Analysis and Computational Fluid Dynamics for the Stabilization and Control of 3-Dimensional Navier-Stokes Fluid Channel Flows by a Wall-Normal Boundary Controller

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Chapter 1

Wall-Normal Control Stabilization Enhancement of the Navier-Stokes Equations 3-D Channel Flow

1.1 Problem Statement and Main Results

1.1.0 The Wall-Normal Stabilization Problem and Recent Work in the Literature on Stabilization of Navier-Stokes Equations

The present work is focused on the study of an incompressible fluid flow (modeled by the Navier-Stokes equations) in the prototype case of a channel flow. The ultimate goal is how to design a suitable wall-normal feedback control as to achieve a predetermined goal. Recognized objectives include: (i) suppressing turbulence; (ii) accelerating the transition from a turbulent flow to a laminar flow; (iii) preventing separation of the flow. A laminar flow is distinguished by a special structure: parallel layers of fluid moving in a regular and deterministic way. Thus, a laminar flow produces considerably less drag, or friction, at the wall-fluid interfaces, than other configurations of the flow. On the other extreme of the spectrum is the turbulent flow: it is characterized by small scale velocity vectors, which, moreover, appear to be random or stochastic. Whether a designer seeks to obtain a laminar flow or a turbulent flow depends on the objectives at hand. If the goal is mixing, then a turbulent flow is the one to be designed or provoked. If, on the other hand, one seeks to reduce energy consumption of a compressor pumping the fluid, then a laminar flow is preferable. Usually, laminar flows are unstable, and unless suitably controlled, will evolve and turn into turbulent flows. Thus, the need of flow control: it consists of using passive and active mechanisms of the flow in a desired, pre-determined manner, and ultimately steer it to a predetermined sought-after configuration.

The present analysis addresses the local exponential stabilization of the Navier-Stokes equations in three dimensions via boundary feedback control. Unlike some other works, the feedback control law this project seeks to design is not distributed over the entire boundary of the channel. Rather, it consists of finitely many feedback controls supported in an arbitrarily small portion of the boundary of the channel. Moreover, this project seeks to implement wall-normal controls.

Work in recent years on local exponential stabilization of the Navier-Stokes equations, in dimensions $d = 2, 3$, has focused on the following three control cases: (i) *interior*, finite-dimensional feedback controls with arbitrary small support [B-T.1], [B-L-T.3]; (ii) *tangential boundary* controls (for $d = 2$, with arbitrarily small support, generically finite-dimensional) [B-L-T.1]–[B-L-T.3], and (iii) *normal boundary* controls (for $d = 2, 3$, with support on the upper boundary, generically finite-dimensional) [M.2],[M.3], [B.2]. In [B-L-K.1], Balogh et al. develops a method for Lyapunov stabilization of a fully non-linear Navier-Stokes channel flow system with Dirichlet no-slip boundary conditions and only local wall-shear stress observation with a tangential boundary controller. This stabilization result however is limited to low values of Reynolds number. The topological level of decay of the solutions is problem- and dimension-dependent. As pointed out in [B-L-T.1]–[B-L-T.3], in the case $d = 2$, the topological level $(H^{\frac{1}{2}-\epsilon}(\Omega))$ of the state space permits one to make use of long-established Riccati theory as reported in [L-T.1], [L-T.2]. The boundary case $d = 2$ was also studied in [R.1], however within the larger class of non-necessarily tangential feedback controls, though with tangential initial conditions. Also, an important exception to the previously addressed cases, there was Barbu's stabilization control design for an *oblique* (i.e. more inclusive control space than either strictly normal or tangential boundary controls) *boundary* controller that achieves stabilizing boundary feedback to the Navier-Stokes system on an *arbitrary* open domain (subject to certain spectrally-related geometric requirements on the domain's geometry) [B.3].

This present work addresses an extension of the results of [Tr.4] in an exponential stabilization enhancement of the Navier-Stokes channel flow from dimension $d = 2$ to dimension $d = 3$ with wall-normal feedback control. In [Tr.4], Triggiani addresses the feedback stabilization of the linear Navier-Stokes model to a parabolic steady-state solution of a 2-dimensional channel. His method follows a careful elimination of pressure, a precise computational spectral analysis of the velocity components, a projection of the velocity dynamics onto spectrally-defined dynamical subspaces, and then a finite-dimensional stabilization control extended to the whole dynamical space. This work, in extending these results from the 2-D case to the 3-D case using the same model (specifically, with Neumann boundary conditions on the tangential components of velocity u and w), will follow the same method, which succeeds in extending the aforementioned analysis to the 3-dimensional case as well.

The theoretical foundations for stability control of the 2 and 3 dimensional channel flow with wall-normal rather than wall-tangential control actuation are motivated by engineering-based considerations [A-K-B.1], [V-K.1]: Several *Automatica-IEEE* papers—the journals of the control engineering community— emphasize the importance of the practical physical readiness of using wall-normal controls [Tr.4]. With regard to application, the technological feasibility for implementing both tangential as well as normal control actuation has been established. Implementable methods for normal actuation includes synthetic jets (see [D-L-S-T], [B-L-K.1, p. 1696]) and for tangential actuation includes synthetic jets (see [S-G], [B-L-K.1, p. 1696]) and rotating disks [K.1].

The analysis in the case of wall-tangential controls with localized Dirichlet boundary conditions allows for the elimination of pressure via use of the Leray (also called Helmotz) projection [B-L-T.1]–[B-L-T.3]. This method does not work for wall-normal controls however, and so the analysis for this case must address the elimination of the pressure via an alternative method. In the analysis of the tangential control case, the readiness of this initial step for use of Leray projector has the effect of causing and transferring a serious difficulty until later along the proof: namely, at the level of implementing a stabilizing tangential feedback control. The failure of the Leray projector's application in the wall-normal control case constitutes a serious obstacle to just get started in the analysis and forces the need to devise a radically different strategy (to be explained a few paragraphs below in "Approach"). [Tr.4] demonstrated that the complementary class of (technologically feasible) wall-normal, feedback controls are still capable of yielding stabilization, for the case of dimension $d = 2$, and the current work establishes the successful extension of these methods to the case of dimension $d = 3$ as well. Driven by the above considerations, the present work along with [Tr.4] seeks to stabilize the Navier-Stokes equations by means of wall-normal feedback boundary controls requiring observation only of the normal velocity component working again in the canonical case (typical in fluid-mechanics laboratories) of a channel flow, now in 3-D rather than 2-D. The approach will follow closely that of the analysis in [Tr.4], and will again exploit the special geometry of the parallelepiped channel domain.

Overview of Methods Used. As in [Tr.4], in sharp contrast with the interior [B-T.1], [B-L-T.3] or tangential boundary stabilization cases [B-L-T.1]–[B-L-T.3], an important feature of the present approach is the elimination of the unknown pressure as in (1.3.1) below by solving the corresponding elliptic problem ((1.1.1.12a–e) or (1.3.3a–e) below) and substituting the result in the equation of the normal velocity component v, as to obtain the abstract equation (1.3.5). This equation is then subject to a precise spectral analysis. Our approach will take the following form. In the 3-D case as well as was in the 2-D case, the wall-normal boundary controlled *linearized* system (1.1.1.5) contains an infinite-dimensional subspace (E^0 in (1.5.1b), explicitly identified) of the solution space $L_2(\Omega)$, where the boundary control turns out to be inactive. Thus, the orthogonal projection of the original boundary controlled *linearized* problem (1.1.1.5) evolves on E^0 as a control-free system. However, for the Neumann boundary conditions used in this analysis, this dynamical projection on E^0 is fortunately intrinsically uniformly stable (with no restrictions on permissible values of Reynolds numbers), though with a constrained decay rate ($\nu\pi^2$): see Section 1.6. Regarding the analysis of the evolution of the boundary controlled dynamics on the complementary infinite-dimensional space (Z in (1.5.3c), also explicitly identified) of $L_2(\Omega)$, beginning with Section 1.7, this paper specializes to the linear problem: (1.1.1.5) with $a \equiv b \equiv 0$; that is, (1.7.1a–s). Then, the following optimal result is presented in this case. When the boundary control is switched off ($V \equiv 0$ in (1.1.1.5j)), the resulting free system on Z is intrinsically exponentially stable, though with a constrained decay rate ($\nu(1 + \pi^2 + (\frac{\pi}{e})^2)$).

This paper explicitly constructs a boundary feedback based only on four controls (the minimal possible number) such that the resulting feedback problem on Z decays with an *arbitrarily preassigned* decay rate ($\nu\gamma_0$, where γ_0 is given arbitrarily in advance). Thus, this conclusion constitutes an arbitrary *enhancement of the margin of stability* (on Z), by a 4-dimensional wall-normal feedback control (in domain dimension $d = 2$, the control-dimension was 2). This result is optimal. Moreover, the required feedback controls are subject to an explicit 'rank condition test' (controllability of the finite-dimensional component of the dynamics, with spectrum of the Laplacian Δ on the right of $(-\gamma_0)$). The resulting analysis is also fully explicit: not only for the normal component v of the velocity, but also for its tangential components u and w, the pressure p and the vorticity ω. Thus, the analysis of the present specialized linear case (1.7.1) is complete.

Literature. We cite here several additional recent articles regarding work on Navier-Stokes boundary control. [V-K.1] considers a 2-D channel of infinite length in the streamwise x-direction, subject to two boundary feedback controls acting on the entire infinite top wall, of convolution type. [B.1] considers a finite channel system, periodic in the streamwise x-direction, with two finite-dimensional feedback controls acting on the top and bottom walls, and [B.2] considers the same in the stochastic setting. [M.2] considers a linearized N-S system in a 2-D channel of with periodic boundary conditions in the streamwise x-direction, subject to a finite-dimensional boundary feedback control acting on the entire upper wall. The only analysis currently available in the literature addressing the stabilization of the N-S channel flow with wall normal boundary controllers in 3-D is [M.3], in which Munteanu extends his 2-D analysis in [M.2] to 3-D, establishing again a finite-dimensional boundary feedback control on the entire upper wall with exponential stabilization on the linearized 3-D N-S system. The models being considered in these articles use Dirichlet no-slip boundary conditions in the tangential component(s) u (and w, if in 3-D) of the velocity, while the analysis in the present paper addresses corresponding boundary conditions of Neumann type. These stabilization results cited here were subject to the constraint of permitting only small values of Reynolds number (with exceptions [M.2], [M.3], and [B.1]).

1.1.1 Original Linearized Boundary Control Model of the 3-D Channel Flow with Periodic B.C. in the Streamwise x and z Directions

The 3-D channel. We consider a 3-D channel flow evolving in the following domain:

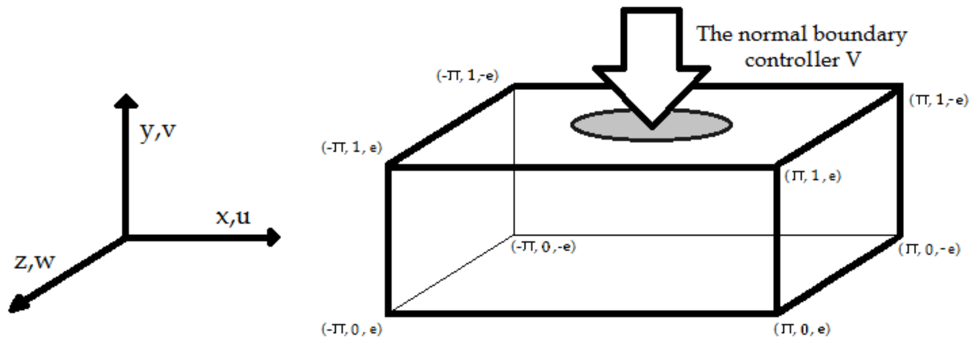
$$\Omega = \{(x, y, z) : \ -\pi \leq x \leq \pi; \ 0 \leq y \leq 1; -e \leq z \leq e\}; \tag{1.1.1.1}$$

with boundary $\Gamma = \Gamma_0 \cup \Gamma_1$, where

$$\begin{aligned} \Gamma_0 &= \{x = \pm\pi; \ 0 \leq y \leq 1; \ -e \leq z \leq e\} \ \cup \ \{z = \pm e; \ -\pi \leq x \leq \pi \ \ 0 \leq y \leq \ 1\} \\ &\quad \cup \{y = 0; \ -\pi \leq x \leq \pi \quad -e \leq z \leq e\}; \qquad (1.1.1.2) \\ \Gamma_1 &= \{y = 1; \ -\pi \leq x \leq \ \pi; \ -e \leq z \leq e\}, \qquad (1.1.1.3) \end{aligned}$$

with periodic boundary conditions (B.C.) in the streamwise x and z-directions. Let

$\{u, v, w\}$ be the velocity vector, with u in the x-direction and w in the z-direction its tangential components and v its normal component in the y-direction.



Original Navier-Stokes boundary control model in $\{u, v, w\}$. For positive parameters a, b, and ν, the parabolic equilibrium profile is given by

$$\begin{aligned}
\overline{U}(y) &= \frac{a}{2\nu}\, y(1-y); \quad \overline{U}'(y) = \frac{a}{2\nu}\,(1-2y); \quad \overline{U}''(y) = -\frac{a}{\nu}, \quad 0 \le y \le 1; \\
\overline{W}(y) &= \frac{b}{2\nu}\, y(1-y); \quad \overline{W}'(y) = \frac{b}{2\nu}\,(1-2y); \quad \overline{W}''(y) = -\frac{b}{\nu}, \quad 0 \le y \le 1.
\end{aligned} \tag{1.1.1.4}$$

With $Q = \Omega \times (0, T]$, the corresponding N-S boundary control problem is then

$$\begin{cases}
u_t - \nu\Delta u + \overline{U}(y)u_x + \overline{U}'(y)v + \overline{W}(y)u_z = p_x & \text{in } Q; \quad (1.1.1.5a) \\
v_t - \nu\Delta v + \overline{U}(y)v_x + \overline{W}(y)v_z = p_y & \text{in } Q; \quad (1.1.1.5b) \\
w_t - \nu\Delta w + \overline{U}(y)w_x + \overline{W}'(y)v + \overline{W}(y)w_z = p_z & \text{in } Q; \quad (1.1.1.5c) \\
\text{div}\{u, v, w\} \equiv u_x + v_y + w_z \equiv 0 & \text{in } Q; \quad (1.1.1.5d)
\end{cases}$$

$$\text{B.C. for } u: \begin{cases} u_y(x,0,z,t) \equiv 0, \quad u_y(x,1,z,t) \equiv 0; & \text{(1.1.1.5e)} \\ u(-\pi,y,z,t) \equiv u(\pi,y,z,t); & \text{(1.1.1.5f)} \\ u_x(-\pi,y,z,t) \equiv u_x(\pi,y,z,t); & \text{(1.1.1.5g)} \\ u(x,y,-e,t) \equiv u(x,y,e,t); & \text{(1.1.1.5h)} \\ u_z(x,y,-e,t) \equiv u_z(x,y,e,t); & \text{(1.1.1.5i)} \end{cases}$$

$$\text{B.C. for } v: \begin{cases} v(x,0,z,t) \equiv 0, \quad v(x,1,z,t) \equiv V(x,z,t); & \text{(1.1.1.5j)} \\ v(-\pi,y,z,t) \equiv v(\pi,y,z,t); & \text{(1.1.1.5k)} \\ v_x(-\pi,y,z,t) \equiv v_x(\pi,y,z,t); & \text{(1.1.1.5l)} \\ v(x,y,-e,t) \equiv v(x,y,e,t); & \text{(1.1.1.5m)} \\ v_z(x,y,-e,t) \equiv v_z(\pi,y,e,t), & \text{(1.1.1.5n)} \end{cases}$$

$$\text{B.C. for } w: \begin{cases} w_y(x,0,z,t) \equiv 0, \quad w_y(x,1,z,t) \equiv 0; & \text{(1.1.1.5o)} \\ w(-\pi,y,z,t) \equiv w(\pi,y,z,t); & \text{(1.1.1.5p)} \\ w_x(-\pi,y,z,t) \equiv w_x(\pi,y,z,t); & \text{(1.1.1.5q)} \\ w(x,y,-e,t) \equiv w(x,y,e,t); & \text{(1.1.1.5r)} \\ w_z(x,y,-e,t) \equiv w_z(x,y,e,t); & \text{(1.1.1.5s)} \end{cases}$$

with boundary control $V(x,z,t)$ acting as a wall normal component on the top wall $y=1$ of the channel. Here, $-p(x,y,z,t)$ is the pressure, the second unknown of the Navier-Stokes system. The constant $\nu > 0$ is the kinematic viscosity. Application of the divergence theorem yields, via (1.1.1.3d), the corresponding boundary constraint:

$$0 = \int_\Omega \text{div}\{u,v,w\} d\Omega = \int_\Gamma \begin{bmatrix} u \\ v \\ w \end{bmatrix} \cdot \nu \, d\Gamma;$$

$$\text{or} \quad \int_{-e}^{e} \int_{-\pi}^{\pi} v|_{y=1} dz dx = \int_{-e}^{e} \int_{-\pi}^{\pi} V(x,z,t) dx dz = 0, \tag{1.1.1.5t}$$

on the wall-normal control $V(x,z,t)$ in (1.1.1.5j), Γ being oriented with an outward unit normal. This is so, since the contribution to $\int_\Gamma \cdot d\Gamma$ of the two pairs of vertical walls at $x = \pm\pi$ and $z = \pm e$ compensate each other by periodicity of u and w, while the contribution over the horizontal wall $y = 0$ vanishes by the B.C. $v|_{y=0} \equiv 0$ in (1.1.1.5j). For large Reynolds number $\frac{1}{\nu}$, the steady-state (stationary) solutions y_e are unstable and cause turbulence in their surroundings.

Original Navier-Stokes boundary control model in $\{\omega, v\}$, ω being the corresponding vorticity. Let

$$\omega(x,y,z,t) \equiv (\omega^1, \omega^2, \omega^3); \quad \omega^1 \equiv w_y - v_z; \quad \omega^2 \equiv u_z - w_x; \quad \omega^3 \equiv v_z - u_y \tag{1.1.1.6}$$

be the vorticity (the vorticity vector is curl$\{u,v,w\}$). It is verified in Appendix A that the *vorticity version* of model (1.1.1.5a–s) replaces the u- and w-equations (1.1.1.5a and c) and its boundary conditions (1.1.5e–i and o–s) by

$$\begin{cases} \omega_t - \nu\Delta\omega + \overline{U}(y)\omega_x - \overline{W}'(y)\nabla(\Psi(\omega)^1) + \overline{U}'(y)\nabla(\Psi(\omega)^3) \\ \qquad + (v\overline{W}''(y), -\omega^1\overline{U}'(y) - \omega^3\overline{W}'(y), -v\overline{U}''(y)) = 0; & (1.1.1.7a) \\ \text{B.C. for } \omega: \begin{cases} \omega^1(x,0,z,t) \equiv 0,\ \omega^1(x,1,z,t) = -V_z(x,z,t); & (1.1.1.7b) \\ \omega^2_y(x,0,z,t) = 0,\ \omega^2_y(x,1,z,t) = 0, & (1.1.1.7c) \\ \omega^3(x,0,z,t) = 0,\ \omega^3(x,1,z,t) = V_x(x,z,t); & (1.1.1.7d) \\ \omega(-\pi,y,z,t) = \omega(\pi,y,z,t); & (1.1.1.7e) \\ \omega(x,y,-e,t) = \omega(x,y,e,t); & (1.1.1.7f) \\ \omega_x(-\pi,y,z,t) = \omega_x(\pi,y,z,t); & (1.1.1.7g) \\ \omega_z(x,y,-e,t) = \omega_z(x,y,e,t), & (1.1.1.7h) \end{cases} \end{cases}$$

to be associated with the v-equation (1.1.1.5b) and corresponding B.C. (1.1.1.5j–n). In this setting involving the pair $\{\omega, v\}$ the only non-homogeneous boundary terms are the wall-normal control $v|_{y=1} \equiv V$, $\omega^3|_{y=1} = V_x$, and $\omega^1|_{y=1} = -V_z$. The velocity u, v, w can be solved for in terms of the vorticity and the controller V (up to a modulation by a 1-dimensional subspace of constant functions in the u and w components). The linear operator Ψ gives u, v, w as a function of vorticity $\omega = (\omega_1, \omega_2, \omega_3)$ and the control V. Ψ is defined by

$$\Psi(\omega, V) = (-A_N^{-1}(\text{curl}(\omega)^1), DV - A_D^{-1}(\text{curl}(\omega)^2), -A_N^{-1}(\text{curl}(\omega)^3)) : \quad (1.1.1.8a)$$

$$(H^s(\Omega), H^s(\Omega), H^s(\Omega)) \times H^q(\Lambda_1) \to (L_2^0(\Omega), L_2(\Omega), L_2^0(\Omega)); \quad (1.1.1.8b)$$

where

$$L_2^0(\Omega) = L_2(\Omega)/\mathcal{N}(A_N), \tag{1.1.1.9}$$

A_N is the Laplacian operator given by (1.11.3), D is the Dirichlet map given by (1.2.1), and A_D is the Laplacian operator given by (1.2.2). Via elliptic theory, we have the following regularity and continuity property

$$\Psi: \text{ continuous } (H^s(\Omega), H^s(\Omega), H^s(\Omega)) \times H^q(\Gamma_1) \to$$

$$(H^{0,s+1}(\Omega), H^{\min\{s+1, q+\frac{1}{2}\}}(\Omega), H^{0,s+1}(\Omega)); \quad 0 < s \in \mathbb{R}, \quad 0 < q \in \mathbb{R}, \tag{1.1.1.10}$$

where

$$H^{0,s+1}(\Omega) = H^{s+1}(\Omega)/\mathcal{N}(A_N), \quad \mathcal{N}(A_N) = \{f \in \mathcal{D}(A_N) \mid A_N(f) = 0\}. \tag{1.1.1.11}$$

It is verified in Appendix A that Ψ does indeed give the velocity u, v, w (modulo the 1-dimensional subspace of constant functions in the u and w components) as a function of vorticity ω and the control V.

Elliptic problem solved by the pressure $p(x, y, z, t)$. It is verified in Appendix A, by use of the divergence free condition (1.1.1.5d), that the pressure $p(x, y, z, t)$

satisfies the following elliptic system (at each t)

$$\begin{cases} p_{xx}+p_{yy}+p_{zz}=2(\overline{U}'(y)v_x+\overline{W}'(y)v_z), \text{ in } Q; & (1.1.1.12\text{a}) \\ \text{B.C. for } p: \begin{cases} p_y|_{y=0}\equiv 0;\ p_y|_{y=1}\equiv V_t-\nu(V_{xx}+V_{zz}); & (1.1.1.12\text{b}) \\ p_x(-\pi,y,z,t)\equiv p_x(\pi,y,z,t), & \\ p_x(x,y,-e,t)\equiv p_x(x,y,e,t); & (1.1.1.12\text{c}) \\ p_y(-\pi,y,z,t)\equiv p_y(\pi,y,z,t), & \\ p_y(x,y,-e,t)\equiv p_y(x,y,e,t); & (1.1.1.12\text{d}) \\ p_z(-\pi,y,z,t)\equiv p_z(\pi,y,e,t), & \\ p_z(x,y,-e,t)\equiv p_z(x,y,e,t). & (1.1.1.12\text{e}) \end{cases} \end{cases}$$

The next *goal* is to eliminate the pressure term p_y by use of the elliptic problem (1.1.1.12a–e) and substitute it into the RHS of eq. (1.1.1.5b), thus obtaining a fully *uncoupled* equation for the velocity normal component v. This will be done in Section 1.3, after the introduction of preliminary background in Section 1.2.

Remark 1.1.1.1. As already noted in Section 1.1.0, we remark that, by failure of satisfying appropriate boundary conditions, in the present setting with *wall-normal control*, we cannot eliminate the pressure from, say, problem (1.1.1.5), in the traditional way of much of the literature, by applying the Leray projector, as it is done in the case of homogeneous 'no-slip' (that is, Dirichlet) boundary conditions [C-F.1], or in the case of *tangential* controls [B-L-T.1], [B-L-T.2]. That is why we use, in this present approach, the alternative method described above, as in [Tr.4].

1.1.2 Description of Stability Enhancing Feedback Problem and Main Results

We first provide a qualitative description of the problem studied in the present work. Next, we state our main technical results. There are two unknowns of the Navier-Stokes equations: the 3-D velocity (u, v, w) and the scalar pressure $-p$.

Linearized case (1.1.1.5a–s). Theorem 1.1.2.1 below shows that there exists an explicitly identified subspace—invariant for the dynamics—where the wall-normal control V has no influence whatsoever, and thus the dynamics evolves as a free system in a simple way. Fortunately, it is exponentially stable, though only with limited rate $(\nu\pi^2)$.

Theorem 1.1.2.1. *Consider the linearized problem (1.1.1.5a–s) with wall-normal control $V(x, z, t)$, $-\pi \leq x \leq \pi$, $-e \leq z \leq e$, acting on the top wall $y = 1$ of Ω. Let E^0 (see (1.5.1b) below) be the infinite-dimensional eigenspace of the operator A_D (see (1.2.2)), defined as the realization of the Laplace operator on $L_2(\Omega)$, with homogeneous Dirichlet Boundary Conditions (B.C.) at both the bottom and top walls $y = \pm 1$, and periodic B.C. in the streamwise directions x and z, corresponding to the eigenvalues $\lambda_{0m0} = -(m\pi^2)$, $m = 1, 2, \ldots$ (see (1.2.3a)) and corresponding normalized eigenvectors $e^0_{0m0} = \frac{1}{\sqrt{2\pi e}} \sin m\pi y$. Then, the orthogonal projection $\Pi^0 v$ of the normal component v of the velocity vector onto the eigenspace E^0 is uncontrolled or control-*

free; more precisely, it satisfies the differential equation

$$\text{on } E^0: \qquad \Pi^0 v_t = \nu A_D^0 \Pi^0 v, \quad A_D^0 = A_D|_{E^0}, \tag{1.1.2.1}$$

and hence has solution $\Pi^0 v(t)$ *satisfying*

$$\text{on } E^0: \qquad \Pi^0 v(t) = e^{\nu A_D^0 t}\Pi^0 v(0) = \sum_{m=1}^{\infty} e^{\nu\lambda_{0m0}t}(\Pi^0 v(0), e_{0m0}^0)_\Omega, \quad t \geq 0; \tag{1.1.2.2}$$

$$\|\Pi^0 v(t)\| \leq e^{-\nu\pi^2 t}\|\Pi^0 v(0)\|, \tag{1.1.2.3}$$

where $\| \quad \|$ *denotes either the* $L_2(\Omega)$*-norm; or else the norm of* $\mathcal{D}((-A_D^0)^{\frac{1}{2}})$*, equivalent to the* $H^1(\Omega)$*-norm.*

The proof of this result is given in Section 1.6.

Analysis of the v-equation. At first, we concentrate on the v-equation (1.1.1.5b) (normal velocity) containing the partial derivative p_y of the pressure, corresponding boundary conditions (1.1.1.5j–n) for v and related elliptic system (1.1.1.12) for p (or related elliptic system (1.3.3a–e) for p_y in Section 1.3). These two elliptic systems, either in p or p_y, are steered by the boundary term $[V_t(x,z,t) - \nu(V_{xx} + V_{zz})(x,z,t)]$, containing in particular the *time derivative* $V_t(t,x,z)$ of the *wall-normal* control $V(t,x,z) = V(t,x,y=1,z)$ acting as a boundary control at the upper wall $y = 1$. This is part of the model of the problem. We take $V = \psi_1(x,z)\varphi_1(t) + \psi_2(x,z)\varphi_2(t) + \psi_3(x,z)\varphi_3(t) + \psi_4(x,z)\varphi_4(t)$, see (1.3.7) below (with $J = 4$). Thus, solving (via elliptic theory) for the pressure term p_y (a task to be performed in Section 1.3) in terms of the data, hence also of $[V_t - \nu(V_{xx} + V_{zz})]$, yields an abstract equation for the normal velocity v which contains also the *time-derivative* V_t of the boundary control $V(t)$:

$$(v - DV)_t = \nu A_D(v - DV) - Lv - \nu D(V_{xx} + V_{zz}), \tag{1.1.2.4}$$

see (1.3.5) below. This method occurs also [L-L-P.1], [L-P-T.1] in the case of an Euler-Bernoulli plate equation acted upon by a boundary control in a special "moment-type" boundary condition as well as in [Tr.4].

Linear case: $a \equiv b \equiv 0$ in (1.1.1.5a–s), hence $L \equiv 0$ in (1.1.2.4) = (1.3.5) below, that is

$$(v - DV)_t = \nu A_D(v - DV) - \nu D(V_{xx} + V_{zz}). \tag{1.1.2.5}$$

As in the aforementioned references, in our present 3-D channel flow problem, the presence of V_t in the (concrete v-PDE-problem (1.3.2a–d), that is, in the) abstract v-equation (1.1.2.5) leads to the conclusion that the *natural state variable* of the problem in question with $a \equiv b \equiv 0$ is, in fact, the variable:

$$h(t) \equiv [v(t) - DV(t)], \tag{1.1.2.6}$$

where D is the Dirichlet map (1.2.1g) (harmonic extension from the boundary to the interior of the corresponding (static) elliptic problem (1.2.1a–f)). Such new variable $h(t)$ is *intrinsic*, as it satisfies the first-order evolution equation (1.1.2.5) with initial condition: $h|_{t=0} = (v - DV)(0) = v(0) - DV(0)$ assigned. It thus requires, for its solution, not only the Initial Condition $v(0)$ at $t = 0$ of the normal velocity, but, moreover, that *also the Initial Condition* $V(0)$ *at* $t = 0$ *of the control* $V(t)$

be pre-assigned. Thus, $h(t)$ is the *defacto* state variable of the present problem. Instead, the term $V_{xx}+V_{zz}=(\partial_{xx}+\partial_{zz})\psi_1(x,z)\varphi_1(t)+(\partial_{xx}+\partial_{zz})\psi_2(x,z)\varphi_2(t)+(\partial_{xx}+\partial_{zz})\psi_3(x,z)\varphi_3(t)+(\partial_{xx}+\partial_{zz})\psi_3(x,z)\varphi_3(t)$ acts like the control influencing the h-dynamics in (1.2.5), with $[\varphi_1(t),\varphi_2(t),\varphi_3(t),\varphi_4(t)]$ to be expressed in feedback form. The approach of the present paper is based on the spectral decomposition of the dominant, negative, self-adjoint operator A_D in (1.1.2.5), which consists of the Laplacian Δ with corresponding boundary conditions. The precise statement is given in Theorem 1.1.2.2 below, an optimal result.

