Quantum Symmetric Pairs and Quantum Supergroups at Roots of 1

Thomas Sale

Evington, Virginia

B.S., Duke University, 2015

M.S., University of Virginia, 2016

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University of Virginia

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Abstract

A quantum group, as conceived by Drinfeld and Jimbo, is the quantization of an enveloping algebra via the quantum parameter v. In analogue with the theory of algebraic groups in prime characteristic, Lusztig laid the foundations of a theory of quantum groups when v has been specialized to a root of 1. Among his fundamental results and constructions are a quantum Frobenius homomorphism, a Steinberg tensor product theorem and the small quantum group.

In this dissertation, we extend the aforementioned results to two related settings. A quantum symmetric pair is the quantization of a symmetric pair of a Lie algebra and its fixed point subalgebra under an involution; the corresponding subalgebra is called an $\imath$quantum group. In the first part of the dissertation, we show that Lusztig's Frobenius homomorphism restricts to a map of $\imath$quantum groups in finite type. We also formulate the small $\imath$quantum group and compute its dimension. A number of elements are shown to be central in the $\imath$quantum group at a root of 1. In ADE type, the action of the $\imath$quantum group at a root of 1 on the quantized adjoint module gives rise to a Lie algebra isomorphic to the symmetric pair subalgebra.

A quantum covering group is an algebra with parameters v and π, where $\pi^2 = 1$. When π is specialized to 1, it is a quantum group of anisotropic Kac-Moody type, and when π is specialized to -1, it is a quantum supergroup. In the second part, we establish analogues of Lusztig's Frobenius homomorphism and Steinberg tensor product theorem for quantum covering groups. Moreover, we formulate the small quantum covering group; in finite type, we compute its dimension. The specialization of these constructions to $\pi = 1$ recovers those of Lusztig.

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Contents

Chapter 0

Introduction

0.1 Quantum symmetric pairs at roots of 1

0.1.1

Quantum groups, $\mathbf{U}$, were defined by Drinfeld [Dri85] and Jimbo [Jim85] as quantizations of the enveloping algebras, $U(\mathfrak{g})$, associated to Lie algebras, $\mathfrak{g}$, of finite or Kac-Moody type, depending upon a parameter v, as a means of generating solutions to the Yang-Baxter equation. Having an integral form ${}_{\mathcal{A}}\mathbf{U}$, i.e. an $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$-algebra in the $\mathbb{Q}(v)$-algebra $\mathbf{U}$ from which the original quantum group may be recovered, the parameter may be specialized freely. Lusztig [Lu90a, Lu90b], [Lu94, Chapter V] observed that when v is specialized to a primitive ℓ-th root of 1, ϵ, changing the base ring to $\mathcal{A}' = \mathbb{Z}[\epsilon]$, that the quantum group, ${}_{\mathcal{A}'}\mathbf{U}$, exhibits properties analgous to those of an algebraic group in prime characteristic. There is a Frobenius homomorphism, $\mathbf{Fr} : {}_{\mathcal{A}'}\mathbf{U} \to U(\mathfrak{g})$, relating the quantum group to the corresponding enveloping algebra, by which simple modules of the latter may be pulled back to simple modules of the former that are known as Frobenius twists.

In finite type, the kernel of the Frobenius homomorphism, $\mathfrak{u}$, is a finite dimensional Hopf algebra called the small quantum group, with numerous applications. For example, its cohomology describes the geometry of the nilpotent cone [GK93]. Finally, in finite type, there is also a Steinberg-type tensor product decomposition of finite dimensional simple modules into a restricted part and a Frobenius twist part. These constructions are of interest in the modular representation theory of algebraic groups and the representation theory of affine Lie algebras (cf. [Lu90a], [AJS94]).

0.1.2

The rest of this section serves as an introduction to Part I of the dissertation.

The classification of symmetric pairs, $(\mathfrak{g}, \mathfrak{k})$, consisting of a Lie algebra, $\mathfrak{g}$, of finite type and the subalgebra, $\mathfrak{k}$ of its fixed points under an involutive automorphism, θ, arose from Cartan's classification of real forms of complex Lie groups. The quantization, $(\mathbf{U}, \mathbf{U}^\imath)$ of their enveloping algebras $(U(\mathfrak{g}), U(\mathfrak{k}))$, known as quantum symmetric pairs, was conceived of by Letzter [Let99] to study homogeneous spaces and reflection equations. Later, Kolb [Kol14] extended the theory to encompass Kac-Moody type. The subalgebra $\mathbf{U}^\imath$ is coideal in $\mathbf{U}$, and is referred to as an $\imath$quantum group.

More recently, a theory of canonical bases for quantum symmetric pairs was developed by Bao and Wang [BW18a, BW18b, BW18c] (see also [BKLW]) leading, among other things, to the resolution of the long-standing open problem of formulating super Kahzdan-Lusztig theory of types B and D. Their work also introduced an integral form, ${}_\mathcal{A}\dot{\mathbf{U}}^\imath$ for the $\imath$quantum group, and a class of elements, $B_{i,\zeta}^{(n)}$, called $\imath$divided powers, depending on simple root i and $\zeta \in X_\imath$ an equivalence class of weights, that generate it as an algebra, in analogy with the divided powers, $E_i^{(n)}, F_i^{(n)}$ in a quantum group. The

$\imath$divided powers and their expansion formulas in terms of the ordinary divided powers due to Wang and Berman [BW18d], facilitated the study of quantum symmetric pairs at roots of 1.

0.1.3

Part I of this dissertation investigates quantum symmetric pairs at roots of 1. It is based on the paper [BS19], with the exceptions of Chapters 5 and 6. One of our goals is to show that Lusztig's Frobenius homomorphism restricts to a map from the $\imath$quantum group of finite type to the symmetric pair enveloping subalgebra.

Theorem A. *The quantum Frobenius morphism restricts to an $\mathcal{A}'$-algebra homomorphism*

$$\mathbf{Fr} : {}_{\mathcal{A}'}\dot{\mathbf{U}}^{\imath} \longrightarrow {}_{\mathcal{A}'}\dot{\mathbf{U}}^{\diamond,\imath}.$$

The latter algebra, ${}_{\mathcal{A}'}\dot{\mathbf{U}}^{\diamond,\imath}$, must be constructed. This construction is the content of Chapter 2, along with determining that the Cartan part will be well-behaved under the Frobenius homomorphism when we work with simply connected root data. There are many simple quantum symmetric pairs, but it becomes apparent that all the computations that we need can be reduced to the few real rank one cases. The main task, then, is to determine the behavior of the map by explicitly computing the image of each $\imath$divided power arising from a real rank one subsystem, which is carried out in Chapter 3.

0.1.4

Having shown that Lusztig's Frobenius homomorphism descends to a homomorphism on the $\imath$quantum group level, we define the small $\imath$quantum group, ${}_{\mathcal{A}'}\dot{\mathbf{u}}^{\imath}$, and determine some of its properties. Namely, that it is coideal in the small quantum group, and it is finite dimensional. In particular, its dimension, when attached to the idempotent 1_ζ, is equivalent to that of a small parabolic quantum group, with $|\Phi^+|$ negative roots and $|\Phi^+_\bullet|$ positive roots. The result is proved by setting up a linear isomorphism between them.

Theorem B. *For any $\zeta \in X_\imath$, ${}_{\mathcal{A}'}\dot{\mathfrak{u}}^\imath 1_\zeta$ is a free $\mathcal{A}'$-module with rank $\ell^{|\Phi^+_\bullet|+|\Phi^+|}$.*

0.1.5

DeConcini, Kac and Procesi ([DKP]) used an alternate version of the quantum group at a root of 1, $\mathbf{U}_\ell$, possessing a large central subalgebra, to formulate a quantum coadjoint action. The central subalgebra is generated by the ℓ-th powers of the standard generators. Analogously, we define a version of the $\imath$quantum group, $\mathbf{U}^\imath_\ell \subset \mathbf{U}_\ell$, and show that the $k\ell$-th $\imath$divided powers suitably modified, $B^{[k\ell]}_{i,ev,odd,\emptyset}$ are central, where $i \in I_\circ$, a particular subset of I, the simple roots. Depending upon certain properites of i, the $\imath$divided powers will sometimes have a dependence upon parity ($B^{[k\ell]}_{i,ev}$ and $B^{[k\ell]}_{i,odd}$) and sometimes not ($B^{[k\ell]}_{i,\emptyset} = B^{[k\ell]}_i$).

Theorem C. *For $i \in \mathbb{I}_\circ$, $B^{[k\ell]}_{i,ev,odd,\emptyset}$ is central in $\mathbf{U}_l$ and $\mathbf{U}^\imath_\ell$.*

In types ADE, we compute the action of the $\imath$quantum group on the quantized adjoint module. In particular, K and P, the quantizations of the symmetric pair subalgebra $\mathfrak{k}$ and the -1 eigenspace of θ, $\mathfrak{p}$, respectively are modules of the $\imath$quantum group. The action of the ℓ-th $\imath$divided powers on P gives rise, when appropriately specialized to a primitive ℓ-th root of 1 to linear maps $b_{i,ev}$ on $\mathfrak{p}$. The complex Lie algebra generated by these elements, L, is isomorphic to $\mathfrak{k}$.

Theorem D. *We have a Lie algebra isomorphism, $L \cong \mathfrak{k}$.*

0.2 Quantum supergroups at roots of 1

This section serves as an introduction to Part II of the dissertation. Note the differences in notation. In particular, $\mathbf{U}$ refers to the quantum covering group in this section.

0.2.1

A quantum covering group, $\mathbf{U}$, introduced in [CHW13] (cf. [HW15]) is an algebra defined by super Cartan datum $(I, \cdot)$, and depending upon parameters v and π, where $\pi^2 = 1$. When v is specialized to 1, it is a quantum group of anisotropic Kac Moody type, and when π is specialized to -1, it is a quantum supergroup (cf. [BKM98]). The parameter π, introduced by Hill and Wang [HW15], encodes the super sign and is the decategorified parity shift functor of the categorified quantum supergroup. This program of categorification has remained active in recent years (cf. [KKT16, KKO14, EL16, BE17]).

A canonical basis for anistropic quantum supergroups, which had previously proven elusive, was achieved in [CHW14] and [Cl14] with the help of the parameter π. Inspired by the application to canonical bases, a project was undertaken to generalize [Lu94] to quantum covering groups. Part II of this dissertation, based on the paper [CSW19], is part of this program, extending [Lu94, Chapter V], concerning quantum groups at roots of 1, to the setting of quantum covering groups.

0.2.2

Unlike the quantum symmetric pairs that were dealt with previously, quantum covering groups have a triangular decomposition, so we can discuss the half quantum covering groups ${}_R\mathbf{f}$ and ${}_R\mathbf{f}^\diamond$, generated by divided powers $\theta_i^{(n)}$ over the $\mathbb{Z}[\epsilon, \sqrt{\pi}]$-algebra R^π. The algebra ${}_R\mathbf{f}^\diamond$ is called quasi-classical and plays the role that the enveloping algebra played in Lusztig's Frobenius homomorphism. Thus, following the general plan of proof from [Lu94, Theorem 35.1.7], we will first obtain a version of the Frobenius homomorphism on the half quantum covering group.

Theorem E. *There is a unique R^π-superalgebra homomorphism*

$$\mathbf{Fr}' : {}_R\mathbf{f}^\diamond \longrightarrow {}_R\mathbf{f}, \qquad \mathbf{Fr}'(\theta_i^{(n)}) = \theta_i^{(n\ell_i)} \quad (\forall i \in I, n \in \mathbb{Z}_{>0}).$$

Just as computations involving quantum groups rely heavily on q-binomial coefficients, our results will require a number of identities involving (q, π) binomials, specialized to a root of 1. Another prerequisite is a presentation of the half quasi-classical quantum group, which is handled in Chapter 8. The Frobenius homomorphism for the half quantum covering group can then be proven by checking that those relations are preserved.

0.2.3

A Steinberg tensor product theorem on the level of half quantum covering groups follows from the previous result. Following [Lu94, Theorem 35.4.2], we give the half quantum covering group as a tensor product of the half small quantum covering group, $\mathfrak{f}$, with the half quasi-classical quantum covering group.

Theorem F. *The R^π-linear map*

$$\chi : {}_R\mathbf{f}^\diamond \otimes_R \mathfrak{f} \to {}_R\mathbf{f}, \qquad x \otimes y \mapsto \mathbf{Fr}'(x)y$$

is an isomorphism of R^π-modules.

We prove a second version of the Frobenius homomorphism, analogous to [Lu94, Theorem 35.1.8] for the half quantum covering group using the previous theorem.

Theorem G. *There is a unique R^π-superalgebra homomorphism* $\mathbf{Fr} : {}_R\mathbf{f} \longrightarrow {}_R\mathbf{f}^\diamond$ *such that, for all* $i \in I, n \in \mathbb{N}$,

$$\mathbf{Fr}(\theta_i^{(n)}) = \begin{cases} \theta_i^{(n/\ell_i)}, & \textit{if } \ell_i \textit{ divides } n, \\ 0, & \textit{otherwise.} \end{cases}$$

We use a modified version of the quantum covering group ${}_R\dot{\mathbf{U}}$ obtained by introducing idempotents $\mathbf{1}_\lambda$ indexed by weights $\lambda \in X$. In analogy with the case of half quantum covering groups, ${}_R\dot{\mathbf{U}}^\diamond$ is the quasi-classical version. Using the previous form of the Frobenius homomorphism, we prove a generalization of [Lu94, Theorem 35.1.9] for the modified quantum covering group.

Theorem H. *There is a unique R^π-superalgebra homomorphism* $\mathbf{Fr} : {}_R\dot{\mathbf{U}} \longrightarrow {}_R\dot{\mathbf{U}}^\diamond$ *such that for all* $i \in I, n \in \mathbb{Z}, \lambda \in X$,

$$\mathbf{Fr}(E_i^{(n)}\mathbf{1}_\lambda) = \begin{cases} \pi_i^{\binom{\ell_i}{2} n/\ell_i} E_i^{(n/\ell_i)}\mathbf{1}_\lambda, & \textit{if } \ell_i \textit{ divides } n \textit{ and } \lambda \in X^\diamond, \\ 0, & \textit{otherwise} \end{cases} \tag{0.2.1}$$

and

$$\mathbf{Fr}(F_i^{(n)}\mathbf{1}_\lambda) = \begin{cases} F_i^{(n/\ell_i)}\mathbf{1}_\lambda, & \textit{if } \ell_i \textit{ divides } n \textit{ and } \lambda \in X^\diamond, \\ 0, & \textit{otherwise.} \end{cases}$$

One should note that the formulas for the latter involve a twist by a super sign on only the upper half quantum covering group.

0.2.4

We formulate the small quantum covering group and determine some of its basic properties. In particular, it is a Hopf-superalgebra that is finite dimensional in finite type.

Theorem I. *The small quantum covering group* ${}_R\mathfrak{u}$ *of type* $\mathfrak{osp}(1|2n)$ *is a finite dimensional* R^π*-module. In particular,*

$$\dim_{R^\pi}({}_R\mathfrak{u}) = \frac{\ell^{2n^2}}{\gcd(2,\ell)^{2n^2-2n}}(2\tilde{\ell})^n = \begin{cases} \ell^{2n^2}(4\ell)^n, & \text{for } \ell \text{ odd,} \\ \dfrac{\ell^{2n^2}}{2^{2n^2-2n}}(2\ell)^n, & \text{for } \ell \text{ even,} \end{cases}$$

when X *is the weight lattice, and similarly,*

$$\dim_{R^\pi}({}_R\mathfrak{u}) = \frac{\ell^{2n^2}}{\gcd(2,\ell)^{2n^2-2n}}2^{n-1}\tilde{\ell}^n = \begin{cases} \ell^{2n^2}2^{2n-1}\ell^n, & \text{for } \ell \text{ odd,} \\ \dfrac{\ell^{2n^2}}{2^{2n^2-2n}}2^{n-1}\ell^n, & \text{for } \ell \text{ even,} \end{cases}$$

when X *is the root lattice.*

0.3 Overview of dissertation

Here we note that the author has written another paper, [Sal19], in which formulas for singular vectors in Verma modules for Lie superalgebras of types $BDFG$ are determined. The results of *loc. cit.* will not be included in this work.

The following is an outline of the dissertation.

In Part I, we develop the foundations of a theory of quantum symmetric pairs at roots of 1. It is based on the paper [BS19], with the exceptions of Chapters 5 and 6.

In Chapter 1, we review some essential facts related to root data and q-binomial coefficients, as well as the modified version of the quantum group and corresponding Frobenius homomorphism that we will use in Part 1. We recall the modified $\imath$quantum group, its integral form and the $\imath$divided powers.

In Chapter 2, we define an $\imath$quantum group that is essentially an idempotented version of the enveloping algebra of the symmetric pair subalgebra. When the root data is simply connected, we show that these idempotents with be mapped in the expected way under Lusztig's Frobenius homomorphism.

In Chapter 3, we prove that Lusztig's Frobenius homomorphism descends to a homomorphism of $\imath$quantum groups. We show that this can be accomplished on the level of real rank one quantum symmetric pairs, and compute the image of each type of $\imath$divided power.

In Chapter 4, we define the small $\imath$quantum group. Furthermore, we show that it is coideal in the small quantum group and compute its dimension.

In Chapter 5, we compute central elements in $\imath$quantum group at a root of 1.

In Chapter 6, we compute maps on the quantized adjoint module induced by the $\imath$divided powers specialized to a root of 1. These maps generate a Lie algebra isomorphic to the corresponding symmetric pair subalgebra.

Part II is based on the paper [CSW19]. In it, we investigate quantum covering groups at roots of 1.

In Chapter 7, we recall (q, π)-binomial coefficients and prove several identities related to them when specialized to a root of 1.

In Chapter 8, we recall super root data and quantum covering groups, as well as the modified form. We also give a presentation of the modified form. Moreover, we determine a presentation for the quasi-classical half quantum covering group.

In Chapter 9, we prove three variants of the Frobenius homomorphism for quantum covering groups, two for the half quantum covering group and one for the modified form. We also show the existence of a Steinberg decomposition of the half quantum covering group.

In Chapter 10, we define the small quantum covering group and show that it is a finite dimensional Hopf superalgebra.

Part I

Quantum symmetric pairs at roots of 1

Chapter 1

Quantum symmetric pairs

In this chapter, we follow the conventions of [Lu94] and [BW18b].

1.1 Cartan data and root data

Let $(I, \cdot)$ be a Cartan datum (cf. [Lu94, §1.1.1]). That is, I is a finite set with a symmetric bilinear form $\nu, \nu' \mapsto \nu \cdot \nu'$ on the free abelian group $\mathbb{Z}[I]$ with values in $\mathbb{Z}$ satisfying

(a) $d_i = \frac{i \cdot i}{2} \in \mathbb{Z}_{>0}$;

(b) $2\frac{i \cdot j}{i \cdot i} \in -\mathbb{N}$ for $i \neq j$ in I.

Throughout Part I, we assume that $(I, \cdot)$ is of finite type, meaning that the corresponding Lie algebra is semisimple.

A root datum of type $(I, \cdot)$ consists of 2 finite rank lattices X, Y with a perfect bilinear pairing $\langle \cdot, \cdot \rangle : Y \times X \to \mathbb{Z}$, 2 embeddings $I \hookrightarrow X$ $(i \mapsto i')$ and $I \hookrightarrow Y$ $(i \mapsto i)$ such that $\langle i, j' \rangle = 2\frac{i \cdot j}{i \cdot i}$, $\forall i, j \in I$.

Let $\Phi \subset X$ (resp. $\Phi^+ \subset X$) be the set of roots (resp. positive roots). Let $\Phi^\vee \subset Y$ (resp. $\Phi^{\vee,+} \subset Y$) be the set of coroots (resp. positive coroots).

Let v be an indeterminate and write $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$. We define, for $n \in \mathbb{Z}$ and $a \in \mathbb{Z}_{\geq 0}$,

$$[n] = \frac{v^n - v^{-n}}{v - v^{-1}}, \quad [n]^! = [n][n-1]...[2][1], \quad \begin{bmatrix} n \\ a \end{bmatrix} = \frac{[n]^!}{[n-a]^![a]^!}.$$

For any $i \in I$, we define $v_i := v^{(i \cdot i)/2}$ and also

$$[n]_i = \frac{v_i^n - v_i^{-n}}{v_i - v_i^{-1}}, \quad [n]_i^! = [n]_i[n-1]_i...[2]_i[1]_i, \quad \begin{bmatrix} n \\ a \end{bmatrix}_i = \frac{[n]_i^!}{[n-a]_i^![a]_i^!}.$$

We also define

$$[n]_{v_i^2} = \frac{v_i^{2n} - v_i^{-2n}}{v_i^2 - v_i^{-2}}, \text{ and similarly define } \quad [n]_{v_i^2}^!, \quad \begin{bmatrix} n \\ a \end{bmatrix}_{v_i^2}.$$

Throughout the first part of the dissertation, we assume that $\ell > 0$ is an odd integer, and ℓ is prime to 3 if the Cartan datum has a factor of G_2. We define a new Cartan datum $(I, \diamond)$ with the same I and the pairing

$$i \diamond j = (i \diamond j)\ell_i \ell_j,$$

where ℓ_i is the smallest positive integer such that $\ell_i(i \cdot i)/2 \in \ell\mathbb{Z}$; cf. [Lu94, §2.2.4].

We further define a new root datum $(Y^\diamond, X^\diamond, \dots)$ of type $(I, \diamond)$ following [Lu94, §2.2.5]. Define $X^\diamond = \{\lambda \in X | \langle i, \lambda\rangle \in \ell_i\mathbb{Z} \text{ for all } i \in I\}$ and $Y^\diamond = \mathrm{Hom}(X^\diamond, \mathbb{Z})$ with the obvious bilinear pairing, which we denote by $\langle \cdot, \cdot \rangle^\diamond$. The map $I \to X^\diamond$ is given by $i \mapsto i'^\diamond = \ell_i i' \in X$. The map $I \to Y^\diamond$ associates to $i \in I$ the element $i^\diamond \in Y^\diamond$ such that $\langle i^\diamond, \zeta \rangle^\diamond = \langle i, \zeta \rangle / \ell_i$ for any $\zeta \in X^\diamond$.

We define $v_i^\diamond := v_i^{(i \diamond i)/2}$. We also define $[n]_i^\diamond$, $([n]_i^\diamond)^!$, $\begin{bmatrix} n \\ a \end{bmatrix}_i^\diamond$ in the obvious way, for $i \in I$.

Lemma 1.1.1. *We have $\langle i^\diamond, j'^\diamond \rangle^\diamond = \langle i, j' \rangle$ for any $i, j \in I$.*

Proof. Recall $\langle i^\diamond, j'^\diamond \rangle^\diamond = \langle i, \ell_j j' \rangle / \ell_i$. If $\langle i, j' \rangle = 0$ or $j = i$, we trivially have $\langle i^\diamond, j'^\diamond \rangle^\diamond = \langle i, j' \rangle$. Otherwise, we could have $\langle i, j' \rangle = 1, 2, 3$ thanks for our finite type assumption. Then since ℓ is odd and prime to 3 if there is a G_2 factor, we must have $\ell_i = \ell_j$. The lemma follows. □

As a consequence of the proof, we must have $\ell = \ell_i$ for all $i \in I$ (cf. [Lu90b, §8.4]).

1.2 Quantum groups

Given a root datum $(Y, X, \dots)$ of type $(I, \cdot)$, let $\mathbf{U}$ be the associated quantum group (cf. [Lu94, §3.1]). It is a unital $\mathbb{Q}(v)$-algebra with generators

$$E_i \quad (i \in I), \quad F_i \quad (i \in I), \quad K_\mu \quad (\mu \in Y),$$

and the following relations for all $i, j \in I, \mu, \mu' \in Y$:

$$K_0 = 1, \quad K_\mu K_{\mu'} = K_{\mu + \mu'}, \tag{i}$$

$$K_\mu E_i = v^{\langle \mu, i' \rangle} E_i K_\mu, \tag{ii}$$

$$K_\mu F_i = v^{-\langle \mu, i' \rangle} F_i K_\mu, \tag{iii}$$

$$E_i F_j - F_j E_i = \delta_{i,j} \frac{\widetilde{K}_i - \widetilde{K}_{-i}}{v_i - v_i^{-1}}, \tag{iv}$$

$$\sum_{n+n'=1-\langle i, j' \rangle} (-1)^{n'} E_i^{(n)} E_j E_i^{(n')} = 0 \tag{v}$$

$$\sum_{n+n'=1-\langle i, j' \rangle} (-1)^{n'} F_i^{(n)} F_j F_i^{(n')} = 0 \tag{vi}$$

where $\widetilde{K}_\nu = \prod_i K_{d_i \nu_i i}$ (in particular, $\widetilde{K}_i = K_{d_i i}$) for $\nu = \sum_i \nu_i i \in \mathbb{Z}[I]$, and $E_i^{(n)} = E_i^n / [n]_i^!$ and $F_i^{(n)} = F_i^n / [n]_i^!$ for $n \in \mathbb{Z}_{\geq 0}$.

Recall [Lu94, §23.1] the modified version of $\mathbf{U}$, denoted by $\dot{\mathbf{U}}$, a non-unital $\mathbb{Q}(v)$-algebra, which is generated by $1_\lambda, E_i 1_\lambda$ and $F_i 1_\lambda$, for $\lambda \in X$ and $i \in I$, with relations:

$$1_\lambda 1_{\lambda'} = \delta_{\lambda, \lambda'} 1_\lambda,$$
$$(E_i 1_\lambda) 1_{\lambda'} = \delta_{\lambda, \lambda'} E_i 1_\lambda, \quad 1_{\lambda'} (E_i 1_\lambda) = \delta_{\lambda', \lambda + i'} E_i 1_\lambda,$$
$$(F_i 1_\lambda) 1_{\lambda'} = \delta_{\lambda, \lambda'} F_i 1_\lambda, \quad 1_{\lambda'} (F_i 1_\lambda) = \delta_{\lambda', \lambda - i'} F_i 1_\lambda,$$
$$(E_i F_j - F_j E_i) 1_\lambda = \delta_{ij} \left[\langle i, \lambda \rangle \right]_i 1_\lambda,$$
$$\textstyle\sum_{n+n'=1-\langle i, j' \rangle} (-1)^{n'} E_i^{(n)} E_j E_i^{(n')} 1_\lambda = 0 \;\; (i \neq j),$$
$$\textstyle\sum_{n+n'=1-\langle i, j' \rangle} (-1)^{n'} F_i^{(n)} F_j F_i^{(n')} 1_\lambda = 0 \;\; (i \neq j),$$

where $i, j \in I$, $\lambda, \lambda' \in X$, and $xy1_\lambda = (x 1_{\lambda + |y|})(y 1_\lambda)$ for $x, y \in \mathbf{U}$.

The algebra $\dot{\mathbf{U}}$ has the direct sum decomposition

$$\dot{\mathbf{U}} = \bigoplus_{\lambda, \lambda' \in X} {}_\lambda \mathbf{U}_{\lambda'}.$$

The $\mathcal{A}$-form ${}_{\mathcal{A}}\dot{\mathbf{U}}$ is the $\mathcal{A}$-subalgebra of $\dot{\mathbf{U}}$ generated by:

$$E_i^{(n)}1_\lambda = 1_{\lambda+ni'}E_i^{(n)} = 1_{\lambda+ni'}E_i^{(n)}1_\lambda \in {}_{\lambda+ni'}\mathbf{U}_\lambda,$$

and

$$F_i^{(n)}1_\lambda = 1_{\lambda-ni'}F_i^{(n)} = 1_{\lambda-ni'}F_i^{(n)}1_\lambda \in {}_{\lambda-ni'}\mathbf{U}_\lambda,$$

for various $n \in \mathbb{Z}_{\geq 0}$, $i \in I$, $\lambda \in X$.

