Convex Sets Associated to C^*-Algebras

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Abstract

For a separable unital C^*-algebra $\mathfrak{A}$ and a separable McDuff II_1-factor M, we show that the space $\mathbb{H}\mathrm{om}_w(\mathfrak{A}, M)$ of weak approximate unitary equivalence classes of unital $*$-homomorphisms $\mathfrak{A} \to M$ may be considered as a closed, bounded, convex subset of a separable Banach space – a variation on N. Brown's convex structure $\mathbb{H}\mathrm{om}(N, R^{\mathcal{U}})$. When $\mathfrak{A}$ is nuclear, $\mathbb{H}\mathrm{om}_w(\mathfrak{A}, M)$ is affinely homeomorphic to the trace space of $\mathfrak{A}$, but in general $\mathbb{H}\mathrm{om}_w(\mathfrak{A}, M)$ and the trace space of $\mathfrak{A}$ do not share the same data (several examples are provided). We characterize extreme points of $\mathbb{H}\mathrm{om}_w(\mathfrak{A}, M)$ in the case where either $\mathfrak{A}$ or M is amenable, and we give two different conditions – one necessary and the other sufficient – for extremality in general. The universality of $C^*(\mathbb{F}_\infty)$ is reflected in the fact that for any unital separable $\mathfrak{A}, \mathbb{H}\mathrm{om}_w(\mathfrak{A}, M)$ may be embedded as a face in $\mathbb{H}\mathrm{om}_w(C^*(\mathbb{F}_\infty), M)$. We also extend Brown's construction to apply more generally to $\mathbb{H}\mathrm{om}(\mathfrak{A}, M^{\mathcal{U}})$. Finally, we return to the context of $\mathbb{H}\mathrm{om}(N, R^{\mathcal{U}})$ and examine the properties of finite dimensional minimal faces in that setting.

The connection between algebraic and convex geometric concepts is the main theme of this thesis, and in studying this connection we uncover some new purely operator algebraic insights.

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Dedication

To Haley and Ford.

Contents

Chapter 1

Introduction

The purpose of this thesis is to introduce and investigate a convex structure on the space $\mathbb{H}\mathrm{om}_w(\mathfrak{A}, M)$ of equivalence classes of $*$-homomorphisms from a unital separable C^*-algebra $\mathfrak{A}$ to a separable McDuff II_1-factor M. Placing a tractable structure on equivalence classes of homomorphisms between operator algebras is no new idea (e.g. $\mathrm{Ext}(\mathfrak{A})$). In fact, N. Brown presented a convex structure on the typically *non-separable* space $\mathbb{H}\mathrm{om}(N, R^{\mathcal{U}})$ of unitary equivalence classes of $*$-homomorphisms from a separable II_1-factor N into the ultrapower of the separable hyperfinite II_1-factor $R^{\mathcal{U}}$ in [8]. In this thesis, we extend the scope to C^*-algebras and replace the approximation mechanism of Brown's construction – the ultrapower – with the mechanism of weak approximate unitary equivalence, allowing us to consider separable target algebras. The result is a *separable* adaptation, $\mathbb{H}\mathrm{om}_w(\mathfrak{A}, M)$, of Brown's $\mathbb{H}\mathrm{om}(N, R^{\mathcal{U}})$ that still retains a convex structure. We also exhibit a convex structure on a generalization, $\mathbb{H}\mathrm{om}(\mathfrak{A}, M^{\mathcal{U}})$, of Brown's $\mathbb{H}\mathrm{om}(N, R^{\mathcal{U}})$. There are interesting connections (and disconnections) between algebraic concepts (e.g. traces, ideals, commutants) and concepts associated with convex geometry (e.g. affine maps, faces, extreme points).

These interactions are explored through both general theorems and specific examples.

The uninitiated reader will be able to find preliminary definitions and examples of the concepts discussed in this introduction in Chapter 2.

Definition. For a C^*-algebra $\mathfrak{A}$ (reviewed in §2.1) and a II_1-factor N (reviewed in §2.2), let $\mathbb{H}\mathrm{om}_w(\mathfrak{A}, N)$ denote the space of unital $*$-homomorphisms $\mathfrak{A} \to N$ modulo the equivalence relation of weak approximate unitary equivalence (reviewed in §2.5). We let $[\pi]$ denote the equivalence class in $\mathbb{H}\mathrm{om}_w(\mathfrak{A}, N)$ of $\pi : \mathfrak{A} \to N$. As explained in Definition 3.0.1, the space $\mathbb{H}\mathrm{om}_w(\mathfrak{A}, N)$ can be naturally metrized in a way similar to that of Definition 1.2 of [8].

The foundation of this thesis is the following theorem.

Theorem 3.1.6. *If M is a separable McDuff II_1-factor, then $\mathbb{H}om_w(\mathfrak{A}, M)$ may be considered as a closed, bounded, convex subset of a separable Banach space.*

As discussed in §2.3, a II_1-factor M is McDuff if and only if $M \cong M \otimes R$, where R denotes the separable hyperfinite II_1-factor (see Example 2.2.14). We establish the above theorem by showing that $\mathbb{H}\mathrm{om}_w(\mathfrak{A}, M)$ satisfies the axioms for a "convex-like structure"as in Definition 2.1 of [8]. The authors of [11] showed that these axioms characterize a closed, bounded, convex subset of a Banach space (convexity will be reviewed in §2.7).

It is natural to ask why we restrict to McDuff targets. The main reason is the existence of isomorphisms $\sigma_M : M \otimes R \to M$ with the following property:

$$\sigma_M \circ (\mathrm{id}_M \otimes 1_R) \sim \mathrm{id}_M$$

($\sim$ denotes weak approximate unitary equivalence). Given $\pi : \mathfrak{A} \to M$, such an isomorphism gives $\sigma_M \circ (\pi \otimes 1_R) \in [\pi]$. So we can always find a representative whose relative commutant unitally contains a copy of R. As we will see from Definition 3.1.4, the operation of taking a convex combination of $[\pi_1]$ and $[\pi_2]$ in $\mathbb{H}\mathrm{om}_w(\mathfrak{A}, M)$ is obtained by slicing each representative $\sigma_M \circ (\pi_1 \otimes 1_R)$ and $\sigma_M \circ (\pi_2 \otimes 1_R)$ by complementary projections of the form $\sigma_M(1_M \otimes p_1)$ and $\sigma_M(1_M \otimes p_2)$ respectively, with both projections contained in both relative commutants. In this way, the structure of a McDuff factor always provides us with representatives whose relative commutants contain the same copy of R and thus have an interval's worth of projections in common. Also, allowing any McDuff factor as a target algebra maintains enough generality so that technical embeddability obstructions do not arise. In fact, requiring a McDuff target is not so much of an obstruction. Thanks to N. Ozawa, we have Theorem 7.1.7 which says that for *any* separable II_1-factor N we may consider $\mathbb{H}\mathrm{om}_w(\mathfrak{A}, N)$ within this convex context by stabilizing the target algebra to obtain a homeomorphic embedding of $\mathbb{H}\mathrm{om}_w(\mathfrak{A}, N)$ as a closed set inside the convex $\mathbb{H}\mathrm{om}_w(\mathfrak{A}, N \otimes R)$.

As mentioned above, $\mathbb{H}\mathrm{om}_w(\mathfrak{A}, M)$ is a variation of the object of study $\mathbb{H}\mathrm{om}(N, R^{\mathcal{U}})$ in N. Brown's paper [8] (reviewed in §2.8). A major distinction between these two objects is that $\mathbb{H}\mathrm{om}_w(\mathfrak{A}, M)$ is always separable (Proposition 3.2) whereas $\mathbb{H}\mathrm{om}(N, R^{\mathcal{U}})$ is either nonseparable or trivial. So by studying $\mathbb{H}\mathrm{om}_w(\mathfrak{A}, M)$, we have the advantage of studying a separable object.

The space $\mathbb{H}\mathrm{om}_w(\mathfrak{A}, M)$ has a connection with the trace space of $\mathfrak{A}$. Let $T(\mathfrak{A})$ denote the trace space of $\mathfrak{A}$; as defined in §2.1,

$$T(\mathfrak{A}) := \{\tau \in \mathfrak{A}^* | \tau(1_{\mathfrak{A}}) = 1, \tau(a^*a) = \tau(aa^*) \geq 0 \ \forall a \in \mathfrak{A}\}.$$

Let $[\pi] \in \mathbb{H}\mathrm{om}_w(\mathfrak{A}, M)$ be the equivalence class of $\pi : \mathfrak{A} \to M$ and τ_M denote the unique faithful tracial state on M. There is a well-defined, affine map given by

$$[\pi] \mapsto \tau_M \circ \pi$$

(see Definition 4.1). We can think of $\tau_M \circ \pi$ as being a trace in $T(\mathfrak{A})$ that "lifts through M," and injectivity of this map means "liftable traces remember their homomorphisms (up to weak approximate unitary equivalence)." The following theorem shows that in the nuclear case, this map is very well-behaved (nuclearity will be reviewed in §2.6).

Theorem 4.1.2. *If $\mathfrak{A}$ is nuclear, then for any McDuff M, $\mathbb{H}om_w(\mathfrak{A}, M)$ is affinely homeomorphic to $T(\mathfrak{A})$ via $[\pi] \mapsto \tau_M \circ \pi$.*

In English, this theorem says: for a nuclear $\mathfrak{A}$, all traces on $\mathfrak{A}$ lift through M and remember their homomorphisms. So in this case $\mathbb{H}\mathrm{om}_w(\mathfrak{A}, M)$ serves as a different perspective from which we may study the trace space of $\mathfrak{A}$, and on the other side of the coin, $T(\mathfrak{A})$ gives insight into understanding $\mathbb{H}\mathrm{om}_w(\mathfrak{A}, M)$ in general. We notice how this compares with $\mathrm{Ext}(\mathfrak{A})$ – when $\mathfrak{A}$ is nuclear, we get that $\mathrm{Ext}(\mathfrak{A})$ is a group (see [2]). Some nontrivial work had to be done to show that there are algebras $\mathfrak{A}$

for which $\mathrm{Ext}(\mathfrak{A})$ is not a group (see [1] and [23]). So as in the program of $\mathrm{Ext}(\mathfrak{A})$, it is natural to ask: is $\mathbb{H}\mathrm{om}_w(\mathfrak{A}, M)$ always the same as $T(\mathfrak{A})$? We present several examples in §4.2 offering various negative answers (an algebra with forgetful traces, and an algebra with so many traces that they cannot all lift through one M). These examples are encouraging in that they show that the collection of $\{\mathbb{H}\mathrm{om}_w(\mathfrak{A}, M)\}_M$ as M varies over the McDuff factors contains information different from $T(\mathfrak{A})$. Notice that we would not get much information if we examined this connection in the context of $\mathbb{H}\mathrm{om}(N, R^{\mathcal{U}})$ because any II_1-factor N has a unique tracial state. Thus, for every $[\pi] \in \mathbb{H}\mathrm{om}(N, R^{\mathcal{U}})$ we get $\tau_{R^{\mathcal{U}}} \circ \pi = \tau_N$.

We go further to show that the class of algebras $\mathfrak{A}$ such that for every McDuff M $\mathbb{H}\mathrm{om}_w(\mathfrak{A}, M)$ is affinely homeomorphic to $T(\mathfrak{A})$ via $[\pi] \mapsto \tau \circ \pi$ is precisely the class of algebras $\mathfrak{A}$ for which given any $T \in T(\mathfrak{A})$, the weak closure of the GNS representation induced by T (see §2.1) is hyperfinite – a class strictly larger than nuclear algebras, see Example 2.6.13. This leads us to a characterization of hyperfiniteness stated in our context of weak approximate unitary equivalence in McDuff factors: a separable, tracial, $R^{\mathcal{U}}$-embeddable von Neumann algebra N is hyperfinite if and only if for every separable McDuff II_1-factor M, any two embeddings $\pi, \rho : N \to M$ are weakly approximately unitarily equivalent.

We turn to consider the convex geometry of $\mathbb{H}\mathrm{om}_w(\mathfrak{A}, M)$. In Proposition 5.2 of [8], Brown showed that given $[\pi] \in \mathbb{H}\mathrm{om}(N, R^{\mathcal{U}})$, $[\pi]$ is extreme if and only if $\pi(N)' \cap R^{\mathcal{U}}$ is a factor. We would like to adapt this characterization to our separable situation. The analogous statement is not available in our context because the relative commutant of the image of a $*$-homomorphism is not in general well-defined under weak approximate unitary equivalence–see Example 2.6.6. We have the following necessary condition for $[\pi]$ to be extreme.

Theorem 5.1.1. *If $[\pi] \in \mathbb{H}om_w(\mathfrak{A}, M)$ is extreme then $W^*(\pi(\mathfrak{A}))$ is a factor.*

The converse of the above theorem holds when $\mathfrak{A}$ is nuclear by Theorem 4.1.2. Also, when $M = R$, the converse holds for general $\mathfrak{A}$. However, the converse of this theorem fails in general.

Since our domains are unital separable C^*-algebras, we have access to nontrivial ideals. Using the contravariance in the first argument, we show that for a closed two-sided ideal J of $\mathfrak{A}$, $\mathbb{H}\mathrm{om}_w(\mathfrak{A}/J, M)$ is a face of $\mathbb{H}\mathrm{om}_w(\mathfrak{A}, M)$. A statement like this is meaningless in the setting of $\mathbb{H}\mathrm{om}(N, R^{\mathcal{U}})$ because II_1-factors are simple. The observation that any unital separable C^*-algebra is a quotient of $C^*(\mathbb{F}_\infty)$ translates into the following surprising fact.

Theorem 5.2.5. *For any unital separable C^*-algebra $\mathfrak{A}$, $\mathbb{H}om_w(\mathfrak{A}, M)$ is a face of $\mathbb{H}om_w(C^*(\mathbb{F}_\infty), M)$.*

We also discuss ultrapowers (see §2.4) in considering $\mathbb{H}\mathrm{om}(\mathfrak{A}, M^{\mathcal{U}})$: the space of all unital $*$-homomorphisms $\mathfrak{A} \to M^{\mathcal{U}}$ modulo unitary equivalence. This is an obvious generalization of $\mathbb{H}\mathrm{om}(N, R^{\mathcal{U}})$ and also supports a convex structure. We extend Brown's characterization of extreme points to apply to $\mathbb{H}\mathrm{om}(\mathfrak{A}, M^{\mathcal{U}})$: $[\pi] \in$

$\mathbb{H}\text{om}(\mathfrak{A}, M^{\mathcal{U}})$ is extreme if and only if $\pi(\mathfrak{A})' \cap M^{\mathcal{U}}$ is a factor. We observe that $\mathbb{H}\text{om}_w(\mathfrak{A}, M)$ may be embedded into $\mathbb{H}\text{om}(\mathfrak{A}, M^{\mathcal{U}})$ via $[\pi] \mapsto [\pi^{\mathcal{U}}]$ where $\pi^{\mathcal{U}}$ denotes π followed by the canonical constant-sequence embedding of M into $M^{\mathcal{U}}$. This is a strict inclusion in general, but we observe that $\mathbb{H}\text{om}_w(\mathfrak{A}, M) \cong \mathbb{H}\text{om}(\mathfrak{A}, M^{\mathcal{U}})$ in the nuclear case. This embedding yields the following sufficient condition for extreme points.

Theorem 6.1.13. *If $\pi^{\mathcal{U}}(\mathfrak{A})' \cap M^{\mathcal{U}}$ is a factor, then $[\pi]$ is extreme in $\mathbb{H}om_w(\mathfrak{A}, M)$.*

The converse holds in the case when either $\mathfrak{A}$ or M is amenable. It is unknown if the converse of Theorem 6.1.13 holds in general. It would hold if one could show that in general, $\mathbb{H}\text{om}_w(\mathfrak{A}, M)$ embeds as a face of $\mathbb{H}\text{om}(\mathfrak{A}, M^{\mathcal{U}})$, and it is known that $\mathbb{H}\text{om}_w(\mathfrak{A}, R)$ embeds as a face of $\mathbb{H}\text{om}(\mathfrak{A}, R^{\mathcal{U}})$. As a consequence of the characterizations of extreme points in the amenable cases, we get an equivalence of two purely algebraic statements with no reference to $\mathbb{H}\text{om}_w(\mathfrak{A}, M)$ (see Corollary 6.2.3 and Theorem 6.3.1). This discussion of relative commutants in ultrapowers along with a helpful comment made by S. White leads us to the following new characterization of the hyperfinite II_1-factor.

Theorem 6.1.8. *Let N be an embeddable separable II_1-factor. The following are equivalent:*

1. *$N = R$;*

2. *For any separable II_1-factor X and any embedding $\pi : N \to X^{\mathcal{U}}$, $\pi(N)' \cap X^{\mathcal{U}}$ is*

a factor;

3. *For any separable II_1-factor X and any embedding $\pi : N \to X^{\mathcal{U}}$, the collection of tracial states $\{\tau(\pi(x)\cdot) : x \in N^+, \tau(x) = 1\}$ is weak-$*$ dense in the trace space of $\pi(N)' \cap X^{\mathcal{U}}$.*

Notice that this is a strengthening of Corollary 5.3 in [8].

We will also discuss some interesting questions regarding the structure of $\mathbb{H}\mathrm{om}(N, R^{\mathcal{U}})$. Given $[\pi] \in \mathbb{H}\mathrm{om}(N, R^{\mathcal{U}})$ we let $F_{[\pi]}$ denote the minimal face in $\mathbb{H}\mathrm{om}(N, R^{\mathcal{U}})$ containing $[\pi]$. The following theorem further demonstrates the connection between geometric properties of $\mathbb{H}\mathrm{om}(N, R^{\mathcal{U}})$ and algebraic properties of the underlying operator algebras.

Theorem 8.2. *Let the embedding $\pi : N \to R^{\mathcal{U}}$ be given.*

1. *If $\dim(\mathcal{Z}(\pi(N)' \cap R^{\mathcal{U}})) = n < \infty$ then $F_{[\pi]}$ is an n-vertex simplex.*

2. *If $\varphi \in t_1[\pi_1] + \cdots + t_n[\pi_n]$ where $0 < t_j < 1$ and $[\pi_j]$ is an extreme point for every $1 \leq j \leq n$, then*

$$\varphi(N)' \cap R^{\mathcal{U}} \cong \oplus_{j=1}^{n} \pi_j(N)' \cap R^{\mathcal{U}}.$$

3. $\dim(F_{[\pi]}) + 1 = \dim(\mathcal{Z}(\pi(N)' \cap R^{\mathcal{U}}))$.

Propostion 5.2 of [8] is the case where $\dim(F_{[\pi]}) + 1 = \dim(\mathcal{Z}(\pi(N)' \cap R^{\mathcal{U}})) = 1$. In order to prove this theorem, we use a sort of $R^{\mathcal{U}}$-version of Schur's lemma.

This thesis is organized as follows.

Chapter 2 Preliminaries: This chapter is devoted to introducing and reviewing the necessary definitions and concepts used throughout this thesis. §2.1 reviews some basic definitions, properties, and examples regarding C^*-algebras. In §2.2, we cover some facts about von Neumann algebras. §2.3 reviews tensor products in both the context of C^*-algebras and the context of von Neumann algebras. We review some introductory material on ultraproducts of certain operator algebras in §2.4. In §2.5 we will discuss some background material regarding approximate unitary equivalence. §2.6 gives an (extremely brief) overview of nuclearity in the context of C^*-algebras and hyperfiniteness in the context of von Neumann algebras. In §2.7, we give a short discussion on the convexity and some of its accompanying notions in the context of functional analysis. We will review some results concerning $\mathbb{H}\mathrm{om}(N, R^{\mathcal{U}})$ from [8] in §2.8.

Chapter 3 The Space $\mathbb{H}\mathbf{om}_w(\mathfrak{A}, M)$: We provide all of the initial definitions for $\mathbb{H}\mathrm{om}_w(\mathfrak{A}, M)$. The convex structure of $\mathbb{H}\mathrm{om}_w(\mathfrak{A}, M)$ is introduced and verified in §3.1. We briefly discuss some surface-level functoriality in §3.2.

Chapter 4 Connection to the Trace Space: Here we introduce the relationship between $\mathbb{H}\mathrm{om}_w(\mathfrak{A}, M)$ and $T(\mathfrak{A})$ as mentioned above. In §4.1 we establish the fact that the relationship is a bijection when $\mathfrak{A}$ is nuclear, and we show that traces remember their homomorphisms when $M = R$. We then discuss in §4.2 several examples showing that $\mathbb{H}\mathrm{om}_w(\mathfrak{A}, M)$ is not the same as $T(\mathfrak{A})$ in general. These examples include the "forgetful trace" and "too many traces" examples mentioned above. In §4.3 we record our alternative separable characterization for separable tracial hyperfinite embeddable von Neumann algebras.

Chapter 5 Convex Geometry: We take a closer look at some of the convex geometry of $\mathbb{H}\mathrm{om}_w(\mathfrak{A}, M)$. In §5.1, Theorem 5.1.1 gives a necessary condition for extremality, and we provide an example showing that its converse is false in general. In §5.2 we show that quotients of $\mathfrak{A}$ give rise to faces in $\mathbb{H}\mathrm{om}_w(\mathfrak{A}, M)$.

Chapter 6 Ultrapower Situation: In §6.1 we generalize $\mathbb{H}\mathrm{om}(N, R^{\mathcal{U}})$ by considering the space $\mathbb{H}\mathrm{om}(\mathfrak{A}, M^{\mathcal{U}})$. We extend the characterization of extreme points in $\mathbb{H}\mathrm{om}(N, R^{\mathcal{U}})$ to a characterization in $\mathbb{H}\mathrm{om}(\mathfrak{A}, M^{\mathcal{U}})$. This provides a sufficient condition for extreme points in $\mathbb{H}\mathrm{om}_w(\mathfrak{A}, M)$ in general. We give a characterization of R in Theorem 6.1.8. The embedding $\mathbb{H}\mathrm{om}_w(\mathfrak{A}, M) \subset \mathbb{H}\mathrm{om}(\mathfrak{A}, M^{\mathcal{U}})$ is defined and discussed. In §6.2 we present a characterization of extreme points in $\mathbb{H}\mathrm{om}_w(\mathfrak{A}, M)$ in the case where all traces of $\mathfrak{A}$ give a hyperfinite GNS construction. In §6.3 we present a characterization of extreme points in $\mathbb{H}\mathrm{om}_w(\mathfrak{A}, R)$.

Chapter 7 More on $\mathbb{H}\mathrm{om}_w(\mathfrak{A}, M)$: In this chapter we present some more interesting facts about the structure and dynamics of $\mathbb{H}\mathrm{om}_w(\mathfrak{A}, M)$. In §7.1 we address the stabilization of non-McDuff target algebras. We show in Theorem 7.1.7 that for any separable II_1-factors N_1 and N_2, $\mathbb{H}\mathrm{om}_w(\mathfrak{A}, N_1)$ embeds homeomorphically into $\mathbb{H}\mathrm{om}_w(\mathfrak{A}, N_1 \otimes N_2)$ as a closed subset. So in particular, $\mathbb{H}\mathrm{om}_w(\mathfrak{A}, N) \subset \mathbb{H}\mathrm{om}_w(\mathfrak{A}, N \otimes R)$. In §7.2 it is shown that when a coassociative comultiplication on

$\mathfrak{A}$ exists (e.g. $\mathfrak{A}$ a compact quantum group), we can define an affine-distributive, associative product on $\mathbb{H}\mathrm{om}_w(\mathfrak{A}, R)$.

Chapter 8 Simplices in $\mathbb{H}\mathbf{om}(N, R^{\mathcal{U}})$: In this chapter we explore more of the structure of $\mathbb{H}\mathrm{om}(N, R^{\mathcal{U}})$. In particular, we focus on analyzing finite dimensional minimal faces of $\mathbb{H}\mathrm{om}(N, R^{\mathcal{U}})$. An $R^{\mathcal{U}}$-version of Schur's lemma is proved on the way to establishing Theorem 8.2.

Chapter 2

Preliminaries

We review here some basic definitions, properties, and examples of the key elements relevant to this thesis.

2.1 C^*-algebras

We will start with C^*-algebras. The books [37], [15], [10], and [46] contain helpful introductions to this subject. Let H be a (complex) Hilbert space with inner product $\langle\cdot|\cdot\rangle$, and let $B(H)$ denote the space of all bounded linear operators on H. A bounded operator $T : H \to H$ is one such that for any vector $\xi \in H$, there is a constant $C \in [0, \infty)$ such that $||T\xi|| \leq C||\xi||$ where $||\cdot||$ is the Hilbert space norm given by $||\xi||^2 = \langle\xi|\xi\rangle$. We can define a norm on $B(H)$ given by

$$||T|| := \sup_{\xi\in H, ||\xi||\leq 1} ||T\xi||.$$

Definition 2.1.1. A (complex) C^*-*algebra* is typically defined in one of the following two equivalent ways.

- (Spatial) Let H be a Hilbert space. A norm closed, $*$-closed subalgebra $\mathfrak{A} \subset$

$B(H)$ is called a C^*-algebra. (A subset $X \subset B(H)$ is $*$-closed if $x \in X \Rightarrow x^* \in X$ where $x \mapsto x^*$ is the adjoint operation on $B(H)$. A $*$-closed subalgebra is often referred to as a $*$-subalgebra.)

- (Abstract) A (complex) algebra $\mathfrak{A}$ is a C^*-algebra if it is a Banach $*$-algebra ($\mathfrak{A}$ is in addition a complete normed linear space with submultiplicative norm and involution $*$) that satisfies the C^*-identity: $||x^*x|| = ||x||^2$ for every $x \in \mathfrak{A}$.

It is a well-known theorem of C^*-algebras that any abstract C^*-algebra can be concretely realized as a norm closed $*$-subalgebra of $B(H)$ for some Hilbert space H.

A C^*-algebra is called *unital* if it contains a multiplicative identity.

Example 2.1.2. The following are some of the first examples of C^*-algebras that an introductory course would cover.

1. For $n \in \mathbb{N}$ the algebra of $n \times n$ matrices with complex entries, denoted as $\mathbb{M}_n$ is a C^*-algebra. The involution $*$ is the operation of taking the conjugate transpose. These algebras are unital.

2. Given a locally compact Hausdorff topological space X, the algebra $C_0(X)$ of complex-valued continuous functions on X vanishing at ∞ is an abelian C^*-algebra. In fact, thanks to the Gelfand transform, any abelian C^*-algebra takes the form of $C_0(X)$ for some locally compact Hausdorff space X. $C_0(X)$ is unital if and only if X is compact. These abelian C^*-algebras are completely identified

by the underlying topological space, so it is a shared philosophy that the study of general C^*-algebras is the study of non-commutative topology.

3. Given any Hilbert space H, $B(H)$ itself is a C^*-algebra. If H_n is n-dimensional, then $B(H_n) \cong \mathbb{M}_n$. These algebras are unital.

4. Given $T \in B(H)$ for a Hilbert space H, we can consider the C^*-algebra generated by T denoted $C^*(T)$. This is the smallest C^*-subalgebra of $B(H)$ containing T. Or more explicitly, it is the norm closure of the algebra of $*$-polynomials without constant terms in T (finite linear combinations of finite products of T and T^*). More generally, given an n-tuple of operators $T_1, \ldots, T_n \in B(H)$, one can analogously define $C^*(T_1, \ldots, T_n)$. These algebras are not unital in general.

Definition 2.1.3. Let $\mathfrak{A}$ and $\mathfrak{B}$ be C^*-algebras. A $*$*-homomorphsim* $\pi : \mathfrak{A} \to \mathfrak{B}$ is a well-defined map satisfying the following properties.

- (linear) For $\lambda \in \mathbb{C}, x, y \in \mathfrak{A}, \pi(\lambda x + y) = \lambda\pi(x) + \pi(y)$.

- (multiplicative) For $x, y \in \mathfrak{A}, \pi(xy) = \pi(x)\pi(y)$.

- ($*$-preserving) For $x \in \mathfrak{A}, \pi(x^*) = \pi(x)^*$.

If $\mathfrak{A}$ and $\mathfrak{B}$ are unital, then $\pi : \mathfrak{A} \to \mathfrak{B}$ is a *unital* $*$-homomorphism if in addition to the three properties above, $\pi(1_{\mathfrak{A}}) = 1_{\mathfrak{B}}$.

Definition 2.1.4. We now define several different properties elements of a C^*-algebra can have. Let $\mathfrak{A}$ be a C^*-algebra.

- An element $a \in \mathfrak{A}$ is called a *contraction* if $||a|| \leq 1$. The set of all contractions in $\mathfrak{A}$ will be denoted $\mathfrak{A}_{\leq 1}$.
- An element $a \in \mathfrak{A}$ is called *normal* if a commutes with its adjoint. That is, $a^*a = aa^*$.
- An element $a \in \mathfrak{A}$ is called *self-adjoint* if $a = a^*$. The set of all self-adjoint elements in $\mathfrak{A}$ will be denoted $\mathfrak{A}^{\text{s.a.}}$.
- An element $a \in \mathfrak{A}$ is called *positive* if there is some $b \in \mathfrak{A}$ such that $a = b^*b$. The set of all positive elements in $\mathfrak{A}$ will be denoted $\mathfrak{A}^+$.
- An element $p \in \mathfrak{A}$ is called a *projection* if $p = p^* = p^2$.
- An element $v \in \mathfrak{A}$ is called a *partial isometry* if v^*v is a projection.
- If $\mathfrak{A}$ is unital, an element $v \in \mathfrak{A}$ is called an *isometry* if $v^*v = 1$; and v is called a *coisometry* is $vv^* = 1$.
- If $\mathfrak{A}$ is unital, an element $u \in \mathfrak{A}$ is called a *unitary* if $u^*u = uu^* = 1$. That is, a unitary is both an isometry and a coisometry. The set of all unitary elements in $\mathfrak{A}$ will be denoted $\mathcal{U}(\mathfrak{A})$.

Example 2.1.5 (Group C^*-algebras)**.** We take the time now to explain how to construct C^*-algebras out of groups. Such C^*-algebras are crucial to the subject and provide many rich examples. We first present the left-regular representation of a discrete group. Let Γ be a discrete group and let $\ell^2(\Gamma)$ be the collection of square-summable complex-valued functions on Γ. That is $f \in \ell^2(\Gamma)$ if and only if $\sum_{g\in\Gamma} |f(g)|^2 < \infty$. This naturally makes $\ell^2(\Gamma)$ a Hilbert space with a canonical orthonormal basis given by $\{\delta_g\}_{g\in\Gamma}$ where δ_g is the indicator function on the set $\{g\}$. We now define a group homomorphism from Γ into the unitary group of $B(\ell^2(\Gamma))$. Let

$$\lambda : \Gamma \to \mathcal{U}(B(\ell^2(\Gamma)))$$

be given by

$$\lambda(g)(\delta_h) = \delta_{gh}.$$

It is a direct exercise to check that for every $g \in \Gamma, \lambda(g)$ is indeed a unitary operator on $\ell^2(\Gamma)$. Furthermore, one can check that given $f \in \ell^2(\Gamma)$ and $g, x \in \Gamma$, we have

$$[\lambda(g)(f)](x) = f(g^{-1}x).$$

We we let $C^*_\lambda(\Gamma)$ be the C^*-algebra generated by the unitaries $\{\lambda(g)\}$. That is,

$$C^*_\lambda(\Gamma) := C^*(\{\lambda(g)\}_{g\in\Gamma}) \subset B(\ell^2(\Gamma)).$$

We call $C^*_\lambda(\Gamma)$ the *reduced group C^*-algebra of* Γ. In the literature, $C^*_\lambda(\Gamma)$ is sometimes written as $C^*_r(\Gamma)$.

There is also a *full* group C^*-algebra of Γ, denoted simply by $C^*(\Gamma)$, given by the C^*-closure of image of the direct sum of *all* unitary representations of Γ.

Another useful notion is the spectrum of an element of a C^*-algebra. Regardless of whether or not a C^*-algebra is unital, one can write down the definition of the spectrum. The non-unital case requires some technicalities, and since this thesis is concerned exclusively with unital C^*-algebras, we only provide the definition for the unital case.

Definition 2.1.6. If $\mathfrak{A}$ is a unital C^*-algebra and $a \in \mathfrak{A}$, then the spectrum of a is given by

$$\mathrm{sp}(a) := \{\lambda \in \mathbb{C} : (a - \lambda \cdot 1) \text{ is not invertible } \}.$$

For any $a \in \mathfrak{A}$, $\mathrm{sp}(a)$ is compact, nonempty, and contained in $\{z \in \mathbb{C} : |z| \leq ||a||\}$. If a is self-adjoint, then $\mathrm{sp}(a) \subset \mathbb{R}$; if a is positive, then $\mathrm{sp}(a) \subset [0, \infty)$; if p is a projection, then $\mathrm{sp}(p) = \{0, 1\}$; if u is a unitary, then $\mathrm{sp}(u) \subset \left\{e^{i\theta} : 0 \leq \theta \leq 2\pi\right\}$.

Let us now turn to discuss some functional analysis of C^*-algebras. This subject is indeed deep and interesting on its own, but here we will only introduce the concepts pertinent to this thesis.

Definition 2.1.7. A *continuous linear functional* f on a C^*-algebra $\mathfrak{A}$ is a continuous linear map $f : \mathfrak{A} \to \mathbb{C}$. The set of all continuous linear functionals on $\mathfrak{A}$ is called the *dual* of $\mathfrak{A}$ and denoted $\mathfrak{A}^*$.

- A continuous linear functional f is called *positive* if for any $a \in \mathfrak{A}^+$, $f(a) \geq 0$.
- A continuous linear functional f is called *faithful* if for any $a \in \mathfrak{A}$, $f(a^*a) = 0$ if and only if $a = 0$.
- A positive continuous linear functional f is called a *state* if $f(1) = 1$ (there are states on non-unital C^*-algebras, but for our context, this definition works). The set of all states on $\mathfrak{A}$ is denoted $S(\mathfrak{A})$.

- A positive continuous linear functional f is called a *tracial state* if it is a state and $f(ab) = f(ba)$ for every $a, b \in \mathfrak{A}$. The set of all tracial states on $\mathfrak{A}$ is denoted $T(\mathfrak{A})$.

Example 2.1.8.
- Let $\mathfrak{A}$ be a C^*-algebra and let H be a Hilbert space such that $\mathfrak{A} \subseteq B(H)$. Let $\xi, \eta \in H$ be vectors. Then we can define the following linear functional f given by

$$f(A) = \langle A\xi|\eta\rangle$$

where $\langle\cdot|\cdot\rangle$ is the inner product on H. If $\xi = \eta$, then f is positive. Furthermore, if $||\xi|| = 1$ then f is a state. Such a state is called a *vector state*.

- The normalized trace on $\mathbb{M}_n$ is a tracial state. The normalized trace is given by

$$\mathrm{tr}((a_{ij})) = \frac{1}{n}\sum_{i=1}^{n} a_{ii}$$

for $(a_{ij}) \in \mathbb{M}_n$.

- Let X be a compact Hausdorff space. Then given any measure μ on the Borel σ-algebra on X, the map $f \mapsto \int f d\mu$ is a positive continuous linear functional. If $\mu(X) = 1$, then $f \mapsto \int f d\mu$ is a tracial state. By Riesz Representation, any continuous linear functional on $C(X)$ can be expressed as integration against some signed measure. So continuous linear functionals may be considered morally as non-commutative integrals.

Notice that if $\mathfrak{A}$ is a unital C^*-algebra and $T \in T(\mathfrak{A})$, then T "ignores"unitary conjugation. That is, for $x \in \mathfrak{A}$ and $u \in \mathcal{U}(\mathfrak{A})$,

$$T(uxu^*) = T(xu^*u) = T(x).$$

We now describe a construction that shows how any abstract C^*-algebra may be realized concretely as a norm closed $*$-subalgebra of $B(H)$ for some Hilbert space H. This construction can be found in [15]. Let $\mathfrak{A}$ be a C^*-algebra and let $f \in S(\mathfrak{A})$. Let $\mathcal{N} := \{a \in \mathfrak{A} : f(a^*a) = 0\}$. One can show that $\mathcal{N}$ is a left ideal of $\mathfrak{A}$. So consider the positive definite sesquilinear form on $\mathfrak{A}/\mathcal{N}$ given by

$$(x + \mathcal{N}|y + \mathcal{N})_f = f(y^*x).$$

This is an inner product on $\mathfrak{A}/\mathcal{N}$, making $\mathfrak{A}/\mathcal{N}$ into a pre-Hilbert space. Let H_f be the completion of $\mathfrak{A}/\mathcal{N}$ under the norm induced by the inner product. We can now define a $*$-representation $\pi_f : \mathfrak{A} \to H_f$ in the following way. Let $a \in \mathfrak{A}$ and $x + \mathcal{N} \in \mathfrak{A}/\mathcal{N}$, then $\pi_f(a)(x+\mathcal{N}) = ax+\mathcal{N}$. This operation naturally extends by continuity so that $\pi_f(A)$ is truly an operator in $B(H_f)$, and moreover, π_f is a $*$-homomorphism. This construction is called the *GNS construction* named for Gelfand, Naimark, and Segal. The representation π_f is called the *GNS representation associated to* f. To concretely realize an abstract C^*-algebra, one takes a direct sum of all GNS representations π_f for every $f \in S(\mathfrak{A})$.

Another important notion associated with C^*-algebras is that of a *completely positive map*. First we must establish some notation. let $\varphi : \mathfrak{A} \to \mathfrak{B}$ be a linear map from a C^*-algebra $\mathfrak{A}$ to a C^*-algebra $\mathfrak{B}$. Let $\mathbb{M}_n(\mathfrak{A})$ denote the C^*-algebra of $n \times n$

matrices with entries from $\mathfrak{A}$. For $n \in \mathbb{N}$, we let $\varphi^{(n)} : \mathbb{M}_n(\mathfrak{A}) \to \mathbb{M}_n(\mathfrak{B})$ be the $n \times n$ amplification of φ given by

$$\varphi^{(n)}((a_{ij})) = (\varphi(a_{ij})).$$

Definition 2.1.9. Let $\mathfrak{A}$ and $\mathfrak{B}$ be C^*-algebras. A linear map $\varphi : \mathfrak{A} \to \mathfrak{B}$ is called completely positive if for every $n \in \mathbb{N}$, $\varphi^{(n)}$ is positive. That is, for every $n \in \mathbb{N}$, $\varphi^{(n)}(\mathbb{M}_n(\mathfrak{A})^+) \subset \mathbb{M}_n(\mathfrak{B})^+$.

Example 2.1.10.

- Any $*$-homomorphism is completely positive.
- Any positive linear functional is completely positive. More generally, any positive linear map with abelian domain or range is completely positive.

2.2 von Neumann Algebras

We next turn to von Neumann algebras. Again, the books [37], [15], [10], and [46] contain valuable information on this subject. In order to discuss von Neumann algebras, we must first discuss some more topologies on $B(H)$. Given a countably infinite dimensional Hilbert space $H, B(H)$ has several distinct modes of convergence. We have already mentioned the norm topology in discussing C^*-algebras. There are six other distinct topologies of interest that one can place on $B(H)$. See Section II.2 of [46] for a description and discussion of these topologies. For the sake of this thesis, we will discuss only the weak and the strong topologies. We will define these topologies by describing what convergence means for each topology respectively.

Definition 2.2.1. • We say that a sequence $\{T_n\}$ in $B(H)$ converges to $T \in B(H)$ *weakly* if for every $\xi, \eta \in H$ we have

$$\langle T_n \xi | \eta \rangle \to \langle T \xi | \eta \rangle \, .$$

The topology induced by this convergence is called the *Weak Operator Topology*, abbreviated WOT. This convergence is often denoted as

$$T = \text{WOT-} \lim_{n \to \infty} T_n .$$

• We say that a sequence $\{T_n\}$ in $B(H)$ converges to $T \in B(H)$ *strongly* if for every $\xi \in H$, we have

$$T_n \xi \to T \xi .$$

The topology induced by this convergence is called the *Strong Operator Topology*, abbreviated SOT. This convergence is often denoted as

$$T = \text{SOT-} \lim_{n \to \infty} T_n .$$

The following definition will seem very unrelated to Definition 2.2.1.

Definition 2.2.2. Given a subset $S \subset B(H)$, the *commutant* S' of S is given by

$$S' := \{A \in B(H) : AX = XA \, , \forall X \in S\} \, .$$

Let $S'' := (S')'$, and so on.

Thanks to the following deep and celebrated theorem by John von Neumann, these two notions are fundamentally related.

Theorem 2.2.3 (Double Commutant Theorem, [37],[48])**.** *Let M be a $*$-closed, unital subalgebra of $B(H)$. Then the following are equivalent.*

1. *$M = M''$;*

2. *M is weakly closed;*

3. *M is strongly closed.*

We can now define a von Neumann algebra.

Definition 2.2.4. A $*$-closed, unital subalgebra M of $B(H)$ is a *von Neumann algebra* if it satisfies the equivalent conditions in Theorem 2.2.3. von Neumann algebras are sometimes referred to as W^*-algebras.

