Broadening the Political Methodologist's Toolkit:
A Population Dynamics Model of Political Science Time Series Data

Kimberly Michelle Ganczak

MA, Old Dominion University, 2015
BA, Old Dominion University, 2013

A Dissertation presented to the Graduate Faculty
of the University of Virginia in Candidacy for the Degree of
Doctor of Philosophy

Woodrow Wilson Department of Politics

University of Virginia
May, 2021

For Sean.

To have been loved so deeply, even though the person who loved us is gone, will give us some protection forever.
- JK Rowling, *Harry Potter and the Sorcerer's Stone*

Rwy'n dy garu di, cariad bach.

ABSTRACT

Political scientists, and social scientists more generally, rely on statistical modeling to understand complex systems and behaviors. However, as interest grows in not only how political actors change over time, but why they change, it is increasingly necessary to explore the use of dynamic models of time. In this dissertation, I introduce, describe, and implement an early version of a new estimation strategy of the Lotka-Volterra, or predator/prey, model that is theoretically and empirically accessible to political scientists.

Contents

List of Figures

Chapter 1

Introduction

Political scientists, and social scientists more generally, rely on statistical modeling to understand complex systems and behaviors. Traditionally, political methodologists rely primarily on generalized linear models (GLM), resulting in a deeply ingrained set of assumptions about how and why events occur in societies. Abbott (1988) coined the term general linear reality (GLR) to characterize these assumptions. GLR is a way of thinking about how the world works and arises from treating linear models as representative of actual societal functions. Crucially, Abbott argues that the assumptions underpinning GLR "prevent the analysis of many problems interesting to theorists and empiricists alike" (Abbott 1988, p. 169). Therefore, it is necessary to explore alternative modeling techniques in order to continue to expand our understanding of the world around us. GLM is a powerful technique, especially when carefully applied. Therefore, my suggestion to diversify the political methodology toolkit is not a derogation of GLR nor GLM. Instead, this project seeks to expand the breadth of models available to empiricists, particularly in cases where the underlying theoretical assumptions of linearity are unreasonable or might be violated.

The most fundamental goal of linear modeling is to identify a regression coefficient or set of coefficients that estimate a best-fit line that lies as close as possible to all of the observed data points (Stock, Watson, et al. 2007). The slope of the best-fit line remains constant across all observations. However, it seems intuitively unreasonable to assume that this characterization is representational of all human behavior (Abbott 1988). Therefore, alternative modeling techniques, specifically those that rely on nonlinear ordinary differential equations (ODE), might open or re-open new pathways of analysis by relaxing the often intractable assumptions that underlie GLM. ODE understand the world in a fundamentally different way compared to GLM. They directly model rates of change of variables of interest and allow flexibility of several of the key assumptions of GLM. There are several configurations of ODE based around different theories. I focus on population dynamics models, and the Lotka-Volterra (LV) model in particular. LV is most commonly used in ecological models to capture the interdependent relationship between predator and prey. Appearing in Equation 1.1, LV are a pair of nonlinear ODE that model the relationship between two interdependent populations. We can conceptualize several types of political relationships that might fit within this framework. Modeling political relationships in this way allows us to think about political time series data in new ways.

$$\begin{cases} \frac{dx}{dt} = \alpha x - \beta xy \\ \frac{dy}{dt} = \delta xy - \gamma y \end{cases} \tag{1.1}$$

The LV model, when correctly applied, is useful in several key ways. First, it allows us to directly model the rates of change of two (or more) interdependent time series. Although political scientists often discuss rate of change, it is rarely modeled directly. Substantively, modeling rate of change is particularly beneficial in capturing how a given set of variables are affecting population growth or decline at any given point in time. Especially when it is theoretically unrealistic to assume that the independent variables change at a constant linear rate over time, it is appropriate to explore alternative nonlinear modeling techniques such as the Lotka-Volterra model. LV, because it is composed of ordinary differential equations that are functions of time, directly models rates of change over time. The most basic assumption of LV is that the rates of change are nonlinear and oscillate over time. The pair of equations measure two unique rates of change: $\frac{dx}{dt}$ and $\frac{dy}{dt}$. The former is understood as the rate of change of a 'prey' population, x, at any given point in time, t and the later is understood as the rate of change of the 'predator' population, y, at time t. Specifically, because the LV model is nonlinear, it allows us to capture complex relationships between actors whose behavior cannot be assumed to obey a constant, linear transformation.

Second, GLR assumes that "the causal meaning of a given attribute cannot, in general, depend on its context in either space or time [and] its effect does not change as other variables change around it, not is its causal effect redefined by its own past" (Abbott 1988, p. 180). While, in practice, this is often remedied by the inclusion of interaction terms or lagged variables into a GLM, GLR as a paradigm has a difficult time explaining their inclusion. GLR as a world view struggles to visualize interaction terms: interaction terms in particular require one to visualize the existence of all cases between two points in time within a multi-dimensional space. Nonetheless, complex interactions define most historical processes (Abbott 1988). The Lotka-Volterra model, however, eliminates the need for such complex visualization.

LV is a model of endogeneity because the variables of interest are inherently dependent upon the time and environment within which they exist. In its most basic form, LV captures the behavior of two populations that grow and shrink in direct response to one another. In order to induce this interconnectedness, the populations of both predator and prey are included in the calculation of both derivatives. Concretely, the prey population's rate of change is a function of its intrinsic growth rate, α, and its death rate from predation, β. On the other hand, the predator's death rate, γ, is intrinsic and its growth rate, δ, is the product of intrinsic growth and growth from predation. While LV is not unique in its ability to handle endogeneity, it is a very straightforward and easy way to visualize in the real world. In more complex predator-prey models, environmental factors further contextualize the relationship and make the rate of change of the two populations dependent upon their environmental context as well as upon one another.

Third, but tangentially, GLR assumes that the "observed sequence of attributes over time does not influence their ultimate result" (Abbott 1988, p. 178). However, this assumption presents a challenge to our intuitive theoretical understanding of both historical and social processes, many of which rely on the supposition that the sequence of events is integral in the outcome. For example, the common historical narrative about the lead-up to the first World War holds that preceding events and their short and long-term effects ultimately led to the collapse of enough international alliances, thus leading to War. The nature of ODE, particularly when they are functions of time, naturally induce dependence on past events. When solving the LV model at discrete time points, x and y are equal to the predator and prey populations at $t-1$ and produce a population estimate at time t. Several common, highly effective models such as ARIMA and Vector Autoregression (VAR) do allow variables to depend on the past. Therefore, if dependence on past behavior or sequence of events is the primary motivation in model selection, it is important to further theoretically assess the data to determine whether or not LV is the appropriate choice.

Finally, LV is capable of producing new quantities of interest. The pair of equations exist in equilibrium under two conditions. First, extinction occurs when both derivatives equal 0. Extinction conditions are substantively interesting to the study of political phenomena when one of the populations of interest is, consciously or not, attempting to eliminate its counterpart in a parasitic relationship. For instance, governments attempting to eradicate terrorism might fall into this category. The second equilibrium, steady state, is the point at which both populations maintain their values. In the basic model utilized in this project, steady state represents indefinite sustainability for both populations and is determined by the values of the four parameters. Again, when applied to political phenomena, we can envision a scenario where two populations that exist in a steady state equilibrium experience an exogenous shock to the system. Using LV, we would be able to measure the effect of that shock on the previously stable equilibrium.

The underlying assumptions of Lotka-Volterra describe both the intrinsic qualities of each population as well as ways they relate to each other. First, we must be able to reasonably assume that the parameters governing the birth and death rates of each population are positive. When this assumption is violated, the two populations are unstable leading to either systemic extinction or the extinction of one side and the exponential growth of the other. Second, the populations cannot be negative because they are drawn from a continuous count distribution. Third, LV is an appropriate choice when we can reasonably hypothesize that each population of interest grows and shrinks in direct response to the other. In other words, if theory supports the idea that causality might be cyclical, or endogenous, then LV might be an appropriate means of modeling that relationship. Finally, LV is a continuous time model; therefore, it is best applied when the populations of interest can be thought of as existing and changing dynamically over time. Chapter 2 develops these and other conditions that predicate the selection of LV as the appropriate model.

Chapter 2 begins by exploring previous uses of LV in empirical social science studies, primarily within sociology, since the late 1970s. This is followed by a discussion about LV in detail, including the differences between GLR and LV and the specific ways in which LV can augment the political science empirical toolkit. I then introduce a new strategy for estimating the Lotka-Volterra parameters that relies on Ordinary Least Squares (OLS) to estimate the four parameter values. Unlike previous estimation methods that rely on a discrete approximation of the LV equations to estimate LV, I circumvent the need to do so by relying on the implicit relationship between predator and prey and directly estimating the Lotka-Volterra parameters. In order to demonstrate the efficacy of this estimation strategy, I test it on simulated LV data under several conditions. I focus on two general sets of simulations. The first set of models is a basic Lotka-Volterra configuration where the populations change only in response to their own intrinsic qualities and to changes in the other population. The second set of models adds complexity into the LV system by making the parameters dependent upon an exogenous covariate. Within each set of models, I also test two different ways of adding noise into the simulated data, demonstrating that the estimation method is extremely sensitive to noise, depending on how noise is added into the system. In later chapters, I utilize this estimation strategy to extract the parameter values of real-world data.

Chapter 3 is a case study based around Ura's (2014) empirical study of the ambiguity surrounding the direction of causality between Supreme Court (SCOTUS) decisions and attitudes of the American public. Scholars are generally split over the direction of causality in that relationship. Those in support of a thermostatic model argue that the Federal government, in general, is responsive to public opinion. The other side of the debate supports a theory of legitimation, where public mood is shaped in response to Supreme Court decision making. Although fundamentally split over the direction of causality, there is uniform agreement that there exists a strong relationship between the two variables. Ura's study represents an initial attempt to settle the debate between the two models. Empirically, he utilizes an Error Correction Model (ECM) because these models measure both short- and long-term effects and finds evidence in support of both models: there is a thermostatic response in the short-term and a response more consistent with legitimation theory over long periods of time.

It is reasonable to assume that there is endogeneity in Ura's data. Intuitively, public mood and SCOTUS decision making likely act in response to one another, to at least some degree. Ura's results support this assertion because they seem to imply a reversal of causality over time. LV is an appropriate choice in model when seeking to extend Ura's work for two reasons. First, the LV equations are functions of time. Because time seems to be an integral determinant in which mechanism is at work in Ura's data, it is appropriate to extend Ura's work by applying his data to a time dependent model. Second, and more importantly, LV measures rates of change over time. This can allow us to visualize how the two variables relate to one another at different points in time as well as how and when Ura's exogenous covariates impact the changes in their relationship. I find support for Ura's conclusions that both the thermostatic model and legitimation theory are active mechanisms underlying the relationship between SCOTUS decision making and public mood.

Chapter 4 is a case study based on Jaeger and Paserman's (2008) study characterizing the relationship between Israel and Palestine during the Second Intifada. The relationship between Israel and Palestine has often been described as a tit-for-tat cycle of violence where "violence by one party causes violence by the other party and vice versa" (Jaeger and Paserman 2008, p. 1591). If this is true, then LV is a theoretically appropriate model for the data. However, through the use of VAR, Jaeger and Paserman find evidence that causality in the relationship is unidirectional: Israel responds predictably to Palestinian violence but the opposite does not hold true.

I chose Jaeger and Paserman's study as a second case study in order to test their refutation of the conventional characterization of the Israeli/Palestinian relationship in a different way. Ultimately, after applying the Israel/Palestine data to the LV model, I find support for Jaeger and Paserman's conclusions about unidirectional causality. Therefore, LV is not the appropriate model for this data. However, not fitting with LV in particular does not exclude the potential for other pairs of ODE to be more theoretically robust models for this data, particularly because the rate of change of violence between the two sides remains a substantively interesting concept that ODE is capable of measuring directly.

Ultimately, I argue that the use of ODE, and LV in particular, is a valuable new way for political scientists to think about the world in conjunction with GLM. Ordinary differential equation frameworks grant us flexibility in the underlying assumptions of GLM and also allow us to directly model rates of change, which are substantively interesting, especially in complex systems. I focus exclusively on the application of Lotka-Volterra to political relationships. However, more broadly, this project serves to demonstrate the utility of broadening empirical modeling to include ordinary differential equations models.

Chapter 2

The Lotka-Volterra Model for Analyzing Political Time-Series Data

In this chapter, I introduce and explore a population dynamics approach as an alternative method to analyze the relationship between highly interdependent political variables. Specifically, I re-conceptualize these types of relationships from the perspective of predator versus prey in a natural environment. The Lotka-Volterra system of ordinary, nonlinear differential equations allow us to simulate the impact of behavioral changes of one side of a highly interdependent relationship upon the overall dynamic of a two-party relationship. Originally developed in the early 20th century to model biological systems, these models have a broad applicability that extends far beyond the natural sciences that can allow researchers to test theory by parameterizing x and deriving quantities we're unable to obtain from more mainstream, linear methods. This method is not only appropriate and useful for any data that makes sense within the analogy of predator and prey, but also useful for data with feedback loops where the quantities of interest are useful and applicable. Although researchers across multiple social science fields have been interested in using the Lotka-Volterra model for several decades, these models are not widely utilized in empirical time series studies.

Along with building upon previous work, this method is a beneficial addition to the political methodology toolkit in three key ways. First, it allows us to navigate around some of the intractable assumptions and problems that underpin more mainstream methods such as endogeneity concerns and autocorrelation associated with including time in a more traditional time series model. In other words, dynamic modeling is a novel approach that strays from the paradigm of the linear model and advances practical time series analysis.

The Lotka-Volterra approach introduced in this dissertation is a workable statistical model using differential equations with a novel estimation strategy. Differential equations are useful in this context because they allow us to look at the rates of change of x and y over time, a topic that is often discussed in time series analysis, but that is rarely modeled directly. Similarly, this approach allows us to model feedback loops directly instead of assuming they introduce endogeneity and bias into the model. Thus, this method might allow this modeling technique to fit more easily with causal inference than more traditional methods because this model explicitly models feedback instead of getting stuck or simply assuming there is no feedback.

Second, this method can provide new and substantive values of interest. Compared to the values we can extract from more mainstream methods such as VAR and ECM, Lotka-Volerra can provide values for a fixed point where both populations remain constant over time (indefinitely in the most simplistic version of the model), extinction conditions, and parameter values for each of the variables of interest. These parameter values directly correlate to the growth and death rates of the data being modeled. This method allows us to forecast future estimates of the values of x and y. Finally, this method answers academically interesting questions, but can serve to inform practical policy questions such as how the implementation of new counter-terrorism strategies or societal changes impact the frequency of domestic terror attacks.

When added alongside the other prominent methods for modeling time series data within political methodology, the Lotka-Volterra method is a new approach that represents a significant paradigm shift by providing researchers an alternative, non-linear perspective in approaching highly interdependent data. Benefits of including this new tool to the political methodology toolkit include an honest and realistic way on analyzing these complex data and the ability to avoid imposing assumptions such as endogeneity or linearity onto those data. The remainder of this chapter is broken down into several key parts. First, I examine challenges often presented by contemporary, competitive time series data. This is followed by a frank discussion about how the Lotka-Volterra method can provide an alternative to current modeling techniques by answering new and different questions as well as a discussion about previous uses of Lotka-Volterra in social science. Third, I introduce and describe the Lotka-Volterra method in detail. Finally, I include a simulation my estimator. Largely due to endogeneity in the estimation strategy, the estimator underperforms and produces biased coefficient estimates. In response to this, it is important to be clear that the estimator I describe in this chapter is meant to serve as a launching point for future work in this area. Therefore, I identify several paths forward for this work that could address or potentially remedy these issues.

2.1 Existing Time Series Analysis Methods

Vector autoregression (VAR) captures linear interdependencies among multivariate time series data.[1] It is a linear function of the past values of a set of k variables. The goal of the VAR is to generate the impulse response functions (IRF) and to estimate Granger causality. VAR is an appropriate approach when researchers are unwilling or unable to make assumptions about the direction of causality and assume that both variables are stationary. First introduced into the political science literature in 1989, VAR has become a go-to method for modeling political science time series variables (Freeman, Williams, and Lin 1989). VAR relies on three key assumptions. First, it assumes that all k time series variables are stationary, meaning that the data fluctuate around the same mean over time and are not cointegrated. However, this stipulation is often untrue in real-world data that is often stochastic and interdependent in nature. If the data are not stationary, researchers must rely on another method such as a Vector Error Correction Model (VECM).

Second, VAR assumes the error terms in the VAR equations are uncorrelated (Box-

[1]For further discussion on VAR, see Lütkepohl (2006) and Box-Steffensmeier, et. al. (2014).

Steffensmeier, Freeman, Hitt, and Pevehouse 2014). Although not "causal" according to the traditional causal inference definition, the Granger test generates a single p-value from an f-test that assesses if the prior history of x affects the current value of y after controlling for the prior history of y (Box-Steffensmeier, Freeman, Hitt, and Pevehouse 2014). In other words, VAR allows for Granger causality to occur in either direction, meaning it is an autodistributed lag model in both directions. However, if an underlying theory suggests a causal order, the modeler must make substantive changes to the model itself by using an auto-distributed lag model where y is regressed on lagged y and lagged x with an indeterminant amount of lags. The Granger causality test is performed by conducting an f-test on all of the coefficients on lagged x to test whether they are all equal to zero at the same time. If x has a significant f-test for all y, and not vice-versa, x is said to Granger cause y. However, in analyzing the results of a Granger causality test, the modeler is looking for a high p-value, challenging the notion that a high p-value is not definitive proof of causality. Granger causality tests do not convey directionality or magnitude of causality of a non-zero effect.

Finally, VAR requires the modeler to determine the lag structure of the data. Not only must the lag structure for x and y be the same, but the number of lags is chosen atheoretically (Box-Steffensmeier, Freeman, Hitt, and Pevehouse 2014). It might be unrealistic to assume the same number of lags on both x and y. This is true largely because if you want to control for the past history of y when assessing the effects of the past history of x on y, intuitively, the number of lags on y should be at least as long as the number of lags on x, but may not be identical. Similarly, the use of lags raises the issue of multicollinearity, thus causing the individual coefficient estimates to have little to no substantive meaning (Aguiar-Conraria, Magalhães, and Soares 2012). However, the selection of lags is, inherently, an atheoretical process. Thus, it might present a problem for political science researchers whose approach to mathematical modeling is rooted in theory and supported by the method.

Another typical tool for modeling time series data are Error Correction Models (ECM), which are used on cointegrated data to estimate the long- and short-term effects of one time series on another when there is assumed to be an underlying stochastic trend between the two time series.[2] They are used to estimate how quickly a dependent variable returns to equilibrium after a shock to the system. This approach is appropriate if the variables are non-stationary and cointegration is present. An ECM consists of two parts. First, a differenced VAR shows the short term effects of the impulse on the response. Second, the model estimates a cointegrating equation, showing the long term effects of the impulse on the response. This method requires the time series to be non-stationary, or completely unrelated but integrated. Therefore, before utilizing an ECM, the modeler must test for cointegration. Two variables are cointegrated if both are non-stationary in and of themselves but have a relationship to one another that is stationary. There is an ongoing debate within political methodology as to whether stationary variables can be thought of as cointegrated.[3]

ECMs are useful for the analysis of non-stationary, cointegrated data. However, despite their utility, ECMs have a few key drawbacks. First, ECMs rely on an atheoretical test of cointegration, a principle that is contradictory to political science methodological research which champions methods work that is theoretically driven. ECMs are also difficult to interpret and can be difficult to estimate, especially when compared to ecological models that produce a clear and concise output. Second, the cointegrating equation is difficult to substantively interpret. Finally, as discussed, a debate is ongoing about whether or not ECMs can apply to stationary variables.

If a researcher is diligent in choosing their model based on the aforementioned criteria and considerations, they obtain a model that describes the linear relationship(s) between their variables. These results specifically answer questions regarding how the change in

[2] For further discussion on ECM, see Lütkepohl (2006) and Box-Steffensmeier, et. al. (2014).

[3] For additional discussion on the ongoing debate, see Keele and Webb (2016) and Grant and Lebo (2016).

one variable affects the change in the other variable(s) and at what point in time that change occurs. In cases where these are the relevant questions of interest, VAR and ECM remain the appropriate method of modeling. However, in special cases that either do not satisfy the underlying assumptions of either model or look to answer different questions such as extinction criteria or fixed point(s), Lotka-Volterra can be used in lieu of or alongside these more common methods to further broaden our understanding of the world.

2.2 Challenges Posed by Modeling Contemporary Time Series Data Sets

The nature of political time series data as complex, random, or redundant poses several challenges to statistical modeling. Several of these challenges are discussed below. While I do not intend to argue that contemporary, mainstream methods of time series analysis are inadequate, I do argue that there are several new questions about competitive time series data that can be answered by utilizing the Lotka-Volterra method, which simply offers a new perspective on these data. To achieve this end, I explore how two major time series models view the world and compare them each to the Lotka-Volterra method to demonstrate how Lotka-Volterra and build upon these common methods to answer new questions. Used alongside these methods, I argue that Lotka-Volterra can broaden our understanding of the world.

First and foremost, time series data is often rife with autocorrelation, which inflates standard errors, thus limiting the power of inference (Stock, Watson, et al. 2007). While often an indication of a misspecified model, the ideal solution is re-specifying the model itself; however, this solution is atheoretical, assuming the initial model was chosen, based on theory, to best fit the data. In the same vein, time series data is prone to spurious correlation, wherein any two variables that trend over time will be correlated in a statistically significant way regardless of whether a causal relationship between the two exists (Stock, Watson, et al. 2007).

Third, coefficients are a narrow means of understanding causal effects for several reasons. First, we are often uncertain of the timing and magnitude of the effects of coefficients, raising the question whether the β change is instantaneous or occurs over time. In traditional regression and GLM models, coefficients are static when allowing them to change over time might more accurately capture reality. A shift from looking to achieve linear slopes to modeling cycles as the principle mode of thinking about competitive data can express fundamentally different conclusions than traditional linear approaches. Especially in the case of feedback loops, which are often present in political time series data, a new approach based on cyclical modeling, such as Lotka-Volterra, offers a more elegant method of modeling because it is fundamentally a model of feedback loops.

Other methods commonly used to analyze time series data include *Vector Autoregression* and *Error Correction Models*. While these models successfully capture their own intricacies of time series data, they may not capture the complex and often random or cyclical relationships between two or more highly interdependent populations, especially in "irregular cycles without fixed periodicity"(Aguiar-Conraria, Magalhães, and Soares 2012, pg. 500). This is not to presume that these methods are fundamentally flawed. Instead, VAR, ECM, and other common methods should be understood as making very specific kinds of statements about very specific types of data and the choice to use an alternate method such as Lotka-Volterra should be driven by which questions the researcher is asking. In other words, Lotka-Volterra provides methodologists an alternative lens through which to analyze political time series data.

In cases of irregular or transient cycles, the new perspective offered by Lotka-Volterra can offer insights about cyclical patterns. In this type of special case, traditional methods might be inappropriate because cycles that are more prominent in Time Period A might remain undetected or are over-detected, thus attributing a "blip", to the entire time series

(Aguiar-Conraria, Magalhães, and Soares 2012). Weakliem (2010) even goes to far as to argue that many advanced methods of analyzing time series data are not "usefully applied to most political [science time series] data" (p. 637). These methods can be complicated to substantively interpret for a variety of reasons. For example, in a VAR model with 15 lags, one would have to report 240 coefficients, all of which are more or less void of substantive meaning (Aguiar-Conraria, Magalhães, and Soares 2012). The following section will examine two such methods, VAR and ECM, in greater detail to highlight ways in which Lotka-Volterra can offer an alternative method to broaden our ability to draw conclusions about competitive time series data. It is important, however, to keep in mind that the observations offered in the following section do not represent 'shortcomings' or 'faults' in these common methods. Instead, these reflections are simply an insight into how Lotka-Volterra can be applied to answer questions in special cases where the data of interest do not fit the assumptions of more common methods.

2.3 Benefits of Lotka-Volterra Models

In contrast to static theories that describe the attributes of individuals or organizations at a discrete time point, dynamic models are useful for understanding how and why individuals or social systems change over time (Tuma and Hannan 1984, p. 4). As theoretical interest in social science has increasingly trended towards exploring systematic change over time, dynamism in empirical modeling is an increasingly valuable tool.

Lotka-Volterra can improve upon common approaches by answering new and interesting substantive questions about cyclical, competitive variables. This section lays out several questions that LV is well-suited to answer and the benefits of using LV as opposed to other, traditional methods, in those specific cases. The Lotka-Volterra approach offers several advantages over previous methods. First, variables are not required to be either stationary or non-stationary, because ecological models can apply to all types of relationships. This eliminates the need for atheoretical and hard-to-interpret tests for stationarity that are inherent in more traditional models. Similarly, LV approaches do not rely on atheoretical selection of model components. Substantively speaking, ecological models are theory-driven, especially when the data work well with the predator/prey analogy as in the case of modeling the relationship between terrorist and counter-terrorism responses. Unlike VAR or ECM, ecological models are not only theoretically driven, but are easier to substantively interpret as a result. For example, VAR relies on the atheoretical selection of lag structure while ECM relies on atheoretical tests of cointegration. Lotka-Volterra, on the other hand, is a model chosen theoretically that tests both primary and secondary theory and is not beholden to the Gauss-Markov assumptions underlying standard OLS approaches.

Second, Lotka-Volterra allows us to model social and political variables of interest as a cyclical process as opposed to a linear process. This interpretation of human behavior is inherently useful in competitive situations where the actions of Side A are determined by the actions of Side B (and vice versa). It is also useful over long periods of time where the periodicity of behavior is not fixed (Aguiar-Conraria, Magalhães, and Soares 2012).

Next, LV does not answer questions about x and y, but about the rates of change of those variables. More common models such as VAR and ARMA are adequate at identifying cycles; however, they are inadequate at examining cycles that vary over time (Aguiar-Conraria, Magalhães, and Soares 2012). Lotka-Volterra, on the other hand, allows us to simulate the impact of behavioral changes on one side of a heavily interdependent relationship upon the overall dynamic of a two (or more) party system and examine how that cyclical behavior changes in intensity across multiple periods.

Finally, Lotka-Volterra allows us to analyze data without the restrictions of the Gauss-Markov assumptions that underlie traditional GLM and other linear approaches. The Lotka-Volterra approach, and the use of ordinary differential equation models in general, provides a theoretical alternative that has been selected to fit the data, allowing researchers to avoid concern with whether the Gauss-Markov assumptions are met or not and then attempting to correct any violations that do occur. In certain relationships, the assumptions underlying linear frameworks may hinder or prohibit modeling and should be relaxed (Abbott 1988).