We define the modified quantum group with scalars in a commutative $\mathcal{A}$-algebra R as follows:

$${}_R\dot{\mathbf{U}} := R \otimes_{\mathcal{A}} {}_{\mathcal{A}}\dot{\mathbf{U}}.$$

We consider the completion $\dot{\mathbf{U}}^\wedge$ of $\dot{\mathbf{U}}$ as follows; cf. [Lu94, §36.2.3]. Recall any element in $\dot{\mathbf{U}}$ can be written uniquely as a (finite) sum $\sum_{\lambda,\lambda'} x_{\lambda,\lambda'}$ for $x_{\lambda,\lambda'} \in {}_\lambda\mathbf{U}_{\lambda'}$. We consider infinite summation of the form

$$\sum_{\lambda,\lambda'} x_{\lambda,\lambda'},$$

as long as there is a finite set $F \in X$ such that $x_{\lambda,\lambda'} = 0$ unless $\lambda - \lambda' \in F$. The algebra structure of $\dot{\mathbf{U}}$ extends naturally to an algebra structure of $\dot{\mathbf{U}}^\wedge$. We similarly define ${}_{\mathcal{A}}\dot{\mathbf{U}}^\wedge$ and ${}_R\dot{\mathbf{U}}^\wedge$ for any $\mathcal{A}$-algebra R.

Let $\mathbf{U}^\diamond$ be the quantum group over the field $\mathbb{Q}(v)$ associated with the root datum $(Y^\diamond, X^\diamond, \dots)$ of type $(I, \diamond)$, generated by (by abuse of notation) E_i, F_i, and K_μ, for all $i \in I$ and $\mu \in Y^\diamond$.

We also abuse notation by writing $E_i^{(n)} := E_i^n/([n]_i^\diamond)^! \in \mathbf{U}^*$ and $F_i^{(n)} := F_i^n/([n]_i^\diamond)^! \in \mathbf{U}^\diamond$. We similarly define $\dot{\mathbf{U}}^\diamond$, ${}_{\mathcal{A}}\dot{\mathbf{U}}^\diamond$, ${}_R\mathbf{U}^\diamond$, ${}_R\dot{\mathbf{U}}^\diamond$.

We define similarly the completions $\dot{\mathbf{U}}^{\diamond,\wedge}$, ${}_{\mathcal{A}}\dot{\mathbf{U}}^{\diamond,\wedge}$, as well as ${}_{R}\dot{\mathbf{U}}^{\diamond,\wedge}$ for any $\mathcal{A}$-algebra R.

Let $\mathcal{A}'$ be the quotient of $\mathcal{A}$ by the two-sided ideal generated by the ℓ-th cyclotomic polynomial $f_\ell \in \mathcal{A}$. Recall that $(f_1, f_2, f_3, \dots) = (v-1, v+1, v^2+v+1, \dots)$. We denote by $\phi : \mathcal{A} \to \mathcal{A}'$ the quotient map.

Theorem 1.2.1. *[Lu94, §35.1.9] There is a homomorphism of $\mathcal{A}'$-algebras $\mathbf{Fr} : {}_{\mathcal{A}'}\dot{\mathbf{U}} \to {}_{\mathcal{A}'}\dot{\mathbf{U}}^{\diamond}$ such that*

$$\mathbf{Fr} : E_i^{(n)} 1_\lambda \mapsto \begin{cases} E_i^{(n/\ell)} 1_\lambda, & \text{for } n \in \ell\mathbb{Z}, \text{ and } \lambda \in X^{\diamond}; \\ 0, & \text{otherwise.} \end{cases}$$

and

$$\mathbf{Fr} : F_i^{(n)} 1_\lambda \mapsto \begin{cases} F_i^{(n/\ell)} 1_\lambda, & \text{for } n \in \ell\mathbb{Z}, \text{ and } \lambda \in X^{\diamond}; \\ 0, & \text{otherwise.} \end{cases}$$

Note that $\mathbf{Fr}$ extends natually to an algebra homomorphism

$$\mathbf{Fr} : {}_{\mathcal{A}'}\dot{\mathbf{U}}^{\wedge} \to {}_{\mathcal{A}'}\dot{\mathbf{U}}^{\diamond,\wedge}.$$

1.3 Quantum symmetric pairs

Let τ be an involution of the Cartan datum $(I, \cdot)$; we allow $\tau = 1$. We further assume that τ extends to an involution on X and an involution on Y, respectively, such that the perfect bilinear pairing is invariant under the involution τ.

Let $I_\bullet \subset I$. We have a subroot datum of $(Y, X, \dots)$ of type $(I_\bullet, \cdot)$. Let $W = \langle s_i | i \in I \rangle$ be the Weyl group and $W_{I_\bullet} = \langle s_i | i \in I_\bullet \rangle$ be the parabolic subgroup with $w_\bullet$ as its longest element. Let $\Phi_\bullet \subset X$ (resp. $\Phi_\bullet^+ \subset X$) be the set of roots (resp. positive roots). Let $\Phi_\bullet^\vee \subset Y$ (resp. $\Phi_\bullet^{\vee,+} \subset Y$) be the set of coroots (resp. positive coroots).

Let $\rho_\bullet^\vee$ be the half sum of all positive coroots in the set $\Phi_\bullet^\vee$, and let $\rho_\bullet$ be the half sum of all positive coroots in the set $\Phi_\bullet$. We shall write

$$I_\circ = I \backslash I_\bullet. \tag{1.3.1}$$

A pair $(I_\bullet, \tau)$ is called *admissible* (cf. [Kol14, Definition 2.3]) if the following conditions are satisfied (with respect to the root datum $(Y, X, \dots)$ of type $(I, \cdot)$):

(1) $\tau(I_\bullet) = I_\bullet$;

(2) The action of τ on $I_\bullet$ coincides with the action of $-w_\bullet$;

(3) If $j \in I_\circ$ and $\tau(j) = j$, then $\langle \rho_\bullet^\vee, j' \rangle \in \mathbb{Z}$.

Let $\theta = -w_\bullet \circ \tau$ be an involution of X and Y. Following [BW18b], we introduce the $\imath$-weight lattice and $\imath$-root lattice

$$\begin{aligned} X_\imath &= X/\breve{X}, \quad \text{where } \breve{X} = \{\lambda - \theta(\lambda) | \lambda \in X\}, \\ Y^\imath &= \{\mu \in Y \big| \theta(\mu) = \mu\}. \end{aligned} \tag{1.3.2}$$

For any $\lambda \in X$, we denote its image in $X_\imath$ by $\overline{\lambda}$.

The involution τ of I induces an automorphism of the $\mathbb{Q}(v)$-algebra $\mathbf{U}$, denoted also by τ, under which $E_i \mapsto E_{\tau i}, F_i \mapsto F_{\tau i}$, and $K_\mu \mapsto K_{\tau\mu}$.

We recall here the definition of quantum symmetric pair $(\mathbf{U}, \mathbf{U}^\imath)$ following [BW18b, §3.3].

Definition 1.3.1. The algebra $\mathbf{U}^\imath$, with parameters

$$\varsigma_i \in \pm v^{\mathbb{Z}}, \quad \kappa_i \in \mathbb{Z}[v, v^{-1}], \qquad \text{for } i \in I_\circ, \tag{1.3.3}$$

is the $\mathbb{Q}(v)$-subalgebra of $\mathbf{U}$ generated by the following elements:

$$F_i + \varsigma_i \mathtt{T}_{w_\bullet}(E_{\tau i})\widetilde{K}_i^{-1} + \kappa_i \widetilde{K}_i^{-1}\,(i \in I_\circ),$$
$$K_\mu\,(\mu \in Y^\imath), \quad F_i\,(i \in I_\bullet), \quad E_i\,(i \in I_\bullet).$$

The parameters are required to satisfy Conditions (1.3.4)-(1.3.7):

$$\begin{aligned} &\kappa_i = 0 \ \text{ unless } \tau(i) = i, \langle i, j'\rangle = 0 \ \forall j \in I_\bullet, \\ &\qquad \text{and } \langle k, i'\rangle \in 2\mathbb{Z}\ \forall k = \tau(k) \in I_\circ \text{ such that } \langle k, j'\rangle = 0 \ \text{ for all } j \in I_\bullet; \end{aligned} \tag{1.3.4}$$

$$\overline{\kappa_i} = \kappa_i; \tag{1.3.5}$$

$$\varsigma_i = \varsigma_{\tau i} \text{ if } i \cdot \theta(i) = 0; \tag{1.3.6}$$

$$\varsigma_{\tau i} = (-1)^{\langle 2\rho^\vee_\bullet, i'\rangle} v_i^{-\langle i, 2\rho_\bullet + w_\bullet \tau i'\rangle} \overline{\varsigma_i}. \tag{1.3.7}$$

Remark 1.3.2. Note that we require $\varsigma_i \in \pm v^{\mathbb{Z}}$, instead of $\varsigma_i \in \mathbb{Z}[v, v^{-1}]$ as in [BW18c]. Note that such ς_i always exists by [BK15, Remark 3.14], thanks to the establishment of [BK15, Conjecture 2.7] in [BW18c, Theorem 4.1].

We recall the modified coideal subalgebra $\dot{\mathbf{U}}^\imath$ from [BW18b, §3.7]. First define for all $\lambda', \lambda'' \in X_\imath$,

$${}_{\lambda'}\mathbf{U}^\imath_{\lambda''} = \mathbf{U}^\imath / \big(\sum_{\mu \in Y^\imath} (K_\mu - q^{\langle \mu, \lambda'\rangle})\mathbf{U}^\imath + \sum_{\mu \in Y^\imath} \mathbf{U}^\imath (K_\mu - q^{\langle \mu, \lambda''\rangle})\big)$$

Let $\pi_{\lambda',\lambda''} : \mathbf{U}^\imath \to {}_{\lambda'}\mathbf{U}^\imath_{\lambda''}$ be the natural projections and write $\pi_{\lambda,\lambda} = 1_\lambda$. Then,

$$\dot{\mathbf{U}}^\imath = \bigoplus_{\lambda',\lambda''\in X_\imath} {}_{\lambda'}\mathbf{U}^\imath_{\lambda''}$$

is a non-unital associative $\mathbb{Q}(v)$-algebra. It follows from [BW18b, §3.7] that $\dot{\mathbf{U}}$ is naturally a $\dot{\mathbf{U}}^\imath$-bimodule.

We recall here the integral from ${}_\mathcal{A}\dot{\mathbf{U}}^\imath$ of the modified coideal subalgebra $\dot{\mathbf{U}}^\imath$.

Definition 1.3.3. [BW18c, Definition 3.10] We define ${}_\mathcal{A}\dot{\mathbf{U}}^\imath$ to be the set of elements $u \in \dot{\mathbf{U}}^\imath$ such that $u \cdot m \in {}_\mathcal{A}\dot{\mathbf{U}}$ for all $m \in {}_\mathcal{A}\dot{\mathbf{U}}$.

The $\imath$-divided powers, $B^{(a)}_{i,\zeta} \in {}_\mathcal{A}\dot{\mathbf{U}}^\imath$ ($i \in I$, $a \in \mathbb{Z}_{>0}$) play a crucial role in Part I. See (3.2.1), (3.3.1), (3.4.1), (3.4.2), (3.4.3), and (3.4.4) for more detailed descriptions. For now, we only need to recall the following.

Proposition 1.3.4. *[BW18c, Theorem 5.1, Theorem 7.2, Corollary 7.5]*

1. *${}_\mathcal{A}\dot{\mathbf{U}}^\imath$ is a free $\mathcal{A}$-module.*
2. *${}_\mathcal{A}\dot{\mathbf{U}}^\imath$ is generated as an $\mathcal{A}$-subalgebra of $\dot{\mathbf{U}}^\imath$ by the $\imath$-divided powers $B^{(a)}_{i,\zeta}$ ($i \in I$) and $E^{(a)}_j 1_\zeta$ ($j \in I_\bullet$) for $\zeta \in X_\imath$ and $a \geq 0$.*

Let $x_{\lambda,\lambda'} \in {}_\lambda\mathbf{U}^\imath_{\lambda'}$. We consider the completion $\dot{\mathbf{U}}^{\imath,\wedge}$ of $\dot{\mathbf{U}}^\imath$ analogous to that of $\dot{\mathbf{U}}$ in §1.2 by allowing infinite summation

$$\sum_{\lambda,\lambda'} x_{\lambda,\lambda'}$$

as long as there is a finite set $G \in X_\imath$ such that $x_{\lambda,\lambda'} = 0$ unless $\lambda - \lambda' \in G$. The following lemma is straightforward.

Lemma 1.3.5. *We have the algebra embedding*

$$\begin{aligned}\imath : \dot{\mathbf{U}}^{\imath,\wedge} &\longrightarrow \dot{\mathbf{U}}^{\wedge},\\ x &\mapsto \sum_{\lambda\in X} x\mathbf{1}_\lambda.\end{aligned}$$

In particular, $\imath$ restricts to embeddings $\imath : \dot{\mathbf{U}}^{\imath} \to \dot{\mathbf{U}}^{\wedge}$, and $\imath : {}_{\mathcal{A}}\dot{\mathbf{U}}^{\imath} \to {}_{\mathcal{A}}\dot{\mathbf{U}}^{\wedge}$.

For a commutative $\mathcal{A}$-algebra R, we define the modified coideal subalgebra with scalars in R as

$$_R\dot{\mathbf{U}}^{\imath} = R \otimes_{\mathcal{A}} {}_{\mathcal{A}}\dot{\mathbf{U}}^{\imath}.$$

We similarly define $_R\dot{\mathbf{U}}^{\imath,\wedge}$.

Proposition 1.3.6. *Let R be a commutative $\mathcal{A}$-algebra. The following induced embedding after base change is injective*

$$\imath : {}_R\dot{\mathbf{U}}^{\imath} \longrightarrow {}_R\dot{\mathbf{U}}^{\wedge}. \tag{1.3.8}$$

Proof. Recall the canonical basis $\dot{\mathbf{B}}^{\imath}$ of $_{\mathcal{A}}\dot{\mathbf{U}}^{\imath}$ from [BW18c, Theorem 7.2]. For any $b_1\diamondsuit^{\imath}_{\zeta}b_2 \in \dot{\mathbf{B}}^{\imath}$ and $\lambda \in X$ such that $\overline{\lambda} = \zeta \in X_{\imath}$, we have

$$b_1\diamondsuit^{\imath}_{\zeta}b_2 1_\lambda = b_1\diamondsuit_{\lambda}b_2 1_\lambda + \text{lower terms}.$$

Here $b_1\diamondsuit_{\lambda}b_2$ denotes Lusztig's canonical basis element ([Lu94, Theorem 25.2.1]) on $_{\mathcal{A}}\dot{\mathbf{U}}$. The proposition follows, since the coefficient of the leading term is 1. □

Chapter 2

The $\imath$quantum group $\mathbf{U}^{\diamond,\imath}$

In this chapter, we define the $\imath$quantum group associated with the root datum $(Y^\diamond, X^\diamond, \dots)$ and the pair $(I_\bullet, \tau)$.

2.1 Root data

Recall the root datum $(Y^\diamond, X^\diamond, \dots)$ in §1.1. Note that since τ is an involution of the Cartan datum $(I, \cdot)$, it is naturally an involution the Cartan datum $(I, \diamond)$. The involution τ restricts to an involution of $X^\diamond \subset X$. The involution τ extends naturally on $Y^\diamond = \mathrm{Hom}(X^\diamond, \mathbb{Z})$, such that the perfect pairing $\langle \cdot, \cdot \rangle^\diamond$ is τ-invariant. Thanks to Lemma 1.1.1, the pair $(I_\bullet, \tau)$ is admissible with respect to the root datum $(Y^\diamond, X^\diamond, \dots)$.

Recall the definition of $X_\imath$ and $Y^\imath$ in (1.3.2). We similarly define $X^\diamond_\imath$ and $Y^{\diamond,\imath}$. In particular, we have

$$X^\diamond_\imath = X^\diamond / \breve{X}^\diamond, \quad \text{where } \breve{X}^\diamond = \{\lambda - \theta(\lambda) | \lambda \in X^\diamond\},$$

and

$$Y^{\diamond,\imath} = \{\mu \in Y^{\diamond} \big| \theta(\mu) = \mu\}.$$

Lemma 2.1.1. *If $\lambda \in X^{\diamond}$, then we have $\theta(\lambda) \in X^{\diamond}$.*

Proof. Note that $X^{\diamond}$ is τ-invariant by our assumption on τ. The sublattice $X^{\diamond}$ is invariant under the Weyl group action as well, thanks to Lemma 1.1.1. Therefore $X^{\diamond}$ is invariant under θ. $\square$

So we have $\check{X}^{\diamond} \subset X^{\diamond} \cap \check{X}$ and the following commutative diagram:

$$\begin{array}{ccc} X^{\diamond} & \longrightarrow & X \\ \downarrow & & \downarrow \\ X^{\diamond}_{\imath} & \longrightarrow & X_{\imath} \end{array}$$

A root datum $(Y, X, \dots)$ is simply connected if X is the weight lattice $\oplus_{i\in\mathbb{I}}\mathbb{Z}\omega_i$ and $Y = Hom(X, \mathbb{Z})$.

Lemma 2.1.2. *If the root datum $(Y, X, \dots)$ is simply connected, then we have $\check{X}^{\diamond} = X^{\diamond} \cap \check{X}$.*

Proof. We need to show that if $\langle i, \lambda - \theta(\lambda)\rangle \in \ell\mathbb{Z}$, then $\lambda - \theta(\lambda) = \mu - \theta(\mu)$ for some $\mu \in X^{\diamond}$.

We write $X = \oplus_{i\in I}\mathbb{Z}\omega_i$, where ω_i denotes the i-th fundamental weight. Let

$$\lambda = \sum_{i\in I} a_i\omega_i.$$

Note that since

$$\theta(\omega_i) = \begin{cases} -\omega_{\tau i}, & \text{if } i \in I_\circ; \\ \omega_i, & \text{if } i \in I_\bullet, \end{cases}$$

we have

$$\lambda - \theta(\lambda) = \sum_{i \in I_\circ} (a_i + a_{\tau i})\omega_i.$$

Since $\langle i, \lambda - \theta(\lambda) \rangle \in \ell\mathbb{Z}$, we have

$$a_i \equiv -a_{\tau i}, \quad \bmod \ell.$$

Since ℓ is odd, then we must have

$$a_i \equiv a_{\tau i} \equiv 0, \quad \bmod \ell, \text{ if } i = \tau i.$$

Therefore by considering τ-orbits in $I_\circ$, we can find $1 - \ell \leq b_i \leq \ell - 1$ for each $i \in I_\circ$ such that

$$b_i + b_{\tau i} = 0, \quad \text{and} \quad b_i \equiv a_i \mod \ell.$$

Now we can simply take

$$\mu = \sum_{i \in I_\circ} a_i\omega_i - \sum_{i \in I_\circ} b_i\omega_i.$$

Then since $\langle i, \mu \rangle \in \ell\mathbb{Z}$ for all $i \in I_\circ$, we have $\mu \in X^\diamond$. Since $b_i + b_{\tau i} = 0$ for all $i \in I_\circ$ by definition, we have

$$\mu - \theta(\mu) = \sum_{i \in I_\circ} (a_i + a_{\tau i} + b_i + b_{\tau i})\omega_i = \lambda - \theta(\lambda).$$

The lemma follows. □

For the rest of Part I, we assume $\breve{X}^{\diamond} = X^{\diamond} \cap \breve{X}$. Thanks to the previous lemma, the equality holds when the root datum $(Y, X, \dots)$ is simply connected.

Lemma 2.1.3. *Let $\lambda \in X$ such that $\overline{\lambda} \in X^{\diamond}_{\imath}$, then $\lambda \in X^{\diamond}$.*

Proof. We have

$$\lambda = \mu + \nu - \theta(\nu), \text{ with } \mu, \nu \in X^{\diamond}.$$

Then obviously we have $\lambda \in X^{\diamond}$. □

2.2 The $\imath$quantum group $\mathbf{U}^{\diamond,\imath}$

We now define the $\imath$quantum group associated with the root datum $(Y^{\diamond}, X^{\diamond}, \dots)$ and the pair $(I_{\bullet}, \tau)$.

Definition 2.2.1. The algebra $\mathbf{U}^{\diamond,\imath}$, with parameters

$$\varsigma_i^{\diamond} = \varsigma_i^{\ell^2}, \quad \kappa_i^{\diamond} = \kappa_i^{\ell^2}, \qquad \text{for } i \in I_{\circ}, \tag{2.2.1}$$

is the $\mathbb{Q}(v)$-subalgebra of $\mathbf{U}^{\diamond}$ generated by the following elements:

$$F_i + \varsigma_i^{\diamond} \mathtt{T}_{w_\bullet}(E_{\tau i}) \widetilde{K}_i^{-1} + \kappa_i^{*} \widetilde{K}_i^{-1} \, (i \in I_{\circ}),$$
$$K_{\mu} \, (\mu \in Y^{\imath,\diamond}), \quad F_i \, (i \in I_{\bullet}), \quad E_i \, (i \in I_{\bullet}).$$

Lemma 2.2.2. *The parameters $\varsigma_i^\diamond$ and $\kappa_i^\diamond$ satisfy the following conditions:*

$$\kappa_i^\diamond = 0 \;\; \textit{unless}\; \tau(i) = i, \langle i^\diamond, j'^\diamond\rangle^\diamond = 0\; \forall j \in I_\bullet,$$

$$\textit{and}\; \langle k^\diamond, i'^\diamond\rangle^\diamond \in 2\mathbb{Z}\; \forall k = \tau(k) \in I_\circ \;\textit{such that}\; \langle k^\diamond, j'^\diamond\rangle^\diamond = 0 \;\; \textit{for all}\; j \in I_\bullet;$$

$$\varsigma_i^\diamond = \varsigma_{\tau i}^\diamond \;\textit{if}\; i \circ \theta(i) = 0;$$

$$\varsigma_{\tau i}^\diamond = (-1)^{\langle 2(\rho_\bullet^\diamond)^\vee, i'^\diamond\rangle^\diamond} (v_i^\diamond)^{-\langle i^\diamond, 2\rho_\bullet^\diamond + w_\bullet \tau i'^\diamond\rangle^\diamond}\, \overline{\varsigma_i^\diamond}.$$

Proof. The lemma follows directly from Lemma 1.1.1, and the fact that $v_i^\diamond = v_i^{\ell^2}$. □

So $(\mathbf{U}^\diamond, \mathbf{U}^{\imath,\diamond})$ is also a quantum symmetric pair as defined in Definition 1.3.1. Hence results from [BW18b, BW18c] applies. Therefore, we can similarly define the following as before (R is any commutative $\mathcal{A}$-algebra)

$$_\mathcal{A}\dot{\mathbf{U}}^{\diamond,\imath},\; _R\dot{\mathbf{U}}^{\diamond,\imath},\; \dot{\mathbf{U}}^{\diamond,\imath,\wedge},\; _\mathcal{A}\dot{\mathbf{U}}^{\diamond,\imath,\wedge},\; _R\dot{\mathbf{U}}^{\diamond,\imath,\wedge}.$$

In particular, we have the following counterpart of Proposition 1.3.6:

$$\imath : {}_R\dot{\mathbf{U}}^{\diamond,\imath,\wedge} \longrightarrow {}_R\dot{\mathbf{U}}^{\diamond,\wedge}. \tag{2.2.2}$$

Remark 2.2.3. If we take l to be even, then the pair $(I_\bullet, \tau)$ may not be admissible with respect to the root datum $(Y^\diamond, X^\diamond, \dots)$. Let us illustrate this phenomenon in the following example.

Let $(I = \{\alpha_1, \alpha_2\}, \cdot)$ be the Cartan datum of type B_2, where α_2 denotes the short root. Let $(Y, X, \dots)$ be any root datum of type $(I, \cdot)$. We take $I_\bullet = \{\alpha_2\}$ and $\tau = \mathrm{id}$. It follows that $(I_\bullet, \tau)$ is admissible, which one can see from the Satake diagram. Let $\ell = 4$. Then $(I = \{\alpha_1, \alpha_2\}, \circ)$ is actually of type C_2 with short root α_1. It is easy to see that $(I_\bullet, \tau)$ is not admissible anymore, which one can again observe from the Satake diagram.

Chapter 3

The Frobenius-Lusztig homomorphism for QSP

3.1 Quantum Frobenius homomorphism

Let $\mathcal{A}'$ be the quotient of $\mathcal{A}$ by the two-sided ideal generated by the ℓ-th cyclotomic polynomial $f_\ell \in \mathcal{A}$ through out this section. (Recall that $(f_1, f_2, f_3, \dots) = (v-1, v+1, v^2+v+1, \dots)$). We denote by $\phi : \mathcal{A} \to \mathcal{A}'$ the quotient map.

Recall the Frobenius morphism in Theorem 1.2.1. Thanks to the embeddings (1.3.8) and (2.2.2), we have

$$\begin{array}{ccc} {}_{\mathcal{A}'}\dot{\mathbf{U}}^{\imath} & \xrightarrow{\imath} & {}_{\mathcal{A}'}\dot{\mathbf{U}}^{\wedge} \\ & & \big\downarrow \mathbf{Fr} \\ {}_{\mathcal{A}'}\dot{\mathbf{U}}^{\diamond,\imath} & \xrightarrow{\imath} & {}_{\mathcal{A}'}\dot{\mathbf{U}}^{\diamond,\wedge} \end{array} .$$

Recall the parameters for the algebra $\mathbf{U}^{\imath}$ in (2.2.1). **For the rest of Part I, we shall only consider the case when the parameter $\kappa_i = 0$.** We remark that this restriction is only relevant to the computation in §3.4. The goal of this chapter is to establish the following theorem:

Theorem 3.1.1. *The quantum Frobenius morphism restricts to an $\mathcal{A}'$-algebra homomorphism*

$$\mathbf{Fr} : {}_{\mathcal{A}'}\dot{\mathbf{U}}^{\imath} \longrightarrow {}_{\mathcal{A}'}\dot{\mathbf{U}}^{\diamond,\imath}.$$

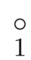
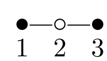
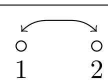
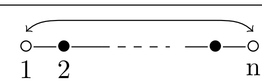
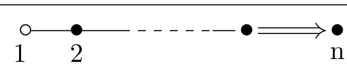
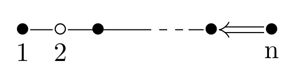
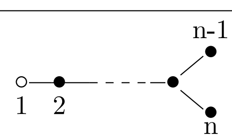
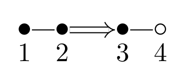
Recall Proposition 1.3.4 that ${}_{\mathcal{A}'}\dot{\mathbf{U}}^{\imath}$ is generated by the $\imath$-divided powers $B_{i,\zeta}^{(a)}$ $(i \in I)$ and $E_j^{(a)}1_\zeta$ $(j \in I_\bullet)$ for $\zeta \in X_i$ and $a \geq 0$. It suffices to prove that

1. $\mathbf{Fr}(\imath(E_j^{(a)}1_\zeta)) = \mathbf{Fr}(\sum_{\lambda\in X} E_j^{(a)}1_\lambda) \in$ Image of $\imath : {}_{\mathcal{A}'}\dot{\mathbf{U}}^{\diamond,\imath} \to {}_{\mathcal{A}'}\dot{\mathbf{U}}^{\diamond,\wedge}$,

2. $\mathbf{Fr}(\imath(B_{i,\zeta}^{(a)})) = \mathbf{Fr}(\sum_{\lambda\in X} B_{i,\zeta}^{(a)}1_\lambda) \in$ Image of $\imath : {}_{\mathcal{A}'}\dot{\mathbf{U}}^{\diamond,\imath} \to {}_{\mathcal{A}'}\dot{\mathbf{U}}^{\diamond,\wedge}$.

The proof reduces to real rank one quantum symmetric pairs. The statement (1) is trivial. We shall prove the statement (2) case by case for real rank one quantum symmetric pairs. We include the Satake diagram for real rank one quantum symmetric pairs for the readers' convenience.

