A New Gauge-Theoretic Construction of 4-Dimensional Hyperkähler ALE Spaces

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Abstract

This thesis provides a new gauge-theoretic construction of 4-dimensional hyperkähler ALE spaces.

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as about myself as I went, failing time and time again, trying again time after time. Ultimately, I am proud of what I have done especially because it is all so easy to forget or overlook that sometimes.

Contents

Chapter 1

Introduction

In this paper, we give a new gauge-theoretic construction of all the 4-dimensional hyperkähler ALE (asymptotically locally Euclidean) spaces. These spaces are originally constructed by Peter Kronheimer in his Ph.D. thesis [21]. They are in one-to-one correspondence with the finite subgroups of $SU(2)$ and have deep connections with representation theory, singularity theory and low-dimensional topology. Topologically, these spaces are plumbings of the 4-ball where the plumbing graph is described by the ADE-type Dynkin diagrams of semi-simple Lie algebras. Geometrically, they are the resolution of singularity of $\mathbb{C}^2/\Gamma$, where Γ is a finite subgroup of $SU(2)$ and the blowup diagram naturally corresponds to the plumbing graph. The interesting connections these spaces share with representation theory, singularity theory and low-dimensional topology are captured by the McKay Correspondence [23]. In Kronheimer's construction, each of them is realized through a hyperkähler reduction of a finite-dimensional vector space. We will review this construction in Section 2.

On the other hand, non-compact hyperkähler spaces frequently arise in gauge theory as the moduli spaces of solutions to gauge theoretic equations. Well-known examples include the Hitchin moduli spaces of solutions to self-duality equations on Riemann surfaces [14], and gravitational instantons as moduli spaces of monopoles [4]. Here we give a new construction of 4-dimensional hyperkähler ALE spaces – a class of gravitational instantons, using a gauge theoretic approach. More specifically, we realize each 4-dimensional hyperkähler ALE space as a moduli space of solutions to a system of equations for a pair consisting of a connection and a section of a vector bundle over an orbifold Riemann surface, modulo a gauge group action. This new construction parallels Kronheimer's original construction in [21] and leads to different directions for generalizations.

Chapter 2

Symplectic and kähler geometry

In this chapter, we give the basic definitions and constructions in symplectic and kähler geometry leading to the construction of ALE spaces given by Kronheimer which will be introduced in detail in chapter 4. We will follow [6] for the discussion.

2.1 Basic definitions in symplectic geometry

Definition 2.1.1 (symplectic vector space)**.**

Let V be a vector space over $\mathbb{R}$. A symplectic form on V is a bilinear map $\omega : V \times V \to \mathbb{R}$ such that

- ω is skew-symmetric, that is, $\omega(v, w) = -\omega(w, v)$.
- ω is non-degenerate: $\omega(v, w) = 0, \forall w$ if and only if $v = 0$.

We say that (V, ω) is a symplectic vector space.

Definition 2.1.2. *A linear map $A : (V, \omega) \to (V', \omega')$ is a (linear) symplectomorphism if it's an isomorphism and $A^*\omega' = \omega$, where $A^*\omega'(v, w) = \omega'(Av, Aw)$.*

Definition 2.1.3 (symplectic manifold)**.** *Let M be a smooth $2n$-dimensional manifold. A symplectic form on M is a differential 2-form $\omega \in \Omega^2(M)$ such that*

- *$\omega_p \in \bigwedge^2 T_p^* M$ is non-denegerate for each $p \in M$.*

- *ω is closed:* $d\omega = 0$.

We say that (M, ω) is a symplectic manifold.

Example 2.1.4. *Below are some examples of symplectic manifolds:*

- $(\mathbb{R}^{2n}, \omega_{std})$
- $(\mathbb{C}P^n, \omega_{FS})$
- (Σ_g, ω_{vol})

2.2 Hamiltonian action and moment maps

Definition 2.2.1 (symplectomorphism). *Let (M_1, ω_1) and (M_2, ω_2) be $2n$-dimensional symplectic manifolds, and let $g : M_1 \to M_2$ be a diffeomorphism. Then g is a symplectomorphism if $g^*\omega_2 = \omega_1$. Let $\mathrm{Sympl}(M, \omega)$ denote the group of symplectomorphisms of (M, ω).*

Definition 2.2.2 (symplectic action). *Let (M, ω) be a symplectic manifold and G a Lie group. Let $\psi : G \to \mathrm{Diff}(M)$ be a smooth action. Then ψ is a symplectic action if $\mathrm{im}(\psi) \subset \mathrm{Sympl}(M, \omega) \subset \mathrm{Diff}(M)$.*

Definition 2.2.3 (Hamiltonian action and moment map). *Let $\psi : G \to \mathrm{Sympl}(M, \omega)$ be a symplectic action on (M, ω). Then ψ is hamiltonian if there exists a map $\mu : M \to \mathbf{g}^*$ satisfying:*

(1) For each $X \in \mathbf{g}$, $d\mu^X = \iota_{X^\sharp}\omega$, where

- *$\mu^X : M \to \mathbb{R}$, $\mu^X(p) = \langle \mu(p), X \rangle$, is the component of μ along X.*
- *$X^\sharp$ is the vector field on M generated by the one-parameter subgroup $\{\exp(tX) | t \in \mathbb{R}\} \subset G$.*

In other words, μ^X is a hamiltonian function for the vector field $X^\sharp$.

(2) $\mu \circ \psi_g = Ad_g^* \circ \mu$, for all $g \in G$, where Ad^* denotes the coadjoint representation of G on $\mathbf{g}^*$.

We say μ is a moment map of ψ.

2.3 Symplectic reduction and moment map equation

Theorem 2.3.1 (Marsden-Weinstein-Meyer). *Let (M, ω, G, μ) be a hamiltonian G-space for a compact Lie group G. Let $i : \mu^{-1}(0) \hookrightarrow M$ be the inclusion map. Assume that G acts freely on $\mu^{-1}(0)$. Then*

- *the orbit space $M_{red} = \mu^{-1}(0)/G$ is a manifold,*
- *$\pi : \mu^{-1}(0) \to M_{red}$ is a principal G-bundle, and*
- *there is a symplectic form ω_{red} on M_{red} satisfying $i^*\omega = \pi^*\omega_{red}$.*

Definition 2.3.2. *The pair (M_{red}, ω_{red}) is called the reduction of (M, ω) with respect to G, μ, or the reduced space, or the symplectic quotient, or the Marsden-Weinstein-Meyer quotient, etc.*

2.4 Basic definitions in kähler geometry

Definition 2.4.1. *A complex structure on a real vector space V is an automorphism $J : V \to V$ such that $J \circ J = -Id_V$. Such a structure gives V the structure of a complex vector space $V \otimes \mathbb{C} \to V$, namely, $v \otimes (s + it) = sv + tJv$.*

Definition 2.4.2. *Let V be a vector space with ω, J a symplectic form and a complex structure. We say that ω and J are compatible if*

- *ω tames J, meaning $\omega(v, Jv) > 0, \forall v \neq 0$.*
- *ω is J-invariant, meaning $\omega(Jv, Jw) = \omega(v, w), \forall v, w \in V$.*

Remark 2.4.3. *If ω and J are compatible in this sense, then define $g_J(v, w) = \omega(v, Jw)$. This is*

- *Symmetric: $g_J(w, v) = g_J(v, w)$.*
- *Positive definite: $g_J(v, v) > 0, \forall v \neq 0$.*

Hence, we get a compatible metric g_J.

Definition 2.4.4 (almost complex manifold). *An almost complex structure on a manifold M is a smooth field of complex structures on the tangent spaces:*

$$x \mapsto J_x : T_xM \to T_xM \text{ linear, and } J_x^2 = -Id.$$

The pair (M, J) is then called an almost complex manifold.

Definition 2.4.5 (complex manifold). *A complex manifold of (complex) dimension n is a set M with a complete complex atlas*

$$\mathcal{A} = \{(\mathcal{U}_\alpha, \mathcal{V}_\alpha, \varphi_\alpha), \alpha \in \text{ index set } I\},$$

where $M = \bigcup_\alpha \mathcal{U}_\alpha$, the $\mathcal{V}_\alpha$'s are open subsets of $\mathbb{C}^n$, and the maps $\varphi_\alpha : \mathcal{U}_\alpha \to \mathcal{V}_\alpha$ are such that the transition maps $\psi_{\alpha\beta}$ are biholomorphic as maps on open subsets of $\mathbb{C}^n$:

$$\begin{tikzcd} & \mathcal{U}_\alpha \bigcap \mathcal{U}_\beta \arrow[dl, "\varphi_\alpha"'] \arrow[dr, "\varphi_\beta"] & \\ \mathcal{V}_{\alpha\beta} \arrow[rr, "\psi_{\alpha\beta}=\varphi_\beta\circ\varphi_\alpha^{-1}"] & & \mathcal{V}_{\beta\alpha} \end{tikzcd}$$

where $\mathcal{V}_{\alpha\beta} = \varphi_\alpha(\mathcal{U}_\alpha \bigcap \mathcal{U}_\beta) \subset \mathbb{C}^n$ and $\mathcal{V}_{\beta\alpha} = \varphi_\beta(\mathcal{U}_\alpha \bigcap \mathcal{U}_\beta) \subset \mathbb{C}^n$, and $\psi_{\alpha\beta}$ being biholomorphic means that $\psi_{\alpha\beta}$ is a bijection and that $\psi_{\alpha\beta}$ and $\psi_{\alpha\beta}^{-1}$ are both holomorphic.

Definition 2.4.6 (integrable almost complex structure). *An almost complex structure J on a manifold M is called integrable if and only if J is induced by a structure of complex manifold on M.*

Definition 2.4.7 (kähler manifold). *A kähler manifold is a symplectic manifold (M, ω) equipped with an integrable compatible almost complex structure. The symplectic form ω us then called a kähler form.*

Definition 2.4.8 (hyperkähler manifold). *A hyperkähler manifold is a Riemannian manifold (M, g) endowed with 3 integrable almost complex structures I, J, K that are kähler with respect to the Riemannian metric g and satisfies the quaternionic relations $I^2 = J^2 = K^2 = IJK = -1$.*

Example 2.4.9 (4-dimensional hyperkähler manifolds). *In the case of 4-dimensional hyperkähler manifolds, the noncompact ones are ALE spaces, ALF spaces, ALG spaces, ALH spaces; the compact ones are T^4 and $K3$.*

2.5 Kähler and hyperkähler reduction

Definition 2.5.1. *Let (M, g, ω) be a kähler manifold. Suppose G is a compact Lie group acting freely on M and preserving both the metric and the symplectic form, then symplectic quotient M_{red} defined previously is a kähler manifold. We call this a kähler reduction.*

Definition 2.5.2. *Let (M, g, ω) be a hyperkähler manifold. Let G be a compact Lie group of isometries acting freely on M and preserving the structures I, J, K. The group G preserves the three kähler forms ω_1, ω_2, ω_3 corresponding to the three complex structures, so we may define three moment maps μ_1, μ_2, μ_3. More invariantly these can be written as a single map:*

$$\mu : M \to \mathbf{g}^* \otimes \mathbb{R}^3.$$

Then, the quotient $\mu^{-1}(0)/G$ is a hyperkähler manifold. We call this a hyperkähler reduction.

Chapter 3

Basic representation theory

In this chapter, we review the basic representation theory of Lie algebras for understanding Kronheimer's construction of ALE spaces in the following chapter. We will follow [16] for the discussion.

3.1 Basic definitions

Definition 3.1.1. *A vector space L over $\mathbb{C}$, with an operation $L \otimes L \to L$, denotes $(x, y) \mapsto [xy]$ and called the bracket or commutator of x and y, is called a Lie algebra over $\mathbb{C}$ if the following axioms are satisfied:*

1. *The bracket operation is bilinear.*
2. *$[xx] = 0$ for all x in L.*
3. *$[x[yz]] + [y[zx]] + [z[xy]] = 0$, for all x, y, z in L.*

The last axiom is called the Jacobi identity.

Definition 3.1.2. *A subspace I of a Lie algebra is called an ideal of L if $x \in L$ and $y \in I$ together imply $[xy] \in I$.*

Definition 3.1.3. *The center of a Lie algebra L, denoted $Z(L)$, is defined as $Z(L) = \{z \in L | [xz] = 0, \forall x \in L\}$. The center $Z(L)$ is an ideal of L.*

Definition 3.1.4. *If L has no ideals except itself and 0, we call L simple.*

Definition 3.1.5. *A representation of a Lie algebra L is a homomorphism $\phi : L \to$ $\mathbf{gl}(V)$, where V is a vector space over $\mathbb{C}$.*

Definition 3.1.6. *Let the derived series of L be defined as $L^{(0)} = L, L^{(i)} = [L^{(i-1)}L^{(i-1)}]$. We say L is solvable if $L^{(n)} = 0$, for some n.*

Proposition 3.1.7. *There exists a unique maximal solvable ideal, called the radical of L, denoted Rad(L). We say L is semisimple if Rad$(L) = 0$.*

Definition 3.1.8. *Let the lower central series of L be defined as $L^0 = L, L^i = [LL^{i-1}]$. We say L is nilpotent if $L^n = 0$, for some n.*

3.2 Semisimple Lie algebras

Definition 3.2.1. *Let L be a Lie algebra, and let $x, y \in L$. Define*

$$\kappa(x, y) = \mathrm{Tr}(\mathrm{ad}x\ \mathrm{ad}y).$$

Then κ is a symmetric bilinear form on L, called the Killing form. The Killing form κ is associative, that is, $\kappa([xy], z) = \kappa(x, [yz])$.

Theorem 3.2.2. *Let L be a Lie algebra. Then L is semisimple if and only if its Killing form is nondegenerate.*

Let V be a $\mathbb{C}$-vector space, and let $x \in \mathrm{End}(V)$. We say x is semisimple if x is diagonalizable; we say x is nilpotent if $x^n = 0$, for some n.

For $x \in \mathrm{End}(V)$, there exists a decomposition, called the (additive) Jordan-Chevalley decomposition of x such that $x = x_s + x_n$, where x_s is semisimple and x_n is nilpotent, and x_s, x_n commute. We also have that any endomorphism that commutes with x commutes with both x_s and x_n.

3.3 Representations of $\mathbf{sl}(2, \mathbb{C})$

First, we specify the generators of $\mathbf{sl}(2, \mathbb{C})$: $x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then, we have $[hx] = 2x$, $[hy] = -2y$, $[xy] = h$.

Let V be an arbitary $\mathbf{sl}(2, \mathbb{C})$-module. We have that h is semisimple and it acts on V diagonally. This yields a decomposition of V as direct sum of eigenspaces

$$V_\lambda = \{v \in V | h \cdot v = \lambda v\}, \lambda \in F.$$

When V_λ isn't 0, we call λ a weight of h in V and we call V_λ a weight space.

Lemma 3.3.1. *If $v \in V_\lambda$, then $x \cdot v \in V_{\lambda+2}$ and $y \cdot v \in V_{\lambda-2}$.*

Provided the previous lemma, we say that a vector v is maximal of weight λ if it lies in some $V_\lambda \neq 0$ such that $V_{\lambda+2} = 0$. The weight of a maximal vector is always a nonnegative integer, and we call it the highest weight of V.

Theorem 3.3.2. *Let V be an irreducible $\mathbf{sl}(2, \mathbb{C})$-module.*

1. *Relative to h, V is the direct sum of weight spaces $V_\mu, \mu = m, m-2, ..., -(m-2), -m$, where $m + 1 = \dim V$ and $\dim V_\mu = 1$ for each μ.*

2. *V has (up to nonzero scalar multiples) a unique maximal vector, whose weight is m.*

3. *The action of $\mathbf{sl}(2, \mathbb{C})$ on V is given explicitly as follows: choose $v_0 \in V_m$ and set $v_{-1} = 0$, $v_i = \frac{1}{i!} y^i \cdot v_0, i \geq 0$, then we have*

    - $h \cdot v_i = (\lambda - 2i)v_i$,
    - $y \cdot v_i = (i + 1)v_{i+1}$,
    - $x \cdot v_i = (\lambda - i + 1)v_{i-1}, i \geq 0.$

    In particular, there exists at most one irreducible $\mathbf{sl}(2, \mathbb{C})$-module (up to isomorphism) of each possible dimension $m + 1$, $m \geq 0$.

Corollary 3.3.3. *Let V be any $\mathbf{sl}(2,\mathbb{C})$-module. Then the eigenvalues of h on V are all integers, and each occurs along with its negative (an equal number of times). Moreover, in any decomposition of V into a direct sum of irreducible submodules, the number of summands is precisely $\dim V_0 + \dim V_1$.*

3.4 Cartan subalgebra

Definition 3.4.1. *Let L be a semisimple Lie algebra. A toral subalgebra of L is a subalgebra of L consisting of semisimple elements.*

We remark that nonzero toral subalgebras exist, and it is abelian. Now fix a maximal toral subalgebra H of L. Then, we have that L is the direct sum of the subspaces

$$L_\alpha = \{x \in L | [hx] = \alpha(h)x, \text{ for all } h \in H\}, \text{ where } \alpha \text{ ranges over } H^*.$$

Notice that L_0 is simply $C_L(H)$ the centralizer of H, which contains H.

Definition 3.4.2. *Let Φ denote the set of all nonzero $\alpha \in H^*$ for which $L_\alpha \neq 0$. The elements of Φ are called the roots of L relative to H. With this notation, we obtain the root space decomposition of L given by*

$$L = C_L(H) \oplus \coprod_{\alpha \in \Phi} L_\alpha.$$

Proposition 3.4.3. *Let H be a maximal toral subalgebra of L. Then $H = C_L(H)$.*

Proposition 3.4.4. *The restriction of the Killing form κ to H is nondegenerate.*

The above proposition allows us to identify H with H^*: to ϕ corresponds the (unique) element $t_\phi \in H$ satisfying $\phi(H) = \kappa(t_\phi, h)$ for all $h \in H$. In particular, Φ corresponds to the subset $\{t_\alpha | \alpha \in \Phi\}$ of H.

Proposition 3.4.5. *1. Φ spans H^*.*

2. *If $\alpha \in \Phi$, then $-\alpha \in \Phi$.*

3. *Let $\alpha \in \Phi$, $x \in L_\alpha$, $y \in L_{-\alpha}$. Then $[xy] = \kappa(x, y)t_\alpha$.*

4. *If $\alpha \in \Phi$, then $[L_\alpha L_{-\alpha}]$ is one dimensional, with basis t_α.*

5. *$\alpha(t_\alpha) = \kappa(t_\alpha, t_\alpha) \neq 0$, for $\alpha \in \Phi$.*

6. *If $\alpha \in \Phi$ and x_α is any nonzero element of L_α, then there exists $y_\alpha \in L_{-\alpha}$ such that x_α, y_α, $h_\alpha = [x_\alpha y_\alpha]$ span a three dimensional simple subalgebra of L isomorphic to $\mathbf{sl}(2, \mathbb{C})$ via $x_\alpha \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $y_\alpha \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $h_\alpha \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.*

7. *$h_\alpha = \frac{2t_\alpha}{\kappa(t_\alpha, t_\alpha)}$; $h_\alpha = -h_{-\alpha}$.*

3.5 Root system

Since the restriction of the Killing form to H is nondegenerate, we can transfer the form to H^* by letting $(\gamma, \delta) = \kappa(t_\gamma, t_\delta)$, for all $\gamma, \delta \in H^*$. We know that Φ spans H^*, so we can choose a basis $\alpha_1, ..., \alpha_l$ of H^* consisting of roots. If $\beta \in \Phi$, the write β uniquely as

$$\beta = \sum_{i=1}^{l} c_i \alpha_i, c_i \in \mathbb{C}.$$

It can be shown that c_i are in fact in $\mathbb{Q}$. Consider the real vector space spanned by the roots equipped with an inner product given by the Killing form, denoted by E.

Theorem 3.5.1. *Let L, H, Φ, E be as above. Then:*

1. *Φ spans E, and 0 does not belong to Φ.*

2. *If $\alpha \in \Phi$, then $-\alpha \in \Phi$, but no other scalar multiple of α is a root.*

3. *If $\alpha, \beta \in \Phi$, then $\beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)} \in \Phi$.*

4. *If $\alpha, \beta \in \Phi$, then $\frac{2(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$.*

3.6 Weyl group and Weyl chamber

We resume the discussion with the same notations developed from the previous subsection. For a nonzero vector $\alpha \in E$, let σ_α denote the reflection generated by α with reflecting hyperplane given by $P_\alpha = \{\beta \in E | (\beta, \alpha) = 0\}$. More explicitly, $\sigma_\alpha(\beta) = \beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha$. We abbreviate $\frac{2(\beta,\alpha)}{(\alpha,\alpha)}$ by $\langle \beta, \alpha \rangle$.

Definition 3.6.1. *A subset Φ of E is called a root system in E if the following properties are satisfied:*

- *(R1) Φ is finite, spans E, and does not contain 0.*
- *(R2) If $\alpha \in \Phi$, the only multiples of $\alpha \in \Phi$ are α and $-\alpha$.*
- *(R3) If $\alpha \in \Phi$, the reflection σ_α leaves Φ invariant.*
- *(R4) If $\alpha, \beta \in \Phi$, then $\langle \beta, \alpha \rangle \in \mathbb{Z}$.*

Definition 3.6.2. *Let Φ be a root system in E. Denote by $\mathcal{W}$ the subgroup of $GL(E)$ generated by the reflections $\sigma_\alpha, \alpha \in \Phi$. We call $\mathcal{W}$ the Weyl group of Φ.*

Definition 3.6.3. *A subset Δ of Φ is called a base if:*

- *(B1) Δ is a basis of E.*
- *(B2) Each root β can be written as $\beta = \sum k_\alpha \alpha (\alpha \in \Delta)$ with integral coefficient k_α all nonnegative or all nonpositive.*

The roots in Δ are then called simple and $|\Delta| = l$. The expression for β in (B2) is unique and we call $\sum_{\alpha \in \Delta} k_\alpha$ the height of β.

Definition 3.6.4. *Let γ be a vector in E.*

1. *Let $\Phi^+(\gamma) = \{\alpha \in \Phi | (\gamma, \alpha) > 0\}$ denote the set of roots lying on the positive side of the hyperplane orthogonal to γ.*
2. *We say $\gamma \in E$ is regular if $\gamma \in E \setminus \bigcup_{\alpha \in \Phi} P_\alpha$, and singular otherwise.*

3. *When γ is regular, Φ decomposes into $\Phi = \Phi^+(\gamma) \cup -\Phi^+(\gamma)$. We say $\alpha \in \Phi^+(\gamma)$ is decomposable if $\alpha = \beta_1 + \beta_2$ with $\beta_1, \beta_2 \in \Phi^+(\gamma)$, and indecomposable otherwise.*

Theorem 3.6.5. *Let $\gamma \in E$ be regular. Then the set $\Delta(\gamma)$ of all indecomposable roots in $\Phi^+(\gamma)$ is a base of Φ, and every base is obtainable in the manner.*

Definition 3.6.6. *The hyperplanes P_α partition E into finitely many regions. We call the connected components of $E \setminus \bigcup_{\alpha \in \Phi} P_\alpha$ the (open) Weyl chambers of E. For each indecomposable γ, let $\mathfrak{C}(\gamma)$ denote the Weyl chamber containing γ. Write $\mathfrak{C}(\Delta) = \mathfrak{C}(\gamma)$ if $\Delta = \Delta(\gamma)$, and call this the fundamental Weyl chamber relative to Δ.*

Theorem 3.6.7. *Let Δ be a base of Φ.*

1. *If $\gamma \in E$ is regular, there exists $\sigma \in \mathcal{W}$ such that $(\sigma(\gamma), \alpha) > 0$ for all $\alpha \in \Delta$, so $\mathcal{W}$ acts transitively on Weyl chambers.*

2. *If Δ' is another base of Φ, then $\sigma(\Delta') = \Delta$ for some $\sigma \in \mathcal{W}$, so $\mathcal{W}$ acts transitively on bases.*

3. *If α is any root, there exists $\sigma \in \mathcal{W}$ such that $\sigma(\alpha) \in \Delta$.*

4. *$\mathcal{W}$ is generated by the σ_α with $\alpha \in \Delta$.*

5. *If $\sigma(\Delta) = \Delta$, $\sigma \in \mathcal{W}$, then $\sigma = 1$, so $\mathcal{W}$ acts simply transitively on bases.*

3.7 ADE type dynkin diagrams

Definition 3.7.1. *We call Φ irreducible if it cannot be partition into the union of two proper subsets such that each root in one set is orthogonal to each root in the other.*

Definition 3.7.2. *Fix an ordering $\alpha_1, ..., \alpha_l$ of the simple roots. The matrix $M_{ij} = (\alpha_i, \alpha_j)$ is called the Cartan matrix of Φ, and its entries are called the Cartan integers.*

Definition 3.7.3. 1. *Define the Coxeter graph of Φ to be a graph having l vertices, with the i-th joined to the j-th ($i \neq j$) by $\langle \alpha_i, \alpha_j \rangle \langle \alpha_j, \alpha_i \rangle$ edges. (The number $\rangle \langle \alpha_j, \alpha_i \rangle$ can only be 0, 1, 2, or 3.)*

2. *When a double or triple edge occurs in the Coxeter graph of Φ, we add an arrow pointing to the shorter of the two roots. We call the resulting figure the Dynkin diagram of Φ.*

Recall that Φ is irreducible if and only if Φ cannot be partitioned into two proper, orthogonal subsets. Hence, Φ is irreducible if and only if its Coxeter graph is connected.

