Coordinated Control of Multi-Agent Systems in the Presence of Communication Delays

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Abstract

Over the past decades, coordinated control of multi-agent systems has received increasing attention for its potential in applications such as cooperation of robots, coordination of unmanned air vehicles, management of distributed database and synchronization of networked oscillators.

Flocking behavior is said to be achieved if both position aggregation and heading alignment are achieved. Collision avoidance is an additional control objective in most flocking applications involving physical subjects. Formation control is a widely used method to create flocking behavior in a multi-agent system. Since formation control requires a virtual leader and a predefined geometric configuration of the flock, the resulting closed-loop multi-agent system is sensitive to agent ordering and individual agent failure. An alternative approach to enforcing the aggregation of agents is to define an artificial potential function. The artificial potential function determines the attractive-repulsive interaction between the agents. Then, a control law based on the gradient of the potential function drives the system into a desired configuration. In this case, no a priori knowledge is needed and the closed-loop system is more robust.

Consensus control is another important problem in coordinated control. It is concerned with reaching a networkwide agreement on some quantities of interest while each agent can only access local information. Many of the existing studies address the consensus of multi-agent systems with linear dynamics, and both linear and nonlinear controllers have been proposed.

Regardless of the control algorithm employed, coordinated control highly relies on interactions among agents and for this reason communication delays are inevitable and should be taken into consideration during the development of the control algorithm. Control of indi-

vidual dynamic systems with time delays has been studied extensively in the literature. In particular, the low gain feedback method has been demonstrated to be effective in developing control laws for the stabilization of linear and nonlinear delayed systems.

In this dissertation, we firstly consider the flocking of nonholonomic vehicles in the presence of communication delays. In both continuous-time and discrete-time scenarios, distributed control laws are developed based on artificial potential functions. Aggregation of positions and alignment of headings are proved separately through the Lyapunov functional approach. Then, we study the consensus of a class of nonlinear affine systems in the presence of communication delays. In both continuous-time and discrete-time settings, distributed control laws are constructed and consensus is proved as well.

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Contents

1 Introduction **1**

1.1 Overview of multi-agent systems 1

1.2 Flocking control of multi-agent systems 2

1.3 Consensus control of multi-agent systems 3

1.4 Time-delay systems 4

1.5 Research objectives 4

1.6 Dissertation outline 5

2 Fundamentals **7**

2.1 Graph theory 7

2.2 Potential function based flocking control 10

2.3 Lie algebra 13

3 Flocking of Nonholonomic Vehicles in the Continuous-time Setting **14**

3.1 Problem statement 14

3.2 Control protocols 16

3.3 Analysis of flocking behavior 17

3.4 Simulation results 31

3.4.1 Flocking with a complete graph 31

3.4.2 Flocking with a disconnected graph 33

3.4.3 Collisions in systems without complete graphs 33

3.5 Summary 35

4 Flocking of Nonholonomic Vehicles in the Discrete-time Setting 36

4.1 Problem statement 36

4.2 Control protocols 37

4.3 Analysis of flocking behavior 38

4.4 Simulation results 51

4.4.1 Flocking without delays 51

4.4.2 Flocking with delays 53

4.5 Summary 54

5 Consensus of Nonlinear Multi-agent Systems in the Continuous-time Setting 55

5.1 Problem statement 55

5.2 Consensus in the absence of communication delays 58

5.3 Consensus in the presence of communication delays 64

5.4 Simulation results 70

5.4.1 Example 1: Linear unforced dynamics without delays 70

5.4.2 Example 2: Nonlinear unforced dynamics without delays 71

5.4.3 Example 3: Nonlinear unforced dynamics with delays 72

5.4.4 Example 4: Consensus under actuator saturation 73

5.5 Summary 74

6 Consensus of Nonlinear Multi-agent Systems in the Discrete-time Setting 75

6.1 Problem statement 75

6.2 Consensus in the absence of communication delays 76

6.3 Consensus in the presence of communication delays 80

6.4 Simulation results 92

6.4.1 Example 1: Consensus without communication delays 92

6.4.2 Example 2: Consensus under actuator saturation 93

6.4.3 Example 3: Consensus with communication delays 94

6.5 Summary 95

7 Conclusions **96**

List of Figures

2.1 Attractive and repulsive forces 11

2.2 Artificial potential function 13

3.1 Dynamics of agent i 15

3.2 Decomposition of artificial force F_i 16

3.3 Simulation with a complete graph, $\tau = 0$ s, $\varepsilon = 1$. 31

3.4 Simulation with a complete graph, $\tau = 0.1$ s, $\varepsilon = 1$. 32

3.5 Simulation with a complete graph, $\tau = 0.1$ s, $\varepsilon = 0.05$. 32

3.6 A disconnected graph. 33

3.7 Simulation with a disconnected graph, $\tau = 0.1$ s, $\varepsilon = 0.05$. 33

3.8 A connected but incomplete graph. 34

3.9 Simulation with a connected but incomplete graph, $\tau = 0.1$ s, $\varepsilon = 0.05$. . . . 34

4.1 Saturated angular velocity. 37

4.2 A connected but incomplete graph. 51

4.3 Simulation with a connected but incomplete graph, $\tau = 0$, $\varepsilon = 0.8$. 52

4.4 Simulation with a connected but incomplete graph, $\tau = 0$, $\varepsilon = 0.05$. 52

4.5 Simulation with a connected but incomplete graph, $\tau = 4$, $\varepsilon = 0.05$. 53

4.6 Simulation with a connected but incomplete graph, $\tau = 4$, $\varepsilon = 0.015$. 53

5.1 The communication topology 70

5.2 Example 1: Linear unforced dynamics, $\tau = 0$ 71

5.3 Example 2: Nonlinear unforced dynamics, $\tau = 0$ 72

5.4 Example 3: Nonlinear unforced dynamics, $\tau = 0.1$, $\varepsilon = 0.5$ 73

5.5 Example 4: Consensus under actuator saturation, $\tau = 0$ 73

6.1 The communication topology 92
6.2 Example 1: Consensus without communication delays, $\tau = 0$ 93
6.3 Example 2: Consensus under actuator saturation, $\tau = 0$ 93
6.4 Example 3: Consensus with communication delays, $\tau = 1$ 94

Chapter 1

Introduction

1.1 Overview of multi-agent systems

There has been exploding amount of study related to multi-agent systems over the past decades, and one can refer to [1] and [2] for a more comprehensive introduction to the related topics and applications. In this research, we only concern the coordinated control of multi-agent systems. In particular, flocking and consensus problems will be investigated.

A multi-agent system consists of a number of interacting intelligent agents that cooperate to solve a problem or to achieve a common objective. Each intelligent agent in the system must be endowed with autonomy and social ability. That is, each agent must be able to observe the environment and other agents in the system and to react based on such observations. Intelligent agents can be any entities that possess such two properties, such as, unmanned vehicles and robotics.

The most significant feature and the biggest advantage of multi-agent systems is that the capability of the system as a whole can be greater than the sum of individuals' capability. As a result, multi-agent systems can be used to solve large-scale or complicated problems that are not solvable by individual systems.

1.2 Flocking control of multi-agent systems

Collective motion of agents in a large community with a common objective is easy to observe in nature. Many types of mammals, birds and fishes rely on coordinated motion in the form of a flock or swarm for survival. The most basic collective behavior of a flock, which involves the aggregation (or gathering) of the agents and the alignment in their direction of motion, is commonly known as flocking [3].

The study of this topic can be traced back to 1986, when the flocking behavior was first simulated by Craig Reynolds with his computer program "Boids" [4]. Craig Reynolds has proposed three simple but crucial steering rules,

- Cohesion: to stay close to neighbor agents;
- Separation: to avoid collisions with neighbor agents;
- Alignment: to match heading with neighbor agents.

Craig Reynolds' rules describe how each agent moves according to the positions and headings of its nearby agents and provide guidelines to much of the later research.

Different approaches for flocking control have been proposed by scientists and engineers according to different application background. One approach to achieving flocking is by specifying the formation geometry of the flock a priori. A control law then brings the agents to their predefined relative positions. This method has been widely employed in applications where maintaining a specific geometric configuration helps reduce the system cost or increase the capability of system, such as search and rescue in large-scale disasters, clustering of small satellites and security patrols. Many authors have explored the formation control method in their work. A model-independent approach is presented in [5] by decoupling the formation problem into the trajectory tracking of each agent. The stable formation of multi-agent systems with communication delays was studied in [6] for linear systems, and in [7, 8] for a class of nonlinear agents. Obstacle avoidance and vehicles following in traffic are studied in [9]. In [10], obstacle avoidance is achieved in flocking by switching the formation pattern. Despite of the advantages of the formation control method, in many cases, defining the formation of a multi-agent system may require a specific ordering of the agents, which can negatively affect the robustness of the closed-loop system to instances of individual agent failure.

An alternative approach to enforcing the aggregation of the agents is to define an artificial potential function. The artificial potential function determines the attractive-repulsive interaction between the agents. A control law based on the gradient of the potential function drives the system into a formation corresponding to minimum energy, which together with connected topologies imply aggregated positions. The flocking behavior of agents that are described by linear systems was studied in [11, 12, 13]. For systems consisting of a class of nonholonomic agents, reference [14] provides the theoretical justification for the flocking behavior in the delay-free scenarios. In [15], flocking with obstacle avoidance is studied by constructing artificial repulsive forces associated with obstacles. In the design of potential function based control laws, it is not necessary to define a specific formation geometry for the system, which makes the closed-loop system much more robust to individual agent failure.

1.3 Consensus control of multi-agent systems

Consensus control is another important problem in coordinated control. Many algorithms used to solve other coordination problems, including formation control and swarm tracking, origin from consensus protocols. Consensus control is concerned with reaching a network-wide agreement on some quantities of interest while each agent can only access local information. Applications of consensus control include wireless sensor networks, management of distributed database and synchronization of networked oscillators.

The consensus control problem of multi-agent systems with linear models has been intensively studied by researchers. Reference [16] studied a double integer model and delayed directed networks. Consensus of higher order multi-agent systems has been studied in [17] and [18]. In [17], a truncated predictor feedback based protocol was developed for multi-agent systems with bounded communication delays. Their further analysis showed that the closed-loop system tolerates arbitrarily large communication delays. The work conducted in [18] also allowed for internal uncertainties and external disturbance.

More recently, there has been a surge of interest in consensus of nonlinear multi-agent systems. In [19], second order consensus problem was studied for multi-agent systems with a class of affine nonlinear dynamics. Reference [20] addressed leader-follower consensus of multi-agent systems whose dynamics are given in a normal form with uncertainties. Two neural networks were employed to estimate nonlinearities in the system. In [21], the author investigated output consensus of a class of affine nonlinear systems that possess input-output passivity. It was demonstrated that output consensus is guaranteed if the storage function of the system is positive definite and radially unbounded.

1.4 Time-delay systems

As mentioned earlier in Sections 1.2 and 1.3, both flocking control and consensus control require collecting information from neighboring agents, which is subject to communication delays. It is well known that time-delays cause problems such as instability. Hence, when we design control protocols for either flocking or consensus control, communication delays should be considered and actions should be taken to eliminate the negative effects of communication delays.

Many methods have been developed for the control of individual dynamic systems with time delays. Introductions to the related issues for linear and nonlinear systems, and some examples of recent advances can be found in [22, 23, 24] and the references therein. In particular, the low gain feedback method [25] has been demonstrated to be effective in developing control laws for the stabilization of linear and nonlinear delayed systems. For example, Lin and Fang developed in [26] a low gain control approach for the stabilization of a class of linear systems with constant input delay. Zhou *et al.* extended in [27] the low gain control approach for the stabilization of linear systems with time-varying input delay, and Yoon and Lin in [28] studied the stabilization of exponentially unstable linear systems with time-varying input delay.

1.5 Research objectives

A summary of some existing research and problems that remain unresolved is as follows.

1. Flocking of nonholonomic vehicles based on artificial potentials has been studied in [14], but communication delays were not considered.

2. State consensus of multi-agent systems has been studied for some nonlinear models but not for the class of general nonlinear systems described by

$$\dot{z}_i = f(z_i) + g(z_i)u_i, \ i = 1, 2, \ldots, N, \tag{1.1}$$

   where $z_i \in \mathbb{R}^n$ is the state vector, $u_i \in \mathbb{R}^m$ is the control input, and $g(z_i) = [g_1(z_i), g_2(z_i), \ldots, g_m(z_i)] \in \mathbb{R}^{n\times m}$. $f(z_i)$ and $g_k(z_i)$, $k = 1, 2, \ldots, m$, are smooth vector fields in $\mathbb{R}^n$.

3. In many cases, controllers operate in discrete-time settings. However, as observed in [29, 30], applications of continuous-time controllers through direct discretization could be very restrictive due to the requirement for small sampling periods. For application purpose, for both flocking and consensus problems, control algorithms should be designed in discrete-time settings as well.

In view of the problems summarized in the above list, in this research we aimed to:

1. Design control protocols for achieving flocking of nonholonomic vehicles based on potential functions and in consideration of communication delays. A low gain method is developed.

2. Design control laws for the consensus of nonlinear multi-agent system (1.1). We first study the delay-free scenario and then consider cases with communication delays.

3. For both flocking and consensus problems, control laws are constructed in discrete-time settings as well.

1.6 Dissertation outline

The remainder of this dissertation will be organized as follows. In Chapter 2, mathematical tools and concepts are introduced. Chapters 3 and 4 provide our solutions to the flocking of nonholonomic vehicles in the continuous-time and discrete-time setting, respectively.

Chapters 5 and 6 consider a class of nonlinear multi-agent system in continuous-time and discrete-time, respectively. Distributed control laws are constructed that drive the system into consensus. Finally, conclusions regarding this work are drawn in Chapter 7, where possible future works are proposed as well. The results reported in this dissertation have been published in [31, 32, 33, 34].

Chapter 2

Fundamentals

2.1 Graph theory

Coordinated control of multi-agent systems, including flocking and consensus, requires the exchange of information among the agents. In this work, we represent the communication network among the agents by an undirected graph.

Definition 1. *An undirected graph* $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ *consists of a nonempty set of nodes* $\mathcal{V} = \{v_1, v_2, \ldots, v_N\}$*, and a set of edges* $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$*. The unordered pair* $(v_i, v_j) \in \mathcal{E}$ *if and only if there exists a bidirectional communication link between node* v_i *and node* v_j*. Let* $\mathcal{N}_i$ *denote the index set of the neighboring nodes of* v_i*, that is,*

$$\mathcal{N}_i = \{1 \leq j \leq N : (v_i, v_j) \in \mathcal{E}\}.$$

Definition 2. *An undirected graph* $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ *is said to be complete if* $(v_i, v_j) \in \mathcal{E}$ *for any* $v_i, v_j \in \mathcal{E}$ *and* $v_i \neq v_j$*. This means that there is an undirected edge between any pair of nodes.*

Definition 3. *In an undirected graph, a path is defined as a sequence of nodes in which any two consecutive nodes are linked by an edge. An undirected graph* $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ *is said to be connected if there exists a path between any* $v_i, v_j \in \mathcal{E}$ *and* $v_i \neq v_j$*.*

Definition 4. *The topology of a graph* $\mathcal{G}$ *can be described by an associated adjacency matrix*

$A_{adj} = [a_{ij}] \in \mathbb{R}^{N\times N}$, *which is defined as*

$$a_{ij} = \begin{cases} 1, & \text{if } (v_i, v_j) \in \mathcal{E}, \\ 0, & \text{otherwise.} \end{cases}$$

We define the Laplacian matrix of $\mathcal{G}$ *as* $L = D - A_{adj}$, *where* $D = \mathrm{diag}\{d_1, d_2, \ldots, d_N\}$ *and* $d_i = \sum_{j=1}^{N} a_{ij}, i = 1, 2, \ldots, N$. *For an undirected graph,* $L = L^T$.

Without loss of generality, we assume that the communication network does not allow self-loops. In other words, $(v_i, v_i) \notin \mathcal{E}$, and $a_{ii} = 0$, for $i = 1, 2, \ldots, N$, in Chapters 3 and 4. The reason for such an assumption will be explained in the next section.

The following lemmas state some useful properties of the Laplacian matrix.

Lemma 1. *[35] The Laplacian matrix* L *of an undirected graph is positive semi-definite.*

Lemma 2. *[36] An undirected graph is connected if and only if zero is a simple eigenvalue of the Laplacian matrix* L *with* $\bar{\mathbf{1}}_N = [1\ 1\ \ldots\ 1]^T \in \mathbb{R}^N$ *being the only corresponding eigenvector.*

Lemma 3. *Let* $L = L^T$ *be the Laplacian matrix of an undirected graph* $\mathcal{G}$. *For any*

$$\upsilon = \begin{bmatrix} \upsilon_1 \\ \upsilon_2 \\ \vdots \\ \upsilon_N \end{bmatrix} \in \mathbb{R}^{Nn}, \quad \omega = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_N \end{bmatrix} \in \mathbb{R}^{Nn},$$

we have

$$2\upsilon^T (L \otimes I_n)\omega = \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}(\upsilon_i - \upsilon_j)^T(\omega_i - \omega_j),$$

where a_{ij} *is the entry of the adjacency matrix of the undirected graph* $\mathcal{G}$.

Proof. By the definition of the Laplacian matrix and straightforward calculations, we obtain

$$\begin{aligned} \upsilon^{\mathrm{T}}(L \otimes I_n)\omega &= \sum_{i=1}^{N} \upsilon_i^{\mathrm{T}} \sum_{j=1}^{N} a_{ij}(\omega_i - \omega_j) \\ &= \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\upsilon_i^{\mathrm{T}}(\omega_i - \omega_j). \end{aligned} \tag{2.1}$$

For an undirected graph, $a_{ij} = a_{ji}$. Thus,

$$\upsilon^{\mathrm{T}}(L \otimes I_n)\omega = \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ji}\upsilon_i^{\mathrm{T}}(\omega_i - \omega_j).$$

Reordering and renaming the summation indices, we have

$$\upsilon^{\mathrm{T}}(L \otimes I_n)\omega = \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\upsilon_j^{\mathrm{T}}(\omega_j - \omega_i). \tag{2.2}$$

Adding both sides of (2.1) and (2.2) gives

$$\begin{aligned} 2\upsilon^{\mathrm{T}}(L \otimes I_n)\omega &= \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\upsilon_i^{\mathrm{T}}(\omega_i - \omega_j) + \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\upsilon_j^{\mathrm{T}}(\omega_j - \omega_i) \\ &= \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}(\upsilon_i - \upsilon_j)^{\mathrm{T}}(\omega_i - \omega_j). \end{aligned}$$

□

Lemma 4. *Let $L = L^T$ be the Laplacian matrix and D be the corresponding degree matrix as defined in Definition 4. Then,*

$$L \leq 2D.$$

Proof. By the definition of Laplacian matrix, we have

$$L - 2D = -D - A.$$

For any $z \in \mathbb{R}^n$,

$$\begin{aligned} z^{\mathrm{T}}(-D - A)z &= \sum_{i=1}^{N} z_i \sum_{j=1}^{N} a_{ij}(-z_i - z_j) \\ &= \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}z_i(-z_i - z_j). \end{aligned} \tag{2.3}$$

Reordering and renaming the summation indices, we have

$$z^{\mathrm{T}}(-D - A)z = \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ji}z_j(-z_i - z_j).$$

For an undirected graph, $a_{ij} = a_{ji}$. Thus,

$$z^{\mathrm{T}}(-D - A)z = \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}z_j(-z_i - z_j). \tag{2.4}$$

Adding both sides of (2.3) and (2.4) gives

$$\begin{aligned} z^{\mathrm{T}}(-D-A)z &= -\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}(z_i+z_j)^2 \\ &\le 0. \end{aligned}$$

Therefore, $-D-A \le 0$, which implies that $L \le 2D$. □

Lemma 5. *Let $L = L^T \in \mathbb{R}^{N\times N}$ be the Laplacian matrix of an undirected graph $\mathcal{G}$. Then, $L^2 \le 4d_{\max}^2 I_N$, where $d_{\max} = \max_{i=1,2,\ldots,N}\{d_i\}$ denotes the maximum degree of the communication graph..*

Proof. For any $x \in \mathbb{R}^N$, by the definition of Laplacian matrix, we have

$$\begin{aligned} x^{\mathrm{T}}L^2x &= \sum_{i=1}^{N}\left(\sum_{j=1}^{N} a_{ij}(x_i-x_j)\right)^2 \\ &\le d_{\max}\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}(x_i-x_j)^2 \\ &= 2d_{\max}x^{\mathrm{T}}Lx, \end{aligned} \tag{2.5}$$

where the second and the third lines are obtained by applying Chebyshev's sum inequality and Lemma 3, respectively.

By Lemma 4, it follows from (2.5) that

$$\begin{aligned} x^{\mathrm{T}}L^2x &\le 4d_{\max}x^{\mathrm{T}}Dx \\ &\le 4d_{\max}^2 x^{\mathrm{T}}x, \end{aligned}$$

which completes the proof. □

2.2 Potential function based flocking control

To explain the underlying scheme of our control algorithm, we start with the simplest case and only consider the interaction between agents i and j. Suppose the two agents are steered by a pair of attractive and repulsive forces which are solely dependent on the distance between two agents. As in physics, their potential is given by the integration of force and conversely force can be calculated from the potential function. If we properly design a potential function, the corresponding attractive and repulsive forces will drive the two agents to a desired configuration.

Definition 5. *The artificial potential function between agents i and j is*

$$V_{ij}(\|r_{ij}\|) = V_a(\|r_{ij}\|) + V_r(\|r_{ij}\|), \tag{2.6}$$

where r_{ij} is the relative position between agents i and j. $V_a(\|r_{ij}\|)$ and $V_r(\|r_{ij}\|)$ are respectively the attractive and repulsive potential components which should be designed such that

- *$V_a(\|r_{ij}\|)$ is a strictly increasing function and has a minimum at $\|r_{ij}\| = 0$;*
- *$V_r(\|r_{ij}\|)$ is a strictly decreasing function and approaches its minimum as $\|r_{ij}\| \to \infty$;*
- *the combined potential $V_a(\|r_{ij}\|) + V_r(\|r_{ij}\|)$ does not have any local maximum and reaches its unique minimum at $\|r_{ij}\| = d$, for some $d > 0$.*

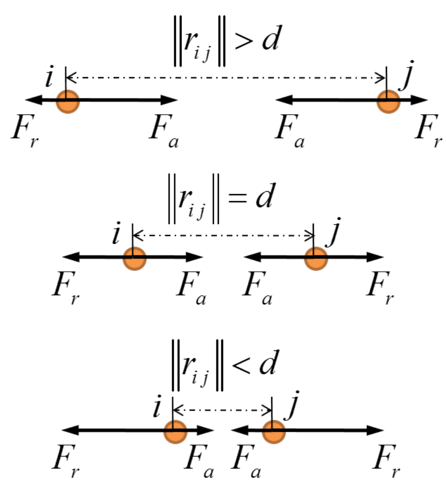


Figure 2.1: Attractive and repulsive forces

Suppose that agent i is driven by a virtual force

$$\begin{aligned} F_{ij} &= -\nabla_{r_i} V_{ij}(\|r_{ij}\|) \\ &= -\left(\nabla_{r_i} V_a(\|r_{ij}\|) + \nabla_{r_i} V_r(\|r_{ij}\|)\right) \\ &= -r_{ij}\left(g_a(\|r_{ij}\|) + g_r(\|r_{ij}\|)\right). \end{aligned} \tag{2.7}$$

The positive valued function $g_a(\|r_{ij}\|)$ and negative valued function $g_r(\|r_{ij}\|)$ represent attraction and repulsion term respectively and balance at the unique distance $\|r_{ij}\| = d$, as illustrated in Figure 2.1. When $\|r_{ij}\| > d$, $g_a(\|r_{ij}\|)$ dominates and the two agents move towards each other. On the other hand, $g_r(\|r_{ij}\|)$ dominates when $\|r_{ij}\| < d$, and then the two agents diverge. Consequently, agents i and j maintain a distance of d.

For a system composed of $N > 2$ agents, the potential of each agent is given by

$$\begin{aligned} V_i &= \sum_{j \in \mathcal{N}_i} V_{ij}(\|r_{ij}\|) \\ &= \sum_{j \in \mathcal{N}_i} V_a(\|r_{ij}\|) + V_r(\|r_{ij}\|). \end{aligned} \tag{2.8}$$

Hence we obtain the summation of virtual forces acting on each agent as

$$\begin{aligned} F_i &= -\nabla_{r_i} V_i \\ &= -\sum_{j \in \mathcal{N}_i} r_{ij} \left(g_a(\|r_{ij}\|) + g_r(\|r_{ij}\|) \right), \end{aligned} \tag{2.9}$$

which steers the system to a configuration corresponding to a local minimum of $\sum_{i=1}^{N} V_i$. In this case there might exist multiple minima. Let the minima be at $\|r_{ij}\| = d_{ij}$, $j \neq i$. Note that, these d_{ij} might not be identical. Besides, we notice that attractive and repulsive forces only exist between two different and neighboring agents, so self-loops are excluded from our communication network when considering the flocking control problems.

In our design, we select

$$V_{ij}(\|r_{ij}\|) = \rho^2 \left(\frac{1}{\left\| \frac{1}{\rho} r_{ij} \right\|^2} + \ln \left(\left\| \frac{1}{\rho} r_{ij} \right\|^2 \right) \right), \tag{2.10}$$

which is illustrated in Figure 2.2. It can be seen that potential function (2.10) takes its unique minimum at $\|r_{ij}\| = \rho$. Taking gradient yields

$$\begin{aligned} -\nabla_{r_i} V_{ij}(t) &= -2 r_{ij}(t) \Pi_{ij}(t) \\ &\triangleq -2 \Upsilon_{ij}(t), \end{aligned} \tag{2.11}$$

where

$$\Pi_{ij}(t) = \frac{\rho^2}{\|r_{ij}(t)\|^2} - \frac{\rho^4}{\|r_{ij}(t)\|^4}. \tag{2.12}$$

where $\frac{\rho^2}{\|r_{ij}\|^2}$ is the attraction term and $-\frac{\rho^4}{\|r_{ij}\|^4}$ is the repulsion term.

The choice of the artificial potential function that satisfies Definition 5 is not unique.

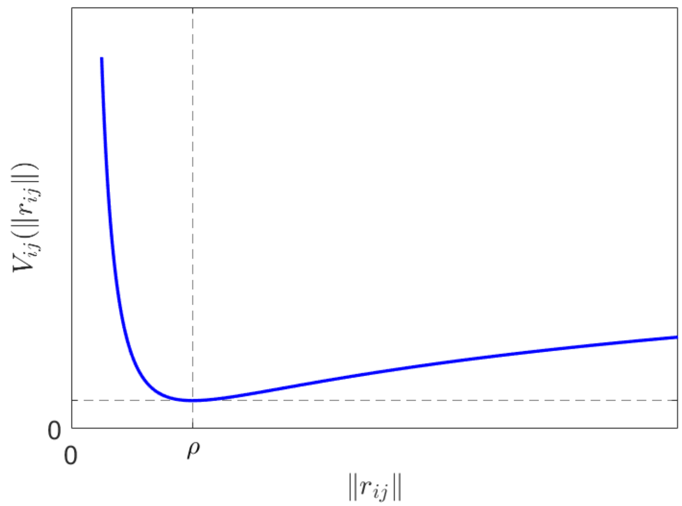


Figure 2.2: Artificial potential function

2.3 Lie algebra

The Lie algebra is useful in the analysis of nonlinear affine systems, and it consists of a vector space and a multiplication on the vector space called "Lie bracket."

Definition 6. *Let $f(z) : \mathbb{R}^n \to \mathbb{R}^n$ and $h(z) : \mathbb{R}^n \to \mathbb{R}^n$ be smooth vector fields. Their Lie bracket is defined by*

$$[f(z), h(z)] = \frac{\partial h}{\partial z} f(z) - \frac{\partial f}{\partial z} h(z).$$

Then, we define following notations inductively

$$ad_f^0 h(z) = h(z),$$
$$ad_f^q h(z) = [f(z), ad_f^{q-1} h(z)],\ q \in \mathbb{N}^+.$$

The Lie derivatives are also defined inductively as

$$L_f^0 h(z) = h(z),$$
$$L_f^q h(z) = \frac{\partial L_f^{q-1} h(z)}{\partial z} f(z),\ q \in \mathbb{N}^+.$$

Chapter 3

Flocking of Nonholonomic Vehicles in the Continuous-time Setting

3.1 Problem statement

Consider a multi-agent system composed of N nonholonomic vehicles, numbered $1, 2, \ldots, N$. The dynamics of agent i are given by [8],

$$\begin{aligned} \dot{x}_i(t) &= v_i(t)\cos\theta_i(t), \\ \dot{y}_i(t) &= v_i(t)\sin\theta_i(t), \\ \dot{\theta}_i(t) &= \omega_i(t). \end{aligned} \tag{3.1}$$

The states and inputs of vehicle i are illustrated in Figure 3.1. The position of agent i is given by the vector $r_i = [x_i,\, y_i]^{\mathrm{T}}$, whereas the orientation is determined by the state θ_i. The control inputs for each agent are the translational velocity $v_i(t)$ and the angular velocity $\omega_i(t)$. For agents i and j, the relative position is determined as

$$r_{ij}(t) = r_i(t) - r_j(t) = \begin{bmatrix} x_i(t) \\ y_i(t) \end{bmatrix} - \begin{bmatrix} x_j(t) \\ y_j(t) \end{bmatrix}. \tag{3.2}$$

The heading error between agent i and agent j is denoted by $e_{ij}(t) = \theta_i(t) - \theta_j(t)$.

We notice that the general control-affine nonlinear model (1.1), considered in Section 1.5,

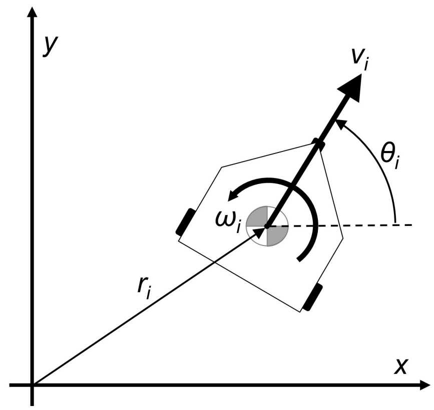


Figure 3.1: Dynamics of agent i

reduces to a nonholonomic vehicle model (3.1) with the following substitutions

$$z_i = \begin{bmatrix} x_i \\ y_i \\ \theta_i \end{bmatrix}, \; u_i = \begin{bmatrix} v_i \\ \omega_i \end{bmatrix}, \; f(z_i) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \; g(z_i) = \begin{bmatrix} \cos\theta_i & 0 \\ \sin\theta_i & 0 \\ 0 & 1 \end{bmatrix}.$$

We are interested in developing a control law, such that the multi-vehicle system (3.1) achieves aggregation of positions and alignment of headings. In addition, collision avoidance is crucial in a system consisting of vehicles. We make the following assumptions regarding the communication network and the initial conditions.

Assumption 1. *The communication topology is described by an undirected and connected graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$, where $\mathcal{V}$ is the indexed set of agents in the system and $\mathcal{E}$ represents the communication links among the agents.*

The assumption of a connected communication graph is necessary to guarantee that all the agents in the system aggregate into a single flock. If the mentioned assumption is not satisfied, then the agents can be divided into smaller groups of connected agents.

Assumption 2. *An agent collects position and heading information of all its neighboring agents with a constant delay $\tau \geq 0$, but knows its own states in real time.*

Assumption 3. *For an arbitrary positive constant c, let the initial conditions of system* (3.1)*, be inside the bounded set*

$$\Omega = \left\{ (r_{ij}, e_{ij}) : \sum_{i=0}^{N} V_i \leq c, \; -2\pi < e_{ij} < 2\pi \right\},$$

where the bound on e_{ij} comes from the condition that $-\pi < \theta_i \le \pi$.

This assumption requires that, before the controllers take effect, no collision occurs and no vehicle is infinitely far from its neighboring vehicles.

3.2 Control protocols

Based on the analysis in Section 2.2, we know that it is feasible to realize cohesion and separation among agents by introducing attractive and repulsive forces. The next step is to integrate these forces into control inputs, the translational velocity v_i and the angular velocity ω_i.

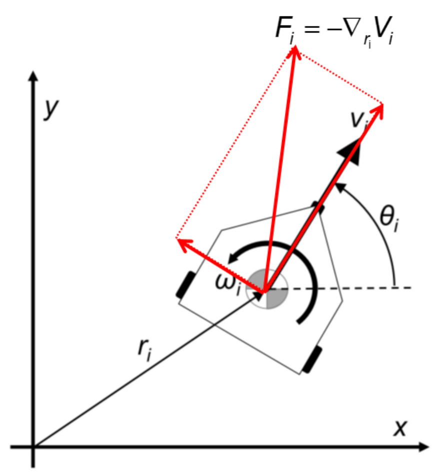


Figure 3.2: Decomposition of artificial force F_i

For any vehicle i, we decompose its steering force F_i along two perpendicular directions as shown in Figure 3.2. The resulting components determine control inputs respectively as

$$\begin{aligned} v_i(t) =& -\varepsilon[\cos\theta_i(t) \;\; \sin\theta_i(t)]\nabla_{r_i} V_i(t-\tau), \\ \omega_i(t) =& -\varepsilon\kappa \sum_{j\in\mathcal{N}_i} e_{ij}(t-\tau) - \varepsilon[-\sin\theta_i(t) \;\; \cos\theta_i(t)]\nabla_{r_i} V_i(t-\tau). \end{aligned} \tag{3.3}$$

The first term in ω_i is introduced for the purpose of heading alignment, and κ is a tunable control parameter. Control inputs are scaled by a low gain parameter $\varepsilon > 0$, in order to compensate the effects of communication delays.