Table 3.1: Satake diagrams of symmetric pairs of real rank one

AI_1		AII_3	
AIII_{11}		AIV, n≥2	
BII, n≥ 2		CII, n≥3	
DII, n≥4		FII	

3.2 The first case: $\tau(i) \neq i$

Let $i \in I_\circ$ be such that $\tau(i) \neq i$. This includes type AIII_{11} and type AIV. We have $\kappa_i = 0$, and $B_i = F_i + \varsigma_i \mathtt{T}_{w_\bullet}(E_{\tau i})\widetilde{K}_i^{-1}$. We write

$$Y_i := \varsigma_i \mathtt{T}_{w_\bullet}(E_{\tau i})\widetilde{K}_i^{-1}, \qquad Y_i^{(n)} = Y_i^n/[n]_i^!.$$

We have $F_i Y_i - v_i^{-2} Y_i F_i = [F_i, Y_i]\widetilde{K}_i^{-1} = 0$. Following [BW18c, §5.5.1], we define

$$B_{i,\zeta}^{(n)} = \frac{B_i^n}{[n]_i!}\mathbf{1}_\zeta = \sum_{a=0}^{n} v_i^{-a(n-a)} Y_i^{(a)} F_i^{(n-a)} \mathbf{1}_\zeta \in {}_{\mathcal{A}}\dot{\mathbf{U}}^\imath. \tag{3.2.1}$$

Proposition 3.2.1. *We have that*

$$\mathbf{Fr} : {}_{\mathcal{A}'}\dot{\mathbf{U}}^\wedge \longrightarrow {}_{\mathcal{A}'}\dot{\mathbf{U}}^{\diamond,\wedge},$$

$$\sum_{\lambda\in X} B_{i,\zeta}^{(n)} 1_\lambda \mapsto \begin{cases} \sum_{\lambda\in X^\diamond} B_{i,\zeta}^{(n/\ell)} 1_\lambda, & \text{if } n \in \ell\mathbb{Z}, \zeta \in X_\imath^\diamond; \\ 0, & \text{otherwise.} \end{cases}$$

In other words, we have

$$\mathbf{Fr} : {}_{\mathcal{A}'}\dot{\mathbf{U}}^\imath \longrightarrow {}_{\mathcal{A}'}\dot{\mathbf{U}}^{\diamond,\imath},$$

$$B_{i,\zeta}^{(n)} \mapsto \begin{cases} B_{i,\zeta}^{(n/\ell)}, & \text{if } n \in \ell\mathbb{Z}, \zeta \in X_\imath^\diamond; \\ 0, & \text{otherwise.} \end{cases}$$

Proof. It suffices to check that

$$\mathbf{Fr}(B_{i,\zeta}^{(n)} 1_\lambda) = B_{i,\zeta}^{(n/\ell)} 1_\lambda, \text{ for } \zeta \in X_\imath^\diamond, \overline{\lambda} = \zeta.$$

We check only the cases where $n = k\ell \in \ell\mathbb{Z}$. The other cases are entirely similar.

We can use the quantum binomial formula ([Lu94, §1.3.5]) to write, for $\lambda \in X^\diamond$ with $\overline{\lambda} = \zeta$,

$$\begin{aligned}\mathbf{Fr}(B_{i,\zeta}^{(n)}\mathbf{1}_\lambda) &= \mathbf{Fr}((\sum_{a=0}^{n} v_i^{-a(n-a)} Y_i^{(a)} F_i^{(n-a)})\mathbf{1}_\lambda) \\ &\overset{\heartsuit}{=} (\sum_{a=0}^{k} (v_i^\diamond)^{-a(n-a)} Y_i^{(a)} F_i^{(n-a)})\mathbf{1}_\lambda \\ &= B_{i,\zeta}^{(k)}\mathbf{1}_\lambda,\end{aligned}$$

where $\heartsuit$ follows from Theorem 1.2.1, and the fact that $\varsigma_i^{l_i^2} = \varsigma_i^\diamond$. □

3.3 The second case: $\tau(i) = i \neq w_\bullet(i)$

Let $i \in I_\circ$ such that $\tau(i) = i \neq w_\bullet(i)$. This includes the types: BII, DII, AII_3, CII, and FII. We have $\kappa_i = 0$, and $B_i = F_i + \varsigma_i \mathtt{T}_{w_\bullet}(E_{\tau i})\widetilde{K}_i^{-1}$.

We write

$$Y_i = \varsigma_i \mathtt{T}_{w_\bullet}(E_i)\widetilde{K}_i^{-1}, \quad b_i^{(n)} = \sum_{a=0}^{n} v_i^{-a(n-a)} Y_i^{(a)} F_i^{(n-a)}.$$

Following ([BW18c, (5.12)]), we define

$$B_{i,\zeta}^{(n)}\mathbf{1}_\lambda = b_i^{(n)}\mathbf{1}_\lambda + \frac{v}{v - v^{-1}} \sum_{k\geq 1} v_i^{\frac{k(k+1)}{2}} \mathfrak{z}_i^{(k)} b_i^{(n-2k)}\mathbf{1}_\lambda, \tag{3.3.1}$$

where

$$\mathfrak{z}_i^{(n)} = -v_i^{\frac{1}{2}n(n-1)} (\sum_{a=0}^{n-1} v_i^{-2n^2+2na-\frac{1}{2}a(a-1)} Y_i^{(n-a)} F_i^{(n-a)} \mathfrak{z}_i^{(a)} - F_i^{(n)} Y_i^{(n)}).$$

Lemma 3.3.1. *We have that*

$$\begin{aligned}\mathbf{Fr} : {}_{\mathcal{A}'}\dot{\mathbf{U}}^{\wedge} &\longrightarrow {}_{\mathcal{A}'}\dot{\mathbf{U}}^{\diamond,\wedge},\\ \sum_{\lambda\in X} b_i^{(n)} 1_\lambda &\mapsto \begin{cases} \sum_{\lambda\in X^\diamond} b_i^{(n/\ell)} 1_\lambda, & \text{if } n \in \ell\mathbb{Z};\\ 0, & \text{otherwise.}\end{cases}\end{aligned}$$

In other words, we have

$$\begin{aligned}\mathbf{Fr} : {}_{\mathcal{A}'}\dot{\mathbf{U}}^{\imath} &\longrightarrow {}_{\mathcal{A}'}\dot{\mathbf{U}}^{\diamond,\imath},\\ b_i^{(n)} 1_\zeta &\mapsto \begin{cases} b_i^{(n/\ell)} 1_\zeta, & \text{if } n \in \ell\mathbb{Z}, \zeta \in X_\imath^\diamond;\\ 0, & \text{otherwise.}\end{cases}\end{aligned}$$

Proof. This is the same computation as Proposition 3.2.1. $\square$

Proposition 3.3.2. *We have that*

$$\begin{aligned}\mathbf{Fr} : {}_{\mathcal{A}'}\dot{\mathbf{U}}^{\wedge} &\longrightarrow {}_{\mathcal{A}'}\dot{\mathbf{U}}^{\diamond,\wedge},\\ \sum_{\lambda\in X} B_{i,\zeta}^{(n)} 1_\lambda &\mapsto \begin{cases} \sum_{\lambda\in X^\diamond} b_i^{(n/\ell)} 1_\lambda, & \text{if } n \in \ell\mathbb{Z}, \zeta \in X_\imath^\diamond;\\ 0, & \text{otherwise.}\end{cases}\end{aligned}$$

In other words, we have

$$\begin{aligned}\mathbf{Fr} : {}_{\mathcal{A}'}\dot{\mathbf{U}}^{\imath} &\longrightarrow {}_{\mathcal{A}'}\dot{\mathbf{U}}^{\diamond,\imath},\\ B_i^{(n)} 1_\zeta &\mapsto \begin{cases} b_i^{(n/\ell)} 1_\zeta, & \text{if } n \in \ell\mathbb{Z}, \zeta \in X_\imath^\diamond;\\ 0, & \text{otherwise.}\end{cases}\end{aligned}$$

Proof. We first have

$$\mathbf{Fr}(\mathfrak{z}_i^{(n)}\mathbf{1}_\lambda) = 0, \qquad \text{for } 1 \le n \le \ell - 1, \lambda \in X.$$

Moreover,

$$\mathbf{Fr}(\mathfrak{z}_i^{(\ell)}\mathbf{1}_\lambda) = -\mathbf{Fr}(Y_i^{(\ell)}F_i^{(\ell)} - F_i^{(\ell)}Y_i^{(\ell)})\mathbf{1}_\lambda = \varsigma_i^\ell[F_i, \mathtt{T}_{w_\bullet}(E_i)]\mathbf{1}_\lambda = 0, \text{ for any } \lambda \in X.$$

The last equality follows from case by case computation. It suffices to notice that in none of these cases is $w_\bullet(i') - i'$ a root.

For $1 \le n \le \ell - 1$,

$$0 = \mathbf{Fr}(\mathfrak{z}_i^{(k\ell)}\mathbf{1}_{\mu'})\mathbf{Fr}(\mathfrak{z}_i^{(n)}\mathbf{1}_\mu) = \begin{bmatrix} k\ell + n \\ n \end{bmatrix}_i \mathbf{Fr}(\mathfrak{z}_i^{(k\ell+n)}\mathbf{1}_\mu)$$

and

$$0 = \mathbf{Fr}(\mathfrak{z}_i^{(\ell)})^k\mathbf{1}_\mu = k!\mathbf{Fr}(\mathfrak{z}_i)^{(k\ell)}\mathbf{1}_\mu.$$

Therefore we have

$$\mathbf{Fr}(\mathfrak{z}_i^{(n)}\mathbf{1}_\mu) = 0 \quad \text{for } n \ge 1.$$

The proposition follows from (3.3.1) and Lemma 3.3.1. □

3.4 The third case: $\tau(i) = i = w_\bullet(i)$

In this final section, we consider the case $\tau(i) = i = w_\bullet(i)$. This is of type AI_1. In this case, we have $B_i = F_i + \varsigma_i E_i \widetilde{K}_i^{-1}$. Following the definition of $\imath$divided powers in this case given in [BW18a], the precise formula for $B_{i,\zeta}^{(n)}$ has been obtained in [BW18d] (recall the parameter $\kappa_i = 0$).

Let $\lambda \in X$ such that $\overline{\lambda} = \zeta$. The parity of $\langle i, \lambda\rangle \in \mathbb{Z}$ depends only on ζ, but not on λ. We shall simply call it the parity of $\langle i, \zeta\rangle$. The computation divides into two cases depends on the parity of $\langle i, \zeta\rangle$. We shall focus on the case when $\langle i, \zeta\rangle$ is even. The odd case is entirely similar.

Proposition 3.4.1. *[BW18d, Proposition 2.8] Let $\langle i, \zeta\rangle$ be even. Let $m \geq 1$, and $\lambda \in X$ such that $\overline{\lambda} = \zeta$. We have*

$$\begin{aligned} B_{i,\zeta}^{(2m)} 1_\lambda = & \sum_{c=0}^{m} \sum_{a=0}^{2m-2c} v_i^{2(a+c)(m-a-\langle i,\lambda\rangle/2)-2ac-\binom{2c+1}{2}} \\ & \cdot \begin{bmatrix} m-c-a-\langle i,\lambda\rangle/2 \\ c \end{bmatrix}_{v_i^2} E_i^{(a)} F_i^{(2m-2c-a)} 1_\lambda, \end{aligned} \tag{3.4.1}$$

$$\begin{aligned} B_{i,\zeta}^{(2m-1)} 1_\lambda = & \sum_{c=0}^{m-1} \sum_{a=0}^{2m-1-2c} v_i^{2(a+c)(m-a-\langle i,\lambda\rangle/2)-2ac-a-\binom{2c+1}{2}} \cdot \\ & \begin{bmatrix} m-c-a-\langle i,\lambda\rangle/2-1 \\ c \end{bmatrix}_{v_i^2} E_i^{(a)} F_i^{(2m-1-2c-a)} 1_\lambda. \end{aligned} \tag{3.4.2}$$

Proposition 3.4.2. *Let $\langle i, \zeta\rangle$ be even. We have, via restriction,*

$$\mathbf{Fr} : {}_{\mathcal{A}'}\dot{\mathbf{U}}^{\wedge} \longrightarrow {}_{\mathcal{A}'}\dot{\mathbf{U}}^{\diamond,\wedge},$$

$$\sum_{\lambda\in X} B_{i,\zeta}^{(n)} 1_\lambda \mapsto \begin{cases} \sum_{\lambda\in X^\diamond} B_{i,\zeta}^{(n/\ell)} 1_\lambda, & \text{for } n\in \ell\mathbb{Z} \text{ and } \zeta\in X_\imath^\diamond; \\ \sum_{\lambda\in X^\diamond} \begin{bmatrix} (\ell-1)/2 \\ b/2 \end{bmatrix}_{v_i^2} B_{i,\zeta}^{(k)} 1_\lambda, & \text{for } n = k\ell+b,\ k \text{ odd}, \\ & b \text{ even},\ 0<b<\ell,\ \zeta\in X_\imath^\diamond; \\ 0, & \text{otherwise.} \end{cases}$$

In other words, we have

$$\mathbf{Fr} : {}_{\mathcal{A}'}\dot{\mathbf{U}}^{\imath} \longrightarrow {}_{\mathcal{A}'}\dot{\mathbf{U}}^{\diamond,\imath},$$

$$B_{i,\zeta}^{(n)} \mapsto \begin{cases} B_{i,\zeta}^{(n/\ell)}, & \text{for } n\in \ell\mathbb{Z} \text{ and } \zeta\in X_\imath^\diamond; \\ \begin{bmatrix} (\ell-1)/2 \\ b/2 \end{bmatrix}_{v_i^2} B_{i,\zeta}^{(k)}, & \text{for } n = k\ell+b,\ k \text{ odd}, \\ & b \text{ even},\ 0<b<\ell,\ \zeta\in X_\imath^\diamond; \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Let $\lambda \in X^\diamond$ such that $\overline{\lambda} = \zeta$. We divide the computation into several cases.

(1) We first consider the case when $n = 2m$ is even, where either $0 < n = 2m < l$ or $n = 2m = k\ell$. Recall (3.4.1) for the expression of $B_{i,\zeta}^{(n)} 1_\lambda$.

(1.1) We assume $0 < n = 2m < \ell$. Then it follows from direct computation and [Lu94, Lemma 34.1.2] that

$$\mathbf{Fr}(B_{i,\zeta}^{(n)} 1_\lambda) = 0.$$

(1.2) We assume $n = 2m = k\ell$. Note that

$$\mathbf{Fr}(E_i^{(a)} F_i^{(2m-2c-a)}) = \begin{cases} E_i^{(a/\ell)} F_i^{((2m-2c-a)/\ell)}, & \text{if } \ell | a \text{ and } \ell | c; \\ 0, & \text{otherwise.} \end{cases}$$

However, if $\ell|a$ and $\ell|c$, we must have $v_i^{2(a+c)(m-a-\langle i,\lambda\rangle/2)-2ac-\binom{2c+1}{2}} = 1$.

Then, we have

$$\begin{aligned} &\mathbf{Fr}(B_{i,\zeta}^{(2k\ell)} 1_\lambda) \\ =& \sum_{c=0}^{k} \sum_{a=0}^{2k-2c} \begin{bmatrix} k\ell - c\ell - a\ell - \langle i, \lambda \rangle/2 \\ c\ell \end{bmatrix}_{v_i^2} E_i^{(a)} F_i^{(2k-2c-a)} 1_\lambda \\ =& \sum_{c=0}^{k} \sum_{a=0}^{2k-2c} \begin{pmatrix} k - c - a - \langle i, \lambda \rangle^\diamond/2 \\ c \end{pmatrix} E_i^{(a)} F_i^{(2k-2c-a)} 1_\lambda \\ =& B_{i,\zeta}^{(2k)} 1_\lambda. \end{aligned}$$

(2) We then consider the case when $n = 2m-1$ is odd, where either $0 < n = 2m - 1 < \ell$ or $n = 2m-1 = (2k-1)\ell$. Recall (3.4.2) for the expression of $B_{i,\zeta}^{(n)} 1_\lambda$ in this case.

(2.1) We assume $n = 2m - 1 < \ell$. Then by direct computation, we must have

$$\mathbf{Fr}(B_{i,\zeta}^{(n)} 1_\lambda) = 0.$$

(2.2) We assume $n = 2m-1 = (2k-1)\ell$ for $2k - 1 > 0$. Then we have

$$m - 1 = (k-1)\ell + (\ell - 1)/2, \quad \text{with } 0 < (\ell-1)/2 < \ell.$$

Then

$$B_{i,\zeta}^{(n)}1_\lambda = \sum_{c=0}^{(k-1)\ell+(\ell-1)/2} \sum_{a=0}^{(2k-1)\ell-2c} v_i^{2(a+c)(m-a-\langle i,\lambda\rangle/2)-2ac-a-\binom{2c+1}{2}} \cdot \begin{bmatrix} m-c-a-\langle i,\lambda\rangle/2-1 \\ c \end{bmatrix}_{v_i^2} E_i^{(a)}F_i^{(2m-1-2c-a)}1_\lambda.$$

Note that

$$\mathbf{Fr}(E_i^{(a)}F_i^{(2m-1-2c-a)}) = \begin{cases} E_i^{(a/\ell)}F_i^{((2m-1-2c-a)/\ell)}, & \text{if } \ell|a \text{ and } \ell|c; \\ 0, & \text{otherwise.} \end{cases}$$

However, if $\ell|a$ and $\ell|c$, we must have $v_i^{2(a+c)(m-a-\langle i,\lambda\rangle/2)-2ac-a-\binom{2c+1}{2}} = 1$.

Therefore

$$\begin{aligned}
&\mathbf{Fr}(B_{i,\zeta}^{(n)}1_\lambda) \\
=&\sum_{c=0}^{(k-1)} \sum_{a=0}^{2k-1-2c} \begin{bmatrix} (k-1)\ell+(\ell-1)/2-c\ell-a\ell-\langle i,\lambda\rangle/2 \\ c\ell \end{bmatrix}_{v_i^2} E_i^{(a)}F_i^{(2k-1-2c-a)}1_\lambda \\
\overset{\spadesuit}{=}&\sum_{c=0}^{(k-1)} \sum_{a=0}^{2k-1-2c} \begin{pmatrix} k-1-c-a-\langle i,\lambda\rangle^\diamond/2 \\ c \end{pmatrix} E_i^{(a)}F_i^{(2k-1-2c-a)}1_\lambda \\
=&B_{i,\zeta}^{(2k-1)}1_\lambda,
\end{aligned}$$

where ♠ follows from [Lu94, Lemma 34.1.2].

(3) At last we consider the case where $n = k\ell + b$ with $0 < b < \ell$. Note that since $\mathcal{A}'$ is an integral domain, it suffices to perform the computation in the field of fractions.

Recall the following induction formula from [BW18d, (2.5)]:

$$B_i B_{i,\zeta}^{(k\ell+c)} = [k\ell+c+1]_{v_i} B_{i,\zeta}^{(k\ell+c+1)}, \qquad \text{if } k\ell+c \text{ is odd;}$$
$$B_i B_{i,\zeta}^{(k\ell+c)} = [k\ell+c+1]_{v_i} B_{i,\zeta}^{(k\ell+c+1)} + [k\ell+c]_{v_i} B_{i,\zeta}^{(k\ell+c-1)}, \qquad \text{if } k\ell+c \text{ is even.}$$

Recall $v_i^\ell = 1$. Note that $[k\ell]_{v_i} = 0$ and $[k\ell+b]_{v_i} = [b]_{v_i} \neq 0$ for all $0 < b < \ell$.

(3.1) When k (hence also $k\ell$) is odd, we have

$$\mathbf{Fr}(B_{i,\zeta}^{(k\ell+b)} 1_\lambda) = \begin{cases} (-1)\frac{[b-1]_{v_i}}{[b]_{v_i}} \mathbf{Fr}(B_{i,\zeta}^{(k\ell+b-2)} 1_\lambda), & \text{if } b \text{ is even;} \\ 0, & \text{if } b \text{ is odd.} \end{cases}$$

Therefore we have

$$\begin{aligned} \mathbf{Fr}(B_{i,\zeta}^{(k\ell+b)} 1_\lambda) &= (-1)^{b/2} \frac{[b-1]_{v_i}}{[b]_{v_i}} \frac{[b-3]_{v_i}}{[b-2]_{v_i}} \cdots \frac{[1]_{v_i}}{[2]_{v_i}} \mathbf{Fr}(B_{i,\zeta}^{(k\ell)} 1_\lambda) \\ &= \frac{[\ell-b+1]_{v_i}}{[b]_{v_i}} \frac{[\ell-b+3]_{v_i}}{[b-2]_{v_i}} \cdots \frac{[\ell-1]_{v_i}}{[2]_{v_i}} \mathbf{Fr}(B_{i,\zeta}^{(k\ell)} 1_\lambda) \\ &= \begin{bmatrix} (\ell-1)/2 \\ b/2 \end{bmatrix}_{v_i^2} \mathbf{Fr}(B_{i,\zeta}^{(k\ell)} 1_\lambda). \end{aligned}$$

(3.2) When k (hence also kl) is even, note that

$$B_i B_{i,\zeta}^{(k\ell)} = [k\ell+1]_{v_i} B_{i,\zeta}^{(k\ell+1)}, \qquad \text{since } [k\ell]_{v_i} = 0.$$

Therefore by a similar computation as above, we have

$$\mathbf{Fr}(B_{i,\zeta}^{(k\ell+b)} 1_\lambda) = 0, \qquad \text{for all } 0 < b < \ell.$$

□

We recall the following formula for ıdivided powers.

Proposition 3.4.3. *[BW18d, Proposition 3.5] Let $\langle i, \zeta\rangle$ be odd. Let $m \geq 0$, and $\lambda \in X$ such that $\overline{\lambda} = \zeta$. We have*

$$B_{i,\zeta}^{(2m)} 1_\lambda = \sum_{c=0}^{m} \sum_{a=0}^{2m-2c} v_i^{2(a+c)(m-a-(\langle i,\lambda\rangle+1)/2)-2ac+a-\binom{2c}{2}} \cdot \begin{bmatrix} m-c-a-(\langle i,\lambda\rangle+1)/2 \\ c \end{bmatrix}_{v_i^2} E_i^{(a)} F_i^{(2m-2c-a)} 1_\lambda, \tag{3.4.3}$$

$$B_{i,\zeta}^{(2m+1)} 1_\lambda = \sum_{c=0}^{m} \sum_{a=0}^{2m+1-2c} v_i^{2(a+c)(m-a-(\langle i,\lambda\rangle+1)/2)-2ac+2a-\binom{2c}{2}} \cdot \begin{bmatrix} m-c-a-(\langle i,\lambda\rangle+1)/2+1 \\ c \end{bmatrix}_{v_i^2} E_i^{(a)} F_i^{(2m+1-2c-a)} 1_\lambda. \tag{3.4.4}$$

Proposition 3.4.4. *Let $\langle i, \zeta\rangle$ be odd. We have*

$$\mathbf{Fr} : {}_{\mathcal{A}'}\dot{\mathbf{U}}^{\wedge} \longrightarrow {}_{\mathcal{A}'}\dot{\mathbf{U}}^{\diamond,\wedge},$$

$$\sum_{\lambda \in X} B_{i,\zeta}^{(n)} 1_\lambda \mapsto \begin{cases} \sum_{\lambda \in X^\diamond} B_{i,\zeta}^{(n/\ell)} 1_\lambda, & \text{for } n \in \ell\mathbb{Z} \text{ and } \zeta \in X_i^\diamond; \\ \sum_{\lambda \in X^\diamond} \begin{bmatrix} (\ell-1)/2 \\ b/2 \end{bmatrix}_{v_i^2} B_{i,\zeta}^{(k)} 1_\lambda, & \text{for } n = k\ell + b,\ k \text{ even}, \\ & b \text{ even},\ 0 < b < \ell,\ \zeta \in X^\diamond; \\ 0, & \text{otherwise.} \end{cases}$$

In other words, we have

$$\mathbf{Fr} : {}_{\mathcal{A}'}\dot{\mathbf{U}}^\imath \longrightarrow {}_{\mathcal{A}'}\dot{\mathbf{U}}^{\diamond,\imath},$$
$$B_{i,\zeta}^{(n)} \mapsto \begin{cases} B_{i,\zeta}^{(n/\ell)}, & \text{for } n \in \ell\mathbb{Z} \text{ and } \lambda \in X_\imath^\diamond; \\ \begin{bmatrix} (\ell-1)/2 \\ b/2 \end{bmatrix}_{v_i^2} B_{i,\zeta}^{(k)}, & \text{for } n = k\ell + b,\ k \text{ even}, \\ & b \text{ even},\ 0 < b < \ell,\ \zeta \in X_\imath^\diamond; \\ 0, & \text{otherwise.} \end{cases}$$

Proof. The computation is entirely similar to that of Proposition 3.4.2, except that here we use (3.4.3) and (3.4.4) and the following recursive formula from [BW18d, (3.2)] for $a \geq 0$. and $\langle i, \zeta \rangle$ odd:

$$B_i B_{i,\zeta}^{(2a)} = [2a+1]_{v_i} B_{i,\zeta}^{(2a+1)},$$
$$B_i B_{i,\zeta}^{(2a+1)} = [2a+2]_{v_i} B_{i,\zeta}^{(2a+2)} + [2a+1]_{v_i} B_{i,\zeta}^{(2a)}.$$

□

Chapter 4

Small quantum symmetric pairs

4.1 The small $\imath$quantum group

Let ${}_{\mathcal{A}'}\dot{\mathfrak{u}}$ be the $\mathcal{A}'$-subalgebra of ${}_{\mathcal{A}'}\dot{\mathbf{U}}$ generated by $E_i^{(n)}1_\lambda$, $F_i^{(n)}1_\lambda$ for various $i \in I$, various n such that $0 \leq n < \ell$ and various $\lambda \in X$. Let ${}_{\mathcal{A}'}\dot{\mathfrak{p}} = {}_{\mathcal{A}'}\dot{\mathfrak{p}}_{I_\bullet}$ be the $\mathcal{A}'$-subalgebra of ${}_{\mathcal{A}'}\dot{\mathbf{U}}$ generated by $E_i^{(n)}1_\lambda$, $F_j^{(n)}1_\lambda$ for various $i \in I_\bullet, j \in I$, various n such that $0 \leq n < \ell$ and various $\lambda \in X$.

Definition 4.1.1. Let ${}_{\mathcal{A}'}\dot{\mathfrak{u}}^\imath$ be the $\mathcal{A}'$-subalgebra of ${}_{\mathcal{A}'}\dot{\mathbf{U}}^\imath$ generated by $B_{i,\zeta}^{(n)}$, $E_j^{(n)}1_\zeta$ for various $i \in I$, $j \in I_\bullet$, $\zeta \in X_\imath$ and n such that $0 \leq n < \ell$. For any $\mathcal{A}'$-commutative ring R, we define ${}_R\dot{\mathfrak{u}}^\imath = R \otimes_{\mathcal{A}'} {}_{\mathcal{A}'}\dot{\mathfrak{u}}^\imath$.

We call $({}_{\mathcal{A}'}\dot{\mathfrak{u}}^\imath, {}_{\mathcal{A}'}\dot{\mathfrak{u}})$ the small quantum symmetric pair. It is clear that ${}_{\mathcal{A}'}\dot{\mathfrak{u}}^\imath$ is a "coideal subalgebra" of ${}_{\mathcal{A}'}\dot{\mathfrak{u}}$, that is, we have (via restriction)

$$\Delta : {}_{\mathcal{A}'}\dot{\mathfrak{u}}^\imath \longrightarrow \prod_{\zeta \in X_\imath^\diamond, \lambda \in X^\diamond} {}_{\mathcal{A}'}\dot{\mathfrak{u}}^\imath 1_\zeta \otimes_{\mathcal{A}'} {}_{\mathcal{A}'}\dot{\mathfrak{u}} 1_\lambda.$$

Remark 4.1.2. Here we abuse the terminology "coideal subalgebra", even though ${}_{\mathcal{A}'}\dot{\mathfrak{u}}^\imath$

is not a subalgebra of ${}_{\mathcal{A}'}\dot{\mathfrak{u}}$.