Theorem 1.1.2.2. *Consider the linear system obtained from (1.1.1.5a–s) by setting $a\equiv b\equiv 0$; that is, consider the system (1.7.1a–s) below, subject to the wall-normal control $V(x,z,t)$ of the form: $V(x,z,t)=\psi_1(x,z)\varphi_1(t)+\psi_2(x,z)\varphi_2(t)+\psi_3(x,z)\varphi_3(t)+\psi_4(x,z)\varphi_4(t)$, $-\pi\le x\le\pi$, $-e\le z\le e$. Let $Z\equiv L_2(\Omega)/E^0$ (see (1.5.1a)). The subspace Z is an eigenspace: $Z\equiv\overline{\text{span}}\{e^i_{nm0}, i=1,2; n,m=1,2,\ldots\}+\overline{\text{span}}\{e^i_{0mk}, i=3,4; m,k=1,2,\ldots\}+\overline{\text{span}}\{e^i_{nmk}, i=5,6,7,8; n,m,k=1,2,\ldots\}$ (see (1.5.1e)), of the operator A_D in (1.2.2a–b), with normalized eigenvectors e^i_{nm0} identified in (1.2.4b–c), corresponding to the double eigenvalues $\lambda_{nm0}=-[n^2+(m\pi)^2]$, $n,m=1,2,i=1,2;\ldots$, e^i_{0mk} identified in (1.2.4d–e), corresponding to the double eigenvalues $\lambda_{0mk}=-[(m\pi)^2+(\frac{\pi}{e}k)^2]$, $m,k=1,2,\ldots$, $i=3,4$; and e^i_{nmk} identified in (1.2.5b–e), corresponding to the quadruple eigenvalues $\lambda_{nmk}=-[n^2+(m\pi)^2+(\frac{\pi}{e}k)^2]$, $n,m,k=1,2,\ldots,i=5,6,7,8$.*

(a) (Free system) With control V switched off, $V\equiv 0$, the orthogonal projection

$I - \Pi^0 v$ onto Z of the velocity component v has the decay

$$\text{on } Z: \qquad \|(I-\Pi^0)v(t)\| \le e^{-\nu(1+\pi^2+\left(\frac{\pi}{e}\right)^2)t}\|(I-\Pi^0)v(0)\|, \quad t \ge 0, \qquad (1.1.2.7)$$

where $\| \quad \|$ denotes either the $L_2(\Omega)$-norm; or else the norm of $\mathcal{D}((-A_D|_Z)^{\frac{1}{2}})$, equivalent to the $H^1(\Omega)$-norm.

(b) (Arbitrary enhancement of stability margin by wall-normal control V) Preassign an arbitrary but fixed constant $\gamma_0 > 0$. Define the finite-dimensional subspace $Z^u_{\gamma_0}$ of Z by (see (1.5.3)):

$$\begin{aligned} Z^u_{\gamma_0} &= \text{span}\{e^i_{nm0},\ i=1,2;\ n,m \text{ s.t. } n^2+(m\pi)^2 < \gamma_0\} \\ &+ \ \text{span}\{e^i_{0mk},\ i=3,4;\ m,k \text{ s.t. } (m\pi)^2+\left(\frac{\pi}{e}k\right)^2 < \gamma_0\} \\ &+ \ \text{span}\{e^i_{nmk},\ i=5,6,7,8;\ n,m,k \text{ s.t. } n^2+(m\pi)^2+\left(\frac{\pi}{e}k\right)^2 < \gamma_0\}. \quad (1.1.2.8) \end{aligned}$$

Let $V(0) = V|_{t=0}$ be also preassigned. Set $q(t) = (I-\Pi^0)v(t)$ for the orthogonal projection of the velocity component v onto the eigenspace Z. Then, q satisfies the abstract equation

$$\text{on } Z: \ [q-DV]_t = \nu A_D[q-DV] - \nu D(V_{xx}+V_{zz}),\ I.C. = q(0)-DV(0), \ (1.1.2.9)$$

where D is the Dirichlet map: $H^s(\Gamma_1) \to H^{s+\frac{1}{2}}(\Omega)$, $s \in \mathbb{R}$, defined by the elliptic problem (1.2.1a–f), see (1.2.1g).

Next, let $\Lambda \equiv \partial_{xx}+\partial_{zz}$. Let $\langle \psi_i, 1\rangle_{\Gamma_1} = 0$, $\langle(\Lambda\psi_i, 1\rangle_{\Gamma_1} = 0$, $i = 1,2,3,4$. Let $\Lambda\psi_1$,

$\Lambda\psi_2$, $\Lambda\psi_3$, $\Lambda\psi_4$ further satisfy the following rank conditions (see (1.8.3) below):

$$\mathit{rank}\begin{bmatrix} \langle \Lambda\psi_1, \sin nx\rangle & \langle \Lambda\psi_2, \sin nx\rangle & \langle \Lambda\psi_3, \sin nx\rangle & \langle \Lambda\psi_4, \sin nx\rangle \\ \langle \Lambda\psi_1, \cos nx\rangle & \langle \Lambda\psi_2, \cos nx\rangle & \langle \Lambda\psi_3, \cos nx\rangle & \langle \Lambda\psi_4, \cos nx\rangle \end{bmatrix} = 2 \quad (1.1.2.10)$$

for all integers $n > 0$ such that $n^2 < \gamma_0 - \pi^2$,

$$\mathit{rank}\begin{bmatrix} \langle \Lambda\psi_1, \sin \frac{\pi}{e}kz\rangle & \langle \Lambda\psi_2, \sin \frac{\pi}{e}kz\rangle & \langle \Lambda\psi_3, \sin \frac{\pi}{e}kz\rangle & \langle \Lambda\psi_4, \sin \frac{\pi}{e}kz\rangle \\ \langle \Lambda\psi_1, \cos \frac{\pi}{e}kz\rangle & \langle \Lambda\psi_2, \cos \frac{\pi}{e}kz\rangle & \langle \Lambda\psi_3, \cos \frac{\pi}{e}kz\rangle & \langle \Lambda\psi_4, \cos \frac{\pi}{e}kz\rangle \end{bmatrix} = 2 \quad (1.1.2.11)$$

for all integers $k > 0$ such that $\left(\frac{\pi}{e}k\right)^2 < \gamma_0 - \pi^2$, and (split over two lines for spacing)

$$\mathit{rank}\left[\begin{array}{c|c} \langle \Lambda\psi_1, \sin nx \ \ \sin \frac{\pi}{e}kz \ \rangle & \langle \Lambda\psi_2, \sin nx \ \ \sin \frac{\pi}{e}kz \ \rangle \\ \langle \Lambda\psi_1, \sin nx \ \ \cos \frac{\pi}{e}kz \ \rangle & \langle \Lambda\psi_2, \sin nx \ \ \cos \frac{\pi}{e}kz \ \rangle \\ \langle \Lambda\psi_1, \cos nx \ \ \sin \frac{\pi}{e}kz \ \rangle & \langle \Lambda\psi_2, \cos nx \ \ \sin \frac{\pi}{e}kz \ \rangle \\ \langle \Lambda\psi_1, \cos nx \ \ \cos \frac{\pi}{e}kz \ \rangle & \langle \Lambda\psi_2, \cos nx \ \ \cos \frac{\pi}{e}kz \ \rangle \end{array}\right|$$

$$\left|\begin{array}{c|c} \langle \Lambda\psi_3, \sin nx \ \ \sin \frac{\pi}{e}kz \ \rangle & \langle \Lambda\psi_4, \sin nx \ \ \sin \frac{\pi}{e}kz \ \rangle \\ \langle \Lambda\psi_3, \sin nx \ \ \cos \frac{\pi}{e}kz \ \rangle & \langle \Lambda\psi_4, \sin nx \ \ \cos \frac{\pi}{e}kz \ \rangle \\ \langle \Lambda\psi_3, \cos nx \ \ \sin \frac{\pi}{e}kz \ \rangle & \langle \Lambda\psi_4, \cos nx \ \ \sin \frac{\pi}{e}kz \ \rangle \\ \langle \Lambda\psi_3, \cos nx \ \ \cos \frac{\pi}{e}kz \ \rangle & \langle \Lambda\psi_4, \cos nx \ \ \cos \frac{\pi}{e}kz \ \rangle \end{array}\right] = 4 \qquad (1.1.2.12)$$

for all integers $n, k > 0$ such that $n^2 + \left(\frac{\pi}{e}k\right)^2 < \gamma_0 - \pi^2$.

Let $P^u_{\gamma_0}$ be the orthogonal projection of Z onto $Z^u_{\gamma_0}$. Then, there exists a feedback *operator* $\tilde{F}^u_{\gamma_0} \in \mathcal{L}(Z^u_{\gamma_0}; \mathbb{R}^4)$, *such that, setting*

$$\begin{bmatrix} \varphi_1(t) \\ \varphi_2(t) \\ \varphi_3(t) \\ \varphi_4(t) \end{bmatrix} = \tilde{F}^u_{\gamma_0} P^u_{\gamma_0}[q(t) - DV(t)], \tag{1.1.2.13}$$

the resulting feedback dynamics satisfies

$$\text{on } Z: \quad [q - DV]_t = \nu A_D [q - DV]$$

$$- \nu D \left((\partial_{xx} + \partial_{zz})[\psi_1, \psi_2, \psi_3, \psi_4] \right) \tilde{F}^u_{\gamma_0} P^u_{\gamma_0}[q(t) - DV(t)], \tag{1.1.2.14}$$

as well as the arbitrarily preassigned decay

$$\|(I - \Pi^0)v(t)\| \equiv \|z(t)\| \leq Const\ e^{-\nu\gamma_0 t}\|q(0) - DV(0)\|,$$

$$t \geq 0;\ q(0) = (I - \Pi^0)v(0), \tag{1.1.2.15}$$

where $\| \quad \|$ *denotes either the* $L_2(\Omega)$*-norm; or else the* $H^1(\Omega)$*-norm.*

The control $V(t) = \psi_1(x,z)\varphi_1(t) + \psi_2(x,z)\varphi_2(t) + \psi_3(x,z)\varphi_3(t) + \psi_4(x,z)\varphi_4(t)$ *obeys the decay*

$$\|V(t)\|_{L_2(\Gamma_1)} + \|V_t(t)\|_{L_2(\Gamma_1)} + \|V_x(t)\|_{L_2(\Gamma_1)} + \|V_{xx}(t)\|_{L_2(\Gamma_1)} + \|V_{xt}(t)\|_{L_2(\Gamma_1)}$$

$$+ \|V_z(t)\|_{L_2(\Gamma_1)} + \|V_{zz}(t)\|_{L_2(\Gamma_1)} + \|V_{zt}(t)\|_{L_2(\Gamma_1)}$$

$$\leq Const\ e^{-\sigma_0 t}\|P^u_{\gamma_0}(q(0) - DV(0))\|_{L_2(\Omega)}, \tag{1.1.2.16}$$

where σ_0 *is an arbitrarily preassigned constant* $\sigma_0 > \nu\gamma_0$ $[\tilde{F}^u_{\gamma_0}$ *depends on* $\sigma_0]$.

The proof is given in Sections 1.7 through 1.10. Implementation of functions $\psi_i(x,z),\ i = 1,2,3,4$ satisfying the hypotheses of Theorem 1.1.2.2 is postponed to Section 1.1.3.

Remark 1.1.2.1. Eq. (1.1.2.9) of Theorem 1.1.2.2(b) shows that the 4-dimensional control $\Phi(t) = [\varphi_1(t), \varphi_2(t), \varphi_3(t), \varphi_4(t)]^{tr}$ is expressed as *a feedback of* $P^u_{\gamma_0}h(t) \equiv P^u_{\gamma_0}[q(t) - DV(t)]$; that is, *of the intrinsic variable* $h(t) \equiv q(t) - DV(t) = (I - \Pi^0)(v - DV(t))$, see (1.5.13b), projected onto the finite-dimensional subspace $Z^u_{\gamma_0}$, of dimension dim $Z^u_{\gamma_0} = N_{\gamma_0}$. □

Analysis of tangential velocity components u and w, pressure p, and vorticity ω. Our next theorem completes the description of the analysis by proving the corresponding exponential decays of: the tangential components u and w of the velocity vector; the pressure p; the vorticity ω.

Theorem 1.1.2.3. *Assume the setting of Theorem 1.2.2(b), so that the arbitrarily preassigned and fast decays (1.1.2.15) for* $(I - \Pi^0)v(t)$ *and (1.1.2.16) for* $V(t)$*, etc., hold true. Then the following additional results hold true:*

(i) the pressure p *decays with the arbitrarily fast decay rate* σ_0 *(see (1.8.5a) for* σ_0*):*

$$\|p(t)\|_{H^{\frac{3}{2}}(\Omega)} \leq Const\ e^{-\sigma_0 t}\|P^u_{\gamma_0}(q(0) - DV(0)\|_{Z^u_{\gamma_0}} \qquad (1.1.2.17)$$

$$= Const\ e^{-\sigma_0 t}\|P^u_{\gamma_0}(I - \Pi^0)(v(0) - DV(0))\|_{Z^u_{\gamma_0}} \qquad (1.1.2.18)$$

[this is proved in Theorem 1.11.1 below];

(ii) the components of vorticity $\omega^i(t)$ *satisfy the following constrained exponential decay rates:*

$$\|\Pi^0\omega^i(t)\| \le e^{-\nu\pi^2 t}\|\Pi^0\omega(0)\|, \quad t \ge 0, \tag{1.1.2.19}$$

where $\| \quad \|$ *denotes either the* $L_2(\Omega)$*-norm; or else the* $\mathcal{D}((-A_D^0)^{\frac{1}{2}})$*-norm, equivalent to the* $H^1(\Omega)$*-norm;*

$$\|(I-\Pi^0)\omega^i(t)\|_Z \le e^{-\nu(1+\pi^2+\left(\frac{\pi}{e}\right)^2)t}[\|(I-\Pi^0)\omega^i(0)\|_Z + \|P_{\gamma_0}^u h(0)\|_{Z_{\gamma_0}^u}], \quad t \ge 0, \tag{1.1.2.20}$$

where, of course, the Z*-norm is the* $L_2(\Omega)$*-norm; moreover,* $h(0) = q(0) - DV(0) = (I-\Pi^0)[v(0) - DV(0)]$*, see (1.5.13b) [this result is shown in Theorem 1.12.1; see also the statement just above Remark 1.13.1];*

(iii) the tangential velocity components u *and* w *satisfy the following exponential decay rate:*

$$\begin{aligned} &\|f(t)\|_{L_2(\Omega)/\mathbb{R}} + \|\nabla f(t)\|_{L_2(\Omega)} \\ &\le \quad Const\; e^{-\nu\pi^2 t}\left[\|f(0)\|_{H^1(\Omega)} + \|v(0)\|_{H^1(\Omega)} + \|h(0)\|_{H^1(\Omega)}\right], \; f = u, w, \end{aligned} \tag{1.1.2.21}$$

$h(0) = q(0) - DV(0) = (I-\Pi^0)[v(0) - DV(0)]$*, see (1.5.13b). [This result is established in Theorem 1.13.1 (by means of two proofs in Section 1.13.]*

Remark 1.1.2.2. Analysis of the stabilization problem of eq. (1.1.2.4) on Z in the linearized case [the relevant case, after Theorem 1.2.1 has been shown] may be carried out in a subsequent companion paper. This will have the *implication* of ultimately yielding a local exponential stabilization result near a steady state solution of the full, nonlinear Navier-Stokes model. In the style of [B-T.1], [B-L-T.1], [B-L-T.2], [B-L-T.3], it will employ the following strategy: the same finite-dimensional feedback control mechanism, or alternatively the same Riccati-based boundary feedback optimal control mechanism that is obtained in optimal control for the linearized model will then be selected and implemented also in the full Navier-Stokes system, thereby generating a locally dissipative feedback control. However, for technical reasons due to the presence of the *time derivative* of the boundary control as it appears in eq. (1.1.1.12b) for the pressure p—yielding the abstract model (1.1.2.4) in the new, intrinsic variable $h = [v - DV]$—the *non-standard* Riccati theory as in [L-L-P.1], [L-P-T.1], [Tr.3] will likely need to be critically used. $\square$

1.1.3 Construction of Required Functions $\psi_1(x,z), \cdots, \psi_4(x,z)$

(a) A simple way of constructing functions $\psi_1(x,z)$, $\psi_2(x,z)$, $\psi_3(x,z)$, $\psi_4(x,z)$, $-\pi \leq x \leq \pi$, $-e \leq z \leq e$, as to fulfill the requirements of Theorem 1.1.2.2(b), is as follows. Let n_0 and k_0 be defined respectively as the largest positive integers such that

$$n_0^2 < \gamma_0 - \pi^2 - \left(\frac{\pi}{e}\right)^2, \text{ and} \tag{1.1.3.1}$$

$$\left(\frac{\pi}{e}k_0\right)^2 < \gamma_0 - \pi^2 - 1, \tag{1.1.3.2}$$

respectively. In the subspace $\mathcal{O}_{(s,s)} = \text{span}(\{\sin nx,\ n = 1, 2, \ldots, n_0\} \cup \{\sin \frac{\pi}{e} kz,\ k = 1, 2, \ldots, k_0\} \cup \{\sin nx \ \ \sin \frac{\pi}{e} kz,\ \ n = 1, 2, \ldots, n_0;\ \ k = 1, 2, \ldots, k_0\})$, there exists a unique smooth function $\psi_1(x, z)$ such that

$$\langle \psi_1, 1\rangle = 0;\ \langle \psi_1, \sin nx\rangle \equiv 1,\ \langle \psi_1, \cos nx\rangle \equiv 0,\ \langle \psi_1, \sin \frac{\pi}{e} z\rangle \equiv 1,\ \langle \psi_1, \cos \frac{\pi}{e} z\rangle \equiv 0;$$

$$\langle \psi_1, \sin nx \sin \frac{\pi}{e} z\rangle \equiv 1;\ \langle \psi_1, \sin nx \cos \frac{\pi}{e} z\rangle \equiv \ \langle \psi_1, \cos nx \sin \frac{\pi}{e} z\rangle$$

$$\equiv \ \langle \psi_1, \cos nx \cos \frac{\pi}{e} z\rangle \equiv 0,\ n = 1, 2, \ldots, n_0,\ k = 1, 2, \ldots, k_0, \tag{1.1.3.3}$$

where $\langle\ \ ,\ \ \rangle = \langle\ \ ,\ \ \rangle_{\Gamma_1} = L_2([-\pi, \pi] \times [-e, e])$-inner product. Indeed, it is given explicitly by

$$\psi_1(x, z) \equiv \frac{1}{\pi e} \sum_{k=1}^{k_0} \sum_{n=1}^{n_0} \sin nx \ \ \sin \frac{\pi}{e} kz \ + \ \frac{1}{2\pi e} \sum_{n=1}^{n_0} \sin nx \ + \ \frac{1}{2\pi e} \sum_{k=1}^{k_0} \sin \frac{\pi}{e} kz,$$

$$-\pi \le x \le \pi,\ -e \le z \le e. \tag{1.1.3.4}$$

Thus,

$$(\partial_{xx} + \partial_{zz})\psi_1(x, z) \ \equiv \ -\frac{1}{\pi e} \sum_{k=1}^{k_0} \sum_{n=1}^{n_0} \left(\left(\frac{\pi}{e} k\right)^2 + n^2 \right) \sin nx \sin \frac{\pi}{e} kz$$

$$- \ \frac{1}{2\pi e} \sum_{n=1}^{n_0} n^2 \sin nx \ - \ \frac{\pi}{2e^3} \sum_{k=1}^{k_0} k^2 \sin \frac{\pi}{e} kz;$$

$$\langle (\partial_{xx} + \partial_{zz})\psi_1,\ 1\rangle = 0;$$

$$\langle(\partial_{xx}+\partial_{zz})\psi_1,\sin nx\rangle \equiv -n^2,\ \ \langle(\partial_{xx}+\partial_{zz})\psi_1,\cos nx\rangle \equiv 0,$$

$$\langle(\partial_{xx}+\partial_{zz})\psi_1,\sin\frac{\pi}{e}kz\rangle \equiv -\left(\frac{\pi}{e}k\right)^2,\ \ \langle(\partial_{xx}+\partial_{zz})\psi_1,\cos\frac{\pi}{e}kz\rangle \equiv 0,$$

$$\langle(\partial_{xx}+\partial_{zz})\psi_1,\ \sin nx\ \ \sin\frac{\pi}{e}kz\rangle = -n^2k^2,$$

$$\langle(\partial_{xx}+\partial_{zz})\psi_1,\ \sin nx\ \ \cos\frac{\pi}{e}kz\rangle \equiv 0,$$

$$\langle(\partial_{xx}+\partial_{zz})\psi_1,\ \cos nx\ \ \sin\frac{\pi}{e}kz\rangle \equiv 0,$$

$$\langle(\partial_{xx}+\partial_{zz})\psi_1,\ \cos nx\ \ \cos\frac{\pi}{e}kz\rangle \equiv 0,$$

$$n=1,2,\dots,n_0,\ \ k=1,2,\dots,k_0.$$

Under similar considerations in the subspaces

$$\begin{aligned}
\mathcal{O}_{(s,c)} &= \mathrm{span}(\{\sin nx,\ \ n=1,2,\dots,n_0\}\ \cup\ \{\cos\frac{\pi}{e}kz,\ \ k=1,2,\dots,k_0\}\ \cup \\
&\qquad \{\sin nx\ \ \cos\frac{\pi}{e}kz,\ \ n=1,2,\dots,n_0;\ \ k=1,2,\dots,k_0\}), \\
\mathcal{O}_{(c,s)} &= \mathrm{span}(\{\cos nx,\ \ n=1,2,\dots,n_0\}\ \cup\ \{\sin\frac{\pi}{e}kz,\ \ k=1,2,\dots,k_0\}\ \cup \\
&\qquad \{\cos nx\ \ \sin\frac{\pi}{e}kz,\ \ n=1,2,\dots,n_0;\ \ k=1,2,\dots,k_0\}), \\
\mathcal{O}_{(c,c)} &= \mathrm{span}(\{\cos nx,\ \ n=1,2,\dots,n_0\}\ \cup\ \{\cos\frac{\pi}{e}kz,\ \ k=1,2,\dots,k_0\}\ \cup \\
&\qquad \{\cos nx\ \ \cos\frac{\pi}{e}kz,\ \ n=1,2,\dots,n_0;\ \ k=1,2,\dots,k_0\}),
\end{aligned}$$

we can define $\psi_2(x,z) \in \mathcal{O}_{(s,c)}$, $\psi_3(x,z) \in \mathcal{O}_{(c,s)}$, and $\psi_4(x,z) \in \mathcal{O}_{(c,c)}$ for $-\pi \le x \le \pi$ and $-e \le z \le e$ by

$$\psi_2(x,z) \equiv \frac{1}{\pi e}\sum_{k=1}^{k_0}\sum_{n=1}^{n_0} \sin nx \;\cos\frac{\pi}{e}kz \;+\; \frac{1}{2\pi e}\sum_{n=1}^{n_0}\sin nx + \frac{1}{2\pi e}\sum_{k=1}^{k_0}\cos\frac{\pi}{e}kz; \tag{1.1.3.5}$$

$$\psi_3(x,z) \equiv \frac{1}{\pi e}\sum_{k=1}^{k_0}\sum_{n=1}^{n_0} \cos nx \;\sin\frac{\pi}{e}kz \;+\; \frac{1}{2\pi e}\sum_{n=1}^{n_0}\cos nx + \frac{1}{2\pi e}\sum_{k=1}^{k_0}\sin\frac{\pi}{e}kz; \tag{1.1.3.6}$$

$$\psi_4(x,z) \equiv \frac{1}{\pi e}\sum_{k=1}^{k_0}\sum_{n=1}^{n_0} \cos nx \;\cos\frac{\pi}{e}kz \;+\; \frac{1}{2\pi e}\sum_{n=1}^{n_0}\cos nx + \frac{1}{2\pi e}\sum_{k=1}^{k_0}\cos\frac{\pi}{e}kz. \tag{1.1.3.7}$$

With these choices of functions $\psi_i,\ i = 1,2,3,4$, we have the matrices from (1.1.2.10–1.1.2.12) reduce to (again, letting $\Lambda = \partial_{xx} + \partial_{zz}$)

$$\left[\begin{array}{c|c|c|c} \langle\Lambda\psi_1, \sin nx\rangle & \langle\Lambda\psi_2, \sin nx\rangle & \langle\Lambda\psi_3, \sin nx\rangle & \langle\Lambda\psi_4, \sin nx\rangle \\ \langle\Lambda\psi_1, \cos nx\rangle & \langle\Lambda\psi_2, \cos nx\rangle & \langle\Lambda\psi_3, \cos nx\rangle & \langle\Lambda\psi_4, \cos nx\rangle \end{array}\right] = \begin{bmatrix} -n^2 & -n^2 & 0 & 0 \\ 0 & 0 & -n^2 & -n^2 \end{bmatrix}, \tag{1.1.3.8}$$

$$\left[\begin{array}{c|c|c|c} \langle \Lambda\psi_1, \sin\frac{\pi}{e}kz\ \rangle & \langle \Lambda\psi_2, \sin\frac{\pi}{e}kz\ \rangle & \langle \Lambda\psi_3, \sin\frac{\pi}{e}kz\ \rangle & \langle \Lambda\psi_4, \sin\frac{\pi}{e}kz\ \rangle \\ \langle \Lambda\psi_1, \cos\frac{\pi}{e}kz\ \rangle & \langle \Lambda\psi_2, \cos\frac{\pi}{e}kz\ \rangle & \langle \Lambda\psi_3, \cos\frac{\pi}{e}kz\ \rangle & \langle \Lambda\psi_4, \cos\frac{\pi}{e}kz\ \rangle \end{array}\right]$$

$$= \begin{bmatrix} -\left(\frac{\pi}{e}k\right)^2 & 0 & -\left(\frac{\pi}{e}k\right)^2 & 0 \\ 0 & -\left(\frac{\pi}{e}k\right)^2 & 0 & -\left(\frac{\pi}{e}k\right)^2 \end{bmatrix}, \tag{1.1.3.9}$$

and (with splitting the following matrix over two lines for spacing)

$$\left[\begin{array}{c|c|} \langle \Lambda\psi_1, \sin nx\ \ \sin\frac{\pi}{e}kz\rangle & \langle \Lambda\psi_2, \sin nx\ \ \sin\frac{\pi}{e}kz\rangle \\ \langle \Lambda\psi_1, \sin nx\ \ \cos\frac{\pi}{e}kz\rangle & \langle \Lambda\psi_2, \sin nx\ \ \cos\frac{\pi}{e}kz\rangle \\ \langle \Lambda\psi_1, \cos nx\ \ \sin\frac{\pi}{e}kz\rangle & \langle \Lambda\psi_2, \cos nx\ \ \sin\frac{\pi}{e}kz\rangle \\ \langle \Lambda\psi_1, \cos nx\ \ \cos\frac{\pi}{e}kz\rangle & \langle \Lambda\psi_2, \cos nx\ \ \cos\frac{\pi}{e}kz\rangle \end{array}\right.$$

$$\left.\begin{array}{|c|c} \langle \Lambda\psi_3, \sin nx\ \ \sin\frac{\pi}{e}kz\rangle & \langle \Lambda\psi_4, \sin nx\ \ \sin\frac{\pi}{e}kz\rangle \\ \langle \Lambda\psi_3, \sin nx\ \ \cos\frac{\pi}{e}kz\rangle & \langle \Lambda\psi_4, \sin nx\ \ \cos\frac{\pi}{e}kz\rangle \\ \langle \Lambda\psi_3, \cos nx\ \ \sin\frac{\pi}{e}kz\rangle & \langle \Lambda\psi_4, \cos nx\ \ \sin\frac{\pi}{e}kz\rangle \\ \langle \Lambda\psi_3, \cos nx\ \ \cos\frac{\pi}{e}kz\rangle & \langle \Lambda\psi_4, \cos nx\ \ \cos\frac{\pi}{e}kz\rangle \end{array}\right]$$

$$= \begin{bmatrix} -\left(\left(\frac{\pi}{e}k\right)^2 + n^2\right) & 0 & 0 & 0 \\ 0 & -\left(\left(\frac{\pi}{e}k\right)^2 + n^2\right) & 0 & 0 \\ 0 & 0 & -\left(\left(\frac{\pi}{e}k\right)^2 + n^2\right) & 0 \\ 0 & 0 & 0 & -\left(\left(\frac{\pi}{e}k\right)^2 + n^2\right) \end{bmatrix} \tag{1.1.3.10}$$

$$\text{for} \quad n = 1, 2, \ldots, n_0, \ \ k = 1, 2, \ldots, k_0.$$

Then, the above functions $\psi_1, \psi_2, \psi_3, \psi_4$, satisfy the three assumptions of Theorem 1.1.2.2(b): $\langle \psi_i, 1\rangle = \langle (\partial_{xx} + \partial_{zz})\psi_i, 1\rangle = 0$ and the rank conditions (1.1.2.10–1.1.2.12).

(b) One can easily modify the above strategy as to obtain infinitely many subsets $\{\psi_1, \psi_2, \psi_3, \psi_4\}$ of the subspace

$$\mathcal{O} = \text{span}(\{\sin nx, \cos nx,\ \ n = 1, 2, \ldots, n_0\}\ \cup\ \{\sin \frac{\pi}{e}kz, \cos \frac{\pi}{e}kz,\ \ k = 1, 2, \ldots, k_0\}$$

$$\cup\ \{\sin nx\ \ \sin \frac{\pi}{e}kz,\ \ \sin nx\ \ \cos \frac{\pi}{e}kz,\ \ \cos nx\ \ \sin \frac{\pi}{e}kz,\ \ \cos nx\ \ \cos \frac{\pi}{e}kz,$$

$$n = 1, 2, \ldots, n_0;\ \ k = 1, 2, \ldots, k_0\}),$$

such that

$$\langle \psi_i, 1 \rangle = 0; \quad \langle \psi_1, \sin nx \rangle = \alpha_{i,1,n}; \quad \langle \psi_i, \cos nx \rangle = \alpha_{i,2,n};$$

$$\langle \psi_i, \sin \frac{\pi}{e}kz \rangle = \beta_{i,1,n}; \quad \langle \psi_i, \cos \frac{\pi}{e}kz \rangle = \beta_{i,2,n};$$

$$\langle \psi_i, \sin \frac{\pi}{e}kz \rangle = \gamma_{i,1,n}; \quad \langle \psi_i, \cos \frac{\pi}{e}kz \rangle = \gamma_{i,1,n};$$

$$\langle \psi_i, \sin \frac{\pi}{e}kz \rangle = \gamma_{i,3,n}; \quad \langle \psi_i, \cos \frac{\pi}{e}kz \rangle = \gamma_{i,4,n};\ \ i = 1, 2, 3, 4. \tag{1.1.3.11}$$

Indeed, ψ_1, ψ_2, ψ_3, and ψ_4 are given by

$$\psi_i(x, z) = \frac{1}{2\pi e} \sum_{n=1}^{n_0} (\alpha_{i,1,n} \sin nx + \alpha_{i,2,n} \cos nx)\ \ +$$

$$\frac{1}{2\pi e} \sum_{k=1}^{k_0} \left(\beta_{i,1,k} \sin \frac{\pi}{e}kz + \beta_{i,2,k} \cos \frac{\pi}{e}kz\right)\ \ +$$

$$\frac{1}{\pi e} \sum_{k=1}^{k_0} \sum_{n=1}^{n_0} (\gamma_{i,1,n,k} \sin nx \sin \frac{\pi}{e}kz\ \ +\ \ \gamma_{i,2,n,k} \sin nx \cos \frac{\pi}{e}kz$$

$$+\ \gamma_{i,3,n,k} \cos nx \sin \frac{\pi}{e}kz\ \ +\ \ \gamma_{i,4,n,k} \cos nx \cos \frac{\pi}{e}kz),\ \ i = 1, 2, 3, 4. \tag{1.1.3.12}$$

Thus

$$(\partial_{xx}+\partial_{zz})\psi_i(x,z) = -\frac{1}{2\pi e}\sum_{n=1}^{n_0} n^2\left(\alpha_{i,1,n}\sin nx+\alpha_{i,2,n}\cos nx\right)$$
$$-\frac{1}{2\pi e}\sum_{k=1}^{k_0}\left(\frac{\pi}{e}k\right)^2\left(\beta_{i,1,k}\sin\frac{\pi}{e}kz+\beta_{i,2,k}\cos\frac{\pi}{e}kz\right)$$
$$-\frac{1}{\pi e}\sum_{k=1}^{k_0}\sum_{n=1}^{n_0}\left(\left(\frac{\pi}{e}k\right)^2+n^2\right)(\gamma_{i,1,n,k}\sin nx\sin\frac{\pi}{e}kz \;+\; \gamma_{i,2,n,k}\sin nx\cos\frac{\pi}{e}kz$$
$$+\;\gamma_{i,3,n,k}\cos nx\sin\frac{\pi}{e}kz \;+\; \gamma_{i,4,n,k}\cos nx\cos\frac{\pi}{e}kz),\;\; i=1,2,3,4. \qquad (1.1.3.13)$$

$$\langle(\partial_{xx}+\partial_{zz})\psi_i,1\rangle = 0. \qquad (1.1.3.14)$$

Finally, the constants $\alpha_{i,j,n}$, $\beta_{i,j,k}$, and $\gamma_{i,p,n,k}$ for $i=1,2,3,4,\ j=1,2,\ p=1,2,3,4,\ n=1,2,\ldots,n_0$, and $k=1,2,\ldots,k_0$ are chosen so that

$$\text{rank}\begin{bmatrix}\alpha_{1,1,n} & \alpha_{2,1,n} & \alpha_{3,1,n} & \alpha_{4,1,n}\\ \alpha_{1,2,n} & \alpha_{2,2,n} & \alpha_{3,2,n} & \alpha_{4,2,n}\end{bmatrix}\equiv 2, \qquad (1.1.3.15)$$

$$\text{rank}\begin{bmatrix}\beta_{1,1,k} & \beta_{2,1,k} & \beta_{3,1,k} & \beta_{4,1,k}\\ \beta_{1,2,k} & \beta_{2,2,k} & \beta_{3,2,k} & \beta_{4,2,k}\end{bmatrix}\equiv 2, \qquad (1.1.3.16)$$

$$\text{rank}\begin{bmatrix}\gamma_{1,1,n,k} & \gamma_{2,1,n,k} & \gamma_{3,1,n,k} & \gamma_{4,1,n,k}\\ \gamma_{1,2,n,k} & \gamma_{2,2,n,k} & \gamma_{3,2,n,k} & \gamma_{4,2,n,k}\\ \gamma_{1,3,n,k} & \gamma_{2,3,n,k} & \gamma_{3,3,n,k} & \gamma_{4,3,n,k}\\ \gamma_{1,4,n,k} & \gamma_{2,4,n,k} & \gamma_{3,4,n,k} & \gamma_{4,4,n,k}\end{bmatrix}\equiv 4. \qquad (1.1.3.17)$$

In the strategy of part (a), we have chosen $\alpha_{i,j,n}$, $\beta_{i,j,k}$, and $\gamma_{i,p,n,k}$ to each be either 0 or 1 to produce equations (1.1.3.12–14).

