, in (7.31)-(7.32) are clearly bar invariant. It follows by inspection that all the explicit defining relations for $\mathbf{U}^\imath$ in (8.1)-(8.7) are bar invariant. The extra factor of π_i in (c) comes from applying $^{-}$ to the right hand side of (8.6). □

9.2 Quasi K-matrix

The goal of this section is the construction of a quasi K-matrix for quasi-split $\mathbf{U}^\imath$ that 'intertwines' the bar involutions $\psi^\imath$ for $\mathbf{U}^\imath$ and ψ for $\mathbf{U}$, which do not agree:

Theorem 9.2. *There exists a unique family of elements* $\Upsilon_\mu \in \mathbf{U}_\mu^+$ *such that* $\Upsilon_0 = 1$ *and* $\Upsilon = \sum_\mu \Upsilon_\mu$ *where* $p(\mu)$ *is even, satisfying the following identity in* $\hat{\mathbf{U}}$*:*

$$\psi_\imath(u)\Upsilon = \Upsilon\psi(u), \quad \textit{for all } u \in \mathbf{U}^\imath. \tag{9.1}$$

Among other things, carrying out the rank one computation was instructive for identifying the condition that Υ has no odd part, so we will cover that before going into a proof of the general case.

Quasi K-matrix for rank 1 (a single odd root)

We know that when we specialize Υ to the non-super rank one case by setting $\pi = 1$, we will obtain the formulas given in [DK18, Lemma 2.6] or [BW18a, (4.1)] (note that a change from E to F in the second reference is required, since the convention there for the embedding formula for B is different). Thus, the general form of Υ is given by

$$\Upsilon = \sum_{N \geq 0} a_N E^{(N)}.$$

where $a_N \in \mathbb{K}(q)^\pi$. From the first identity in [CHW13, Lemma 2.2.3] with $M = 1$, we have

$$E^{(N)}F - \pi^N F E^{(N)} = \pi^{N-1} \begin{bmatrix} K; 1-N \\ 1 \end{bmatrix} E^{(N-1)} = \pi E^{(N-1)} \frac{(\pi q)^{1-N} JK - q^{N-1}K^{-1}}{\pi q - q^{-1}}.$$

We need to separate the computation for the condition $B\Upsilon = \Upsilon\overline{B}$ when N is even from when N is odd. When $N = 2k$ is even, we have

$$a_{2k}(E^{(2k)}F - \pi^N F E^{(2k)}) = a_{2k-2}(cq^2 K^{-1} E E^{(2k-2)} - \overline{c} E E^{(2k-2)} JK)$$

and so using [CHW13, Lemma 2.2.3] and comparing coefficients of $E^{(2k-1)}JK$ and $E^{(2k-1)}K^{-1}$ respectively yield the (over-determined) system of solutions

$$a_{2k} = -c\pi q^2(\pi q - q^{-1})q^{1-2k}[2k-1]a_{2k-2}$$

and

$$a_{2k} = -c\pi q^2(\pi q - q^{-1})q^{2k-1}q^{2(1-2k)}[2k-1]a_{2k-2}.$$

Hence for k even,

$$a_{2k} = (-c\pi q^2)^k(\pi q - q^{-1})^k q^{-k^2}[2k-1]^{!!}$$

where $[2k-1]^{!!} = [2k-1]\cdot[2k-3]\cdot\ldots\cdot[1]$ (normalization: $a_0 = 1$).

For $N = 2k+1$ odd, we also obtain an over-determined system of two solutions:

$$\begin{aligned} a_{2k+1} &= (-c\pi q^2)(\pi q - q^{-1})q^{-2k}[2k]a_{2k-1} \\ &= (-c\pi q^2)^{k+1}(\pi q - q^{-1})^{k+1}q^{-2\binom{k+1}{2}}[2k]^{!!}a_{-1} \end{aligned}$$

where $[2k]^{!!} = [2k]\cdot[2k-2]\cdot\ldots\cdot[2]$. Since $a_{-1} = 0$, we see that Υ has no odd part.

So we have (cf. [DK18, Lemma 2.6] when $\pi = 1$)

$$\Upsilon = \sum_{k\geq 0}(-c\pi q^2)^k(\pi q - q^{-1})^k q^{-k^2}[2k-1]^{!!}E^{(2k)}.$$

An equivalent, systematic approach to the definition of Υ involves the twisted derivations defined by Lusztig in [Lu94, 1.2.13], and can be found in [BaK19, Proposition 6.3] or [DK18, Lemma 3.8]. For quantum covering algebras the twisted derivations r_i and ${}_ir$ is defined in [CHW13, §1.5]. Following this, we may define Υ to be the (unique) solution to the system of equations:

$${}_1r(\Upsilon) = -c\pi q^2(\pi q - q^{-1})E\Upsilon, \text{ and} \tag{9.2}$$

$$r_1(\Upsilon) = -c\pi q^2(\pi q - q^{-1})\Upsilon E, \tag{9.3}$$

The existence of such a solution, and hence the existence of Υ, can be verified just as in

[DK18, Lemma 3.8], using the fact that ${}_1r(E^{(2k)}) = q^{2k-1}E^{(2k-1)}$ for the first equation:

$$\begin{aligned}
{}_1r(\Upsilon_{2k}) &= {}_1r(a_{2k}E^{(2k)}) \\
&= a_{2k}q^{2k-1}E^{(2k-1)} \\
&= -c\pi q^2(\pi q - q^{-1})a_{2k-2}[2k-1]\frac{EE^{(2k-2)}}{[2k-1]} \\
&= (-c\pi q^2)(\pi q - q^{-1})E\Upsilon_{2k-2},
\end{aligned}$$

and using $r_1(E^{(2k)}) = q^{2k-1}E^{(2k-1)}(= {}_1r(E^{(2k)}))$ for the second. Note that this definition also implies that Υ has no odd part, because

$$\begin{aligned}
{}_1r(\Upsilon_{2k+1}) &= {}_1r(a_{2k+1}\sigma E^{(2k+1)}) \\
&= a_{2k+1}q^{2k}\sigma E^{(2k)} \\
&= -c\pi q^2(\pi q - q^{-1})a_{2k-1}\pi[2k]E\sigma\frac{E^{(2k-1)}}{[2k]} \\
&= \pi(-c\pi q^2)(\pi q - q^{-1})E\Upsilon_{2k-1},
\end{aligned}$$

and $\Upsilon_{-1} = 0$. The equivalence of the definition with the previous construction is a direct application of [CHW13, Proposition 2.2.2], which is a π-analogues of [Lu94, Proposition 3.1.6].

Remark 9.3. When attempting to directly apply [CHW13, Proposition 2.2.2], we ran into the following issue: since $\overline{B_i} = F_i + \overline{c_i}E_iJ_iK_i$ in $\mathbf{U}$, we would like to have $\Upsilon = \sum_\mu \Upsilon_\mu \in \hat{\mathbf{U}}^+$ satisfying

$$(F_i + c_iE_iK_i^{-1})\Upsilon = \Upsilon(F_i + \overline{c_i}E_iJ_iK_i),$$

but on the other hand we have

$$F_i\Upsilon_\mu - \Upsilon_\mu F_i = \Upsilon_{\mu-2i}\overline{c_i}E_iJ_iK_i - c_iE_iK_i^{-1}\Upsilon_{\mu-2i}$$

for which Proposition 2.2.2 is inadmissible when $p(\mu) = \overline{1}$, a factor of π_i is missing.

Borrowing inspiration from [BKK00], we attempted a workaround by introducing a parity operator σ to our algebra such that

$$\sigma E_i = \pi^{p(i)} E_i \sigma, \quad \sigma F_i = \pi^{p(i)} F_i \sigma, \quad \sigma K_\mu = K_\mu \sigma \text{ and } \sigma J_\mu = J_\mu \sigma$$

and separating odd and even parts $\Upsilon = \Upsilon_{\bar{0}} + \sigma \Upsilon_{\bar{1}}$, but when carrying out the computation above we find that the terms with σ vanish anyway, and so conclude that Υ must have no odd terms.