Recall that ${}_{\mathcal{A}}\dot{\mathbf{P}}$ is the parabolic subalgebra of ${}_{\mathcal{A}}\dot{\mathbf{U}}$ generated by $E_i^{(n)}1_\lambda$ and $F_j^{(n)}1_\lambda$ for all $n \in \mathbb{Z}_{\geq 0}$, $i \in I_\bullet$ and $j \in I$. Moreover, we define ${}_{\mathcal{A}'}\dot{\mathbf{P}} = \mathcal{A}' \otimes_{\mathcal{A}} {}_{\mathcal{A}}\dot{\mathbf{P}}$.

Theorem 4.1.3. *For any* $\zeta \in X_\imath$, ${}_{\mathcal{A}'}\dot{\mathfrak{u}}^\imath 1_\zeta$ *is a free* $\mathcal{A}'$*-module with rank* $\ell^{|\Phi_\bullet^+|+|\Phi^+|}$.

Proof. We have the following linear isomorphism via a base change from [BW18b, Corollary 6.20]

$$p_{\imath,\lambda} : \ {}_{\mathcal{A}'}\dot{\mathbf{U}}^\imath 1_\zeta \xrightarrow{\cong} {}_{\mathcal{A}'}\dot{\mathbf{P}}1_\lambda \ , \qquad \overline{\lambda} = \zeta, \lambda \in X.$$

Via restriction, we have the map

$$p_{\imath,\lambda} : {}_{\mathcal{A}'}\dot{\mathfrak{u}}^\imath 1_\zeta \longrightarrow {}_{\mathcal{A}'}\dot{\mathfrak{p}}1_\lambda,$$

whose surjectivity can be obtained similar to [BW18b, Corollary 6.20]. We know the image lies in ${}_{\mathcal{A}'}\dot{\mathfrak{p}}1_\lambda$ thanks to the precise formulas of $B_{i,\zeta}^{(n)}$ obtained in [BW18b, BW18d].

The theorem follows immediately from the fact that ${}_{\mathcal{A}'}\dot{\mathfrak{p}}1_\lambda$ is a free $\mathcal{A}'$-module with rank $\ell^{|\Phi_\bullet^+|+|\Phi^+|}$ thanks to [Lu90b, Theorem 8.3]. □

Chapter 5

Central elements in $\mathbf{U}^\imath_\ell$

5.1 The $\imath$quantum group $\mathbf{U}^\imath_\ell$

In this chapter, we now allow any finite root datum $(Y, X, \dots)$, as opposed to only the simply connected kind. In particular, we maintain the assumption of Chapter 3 that $B_i = F_i + \varsigma_i T_{w_\bullet}(E_{\tau(i)}\tilde{K}_i^{-1})$ for $i \in I_\circ$, but we now require $\varsigma_i \in v^{\mathbb{Z}}$. Moreover, we require that $\ell \neq 1$ in this chapter.

Let $i \in I_\circ$. Recall that the $\imath$divided powers may be written $B_{i,\zeta}^{(n)} = x1_\zeta$ for some $x \in \mathbf{U}^\imath$. From the various formulas for $\imath$divided powers reviewed in Chapter 3 (cf. [BW18c, §5.5.1,(5.12)]), one can see that x remains constant as ζ varies, except when $\tau(i) = i = w_\bullet(i)$. We then define the $\imath$divided powers in $\mathbf{U}^\imath$ as $B_i^{(n)} = x$.

Now suppose that $\tau(i) = i = w_\bullet(i)$ and $\langle i, \zeta\rangle$ is even (resp. odd). Then, x remains constant as ζ varies so long that the parity of $\langle i, \zeta\rangle$ remains constant (cf. [BW18d, Theorems 2.5, 3.1]). Hence, we define the $\imath$divided powers in $\mathbf{U}^\imath$ as $B_{i,ev}^{(n)} = x$ (resp. $B_{i,odd}^{(n)} = x$).

Finally, define $B_{i,ev,odd,\varnothing}^{[n]} = [n]_i^! B_{i,ev,odd,\varnothing}^{(n)}$. We write $\mathbf{U}_{\mathbb{C}} = \mathbb{C}(v) \otimes_{\mathbb{Q}(v)} \mathbf{U}$ and denote by $\mathbf{U}^\imath_{\mathbb{C}}$ the subalgebra generated by E_i, F_i, B_j and K_μ for all $i \in \mathbf{I}_\bullet$, $j \in I_\circ$ and $\mu \in Y^\imath$.

Let $\mathbf{A} = \mathbb{C}[v, v^{-1}]$, and define $\mathbf{U}_\mathbf{A}$ to be the $\mathbf{A}$-algebra in $\mathbf{U}$ generated by $E_i, F_i, (\tilde{K}_i - \tilde{K}_i^{-1})/(v_i - v_i^{-1})$ and K_μ for all $i \in I$ and $\mu \in Y$.

Define $\mathbf{U}^\imath_\mathbf{A}$ to be the $\mathbf{A}$-subalgebra of $\mathbf{U}_\mathbf{A}$ generated by $E_i, F_i, (\tilde{K}_i - \tilde{K}_i^{-1})/(v_i - v_i^{-1})$, B_j and K_μ for all $i \in \mathbf{I}_\bullet$, $j \in I_\circ$ and $\mu \in Y^\imath$. The elements $B^{[n]}_{i,ev,odd,\varnothing}$ are in $\mathbf{U}^\imath_\mathbf{A}$.

Let $f_\ell(v)$ be the ℓ-th cyclotomic polynomial. Define $\mathbf{U}_\ell = \mathbf{U}_\mathbf{A}/(f_\ell(v))$ (cf. [DKP, §0.5]), and define $\mathbf{U}^\imath_\ell$ to be the subalgebra of $\mathbf{U}_\ell$ generated by $E_i, F_i, (\tilde{K}_i - \tilde{K}_i^{-1})/(v_i - v_i^{-1})$, B_j and K_μ for all $i \in \mathbf{I}_\bullet$, $j \in I_\circ$ and $\mu \in Y^\imath$.

5.2 The $\imath$divided power $B_i^{[k\ell]}$, $\tau(i) \neq i$

Lemma 5.2.1. *Let $i \in I_\circ$ be such that $\tau(i) \neq i$. Then, we have*

$$B_i^{[k\ell]} = (F_i^\ell + T_{w_\bullet}(E_{\tau(i)})^\ell \tilde{K}_i^{-\ell})^k.$$

Hence, $B_i^{[k\ell]}$ is central in $\mathbf{U}_\ell$ and $\mathbf{U}^\imath_\ell$.

Proof. In $\mathbf{U}_\ell$, we have

$$\begin{aligned} B_i^{[k\ell]} &= \sum_{n=0}^{k\ell} \begin{bmatrix} k\ell \\ n \end{bmatrix}_i F_i^n (\varsigma_i T_{w_\bullet}(E_{\tau(i)}) \tilde{K}_i^{-1})^{k\ell-n} \\ &= \sum_{n=0}^{k\ell} \begin{bmatrix} k\ell \\ n \end{bmatrix}_i F_i^n \varsigma_i^{k\ell-n} v_i^{-\langle i, w_\bullet(\tau(i))' \rangle \binom{k\ell-n}{2}} T_{w_\bullet}(E_{\tau(i)})^{k\ell-n} \tilde{K}_i^{-k\ell+n} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=0}^{k} \binom{k}{n} F_i^{nl} \varsigma_i^{k\ell-n\ell} v_i^{-\langle i, w_\bullet(\tau(i)')\rangle \binom{k\ell-n\ell}{2}} T_{w_\bullet}(E_{\tau(i)})^{k\ell-n\ell} \tilde{K}_i^{-k\ell+n\ell} \\
&= \sum_{n=0}^{k} \binom{k}{n} F_i^{n\ell} T_{w_\bullet}(E_{\tau(i)})^{k\ell-n\ell} \tilde{K}_i^{-k\ell+n\ell} \\
&= (F_i^\ell + T_{w_\bullet}(E_{\tau(i)})^\ell \tilde{K}_i^{-\ell})^k.
\end{aligned}
$$

As F_i^ℓ, E_i^ℓ and K_i^ℓ are central in $\mathbf{U}_\ell$ (cf. [DKP, §3.1]), so is $B_i^{[k\ell]}$. □

5.3 The ıdivided power $B_i^{[k\ell]}$, $\tau(i) = i \neq w_\bullet(i)$

Lemma 5.3.1. *Let $i \in I_\circ$ be such that $\tau(i) = i$ and $w_\bullet(i) \neq i$. Then, we have*

$$
B_i^{[k\ell]} = (F_i^\ell + T_{w_\bullet}(E_i)^\ell \tilde{K}_i^{-\ell})^k.
$$

Hence, $B_i^{[k\ell]}$ is central in $\mathbf{U}_\ell$ and $\mathbf{U}_\ell^\imath$.

Proof. Recalling [BW18c, (5.12)], we have in $\mathbf{U}_\mathbf{A}$

$$
\begin{aligned}
&(v - v^{-1}) B_i^{[k\ell]} \\
&\qquad = (v - v^{-1})[k\ell]_i^! b_i^{(k\ell)} + v \sum_{a \geq 1} v_i^{\binom{a+1}{2}} [k\ell]_i^! \mathfrak{z}_i^{(a)} b_i^{(k\ell-2a)} \\
&\qquad = (v - v^{-1})[k\ell]_i^! b_i^{(k\ell)} + v \sum_{a \geq 1} \sum_{i=0}^{k\ell-2a} v_i^{\binom{a+1}{2}} \frac{[k\ell]_i^!}{[a]_i^! [k\ell - 2a - c]_i^! [c]_i^!} \\
&\qquad \times \mathfrak{z}_i^a F_i^{k\ell-2a-b} (\varsigma_i T_{w_\bullet}(E_i) \tilde{K}_i^{-1})^b
\end{aligned}
$$

$$= (v - v^{-1})[k\ell]_i^! b_i^{(k\ell)} + \sum_{a\geq 1} \sum_{i=0}^{k\ell-2a} v \cdot v_i^{\binom{a+1}{2}} \begin{bmatrix} k\ell \\ c \end{bmatrix}_i \begin{bmatrix} k\ell - c \\ a \end{bmatrix}_i [k\ell - c - a]_i$$

$$\times [k\ell - c - a - 1]_i ... [k\ell - c - 2a + 1]_i) \mathfrak{Z}_i^a F_i^{k\ell - 2a - b} (\varsigma_i T_{w_\bullet}(E_i) \tilde{K}_i^{-1})^b.$$

In $\mathbf{U}_\ell$, by the computation of the previous proof, we have

$$[k\ell]_i^! b_i^{(k\ell)} = (F_i^\ell + T_{w_\bullet}(E_i)^\ell \tilde{K}_i^{-\ell})^k.$$

Moreover, when v is an ℓ-th root of unity, using [Lu94, Lemma 34.1.2], we have

$$\begin{bmatrix} k\ell \\ c \end{bmatrix}_i \begin{bmatrix} k\ell - c \\ a \end{bmatrix}_i ([k\ell - c - a]_i [k\ell - c - a - 1]_i ... [k\ell - c - 2a + 1]_i) = 0$$

for $a \geq 1$, as we have

$$\begin{bmatrix} k\ell \\ c \end{bmatrix}_i \begin{bmatrix} k\ell - c \\ a \end{bmatrix}_i = 0$$

unless $a, c \in \ell\mathbb{Z}$, but in that case $[k\ell - c - a]_i = 0$.

Therefore in $\mathbf{U}_\ell$, we have

$$B_i^{[k\ell]} = (F_i^\ell + T_{w_\bullet}(E_i)^\ell \tilde{K}_i^{-\ell})^k.$$

□

5.4 The ıdivided power $B^{[k\ell]}_{i,ev/odd}$, $\tau(i) = i = w_\bullet(i)$

Following [BW18d, §2], we define for $n \geq 0$ and $a \in \mathbb{Z}$

$$\begin{bmatrix} h; a \\ n \end{bmatrix} = \prod_{j=1}^{n} \frac{v_i^{4a+4j-4} K_i^{-2} - 1}{v_i^{4j} - 1}$$

and

$$\check{E}_i^{(n)} = (\varsigma_i E_i K_i^{-1})^n / [n]_i^!$$

We recall the ıdivided power formulas of [BW18d, Theorem 2.5]:

Lemma 5.4.1. *[BW18d, Theorem 2.5] For $m \geq 1$ we have*

$$B^{(2m)}_{i,ev} = \sum_{c=0}^{m} \sum_{a=0}^{2m-2c} v_i^{\binom{2c}{2} - a(2m-2c-a)} \check{E}_i^{(a)} \begin{bmatrix} h; 1-m \\ c \end{bmatrix} F_i^{(2m-2c-a)}, \tag{5.4.1}$$

$$B^{(2m-1)}_{i,ev} = \sum_{c=0}^{m-1} \sum_{a=0}^{2m-1-2c} v_i^{\binom{2c+1}{2} - a(2m-1-2c-a)} \check{E}_i^{(a)} \begin{bmatrix} h; 1-m \\ c \end{bmatrix} F_i^{(2m-1-2c-a)}. \tag{5.4.2}$$

Lemma 5.4.2. *Let $i \in I_\circ$ be such that $\tau(i) = i$ and $w_\bullet(i) = i$. Then, we have*

$$B^{[k\ell]}_{i,ev} = (F_i^\ell + E_i^\ell \tilde{K}_i^{-\ell})^k.$$

Hence, $B^{[k\ell]}_{i,ev}$ is central in $\mathbf{U}_\ell$ and $\mathbf{U}^\imath_\ell$.

Proof. When k is even, we have in $\mathbf{U}_\mathbf{A}$

$$
\begin{aligned}
B^{[k\ell]}_{i,ev} &= \sum_{c=0}^{k\ell/2} \sum_{a=0}^{k\ell-2c} \frac{[k\ell]^!_i}{[a]^!_i [k\ell-2c-a]^!_i} v_i^{\binom{2c}{2}-a(k\ell-2c-a)} (\varsigma_i E_i \tilde{K}_i^{-1})^a \\
&\quad \times \prod_{j=1}^{c} \frac{v_i^{4(1-k\ell/2)+4j-4} K_i^{-2} - 1}{v_i^{4j}-1} F_i^{k\ell-2c-a} \\
&= \sum_{c=0}^{k\ell/2} \sum_{a=0}^{k\ell-2c} \begin{bmatrix} k\ell \\ a \end{bmatrix}_i ([k\ell-a]_i [k\ell-a-1]_i \ldots [k\ell-a-2c+1]_i) \\
&\quad \times v_i^{\binom{2c}{2}-a(k\ell-2c-a)} (\varsigma_i E_i \tilde{K}_i^{-1})^a \prod_{j=1}^{c} \frac{v_i^{4(1-k\ell/2)+4j-4} K_i^{-2} - 1}{v_i^{4j}-1} F_i^{k\ell-2c-a}.
\end{aligned}
$$

Note that $B^{[k\ell]}_{i,ev}$ is, in fact, in $\mathbf{U}_\mathbf{A}$, by the recursive formulas used in Chapter 3 (cf. [BW18d, (2.4)]).

Using [Lu94, Lemma 34.1.2], we have when v is an ℓ-th root of unity that

$$
\begin{bmatrix} k\ell \\ a \end{bmatrix}_i \frac{[k\ell-a]_i [k\ell-a-1]_i \ldots [k\ell-a-2c+1]_i}{(v_i^4-1)(v_i^8-1)\ldots(v_i^{4c}-1)} = 0
$$

for $1 \le c \le k\ell/2$ and $0 \le a \le k\ell - 2c$, as

$$
\begin{bmatrix} k\ell \\ a \end{bmatrix}_i = 0
$$

unless $a \in \ell\mathbb{Z}$ in which case $[k\ell-a]_i[k\ell-a-1]_i \ldots [k\ell-a-2c+1]_i$ has $\lfloor (2c+\ell-1)/\ell \rfloor$ factors of $v_i^\ell - 1$, whereas $(v_i^4-1)(v_i^8-1)\ldots(v_i^{4c}-1)$ has $\lfloor c/\ell \rfloor$ factors of $v_i^\ell - 1$. Hence,

$$
\frac{[k\ell-a]_i [k\ell-a-1]_i \ldots [k\ell-a-2c+1]_i}{(v_i^4-1)(v_i^8-1)\ldots(v_i^{4c}-1)} = 0.
$$

Thus, in $\mathbf{U}_\ell$, we have

$$\begin{aligned}
B^{[k\ell]}_{i,ev} &= \sum_{a=0}^{k\ell} \begin{bmatrix} k\ell \\ a \end{bmatrix}_i v_i^{-a(k\ell-a)} (\varsigma_i E_i \tilde{K}_i^{-1})^a \begin{bmatrix} h;1-k\ell/2 \\ 0 \end{bmatrix}_i F_i^{kl-a} \\
&= \sum_{a=0}^{k\ell} \begin{bmatrix} k\ell \\ a \end{bmatrix}_i v_i^{-a(k\ell-a)} \varsigma_i^a v_i^{-2\binom{a}{2}} E_i^a \tilde{K}_i^{-a} F_i^{k\ell-a} \\
&= \sum_{a=0}^{k} \binom{k}{a} v_i^{-a\ell(k\ell-a\ell)} \varsigma_i^{a\ell} v_i^{-2\binom{a\ell}{2}} E_i^{a\ell} \tilde{K}_i^{-a\ell} F_i^{k\ell-a\ell} \\
&= (F_i^\ell + E_i^\ell \tilde{K}_i^{-\ell})^k.
\end{aligned}$$

Similarly, for k odd, we have in $\mathbf{U}_\mathbf{A}$

$$\begin{aligned}
B^{[k\ell]}_{i,ev} = \sum_{c=0}^{(k\ell-1)/2} \sum_{a=0}^{k\ell-2c} \begin{bmatrix} k\ell \\ a \end{bmatrix}_i & ([k\ell-a]_i[k\ell-a-1]_i ... [k\ell-a-2c+1]_i) \\
\times\, v_i^{\binom{2c+1}{2}-a(k\ell-2c-a)} (\varsigma_i E_i \tilde{K}_i^{-1})^a & \prod_{j=1}^{c} \frac{v_i^{4(1-k\ell+1)/2+4j-4} K_i^{-2} - 1}{v_i^{4j}-1} F_i^{k\ell-2c-a}.
\end{aligned}$$

By the same reasoning as before, we have

$$\begin{bmatrix} k\ell \\ a \end{bmatrix}_i \frac{[k\ell-a]_i[k\ell-a-1]_i ... [k\ell-a-2c+1]_i}{(v_i^4-1)(v_i^8-1)...(v_i^{4c}-1)} = 0$$

for $1 \le c \le (k\ell-1)/2$ and $0 \le a \le k\ell-2c$.

Thus, in $\mathbf{U}_\ell$, we have

$$\begin{aligned}B^{[k\ell]}_{i,ev} &= \sum_{a=0}^{k\ell} \begin{bmatrix} k\ell \\ a \end{bmatrix}_i v_i^{-a(k\ell-a)} (\varsigma_i E_i \tilde{K}_i^{-1})^a \begin{bmatrix} h; 1-(k\ell+1)/2 \\ 0 \end{bmatrix}_i F_i^{k\ell-a} \\ &= \sum_{a=0}^{k\ell} \begin{bmatrix} k\ell \\ a \end{bmatrix}_i v_i^{-a(k\ell-a)} \varsigma_i^a v_i^{-2\binom{a}{2}} E_i^a \tilde{K}_i^{-a} F_i^{k\ell-a} \\ &= \sum_{a=0}^{k} \binom{k}{a} v_i^{-a\ell(k\ell-a\ell)} \varsigma_i^{a\ell} v_i^{-2\binom{a\ell}{2}} E_i^{a\ell} \tilde{K}_i^{-a\ell} F_i^{k\ell-a\ell} \\ &= (F_i^\ell + E_i^\ell \tilde{K}_i^{-\ell})^k.\end{aligned}$$

□

Lemma 5.4.3. *Let $i \in I_\circ$ be such that $\tau(i) = i$ and $w_\bullet(i) = i$. Then, we have*

$$B^{[k\ell]}_{i,odd} = (F_i^\ell + E_i^\ell \tilde{K}_i^{-\ell})^k.$$

Hence, $B^{[k\ell]}_{i,odd}$ is central in $\mathbf{U}_\ell$ and $\mathbf{U}^\imath_\ell$.

Proof. The proof is entirely similar to that of the previous lemma except we use the formulas of [BW18d, Theorem 3,1].

□

Combining the prior lemmas, we have the following.

Theorem 5.4.4. *For $i \in I_\circ$, $B^{[k\ell]}_{i,ev,odd,\varnothing}$ is central in $\mathbf{U}_\ell$ and $\mathbf{U}^\imath_\ell$.*

Chapter 6

$\imath$Quantum adjoint action

6.1 $\imath$Quantum adjoint action for generic v

We maintain the assumptions of the previous chapter throughout this chapter, and additionally we require that the $\imath$root data be simple, simply laced and split. That is, $(I, \cdot)$ is of type A_n, D_n or E_n, $I = I_\circ$ and τ is the identity.

Recall the definitions of $\mathbf{U}_{\mathbb{C},\mathbf{A},\ell}$ and $\mathbf{U}^\imath_{\mathbb{C},\mathbf{A},\ell}$ from the beginning of §5.1. Note that in the split case case, $\mathbf{U}^\imath_{\mathbb{C},\mathbf{A},\ell}$ are generated by the B_i for all $i \in I$.

We recall the quantized adjoint module for $\mathbf{U}_\mathbb{C}$, M, from [Jan96, §5A.2]. It has basis

$$\{x_\gamma, y_\gamma = x_{-\gamma}, h_i | \gamma \in \Phi^+, i \in I\}$$

satisfying the following:

$$E_i : x_\gamma \mapsto \begin{cases} x_{\gamma+i} & \text{if } \langle i, \gamma' \rangle = -1 \\ 0 & \text{otherwise,} \end{cases} \qquad F_i : x_\gamma \mapsto \begin{cases} x_{\gamma-i} & \text{if } \langle i, \gamma' \rangle = 1 \\ 0 & \text{otherwise,} \end{cases}$$

$$E_i : h_j \mapsto \begin{cases} [2]x_i & \text{if } i = j \\ x_i & \text{if } \langle i, j' \rangle = -1 \\ 0 & \text{otherwise,} \end{cases} \qquad F_i : h_j \mapsto \begin{cases} [2]x_{-i} & \text{if } i = j \\ x_{-i} & \text{if } \langle i, j' \rangle = -1 \\ 0 & \text{otherwise,} \end{cases}$$

and

$$K_\mu.x_\gamma = v^{\mu\cdot\gamma} x_\gamma, \quad K_\nu.h_i = h_i.$$

Define K to be the span of $\{b_\gamma := y_\gamma - v x_\gamma | \gamma \in \Phi^+\}$ and let P be the span of $\{p_\gamma := y_\gamma + v^{-1} x_\gamma, h_i | \gamma \in \Phi^+, i \in \mathbb{I}\}$. Then, $M = K \oplus P$, as a vector space.

Proposition 6.1.1. *The subspaces, K and P, of M are $\mathbf{U}^\imath_{\mathbb{C}}$ modules. The action of $\mathbf{U}^\imath_{\mathbb{C}}$ on K and P is given by the following formulas*

$$B_i : b_\gamma \mapsto \begin{cases} b_{\gamma+i} & \textit{if } \gamma + i \in \Phi^+ \\ b_{\gamma-i} & \textit{if } \gamma - i \in \Phi^+ \\ 0 & \textit{otherwise,} \end{cases}$$

and

$$B_i : p_\gamma \mapsto \begin{cases} p_{\gamma+i} & \textit{if } \gamma + i \in \Phi^+ \\ p_{\gamma-i} & \textit{if } \gamma - i \in \Phi^+ \\ [2]h_i & \textit{if } \gamma = i \\ 0 & \textit{otherwise,} \end{cases} \qquad B_i : h_j \mapsto \begin{cases} [2]p_i & \textit{if } j = i \\ p_i & \textit{if } \langle i', j \rangle = -1 \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. These are by direct computation. Let $\gamma \in \Phi^+$ throughout. For example, for $\gamma + i \in \Phi^+$, we have

$$\begin{aligned} B_i.p_\gamma &= (F_i + v^{-1}E_iK_i^{-1}).(y_\gamma + v^{-1}x_\gamma) \\ &= F_i.y_\gamma + v^{-1}F_i.x_\gamma + v^{-1}E_iK_i^{-1}.y_\gamma + v^{-2}E_iK_i^{-1}.x_\gamma \\ &= y_{\gamma+i} + 0 + 0 + v^{-1}x_{\gamma+i} = p_{\gamma+i}. \end{aligned}$$

For $\gamma - i \in \Phi^-$, we have

$$\begin{aligned} B_i.p_\gamma &= (F_i + v^{-1}E_iK_i^{-1}).(y_\gamma + v^{-1}x_\gamma) \\ &= F_i.y_\gamma + v^{-1}F_i.x_\gamma + v^{-1}E_iK_i^{-1}.y_\gamma + v^{-2}E_iK_i^{-1}.x_\gamma \\ &= 0 + q^{-1}x_{\gamma-i} + y_{\gamma-i} + 0 = p_{\gamma-i}. \end{aligned}$$

When $\gamma = i$, we have

$$\begin{aligned}
B_i.p_\gamma &= (F_i + v^{-1}E_iK_i^{-1}).(y_\gamma + v^{-1}x_\gamma) \\
&= F_i.y_\gamma + v^{-1}F_i.x_\gamma + v^{-1}E_iK_i^{-1}.y_\gamma + v^{-2}E_iK_i^{-1}.x_\gamma \\
&= 0 + v^{-1}h_i + vh_i + 0 = [2]h_i.
\end{aligned}$$

Finally, for γ not satisfying any of the previous conditions, we have

$$\begin{aligned}
B_i.p_\gamma &= (F_i + v^{-1}E_iK_i^{-1}).(y_\gamma + v^{-1}x_\gamma) \\
&= F_i.y_\gamma + v^{-1}F_i.x_\gamma + v^{-1}E_iK_i^{-1}.y_\gamma + v^{-2}E_iK_i^{-1}.x_\gamma \\
&= 0.
\end{aligned}$$

The other cases are computed similarly.