3.3 Analysis of flocking behavior

First, the following lemmas are introduced to help in the derivation of further results. Lemma 6 is a special case of the Leibniz integral rule [37], while Lemmas 7 and 8 are special cases of Chebyshev's sum inequality and Jensen's inequality [38], respectively.

Lemma 6. *Let $f(s)$ be a continuous function in some region of $a(t) \leq s \leq b(t)$ and $t_0 \leq t \leq t_1$. Also suppose that the functions $a(t)$ and $b(t)$ are both continuous and both have continuous derivatives for $t_0 \leq t \leq t_1$. Then for $t_0 \leq t \leq t_1$,*

$$\frac{d}{dt}\int_{a(t)}^{b(t)} f(s)ds = f(b(t))\frac{d}{dt}b(t) - f(a(t))\frac{d}{dt}a(t). \tag{3.4}$$

Lemma 7. *Consider a series of real vectors a_i where $i = 1, 2, \cdots, N$. Then,*

$$\left(\sum_{i=1}^{N} a_i\right)^T \left(\sum_{i=1}^{N} a_i\right) \leq N \sum_{i=1}^{N} a_i^T a_i. \tag{3.5}$$

Lemma 8. *Let a and b be real numbers, and $f(s)$ be an integrable real-valued function. Then*

$$\left(\int_a^b f(s)ds\right)^2 \leq (b-a)\int_a^b f^2(s)ds. \tag{3.6}$$

Lemma 9. *Define a function, in terms of any given integrable function $f(t)$, as follows,*

$$h(t) = \int_0^\tau \int_{t-s_1}^{t} f(s_2)ds_2 ds_1,$$

where $\tau \in \mathbb{R}^+$ is a constant. Then,

$$\frac{\mathrm{d}}{\mathrm{d}t}h(t) = \tau f(t) - \int_{t-\tau}^{t} f(s_2)ds_2.$$

Proof. Switching the order of integration, we can rewrite $h(t)$ as

$$\begin{aligned} h(t) &= \int_{t-\tau}^{t}\int_{t-s_2}^{\tau} f(s_2)ds_1 ds_2 \\ &= \int_{t-\tau}^{t} (\tau - t + s_2) f(s_2) ds_2, \end{aligned}$$

from which we have, by Lemma 6,

$$\frac{\mathrm{d}}{\mathrm{d}t}h(t) = -\int_{t-\tau}^{t} f(s_2)ds_2 + (\tau - t + t)f(t)$$

$$
\begin{aligned}
&-(\tau - t + t - \tau) f(t-\tau) \\
=&\tau f(t) - \int_{t-\tau}^{t} f(s_2) ds_2.
\end{aligned}
$$

□

The behavior of the multi-vehicle system (3.1) under the proposed low gain control laws (3.3) is established in the following three Lemmas, and the main theorem for this chapter is given afterwards.

Lemma 10. *Consider the multi-agent system in* (3.1) *under the decentralized control laws* (3.3) *and communication topology described by an undirected and connected graph* $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$*. Let* $\mathcal{I}$ *be the set of all complete trajectories of the closed-loop system with all agents* i *satisfying the condition*

$$
\begin{bmatrix} \cos\theta_i(t) & \sin\theta_i(t) \end{bmatrix} \sum_{j\in\mathcal{N}_i} r_{ij}(t-\tau)\Pi_{ij}(t-\tau) = 0. \tag{3.7}
$$

If there exists a subset $\mathcal{V}_2 \subseteq \mathcal{V}$ *such that a trajectory in* $\mathcal{I}$ *satisfies* $\sum_{j\in\mathcal{N}_i} r_{ij}\Pi_{ij} \neq 0$*, for all* $n_i \in \mathcal{V}_2$*, then the same trajectory approaches a steady state solution such that* $\dot{\theta}_i = 0$ *for all agent* i*.*

Proof. First of all, it is observed from (3.3) that $v_i = 0$ for all trajectories satisfying (3.7). Therefore $r_{ij}\Pi_{ij}$ is constant for all agents i and j, and every trajectory in $\mathcal{I}$. Let $\{\mathcal{V}_1,\ \mathcal{V}_2\}$ be a partition of $\mathcal{V}$ such that $\sum_{j\in\mathcal{N}_i} r_{ij}\Pi_{ij} = 0$ for agents i such that $n_i \in \mathcal{V}_1$, and $\sum_{j\in\mathcal{N}_i} r_{ij}\Pi_{ij} \neq 0$ for agents i such that $n_i \in \mathcal{V}_2$. Without loss of generality, assume the agents are ordered such that $n_i \in \mathcal{V}_1$ for agents $i \leq \tilde{n}$ and $n_i \in \mathcal{V}_2$ for agents $i > \tilde{n}$ for some positive integer $\tilde{n}$. Then the vector of agent orientations is defined as,

$$
\theta = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_N \end{bmatrix} = \begin{bmatrix} \tilde{\theta}_1 \\ \tilde{\theta}_2 \end{bmatrix}, \tag{3.8}
$$

where $\tilde{\theta}_1 \in \mathbb{R}^{\tilde{n}}$ and $\tilde{\theta}_2 \in \mathbb{R}^{N-\tilde{n}}$. Because $r_{ij}\Pi_{ij}$ is constant for all agents i and j, it follows from (3.7) that $\dot{\tilde{\theta}}_2 = 0$.

Let the Laplacian matrix of $\mathcal{G}$ be given as

$$L = \begin{bmatrix} L_1 & L_3 \\ L_3^{\mathrm{T}} & L_2 \end{bmatrix}, \tag{3.9}$$

where $L_1 \in \mathbb{R}^{\tilde{n}\times\tilde{n}}$ and $L_1 > 0$ under the assumption that $\mathcal{G}$ is connected. The equation for the orientation $\tilde{\theta}_1$ then becomes, $\dot{\tilde{\theta}}_1(t) = -\varepsilon\kappa L_1\tilde{\theta}_1(t-\tau) - \varepsilon\kappa L_3\tilde{\theta}_2$, and the second time derivative of $\tilde{\theta}_1(t)$ is found to be

$$\begin{aligned}\ddot{\tilde{\theta}}_1(t) &= -\varepsilon\kappa L_1\dot{\tilde{\theta}}_1(t-\tau) \\ &= -\varepsilon\kappa L_1 \mathrm{e}^{\varepsilon\kappa L_1\tau}\dot{\tilde{\theta}}_1(t).\end{aligned} \tag{3.10}$$

From the facts that $L_1 > 0$ and $\mathrm{e}^{\varepsilon\kappa L_1\tau} > 0$, it is obtained that $L_1\mathrm{e}^{\varepsilon\kappa L_1\tau} > 0$ and (3.10) is asymptotically stable, and the trajectories of (3.10) asymptotically approach $\dot{\tilde{\theta}}_1 = 0$. □

Lemma 11. *Consider the multi-agent system* (3.1) *with communication delay* τ *and initial conditions in* Ω. *Given the artificial potential function*

$$V_{r,1} = \sum_{i=1}^{N} V_i(t), \tag{3.11}$$

there exists an $\bar{\varepsilon}_1 > 0$ *such that, for any* $0 < \varepsilon \le \bar{\varepsilon}_1$, *the low gain control laws* (3.3) *steer the system into a formation corresponding to a minimum of* $V_{r,1}$.

Proof. Consider the artificial potential function (3.11), which can be expressed as

$$V_{r,1}(t) = \sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i} V_{ij}(t). \tag{3.12}$$

The time derivative of the function $V_{r,1}$ along the trajectories of (3.1) under the control law (3.3) is given by

$$\begin{aligned}\dot{V}_{r,1}(t) =& \sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i} 2\Upsilon_{ij}^{\mathrm{T}}(t)\dot{r}_{ij}(t), \\ =& 2\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\Upsilon_{ij}^{\mathrm{T}}(t)\Bigg(-2\varepsilon\Phi_i(t)\sum_{k\in\mathcal{N}_i}\Upsilon_{ik}(t-\tau) \\ &+2\varepsilon\Phi_j(t)\sum_{m\in\mathcal{N}_j}\Upsilon_{jm}(t-\tau)\Bigg),\end{aligned} \tag{3.13}$$

where the time-varying positive semi-definite matrix $\Phi_i(t)$ is defined as

$$\Phi_i(t) = \begin{bmatrix} \cos\theta_i(t) \\ \sin\theta_i(t) \end{bmatrix} \begin{bmatrix} \cos\theta_i(t) & \sin\theta_i(t) \end{bmatrix}. \tag{3.14}$$

The right-hand side of equality (3.13) can be rearranged to obtain

$$\begin{aligned} \dot{V}_{r,1} = &-4\varepsilon \sum_{i=1}^{N} \sum_{j\in\mathcal{N}_i} \Upsilon_{ij}^{\mathrm{T}}(t)\Phi_i(t) \sum_{k\in\mathcal{N}_i} \Upsilon_{ik}(t-\tau) \\ &+4\varepsilon \sum_{i=1}^{N} \sum_{j\in\mathcal{N}_i} \Upsilon_{ij}^{\mathrm{T}}(t)\Phi_j(t) \sum_{m\in\mathcal{N}_j} \Upsilon_{jm}(t-\tau), \end{aligned} \tag{3.15}$$

which, by a reordering of the summation signs and in view of the fact that $r_{ij}(t) = -r_{ji}(t)$, simplifies to

$$\dot{V}_{r,1} = -8\varepsilon \sum_{i=1}^{N} \sum_{j\in\mathcal{N}_i} \Upsilon_{ij}^{\mathrm{T}}(t)\Phi_i \sum_{k\in\mathcal{N}_i} \Upsilon_{ik}(t-\tau). \tag{3.16}$$

It is observed in (3.16) that, within the summation about index i, the summation about index j to the left of Φ_i is given at time t and the summation about index k to the right of Φ_i is at the delayed time $t-\tau$.

From (2.12), $\Upsilon_{ij}(t)$ can be rewritten as

$$\Upsilon_{ij}(t) = \Upsilon_{ij}(t-\tau) + \int_{t-\tau}^{t} \lambda_{ij}(\sigma)\mathrm{d}\sigma, \tag{3.17}$$

where

$$\lambda_{ij}(t) = \Lambda_{ij}(t)\dot{r}_{ij}(t), \tag{3.18}$$

and

$$\Lambda_{ij}(t) = \Pi_{ij}(t)I + 2\left(\frac{2\rho^4}{\|r_{ij}(t)\|^6} - \frac{\rho^2}{\|r_{ij}(t)\|^4}\right) r_{ij}(t) r_{ij}^{\mathrm{T}}(t).$$

Substituting the right-hand side of (3.17) into (3.16) results in

$$\begin{aligned} \dot{V}_{r,1} = &-8\varepsilon \sum_{i=1}^{N} \sum_{j\in\mathcal{N}_i} \Upsilon_{ij}^{\mathrm{T}}(t-\tau)\Phi_i \sum_{k\in\mathcal{N}_i} \Upsilon_{ik}(t-\tau) \\ &-8\varepsilon \sum_{i=1}^{N} \sum_{j\in\mathcal{N}_i} \int_{t-\tau}^{t} \lambda_{ij}^{\mathrm{T}}(\sigma)\mathrm{d}\sigma\, \Phi_i \sum_{k\in\mathcal{N}_i} \Upsilon_{ik}(t-\tau). \end{aligned} \tag{3.19}$$

The matrix product within the second summation term about index i can be further expanded to obtain

$$\dot{V}_{r,1} \leq -8\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\Upsilon_{ij}^{\mathrm{T}}(t-\tau)(\varepsilon\Phi_i - 2\varepsilon^2\Phi_i^2)\sum_{k\in\mathcal{N}_i}\Upsilon_{ik}(t-\tau)$$
$$+\sum_{i=1}^{N}\int_{t-\tau}^{t}\sum_{j\in\mathcal{N}_i}\lambda_{ij}^{\mathrm{T}}(\sigma)\mathrm{d}\sigma\int_{t-\tau}^{t}\sum_{k\in\mathcal{N}_i}\lambda_{ik}(\sigma)\mathrm{d}\sigma.$$

Applying the result of Lemma 8 to the product of integrals in the above inequality gives

$$\dot{V}_{r,1} \leq -8\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\Upsilon_{ij}^{\mathrm{T}}(t-\tau)(\varepsilon\Phi_i - 2\varepsilon^2\Phi_i^2)\sum_{j\in\mathcal{N}_i}\Upsilon_{ik}(t-\tau)$$
$$+\tau\sum_{i=1}^{N}\int_{t-\tau}^{t}\sum_{j\in\mathcal{N}_i}\lambda_{ij}^{\mathrm{T}}(\sigma)\sum_{k\in\mathcal{N}_i}\lambda_{ik}(\sigma)\mathrm{d}\sigma. \tag{3.20}$$

Define a second potential functional as

$$V_{r,2}(t) = \sum_{i=1}^{N}\tau\int_{0}^{\tau}\int_{t-s_1}^{t}\sum_{j\in\mathcal{N}_i}\lambda_{ij}^{\mathrm{T}}(s_2)\sum_{k\in\mathcal{N}_i}\lambda_{ik}(s_2)\mathrm{d}s_2\mathrm{d}s_1. \tag{3.21}$$

Using the result from Lemma 9, we find the time derivative along the trajectories of (3.1) and (3.3) to be

$$\dot{V}_{r,2} = \tau^2\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\lambda_{ij}^{\mathrm{T}}(t)\sum_{k\in\mathcal{N}_i}\lambda_{ik}(t) - \tau\sum_{i=1}^{N}\int_{t-\tau}^{t}\sum_{j\in\mathcal{N}_i}\lambda_{ij}^{\mathrm{T}}(s)\sum_{k\in\mathcal{N}_i}\lambda_{ik}(s)\mathrm{d}s. \tag{3.22}$$

Finally, define the total potential functional

$$V_r = V_{r,1} + V_{r,2}. \tag{3.23}$$

The time derivative along the trajectories of (3.1) and (3.3) is found by combining (3.20) and (3.22) as

$$\dot{V}_{r} \leq -8\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\Upsilon_{ij}^{\mathrm{T}}(t-\tau)(\varepsilon\Phi_i - 2\varepsilon^2\Phi_i^2)\sum_{k\in\mathcal{N}_i}\Upsilon_{ik}(t-\tau)$$
$$+\tau^2\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\lambda_{ij}^{\mathrm{T}}(t)\sum_{k\in\mathcal{N}_i}\lambda_{ik}(t). \tag{3.24}$$

The function $\lambda_{ij}(t)$ was defined in (3.18), and the summation about index j is expressed as

$$\sum_{j\in\mathcal{N}_i}\lambda_{ij}(t) = -2\varepsilon\Lambda_i\Phi_i\sum_{k\in\mathcal{N}_i}\Upsilon_{ik}(t-\tau) + 2\varepsilon\sum_{j\in\mathcal{N}_i}\Lambda_{ij}\Phi_j\sum_{k\in\mathcal{N}_j}\Upsilon_{jk}(t-\tau), \tag{3.25}$$

where Λ_i is defined to simplify notation as

$$\Lambda_i = \sum_{j\in\mathcal{N}_i}\Lambda_{ij}. \tag{3.26}$$

By Lemma 7, the product of (3.25) with its transpose is found to be

$$\begin{aligned}\sum_{j\in\mathcal{N}_i}\lambda_{ij}^{\mathrm{T}}(t)\sum_{m\in\mathcal{N}_i}\lambda_{im}(t) \le{}& 8\varepsilon^2\sum_{j\in\mathcal{N}_i}\Upsilon_{ij}^{\mathrm{T}}(t-\tau)\Phi_i\Lambda_i^2\Phi_i\sum_{m\in\mathcal{N}_i}\Upsilon_{im}(t-\tau)\\ &+ 8\varepsilon^2\left(\sum_{j\in\mathcal{N}_i}\Lambda_{ij}\Phi_j\sum_{k\in\mathcal{N}_j}\Upsilon_{jk}(t-\tau)\right)^{\mathrm{T}}\\ &\times\left(\sum_{m\in\mathcal{N}_i}\Lambda_{im}\Phi_m\sum_{k\in\mathcal{N}_m}\Upsilon_{mk}(t-\tau)\right).\end{aligned} \tag{3.27}$$

The result of Lemma 7 can be applied once again on the second term on the right-hand side of the above inequality to obtain

$$\begin{aligned}\sum_{j\in\mathcal{N}_i}\lambda_{ij}^{\mathrm{T}}(t)\sum_{m\in\mathcal{N}_i}\lambda_{im}(t) \le{}& 8\varepsilon^2\sum_{j\in\mathcal{N}_i}\Upsilon_{ij}^{\mathrm{T}}(t-\tau)\Phi_i\Lambda_i^2\Phi_i\sum_{m\in\mathcal{N}_i}\Upsilon_{im}(t-\tau)\\ &+ 8\varepsilon^2(N-1)\sum_{j\in\mathcal{N}_i}\sum_{k\in\mathcal{N}_j}\Upsilon_{jk}^{\mathrm{T}}(t-\tau)\\ &\times\Phi_j\Lambda_{ij}^2\Phi_j\sum_{k\in\mathcal{N}_j}\Upsilon_{jk}^{\mathrm{T}}(t-\tau).\end{aligned} \tag{3.28}$$

By summing the left-hand side of the above inequality about the index i, the following expression is obtained

$$\begin{aligned}\sum_{i=1}^{N}\left(\sum_{j\in\mathcal{N}_i}\lambda_{ij}^{\mathrm{T}}(t)\sum_{m\in\mathcal{N}_i}\lambda_{im}(t)\right) \le{}& 8\varepsilon^2\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\Upsilon_{ij}^{\mathrm{T}}(t-\tau)\Phi_i\Lambda_i^2\Phi_i\sum_{m\in\mathcal{N}_i}\Upsilon_{im}(t-\tau)\\ &+ 8\varepsilon^2(N-1)\sum_{j=1}^{N}\left(\sum_{k\in\mathcal{N}_j}\Upsilon_{jk}^{\mathrm{T}}(t-\tau)\Phi_j\right.\\ &\left.\times\sum_{i\in\mathcal{N}_j}\Lambda_{ij}^2\Phi_j\sum_{k\in\mathcal{N}_j}\Upsilon_{jk}(t-\tau)\right).\end{aligned} \tag{3.29}$$

The above expression can be further simplified by a reordering of the summation indices and the definition of a constant η such that

$$\eta I \geq (N-1) \sum_{j \in \mathcal{N}_i} \Lambda_{ij}^2, \text{ for } i = 1, 2, \ldots, N. \tag{3.30}$$

Such constant η exists for any $(r_{ij}, e_{ij}) \in \Omega$. The result of substituting the above inequality into (3.29) is

$$\sum_{i=1}^{N} \left(\sum_{j \in \mathcal{N}_i} \lambda_{ij}^{\mathrm{T}}(t) \sum_{m \in \mathcal{N}_i} \lambda_{im}(t) \right) \leq 8N\eta\varepsilon^2 \sum_{i=1}^{N} \sum_{j \in \mathcal{N}_i} \Upsilon_{ij}^{\mathrm{T}}(t-\tau)\Phi_i^2 \sum_{m \in \mathcal{N}_i} \Upsilon_{im}(t-\tau). \tag{3.31}$$

Finally, by combining (3.24) and (3.31), we obtain the derivative of $V_r(t)$ as

$$\dot{V}_r \leq -8 \sum_{i=1}^{N} \sum_{j \in \mathcal{N}_i} \Upsilon_{ij}^{\mathrm{T}}(t-\tau)(\varepsilon\Phi_i - 2\varepsilon^2\Phi_i^2 - N\varepsilon^2\tau^2\eta\Phi_i^2) \sum_{k \in \mathcal{N}_i} \Upsilon_{ik}(t-\tau). \tag{3.32}$$

We note that $\Phi_i = \Phi_i^2$ is positive semi-definite. Define a positive $\bar{\varepsilon}_1$ such that

$$\bar{\varepsilon}_1 < \frac{1}{2 + \tau^2 N\eta}, \tag{3.33}$$

and let $0 < \varepsilon \leq \bar{\varepsilon}_1$. Then it follows that

$$\varepsilon\Phi_i - 2\varepsilon^2\Phi_i^2 - N\varepsilon^2\tau^2\eta\Phi_i^2 \geq 0, \tag{3.34}$$

and $\dot{V}_r \leq 0$.

From (3.32) it can be deduced that $\dot{V}_r$ may be equal to zero only if, based on the definition of Φ_i in (3.14),

$$\begin{bmatrix} \cos\theta_i(t) & \sin\theta_i(t) \end{bmatrix} \sum_{j \in \mathcal{N}_i} r_{ij}(t-\tau)\Pi_{ij}(t-\tau) = 0, \tag{3.35}$$

for all positive integer $i \leq N$. Assuming that $\sum_{j \in \mathcal{N}_i} r_{ij}\Pi_{ij}$ is not zero for all agent i, Lemma 10 gives that any complete trajectory of the system (3.1) under the control laws (3.3) satisfying (3.35) corresponds to an equilibrium point of the closed-loop system if $\omega_i = 0$ for all agents i. Based on (3.3), this is equivalent to

$$\sum_{j \in \mathcal{N}_k} r_{kj}\Pi_{kj} = \frac{\kappa}{2} \begin{bmatrix} \sin\theta_k \\ -\cos\theta_k \end{bmatrix} \sum_{j \in \mathcal{N}_k} e_{kj}, \tag{3.36}$$

for all positive integer $k \leq N$. Because $r_{ij} = -r_{ji}$, the sum of the left-hand side of (3.36) over all $0 < k \leq N$ equals zero, and

$$\sum_{k=1}^{N} \frac{\kappa}{2} \begin{bmatrix} \sin\theta_k \\ -\cos\theta_k \end{bmatrix} \sum_{j\in\mathcal{N}_k} e_{kj} = 0. \tag{3.37}$$

For any agent i, define $\vec{v}_i$ and $\vec{v}_{-i}$ as

$$\vec{v}_i = \begin{bmatrix} \sin\theta_i \\ -\cos\theta_i \end{bmatrix} \sum_{j\in\mathcal{N}_i} e_{ij}, \tag{3.38}$$

$$\vec{v}_{-i} = \sum_{\substack{k=1 \\ k\neq i}}^{N} \begin{bmatrix} \sin\theta_k \\ -\cos\theta_k \end{bmatrix} \sum_{j\in\mathcal{N}_k} e_{kj}. \tag{3.39}$$

It follows from (3.37) that $\vec{v}_i + \vec{v}_{-i} = 0$ and

$$\|\vec{v}_i\|^2 = \|\vec{v}_{-i}\|^2. \tag{3.40}$$

The right-hand side of (3.40) is computed from (3.39) to be

$$\begin{aligned} \|\vec{v}_{-i}\|^2 &= \sum_{\substack{k=1 \\ k\neq i}}^{N} \sum_{\substack{l=1 \\ l\neq i}}^{N} \cos e_{kl} \sum_{j\in\mathcal{N}_k} e_{kj} \sum_{j\in\mathcal{N}_l} e_{lj}, \\ &= -\|\vec{v}_i\|^2 - 2\sum_{\substack{k=1 \\ k\neq i}}^{N} \cos e_{kl} \sum_{j\in\mathcal{N}_i} e_{ij} \sum_{j\in\mathcal{N}_k} e_{kj}, \end{aligned} \tag{3.41}$$

where the second line of the above equation comes from (3.37) and the fact that

$$\begin{aligned} \|\vec{v}_i + \vec{v}_{-1}\|^2 &= \sum_{k=1}^{N} \sum_{l=1}^{N} \cos e_{kl} \sum_{j\in\mathcal{N}_k} e_{kj} \sum_{j\in\mathcal{N}_l} e_{lj}, \\ &\equiv 0. \end{aligned}$$

By combining the results in (3.40) and (3.41), it is obtained that for any agent i,

$$\sum_{j\in\mathcal{N}_i} e_{ij} = -\sum_{\substack{k=1 \\ k\neq i}}^{N} \cos e_{ik} \sum_{j\in\mathcal{N}_k} e_{kj}. \tag{3.42}$$

For any agent m adjacent to an agent i, the partial derivative of (3.42) about relative orientation e_{im} is determined to be

$$1 = \sin e_{im} \sum_{j\in\mathcal{N}_m} e_{mj} + \cos e_{im}. \tag{3.43}$$

If $\sin e_{im} = 0$, then it follows from the above equation that $\cos e_{im} = 1$, which is true only if the orientations of agents i and m are aligned. On the other hand, if $\sin e_{im} \neq 0$, then

$$\sum_{j \in \mathcal{N}_m} e_{mj} = \frac{1 - \cos e_{im}}{\sin e_{im}}, \tag{3.44}$$

for any agent $m \in \mathcal{N}_i$. By the same token, $i \in \mathcal{N}_m$, and

$$\sum_{j \in \mathcal{N}_i} e_{ij} = \frac{1 - \cos e_{mi}}{\sin e_{mi}}. \tag{3.45}$$

This implies that for the two adjacent agents i and m,

$$\sum_{j \in \mathcal{N}_i} e_{ij} + \sum_{j \in \mathcal{N}_m} e_{mj} = 0. \tag{3.46}$$

Consider the subgraph $\bar{\mathcal{G}} = \{\mathcal{V}, \bar{\mathcal{E}}\}$ of $\mathcal{G}$, where $\bar{\mathcal{E}} = \{(n_i, n_j) \in \mathcal{E} | \sin e_{ij} \neq 0\}$, and define $\bar{\mathcal{N}}_i = \{j | (n_i, n_j) \in \bar{\mathcal{E}}\}$ and $\bar{L}$ to be the Laplacian matrix of $\bar{\mathcal{G}}$. Without loss of generality, we assume that $\bar{\mathcal{G}}$ is a connected graph. For the case where the last assumption on $\bar{\mathcal{G}}$ is not satisfied, the remainder of the current proof can be repeated for each connected subgroup of $\bar{\mathcal{G}}$, in combination with the fact that the orientations of agents i and j are aligned if $(i, j) \in \mathcal{E} \backslash \bar{\mathcal{E}} = \{(i, j) \in \mathcal{E} | (i, j) \notin \bar{\mathcal{E}}\}$.

For any adjacent agents i and m in $\bar{\mathcal{G}}$, it follows from the same argument as in (3.46) that

$$\sum_{j \in \bar{\mathcal{N}}_i} e_{ij} + \sum_{j \in \bar{\mathcal{N}}_m} e_{mj} = 0. \tag{3.47}$$

Because $\bar{\mathcal{G}}$ is connected, there are at least $N - 1$ edges in $\bar{\mathcal{E}}$ resulting in $N - 1$ linearly independent equations in the form of (3.47) for any adjacent agents i and j in $\bar{\mathcal{G}}$. Furthermore, because $\bar{\mathcal{G}}$ is undirected, it is true that

$$\sum_{i=1}^{N} \sum_{j \in \bar{\mathcal{N}}_i} e_{ij} = 0. \tag{3.48}$$

By combining (3.47) and (3.48), N linearly independent equations are obtained with the unique solution $\sum_{j \in \bar{\mathcal{N}}_i} e_{ij} = 0$ for all agents i or,

$$\bar{L}\theta = 0, \tag{3.49}$$

where $\theta = [\theta_1, \theta_2, \ldots, \theta_N]^{\mathrm{T}}$. Because $\bar{\mathcal{G}}$ is undirected and connected, by Lemma 2, $\bar{L}$ has a single eigenvalue equal to zero. A nontrivial solution of (3.49) is $\theta_i = z$ for all agents i and some $z \in (-\pi, \pi]$, thus all the agent orientations are aligned. Referring back to (3.36), the alignment of the orientations of all agents results in that

$$\sum_{j\in\mathcal{N}_i} r_{ij}\Pi_{ij} = 0 \tag{3.50}$$

must be true for all i in order to correspond to an equilibrium point of the closed-loop system. Therefore, by LaSalle's invariance principle, the multi-agent system is driven to the equilibrium state (3.50). It then follows from (2.11) that the relative positions of the agents satisfying (3.50) correspond to a minimum of (3.11). This completes the proof. □

The previous two lemmas demonstrate the convergence of the agents to a formation. To complete the flocking behavior, it is necessary to demonstrate that the orientations of the agents become aligned. The next lemma guarantees the asymptotic alignment of the agents as they converge to the formation of minimum artificial potential energy.

Lemma 12. *Consider the multi-agent system* (3.1) *with a communication delay* τ *and initial conditions in* Ω*. Then there exists an* $\bar{\varepsilon}_2 > 0$ *such that, for any* $0 < \varepsilon \leq \bar{\varepsilon}_2$*, the low gain control laws* (3.3) *aligns the orientations of the agents as* $\nabla_{r_i} V_i(t)$ *asymptotically approaches zero for all* i*.*

Proof. Consider the positive definite function

$$V_{e,1} = \sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i} e_{ij}^2(t). \tag{3.51}$$

The time derivative of this function along the trajectories of (3.1) under the control laws (3.3) is found to be

$$\dot{V}_{e,1} = 2\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i} e_{ij}(t)\dot{e}_{ij}(t). \tag{3.52}$$

The dynamics of the difference between the orientation of agent i and j is determined from (3.3) as

$$\dot{e}_{ij}(t) = \varepsilon\left(-\kappa\sum_{k\in\mathcal{N}_i} e_{ik}(t-\tau) - \Psi_i\nabla_{r_i}V_i(t-\tau)\right.$$

$$+\kappa \sum_{k\in\mathcal{N}_j} e_{jk}(t-\tau) + \Psi_j \nabla_{r_j} V_j(t-\tau) \Bigg), \tag{3.53}$$

where $\Psi_i(t) = [-\sin\theta_i(t) \ \ \cos\theta_i(t)]$ is defined to simplify the notation. The value of Ψ_i is always obtained at the current time t, and thus the time variable will be dropped in the remainder of the proof.

By combining (3.52) and the dynamics equation of e_{ij} in (3.53), we can express the derivative of $\dot{V}_{e,1}$ as

$$\begin{aligned}\dot{V}_{e,1} \leq & -4\varepsilon\kappa \sum_{i=1}^{N} \sum_{j\in\mathcal{N}_i} e_{ij}(t) \sum_{k\in\mathcal{N}_i} e_{ik}(t-\tau) \\ & + 2\varepsilon \sum_{i=1}^{N} \sum_{j\in\mathcal{N}_i} e_{ij}(t) \big(-\Psi_i \nabla_{r_i} V_i(t-\tau) + \Psi_j \nabla_{r_j} V_j(t-\tau) \big). \end{aligned} \tag{3.54}$$

Once again, the above derivative contains terms given at time t and $t-\tau$. In this proof, the information about the dynamics of the closed-loop system is employed in order to express all terms at the same time value.