Remark 2.2.5. It is a quick exercise to see that norm convergence implies both WOT and SOT convergence. In particular, this shows that von Neumann algebras are norm-closed. So *von Neumann algebras are C^*-algebras.* In fact, it was shown in [41] that a von Neumann algebra can also be characterized as a C^*-algebra that is a dual Banach space.

Example 2.2.6.

- For any Hilbert space H, $B(H)$ itself is a von Neumann algebra. In particular, for any $n \in \mathbb{N}, \mathbb{M}_n$ is a von Neumann algebra.

- Let $(X, \mathcal{M}, \mu)$ be a measure space. Then $L^\infty(X, \mu)$ is a von Neumann algebra. One can represent $L^\infty(X, \mu)$ as multiplication operators on the Hilbert space $L^2(X, \mu)$. Any abelian von Neumann algebra takes this form. So, in comparison with the C^*-algebra situation, it is a shared philosophy that the study of von Neumann algebras is the study of non-commutative measure theory.

- If M is a von Neumann algebra and $S \subset M$ is a subset of M, then we let $W^*(S)$ denote the von Neumann algebra generated by S. One can view $W^*(S) \subset M$ as the smallest von Neumann subalgebra of M containing S. $W^*(S)$ can be obtained by taking the WOT-closure of the algebra of $*$-polynomials with entries from S.

- As in the C^* case, given a discrete group Γ, one associates to Γ a von Neumann algebra. The *group von Neumann algebra of* Γ is given by $(\{\lambda(g)\}_{g\in\Gamma})'' \subset B(\ell^2(\Gamma))$ where $\lambda : \Gamma \to \mathcal{U}(B(\ell^2(\Gamma)))$ is the left-regular representation as defined in Example 2.1.5. Group von Neumann algebras provide important and deep examples in the theory of von Neumann algebras.

If $M \subset B(H)$ is a von Neumann algebra, and H is a separable Hilbert space, then we say that M is *separably acting*.

Definition 2.2.7. Given a von Neumann algebra M, the center of M, denoted $\mathcal{Z}(M)$ is given by

$$\mathcal{Z}(M) := \{z \in M : zx = xz \; \forall x \in M\}.$$

Clearly, $\mathcal{Z}(M) = M \cap M'$.

Definition 2.2.8. A von Neumann algebra M is called a *factor* if the center of M is isomorphic to $\mathbb{C}$.

Definition 2.2.9. A von Neumann algebra M is called *finite* if there is no partial isometry v such that $v^*v = 1_M$ but $vv^* \neq 1_M$.

Example 2.2.10.

- $\mathbb{M}_n$ is a finite factor.
- For H an infinite dimensional Hilbert space, $B(H)$ is factor, but it is not finite.

All von Neumann algebras are classified by types: I, II, and III. For the sake of brevity, we will not define these types in generality. See Section V.1 of [46] for full definitions. We will primarily be concerned with factor von Neumann algebras of type II that are also finite. Such factors are called type II_1-factors. We define a type II_1-factor as follows.

Definition 2.2.11. A *type II_1-factor* is an infinite dimensional factor von Neumann algebra that is finite in the sense of Definition 2.2.9.

We can equivalently define a II_1-factor to be an infinite dimensional factor von Neumann algebra that admits a unique faithful tracial state. If N is a II_1-factor, then we typically denote this unique tracial state as τ_N or just τ when no confusion may occur.

If N is a II_1-factor, then we may use its unique tracial state τ to define a norm on N. The so-called trace norm, denoted $||\cdot||_2$, is given by

$$||x||_2 = \sqrt{\tau(x^*x)}.$$

While N is not complete with respect to $||\cdot||_2$, we have on bounded subsets of N that $||\cdot||_2$-convergence coincides with SOT convergence. If N is separably acting, then N

is separable with respect to the topology coming from $||\cdot||_2$. For this reason, we will sometimes simply use the word *separable* in place of *separably acting* in the context of II_1-factors.

It can be shown that a von Neumann algebra is generated by its projections. Thus, the theory of projections in von Neumann algebras is important to the subject as a whole. A fundamental notion associated to projections is *Murray-von Neumann equivalence.*

Definition 2.2.12. Let M be a von Neumann algebra. Two projections $p, q \in M$ are *Murray-von Neumann equivalent* in M if there is a partial isometry $v \in M$ such that $p = v^*v$ and $q = vv^*$. This equivalence relation is sometimes denoted as $p \sim_{\text{MvN}} q$.

Example 2.2.13.

- If $M = B(H)$, then the Murray-von Neumann equivalence class of a projections is completely determined by the rank of the projection.
- If M is a II_1 factor, then Murrray-von Neumann equivalence is completely determined by the value of the trace on the projections. That is, for two projections $p, q \in M, p \sim_{\text{MvN}} q$ if and only if $\tau(p) = \tau(q)$ where τ is the unique tracial state on M.

We now present some constructions of II_1-factors.

Example 2.2.14.

1. (The hyperfinite II_1-factor) Consider the algebraic direct limit Q of the sequence of algebras $\{\mathbb{M}_{2^n}\}$ with connecting maps $\varphi_{mn} : \mathbb{M}_{2^n} \to$

$\mathbb{M}_{2^m}(n < m)$ given by

$$\varphi_{(n+1)n}(a) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$$

and if $m > n+1$, then $\varphi_{mn} = \varphi_{m(m-1)} \circ \cdots \circ \varphi_{(n+1)n}$. So Q is the infinite nested union of the matrix algebras $\mathbb{M}_{2^n}$; that is,

$$Q = \cup_{n=1}^{\infty} \mathbb{M}_{2^n}.$$

Each $\mathbb{M}_{2^n}$ admits a unique faithful tracial state τ_n (the normalized trace), and since $\tau_m \circ \varphi_{mn} = \tau_n$, these traces induce a trace τ on Q. In particular, if $x \in Q$, then $x \in \mathbb{M}_{2^n}$ for some n, so $\tau(x) = \tau_n(x)$. Since all the τ_n's are faithful, we have that Q with positive-definite inner product $\langle \cdot | \cdot \rangle_\tau$ given by

$$\langle x | y \rangle_\tau := \tau(y^* x)$$

is a pre-Hilbert space. Let H be the Hilbert space obtained by taking the completion of Q with respect to the norm induced by the inner product $\langle \cdot | \cdot \rangle_\tau$. As in the GNS construction, we can view Q as a unital $*$-subalgebra of $B(H)$ by having it act (densely) on H by left multiplication. Then we let $R = Q''$ be the weak closure of Q in this representation. Since each τ_n is the unique faithful tracial state on $\mathbb{M}_{2^n}$, we get can extend τ by continuity to be the unique faithful tracial state on R. Thus R is an infinite dimensional factor von Neumann algebra with a unique faithful tracial state τ. This II_1-factor R is known as the *hyperfinite II_1-factor*. See §2.6 for a definition of hyperfinite.

Murray and von Neumann showed that up to isomorphism, there is only one separable hyperfinite II_1-factor.

2. (Group II_1-factors) If Γ is a group such that every non-trivial conjugacy class is infinite (an *infinite conjugacy class* or *i.c.c.* group), then $L(\Gamma)$ is a II_1 factor with trace given by $\tau(x) = \langle x\delta_e|\delta_e\rangle_{L^2(\Gamma)}$. e.g. $\Gamma = \mathbb{F}_n(n \in \{2, 3, \cdots, \infty\})$ the free group with n letters, $\Gamma = S_\infty$ the group of finite permutations of $\mathbb{N}$, or $\Gamma = SL(n, \mathbb{Z})$. It can be shown that $L(S_\infty)$ is hyperfinite. So $L(S_\infty) \cong R$ from (1) above. Historically, $L(\mathbb{F}_2)$ was the first II_1-factor shown to be distinct from R ([30]).

2.3 Tensor Products of Operator Algebras

The operation of taking a tensor product of C^*-algebras or von Neumann algebras has been a topic of intense study for many decades. The following treatment of the topic can be found in Chapter 3 of [10].

Let $\mathfrak{A}$ and $\mathfrak{B}$ be C^*-algebras. We begin by taking the algebraic tensor product $\mathfrak{A} \odot \mathfrak{B}$ which is given by the following universal property. For any vector space Z and any bilinear map $\sigma : \mathfrak{A} \times \mathfrak{B} \to Z$, there is a unique linear map $\dot{\sigma} : \mathfrak{A} \odot \mathfrak{B} \to Z$ such that for every $a \in \mathfrak{A}$ and $b \in \mathfrak{B}, \dot{\sigma}(a \otimes b) = \sigma(a, b)$. The algebraic tensor product $\mathfrak{A} \odot \mathfrak{B}$ is a $*$-algebra, but it has not been topologized by any sort of norm–it is the linear span of the simple tensors. It turns out that we have options when it comes to choosing a C^*-norm (one satisfying $||x^*x|| = ||x||^2$).

Definition 2.3.1 ([10])**.** The largest C^*-norm we can place on $\mathfrak{A} \odot \mathfrak{B}$ is called the *maximal C^*-norm* $||\cdot||_{\max}$. Given $x \in \mathfrak{A} \odot \mathfrak{B}$, define

$$||x||_{\max} := \sup\{||\pi(x)|| : \pi : \mathfrak{A} \odot \mathfrak{B} \to B(H) \text{ a (cyclic) } *\text{-homomorphism}\}$$

where $\pi : X \to B(H)$ is a cyclic $*$-homomorphism if there is a vector $\xi \in B(H)$ such that $\{\pi(x)\xi : x \in X\}$ is dense in H. Let $\mathfrak{A} \otimes_{\max} \mathfrak{B}$ denote the completion of $\mathfrak{A} \odot \mathfrak{B}$ with respect to $||\cdot||_{\max}$.

Definition 2.3.2 ([10])**.** On the other side of the coin, the smallest C^*-norm we can place on $\mathfrak{A} \odot \mathfrak{B}$ is called the *spatial* or the *minimal C^*-norm* $||\cdot||_{\min}$. Let $\pi : \mathfrak{A} \to B(H)$ and $\sigma : \mathfrak{B} \to B(K)$ be faithful representations. Given $\sum a_i \otimes b_i \in \mathfrak{A} \odot \mathfrak{B}$, define

$$||\sum a_i \otimes b_i||_{\min} := ||\sum \pi(a_i) \otimes \sigma(b_i)||_{B(H \otimes K)}$$

where tensor products of Hilbert spaces and of representations can be made precise. The completion of $\mathfrak{A} \odot \mathfrak{B}$ with respect to $||\cdot||_{\min}$ is denoted $\mathfrak{A} \otimes_{\min} \mathfrak{B}$.

Example 2.3.3 ([10])**.** For any $n \in \mathbb{N}$ and any C^*-algebra $\mathfrak{A}$ we have that $\mathfrak{A} \otimes_{\min} \mathbb{M}_n \cong \mathfrak{A} \otimes_{\max} \mathbb{M}_n \cong \mathbb{M}_n(\mathfrak{A})$.

When dealing with tensor products of von Neumann algebras, there are fewer choices.

Definition 2.3.4 ([10])**.** Let $M \subset B(H)$ and $N \subset B(K)$ be von Neumann algebras. The *von Neumann algebraic tensor product* $M \overline{\otimes} N$ is defined to be the von Neumann algebra generated by

$$M \otimes N := \left\{ \sum_{i=1}^{n} x_i \otimes y_i : n \in \mathbb{N}, x_i \in M, y_i \in M \right\} \subset B(H \otimes K)$$

where the tensor product of two operators on the tensor product of two Hilbert spaces can be made precise.

We will primarily be dealing in tensor products of separable II_1-factors. Given two II_1-factors N_1 and N_2, we can describe $N_1 \overline{\otimes} N_2$ as follows. Let

$$(N_1 \odot N_2)_{\leq r} := \{x \in N_1 \odot N_2 \subset B(H \otimes K) : ||x|| \leq r\}.$$

Then we have

$$N_1 \overline{\otimes} N_2 = \cup_{r=1}^{\infty} \overline{(N_1 \odot N_2)_{\leq r}}^{||\cdot||_2}$$

where the trace on $N_1 \odot N_2$ is given by $\tau = \tau_1 \otimes \tau_2$ where τ_i is the unique tracial state on $N_i, i = 1, 2$.

This convex structure presented in this thesis makes use of a class of II_1-factors called *McDuff* II_1-factors. Before giving a definition of a McDuff II_1-factor, we need a preliminary definition.

Definition 2.3.5 ([29])**.** Let N be a II_1-factor. Let $[x, y] := xy - yx$ denote the commutator of x and y. A bounded sequence $\{t_k\} \subset N$ is called a *central sequence* in N if

$$||[t_k, x]||_2 \to 0$$

for every $x \in N$. A central sequence $\{x_k\} \subset N$ is called a *hypercentral sequence* in N if for every central sequence $\{t_k\}$,

$$||[s_k, t_k]||_2 \to 0.$$

Definition 2.3.6 ([29])**.** A II_1-factor M is called *McDuff* if M contains a central sequence that is not hypercentral.

Example 2.3.7. Let $\{(N_n, \tau_n)\}$ be a sequence of II_1-factors. The infinite tensor product $M := \otimes_{n=1}^{\infty} N_n$ is a McDuff II_1-factor. To see this, we must find a non-hypercentral central sequence. Let $a_n, b_n \in N_n$ be contractions such that $||[a_n, b_n]||_2 \geq \frac{1}{2}$ for every n. Then clearly,

$$\left\{1_{N_1} \otimes \cdots \otimes 1_{N_{n-1}} \otimes a_n \otimes 1_{N_{n+1}} \otimes \cdots\right\}_{n=1}^{\infty}$$

and

$$\left\{1_{N_1} \otimes \cdots \otimes 1_{N_{n-1}} \otimes b_n \otimes 1_{N_{n+1}} \otimes \cdots\right\}_{n=1}^{\infty}$$

are non-hypercentral central sequences.

In [29], McDuff proved the following celebrated theorem giving a structural characterization of McDuff II_1-factors.

Theorem 2.3.8. *[29] A separable II_1-factor M is McDuff if and only if $M \cong M \overline{\otimes} R$ where R denotes the separable hyperfinite II_1-factor.*

Remark 2.3.9. In the remainder of the thesis, we will simply use the notation $M \otimes N$ when discussing the von Neumann algebraic tensor product of II_1-factors.

2.4 Ultraproducts

In operator algebras, ultraproducts and utlrapowers play the dual roles of being useful tools for proofs and being interesting objects on their own. We will see both roles in this thesis. There are many publications regarding this subject; for a few, see [19], [20], [21], [38], [5], [29], and [12]. This section is meant to give the basic definitions needed in order to discuss ultrapowers. We will mainly be pulling from the material in Appendix A of [10].

Definition 2.4.1. Let I be a set. An *ultrafilter* on I is a nonempty family $\mathcal{U}$ of subsets of I that satisfies the following properties:

1. (nontriviality) $\emptyset \notin \mathcal{U}$;

2. (finite intersection property) if $I_0, I_1 \in \mathcal{U}$, then there is a $J \in \mathcal{U}$ such that $J \subset I_0 \cap I_1$;

3. (directedness) if $I_0 \in \mathcal{U}$ and $I_0 \subset I_1 \subset I$, then $I_1 \in \mathcal{U}$;

4. (maximality) for any $I_0 \subset I$, either $I_0 \in \mathcal{U}$ or $I \setminus I_0 \in \mathcal{U}$.

If $\mathcal{U}$ satisfies (1) and (2), it is called a *filter base.* If $\mathcal{U}$ satisfies (1), (2), and (3), it is called a *filter.*

It helps if one considers the elements of an ultrafilter $\mathcal{U}$ to be the "large" subsets of I.

Example 2.4.2.

- For any set I, the *principal ultrafilter* generated by $i_0 \in I$ is the family of all subsets which contain i_0.

- Let $I = \mathbb{N}$. The *cofinal filter base* on $\mathbb{N}$ is the collection of all subsets of the form $\{n \in \mathbb{N} : n \geq N\}$ for some $N \in \mathbb{N}$. There is a general theorem that says that given a filter base $\mathcal{U}'$ on I, there is an ultrafilter $\mathcal{U}$ on I which contains $\mathcal{U}'$. A *free* ultrafilter on $\mathbb{N}$ is an ultrafilter which contains the cofinal filter base. Free ultrafilters cannot be principal too.

Remark 2.4.3. While our definition of an ultrafilter works for general sets, in this thesis, we will exclusively discuss free ultrafilters on $\mathbb{N}$.

Ultrafilters can be used to describe convergence. Next, we define what it means to "converge along an ultrafilter."

Definition 2.4.4. Let X be a topological space, and let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$. A sequence $\{x_n\}$ in X is said to *converge along* $\mathcal{U}$ if for any open set in $A \subset X$, the set $\{n \in \mathbb{N} : x_n \in A\}$ is a member of $\mathcal{U}$. The limit point of this convergence is denoted

$$\lim_{n \to \mathcal{U}} x_n$$

or

$$\lim_{\mathcal{U}} x_n.$$

We are now ready to define the tracial ultraproduct of II_1-factors. Fix a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$, and let $\{M_n\}$ be a collection of II_1 -factors with tracial states τ_n. Let $\prod_{n \in \mathbb{N}} M_n$ denote the algebra of norm-bounded sequences (x_n) such that $x_n \in M_n$

for every $n \in \mathbb{N}$. Let $N_{\mathcal{U}}^{(2)}$ be given by

$$N_{\mathcal{U}}^{(2)} := \left\{ (x_n) \in \prod_{n\in\mathbb{N}} M_n : \lim_{n\to\mathcal{U}} ||x_n||_2 = 0 \right\}$$

where $||x_n||_2^2 = \tau_n(x_n^* x_n)$ is the respective trace norm. It can be shown that $N_{\mathcal{U}}^{(2)}$ is a norm-closed ideal of $\prod_{n\in\mathbb{N}} M_n$. So we define the *tracial ultraproduct* of $\{(M_n, \tau_n)\}$ to be

$$\prod_{\mathcal{U}} M_n := \Big(\prod_{n\in\mathbb{N}} M_n\Big)/N_{\mathcal{U}}^{(2)}.$$

If $M_n = M$ for every n, then we call $\prod_{\mathcal{U}} M_n$ the *tracial ultrapower* (or just *ultrapower* when the context is clear) of M and denote it as $M^{\mathcal{U}}$. We denote a coset $(x_n) + N_{\mathcal{U}}^{(2)}$ as $(x_n)_{\mathcal{U}}$. The ultraproduct $\prod_{\mathcal{U}} M_n$ is a II_1 factor with trace $\tau^{\mathcal{U}}((x_n)_{\mathcal{U}}) := \lim_{\mathcal{U}} \tau_n(x_n)$; and $M^{\mathcal{U}}$ is a II_1-factor with trace $\tau^{\mathcal{U}}((x_n)_{\mathcal{U}}) := \lim_{\mathcal{U}} \tau(x_n)$.

Example 2.4.5. Let $\{k(n)\}$ be an increasing sequence of natural numbers. One can follow the exact same construction as above with $M_n = \mathbb{M}_{k(n)}$ and $\tau_n = \text{tr}_{k(n)}$ (where tr_m is the unique tracial state on $\mathbb{M}_m$) to obtain the ultraproduct $\prod_{\mathcal{U}} \mathbb{M}_{k(n)}$. If R denotes the separable hyperfinite II_1-factor, it turns out that $R^{\mathcal{U}} \cong \prod_{\mathcal{U}} \mathbb{M}_{k(n)}$.

Given an element $x \in \prod_{\mathcal{U}} M_n$, a *lift* of x is a sequence $(x_n) \in \prod_{n\in\mathbb{N}} M_n$ such that $x = (x_n)_{\mathcal{U}}$. The following proposition is very useful in that it shows that we can lift certain properties of elements. This proposition follows from a theorem due to Hadwin and Li in [26] which appears in §2.6 as Theorem 2.6.15.

Proposition 2.4.6. *Let $\{(M_n, \tau_n)\}$ be a collection of II_1-factors. If $x \in \prod_{\mathcal{U}} M_n$ is **[normal, self-adjoint, positive, unitary, a projection, or a partial isometry]**, then there is a lift $(x_n) \in \prod_{n\in\mathbb{N}} M_n$ of x such that for each n, x_n is **[normal,***

self-adjoint, positive, unitary, a projection, or a partial isometry] *respectively.*

2.5 Approximate Unitary Equivalence

The notion of weak approximate unitary equivalence is central to the work in this thesis. Some publications regarding this topic are [47], [2], [24], [25], [16], [44], and [45]. We present some background on the topic in this section. Much of the initial discussion is pulled from [15].

Two operators $S, T \in B(H)$ are unitarily equivalent if there is a unitary $U \in B(H)$ such that $T = USU^*$. In this case, T and S are philosophically the same operator: if one chooses an orthonormal basis $\{\xi_i\}$ on $B(H)$, then the matrix representation of S with respect to $\{\xi_i\}$ will be exactly the same as the matrix representation of T with respect to the orthonormal basis $\{U\xi_i\}$. The *unitary orbit* of an operator $T \in B(H)$ is given by

$$\mathcal{U}(T) := \{UTU^* : U \text{ unitary}\}.$$

Unitaries in $B(H)$ encode the symmetries of H. So morally, two operators sharing a unitary orbit operate the same way on different rotations of the Hilbert space.

Unitary equivalence of operators implies the exact same observable data (evaluations of the form $\langle T\xi|\eta\rangle$) associated to those operators. In [15], the situation that two operators have the same observable data is described as "no finite set of measurements determined by vectors can distinguish the two operators." There is in fact a weaker equivalence relation on operators that implies that there is no difference between the operators in terms of observable data. We say that two operators $S, T \in B(H)$ are *approximately unitarily equivalent* (denoted $S \sim_a T$) if there is a sequence of unitary operators U_n such that

$$||T - U_n S U_n^*|| \to 0.$$

Note that being approximately unitarily equivalent is the same as sharing a norm-closed unitary orbit.

We will sometimes use the following notation. Given a unitary $U \in B(H)$, let $\mathrm{Ad}(U)$ denote the map

$$\mathrm{Ad}(U) : B(H) \to B(H)$$

given by

$$\mathrm{Ad}(U)(T) = UTU^*.$$

Example 2.5.1. Approximate unitary equivalence is a strictly weaker relation than that of unitary equivalence. Let H be a separable infinite dimensional Hilbert space. Fix an orthonormal basis $\{\xi_i\}$. Let $S \in B(H)$ be the operator with its $\{\xi_i\}$ matrix representation given by

$$S = \begin{pmatrix} 1 & & & & & \\ & \frac{1}{2} & & & & \\ & & \frac{1}{4} & & & \\ & & & \ddots & & \\ & & & & \frac{1}{2^{i-1}} & \\ & & & & & \ddots \end{pmatrix}.$$

Let $U_n \in B(H)$ be the unitary with its $\{\xi_i\}$ matrix representation given by

$$U_n = \begin{pmatrix} P_n & & & \\ & 1 & & \\ & & 1 & \\ & & & \ddots \end{pmatrix}$$

where P_n is the $n \times n$ permutation matrix given by

$$P_n = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 1 & \ddots & & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots\ 0 & 1 & 0 \end{pmatrix}.$$

Thus,

$$U_n S U_n^* = \begin{pmatrix} \frac{1}{2^{n-1}} & & & & & & \\ & 1 & & & & & \\ & & \frac{1}{2} & & & & \\ & & & \ddots & & & \\ & & & & \frac{1}{2^{n-2}} & & \\ & & & & & \frac{1}{2^{n+1}} & \\ & & & & & & \frac{1}{2^{n+2}} \\ & & & & & & & \ddots \end{pmatrix}.$$

So we see that

$$U_nSU_n^* \to T := \begin{pmatrix} 0 & & & & & \\ & 1 & & & & \\ & & \frac{1}{2} & & & \\ & & & \frac{1}{4} & & \\ & & & & \ddots & \\ & & & & & \frac{1}{2^i} \\ & & & & & & \ddots \end{pmatrix}.$$

But it is clear to see that S and T are not unitarily equivalent because S has trivial kernel and T has a one-dimensional kernel.

Let $\mathfrak{A}$ be a C^*-algebra, and let $\rho, \sigma : \mathfrak{A} \to B(H)$ be $*$-representations of $\mathfrak{A}$. We say that ρ and σ are approximately unitarily equivalent ($\rho \sim_a \sigma$) if there is a sequence of unitaries $\{U_n\} \subset B(H)$ such that for any $a \in \mathfrak{A}$,

$$\rho(a) = \lim_{n\to\infty} U_n\sigma(a)U_n^*$$

where the limit is taken in the norm topology. By changing the topology in which the convergence occurs, we can define *weak approximate unitary equivalence* as follows: ρ and σ are weakly approximately unitarily equivalent ($\rho \sim_{wa} \sigma$) if there are two sequences of unitaries $\{U_n\}$ and $\{V_n\}$ in $B(H)$ such that for every $a \in \mathfrak{A}$,

$$\sigma(a) = \text{WOT-}\lim_{n\to\infty} U_n\rho(a)U_n^*$$

and

$$\rho(a) = \text{WOT-}\lim_{n\to\infty} V_n\sigma(a)V_n^*.$$

The equivalence relations [approximate unitary equivalence] and [weak approximate unitary equivalence] are in fact equivalent to one another. This fact is a consequence of the Weyl-von Neumann-Berg-Voiculescu theorem. The general non-commutative version of the theorem first appeared in [47]. It appears as Theorem II.5.8 in [15] and reads as follows.

Theorem 2.5.2. *Let $\mathfrak{A}$ be a separable C^*-algebra, and let σ and ρ be non-degenerate representations of $\mathfrak{A}$ on a separable Hilbert space H ($\sigma(\mathfrak{A})H$ and $\rho(\mathfrak{A})H$ are dense in H). Then the following are equivalent.*

1. $\sigma \sim_a \rho$
2. $\sigma \sim_{wa} \rho$
3. $rank(\sigma(a)) = rank(\rho(a))$ *for every* $a \in \mathfrak{A}$.

Arveson provides a nice survey of these results in [2]; and Hadwin showed in [24] that this theorem holds for non-separable representations.

In the last decade or so, there has been growing interest in approximate unitary equivalence in von Neumann algebras rather than in $B(H)$. In [44], Sherman discusses the closures in various topologies of unitary orbits of normal operators in von Neumann algebras. In [16], Ding and Hadwin investigated a version of the Weyl-von Neumann-Berg-Voiculescu theorem where the target of the representations is a von Neumann algebra M instead of $B(H)$. Of course, in this context the unitaries implementing the (weak) approximate unitary equivalence of the representations must come from M rather than $B(H)$. For the Ding-Hadwin version of the Weyl-von Neumann-Berg-Voiculescu theorem, the notion of rank must be replaced with that of *M-rank*: the Murray-von Neumann equivalence class in M of the range projection. Though [16] deals in much more generality, we will discuss some of the results of the paper in the context where $\pi, \rho : \mathfrak{A} \to M$ are unital $*$-homomorphisms from a C^*-algebra $\mathfrak{A}$ to a finite factor von Neumann algebra M. One of the main questions addressed in [16] is: "What are the C^*-algebras for which the notions of equal M-rank, approximate unitary equivalence, and weak approximate unitary equivalence (or a sub-pair) are equivalent?" Before reporting some of the results from [16] we first mention that in the context of considering a finite factor von Neumann algebra as a target, the notion of M-rank can be simplified as follows.

Proposition 2.5.3 ([16])**.** *Let $\mathfrak{A}$ be a C^*-algebra, let M be a finite factor von Neumann algebra with unique tracial state τ, and let $\pi, \rho : \mathfrak{A} \to M$ be unital $*$-homomorphisms. Then $(M - \mathit{rank}) \circ \pi = (M - \mathit{rank}) \circ \rho$ if and only if $\tau \circ \pi = \tau \circ \rho$.*

A C^*-algebra $\mathfrak{A}$ is called *approximately homogeneous* or *AH* if $\mathfrak{A}$ is the C^* direct limit (see [49] or [40]) of algebras of the form $\mathbb{M}_n \otimes C(X)$. The class of AH algebras admits the following version of the Weyl-von Neumann-Berg-Voiculescu theorem.

Theorem 2.5.4 ([16])**.** *Let $\mathfrak{A}$ be a C^*-algebra, let M be a finite factor von Neumann algebra with unique tracial state τ, and let $\pi, \rho : \mathfrak{A} \to M$ be unital $*$-homomorphisms. If $\mathfrak{A}$ is AH, then the following are equivalent.*

1. $\pi \sim_a \rho$*;*

2. $\pi \sim_{wa} \rho$*;*

3. $\tau \circ \pi = \tau \circ \rho$*.*

In §2.6 we define and discuss the class of so-called nuclear C^*-algebras. Nuclear algebras are also well-behaved under this analysis in that if $\mathfrak{A}$ is nuclear, M is a finite factor von Neumann algebra with unique tracial state τ, and $\pi, \rho : \mathfrak{A} \to M$ are $*$-homomorphisms then $\pi \sim_{wa} \rho$ if and only if $\tau \circ \pi = \tau \circ \rho$ (see Theorem 2.6.10). In general, it is unknown if $\sim_a$ is the same as $\sim_{wa}$ in the context of representations of C^*-algebras in finite factor von Neumann algebras.

In this thesis, we will study weak approximate unitary equivalence of unital $*$-homomorphisms from a unital separable C^*-algebra into a separable II_1-factor von Neumann algebra N. In this context, one can define weak approximate unitary equivalence using the trace norm as follows.

Definition 2.5.5. Given a unital separable C^*-algebra $\mathfrak{A}$ and a separable II_1-factor N, two unital $*$-homomorphisms $\pi, \rho : \mathfrak{A} \to N$ are weakly approximately unitarily equivalent if there is a sequence of unitaries $\{u_n\} \subset \mathcal{U}(N)$ such that for every $a \in \mathfrak{A}$,

$$\lim_{n\to\infty} ||\pi(a) - u_n\rho(a)u_n^*||_2 = 0$$

where $||x||_2^2 = \tau(x^*x)$ for τ the unique tracial state on N. For the rest of the thesis, we will simply denote weak approximate unitary equivalence with the symbol $\sim$. We sometimes use the abbreviation w.a.u.e. for weak approximate unitary equivalence. It will be useful to keep the following equivalent formulation of this definition in mind. For $\pi, \rho : \mathfrak{A} \to N$, $\pi \sim \rho$ if and only if for every finite subset $F \subset \mathfrak{A}_{\leq 1}$ and every $\varepsilon > 0$ there is a unitary $u \in \mathcal{U}(N)$ such that

$$||\pi(a) - u\rho(a)u^*||_2 < \varepsilon$$

for every $a \in F$.

We conclude this section by discussing weak approximate unitary equivalence of unital $*$-homomorphisms into an ultraproduct of II_1-factor von Neumann algebras. The following theorem from [45] can be seen as an advertisement for working in an ultraproduct/power rather than the original root algebra.

Theorem 2.5.6 ([45])**.** *Let $\mathfrak{A}$ be a separable unital C^*-algebra, $\{(M_n, \tau_n)\}$ be a collection of II_1-factors, and $\pi, \rho : \mathfrak{A} \to \prod_{\mathcal{U}} M_n$ where $\mathcal{U}$ is a free ultrafilter. Then π and ρ are weakly approximately unitarily equivalent if and only if π and ρ are unitarily equivalent (there exists $u \in \mathcal{U}\Big(\prod_{\mathcal{U}} M_n\Big)$ such that for every $a \in \mathfrak{A}, \pi(a) = u\rho(a)u^*$).*

Remark 2.5.7. Theorem 2.5.6 was originally stated for an ultrapower $M^{\mathcal{U}}$ as the target, but the argument easily applies to the more general case with an ultraproduct as the target. This theorem says that in an ultraproduct, approximate unitary equivalence of homomorphisms on separable algebras is the same as exact unitary equivalence. This is an attractive property because it allows one to avoid any (annoying) technicalities involving the approximation arguments intrinsic to the subject of weak approximate unitary equivalence.

2.6 Amenability and W.A.U.E.

In this section, we will discuss the relationship between weak approximate unitary equivalence and operator algebras which can be nicely approximated by finite dimensional algebras. On the von Neumann algebras side of things, such algebras are called *hyperfinite*; and for C^*-algebras, such algebras are called *nuclear*. The term *amenable* is often used in place of either of these terms. We will define these properties and discuss how they relate to weak approximate unitary equivalence. The results discussed in this section are well-known. We record them here for the sake of completeness.

Let us first give a definition of a hyperfinite von Neumann algebra.

Definition 2.6.1. A separably acting von Neumann algebra M is called *hyperfinite* if there is an ascending sequence $A_1 \subseteq A_2 \subseteq \cdots \subseteq M$ of finite dimensional subalgebras such that their union $\cup A_n$ is weakly dense in M. (For non separably acting von Neumann algebras, replace "sequence" with "net.")

Thanks largely to the celebrated 1976 paper [13] by Connes, this property is equivalent to the following properties:

- M is injective: any completely positive map from a unital self-adjoint closed subspace of a unital C^*-algebra $\mathfrak{A}$ to M can be extended to a completely positive map from $\mathfrak{A}$ to M. (see [34])

- M has property P of Schwartz: Say M acts on the Hilbert space H. For any $T \in B(H)$, the weak closure of the convex hull of the unitary orbit of T contains an element of M'. (see [43])

- M is semi-discrete: the identity map $\mathrm{id}_M : M \to M$ is a weak pointwise limit of maps that factor through finite dimensional algebras. (see [17])

Recall that an *embedding* is a unital trace-preserving injective $*$-homomorphism. Let $\mathbb{M}_n$ denote the algebra of $n \times n$ matrices with complex entries. This first lemma is fundamental to the relationship between amenability in operator algebras and weak approximate unitary equivalence.

Lemma 2.6.2. *Let N be a II_1-factor. For any $n \in \mathbb{N}$, any two embeddings $\pi, \rho : \mathbb{M}_n \to N$ are unitarily equivalent.*

Proof. We must show that there is a unitary $u \in \mathcal{U}(N)$ such that for every $a \in \mathbb{M}_n, \pi(a) = u\rho(a)u^*$. For $1 \leq i, j \leq n$, let e_{ij} denote the matrix unit with a 1 in the ij-entry and zeros everywhere else. Consider $\pi(e_{11})$, a projection of trace $\dfrac{1}{n}$ in N. The projection $\rho(e_{11})$ also has trace $\frac{1}{n}$. Since the value of the trace completely determines Murray-von Neumann equivalence classes of projections in a II_1-factor, we have that $\pi(e_{11}) \sim_{\text{MvN}} \rho(e_{11})$. Let $v_1 \in N$ be a partial isometry such that

$$v_1^* v_1 = \pi(e_{11})$$

and

$$v_1 v_1^* = \rho(e_{11}).$$

We will now define partial isometries $v_2, \ldots v_n \in N$ with $v_k^* v_k = \pi(e_{kk})$ and $v_k v_k^* = \rho(e_{kk})$ for every $2 \leq k \leq n$. For every $2 \leq k \leq n$, put

$$v_k := \rho(e_{k1}) v_1 \pi(e_{1k}).$$

Then

$$\begin{aligned} v_k^* v_k &= \pi(e_{k1}) v_1^* \rho(e_{1k} e_{k1}) v_1 \pi(e_{1k}) \\ &= \pi(e_{k1}) v_1^* \rho(e_{11}) v_1 \pi(e_{1k}) \\ &= \pi(e_{k1}) v_1^* v_1 v_1^* v_1 \pi(e_{1k}) \\ &= \pi(e_{k1} e_{11} e_{1k}) \\ &= \pi(e_{kk}); \end{aligned}$$

and

$$\begin{aligned} v_k v_k^* &= \rho(e_{k1}) v_1 \pi(e_{1k} e_{k1}) v_1^* \rho(e_{1k}) \\ &= \rho(e_{k1}) v_1 \pi(e_{11}) v_1^* \rho(e_{1k}) \\ &= \rho(e_{k1}) v_1 v_1^* v_1 v_1^* \rho(e_{1k}) \\ &= \rho(e_{k1} e_{11} e_{1k}) \\ &= \rho(e_{kk}). \end{aligned}$$

Now set

$$u := \sum_{k=1}^{n} v_k^*.$$

It is an easy observation that u is a unitary in N. It will suffice to show the unitary equivalence on matrix units, because the matrix units generate $\mathbb{M}_n$. Fix $1 \leq i, j \leq n$. Then

$$\begin{aligned} u\rho(e_{ij})u^* &= \Big(\sum_{k=1}^{n} v_k^*\Big)\rho(e_{ij})\Big(\sum_{k=1}^{n} v_k\Big) \\ &= v_i^*\rho(e_{ij})v_j \\ &= \pi(e_{i1})v_1^*\rho(e_{1i}e_{ij}e_{j1})v_1\pi(e_{1j}) \\ &= \pi(e_{i1})v_1^*\rho(e_{11})v_1\pi(e_{1j}) \\ &= \pi(e_{i1})v_1^*v_1v_1^*v_1\pi(e_{1j}) \\ &= \pi(e_{i1}e_{11}e_{1j}) \\ &= \pi(e_{ij}). \end{aligned}$$

□

From Lemma 2.6.2, we get some nice consequences fairly quickly.

Proposition 2.6.3. *Let A be a finite dimensional von Neumann algebra, let N be a II_1-factor, and let $\pi, \rho : A \to N$ be $*$-homomorphisms. Then the following are equivalent.*

1. *π and ρ are unitarily equivalent;*
2. *$\tau \circ \pi = \tau \circ \rho$.*

Proof. This follows from the facts that (1) any finite dimensional von Neumann algebra is a finite direct sum of matrix algebras and (2) for any projection $p \in N$, the corner algebra pNp is still a II_1-factor. □

Proposition 2.6.4. *Let M be a separable finite hyperfinite von Neumann algebra, let N be a II_1-factor, and let $\pi, \rho : M \to N$ be $*$-homomorphisms. Then the following are equivalent.*

1. *π and ρ are weakly approximately unitarily equivalent;*

2. *$\tau \circ \pi = \tau \circ \rho$.*

Proof. ($1 \Rightarrow 2$): The fact that the trace is $||\cdot||_2$-continuous and ignores unitary conjugation makes this implication obvious.

($2 \Rightarrow 1$): We will use the finite-subset formulation of weak approximate unitary equivalence to prove this direction. Fix $\varepsilon > 0$, and let $x_1, \ldots, x_n \in M$. Because M is hyperfinite, there is a finite dimensional von Neumann subalgebra $A \subset M$ such that there are elements $y_1, \ldots, y_n \in A$ with

$$||\pi(x_k) - \pi(y_k)||_2 < \frac{\varepsilon}{2}$$

and

$$||\rho(x_k) - \pi(y_k)||_2 < \frac{\varepsilon}{2}$$

for every $1 \leq k \leq n$.

Consider the homomorphisms

$$\pi|_A : A \to N$$

and

$$\rho|_A : A \to N.$$

By assumption, $\tau \circ \pi|_A = \tau \circ \rho|_A$. So by Proposition 2.6.3, there is a unitary $u \in N$ such that $\pi|_A = \mathrm{Ad}(u) \circ \rho|_A$. Thus we have for every $1 \leq k \leq n$,

$$\begin{aligned} ||\pi(x_k) - u\rho(x_k)u^*||_2 &\leq ||\pi(x_k) - \pi(y_k)||_2 + ||\pi(y_k) - u\rho(y_k)u^*||_2 \\ &+ ||u\rho(y_k)u^* - u\rho(x_k)u^*||_2 \\ &= ||\pi(x_k) - \pi(y_k)||_2 + ||\pi(y_k) - u\rho(y_k)u^*||_2 \\ &+ ||\rho(y_k) - \rho(x_k)||_2 \\ &= ||\pi(x_k) - \pi(y_k)||_2 + ||\rho(y_k) - \rho(x_k)||_2 \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

□

This gives the following corollary.

Corollary 2.6.5. *Any unital endomorphism $\pi : R \to R$ is approximately inner (i.e. weakly approximately unitarily equivalent to the identity endomorphism).*

Example 2.6.6. It is easy to see that the II_1-factor $R \otimes R$ is hyperfinite. So by uniqueness, we have that $R \otimes R \cong R$. Let $\epsilon : R \otimes R \to R$ be an isomorphism. Consider the map $\mathrm{id}_R \otimes 1_R : R \to R \otimes R$ given by $\mathrm{id}_R \otimes 1_R(x) = x \otimes 1_R$. By Corollary 2.6.5, we have that $\epsilon \circ (\mathrm{id}_R \otimes 1_R) \sim \mathrm{id}_R$. The *relative commutant* of a homomorphism $\pi : A \to B$ is given by $\{b \in B : b\pi(a) = \pi(a)b, \ \forall a \in A\}$ and is denoted by $\pi(A)' \cap B$. It is clear that if π and ρ are unitarily equivalent, then π and ρ have isomorphic (via a unitary) relative commutants. This example shows that w.a.u.e does not preserve the isomorphism class of relative commutants. Indeed, because R is a factor, we have that $\mathrm{id}_R(R)' \cap R \cong \mathbb{C}$, but $(\epsilon \circ (\mathrm{id}_R \otimes 1_R))(R)' \cap R \cong R$.

A finite tracial von Neumann algebra N is called *embeddable* if there exists an embedding (unital, trace-preserving, injective $*$-homomorphism) $\pi : N \to R^{\mathcal{U}}$. In [27], Jung gave a fundamental characterization of a separable tracial finite hyperfinite embeddable von Neumann algebra. We will discuss this characterization in §4.3, but it is worth mentioning here.