Quasi K-matrix for quasi-split $\mathbf{U}^\imath$

We will now prove Theorem 9.2 for general quasi-split $\mathbf{U}^\imath$. First, using [CHW13, Proposition 2.2.2] the condition that $\Upsilon = \sum_\mu \sigma^{p(\mu)} \Upsilon_\mu \in \hat{\mathbf{U}}^+$ satisfies the identity

$$(F_i + c_i E_i K_i^{-1}) \Upsilon = \Upsilon (F_i + \overline{c_i} E_i J_i K_i)$$

is the equivalent to the conditions that Υ_μ satisfy both of the following system of equations

$$r_i(\Upsilon_\mu) = -(\pi_i q_i - q_i^{-1})(c_i \pi_i q_i^2) \Upsilon_{\mu - 2i} E_i, \text{ and} \tag{9.4}$$

$${}_i r(\Upsilon_\mu) = -(\pi_i q_i - q_i^{-1})(c_i \pi_i q_i^2) E_i \Upsilon_{\mu - 2i}, \tag{9.5}$$

here we have used the the bar-consistency condition i.e. $p(i) \equiv d_i \pmod 2$, which gives us the identification $\pi_i^{p(i)} = \pi^{p(i)^2} = \pi^{p(i)} \pi_i$

With this, we can use the methods in [BW18a, §2.4] (cf. also [BaK19, §6.2]) to construct Υ. Recall the non-degenerate symmetric bilinear form $(\cdot, \cdot)$ on the algebra $'\mathbf{f}$ defined in [CHW13, Proposition 1.4.1].

Just as in [BW18a, (2.11)-(2.14)], the system (9.4)–(9.5) is equivalent to

$$(\Upsilon_\mu, E_i z) = -c_i q_i^3 (1 - \pi_i q_i^{-2})^{-1} (\Upsilon_{\mu-2i}, {}_i r(z)) \tag{9.6}$$

$$(\Upsilon_\mu, z E_i) = -c_i q_i^3 (1 - \pi_i q_i^{-2})^{-1} (\Upsilon_{\mu-2i}, r_i(z)), \tag{9.7}$$

which we can see from the brief computation

$$\begin{aligned}
(9.4) &\iff (r_i(\Upsilon_\mu), z) = -(\pi_i q_i - q_i^{-1})(c_i \pi_i q_i^2)(\Upsilon_{\mu-2i} E_i, z) \\
&\overset{(1.4.1)}{\iff} (\Upsilon_\mu, z E_i) = -(\pi_i q_i - q_i^{-1})(c_i \pi_i q_i^2)(E_i, E_i)^2 (\Upsilon_{\mu-2i}, r_i(z)) \\
&\iff (9.6);
\end{aligned}$$

for (9.4) $\iff$ (9.6), and a similar one for (9.5) $\iff$ (9.7).

Thus we may inductively define Υ_L^* and Υ_R^* in ${}'\mathbf{f}^*$, the non-restricted dual of ${}'\mathbf{f}$, such that $\Upsilon_L^*(1) = \Upsilon_R^*(1) = 1$ and

$$\Upsilon_L^*(E_i z) = -c_i q_i^3 (1 - \pi_i q_i^{-2})^{-1} \Upsilon_L^*({}_i r(z)) \tag{9.8}$$

$$\Upsilon_R^*(z E_i) = -c_i q_i^3 (1 - \pi_i q_i^{-2})^{-1} \Upsilon_R^*(r_i(z)). \tag{9.9}$$

Note that for all $i, j \in I$, we have from ${}_i r(1) = 0$ and ${}_i r(E_j) = \delta_{ij}$ that

$$\Upsilon_L^*(E_i) = 0 \quad \text{and} \quad \Upsilon_L^*(E_i E_j) = -c_i q_i^3 (1 - \pi q_i^{-2})^{-1} \delta_{ij},$$

and similarly for Υ_R^*.

Lemma 9.4. *Let ${}'\mathbf{f}_\mu$ denote the μ weight space in the weight space-decomposition of ${}'\mathbf{f}$. For $x \in {}'\mathbf{f}_\mu$, if either $p(\mu)$ or $\mathrm{ht}(\mu)$ is odd, then $\Upsilon_L^*(x) = \Upsilon_R^*(x) = 0$*

Proof. We show this for odd $p(\mu)$ by induction on $\mathrm{ht}(\mu)$ (the statement for odd $\mathrm{ht}(\mu)$ is similar). The base cases $\mathrm{ht}(\mu) = 1, 3$ are given above. For homogeneous such $x \in {}'\mathbf{f}_\mu$, $x = E_i z$

for some $z \in {}'\mathbf{f}_\nu$ so ${}_ir(z) \in {}'\mathbf{f}_{\nu-i}$ where $p(\nu - i)$ is odd ($p(\nu)$ and $p(i)$ have opposite parity since $p(\mu) = p(\nu) + p(i)$ is odd), and so by induction hypothesis, $\Upsilon_L^*({}_ir(z)) = 0$, and hence by (9.8), $\Upsilon_L^*(x) = 0$ as well (similarly for Υ_R^*). $\square$

Lemma 9.5. *We have* $\Upsilon_L^* = \Upsilon_R^*$.

Proof. We will show that $\Upsilon_L^*(x) = \Upsilon_R^*(x)$ for all homogeneous $x \in {}'\mathbf{f}_\mu$ by induction on $\mathrm{ht}(\mu)$, using Lemma 2.1 above. The base cases $\mathrm{ht}(|x|) = 0$ or 1 are trivial from the definition. Suppose that the identity holds for all homogeneous elements with height no greater than k for $k \geq 1$, and let $x = E_i x' E_j$ with $\mathrm{ht}(|x|) = k + 1 \geq 2$ for some $i, j \in I$. Let $\xi_k = -c_k q_k^3 (1 - \pi_k q_k^{-2})^{-1}$. Then,

$$\begin{aligned}\Upsilon_L^*(E_i x' E_j) &= \xi_i \Upsilon_L^*({}_ir(x' E_j)) \\ &= \xi_i \left(\Upsilon_L^*({}_ir(x') E_j) + \pi^{p(x')p(i)} q^{|x'| \cdot i} \Upsilon_L^*(x' {}_ir(E_j)) \right)\end{aligned}$$

and

$$\begin{aligned}\Upsilon_R^*(E_i x' E_j) &= \xi_j \Upsilon_R^*(r_j(E_i x')) \\ &= \xi_j \left(\Upsilon_R^*(E_i r_j(x')) + \pi^{p(x')p(j)} q^{|x'| \cdot j} \Upsilon_R^*(x' r_j(E_i)) \right).\end{aligned}$$

The second terms of both of the final expressions above vanish unless $i = j$, in which case they are both equal (by application of the induction hypothesis to x' of height $k - 1$), so it remains to show that

$$\xi_i \Upsilon_L^*({}_ir(x') E_j) = \xi_j \Upsilon_R^*(E_i r_j(x')).$$

This can be done by applying the induction hypothesis to ${}_ir(x') E_j$ and $E_i r_j(x')$ to obtain

$$\xi_i \Upsilon_L^*({}_ir(x') E_j) = \xi_i \Upsilon_R^*({}_ir(x') E_j) \overset{(9.9)}{=} \xi_i \xi_j \Upsilon_R^*(r_j \circ {}_ir(x'))$$

and

$$\xi_j \Upsilon_R^*(E_i r_j(x')) = \xi_j \Upsilon_L^*(E_i r_j(x')) \overset{(9.8)}{=} \xi_i \xi_j \Upsilon_L^*({}_i r \circ r_j(x')),$$

and so from the fact that $r_j \circ {}_i r = {}_i r \circ r_j$ by Lemma 2.2, and the induction hypothesis since $r_j \circ {}_i r(x') = {}_i r \circ r_j(x') \in {}'\mathbf{f}_{|x'|-i-j}$ the desired result follows. □

Thus, we can denote $\Upsilon_L^* = \Upsilon_R^*$ by Υ^*.

Let $I = \langle S_{ij} \rangle$, the ideal generated by the Serre relators $S_{ij} := F_{ij}(E_i, E_j)$ (where F_{ij} is defined above in (2.2)), so that the half quantum group $\mathbf{U}^+$ is isomorphic to ${}'\mathbf{f}/I$. We will now show that Υ^* vanishes on I, and so descends to an element in $(\mathbf{U}^+)^*$, the unrestricted dual of $\mathbf{U}^+$.

Lemma 9.6. $\Upsilon^*(I) = 0$ *and hence* Υ^* *belongs in* $(\mathbf{U}^+)^*$.

Proof. For finite type corresponding to $B(0,n)$, $|S_{ij}|$ has height 3 when $(i,j) \neq (n, n-1)$, and $p(S_{n,n-1})$ is odd, so by 9.4, we have that $\Upsilon^*(S_{ij}) = 0$. By the same induction argument in [BW18a, Lemma 2.17], this holds for the ideal $I = \langle S_{ij} \rangle$ they generate, and so $\Upsilon^*(I) = 0$.