□

Lemma 6.1.2. *The action of ıdivided powers in $\mathbf{U}^\imath_\mathbb{C}$ on P is given by the following formulas.*

We have

$$[2]B^{(2)}_{i,ev} : p_\gamma \mapsto \begin{cases} p_\gamma & \textit{if } \gamma + i \in \Phi^+ \\ p_\gamma & \textit{if } \gamma - i \in \Phi^+ \\ [2]^2 p_i & \textit{if } \gamma = i \\ 0 & \textit{otherwise,} \end{cases}$$

and

$$[2]B_{i,ev}^{(2)} : h_j \mapsto \begin{cases} [2]^2 h_i & \text{if } j = i \\ [2]h_i & \text{if } \langle i', j\rangle = -1 \\ 0 & \text{otherwise.} \end{cases}$$

For $k \geq 2$, we have

$$[2k]!B_{i,ev}^{(2k)} : p_\gamma \mapsto \begin{cases} (1-[2]^2)...(1-[2k-2]^2)p_\gamma & \text{if } \gamma + i \in \Phi^+ \\ (1-[2]^2)...(1-[2k-2]^2)p_\gamma & \text{if } \gamma - i \in \Phi^+ \\ 0 & \text{if } \gamma = i \\ 0 & \text{otherwise,} \end{cases}$$

and

$$[2k]!B_{i,ev}^{(2k)} : h_j \mapsto \begin{cases} 0 & \text{if } j = i \\ 0 & \text{if } \langle i', j\rangle = -1 \\ 0 & \text{otherwise.} \end{cases}$$

For $k \geq 2$, we have

$$[2k-1]!B_{i,ev}^{(2k-1)} : p_\gamma \mapsto \begin{cases} (1-[2]^2)...(1-[2k-2]^2)p_{\gamma+i} & \text{if } \gamma + i \in \Phi^+ \\ (1-[2]^2)...(1-[2k-2]^2)p_{\gamma-i} & \text{if } \gamma - i \in \Phi^+ \\ 0 & \text{if } \gamma = i \\ 0 & \text{otherwise,} \end{cases}$$

and

$$[2k-1]!B_{i,ev}^{(2k-1)} : h_j \mapsto \begin{cases} 0 & \text{if } j = i \\ 0 & \text{if } \langle i', j\rangle = -1 \\ 0 & \text{otherwise.} \end{cases}$$

For $k \geq 1$, we have

$$[2k]!B_{i,odd}^{(2k)} : p_\gamma \mapsto \begin{cases} 0 & \textit{if } \gamma + i \in \Phi^+ \\ 0 & \textit{if } \gamma - i \in \Phi^+ \\ ([2]^2-[1]^2)...([2]^2-[2k-1]^2)p_\gamma & \textit{if } \gamma = i \\ (-1)^k[1]^2[3]^2...[2k-1]^2 p_\gamma & \textit{otherwise,} \end{cases}$$

and

$$[2k]!B_{i,odd}^{(2k)} : h_j \mapsto \begin{cases} ([2]^2-[1]^2)...([2]^2-[2k-1]^2)h_i & \textit{if } j = i \\ ([2]-[1]^2)...([2]-[2k-1]^2)h_i + (-1)^k[1]^2[3]^2...[2k-1]^2(h_j+h_i) & \textit{if } \langle i', j\rangle = -1 \\ (-1)^k[1]^2[3]^2...[2k-1]^2 h_j & \textit{otherwise.} \end{cases}$$

For $k \geq 0$, we have

$$[2k+1]!B_{i,odd}^{(2k+1)} : p_\gamma \mapsto \begin{cases} 0 & \textit{if } \gamma + i \in \Phi_+ \\ 0 & \textit{if } \gamma - i \in \Phi_+ \\ [2]([2]^2-[1]^2)...([2]^2-[2k-1]^2)h_i & \textit{if } \gamma = i \\ 0 & \textit{otherwise,} \end{cases}$$

$$[2k+1]!B_{i,odd}^{(2k+1)} : h_j \mapsto \begin{cases} [2]([2]^2-[1]^2)...([2]^2-[2k-1]^2)p_i & \textit{if } j = i \\ ([2]^2-[1]^2)...([2]^2-[2k-1]^2)p_i & \textit{if } \langle i', j\rangle = -1 \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. From the previous lemma, we have

$$(B_i^2 - [k]^2) : p_\gamma \mapsto \begin{cases} (1-[k]^2)p_\gamma & \text{if } \gamma + i \in \Phi^+ \\ (1-[k]^2)p_\gamma & \text{if } \gamma - i \in \Phi^+ \\ ([2]^2-[k]^2)p_\gamma & \text{if } \gamma = i \\ -[k]^2 p_\gamma & \text{otherwise,} \end{cases}$$

and

$$(B_i^2 - [k]^2) : h_j \mapsto \begin{cases} ([2]^2-[k]^2)h_i & \text{if } j = i \\ [2]h_i - [k]^2 h_j & \text{if } \langle i', j\rangle = -1 \\ -[k]^2 h_j & \text{otherwise.} \end{cases}$$

Recall the formulas [BW18d, (2.4),(3.1)] for the ıdivided powers:

$$[2k]!B_{i,ev}^{(2k)} = (B_i^2 - [2]^2)(B_i^2 - [4]^2)...(B_i^2 - [2k-2]^2)B_i^2,$$

$$[2k-1]!B_{i,ev}^{(2k-1)} = (B_i^2 - [2]^2)(B_i^2 - [4]^2)...(B_i^2 - [2k-2]^2)B_i,$$

$$[2k]!B_{i,odd}^{(2k)} = (B_i^2 - [1]^2)(B_i^2 - [3]^2)...(B_i^2 - [2k-1]^2),$$

and

$$[2k+1]!B_{i,odd}^{(2k+1)} = (B_i^2 - [1]^2)(B_i^2 - [3]^2)...(B_i^2 - [2k-1]^2)B_i.$$

Let $k \geq 2$. For $\gamma \in \Phi^+$, such that $\gamma \pm i \in \Phi^+$, we have

$$\begin{aligned}[2k-1]!B_{i,ev}^{(2k-1)}.p_\gamma &= (B_i^2-[2]^2)(B_i^2-[4]^2)...(B_i^2-[2k-2]^2)B_i.p_\gamma \\ &= (B_i^2-[2]^2)(B_i^2-[4]^2)...(B_i^2-[2k-2]^2).p_{\gamma\pm i} \\ &= (1-[2]^2)(1-[4]^2)...(1-[2k-2]^2)p_{\gamma\pm i}.\end{aligned}$$

For $\gamma = i$, we have

$$\begin{aligned}[2k-1]!B_{i,ev}^{(2k-1)}.p_\gamma &= (B_i^2-[2]^2)(B_i^2-[4]^2)...(B_i^2-[2k-2]^2)B_i.p_\gamma \\ &= [2](B_i^2-[2]^2)(B_i^2-[4]^2)...(B_i^2-[2k-2]^2).h_i \\ &= [2]([2]^2-[2]^2)([2]^2-[4]^2)...([2]^2-[2k-2]^2)h_i = 0.\end{aligned}$$

Finally for all other γ, we have

$$\begin{aligned}[2k-1]!B_{i,ev}^{(2k-1)}.p_\gamma &= (B_i^2-[2]^2)(B_i^2-[4]^2)...(B_i^2-[2k-2]^2)B_i.p_\gamma \\ &= (B_i^2-[2]^2)(B_i^2-[4]^2)...(B_i^2-[2k-2]^2).0 = 0.\end{aligned}$$

The other cases are computed similarly.

□

6.2 ıQuantum adjoint action at a root of 1

Let $\ell > 1$ be an odd integer and let P_ℓ be the $\mathbb{C}$-vector space with basis denoted by

$$\{p_\gamma, h_i | \gamma \in \Phi^+, i \in I\}$$

by abuse of notation. By the previous lemma, $B^{(\ell)}_{i,ev/odd} \in \mathbf{U}^\imath_\mathbb{C}$ maps each $w \in \{p_\gamma, h_i | \gamma \in \Phi^+, i \in I\} \subset P$ to an element of the form cz, where $c \in \mathbb{C}(v)$ and $z \in \{p_\gamma, h_i | \gamma \in \Phi^+, i \in I\}$. We define the linear map $b_{i,ev/odd}$ on P_ℓ by $b_{i,ev/odd}.w = c'z$, where c' is c when v is specialized to a primitive ℓ-th root of unity. The element c' and thus the action are well-defined by the following lemma.

Lemma 6.2.1. *The maps $b_{i,ev/odd}$ on P_ℓ are well-defined and are given by the following formulas:*

$$b_{i,ev} : p_\gamma \mapsto \begin{cases} p_{\gamma+i} & \textit{if } \gamma + i \in \Phi^+ \\ p_{\gamma-i} & \textit{if } \gamma - i \in \Phi^+ \\ 0 & \textit{if } \gamma = i \\ 0 & \textit{otherwise,} \end{cases}$$

$$b_{i,ev} : h_j \mapsto \begin{cases} 0 & \textit{if } j = i \\ 0 & \textit{if } \langle i', j \rangle = -1 \\ 0 & \textit{otherwise,} \end{cases}$$

$$b_{i,odd} : p_\gamma \mapsto \begin{cases} 0 & \textit{if } \gamma + i \in \Phi^+ \\ 0 & \textit{if } \gamma - i \in \Phi^+ \\ h_i & \textit{if } \gamma = i \\ 0 & \textit{otherwise,} \end{cases}$$

and

$$b_{i,odd} : h_j \mapsto \begin{cases} p_i & \text{if } j = i \\ [2]^{-1} p_i & \text{if } \langle i', j \rangle = -1 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We will need the following computation. Note that for v not having been specialized to a root of 1 we have

$$\begin{aligned}
[2a+1][2a-1]+1 &= (v^{2a} + v^{2a-2} + ... + v^2 + 1 + v^{-2} + .. + .v^{2-2a} + v^{-2a}) \\
&\times (v^{2a-2} + v^{2a-4} + ... + v^2 + 1 + v^{-2} + ... + v^{4-2a} + v^{2-2a}) + 1 \\
&= v^{4a-2} + 2v^{4a-4} + 3v^{4a-6} + ... + (2a-1)v^2 + 2a \\
&+ (2a-1)v^{-2} + ... + 3v^{6-4a} + 2v^{4-4a} + v^{2-4a} \\
&= [2a]^2.
\end{aligned}$$

Hence, we have

$$\frac{(1-[2a]^2)(1-[2a-2]^2)...(1-[2]^2)}{[2a+1][2a]...[3][2]} = (-1)^a \frac{[2a-1][2a-3]...[3][1]}{[2a][2a-2]...[4][2]}.$$

Taking $\ell = 2a+1$ and v to be a primitive ℓ-th root of 1, we have

$$\begin{aligned}
(-1)^{(\ell-1)/2} \frac{[\ell-2][\ell-4]...[3][1]}{[\ell-1][\ell-3]...[4][2]} &= \frac{[2-\ell][4-\ell]...[-3][-1]}{[\ell-1][\ell-3]...[4][2]} \\
&= \frac{[2][4]...[\ell-3][\ell-1]}{[\ell-1][\ell-3]...[4][2]} = 1.
\end{aligned}$$

Similarly, note that for v not having been specialized we have

$$[2a+1][2a-3]+[2]^2=[2a-1]^2.$$

So, we have

$$\begin{aligned}&\frac{([2]^2-[2a-1]^2)([2]^2-[2a-3]^2)...([2]^2-[1]^2)}{[2a+1][2a]...[3][2]}\\&=(-1)^a\frac{([2a+1][2a-3])([2a-1][2a-5])...([3][-1])}{[2a+1][2a]...[3][2]}\\&=(-1)^{a-1}\frac{[2a-3][2a-5]...[3][1]}{[2a][2a-2]...[4][2]}.\end{aligned}$$

Taking $\ell=2a+1$ and v to be a primitive ℓ-th root of 1, we have

$$\begin{aligned}(-1)^{a-1}[2]\frac{[\ell-4][\ell-6]...[3][1]}{[\ell-1][\ell-3]...[4][2]}&=[2]\frac{[4-\ell][6-\ell]...[-3][-1]}{[\ell-1][\ell-3]...[4][2]}\\&=\frac{[2][4]...[\ell-3][\ell-1]}{[\ell-1][\ell-3]...[4][2]}=1.\end{aligned}$$

Combining these computations with the previous lemma, one has the desired result.

□

We define for $t\in\mathbb{C}$,

$$e_{i,odd/ev}(t):=exp(tb_{iev/odd}).$$

Corollary 6.2.2. *The following formulas give the action of $e_{i,ev,odd}(t)$ on P_ℓ:*

$$e_{i,ev}(t) : p_\gamma \mapsto \begin{cases} \cosh(t)p_\gamma + \sinh(t)p_{\gamma+i} & \text{if } \gamma + i \in \Phi^+ \\ \cosh(t)p_\gamma + \sinh(t)p_{\gamma-i} & \text{if } \gamma - i \in \Phi^+ \\ p_\gamma & \text{if } \gamma = i \\ p_\gamma & \text{otherwise,} \end{cases}$$

$$e_{i,ev}(t) : h_j \mapsto \begin{cases} h_j & \text{if } j = i \\ h_j & \text{if } \langle i', j \rangle = -1 \\ h_j & \text{otherwise,} \end{cases}$$

$$e_{i,odd}(t) : p_\gamma \mapsto \begin{cases} p_\gamma & \text{if } \gamma + i \in \Phi^+ \\ p_\gamma & \text{if } \gamma - i \in \Phi^+ \\ \cosh(t)p_\gamma + \sinh(t)h_i & \text{if } \gamma = i \\ p_\gamma & \text{otherwise,} \end{cases}$$

and

$$e_{i,odd}(t) : h_j \mapsto \begin{cases} \cosh(t)h_i + \sinh(t)p_i & \text{if } j = i \\ [2]^{-1}(\cosh(t)h_i + \sinh(t)p_i - h_i) + h_j & \text{if } \langle i', j \rangle = -1 \\ h_j & \text{otherwise.} \end{cases}$$

Proof. Suppose that $\gamma \in \Phi^+$ and $\gamma \pm i \in \Phi^+$. Then, using the previous lemma, we have

$$\begin{aligned} e_{i,ev}(t).p_\gamma &= (1 + tb_{i,ev} + t^2(b_{i,ev})^2/2 + t^3(b_{i,ev})^3/6 + t^4(b_{i,ev})^4/24 + ...).p_\gamma \\ &= (1 + t^2/2 + ..)p_\gamma + (t + t^3/6 + ...)p_{\gamma\pm i} \\ &= \cosh(t)p_\gamma + \sinh(t)p_{\gamma\pm i}. \end{aligned}$$

The other cases are computed similarly. □

6.3 The Lie algebra L

Consider the complex Lie algebra, L, generated by the maps on the module P_ℓ, $\{b_{i,ev} | i \in I\}$. Let $(\mathfrak{g}, \mathfrak{k})$ be the symmetric pair of complex Lie algebras of the same type as the quantum symmetric pair $(\mathbf{U}, \mathbf{U}^\imath)$, and let $(U(\mathfrak{g}), U(\mathfrak{k}))$ be their enveloping algebras over $\mathbb{C}$.

Theorem 6.3.1. *We have a Lie algebra isomorphism, $L \cong \mathfrak{k}$.*

Proof. Let $\tilde{\mathbf{U}}_\mathbf{A}$ be the $\mathbf{A}$-subalgebra of $\mathbf{U}_\mathbf{A}$ generated by E_i, F_i and K_μ, for all $i \in I$ and $\mu \in Y$. Similarly, let $\tilde{\mathbf{U}}^\imath_\mathbf{A}$ be the $\mathbf{A}$-subalgbra of $\mathbf{U}^\imath_\mathbf{A}$ generated by E_i, F_i, B_j and K_μ, for all $i \in I_\bullet, j \in I_\circ$ and $\mu \in Y^\imath$. Define $\mathbf{U}_1 = \tilde{\mathbf{U}}_\mathbf{A}/(v-1) \cong U(\mathfrak{g})$ and $\mathbf{U}^\imath_1 = \tilde{\mathbf{U}}^\imath_\mathbf{A}/(v-1) \cong U(\mathfrak{k})$. Let $K_\mathbf{A}$ be the restriction of the module K from the ring $\mathbb{C}(v)$ to the ring $\mathbf{A}$, and let $K_1 = \mathbb{C} \otimes_\mathbf{A} K_\mathbf{A}$, where v acts on $\mathbb{C}$ as 1. The action of the $B_i \in \mathbf{U}^\imath_1$ on K_1 is equivalent to the adjoint action of $\mathfrak{k}$ on itself and is given by Proposition 6.1.1:

$$B_i : b_\gamma \mapsto \begin{cases} b_{\gamma+i} & \text{if } \gamma + i \in \Phi^+ \\ b_{\gamma-i} & \text{if } \gamma - i \in \Phi^+ \\ 0 & \text{otherwise.} \end{cases}$$

Let $\tilde{P}_\ell$ be the span of the $p_\gamma \in P_\ell$. This is an L submodule with the following action of L by Lemma 6.2.1:

$$b_{i,ev} : p_\gamma \mapsto \begin{cases} p_{\gamma+i} & \text{if } \gamma + i \in \Phi^+ \\ p_{\gamma-i} & \text{if } \gamma - i \in \Phi^+ \\ 0 & \text{otherwise.} \end{cases}$$

In types A_n, D_n, E_6, E_7 and E_8, $\mathfrak{k}$ is isomorphic to $\mathfrak{so}_{n+1}$, $\mathfrak{so}_n \oplus \mathfrak{so}_n$, $\mathfrak{sp}_8$, $\mathfrak{sl}_8$ and $\mathfrak{so}_{16}$ respectively. As $\mathfrak{k}$ is semisimple, its adjoint representation is faithful because its kernel is the center of $\mathfrak{k}$, which is $\{0\}$. Hence, as $\mathfrak{k}$ and L have identical actions, they are isomorphic as Lie algebras. $\square$

Part II

Quantum supergroups at roots of 1

Chapter 7

(q,π)-binomials at roots of 1

In Part II, we establish notations and conventions independently of Pat I.

In this chapter, we establish several basic formulas of the (q,π)-binomial coefficients at roots of 1. They specialize to the formulas in [Lu94, Chapter 34] at $\pi = 1$.

7.1 (q,π)-integers

Let π and q be formal indeterminants such that $\pi^2 = 1$. Fix $\sqrt{\pi}$ such that $\sqrt{\pi}^2 = \pi$. In contrast to earlier papers on the quantum covering groups [CHW13, CHW14, CFLW, Cl14], it is often helpful and sometimes crucial for the ground rings considered in Part II to contain $\sqrt{\pi}$, and for the sake of simplicity we choose to do so uniformly from the outset. For any ring S with 1, define the new ring

$$S^\pi = S \otimes_{\mathbb{Z}} \mathbb{Z}[\sqrt{\pi}].$$

We shall use often the following two rings:

$$\mathcal{A} = \mathbb{Z}[q, q^{-1}], \qquad \mathcal{A}^{\pi} = \mathbb{Z}[q, q^{-1}, \sqrt{\pi}].$$

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$. For $a \in \mathbb{Z}$ and $n \in \mathbb{N}$, we define the (q, π)-*integer*

$$[a]_{q,\pi} = \frac{(\pi q)^a - q^{-a}}{\pi q - q^{-1}} \in \mathcal{A}^{\pi},$$

and then define the corresponding (q, π)-factorials and (q, π)-binomial coefficients by

$$[n]^!_{q,\pi} = \prod_{i=1}^{n} [i]_{q,\pi}, \qquad \begin{bmatrix} a \\ n \end{bmatrix}_{q,\pi} = \frac{\prod_{i=1}^{n} [a+1-i]_{q,\pi}}{[n]^!_{q,\pi}}.$$

For an indeterminant v, we denote the v-integers

$$[a]_v = \frac{v^a - v^{-a}}{v - v^{-1}}$$

and we similarly define the v-factorials $[n]^!_v$ and v-binomial coefficients $\begin{bmatrix} a \\ n \end{bmatrix}_v$. We denote by $\binom{a}{n}$ the classical binomial coefficients.

In Part II, the notation v is auxiliary, and we will identify

$$v := \sqrt{\pi} q,$$

and hence, for $n, t \in \mathbb{N}$,

$$[n]_{q,\pi} = \sqrt{\pi}^{n-1}[n]_v, \qquad [n]^!_{q,\pi} = \sqrt{\pi}^{n(n-1)/2}[n]^!_v,$$

$$\begin{bmatrix} n \\ t \end{bmatrix}_{q,\pi} = \sqrt{\pi}^{(n-t)t} \begin{bmatrix} n \\ t \end{bmatrix}_v . \tag{7.1.1}$$

Fix $\ell \in \mathbb{Z}_{>0}$ and let $\ell' = \ell$ or 2ℓ if ℓ is odd and let $\ell' = 2\ell$ if ℓ is even. Let

$$\mathcal{A}' = \mathcal{A}/\langle f(q) \rangle,$$

where $\mathcal{A}/\langle f(q) \rangle$ denotes the ideal generated by the ℓ'-th cyclotomic polynomial $f(q)$; we denote by $\varepsilon \in \mathcal{A}'$ the image of $q \in \mathcal{A}$. Take R to be an $\mathcal{A}'$-algebra with 1 (and so also an $\mathcal{A}$-algebra). Introduce the following root of 1 in R^π:

$$\mathbf{q} = \sqrt{\pi}\varepsilon \in R^\pi. \tag{7.1.2}$$

Then the element

$$\mathbf{v} := \sqrt{\pi}\mathbf{q} \in R^\pi$$

satisfies that

$$\mathbf{v}^{2\ell} = 1, \qquad \mathbf{v}^{2t} \neq 1 \quad (\forall t \in \mathbb{Z}, \ell > t > 0). \tag{7.1.3}$$

Consider the specialization homomorphism $\phi : \mathcal{A}^\pi \to R^\pi$ which sends q to $\mathbf{q}$ and $\sqrt{\pi}$ to $\sqrt{\pi}$. We shall denote by $[n]_{\mathbf{q},\pi}$ and $\begin{bmatrix} n \\ t \end{bmatrix}_{\mathbf{q},\pi}$ the images of $[n]_{q,\pi}$ and $\begin{bmatrix} n \\ t \end{bmatrix}_{q,\pi}$ under ϕ respectively, and so on.

7.2 (q, π)-binomial formulas

The following lemma is an analogue of [Lu94, Lemma 34.1.2], which can be in turn recovered by setting $\pi = 1$ below.

Lemma 7.2.1. *(a) If $t \in \mathbb{Z}_{>0}$ is not divisible by ℓ and $n \in \mathbb{Z}$ is divisible by ℓ, then* $\begin{bmatrix} n \\ t \end{bmatrix}_{\mathbf{q},\pi} = 0.$

(b) If $n_1 \in \mathbb{Z}$ and $t_1 \in \mathbb{N}$, then we have

$$\begin{bmatrix} \ell n_1 \\ \ell t_1 \end{bmatrix}_{\mathbf{q},\pi} = \pi^{\ell^2 t_1 (n_1-(t_1-1)/2)} \mathbf{q}^{\ell^2 t_1 (n_1+1)} \binom{n_1}{t_1}.$$

(c) Let $n \in \mathbb{Z}$ and $t \in \mathbb{N}$. Write $n = n_0 + \ell n_1$ with $n_0, n_1 \in \mathbb{Z}$ such that $0 \le n_0 \le \ell - 1$ and write $t = t_0 + \ell t_1$ with $t_0, t_1 \in \mathbb{N}$ such that $0 \le t_0 \le \ell - 1$. Then we have

$$\begin{bmatrix} n \\ t \end{bmatrix}_{\mathbf{q},\pi} = \pi^{\ell(n_0-t_0)t_1+\ell^2(n_1-(t_1-1)/2)t_1} \mathbf{q}^{\ell(n_0t_1-n_1t_0)+\ell^2(n_1+1)t_1} \begin{bmatrix} n_0 \\ t_0 \end{bmatrix}_{\mathbf{q},\pi} \binom{n_1}{t_1}.$$

Proof. One proof would be by imitating the arguments for [Lu94, Lemma 34.1.2]. Below we shall use an alternative and quicker approach, which is to convert [Lu94, Lemma 34.1.2] into our current statements using (7.1.1) via the substitution $\mathbf{v} = \sqrt{\pi}\mathbf{q}$. Part (a) immediately follows from [Lu94, Lemma 34.1.2(a)].

(b) By applying [Lu94, Lemma 34.1.2(b)] to $\begin{bmatrix} \ell n_1 \\ \ell t_1 \end{bmatrix}_{\mathbf{v}}$ and using (7.1.1), we have

$$\begin{bmatrix} \ell n_1 \\ \ell t_1 \end{bmatrix}_{\mathbf{q},\pi} = \sqrt{\pi}^{\ell t_1(\ell n_1 - \ell t_1)} \begin{bmatrix} \ell n_1 \\ \ell t_1 \end{bmatrix}_{\mathbf{v}} = \sqrt{\pi}^{\ell^2 t_1(n_1-t_1)} \mathbf{v}^{\ell^2 t_1(n_1+1)} \binom{n_1}{t_1},$$

which can be easily shown to be equal to the formula as stated in the lemma.

(c) Note that

$$\sqrt{\pi}^{(n-t)t} = \sqrt{\pi}^{\ell((n_0-t_0)t_1+(n_1-t_1)t_0)} \sqrt{\pi}^{\ell^2(n_1-t_1)t_1} \sqrt{\pi}^{(n_0-t_0)t_0}. \tag{7.2.1}$$

By applying [Lu94, Lemma 34.1.2(c)] to $\begin{bmatrix} n \\ t \end{bmatrix}_{\mathbf{v}}$ and using (7.1.1)-(7.2.1), we have

$$\begin{aligned}
\begin{bmatrix} n \\ t \end{bmatrix}_{\mathbf{q},\pi} &= \sqrt{\pi}^{(n-t)t} \begin{bmatrix} n \\ t \end{bmatrix}_{\mathbf{v}} \\
&= \sqrt{\pi}^{(n-t)t} \mathbf{v}^{\ell(n_0 t_1 - n_1 t_0) + \ell^2(n_1+1)t_1} \begin{bmatrix} n_0 \\ t_0 \end{bmatrix}_{\mathbf{v}} \binom{n_1}{t_1} \\
&= \sqrt{\pi}^{\ell((n_0-t_0)t_1+(n_1-t_1)t_0)} \sqrt{\pi}^{\ell^2(n_1-t_1)t_1} \sqrt{\pi}^{\ell(n_0 t_1 - n_1 t_0) + \ell^2(n_1+1)t_1} \\
&\qquad \times \mathbf{q}^{\ell(n_0 t_1 - n_1 t_0) + \ell^2(n_1+1)t_1} \left(\sqrt{\pi}^{(n_0-t_0)t_0} \begin{bmatrix} n_0 \\ t_0 \end{bmatrix}_{\mathbf{v}} \right) \binom{n_1}{t_1} \\
&= \pi^{\ell(n_0-t_0)t_1 + \ell^2(n_1-(t_1-1)/2)t_1} \mathbf{q}^{\ell(n_0 t_1 - n_1 t_0) + \ell^2(n_1+1)t_1} \begin{bmatrix} n_0 \\ t_0 \end{bmatrix}_{\mathbf{q},\pi} \binom{n_1}{t_1}.
\end{aligned}$$

The lemma is proved. □

Note that, due to our choice of $\mathbf{q} = \sqrt{\pi}\varepsilon$, we also have an analogue of equation (e) in the proof of [Lu94, Lemma 34.1.2]:

$$\mathbf{v}^{\ell^2+\ell} = \pi^{(\ell+1)\ell/2}\mathbf{q}^{\ell^2+\ell} = (-1)^{\ell+1}. \tag{7.2.2}$$

The following is an analogue of [Lu94, §34.1.3(a)].

Lemma 7.2.2. *Let $b \geq 0$. Then*

$$\frac{[\ell b]^!_{\mathbf{q},\pi}}{([\ell]^!_{\mathbf{q},\pi})^b} = b!(\pi\mathbf{q})^{\ell^2 b(b-1)/2}.$$

Proof. Recall $\mathbf{v} = \sqrt{\pi}\mathbf{q}$. Using (7.1.1) and [Lu94, §34.1.3(a)], we have

$$\begin{aligned}[\ell b]^!_{\mathbf{q},\pi}/([\ell]^!_{\mathbf{q},\pi})^b &= \sqrt{\pi}^{\ell b(\ell b-1)/2 - b\ell(\ell-1)/2}[\ell b]^!_{\mathbf{v}}/([\ell]^!_{\mathbf{v}})^b \\ &= \sqrt{\pi}^{\ell^2 b(b-1)/2} b!\mathbf{v}^{\ell^2 b(b-1)/2} = b!(\pi\mathbf{q})^{\ell^2 b(b-1)/2}.\end{aligned}$$

The lemma is proved. □

Below is a π-enhanced version of [Lu94, Lemma 34.1.4].

Lemma 7.2.3. *Suppose that $0 \leq r \leq a < \ell$. Then,*

$$\sum_{s=0}^{\ell-a-1} (-1)^{\ell-r+1+s}\pi^{\binom{s+1}{2}+s(r-\ell)}\mathbf{q}^{-(\ell-r)(a-\ell+1+s)+s}\begin{bmatrix}\ell-r\\ s\end{bmatrix}_{\mathbf{q},\pi} = \pi^{\binom{r}{2}-\binom{l}{2}-a(r-l)}\mathbf{q}^{\ell(a-r)}\begin{bmatrix}a\\ r\end{bmatrix}_{\mathbf{q},\pi}.$$

Proof. Plugging $\mathbf{v} = \sqrt{\pi}\mathbf{q}$ into [Lu94, Lemma 34.1.4] and using (7.1.1), we obtain

$$\sum_{s=0}^{\ell-a-1} (-1)^{\ell-r+1+s} \sqrt{\pi}^{-(\ell-r)(a-\ell+1+s)+s+s(s-\ell+r)} \mathbf{q}^{-(\ell-r)(a-\ell+1+s)+s} \begin{bmatrix} \ell - r \\ s \end{bmatrix}_{\mathbf{q},\pi} = \sqrt{\pi}^{\ell(a-r)+r(r-a)} \mathbf{q}^{\ell(a-r)} \begin{bmatrix} a \\ r \end{bmatrix}_{\mathbf{q},\pi}.$$

Rearranging the $\sqrt{\pi}$ terms, we have

$$\sum_{s=0}^{\ell-a-1} (-1)^{\ell-r+1+s} \sqrt{\pi}^{s(s+1)+2s(r-\ell)} \mathbf{q}^{-(\ell-r)(a-\ell+1+s)+s} \begin{bmatrix} \ell - r \\ s \end{bmatrix}_{\mathbf{q},\pi} = \sqrt{\pi}^{r(r-1)-\ell(\ell-1)-2a(r-l)} \mathbf{q}^{\ell(a-r)} \begin{bmatrix} a \\ r \end{bmatrix}_{\mathbf{q},\pi}.$$

from which the desired formula is immediate. □

Chapter 8

Quantum covering groups at roots of 1

8.1 Cartan and root data

In this chapter we recall the notion of super Cartan/root datum and the quantum covering groups. Then we obtain presentations of the modified quantum covering groups and their quasi-classical counterpart.