Theorem 4.3.1 ([27]). *Let N be a separable tracial finite embeddable von Neumann algebra. The following are equivalent.*

1. *N is hyperfinite;*

2. *any two embeddings $\pi, \rho : N \to R^{\mathcal{U}}$ are unitarily equivalent.*

In §4.3, we show how we can rephrase this characterization in a separable context. In gaining separability, we must pass to the weaker equivalence relation of weak approximate unitary equivalence.

Theorem 4.3.2. *Let N be a separable tracial finite embeddable von Neumann algebra. The following are equivalent.*

1. *N is hyperfinite;*

2. *for any McDuff II_1-factor M, any two embeddings $\pi, \rho : N \to M$ are weakly approximately unitarily equivalent.*

Now we move to discuss amenability in the context of C^*-algebras.

Definition 2.6.7. [10] A C^*-algebra $\mathfrak{A}$ is *nuclear* if there is a sequence of matrix algebras $\mathbb{M}_{k(n)}$ and completely positive maps $\varphi_n : \mathfrak{A} \to \mathbb{M}_{k(n)}$ and $\psi_n : \mathbb{M}_{k(n)} \to \mathfrak{A}$ such that $\psi_n \circ \varphi_n$ approximates the identity $\mathrm{id}_{\mathfrak{A}} : \mathfrak{A} \to \mathfrak{A}$. That is, for every $a \in \mathfrak{A}$,

$$\lim_{n\to\infty} ||\psi_n \circ \varphi_n(a) - a|| = 0.$$

This can be interpreted as the C^*-version of the semidiscrete property mentioned above. The original definition of nuclearity is in the context of tensor products. A C^*-algebra $\mathfrak{A}$ is nuclear if for any other C^*-algebra $\mathfrak{B}$,

$$\mathfrak{A} \otimes_{\min} \mathfrak{B} \cong \mathfrak{A} \otimes_{\max} \mathfrak{B}.$$

There is also a C^*-version of hyperfiniteness called *approximately finite dimensional* or AFD ($\mathfrak{A}$ is AFD if it is the norm closure of an increasing union of finite dimensional subalgebras). Unlike the von Neumann case, being AFD and being nuclear are not equivalent. The class of nuclear algebras strictly contains the class of AFD algebras.

The following lemma is a useful characterization of nuclearity.

Lemma 2.6.8 ([10])**.** *Let $\mathfrak{A}$ be a C^*-algebra. The following are equivalent.*

1. *$\mathfrak{A}$ is nuclear;*

2. *$\mathfrak{A}^{**}$ is semidiscrete (as a von Neumann algebra);*

3. *$\mathfrak{A}^{**}$ is hyperfinite (as a von Neumann algebra).*

Proposition 2.6.9. *Let $\mathfrak{A}$ be a separable unital nuclear algebra, and let T be a tracial state on $\mathfrak{A}$. Let π_T be the induced GNS representation associated with T. Then $\pi_T(\mathfrak{A})''$ is hyperfinite.*

Proof. Let $M = \pi_T(\mathfrak{A})''$. By Lemma III.2.2 of [46], the double dual $\mathfrak{A}^{**}$ surjects onto M. Since $\mathfrak{A}$ is nuclear, we get that $\mathfrak{A}^{**}$ is hyperfinite. It is an easy exercise to show that the homomorphic image of a hyperfinite von Neumann algebra is itself hyperfinite. So M is hyperfinite. □

In particular, Lemma 2.6.8 and Proposition 2.6.9 can be used to prove the following theorem from [16] relating traces on nuclear C^*-algebras to weak approximate unitary equivalence–a major piece the argument for Theorem 4.1.2 from §4.1. The proof we present for the following theorem is different from the one appearing in [16].

Theorem 2.6.10 ([16])**.** *Let $\mathfrak{A}$ be a separable unital nuclear C^*-algebra and let N be a separable II_1-factor. If $\pi, \rho : \mathfrak{A} \to N$ are unital $*$-homomorphisms, then $\tau \circ \pi = \tau \circ \rho$ if and only if π and ρ are weakly approximately unitarily equivalent.*

Proof. As argued before, the reverse implication is obvious.

($\Rightarrow$): Consider the algebras $W^*(\pi(\mathfrak{A}))$ and $W^*(\rho(\mathfrak{A}))$. Since $\mathfrak{A}$ is nuclear, both of these algebras are hyperfinite by Proposition 2.6.9. The assumption that $\tau \circ \pi = \tau \circ \rho$ gives that the map

$$\varphi : W^*(\pi(\mathfrak{A})) \to W^*(\rho(\mathfrak{A}))$$

SOT-densely defined by

$$\varphi(\pi(a)) = \rho(a), a \in \mathfrak{A}$$

is a well-defined $*$-isomorphism. So we have the maps $\mathrm{id}_{W^*(\pi(\mathfrak{A}))}, \varphi : W^*(\pi(\mathfrak{A})) \to N$ are such that $\tau \circ \mathrm{id}_{W^*(\pi(\mathfrak{A}))} = \tau \circ \varphi$. And by Propostion 2.6.4, since $W^*(\pi(\mathfrak{A}))$ is hyperfinite, we have that $\mathrm{id}_{W^*(\pi(\mathfrak{A}))}$ and φ are weakly approximately unitarily equivalent. It then follows that $\pi \sim \rho$. □

The main property of nuclear algebras at play in this discussion is the fact that all of their traces give hyperfinite GNS constructions. Consider the following definition.

Definition 2.6.11 (Definition 3.2.1, [7])**.** A trace $T \in T(\mathfrak{A})$ is called *uniform amenable* if there exists a sequence of unital completely positive maps $\varphi_n : \mathfrak{A} \to \mathbb{M}_{k(n)}$ such that

$$\lim_n ||\varphi_n(ab) - \varphi_n(a)\varphi_n(b)||_2 = 0$$

for all $a, b \in \mathfrak{A}$, and

$$\lim_n ||T - \mathrm{tr}_{k(n)} \circ \varphi_n||_{\mathfrak{A}^*} = 0$$

where $|| \cdot ||_{\mathfrak{A}^*}$ is the natural norm on the dual of $\mathfrak{A}$. Let $\mathrm{UAT}(\mathfrak{A})$ denote the set of all such traces.

We have the following fact about uniformly amenable traces thanks to Theorem 3.2.2 of [7].

Theorem 2.6.12 ([7]). *Let $\mathfrak{A}$ be a separable unital C^*-algebra, and let T be a tracial state on $\mathfrak{A}$. Let π_T be the induced GNS representation associated with T. Then $T \in UAT(\mathfrak{A})$ if and only if $\pi_T(\mathfrak{A})''$ is hyperfinite.*

So according to Theorem 2.6.12, the result of Thorem 2.6.10 applies to any algebra $\mathfrak{A}$ such that $T(\mathfrak{A}) = \text{UAT}(\mathfrak{A})$.

Example 2.6.13. The class of algebras for which $T(\mathfrak{A}) = \text{UAT}(\mathfrak{A})$ is strictly larger than the class of nuclear algebras. Dadarlat's example of a non-nuclear subalgebra of an AF-algebra in [14] is an example of a non-nuclear algebra whose tracial GNS representations are hyperfinite.

Because weak approximate unitary equivalence becomes exact unitary equivalence in an ultrapower (when considering a separable subalgebra), we can write an ultrapower version of Theorem 2.6.10. The following theorem appears in [45] and is an immediate corollary of Theorems 2.5.6 and 2.6.10.

Theorem 2.6.14 ([45]). *Let $\mathfrak{A}$ be a separable unital C^*-algebra such that $T(\mathfrak{A}) = UAT(\mathfrak{A})$, and let N be a separable II_1-factor. If $\pi, \rho : \mathfrak{A} \to N^{\mathcal{U}}$ are unital $*$-homomorphisms, then $\tau \circ \pi = \tau \circ \rho$ if and only if π and ρ are unitarily equivalent.*

The following theorem originally appeared in Hadwin and Li's paper [26] and naturally follows Theorem 2.6.14 in that Theorem 2.6.14 can be used to give a proof much more concise than the original.

Theorem 2.6.15 (Theorem 4.10, [26]). *Let $\{M_i\}$ be a collection of II_1-factors with traces τ_i. If $\mathfrak{A}$ is either countably generated hyperfinite von Neumann algebra or a separable unital C^*-algebra such that $T(\mathfrak{A}) = UAT(\mathfrak{A})$, then for any unital $*$-homomorphism*

$$\pi : \mathfrak{A} \to \prod_{\mathcal{U}} M_i$$

there exist unital $*$-homomorphisms $\pi_i : \mathfrak{A} \to M_i$ such that for every $a \in \mathfrak{A}$,

$$\pi(a) = (\pi_i(a))_{\mathcal{U}}$$

and

$$\tau_{\mathcal{U}} \circ \pi = \tau_i \circ \pi_i$$

for every i where $\tau_{\mathcal{U}}$ denotes the trace on the ultraproduct.

In particular, homomorphisms from separable nuclear C^*-algebras into ultraproducts of II_1-factors lift to coordinate-wise $*$-homomorphisms.

Proof. Let $\pi : \mathfrak{A} \to \prod_{\mathcal{U}} M_i$ be given. Let $T = \tau_{\mathcal{U}} \circ \pi$. Since T induces a hyperfinite GNS construction, we can find unital $*$-homomorphisms $\rho_i : \mathfrak{A} \to M_i$ so that $T = \tau_i \circ \rho_i$. By uniqueness of GNS constructions we have that π and $(\rho_i)_{\mathcal{U}}$ both have hyperfinite images then using Theorem 2.6.14, we get that π and $(\rho_i)_{\mathcal{U}}$ are unitarily equivalent. Let u be a unitary in $\prod_{\mathcal{U}} M_i$ such that $\pi = u(\rho_i)_{\mathcal{U}}u^*$. We may write $u = (u_i)_{\mathcal{U}}$ where each u_i is a unitary in M_i. Then we have $\pi = (u_i\rho_i u_i^*)_{\mathcal{U}}$. So put $\pi_i = u_i\rho_i u_i^*$, and we are done. □

2.7 Convexity

In this section, we will briefly review some concepts and results concerning convexity in a functional analytic setting. Most of this material can be found in [37].

Definition 2.7.1. 1. Let V be a linear space (e.g.vector space, topological vector space, Banach space, Hilbert space, C^*-algebra, von Neumann algebra, ...). A subset $C \subset V$ is called *convex* if for any $t \in [0,1], x, y \in C$ we have $tx+(1-t)y \in C$. In plain English, a set C is convex if any average of any two elements in C remains in C.

2. Let C be a convex subset of a linear space V. A convex subset F of C is called a *face* if for any $t \in (0,1), tx + (1-t)y \in F$ implies that $x, y \in F$.

3. Let C be a convex subset of a linear space V. An element $z \in C$ is called an *extreme point* if $\{z\}$ is a face of C. That is, for any $t \in (0,1), tx + (1-t)y = z \Rightarrow x = y = z$. The set of extreme points of C is sometimes denoted as $\partial_e(C)$.

4. Given a subset S of V, the *convex hull* of S is the smallest convex subset of V that contains S, denoted $\text{conv}(S)$. It can be obtained by taking the collection of all convex combinations of elements from S. If there is a topology on V, it is sometimes useful to consider the closed convex hull $\overline{\text{conv}(S)}$ of S in V.

5. Let C, D be two convex sets. A map $\varphi : C \to D$ is called *affine* if for any $t \in [0,1], x, y \in C, \varphi(tx + (1-t)y) = t\varphi(x) + (1-t)\varphi(y)$. Affine maps preserve

convex combinations.

Example 2.7.2. 1. A solid disc is convex, but just the boundary circle is not. The real line $\mathbb{R}$ is convex.

2. Any edge of a solid square is a face. The diagonal line connecting two non-adjacent corners of a square is not a face. The real line $\mathbb{R}$ as a subset of itself is a face.

3. Any corner of a solid square is an extreme point. Any point on the boundary circle of a solid disc is an extreme point. The real line $\mathbb{R}$ has no extreme points.

4. In $\mathbb{R}^2$, the convex hull of $(0,0),(1,0)$, and $(0,1)$ is a solid right triangle.

5. Let C be the closed line segment in $\mathbb{R}^2$ between $(0,0)$ and $(1,0)$, and let D be the closed line segment in $\mathbb{R}^2$ between $(0,1)$ and $(1,3)$. Then the map $\varphi : C \to D$ given by $\varphi(x,0) = (x, 2x+1)$ for $x \in [0,1]$ is affine.

Naturally, as the linear space increases in complexity, studying the convex geometry of its convex subsets can become more difficult–faces and extreme points become harder to find and identify. The following theorem is invaluable when exploring convexity in the context of functional analysis.

Theorem 2.7.3 (Krein-Milman Theorem, [37])**.** *Let V be a topological vector space, and let K be a compact convex subset of V. Then $K = \overline{conv(\partial_e(K))}$. That is, a compact convex subset is the closed convex hull of its extreme points.*

This theorem is true in more generality than the version presented here, but the present version is sufficient for the purposes of this thesis. One of the major consequences of the Krein-Milman Theorem is that if K is a compact convex subset, then K is guaranteed to have extreme points.

Example 2.7.4. Let $\mathfrak{A}$ be a unital C^*-algebra. The space of tracial states $T(\mathfrak{A})$ is convex and compact under an appropriate topology. Thus, by the Krein-Milman Theorem, $T(\mathfrak{A})$ is the closed convex hull of its extreme points. In particular, if $T(\mathfrak{A})$ is nonempty, then there exist extreme tracial states. It can be shown that an extreme tracial state T gives rise to a GNS representation π_T such that $\pi_T(\mathfrak{A})''$ is a factor.

2.8 Survey of $\mathbb{H}\mathrm{om}(N, R^{\mathcal{U}})$

Let N be a separable embeddable II_1-factor. We let $\mathbb{H}\mathrm{om}(N, R^{\mathcal{U}})$ denote the set of $*$-homomorphisms $\pi : N \to R^{\mathcal{U}}$ modulo unitary equivalence. Given a $*$-homomorphism $\pi : N \to R^{\mathcal{U}}$, we let $[\pi]$ denote the unitary equivalence class of π. In [8], it was shown that a convex structure can be placed on $\mathbb{H}\mathrm{om}(N, R^{\mathcal{U}})$. That is, one can take averages of unitary equivalence classes of embeddings of N into $R^{\mathcal{U}}$. The purpose of this section is to go over the construction of and basic properties concerning the convex structure on $\mathbb{H}\mathrm{om}(N, R^{\mathcal{U}})$ established in [8].

The topology of $\mathbb{H}\mathrm{om}(N, R^{\mathcal{U}})$ can be described as pointwise $||\cdot||_2$-convergence along representatives. More precisely, $[\pi_n] \to [\pi]$ in $\mathbb{H}\mathrm{om}(N, R^{\mathcal{U}})$ if there are representatives

$\pi_n' \in [\pi_n]$ such that for every $x \in N$, $||\pi_n'(x) - \pi(x)||_2 \to 0$. This can be metrized as follows. Let $\{x_n\}$ be a generating subset of the unit ball of N and define the metric d on $\mathbb{H}\mathrm{om}(N, R^{\mathcal{U}})$ to be

$$d([\pi],[\rho]) = \inf_{u \in \mathcal{U}(R^{\mathcal{U}})} \Big(\sum_{n=1}^{\infty} \frac{1}{2^{2n}} ||\pi(x_n) - u\rho(x_n)u^*||_2^2 \Big)^{\frac{1}{2}}.$$

In the appendix of [8], it is shown that if $N \ncong R$ then $\mathbb{H}\mathrm{om}(N, R^{\mathcal{U}})$ is not second countable in this topology.

We start with a technical proposition that is fundamental in the structure and analysis of $\mathbb{H}\mathrm{om}(N, R^{\mathcal{U}})$. The following proposition appears as Proposition 3.1.2 of [8].

Proposition 2.8.1. *Let $p, q \in R^{\mathcal{U}}$ be projections of the same trace, $M \subset pR^{\mathcal{U}}p$ be a* separable *von Neumann subalgebra and $\varphi : pR^{\mathcal{U}}p \to qR^{\mathcal{U}}q$ be a unital $*$-homomorphism. Assume there exist projections $p_i, q_i \in R, i \in \mathbb{N}$ with $\tau(p_i) = \tau(q_i) = \tau(p)$ for every $i \in \mathbb{N}$ such that $(p_i)_{\mathcal{U}} = p$ and $(q_i)_{\mathcal{U}} = q$, and there exist $*$-homomorphisms $\varphi_i : p_iRp_i \to q_iRq_i$ such that $\varphi = (\varphi_i)_{\mathcal{U}}$. Then there exists a partial isometry $v \in R^{\mathcal{U}}$ with $v^*v = p$ and $vv^* = q$ such that $\varphi(x) = vxv^*$ for every $x \in M$.*

This proposition will be used several times in Chapter 8. We will say that such a homomorphism φ lifts to fiberwise or coordinatewise homomorphisms. Let $\sigma : R \otimes R \to R$ be an isomorphism, and to allow an abuse of notation, let $\sigma : (R \otimes R)^{\mathcal{U}} \to R^{\mathcal{U}}$ also denote the induced isomorphism between ultrapowers. As a consequence of

Proposition 2.8.1, we get the following fact.

Proposition 2.8.2. *Let* $\pi : N \to R^{\mathcal{U}}$ *be given. Then* $[\pi] = [\sigma(1 \otimes \pi)]$ *where* $\sigma(1 \otimes \pi)(x) = \sigma(1 \otimes \pi(x))$.

The following definition provides one of the ingredients for the convex structure on $\mathbb{H}\text{om}(N, R^{\mathcal{U}})$.

Definition 2.8.3. Let $p \in R^{\mathcal{U}}$ be a projection such that $p = (p_i)_{\mathcal{U}}$ where p_i is a projection in R with $\tau(p_i) = \tau(p)$ for each $i \in \mathbb{N}$. An isomorphism $\theta_p : pR^{\mathcal{U}}p \to R^{\mathcal{U}}$ is called a *standard isomorphism* if it lifts to coordinate-wise isomorphisms $p_iRp_i \to R$.

Before exhibiting a convex structure on $\mathbb{H}\text{om}(N, R^{\mathcal{U}})$, Brown had to establish in [8] what it means to have a convex structure outside of the context of linear space. Brown gave five axioms in Definition 2.1 of [8] that should be expected of a bounded convex subset of a linear space.

Definition 2.8.4. [8] If X is a complete bounded metric space, then X has a *convex-like structure* if

1. (commutativity) $t_1x_1 + \cdots + t_nx_n = t_{\alpha(1)}x_{\alpha(1)} + \cdots + t_{\alpha(n)}x_{\alpha(n)}$ for every permutation $\alpha \in S_n$.

2. (linearity) if $x_1 = x_2$ then $t_1x_1 + t_2x_2 + t_3x_3 + \cdots + t_nx_n = (t_1 + t_2)x_1 + t_3x_3 + \cdots + t_nx_n$.

3. (scalar identity) if $t_i = 1$ then $t_1x_1 + \cdots + t_nx_n = x_i$.

4. (metric compatibility) $d((t_1x_1 + \cdots + t_nx_n), (t'_1x_1 + \cdots + t'_nx_n)) \leq C \sum |t_i - t'_i|$ and $d((t_1x_1 + \cdots + t_nx_n), (t_1y_1 + \cdots + t_ny_n)) \leq \sum t_i d(x_i, y_i)$.

5. (algebraic compatibility)

$$s\Big(\sum_{i=1}^{n} t_i x_i\Big) + (1-s)\Big(\sum_{j=1}^{m} t''_j z_j\Big) = \sum_{i=1}^{n} s t_i x_i + \sum_{j=1}^{m} (1-s) t''_j z_j$$

where $x_1, \ldots, x_n, y_1, \ldots, y_n, z_1, \ldots, z_m \in X$ and $0 \leq t_1, \ldots, t_n, t'_1, \ldots, t'_n, t''_1, \ldots, t''_m \leq 1$ with $\sum_{i=1}^{n} t_i = \sum_{i=1}^{n} t'_i = \sum_{i=1}^{m} t''_i = 1$.

Remark 2.8.5. The notation using the "+"sign of course needs to be made precise because a priori, X may not have any notion of a "sum."To do so, let X^n be the n-fold Cartesian product and let Δ_n denote the set of probability measures on the n-point set $\{1, \ldots, n\}$ with the ℓ_1-metric. Then for each $n \in \mathbb{N}$ and $\mu \in \Delta_n$, there is a continuous map $\gamma_\mu : X^n \to X$ that satisfies the above axioms when we set the notation as

$$\gamma_\mu(x_1, \ldots, x_n) =: \mu(1)x_1 + \mu(2)x_2 + \cdots + \mu(n)x_n.$$

In [11], Capraro and Fritz showed that closed bounded convex subsets of Banach spaces are characterized by these axioms defining a convex-like structure. That is, any complete bounded metric space with a convex-like structure can be realized as a closed bounded convex subset of a Banach space.

We are now ready to define convex combinations in $\mathbb{H}\mathrm{om}(N, R^{\mathcal{U}})$.

Definition 2.8.6. Given $[\pi_1], \ldots, [\pi_n] \in \mathbb{H}\mathrm{om}(N, R^{\mathcal{U}})$ and $0 \leq t_1, \ldots, t_n \leq 1$ with $\sum t_i = 1$, we define

$$t_1[\pi_1] + \cdots + t_n[\pi_n] := [\theta_{p_1}^{-1} \circ \pi_1 + \cdots + \theta_{p_n}^{-1} \circ \pi_n]$$

where $\tau(p_i) = t_i$ for every $1 \leq i \leq n$ and θ_{p_i} is a standard isomorphism.

Remark 2.8.7. Thanks to Proposition 2.8.1, this operation is well-defined. Proposition 2.8.1 can also be used to show that

$$t_1[\pi_1] + \cdots t_n[\pi_n] = [\sigma(p_1 \otimes \pi_1) + \cdots + \sigma(p_n \otimes \pi_n)]$$

where $p_1, \ldots, p_n$ are projections with traces $t_1, \ldots, t_n$ respectively and $\sigma \circ (p_k \otimes \pi_k)(x) = \sigma \circ (p_k \otimes \pi_k(x))$.

Theorem 2.8.8 ([8])**.** $(\mathbb{H}om(N, R^{\mathcal{U}}), d)$ *is a complete metric space with a convex-like structure with convex combinations defined as in Definition 2.8.6*

The next concept is very useful in studying the convex geometry of $\mathbb{H}\mathrm{om}(N, R^{\mathcal{U}})$.

Definition 2.8.9. Let $\pi : N \to R^{\mathcal{U}}$ be given. For a projection $p \in \pi(N)' \cap R^{\mathcal{U}}$, we define the *cutdown of* π *by* p to be the map π_p given by $\pi_p(x) = \theta_p(p\pi(x))$ where θ_p is a standard isomorphism. It can be shown that $[\pi_p]$ is independent of the choice of the standard isomorphism.

The following proposition records important facts regarding this operation of taking a cutdown.

Proposition 2.8.10 ([8])**.** *Let $\pi : N \to R^{\mathcal{U}}$ be given.*

1. *Let $p \in \pi(N)' \cap R^{\mathcal{U}}$ be a projection. If $u \in R^{\mathcal{U}}$ is a unitary, then*

$$[\pi_p] = [(Ad(u) \circ \pi)_{upu^*}].$$

2. *For any projection $p \in R^{\mathcal{U}}$,*

$$[\pi] = [\sigma(1 \otimes \pi)_{\sigma(p \otimes 1)}].$$

3. (a) *Given any $p \in \pi(N)' \cap R^{\mathcal{U}}$, $[\pi] = \tau(p)[\pi_p] + \tau(p^{\perp})[\pi_{p^{\perp}}]$.*

   (b) *If $[\pi] = t[\rho_1] + (1-t)[\rho_2]$ then there is a projection $p \in \pi(N)' \cap R^{\mathcal{U}}$ with trace t such that $[\rho_1] = [\pi_p]$ and $[\rho_2] = [\pi_{p^{\perp}}]$.*

4. *Let $p, q \in \pi(N)' \cap R^{\mathcal{U}}$ be projections with the same trace. Then the following are equivalent.*

   (a) $[\pi_p] = [\pi_q]$

   (b) *p and q are Murray-von Neumann equivalent in $\pi(N)' \cap R^{\mathcal{U}}$.*

With Proposition 2.8.10 established, we can now present Brown's characterization of extreme points in $\mathbb{H}\text{om}(N, R^{\mathcal{U}})$.

Theorem 2.8.11 ([8])**.** *Given $\pi : N \to R^{\mathcal{U}}$, $[\pi]$ is extreme in $\mathbb{H}om(N, R^{\mathcal{U}})$ if and only if $\pi(N)' \cap R^{\mathcal{U}}$ is a factor.*

Proof. (⇒): We will show that any two projections in $\pi(N)' \cap R^{\mathcal{U}}$ are Murray-von Neumann equivalent if and only if they have the same trace. Assume let $p, q \in \pi(N)' \cap R^{\mathcal{U}}$ be projections with $\tau(p) = \tau(q) = t$. By part (3a) of Proposition 2.8.10, we have

$$\begin{aligned}[\pi] &= t[\pi_p] + (1-t)[\pi_{p^\perp}] \\ &= t[\pi_q] + (1-t)[\pi_{q^\perp}].\end{aligned}$$

And since $[\pi]$ is extreme, $[\pi_p] = [\pi] = [\pi_q]$. And by part (4) of Proposition 2.8.10 p and q are Murray-von Neumann equivalent in $\pi(N)' \cap R^{\mathcal{U}}$.

(⇐): Assume $[\pi] = t[\rho_1] + (1-t)[\rho_2]$. Then by part (3b) of Proposition 2.8.10, there is a projection $p \in \pi(N)' \cap R^{\mathcal{U}}$ with trace t such that $[\rho_1] = [\pi_p]$ and $[\rho_2] = [\pi_{p^\perp}]$. Let $q \in \pi(N)' \cap R^{\mathcal{U}}$ be a projection with trace t such that $[\pi] = [\pi_q]$ (this is possible by parts (1) and (2) of Proposition 2.8.10). Since $\pi(N)' \cap R^{\mathcal{U}}$ is a factor, then p and q are Murray-von Neumann equivalent. Thus, $[\rho_1] = [\pi_p] = [\pi_q] = [\pi]$. And similarly, $[\pi] = [\rho_2]$. □

Brown follows this characterization with a quick corollary that follows from Theorems 4.3.1 and 2.8.11.

Corollary 2.8.12 ([8])**.** *R is the unique (embeddable) separable II_1-factor with the property that every embedding into $R^{\mathcal{U}}$ has factorial commutant.*

We will show generalizations of Theorem 2.8.11 and Corollary 2.8.12 in Chapter 6. See Theorems 6.1.4 and 6.1.8.

This completes the overview of the basic facts about $\mathbb{H}\mathrm{om}(N, R^{\mathcal{U}})$ established in [8]. We should mention that in [9], for N a separable II_1-factor, Brown and Capraro constructed a real Banach space that naturally contains $\mathbb{H}\mathrm{om}(N, M^{\mathcal{U}})$. This space is constructed by applying the Grothendieck construction to a cancellative semigroup structure on the space of $*$-homomorphisms of N into amplifications of $M^{\mathcal{U}}$. Also, in [35] and [36], Pănescu exhibits and investigates a similar convex structure on the space of sofic representations of a given sofic group. These variations are interesting on their own, but examining such things is not within the scope of this thesis.

Chapter 8 will return to the context of $\mathbb{H}\mathrm{om}(N, R^{\mathcal{U}})$ and call upon many of the definitions and results laid out in this section.

Chapter 3

The Space $\mathbb{H}\mathrm{om}_w(\mathfrak{A}, M)$

Unless otherwise noted, $\mathfrak{A}$ will denote a separable unital C^*-algebra, N will denote a separable II_1-factor, M will denote a separable McDuff II_1-factor, and R will denote the separable hyperfinite II_1-factor.

We consider the topology of pointwise convergence (of equivalence classes) for $\mathbb{H}\mathrm{om}_w(\mathfrak{A}, N)$. That is, for $[\pi_n], [\pi] \in \mathbb{H}\mathrm{om}_w(\mathfrak{A}, N)$, $[\pi_n] \to [\pi]$ if there are representatives $\pi_n' \in [\pi_n]$ such that $\pi_n'(a) \to \pi(a)$ under the $||\cdot||_2$-norm for every $a \in \mathfrak{A}$. This topology can be metrized in the following way.

Definition 3.1. For $[\pi], [\rho] \in \mathbb{H}\mathrm{om}_w(\mathfrak{A}, N)$, let $\{a_n\}$ be a countable generating set in $\mathfrak{A}_{\leq 1}$ and define the metric (same as in Definition 1.2 of [8])

$$d([\pi], [\rho]) = \inf_{u \in \mathcal{U}(N)} \Big(\sum_{n=1}^{\infty} \frac{1}{2^{2n}} ||\pi(a_n) - u\rho(a_n)u^*||_2^2 \Big)^{\frac{1}{2}}. \tag{3.0.1}$$

This is quickly seen to be a metric that induces the topology described above. We note that the objects of study in [8] are typically not second countable with respect to the corresponding metric, but in our situation we have the following fact.

Proposition 3.2. $\mathbb{H}om_w(\mathfrak{A}, N)$ *is complete and separable under the metric* d.

Proof. Completeness follows from an argument identical to one found in the proof of Proposition 4.6 of [8].

For separability under d, let $(N_{\leq 1})^{\mathbb{N}}/\sim$ denote the set of all sequences in the unit ball of N modulo the equivalence relation given by $\{x_n\} \sim \{y_n\}$ if there is a sequence of unitaries $\{u_p\} \subset \mathcal{U}(N)$ such that for every n we have $x_n = \lim_p u_p y_n u_p^*$ where the limit is taken in the $||\cdot||_2$-norm. Let $[\{x_n\}]$ denote the equivalence class of $\{x_n\}$ under this equivalence relation. Consider a metric d' on $(N_{\leq 1})^{\mathbb{N}}/\sim$ given by

$$d'([\{x_n\}],[\{y_n\}]) = \inf_{u\in\mathcal{U}(N)} \left(\sum_{n=1}^{\infty} \frac{1}{2^{2n}}||x_n - uy_nu^*||_2^2\right)^{\frac{1}{2}}.$$

We claim that $(N_{\leq 1})^{\mathbb{N}}/\sim$ is separable under d'.

Let $\{m_n\}$ be $||\cdot||_2$-dense in $N_{\leq 1}$. Fix $\varepsilon > 0$ and $[\{x_n\}] \in (N_{\leq 1})^{\mathbb{N}}/\sim$. Let $K \in \mathbb{N}$ be such that

$$4\sum_{n=K+1}^{\infty} \frac{1}{2^{2n}} < \frac{\varepsilon^2}{2},$$

and let $f : \{1,\ldots,K\} \to \mathbb{N}$ be such that

$$\frac{1}{2^{2n}}||x_n - m_{f(n)}||_2^2 < \frac{\varepsilon^2}{2K}, \forall 1 \leq n \leq K.$$

For such a K and f, put $\{z_{K,f,n}\} \subset N_{\leq 1}$ with

$$z_{K,f,n} = \begin{cases} m_{f(n)} & \text{if} \quad 1 \leq n \leq K \\ m_{n+K'} & \text{if} \qquad\quad n > K \end{cases}$$

where $K' = \max\{f(n) : 1 \leq n \leq K\}$.

Then we have

$$d'([\{x_n\}],[\{z_{K,f,n}\}])^2 \leq \sum_{n=1}^{\infty} \frac{1}{2^{2n}}||x_n - z_{K,f,n}||_2^2$$
$$= \sum_{n=1}^{K} \frac{1}{2^{2n}}||x_n - z_{K,f,n}||_2^2 + \sum_{n=K+1}^{\infty} \frac{1}{2^{2n}}||x_n - z_{K,f,n}||_2^2$$
$$< K \cdot \frac{\varepsilon^2}{2K} + \frac{\varepsilon^2}{2} = \varepsilon^2.$$

Thus $d'([\{x_n\}],[\{z_{K,f,n}\}]) < \varepsilon$.

So

$$\{[\{z_{K,f,n}\}] | K \in \mathbb{N}, f : \{1,\ldots,K\} \to \mathbb{N}\} = \bigcup_{K=1}^{\infty} \bigcup_{f:\{1,\ldots,K\}\to\mathbb{N}} \{[\{z_{K,f,n}\}]\}$$

is dense and countable. Thus $(N_{\leq 1})^{\mathbb{N}}/\sim$ is separable under the metric d'.

By fixing a generating sequence $\{a_n\}$ in $\mathfrak{A}_{\leq 1}$, we get the metric d on $\mathbb{H}\mathrm{om}_w(\mathfrak{A}, N)$ as defined in (3.0.1). We can consider the metric space $(\mathbb{H}\mathrm{om}_w(\mathfrak{A}, N), d)$ as a subspace of the metric space $((N_{\leq 1})^{\mathbb{N}}/\sim, d')$ by identifying $[\pi] \in \mathbb{H}\mathrm{om}_w(\mathfrak{A}, N)$ with $[\{\pi(a_n)\}] \in (N_{\leq 1})^{\mathbb{N}}/\sim$. Since subspaces of separable metric spaces are separable, the proof is complete. □

Remark 3.3. This metric is not canonical–it depends on the choice of the generating sequence. It will sometimes be useful to choose our generating sequence to be a sequence of unitaries (always possible in a unital C^*-algebra)–see Theorem 7.1.7.

We will see later in Example 4.2.2 that $\mathbb{H}\mathrm{om}_w(\mathfrak{A}, M)$ is not necessarily compact.

3.1 Convex Structure

We now turn to define a convex structure on $\mathbb{H}\mathrm{om}_w(\mathfrak{A}, M)$ for M McDuff. In [8] Brown uses certain isomorphisms between corner algebras $pR^{\mathcal{U}}p$ and $R^{\mathcal{U}}$ to define a convex structure. We take a slightly different approach in order to define convex combinations in $\mathbb{H}\mathrm{om}_w(\mathfrak{A}, M)$. We must first introduce some terminology.

Definition 3.1.1. For a McDuff II$_1$-factor, a *regular isomorphism* $\sigma : M \otimes R \to M$ is an isomorphism such that $\sigma \circ (\mathrm{id}_M \otimes 1_R) \sim \mathrm{id}_M$ where $(\mathrm{id}_M \otimes 1_R)(x) = x \otimes 1_R$ for $x \in M$. Denote the set of regular isomorphisms of M as REG(M).

Proposition 3.1.2. *Let M be a McDuff II$_1$-factor.*

1. *REG$(M) \neq \emptyset$.*

2. *Any two regular isomorphisms $\sigma_M, s_M : M \otimes R \to M$ are weakly approximately unitarily equivalent.*

3. *The following are equivalent.*

   (a) *For every isomorphism $\nu : M \otimes R \to M, \nu \in$ REG(M);*

   (b) *$\overline{Inn(M)} = Aut(M)$. (The closure is in the point-$||\cdot||_2$ topology).*

Proof. (1): We will construct an isomorphism $\sigma_M : M \otimes R \to M$ such that $\mathrm{id}_M \sim \sigma_M(\mathrm{id}_M \otimes 1_R)$. Let $\nu : M \otimes R \to M$ and $\epsilon : R \otimes R \to R$ be isomorphisms. By Corollary 2.6.5, any unital endomorphism of R is approximately inner. We apply this fact to the map $\epsilon \circ (\mathrm{id}_R \otimes 1_R) : R \to R$ getting that $\epsilon \circ (\mathrm{id}_R \otimes 1_R) \sim \mathrm{id}_R$. Let

$$\sigma_M := \nu \circ (\mathrm{id}_M \otimes \epsilon) \circ (\nu^{-1} \otimes \mathrm{id}_R)$$

and consider

$$\begin{aligned}
\sigma_M(\mathrm{id}_M \otimes 1_R) &= \nu \circ (\mathrm{id}_M \otimes \epsilon) \circ (\nu^{-1} \otimes \mathrm{id}_R) \circ (\mathrm{id}_M \otimes 1_R) \\
&= \nu \circ (\mathrm{id}_M \otimes \epsilon) \circ (\nu^{-1} \otimes 1_R) \\
&= \nu \circ (\mathrm{id}_M \otimes \epsilon) \circ (\mathrm{id}_M \otimes \mathrm{id}_R \otimes \mathrm{id}_R) \circ (\nu^{-1} \otimes 1_R) \\
&= \nu \circ (\mathrm{id}_M \otimes \epsilon) \circ (\mathrm{id}_M \otimes \mathrm{id}_R \otimes 1_R) \circ \nu^{-1} \\
&= \nu \circ (\mathrm{id}_M \otimes (\epsilon \circ (\mathrm{id}_R \otimes 1_R))) \circ \nu^{-1} \\
&\sim \nu \circ (\mathrm{id}_M \otimes \mathrm{id}_R) \circ \nu^{-1} \\
&= \mathrm{id}_M.
\end{aligned}$$

(2): From Definition 3.1.1 we have that $\sigma_M^{-1} \sim \mathrm{id}_M \otimes 1_R \sim s_M^{-1}$. Then it is a straightforward exercise to see that this implies that $\sigma_M \sim s_M$.

(3): $(a \Rightarrow b)$: Let $\alpha \in \mathrm{Aut}(M)$, and let $\nu : M \otimes R \to M$ be an isomorphism. Define $\nu_\alpha := \alpha \circ \nu$. By assumption and part (2), $\nu \sim \nu_\alpha$, or equivalently, $\nu^{-1} \sim \nu_\alpha^{-1}$. So we get

$$\begin{aligned}
\alpha &= \alpha \circ \nu \circ \nu^{-1} \\
&= \nu_\alpha \circ \nu^{-1} \\
&\sim \nu_\alpha \circ \nu_\alpha^{-1} \\
&= \mathrm{id}_M.
\end{aligned}$$

$(b \Rightarrow a)$: Let $\nu : M \otimes R \to M$ be an isomorphism, and let $\sigma \in \mathrm{REG}(M)$. Then $\nu \circ \sigma^{-1} \in \mathrm{Aut}(M)$. Thus

$$\begin{aligned} \nu \circ \sigma^{-1} \sim \mathrm{id}_M &\Rightarrow \nu^{-1} \sim \sigma^{-1} \\ &\Rightarrow \nu^{-1} \sim \mathrm{id}_M \otimes 1_R \\ &\Rightarrow \nu \in \mathrm{REG}(M). \end{aligned}$$

□

Remark 3.1.3. In Theorem 8 of [42], Sakai gave an example of a McDuff factor ($\otimes_{\mathbb{Z}} L(\mathbb{F}_2)$) that fails condition (3b) of Proposition 3.1.2. Also, by Corollary 3.3 of [13], the McDuff factor $L(\mathbb{F}_2) \otimes R$ also fails condition (3b) of Proposition 3.1.2. So it is nontrivial for us to restrict to regular isomorphisms in this thesis.

For $\pi : \mathfrak{A} \to M$ and p a projection in R, let $\pi \otimes p : \mathfrak{A} \to M \otimes R$ be given by $(\pi \otimes p)(a) = \pi(a) \otimes p$ as similarly described in Remark 2.8.7. We now define convex combinations in $\mathbb{H}\mathrm{om}_w(\mathfrak{A}, M)$.

Definition 3.1.4. Given $[\pi], [\rho] \in \mathbb{H}\mathrm{om}_w(\mathfrak{A}, M)$ and $t \in [0, 1]$ we define

$$t[\pi] + (1 - t)[\rho] := [\sigma_M(\pi \otimes p) + \sigma_M(\rho \otimes p^{\perp})] \tag{3.1.1}$$

where $p \in \mathcal{P}(R)$ with $\tau(p) = t$ and $\sigma_M : M \otimes R \to M$ is a regular isomorphism.

Compare Definition 3.1.4 with Remark 2.8.7. Clearly, this definition extends to taking convex combinations of n equivalence classes. The following picture is helpful in visualizing this operation.

$$t[\pi] + (1-t)[\rho] \mapsto \left[\sigma_M \left(\begin{array}{c|c} \pi \otimes p & 0 \\ \hline 0 & \rho \otimes p^{\perp} \end{array} \right) \right]$$

where the block decomposition corresponds to the decomposition via $1_M \otimes p$ and $1_M \otimes p^{\perp}$.