Then, the corresponding set $\{\psi_1, \psi_2, \psi_3, \psi_4\}$ satisfies all three assumptions of Theorem 1.2.2(b): $\langle \psi_i, 1\rangle = 0$, $\langle (\partial_{xx} + \partial_{zz})\psi_i, 1\rangle = 0$, and the rank conditions (1.1.2.10–1.1.2.12).

(c) **Selection of required functions ψ_1, ψ_2, ψ_3, and ψ_4 with arbitrarily small support on $[-\pi, \pi] \times [e, e]$.** We may select ψ_1, ψ_2, ψ_3, and ψ_4 to be smooth functions, with ψ_1 even with respect to x and z, ψ_2 odd with respect to x and even with respect to z, ψ_3 even with respect to x and odd with respect to z, and ψ_4 odd with respect to both x and z, with arbitrarily small support around $(x, z) = (0, 0)$ in $[-\pi, \pi] \times [e, e]$. Thus ψ_1 is given by a sine-sine-expansion, ψ_2 by a sine-cosine-expansion, ψ_3 by a cosine-sine-expansion, and ψ_4 by a cosine-cosine-expansion

$$\psi_1(x,z) = \text{Const} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{k=1}^{\infty} b_n \cos\frac{\pi}{e}kz + \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} c_n \cos nx \cos\frac{\pi}{e}kz;$$

$$(\partial_{xx} + \partial_{zz})\psi_1 = -\sum_{n=1}^{\infty} n^2 a_n \cos nx \;-\; \sum_{k=1}^{\infty}\left(\frac{\pi}{e}k\right)^2 b_n \cos\frac{\pi}{e}kz$$

$$-\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\left(\left(\frac{\pi}{e}k\right)^2 + n^2\right) c_n \cos nx \cos\frac{\pi}{e}kz; \qquad (1.1.3.18)$$

$$\psi_2(x,z) \;=\; \sum_{n=1}^{\infty} a_n \sin nx \;+\; \sum_{k=1}^{\infty} b_n \cos\frac{\pi}{e}kz \;+\; \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} c_n \sin nx \cos\frac{\pi}{e}kz;$$

$$(\partial_{xx} + \partial_{zz})\psi_1 = -\sum_{n=1}^{\infty} n^2 a_n \sin nx \;-\; \sum_{k=1}^{\infty}\left(\frac{\pi}{e}k\right)^2 b_n \cos\frac{\pi}{e}kz$$

$$-\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\left(\left(\frac{\pi}{e}k\right)^2 + n^2\right) c_n \sin nx \cos\frac{\pi}{e}kz; \qquad (1.1.3.19)$$

$$\psi_3(x,z) \;=\; \sum_{n=1}^{\infty} a_n \cos nx \;+\; \sum_{k=1}^{\infty} b_n \sin \frac{\pi}{e}kz \;+\; \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} c_n \cos nx \sin \frac{\pi}{e}kz;$$

$$(\partial_{xx}+\partial_{zz})\psi_1 = -\sum_{n=1}^{\infty} n^2 a_n \cos nx \;-\; \sum_{k=1}^{\infty}\left(\frac{\pi}{e}k\right)^2 b_n \sin \frac{\pi}{e}kz$$
$$-\;\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\left(\left(\frac{\pi}{e}k\right)^2+n^2\right) c_n \cos nx \sin \frac{\pi}{e}kz; \qquad (1.1.3.20)$$

$$\psi_4(x,z) \;=\; \sum_{n=1}^{\infty} a_n \sin nx \;+\; \sum_{k=1}^{\infty} b_n \sin \frac{\pi}{e}kz \;+\; \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} c_n \sin nx \sin \frac{\pi}{e}kz;$$

$$(\partial_{xx}+\partial_{zz})\psi_1 = -\sum_{n=1}^{\infty} n^2 a_n \sin nx \;-\; \sum_{k=1}^{\infty}\left(\frac{\pi}{e}k\right)^2 b_n \sin \frac{\pi}{e}kz$$
$$-\;\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\left(\left(\frac{\pi}{e}k\right)^2+n^2\right) c_n \sin nx \sin \frac{\pi}{e}kz; \qquad (1.1.3.21)$$

by term-by-term differentiation, which is legal by smoothness. Note that ψ_1 must additionally be chosen so that the *Const* term in (1.1.3.22) is equal to zero, or equivalently, that $\langle \psi_1, 1\rangle = 0$. As an example of our choices of ψ_i, for instance, we may take variations of the classical function in $C_0^\infty(\mathbb{R}^2)$: $f(x,z) = \exp((x^2+z^2-1)^{-1})$ for $x^2+z^2 < 1$ and $f(x,z) \equiv 0$ for $x^2+z^2 \geq 1$ and multiplied by sine and cosine functions. Examples include:

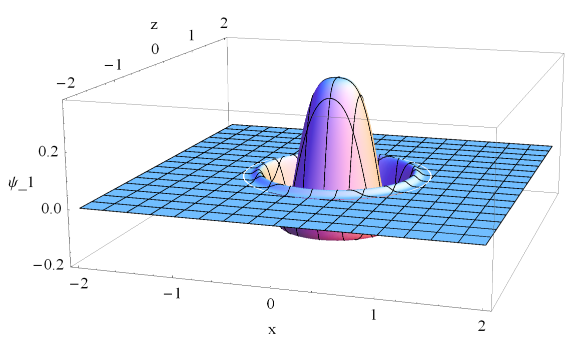
$$\psi_1(x,z) = \begin{cases} \cos\left(\frac{\delta}{\epsilon^2}(x^2+z^2)\right)\exp((x^2+z^2-\epsilon^2))^{-1}, & x^2+z^2<\epsilon^2 \\ 0 & x^2+z^2 \geq \epsilon^2 \end{cases}$$

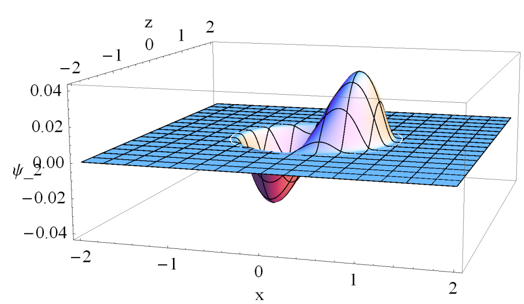
$$\psi_2(x,z) = \begin{cases} \sin\left(\frac{x}{\pi\epsilon}\right)\exp((x^2+z^2-\epsilon^2))^{-1}, & x^2+z^2<\epsilon^2 \\ 0 & x^2+z^2 \geq \epsilon^2 \end{cases}$$

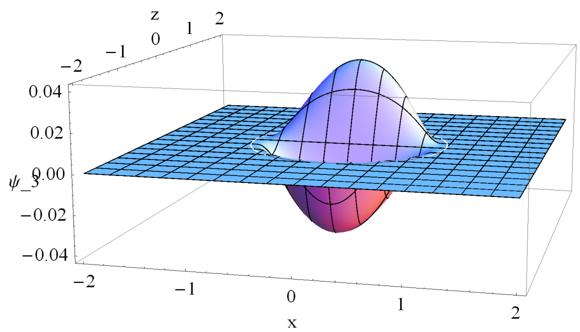
$$\psi_3(x,z) = \begin{cases} \sin\left(\frac{z}{\pi\epsilon}\right)\exp((x^2+z^2-\epsilon^2))^{-1}, & x^2+z^2<\epsilon^2 \\ 0 & x^2+z^2 \geq \epsilon^2 \end{cases}$$

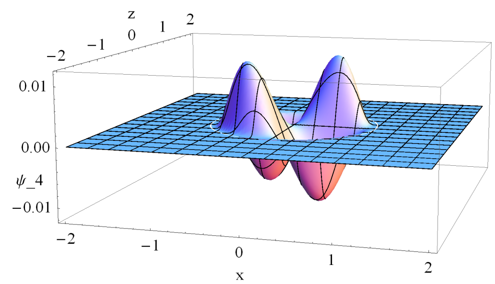
$$\psi_4(x,z) = \begin{cases} \sin\left(\frac{xz}{\pi\epsilon}\right)\exp((x^2+z^2-\epsilon^2))^{-1}, & x^2+z^2<\epsilon^2 \\ 0 & x^2+z^2 \geq \epsilon^2 \end{cases}$$

with δ chosen so that $\langle\psi_1, 1\rangle = 0$, i.e. by $\delta = 7.522079004563391$ (value found numerically up to 16 digits). The graphs of these ψ_i are given (with x and z being scaled by ϵ).









By (1.1.3.23–26), we have $\langle\psi_i, 1\rangle = \langle(\partial_{xx} + \partial_{zz})\psi_i, 1\rangle = 0$, $i = 1, 2, 3, 4$, and, moreover, we can satisfy the rank conditions (1.1.2.10–12).

1.2 The Dirichlet Map D and the Corresponding Principal Part Operator A_D. Spectral Properties of A_D

We first introduce the Dirichlet map D:

$$h \overset{\text{def}}{=} Dg \Longleftrightarrow \begin{cases} & h_{xx} + h_{yy} + h_{zz} \equiv 0 \quad \text{in } \Omega; & (1.2.1\text{a}) \\ \text{B.C.} & h(x,0,z) \equiv 0; \ h(x,1,z) = g(x,z) \quad \text{on } \Gamma_0, \Gamma_1; & (1.2.1\text{b}) \\ & \begin{cases} h(-\pi,y,z) \equiv h(\pi,y,z); & (1.2.1\text{c}) \\ h_x(-\pi,y,z) \equiv h_x(\pi,y,z); & (1.2.1\text{d}) \end{cases} \\ & h(x,y,-e) \equiv h(x,y,e); & (1.2.1\text{e}) \\ & h_z(x.y.-e) \equiv h_z(x,y,e), & (1.2.1\text{f}) \end{cases}$$

as yielding the solution of the elliptic problem (1.2.1a–f) with periodic B.C. in the x-direction, homogeneous on the bottom wall $y = 0$, and with the non-homogeneous term g on the top wall Γ_1, where $y = 1$. The usual regularity property holds

$$D: \text{ continuous } H^s(\Gamma_1) \to H^{s+\frac{1}{2}}(\Omega), \quad 0 \le s \in \mathbb{R}. \tag{1.2.1g}$$

We next introduce the operator

$$A_D f = \Delta f: \ L_2(\Omega) \supset \mathcal{D}(A_D) \to L_2(\Omega); \tag{1.2.2a}$$

$$\mathcal{D}(A_D) = \{f \in H^2(\Omega): \ f|_{y=0} = f|_{y=1} \equiv 0; \ f|_{x=-\pi} \equiv f|_{x=\pi}; \ f_x|_{x=-\pi} = f_x|_{x=\pi};$$

$$f|_{z=-e} = f|_{z=e}; \ f_z|_{z=-e} \equiv f_z|_{z=e}\}. \tag{1.2.2b}$$

Proposition 1.2.1. *(a) The operator A_D in (1.2.2) is strictly negative definite, self-adjoint and has compact resolvent.*

(b) The spectrum of A_D, which coincides with its point spectrum, consists of the following branches:

(b_1) a branch of simple eigenvalues λ_{0m0} (geometric and algebraic multiplicity equal to one), with corresponding normalized eigenfunctions $e^0_{0m0}(x,y,z)$, $-\pi \le x \le \pi$, $0 \le y \le 1$, $-e \le z \le e$,

$$A_D \; e^0_{0m0} = \lambda_{0m0} \; e^0_{0m0}, \qquad m = 1, 2, 3, \ldots \tag{1.2.3a}$$

given by (note that the terms e^0_{0m0} are independent of x and z):

$$\lambda_{0m0} = -(m\pi)^2; \quad e^0_{0m0}(x,y,z) \equiv \frac{1}{\sqrt{2\pi e}} \sin m\pi y, \quad m = 1, 2, 3, \ldots; \tag{1.2.3b}$$

(b_2) a branch of double eigenvalues λ_{nm0} and λ_{0mk} (geometric and algebraic multiplicity equal to two), with corresponding normalized eigenfunctions $e^i_{nm0}(x,y,z)$ and $e^i_{0mk}(x,y,z)$, $-\pi \le x \le \pi$, $0 \le y \le 1$, $-e \le z \le e$,

$$A_D \; e^i_{nmk} = \lambda_{nmk} \; e^i_{nmk}; \qquad i = 1, 2; \; \textit{either } n = 0 \textit{ or } k = 0;$$

$$n, k = 0, 1, 2; \ldots; \; m = 1, 2, 3, \ldots \tag{1.2.4a}$$

given by

$$\lambda_{nm0} = -[n^2 + (m\pi)^2]; \qquad \begin{cases} e^1_{nm0}(x,y,z) \equiv \frac{1}{\sqrt{\pi e}} \sin nx \sin m\pi y; & (1.2.4b) \\ e^2_{nm0}(x,y,z) \equiv \frac{1}{\sqrt{\pi e}} \cos nx \sin m\pi y; & (1.2.4c) \end{cases}$$

and

$$\lambda_{0mk} = -[(\frac{\pi}{e}k)^2 + (m\pi)^2]; \quad \begin{cases} e^3_{omk}(x,y,z) \equiv \frac{1}{\sqrt{\pi e}} \sin \frac{\pi}{e} kz \sin m\pi y; & (1.2.4\text{d}) \\ e^4_{omk}(x,y,z) \equiv \frac{1}{\sqrt{\pi e}} \cos \frac{\pi}{e} kz \sin m\pi y; & (1.2.4\text{e}) \end{cases}$$

(b_3) a branch of quadruple eigenvalues λ_{nmk} (geometric and algebraic multiplicity equal to four), with corresponding normalized eigenfunctions $e^i_{nmk}(x,y,z)$ and $e^i_{nmk}(x,y,z)$, $-\pi \le x \le \pi$, $0 \le y \le 1$, $-e \le z \le e$,

$$A_D \; e^i_{nmk} = \lambda_{nmk} \; e^i_{nmk}; \qquad i = 5,6,7,8; \quad n,m,k = 1,2,3,\dots \tag{1.2.5a}$$

given by

$$\lambda_{nmk} = -[n^2 + (\frac{\pi}{e}k)^2 + (m\pi)^2];$$

$$\begin{cases} e^5_{nmk}(x,y,z) \equiv \sqrt{\frac{2}{\pi e}} \sin nx \sin \frac{\pi}{e} kz \sin m\pi y; & (1.2.5\text{b}) \\ e^6_{nmk}(x,y,z) \equiv \sqrt{\frac{2}{\pi e}} \sin nx \cos \frac{\pi}{e} kz \sin m\pi y; & (1.2.5\text{c}) \\ e^7_{nmk}(x,y,z) \equiv \sqrt{\frac{2}{\pi e}} \cos nx \sin \frac{\pi}{e} kz \sin m\pi y; & (1.2.5\text{d}) \\ e^8_{nmk}(x,y,z) \equiv \sqrt{\frac{2}{\pi e}} \cos nx \cos \frac{\pi}{e} kz \sin m\pi y; & (1.2.5\text{e}) \end{cases}$$

(c) The eigenvectors $\{e^0_{0m0}\}^\infty_{m=1} \cup \{e^i_{nm0},\; i = 1,2\}^\infty_{n,m=1} \cup \{e^i_{0mk},\; i = 3,4\}^\infty_{m,k=1} \cup \{e^i_{nmk},\; i = 5,6,7,8\}^\infty_{n,m,k=1}$ form an orthonormal basis on $L_2(\Omega)$.

Proof. This result is standard. See, e.g., [Z.1, p. 187 Second Edit; p. 144 First Edit], [A.2, p. 56] for the case of periodic B.C. in the x and z-directions yielding double eigenvalues for $n = 0$ or $k = 0$ and $n,k = 1,2,\dots$, a simple eigenvalue for $n,k = 0$, and quadruple eigenvalues for $n,k = 1,2,\dots$. These are then combined with the eigenvalues contributed by the zero Dirichlet B.C. in the y-direction, $y = 0$ and $y = 1$. □

We note that since the coefficients in equations (1.2.3b), (1.2.4b–e), and (1.2.5b–e) (namely 1, $\frac{\pi}{e}$, and π) are linearly independent with respect to multiplication by integers, it follows that there is a bijection between eigenvalues and their indices (n, m, k). This is the reason for requiring the size of the domain being considered to have integer-linearly independent x, y, and z lengths.

We shall also need the following result.

Proposition 1.2.2. *With reference to the negative self-adjoint operator A_D in (1.2.2), we have*

$$\|(-A_D)^{\frac{1}{2}} f\|^2_{L_2(\Omega)} = \int_\Omega |\nabla f|^2 d\Omega, \qquad f \in \mathcal{D}((-A_D)^{\frac{1}{2}}); \tag{1.2.6}$$

$$\mathcal{D}((-A_D)^{\frac{1}{2}}) = \left\{ f \in H^1(\Omega) : f|_{y=0} = f|_{y=1} = 0,\ f|_{x=-\pi} = f|_{x=\pi},\ f|_{z=-e} = f|_{z=e} \right\}. \tag{1.2.7}$$

Proof. At first, let $f \in \mathcal{D}(A_D)$. By Green's first theorem, via (1.2.2), we obtain

$$(A_D f, f)_{L_2(\Omega)} = \int_\Omega \Delta f f \, d\Omega = \int_\Gamma \frac{\partial f}{\partial \nu} f \, d\Gamma - \int_\Omega |\nabla f|^2 d\Omega = -\int_\Omega |\nabla f|^2 d\Omega, \tag{1.2.8}$$

since the boundary integral vanishes by (1.2.2b). Next, we extend the validity of (1.2.8) to all $f \in \mathcal{D}((-A_D)^{\frac{1}{2}})$ by density and (1.2.6) is verified. □

1.3 The Final v-Equation after Elimination of the Pressure Term p_y. Its Abstract Version

The next result achieves the goal of eliminating the pressure term p_y from the v-equation (1.1.1.5b), as anticipated below (1.1.1.12). Refer also to Remark 1.1.1.1.

Proposition 1.3.1. *(a) With reference to the p-problem (1.1.1.12), we have*

$$p_y = DV_t - \nu D(V_{xx} + V_{zz}) + 2A_D^{-1}[\partial_x \partial_y (\overline{U}'(y)v) + \partial_z \partial_y (\overline{W}'(y)v)], \tag{1.3.1}$$

where the operators D and A_D are defined in (1.2.1) and (1.2.2).

(b) Substituting (1.3.1) into the RHS of eq. (1.1.1.5b) yields the final version of the v-problem

$$\begin{cases} v_t - \nu \Delta v + \partial_x(\overline{U}(y)v) + \partial_z(\overline{W}(y)v) - 2A_D^{-1}[\partial_x \partial_y (\overline{U}'(y)v) \\ \qquad +\partial_z \partial_y (\overline{W}'(y)v)] \; = \; DV_t - \nu D(V_{xx} + V_{zz}) & (1.3.2a) \\ v(x,0,z,t) \equiv 0, \;\; v(x,1,z,t) = V(x,z,t) & (1.3.2b) \\ v(-\pi,y,z,t) \equiv v(\pi,y,z,t); \;\; v_x(-\pi,y,z,t) \equiv v_x(\pi,y,z,t) & (1.3.2c) \\ v(x,y,-e,t) \equiv v(x,y,e,t); \;\; v_x(x,y,-e,t) \equiv v_x(x,y,e,t), & (1.3.2d) \end{cases}$$

where the pressure term has been eliminated.

Proof. We return to the p-problem (1.1.1.12), take the partial derivative ∂_y in y in (1.1.1.12a) and (1.1.1.12c–e), and obtain (in Q)

$$\begin{cases} (p_y)_{xx} + (p_y)_{yy} + (p_y)_{zz} = 2[\partial_x\partial_y(\overline{U}'(y)v) + \partial_z\partial_y(\overline{W}'(y)v)]; & (1.3.3a) \\ \text{B.C.} \begin{cases} p_y(x,0,z,t) \equiv 0; \quad p_y(x,1,z,t) \equiv V_t(t,x,z) - \nu(V_{xx} + V_{zz}); & (1.3.3b) \\ p_y(-\pi,y,z,t) = p_y(\pi,y,z,t); & (1.3.3c) \\ (p_y)_x(-\pi,y,z,t) = (p_y)_x(\pi,y,z,t); & (1.3.3d) \\ (p_y)_z(-x,y,-e,t) = (p_y)_z(x,y,e,t). & (1.3.3e) \end{cases} \end{cases}$$

The elliptic problem (1.3.3) in p_y has two non-homogeneous terms: one on the RHS of eq. (1.3.3a) and one in the B.C. (1.3.3b) for $y = 1$. Its solution is given precisely by (1.3.1), in light of the definitions of D and A_D in (1.2.1), (1.2.2), where $A_D^{-1} \in \mathcal{L}(L_2(\Omega))$ by Proposition 1.2.1. Substituting (1.3.1) into the RHS of (1.1.1.5b) yields (1.3.2a). □

In eq. (1.3.2a), we have put into evidence the differentiation operator ∂_x in x and ∂_z in z in the four (lower-order) terms that perturb the Laplacian: the reason will be seen, e.g., in Section 1.6. Here, starting with the v-problem (1.3.2a), we now rewrite it in an abstract form, by use of the Dirichlet map D in (1.2.1). To this end, we first introduce the (lower-order) operator

$$Lf = \mathcal{L}f, \; \mathcal{D}(L) = \{f \in H^1(\Omega) : \; f|_{y=0} = 0; \; f|_{x=-\pi} \equiv f|_{x=\pi}; \; f_x|_{x=-\pi} = f_x|_{x=\pi};$$

$$f|_{z=-e} \equiv f|_{z=e}; \; f_z|_{z=-e} = f_z|_{x=e}\}; \tag{1.3.4a}$$

$$(\mathcal{L}f)(x,y,z) = \left\{\partial_x(\overline{U}(y)f) + \partial_z(\overline{W}(y)f) - 2A_D^{-1}[\partial_x\partial_y(\overline{U}'(y)f + \partial_z\partial_y(\overline{W}'(y)f]\right\}. \tag{1.3.4b}$$

Proposition 1.3.2. *Problem (1.3.2a-b-c-d) can be rewritten abstractly as*

$$(v - DV)_t = \nu A_D(v - DV) - Lv - \nu(DV_{xx} + DV_{zz}), \tag{1.3.5}$$

with D, A_D, L defined by (1.2.1), (1.2.2), (1.3.4).

Proof. By definition of D in (1.2.1), we have $\Delta v = \Delta(v - DV)$ in Ω. Furthermore, $(v - DV)_{y=1} = V - V = 0 = (v - DV)_{y=0}$ on the top wall $y = 1$, and bottom wall $y = 0$, by (1.1.1.5j) and (1.2.1b). Thus, by use also of (1.1.1.5l–n) and (1.2.1c–f), problem (1.3.2) may be rewritten as

$$\begin{cases} (v - DV)_t - \nu\Delta(v - DV) - 2A_D^{-1}[\partial_x\partial_y(\overline{U}'(y)v) + \partial_z\partial_y(\overline{W}'(y)v)] & \\ \qquad +\partial_x(\overline{U}(y)v) + \partial_z(\overline{W}(y)v) \equiv -\nu(DV_{xx} + DV_{zz}) \text{ in } Q; & (1.3.6a) \\ (v - DV)|_{y=0} = 0, \ (v - DV)|_{y=1} = 0; & (1.3.6b) \\ (v - DV)|_{x=-\pi} \equiv (v - DV)|_{x=\pi}; \ (v - DV)_x|_{x=-\pi} = (v - DV)_x|_{x=\pi}; & (1.3.6c) \\ (v - DV)|_{z=-e} \equiv (v - DV)|_{z=e}; \ (v - DV)_z|_{z=-e} = (v - DV)_z|_{z=e}. & (1.3.6d) \end{cases}$$

Thus, $(v - DV)$ satisfies the B.C.'s of the operator A_D in (1.2.2b). Then, problem (1.3.6a-b-c-d) may be rewritten abstractly as in (1.3.5), by invoking (1.2.2) for A_D and (1.3.4) for L, with v having the B.C. in (1.3.2b–d). □

Selection of structure of the control $V(x,t)$. As already noted in Section 1.1.2, henceforth we shall specialize the boundary control $V(x,z,t)$ acting on the wall

Γ_1 $(y=1)$ in (1.1.1.5j) to be J-dimensional, that is, of the (for now, open loop) form

$$\begin{cases} V(x,z,t) = \varphi_1(t)\psi_1(x,z) + \cdots + \varphi_J(t)\psi_J(x,z) & (1.3.7a) \\ \langle \psi_j, 1\rangle_{\Gamma_1} = \int_{-e}^{e}\int_{-\pi}^{\pi} \psi_j(x,z)dxdz = 0, \quad j = 1,\dots,J, & (1.3.7b) \end{cases}$$

with J to be determined. Condition (1.3.7b) follows in light of (1.1.1.5d) (i.e., the divergence-free condition).

Initial condition of intrinsic variable: $q(0)$**.** For our subsequent development, we note that

$$\begin{cases} v|_{t=0} \in \mathcal{D}((-A_D)^{\frac{1}{2}}) \subset H^1(\Omega);\ \psi_j \in H^{\frac{1}{2}}(\Gamma_1),\ \text{hence } DV|_{t=0} \in H^1(\Omega) & (1.3.8a) \\ \Rightarrow & \\ q|_{t=0} = v|_{t=0} - DV|_{t=0} \in \mathcal{D}((-A_D)^{\frac{1}{2}}) \subset H^1(\Omega). & (1.3.8b) \end{cases}$$

Indeed, $DV|_{t=0} \in H^1(\Omega)$ follows by (1.2.1g). Thus, $q|_{t=0} \in H^1(\Omega)$ and satisfies the B.C. of $\mathcal{D}((-A_D)^{\frac{1}{2}})$ in (1.2.7):

$$(q|_{t=0})|_{y=0} = (v|_{t=0})|_{y=0} - (DV|_{t=0})|_{y=0} = 0 - 0 = 0; \tag{1.3.9a}$$

$$(q|_{t=0})|_{y=1} = (v|_{t=0})|_{y=1} - (DV|_{t=0})|_{y=1} = V|_{t=0} - V|_{t=0} = 0; \tag{1.3.9b}$$

$$\begin{aligned} (q|_{t=0})|_{x=-\pi} &= (v|_{t=0})|_{x=-\pi} - (DV|_{t=0})|_{x=-\pi} \\ &= (v|_{t=0})|_{x=\pi} - (DV|_{t=0})|_{x=\pi} = (q|_{t=0})|_{x=\pi}; \end{aligned} \tag{1.3.9c}$$

$$\begin{aligned} (q|_{t=0})|_{z=-e} &= (v|_{t=0})|_{z=-e} - (DV|_{t=0})|_{z=-e} \\ &= (v|_{t=0})|_{z=e} - (DV|_{t=0})|_{z=e} = (q|_{t=0})|_{z=e}, \end{aligned} \tag{1.3.9d}$$

recalling (1.1.1.5j–n), (1.2.1a–f). Thus, (1.3.8b) is proved.