The following is an analogue of [Lu94, §2.2.4-5].

We recall the notions of Cartan and root data that were introduced in Part I. A *Cartan datum* is a pair $(I, \cdot)$ consisting of a finite set I and a symmetric bilinear form $\nu, \nu' \mapsto \nu \cdot \nu'$ on the free abelian group $\mathbb{Z}[I]$ with values in $\mathbb{Z}$ satisfying

(a) $d_i = \frac{i \cdot i}{2} \in \mathbb{Z}_{>0}$;

(b) $2\frac{i \cdot j}{i \cdot i} \in -\mathbb{N}$ for $i \neq j$ in I.

If the datum can be decomposed as $I = I_0 \coprod I_1$ such that

(c) $I_1 \neq \emptyset$,

(d) $2\frac{i\cdot j}{i\cdot i} \in 2\mathbb{Z}$ if $i \in I_1$,

then it is called a *super Cartan datum*; cf. [CHW13]. We denote the parity $p(i) = 0$ for $i \in I_0$ and $p(i) = 1$ for $i \in I_1$.

Following [CHW13], we will always assume a super Cartan datum satisfies the additional *bar-consistent* condition:

(e) $\frac{i\cdot i}{2} \equiv p(i) \mod 2, \quad \forall i \in I.$

A root datum of type $(I, \cdot)$ consists of 2 finite rank lattices X, Y with a perfect bilinear pairing $\langle\cdot,\cdot\rangle : Y \times X \to \mathbb{Z}$, 2 embeddings $I \hookrightarrow X$ $(i \mapsto i')$ and $I \hookrightarrow Y$ $(i \mapsto i)$ such that $\langle i, j'\rangle = 2\frac{i\cdot j}{i\cdot i}$, $\forall i, j \in I$. Moreover, we will assume throughout Part II that the root datum is X-*regular*, i.e., that the simple roots are linearly independent in X.

Define

$$\ell_i = \min\{r \in \mathbb{Z}_{>0} \mid r(i \cdot i)/2 \in \ell\mathbb{Z}\}.$$

The next lemma follows by the definition of ℓ_i and the bar-consistency condition of I.

Lemma 8.1.1. *For each $i \in I_1$, ℓ_i has the same parity as ℓ.*

Then $(I, \diamond)$ is a new root datum by [Lu94, 2.2.4], where we let

$$i \diamond j = (i \cdot j)\ell_i\ell_j, \quad \forall i, j \in I.$$

Note that if ℓ is odd, then $(I, \diamond)$ is a super Cartan datum with the same parity decomposition $I = I_0 \cup I_1$ as for $(I, \cdot)$ by Lemma 8.1.1; if ℓ is even, then $(I, \diamond)$ is a (non-super) Cartan datum with $I_1 = \emptyset$.

We shall write $Y^\diamond, X^\diamond$ in this paper what Lusztig [Lu94, 2.2.5] denoted by Y^*, X^* respectively, and we will use superscript $^\diamond$ in related notation associated to $(Y^\diamond, X^\diamond, I, \diamond)$ below. More explicitly, we set $X^\diamond = \{\zeta \in X | \langle i, \zeta\rangle \in \ell_i\mathbb{Z}, \forall i \in I\}$ and $Y^\diamond = \mathrm{Hom}_{\mathbb{Z}}(X^\diamond, \mathbb{Z})$ with the obvious pairing. The embedding $I \hookrightarrow X^\diamond$ is given by $i \mapsto i'^\diamond = \ell_i i' \in X$, while embedding $I \hookrightarrow Y^\diamond$ is given by $i \mapsto i^\diamond \in Y^\diamond$ whose value at any $\zeta \in X^\diamond$ is $\langle i, \zeta\rangle/\ell_i$. It follows that $\langle i^\diamond, j'^\diamond\rangle = 2i \diamond j/i \diamond i$.

If ℓ is odd, then $(Y^\diamond, X^\diamond, \cdots)$ is a new super root datum satisfying (a)-(d) above and in addition the bar-consistency condition (e). Indeed, we have $2\frac{i\diamond j}{i\diamond i} = 2\frac{i\cdot j}{i\cdot i}\frac{\ell_j}{\ell_i} \in 2\mathbb{Z}$ by Lemma 8.1.1, whence (d), and $\frac{i\diamond i}{2} = \frac{i\cdot i}{2}\ell_i^2 \equiv p(i) \mod 2$ by Lemma 8.1.1, whence (e). If ℓ is even, then $(Y^\diamond, X^\diamond, \cdots)$ is a new (non-super) root datum just as in [Lu94, 2.2.5].

8.2 Quantum covering groups

By [CHW13, Propositions 1.4.1, 3.4.1], the unital $\mathbb{Q}(q)^\pi$-superalgebra $\mathbf{f}$ is generated by θ_i $(i \in I)$ subject to the super Serre relations

$$\sum_{n+n'=1-\langle i,j'\rangle} (-1)^{n'} \pi_i^{n'p(j)+\binom{n'}{2}} \theta_i^{(n)}\theta_j\theta_i^{(n')} = 0$$

for any $i \neq j$ in I; here a generator θ_i is even if and only if $i \in I_0$. There is an $\mathcal{A}^\pi$-form for $\mathbf{f}$, which we call ${}_{\mathcal{A}}\mathbf{f}$. It is generated by the divided powers $\theta_i^{(n)} = \theta_i^n/[n]^!_{q_i,\pi_i}$ for all $i \in I, n \geq 1$. As R^π is an $\mathcal{A}^\pi$-algebra (cf. §7.1), by a base change we define ${}_R\mathbf{f} = R^\pi \otimes_{\mathcal{A}^\pi} {}_{\mathcal{A}}\mathbf{f}$. The algebras $'\mathbf{f}^\diamond$, $\mathbf{f}^\diamond$ and ${}_R\mathbf{f}^\diamond$ are defined in the same way using the Cartan datum $(I, \diamond)$.

Let $\mathbf{U}$ denote the quantum covering group associated to the root datum $(Y, X, ...)$ introduced in [CHW13]. By [CHW13, Proposition 3.4.2], $\mathbf{U}$ is a unital $\mathbb{Q}(q)^\pi$-superalgebra with generators

$$E_i \quad (i \in I), \quad F_i \quad (i \in I), \quad J_\mu \quad (\mu \in Y), \quad K_\mu \quad (\mu \in Y),$$

subject to the relations (a)-(f) below for all $i, j \in I, \mu, \mu' \in Y$:

$$K_0 = 1, \quad K_\mu K_{\mu'} = K_{\mu+\mu'}, \tag{a}$$

$$J_{2\mu} = 1, \quad J_\mu J_{\mu'} = J_{\mu+\mu'}, \quad J_\mu K_{\mu'} = K_{\mu'} J_\mu,$$

$$K_\mu E_i = q^{\langle \mu, i' \rangle} E_i K_\mu, \quad J_\mu E_i = \pi^{\langle \mu, i' \rangle} E_i J_\mu, \tag{b}$$

$$K_\mu F_i = q^{-\langle \mu, i' \rangle} F_i K_\mu, \quad J_\mu F_i = \pi^{-\langle \mu, i' \rangle} F_i J_\mu, \tag{c}$$

$$E_i F_j - \pi^{p(i)p(j)} F_j E_i = \delta_{i,j} \frac{\widetilde{J}_i \widetilde{K}_i - \widetilde{K}_{-i}}{\pi_i q_i - q_i^{-1}}, \tag{d}$$

$$\sum_{n+n'=1-\langle i, j' \rangle} (-1)^{n'} \pi_i^{n' p(j) + \binom{n'}{2}} E_i^{(n)} E_j E_i^{(n')} = 0 \tag{e}$$

$$\sum_{n+n'=1-\langle i, j' \rangle} (-1)^{n'} \pi_i^{n' p(j) + \binom{n'}{2}} F_i^{(n)} F_j F_i^{(n')} = 0 \tag{f}$$

where for any element $\nu = \sum_i \nu_i i \in \mathbb{Z}[I]$ we have set $\widetilde{K}_\nu = \prod_i K_{d_i \nu_i i}$, $\widetilde{J}_\nu = \prod_i J_{d_i \nu_i i}$. In particular, $\widetilde{K}_i = K_{d_i i}$, $\widetilde{J}_i = J_{d_i i}$. (Under the bar-consistent condition (e), $\widetilde{J}_i = 1$ for $i \in I_{\overline{0}}$ while $\widetilde{J}_i = J_i$ for $i \in I_{\mathbf{1}}$.) We endow $\mathbf{U}$ with a $\mathbb{Z}[I]$-grading $|\cdot|$ by setting $|E_i| = i, \quad |F_i| = -i, \quad |J_\mu| = |K_\mu| = 0$. The parity on $\mathbf{U}$ is given by $p(E_i) = p(F_i) = p(i)$ and $p(K_\mu) = p(J_\mu) = 0$,

The algebra $\mathbf{U}$ has an $\mathcal{A}^\pi$-form ${}_\mathcal{A}\mathbf{U}$. By a base change, we obtain ${}_R\mathbf{U} = R^\pi \otimes_{\mathcal{A}^\pi} {}_\mathcal{A}\mathbf{U}$.

Let ${}_R\mathbf{U}^+$ (resp. ${}_R\mathbf{U}^-$) denote the subalgebra of ${}_R\mathbf{U}$ generated by the $E_i^{(n)} = E_i^n/[n]^!_{\mathbf{q}_i,\pi_i}$ (resp. $F_i = F_i^n/[n]^!_{\mathbf{q}_i,\pi_i}$). As a R^π-algebra ${}_R\mathbf{f}$ is isomorphic to ${}_R\mathbf{U}^+$ (resp. ${}_R\mathbf{U}^-$) via the map $x \mapsto x^+$ (resp. $x \mapsto x^-$), where $(\theta_i^{(n)})^+ = E_i^{(n)}$ (resp. $(\theta_i^{(n)})^- = F_i^{(n)}$.

Denote by $X^+ = \{\lambda \in X \mid \langle i, \lambda\rangle \in \mathbb{N}, \forall i \in I\}$, the set of dominant integral weights.

For $\lambda \in X$, let $M(\lambda)$ be the Verma module of $\mathbf{U}$, and we can naturally identify $M(\lambda) = \mathbf{f}$ as $\mathbb{Q}(q)^\pi$-modules. The ${}_\mathcal{A}\mathbf{U}$-submodule ${}_\mathcal{A}M(\lambda)$ can be identified with ${}_\mathcal{A}\mathbf{f}$ as $\mathcal{A}^\pi$-free modules. For $\lambda \in X^+$, we define the integrable $\mathbf{U}$-module $V(\lambda) = M(\lambda)/J_\lambda$, where J_λ is the left $\mathbf{f}$-module generated by $\theta_i^{\langle i,\lambda\rangle+1}$ for all $i \in I$. Let ${}_RM(\lambda) = R^\pi \otimes_{\mathcal{A}^\pi} {}_\mathcal{A}M(\lambda)$ for $\lambda \in X$, and ${}_RV(\lambda) = R^\pi \otimes_{\mathcal{A}^\pi} {}_\mathcal{A}V(\lambda)$ for $\lambda \in X^+$.

The algebra $\mathbf{U}^\diamond$ is defined in the same way as $\mathbf{U}$ based on the root datum $(Y^\diamond, X^\diamond, \ldots)$.

Recall from [CFLW, Definition 4.2] that the modified quantum covering group $\dot{\mathbf{U}}$ is a $\mathbb{Q}(q)^\pi$-algebra without unit which is generated by the symbols $1_\lambda, E_i1_\lambda$ and F_i1_λ, for $\lambda \in X$ and $i \in I$, subject to the relations:

$$
\begin{gathered}
1_\lambda 1_{\lambda'} = \delta_{\lambda,\lambda'} 1_\lambda, \\
(E_i1_\lambda)1_{\lambda'} = \delta_{\lambda,\lambda'} E_i1_\lambda, \quad 1_{\lambda'}(E_i1_\lambda) = \delta_{\lambda',\lambda+i'} E_i1_\lambda, \\
(F_i1_\lambda)1_{\lambda'} = \delta_{\lambda,\lambda'} F_i1_\lambda, \quad 1_{\lambda'}(F_i1_\lambda) = \delta_{\lambda',\lambda-i'} F_i1_\lambda, \\
(E_iF_j - \pi^{p(i)p(j)} F_jE_i)1_\lambda = \delta_{ij} \, [\langle i, \lambda\rangle]_{v_i,\pi_i} \, 1_\lambda, \\
\textstyle\sum_{n+n'=1-\langle i,j'\rangle} (-1)^{n'} \pi_i^{n'p(j)+\binom{n'}{2}} E_i^{(n)} E_j E_i^{(n')} 1_\lambda = 0 \;\; (i \neq j), \\
\textstyle\sum_{n+n'=1-\langle i,j'\rangle} (-1)^{n'} \pi_i^{n'p(j)+\binom{n'}{2}} F_i^{(n)} F_j F_i^{(n')} 1_\lambda = 0 \;\; (i \neq j),
\end{gathered}
$$

where $i, j \in I$, $\lambda, \lambda' \in X$, and we use the notation $xy1_\lambda = (x1_{\lambda+|y|})(y1_\lambda)$ for $x, y \in \mathbf{U}$.

The modified quantum covering group $\dot{\mathbf{U}}$ admits an $\mathcal{A}^\pi$-form, ${}_\mathcal{A}\dot{\mathbf{U}}$ and so we can define ${}_R\dot{\mathbf{U}} = R^\pi \otimes_{\mathcal{A}^\pi} {}_\mathcal{A}\dot{\mathbf{U}}$. Let us give a presentation for ${}_R\dot{\mathbf{U}}$.

Lemma 8.2.1. *The modified quantum covering group* ${}_R\dot{\mathbf{U}}$ *is generated as an* R^π*-algebra by* $x^+\mathbf{1}_\lambda x'^-$ *or equivalently by* $x^-\mathbf{1}_\lambda x'^+$*, where* $x \in {}_R\mathbf{f}_\mu, x' \in {}_R\mathbf{f}_\nu$ *and* $\lambda \in X$*, subject to the following relations:*

$$(\theta_i^{(N)})^+\mathbf{1}_\lambda(\theta_i^{(M)})^- \\ = \sum_{t\geq 0} \pi_i^{MN-\binom{t+1}{2}}(\theta_i^{(M-t)})^- \begin{bmatrix} M+N+\langle i,\lambda\rangle \\ t \end{bmatrix}_{\mathbf{q}_i,\pi_i} \mathbf{1}_{\lambda+(M+N-t)i'}(\theta_i^{(N-t)})^+,$$

$$(\theta_i^{(N)})^-\mathbf{1}_\lambda(\theta_i^{(M)})^+ \\ = \sum_{t\geq 0} \pi_i^{MN+t\langle i,\lambda\rangle-\binom{t}{2}}(\theta_i^{(M-t)})^+ \begin{bmatrix} M+N-\langle i,\lambda\rangle \\ t \end{bmatrix}_{\mathbf{q}_i,\pi_i} \mathbf{1}_{\lambda-(M+N-t)i'}(\theta_i^{(N-t)})^-,$$

$$(\theta_i^{(N)})^+(\theta_j^{(M)})^-\mathbf{1}_\lambda = \pi^{MNp(i)p(j)}(\theta_j^{(M)})^-(\theta_i^{(N)})^+\mathbf{1}_\lambda, \ \textit{for } i \neq j,$$

$$x^+\mathbf{1}_\lambda = \mathbf{1}_{\lambda+\mu}x^+, \quad x^-\mathbf{1}_\lambda = \mathbf{1}_{\lambda-\mu}x^-,$$

$$(x^+\mathbf{1}_\lambda)(\mathbf{1}_{\lambda'}x'^-) = \delta_{\lambda,\lambda'}x^+\mathbf{1}_\lambda x'^-, \quad (x^-\mathbf{1}_\lambda)(\mathbf{1}_{\lambda'}x'^+) = \delta_{\lambda,\lambda'}x^-\mathbf{1}_\lambda x'^+,$$

$$(x^+\mathbf{1}_\lambda)(\mathbf{1}_{\lambda'}x'^-) = \delta_{\lambda,\lambda'}\mathbf{1}_{\lambda+\mu}x^+x'^-, \quad (x^-\mathbf{1}_\lambda)(\mathbf{1}_{\lambda'}x'^+) = \delta_{\lambda,\lambda'}\mathbf{1}_{\lambda-\mu}x^-x'^+,$$

$$(rx+r'x')^\pm\mathbf{1}_\lambda = rx^\pm\mathbf{1}_\lambda + r'x'^\pm\mathbf{1}_\lambda, \ \textit{where } r,r' \in R^\pi.$$

Proof. This is proved in the same way as [Lu94, §31.1.3]. Let A be the R^π-algebra with the above generators and relations. All of these relations are known to hold in ${}_R\dot{\mathbf{U}}$. The first three are shown to hold in ${}_R\dot{\mathbf{U}}$ by a direct application of [CHW13, Lemma 2.2.3] as in [Cl14, Lemma 4] while the remaining ones are clear. However, there was an error in the second relation of [Cl14, Lemma 4], so we derive that relation from [CHW13, Lemma 2.2.3] here. We have

$$
\begin{aligned}
&(\theta_i^{(N)})^-\mathbf{1}_\lambda(\theta_i^{(M)})^+\\
&=(\theta_i^{(N)})^-(\theta_i^{(M)})^+\mathbf{1}_{\lambda-Mi'}\\
&=\sum_{t\geq 0}(-1)^t\pi_i^{(M-t)(N-t)-t^2}(\theta_i^{(M-t)})^+\begin{bmatrix}\tilde{K}_i;M+N-(t+1)\\ t\end{bmatrix}_{\mathbf{q}_i,\pi_i}(\theta_i^{(N-t)})^-\mathbf{1}_{\lambda-Mi'}\\
&=\sum_{t\geq 0}(-1)^t\pi_i^{(M-t)(N-t)-t^2}(\theta_i^{(M-t)})^+\begin{bmatrix}\langle i,\lambda\rangle-M-N+t-1\\ t\end{bmatrix}_{\mathbf{q}_i,\pi_i}\mathbf{1}_{\lambda-(M+N-t)i'}(\theta_i^{(N-t)})^-\\
&=\sum_{t\geq 0}\pi_i^{MN+t\langle i,\lambda\rangle-\binom{t}{2}}(\theta_i^{(M-t)})^+\begin{bmatrix}M+N-\langle i,\lambda\rangle\\ t\end{bmatrix}_{\mathbf{q}_i,\pi_i}\mathbf{1}_{\lambda-(M+N-t)i'}(\theta_i^{(N-t)})^-
\end{aligned}
$$

where in the last step, we used [CHW13, (1.10)] with $a=M+N-\langle i,\lambda\rangle$. Hence the natural homomorphism $A\longrightarrow {}_R\dot{\mathbf{U}}$ is surjective. Let $\mathbf{S}$ be an R^π-basis of ${}_R\mathbf{f}$ consisting of weight vectors. Then $\{x^+\mathbf{1}_\lambda x'^-|x,x'\in\mathbf{S},\lambda\in X\}$ can be seen to be an R^π-basis for A, and it is known to be one for ${}_R\dot{\mathbf{U}}$ (cf. [Cl14, Lemma 5]). Thus, the natural homomorphism is, in fact, an isomorphism. □

8.3 Quasi-classical quantum covering groups

The algebra $\dot{\mathbf{U}}^\diamond$ is defined in the same way using $\mathbf{U}^\diamond$ and $(Y^\diamond, X^\diamond, ...)$, and so it also has an $\mathcal{A}^\pi$-form ${}_\mathcal{A}\dot{\mathbf{U}}^\diamond$ and we can define ${}_R\dot{\mathbf{U}}^\diamond=R^\pi\otimes_{\mathcal{A}^\pi}{}_\mathcal{A}\dot{\mathbf{U}}^\diamond$.

Remark 8.3.1. If ℓ is even, then ${}_R\mathbf{f}^\diamond$ is a (non-super) algebra; if ℓ is odd, then the θ_i in ${}_R\mathbf{f}^\diamond$ and ${}_R\mathbf{f}$ for any given i have the same parity.

For $i \in I$, we denote

$$q_i^\diamond = q^{i\diamond i/2} = q_i^{\ell_i^2}, \qquad \mathbf{q}_i^\diamond = \mathbf{q}^{i\diamond i/2} = \mathbf{q}_i^{\ell_i^2}, \qquad \pi_i^\diamond = \pi^{i\diamond i/2} = \pi_i^{\ell_i^2}. \tag{8.3.1}$$

Lemma 8.3.2. *Let $i \in I_1$.*

(a) If ℓ is odd, then $\pi_i^\diamond = \pi_i$.

(b) If ℓ is even, then $\pi_i^\diamond = 1$.

Proof. Recall from Lemma 8.1.1 that ℓ_i must have the same parity as ℓ. The claim on $\pi_i^\diamond$ follows now from (8.3.1). □

For each $i \in I$, we have

$$\pi_i^\diamond \mathbf{q}_i^{\diamond 2} = (\pi_i \mathbf{q}_i^2)^{\ell_i^2} = 1. \tag{8.3.2}$$

Following Lusztig [Lu94], we will refer to the quantum supergroup ${}_R\mathbf{f}^\diamond$ associated to $(Y^\diamond, X^\diamond, \cdots)$ as *quasi-classical*; cf. (8.3.2).

Proposition 8.3.3. *Let R be the fraction field of $\mathcal{A}'$. The quasi-classical algebra ${}_R\mathbf{f}^\diamond$ is isomorphic to ${}_R\tilde{\mathbf{f}}^\diamond$, the R^π-algebra generated by θ_i, $i \in I$, subject to the super Serre relations:*

$$\sum_{n+n'=1-\langle i,j'\rangle^\diamond} (-1)^{n'} (\pi_i^\diamond)^{np(j)+\binom{n}{2}} \theta_i^{(n)} \theta_j \theta_i^{(n')} = 0 \qquad (i \neq j \in I).$$

Proof. When $\pi_i = 1$ or ℓ is even, $\pi_i^\diamond = 1$ and $\mathbf{q}_i^\diamond = \pm 1$ for each $i \in I$. Hence, in this case the lemma reduces to [Lu94, §33.2].

Now let ℓ be odd and $\pi = -1$. We make use of the *weight-preserving* automorphism $\dot{\Psi}$ of ${}_R\dot{\mathbf{U}}^\diamond$ (called a twistor) given in [CFLW, Theorem 4.3] when the base ring contains

$\sqrt{-1}$. We will only recall the basic property of $\dot{\Psi}$ which we need, and refer to *loc. cit.* for details. Note that for all $i \in I$, $\mathbf{q}_i^\diamond$ is a power of $\sqrt{-1}$ with at least one of the $\mathbf{q}_i^\diamond = \pm\sqrt{-1}$. Thus, $\pm\sqrt{-1}$ will play the role played by the v in [CFLW, Theorem 4.3], which we will denote by $\tilde{v}$ in this proof so as not to confuse it with the v defined in this paper. Recall $\dot{\Psi}$ takes π to $-\pi$ and $\tilde{v}$ to $\sqrt{-1}\tilde{v}$. When we specialize $\pi = -1$ and $\tilde{v} = \pm\sqrt{-1}$, we obtain an R-linear isomorphism of that specialization of ${}_R\dot{\mathbf{U}}^\diamond$, denoted by ${}_R\dot{\mathbf{U}}^\diamond|_{-1}$, with the (quasi-classical) modified quantum group corresponding to the specialization $\pi = 1$ and $\mathbf{q}_j^\diamond = \pm 1$, denoted by ${}_R\dot{\mathbf{U}}^\diamond|_1$.

Write

▷ ${}_{R_{-1}}\mathbf{f}$ for the half quantum (super)group over R corresponding to the former (i.e., $\pi = -1$);

▷ ${}_{R_1}\mathbf{f}^\diamond$ for the half (quasi-classical) quantum group over R corresponding to the latter (i.e., $\pi = 1$); cf. [Lu94, 33.2].

Recall that ${}_R\mathbf{f}^\diamond$ is a direct sum of finite-dimensional weight spaces ${}_R\mathbf{f}_\nu^\diamond$, where $\nu \in \mathbb{Z}_{\geq 0}[I]$. The weight-preserving isomorphism $\dot{\Psi}$ above implies that

$$\dim_{R^\pi}({}_R\mathbf{f}_\nu^\diamond) = \dim_R({}_{R_{-1}}\mathbf{f}_\nu^\diamond) = \dim_R {}_{R_1}\mathbf{f}_\nu^\diamond, \quad \forall \nu.$$

As ${}_{R_1}\mathbf{f}^\diamond$ is quasi-classical in the sense of [Lu94, 33.2], we have $\dim_R {}_{R_1}\mathbf{f}_\nu^\diamond = \dim_R {}_{R_1}\mathbf{f}_\nu$ for all ν, by [Lu94, 33.2.2], where ${}_{R_1}\mathbf{f}$ is the enveloping algebra of the half KM algebra over R. Hence we have

$$\dim_{R^\pi}({}_R\mathbf{f}_\nu^\diamond) = \dim_R({}_{R_1}\mathbf{f}_\nu), \quad \forall \nu. \tag{8.3.3}$$

Since the super Serre relations hold in ${}_R\mathbf{f}^\diamond$ (cf. [CHW13, Proposition 1.7.3]) we have a surjective algebra homomorphism $\varphi : {}_R\tilde{\mathbf{f}}^\diamond \longrightarrow {}_R\mathbf{f}^\diamond$ mapping $\theta_i \mapsto \theta_i$ for all i. Then φ maps each weight space ${}_R\tilde{\mathbf{f}}^\diamond_\nu$ onto the corresponding weight space ${}_R\mathbf{f}^\diamond_\nu$. As ${}_R\tilde{\mathbf{f}}^\diamond$ has a Serre-type presentation by definition, it follows by [KKO14, CHW14] that $\dim_{R^\pi}({}_R\tilde{\mathbf{f}}_\nu) = \dim_R({}_{R_1}\mathbf{f}_\nu)$ for each ν. This together with (8.3.3) implies that $\dim_{R^\pi}({}_R\tilde{\mathbf{f}}_\nu) = \dim_{R^\pi}({}_R\mathbf{f}^\diamond_\nu)$. Therefore φ is a linear isomorphism on each weight space and thus an isomorphism. □

Below we provide an analogue of [Lu94, 35.1.5].

Lemma 8.3.4. *Assume that both $n \in \mathbb{Z}$ and $t \in \mathbb{N}$ are divisible by ℓ_i. Then*

$$\begin{bmatrix} n \\ t \end{bmatrix}_{\mathbf{q}_i,\pi_i} = \begin{bmatrix} n/\ell_i \\ t/\ell_i \end{bmatrix}_{\mathbf{q}_i^\diamond,\pi_i^\diamond}.$$

(Setting $\pi = 1$ in the above formula recovers [Lu94, 35.1.5].)