Proposition 3.1.5. *The formula* (3.1.1) *is well-defined. That is, for σ_M and s_M regular isomorphisms, $p, q \in \mathcal{P}(R)$ with $\tau(p) = \tau(q) = t$, and $[\pi_1] = [\pi_2], [\rho_1] = [\rho_2]$, then*

$$[\sigma_M(\pi_1 \otimes p) + \sigma_M(\rho_1 \otimes p^{\perp})] = [s_M(\pi_2 \otimes q) + s_M(\rho_2 \otimes q^{\perp})].$$

Proof. By Proposition 3.1.2 (2) we have

$$[\sigma_M(\pi_2 \otimes q) + \sigma_M(\rho_2 \otimes q^{\perp})] = [s_M(\pi_2 \otimes q) + s_M(\rho_2 \otimes q^{\perp})]. \tag{3.1.2}$$

Let $v, w \in R$ be partial isometries such that

$$v^*v = p, \qquad vv^* = q,$$

$$w^*w = p^{\perp}, \qquad ww^* = q^{\perp}.$$

Then $u := \sigma_M(1_M \otimes (v + w))$ is a unitary with

$$\sigma_M(\pi_2 \otimes p) + \sigma_M(\rho_2 \otimes p^{\perp}) = u^*(\sigma_M(\pi_2 \otimes q) + \sigma_M(\rho_2 \otimes q^{\perp}))u. \tag{3.1.3}$$

Let $\{u_n\}, \{v_n\} \subset \mathcal{U}(M)$ be such that

$$\pi_1(a) = \lim_n u_n \pi_2(a) u_n^*,$$

$$\rho_1(a) = \lim_n v_n \rho_2(a) v_n^*$$

for every $a \in \mathfrak{A}$. Let $w_n = \sigma_M(u_n \otimes p) + \sigma_M(v_n \otimes p^\perp)$. Then $\{w_n\} \subset \mathcal{U}(M)$ with

$$\sigma_M(\pi_1 \otimes p) + \sigma_M(\rho_1 \otimes p^\perp) = \lim_n w_n(\sigma_M(\pi_2 \otimes p) + \sigma_M(\rho_2 \otimes p^\perp))w_n^*. \tag{3.1.4}$$

So by (3.1.4), (3.1.3), and (3.1.2) respectively we have

$$\begin{aligned}[\sigma_M(\pi_1 \otimes p) + \sigma_M(\rho_1 \otimes p^\perp] &= [\sigma_M(\pi_2 \otimes p) + \sigma_M(\rho_2 \otimes p^\perp)] \\ &= [\sigma_M(\pi_2 \otimes q) + \sigma_M(\rho_2 \otimes q^\perp)] \\ &= [s_M(\pi_2 \otimes q) + s_M(\rho_2 \otimes q^\perp)].\end{aligned}$$

□

We are now ready to prove the following theorem.

Theorem 3.1.6. *With convex combinations defined as in Definition 3.1.4, $\mathbb{H}om_w(\mathfrak{A}, M)$ satisfies the axioms of Brown's convex-like-structure (Definition 2.1 of [8]). Therefore, by Proposition 3.2 and the main result of [11], $\mathbb{H}om_w(\mathfrak{A}, M)$ may be considered as a closed, bounded, convex subset of a separable Banach space.*

Proof. Given $[\pi_1], \dots, [\pi_n] \in \mathbb{H}\mathrm{om}_w(\mathfrak{A}, M)$ and $0 \leq t_1, \dots t_n \leq 1$ such that $\sum t_i = 1$, we must show that $\mathbb{H}\mathrm{om}_w(\mathfrak{A}, M)$ is complete under d and that Definition 3.1.4 satisfies the following axioms.

1. (commutativity) $t_1[\pi_1] + \cdots + t_n[\pi_n] = t_{\alpha(1)}[\pi_{\alpha(1)}] + \cdots + t_{\alpha(n)}[\pi_{\alpha(n)}]$ for every permutation $\alpha \in S_n$.

2. (linearity) if $[\pi_1] = [\pi_2]$ then $t_1[\pi_1] + t_2[\pi_2] + t_3[\pi_3] + \cdots + t_n[\pi_n] = (t_1 + t_2)[\pi_1] + t_3[\pi_3] + \cdots + t_n[\pi_n]$.

3. (scalar identity) if $t_i = 1$ then $t_1[\pi_1] + \cdots + t_n[\pi_n] = [\pi_i]$.

4. (metric compatibility) $d((t_1[\pi_1]+\cdots+t_n[\pi_n]),(t'_1[\pi_1]+\cdots+t'_n[\pi_n])) \leq C\sum|t_i - t'_i|$ and $d((t_1[\pi_1] + \cdots + t_n[\pi_n]),(t_1[\pi'_1] + \cdots + t_n[\pi'_n])) \leq \sum t_i d([\pi_i],[\pi'_i])$.

5. (algebraic compatibility)

$$s\Big(\sum_{i=1}^{n} t_i[\pi_i]\Big) + (1-s)\Big(\sum_{j=1}^{m} t'_j[\pi'_j]\Big) = \sum_{i=1}^{n} st_i[\pi_i] + \sum_{j=1}^{m}(1-s)t'_j[\pi'_j].$$

We have completeness by Proposition 3.2. Metric compatibility follows from an argument identical to the one found in Proposition 4.6 of [8].

Commutativity and scalar identity are automatic.

We check linearity. That is, if $[\pi_1] = [\pi_2]$ then

$$t_1[\pi_1] + t_2[\pi_2] + \cdots + t_n[\pi_n] = (t_1 + t_2)[\pi_1] + \cdots + t_n[\pi_n].$$

By definition, for σ_M a regular isomorphism, we have that

$$\begin{aligned} t_1[\pi_1] + t_2[\pi_1] + \cdots + t_n[\pi_n] &= [\sigma_M(\pi_1 \otimes p_1) + \sigma_M(\pi_1 \otimes p_2) + \cdots + \pi_n \otimes p_n] \\ &= [\sigma_M(\pi_1 \otimes (p_1 + p_2)) + \cdots + \pi_n \otimes p_n] \\ &= (t_1 + t_2)[\pi_1] + \cdots + t_n[\pi_n]. \end{aligned}$$

We next check algebraic compatibility. That is, for $0 \leq t_i, t'_j, s \leq 1$ with $\sum t_i = \sum t'_j = 1$, then

$$s\Big(\sum t_i[\pi_i]\Big) + (1-s)\Big(\sum t'_j[\pi'_j]\Big) = \sum st_i[\pi_i] + \sum(1-s)t'_j[\pi'_j].$$

We have that

$$s\Big(\sum t_i[\pi_i]\Big) + (1-s)\Big(\sum t'_j[\pi'_j]\Big)$$

$$
\begin{aligned}
&= s\Big[\sum \sigma_M(\pi_i \otimes p_i)\Big] + (1-s)\Big[\sum \sigma_M(\pi'_j \otimes p'_j)\Big] \\
&= \Big[\sum \sigma_M(\sigma_M(\pi_i \otimes p_i) \otimes p_s) + \sum \sigma_M(\sigma_M(\pi'_j \otimes p'_j) \otimes p_{1-s})\Big] \\
&= \Big[\sigma_M((\sigma_M \otimes \mathrm{id}_R)\Big(\sum \pi_i \otimes p_i \otimes p_s + \sum \pi'_j \otimes p'_j \otimes p_{1-s}\Big)\Big]
\end{aligned}
$$

for projections $p_s, p_{1-s}, p_i, p'_j \in \mathcal{P}(R)$ with $\tau(p_s) = s, \tau(p_{1-s}), \tau(p_i) = t_i, \tau(p'_j) = t'_j$ with $p_{1-s} = 1 - p_s$, the projections $\{p_i\}$ pairwise orthogonal, and the projections $\{p'_j\}$ pairwise orthogonal.

From Definition 3.1.1 we know that σ_M^{-1} is weakly approximately unitarily equivalent to $\mathrm{id}_M \otimes 1_R$. Also $\epsilon \circ (1_R \otimes \mathrm{id}_R)$ is weakly approximately unitarily equivalent to id_R because it is a unital endomorphism of R. So we get the following equivalences with respect to weak approximate unitary equivalence.

$$
\begin{aligned}
(\mathrm{id}_M \otimes \epsilon) \circ (\sigma_M^{-1} \otimes \mathrm{id}_R) &\sim (\mathrm{id}_M \otimes \epsilon) \circ (\mathrm{id}_M \otimes 1_R \otimes \mathrm{id}_R) \\
&= \mathrm{id}_M \otimes (\epsilon \circ (1_R \otimes \mathrm{id}_R)) \\
&\sim \mathrm{id}_M \otimes \mathrm{id}_R \\
&= \mathrm{id}_{M \otimes R}
\end{aligned}
$$

It follows that

$$(\mathrm{id}_M \otimes \epsilon) \sim (\sigma_M \otimes \mathrm{id}_R).$$

Thus we get that

$$s\Big(\sum t_i[\pi_i]\Big) + (1-s)\Big(\sum t'_j[\pi'_j]\Big)$$

$$\begin{aligned}
&= \Big[\sigma_M\Big((\sigma_M \otimes \mathrm{id}_R)\Big(\sum \pi_i \otimes p_i \otimes p_s + \sum \pi'_j \otimes p'_j \otimes p_{1-s}\Big)\Big)\Big] \\
&= \Big[\sigma_M\Big((\mathrm{id}_M \otimes \epsilon)\Big(\sum \pi_i \otimes p_i \otimes p_s + \sum \pi'_j \otimes p'_j \otimes p_{1-s}\Big)\Big)\Big] \\
&= \Big[\sum \sigma_M(\pi_i \otimes \epsilon(p_i \otimes p_s)) + \sum \sigma_M(\pi'_j \otimes \epsilon(p'_j \otimes p_{1-s}))\Big] \\
&= \sum st_i[\pi_i] + \sum (1-s)t'_j[\pi'_j]
\end{aligned}$$

since the operation is well-defined. □

3.2 Functoriality

A $*$-homomorphism $\varphi : \mathfrak{A} \to \mathfrak{B}$ induces an affine map $\varphi^* : \mathbb{H}\mathrm{om}_w(\mathfrak{B}, M) \to \mathbb{H}\mathrm{om}_w(\mathfrak{A}, M)$ given by

$$\varphi^*([\pi]) = [\pi \circ \varphi].$$

Proposition 3.2.1. *The induced map φ^* is well-defined, continuous, and affine.*

Proof. Well-Defined: Let $[\pi] = [\rho] \in \mathbb{H}\mathrm{om}_w(\mathfrak{B}, M)$. So there is a sequence of unitaries $\{u_n\} \subset M$ such that $\rho(b) = \lim_n u_n \pi(b) u_n^*$ for any $b \in \mathfrak{B}$. Thus, for $a \in \mathfrak{A}$, we have that $\rho(\varphi(a)) = \lim_n u_n \pi(\varphi(a)) u_n^*$. Therefore $\varphi^*([\pi]) = [\pi \circ \varphi] = [\rho \circ \varphi] = \varphi^*([\rho])$. Continuity is just as quick to see.

Affine:

$$\begin{aligned}\varphi^*(t[\pi]+(1-t)[\rho]) &= \varphi^*([\sigma_M(\pi\otimes p)+\sigma_M(\rho\otimes p^\perp)])\\ &= [(\sigma_M(\pi\otimes p)+\sigma_M(\rho\otimes p^\perp))\circ\varphi]\\ &= [\sigma_M((\pi\circ\varphi)\otimes p)+\sigma_M((\rho\circ\varphi)\otimes p^\perp)]\\ &= t[\pi\circ\varphi]+(1-t)[\rho\circ\varphi]\\ &= t\varphi^*([\pi])+(1-t)\varphi^*([\rho]).\end{aligned}$$ □

The chain rule and preservation of identity are obvious observations; so we see that $\mathbb{H}\mathrm{om}_w(\cdot, M)$ is a contravariant functor from the category of C^*-algebras to the category of affine metrizable spaces.

Example 3.2.2. We exhibit an injective homomorphism φ such that φ^* fails to be surjective. Consider $\varphi : \mathbb{C}\oplus\mathbb{M}_2 \to \mathbb{C}\oplus\mathbb{M}_3$ given by

$$\lambda\oplus\begin{pmatrix} a & b\\ c & d\end{pmatrix}\mapsto\lambda\oplus\begin{pmatrix} a & b & 0\\ c & d & 0\\ 0 & 0 & \lambda\end{pmatrix}.$$

By Lemma 2.6.2 we know that up to unitary equivalence, $*$-homomorphisms from finite dimensional algebras into II_1 factors are completely determined by their induced traces. Therefore the induced map $\varphi^* : \mathbb{H}\mathrm{om}_w(\mathbb{C}\oplus\mathbb{M}_3, M)\to\mathbb{H}\mathrm{om}_w(\mathbb{C}\oplus\mathbb{M}_2, M)$ may be understood as $\varphi^* : T(\mathbb{C}\oplus\mathbb{M}_3)\to T(\mathbb{C}\oplus\mathbb{M}_2)$ where given $f\in T(\mathbb{C}\oplus\mathbb{M}_3)$ we have $\varphi^*(f) = f\circ\varphi$. Since both algebras have two summands, both trace spaces are the two-vertex simplex (i.e. the unit interval). The fact that φ^* is affine allows us to only check the images of the extreme points (endpoints) under φ^* in order to see the image $\varphi^*(T(\mathbb{C} \oplus \mathbb{M}_3))$. One endpoint of $T(\mathbb{C} \oplus \mathbb{M}_3)$ is the trace

$$f_1(\lambda \oplus (a_{ij})) = \lambda.$$

We see that

$$\varphi^*(f_1)\left(\lambda \oplus \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \lambda.$$

The other endpoint of $T(\mathbb{C} \oplus \mathbb{M}_3)$ is

$$f_2(\lambda \oplus (a_{ij})) = \mathrm{tr}_3(a_{ij})$$

where tr_3 is the (unique) normalized trace on $\mathbb{M}_3$. We get that

$$\varphi^*(f_2)\left(\lambda \oplus \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \frac{1}{3}\lambda + \frac{2}{3}\mathrm{tr}_2\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)$$

where tr_2 is the normalized trace on $\mathbb{M}_2$. So the image $\varphi^*(T(\mathbb{C} \oplus \mathbb{M}_3))$ is the convex hull of $\varphi^*(f_1)$ and $\varphi^*(f_2)$. From this it is clear that the extreme trace

$$\lambda \oplus \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \mathrm{tr}_2\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)$$

does not lie in the image $\varphi^*(T(\mathbb{C} \oplus \mathbb{M}_3))$. Hence φ^* is not surjective.

On the other hand, we have the following fact.

Proposition 3.2.3. *If φ is surjective, then φ^* is an affine homeomorphism onto its image.*

Proof. We must show that φ^* is injective and that $(\varphi^*)^{-1}$ is continuous on $\varphi^*(\mathbb{H}\mathrm{om}_w(\mathfrak{B}, M))$. Showing that φ^* is injective is a simple exercise and will be left to the reader. To show the continuity of $(\varphi^*)^{-1}$ assume that $\varphi^*([\pi_n]) \to \varphi^*([\pi])$ in $\mathbb{H}\mathrm{om}_w(\mathfrak{A}, M)$. This is the same as saying $[\pi_n \circ \varphi] \to [\pi \circ \varphi]$. We will demonstrate that $[\pi_n] \to [\pi]$. Let $\gamma_n \in [\pi_n \circ \varphi]$ be representatives such that $\gamma_n(a) \to \pi(\varphi(a))$ in M under the trace norm. Since $\gamma_n \in [\pi_n \circ \varphi]$ we get that $\ker(\varphi) \subseteq \ker(\gamma_n)$ for every n. Thus by isomorphism theorems, we can write $\gamma_n = \delta_n \circ \varphi$ for some $*$-homomorphism $\delta_n : \mathfrak{B} \to M$ for every n. For $b \in \mathfrak{B}$, we have that $b = \varphi(a)$ for some $a \in \mathfrak{A}$; thus

$$\delta_n(b) = \delta_n(\varphi(a)) = \gamma_n(a) \to \pi(\varphi(a)) = \pi(b).$$

It remains to show that $[\delta_n] = [\pi_n]$ for every n. Fix n; there exists $\{u_k(n)\} \subset \mathcal{U}(M)$ such that for every $a \in \mathfrak{A}$,

$$u_k(n)\gamma_n(a)u_k(n)^* \to \pi_n(\varphi(a)).$$

Let $b \in \mathfrak{B}$, and let $a \in \mathfrak{A}$ be such that $\varphi(a) = b$. Then

$$\begin{aligned} u_k(n)\delta_n(b)u_k(n)^* &= u_k(n)\delta_n(\varphi(a))u_k(n)^* \\ &= u_k(n)\gamma_n(a)u_k(n)^* \\ &\to \pi_n(\varphi(a)) \\ &= \pi_n(b). \end{aligned}$$

□

Similarly, a unital $*$-homomorphism $\psi : M_1 \to M_2$ between McDuff factors M_1 and M_2 also induces a map

$$\psi_* : \mathbb{H}\mathrm{om}_w(\mathfrak{A}, M_1) \to \mathbb{H}\mathrm{om}_w(\mathfrak{A}, M_2)$$

given by $\psi_*([\pi]) = [\psi \circ \pi]$.

Proposition 3.2.4. *The induced map ψ_* is well-defined, continuous, and affine.*

Proof. The fact that ψ is a unital $*$-homomorphism guarantees that ψ_* is well-defined. Continuity is also routine.

To show that ψ_* is affine, we will show that for $[\pi], [\rho] \in \mathbb{H}\mathrm{om}_w(\mathfrak{A}, M_1)$ and for a projection $p \in \mathcal{P}(R)$ we have

$$[\psi(\sigma_{M_1}(\pi \otimes p) + \sigma_{M_1}(\rho \otimes p^\perp))] = [\sigma_{M_2}((\psi \circ \pi) \otimes p) + \sigma_{M_2}((\psi \circ \rho) \otimes p^\perp)]$$

or

$$[\psi \circ \sigma_{M_1}(\pi \otimes p + \rho \otimes p^\perp)] = [\sigma_{M_2} \circ (\psi \otimes \mathrm{id}_R)(\pi \otimes p + \rho \otimes p^\perp)]$$

Here σ_{M_1} and σ_{M_2} are regular isomorphisms.

Thus it suffices to show that $\psi \circ \sigma_{M_1} \sim \sigma_{M_2} \circ (\psi \otimes \mathrm{id}_R)$; or equivalently, $\sigma_{M_2}^{-1} \circ \psi \sim (\psi \otimes \mathrm{id}_R) \circ \sigma_{M_1}^{-1}$. Since σ_{M_1} and σ_{M_2} are regular isomorphisms,

$$\begin{aligned} \sigma_{M_2}^{-1} \circ \psi &\sim (\mathrm{id}_{M_2} \otimes 1_R) \circ \psi \\ &= \psi \otimes 1_R \\ &= (\psi \otimes \mathrm{id}_R) \circ (\mathrm{id}_{M_1} \otimes 1_R) \\ &\sim (\psi \otimes \mathrm{id}_R) \circ \sigma_{M_2}^{-1}. \end{aligned}$$

□

One can easily see that $\mathbb{H}\mathrm{om}_w(\mathfrak{A}, \cdot)$ satisfies the chain rule and preserves identities; so $\mathbb{H}\mathrm{om}_w(\mathfrak{A}, \cdot)$ is a covariant functor from the category of McDuff II_1 factors to the category of affine metrizable spaces.

Chapter 4

Connection to the Trace Space

Given $[\pi] \in \mathbb{H}\mathrm{om}_w(\mathfrak{A}, M)$, we can assign to it a trace on $\mathfrak{A}$ in the following natural way.

Definition 4.1. For a separable unital tracial C^*-algebra $\mathfrak{A}$ and a McDuff factor M, let $\alpha_{(\mathfrak{A},M)} : \mathbb{H}\mathrm{om}_w(\mathfrak{A}, M) \to T(\mathfrak{A})$ be the map given by $\alpha_{(\mathfrak{A},M)}([\pi]) = \tau_M \circ \pi$ where τ_M is the unique tracial state of M, and $T(\mathfrak{A})$ denotes the trace space of $\mathfrak{A}$ (see Definition 2.1.7).

Proposition 4.2. *For any separable unital tracial C^*-algebra $\mathfrak{A}$ and McDuff factor M the map $\alpha_{(\mathfrak{A},M)} : \mathbb{H}om_w(\mathfrak{A}, M) \to T(\mathfrak{A})$ is well-defined, continuous (from the d-metric to the weak-$*$ topology), and affine.*

Proof. That $\alpha_{(\mathfrak{A},M)}$ is well-defined and continuous follows from the continuity of τ_M. To see that $\alpha_{(\mathfrak{A},M)}$ is affine, let $[\pi], [\rho] \in \mathbb{H}\mathrm{om}_w(\mathfrak{A}, M)$ and $t \in [0, 1]$. Then for $a \in \mathfrak{A}$

and $p \in \mathcal{P}(R)$ with $\tau(p) = t$, letting α stand for $\alpha_{(\mathfrak{A},M)}$ we have

$$\begin{aligned}\alpha(t[\pi] + (1-t)[\rho])(a) &= \alpha(\sigma_M(\pi \otimes p + \rho \otimes p^\perp))(a) \\ &= \tau(\sigma_M(\pi(a) \otimes p)) + \tau(\sigma_M(\rho(a) \otimes p^\perp)) \\ &= t\tau(\pi(a)) + (1-t)\tau(\rho(a)) \qquad (4.0.1) \\ &= (t\alpha([\pi]) + (1-t)\alpha([\rho]))(a).\end{aligned}$$

Where (4.0.1) follows from the fact that $\tau(\sigma_M(\cdot \otimes p))$ is a trace on M that evaluates to t at 1_M, and thus by the uniqueness of trace, we get that $\tau(\sigma_M(\cdot \otimes p)) = t\tau_M(\cdot)$. □

4.1 Nuclear and Hyperfinite Cases

In this section, we investigate the cases in which $\alpha_{(\mathfrak{A},M)}$ is an affine homeomorphism for every McDuff M. We first prove a technical lemma.

Lemma 4.1.1. *Let $\pi, \pi_k : \mathfrak{A} \to M$ be $*$-homomorphisms for $k \in \mathbb{N}$ and consider the $*$-homomorphisms*

$$(\pi_k)_{\mathcal{U}} : \mathfrak{A} \to M^{\mathcal{U}} \quad \textit{and} \quad \pi^{\mathcal{U}} : \mathfrak{A} \to M^{\mathcal{U}}$$

given by

$$(\pi_k)_{\mathcal{U}}(a) = (\pi_k(a))_{\mathcal{U}} \quad \textit{and} \quad \pi^{\mathcal{U}}(a) = (\pi(a))_{\mathcal{U}}$$

where $\mathcal{U}$ is a free ultrafilter on $\mathbb{N}$. If $(\pi_k)_{\mathcal{U}}$ is unitarily equivalent to $\pi^{\mathcal{U}}$, then there is a subsequence $\{k_j\}$ such that $[\pi_{k_j}] \to [\pi]$ in $\mathbb{H}om_w(\mathfrak{A}, M)$.

Proof. Let $u = (u_k)_{\mathcal{U}} \in \mathcal{U}(M^{\mathcal{U}})$ be such that for every $a \in \mathfrak{A}$,

$$(\pi(a))_{\mathcal{U}} = (u_k \pi_k(a) u_k^*)_{\mathcal{U}}.$$

Let $\{a_i\} \subseteq \mathfrak{A}_{\leq 1}$ be dense in the unit ball of $\mathfrak{A}$. Put

$$A_j := \bigcap_{1 \leq i \leq j} \left\{ k : ||u_k \pi_k(a_i) u_k^* - \pi(a_i)||_2 < \frac{1}{j} \right\}.$$

We have $A_j \in \mathcal{U}$ for every j. Pick $k_1 \in A_1$, and for $j > 1$, pick $k_j \in A_j \cap \{k > k_{j-1}\}$. We claim that for every $a \in \mathfrak{A}_{\leq 1}$,

$$u_{k_j} \pi_{k_j}(a) u_{k_j}^* \to \pi(a)$$

as $j \to \infty$. Fix $a \in \mathfrak{A}_{\leq 1}$ and $\varepsilon > 0$. Let $i \in \mathbb{N}$ be such that $||a - a_i|| < \frac{\varepsilon}{4}$. Let $J \in \mathbb{N}$ be such that $i \leq J$ and $\frac{1}{J} < \frac{\varepsilon}{2}$. Then for $j > J$, since $k_j \in A_j$, we have

$$\begin{aligned} ||u_{k_j} \pi_{k_j}(a) u_{k_j}^* - \pi(a)||_2 &\leq ||u_{k_j} \pi_{k_j}(a) u_{k_j}^* - u_{k_j} \pi_{k_j}(a_i) u_{k_j}^*||_2 + \\ &\quad ||u_{k_j} \pi_{k_j}(a_i) u_{k_j}^* - \pi(a_i)||_2 + ||\pi(a_i) - \pi(a)||_2 \\ &< \frac{\varepsilon}{4} + \frac{1}{j} + \frac{\varepsilon}{4} \\ &< \varepsilon. \end{aligned}$$

Since $\mathrm{Ad}(u_{k_j}) \circ \pi_{k_j} \in [\pi_{k_j}]$, this gives $[\pi_{k_j}] \to [\pi]$. □

Theorem 4.1.2. *If $\mathfrak{A}$ is nuclear, then $\alpha_{(\mathfrak{A},M)}$ is an affine homeomorphism for any McDuff M. In particular, $\mathbb{H}om_w(\mathfrak{A}, M)$ is affinely homeomorphic to $T(\mathfrak{A})$.*

Proof. **Injective:** When $\mathfrak{A}$ is nuclear we have

$$(\alpha([\pi]) = \alpha([\rho])) \Leftrightarrow (\tau \circ \pi = \tau \circ \rho)$$

$$\Leftrightarrow ([\pi] = [\rho]). \tag{4.1.1}$$

Here (4.1.1) follows from Theorem 2.6.10.

Surjective: By Proposition 2.6.9, every trace of $\mathfrak{A}$ gives a finite hyperfinite GNS closure, and it is well known that in this case every trace lifts through $R \subset M$. Thus $\alpha_{(\mathfrak{A},M)}$ is surjective.

Bicontinuous: Let $T_n \to T$ in $T(\mathfrak{A})$. Let $[\pi_n] = \alpha^{-1}_{(\mathfrak{A},M)}(T_n)$ and $[\pi] = \alpha^{-1}_{(\mathfrak{A},M)}(T)$. We must show that $[\pi_n] \to [\pi]$. We will show this by appealing to the following standard topological fact: for a sequence $\{a_n\}$, if every subsequence $\{a_{n(k)}\}$ has a sub-subsequence $\{a_{n(k_j)}\}$ converging to a, then $\{a_n\}$ converges to a. Let $\{[\pi_{n(k)}]\}$ be a subsequence of $\{[\pi_n]\}$. Now consider the homomorphism $(\pi_{n(k)})_{\mathcal{U}} : \mathfrak{A} \to M^{\mathcal{U}}$ where $\mathcal{U}$ is a free ultrafilter on $\mathbb{N}$. By the convergence of the induced traces, we get that

$$\tau_{M^{\mathcal{U}}} \circ (\pi_{n(k)})_{\mathcal{U}} = \tau_{M^{\mathcal{U}}} \circ \pi^{\mathcal{U}}.$$

And by Theorem 2.6.14, we get that $(\pi_{n(k)})_{\mathcal{U}}$ is unitarily equivalent to $\pi^{\mathcal{U}}$. So by Lemma 4.1.1, there is a sub-subsequence $\{n(k_j)\}$ such that $[\pi_{n(k_j)}] \to [\pi]$. □

Example 4.1.3. Theorem 3.10 of [3] says that for any metrizable Choquet simplex Δ, there exists a simple unital AF algebra $\mathfrak{B}$ such that $T(\mathfrak{B})$ is affinely homeomorphic to Δ. Combining this fantastic result with Theorem 4.1.2 tells us that any (separable) metrizable Choquet simplex Δ can arise as $\mathbb{H}\mathrm{om}_w(\mathfrak{B}, M)$ for some (separable) $\mathfrak{B}$.

We now work to characterize the algebras $\mathfrak{A}$ for which $\alpha_{(\mathfrak{A},M)}$ is an affine homeomorphism for every McDuff M.

Definition 4.1.4. Let

$$T(\mathfrak{A}, M) := \{\in T(\mathfrak{A}) : \text{ there exists } \pi : \mathfrak{A} \to M \text{ such that } T = \tau_M \circ \pi\}.$$

For $T \in T(\mathfrak{A}, M)$, we say "T lifts through M."

Theorem 4.1.5. *For any $\mathfrak{A}$, the map $\alpha_{(\mathfrak{A},R)} : \mathbb{H}om_w(\mathfrak{A}, R) \to T(\mathfrak{A})$ is always a homeomorphism onto its image. In particular, $\mathbb{H}om_w(\mathfrak{A}, R) \cong T(\mathfrak{A}, R)$ (affine homeomorphism).*

Proof. Let $\pi, \rho : \mathfrak{A} \to R$ be unital $*$-homomorphisms such that $\alpha_{(\mathfrak{A},R)}([\pi]) = \alpha_{(\mathfrak{A},R)}([\rho])$. We must show that $[\pi] = [\rho]$.

Consider the following map

$$\varphi : W^*(\pi(\mathfrak{A})) \to W^*(\rho(\mathfrak{A}))$$

densely defined by

$$\varphi(\pi(a)) = \rho(a).$$

The assumption that $\tau \circ \pi = \tau \circ \rho$ gives that all the $*$-moments in $\pi(\mathfrak{A})$ agree with those of $\rho(\mathfrak{A})$. So this is a well-defined $*$-isomorphism. In fact, $\varphi : W^*(\pi(\mathfrak{A})) \to W^*(\rho(\mathfrak{A})) \subseteq R$ is an embedding. Now $W^*(\pi(\mathfrak{A})) \subseteq R$ is hyperfinite, so Proposition 2.6.4 gives that $\varphi \sim \mathrm{id}_{W^*(\pi(\mathfrak{A}))}$. Thus $[\pi] = [\rho]$, and $\alpha_{(\mathfrak{A},R)}$ is injective.

It remains to show that $\alpha^{-1}_{(\mathfrak{A},R)}$ is continuous on $T(\mathfrak{A},R)$. We proceed similarly to the bicontinuous part of the proof of Theorem 4.1.2. Let $T_n \to T$ in $T(\mathfrak{A},R)$ in the weak-$*$ sense. And let $\pi_n, \pi : \mathfrak{A} \to R$ be such that $\tau \circ \pi_n = T_n$ and $\tau \circ \pi = T$. Let $\{n(k)\}$ be a subsequence. Consider $(\pi_{n(k)})_{\mathcal{U}}$ and $\pi^{\mathcal{U}}$ for a free ultrafilter $\mathcal{U}$ of $\mathbb{N}$. By assumption, $\tau_{R^{\mathcal{U}}} \circ (\pi_{n(k)})_{\mathcal{U}} = T$. So by the uniqueness of the GNS construction, $W^*((\pi_{n(k)})_{\mathcal{U}}(\mathfrak{A})) \cong W^*(\pi^{\mathcal{U}}(\mathfrak{A}))$ is hyperfinite. As above in the proof of the injectivity of $\alpha_{(\mathfrak{A},R)}$, the fact that $\tau_{R^{\mathcal{U}}} \circ (\pi_{n(k)})_{\mathcal{U}} = \tau_{R^{\mathcal{U}}} \circ \pi^{\mathcal{U}}$ gives that the map

$$\psi : W^*((\pi_{n(k)})_{\mathcal{U}}(\mathfrak{A})) \to W^*(\pi^{\mathcal{U}}(\mathfrak{A})) \subset R^{\mathcal{U}}$$

given by

$$\psi((\pi_{n(k)})_{\mathcal{U}}(a)) = \pi^{\mathcal{U}}(a)$$

is an embedding. Thus, by Theorem 4.3.1, $\mathrm{id}_{W^*((\pi_{n(k)})_{\mathcal{U}}(\mathfrak{A}))}$ is unitarily equivalent to ψ. It follows immediately that $(\pi_{n(k)})_{\mathcal{U}}$ is unitarily equivalent to $\pi^{\mathcal{U}}$. Then Lemma 4.1.1 tells us that there is a sub-subsequence $\{n(k_j)\}$ such that $[\pi_{n(k_j)}] \to [\pi]$. So we have shown that for any subsequence $\left\{[\pi_{n(k)}]\right\}$ of $\{[\pi_n]\}$ there is a sub-subsequence $\left\{[\pi_{n(k_j)}]\right\} \subset \left\{[\pi_{n(k)}]\right\}$ such that $[\pi_{n(k_j)}] \to [\pi]$. Thus

$$\alpha^{-1}_{(\mathfrak{A},R)}(T_n) = [\pi_n] \to [\pi] = \alpha^{-1}_{(\mathfrak{A},R)}(T). \qquad \square$$

From Theorem 4.1.5 we have that $\mathbb{H}\mathrm{om}_w(\mathfrak{A},R) \cong T(\mathfrak{A},R)$ as convex sets. Note that by Theorem 2.6.12, $T(\mathfrak{A},R) = \mathrm{UAT}(\mathfrak{A})$. We can now give our characterization theorem.

Theorem 4.1.6. *The following are equivalent:*

1. $\alpha_{(\mathfrak{A},M)}$ *is an affine homeomorphism for every McDuff* M*;*

2. $\alpha_{(\mathfrak{A},R)}$ *is an affine homeomorphism;*

3. $\alpha_{(\mathfrak{A},R)}$ *is surjective;*

4. $UAT(\mathfrak{A}) = T(\mathfrak{A}, R) = T(\mathfrak{A})$.

Proof. The implications $((1) \Rightarrow (2))$ and $((2) \Rightarrow (3))$ are obviously true.

The observation that $\alpha_{(\mathfrak{A},R)}(\mathbb{H}\text{om}_w(\mathfrak{A}, R)) = T(\mathfrak{A}, R) = \text{UAT}(\mathfrak{A})$ shows the equivalence $((3) \Leftrightarrow (4))$.

It remains to show $((4) \Rightarrow (1))$: If M is such that $\alpha_{(\mathfrak{A},M)}$ is not injective, then there are homomorphisms

$$\pi, \rho : \mathfrak{A} \to M$$

such that $\tau_M \circ \pi = \tau_M \circ \rho$ but $[\pi] \neq [\rho]$. Then by Theorem 4.1.5 we have that $\tau \circ \pi \notin T(\mathfrak{A}, R)$ – a contradiction. So $\alpha_{(\mathfrak{A},M)}$ must be injective for every M. If M is such that $\alpha_{(\mathfrak{A},M)}$ is not surjective then there exists $T \in T(\mathfrak{A})$ such that T does not lift through M; thus T cannot lift through R either. So again $T(\mathfrak{A}, R) \neq T(\mathfrak{A})$ – a contradiction. So $\alpha_{(\mathfrak{A},M)}$ must be bijective for every M. To show that $\alpha^{-1}_{(\mathfrak{A},M)}$ is continuous for every M, we use the assumption that $T(\mathfrak{A}) = T(\mathfrak{A}, R)$ along with an argument identical to the justification of the continuity of $\alpha^{-1}_{(\mathfrak{A},R)}$ in the proof of Theorem 4.1.5. □

Thus, the class of algebras $\mathfrak{A}$ for which $\alpha_{(\mathfrak{A},M)}$ is an affine homeomorphism for all McDuff M is exactly the class of all $\mathfrak{A}$ such that for any trace $T \in T(\mathfrak{A})$, the GNS representation of $\mathfrak{A}$ induced by T has a hyperfinite von Neumann closure. Recall that in Example 2.6.13 we noted that this class of algebras is strictly larger than that of nuclear algebras.

4.2 Examples

By Theorem 4.1.2 we know that $\alpha_{(\mathfrak{A},M)}$ can be both injective and surjective. The following examples will demonstrate that the other three cases where one or both properties fail are possible. This suggests that $\mathbb{H}\mathrm{om}_w(\mathfrak{A}, M)$ is a rich object with deep and interesting subtleties.

Example 4.2.1. "Forgetful Trace." This example shows that $\alpha_{(\mathfrak{A},M)}$ is not always injective – confirming that $\mathbb{H}\mathrm{om}_w(\mathfrak{A}, M)$ carries information different from that of $T(\mathfrak{A})$. The strategy of this example also reveals the usefulness of non-approximately inner automorphisms of McDuff factors. Let M be a McDuff factor with a non-approximately inner automorphism φ (e.g. $M = \otimes_{\mathbb{Z}} L(\mathbb{F}_2)$ satisfies this property as mentioned in Remark 3.1.3). Let $\mathfrak{A}$ be a separable, $|| \cdot ||_2$-dense C^*-subalgebra of M. Let $\pi : \mathfrak{A} \to M$ be the identity inclusion, and let $\rho = \varphi \circ \pi$. Then we claim

$$\tau_M \circ \rho = \tau_M \circ \pi,$$

but

$$[\pi] \neq [\rho].$$

Since $\tau_M \circ \varphi$ is also a trace on the II_1 factor M, we have $\tau_M = \tau_M \circ \varphi$. Then $\tau_M(\pi(a)) = \tau_M(\varphi(\pi(a))) = \tau_M(\rho(a))$. Now suppose for the sake of contradiction that $[\pi] = [\rho]$. Then there is a sequence of unitaries $\{u_n\}$ in M such that for every $a \in \mathfrak{A}$ we have

$$\lim_n ||\varphi(\pi(a)) - u_n \pi(a) u_n^*||_2 = \lim_n ||\varphi(a) - u_n a u_n^*||_2 = 0.$$

Then the $||\cdot||_2$-density of $\pi(\mathfrak{A}) = \mathfrak{A} \subset M$ implies that for every $x \in M$ we have

$$\lim_n ||\varphi(x) - u_n x u_n^*||_2 = 0$$

meaning that φ is approximately inner – a contradiction.

If we further insist that $\mathfrak{A}$ has a unique trace (by throwing in enough unitaries via Dixmier approximation – see Lemma 4.2.3), then this is an example of $\alpha_{(\mathfrak{A},M)}$ failing to be injective while remaining surjective.

Example 4.2.2. By Proposition 3.5.1 of [7] we have that $T(\mathfrak{A}, R)$ is a weakly closed subset of $T(\mathfrak{A})$. It is not true in general, however, that $T(\mathfrak{A}, R)$ is weak-$*$ closed in $T(\mathfrak{A})$. By Remark 4.1.7 of [7] if Γ is a non-amenable, residually finite, discrete group (e.g. $\mathbb{F}_n$) then the $T(C^*(\Gamma), R)$ is not weak-$*$ closed. So for $\mathfrak{A} = C^*(\Gamma)$ where Γ is a non-amenable, residually finite, discrete group, we have that $\alpha_{(\mathfrak{A},R)}$ fails to be surjective while remaining injective. Furthermore, $\mathbb{H}\mathrm{om}_w(\mathfrak{A}, R)$ is not compact: if it were, then by continuity $\alpha_{(\mathfrak{A},R)}(\mathbb{H}\mathrm{om}_w(\mathfrak{A},R)) = T(\mathfrak{A},R) = \mathrm{UAT}(\mathfrak{A})$ would also be weak-$*$ compact and thus weak-$*$ closed – a contradiction.

For the next example, we will need the following lemma which is most likely known to experts. We include a proof for the sake of completeness. We thank N. Brown for suggesting the proof of the following lemma.

Lemma 4.2.3. *If Y is a separable von Neumann subalgebra of a II_1 factor X with $1_Y = 1_X$, then there is a separable II_1 factor M contained unitally in X that contains Y: $Y \subset M \subset X$.*

Proof. This proof will heavily rely on Dixmier's approximation theorem: For N a finite von Neumann algebra and $x \in N$ then we have

$$\overline{\mathrm{conv}}\,\{uxu^*|u \in \mathcal{U}(N)\} \cap \mathcal{Z}(N) = \{T(x)\}$$

where T is the unique center-valued trace, and the closure is in the norm topology.

We will recursively construct an increasing sequence $\{Y_i\}_{i=0}^{\infty}$ of separable subalgebras of X and claim that $M = W^*(\cup_{i=0}^{\infty} Y_i)$ is the desired factor. Let $Y_0 = Y$. We will assume that Y_i has been constructed and go about constructing Y_{i+1}. Let $\{y(i,j)\}_{j=1}^{\infty} \subset (Y_i)_{\leq 1}$ be weakly dense in $(Y_i)_{\leq 1}$. By Dixmier, for any j and for any $k \in \mathbb{N}$ there are unitaries $u_1(i,j,k), \ldots, u_{n(i,j,k)}(i,j,k) \in \mathcal{U}(X)$ and scalars $\alpha_1(i,j,k), \ldots, \alpha_{n(i,j,k)}(i,j,k) \in (0,1)$ with $\sum_{p=1}^{n(i,j,k)} \alpha_p(i,j,k) = 1$ such that

$$\Big|\Big| \sum_{p=1}^{n(i,j,k)} \alpha_p(i,j,k) u_p(i,j,k) y(i,j) u_p(i,j,k)^* - \tau(y(i,j)) I \Big|\Big| < \frac{1}{k}.$$

Then we let

$$Y_{i+1} = W^*(Y_i \cup (\cup_{j=1}^{\infty} \cup_{k=1}^{\infty} \left\{u_1(i,j,k), \ldots, u_{n(i,j,k)}(i,j,k)\right\})).$$

Let $M := W^*(\cup_{i=0}^{\infty} Y_i)$. To show that M is a factor we will show that it has a unique unital trace (given by restriction of the unital trace τ on X). Let T be a unital trace on M and let $m \in M_{\leq 1}$. It will suffice to show that for any $\varepsilon > 0, |T(m) - \tau(m)| < \varepsilon$. Fix $\varepsilon > 0$. Let $K \in \mathbb{N}$ be such that $\frac{1}{K} < \varepsilon$. And let $i(m), j(m)$ be such that $y(i(m), j(m)) \in Y_{i(m)}$ with $|T(m) - T(y(i(m), j(m))| < \frac{\varepsilon}{3}$ and $|\tau(m) - \tau(i(m), j(m))| < \frac{\varepsilon}{3}$ (guaranteed by the weak continuity of both traces).