1.4 The Trace Operator D^*A_D

The following trace result holds true, with the usual proof (e.g., [L-T.1]) adapted to the present B.C.'s of A_D in (1.2.2b). Recalling the top wall Γ_1 ($y = 1$), we have:

Proposition 1.4.1. *Let $f \in \mathcal{D}(A_D)$, as defined in (1.2.2b). Then, the following trace result holds true*

$$D^*A_D f = \begin{cases} \left.\dfrac{\partial f}{\partial y}\right|_{y=1} & on\ \Gamma_1 = \{y = 1,\ -\pi \le x \le \pi\ -e \le z \le e\} \\ 0 & on\ \Gamma_0 = \{y = 0,\ -\pi \le x \le \pi,\ -e \le z \le e; \quad x = \pm\pi, \\ & \quad 0 \le y \le 1,\ -e \le z \le e; \quad z = \pm e, -\pi \le x \le \pi,\ 0 \le y \le 1\}. \end{cases} \tag{1.4.1}$$

Corollary 1.4.2. *With reference to the (normalized) eigenvectors e^i_{nm} of the operator A_D, identified in (1.2.3b), (1.2.4b–e), (1.2.5b–e); Proposition 1.4.1 specializes to the following results:*

(i)

$$D^*A_D e^0_{0m0} = \begin{cases} \dfrac{1}{\sqrt{2\pi e}}\ \dfrac{\partial}{\partial y}(\sin m\pi y)|_{y=1} = \sqrt{\dfrac{\pi}{2e}}\, m(-1)^m,\ \ m = 1, 2, \ldots & on\ \Gamma_1 \\ 0 & on\ \Gamma_0; \end{cases} \tag{1.4.2a}$$

(ii) for $n, m = 1, 2, \ldots,$

$$D^*A_D e^1_{nm0} = \begin{cases} \dfrac{1}{\sqrt{\pi e}} \sin nx\, \dfrac{\partial}{\partial y}(\sin m\pi y)|_{y=1} = \sqrt{\dfrac{\pi}{e}}\, m(-1)^m \sin nx & on\ \Gamma_1 \\ 0 & on\ \Gamma_0; \end{cases} \tag{1.4.2b}$$

(iii) for $n, m = 1, 2, \ldots,$

$$D^* A_D e^2_{nm0} = \begin{cases} \dfrac{1}{\sqrt{\pi e}} \cos nx \, \dfrac{\partial}{\partial y} \left(\sin m\pi y \right)|_{y=1} = \sqrt{\dfrac{\pi}{e}}\, m(-1)^m \cos nx & \text{on } \Gamma_1 \\ 0 & \text{on } \Gamma_0. \end{cases} \tag{1.4.2c}$$

(iv) for $m, k = 1, 2, \ldots,$

$$D^* A_D e^3_{0mk} = \begin{cases} \dfrac{1}{\sqrt{\pi e}} \sin(\dfrac{\pi}{e} kz) \, \dfrac{\partial}{\partial y} \left(\sin m\pi y \right)|_{y=1} = \sqrt{\dfrac{\pi}{e}}\, m(-1)^m \sin(\dfrac{\pi}{e} kz) & \text{on } \Gamma_1 \\ 0 & \text{on } \Gamma_0. \end{cases} \tag{1.4.2d}$$

(v) for $n, m, k = 1, 2, \ldots,$

$$D^* A_D e^4_{0mk} = \begin{cases} \dfrac{1}{\sqrt{\pi e}} \cos(\dfrac{\pi}{e} kz) \, \dfrac{\partial}{\partial y} \left(\sin m\pi y \right)|_{y=1} = \sqrt{\dfrac{\pi}{e}}\, m(-1)^m \cos(\dfrac{\pi}{e} kz) & \text{on } \Gamma_1 \\ 0 & \text{on } \Gamma_0. \end{cases} \tag{1.4.2e}$$

(vi) for $n, m, k = 1, 2, \ldots,$

$$D^* A_D e^5_{nmk} = \begin{cases} \sqrt{\dfrac{2}{\pi e}} \sin nx \; \sin(\dfrac{\pi}{e} kz) \, \dfrac{\partial}{\partial y} \left(\sin m\pi y \right)|_{y=1} = & \\ \qquad\qquad \sqrt{\tfrac{2\pi}{e}}\, m(-1)^m \sin(\tfrac{\pi}{e} kz) \sin nx & \text{on } \Gamma_1 \\ 0 & \text{on } \Gamma_0. \end{cases} \tag{1.4.2f}$$

(vii) for $n, m, k = 1, 2, \ldots,$

$$D^* A_D e^6_{nmk} = \begin{cases} \sqrt{\dfrac{2}{\pi e}} \sin nx \; \cos(\dfrac{\pi}{e} kz) \, \dfrac{\partial}{\partial y} \left(\sin m\pi y \right)|_{y=1} = & \\ \qquad\qquad \sqrt{\tfrac{2\pi}{e}}\, m(-1)^m \cos(\tfrac{\pi}{e} kz) \sin nx & \text{on } \Gamma_1 \\ 0 & \text{on } \Gamma_0. \end{cases} \tag{1.4.2g}$$

(viii) for $n, m, k = 1, 2, \ldots,$

$$D^* A_D e^7_{nmk} = \begin{cases} \sqrt{\frac{2}{\pi e}} \cos nx \, \sin(\frac{\pi}{e} kz) \, \frac{\partial}{\partial y} \, (\sin m\pi y)|_{y=1} = \\ \qquad \sqrt{\frac{2\pi}{e}} \, m(-1)^m \sin(\frac{\pi}{e} kz) \cos nx & \text{on } \Gamma_1 \\ 0 & \text{on } \Gamma_0. \end{cases} \tag{1.4.2h}$$

(ix) for $n, m, k = 1, 2, \ldots,$

$$D^* A_D e^8_{nmk} = \begin{cases} \sqrt{\frac{2}{\pi e}} \cos nx \, \cos(\frac{\pi}{e} kz) \, \frac{\partial}{\partial y} \, (\sin m\pi y)|_{y=1} = \\ \qquad \sqrt{\frac{2\pi}{e}} \, m(-1)^m \cos(\frac{\pi}{e} kz) \cos nx & \text{on } \Gamma_1 \\ 0 & \text{on } \Gamma_0. \end{cases} \tag{1.4.2i}$$

Next, using $D^* e^i_{nm} = D^* A_D A_D^{-1} e^i_{nm}$*, as well as (i)–(iii) and (1.2.3a), (1.2.4a), we obtain, with* $\lambda_{nmk} = -[n^2 + (m\pi)^2 + (k\frac{\pi}{e})^2]$*,* $n, k = 0, 1, \ldots,$ $m = 1, 2, \ldots$*: (x)*

$$D^* e^0_{0m0} = \begin{cases} \sqrt{\frac{\pi}{2e}} \, \frac{m}{\lambda_{0m0}} (-1)^m, \; m = 1, 2, \ldots & \text{on } \Gamma_1 \\ 0 & \text{on } \Gamma_0; \end{cases} \tag{1.4.3a}$$

(xi)

$$D^* e^1_{nm0} = \begin{cases} \sqrt{\frac{\pi}{e}} \, \frac{m}{\lambda_{nm0}} (-1)^m \sin nx, \; n, m = 1, 2, \ldots & \text{on } \Gamma_1 \\ 0 & \text{on } \Gamma_0; \end{cases} \tag{1.4.3b}$$

(xii)

$$D^* e^2_{nm0} = \begin{cases} \sqrt{\frac{\pi}{e}} \, \frac{m}{\lambda_{nm0}} (-1)^m \cos nx, \; n, m = 1, 2, \ldots & \text{on } \Gamma_1 \\ 0 & \text{on } \Gamma_0; \end{cases} \tag{1.4.3c}$$

(xiii)

$$D^* e^3_{0mk} = \begin{cases} \sqrt{\frac{\pi}{e}}\,\frac{m}{\lambda_{0mk}}(-1)^m \sin(\frac{\pi}{e}kz), \;\; m,k=1,2,\dots & on\ \Gamma_1 \\ \\ 0 & on\ \Gamma_0; \end{cases} \tag{1.4.3d}$$

(xiv)

$$D^* e^4_{0mk} = \begin{cases} \sqrt{\frac{\pi}{e}}\,\frac{m}{\lambda_{0mk}}(-1)^m \cos(\frac{\pi}{e}kz), \;\; m,k=1,2,\dots & on\ \Gamma_1 \\ \\ 0 & on\ \Gamma_0; \end{cases} \tag{1.4.3e}$$

(xv)

$$D^* e^5_{nmk} = \begin{cases} \sqrt{\frac{2\pi}{e}}\,\frac{m}{\lambda_{nmk}}(-1)^m \sin(\frac{\pi}{e}kz)\sin nx, \;\; n,m,k=1,2,\dots & on\ \Gamma_1 \\ \\ 0 & on\ \Gamma_0; \end{cases} \tag{1.4.3f}$$

(xvi)

$$D^* e^6_{nmk} = \begin{cases} \sqrt{\frac{2\pi}{e}}\,\frac{m}{\lambda_{nmk}}(-1)^m \cos(\frac{\pi}{e}kz)\sin nx, \;\; n,m,k=1,2,\dots & on\ \Gamma_1 \\ \\ 0 & on\ \Gamma_0; \end{cases} \tag{1.4.3g}$$

(xvii)

$$D^* e^7_{nmk} = \begin{cases} \sqrt{\frac{2\pi}{e}}\,\frac{m}{\lambda_{nmk}}(-1)^m \sin(\frac{\pi}{e}kz)\cos nx, \;\; n,m,k=1,2,\dots & on\ \Gamma_1 \\ \\ 0 & on\ \Gamma_0; \end{cases} \tag{1.4.3h}$$

(xviii)

$$D^* e^8_{nmk} = \begin{cases} \sqrt{\frac{2\pi}{e}}\,\frac{m}{\lambda_{nmk}}(-1)^m \cos(\frac{\pi}{e}kz)\cos nx, \;\; n,m,k=1,2,\dots & on\ \Gamma_1 \\ \\ 0 & on\ \Gamma_0; \end{cases} \tag{1.4.3i}$$

1.5 Orthogonal Spectral Decomposition $E = E^0 + Z$ of the state space $E \equiv L_2(\Omega)$. The v-equation (1.3.5) projected onto E^0 and Z

Henceforth, we shall often set $E \equiv L_2(\Omega)$. The eigenvectors $\{e^0_{0m0}\}^\infty_{m=1} \cup \{e^i_{nm0},\ i = 1,2\}^\infty_{n,m=1} \cup \{e^i_{0mk},\ i = 3,4\}^\infty_{m,k=1} \cup \{e^i_{nmk},\ i = 5,6,7,8\}^\infty_{n,m,k=1}$ explicitly identified in Proposition 1.2.1 form an orthonormal basis on $E \equiv L_2(\Omega)$. Thus, $E \equiv L_2(\Omega)$ decomposes as an orthogonal sum as follows:

$$L_2(\Omega) \equiv E \equiv E^0 + Z, \quad Z \equiv E^a + E^b + E^c \equiv E^{ab} + E^c \equiv E^{abc}; \tag{1.5.1a}$$

$$E^0 \equiv \overline{\text{span}}\{e^0_{0m0}\}^\infty_{m=1};\ E^i \equiv \overline{\text{span}}\{e^i_{nm0}\}^\infty_{n,m=1},\ i = 1,2; \tag{1.5.1b}$$

$$E^i \equiv \overline{\text{span}}\{e^i_{0mk}\}^\infty_{m,k=1},\ i = 3,4;\ E^i \equiv \overline{\text{span}}\{e^i_{nmk}\}^\infty_{n,m,k=1},\ i = 5,6,7,8; \tag{1.5.1c}$$

$$E^a \equiv \overline{\text{span}}\{E^1 \cup E^2\};\ \ E^b \equiv \overline{\text{span}}\{E^3 \cup E^4\};\ \ E^c \equiv \overline{\text{span}}\{E^5 \cup E^6 \cup E^7 \cup E^8\}; \tag{1.5.1d}$$

$$E^{ab} \equiv \overline{\text{span}}\{E^a \cup E^b\}; \quad Z \equiv E^{abc} \equiv \overline{\text{span}}\{E^a \cup E^b \cup E^c\}; \tag{1.5.1e}$$

with corresponding eigenvector expansions:

$$E^0 \ni \xi = \sum_{m=1}^\infty (\xi, e^0_{0m0})_\Omega\ e^0_{0m0}; \quad Z \ni g = \sum_{\substack{i=1,2\\ n,m=1}}^\infty (g, e^i_{nm0})_\Omega\ e^i_{nm0}$$

$$+ \sum_{\substack{i=3,4 \\ m,k=1}}^{\infty} (g, e^i_{0mk})_\Omega \; e^i_{0mk} \; + \sum_{\substack{i=5,6,7,8 \\ n,m,k=1}}^{\infty} (g, e^i_{nmk})_\Omega \; e^i_{nmk}, \tag{1.5.1f}$$

where $(\quad,\quad)_\Omega$ is the $L_2(\Omega)$ inner-product. We shall next (orthogonally) project the abstract v-equation (1.3.5) onto E^0 and Z. However, a further decomposition of Z into two orthogonal subspaces, one finite-dimensional, the other infinite-dimensional—will play a critical role, in the style of [Tr.1], [Tr.2].

Orthogonal decomposition: $Z \equiv Z^s_{\gamma_0} + Z^u_{\gamma_0}$ of $Z \equiv E^{abc}$ with respect to $\gamma_0 > 0$. Henceforth, we let $\gamma_0 > 1 + \pi^2 + \frac{\pi}{e}^2$ (see (1.2.4–5)) be an arbitrary positive number, fixed throughout once and for all, such that it separates the eigenvalues $\{\lambda_{nmk} = -[n^2 + (m\pi)^2 + (k\frac{\pi}{e})^2],\ n,k = 0,1,2,\dots,\ m = 1,2,3,\dots\ n > 0 \text{ or } k > 0\}$ into two disjoint sets:

(i) 3 sets of eigenvalues λ_{nmk} of finite sizes $N^a_{\gamma_0}, N^b_{\gamma_0}$, and $N^c_{\gamma_0}$ (ignoring multiplicities), identified with indices $(n,m,k) \in \mathcal{U}^a_{\gamma_0}, \mathcal{U}^b_{\gamma_0}$, and $\mathcal{U}^c_{\gamma_0}$, such that

$$-\gamma_0 < \lambda_{nmk}, \qquad \lambda_{nmk} \in E^a, \qquad (n,m,k) \in \mathcal{U}^a_{\gamma_0}; \tag{1.5.2a}$$

$$-\gamma_0 < \lambda_{nmk}, \qquad \lambda_{nmk} \in E^b, \qquad (n,m,k) \in \mathcal{U}^b_{\gamma_0}; \tag{1.5.2b}$$

$$-\gamma_0 < \lambda_{nmk}, \qquad \lambda_{nmk} \in E^c, \qquad (n,m,k) \in \mathcal{U}^c_{\gamma_0}. \tag{1.5.2c}$$

Also, set

$$\mathcal{U}_{\gamma_0} \equiv \mathcal{U}^a_{\gamma_0} \cup \mathcal{U}^b_{\gamma_0} \cup \mathcal{U}^c_{\gamma_0}; \quad \mathcal{S}_{\gamma_0} \equiv \mathcal{S}^a_{\gamma_0} \cup \mathcal{S}^b_{\gamma_0} \cup \mathcal{S}^c_{\gamma_0}, \tag{1.5.2d}$$

and

$$N_{\gamma_0} \;\equiv\; 2N^a_{\gamma_0} \;+ 2\; N^b_{\gamma_0} \;+\; 4N^c_{\gamma_0}, \tag{1.5.2e}$$

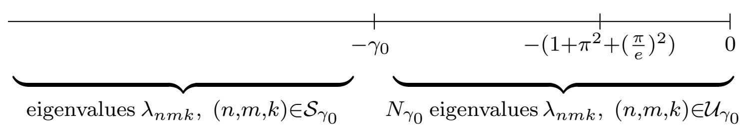
i.e. set N_{γ_0} equal to the total number of eigenvalues λ satisfying $-\gamma_0 < \lambda$, counting multiplicities.

(ii) 3 sets whose union is all the remaining infinitely many eigenvalues λ_{nmk}, identified with indices $(n, m, k) \in \mathcal{S}^a_{\gamma_0}, \mathcal{S}^b_{\gamma_0}$, and $\mathcal{S}^c_{\gamma_0}$, such that

$$\lambda_{nmk} < -\gamma_0, \qquad \lambda_{nmk} \in E^a, \qquad (n, m, k) \in \mathcal{S}^a_{\gamma_0}; \tag{1.5.2f}$$

$$\lambda_{nmk} < -\gamma_0, \qquad \lambda_{nmk} \in E^b, \qquad (n, m, k) \in \mathcal{S}^b_{\gamma_0}; \tag{1.5.2g}$$

$$\lambda_{nmk} < -\gamma_0, \qquad \lambda_{nmk} \in E^c, \qquad (n, m, k) \in \mathcal{S}^c_{\gamma_0}. \tag{1.5.2h}$$



Now we will choose an order for the eigenvalues λ_{nmk} with $(n, m, k) \in \mathcal{U}^a_{\gamma_0}, \mathcal{U}^b_{\gamma_0}, \mathcal{U}^c_{\gamma_0}$, and $\mathcal{U}_{\gamma_0}$, which will be the implied orders from here onwards (unless explicity stated otherwise).

Choose and fix an ordering for the eignevalues λ_{nmk} with $(n, m, k) \in \mathcal{U}^a_{\gamma_0}$,

$$\lambda^a_1, \lambda^a_2, \ldots, \lambda^a_{N^a_{\gamma_0}}, \tag{1.5.2i}$$

next for the eignevalues λ_{nmk} with $(n, m, k) \in \mathcal{U}^b_{\gamma_0}$,

$$\lambda^b_1, \lambda^b_2, \ldots, \lambda^b_{N^b_{\gamma_0}}, \tag{1.5.2j}$$

and finally for the eignevalues λ_{nmk} with $(n,m,k)\in\mathcal{U}^c_{\gamma_0}$,

$$\lambda^a_1, \lambda^c_2, \ldots, \lambda^c_{N^c_{\gamma_0}}. \tag{1.5.2k}$$

Let $\sigma(\lambda) \equiv (\sigma_{\mathfrak{n}}(\lambda)\sigma_{\mathfrak{m}}(\lambda)\sigma_{\mathfrak{k}}(\lambda))$ be the bijection taking eigenvalues λ of A_D to their corresponding indices (n,m,k) in terms of the notation λ_{nmk}. For example, this means $\lambda_{nmk} = \lambda_{\sigma_{\mathfrak{n}}(\lambda_{nmk})\sigma_{\mathfrak{m}}(\lambda_{nmk})\sigma_{\mathfrak{k}}(\lambda_{nmk})}$. We thus set the following ordering for the eignevectors e_{nmk} with $(n,m,k)\in\mathcal{U}_{\gamma_0}$ to be

$$\begin{gathered}
e^1_{\sigma_{\mathfrak{n}}(\lambda^a_1)\sigma_{\mathfrak{m}}(\lambda^a_1)0},\ e^2_{\sigma_{\mathfrak{n}}(\lambda^a_1)\sigma_{\mathfrak{m}}(\lambda^a_1)0},\ e^1_{\sigma_{\mathfrak{n}}(\lambda^a_2)\sigma_{\mathfrak{m}}(\lambda^a_2)0},\ e^2_{\sigma_{\mathfrak{n}}(\lambda^a_2)\sigma_{\mathfrak{m}}(\lambda^a_2)0},\\
\ldots,\ e^1_{\sigma_{\mathfrak{n}}(\lambda^a_{N^a_{\gamma_0}})\sigma_{\mathfrak{m}}(\lambda^a_{N^a_{\gamma_0}})0},\ e^2_{\sigma_{\mathfrak{n}}(\lambda^a_{N^a_{\gamma_0}})\sigma_{\mathfrak{m}}(\lambda^a_{N^a_{\gamma_0}})0},\\
e^3_{0\sigma_{\mathfrak{m}}(\lambda^b_1)\sigma_{\mathfrak{k}}(\lambda^b_1)},\ e^4_{0\sigma_{\mathfrak{m}}(\lambda^b_1)\sigma_{\mathfrak{k}}(\lambda^b_1)},\ e^3_{0\sigma_{\mathfrak{m}}(\lambda^b_2)\sigma_{\mathfrak{k}}(\lambda^b_2)},\ e^4_{0\sigma_{\mathfrak{m}}(\lambda^b_2)\sigma_{\mathfrak{k}}(\lambda^b_2)},\\
\ldots,\ e^3_{0\sigma_{\mathfrak{m}}(\lambda^b_{N^b_{\gamma_0}})\sigma_{\mathfrak{k}}(\lambda^b_{N^b_{\gamma_0}})},\ e^4_{0\sigma_{\mathfrak{m}}(\lambda^b_{N^b_{\gamma_0}})\sigma_{\mathfrak{k}}(\lambda^b_{N^b_{\gamma_0}})},\\
e^5_{\sigma_{\mathfrak{n}}(\lambda^c_1)\sigma_{\mathfrak{m}}(\lambda^c_1)\sigma_{\mathfrak{k}}(\lambda^c_1)},\ e^6_{\sigma_{\mathfrak{n}}(\lambda^c_1)\sigma_{\mathfrak{m}}(\lambda^c_1)\sigma_{\mathfrak{k}}(\lambda^c_1)},\ e^7_{\sigma_{\mathfrak{n}}(\lambda^c_1)\sigma_{\mathfrak{m}}(\lambda^c_1)\sigma_{\mathfrak{k}}(\lambda^c_1)},\ e^8_{\sigma_{\mathfrak{n}}(\lambda^c_1)\sigma_{\mathfrak{m}}(\lambda^c_1)\sigma_{\mathfrak{k}}(\lambda^c_1)},\\
e^5_{\sigma_{\mathfrak{n}}(\lambda^c_2)\sigma_{\mathfrak{m}}(\lambda^c_2)\sigma_{\mathfrak{k}}(\lambda^c_2)},\ e^6_{\sigma_{\mathfrak{n}}(\lambda^c_2)\sigma_{\mathfrak{m}}(\lambda^c_2)\sigma_{\mathfrak{k}}(\lambda^c_2)},\ e^7_{\sigma_{\mathfrak{n}}(\lambda^c_2)\sigma_{\mathfrak{m}}(\lambda^c_2)\sigma_{\mathfrak{k}}(\lambda^c_2)},\ e^8_{\sigma_{\mathfrak{n}}(\lambda^c_2)\sigma_{\mathfrak{m}}(\lambda^c_2)\sigma_{\mathfrak{k}}(\lambda^c_2)},\\
\ldots,\ e^5_{\sigma_{\mathfrak{n}}(\lambda^c_{N^c_{\gamma_0}})\sigma_{\mathfrak{m}}(\lambda^c_{N^c_{\gamma_0}})\sigma_{\mathfrak{k}}(\lambda^c_{N^c_{\gamma_0}})},\ e^6_{\sigma_{\mathfrak{n}}(\lambda^c_{N^c_{\gamma_0}})\sigma_{\mathfrak{m}}(\lambda^c_{N^c_{\gamma_0}})\sigma_{\mathfrak{k}}(\lambda^c_{N^c_{\gamma_0}})},\\
e^7_{\sigma_{\mathfrak{n}}(\lambda^c_{N^c_{\gamma_0}})\sigma_{\mathfrak{m}}(\lambda^c_{N^c_{\gamma_0}})\sigma_{\mathfrak{k}}(\lambda^c_{N^c_{\gamma_0}})},\ e^8_{\sigma_{\mathfrak{n}}(\lambda^c_{N^c_{\gamma_0}})\sigma_{\mathfrak{m}}(\lambda^c_{N^c_{\gamma_0}})\sigma_{\mathfrak{k}}(\lambda^c_{N^c_{\gamma_0}})}.
\end{gathered} \tag{1.5.2l}$$

We then set

$$\begin{aligned}
Z^{ua}_{\gamma_0} &= \operatorname{span}\{e^i_{nm0},\ i=1,2;\ (n,m,0)\in\mathcal{U}^a_{\gamma_0}\};\\
Z^{ub}_{\gamma_0} &= \operatorname{span}\{e^i_{0mk},\ i=3,4;\ (0,m,k)\in\mathcal{U}^b_{\gamma_0}\}; \qquad (1.5.3a)\\
Z^{uc}_{\gamma_0} &= \operatorname{span}\{e^i_{nmk},\ i=5,6,7,8;\ (n,m,k)\in\mathcal{U}^c_{\gamma_0}\};
\end{aligned}$$

$$
\begin{aligned}
Z_{\gamma_0}^{sa} &= \overline{\text{span}}\{e^i_{nm0},\ i=1,2;\ (n,m,0)\in \mathcal{S}_{\gamma_0}\};\\
Z_{\gamma_0}^{sb} &= \overline{\text{span}}\{e^i_{0mk},\ i=3,4;\ (0,m,k)\in \mathcal{S}^b_{\gamma_0}\}; \qquad (1.5.3b)\\
Z_{\gamma_0}^{sc} &= \overline{\text{span}}\{e^i_{nmk},\ i=5,6,7,8;\ (n,m,k)\in \mathcal{S}^c_{\gamma_0}\};
\end{aligned}
$$

$$
\begin{aligned}
Z^s_{\gamma_0} &\equiv Z^{sa}_{\gamma_0}\oplus Z^{sb}_{\gamma_0}\oplus Z^{sc}_{\gamma_0};\quad Z^u_{\gamma_0}\equiv Z^{ua}_{\gamma_0}\oplus Z^{ub}_{\gamma_0}\oplus Z^{uc}_{\gamma_0}(\text{orthoganal sum});\\
Z &\equiv Z^s_{\gamma_0}\oplus Z^u_{\gamma_0}(\text{orthogonal sum});\quad N_{\gamma_0}=\dim Z^u_{\gamma_0},
\end{aligned} \qquad (1.5.3c)
$$

so that the following eigenvector expansions hold true:

$$
\begin{aligned}
Z^u_{\gamma_0}\ni g^u &\equiv \sum_{\substack{i=1,2\\(n,m,0)\in\mathcal{U}^a_{\gamma_0}}}(g^u,e^i_{nm0})_\Omega e^i_{nm0} + \sum_{\substack{i=3,4\\(0,m,k)\in\mathcal{U}^b_{\gamma_0}}}(g^u,e^i_{0mk})_\Omega e^i_{0mk} +\\
&\qquad \sum_{\substack{i=5,6,7,8\\(n,m,k)\in\mathcal{U}^c_{\gamma_0}}}(g^u,e^i_{nmk})_\Omega e^i_{nmk};\\
Z^s_{\gamma_0}\ni g^s &\equiv \sum_{\substack{i=1,2\\(n,m,0)\in\mathcal{S}^a_{\gamma_0}}}(g^s,e^i_{nm0})_\Omega e^i_{nm0} + \sum_{\substack{i=3,4\\(0,m,k)\in\mathcal{S}^b_{\gamma_0}}}(g^s,e^i_{0mk})_\Omega e^i_{0mk} +\\
&\qquad \sum_{\substack{i=5,6,7,8\\(n,m,k)\in\mathcal{S}^c_{\gamma_0}}}(g^s,e^i_{nmk})_\Omega e^i_{nmk}.
\end{aligned} \qquad (1.5.3d)
$$

Corresponding orthogonal projectors. In the sequel, we shall introduce the following orthogonal projectors. On $L_2(\Omega)\equiv E$:

$$
\begin{cases}
\Pi^0 = \text{orthogonal projector: } L_2(\Omega)\equiv E \text{ onto } E^0, & (1.5.4a)\\
(I-\Pi^0) = \text{orthogonal projector: } L_2(\Omega)\equiv E \text{ onto } E^{abc}\equiv Z, & (1.5.4b)
\end{cases}
$$

$I=$ identity on E, with E^0 and Z defined in (1.5.1b–e); and on Z:

$$
\begin{cases}
P^u_{\gamma_0} = \text{orthogonal projector } Z\equiv E^{abc} \text{ onto } Z^u_{\gamma_0} & (1.5.5a)\\
P^s_{\gamma_0}\equiv (I_Z-P^u_{\gamma_0}) = \text{orthogonal projector } Z\equiv E^{abc} \text{ onto } Z^u_{\gamma_0}. & (1.5.5b)
\end{cases}
$$

The projectors Π^0 and $P^u_{\gamma_0}$ commute with A_D. The indices "s," "$\mathcal{S}$," and "u," "$\mathcal{U}$" may be recalled as "satisfactory" (or very "stable") and "unsatisfactory" (or "unstable" by abuse of language).

Projected versions of v-equation (1.3.5). First, onto E^0, Z, and then onto $Z^u_{\gamma_0}$ and $Z^s_{\gamma_0}$. We orthogonally project the abstract v-equation (1.3.5) first onto E^0 and Z. We obtain:

$$\text{on } E^0: \ [\Pi^0(v - DV)]_t = \nu A^0_D[\Pi^0(v - DV)] - \Pi^0 Lv - \nu \Pi^0 D(V_{xx} + V_{zz}), \quad (1.5.6a)$$

$$A^0_D = A_D|_{E^0} = \text{restriction of } A_D \text{ on } E^0; \quad A_D|_Z = \text{restriction of } A_D \text{ on } Z; \quad (1.5.6b)$$

$$\text{on } Z: \ [(I - \Pi^0)(v - DV)]_t$$
$$= \nu A_D|_Z[(I - \Pi^0)(v - DV)] - (I - \Pi^0)Lv - \nu(I - \Pi^0)D(V_{xx} + V_{zz}) \quad (1.5.7)$$

(throughout, I denotes the identity on $L_2(\Omega) \equiv E$).

Orientation. The dynamics (1.5.6a) on E^0 will be studied in Section 1.6 in the present case of the linearized model (1.1.1.5), considered up to now. Instead, as to the complementary dynamics (1.5.7) on Z, we shall proceed as follows, as already outlined in the introduction. In the present paper, we shall consider the special case $a \equiv b \equiv 0$: this is the linear version of the linearized model (1.1.1.5), whereby then $L \equiv 0$. Here we shall test and employ a technique which will yield a preassigned enhancement of the margin of stability of the dynamic (1.5.7) (with $L \equiv 0$) on Z, through an explicit and fully verifiable analysis. This will require just $J = 4$ controllers $\psi_1(x, z), \psi_2(x, z), \psi_3(x, z)$,and $\psi_4(x, z)$ with arbitrarily small support on $-\pi \leq x \leq \pi$ and $-e \leq z \leq e$ (that is, on Γ_1), subject to traditional 'rank conditions,' even though the model is not traditional, as it contains the control V, its time derivative V_t and its space Laplacian $V_{xx} + V_{zz}$. The reason why four controllers, $J = 4$, are sufficient is because four is the maximum algebraic/geometric multiplicity of the eigenvalues λ_{nmk} in (1.2.4–5) on Z.

The analysis on the space Z of eq. (1.5.7) in the original linearized case $a \not\equiv 0$ and $b \not\equiv 0$ in (1.1.1.5) will then be treated in a separate paper, to take advantage—though in a less transparent form—of the technique tested in the present paper. □

Now, however, we provide a further property of the operator D, which will allow us to simplify the Z-dynamics (1.5.7).

Proposition 1.5.1. *Let $g \in L_2(\Gamma_1)$ satisfy the orthogonality condition:*

$$\langle g, 1\rangle_{\Gamma_1} = \int_{-e}^{e} \int_{-\pi}^{\pi} g(x, z)\ dxdz = 0, \tag{1.5.8a}$$

where $\langle \quad , \quad \rangle_{\Gamma_1}$ is the $L_2(\Gamma_1)$-inner product. Then:

(a)

$$Dg \in Z \equiv E^{abc}, \tag{1.5.8b}$$

so that

$$\Pi^0 Dg = 0 \quad \textit{and} \quad (I - \Pi^0)Dg = Dg, \tag{1.5.8c}$$

where the Dirichlet map D is defined in (1.2.1).