Given the system dynamics in (3.1), $e_{ij}(t)$ can be rewritten as

$$e_{ij}(t) = e_{ij}(t-\tau) + \int_{t-\tau}^{t} \dot{e}_{ij}(\sigma)\mathrm{d}\sigma, \tag{3.55}$$

where $\dot{e}_{ij}$ is given in (3.53). Substituting the equality (3.55) into (3.54) and reordering the summation indices result in

$$\begin{aligned}\dot{V}_{e,1} \leq & -4\varepsilon\kappa \sum_{i=1}^{N} \left(\sum_{j\in\mathcal{N}_i} e_{ij}(t-\tau) \right)^2 - 4\varepsilon\kappa \sum_{i=1}^{N} \int_{t-\tau}^{t} \sum_{j\in\mathcal{N}_i} \dot{e}_{ij}(\sigma)\mathrm{d}\sigma \sum_{k\in\mathcal{N}_i} e_{ik}(t-\tau) \\ & -4\varepsilon \sum_{i=1}^{N} \sum_{j\in\mathcal{N}_i} e_{ij}(t-\tau) \Psi_i \nabla_{r_i} V_i(t-\tau) \\ & -4\varepsilon \sum_{i=1}^{N} \int_{t-\tau}^{t} \sum_{j\in\mathcal{N}_i} \dot{e}_{ij}(\sigma)\mathrm{d}\sigma \Psi_i \nabla_{r_i} V_i(t-\tau). \end{aligned} \tag{3.56}$$

The above expression can be further simplified by recalling that $2ab \leq a^2 + b^2$ for $a, b \in \mathbb{R}$. After applying this relationship to the second, third and fourth terms on the right-hand side of (3.56), the quadratic terms are collected, and Lemma 8 is employed to rewrite the square of the integral on $\dot{e}_{ij}$. The resulting expression is

$$\dot{V}_{e,1} \leq -4(\varepsilon\kappa - 2\varepsilon^2\kappa^2) \sum_{i=1}^{N} \left(\sum_{j\in\mathcal{N}_i} e_{ij}(t-\tau) \right)^2$$

$$+\left(\frac{1}{\kappa^2}+4\varepsilon^2\right)\sum_{i=1}^{N}\nabla_{r_i}V_i^{\mathrm{T}}(t-\tau)\Psi_i^{\mathrm{T}}\Psi_i\nabla_{r_i}V_i(t-\tau)$$

$$+2\tau\sum_{i=1}^{N}\int_{t-\tau}^{t}\left(\sum_{j\in\mathcal{N}_i}\dot{e}_{ij}(\sigma)\right)^2\mathrm{d}\sigma. \tag{3.57}$$

Consider a second positive definite functional

$$V_{e,2}=2\tau\sum_{i=1}^{N}\int_0^{\tau}\int_{t-s_1}^{t}\left(\sum_{j\in\mathcal{N}_i}\dot{e}_{ij}(s_2)\right)^2\mathrm{d}s_2\mathrm{d}s_1. \tag{3.58}$$

The time derivative of this functional along the trajectories of (3.1) under the control law (3.3) is computed by using the result of Lemma 9

$$\dot{V}_{e,2}=2\tau^2\sum_{i=1}^{N}\left(\sum_{j\in\mathcal{N}_i}\dot{e}_{ij}(t)\right)^2-2\tau\sum_{i=1}^{N}\int_{t-\tau}^{t}\left(\sum_{j\in\mathcal{N}_i}\dot{e}_{ij}(\sigma)\right)^2\mathrm{d}\sigma. \tag{3.59}$$

In view of Lemma 7, the square of the sum of $\dot{e}_{ij}(t)$ can be expressed as

$$\begin{aligned}\left(\sum_{j\in\mathcal{N}_i}\dot{e}_{ij}(t)\right)^2 &\le (N-1)\sum_{j\in\mathcal{N}_i}\dot{e}_{ij}^2\\ &\le 4\varepsilon^2(N-1)\sum_{j\in\mathcal{N}_i}\Bigg(\left(\kappa\sum_{k\in\mathcal{N}_i}e_{ik}(t-\tau)\right)^2+\left(\kappa\sum_{k\in\mathcal{N}_j}e_{jk}(t-\tau)\right)^2\\ &\quad+\nabla_{r_i}V_i^{\mathrm{T}}(t-\tau)\Psi_i^{\mathrm{T}}\Psi_i\nabla_{r_i}V_i(t-\tau)\\ &\quad+\nabla_{r_j}V_j^{\mathrm{T}}(t-\tau)\Psi_j^{\mathrm{T}}\Psi_j\nabla_{r_j}V_j(t-\tau)\Bigg).\end{aligned} \tag{3.60}$$

The summation of (3.60) about index i then becomes,

$$\begin{aligned}\sum_{i=1}^{N}\left(\sum_{j\in\mathcal{N}_i}\dot{e}_{ij}(t)\right)^2 \le 8\varepsilon^2(N-1)^2\Bigg(&\kappa^2\sum_{i=1}^{N}\left(\sum_{j\in\mathcal{N}_i}e_{ij}(t-\tau)\right)^2\\ &+\sum_{i=1}^{N}\nabla_{r_i}V_i^{\mathrm{T}}(t-\tau)\Psi_i^{\mathrm{T}}\Psi_i\nabla_{r_i}V_i(t-\tau)\Bigg),\end{aligned} \tag{3.61}$$

after the summation indices are reordered and similar terms are collected.

Finally, consider the positive definite functional

$$V_e=V_{e,1}+V_{e,2}.$$

The time derivative of V_e is obtained by combining (3.57), (3.59) and (3.61) as

$$\dot{V}_e \leq -4\left(\varepsilon\kappa - 2\varepsilon^2\kappa^2 - 4\varepsilon^2\tau^2(N-1)^2\kappa^2\right)\sum_{i=1}^{N}\left(\sum_{j\in\mathcal{N}_i} e_{ij}(t-\tau)\right)^2 + \left(\frac{1}{\kappa^2} + 4\varepsilon^2 + 16\varepsilon^2\tau^2(N-1)^2\right)\sum_{i=1}^{N}\nabla_{r_i}V_i^{\mathrm{T}}(t-\tau)\Psi_i^{\mathrm{T}}\Psi_i\nabla_{r_i}V_i(t-\tau). \tag{3.62}$$

Under the assumption that $\nabla_{r_i}V_i$ asymptotically approaches zero for all i, for any given $\gamma > 0$ there exists a $T_1 > 0$ such that for $t > T_1$,

$$\dot{V}_e \leq -4\left(\varepsilon\kappa - 2\varepsilon^2\kappa^2 - 4\varepsilon^2\tau^2(N-1)^2\kappa^2\right)\sum_{i=1}^{N}\left(\sum_{j\in\mathcal{N}_i} e_{ij}(t-\tau)\right)^2 + \gamma^2. \tag{3.63}$$

Define a positive constant $\bar{\varepsilon}_2 \leq \bar{\varepsilon}_1$ such that

$$\bar{\varepsilon}_2 < \frac{1}{\kappa\left(2 + 4(N-1)^2\tau^2\right)}. \tag{3.64}$$

Then for any positive $\varepsilon \leq \bar{\varepsilon}_2$, the right-hand side of (3.63) is negative if $\left|\sum_{j\in\mathcal{N}_i} e_{ij}(t-\tau)\right| > \psi\gamma$, for any agent i, where the positive constant ψ is computed as

$$\psi = \frac{1}{2}\sqrt{\left(\varepsilon\kappa - 2\varepsilon^2\kappa^2 - 4\varepsilon^2(N-1)^2\kappa^2\tau^2\right)^{-1}}.$$

This implies that there exists a $T \geq T_1$ such that for $t > T$, $\left|\sum_{j\in\mathcal{N}_i} e_{ij}(t)\right| \leq \psi\gamma$ for all i. Moreover, since γ can be arbitrarily small and ψ is a constant, this implies that $\sum_{j\in\mathcal{N}_i} e_{ij}(t) \to 0$ as $t \to \infty$ for all i. Thus, we obtain that $L\theta \to 0$ as $t \to \infty$, where $\theta = [\theta_1, \theta_2, \ldots, \theta_N]^{\mathrm{T}}$, and L is the Laplacian matrix of the graph $\mathcal{G}$. Because $\mathcal{G}$ is undirected and connected, by Lemma 2, solving equation $L\theta = 0$ gives a nontrivial solution $\theta_i = z$ for all i, where $z \in \mathbb{R}$ is a free variable. Therefore, the orientations of all agents must align as $t \to \infty$. □

Lemma 11 demonstrates the aggregation behavior between the agents in the closed-loop multi-agent system. Lemma 12 proves that the orientations of the agents become aligned as the agents converge to a minimum energy formation. The complete flocking behavior of the closed-loop system can be shown by combining the results from these two lemmas.

Theorem 1. *Consider the multi-vehicle system (3.1), with a constant communication delay τ. If both Assumptions 1 and 3 hold, then, for a sufficiently small ε, the distributed control* *laws (3.3) steer the system to a formation corresponding to a minimum of* $\sum_{i=1}^{N} V_i$ *and a common orientation.*

Proof. Let $\varepsilon \leq \bar{\varepsilon}_2$. Then by Lemma 11, the control laws (3.3) steer the multi-agent system into a formation corresponding to a minimum of (3.11) and $\nabla_{r_i} V_i(t)$ approaches zero for all i. Also, by Lemma 12 the orientations of all agents become aligned as the closed-loop multi-agent system approaches the final formation. This completes the proof. □

For the delay-free scenarios, i.e. $\tau = 0$, with any arbitrary $\varepsilon > 0$, the statement in Theorem 1 is true for any choice of $\varepsilon > 0$.

For the cases where $\tau > 0$, we can find an upper bound $\bar{\varepsilon}_{\text{ct}}$ such that for any $\varepsilon \in (0, \bar{\varepsilon}_{\text{ct}})$, the statement in Theorem 1 is always true. The upper bound $\bar{\varepsilon}_{\text{ct}}$ relies on the communication delay τ, the control parameter κ, the graph connectivity and the potential functions. We are not surprised to find that either greater communication delays or a larger κ will lead to a smaller $\bar{\varepsilon}_{\text{ct}}$. However, it is interesting that a communication graph with stronger connectivity will also result in a smaller $\bar{\varepsilon}_{\text{ct}}$. A reasonable explanation of such result is that stronger connectivity implies higher dependence on the received information and therefore greater effects from communication delays. Thus, we need smaller ε.

We should also notice that the achievement of a minimum total potential does not necessarily indicate position aggregation unless with a connected graph.

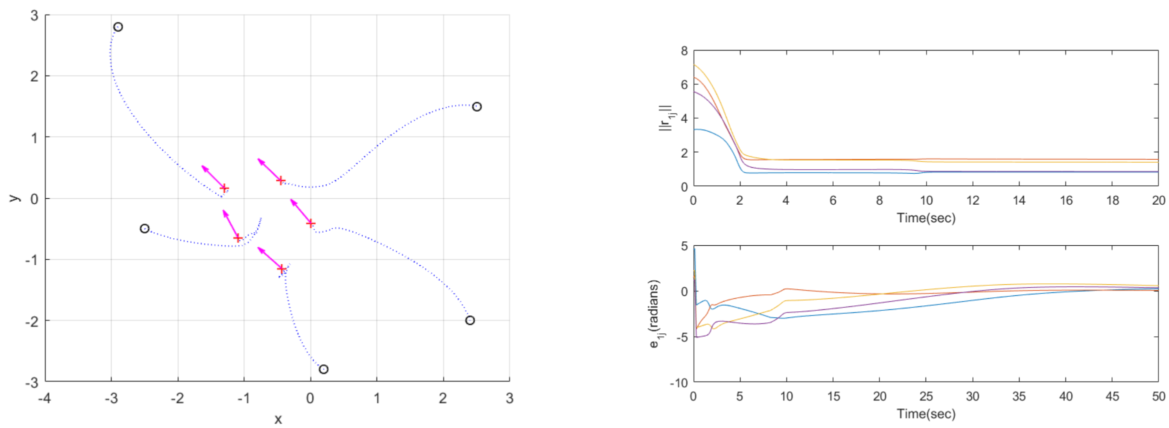
Collision avoidance between agents is another objective in this study. Based on the previous theoretical analysis, the total potential of a system satisfying both Assumptions 1 and 3 remains bounded if ε is sufficiently small, and thus the distance between two vehicles with a communication link cannot be zero. A consequence of this is that collision is prevented between any two neighboring vehicles. Since any vehicle j that is not in $\mathcal{N}_i$ does not affect the artificial potential of vehicle i, there are chances that vehicles i and j crash into each other. Following the same idea, a direct result of Theorem 1 is that collision avoidance is guaranteed for system (3.1) under the control laws (3.3) only when the communication graph is complete.

3.4 Simulation results

The effectiveness of the proposed continuous-time control laws is verified in this section through simulation. Consider five agents with dynamics (3.1) and steered by the control laws (3.3) with $\rho = 1$ and $\kappa = 0.01$. In the trajectory plots, circles and crosses give the initial positions the final positions, respectively.

3.4.1 Flocking with a complete graph

We first consider the case where the communication graph is complete and each agent collects information from it neighbors in real time, i.e. $\tau = 0$ s. The initial positions of the five agents are respectively $[-2.9, 2.8]^{\mathrm{T}}$, $[-2.5, -0.5]^{\mathrm{T}}$, $[0.2, -2.8]^{\mathrm{T}}$, $[2.4, -2]^{\mathrm{T}}$, and $[2.5, 1.5]^{\mathrm{T}}$. The initial headings of the five agents are respectively -2.25 rad, -0.49 rad, 2.61 rad, 1.84 rad, and 2.88 rad.

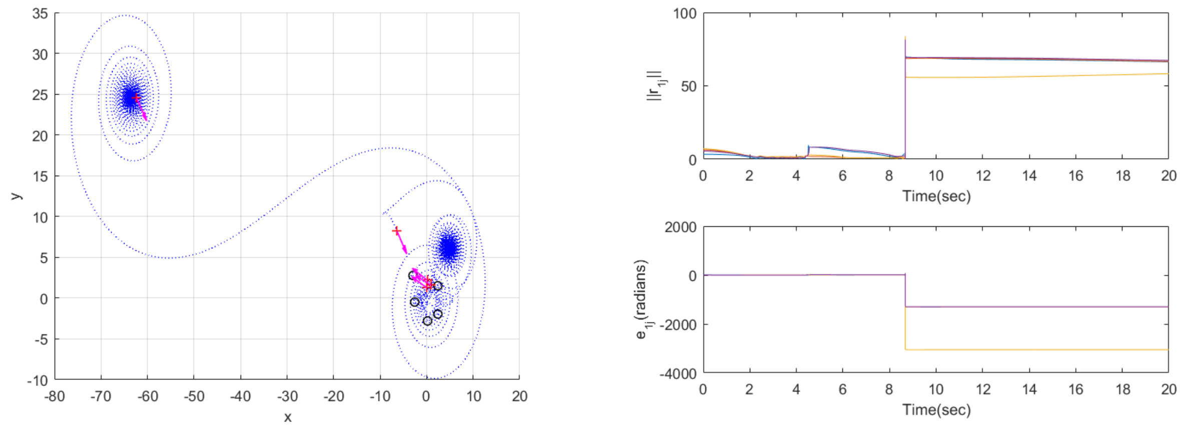


(a) Trajectories and headings. (b) Relative distances and headings.

Figure 3.3: Simulation with a complete graph, $\tau = 0$ s, $\varepsilon = 1$.

As shown in Figure 3.3, the agents converge to a formation that the relative distances among agents is approximately 1 as we specified before. In addition, the relative headings of the agents asymptotically converge to zero as the agents approach the final formation.

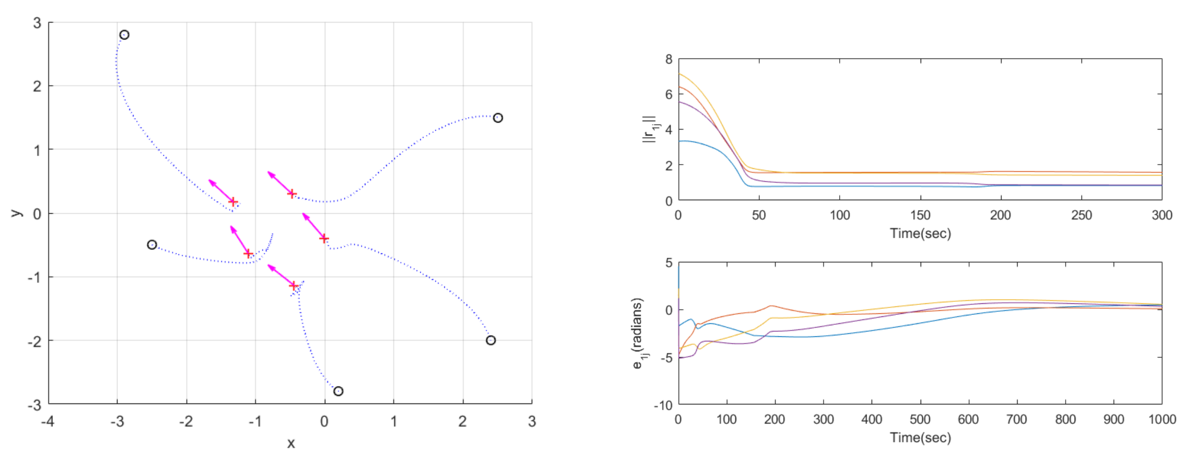
Then, we introduce communication delays, $\tau = 0.1$ s, and still have $\varepsilon = 1$ as in the delay-free scenario. It can be seen in Figure 3.4 that communication delays cause problems and the system becomes unstable.



(a) Trajectories and headings. (b) Relative distances and headings.

Figure 3.4: Simulation with a complete graph, $\tau = 0.1$ s, $\varepsilon = 1$.

After we tune the low gain parameter to a smaller value, $\varepsilon = 0.05$, the system regains a flocking behavior as shown in Figure 3.5. Comparing the results in Figures 3.3−3.5, we verify that the proposed control laws (3.3) are effective.



(a) Trajectories and headings. (b) Relative distances and headings.

Figure 3.5: Simulation with a complete graph, $\tau = 0.1$ s, $\varepsilon = 0.05$.

3.4.2 Flocking with a disconnected graph

The necessity of a connected communication graph can be verified by simulation as well. Consider a disconnected graph, as shown in Figure 3.6, which consists of tow connected components. We choose the initial positions of the five agents as $[-2.9, 2.8]^{\mathrm{T}}$, $[-2.5, -0.5]^{\mathrm{T}}$, $[0.2, -2.8]^{\mathrm{T}}$, $[2.4, -2]^{\mathrm{T}}$, and $[2.5, 1.5]^{\mathrm{T}}$, respectively. The initial headings of the five agents are respectively $-2.25\,\mathrm{rad}$, $-0.49\,\mathrm{rad}$, $2.61\,\mathrm{rad}$, $1.84\,\mathrm{rad}$, and $2.88\,\mathrm{rad}$. We observe in Figure 3.7 that the agents converge to 2 separate flocks.

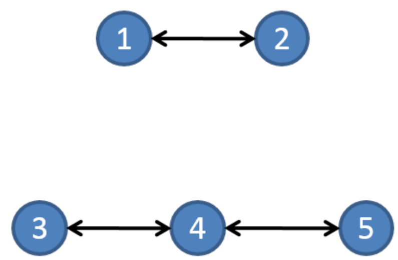
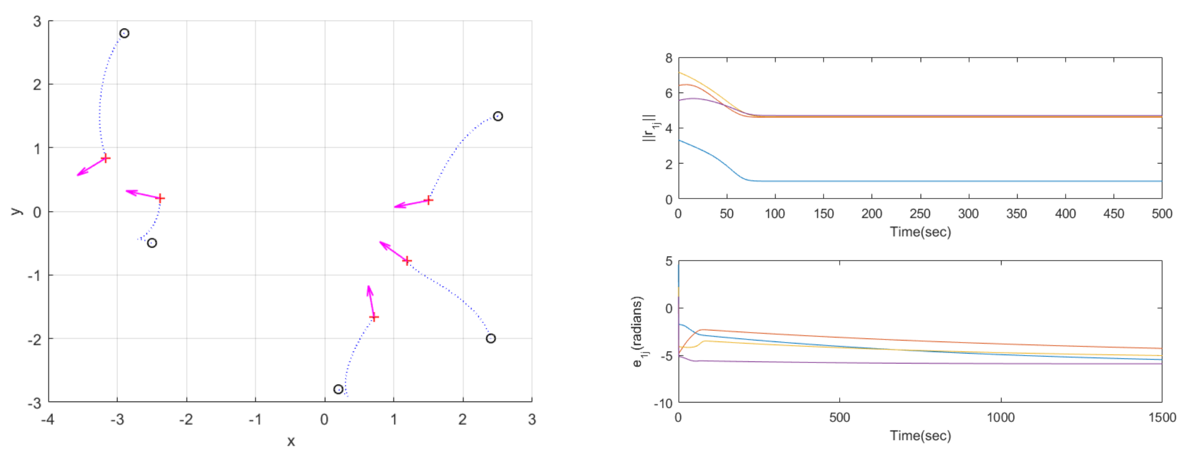


Figure 3.6: A disconnected graph.



(a) Trajectories and headings. (b) Relative distances and headings.

Figure 3.7: Simulation with a disconnected graph, $\tau = 0.1$ s, $\varepsilon = 0.05$.

3.4.3 Collisions in systems without complete graphs

As we mentioned in Section 3.3, collision avoidance is guaranteed only when the communication graph is complete. We now consider a connected but incomplete graph as shown in Figure 3.8. To carry out simulation, the initial positions of the five agents are selected as $[-2.9, 2.8]^{\mathrm{T}}$, $[-2.5, -0.5]^{\mathrm{T}}$, $[0.2, -2.8]^{\mathrm{T}}$, $[2.4, -2]^{\mathrm{T}}$, and $[2.5, 1.5]^{\mathrm{T}}$, respectively. The initial headings of the five agents are respectively $-2.25\,\mathrm{rad}$, $-0.49\,\mathrm{rad}$, $2.61\,\mathrm{rad}$, $1.84\,\mathrm{rad}$, and $2.88\,\mathrm{rad}$.

As expected, flocking behavior is observed in Figure 3.9. Different from examples with complete graphs, we also observe in Figure 3.9 that two of the agents collide and their positions overlap in the final formation. Further investigation shows that these collided agents do not share a communication link, which means that they are blind to each other and collision cannot be avoided.

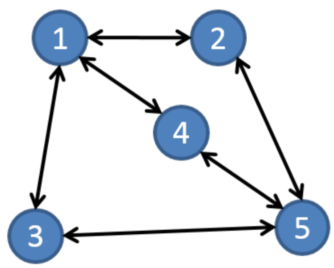
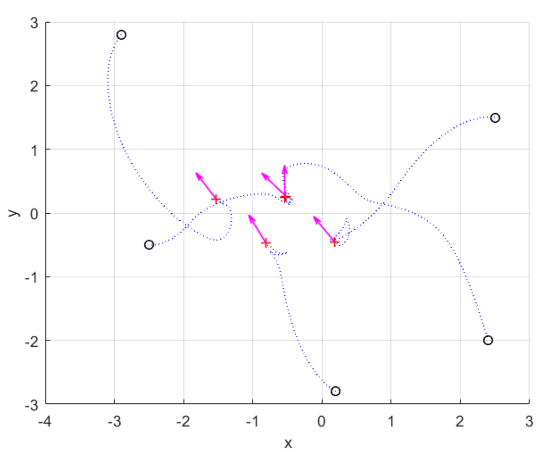
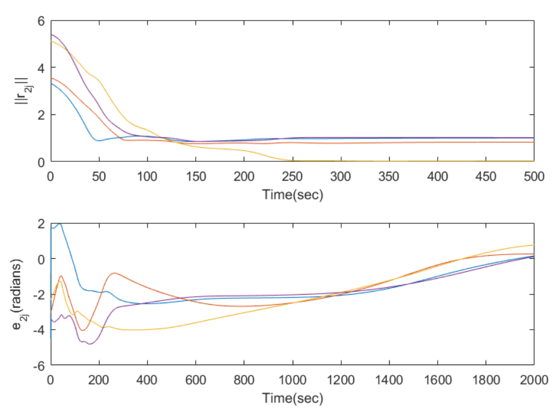


Figure 3.8: A connected but incomplete graph.



(a) Trajectories and headings.



(b) Relative distances and headings.

Figure 3.9: Simulation with a connected but incomplete graph, $\tau = 0.1$ s, $\varepsilon = 0.05$.

3.5 Summary

In this chapter, flocking control of multi-vehicle systems with nonholonomic dynamics and large communication delays was studied for the continuous-time scenarios. From an artificial potential function describing the attractive-repulsive interaction between the vehicles, a decentralized low gain control law was constructed. Before this work, potential function based flocking has not been studied for multi-agent systems with nonholonomic dynamics in the presence of communication delay. Through the Lyapunov functional approach, we proved that, as long as the communication topology is a connected undirected graph and initial potential is finite, for any arbitrarily large communication delay we can always find some low gain parameter such that aggregated positions and aligned headings will be achieved. Numerical examples demonstrated the effectiveness of the proposed control protocols. This chapter also demonstrates that the artificial potential function is a very useful and intuitive tool for coordinated control design. With given control objectives, we design the artificial potential function accordingly. Besides flocking, many other coordinated behaviors, including consensus and swarm tracking, can be achieved through such approach.

Chapter 4

Flocking of Nonholonomic Vehicles in the Discrete-time Setting

As mentioned in Section 1.5, direct discretization of a continuous-time controller requires small sampling periods. For the application purpose, we study the discrete-time flocking control problems in this chapter.

4.1 Problem statement

The continuous-time nonholonomic vehicle model (3.3) is discretized through a zero-order hold with an arbitrary sampling period T. The discrete-time model of multi-vehicle system is obtained as follows,

$$\begin{aligned} x_i(k+1) &= x_i(k) + Tv_i(k)\cos\theta_i(k), \\ y_i(k+1) &= y_i(k) + Tv_i(k)\sin\theta_i(k), \\ \theta_i(k+1) &= \theta_i(k) + T\omega_i(k), \end{aligned} \tag{4.1}$$

where the discrete-time states and inputs are defined as $x(k) = x(kT)$, $y(k) = y(kT)$, $\theta(k) = \theta(kT)$, $v_i(k) = v(kT)$, and $\omega(k) = \omega(kT)$, respectively. Similarly as in the continuous-time setting, we define the position vector $r_i = [x_i, y_i]^{\mathrm{T}}$, the relative position vector $r_{ij} = r_i - r_j$ and the heading error $e_{ij} = \theta_i - \theta_j$.

We assume that all agents have their states and inputs updated simultaneously, and so have the information on the relative position and heading error of their neighboring agents.

We also assume that the communication delay between two neighboring agents is a multiple of the sampling period, i.e., τT, where $\tau \in \mathbb{N}$. As in the continuous-time case, we also assume an undirected and connected communication graph as well as bounded initial potential. In the following section, we will develop a potential function based low gain control law which creates the flocking behavior of the multi-agent system in the discrete-time setting.

4.2 Control protocols

We propose the following discrete-time potential function based low gain control laws,

$$\begin{aligned} v_i(k) =& -\varepsilon \left[\cos\theta_i(k) \;\; \sin\theta_i(k)\right] \nabla_{r_i} V_i(k-\tau), \\ \tilde{\omega}_i(k) =& -\varepsilon\kappa \sum_{j\in\mathcal{N}_i} e_{ij}(k-\tau) - \varepsilon \left[-\sin\theta_i(k) \;\; \cos\theta_i(k)\right] \nabla_{r_i} V_i(k-\tau), \\ \omega_i(k) =& \operatorname{sign}(\tilde{\omega}_i(k)) \min\left\{|\tilde{\omega}_i(k)|, \frac{\pi}{2T}\right\}. \end{aligned} \tag{4.2}$$

We note that discrete-time control laws (4.2) are very similar to continuous-time control laws (3.3) except that the discrete-time angular velocity reaches saturation at $\pm\frac{\pi}{2T}$ as shown in Figure 4.1.

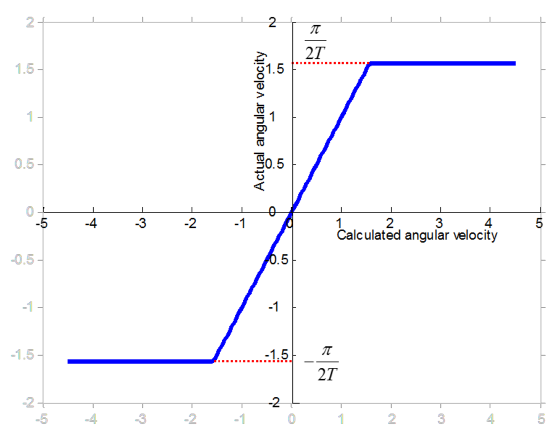


Figure 4.1: Saturated angular velocity.

When a vehicle reaches its steady states, for $i = 1, 2, \ldots, N$, the following equations

$$\left[\cos\theta_i(t) \;\; \sin\theta_i(t)\right] \nabla_{r_i} V_i(t-\tau) \equiv 0, \tag{4.3}$$

and

$$\left[\cos\theta_i(k) \;\; \sin\theta_i(k)\right] \nabla_{r_i} V_i(k-\tau) \equiv 0, \tag{4.4}$$

must hold for the continuous-time and the discrete-time scenarios respectively. Equations (4.3) and (4.4) indicate that the steering force for each vehicle is either zero or perpendicular to its heading. Note that (4.3) is true over a continuous time span, but (4.4) holds only at discrete points in time. From (4.3), it is derived that $\dot{\theta}_i(t) = 0$. However, from (4.4), we can only obtain $\theta_i(k+1) - \theta_i(k) = m\pi$ and m can be any integer. In other words, in the continuous-time case, all vehicles asymptotically approach a steady state where angular velocity equals to zero, while in the discrete-time case, without the saturation on angular velocity, the headings of vehicles may stay oscillating among angles with a common difference π after the vehicles reach their steady states.

4.3 Analysis of flocking behavior

A discrete-time version of Lemma 9 is needed for developing further results.

Lemma 13. *Define a new discrete-time function in terms of any given discrete-time function $f(k)$ as follows,*

$$h(k) = \sum_{s_1=0}^{\tau} \sum_{s_2=k-s_1}^{k-1} f(s_2),$$

where $\tau \in \mathbb{N}$ is a given constant. Then, the variation of $h(k)$ equals

$$h(k+1) - h(k) = \tau f(k) - \sum_{s_2=k-\tau}^{k-1} f(s_2).$$

Proof. Reordering the summation indices, we can rewrite $h(k)$ as

$$h(k) = \sum_{s_2=k-\tau}^{k-1} \sum_{s_1=k-s_2}^{\tau} f(s_2). \tag{4.5}$$

Since the function f in (4.5) does not depend on s_1, (4.5) can be simplified as

$$\begin{aligned} h(k) = &\sum_{s_2=k-\tau}^{k-1} (\tau - k + s_2 + 1) f(s_2), \\ =&(\tau - k + 1) \sum_{s_2=k-\tau}^{k-1} f(s_2) + \sum_{s_2=k-\tau}^{k-1} s_2 f(s_2). \end{aligned}$$

Then, the variation of $h(k)$ is computed as

$$\begin{aligned} h(k+1)-h(k) =&(\tau-k)\sum_{s_2=k-\tau+1}^{k} f(s_2) - (\tau-k+1)\sum_{s_2=k-\tau}^{k-1} f(s_2) + \sum_{s_2=k-\tau+1}^{k} s_2 f(s_2) - \sum_{s_2=k-\tau}^{k-1} s_2 f(s_2), \\ =&(\tau-k)(f(k)-f(k-\tau)) - \sum_{s_2=k-\tau}^{k-1} f(s_2) + kf(k) - (k-\tau)f(k-\tau), \\ =&\tau f(k) - \sum_{s_2=k-\tau}^{k-1} f(s_2). \end{aligned} \tag{4.6}$$

□

The next lemma demonstrates, for a sufficiently small value of the low gain parameter ε, system (4.1) achieves position aggregation under the control law (4.2), even in the presence of an arbitrarily large communication delay τ.

Lemma 14. *Consider multi-agent system (4.1) with communication delay τ. If both Assumptions 1 and 3 are satisfied, there exists an $\bar{\varepsilon}_1 > 0$ such that, for any $0 < \varepsilon < \bar{\varepsilon}_1$, the low gain control laws (4.2) drive the system into a formation corresponding to a minimum of $\sum_{i=0}^{N} V_i$.*

Proof. By Taylor series expansion, we have

$$V_{ij}(k+1) = V_{ij}(k) + \nabla_{r_{ij}}^{\mathrm{T}} V_{ij}(k)(r_{ij}(k+1) - r_{ij}(k)) + R_{ij}(k), \tag{4.7}$$

where $R_{ij}(k)$ is the Lagrange remainder.

By Assumption 3, the initial states of the system are bounded in the set Ω, and the second and higher order derivatives of V_{ij} are also bounded. Thus, there always exists an $\alpha > 0$ that satisfies

$$R_{ij}(k) \le \alpha(r_{ij}(k+1) - r_{ij}(k))^{\mathrm{T}}(r_{ij}(k+1) - r_{ij}(k)), \tag{4.8}$$

for all i and j.