For brevity let

$$y = y(i(m), j(m)), \qquad u_p = u_p(i(m), j(m), 3K),$$

$$\alpha_p = \alpha_p(i(m), j(m), 3K), \qquad n = n(i(m), j(m), 3K)$$

and consider

$$\begin{aligned} |T(y) - \tau(y)| &= \left|T\Big(\sum_{p=1}^{n} \alpha_p u_p y u_p^*\Big) - \tau(y)\right| \\ &= \left|T\Big(\sum_{p=1}^{n} \alpha_p u_p y u_p^* - \tau(y) I\Big)\right| \\ &\leq \left|\left|\sum_{p=1}^{n} \alpha_p u_p y u_p^* - \tau(y) I\right|\right| \\ &< \frac{1}{3K} \\ &< \frac{\varepsilon}{3}. \end{aligned}$$

Thus we have

$$\begin{aligned}|T(m)-\tau(m)| &\leq |T(m)-T(y)|+|T(y)-\tau(y)|+|\tau(y)-\tau(m)|\\ &< \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}\\ &= \varepsilon,\end{aligned}$$

and we are done. □

Definition 4.2.4. We say that a tracial von Neumann algebra N is *embeddable* if there exists a trace-preserving unital embedding $\pi : N \to R^{\mathcal{U}}$.

Example 4.2.5. "Too Many Traces." This example will provide an algebra $\mathfrak{A}$ such that $\alpha_{(\mathfrak{A},M)}$ simultaneously fails injectivity and surjectivity for some M. This idea was suggested by N. Brown. Let N be an embeddable separable non-hyperfinite II_1-factor. Let $\pi, \rho : N \to R^{\mathcal{U}}$ be two embeddings that are not unitarily equivalent (this is guaranteed by Theorem 4.3.1). Let $Y := W^*(\pi(N) \cup \rho(N))$, and let $X = R^{\mathcal{U}}$. We have that the separable algebra Y is contained in the (nonseparable) II_1 factor X; so by Lemma 4.2.3 there is a separable II_1 factor M such that $Y \subset M \subset X$. We claim that π and ρ are not weakly approximately unitarily equivalent in M. If they were weakly approximately unitarily equivalent in M then they will also be weakly approximately unitarily equivalent in $R^{\mathcal{U}}$; but then by Theorem 2.5.6 we have that π and ρ are unitarily equivalent in $R^{\mathcal{U}}$ – a contradiction. Consider $\pi \otimes 1_R, \rho \otimes 1_R : N \to M \otimes R$. By Theorem 7.1.4, since π and ρ are inequivalent, we have that $\pi \otimes 1_R$ is not weakly approximately unitarily equivalent to $\rho \otimes 1_R$ (be assured that the proof of Theorem 7.1.4 does not depend on this example).

Now let $\mathfrak{A} = C^*(\mathbb{F}_\infty)$. And let $\zeta : C^*(\mathbb{F}_\infty) \to N$ be a $*$-monomorphism with weakly dense image as guaranteed by Proposition 3.1 of [6]. Let $\hat{\pi}$ and $\hat{\rho}$ be given by

$$\hat{\pi} = (\pi \otimes 1_R) \circ \zeta : C^*(\mathbb{F}_\infty) \to N \to M \otimes R$$

and

$$\hat{\rho} = (\rho \otimes 1_R) \circ \zeta : C^*(\mathbb{F}_\infty) \to N \to M \otimes R.$$

Then we clearly have that $[\hat{\pi}] \neq [\hat{\rho}]$ but $\alpha_{(\mathfrak{A}, M\otimes R)}([\hat{\pi}]) = \alpha_{(\mathfrak{A}, M\otimes R)}([\hat{\rho}])$.

Another consequence of Proposition 3.1 of [6] is that $\mathfrak{A} = C^*(\mathbb{F}_\infty)$ enjoys the property that for any McDuff factor S there is a trace $T_S \in T(\mathfrak{A})$ such that $\pi_{T_S}(\mathfrak{A})'' \cong S$ (where π_{T_S} is the GNS representation corresponding with T_S). By [32], there is no separable universal II_1 factor, and so we can conclude that there is no separable universal McDuff factor. So let S be such that S does not embed into $M \otimes R$. Then we have that $T_S \in T(\mathfrak{A})$ as described above does not lift through $M \otimes R$. Thus $T_S \notin \alpha_{(\mathfrak{A}, M\otimes R)}(\mathbb{H}\mathrm{om}_w(\mathfrak{A}, M \otimes R))$. So we have that $\alpha_{(\mathfrak{A}, M\otimes R)}$ is neither injective nor surjective.

4.3 An Alternative Characterization of Hyperfiniteness

Investigating the connection between weak approximate unitary equivalence and preservation of a given trace has led us to a characterization of a finite tracial embeddable hyperfinite von Neumann algebra that we believe to be new.

A result of Jung gives a characterization of hyperfiniteness using embeddings into $R^{\mathcal{U}}$. We state it as follows.

Theorem 4.3.1 (Lemma 2.9, [27]). *A separable tracial embeddable von Neumann algebra N is hyperfinite if and only if any two embeddings $\pi, \rho : N \to R^{\mathcal{U}}$ are conjugate by a unitary in $R^{\mathcal{U}}$.*

The characterization we present in the following theorem frames Jung's result in terms of embeddings into separable algebras – removing (most of) the ultrapower language from the characterization. This characterization may be known to experts, but we have not seen it in the literature.

Theorem 4.3.2. *Let N be a separable tracial embeddable von Neumann algebra. Then N is hyperfinite if and only if for every separable McDuff II_1-factor M, any two embeddings $\pi, \rho : N \to M$ are weakly approximately unitarily equivalent.*

Proof. **($\Rightarrow$):** This follows directly from Proposition 2.6.4.

($\Leftarrow$): The argument here is similar to the one found in Example 4.2.5. Assume that N is not hyperfinite. Then Jung's result says that there exist two embeddings $\pi, \rho :$

$N \to R^{\mathcal{U}}$ such that π and ρ are not unitarily conjugate. Let $Y := W^*(\pi(N) \cup \rho(N))$. Then by Lemma 4.2.3 there is a separable II_1-factor $M_0 \subset R^{\mathcal{U}}$ containing Y. Just as in Example 4.2.5, we may argue that π and ρ are not weakly approximately unitarily equivalent in M_0. And by Theorem 7.1.4 (whose proof does not rely on this result), we have that $\pi \otimes 1_R : N \to M_0 \otimes R$ is not weakly approximately unitarily equivalent to $\rho \otimes 1_R : N \to M_0 \otimes R$. So putting $M := M_0 \otimes R$, we are done. □

Remark 4.3.3. Jung's approach to the characterization in [27] hinges on the concept of tubularity: a condition on neighborhoods of unitary orbits of the microstate spaces for the generators of the algebra. We remark here that this concept of tubularity was preceded a decade earlier in [25] by Hadwin's concept of dimension ratio. The dimension ratio is a quantity associated to the self adjoint generators of a tracial C^*-algebra. This dimension ratio quantifies tubularity in the sense that the dimension ratio of the generators is 0 if and only if the generators are tubular. See [27] and [25] for the relevant definitions and theorems.

Chapter 5

The Convex Geometry of $\mathbb{H}\mathbf{om}_w(\mathfrak{A}, M)$

In the first part of this chapter we discuss a necessary condition for extreme points of $\mathbb{H}\mathrm{om}_w(\mathfrak{A}, M)$. A complete characterization of extreme points remains open, and thus the question of existence of extreme points in $\mathbb{H}\mathrm{om}_w(\mathfrak{A}, M)$ is also open. In Chapter 6 we provide characterizations for extreme points in broad cases. The second part of this section discusses a natural relationship between quotients of $\mathfrak{A}$ and faces of $\mathbb{H}\mathrm{om}_w(\mathfrak{A}, M)$. The discussion there provides a sufficient condition for the existence of extreme points.

5.1 Factorial Closure

We now proceed to establish a necessary (but not sufficient: see Example 5.1.2) condition for extreme points in $\mathbb{H}\mathrm{om}_w(\mathfrak{A}, M)$. Although relative commutants are not well-defined under weak approximate unitary equivalence in general, it is easy to see that the von Neumann closure of the image of a $*$-homomorphism is well-defined under weak approximate unitary equivalence up to $*$-isomorphism. We state the necessary condition for extreme points as follows.

Theorem 5.1.1. *If* $[\pi] \in \mathbb{H}om_w(\mathfrak{A}, M)$ *is extreme, then* $W^*(\pi(\mathfrak{A}))$ *is a factor.*

Proof. Using the isomorphism $\mathbb{H}\mathrm{om}_w(\mathfrak{A}, M) \cong \mathbb{H}\mathrm{om}_w(\mathfrak{A}, M \otimes R)$ we will show that if $[\pi] \in \mathbb{H}\mathrm{om}_w(\mathfrak{A}, M \otimes R)$ is extreme then $W^*(\pi(\mathfrak{A}))$ is a factor. We will argue by contrapositive and assume that $W^*(\pi(\mathfrak{A}))$ is not a factor. Then there exists a nontrivial central projection $z \in W^*(\pi(\mathfrak{A}))$. In particular $0 < \tau(z) < 1$. We will now construct inequivalent $*$-homomorphisms ρ_1 and ρ_2 using z and $z^\perp$ in the following way. Let $p \in R$ be a projection such that $\tau(p) = \tau(z)$. Let $\sigma_{M\otimes R} : M \otimes R \otimes R \to M \otimes R, \sigma_M : M \otimes R \to M$, and $\epsilon : R \otimes R$ be regular isomorphisms. Let $v, w \in M \otimes R \otimes R$ be partial isometries with

$$v^*v = \sigma^{-1}_{M\otimes R}(z), \qquad vv^* = 1_M \otimes 1_R \otimes p,$$

$$w^*w = \sigma^{-1}_{M\otimes R}(z^\perp), \qquad ww^* = 1_M \otimes 1_R \otimes p^\perp.$$

We have $v + w \in \mathcal{U}(M \otimes R \otimes R)$. Let $Ad(u)(x) = uxu^*$. For $q \in \{p, p^\perp\}$, let $T_q : qRq \to R$ be an isomorphism. We are now ready to define $\rho_1, \rho_2 : \mathfrak{A} \to M \otimes R$ by the following formulas.

$$\rho_1(a) = (\mathrm{id}_M \otimes \epsilon) \circ (\mathrm{id}_M \otimes \mathrm{id}_R \otimes T_p) \circ Ad(v + w) \circ \sigma^{-1}_{M\otimes R}(z\pi(a))$$

$$\rho_2(a) = (\mathrm{id}_M \otimes \epsilon) \circ (\mathrm{id}_M \otimes \mathrm{id}_R \otimes T_p^\perp) \circ Ad(v + w) \circ \sigma^{-1}_{M\otimes R}(z^\perp\pi(a)).$$

By construction, we have $[\rho_1], [\rho_2] \in \mathbb{H}\mathrm{om}_w(\mathfrak{A}, M \otimes R)$.

We claim that we will be done if we show the following two statements are true:

1. If $t = \tau(z)$ then $t[\rho_1] + (1-t)[\rho_2] = [\pi]$.

2. $[\rho_1] \neq [\pi]$.

Indeed, if these two statements hold, then $[\pi]$ is not an extreme point.

(1): By definition, we have that

$$t[\rho_1] + (1-t)[\rho_2] = [\sigma_{M\otimes R}(\rho_1 \otimes p + \rho_2 \otimes p^{\perp})].$$

And we have

$$\rho_1(a) \otimes p = \big((\mathrm{id}_{M\otimes R} \otimes p) \circ (\mathrm{id}_M \otimes \epsilon) \circ (\mathrm{id}_M \otimes \mathrm{id}_R \otimes T_p)\big) \circ Ad(v+w) \circ \sigma^{-1}_{M\otimes R}(z\pi(a)).$$

Notice that

$$\big((\mathrm{id}_{M\otimes R} \otimes p) \circ (\mathrm{id}_M \otimes \epsilon) \circ (\mathrm{id}_M \otimes \mathrm{id}_R \otimes T_p)\big) = \mathrm{id}_M \otimes f_p : M \otimes (R \otimes pRp) \to M \otimes (R \otimes pRp)$$

where

$$f_p = (\mathrm{id}_R \otimes p) \circ \epsilon \circ (\mathrm{id}_R \otimes T_p) : R \otimes pRp \to R \otimes pRp$$

is a unital $*$-homomorphism. Thus we have

$$f_p(x) = \lim_j a_j x a_j^*$$

where $a_j \in \mathcal{U}(R \otimes pRp)$. Note that $a_j^* a_j = a_j a_j^* = 1_R \otimes p$. So we have

$$\big((\mathrm{id}_{M\otimes R} \otimes p) \circ (\mathrm{id}_M \otimes \epsilon) \circ (\mathrm{id}_M \otimes \mathrm{id}_R \otimes T_p)\big)(y) = \lim_j (1_M \otimes a_j) y (1_M \otimes a_j)^*.$$

Similarly,

$$\rho_2(a)\otimes p^{\perp} = \big((\mathrm{id}_{M\otimes R}\otimes p^{\perp})\circ(\mathrm{id}_M\otimes\epsilon)\circ(\mathrm{id}_M\otimes\mathrm{id}_R\otimes T_p^{\perp})\big)\circ Ad(v+w)\circ\sigma_{M\otimes R}^{-1}(z^{\perp}\pi(a)),$$

and there is a sequence $\{b_j\}\subset R\otimes R$ with $b_j^*b_j = b_jb_j^* = 1\otimes p^{\perp}$ such that

$$\big((\mathrm{id}_{M\otimes R}\otimes p^{\perp})\circ(\mathrm{id}_M\otimes\epsilon)\circ(\mathrm{id}_M\otimes\mathrm{id}_R\otimes T_{p^{\perp}})\big)(y) = \lim_j (1_M\otimes b_j)y(1_M\otimes b_j)^*.$$

We now have that $(1_M\otimes a_j + 1_M\otimes b_j)$ is a unitary for every j and so

$$\rho_1(a)\otimes p + \rho_2(a)\otimes p^{\perp}$$

$$= \lim_j Ad(1_M\otimes a_j + 1_M\otimes b_j)\circ Ad(v+w)\circ\sigma_{M\otimes R}^{-1}(z\pi(a)+z^{\perp}\pi(a))$$

$$= \lim_j Ad(1_M\otimes a_j + 1_M\otimes b_j)\circ Ad(v+w)\circ\sigma_{M\otimes R}^{-1}(\pi(a)).$$

Thus,

$$\begin{aligned}[t[\rho_1]+(1-t)[\rho_2]] &= [\sigma_{M\otimes R}(\rho_1\otimes p + \rho_2\otimes p^{\perp})]\\ &= [\sigma_{M\otimes R}\circ Ad(v+w)\circ\sigma_{M\otimes R}^{-1}\circ\pi]\\ &= [\sigma_{M\otimes R}\circ\sigma_{M\otimes R}^{-1}\circ\pi]\\ &= [\pi].\end{aligned}$$

So (1) has been verified.

(2): To show that $[\pi]\neq[\rho_1]$ we will exhibit an element x with $||\pi(x)||_2\neq||\rho_1(x)||_2$. Let $\varepsilon>0$ be such that $\varepsilon<\dfrac{\sqrt{1-t}}{1+\frac{1}{\sqrt{t}}}$. Let $x\in\mathfrak{A}$ be such that $||\pi(x)-(1-z)||_2<\varepsilon$.

Then we have

$$\begin{aligned} ||z\pi(x)||_2 &= ||z\pi(x) + z(1-z)||_2 \\ &= ||z(\pi(x) - (1-z))||_2 \\ &\leq ||z|| \cdot ||\pi(x) - (1-z)||_2 \\ &< \varepsilon \end{aligned} \tag{5.1.1}$$

and

$$\begin{aligned} ||\pi(x)||_2 &= ||(\pi(x) - (1-z)) + (1-z)||_2 \\ &\geq ||(1-z)||_2 - ||\pi(x) - (1-z)||_2 \\ &> \sqrt{1-t} - \varepsilon. \end{aligned} \tag{5.1.2}$$

Let $\varphi : z(M \otimes R)z \to M \otimes R$ be given by

$$\varphi = (\mathrm{id}_M \otimes \epsilon) \circ (\mathrm{id}_M \otimes \mathrm{id}_R \otimes T_p) \circ Ad(v+w) \circ \sigma^{-1}_{M\otimes R}.$$

So $\rho_1(x) = \varphi(z\pi(x))$. If $[\pi] = [\rho_1]$ then there exists a sequence of unitaries $\{u_n\} \subset \mathcal{U}(M \otimes R)$ such that

$$\lim_{n\to\infty} u_n \pi(x) u_n^* = \rho_1(x) = \varphi(z\pi(x)).$$

So we have the following equation of norms

$$||\pi(x)||_2 = ||\lim_{n\to\infty} u_n \pi(x) u_n^*||_2 = ||\varphi(z\pi(x))||_2.$$

Then according to (5.1.2) we have on one hand

$$||\pi(x)||_2 > \sqrt{1-t} - \varepsilon,$$

while on the other hand, by (5.1.1) we have

$$||\varphi(z\pi(x))||_2 = \frac{1}{\sqrt{t}}||z\pi(x)||_2 < \frac{\varepsilon}{\sqrt{t}}.$$

This gives the following implications

$$\begin{aligned}\sqrt{1-t} - \varepsilon < \frac{\varepsilon}{\sqrt{t}} &\Rightarrow \sqrt{1-t} < \Big(1 + \frac{1}{\sqrt{t}}\Big)\varepsilon \\ &\Rightarrow \frac{\sqrt{1-t}}{1+\frac{1}{\sqrt{t}}} < \varepsilon.\end{aligned}$$

This last line is a contradiction to the assumption that $\varepsilon < \dfrac{\sqrt{1-t}}{1+\frac{1}{\sqrt{t}}}$. So we can conclude that $[\pi] \neq [\rho_1]$, and this completes the proof. □

Example 5.1.2. In this example, we will show that the converse of the above theorem does not hold in general. Let M be a McDuff factor with an automorphism $\beta \in \mathrm{Aut}(M) \setminus \overline{\mathrm{Inn}}(M)$. As in Example 4.2.1 we may consider the nonhyperfinite McDuff factor $\otimes L(\mathbb{F}_2)$; the fact that the inner automorphisms are not dense in the automorphism group of M is the main player in this argument. Let $\mathfrak{A}$ be a dense C^*-subalgebra of M. Let $\rho_1 = \mathrm{id}_{\mathfrak{A}}$ be the identity inclusion of $\mathfrak{A}$ in M, and let $\rho_2 = \beta \circ \rho_1$. Then from Example 4.2.1 we have that $[\rho_1] \neq [\rho_2]$. Thus $[\pi] = \frac{1}{2}[\rho_1] + \frac{1}{2}[\rho_2]$ is not an extreme point in $\mathbb{H}\mathrm{om}_w(\mathfrak{A}, M)$. Let p be a projection in R so that $\tau(p) = \frac{1}{2}$, and thus $\sigma_M(\rho_1 \otimes p + \rho_2 \otimes p^\perp)$ is a representative of $[\pi]$. We will show that $W^*((\rho_1 \otimes p + \rho_2 \otimes p^\perp)(\mathfrak{A})) \cong W^*(\mathfrak{A}) \cong M$, and thus giving an example of a non-extreme point with a factorial closure of its image. Consider the map

$$\varphi : \mathfrak{A} \to (\rho_1 \otimes p + \rho_2 \otimes p^\perp)(\mathfrak{A})$$

given by

$$\varphi(a) = \rho_1(a) \otimes p + \rho_2(a) \otimes p^{\perp}$$
$$= a \otimes p + \beta(a) \otimes p^{\perp}.$$

The map φ is clearly a bijective $*$-homomorphism. The following computation shows that φ is also isometric with respect to $||\cdot||_2$.

$$\begin{aligned} ||\varphi(a)||_2^2 &= \tau((a \otimes p + \beta(a) \otimes p^{\perp})^*(a \otimes p + \beta(a) \otimes p^{\perp})) \\ &= \tau(a^*a \otimes p + \beta(a^*a) \otimes p^{\perp}) \\ &= \frac{1}{2}\tau(a^*a) + \frac{1}{2}\tau(a^*a) \\ &= ||a||_2^2. \end{aligned}$$

Evidently, φ extends to an isomorphism between $M = W^*(\mathfrak{A})$ and $W^*((\rho_1 \otimes p + \rho_2 \otimes p^{\perp})(\mathfrak{A}))$.

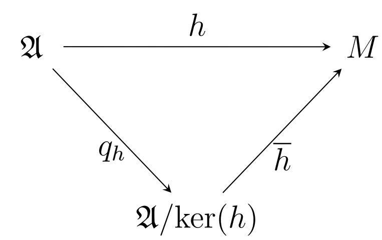
While the converse of Theorem 5.1.1 fails in general, if we combine Theorem 4.1.2 with the observation that a trace is extreme if and only if it gives a factorial GNS construction, then the converse holds in the nuclear case. Thus we have the following theorem characterizing extreme points of $\mathbb{H}\mathrm{om}_w(\mathfrak{A}, M)$ when $\mathfrak{A}$ is nuclear.

Theorem 5.1.3. *If $\mathfrak{A}$ is nuclear, then $[\pi] \in \mathbb{H}om_w(\mathfrak{A}, M)$ is extreme if and only if $W^*(\pi(\mathfrak{A}))$ is a factor.*

We will extend this characterization in §6.2, and we will see in §6.3 that this characterization of extreme points in $\mathbb{H}\mathrm{om}_w(\mathfrak{A}, M)$ holds for a general $\mathfrak{A}$ when $M = R$.

5.2 Quotients

We have access to quotients of C^*-algebras; this access is unavailable in the setting of [8] because II_1 factors are simple. Given a $*$-homomorphism $h : \mathfrak{A} \to M$, let $q_h : \mathfrak{A} \to \mathfrak{A}/\ker(h)$ be the canonical quotient map; and let $\overline{h} : \mathfrak{A}/\ker(h) \to M$ be the natural $*$-homomorphism that makes the following diagram commute.



In particular $h = \overline{h} \circ q_h$. The map q_h induces a map $q_h^* : \mathbb{H}\mathrm{om}_w(\mathfrak{A}/\ker(h), M) \to \mathbb{H}\mathrm{om}_w(\mathfrak{A}, M)$ (see Proposition 3.2.1) with

$$q_h^*([\overline{h}]) = [\overline{h} \circ q_h] = [h]. \tag{5.2.1}$$

Definition 5.2.1. A *singly exposed face* of a closed bounded convex subset C of a Banach space is a face that can be described as $\{x \in C : h(x) = M\}$ where $h : C \to \mathbb{R}$ is a continuous affine functional and $M = \max\{h(x) : x \in C\}$.

Definition 5.2.2. Given $a \in \mathfrak{A}$, we define a natural continuous affine functional f_a on $\mathbb{H}\mathrm{om}_w(\mathfrak{A}, M)$ given by $f_a([\pi]) = \tau_M(\pi(a))$.

We leave it to the reader to check that this satisfies the definition of a continuous affine functional.

Lemma 5.2.3. *Let $J \trianglelefteq \mathfrak{A}$ be a closed, two-sided ideal, and let $q : \mathfrak{A} \to \mathfrak{A}/J$ be the canonical quotient map. Then we have that $q^*(\mathbb{H}om_w(\mathfrak{A}/J, M))$ is a singly exposed face of $\mathbb{H}om_w(\mathfrak{A}, M)$.*

Proof. Without loss of generality, assume that J is generated by $\{a_n\}_{n=1}^{\infty} \subset (J)_{\leq 1}^{+}$. Put

$$a := \sum_{n=1}^{\infty} \frac{1}{2^n} a_n.$$

Consider f_{-a}. By the positivity of τ_M we get that $f_{-a}([\pi]) \leq 0$ for every $[\pi] \in \mathbb{H}\mathrm{om}_w(\mathfrak{A}, M)$. It is a quick observation to see that

$$q^*(\mathbb{H}\mathrm{om}_w(\mathfrak{A}/J, M)) = \{[\pi] \in \mathbb{H}\mathrm{om}_w(\mathfrak{A}, M) : f_{-a}([\pi]) = 0\}. \qquad \square$$

The quotient map $q : \mathfrak{A} \to \mathfrak{A}/J$ is surjective, so by Proposition 3.2.3 we get that q^* is a homeomorphism onto its image. Thus we may regard $\mathbb{H}\mathrm{om}_w(\mathfrak{A}/J, M)$ as a face of $\mathbb{H}\mathrm{om}_w(\mathfrak{A}, M)$ by identifying it with its image $q^*(\mathbb{H}\mathrm{om}_w(\mathfrak{A}/J, M))$. So every quotient of $\mathfrak{A}$ gives a face of $\mathbb{H}\mathrm{om}_w(\mathfrak{A}, M)$. Conversely, by Example 4.1.3, any metrizable Choquet simplex arises as $\mathbb{H}\mathrm{om}_w(\mathfrak{A}, M)$ for some simple AF-algebra; so in this situation, no nontrivial face of $\mathbb{H}\mathrm{om}_w(\mathfrak{A}, M)$ is induced by a quotient of $\mathfrak{A}$.

We can use this discussion to give a sufficient condition for the existence of extreme points in $\mathbb{H}\mathrm{om}_w(\mathfrak{A}, M)$.

Theorem 5.2.4. *If $\mathfrak{A}$ has a finite nuclear quotient, then $\mathbb{H}om_w(\mathfrak{A}, M)$ has extreme points.*

Proof. If $\mathfrak{A}/J$ is finite and nuclear, then $\mathbb{H}\mathrm{om}_w(\mathfrak{A}/J, M) \cong T(\mathfrak{A}/J)$; and thus $\mathbb{H}\mathrm{om}_w(\mathfrak{A}/J, M)$ has extreme points. An extreme point of a face is extreme. □

If we combine Lemma 5.2.3 with the observation that any $\mathfrak{A}$ is a quotient of $C^*(\mathbb{F}_\infty)$, we get the following theorem.

Theorem 5.2.5. *For any $\mathfrak{A}$, an affinely homeomorphic copy of $\mathbb{H}om_w(\mathfrak{A}, M)$ appears as a face of $\mathbb{H}om_w(C^*(\mathbb{F}_\infty), M)$.*

Remark 5.2.6. Theorem 5.2.4 guarantees that $\mathbb{H}\mathrm{om}_w(C^*(\mathbb{F}_\infty), M)$ has extreme points. Also, when we take Example 4.1.3 into account, this property enjoyed by $\mathbb{H}\mathrm{om}_w(C^*(\mathbb{F}_\infty, M)$ is very similar to one characterizing property of the so-called Poulsen simplex: A metrizable Choquet simplex S is the Poulsen simplex if and only if S contains every metrizable Choquet simplex as a face and for any two faces F_1 and F_2 with $\dim F_1 = \dim F_2 < \infty$ there is an affine homeomorphism σ of S onto itself with $\sigma(F_1) = F_2$. The Poulsen simplex was originally defined to be a metrizable Choquet simplex whose extreme points are dense. See [39] and [28] for more information.

Chapter 6

Ultrapower Situation

6.1 The Space $\mathbb{H}\mathrm{om}(\mathfrak{A}, M^{\mathcal{U}})$

Let $\mathcal{U}$ denote a free ultrafilter on $\mathbb{N}$. We now take the opportunity to extend Brown's construction of the convex structure on $\mathbb{H}\mathrm{om}(N, R^{\mathcal{U}})$ to a convex structure on $\mathbb{H}\mathrm{om}(\mathfrak{A}, M^{\mathcal{U}})$: the space of unital $*$-homomorphisms $\mathfrak{A} \to M^{\mathcal{U}}$ modulo unitary equivalence (M is still a separable McDuff II_1-factor). This space has the same metric as its predecessor. For $\sigma_M : M \otimes R \to M$ a regular isomorphism, we let

$$(\sigma_M)^{\mathcal{U}} : (M \otimes R)^{\mathcal{U}} \to M^{\mathcal{U}}$$

be given by

$$(\sigma_M)^{\mathcal{U}}((x_i)_{\mathcal{U}}) = (\sigma_M(x_i))_{\mathcal{U}}.$$

There is a natural way to embed M into $M^{\mathcal{U}}$ as cosets of constant sequences: $x \in M \mapsto (x)_{\mathcal{U}} \in M^{\mathcal{U}}$. Also, as in Remark 3.2.4 of [8], there is a natural embedding of $M^{\mathcal{U}} \otimes R^{\mathcal{U}}$ in $(M \otimes R)^{\mathcal{U}}$. So for $x \in M^{\mathcal{U}}$ and $y \in R^{\mathcal{U}}$, the expression $(\sigma_M)^{\mathcal{U}}(x \otimes y)$ makes sense once we consider $x \otimes y \in (M \otimes R)^{\mathcal{U}}$ via this natural embedding. We can now define convex combinations in $\mathbb{H}\mathrm{om}(\mathfrak{A}, M^{\mathcal{U}})$ in a way similar to Definition 3.1.4.

Definition 6.1.1. For $[\pi], [\rho] \in \mathbb{H}\mathrm{om}(\mathfrak{A}, M^{\mathcal{U}})$ and $t \in [0,1]$, define

$$t[\pi] + (1-t)[\rho] = [\sigma_M^{\mathcal{U}}(\pi \otimes (p)_{\mathcal{U}} + \rho \otimes (p^{\perp})_{\mathcal{U}})] \tag{6.1.1}$$

where $\sigma_M : M \otimes R \to M$ is a regular isomorphism and $p \in R$ is a projection with $\tau(p) = t$.

We leave it to the reader to check that this convex combination is well-defined and satisfies the axioms of a convex-like structure (the proofs are analogous). This generalization of $\mathbb{H}\mathrm{om}(N, R^{\mathcal{U}})$ from [8] also retains the characterization of extreme points. One can see this by following the same reasoning as in [8] and looking at "cut-downs" of homomorphisms by projections in the relative commutant.

Definition 6.1.2. For technical reasons, we consider homomorphisms $\pi : \mathfrak{A} \to (M \otimes R)^{\mathcal{U}}$. Given $\pi : \mathfrak{A} \to (M \otimes R)^{\mathcal{U}}$, let $q \in \pi(\mathfrak{A})' \cap (M \otimes R)^{\mathcal{U}}$ be a projection. We now define the cut-down

$$\pi_q : \mathfrak{A} \to (M \otimes R)^{\mathcal{U}}$$

of π by q. Let $p \in R$ be a projection in R with $\tau(p) = \tau(q)$. Let $v \in (M \otimes R \otimes R)^{\mathcal{U}}$ be a partial isometries such that

$$v^*v = (\sigma_{(M\otimes R)}^{\mathcal{U}})^{-1}(q), \qquad vv^* = (1 \otimes 1 \otimes p)_{\mathcal{U}}.$$

Let $\theta_p : pRp \to R$ be an isomorphism. Then we let

$$\pi_q(\cdot) := (\sigma_{(M\otimes R)}^{\mathcal{U}}) \circ (\mathrm{id}_M \otimes \mathrm{id}_R \otimes \theta_p)^{\mathcal{U}} \circ (\mathrm{Ad}\, v) \circ (\sigma_{(M\otimes R)}^{\mathcal{U}})^{-1}(q\pi(\cdot)).$$

Proposition 6.1.3. *Using the above definition of cut-downs, one can verify the following six statements.*

1. $[\pi_q]$ *is independent of choices made in the definition*

2. *For a unitary* u, $[\pi_q] = [(Adu \circ \pi)_{uqu^*}]$

3. (a) *Given any* $q \in \pi(\mathfrak{A})' \cap M^{\mathcal{U}}, [\pi] = \tau(q)[\pi_q] + \tau(q^\perp)[\pi_{q^\perp}]$.

   (b) *If* $[\pi] = t[\rho_1] + (1-t)[\rho_2]$ *then there is a projection* $q \in \pi(\mathfrak{A})' \cap M^{\mathcal{U}}$ *with trace* t *such that* $[\rho_1] = [\pi_q]$ *and* $[\rho_2] = [\pi_{q^\perp}]$. *In particular, since for any* $t \in (0,1), [\pi] = t[\pi] + (1-t)[\pi]$ *then we have that for any* $t \in (0,1)$ *there is a projection* $q \in \pi(\mathfrak{A})' \cap M^{\mathcal{U}}$ *with* $\tau(q) = t$ *such that* $[\pi] = [\pi_q]$.

4. *Given projections* $q_1, q_2 \in \pi(\mathfrak{A})' \cap M^{\mathcal{U}}$ *with* $\tau(q_1) = \tau(q_2)$, *we have that* $[\pi_{q_1}] = [\pi_{q_2}]$ *if and only if* q_1 *is Murray-von Neumann equivalent to* q_2 *in* $\pi(\mathfrak{A})' \cap M^{\mathcal{U}}$.

5. $\pi(\mathfrak{A})' \cap (M \otimes R)^{\mathcal{U}}$ *is diffuse.*

6. *A finite diffuse von Neumann subalgebra* $A \subset M^{\mathcal{U}}$ *is a factor if and only if for projections* $p, q \in A, \tau(p) = \tau(q) \Rightarrow p$ *is Murray-von Neumann equivalent to* q.

Proof. Proof of (1): To show this, we must show that $[\pi_q]$ is independent of choice of $\sigma_{(M\otimes R)}$, θ_p, v, and p. The independence of choice of $\sigma_{(M\otimes R)}$ follows from the combination of the fact that any two regular isomorphisms are weakly approximately unitarily equivalent with Theorem 2.5.6. Once p is selected, the independence of choice of θ_p follows from the combination of the fact that any endomorphism of

$R(\cong pRp)$ is approximately inner with Theorem 2.5.6, and the independence of choice of v is clear. It remains to show that $[\pi_q]$ is independent of the choice of p. Let $p, p' \in R$ be projections with $\tau(p) = \tau(p') = \tau(q)$. Then there exists a partial isometry $y \in R$ such that $y^*y = p$ and $yy^* = p'$. Let v be the partial isometry in the definition of π_q according to the choice of p, and let v' be the partial isometry in the definition of π_q according to the choice of p'. By Proposition 2.8.1, there exists a unitary $a \in R^{\mathcal{U}}$ such that

$$\mathrm{Ad}(1 \otimes 1 \otimes a) \circ (\mathrm{id}_M \otimes \mathrm{id}_R \otimes \theta_{p'})^{\mathcal{U}} \circ \mathrm{Ad}((1 \otimes 1 \otimes y)_{\mathcal{U}})(x) = (\mathrm{id}_M \otimes \mathrm{id}_R \otimes \theta_p)^{\mathcal{U}}(x)$$

for every $x \in \mathrm{Ad}(v) \circ (\sigma^{\mathcal{U}}_{M \otimes R})^{-1}(q \cdot \pi(\mathfrak{A}))$. So we have

$$\sigma^{\mathcal{U}}_{(M\otimes R)} \circ (\mathrm{id}_M \otimes \mathrm{id}_R \otimes \theta_p)^{\mathcal{U}} \circ \mathrm{Ad}(v) \circ (\sigma^{\mathcal{U}}_{(M\otimes R)})^{-1}(q\pi(x))$$

$$= \sigma^{\mathcal{U}}_{(M\otimes R)} \circ \mathrm{Ad}(1 \otimes 1 \otimes a) \circ (\mathrm{id}_M \otimes \mathrm{id}_R \otimes \theta_{p'})^{\mathcal{U}} \circ \mathrm{Ad}((1 \otimes 1 \otimes y)_{\mathcal{U}}) \circ \mathrm{Ad}(v) \circ (\sigma^{\mathcal{U}}_{(M\otimes R)})^{-1}(q\pi(x))$$

which is unitarily equivalent by the independence of choice of the partial isometry established above to

$$\sigma^{\mathcal{U}}_{(M\otimes R)} \circ (\mathrm{id}_M \otimes \mathrm{id}_R \otimes \theta_{p'})^{\mathcal{U}} \circ \mathrm{Ad}(v') \circ (\sigma^{\mathcal{U}}_{(M\otimes R)})^{-1}(q\pi(x))$$

by the selection of y.