The LV approach is an entirely novel way of modeling time series in political science. While I recognize that it is not a catch-all for all applications, it is a good fit for a specific class of problems, namely problems that involve highly interdependent relationships between two or more sets of actors. For example, expanding beyond terrorism and conflict studies, predator-prey models might be useful to study competition among political candidates, the legislative process, and inter-agency competition among government bureaucracies.

LV models can answer inherently different questions than more traditional methods. For example, LV models are capable of modeling systemic equilibrium in two ways, both occurring when the rate of change is set equal to zero. This can result from one of two conditions. First, both populations can be sustained at their current respective population levels for an indefinite period of time: $\{y = \frac{\alpha}{\beta}, x = \frac{\gamma}{\delta}\}$ where the birth rate of each population is proportional only to its death rate. This stability condition results in a consistent ebb and flow of population sizes that stays equal, on average, over time.

Second, and perhaps most interesting, is the event of extinction. In the event of extinction, the two equations reduce to $\{y = 0, x = 0\}$, implying that the two populations will be indefinitely sustained at zero members. Of particular interest are the conditions that lead to extinction. For instance, we might be able to model a scenario in which the predator begins eating significantly more prey than usual, and determine whether this alone would force the prey's extinction. However, unless the prey are artificially manipulated into extinction, both populations can get infinitesimally close to zero and still recover in number, making extinction extremely difficult, if not impossible, to obtain naturally in the simplified version of the model (Begon, Harper, Townsend, et al. 1986).

2.4 The Lotka-Volterra Model in Detail

Developed in the early 20th century separately by both Alfred J. Lotka (Lotka 1910) and Vito Volterra (Volterra 1938), the Lotka-Volterra equations are a pair of first-order, nonlinear, differential equations. This set of ordinary differential equations (ODE's) is often used in mathematical biology and ecology to describe the dynamics of the interaction between two interdependent species, one the predator and one the prey (Gotelli 2008). The set of ODE's has a continuous and deterministic solution, meaning the ebb and flow of the predator and prey populations overlap and the solution to the equations oscillates on an ellipse (Begon, Harper, Townsend, et al. 1986).

The Lotka-Volterra approach has been in use, albeit sparsely, in social science beginning with Hannan & Freeman (1977); however, the model was not empirically implemented until 1981 (Carroll 1981). In political science specifically, Francisco (1995) adapted the Lotka-Volterra model to empirically evaluate the interdependent relationship between coercion and protest in coercive states. Other than this single application, Lotka-Volterra has not been a widely utilized tool in political science for two key reasons. First, Lotka-Volterra is technically advanced, and can be difficult to properly implement. I intend to resolve this issue by writing an accessible R package to allow researchers to easily apply this method to their own sets of data. However, I argue that the key reason this method has not permeated political science methodology more broadly is because it doesn't fit within the traditional, dominant paradigm for model selection in political methodology. Instead of automatically beginning with OLS, evaluating whether the Gauss-Markov assumptions are met or not, and accounting for violations of those assumptions, the Lotka-Volterra approach is a theoretical model that is selected to fit the data. The set of equations is reproduced below:

$$
\begin{aligned}
\frac{dx}{dt} &= \alpha x_t - \beta x_t y_t \\
\frac{dy}{dt} &= \delta x_t y_t - \gamma y_t
\end{aligned}
\tag{2.1}
$$

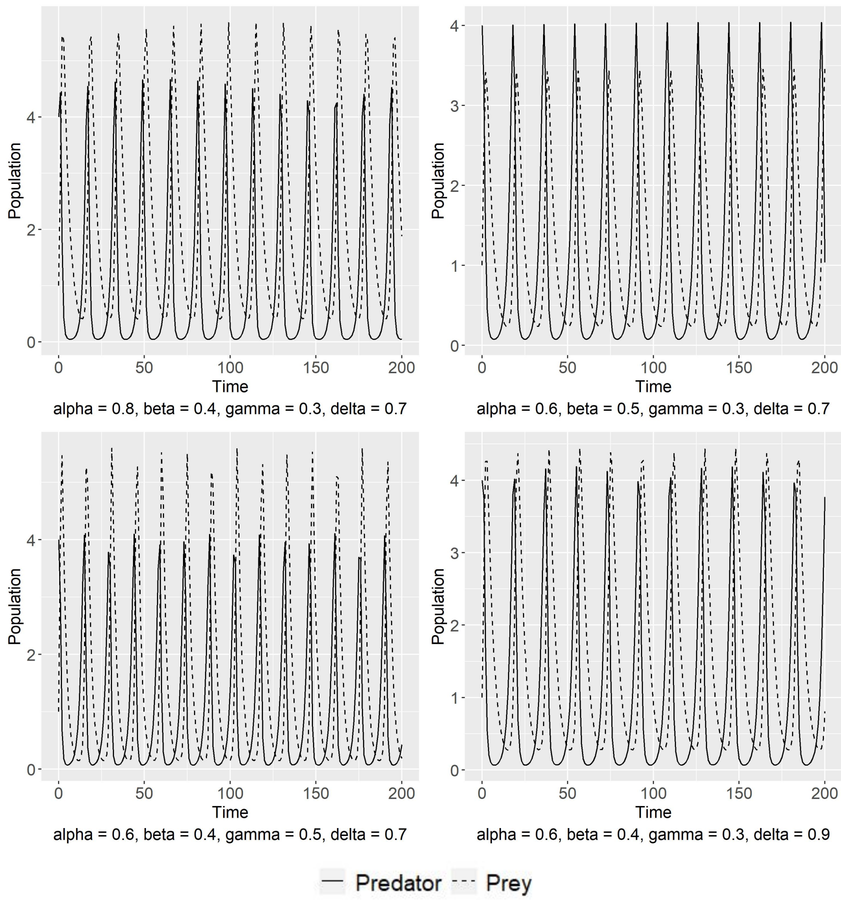
where:

- $\frac{dx}{dt}$ and $\frac{dy}{dt}$ represent the growth rates of prey and predator species, respectively, over time;
- x_t and y_t represent the population of the prey predator species, respectively; and
- α, β, γ, and δ are non-negative constants that represent the growth and death rates of prey and predator, respectively.

The prey population's rate of change is determined by its own natural birth or generation rate minus the rate of predation, and the predator population's rate of change is determined by subtracting its natural death rate from its rate of predation. Figure 2.1 graphically depicts how changing each parameter with fixed starting values alters the shape of the graph.

The Lotka-Volterra equations rely on five key assumptions. First, the rate of change of either population, predator or prey, is proportional to its size. Second, the prey has an unlimited food supply. Third, without predation, the prey population would increase exponentially and without prey, the predator population would decrease exponentially. Fourth, the size of the predator population is entirely dependent upon the size of the prey population. Finally, the external environment is inconsequential to the relationship (Begon, Harper, Townsend, et al. 1986). Assumptions four and five are conceptually violated when dealing with political data; however, I have adapted the Lotka-Volterra equations to accept the input of exogenous covariates so that the model may more accurately mimic real-world scenarios. In order to achieve this end, each of the four parameters is not only

Figure 2.1: Effects of Altering Lotka-Volterra Parameters



estimated to be a function of time, but also the function of a at least one exogenous variable.

In keeping with these assumptions, it is important to note that the prey parameters must relate to one another in a specific way to maintain stability within the system: $\{\alpha > \beta\}$. The birth rate of the prey (α) must be greater than the death rate (β) because, in the absence of predation, the prey population increases exponentially. On the other hand, as long as the prey population is sufficiently large, the birth rate of the predator (δ) is not required to be greater than its death rate (γ) because there is ample food (prey) to sustain the predator over time.

The set of differential equations can result in equilibrium in three ways: the exponential growth of the prey in the absence of the predator, extinction, and a fixed point where neither population changes over time (Skvortsov, Ristic, and Kamenev 2018). Each of these values is interesting, but the latter two are the most substantively interesting in the study of political science. Extinction events occur when both populations reach 0: $\{y = 0, x = 0\}$. When this occurs, both populations remain extinct forever. This fixed point is unstable and represents a saddle point, thus making natural extinction of both species difficult, suggesting artificial manipulation of the populations is necessary to force total extinction (Kinoshita 2013). This quantity is interesting for studying social science because it can allow us to examine how external stimuli can affect the continued existence of one side of an interdependent dataset.

$$\left\{y = \frac{\alpha}{\beta}, x = \frac{\gamma}{\delta}\right\} \tag{2.2}$$

The third solution, the population equilibrium or steady state, shown in Equation 2.2, "corresponds to the balanced coexistence of the two species with oscillating but stable populations" (Skvortsov, Ristic, and Kamenev 2018, pg. 373). The eigenvalues of the Jacobian matrix associated with this fixed point are conjugates of one another, resulting in an ellipse around the fixed point. These orbits will continue to circle around the fixed point without converging upon it (Kinoshita 2013). In other words, the fixed point represents population stability. The steady state is reminiscent of a Nash equilibrium, where neither side has incentive to change its behavior because doing so would ultimately disadvantage that side. It is also important to note that, even when α is fixed, we can still calculate the value of the steady state. Equation 2.3 demonstrates the proof of this assertion:

$$\left\{y = \frac{\hat{\alpha}}{\hat{\beta}}, x = \frac{\hat{\gamma}}{\hat{\delta}}\right\} \tag{2.3a}$$

$$\left\{y = \frac{\frac{\alpha}{\alpha}}{\frac{\beta}{\alpha}}, x = \frac{\frac{\gamma}{\alpha}}{\frac{\delta}{\alpha}}\right\} \tag{2.3b}$$

$$\left\{y = \frac{1}{\frac{\beta}{\alpha}}, x = \frac{\frac{\gamma}{\alpha}}{\frac{\delta}{\alpha}}\right\} \tag{2.3c}$$

$$\left\{y = \frac{1}{\hat{\beta} + \theta_\beta z_i}, x = \frac{\hat{\gamma} + \theta_\gamma z_i}{\hat{\delta} + \theta_\delta z_i}\right\} \tag{2.3d}$$

This particular specification of Lotka-Volterra is the appropriate model choice when several criteria are met. First, and most fundamentally, we must be able to reasonably assume that the two populations of interest grow and shrink in direct response to one another. Because LV is a model of competition, it is important that the two populations of interest are not only competing for resources, but that growth in the prey population directly causes growth in the predator population, which in turn causes the prey's population to decline followed by a decline in the predator population. If we cannot hypothesize that the relationship between the two populations can be analogized to predator/prey, but may have a mutualistic, commensalistic, or parasitic relationship, we would need to fundamentally reformulate the model. Tangentially, we have to analyze the goals of both populations of interest.

In the LV model discussed here, the two populations are exclusively interested in survival: the predator population eats the prey to survive and thrive and the prey reproduces at a rate to survive predation. However, if the goal of one population is to eradicate the other or help the other thrive, the relationship is no longer characterized as predator/prey. In order to make this determination, we have to delve into the underlying theory about any dyadic relationship of interest to truly understand the motivations of both sides. For example, in Chapter 4, I analyze the relationship between Israel and Palestine during the Second Intifada. Although it has often been characterized as a tit-for-tat cycle of violence, we may also hypothesize that the goal of the Israelis stretches beyond simple survival with regards to Palestine. I argue, in this particular case, that Israel's level of violence is motivated by more than surviving Palestinian violence, making it a poor candidate for the LV model.

A simple, but not infallible, way of testing whether a particular relationship is amenable to LV is to examine the phase-space plot of the data. If the relationship between the proposed predator and prey populations do not form any sort of elliptical orbits, they are likely incompatible with the LV method. However, visual inspection will not work if, for instance, there are multiple steady states and multiple ellipses that overlap or confound. In this case, it might also benefit us to run a preliminary Granger test for causality. For example, a Granger test of the Ura data, which is a good fit for LV, returns results consistent with causality in both directions. On the other hand, a Granger test of the Jaeger and Paserman data, which I show is not amenable to LV, is unable to reject the null hypothesis to identify causality in either direction. Visual inspection and the Granger test should not be the final determinant of whether LV is appropriate or not; however, they can serve as an additional means of identifying the underlying causal structure of the data and whether or not it is consistent with the Lotka-Volterra assumptions.

Finally, the data for both the predator and prey populations must be understood as continuous counts and constrained above zero. LV originated as way to model predator/prey relationships in nature; therefore, it doesn't make sense to think about a negative population. However, this isn't to say that data that includes negative values is fully incompatible with LV. For example, in Chapter 3, I examine the relationship between Supreme Court decision making and public mood. Both are composite indices as opposed to concrete physical populations. Therefore, as long as we maintain the same scale of the data and adjust our understanding of the different values in the indices, we can reasonably transform them to satisfy the data requirements of LV. It is also conceivable to develop a model of competition that is not beholden to the requirement of positive numbers, although this is outside the scope of this particular project.

2.5 Previous Uses of Lotka-Volterra in Social Science

Historically, sociologists have utilized Lotka-Volterra models to study interorganizational ecology (Carroll 1981). Much of the subsequent work with Lotka-Volterra is built upon work by Ayala, Gilpin and Ehrenfeld (1973), who performed population ecology experiments on fruit fly populations. What distinguishes this particular study over previous work, thus making it a solid foundation for later sociologists, is the use of simple OLS regression to solve a series of eleven ODE models, eight of which utilize Lotka-Volterra as a special case. While the OLS approach is useful in the case of inter-species competition in biology, Nielsen and Hannan (1977) identify that, in human social organizations, autocorrelation and heteroskedasticity in the data make OLS an inappropriate approach. In order to correct these issues, Nielsen and Hannan suggest the use of weighted generalized least squares (WGLS) and interpret the various dynamic parameters of social organizations as dependent variables (Nielsen and Hannan 1977, p. 479).

Further expanding upon Ayala, et. al., are Hannan and Freeman (1977) and Meyer et. al. (1977). As two of the earliest theoretical applications of the Lotka-Volterra system of equations to human social organizations, all three suggest the use of a logistic model of growth as opposed to the traditional assumption of exponential growth. In their earlier theoretical application, Hannan and Freeman identify several constraints of the application of purely ecological population models to human social organizations. For instance, some of those constraints that are unique to human social organizations include the incomplete amount of information passed on to organizational decision makers, internal politics, limits on resource pools, and organizational constraints generated from their own organizational histories (Hannan and Freeman 1977, p. 931).

While Hannan and Freeman speak more abstractly, Meyer et. al. theoretically apply ecological population modeling to the expansion of enrollment in the American school system. This concrete example mainly serves to demonstrate the importance of taking into account the unique properties of human-generated data. As opposed to biological applications of population dynamic models where information is passed generationally through genetics, information in human-based systems is inherently more complex. Thus, we must take the time to examine trends in the data at hand in order to identify the correct model specification. Notably, in this specific case, Meyer, et. al. observe that their data is diffused along a logistic curve, thus theorizing that a logistic version of the Lotka-Volterra equations is more appropriate for analyzing organizational growth as compared to the traditional, exponential curve of biological Lotka-Volterra models (Meyer, Ramirez, Rubinson, and Boli-Bennett 1977).

Aldrich (1979) expanded upon the use of logistic-based formulas for organizational growth. He largely rationalizes this argument using the idea of carrying capacity, or the maximum sustainable number of organizations within a given environment. In other words, any given environment can only accommodate a set number of organizations and once that upper limit is reached, growth slows and eventually flattens into the steady state where no more growth can occur (Aldrich 1979, p. 64). When considering inter-organizational competition specifically, Aldrich argues that dependencies that occur between organizations are largely based on resource scarcity. For example, when an organization has a monopoly on an otherwise scarce resource, competing organizations will become dependent upon the monopolizer. However, the monopolizer is only realistically able to distribute a set number of resources to its dependents, the environmental carrying capacity, which resembles a logistic curve with rapid initial growth of the number of dependents that eventually tapers off and stops growing (Aldrich 1979, p. 266).

Nielsen and Hannan (1977) estimated an empirical study of the expansion of education in the United States using a discrete time linear partial-adjustment model. They analyzed the models with weighted generalized least squares regression on panel data with a 5-year lag in order to estimate the speed of adjustment of each level of education as a function of not only its own population, but the population of the other two levels as well. However, their estimates of the adjustment parameters at each educational level were sometimes negative, unrealistically implying explosive growth in the education system (Tuma and Hannan 1984, p. 496).

Carroll (1981) built directly upon the foundation built by Nielsen and Hannan. He argued that Lotka-Volterra is more appropriate than the linear partial-adjustment model because "it emphasizes resource constraints, interorganizational competition, and temporal disequilibrium" (Carroll 1981, p. 586). Carroll reformulated Nielsen and Hannan's linear partial-adjustment model as a discrete approximation of the LV model. This strategy, Carroll argued, was necessary because the integral solution for LV has not yet been found. In order to approximate the two derivatives, Carroll identified "a numerically tractable, discrete time approximation to the integral solution of the model" for each population of interest that "converge[d] to the differential equation as the time interval becomes infinitesimally small" (Carroll 1981, p. 590). He proceeded to estimate each of these models using nonlinear WGLS "applied to pooled time-series...cross-sectional data" in order to remedy complications attributed to autocorrelation and heteroskedasticity (Carroll 1981, p. 591).

Carroll also focuses heavily on the role external variables play on organizational growth. Therefore, he builds upon the models developed by Ayala et. al. (1973) that are designed to incorporate external, environmental variables. Incorporating these external, independent variables inherently transforms the original Lotka-Volterra set of equations to rely on more than just the internal dynamics between two organizations. This is necessary largely due to the fact that human social organizations rely upon a more diverse and complex set of variables beyond internal, population competition. The estimation of the coefficients associated with these external variables is complicated by the fact that the integral solution to the Lotka-Volterra system of equations has never been found, although it is known to exist. In an attempt to mitigate this dilemma, he estimates parameters with an approximate instead of an exact solution. This approach, he argues, is significantly more feasible, flexible, and generalizable (Carroll 1981, p. 598).

Following the 1980's, the use of the Lotka-Volterra method within sociology has largely fallen out of favor, likely as the result of a combination of two competing factors. First, at the time, complications arose from the lack of an integral solution to the set of equations as well as the difficulties associated with estimating parameters within that constraint (Ünver 2008). Similarly, in the 1980s, estimating and optimizing the approximate Lotka-Volterra equations was technologically cumbersome. However, even as technology and estimation time improved, LV specifically did not make a major comeback. Instead, the discipline transitions to LV-inspired survival models that focused on density-dependent growth. This shift may have been spurred on by the tight coupling between theory and method that is inherent in the survival framework.

These empirical studies all have in common the use of exact discrete approximations of the Lotka-Volterra equations. Because there is no known explicit integral solution for the Lotka-Volterra equations, it is necessary to approximate the equations. In general, discrete approximations "approximate the dynamic differential equations (DDE) with a difference equation that is convenient for numerical analysis and...approaches the DDE as the discrete interval in the difference equation becomes infinitesimal" (Tuma and Hannan 1984, p. 485). In other words, all of these authors are approximating functions for x and y, whether from a non-linear logistic model (Nielsen and Hannan, 1977) or from the Lotka-Volterra models themselves (Carroll 1981, Tuma 1984).[4]

What makes the approach in this project distinct from these previous studies is the use of the implicit relationship between predator and prey. My strategy of estimating the Lotka-Volterra parameters is not a discrete approximation of the Lotka-Volterra equations.

[4]For a full discussion on the underlying mathematics of different discrete approximation techniques, see Tuma (1984).

Instead, it is based on the calculation of a constant, C, and is the integral solution to the phase-state of the relationship. In other words, the estimated coefficients do not tell us anything directly about the rates of change of x and y; instead, it directly informs us of how the two populations grow and shrink in response to one another. The purpose, then, is to circumvent the need to approximate the ODE and to directly estimate the Lotka-Volterra parameters. It is reasonable to ask, then, where the value lies in a workaround that does not directly estimate $\frac{dx}{dt}$ and $\frac{dy}{dt}$. However, if we have accurate estimates of the LV parameters, we can still draw informed conclusions about quantities of interest such as extinction conditions, steady state, and the phase-space. These parameter estimates can also be fed into the Lotka-Volterra set of equations in order to directly estimate $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

2.6 Estimating The Generalized Lotka-Volterra Model

I have adapted and expanded upon the basic Lotka-Volterra model to include exogenous variables and dynamic parameter values. I have coined this new iteration of the LV model the Generalized Lotka-Volterra (G-LV) Model. This new approach to LV modeling allows us to introduce control variables and external data into the calculation of the four LV parameters, thus allowing us to make different types of conclusions about cyclical, cointegrated time series data that are influenced by their environment as well as by the natural ebb and flow of the populations of the predator and prey. This approach allows us to answer several questions that other methods do not answer. For example, we can estimate how alterations in environmental factors, such as policy changes, economic conditions, or natural phenomena affect patterns of behavior. Specifically, by introducing covariates into the Lotka-Volterra equations, the birth and death rates of the variables are allowed to change over time, dependent upon the exogenous covariates. This introduces significant flexibility into the model by dispelling the underlying assumption that LV produces a single, repetitive cycle and allowing researchers to model any kind of feedback loop of interest.

The estimation strategy offered in this section offers a quicker and more direct approach to estimating Lotka-Volterra parameters. In order to demonstrate the efficacy of the Generalized Lotka-Volterra method as a means of analyzing cyclical, interdependent, and messy political data, I implement a simulation on artificially generated data with known starting values for the parameters.[5] The purpose of the simulation is to demonstrate that a linear model, derived from the calculation of the constant of the set of ordinary differential equations does, in fact, come close to the true values of the pre-established parameters from which the simulated data was created, regardless of how much noise is introduced into the data.

$$\begin{cases} \frac{dx}{dt} = \alpha x_t - \beta x_t y_t \\ \frac{dy}{dt} = \delta x_t y_t - \gamma y_t \end{cases} \tag{2.4a}$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\delta xy - \gamma y}{\alpha x - \beta xy} \tag{2.4b}$$

$$\frac{dy}{dx} = \left(\frac{y}{x}\right)\left(\frac{\delta x - \gamma}{\alpha - \beta y}\right) \tag{2.4c}$$

$$x(\alpha - \beta y)dy = y(\delta x - \gamma)dx \tag{2.4d}$$

$$\frac{\alpha - \beta y}{y}dy = \frac{\delta x - \gamma}{x}dx \tag{2.4e}$$

$$\int \frac{\alpha}{y} - \beta dy = \int \delta - \frac{\gamma}{x}dx \tag{2.4f}$$

$$\alpha \log(y) - \beta y + c_1 = \delta x - \gamma \log(x) + c_2 \tag{2.4g}$$

$$C = \beta y - \alpha \log(y) + \delta x - \gamma \log(x) \tag{2.4h}$$

The goal of the simulation is to ultimately arrive at estimates for three of the four

[5] See Appendices A-B for the annotated *Rmarkdown* simulation code.

Lotka-Volterra parameters from simulated data. However, this is impossible because, when attempting to choose parameter estimates that minimize the variance of the constant equation, without making assumptions about the properties of x and y, we must set all parameters equal to 0, thus resulting in a trivial solution. One effective solution is to divide both sides of the equation by one of the parameters (e.g. α). This solution avoids a trivial outcome because, if we set $\alpha = 0$, when algebraically rearranging the equation, dividing by 0 breaks the equations:

$$C = \beta y - \alpha \log(y) + \delta x - \gamma \log(x) \tag{2.5a}$$

$$\frac{C}{\alpha} = \frac{\beta y - \alpha \log(y) + \delta x - \gamma \log(x)}{\alpha} \tag{2.5b}$$

$$\frac{C}{\alpha} = \frac{\beta y}{\alpha} - \frac{\alpha \log(y)}{\alpha} + \frac{\delta(x)}{\alpha} - \frac{\gamma \log(x)}{\alpha} \tag{2.5c}$$

$$C^* = \beta^* y - \log(y) + \delta^* x - \gamma^* \log(x) \tag{2.5d}$$

where $C^* = \frac{C}{\alpha}$, $\beta^* = \frac{\beta}{\alpha}$, $\delta^* = \frac{\delta}{\alpha}$, and $\gamma^* = \frac{\gamma}{\alpha}$.

We can further algebraically rearrange the formula for the constant to allow researchers to estimate the parameters using OLS regression. By doing so, we not only make the estimation strategy more user-friendly and familiar, but this allows us to easily extract standard errors and calculate coverage. In the output of the new regression formula, the regression coefficients correspond to the Lotka-Volterra parameter estimates:

$$\log(y) = \beta^* y + \delta^* x - \gamma^* \log(x) - C^* \tag{2.6}$$

Although the equation can be rearranged and run as a linear regression, this model is not a traditional regression model, largely due to the endogeneity introduced by including $\log(y)$ on the left side of the equation and y on the right. This endogeneity ultimately biases the estimates of the coefficients and causes imprecision in the model.

To extend the model further, I have adapted the above estimation strategy to also include one or more external covariates in the estimation of three non-fixed base parameters. Each parameter in the simulation, β^*, δ^*, and γ^*, is constructed as an independent linear model that includes a general intercept and a coefficient(s) on the external variable(s). An example of parameter construction follows:

$$\beta^* = \frac{\hat{\beta} + \theta_\beta z_t}{\alpha} \tag{2.7}$$

where:

- β^* is the original LV parameter;
- $\hat{\beta}$ is the main effect of the parameter β;
- θ_β is the coefficient on the external data; and
- z_t is the external data.

This same process is carried out to estimate the values of the three non-fixed parameters as a function of one or more external variables. After each parameter is estimated, the Lotka-Volterra equation is applied using the linearly-estimated parameters to represent β^*, δ^*, and γ^*, with α set constant at 1. The inclusion of an external covariate currently requires that covariate to be categorical instead of continuous. Using this estimation strategy, if the external covariate is continuous, the number of coefficients increases exponentially with each included external variable. This occurs because, for each unique value of the external variable, the constant term changes.

Although this is a novel estimation strategy that is quick, easy, and reliable in estimating the values of interest, the means of including external data is a notable limitation of this method. It requires us to potentially over-simplify external data, thus potentially losing nuance in the overall analysis. In the next section, I walk through the steps of the simulation that ultimately prove that this estimation strategy is an effective approach to analyzing data that is amenable to the Lotka-Volterra approach.

2.7 Simulation of the G-LV Method

The goal of simulation is to determine how often, on average, the estimator arrives at the correct parameter estimates. Typical linear models take the form $Y = \beta_0 + \beta_j X_j$ where Y is the dependent variable (the effect), X_j is the independent variable (the cause), and β_j is the coefficient on X_j that describes the effect of X_j on Y. Adding noise into this classic linear model is fairly straightforward. In these models, X_j are assumed exogenous and y endogenous; therefore, it's natural to think of X_j as the 'cause' and Y as the 'effect' and to add noise onto Y. Substantively, we are uncertain about the actual effect that X_j has on y. In other words, we are certain about the values of our independent variables and uncertain about the values of the dependent variable. Consequently, it is intuitive to add noise onto Y because this linear model should estimate, within a margin of error, the true effect of X_j on Y on average.