For quasi-split $\mathbf{U}^\imath$ in general, we need to show that $\Upsilon^*(S_{ij}) = 0$ for general Serre relators. With Lemma 9.4 this is already addressed for the case $\mathrm{ht}(S_{ij})$ odd, and so it remains to show this for $\mathrm{ht}(S_{ij})$ even. This can be done by showing that terms of the form

$$\Upsilon^*(E_i^a E_j E_i^b) \text{ for } j \neq i \text{ and } a+b+1 \text{ even} \tag{9.10}$$

vanish by induction using (9.8) or (9.9). For instance if $a > 1$, we use (9.8) to show that (using $\xi_k = -c_k q_k^3 (1 - \pi_k q_k^{-2})^{-1}$ as above)

$$\begin{aligned}
\Upsilon^*(E_i^a E_j E_i^b) &= \xi_i \Upsilon^*({}_i r(E_i^{a-1} E_j E_i^b)) \\
&= \Upsilon^*({}_i r(E_i^{a-1} E_j) E_i^b + \pi_i^{p(ai+j)} q^{(ai+j)\cdot i} E_i^{a-1} E_{ji} r(E_i^b)) \\
&= \Upsilon^*({}_i r(E_i^{a-1}) E_j E_i^b + \pi_i^{p(ai+j)} q^{(ai+j)\cdot i} E_i^{a-1} E_{ji} r(E_i^b))
\end{aligned}$$

and each of the two terms are of the form (9.10), and so we can apply the induction hypothesis (the base case $\Upsilon^*(E_iE_j) = 0$ for $i \neq j$ is computed above, and for if we are not in the base case and $a = 1$ we must therefore have $b > 1$ and so we can use 9.9 on the other side. $\square$

With these lemmas, we can now construct Υ in the same way as [BW18a, Theorem 2.10]:

Proof of Theorem 9.2. Let $B = \{b\}$ be a basis of $\mathbf{U}^-$ such that $B_\mu = B \cap \mathbf{U}^+_{-\mu}$ is a basis for $\mathbf{U}^+_{-\mu}$, and let $B^* = \{b^*\}$ be the dual basis of B with respect to $(\cdot, \cdot)$ and let

$$\Upsilon := \sum_{b \in B} \Upsilon^*(b^*)b = \sum_{\mu} \Upsilon_\mu \in \hat{\mathbf{U}}^+ \tag{9.11}$$

(recall that there are no terms here with $p(\mu) = 1$). As functions on $\mathbf{U}^+$, we have $(\Upsilon, \cdot) = \Upsilon^*$, and $\Upsilon_0 = 1$. Also Υ satisfies the identities in (9.4) and (9.5) by construction, since Υ^* satisfies the equivalent identities in (9.8) and (9.9)).

From this construction we also see that $r_i(\Upsilon_\mu)$ is determined by Υ_ν with weight $\nu \prec \mu$. Together with Lemma 2.1, this implies the uniqueness of Υ. $\square$

As in [BW18a, Corollary 2.13], it follows that Υ is invertible, and in fact $\overline{\Upsilon} = \Upsilon^{-1}$:

Corollary 9.7. *We have* $\Upsilon \cdot \overline{\Upsilon} = 1$.

Proof. Multiplying Υ^{-1} on both sides of the identity (9.1) in Theorem 9.2, we have

$$\Upsilon^{-1}\imath(\overline{u}) = \overline{\imath(u)}\Upsilon^{-1}, \qquad \forall u \in \mathbf{U}^\imath.$$

Applying the bar involution $\bar{\ }$ to the above identity and replacing $\overline{u}$ by u, we have

$$\overline{\Upsilon}^{-1}\overline{\imath(u)} = \imath(\overline{u})\overline{\Upsilon}^{-1}, \qquad \forall u \in \mathbf{U}^\imath.$$

Hence $\overline{\Upsilon}^{-1}$ (in place of Υ) satisfies the identity (9.1) as well. Hence, by the uniqueness of Υ in Theorem 9.2, we must have $\overline{\Upsilon}^{-1} = \Upsilon$. $\square$

9.3 Integrality of the action of Υ

In this section we will prove an integrality property for Υ; in particular, we will show that Υ preserves the integral forms of various based modules and their tensor products, with Section 6 of [BW18c] as a general outline; see also Section 2 of [BW16].

Definitions

We will define based $\mathbf{U}$-modules $(M, B(M))$ in the same way as [BW16, §2] i.e. M is an integrable $\mathbf{U}$-module with a distinguished basis $B(M)$ satisfying conditions (a)–(d) of [Lu94, 27.1.2], with integrality replacing the finite-dimensionality.

Remark 9.8. We will note here that the use of the term 'basis' in the context of quantum covering groups will be understood to refer to a π-basis in the sense of [Cl14, §2.6]. A π-basis of an R^π-module M is also an R-basis of M. In Theorem 1 of *loc. cit.*, a π-basis of $\mathbf{f}$ is given; when $\pi = 1$ this specializes to the Lusztig-Kashiwara canonical basis, and when $\pi = -1$, this specializes to the signed basis of [Lu94, Chapter 14].

For the remainder of this section and the following chapter we will suppress the superscript π for $\mathcal{A}^\pi$ when referring to integral forms of algebras and modules, so e.g. ${}_{\mathcal{A}}\mathbf{U}$ refers to ${}_{\mathcal{A}^\pi}\mathbf{U}$ and ${}_{\mathcal{A}}M$ refers to ${}_{\mathcal{A}^\pi}M$. We will find useful in this section the following analogue of [BW16, Lemma 2.2]:

Lemma 9.9. *Let $(M, B(M))$ be a based $\mathbf{U}$-module and let $\lambda \in X$. Then,*

1. *for $b \in B(M)$, the $\mathbb{K}(q)^\pi$-linear map $\pi_b : \mathbf{U}^-\mathbf{1}_{\overline{|b|+\lambda}} \longrightarrow M \otimes M(\lambda),\ u \mapsto u(b \otimes \eta_\lambda))$, restricts to an $\mathcal{A}^\pi$-linear map $\pi_b : {}_{\mathcal{A}}\mathbf{U}^-\mathbf{1}_{\overline{|b|+\lambda}} \longrightarrow {}_{\mathcal{A}}M \otimes_{\mathcal{A}^\pi} {}_{\mathcal{A}}M(\lambda)$;*

2. *we have $\sum_{b \in B(M)} \pi_b({}_{\mathcal{A}}\mathbf{U}^-\mathbf{1}_{\overline{|b|+\lambda}}) = {}_{\mathcal{A}}M \otimes_{\mathcal{A}^\pi} {}_{\mathcal{A}}M(\lambda)$.*

Proof. The proof is the same as the one for [BW16, Lemma 2.2]: The comultiplication has the same form as [BW16, (2.1)], the filtration on ${}_{\mathcal{A}}\mathbf{f}$ is the same, and the appropriate analogue to [BW16, (2.2)] can be obtained from [Cl14, (3.2)-(3.3)]. □

Remark 9.10. Many of the results for based $\mathbf{U}$-modules for $\mathbf{U}$ of Kac-Moody type established in [BW16, §2] also apply for quantum covering algebras. In particular, with straightforward modifications, the same arguments therein give us versions of Lemma 2.3 (Corollary 9.15 below), Theorem 2.7, Theorem 2.9 and Proposition 2.11.

Definition 9.11. Just as in Definition 3.10 of *loc. cit.*, we define ${}_{\mathcal{A}}\dot{\mathbf{U}}^\imath$ to be the set of elements $u \in \dot{\mathbf{U}}^\imath$, such that $u \cdot m \in {}_{\mathcal{A}}\dot{\mathbf{U}}$ for all $m \in {}_{\mathcal{A}}\dot{\mathbf{U}}$. Then ${}_{\mathcal{A}}\dot{\mathbf{U}}^\imath$ is an $\mathcal{A}^\pi$-subalgebra of $\dot{\mathbf{U}}^\imath$ which contains all the idempotents $\mathbf{1}_\zeta$ ($\zeta \in X_\imath$), and ${}_{\mathcal{A}}\dot{\mathbf{U}}^\imath = \bigoplus_{\zeta \in X_\imath} {}_{\mathcal{A}}\dot{\mathbf{U}}^\imath \mathbf{1}_\zeta$.