Proof of (2): We have

$$(\mathrm{Ad}(u) \circ \pi)_{uqu^*}(\cdot) = \sigma^{\mathcal{U}}_{(M\otimes R)} \circ (\mathrm{id}_M \otimes \mathrm{id}_R \otimes \theta_p)^{\mathcal{U}} \circ \mathrm{Ad}(v) \circ (\sigma^{\mathcal{U}}_{(M\otimes R)})^{-1}(uqu^*u\pi(\cdot)u^*)$$

where p and v are appropriately chosen per the definition of a cut-down. Let $u' :=$

$(\sigma^{\mathcal{U}}_{(M\otimes R)})^{-1}(u)$. Then we have

$$(\mathrm{Ad}(u)\circ\pi)_{uqu^*}(\cdot) = \sigma^{\mathcal{U}}_{(M\otimes R)} \circ (\mathrm{id}_M \otimes \mathrm{id}_R \otimes \theta_p)^{\mathcal{U}} \circ \mathrm{Ad}(vu') \circ (\sigma^{\mathcal{U}}_{(M\otimes R)})^{-1}(q\pi(\cdot)).$$

Note that

$$(vu')^* vu' = u'^* v^* vu' = u'^*((\sigma^{\mathcal{U}}_{(M\otimes R)})^{-1}(uqu^*))u' = (\sigma^{\mathcal{U}}_{(M\otimes R)})^{-1}(q)$$

and

$$vu'(vu')^* = vu'u'^*v^* = vv^* = (1\otimes 1\otimes p)_{\mathcal{U}}.$$

So by part (1),

$$[\pi_q] = [(\mathrm{Ad}(u)\circ\pi)_{uqu^*}].$$

Proof of (3a): We have, for appropriately chosen $p \in R$ and $v, w \in R^{\mathcal{U}}$ per the definition,

$$\pi_q = \sigma^{\mathcal{U}}_{(M\otimes R)} \circ (\mathrm{id}_M \otimes \mathrm{id}_R \otimes \theta_p)^{\mathcal{U}} \circ \mathrm{Ad}(v) \circ (\sigma^{\mathcal{U}}_{(M\otimes R)})^{-1} \circ (q\cdot\pi)$$

and

$$\pi_{q^\perp} = \sigma^{\mathcal{U}}_{(M\otimes R)} \circ (\mathrm{id}_M \otimes \mathrm{id}_R \otimes \theta_{p^\perp})^{\mathcal{U}} \circ \mathrm{Ad}(w) \circ (\sigma^{\mathcal{U}}_{(M\otimes R)})^{-1} \circ (q^\perp\cdot\pi).$$

Also,

$$\tau(q)[\pi_q] + \tau(q^\perp)[\pi_{q^\perp}] = [\sigma^{\mathcal{U}}_{(M\otimes R)}(\pi_q \otimes (p)_{\mathcal{U}} + \pi_{q^\perp} \otimes (p^\perp)_{\mathcal{U}})].$$

For this argument, we must specify our choice of $\sigma_{(M\otimes R)}$. Let $\sigma_{(M\otimes R)} = \mathrm{id}_M \otimes \epsilon$ where $\epsilon : R\otimes R \to R$ is a $*$-isomorphism. The possibility for this choice of $\sigma_{(M\otimes R)}$ is the reason for working in $M \otimes R$ rather than M. Now, observe that

$$\sigma^{\mathcal{U}}_{(M\otimes R)}(\pi_q \otimes (p)_{\mathcal{U}})$$

$$= \sigma^{\mathcal{U}}_{(M\otimes R)} \circ (\mathrm{id}_M \otimes \mathrm{id}_R \otimes p)^{\mathcal{U}} \circ \pi_q$$

$$= \sigma^{\mathcal{U}}_{(M\otimes R)} \circ (\mathrm{id}_M \otimes \mathrm{id}_R \otimes p)^{\mathcal{U}} \circ \sigma^{\mathcal{U}}_{(M\otimes R)} \circ (\mathrm{id}_M \otimes \mathrm{id}_R \otimes \theta_p)^{\mathcal{U}}$$

$$\circ \mathrm{Ad}(v) \circ (\sigma^{\mathcal{U}}_{(M\otimes R)})^{-1} \circ (q \cdot \pi)$$

$$= \sigma^{\mathcal{U}}_{(M\otimes R)} \circ (\mathrm{id}_M \otimes ((\mathrm{id}_R \otimes p) \circ \epsilon \circ (\mathrm{id}_R \otimes \theta_p)))^{\mathcal{U}} \circ \mathrm{Ad}(v) \circ (\sigma^{\mathcal{U}}_{(M\otimes R)})^{-1} \circ (q \cdot \pi).$$

Note that

$$(\mathrm{id}_R \otimes p) \circ \epsilon \circ (\mathrm{id}_R \otimes \theta_p) : (1 \otimes p)(R \otimes R)(1 \otimes p) \to (1 \otimes p)(R \otimes R)(1 \otimes p)$$

is a unital $*$-homomorphism. So by Proposition 2.8.1, there is a partial isometry $v' \in (M \otimes R \otimes R)^{\mathcal{U}}$ such that $v'^* v' = v' v'^* = (1 \otimes 1 \otimes p)_{\mathcal{U}}$ and $(\mathrm{id}_M \otimes ((\mathrm{id}_R \otimes p) \circ \epsilon \circ (\mathrm{id}_R \otimes \theta_p)))^{\mathcal{U}}(x) = v' x v'^*$ for every $x \in \mathrm{Ad}(v) \circ (\sigma^{\mathcal{U}}_{(M\otimes R)})^{-1} \circ (q \cdot \pi)(\mathfrak{A})$. Thus,

$$\sigma^{\mathcal{U}}_{(M\otimes R)}(\pi_q \otimes (p)_{\mathcal{U}}) = \sigma^{\mathcal{U}}_{(M\otimes R)} \circ \mathrm{Ad}(v') \circ \mathrm{Ad}(v) \circ (\sigma^{\mathcal{U}}_{(M\otimes R)})^{-1} \circ (q \cdot \pi).$$

Similarly, there is a partial isometry $w' \in (M \otimes R \otimes R)^{\mathcal{U}}$ such that $w'^* w' = w' w'^* = (1 \otimes 1 \otimes p^{\perp})_{\mathcal{U}}$ and $(\mathrm{id}_M \otimes ((\mathrm{id}_R \otimes p^{\perp}) \circ \epsilon \circ (\mathrm{id}_R \otimes \theta_{p^{\perp}})))^{\mathcal{U}}(x) = w' x w'^*$ for every $x \in \mathrm{Ad}(w) \circ (\sigma^{\mathcal{U}}_{(M\otimes R)})^{-1} \circ (q^{\perp} \cdot \pi)(\mathfrak{A})$. And so

$$\sigma^{\mathcal{U}}_{(M\otimes R)}(\pi_{q^{\perp}} \otimes (p^{\perp})_{\mathcal{U}} = \sigma^{\mathcal{U}}_{(M\otimes R)} \circ \mathrm{Ad}(w') \circ \mathrm{Ad}(w) \circ (\sigma^{\mathcal{U}}_{(M\otimes R)})^{-1} \circ (q^{\perp} \cdot \pi).$$

Therefore,

$$\sigma^{\mathcal{U}}_{(M\otimes R)}(\pi_q \otimes (p)_{\mathcal{U}} + \pi_{q^{\perp}} \otimes (p^{\perp})_{\mathcal{U}})$$

$$= \sigma^{\mathcal{U}}_{(M\otimes R)} \circ \mathrm{Ad}(v') \circ \mathrm{Ad}(v) \circ (\sigma^{\mathcal{U}}_{(M\otimes R)})^{-1} \circ (q\cdot\pi)$$

$$+ \sigma^{\mathcal{U}}_{(M\otimes R)} \circ \mathrm{Ad}(w') \circ \mathrm{Ad}(w) \circ (\sigma^{\mathcal{U}}_{(M\otimes R)})^{-1} \circ (q^{\perp}\cdot\pi)$$

$$= \sigma^{\mathcal{U}}_{(M\otimes R)} \circ \mathrm{Ad}(v'+w') \circ (\mathrm{Ad}(v) \circ (\sigma^{\mathcal{U}}_{(M\otimes R)})^{-1} \circ (q\cdot\pi)$$

$$+ \mathrm{Ad}(w) \circ (\sigma^{\mathcal{U}}_{(M\otimes R)})^{-1} \circ (q^{\perp}\cdot\pi))$$

$$= \sigma^{\mathcal{U}}_{(M\otimes R)} \circ \mathrm{Ad}(v'+w') \circ \mathrm{Ad}(v+w) \circ (\sigma^{\mathcal{U}}_{(M\otimes R)})^{-1} \circ (q\cdot\pi + q^{\perp}\cdot\pi)$$

$$= \sigma^{\mathcal{U}}_{(M\otimes R)} \circ \mathrm{Ad}(v'+w') \circ \mathrm{Ad}(v+w) \circ (\sigma^{\mathcal{U}}_{(M\otimes R)})^{-1} \circ \pi$$

$$\sim \pi$$

Proof of (3b): Suppose $[\pi] = t[\rho_1] + (1-t)[\rho_2]$. Then for $p \in R$ with $\tau(p) = t$, we have

$$[\pi] = [\sigma^{\mathcal{U}}_{(M\otimes R)}(\rho_1 \otimes (p)_{\mathcal{U}} + \rho_2 \otimes (p^{\perp})_{\mathcal{U}})].$$

Let $\zeta = \sigma^{\mathcal{U}}_{(M\otimes R)}(\rho_1 \otimes (p)_{\mathcal{U}} + \rho_2 \otimes (p^{\perp})_{\mathcal{U}})$, and let $u \in R^{\mathcal{U}}$ be a unitary such that $\pi = \mathrm{Ad}(u)\circ\zeta$. By construction, we have that $\sigma^{\mathcal{U}}_{(M\otimes R)}(1\otimes 1\otimes (p)_{\mathcal{U}}) \in \zeta(\mathfrak{A})' \cap R^{\mathcal{U}}$. So we can consider the cutdown of ζ by $\sigma^{\mathcal{U}}_{(M\otimes R)}(1\otimes 1\otimes (p)_{\mathcal{U}})$:

$$\begin{aligned}
\zeta_{\sigma^{\mathcal{U}}_{(M\otimes R)}(1\otimes 1\otimes (p)_{\mathcal{U}})} &= \sigma^{\mathcal{U}}_{(M\otimes R)} \circ (\mathrm{id}_M \otimes \mathrm{id}_R \otimes \theta_p)^{\mathcal{U}} \circ \mathrm{Ad}(1) \\
&\circ (\sigma^{\mathcal{U}}_{(M\otimes R)})^{-1} \circ (\sigma^{\mathcal{U}}_{(M\otimes R)}(1\otimes 1\otimes (p)_{\mathcal{U}})\cdot\zeta) \\
&= \sigma^{\mathcal{U}}_{(M\otimes R)} \circ (\mathrm{id}_M \otimes \mathrm{id}_R \otimes \theta_p)^{\mathcal{U}} \circ (\rho_1 \otimes (p)_{\mathcal{U}}) \\
&= (\mathrm{id}_M \otimes (\epsilon \circ (\mathrm{id}_R \otimes \theta_p) \circ (\mathrm{id}_R \otimes p)))^{\mathcal{U}} \circ \rho_1.
\end{aligned}$$

Again, by Proposition 2.8.1, there is a unitary $v \in (M\otimes R)^{\mathcal{U}}$ such that

$$(\mathrm{id}_M \otimes (\epsilon \circ (\mathrm{id}_R \otimes \theta_p) \circ (\mathrm{id}_R \otimes p)))^{\mathcal{U}}(x) = vxv^*$$

for every $x \in \rho_1(\mathfrak{A})$. Thus,

$$\begin{aligned}[\rho_1] &= [\zeta_{\sigma^{\mathcal{U}}_{(M\otimes R)}(1\otimes 1\otimes (p)_{\mathcal{U}})}] \\ &= [(\mathrm{Ad}(u)\circ\pi)_{\sigma^{\mathcal{U}}_{(M\otimes R)}(1\otimes 1\otimes (p)_{\mathcal{U}})}] \\ &= [\pi_{u^*\sigma^{\mathcal{U}}_{(M\otimes R)}(1\otimes 1\otimes (p)_{\mathcal{U}})u}].\end{aligned}$$

The same argument works for the complement.

Proof of (4): Let $p \in R$ and $v_1, v_2 \in R^{\mathcal{U}}$ be appropriately chosen such that

$$\pi_{q_1} = \sigma^{\mathcal{U}}_{(M\otimes R)} \circ (\mathrm{id}_M \otimes \mathrm{id}_R \otimes \theta_p)^{\mathcal{U}} \circ \mathrm{Ad}(v_1) \circ (\sigma^{\mathcal{U}}_{(M\otimes R)})^{-1} \circ (q_1 \cdot \pi)$$

and

$$\pi_{q_2} = \sigma^{\mathcal{U}}_{(M\otimes R)} \circ (\mathrm{id}_M \otimes \mathrm{id}_R \otimes \theta_p)^{\mathcal{U}} \circ \mathrm{Ad}(v_2) \circ (\sigma^{\mathcal{U}}_{(M\otimes R)})^{-1} \circ (q_2 \cdot \pi)$$

($\Leftarrow$): Assume there is a partial isometry $z \in \pi(\mathfrak{A})' \cap (M \otimes R)^{\mathcal{U}}$ such that $z^*z = q_1$ and $zz^* = q_2$. It follows that

$$zq_1\pi(a)z^* = q_2\pi(a)$$

for every $a \in \mathfrak{A}$. Then

$$\begin{aligned}\pi_{q_2} &= \sigma^{\mathcal{U}}_{(M\otimes R)} \circ (\mathrm{id}_M \otimes \mathrm{id}_R \otimes \theta_p)^{\mathcal{U}} \circ \mathrm{Ad}(v_2) \circ (\sigma^{\mathcal{U}}_{(M\otimes R)})^{-1} \circ (q_2 \cdot \pi) \\ &= \sigma^{\mathcal{U}}_{(M\otimes R)} \circ (\mathrm{id}_M \otimes \mathrm{id}_R \otimes \theta_p)^{\mathcal{U}} \circ \mathrm{Ad}(v_2) \circ (\sigma^{\mathcal{U}}_{(M\otimes R)})^{-1} \circ \mathrm{Ad}(z) \circ (q_1 \cdot \pi) \\ &= \sigma^{\mathcal{U}}_{(M\otimes R)} \circ (\mathrm{id}_M \otimes \mathrm{id}_R \otimes \theta_p)^{\mathcal{U}} \circ \mathrm{Ad}(v_2(\sigma^{\mathcal{U}}_{(M\otimes R)})^{-1}(z)) \circ (\sigma^{\mathcal{U}}_{(M\otimes R)})^{-1} \circ (q_1 \cdot \pi).\end{aligned}$$

By part (1), it suffices to show that

$$(v_2(\sigma^{\mathcal{U}}_{(M\otimes R)})^{-1}(z))^*(v_2(\sigma^{\mathcal{U}}_{(M\otimes R)})^{-1}(z)) = (\sigma_{(M\otimes R)})^{-1}(q_1)$$

and

$$(v_2(\sigma^{\mathcal{U}}_{(M\otimes R)})^{-1}(z))(v_2(\sigma^{\mathcal{U}}_{(M\otimes R)})^{-1}(z))^* = 1\otimes 1\otimes (p)_{\mathcal{U}}.$$

Indeed,

$$\begin{aligned}(v_2(\sigma^{\mathcal{U}}_{(M\otimes R)})^{-1}(z))^*(v_2(\sigma^{\mathcal{U}}_{(M\otimes R)})^{-1}(z)) &= (\sigma^{\mathcal{U}}_{(M\otimes R)})^{-1}(z^*)v_2^*v_2(\sigma^{\mathcal{U}}_{(M\otimes R)})^{-1}(z)\\ &= (\sigma^{\mathcal{U}}_{(M\otimes R)})^{-1}(z^*)(\sigma^{\mathcal{U}}_{(M\otimes R)})^{-1}(q_2)(\sigma^{\mathcal{U}}_{(M\otimes R)})^{-1}(z)\\ &= (\sigma^{\mathcal{U}}_{(M\otimes R)})^{-1}(z^*q_2z)\\ &= (\sigma^{\mathcal{U}}_{(M\otimes R)})^{-1}(q_1)\end{aligned}$$

and

$$\begin{aligned}(v_2(\sigma^{\mathcal{U}}_{(M\otimes R)})^{-1}(z))(v_2(\sigma^{\mathcal{U}}_{(M\otimes R)})^{-1}(z))^* &= (v_2(\sigma^{\mathcal{U}}_{(M\otimes R)})^{-1}(z))(\sigma^{\mathcal{U}}_{(M\otimes R)}) - 1(z^*)v_2^*\\ &= v_2\sigma^{\mathcal{U}}_{(M\otimes R)})^{-1}(q_2)v_2^*\\ &= 1\otimes 1\otimes (p)_{\mathcal{U}}.\end{aligned}$$

Therefore, by the independence of choice established in part (1), we have that $[\pi_{q_1}] = [\pi_{q_2}]$.

($\Rightarrow$): Assume that $[\pi_{q_1}] = [\pi_{q_2}]$. Let $u \in (M\otimes R)^{\mathcal{U}}$ be a unitary such that $\pi_{q_1} = \mathrm{Ad}(u)\circ \pi_{q_2}$. That is, for appropriately chosen $p\in R$ and $v_1, v_2 \in (M\otimes R\otimes R)^{\mathcal{U}}$,

$$\sigma^{\mathcal{U}}_{(M\otimes R)}\circ (\mathrm{id}_M\otimes \mathrm{id}_R\otimes \theta_p)^{\mathcal{U}}\circ \mathrm{Ad}(v_1)\circ (\sigma^{\mathcal{U}}_{(M\otimes R)})^{-1}\circ (q_1\cdot \pi)$$

$$= \mathrm{Ad}(u)\circ \sigma^{\mathcal{U}}_{(M\otimes R)}\circ (\mathrm{id}_M\otimes \mathrm{id}_R\otimes \theta_p)^{\mathcal{U}}\circ \mathrm{Ad}(v_2)\circ (\sigma^{\mathcal{U}}_{(M\otimes R)})^{-1}\circ (q_2\cdot \pi).$$

Let

$$u' := \sigma^{\mathcal{U}}_{(M\otimes R)}\circ \mathrm{Ad}(v_2^*)\circ (\mathrm{id}_M\otimes \mathrm{id}_R\otimes \theta_p^{-1})^{\mathcal{U}}\circ (\sigma^{\mathcal{U}}_{(M\otimes R)})^{-1}(u).$$

(Note that $u'^*u' = u'u'^* = q_2$.) So this gives

$$\sigma^{\mathcal{U}}_{(M\otimes R)} \circ (\mathrm{id}_M \otimes \mathrm{id}_R \otimes \theta_p)^{\mathcal{U}} \circ \mathrm{Ad}(v_1) \circ (\sigma^{\mathcal{U}}_{(M\otimes R)})^{-1} \circ (q_1 \cdot \pi)$$

$$= \sigma^{\mathcal{U}}_{(M\otimes R)} \circ (\mathrm{id}_M \otimes \mathrm{id}_R \otimes \theta_p)^{\mathcal{U}} \circ \mathrm{Ad}(v_2) \circ (\sigma^{\mathcal{U}}_{(M\otimes R)})^{-1} \circ \mathrm{Ad}(u') \circ (q_2 \cdot \pi).$$

Then

$$\begin{aligned} q_1 \cdot \pi &= \sigma^{\mathcal{U}}_{(M\otimes R)} \circ \mathrm{Ad}(v_1^*) \circ (\mathrm{id}_M \otimes \mathrm{id}_R \otimes \theta_p^{-1})^{\mathcal{U}} \circ (\sigma^{\mathcal{U}}_{(M\otimes R)})^{-1} \circ \sigma^{\mathcal{U}}_{(M\otimes R)} \\ &\circ (\mathrm{id}_M \otimes \mathrm{id}_R \otimes \theta_p)^{\mathcal{U}} \circ \mathrm{Ad}(v_2) \circ (\sigma^{\mathcal{U}}_{(M\otimes R)})^{-1} \circ \mathrm{Ad}(u') \circ (q_2 \cdot \pi) \\ &= \sigma^{\mathcal{U}}_{(M\otimes R)} \circ \mathrm{Ad}(v_1^*) \circ \mathrm{Ad}(v_2) \circ (\sigma^{\mathcal{U}}_{(M\otimes R)})^{-1} \circ \mathrm{Ad}(u') \circ (q_2 \cdot \pi) \\ &= \sigma^{\mathcal{U}}_{(M\otimes R)} \circ \sigma^{\mathcal{U}}_{(M\otimes R)})^{-1} \circ \mathrm{Ad}(\sigma^{\mathcal{U}}_{(M\otimes R)}(v_1^* v_2) u') \circ (q_2 \cdot \pi) \\ &= \mathrm{Ad}(\sigma^{\mathcal{U}}_{(M\otimes R)}(v_1^* v_2) u') \circ (q_2 \cdot \pi). \end{aligned}$$

So taking $z = \sigma^{\mathcal{U}}_{(M\otimes R)}(v_1^* v_2)u'$ will give $z^*z = q_2$ and $zz^* = q_1$. Indeed,

$$\begin{aligned} z^*z &= u'^*\sigma^{\mathcal{U}}_{(M\otimes R)}(v_2^* v_1 v_1^* v_2)u' \\ &= u'^*\sigma^{\mathcal{U}}_{(M\otimes R)}(v_2^*(1\otimes 1\otimes (p)_{\mathcal{U}})v_2)u' \\ &= u'^* q_2 u' \\ &= q_2 \end{aligned}$$

and

$$\begin{aligned} zz^* &= \sigma^{\mathcal{U}}_{(M\otimes R)}(v_1^* v_2)u'u'^*\sigma^{\mathcal{U}}_{(M\otimes R)}(v_2^* v_1) \\ &= \sigma^{\mathcal{U}}_{(M\otimes R)}(v_1^* v_2)q_2\sigma^{\mathcal{U}}_{(M\otimes R)}(v_2^* v_1) \\ &= \sigma^{\mathcal{U}}_{(M\otimes R)}(v_1^*(1\otimes 1\otimes (p)_{\mathcal{U}})v_1) \\ &= q_1. \end{aligned}$$

Proof of (5): We will show that $\pi(\mathfrak{A})' \cap (M\otimes R)^{\mathcal{U}}$ has no minimal projections. Let $p \in \pi(\mathfrak{A})' \cap (M\otimes R)^{\mathcal{U}}$ be a nonzero projection. Let $u \in (M\otimes R)^{\mathcal{U}}$ be a unitary such that $\sigma^{\mathcal{U}}_{(M\otimes R)}(x\otimes 1) = uxu^*$ for every $x \in W^*(\pi(\mathfrak{A}) \cup \{p\})$. Let $q \in R$ be a nonzero projection with $\tau(q) < 1$. Then we have

$$\sigma^{\mathcal{U}}_{(M\otimes R)}(p\otimes q) < \sigma^{\mathcal{U}}_{(M\otimes R)}(p\otimes 1).$$

And clearly,

$$\sigma^{\mathcal{U}}_{(M\otimes R)}(p\otimes q) \in \sigma^{\mathcal{U}}_{(M\otimes R)} \circ (\pi\otimes 1)(\mathfrak{A})' \cap (M\otimes R)^{\mathcal{U}} = \mathrm{Ad}(u)\pi(\mathfrak{A})' \cap (M\otimes R)^{\mathcal{U}}.$$

Thus,

$$u^*\sigma^{\mathcal{U}}_{(M\otimes R)}(p\otimes q)u \in \pi(\mathfrak{A})' \cap (M\otimes R)^{\mathcal{U}}$$

is a projection with

$$u^*\sigma^{\mathcal{U}}_{(M\otimes R)}(p\otimes q)u < u^*\sigma^{\mathcal{U}}_{(M\otimes R)}(p\otimes 1)u = p.$$

So p cannot be minimal.

(6) is well-known. □

With the above proposition established, we have the following theorem.

Theorem 6.1.4. *The equivalence class* $[\pi] \in \mathbb{H}om(\mathfrak{A}, M^{\mathcal{U}})$ *is extreme if and only if* $\pi(\mathfrak{A})' \cap M^{\mathcal{U}}$ *is a factor.*

Proof. ($\Rightarrow$): Assume $[\pi]$ is extreme. Let $q_1, q_2 \in \pi(\mathfrak{A})' \cap (M \otimes R)^{\mathcal{U}}$ be projections with $\tau(q_1) = \tau(q_2)$. Then by part (3a) of Proposition 6.1.3, we have that

$$\begin{aligned}[\pi] &= \tau(q_1)[\pi_{q_1}] + \tau(q_1^{\perp})[\pi_{q_1^{\perp}}] \\ &= \tau(q_2)[\pi_{q_2}] + \tau(q_2^{\perp})[\pi_{q_2^{\perp}}].\end{aligned}$$

And since $[\pi]$ is extreme, we have that $[\pi] = [\pi_{q_1}] = [\pi_{q_2}]$. And so by part (4) of Proposition 6.1.3, q_1 is Murray-von Neumann equivalent to q_2 in $\pi(\mathfrak{A})' \cap (M \otimes R)^{\mathcal{U}}$. Thus by parts (5) and (6) in Proposition 6.1.3, $\pi(\mathfrak{A})' \cap (M \otimes R)^{\mathcal{U}}$ is a factor.

($\Leftarrow$): Assume that $\pi(\mathfrak{A})' \cap (M \otimes R)^{\mathcal{U}}$ is a factor. Let

$$[\pi] = t[\rho_1] + (1-t)[\rho_2]$$

with $t \in (0,1)$. By part (3b) of Proposition 6.1.3 we have that $[\rho_1] = [\pi_q]$ and $[\rho_2] = [\pi_{q^{\perp}}]$ for a projection $q \in \pi(\mathfrak{A})' \cap (M \otimes R)^{\mathcal{U}}$ with $\tau(q) = t$. Also by part (3b) of Proposition 6.1.3 there is a projection $q' \in \pi(\mathfrak{A})' \cap (M \otimes R)^{\mathcal{U}}$ with $\tau(q') = t$ and $[\pi_{q'}] = [\pi]$. Since $\pi(\mathfrak{A})' \cap (M \otimes R)^{\mathcal{U}}$ is a factor, we have that q and q' are Murray-von Neumann equivalent in $\pi(\mathfrak{A})' \cap (M \otimes R)^{\mathcal{U}}$. So by part (4) of Proposition 6.1.3 we have

$$[\rho_1] = [\pi_q] = [\pi_{q'}] = [\pi]$$

and similarly

$$[\rho_2] = [\pi]. \qquad \square$$

Remark 6.1.5. The existence of extreme points in $\mathbb{H}\text{om}(\mathfrak{A}, M^{\mathcal{U}})$ remains an open problem. The existence of extreme points in the context of $\mathbb{H}\text{om}(N, R^{\mathcal{U}})$ as in [8] is a well-known open question. The most recent work done on this question can be found in [12].

Given any McDuff M and any separable finite hyperfinite factor N, we have that any embedding of N into $M^{\mathcal{U}}$ is unique up to unitary equivalence by Theorem 2.6.14. That is, $\mathbb{H}\text{om}(N, M^{\mathcal{U}})$ is a single point. Combining this observation with the above theorem gives the following consequence.

Corollary 6.1.6. *For any McDuff M and any finite hyperfinite factor N, any embedding $\pi : N \to M^{\mathcal{U}}$ has the property that its relative commutant $\pi(N)' \cap M^{\mathcal{U}}$ is a factor.*

Thanks to an observation by S. White, a more general version of the above corollary is available. In particular, we do not need to require that M is McDuff.

Theorem 6.1.7. *For any separable II_1-factor X and any separable finite hyperfinite factor N, any embedding $\pi : N \to X^{\mathcal{U}}$ has the property that its relative commutant $\pi(N)' \cap X^{\mathcal{U}}$ is a factor.*

Proof. Throughout this proof we will abuse notation by letting τ denote both the trace on N and the trace on $X^{\mathcal{U}}$. The proof of this theorem essentially follows from Lemma 3.21 of [4]. The lemma says, among other things, that the collection of tracial states

$$\left\{\tau(\pi(x)\cdot) : x \in N^+, \tau(x) = 1\right\}$$

on $\pi(N)' \cap X^{\mathcal{U}}$ is weak-$*$ dense in the trace space of $\pi(N)' \cap X^{\mathcal{U}}$. For any fixed $x \in N^+$ with $\tau(x) = 1$, Dixmier approximation gives that for any $\varepsilon > 0$, there exist unitaries $u_1, \ldots, u_n \in \mathcal{U}(N)$ and numbers $0 \leq \lambda_1, \ldots, \lambda_n \leq 1$ with $\sum \lambda_j = 1$ such that

$$\left|\left|\sum_{j=1}^{n} \lambda_j u_j x u_j^* - 1_N\right|\right| < \varepsilon.$$

Note that for such an $x \in N, \tau(\pi(x)\cdot) = \tau(\pi(\sum_j \lambda_j u_j x u_j^*)\cdot)$.

We will now show that $\pi(N)' \cap X^{\mathcal{U}}$ is a factor by showing that it has a unique normalized trace (in particular, the trace induced by the unique trace on $X^{\mathcal{U}}$). Let T be a tracial state on $\pi(N)' \cap X^{\mathcal{U}}$. Fix $\delta > 0$ and $y \in \pi(N)' \cap X^{\mathcal{U}}$ with $||y|| \leq 1$. By Lemma 3.21 of [4] there is an $x_y \in N^+$ with $\tau(x_y) = 1$ such that

$$|T(y) - \tau(\pi(x_y)y)| < \frac{\delta}{2}.$$

For such an x_y, let $u_1, \ldots, u_n \in \mathcal{U}(N)$ and $0 \leq \lambda_1, \ldots, \lambda_n \leq 1$ with $\sum_j \lambda_j = 1$ be such that

$$\Big|\Big|\sum_{j=1}^{n}\lambda_j u_j x_y u_j^* - 1_N\Big|\Big| < \frac{\delta}{2}.$$

Then we have

$$\begin{aligned}\Big|\tau\Big(\pi\Big(\sum_j \lambda_j u_j x_y u_j^*\Big)y\Big) - \tau(y)\Big| &= \Big|\tau\Big(\pi\Big(\sum_j \lambda_j u_j x_y u_j^* - 1_N\Big)y\Big)\Big| \\ &\leq \Big|\Big|\pi\Big(\sum_j \lambda_j u_j x_y u_j^* - 1_N\Big)y\Big|\Big| \\ &\leq \Big|\Big|\sum_j \lambda_j u_j x_y u_j^* - 1_N\Big|\Big| \cdot ||y|| \\ &< \frac{\delta}{2}.\end{aligned}$$

Thus,

$$\begin{aligned}|T(y) - \tau(y)| &\leq |T(y) - \tau(\pi(x_y)y)| + |\tau(\pi(x_y)y) - \tau(y)| \\ &= |T(y) - \tau(\pi(x_y)y)| + \Big|\tau\Big(\pi\Big(\sum_j \lambda_j u_j x_y u_j^*\Big)y\Big) - \tau(y)\Big| \\ &< \delta.\end{aligned}$$

□

The case where $N = X = R$ is already well-known, but we have not seen the statement as it appears above in the literature. A similar discussion does appear in Section 2 of [38] (in particular Theorem 2.1 and Conjecture 2.3.1). There, Popa addresses bicentralizers in ultraproduct von Neumann algebras, while we are concerned with relative commutants (centralizers).

Corollary 5.3 of [8] says "R is the unique separable II_1-factor with the property that every embedding into $R^{\mathcal{U}}$ has factorial commutant." Theorem 6.1.7 leads us to the following stronger version of Brown's statement.

Theorem 6.1.8. *Let N be an embeddable separable II_1-factor. The following are equivalent:*

1. *$N \cong R$;*
2. *For any separable II_1-factor X and any embedding $\pi : N \to X^{\mathcal{U}}$, $\pi(N)' \cap X^{\mathcal{U}}$ is a factor;*
3. *For any separable II_1-factor X and any embedding $\pi : N \to X^{\mathcal{U}}$, the collection of tracial states $\{\tau(\pi(x)\cdot) : x \in N^+, \tau(x) = 1\}$ is weak-$*$ dense in the trace space of $\pi(N)' \cap X^{\mathcal{U}}$.*

Proof. ((1)⇒ (3)): This follows from Lemma 3.21 of [4].

((3)⇒ (2)): This follows from the proof of Theorem 6.1.7.

((2)⇒ (1)): This follows from Corollary 5.3 of [8]. □

The characterization in Theorem 6.1.4 of extreme points in the ultrapower case reveals some information on extreme points in the separable $\mathbb{H}\text{om}_w(\mathfrak{A}, M)$ setting. Using the canonical constant-sequence embedding of M into $M^{\mathcal{U}}$ we get the following map.

Definition 6.1.9. Let

$$\beta_{(\mathfrak{A},M)} : \mathbb{H}\text{om}_w(\mathfrak{A}, M) \to \mathbb{H}\text{om}(\mathfrak{A}, M^{\mathcal{U}})$$

be given by

$$\beta_{(\mathfrak{A},M)}([\pi]) = [\pi^{\mathcal{U}}]$$

where $\pi^{\mathcal{U}}(a) = (\pi(a))_{\mathcal{U}}$ for $a \in \mathfrak{A}$.

Proposition 6.1.10. *The map $\beta_{(\mathfrak{A},M)}$ is a well-defined affine homeomorphism onto its image.*

Proof. That $\beta_{(\mathfrak{A},M)}$ is continuous and affine is an easy check. Well-defined and injective follow from Theorem 3.1 of [45].

It remains to show that if $\beta_{(\mathfrak{A},M)}([\pi_n]) \to \beta_{\mathfrak{A},M)}([\pi])$ in $\mathbb{H}\mathrm{om}(\mathfrak{A}, M^{\mathcal{U}})$ then $[\pi_n] \to [\pi]$ in $\mathbb{H}\mathrm{om}_w(\mathfrak{A}, M)$. So suppose that

$$[\pi_n^{\mathcal{U}}] \to [\pi^{\mathcal{U}}].$$

That means that there exists homomorphisms $\varphi_n \in [(\pi_n)^{\mathcal{U}}]$ such that

$$\varphi_n(a) \to (\pi)^{\mathcal{U}}(a)$$

for every $a \in \mathfrak{A}$ in the trace norm on $R^{\mathcal{U}}$. Now for n fixed, $\varphi_n \in [(\pi_n)^{\mathcal{U}}]$ means that there is a unitary $u_n \in \mathcal{U}(R^{\mathcal{U}})$ such that $\varphi_n(a) = u_n\pi_n(a)u_n^*$. Without loss of generality, say that $u_n = (u_{n,j})_{\mathcal{U}}$. So we have that $\varphi_n(a) = (u_{n,j}\pi_n(a)u_{n,j}^*)_{\mathcal{U}}$.

We follow an argument similar to the one found in the proof of Lemma 4.1.1. Let $\{a_i\} \subset \mathfrak{A}_{\leq 1}$ be dense in the unit ball of $\mathfrak{A}$. Recursively construct a sequence of positive integers $\{N_k\}$ in the following way. Let $N_1 \in \mathbb{N}$ be such that for every $n \geq N_1$,

$$||\varphi_n(a_1) - \pi(a_1)||_2 < \frac{1}{2}.$$

Let $N_2 > N_1$ be such that for every $n \geq N_2$ and $i = 1, 2$

$$||\varphi_n(a_i) - \pi(a_i)||_2 < \frac{1}{4}.$$

In general, let $N_k > N_{k-1}$ be such that for every $n \geq N_k$ and $1 \leq i \leq k$

$$||\varphi_n(a_i) - \pi(a_i)||_2 < \frac{1}{2k}.$$

For $N_k \leq n < N_{k+1}$ and $1 \leq i \leq k$, let

$$L(n,i) = ||\varphi_n(a_i) - \pi(a_i)||_2 = \lim_{j \to \mathcal{U}} ||u_{n,j}\pi_n(a_i)u_{n,j}^* - \pi(a_i)||_2.$$

By our construction, we have $0 \leq L(n,i) < \frac{1}{2k}$. By definition

$$\left\{ j : \left| ||u_{n,j}\pi_n(a_i)u_{n,j}^*||_2 - L(n,i) \right| < \frac{1}{2k} \right\} \in \mathcal{U}.$$

And since

$$\left\{ j : \left| ||u_{n,j}\pi_n(a_i)u_{n,j}^*||_2 - L(n,i) \right| < \frac{1}{2k} \right\} \subseteq \left\{ j : ||u_{n,j}\pi_n(a_i)u_{n,j}^* - \pi(a_i)||_2 < \frac{1}{k} \right\},$$

we get

$$\left\{ j : ||u_{n,j}\pi_n(a_i)u_{n,j}^* - \pi(a_i)||_2 < \frac{1}{k} \right\} \in \mathcal{U}.$$

So for $N_k \leq n < N_{k+1}$, the intersection

$$A_n := \bigcap_{1 \leq i \leq k} \left\{ j : ||u_{n,j}\pi_n(a_i)u_{n,j}^* - \pi(a_i)||_2 < \frac{1}{k} \right\}$$

is in the ultrafilter $\mathcal{U}$, and hence is nonempty. Pick $j(1) \in A_1$ and for $n > 1$, let $j(n) \in A_n \cap \{j > j(n-1)\}$ (also nonempty since it is an element of the ultrafilter $\mathcal{U}$).

Let $v_n = u_{n,j(n)}$. We will now show that for $a \in \mathfrak{A}_{\leq 1}, v_n\pi_n(a)v_n^* \to \pi(a)$. Fix $\varepsilon > 0$. Let $i' \in \mathbb{N}$ be such that $||a - a_{i'}|| < \frac{\varepsilon}{4}$. Let $k \in \mathbb{N}$ be such that $i' \leq k$ and $\frac{1}{k} < \frac{\varepsilon}{2}$. Let $n \geq N_k$; thus $n \in [N_{k+c}, N_{k+c+1})$ for some $c \geq 0$. Then

$$j(n) \in \bigcap_{1 \leq i \leq k+c} \left\{ j : ||u_{n,j}\pi_n(a_i)u_{n,j}^* - \pi(a_i)||_2 < \frac{1}{k+c} \right\}.$$

So we have

$$\begin{aligned} ||v_n\pi_n(a)v_n^* - \pi(a)||_2 &\leq ||v_n\pi_n(a)v_n^* - v_n\pi_n(a_{i'})v_n^*||_2 + ||v_n\pi_n(a_{i'})v_n^* - \pi(a_{i'})||_2 + \\ &\quad ||\pi(a_{i'}) - \pi(a)||_2 \\ &< \frac{\varepsilon}{4} + ||u_{n,j(n)}\pi_n(a_{i'})u_{n,j(n)}^* - \pi(a_{i'})||_2 + \frac{\varepsilon}{4} \\ &< \frac{\varepsilon}{4} + \frac{1}{k+c} + \frac{\varepsilon}{4} \\ &< \varepsilon. \end{aligned}$$

Since $(\mathrm{Ad}(v_n) \circ \pi_n) \in [\pi_n]$, we have $[\pi_n] \to [\pi]$. Thus $\beta_{(\mathfrak{A},R)}$ is an affine homeomorphism onto its image. $\square$

We can also define a map $\tilde{\alpha}_{(\mathfrak{A},M)} : \mathbb{H}\mathrm{om}(\mathfrak{A}, M^{\mathcal{U}}) \to T(\mathfrak{A})$ analogous to $\alpha_{(\mathfrak{A},M)}$ in the following way.

Definition 6.1.11. Let

$$\tilde{\alpha}_{(\mathfrak{A},M)} : \mathbb{H}\mathrm{om}(\mathfrak{A}, M^{\mathcal{U}}) \to T(\mathfrak{A})$$

given by

$$\tilde{\alpha}_{(\mathfrak{A},M)}([\pi]) = \tau_{M^{\mathcal{U}}} \circ \pi$$

where $\tau_{M^{\mathcal{U}}}$ is the unique normalized trace of $M^{\mathcal{U}}$.

The maps $\alpha_{(\mathfrak{A},M)}$ and $\tilde{\alpha}_{(\mathfrak{A},M)}$ relate naturally to one another via the map $\beta_{(\mathfrak{A},M)}$. In particular, we have that the diagram

$$\begin{array}{ccc} \mathbb{H}\mathrm{om}_w(\mathfrak{A}, M) & \xrightarrow{\alpha_{(\mathfrak{A},M)}} & T(\mathfrak{A}) \\ \beta_{(\mathfrak{A},M)} \big\downarrow & \nearrow \tilde{\alpha}_{(\mathfrak{A},M)} & \\ \mathbb{H}\mathrm{om}(\mathfrak{A}, M^{\mathcal{U}}) & & \end{array} \tag{6.1.2}$$

commutes.

It is natural to ask "is $\beta_{(\mathfrak{A},M)}$ ever surjective?" The answer is: sometimes. The following example will show that $\beta_{(\mathfrak{A},M)}$ is not surjective in general.

Example 6.1.12. This example will show that $\mathbb{H}\mathrm{om}(\mathfrak{A}, M^{\mathcal{U}})$ can strictly contain $\mathbb{H}\mathrm{om}_w(\mathfrak{A}, M)$; or in the notation of Definition 6.1.9, $\beta_{(\mathfrak{A},M)}$ can fail to be surjective. Let $\mathfrak{A} = C^*(\mathbb{F}_\infty)$ and $M = R$. Furthermore, let N be a non-hyperfinite separable embeddable II_1-factor. By Proposition 3.1 of [6], $\mathfrak{A}$ can be embedded into N (say via $\zeta : \mathfrak{A} \to N$) such that it is weakly dense in N. Consider the map $\zeta^* : \mathbb{H}\mathrm{om}(N, R^{\mathcal{U}}) \to \mathbb{H}\mathrm{om}(\mathfrak{A}, R^{\mathcal{U}})$ given by

$$\zeta^*([\pi]) = [\pi \circ \rho].$$

Just as in Proposition 3.2.1, ζ^* is well-defined, continuous, and affine. It is not hard to see that ζ^* is additionally injective. Now, by Theorem A.1 of [8], we know that $\mathbb{H}\mathrm{om}(N, R^{\mathcal{U}})$ is nonseparable. So, since $\zeta^*(\mathbb{H}\mathrm{om}(N, R^{\mathcal{U}})) \subset \mathbb{H}\mathrm{om}(\mathfrak{A}, R^{\mathcal{U}})$, this means that $\mathbb{H}\mathrm{om}(\mathfrak{A}, R^{\mathcal{U}})$ is nonseparable. On the other hand, $\mathbb{H}\mathrm{om}_w(\mathfrak{A}, R)$ is separable. So by cardinality considerations,

$$\beta_{(\mathfrak{A},R)}(\mathbb{H}\text{om}_w(\mathfrak{A},R)) \subsetneq \mathbb{H}\text{om}(\mathfrak{A},R^{\mathcal{U}}).$$

With this embedding $\beta_{(\mathfrak{A},M)}$ established, we immediately get a sufficient condition for extreme points in $\mathbb{H}\text{om}_w(\mathfrak{A},M)$.

Theorem 6.1.13. *For $\pi:\mathfrak{A}\to M$, if $\pi^{\mathcal{U}}(\mathfrak{A})'\cap M^{\mathcal{U}}$ is a factor, then $[\pi]\in\mathbb{H}om_w(\mathfrak{A},M)$ is extreme.*

The converse of the above statement would be true if we can show that in general $\beta_{(\mathfrak{A},M)}(\mathbb{H}\text{om}_w(\mathfrak{A},M))$ is a face of $\mathbb{H}\text{om}(\mathfrak{A},M^{\mathcal{U}})$. This question comes down to asking if the cut-down of a constant-sequence homomorphism $\pi^{\mathcal{U}}$ is itself a constant-sequence homomorphism. In the case where $M=R$, the answer is yes. That is, we have the following theorem.

Theorem 6.1.14. *$\beta_{(\mathfrak{A},R)}(\mathbb{H}om_w(\mathfrak{A},R))$ is a face of $\mathbb{H}om(\mathfrak{A},R^{\mathcal{U}})$.*

Proof. Let $[\pi]\in\mathbb{H}\text{om}_w(\mathfrak{A},R)$ be given, and suppose for $t\in(0,1)$ and $[\rho_1],[\rho_2]\in\mathbb{H}\text{om}(\mathfrak{A},R^{\mathcal{U}})$ that $\beta_{(\mathfrak{A},R)}([\pi])=[\pi^{\mathcal{U}}]=t[\rho_1]+(1-t)[\rho_2]$. We must show that $[\rho_i]\in\beta_{(\mathfrak{A},R)}(\mathbb{H}\text{om}_w(\mathfrak{A},R))$ for $i=1,2$.

Because the map $\tilde{\alpha}_{(\mathfrak{A},R)}$ is affine, we have that

$$\tilde{\alpha}_{(\mathfrak{A},R)}([\pi^{\mathcal{U}}])=t\tilde{\alpha}_{(\mathfrak{A},R)}([\rho_1])+(1-t)\tilde{\alpha}_{(\mathfrak{A},R)}([\rho_2]).$$

Since $[\pi]\in\mathbb{H}\text{om}_w(\mathfrak{A},R)$, we have that $\tilde{\alpha}_{(\mathfrak{A},R)}([\pi^{\mathcal{U}}])\in\text{UAT}(\mathfrak{A})$. And from [7] we know that $\text{UAT}(\mathfrak{A})$ is a face of $T(\mathfrak{A})$. Thus, we have that $\tilde{\alpha}_{(\mathfrak{A},R)}([\rho_i])\in\text{UAT}(\mathfrak{A})$ for $i=1,2$.

By the uniqueness of the GNS construction, this implies that $W^*(\rho_i(\mathfrak{A}))$ is hyperfinite for $i = 1, 2$. Thus, for $i = 1, 2$ there exists an embedding $\gamma_i : W^*(\rho_i(\mathfrak{A})) \to R$. Then consider the maps

$$(\gamma_1 \circ \rho_1) : \mathfrak{A} \to R$$

and

$$(\gamma_2 \circ \rho_2) : \mathfrak{A} \to R.$$

Theorem 4.3.1 immediately shows that $\beta_{(\mathfrak{A},R)}([\gamma_i \circ \rho_i]) = [\rho_i]$ for $i = 1, 2$. □

We now give an example of a face in $\mathbb{H}\mathrm{om}_w(C^*(\mathbb{F}_\infty), M)$ that does not come from an ideal (cf. §5.2).