Proposition 3.7.4. *Φ decomposes (uniquely) as the union of irreducible root systems Φ_i (in subspaces E_i of E) such that $E = E_1 \oplus ... \oplus E_t$ (orthogonal direct sum).*

Theorem 3.7.5. *If Φ is an irreducible root system of rank l, its Dynkin diagram is one of the following (l vertices in each case): $A_l(l \geq 1)$, $B_l(l \geq 2)$, $C_l(l \geq 3)$, $D_l(l \geq 4)$, E_6, E_7, E_8, G_2.*

Chapter 4

Kronheimer's construction of ALE spaces

We use the following section to give a review of Kronheimer's construction of ALE spaces in [21] which will be of great importance the main gauge-theoretic construction which will be given in the last chapter. Additional basic representation theory of finite groups can be found in [10].

4.1 Kronheimer's construction of ALE spaces

We review Kronheimer's construction of ALE spaces via hyperkähler reduction in [21] in this subsection.

Let Γ be a finite subgroup of $SU(2)$ and let R be its regular representation. Let $Q \cong \mathbb{C}^2$ be the canonical 2-dimensional representation of $SU(2)$ and let $P = Q \otimes End(R)$, where $End(R)$ denote the endomorphism space of R. Let $M = P^\Gamma$ be the space of Γ-invariant elements in P. After fixing a Γ-invariant hermitian metric on R, P and M can be regarded as right $\mathbb{H}$-modules. Now, choose an orthonormal basis on Q, then we can write an element in P as a pair of matrices (α, β) with $\alpha, \beta \in End(R)$, and the action of J on P is given by

$$J(\alpha, \beta) = (-\beta^*, \alpha^*).$$

Since the action of Γ on P is $\mathbb{H}$-linear, the subspace M is then an $\mathbb{H}$-submodule, which can be regarded as a flat hyperkähler manifold. Explicitly, a pair (α, β) is in M if for each

$$\gamma = \begin{pmatrix} u & v \\ -v^* & u^* \end{pmatrix},$$

where v^* and u^* denote the complex conjugate of v and u, respectively, we have

$$R(\gamma^{-1})\alpha R(\gamma) = u\alpha + v\beta, \tag{4.1.1}$$

$$R(\gamma^{-1})\beta R(\gamma) = -v^*\alpha + u^*\beta. \tag{4.1.2}$$

Let $U(R)$ denote the group of unitary transformations of R and let F be the subgroup formed by elements in $U(R)$ that commute with the Γ-action on R. The natural action of F on P is given by the following: for $f \in F$,

$$(\alpha, \beta) \mapsto (f\alpha f^{-1}, f\beta f^{-1}).$$

Again, the action of F on P is $\mathbb{H}$-linear and preserves M. On the other hand, since F acts by conjugation, the scalar subgroup $T \subset F$ acts trivially, and hence, we get an action of F/T on M that preserves I, J, K.

Now, let $\mathbf{f}/\mathbf{t}$ be the Lie algebra of F/T and identify $(\mathbf{f}/\mathbf{t})^*$ with the traceless elements of $\mathbf{f} \subset End(R)$. As the action of F/T on M is Hamiltonian with respect to I, J, K, we obtain the following moment maps:

$$\mu_1(\alpha, \beta) = \frac{i}{2}([\alpha, \alpha^*] + [\beta, \beta^*]),$$

$$\mu_2(\alpha, \beta) = \frac{1}{2}([\alpha, \beta] + [\alpha^*, \beta^*]),$$

$$\mu_3(\alpha, \beta) = \frac{i}{2}(-[\alpha, \beta] + [\alpha^*, \beta^*]).$$

Let $\mu = (\mu_2, \mu_2, \mu_3) : M \to \mathbb{R}^3 \otimes (\mathbf{f}/\mathbf{t})^*$. Let Z denote the center of $(\mathbf{f}/\mathbf{t})^*$ and let

$\zeta = (\zeta_1, \zeta_2, \zeta_3) \in \mathbb{R}^3 \otimes Z$. For ζ lying in the "good set", we get $X_\zeta = \mu^{-1}(\zeta)/F$ is a smooth 4-manifold diffeomorphic to $\widetilde{\mathbb{C}^2/\Gamma}$.

Proposition 4.1.1 (cf. Proposition 2.1. in [21]). *Suppose that F acts freely on $\mu^{-1}(\zeta)$. Then*

1. *$d\mu$ has full rank at all points of $\mu^{-1}(\zeta)$, so that X_ζ is a nonsingular manifold of* $\dim M - 2\dim F$ *(*resp. $\dim M - 4\dim F$*),*
2. *the metric g and complex structures I (*resp. *I, J, K) descend to X_ζ, and equipped with these, X_ζ is kähler (*resp. *hyperkähler).*

Now, we review some basic representation theory regarding to the McKay Correspondence [23] mentioned in [21]. Let $R_0, ..., R_r$ be the irreducible representations of Γ with R_0 the trivial representation, and let

$$Q \otimes R_i = \bigoplus_j a_{ij} R_j$$

be the decomposition of $Q \otimes R_i$ into irreducibles. The representations $R_1, ..., R_r$ correspond to the set of simple roots $\xi_1, ..., \xi_r$ for the associated root system of one of the ADE-type Dynkin diagrams. Furthermore, if $\xi_0 = -\sum_1^r n_i \xi_i$ is the negative of the highest root, then we have that for all i,

$$n_i = \dim R_i.$$

Hence, the regular representation R decomposes as

$$R = \bigoplus_i \mathbb{C}^{n_i} \otimes R_i,$$

and M decomposes as

$$M = \bigoplus_{i,j} a_{ij} Hom(\mathbb{C}^{n_i}, \mathbb{C}^{n_j}),$$

and F can be written as

$$F = \times_i U(n_i).$$

Consequently, we get

$$\dim_{\mathbb{R}} M = \sum_{i,j} 2a_{ij}n_i n_j = \sum_i 4n_i^2 = 4|\Gamma|,$$

and

$$\dim_{\mathbb{R}} F = \sum_i n_i^2 = |\Gamma|.$$

The center of the Lie algebra **f** is spanned by the elements $\sqrt{-1}\pi_i$, where π_i is the projection $\pi_i : R \to \mathbb{C}^{n_i} \otimes R_i$ $(i = 0, ..., r)$. Let h be the real Cartan algebra associated to the Dynkin diagram, then there is a linear map l from the center of **f** to h^* defined by the following:

$$l : \sqrt{-1}\pi_i \mapsto n_i \xi_i.$$

The kernel of l is the one-dimensional subalgebra $\mathbf{t} \subset \mathbf{f}$, so on the dual space, we get an isomorphism

$$\iota : Z \to h.$$

For each root ξ, we write

$$D_\xi = \ker(\xi \circ \iota).$$

Proposition 4.1.2 (cf. Proposition 2.8. in [21])**.** *If F/T does not act freely on $\mu^{-1}(\zeta)$, then ζ lies in one of the codimensional-3 subspaces $\mathbb{R}^3 \otimes D_\xi \subset \mathbb{R} \otimes Z$, where ξ is a root.*

Hence, the “good set" mentioned earlier in the subsection refers to the following:

$$(\mathbb{R}^3 \otimes Z)^\circ = (\mathbb{R}^3 \otimes Z) \setminus \bigcup_\xi (\mathbb{R}^3 \otimes D_\xi).$$

4.2 Relevant theorems

The following theorems are also proven in [21] and [22], and together, they give a complete construction and classification of ALE spaces. For all the theorems below in this subsection, let (X, g) be a 4-dimensional hyperkähler manifold.

Theorem 4.2.1 (cf. Theorem 1.1. in [21])**.** *Let three cohomology classes $\alpha_1, \alpha_2, \alpha_3 \in H^2(X;\mathbb{R})$ be given which satisfy the nondegeneracy condition $(*)$:*

- *for each $\Sigma \in H_2(X;\mathbb{Z})$ with $\Sigma \cdot \Sigma = -2$, there exists $i \in \{1,2,3\}$ with $\alpha_i(\Sigma) \neq 0$.*

Then there exists on X an ALE hyperkähler structure for which the cohomology classes of the kähler form $[\omega_i]$ are the given α_i.

Theorem 4.2.2 (cf. Theorem 1.2. in [21])**.** *Every ALE hypherkähler 4-manifold is diffeomorphic to the minimal resolution of $\mathbb{C}^2/\Gamma$ for some $\Gamma \subset SU(2)$, and the cohomology classes of the kähler forms on such a manifold must satisfy the nondegeneracy condition $(*)$.*

Theorem 4.2.3 (cf. Theorem 1.3. in [21])**.** *If X_1 and X_2 are two ALE hyperkähler 4-manifolds, and there is a diffeomorphism $X_1 \to X_2$ under which the cohomology classes of the kähler forms agree, then X_1 and X_2 are isometric.*

Chapter 5

Basic gauge theory

We follow [6], [19] and [20] for the discussion of basic gauge theory in the following several sections. For the discussion of orbifolds and orbifold bundles, we mostly follow [3].

5.1 Vector bundles and principal bundles

Definition 5.1.1 (principal bundle). *A principal G-bundle over B is a manifold P with a smooth map $\pi : P \to B$ such that*

- *G acts freely on P (on the left),*
- *B is the orbit space for this action and π is the point-orbit projection, and*
- *there is an open covering of B such that to each set $\mathcal{U}$ in that covering corresponds a map $\varphi_{\mathcal{U}} : \pi^{-1}(\mathcal{U}) \to \mathcal{U} \times G$ with $\varphi_{\mathcal{U}}(p) = (\pi(p), s_{\mathcal{U}}(p))$, and $s_{\mathcal{U}}(g \cdot p) = g \cdot s_{\mathcal{U}}(p)$, for all $p \in \pi^{-1}(\mathcal{U})$.*

The G-valued maps $s_{\mathcal{U}}$ are determined by the corresponding $\varphi_{\mathcal{U}}$. The third condition is called the property of being locally trivial.

Example 5.1.2. *S^3 as a principal S^1-bundle with base S^2 via the Hoxpf fibration. The explicit construction is as follows:*

$$S^3 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\}.$$

The S^1-action on S^3 is given by the following: for $g = e^{i\theta} \in S^1$,

$$(z_1, z_2)g = (e^{-i\theta} z_1, e^{-i\theta} z_2).$$

If we think of S^2 as $\mathbb{C}P^1$ with the standard homogeneous coordinates, we get the following projection map for the Hopf fibration: $\pi : S^3 \to \mathbb{C}P^1$, $(z_1, z_2) \mapsto [z_1 : z_2]$.

Definition 5.1.3 (associated bundles)**.** *Let V be a complex vector space and S^1 acts on V. Let $E(V)$ be defined as follows: $E(V) = S^3 \times V/\sim$, where $[p, v] \sim [pg^{-1}, gv]$, for all $g \in S^1$. Thus, we can think of $E(V)$ as an associated bundle of S^3 with base S^2 and fiber V.*

Example 5.1.4 (hyperplane bundle)**.** *Let $V = \mathbb{C}$. Then E(V) is a complex line bundle over $S^2 = \mathbb{C}P^1$. Recall, the hyperplane bundle H over $\mathbb{C}P^1$ is a complex line bundle given by the following trivialization: let $U_1 = \{z_1 \neq 0\}$ and $U_2 = \{z_2 \neq 0\}$ be the two charts covering $\mathbb{C}P^1$. On $U_1 \cap U_2$, the transition map is given by $g_{12} = \frac{z_2}{z_1}$ and on $U_2 \cap U_1$, the transition map is given by $g_{21} = \frac{z_1}{z_2}$.*

Now, let z be a complex number. Using the previous trivialization for H, we can show that the following map $\phi : E(V) \to H$, where $\phi([(z_1, z_2), z]) = ([z_1 : z_2], z)$, is a well-defined bundle isomorphism. Hence, the hyperplane bundle over $\mathbb{C}P^1$ is an associated bundle of S^3.

5.2 Sections and connections

Definition 5.2.1 (Section of a vector bundle)**.** *Let $\pi : E \to M$ be a vector bundle over M, a section of E is a smooth map $s : M \to E$ with $\pi \circ s = id_M$.*

Definition 5.2.2 (section of a principal bundle)**.** *Let $\pi : P \to M$ be a principal G-bundle over M, a section of P is a holomorphic map $s : M \to P$ with $\pi \circ s = id_M$.*

Remark 5.2.3. *A principal bundle admits a section if and only if it is trivial.*

Definition 5.2.4 (connection on a principal bundle)**.** *We give two equivalent definitions here:*

1. Let P be a principal G-bundle and let g *denote the Lie-algebra of G. Fix a basis $X_1, ..., X_k$ of* g*, let $X_1^\sharp, ..., X_k^\sharp$ denote the linearly independent vector fields generated by the one-parameter groups $\{\exp(tX_i(e)) | t \in \mathbb{R}\}$. A connection form on P is a choice of splitting $TP = V \oplus H$ such that H is a G-invariant horizontal subbundle of TP complementary to the vertical bundle V, where V is the vertical subbundle generated by $X_1^\sharp, ..., X_k^\sharp$.*

2. A connection form on a principal G-bundle P is a Lie-algebra-valued 1-form $A = \sum_{i=1}^k A_i \otimes X_i \in \Omega^1(P) \otimes$ g *such that:*

(a) A is G-invariant, with respect to the product action of G on $\Omega^1(P)$ (the pullback action induced by the action on P: $g \cdot \alpha = g^*\alpha$) and on g *(the adjoint representation)*

(b) A is vertical, in the sense that $\iota_{X^\sharp} A = X$ for any $X \in$ g*.*

Definition 5.2.5 (connection on a vector bundle)**.** *Let M be a smooth manifold and $E \to M$ a smooth complex vector bundle. A connection on E is a linear map $\nabla : C^\infty(M, E) \to C^\infty(M, T^*M \otimes E)$ satisfying the Leibiniz rule: for $f \in C^\infty(M)$ and $\sigma \in C^\infty(M, E)$, $\nabla(f\sigma) = df \otimes \sigma + f\nabla\sigma$.*

Definition 5.2.6 (curvature 2-form)**.** *The curvature of ∇ is $F^\nabla \in \Omega^2(M; End(E))$ given by $F(X, Y)\sigma = \nabla_X\nabla_Y\sigma - \nabla_Y\nabla_X\sigma - \nabla_{[X,Y]}\sigma$. Locally, let $\{e_1, ...e_n\}$ be a local frame for E, then the connection 1-forms for ∇ are given by $A_i^j \in \Omega^1(M)$, where $\nabla_{e_i} = A_i^j \otimes e_j$. In other words, ∇ is locally given by a matrix of 1-forms $A \in \Omega^1(M; End(E))$ and hence F^∇ is locally given by $F = dA + A \wedge A$.*

Definition 5.2.7 (covariant exterior derivative)**.** *Given a connection ∇ on E, let d_∇ be defined as follows: for $\omega \in \Omega^p(M)$, $\sigma \in C^\infty(M, E)$, $d_\nabla(\omega \otimes \sigma) = d\omega \otimes \sigma + (-1)^p\omega \wedge \nabla\sigma$. This gives rise to the following sequence:*

$$... \to \Omega^{p-1}(M; E) \to \Omega^p(M; E) \to \Omega^{p+1}(M; E) \to ...$$

In particular, for $\beta \in \Omega^p(M; E)$, $d_\nabla \circ d_\nabla(\beta) = F^\nabla \wedge \beta$.

Definition 5.2.8 (Hermitian connection)**.** *Let $h : E \otimes \bar{E} \to \mathbb{C}$ be a hermitian inner product on E. A connection ∇ on E is hermitian with respect to h if for all sections*

$\sigma, \tau \in C^\infty(M, E)$, we have $X(h(\sigma,\tau)) = h(\nabla_X \sigma, \tau) + h(\sigma, \nabla_X \tau)$.

Lemma 5.2.9. *(1) The space $\mathcal{A}(P)$ of connections on a principal bundle P over M is modeled on the affine space $\Omega^1(M; ad(P))$.*

(2) The space $\mathcal{A}(E)$ of connections on E is modeled on the affine space $\Omega^1(M; End(E))$.

(3)The space $\mathcal{A}^h(E)$ of hermitian connections on E is modeled on the affine space $\Omega^1(M; skewEnd(E))$ where $skewEnd(E) = \{B \in End(E) | h(Bv, w) + h(v, Bw) = 0, v, w \in E\}$.

Below, we illustrate a way to construct a connection on an associated bundle from a connection on the principal bundle:

Given a connection A on a principal G-bundle $P \to M$, as in a previous definition, we can think of A as an element in $\Omega^1(P, \mathbf{g})$ such that A is G-invariant and vertical. Hence, locally on a trivializing neighborhood $U \subset M$, we can push A down and write A as $A_U \in \Omega^1(U, \mathbf{g})$; in other words, A is locally a 1-forms on U with values in $\mathbf{g}$. Now let V be a G-representation, that is, we have $\rho : G \to Gl(V)$, and let $E(V)$ be the associated bundle of P with fiber V. We get a connection ∇^A on $E(V)$ induced by A as follows: locally on U, ∇^A can be expressed as a matrix of 1-forms given by $\alpha_U = \rho_* \circ A_U \in \Omega^1(U, End(V))$, where $\rho_* : \mathbf{g} = T_e G \to T_e Gl(V) = gl(V) = End(V)$. Hence, we can think of the space of connections on P as a subset of the space of connections on an associated bundle.

On the other hand, for the case of $P = S^3$ and $G = S^1 \cong U(1)$, we can identify the space of hermitian connections on $H \cong E(\mathbb{C})$ precisely with the space of connections on S^3, that is, $\mathcal{A}^h(H) \cong \mathcal{A}(S^3)$.

5.3 Holomorphic structures

Definition 5.3.1 (holomorphic vector bundle). *Let M be a complex manifold and $E \to M$ a complex vector bundle over M. Then E is holomorphic if either one of the following equivalent conditions holds:*

(1) E is a complex manifold, $\pi : E \to M$ is holomorphic and local trivializations $\{\varphi_\alpha\}$ can be chosen to be holomorphic as well.

(2) The transition maps $g_{\alpha\beta} : U_\alpha \cap U_\beta \to Gl_n(\mathbb{C})$ can be chosen to be holomorphic.

Example 5.3.2. *With the transition maps given in Example 4.5, the hyperplane bundle H over $\mathbb{C}P^1$ is a holomorphic line bundle.*

Definition 5.3.3 (holomorphic section)**.** *Let $\pi : E \to M$ be a holomorphic vector bundle over M, a section of E is a holomorphic map $s : M \to E$ with $\pi \circ s = id_M$.*

Before we proceed, note that for a complex manifold M, we have the following linear operators: $\partial : \Omega^{p,q}(M) \to \Omega^{p+1,q}(M)$ and $\bar{\partial} : \Omega^{p,q}(M) \to \Omega^{p,q+1}(M)$. Now, let $E \to M$ be a holomorphic vector bundle over M. Then there exists a unique linear operator $\bar{\partial}_E : \Omega^{p,q}(M; E) \to \Omega^{p,q+1}(M; E)$ satisfying $\bar{\partial}(f\omega) = \bar{\partial} f \wedge \omega + f\bar{\partial}_E \omega$, for $f \in C^\infty(M)$ and $\omega \in \Omega^{p,q}(M; E)$; moreover, we have $\bar{\partial}_E^2 = 0$. Given ∇ a connection on E, we can write $d_\nabla = \partial_\nabla + \bar{\partial}_\nabla : \Omega^0(M; E) \to \Omega^{1,0}(M; E) \oplus \Omega^{0,1}(M; E)$, where $\partial_\nabla = \pi^{1,0} \circ d_\nabla$, $\bar{\partial}_\nabla = \pi^{0,1} \circ d_\nabla$.

Definition 5.3.4 (holomorphic connection)**.** *Let E be a holomorphic vector bundle. A connection ∇ on E is compatiable with the holomorphic structure if $\bar{\partial}_\nabla = \bar{\partial}_E$.*

Lemma 5.3.5. *Let E be a complex vector bundle and fix a Hermitian structure on E. Then there is a one-to-one correspondence between the Hermitian connections and the holomorphic structures on E.*

5.4 Actions on the configuration space

Definition 5.4.1 (gauge group)**.** *Let P be a principal fiber bundle with base M and fiber G. Then a diffeomorphism $f : P \to P$ is a gauge transformation if f commutes with the G-action on P and the induced map on the base is the identity, that is $f(pg) = f(p)g$. The group of all gauge transformations is called the gauge group, denoted $\mathcal{G}$.*

Definition 5.4.2 (complexified gauge group)**.** *The complexifed group group is defined as the complexification of $\mathcal{G}$, that is, $\mathcal{G} \otimes \mathbb{C}$.*

Example 5.4.3 (the gauge group of S^3). *In the case of S^3, the gauge group $\mathcal{G}$ is given by $\mathcal{G} = Maps(S^2, S^1)$ and the Lie algebra of $\mathcal{G}$ is given by $Lie\mathcal{G} = Maps(S^2, \mathbb{R}) = C^\infty(S^2)$. Let $\mathcal{G}_0$ be the based gauge group of S^3, $\mathcal{G}_0 = \{g : S^2 \to S^1 \mid g(x_0) = 1, x_0 \in S^2\}$, and $\mathcal{G}_0^c$ the complexified based gauge group, $\mathcal{G}_0^c = \{u : S^2 \to \mathbb{C}^* \mid u(x_0) = 1, x_0 \in S^2\}$.*

Lemma 5.4.4. *Let $\mathcal{G}$ be the gauge group and let $\mathcal{G}^c$ be the complexified gauge group of P. Then $\mathcal{G}$ and $\mathcal{G}^c$ act on the space of connections $\mathcal{A}(P)$ as follows: for $A \in \Omega^1(M; ad(P))$ and $g \in \mathcal{G}$, $g \cdot A = gd_Ag^{-1} + gAg^{-1}$. Similarly, for $u \in \mathcal{G}^c$, $u \cdot A = u^{-1} \circ \bar{\partial}_A \circ u - u^{*-1} \circ \partial_A \circ u^*$.*

Below, we introduce a symplectic structure on $\mathcal{A}^h(H) \times C^\infty(S^2, H)$ taken from [25]. The symplectic 2-form Ω on $\mathcal{A}^h(H) \times C^\infty(S^2, H)$ is given as follows: for $\alpha_1, \alpha_2 \in T_A\mathcal{A}^h(H)$, $\theta_1, \theta_2 \in T_\Theta C^\infty(S^2, H)$, $\Omega((\alpha_1, \theta_1), (\alpha_2, \theta_2)) = -\int_{S^2} \alpha_1 \wedge \alpha_2 + \int_{S^2} \mathrm{Im}\langle\theta_1, \theta_2\rangle\omega_{vol}$

5.5 Orbifold vector bundles

For this section, we follow mainly [3] for the discussion of orbifolds and orbifold bundles.