Substantively, error in LV data is really error in counts; however, the counts of both x and y are inextricably linked because the DGP of both x and y are defined as endogenous processes causing both x and y to act as both cause and effect. Therefore, in order to preserve this relationship, we must carefully assess different ways of introducing error into the system before proceeding. I argue that we can add noise into the simulation in two different ways: during the generation of the simulation data itself (internal) or after the simulation data has been generated (external).

Internal noise in the Lotka-Volterra system is most commonly modeled using a stochastic Lotka-Volterra model.[6] Stochastic LV accounts for random noise in the environment, which is "the chief cause of fluctuations in the ecosystem" and can cause major changes in the dynamic of the system (Aratò 2003, p. 710). The stochastic LV system is modeled using stochastic differential equations derived from the deterministic LV model and have

[6]For more information, see Tuma and Hannan (1984), Bahar and Mao (2003), Mao, Yuan, and Zou (2004), and Baosheng, Shigeng, and Yang (2011).

taken several forms including simple diffusion processes (Aratò, 2003) and stochastic delay differential equations (Mao, Yuan, and Zou, 2004). When thinking about real-world data, especially in political science, noise in the environment is omnipresent and affects all types of actors. We can think of this noise as it pertains to political relationships as actors with goals that exceed simple survival, mitigating factors in a relationship such as economic conditions, or regime change, among many others. Despite the utility of introducing and modeling stochastic noise in the environment, I have chosen to introduce error on the count data both because this is a more familiar and accessible approach for political scientists and because measurement error in political time series data is so common a phenomenon.

Political time series data is often rife with measurement error. One of the most blatant examples of this exists in the literature on terrorism. A vast portion of empirical studies on terrorism rely on data from the National Consortium for the Study of Terrorism and Responses to Terrorism's (START) Global Terrorism Database (GTD). However, during the digitization process, the data for 1993 were lost while transferring handwritten data to a new office space and have never been recovered in full (START 2020). This missing data has impeded the ability of researchers to run large-n studies across the entirety of GTD's date range (Acosta and Ramos 2017). In an effort to remedy this issue, several researchers have constructed their own time-series of terrorist events in 1993. While these substitute time series are invaluable assets to terrorism studies and I do not intend to denigrate them, we should still consider them as 'best guess' efforts at recreating the lost data and potentially suffering from measurement error, making them systematically different from the GTD as a whole.

In order to simulate measurement error in this simulation, we can add noise post-hoc onto the deterministic values of x and y, which have been generated directly from the Lotka-Volterra equations with known parameters. By incorporating noise in this way, we can directly control how the counts are biased. The key argument in favor of introducing noise into the system in this way is that the non-noisy data are drawn from a deterministic Lotka-Volterra population; however, as the result of imperfect observation, we are uncertain about the accuracy of our observed counts. The goal, then, is to determine how well the estimation strategy captures the true parameter values, on average, given this type of noisy data. We can simulate measurement error in three ways: by adding noise onto x, onto y, and onto both.

$$\log(y) = \beta^* y + \delta^* x - \gamma^* \log(x) - C^* \tag{2.8}$$

First, let's consider introducing noise solely into the count of the prey population, x. Substantively, this type of behavior occurs in social science when we imprecisely count the prey population, x, but are confident in our count of y. For example, we might observe this type of bias if we are regressing a count of internally displaced persons (IDP) on a country's GDP. GDP is easily quantifiable; however, an accurate count of IDP can be exceptionally difficult, especially in war-torn areas. By adding noise only onto the x variable, we induce errors-in-variables bias, which occurs when the independent variable (IV) is imprecisely measured. When the imprecise variable is included as a regressor, the regression model includes an error term that includes the difference between the actual and observed values of the independent variable (IV). If the observed value of the IV is correlated with that error term, endogeneity exists in the model and the coefficient estimates on x and $\log(x)$ will be both biased and inconsistent (Stock, Watson, et al. 2007, p. 320). Additionally, adding noise onto only x also preserves the multicollinearity in the model that is caused by including both x and $\log(x)$ as regressors. The estimates of γ^* and δ^* both have a variance inflation factor (VIF) of above 20, indicating high multicollinearity. Highly correlated independent variables decrease the precision of the parameter estimates, can produce estimates with the incorrect sign, and make the coefficient estimates highly sensitive to small changes in the model. Therefore, if we add noise only onto the x variable, we expect to observe poor coverage of γ^* and δ^* and near perfect estimation of y. We observe this behavior in the sample simulation results in Table 2.1.

Table 2.1: Sample Simulation Results

	(1) Noise on x			(2) Noise on y		
	RMSE	Coverage	Bias	RMSE	Coverage	Bias
β^*	0.001	99.88%	$-4.3 \times 10^{-}5$	0.004	62.95%	0.003
γ^*	0.030	3.64%	$-2.8 \times 10^{-}2$	0.017	93.41%	0.005
δ^*	0.007	3.76%	$-6.7 \times 10^{-}3$	0.004	93.79%	0.001

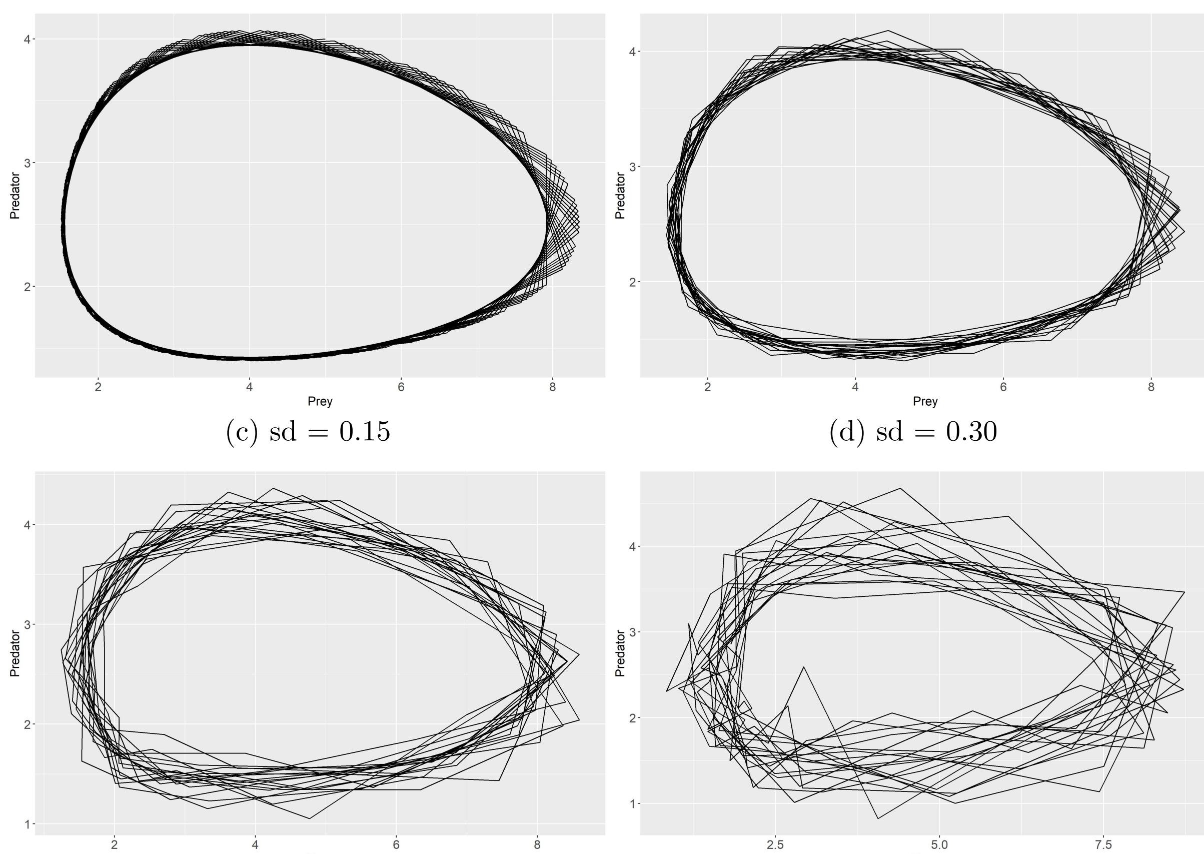
10,000 Trials, random additive noise with a standard deviation of 0.15.

Adding noise solely onto the predator population, y, reduces the precision of the entire model by preserving the endogeneity that is induced by including y on both the left and right hand side of the linear model. Including y on the right hand side and $\log(y)$ on the left hand side of the linear equation induces endogeneity because, when y is acting as a predictor for a transformed version of itself, it will inevitably be correlated with the model's error term. Therefore, we expect reduced precision in the model overall because endogeneity increases the standard errors of the coefficients, leading to larger confidence intervals. Not including error on x preserves the amount of the dependent variable that x and $\log(x)$ can explain, thus not significantly misspecifying their coefficient estimates.

Finally, we can add noise onto both x and y, which is the strategy I use for the full simulation. This is the preferable path forward for two reasons. First, it preserves the endogeneity and multicollinearity built into the model by including y on both sides of the equation as well as including x in two ways on the right hand side of the equation. Therefore, we remain faithful to the structure of the linear model as it's written. Second, theoretically, including error in both x and y aligns with the fact that social science time series data is rarely perfectly measured, especially when the variables of interest only tacitly measure the phenomena. While this is the most faithful way of including error into the linear model, it does pose challenges. Because of the aforementioned endogeneity and multicollinearity concerns, we will expect to observe both bias and reduced coverage in the simulation results. This is an important focal point and implies one of two hypotheses about the quality of the estimation strategy. First, the estimator itself could be flawed: inducing endogeneity by rearranging the implicit relationship with $\log(y)$ on the left hand side of the linear equation is, in fact, a poor way to approach model estimation. Second, it could also suggest that adding noise post-hoc into x and y is a poor way to think about error in the model. In other words, it might make more sense to introduce error into the creation of x and y instead of adding it after the fact. In other words, the way we think about noise in this model is important and isn't necessarily straightforward.

Figure 2.2: The Effect of Different Levels of Random Noise

(a) No Noise (b) sd = 0.05



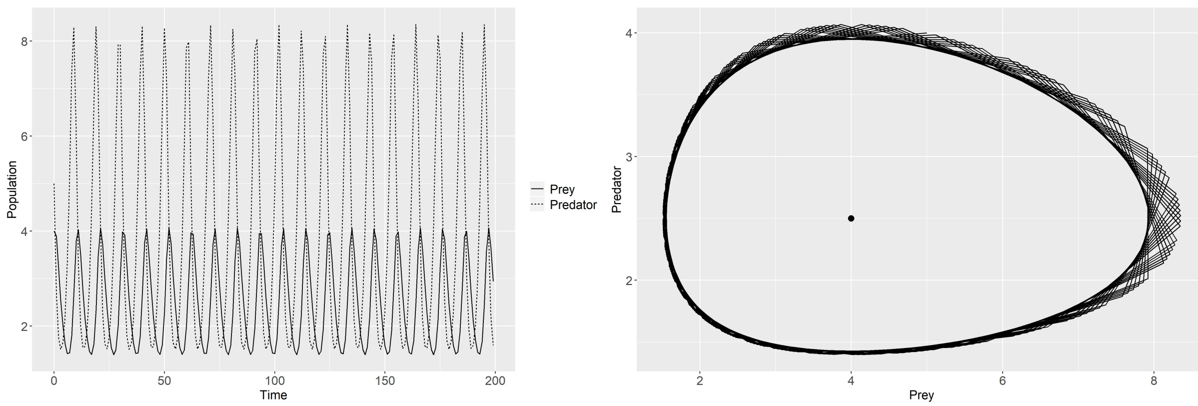
(c) sd = 0.15 (d) sd = 0.30

The final determination we must make with regard to noise before proceeding is how much noise to introduce into the system. Because the estimator is based on the implicit relationship between x and y in the phase-space, in order to visualize the noise, Figure 2.2 plots the phase-space plots of x versus y with no noise and then at three different levels of random noise. This visualization is important to contextualize how noise impacts the relationship between predator and prey and reflects the noisiness of real world data. For the purposes of this simulation, I have chosen to generate noise from a multivariate random normal distribution with a standard deviation of 0.05 and 0.15.

In order to demonstrate the efficacy and ease of this estimation strategy, I conduct four independent simulations at different levels of noise and with different numbers of covariates. The simulations are carried out in several steps: data generation, simulation, and output. Each simulation is run, in full, four times: once each for number of external variables included in the simulation (either zero or one) and, within those, once for each level of noise. I will begin by discussing the simulation that includes zero covariates.

The simulation data were generated in several steps. In the case of the zero-covariate simulation, first, I assign fixed values for the four LV parameters: $\alpha = 1, \beta^* = 0.4, \gamma^* = 0.4$, and $\delta^* = 0.1$. The simulation's goal is to generate parameters that, as accurately as possible, produce the output data from the 'ode' command in the 'deSolve' package in R. The 'ode' command takes several inputs: starting values of the parameters, initial state (population at $t = 0$), and number of time steps over which to estimate. The 'ode' command is a built-in solver for ordinary differential equations. I utilize the 'lsode' solver built into the package (Soetaert, Petzoldt, and Setzer 2010). In cases of abnormal parameters or extinction, the Lotka-Volterra system of equations is stiff. The default solver for the 'ode' function is the 'lsoda' function which makes a choice to evaluate the system as stiff or non-stiff. However, to have greater control over the solver, I chose to utilize 'lsode,' which allows users to specify whether the system is stiff or not (Soetaert, Petzoldt, and Setzer 2010).

Figure 2.3: Lotka-Volterra Output, Zero External Variables



(a) Estimated Population Level

(b) Phase-Space Plot with Steady State

The actual Lotka-Volterra equations are built into a custom Lotka-Volterra model function, ‘LotVmod.’ The initial state is equal to $(x = 5, y = 4)$, time is equal to 200 steps, and *Pars* variable includes the values of the four Lotka-Volterra parameters. The function returns 200 observations for the populations of x and y. Figure 2.3a displays the output of the data generation process with zero covariates and Figure 2.3b displays the phase-space plot of the simulated data along with the steady state at $\{y = 2.5, x = 4.0\}$

After data generation, the actual simulation begins.[7]. For each of 100,000 trials, the simulation adds random noise into both x and y consistent with the above discussion. I then run the linear model and extract standard errors and coefficient estimates to be used later for the calculation of fit statistics including root mean squared error (RMSE), coverage, and bias. The simulation ends by producing an output report including the the RMSE and the percent coverage. Those results, for both levels of noise, are available in Table 2.2. Notice that, even for very small amounts of noise, the estimator does not achieve 95% coverage, further highlighting the issues of endogeneity identified previously.

The final simulation includes a single, exogenous dummy covariate. The data for the single covariate simulation were generated in several different steps and is unique from

[7] See Appendix A for the annotated R code of the zero-covariate simulation

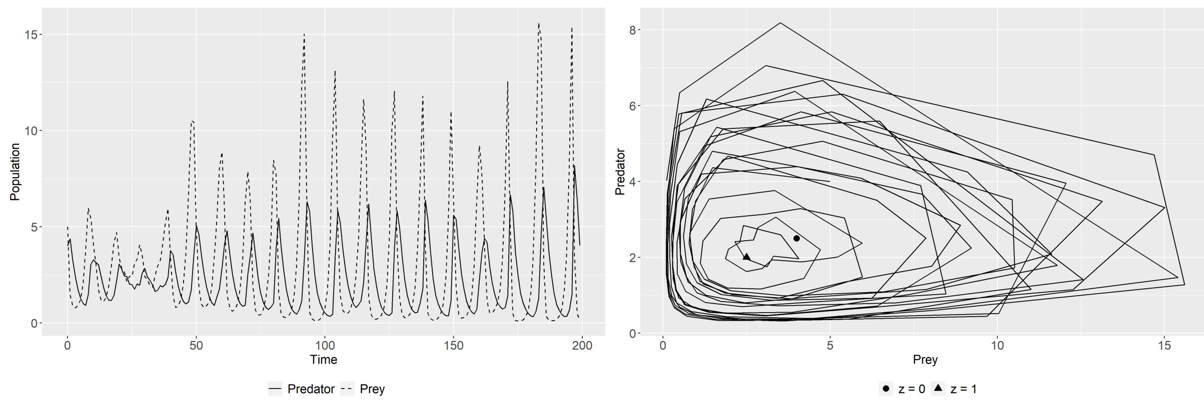
Table 2.2: Simulation Results, No External Variables

	(1) sd = 0.05			(2) sd = 0.15		
	RMSE	Coverage	Bias	RMSE	Coverage	Bias
β^*	0.001	90.47%	0.0003	0.004	68.71%	0.003
γ^*	0.006	90.78%	−0.003	0.029	67.79%	−0.023
δ^*	0.002	92.34%	−0.001	0.007	72.92%	−0.005

the zero-covariate simulation in various ways. First, in order to include an exogenous variable into the simulation, I adjust the equation to include that external variable as an interaction term in the calculation of the three non-fixed parameters. Including the exogenous variable as an interaction term makes each non-fixed parameter a function of the external variable. In other words, non-fixed parameters are a function of changes in the external variable (Stock, Watson, et al. 2007). Therefore, the three non-fixed coefficients reflect the effects of both intrinsic fluctuations in population and variations in exogenous covariate. In order to pass the exogenous variable through the regression, we also must include a categorical indicator, μ, that serves to identify 'eras', or periods where the value of z_t switches from 0 to 1. Although it is not necessarily required to subtract out the interaction term, I have chosen to do so because, when it is included, it negates one of the era dummies due to perfect collinearity. It is important to note that, the more eras that are included in dummy variable μ, the more degrees of freedom are consumed. This will become especially important when applying this method to real-world data, particularly if that data has a small sample size and will be addressed more in-depth in following chapters. The linear model associated with the single-covariate simulation is available in Equation 2.9.

$$\alpha \log(y) = \hat{\beta} y + \theta_\beta yz + \hat{\delta} x + \theta_\delta xz - \hat{\gamma} \log(x) + \theta_\gamma \log(x) z - z + \mu - C^* \tag{2.9}$$

Figure 2.4: Lotka-Volterra Output, One External Variable



(a) Estimated Population Level (b) Phase-Space Plot with Steady States

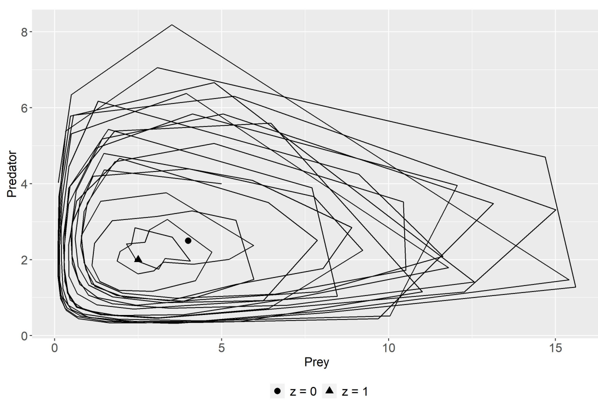
The estimation procedure begins by assigning known values to the main effects coefficients on the parameters ($\alpha = 1, \hat{\beta} = 0.4, \hat{\gamma} = 0.4$, and $\hat{\delta} = 0.2$) as well as the coefficients on the interaction term ($\theta_\beta = 0.1$, $\theta_\gamma = 0.1$, and $\theta_\delta = 0.1$). Next, I generated a random dummy variable, z_t, to serve as the external variable. I then generated the simulation data by running the 'ode' command within a loop that operated one time-step at a time. Each iteration of the loop used the output from the previous iteration as the starting state value to be fed into the ODE solver. This allows the corresponding value of the exogenous variable to go into the calculation of the equivalent population value at each time step. The state and time variables remain constant between the zero and one covariate simulations. Figure 2.4a displays the output of the data generation process with one covariate and Figure 2.4b shows the phase-space plot of that data.

Note that in Figure 2.4b, the phase-space plot is significantly less elegant in appearance than that of the zero-covariate simulation. This is because, when we incorporate an exogenous covariate, the steady state fluctuates and the phase-space becomes more complicated. When populations shift as the result of a change in z_t, the steady state shifts as well: when $z_t = 0$, the steady state is at $\{y = 2.5, x = 4.0\}$ and when $z_t = 1$, the steady state is at $\{y = 2, x = 2.5\}$.

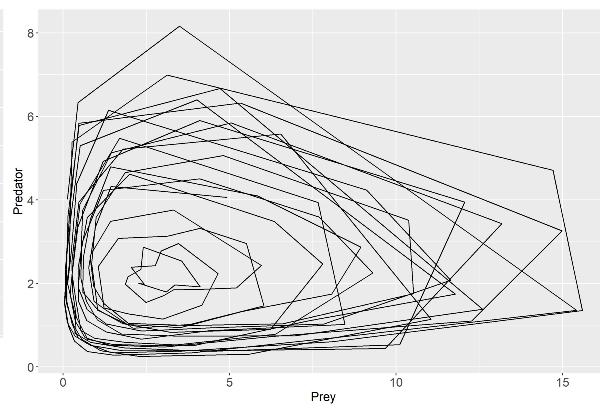
I again test the estimator across 100,000 trials and, like in the zero-covariate simulation, random noise is added in to the deterministic values of x and y at the beginning of each trial. Notice that, while it is still evident that the phase plots are getting noisier, the change between levels of noise is not as dramatic as in the zero-covariate simulation.

Figure 2.5: The Effect of Different Levels of Random Noise

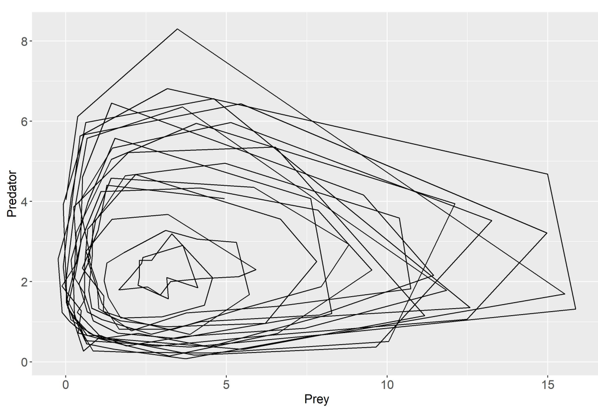
(a) No Noise



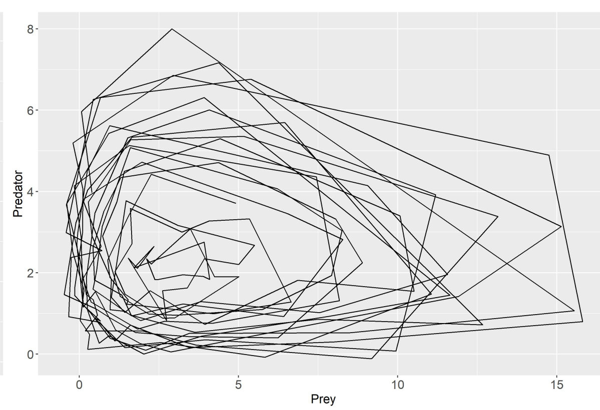
(b) sd = 0.05



(c) sd = 0.15



(d) sd = 0.30



Each trial then calculates the linear model and extracts the standard errors and RMSE. The results for the single covariate simulation, for both levels of noise, are available in Table 2.3. Notice, again, that we have coverage below 95% as well as bias in the estimates. Both are the direct result of the endogeneity built into the model.

Table 2.3: Simulation Results, One External Variable

	(1) sd = 0.05			(2) sd = 0.15		
	RMSE	Coverage	Bias	RMSE	Coverage	Bias
$\hat{\beta}$	0.006	88.20%	−0.001	0.018	79.55%	−0.008
$\hat{\gamma}$	0.012	89.15%	0.196	0.039	80.22%	−0.026
$\hat{\delta}$	0.004	87.39%	−0.201	0.014	70.26%	−0.010
θ_β	0.014	81.12%	−0.008	0.066	35.18%	−0.057
θ_γ	0.035	82.31%	−0.015	0.148	48.98%	−0.123
θ_δ	0.013	78.72%	−0.007	0.059	37.26%	−0.051

2.8 Discussion

This new estimation strategy is a new, substantively interesting, easy, and effective way to estimate Lotka-Volterra models that will advance political science methodology in several ways. First, and most importantly, this new estimation method is user-friendly and familiar in its execution: an interested researcher simply runs the linear model the same as any other regression, directly interpreting the regression coefficients as the parameter values that correspond to the four Lotka-Volterra parameters. However, there are two major issues with the estimator in its current form. First, the manner of including noise into the estimation strategy is only one way of introducing uncertainty into the model. This, then, leads to two clear paths forward for this project. While it is intuitive to political scientists to include noise on the data itself, we can also conceptualize uncertainty in terms of the four Lotka-Volterra parameters and also represents one major way forward for this project. We could also explore the use of stochastic differential equations to introduce noise in to the DGP of the predator and prey populations.

The most glaring problem in this model is caused by the endogeneity induced by including $\log(y)$ on the left hand side of the linear model and y on the right hand side. This leads to biased parameter estimates and poor precision from inflated standard errors. In order to address this concern, future work might explore ways to address the endogeneity. Until that point, it is important to keep in mind that the parameter estimates obtained from this model applied to real-world data are likely incorrect, to some degree. Finally, it would be valuable both theoretically and empirically to assess estimators beyond OLS such as non-linear least squares. This would be a beneficial next step, especially when we can reasonably assume that the relationship between predator and prey is not linear.

Despite these issues in the estimation strategy, LV as an approach is beneficial for describing dynamic processes in political science. Endogeneity caused by feedback loops is often found in time series data and may presents challenges to estimating more traditional time series models. Not only does endogeneity introduce bias into linear models, but it violates the assumption of the exogeneity of independent variables that underlies regression analysis. However, Lotka-Volterra is a useful, theoretically robust alternative that directly models endogeneity and highly interdependent variables. Further, feedback loops are not a concern within the Lotka-Volterra paradigm because this method directly models them. The Lotka-Volterra system of equations conceptually approaches two variables as not only interdependent, but directly responsible for changes in one another's population. In other words, it is an explicit model of feedback. The model's use of differential equations allows the researcher to directly model how two or more interdependent variables change over time and in relation to one another, thus providing much needed flexibility in some of the intractable assumptions underlying other methods of time series analysis. While Lotka-Volterra does not replace any current methods of time series analysis, it does offer a new perspective and can broaden our understanding of complex, competitive data sets.

Finally, this method generates previously unresearched, but substantively interesting, values of interest. Extinction, fixed points, and futures are all of substantive interest to researchers in the social sciences. The steady state is of particular interest when applied to competitive political science time series data. If we observe a relationship moving from volatility towards stability, the underlying mechanism may be a Lotka-Volterra process where the predator and prey are moving towards the steady state. This is useful for prediction by estimating what types of stimuli might interrupt volatile processes and begin shifting the relationship towards stability. For example, alluding to the analysis in the upcoming chapters, this method might allow researchers to more accurately forecast how terrorist cells are most likely to respond to the implementation of critical legislation, changes in international trade, or military decisions by the governments actively fighting against them. Lotka-Volterra, then, is a new addition to the time series analysis toolkit. I intend to demonstrate the assertion that the estimation strategy developed earlier in this chapter is a useful and user-friendly way of advancing the understanding of several literatures within the social sciences. Examples might include literatures that deal with interdependent variables such as studies of domestic terrorism, voting behavior, and international trade.