Consider a potential function

$$V_{r,1}(k) = \sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i} V_{ij}(k). \tag{4.9}$$

Combining (4.7) and (4.8) yields

$$\begin{aligned}\Delta V_{r,1}(k) =& V_{r,1}(k+1) - V_{r,1}(k),\\ \leq& \sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i} \nabla_{r_{ij}}^{\mathrm{T}} V_{ij}(k)(r_{ij}(k+1) - r_{ij}(k))\\ &+ \alpha\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}(r_{ij}(k+1) - r_{ij}(k))^{\mathrm{T}}(r_{ij}(k+1) - r_{ij}(k)). \end{aligned} \tag{4.10}$$

In view of (4.1) and (4.2), we have

$$r_{ij}(k+1) = r_{ij}(k) - \varepsilon T\Phi_i(k)\nabla_{r_i} V_i(k-\tau) + \varepsilon T\Phi_j(k)\nabla_{r_j} V_j(k-\tau), \tag{4.11}$$

where $\Phi_i(k)$ is a positive semi-definite matrix and is defined as

$$\Phi_i(k) = \begin{bmatrix}\cos\theta_i(k)\\ \sin\theta_i(k)\end{bmatrix}\begin{bmatrix}\cos\theta_i(k) & \sin\theta_i(k)\end{bmatrix}.$$

For simplicity, we introduce the following notation,

$$\nabla_{r_i} V_{ij}(k) = 2r_{ij}(k)\Pi_{ij}(k), \tag{4.12}$$

where

$$\Pi_{ij}(k) = \frac{\rho^2}{\|r_{ij}(k)\|^2} - \frac{\rho^4}{\|r_{ij}(k)\|^4}. \tag{4.13}$$

Substituting (4.11) and (4.12) into (4.10) yields

$$\begin{aligned}\Delta V_{r,1}(k) \leq& \sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i} 2r_{ij}^{\mathrm{T}}(k)\Pi_{ij}(k)\Bigg(-2\varepsilon T\Phi_i(k)\sum_{l\in\mathcal{N}_i} r_{il}(k-\tau)\Pi_{il}(k-\tau)\\ &+ 2\varepsilon T\Phi_j(k)\sum_{l\in\mathcal{N}_j} r_{jl}(k-\tau)\Pi_{jl}(k-\tau)\Bigg) + \alpha\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i} 2\varepsilon T\\ &\times\Bigg(-\Phi_i(k)\sum_{l\in\mathcal{N}_i} r_{il}(k-\tau)\Pi_{il}(k-\tau) + \Phi_j(k)\sum_{l\in\mathcal{N}_j} r_{jl}(k-\tau)\Pi_{jl}(k-\tau)\Bigg)^{\mathrm{T}}\\ &\times 2\varepsilon T\Bigg(-\Phi_i(k)\sum_{l\in\mathcal{N}_i} r_{il}(k-\tau)\Pi_{il}(k-\tau) + \Phi_j(k)\sum_{l\in\mathcal{N}_j} r_{jl}(k-\tau)\Pi_{jl}(k-\tau)\Bigg).\end{aligned}$$

Applying Lemma 7 to the second term on the right-hand side of the above inequality and reordering summation indices gives

$$\Delta V_{r,1}(k) \leq -8\varepsilon T\sum_{i=1}^{N}\Bigg(\sum_{j\in\mathcal{N}_i} r_{ij}(k)\Pi_{ij}(k)\Bigg)^{\mathrm{T}}\Phi_i(k)\Bigg(\sum_{j\in\mathcal{N}_i} r_{ij}(k-\tau)\Pi_{ij}(k-\tau)\Bigg)$$

$$+ 16\alpha\varepsilon^2 T^2 d_{\max} \sum_{i=1}^{N} \left(\sum_{j\in\mathcal{N}_i} r_{ij}(k-\tau)\Pi_{ij}(k-\tau) \right)^{\mathrm{T}}$$

$$\times \Phi_i^2(k) \left(\sum_{j\in\mathcal{N}_i} r_{ij}(k-\tau)\Pi_{ij}(k-\tau) \right), \tag{4.14}$$

where $d_{\max} = \max_{i=1,2,\dots,N}\{d_i\}$ and d_i is defined in Definition 4. By the Mean Value Theorem, we have

$$r_{ij}(k)\Pi_{ij}(k) = r_{ij}(k-\tau)\Pi_{ij}(k-\tau) + \Lambda(\hat{r}_{ij})(r_{ij}(k) - r_{ij}(k-\tau)), \tag{4.15}$$

where $\hat{r}_{ij} = r_{ij}(k-\tau) + c_{ij}(r_{ij}(k) - r_{ij}(k-\tau))$, for some $c_{ij} \in (0,1)$, and

$$\Lambda(\hat{r}_{ij}) = \left(\frac{\rho^2}{\|\hat{r}_{ij}\|^2} - \frac{\rho^4}{\|\hat{r}_{ij}\|^4} \right) I + 2\left(\frac{2\rho^4}{\|\hat{r}_{ij}\|^6} - \frac{\rho^2}{\|\hat{r}_{ij}\|^4} \right) \hat{r}_{ij}\hat{r}_{ij}^{\mathrm{T}}.$$

Define $\lambda_{ij}(k) = \Lambda(\hat{r}_{ij})(r_{ij}(k+1) - r_{ij}(k))$, then (4.15) can be rewritten as

$$r_{ij}(k)\Pi_{ij}(k) = r_{ij}(k-\tau)\Pi_{ij}(k-\tau) + \sum_{\sigma=k-\tau}^{k-1} \lambda_{ij}(\sigma). \tag{4.16}$$

Substituting (4.16) into (4.14), we obtain

$$\begin{aligned}
\Delta V_{r,1}(k) \le & - 8\varepsilon T \sum_{i=1}^{N} \left(\sum_{j\in\mathcal{N}_i} r_{ij}(k-\tau)\Pi_{ij}(k-\tau) \right)^{\mathrm{T}} \Phi_i(k) \left(\sum_{j\in\mathcal{N}_i} r_{ij}(k-\tau)\Pi_{ij}(k-\tau) \right) \\
& - 8\varepsilon T \sum_{i=1}^{N} \left(\sum_{j\in\mathcal{N}_i} \sum_{\sigma=k-\tau}^{k-1} \lambda_{ij}(\sigma) \right)^{\mathrm{T}} \Phi_i(k) \left(\sum_{j\in\mathcal{N}_i} r_{ij}(k-\tau)\Pi_{ij}(k-\tau) \right) \\
& + 16\alpha\varepsilon^2 T^2 d_{\max} \sum_{i=1}^{N} \left(\sum_{j\in\mathcal{N}_i} r_{ij}(k-\tau)\Pi_{ij}(k-\tau) \right)^{\mathrm{T}} \Phi_i^2(k) \left(\sum_{j\in\mathcal{N}_i} r_{ij}(k-\tau)\Pi_{ij}(k-\tau) \right).
\end{aligned}$$

Applying Lemma 7 to the second term on the right-hand side of the above inequality gives

$$\begin{aligned}
\Delta V_{r,1}(k) \le & - 8\varepsilon T \sum_{i=1}^{N} \left(\sum_{j\in\mathcal{N}_i} r_{ij}(k-\tau)\Pi_{ij}(k-\tau) \right)^{\mathrm{T}} \Phi_i(k) \left(\sum_{j\in\mathcal{N}_i} r_{ij}(k-\tau)\Pi_{ij}(k-\tau) \right) \\
& + 16\varepsilon^2 T^2 \sum_{i=1}^{N} \left(\sum_{j\in\mathcal{N}_i} r_{ij}(k-\tau)\Pi_{ij}(k-\tau) \right)^{\mathrm{T}} \Phi_i^2(k) \left(\sum_{j\in\mathcal{N}_i} r_{ij}(k-\tau)\Pi_{ij}(k-\tau) \right) \\
& + 16\alpha\varepsilon^2 T^2 d_{\max} \sum_{i=1}^{N} \left(\sum_{j\in\mathcal{N}_i} r_{ij}(k-\tau)\Pi_{ij}(k-\tau) \right)^{\mathrm{T}} \Phi_i^2(k) \left(\sum_{j\in\mathcal{N}_i} r_{ij}(k-\tau)\Pi_{ij}(k-\tau) \right) \\
& + \sum_{i=1}^{N} \left(\sum_{\sigma=k-\tau}^{k-1} \sum_{j\in\mathcal{N}_i} \lambda_{ij}(\sigma) \right)^{\mathrm{T}} \left(\sum_{\sigma=k-\tau}^{k-1} \sum_{j\in\mathcal{N}_i} \lambda_{ij}(\sigma) \right).
\end{aligned}$$

Lemma 7 can be applied again to the last term on the right-hand side of the above inequality to obtain

$$\begin{aligned}\Delta V_{r,1}(k) \leq & - 8\varepsilon T \sum_{i=1}^{N} \left(\sum_{j\in\mathcal{N}_i} r_{ij}(k-\tau)\Pi_{ij}(k-\tau) \right)^{\mathrm{T}} \\ & \times \left(\Phi_i(k) - 2\alpha\varepsilon T d_{\max}\Phi_i^2(k) - 2\varepsilon T \Phi_i^2(k)\right) \left(\sum_{j\in\mathcal{N}_i} r_{ij}(k-\tau)\Pi_{ij}(k-\tau) \right) \\ & + \tau \sum_{i=1}^{N} \sum_{\sigma=k-\tau}^{k-1} \left(\sum_{j\in\mathcal{N}_i} \lambda_{ij}(\sigma) \right)^{\mathrm{T}} \left(\sum_{j\in\mathcal{N}_i} \lambda_{ij}(\sigma) \right). \end{aligned} \tag{4.17}$$

Consider a second potential function

$$V_{r,2}(k) = \tau \sum_{i=1}^{N} \sum_{s_1=0}^{\tau} \sum_{s_2=k-s_1}^{k-1} \left(\sum_{j\in\mathcal{N}_i} \lambda_{ij}(s_2) \right)^{\mathrm{T}} \left(\sum_{j\in\mathcal{N}_i} \lambda_{ij}(s_2) \right).$$

In view of Lemma 13, the variation of $V_{r,2}(k)$ is computed as

$$\begin{aligned}\Delta V_{r,2}(k) &= V_{r,2}(k+1) - V_{r,2}(k), \\ &= \tau^2 \sum_{i=1}^{N} \left(\sum_{j\in\mathcal{N}_i} \lambda_{ij}(k) \right)^{\mathrm{T}} \left(\sum_{j\in\mathcal{N}_i} \lambda_{ij}(k) \right) \\ & \quad - \tau \sum_{i=1}^{N} \sum_{\sigma=k-\tau}^{k-1} \left(\sum_{j\in\mathcal{N}_i} \lambda_{ij}(\sigma) \right)^{\mathrm{T}} \left(\sum_{j\in\mathcal{N}_i} \lambda_{ij}(\sigma) \right). \end{aligned} \tag{4.18}$$

Finally, we define a proper Lyapunov function $V_r(k) = V_{r,1}(k) + V_{r,2}(k)$, the variation of which can be easily determined by combining (4.17) and (4.18) as follows,

$$\begin{aligned}\Delta V_r(k) \leq & - 8\varepsilon T \sum_{i=1}^{N} \left(\sum_{j\in\mathcal{N}_i} r_{ij}(k-\tau)\Pi_{ij}(k-\tau) \right)^{\mathrm{T}} \left(\Phi_i(k) - 2\alpha\varepsilon T d_{\max}\Phi_i^2(k) - 2\varepsilon T \Phi_i^2(k)\right) \\ & \times \left(\sum_{j\in\mathcal{N}_i} r_{ij}(k-\tau)\Pi_{ij}(k-\tau) \right) + \tau^2 \sum_{i=1}^{N} \left(\sum_{j\in\mathcal{N}_i} \lambda_{ij}(k) \right)^{\mathrm{T}} \left(\sum_{j\in\mathcal{N}_i} \lambda_{ij}(k) \right). \end{aligned}$$

In view of Lemma 7, the above inequality implies that

$$\begin{aligned}\Delta V_r(k) \leq & - 8\varepsilon T \sum_{i=1}^{N} \left(\sum_{j\in\mathcal{N}_i} r_{ij}(k-\tau)\Pi_{ij}(k-\tau) \right)^{\mathrm{T}} \left(\Phi_i(k) - 2\alpha\varepsilon T d_{\max}\Phi_i^2(k) - 2\varepsilon T \Phi_i^2(k)\right) \\ & \times \left(\sum_{j\in\mathcal{N}_i} r_{ij}(k-\tau)\Pi_{ij}(k-\tau) \right) + \tau^2 d_{\max} \sum_{i=1}^{N} \sum_{j\in\mathcal{N}_i} \lambda_{ij}^{\mathrm{T}}(k)\lambda_{ij}(k). \end{aligned} \tag{4.19}$$

With the aid of (4.11), $\lambda_{ij}(k)$ can be rewritten as,

$$\lambda_{ij}(k) = 2\varepsilon T\Lambda(\hat{r}_{ij}) \left(-\Phi_i(k) \sum_{l\in\mathcal{N}_i} r_{il}(k-\tau)\Pi_{il}(k-\tau) + \Phi_j(k) \sum_{l\in\mathcal{N}_j} r_{jl}(k-\tau)\Pi_{jl}(k-\tau) \right),$$

and Lemma 7 can be used to demonstrate that

$$\begin{aligned}\lambda_{ij}^{\mathrm{T}}(k)\lambda_{ij}(k) \leq\ & 8\varepsilon^2T^2 \left(\sum_{l\in\mathcal{N}_i} r_{il}(k-\tau)\Pi_{il}(k-\tau) \right)^{\mathrm{T}} \Phi_i(k)\Lambda^2(\hat{r}_{ij})\Phi_i(k) \\ & \times \left(\sum_{l\in\mathcal{N}_i} r_{il}(k-\tau)\Pi_{il}(k-\tau) \right) + 8\varepsilon^2T^2 \left(\sum_{l\in\mathcal{N}_j} r_{jl}(k-\tau)\Pi_{jl}(k-\tau) \right)^{\mathrm{T}} \\ & \times \Phi_j(k)\Lambda^2(\hat{r}_{ij})\Phi_j(k) \left(\sum_{l\in\mathcal{N}_j} r_{jl}(k-\tau)\Pi_{jl}(k-\tau) \right). \end{aligned} \tag{4.20}$$

By Assumption 3, there exists some constant $\eta > 0$ such that $\eta I \geq \Lambda^2(\hat{r}_{ij})$ for any i and $j \neq i$. After summing both sides of (4.20) over i and j, we deduce that

$$\begin{aligned}\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i} \lambda_{ij}^{\mathrm{T}}(k)\lambda_{ij}(k) \leq 16\varepsilon^2T^2\eta d_{\max} \sum_{i=1}^{N} & \left(\sum_{j\in\mathcal{N}_i} r_{ij}(k-\tau)\Pi_{ij}(k-\tau) \right)^{\mathrm{T}} \Phi_i^2(k) \\ & \times \left(\sum_{j\in\mathcal{N}_i} r_{ij}(k-\tau)\Pi_{ij}(k-\tau) \right). \end{aligned} \tag{4.21}$$

Substituting (4.21) into (4.19) and combining the fact that $\Phi_i^2(k) = \Phi_i(k)$ is positive semi-definite give

$$\begin{aligned}\Delta V_r(k) \leq & -8\varepsilon T\big(1 - 2\alpha\varepsilon T d_{\max} - 2\varepsilon T - 2\varepsilon T\eta\tau^2 d_{\max}^2\big) \sum_{i=1}^{N} \left(\sum_{j\in\mathcal{N}_i} r_{ij}(k-\tau)\Pi_{ij}(k-\tau) \right)^{\mathrm{T}} \Phi_i(k) \\ & \times \left(\sum_{j\in\mathcal{N}_i} r_{ij}(k-\tau)\Pi_{ij}(k-\tau) \right). \end{aligned} \tag{4.22}$$

Define the following constant

$$\bar{\varepsilon}_1 = \frac{1}{2T(1 + \alpha d_{\max} + \eta\tau^2 d_{\max}^2)}. \tag{4.23}$$

Then it follows that $\Delta V_r(k) \leq 0$, for any $0 < \varepsilon < \bar{\varepsilon}_1$.

From (4.22), we can easily see that $\Delta V_r(k) \equiv 0$ if and only if

$$\begin{bmatrix}\cos\theta_i(k) & \sin\theta_i(k)\end{bmatrix} \sum_{j\in\mathcal{N}_i} r_{ij}(k-\tau)\Pi_{ij}(k-\tau) \equiv 0, \tag{4.24}$$

for all agent i and all $k \geq 0$, and hence the following is true as well,

$$\Big[\cos\theta_i(k+1)\ \ \sin\theta_i(k+1)\Big] \sum_{j\in\mathcal{N}_i} r_{ij}(k+1-\tau)\Pi_{ij}(k+1-\tau) \equiv 0. \tag{4.25}$$

Equation (4.24) indicates that $v_i(k) = 0$ for all agent i. Any complete trajectory of system (4.1) satisfying (4.24) also possesses the property that $r_{ij}(k+1) = r_{ij}(k)$. Therefore, we have

$$\sum_{j\in\mathcal{N}_i} r_{ij}(k+1-\tau)\Pi_{ij}(k+1-\tau) = \sum_{j\in\mathcal{N}_i} r_{ij}(k-\tau)\Pi_{ij}(k-\tau),$$

which, together with (4.25), implies that

$$\Big[\cos\theta_i(k+1)\ \ \sin\theta_i(k+1)\Big] \sum_{j\in\mathcal{N}_i} r_{ij}(k-\tau)\Pi_{ij}(k-\tau) = 0, \tag{4.26}$$

for any agent i. Suppose that $\sum_{j\in\mathcal{N}_i} r_{ij}\Pi_{ij} \neq 0$ for all agent i. Then, after comparing (4.24) and (4.26), we find that

$$\theta_i(k+1) - \theta_i(k) = m\pi,$$

where m could be any integer. According to the proposed control laws (4.2), it is true that

$$|\theta_i(k+1) - \theta_i(k)| \leq \frac{\pi}{2}.$$

Consequently, we have $\theta_i(k+1) = \theta_i(k)$, and $\omega_i(k) = 0$, for all agent i.

Finally, following similar arguments as in the proof of Lemma 11, we conclude that $\omega_i(k) = 0$ if and only if

$$\sum_{j\in\mathcal{N}_i} r_{ij}\Pi_{ij} = 0, \tag{4.27}$$

for all agent i. Therefore, by LaSalle's Invariance Principle, system (4.1) is driven into a formation that satisfies (4.27). In the meantime, it can be easily verified from (4.12) that any equilibrium satisfying (4.27) corresponds to a minimum of $\sum_{i=0}^{N} V_i$. This completes the proof. □

Remark 1. *In the above proof, we derived a sufficient condition on the low gain parameter* ε *for position aggregation in the presence of communication delay* τ*. For the delay-free cases where* $\tau = 0$*, the first term in (4.14) becomes negative semi-definite and the second term is positive semi-definite. Then, without further manipulations we obtain that* $\Delta V_{r,1} \leq 0$ *when* $\varepsilon < \frac{1}{2\alpha T d_{max}}$.

Lemma 14 proves that system (4.1) converges to a desired formation under the distributed low gain control laws (4.2). In the next two lemmas, it will be shown that heading alignment is also guaranteed.

Lemma 15. *Consider the following system*

$$z(k+1) = z(k) + u_1(k),$$

where $z \in \mathbb{R}$ is the state variable and $u_1 \in \mathbb{R}$ is the control input. Let $V(z(k)) = z^2(k)$. Suppose that there exists some feedback control law $u_1(z(k))$ under which

$$\Delta V(k) = V(z(k+1)) - V(z(k)) < 0, \quad z(k) \neq 0.$$

Then,

$$\Delta V(k) < 0, \quad z(k) \neq 0,$$

if we replace $u_1(z(k))$ with $u_2(z(k)) = \operatorname{sign}(u_1(z(k))) \min\{|u_1(z(k))|, \beta\}$, where β is a positive constant.

Proof. We know that

$$\begin{aligned} \Delta V(k) =& z^2(k+1) - z^2(k), \\ =& 2z(k)u_1(z(k)) + u_1^2(z(k)) < 0, \quad z(k) \neq 0. \end{aligned} \tag{4.28}$$

The saturated control input $u_2(z(k))$, $z(k) \neq 0$, can be rewritten as

$$\begin{aligned} u_2(z(k)) =& a u_1(z(k)), \\ a =& \frac{\min(|u_1(z(k))|, \beta)}{|u_1(z(k))|}, \end{aligned}$$

and $a \in (0, 1]$. If the system is steered by $u_2(z(k))$ instead, the variation of $V(z(k))$ becomes

$$\begin{aligned} \Delta V(k) =& 2z(k)u_2(z(k)) + u_2^2(z(k)), \\ =& 2az(k)u_1(z(k)) + a^2u_1^2(z(k)), \\ \leq& a\left(2z(k)u_1(z(k)) + u_1^2(z(k))\right). \end{aligned}$$

In view of Equation (4.28), we have

$$\Delta V(k) < 0, \quad z(k) \neq 0.$$

□

Lemma 16. *Consider multi-agent system (4.1) with communication delay τ. If both Assumptions 1 and 3 are satisfied, there exists an $\bar{\varepsilon}_2 > 0$ such that, for any $0 < \varepsilon < \bar{\varepsilon}_2$, the low gain control laws (4.2) aligns the orientations of all agents as the formation convergence is achieved asymptotically.*

Proof. Consider a positive definite function

$$V_{e,1}(k) = \sum_{i=1}^{N} \sum_{j \in \mathcal{N}_i} e_{ij}^2(k). \tag{4.29}$$

In view of Lemma 15, it is equivalent to prove that heading alignment will be achieved if the system (4.1) is steered by $\tilde{\omega}_i$ instead, where $\tilde{\omega}_i$ is defined in (4.2). In other words, we consider the following dynamics of orientation errors,

$$\begin{aligned} e_{ij}(k+1) = & e_{ij}(k) + \varepsilon T \left(-\kappa \sum_{l \in \mathcal{N}_i} e_{il}(k-\tau) - \Psi_i(k) \nabla_{r_i} V_i(k-\tau) \right. \\ & \left. + \kappa \sum_{l \in \mathcal{N}_j} e_{jl}(k-\tau) + \Psi_j(k) \nabla_{r_j} V_j(k-\tau) \right), \end{aligned} \tag{4.30}$$

where we have introduced the following definition to simplify notation,

$$\Psi_i(k) = \begin{bmatrix} -\sin\theta_i(k) & \cos\theta_i(k) \end{bmatrix}.$$

Let $e_{ij}(k+1) = (e_{ij}(k+1) - e_{ij}(k)) + e_{ij}(k)$. After simple manipulations we obtain the following equality,

$$e_{ij}^2(k+1) = e_{ij}^2(k) + 2e_{ij}(k)(e_{ij}(k+1) - e_{ij}(k)) + (e_{ij}(k+1) - e_{ij}(k))^2. \tag{4.31}$$

Combining (4.30) and (4.31) yields

$$\begin{aligned} \Delta V_{e,1}(k) = & V_{e,1}(k+1) - V_{e,1}(k), \\ = & 2\varepsilon T \sum_{i=1}^{N} \sum_{j \in \mathcal{N}_i} e_{ij}(k) \left(-\kappa \sum_{l \in \mathcal{N}_i} e_{il}(k-\tau) - \Psi_i(k) \nabla_{r_i} V_i(k-\tau) \right. \\ & \left. + \kappa \sum_{l \in \mathcal{N}_j} e_{jl}(k-\tau) + \Psi_j(k) \nabla_{r_j} V_j(k-\tau) \right) + \sum_{i=1}^{N} \sum_{j \in \mathcal{N}_i} \Delta e_{ij}^2(k), \end{aligned} \tag{4.32}$$

where $\Delta e_{ij}(k) = e_{ij}(k+1) - e_{ij}(k)$. By reordering summation indices and collecting similar terms, (4.32) can be simplified as

$$\begin{aligned}\Delta V_{e,1}(k) =& -4\varepsilon\kappa T \sum_{i=1}^{N} \sum_{j\in\mathcal{N}_i} e_{ij}(k) \sum_{l\in\mathcal{N}_i} e_{il}(k-\tau) - 4\varepsilon T \sum_{i=1}^{N} \sum_{j\in\mathcal{N}_i} e_{ij}(k)\Psi_i(k)\nabla_{r_i} V_i(k-\tau) \\ &+ \sum_{i=1}^{N} \sum_{j\in\mathcal{N}_i} \Delta e_{ij}^2(k). \end{aligned} \tag{4.33}$$

Express $e_{ij}(k)$ in terms of $e_{ij}(k-\tau)$ and Δe_{ij} as follows,

$$e_{ij}(k) = e_{ij}(k-\tau) + \sum_{\sigma=k-\tau}^{k-1} \Delta e_{ij}(\sigma).$$

Substituting the above expression into (4.33) results in

$$\begin{aligned}\Delta V_{e,1}(k) =& -4\varepsilon\kappa T \sum_{i=1}^{N} \left(\sum_{j\in\mathcal{N}_i} e_{ij}(k-\tau) \right)^2 - 4\varepsilon\kappa T \sum_{i=1}^{N} \sum_{\sigma=k-\tau}^{k-1} \sum_{j\in\mathcal{N}_i} \Delta e_{ij}(\sigma) \sum_{l\in\mathcal{N}_i} e_{il}(k-\tau) \\ &- 4\varepsilon T \sum_{i=1}^{N} \sum_{j\in\mathcal{N}_i} e_{ij}(k-\tau)\Psi_i(k)\nabla_{r_i} V_i(k-\tau) \\ &- 4\varepsilon T \sum_{i=1}^{N} \sum_{\sigma=k-\tau}^{k-1} \sum_{j\in\mathcal{N}_i} \Delta e_{ij}(\sigma)\Psi_i(k)\nabla_{r_i} V_i(k-\tau) + \sum_{i=1}^{N} \sum_{j\in\mathcal{N}_i} \Delta e_{ij}^2(k). \end{aligned} \tag{4.34}$$

Recalling that $2ab \le a^2 + b^2$ for any $a, b \in \mathbb{R}$, and applying it to the non-quadratic terms in (4.34), we obtain

$$\begin{aligned}\Delta V_{e,1}(k) \le& \left(-4\varepsilon\kappa T + 5\varepsilon^2\kappa^2 T^2\right) \sum_{i=1}^{N} \left(\sum_{j\in\mathcal{N}_i} e_{ij}(k-\tau) \right)^2 \\ &+ \left(\frac{4}{\kappa^2} + 4\varepsilon^2 T^2 \right) \sum_{i=1}^{N} (\Psi_i(k)\nabla_{r_i} V_i(k-\tau))^2 \\ &+ 2\sum_{i=1}^{N} \left(\sum_{\sigma=k-\tau}^{k-1} \sum_{j\in\mathcal{N}_i} \Delta e_{ij}(\sigma) \right)^2 + \sum_{i=1}^{N} \sum_{j\in\mathcal{N}_i} \Delta e_{ij}^2(k). \end{aligned}$$

In view of Lemma 7, it follows from the above inequality that

$$\begin{aligned}\Delta V_{e,1}(k) \le& (-4\varepsilon\kappa T + 5\varepsilon^2\kappa^2 T^2) \sum_{i=1}^{N} \left(\sum_{j\in\mathcal{N}_i} e_{ij}(k-\tau) \right)^2 \\ &+ \left(\frac{4}{\kappa^2} + 4\varepsilon^2 T^2 \right) \sum_{i=1}^{N} (\Psi_i(k)\nabla_{r_i} V_i(k-\tau))^2 \end{aligned}$$

$$+ 2\tau \sum_{i=1}^{N} \sum_{\sigma=k-\tau}^{k-1} \left(\sum_{j\in\mathcal{N}_i} \Delta e_{ij}(\sigma) \right)^2 + \sum_{i=1}^{N} \sum_{j\in\mathcal{N}_i} \Delta e_{ij}^2(k). \tag{4.35}$$

Define a second positive definite function

$$V_{e,2}(k) = 2\tau \sum_{i=1}^{N} \sum_{s_1=0}^{\tau} \sum_{s_2=k-s_1}^{k-1} \left(\sum_{j\in\mathcal{N}_i} \Delta e_{ij}(s_2) \right)^2 .$$

By Lemma 13, the variation of $V_{e,2}(k)$ is computed as

$$\begin{aligned} \Delta V_{e,2}(k) &= V_{e,2}(k+1) - V_{e,2}(k) \\ &= 2\tau^2 \sum_{i=1}^{N} \left(\sum_{j\in\mathcal{N}_i} \Delta e_{ij}(k) \right)^2 - 2\tau \sum_{i=1}^{N} \sum_{\sigma=k-\tau}^{k-1} \left(\sum_{j\in\mathcal{N}_i} \Delta e_{ij}(\sigma) \right)^2 . \end{aligned} \tag{4.36}$$

Finally, consider the Lyapunov function $V_e(k) = V_{e,1}(k) + V_{e,2}(k)$. Its variation is determined by combining (4.35) and (4.36) as follows.

$$\begin{aligned} \Delta V_e(k) \le{}& (-4\varepsilon\kappa T + 5\varepsilon^2\kappa^2 T^2) \sum_{i=1}^{N} \left(\sum_{j\in\mathcal{N}_i} e_{ij}(k-\tau) \right)^2 \\ &+ \left(\frac{4}{\kappa^2} + 4\varepsilon^2 T^2 \right) \sum_{i=1}^{N} (\Psi_i(k) \nabla_{r_i} V_i(k-\tau))^2 \\ &+ 2\tau^2 \sum_{i=1}^{N} \left(\sum_{j\in\mathcal{N}_i} \Delta e_{ij}(k) \right)^2 + \sum_{i=1}^{N} \sum_{j\in\mathcal{N}_i} \Delta e_{ij}^2(k). \end{aligned} \tag{4.37}$$

By Lemma 7, the last two terms in (4.37) can be simplified and combined

$$\begin{aligned} \Delta V_e(k) \le{}& (-4\varepsilon\kappa T + 5\varepsilon^2\kappa^2 T^2) \sum_{i=1}^{N} \left(\sum_{j\in\mathcal{N}_i} e_{ij}(k-\tau) \right)^2 \\ &+ \left(\frac{4}{\kappa^2} + 4\varepsilon^2 T^2 \right) \sum_{i=1}^{N} (\Psi_i(k) \nabla_{r_i} V_i(k-\tau))^2 \\ &+ \left(2\tau^2 d_{\max} + 1 \right) \sum_{i=1}^{N} \sum_{j\in\mathcal{N}_i} \Delta e_{ij}^2(k). \end{aligned} \tag{4.38}$$

Applying Lemma 7 to the following definition of $\Delta e_{ij}(k)$

$$\Delta e_{ij}(k) = \varepsilon T \left(-\kappa \sum_{l\in\mathcal{N}_i} e_{il}(k-\tau) - \Psi_i(k) \nabla_{r_i} V_i(k-\tau) + \kappa \sum_{l\in\mathcal{N}_j} e_{jl}(k-\tau) + \Psi_j(k) \nabla_{r_j} V_j(k-\tau) \right),$$

we have

$$\Delta e_{ij}^2(k) \leq 4\varepsilon^2\kappa^2T^2\left(\sum_{l\in\mathcal{N}_i} e_{il}(k-\tau)\right)^2 + 4\varepsilon^2\kappa^2T^2\left(\sum_{l\in\mathcal{N}_j} e_{jl}(k-\tau)\right)^2$$
$$+ 4\varepsilon^2T^2\left(\Psi_i(k)\nabla_{r_i}V_i(k-\tau)\right)^2 + 4\varepsilon^2T^2\left(\Psi_j(k)\nabla_{r_j}V_j(k-\tau)\right)^2. \tag{4.39}$$

Summing both sides of (4.39) over i and j and reordering the summation indices, we have the resulting inequality as

$$\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\Delta e_{ij}^2(k) \leq 8\varepsilon^2\kappa^2T^2d_{\max}\sum_{i=1}^{N}\left(\sum_{j\in\mathcal{N}_i} e_{ij}(k-\tau)\right)^2$$
$$+ 8\varepsilon^2T^2d_{\max}\sum_{i=1}^{N}\left(\Psi_i(k)\nabla_{r_i}V_i(k-\tau)\right)^2. \tag{4.40}$$

Substituting (4.40) into (4.38) yields

$$\Delta V_e(k) \leq (-4\varepsilon\kappa T + (8d_{\max}+5)\varepsilon^2\kappa^2T^2 + 16\varepsilon^2\kappa^2T^2\tau^2d_{\max}^2)\sum_{i=1}^{N}\left(\sum_{j\in\mathcal{N}_i} e_{ij}(k-\tau)\right)^2$$
$$+\left(\frac{4}{\kappa^2} + (8d_{\max}+4)\varepsilon^2T^2 + 16\varepsilon^2T^2\tau^2d_{\max}^2\right)\sum_{i=1}^{N}(\Psi_i(k)\nabla_{r_i}V_i(k-\tau))^2. \tag{4.41}$$

It is proved in Lemma 14 that $\nabla_{r_i}V_i(k-\tau)$ asymptotically approaches 0 as $k\to\infty$ for all agent i, if $\varepsilon < \bar{\varepsilon}_1$. This indicates that, for any given $\gamma > 0$, there exists some $K_1 > 0$ such that for $k \geq K_1$,

$$\Delta V_e(k) \leq (-4\varepsilon\kappa T + (8d_{\max}+5)\varepsilon^2\kappa^2T^2 + 16\varepsilon^2\kappa^2T^2\tau^2d_{\max}^2)$$
$$\times\sum_{i=1}^{N}\left(\sum_{j\in\mathcal{N}_i} e_{ij}(k-\tau)\right)^2 + \gamma^2. \tag{4.42}$$

Define the following positive constant

$$\bar{\varepsilon}_2 = \frac{4}{\kappa T(16\tau^2d_{\max}^2 + 8d_{\max}+5)}.$$

In (4.42) we easily observe that, for any $0 < \varepsilon < \min\{\bar{\varepsilon}_1, \bar{\varepsilon}_2\}$, $\Delta V_e(k) < 0$ when

$$\left|\sum_{j\in\mathcal{N}_i} e_{ij}(k-\tau)\right| > \frac{\gamma}{\sqrt{4\varepsilon\kappa T - (8d_{\max}+5)\varepsilon^2\kappa^2T^2 - 16\varepsilon^2\kappa^2T^2\tau^2d_{\max}^2}},$$

for any agent i. Since the dominator in the above expression is a constant and γ can be arbitrarily small, this implies that $\sum_{j \in \mathcal{N}_i} e_{ij}(k-\tau) \to 0$ as $k \to \infty$ for all agent i. Consequently $L\theta \to 0$ as $k \to \infty$, where $\theta = [\theta_1, \theta_2, \ldots, \theta_N]^{\mathrm{T}}$ and L is the Laplacian matrix defined in Definition 4. Under the assumption that the communication network is undirected and connected, by Lemma 2, the only nontrivial solution of $L\theta = 0$ is $\theta = [b, b, \ldots, b]^{\mathrm{T}}$, for some constant b dependent on the initial conditions. Hence, orientation alignment is guaranteed. □

Flocking behavior includes position aggregation and heading alignment. Combining Lemmas 14 and 16 gives the following theorem, which demonstrates the behavior of the multi-vehicle system (4.1) under the proposed discrete-time control laws (4.2).