Definition 5.5.1 (orbifold chart). *Let M be a topological space. An orbifold chart $(\tilde{U}, \Gamma, \phi)$ of dimension n for an open set $U \subset M$ consists of a connected open subset $\tilde{U} \subset \mathbb{R}^n$, a finite group Γ acting smoothly and effectively on $\tilde{U}$ and a continuous Γ-invariant map $\phi : \tilde{U} \to M$ that induces a homeomorphism between $\tilde{U}/\Gamma$ and U.*

Definition 5.5.2 (orbifold embedding). *An embedding $\lambda : (\tilde{U_1}, \Gamma_1, \phi_1) \to (\tilde{U_2}, \Gamma_2, \phi_2)$ between two orbifold charts is a smooth embedding $\lambda : \tilde{U_1} \hookrightarrow \tilde{U_2}$ that satisfies $\phi_2 \circ \lambda = \phi_1$.*

Definition 5.5.3 (orbifold atlas). *An orbifold atlas for M is a collection of orbifold charts $\{(\tilde{U_i}, \Gamma_i, \phi_i)\}$ that covers M and are locally compatible in the following sense:*

for any two charts $(\tilde{U}_i, \Gamma_i, \phi_i)$*,* $i = 1, 2$*, and* $x \in U_1 \cap U_2$*, there is an open neighborhood* $U_3 \subset U_1 \cap U_2$ *containing* x *and an orbifold chart* $(\tilde{U}_3, \Gamma_3, \phi_3)$ *for* U_3 *that admits embeddings in* $(\tilde{U}_i, \Gamma_i, \phi_i)$*,* $i = 1, 2$*.*

Definition 5.5.4 (orbifold)**.** *An orbifold* $\mathcal{O}$ *is an underlying topological space* $|\mathcal{O}| = M$ *together with an orbifold atlas* $\mathcal{A}$*.*

Definition 5.5.5 (orbifold smooth map)**.** *Let* $\mathcal{O}$ *and* $\mathcal{P}$ *be orbifolds and let* $|f| : |\mathcal{O}| \to |\mathcal{P}|$ *be a continuous map between the underlying topological spaces. We say that* $|f|$ *is smooth at* $x \in |\mathcal{O}|$ *when there are charts* $(\tilde{U}, \Gamma_x, \phi)$ *and* $(\tilde{V}, \Gamma_{|f|(x)}, \psi)$ *around* x *and* $|f|(x)$*, respectively, such that* $|f|(U) \subset V$ *and there exists a smooth local lift of* $|f|$ *at* x*, that is, a homomorphism* $\bar{f}_x : \Gamma_x \to \Gamma_{|f|(x)}$ *together with a smooth map* $\tilde{f}_x : \tilde{U} \to \tilde{V}$ *such that* $\tilde{f}_x(gy) = \bar{f}_x(g)\tilde{f}_x(\tilde{y})$*, for each* $g \in \Gamma_x$*,* $\tilde{y} \in \tilde{U}$ *and the following diagram commutes.*

$$\begin{tikzcd} \tilde{U} \arrow[r, "\tilde{f}_x"] \arrow[d, "\phi"'] & \tilde{V} \arrow[d, "\psi"] \\ U \arrow[r, "|f|"'] & V \end{tikzcd}$$

A smooth map $f : \mathcal{O} \to \mathcal{P}$ *consists of a continuous map* $|f| : |\mathcal{O}| \to |\mathcal{P}|$ *that is smooth at every* $x \in |\mathcal{O}|$*.*

Definition 5.5.6 (fiber orbibundle)**.** *Let* $\mathcal{E}$ *and* $\mathcal{B}$ *be smooth orbifolds. A smooth map* $\pi : \mathcal{E} \to \mathcal{B}$ *is a fiber orbibfold if* $|\pi|$ *is surjective and there is a third orbifold* $\mathcal{F}$ *such that, for all* $x \in |\mathcal{B}|$*, there is an orbifold chart* $(\tilde{U}, \Gamma_x, \phi)$ *around* x*, an action of* Γ_x *on* $\mathcal{F}$ *and a diffeomorphism* $(\mathcal{F} \times \tilde{U})/\Gamma_x \to \mathcal{E}|_{|\pi|^{-1}(U)}$ *such that the following diagram commutes, where* Γ_x *denotes the stabilizer subgroup of some* $\tilde{x}$ *sitting over* x*.*

$$\begin{tikzcd} (\mathcal{F} \times \tilde{U})/\Gamma_x \arrow[r] \arrow[d] & \mathcal{E} \arrow[d] \\ \tilde{U}/\Gamma_x \arrow[r] & \mathcal{B} \end{tikzcd}$$

Definition 5.5.7 (vector orbibundle)**.** *When* $\mathcal{F}$ *is a* k*-dimensional vector space with* *a linear action of Γ_x, then $\pi : \mathcal{E} \to \mathcal{B}$ is a vector orbibundle.*

Example 5.5.8 (tangent orbibundle and differential)**.** *(1) We can define the tangent bundle of an orbifold as follows. Let $(\tilde{U}, \Gamma, \phi)$ be an orbifold chart. Observe that the Γ-action on $\tilde{U}$ induces an a Γ-action on $T\tilde{U}$ as follows: $\gamma(\tilde{x}, v) = (\gamma(\tilde{x}), d\gamma_{\tilde{x}} v)$. This gives rise to an orbifold chart $(T\tilde{U}, \Gamma, \varphi)$, where $\varphi : T\tilde{U} \to TU = T\tilde{U}/\Gamma$. We also get a projection map $|\pi| : TU \to U$. For $x = \phi(\tilde{x})$, $|\pi|^{-1}(x) \cong T_{\tilde{x}}\tilde{U}/\Gamma_x$.*

(2) Similarly, we can construct the cotangent bundle of an orbifold. Again, let $(\tilde{U}, \Gamma, \phi)$ be an orbifold chart. Then the Γ-action on $T^*\tilde{U}$ is given by $\gamma(\tilde{x}, \eta) = (\gamma(\tilde{x}), \eta \circ d\gamma^{-1}_{\gamma(\tilde{x})})$. Using these charts, we produce $T^*\mathcal{O}$ as a vector orbibundle over $\mathcal{O}$.

(3) Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be two vector orbibundles over $\mathcal{B}$ each given by the quotient of a Γ-action. Then we can define the following Γ-action: $\gamma(\tilde{x}, v \otimes w) = (\gamma(\tilde{x}), \gamma(v) \otimes \gamma(w))$. This gives rise to $\mathcal{E}_1 \otimes \mathcal{E}_2$ as a vector orbibundle over $\mathcal{B}$. The same construction also works for $\mathcal{E}_1 \oplus \mathcal{E}_2$.

5.6 Additional structures on a vector orbibundle

Definition 5.6.1 (complex orbifold chart)**.** *A complex orbifold chart is an orbifold chart $(\tilde{U}, \Gamma, \phi)$ where $\tilde{U} \subset \mathbb{C}^n$ and Γ acts not just smoothly, but in fact holomorphically.*

We can also make similar definitions for holomorphic embedding and holomorphic orbifold atlas. Hence, we arrive at the following definition:

Definition 5.6.2 (complex orbifold)**.** *A complex orbifold $\mathcal{O}$ is an underlying topological space $|\mathcal{O}| = M$ together with a complex orbifold atlas $\mathcal{A}$.*

A holomorphic map between two complex orbifolds can be defined similar to a smooth map between orbifolds where we replace $smooth$ with $holomorphic$.

Definition 5.6.3 (orbibundle section)**.** *Let $\pi : \mathcal{E} \to \mathcal{B}$ be a complex vector orbibundle. Then a section of $\mathcal{E}$ is an orbifold smooth map $s : \mathcal{B} \to \mathcal{E}$ such that $\pi \circ s = id_{\mathcal{B}}$.*

Definition 5.6.4 (orbibundle connection)**.** *Given a vector orbibundle $\mathcal{E} \to \mathcal{B}$, a connection on $\mathcal{E}$ is a linear map $\nabla : C^\infty(\mathcal{B}, \mathcal{E}) \to C^\infty(\mathcal{B}, T^*\mathcal{B} \otimes \mathcal{E})$ satisfying the Leibiniz rule: for $f \in C^\infty(\mathcal{B})$ and $\sigma \in C^\infty(\mathcal{B}, \mathcal{E})$, $\nabla(f\sigma) = df \otimes \sigma + f\nabla\sigma$.*

Definition 5.6.5 (Hermitian structure on an orbibundle)**.** *Let $\mathcal{E} \to \mathcal{B}$ be an orbibundle with complex fibers. Then a hermitian structure on $\mathcal{E}$ is a section h of $(\mathcal{E} \otimes \bar{\mathcal{E}})^*$ such that for $x \in \mathcal{B}$, $h_x(\eta, \bar{\zeta}) = \overline{h_x(\zeta, \bar{\eta})}$, $\eta, \zeta \in \mathcal{E}_x$ and $h_x(\eta, \bar{\eta}) > 0$, for $\eta \neq 0$.*

Definition 5.6.6 (holomorphic orbibundle)**.** *Let $\mathcal{B}$ be a complex orbifold and $\mathcal{E} \to \mathcal{B}$ a complex orbibundle over $\mathcal{B}$. Then $\mathcal{E}$ is holomorphic if $\mathcal{E}$ is a complex orbifold, $|\pi| : \mathcal{E} \to \mathcal{B}$ is holomorphic and local trivializations $\{\varphi_\alpha\}$ can be chosen to be holomorphic as well.*

Definition 5.6.7 (holomoprhic orbibundle section)**.** *Let $\pi : \mathcal{E} \to \mathcal{B}$ be a holomorphic orbibundle. Then a holomorphic section of $\mathcal{E}$ is an orbifold holomorphic map $s : \mathcal{B} \to \mathcal{E}$ such that $\pi \circ s = id_{\mathcal{B}}$.*

Definition 5.6.8 (global quotient)**.** *(1) An orbifold $\mathcal{O}$ is a global quotient of a manifold M by a finite group Γ if Γ acts smoothly on M (note that we do not assume effectiveness of the action). In other words, $\mathcal{O}$ can be given an orbifold atlas consisting of charts of the form $(\tilde{U}, \Gamma, \phi)$.*

(2) An orbibundle $\pi : \mathcal{E} \to \mathcal{B}$ is a global quotient of a vector bundle $\tilde{\pi} : E \to B$ if $\mathcal{B}$ is a global quotient of B by Γ and for each $x \in B$, Γ_x acts smoothly on the fiber V_x of E over x. In this case, $\mathcal{E}$ is also a global quotient of E, for which the Γ-action respects the bundle structure of E, that is, $\tilde{\pi} \circ \gamma = \gamma \circ \tilde{\pi}$, for all $\gamma \in \Gamma$.

Definition 5.6.9 (smooth point)**.** *Let x be a point in an orbifold $\mathcal{O}$. Then x is a smooth point if locally around x, the group action induces a covering map.*

Lemma 5.6.10. *Let $\mathcal{E} \to \mathcal{O}$ be a global quotient of a vector bundle $E \to M$ by a finite group Γ such that at a smooth point $x \in \mathcal{O}$, Γ_x acts trivially on the fiber V_x. Then:*

(1) The space of vector orbibundle sections of $\mathcal{E}$ can be identified with the space of Γ-equivariant sections of E.

(2) The space of vector orbibundle connections of $\mathcal{E}$ can be identified with the space of Γ-equivariant connections of E.

Proof. (1) Let s be a Γ-equivariant section of E, that is, for $x \in M$, $s(\gamma(x)) = \gamma(s(x))$. Hence, s descends to a map between the underlying orbifolds, $|s| : |\mathcal{O}| \to$

$|\mathcal{E}|$. Observe also that $|\pi| \circ |s| = id_{\mathcal{O}}$. We want to check that $|s|$ is an orbifold smooth map, which amounts to finding a Γ-equivariant local lift of $|s|$, but s satisfies the required properties precisely.

Conversely, let s be a section of $\mathcal{E}$, as at a smooth point x, Γ_x acts trivially on the fiber, we get a well-defined local lift of s which is Γ-equivariant by construction.

(2) Recall, a connection on E is a linear map $\nabla : C^\infty(M, E) \to C^\infty(M, T^*M \otimes E)$ satisfying the Leibiniz rule: for $f \in C^\infty(M)$ and $\sigma \in C^\infty(M, E)$, $\nabla(f\sigma) = df \otimes \sigma + f\nabla\sigma$. Let ∇ be a Γ-equivariant connection of E, that is, $\nabla\gamma(\sigma) = \gamma(\nabla\sigma)$. Now, let σ be a Γ-equivariant section. Then $\nabla\gamma(\sigma) = \nabla\sigma = \gamma(\nabla\sigma)$. Hence, ∇ acts on the space of Γ-equivariant sections on E. Moreover, for a Γ-equivariant function f, $\nabla(\gamma(f\sigma)) = df \otimes \sigma + f\nabla\sigma = \gamma(\nabla(f\sigma))$, so ∇ also satisfies the Leibiniz rule. Thus, ∇ descends to a connection $\nabla^{\mathcal{E}}$ on $\mathcal{E}$, where $\nabla^{\mathcal{E}} : C^\infty(\mathcal{O}, \mathcal{E}) \to C^\infty(\mathcal{O}, T^*\mathcal{O} \otimes \mathcal{E})$. The converse holds as well similar to the previous case.

$\square$

Lemma 5.6.11. *Let $\mathcal{E} \to \mathcal{O}$ be a complex vector orbibundle arising as the global quotient of a complex vector bundle $E \to M$ by a finite group Γ such that at each smooth point $x \in \mathcal{O}$, Γ_x acts trivially on the fiber V_x. Fix a Γ-equivariant hermitian structure on E which descends to a hermitian structure on $\mathcal{E}$. Then there is a one-to-one correspondence between the hermitian connections and the holomorphic structures on $\mathcal{E}$.*

Proof. We know that there is a one-to-one correspondence between the hermitian connections on E and the $\bar{\partial}$-operators on E. By similar arguments as in the previous lemma, the hermitian structure on $\mathcal{E}$ comes from a Γ-equivariant hermitian structure on E. Let ∇ be a Γ-equivariant hermitian connection on E. Then ∇ induces an exterior covariant derivative $d_\nabla = \partial_\nabla + \bar{\partial}_\nabla : \Omega^0(M; E) \to \Omega^{1,0}(M; E) \oplus \Omega^{0,1}(M; E)$, where $\partial_\nabla = \pi^{1,0} \circ d_\nabla$, $\bar{\partial}_\nabla = \pi^{0,1} \circ d_\nabla$. In particular, $\bar{\partial}_\nabla = \pi^{0,1} \circ d_\nabla : \Omega^0(M; E) \to \Omega^{0,1}(M; E)$. Let $\sigma \in \Omega^0(M; E)$. Then $\bar{\partial}_\nabla(\gamma(\sigma)) = \pi^{0,1} \circ d_\nabla(\gamma(\sigma)) = \gamma(\pi^{0,1} \circ d_\nabla\sigma)) = \gamma(\bar{\partial}_\nabla(\sigma))$. This shows that Γ-equivariant hermitian connections correspond to Γ-equivariant $\bar{\partial}$-operators which further implies that there is a one-to-one correspondence between the hermitian connections and the holomorphic structures on $\mathcal{E}$. $\square$

Chapter 6

Analytic tools in gauge theory

We review the analytic foundations of gauge theory in this chapter. We will use [24] as a reference.

6.1 Linear differential operators

Definition 6.1.1. *Given $x \in M$, let $\mathbf{m}_x \subset C^\infty(M)$ be the ideal consisting of $f \in C^\infty(M)$ such that $f(x) = 0$.*

Definition 6.1.2. *Let $E, F \to M$ be vector bundles. A linear map $D : \Gamma(E) \to \Gamma(F)$ is a linear differential operator (LDO) of order k if k is the smallest integer such that for each $x \in M$, and every $f \in (\mathbf{m}_x)^{k+1}$, and every section $\sigma \in \Gamma(E)$, we have*

$$D(f\sigma)|_x = 0.$$

Example 6.1.3. *A linear differential operator on $\mathbb{R}^n$ of order k can be expressed as a combination of partial derivatives of order less than or equal to k. If $I = \{i_1, ..., i_k\}$ us a multi-index. We write*

$$\partial^I \phi = \frac{\partial^k}{\partial_{i_1} ... \partial_{i_k}}(\phi).$$

Then an order k LDO can be written as

$$D\phi = \sum_{|I|\le k} a_I \partial^I \phi,$$

where a_I is a (smooth) matrix-valued function.

Definition 6.1.4. *Let $D : \Gamma(E) \to \Gamma(F)$ be a k-th order LDO. Fix $x \in M$ and covectors $\alpha_1, ..., \alpha_k \in T^*M$, we define a map $\sigma_D(\alpha_1 \otimes ... \otimes \alpha_k) : E_x \to F_x$ as follows:*

Choose any $f_1, ..., f_k \in \mathbf{m}_x$ such that $(df_j)_x = \alpha_j$. Let $v \in E_x$ and choose any $\phi \in \Gamma(E)$ such that $\phi(x) = v$. Then let

$$\sigma_D(\alpha_1 \otimes ... \otimes \alpha_k)(v) = \frac{1}{k!} D(f_1 ... f_k \phi)|_k \in F_x.$$

Remark 6.1.5. *This defines a linear map $E_x \in F_x$ independent of choices of $f_1, ..., f_k$ and ϕ. In fact, σ_D defines a bundle map*

$$Sym^k(T^*M) \to Hom(E, F).$$

On the other hand, there is an identification of

$$Sym^k(V^*) = \{\textit{symmetric multilinear maps } V \times ... \times V \to \mathbb{R}\}$$

with

$$\mathcal{P}_k(V) = \{\textit{homogeneous functions } \rho : V \to \mathbb{R} \textit{ of degree } k, \rho(tv) = t^k p(v)\},$$

given by the following correspondence

$$Sym^k(V^*) \to \mathcal{P}_k(V),$$

$$\sigma \mapsto \rho : \rho(v) = \sigma(v \otimes ... \otimes v).$$

The inverse map is given by "polarization".

Definition 6.1.6. *A linear differential operator $D : \Gamma(E) \to \Gamma(E)$ is elliptic if the map $\sigma_D(\alpha) : E_x \to E_x$ is an isomorphism for all nonzero $\alpha \in T_x^*M$. Here, $\sigma_D(\alpha) = \sigma_D(\alpha \otimes ... \otimes \alpha)$.*

Proposition 6.1.7. *Let $D : \Gamma(E) \to \Gamma(E)$ be a k-th order LDO, and choose a connection on E. Then*

$$D\phi = (\sigma_D \circ \nabla^k)\phi + K(\phi),$$

where K is a linear differential operator of order less than or equal to $k-1$.

Thus, any k-th order LDO can be expressed as:

$$D = \sigma_D \circ \nabla^k + \sigma_D^{(k-1)} \circ \nabla^{k-1} + ... + \sigma_D^{(1)} \circ \nabla + \sigma_D^{(0)},$$

for some collection of lower order symbols $\sigma_D^{(l)}$.

6.2 Functional framework

We work on a compact, oriented, Riemannian manifold (M, g), and a vector bundle $E \to M$ equipped with a metric and compatible connection. For a section $\phi \in \Gamma(E)$, we can consider its k-th covariant derivative $\nabla^k\phi$, which has a norm $|\nabla^k\phi| \in C^\infty(M)$. As M is oriented, we also have a volume form μ_g on M.

Definition 6.2.1. *For $p \geq 1$, an L^p norm of a section $\phi \in \Gamma(E)$ is defined as follows:*

$$\|\phi\|_p = (\int_M |\phi|^p \mu_g)^{\frac{1}{p}}.$$

Definition 6.2.2. *Given integer $k \geq 0$ and p as above, the (p, k) Sobolev norm of $\phi \in \Gamma(E)$ is given by*

$$\|\phi\|_{p,k} = \|\phi\|_p + \|\nabla\phi\|_p + \|\nabla^2\phi\|_p + ... + \|\nabla^k\phi\|_p.$$

The Sobolev space $L^{p,k}(E)$ is the completion of $\Gamma(E)$ with respect to this norm.

Remark 6.2.3. *1. $L^{p,k}(E)$ is a Banach space.*

2. The norm $\|\cdot\|_{p,k}$ depends on the choices of metric and connection ong M and E. If M is compact, the different choices give equivalent norms, that is, there exist constants $A, B > 0$ such that for all $\phi \in \Gamma(E)$, we have

$$A\|\phi\|_{p,k}^1 \leq \|\phi\|_{p,k}^2 \leq B\|\phi\|_{p,k}^1.$$

3. If $p = 2$, the spaces $L^{2,k}(E)$ are Hilbert spaces, that is, the norm arises from an inner product:

$$\langle \phi, \psi \rangle_k = \int \langle \phi, \psi \rangle + \int \langle \nabla\phi, \nabla\psi \rangle + ... + \int \langle \nabla^k\phi, \nabla^k\psi \rangle.$$

6.3 Sobolev embedding theorem

We resume the same setup as in the previous subsection. Let (p, k), (q, l) be given, such that $k > l$ and such that

$$k - \frac{n}{p} \geq l - \frac{n}{q}, \tag{6.3.1}$$

where n is the dimension of M.

Definition 6.3.1. *A bounded linear map $A : V \to W$ between Banach spaces is compact if whenever $\{v_n\} \subset V$ is a bounded sequence, the image sequence $\{A(v_n)\} \subset W$ has a convergent subsequence.*

Remark 6.3.2. *1. The identity map $L^{p,k}(E) \hookrightarrow L^{q,l}(E)$ is a continuous inclusion if (3) is satisfied. If the inequality in (3) is strict, the inclusion is a compact linear map.*

2. The existence of such continuous inclusion is equivalent to the existence of $C > 0$ such that

$$\|\phi\|_{q,l} \leq C\|\phi\|_{p,k}, \forall \phi \in \Gamma(E).$$

Now, let $\mathcal{C}^k(E)$ be the space of $\mathcal{C}^k$-sections of E. It can be made into a Banach space as follows: for $\phi \in \mathcal{C}^0(E)$, let

$$\|\phi\|_{\mathcal{C}^0} = \sup_{x \in M} |\phi(x)|,$$

and let

$$\|\phi\|_{\mathcal{C}^k} = \sum_{i=0}^{k} \|\nabla^i \phi\|_{\mathcal{C}^0}.$$

Let $\mathcal{C}^{k,\alpha}(E)$ be a Hölder space where $\alpha \in [0,1]$, that is, ϕ is in $\mathcal{C}^{k,\alpha}(E)$ if ϕ has continuous derivatives up to order k and the k-th derivative satisfies the Hölder condition with exponent α given by

$$|f(x) - f(y)| \leq C\|x - y\|^{\alpha}.$$

For given k, p, the strength of $L^{p,k}$ is the quantity $k - \frac{n}{p}$; for $\mathcal{C}^{k,\alpha}$, the strength is $k + \alpha$.

Theorem 6.3.3 (Sobolev embedding theorem).

1. *If $k \geq l$ and $k - \frac{n}{p} \geq l - \frac{n}{q}$, then the identity extends to a continuous map from $L^{p,k}(E) \hookrightarrow L^{q,l}(E)$. If the inequality is strict, then this is a compact embedding.*

2. *If $k \geq l$ and $k - \frac{n}{p} \geq l + \alpha$, then $L^{p,k}(E) \hookrightarrow \mathcal{C}^{l,\alpha}(E)$, and if the inequality is strict, this is a compact embedding. Note, here we must have $\alpha > 0$.*

6.4 Elliptic Regularity

Definition 6.4.1. *Let $D : \Gamma(E) \to \Gamma(F)$ be a linear differential operator over a (compact) Riemannian manifold, and let $\phi \in L^1(E)$ and let $\psi \in L^1(F)$. We say ϕ is a weak solution of the equation*

$$D\phi = \psi$$

if for any smooth section $s \in \Gamma(F)$, we have

$$\int_M \langle D^* s, \phi \rangle = \int_M \langle s, \psi \rangle.$$

Definition 6.4.2. *Let U be an open subset of $\mathbb{R}^n$ and let u, v be locally integrable functions in $L^1_{loc}(U)$. Let α be a multi-index, we say v is the α-th weak derivative of u if*

$$\int_U u D^\alpha \varphi = (-1)^{|\alpha|} \int_U v \varphi,$$

for all infinitely differentiable functions φ with compact support in U.