2.9 Conclusion

In this chapter, I introduced a new estimation strategy for the Lotka-Volterra set of ordinary differential equations that can be used as an alternative method of analyzing time series data. Despite previous interest in this specific type of modeling, interest has dwindled over the last thirty years, likely due to the difficult and cumbersome estimation strategies previously indicated that made this approach difficult to apply. I have demonstrated, through the use of four simulations, that estimating these models can be a simple and effective process.

The remaining two chapters introduce and explore the role that the Generalized Lotka-Volterra method can play in expanding the conclusions of two different articles, one in American politics and the other in transnational terrorism. Each of the two articles employs one of the two main time series analysis methods discussed earlier in this chapter. Ura (2014) utilizes an ECM to assess the relationship between the Supreme Court of the US and public opinion while Jaeger and Paserman (2008) utilize a VAR model to characterize and analyze the Israeli-Palestinian relationship during the Second Intifada. I intend to build upon the conclusions drawn by these scholars by using the Lotka-Volterra approach to answer new questions. The results of these two case studies will aim to demonstrate the model's utility to analyzing real-world political science time series data.

Chapter 3

Case Study 1: Public Mood and Supreme Court Decision-making

There is an academically established and significant relationship between public opinion and Supreme Court decision-making in the United States. However, there is an ongoing debate among scholars about the underlying mechanism driving the relationship. The field is generally split between those who advocate for a thermostatic model of public opinion and those advocating legitimation theory. Ura (2014) was the first to make an attempt to settle the debate between these two theories by empirically assessing the competing predictions of the two models. In order to achieve this end, Ura utilizes Error Correction Models (ECM) because they measure both short- and long-term effects for each independent variable in the model (Ura 2014a, pg. 111). Ultimately, Ura found evidence to support thermostatic behavior in the short-term and legitimation behavior over long periods of time. Ura's work is often cited in ongoing research on Supreme Court legitimacy, decision-making, and the relationship between SCOTUS' rulings and public opinion.

I chose this particular study because it is a conceptually good fit for Lotka-Volterra (LV) modeling. As discussed in Chapter 2, the most important criteria in determining whether or not LV is a good fit for a particular relationship is identifying whether the two populations of interest are, in fact, in direct competition and whether they directly influence the growth and decline of one another. Because, in this case, we are discussing composite indices, it does not make sense to talk about the goals of one population versus the other. However, it does make sense to think about public mood and Supreme Court behavior as a chicken-and-egg problem: we know public mood is responsive to the behavior of the government, but the government is also responsive to public opinion. If this is true, LV is a theoretically appropriate model choice. Finally, as discussed in Chapter 2, a Granger test on the direction of causality in the data rejects the null hypothesis and indicates bidirectional causality.

In this chapter, I briefly introduce the debate within the literature surrounding the characterization of the relationship between public mood and SCOTUS decision making. I follow this with an exploration of Ura's methods and conclusions followed by an introduction of the Lotka-Volterra method as an alternative approach to analyzing his data. To be clear, this new analysis does not seek to disprove nor critique Ura's methods. Instead, it serves as an extension of Ura's work by attempting to draw new conclusions and to answer new questions about the competitive relationship between Supreme Court decision making and fluctuations in public opinion. I find that LV is a robust way of modeling the dynamic relationship between Supreme Court behavior and public mood; however, due the issues with estimation strategy identified in Chapter 2, largely endogeneity, the parameter estimates in this chapter are biased.

3.1 Background

In 2014, there existed strong evidence and a general consensus in favor of the thermostatic model (Erikson, MacKuen, and Stimson 2002).[1] This model predicts two key behaviors. First, that the public's preferences regarding policy change fluctuate to some measurable degree as the result of ongoing changes in the public policy environment. In other words, the population is sensitive to the public policy environment and adapts their opinions in response to fluctuations in the policy environment. According to the thermostatic model, this reciprocal relationship between public opinion and decision-making by all branches of the Federal government is the driving force underlying fluctuations in public

[1] Also see Flemming, Bohte and Wood (1997) and Stimson (1999).

opinion (Wlezien 1995; Wlezien 1996). Therefore, the thermostatic model is responsible for explaining at least a portion of the national government's responsiveness to changes in public opinion (Erikson, MacKuen, and Stimson 2002).

Second, the thermostatic model predicts a negative relationship between the ideological direction of Supreme Court decisions and fluctuations in public opinion (Ura 2014a, p. 110). More concretely, the relationship can be described as cyclical: as Supreme Court opinions become more liberal, the public's desire for more liberal policies becomes increasingly sated, peaks, and eventually reverses and results in an increasingly conservative public mood (Ura 2014a, p. 110).

Legitimation theory directly contradicts the thermostatic model in its prediction of the direction of causality between public mood and SCOTUS behavior. First suggested in 1957, legitimation theory largely emerged in direct response to the thermostatic model. The latter is traditionally applied to elected branches of government; however, public mood behaves differently in response to the actions of political appointees as opposed to the actions of elected officials (Dahl 1957). Legitimation theory proceeds under two assumptions that are in direct contrast with the thermostatic model. First, the close association of the Court to the Constitution and other powerful symbols of institutional legitimacy and justice is strong enough to attract public attitudes towards decisions made by the Supreme Court (Dahl 1957). In the broader scheme, this association with these symbols has allowed the Court to retain its legitimacy among the public even when the Court's decisions are unpopular in the eyes of the public (Gibson and Nelson 2015, p. 173).

Second, legitimation theory predicts a positive relationship between the ideological leanings of SCOTUS' decisions and fluctuations in public opinion (Ura 2014a, p. 110). Supreme Court decisions are intrinsically persuasive and, as a result, shape public attitudes even on extremely polarizing and controversial issues such as abortion (Hoekstra 2003, p. 90). Over time, the effects of SCOTUS decisions accumulate, drawing public opinion closer in line with the Supreme Court. Especially when Court decisions are reflective of common trends in public opinion across issues, this cumulative effect amounts to a change in public mood to more closely align with SCOTUS' policy positions (Martin and Quinn 2002; McGuire and Stimson 2004).

In order to begin the process of settling the debate between these two theories of public opinion responsiveness to SCOTUS decision-making, Ura (2014) employs a single-equation, bivariate error correction model (ECM) to analyze public mood between 1956 and 2009. This ECM, developed by Bardsen (1989) takes the form of Equation 3.1:

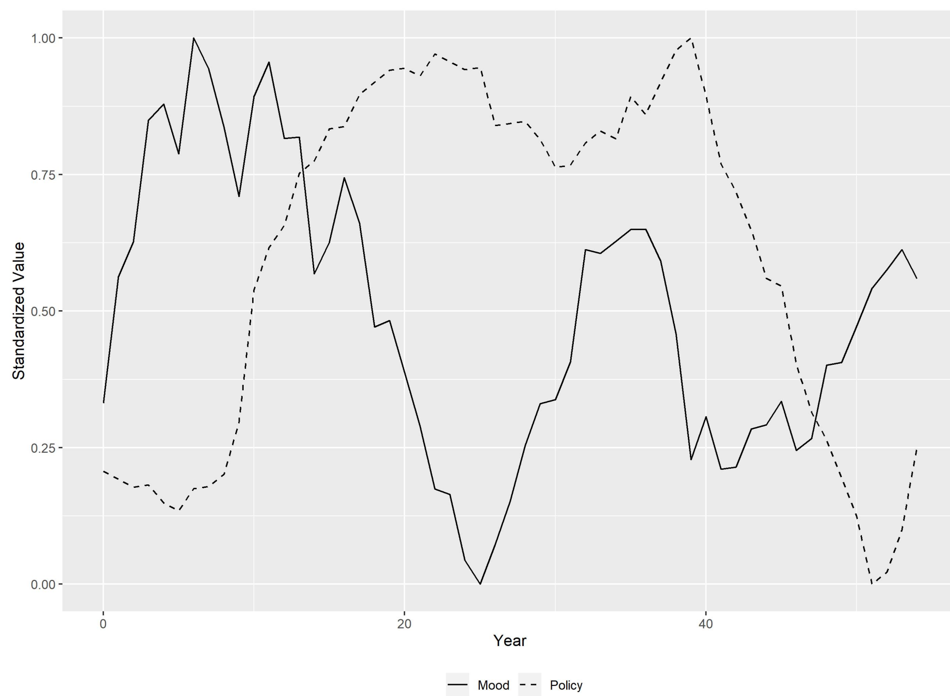
$$\Delta Y_t = \alpha_0 + \alpha_1^* Y_{t-1} + \beta_1^* \Delta X_t + \beta_2^* X_{t-1} + \epsilon_t \tag{3.1}$$

"where α_1 indicates the speed of the re-equilibration of Y to a deviation from its equilibrium with X, β_2 reflects the long-run effect of changes in X on Y, and β_1 indicates the contemporaneous relationship between a change in X and and a change in Y" (Ura 2014a, p. 116).

Ura's justification for the decision to utilize an ECM over an autoregressive distributed lag model is four-fold. First, ECMs indicate both the direction and the magnitude of each independent variable on shifts in the public's mood. Second, error correction models serve to measure the temporal dynamics of predictive relationships (Ura 2014a, p. 116). Third, ECMs implemented via OLS have proven successful in identifying underlying data generation processes even when sample sizes are small (Keele and DeBoef 2008). Finally, Ura is speaking to the debate that ECMs, while traditionally applied to cointegrated time series data, may be effectively utilized to model data with no cointegration, regardless of the data's stationarity (Keele and DeBoef 2008).

Ura's data consists of the dependent variable, public mood, as well as four indicators he theorizes contribute to changes in public mood. The mood variable is an index that measures the liberalism of public opinion that is drawn directly from Stimson's (2009) annual mood index. Figure 3.1 displays the two key variables of interest, standardized between 0 and 1 in order to more clearly show the relationship at each period of time.

Figure 3.1: Standardized Time Series Plot of the Two Variables of Interest



The first two independent variables, inflation and unemployment, are very straightforward measurements, both derived from Bureau of Labor and Statistics reports. Inflation "is the change in the Consumer Price Index (January to December) in each year" (Ura 2014a, p. 116). Unemployment is the "average annual rate of unemployment" (Ura 2014a, p. 116).

The remaining two independent variables are more complex in structure. The third independent variable, the public policy index, begins with Mayhew's (1991, 2011) list of "major or important pieces of legislation passed in each year (selected based on media coverage of Congress" (Ura 2014a, p. 116). Each law is coded as either liberal or conservative, and the net number of laws that are coded as liberal serve as an annual indicator of the policy liberalism created by Congress. Once Ura determined that value, he utilized it alongside the President's policy liberalism to calculate the final index. The final index scores "...each year's policy outputs as the difference between its value and the mean of the annual series an then taking the sum of the resulting series at each point in time" (Ura 2014a, p. 116).

The final independent variable, named the 'caselaw index,' is an aggregate index of the ideological content of SCOTUS' decisions (Ura 2014a, p. 116). Mirroring the policy index, Ura (2014) identifies important Supreme Court cases, defined by Epstein and Segal (2000) as those whose decisions are printed on the front page of the *New York Times*. Ura proceeds to count the total number of important decisions per year. Finally, Ura "construct[s] a cumulative measure of liberalism in the Supreme Court's decisions by rescaling the net number of liberal decisions in each period as its deviation from the mean value of the annual Supreme Court liberalism series and taking the sum of the series at each point in time" (Ura 2014a, p. 116). Table 3.1 displays the basic descriptive statistics of each variable in the model.

Table 3.1: Descriptive Statistics

Statistic	N	Mean	St. Dev.	Min	Pctl(25)	Pctl(75)	Max
Mood	55	58.3	4.2	50.0	54.9	60.9	66.7
Policy Index	55	19.2	17.7	−13.6	−1.2	34.1	40.8
Unemployment	55	5.8	1.4	3.5	4.8	6.8	9.7
Inflation	55	3.9	2.9	0.1	1.9	4.5	13.3
Caselaw Index	55	43.8	36.8	−1.4	6.8	74.0	114.5

In order to reflect the theoretical claims of the thermostatic model, the caselaw index and policy index are cumulative measures of policymaking. The thermostatic model, which predicts that the strength of the public's desire for changes in policy liberalism "is, in part, a function of the total, issue-by-issue divergence between public policy and the public's preferences" (Ura 2014a, p. 116). Legitimation theory, on the other hand, "predicts that the Supreme Court may support or undermine policies by validating or vetoing them in its decisions" (Ura 2014a, p. 116). Ultimately, the justification for utilizing composite indices for these two variables is the idea that, in both theories, the total liberalism or conservatism of SCOTUS' rulings is what should ultimately act on mood (Ura 2014a, p. 116).

Ura performed his analysis in Stata; however, because my own analysis is conducted in R, I precisely replicated the results presented in Table 1 (Ura 2014a, p. 118). In a minimal amount of code, I used the 'ecm' R package to estimate (Bansal 2019). The 'ecm' package uses a modified version of the traditional ECM:

$$\Delta Y = \beta_0 + \beta_1 \Delta x_{1,t} + ... + \beta_i \Delta x_{i,t} + \gamma y_{t-1} + \gamma_1 x_{1,t-1} + ... + \gamma_i x_{i,t-1}, \tag{3.2}$$

$$where \gamma_i = -\gamma \alpha_i$$

This modification is necessary in order to model the ECM using the OLS function in base R (Bansal 2019). Therefore, this version of the equation falls perfectly in line with Ura's own analysis, which is implemented by "estimating an OLS model of the first difference of mood expressed as a function of the first lag of mood (error correction) as well as the first difference (short-run effect) and first lag (long-run effect) of the caselaw index (Supreme Court liberalism), policy (congressional liberalism), inflation, and unemployment" (Ura 2014a, p.117). A reproduction of the majority of Ura's results are available in Table 3.2 side-by-side the results produced by running the same model in R.

One important potential caveat that Ura (2014) notes is a legitimate concern about the endogeneity of Supreme Court decision-making and public mood. In order to account for that endogeneity, Ura reestimates the ECM using instrumental variable (IV) regression. Because this model produced "comparable estimates of the short-run and long-run effects of Supreme Court liberalism for public mood" (Ura 2014a, p. 118). These estimates are indicative of endogeneity having no substantial effect on Ura's conclusions drawn from the OLS solution to the ECM. Consequently, I have omitted those results from Table 3.2.[2]

[2]See Ura (2014m p. 118) for the full table.

Table 3.2: Error Correction Model of Annual Mood (1956-2009): Replication Results

Predictors (Expected Sign)	Effects (Ura 2014)†	Effects (Replication)
Long-Run Effects		
Caselaw Index$_{t-1}$ (+/−)	0.02*	0.024*
	(0.01)	(0.011)
Policy Index$_{t-1}$ (−)	-0.07*	-0.073*
	(0.02)	(0.021)
Inflation$_{t-1}$ (−)	-0.29*	-0.291*
	(0.13)	(0.127)
Unemployment$_{t-1}$ (+)	-0.24	-0.238
	(0.19)	(0.19)
Short-Run Effects		
Δ Caselaw Index$_t$ (+/−)	-0.09*	-0.092*
	(0.04)	(0.039)
Δ Policy Index$_t$ (+)	0.07	0.068
	(0.07)	(0.072)
Δ Inflation$_t$ (−)	-0.30*	-0.305*
	(0.13)	(0.129)
Δ Unemployment$_{t-1}$ (+)	-0.23‡	-0.321
	(0.19)	(0.269)
Error Correction and Constant		
Error Correction (Mood$_{t-1}$)	-0.28*	-0.284*
	(0.08)	(0.078)
Constant	19.49* *	19.492*
	(5.14)	(5.142)
R^2	0.42	0.42

Note: Standard errors in parentheses. $N = 54$

† Table replicated from Table 1 on pg. 118 of Ura (2014).

‡ The results table in (Ura 2014a, p. 118) list the effect of Δ Unemployment$_{t-1}$ (+) as -0.23 (0.19); however, rerunning Ura's original code in Stata estimates the effect to be equal to -0.32 (0.269), thus matching the replication results in R.

The results of Ura's ECMs indicate the presence of both short- and long-term relationships in the data. In the short term, the negative coefficient on the relationship between SCOTUS liberalism and public mood supports the thermostatic model. However, the positive long-term effect lends credence to the legitimization response hypothesis. In other words, the relationship between the Supreme Court and the public mood is characterized by a complex interaction. Although there is likely to be public backlash immediately following SCOTUS decisions, in the long-run, the public mood begins to shift closer toward the ideological position of the court (Ura 2014a, p. 118).

Ura's (2014) work has been integral in informing several research areas. First, Ura informs the literature on the legitimacy of the Supreme Court, specifically in regards to how public opinion reacts to unpopular Court decisions (Gibson and Nelson 2015). Generally speaking, SCOTUS retains enough legitimacy among the public that, despite public dissatisfaction with any single ruling, its legitimacy does not waver (Gibson and Nelson 2016).

Second, Ura has advanced research on how the Court articulates and hands down rulings and opinions (Black, Owens, Wedeking, and Wohlfarth 2016b). Specifically, this area of research focuses on how Supreme Court Justices alter the clarity of their written opinions in response to the anticipated reaction of public opinion (Black, Owens, Wedeking, and Wohlfarth 2016b). The argument that Justices write clearer, more concise, and more accessible opinions when their rulings conflict with popular public sentiment supports the idea that public opinion does, in fact, directly influence the Court (Black, Owens, Wedeking, and Wohlfarth 2016a).

Finally, Ura (2014) has contributed to research on the complex relationship between SCOTUS and public opinion. Bryan and Kromphardt (2016) cite Ura's work in arguing that individual SCOTUS Justices are uniquely sensitive to public opinion, especially when public support of SCOTUS is low and/or when the salience of any given case is high. More broadly, the overall public's view of the Court's legitimacy does not necessarily decline in response to individual opinions that are contrary to the interests of individuals (Badas 2016).

To continue to move the debate forward, I suggest the use of Lotka-Volterra as a means of answering new, interesting questions about the dynamic relationship between SCOTUS and public mood. To be clear, I am not suggesting LV as a means of improving upon Ura's methods, but as a way of generating several new quantities of interest. First, Lotka-

Volterra can identify the steady-state of the Supreme Court - public opinion relationship. In other words, Lotka-Volterra offers another way to measure under what conditions the two variables balance each other out and exist in more or less stable equilibrium. This is conceptually beneficial because it allows us to explore the environmental conditions under which there is relative calm between public mood and Supreme Court behavior.

On the other side of the same coin, LV is poised to estimate the conditions under which SCOTUS is a driver of public opinion or vice versa, and whether a positive or negative relationship exists between the two. This is simply another way to speak to the debate between the thermostatic model and legitimation theory. By approaching the debate from a different angle, I suggest we might be able to provide further evidence in favor of Ura's conclusions. This quality of LV makes it a useful forecasting tool by allowing researchers to evaluate what may have occurred as the result of different decisions in the past and how fluctuations in decision making in the future relate to public opinion.

Finally, LV is another way to address the endogeneity that theoretically exists in Ura's data because Lotka-Volterra is, intrinsically, a model of endogeneity.

3.2 The LV Approach to Analyzing the Supreme Court-Public Mood Relationship

The first step in applying the Lotka-Volterra method to Ura's (2014) data, is to load, arrange, and clean the original data (Ura 2014b). The original data file is in Stata format; therefore, using Stata, I convert the data file into a comma-separated file, which is more easily and accurately loaded into and manipulated in R. In order to demonstrate the efficacy of the LV method using Ura's data, I only include one covariate. While this does present a significant limitation in my conclusions, the goal of this section is two-fold. First, to demonstrate the ease in which this method can be applied to real-world data. Second, I present cursory values of interest that are unique to the Lotka-Volterra method such as steady state. Future iterations of this study will include all three external covariates that Ura includes in his analysis.

In Ura's ECM, the dependent variable is Mood; therefore, in line with the OLS estimation strategy I utilize to estimate the LV parameters, I have assigned Mood to the predator and Ura's independent variable of interest, Policy, to the prey. Because the OLS estimation strategy I employ involves the logistic transformation of both of these populations, both must be greater than 0. Policy, however, has a minimum value of about -14; therefore, I use a simple transformation to uniformly increase the variable's value. This is a valid option because policy is an index and, as long as we preserve the scale of the variable and the relationships between observations, transforming it does not change the variable's underlying meaning.

$$Policy = policy - min(policy) + 1 \tag{3.3}$$

Equation 3.4 is the regression formula used to estimate the parameter values with one external variable.

$$\log(y) = \hat{\beta}y + \theta_{\beta z_1} y z_1 - \hat{\gamma}\log(x) - \theta_{\gamma z_1}\log(x) z_1 + \hat{\delta}x + \theta_{\delta z_1} x z_1 - z_1 + \mu - C^* \tag{3.4}$$

where:

- x and y are the initial starting values for Policy (prey) and Mood (predator), respectively;
- $\hat{\beta}$, $\hat{\gamma}$, and $\hat{\delta}$ are the main effects coefficients on the traditional LV parameters divided by α;
- θ_β, θ_γ, and θ_δ are the coefficients on the independent variable and also divided by α;

- z_{1t} is the value of the the exogenous covariate; and
- μ is a control dummy variable that indicates when the value of z_t changes.

The value of both Mood and Policy at $t = 0$ are assigned as the starting state values for prey and predator, respectively. Finally, I also include one of three exogenous variables used in Ura's analysis: inflation. However, because the estimation method is carried out using OLS and the coefficients on those variables are the quantities of interest, the exogenous variable is treated as a binary dummy coefficient. A binary variable interaction regression model is the appropriate measurement tool in this case because the coefficients on the exogenous variables are interdependent, where the estimated "effect of changing one of the binary independent variables..." influences "...the value of the other binary variable[s]" (Stock, Watson, et al. 2007, p. 278). This estimation strategy also allows me to effectively control the output of the regression and avoid a veritable explosion of coefficient values. To assign binary values, I assign values below each variable's median as equal to 0 and those above equal to 1. Table 3.3 displays a summary of the transformed variables. In the estimation, the variables are treated as factors.

Table 3.3: Summary of Manipulated Covariates

Variable	Median Value	$Nz = 0$	$Nz = 1$
Inflation	3.30	27	28
Caselaw Index	34.33	27	28
Unemployment Rate	5.60	27	28

The inclusion of only 55 observations can be problematic and reduce precision by widening the confidence intervals and offsetting some of the problems with endogeneity. However, this does not mean we get better estimates. Instead, we likely have worse estimates because of such a small N. While sample size is an issue, the most obvious concern with this estimation strategy is the endogeneity introduced by including y on both sides of the regression equation. Endogeneity biases our coefficient estimates and inflates standard errors, reducing the precision of the model. While we can conceive some workarounds or remedies to the issue, these are outside the scope of this particular project. However, in the discussion section below, I touch on some of these paths forward.

Once the data is ready to be applied to the Lotka-Volterra method, I run the linear model and extract and assign the coefficient values that correspond to each parameter. However, it is important to note: the OLS estimator does not provide an estimate for the parameter α. Therefore, if we are interested in prediction or steady state and extinction conditions, we must estimate α. In order to achieve this, I construct an optimization routine that carries out several steps. The routine is fed a random initial value of α. Each iteration of the optimizer divides the parameter estimates obtained from the linear model by the updated value of α, solves the pair of ODE, and then calculates the sum of squared errors (SSE) between the observed data and the predicted values obtained from solving the LV system. The optimizer concludes when it has minimized the SSE. Estimates for the exogenous covariates are available in Table 3.4. The results in Table 3.4 reflect the raw estimates of the coefficients and preserve the directionality estimated by the linear model. However, before inserting the parameters into the Lotka-Volterra equation, we must reverse the signs on $\hat{\gamma}$ and θ_γ. This is because, in the linear model in Equation 3.4, γ and θ_γ, both have negative signs.

It is imperative to note that Model 2 represents a special case because the coefficient on $\hat{\delta}$ is negative and the coefficient on $\hat{\gamma}$ is positive. This violates the most basic assumption of non-negative parameters. A negative $\hat{\delta}$ describes a negative growth rate of the predator. With a negative growth rate and positive death rate, we expect exponential decline of the predator population. Because the populations are constrained $[0, \infty)$, if any of the parameters fall below 0, so do population values, thus further evidence that this ratio of parameters leads to extinction. Substantively, this ratio of parameters suggests an unstable system where, regardless of the population of the prey, the predator will quickly go extinct and the prey will grow exponentially. Within a system with this specification of

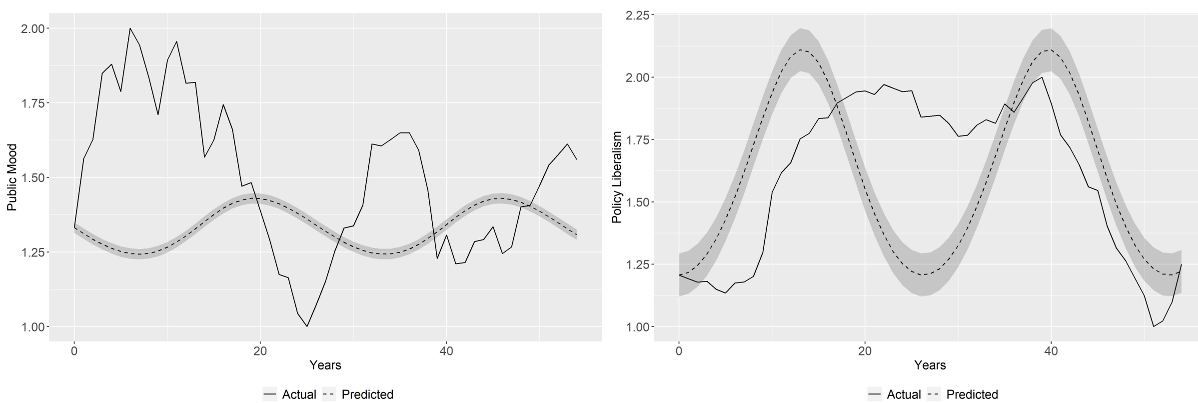
Table 3.4: Estimated L-V Coefficients

	(1)	(2)	(3)	(4)
Coefficient				
α	0.95	0.93	0.92	0.98
$\hat{\beta}$	0.71	0.67	0.70	0.62
	(0.010)	(0.017)	(0.019)	(0.020)
$\theta_{\beta inflation}$	-	0.11	-	-
	-	(0.022)	-	-
$\theta_{\beta caselaw}$	-	-	0.08	-
	-	-	(0.023)	-
$\theta_{\beta unemployment}$	-	-	-	0.14
	-	-	-	(0.024)
$\hat{\gamma}$	-0.06	0.48	-0.22	-0.23
	(0.151)	(0.377)	(0.179)	(0.273)
$\theta_{\gamma inflation}$	-	0.35	-	-
	-	(0.869)	-	-
$\theta_{\gamma caselaw}$	-	-	-0.04	-
	-	-	(0.400)	-
$\theta_{\gamma unemployment}$	-	-	-	0.05
	-	-	-	(0.772)
$\hat{\delta}$	0.04	-0.38	0.14	0.16
	(0.102)	(0.277)	(0.121)	(0.180)
$\theta_{\delta inflation}$	-	0.32	-	-
	-	(0.511)	-	-
$\theta_{\delta caselaw}$	-	-	0.07	-
	-	-	(0.260)	-
$\theta_{\delta unemployment}$	-	-	-	-0.08
	-	-	-	(0.432)
Steady State $\{y, x\}$				
0 Covariates	$\{1.33, 1.62\}$	-	-	-
1 Covariate				
$z_t = 0$	-	$\{1.38, 1.27\}$	$\{1.30, 1.61\}$	$\{1.58, 1.44\}$
$z_t = 1$	-	$\{1.19, 2.40\}$	$\{1.18, 1.25\}$	$\{1.29, 2.33\}$
Standard Errors in Parentheses				

coefficients, there exists no dependence of one population upon the other. In other words, this particular specification of Lotka-Volterra described above is likely not an appropriate model for this particular configuration of predictors. In the case of Model 2, this might be occurring because inflation and other economic factors have a major impact on public mood that does not, in turn, directly impact Supreme Court behavior. We can observe this behavior in the estimates of the θ coefficients: the estimate of θ_β is significantly lower than the estimates of θ_γ and θ_δ indicating a much stronger effect on the coefficients for public mood than for the Supreme Court.