Example 6.1.15. Let $M = R$, and let $\zeta : C^*(\mathbb{F}_\infty) \to R$ be an injective $*$-homomorphism such that $\zeta(C^*(\mathbb{F}_\infty))$ is weakly dense in R as provided by Proposition 3.1 of [6]. Then we consider

$$\zeta^{\mathcal{U}} : C^*(\mathbb{F}_\infty) \to R^{\mathcal{U}}.$$

Consider $R \subset R^{\mathcal{U}}$ via the constant embedding. Since ζ has a dense image in R, we get that

$$\zeta^{\mathcal{U}}(C^*(\mathbb{F}_\infty))' \cap R^{\mathcal{U}} = R' \cap R^{\mathcal{U}}.$$

It is well-known that $R' \cap R^{\mathcal{U}}$ is a factor. Thus $\zeta^{\mathcal{U}}(C^*(\mathbb{F}_\infty))' \cap R^{\mathcal{U}}$ is a factor, and by Theorem 6.1.13, we get that $[\zeta]$ is extreme. So $\{[\zeta]\}$ is a face of $\mathbb{H}\mathrm{om}_w(C^*(\mathbb{F}_\infty), R)$ that does not factor through a quotient map.

6.2 Extreme Points in $\mathbb{H}\mathbf{om}_w(\mathfrak{A}, M)$: Amenability in First Argument

We first note the following theorem.

Theorem 6.2.1. *If $T(\mathfrak{A}) = UAT(\mathfrak{A})$ then for any McDuff M,*

$$\mathbb{H}om_w(\mathfrak{A}, M) \cong \mathbb{H}om(\mathfrak{A}, M^{\mathcal{U}})$$

via $\beta_{(\mathfrak{A},M)}$. In particular, if $\mathfrak{A}$ is nuclear, then $\mathbb{H}om_w(\mathfrak{A}, M) \cong \mathbb{H}om(\mathfrak{A}, M^{\mathcal{U}})$.

Proof. By Proposition 6.1.10, it suffices to show that $\beta_{(\mathfrak{A},M)}$ is surjective. We will let τ denote the unique tracial state on both M and $M^{\mathcal{U}}$. Let $\pi : \mathfrak{A} \to M^{\mathcal{U}}$ be given. Let $T \in T(\mathfrak{A})$ be given by $T = \tau \circ \pi$. Since $T \in \mathrm{UAT}(\mathfrak{A})$, there is a $\rho : \mathfrak{A} \to M$ so that $T = \tau \circ \rho$. Now, consider $\rho^{\mathcal{U}} : \mathfrak{A} \to M^{\mathcal{U}}$. Since the images of π and ρ are both hyperfinite, the argument from Theorem 4.1.5 gives that $\pi \sim \rho$. By Theorem 2.5.6 we get that π and ρ are unitarily equivalent. Thus $\beta_{(\mathfrak{A},M)}([\rho]) = [\rho^{\mathcal{U}}] = [\pi]$. $\square$

With the notation introduced in Definition 6.1.11 , we have the following corollary to Theorem 6.2.1

Corollary 6.2.2. *If $T(\mathfrak{A}) = UAT(\mathfrak{A})$, then for any McDuff M, $\mathbb{H}om(\mathfrak{A}, M^{\mathcal{U}}) \cong T(\mathfrak{A})$ via $\tilde{\alpha}_{(\mathfrak{A},M)}$.*

So we can immediately observe the following characterizations of extreme points in $T(\mathfrak{A})$ and $\mathbb{H}\mathrm{om}_w(\mathfrak{A}, M)$ when $T(\mathfrak{A}) = \mathrm{UAT}(\mathfrak{A})$.

Corollary 6.2.3. *Let $\mathfrak{A}$ be such that $T(\mathfrak{A}) = UAT(\mathfrak{A})$.*

1. *$T \in T(\mathfrak{A})$ is extreme if and only if $\pi_T(\mathfrak{A})' \cap X^{\mathcal{U}}$ is a factor where $\pi_T : \mathfrak{A} \to X^{\mathcal{U}}$ is a lift of T through $X^{\mathcal{U}}$ for any separable II_1-factor X.*

2. *The following are equivalent.*

   (a) *$[\pi] \in \mathbb{H}om_w(\mathfrak{A}, M)$ is extreme;*

   (b) *$\pi^{\mathcal{U}}(\mathfrak{A})' \cap M^{\mathcal{U}}$ is a factor;*

   (c) *$W^*(\pi(\mathfrak{A})) \subset M$ is a factor.*

Note that the equivalence between (2b) and (2c) is a purely algebraic statement with no reference to $\mathbb{H}\mathrm{om}_w(\mathfrak{A}, M)$.

6.3 Extreme Points in $\mathbb{H}\mathbf{om}_w(\mathfrak{A}, M)$: Amenability in Second Argument

A satisfying characterization of extreme points is also available when we shift our amenability assumption to the second argument of $\mathbb{H}\mathrm{om}_w(\mathfrak{A}, M)$. We state this in the following theorem.

Theorem 6.3.1. *Let $\mathfrak{A}$ be a (not necessarily nuclear) separable unital C^*-algebra. Then given a $*$-homomorphism $\pi : \mathfrak{A} \to R$, the following are equivalent.*

1. *$[\pi] \in \mathbb{H}om_w(\mathfrak{A}, R)$ is extreme;*

2. $W^*(\pi(\mathfrak{A})) \subset R$ *is a factor;*

3. $\pi^{\mathcal{U}}(\mathfrak{A})' \cap R^{\mathcal{U}}$ *is a factor.*

Proof. ((1) $\Rightarrow$ (2)): This is just Theorem 5.1.1.

((2) $\Rightarrow$ (3)): We have that

$$R \supset W^*(\pi(\mathfrak{A})) \cong W^*(\pi^{\mathcal{U}}(\mathfrak{A})) \subset R \subset R^{\mathcal{U}}$$

and

$$W^*(\pi^{\mathcal{U}}(\mathfrak{A}))' \cap R^{\mathcal{U}} = \pi^{\mathcal{U}}(\mathfrak{A})' \cap R^{\mathcal{U}}.$$

And since the factor $W^*(\pi(\mathfrak{A})) \subset R$ must be separable, finite, and hyperfinite, Corollary 6.1.6 or Theorem 6.1.7 implies that $\pi^{\mathcal{U}}(\mathfrak{A})' \cap R^{\mathcal{U}}$ must be a factor.

((3) $\Rightarrow$ (1)): This is just Theorem 6.1.13. □

Again, notice that the equivalence of (2) and (3) is a purely algebraic statement.

Remark 6.3.2. In Example 6.4(2) of [21], the existence of a locally universal separable II_1-factor S was established. This S has the property that any separable II_1-factor embeds into $S^{\mathcal{U}}$. Tensoring S with R preserves this property, so we may assume that S is McDuff. Therefore, we may consider the convex structure $\mathbb{H}\mathrm{om}(N, S^{\mathcal{U}})$ for *any* separable II_1-factor N without any additional embeddability assumptions.

Chapter 7

More on $\mathbb{H}\mathrm{om}_w(\mathfrak{A}, M)$

7.1 Stabilization

The "McDuffness"of the codomain of $\pi : \mathfrak{A} \to M$ allows us to coherently define the convex structure on $\mathbb{H}\mathrm{om}_w(\mathfrak{A}, M)$. Only considering McDuff codomains seems to provide some restrictions on our theory and collection of examples. Unfortunately, without a tensor factor of R in the target, it is unclear how to define a convex structure on $\mathbb{H}\mathrm{om}_w(\mathfrak{A}, N)$ for a non-McDuff N.

A natural way around this obstruction is to stabilize a given non-McDuff codomain. That is, given a non-McDuff factor N and a $*$-homomorphism

$$\pi : \mathfrak{A} \to N,$$

we compose π with the embedding

$$\mathrm{id}_N \otimes 1_R : N \to N \otimes R.$$

Using the notation from §3.2, this composition induces the map

$$(\mathrm{id}_N \otimes 1_R)_* : \mathbb{H}\mathrm{om}_w(\mathfrak{A}, N) \to \mathbb{H}\mathrm{om}_w(\mathfrak{A}, N \otimes R).$$

That is,

$$(\mathrm{id}_N \otimes 1_R)_*([\pi]) = [(\mathrm{id}_N \otimes 1_R) \circ \pi] = [\pi \otimes 1_R]$$

where

$$(\pi \otimes 1_R)(a) = \pi(a) \otimes 1_R.$$

It turns out that $(\mathrm{id}_N \otimes 1_R)_*$ is well-defined and injective:

$$(\pi \otimes 1_R \sim \rho \otimes 1_R) \Leftrightarrow (\pi \sim \rho).$$

Well-defined is a clear observation. Showing that $(\mathrm{id}_N \otimes 1_R)_*$ is injective is not obvious at all. The author would like to thank N. Ozawa for suggesting the proof of Theorem 7.1.4. First we need the following fact established by Haagerup in Section 4 of [22] concerning the notion of δ-related n-tuples of unitaries.

Definition 7.1.1 ([22])**.** Let N be a II_1-factor. For $n \in \mathbb{N}$ and $\delta > 0$, two n-tuples $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_n)$ of unitaries in N are *δ-related* if there is a sequence $\{a_j\} \subset N$ with

$$\sum_j a_j^* a_j = 1 = \sum_j a_j a_j^* \tag{7.1.1}$$

such that for every $1 \leq k \leq n$,

$$\sum_j ||a_j u_k - v_k a_j||_2^2 < \delta. \tag{7.1.2}$$

We say that $\{a_j\}$ is a sequence that witnesses that $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_n)$ are δ-related.

Theorem 7.1.2 ([22]). *Let N be a II_1-factor. For every $n \in \mathbb{N}$ and every $\varepsilon > 0$, there exists a $\delta(n,\varepsilon) > 0$ such that for any two $\delta(n,\varepsilon)$-related n-tuples of unitaries $(u_1,\dots,u_n)$ and $(v_1,\dots,v_n)$ in N, there exists a unitary $w \in N$ such that for every $1 \leq k \leq n$*

$$||wu_k - v_k w||_2 < \varepsilon.$$

Next we establish the following lemma.

Lemma 7.1.3. *Let N_1 and N_2 be separable II_1-factors, and let $(u_1,\dots,u_n)$ and $(v_1,\dots,v_n)$ be two n-tuples of unitaries in N_1. Fix $\delta > 0$, and let $z \in N_1 \otimes N_2$ be a unitary of the form*

$$z = \sum_{j=1}^{\infty} a_j \otimes b_j$$

where $\{b_j\} \subset N_2$ is an orthonormal basis in $L^2(N_2)$. If z is such that for every $1 \leq k \leq n$,

$$||z(u_k \otimes 1_{N_2}) - (v_k \otimes 1_{N_2})z||_2^2 < \delta,$$

then $(u_1,\dots,u_n)$ and $(v_1,\dots,v_n)$ are δ-related. Furthermore, $\{a_j\}$ is a sequence that witnesses that $(u_1,\dots,u_n)$ and $(v_1,\dots,v_n)$ are δ-related.

Proof. It suffices to show that $\{a_j\}$ is the sequence that witnesses that $(u_1,\dots,u_n)$ and $(v_1,\dots,v_n)$ are δ-related. First we show that the summing condition (7.1.1) is satisfied. This is a consequence of the fact that $\sum_j a_j \otimes b_j$ is a unitary. Let $\mathbb{E}_1$ be the canonical normal conditional expectation

$$\mathbb{E}_1 : N_1 \otimes N_2 \to N_1 \otimes \mathbb{C}1_{N_2}$$

onto the first tensor factor. So on simple tensors, $\mathbb{E}_1(x \otimes y) = x \otimes \tau(y)$. So we have

$$\begin{aligned}
1 &= \mathbb{E}_1(z^* z) \\
&= \mathbb{E}_1\Big(\sum_{j',j} a_{j'}^* a_j \otimes b_{j'}^* b_j\Big) \\
&= \sum_{j',j} a_{j'}^* a_j \otimes \tau(b_{j'}^* b_j) \\
&= \sum_{j} a_j^* a_j \otimes 1.
\end{aligned}$$

Thus $\sum_j a_j^* a_j = 1$ and by a symmetric argument, $\sum_j a_j a_j^* = 1$.

To check (7.1.2), fix $1 \leq k \leq n$ and observe

$$\begin{aligned}
\sum_j \Big|\Big| a_j u_k - v_k a_j \Big|\Big|_2^2 &= \sum_j \tau\Big(\big(a_j u_k - v_k a_j\big)^* \big(a_j u_k - v_k a_j\big)\Big) \\
&= \sum_{j',j} \tau\Big(\big((a_{j'} u_k - v_k a_{j'})^* (a_j u_k - v_k a_j)\big) \otimes b_{j'}^* b_j\Big) \\
&= \tau\Big(\Big(\sum_{j'} (a_{j'} u_k - v_k a_{j'}) \otimes b_{j'}\Big)^* \Big(\sum_j (a_j u_k - v_k a_j) \otimes b_j\Big)\Big) \\
&= ||z(u_k \otimes 1_{N_2}) - (v_k \otimes 1_{N_2}) z||_2^2 \\
&< \delta.
\end{aligned}$$

□

Theorem 7.1.4. *Let N_1 and N_2 be arbitrary separable II_1-factors. Given $*$-homomorphisms $\pi, \rho : \mathfrak{A} \to N_1$, consider $\pi \otimes 1_{N_2}, \rho \otimes 1_{N_2} : \mathfrak{A} \to N_1 \otimes N_2$. If $\pi \otimes 1_{N_2} \sim \rho \otimes 1_{N_2}$ then $\pi \sim \rho$.*

Proof. Because a unital C^*-algebra is generated by its unitaries, it suffices to show that for any $\varepsilon > 0$ and any set of unitaries $u_1, \ldots, u_n \in \mathfrak{A}$, there exists a unitary $w \in N$ such that for every $1 \leq k \leq n$,

$$||w\pi(u_k) - \rho(u_k)w||_2 < \varepsilon.$$

Fix $\varepsilon > 0$ and unitaries $u_1, \ldots, u_n \in \mathfrak{A}$. Let $\delta(n, \varepsilon) > 0$ be such that if $(v_1, \ldots, v_n)$ and $(v'_1, \ldots, v'_n)$ are $\delta(n, \varepsilon)$-related n-tuples of unitaries in N_1, then there is a unitary $w \in N_1$ such that for every $1 \leq k \leq n$,

$$||wv_k - v'_k w||_2 < \varepsilon$$

as guaranteed by Theorem 7.1.2. Thus we will be done if we show that $(\pi(u_1), \ldots, \pi(u_n))$ and $(\rho(u_1), \ldots, \rho(u_n))$ are $\delta(n, \varepsilon)$-related.

Since $\pi \otimes 1_{N_2} \sim \rho \otimes 1_{N_2}$, we can find a unitary $z \in N_1 \otimes N_2$ such that for every $1 \leq k \leq n$,

$$||z(\pi(u_k) \otimes 1_{N_2}) - (\rho(u_k) \otimes 1_{N_2})z||_2^2 < \delta(n, \varepsilon).$$

By standard approximation arguments we may assume that

$$z = \sum_{j=1}^{\infty} a_j \otimes b_j$$

where $\{b_j\} \subset N_2$ is an orthonormal basis in $L^2(N_2)$ (guaranteed by Gram-Schmidt). Then by Lemma 7.1.3 we have that $(\pi(u_1), \ldots, \pi(u_n))$ and $(\rho(u_1), \ldots, \rho(u_n))$ are $\delta(n, \varepsilon)$-related. $\square$

We can also use the proof strategy from Theorem 7.1.4 to provide an alternative proof to Corollary 3.3 in [13]: Let N_1 and N_2 be separable II_1 factors, and let $\theta_i \in \mathrm{Aut}(N_i)$, $i = 1, 2$. Then $\theta_1 \otimes \theta_2 \in \mathrm{Aut}(N_1 \otimes N_2)$ is approximately inner if and only if θ_1 and θ_2 are both approximately inner.

Example 7.1.5. It is well-known that $L(\mathbb{F}_2)$ has non-approximately inner automorphisms (e.g. exchanging generators). Let β be such a non-approximately inner automorphism of $L(\mathbb{F}_2)$. Let $\iota : C_r^*(\mathbb{F}_2) \to L(\mathbb{F}_2)$ be the canonical embedding, then we have

$$\pi := \iota \nsim \beta \circ \iota =: \rho.$$

Then by Theorem 7.1.4 we have that $[\pi \otimes 1_R] \neq [\rho \otimes 1_R]$. Since $\mathbb{H}\mathrm{om}_w(C_r^*(\mathbb{F}_2), L(\mathbb{F}_2) \otimes R)$ is convex, there is at least an interval's worth,

$$\{t[\pi \otimes 1_R] + (1-t)[\rho \otimes 1_R] : t \in [0, 1]\},$$

of inequivalent $*$-homomorphisms of $C_r^*(\mathbb{F}_2)$ into $L(\mathbb{F}_2) \otimes R$.

For the next theorem, we will use the following standard fact about complete metric spaces.

Proposition 7.1.6. *Let (X, d) and (Y, d') be complete metric spaces. If $\varphi : X \to Y$ satisfies the following conditions*

1. *φ is continuous,*

2. *φ is injective,*

3. $\{\varphi(x_n)\}$ *is Cauchy in* $d' \Rightarrow \{x_n\}$ *is Cauchy in* d*;*

then φ *is a homeomorphism onto its image and* $\varphi(X)$ *is closed in* Y*.*

Theorem 7.1.7. *For any two separable* II_1*-factors* N_1 *and* N_2*, the map*

$$(id_{N_1} \otimes 1_{N_2})_* : \mathbb{H}om_w(\mathfrak{A}, N_1) \to \mathbb{H}om_w(\mathfrak{A}, N_1 \otimes N_2)$$

is a homeomorphism onto its image, which is closed in $\mathbb{H}om_w(\mathfrak{A}, N_1 \otimes N_2)$*. In particular, we may consider* $\mathbb{H}om_w(\mathfrak{A}, N_1)$ *as a closed subset of* $\mathbb{H}om_w(\mathfrak{A}, N_1 \otimes N_2)$*.*

Proof. Let $\{u_k\}$ be a sequence of unitaries that generate $\mathfrak{A}$. As in Remark 3.3, we can use these unitaries to define metrics d_{N_1} and $d_{N_1 \otimes N_2}$ on $\mathbb{H}\mathrm{om}_w(\mathfrak{A}, N_1)$ and $\mathbb{H}\mathrm{om}_w(\mathfrak{A}, N_1 \otimes N_2)$ respectively. That is,

$$d_{N_1}([\pi],[\rho]) = \inf_{v \in \mathcal{U}(N_1)} \Big(\sum_{k=1}^{\infty} \frac{1}{2^{2k}} ||v\pi(u_k) - \rho(u_k)v||_2^2 \Big)^{\frac{1}{2}}$$

and

$$d_{N_1 \otimes N_2}([\pi],[\rho]) = \inf_{z \in \mathcal{U}(N_1 \otimes N_2)} \Big(\sum_{k=1}^{\infty} \frac{1}{2^{2k}} ||z\pi(u_k) - \rho(u_k)z||_2^2 \Big)^{\frac{1}{2}}.$$

Theorem 7.1.4 shows that $(\mathrm{id}_{N_1} \otimes 1_{N_2})_*$ is injective, and $(\mathrm{id}_{N_1} \otimes 1_{N_2})_*$ is continuous by Proposition 3.2.4. Now, by Proposition 7.1.6, it suffices to show that if $\{[\pi_n \otimes 1_{N_2}]\}$ is Cauchy in $d_{N_1 \otimes N_2}$ then $\{[\pi_n]\}$ is Cauchy in d_{N_1}. Fix $\varepsilon > 0$. Let $J \in \mathbb{N}$ be such that $\sum_{k=J+1}^{\infty} \frac{4}{2^{2k}} < \frac{\varepsilon^2}{2}$. By Theorem 7.1.2 there is a $\delta\Big(J, \frac{\varepsilon}{\sqrt{2J}}\Big)$ such that for any pair of $\delta\Big(J, \frac{\varepsilon}{\sqrt{2J}}\Big)$-related n-tuples of unitaries $(w_1, \ldots, w_J)$ and $(w_1', \ldots, w_J')$ in N_1 there is a unitary $v \in N_1$ such that for every $1 \le k \le J$,

$$||vw_k - w_k'v||_2 < \frac{\varepsilon}{\sqrt{2J}}$$

or

$$||vw_k - w'_k v||_2^2 < \frac{\varepsilon^2}{2J}.$$

Now let $K \in \mathbb{N}$ be such that for $n, m \geq K$ we have

$$d_{N_1 \otimes N_2}([\pi_n \otimes 1_{N_2}], [\pi_m \otimes 1_{N_2}])^2 < \frac{\delta\Big(J, \frac{\varepsilon}{\sqrt{2J}}\Big)}{2^{2J}}.$$

We will show that for any $n, m \geq K, d_{N_1}([\pi_n], [\pi_m]) < \varepsilon$. Fix $n, m \geq K$. From the definition of $d_{N_1 \otimes N_2}$ there is a unitary $z \in N_1 \otimes N_2$ of the form $z = \sum_j a_j \otimes b_j$ with $\{b_j\}$ an orthonormal basis in $L^2(N_2)$ such that

$$\sum_{k=1}^{\infty} \frac{1}{2^{2k}} ||z((\pi_n \otimes 1_{N_2})(u_k)) - ((\pi_m \otimes 1_{N_2})(u_k))z||_2^2 < \frac{\delta\Big(J, \frac{\varepsilon}{\sqrt{2J}}\Big)}{2^{2J}}.$$

So, for $1 \leq k' \leq J$ we have

$$\begin{aligned} &\frac{1}{2^{2k'}} ||z((\pi_n \otimes 1_{N_2})(u_{k'})) - ((\pi_m \otimes 1_{N_2})(u_{k'}))z||_2^2 \\ &\leq \sum_{k=1}^{\infty} \frac{1}{2^{2k}} ||z((\pi_n \otimes 1_{N_2})(u_k)) - ((\pi_m \otimes 1_{N_2})(u_k))z||_2^2 \\ &< \frac{\delta\Big(J, \frac{\varepsilon}{\sqrt{2J}}\Big)}{2^{2J}}. \end{aligned}$$

Therefore for every $1 \leq k \leq J$,

$$||z((\pi_n \otimes 1_{N_2})(u_k)) - ((\pi_m \otimes 1_{N_2})(u_k))z||_2^2 < 2^{2k} \frac{\delta\Big(J, \frac{\varepsilon}{\sqrt{2J}}\Big)}{2^{2J}} \leq \delta\Big(J, \frac{\varepsilon}{\sqrt{2J}}\Big).$$

Thus by Lemma 7.1.3 we have that $(\pi_n(u_1), \ldots, \pi_n(u_J))$ and $(\pi_m(u_1), \ldots, \pi_m(u_J))$ are $\delta\Big(J, \frac{\varepsilon}{\sqrt{2J}}\Big)$-related. By Theorem 7.1.2, there is a unitary $v \in N_1$ such that for every $1 \leq k \leq J$,

$$||v\pi_n(u_k) - \pi_m(u_k)v||_2^2 < \frac{\varepsilon^2}{2J}.$$

So to complete the proof we observe that

$$\begin{aligned}
d_{N_1}([\pi_n],[\pi_m])^2 &\leq \sum_{k=1}^{\infty} \frac{1}{2^{2k}} ||v\pi_n(u_k) - \pi_m(u_k)v||_2^2 \\
&= \sum_{k=1}^{J} \frac{1}{2^{2k}} ||v\pi_n(u_k) - \pi_m(u_k)v||_2^2 \\
&\quad + \sum_{k=J+1}^{\infty} \frac{1}{2^{2k}} ||v\pi_n(u_k) - \pi_m(u_k)v||_2^2 \\
&< J \cdot \frac{\varepsilon^2}{2J} + \frac{\varepsilon^2}{2} \\
&= \varepsilon^2.
\end{aligned}$$

□

Therefore, if N is an arbitrary separable II_1-factor, we may consider $\mathbb{H}\mathrm{om}_w(\mathfrak{A}, N)$ as a closed subset of the convex set $\mathbb{H}\mathrm{om}_w(\mathfrak{A}, N \otimes R)$.

Example 7.1.8. When $\mathfrak{A}$ is such that $\mathrm{UAT}(\mathfrak{A}) = T(\mathfrak{A})$, $\mathbb{H}\mathrm{om}_w(\mathfrak{A}, N) \cong T(\mathfrak{A})$; so

$$(\mathrm{id}_N \otimes 1_R)_*(\mathbb{H}\mathrm{om}_w(\mathfrak{A}, N)) = \mathbb{H}\mathrm{om}_w(\mathfrak{A}, N \otimes R).$$

Example 7.1.9. If N is a non-hyperfinite solid II_1-factor, for example $L(\mathbb{F}_2)$ or $L(\mathbb{Z}^2 \rtimes \mathrm{SL}(2,\mathbb{Z}))$, then all of its subfactors are also solid, see [31] and [33]. Non-hyperfinite solid factors are prime. So we know that $N \otimes R$ does not embed into N. Thus, if we let $\mathfrak{A}$ be a separable dense C^*-subalgebra of $N \otimes R$ with a unique trace, then $\mathbb{H}\mathrm{om}_w(\mathfrak{A}, N)$ is empty, but $\mathbb{H}\mathrm{om}_w(\mathfrak{A}, N \otimes R)$ is nonempty. Such a dense monotracial C^*-subalgebra exists by applying the argument found in Lemma 4.2.3 except all W^*'s should be replaced with C^*'s and all instances of "weak" with "norm." Therefore, in contrast to the situation of Example 7.1.8, we have that $(\mathrm{id}_M \otimes 1_R)_*(\mathbb{H}\mathrm{om}_w(\mathfrak{A}, N))$ is empty.

It would be interesting to know more about the way $\mathbb{H}\mathrm{om}_w(\mathfrak{A}, N)$ sits inside $\mathbb{H}\mathrm{om}_w(\mathfrak{A}, N \otimes M)$ for M McDuff and $\mathfrak{A}$ such that $\mathrm{UAT}(\mathfrak{A}) \neq T(\mathfrak{A})$. In particular, it would be nice to find an example of a separable II_1-factor N and a separable C^*-algebra $\mathfrak{A}$ such that $\mathbb{H}\mathrm{om}_w(\mathfrak{A}, N)$ fails to be convex.

7.2 A Product

Remark 7.3 at the end of [8] suggests a way to define a product on $\mathbb{H}\mathrm{om}(N, R^{\mathcal{U}})$ when there exists a comultiplication on $\mathfrak{A}$. We show here that we can also define a product on $\mathbb{H}\mathrm{om}_w(\mathfrak{A}, R)$ under the same assumptions. Before restricting ourselves, we discuss this idea in more generality. We can define a binary operation in the following way.

Definition 7.2.1. Given C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$ and McDuff II_1-factors M and N, we define the binary operation

$$\bullet : \mathbb{H}\mathrm{om}_w(\mathfrak{A}, M) \times \mathbb{H}\mathrm{om}_w(\mathfrak{B}, N) \to \mathbb{H}\mathrm{om}_w(\mathfrak{A} \otimes_{\min} \mathfrak{B}, M \otimes N)$$

to be given by

$$[\pi] \bullet [\rho] = [\pi \otimes \rho].$$

Proposition 7.2.2. *The map $\bullet$ is well-defined, jointly continuous, and affine-distributive on both sides–that is, the following equations hold.*

$$(t[\pi_1] + (1-t)[\pi_2]) \bullet [\rho] = t([\pi_1] \bullet [\rho]) + (1-t)([\pi_2] \bullet [\rho]),$$

$$[\pi] \bullet (s[\rho_1] + (1-s)[\rho_2]) = s([\pi] \bullet [\rho_1]) + (1-s)([\pi] \bullet [\rho_2])$$

Proof. Well-defined and continuous is routine.

We will show one side of the affine-distributive claim, and the other side will follow by a symmetric argument. We show that

$$[(\sigma_M \otimes \mathrm{id}_N)(\pi_1 \otimes p \otimes \rho + \pi_2 \otimes p^\perp \otimes \rho)] = [\sigma_{M\otimes N}(\pi_1 \otimes \rho \otimes p + \pi_2 \otimes \rho \otimes p^\perp)]$$

for σ_M and $\sigma_{M\otimes N}$ regular isomorphisms.

Let $\mathscr{D} : M \otimes N \otimes R \to M \otimes R \otimes N$ be the canonical isomorphism from the commutativity of tensor products. Define $s_{M\otimes N} : M \otimes N \otimes R \to M \otimes N$ by $s_{M\otimes N} := (\sigma_M \otimes \mathrm{id}_N) \circ \mathscr{D}$. Then we have that

$$\begin{aligned} s_{M\otimes N}^{-1} &= \mathscr{D}^{-1} \circ (\sigma_M^{-1} \otimes \mathrm{id}_N) \\ &\sim \mathscr{D}^{-1} \circ (\mathrm{id}_M \otimes 1_R \otimes \mathrm{id}_N) \\ &= \mathrm{id}_M \otimes \mathrm{id}_N \otimes 1_R \\ &= \mathrm{id}_{M\otimes N} \otimes 1_R. \end{aligned}$$

So by Proposition 3.1.2 (2), $s_{M\otimes N} \sim \sigma_{M\otimes N}$. So we get

$$\begin{aligned} [(\sigma_M \otimes \mathrm{id}_N)(\pi_1 \otimes p \otimes \rho + \pi_2 \otimes p^\perp \otimes \rho)] &= [s_{M\otimes N}(\pi_1 \otimes \rho \otimes p + \pi_2 \otimes \rho \otimes p^\perp)] \\ &= [\sigma_{M\otimes N}(\pi_1 \otimes \rho \otimes p + \pi_2 \otimes \rho \otimes p^\perp)]. \end{aligned}$$

□

Example 7.2.3. In the case that $\mathfrak{A}$ and $\mathfrak{B}$ are nuclear, $\bullet$ is simply the tensor product on traces.

Now, if there is a unital $*$-homomorphism $\gamma : \mathfrak{A} \to \mathfrak{A} \otimes_{\min} \mathfrak{A}$ then we may use the functoriality discussed in §3.2 to define a product on $\mathbb{H}\mathrm{om}_w(\mathfrak{A}, R)$ as follows. Let

$$\bullet_\gamma : \mathbb{H}\mathrm{om}_w(\mathfrak{A}, R) \times \mathbb{H}\mathrm{om}_w(\mathfrak{A}, R) \to \mathbb{H}\mathrm{om}_w(\mathfrak{A}, R)$$

be given by

$$\bullet_\gamma = \epsilon_* \circ \gamma^* \circ \bullet$$

where $\epsilon : R \otimes R \to R$ is an isomorphism. That is,

$$[\pi] \bullet_\gamma [\rho] = [\epsilon \circ (\pi \otimes \rho) \circ \gamma].$$

We remark that $\bullet_\gamma$ is independent of the choice of ϵ.

We do not have a priori that $\bullet_\gamma$ is associative. However, if we further assume that $\gamma : \mathfrak{A} \to \mathfrak{A} \otimes_{\min} \mathfrak{A}$ is coassociative, then we do get that $\bullet_\gamma$ is associative.

Definition 7.2.4. A $*$-homomorphism $\gamma : \mathfrak{A} \to \mathfrak{A} \otimes_{\min} \mathfrak{A}$ is *coassociative* if

$$(\mathrm{id}_{\mathfrak{A}} \otimes \gamma) \circ \gamma = (\gamma \otimes \mathrm{id}_{\mathfrak{A}}) \circ \gamma.$$

Proposition 7.2.5. *If $\gamma : \mathfrak{A} \to \mathfrak{A} \otimes_{min} \mathfrak{A}$ is a coassociative $*$-homomorphism, then $\bullet_\gamma$ is associative.*

Proof. We must show

$$[\pi_1] \bullet_\gamma ([\pi_2] \bullet_\gamma [\pi_3]) = ([\pi_1] \bullet_\gamma [\pi_2]) \bullet_\gamma [\pi_3]. \tag{7.2.1}$$

That is

$$[\sigma_R \circ (\pi_1 \otimes (\sigma_R \circ (\pi_2 \otimes \pi_3) \circ \gamma)) \circ \gamma] = [\sigma_R \circ ((\sigma_R \circ (\pi_1 \otimes \pi_2) \circ \gamma) \otimes \pi_3) \circ \gamma],$$

where σ_R is a regular isomorphism. (We can take $\sigma_R = \epsilon$ if we like.)

We have

$$\sigma_R \circ (\pi_1 \otimes (\sigma_R \circ (\pi_2 \otimes \pi_3) \circ \gamma)) \circ \gamma$$

$$= \sigma_R \circ ((\sigma_R \circ \sigma_R^{-1} \circ \pi_1 \circ \mathrm{id}_{\mathfrak{A}}) \otimes (\sigma_R \circ (\pi_2 \otimes \pi_3) \circ \gamma)) \circ \gamma$$

$$= \sigma_R \circ (\sigma_R \otimes \sigma_R) \circ ((\sigma_R^{-1} \circ \pi_1) \otimes (\pi_2 \otimes \pi_3)) \circ (\mathrm{id}_{\mathfrak{A}} \otimes \gamma) \circ \gamma$$

$$\sim \sigma_R \circ (\sigma_R \otimes \sigma_R) \circ ((\pi_1 \otimes 1_R) \otimes (\pi_2 \otimes \pi_3)) \circ (\mathrm{id}_{\mathfrak{A}} \otimes \gamma) \circ \gamma \tag{7.2.2}$$

$$= \sigma_R \circ (\sigma_R \otimes \sigma_R) \circ ((\pi_1 \otimes 1_R) \otimes (\pi_2 \otimes \pi_3)) \circ (\gamma \otimes \mathrm{id}_{\mathfrak{A}}) \circ \gamma \tag{7.2.3}$$

$$\sim \sigma_R \circ (\sigma_R \otimes \sigma_R) \circ ((\pi_1 \otimes \pi_2) \otimes (\pi_3 \otimes 1_R)) \circ (\gamma \otimes \mathrm{id}_{\mathfrak{A}}) \circ \gamma \tag{7.2.4}$$

$$\sim \sigma_R \circ (\sigma_R \otimes \sigma_R) \circ ((\pi_1 \otimes \pi_2) \otimes (\sigma_R^{-1} \circ \pi_3)) \circ (\gamma \otimes \mathrm{id}_{\mathfrak{A}}) \circ \gamma \tag{7.2.5}$$

$$= \sigma_R \circ ((\sigma_R \circ (\pi_1 \otimes \pi_2) \circ \gamma) \otimes \pi_3) \circ \gamma.$$

Here (7.2.2) and (7.2.5) follow from the fact that σ_R is a regular isomorphism, and (7.2.3) follows because γ is coassociative. To verify (7.2.4) we consider $(\pi_1 \otimes 1_R) \otimes (\pi_2 \otimes \pi_3)$ as an element of $R^{\otimes 4}$. Note that since all $*$-endomorphisms of R are approximately inner, we have that $\sigma_R(\mathrm{id}_R \otimes 1_R) \sim \mathrm{id}_R \sim \sigma_R(1_R \otimes \mathrm{id}_R)$. Thus we have $\mathrm{id}_R \otimes 1_R \sim 1_R \otimes \mathrm{id}_R$ and

$$\begin{aligned}(\pi_1 \otimes 1_R) \otimes (\pi_2 \otimes \pi_3) &= \pi_1 \otimes (1_R \otimes \pi_2) \otimes \pi_3 \\ &\sim \pi_1 \otimes (\pi_2 \otimes 1_R) \otimes \pi_3 \\ &= (\pi_1 \otimes \pi_2) \otimes (1_R \otimes \pi_3) \\ &\sim (\pi_1 \otimes \pi_2) \otimes (\pi_3 \otimes 1_R),\end{aligned}$$

verifying (7.2.4). So (7.2.1) has been demonstrated, and the proof is complete. □

Example 7.2.6. If $\mathfrak{A}$ is a compact quantum group (cf. [50]) and we take γ to be the comultiplication Δ (denoted as Φ in [50]), then we are in the situation of Proposition 7.2.5.

Example 7.2.7. If $\mathfrak{A}$ is a nuclear compact quantum group with comultiplication Δ and tracial Haar state it can be shown that under this product, the representative of the Haar state behaves as a zero. That is, if $[H] \in \mathbb{H}\mathrm{om}_w(\mathfrak{A}, R)$ is a lift of the Haar state through R, then for any $[\pi] \in \mathbb{H}\mathrm{om}_w(\mathfrak{A}, R), [H] \bullet_\Delta [\pi] = [\pi] \bullet_\Delta [H] = [H]$.

We go a bit further to observe that in the case that $\mathfrak{A}$ is a compact quantum group with comultiplication Δ, $\mathbb{H}\mathrm{om}_w(\mathfrak{A}, M)$ can be seen to behave as a sort of right (or left) $\mathbb{H}\mathrm{om}_w(\mathfrak{A}, R)$-module with the action given by

$$\bullet_\Delta^M = (\sigma_M)_* \circ \Delta^* \circ \bullet$$

where σ_M is a regular isomorphism. That is, for $[\pi] \in \mathbb{H}\mathrm{om}_w(\mathfrak{A}, M)$ and $[\rho] \in \mathbb{H}\mathrm{om}_w(\mathfrak{A}, R)$ we have

$$[\pi] \bullet_\Delta^M [\rho] = [\sigma_M \circ (\pi \otimes \rho) \circ \Delta].$$

Remark 7.2.8. It would be interesting to study the algebraic properties of this product. Can we find examples where $\bullet_\gamma$ is not associative? In the associative case, how do powers $[\pi]^n$ behave? How do faces react to this product?

This approach should extend painlessly to the object $\mathbb{H}\mathrm{om}(\mathfrak{A}, M^{\mathcal{U}})$ using the results from Chapter 6.

Chapter 8

Simplices in $\mathbb{H}\mathrm{om}(N, R^{\mathcal{U}})$

In this chapter, we will work in Brown's original context from [8] (see §2.8 for a brief survey). Let N be a separable II_1-factor, and let R denote the separable hyperfinite II_1-factor. We denote by $\mathbb{H}\mathrm{om}(N, R^{\mathcal{U}})$ the collection of unitary equivalence classes of $*$-homomorphisms $N \to R^{\mathcal{U}}$ where $\mathcal{U}$ is a free ultrafilter on the natural numbers. We let $[\pi]$ denote the equivalence class of the $*$-homomorphism $\pi : N \to R^{\mathcal{U}}$. It was shown in [8] and [11] that $\mathbb{H}\mathrm{om}(N, R^{\mathcal{U}})$ can be considered as closed bounded convex subset of a Banach space.

Definition 8.1. Let $F_{[\pi]}$ denote the minimal face in $\mathbb{H}\mathrm{om}(N, R^{\mathcal{U}})$ containing $[\pi]$. $F_{[\pi]}$ is obtained by intersecting all faces in $\mathbb{H}\mathrm{om}(N, R^{\mathcal{U}})$ that contain $[\pi]$. Let $\dim(F_{[\pi]})$ be the dimension of the minimal face, given by the smallest n such that $F_{[\pi]}$ affinely embeds into $\mathbb{R}^n$; if there is no such n, then we say $\dim(F_{[\pi]}) = \infty$. As a convention, $\dim(F_{[\pi]}) = 0$ if and only if $F_{[\pi]} = \{\bullet\}$ is a singleton.

The work of this chapter builds upon the results of [8] in order to further establish a connection between the convex geometry of $\mathbb{H}\mathrm{om}(N, R^{\mathcal{U}})$ and the algebraic data of the embeddings of N into $R^{\mathcal{U}}$. In particular, given $\pi : N \to R^{\mathcal{U}}$, we will examine the relationship between $F_{[\pi]}$ and the relative commutant $\pi(N)' \cap R^{\mathcal{U}}$. The work of this chapter culminates in the following theorem.

Theorem 8.2. *Let the embedding $\pi : N \to R^{\mathcal{U}}$ be given.*

1. *If $\dim(\mathcal{Z}(\pi(N)' \cap R^{\mathcal{U}})) = n < \infty$ then $F_{[\pi]}$ is an n-vertex simplex.*

2. *If $\varphi \in t_1[\pi_1] + \cdots + t_n[\pi_n]$ where $0 < t_j < 1$ and $[\pi_j]$ is an extreme point for every $1 \leq j \leq n$, then*

$$\varphi(N)' \cap R^{\mathcal{U}} \cong \oplus_{j=1}^{n} \pi_j(N)' \cap R^{\mathcal{U}}.$$

3. $\dim(F_{[\pi]}) + 1 = \dim(\mathcal{Z}(\pi(N)' \cap R^{\mathcal{U}}))$.