Theorem 6.4.3. *Let $D : \Gamma(E) \to \Gamma(F)$ be a k-th order elliptic linear differential operator on a compact manifold, and suppose $\phi \in L^p(E)$ is a weak solution to $D\phi = \psi$, where $\psi \in L^{p,l}(F)$. Then $\phi \in L^{p,k+l}(E)$ and furthermore:*

$$\|\phi\|_{p,k+l} \leq C(\|D\phi\|_{p,l} + \|\phi\|_p),$$

for some constant C depending on D, l, p.

Likewise, if ϕ lies in $\mathcal{C}^{0,\alpha}(E)$ for some $\alpha > 0$ and $\psi \in \mathcal{C}^{l,\alpha}(F)$, then ϕ lies in $\mathcal{C}^{k+l,\alpha}(E)$ and

$$\|\phi\|_{\mathcal{C}^{k+l,\alpha}} \leq C(\|D\phi\|_{\mathcal{C}^{l,\alpha}} + \|\phi\|_{\mathcal{C}^{0,\alpha}}).$$

Remark 6.4.4. *Suppose $D\phi = 0$. Since $0 \in L^{p,l}$, for all p, l, elliptic regularity implies that $\phi \in L^{2,k+l}$, for all l. Then, by the Sobolev embedding theorem, we have that $\phi \in \mathcal{C}^{l,\alpha}$, for all l. Hence, ϕ is smooth.*

Chapter 7

Various constructions of hyperkähler spaces

In this chapter, we give a few different constructions of hyperkähler spaces.

7.1 Hitchin's construction of moduli spaces of solutions to the self-duality equations on Riemann surfaces

Hitchin constructs moduli spaces of a special class of solutions to the self-dual Yang-Mills equations through dimension reduction in [14]. The moduli spaces can be naturally equipped with hyperkähler structures. We recall his construction in this section.

Let M be a Riemann surface. Let G be $SU(2)$ or $SO(3)$. Let P be a principal G-bundle over M. Let A be a connection of P and let Φ be a Higgs field, that is, a $(1,0)$-form on M with values in the (complex) Lie algebra bundle of P. Equivalently, we can consider the associated vector bundle V of P which is a holomorphic rank-2 vector bundle over M together with a holomorphic section Φ of End$V \otimes K$, where K is the canonical bundle of M. Hence, we can write down and consider the following equations:

$$d''_A \Phi = 0 \tag{7.1.1}$$

$$F(A) + [\Phi, \Phi^*] = 0. \tag{7.1.2}$$

Theorem 7.1.1 (cf. Theorem (2.1) in [14])**.** *Let (A, Φ) satisfy the $SO(3)$ self-duality equations on a compact Riemann surface M and let V be the associated rank-2 complex vector bundle. If $L \subset V$ is a Φ-invariant subbundle, then*

1. *deg$(L) \leq \frac{1}{2}$deg$(\Lambda^2 V)$, and*

2. *if equality holds then (A, Φ) reduces to a $U(1)$ solution.*

Theorem 7.1.2 (cf. Theorem (2.7) in [14])**.** *Let (A_1, Φ_1), (A_2, Φ_2) be two solutions of the self-duality equations on a principal $SO(3)$ bundle over a Riemann surface M. Let V be the associated rank-2 complex vector bundle and assume that there is an isomorphism*

$$h : V \to V$$

such that

$$d''_{A_2} h = h d''_{A_1}, \tag{7.1.3}$$

$$\Phi_2 h = h \Phi_1. \tag{7.1.4}$$

Then (A_1, Φ_1), (A_2, Φ_2) are gauge-equivalent solutions.

Now, we introduce the stability condition that will give rise a correspondence between holomorphic vector bundles coupled with a holomorphic 1-form and solutions to the above equations 7.1.1 and 7.1.2.

Definition 7.1.3 (cf. Definition (3.1) in [14])**.** *Let V be a rank-2 holomorphic vector bundle over a compact Riemann surface M and Φ a holomorphic section of End$V \otimes K$*

where K is the canonical bundle of M. The pair (V, Φ) is defined to be stable if, for every Φ-invariant rank-1 subbundle L of V,

$$\mathit{deg} L < \frac{1}{2}\mathit{deg}(\Lambda^2 V).$$

Proposition 7.1.4 (cf. Proposition (3.3) in [14]). *Let M be a compact Riemann surface of genus $g > 1$. A rank-2 vector bundle V occurs in a stable pair (V, Φ) if and only if one of the following holds:*

1. *V is stable;*
2. *V is semi-stable and $g > 2$;*
3. *if V is semi-stable and $g = 2$ then $V \cong U \otimes L$ where U is either decomposable or an extension of the trivial bundle by itself;*
4. *V is not semi-stable and $\dim H^0(M; L_V^{-2} K \otimes V)$ is greater than 1, where L_V is the canonical subbundle;*
5. *V is decomposable as $V = L_v \oplus (L_V^* \otimes \Lambda^2 V)$ and $\dim H^0(M; L_V^{-2} K \otimes \Lambda^2 V) = 1$.*

Proposition 7.1.5 (cf. Proposition (3.4) in [14]). *Let M be a compact Riemann surface of genus $g > 1$. A rank-2 vector bundle V occurs in a stable pair (V, Φ) if and only if there is a Zariski open subset $U \subset H^0(M; \mathit{End} V \otimes K)$ such that if $\Phi \in U$, then Φ leaves invariant no proper subbundles.*

Proposition 7.1.6 (cf. Proposition (3.15) in [14]). *Let (V_1, Φ_1) and (V_2, Φ_2) be stable pairs with $\Lambda^2 V_1 \cong \Lambda^2 V_2$ and $\Psi : V_1 \to V_2$ a non-zero homomorphism such that $\Psi \Phi_1 = \Phi_2 \Psi$. Then Ψ is an isomorphism. If $(V_1, \Phi_1) = (V_2, \Phi_2)$, then Ψ is a scalar multiplication.*

The following theorem is the key theorem that establishes the aforementioned correspondence between holomorphic data and solutions to gauge-theoretic equations.

Theorem 7.1.7 (cf. Theorem (4.3) in [14]). *Let A be an $SO(3)$ connection on a bundle P over a compact Riemann surface M of genus $g > 1$, and let $\Phi \in \Omega^{1,0}(M; adP \otimes \mathbb{C})$ satisfy $d''_A\Phi = 0$. Let V be an associated rank-2 vector bundle with complex structure determined by A. If (V, Φ) is a stable pair, then there exists an automorphism of V of determinant 1, unique modulo $SO(3)$ gauge transformations, which takes (A, Φ) to a solution of the equation $F(A) + [\Phi, \Phi^*] = 0$.*

Corollary 7.1.8 (cf. Corollary (4.22) in [14], Narashimhan and Seshadri). *Every stable rank-2 bundle V over a compact Riemann surface M of genus $g > 1$ is associated to a flat $SO(3)$ connection, unique up to gauge transformations.*

By studying the elliptic complex given by equations 7.1.1, 7.1.2 and the gauge group action, one can show that the moduli space is smooth and calculate its dimension.

Theorem 7.1.9 (cf. Theorem (5.7) in [14]). *Let V be a rank-2 vector bundle of odd degree over a compact Riemann surface M of genus $g > 1$, and let $\mathcal{M}$ be the moduli space of solutions to the self-duality equations on V, with fixed induced connection on $\Lambda^2 V$. Then $\mathcal{M}$ is a smooth manifold of dimension $12(g-1)$.*

Theorem 7.1.10 (cf. Theorem (5.8) in [14]). *Let M be a compact Riemann surface of genus $g > 1$. The moduli space of all stable pairs (V, Φ), where V is a rank-2 holomorphic vector bundle of fixed determinant and odd degree, and Φ is a trace-free holomorphic section of End$V \otimes K$, is a smooth manifold of real dimension $12(g-1)$.*

Theorem 7.1.11 (cf. Theorem (6.1) in [14]). *Let M be a compact Riemann surface of genus $g > 1$ and $\mathcal{M}$ the moduli space of solutions to the self-duality equations on a rank-2 vector bundle V of odd degree. Then the natural metric on $\mathcal{M}$ is complete.*

By thinking of equation 7.1.1 as a complex moment map and coupling it with equation 7.1.2, one can interpret equations 7.1.1 and 7.1.2 together giving rising to a hyperkähler moment map equation and hence obtain the moduli space as a hyperkähler reduction. As a result, the moduli space is hyperkähler.

Theorem 7.1.12 (cf. Theorem (6.7) in [14])**.** *Let M be a compact Riemann surface of genus $g > 1$ and $\mathcal{M}$ the moduli space of irreducible solutions to the $SO(3)$ self-duality equations. Then the natural metric on the $12(g-1)$-dimensional manifold $\mathcal{M}$ is hyperkählerian.*

Finally, one can use the following Morse-Bott function coming from the S^1-symmetry to study the topology of the moduli space.

Consider the following function

$$f(A, \Phi) = 2i \int_M \mathrm{Tr}(\Phi\Phi^*) = \|\Phi\|^2_{L^2}.$$

We can think of f as a Morse function on $\mathcal{M}$, and use it to study the topology of $\mathcal{M}$.

Proposition 7.1.13 (cf. Proposition (7.1) in [14])**.** *The function $f = \|\Phi\|^2_{L^2}$ on $\mathcal{M}$ has the following properties.*

1. *f is proper.*

2. *f has critical values 0 and $(d - \frac{1}{2})\pi$ where d is a positive integer less than or equal to $g-1$.*

3. *$f^{-1}(0)$ is a non-degenerate critical manifold of index 0, and is diffeomorphic to the moduli space of stable rank-2 bundles of odd degree and fixed determinant over M.*

4. *$f^{-1}((d-\frac{1}{2})\pi)$ is a non-degenerate critical manifold of index $2(g+2d-2)$, and is diffeomorphic to a 2^{2g}-fold covering of the $(2g-2d-1)$-fold symmetric product $S^{2g-2d-1}M$ of the Riemann surface. The covering is the pullback of the covering Jac$(M) \to$ Jac(M) given by $x \mapsto 2x$ under the natural map $S^{2g-2d-1}M \to$ Jac(M) which associates to a $(2g-2d-2) - tuple$ of points of M its divisor class.*

7.2 Moduli spaces of monopoles

In this section, we give another construction of hyperkähler spaces via monopoles. The program of constructing hyperkähler spaces as moduli spaces of monopoles is originally proposed by Cherkis-Kapustin [4]. We follow the Ph.D. thesis of Lorenzo Foscolo for this construction [8].

Let (X, g) be an oriented Riemannian 3-manifold and let $P \to X$ be a principal G-bundle, where G is a compact Lie group. In practice, G will be taken to be $U(1)$, $SU(2)$, $U(2)$ or $SO(3)$. We can equivalently work with associated vector bundles to P, as in the previous subsection.

Definition 7.2.1. *Magnetic monopoles are gauge equivalence classes of solutions (A, Φ) to the* Bogomolny equation

$$*F_A = d_A\Phi, \tag{7.2.1}$$

where $*$ denotes the Hodge star operator of (X, g), F_A is the curvature form of a connection A on the principal bundle P, and Φ is a section of the adjoint bundle ad$P$. The gauge group is Aut(P), and the equivalence is with respect to the gauge group action.

Just as we can think of Hitchin's construction of moduli spaces of solutions to anti-self-dual equations on Riemann surfaces as a dimensional reduction of the Yang-Mills equation on 4-manifolds, we can think of the Bogomolny equation as well as a dimensional reduction as follows:

Consider the 4-manifold $X \times \mathbb{R}_s$, then (A, Φ) is a solution to (6) if and only if $\hat{A} = A + \Phi \otimes ds$ is an anti-self-dual (ASD) connection on $X \times \mathbb{R}_s$ invariant under translations along the s-axis, where $X \times \mathbb{R}_s$ is equipped with the product metric and the volume form $ds \wedge d\mathrm{vol}_g$, and $\hat{A}$ is ASD if $*_4 F_{\hat{A}} = -F_{\hat{A}}$.

When X is compact, smooth monopoles are trivial in the sense that A is a flat connection and Φ is a parallel section. However, on the other hand, this special case yields the interesting study of instanton Floer theory. Hence, to find non-trivial solutions to the Bogomolny equation, one needs to consider the cases where X is non-compact or monopoles are allowed to have singularities.

Now, we specialize to the case where $X = \mathbb{R}^3$ and $G = SU(2)$.

Definition 7.2.2. *Let the Yang-Mills-Higgs energy of a pair* (A, Φ) *be defined as follows:*

$$\mathcal{A}(A, \Phi) = \frac{1}{2} \int_X |F_A|^2 + |d_A \Phi|^2.$$

We will assume that the energy $\mathcal{A}(A, \Phi)$ is finite and impose the boundary condition

$$\lim_{|x| \to +\infty} |\Phi| = 1.$$

Definition 7.2.3. *Let the charge* $k \in \mathbb{Z}_{\geq 0}$ *of a solution* (A, Φ) *to (6) be defined as the following quantity:*

$$\frac{1}{4} \lim_{R \to \infty} \int_{\partial B_R} \langle \Phi, F_A \rangle.$$

Note that this quantity is also equal to minus the degree of the map

$$|\Phi|^{-1} \Phi : \partial B_R \to S^2,$$

for large enough R*.*

Fix some $k \in \mathbb{Z}_{\geq 0}$, and let $\mathcal{C}_k$ be the space of smooth pairs (A, Φ) on the trivial $SU(2)$-bundle on $\mathbb{R}^3$ with finite energy, charge k and such that $\lim_{|x| \to +\infty} |\Phi| = 1$. Let

$$g \in \mathcal{G} = Map(\mathbb{R}^3, SU(2)),$$

then g acts on a pair $(A, \Phi) \in \mathcal{C}^k$ as follows:

$$g \cdot (A, \Phi) = (A - d_A g g^{-1}, g \Phi g^{-1}).$$

Let $c = (A, \Phi)$, then we can also denote the above action as

$$g \cdot c = c + (d_1 g) g^{-1},$$

where $d_1 g = -(d_A g, [\Phi, g])$. Now let the gauge group $\mathcal{G}$ be the space of bounded gauge transformations such that $(d_1 g)g^{-1} \in L^2$, and let $\mathcal{G}_0$ be the subspace of gauge transformations which are asymptotic to the identity.

We can regard the Bogomolny equation as a map $\Psi : \mathcal{C}_k \to \Omega^1(\mathbb{R}^3; \mathbf{su}(2))$, and we can define the following moduli spaces of monopoles on $\mathbb{R}^3$ with charge k:

$$M_k = \Psi^{-1}(0)/\mathcal{G}_0,$$

$$N_k = \Psi^{-1}(0)/\mathcal{G}.$$

Now, we want to sketch the arguments for showing that M_k is a hyperkähler manifold. We will think of M_k as the quotient of a hyperkähler reduction in the following sense: we regard the map $\Psi : \mathcal{C}_k \to \Omega^1(\mathbb{R}^3; \mathbf{su}(2))$ as a hyperkähler moment map for the action of $\mathcal{G}_0$ on $\mathcal{C}_k$ and the Bogomolny equation as the vanishing of the moment map, where the kähler forms on $\Psi^{-1}(0)$ are given by the following:

$$\omega_h(\xi, \xi') = \int_{\mathbb{R}^3} \langle \gamma(dx_h)\xi, \xi' \rangle,$$

with $\gamma(dx_h), h = 1, 2, 3,$ denoting the almost complex structures on $\Psi^{-1}(0)$. Hence, M_k can be thought of as the corresponding hyperkähler quotient which is itself a hyperkähler manifold.

7.3 Gibbons-Hawking construction

The Gibbons-Hawking construction for hyperkähler manifolds is a non-gauge-theoretic construction that yields hyperkähler manifolds with a $U(1)$-symmetry. We will describe this construction by following the Ph.D. thesis of Saman Habibi Esfahani [7]. In this section, let X be a 4-manifold with a $U(1)$-action.

Definition 7.3.1 (The Gibbons-Hawking Ansatz). *Let $U \subset \mathbb{R}^3$ be an open subset with coordinates u_1, u_2, u_3, and let $p_1, ..., p_n$ be n distinct points in U. Let $\pi : X \to U \setminus \{p_1, ..., p_n\}$ be a principal $U(1)$-bundle. Let t be the coordinate along the fibers, normalized to have period 2π, with ∂t the corresponding vector field of the S^1-action,*

θ a connection 1-form on X such that $\theta(\partial t) = i$. Let β be the curvature 2-form defined by $d\theta = \pi^*(\beta)$, and $V : U \to \mathbb{R}$ a positive harmonic real-valued function such that

$$*dV = \frac{1}{2\pi i}\beta.$$

The hyperkähler metric on X can be expressed as follows:

$$g_V = V\sum_{i=1}^{3} du_i^2 + V^{-1}\theta_0^2 \in \Gamma(T^*X \otimes T^*X).$$

The kähler forms are the following:

$$\omega_1 = du_1 \wedge \theta_0 + V du_2 \wedge du_3,$$

$$\omega_2 = du_2 \wedge \theta_0 + V du_3 \wedge du_1,$$

$$\omega_3 = du_3 \wedge \theta_0 + V du_1 \wedge du_2.$$

The corresponding almost complex structures are given by:

$$I(du_2) = -du_3, I(du_1) = -\frac{1}{V}d\theta_0,$$

$$J(du_3) = -du_1, J(du_2) = -\frac{1}{V}d\theta_0,$$

$$K(du_1) = -du_2, K(du_3) = -\frac{1}{V}d\theta_0.$$

Now, we give two examples of the constructions.

Example 7.3.2 (Multi-Eguchi-Hanson spaces)**.** *Let $p_1, ..., p_n$ be distinct points in $\mathbb{R}^3$ and let $U = \mathbb{R}^3 \setminus \{p_1, ..., p_n\}$. Let $V : U \to \mathbb{R}$ be the harmonic function defined by*

$$V(x) = \sum_{i=1}^{n} \frac{1}{4\pi|x - p_i|}.$$

The induced metric g_V on X can be extended smoothly to $\pi^{-1}(x_i)$, for all i. The resulting hyperkähler manifold $\overline{X}$ is cannled a multi-Eguchi-Hanson space. A multi-

Eguchi-Hanson metric is ALE.

Example 7.3.3 (Multi-Taub-NUT Spaces)**.** *Let $m > 0$. Let $p_1, ..., p_n$ be distinct points in $\mathbb{R}^3$ and let $U = \mathbb{R}^3 \setminus \{p_1, ..., p_n\}$. Let $V : U \to \mathbb{R}$ be the harmonic function defined by*

$$V(x) = m + \sum_{i=1}^{n} \frac{1}{4\pi |x - p_i|}.$$

The induced metric g_V on X can be extended smoothly to $\pi^{-1}(x_i)$, for all i. The resulting hyperkähler manifold $\overline{X}$ is cannled a multi-Taub-NUT space. A multi-Eguchi-Hanson metric is ALF.

Chapter 8

A new gauge-theoretic construction of 4-dimensional hyperkähler ALE spaces

In this chapter, we present the new results of this thesis.

8.1 Basic setups for the gauge-theoretic construction

We start off by considering S^3 as a principal S^1-bundle over S^2 via the dual Hopf fibration. The explicit construction is as follows:

$$S^3 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\}.$$

The S^1-action on S^3 is given by the following: for $g = e^{i\theta} \in S^1$,

$$(z_1, z_2)g = (z_1 e^{-i\theta}, z_2 e^{-i\theta}).$$

If we think of the base S^2 as sitting inside $\mathbb{R}^3$, we can write down the projection map explicitly which will be useful later on: let $\pi : S^3 \to S^2$ be the projection map where $\pi(z_1, z_2) = (2z_1 z_2^*, |z_1|^2 - |z_2|^2)$. In terms of real coordinates, $\pi(a, b, c, d) =$

$(2(ac+bd), bc-ad, a^2+b^2-c^2-d^2)$. Equivalently, we can think of π as a map from S^3 to $\mathbb{C}P^1$ given by $\pi : S^3 \to \mathbb{C}P^1$ with $\pi(z_1, z_2) = [z_1 : z_2]$.

Now, we turn to the associated bundles of S^3. A complex vector space V with an S^1-action on V determines a vector bundle associated to S^3 with fiber V. Below, we consider the specific S^1-action on a complex vector space V given by the scalar multiplication.

Definition 8.1.1. *Let V be a complex vector space with S^1-action given by scalar multiplication. Then $E(V)$ is defined as $E(V) = S^3 \times V/\sim$, where $[p, v] \sim [pg, g^{-1}v]$, for all $g \in S^1$, and $E(\overline{V})$ is defined as $E(\overline{V}) = S^3 \times V/\sim$, where $[p, v] \sim [pg, gv]$, for all $g \in S^1$.*

There are three important examples that we will be working with closely, i.e., the hyperplane bundle, the tautological bundle over S^2 and the associate bundle with fiber $V = End(R)$, where R is the regular representation of a finite subgroup Γ of $SU(2)$.

Example 8.1.2 (The hyperplane bundle)**.** *Let $V = \mathbb{C}$. Then by the previous definition, $E(\mathbb{C})$ is isomorphic to the hyperplane bundle H over $\mathbb{C}P^1$.*

Example 8.1.3. *Let Γ be a finite subgroup of $SU(2)$, and let R be the regular representation of Γ with hermitian metric chosen so that the canonical basis $\{e_\gamma | \gamma \in \Gamma\}$ is unitary. We see that $E(V)$ splits orthogonally into a direct sum of hyperplane bundles, that is, $E(V) = \oplus_i H_i$, where each $H_i = E(\mathbb{C} \cdot e_i)$ is isomorphic to the hyperplane bundle H.*

8.1.1 The Γ-action and orbifold vector bundles

Let Γ be a finite subgroup of $SU(2)$ as before, and let V be a representation of Γ given by $r : \Gamma \to GL_{\mathbb{C}}(V)$. We want to build an orbifold vector bundle incorporating the Γ-representation. To this end, we introduce the following definition.

Definition 8.1.4.

1. *Suppose either Γ doesn't contain $-1 \in SU(2)$ or Γ contains -1 and $r(-1) = -1 \in GL_{\mathbb{C}}(V)$. Let $E(V)^\Gamma_r$ be defined as follows: $E(V)^\Gamma_r =$*

$S^3 \times V/\sim$, where $[p, v] \sim [pg, g^{-1}v] \sim [p\gamma, \gamma^{-1}v]$, for all $g \in S^1$ and $\gamma \in \Gamma$, where $\gamma^{-1}v = r(\gamma^{-1})v$.

2. *Suppose Γ contains $-1 \in SU(2)$ and $r(-1) \neq -1 \in GL_{\mathbb{C}}(V)$, we decompose V into the eigenspaces of $r(-1)$, and write $V = V_0 \oplus V_1$, where $r(-1)$ acts as 1 on V_0 and acts as -1 on V_1. Define $E(V)^\Gamma_r$ to be $E(V)^\Gamma_r = S^3 \times V_1/\sim$, where $[p, v] \sim [pg, g^{-1}v] \sim [p\gamma, \gamma^{-1}v]$, for all $g \in S^1$ and $\gamma \in \Gamma$, where $\gamma^{-1}v = r(\gamma^{-1})v$.*

   We will oftentimes abbreviate $E(V)^\Gamma_r$ as $E(\Gamma)$.

Remark 8.1.5. 1. *We can think of $E(\Gamma)$ as an orbifold vector bundle over S^2/Γ.*

2. *Let $C^\infty(S^2, E(V))$ denote the space of sections of $E(V)$ and let $C^\infty(S^2, E(V))^\Gamma$ denote the space of Γ-invariant sections of $E(V)$. With the above definition, we always have that $C^\infty(S^2, E(V))^\Gamma \cong C^\infty_{orb}(S^2/\Gamma, E(V)^\Gamma_r) = C^\infty_{orb}(S^2/\Gamma, E(\Gamma))$. Note that we will begin to drop the subscript and simply use $C^\infty(S^2/\Gamma, E(\Gamma))$ or $C^\infty(E(\Gamma))$ to denote the space of orbifold sections of $E(\Gamma)$ or equivalently the Γ-invariant sections of $E(V)$ in the coming sections.*

3. *If we let V be equal to the endomorphism space $End(R)$ of the regular representation R of Γ. Then we have $\gamma^{-1}v = R(\gamma^{-1})vR(\gamma)$. Recall that in Kronheimer's construction, when forming $M = P^\Gamma = (Q \otimes V)^\Gamma$, the element $-1 \in \Gamma$ acts on Q by scalar multiplication and on V by the Γ-representation $r(-1)$. Hence, if an element $\sum q \otimes v$ is Γ-invariant, we must have $\sum q \otimes v = \sum(-q) \otimes r(-1)v$, so $r(-1)$ must act as $-1 \in GL_{\mathbb{C}}(V)$ on V, so $\sum q \otimes v$ lies in $Q \otimes V_1$. In other words, $P^\Gamma = (Q \otimes V_1)^\Gamma$.*

Now, we want to equip the bundles with a pointwise metric. Let $E(V)$ and $E(\Gamma)$ be defined as above.