Theorem 2. *Consider the discrete-time multi-vehicle system (4.1), with a communication delay $\tau \in \mathbb{N}$. If both Assumptions 1 and 3 hold, then, for a sufficiently small ε, the discrete-time distributed control laws (4.2) steer the system into a formation corresponding to a minimum of $\sum_{i=1}^{N} V_i$ and a common orientation.*

Proof. Properly choose some positive $\varepsilon < \min\{\bar{\varepsilon}_1, \bar{\varepsilon}_2\}$, where $\bar{\varepsilon}_1$ and $\bar{\varepsilon}_2$ are specified in Lemmas 14 and 16 respectively. By Lemma 14, the positions of all agents gradually converge to a formation corresponding to a minimum of $\sum_{i=1}^{N} V_i$ and $\nabla_{r_i} V_i \to 0$ as $k \to \infty$ for $i = 1, 2, \ldots, N$. Then, by Lemma 16, the orientations of all agents converge asymptotically to a common value as the desired formation is approached. □

Unlike in the continuous-time case where ε can be any arbitrary positive number when $\tau = 0$, in discrete-time setting, even in the absence of communication delays, ε still needs to be bounded by some $\bar{\varepsilon}_{\mathrm{dt1}} > 0$ to achieve a stable flocking behavior. When a constant communication delay $\tau \in \mathbb{N}^+$ is considered, there exists some $\bar{\varepsilon}_{\mathrm{dt2}} > 0$ such that for any $\varepsilon \in (0, \bar{\varepsilon}_{\mathrm{dt2}})$ the statement in Theorem 2 is always true. Obviously, $\bar{\varepsilon}_{\mathrm{dt2}} < \bar{\varepsilon}_{\mathrm{dt1}}$.

Another difference between the discrete-time and continuous-time cases is that $\bar{\varepsilon}_{\mathrm{dt1}}$ and $\bar{\varepsilon}_{\mathrm{dt2}}$ depend on the sampling period T as well, in addition to communication delay τ, the control parameter κ, graph connectivity and potential functions.

As in the continuous-time case, a minimum of $\sum_{i=1}^{N} V_i$ indicates aggregated positions only when communication graph is connected, and collision avoidance is guaranteed only under a complete graph, i.e., each vehicle knows the positions and headings of all the other vehicles.

4.4 Simulation results

The effectiveness of the proposed discrete-time control laws is verified in this section through numerical simulation, with a connected but incomplete graph as shown in Figure 4.2. Consider five agents with dynamics given in (4.1) and steered by the control laws (4.2). We choose the control parameters $\rho = 1$ and $\kappa = 0.05$, and the sampling period is $T = 1$ s.

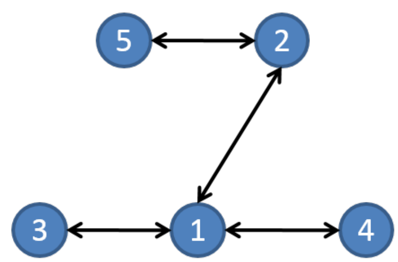


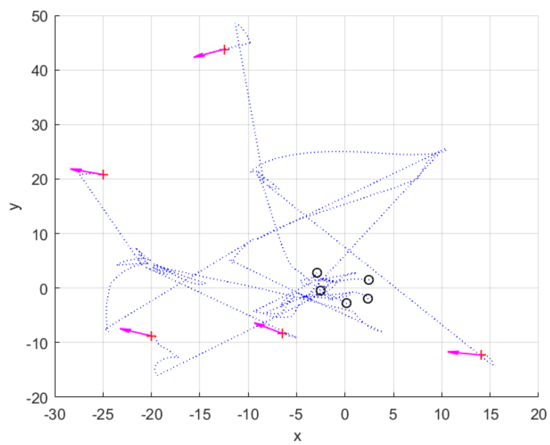
Figure 4.2: A connected but incomplete graph.

4.4.1 Flocking without delays

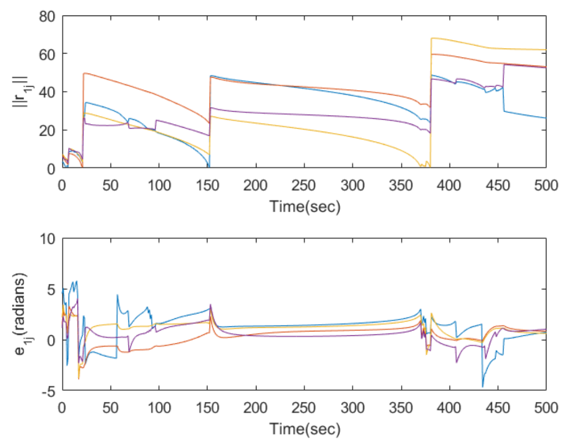
First, a delay-free scenario is considered. The initial positions $r_i(0)$ of the five agents are respectively $[-2.9, 2.8]^{\mathrm{T}}$, $[-2.5, -0.5]^{\mathrm{T}}$, $[0.2, -2.8]^{\mathrm{T}}$, $[2.4, -2]^{\mathrm{T}}$, and $[2.5, 1.5]^{\mathrm{T}}$. The initial orientations $\theta_i(0)$ of the five agents are respectively -2.25 rad, -0.49 rad, 2.61 rad, 1.84 rad, and 2.88 rad.

In the continuous-time settings, ε can be any positive number to achieve flocking behavior. However, in discrete-time settings, as observed in Figure 4.3, the agents do not converge to a close vicinity even with $\varepsilon = 0.8$.

If we choose a smaller value of the low gain parameter, such as $\varepsilon = 0.05$, the agents achieve both aggregated positions and aligned headings, as plotted in Figure 4.4. This means that in the absence of communication delays the low gain parameter still need to be sufficiently small in order to create a stable flocking behavior.

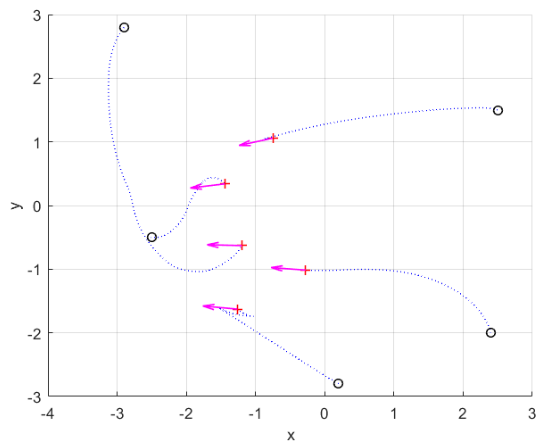


(a) Trajectories and headings.

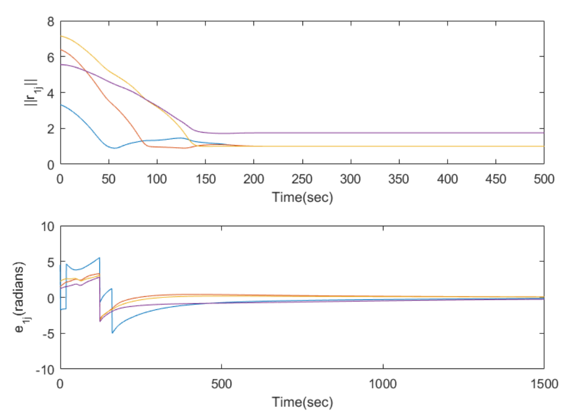


(b) Relative distances and headings.

Figure 4.3: Simulation with a connected but incomplete graph, $\tau = 0$, $\varepsilon = 0.8$.



(a) Trajectories and headings.

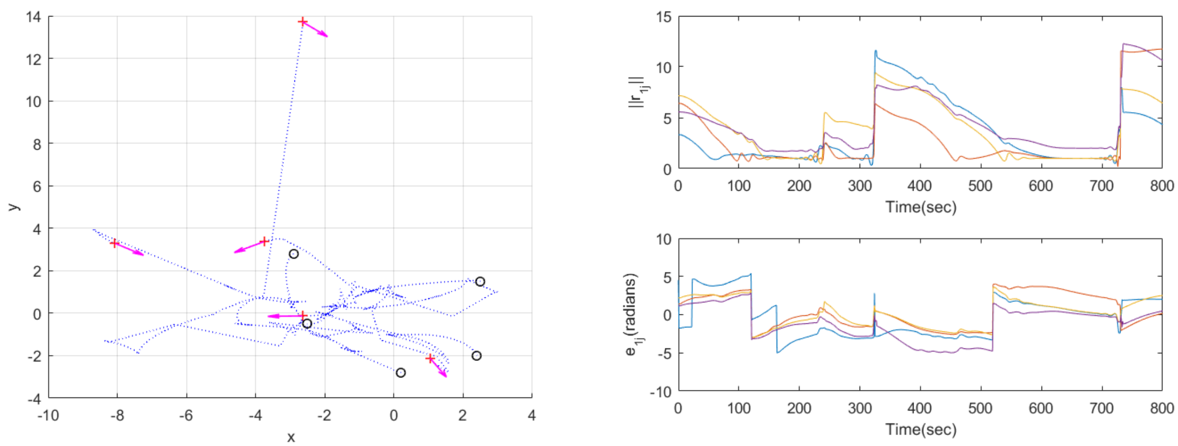


(b) Relative distances and headings.

Figure 4.4: Simulation with a connected but incomplete graph, $\tau = 0$, $\varepsilon = 0.05$.

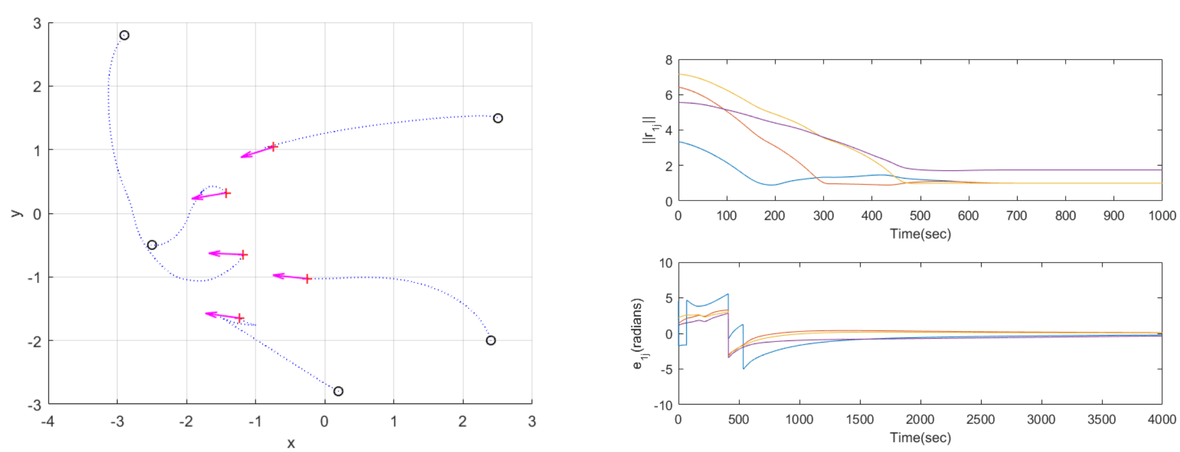
4.4.2 Flocking with delays

Next, we introduce communication delays $\tau = 4$ and examine the effects of delays on the closed-loop multi-agent system. Again, the initial positions of the five agents are chosen as $[-2.9, 2.8]^{\mathrm{T}}$, $[-2.5, -0.5]^{\mathrm{T}}$, $[0.2, -2.8]^{\mathrm{T}}$, $[2.4, -2]^{\mathrm{T}}$, and $[2.5, 1.5]^{\mathrm{T}}$, respectively. The initial orientations $\theta_i(0)$ of the five agents are respectively -2.25 rad, -0.49 rad, 2.61 rad, 1.84 rad, and 2.88 rad.



(a) Trajectories and headings. (b) Relative distances and headings.

Figure 4.5: Simulation with a connected but incomplete graph, $\tau = 4$, $\varepsilon = 0.05$.



(a) Trajectories and headings. (b) Relative distances and headings.

Figure 4.6: Simulation with a connected but incomplete graph, $\tau = 4$, $\varepsilon = 0.015$.

In the delay-free cases, agents converge to a flock when $\varepsilon = 0.05$. In the delayed system, we notice that a smaller low gain parameter is required. In Figure 4.5 where $\varepsilon = 0.05$, it can be seen that agents are not able to aggregate. Again, we reduce the low gain parameter to $\varepsilon = 0.015$ and then flocking behavior is obtained, as shown in Figure 4.6.

4.5 Summary

In this chapter, flocking control of multi-vehicle systems with nonholonomic dynamics and large communication delays was studied in the discrete-time setting. A decentralized low gain control law was constructed based on an artificial potential function. We proved that the closed-loop multi-agent system possesses the desired flocking behavior, including position aggregation and orientation alignment, in the presence of arbitrarily large communication delays. In addition, the effectiveness of the proposed controllers was demonstrated by numerical examples. Discrete-time flocking of nonholonomic vehicles through a potential function based method was first studied in this work.

Chapter 5

Consensus of Nonlinear Multi-agent Systems in the Continuous-time Setting

5.1 Problem statement

We consider a multi-agent system consisting of N agents, each described by,

$$\dot{x}_i = f(x_i) + g(x_i)u_i, \tag{5.1}$$

where, for $i = 1, 2, \ldots, N$, $x_i \in \mathbb{R}^n$ is the state vector of agent i, $u_i \in \mathbb{R}^m$ is the control input, $g(x_i) = [g_1(x_i),\ g_2(x_i),\ \ldots,\ g_m(x_i)] \in \mathbb{R}^{n\times m}$, and $f(x_i)$ and $g_k(x_i)$, $k = 1, 2, \ldots, m$, are smooth vector fields in $\mathbb{R}^n$.

Definition 7. *Global consensus of a multi-agent system is said to be achieved if the states of all agents are synchronized, i.e., for all $x_i(0) \in \mathbb{R}^n$, $i = 1, 2, \ldots, N$, $\lim_{t\to\infty} \|x_i(t) - x_j(t)\| = 0$, $i, j = 1, 2, \ldots, N$.*

Neighborhood consensus errors are defined as

$$e_i = \sum_{j=1}^{N} a_{ij}(x_i - x_j), \tag{5.2}$$

We introduce the following notations,

$$x = [x_1^{\mathrm{T}},\, x_2^{\mathrm{T}},\, \ldots,\, x_N^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{Nn}, \tag{5.3}$$

$$e = [e_1^{\mathrm{T}},\, e_2^{\mathrm{T}},\, \ldots,\, e_N^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{Nn}, \tag{5.4}$$

$$u = [u_1^{\mathrm{T}},\, u_2^{\mathrm{T}},\, \ldots,\, u_N^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{Nm}, \tag{5.5}$$

$$F(x) = [f^{\mathrm{T}}(x_1),\, f^{\mathrm{T}}(x_2),\, \ldots,\, f^{\mathrm{T}}(x_N)]^{\mathrm{T}} \in \mathbb{R}^{Nn}, \tag{5.6}$$

$$G(x) = \begin{bmatrix} g(x_1) & & & \\ & g(x_2) & & \\ & & \ddots & \\ & & & g(x_N) \end{bmatrix} \in \mathbb{R}^{Nn \times Nm}. \tag{5.7}$$

Then the dynamics of the multi-agent system (5.1) can be expressed more compactly as

$$\dot{x} = F(x) + G(x)u. \tag{5.8}$$

The consensus errors (5.2) can be rewritten as

$$e = (L \otimes I_n)x, \tag{5.9}$$

$$\dot{e} = (L \otimes I_n)\left(F(x) + G(x)u\right). \tag{5.10}$$

In view of Assumption 1, communication graph of the multi-agent system is undirected and connected. This assumption indicates that consensus is achieved if and only if all the elements in $e(t)$ approach zero as t goes to infinity.

Remark 2. *Since the Laplacian matrix L is symmetric, there exists some orthogonal matrix $T \in \mathbb{R}^{N \times N}$ such that*

$$T^T L T = \begin{bmatrix} 0 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_N \end{bmatrix} \triangleq J_L.$$

By Lemma 1, we know that L is positive semi-definite and $\lambda_2, \lambda_3, \ldots, \lambda_N > 0$. Therefore, we *can always construct a positive definite matrix M by $M = TJ_MT^T$, where*

$$J_M = \begin{bmatrix} \frac{1}{\lambda_1} & & & \\ & \frac{1}{\lambda_2} & & \\ & & \ddots & \\ & & & \frac{1}{\lambda_N} \end{bmatrix},$$

with any $\lambda_1 > 0$. Noting that $J_L J_M J_L = J_L$, we have

$$\begin{aligned} LML &= TJ_LT^TTJ_MT^TTJ_LT^T \\ &= TJ_LT^T \\ &= L. \end{aligned}$$

Consequently, we have

$$\begin{aligned} e^T(ML \otimes I_n) &= x^T(LML \otimes I_n) \\ &= x^T(L \otimes I_n) \\ &= e^T. \end{aligned} \tag{5.11}$$

Assumption 4. *There exists a positive definite matrix $P \in \mathbb{R}^{n\times n}$, such that for all $x_i, x_j \in \mathbb{R}^n$,*

$$(x_i - x_j)^T P[f(x_i) - f(x_j)] \leq 0.$$

Suppose that system (5.1) has linear unforced dynamics, that is, $f(x_i) = Ax_i$, $i = 1, 2, \ldots, N$. If each individual system is open-loop marginally stable, we can find a matrix $P > 0$ which satisfies Assumption 4 by solving the Lyapunov equation $A^{\mathrm{T}}P + PA = Q$, for some $Q \leq 0$. For systems that have nonlinear $f(x_i)$, this assumption requires the entire multi-agent system to be "open-loop cooperatively marginally stable." That is, in the absence of control inputs the distance between any two agents remains bounded.

Assumption 5. *For any $z \in \mathbb{R}^n$, we have*

$$\dim \operatorname{span}\{ad_f^q g_k(z) \colon 0 \leq q \leq n-1,\ 1 \leq k \leq m\} = n.$$

Assumption 5 is a controllability-like condition. If an affine in control system is open-loop marginally stable and this controllability-like condition is satisfied, then the system is globally asymptotically stabilizable [39].

5.2 Consensus in the absence of communication delays

Since consensus of system (5.1) has not been studied in the existing works, we first examine the case where communication delays are not considered and propose the following distributed control laws,

$$\begin{aligned} u_i(t) &= -g^{\mathrm{T}}(x_i(t))Pe_i(t) \\ &= -g^{\mathrm{T}}(x_i(t))P\sum_{j=1}^{N} a_{ij}(x_i(t) - x_j(t)), \end{aligned} \tag{5.12}$$

where the definition of the matrix P is as defined in Assumption 4. Using the notations in (5.3)-(5.7), we obtain

$$u(t) = -G^{\mathrm{T}}(x(t))(I_N \otimes P)e(t). \tag{5.13}$$

The following lemmas are important in the analysis of the closed-loop multi-agent system. Lemma 17 implies that, under Assumption 4, the consensus errors remain bounded in the absence of control inputs. Lemma 18 is a consequence of the controllability-like condition, as introduced in Assumption 5, to the overall multi-agent system.

Lemma 17. *If Assumption 4 is satisfied, then,*

$$e^T(I_N \otimes P)F(x) \le 0,$$

where the matrix $P = P^T$ is as defined in Assumption 4.

Proof. We note that

$$\begin{aligned} (I_N \otimes P)F(x) &= [f^{\mathrm{T}}(x_1)P,\ f^{\mathrm{T}}(x_2)P,\ \ldots,\ f^{\mathrm{T}}(x_N)P]^{\mathrm{T}} \\ &\triangleq F_P(x). \end{aligned}$$

Then

$$\begin{aligned} e^{\mathrm{T}}(I_N \otimes P)F(x) &= e^{\mathrm{T}}F_P(x) \\ &= x^{\mathrm{T}}(L \otimes I_n)F_P(x). \end{aligned} \tag{5.14}$$

Applying Lemma 3 to (5.14) gives

$$e^{\mathrm{T}}(I_N \otimes P)F(x) = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}(x_i - x_j)^{\mathrm{T}}P\left(f(x_i) - f(x_j)\right).$$

By Assumption 4, we have $e^{\mathrm{T}}(I_N \otimes P)F(x) \le 0$. □

Lemma 18. *If Assumption 5 is satisfied, then, for any $x \in \mathbb{R}^{Nn}$,* $\dim \operatorname{span}\{ad_F^q G_r(x) \colon 0 \leq q \leq n-1,\ 1 \leq r \leq Nm\} = Nn$, *where $G_r(x)$ denotes the rth column of $G(x)$.*

Proof. We define the following notations,

$$\mathbf{0}_{n\times 1} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^n,\ \mathbf{0}_{n\times n} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{n\times n}.$$

Then for all $i = 1, 2, \ldots, N$, $k = 1, 2, \ldots, m$,

$$G_{(i-1)m+k}(x) = \begin{bmatrix} \mathbf{0}_{n\times 1} \\ \vdots \\ \mathbf{0}_{n\times 1} \\ g_k(x_i) \\ \mathbf{0}_{n\times 1} \\ \vdots \\ \mathbf{0}_{n\times 1} \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix} a \\ a \\ a \end{matrix}}\right\} i-1 \\ \\ \\ \\ \\ \end{matrix}.$$

By the definition of Lie brackets, we have

$$ad_F^0 G_{(i-1)m+k}(x) = \begin{bmatrix} \mathbf{0}_{n\times 1} \\ \vdots \\ \mathbf{0}_{n\times 1} \\ g_k(x_i) \\ \mathbf{0}_{n\times 1} \\ \vdots \\ \mathbf{0}_{n\times 1} \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix} a \\ a \\ a \end{matrix}}\right\} i-1 \\ \\ \\ \\ \\ \end{matrix},$$

$$
ad_F G_{(i-1)m+k}(x) = \begin{array}{l} i-1 \left\{ \begin{array}{c} \\ \\ \\ \end{array} \right. \\ \\ \\ \\ \\ \end{array} \begin{bmatrix} \mathbf{0}_{n\times n} \cdots \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} \cdots \mathbf{0}_{n\times n} \\ \vdots \quad \ddots \quad \vdots & \vdots & \vdots \quad \ddots \quad \vdots \\ \mathbf{0}_{n\times n} \cdots \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} \cdots \mathbf{0}_{n\times n} \\ \mathbf{0}_{n\times n} \cdots \mathbf{0}_{n\times n} & \frac{\partial g_k(x_i)}{\partial x_i} & \mathbf{0}_{n\times n} \cdots \mathbf{0}_{n\times n} \\ \mathbf{0}_{n\times n} \cdots \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} \cdots \mathbf{0}_{n\times n} \\ \vdots \quad \ddots \quad \vdots & \vdots & \vdots \quad \ddots \quad \vdots \\ \underbrace{\mathbf{0}_{n\times n} \cdots \mathbf{0}_{n\times n}}_{i-1} & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} \cdots \mathbf{0}_{n\times n} \end{bmatrix} \begin{bmatrix} f(x_1) \\ \vdots \\ f(x_i) \\ \vdots \\ f(x_N) \end{bmatrix}
$$

$$
- \begin{bmatrix} \frac{\partial f(x_1)}{\partial x_1} & & & \\ & \frac{\partial f(x_2)}{\partial x_2} & & \\ & & \ddots & \\ & & & \frac{\partial f(x_N)}{\partial x_N} \end{bmatrix} \begin{bmatrix} \mathbf{0}_{n\times 1} \\ \vdots \\ \mathbf{0}_{n\times 1} \\ g_k(x_i) \\ \mathbf{0}_{n\times 1} \\ \vdots \\ \mathbf{0}_{n\times 1} \end{bmatrix} \begin{array}{l} \left. \begin{array}{c} \\ \\ \\ \end{array} \right\} i-1 \\ \\ \\ \\ \\ \end{array}
$$

$$
= \begin{bmatrix} \mathbf{0}_{n\times 1} \\ \vdots \\ \mathbf{0}_{n\times 1} \\ \frac{\partial g_k(x_i)}{\partial x_i} f(x_i) \\ \mathbf{0}_{n\times 1} \\ \vdots \\ \mathbf{0}_{n\times 1} \end{bmatrix} - \begin{bmatrix} \mathbf{0}_{n\times 1} \\ \vdots \\ \mathbf{0}_{n\times 1} \\ \frac{\partial f(x_i)}{\partial x_i} g_k(x_i) \\ \mathbf{0}_{n\times 1} \\ \vdots \\ \mathbf{0}_{n\times 1} \end{bmatrix} \begin{array}{l} \left. \begin{array}{c} \\ \\ \\ \end{array} \right\} i-1 \\ \\ \\ \\ \\ \end{array}
$$

$$
= \begin{bmatrix} \mathbf{0}_{n\times 1} \\ \vdots \\ \mathbf{0}_{n\times 1} \\ ad_f g_k(x_i) \\ \mathbf{0}_{n\times 1} \\ \vdots \\ \mathbf{0}_{n\times 1} \end{bmatrix} \begin{array}{l} \left. \begin{array}{c} \\ \\ \\ \end{array} \right\} i-1 \\ \\ \\ \\ \\ \end{array} .
$$

Similarly, it is straightforward to show that, for all $i = 1, 2, \ldots, N,\ k = 1, 2, \ldots, m,\ q \in \mathbb{N}$,

$$ad_F^q G_{(i-1)m+k}(x) = \begin{bmatrix} \mathbf{0}_{n\times 1} \\ \vdots \\ \mathbf{0}_{n\times 1} \\ ad_f^q g_k(x_i) \\ \mathbf{0}_{n\times 1} \\ \vdots \\ \mathbf{0}_{n\times 1} \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}0\\ \vdots\\ 0\end{matrix}}\right\} i-1 \\ \\ \\ \\ \\ \end{matrix}. \tag{5.15}$$

By Assumption 5 and (5.15), we have, for any $x \in \mathbb{R}^{Nn}$,

$$\dim \operatorname{span}\{ad_F^q G_r(x) \colon 0 \le q \le n-1,\ 1 \le r \le Nm\} = Nn.$$

□

The behavior of the multi-agent system (5.1) under the proposed control laws (5.12), in the absence of communication delays, is established in the following theorem.

Theorem 3. *Consider a multi-agent system whose dynamics are given by (5.1). Suppose that Assumptions 1, 4 and 5 hold. Then the control laws (5.12) steer the multi-agent system into global consensus. That is,*

$$\lim_{t\to\infty} \|x_i(t) - x_j(t)\| = 0,\ i, j = 1, 2, \ldots, N.$$

Proof. Consider the following Lyapunov function

$$V_{\mathrm{M}}(e) = e^{\mathrm{T}}(t)(M \otimes P)e(t),$$

where matrices $M > 0$ and $P > 0$ are defined in Remark 2 and Assumption 4, respectively. For brevity, we drop all the time variables in the remainder of this proof.

Taking time derivative along the trajectories of (5.10) yields

$$\begin{aligned} \dot{V}_{\mathrm{M}} &= 2e^{\mathrm{T}}(M \otimes P)\dot{e} \\ &= 2e^{\mathrm{T}}(M \otimes P)(L \otimes I_n)\left(F(x) + G(x)u\right) \\ &= 2e^{\mathrm{T}}(ML \otimes I_n)(I_N \otimes P)\left(F(x) + G(x)u\right). \end{aligned} \tag{5.16}$$

In view of (5.11), we have

$$\dot{V}_{\mathrm{M}} = 2e^{\mathrm{T}}(I_N \otimes P)\left(F(x) + G(x)u\right). \tag{5.17}$$

Substituting (5.13) into (5.17) gives

$$\dot{V}_{\mathrm{M}} = 2e^{\mathrm{T}}(I_N \otimes P)F(x) - 2e^{\mathrm{T}}(I_N \otimes P)G(x)G^{\mathrm{T}}(x)(I_N \otimes P)e. \tag{5.18}$$

By Lemma 17, we conclude that $\dot{V}_{\mathrm{M}} \leq 0$.

Since both terms on the right-hand side of (5.18) are non-positive, $\dot{V}_{\mathrm{M}} = 0$ if and only if $2e^{\mathrm{T}}(I_N \otimes P)F(x) = 0$ and $2e^{\mathrm{T}}(I_N \otimes P)G_r(x) = 0,\ 1 \leq r \leq Nm$, which are respectively equivalent to

$$2x^{\mathrm{T}}(L \otimes P)F(x) = 0, \tag{5.19}$$

and

$$2x^{\mathrm{T}}(L \otimes P)G_r(x) = 0,\ 1 \leq r \leq Nm. \tag{5.20}$$

Note that $V_{\mathrm{M}}(e) = x^{\mathrm{T}}(LML \otimes P)x = x^{\mathrm{T}}(L \otimes P)x$. Then,

$$\frac{\partial V_{\mathrm{M}}(e)}{\partial x} = 2x^{\mathrm{T}}(L \otimes P). \tag{5.21}$$

Substitution of (5.21) into (5.19) and (5.20) results in

$$\frac{\partial V_{\mathrm{M}}(e)}{\partial x}F(x) = 0, \tag{5.22}$$

and

$$\frac{\partial V_{\mathrm{M}}(e)}{\partial x}G_r(x) = 0,\ 1 \leq r \leq Nm, \tag{5.23}$$

respectively. It is clear now $\dot{V}_{\mathrm{M}} = 0$ if and only if

$$L_F V_{\mathrm{M}}(e) = 0, \tag{5.24}$$

and

$$L_{G_r} V_{\mathrm{M}}(e) = 0,\ 1 \leq r \leq Nm. \tag{5.25}$$

Here we have abused the notations by using the Lie derivative notations $L_F V_{\mathrm{M}}(e)$ and $L_{G_r} V_{\mathrm{M}}(e)$ to denote $\frac{\partial V_{\mathrm{M}}(e)}{\partial x}F(x)$ and $\frac{\partial V_{\mathrm{M}}(e)}{\partial x}G_r(x)$, respectively. We will also use similar notations in the rest of the proof.

Under (5.25), equation (5.8) reduces to $\dot{x} = F(x)$.

Let S be the largest invariant subset of $\{e \in \mathbb{R}^{Nn} \mid \dot{V}_{\mathrm{M}} = 0\}$. Hence, for any $e(0) \in S$, $e(t, x(0))$ stays in S forever, where $e(t, x(0)) = (L \otimes I_n)x(t, x(0))$ and $x(t, x(0))$ is the trajectory of $\dot{x} = F(x)$ starting from $x(0)$. This, together with (5.24) and (5.25), imply that, for any $e(0) \in S$,

$$L_F V_{\mathrm{M}}(e(t, x(0))) = 0,\ t \geq 0, \tag{5.26}$$

and

$$L_{G_r} V_{\mathrm{M}}(e(t, x(0))) = 0,\ t \geq 0, 1 \leq r \leq Nm. \tag{5.27}$$

Define $\xi = ad_F G_r(x(t, x(0)))$, $1 \leq r \leq Nm$. In view of (5.26) and (5.27), it follows that

$$\begin{aligned} L_\xi V_{\mathrm{M}}(e(t, x(0))) &= L_F L_{G_r} V_{\mathrm{M}}(e(t, x(0))) - L_{G_r} L_F V_{\mathrm{M}}(e(t, x(0))) \\ &= 0. \end{aligned} \tag{5.28}$$

Similar manipulations lead to

$$L_\xi V_{\mathrm{M}}(e(t, x(0))) = 0,\ \forall e(0) \in S, \tag{5.29}$$

where $\xi = ad_F^q G_r(x(t, x(0)))$, $q \in \mathbb{N}$, $1 \leq r \leq Nm$. Under Assumption 5, it follows from Lemma 18 that $\dim \operatorname{span}\{ad_F^q G_r(x) \colon 0 \leq q \leq n-1, 1 \leq r \leq Nm\} = Nn$, $\forall x \in \mathbb{R}^{Nn}$. This, along with (5.29), imply that

$$\begin{aligned} S &= \left\{ e \in \mathbb{R}^{Nn} \mid \frac{\partial V_{\mathrm{M}}(e)}{\partial x} = \mathbf{0}_{1 \times Nn} \right\} \\ &= \left\{ e \in \mathbb{R}^{Nn} \mid 2x^{\mathrm{T}}(L \otimes P) = \mathbf{0}_{1 \times Nn} \right\} \\ &= \left\{ e \in \mathbb{R}^{Nn} \mid (I_N \otimes P)e = \mathbf{0}_{Nn \times 1} \right\}. \end{aligned} \tag{5.30}$$

Since P is nonsingular, $S = \{e = \mathbf{0}_{Nn \times 1}\}$. Therefore, by LaSalle's Invariance Principle, $\lim_{t \to \infty} e(t) = 0$, and consensus is achieved. □

Remark 3. *If there is a constraint on the magnitude of the control inputs of the agents, we can modify the control input to each agent i as*

$$u_{ik} = \operatorname{sign}(-g_k^T(x_i)Pe_i) \min(|g_k^T(x_i)Pe_i|, \alpha), \tag{5.31}$$

where $u_{ik}, k = 1, 2, \ldots, m$, is the kth element in u_i, and $\alpha > 0$ is the bound on the inputs. The bounded control inputs (5.31) then bring the multi-agent system into consensus if Assumptions 1, 4 and 5 are satisfied.