Moreover, for $u \in \dot{\mathbf{U}}^\imath$, we have $u \in {}_{\mathcal{A}}\dot{\mathbf{U}}$ if and only if $u \cdot \mathbf{1}_\lambda \in {}_{\mathcal{A}}\dot{\mathbf{U}}$ for each $\lambda \in X$ (cf. [BW18b, Lemma 3.20])

Theorem 9.12. *For any $i \in I$ and $\mu \in X_\imath$, there exists an element $B^{(n)}_{i,\zeta} \in {}_{\mathcal{A}}\dot{\mathbf{U}}^\imath \mathbf{1}_\zeta$. In particular, these elements satisfy the following 2 properties:*

1. $\psi_\imath(B^{(n)}_{i,\zeta}) = B^{(n)}_{i,\zeta}$;
2. $B^{(n)}_{i,\zeta}\mathbf{1}_\lambda = F^{(n)}_i \mathbf{1}_\lambda + \sum_{a<n} F^{(a)}_i {}_{\mathcal{A}}\mathbf{U}^+ \mathbf{1}_\lambda$, *for* $\mathbf{1}_\lambda \in {}_{\mathcal{A}}\dot{\mathbf{U}}^\imath$ *with* $\overline{\lambda} = \zeta$.

Definition 9.13. Let ${}'_{\mathcal{A}}\dot{\mathbf{U}}^\imath$ be the $\mathcal{A}^\pi$-subalgebra of ${}_{\mathcal{A}}\dot{\mathbf{U}}^\imath$ generated by the $\imath$divided powers $B^{(n)}_{i,\zeta}$ ($i \in I$) for all $n \geq 1$ and $\zeta \in X_\imath$.

Recall for $\lambda \in X$, we denote by $M(\lambda)$ the Verma module of highest weight λ (see [CHW13, §2.6]). We denote the highest weight vector by η_λ. The following is an analogue of [BW18c, Lemma 6.3].

Lemma 9.14. *Let $(M, B(M))$ be a based $\mathbf{U}$-module. Let $\lambda \in X$. Then,*

1. *for $b \in B(M)$, the $\mathbb{K}(q)$-linear map $\pi_b : \dot{\mathbf{U}}^\imath \mathbf{1}_{\overline{|b|+\lambda}} \longrightarrow M \otimes M(\lambda)$, $u \mapsto u(b \otimes \eta_\lambda))$, restricts to an $\mathcal{A}^\pi$-linear map $\pi_b : {}'_{\mathcal{A}}\dot{\mathbf{U}}^\imath \mathbf{1}_{\overline{|b|+\lambda}} \longrightarrow {}_{\mathcal{A}}M \otimes_{\mathcal{A}^\pi} {}_{\mathcal{A}}M(\lambda)$;*
2. *we have $\sum_{b \in B(M)} \pi_b({}'_{\mathcal{A}}\dot{\mathbf{U}}^\imath \mathbf{1}_{\overline{|b|+\lambda}}) = {}_{\mathcal{A}}M \otimes_{\mathcal{A}^\pi} {}_{\mathcal{A}}M(\lambda)$.*

Proof. Recall ${}'_{\mathcal{A}}\dot{\mathbf{U}}^\imath \subset {}_{\mathcal{A}}\dot{\mathbf{U}}^\imath$. Part (1) follows from Definition 9.11. Part (2) is proven in the same way as *loc. cit.* By part (1) we have $\sum_{b\in B(M)} \pi_b({}'_{\mathcal{A}}\dot{\mathbf{U}}^\imath \mathbf{1}_{\overline{|b|+\lambda}}) \subset {}_{\mathcal{A}}M \otimes_{\mathcal{A}^\pi} {}_{\mathcal{A}}M(\lambda)$, and ${}_{\mathcal{A}}\mathbf{U}^-$ has the increasing filtration

$$\mathcal{A}^\pi = {}_{\mathcal{A}}\mathbf{U}^-_{\leq 0} \subseteq {}_{\mathcal{A}}\mathbf{U}^-_{\leq 1} \subseteq \cdots \subseteq {}_{\mathcal{A}}\mathbf{U}^-_{\leq N} \subseteq \cdots$$

where ${}_{\mathcal{A}}\mathbf{U}^-_{\leq N}$ is the $\mathcal{A}^\pi$-span of $\{F_{i_1}^{(a_1)} \dots F_{i_n}^{(a_n)} | a_1 + \ldots + a_n \leq N, i_1, \ldots, i_n \in I\}$, which induces an increasing filtration $\{{}_{\mathcal{A}}M(\lambda)_{\leq N}\}$ on ${}_{\mathcal{A}}M(\lambda)$.

We can prove by induction on N that ${}_{\mathcal{A}}M \otimes_{\mathcal{A}^\pi} {}_{\mathcal{A}}M(\lambda)_{\leq N} \subset \sum_{b\in B(M)} \pi_b({}'_{\mathcal{A}}\dot{\mathbf{U}}^\imath \mathbf{1}_{\overline{|b|+\lambda}})$: Let $b \otimes \big(F_{i_1}^{(a_1)} \dots F_{i_n}^{(a_n)} \eta_\lambda\big) \in {}_{\mathcal{A}}M \otimes_{\mathcal{A}^\pi} {}_{\mathcal{A}}M(\lambda)_{\leq N}$. Since the form of $\Delta\big(B_{i_1,\zeta}^{(a_1)}\big)$ has a 'leading term' $1 \otimes F_{i_1,\zeta}^{(a_1)}$ plus terms lower in filtration degree, we can use Theorem 9.12 to conclude that with appropriately chosen $\zeta \in X^\imath$ (see [BW16, Lemma 2.2]), we have

$$B_{i_1,\zeta}^{(a_1)}\Big(b \otimes \big(F_{i_2}^{(a_2)} \dots F_{i_n}^{(a_n)} \eta_\lambda\big)\Big) \in b \otimes \big(F_{i_1}^{(a_1)} \dots F_{i_n}^{(a_n)} \eta_\lambda\big) + {}_{\mathcal{A}}M \otimes_{\mathcal{A}^\pi} {}_{\mathcal{A}}M(\lambda)_{\leq N-1}.$$

The lemma follows. □

For $\lambda \in X^+$, we abuse the notation and denote also by η_λ the image of η_λ under the projection $p_\lambda : M(\lambda) \to L(\lambda)$. Note that p_λ restricts to $p_\lambda : {}_{\mathcal{A}}M(\lambda) \to {}_{\mathcal{A}}L(\lambda)$. The next corollary follows from Lemma 9.14.

Corollary 9.15. *Let $\lambda \in X^+$, and let $(M, B(M))$ be a based $\mathbf{U}$-module. Then,*

1. *for $b \in B(M)$, the $\mathbb{K}(q)$-linear map $\pi_b : \mathbf{U}^\imath \mathbf{1}_{\overline{|b|+\lambda}} \longrightarrow M \otimes L(\lambda)$, $u \mapsto u(b \otimes \eta_\lambda)$, restricts to an $\mathcal{A}^\pi$-linear map $\pi_b : {}'_{\mathcal{A}}\mathbf{U}^\imath \mathbf{1}_{\overline{|b|+\lambda}} \longrightarrow {}_{\mathcal{A}}M \otimes_{\mathcal{A}^\pi} {}_{\mathcal{A}}L(\lambda)$;*

2. *we have $\sum_{b\in B(M)} \pi_b({}'_{\mathcal{A}}\mathbf{U}^\imath \mathbf{1}_{\overline{|b|+\lambda}}) = {}_{\mathcal{A}}M \otimes_{\mathcal{A}^\pi} {}_{\mathcal{A}}L(\lambda)$.*

Integrality of actions of Υ

We now show that the quasi K-matrix $\Upsilon \in \widehat{\mathbf{U}}^+$ induces a well-defined $\mathbb{K}(q)^\pi$-linear map on $M \otimes L(\lambda)$:

$$\Upsilon : M \otimes L(\lambda) \longrightarrow M \otimes L(\lambda), \tag{9.12}$$

for any $\lambda \in X^+$ and any weight $\mathbf{U}$-module M whose weights are bounded above.

Recall [BW18b, §5.1] that a $\mathbf{U}^\imath$-module M equipped with an anti-linear involution $\psi_\imath$ is called *involutive* (or *$\imath$-involutive*) if

$$\psi_\imath(um) = \psi_\imath(u)\psi_\imath(m), \qquad \text{for all } u \in \mathbf{U}^\imath, m \in M.$$

Proposition 9.16. *Let (M, B) be a based $\mathbf{U}$-module whose weights are bounded above. We denote the bar involution on M by ψ. Then M is an $\imath$-involutive $\mathbf{U}^\imath$-module with involution*

$$\psi_\imath := \Upsilon \circ \psi. \tag{9.13}$$

Proof. Just as in [BW18c], since the weights of M are bounded above, the action of $\Upsilon : M \to M$ is well defined. The rest of the argument is analogous to the one found in the proof of [BW18b, Proposition 5.1] (also [BW18a, Proposition 3.10]): using Theorem 9.2, we have

$$\psi_\imath(um) = \Upsilon\psi(um) = \Upsilon\psi(u)\psi(m) = \psi_\imath(u)\Upsilon\psi(m) = \psi_\imath(u)\psi_\imath(m)$$

as required. $\square$

Let (M, B) be a based $\mathbf{U}$-module whose weights are bounded above. Assume $\Upsilon : M \to M$ preserves the $\mathcal{A}^\pi$-submodule ${}_{\mathcal{A}}M$.