Definition 8.1.6. 1. *Let h_V be a hermitian metric on V. Then the pointwise hermitian metric $h_{E(V)}$ on $E(V)$ with respect to h_V is given by $h_{E(V)}([p, v_1], [p, v_2])_x = h_V(v_1, v_2)$, where $p \in S^3$ lies in the fiber over $x \in S^2$.*

2. *Suppose h_V is also Γ-invariant, then h_V gives rise to a pointwise hermitian metric on $E(\Gamma)$ again given by $h_{E(V)}([p, v_1], [p, v_2])_x = h_V(v_1, v_2)$, where $p \in S^3$ lies in the fiber over $x \in S^2$.*

Remark 8.1.7. *With the above definition, we can identify $E(\overline{V})$ with the dual bundle $E(V)^*$, where $[p, v]^*([p, v']) = h_V([p, v'], [p, v]) = [p, v^*]([p, v'])$, for $[p, v^*] \in E(\overline{V})$, and the metric on $E(\overline{V})$ is given by taking $h_{E(\overline{V})}([p, v_1^*], [p, v_2^*])_x = h_{\overline{V}}(v_1^*, v_2^*) = h_V(v_2, v_1)$. On the other hand, we can also express $h_{E(V)}$ in terms of the trace, that is, let*

$$h_{E(V)}([p, v_1], [p, v_2])_x = \mathrm{Tr}([p, v_1], [p, v_2]^*)_x = \mathrm{Tr}(v_1 v_2^*).$$

As a result, we also get $E(\Gamma)^*$.

8.2 The gauge-theoretic framework

We are ready to introduce the gauge-theoretic framework in this paper. We will mainly be working with the orbifold vector bundle $E(\Gamma)$ that we have defined previously for the main construction. We fix a holomorphic structure on $E(\mathbb{C}) = H$, and denote it by $\bar{\partial}$. For the remaining of the section, we assume $V = End(S)$ to be the endomorphism space of some Γ-representation S. We fix a Γ-invariant hermitian structure h_V on V and hence get pointwise metrics on $E(V)$ and $E(\Gamma)$. We take ω_{vol} to be the Fubini-Study form on $\mathbb{C}P^1$.

Let A_0 be the unique Chern connection on H compatible with the holomorphic structure $\bar{\partial}$ and the hermitian structure descending from $E(V)$. Note that A_0 will be Γ-invariant as it is invariant under $SU(2)$.

Let P be the bundle of automorphisms of $E(V)$. Then P is in fact the trivial bundle $S^2 \times GL_{\mathbb{C}}(V)$. Now, let $F \subset U(S)$ be the subgroup of unitary transformations of S that commute with the Γ-action, and let T be the scalar subgroup sitting inside F. Then we can think of $\tilde{P}$ defined such that $\tilde{P} = S^2 \times F/T$ as a subbundle of P, as we can think of F/T as lying inside $GL_{\mathbb{C}}(V)$ by acting on V by conjugation. As F is the subgroup of $U(S)$ with elements that commute with the Γ-action, we also get that $\tilde{P}^\Gamma$ defined as $\tilde{P}^\Gamma = S^2/\Gamma \times F/T$ is a subbundle of the bundle automorphisms of $E(\Gamma)$. This motivates the following definition.

Definition 8.2.1. *Let $V = End(S)$ be the endomorphism space of some Γ-representation S. Let $F \subset U(S)$ be the unitary transformations of S that commute with the Γ-action, and let T be the scalar subgroup sitting inside F.*

1. *Let the gauge group $\mathcal{G}^{F,\Gamma}$ of $E(\Gamma)$ be defined as $\mathcal{G}^{F,\Gamma} = Map(S^2/\Gamma, F/T)$. Let $\mathbf{g}^{F,\Gamma}$ denote the Lie algebra of $\mathcal{G}^{F,\Gamma}$. We use ρ to denote an element in $\mathcal{G}^{F,\Gamma}$, and we use Y to denote an element in $\mathbf{g}^{F,\Gamma}$.*

2. *Let $\mathcal{G}^{F,\Gamma}_{\mathbb{C}}$ denote the complexification of $\mathcal{G}^{F,\Gamma}$, that is, $\mathcal{G}^{F,\Gamma}_{\mathbb{C}} = Map(S^2/\Gamma, F^c/\mathbb{C}^*)$, where $F^c = GL_{\mathbb{C}}(V)^\Gamma$ denotes the complex linear transformations of S that commute with the Γ-action. We use κ to denote an element in $\mathcal{G}^{F,\Gamma}_{\mathbb{C}}$.*

Definition 8.2.2. *We define the configuration space to be $\mathcal{A}^F \times C^\infty(S^2/\Gamma, E(\Gamma))$ where $\mathcal{A}^F$ and $C^\infty(S^2/\Gamma, E(\Gamma))$ are defined as follows.*

1. *Let $\mathcal{A}^F$ be the space of connections on $E(\Gamma)$ given by*

$$\mathcal{A}^F = \{A_0 + \kappa^* \partial \kappa^{*-1} + \kappa^{-1} \bar{\partial} \kappa \mid \kappa \in \mathcal{G}^{F,\Gamma}_{\mathbb{C}}\},$$

   where A_0 is the aforementioned Chern connection on H or equivalently S^3 thought of as the induced connection on $E(\Gamma)$. We will always denote $\kappa^* \partial \kappa^{*-1} + \kappa^{-1} \bar{\partial} \kappa$ by B, and sometimes we omit the base connection A_0.

2. *Let $C^\infty(S^2/\Gamma, E(\Gamma))$ denote the abbreviation for the space of orbifold vector bundle sections $C^\infty_{orb}(S^2/\Gamma, E(\Gamma))$.*

Remark 8.2.3. 1. *Notice that $\mathcal{G}^{F,\Gamma}$ is the subgroup of the group of unitary gauge automorphisms of $E(\Gamma) = E(V)^\Gamma = E(End(S))^\Gamma$ induced by the automorphisms of $E(S)^\Gamma$. And the action of $\rho \in \mathcal{G}^{F,\Gamma}$ on $\mathcal{A}^F \times C^\infty(S^2/\Gamma, E(\Gamma))$ is given by the following: for a pair $(B, \Theta) \in \mathcal{A}^F \times C^\infty(S^2/\Gamma, E(\Gamma))$,*

$$\rho \cdot (B, \Theta) = (B + \rho d_B \rho^{-1}, \rho \Theta \rho^{-1}).$$

Note that here we omit the base connection A_0 as ρ fixes A_0.

2. *The action of the connection form B on a section Θ comes from the representation of the Lie algebra of F^c on V induced from the representation S.*

3. *The key point of the definition of $\mathcal{A}^F$ is that it can be thought of as the complex gauge orbit containing A_0, which will become important in the later sections.*

Definition 8.2.4 (Symplectic structure on $\mathcal{A}^F \times C^\infty(S^2/\Gamma, E(\Gamma))$)**.** *Let $(B_1, \Theta_1), (B_2, \Theta) \in \mathcal{A}^F \times C^\infty(S^2/\Gamma, E(\Gamma))$, let a symplectic 2-form $\mathbf{\Omega}$ on $\mathcal{A}^F \times C^\infty(S^2/\Gamma, E(\Gamma))$ be defined as follows:*

$$\mathbf{\Omega}((B_1, \Theta_1), (B_2, \Theta)) = \int_{S^2/\Gamma} B_1 \wedge B_2 + \int_{S^2/\Gamma} -Im\langle \Theta_1, \Theta_2 \rangle \omega_{vol}.$$

Definition 8.2.5. 1. *Let $\mathcal{G}_0^{F,\Gamma}$ denote the based subgroup of $\mathcal{G}^{F,\Gamma}$, that is*

$$\mathcal{G}_0^{F,\Gamma} = \{\rho \in \mathcal{G}^{F,\Gamma} | \rho(x) = 1, \textit{for some fixed base point } x \in S^2/\Gamma\}.$$

We also get the complexified version $\mathcal{G}_{0,\mathbb{C}}^{F,\Gamma}$ for the above definition.

2. *Let $\mathcal{G}_\tau^{F,\Gamma}$ denote the antipodal-invariant subgroup of $\mathcal{G}^{F,\Gamma}$ and let $\Omega^2_\tau(S^2/\Gamma; \mathbf{f}/\mathbf{t})$ denote the antipodal-invariant subgroup of $\Omega^2(S^2/\Gamma; \mathbf{f}/\mathbf{t})$, where $\tau : S^2 \to S^2$ is the antipodal map given by $x = (a, b, c) \mapsto \tau(x) = (-a, -b, -c)$. We can also think of τ as a map from $\mathbb{C}P^1$ to $\mathbb{C}P^1$ with $\tau : \mathbb{C}P^1 \to \mathbb{C}P^1, [z_1 : z_2] \mapsto [-\bar{z}_2 : \bar{z}_1]$. We remark here that τ commutes with the Γ-action and hence descends to a map $\tau : S^2/\Gamma \to S^2/\Gamma$.*

Below, we define the L^2 inner product on various spaces.

Definition 8.2.6. 1. *Let Θ_1, Θ_2 be two sections of $E(\Gamma)$. We define the L^2 inner product of Θ_1 and Θ_2 to be*

$$\langle \Theta_1, \Theta_2 \rangle_{L_2} = \int_{S^2/\Gamma} \langle \Theta_1, \Theta_2 \rangle \omega_{vol} = \int_{S^2/\Gamma} \mathrm{Tr}(\Theta_1 \Theta_2^*) \omega_{vol},$$

where $\langle \Theta_1, \Theta_2 \rangle_x = h_{E(V)}(\Theta_1(x), \Theta_2(x))_x = \mathrm{Tr}(\Theta_1(x)\Theta_2^*(x))_x$.

2. *We identify* $\Omega^0(S^2/\Gamma; \mathbf{f}/\mathbf{t})$ *and* $\Omega^2(S^2/\Gamma; \mathbf{f}/\mathbf{t})$ *as dual spaces through the following integration: let* $\phi_1 \in \Omega^0(S^2/\Gamma; \mathbf{f}/\mathbf{t})$ *and* $\phi_2 \in \Omega^2(S^2/\Gamma; \mathbf{f}/\mathbf{t})$*, then* $\phi_2(\phi_1) = \int_{S^2/\Gamma} \langle \phi_1, \phi_2 \rangle$*, where we think of* ϕ_2 *as an element in* $\Omega^0(S^2/\Gamma; \mathbf{f}/\mathbf{t})$ *multiplied by the volume form* ω_{vol}*, and the inner product is pointwisely given by the inner product on* $\mathbf{f}/\mathbf{t}$*.*

8.3 An Overview of the Gauge-Theoretic Construction

In this section, we describe the main gauge-theoretic construction of the ALE spaces while leaving some details of the construction and most proofs to the future sections. We make an important remark that from this point on and throughout the rest of the paper, we take the Γ-representation S to be the regular representation R of Γ unless otherwise specified, and carry on with the same notations introduced in the previous sections. In particular, we have $V = End(R)$.

8.3.1 Symplectic reduction

Recall that in the previous section, we define the gauge group to be $\mathcal{G}^{F,\Gamma} = Map(S^2/\Gamma, F/T)$ acting on the configuration space $\mathcal{A}^F \times C^\infty(S^2/\Gamma, E(\Gamma))$ under the following action: for $\rho \in \mathcal{G}^{F,\Gamma}$, and $(B, \Theta) \in \mathcal{A}^F \times C^\infty(S^2/\Gamma, E(\Gamma))$,

$$\rho \cdot (B, \Theta) = (B + \rho d_B \rho^{-1}, \rho \Theta \rho^{-1}).$$

Proposition 8.3.1. *The above gauge group action on* $\mathcal{A}^F \times C^\infty(S^2/\Gamma, E(\Gamma))$ *is Hamiltonian and gives rise to the following moment map:*

$$\tilde{\mu}_1 : \mathcal{A}^F \times C^\infty(S^2/\Gamma, E(\Gamma)) \to \Omega^2(S^2/\Gamma; \mathbf{f}/\mathbf{t}),$$

$$(B, \Theta) \mapsto F_B - \frac{i}{2}[\Theta, \Theta^*]\omega_{vol}.$$

Remark 8.3.2. 1. *Notice that* B *alone isn't a connection whereas* $A_0 + B$ *is a connection on* $E(\Gamma)$*. Hence, we can write* $F_{A_0+B} = F_{A_0} + F_B$*, and* $\bar{\partial}_{A_0+B} = \bar{\partial}_{A_0} + \bar{\partial}_B$*.*

2. *With the preceding proposition in place, we can write down the following equations: for $\tilde{\zeta}_1 \in Z$, where Z is the center of $(\mathbf{f}/\mathbf{t})^*$ thought of as traceless matrices in $\mathbf{f}/\mathbf{t}$, we consider*

$$\bar{\partial}_{A_0+B}\Theta = 0 \tag{8.3.1}$$

$$F_B - \frac{i}{2}[\Theta, \Theta^*]\omega_{vol} = \tilde{\zeta}_1 \cdot \omega_{vol} \tag{8.3.2}$$

The above equations motivate the following definition.

Definition 8.3.3. *For an element $\tilde{\zeta}_1 \in Z$, let $\mathcal{M}(\Gamma, \tilde{\zeta}_1)$ be the moduli space of solutions to 8.3.1 and 8.3.2 that lie in the configuration space $\mathcal{A}^F \times C^\infty(S^2/\Gamma, E(\Gamma))$ modulo the gauge group action, that is,*

$$\mathcal{M}(\Gamma, \tilde{\zeta}_1) = \{(A_0 + B, \Theta) \in \mathcal{A}^F \times C^\infty(S^2/\Gamma, E(\Gamma)) \mid (8.3.1) - (8.3.2)\}/\mathcal{G}^{F,\Gamma}.$$

Proposition 8.3.4. *For choices of $\tilde{\zeta}_1$ such that $\mathcal{G}^{F,\Gamma}$ acts freely on the space of solutions to 8.3.1 and 8.3.2 in $\mathcal{A}^F \times C^\infty(S^2/\Gamma, E(\Gamma))$, $\mathcal{M}(\Gamma, \tilde{\zeta}_1)$ can be identified with $\mu_1(\tilde{\zeta}_1)^{-1}/F$ in [21].*

Remark 8.3.5. *We will discuss the conditions assumed in the above proposition in the future sections in detail and we will prove the proposition in Section 7.*

8.3.2 Further reduction

Everything regarding to the hyperkähler structure on $C^\infty(S^2/\Gamma, E(\Gamma))$ appearing in this subsection will be discussed in detail in Section 4. Here we give a brief overview. It turns out that $C^\infty(S^2/\Gamma, E(\Gamma))$ can be given a hyperkähler structure.

Before we write down the kähler forms, we first introduce some notations. For a section Θ on $C^\infty(S^2/\Gamma, E(\Gamma))$, we identify Θ with an S^1- and Γ-equivariant map $\lambda : S^3 \to End(R)$, and hence we can express Θ as $\Theta : x \mapsto [p, \lambda(p)]$, for $x \in S^2$ and $p \in \pi^{-1}(x) \subset S^3$.

There is a complex structure J, in addition to the standard complex structure I, on the space of sections $C^\infty(S^2/\Gamma, E(\Gamma))$, which we can express as follows. Let $\Theta : x \mapsto [p, \lambda(p)]$ be given, the action of J on Θ is given by $J\Theta : x \mapsto [p, -\lambda(J(p))^*]$, where $p \in S^3$ and J on S^3 is just the usual quaternion action.

Proposition 8.3.6. *There are three symplectic forms on $C^\infty(S^2/\Gamma, E(\Gamma))$ compatible with complex structures I, J, K, respectively:*

$$\omega_1(\Theta_1, \Theta_2) = \int_{S^2/\Gamma} -Im\langle\Theta_1, \Theta_2\rangle\omega_{vol},$$

$$\omega_2(\Theta_1, \Theta_2) = \int_{S^2/\Gamma} Re\langle J\Theta_1, \Theta_2\rangle\omega_{vol},$$

$$\omega_3(\Theta_1, \Theta_2) = \int_{S^2/\Gamma} -Im\langle J\Theta_1, \Theta_2\rangle\omega_{vol},$$

and a hyperkähler metric g_h such that

$$g_h(\Theta_1, \Theta_2) = \int_{S^2/\Gamma} Re\langle\Theta_1, \Theta_2\rangle\omega_{vol},$$

together giving rise to a hyperkähler structure on $C^\infty(S^2/\Gamma, E(\Gamma))$.

We will prove the above proposition in Section 4. It turns out that the action of the τ-invariant gauge group $\mathcal{G}_\tau^{F,\Gamma}$ on the space of sections of $E(\Gamma)$ with respect to each one of the three symplectic forms is again Hamiltonian. Hence, we can write down the following additional moment maps and operate a further reduction on the configuration space:

- $\tilde{\mu}_2 : C^\infty(S^2/\Gamma, E(\Gamma)) \to \Omega^2(S^2/\Gamma; \mathbf{f}/\mathbf{t}), \Theta \mapsto -\frac{1}{4}([J\Theta, \Theta^*] - [\Theta, J\Theta^*])\omega_{vol}$,
- $\tilde{\mu}_3 : C^\infty(S^2/\Gamma, E(\Gamma)) \to \Omega^2(S^2/\Gamma; \mathbf{f}/\mathbf{t}), \Theta \mapsto -\frac{i}{4}([J\Theta, \Theta^*] + [\Theta, J\Theta^*])\omega_{vol}$.

We get two additional moment map equations: let $\tilde{\zeta}_2, \tilde{\zeta}_3 \in Z$, consider

$$-\frac{1}{4}([J\Theta, \Theta^*] - [\Theta, J\Theta^*])\omega_{vol} = \tilde{\zeta}_2 \cdot \omega_{vol}, \tag{8.3.3}$$

$$-\frac{i}{4}([J\Theta, \Theta^*] + [\Theta, J\Theta^*])\omega_{vol} = \tilde{\zeta}_3 \cdot \omega_{vol}. \tag{8.3.4}$$

Definition 8.3.7. *Let $\mathcal{A}^F_\tau \subset \mathcal{A}^F$ be the subspace of connections in $\mathcal{A}^F$ on $E(\Gamma)$ given by*

$$\mathcal{A}^F_\tau = \{A_0 + \kappa^* \partial \kappa^{*-1} + \kappa^{-1} \bar{\partial} \kappa \mid \kappa \in \mathcal{G}^{F,\Gamma}_{\tau,\mathbb{C}}\},$$

where A_0 is again the base Chern connection on $E(\Gamma)$, and $\mathcal{G}^{F,\Gamma}_{\tau,\mathbb{C}}$ is the complexification of the τ-invariant subgroup $\mathcal{G}^{F,\Gamma}_\tau$.

Theorem 8.3.8. *Let $\tilde{\zeta} = (\tilde{\zeta}_1, \tilde{\zeta}_2, \tilde{\zeta}_3)$, where for all i, $\tilde{\zeta}_i \in Z$. Let*

$$\mathcal{X}_{\tilde{\zeta}} = \{(B, \Theta) \in \mathcal{A}^F_\tau \times C^\infty(S^2/\Gamma, E(\Gamma)) | (3.1) - (3.4)\}/\mathcal{G}^{F,\Gamma}_\tau.$$

Then for a suitable choice of $\tilde{\zeta}$, $\mathcal{X}_{\tilde{\zeta}}$ is diffeomorphic to the resolution of singularity $\widetilde{\mathbb{C}^2/\Gamma}$. Furthermore, for $\zeta = \tilde{\zeta}^* = -\tilde{\zeta}$, there exists a map Φ taking X_ζ in [21] to $\mathcal{X}_{\tilde{\zeta}}$ and a natural choice of metric on $\mathcal{X}_{\tilde{\zeta}}$ such that Φ is an isometry.

Remark 8.3.9. *We will make the statement of "a suitable choice of $\tilde{\zeta}$" precise in Section 7 where we also prove the theorem.*

8.3.3 Proof of Proposition 8.3.1

Here in this subsection, we give the proof of Proposition 8.3.1, which involves simply standard calculations.

Proof of Proposition 8.3.1. We will show that $F_B - \frac{i}{2}[\Theta, \Theta^*]\omega_{vol}$ is a moment map on $\Omega^1(S^2/\Gamma; \mathbf{f}/\mathbf{t}) \times C^\infty(S^2/\Gamma, E(\Gamma))$ induced by the action of $\mathcal{G}^{F,\Gamma}$. We need to check the two properties of a moment map.

Let $Y : S^2/\Gamma \to \mathbf{f}/\mathbf{t}$ be in $\mathbf{g}^{F,\Gamma}$, and let $Y^\sharp$ be the vector field on $\Omega^1(S^2/\Gamma; \mathbf{f}/\mathbf{t}) \times C^\infty(S^2/\Gamma, E(\Gamma))$ generated by Y. Then $Y^\sharp(B, \Theta)$ is given by

$$\frac{d}{dt}|_{t=0}(B + \exp(tY) d_B \exp(-tY), \exp(tY)\Theta\exp(-tY)) = (-d_B Y, [Y, \Theta]).$$

Hence, we have

$$\iota_{Y^\sharp}\omega_{(B,\Theta)}(B',\Theta') =$$

$$\int_{S^2/\Gamma} \mathrm{Tr}(-d_B Y \wedge B') - \int_{S^2/\Gamma} Im\langle [Y,\Theta],\Theta'\rangle \omega_{vol} =$$

$$\int_{S^2/\Gamma} \mathrm{Tr}([Y,B]\wedge B' - dY \wedge B') - \int_{S^2/\Gamma} Im\langle [Y,\Theta],\Theta'\rangle \omega_{vol}.$$

Meanwhile, let $(B_t,\Theta_t)_{t\in[0,1]}$ be a path in $\Omega^1(S^2/\Gamma;\mathbf{f}/\mathbf{t}) \times C^\infty(S^2/\Gamma, E(\Gamma))$ such that $(B_0,\Theta_0) = (B,\Theta)$ and $\frac{d}{dt}|_{t=0}(B_t,\Theta_t) = (B',\Theta')$. Then we also have

$$d\tilde{\mu}^Y_{1(B,\Theta)}(B',\Theta') =$$

$$\frac{d}{dt}|_{t=0}\int_{S^2/\Gamma} \mathrm{Tr}(Y \wedge F_{B_t}) - \frac{d}{dt}|_{t=0}\int_{S^2/\Gamma} \langle Y, \frac{i}{2}[\Theta_t,\Theta_t^*]\rangle \omega_{vol} =$$

$$\frac{d}{dt}|_{t=0}\int_{S^2/\Gamma} \mathrm{Tr}(Y \wedge F_{B_t}) - \frac{d}{dt}|_{t=0}\int_{S^2/\Gamma} -\frac{i}{2}\langle Y, [\Theta_t,\Theta_t^*]\rangle \omega_{vol} =$$

$$\int_{S^2/\Gamma} \mathrm{Tr}(Y \wedge (dB' + B'\wedge B + B\wedge B')) - \int_{S^2/\Gamma} -\frac{i}{2}\langle Y, [\Theta',\Theta^*] + [\Theta,\Theta'^*]\rangle \omega_{vol}.$$

Hence, we have $\iota_{Y^\sharp}\omega_{(B,\Theta)}(B',\Theta') = d\tilde{\mu}^Y_{1(B,\Theta)}(B',\Theta')$.