Figure 3.2: Actual vs. Predicted Values: Zero Covariates

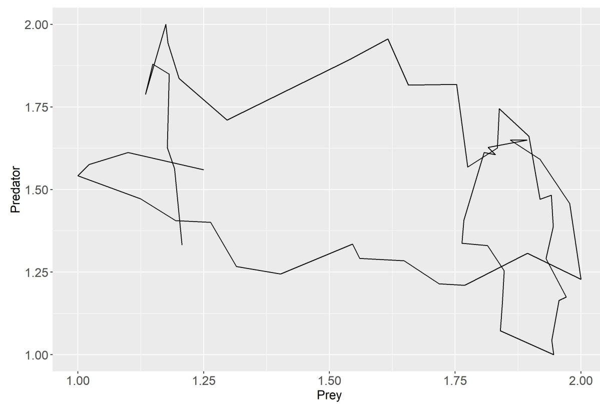
Model 1: Zero Covariates



In order to test the fit of the parameter estimates, I solve the Lotka-Volterra equations using the estimated parameter values and plot the predicted versus actual values of the independent and dependent variables. Figure 3.2 plots the predicted values of Model 1, the zero-covariate model. The predicted results indicate an inaccurate fit. Substantively speaking, this suggests that the system is not self-perpetuating and is driven by one or more additional covariates of interest.

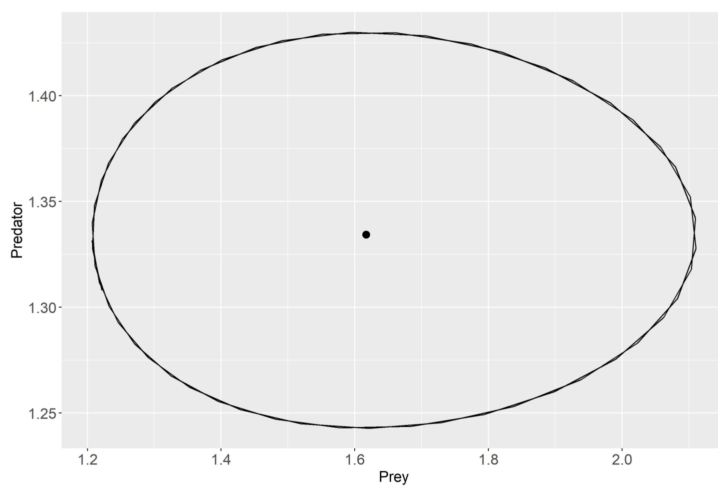
This assertion is further supported by the phase-space plot of the raw data in Figure 3.3. There are two distinct orbits in the data. This phenomenon happens as the result of a shock to or fluctuation in the system (Kinoshita 2013). Therefore, I argue that one or more external covariates are contributing to the relationship between x and y in a meaningful way. Visually, the need for at least one additional covariate is obvious in the

Figure 3.3: Phase-Space Plot of Observed Data



phase-space plot of Model 1 in Figure 3.4. The regular orbit around a single fixed point is not an accurate representation of the observed system.

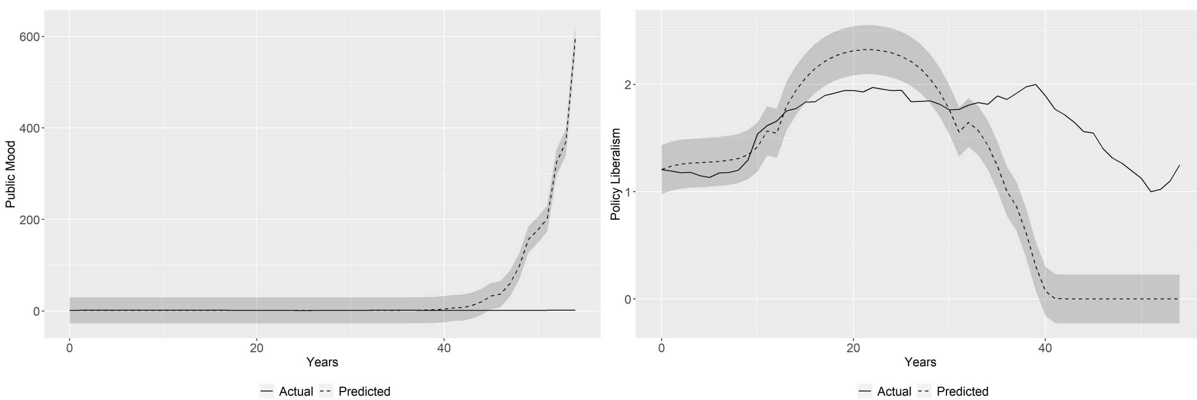
Figure 3.4: Phase Plot of Model 1



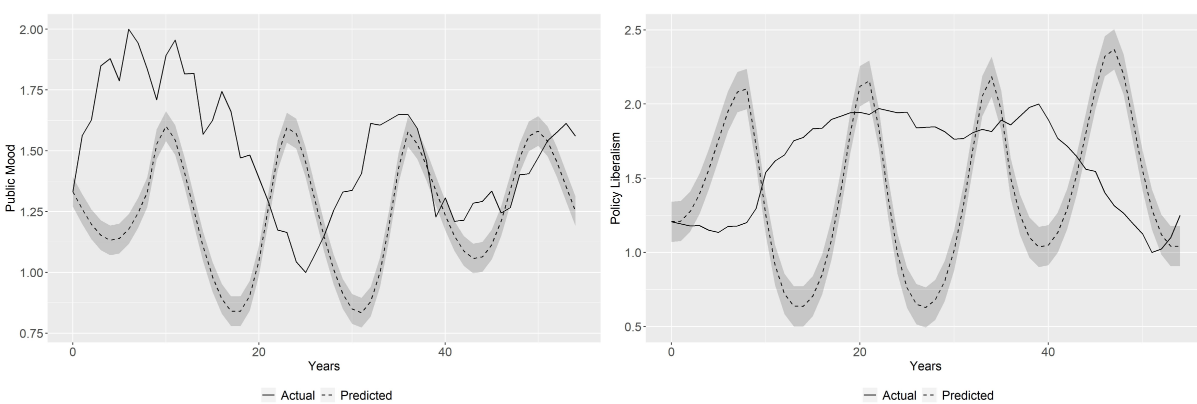
Models 2-4 are efforts at identifying possible causes of the observed phase shift in the observed data and therefore incorporate one of Ura's (2014) original covariates: inflation rate, caselaw index, or unemployment rate. Figure 3.5 displays the plots of the predicted values of Models 2-4 with 95% confidence intervals. The plots of Models 3 and 4 show periodic trends in the predicted results that seem to occasionally oscillate in tandem with their respective known values, the predicted values are often over-estimated. However, the most important observation from these plots is the way in which the exogenous variables shape the predicted values. In Figures 3.2 and 3.2, we have very crude predicted values for both policy and mood, although Figure 3.2 seems to capture the two obvious local maxima, albeit not very accurately.

Figure 3.5: Actual vs. Predicted Values: 1 Covariate

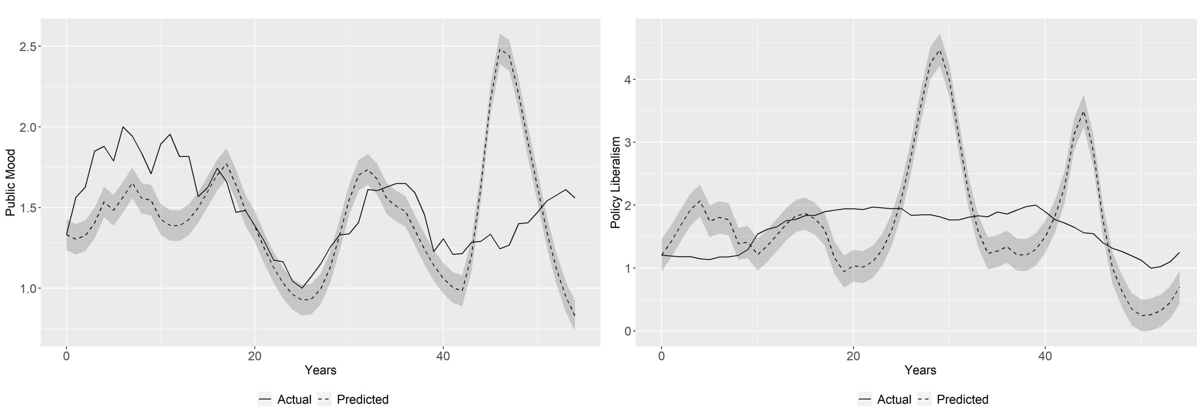
Model 2: Inflation



Model 3: Caselaw Index

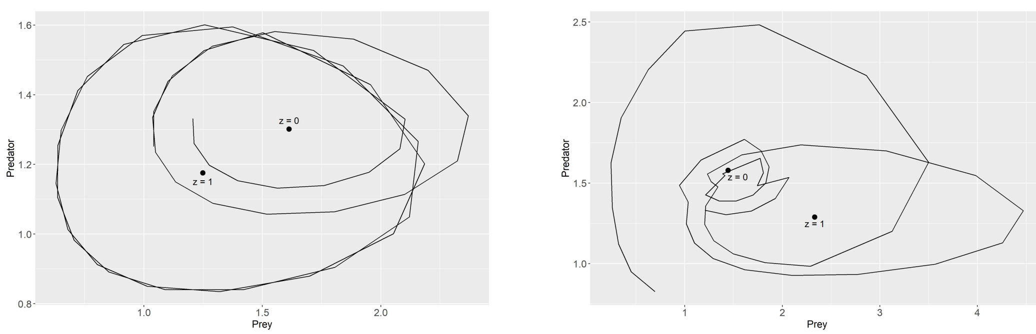


Model 4: Unemployment Rate



These graphs show significant differences in the shape of the predicted values. And, while none of the models predict the actual values particularly well overall, there are trends present in portions of each graph of predicted values that seem to reflect trends in the actual values. For example, from $t = 40$ onward in Figure 3.5, we see a trough that simultaneously appears in the actual data, whereas this behavior is nonexistent in Figure 3.5. In other words, we can get a sense of the ways in which these exogenous covariates interact and shape the observed population data. Substantively, this suggests that omitted variables are important in predicting the relationship between public mood and policy liberalism.

Figure 3.6: Phase-Space Plots with Steady States

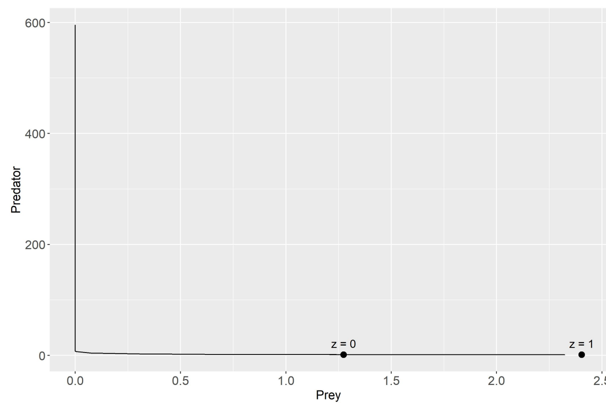


(a) Model 3: Caselaw Index (b) Model 4: Unemployment Rate

The phase-space plots of Models 3 and 4 suggest relative stability in systems that include either caselaw index or unemployment as predictors. Both have two obvious orbits, with the phase plot of Model 3 demonstrating the most stability. Figure 3.6a implies that the caselaw index does not induce any instability in the relationship and does not demand the addition of any other covariates: there are only two orbits and both circle around their respective fixed points with regularity.

The phase-space plot of Model 4 also suggests some measure of stability around the two fixed points; however, there are three distinct orbits. Substantively speaking, this suggests that the relationship between x and y is more sensitive and responsive to shifts in the unemployment rate than to changes in the caselaw index. The orbits in Figure 3.6b imply that frequent or sharp changes in the unemployment rate deal a significant shock to the system, and that the system could potentially lack sustainability in the long-term. This volatility may potentially be corrected by including additional predictor variables in the Lotka-Volterra model.

Figure 3.7: Phase Plot of Model 2



The graphs of the predicted values of Model 2 are interesting and unique for several reasons. When the signs of $\hat{\delta}$ and $\hat{\gamma}$ are reversed, we expect the predator to go extinct and the prey to grow exponentially. However, we observe the opposite behavior in the plots of the predicted values. This suggests two things. First, that the inclusion of the inflation rate forces the system into instability and eventual collapse. This collapse is evident in the phase-space plot of Model 2 in Figure 3.7. This suggests that, on its own, inflation does not contribute to the self-perpetuating nature of the system. Instead, inflation alone forces x and y apart. Substantively, this would suggest that, as inflation increases, policy mood rapidly becomes more liberal and SCOTUS becomes more conservative and they do not come back together. Because we do not observe this of behavior in reality, we can conclude that inflation has a polarizing effect on the two populations that might be mitigated by additional covariates of interest.

3.3 Discussion

The Lotka-Volterra model's simplest purpose is to establish how two populations interact with one another. My extension extends this purpose to also include how exogenous covariates affect the overall system. In utilizing the Lotka-Volterra method to analyze Ura's (2014) data, I find evidence that supports his conclusion that both thermostatic and legitimation theories can (and do) exist within the same dataset and are not mutually exclusive phenomena. In other words, the relationship between public mood and the liberalism of SCOTUS decisions is endogenous and ebbs and flows in a cyclical way. Further, evidence from the phase plots does imply that public mood and SCOTUS policy liberalism exist in relative stasis throughout the study period. This suggests that the relationship is symbiotic, where changes in one inevitably impact the other. This assertion is consistent with a significant portion of the literature that provides evidence of the endogeneity of public mood and Supreme Court decision making. [3]

This symbiotic relationship is fragile when introducing exogenous covariates and the system is extremely sensitive to shocks. Exogenous data included in the estimation of the Lotka-Volterra model significantly shifts the shape of the predicted values over time and alters the steady state(s) of the overall system. The system is so sensitive to exogenous data that the inclusion of inflation as an indicator forced the extinction of an otherwise stable system. That is not to say that inflation is an insignificant covariate. Instead, it implies that inflation, on its own, results in the complete divergence of policy liberalism and public mood. This is likely because inflation directly impacts the individuals whose opinions compose the public mood and does not impact the Supreme Court in the same way, thus forcing the relationship out of relative stasis. Substantively, that would suggest that high rates of inflation are associated with a sharp, upward (liberal) trend in SCOTUS

[3] For further information, see Casillas, Enns, and Wholfarth (2011), McGuire and Stimson (2004) and Stimson, MacKuen, and Erikson (1995).

policy-making and a sharp, downward (conservative) trend in the public mood. While this hypothesis is in line with the results of Ura's error correction model, in order to fully explore this hypothesis within the Lotka-Volterra context, we would need to employ a model with inputs that are not constrained above 0.

The application described in this chapter is limited in several ways, many of which could be addressed in extensions of this project. First, my conclusions are limited by sample size. Ura's (2014) data only covers 55 time points, largely due to limits on sample size in the construction of the mood index before 1956 (Ura 2014a). While that is a long enough study period to generate results in the basic Lotka-Volterra context, it quickly leads to using up degrees of freedom in the estimation of the coefficients when multiple covariates are included in the model and leads to imprecise coefficients.

Second, the real-world relationship between public mood and SCOTUS decision-making does not occur in a vacuum, and this approach to modeling that relationship is limited by the fact that it only incorporates covariates that affect both x and y. An interesting extension of this project would introduce additional complexity into the environment. An example would be conditioning the model on one or more external covariates such as political climate, including a competition coefficient, and introducing a carrying capacity to the environment. These types of complexities would help to remove the relationship from the vacuum and place it into the context of a more realistic environment. Similarly, it would be scholastically interesting to add additional interaction terms to the model in order to see how different combinations of variables affect the dynamics of the relationship. A possible extension of this particular project might incorporate additional covariates by modifying Equation 3.4:

$$\alpha \log(y) = \hat{\beta} y + \theta_{\beta z_{1t}} y z_{1t} + \theta_{\beta z_{2t}} y z_{2t} + \ldots + \hat{\delta} x + \theta_{\delta z_{1t}} x z_{2t} + \theta_{\delta z_{1t}} x z_{2t} - z_{1t} - z_{2t} - z_{1t} z_{2t} + \mu - C^* \tag{3.5}$$

where z_{jt} is a second exogenous variable. Following the convention for constructing Equation 3.5, it is possible to add any number of exogenous covariates to the estimator.

Finally, as discussed in Chapter 2, the model loses a significant amount of predictive power because we are forced to transform the exogenous covariates into binary dummy variables for ease of computation, readability, and interpretation. Because these variables are generalized in such a crude way, we lose significant amounts of information and are unable to extract the most accurate coefficient estimates. Future iterations of this project might focus on an elegant way of incorporating continuous exogenous variables into the system.

Despite these limitations, this type of analysis can inform the broader literature on the dynamics between public opinion and government decision making at all levels of government. Lotka-Volterra can address questions specifically about how different local, state, and/or national conditions impact the relationship between the public and the government. Further, this type of model could serve to predict how changes in those conditions would alter the system by identifying what types of shocks impact the dynamic and cause it to shift to a new equilibrium or out of equilibrium entirely. For example, it might be applied to the elected branches of government, specifically the legislative branch to inform political strategists how changes in a given law or regulation might impact public mood and possibly, by extension, a candidate's favorability with a given sector of the public.

3.4 Conclusion

In this chapter, I analyzed Ura's (2014) data on the relationship between public mood and policy liberalism of the Supreme Court using the Lotka-Volterra method. I find early evidence to support Ura's conclusions that both the thermostatic and legitimation models are not necessarily mutually exclusive processes. Instead, they characterize the relationship at different periods of time. Despite several limitations to this specific approach, this study also highlights the sensitivity of the Lotka-Volterra model to both small changes in parameter values as well as different exogenous covariates. Substantive future work for this case study might include narrowing down the types of SCOTUS legislation of interest, narrowing down public mood, or examining the relationship between other branches or levels of government and public mood.

Chapter 4

Case Study 2: Israel and Palestine in the Second Intifada

An ongoing debate within the broader academic dialog about the Israeli-Palestinian conflict wrestles with whether or not the relationship can be characterized as a tit-for-tat cycle of violence or not. Jaeger and Paserman (2008) were really the first to argue against the notion of the relationship as inherently cyclical. They contribute directly to the debate by empirically analyzing the relationship between Palestinian and Israeli violence during the Second Intifada in order to determine whether the "pattern of violence in the conflict [could]...be characterized as a cycle, in which violence by one party causes violence by the other party and vice versa, or whether causality is unidirectional" (Jaeger and Paserman 2008, p. 1591). They ultimately conclude that, while Israel responds in a predictable way to Palestinian violence, the reverse is not substantiated by the data.

I chose this particular case study because the debate in the literature can be characterized as a debate as to whether this relationship fits the assumptions of Lotka-Volterra or not. For those arguing that the relationship is, in fact, a cycle of violence, then the underlying assumptions of Lotka-Volterra should not be violated. However, I find preliminary evidence to support Jaeger and Paserman's conclusions that the relationships is, in fact, not cyclical. In fact, I argue that disparities in the goals, technology, and organization of Israel and Palestine fundamentally violate the Lotka-Volterra assumptions, making it an inappropriate model choice. This information is interesting in an of itself, because it highlights the need for rigorous theoretical justification prior to the selection of the Lotka-Volterra model for competitive relationships. Finally, as discussed in Chapter 2, a Granger test on the direction of causality in the data fails reject the null hypothesis and does not confirm causality in either direction, lending further evidence to support the conclusion that LV is a poor choice for this particular dataset.

In this chapter, I briefly introduce the literature that characterizes the relationship between Israel and Palestine in order to build a foundation for evaluating the characteristics of the relationship overall during the study period. I follow this with a presentation and analysis of Jaeger and Paserman's (2008) methods and conclusions about the direction of causality of violence during the Second Intifada. I then present Lotka-Volterra as both an extension of and a different perspective in approaching their main research question. Finally, I apply Jaeger and Paserman's data to the Lotka-Volterra framework developed in Chapter 2 and discuss the results. The discussion of the results focuses mainly on both describing how and examining why, precisely, Lotka-Volterra fails when it is applied to a case that violates its underlying assumptions.

4.1 Background

The Second (al-Aqsa) Intifada began in September 2000 following Israeli Defense Minister Ariel Sharon's visit to the Temple Mount (United Nations 2020). Widely considered an affront to Islamic faith and traditions, Palestinian extremists, largely led by Hamas and Palestine Islamic Jihad (PIJ), launched a sustained violent campaign against Israeli targets that lasted until January 2005 (Beitler 2004). The conflict, often characterized by a large-scale military response by Israel and sustained guerrilla warfare tactics by Palestinian militant groups, resulted in thousands of deaths on both sides of the conflict.

In the years since the Second Intifada began, a growing body of literature has focused on empirically analyzing the dynamics of the Israel-Palestine conflict, particularly with regards to terrorism (Jaeger and Paserman 2008). These works largely focus on the ways in which the conflict affects politics and economics as well as the efficacy of counter-terrorism tactics. Getmansky and Zeitzoff (2014) examine the ways in which the threat of terrorism, measured as a function of changes in rocket range, affects voting behavior in Israeli national elections. Berrebi and Klor (2006, 2008) similarly conclude that a higher expected level of terrorism increases support of right-wing parties. Eckstein and Tsiddon (2004) describe the effect terrorism has on financial markets, arguing that ongoing terrorism is responsible for a decrease in per capita consumption.

The most robust section of this literature measures the direct effects of different counterterrorism strategies on terrorism production (Zussman and Zussman 2006, p. 193). Kaplan, et. al. (2005) focus on how targeted killings by Israel increases the recruitment of Palestinians to terrorist organizations and ultimately results in an increased rate of suicide bombings. On the other hand, they find that preventative arrests are a much more effective counterterrorism tactic at reducing suicide bombings (Kaplan, Mintz, Mishal, and Samban 2005, p. 226).

Zussman and Zussman (2006) further the debate by using the Israeli stock market as an indicator of the efficacy of targeted killing of Palestinian terrorists. They identify a strong reaction from the market following the assassinations of senior Palestinian leaders, declining "following assassinations targeting senior political leaders [and rising] following assassinations of senior military leaders" (Zussman and Zussman 2006, p. 204). Dugan and Chenoweth (2012) incorporate rational choice into the debate by arguing that Israel should shift their counterterrorism response to focus on more than punishment. They argue that states, in general, should also heavily focus on raising the expected utility of abstaining from terrorism (Dugan and Chenoweth 2012, p. 598).

The majority of the literature is Israel-centric, focusing heavily on how terrorism, and the conflict more broadly, shape Israeli politics and economics. However, tangential to Dugan and Chenoweth's use of rational choice, Berrebi and Lakdawalla (2007) model the spatio-temporal dynamics that influence terror organizations' choices in targets and timing. They argue that Palestinian terrorist organizations rationally choose their targets by considering accessibility, political importance, and Jewish population. Similarly, unless a potential target is a regional or national capital, the time between attacks corresponds to a decrease in an area's overall risk of future terrorism events.

While a significant portion of the discussion surrounding the dynamics of the conflict falls into one of these three categories, a broader question underlies the entire debate: can scholars even characterize the conflict as 'cyclical?' Several scholars have characterized this period as a vicious cycle of violence where violence by one side begets violence by the other side and vice versa. For example, Berrebi and Klor (2006) characterize the the Second Intifada as fluctuating periods of relative calm and violence that ebb and flow in a cyclical pattern. Although this is the most common understanding of the Israel-Palestine relationship, an emerging body of literature argues that this classification is a mischaracterization (Goldstein, Pevehouse, Gerner, and Telhami 2001). Jaeger and Paserman (2006) argue that describing the conflict as an endless cycle of violence followed by relative calm is a vast oversimplification of the conflict's dynamics. Instead, they conclude that Israel responds in a predictable and systematic way to Palestinian terrorism while the Palestinians do not seem to act in direct response to Israeli violence (Jaeger and Paserman 2006, p. 45).

Jaeger and Paserman (2008) build upon these articles by directly addressing the fundamental question of whether violence between Palestinians and Israelis "affects the incidence and intensity of each side's reaction" (Jaeger and Paserman 2008, p. 1591). They conclude that causality flows from "violence committed by Palestinians to violence committed by Israelis, and not vice versa" (Jaeger and Paserman 2008, p. 45). In other words, the Palestinians might deliberately randomize when they respond to Israeli violence. In order to demonstrate these conclusions, they model Israeli and Palestinian violence between September 2000 and January 2005 using a series of Vector Autoregression models (VAR).