Proposition 9.17. *The $\mathbb{K}(q)^\pi$-linear map $\psi_\imath := \Upsilon \circ \psi$ preserves the $\mathcal{A}^\pi$-submodule ${}_{\mathcal{A}}M \otimes_{\mathcal{A}^\pi} {}_{\mathcal{A}}L(\lambda)$, for any $\lambda \in X^+$.*

Proof. The proof is again very similar: the $\mathbf{U}$-module $M \otimes L(\lambda)$ is involutive with the involution $\psi := \Theta \circ (\overline{\ } \otimes \overline{\ })$, where Θ is the quasi $\mathcal{R}$-matrix defined in [CHW13, Theorem 3.1.1]. It follows by a direct analogue of the argument in [BW16, Proposition 2.4] that ψ preserves the $\mathcal{A}\mathcal{A}^\pi$-submodule ${}_{\mathcal{A}}M \otimes_{\mathcal{A}^\pi} {}_{\mathcal{A}}L(\lambda)$.

The argument will be reproduced here: the statement is that for $\lambda \in X^+$ and $(M, B(M))$ be a based $\mathbf{U}$-module, the $\mathbb{K}(q)$-linear map

$$\Theta : M \otimes L(\lambda) \to M \otimes L(\lambda)$$

preserves the $\mathcal{A}^\pi$-submodule ${}_{\mathcal{A}}M \otimes_{\mathcal{A}^\pi} {}_{\mathcal{A}}L(\lambda)$.

We will write $\overline{\ }$ for $\overline{\ } \otimes \overline{\ }$, which preserves the $\mathcal{A}^\pi$-lattice ${}_{\mathcal{A}}M \otimes_{\mathcal{A}^\pi} {}_{\mathcal{A}}L(\lambda)$. Thus, any $x \in {}_{\mathcal{A}}M \otimes_{\mathcal{A}^\pi} {}_{\mathcal{A}}L(\lambda)$ can be recognized as $x = \overline{x'}$ for some $x' \in {}_{\mathcal{A}}M \otimes_{\mathcal{A}^\pi} {}_{\mathcal{A}}L(\lambda)$. By Lemma 9.9, $x' = \sum_i \pi_{b_i}(u_i')$ (a finite sum), for some $b_i \in B(M)$ and $u_i' \in {}_{\mathcal{A}}\mathbf{U}^-\mathbf{1}_{|b_i|+\lambda}$. Since ${}_{\mathcal{A}}\mathbf{U}^-\mathbf{1}_{|b_i|+\lambda}$ is preserved by the bar involution on $\dot{\mathbf{U}}$, we have $u_i' = \overline{u_i}$ for some $u_i \in {}_{\mathcal{A}}\mathbf{U}^-\mathbf{1}_{|b_i|+\lambda}$. Hence,

$$x = \overline{x'} = \sum_i \overline{\overline{u_i}(b_i \otimes \eta_\lambda)}.$$

From [CHW13, Theorem 3.1.1], the quasi $\mathcal{R}$-matrix for has the property that

$$u\Theta(m \otimes m') = \Theta(\overline{\overline{u}(\overline{m} \otimes \overline{m'})}),$$

for $u \in \dot{\mathbf{U}}$, $m \in M$ and $m' \in L(\lambda)$. Taking $m = b_i = \overline{b_i}$ and $m' = \eta_\lambda = \overline{\eta_\lambda}$, this gives

$$u(b_i \otimes \eta_\lambda) = \Theta(\overline{\overline{u}(\overline{b_i} \otimes \overline{\eta_\lambda})})$$

since $\Theta(\overline{b_i} \otimes \overline{\eta_\lambda}) = \overline{b_i} \otimes \overline{\eta_\lambda}$ (which follows from the fact that Θ lies in a completion of $\mathbf{U}^- \otimes \mathbf{U}^+$,

cf. [CHW13, Theorem 3.1.1]). Hence,

$$\Theta(x) = \sum_i \Theta(\overline{\overline{u}(\overline{b_i} \otimes \overline{\eta_\lambda})}) = u_i(b_i \otimes \eta_\lambda) = \sum_i \pi_{b_i}(u_i),$$

where the latter lies in ${}_{\mathcal{A}}M \otimes_{\mathcal{A}} {}_{\mathcal{A}^\pi}L(\lambda)$ by Lemma 9.9, which completes the proof.

Regarded as $\mathbf{U}^\imath$-module $M \otimes L(\lambda)$ is $\imath$-involutive with the involution $\psi_\imath := \Upsilon \circ \psi$. We can now prove that $\psi_\imath$ preserves the $\mathcal{A}^\pi$-submodule ${}_{\mathcal{A}}M \otimes_{\mathcal{A}^\pi} {}_{\mathcal{A}}L(\lambda)$.

By Corollary 9.15(2), for any $x \in {}_{\mathcal{A}}M \otimes_{\mathcal{A}^\pi} {}_{\mathcal{A}}L(\lambda)$, we can write $x = \sum_k u_k(b_k \otimes \eta_\lambda)$, for $u_k \in {}'_{\mathcal{A}}\dot{\mathbf{U}}^\imath$ and $b_k \in B$. Since $M \otimes L(\lambda)$ is $\imath$-involutive, we have

$$\psi_\imath(x) = \sum_k \psi_\imath(u_k)\psi_\imath(b_k \otimes \eta_\lambda) = \sum_k \psi_\imath(u_k)\Upsilon\psi(b_k \otimes \eta_\lambda) = \sum_k \psi_\imath(u_k)(\Upsilon b_k \otimes \eta_\lambda), \tag{9.14}$$

where for the last equality we used the fact $\Delta(\Upsilon) \in \Upsilon \otimes 1 + \mathbf{U} \otimes \mathbf{U}^+_{>0}$ (from the formulas (2.5) for Δ and (9.1) for Υ directly), together with the fact that $\psi(b_k \otimes \eta_\lambda) = \Theta(b_k \otimes \eta_\lambda) = b_k \otimes \eta_\lambda$ since $\Theta = \sum_\nu \Theta_\nu$, where $\Theta_\nu = \mathbf{U}^-_\nu \otimes \mathbf{U}^+_\nu$ and $\Theta_0 = 1 \otimes 1$. By assumption we have $\Upsilon b_k \in {}_{\mathcal{A}}M$ and it follows by definition of ${}'_{\mathcal{A}}\dot{\mathbf{U}}^\imath$ that $\psi_\imath(u_k) \in {}_{\mathcal{A}}\dot{\mathbf{U}}^\imath$. Applying Corollary 9.15(2) again to (9.14), we see that $\psi_\imath(x) \in {}_{\mathcal{A}}M \otimes_{\mathcal{A}^\pi} {}_{\mathcal{A}}L(\lambda)$, and so the proposition follows. □

Corollary 9.18. *The quasi K-matrix Υ preserves the $\mathcal{A}^\pi$-submodule ${}_{\mathcal{A}}M \otimes_{\mathcal{A}^\pi} {}_{\mathcal{A}}L(\lambda)$. In particular, Υ preserves the $\mathcal{A}^\pi$-submodule ${}_{\mathcal{A}}L(\lambda)$ of $L(\lambda)$.*

Proof. Recall that $\Upsilon = \psi_\imath \circ \psi$. The corollary follows from Proposition 9.17 and the fact that ψ preserves the $\mathcal{A}^\pi$-submodule ${}_{\mathcal{A}}M \otimes_{\mathcal{A}^\pi} {}_{\mathcal{A}}L(\lambda)$. □

Corollary 9.19. *Let $\lambda_i \in X^+$ for $1 \le i \le \ell$. The involution $\psi_\imath$ on the $\imath$-involutive $\mathbf{U}^\imath$-module $L(\lambda_1) \otimes \ldots \otimes L(\lambda_\ell)$ preserves the $\mathcal{A}^\pi$-submodule ${}_{\mathcal{A}}L(\lambda_1) \otimes_{\mathcal{A}^\pi} \ldots \otimes_{\mathcal{A}^\pi} {}_{\mathcal{A}}L(\lambda_\ell)$.*

Proof. The module $L(\lambda_1) \otimes \ldots \otimes L(\lambda_\ell)$ is a based $\mathbf{U}$-module whose weights are bounded above, and so the corollary follows by applying Proposition 9.17 consecutively. □

For finite type, the quasi K-matrix Υ is itself integral:

Corollary 9.20. *Assume* $(\mathbf{U}, \mathbf{U}^\imath)$ *is of finite type. Write* $\Upsilon = \sum_{\mu \in \mathbb{Z}\Pi} \Upsilon_\mu$. *Then we have* $\Upsilon_\mu \in {}_{\mathcal{A}}\mathbf{U}^+$ *for each* μ.