(b) The following eigenvector expansion holds true:

$$\begin{aligned} Dg \;=\; & \sum_{n,m=1}^{\infty} \sqrt{\frac{\pi}{e}}\,\frac{m}{\lambda_{nm0}}(-1)^m \langle g, \sin nx\rangle_{\Gamma_1} e^1_{nm0} + \\ & \sum_{n,m=1}^{\infty} \sqrt{\frac{\pi}{e}}\,\frac{m}{\lambda_{nm0}}(-1)^m \langle g, \cos nx\rangle_{\Gamma_1} e^2_{nm0} + \\ & \sum_{m,k=1}^{\infty} \sqrt{\frac{\pi}{e}}\,\frac{m}{\lambda_{0mk}}(-1)^m \langle g, \sin \frac{\pi}{e}kz\rangle_{\Gamma_1} e^3_{0mk} + \\ & \sum_{m,k=1}^{\infty} \sqrt{\frac{\pi}{e}}\,\frac{m}{\lambda_{0mk}}(-1)^m \langle g, \cos \frac{\pi}{e}kz\rangle_{\Gamma_1} e^4_{0mk} + \\ & \sum_{n,m,k=1}^{\infty} \sqrt{\frac{2\pi}{e}}\,\frac{m}{\lambda_{nmk}}(-1)^m \langle g, \sin nx \sin \frac{\pi}{e}kz\rangle_{\Gamma_1} e^5_{nmk} + \\ & \sum_{n,m,k=1}^{\infty} \sqrt{\frac{2\pi}{e}}\,\frac{m}{\lambda_{nmk}}(-1)^m \langle g, \sin nx \cos \frac{\pi}{e}kz\rangle_{\Gamma_1} e^6_{nmk} + \\ & \sum_{n,m,k=1}^{\infty} \sqrt{\frac{2\pi}{e}}\,\frac{m}{\lambda_{nmk}}(-1)^m \langle g, \cos nx \sin \frac{\pi}{e}kz\rangle_{\Gamma_1} e^7_{nmk} + \\ & \sum_{n,m,k=1}^{\infty} \sqrt{\frac{2\pi}{e}}\,\frac{m}{\lambda_{nmk}}(-1)^m \langle g, \cos nx \cos \frac{\pi}{e}kz\rangle_{\Gamma_1} e^8_{nmk}. \end{aligned} \tag{1.5.9}$$

Proof. (a) Recalling Π^0 in (1.5.4a) and E^0 in (1.5.1b), we shall show that $\Pi^0 Dg = 0$. In fact, by (1.5.1b),

$$\Pi^0 Dg = \sum_{m=1}^{\infty} (Dg, e^0_{0m0})_\Omega \; e^0_{0m0} = \sum_{m=1}^{\infty} \langle g, D^* e^0_{0m0}\rangle_{\Gamma_1} e^0_{0m0} \tag{1.5.10}$$

$$\text{(by (1.4.3a))} \quad = \sum_{m=1}^{\infty} \frac{1}{\sqrt{2}}\,\frac{m}{\lambda_{0m0}}\,(-1)^m \langle g, 1\rangle_{\Gamma_1} e^0_{0m0} = 0, \tag{1.5.11}$$

as desired, using (1.4.3a) and the assumption $\langle g, 1\rangle_{\Gamma_1} = 0$ in (1.5.8a).

(b) Similarly, using this time (1.4.3b–i), we readily obtain (1.5.9). □

Henceforth, we introduce the new variable

$$q = (I - \Pi^0)v \in Z, \tag{1.5.12}$$

and obtain the following simplifications on the projected dynamics (1.5.7) on Z, $\psi_j(x, z)$, $j = 1, 2, \dots, J$ in the definition of $V(x, z, t)$ in (1.3.7), selected as to satisfy: $\langle \psi_j, 1 \rangle_{\Gamma_1} = 0$, $j = 1, 2, \dots, J$, as in Proposition 1.5.1 and as required in (1.3.7b).

Corollary 1.5.2. *With reference to the control V in (1.3.7), let $\psi_j \in H^2(\Gamma_1)$, $j = 1, 2, \dots, J$, satisfy $\langle \psi_j, 1 \rangle_{\Gamma_1} \equiv 0$ and $\langle (\partial_x x + \partial_z z)\psi_j, 1 \rangle_{\Gamma_1} \equiv 0$. Set $\Psi = [\psi_1, \dots, \psi_J]$. Then, recalling (1.3.7):*

(a)

$$\Pi^0 D\Psi = 0; \quad \Pi^0 DV = 0; \quad \Pi^0 D(V_{xx} + V_{zz}) = 0; \tag{1.5.13a}$$

$$(I - \Pi^0)D\Psi = D\Psi; \quad (I - \Pi^0)DV = DV; \quad (I - \Pi^0)D(V_{xx} + V_{zz}) = D(V_{xx} + V_{zz}), \tag{1.5.13b}$$

and recalling also (1.5.12)

$$(I - \Pi^0)(v - DV) = q - DV. \tag{1.5.13c}$$

(b) The Z-equation (1.5.7) can be rewritten more simply as

$$\text{on } Z: \ [q - DV]_t = \nu A_D|_Z[q - DV] - (I - \Pi^0)Lv - \nu D(V_{xx} + V_{zz}). \tag{1.5.14}$$

Proof. We use Proposition 1.5.1(a) and (1.5.12). □

1.6 The Projection $\Pi^0(v-DV) \equiv \Pi^0 v$ on E^0 in (1.5.6a) is Intrinsically Exponentially Stable with Rate $(\nu\pi^2)$

In this section we study the dynamics (1.5.6a) on E^0 in the variable $\Pi^0(v-DV) = \Pi^0 v$ by (1.5.13a); and show that, on E^0, the operator L in (1.3.4) acts like the zero operator. As a consequence, $\Pi^0 v(t)$ is intrinsically exponentially stable, see (1.6.6) below.

Proposition 1.6.1. *(a) With reference to the operators L defined in (1.3.4) and Π^0 in (1.5.4a), we have*

$$\Pi^0 Lv = 0, \qquad v \in \mathcal{D}(L). \tag{1.6.1}$$

(b) Thus, under the conditions $\langle \psi_j, 1\rangle_{\Gamma_1} \equiv 0$, $\langle (\partial_{xx} + \partial_{zz})\psi_j, 1\rangle_{\Gamma_1} \equiv 0$, $j = 1, \ldots, J$ (see (1.3.7b) and (1.1.1.5t)), the dynamics (1.5.6a) on E^0 reduces to

$$\text{on } E^0: \ \Pi^0 v_t = \nu A_D^0 \Pi^0 v, \tag{1.6.2}$$

whose solution is

$$\text{on } E^0: \ \Pi^0 v(t) = e^{\nu A_D^0 t}\Pi^0 v(0); \tag{1.6.3}$$

$$e^{\nu A_D^0 t}\xi = \sum_{m=1}^{\infty} e^{\nu \lambda_{0m0} t}(\xi, e_{0m0}^0)_\Omega \ e_{0m0}^0, \ t \geq 0, \ \xi \in E^0. \tag{1.6.4}$$

Thus,

$$\|\Pi^0 v(t)\|_{E^0} = \|e^{\nu A_D^0 t}\Pi^0 v(0)\|_{E^0} \tag{1.6.5}$$

$$\leq e^{-\nu\pi^2 t}\|\Pi^0 v(0)\|_{E^0},\ t \geq 0. \tag{1.6.6}$$

(c) Moreover, if $v(0) \in \mathcal{D}((-A_D)^{\frac{1}{2}}) \subset H^1(\Omega)$ *(see (1.2.5)), or* $\Pi^0 v(0) \in \mathcal{D}((-A_D^0)^{\frac{1}{2}})$*, then*

$$\|\Pi^0 v(t)\|_{\mathcal{D}((-A_D^0)^{\frac{1}{2}})} = \|\Pi^0 v(t)\|_{H^1(\Omega)} = \|e^{\nu A_D^0 t}\Pi^0(-A_D)^{\frac{1}{2}} v(0)\|_{\mathcal{D}((-A_D^0)^{\frac{1}{2}})} \tag{1.6.7}$$

$$\leq e^{-\nu\pi^2 t}\|\Pi^0 v(0)\|_{\mathcal{D}((-A_D^0)^{\frac{1}{2}})} = e^{-\nu\pi^2 t}\|\Pi^0 v(0)\|_{H^1(\Omega)}. \tag{1.6.8}$$

Proof. (a) Recalling the definition of the operator L in (1.3.4), we need to show that:

$$\Pi^0(\partial_x(\overline{U}(y)v) + \partial_z(\overline{W}(y)v)) = 0; \tag{1.6.9a}$$

$$\Pi^0 A_D^{-1}(\partial_x[\partial_y(\overline{U}'(y)v)] + \partial_z[\partial_y(\overline{W}'(y)v)]) = 0,\ \forall\ v \in \mathcal{D}(L), \tag{1.6.9b}$$

after which identity (1.6.1) follows from (1.3.4).

Proof of LHS identity in (1.6.9). Invoking the expansion (1.5.1f) (LHS) on E^0, with $\xi = \Pi^0(\partial_x(\overline{U}(y)v) + \partial_z(\overline{W}(y)v))$, $v \in \mathcal{D}(L)$, we obtain as $(\Pi^0)^* = \Pi^0$ and

$\Pi^0 e^0_{0m0} = e^0_{0m0}$:

$$\Pi^0(\partial_x(\overline{U}v) + \partial_z(\overline{W}v)) = \sum_{m=1}^{\infty}(\Pi^0(\partial_x(\overline{U}v) + \partial_z(\overline{W}v)), e^0_{0m0})_\Omega \; e^0_{0m0} \tag{1.6.10}$$

$$= \sum_{m=1}^{\infty}(\partial_x(\overline{U}v) + \partial_z(\overline{W}v), e^0_{0m0})_\Omega \; e^0_{0m0} \tag{1.6.11}$$

$$= -\sum_{m=1}^{\infty}[(\overline{U}v, \partial_x e^0_{0m0})_\Omega + (\overline{W}v, \partial_z e^0_{0m0})_\Omega] \; e^0_{0m0} \equiv 0, \tag{1.6.12}$$

as desired, as the vanishing in (1.6.12) occurs since $\partial_x e^0_{0m0} \equiv \partial_z e^0_{0m0} \equiv 0$ (the e^0_{0m0} do not depend on x or z, see (1.2.3b). In the integration by parts in x and z leading from (1.6.11) to (1.6.12), the boundary terms vanish due to the periodic conditions in x and z of $v \in \mathcal{D}(L)$: $v(-\pi, y, z, t) = v(\pi, y, z, t)$ and $v(x, y, -e, t) = v(x, y, e, t)$, see (1.3.4a); while e^0_{0m0} is constant in x and z, so that $[(\overline{U}(y)v, e^0_{0m0})]^{x=\pi}_{x=-\pi} = 0$ and $[(\overline{W}(y)v, e^0_{0m0})]^{z=e}_{z=-e} = 0$.

Proof of RHS identity in (1.6.9). Specializing now the expansion (1.5.1f) (LHS) with $\xi = \Pi^0 A_D^{-1}(\partial_x[\partial_y \overline{U}'(y)v] + \partial_z[\partial_y \overline{W}'(y)v])$, $v \in \mathcal{D}(L)$, we likewise obtain since $A_D^{-1} e^0_{0m0} = \frac{1}{\lambda_{0m0}} e^0_{0m0}$ by (1.2.3a), with A_D self-adjoint by Proposition 1.2.1:

$$\Pi^0 A_D^{-1}(\partial_x[\partial_y \overline{U}'(y)v] + \partial_z[\partial_y \overline{W}'(y)v])$$

$$= \sum_{m=1}^{\infty}(\Pi^0 A_D^{-1}(\partial_x[\partial_y \overline{U}'(y)v] + \partial_z[\partial_y \overline{W}'(y)v]), e^0_{0m0})_\Omega \; e^0_{0m0} \tag{1.6.13}$$

$$= \sum_{m=1}^{\infty}\frac{1}{\lambda_{0m0}}(\partial_x[\partial_y \overline{U}'(y)v] + \partial_z[\partial_y \overline{W}'(y)v], \; e^0_{0m0})_\Omega \; e^0_{0m0} \tag{1.6.14}$$

$$= -\sum_{m=1}^{\infty}\frac{1}{\lambda_{0m0}}\Big(([\partial_y(\overline{U}'v)], \partial_x e^0_{0m0})_\Omega + (\partial_y(\overline{W}'v), \partial_z e^0_{0m0})_\Omega\Big) e^0_{0m0} \equiv 0, \tag{1.6.15}$$

as desired, again since $\partial_x e^0_{0m0} \equiv \partial_z e^0_{0m0} \equiv 0$. Again, in the integration by parts in x and z leading from (1.6.14) to (1.6.15), the boundary terms vanish, due to periodic conditions in x and z of $v \in \mathcal{D}(L)$: $v(-\pi, y, z, t) = v(\pi, y, z, t)$ and $v(x, y, -e, t) = v(x, y, e, t)$, see (1.3.4a), and hence of v_y : $v_y(-\pi, y, z, t) = v_y(-\pi, y, z, t)$ and $v_y(x, y, -e, t) = v_y(x, y, e, t)$; while e^0_{0m0} is constant in x and z, so that $\left[(\partial_y(\overline{U}'(y)v)], e^0_{0m0})_\Omega\right]_{x=-\pi}^{x=\pi} = 0$ and $\left[(\partial_y(\overline{U}'(y)v)], e^0_{0m0})_\Omega\right]_{z=-e}^{z=e} = 0$.

In conclusion, identities (1.6.9) imply identity (1.6.1), by (1.3.4b).

(b) Part (b) is an immediate consequence of Part (a) and (1.5.13a): $\Pi^0 DV = 0$, $\Pi^0 D(V_{xx} + V_{zz}) = 0$. Thus, (1.5.6a) leads to (1.6.2), whose solution is plainly (1.6.3) and satisfies the exponential bound (1.6.6), since $\lambda_{0m0} = -(m\pi)^2$, $m = 1, 2, \ldots$ by (1.2.3b). Part (c) is self-explanatory. □

Proposition 1.6.2. *Let v be the normal velocity component in (1.1.1.5/b), Π^0 the projector in (1.5.4a) and L the first-order differential operator in (1.3.4). Then*

(a)

$$L\Pi^0 v = 0. \tag{1.6.16}$$

(b) Let $\langle \psi_j, 1\rangle_{\Gamma_1} \equiv \langle (\partial_{xx}+\partial_{zz})\psi_j, 1\rangle_{\Gamma_1} \equiv 0$, $j = 1, \ldots, J$. Then, Z-equation (1.5.14) simplifies to

$$\text{on } Z: \ [q - DV]_t = \nu A_D|_Z[q - DV] - (I - \Pi^0)Lq - \nu D(V_{xx} + V_{zz}). \tag{1.6.17}$$

Proof. (a) To begin with, we have $Le^0_{0m0} \equiv 0$, $m = 1, 2, \ldots$. This is so due to the form of L in (1.3.4b) which includes ∂_x and ∂_z, while the eigenvectors e^0_{0m0} in (1.2.3b) are independent of x and z. Then, by definition of Π^0, we have

$$L\Pi^0 v = \lim_{M\to\infty} \sum_{m=1}^{M} (\Pi^0 v, e^0_{0m0})_\Omega L e^0_{0m0} = 0. \tag{1.6.18}$$

(b) We write $v = \Pi^0 v + (I-\Pi^0)v$, hence $Lv = L\Pi^0 v + L(I-\Pi^0)v = Lq$, by (1.6.16) and (1.5.12). Thus, under present assumptions, (1.5.14) simplifies to (1.6.17). □

1.7 Preliminary Analysis of the Dynamics (1.5.14) on Z in the Linear Case $a \equiv b \equiv 0$ in (1.1.1.5)

Henceforth, in this paper we shall focus on the *linear* model (1.1.1.5) with $a \equiv b \equiv 0$ (see introduction and orientation in Section 1.5), rewritten here for convenience:

$$\begin{cases} u_t - \nu\Delta u = p_x \quad \text{in } Q; & (1.7.1a) \\ v_t - \nu\Delta v = p_y \quad \text{in } Q; & (1.7.1b) \\ w_t - \nu\Delta w = p_z \quad \text{in } Q; & (1.7.1c) \\ u_x + v_y + w_z \equiv 0 \quad \text{in } Q; & (1.7.1d) \end{cases}$$

$$\text{B.C. for } u: \begin{cases} u_y(x,0,z,t) \equiv 0, \; u_y(x,1,z,t) \equiv 0; & (1.7.1e) \\ u(-\pi,y,z,t) = u(\pi,y,z,t); & (1.7.1f) \\ u_x(-\pi,y,z,t) \equiv u_x(\pi,y,z,t); & (1.7.1g) \\ u(x,y,-e,t) = u(x,y,e,t); & (1.7.1h) \\ u_z(x,y,-e,t) \equiv u_z(\pi,y,e,t); & (1.7.1i) \end{cases}$$

$$\text{B.C. for } v: \begin{cases} v(x,0,z,t) \equiv 0, \;\; v(x,1,z,t) = V(x,z,t); & (1.7.1j) \\ v(-\pi,y,z,t) \equiv v(\pi,y,z,t); & (1.7.1k) \\ v_x(-\pi,y,z,t) \equiv v_x(\pi,y,z,t); & (1.7.1l) \\ v(x,y,-e,t) \equiv v(x,y,e,t); & (1.7.1m) \\ v_z(x,y,-e,t) \equiv v_z(\pi,y,e,t); & (1.7.1n) \end{cases}$$

$$\text{B.C. for } v: \begin{cases} w(x,0,z,t) \equiv 0, \;\; w(x,1,z,t) \equiv 0; & (1.7.1o) \\ w(-\pi,y,z,t) \equiv w(\pi,y,z,t); & (1.7.1p) \\ w_x(-\pi,y,z,t) \equiv w_x(\pi,y,z,t); & (1.7.1q) \\ w(x,y,-e,t) \equiv w(x,y,e,t); & (1.7.1r) \\ w_z(x,y,-e,t) \equiv w_z(\pi,y,e,t). & (1.7.1s) \end{cases}$$

In this case, where ($a \equiv b \equiv 0$ hence) $L \equiv 0$, the projection $q = (I - \Pi^0)v$ in (1.5.12) satisfies the following specialization of the abstract dynamics (1.5.14) or (1.6.17) on the space Z:

$$\text{on } Z : [q - DV]_t = \nu A_D[q - DV] - \nu D(V_{xx} + V_{zz}), \tag{1.7.2}$$

under the assumptions $\langle \psi_j, 1 \rangle_{\Gamma_1} = \langle (\partial_{xx} + \partial_{zz})\psi_j, 1 \rangle_{\Gamma_1} \equiv 0$ of Corollary 1.5.2 or Proposition 1.6.2(b), where actually A_D is restricted over Z.

The intrinsic variable $h = q - DV$. The abstract eq. (1.7.2) suggests the introduction of a new variable

$$h(t) \equiv q(t) - DV(t) = (I - \Pi^0)(v - DV) \in Z, \tag{1.7.3}$$

recalling $q = (I - \Pi^0)v$ from (1.5.12) and $(I - \Pi^0)DV = DV$ from (1.5.13b), for V having the finite-dimensional structure (1.3.7), with $\langle \psi_j, 1 \rangle_{\Gamma_1} \equiv 0,\ \langle (\partial_{xx} + \partial_{zz})\psi_j, 1 \rangle_{\Gamma_1} \equiv 0$, as required by Corollary 1.5.2. With respect to the intrinsic variable h in (1.7.3), we rewrite the abstract dynamics (1.7.2) as

$$\text{on } Z: \ h_t = \nu A_D h - \nu D(V_{xx} + V_{zz}), \qquad h(0) = q(0) - DV(0). \tag{1.7.4}$$

Eq. (1.7.4) requires the I.C. $q(0) = (I - \Pi^0)v(0)$, as well as *the value* $V(0)$ *of the control variable at* $t = 0$. Instead, $(V_{xx} + V_{zz}) = \sum_{j=1}^{J} (\partial_{xx} + \partial_{zz})\psi_j(x,z)\varphi_j(t)$ acts as the control on eq. (1.7.4).

Goal. Regarding eq. (1.7.4), we pose the following problem. Given the data $\{\nu > 0, v(0), V(0)\}$ and an arbitrary positive number $\gamma_0 > 0$ as in (1.5.2), we seek to express the control $(V_{xx} + V_{zz})$ in a suitable finite-dimensional feedback form of h (given quantitatively below in (1.7.7)), so that we have semigroup well-posedness and the resulting feedback dynamics decays exponentially to zero with rate $(\nu\gamma_0)$. □

To this end, we return to the orthogonal projectors $P^u_{\gamma_0}$ and $P^s_{\gamma_0}$ in (1.5.5a–b), and corresponding complementary subspaces $Z^u_{\gamma_0}$ and $Z^s_{\gamma_0}$ of Z in (1.5.3c), and, next, we further orthogonally project eq. (1.7.4) onto these, thus obtaining:

$$\text{on } Z^u_{\gamma_0}: [P^u_{\gamma_0} h]_t = \nu A^u_D [P^u_{\gamma_0} h] - \nu P^u_{\gamma_0} D(V_{xx} + V_{zz}); \tag{1.7.5a}$$

$$A_D^u = A_D|_{Z_{\gamma_0}^u} = \text{restriction of } A_D \text{ on } Z_{\gamma_0}^u; \tag{1.7.5b}$$

$$\text{on } Z_{\gamma_0}^s : [P_{\gamma_0}^s h]_t = \nu A_D^s [P_{\gamma_0}^s h] - \nu P_{\gamma_0}^s D(V_{xx} + V_{zz}); \tag{1.7.6a}$$

$$A_D^s = A_D|_{Z_{\gamma_0}^s} = \text{restriction of } A_D \text{ on } Z_{\gamma_0}^s; \tag{1.7.6b}$$

$$e^{\nu A_D^s t}\chi = \sum_{\substack{i=1,2 \\ (n,m,0)\in\mathcal{S}_{\gamma_0}^a}} e^{-\nu[n^2+(m\pi)^2]t}(\chi, e_{nm0}^i)_{L_2(\Omega)} e_{nm0}^i + \sum_{\substack{i=3,4 \\ (0,m,k)\in\mathcal{S}_{\gamma_0}^b}} e^{-\nu[(m\pi)^2+(\frac{\pi}{e})^2]t}(\chi, e_{0mk}^i)_{L_2(\Omega)} e_{0mk}^i + \sum_{\substack{i=5,6,7,8 \\ (n,m,k)\in\mathcal{S}_{\gamma_0}^c}} e^{-\nu[n^2+(m\pi)^2+(\frac{\pi}{e})^2]t}(\chi, e_{nmk}^i)_{L_2(\Omega)} e_{nmk}^i, \quad \chi \in Z_{\gamma_0}^s, \tag{1.7.6c}$$

as we recall (below (1.5.5b)) that the eigenvectors-based projectors $P_{\gamma_0}^u$ and $P_{\gamma_0}^s$ commute with A_D.

Selected finite-dimensional feedback control. The open-loop original control V in (1.3.7a–b), is hereby specialized as a feedback closed loop control $V_{xx} + V_{zz}$ for eq. (1.7.4), as follows (again letting $\Lambda = \partial_{xx} + \partial_{zz}$):

$$(V_{xx}+V_{zz})(x,z,t) = [\Lambda\psi_1(x,z), \Lambda\psi_2(x,z), \ldots, \Lambda\psi_J(x,z)] \begin{bmatrix} \varphi_1(t) \\ \varphi_2(t) \\ \vdots \\ \varphi_J(t) \end{bmatrix} = (\Lambda\Psi(x,z)F_{\gamma_0}^u)P_{\gamma_0}^u h(t); \tag{1.7.7}$$

$$\Lambda\Psi(x,z) = [\Lambda\psi_1(x,z),\dots,\Lambda\psi_J(x,z)];\;\; \Phi(t) = \begin{bmatrix} \varphi_1(t) \\ \vdots \\ \varphi_J(t) \end{bmatrix} = F^u_{\gamma_0} P^u_{\gamma_0} h(t),\; \langle \Lambda\psi_j, 1\rangle_{\Gamma_1} \equiv 0, \tag{1.7.8}$$

for an operator $F^u_{\gamma_0} \in \mathcal{L}(Z^u_{\gamma_0}; \mathbb{R}^J)$ or a matrix $F^u_{\gamma_0} : J \times N_{\gamma_0}$, $N_{\gamma_0} = \dim\, Z^u_{\gamma_0}$, see (1.5.3c). Thus, via (1.7.7), (1.7.8), the finite-dimensional control $V_{xx} + V_{zz}$ on eq. (1.7.2) is expressed in feedback form in terms of the *intrinsic variable* $h(t)$ in (1.7.3). Accordingly, we rewrite the finite-dimensional system (1.7.5a), under the standing assumptions $\langle \psi_j, 1\rangle_{\Gamma_1} \equiv \langle (\partial_{xx} + \partial_{zz})\psi_j, 1\rangle_{\Gamma_1} \equiv 0$, as

$$\text{on } Z^u_{\gamma_0} :\;\; [P^u_{\gamma_0} h]_t = \nu A^u_D [P^u_{\gamma_0} h] + \nu B^u_{\gamma_0} \Phi(t),\;\; \Phi(t) = \begin{bmatrix} \varphi_1(t) \\ \varphi_2(t) \\ \varphi_3(t) \\ \varphi_4(t) \end{bmatrix}, \tag{1.7.9}$$

with $J = 4$, where we have set

$$B^u_{\gamma_0} \equiv -P^u_{\gamma_0} D(\partial_{xx} + \partial_{zz})\Psi(x,z) \in \mathcal{L}(\mathbb{R}^J; Z^u_{\gamma_0}), \quad J = 4. \tag{1.7.10}$$

In the order chosen in (1.5.2l) for the orthonormal basis of eigenvectors e_{nmk} of the operator A^u_D for $(n,m,k) \in \mathcal{U}_{\gamma_0}$, A^u_D has the following matrix representation (see

1.5.2i–k)), given in block diagonal form:

$$A_D^u = \begin{bmatrix} A_1 & & & & & & & & & & & 0 \\ & A_2 & & & & & & & & & & \\ & & \ddots & & & & & & & & & \\ & & & A_{N_{\gamma_0}^a} & & & & & & & & \\ & & & & B_1 & & & & & & & \\ & & & & & B_2 & & & & & & \\ & & & & & & \ddots & & & & & \\ & & & & & & & B_{N_{\gamma_0}^b} & & & & \\ & & & & & & & & C_1 & & & \\ & & & & & & & & & C_2 & & \\ & 0 & & & & & & & & & \ddots & \\ & & & & & & & & & & & C_{N_{\gamma_0}^c} \end{bmatrix} \tag{1.7.11a}$$

with the diagonal submatrices A_i, B_i, and C_i are given by

$$A_i = \begin{bmatrix} \lambda_i^a & 0 \\ 0 & \lambda_i^a \end{bmatrix}, \tag{1.7.11b}$$

$$B_i = \begin{bmatrix} \lambda_i^b & 0 \\ 0 & \lambda_i^b \end{bmatrix}, \tag{1.7.11c}$$

$$C_i = \begin{bmatrix} \lambda_i^c & 0 & 0 & 0 \\ 0 & \lambda_i^c & 0 & 0 \\ 0 & 0 & \lambda_i^c & 0 \\ 0 & 0 & 0 & \lambda_i^c \end{bmatrix}. \tag{1.7.11d}$$

As noted in (1.7.10), henceforth, we select $J = 4$ so that (the original control V in (1.3.7) as acting on (1.1.1.5j) or (1.7.1j) or the control $V_{xx} + V_{zz}$ in (1.7.7) as acting on eq. (1.7.2) are four-dimensional. Accordingly, we have

Proposition 1.7.1. *Let $J = 4$, $\Psi = [\psi_1, \psi_2, \psi_3, \psi_4]$, where, by Corollary 5.2, $\langle \psi_j, 1 \rangle_{\Gamma_1} \equiv \langle (\partial_{xx} + \partial_{zz})\psi_j, 1 \rangle_{\Gamma_1} \equiv 0$. Again, let $\Lambda = \partial_{xx} + \partial_{zz}$. With reference to the same orthonormal basis of eigenvectors of A_D^u used for (1.7.11), the matrix representation of the operator $B_{\gamma_0}^u$ in (1.7.10) is (with the matrix split with the first two columns on one page and then the second two columns on the following page):*