5.3 Consensus in the presence of communication delays

In this section, we take communication delays into consideration and restrict $g(x_i)$ in (5.1) to a constant matrix, that is,

$$\dot{x}_i = f(x_i) + Bu_i. \tag{5.32}$$

We assume the communication delays to be a constant τ. Then, we propose the following distributed low gain control laws,

$$u_i(t) = -\varepsilon B^{\mathrm{T}} P e_i(t-\tau), \tag{5.33}$$

where ε is a low gain parameter. Furthermore, with the help of the notations in (5.3)-(5.7), we have

$$u(t) = -\varepsilon(I_N \otimes B^{\mathrm{T}} P)e(t-\tau). \tag{5.34}$$

The dynamics of the consensus errors are given by

$$\dot{e}(t) = (L \otimes I_n)\left(F(x(t)) - \varepsilon(I_N \otimes BB^{\mathrm{T}} P)e(t-\tau)\right). \tag{5.35}$$

Assumption 6. *There exists a constant $\beta > 0$ such that,*

$$(f(x_i) - f(x_j))^T PBB^T P\,(f(x_i) - f(x_j)) \leq \beta(x_i - x_j)^T PBB^T P(x_i - x_j),\ \forall x_i, x_j \in \mathbb{R}^n,$$

where the matrix P is as defined in Assumption 4.

The following lemma states a important property of positive semi-definite matrix.

Lemma 19. *[40] Consider two positive semi-definite matrices A and B. The product of the two matrices, AB, is positive semi-definite if and only if AB is symmetric.*

Then, we establish further results regarding the behavior of the closed-loop multi-agent system (5.32).

Theorem 4. *Consider a multi-agent system whose agent dynamics are described by (5.32). Suppose that Assumptions 1, 4, 5 and 6 are all satisfied. There exists a constant $\bar{\tau} > 0$ such that for any $\tau \in [0, \bar{\tau}]$, the control laws (5.33) steer the system into global consensus as long as the low gain parameter ε is sufficiently small. That is, for any $\tau \in [0, \bar{\tau}]$, there exists an $\bar{\varepsilon}$ such that for any $\varepsilon \in (0, \bar{\varepsilon}]$, $\lim_{t\to\infty} e(t) = 0$ for all $x_i(0) \in \mathbb{R}^n$, $i = 1, 2, \ldots, N$.*

Proof. We consider the following Lyapunov function,

$$V_{\mathrm{M}}(e) = V_1(e) + V_2(e) + V_3(e), \tag{5.36}$$

with

$$\begin{aligned}
V_1(e) &= e^{\mathrm{T}}(t)(M \otimes P)e(t), \\
V_2(e) &= \varepsilon\tau \int_0^{\tau} \int_{t-s_1}^{t} \dot{e}^{\mathrm{T}}(s_2)\Phi\dot{e}(s_2)ds_2 ds_1, \\
V_3(e) &= \left(\sum_{k=0}^{\infty} aq^k\right) \int_0^{\tau} \int_{t-s_1}^{t} \dot{e}^{\mathrm{T}}(s_2)\Psi\dot{e}(s_2)ds_2 ds_1,
\end{aligned}$$

where $a = 8\varepsilon\tau^3 N\beta$, $q = 4\tau^2\beta$, and $\Phi = I_N \otimes PBB^{\mathrm{T}}P$ and $\Psi = M \otimes PBB^{\mathrm{T}}P$ are positive semi-definite matrices. To guarantee the convergence in V_3, we require $q < 1$, i.e., $\tau < \frac{1}{2\sqrt{\beta}}$. We note that V_1 is the Lyapunov function used in the proof of Theorem 3 for the delay-free case. Because of the presence of communication delays, a positive semi-definite term appears in the time derivative of V_1 and in order to cancel that term we introduce V_2. However, the time derivative of V_2 in turn contains another positive semi-definite term, so we introduce the first term in V_3. Similarly, every time we add a new term to the Lyapunov function to cancel the positive semi-definite term in the derivative resulting from the previously introduced term in the Lyapunov function, the new term itself resulting in a new positive semi-definite term in the derivative. This is the reason why V_3 contains infinitely many terms.

Taking time derivative of V_1 yields

$$\dot{V}_1 = 2e^{\mathrm{T}}(t)(M \otimes P)\dot{e}(t), \tag{5.37}$$

which, together with (5.35), give

$$\begin{aligned}
\dot{V}_1 =& 2e^{\mathrm{T}}(t)(ML \otimes P)F(x(t)) - 2\varepsilon e^{\mathrm{T}}(t)(ML \otimes PBB^{\mathrm{T}}P)e(t-\tau) \\
=& 2e^{\mathrm{T}}(t)(I_N \otimes P)F(x(t)) - 2\varepsilon e^{\mathrm{T}}(t)(I_N \otimes PBB^{\mathrm{T}}P)e(t-\tau).
\end{aligned} \tag{5.38}$$

Note that

$$e(t) = e(t-\tau) + \int_{t-\tau}^{t} \dot{e}(\sigma)d\sigma. \tag{5.39}$$

Equation (5.38) can then be continued as

$$\dot{V}_1 = 2e^{\mathrm{T}}(t)(I_N \otimes P)F(x(t)) - 2\varepsilon e^{\mathrm{T}}(t-\tau)(I_N \otimes PBB^{\mathrm{T}}P)e(t-\tau)$$

$$- 2\varepsilon e^{\mathrm{T}}(t-\tau)(I_N \otimes PBB^{\mathrm{T}}P)\int_{t-\tau}^{t} \dot{e}(\sigma)d\sigma$$

$$\leq 2e^{\mathrm{T}}(t)(I_N \otimes P)F(x(t)) - \varepsilon e^{\mathrm{T}}(t-\tau)\Phi e(t-\tau) + \varepsilon \int_{t-\tau}^{t} \dot{e}^{\mathrm{T}}(\sigma)d\sigma \Phi \int_{t-\tau}^{t} \dot{e}(\sigma)d\sigma.$$

By Lemma 8, it follows that

$$\dot{V}_1 \leq 2e^{\mathrm{T}}(t)(I_N \otimes P)F(x(t)) - \varepsilon e^{\mathrm{T}}(t-\tau)\Phi e(t-\tau) + \varepsilon\tau \int_{t-\tau}^{t} \dot{e}^{\mathrm{T}}(\sigma)\Phi\dot{e}(\sigma)d\sigma. \tag{5.40}$$

For V_2, by Lemma 9, we have

$$\dot{V}_2 = \varepsilon\tau^2 \dot{e}^{\mathrm{T}}(t)\Phi\dot{e}(t) - \varepsilon\tau \int_{t-\tau}^{t} \dot{e}^{\mathrm{T}}(\sigma)\Phi\dot{e}(\sigma)d\sigma. \tag{5.41}$$

In view of (5.35), it follows from Lemma 7 that

$$\dot{V}_2 \leq 2\varepsilon\tau^2 F^{\mathrm{T}}(x(t))(L \otimes I_n)\Phi(L \otimes I_n)F(x(t)) + 2\varepsilon^3\tau^2 e^{\mathrm{T}}(t-\tau)\Phi(L^2 \otimes BB^{\mathrm{T}})\Phi e(t-\tau)$$
$$- \varepsilon\tau \int_{t-\tau}^{t} \dot{e}^{\mathrm{T}}(\sigma)\Phi\dot{e}(\sigma)d\sigma. \tag{5.42}$$

We expand the first term in the above inequality and obtain

$$\begin{aligned}
&F^{\mathrm{T}}(x(t))(L \otimes I_n)\Phi(L \otimes I_n)F(x(t)) \\
=&\sum_{i=1}^{N} \left\| \sum_{j=1}^{N} a_{ij}B^{\mathrm{T}}P\left[f(x_i) - f(x_j)\right] \right\|^2 \\
\leq& N\sum_{i=1}^{N}\sum_{j=1}^{N} \left\| a_{ij}B^{\mathrm{T}}P\left[f(x_i) - f(x_j)\right] \right\|^2,
\end{aligned} \tag{5.43}$$

where we have applied Lemma 7. By Assumption 6, it follows from (5.43) that

$$\begin{aligned}
&F^{\mathrm{T}}(x(t))(L \otimes I_n)\Phi(L \otimes I_n)F(x(t)) \\
\leq& N\beta\sum_{i=1}^{N}\sum_{j=1}^{N} \left\| a_{ij}B^{\mathrm{T}}P\left(x_i - x_j\right)\right] \right\|^2 \\
=&2N\beta x^{\mathrm{T}}(t)(L \otimes PBB^{\mathrm{T}}P)x(t) \\
=&2N\beta e^{\mathrm{T}}(t)(M \otimes PBB^{\mathrm{T}}P)e(t).
\end{aligned} \tag{5.44}$$

Substituting (5.39) and (5.44) into (5.42), and applying Lemmas 7 and 8, we obtain

$$\dot{V}_2 \leq 8\varepsilon\tau^2 N\beta e^{\mathrm{T}}(t-\tau)\Psi e(t-\tau) + 8\varepsilon\tau^3 N\beta \int_{t-\tau}^{t} \dot{e}^{\mathrm{T}}(\sigma)\Psi\dot{e}(\sigma)d\sigma$$

$$+ 2\varepsilon^3\tau^2 e^{\mathrm{T}}(t-\tau)\Phi(L^2 \otimes BB^{\mathrm{T}})\Phi e(t-\tau) - \varepsilon\tau \int_{t-\tau}^{t} \dot{e}^{\mathrm{T}}(\sigma)\Phi\dot{e}(\sigma)d\sigma. \quad (5.45)$$

Expanding the third term on the right-hand side of (5.45) to a summation form, as what we did in (5.43), and applying Lemmas 3 and 7, we have

$$\begin{aligned}\dot{V}_2 \le & 8\varepsilon\tau^2 N\beta e^{\mathrm{T}}(t-\tau)\Psi e(t-\tau) + 8\varepsilon\tau^3 N\beta \int_{t-\tau}^{t} \dot{e}^{\mathrm{T}}(\sigma)\Psi\dot{e}(\sigma)d\sigma \\ & + 4\varepsilon^3\tau^2 N e^{\mathrm{T}}(t-\tau)\Phi(L \otimes BB^{\mathrm{T}})\Phi e(t-\tau) - \varepsilon\tau \int_{t-\tau}^{t} \dot{e}^{\mathrm{T}}(\sigma)\Phi\dot{e}(\sigma)d\sigma. \quad (5.46)\end{aligned}$$

By Lemma 9, the time derivative of V_3 is given by

$$\dot{V}_3 = \sum_{k=0}^{\infty} aq^k\tau\dot{e}^{\mathrm{T}}(t)\Psi\dot{e}(t) - \sum_{k=0}^{\infty} aq^k \int_{t-\tau}^{t} \dot{e}^{\mathrm{T}}(\sigma)\Psi\dot{e}(\sigma)d\sigma. \quad (5.47)$$

Again, recalling (5.35), it follows from Lemma 7 that

$$\begin{aligned}\dot{V}_3 \le & \sum_{k=0}^{\infty} 2aq^k\tau F^{\mathrm{T}}(x(t))(L \otimes PBB^{\mathrm{T}}P)F(x(t)) \\ & + \sum_{k=0}^{\infty} 2aq^k\varepsilon^2\tau e^{\mathrm{T}}(t-\tau)\Phi(L \otimes BB^{\mathrm{T}})\Phi e(t-\tau) - \sum_{k=0}^{\infty} aq^k \int_{t-\tau}^{t} \dot{e}^{\mathrm{T}}(\sigma)\Psi\dot{e}(\sigma)d\sigma. \quad (5.48)\end{aligned}$$

By Assumption 6, (5.48) can be continued as

$$\begin{aligned}\dot{V}_3 \le & \sum_{k=0}^{\infty} 2aq^k\tau\beta x^{\mathrm{T}}(t)(L \otimes PBB^{\mathrm{T}}P)x(t) \\ & + \sum_{k=0}^{\infty} 2aq^k\varepsilon^2\tau e^{\mathrm{T}}(t-\tau)\Phi(L \otimes BB^{\mathrm{T}})\Phi e(t-\tau) - \sum_{k=0}^{\infty} aq^k \int_{t-\tau}^{t} \dot{e}^{\mathrm{T}}(\sigma)\Psi\dot{e}(\sigma)d\sigma \\ = & \sum_{k=0}^{\infty} 2aq^k\tau\beta e^{\mathrm{T}}(t)\Psi e(t) \\ & + \sum_{k=0}^{\infty} 2aq^k\varepsilon^2\tau e^{\mathrm{T}}(t-\tau)\Phi(L \otimes BB^{\mathrm{T}})\Phi e(t-\tau) - \sum_{k=0}^{\infty} aq^k \int_{t-\tau}^{t} \dot{e}^{\mathrm{T}}(\sigma)\Psi\dot{e}(\sigma)d\sigma. \quad (5.49)\end{aligned}$$

Substituting (5.39) into (5.49) and applying Lemmas 7 and 8, we have

$$\begin{aligned}\dot{V}_3 \le & \sum_{k=0}^{\infty} 4aq^k\tau\beta e^{\mathrm{T}}(t-\tau)\Psi e(t-\tau) + \sum_{k=0}^{\infty} 4aq^k\tau^2\beta \int_{t-\tau}^{t} \dot{e}^{\mathrm{T}}(\sigma)\Psi\dot{e}(\sigma)d\sigma \\ & + \sum_{k=0}^{\infty} 2aq^k\varepsilon^2\tau e^{\mathrm{T}}(t-\tau)\Phi(L \otimes BB^{\mathrm{T}})\Phi e(t-\tau) - \sum_{k=0}^{\infty} aq^k \int_{t-\tau}^{t} \dot{e}^{\mathrm{T}}(\sigma)\Psi\dot{e}(\sigma)d\sigma, \quad (5.50)\end{aligned}$$

which, by collection of similar terms, simplifies to

$$\dot{V}_3 \leq \sum_{k=0}^{\infty} 4aq^k \tau\beta e^{\mathrm{T}}(t-\tau)\Psi e(t-\tau) + \sum_{k=0}^{\infty} 2aq^k \varepsilon^2 \tau e^{\mathrm{T}}(t-\tau)\Phi(L\otimes BB^{\mathrm{T}})\Phi e(t-\tau)$$
$$- 8\varepsilon\tau^3 N\beta \int_{t-\tau}^{t} \dot{e}^{\mathrm{T}}(\sigma)\Psi\dot{e}(\sigma)d\sigma. \tag{5.51}$$

Summing both sides of (5.40), (5.46) and (5.51) yields

$$\begin{aligned}\dot{V}_{\mathrm{M}} \leq & 2e^{\mathrm{T}}(t)(I_N\otimes P)F(x(t)) - \varepsilon e^{\mathrm{T}}(t-\tau)\Phi e(t-\tau) + 8\varepsilon\tau^2 N\beta e^{\mathrm{T}}(t-\tau)\Psi e(t-\tau)\\
&+ 4\varepsilon^3\tau^2 N e^{\mathrm{T}}(t-\tau)\Phi(L\otimes BB^{\mathrm{T}})\Phi e(t-\tau) + \sum_{k=0}^{\infty} 4aq^k\tau\beta e^{\mathrm{T}}(t-\tau)\Psi e(t-\tau)\\
&+ \sum_{k=0}^{\infty} 2aq^k\varepsilon^2\tau e^{\mathrm{T}}(t-\tau)\Phi(L\otimes BB^{\mathrm{T}})\Phi e(t-\tau)\\
=& 2e^{\mathrm{T}}(t)(I_N\otimes P)F(x(t)) - \varepsilon e^{\mathrm{T}}(t-\tau)\Phi e(t-\tau) + \sum_{k=0}^{\infty} 8\varepsilon\tau^2 N\beta q^k e^{\mathrm{T}}(t-\tau)\Psi e(t-\tau)\\
&+ \sum_{k=0}^{\infty} 4\varepsilon^3\tau^2 N q^k e^{\mathrm{T}}(t-\tau)\Phi(L\otimes BB^{\mathrm{T}})\Phi e(t-\tau).\end{aligned} \tag{5.52}$$

As mentioned before, when $q < 1$, i.e., $\tau < \frac{1}{2\sqrt{\beta}}$, the summations of geometric series in (5.52) are finite and therefore (5.52) can be rewritten as

$$\begin{aligned}\dot{V}_{\mathrm{M}} \leq & 2e^{\mathrm{T}}(t)(I_N\otimes P)F(x(t)) - \varepsilon e^{\mathrm{T}}(t-\tau)\Phi e(t-\tau)\\
&+ \frac{4\varepsilon^3\tau^2 N}{1-4\tau^2\beta} e^{\mathrm{T}}(t-\tau)\Phi(L\otimes BB^{\mathrm{T}})\Phi e(t-\tau)\\
&+ \frac{8\varepsilon\tau^2 N\beta}{1-4\tau^2\beta} e^{\mathrm{T}}(t-\tau)(M\otimes I_n)\Phi e(t-\tau).\end{aligned} \tag{5.53}$$

Since B, L and M are known and constant, there exist positive constants η_1 and η_2 such that,

$$L\otimes BB^{\mathrm{T}} \leq \eta_1 I_{Nn}, \tag{5.54}$$

$$M\otimes I_n \leq \eta_2 I_{Nn}. \tag{5.55}$$

Substituting (5.54) and (5.55) into (5.53), we have

$$\begin{aligned}\dot{V}_{\mathrm{M}} \leq & 2e^{\mathrm{T}}(t)(I_N\otimes P)F(x(t)) - \varepsilon e^{\mathrm{T}}(t-\tau)\Phi e(t-\tau)\\
&+ \frac{4\varepsilon^3\tau^2 N\eta_1}{1-4\tau^2\beta} e^{\mathrm{T}}(t-\tau)\Phi^2 e(t-\tau) + \frac{8\varepsilon\tau^2 N\beta\eta_2}{1-4\tau^2\beta} e^{\mathrm{T}}(t-\tau)\Phi e(t-\tau).\end{aligned} \tag{5.56}$$

We have proved that $2e^{\mathrm{T}}(t)(I_N \otimes P)F(x(t)) \leq 0$ in Lemma 17. It is straightforward to obtain from (5.56) that $\dot{V}_{\mathrm{M}} \leq 0$ when

$$\left(\varepsilon - \frac{8\varepsilon\tau^2 N\beta\eta_2}{1-4\tau^2\beta}\right)\Phi - \frac{4\varepsilon^3\tau^2 N\eta_1}{1-4\tau^2\beta}\Phi^2 \geq 0. \tag{5.57}$$

Since $\Phi \geq 0$ and the left-hand side of (5.57) is symmetric, by Lemma 19, (5.57) is true when

$$\left(\varepsilon - \frac{8\varepsilon\tau^2 N\beta\eta_2}{1-4\tau^2\beta}\right)I_{Nn} - \frac{4\varepsilon^3\tau^2 N\eta_1}{1-4\tau^2\beta}\Phi \geq 0. \tag{5.58}$$

In order to satisfy (5.58), we need the following two conditions

$$\varepsilon - \frac{8\varepsilon\tau^2 N\beta\eta_2}{1-4\tau^2\beta} > 0, \tag{5.59}$$

and

$$\varepsilon - \frac{8\varepsilon\tau^2 N\beta\eta_2}{1-4\tau^2\beta} - \frac{4\varepsilon^3\tau^2 N\eta_1}{1-4\tau^2\beta}\lambda_{\max}(\Phi) > 0, \tag{5.60}$$

where $\lambda_{\max}(\Phi)$ is the maximum eigenvalue of the positive semi-definite matrix $\Phi = I_N \otimes PBB^{\mathrm{T}}P$. According to the preceding derivations, $\dot{V}_{\mathrm{M}} \leq 0$ when both (5.59) and (5.60) are satisfied. Then, solving (5.59) and (5.60), we obtain $\tau \leq \bar{\tau}$ and $\varepsilon \leq \bar{\varepsilon}$, where $\bar{\tau}$ and $\bar{\varepsilon}$ are such that,

$$\bar{\tau} < \sqrt{\frac{1}{4\beta(2N\eta_2+1)}},$$
$$\bar{\varepsilon} < \sqrt{\frac{1-4\tau^2\beta(2N\eta_2+1)}{4\tau^2 N\eta_1\lambda_{\max}(\Phi)}}.$$

Clearly, $\bar{\tau} < \frac{1}{2\sqrt{\beta}}$.

From (5.56), we observe that $\dot{V}_{\mathrm{M}} = 0$ if and only if

$$2e^{\mathrm{T}}(t)(I_N \otimes P)F(x(t)) = 0, \tag{5.61}$$

and

$$e^{\mathrm{T}}(t-\tau)\left(\left(\varepsilon - \frac{8\varepsilon\tau^2 N\beta\eta_2}{1-4\tau^2\beta}\right)\Phi - \frac{4\varepsilon^3\tau^2 N\eta_1}{1-4\tau^2\beta}\Phi^2\right)e(t-\tau) = 0, \tag{5.62}$$

which is equivalent to

$$e^{\mathrm{T}}(t-\tau)(I_N \otimes PB)\Gamma(I_N \otimes B^{\mathrm{T}}P)e(t-\tau) = 0, \tag{5.63}$$

where

$$\Gamma = \left(\varepsilon - \frac{8\varepsilon\tau^2 N\beta\eta_2}{1-4\tau^2\beta}\right) I_{Nm} - \frac{4\varepsilon^3\tau^2 N\eta_1}{1-4\tau^2\beta}\left(I_N \otimes B^{\mathrm{T}}PPB\right). \tag{5.64}$$

Since Φ and $(I_N \otimes B^{\mathrm{T}}PPB)$ have the same nonzero eigenvalues, under (5.59) and (5.60), we have $\Gamma > 0$. Then, (5.63) is true if and only if

$$e^{\mathrm{T}}(t-\tau)(I_N \otimes PB) = \mathbf{0}_{1\times Nm}. \tag{5.65}$$

In view of (5.61) and (5.65), following similar derivations as in the proof of Theorem 3, we conclude that the largest invariant set in which $\dot{V}_{\mathrm{M}} = 0$ is $S = \{e = \mathbf{0}_{Nn\times 1}\}$. Therefore, by LaSalle's Invariance Principle, $\lim_{t\to\infty} e(t) = 0$. This completes the proof. □

5.4 Simulation results

The performance of the proposed control laws is demonstrated by numerical simulation. In our simulation, we consider multi-agent systems with five agents and a communication topology shown in Fig. 5.1, which is an undirected and connected graph.

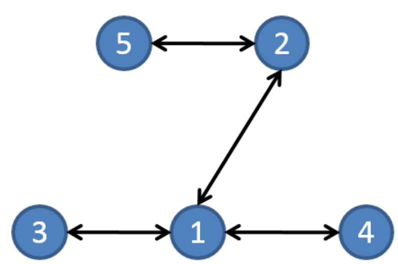


Figure 5.1: The communication topology

5.4.1 Example 1: Linear unforced dynamics without delays

The dynamics of agent i, $i = 1, 2, \ldots, 5$ are given as follows

$$\dot{x}_i = Ax_i + g(x_i)u_i, \tag{5.66}$$

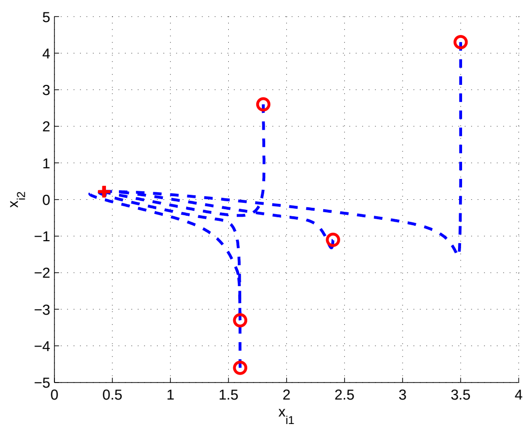
where $x_i \in \mathbb{R}^2$, $u_i \in \mathbb{R}$,

$$A = \begin{bmatrix} -1.5 & 3 \\ 1 & -2 \end{bmatrix}, \; g(x_i) = \begin{bmatrix} 0 \\ 1 + x_{i1}^2 + x_{i2}^2 \end{bmatrix}.$$

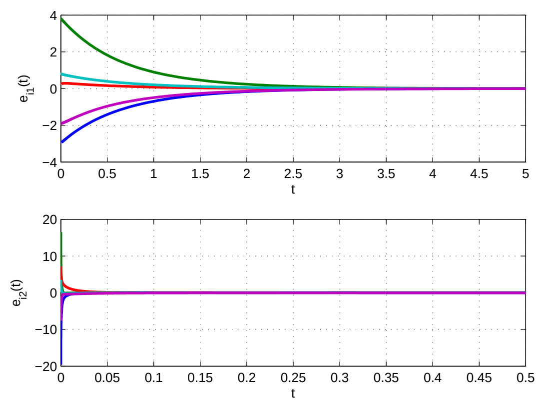
It can be easily verified that Assumption 5 holds and Assumption 4 is satisfied with

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}.$$

Shown in Fig. 5.2a and 5.2b are trajectories of the agents and the consensus errors. In the simulation, the initial states of the agents are randomly chosen as $x_1(0) = [1.6, -4.6]^{\mathrm{T}}$, $x_2(0) = [3.5, 4.3]^{\mathrm{T}}$, $x_3(0) = [1.8, 2.6]^{\mathrm{T}}$, $x_4(0) = [2.4, -1.1]^{\mathrm{T}}$, and $x_5(0) = [1.6, -3.3]^{\mathrm{T}}$. As observed in Fig. 5.2a, where circles indicate the initial states and crosses represent the states when simulation terminates, the sates of the agents are synchronized. We can see more clearly in Fig. 5.2b that consensus errors converge to zero as the system evolves.



(a) Trajectories of agents



(b) Consensus errors

Figure 5.2: Example 1: Linear unforced dynamics, $\tau = 0$

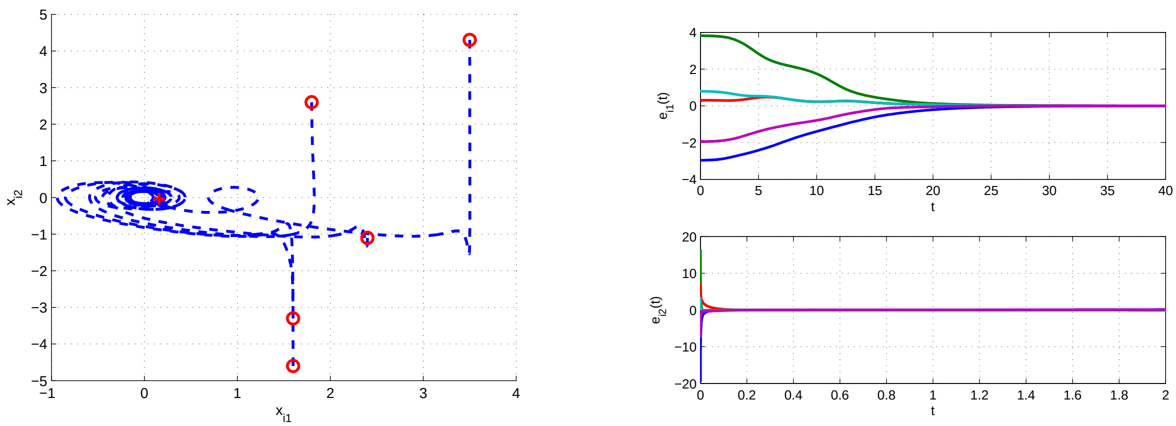
5.4.2 Example 2: Nonlinear unforced dynamics without delays

The dynamics of agent i, $i = 1, 2, \ldots, 5$, are given by,

$$\begin{aligned} \dot{x}_{i1} &= x_{i2}, \\ \dot{x}_{i2} &= -x_{i1} - x_{i2}^3 + (1 + x_{i1}^2 + x_{i2}^2)u_i. \end{aligned} \tag{5.67}$$

It can be easily verified that both Assumptions 4 and 5 are satisfied, with $P = I_2$.

Trajectories of the agents under the influence of the control laws (5.12) are shown in Fig. 5.3a, where circles and crosses respectively indicate the initial states and the states when simulation terminates. The evolution of the errors are shown in Fig. 5.3b. In the simulation, we have chosen the initial conditions to be $x_1(0) = [1.6, -4.6]^{\mathrm{T}}$, $x_2(0) = [3.5, 4.3]^{\mathrm{T}}$, $x_3(0) = [1.8, 2.6]^{\mathrm{T}}$, $x_4(0) = [2.4, -1.1]^{\mathrm{T}}$, and $x_5(0) = [1.6, -3.3]^{\mathrm{T}}$. Again, the states of all the agents reach an agreement and errors approach zero.



(a) Trajectories of agents

(b) Consensus errors

Figure 5.3: Example 2: Nonlinear unforced dynamics, $\tau = 0$

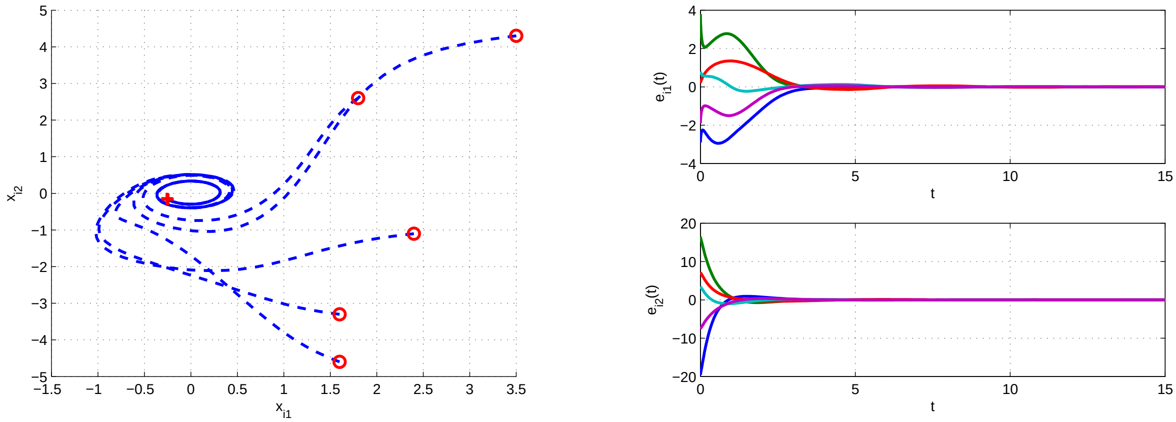
5.4.3 Example 3: Nonlinear unforced dynamics with delays

The dynamics of agent i, $i = 1, 2, \ldots, 5$, are given by,

$$\begin{aligned} \dot{x}_{i1} &= -x_{i1}^3 + x_{i2}, \\ \dot{x}_{i2} &= -x_{i1} + u_i, \end{aligned} \tag{5.68}$$

which satisfy all the proposed assumptions, with $P = I_2$ and $\beta = 1$. The communication delay is $\tau = 0.1$. The low gain parameter is thus chosen to be $\varepsilon = 0.5$.

We have chosen the initial conditions to be $x_1(t) = [1.6, -4.6]^{\mathrm{T}}$, $x_2(t) = [3.5, 4.3]^{\mathrm{T}}$, $x_3(t) = [1.8, 2.6]^{\mathrm{T}}$, $x_4(t) = [2.4, -1.1]^{\mathrm{T}}$, and $x_5(t) = [1.6, -3.3]^{\mathrm{T}}$, for $t \in [-\tau, 0]$. Simulation results are shown in Figs. 5.4a and 5.4b, from which consensus is observed.

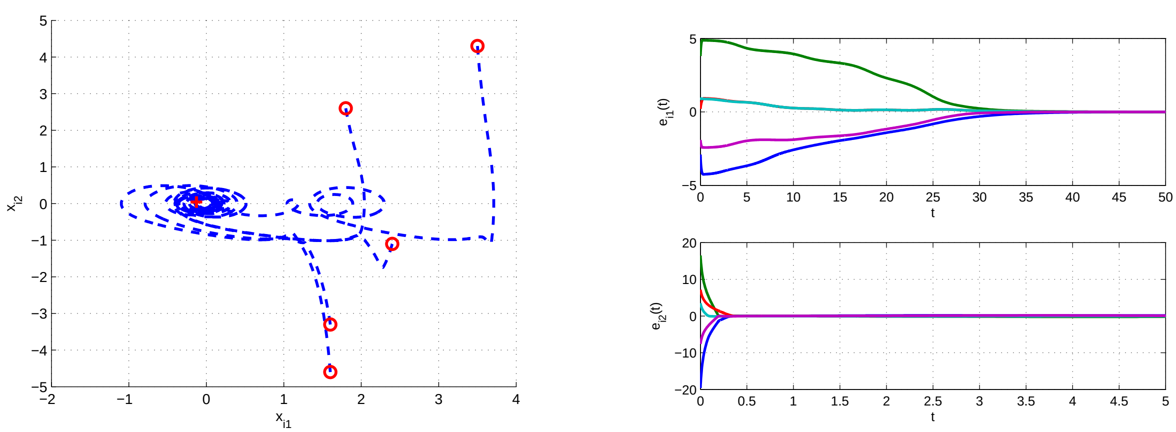


(a) Trajectories of agents (b) Consensus errors

Figure 5.4: Example 3: Nonlinear unforced dynamics, $\tau = 0.1$, $\varepsilon = 0.5$

5.4.4 Example 4: Consensus under actuator saturation

Consider the multi-agent system described by (5.67) and assume that the magnitude of the control input of each agent is limited by 1. Following Remark 3, the appropriately modified bounded control laws, with $\alpha = 1$, would ensure consensus. In simulation, we choose the initial states of the agents as $x_1(0) = [1.6, -4.6]^{\mathrm{T}}$, $x_2(0) = [3.5, 4.3]^{\mathrm{T}}$, $x_3(0) = [1.8, 2.6]^{\mathrm{T}}$, $x_4(0) = [2.4, -1.1]^{\mathrm{T}}$, and $x_5(0) = [1.6, -3.3]^{\mathrm{T}}$. The simulation results in Figs. 5.5a and 5.5b show that the consensus errors converge to zero and the states of agents are synchronized.