We also need to check the equivariance condition, that is, $\tilde{\mu}_1 \circ \psi_\rho = Ad^*_\rho \circ \tilde{\mu}_1$. Let ρ be an element in the unitary gauge group $\mathcal{G}^{F,\Gamma}$, and let

$$\psi_\rho : \Omega^1(S^2/\Gamma;\mathbf{f}/\mathbf{t}) \times C^\infty(S^2/\Gamma, E(\Gamma)) \to \Omega^1(S^2/\Gamma;\mathbf{f}/\mathbf{t}) \times C^\infty(S^2/\Gamma, E(\Gamma))$$

be the diffeomorphism on the configuration space induced by ρ. We have

$$\tilde{\mu}_1 \circ \psi_\rho(B,\Theta) = F(B + \rho d_B \rho^{-1}) - \frac{i}{2}[\rho\Theta\rho^{-1}, (\rho\Theta\rho^{-1})^*]\omega_{vol}.$$

Meanwhile,

$$Ad^*_\rho \circ \tilde{\mu}_1(B,\Theta) = \rho F_B \rho^{-1} - \frac{i}{2}\rho[\Theta,\Theta^*]\rho^{-1}\omega_{vol}.$$

Since the gauge action on curvature is conjugation and $\rho^{-1} = \rho^*$, we have the desired equality

$$\tilde{\mu}_1 \circ \psi_\rho(B, \Theta) = Ad^*_\rho \circ \tilde{\mu}_1(B, \Theta).$$

□

8.4 Hyperkähler structure on $C^\infty(S^2/\Gamma, E(\Gamma))$

8.4.1 The quaternions

Let $SU(2)$ denote the 2-dimensional special unitary group. Explicitly, $SU(2) = \{\gamma \in Gl_2(\mathbb{C}) | \gamma = \begin{pmatrix} u & v \\ -v^* & u^* \end{pmatrix}, |u|^2 + |v|^2 = 1\}$. Note that u^* refers to complex conjugation.

Below, we will set up another piece of conventions, that is, to endow $\mathbb{C}^2$ with a right $\mathbb{H}$-module structure. Write (z_1, z_2) for a point in $\mathbb{C}^2$, where I, J, K act on $\mathbb{C}^2$ as follows: $I(z_1, z_2) = (iz_1, iz_2)$, $J(z_1, z_2) = (-z_2^*, z_1^*)$, $K(z_1, z_2) = (-iz_2^*, iz_1^*)$.

Note we also have $SU(2)$ acting on the right on $\mathbb{C}^2$: Let $\gamma \in SU(2)$, $\gamma = \begin{pmatrix} u & v \\ -v^* & u^* \end{pmatrix}$, then

$$J((z_1, z_2)\gamma) = J(uz_1 - v^*z_2, vz_1 + u^*z_2) = (-v^*z_1^* - uz_2^*, u^*z_1^* - vz_2^*),$$

and

$$(J(z_1, z_2))\gamma = (-z_2^*, z_1^*)\gamma = (-v^*z_1^* - uz_2^*, u^*z_1^* - vz_2^*),$$

so the $SU(2)$-action commutes with the J-action. Hence, the $SU(2)$-action on $\mathbb{C}^2$ commutes with all the I-, J-, K-actions.

If we restrict the actions to S^3 thought of as the unit quaternions, then we make the following observations:

Lemma 8.4.1. *The S^1-action on S^3 coming from the dual Hopf fibration commutes with I, and for $p \in S^3$, $g \in S^1$, $J(pg) = J(p)g^*$, $K(pg) = K(p)g^*$.*

Lemma 8.4.2. *Consider S^3 as the principal S^1-bundle via the dual Hopf fibration.*

Let $\pi : S^3 \to \mathbb{C}P^1$ be the projection map where $\pi(z_1, z_2) = [z_1 : z_2]$. Then I acts as the identity and J, K act as the natural involution on the base $\mathbb{C}P^1$ given by $\tau : \mathbb{C}P^1 \to \mathbb{C}P^1$, $[z_1 : z_2] \mapsto [-z_2^* : z_1^*]$.

8.4.2 Quaternionic structures on associated bundles and spaces of sections

We have previously introduced the bundle $E(V)$ and $E(\Gamma)$. In this subsection, we introduce quaternionic structures on these bundles and their spaces of sections.

We begin with $E(V)$. Notice that we can define the quaternion actions on $E(V)$ in the following way:

$$I[p,v] = [-I(p), v] = [p, iv],$$

$$J[p,v] = [J(p), v^*],$$

$$K[p,v] = [-K(p), v^*],$$

with $[p,v] \in E(V)$.

It's straightforward to check that the I-, J-, K-actions defined above satisfy the properties for quaternionic actions. Hence, we have equipped $E(V)$ with a quaternionic structure.

Now, we move on to $E(\Gamma)$. In the previous subsection, we have shown that the Γ-action and the J-action commute on $\mathbb{C}^2$. Observe that we have that the Γ-action commutes with the quaternion actions on the level of $E(V)$ as well; more precisely, we have that

$$J(\gamma[p,v]) = J[p\gamma, \gamma^{-1}v] = J[p\gamma, R(\gamma^{-1})vR(\gamma)]$$

$$= [J(p\gamma), (R(\gamma^{-1})vR(\gamma))^*] = [J(p)\gamma, R(\gamma)^* v^* R(\gamma^*)^*]$$

$$= [J(p)\gamma, R(\gamma^*)v^*R(\gamma)] = [J(p)\gamma, R(\gamma^{-1})v^*R(\gamma)] = \gamma(J[p,v]),$$

given that $\gamma \in SU(2)$ and $R : \Gamma \to U(R) \subset End(R)$ is the regular representation.

Hence, the quaternion actions descend to $E(\Gamma)$. We remark that the J- and K-

actions on $E(V)$ and $E(\Gamma)$ act on the base by τ which we have introduced previously.

Proposition 8.4.3. *The map $I : E(V) \to E(V)$ is an isometry with respect to the hermitian metric on $E(V)$, and $J, K : E(V) \to E(V)$ are skew-isometries in the sense that $\langle J[p, v_1], J[p, v_2]\rangle = \overline{\langle v_1, v_2\rangle}$, $\langle K[p, v_1], K[p, v_2]\rangle = \overline{\langle v_1, v_2\rangle}$, for $[p, v_1], [p, v_2] \in E(V)_x$, for all $x \in S^2$.*

From here, by pullbacks, we can make the spaces of sections $C^\infty(E(V))$ and $C^\infty(E(\Gamma))$ into right $\mathbb{H}$-modules. We will focus on $C^\infty(E(\Gamma))$ here but the statements for $C^\infty(E(V))$ are exactly the same.

Proposition 8.4.4. *The space of sections $C^\infty(E(\Gamma))$ of $E(\Gamma)$ is an infinite-dimensional right $\mathbb{H}$-module with the following quaternionic actions: for Θ a section of $E(\Gamma)$, we identify Θ with a map $\lambda : S^3 \to End(R)$ equivariant with respect to the S^1- and Γ-action, and we define that for $\Theta : x \mapsto [p, \lambda(p)]$,*

$$I\Theta : x \mapsto [p, i\lambda(p)],$$

$$J\Theta : x \mapsto [p, -\lambda(J(p))^*],$$

$$K\Theta : x \mapsto [p, \lambda(K(p))^*],$$

where $J(p)$ and $K(p)$ are the usual J-, K- actions on S^3.

We leave out the proofs for the above propositions as they involve simply using and checking the properties of quaternionic actions. Also, Proposition 8.4.4 holds for the space of sections $C^\infty(E(V))$ of $E(V)$ with appropriate modifications of adjectives.

8.4.3 Hyperkähler structure on the space of sections $C^\infty(E(\Gamma))$

With the previous observations involving the quaternion actions, we are now ready to introduce the hyperkähler structure on the space of sections $C^\infty(E(\Gamma))$ that will be relevant to the construction. We remark that the same analysis below will give rise to hyperkähler structures to $C^\infty(E(V))$ as well; in fact, we can even replace the regular representation R with any Γ-representation S and obtain a hyperkähler structure on $C^\infty(E(End(S)))$, as we use no specific properties of the regular representation R for defining the hyperkähler structure.

Recall that in the previous subsection, we have that for $\Theta : x \mapsto [p, \lambda(p)]$, the action of J on Θ is such that

$$J\Theta : x \mapsto [p, -\lambda(J(p))^*],$$

where $J(p)$ is the usual J-action on S^3.

We now give the proof of Proposition 8.3.6.

Proof of Proposition 8.3.6. We focus on ω_3. First we make the observation that

$$\omega_3(\Theta_1, \Theta_2) = \int_{S^2/\Gamma} -Im\langle J\Theta_1, \Theta_2\rangle \omega_{vol} = \int_{S^2/\Gamma} -Im\langle -J\Theta_2, \Theta_1\rangle \omega_{vol}$$

$$= \int_{S^2/\Gamma} -Im\langle J\Theta_2, \Theta_1\rangle \tau^* \omega_{vol} = -\omega_3(\Theta_2, \Theta_1).$$

Indeed, for $\Theta_1 : x \mapsto [p, \lambda_1(p)]$ and $\Theta_2 : x \mapsto [p, \lambda_2(p)]$, we have

$$\langle J\Theta_1, \Theta_2\rangle_x = \mathrm{Tr}(-\lambda_1(J(p))^* \lambda_2(p)^*)$$

and

$$\langle -J\Theta_2, \Theta_1\rangle_x = \mathrm{Tr}(\lambda_2(J(p))^* \lambda_1(p)^*) = \mathrm{Tr}(\lambda_1(p)^* \lambda_2(J(p))^*).$$

Since J acts on S^2/Γ by τ which has the property that $\tau^*\omega_{vol} = -\omega_{vol}$, we have the desired equality after integration. This gives ω_3 the skew-symmetric property of a symplectic form. The same can be shown for ω_2. The properties of ω_2 and ω_3 being closed and non-degenerate are obvious. We hence can also write down the compatible hyperkähler metric g_h on $C^\infty(E(\Gamma))$:

$$g_h(\Theta_1, \Theta_2) = \int_{S^2/\Gamma} Re\langle \Theta_1, \Theta_2\rangle \omega_{vol},$$

and it's evident to see that g_h is compatible with the complex structures and the symplectic forms.

$\square$

Next, we want to justify the two additional moment map equations, 8.3.3 and 8.3.4. To start with, we make the observation that for $\Theta : x \mapsto [p, \lambda(p)]$ and $Y : S^2/\Gamma \to \mathbf{f}/\mathbf{t}$ an element in $\mathrm{g}^{F,\Gamma}$, we have

$$Y\Theta - \Theta Y : x \mapsto [p, Y(x)\lambda(p) - \lambda(p)Y(x)]$$

and

$$J(Y\Theta - \Theta Y) : x \mapsto [p, -\lambda(J(p))^* Y(\tau(x))^* + Y(\tau(x))^* \lambda(J(p))^*].$$

Thus, we can think of $J(Y\Theta - \Theta Y) = [J\Theta, (\tau^* Y)^*]$, where τ denotes the involution we have introduced previously. Meanwhile, for $YJ\Theta - J\Theta Y$, we have

$$YJ\Theta - J\Theta Y : x \mapsto [p, -Y(x)\lambda(J(p))^* + \lambda(J(p))^* Y(x)].$$

Hence, for $Y : S^2/\Gamma \to \mathbf{f}/\mathbf{t}$ invariant under τ, that is, $Y(x) = Y(\tau(x)), \forall x \in S^2/\Gamma$, we have

$$J(Y\Theta - \Theta Y) = [J\Theta, (\tau^* Y)^*] = [J\Theta, -Y] = [Y, J\Theta] = YJ\Theta - J\Theta Y. \tag{8.4.1}$$

Proposition 8.4.5. *The action of the τ-invariant subgroup $\mathcal{G}_\tau^{F,\Gamma}$ of $\mathcal{G}^{F,\Gamma}$ on $C^\infty(E(\Gamma))$ is Hamiltonian with respect to the symplectic forms ω_2 and ω_3 and give rise to the following moment maps:*

$$\tilde{\mu}_2 : C^\infty(S^2/\Gamma, E(\Gamma)) \to \Omega^2(S^2/\Gamma; \mathbf{f}/\mathbf{t}),$$

$$\Theta \mapsto -\frac{1}{4}([J\Theta, \Theta^*] - [\Theta, J\Theta^*])\omega_{vol},$$

and

$$\tilde{\mu}_3 : C^\infty(S^2/\Gamma, E(\Gamma)) \to \Omega^2(S^2/\Gamma; \mathbf{f}/\mathbf{t}),$$

$$\Theta \mapsto -\frac{i}{4}([J\Theta, \Theta^*] + [\Theta, J\Theta^*])\omega_{vol}.$$

Proof. Again, we first focus on ω_3. Similar to the proof of Proposition 3.1, we let $Y : S^2/\Gamma \to \mathfrak{f}/\mathfrak{t}$ be a τ-invariant element in $\mathrm{g}^{F,\Gamma}$ and let $Y^\sharp$ denote the vector field on $C^\infty(E(\Gamma))$ induced by Y.

Now, let's compute $\iota_{Y^\sharp}\omega_{3\Theta}(\Theta')$. We have

$$\iota_{Y^\sharp}\omega_{3\Theta}(\Theta') = \int_{S^2/\Gamma} -Im\langle J[Y,\Theta], \Theta'\rangle\omega_{vol} = \int_{S^2/\Gamma} -Im\langle J(Y\Theta - \Theta Y), \Theta'\rangle\omega_{vol}.$$

Hence, by 8.4.1, we have

$$\int_{S^2/\Gamma} -Im\langle J(Y\Theta - \Theta Y), \Theta'\rangle\omega_{vol} =$$

$$\int_{S^2/\Gamma} -Im\langle [J\Theta, Y^*], \Theta'\rangle\omega_{vol} = \int_{S^2/\Gamma} Im\mathrm{Tr}([Y^*, J\Theta]\Theta'^*)\omega_{vol}$$

$$= \int_{S^2/\Gamma} \frac{i}{2}\mathrm{Tr}([\Theta', J\Theta^*]Y^* + [J\Theta, \Theta'^*]Y^*)\omega_{vol}$$

$$= \int_{S^2/\Gamma} \frac{i}{2}(\langle[\Theta', J\Theta^*], Y\rangle + \langle[J\Theta, \Theta'^*], Y\rangle)\omega_{vol}.$$

Meanwhile, by the skew-symmetric property of ω_3, we also have

$$\int_{S^2/\Gamma} -Im\langle J[Y,\Theta], \Theta'\rangle\omega_{vol} = \int_{S^2/\Gamma} -Im\langle -J\Theta', [Y,\Theta]\rangle\omega_{vol}$$

$$= \int_{S^2/\Gamma} Im\mathrm{Tr}([Y,\Theta^*]J\Theta')\omega_{vol} = \int_{S^2/\Gamma} \frac{i}{2}\mathrm{Tr}([J\Theta'^*, \Theta]Y + [\Theta^*, J\Theta']Y)\omega_{vol}$$

$$= \int_{S^2/\Gamma} \frac{i}{2}(\langle[\Theta, J\Theta'^*], Y\rangle + \langle[J\Theta', \Theta^*], Y\rangle)\omega_{vol}.$$

Now, we obtain the following:

$$2\iota_{Y^\sharp}\omega_{3\Theta}(\Theta') =$$

$$\int_{S^2/\Gamma} \frac{i}{2}\langle[\Theta', J\Theta^*] + [\Theta, J\Theta'^*], Y\rangle\omega_{vol} + \int_{S^2/\Gamma} \frac{i}{2}\langle[J\Theta, \Theta'^*] + [J\Theta', \Theta^*], Y\rangle\omega_{vol}.$$

On the other hand, let Θ_t with $t \in [0,1]$ be a path in $C^\infty(S^2/\Gamma, E(\Gamma))$ such that $\Theta_0 = \Theta$ and $\frac{d}{dt}|_{t=0}\Theta_t = \Theta'$. Then we have

$$d\tilde{\mu}^Y_{3_\Theta}(\Theta') = \frac{d}{dt}|_{t=0} \int_{S^2/\Gamma} -\langle Y, \frac{i}{4}([J\Theta_t, \Theta_t^*] + [\Theta_t, J\Theta_t^*])\rangle \omega_{vol}$$

$$= \int_{S^2/\Gamma} -\langle Y, \frac{i}{4}([J\Theta', \Theta^*] + [J\Theta, \Theta'^*])\rangle \omega_{vol} + \int_{S^2/\Gamma} -\langle Y, \frac{i}{4}([\Theta', J\Theta^*] + [\Theta, J\Theta'^*])\rangle \omega_{vol}.$$

The above computations verify that

$$\tilde{\mu}_3(\Theta) = -\frac{i}{4}([J\Theta, \Theta^*] + [\Theta, J\Theta^*])\omega_{vol}.$$

By very similar computations, we also get that for

$$\omega_2(\Theta_1, \Theta_2) = \int_{S^2/\Gamma} Re\langle J\Theta_1, \Theta_2\rangle \omega_{vol},$$

we have

$$\tilde{\mu}_2(\Theta) = -\frac{1}{4}([J\Theta, \Theta^*] - [\Theta, J\Theta^*])\omega_{vol}.$$

We leave out the proof for the equivariance condition as it is essentially the same as that of Proposition 8.3.1.

□

Remark 8.4.6. *1. Note, here we need to restrict the gauge group action to the τ-invariant subgroup $\mathcal{G}_\tau^{F,\Gamma}$ which is different from the previous setup.*

2. Observe that $\frac{i}{4}([J\Theta, \Theta^*] + [\Theta, J\Theta^*])$ and $\frac{1}{4}([J\Theta, \Theta^*] - [\Theta, J\Theta^*])$ are both τ-invariant and hence the new moment maps map into the correct space.

Lemma 8.4.7. *If Θ is holomorphic with respect to a fixed holomorphic structure on $E(\Gamma)$ and is identified with a pair of matrices (α, β), then $J\Theta = J(\alpha, \beta) = (-\beta^*, \alpha^*)$.*

Proof. As before, we express Θ as $\Theta : x \mapsto [p, \lambda(p)]$, where $\lambda : S^3 \to End(R)$ is S^1- and Γ-equivariant. Since Θ is holomorphic, λ can be extended to a complex linear map $\lambda : \mathbb{C}^2 \to End(R)$. Hence, λ can be thought of as a pair of matrices (α, β) such that $\lambda(z_1, z_2) = z_1\alpha + z_2\beta$, for $(z_1, z_2) \in \mathbb{C}^2$.

On the other hand, we have $J\Theta : x \mapsto [p, -\lambda(J(p))^*]$. This give us

$$-\lambda(J(z_1, z_2))^* = -\lambda(-z_2^*, z_1^*)^* = -(-z_2^*\alpha + z_1^*\beta)^* = -z_1\beta^* + z_2\alpha^*.$$

This precisely says that $J\Theta$ reduces to $(-\beta^*, \alpha^*)$. □

Remark 8.4.8. *1. Provided with the previous lemma, we observe that for $\tilde{\mu}_3$, we have*

$$\begin{aligned}
&\frac{i}{4}([J\Theta, \Theta^*] + [\Theta, J\Theta^*]) \\
&= \frac{i}{4}([-\beta^*, \alpha^*] + [\alpha^*, \beta^*] + [\alpha, -\beta] + [\beta, \alpha]) \\
&= \frac{i}{4}(2[\alpha^*, \beta^*] - 2[\alpha, \beta]) = \frac{i}{2}([\alpha^*, \beta^*] - [\alpha, \beta]),
\end{aligned}$$

but this is precisely the third moment map μ_3 in Kronheimer's setup [21]; similar calculations show that $\tilde{\mu}_2$ also reduces to μ_2 in Kronheimer's setup [21]. This observation will become a key element in the proof of Theorem 3.6.

2. We remark that the same analysis presented in this section will give rise to hyperkähler structures to $C^\infty(E(End(S)))$ and $C^\infty(E(End(S))^\Gamma_r)$ if we replace the regular representation R with any Γ-representation r on S with an appropriately chosen hermitian structure to obtain a hyperkähler structure on $C^\infty(E(End(S)))$ and $C^\infty(E(End(S))^\Gamma_r)$, as we use no specific properties of the regular representation R for defining the hyperkähler structure.

8.5 Uniqueness theorems

In this section, we analyze both the unitary gauge group action and the complex gauge group action on the configuration space $\mathcal{A}^F \times C^\infty(E(\Gamma))$. In particular, we prove two uniqueness theorems: the first one states that any solution to 8.3.1 and 8.3.2 lying in $\mathcal{A}^F \times C^\infty(E(\Gamma))$ that are $\mathcal{G}^{F,\Gamma}_{\mathbb{C}}$-equivalent are also $\mathcal{G}^{F,\Gamma}$-equivalent, which is a standard occurrence in gauge theory. The second uniqueness theorem can be thought of as a corollary of the first one, which states that any solution to

8.3.1 – 8.3.4 lying in $\mathcal{A}_\tau^F \times C^\infty(E(\Gamma))$ that are $\mathcal{G}_{\tau,\mathbb{C}}^{F,\Gamma}$-equivalent must also be $\mathcal{G}_\tau^{F,\Gamma}$-equivalent.

Lemma 8.5.1. *Up to automorphisms of $E(\Gamma)$, the space $\mathcal{A}^F$ defines a single holomorphic structure on $E(\Gamma)$, identifying $E(\Gamma)$ with the direct sum of hyperplane bundles holomorphically.*

Proof. By construction, A_0 is taken to be the Chern connection giving rise to the holomorphic structure on $E(\Gamma)$ such that $E(\Gamma)$ splits holomorphically as a direct sum of hyperplane bundles. As $\mathcal{A}^F$ is simply defined to be the complex orbit containing A_0, we must have that $\mathcal{A}^F$ defines a single holomorphic structure identifying $E(\Gamma)$ with the direct sum of hyperplane bundles holomorphically, as stated in the lemma.

□

Lemma 8.5.2. *The based complex gauge group acts freely on $\mathcal{A}^F$, and the stabilizer of B in the complex gauge group is isomorphic to the constant subgroup.*

Remark 8.5.3. *The two preceding lemmas can both be formulated where we replace $\mathcal{A}^F$ with $\mathcal{A}_\tau^F$ and use the corresponding τ-invariant gauge groups.*

Definition 8.5.4. *Let Q be the canonical 2-dimensional representation of $SU(2)$. Let $Hom(Q,V)^\Gamma$ denote the Γ-invariant subset of $Hom(Q,V)$, consisting of all maps that commute with the Γ-actions on Q and V, that is, for $f \in Hom(Q,V)$, $f(\gamma(z)) = \gamma(f(z))$, where $\gamma \in \Gamma$ and $z \in Q$.*

Lemma 8.5.5. *The space $Hom(Q,V)$ is isomorphic to the space of holomorphic sections of $E(V)$ with respect to A_0. The space $Hom(Q,V)^\Gamma$ is isomorphic to the space of holomorphic sections of $E(\Gamma)$ with respect to A_0.*

Remark 8.5.6. *1. It is easy to see that $M \cong Hom(Q,V)^\Gamma$, and hence by the previous lemma, we can think of M as the space of holomorphic sections of $E(\Gamma)$ with respect to the fixed connection A_0.*

2. *The above lemma gives rise to a map*

$$\Psi : M \to \mathcal{A}^F \times C^\infty(E(\Gamma))$$

$$\lambda \mapsto (A_0, \Theta : x \mapsto [p, \lambda(p)]),$$

with the property that Ψ is an isomorphism onto its image. In addition, Ψ can be naturally regarded as an isometry onto its image. To see this, we observe that the hyperkähler metric g_h given in Proposition 3.6 restricted to the set $\{\Theta \in C^\infty(E(\Gamma)) | \bar{\partial}_{A_0}\Theta = 0\}$ agrees with the natural flat hyperkähler metric on M. Hence, Ψ is an isometry onto its image.

3. *A holomorphic section of $E(\Gamma)$ with respect to the fixed connection A_0 can be expressed as a pair of matrices (α, β) where (α, β) is Γ-invariant as in [21].*

We omit the proofs for the two preceding lemmas as the proofs can be found in or follow from standard references such as [20], [19], and [13].

Lemma 8.5.7. *There is a map*

$$\tilde{\Psi} : M \to \{(A_0 + B, \Theta) \in \mathcal{A}^F \times C^\infty(E(\Gamma)) | \bar{\partial}_{A_0+B}\Theta = 0\}/\mathcal{G}^{F,\Gamma}_{0,\mathbb{C}}$$

such that is $\tilde{\Psi}$ an isomorphism, where M comes from Kronheimer's construction in [21], and there exists a residual F^c action on both sides which also coincides.