The main data of interest was collected from B'Tselem and includes a comprehensive listing of every fatality from both sides during the Second Intifada (B'Tselem 2020). Of

Table 4.1: Descriptive Statistics

Statistic	N	Mean	St. Dev.	Min	Pctl(25)	Pctl(75)	Max
Palestinian Fatalities	53	61.6	46.6	4	37	66	250
Israeli Fatalities	53	18.8	19.4	1	8	27	125
Barrier Length (100 km)	53	61.9	81.1	0	0	167.12	197.9

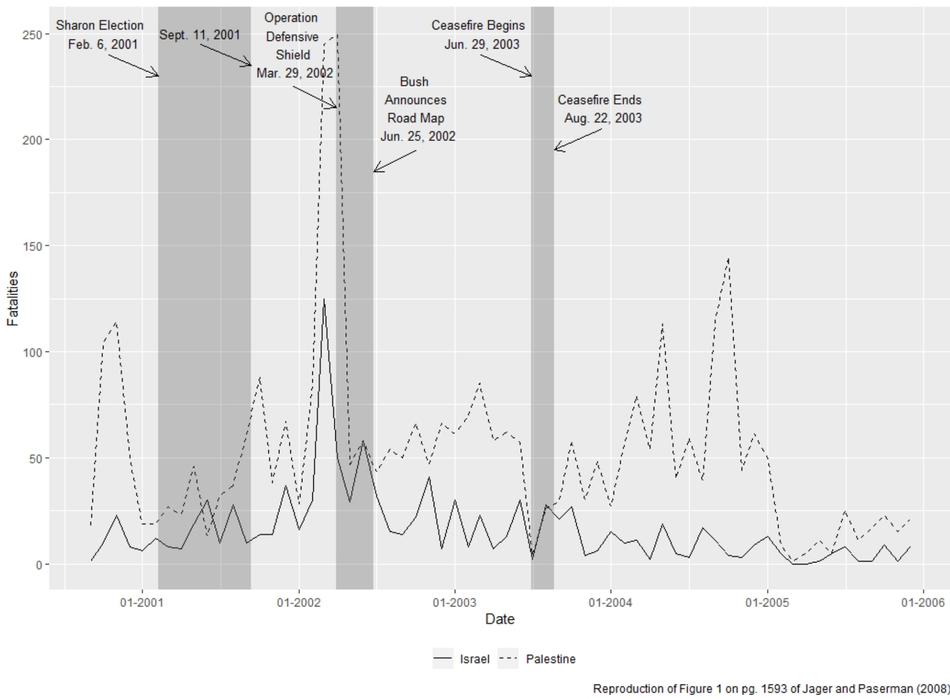
the 4,238 fatalities included in the final dataset, 994 are Israeli and 3,244 are Palestinian. The Israeli data "includes all civilian and members of the security forces killed during [the study] period, either in Israel...or in the Territories, as well as foreign civilians killed by Palestinians. The Palestinian fatality count includes all civilians and members of the security forces, as well as foreign civilians killed by Israeli security forces and civilians" (Jaeger and Paserman 2008, p. 1592). Substantively speaking, the number of Palestinian fatalities is an indicator of Israeli violence and vice versa. The authors also subdivide the study period into seven unique periods.[1] Figure 4.1 displays the monthly fatalities on both sides and the seven phases of the conflict.

The final covariate of interest is the length of the Israeli-built separation barrier that separates Israel from much of the West Bank. Construction of the barrier began in 2002 and its purpose and route remain hotly debated between supporters and opponents (Just Vision 2020).[2] The data for this covariate was derived from data provided by the Israeli Ministry of defense based on the date of completion and length of each segment (Jaeger and Paserman 2008).

Jaeger and Paserman identify three main, underlying factors that might drive the conflict: incapacitation, deterrence, and vengeance. In other words, violence by one side works in one of three ways. First, violence by one side could limit its opponent's ability to retaliate. Second, violence by one side could cause sufficient fear in the opponent to deter future violence. Finally, violence could cause the opponent to retaliate (Jaeger and

[1] Jaeger and Paserman (2005) provide a detailed description of each of these periods.
[2] Also see B'Tselem (2017).

Figure 4.1: Monthly Fatalities



Paserman 2008, p. 1593). In order to model these dynamic links, the authors utilize a VAR framework, available in Equation 4.1

$$\begin{bmatrix} Pal_t \\ Isr_t \end{bmatrix} = A_0 + A_1 \begin{bmatrix} Pal_{t-1} \\ Isr_{t-1} \end{bmatrix} + ... + A_p \begin{bmatrix} Pal_{t-p} \\ Isr_{t-p} \end{bmatrix} + BX_t + \epsilon_t, \tag{4.1}$$

"where the A_j's and B are matrices of coefficients, X_t is a vector of exogenous variables that may shift the reaction function up or down, and ϵ_t is the vector error term" (Jaeger and Paserman 2008, p. 1594). The dependent variable in the reaction function is the number of fatalities of the opposite group. The VAR is is specified two ways. In the first VAR, Pal_t and Isr_t are dummy variables representing whether or not a fatality occurred on day t. In the second specification, Pal_t and Isr_t are equal to aggregate number of fatalities for any given day. "All models [were] estimated equation by equation with ordinary least squares and heteroskedasticity-consistent standard errors" (Jaeger and Paserman 2008, p. 1594). However, before executing the VAR models, the authors present two nonparametric impulse response functions (IRF) for Israel and Palestine. They are replicated in Equations 4.2 and 4.3, respectively, "where P_s and I_s are the number of Palestinian and Israeli fatalities on day s (Jaeger and Paserman 2008, p. 1595).

$$IsrRF_t = \left(\frac{\sum_{s:I_s>0} I_s}{\sum_{s:I_s>0} 1} \right)^{-1} \left(\frac{\sum_{s:I_{s-t}>0} P_s}{\sum_{s:I_{s-t}>0} 1} - \frac{1}{T} \sum_s P_s \right) \tag{4.2}$$

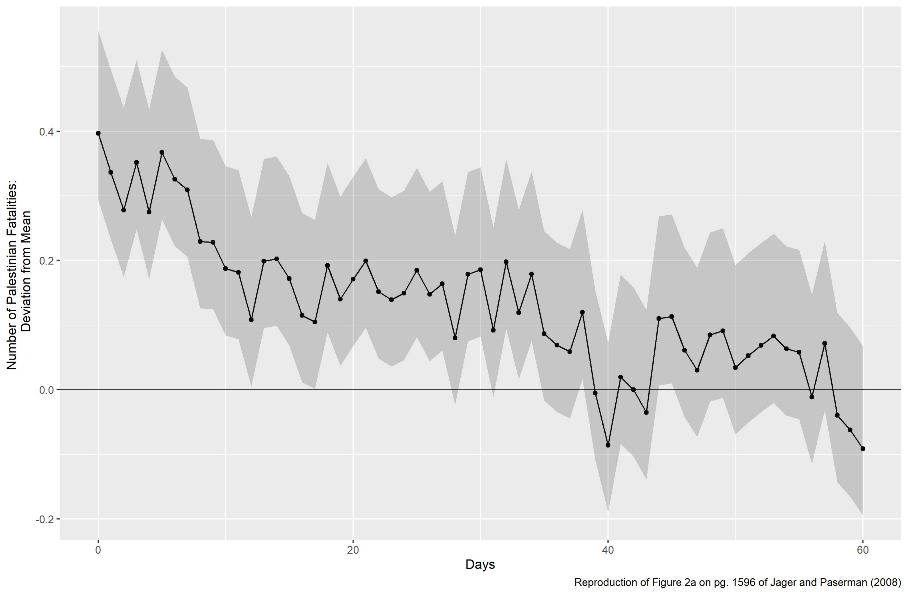
$$PalRF_t = \left(\frac{\sum_{s:P_s>0} P_s}{\sum_{s:P_s>0} 1} \right)^{-1} \left(\frac{\sum_{s:P_{s-t}>0} I_s}{\sum_{s:P_{s-t}>0} 1} - \frac{1}{T} \sum_s I_s \right) \tag{4.3}$$

Figure 4.2 displays my replications of the author's empirical response function plots, as calculated in R.

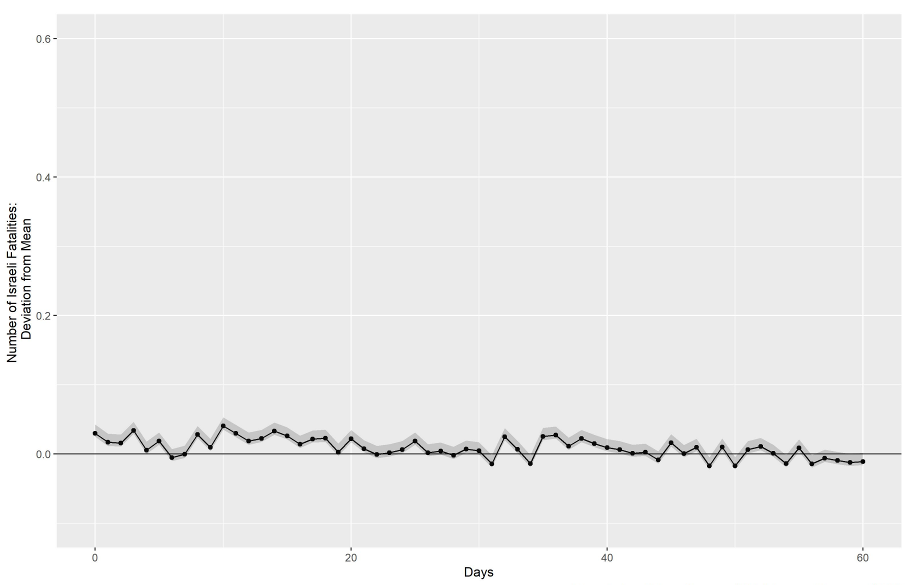
Along with the IRF plots, Jaeger and Paserman calculate a set of four VAR models. Each model is either an Israeli or Palestinian impulse function and, under each, is a model of incidence of a fatality on a given day or a number of fatalities on that day. A table of replicated results is available in Appendix C and displays only the results that Jaeger and Paserman highlight: the coefficients on lagged Israeli and Palestinian fatalities for the Israeli and Palestinian reaction functions, respectively. The results in this table, along with the IRF plots lead the authors to conclude that violence between Israel and Palestine is a tit-for-tat cycle of violence, at least during the Second Intifada. Instead, they find that with regards to levels of past violence, Israel reacts in a more predictable way to Palestinian violence. On the other hand, Palestinian violence is much more difficult to predict. The authors further conclude that no net deterrent nor incapacitation effects are present in the relationship (Jaeger and Paserman 2008, p. 1598).

The goal of the next section is to contribute to the debate about the nature of Israeli-Palestinian violence by applying Lotka-Volterra, assisted by my OLS estimator. I suggest

Figure 4.2: Empirical IRF Plots†



(a) Israeli Empirical Response Function



(b) Palestinian Empirical Response Function

†Confidence Intervals differ slightly from Jaeger and Paserman due to the translation from Stata to R.

the use of Lotka-Volterra as a means of both answering new, interesting questions and looking at the conflict from a different perspective. I intend to use the LV method described in Chapter 2 to take a deeper dive into the endogeneity that may or may not exist within the relationship as well as when or how the relationship might become or remain stable or lose stability all together. Similarly, I look to Lotka-Volterra to aid in characterizing the nature of the relationship between Israeli and Palestinian violence as cyclical or non-cyclical, per the ongoing debate in the literature.

4.2 The LV Approach to Analyzing the Israeli-Palestinian Relationship

Prior to any application, we must make an educated decision of which variable of interest to assign to the predator (y) and which to the prey (x). This choice was difficult because, at different stages of the conflict, both sides demonstrated behavior consistent with the assumptions that characterize predator behavior. I will address this particular concern in depth later in the chapter when discussing the results of the LV models. Despite the difficulty in choosing which side to attribute to predator and which to prey, in order to carry out the application of the LV model on Jaeger and Paserman's data, we must assign values to x and y.

I settled on assigning the monthly Palestinian death toll, an indicator of Israeli violence towards Palestinians, as the prey.[3] I assign Israeli violence (total Palestinian deaths per month) to the 'prey' population largely because, in order to maintain a stable Lotka-Volterra relationship, the intrinsic reproduction rate (α) of the prey must be greater than the growth rate (δ) of the predator. When this ratio is reversed, it is implied that the

[3]The assignment of predator and prey is neither an political statement nor an attempt to place guilt or blame on either party to the conflict. The term 'prey' is *not* synonymous with 'victim.' For more information on civilian deaths and tactics employed by both sides, see Amnesty International (2001), Moghadam (2003), Duschinsky (2011), Manekin (2013), and B'Tslem (2020).

prey cannot reproduce in sufficient numbers to satisfy the predator. This in turn results in a higher death or emigration rate (γ) of the predator, violating the ratio of $\delta > \gamma$ and ultimately leading to the extinction of the predator.

With the decision of which populations to assign to x and y made, I begin the process of preparing the data for application to the LV method and OLS estimator by loading their monthly data into R. I extract only the observations that occur during the study period from September 2000 to January 2005. Unlike the data used in Chapter 3, no additional data manipulation of x or y is necessary before proceeding with the analysis because both populations are both continuous count variables and, therefore, naturally constrained above zero.

In order to construct the exogenous covariates, I generate a set of seven of 'period' dummy variables, one for each period in the study, that equal 1 during the time frame that period covers and 0 otherwise. However, for this particular study I only look at two out of the seven periods. The first period of interest runs from the September 11, 2001 terrorist attacks until the inception of Operation Defensive Shield at the end of March, 2002. Many observers believed that the terror attacks and subsequent US response effectively gave the Israeli government permission "to pursue more proactive measures against militant and terrorist groups, including incursions into the Palestinian-administered Territories" (Jaeger and Paserman 2005, p. 5). Measures included the beginning of the long-term Israeli-enforced confinement of Palestinian Authority President Yasser Arafat in December 2001 until October 2004 (Jaeger and Paserman 2005).

The second period I focus on is the time between President George W. Bush's Middle East speech on June 24, 2002 and the beginning of the ceasefire on June 29, 2003. The speech marked a renewed US, and eventually global, effort to broker peace and negotiate a settlement to violence in the region (Jaeger and Paserman 2005, p. 7). I chose this period largely because it is characterized by multinational attempts to move both parties to the conflict closer to the negotiating table, as well as generally stable levels of violence.

It also represents the longest of the periods identified in Jaeger and Paserman's dataset, covering 12 of the 53 months.

While these are the two periods I focus on in this study, this is not to say that other periods during the conflict are unimportant. Therefore, extensions of this project could and should seek to incorporate the rest of the periods individually or all of the periods. Unfortunately, in the current specification of the OLS estimator, including seven predictor variables is unwieldy, and leads to the inclusion of 127 total interaction terms.

The manipulation of the barrier wall covariate falls in line with the variable manipulated employed in Chapter 3: all values less than median are assigned a 0 and those above are assigned a 1. To complement these variables, I also construct two dummy control variables. The first indicates when the period switches and the second indicates when new construction has been completed on the barrier wall. Equation 4.4 is the regression formula used to estimate the parameter values with one external variable.

$$\log(y) = \hat{\beta}y + \theta_{\beta z_{1t}} y z_{1t} - \hat{\gamma}\log(x) - \theta_{\gamma z_{1t}}\log(x) z_{1t} + \hat{\delta}x + \theta_{\delta z_{1t}} x z_{1t} - z_{1t} + \mu - C^* \quad (4.4)$$

where:

- x and y are the initial starting values for Total Israeli deaths per month (prey) and Total Palestinian deaths per month (predator), respectively;
- $\hat{\beta}$, $\hat{\gamma}$, and $\hat{\delta}$ are the main effects coefficients on the traditional LV parameters, divided by α;
- θ_β, θ_γ, and θ_δ are the coefficients on the independent variable, divided by α;
- z_{1t} is the value of the the exogenous covariate; and
- μ is a control dummy variable that indicates when the value of z_t changes.

In total, I run five different OLS models. The results of these models are available in

Table 4.2. Model 1 includes zero covariates, models 2 - 4 each include a different single covariate, and model 5 includes both Period 3 and Period 5 as predictors.

There are two extremely interesting observations from these coefficient estimates. First are the relatively small size of the $\hat{\beta}$ and $\hat{\delta}$ coefficients. Substantively, this has implications for the populations of both x and y. For the prey population, x (Israeli violence), the small predicted values of $\hat{\beta}$ indicate a near-zero death rate from predation. In other words, small values of $\hat{\beta}$ imply that the Palestinian death toll at time $t+1$ is minimally dependent upon both its own population and the number of Israeli deaths at time t. Over time, such small values of $\hat{\beta}$ would likely result in the long-term growth of the prey. The opposite holds true for the predator population, y (Palestinian violence). The near-zero value of $\hat{\delta}$ indicates near-zero growth of the predator population over time as the result of consuming prey. In other words, small values of $\hat{\delta}$ imply that the number of Israeli deaths at time $t+1$ is minimally dependent upon both its own population and the number of Palestinian deaths t. Overall, these parameter estimates imply eventual extinction of the predator and exponential growth of the prey.

The second observation is the reversal of the signs of the coefficients of $\hat{\delta}$ and $\hat{\gamma}$. The reversed signs of these two parameters violate the assumption that all four coefficients are positive. A negative gamma and positive delta suggest both negative population growth of the predator as well as a positive death rate. Substantively, this suggests that, within a pure Lotka-Volterra system, that the predator is unable to sustain its population, although extinction is not necessarily immediate. I suspect, then, that the DGP of the system is driven by one or more mechanisms that the current specification of the Lotka-Volterra model is incapable of capturing due to its own underlying assumptions. If the estimated growth rate of the predator is negative and the death rate positive, but we do not observe behavior consistent with these predictions, then some other process or set of processes must be at work keeping the predator population from extinction or the relationship is not appropriate or LV modeling.

Table 4.2: Estimated L-V Coefficients

	(1)	(2)	(3)	(4)	(5)
Coefficient					
α	0.32	0.60	0.64	0.61	1.96
$\hat{\beta}$	0.14	0.06	0.11	0.06	0.04
	(0.004)	(0.005)	(0.006)	(0.006)	(0.008)
$\theta_{\beta barrier}$	-	0.14	-	-	-
	-	(0.024)	-	-	-
$\theta_{\beta period3}$	-	-	-0.05	-	-0.02
	-	-	(0.017)	-	(0.178)
$\theta_{\beta period5}$	-	-	-	0.03	-0.01
	-	-	-	(0.018)	(0.014)
$\hat{\gamma}^{\dagger}$	2.25	1.30	0.72	1.38	0.21
	(0.194)	(0.288)	(0.202)	(0.263)	(0.204)
$\theta^{\dagger}_{\gamma barrier}$	-	-1.47	-	-	-
	-	(1.109)	-	-	-
$\theta^{\dagger}_{\gamma period3}$	-	-	0.97	-	0.35
	-	-	(0.858)	-	(0.854)
$\theta^{\dagger}_{\gamma period5}$	-	-	-	-1.71	-0.31
	-	-	-	(8.093)	(5.756)
$\hat{\delta}$	-0.04	-0.02	-0.008	-0.02	-0.002
	(0.003)	(0.004)	(0.003)	(0.004)	(0.003)
$\theta_{\delta barrier}$	-	0.02	-	-	-
	-	(0.016)	-	-	-
$\theta_{\delta period3}$	-	-	-0.02	-	-0.008
	-	-	(0.015)	-	(0.015)
$\theta_{\delta period5}$	-	-	-	0.03	0.006
	-	-	-	(0.130)	(0.092)
Steady State $\{y, x\}$					
0 Covariates	$\{58.97, 2.28\}$	-	-	-	-
1 Covariate					
$z_t = 0$	-	$\{10.22, 73.98\}$	$\{5.93, 93.46\}$	$\{9.83, 69.86\}$	-
$z_t = 1$	-	$\{2.98, 67.21\}$	$\{10.85, 54.40\}$	$\{6.80, 31.15\}$	-
2 Covariates					
$z_{t1} = 0, z_{t2} = 0$	-	-	-	-	$\{49.27, 86.49\}$
$z_{t1} = 1, z_{t2} = 0$	-	-	-	-	$\{101.55, 54.40\}$
$z_{t1} = 0, z_{t2} = 1$	-	-	-	-	$\{70.09, 31.15\}$
$z_{t1} = 1, z_{t2} = 1$	-	-	-	-	$\{262.03, 54.45\}$

Standard Errors in Parentheses

† Before use in the Lotka-Volterra model, the sign of all γ and θ_{γ} coefficient estimates is reversed.

Assuming the relationship does not violate the underlying LV assumptions and we observe similarly explosive behavior in the parameters, I would suspect an omitted variable bias is at work, such that we are unable to accurately apply this particular framework to these data. I hypothesize that, with the correct covariate or series of covariates, we would obtain parameter estimates with the correct signs and more accurately model the system as a while. One particular remedy of interest might be the inclusion of exogenous covariates that only directly affect the population of either predator or prey and not both. Similar to the inclusion of substrate in biological models, a third differential equation in the system that directly affects the birth rate of the prey, α, it is entirely feasible to re-write the LV system of equations to include species-unique predictors. Such an adjustment to the set of differential equations might allows us to model populations that are either self-sustaining to a degree (cannibalistic) or those that are, to some degree, reliant upon different data generating processes. In the case of Israel/Palestine, it is logical to assume that, because both parties to the conflict are motivated by different political, social, and economic factors, the underlying DGP of the overall system is not driven by the same exogenous data. While making such an adjustment would increase the complexity of the model, it might be worth considering for special subsets of relationships, such as the Israeli/Palestinian case, where the estimated signs of the coefficients are contrary to what we expect. Eat your heart out, Occam.

Second, this particular relationship might just be driven by a data generating process that is not conducive to this particular specification of Lotka-Volterra more generally. It is possible to recreate this phenomenon with simulated data by applying data created from sine and cosine waves with different frequencies:

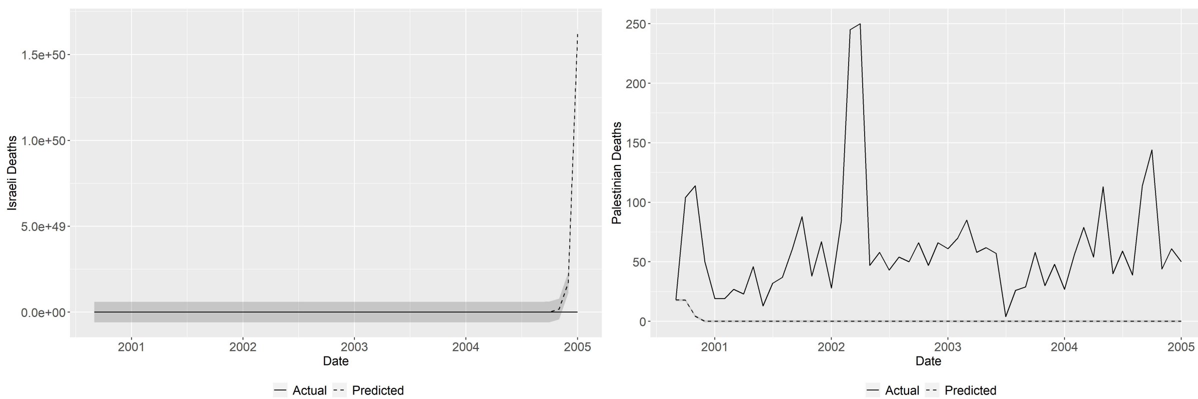
$$
\begin{aligned}
y &= \sin(i_1 t)^2 \\
x &= \cos(i_2 t)^2
\end{aligned}
\tag{4.5}
$$

where i_j is a set of two different values and t is the number of waves to generate. This suggests that certain data generating processes defy, at a fundamental level, characterization by this specific, and simplistic, specification of Lotka-Volterra.

However, in this particular case, I argue that omitted variables are not causing the behavior we observe in the parameters. Instead, I hypothesize that the underlying assumptions of Lotka-Volterra are violated, in some meaningful way, thus causing explosive behavior in the predicted values. In other words, the relationship is not, in fact, cyclical and the behavior of both sides is not driven by survival. Jaeger and Paserman argue that the relationship is better characterized by unidirectional causality that is the result of disparities in the organization and accessible technologies of either side. I agree, but argue there may be an even more fundamental mechanism at work: the goals of either side are multidimensional and not dominated by continuing a sustained campaign of violence. In other words, neither side is perpetuating violence simply for the sake of doing so and exclusively in response to the other's level of violence. Instead, I argue that both sides have goals that supersede the need to continue the violence such as Palestine's goal of internationally recognized statehood or Israel posturing both regionally and internationally. Just because two populations of interest seem to be acting in a tit-for-tat way does not mean they actually are. In other words, when a behavior is a means to an end unrelated or only tangentially related to the observed behavior and two populations are engaging in correlated actions, the model's assumptions are violated and LV is an inappropriate choice of model.

For the remainder of this section, I utilize the predicted coefficients to solve the set of ODE to demonstrate exactly how LV fails when it is applied to a unsuitable relationship. In order to generate predicted values based on the coefficient estimates, we must first determine the value of α. In order to achieve this, I utilize the same optimization routine as in Chapter 3 to estimate a value for α that minimizes the sum of squared errors of the x and y data combined. With α in hand, I assign the initial values of x and y to the corresponding first observations in the dataset. I then utilize the coefficient estimates to solve the Lotka-Volterra set of differential equations one time-step at a time to generate predicted data that is dependent upon the value of the external data. Before discussing the results in depth, it is important to keep in mind that all of the results are always predicated on the past values of each population. In other words, when we discuss the predicted impact of any given covariate, that covariate is working in tandem with population values in the differential equations.

Figure 4.3: Actual vs. Predicted Values: Model 1

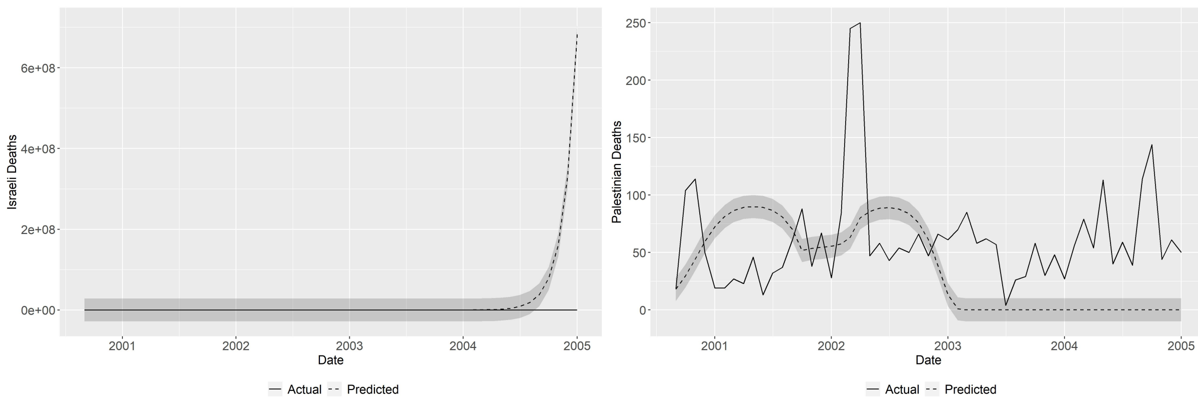


Graphically, the predicted population values are plotted against the actual values in Figure 4.3. It is important to note here the total reversal of the expected behavior of the predicted values. It appears from the plot of the predicted values of both x and y that the prey is behaving more like a predator, as it eventually reaches extinction, and the predator is behaving more like the prey with exponential growth.[4] This is substantively interesting in that it might indicate that, under certain conditions, the 'predator' may behave more like 'prey' and vice versa.