Proof. This follows directly from Corollary 9.18, by applying Υ to the lowest weight vector $\xi_{-w_0\lambda} \in {}_{\mathcal{A}}L(\lambda)$, for $\lambda \gg 0$ (i.e., $\lambda \in X^+$ such that $\langle i, \lambda \rangle \gg 0$ for each i). $\square$

Chapter 10

Canonical basis on $\mathbf{U}^\imath$

In this chapter, we will define based modules for the $\imath$quantum covering groups, and develop canonical basis for these modules using the properties for the quasi K-matrix Υ established in the previous chapter.

10.1 Canonical basis for $\mathbf{U}^\imath$-modules

We call a $\mathbf{U}^\imath$-module M a weight $\mathbf{U}^\imath$-module if M admits a direct sum decomposition $M = \oplus_{\lambda \in X_\imath} M_\lambda$ such that, for any $\mu \in Y^\imath$, $\lambda \in X_\imath$, $m \in M_\lambda$, we have $K_\mu m = q^{\langle \mu, \lambda \rangle} m$. We will make the following definition of based $\mathbf{U}^\imath$-modules (based on [BWW18, Definition 1]).

Definition 10.1. Let M be a weight $\mathbf{U}^\imath$-module over $\mathbb{K}(q)^\pi$ with a given $\mathbb{K}(q)^\pi$-basis $\mathbb{B}^\imath$. The pair $(M, \mathbb{B}^\imath)$ is called a based $\mathbf{U}^\imath$-module if the following conditions are satisfied:

1. $\mathbb{B}^\imath \cap M_\nu$ is a basis of M_ν, for any $\nu \in X_\imath$;

2. The $\mathcal{A}^\pi$-submodule ${}_{\mathcal{A}}M$ generated by $\mathbb{B}^\imath$ is stable under ${}_{\mathcal{A}}\dot{\mathbf{U}}^\imath$;

3. M is $\imath$-involutive; that is, the $\mathbb{K}^\pi$-linear involution $\psi_\imath : M \to M$ defined by $\psi_\imath(q) = q^{-1}, \psi_\imath(b) = b$ for all $b \in \mathbb{B}^\imath$ is compatible with the $\dot{\mathbf{U}}^\imath$-action, i.e., $\psi_\imath(um) = \psi_\imath(u)\psi_\imath(m)$, for all $u \in \dot{\mathbf{U}}^\imath, m \in M$;

4. Let $\mathbf{A} = \mathbb{K}[[q^{-1}]]^\pi \cap \mathbb{K}(q)^\pi$. Let $L(M)$ be the $\mathbf{A}$-submodule of M generated by $\mathbb{B}^\imath$. Then the image of $\mathbb{B}^\imath$ in $L(M)/q^{-1}L(M)$ forms a $\mathbb{K}^\pi$-basis in $L(M)/q^{-1}L(M)$.

We shall denote by $\mathcal{L}(M)$ the $\mathbb{Z}[q^{-1}]^\pi$-span of $\mathbb{B}^\imath$; then $\mathbb{B}^\imath$ forms a $\mathbb{Z}[q^{-1}]^\pi$-basis for $\mathcal{L}(M)$. We also define based $\mathbf{U}^\imath$-submodules and based quotient $\mathbf{U}^\imath$-modules in the obvious way.

By a standard argument using [Cl14, Lemma 9] (cf. [Lu94, Lemma 24.2.1]), we have the following generalization of [BW18c, Theorem 6.12] (cf. [BW18b, Theorem 5.7]): Let $\leq$ be the partial order defined in (2.1) i.e. $\lambda \leq \lambda'$ if and only if $\lambda' - \lambda \in \mathbb{N}[I]$.

Theorem 10.2. *Let (M,B) be a based* $\mathbf{U}$*-module whose weights are bounded above. Assume that the* $\mathcal{A}^\pi$*-submodule* ${}_{\mathcal{A}}M$ *is preserved by the involution* $\psi_\imath$ *of* M*.*

1. *The* $\mathbf{U}^\imath$*-module* M *admits a unique basis (called the* $\imath$*-canonical basis)* $B^\imath := \{b^\imath | b \in B\}$*, which is* $\psi_\imath$*-invariant and of the form*

$$b^\imath = b + \sum_{b' \in B, b' < b} t_{b;b'} b', \quad \textit{for} \quad t_{b;b'} \in q^{-1}\mathbb{Z}^\pi[q^{-1}]. \tag{10.1}$$

2. $B^\imath$ *forms an* $\mathcal{A}^\pi$*-basis for the* $\mathcal{A}^\pi$*-lattice* ${}_{\mathcal{A}}M$ *(generated by* B*), and forms a* $\mathbb{Z}^\pi[q^{-1}]$*-basis for the* $\mathbb{Z}^\pi[q^{-1}]$*-lattice* $\mathcal{M}$ *(generated by* B*).*

3. $(M, B^\imath)$ *is a based* $\mathbf{U}^\imath$ *module, and we call* $B^\imath$ *the* $\imath$*-canonical basis of* M*.*

Recall the based $\mathbf{U}$-module $L(\lambda, \mu) := \mathbf{U}(\eta_\lambda \otimes \eta_\mu) \subset L(\lambda) \otimes L(\mu)$.

Corollary 10.3. *Let* $\lambda, \mu \in X^+$*, and* $w \in W$*.*

1. $L(\lambda) \otimes \lambda(\mu)$ *is a based* $\mathbf{U}^\imath$*-module, with the* $\imath$*canonical basis defined as Theorem 10.2.*

2. $L(\lambda, \mu)$ *is a based* $\mathbf{U}^\imath$*-submodule of* $L(\lambda) \otimes \lambda(\mu)$*.*

Proof. It suffices to verify the assumptions of Theorem 10.2. It is clear both $L(w\lambda, \mu)$ and $L(\lambda) \otimes L(\mu)$ have weights bounded above. It follows from Corollary 9.19 that $\psi_\imath$ preserves

the $\mathcal{A}^\pi$-submodule ${}_{\mathcal{A}}L(\lambda) \otimes_{\mathcal{A}} {}_{\mathcal{A}}L(\mu)$, hence also ${}_{\mathcal{A}}L(w\lambda, \mu)$. Therefore both $L(\lambda) \otimes L(\mu)$ and $L(w\lambda, \mu)$ are based $\mathbf{U}^\imath$-modules. It is immediate that $L(w\lambda, \mu)$ is a based $\mathbf{U}^\imath$-submodule of $L(\lambda) \otimes \lambda(\mu)$. $\square$

Next, we will develop canonical basis for tensor products of based $\mathbf{U}^\imath$-modules. A first step in this direction is the construction of quasi R-matrix $\Theta^\imath$ for $\mathbf{U}^\imath$ from the quasi K-matrix in Chapter 9.

The quasi R-matrix $\Theta^\imath$ for $\mathbf{U}^\imath$

Let $\widehat{\mathbf{U} \otimes \mathbf{U}}$ be the completion of the $\mathbb{K}(q)^\pi$-vector space $\mathbf{U} \otimes \mathbf{U}$ with respect to the descending sequence of subspaces

$$\mathbf{U} \otimes \mathbf{U}^- \mathbf{U}^0 \big(\sum_{\mathrm{ht}(\mu) \geq N} \mathbf{U}^+_\mu \big) + \mathbf{U}^+ \mathbf{U}^0 \big(\sum_{\mathrm{ht}(\mu) \geq N} \mathbf{U}^-_\mu \big) \otimes \mathbf{U}, \text{ for } N \geq 1, \mu \in \mathbb{Z}I.$$

We have the obvious embedding of $\mathbf{U} \otimes \mathbf{U}$ into $\widehat{\mathbf{U} \otimes \mathbf{U}}$. By continuity the $\mathbb{K}(q)^\pi$-algebra structure on $\mathbf{U} \otimes \mathbf{U}$ extends to a $\mathbb{K}(q)^\pi$-algebra structure on $\widehat{\mathbf{U} \otimes \mathbf{U}}$. Recall the quasi $\mathcal{R}$-matrix Θ defined in [CHW13, Theorem 3.1.1] which lies in $\widehat{\mathbf{U} \otimes \mathbf{U}}$. It follows from Theorem 9.2 that $\Upsilon^{-1} \otimes \mathrm{id}$ and $\Delta(\Upsilon)$ are both in $\widehat{\mathbf{U} \otimes \mathbf{U}}$.