This theorem is a generalization of Theorem 2.8.11–different from the generalization given by Theorem 6.1.4–in the sense that Theorem 2.8.11 gives part (3) of Theorem 8.2 in the case where $\dim(F_{[\pi]}) + 1 = \dim(\mathcal{Z}(\pi(N)' \cap R^{\mathcal{U}})) = 1$. As mentioned before, the question of existence of extreme points in $\mathbb{H}\text{om}(N, R^{\mathcal{U}})$ is an equivalent formulation of a well-known open question (given any separable II_1-factor N, is there an embedding $\pi : N \to R^{\mathcal{U}}$ such that $\pi(N)' \cap R^{\mathcal{U}}$ is a factor?). Theorem 8.2 informs us about the convex geometry of $\mathbb{H}\text{om}(N, R^{\mathcal{U}})$ and thus gives us deeper insight into this open question. In particular, we have the following corollary.

Corollary 8.3. *The following are equivalent.*

- *There is an embedding $\pi : N \to R^{\mathcal{U}}$ such that $\pi(N)' \cap R^{\mathcal{U}}$ is a factor.*

- *There is an embedding $\rho : N \to R^{\mathcal{U}}$ such that $\mathcal{Z}(\pi(N)' \cap R^{\mathcal{U}})$ is finite dimensional.*

8.1 Proof of Part (1) of Theorem 8.2

From now on, fix $\pi : N \to R^{\mathcal{U}}$. To allow an abuse of notation, let $\sigma : (R \otimes R)^{\mathcal{U}} \to R^{\mathcal{U}}$ denote the isomorphism induced by an isomorphism $\sigma : R \otimes R \to R$. The following Proposition shows that given a cutdown π_p of π, it is always unitarily equivalent to a cutdown of π by a projection of smaller trace. This can be considered as a rescaling propostion.

Proposition 8.1.1. *Let p be a projection in $\pi(N)' \cap R^{\mathcal{U}}$. Then for any nonzero projection $Q \in R^{\mathcal{U}}$, we have*

$$[\pi_p] = [\sigma(1 \otimes \pi)_{\sigma(1 \otimes p)}] = [\sigma(1 \otimes \pi)_{\sigma(Q \otimes p)}].$$

Proof. To show the first equality, by Proposition 2.8.1 there is a unitary $u \in R^{\mathcal{U}}$ so that $\sigma(1 \otimes \pi)(x) = uxu^*$ for every $x \in W^*(\pi(N) \cup \{p\})$. Then by Proposition 2.8.10, we have

$$[\pi_p] = [(\mathrm{Ad}u \circ \pi)_{upu^*}] = [\sigma(1 \otimes \pi)_{\sigma(1 \otimes p)}].$$

For the second equality, we will appeal to the fact that we can take $\theta_{\sigma(Q' \otimes p)} =$

$\sigma \circ (\theta_{Q'} \otimes \theta_p) \circ \sigma^{-1}$ for any projection Q' (see Definition 3.3.2 in [8]). Thus we have

$$\begin{aligned}
\sigma(1 \otimes \pi)_{\sigma(Q\otimes p)} &= \theta_{\sigma(Q\otimes p)}(\sigma(Q \otimes p)\sigma(1 \otimes \pi)) \\
&= \theta_{\sigma(Q\otimes p)}(\sigma(Q \otimes p\pi)) \\
&= \sigma \circ (\theta_Q \otimes \theta_p) \circ \sigma^{-1}(\sigma(Q \otimes p\pi)) \\
&= \sigma \circ (\theta_Q \otimes \theta_p)(Q \otimes p\pi) \\
&= \sigma(1 \otimes \theta_p(p\pi)) \\
&= \sigma \circ (1 \otimes \theta_p) \circ \sigma^{-1}(\sigma(1 \otimes p)\sigma(1 \otimes \pi)) \\
&= \theta_{\sigma(1\otimes p)}(\sigma(1 \otimes p)\sigma(1 \otimes \pi)) \\
&= \sigma(1 \otimes \pi)_{\sigma(1\otimes p)}.
\end{aligned}$$

□

The next proposition addresses convex combinations of cutdowns.

Proposition 8.1.2. *Let $p, q \in \pi(N)' \cap R^{\mathcal{U}}$ be projections with $\tau(p) = \tau(q)$.*

$$t[\pi_p] + (1-t)[\pi_q] = [\sigma(1 \otimes \pi)_{(\sigma(S\otimes p)+\sigma(S^{\perp}\otimes q))}]$$

for any projection $S \in R^{\mathcal{U}}$ with $\tau(S) = t$.

Proof. By Example 4.5 of [8] we have that

$$t[\pi_p] + (1-t)[\pi_q] = [\sigma(S \otimes \pi_p) + \sigma(S^{\perp} \otimes \pi_q)]$$

for any projection $S \in R^{\mathcal{U}}$ with $\tau(S) = t$. So we must show

$$[\sigma(1 \otimes \pi)_{\sigma(S\otimes p)+\sigma(S^{\perp}\otimes q)}] = [\sigma(S \otimes \pi_p) + \sigma(S^{\perp} \otimes \pi_q)].$$

By definition

$$\sigma(1 \otimes \pi)_{\sigma(S\otimes p)+\sigma(S^\perp\otimes q)} = \theta_{\sigma(S\otimes p)+\sigma(S^\perp\otimes q)}(\sigma(S \otimes p\pi) + \sigma(S^\perp \otimes q\pi))$$

and

$$\sigma(S \otimes \pi_p) + \sigma(S^\perp \otimes \pi_q) = \sigma(S \otimes \theta_p(p\pi)) + \sigma(S^\perp \otimes \theta_q(q\pi)).$$

Now note that,

$$\begin{aligned}\tau(\theta_{\sigma(S\otimes p)+\sigma(S^\perp\otimes q)}(\sigma(S \otimes p)) &= \frac{\tau(S)\cdot\tau(p)}{\tau(\sigma(S\otimes p)+\sigma(S^\perp\otimes q))} \\ &= \frac{\tau(S)\cdot\tau(p)}{\tau(p)} \\ &= \tau(S).\end{aligned}$$

Let $p' := \theta_{\sigma(S\otimes p)+\sigma(S^\perp\otimes q)}(\sigma(S \otimes p))$ and consider

$$\psi : p'R^{\mathcal{U}}p' \to \sigma(S \otimes 1)R^{\mathcal{U}}\sigma(S \otimes 1)$$

given by

$$\psi = \sigma \circ (S \otimes \mathrm{id}) \circ \theta_{\sigma(S\otimes p)} \circ \theta^{-1}_{\sigma(S\otimes p)+\sigma(S^\perp\otimes q)}\big|_{p'R^{\mathcal{U}}p'}.$$

Let $u \in R^{\mathcal{U}}$ be a unitary such that for every $a \in N$,

$$\sigma(1 \otimes \theta_p(p\pi(a))) = u\theta_p(p\pi(a))u^*$$

as provided by Proposition 2.8.2.

So, for $a \in N$ we have

$$\psi(\theta_{\sigma(S\otimes p)+\sigma(S^\perp\otimes q)}(\sigma(S \otimes p\pi(a))))$$

$$= \sigma \circ (S \otimes \mathrm{id}) \circ \theta_{\sigma(S\otimes p)}(\sigma(S \otimes p\pi(a)))$$

$$= \sigma \circ (S \otimes \mathrm{id}) \circ \sigma \circ (\theta_S \otimes \theta_p) \circ \sigma^{-1}(\sigma(S \otimes p\pi(a))) \tag{8.1.1}$$

$$= \sigma \circ (S \otimes \mathrm{id}) \circ \sigma \circ (\theta_S \otimes \theta_p)(S \otimes p\pi(a))$$

$$= \sigma \circ (S \otimes \mathrm{id})(\sigma(1 \otimes \theta_p(p\pi(a))))$$

$$= \sigma(S \otimes \sigma(1 \otimes \theta_p(p\pi(a))))$$

$$= \sigma(S \otimes u\theta_p(p\pi(a))u^*)$$

$$= \sigma(S \otimes u)\sigma(S \otimes \theta_p(p\pi(a)))\sigma(S \otimes u^*)$$

where (8.1.1) follows from the fact that $\theta_{\sigma(S\otimes p)} = \sigma \circ (\theta_S \otimes \theta_p) \circ \sigma^{-1}$.

Evidently, ψ is a unital $*$-homomorphism that lifts to coordinate-wise homorphisms. Then by Proposition 2.8.1 there is a partial isometry $v \in R^{\mathcal{U}}$ such that $v^*v = p', vv^* = \sigma(S \otimes 1)$, and $\psi(x) = vxv^*$ for every

$$x \in \theta_{\sigma(S\otimes p)+\sigma(S^\perp \otimes q)}(\sigma(S \otimes p\pi(N)))$$

(a separable subalgebra). Therefore, for every $a \in N$,

$$\begin{aligned}\sigma(S \otimes u)\sigma(S \otimes \theta_p(p\pi(a)))\sigma(S \otimes u^*) &= \psi(\theta_{\sigma(S\otimes p)+\sigma(S^\perp\otimes q)}(\sigma(S \otimes p\pi(a))) \\ &= v\theta_{\sigma(S\otimes p)+\sigma(S^\perp\otimes q)}(\sigma(S \otimes p\pi(a))v^*.\end{aligned}$$

It follows that

$$v^*\sigma(S \otimes u)\sigma(S \otimes \theta_p(p\pi(a)))\sigma(S \otimes u^*)v = \theta_{\sigma(S\otimes p)+\sigma(S^\perp\otimes q)}(\sigma(S \otimes p\pi(a))).$$

Let $v' := v^*\sigma(S \otimes u)$. Then $v'^*v' = \sigma(S \otimes 1)$ and $v'v'^* = \theta_{\sigma(S\otimes p)+\sigma(S^\perp\otimes q)}(S \otimes p)$. Thus

$$v'\sigma(S \otimes \theta_p(p\pi(a)))v'^* = \theta_{\sigma(S\otimes p)+\sigma(S^\perp\otimes q)}(\sigma(S \otimes p\pi(a)))$$

for every $a \in N$.

Similarly, there is a partial isometry $w' \in R^{\mathcal{U}}$ with $w'^*w' = \sigma(S^\perp \otimes 1)$ and $w'w'^* = \theta_{\sigma(S\otimes p)+\sigma(S^\perp\otimes q)}(\sigma(S^\perp \otimes q))$ such that

$$w'\sigma(S^\perp \otimes \theta_q(q\pi(a)))w'^* = \theta_{\sigma(S\otimes p)+\sigma(S^\perp\otimes q)}(\sigma(S^\perp \otimes q\pi(a)))$$

for every $a \in N$.

Thus, if $u' = v' + w'$ then u' is a unitary such that

$$u'(\sigma(S \otimes \theta_p(p\pi(a))) + \sigma(S^\perp \otimes \theta_q(q\pi(a))))u'^*$$

$$= \theta_{\sigma(S\otimes p)+\sigma(S^\perp\otimes q)}(\sigma(S \otimes p\pi(a)) + \sigma(S^\perp \otimes q\pi(a))).$$ □

Note that thanks to the rescaling Proposition 8.1.1, the requirement that p and q have matching traces in Proposition 8.1.2 is not an obstruction at all.

Proposition 8.1.3.

$$F_{[\pi]} = \left\{[\pi_p] : p \in \pi(N)' \cap R^{\mathcal{U}},\ \textit{a nonzero projection}\right\}$$

Proof. $\subseteq$: We will show that

$$A := \left\{[\pi_p] : p \in \pi(N)' \cap R^{\mathcal{U}},\ \text{a projection }, p \neq 0\right\}$$

is a face. Then this inclusion will hold due to minimality of $F_{[\pi]}$. By Proposition 8.1.2 we have that A is convex. Now if $t[\rho_1] + (1-t)[\rho_2] = [\pi_p]$. So we have

$$[\pi] = \tau(p)[\pi_p] + \tau(p^\perp)[\pi_{p^\perp}]$$

$$= t\tau(p)[\rho_1] + (1-t)\tau(p)[\rho_2] + \tau(p^\perp)[\pi_{p^\perp}].$$

by Proposition 2.8.10. Also by Proposition 2.8.10, we have that $[\rho_i] = [\pi_{q_i}]$ for $q_i \in \pi(N)' \cap R^{\mathcal{U}}$ for $i = 1, 2$. So indeed, A is a face. Thus $F_{[\pi]} \subseteq A$.

$\supseteq$: By Proposition 2.8.10, we have that $\tau(p)[\pi_p] + \tau(p^\perp)[\pi_{p^\perp}] = [\pi] \in F_{[\pi]}$ for any $p \in \pi(N)' \cap R^{\mathcal{U}}$. And since $F_{[\pi]}$ is a face, we have that $[\pi_p] \in F_{[\pi]}$ for any $p \in \pi(N)' \cap R^{\mathcal{U}}$. □

Proposition 8.1.4. *If $\mathcal{Z}(\pi(N)' \cap R^{\mathcal{U}})$ is separable then*

1. $\mathcal{Z}(\sigma(1 \otimes \pi)(N)' \cap R^{\mathcal{U}}) = \sigma(\mathbb{C} \otimes \mathcal{Z}(\pi(N)' \cap R^{\mathcal{U}}))$.
2. *If z is a minimal central projection in $\mathcal{Z}(\pi(N)' \cap R^{\mathcal{U}})$ then $\sigma(1 \otimes z)$ is minimal in $\mathcal{Z}(\sigma(1 \otimes \pi)(N)' \cap R^{\mathcal{U}})$.*

Proof. (1): We have that $X := W^*(\pi(N) \cup \mathcal{Z}(\pi(N)' \cap R^{\mathcal{U}}))$ is separable. So by Proposition 2.8.1, there is a unitary $u \in R^{\mathcal{U}}$ such that for every $x \in X$, $\sigma(1 \otimes x) = uxu^*$. It follows that

$$\sigma(1 \otimes \pi)(N)' \cap R^{\mathcal{U}} = \mathrm{Ad}(u)(\pi(N)' \cap R^{\mathcal{U}}).$$

So we have

$$\begin{aligned}\mathcal{Z}(\sigma(1 \otimes \pi)(N)' \cap R^{\mathcal{U}}) &= \mathrm{Ad}(u)\mathcal{Z}(\pi(N)' \cap R^{\mathcal{U}}) \\ &= \sigma(1 \otimes \mathcal{Z}(\pi(N)' \cap R^{\mathcal{U}})).\end{aligned}$$

(2): Let u be as in the proof of part (1). Then since $z \in \mathcal{Z}(\pi(N)' \cap R^{\mathcal{U}})$ is minimal, it follows that $\sigma(1 \otimes z) = uzu^*$ is minimal. □

The following lemma shows that in the case that the center of the relative commutant of π is finite dimensional, there is a number $0 < t_0 < 1$ such that every element of $F_{[\pi]}$ may be expressed as a cutdown of π by a projection with trace t_0.

Lemma 8.1.5. *Let $\mathcal{Z}(\pi(N)' \cap R^{\mathcal{U}})$ be finite dimensional with minimal central projections $z_1, \ldots, z_n$, and let $t_0 \leq \min\{\tau(z_1), \ldots, \tau(z_n)\}$. Then*

$$F_{[\pi]} = \left\{[\pi_p] : p \in \pi(N)' \cap R^{\mathcal{U}},\ \textit{a projection}\ , \tau(p) = t_0\right\}$$

Proof. Let $q \in \pi(N)' \cap R^{\mathcal{U}}$ be a projection and let $t' = \tau(q)$. Put

$$A_{t_0} = \left\{[\pi_p] : p \in \pi(N)' \cap R^{\mathcal{U}},\ \text{a projection}\ , \tau(p) = t_0\right\}.$$

Assume that $t' > t_0$. Let $Q \in R^{\mathcal{U}}$ be a projection with $\tau(Q) = \frac{t_0}{t'}$, and let $u \in R^{\mathcal{U}}$ be a unitary such that $\sigma(1 \otimes x) = uxu^*$ for every $x \in W^*(\pi(N) \cup \mathcal{Z}(\pi(N)' \cap R^{\mathcal{U}}))$, then by Proposition 8.1.1,

$$[\pi_q] = [\sigma(1 \otimes \pi)_{\sigma(Q \otimes q)}] = [\pi_{u^*\sigma(Q \otimes q)u}] \in A_{t_0}.$$

Now let $t' < t_0$. Let $p \in \pi(N)' \cap R^{\mathcal{U}}$ be a projection such that $\tau(pz_i) = \frac{t_0}{t'}\tau(qz_i)$ for every $1 \leq i \leq n$. That is, the center-valued trace of p is a $\frac{t_0}{t'}$-scaling of the center-valued trace of q. Clearly, $\tau(p) = t_0$. Let $Q \in R^{\mathcal{U}}$ be a projection such that $\tau(Q) = \frac{t'}{t_0}$. By Proposition 8.1.4, the minimal central projections in $\sigma(1 \otimes \pi)(N)' \cap R^{\mathcal{U}}$

are $\{\sigma(1 \otimes z_i)\}_{i=1}^n$. Observe that for every $1 \le i \le n$ we have

$$\begin{aligned}\tau(\sigma(Q \otimes p)\sigma(1 \otimes z_i)) &= \tau(\sigma(Q \otimes pz_i)) \\ &= \tau(Q) \cdot \tau(pz_i) \\ &= \frac{t'}{t_0} \cdot \frac{t_0}{t'}\tau(qz_i) \\ &= \tau(qz_i) \\ &= \tau(\sigma(1 \otimes q)\sigma(1 \otimes z_i)).\end{aligned}$$

Thus $\sigma(Q \otimes p)$ is Murray-von Neumann equivalent to $\sigma(1 \otimes q)$ in $\sigma(1 \otimes \pi)(N)' \cap R^{\mathcal{U}}$. By Propositions 2.8.10 and 8.1.1 we get that

$$\begin{aligned}[\pi_q] &= [\sigma(1 \otimes \pi)_{\sigma(1 \otimes q)}] \\ &= [\sigma(1 \otimes \pi)_{\sigma(Q \otimes p)}] \\ &= [\sigma(1 \otimes \pi)_{\sigma(1 \otimes p)}] \\ &= [\pi_p] \in A_{t_0}.\end{aligned}$$

□

We are now sufficiently prepared to prove part (1) of Theorem 8.2.

Proof. (of part (1) of Theorem 8.2) We will show that if $\mathcal{Z}(\pi(N)' \cap R^{\mathcal{U}})$ is n-dimensional with $n < \infty$ then $F_{[\pi]}$ is affinely isomorphic to the n-vertex simplex given by

$$\mathcal{X}_{t_0} := \left\{(x_1, \ldots, x_n) : 0 \le x_i \le t_0 \quad \forall 1 \le i \le n, \sum_{i=1}^n x_i = t_0\right\}.$$

By Lemma 8.1.5, we may identify $F_{[\pi]}$ with

$$A_{t_0} := \left\{[\pi_p] : p \in \pi(N)' \cap R^{\mathcal{U}},\ \text{a projection },\tau(p) = t_0\right\}.$$

Consider the map

$$\varphi : A_{t_0} \to \mathcal{X}_{t_0}$$

given by

$$\varphi([\pi_p]) = (\tau(pz_1), \ldots, \tau(pz_n))$$

where $z_1, \ldots z_n$ are the minimal central projections of $\pi(N)' \cap R^{\mathcal{U}}$. Proposition 2.8.10 ensures that φ is well-defined and injective. Given any $(x_1, \ldots, x_n) \in \mathcal{X}_{t_0}$, it is well-known that there is a projection $p \in \pi(N)' \cap R^{\mathcal{U}}$ such that $(\tau(pz_1), \ldots, \tau(pz_n)) = (x_1, \ldots, x_n)$; thus, φ is surjective. It remains to show that φ is affine. Since $W^*(\pi(N) \cup \mathcal{Z}(\pi(N)' \cap R^{\mathcal{U}}))$ is separable, there is a unitary $u \in R^{\mathcal{U}}$ so that $\sigma(1 \otimes x) = uxu^*$ for every $x \in W^*(\pi(N) \cup \mathcal{Z}(\pi(N)' \cap R^{\mathcal{U}}))$ as in Proposition 8.1.4. Now, by Proposition 8.1.2,

$$t[\pi_p] + (1-t)[\pi_q] = [\sigma(1 \otimes \pi)_{\sigma(S \otimes p) + \sigma(S^\perp \otimes q)}] = [\pi_{u^*(\sigma(S \otimes p) + \sigma(S^\perp \otimes q))u}]$$

where $S \in R^{\mathcal{U}}$ is a projection such that $\tau(S) = t$. Furthermore, for every $1 \leq i \leq n$, we have that

$$\begin{aligned}
\tau(u^*(\sigma(S \otimes p) + \sigma(S^\perp \otimes q))uz_i) &= \tau((\sigma(S \otimes p) + \sigma(S^\perp \otimes q))uz_iu^*) \\
&= \tau((\sigma(S \otimes p) + \sigma(S^\perp \otimes q))\sigma(1 \otimes z_i)) \\
&= \tau(\sigma(S \otimes pz_i)) + \tau(\sigma(S^\perp \otimes qz_i)) \\
&= \tau(S) \cdot \tau(pz_i) + \tau(S^\perp) \cdot \tau(qz_i) \\
&= t\tau(pz_i) + (1-t)\tau(qz_i).
\end{aligned}$$

So

$$\varphi(t[\pi_p] + (1-t)[\pi_q]) = \varphi([\pi_{u^*(\sigma(S\otimes p)+\sigma(S^\perp\otimes q))u}]$$

$$= t\varphi([\pi_p]) + (1-t)\varphi([\pi_q]).$$

□

8.2 A Form of Schur's Lemma for $R^{\mathcal{U}}$

Before we establish part (2) of Theorem 8.2 we must prove an intuitive yet difficult lemma. Since Lemma 8.2.1 addresses intertwiners of unital representations of N in $R^{\mathcal{U}}$, we can consider this as a sort of $R^{\mathcal{U}}$-version of Schur's lemma. An argument similar to the argument presented in Lemma 8.2.1 appears in [18]. Thanks to Nate Brown for a helpful discussion regarding this lemma and to Stuart White for suggesting the proof of this lemma.

Lemma 8.2.1. *Let $[\pi], [\rho] \in \mathbb{H}om(N, R^{\mathcal{U}})$ be extreme points. If there is a nonzero $x \in R^{\mathcal{U}}$ such that $\pi(a)x = x\rho(a)$ for every $a \in N$ (that is, x intertwines π and ρ), then $[\pi] = [\rho]$.*

Proof. Let $x = v|x|$ be the polar decomposition of x (here $|x| = (x^*x)^{\frac{1}{2}}$). We first claim that v also intertwines π and ρ. Note that $x^*x \in \rho(N)' \cap R^{\mathcal{U}}$ and so $|x| \in \rho(N)' \cap R^{\mathcal{U}}$. Thus we have for every $a \in N$

$$\pi(a)v|x| = v|x|\rho(a)$$

$$= v\rho(a)|x|.$$

So for every $a \in N$, $\pi(a)v = v\rho(a)$ on $\overline{\text{range}|x|}$. Also, v^*v is the projection onto $\overline{\text{range}|x|}$, and thus $v^*v \in W^*(|x|) \subset \rho(N)' \cap R^{\mathcal{U}}$. Thus, for every $a \in N$,

$$\begin{aligned}\pi(a)v &= \pi(a)v(v^*v) \\ &= v\rho(a)v^*v \\ &= vv^*v\rho(a) \\ &= v\rho(a).\end{aligned}$$

Consider the set

$$S := \left\{w : w \in R^{\mathcal{U}} \text{ a partial isometry and } \pi(a)w = w\rho(a), \quad \forall a \in N\right\}.$$

By above, S is nonempty. Define the following partial order on S:

$$v \leq w \Leftrightarrow wv^*v = v (\Leftrightarrow v^* = v^*vw^*).$$

Let $w_1 \leq w_2 \leq w_3 \leq \ldots$ be an increasing chain of elements in S. We will show that this chain has an upper bound. To do this we will show that $\{w_n\}$ is $||\cdot||_2$-Cauchy. Note that $\{\tau(w_n^*w_n)\}$ is a monotone, bounded sequence of real numbers, so it is a convergent sequence. Let $n \leq m$; by the definition of the ordering,

$$\tau(w_m^*w_n) = \tau(w_nw_n^*)$$

and

$$\tau(w_n^*w_m) = \tau(w_nw_n^*).$$

Thus,

$$\begin{aligned}||w_n - w_m||_2^2 &= \tau((w_n - w_m)^*(w_n - w_m)) \\ &= \tau(w_n^* w_n - w_m^* w_n - w_n^* w_m + w_m^* w_m) \\ &= \tau(w_n^* w_n) - \tau(w_n w_n^*) - \tau(w_n w_n^*) - \tau(w_m^* w_m) \\ &= \tau(w_m^* w_m) - \tau(w_n^* w_n).\end{aligned}$$

And since $\{\tau(w_n^* w_n)\}$ is convergent, this shows that $\{w_n\}$ is $||\cdot||_2$-Cauchy. So let w be the $||\cdot||_2$-limit of $\{w_n\}$. Clearly $w \in S$ and w is an upper bound of the chain $w_1 \leq w_2 \leq \cdots$. So by Zorn's lemma, there is a maximal (with respect to this ordering) $v \in S$.

Assume by way of contradiction that v is not a unitary. Then $\tau(v^*v) < 1$. Also note that $v^*v \in \rho(N)' \cap R^{\mathcal{U}}$. Let $p \in \rho(N)' \cap R^{\mathcal{U}}$ be a nonzero projection orthogonal to v^*v and such that $\tau(p) \leq \tau(v^*v)$. Let $w \in \rho(N)' \cap R^{\mathcal{U}}$ be such that $w^*w = p$ and $ww^* \leq v^*v$ (this is possible because $\rho(N)' \cap R^{\mathcal{U}}$ is a factor). Note that for every $a \in N$

$$\begin{aligned}vw\rho(a) &= v\rho(a)w \\ &= \pi(a)vw\end{aligned}$$

which implies that $(vw)(vw)^* \in \pi(N)' \cap R^{\mathcal{U}}$. Now let $y \in \pi(N)' \cap R^{\mathcal{U}}$ be such that $y^*y = (vw)(vw)^*$ and $yy^* \leq 1 - vv^*$ (this is possible becuase $\pi(N)' \cap R^{\mathcal{U}}$ is a factor). Now consider

$$v + yvw.$$

First note that

$$\begin{aligned}||yvw||_2^2 &= \tau(w^*v^*y^*yvw)\\ &= \tau(w^*v^*vww^*v^*vw)\\ &= \tau(w^*v^*vw)\\ &= \tau(w^*w)\\ &= \tau(p)\\ &\neq 0,\end{aligned}$$

so $v \neq v + yvw$. Next, we see that $v + yvw \in S$: for every $a \in N$,

$$\begin{aligned}\pi(a)(v + yvw) &= \pi(a)v + \pi(a)yvw\\ &= v\rho(a) + y\pi(a)vw\\ &= v\rho(a) + yv\rho(a)w\\ &= v\rho(a) + yvw\rho(a)\\ &= (v + yvw)\rho(a).\end{aligned}$$

Lastly, we observe that

$$\begin{aligned}(v + yvw)v^*v &= v + yvwv^*v\\ &= v + yvww^*wv^*v\\ &= v + 0,\end{aligned}$$

so $v \leq v + yvw$ in the ordering on S. So maximality of v implies that $v = v + yvw$, but this is absurd because $yvw \neq 0$. So v must be a unitary, and so $\pi \sim \rho$. $\square$

Next we record the following easy lemma. This is essentially a scaled version of Lemma 8.2.1.

Lemma 8.2.2. *Let $p, q \in R^{\mathcal{U}}$ be mutually orthogonal projections with $\tau(p) = \tau(q)$. Let $[\pi], [\rho] \in \mathbb{H}om(N, R^{\mathcal{U}})$ be distinct extreme points. If $x \in (p+q)R^{\mathcal{U}}(p+q)$ intertwines $\theta_p^{-1} \circ \pi$ and $\theta_q^{-1} \circ \rho$, then $x = 0$.*

Proof. We are assuming that for every $a \in N$,

$$\theta_p^{-1}(\pi(a))(p+q)x(p+q) = (p+q)x(p+q)\theta_q^{-1}(\rho(a)). \tag{8.2.1}$$

Then by (8.2.1) we have

$$\begin{aligned} pxp &= p\theta_p^{-1}(1)(p+q)x(p+q)p \\ &= p(p+q)x(p+q)\theta_q^{-1}(1)p \\ &= p(p+q)x(p+q)qp \\ &= 0. \end{aligned}$$

And similarly, $qxq = 0$. So $x = pxq + qxp$. Thus, for every $a \in N$

$$\begin{aligned} \theta_p^{-1}(\pi(a))(pxq) &= \theta_p^{-1}(\pi(a))(pxq + qxp) \\ &= \theta_p^{-1}(\pi(a))(x) \\ &= (x)\theta_q^{-1}(\rho(a)) \\ &= (pxq + qxp)\theta_q^{-1}(\rho(a)) \\ &= pxq\theta_q^{-1}(\rho(a)). \end{aligned}$$

Let $v \in R^{\mathcal{U}}$ be a partial isometry such that $v^*v = p$ and $vv^* = q$. Then for every $a \in N$,

$$\begin{aligned}\pi(a)\theta_p(pxqv) &= \theta_p(\theta_p^{-1}(\pi(a))(pxq)v)\\ &= \theta_p(pxq\theta_q^{-1}(\rho(a))v)\\ &= \theta_p(pxqv)\theta_p(v^*\theta_q^{-1}(\rho(a))v).\end{aligned}$$

Notice that $\theta_p \circ \mathrm{Ad}(v^*) \circ \theta_q^{-1} : R^{\mathcal{U}} \to R^{\mathcal{U}}$ is a unital $*$-homomorphism that lifts to homomorphisms fiberwise. So by Proposition 2.8.1 we have that

$$\theta_p(v^*\theta_q^{-1}(\rho(\cdot))v \sim \rho(\cdot).$$

Thus $\theta_p(pxqv)$ intertwines π and $\theta_p \circ \mathrm{Ad}(v^*) \circ \theta_q^{-1} \circ \rho \sim \rho$. So by Lemma 8.2.1, $0 = \theta_p(pxqv)$. Since θ_p is an isomorphism, we get that $0 = pxqv$. Then multiplying on the right by v^* yields

$$0 = pxqvv^* = pxq.$$

By taking adjoints, one can show in an identical way that $qxp = 0$. Thus $x = 0$. □

8.3 Proof of Parts (2) and (3) of Theorem 8.2

Now we are ready to prove part (2) of the Theorem.

Proof. (of part (2) of Theorem 8.2)

We prove this part of the theorem in the case where $n = 2$. All other cases are direct generalizations of this one. So we must show that if $[\pi], [\rho] \in \mathbb{H}\mathrm{om}(N, R^{\mathcal{U}})$ are distinct extreme points and if $\varphi \in t[\pi] + (1-t)[\rho]$ for $0 < t < 1$, then

$$\varphi(N)' \cap R^{\mathcal{U}} \cong \pi(N)' \cap R^{\mathcal{U}} \oplus \rho(N)' \cap R^{\mathcal{U}}.$$

We will further subdivide the problem into two cases.

Case I: t is rational. Let $t = \frac{k}{N}$ for positive integers k and N. Then $1 - t = \frac{N-k}{N}$. Let $p \in R^{\mathcal{U}}$ be a projection such that $\tau(p) = \frac{k}{N}$. Without loss of generality, let

$$\varphi = \theta_p^{-1} \circ \pi + \theta_{p^\perp}^{-1} \circ \rho.$$

Let $p_1, \ldots, p_k \leq p$ be mutually orthogonal projections with $\tau(p_i) = \frac{1}{N}$; and let $v \in R^{\mathcal{U}}$ be a partial isometry with $v^*v = vv^* = p$ such that

$$\theta_p^{-1} \circ \pi = \mathrm{Ad}(v) \circ \Big(\sum_{i=1}^{k} \theta_{p_i}^{-1} \circ \pi \Big).$$

Similarly, let $q_1, \ldots, q_{N-k} \leq p^\perp$ be mutually orthogonal projections with $\tau(q_j) = \frac{1}{N}$; and let $w \in R^{\mathcal{U}}$ be a partial isometry with $w^*w = ww^* = p^\perp$ such that

$$\theta_{p^\perp}^{-1} \circ \rho = \mathrm{Ad}(w) \circ \Big(\sum_{j=1}^{N-k} \theta_{q_j}^{-1} \circ \rho \Big).$$

Fix $x \in \varphi(N)' \cap R^{\mathcal{U}}$. It will suffice to show that $x = pxp + p^\perp x p^\perp$. Observe that

$$p = \sum_{i=1}^{k} v p_i v^*$$

and

$$p^\perp = \sum_{j=1}^{N-k} w q_j w^*.$$

So we get that

$$pxp^{\perp} = \sum_{i=1}^{k}\sum_{j=1}^{N-k}(vp_iv^*)x(wq_jw^*)$$

and

$$p^{\perp}xp = \sum_{i=1}^{k}\sum_{j=1}^{N-k}(wq_jw^*)x(vp_iv^*).$$

Next, we claim that for every $1 \le i \le k$ and $1 \le j \le N-k$, $(vp_iv^*)x(wq_jw^*)$ intertwines $v(\theta_{p_i}^{-1} \circ \pi)v^*$ and $w(\theta_{q_j}^{-1} \circ \rho)w^*$. For every $a \in N$, we have by assumption that $x\varphi(a) = \varphi(a)x$. Expanding x gives the following equation

$$\Big(pxp + \sum_{i=1}^{k}\sum_{j=1}^{N-k}(vp_iv^*)x(wq_jw^*) + \sum_{i=1}^{k}\sum_{j=1}^{N-k}(wq_jw^*)x(vp_iv^*) + p^{\perp}xp^{\perp}\Big)\cdot$$
$$\Big(\sum_{i=1}^{k} v\theta_{p_i}^{-1}(\pi(a))v^* + \sum_{j=1}^{N-k} w\theta_{q_j}^{-1}(\rho(a))w^*\Big)$$
$$= \Big(\sum_{i=1}^{k} v\theta_{p_i}^{-1}(\pi(a))v^* + \sum_{j=1}^{N-k} w\theta_{q_j}^{-1}(\rho(a))w^*\Big)\cdot$$
$$\Big(pxp + \sum_{i=1}^{k}\sum_{j=1}^{N-k}(vp_iv^*)x(wq_jw^*) + \sum_{i=1}^{k}\sum_{j=1}^{N-k}(wq_jw^*)x(vp_iv^*) + p^{\perp}xp^{\perp}\Big).$$

Multiplying the above equation on the left by vp_iv^* and on the right by wq_jw^* yields

$$(vp_iv^*)x(wq_jw^*)(w\theta_{q_j}^{-1}(\rho(a))w^*) = (v\theta_{p_i}^{-1}(\pi(a))v^*)(vp_iv^*)x(wq_jw^*)$$

as claimed. Then by Lemma 8.2.2 we have that $(vp_iv^*)x(wq_jw^*) = 0$. Similarly, one can show that $(wq_jw^*)x(vp_iv^*) = 0$ for every $1 \le i \le k$ and $1 \le j \le N-k$. So $x = pxp + p^{\perp}xp^{\perp}$.

Case II: t is irrational. Fix $N \in \mathbb{N}$ and let $1 \le k \le N-1$ be such that $\dfrac{k}{N} < t < \dfrac{k+1}{N}$.

Let $p \in R^{\mathcal{U}}$ be a projection with $\tau(p) = t$ and let

$$\varphi = \theta_p^{-1} \circ \pi + \theta_{p^\perp}^{-1} \circ \rho.$$

Let $p_1, \ldots, p_k, \tilde{p} \leq p$ be mutually orthogonal projections such that $\tau(p_i) = \dfrac{1}{N}$ for $1 \leq i \leq k$ and $\tau(\tilde{p}) = t - \dfrac{k}{N} (< \dfrac{1}{N})$; and let v be a partial isometry with $v^*v = vv^* = p$ such that

$$\theta_p^{-1} \circ \pi = \mathrm{Ad}(v) \circ \Big(\sum_{i=1}^{k} \theta_{p_i}^{-1} \circ \pi + \theta_{\tilde{p}} \circ \pi \Big).$$

Similarly, let $q_1, \ldots, q_{N-k-1}, \tilde{q} \leq p^\perp$ be mutually orthogonal projections such that $\tau(q_j) = \dfrac{1}{N}$ for every $1 \leq j \leq N-k-1$ and $\tau(\tilde{q}) = \dfrac{k+1}{N} - t$; and let w be a partial isometry with $w^*w = ww^* = p^\perp$ such that

$$\theta_{p^\perp}^{-1} \circ \rho = \mathrm{Ad}(w) \circ \Big(\sum_{j=1}^{N-k-1} \theta_{q_j}^{-1} \circ \rho + \theta_{\tilde{q}}^{-1} \circ \rho \Big).$$

Fix $x \in \varphi(N)' \cap R^{\mathcal{U}}$ with $||x|| \leq 1$. As before, it will suffice to show that $x = pxp + p^\perp x p^\perp$. By an argument identical to the one in Case I, we have that

$$\sum_{i=1}^{k} \sum_{j=1}^{N-k-1} (vp_iv^*)x(wq_jw^*) = \sum_{i=1}^{k} \sum_{j=1}^{N-k-1} (wq_jw^*)x(vp_iv^*) = 0.$$

So

$$pxp^\perp = px(w\tilde{q}w^*) + (v\tilde{p}v^*)x(p^\perp - w\tilde{q}w^*)$$

and

$$p^\perp xp = (w\tilde{q}w^*)xp + (p^\perp - w\tilde{q}w^*)x(v\tilde{p}v^*).$$

Thus,

$$\begin{aligned} ||pxp^{\perp} + p^{\perp}xp||_2 &\leq ||px(w\tilde{q}w^*)||_2 + ||(v\tilde{p}v^*)x(p^{\perp} - w\tilde{q}w^*)||_2 + ||(w\tilde{q}w^*)xp||_2 \\ &+ ||(p^{\perp} - w\tilde{q}w^*)x(v\tilde{p}v^*)||_2 \\ &\leq ||w\tilde{q}w^*||_2 + ||v\tilde{p}v^*||_2 + ||w\tilde{q}w^*||_2 + ||v\tilde{p}v^*||_2 \\ &< 4\sqrt{\frac{1}{N}}. \end{aligned}$$

Since $N \in \mathbb{N}$ was arbitrary, this shows that $pxp^{\perp} + p^{\perp}xp = 0$. Thus, $x = pxp + p^{\perp}xp^{\perp}$. □

Proof. (of part (3) of Theorem 8.2) This statement follows from (1) and (2). Let $\dim(\mathcal{Z}(\pi(N)' \cap R^{\mathcal{U}})) = n$. If $n < \infty$, then by (1) we have that $F_{[\pi]}$ is an n-vertex simplex and thus $\dim(F_{[\pi]}) = n-1$. If $n = \infty$ but $\dim(F_{[\pi]}) < \infty$, then $[\pi]$ is an average of finitely many extreme points. And this would imply by (2) that $\dim(\mathcal{Z}(\pi(N)' \cap R^{\mathcal{U}}))$ is finite–a contradiction. So we must have $\dim(F_{[\pi]}) = \infty$. □

The following corollary indicates a sort of linear independence between extreme points.

Corollary 8.3.1. *The convex hull of n extreme points in $\mathbb{H}om(N, R^{\mathcal{U}})$ is always an n-vertex simplex.*

So for example, the convex hull of four extreme points cannot be a square–it must be a tetrahedron.

Example 8.3.2. In Corollaries 6.10 and 6.11 of [8], Brown exhibits II_1-factors with the property that for such a II_1-factor N, $\mathbb{H}\mathrm{om}(N, R^{\mathcal{U}})$ has infinitely many extreme points with a cluster point. So for such a II_1-factor N and any $n \in \mathbb{N}$, by Theorem 8.2, there is a face in $\mathbb{H}\mathrm{om}(N, R^{\mathcal{U}})$ taking the form of an n-vertex simplex. Given a sequence of extreme points $[\pi_n] \in \mathbb{H}\mathrm{om}(N, R^{\mathcal{U}})$ such that $[\pi_n] \to [\pi]$, it would be interesting to have a description of

$$\overline{\mathrm{conv}}(\{[\pi_n]\}_{n=1}^{\infty} \cup \{[\pi]\}).$$

Remark 8.3.3. An interesting property of a simplex is that the convex hull of any finite number of extreme points is a face. Although $\mathbb{H}\mathrm{om}(N, R^{\mathcal{U}})$ is rarely a simplex, in the cases where extreme points exist, it is a consequence of Theorem 8.2 that $\mathbb{H}\mathrm{om}(N, R^{\mathcal{U}})$ shares this property.

Remark 8.3.4. The content of this chapter was in the context of $\mathbb{H}\mathrm{om}(N, R^{\mathcal{U}})$ in order to address the structure of that well-known object. At no point was it used that N is a II_1-factor, so all of the results in this chapter apply to $\mathbb{H}\mathrm{om}(\mathfrak{A}, R^{\mathcal{U}})$ for any separable unital C^*-algebra $\mathfrak{A}$. Though it would require even more technical notation, it is reasonable to expect that these results extend even further to $\mathbb{H}\mathrm{om}(\mathfrak{A}, M^{\mathcal{U}})$ for any separable unital C^*-algebra $\mathfrak{A}$ and any separable McDuff II_1-factor M.

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