(a) Trajectories of agents (b) Consensus errors

Figure 5.5: Example 4: Consensus under actuator saturation, $\tau = 0$

5.5 Summary

Consensus behavior of multi-agent systems with continuous-time control-affine nonlinear dynamics was studied in this chapter. We proposed distributed control laws based on neighborhood consensus errors, that achieve global consensus both in the absence and in the presence of communication delays. In addition, a theoretical upper bound on the tolerable communication delays is provided. The performance of the proposed control laws is demonstrated by numerical simulation.

Chapter 6

Consensus of Nonlinear Multi-agent Systems in the Discrete-time Setting

Recalling that applications of continuous-time controllers through direct discretization could be very restrictive due to the requirement for small sampling periods. In order to fulfill the demand of implementing consensus control in a discrete time setting, we study the discrete-time consensus control problems in this chapter.

6.1 Problem statement

We consider a discrete-time multi-agent system that consists of N agents, each described by,

$$x_i(k+1) = Ax_i(k) + g(x_i(k))u_i(k), \tag{6.1}$$

where $x_i \in \mathbb{R}^n$ and $u_i \in \mathbb{R}^m$ are the state vector of agent i and the control input, respectively, $A \in \mathbb{R}^{n\times n}$ and $g(x_i) \in \mathbb{R}^{n\times m}$.

Similarly, we have the following definition of global consensus in the discrete-time setting.

Definition 8. *Global consensus of a multi-agent system is said to be achieved if the states of all agents are synchronized, i.e., for all* $x_i(0) \in \mathbb{R}^n$, $i = 1, 2, \ldots, N$, $\lim_{k\to\infty} \|x_i(k)-x_j(k)\| = 0,\ i, j = 1, 2, \ldots, N$.

Neighborhood consensus errors are defined as

$$e_i(k) = \sum_{j=1}^{N} a_{ij}(x_i(k) - x_j(k)),\ i = 1, 2, \ldots, N. \tag{6.2}$$

Then, with notations (5.3)–(5.7) the dynamics of the multi-agent system (6.1) can be expressed more compactly as

$$x(k+1) = (I_N \otimes A)x(k) + G(x(k))u(k). \tag{6.3}$$

The consensus errors (6.2) and its dynamics can be rewritten as

$$e(k) = (L \otimes I_n)x(k), \tag{6.4}$$

and

$$e(k+1) = (I_N \otimes A)e(k) + (L \otimes I_n)G(x(k))u(k), \tag{6.5}$$

respectively.

Assumption 7. *The agent dynamics is open-loop marginally stable and $P \in \mathbb{R}^{n\times n}$ is a positive definite matrix such that*

$$A^T PA - P \leq 0.$$

Assumption 8. *For any $z \in \mathbb{R}^n$, we have*

$$\dim\ \text{rowsp}\left\{ \left(A^l\right)^T \left(A^T PA - P\right) A^l,\ \ g^T \left(A^l z\right) PA^{l+1} \colon l = 0, 1, 2, \ldots \right\} = n,$$

where rowsp *denotes row space.*

6.2 Consensus in the absence of communication delays

In this section, we first study a simpler case where each agent collects the states of its neighboring agents instantly. That is, communication delays are not considered. In the next section, we will study global consensus control of multi-agent systems that are subject to communication delays.

We propose the following distributed control laws to enforce the state consensus,

$$u_i(k) = -2\left(I_m + 2d_i g^{\mathrm{T}}(x_i(k))Pg(x_i(k))\right)^{-1} g^{\mathrm{T}}(x_i(k))PA\sum_{j=1}^{N} a_{ij}\left(x_i(k) - x_j(k)\right), \quad (6.6)$$

where the definition of the matrix P is given in Assumption 7. Using the notations in (5.3)-(5.7), we obtain

$$u(k) = -2\left(I_{Nm} + 2G^{\mathrm{T}}(x(k))(D \otimes P)G(x(k))\right)^{-1} G^{\mathrm{T}}(x(k))(I_N \otimes PA)e(k). \quad (6.7)$$

The behavior of the discrete-time multi-agent system (6.1) under the proposed control laws (6.6) is established in the following theorem.

Theorem 5. *Consider a multi-agent system whose dynamics are given by (6.1). Suppose that Assumptions 1, 7 and 8 hold. Then the control laws (6.6) steer the multi-agent system into global consensus. That is, under the control laws (6.6),*

$$\lim_{k\to\infty} \|x_i(k) - x_j(k)\| = 0,\ i, j = 1, 2, \ldots, N,$$

for all $x_i(0) \in \mathbb{R}^n$, $i = 1, 2, \ldots, N$.

Proof. Consider the following Lyapunov function

$$V(e) = e^{\mathrm{T}}(k)(M \otimes P)e(k),$$

where matrices $M > 0$ and $P > 0$ are as defined in Remark 2 and Assumption 7, respectively.

The difference of $V(e)$ along the trajectories of (6.5) is evaluated as

$$\begin{aligned}\Delta V &= e^{\mathrm{T}}(k+1)(M \otimes P)e(k+1) - e^{\mathrm{T}}(k)(M \otimes P)e(k)\\ &= e^{\mathrm{T}}(k)\left(M \otimes \left(A^{\mathrm{T}}PA - P\right)\right)e(k) + 2u^{\mathrm{T}}(k)G^{\mathrm{T}}(x(k))(LM \otimes PA)e(k)\\ &\quad + u^{\mathrm{T}}(k)G^{\mathrm{T}}(x(k))(LML \otimes P)G(x(k))u(k). \end{aligned} \quad (6.8)$$

Recalling that $LML = L$ and $e^{\mathrm{T}}(ML \otimes I_n) = e^{\mathrm{T}}$, we have

$$\begin{aligned}\Delta V &= e^{\mathrm{T}}(k)\left(M \otimes \left(A^{\mathrm{T}}PA - P\right)\right)e(k) + 2u^{\mathrm{T}}(k)G^{\mathrm{T}}(x(k))(I_N \otimes PA)e(k)\\ &\quad + u^{\mathrm{T}}(k)G^{\mathrm{T}}(x(k))(L \otimes P)G(x(k))u(k). \end{aligned} \quad (6.9)$$

Applying Lemma 4 to the third term on the right-hand side of (6.9) gives

$$\Delta V \leq e^{\mathrm{T}}(k)\left(M \otimes \left(A^{\mathrm{T}}PA - P\right)\right)e(k) + 2u^{\mathrm{T}}(k)G^{\mathrm{T}}(x(k))(I_N \otimes PA)e(k) \\ + 2u^{\mathrm{T}}(k)G^{\mathrm{T}}(x(k))(D \otimes P)G(x(k))u(k). \tag{6.10}$$

Substituting (6.7) into (6.10) gives

$$\begin{aligned}\Delta V \leq\ & e^{\mathrm{T}}(k)\left(M \otimes \left(A^{\mathrm{T}}PA - P\right)\right)e(k) \\ & - 4e^{\mathrm{T}}(k)\left(I_N \otimes A^{\mathrm{T}}P\right)G(x(k))\left(I_{Nm} + 2G^{\mathrm{T}}(x(k))(D \otimes P)G(x(k))\right)^{-1} \\ & \times G^{\mathrm{T}}(x(k))\left(I_N \otimes PA\right)e(k) \\ \leq\ & 0. \end{aligned} \tag{6.11}$$

Since both terms on the right-hand side of (6.11) are non-positive, $\Delta V \equiv 0$ implies

$$e^{\mathrm{T}}(k)\left(M \otimes \left(A^{\mathrm{T}}PA - P\right)\right)e(k) \equiv 0,$$

and

$$G^{\mathrm{T}}(x(k))(I_N \otimes PA)e(k) \equiv \mathbf{0}_{Nm\times 1},$$

which are equivalent to

$$x^{\mathrm{T}}(k)\left(L \otimes \left(A^{\mathrm{T}}PA - P\right)\right)x(k) \equiv 0, \tag{6.12}$$

and

$$G^{\mathrm{T}}(x(k))(L \otimes PA)x(k) \equiv \mathbf{0}_{Nm\times 1}, \tag{6.13}$$

respectively. Under (6.13), we have $u(k) \equiv \mathbf{0}_{Nm\times 1}$ and the solution to (6.3) is given by

$$x(k) = \left(I_N \otimes A^k\right)x(0). \tag{6.14}$$

Substitution of (6.14) into (6.12) and (6.13) results in,

$$x^{\mathrm{T}}(0)\left(L \otimes \left(A^k\right)^{\mathrm{T}}\left(A^{\mathrm{T}}PA - P\right)A^k\right)x(0) = 0,\ k \geq 0,$$

and,

$$G^{\mathrm{T}}\left(\left(I_N \otimes A^k\right)x(0)\right)(L \otimes PA^{k+1})x(0) = \mathbf{0}_{Nm\times 1},\ k \geq 0,$$

respectively, which can be further expanded to,

$$\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}(x_i(0) - x_j(0))^{\mathrm{T}}\left(A^k\right)^{\mathrm{T}}\left(A^{\mathrm{T}}PA - P\right)A^k(x_i(0) - x_j(0)) = 0,\ k \geq 0, \tag{6.15}$$

and,

$$g^{\mathrm{T}}\left(A^k x_i(k)\right) P A^{k+1} e_i(0) = \mathbf{0}_{m\times 1},\ k \geq 0,$$
$$i = 1, 2, \ldots, N, \tag{6.16}$$

respectively. Note that each term on the left-hand side of (6.15) is non-positive. Then (6.15) is equivalent to

$$a_{ij}(x_i(0) - x_j(0))^{\mathrm{T}} \left(A^k\right)^{\mathrm{T}} \left(A^{\mathrm{T}} P A - P\right) A^k (x_i(0) - x_j(0)) = 0,\ \forall i, j = 1, 2, \ldots, N,$$

Since $\left(A^k\right)^{\mathrm{T}} \left(A^{\mathrm{T}} P A - P\right) A^k$ is negative semi-definite, it follows that

$$a_{ij} \left(A^k\right)^{\mathrm{T}} \left(A^{\mathrm{T}} P A - P\right) A^k (x_i(0) - x_j(0)) = \mathbf{0}_{n\times 1},$$
$$\forall i, j = 1, 2, \ldots, N,$$

which further implies that

$$\left(A^k\right)^{\mathrm{T}} \left(A^{\mathrm{T}} P A - P\right) A^k e_i(0) = \mathbf{0}_{n\times 1},\ i = 1, 2, \ldots, N. \tag{6.17}$$

In view of Assumption 8, the space spanned by the rows in the matrices $\left(A^k\right)^{\mathrm{T}} \left(A^{\mathrm{T}} P A - P\right) A^k$ and $g^{\mathrm{T}}\left(A^k x_i(k)\right) P A^{k+1}$, $k \geq 0$, has a dimension of n. Then, from (6.16) and (6.17), we conclude that $\Delta V \equiv 0$ only when $e(0) = \mathbf{0}_{Nn\times 1}$, which implies $e(k) = \mathbf{0}_{Nn\times 1}$, $k \geq 0$. Therefore, by LaSalle's Invariance Principle, for any $e(0) \in \mathbb{R}^{Nn}$, $\lim_{k\to\infty} e(k) = \mathbf{0}_{Nn\times 1}$ and global consensus is achieved. □

Remark 4. *If there is a constraint on the magnitude of the control inputs of the agents, we can modify the control inputs to each agent i as*

$$\tilde{u}_{ik} = \mathrm{sign}(u_{ik}) \min(|u_{ik}|, \alpha),\ k = 1, 2, \ldots, m, \tag{6.18}$$

where $\tilde{u}_{ik}$ is the kth element in the constrained inputs to agent i, u_{ik} is the kth element in u_i as defined in (6.6), and $\alpha > 0$ is the bound on the inputs. The bounded control inputs (6.18) then bring the multi-agent system into consensus if Assumptions 1, 7 and 8 are satisfied.

6.3 Consensus in the presence of communication delays

In this section, we assume the communication delay to be τ steps and $\tau \in \mathbb{N}^+$ is constant. Communication delays increase the complexity in the analysis of the system behavior, and we need to make following assumptions before establishing further results.

Assumption 9. *There exists a matrix $B \in \mathbb{R}^{n\times m}$ and a positive constant α such that*

$$\alpha BB^T \leq g(z)g^T(z) \leq BB^T,\ \forall z \in \mathbb{R}^n.$$

Remark 5. *For any $z \in \mathbb{R}^n$ and any positive definite matrix $Q \in \mathbb{R}^{m\times m}$, if Assumption 9 is satisfied, the following inequalities always hold,*

$$\begin{aligned} g(z)Qg^T(z) &\leq \frac{\lambda_{\max}(Q)}{\lambda_{\min}(Q)} BQB^T, \\ g(z)Qg^T(z) &\geq \frac{\alpha\lambda_{\min}(Q)}{\lambda_{\max}(Q)} BQB^T. \end{aligned} \tag{6.19}$$

Since asymptotically stable eigenvalues do not affect the stability of the system, without loss of generality, we make an assumption as follows.

Assumption 10. *Let matrix B be as defined in Assumption 9. The matrix pair (A, B) is controllable and all eigenvalues of A are located on the unit circle.*

We propose the following low gain control laws for consensus in the presence of communication delays,

$$u_i(k) = -\left(\gamma_2 B^{\mathrm{T}} P B + \gamma_3 I_m\right)^{-1} g^{\mathrm{T}}(x_i(k)) P A \sum_{j=1}^{N} a_{ij}\left(x_i(k-\tau) - x_j(k-\tau)\right), \tag{6.20}$$

which, with the notations in (5.3)-(5.7), can be rewritten as

$$u(k) = -\left(I_N \otimes \left(\gamma_2 B^{\mathrm{T}} P B + \gamma_3 I_m\right)^{-1}\right) G^{\mathrm{T}}(x(k))(I_N \otimes PA)e(k-\tau). \tag{6.21}$$

where B is the matrix defined in Assumption 9, and the positive definite matrix P is the unique positive definite solution to the following discrete-time algebraic Riccati equation,

$$P = A^{\mathrm{T}} P A - A^{\mathrm{T}} P B \left(B^{\mathrm{T}} P B + \gamma_1 I_m\right)^{-1} B^{\mathrm{T}} P A + \varepsilon P. \tag{6.22}$$

There are multiple parameters in (6.21) and (6.22). The parameter $\varepsilon \in (0,1)$ is a low gain parameter, and the parameters γ_1, γ_2 and γ_3 are designed such that

$$\begin{aligned} \gamma_1 &= \frac{\lambda_{\max}(M)}{\alpha(1-\varepsilon)^n}\gamma_3, \\ \gamma_2 &= \frac{\gamma_3}{\gamma_1} = \frac{\alpha(1-\varepsilon)^n}{\lambda_{\max}(M)}, \\ \gamma_3 &\geq \frac{2d_{\max}\lambda_{\max}(P)\lambda_{\max}(BB^{\mathrm{T}})}{(1-\varepsilon)^n}, \end{aligned} \tag{6.23}$$

where $d_{\max} = \max_{i=1,2,\ldots,N}\{d_i\}$ denotes the maximum degree of the communication graph.

Remark 6. *If the matrix pair (A, B) is controllable, for any $\varepsilon \in (0,1)$, there always exists a unique positive definite solution to discrete-time algebraic Riccati equation (6.22), and such a solution can also be obtained from the parametric Lyapunov equation as follows, with $W = P^{-1}$,*

$$W - \frac{1}{1-\varepsilon}AWA^T = -\frac{1}{\gamma_1}BB^T. \tag{6.24}$$

We establish following lemmas, which are essential in proving global consensus under communication delays, in the discrete-time setting.

Lemma 20. *Consider a matrix pair (A, B) that satisfies Assumption 10, and let P be the positive definite solution to (6.22), with $\varepsilon \in (0,1)$. Then, the following inequalities hold,*

$$\lambda_{\max}(B^TPB) \leq \mathrm{tr}(B^TPB) \leq \gamma_1\left(\frac{1}{(1-\varepsilon)^n} - 1\right), \tag{6.25}$$

$$A^TPB\left(B^TPB + \gamma_1 I_m\right)^{-1}B^TPA \leq \frac{1-(1-\varepsilon)^n}{(1-\varepsilon)^{n-1}}P, \tag{6.26}$$

$$(A - I_n)^T P (A - I_n) \leq \phi P, \tag{6.27}$$

where

$$\phi = \frac{2}{(1-\varepsilon)^{n-1}} - 2 - (n-2)\varepsilon + 2\left(n - \mathrm{tr}(A)\right).$$

Proof. We first prove (6.25). It is clear that

$$\det\left(B^{\mathrm{T}}PB + \gamma_1 I_m\right) = \gamma_1^m \det\left(\frac{1}{\gamma_1}B^{\mathrm{T}}PB + I_m\right).$$

By Sylvester's determinant identity, it follows that

$$\det\left(B^{\mathrm{T}}PB + \gamma_1 I_m\right) = \gamma_1^m \det\left(\frac{1}{\gamma_1}PBB^{\mathrm{T}} + I_n\right). \tag{6.28}$$

Multiplying both sides of (6.24) by P gives

$$I_n + \frac{1}{\gamma_1} PBB^{\mathrm{T}} = \frac{1}{1-\varepsilon} PAP^{-1}A^{\mathrm{T}}. \tag{6.29}$$

Substituting (6.29) into (6.28), we have

$$\begin{aligned} \det\left(B^{\mathrm{T}}PB + \gamma_1 I_m\right) &= \gamma_1^m \det\left(\frac{1}{1-\varepsilon} PAP^{-1}A^{\mathrm{T}}\right) \\ &= \frac{\gamma_1^m}{(1-\varepsilon)^n} \left(\det(A)\right)^2 \\ &= \frac{\gamma_1^m}{(1-\varepsilon)^n}. \end{aligned} \tag{6.30}$$

Denote the eigenvalues of $B^{\mathrm{T}}PB$ by $\lambda_1, \lambda_2, \ldots, \lambda_m$. Then, by the definition of determinant, we have

$$\begin{aligned} & \det\left(B^{\mathrm{T}}PB + \gamma_1 I_m\right) \\ = & \prod_{i=1}^{m} (\lambda_i + \gamma_1) \\ = & \gamma_1^m + \gamma_1^{m-1} \sum_{i=1}^{m} \lambda_i + \gamma_1^{m-2} \sum_{i \neq j} \lambda_i \lambda_j + \ldots + \prod_{i=1}^{m} \lambda_i, \end{aligned}$$

which, together with (6.30), gives

$$\gamma_1^m + \gamma_1^{m-1} \sum_{i=1}^{m} \lambda_i + \gamma_1^{m-2} \sum_{i \neq j} \lambda_i \lambda_j + \ldots + \prod_{i=1}^{m} \lambda_i = \frac{\gamma_1^m}{(1-\varepsilon)^n}.$$

Noting that $\lambda_i \geq 0$ for $i = 1, 2, \ldots, m$, we get

$$\max\{\lambda_1, \lambda_2, \ldots, \lambda_m\} \leq \sum_{i=1}^{m} \lambda_i \leq \gamma_1 \left(\frac{1}{(1-\varepsilon)^n} - 1\right).$$

Then, we prove (6.26). Note that

$$\begin{aligned} PB\left(B^{\mathrm{T}}PB + \gamma_1 I_m\right)^{-1} B^{\mathrm{T}}P &= P^{\frac{1}{2}} \left(P^{\frac{1}{2}} B \left(B^{\mathrm{T}}PB + \gamma_1 I_m\right)^{-1} B^{\mathrm{T}} P^{\frac{1}{2}}\right) P^{\frac{1}{2}} \\ &\leq \lambda_{\max}\left(P^{\frac{1}{2}} B \left(B^{\mathrm{T}}PB + \gamma_1 I_m\right)^{-1} B^{\mathrm{T}} P^{\frac{1}{2}}\right) P. \end{aligned}$$

For any matrix S, SS^{T} and $S^{\mathrm{T}}S$ share the same non-zero eigenvalues, and thus $\lambda_{\max}\left(SS^{\mathrm{T}}\right) = \lambda_{\max}\left(S^{\mathrm{T}}S\right)$. Then, it follows that

$$PB\left(B^{\mathrm{T}}PB + \gamma_1 I_m\right)^{-1} B^{\mathrm{T}}P \leq \lambda_{\max}\left(R^{-\frac{1}{2}} B^{\mathrm{T}} PBR^{-\frac{1}{2}}\right) P,$$

where $R = B^{\mathrm{T}}PB + \gamma_1 I_m$. Let λ be an eigenvalue of $R^{-\frac{1}{2}}B^{\mathrm{T}}PBR^{-\frac{1}{2}}$. By the definition of eigenvalues, we have

$$\det\left(\lambda I_m - R^{-\frac{1}{2}}B^{\mathrm{T}}PBR^{-\frac{1}{2}}\right) = 0. \tag{6.31}$$

Multiplying to left and right of both sides of the above equation by $\det(R^{\frac{1}{2}})$, we obtain

$$\det\left(\lambda\left(B^{\mathrm{T}}PB + \gamma_1 I_m\right) - B^{\mathrm{T}}PB\right) = 0,$$

which can be further simplified as

$$\det\left(\frac{\gamma_1\lambda}{1-\lambda}I_m - B^{\mathrm{T}}PB\right) = 0. \tag{6.32}$$

Equations (6.31) and (6.32) imply that if λ is an eigenvalue of $R^{-\frac{1}{2}}B^{\mathrm{T}}PBR^{-\frac{1}{2}}$ then $\frac{\gamma_1\lambda}{1-\lambda}$ is an eigenvalue of $B^{\mathrm{T}}PB$. In other words, if λ is an eigenvalue of $B^{\mathrm{T}}PB$ then $\frac{\lambda}{\gamma_1+\lambda}$ is an eigenvalue of $R^{-\frac{1}{2}}B^{\mathrm{T}}PBR^{-\frac{1}{2}}$. Note that $\frac{\lambda}{\gamma_1+\lambda}$ is strictly increasing with respect to λ, when $\lambda > 0$. Then, we have

$$\begin{aligned}\lambda_{\max}\left(R^{-\frac{1}{2}}B^{\mathrm{T}}PBR^{-\frac{1}{2}}\right) &= \frac{\lambda_{\max}\left(B^{\mathrm{T}}PB\right)}{\gamma_1 + \lambda_{\max}\left(B^{\mathrm{T}}PB\right)} \\ &\leq 1 - (1-\varepsilon)^n.\end{aligned}$$

Therefore, it follows that

$$PB\left(B^{\mathrm{T}}PB + \gamma_1 I_m\right)^{-1}B^{\mathrm{T}}P \leq \left(1 - (1-\varepsilon)^n\right)P.$$

Multiplying to left and right of both sides of (6.22) by $A^{-\mathrm{T}}$ and A^{-1}, respectively, we obtain

$$\begin{aligned}P - (1-\varepsilon)A^{-\mathrm{T}}PA^{-1} &= PB\left(B^{\mathrm{T}}PB + \gamma_1 I_m\right)^{-1}B^{\mathrm{T}}P \\ &\leq \left(1 - (1-\varepsilon)^n\right)P,\end{aligned}$$

from which it follows that

$$A^{\mathrm{T}}PA \leq \frac{1}{(1-\varepsilon)^{n-1}}P. \tag{6.33}$$

Substitution of (6.33) into (6.22) yields

$$\begin{aligned}A^{\mathrm{T}}PB\left(B^{\mathrm{T}}PB + \gamma_1 I_m\right)^{-1}B^{\mathrm{T}}PA &= A^{\mathrm{T}}PA - (1-\varepsilon)P \\ &\leq \frac{1-(1-\varepsilon)^n}{(1-\varepsilon)^{n-1}}P.\end{aligned}$$

Finally, we prove (6.27). Obviously,

$$(A - I_n)^{\mathrm{T}} P (A - I_n) = A^{\mathrm{T}} P A - A^{\mathrm{T}} P - PA + P. \tag{6.34}$$

The parametric Lyapunov equation (6.24) can be rearranged as

$$P^{-1} - \frac{1}{1-\varepsilon}(A - I_n + I_n)P^{-1}(A - I_n + I_n)^{\mathrm{T}} = -\frac{1}{\gamma_1}BB^{\mathrm{T}},$$

which can be expanded to

$$(\varepsilon - 2)P^{-1} + (A - I_n)P^{-1}(A - I_n)^{\mathrm{T}} + AP^{-1} + P^{-1}A^{\mathrm{T}} = \frac{1-\varepsilon}{\gamma_1}BB^{\mathrm{T}}. \tag{6.35}$$

Multiplying to left and right of both sides of (6.35) by P, we have

$$PA + A^{\mathrm{T}}P = \frac{1-\varepsilon}{\gamma_1}PBB^{\mathrm{T}}P + (2-\varepsilon)P - P(A - I_n)P^{-1}(A - I_n)^{\mathrm{T}}P. \tag{6.36}$$

Substitution of (6.36) into (6.34) gives

$$\begin{aligned} & (A - I_n)^{\mathrm{T}} P (A - I_n) \\ = & A^{\mathrm{T}}PA - \frac{1-\varepsilon}{\gamma_1}PBB^{\mathrm{T}}P - (1-\varepsilon)P + P(A - I_n)P^{-1}(A - I_n)^{\mathrm{T}}P \\ \leq & A^{\mathrm{T}}PA - (1-\varepsilon)P + P(A - I_n)P^{-1}(A - I_n)^{\mathrm{T}}P. \end{aligned} \tag{6.37}$$

Note that

$$\begin{aligned} P(A - I_n)P^{-1}(A - I_n)^{\mathrm{T}}P &= P^{\frac{1}{2}}\left(P^{\frac{1}{2}}(A - I_n)P^{-1}(A - I_n)^{\mathrm{T}}P^{\frac{1}{2}}\right)P^{\frac{1}{2}} \\ &\leq \operatorname{tr}\left(P^{\frac{1}{2}}(A - I_n)P^{-1}(A - I_n)^{\mathrm{T}}P^{\frac{1}{2}}\right)P \\ &= \operatorname{tr}\left((A - I_n)P^{-1}(A - I_n)^{\mathrm{T}}P\right)P. \end{aligned} \tag{6.38}$$

Multiplying both sides of (6.35) by P and taking trace of both sides of the resulting equation, we obtain

$$(\varepsilon - 2)\operatorname{tr}(I_n) + \operatorname{tr}\left((A - I_n)P^{-1}(A - I_n)^{\mathrm{T}}P\right) + \operatorname{tr}(A) + \operatorname{tr}\left(P^{-1}A^{\mathrm{T}}P\right) = \frac{1-\varepsilon}{\gamma_1}\operatorname{tr}\left(BB^{\mathrm{T}}P\right),$$

from which it follows that

$$\operatorname{tr}\left((A - I_n)P^{-1}(A - I_n)^{\mathrm{T}}P\right) = \frac{1-\varepsilon}{\gamma_1}\operatorname{tr}\left(B^{\mathrm{T}}PB\right) - 2\operatorname{tr}(A) + (2-\varepsilon)n$$

$$\leq \frac{1-(1-\varepsilon)^n}{(1-\varepsilon)^{n-1}} - 2\operatorname{tr}(A) + (2-\varepsilon)n. \tag{6.39}$$

Substituting (6.33), (6.38) and (6.39) into (6.37) yields

$$\begin{aligned}(A-I_n)^{\mathrm{T}}P(A-I_n) \leq{}& \frac{1}{(1-\varepsilon)^{n-1}}P - (1-\varepsilon)P \\ &+ \left(\frac{1-(1-\varepsilon)^n}{(1-\varepsilon)^{n-1}} - 2\operatorname{tr}(A) + (2-\varepsilon)n\right)P \\ ={}& \left(\frac{2}{(1-\varepsilon)^{n-1}} - 2 - (n-2)\varepsilon + 2\left(n - \operatorname{tr}(A)\right)\right)P.\end{aligned}$$

□

Lemma 21. *Consider the following continuous function of $\varepsilon \in (0,1)$,*

$$\begin{aligned}H(\varepsilon) ={}& \frac{6\lambda_{\max}(M)}{\alpha\lambda_{\min}(M)}\frac{1-(1-\varepsilon)^n}{\varepsilon(1-\varepsilon)^{3n-1}}\left(\frac{2}{(1-\varepsilon)^{n-1}} - (n-2)\varepsilon - 2 + 2(n-\operatorname{tr}(A))\right. \\ &\left.+\frac{2d_{\max}\lambda_{\max}(M)}{\alpha}\frac{1-(1-\varepsilon)^n}{(1-\varepsilon)^{3n-1}}\right).\end{aligned}$$

If all eigenvalues of matrix A are located on the unit circle, $H(\varepsilon)$ is strictly increasing.

Proof. We define the following continuous functions

$$\begin{aligned}h_1(\varepsilon) &= \frac{1-(1-\varepsilon)^n}{\varepsilon(1-\varepsilon)^{n-1}}, \\ h_2(\varepsilon) &= \frac{1}{(1-\varepsilon)^{2n}}, \\ h_3(\varepsilon) &= h_1(\varepsilon)h_2(\varepsilon), \\ h_4(\varepsilon) &= \frac{2}{(1-\varepsilon)^{n-1}} - (n-2)\varepsilon.\end{aligned}$$

Taking derivative of $h_1(\varepsilon)$, we get

$$\frac{dh_1(\varepsilon)}{d\varepsilon} = \frac{(1-\varepsilon)^n - 1 + n\varepsilon}{\varepsilon^2(1-\varepsilon)^n}.$$

We can easily verify that $(1-\varepsilon)^n - 1 + n\varepsilon$ is strictly increasing in $(0,1)$ and thus $(1-\varepsilon)^n - 1 + n\varepsilon > 0$. Therefore, $h_1(\varepsilon)$ is strictly increasing and $h_1(\varepsilon) > \lim_{\varepsilon\to 0^+} h_1(\varepsilon) = n$, for any $\varepsilon \in (0,1)$. Obviously, $h_2(\varepsilon)$ is positive and strictly increasing in $(0,1)$ as well. Consequently,

$$\frac{dh_3(\varepsilon)}{d\varepsilon} = \frac{dh_1(\varepsilon)}{d\varepsilon}h_2(\varepsilon) + \frac{dh_2(\varepsilon)}{d\varepsilon}h_1(\varepsilon)$$

$$> 0,$$

and $h_3(\varepsilon)$ is strictly increasing. Thus, $h_3(\varepsilon) > \lim_{\varepsilon \to 0^+} h_3(\varepsilon) = n$. Again, taking derivative of $h_4(\varepsilon)$ gives

$$\frac{dh_4(\varepsilon)}{d\varepsilon} = \frac{2(n-1)}{(1-\varepsilon)^n} - n + 2,$$

which is strictly increasing with respect to ε. Then, we have

$$\frac{dh_4(\varepsilon)}{d\varepsilon} > \frac{dh_4}{d\varepsilon}(0) = n.$$

Thus, $h_4(\varepsilon)$ is strictly increasing as well, and $h_4(\varepsilon) > h_4(0) = 2$. We note that

$$H(\varepsilon) = \frac{6\lambda_{\max}(M)}{\alpha\lambda_{\min}(M)} h_3(\varepsilon)\left(h_4(\varepsilon) - 2 + 2(n - \mathrm{tr}(A)) + \frac{2d_{\max}\lambda_{\max}(M)}{\alpha} h_3(\varepsilon)\right).$$

Clearly, $h_4(\varepsilon) - 2 + 2(n - \mathrm{tr}(A)) + 2d_{\max}\lambda_{\max}(M)\alpha^{-1}h_3(\varepsilon)$ is strictly increasing and positive for any $\varepsilon \in (0, 1)$. Therefore, $H(\varepsilon)$ is strictly increasing in $(0, 1)$. □

Then, in the following theorem, we establish the behavior of the closed-loop multi-agent system under the proposed low gain control laws (6.20).