Proof. By Lemma 7.2.1(b), we have

$$\begin{bmatrix} n \\ t \end{bmatrix}_{\mathbf{q}_i,\pi_i} = \pi_i^{t(n-(t-\ell_i)/2)}\mathbf{q}_i^{t(n+\ell_i)}\binom{n/\ell_i}{t/\ell_i}.$$

Note that $\pi_i^\diamond \mathbf{q}_i^{\diamond 2} = (\pi\mathbf{q}^2)^{\frac{i\cdot i}{2}\ell_i^2}$. Since $(\pi\mathbf{q}^2)^{2\ell} = 1$ and ℓ divides $\frac{i\cdot i}{2}\ell_i^2$ by the definition of ℓ_i, we have $(\pi_i^\diamond \mathbf{q}_i^{\diamond 2})^2 = 1$. Hence by (8.3.1) and Lemma 7.2.1(b) with $\ell = 1$ we have

$$\begin{bmatrix} n/\ell_i \\ t/\ell_i \end{bmatrix}_{\mathbf{q}_i^\diamond,\pi_i^\diamond} = \pi_i^{t(n-(t-\ell_i)/2)}\mathbf{q}_i^{t(n+\ell_i)}\binom{n/\ell_i}{t/\ell_i}.$$

The lemma follows. □

Chapter 9

The Frobenius-Lusztig homomorphism for QCG

9.1 The Frobenius-Lusztig homomorphism Fr′

In this chapter we establish the Frobenius-Lusztig homomorphism between the quasi-classical covering group and the quantum covering group at roots of 1. We also formulate Lusztig-Steinberg tensor product theorem in this setting.

Following [Lu94, 35.1.2], in this and following sections we shall assume

(a) for any $i \neq j \in I$ with $\ell_j \geq 2$, we have $\ell_i \geq -\langle i, j' \rangle + 1$.

(b) $(I, \cdot)$ has no odd cycles.

Below is a generalization of [Lu94, Theorem 35.1.8], which will be proved later in this section.

Theorem 9.1.1. *There is a unique R^π-superalgebra homomorphism*

$$\mathbf{Fr}' : {}_R\mathbf{f}^\diamond \longrightarrow {}_R\mathbf{f}, \qquad \mathbf{Fr}'(\theta_i^{(n)}) = \theta_i^{(n\ell_i)} \quad (\forall i \in I, n \in \mathbb{Z}_{>0}).$$

(Be aware that the two θ_i's above belong to different algebras and hence are different. Theorem 9.1.1 is consistent with Remark 8.3.1.)

The rest of the section is devoted to a proof of Theorem 9.1.1. The same remark as in [Lu94, 35.1.11] allows us to reduce the proof to the case when R is the quotient field of $\mathcal{A}'$, which we will assume in the remainder of this and the next chapter.

Recall from (7.1.3) that $\pi^\ell \mathbf{q}^{2\ell} = 1$ and $\pi^t \mathbf{q}^{2t} \neq 1$ for $0 < t < \ell$. By the definition of ℓ_i, we have $\pi_i^\ell \mathbf{q}_i^{2\ell} = 1$ and $\pi_i^t \mathbf{q}_i^{2t} \neq 1$ for $0 < t < \ell_i$. Then $[t]_{\mathbf{q}_i}^\pi!$ is invertible in R^π, for $0 < t < \ell_i$.

The following is an analogue of [Lu94, Lemma 35.2.2] and the proof uses now Lemmas 7.2.1 and 7.2.2.

Lemma 9.1.2. *The R^π-superalgebra ${}_R\mathbf{f}$ is generated by the elements $\theta_i^{(\ell_i)}$ for all $i \in I$ and the elements θ_i for $i \in I$ with $\ell_i \geq 2$.*

Proof. By definition the algebra ${}_R\mathbf{f}$ is generated by $\theta_i^{(n)}$ for all $i \in I$ and $n \geq 0$. We can write $n = a + \ell_i b$, for $0 \leq a < \ell_i$ and $b \in \mathbb{N}$. We note the following three identities in ${}_R\mathbf{f}$:

$$\theta_i^{(a+\ell_i b)} = \mathbf{q}_i^{\ell_i ab} \theta_i^{(a)} \theta_i^{(\ell_i b)}, \tag{9.1.1}$$

$$\theta_i^{(a)} = [a]_{\mathbf{q}_i,\pi_i}^{-1} \theta_i^a, \tag{9.1.2}$$

$$\theta_i^{(\ell_i b)} = (b!)^{-1} (\pi_i \mathbf{q}_i)^{-\ell_i^2 \binom{b}{2}} (\theta_i^{(\ell_i)})^b, \tag{9.1.3}$$

where (9.1.1) follows by Lemma 7.2.1 and (9.1.3) follows by Lemma 7.2.2, respectively.

(Note that a sign in the power of $\mathbf{v}_i$ in the identity (b) in [Lu94, proof of Lemma 35.2.2] is optional, but the sign cannot be dropped from the power of $\mathbf{q}_i$ in (9.1.3).) The lemma follows. □

Proof of Theorem 9.1.1. The uniqueness is clear.

By Lemma 7.2.2 (with $\ell = 1$), we have

$$[n]^!_{\mathbf{q}_i^\diamond,\pi_i^\diamond} = (\pi_i \mathbf{q}_i)^{\ell_i^2 \binom{n}{2}} n!. \tag{9.1.4}$$

We first observe that the existence of a homomorphism $\mathbf{Fr}'$ such that $\mathbf{Fr}'(\theta_i) = \theta_i^{(\ell_i)}$ implies that $\mathbf{Fr}'(\theta_i^{(n)}) = \theta_i^{(n\ell_i)}$ for all $n \geq 0$. Indeed, using (9.1.3)-(9.1.4) we have

$$\mathbf{Fr}'(\theta_i^{(n)}) = ([n]_{\mathbf{q}_i^\diamond,\pi_i^\diamond}!)^{-1}\mathbf{Fr}'(\theta_i)^n = \big((\pi_i\mathbf{q}_i)^{\ell_i^2 n(n-1)/2} n!\big)^{-1}\mathbf{Fr}'(\theta_i)^n = \theta_i^{(n\ell_i)}.$$

Hence it remains to show that there exists an algebra homomorphism $\mathbf{Fr}' : {}_R\mathbf{f}^\diamond \to {}_R\mathbf{f}$ such that $\theta_i \to \theta_i^{(\ell_i)}, \forall i \in I$. By Proposition 8.3.3 (also cf. [CHW13]), the algebra ${}_R\mathbf{f}^\diamond$ has the following defining relations:

$$\sum_{n+n'=1-\langle i,j'\rangle^\diamond} (-1)^{n'} (\pi_i^\diamond)^{np(j)+\binom{n}{2}} \theta_i^{(n)}\theta_j\theta_i^{(n')} = 0 \qquad (i \neq j \in I).$$

By (9.1.4) it suffices to check the following identity in ${}_R\mathbf{f}$: for $i \neq j \in I$,

$$\sum_{n+n'=1-\langle i,j'\rangle \ell_j/\ell_i} (-1)^{n'} \pi_i^{\ell_i^2(np(j)+n(n-1)/2)} (\pi_i\mathbf{q}_i)^{-\ell_i^2\binom{n}{2}} (\pi_i\mathbf{q}_i)^{-\ell_i^2\binom{n'}{2}} \frac{(\theta_i^{(\ell_i)})^n}{n!} \theta_j^{(\ell_j)} \frac{(\theta_i^{(\ell_i)})^{n'}}{n'!} = 0,$$

which, by the identity (9.1.3), is equivalent to checking the following identity in ${}_R\mathbf{f}$:

$$\sum_{n+n'=1-\langle i,j'\rangle \ell_j/\ell_i} (-1)^{n'} \pi_i^{\ell_i^2(np(j)+n(n-1)/2)} \theta_i^{(\ell_i n)} \theta_j^{(\ell_j)} \theta_i^{(\ell_i n')} = 0. \tag{9.1.5}$$

It remains to prove (9.1.5). Set $\alpha = -\langle i, j'\rangle$. For any $0 \le t \le \ell_i - 1$, we set

$$g_t = \sum_{\substack{r,s \\ r+s=\ell_j\alpha+\ell_i-t}} (-1)^r \pi_i^{\ell_j rp(j)+r(r-1)/2} q_i^{r(\ell_i-1-t)} \theta_i^{(r)} \theta_j^{(\ell_j)} \theta_i^{(s)} \in {}_{\mathcal{A}}\mathbf{f}.$$

This is basically $f'_{i,j;\ell_j,\ell_j\alpha+\ell_i-t}$ in [CHW13, 4.1.1(d)] in the notation of θ's. By the higher super Serre relations (see [CHW13, Proposition 4.2.4] and [CHW13, 4.1.1(e)]), we have $g_t = 0$ for all $0 \le t \le \ell_i - 1$. Set

$$g = \sum_{t=0}^{\ell_i-1} (-1)^t \pi_i^{t(t-1)/2} \mathbf{q}_i^{\ell_j\alpha t+\ell_i t-t} g_t \theta_i^{(t)},$$

which must be 0. On the other hand, setting $s' = s + t$, we have

$$(0 =)\ g = \sum_{\substack{r,s' \\ r+s'=\ell_j\alpha+\ell_i}} c_{r,s'} \theta_i^{(r)} \theta_j^{(\ell_j)} \theta_i^{(s')}, \tag{9.1.6}$$

where

$$c_{r,s'} = \sum_{t=0}^{\ell_i-1} (-1)^{r+t} \pi_i^{\ell_j rp(j)+r(r-1)/2+t(t-1)/2} q_i^{r(\ell_i-1-t)+\ell_j\alpha t+\ell_i t-t} \begin{bmatrix} s' \\ t \end{bmatrix}_{q_i,\pi_i}.$$

Taking the image of the identity (9.1.6) under the map ${}_{\mathcal{A}}\mathbf{f} \to {}_{R}\mathbf{f}$, we have

$$\sum_{\substack{r,s' \\ r+s'=\ell_j\alpha+\ell_i}} \phi(c_{r,s'})\theta_i^{(r)}\theta_j^{(\ell_j)}\theta_i^{(s')} = 0 \ \in {}_{R}\mathbf{f}.$$

For a fixed s', we write $s' = a + \ell_i n$, where $a, n \in \mathbb{Z}$ and $0 \le a \le \ell_i - 1$. Note by Lemma 7.2.1(c) that $\begin{bmatrix} s' \\ t \end{bmatrix}_{\mathbf{q}_i,\pi_i} = \mathbf{q}_i^{-\ell_i n t} \begin{bmatrix} a \\ t \end{bmatrix}_{\mathbf{q}_i,\pi_i}$. Now using $r + s' = \ell_j\alpha + \ell_i$ we compute

$$\begin{aligned}
\phi(c_{r,s'}) &= (-1)^r \mathbf{q}_i^{r(\ell_i-1)} \sum_{t=0}^{\ell_i-1} (-1)^t \pi_i^{\ell_j r p(j)+r(r-1)/2+t(t-1)/2} \mathbf{q}_i^{t(s'-1)-\ell_i n t} \begin{bmatrix} a \\ t \end{bmatrix}_{\mathbf{q}_i,\pi_i} \\
&= (-1)^r \mathbf{q}_i^{r(\ell_i-1)} \sum_{t=0}^{a} (-1)^t \pi_i^{\ell_j r p(j)+r(r-1)/2+t(t-1)/2} \mathbf{q}_i^{t(a-1)} \begin{bmatrix} a \\ t \end{bmatrix}_{\mathbf{q}_i,\pi_i} \\
&\overset{(a)}{=} \delta_{a,0}(-1)^{\ell_j\alpha+\ell_i-\ell_i n} \pi_i^{\ell_j r p(j)+r(r-1)/2} \mathbf{q}_i^{(\ell_i-1)(\ell_j\alpha+\ell_i-\ell_i n)} \\
&\overset{(b)}{=} \delta_{a,0}(-1)^{\alpha\ell_j/\ell_i+1-n} \pi_i^{\ell_j r p(j)+r(r-1)/2-r(\ell_i-1)/2}.
\end{aligned} \tag{9.1.7}$$

The identity (a) above follows by the identity $\sum_{t=0}^{a}(-1)^t \pi_i^{t(t-1)/2}\mathbf{q}_i^{t(a-1)} \begin{bmatrix} a \\ t \end{bmatrix}_{\mathbf{q}_i,\pi_i} = \delta_{a,0}$ (see [CHW13, 1.4.4]), and (b) follows by the identity $\pi_i^{(\ell_i-1)\ell_i/2}\mathbf{q}_i^{\ell_i^2-\ell_i} = (-1)^{\ell_i+1}$ (which is an i-version of (7.2.2) with the help of $\pi_i^{\ell_i}\mathbf{q}_i^{2\ell_i} = 1$).

Inserting (9.1.7) into (9.1.6) and comparing with (9.1.5), we reduce the proof of (9.1.5) to verifying that $\pi_i^{\ell_i^2(np(j)+n(n-1)/2)} = \pi_i^{\ell_j\ell_i np(j)+\ell_i n(\ell_i n-1)/2-\ell_i n(\ell_i-1)/2}$, which is equivalent to verifying $\pi_i^{\ell_i^2 np(j)} = \pi_i^{\ell_j\ell_i np(j)}$. The latter identity is trivial unless both i and j are in I_1; when both i and j are in I_1, the identity follows from Lemma 8.1.1.

Therefore, we have proved (9.1.5) and hence Theorem 9.1.1. □

9.2 The Steinberg tensor product theorem for QCG

We develop in this section the analogue of [Lu94, 35.3]; recall we are still working under the assumption that R is the quotient field of $\mathcal{A}'$.

Proposition 9.2.1. *Let $\lambda \in X^\diamond$, i.e., $\langle i, \lambda \rangle \in \ell_i\mathbb{Z}$ for all $i \in I$. Let M denote the simple highest weight module with highest weight λ in the category of R^π-free weight $\mathbf{U}$-modules, and let η be a highest weight vector of M^λ.*

(a) If $\zeta \in X$ satisfies $M^\zeta \neq 0$, then $\zeta = \lambda - \sum_i \ell_i n_i i'$, where $n_i \in \mathbb{N}$. In particular, $\langle i, \zeta \rangle \in \ell_i\mathbb{Z}$ for all $i \in I$.

(b) If $i \in I$ is such that $\ell_i \geq 2$, then E_i, F_i act as zero on M.

(c) For any $r \geq 0$, let M'_r be the subspace of M spanned by $F_{i_1}^{(\ell_{i_1})} F_{i_2}^{(\ell_{i_2})} \dots F_{i_r}^{(\ell_{i_r})}\eta$ for various sequences $i_1, i_2, \dots, i_r$ in I. Let $M' = \sum_r M'_r$. Then $M' = M$.

Proof. The proof is completely analogous to [Lu94]. All computations are similar except that we are now working over R^π instead of R; and the results follow from Lemma 7.2.1, [CHW13, (4.1) and Proposition 4.2.4], and Lemma 9.1.2.

First, we show that

(d) $E_i M'_r = 0$, $F_i M'_r = 0$ for any $i \in I$ such that $\ell_i \geq 2$,

which is similarly proved by induction on $r \geq 0$. The base case $r = 0$ follows from the fact that $\begin{bmatrix} \langle i, \lambda \rangle \\ t \end{bmatrix}_{\mathbf{q}_i, \pi_i} = 0$ since $\lambda \in X^\diamond$ (using Lemma 7.2.1) and the fact that $E_j^{(n)} F_i \eta$ is an R^π-linear combination of $F_i E_j^{(n)}$ and $E_j^{(n-1)}$. For the inductive step, we want to show that $E_iF_j^{(\ell_j)}m = 0$ and $F_iF_j^{(\ell_j)}m = 0$ for any $i, j \in I$ such that $\ell_i \geq 2$ and any $m \in M'_{r-1}\zeta$. For the first one we use the fact that $E_iF_j^{(\ell_j)}m$ is an R^π-linear combination of $F_j^{(\ell_j)}E_im$ and $F_j^{\ell_j-1}$ in the case $\ell_j \geq 2$, and for $\ell_j = 1$ we again use $\begin{bmatrix} \langle i, \lambda \rangle \\ t \end{bmatrix}_{\mathbf{q}_i, \pi_i} = 0$ from Lemma 7.2.1. For the second one, we may use [CHW13, (4.1) and Proposition 4.2.4] to write $F_iF_j^{(\ell_j)}m$ as a R^π-linear combination of $F_j^{(\ell_j - r)}F_iF_j^{(r)}m$ for various r with $0 \leq r < \ell_j$, and for such r we have $F_iF_j^{(r)}m = 0$ by the induction hypothesis.

Next, we may show by induction on $r \geq 0$ that

(e) $E_i^{(l_i)}M'_r \subset M'_{r-1}$ for any $i \in I$,

(by convention $M'_{-1} = 0$); again for $m' \in M'_{r-1}$ we can use the fact that $E_i^{(l_i)}F_j^{(\ell_j)}m'$ is an R^π-linear combination of $F_j^{(\ell_j)}E_i^{(\ell_i)}m'$ (which is in M'_{r-1} by the induction hypothesis), and elements of the form $F_j^{(\ell_j - t)}E_i^{(\ell_i - t)}m'$ with $t > 0$ and $t \leq \ell_i, t \leq \ell_j$ (which as before are zero if $t < \ell_i$ or if $t = \ell_i$ and $t < \ell_j$, by (d), and are in M'_{r-1} if $t = \ell_i = \ell_j$).

The statements (d), (e) together with Lemma 9.1.2 show that $\sum_r M'_r$ is an ${}_R\dot{\mathbf{U}}$-submodules of M, and by simplicity of M it follows that $M = \sum_r M'_r$, from which (a) and (b) also follow. □

Corollary 9.2.2. *There is a unique weight ${}_R\dot{\mathbf{U}}^\diamond$-module structure on M (as in Proposition 9.2.1) in which the ζ-weight space is the same as that in the ${}_R\dot{\mathbf{U}}^\diamond$-modules M, for any $\zeta \in X^\diamond \subset X$, and such that $E_i, F_i \in {}_R\mathbf{f}^\diamond$ act as $E_i^{(\ell_i)}, F_i^{(\ell_i)} \in {}_R\mathbf{f}$. Moreover, this is a simple (R^π-free) highest weight module for ${}_R\dot{\mathbf{U}}^\diamond$ with highest weight $\lambda \in X^\diamond$.*

Proof. We define operators $e_i, f_i : M \to M$ for $i \in I$ by $e_i = E_i^{(\ell_i)}$, $f_i = F_i^{(\ell_i)}$. Using Theorem 9.1.1 we see that e_i and f_i satisfy the Serre-type relations of ${}_R\mathbf{f}^\diamond$.

If $\zeta \in X \setminus X^\diamond$ we have $M^\zeta = 0$ by Proposition 9.2.1(a) above. If $\zeta \in X^\diamond$ and

$m \in M^{\zeta}$, then we have that $(e_i f_j - f_j e_i)(m)$ is equal to $\delta_{i,j} \begin{bmatrix} \langle i, \lambda \rangle \\ \ell_i \end{bmatrix}_{\mathbf{q}_i, \pi_i} \cdot m$ plus an R^{π}-linear combination of elements of the form $F_i^{\ell_i - t} E_i^{\ell_i - t}(m)$ with $0 < t < \ell_i$ (this follows by [Cl14, Lemma 4]) which are zero by Proposition 9.2.1(b). Since $\langle i, \zeta \rangle \in \ell_i \mathbb{Z}$, we see from Lemma 8.3.4 that

$$\begin{bmatrix} \langle i, \lambda \rangle \\ \ell_i \end{bmatrix}_{\mathbf{q}_i, \pi_i} = \begin{bmatrix} \langle i, \lambda \rangle / \ell_i \\ 1 \end{bmatrix}_{\mathbf{q}_i^{\diamond}, \pi_i^{\diamond}}$$

and so $(e_i f_j - f_j e_i)m = \delta_{i,j} [\langle i, \lambda \rangle / \ell_i]_{\mathbf{q}_i^{\diamond}, \pi_i^{\diamond}} \cdot m$. We also have that $e_i(M^{\zeta}) \subset M^{\zeta + \ell_i i'}$ and $f_i(M^{\zeta}) \subset M^{\zeta - \ell_i i'}$. Thus, we have a unital ${}_R\dot{\mathbf{U}}^{\diamond}$-module structure on M, and by Proposition 9.2.1(c) this is a highest weight module of ${}_R\dot{\mathbf{U}}^{\diamond}$ with highest weight λ and simplicity also follows using Lemma 9.1.2 in the same argument as in [Lu94]. □

Now we are ready to state our analogue of the main result of [Lu94, 35.4] on a tensor product decomposition. Let $\mathfrak{f}$ be the R-subalgebra of ${}_R\mathbf{f}$ generated by the elements θ_i for various i such that $\ell_i \geq 2$. We have $\mathfrak{f} = \oplus_{\nu} \mathfrak{f}_{\nu}$ where $\mathfrak{f} = {}_R\mathbf{f}_{\nu} \cap \mathfrak{f}$.

Theorem 9.2.3 (Lusztig-Steinberg tensor product theorem)**.** *The R^{π}-linear map*

$$\chi : {}_R\mathbf{f}^{\diamond} \otimes_R \mathfrak{f} \to {}_R\mathbf{f}, \qquad x \otimes y \mapsto \mathbf{Fr}'(x) y$$

is an isomorphism of R^{π}-modules.

Proof. First, we make the following statement which is similar to (but slightly less precise than) [Lu94, 35.4.2(a)].

Claim. For any $i \in I$ and $y \in \mathfrak{f}_{\nu}$, there exists some $a(y), b(y) \in \mathbb{Z}$ such that the difference $\theta_i^{(\ell_i)} y - \pi_i^{a(y)} \mathbf{q}_i^{b(y)} y \theta_i^{(\ell_i)}$ belongs to $\mathfrak{f}$.

For $y = y'y''$ one easily reduces the Claim to the same type of claim for y' and y''. Hence it suffices to show this Claim when y is a generator of $\mathfrak{f}$ i.e. $y = \theta_j$ where $\ell_j \geq 2$. Recall our assumption (a) in §9.1 that $\ell_i \geq -\langle i, j' \rangle + 1$. Hence, we may use the higher Serre relation in [CHW13, (4.1) and Proposition 4.2.4] (but with θ_i's instead of F_i's) to show that for some $a(j), b(j)$, the difference $\theta_i^{(\ell_i)}\theta_j - \pi_i^{a(j)}\mathbf{q}_i^{b(j)}\theta_j\theta_i^{(\ell_i)}$ is an R^π-linear combination of products of the form $\theta_i^{(r)}\theta_j\theta_i^{(\ell_i-r)}$ with $0 < r < \ell_i$, which are contained in $\mathfrak{f}$ by definition. The Claim is proved.

By Lemma 9.1.2, ${}_R\mathbf{f}$ is generated by $\theta_i^{(\ell_i)}$ and θ_j with $\ell_j \geq 2$. The surjectivity of χ follows as the Claim allows us to move factors θ_j to the right which produces lower terms in $\mathfrak{f}$.

The injectivity is proved by exactly the same argument as in [Lu94, 35.4.2] using now Proposition 9.2.1 and Corollary 9.2.2; the details will be skipped. □

The following is an analogue of [Lu94, Proposition 35.4.4], which follows by the same argument now using the anti-involution σ of ${}_R\mathbf{f}$ which fixes each θ_i (cf. [CHW13, §1.4]). We omit the detail to avoid much repetition.

Proposition 9.2.4. *Assume that the root datum is simply connected. Then, there is a unique $\lambda \in X^+$ such that $\langle i, \lambda \rangle = \ell_i - 1$ for all i. Let η be the canonical generator of ${}_RV(\lambda)$. The map $x \mapsto x^-\eta$ is an R^π-linear isomorphism $\mathfrak{f} \longrightarrow {}_RV(\lambda)$.*

9.3 The Frobenius-Lusztig homomorphism Fr

The following is a generalization of [Lu94, Theorem 35.1.7]. As with Theorem 9.1.1, we may reduce the proof to the case when R is the quotient field of $\mathcal{A}'$ (cf. [Lu94, 35.1.11]).

Theorem 9.3.1. *There is a unique R^π-superalgebra homomorphism $\mathbf{Fr} : {}_R\mathbf{f} \longrightarrow {}_R\mathbf{f}^\diamond$ such that, for all $i \in I, n \in \mathbb{N}$,*

$$\mathbf{Fr}(\theta_i^{(n)}) = \begin{cases} \theta_i^{(n/\ell_i)}, & \text{if } \ell_i \text{ divides } n, \\ 0, & \text{otherwise.} \end{cases}$$

(We call **Fr** the Frobenius-Lustig homomorphism.)

Proof. The proof proceeds essentially like that of [Lu94, Theorem 35.1.7]. Uniqueness is clear; we need only prove the existence. By Theorem 9.2.3, there is an R^π-linear map $P : {}_R\mathbf{f} \longrightarrow {}_R\mathbf{f}^\diamond$, such that for all $i_k \in I$ and for $j_p \in I$ where $\ell_{j_p} \geq 2$

$$P(\theta_{i_1}^{(\ell_{i_1})} ... \theta_{i_n}^{(\ell_{i_n})} \theta_{j_1} ... \theta_{j_r}) = \begin{cases} \theta_{i_1} ... \theta_{i_n}, & \text{if } r = 0, \\ 0, & \text{otherwise.} \end{cases}$$

We now check that P is a homomorphism of R^π-algebras. Because ${}_R\mathbf{f}$ is generated as an R^π-module by elements of the form $x = \theta_{i_1}^{(\ell_{i_1})} ... \theta_{i_n}^{(\ell_{i_n})} \theta_{j_1} ... \theta_{j_r}$, we need to check that for any such x,

$$P(x\theta_j) = P(x)P(\theta_j) \tag{9.3.1}$$

for $j \in I$ such that $\ell_j \geq 2$ and

$$P(x\theta_i^{(\ell_i)}) = P(x)P(\theta_i^{(\ell_i)}) \tag{9.3.2}$$

for all $i \in I$. As (9.3.1) is obvious, we will concern ourselves with (9.3.2). Note that (9.3.2) is clear when $r = 0$. Assume now $r > 0$. Let us write $x' = \theta_{i_1}^{(\ell_{i_1})} ... \theta_{i_n}^{(\ell_{i_n})} \theta_{j_1} ... \theta_{j_{r-1}}$

and $\theta_j = \theta_{j_r}$ so that $x = x'\theta_j$. For $i = j$, we have $P(x)P(\theta_i^{(\ell_i)}) = 0$ and

$$P(x\theta_i^{(\ell_i)}) = P(x'\theta_i\theta_i^{(\ell_i)}) = P(x'\theta_i^{(\ell_i)}\theta_i) = P(x'\theta_i^{(\ell_i)})P(\theta_i) = 0,$$

where the third equality is due to (9.3.1). Now suppose that $i \neq j$. As $\ell_i > -\langle i, j'\rangle$, we may use the higher order Serre relations for quantum covering groups (cf. [CHW13, (4.1) and Proposition 4.2.4]) to write $\theta_j\theta_i^{(\ell_i)}$ as a linear combination of terms of the form $\theta_i^{(m)}\theta_j\theta_i^{(n)}$ where $m+n = \ell_i$ and $m \geq 1$. Because of (9.1.2) and (9.3.1), $P(x'\theta_i^{(m)}\theta_j\theta_i^{(n)}) = 0$ for $1 \leq m < \ell_i$, and $P(x'\theta_i^{(\ell_i)}\theta_j) = 0$.