Proof. By Lemma 8.5.2, we know that $\mathcal{G}^{F,\Gamma}_{0,\mathbb{C}}$ acts freely and transitively on the space of connections. Hence, we can take $\tilde{\Psi}$ to be the following composition of maps: let $\mathcal{C}$ denote $\{(A_0 + B, \Theta) \in \mathcal{A}^F \times C^\infty(E(\Gamma)) | \bar{\partial}_{A_0+B}\Theta = 0\}$, and consider

$$\tilde{\Psi} : M \to \mathcal{C} \to \mathcal{C}/\mathcal{G}^{F,\Gamma}_{0,\mathbb{C}},$$

$$(\alpha, \beta) = \lambda \mapsto (A_0, \Theta : x \mapsto [p, \lambda(p)]) \mapsto [(A_0, \Theta : x \mapsto [p, \lambda(p)])],$$

where $[(A_0, \Theta : x \mapsto [p, \lambda(p)])]$ denotes the gauge orbit containing the chosen representative. Previous arguments suggest that $\tilde{\Psi}$ is an isomorphism. It follows naturally that the residual F^c action on both M and $\mathcal{C}/\mathcal{G}^{F,\Gamma}_{0,\mathbb{C}}$ coincides.

$\square$

Remark 8.5.8. *We let $\tilde{\Psi}_\tau$ denote the map*

$$\tilde{\Psi}_\tau : M \to \{(A_0 + B, \Theta) \in \mathcal{A}^F_\tau \times C^\infty(E(\Gamma)) | \bar{\partial}_{A_0+B}\Theta = 0\}/\mathcal{G}^{F,\Gamma}_{\tau,0,\mathbb{C}}.$$

We have that $\tilde{\Psi}_\tau$ is again an isomorphism following the same arguments as in the previous lemma.

Before proceeding, we set up some linear algebra that will be of use later. Recall that $E(V)$ is the vector bundle associated to S^3 on the Γ-representation $V = End(R)$. We have the following two maps induced by left and right multiplication on V:

$$c_l : V \to End(V), c_l(\phi)(\psi) = \phi \circ \psi$$

and

$$c_r : V \to End(V), c_l(\phi)(\psi) = \psi \circ \phi.$$

Since both c_l and c_r commute with the S^1-action, they give rise to bundle maps:

$$c_l, c_r : E(V) \to E(End(V)).$$

Hence, given $\phi, \psi \in E(V)_x$, we have the following composition:

$$E(V)_x \otimes E(V)^*_x \to E(End(V))_x \otimes E(End(V))^*_x \to \underline{End(V)}_x,$$

$$\phi \otimes \psi^* \mapsto c_l(\phi) \otimes c_l(\psi^*) \mapsto [c_l(\phi), c_l(\psi^*)].$$

On the other hand, we also have

$$E(V)_x \otimes E(V)^*_x \to \underline{End(R)}_x \to \underline{End(End(R))}_x = \underline{End(V)}_x,$$

$$\phi \otimes \psi^* \mapsto [\phi, \psi^*] \mapsto c_l([\phi, \psi^*]),$$

where we also have

$$[c_l(\phi), c_l(\psi^*)] = c_l([\phi, \psi^*]).$$

Similarly, there are maps such as

$$E(End(R)) \otimes \underline{End(R)} \to E(End(R)),$$

$$\underline{End(R)} \otimes E(End(R)) \to E(End(R)),$$

$$\underline{End(R)} \otimes E(End(R)) \otimes \underline{End(R)} \to E(End(R)),$$

modeled locally on maps such as

$$End(R) \otimes End(R) \to End(R), \phi \otimes \psi \mapsto \phi \circ \psi.$$

Lemma 8.5.9 (Uniqueness theorem 1)**.** *Let (B_1, Θ_1) and (B_2, Θ_2) be two solutions to 8.3.1 and 8.3.2 in $\mathcal{A}^F \times C^\infty(E(\Gamma))$ that lie on the same complex orbit, that is, there exists a complex automorphism of $E(\Gamma)$ taking (B_1, Θ_1) to (B_2, Θ_2). Then (B_1, Θ_1) and (B_2, Θ_2) are unitarily equivalent.*

Proof. This proof is modeled on Hitchin's proof of Theorem (2.7) in [14]. Let $\kappa : E(\Gamma) \to E(\Gamma)$ be the complex automorphism satisfying $\Theta_1 \kappa = \kappa \Theta_2$ and $\bar{\partial}_{B_1} \kappa = \kappa \bar{\partial}_{B_2}$. We also have

$$\bar{\partial}_{A_0+B_1} \Theta_1 = \bar{\partial}_{A_0+B_2} \Theta_2 = 0$$

and

$$F_{B_1} - \frac{i}{2}[\Theta_1, \Theta_1^*]\omega_{vol} = F_{B_2} - \frac{i}{2}[\Theta_2, \Theta_2^*]\omega_{vol} = \sigma.$$

Now we define two bundles: let

$$W = End(E(\Gamma)) \cong E(\Gamma) \otimes E(\Gamma)^*,$$

and let

$$W^\circ = E(End(V))^\Gamma.$$

We remark that both W and W° have the same fibers isomorphic to $End(V)$, but W is a trivial bundle whereas W° is again an associated bundle of S^3. We can think of κ as a section of W. We also have that Θ_1 and Θ_2 together define a section

$$\boldsymbol{\Theta} = c_l(\Theta_1) - c_r(\Theta_2)$$

of W°, and B_1 and B_2 together define a connection

$$\boldsymbol{B} = B_1 \otimes id - id \otimes B_2^*$$

on both W and W°, as $End(W)$ and $End(W^\circ)$ are both isomorphic to $End(End(V))^\Gamma$. As we have

$$\kappa\Theta_1 = \Theta_2\kappa,$$

we must have that

$$\boldsymbol{\Theta}\kappa = (c_l(\Theta_1) - c_r(\Theta_2))\kappa = 0.$$

We observe that the pair $(\boldsymbol{B}, \boldsymbol{\Theta})$ satisfies the equations

$$\bar{\partial}_{\boldsymbol{B}}\boldsymbol{\Theta} = 0$$

and

$$F_{\boldsymbol{B}} - \frac{i}{2}[\boldsymbol{\Theta}, \boldsymbol{\Theta}^*]\omega_{vol} = \mathrm{ad}(\sigma)$$

on W°, where $[\boldsymbol{\Theta}, \boldsymbol{\Theta}^*] = c_l([\Theta_1, \Theta_1^*]) - c_r([\Theta_2, \Theta_2^*])$.

To proceed, we now think of κ as a holomorphic section of W with respect to $\boldsymbol{B}$, that is, $\bar{\partial}_{\boldsymbol{B}}\kappa = 0$, as $\bar{\partial}_{B_1}\kappa = \kappa\bar{\partial}_{B_2}$. Before we continue further, we first want to prove a useful identity. Consider

$$\bar{\partial}\langle\partial_{\boldsymbol{B}}\kappa, \kappa\rangle = \langle\bar{\partial}_{\boldsymbol{B}}\partial_{\boldsymbol{B}}\kappa, \kappa\rangle - \langle\partial_{\boldsymbol{B}}\kappa, \partial_{\boldsymbol{B}}\kappa\rangle.$$

Since $F_{\boldsymbol{B}} = \bar{\partial}_{\boldsymbol{B}}\partial_{\boldsymbol{B}} + \partial_{\boldsymbol{B}}\bar{\partial}_{\boldsymbol{B}}$ and $\bar{\partial}_{\boldsymbol{B}}\kappa = 0$, we have

$$\bar{\partial}\langle\partial_{\boldsymbol{B}}\kappa, \kappa\rangle = \langle F_{\boldsymbol{B}}\kappa, \kappa\rangle - \langle\partial_{\boldsymbol{B}}\kappa, \partial_{\boldsymbol{B}}\kappa\rangle.$$

Now we integrate on both sides and get

$$\int_{S^2/\Gamma} \bar{\partial}\langle\partial_{\boldsymbol{B}}\kappa, \kappa\rangle + \int_{S^2/\Gamma} \langle\partial_{\boldsymbol{B}}\kappa, \partial_{\boldsymbol{B}}\kappa\rangle = \int_{S^2/\Gamma} \langle F_{\boldsymbol{B}}\kappa, \kappa\rangle,$$

and by Stokes' theorem, we get

$$0 \leq \int_{S^2/\Gamma} \langle\partial_{\boldsymbol{B}}\kappa, \partial_{\boldsymbol{B}}\kappa\rangle = \int_{S^2/\Gamma} \langle F_{\boldsymbol{B}}\kappa, \kappa\rangle.$$

Hence, we have

$$\int_{S^2/\Gamma} \langle\partial_{\boldsymbol{B}}\kappa, \partial_{\boldsymbol{B}}\kappa\rangle = \int_{S^2/\Gamma} \langle F_{\boldsymbol{B}}\kappa, \kappa\rangle =$$

$$\int_{S^2/\Gamma} \frac{i}{2}\langle[\boldsymbol{\Theta}, \boldsymbol{\Theta}^*]\kappa, \kappa\rangle\omega_{vol} - \int_{S^2/\Gamma} \langle \mathrm{ad}(\sigma)\kappa, \kappa\rangle.$$

Since σ takes values in the center Z, we have that κ commutes with σ, i.e., $\mathrm{ad}(\sigma)\kappa = 0$, and hence the following equation

$$-\int_{S^2/\Gamma} \langle \mathrm{ad}(\sigma)\kappa, \kappa\rangle = 0$$

holds as $\sigma \otimes 1(\kappa) = 1 \otimes \sigma^T(\kappa)$, which can be shown using essentially the same arguments as in showing $\boldsymbol{\Theta}\kappa = 0$.

As we have shown that $\boldsymbol{\Theta}\kappa = 0$, we also obtain

$$\langle[\boldsymbol{\Theta}, \boldsymbol{\Theta}^*]\kappa, \kappa\rangle = \langle\boldsymbol{\Theta}\boldsymbol{\Theta}^*\kappa, \kappa\rangle = \langle\boldsymbol{\Theta}^*\kappa, \boldsymbol{\Theta}^*\kappa\rangle \geq 0$$

and hence must be purely real. Consequently, $\frac{i}{2}\langle[\boldsymbol{\Theta}, \boldsymbol{\Theta}^*]\kappa, \kappa\rangle$ must be purely imaginary, so it must be 0. This gives us that $\partial_{\boldsymbol{B}}\kappa = 0$.

Putting everything together, we have $\partial_{\boldsymbol{B}}\kappa = \bar{\partial}_{\boldsymbol{B}}\kappa = 0$, $\boldsymbol{\Theta}\kappa = \boldsymbol{\Theta}^*\kappa = 0$. Let $\rho = \kappa(\kappa^*\kappa)^{-\frac{1}{2}}$ then we must have $d_{\boldsymbol{B}}\rho = 0$. Since $\boldsymbol{\Theta}\kappa = \boldsymbol{\Theta}^*\kappa = 0$, we have $\kappa^*\Theta_2 = \Theta_1\kappa^*$

and $\kappa\Theta_2 = \Theta_1\kappa$, which implies $\rho\Theta_2 = \Theta_1\rho$. Hence, we obtain the desire statement that (B_1, Θ_1) and (B_2, Θ_2) lie on the same unitary gauge orbit.

$\square$

Corollary 8.5.10 (Uniqueness theorem 2). *Let (B_1, Θ_1) and (B_2, Θ_2) be two solutions to 8.3.1 – 8.3.4 in $\mathcal{A}^F_\tau \times C^\infty(E(\Gamma))$ that lie on the same $\mathcal{G}^{F,\Gamma}_{\tau,\mathbb{C}}$-orbit, that is, there exists a complex automorphism of $E(\Gamma)$ in $\mathcal{G}^{F,\Gamma}_{\tau,\mathbb{C}}$ that takes (B_1, Θ_1) to (B_2, Θ_2). Then (B_1, Θ_1) and (B_2, Θ_2) lie on the same $\mathcal{G}^{F,\Gamma}_\tau$-orbit.*

Proof. Let κ be such a complex automorphism. By the same arguments as in the previous lemma, we can modify κ and obtain a unitary gauge element $\rho = \kappa(\kappa^*\kappa)^{\frac{1}{2}}$ that also sends (B_1, Θ_1) to (B_2, Θ_2). We must also have that ρ is τ-invariant as κ is τ-invariant. Hence, ρ lies in $\mathcal{G}^{F,\Gamma}_\tau$. $\square$

Before we proceed to the next section, we prove the following proposition which analyzes the stabilizer group of a holomorphic section Θ.

Proposition 8.5.11. *If Θ has trivial stabilizer in $Stab(B)$ with $\bar{\partial}_{A_0+B}\Theta = 0$, then Θ has trivial stabilizer in $\mathcal{G}^{F,\Gamma}$.*

Proof. Let $\kappa : E(\Gamma) \to E(\Gamma)$ be a complex automorphism on $E(\Gamma)$ taking A_0 to $A_0 + B$. In other words, we have $B = \kappa^{-1}\bar{\partial}\kappa + \kappa^*\partial\kappa^{*-1}$. Consider $\kappa^{-1}\Theta\kappa$, it is a holomorphic section of $E(\Gamma)$ with respect to A_0. Hence, we can rewrite $\kappa^{-1}\Theta\kappa$ as a pair of matrices (α, β). The identification is as follows: for $x \in S^2$, $\kappa^{-1}\Theta\kappa : x \mapsto [p, \lambda(p)]$, where $\lambda : S^3 \to End(R)$ is given by $\lambda(z_1, z_2) = z_1\alpha + z_2\beta$.

Since (α, β) is Γ-invariant, we have that for $\gamma = \begin{pmatrix} u & v \\ -v^* & u^* \end{pmatrix}$, the pair (α, β) must satisfy

$$R(\gamma^{-1})\alpha R(\gamma) = u\alpha + v\beta \tag{8.5.1}$$

and

$$R(\gamma^{-1})\beta R(\gamma) = -v^*\alpha + u^*\beta \tag{8.5.2}$$

as in [21]. Notice that if $v \neq 0$, then β is uniquely given by $\beta = v^{-1}R(\gamma^{-1})\alpha R(\gamma) - v^{-1}u\alpha$. On the other hand, if $v = 0$ for all $\gamma \in \Gamma$, then it implies that Γ is a cyclic subgroup. Hence, we break the proof into two cases.

Case 1: Γ is not cyclic.

In this case, we have that $v \neq 0$ and $\beta = v^{-1}R(\gamma^{-1})\alpha R(\gamma) - v^{-1}u\alpha$. First, we want to show that (α, β) has trivial stabilizer in F/T if and only if α has trivial stabilizer in F/T. We can assume that α and β are both nonzero as by 8.5.1 and 8.5.2, it's easy to see that if either α or β is 0, then both have to be 0.

We observe that (α, β) has trivial stabilizer in F/T if and only if α has trivial stabilizer in F/T: if α has trivial stabilizer in F/T, then clearly (α, β) has trivial stabilizer in F/T; on the other hand, if some element f stabilizes α, then it stabilizes β as well by the equality $\beta = v^{-1}R(\gamma^{-1})\alpha R(\gamma) - v^{-1}u\alpha$, so f stabilizes (α, β). With the preceding arguments, we can rephrase the assumption that (α, β) has trivial stabilizer in F/T as simply that α has trivial stabilizer in F/T.

Now, at a point p thought of as a pair (z_1, z_2), we can use some $\gamma \in \Gamma$ to get the following equality

$$f(z_1\alpha + z_2\beta)f^{-1} = f(z_1\alpha - z_2v^{-1}u\alpha + z_2v^{-1}R(\gamma^{-1})\alpha R(\gamma))f^{-1}$$

$$= z_1 f\alpha f^{-1} - z_2v^{-1}uf\alpha f^{-1} + z_2v^{-1}R(\gamma^{-1})(f\alpha f^{-1})R(\gamma).$$

Assume that we are given $f\alpha f^{-1} \neq \alpha$, for all $f \in F/T$, we want to show that for any pair of points (z_1, z_2) and for all $f \in F/T$, we always have the following:

$$z_1\alpha - z_2v^{-1}u\alpha + z_2v^{-1}R(\gamma^{-1})\alpha R(\gamma) \neq z_1 f\alpha f^{-1} - z_2v^{-1}uf\alpha f^{-1} + z_2v^{-1}R(\gamma^{-1})(f\alpha f^{-1})R(\gamma).$$

To achieve this end, let L_γ be the linear map defined as follows: for a pair $(c, d) \in M$, consider

$$L_\gamma : c \mapsto z_1c - z_2v^{-1}uc + z_2v^{-1}R(\gamma^{-1})cR(\gamma).$$

Then we need to show $L_\gamma(\alpha) \neq L_\gamma(f\alpha f^{-1})$. As we know that $\alpha \neq f\alpha f^{-1}$, it suffices to show that

$$\bigcap_{\gamma \in \Gamma} \ker(L_\gamma) = 0.$$

We can assume that $z_2 \neq 0$ as the inequality is clearly satisfied when $z_2 = 0$. Hence, c lies in the kernel of L_γ if

$$\frac{z_2 v^{-1} u - z_1}{z_2 v^{-1}} c = R(\gamma^{-1}) c R(\gamma).$$

This implies that c must be a scalar multiple of d, that is, $d = qc$; in particular, by applying 8.5.1 and 8.5.2 to the pair (c, d), we must have $(qu + q^2 v + v^* - u^* q)c = 0$. Notice that this equality must be satisfied for any choice of $\gamma \in \Gamma$ with $v \neq 0$, and since q and c are fixed, we see that this equality can only hold when $c = 0$. As a result, $z_1 \alpha + z_2 \beta$ has trivial stabilizer for all (z_1, z_2), which gives us that (α, β) has trivial stabilizer in $\mathcal{G}^{F,\Gamma}$.

Case 2: Γ is cyclic.

When Γ is a cyclic subgroup, we can write down α and β explicitly and describe the action of Γ and F/T explicitly as well. We use the decomposition of M in terms of simply-laced Dynkin diagram given in [21] and reviewed in Section 2.1:

$$M = \bigoplus_{i,j} a_{ij} Hom(\mathbb{C}^{n_i}, \mathbb{C}^{n_j}).$$

We also have that

$$F = \times_i U(n_i).$$

For the case where Γ is cyclic, $n_i = 1$ for all i, and

$$M = (\bigoplus_i Hom(\mathbb{C}^{n_i}, \mathbb{C}^{n_{i+1}})) \oplus (\bigoplus_j Hom(\mathbb{C}^{n_{j+1}}, \mathbb{C}^{n_j})).$$

We can regard $\alpha \in \bigoplus_i Hom(\mathbb{C}^{n_i}, \mathbb{C}^{n_{i+1}})$ and $\beta \in \bigoplus_j Hom(\mathbb{C}^{n_{j+1}}, \mathbb{C}^{n_j})$. Hence, we can write $\alpha = (a_1, ..., a_n)$ and $\beta = (b_1, ..., b_n)$, and F acts on $\mathbb{C}^{n_i}$ and $\mathbb{C}^{n_j}$ by scalar multiplification.

For (α, β) to have trivial stabilizer in F/T, we must have that for all $i \in \{1, ..., n\}$, at least one of a_i and b_i is not 0. For $z_1 \alpha + z_2 \beta$ to have trivial stabilizer in F/T at (z_1, z_2), we must have that for all $i \in \{1, ..., n\}$, at least one of $z_1 a_i$ and $z_2 b_i$ is not 0.

But this can only happen when either z_1 or z_2 is 0. This means that the stabilizer of (α, β) in $\mathcal{G}^{F,\Gamma}$ must be the identity away from $(0, z_2)$ and $(z_1, 0)$, and hence it must be the identity by continuity.

Hence, we have shown that if $\lambda(p)$ has trivial stabilizer at a single p, then for any other p', $\lambda(p')$ also has trivial stabilizer. This is equivalent to saying that if $\kappa^{-1}\Theta\kappa$ has trivial stabilizer in F/T, then it has trivial stabilizer in $\mathcal{G}^{F,\Gamma}$. By pushing forward using κ, we get the desired statement of the lemma.

□

Corollary 8.5.12. *If Θ has trivial stabilizer in $Stab(B)$ with $\bar{\partial}_{A_0+B}\Theta = 0$, then Θ has trivial stabilizer in $\mathcal{G}_\tau^{F,\Gamma}$.*

8.6 Smoothness and dimension calculations

In this section, we show that the moduli space is a smooth finite-dimensional manifold and calculate its dimension which will be useful for proving Theorem 8.3.8. To achieve this end, we first introduce the following lemma.

Lemma 8.6.1. *If (B, Θ) and (B', Θ') are two solutions to 8.3.1 and 8.3.2 in $\mathcal{A}^F \times C^\infty(E(\Gamma))$ with B and B' not $\mathcal{G}^{F,\Gamma}$-equivalent, then (B', Θ') is separated from the subset of solutions such that the connection part is $\mathcal{G}^{F,\Gamma}$-equivalent to B.*

Proof. Suppose we have two solutions (B, Θ) and (B', Θ') such that B is not $\mathcal{G}^{F,\Gamma}$-equivalent to B'. We proceed by contradiction. Suppose that there exists a sequence of solutions $\{(B_n, \Theta_n)\}_n$ such that $(B_1, \Theta_1) = (B, \Theta)$ and $\{(B_n, \Theta_n)\}_n$ converges weakly in L_1^2 to (B', Θ') with B_n lying on the same $\mathcal{G}^{F,\Gamma}$-orbit as B, for all n. Then we get a sequence of gauge elements lying in $\mathcal{G}^{F,\Gamma}$, denoted by $\{\rho_n\}$, such that $\rho_n \cdot B = B_n$, for all n. (Note that we don't assume $\rho_n \cdot \Theta = \Theta_n$.) We want to show that $\{\rho_n\}$ converges weakly to some ρ. To this end, we follow Hitchin's proof of Theorem (2.7) in [14]. Consider the following:

$$\bar{\partial}_{B_1 B_n} : \Omega^0(S^2/\Gamma; E(\Gamma)^* \otimes E(\Gamma)) \to \Omega^{0,1}(S^2/\Gamma; E(\Gamma)^* \otimes E(\Gamma)),$$

where B_n acts on the $E(\Gamma)^*$ factor, and B_1 acts on the $E(\Gamma)$ factor. Hence, $\bar{\partial}_{B_1 B'} = \bar{\partial}_{B_1 B_n} + t_n$ where $t_n \to 0$ weakly in L^2_1. As before, ρ_n is the sequence of unitary gauge elements taking B to B_n, and $\|\rho_n\|_{L^2} = 1$.

We also have

$$\rho_n \cdot B_1 = \rho_n^* \circ \partial_{B_1} \circ \rho_n^{*-1} + \rho_n^{-1} \circ \bar{\partial}_{B_1} \circ \rho_n = \partial_{B_n} + \bar{\partial}_{B_n}.$$

Hence, $\rho_n^* \circ \partial_{B_1} \circ \rho_n^{*-1} = \partial_{B_n}$ and $\rho_n^{-1} \circ \bar{\partial}_{B_1} \circ \rho_n = \bar{\partial}_{B_n}$, so we have $\bar{\partial}_{B_1} \circ \rho_n - \rho_n \circ \bar{\partial}_{B_n} = 0$, but this is equivalent to $\bar{\partial}_{B_1 B_n} \rho_n = 0$.

Now, the elliptic estimate for $\bar{\partial}_{B_1 B_n}$ gives us

$$\|\rho_n\|_{L^2_1} \leq C(\|[t_n, \rho_n]\|_{L^2} + \|\rho_n\|_{L^2}) = C(\|[t_n, \rho_n]\|_{L^2} + 1) \leq K_1 \|t_n\|_{L^4} \|\rho_n\|_{L^4} + K_2.$$

Since $L^2_1 \subset L^4$ compactly, we have that $\|\rho_n\|_{L^2_1}$ is bounded and hence has a weakly convergent subsequence. Since $L^2_1 \subset L^2$ is compact and $\|\rho_n\|_{L^2} = 1$, the weak limit ρ is non-zero.

Hence, we have $\rho \cdot B = B'$. Since by construction, B and B' lie on the same complex orbit, ρ must be a complex automorphism. Now since weak convergence implies pointwise convergence, that is, $\rho_n(x) \to \rho(x)$, for all $x \in S^2/\Gamma$, and F/T is compact, we must have $\rho(x) \in F/T$, for all x. Hence, ρ lies in $\mathcal{G}^{F,\Gamma}$, but this is a contradiction.