Theorem 6. *Consider a multi-agent system whose dynamics are given by (5.1). Suppose that Assumptions 1, 9, and 10 are all satisfied. Then there exists a constant* $\bar{\tau} > 0$ *such that for any* $\tau \in [0, \bar{\tau})$*, the low gain control laws (6.20) steer the multi-agent system into global consensus if the low gain parameter* ε *is small enough. That is for any* $\tau \in [0, \bar{\tau})$*, there exists an* $\bar{\varepsilon} \in (0, 1)$ *such that for any* $\varepsilon \in (0, \bar{\varepsilon}]$*, under the control laws (6.20),* $\lim_{k\to\infty} \|x_i(k) - x_j(k)\| = 0$*, for all* $x_i(0) \in \mathbb{R}^n$*,* $i, j = 1, 2, \ldots, N$*.*

Proof. Consider a Lyapunov function,

$$V(e(k)) = e^{\mathrm{T}}(k)(M \otimes P)e(k). \tag{6.40}$$

In view of (6.10), we have

$$\Delta V \leq e^{\mathrm{T}}(k)\left(M \otimes \left(A^{\mathrm{T}}PA - P\right)\right)e(k) + 2u^{\mathrm{T}}(k)G^{\mathrm{T}}(x(k))(I_N \otimes PA)e(k) \\ + 2u^{\mathrm{T}}(k)G^{\mathrm{T}}(x(k))(D \otimes P)G(x(k))u(k). \tag{6.41}$$

Substitution of (6.21) and (6.22) into (6.41) yields

$$\begin{aligned}\Delta V \leq\, & e^{\mathrm{T}}(k)\left(M \otimes A^{\mathrm{T}} P B\left(B^{\mathrm{T}} P B+\gamma_1 I_m\right)^{-1} B^{\mathrm{T}} P A\right) e(k)-\varepsilon e^{\mathrm{T}}(k)(M \otimes P) e(k) \\ & -2 e^{\mathrm{T}}(k-\tau)\left(I_N \otimes A^{\mathrm{T}} P\right) \Omega\left(I_N \otimes P A\right) e(k) \\ & +2 e^{\mathrm{T}}(k-\tau)\left(I_N \otimes A^{\mathrm{T}} P\right) \Omega(D \otimes P) \Omega\left(I_N \otimes P A\right) e(k-\tau),\end{aligned} \tag{6.42}$$

where $\Omega = G(x(k))\left(I_N \otimes \left(\gamma_2 B^{\mathrm{T}} P B + \gamma_3 I_m\right)^{-1}\right) G^{\mathrm{T}}(x(k))$. We define

$$\Delta e(k) = e(k+1) - e(k).$$

Noting that

$$e(k) = e(k-\tau) + \sum_{\sigma=k-\tau}^{k-1} \Delta e(\sigma),$$

we continue (6.42) as

$$\begin{aligned}\Delta V \leq\, & e^{\mathrm{T}}(k)\left(M \otimes A^{\mathrm{T}} P B\left(B^{\mathrm{T}} P B+\gamma_1 I_m\right)^{-1} B^{\mathrm{T}} P A\right) e(k)-\varepsilon e^{\mathrm{T}}(k)(M \otimes P) e(k) \\ & - e^{\mathrm{T}}(k)\left(I_N \otimes A^{\mathrm{T}} P\right) \Omega\left(I_N \otimes P A\right) e(k) \\ & - e^{\mathrm{T}}(k-\tau)\left(I_N \otimes A^{\mathrm{T}} P\right) \Omega\left(I_N \otimes P A\right) e(k-\tau) \\ & + \sum_{\sigma=k-\tau}^{k-1} \Delta e^{\mathrm{T}}(\sigma)\left(I_N \otimes A^{\mathrm{T}} P\right) \Omega\left(I_N \otimes P A\right) \sum_{\sigma=k-\tau}^{k-1} \Delta e(\sigma) \\ & +2 e^{\mathrm{T}}(k-\tau)\left(I_N \otimes A^{\mathrm{T}} P\right) \Omega(D \otimes P) \Omega\left(I_N \otimes P A\right) e(k-\tau) \\ \leq\, & \lambda_{\max}(M) e^{\mathrm{T}}(k)\left(I_N \otimes A^{\mathrm{T}} P B\left(B^{\mathrm{T}} P B+\gamma_1 I_m\right)^{-1} B^{\mathrm{T}} P A\right) e(k)-\varepsilon e^{\mathrm{T}}(k)(M \otimes P) e(k) \\ & - e^{\mathrm{T}}(k)\left(I_N \otimes A^{\mathrm{T}} P\right) \Omega\left(I_N \otimes P A\right) e(k) \\ & - e^{\mathrm{T}}(k-\tau)\left(I_N \otimes A^{\mathrm{T}} P\right) \Omega\left(I_N \otimes P A\right) e(k-\tau) \\ & + \sum_{\sigma=k-\tau}^{k-1} \Delta e^{\mathrm{T}}(\sigma)\left(I_N \otimes A^{\mathrm{T}} P\right) \Omega\left(I_N \otimes P A\right) \sum_{\sigma=k-\tau}^{k-1} \Delta e(\sigma) \\ & +2 d_{\max} \lambda_{\max}(P) e^{\mathrm{T}}(k-\tau)\left(I_N \otimes A^{\mathrm{T}} P\right) \Omega^2\left(I_N \otimes P A\right) e(k-\tau).\end{aligned} \tag{6.43}$$

Under Assumption 9, we have

$$\begin{aligned}& - e^{\mathrm{T}}(k)\left(I_N \otimes A^{\mathrm{T}} P\right) \Omega\left(I_N \otimes P A\right) e(k) \\ \leq\, & -\alpha \frac{\gamma_2 \lambda_{\min}\left(B^{\mathrm{T}} P B\right)+\gamma_3}{\gamma_2 \lambda_{\max}\left(B^{\mathrm{T}} P B\right)+\gamma_3} e^{\mathrm{T}}(k)\left(I_N \otimes A^{\mathrm{T}} P B\left(\gamma_2 B^{\mathrm{T}} P B+\gamma_3 I_m\right)^{-1} B^{\mathrm{T}} P A\right) e(k)\end{aligned}$$

$$= -\frac{\alpha\gamma_1}{\gamma_3}\cdot\frac{\lambda_{\min}\left(B^{\mathrm{T}}PB\right)+\gamma_1}{\lambda_{\max}\left(B^{\mathrm{T}}PB\right)+\gamma_1}e^{\mathrm{T}}(k)\Big(I_N\otimes A^{\mathrm{T}}PB\left(B^{\mathrm{T}}PB+\gamma_1 I_m\right)^{-1}B^{\mathrm{T}}PA\Big)e(k).$$

Inserting (6.23) and (6.25) into the above inequality, we obtain

$$\begin{aligned} &-e^{\mathrm{T}}(k)\left(I_N\otimes A^{\mathrm{T}}P\right)\Omega\left(I_N\otimes PA\right)e(k)\\ \le\ &-\lambda_{\max}(M)e^{\mathrm{T}}(k)\Big(I_N\otimes A^{\mathrm{T}}PB\left(B^{\mathrm{T}}PB+\gamma_1 I_m\right)^{-1}B^{\mathrm{T}}PA\Big)e(k). \end{aligned} \tag{6.44}$$

Similarly, under Assumption 9, we have

$$\begin{aligned} 2d_{\max}\lambda_{\max}(P)\Omega &\le 2d_{\max}\lambda_{\max}(P)\frac{\gamma_1}{\gamma_3}\cdot\frac{\lambda_{\max}\left(B^{\mathrm{T}}PB\right)+\gamma_1}{\lambda_{\min}\left(B^{\mathrm{T}}PB\right)+\gamma_1}\Big(I_N\otimes B\left(B^{\mathrm{T}}PB+\gamma_1 I_m\right)^{-1}B^{\mathrm{T}}\Big)\\ &\le \frac{2d_{\max}\lambda_{\max}(P)\lambda_{\max}(M)}{\alpha(1-\varepsilon)^{2n}}\Big(I_N\otimes B\left(B^{\mathrm{T}}PB+\gamma_1 I_m\right)^{-1}B^{\mathrm{T}}\Big)\\ &\le \frac{2d_{\max}\lambda_{\max}(P)\lambda_{\max}(M)}{\alpha(1-\varepsilon)^{2n}}\cdot\frac{\lambda_{\max}(BB^{\mathrm{T}})}{\gamma_1}I_{Nn}\\ &\le \frac{2d_{\max}\lambda_{\max}(P)}{(1-\varepsilon)^{n}}\cdot\frac{\lambda_{\max}(BB^{\mathrm{T}})}{\gamma_3}I_{Nn}\\ &\le I_{Nn}, \end{aligned}$$

which implies that

$$2d_{\max}\lambda_{\max}(P)\Omega^2\le\Omega. \tag{6.45}$$

Substituting (6.44) and (6.45) into (6.43) and collecting similar terms yields

$$\Delta V\le -\varepsilon e^{\mathrm{T}}(k)(M\otimes P)e(k)+\sum_{\sigma=k-\tau}^{k-1}\Delta e^{\mathrm{T}}(\sigma)\left(I_N\otimes A^{\mathrm{T}}P\right)\Omega\left(I_N\otimes PA\right)\sum_{\sigma=k-\tau}^{k-1}\Delta e(\sigma).$$

By Lemma 7, it follows that

$$\Delta V\le -\varepsilon e^{\mathrm{T}}(k)(M\otimes P)e(k)+\tau\sum_{\sigma=k-\tau}^{k-1}\Delta e^{\mathrm{T}}(\sigma)\left(I_N\otimes A^{\mathrm{T}}P\right)\Omega\left(I_N\otimes PA\right)\Delta e(\sigma).$$

Again, in view of Assumption 9, we continue the above inequality as

$$\begin{aligned} \Delta V\le &-\varepsilon e^{\mathrm{T}}(k)(M\otimes P)e(k)+\tau\frac{\gamma_1}{\gamma_3}\cdot\frac{\lambda_{\max}\left(B^{\mathrm{T}}PB\right)+\gamma_1}{\lambda_{\min}\left(B^{\mathrm{T}}PB\right)+\gamma_1}\sum_{\sigma=k-\tau}^{k-1}\Delta e^{\mathrm{T}}(\sigma)\\ &\times\Big(I_N\otimes A^{\mathrm{T}}PB\left(B^{\mathrm{T}}PB+\gamma_1 I_m\right)^{-1}B^{\mathrm{T}}PA\Big)\Delta e(\sigma). \end{aligned}$$

With (6.26), we can further simplify the above inequality as,

$$\Delta V\le -\varepsilon e^{\mathrm{T}}(k)(M\otimes P)e(k)$$

$$+\tau\frac{\gamma_1}{\gamma_3}\cdot\frac{\lambda_{\max}\left(B^{\mathrm{T}}PB\right)+\gamma_1}{\lambda_{\min}\left(B^{\mathrm{T}}PB\right)+\gamma_1}\cdot\frac{1-(1-\varepsilon)^n}{(1-\varepsilon)^{n-1}}\sum_{\sigma=k-\tau}^{k-1}\Delta e^{\mathrm{T}}(\sigma)\left(I_N\otimes P\right)\Delta e(\sigma). \tag{6.46}$$

In view of (6.5), we have

$$\Delta e(k)=(I_N\otimes(A-I_n))e(k)+(L\otimes I_n)G(x(k))u(k). \tag{6.47}$$

Equation (6.47) along with Lemma 7 imply

$$\begin{aligned}
&\sum_{\sigma=k-\tau}^{k-1}\Delta e^{\mathrm{T}}(\sigma)\left(I_N\otimes P\right)\Delta e(\sigma)\\
&\le 2\sum_{\sigma=k-\tau}^{k-1}\Delta e^{\mathrm{T}}(\sigma)\left(I_N\otimes(A-I_n)^{\mathrm{T}}P(A-I_n)\right)\Delta e(\sigma)\\
&\quad+2\sum_{\sigma=k-\tau}^{k-1}u^{\mathrm{T}}(\sigma)G^{\mathrm{T}}(x(\sigma))\left(L^2\otimes P\right)G(x(\sigma))u(\sigma)\\
&\le 2\phi\sum_{\sigma=k-\tau}^{k-1}\Delta e^{\mathrm{T}}(\sigma)\left(I_N\otimes P\right)\Delta e(\sigma)\\
&\quad+8d_{\max}^2\lambda_{\max}(P)\sum_{\sigma=k-\tau}^{k-1}e^{\mathrm{T}}(\sigma-\tau)(I_N\otimes A^{\mathrm{T}}P)\Omega^2(I_N\otimes PA)e(\sigma-\tau).
\end{aligned}$$

Recalling that $2d_{\max}\lambda_{\max}(P)\Omega^2\le\Omega$, we get

$$\begin{aligned}
&\sum_{\sigma=k-\tau}^{k-1}\Delta e^{\mathrm{T}}(\sigma)\left(I_N\otimes P\right)\Delta e(\sigma)\\
&\le 2\phi\sum_{\sigma=k-\tau}^{k-1}\Delta e^{\mathrm{T}}(\sigma)\left(I_N\otimes P\right)\Delta e(\sigma)\\
&\quad+4d_{\max}\sum_{\sigma=k-\tau}^{k-1}e^{\mathrm{T}}(\sigma-\tau)(I_N\otimes A^{\mathrm{T}}P)\Omega(I_N\otimes PA)e(\sigma-\tau).
\end{aligned}$$

Once again, under Assumption 9, substituting (6.26) to the above inequality, we have

$$\begin{aligned}
&\sum_{\sigma=k-\tau}^{k-1}\Delta e^{\mathrm{T}}(\sigma)\left(I_N\otimes P\right)\Delta e(\sigma)\\
&\le 2\phi\sum_{\sigma=k-\tau}^{k-1}\Delta e^{\mathrm{T}}(\sigma)\left(I_N\otimes P\right)\Delta e(\sigma)\\
&\quad+4d_{\max}\frac{\gamma_1}{\gamma_3}\frac{\lambda_{\max}\left(B^{\mathrm{T}}PB\right)+\gamma_1}{\lambda_{\min}\left(B^{\mathrm{T}}PB\right)+\gamma_1}\sum_{\sigma=k-\tau}^{k-1}e^{\mathrm{T}}(\sigma-\tau)
\end{aligned}$$

$$
\begin{aligned}
&\times \left(I_N \otimes A^{\mathrm{T}} PB \left(B^{\mathrm{T}} PB + \gamma_1 I_m\right)^{-1} B^{\mathrm{T}} PA\right) e(\sigma - \tau) \\
&\leq 2\phi \sum_{\sigma=k-\tau}^{k-1} \Delta e^{\mathrm{T}}(\sigma) \left(I_N \otimes P\right) \Delta e(\sigma) \\
&\quad + 4 d_{\max} \frac{\gamma_1}{\gamma_3} \frac{\lambda_{\max}\left(B^{\mathrm{T}} PB\right) + \gamma_1}{\lambda_{\min}\left(B^{\mathrm{T}} PB\right) + \gamma_1} \cdot \frac{1 - (1-\varepsilon)^n}{(1-\varepsilon)^{n-1}} \\
&\quad \times \sum_{\sigma=k-\tau}^{k-1} e^{\mathrm{T}}(\sigma - \tau) \left(I_N \otimes P\right) e(\sigma - \tau).
\end{aligned} \tag{6.48}
$$

Inserting (6.48) into (6.46) gives

$$
\begin{aligned}
\Delta V \leq & - \varepsilon e^{\mathrm{T}}(k)(M \otimes P) e(k) \\
& + \tau \frac{\gamma_1}{\gamma_3} \cdot \frac{\lambda_{\max}\left(B^{\mathrm{T}} PB\right) + \gamma_1}{\lambda_{\min}\left(B^{\mathrm{T}} PB\right) + \gamma_1} \cdot \frac{1 - (1-\varepsilon)^n}{(1-\varepsilon)^{n-1}} \Bigg(2\phi \sum_{\sigma=k-\tau}^{k-1} e^{\mathrm{T}}(\sigma) \left(I_N \otimes P\right) e(\sigma) + 4 d_{\max} \frac{\gamma_1}{\gamma_3} \\
& \times \frac{\lambda_{\max}\left(B^{\mathrm{T}} PB\right) + \gamma_1}{\lambda_{\min}\left(B^{\mathrm{T}} PB\right) + \gamma_1} \cdot \frac{1 - (1-\varepsilon)^n}{(1-\varepsilon)^{n-1}} \sum_{\sigma=k-\tau}^{k-1} e^{\mathrm{T}}(\sigma - \tau) \left(I_N \otimes P\right) e(\sigma - \tau) \Bigg).
\end{aligned} \tag{6.49}
$$

For any integer $l \in [-2\tau, 0]$ and a constant $\rho > 1$, if we have $V(e(k+l)) < \rho V(e(k))$, or equivalently $e^{\mathrm{T}}(k+l)(M \otimes P)e(k+l) < \rho e^{\mathrm{T}}(k)(M \otimes P)e(k)$, then,

$$
e^{\mathrm{T}}(k+l)(I_N \otimes P)e(k+l) < \frac{\rho}{\lambda_{\min}(M)} e^{\mathrm{T}}(k)(M \otimes P)e(k).
$$

Inserting the above condition into (6.49), we have

$$
\begin{aligned}
\Delta V \leq - & \left(\varepsilon - 2\tau^2 \frac{\gamma_1}{\gamma_3} \cdot \frac{\lambda_{\max}\left(B^{\mathrm{T}} PB\right) + \gamma_1}{\lambda_{\min}\left(B^{\mathrm{T}} PB\right) + \gamma_1} \cdot \frac{1 - (1-\varepsilon)^n}{(1-\varepsilon)^{n-1}} \cdot \frac{\rho}{\lambda_{\min}(M)} \right. \\
& \left. \times \left(\phi + 2 d_{\max} \frac{\gamma_1}{\gamma_3} \cdot \frac{\lambda_{\max}\left(B^{\mathrm{T}} PB\right) + \gamma_1}{\lambda_{\min}\left(B^{\mathrm{T}} PB\right) + \gamma_1} \frac{1 - (1-\varepsilon)^n}{(1-\varepsilon)^{n-1}} \right) \right) e^{\mathrm{T}}(k)(M \otimes P)e(k).
\end{aligned} \tag{6.50}
$$

Let $\rho = 3/2$, if the following condition

$$
\begin{aligned}
& \varepsilon - 3\tau^2 \frac{\gamma_1}{\gamma_3} \cdot \frac{\lambda_{\max}\left(B^{\mathrm{T}} PB\right) + \gamma_1}{\lambda_{\min}\left(B^{\mathrm{T}} PB\right) + \gamma_1} \cdot \frac{1 - (1-\varepsilon)^n}{(1-\varepsilon)^{n-1}} \\
& \times \frac{1}{\lambda_{\min}(M)} \left(\phi + 2 d_{\max} \frac{\gamma_1}{\gamma_3} \cdot \frac{\lambda_{\max}\left(B^{\mathrm{T}} PB\right) + \gamma_1}{\lambda_{\min}\left(B^{\mathrm{T}} PB\right) + \gamma_1} \frac{1 - (1-\varepsilon)^n}{(1-\varepsilon)^{n-1}} \right) \geq \frac{\varepsilon}{2},
\end{aligned}
$$

or equivalently,

$$
\frac{1}{\tau^2} \geq \frac{6\lambda_{\max}(M)}{\alpha \lambda_{\min}(M)} \cdot \frac{\lambda_{\max}\left(B^{\mathrm{T}} PB\right) + \gamma_1}{\lambda_{\min}\left(B^{\mathrm{T}} PB\right) + \gamma_1} \cdot \frac{1 - (1-\varepsilon)^n}{\varepsilon(1-\varepsilon)^{2n-1}}
$$

$$\times \left(\frac{2}{(1-\varepsilon)^{n-1}} - (n-2)\varepsilon - 2 + 2(n - \operatorname{tr}(A)) \right.$$
$$\left. + 2d_{\max}\frac{\lambda_{\max}(M)}{\alpha} \cdot \frac{\lambda_{\max}\left(B^{\mathrm{T}}PB\right) + \gamma_1}{\lambda_{\min}\left(B^{\mathrm{T}}PB\right) + \gamma_1} \cdot \frac{1-(1-\varepsilon)^n}{(1-\varepsilon)^{2n-1}} \right), \tag{6.51}$$

is satisfied, then it follows from (6.50) that

$$\Delta V \leq -\frac{\varepsilon}{2} V(e(k)),$$

which, by Razumikhin Stability Theorem, implies that for any $e(0) \in \mathbb{R}^{Nn}$, $\lim_{k\to\infty} e(k) = \mathbf{0}_{Nn\times 1}$, and global consensus is achieved.

We now examine condition (6.51). It is easy to verify that the right-hand side of (6.51) and $\left(1-(1-\varepsilon)^n\right)/(1-\varepsilon)^{2n-1}$ is always positive for any $\varepsilon \in (0,1)$. Therefore, (6.51) is satisfied, if the following inequality holds,

$$\frac{1}{\tau^2} \geq \frac{6\lambda_{\max}(M)}{\alpha\lambda_{\min}(M)} \cdot \frac{1-(1-\varepsilon)^n}{\varepsilon(1-\varepsilon)^{3n-1}} \left(\frac{2}{(1-\varepsilon)^{n-1}} - (n-2)\varepsilon \right.$$
$$\left. - 2 + 2(n - \operatorname{tr}(A)) + \frac{2d_{\max}\lambda_{\max}(M)}{\alpha} \frac{1-(1-\varepsilon)^n}{(1-\varepsilon)^{3n-1}} \right). \tag{6.52}$$

Notice that the right-hand side of (6.52) is $H(\varepsilon)$ which is defined in Lemma 21 and is proved to be strictly increasing in $(0,1)$. In addition, $\lim_{\varepsilon\to 0^+} H(\varepsilon) = \frac{12\lambda_{\max}(M)}{\alpha\lambda_{\min}(M)} n(n - \operatorname{tr}(A))$, and $\lim_{\varepsilon\to 1^-} H(\varepsilon) = +\infty$. Then, we conclude that for any $\tau < \bar{\tau}$, $\bar{\tau} = \sqrt{\frac{\alpha\lambda_{\min}(M)}{12\lambda_{\max}(M) n(n-\operatorname{tr}(A))}}$, we can always solve for a $\bar{\varepsilon} \in (0,1)$ from $\tau^{-2} = H(\bar{\varepsilon})$, and for any $\tau \in [0, \bar{\tau})$ and $\varepsilon \in (0, \bar{\varepsilon}]$, (6.51) is always satisfied. This completes the proof. $\square$

When all eigenvalues of A are located at $z = 1$, $n - \operatorname{tr}(A) = 0$ and hence $\bar{\tau} = +\infty$, which means that the closed-loop multi-agent system tolerates arbitrarily large communication delays. In addition, we note that $\bar{\tau}$ depends on the minimum and maximum eigenvalues of the positive definite matrix M, one of whose eigenvalues is determined by the design. In order to maximize $\bar{\tau}$, we would like to minimize $\lambda_{\max}(M)$ and maximize $\lambda_{\min}(M)$. From Remark 2, we see that eigenvalues of M are also dependent on eigenvalues of L which come from the given communication graph. At best, we can design M such that $\lambda_{\max}(M) = 1/\lambda_{\min +}(L)$ and $\lambda_{\min}(M) = 1/\lambda_{\max}(L)$, where $\lambda_{\min +}(L)$ denotes the minimum non-zero eigenvalue of L.

6.4 Simulation results

In this section, the performance of the proposed control laws is demonstrated by numerical simulation. In our simulation, we consider multi-agent systems with five agents and a communication topology shown in Fig. 6.1, which is a connected undirected graph.

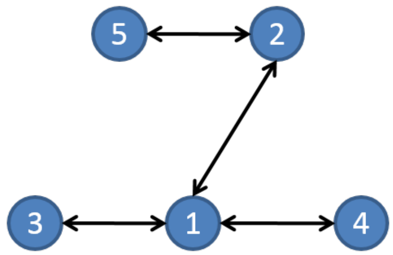


Figure 6.1: The communication topology

6.4.1 Example 1: Consensus without communication delays

The dynamics of agent i, $i = 1, 2, \ldots, 5$, are given as follows

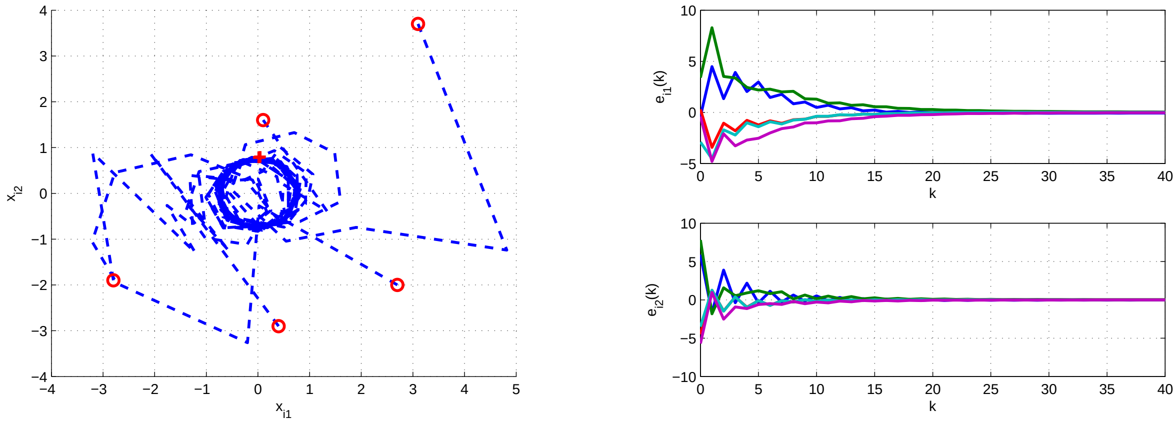
$$x_i(k+1) = Ax_i(k) + g(x_i(k))u_i(k), \tag{6.53}$$

where $x_i \in \mathbb{R}^2$, $u_i \in \mathbb{R}$,

$$A = \begin{bmatrix} 0.6 & 0.8 \\ -0.8 & 0.6 \end{bmatrix}, \ g(x_i) = \begin{bmatrix} 0 \\ 1 + x_{i1}^2 + x_{i2}^2 \end{bmatrix}.$$

It can be easily verified that Assumptions 7 and 8 are satisfied with $P = I_2$.

Shown in Figs. 6.2a and 6.2b are trajectories of the agents and the consensus errors. To carry out simulation, the initial states of the agents are randomly chosen as $x_1(0) = [0.1, 1.6]^{\mathrm{T}}$, $x_2(0) = [3.1, 3.7]^{\mathrm{T}}$, $x_3(0) = [0.4, -2.9]^{\mathrm{T}}$, $x_4(0) = [-2.8, -1.9]^{\mathrm{T}}$, and $x_5(0) = [2.7, -2]^{\mathrm{T}}$. As observed in Fig. 6.2a, where circles indicate the initial states and crosses represent the states when simulation terminates, the sates of the agents are synchronized. We can see more clearly in Fig. 6.2b that consensus errors converge to zero as the system evolves.

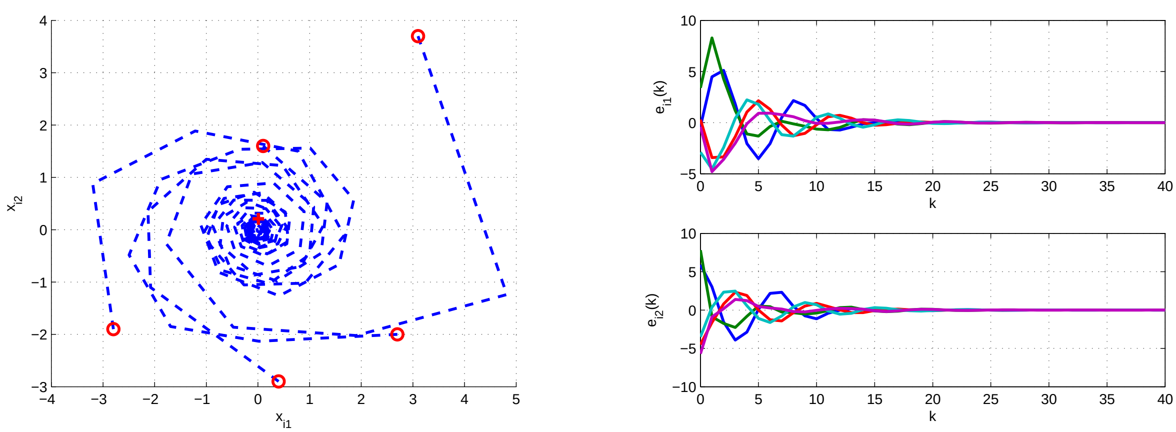


(a) Trajectories of agents (b) Consensus errors

Figure 6.2: Example 1: Consensus without communication delays, $\tau = 0$

6.4.2 Example 2: Consensus under actuator saturation

Consider the multi-agent system described by (6.53) and assume that the magnitude of the control input of each agent is limited by 0.1. Following Remark 4, the appropriately modified bounded control laws, with $\alpha = 0.1$, would ensure consensus. In the simulation, we choose the initial states of the agents as $x_1(0) = [0.1, 1.6]^{\mathrm{T}}$, $x_2(0) = [3.1, 3.7]^{\mathrm{T}}$, $x_3(0) = [0.4, -2.9]^{\mathrm{T}}$, $x_4(0) = [-2.8, -1.9]^{\mathrm{T}}$, and $x_5(0) = [2.7, -2]^{\mathrm{T}}$. The simulation results in Figs. 6.3a and 6.3b show that the consensus errors converge to zero and the states of agents are synchronized.



(a) Trajectories of agents (b) Consensus errors

Figure 6.3: Example 2: Consensus under actuator saturation, $\tau = 0$

6.4.3 Example 3: Consensus with communication delays

We now consider a scenario with communication delays. The dynamics of agent i, $i = 1, 2, \ldots, 5$, are given as follows

$$x_i(k+1) = Ax_i(k) + g(x_i(k))u_i(k), \tag{6.54}$$

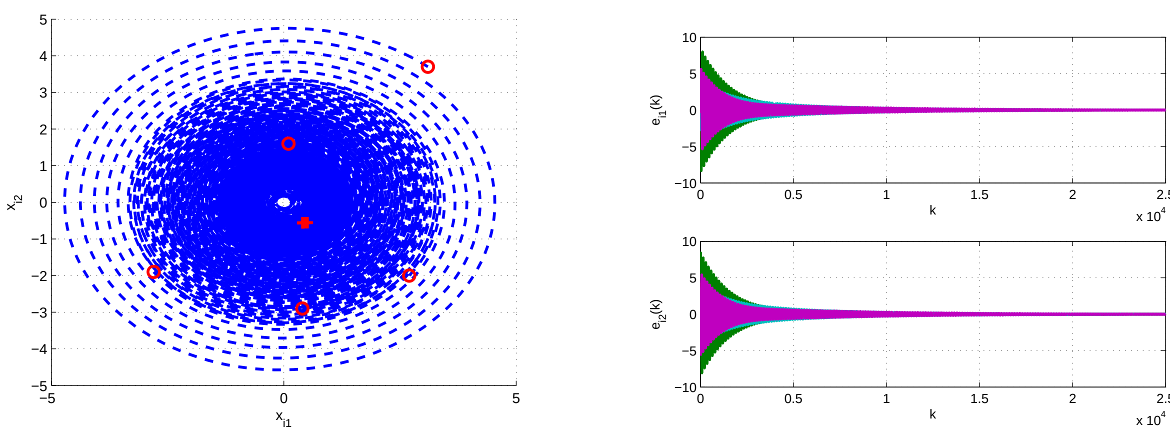
where $x_i \in \mathbb{R}^2$, $u_i \in \mathbb{R}^2$,

$$A = \begin{bmatrix} 0.999 & -0.0447 \\ 0.0447 & 0.999 \end{bmatrix}, \ g(x_i) = \begin{bmatrix} 0.9\cos x_{i2} & \sin x_{i2} \\ 1.8\cos x_{i2} & 2\sin x_{i2} \end{bmatrix}.$$

It can be easily verified that Assumptions 9 and 10 are satisfied with

$$B = \begin{bmatrix} 0.7071 & 0.7071 \\ 1.414 & 1.414 \end{bmatrix}, \ \alpha = 0.81.$$

In this case, we have $\bar{\tau} = 1.4489$ and $\bar{\varepsilon} = 1.437 \times 10^{-4}$. In the simulation, we take $\tau = 1$ and $\varepsilon = 1.437 \times 10^{-4}$. Other control parameters are selected as $\gamma_1 = 7.1412$, $\gamma_2 = 0.4201$ and $\gamma_3 = 3$, which satisfy (6.23). Again, the initial states of the agents are chosen as $x_1(0) = [0.1, 1.6]^{\mathrm{T}}$, $x_2(0) = [3.1, 3.7]^{\mathrm{T}}$, $x_3(0) = [0.4, -2.9]^{\mathrm{T}}$, $x_4(0) = [-2.8, -1.9]^{\mathrm{T}}$, and $x_5(0) = [2.7, -2]^{\mathrm{T}}$. In Figs. 6.4a and 6.4b, we observe that all agents reach state consensus and consensus errors converge to zero asymptotically.



(a) Trajectories of agents

(b) Consensus errors

Figure 6.4: Example 3: Consensus with communication delays, $\tau = 1$

6.5 Summary

This chapter studied the consensus control problem in the discrete-time setting, and each agent is a nonlinear system affine in control. We proposed discrete-time distributed control laws based on neighborhood consensus errors for systems with and without communication delays. Consensus under the proposed control laws was proved and was observed in simulation as well. It was demonstrated that the closed-loop multi-agent system tolerate arbitrarily large communication delays when all eigenvalues of matrix A are located at $z = 1$. For the case where not all eigenvalues of matrix A are located at $z = 1$, a theoretical upper bound on the tolerable delays was derived.

Chapter 7

Conclusions

This dissertation studied the coordinated control problems, including flocking control and consensus control, of nonlinear multi-agent systems in the presence of communication delays. Low gain design method was demonstrated to be effective in counteracting the effect of communication delays without increasing the complexity of controller structures.

We proposed artificial function based low gain control laws for the flocking of nonholonomic vehicles in both continuous-time and discrete-time settings. It was proved that in both cases, the closed-loop multi-agent system can tolerate arbitrarily large communication delays when the low gain parameter is small enough. In addition, collision avoidance was proved if the communication graph is complete.

In this dissertation, we also proposed consensus error based low gain control laws for the consensus of nonlinear affine agents in both continuous-time and discrete-time settings. For the consensus control problems, in most cases, the closed-loop multi-agent system cannot tolerate arbitrarily large communication delays and a theoretical upper bound on the tolerable communication delays was derived.

For each of the coordinated control problems concerned in this dissertation, simulation results were presented to illustrate the performance of the proposed coordinated control laws.

In this dissertation, the communication graph was assumed to be undirected, connected and fixed, and the communication delays are assumed to be constant. More general communication graphs and time-varying communication delays remain to be considered.

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