We define

$$\Theta^\imath = \Delta(\Upsilon) \cdot \Theta \cdot (\Upsilon^{-1} \otimes \mathrm{id}) \in \widehat{\mathbf{U} \otimes \mathbf{U}}. \tag{10.2}$$

Proposition 10.4 (cf. [BW18a, Proposition 3.2])**.** *For any $b \in \mathbf{U}^\imath$, we have in $\widehat{\mathbf{U} \otimes \mathbf{U}}$ the following identity:*

$$\Delta(\psi_\imath(b)) \cdot \Theta^\imath = \Theta^\imath \cdot (\psi_\imath \otimes \psi) \circ \Delta(b)$$

Proof. Let $b \in \mathbf{U}^\imath$. Using the intertwiner relations, we make the following calculation:

$$\begin{aligned}
\Theta^\imath \cdot (\psi_\imath \otimes \psi) \circ \Delta(b) &= \Delta(\Upsilon) \cdot \Theta \cdot (\Upsilon^{-1} \otimes 1) \cdot (\psi_\imath \otimes \psi) \circ \Delta(b) \\
&= \Delta(\Upsilon) \cdot \Theta \cdot (\psi \otimes \psi) \circ \Delta(b) \cdot (\Upsilon^{-1} \otimes 1) \quad \text{using Theorem 9.2} \\
&= \Delta(\Upsilon) \cdot \Delta(\psi(b)) \cdot \Theta \cdot (\Upsilon^{-1} \otimes 1) \quad \text{using [CHW13, Theorem 3.1.1]} \\
&= \Delta(\psi_\imath(b)) \cdot \Delta(\Upsilon) \cdot \Theta \cdot (\Upsilon^{-1} \otimes 1) \quad \text{using Theorem 9.2 again,}
\end{aligned}$$

thus proving the proposition. □

We can write

$$\Theta^\imath = \sum_{\mu \in \mathbb{N}I} \Theta^\imath_\mu, \qquad \text{where } \Theta^\imath_\mu \in \mathbf{U} \otimes \mathbf{U}^+_\mu. \tag{10.3}$$

Lemma 10.5. *The first and second tensor factors of each term in $\Theta^\imath_\mu \in \mathbf{U} \otimes \mathbf{U}^+_\mu$ share the same parity.*

Proof. As we saw above, $p(\Upsilon) = p(\Upsilon^{-1}) = 0$ and so $\Delta(\Upsilon)$ has the property that the first and second tensor factors of its terms share the same parity. By [CHW13, Theorem 3.1.1(b)], Θ_ν also has this property, and so $\Theta^\imath = \Delta(\Upsilon) \cdot \Theta \cdot (\Upsilon^{-1} \otimes \mathrm{id})$ does as well. □

The following result is an analogue of [Ko17, Proposition 3.6], which first appeared in [BW18a, Proposition 3.5] for the quantum symmetric pairs of (quasi-split) type AIII/AIV.

Lemma 10.6. *We have $\Theta^\imath_\mu \in \mathbf{U}^\imath \otimes \mathbf{U}^+_\mu$, for all μ. In particular, we have $\Theta^\imath_0 = 1 \otimes 1$.*

Proof. For any $i \in I$ one has

$$\Delta(B_i) = B_i \otimes K_i^{-1} + 1 \otimes F_i + c_i J_i \otimes E_i K_i^{-1}.$$

Hence Proposition 10.4 implies that

$$\big(B_i \otimes K_i^{-1} + 1 \otimes F_i + c_i J_i \otimes E_i K_i^{-1}\big) \cdot \Theta^\imath = \Theta^\imath \cdot \big(B_i \otimes J_i K_i + 1 \otimes F_i + \overline{c_i} J_i \otimes J_i K_i E_i\big).$$

Rearranging this we obtain

$$\Theta^\imath(1\otimes F_i)-(1\otimes F_i)\Theta^\imath=(B_i\otimes K_i^{-1}+c_iJ_i\otimes E_iK_i^{-1})\Theta^\imath-\Theta^\imath(B_i\otimes J_iK_i+\overline{c_i}J_i\otimes J_iK_iE_i) \tag{10.4}$$

Recall [CHW13, Proposition 2.2.2(a)] concerning the skew-derivation r_i. In each level μ the left hand side is the sum of terms of the form

$$\begin{aligned}
&\big((\Theta^\imath_\mu)_1\otimes(\Theta^\imath_\mu)_2\big)(1\otimes F_i)-(1\otimes F_i)\big((\Theta^\imath_\mu)_1\otimes(\Theta^\imath_\mu)_2\big)\\
&\quad=(\Theta^\imath_\mu)_1\otimes(\Theta^\imath_\mu)_2F_i-\pi_i^{p_1}(\Theta^\imath_\mu)_1\otimes F_i(\Theta^\imath_\mu)_2 \quad \text{where } p_k:=p((\Theta^\imath_\mu)_k),\ k=1,2\\
&\quad=(\Theta^\imath_\mu)_1\otimes[(\Theta^\imath_\mu)_2,F_i], \quad \text{since } \pi_i^{p_1}=\pi_i^{p_2} \text{ by Lemma 10.5}\\
&\quad=(\Theta^\imath_\mu)_1\otimes\left(\frac{r_i((\Theta^\imath_\mu)_2)J_iK_i-K_{-i}\pi_i^{p_2-p(i)}{}_ir((\Theta^\imath_\mu)_2)}{\pi_iq_i-q_i^{-1}}\right) \quad \text{by [CHW13, Proposition 2.2.2(a)]}
\end{aligned}$$

Comparing this to terms on the right hand side of (10.4) with a factor of $1\otimes J_iK_i$, we see that

$$(1\otimes r_i)(\Theta^\imath_\mu)=-(\pi_iq_i-q_i^{-1})\Theta^\imath(B_i\otimes 1+\overline{c_i}q_i^2J_i\otimes E_i) \tag{10.5}$$

Then, the same induction argument as in [Ko17, Proposition 3.6] completes the proof, this time using our Lemma 2.2 above as the appropriate analogue in the quantum covering group setting. □

The following is an version of [BWW18, Lemma 3], used in the proof of the next theorem:

Lemma 10.7. *We have* $\Theta^\imath_\mu\in{}_{\mathcal{A}^\pi}\mathbf{U}\otimes_{\mathcal{A}}{}_{\mathcal{A}}\mathbf{U}^+_\mu$ *for all* μ.

Proof. Since the definition of $\Theta^\imath$ has the same form, the argument is analogous to the proof of [BWW18, Lemma 3]; we have integrality of Θ by [CHW13, Theorem 3.1.1], and integrality of the action of Υ in Theorem 9.20. □

Theorem 10.8. *Let M be a based $\mathbf{U}^\imath$-module, and $\lambda \in X^+$. Then $\psi_\imath \overset{\text{def}}{=} \Theta^\imath \circ (\psi_\imath \otimes \psi)$ is an anti-linear involution on $M \otimes L(\lambda)$, and $M \otimes L(\lambda)$ is a based $\mathbf{U}^\imath$-module with a bar involution $\psi_\imath$.*

Proof. The anti-linear operator $\psi_\imath = \Theta^\imath \circ (\psi_\imath \otimes \psi) : M \otimes L(\lambda) \to M \otimes L(\lambda)$ is well defined thanks to Lemma 10.6 and the fact that the weights of $L(\lambda)$ are bounded above. Then entirely similar to [BW18a, Proposition 3.13], we see that $\psi_\imath^2 = 1$ and $M \otimes L(\lambda)$ is $\imath$-involutive in the sense of Definition 10.1(3).

The proof that $\psi_\imath$ preserves the $\mathcal{A}^\pi$-submodule ${}_{\mathcal{A}}M \otimes_{\mathcal{A}^\pi} {}_{\mathcal{A}}L(\lambda)$ is the same as the proof of Proposition 9.17. By assumption, $(M, \mathbb{B}^\imath(M))$ is a based $\mathbf{U}^\imath$-module. For any $b \in \mathbb{B}^\imath(M)$, define

$$\pi_b : {}_{\mathcal{A}}\dot{\mathbf{U}}^\imath \to {}_{\mathcal{A}}M \otimes_{\mathcal{A}^\pi} {}_{\mathcal{A}}L(\lambda), \quad u \mapsto u(b \otimes \eta_\lambda).$$

Then, π_b is well defined, since by Definition 9.11 and the following remark the coproduct preserves the integral forms, that is, $\Delta(u)(\mathbf{1}_\mu \otimes \mathbf{1}_\nu)$ preserves ${}_{\mathcal{A}}M \otimes_{\mathcal{A}^\pi} {}_{\mathcal{A}}L(\lambda)$, for any $\mu \in X^\imath$ and $\nu \in X$.