$\square$

Corollary 8.6.2. *If (B, Θ) and (B', Θ') are two solutions to 8.3.1 – 8.3.4 in $\mathcal{A}^F_\tau \times C^\infty(E(\Gamma))$ with B and B' not $\mathcal{G}^{F,\Gamma}_\tau$-equivalent, then (B', Θ') is separated from the subset of solutions such that the connection part is $\mathcal{G}^{F,\Gamma}_\tau$-equivalent to B.*

Proof. We assume otherwise and again follow the same arguments as in the previous lemma with the further assumption that all the gauge transformations are τ-invariant, that is, they lie in $\mathcal{G}^{F,\Gamma}_\tau$. Hence, we obtain a limit ρ that lies in $\mathcal{G}^{F,\Gamma}_\tau$ and hence obtains a contradiction.

$\square$

Corollary 8.6.3. 1. *Solutions to 8.3.1 and 8.3.2 in $\mathcal{A}^F \times C^\infty(E(\Gamma))$ with the connection part B not $\mathcal{G}^{F,\Gamma}$-equivalent lie in different connected components of the moduli space $\mathcal{M}(\Gamma, \tilde{\zeta}_1)$.*

2. *Solutions to 8.3.1 – 8.3.4 in $\mathcal{A}^F_\tau \times C^\infty(E(\Gamma))$ with the connection part B not $\mathcal{G}^{F,\Gamma}_\tau$-equivalent lie in different connected components of the moduli space $\mathcal{X}_{\tilde{\zeta}}$.*

Proposition 8.6.4. 1. *Suppose (B, Θ) is a solution to 8.3.1 and 8.3.2 in $\mathcal{A}^F \times C^\infty(E(\Gamma))$ with trivial stabilizer in $\mathcal{G}^{F,\Gamma}$, the moduli space $\mathcal{M}(\Gamma, \tilde{\zeta}_1)$ at the orbit of (B, Θ) is smooth of dimension $2|\Gamma| + 2$.*

2. *If (B, Θ) is a solution to 8.3.1 – 8.3.4 in $\mathcal{A}^F_\tau \times C^\infty(E(\Gamma))$ with trivial stabilizer in $\mathcal{G}^{F,\Gamma}_\tau$, the moduli space $\mathcal{X}_{\tilde{\zeta}}$ at the orbit of (B, Θ) is smooth of dimension 4.*

Proof. 1. Consider the set of sections $\mathcal{S} = \{\Theta \in C^\infty E(\Gamma) | \bar{\partial}_{A_0+B}\Theta = 0\}$. The stabilizer group $Stab(B)$ of B in $\mathcal{G}^{F,\Gamma}$ acts on $\mathcal{S}$. By Lemma 8.5.2 and Lemma 8.5.7 (with small adaptations of the proof), we have that $\mathcal{S}$ is isomorphic to $M = P^\Gamma$ and $Stab(B)$ is isomorphic to F. Hence, we can restrict the symplectic structure compatible with I on $C^\infty E(\Gamma)$ to $\mathcal{S}$ and obtain a Hamiltonian action of $Stab(B)$ on $\mathcal{S}$ with respect to the restrictions of I on $\mathcal{S}$. We also know that $Stab(B)$ acts freely at $\Theta \in \mathcal{S}$ as $\mathcal{G}^{F,\Gamma}$ acts freely at (B, Θ). On the other hand, by Lemma 8.6.1 and Corollary 8.6.3, every point in the connected component of $\mathcal{M}(\Gamma, \tilde{\zeta}_1)$ containing the orbit of (B, Θ) has a unique representative lying in $\mathcal{S}$. Hence, the smoothness and the dimension of $\mathcal{M}(\Gamma, \tilde{\zeta}_1)$ at $[(B, \Theta)]$ follow from Proposition 2.1 in [21].

2. First, we observe that the action of J commutes with the action of ρ when ρ lies in $\mathcal{G}^{F,\Gamma}_\tau$. Hence, we can restrict the hyperkähler structure on $C^\infty E(\Gamma)$ to $\mathcal{S}$ and obtain a Hamiltonian action of $Stab(B)$ on $\mathcal{S}$ with respect to the restrictions of I, J, and K on $\mathcal{S}$. We also know that $Stab(B)$ acts freely at $\Theta \in \mathcal{S}$ as $\mathcal{G}^{F,\Gamma}_\tau$ acts freely at (B, Θ). On the other hand, by Corollary 8.6.2 and Corollary 8.6.3, every point in the connected component of $\mathcal{X}_{\tilde{\zeta}}$ containing the

orbit of (B,Θ) has a unique representative lying in $\mathcal{S}$. Hence, the smoothness and the dimension of $\mathcal{X}_{\tilde{\zeta}}$ at $[(B,\Theta)]$ again follow from Proposition 2.1 in [21].

□

8.7 Proof of Theorem 8.3.8

8.7.1 A criterion for obtaining free $\mathcal{G}_\tau^{F,\Gamma}$-action

Now we want to give a criterion for when the $\mathcal{G}_\tau^{F,\Gamma}$-action is free on $\tilde{\mu}^{-1}(\tilde{\zeta})$.

We adapt the notations introduced in [21] and in Section 2.1 to our setting. Consider projection maps

$$\pi_i : R \to \mathbb{C}^{n_i} \otimes R_i.$$

Now, let $\hat{Z}$ denote the center of $\mathbf{f}$. Then $\Omega^0(S^2/\Gamma;\hat{Z})$ is spanned by elements $\sqrt{-1}\pi_i$, that is, smooth sections such that at each point the endomorphism is a scalar multiple of the projection map. Let h denote the real Cartan algebra associated to the Dynkin diagram, we then get a linear map l from $\Omega^0(S^2/\Gamma;\hat{Z})$ to $\Omega^0(S^2/\Gamma;h^*)$ given by

$$l : \sqrt{-1}\pi_i \mapsto n_i\xi_i,$$

and hence l induces a map $\tilde{l}$ from $\Omega^0(S^2/\Gamma;Z)$ to $\Omega^0(S^2/\Gamma;h)$ which is an isomporhism.

Let ξ be a root, not necessarily simple. We define $\tilde{D}_\xi$ to be $\ker(\xi\circ\tilde{l})$, where we regard ξ as a constant element in $\Omega^0(S^2/\Gamma;h^*)$.

Lemma 8.7.1. *Let (B,Θ) be a solution to 8.3.1 – 8.3.4 in $\mathcal{A}_\tau^F\times C^\infty(E(\Gamma))$. If $\mathcal{G}_\tau^{F,\Gamma}$ does not act freely on (B,Θ), then $\tilde{\zeta}$ lies in $\mathbb{R}^3\otimes\tilde{D}_\xi$.*

Proof. This proof is a reformulation of Kronheimer's original proof of Proposition 2.8 in [21] in our setting. Suppose $(B,\Theta)\in\mu^{-1}(\tilde{\zeta})$ is fixed by some $\rho\in\mathcal{G}_\tau^{F,\Gamma}$. In particular, ρ lies in $Stab(B)$ and fixes Θ. Then we can rewrite ρ as

$$\rho = \kappa\rho_0\kappa^{-1},$$

where ρ_0 is a constant in the complexification of F/T and

$$\kappa : E(\Gamma) \to E(\Gamma)$$

is a complex automorphism with

$$\kappa^{-1}\bar{\partial}\kappa + \kappa^*\partial\kappa^{*-1} = B.$$

We can find a lift $\tilde{\rho}_0$ of ρ_0 in the complexification of F and decompose R into the eigenspaces of $\tilde{\rho}_0$ and obtain at least two Γ-invariant parts

$$R = R' \oplus R''.$$

We have that $E(End(R'))$ is naturally a holomorphic subbundle of $E(\Gamma)$ with respect to A_0. This gives rise to a holomorphic subbundle $\tilde{E}$ of $E(\Gamma)$ with respect to B where the fiber of $\tilde{E}$ over each point x is isomorphic to $End(R')$. Explicitly, $\tilde{E}$ is the image of $E(End(R'))$ under κ.

Without loss of generality we assume that Θ is a holomorphic section of $\tilde{E}$ with a free action by $Map(S^2/\Gamma, F'/T')$, where $Map(S^2/\Gamma, F'/T')$ is the natural gauge group acting on $\tilde{E}$. In other words, $\tilde{E}$ is the smallest holomorphic subbundle of $E(\Gamma)$ such that Θ is a holomorphic section of $\tilde{E}$ and there is no proper subbundle of $\tilde{E}$ of which Θ is a section. We observe that $\tilde{E}$ is Γ-invariant. In particular, $\tilde{E}$ is isomorphic to $E(End(R'))^\Gamma$.

By Proposition 8.6.4, we know that the condition that $Map(S^2/\Gamma, F'/T')$ acts freely on Θ means that the moduli space of the reduction by $Map(S^2/\Gamma, F'/T')$ on pairs on $\tilde{E}$ is a smooth manifold at at least one point, with dimension

$$\dim_{\mathbb{R}}(Hom(\mathbb{C}^2, End(R'))^\Gamma) - 4\dim_{\mathbb{R}}(F'/T') \geq 0.$$

This translates to

$$\dim_{\mathbb{C}}(Hom(\mathbb{C}^2, End(R'))^{\Gamma}) - 2\dim_{\mathbb{C}}(End(R')^{\Gamma}) + 2 \geq 0,$$

and hence we have

$$2\dim_{\mathbb{C}}(End(R')^{\Gamma}) - \dim_{\mathbb{C}}(Hom(\mathbb{C}^2, End(R'))^{\Gamma}) \leq 2.$$

Now further decompose R' into irreducibles $R' = \oplus n_i' R_i$, then the above inequality is the same as the following:

$$2\sum_i (n_i')^2 - \sum_{i,j} a_{i,j} n_i' n_j' \leq 2.$$

Equivalently,

$$\sum_{i,j} c_{i,j} n_i' n_j' \leq 2,$$

where $\bar{C} = (c_{i,j})$ is the extended Cartan matrix. Now let ξ be defined by

$$\xi = \sum_0^r n_i' \xi_i.$$

The inequalities suggest that

$$\|\xi\|^2 \leq 2,$$

which implies that ξ is a root.

Let $\pi_B : E(\Gamma) \to \tilde{E}$ be the projection from $E(\Gamma)$ to $\tilde{E}$. We then have that π_B induces an element $\tilde{\pi} \in \Omega^0(S^2/\Gamma; \mathbf{f})$ such that $\tilde{\pi}(x) \in End(R)$ is given by

$$\tilde{\pi}(x) : R_x \to R_x',$$

where R_x is isomorphic to R, and R_x' is a subrepresentation of R_x which is also isomorphic to R', for all x. Notice that, $\tilde{\pi}$ is identified with $\kappa \cdot \xi = \kappa\xi\kappa^{-1} = \xi$ under l, as ξ is in the center.

We have that $\tilde{\pi}$ acts trivially on Θ, that is, $[\tilde{\pi}, \Theta] = 0$, as it is the identity on $\tilde{E}$. Now consider $\tilde{\zeta}(\tilde{\pi})$. We compute $\tilde{\zeta}_1(\tilde{\pi})$ here:

$$\tilde{\zeta}_1(\tilde{\pi}) = \int_{S^2/\Gamma} \mathrm{Tr}(\tilde{\pi} F_B) - \frac{i}{2} \int_{S^2/\Gamma} \mathrm{Tr}(\tilde{\pi}[\Theta, \Theta^*])\omega_{vol}.$$

We know that $\int_{S^2/\Gamma} \mathrm{Tr}(F_{A_0+B}) = \int_{S^2/\Gamma} \mathrm{Tr}(F_{A_0}) + \mathrm{Tr}(F_B) = \frac{i}{2\pi} \cdot c_1(E(\Gamma))$. By construction, the integral of $c_1(E(\Gamma))$ concentrates on A_0, that is, $\int_{S^2/\Gamma} \mathrm{Tr}(F_{A_0}) = \frac{i}{2\pi} \cdot c_1(E(\Gamma))$. Hence, we have that $\int_{S^2/\Gamma} \mathrm{Tr}(F_B) = 0$. Since $\tilde{E}$ is a holomorphic subbundle of $E(\Gamma)$ and $\tilde{\pi} F_B$ is the projection of F_B onto $\tilde{E}$, we must have that on the subbundle $\tilde{E}$,

$$\int_{S^2/\Gamma} \mathrm{Tr}(\tilde{\pi} F_{A_0+B}) = \int_{S^2/\Gamma} \mathrm{Tr}(\tilde{\pi} F_{A_0}) + \mathrm{Tr}(\tilde{\pi} F_B) = \frac{i}{2\pi} \cdot c_1(\tilde{E})$$

$$= \int_{S^2/\Gamma} \mathrm{Tr}(\tilde{\pi} F_{A_0}).$$

Hence, $\int_{S^2/\Gamma} \mathrm{Tr}(\tilde{\pi} F_B) = 0$.

We have shown that the first integrand is 0. On the other hand, since $[\tilde{\pi}, \Theta] = 0$, we have

$$\mathrm{Tr}(\tilde{\pi}[\Theta, \Theta^*]) = \mathrm{Tr}(\tilde{\pi}\Theta\Theta^* - \tilde{\pi}\Theta^*\Theta)$$

$$= \mathrm{Tr}(\tilde{\pi}\Theta\Theta^* - \Theta\tilde{\pi}\Theta^*) = 0.$$

Hence, $\tilde{\zeta}_1(\tilde{\pi}) = 0$, that is to say, $\tilde{\zeta}_1 \in \tilde{D}_\xi$. Similarly, $\tilde{\zeta}_2(\tilde{\pi}) = \tilde{\zeta}_3(\tilde{\pi}) = 0$. As a result, we have $\tilde{\zeta} \in \mathbb{R}^3 \otimes \tilde{D}_\xi$.

□

Corollary 8.7.2. *For ζ not lying in D_ξ as in [21] and $\tilde{\zeta} = -\zeta$ thought of as a constant element in $\Omega^2(S^2/\Gamma; Z)$, $\mathcal{G}_\tau^{F,\Gamma}$ acts freely on $\tilde{\mu}^{-1}(\tilde{\zeta})$.*

Proof. If ζ doesn't lie in D_ξ as in [21], then $\tilde{\zeta} = -\zeta$ thought of as a constant element in $\Omega^2(S^2/\Gamma; Z)$ doesn't lie in $\tilde{D}_\xi$. Hence, by the previous lemma, $\mathcal{G}_\tau^{F,\Gamma}$ acts freely on $\tilde{\mu}^{-1}(\tilde{\zeta})$. □

8.7.2 Proof of Theorem 8.3.8 Part I

In this subsection, we prove one direction of Theorem 8.3.8 where we show the moduli space obtained by the gauge-theoretic construction contains the 4-dimensional hyperkähler ALE space given by Kronheimer's construction. To do this, we first explicitly identify certain solutions to the equations given previously with solutions to the equations given in Kronheimer's work and hence showing that the moduli space contains the corresponding 4-dimensional hyperkähler ALE space. Then by the uniqueness results, smoothness results and dimension calculations, we conclude that there cannot be any additional solutions other than the ones corresponding to the points of the 4-dimensional hyperkähler ALE space. Hence, we identify the moduli space with a 4-dimensional hyperkähler ALE space.

Proof of Theorem 8.3.8 Part I.

Lemma 8.7.3. *For $\tilde{\zeta} = \zeta^* = -\zeta$, there is a map $\Phi : X_\zeta \to \mathcal{X}_{\tilde{\zeta}}$ which is an embedding and there is a natural choice of metric on $\mathcal{X}_{\tilde{\zeta}}$ such that Φ is an isometry onto its image.*

Proof. We set $B = 0$, then the equations reduce to the following:

$$\bar{\partial}_{A_0}\Theta = 0,$$

$$-\frac{i}{2}[\Theta, \Theta^*]\omega_{vol} = \tilde{\zeta}_1 \cdot \omega_{vol} = -\zeta_1 \cdot \omega_{vol},$$

$$-\frac{1}{4}([J\Theta, \Theta^*] - [\Theta, J\Theta^*])\omega_{vol} = \tilde{\zeta}_2 \cdot \omega_{vol} = -\zeta_2 \cdot \omega_{vol},$$

$$-\frac{i}{4}([J\Theta, \Theta^*] + [\Theta, J\Theta^*])\omega_{vol} = \tilde{\zeta}_3 \cdot \omega_{vol} = -\zeta_3 \cdot \omega_{vol}.$$

Now since in this case, we can think of Θ as a pair of matrices (α, β), the equations can be further rewritten as the following (here we are implictly dropping the volume 2-form on both sides):

$$\frac{i}{2}([\alpha, \alpha^*] + [\beta, \beta^*]) = \zeta_1$$

$$\frac{1}{2}([\alpha, \beta] + [\alpha^*, \beta^*]) = \zeta_2$$

$$\frac{i}{2}([\alpha, \beta] - [\alpha^*, \beta^*]) = \zeta_3.$$

These are precisely Kronheimer's moment map equations and hence by the results of Kronheimer, and we get a solution to the equations. By Lemma 8.5.10, we know that if a $\mathcal{G}_{\tau,\mathbb{C}}^{F,\Gamma}$-orbit contains a solution coming from X_ζ, it is also the unique solution on that orbit. On the other hand, we also want to argue that two distinct solutions coming from X_ζ will remain distinct in the new moduli space. Suppose there are two solutions coming from X_ζ that become identified by $\mathcal{G}_\tau^{F,\Gamma}$, then they must lie on the same $\mathcal{G}_{\tau,\mathbb{C}}^{F,\Gamma}$-orbit as well. Recall that we have

$$\{(A_0 + B, \Theta) \in \mathcal{A}_\tau^F \times C^\infty(E(\Gamma)) | \bar{\partial}_{A_0+B}\Theta = 0\}/\mathcal{G}_{\tau,0,\mathbb{C}}^{F,\Gamma} \cong M.$$

Hence, two solutions lie on the same $\mathcal{G}_{\tau,\mathbb{C}}^{F,\Gamma}$-orbit if and only if they also lie on the same F^c-orbit, which would imply that they are also on the same F-orbit. Hence, we define Φ to be the bottom horizontal map that makes the following diagram commute:

$$\begin{tikzcd}
M \arrow[r, "\Psi"] & \mathcal{A}_\tau^F \times C^\infty(E(\Gamma)) \\
\mu^{-1}(\zeta) \arrow[u, "\iota"] \arrow[r, "\Psi|_{\mu^{-1}(\zeta)}"] \arrow[d, "proj"] & \tilde{\mu}^{-1}(\tilde{\zeta}) \arrow[u, "\iota"'] \arrow[d, "proj"] \\
X_\zeta = \mu^{-1}(\zeta)/F \arrow[r, "\Phi"] & \mathcal{X}_{\tilde{\zeta}} = \tilde{\mu}^{-1}(\tilde{\zeta})/\mathcal{G}^{F,\Gamma}
\end{tikzcd}$$

That Φ can be regarded as an isometry onto its image comes from the fact that $\Psi|_{\mu^{-1}(\zeta)}$ is naturally an isometry onto its image, and we can define a metric on $\mathcal{X}_{\tilde{\zeta}}$ as follows: for $[(B_1, \Theta_1)], [(B_2, \Theta_2)] \in \mathrm{im}(\Phi)$, define

$$d([(B_1, \Theta_1)], [(B_2, \Theta_2)]) = (\inf_{f \in F} \int_{S^2/\Gamma} Re\langle f\Theta_1' f^{-1} - \Theta_2', f\Theta_1' f^{-1} - \Theta_2' \rangle \omega_{vol})^{\frac{1}{2}},$$

where Θ_1', Θ_2' are such that for some $\rho_1, \rho_2 \in \mathcal{G}_\tau^{F,\Gamma}$, we have $\rho_1 \cdot (B_1, \Theta_1) = (0, \Theta_1')$ as well as $\rho_2 \cdot (B_2, \Theta_2) = (0, \Theta_2')$. We see that d is well-defined on the image of Φ, and that Φ is an isometry onto its image. $\square$

□

8.7.3 Proof of Theorem 8.3.8 Part II

In this subsection, we prove the other direction of Theorem 8.3.8, that is, we show that the moduli space $\mathcal{X}_{\tilde{\zeta}}$ obtained by the gauge-theoretic construction is indeed equal to the 4-dimensional hyperkähler ALE space X_ζ given by Kronheimer's construction in [21]. To this end, we first prove the following lemma.

Lemma 8.7.4. *The complement of X_ζ contained in the gauge-theoretic moduli space $\mathcal{X}_{\tilde{\zeta}}$ is of higher codimension.*

Proof. First, in the setup of [21], by result of Kirwan [18] as cited also in [21], a stable orbit (closed and of maximal dimension) of M under the action of F^c contains a solution to the equation $\frac{i}{2}([\alpha,\alpha^*]+[\beta,\beta^*])=0$. Now, for any choice of ζ_1, since $|\mu_1-\zeta_1|^2$ is proper on the F^c-orbit containing a solution to $\frac{i}{2}([\alpha,\alpha^*]+[\beta,\beta^*])=0$, and F/T acts freely on a stable orbit, we have that the complex orbit also contains a solution to $\frac{i}{2}([\alpha,\alpha^*]+[\beta,\beta^*])=\zeta_1$. As the stable orbits are open and dense, the F^c-orbits not containing a solution to $\frac{i}{2}([\alpha,\alpha^*]+[\beta,\beta^*])=\zeta_1$ is of higher codimension.

On the other hand, a solution in $\mathcal{X}_{\tilde{\zeta}}$ that does not a priori come from a solution in X_ζ must have the form (B,Θ) with B not $\mathcal{G}_\tau^{F,\Gamma}$-equivalent to 0. Hence, it lies in a different connected component from the one containing the solutions coming from X_ζ and is contained in a non-stable orbit of M when we identify the F^c-obits of M in [21] with the $\mathcal{G}_{\tau,\mathbb{C}}^{F,\Gamma}$-orbits of $\mathcal{C}$ by Lemma 8.5.7 and Remark 8.5.8. This tells us that the $\mathcal{G}_{\tau,\mathbb{C}}^{F,\Gamma}$-orbits that do not a priori contain a solution coming from Kronheimer's construction must be of higher codimension in the moduli space.

□

Proof of main theorem Part II. We want to argue that there are no additional solutions in the gauge-theoretic moduli space $\mathcal{X}_{\tilde{\zeta}}$ than the solutions coming from X_ζ in [21]. We know if the gauge group acts freely at a solution, then it must come from Kronheimer's construction, by the previous lemma and dimension calculations. But by Lemma 8.7.1, we know that the gauge group $\mathcal{G}_{\tau}^{F,\Gamma}$ acts freely on the space of solutions when ζ not lying in D_{ξ}, which is precisely the assumption we have. Hence, all the solutions in $\mathcal{X}_{\tilde{\zeta}}$ must come from X_{ζ}. Hence, they are equal, and $\Phi : X_{\zeta} \to \mathcal{X}_{\tilde{\zeta}}$ is an isometry. □

We have concluded the proof of the main theorem, and we will end this section by providing the proof of Proposition 8.3.4.

Proof of Proposition 8.3.4. This proof follows essentially the same arguments as those of the proof of Theorem 8.3.8. First, observe that 8.3.1 and 8.3.2 reduce to $\frac{i}{2}([\alpha, \alpha^*] + [\beta, \beta^*]) = \zeta_1$ when we set $B = 0$. Hence, by Lemma 8.5.7 and 8.5.9, we again have that the space of solutions satisfying $\frac{i}{2}([\alpha, \alpha^*] + [\beta, \beta^*]) = \zeta_1$ lies inside $\mathcal{M}(\Gamma, \tilde{\zeta}_1)$ as a subset. Since we assume that we are choosing $\tilde{\zeta}_1$ such that the action of the gauge group $\mathcal{G}^{F,\Gamma}$ on the space of solutions to 8.3.1 and 8.3.2 is free, we then know that $\mathcal{M}(\Gamma, \tilde{\zeta}_1)$ is smooth. Hence again, by Proposition 6.4, we know that there cannot be any additional solutions in $\mathcal{M}(\Gamma, \tilde{\zeta}_1)$, and we get the desired conclusion. □

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