Model 3, which includes the period from September 11, 2001 to the start of Operation Defensive shield, produces predicted coefficient values in line with what we would expect from reversed signs of $\hat{\delta}$ and $\hat{\gamma}$: the predicted extinction of the predator and exponential growth of the prey. In fact, the shape of the graph of the predator only changes slightly.

[4]This behavior is still observed when x and y are switched.

Figure 4.4: Actual vs. Predicted Values: Model 3

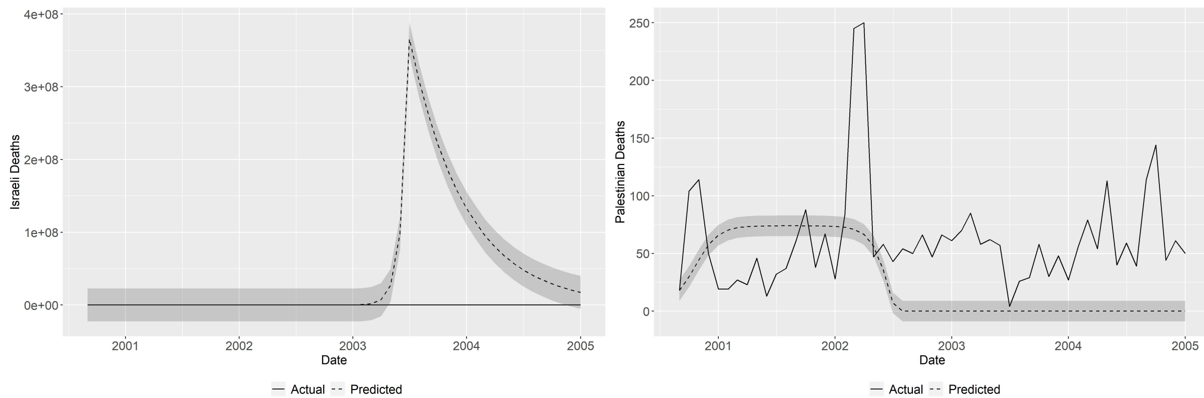


The biggest change from Model 1 and Model 3 is the limit of the y-axis: the limit is much smaller once a covariate is added. While the growth of the predator is still exponential, it isn't as explosively exponential as Model 1. This behavior is likely caused by the much larger (negative) values of $\hat{\gamma}$ compared to the values of the $\hat{\delta}$ coefficients produced when a covariate is included in the model. Plotted in Figure 4.4, the predicted values from the differential equations seem to suggest that, during period 3, x behaved in line with expected prey behavior and y in line with expected predator behavior over time. Going back to Figure 4.1, the observed values during that time period are consistent with this assertion.

Another interesting feature of this, and other following, set of predicted values is non-immediate extinction of the prey population. Generally, when a Lotka-Volterra system collapses, the predator goes extinct almost immediately. However, in these special cases, when the starting value of y is big enough relative to x and the value of $\hat{\gamma}$ is negative enough, the rate of change is slow enough to maintain the population over a longer period of time.

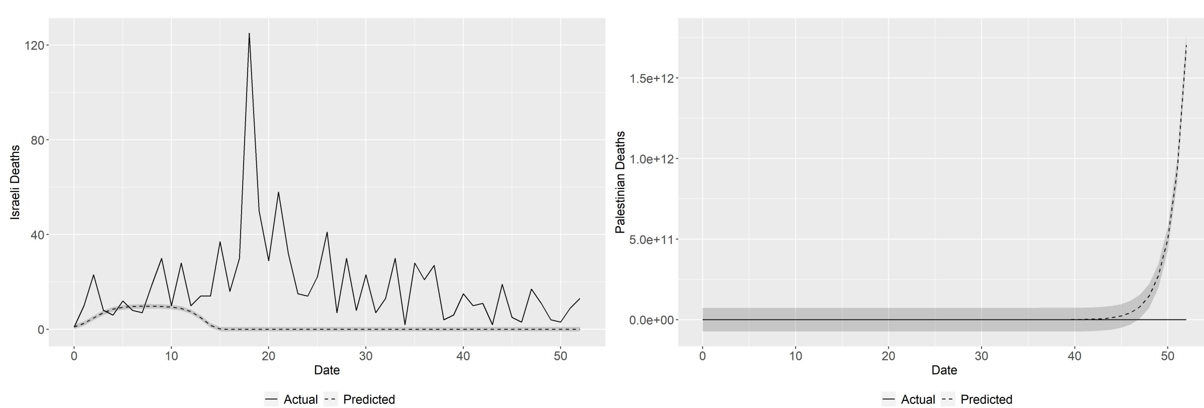
Model 2 includes the covariate for completion of the separation barrier. What is most interesting about these predicted results is that, although they seem to be trending towards the behavior observed in Models 1 and 3 at the end of the study period, we do

Figure 4.5: Actual vs. Predicted Values: Model 2



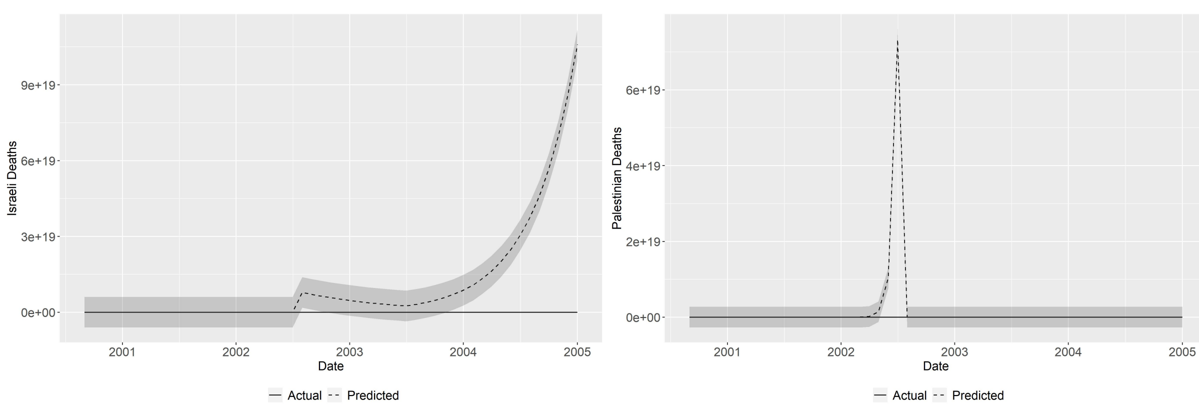
not observe the sharp, exponential changes present in either of those models. Instead, we observe no significant predicted impact in the number of Israeli deaths. This suggests that the barrier was not a major contributor to Palestinian violence. However, the interesting shape of Palestinian deaths suggests that the barrier might have contributed to Israeli violence towards Palestine initially, but that it eventually curbed Israeli violence. It is important to remember that this covariate was manipulated into a dummy variable based on its median value. As a result, we lose potentially valuable information and have less accurate coefficient values.

Figure 4.6: Actual vs. Predicted Values: Model 4



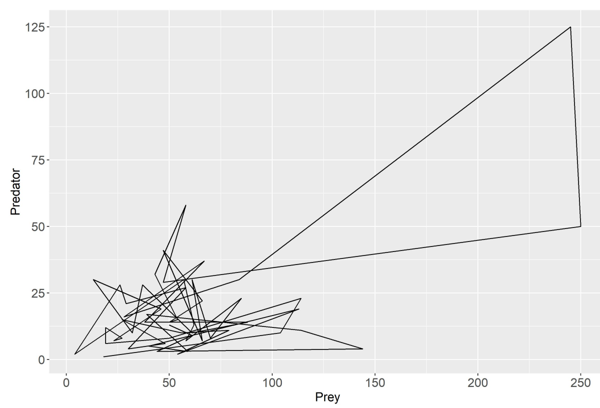
The predicted values from Model 4, plotted in Figure 4.6, exhibits behavior consistent with what we would expect from a more traditional predator/prey system. Model 4 includes the period from the end of Operation Defensive Shield to the beginning of the ceasefire. This period is characterized by a closer ratio of Israeli to Palestinian fatalities. This type of predicted behavior might substantively suggest that, during this particular period of the overall conflict, the roles of predator and prey were reversed and that Israeli violence was behaving in a way more consistent with expected prey behavior and vice versa. Logically, this could make sense: during this period, a below average number of Palestinians were killed (about 60 compared to the overall 61.5) and the graph of the actual values appears to behave differently than in period 3, for example. Instead of observing a rise in Palestinian fatalities at $t = 0$ preceding a rise in Israeli fatalities at $t+1$ as we would expect from a traditional predator-prey type of relationship, the behavior of the curves appears to switch. This type of behavior is inconsistent with Israeli fatalities representing the predator and Palestinian fatalities representing the prey during this time period.

Figure 4.7: Actual vs. Predicted Values: Model 5



The final model, Model 5, contains two covariates: both period 3 and period 5. I chose these two periods for a two-covariate model because, individually, they produce opposite predicted behavior. While the predicted results more closely resemble those of Model 3 with regards to overall trend, it is interesting to note that there is a marked shift in several aspects of the two plots. In the predator plot, there is a shift in both the limit of the exponential growth at the end of the study period as well as some noticeable variation across the entire study period. The prey plot most closely resembles the prey plot from Model 4, but with marked changes largely during the time period that spans period 3. Substantively, this is likely occurring because the effect of period 3 is actively shaping the results during period 3. When the period switches to period 5, that effect becomes dominant. In order to accurately estimate the role each time period plays in affecting the overall relationship, it is necessary to include all seven periods in the modeling process. As discussed previously, doing so with the current specification of the OLS estimator is unwieldy and impractical.

Figure 4.8: Phase Plot, Raw Data

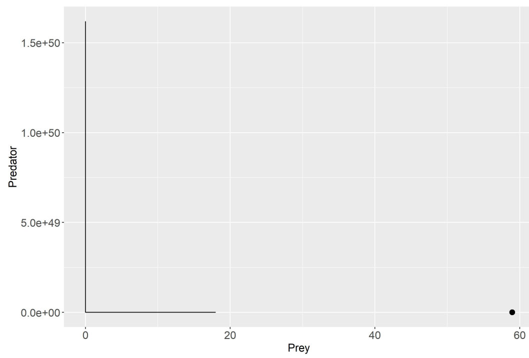


Because the actual data do not follow predicted trends of a Lotka-Volterra system and are, in fact, explosive, I reiterate the argument that this dyadic relationship is not suited for LV analysis. Lending credence to this hypothesis is the phase plot of the raw data (Figure 4.8). Unlike the phase plot of Ura's raw data (Figure 3.3), there are no distinct elliptical orbits. Therefore, the highly erratic phase plot indicates that the system itself is not only inherently unstable but does not behave like phase-state plots of known predator/prey populations.

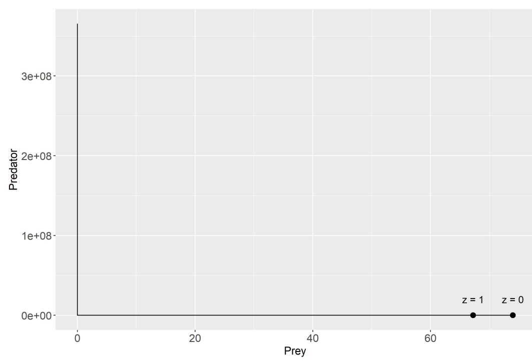
The most important takeaway from the phase-space plots is that, there is no elliptical behavior in the predicted values, and therefore, no evidence that LV is an appropriate

Figure 4.9: Phase-Space Plots with Steady States

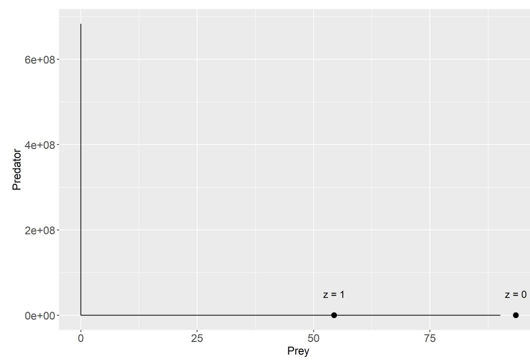
(a) Model 1: Zero Covariates



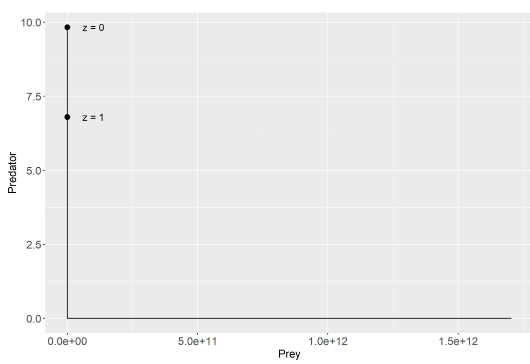
(b) Model 2: Barrier Wall



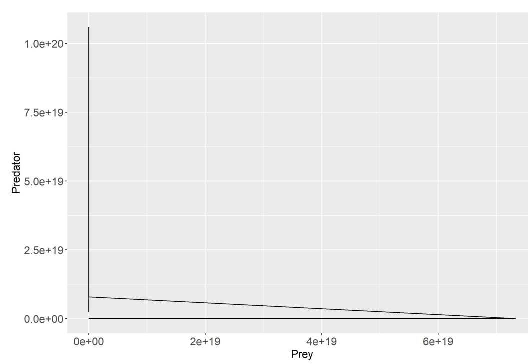
(c) Model 3: Period 3



(d) Model 4: Period 5



(e) Model 5: Period 3 and Period 5



choice of modeling technique.

4.3 Discussion

The results in the previous section lend credence to Jaeger and Paserman's overall conclusion that the relationship cannot be accurately "characterized as a self-perpetuating cycle of violence" (Jaeger and Paserman 2008, p.1603). The observed instability in the system might be caused by the nature or underlying goals of each party to the conflict. The Israeli Defense Forces (IDF) are highly organized and have access to advanced technology and intelligence resources. This allows the Israeli government to inflict violence and fatalities against Palestinians at the government's will (Jaeger and Paserman 2008). The Israeli side is both sanctioned and funded by the Israeli government, perhaps explaining the abnormal behavior in the predicted values of the prey. The government support of the IDF might be acting like a species-unique covariate that allows the Israeli side to continue to attack despite the prediction of the covariates that violence would cease.

In contrast, the decentralized and technologically disadvantaged nature of the Palestinian side of the conflict makes a coordinated response to Israeli violence both unpredictable and unorganized (Jaeger and Paserman 2008, p. 1602). This suggests that something other than past Israeli violence is driving Palestinian behavior. Jaeger and Paserman further argue that the Palestinians act unpredictably because any coordinated attack by Palestinian forces could be easily thwarted by the Israelis. Therefore, they might be incentivized to behave randomly in order to inflict the highest possible cost on Israel until its demands are met (Jaeger and Paserman 2008; Schelling 1980).

These characteristics of each side suggest that each side is behaving as more than the simple product of past levels of violence against it. If goals exist above and beyond sustaining a violent campaign because the other side is doing do, which is likely, then LV is simply not the correct model to choose. When LV is applied to relationships that violate its underlying assumptions, especially those that aren't truly endogenously causal, we are likely to observe nonsensical coefficients akin to those in Table 4.2.

More fundamentally, it makes sense in a terrorism/counter-terrorism context to imagine spans of time with zero attacks from either side. However, Lotka-Volterra is not equipped to handle this type of temporary extinction behavior. If this behavior is observed in a dataset, then this specification of LV is inappropriate and would lead to explosively incorrect estimated population counts following periods of extinction. While this particular dataset does not have any periods of non-action, it is

While it makes sense that, to some extent, Palestine and Israel feed off of one another's violent behavior whether for revenge or deterrence, the underlying goals of the two sides complicates our understanding of the relationship and prevents application to the Lotka-Volterra framework. LV is, however, but one of many dynamic time series models, all with unique underlying assumptions and constraints. Therefore, the relationship is not incompatible with a dynamic time series model in general, especially if we are seeking answers to questions about how or why the behavior of one or both sides changes over time. In this case, it would be worthwhile to explore other dynamic time models or to think about how to construct new ones to describe either one or both sides.

4.4 Conclusion

In this chapter, I set out to explore the debate around whether or not the relationship between Israel and Palestine during the Second Intifada is a self-perpetuating cycle of violence. I found evidence to support Jaeger and Paserman's (2008) conclusion that the relationship is not, in fact, simply characterized as a tit-for-tat system of violence and argued that this is the result of different goal structures of both parties to the conflict. As a result, I found that Lotka-Volterra is an inappropriate choice to model this relationship. The results of applying the Israel/Palestine data to the Lotka-Volterra estimator confirmed this hypothesis by producing nonsensical parameter estimates and explosive predicted values. Despite the poor fit of this particular dataset to the LV framework, future work might look at more narrow aspects of the Israel/Palestine relationship or might look at deaths as a percentage of total population.

Chapter 5

Conclusion

Political scientists have consistently relied on the use of a linear framework to analyze the vast majority of relevant data in the field. While linear modeling is a tried and true approach to political science data, in this dissertation, I offered an alternative approach for competitive time series data as a complement to more familiar methods. The non-linear, dynamic approach of Lotka-Volterra models has the potential to add to the political science empirical toolkit in several ways.

First, dynamic modeling, and LV in particular, allows empiricists to describe and explain change over time. As interest in time series continues to grow within the field, particularly with regards to how and why actors change over time, it will be increasingly beneficial to move beyond discussions about rates of change to actually modeling them (Tuma and Hannan 1984). Further, a dynamic modeling approach is especially useful when theory suggests that the rates of change of populations of interest over time aren't linear or aren't constant. Second, Lotka-Volterra models allow us to estimate new quantities of interest such as extinction and steady state to further describe substantively interesting political relationships. These quantities can further our understanding of complex political relationships and how exogenous shocks can shift a relationship either into or out of equilibrium or even existence. Finally, LV in general is a model that, by its very construction, describes endogenous relationships because the populations of both predator and prey are included in both differential equations. Therefore, appropriate dyadic relationships increase and decrease in size in direct response to one another. This type of behavior is not uncommon in the political and broader social world, although several other criteria beyond the endogenous nature of causality must be considered when choosing Lotka-Volterra as a modeling tool.

The choice of LV as a model requires a rigorous understanding of the populations of interest. Both populations must be reasonable characterized as existing within a predator/prey framework. To meet this criteria, the predator population must rely upon the existence of prey to survive, both populations must grow and shrink in direct response to the other, survival must be the primary goal of both populations, and all populations must be greater than 0. These strict assumptions greatly limit the populations that are suited for modeling within the LV framework. Any conceptual or actual violation of any of them is ample cause to forgo the LV model. For example, the populations of interest in the first case study do not violate the theoretical assumptions of Lotka-Volterra: public opinion is widely accepted as responsive to government behavior and vice versa. However, when we add a variable that disproportionately affects one of the populations, as the inclusion of inflation disproportionately affected public mood, the system is no longer amenable to LV. This, then, further limits the choice of variables to include in the LV model to those that have a more uniform effect on both populations of interest. In other words, any exogenous covariates we choose to include in an LV model must affect both predator and prey in fairly similar ways. Future research into this particular issue would be valuable, particularly in establishing where the threshold may lie and in examining other ways of introducing exogenous covariates into the model.

It is important to note, however, that while the variables in the Ura case study do not necessarily violate the assumptions about both populations, that this specification of LV is, in general, overly simplistic. By allowing for exponential growth or decline of either population, the basic LV model is not realistic. This leads to an important improvement future work could and should focus on: the inclusion of environmental constraints in the set of ODE. The most common modification to LV in this realm is the introduction of a carrying capacity for the environment, or an upper limit on population growth. Capping growth avoids exponential growth and is therefore more realistic. Other environmental constraints include limiting the prey's food source (substrate) and constraints on predator feeding behavior.

By contrast, in Chapter 4, LV was an inappropriate model choice to describe the relationship of Israel and Palestine. Although a cursory understanding of that relationship seems to characterize it as a tit-for-tat cycle of violence, where violence by one side begets violence by the other, I find preliminary evidence to support Jaeger and Paserman's (2008) conclusion that this is not the case at all. Perhaps as a result of asymmetry in the decision making processes of both sides or the disparity in the technology available, there does not appear to be a cyclical relationship between the two sides (Jaeger and Paserman 2008). We might also hypothesize that Israel was not necessarily acting to only preserve its survival, but may have had multiple competing or superseding goals, such as posturing on the international stage. Because the relationship cannot be reasonably assumed to be a cycle of violence, the theoretical assumptions of the LV model are violated, making it an inappropriate choice of model and the reason why the results reported in Chapter 4 are explosive. Even though this particular case does not fit will within the LV framework, there does exist a relationship between Israeli violence and Palestinian violence that, perhaps, changes over time. Thus, it is entirely conceivable to identify a different set of differential equations that account for partial independence of the individual sides or to model different characterizations of relationships.

Apart from concerns about violations of the model's theoretical assumptions, there are several other limitations to the strategy developed in this project resulting from the construction of the estimation strategy itself. First and foremost, the model is plagued with endogeneity. As discussed in detail in Chapter 2, including y on both the left and right hand side of the estimation equation inflates standard errors and biases the coefficient estimates of the linear model. While endogeneity is most commonly addressed using Instrumental Variable (IV) regression, which is often used in econometrics to clarify causality along with magnitude in a linear model.[1].

However, endogeneity in the implicit relationship model is not caused simply by an independent variable correlating with the error term. Instead, the main source of endogeneity is including y on the right hand side and $\log(y)$ on the left hand side of the equation. This makes both the actual and theoretical choice of an instrument exceedingly difficult because it seems impossible to divorce, in any way, y from a transformed version of itself, which would violate the exclusion restriction of IV (Ebbes, Papies, and van Heerde 2016). One possible solution to the endogeneity problem would be to further rearrange the implicit equation such that both y and $\log(y)$ are on one side and x and $\log(x)$ are on the other. While this would require a complicated estimation loop, it might be a valuable way forward for dealing with endogeneity.

When including exogenous data in the models, in order to limit the degrees of freedom eaten up and to estimate a non-explosive number of coefficients, I was forced to simplify the exogenous data into dummy variables indicating whether an observation was above or below the sample mean. Unfortunately, this dramatically limits the information we can learn about the effects of these exogenous covariates because it removes all nuance and interesting variation from the data. When a regression attempts to estimate too many coefficients, the models may suffer from overfitting, especially when the sample size is relatively small. Therefore, future work should explore ways of incorporating important continuous or more refined categorical covariates into the model in a way that avoids overfitting but preserves the interesting variation of the exogenous covariates.

Next, it is worthwhile to once more consider the inclusion of noise into the simulation in Chapter 2. Although it is accessible to political scientists to add noise onto the simulated data itself after data generation, this is not the only means of incorporating noise into the system. This leads to two interesting paths forward for this work. First, in keeping with

[1]For a clear an readily accessible explanation of Instrumental Variables regression, I recommend Ebbes, Papies, and van Heerde (2016)

the notion of including noise in the predator and prey populations themselves, it would be useful for future work to explore the use of stochastic Lotka-Volterra models. These types of LV model population development in a "random environment where random factors" like white noise influence the growth or death rates of one or both populations (Du and Sam 2006, p. 83). This would be extremely useful, especially considering the fact that social actors do not exist within a vacuum and are undoubtedly influenced, to some degree, by environmental conditions.

It would also be of interest to explore the LV models from a Bayesian perspective, by including uncertainty in the parameters themselves or in $\frac{dx}{dt}$ and $\frac{dy}{dy}$ as a whole. This could be beneficial to expanding our overall understanding of how the LV system behaves, especially in light of the fact that starting conditions dramatically influence the outcome of LV. Therefore, we could perhaps learn even more information by altering starting conditions or allowing them to vary throughout (Pascual and Kareiva 1996).

Finally, this study relied on ordinary least squares (OLS) to estimate the linear model. However, I strongly believe that future work in this area should survey other strategies such as non-linear least squares (NLS). I argue that NLS is an appropriate future approach for two reasons. First, although I do not utilize a discrete approximation of the differential equations, prior sociology work in this area utilized non-linear solvers to varying degrees of success.[2] Thus, there is precedent in solving these types of dynamic models in this way. Second, but tangentially, and more importantly, the graph of predator versus prey is elliptical, not linear, and as the parameter values deviate further from their ideal value, the orbits deviate, often radically, from that shape. Non-linear least squares is equipped to handle this type of behavior in a multivariate space in order to estimate the parameters that most closely resemble those which result in stability of the elliptical orbits. Therefore, I think it is a worthwhile effort to explore the theoretical and empirical implications of solving the implicit relationship model using NLS.

[2] See Nielsen and Hannan (1977) and Carroll (1981).

In all, despite several flaws in its current form, I have presented and demonstrated the potential for dynamic modeling, and LV specifically, to be a valuable addition to established time series modeling techniques in political methodology. I believe that, with further work as outlined above and throughout, this method has the potential to answer new and unique questions about dynamic political behavior over time.

Bibliography

Abbott, A. (1988). Transcending general linear reality. *Sociological theory*, 169–186.

Acosta, B. and K. Ramos (2017). Introducing the 1993 terrorism and political violence dataset. *Studies in Conflict & Terrorism 40*(3), 232–247.

Aguiar-Conraria, L., P. C. Magalhães, and M. J. Soares (2012). Cycles in Politics: Wavelet Analysis of Political Time Series. *American Journal of Political Science 56*(2), 500–518.

Aldrich, H. (1979). *Organizations and Environments.* Englewood Cliffs, NJ: Prentice-Hall.

Amnesty International (2001). *Broken Lives: A Year of Intifada.* London, United Kingdom: Amnesty International Publications.

Aratò, M. (2003). A famous nonlinear stochastic equation (lotka-volterra model with diffusion). *Mathematical and Computer Modelling 38*(7-9), 709–726.

Ayala, F. J., M. E. Gilpin, and J. G. Ehrenfeld (1973). Competition between species: theoretical models and experimental tests. *Theoretical Population Biology 4*(3), 331–356.

Badas, A. (2016). The public's motivated response to supreme court decision-making. *Justice System Journal 37*(4), 318–330.

Bahar, A. and X. Mao (2004). Stochastic delay lotka–volterra model. *Journal of Mathematical Analysis and Applications 292*(2), 364–380.

Bansal, G. (2019). *ecm: Build Error Correction Models.* R package version 4.4.0.

Baosheng, L., H. Shigeng, and F. Yang (2011). stochastic delay lotka-volterra model. *Journal of Inequalities and Applications 2011*(1), 914270.

Bardsen, J. (1989). The estimation of long-run coefficients from error correction models. *Oxford Bulletin of Economics and Statistics 54*, 225–255.

Begon, M., J. L. Harper, C. R. Townsend, et al. (1986). *Ecology. Individuals, populations and communities.* Blackwell scientific publications.