$$B_{\gamma_0}^u =$$

$$
\left[\begin{array}{c|c}
\tau_1(\lambda_1^a)\langle\Lambda\psi_1,\sin\sigma_{\mathfrak{n}}(\lambda_1^a)x\ \rangle & \tau_1(\lambda_1^a)\langle\Lambda\psi_2,\sin\sigma_{\mathfrak{n}}(\lambda_1^a)x\ \rangle \\
\tau_1(\lambda_1^a)\langle\Lambda\psi_1,\cos\sigma_{\mathfrak{n}}(\lambda_1^a)x\ \rangle & \tau_1(\lambda_1^a)\langle\Lambda\psi_2,\cos\sigma_{\mathfrak{n}}(\lambda_1^a)x\ \rangle \\
\tau_1(\lambda_2^a)\langle\Lambda\psi_1,\sin\sigma_{\mathfrak{n}}(\lambda_2^a)x\ \rangle & \tau_1(\lambda_2^a)\langle\Lambda\psi_2,\sin\sigma_{\mathfrak{n}}(\lambda_2^a)x\ \rangle \\
\tau_1(\lambda_2^a)\langle\Lambda\psi_1,\cos\sigma_{\mathfrak{n}}(\lambda_2^a)x\ \rangle & \tau_1(\lambda_2^a)\langle\Lambda\psi_2,\cos\sigma_{\mathfrak{n}}(\lambda_2^a)x\ \rangle \\
\cdots & \cdots \\
\tau_1(\lambda^a_{N^a_{\gamma_0}})\langle\Lambda\psi_1,\sin\sigma_{\mathfrak{n}}(\lambda^a_{N^a_{\gamma_0}})x\ \rangle & \tau_1(\lambda^a_{N^a_{\gamma_0}})\langle\Lambda\psi_2,\sin\sigma_{\mathfrak{n}}(\lambda^a_{N^a_{\gamma_0}})x\ \rangle \\
\tau_1(\lambda^a_{N^a_{\gamma_0}})\langle\Lambda\psi_1,\cos\sigma_{\mathfrak{n}}(\lambda^a_{N^a_{\gamma_0}})x\ \rangle & \tau_1(\lambda^a_{N^a_{\gamma_0}})\langle\Lambda\psi_2,\cos\sigma_{\mathfrak{n}}(\lambda^a_{N^a_{\gamma_0}})x\ \rangle \\
\hdashline
\tau_1(\lambda_1^b)\langle\Lambda\psi_1,\sin\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda_1^b)z\ \rangle & \tau_1(\lambda_1^b)\langle\Lambda\psi_2,\sin\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda_1^b)z\ \rangle \\
\tau_1(\lambda_1^b)\langle\Lambda\psi_1,\cos\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda_1^b)z\ \rangle & \tau_1(\lambda_1^b)\langle\Lambda\psi_2,\cos\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda_1^b)z\ \rangle \\
\tau_1(\lambda_2^b)\langle\Lambda\psi_1,\sin\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda_2^b)z\ \rangle & \tau_1(\lambda_2^b)\langle\Lambda\psi_2,\sin\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda_2^b)z\ \rangle \\
\tau_1(\lambda_2^b)\langle\Lambda\psi_1,\cos\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda_2^b)z\ \rangle & \tau_1(\lambda_2^b)\langle\Lambda\psi_2,\cos\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda_2^b)z\ \rangle \\
\cdots & \cdots \\
\tau_1(\lambda^b_{N^b_{\gamma_0}})\langle\Lambda\psi_1,\sin\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda^b_{N^b_{\gamma_0}})z\ \rangle & \tau_1(\lambda^b_{N^b_{\gamma_0}})\langle\Lambda\psi_2,\sin\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda^b_{N^b_{\gamma_0}})z\ \rangle \\
\tau_1(\lambda^b_{N^b_{\gamma_0}})\langle\Lambda\psi_1,\cos\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda^b_{N^b_{\gamma_0}})x\ \rangle & \tau_1(\lambda^b_{N^b_{\gamma_0}})\langle\Lambda\psi_2,\cos\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda^b_{N^b_{\gamma_0}})x\ \rangle \\
\hdashline
\tau_2(\lambda_1^c)\langle\Lambda\psi_1,\sin\sigma_{\mathfrak{n}}(\lambda_1^c)x\ \ \sin\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda_1^c)z\ \rangle & \tau_2(\lambda_1^c)\langle\Lambda\psi_2,\sin\sigma_{\mathfrak{n}}(\lambda_1^c)x\ \ \sin\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda_1^c)z\ \rangle \\
\tau_2(\lambda_1^c)\langle\Lambda\psi_1,\sin\sigma_{\mathfrak{n}}(\lambda_1^c)x\ \ \cos\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda_1^c)z\ \rangle & \tau_2(\lambda_1^c)\langle\Lambda\psi_2,\sin\sigma_{\mathfrak{n}}(\lambda_1^c)x\ \ \cos\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda_1^c)z\ \rangle \\
\tau_2(\lambda_1^c)\langle\Lambda\psi_1,\cos\sigma_{\mathfrak{n}}(\lambda_1^c)x\ \ \sin\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda_1^c)z\ \rangle & \tau_2(\lambda_1^c)\langle\Lambda\psi_2,\cos\sigma_{\mathfrak{n}}(\lambda_1^c)x\ \ \sin\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda_1^c)z\ \rangle \\
\tau_2(\lambda_1^c)\langle\Lambda\psi_1,\cos\sigma_{\mathfrak{n}}(\lambda_1^c)x\ \ \cos\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda_1^c)z\ \rangle & \tau_2(\lambda_1^c)\langle\Lambda\psi_2,\cos\sigma_{\mathfrak{n}}(\lambda_1^c)x\ \ \cos\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda_1^c)z\ \rangle \\
\tau_2(\lambda_2^c)\langle\Lambda\psi_1,\sin\sigma_{\mathfrak{n}}(\lambda_2^c)x\ \ \sin\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda_2^c)z\ \rangle & \tau_2(\lambda_2^c)\langle\Lambda\psi_2,\sin\sigma_{\mathfrak{n}}(\lambda_2^c)x\ \ \sin\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda_2^c)z\ \rangle \\
\tau_2(\lambda_2^c)\langle\Lambda\psi_1,\sin\sigma_{\mathfrak{n}}(\lambda_2^c)x\ \ \cos\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda_2^c)z\ \rangle & \tau_2(\lambda_2^c)\langle\Lambda\psi_2,\sin\sigma_{\mathfrak{n}}(\lambda_2^c)x\ \ \cos\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda_2^c)z\ \rangle \\
\tau_2(\lambda_2^c)\langle\Lambda\psi_1,\cos\sigma_{\mathfrak{n}}(\lambda_2^c)x\ \ \sin\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda_2^c)z\ \rangle & \tau_2(\lambda_2^c)\langle\Lambda\psi_2,\cos\sigma_{\mathfrak{n}}(\lambda_2^c)x\ \ \sin\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda_2^c)z\ \rangle \\
\tau_2(\lambda_2^c)\langle\Lambda\psi_1,\cos\sigma_{\mathfrak{n}}(\lambda_2^c)x\ \ \cos\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda_2^c)z\ \rangle & \tau_2(\lambda_2^c)\langle\Lambda\psi_2,\cos\sigma_{\mathfrak{n}}(\lambda_2^c)x\ \ \cos\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda_2^c)z\ \rangle \\
\cdots & \cdots \\
\tau_2(\lambda^c_{N^c_{\gamma_0}})\langle\Lambda\psi_1,\sin\sigma_{\mathfrak{n}}(\lambda^c_{N^c_{\gamma_0}})x\ \ \sin\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda^c_{N^c_{\gamma_0}})z\rangle & \tau_2(\lambda^c_{N^c_{\gamma_0}})\langle\Lambda\psi_2,\sin\sigma_{\mathfrak{n}}(\lambda^c_{N^c_{\gamma_0}})x\ \ \sin\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda^c_{N^c_{\gamma_0}})z\rangle \\
\tau_2(\lambda^c_{N^c_{\gamma_0}})\langle\Lambda\psi_1,\sin\sigma_{\mathfrak{n}}(\lambda^c_{N^c_{\gamma_0}})x\ \ \cos\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda^c_{N^c_{\gamma_0}})z\rangle & \tau_2(\lambda^c_{N^c_{\gamma_0}})\langle\Lambda\psi_2,\sin\sigma_{\mathfrak{n}}(\lambda^c_{N^c_{\gamma_0}})x\ \ \cos\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda^c_{N^c_{\gamma_0}})z\rangle \\
\tau_2(\lambda^c_{N^c_{\gamma_0}})\langle\Lambda\psi_1,\cos\sigma_{\mathfrak{n}}(\lambda^c_{N^c_{\gamma_0}})x\ \ \sin\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda^c_{N^c_{\gamma_0}})z\rangle & \tau_2(\lambda^c_{N^c_{\gamma_0}})\langle\Lambda\psi_2,\cos\sigma_{\mathfrak{n}}(\lambda^c_{N^c_{\gamma_0}})x\ \ \sin\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda^c_{N^c_{\gamma_0}})z\rangle \\
\tau_2(\lambda^c_{N^c_{\gamma_0}})\langle\Lambda\psi_1,\cos\sigma_{\mathfrak{n}}(\lambda^c_{N^c_{\gamma_0}})x\ \ \cos\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda^c_{N^c_{\gamma_0}})z\rangle & \tau_2(\lambda^c_{N^c_{\gamma_0}})\langle\Lambda\psi_2,\cos\sigma_{\mathfrak{n}}(\lambda^c_{N^c_{\gamma_0}})x\ \ \cos\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda^c_{N^c_{\gamma_0}})z\rangle
\end{array}\right|
$$

$$
\left|\begin{array}{c|c}
\tau_1(\lambda_1^a)\langle\Lambda\psi_3,\sin\sigma_{\mathfrak{n}}(\lambda_1^a)x\ \rangle & \tau_1(\lambda_1^a)\langle\Lambda\psi_4,\sin\sigma_{\mathfrak{n}}(\lambda_1^a)x\ \rangle \\
\tau_1(\lambda_1^a)\langle\Lambda\psi_3,\cos\sigma_{\mathfrak{n}}(\lambda_1^a)x\ \rangle & \tau_1(\lambda_1^a)\langle\Lambda\psi_4,\cos\sigma_{\mathfrak{n}}(\lambda_1^a)x\ \rangle \\
\tau_1(\lambda_2^a)\langle\Lambda\psi_3,\sin\sigma_{\mathfrak{n}}(\lambda_2^a)x\ \rangle & \tau_1(\lambda_2^a)\langle\Lambda\psi_4,\sin\sigma_{\mathfrak{n}}(\lambda_2^a)x\ \rangle \\
\tau_1(\lambda_2^a)\langle\Lambda\psi_3,\cos\sigma_{\mathfrak{n}}(\lambda_2^a)x\ \rangle & \tau_1(\lambda_2^a)\langle\Lambda\psi_4,\cos\sigma_{\mathfrak{n}}(\lambda_2^a)x\ \rangle \\
\cdots & \cdots \\
\tau_1(\lambda_{N^a_{\gamma_0}}^a)\langle\Lambda\psi_3,\sin\sigma_{\mathfrak{n}}(\lambda_{N^a_{\gamma_0}}^a)x\ \rangle & \tau_1(\lambda_{N^a_{\gamma_0}}^a)\langle\Lambda\psi_4,\sin\sigma_{\mathfrak{n}}(\lambda_{N^a_{\gamma_0}}^a)x\ \rangle \\
\tau_1(\lambda_{N^a_{\gamma_0}}^a)\langle\Lambda\psi_3,\cos\sigma_{\mathfrak{n}}(\lambda_{N^a_{\gamma_0}}^a)x\ \rangle & \tau_1(\lambda_{N^a_{\gamma_0}}^a)\langle\Lambda\psi_4,\cos\sigma_{\mathfrak{n}}(\lambda_{N^a_{\gamma_0}}^a)x\ \rangle \\
\hdashline
\tau_1(\lambda_1^b)\langle\Lambda\psi_3,\sin\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda_1^b)z\ \rangle & \tau_1(\lambda_1^b)\langle\Lambda\psi_4,\sin\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda_1^b)z\ \rangle \\
\tau_1(\lambda_1^b)\langle\Lambda\psi_3,\cos\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda_1^b)z\ \rangle & \tau_1(\lambda_1^b)\langle\Lambda\psi_4,\cos\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda_1^b)z\ \rangle \\
\tau_1(\lambda_2^b)\langle\Lambda\psi_3,\sin\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda_2^b)z\ \rangle & \tau_1(\lambda_2^b)\langle\Lambda\psi_4,\sin\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda_2^b)z\ \rangle \\
\tau_1(\lambda_2^b)\langle\Lambda\psi_3,\cos\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda_2^b)z\ \rangle & \tau_1(\lambda_2^b)\langle\Lambda\psi_4,\cos\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda_2^b)z\ \rangle \\
\cdots & \cdots \\
\tau_1(\lambda_{N^b_{\gamma_0}}^b)\langle\Lambda\psi_3,\sin\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda_{N^b_{\gamma_0}}^b)z\ \rangle & \tau_1(\lambda_{N^b_{\gamma_0}}^b)\langle\Lambda\psi_4,\sin\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda_{N^b_{\gamma_0}}^b)z\ \rangle \\
\tau_1(\lambda_{N^b_{\gamma_0}}^b)\langle\Lambda\psi_3,\cos\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda_{N^b_{\gamma_0}}^b)x\ \rangle & \tau_1(\lambda_{N^b_{\gamma_0}}^b)\langle\Lambda\psi_4,\cos\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda_{N^b_{\gamma_0}}^b)x\ \rangle \\
\hdashline
\tau_2(\lambda_1^c)\langle\Lambda\psi_3,\sin\sigma_{\mathfrak{n}}(\lambda_1^c)x\ \ \sin\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda_1^c)z\ \rangle & \tau_2(\lambda_1^c)\langle\Lambda\psi_4,\sin\sigma_{\mathfrak{n}}(\lambda_1^c)x\ \ \sin\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda_1^c)z\ \rangle \\
\tau_2(\lambda_1^c)\langle\Lambda\psi_3,\sin\sigma_{\mathfrak{n}}(\lambda_1^c)x\ \ \cos\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda_1^c)z\ \rangle & \tau_2(\lambda_1^c)\langle\Lambda\psi_4,\sin\sigma_{\mathfrak{n}}(\lambda_1^c)x\ \ \cos\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda_1^c)z\ \rangle \\
\tau_2(\lambda_1^c)\langle\Lambda\psi_3,\cos\sigma_{\mathfrak{n}}(\lambda_1^c)x\ \ \sin\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda_1^c)z\ \rangle & \tau_2(\lambda_1^c)\langle\Lambda\psi_4,\cos\sigma_{\mathfrak{n}}(\lambda_1^c)x\ \ \sin\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda_1^c)z\ \rangle \\
\tau_2(\lambda_1^c)\langle\Lambda\psi_3,\cos\sigma_{\mathfrak{n}}(\lambda_1^c)x\ \ \cos\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda_1^c)z\ \rangle & \tau_2(\lambda_1^c)\langle\Lambda\psi_4,\cos\sigma_{\mathfrak{n}}(\lambda_1^c)x\ \ \cos\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda_1^c)z\ \rangle \\
\tau_2(\lambda_2^c)\langle\Lambda\psi_3,\sin\sigma_{\mathfrak{n}}(\lambda_2^c)x\ \ \sin\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda_2^c)z\ \rangle & \tau_2(\lambda_2^c)\langle\Lambda\psi_4,\sin\sigma_{\mathfrak{n}}(\lambda_2^c)x\ \ \sin\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda_2^c)z\ \rangle \\
\tau_2(\lambda_2^c)\langle\Lambda\psi_3,\sin\sigma_{\mathfrak{n}}(\lambda_2^c)x\ \ \cos\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda_2^c)z\ \rangle & \tau_2(\lambda_2^c)\langle\Lambda\psi_4,\sin\sigma_{\mathfrak{n}}(\lambda_2^c)x\ \ \cos\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda_2^c)z\ \rangle \\
\tau_2(\lambda_2^c)\langle\Lambda\psi_3,\cos\sigma_{\mathfrak{n}}(\lambda_2^c)x\ \ \sin\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda_2^c)z\ \rangle & \tau_2(\lambda_2^c)\langle\Lambda\psi_4,\cos\sigma_{\mathfrak{n}}(\lambda_2^c)x\ \ \sin\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda_2^c)z\ \rangle \\
\tau_2(\lambda_2^c)\langle\Lambda\psi_3,\cos\sigma_{\mathfrak{n}}(\lambda_2^c)x\ \ \cos\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda_2^c)z\ \rangle & \tau_2(\lambda_2^c)\langle\Lambda\psi_4,\cos\sigma_{\mathfrak{n}}(\lambda_2^c)x\ \ \cos\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda_2^c)z\ \rangle \\
\cdots & \cdots \\
\tau_2(\lambda_{N^c_{\gamma_0}}^c)\langle\Lambda\psi_3,\sin\sigma_{\mathfrak{n}}(\lambda_{N^c_{\gamma_0}}^c)x\ \ \sin\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda_{N^c_{\gamma_0}}^c)z\rangle & \tau_2(\lambda_{N^c_{\gamma_0}}^c)\langle\Lambda\psi_4,\sin\sigma_{\mathfrak{n}}(\lambda_{N^c_{\gamma_0}}^c)x\ \ \sin\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda_{N^c_{\gamma_0}}^c)z\rangle \\
\tau_2(\lambda_{N^c_{\gamma_0}}^c)\langle\Lambda\psi_3,\sin\sigma_{\mathfrak{n}}(\lambda_{N^c_{\gamma_0}}^c)x\ \ \cos\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda_{N^c_{\gamma_0}}^c)z\rangle & \tau_2(\lambda_{N^c_{\gamma_0}}^c)\langle\Lambda\psi_4,\sin\sigma_{\mathfrak{n}}(\lambda_{N^c_{\gamma_0}}^c)x\ \ \cos\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda_{N^c_{\gamma_0}}^c)z\rangle \\
\tau_2(\lambda_{N^c_{\gamma_0}}^c)\langle\Lambda\psi_3,\cos\sigma_{\mathfrak{n}}(\lambda_{N^c_{\gamma_0}}^c)x\ \ \sin\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda_{N^c_{\gamma_0}}^c)z\rangle & \tau_2(\lambda_{N^c_{\gamma_0}}^c)\langle\Lambda\psi_4,\cos\sigma_{\mathfrak{n}}(\lambda_{N^c_{\gamma_0}}^c)x\ \ \sin\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda_{N^c_{\gamma_0}}^c)z\rangle \\
\tau_2(\lambda_{N^c_{\gamma_0}}^c)\langle\Lambda\psi_3,\cos\sigma_{\mathfrak{n}}(\lambda_{N^c_{\gamma_0}}^c)x\ \ \cos\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda_{N^c_{\gamma_0}}^c)z\rangle & \tau_2(\lambda_{N^c_{\gamma_0}}^c)\langle\Lambda\psi_4,\cos\sigma_{\mathfrak{n}}(\lambda_{N^c_{\gamma_0}}^c)x\ \ \cos\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda_{N^c_{\gamma_0}}^c)z\rangle
\end{array}\right],
$$

where $\langle\ ,\ \rangle$ *is the* $L_2(\Gamma_1)$*-inner product* $\langle\ ,\ \rangle_{\Gamma_1}$*, and where we have set*

$$\tau_1(\lambda) \ = \ \tfrac{\sigma_{\mathfrak{m}}(\lambda)}{\lambda}\sqrt{\tfrac{\pi}{e}}(-1)^{\sigma_{\mathfrak{m}}(\lambda)+1},$$

$$\tau_2(\lambda) \ = \ \tfrac{\sigma_{\mathfrak{m}}(\lambda)}{\lambda}\sqrt{\tfrac{2\pi}{e}}(-1)^{\sigma_{\mathfrak{m}}(\lambda)+1},$$

where $\sigma_{\mathfrak{n}}, \sigma_{\mathfrak{m}}$*, and* $\sigma_{\mathfrak{k}}$ *are as in the paragraph following (1.5.2k).*

Proof. Let $\int_{-e}^{e}\int_{-\pi}^{\pi}\psi_j(x,z)\ dxdz \equiv 0$, and $\int_{-e}^{e}\int_{-\pi}^{\pi}(\partial_{xx}+\partial_{zz})\psi_j(x,z)\ dxdz \equiv 0$, for $\psi_j \in H^2(\Gamma^1)$, $j = 1,2,3,4$, as required in Corollary 1.5.2 and Proposition 1.6.2(b), so that we can invoke expansion (1.5.9) of Proposition 1.5.1 for $g = (\partial_{xx}+\partial_{zz})\psi_j$, as restricted over $(n,m,k)\in\mathcal{U}_{\gamma_0}$. We obtain

$$
\begin{aligned}
&P^u_{\gamma_0}D((\partial_{xx}+\partial_{zz})\psi_j) = \\
&\quad \sum_{(n,m,0)\in\mathcal{U}^a_{\gamma_0}} \sqrt{\frac{\pi}{e}}\frac{m}{\lambda_{nm0}}(-1)^m\langle(\partial_{xx}+\partial_{zz})\psi_j, \sin nx\rangle_{\Gamma_1} e^1_{nm0} + \\
&\quad \sum_{(n,m,0)\in\mathcal{U}^a_{\gamma_0}} \sqrt{\frac{\pi}{e}}\frac{m}{\lambda_{nm0}}(-1)^m\langle(\partial_{xx}+\partial_{zz})\psi_j, \cos nx\rangle_{\Gamma_1} e^2_{nm0} \\
&\quad + \sum_{(0,m,k)\in\mathcal{U}^b_{\gamma_0}} \sqrt{\frac{\pi}{e}}\frac{m}{\lambda_{0mk}}(-1)^m\langle(\partial_{xx}+\partial_{zz})\psi_j, \sin \frac{\pi}{e}kz\rangle_{\Gamma_1} e^3_{0mk} + \\
&\quad \sum_{(0,m,k)\in\mathcal{U}^b_{\gamma_0}} \sqrt{\frac{\pi}{e}}\frac{m}{\lambda_{0mk}}(-1)^m\langle(\partial_{xx}+\partial_{zz})\psi_j, \cos \frac{\pi}{e}kz\rangle_{\Gamma_1} e^4_{0mk} \\
&\quad + \sum_{(n,m,k)\in\mathcal{U}^c_{\gamma_0}} \sqrt{\frac{2\pi}{e}}\frac{m}{\lambda_{nmk}}(-1)^m\langle(\partial_{xx}+\partial_{zz})\psi_j, \sin nx\ \sin \frac{\pi}{e}kz\rangle_{\Gamma_1} e^5_{nmk} + \\
&\quad \sum_{(n,m,k)\in\mathcal{U}^c_{\gamma_0}} \sqrt{\frac{2\pi}{e}}\frac{m}{\lambda_{nmk}}(-1)^m\langle(\partial_{xx}+\partial_{zz})\psi_j, \sin nx\ \cos \frac{\pi}{e}kz\rangle_{\Gamma_1} e^6_{nmk} \\
&\quad + \sum_{(n,m,k)\in\mathcal{U}^c_{\gamma_0}} \sqrt{\frac{2\pi}{e}}\frac{m}{\lambda_{nmk}}(-1)^m\langle(\partial_{xx}+\partial_{zz})\psi_j, \cos nx\ \sin \frac{\pi}{e}kz\rangle_{\Gamma_1} e^7_{nmk} + \\
&\quad \sum_{(n,m,k)\in\mathcal{U}^c_{\gamma_0}} \sqrt{\frac{2\pi}{e}}\frac{m}{\lambda_{nmk}}(-1)^m\langle(\partial_{xx}+\partial_{zz})\psi_j, \cos nx\ \cos \frac{\pi}{e}kz\rangle_{\Gamma_1} e^8_{nmk} \qquad (1.7.13)
\end{aligned}
$$

$$= -\sum_{i=1}^{N^a_{\gamma_0}} (\tau_1(\lambda^a_i) \langle (\partial_{xx} + \partial_{zz})\psi_j, \sin \sigma_{\mathfrak{n}}(\lambda^a_i)x \ \rangle_{\Gamma_1} e^1_{\sigma_{\mathfrak{n}}(\lambda^a_i)\sigma_{\mathfrak{m}}(\lambda^a_i)0} +$$

$$-\sum_{i=1}^{N^a_{\gamma_0}} (\tau_1(\lambda^a_i) \langle (\partial_{xx} + \partial_{zz})\psi_j, \cos \sigma_{\mathfrak{n}}(\lambda^a_i)x \ \rangle_{\Gamma_1} e^2_{\sigma_{\mathfrak{n}}(\lambda^a_i)\sigma_{\mathfrak{m}}(\lambda^a_i)0}$$

$$-\sum_{i=1}^{N^b_{\gamma_0}} (\tau_1(\lambda^b_i) \langle (\partial_{xx} + \partial_{zz})\psi_j, \sin \pi e \sigma_{\mathfrak{k}}(\lambda^b_i)z \ \rangle_{\Gamma_1} e^3_{0\sigma_{\mathfrak{m}}(\lambda^b_i)\sigma_{\mathfrak{k}}(\lambda^b_i)} +$$

$$-\sum_{i=1}^{N^b_{\gamma_0}} (\tau_1(\lambda^b_i) \langle (\partial_{xx} + \partial_{zz})\psi_j, \cos \pi e \sigma_{\mathfrak{k}}(\lambda^b_i)z \ \rangle_{\Gamma_1} e^4_{0\sigma_{\mathfrak{m}}(\lambda^b_i)\sigma_{\mathfrak{k}}(\lambda^b_i)}$$

$$-\sum_{i=1}^{N^c_{\gamma_0}} (\tau_2(\lambda^c_i) \langle (\partial_{xx} + \partial_{zz})\psi_j, \sin \sigma_{\mathfrak{n}}(\lambda^c_i)x \ \sin \pi e \sigma_{\mathfrak{k}}(\lambda^c_i)z \ \rangle_{\Gamma_1} e^5_{\sigma_{\mathfrak{n}}(\lambda^c_i)\sigma_{\mathfrak{m}}(\lambda^c_i)\sigma_{\mathfrak{k}}(\lambda^c_i)}$$

$$-\sum_{i=1}^{N^c_{\gamma_0}} (\tau_2(\lambda^c_i) \langle (\partial_{xx} + \partial_{zz})\psi_j, \sin \sigma_{\mathfrak{n}}(\lambda^c_i)x \ \cos \pi e \sigma_{\mathfrak{k}}(\lambda^c_i)z \ \rangle_{\Gamma_1} e^6_{\sigma_{\mathfrak{n}}(\lambda^c_i)\sigma_{\mathfrak{m}}(\lambda^c_i)\sigma_{\mathfrak{k}}(\lambda^c_i)}$$

$$-\sum_{i=1}^{N^c_{\gamma_0}} (\tau_2(\lambda^c_i) \langle (\partial_{xx} + \partial_{zz})\psi_j, \cos \sigma_{\mathfrak{n}}(\lambda^c_i)x \ \sin \pi e \sigma_{\mathfrak{k}}(\lambda^c_i)z \ \rangle_{\Gamma_1} e^7_{\sigma_{\mathfrak{n}}(\lambda^c_i)\sigma_{\mathfrak{m}}(\lambda^c_i)\sigma_{\mathfrak{k}}(\lambda^c_i)}$$

$$-\sum_{i=1}^{N^c_{\gamma_0}} (\tau_2(\lambda^c_i) \langle (\partial_{xx} + \partial_{zz})\psi_j, \cos \sigma_{\mathfrak{n}}(\lambda^c_i)x \ \cos \pi e \sigma_{\mathfrak{k}}(\lambda^c_i)z \ \rangle_{\Gamma_1} e^8_{\sigma_{\mathfrak{n}}(\lambda^c_i)\sigma_{\mathfrak{m}}(\lambda^c_i)\sigma_{\mathfrak{k}}(\lambda^c_i)}.$$

$$+ \sum_{(n,m)\in\mathcal{U}_{\gamma_0}} \sqrt{\pi}\frac{m}{\lambda_{nm}} (-1)^m \langle \psi''_j, \cos nx \rangle_{\Gamma_1} e^1_{nm} \in Z^u_{\gamma_0}. \tag{1.7.14}$$

Next, in $\mathcal{U}_{\gamma_0}$, use the order selected in (1.5.2i–k). Then (1.7.14) and the definition (1.7.10) for $B^u_{\gamma_0} = -P^u_{\gamma_0} D(\partial_{xx} + \partial_{zz})\Psi(x, z)$ yield (1.7.12). □

1.8 Linear Case $a \equiv b \equiv 0$ in (1.1.1.5): Eigenvalues (Pole) Assignment to $\{A_D^u, B_{\gamma_0}^u\}$ on the State Space $Z_{\gamma_0}^u$ Via a Suitable Feedback Operator $\tilde{F}_{\gamma_0}^u \in \mathcal{L}(Z_{\gamma_0}^u; \mathbb{R}^J)$, $J = 4$

The present section deals with the finite-dimensional system (1.7.9) in the variable $[P_{\gamma_0}^u h]$ on the space $Z_{\gamma_0}^u$. Up to this stage, the feedback operator $F_{\gamma_0}^u \in \mathcal{L}(Z_{\gamma_0}^u; \mathbb{R}^4)$, $J = 4$ in (1.7.8) has not been specified. In the present section, we make a selection, so that the chosen operator $\tilde{F}_{\gamma_0}^u$ is *a-fortiori* enhancing stability. More precisely, under general and readily verifiable conditions on the vectors ψ_j, $j = 1, 2, 3, 4$, see (1.8.3) below, we shall assert the existence of feedback operators $\tilde{F}_{\gamma_0}^u \in \mathcal{L}(Z_{\gamma_0}^u; \mathbb{R}^4)$ such that the corresponding operator $A_D^u + B_{\gamma_0}^u \tilde{F}_{\gamma_0}^u$ in (1.7.9) has an arbitrarily preassigned set of eigenvalues ('pole assignment'). More precisely, we consider the following linear dynamical system

$$\text{on } Z_{\gamma_0}^u: \ \chi_t = \nu A_D^u \chi + \nu B_{\gamma_0}^u \mu, \quad \chi = P_{\gamma_0}^u w \in Z_{\gamma_0}^u, \tag{1.8.1}$$

with a four-dimensional control $\mu(t) = [\mu_1(t), \mu_2(t), \mu_3(t), \mu_4(t)]^T$. The operator $B_{\gamma_0}^u$ is defined in (1.7.10) and has the matrix representation given by (1.7.12), under Proposition 1.7.1, with the critical complementary condition that $\tau_1(\lambda_{nmk})$ and $\tau_2(\lambda_{nmk}) \neq 0$ for all λ_{nmk} with $(n, m, k) \in \mathcal{U}_{\gamma_0}$.

Proposition 1.8.1. *Let* $\psi_j \in H^2(\Gamma_1)$, $\langle \psi_j, 1 \rangle_{\Gamma_1} \equiv 0$, $\langle (\partial_{xx} + \partial_{zz})\psi_j, 1 \rangle_{\Gamma_1} \equiv 0$, $j =$

1, 2, 3, 4, so that Corollary 5.2 and Proposition 1.6.2(b) hold true, and the results of Section 1.7 apply.