Note that $\psi_\imath(b \otimes \eta_\lambda) = b \otimes \eta_\lambda$ for any $b \in \mathbb{B}^\imath(M)$. Following the proof of Lemma 9.14, we have $\sum_{b \in \mathbb{B}^\imath(M)} \pi_b({}'_{\mathcal{A}}\dot{\mathbf{U}}^\imath) = {}_{\mathcal{A}}M \otimes_{\mathcal{A}^\pi} {}_{\mathcal{A}}L(\lambda)$. Hence we also have $\sum_{b \in \mathbb{B}^\imath(M)} \pi_b({}_{\mathcal{A}}\dot{\mathbf{U}}^\imath) = {}_{\mathcal{A}}M \otimes_{\mathcal{A}^\pi} {}_{\mathcal{A}}L(\lambda)$, since ${}'_{\mathcal{A}}\dot{\mathbf{U}}^\imath \subset {}_{\mathcal{A}}\dot{\mathbf{U}}^\imath$. By the same argument as before, we may conclude that $\psi_\imath$ preserves the $\mathcal{A}$-submodule ${}_{\mathcal{A}}M \otimes_{\mathcal{A}^\pi} {}_{\mathcal{A}}L(\lambda)$.

We write $\mathbb{B} = \{b^- \eta_\lambda | b \in \mathbb{B}(\lambda)\}$ for the canonical basis of $L(\lambda)$. We can now conclude that:

1. for $b_1 \in \mathbb{B}^\imath, b_2 \in \mathbb{B}$, there exists a unique element $b_1 \diamondsuit_\imath b_2$ which is $\psi_\imath$-invariant such that $b_1 \diamondsuit_\imath b_2 \in b_1 \otimes b_2 + q^{-1}\mathbb{Z}^\pi[q^{-1}]\mathbb{B}^\imath \otimes \mathbb{B}$;

2. we have $b_1 \diamondsuit_\imath b_2 \in b_1 \otimes b_2 + \sum_{(b_1', b_2') \in \mathbb{B}^\imath \times \mathbb{B}, |b_2'| < |b_2|} q^{-1}\mathbb{Z}^\pi[q^{-1}]\, b_1' \otimes b_2'$;

3. $\mathbb{B}^\imath \diamondsuit_\imath \mathbb{B} := \{b_1 \diamondsuit_\imath b_2 \mid b_1 \in \mathbb{B}^\imath, b_2 \in \mathbb{B}\}$ forms a $\mathbb{K}(q)^\pi$-basis for $M \otimes L(\lambda)$, an $\mathcal{A}^\pi$-basis for ${}_{\mathcal{A}}M \otimes_{\mathcal{A}\ \mathcal{A}^\pi} L(\lambda)$, and a $\mathbb{Z}^\pi[q^{-1}]$-basis for $\mathcal{L}(M) \otimes_{\mathbb{Z}^\pi[q^{-1}]} \mathcal{L}(\lambda)$;

4. $(M \otimes L(\lambda), \mathbb{B}^\imath \diamondsuit_\imath \mathbb{B})$ is a based $\mathbf{U}^\imath$-module,

following the same arguments as for [BWW18, Theorem 4] i.e. using Lemma 10.6 and Lemma 10.7 and [Cl14, Lemma 9]. □

10.2 Canonical basis on $\dot{\mathbf{U}}^\imath$

In this section, we formulate the main definition and theorems on canonical bases for the modified $\imath$quantum groups. The formulations specialize at $\pi = 1$ to [BW18c, Section 7], which are in turn generalizations of the finite type counterparts in [BW18b, Section 6].

The modified $\imath$quantum groups

Recall the partial order $\leq$ on the weight lattice X in (2.1). The following proposition is a version of [BW18c, Proposition 7.1] in the quantum covering case.

Proposition 10.9. *Let $\lambda, \mu \in X^+$.*

1. *The $\imath$-canonical basis of the $\mathbf{U}^\imath$-module $L^\imath(\lambda, \mu) := L(\lambda + \mu)$ is the basis*

$$\mathbb{B}^\imath(\lambda, \mu) = \left\{(b_1 \diamondsuit_{\zeta_\imath} b_2)^\imath_{\lambda,\mu} | (b_1, b_2) \in \mathbb{B}^\imath \times \mathbb{B}\right\} \backslash \{0\},$$

where $(b_1 \diamondsuit_{\zeta_\imath} b_2)^\imath_{\lambda,\mu}$ is $\psi_\imath$-invariant and lies in

$$(b_1 \diamondsuit_\zeta b_2)(\eta_\lambda \otimes \eta_\mu) + \sum_{|b_1'|+|b_2'| \leq |b_1|+|b_2|} q^{-1}\mathbb{Z}[q^{-1}](b_1' \diamondsuit_\zeta b_2')(\eta_\lambda \otimes \eta_\mu).$$

2. *We have the projective system $\left\{L^\imath(\lambda + \nu^\tau, \mu + \nu)\right\}_{\nu \in X^+}$ of $\mathbf{U}^\imath$-modules, where*

$$\pi_{\nu+\nu_1,\nu_1} : L^\imath(\lambda + \nu^\tau + \nu_1^\tau, \mu + \nu + \nu_1) \longrightarrow L^\imath(\lambda + \nu^\tau, \mu + \nu), \quad \nu, \nu_1 \in X^+,$$

is the unique homomorphism of $\mathbf{U}^\imath$*-modules such that*

$$\pi(\eta_{\lambda+\nu^\tau+\nu_1^\tau} \otimes \eta_{\mu+\nu+\nu_1}) = \eta_{\lambda+\nu^\tau} \otimes \eta_{\mu+\nu}.$$

3. *The projective system in (2) is asymptotically based in the following sense: for fixed* $(b_1, b_2) \in \mathbb{B}^\imath \times \mathbb{B}$ *and any* $\nu_1 \in X^+$*, as long as* $\nu \gg 0$*, we have*

$$\pi_{\nu+\nu_1,\nu_1}\big((b_1 \diamondsuit_{\zeta_\imath} b_2)^\imath_{\lambda+\nu^\tau+\nu_1^\tau,\mu+\nu+\nu_1}\big) = \big((b_1 \diamondsuit_{\zeta_\imath} b_2)^\imath_{\lambda+\nu^\tau,\mu+\nu}\big).$$

Proof. Claim (1) is just a reformulation of Corollary 10.3. Claim (2) follows by the same proof as [BW18b, Proposition 6.12], replacing the $\mathcal{R}$-matrix therein with the one using the $\mathcal{R}$-matrix from [CHW13, Theorem 3.1.1]. Claim (3) is the same as [BW18b, Proposition 6.16], and in the quasi-split case here, we can do without the mild modification needed in [BW18c] since the module $L(\nu^\tau + \nu)$ is finite dimensional. □

The proof in the following version of [BW18c, Theorem 7.2] (see also [BW18b, Theorem 6.17]) rests solely on a version of Proposition 10.9; the same arguments thus lead to the $\imath$-canonical basis for $\dot{\mathbf{U}}^\imath$:

Theorem 10.10. *Let* $\zeta_\imath \in X_\imath$ *and* $(b_1, b_2) \in B \times B$*.*

1. *There is a unique element* $u = b_1 \diamondsuit^\imath_{\zeta_\imath} b_2 \in \dot{\mathbf{U}}^\imath$ *such that*

$$u(\eta_\lambda \otimes \eta_\mu) = (b_1 \diamondsuit_{\zeta_\imath} b_2)^\imath_{\lambda,\mu} \in L^\imath(\lambda, \mu) := L(\lambda + \mu),$$

   for all $\lambda, \mu \gg 0$ *with* $\overline{\lambda + \mu} = \zeta_\imath$*.*

2. *The element* $b_1 \diamondsuit^\imath_{\zeta_\imath} b_2$ *is* $\psi_\imath$*-invariant.*

3. *The set* $\dot{\mathbb{B}}^\imath = \{b_1 \diamondsuit^\imath_{\zeta_\imath} b_2 \big| \zeta_\imath \in X_\imath, (b_1, b_2) \in B \times B\}$ *forms a* $\mathbb{K}(q)^\pi$*-basis of* $\dot{\mathbf{U}}^\imath$ *and an* $\mathcal{A}^\pi$*-basis of* ${}_{\mathcal{A}}\dot{\mathbf{U}}^\imath$*.*

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