Beitler, R. (2004). *The path to mass rebellion: a tale of two Intifadas.* New York, New York: Lexington Books.

Berrebi, C. and E. F. Klor (2006). On terrorism and electoral outcomes: Theory and evidence from the israeli-palestinian conflict. *Journal of conflict resolution 50*(6), 899–925.

Berrebi, C. and E. F. Klor (2008). Are voters sensitive to terrorism? direct evidence from the israeli electorate. *American Political Science Review*, 279–301.

Berrebi, C. and D. Lakdawalla (2007). How does terrorism risk vary across space and time? an analysis based on the israeli experience. *Defence and Peace Economics 18*(2), 113–131.

Black, R. C., R. J. Owens, J. Wedeking, and P. C. Wohlfarth (2016a). The influence of public sentiment on supreme court opinion clarity. *Law & Society Review 50*(3), 703–732.

Black, R. C., R. J. Owens, J. Wedeking, and P. C. Wohlfarth (2016b). *US Supreme Court opinions and their audiences.* Cambridge University Press.

Box-Steffensmeier, J. M., J. R. Freeman, M. P. Hitt, and J. C. Pevehouse (2014). *Time series analysis for the social sciences.* Cambridge University Press.

Bryan, A. C. and C. D. Kromphardt (2016). Public opinion, public support, and counter-attitudinal voting on the us supreme court. *Justice System Journal 37*(4), 298–317.

B'Tselem (2017). The separation barrier. https://www.btselem.org/separation_barrier.

B'Tselem (2020). Fatalities before Operation "Cast Lead". https://www.btselem.org/statistics/fatalities/before-cast-lead/by-date-of-event. [Data File].

Carroll, G. R. (1981). Dynamics of organizational expansion in national systems of education. *American Sociological Review*, 585–599.

Casillas, C. J., P. K. Enns, and P. C. Wohlfarth (2011). How public opinion constrains the us supreme court. *American Journal of Political Science 55*(1), 74–88.

Cox, D. R. (1955). Some statistical methods connected with series of events. *Journal of the Royal Statistical Society. Series B (Methodological)*, 129–164.

Dahl, R. A. (1957). Decision-making in a democracy: The supreme court as a national policy-maker. *J. Pub. L. 6*, 279.

Du, N. H. and V. H. Sam (2006). Dynamics of a stochastic lotka–volterra model perturbed by white noise. *Journal of mathematical analysis and applications 324*(1), 82–97.

Dugan, L. and E. Chenoweth (2012). Moving beyond deterrence: The effectiveness of raising the expected utility of abstaining from terrorism in israel. *American Sociological Review 77*(4), 597–624.

Duschinsky, R. (2011). Slaughtered innocents: child victims in political discourse during the second intifada and gaza conflict. *Social Semiotics 21*(1), 33–51.

Ebbes, P., D. Papies, and H. J. van Heerde (2016). Dealing with endogeneity: A nontechnical guide for marketing researchers. In C. Homburg, M. Klarmann, and A. Vomberg (Eds.), *Handbook of market research*, pp. 1–37. Springer International Publishing.

Eckstein, Z. and D. Tsiddon (2004). Macroeconomic consequences of terror: theory and the case of israel. *Journal of monetary economics 51*(5), 971–1002.

Epstein, L. and J. A. Segal (2000). Measuring issue salience. *American Journal of Political Science 44*(1), 66–83.

Erikson, R. S., M. B. MacKuen, and J. A. Stimson (2002). *The macro polity*. Cambridge University Press.

Flemming, R. B. and B. D. Wood (1997). The public and the supreme court: Individual justice responsiveness to american policy moods. *American Journal of Political Science*, 468–498.

Francisco, R. A. (1995). The relationship between coercion and protest: An empirical evaluation in three coercive states. *Journal of Conflict Resolution 39*(2), 263–282.

Franklin, C. H. and L. C. Kosaki (1989). Republican schoolmaster: The us supreme court, public opinion, and abortion. *American Political Science Review 83*(3), 751–771.

Freeman, J. R., J. T. Williams, and T.-m. Lin (1989). Vector autoregression and the study of politics. *American Journal of Political Science*, 842–877.

Getmansky, A. and T. Zeitzoff (2014). Terrorism and voting: The effect of rocket threat on voting in israeli elections. *American Political Science Review*, 588–604.

Gibson, J. L. and M. J. Nelson (2015). Is the us supreme court's legitimacy grounded in performance satisfaction and ideology? *American Journal of Political Science 59*(1), 162–174.

Gibson, J. L. and M. J. Nelson (2016). Change in institutional support for the us supreme court: Is the court's legitimacy imperiled by the decisions it makes? *Public Opinion Quarterly 80*(3), 622–641.

Goldstein, J. S., J. C. Pevehouse, D. J. Gerner, and S. Telhami (2001). Reciprocity, triangularity, and cooperation in the middle east, 1979-97. *Journal of Conflict Resolution 45*(5), 594–620.

Gotelli, Nicholas J, e. a. (2008). *A primer of ecology*. Number 577.88 G6. Sinauer Associates Sunderland, Massachusetts, USA.

Grant, T. and M. J. Lebo (2016). Error correction methods with political time series. *Political Analysis 24*(1), 3–30.

Hannan, M. T. (1988). Organizational population dynamics and social change. *European Sociological Review 4*(2), 95–109.

Hannan, M. T. and J. Freeman (1977). The population ecology of organizations. *American journal of sociology 82*(5), 929–964.

Hoekstra, V. J. (2003). *Public reaction to Supreme Court decisions*. New York NY: Cambridge University Press.

Human Rights Watch (2005). Israel/occupied palestinian territories: Events of 2004. `https://www.hrw.org/world-report/2005/country-chapters/israel/palestine#`.

Jaeger, D. A. and M. D. Paserman (2005). The cycle of violence? an empirical analysis of fatalities in the palestinian-israeli conflict. Institute for the Study of Labor Discussion Paper 1808.

Jaeger, D. A. and M. D. Paserman (2006). Israel, the palestinian factions, and the cycle of violence. *American Economic Review 96*(2), 45–49.

Jaeger, D. A. and M. D. Paserman (2008). The cycle of violence? an empirical analysis of fatalities in the palestinian-israeli conflict. *American Economic Review 98*(4), 1591–1604.

Just Vision (2020). Second intifada. `https://justvision.org/glossary/second-intifada`.

Kaplan, E. H., A. Mintz, S. Mishal, and C. Samban (2005). What happened to suicide bombings in israel? insights from a terror stock model. *Studies in Conflict & Terrorism 28*(3), 225–235.

Keele, L. and S. DeBoef (2008). Taking time seriously: Dynamic regression. *American Journal of Political Science 52*(1), 184–200.

Keele, L., S. Linn, and C. M. Webb (2016). Treating time with all due seriousness. *Political Analysis 24*(1), 31–41.

Kinoshita, S. (2013). Introduction to nonequilibrium phenomena. In *Pattern Formations and Oscillatory Phenomena*, pp. 1–59. Elsevier.

Kot, M. (2001). *Elements of mathematical ecology*. Cambridge University Press.

Lotka, A. J. (1910). Contribution to the theory of periodic reactions. *The Journal of Physical Chemistry 14*(3), 271–274.

Lütkepohl, H. (2006). *New introduction to multiple time series analysis*. Berlin, Germany: Springer publications.

Manekin, D. (2013). Violence against civilians in the second intifada: The moderating effect of armed group structure on opportunistic violence. *Comparative Political Studies 46*(10), 1273–1300.

Mao, X., C. Yuan, and J. Zou (2005). Stochastic differential delay equations of population dynamics. *Journal of Mathematical Analysis and Applications 304*(1), 296–320.

Martin, A. D. and K. M. Quinn (2002). Dynamic ideal point estimation via markov chain monte carlo for the us supreme court, 1953–1999. *Political analysis 10*(2), 134–153.

Mayhew, D. R. (1991). *Divided we govern: party control, law-making, and investigations*. New Haven, CT: Yale University Press.

Mayhew, D. R. (2011). *Datasets for Divided We Govern.* http://pantheon.yale.edu/dmayhew/data3.html.

McGuire, K. T. and J. A. Stimson (2004). The least dangerous branch revisited: New evidence on supreme court responsiveness to public preferences. *The Journal of Politics 66*(4), 1018–1035.

McPherson, J. M. and L. Smith-Lovin (1988). A comparative ecology of five nations: testing a model of competition among voluntary organizations. *Ecological models of organizations*, 85–109.

McPherson, M. (1983). An ecology of affiliation. *American Sociological Review*, 519–532.

Meyer, J. W., F. O. Ramirez, R. Rubinson, and J. Boli-Bennett (1977). The world educational revolution, 1950-1970. *Sociology of education*, 242–258.

Michalski, R. S. (2000). Learnable evolution model: Evolutionary processes guided by machine learning. *Machine learning 38*(1-2), 9–40.

Moghadam, A. (2003). Palestinian suicide terrorism in the second intifada: Motivations and organizational aspects. *Studies in conflict and terrorism 26*(2), 65–92.

Møller, J., A. R. Syversveen, and R. P. Waagepetersen (1998). Log gaussian cox processes. *Scandinavian journal of statistics 25*(3), 451–482.

Moreno, J. L. (1953). *Who Shall Survive?: Foundations of Sociometry.* Beacon House.

Nash Jr, J. F. (1950). The bargaining problem. *Econometrica: Journal of the econometric society*, 155–162.

Nielsen, F. and M. T. Hannan (1977). The expansion of national educational systems: Tests of a population ecology model. *American Sociological Review*, 479–490.

Pascual, M. A. and P. Kareiva (1996). Predicting the outcome of competition using experimental data: maximum likelihood and bayesian approaches. *Ecology 77*(2), 337–349.

Paul, S., S. P. Mondal, and P. Bhattacharya (2016). Numerical solution of lotka volterra prey predator model by using runge–kutta–fehlberg method and laplace adomian decomposition method. *Alexandria Engineering Journal 55*(1), 613–617.

Rosendorff, B. P. and T. Sandler (2004). Too much of a good thing? the proactive response dilemma. *Journal of Conflict Resolution 48*(5), 657–671.

Schelling, T. C. (1980). *The strategy of conflict.* Harvard university press.

Skvortsov, A., B. Ristic, and A. Kamenev (2018). Predicting population extinction from early observations of the lotka–volterra system. *Applied Mathematics and Computation 320*, 371–379.

Soetaert, K., T. Petzoldt, and R. W. Setzer (2010). Solving differential equations in r: Package desolve. *Journal of Statistical Software 33*(9), 1–25.

Sørensen, J. B. (2004). Recruitment-based competition between industries: a community ecology. *Industrial and corporate change 13*(1), 149–170.

START (2020). History of the gtd. `https://www.start.umd.edu/gtd/about/`.

Stimson, J. (2009). *Mood Data File (Released May 29, 2009).* http://www.unc.edu/ jstimson/time/.

Stimson, J. A. (1999). *Public Opinion in America: Moods, Cycles, and Swings* (2 ed.). Boulder CO: Westview Press.

Stimson, J. A., M. B. MacKuen, and R. S. Erikson (1995). Dynamic representation. *American political science review*, 543–565.

Stock, J. H., M. W. Watson, et al. (2007). *Introduction to Econometrics* (2 ed.). Addison Wesley Boston.

Tuma, N. B. and M. T. Hannan (1984). *Social dynamics models and methods*. Academic Press, Inc.

United Nations (2020). History of the question of palestine. `https://www.un.org/unispal/history/`.

Ünver, H. Ö. (2008). *A comparative study of Lotka-Volterra and system dynamics models for simulation of technology industry dynamics*. Ph. D. thesis, Massachusetts Institute of Technology.

Ura, J. D. (2014a). Backlash and legitimation: Macro political responses to supreme court decisions. *American Journal of Political Science 58*(1), 110–126.

Ura, J. D. (2014b). Replication data for: Backlash and Legitimation: Macro Political Responses to Supreme Court Decisions. `https://www.btselem.org/separation_barrier`.

Volterra, V. (1938). Population growth, equilibria, and extinction under specified breeding conditions: a development and extension of the theory of the logistic curve. *Human Biology 10*(1), 1–11.

Weakliem, D. L. (2010). Time series analysis of political change. In K. T. Leicht and J. C. Jenkins (Eds.), *Handbook of Politics: State and Society in Global Perspective*, pp. 637–651. New York: Springer.

Wlezien, C. (1995). The public as thermostat: Dynamics of preferences for spending. *American journal of political science*, 981–1000.

Wlezien, C. (1996). Dynamics of representation: The case of us spending on defence. *British Journal of Political Science 26*(1), 81–103.

Zussman, A. and N. Zussman (2006). Assassinations: Evaluating the effectiveness of an israeli counterterrorism policy using stock market data. *Journal of Economic Perspectives 20*(2), 193–206.

Appendix A: Zero Covariate Simulation

This appendix provides the code for the simulation of the Generalized Lotka-Volterra Method with zero covariates. The simulation begins with loading in the 'deSolve' and 'MASS' packages. The Control Panel below sets the initial values required for generating Lotka-Volterra data: the number of simuations to run (100,000), the known values of the Lotka-Volterra parameters (pars), the intial population of both predator and prey (yini), and the number of timesteps (times). The variable 'sd' is used to add noise into the simulation and can be changed to incorporate different levels of noise. The reported results in Chapter 2 reflect running the following simulation with a standard deviation of both 0.05 and 0.15. The higher the standard deviation, the noisier the data.

```
library(deSolve)
library(MASS)

iterations <- 100000

n <- 200

pars  <- c(alpha = 1, beta  = 0.4, gamma  = 0.4, delta = 0.1)
yini  <- c(x = 5, y = 4)
times <- seq(0, n-1, by = 1)

sd <- 0.05
```

I then begin the construction of the noise that will be added on to x and y in the simulation. I generate a matrix with the standard deviation of the errors along the diagonal, allow for correlation between the errors, and generate a variance-covariance matrix that will ultimately be used in the creation of the error term.

```
S <- diag(c(sd, sd))
r_yx <- 0 #correlation

R_error <- matrix(c(1, r_yx, r_yx, 1), nrow = 2)
V_error <- S %*% R_error %*% S
```

Next, I define the Lotka-Volterra function:

```
LVmod <- function(Time, State, Pars) {
  with(as.list(c(State, Pars)), {
    dx <- alpha*x - beta*x*y
    dy <- delta*x*y - gamma*y
    return(list(c(dx, dy)))
  })
}
```

Next, I generate the simulated data generated using the parameters listed above. 'out' is a data frame with 200 observations:

```
out   <- data.frame(ode(yini, times, LVmod, pars))
```

With the simulated data generated, I initialize the results vectors that will be used to store the values extracted from the OLS regression and begin the simulation. The simulation runs 100,000 times and generates as many estimates of β^*, γ^*, and δ^*. In line with the discussion in Chapter 2, I generate a matrix of errors from a multivariate normal distribution to be added on to both x and y. I then run the OLS regression and extract the coefficient estimates and standard errors. Finally, I set the estimate of γ to negative because, in the linear model, γ is negative.

```
beta.est <- as.numeric()
beta.se <- as.numeric()

gamma.est <- as.numeric()
```

```r
gamma.se <- as.numeric()

delta.est <- as.numeric()
delta.se <- as.numeric()

for(i in 1:iterations){

  e_mat <- mvrnorm(n, rep(0, 2), V_error)

  y <- out$y + e_mat[,1]
  log.y <- log(y)

  x <- out$x + e_mat[,2]
  log.x <- log(x)

  test.reg <- lm(log.y ~ x + log.x + y)

  coefs <- coef(test.reg)
  ses <- sqrt(diag(vcov(test.reg)))

  beta.est <- c(beta.est, coefs[4])
  gamma.est <- c(gamma.est, coefs[3])
  delta.est <- c(delta.est, coefs[2])
  beta.se <- c(beta.se, ses[4])
  gamma.se <- c(gamma.se, ses[3])
  delta.se <- c(delta.se, ses[2])
}

gamma.est <- -gamma.est
```

Finally, I use the results from the simulation to calculate the bias, root mean squared error (RMSE) and coverage:

```r
bias <- c(beta_star = mean(beta.est) - (pars[['beta']] / 1),
          gamma_star = mean(gamma.est) - (pars[['gamma']] / 1),
          delta_star = mean(delta.est) - (pars[['delta']] / 1))

rmse <- c(sqrt(mean((beta.est - pars['beta'])^2)),
          sqrt(mean((gamma.est - pars['gamma'])^2)),
          sqrt(mean((delta.est - pars['delta'])^2)))

coverage <- c(100*mean(((beta.est - qnorm(.975)*beta.se) < pars['beta']) &
                         ((beta.est + qnorm(.975)*beta.se) > pars['beta'])),
              100*mean(((gamma.est - qnorm(.975)*gamma.se) < pars['gamma']) &
                         ((gamma.est + qnorm(.975)*gamma.se) > pars['gamma'])),
              100*mean(((delta.est - qnorm(.975)*delta.se) < pars['delta']) &
                         ((delta.est + qnorm(.975)*delta.se) > pars['delta'])))
```

Appendix B: One Covariate Simulation

This appendix offers the code for the simulation of the Generalized Lotka-Volterra Method with one covariate. The simulation begins with loading in the 'deSolve' package. The Control Panel sets the initial values required for the single covariate Lotka-Volterra model including the number of simuations to run, the known values of the Lotka-Volterra parameters used for data generation, the intial population of both predator and prey, and the number of timesteps (200). The variable 'sd' is used to add noise into the simulation and can be changed to incorporate different levels of noise. The reported results in Chapter 2 reflect running the following simulation with a standard deviation of both 0.15 and 0.3. I also initialize a data frame that will, at the end of the data generating process, contain the predicted values of x and y.

```
library(deSolve)
library(dplyr)
library(tidyverse)

iterations <- 100000

n <- 200

sd <- 0.05

times <- seq(0, n-1, by = 1) #time index
state <- c(x = 5, y = 4) #beginning state values

alpha <- 1

beta_star <- 0.4
gamma_star <- 0.4
delta_star <- 0.1

theta_b <- 0.1
theta_d <- 0.1
theta_g <- 0.1

res <- data.frame(times = times, x = NA, y = NA)
res[1,2:3] <- state
```

The next step is creating the external data. In order to do this, I draw a random sample of 0's and 1's with replacement. I set a seed to ensure the same z_t is selected each time the entire program is run.

```
set.seed(6284)
z_t <- sample(0:1, n, replace=TRUE)
summary(z_t)
set.seed(NULL)
```

I then begin the construction of the noise that will be added on to x and y in the simulation. I generate a matrix with the standard deviation of the errors along the diagonal, allow for correlation between the errors, and generate a variance-covariance matrix that will ultimately be used in the creation of the error term.

```
S <- diag(c(sd, sd))
r_yx <- 0 #correlation

R_error <- matrix(c(1, r_yx, r_yx, 1), nrow = 2)
V_error <- S %*% R_error %*% S
```

Next, I define the Lotka-Volterra function. The Lotka-Volterra function is designed to be run one step at a time:

```
LVmod <- function (Time, State, pars) {
  with(as.list(c(State, pars)), {
    dx <- State[1]*(alpha - beta_star*State[2] - theta_b*z*State[2])
    dy <- State[2]*(delta_star*State[1] + theta_d*z*State[1] - gamma_star - theta_g*z)
    return(list(c(dx, dy)))
  })
}
```

In this section, I generate the simulated data in a loop that calculates the values of x and y at each time step. In other words, in the first iterations of the loop, t = 1 (remember, the Lotka Volterra system state values are the values of x and y at $t = 1$), z is assigned its value at $t = 1$ and the state values are assigned their values at $t - 1$. This ensures that, with each iteration of the loop, the starting values are the output from the previous iteration. At the end of the for-loop process, we are left with 200 predicted population values for x and for y. All predicted values are placed in the previously initialized results vector 'res.'

```
for(i in 1:199){
  z <- z_t[i]
  state <- c(x = res[i,2], y = res[i,3])
  pars <- c(alpha = alpha, beta_star = beta_star, theta_b = theta_b,
            gamma_star = gamma_star, theta_g = theta_g,
            delta_star = delta_star, theta_d = theta_d, z = z)

  out <- data.frame(ode(state, seq(0,1,1), LVmod, pars))

  res[i+1, 2:3] <- out[2,2:3]
}
```

With the simulated data generated, I generate a dummy variable that indicates every time the value of z changes followed by a data frame for use in the linear model:

```
#Initialize results and standard errors vectors: beta*, gamma*, delta*
beta.est <- as.numeric()
beta.se <- as.numeric()
gamma.est <- as.numeric()
gamma.se <- as.numeric()
delta.est <- as.numeric()
delta.se <- as.numeric()

#Coefficients and standard errors on the external data
theta_beta.est <- as.numeric()
theta_beta.se <- as.numeric()
theta_gamma.est <- as.numeric()
theta_gamma.se <- as.numeric()
theta_delta.est <- as.numeric()
theta_delta.se <- as.numeric()

era <- mutate(res, era = c(0,cumsum(abs(diff(z_t)))))

dat <- data.frame(x = res$x, y = res$y, log.y = log(res$y), log.x = log(res$x),
                  z_t = z_t, era = as.factor(era[,4]))
```

The simulation runs 100,000 times and generates as many estimates of β^*, γ^*, δ^*, and the three θ coefficients. In line with the discussion in Chapter 2, I generate a matrix of errors from a multivariate normal distribution to be added on to both x and y. I then run the linear model with interaction terms for the external data and a dummy factor variable indicating when the value of z changes from 0 to 1 and extract the coefficients and standard errors. Because, in the linear model, γ is negative, I reverse the signs for γ^* and θ_γ once the simulation is complete.

```
for(i in 1:iterations){

  e_mat <- mvrnorm(n, rep(0, 2), V_error) #n x 2 matrix of correlated errors

  log.y <- log(dat$y) + e_mat[,1]
  y <- exp(log.y)

  log.x <- log(dat$x) + e_mat[,2]
  x <- exp(log.x)

  test.reg <- lm(log.y ~ x + x*z_t + log.x + log.x*z_t + y + y*z_t - z_t + dat$era)

  coefs <- coef(test.reg)
  ses <- sqrt(diag(vcov(test.reg)))

  beta.est <- c(beta.est, coefs[4])
  gamma.est <- c(gamma.est, coefs[3])
  delta.est <- c(delta.est, coefs[2])

  theta_beta.est <- c(theta_beta.est, coefs[98])
  theta_gamma.est <- c(theta_gamma.est, coefs[97])
  theta_delta.est <- c(theta_delta.est, coefs[96])

  beta.se <- c(beta.se, ses[4])
  gamma.se <- c(gamma.se, ses[3])
  delta.se <- c(delta.se, ses[2])

  theta_beta.se <- c(theta_beta.se, ses[98])
  theta_gamma.se <- c(theta_gamma.se, ses[97])
  theta_delta.se <- c(theta_delta.se, ses[96])

  rm(test.reg)
}

gamma.est <- -gamma.est
theta_gamma.est <- -theta_gamma.est
```

Finally, I use the results from the simulation to calculate the bias, root mean squared error (RMSE), and coverage:

```
bias <- c(beta_star = mean(beta.est) - (.4 / 1),
          gamma_star = mean(gamma.est) - (.2 / 1),
          delta_star = mean(delta.est) - (.3 / 1),
          theta_beta = mean(theta_beta.est) - (.1/1),
          theta_gamma = mean(theta_gamma.est) - (.1/1),
          theta_delta = mean(theta_delta.est) - (.1/1))


rmse <- c(sqrt(mean((beta.est - beta_star)^2)),
          sqrt(mean((gamma.est - gamma_star)^2)),
          sqrt(mean((delta.est - delta_star)^2)),
```

```
          sqrt(mean((theta_beta.est - theta_b)^2)),
          sqrt(mean((theta_gamma.est - theta_g)^2)),
          sqrt(mean((theta_delta.est - theta_d)^2)))


coverage <- c(100*mean(((beta.est - qnorm(.975)*beta.se) < beta_star) &
                         ((beta.est + qnorm(.975)*beta.se) > beta_star)),
              100*mean(((gamma.est - qnorm(.975)*gamma.se) < gamma_star) &
                         ((gamma.est + qnorm(.975)*gamma.se) > gamma_star)),
              100*mean(((delta.est - qnorm(.975)*delta.se) < delta_star) &
                         ((delta.est + qnorm(.975)*delta.se) > delta_star)),

              100*mean(((theta_beta.est - qnorm(.975)*theta_beta.se) < theta_b) &
                         ((theta_beta.est + qnorm(.975)*theta_beta.se) > theta_b)),
              100*mean(((theta_gamma.est - qnorm(.975)*theta_gamma.se) < theta_g) &
                         ((theta_gamma.est + qnorm(.975)*theta_gamma.se) > theta_g)),
              100*mean(((theta_delta.est - qnorm(.975)*theta_delta.se) < theta_d) &
                         ((theta_delta.est + qnorm(.975)*theta_delta.se) > theta_d)))
```

Appendix C: Jaeger and Paserman Replication Results

	Jaeger & Paserman†		Replication‡	
	Incidence	Number	Incidence	Number
Israeli IRF:				
Israeli Fatalities				
$t-1$	0.071	0.128	0.072	0.123
$t-2$	-0.001	0.066	-0.001	0.061
$t-3$	0.044	0.096	0.040	0.092
$t-4$	0.060	0.051	0.062	0.050
$t-5$	0.078	0.223	0.076	0.220
$t-6$	-0.010	0.050	-0.013	0.050
$t-7$	0.014	0.054	0.016	0.050
$t-8$	0.047	0.138	0.049	0.131
$t-9$	0.072	-0.023	0.070	-0.040
$t-10$	0.054	0.049	0.061	0.043
$t-11$	0.031	-0.070	0.033	-0.080
$t-12$	-0.004	0.002	-0.005	-0.006
$t-13$	0.008	0.024	0.011	0.020
$t-14$	0.006	0.008	0.005	-0.001
Palestinian IRF:				
Palestinian Fatalities				
$t-1$	-0.009	0.026	-0.009	0.026
$t-2$	0.022	0.027	0.020	0.026
$t-3$	0.006	0.000	0.010	0.001
$t-4$	-0.023	-0.009	-0.020	-0.008
$t-5$	0.031	0.014	0.034	0.014
$t-6$	-0.027	-0.011	-0.024	-0.011
$t-7$	-0.020	-0.029	-0.020	-0.029
$t-8$	0.012	0.064	0.012	0.064
$t-9$	-0.009	0.005	-0.008	0.005
$t-10$	0.004	0.009	0.007	0.009
$t-11$	0.008	0.012	0.010	0.012
$t-12$	-0.014	-0.026	-0.013	-0.027
$t-13$	0.006	-0.020	0.007	-0.020
$t-14$	0.016	0.027	0.020	0.027

†Jaeger and Paserman's (2008) results are taken directly from Table 1 on pg. 1597.

‡Note: Unable to find a replica function of Jaeger & Paserman's Stata code in R.