(a) The finite-dimensional system (1.7.9) with $J = 4$ on $Z^u_{\gamma_0}$—in short, the pair $\nu\{A^u_D, B^u_{\gamma_0}\}$—is controllable, that is, it satisfies Kalman's rank condition

$$\text{rank}\ [B^u_{\gamma_0}, A^u_D B^u_{\gamma_0}, \ldots, (A^u_D)^{N_{\gamma_0}-1} B^u_{\gamma_0}] = \text{full} = N_{\gamma_0} = \dim\ Z^u_{\gamma_0}, \tag{1.8.2a}$$

or else the equivalent Hautus condition

$$\text{rank}\ [A^u_D - \lambda_{nmk} I, B^u_{\gamma_0}] = N_{\gamma_0}\ \text{for all eigenvalues}\ \lambda_{nmk}\ \text{of}\ A^u_D, \tag{1.8.2b}$$

if and only if the following rank conditions hold true for the boundary functions $\psi_j \in H^2(\Gamma_1)$ (again letting $\Lambda = \partial_{xx} + \partial_{zz}$, with splitting each matrix over two lines for spacing)

$$\text{rank} \left[\begin{array}{c|c|} \langle \Lambda\psi_1, \sin\sigma_{\mathfrak{n}}(\lambda^a_1)x\ \rangle & \langle \Lambda\psi_2, \sin\sigma_{\mathfrak{n}}(\lambda^a_1)x\ \rangle \\ \langle \Lambda\psi_1, \cos\sigma_{\mathfrak{n}}(\lambda^a_1)x\ \rangle & \langle \Lambda\psi_2, \cos\sigma_{\mathfrak{n}}(\lambda^a_1)x\ \rangle \end{array} \right.$$
$$\left. \begin{array}{|c|c} \langle \Lambda\psi_3, \sin\sigma_{\mathfrak{n}}(\lambda^a_1)x\ \rangle & \langle \Lambda\psi_4, \sin\sigma_{\mathfrak{n}}(\lambda^a_1)x\ \rangle \\ \langle \Lambda\psi_3, \cos\sigma_{\mathfrak{n}}(\lambda^a_1)x\ \rangle & \langle \Lambda\psi_4, \cos\sigma_{\mathfrak{n}}(\lambda^a_1)x\ \rangle \end{array} \right] = 2; \tag{1.8.3a}$$

$$\text{rank} \left[\begin{array}{c|c|} \langle \Lambda\psi_1, \sin\sigma_{\mathfrak{n}}(\lambda^a_2)x\ \rangle & \langle \Lambda\psi_2, \sin\sigma_{\mathfrak{n}}(\lambda^a_2)x\ \rangle \\ \langle \Lambda\psi_1, \cos\sigma_{\mathfrak{n}}(\lambda^a_2)x\ \rangle & \langle \Lambda\psi_2, \cos\sigma_{\mathfrak{n}}(\lambda^a_2)x\ \rangle \end{array} \right.$$
$$\left. \begin{array}{|c|c} \langle \Lambda\psi_3, \sin\sigma_{\mathfrak{n}}(\lambda^a_2)x\ \rangle & \langle \Lambda\psi_4, \sin\sigma_{\mathfrak{n}}(\lambda^a_2)x\ \rangle \\ \langle \Lambda\psi_3, \cos\sigma_{\mathfrak{n}}(\lambda^a_2)x\ \rangle & \langle \Lambda\psi_4, \cos\sigma_{\mathfrak{n}}(\lambda^a_2)x\ \rangle \end{array} \right] = 2; \tag{1.8.3b}$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots$$

$$rank\left[\begin{array}{c|c|} \langle\Lambda\psi_1, \sin\sigma_{\mathfrak{n}}(\lambda^a_{N^a_{\gamma_0}})x\rangle & \langle\Lambda\psi_2, \sin\sigma_{\mathfrak{n}}(\lambda^a_{N^a_{\gamma_0}})x\rangle \\ \langle\Lambda\psi_1, \cos\sigma_{\mathfrak{n}}(\lambda^a_{N^a_{\gamma_0}})x\rangle & \langle\Lambda\psi_2, \cos\sigma_{\mathfrak{n}}(\lambda^a_{N^a_{\gamma_0}})x\rangle \end{array}\right.$$

$$\left.\begin{array}{|c|c} \langle\Lambda\psi_3, \sin\sigma_{\mathfrak{n}}(\lambda^a_{N^a_{\gamma_0}})x\rangle & \langle\Lambda\psi_4, \sin\sigma_{\mathfrak{n}}(\lambda^a_{N^a_{\gamma_0}})x\rangle \\ \langle\Lambda\psi_3, \cos\sigma_{\mathfrak{n}}(\lambda^a_{N^a_{\gamma_0}})x\rangle & \langle\Lambda\psi_4, \cos\sigma_{\mathfrak{n}}(\lambda^a_{N^a_{\gamma_0}})x\rangle \end{array}\right] = 2; \quad (1.8.3c)$$

— —

$$rank\left[\begin{array}{c|c|} \langle\Lambda\psi_1, \sin\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda^b_1)z\ \rangle & \langle\Lambda\psi_2, \sin\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda^b_1)z\ \rangle \\ \langle\Lambda\psi_1, \cos\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda^b_1)z\ \rangle & \langle\Lambda\psi_2, \cos\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda^b_1)z\ \rangle \end{array}\right.$$

$$\left.\begin{array}{|c|c} \langle\Lambda\psi_3, \sin\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda^b_1)z\ \rangle & \langle\Lambda\psi_4, \sin\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda^b_1)z\ \rangle \\ \langle\Lambda\psi_3, \cos\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda^b_1)z\ \rangle & \langle\Lambda\psi_4, \cos\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda^b_1)z\ \rangle \end{array}\right] = 2; \quad (1.8.3d)$$

$$rank\left[\begin{array}{c|c|} \langle\Lambda\psi_1, \sin\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda^b_2)z\ \rangle & \langle\Lambda\psi_2, \sin\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda^b_2)z\ \rangle \\ \langle\Lambda\psi_1, \cos\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda^b_2)z\ \rangle & \langle\Lambda\psi_2, \cos\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda^b_2)z\ \rangle \end{array}\right.$$

$$\left.\begin{array}{|c|c} \langle\Lambda\psi_3, \sin\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda^b_2)z\ \rangle & \langle\Lambda\psi_4, \sin\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda^b_2)z\ \rangle \\ \langle\Lambda\psi_3, \cos\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda^b_2)z\ \rangle & \langle\Lambda\psi_4, \cos\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda^b_2)z\ \rangle \end{array}\right] = 2; \quad (1.8.3e)$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots$$

$$rank\left[\begin{array}{c|c|} \langle\Lambda\psi_1, \sin\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda^b_{N^b_{\gamma_0}})z\rangle & \langle\Lambda\psi_2, \sin\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda^b_{N^b_{\gamma_0}})z\rangle \\ \langle\Lambda\psi_1, \cos\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda^b_{N^b_{\gamma_0}})x\rangle & \langle\Lambda\psi_2, \cos\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda^b_{N^b_{\gamma_0}})x\rangle \end{array}\right.$$

$$\left.\begin{array}{|c|c} \langle\Lambda\psi_3, \sin\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda^b_{N^b_{\gamma_0}})z\rangle & \langle\Lambda\psi_4, \sin\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda^b_{N^b_{\gamma_0}})z\rangle \\ \langle\Lambda\psi_3, \cos\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda^b_{N^b_{\gamma_0}})x\rangle & \langle\Lambda\psi_4, \cos\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda^b_{N^b_{\gamma_0}})x\rangle \end{array}\right] = 2; \quad (1.8.3f)$$

$$- -$$

$$rank \begin{bmatrix} \langle\Lambda\psi_1, \sin\sigma_{\mathfrak{n}}(\lambda_1^c)x \;\; \sin\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda_1^c)z\rangle & \langle\Lambda\psi_2, \sin\sigma_{\mathfrak{n}}(\lambda_1^c)x \;\; \sin\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda_1^c)z\rangle & \langle\Lambda\psi_3, \sin\sigma_{\mathfrak{n}}(\lambda_1^c)x \;\; \sin\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda_1^c)z\rangle & \langle\Lambda\psi_4, \sin\sigma_{\mathfrak{n}}(\lambda_1^c)x \;\; \sin\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda_1^c)z\rangle \\ \langle\Lambda\psi_1, \sin\sigma_{\mathfrak{n}}(\lambda_1^c)x \;\; \cos\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda_1^c)z\rangle & \langle\Lambda\psi_2, \sin\sigma_{\mathfrak{n}}(\lambda_1^c)x \;\; \cos\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda_1^c)z\rangle & \langle\Lambda\psi_3, \sin\sigma_{\mathfrak{n}}(\lambda_1^c)x \;\; \cos\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda_1^c)z\rangle & \langle\Lambda\psi_4, \sin\sigma_{\mathfrak{n}}(\lambda_1^c)x \;\; \cos\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda_1^c)z\rangle \\ \langle\Lambda\psi_1, \cos\sigma_{\mathfrak{n}}(\lambda_1^c)x \;\; \sin\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda_1^c)z\rangle & \langle\Lambda\psi_2, \cos\sigma_{\mathfrak{n}}(\lambda_1^c)x \;\; \sin\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda_1^c)z\rangle & \langle\Lambda\psi_3, \cos\sigma_{\mathfrak{n}}(\lambda_1^c)x \;\; \sin\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda_1^c)z\rangle & \langle\Lambda\psi_4, \cos\sigma_{\mathfrak{n}}(\lambda_1^c)x \;\; \sin\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda_1^c)z\rangle \\ \langle\Lambda\psi_1, \cos\sigma_{\mathfrak{n}}(\lambda_1^c)x \;\; \cos\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda_1^c)z\rangle & \langle\Lambda\psi_2, \cos\sigma_{\mathfrak{n}}(\lambda_1^c)x \;\; \cos\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda_1^c)z\rangle & \langle\Lambda\psi_3, \cos\sigma_{\mathfrak{n}}(\lambda_1^c)x \;\; \cos\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda_1^c)z\rangle & \langle\Lambda\psi_4, \cos\sigma_{\mathfrak{n}}(\lambda_1^c)x \;\; \cos\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda_1^c)z\rangle \end{bmatrix} = 4; \quad (1.8.3g)$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots$$

$$rank \begin{bmatrix} \langle\Lambda\psi_1, \sin\sigma_{\mathfrak{n}}(\lambda_2^c)x \;\; \sin\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda_2^c)z\rangle & \langle\Lambda\psi_2, \sin\sigma_{\mathfrak{n}}(\lambda_2^c)x \;\; \sin\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda_2^c)z\rangle & \langle\Lambda\psi_3, \sin\sigma_{\mathfrak{n}}(\lambda_2^c)x \;\; \sin\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda_2^c)z\rangle & \langle\Lambda\psi_4, \sin\sigma_{\mathfrak{n}}(\lambda_2^c)x \;\; \sin\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda_2^c)z\rangle \\ \langle\Lambda\psi_1, \sin\sigma_{\mathfrak{n}}(\lambda_2^c)x \;\; \cos\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda_2^c)z\rangle & \langle\Lambda\psi_2, \sin\sigma_{\mathfrak{n}}(\lambda_2^c)x \;\; \cos\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda_2^c)z\rangle & \langle\Lambda\psi_3, \sin\sigma_{\mathfrak{n}}(\lambda_2^c)x \;\; \cos\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda_2^c)z\rangle & \langle\Lambda\psi_4, \sin\sigma_{\mathfrak{n}}(\lambda_2^c)x \;\; \cos\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda_2^c)z\rangle \\ \langle\Lambda\psi_1, \cos\sigma_{\mathfrak{n}}(\lambda_2^c)x \;\; \sin\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda_2^c)z\rangle & \langle\Lambda\psi_2, \cos\sigma_{\mathfrak{n}}(\lambda_2^c)x \;\; \sin\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda_2^c)z\rangle & \langle\Lambda\psi_3, \cos\sigma_{\mathfrak{n}}(\lambda_2^c)x \;\; \sin\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda_2^c)z\rangle & \langle\Lambda\psi_4, \cos\sigma_{\mathfrak{n}}(\lambda_2^c)x \;\; \sin\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda_2^c)z\rangle \\ \langle\Lambda\psi_1, \cos\sigma_{\mathfrak{n}}(\lambda_2^c)x \;\; \cos\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda_2^c)z\rangle & \langle\Lambda\psi_2, \cos\sigma_{\mathfrak{n}}(\lambda_2^c)x \;\; \cos\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda_2^c)z\rangle & \langle\Lambda\psi_3, \cos\sigma_{\mathfrak{n}}(\lambda_2^c)x \;\; \cos\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda_2^c)z\rangle & \langle\Lambda\psi_4, \cos\sigma_{\mathfrak{n}}(\lambda_2^c)x \;\; \cos\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda_2^c)z\rangle \end{bmatrix} = 4; \quad (1.8.3h)$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots$$

$$rank \left[\begin{array}{c|c|c|c} \langle \Lambda\psi_1, \sin\sigma_{\mathfrak{n}}(\lambda^c_{N^c_{\gamma_0}})x \ \ \sin\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda^c_{N^c_{\gamma_0}})z\rangle & \langle \Lambda\psi_2, \sin\sigma_{\mathfrak{n}}(\lambda^c_{N^c_{\gamma_0}})x \ \ \sin\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda^c_{N^c_{\gamma_0}})z\rangle & \langle \Lambda\psi_3, \sin\sigma_{\mathfrak{n}}(\lambda^c_{N^c_{\gamma_0}})x \ \ \sin\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda^c_{N^c_{\gamma_0}})z\rangle & \langle \Lambda\psi_4, \sin\sigma_{\mathfrak{n}}(\lambda^c_{N^c_{\gamma_0}})x \ \ \sin\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda^c_{N^c_{\gamma_0}})z\rangle \\ \langle \Lambda\psi_1, \sin\sigma_{\mathfrak{n}}(\lambda^c_{N^c_{\gamma_0}})x \ \ \cos\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda^c_{N^c_{\gamma_0}})z\rangle & \langle \Lambda\psi_2, \sin\sigma_{\mathfrak{n}}(\lambda^c_{N^c_{\gamma_0}})x \ \ \cos\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda^c_{N^c_{\gamma_0}})z\rangle & \langle \Lambda\psi_3, \sin\sigma_{\mathfrak{n}}(\lambda^c_{N^c_{\gamma_0}})x \ \ \cos\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda^c_{N^c_{\gamma_0}})z\rangle & \langle \Lambda\psi_4, \sin\sigma_{\mathfrak{n}}(\lambda^c_{N^c_{\gamma_0}})x \ \ \cos\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda^c_{N^c_{\gamma_0}})z\rangle \\ \langle \Lambda\psi_1, \cos\sigma_{\mathfrak{n}}(\lambda^c_{N^c_{\gamma_0}})x \ \ \sin\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda^c_{N^c_{\gamma_0}})z\rangle & \langle \Lambda\psi_2, \cos\sigma_{\mathfrak{n}}(\lambda^c_{N^c_{\gamma_0}})x \ \ \sin\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda^c_{N^c_{\gamma_0}})z\rangle & \langle \Lambda\psi_3, \cos\sigma_{\mathfrak{n}}(\lambda^c_{N^c_{\gamma_0}})x \ \ \sin\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda^c_{N^c_{\gamma_0}})z\rangle & \langle \Lambda\psi_4, \cos\sigma_{\mathfrak{n}}(\lambda^c_{N^c_{\gamma_0}})x \ \ \sin\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda^c_{N^c_{\gamma_0}})z\rangle \\ \langle \Lambda\psi_1, \cos\sigma_{\mathfrak{n}}(\lambda^c_{N^c_{\gamma_0}})x \ \ \cos\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda^c_{N^c_{\gamma_0}})z\rangle & \langle \Lambda\psi_2, \cos\sigma_{\mathfrak{n}}(\lambda^c_{N^c_{\gamma_0}})x \ \ \cos\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda^c_{N^c_{\gamma_0}})z\rangle & \langle \Lambda\psi_3, \cos\sigma_{\mathfrak{n}}(\lambda^c_{N^c_{\gamma_0}})x \ \ \cos\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda^c_{N^c_{\gamma_0}})z\rangle & \langle \Lambda\psi_4, \cos\sigma_{\mathfrak{n}}(\lambda^c_{N^c_{\gamma_0}})x \ \ \cos\frac{\pi}{e}\sigma_{\mathfrak{k}}(\lambda^c_{N^c_{\gamma_0}})z\rangle \end{array}\right] = 4; \quad (1.8.3\text{i})$$

where $\langle\ ,\ \rangle$ *is the* $L_2(\Gamma_1)$*-inner product* $\langle\ ,\ \rangle_{\Gamma_1}$.

(b) Let $\sigma_0 > 0$ *be arbitrarily preassigned, in particular, say* $\sigma_0 > \nu\gamma_0$, γ_0 *in (1.5.2). Again, with* $J = 4$, *assume that* $(\partial_{xx} + \partial_{zz})\psi_1, \ \cdots\ , \ (\partial_{xx} + \partial_{zz})\psi_4$ *satisfy the rank conditions (1.8.3). Then (by a well-known Popov's result (1964)), there exists an operator* $\tilde{F}^u_{\gamma_0} \in \mathcal{L}(Z^u_{\gamma_0}; \mathbb{R}^4)$ *such that the corresponding operator* $[A^u_D + B^u_{\gamma_0}\tilde{F}^u_{\gamma_0}]$ *has a preassigned set of eigenvalues, in particular, say, all lie on the left of the vertical line* $Re\ \lambda = -\sigma_1 = -\frac{\sigma_0}{\nu}$ *in the complex plane. [*$\tilde{F}^u_{\gamma_0}$ *depends, of course, on the preassigned set of eigenvalues, in particular on* σ_1*.] Thus, with* $\mu = \tilde{F}^u_{\gamma_0}\chi$, *the corresponding feedback system*

$$\text{on } Z^u_{\gamma_0}: \qquad \chi_t = \nu[A^u_D + B^u_{\gamma_0}\tilde{F}^u_{\gamma_0}]\chi, \quad \chi(0) = \chi_0, \quad \chi = P^u_{\gamma_0}h, \qquad (1.8.4\text{a})$$

with solution

$$(P^u_{\gamma_0}h)(t) = \chi(t) = e^{\nu[A^u_D + B^u_{\gamma_0}\tilde{F}^u_{\gamma_0}]t}\chi_0, \quad t \geq 0, \qquad (1.8.4\text{b})$$

is exponentially stable with preassigned decay rate $-\sigma_0$*:*

$$\left\| e^{\nu[A_D^u + B_{\gamma_0}^u \tilde{F}_{\gamma_0}^u]t} \right\|_{\mathcal{L}(Z_{\gamma_0}^u)} \leq C_{\sigma_0} e^{-\sigma_0 t}, \quad t \geq 0, \tag{1.8.5a}$$

where the constant C_{σ_0} *depends on* σ_0 *so that via (1.8.4),*

$$\left\| (P_{\gamma_0}^u h)(t) \right\|_{Z_{\gamma_0}^u} \leq C_{\sigma_0} e^{-\sigma_0 t} \left\| (P_{\gamma_0}^u h)(0) \right\|_{Z_{\gamma_0}^u}, \quad t \geq 0. \tag{1.8.5b}$$

(c) The corresponding feedback control $V_{xx} + V_{zz}$*, given by (1.7.7),* $J = 4$*, with* $F_{\gamma_0}^u$ *specialized by the operator* $\tilde{F}_{\gamma_0}^u$ *above, then satisfies*

$$\begin{aligned} \|(V_{xx} + V_{zz})(t)\|_{L_2(\Gamma_1)} &= \|((\partial_{xx} + \partial_{zz})\Psi \tilde{F}_{\gamma_0}^u)(P_{\gamma_0}^u h)(t)\|_{L_2(\Gamma_1)} \\ \leq \ & \tilde{C}_{\sigma_0} e^{-\sigma_0 t} \|P_{\gamma_0}^u h(0)\|_{Z_{\gamma_0}^u}, t \geq 0, \ \tilde{C}_{\sigma_0} = \|(\partial_{xx} + \partial_{zz})\Psi \tilde{F}_{\gamma_0}^u\| C_{\sigma_0}. \end{aligned} \tag{1.8.6}$$

Proof. (a) For A_D^u as in (1.7.11) and $B_{\gamma_0}^u$ as in (1.7.12), where, moreover, $\tau_1(\lambda) \neq 0$ and $\tau_2(\lambda) \neq 0$ for all $\lambda = \lambda_{nmk}$ with $(n, m, k) \in \mathcal{U}_{\gamma_0}$ in (1.7.13), Kalman's condition (1.8.2) reduces, as is well known, to a generalized Vandermonde determinant whereby (1.8.2) holds true if and only if the set conditions (1.8.3) are fulfilled.

(b) This part follows from (a) via the standard Popov's criterion (1964). Part (c) is clear. □

1.9 Linear Case $a \equiv b \equiv 0$ in (1.1.1.5): Exponential Decay in $Z^s_{\gamma_0}$ and in $\mathcal{D}((-A^s_D)^{\frac{1}{2}})$ of $(P^s_{\gamma_0}h)(t)$ in (1.7.6) with Rate $(\nu\gamma_0)$

Having forced by feedback control $V_{xx} + V_{zz} = ((\partial_{xx} + \partial_{zz})\Psi\tilde{F}^u_{\gamma_0})P^u_{\gamma_0}h$ the arbitrary exponential decay of the finite-dimensional projection $P^u_{\gamma_0}h$ on $Z^u_{\gamma_0}$ in Proposition 1.8.1, we now analyze the consequences on the infinite-dimensional dynamics (1.7.6) for $P^s_{\gamma_0}h$ on $Z^s_{\gamma_0}$. First, we rewrite (1.7.6a) in feedback form as

$$\text{on } Z^s_{\gamma_0}: \qquad [P^s_{\gamma_0}h]_t = \nu A^s_D[P^s_{\gamma_0}h] - \nu P^s_{\gamma_0}D(V_{xx} + V_{zz}) \tag{1.9.1a}$$

$$= \nu A^s_D[P^s_{\gamma_0}h] - \nu P^s_{\gamma_0}D(\partial_{xx} + \partial_{zz})\Psi\tilde{F}^u_{\gamma_0}(P^u_{\gamma_0}w). \tag{1.9.1b}$$

Its solution is

$$(P^s_{\gamma_0}h)(t) = e^{\nu A^s_D t}(P^s_{\gamma_0}h)(0) + \int_0^t e^{\nu A^s_D(t-\tau)}[-\nu P^s_{\gamma_0}D(\partial_{xx} + \partial_{zz})\Psi\tilde{F}^u_{\gamma_0}](P^s_{\gamma_0}h)(\tau)d\tau. \tag{1.9.2}$$

We then have

Proposition 1.9.1. *Assume the hypotheses of Proposition 1.8.1. That is, let* $\psi_j \in H^2(\Gamma_1)$, $\langle\psi_j, 1\rangle_{\Gamma_1} \equiv 0$, $\langle(\partial_{xx} + \partial_{zz})\psi_j, 1\rangle_{\Gamma_1} \equiv 0$, $j = 1, 2, 3, 4$, *satisfy the set of rank conditions (1.8.3). Then, Proposition 1.8.1 applies and yields the feedback operator* $\tilde{F}^u_{\gamma_0}$, *in turn responsible for the feedback decays (1.8.5a–b), (1.8.6). Then,* $(P^s_{\gamma_0}h)$ *in (1.9.1), or (1.9.2), satisfies the following decay:*

(a) If $(P^s_{\gamma_0}h)(0) \in Z^s_{\gamma_0}$*, then:*

$$\|(P^s_{\gamma_0}h)(t)\|_{Z^s_{\gamma_0}} \leq const_{\sigma_0,\gamma_0,\nu} e^{-\nu\gamma_0 t}[\|(P^u_{\gamma_0}h)(0)\|_{Z^u_{\gamma_0}} + \|(P^s_{\gamma_0}h)(0)\|_{Z^s_{\gamma_0}}], \ t \geq 0. \quad (1.9.3)$$

(b) If $(-A^s_D)^{\frac{1}{2}}(P^s_{\gamma_0}h)(0) \in Z^s_{\gamma_0}$ *(and recall (1.2.6)), then:*

$$\begin{aligned} \|(-A^s_D)^{\frac{1}{2}}(P^s_{\gamma_0}h)(t)\|_{Z^s_{\gamma_0}} &\doteq \|(P^s_{\gamma_0}h)(t)\|_{H^1(\Omega)} \\ &\leq \ const_{\sigma_0,\gamma_0,\nu} \ e^{-\nu\gamma_0 t}[\|(-A^s_D)^{\frac{1}{2}}(P^s_{\gamma_0}h)(0)\|_{Z^s_{\gamma_0}} + \|(P^u_{\gamma_0}h)(0)\|_{Z^u_{\gamma_0}}], \ t \geq 0. \end{aligned} \quad (1.9.4)$$

Proof. (a) We recall expansion (1.7.6c) for $e^{\nu A^s_D t}$, where $(n,m,k) \in \mathcal{S}_{\gamma_0}$ means: $\lambda_{nmk} = -[n^2 + (m\pi)^2 + (\frac{\pi}{e}k)^2] < -\gamma_0$, so that

$$\|e^{\nu A^s_D t}\chi\|_{Z^s_{\gamma_0}} = \| \sum_{\substack{i=1,2 \\ (n,m,0)\in\mathcal{S}^a_{\gamma_0}}} e^{-\nu[n^2+(m\pi)^2]t}(\chi, e^i_{nm0})_{L_2(\Omega)}e^i_{nm0} \ + \quad (1.9.5)$$

$$\sum_{\substack{i=3,4 \\ (0,m,k)\in\mathcal{S}^b_{\gamma_0}}} e^{-\nu[(m\pi)^2+(\frac{\pi}{e})^2]t}(\chi, e^i_{0mk})_{L_2(\Omega)}e^i_{0mk} \ + \quad (1.9.6)$$

$$\sum_{\substack{i=5,6,7,8 \\ (n,m,k)\in\mathcal{S}^c_{\gamma_0}}} e^{-\nu[n^2+(m\pi)^2+(\frac{\pi}{e})^2]t}(\chi, e^i_{nmk})_{L_2(\Omega)}e^i_{nmk}\|_{Z^s_{\gamma_0}} \leq e^{-\nu\gamma_0 t}\|\chi\|_{Z^s_{\gamma_0}}, \quad t \geq 0; \quad (1.9.7)$$

$$\|(-A^s_D)^{\frac{1}{2}}e^{\nu A^s_D t}\|_{\mathcal{L}(Z^s_{\gamma_0})} \leq \frac{1}{\sqrt{\nu}} \ \frac{e^{-\nu\gamma_0 t}}{t^{\frac{1}{2}}}, \quad t > 0, \quad (1.9.8)$$

for the exponentially stable, self-adjoint, analytic semigroup $e^{\nu A^s_D t}$ on $Z^s_{\gamma_0}$. Moreover, invoking the exponential decay (1.8.6) for $V_{xx} + V_{zz}$, we obtain

$$\begin{aligned} \|\nu P^s_{\gamma_0} D(V_{xx} + V_{zz})(t)\|_{Z^s_{\gamma_0}} \ &= \ \|\nu P^s_{\gamma_0} D((\partial_{xx} + \partial_{zz})\Psi \tilde{F}^u_{\gamma_0})(P^u_{\gamma_0}h)(t)\|_{Z^s_{\gamma_0}} \\ &\leq \ C_{1,\sigma_0,\nu}e^{-\sigma_0 t}\|(P^u_{\gamma_0}h)(0)\|_{Z^u_{\gamma_0}}, \quad t \geq 0. \end{aligned} \quad (1.9.9)$$

Next, using (1.9.5), (1.9.7) in the variation of parameter formula (1.9.2), we obtain

$$\begin{aligned} \|(P^s_{\gamma_0}h)(t)\|_{Z^s_{\gamma_0}} &\leq e^{-\nu\gamma_0 t}\|(P^s_{\gamma_0}h)(0)\|_{Z^s_{\gamma_0}} \\ &\quad + C_{1,\sigma_0,\nu}\int_0^t e^{-\nu\gamma_0(t-\tau)}e^{-\sigma_0\tau}d\tau\|(P^u_{\gamma_0}h)(0)\|_{Z^u_{\gamma_0}} \quad (1.9.10) \\ &\leq e^{-\nu\gamma_0 t}\left\{\|(P^s_{\gamma_0}h)(0)\| + \frac{C_{1,\sigma_0,\nu}\|(P^s_{\gamma_0}h)(0)\|}{\sigma_0 - \nu\gamma_0}\right\},\ t \geq 0, \quad (1.9.11) \end{aligned}$$

since, with $\sigma_0 - \nu\gamma_0 = (\sigma_1 - \gamma_0)\nu > 0$ (Proposition 1.8.1(b)); we have

$$\begin{aligned} \int_0^t e^{-\nu\gamma_0(t-\tau)}e^{-\sigma_0\tau}d\tau &= e^{-\nu\gamma_0 t}\int_0^t e^{-(\sigma_0-\nu\gamma_0)\tau}d\tau \\ &= e^{-\nu\gamma_0 t}\left[\frac{1-e^{-(\sigma_0-\nu\gamma_0)t}}{\sigma_0-\nu\gamma_0}\right] \quad (1.9.12) \\ &= \frac{e^{-\nu\gamma_0 t} - e^{\sigma_0 t}}{\sigma_0 - \nu\gamma_0} \leq \frac{e^{-\nu\gamma_0 t}}{\sigma_0 - \nu\gamma_0}. \quad (1.9.13) \end{aligned}$$

Thus, (1.9.9) established (1.9.3), as desired.

(b) Applying $(-A^s_D)^{\frac{1}{2}}$ on both sides of identity (1.9.2) yields via (1.7.7)

$$\begin{aligned} (-A^s_D)^{\frac{1}{2}}(P^s_{\gamma_0}h)(t) = &\ e^{\nu A^s_D t}(-A^s_D)^{\frac{1}{2}}(P^s_{\gamma_0}h)(0) \\ &+ \int_0^t (-A^s_D)^{\frac{1}{2}}e^{\nu A^s_D(t-\tau)}[-\nu P^s_{\gamma_0}D(V_{xx}+V_{zz})(\tau)]d\tau, \quad (1.9.14) \end{aligned}$$

whose norm-version, by analyticity of the semigroup $e^{\nu A^s_D t}$, is via (1.9.6), (1.9.7):

$$\begin{aligned} \|(-A^s_D)^{\frac{1}{2}}(P^s_{\gamma_0}h)(t)\|_{Z^s_{\gamma_0}} &\leq e^{-\nu\gamma_0 t}\|(-A^s_D)^{\frac{1}{2}}(P^s_{\gamma_0}h)(0)\|_{Z^s_{\gamma_0}} \\ &\quad + C_{1,\sigma_0,\nu}\int_0^t \frac{e^{-\nu\gamma_0(t-\tau)}e^{-\sigma_0\tau}}{\sqrt{\nu}(t-\tau)^{\frac{1}{2}}}d\tau\|(P^u_{\gamma_0}h)(0)\|_{Z^u_{\gamma_0}}, \end{aligned} \quad (1.9.15)$$

where, setting first $t-\tau=r$, and next $[\sigma_0-\nu\gamma_0]r=\theta$, $[\sigma_0-\nu\gamma_0]dr=d\theta$, we obtain

$$\begin{aligned}\int_0^t \frac{e^{-\nu\gamma_0(t-\tau)}e^{-\sigma_0\tau}}{\sqrt{\nu}(t-\tau)^{\frac{1}{2}}}\,d\tau &= e^{-\sigma_0 t}\int_0^t \frac{e^{(\sigma_0-\nu\gamma_0)r}}{\sqrt{\nu}\sqrt{r}}\,dr \\ &= \frac{e^{-\sigma_0 t}}{\sqrt{\sigma_0-\nu\gamma_0}}\int_0^{(\sigma_0-\nu\gamma_0)t}\frac{e^{\theta}}{\sqrt{\nu}\sqrt{\theta}}\,d\theta. \qquad (1.9.16)\end{aligned}$$

For $(\sigma_0-\nu\gamma_0)t\le 1$, we have

$$\int_0^{(\sigma_0-\nu\gamma_0)t}\frac{e^{\theta}}{\sqrt{\theta}}\,d\theta \le e\int_0^{(\sigma_0-\nu\gamma_0)t}\theta^{-\frac{1}{2}}d\theta = 2e\sqrt{\sigma_0-\nu\gamma_0}\,\sqrt{t}\le 2e. \qquad (1.9.17)$$

For $(\sigma_0-\nu\gamma_0)t>1$, we have, invoking (1.9.15),

$$\begin{aligned}\int_0^{(\sigma_0-\nu\gamma_0)t}\frac{e^{\theta}}{\sqrt{\theta}}\,d\theta &\le \int_0^1\frac{e^{\theta}}{\sqrt{\theta}}\,d\theta+\int_1^{(\sigma_0-\nu\gamma_0)t}e^{\theta}d\theta \\ &\le 2e+e^{(\sigma_0-\nu\gamma_0)t}-e=e^{(\sigma_0-\nu\gamma_0)t}+e. \qquad (1.9.18)\end{aligned}$$

Using estimates (1.9.15) and (1.9.16) into (1.9.14), we obtain

$$\int_0^t \frac{e^{-\nu\gamma_0(t-\tau)}e^{-\sigma_0\tau}}{\sqrt{\nu}(t-\tau)^{\frac{1}{2}}}\,d\tau \le \frac{2e}{\sqrt{\nu}}\,\frac{e^{-\sigma_0 t}}{\sqrt{\sigma_0-\nu\gamma_0}},\ (\sigma_0-\nu\gamma_0)t\le 1; \qquad (1.9.19)$$

$$\int_0^t \frac{e^{-\nu\gamma_0(t-\tau)}e^{-\sigma_0\tau}}{\sqrt{\nu}(t-\tau)^{\frac{1}{2}}}\,d\tau \le \frac{e^{-\nu\gamma_0 t}+ee^{-\sigma_0 t}}{\sqrt{\nu}\,\sqrt{\sigma_0-\nu\gamma_0}}\le\frac{2e}{\sqrt{\nu}}\,\frac{e^{-\nu\gamma_0 t}}{\sqrt{\sigma_0-\nu\gamma_0}},\ (\sigma_0-\nu\gamma_0)t>1, \qquad (1.9.20)$$

since $\sigma_0>\nu\gamma_0$ (Proposition 1.8.1(b)). Then, combining (1.9.17) and (1.9.18), we obtain

$$\int_0^t \frac{e^{-\nu\gamma_0(t-\tau)}e^{-\sigma_0\tau}}{\sqrt{\nu}(t-\tau)^{\frac{1}{2}}}\,d\tau \le \frac{2e}{\sqrt{\nu}}\,\frac{e^{-\nu\gamma_0 t}}{\sqrt{\sigma_0-\nu\gamma_0}},\ t\ge 0. \qquad (1.9.21)$$

Finally, using (1.9.19) into (1.9.13) yields

$$\begin{aligned}\|(-A_D^s)^{\frac{1}{2}}(P_{\gamma_0}^s h)(t)\|_{Z_{\gamma_0}^s} &\le e^{-\nu\gamma_0 t}\|(-A_D^s)^{\frac{1}{2}}(P_{\gamma_0}^s h)(0)\|_{Z_{\gamma_0}^s} \\ &+ \frac{2eC_{1,\sigma_0,\nu}}{\sqrt{\nu}\,\sqrt{\sigma_0-\nu\gamma_0}}\,e^{-\nu\gamma_0 t}\|(P_{\gamma_0}^u h)(0)\|_{Z_{\gamma_0}^u},\ t\ge 0,\end{aligned} \qquad (1.9.22)$$

and (1.9.4) is established. The proof of Proposition 1.9.1 is complete. □

1.10 Linear Case $a \equiv b \equiv 0$ in (1.1.1.5). Preassigned Exponential Decay of Intrinsic Variable $h(t) = [q(t) - DV(t)]$ in (1.7.3). From $h(t)$ to Original Variable $q(t) = (I - \Pi^0)v(t)$ in (1.5.12)

As a corollary of both Proposition 1.8.1 and Proposition 1.9.1, we obtain

Theorem 1.10.1. *Assume the hypotheses of Proposition 1.8.1 (same as those of Proposition 1.9.1) on the functions* $\Psi(x,z) = [\psi_1(x,z), \psi_2(x,z), \psi_3(x,z), \psi_4(x,z)]$, *and let* $\tilde{F}^u_{\gamma_0}$ *be the (stabilizing) feedback operator provided by Proposition 1.8.1(b) and used in Proposition 1.9.1. Then, the intrinsic variable* $h(t) = (P^u_{\gamma_0}h)(t) + (P^s_{\gamma_0}h)(t)$ *satisfies the following preassigned exponential decay:*

(a)

$$\|h(t)\|_Z \leq Const_{\sigma_0,\gamma_0,\nu} e^{-\nu\gamma_0 t}\|h(0)\|_Z, \quad t \geq 0,\ h(0) \in Z \subset L_2(\Omega); \tag{1.10.1}$$

(b)

$$\|h(t)\|_{H^1(\Omega)} \leq Const_{\sigma_0,\gamma_0,\nu}\, e^{-\nu\gamma_0 t}\|h(0)\|_{H^1(\Omega)}, \quad t \geq 0,\ h(0) \in H^1(\Omega). \tag{1.10.2}$$

Proof. (a) We invoke the decay (1.8.5b) for the finite-dimensional component $(P^u_{\gamma_0}h)(t)$ (with $\sigma_0 = \nu\sigma_1$ as in Proposition 1.8.1b)) as well as the decay (1.9.3) in $Z^s_{\gamma_0} \subset L_2(\Omega)$ for the complementary infinite-dimensional component $P^s_{\gamma_0} h$, where $h(t) = (P^u_{\gamma_0} h)(t) + (P^s_{\gamma_0} h)(t)$. We thus obtain

$$\begin{aligned} \|h(t)\|_Z &\leq \|(P^u_{\gamma_0} h)(t)\|_{Z^u_{\gamma_0}} + \|(P^s_{\gamma_0} h)(t)\|_{Z^s_{\gamma_0}} & (1.10.3) \\ &\leq C_{\sigma_0} e^{-\sigma_0 t} \|(P^u_{\gamma_0} h)(0)\|_{Z^u_{\gamma_0}} \\ &\quad + \text{Const}_{\sigma_0,\gamma_0,\nu}\, e^{-\nu\gamma_0 t} [\|(P^u_{\gamma_0} h)(0)\|_{Z^u_{\gamma_0}} + \|(P^s_{\gamma_0} h)(0)\|_{Z^s_{\gamma_0}}] \\ &\leq \text{Const}_{\sigma_0,\gamma_0,\nu} e^{-\nu\gamma_0 t} \|h(0)\|_Z, \quad t \geq 0, & (1.10.4) \end{aligned}$$

since $\sigma_0 = \nu\sigma_1 > \nu\gamma_0$ by Proposition 1.8.1(b).