Now that we know that P is an R^π-algebra homomorphism, it remains to compute $P(\theta_i^{(n)})$ for all $n \in \mathbb{Z}_{\geq 0}$. Write $n = b\ell_i + a$, where $0 \leq a < \ell_i$ and $b \in \mathbb{Z}_{\geq 0}$. Using (9.1.1), (9.1.2) and (9.1.3), for $a > 0$ we have

$$P(\theta^{(b\ell_i+a)}) = \mathbf{q}_i^{\ell_i ab}P(\theta_i^{(a)})P(\theta_i^{(b\ell_i)}) = \mathbf{q}_i^{\ell_i ab}([a]^!_{\mathbf{q}_i,\pi_i})^{-1}P(\theta_i^a)P(\theta_i^{(b\ell_i)}) = 0.$$

Similarly, for $a = 0$ we have

$$\begin{aligned} P(\theta_i^{(b\ell_i)}) &= (b!)^{-1}(\pi_i\mathbf{q}_i)^{-\ell_i^2\binom{b}{2}}P(\theta_i^{(\ell_i)})^b \\ &= (b!)^{-1}(\pi_i^\diamond\mathbf{q}_i^\diamond)^{-\binom{b}{2}}\theta_i^b = ([b]^!_{\mathbf{q}_i^\diamond,\pi_i^\diamond})^{-1}\theta_i^b = \theta_i^{(b)}, \end{aligned}$$

where, in the third equality we used Lemma 7.2.2, with $\ell = 1$. Hence, P is the desired homomorphism **Fr**. □

9.4 The Frobenius-Lusztig homomorphism on ${}_R\dot{\mathbf{U}}$

We extend the Frobenius-Lusztig homomorphism $\mathbf{Fr} : {}_R\mathbf{f} \longrightarrow {}_R\mathbf{f}^\diamond$ in Theorem 9.3.1 to ${}_R\dot{\mathbf{U}}$. In contrast to the quantum group setting, we have to twist $\mathbf{Fr}$ slightly on one half of the quantum covering group.

Theorem 9.4.1. *There is a unique R^π-superalgebra homomorphism $\mathbf{Fr} : {}_R\dot{\mathbf{U}} \longrightarrow {}_R\dot{\mathbf{U}}^\diamond$ such that for all $i \in I, n \in \mathbb{Z}, \lambda \in X$,*

$$\mathbf{Fr}(E_i^{(n)}\mathbf{1}_\lambda) = \begin{cases} \pi_i^{\binom{\ell_i}{2}n/\ell_i} E_i^{(n/\ell_i)}\mathbf{1}_\lambda, & \text{if } \ell_i \text{ divides } n \text{ and } \lambda \in X^\diamond, \\ 0, & \text{otherwise} \end{cases} \tag{9.4.1}$$

and

$$\mathbf{Fr}(F_i^{(n)}\mathbf{1}_\lambda) = \begin{cases} F_i^{(n/\ell_i)}\mathbf{1}_\lambda, & \text{if } \ell_i \text{ divides } n \text{ and } \lambda \in X^\diamond, \\ 0, & \text{otherwise.} \end{cases}$$

(We also call $\mathbf{Fr}$ in this theorem the Frobenius-Lustig homomorphism.)

Proof. Let $\mathbf{Fr} : {}_R\mathbf{f} \longrightarrow {}_R\mathbf{f}^\diamond$ be the homomorphism from Theorem 9.3.1. Consider the homomorphism $\tilde{\mathbf{Fr}} = \psi \circ \mathbf{Fr}$, where $\psi : {}_R\mathbf{f}^\diamond \longrightarrow {}_R\mathbf{f}^\diamond$ is the algebra automorphism such that $\theta_i^{(n)} \mapsto \pi_i^n\theta_i^{(n)}$. The proof, much like that of [Lu94, Theorem 35.1.9], amounts to checking that for $x, x' \in {}_R\mathbf{f}$ the assignment

$$x^+\mathbf{1}_\lambda x'^- \mapsto \tilde{\mathbf{Fr}}(x^+)\mathbf{1}_\lambda Fr(x'^-), \quad x^-\mathbf{1}_\lambda x'^+ \mapsto \mathbf{Fr}(x^-)\mathbf{1}_\lambda\tilde{\mathbf{Fr}}(x'^+),$$

for $\lambda \in X^\diamond$, and

$$x^+\mathbf{1}_\lambda x'^- \mapsto 0, \quad x^-\mathbf{1}_\lambda x'^+ \mapsto 0,$$

for $\lambda \in X\backslash X^\diamond$ satisfies the the appropriate relations. These are the relations of Lemma 8.2.1 for ${}_R\dot{\mathbf{U}}$ and for ${}_R\dot{\mathbf{U}}^\diamond$, using Lemma 8.3.4 to deal with the $(\mathbf{q},\pi)$-binomial coefficients. The use of the homomorphism $\tilde{\mathbf{Fr}}$ (in place of $\mathbf{Fr}$) on $\mathbf{U}^+$ is necessitated by the first and second relations in Lemma 8.2.1. Both sides of the first relation are mapped to zero by $\mathbf{Fr}$ unless $N, M \in \ell_i\mathbb{Z}$ and $\lambda \in X^\diamond$, so we focus on this case. Recalling $\mathbf{q}_i^\diamond, \pi_i^\diamond$ from (8.3.1), we have

$$\mathbf{Fr}\left(\sum_{t\geq 0}\pi_i^{MN-\binom{t+1}{2}}F_i^{(M-t)}\begin{bmatrix}M+N+\langle i,\lambda\rangle\\ t\end{bmatrix}_{\mathbf{q}_i,\pi_i}\mathbf{1}_{\lambda+(M+N-t)i'}E_i^{(N-t)}\right)$$
$$=\sum_{t\geq 0}\pi_i^{MN-\binom{t+1}{2}}\mathbf{Fr}(F_i^{(M-t)})\begin{bmatrix}M+N+\langle i,\lambda\rangle\\ t\end{bmatrix}_{\mathbf{q}_i,\pi_i}\mathbf{1}_{\lambda+(M+N-t)i'}Fr(E_i^{(N-t)})$$

$$=\sum_{t\geq 0, t\in\ell_i\mathbb{Z}}(\pi_i^\diamond)^{(M/\ell_i)(N/\ell_i)-\binom{t/\ell_i+1}{2}}\pi_i^{t/\ell_i\binom{\ell_i}{2}}F_i^{((M-t)/\ell_i)}\begin{bmatrix}(M+N+\langle i,\lambda\rangle)/\ell_i\\ t/\ell_i\end{bmatrix}_{\mathbf{q}_i^\diamond,\pi_i^\diamond}$$
$$\cdot\,\mathbf{1}_{\lambda+(M+N-t)i'}\pi_i^{(N-t)/\ell_i\binom{\ell_i}{2}}E_i^{((N-t)/\ell_i)}$$
$$=\pi_i^{N/\ell_i\binom{\ell_i}{2}}\sum_{t\geq 0, t\in\ell_i\mathbb{Z}}(\pi_i^\diamond)^{(M/\ell_i)(N/\ell_i)-\binom{t/\ell_i+1}{2}}F_i^{((M-t)/\ell_i)}\begin{bmatrix}(M+N+\langle i,\lambda\rangle)/\ell_i\\ t/\ell_i\end{bmatrix}_{\mathbf{q}_i^\diamond,\pi_i^\diamond}$$
$$\cdot\,\mathbf{1}_{\lambda+(M+N-t)i'}E_i^{((N-t)/\ell_i)}$$
$$=\pi_i^{N/\ell_i\binom{\ell_i}{2}}E_i^{(N/\ell_i)}\mathbf{1}_\lambda F_i^{(M/\ell_i)}$$
$$=\mathbf{Fr}\big(E_i^{(N)}\mathbf{1}_\lambda F_i^{(M)}\big),$$

where we have used $\pi_i^{-\binom{t+1}{2}} = (\pi_i^\diamond)^{-\binom{t/\ell_i+1}{2}}\pi_i^{t/\ell_i\binom{\ell_i}{2}}$ and Lemma 8.3.4 in the second equality above.

The verification of the second relation of Lemma 8.2.1 is entirely similar, and the other relations therein are straightforward. □

Chapter 10

Small quantum covering groups

10.1 Small quantum covering groups

In this chapter, we construct and study the small quantum covering groups. We take $R^\pi = \mathbb{Q}(\mathbf{q})^\pi$, where $\mathbf{q}$ is as in (7.1.2).

Let ${}_R\dot{\mathfrak{u}}$ be the subalgebra of ${}_R\dot{\mathbf{U}}$ generated by $E_i\mathbf{1}_\lambda$ and $F_i\mathbf{1}_\lambda$ for all $i \in I$ with $\ell_i \geq 2$ and $\lambda \in X$. It is clear then, that ${}_R\dot{\mathfrak{u}}$ is spanned by terms of the form $x^+\mathbf{1}_\lambda x'^-$ where $x, x' \in \mathfrak{f}$. We follow the construction of [Lu94, §36.2.3] in extending ${}_R\dot{\mathbf{U}}$ to a new algebra ${}_R\hat{\mathbf{U}}$. Any element of ${}_R\dot{\mathbf{U}}$ can be written as a sum of the form $\sum_{\lambda,\mu\in X} x_{\lambda,\mu}$ where $x_{\lambda,\mu} \in \mathbf{1}_\lambda {}_R\dot{\mathbf{U}}\mathbf{1}_\mu$ is zero for all but finitely many pairs λ, μ. We relax this condition in ${}_R\hat{\mathbf{U}}$ by allowing such sums to have infinitely many nonzero terms provided that the corresponding $\lambda - \mu$ are contained in a finite subset of X. The algebra structure extends in the obvious way. We define ${}_R\hat{\mathfrak{u}}$ to be the subalgebra of ${}_R\hat{\mathbf{U}}$ with $x_{\lambda,\mu} \in \mathbf{1}_\lambda {}_R\dot{\mathfrak{u}}\mathbf{1}_\mu$.

Let $2\tilde{\ell}$ be the smallest positive integer such that $\mathbf{q}^{2\tilde{\ell}} = 1$. Hence, $\tilde{\ell} = 2\ell$ for ℓ odd and $\tilde{\ell} = \ell$ for ℓ even. We define the cosets

$$\mathbf{c_a} = \{\lambda \in X \mid \langle i, \lambda \rangle \equiv a_i \pmod{2\tilde{\ell}}, \quad \forall i \in I\}, \tag{10.1.1}$$

for $\mathbf{a} = (a_i | i \in I)$ with $0 \leq a_i \leq 2\tilde{\ell} - 1$. Note that there are at most $(2\tilde{\ell})^{|I|}$ such cosets and they partition X. Moreover, for each coset $\mathbf{c}$, $\mathbf{1_c} := \sum_{\lambda \in \mathbf{c}} \mathbf{1}_\lambda$ is an element of ${}_R\hat{\mathfrak{u}}$.

Let ${}_R\mathfrak{u}$ (resp. ${}_R\mathfrak{u}'$) be the R^π-submodule of ${}_R\hat{\mathfrak{u}}$ generated by the elements $x^+\mathbf{1_c}x'^-$ (resp. $x^-\mathbf{1_c}x'^+$) where $x, x' \in \mathfrak{f}$. The following is an analogue of [Lu94, Lemma 36.2.4].

Lemma 10.1.1. *1. For any $u \in {}_R\mathfrak{u}$ and $0 \leq M \leq \ell_i - 1$, $F_i^{(M)}u$ lies in ${}_R\mathfrak{u}$.*

2. We have ${}_R\mathfrak{u} = {}_R\mathfrak{u}'$, and ${}_R\mathfrak{u}$ is a subalgebra of ${}_R\hat{\mathfrak{u}}$.

The algebra ${}_R\mathfrak{u}$ is called the *small quantum covering group*.

Proof. We follow the proof in [Lu94]. We prove the first statement by induction on p, where our $u = E_{i_1}^{(n_1)} \ldots E_{i_p}^{(n_p)} x'^-$. The result is obvious for $p = 0$, so we now consider $p \geq 1$ and rewrite u as

$$u = \mathbf{1_{c'}} E_{i_1}^{(n_1)} x_1^+ x'^-$$

where $x_1 = \theta_{i_2}^{(n_2)} \ldots \theta_{i_p}^{(n_p)}$. When $i \neq i_1$, the result is immediate, so we consider $i = i_1$. In that case, using the relations of Lemma 8.2.1, we have

$$F_i^{(M)}u = \sum_{\lambda \in \mathbf{c'}} \sum_{t \leq n_1, t \leq M} \pi_i^{MN + t\langle i, \lambda \rangle - \binom{t}{2}} \begin{bmatrix} n_1 + M - \langle i, \lambda \rangle \\ t \end{bmatrix}_{\mathbf{q}_i, \pi_i} \cdot E_i^{(a_1 - t)} \mathbf{1}_{\lambda - (n_1 + M - t)i'} F_i^{(M-t)} x_1^+ x'^-.$$

Fix $\mu \in \mathbf{c'}$. Then for any $\lambda \in \mathbf{c'}$, $n_1 + M - \langle i, \lambda \rangle \equiv n_1 + M - \langle i, \mu \rangle \bmod (\ell_i)$. Using

Lemma 7.2.1 and noting that $t < \ell_i$, we have that

$$\begin{bmatrix} n_1 + M - \langle i, \lambda\rangle \\ t \end{bmatrix}_{\mathbf{q}_i,\pi_i} = \mathbf{q}_i^{-\ell_i t(\langle i,\lambda\rangle - \langle i,\mu\rangle)} \begin{bmatrix} n_1 + M - \langle i, \mu\rangle \\ t \end{bmatrix}_{\mathbf{q}_i,\pi_i} = \begin{bmatrix} n_1 + M - \langle i, \mu\rangle \\ t \end{bmatrix}_{\mathbf{q}_i,\pi_i},$$

where we used in the second equality the condition that $\langle i,\lambda\rangle - \langle i,\mu\rangle \equiv 0 \bmod (2\tilde{\ell})$. Hence, $F_i^{(M)}u$ is equal to

$$\sum_{t\le n_1, t\le M} \pi_i^{MN+t\langle i,\mu\rangle - \binom{t}{2}} \begin{bmatrix} n_1 + M - \langle i, \mu\rangle \\ t \end{bmatrix}_{\mathbf{q}_i,\pi_i} E_i^{(a_1-t)} \Big(\sum_{\lambda\in\mathbf{c}'} \mathbf{1}_{\lambda-(n_1+M-t)i'}\Big) F_i^{(M-t)} x_1^+ x'^-$$

$$= \sum_{t\le n_1, t\le M} \pi_i^{MN+t\langle i,\mu\rangle - \binom{t}{2}} \begin{bmatrix} n_1 + M - \langle i, \mu\rangle \\ t \end{bmatrix}_{\mathbf{q}_i,\pi_i} E_i^{(a_1-t)} \mathbf{1}_{\mathbf{c}''} F_i^{(M-t)} x_1^+ x'^-,$$

for some other $\mathbf{c}''$. Hence, $F_i^{(M)}u \in {}_R\mathfrak{u}$ by induction. Finally, the second statement is shown by repeated application of this result as in [Lu94, Lemma 36.2.4]. □

10.2 Comultiplication

Recall there are a comultiplication Δ and an antipode S on $\mathbf{U}$ as defined in [CHW13, Lemmas 2.2.1, 2.4.1]. Write ${}_\lambda\mathbf{U}_\mu$ for the subspace of ${}_R\dot{\mathbf{U}}$ spanned by elements of the form $\mathbf{1}_\lambda x \mathbf{1}_\mu$, where $x \in {}_R\mathbf{U}$ and write $p_{\lambda,\mu}$ for the canonical projection ${}_R\mathbf{U} \to {}_\lambda\mathbf{U}_\mu$. As in [Lu94, 23.1.5, 23.1.6], Δ and S induce R^π-linear maps

$$\Delta_{\lambda,\mu,\lambda',\mu'} : {}_{\lambda+\lambda'}\mathbf{U}_{\mu+\mu'} \longrightarrow {}_\lambda\mathbf{U}_\mu \otimes {}_{\lambda'}\mathbf{U}_{\mu'}$$

given by $\Delta_{\lambda,\mu,\lambda',\mu'}(p_{\lambda+\lambda',\mu+\mu'}(x)) = (p_{\lambda,\mu} \otimes p_{\lambda',\mu'})(\Delta(x))$, for $\lambda, \mu, \lambda', \mu' \in X$, and

$$\dot{S} : {}_R\dot{\mathbf{U}} \longrightarrow {}_R\dot{\mathbf{U}}$$

defined by $\dot{S}(\mathbf{1}_\lambda x \mathbf{1}_\mu) = \mathbf{1}_{-\mu} S(x) \mathbf{1}_{-\lambda}$ for $x \in {}_R\mathbf{U}$. For example, $\Delta(E_i) = E_i \otimes 1 + \tilde{J}_i \tilde{K}_i \otimes E_i$ in ${}_R\mathbf{U}$, and hence we obtain

$$\Delta_{\lambda-\nu+i',\lambda-\nu,\nu,\nu}(E_i \mathbf{1}_\lambda) = p_{\lambda-\nu+i',\lambda-\nu} \otimes p_{\nu,\nu}(E_i \otimes 1 + \tilde{J}_i \tilde{K}_i \otimes E_i) = E_i \mathbf{1}_{\lambda-\nu} \otimes \mathbf{1}_\nu.$$

This collection of maps is called the comultiplication on ${}_R\dot{\mathbf{U}}$, and it can be formally regarded as a single linear map

$$\dot{\Delta} = \prod_{\lambda,\mu,\lambda',\mu' \in X} \hat{\Delta}_{\lambda,\mu,\lambda',\mu'} : {}_R\dot{\mathbf{U}} \longrightarrow \prod_{\lambda,\mu,\lambda',\mu' \in X} {}_\lambda\mathbf{U}_\mu \otimes {}_{\lambda'}\mathbf{U}_{\mu'}.$$

A comultiplication $\dot{\Delta}^\diamond$ on ${}_R\dot{\mathbf{U}}^\diamond$ can be defined in the same way.

Proposition 10.2.1. *The Frobenius-Lusztig homomorphism* $\mathbf{Fr}$ *is compatible with the comultiplications on* ${}_R\dot{\mathbf{U}}$ *and* ${}_R\dot{\mathbf{U}}^\diamond$*, i.e.,* $\dot{\Delta}^\diamond \circ \mathbf{Fr} = (\mathbf{Fr} \otimes \mathbf{Fr}) \circ \dot{\Delta}$.

(In the usual quantum group setting this was noted by [Lu94, 35.1.10].)

Proof. It suffices to check on the generators $E_i^{(n)} \mathbf{1}_\lambda$ and $F_i^{(n)} \mathbf{1}_\lambda$. Let $n = m\ell_i \in \ell_i\mathbb{Z}$, and recall that $\mathbf{Fr}(E_i^{(m\ell_i)} \mathbf{1}_\lambda) = \pi_i^{\binom{\ell_i}{2} m} E_i^{(m)} \mathbf{1}_\lambda$ in ${}_R\dot{\mathbf{U}}^\diamond$. Using the formula (above [CHW13, Proposition 2.2.2])

$$\Delta(E_i^{(m)}) = \sum_{p+r=m} (\pi_i q_i)^{pr} E_i^{(p)} (\tilde{J}_i \tilde{K}_i)^r \otimes E_i^{(r)}$$

we see that the nonzero parts in $\dot{\Delta}^\diamond(\mathbf{Fr}(E_i^{(m\ell_i)}\mathbf{1}_\lambda))$ computed via (9.4.1) are of the form

$$\pi_i^{\binom{\ell_i}{2}m}(\pi_i^\diamond q_i^\diamond)^{(p+\langle i,\nu\rangle^\diamond)r}E_i^{(p)}\mathbf{1}_\nu \otimes E_i^{(r)}\mathbf{1}_{\lambda-\nu}, \qquad p+r=m$$

for various $\nu \in X^\diamond$, which coincides with $\mathbf{Fr}\otimes\mathbf{Fr}$ applied to terms in $\dot{\Delta}(E_i^{(m\ell_i)}\mathbf{1}_\lambda))$ of the form

$$(\pi_i q_i)^{(p\ell_i+\langle i,\nu\rangle)(r\ell_i)}E_i^{(p\ell_i)}\mathbf{1}_\nu \otimes E_i^{(r\ell_i)}\mathbf{1}_{\lambda-\nu}, \qquad p+r=m,$$

where we note there is a factor contributing from (9.4.1) which matches up with the previous part thanks to $\pi_i^{\binom{\ell_i}{2}p+\binom{\ell_i}{2}r} = \pi_i^{\binom{\ell_i}{2}m}$; the remaining terms are zero under $\mathbf{Fr}\otimes\mathbf{Fr}$ since at least one of the divided powers of E_i appearing in either tensor factor must be not divisible by ℓ_i.

On the other hand, if n is not divisible by ℓ_i, then the right hand side will also be zero, since all the non-zero parts of $\dot{\Delta}(E_i^{(n)}\mathbf{1}_\lambda))$ will have a tensor factor containing some divided power of E_i not divisible by ℓ_i.

A similar verification takes care of $F_i^{(n)}1_\lambda$. □

10.3 Hopf superalgebra structure of $_R\mathfrak{u}$

The maps $\dot{\Delta}$ and $\dot{S}$ restrict to maps on $_R\dot{\mathfrak{u}}$, which extend to R^π-linear maps $\hat{\Delta}$ and $\hat{S}$ on $_R\hat{\mathfrak{u}}$ in the obvious way. Henceforth, when we refer to $\hat{\Delta}$ and $\hat{S}$ we mean the restrictions to $_R\mathfrak{u}$.

Additionally, for any basis $\mathbf{B}$ of $\mathfrak{f}$ consisting of weight vectors, with unique zero weight element equal to 1, we define an R^π-linear map $\hat{e} : {}_R\mathfrak{u} \to R^\pi$ by:

$$\hat{e}(rb^+b'^-\mathbf{1}_{\mathbf{c_a}}) = \begin{cases} r, & \text{if } b, b' = 1 \text{ and } \mathbf{a} = \mathbf{0}, \\ 0, & \text{otherwise.} \end{cases}$$

where $b, b' \in \mathbf{B}$, $r \in R^\pi$, and $\mathbf{c_a}$ in (10.1.1).

Define the following elements:

$$K_i = \sum_{\lambda \in X} \mathbf{q}^{\langle i, \lambda \rangle} \mathbf{1}_\lambda, \quad J_i = \sum_{\lambda \in X} \pi^{\langle i, \lambda \rangle} \mathbf{1}_\lambda, \quad 1 = \sum_{\lambda \in X} \mathbf{1}_\lambda. \tag{10.3.1}$$

Proposition 10.3.1.

1. *The R^π-algebra ${}_R\mathfrak{u}$ has a generating set $\{E_i, F_i\ (\forall i \text{ with } \ell_i \geq 2), K_i, J_i\ (\forall i \in I)\}$.*

2. *$({}_R\mathfrak{u}, \hat{\Delta}, \hat{e}, \hat{S})$ forms a Hopf superalgebra.*

Proof. The elements in (10.3.1) can be written as

$$K_i = \sum_{\mathbf{c}} \mathbf{q}_{\mathbf{c},i} \mathbf{1}_{\mathbf{c}}, \quad J_i = \sum_{\mathbf{c}} \pi_{\mathbf{c},i} \mathbf{1}_{\mathbf{c}}, \quad 1 = \sum_{\mathbf{c}} \mathbf{1}_{\mathbf{c}},$$

where we have defined $\mathbf{q}_{\mathbf{c},i} = \mathbf{q}^{\langle i, \lambda \rangle}$ and $\pi_{\mathbf{c},i} = \pi^{\langle i, \lambda \rangle}$ for any $\lambda \in \mathbf{c}$. This implies that these elements are also in ${}_R\mathfrak{u}$. Moreover, we have

$$\mathbf{1}_{\mathbf{c}} = \prod_{i \in I} (2\tilde{\ell})^{-1} (1 + \pi_{\mathbf{c},i} J_i)(1 + \mathbf{q}_{\mathbf{c},i}^{-1} K_i + \mathbf{q}_{\mathbf{c},i}^{-2} K_i^2 + \ldots + \mathbf{q}_{\mathbf{c},i}^{1-\tilde{\ell}} K_i^{\tilde{\ell}-1}).$$

This proves (1).

A direct computation using these generators shows that $\hat{\Delta}$, $\hat{e}$ and $\hat{S}$ are given by the same formulas as Δ, e and S, the former maps inherit the following properties of the latter: $\hat{\Delta}$ is a homomorphism which satisfies the coassociativity (cf. [CHW13, Lemmas 2.2.1 and 2.2.3]), $\hat{e}$ is a homomorphism (cf. [CHW13, Lemma 2.2.3]), and $\hat{S}(xy) = \pi^{p(x)p(y)}\hat{S}(y)\hat{S}(x)$ (cf. [CHW13, Lemma 2.4.1]). Moreover, the image of $\hat{\Delta}$ (respectively, $\hat{S}$) lies in ${}_R\mathfrak{u} \otimes {}_R\mathfrak{u}$ (respectively, ${}_R\mathfrak{u}$). Hence (2) holds. □

10.4 Dimension of ${}_R\mathfrak{u}$

We consider the Cartan datum associated to the Lie superalgebra $\mathfrak{osp}(1|2n)$, where $n = |I|$, with the following Dynkin diagram:

The black node denotes the (only) odd simple root. We set

$$i \cdot i = \begin{cases} 2, & \text{if } i \text{ is odd,} \\ 4, & \text{if } i \text{ is even.} \end{cases}$$

The above Cartan datum on I is a super Cartan datum satisfying the bar-consistent condition in the sense of §8.1.

Theorem 10.4.1. *The small quantum covering group ${}_R\mathfrak{u}$ of type $\mathfrak{osp}(1|2n)$ is a finite dimensional R^π-module. In particular,*

$$\dim_{R^\pi}({}_R\mathfrak{u}) = \frac{\ell^{2n^2}}{\gcd(2,\ell)^{2n^2-2n}}(2\tilde{\ell})^n = \begin{cases} \ell^{2n^2}(4\ell)^n, & \textit{for } \ell \textit{ odd,} \\ \dfrac{\ell^{2n^2}}{2^{2n^2-2n}}(2\ell)^n, & \textit{for } \ell \textit{ even,} \end{cases}$$

when X is the weight lattice, and similarly,

$$\dim_{R^\pi}({}_R\mathfrak{u}) = \frac{\ell^{2n^2}}{\gcd(2,\ell)^{2n^2-2n}} 2^{n-1}\tilde{\ell}^n = \begin{cases} \ell^{2n^2} 2^{2n-1} \ell^n, & \text{for } \ell \text{ odd,} \\ \dfrac{\ell^{2n^2}}{2^{2n^2-2n}} 2^{n-1}\ell^n, & \text{for } \ell \text{ even,} \end{cases}$$

when X is the root lattice.

Proof. Note that ${}_R\mathfrak{u}$ is a $\mathfrak{f} \otimes \mathfrak{f}^{\mathrm{opp}}$ module with basis given by the $\mathbf{1_c}$ defined above. This basis has at most $(2\tilde{\ell})^n$ elements for any X. In particular, it has $(2\tilde{\ell})^n$ elements when X is the weight lattice, and $2^{n-1}\tilde{\ell}^n$ elements when X is the root lattice, as the root lattice is index 2 in the weight lattice. Moreover, by Proposition 9.2.4, we have that $\dim_{R^\pi}(\mathfrak{f}^{\pm}) = \dim_{R^\pi}({}_RV(\lambda))$, where λ is the unique weight such that $\langle i, \lambda\rangle = \ell_i - 1$ for each $i \in I$. Let $V(\lambda)_1$ (respectively, $V(\lambda)_{-1}$) be the quotient of the Verma module of highest weight λ by its maximal ideal for the quantum group (resp. quantum supergroup) to which the quantum covering group specializes at $\pi = 1$ (respectively, $\pi = -1$) with base field $R = \mathbb{Q}(\varepsilon)$ (recall from §7.1 that ε is an ℓ'-th root of unity). Because

$${}_RV(\lambda) = (\pi+1)_RV(\lambda) \oplus (\pi-1)_RV(\lambda) \cong V(\lambda)_1 \oplus V(\lambda)_{-1}$$

and the characters of $V(\lambda)_1$ and $V(\lambda)_{-1}$ coincide for dominant weights (cf. [KKO14], [CHW14, Remark 2.5]), we have

$$\dim_{R^\pi} \mathfrak{f}^{\pm} = \dim_{R^\pi}\ {}_RV(\lambda) = \dim_R V(\lambda)_1 = \dim_R \mathfrak{f}_1^{\pm} = \frac{\ell^{n^2}}{\gcd(2,\ell)^{n^2-n}}$$

where $\mathfrak{f}_1$ is the (non-super) half small quantum group, i.e., $\mathfrak{f}$ specialized at $\pi = 1$. The last equality is due to [Lu90b, Theorem 8.3(iv)].

□

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