(b) Similarly, this time invoking (1.8.5b) and (1.9.4):

$$\begin{aligned} \|h(t)\|_{H^1(\Omega)} &\leq \|(P^u_{\gamma_0} h)(t)\|_{Z^u_{\gamma_0}} + \|(-A^s_D)^{\frac{1}{2}} (P^s_{\gamma_0} h)(t)\|_{Z^s_{\gamma_0}} & (1.10.5) \\ &\leq \text{Const}_{\sigma_0,\gamma_0,\nu} e^{-\nu\gamma_0 t} \|h(0)\|_{H^1(\Omega)}, \quad t \geq 0. & (1.10.6) \end{aligned}$$

Then, (1.10.4) and (1.10.6) establish conclusions (1.10.1) and (1.10.2), as desired. □

Return from $h(t)$ to $q(t)$. We return to the definition (1.7.3) of the intrinsic variable $h(t)$ and write

$$q(t) = h(t) + DV(t) = h(t) + D\Psi(x,z) \begin{bmatrix} \varphi_1(t) \\ \varphi_2(t) \\ \varphi_3(t) \\ \varphi_4(t) \end{bmatrix} = h(t) + D\Psi(x,z) \tilde{F}^u_{\gamma_0} (P^u_{\gamma_0} h)(t), \tag{1.10.7}$$

after recalling the structure of $V(t)$ in (1.3.7) with $J=4$, and of $\Phi(t)$ in (1.7.8), with feedback matrix $\tilde{F}^u_{\gamma_0}$ provided by Proposition 1.8.1(b). We then have

Theorem 1.10.2. *Assume the hypotheses of Proposition 1.8.1 (same as those of Proposition 1.9.1). Then, we have the following preassigned decay for $q(t) = (I - \Pi^0)v(t) \in Z$:*

(a)

$$\|q(t)\|_Z \le Const_{\sigma_0,\gamma_0,\nu} e^{-\nu\gamma_0 t} \|h(0)\|_Z, \quad t \ge 0, \; h(0) \in Z \subset L_2(\Omega); \tag{1.10.8}$$

(b)

$$\|q(t)\|_{H^1(\Omega)} \le Const_{\sigma_0,\gamma_0,\nu} e^{-\nu\gamma_0 t} \|h(0)\|_{H^1(\Omega)}, \quad t \ge 0, \; h(0) \in H^1(\Omega). \tag{1.10.9}$$

Proof. By assumption we have *a-fortiori* $\psi_j \in H^{\frac{1}{2}}(\Gamma_1)$, hence $D\psi_j \in H^1(\Omega)$ by recalling (1.2.1g). Next, returning to identity (1.10.7) and invoking here (1.10.1), (1.10.2) of Theorem 1.10.1 for the decay of $h(t)$, and (1.8.5b) for the decay of $(P^u_{\gamma_0} w)(t)$, where $\sigma_0 = \nu\sigma_1 > \nu\gamma_0$ by Proposition 1.8.1(b), we obtain readily (1.10.8) and (1.10.9). □

As a corollary of Theorem 1.10.2, eq. (1.10.9) for $q(t) = (I - \Pi^0)v(t)$, we obtain a preassigned exponential decay of v_x and v_z.

Corollary 1.10.3. *Assume the hypotheses of Theorem 1.10.2.*

(a) We have

$$v_x = q_x = \partial_x (I - \Pi^0) v = (I - \Pi^0) v_x, \quad \textit{since } \partial_x(\Pi^0 v) = \Pi^0 v_x = 0. \;\; \textit{Similarly, } v_z = q_z. \tag{1.10.10}$$

(b) Furthermore,

$$\|v_x\|_{L_2(\Omega)} = \|q_x\|_{L_2(\Omega)} \leq Const_{\sigma_0,\gamma_0,\nu} e^{-\nu\gamma_0 t}\|h(0)\|_{H^1(\Omega)}, \quad t \geq 0,$$

$$\|v_z\|_{L_2(\Omega)} = \|q_z\|_{L_2(\Omega)} \leq Const_{\sigma_0,\gamma_0,\nu} e^{-\nu\gamma_0 t}\|h(0)\|_{H^1(\Omega)}, \quad t \geq 0. \quad (1.10.11)$$

Proof. (a) From $v = \Pi^0 v + (I - \Pi^0)v$, $q = (I - \Pi^0)v$, and $v_x = \Pi^0 v_x + (I - \Pi^0)v_x$, we obtain (1.10.10), where indeed, $\Pi^0 \partial_x v = 0$ precisely as in the proof of (1.6.12). Similarly, $\Pi^0 \partial_z v = 0$ and the same argument for v_z follows.

(b) From (1.10.10), as $v_x = q_x$ and $v_z = q_z$, we obtain (1.10.11) by (1.10.9) of Theorem 1.10.2. □

1.11 Arbitrarily Preassigned Exponential Decay of p in $H^{\frac{3}{2}}(\Omega)$

Regarding the pressure $(-p)$, the results of previous sections imply

Theorem 1.11.1. *Assume the hypotheses of Proposition 1.8.1 (same as those of Proposition 1.9.1 and Theorem 1.10.1 or 1.10.2). Then*

$$\|p(t)\|_{H^{\frac{3}{2}}(\Omega)} \leq Const\ e^{-\sigma_0 t}\|(P^u_{\gamma_0} h)(0)\|_{Z^u_{\gamma_0}}, \quad (1.11.1)$$

where Const depends on $\sigma_0, \Psi, \tilde{F}^u_{\gamma_0}, \nu, A^u_D, P^u_{\gamma_0} D$.

Proof. Step 1. We return to problem (1.1.1.12) with $a \equiv b \equiv 0$ (hence $\overline{U}(y) \equiv \overline{W}(y) = 0$, as in the present setting since Section 1.7). Its solution is then given by (compare with (1.3.1))

$$p = NV_t - \nu N(V_{xx} + V_{zz}). \tag{1.11.2}$$

In the present setting, let A_N be the Neumann counterpart of the Dirichlet operator A_D in (1.2.2); that is

$$A_N f = \Delta f : L_2^0(\Omega) \supset \mathcal{D}(A_N) \to L_2^0(\Omega); \tag{1.11.3a}$$

$$\mathcal{D}(A_N) = \{f \in H^2(\Omega)/\mathcal{N}(A_N) : \ f_y|_{y=0} = f_y|_{y=1} = 0;\ f|_{x=-\pi} = f|_{x=\pi};$$
$$f_x|_{x=-\pi} = f_x|_{x=\pi}; f|_{z=-e} = f|_{z=e}; f_z|_{z=-e} = f_z|_{z=e}\}, \tag{1.11.3b}$$

$$L_2^0(\Omega) = L_2(\Omega)/\mathcal{N}(A_N), \tag{1.11.3c}$$

the factor space of $L_2(\Omega)$ modulo the 1-dimensional null space $\mathcal{N}(A_N)$ of constant functions. Moreover, the Neumann map N is the counterpart of the Dirichlet map D in (1.2.1):

$$f = Ng \iff \begin{cases} f_{xx} + f_{yy} + f_{zz} = 0, \text{ in } \Omega; & (1.11.4a) \\ f_y(x,0,z) \equiv 0;\ f_y(x,1,z) = g \text{ on } \Gamma_0, \Gamma_1; & (1.11.4b) \\ \text{B.C.} \begin{cases} f(-\pi,y,z) = f(\pi,y,z); & (1.11.4c) \\ f_x(-\pi,y,z) = f_x(\pi,y,z); & (1.11.4d) \end{cases} \\ f(x,y,-e) = f(x,y,e); & (1.11.4e) \\ f_z(x,y,-e) = f_z(x,y,e), & (1.11.4f) \end{cases}$$

$$N : H^s(\Gamma_1) \to H^{\frac{3}{2}+s}(\Omega)/\mathcal{N}(A_N), \quad s \geq 0. \tag{1.11.4g}$$

Step 2. From (1.3.7) (with $J=4$), (1.7.8) and (1.7.5), we obtain

$$V = \Psi(x,z)\begin{bmatrix}\varphi_1(t)\\ \varphi_2(t)\\ \varphi_3(t)\\ \varphi_4(t)\end{bmatrix} = \Psi(x,z)\tilde{F}^u_{\gamma_0}P^u_{\gamma_0}h(t); \tag{1.11.5}$$

$$V_t = \Psi(x,z)\tilde{F}^u_{\gamma_0}[P^u_{\gamma_0}h(t)]_t = \Psi(x,z)\tilde{F}^u_{\gamma_0}\big\{\nu A^u_D[P^u_{\gamma_0}h] - \nu P^u_{\gamma_0}D(V_{xx}+V_{zz})\big\}, \tag{1.11.6}$$

where $\tilde{F}^u_{\gamma_0}$ is the feedback operator provided by Proposition 1.8.1(b).

Step 3. (Under the stated assumptions on $\Psi(x,z) = [\psi_1(x,z), \psi_2(x,z), \psi_3(x,z), \psi_4(x,z)]$.) From (1.11.6) we estimate, by virtue also of the decays (1.8.5b) and (1.8.6) for $(P^u_{\gamma_0}h)$ and $V_{xx}+V_{zz}$, respectively:

$$\|V_t(t)\|_{L_2(\Gamma_1)} \le \text{Const}\left[\|(P^u_{\gamma_0}h)(t)\|_{Z^u_{\gamma_0}} + \|(V_{xx}+V_{zz})(t)\|_{L_2(\Gamma_1)}\right] \tag{1.11.7}$$

$$\le \text{Const}e^{-\sigma_0 t}\|(P^u_{\gamma_0}w)(0)\|_{Z^u_{\gamma_0}}, \tag{1.11.8}$$

where Const depends on σ_0, $\Psi\tilde{F}_{\gamma_0}$, ν, A^u_D, $(P^u_{\gamma_0}D)$. Substituting (1.11.8) for V_t and (1.8.6) for $(V_{xx}+V_{zz})$ into (1.11.2) and recalling *a-fortiori* from (1.11.4g) that

$$N: \text{ continuous } L_2(\Gamma_1) \to H^{\frac{3}{2}}(\Omega),$$

we finally obtain (1.11.1). □

1.12 Linear Case $a \equiv b \equiv 0$ in (1.1.1.5): Exponential Decay of Vorticity ω in $L_2(\Omega)$ with Rate $(\nu\pi^2)$

We return to the vorticity problem (1.1.1.7) (with $a \equiv b \equiv 0$ so that $\overline{U}(y) \equiv \overline{U}''(y) \equiv \overline{W}(y) \equiv \overline{W}''(y) \equiv 0$).

Theorem 1.12.1. *(i) The abstract model of the ω-problem (1.1.1.7a–h) in vector form with $a \equiv b \equiv 0$ is*

$$\omega_t = \nu A_D \omega + \nu A_D D(-V_z, 0, V_x) \in ([\mathcal{D}(A_D)]')^3, \tag{1.12.1}$$

where in the second term, A_D is actually the isomorphic extension $A_D : L_2(\Omega) \to [\mathcal{D}(A_D^*)]'$ of the original self-adjoint operator A_D.

(ii) Let $\langle \frac{\partial}{\partial_x}\psi_j, 1\rangle_{\Gamma_1} = \langle \frac{\partial}{\partial_z}\psi_j, 1\rangle_{\Gamma_1} = 0,\ j = 1, 2, 3, 4$. Then, the projection $\Pi^0\omega$ of ω onto the eigenspace $(E^0)^3$ in (1.5.1b) satisfies

$$\text{on } (E^0)^3 :\ (\Pi^0\omega)_t = \nu A_D^0(\Pi^0\omega), \quad A_D^0 = A_D|_{E^0},\ \text{see (1.5.6b)}, \tag{1.12.2}$$

whose solution is

$$\text{on } (E^0)^3 :\ \Pi^0\omega(t) = e^{\nu A_D^0 t}\Pi^0\omega(0), t \geq 0,\ \Pi^0\omega(0) \in (E^0)^3, \tag{1.12.3}$$

and whose components are given by

$$\text{on } E^0 := \sum_{m=1}^{\infty} e^{\nu\lambda_{0m0}t}(\Pi^0\omega^i(0), e^0_{0m0})_\Omega e^0_{0m0},\ i = 1, 2, 3,\ t \geq 0,\ \Pi^0\omega(0) \in E^0. \tag{1.12.4}$$

Thus $\Pi^0\omega$ is control-free. Thus the three component corresponding to $i = 1, 2, 3$ satisfy the exponential decays

$$\begin{aligned} &\|\Pi^0\omega^i(t)\|_{E^0} \\ &\quad \leq \ e^{-\nu\pi^2 t}\|\Pi^0\omega^i(0)\|_{E^0}, \ t \geq 0, \ \Pi^0\omega^i(0) \in E^0 \end{aligned} \tag{1.12.5}$$

$$\begin{aligned} &\|\Pi^0\omega^i(t)\|_{\mathcal{D}((-A_D^0)^{\frac{1}{2}})} \\ &\quad \leq \ e^{-\nu\pi^2 t}\|\Pi^0\omega^i(0)\|_{\mathcal{D}((-A_D^0)^{\frac{1}{2}})}, \ \Pi^0\omega^i(0) \in \mathcal{D}((-A_D^0)^{\frac{1}{2}}) \subset H^1(\Omega). \end{aligned} \tag{1.12.6}$$

(iii) In addition to (ii), assume the hypotheses of Proposition 1.8.1: thus, also $\psi_j \in H^2(\Gamma_1)$, $\langle \psi_j, 1\rangle_{\Gamma_1} \equiv 0$, $\langle (\partial_{xx} + \partial_{zz})\psi_j, 1\rangle_{\Gamma_1} = 0$ and the rank conditions (1.8.3). Then the projection $(I - \Pi^0)\omega$ of ω onto the eigenspace $(E^{abc})^3 \equiv Z^3$ in (1.5.4b) satisfies

$$\text{in } Z^3: \ [(I - \Pi^0)\omega]_t = \nu A_D[(I - \Pi^0)\omega] + \nu A_D D(-V_z, 0, V_x) \in ([\mathcal{D}(A_D)]')^3, \tag{1.12.7}$$

where actually $A_D = A_D|_Z$, whose solution has components

$$
\begin{aligned}
[(I-\Pi^0)\omega^1(t)] &= e^{\nu A_D t}[(I-\Pi^0)\omega^1(0)] - \int_0^t \nu A_D e^{\nu A_D (t-\tau)} DV_z(\tau) d\tau && (1.12.8)\\
&= e^{\nu A_D t}[(I-\Pi^0)\omega^1(0)] - e^{\nu A_D t} DV_z(0) + DV_z(t)\\
&- \int_0^t e^{\nu A_D (t-\tau)} DV_{zt}(\tau) d\tau, && (1.12.9)\\
[(I-\Pi^0)\omega^2(t)] &= e^{\nu A_D t}[(I-\Pi^0)\omega^2(0)], && (1.12.10)\\
[(I-\Pi^0)\omega^3(t)] &= e^{\nu A_D t}[(I-\Pi^0)\omega^3(0)] + \int_0^t \nu A_D e^{\nu A_D (t-\tau)} DV_x(\tau) d\tau && (1.12.11)\\
&= e^{\nu A_D t}[(I-\Pi^0)\omega^3(0)] + e^{\nu A_D t} DV_x(0) - DV_x(t)\\
&+ \int_0^t e^{\nu A_D (t-\tau)} DV_{xt}(\tau) d\tau. && (1.12.12)
\end{aligned}
$$

They satisfy the exponential decays

$$
\|(I-\Pi^0)\omega^i(t)\|_Z \;\le\; \text{Const } e^{-\nu(1+\pi^2+\left(\frac{\pi}{e}\right)^2)t}\left[\|(I-\Pi^0)\omega^i(0)\|_Z + \|(P^u_{\gamma_0}h)(0)\|_{Z^u_{\gamma_0}}\right], \quad t \ge 0,\ i = 1,3; \qquad (1.12.13)
$$

$$
\|(I-\Pi^0)\omega^2(t)\|_Z \;\le\; \text{Const } e^{-\nu(1+\pi^2+\left(\frac{\pi}{e}\right)^2)t}|(I-\Pi^0)\omega^2(0)\|_Z, \quad t \ge 0, \qquad (1.12.14)
$$

where Const *depends on* $\|D\|, \sigma_0,$ *and* C_{σ_0}.

Proof. (i) As in the proof of Proposition 1.3.2, the ω-problem (1.1.7a–h) with $a \equiv b \equiv 0$ can be rewritten in vector form as

$$\begin{cases} \omega_t - \nu\Delta(\omega - D(-V_z, 0, V_x)) = 0, \quad \text{in } Q^3; & (1.12.15a) \\ (\omega - D(-V_z,0,V_x))|_{y=0} = 0;\ (\omega - D(-V_z,0,V_x))|_{y=1} = 0; & (1.12.15b) \\ (\omega - D(-V_z,0,V_x))|_{x=-\pi} \equiv (\omega - D(-V_z,0,V_x))|_{x=\pi}; & \\ \quad (\omega - D(-V_z,0,V_x))_x|_{x=-\pi} = (\omega - D(-V_z,0,V_x))_x|_{x=\pi}, & (1.12.15c) \\ (\omega - D(-V_z,0,V_x))|_{z=-e} \equiv (\omega - D(-V_z,0,V_x))|_{z=e}; & \\ \quad (\omega - D(-V_z,0,V_x))_z|_{z=-e} = (\omega - D(-V_z,0,V_x))_z|_{z=e}, & (1.12.15d) \end{cases}$$

recalling the definition of D in (1.2.1). Then, the abstract version of (1.12.15a–d) is precisely: $\omega_t - \nu A_D(\omega - D(-V_z, 0, V_x) = 0$ in $L_2(\Omega)$, hence (1.12.1) in $[\mathcal{D}(A_D^*)]' = [\mathcal{D}(A_D)]'$, after the indicated extension of A_D.

(ii) Under the assumption $\langle \frac{\partial}{\partial x}\psi_j, 1\rangle_{\Gamma_1} = \langle \frac{\partial}{\partial z}\psi_j, 1\rangle_{\Gamma_1} = 0$ we have $\Pi^0 D(\frac{\partial}{\partial x}\psi_j) = \Pi^0 D(\frac{\partial}{\partial z}\psi_j) = 0$, $(I - \Pi^0)D(\frac{\partial}{\partial x}\psi_j) = D(\frac{\partial}{\partial x}\psi_j)$ and $(I - \Pi^0)D(\frac{\partial}{\partial z}\psi_j) = D(\frac{\partial}{\partial z}\psi_j)$ by (1.5.8c) of Proposition 1.5.1: thus, applying the projector Π^0 to (12.1) yields (1.12.2), from which (1.12.3)–(1.12.6) readily follow.

(iii) Similarly, $(I - \Pi^0)$ applied to (1.12.1) yields (1.12.7), whose variation of parameter formulas (1.12.8) and (1.12.11) yield (1.12.9) and (1.12.12) after integration by parts in t. We now establish the exponential decays (1.12.13) and (1.12.14) from (1.12.9), (1.12.12), and (1.12.10). First, for $\zeta \in Z$ we have

$$e^{\nu A_D t}\zeta = \sum_{\substack{i=1,2 \\ n,m=1}}^{\infty} e^{\nu\lambda_{nm0}t}(\zeta, e^i_{nm0})_\Omega e^i_{nm0} + \sum_{\substack{i=3,4 \\ m,k=1}}^{\infty} e^{\nu\lambda_{0mk}t}(\zeta, e^i_{0mk})_\Omega e^i_{0mk}$$

$$+\sum_{\substack{i=5,6,7,8\\ n,m,k=1}}^{\infty} e^{\nu\lambda_{nmk}t}(\zeta, e^i_{nmk})_\Omega e^i_{nmk}; \quad \|e^{\nu A_D t}\zeta\|_Z \;\leq\; e^{-\nu(1+\pi^2+\left(\frac{\pi}{e}\right)^2)t}\|\zeta\|_Z, \quad t \geq 0, \quad (1.12.16)$$

recalling λ_{nmk} in (1.2.4) and (1.2.5) and selecting $n = m = k = 1$. Next, from (1.3.7) (with $J = 4$) and (1.7.8),

$$V_z(t) = \frac{\partial}{\partial z}\Psi(x,z)\tilde{F}^u_{\gamma_0}P^u_{\gamma_0}h(t) \qquad (1.12.17)$$

with $\tilde{F}^u_{\gamma_0}$ the feedback operator provided by Proposition 1.8.1(b) under present assumptions in (iii). Hence, recalling (1.8.5b) in (1.12.17) yields

$$\|V_z(t)\|_{L_2(\Gamma_1)} \leq \text{const}\|P^u_{\gamma_0}h(t)\|_{Z^u_{\gamma_0}} \leq \text{Const}\; e^{-\sigma_0 t}\|P^u_{\gamma_0}h(0)\|_{Z^u_{\gamma_0}},\; t \geq 0, \qquad (1.12.18)$$

with const $= \|\frac{\partial}{\partial x}\Psi\tilde{F}^u_{\gamma_0}\|$ and Const $=$ const C_{σ_0}. Next again, by (1.3.7) (with $J = 4$), (1.7.8) and (1.7.5), we obtain from (1.12.17):

$$V_{zt}(t) = \frac{\partial}{\partial z}\Psi(x,z)\tilde{F}^u_{\gamma_0}[P^u_{\gamma_0}h(t)]_t = \frac{\partial}{\partial z}\Psi(x,z)\tilde{F}^u_{\gamma_0}\{\nu A^u_D[P^u_{\gamma_0}h] - \nu P^u_{\gamma_0}D(V_{xx}+V_{zz})\} \qquad (1.12.19)$$

(compare with (1.11.6)). Thus, we apply on (1.12.19) the decays (1.8.5b) and (1.8.6) for $(P^u_{\gamma_0}h)$ and $V_{xx}+V_{zz}$ and obtain (compare with (1.11.7), (1.11.8)):

$$\begin{aligned}\|V_{zt}(t)\|_{L_2(\Gamma_1)} &\leq \text{Const}\left[\|(P^u_{\gamma_0}h)(t)\|_{Z^u_{\gamma_0}} + \|(V_{xx}+V_{zz})(t)\|_{L_2(\Gamma_1)}\right] &(1.12.20)\\ &\leq \text{const}\; e^{-\sigma_0 t}\|(P^u_{\gamma_0}h)(0)\|_{Z^u_{\gamma_0}}. &(1.12.21)\end{aligned}$$

Thus, using (1.12.16), (1.12.18), (1.12.21) in estimating (1.12.9), we obtain, since

$\|DV_z(0)\|_Z \le \|D\frac{\partial}{\partial z}\Psi\tilde{F}^u_{\gamma_0}\| \; \|P^u_{\gamma_0}h(0)\|_{Z^u_{\gamma_0}}$ from (1.12.17):

$$\begin{aligned}\|(I-\Pi^0)\omega^1(t)\|_Z &\le e^{-\nu(1+\pi^2+\left(\frac{\pi}{e}\right)^2)t}\left[\|(I-\Pi^0)\omega^1(0)\|_Z + \text{const}\|P^u_{\gamma_0}h(0)\|_{Z^u_{\gamma_0}}\right] \\ &+ \|D\| \text{ Const } e^{-\sigma_0 t}\|P^u_{\gamma_0}h(0)\|_{Z^u_{\gamma_0}} \\ &+ \text{const}\int_0^t e^{-\nu(1+\pi^2+\left(\frac{\pi}{e}\right)^2)(t-\tau)}e^{-\sigma_0\tau}d\tau\|P^u_{\gamma_0}h(0)\|_{Z^u_{\gamma_0}}, \quad (1.12.22)\end{aligned}$$

where, recalling an estimate such as (1.9.12) for the last term in (1.12.22), with γ_0 replaced by $(1+\pi^2+\left(\frac{\pi}{e}\right)^2)$ (where $\sigma_0 > \nu(1+\pi^2+\left(\frac{\pi}{e}\right)^2)$), we have

$$\int_0^t e^{-\nu(1+\pi^2+\left(\frac{\pi}{e}\right)^2)(t-\tau)}e^{-\sigma_0\tau}d\tau \le \frac{e^{-\nu\gamma_0 t}}{\sigma_0-\nu(1+\pi^2+\left(\frac{\pi}{e}\right)^2)}, \quad t\ge 0. \qquad (1.12.23)$$

Then, (1.12.22), (1.12.23) yield (1.12.13) with $i=1$, as desired. (1.12.13) with $i=3$ and (1.12.14) are established similarly. □

1.13 Linear Case $a \equiv b \equiv 0$ in (1.1.1.5): Exponential Decay of u and w in $H^1(\Omega)$

First approach: From ω and v to u and w. From the exponential decays in (1.12.5), (1.12.13), and (1.12.14) of Proposition 1.12.1 for the vorticity ω, see (1.1.1.6), as well as the continuity of elliptic solution operator Ψ defined in (1.1.1.8) giving the components of velocity (u,v,w) (modulo the one dimensional null space of A_N from the first and third components) as a function of vorticity ω (see (1.1.1.10)), we obtain the following result as a direct corollary.

Theorem 1.13.1. *Assume the setting and the hypotheses of Theorem 1.12.1(iii) (which include* $\langle \psi_j, 1\rangle_{\Gamma_1} = \langle \frac{\partial}{\partial x}\psi_j, 1\rangle_{\Gamma_1} = \langle \frac{\partial}{\partial z}\psi_j, 1\rangle_{\Gamma_1} = \langle (\partial_{xx} + \partial_{zz}\psi_j, 1\rangle_{\Gamma_1} = 0$ *plus the rank conditions (1.8.3)). Then*

$$\|u\|_{L_2(\Omega)/\mathbb{R}} + \|\nabla u\|_{L_2(\Omega)}$$

$$\leq Const\ e^{-\nu\pi^2 t}\left[\|u(0)\|_{H^1(\Omega)} + \|v(0)\|_{H^1(\Omega)} + \|w(0)\|_{H^1(\Omega)} + \|h(0)\|_{H^1(\Omega)}\right], \quad (1.13.1)$$

$$\|w\|_{L_2(\Omega)/\mathbb{R}} + \|\nabla w\|_{L_2(\Omega)}$$

$$\leq Const\ e^{-\nu\pi^2 t}\left[\|u(0)\|_{H^1(\Omega)} + \|v(0)\|_{H^1(\Omega)} + \|w(0)\|_{H^1(\Omega)} + \|h(0)\|_{H^1(\Omega)}\right], \quad (1.13.2)$$

where

$$\|f\|_{L_2(\Omega)/\mathbb{R}} = \|f - aver(u)\|_{L_2(\Omega)}, \quad aver(f) = \frac{\int_\Omega f\, d\Omega}{|\Omega|}. \quad (1.13.3)$$

Proof. We have

$$\|u\|_{L_2(\Omega)/\mathbb{R}} + \|\nabla u\|_{L_2(\Omega)} \equiv \|u\|_{H^{0,1}(\Omega)} \quad (1.13.4)$$

$$\leq \|\Psi^1\|_{(L_2(\Omega))^3 \to H^{0,1}} \|\omega\|_{L_2(\Omega)^3} \quad (1.13.5)$$

$$\leq \|\Psi^1\|(\|\Pi^0\omega\|_{L_2(\Omega)^3} + \|(I - \Pi^0)\omega\|_{L_2(\Omega)^3} \quad (1.13.6)$$

where $\|\Psi^1\|$ denotes $\|\Psi^1\|_{(L_2(\Omega))^3 \to H^{0,1}}$, the operator norm of Ψ^1 from $(L_2(\Omega))^3$ to $H^{0,1}$ guaranteed by the continuity of Ψ with $s = 0$ in (1.1.1.10). Then, recalling the exponential bounds (1.12.5) on $\Pi^0\omega^i$ and (1.12.13–14) on ω^i, we obtain

$$\|u\|_{L_2(\Omega)/\mathbb{R}} + \|\nabla u\|_{L_2(\Omega)}$$

$$\leq \ \|\Psi^1\| \ \left(e^{-\nu\pi^2 t}\|\Pi^0\omega^(0)\|_{L_2(\Omega)^3} + \|(I-\Pi^0)\omega\|_{L_2(\Omega)^3}\right) \tag{1.13.7}$$

$$\leq \ \|\Psi^1\| \ (e^{-\nu\pi^2 t}\|\Pi^0\omega(0)\|_{L_2(\Omega)^3} +$$

$$Const_1 \ e^{-\nu(1+\pi^2+\left(\frac{\pi}{e}\right)^2)t}\left[\|(I-\Pi^0)\omega(0)\|_{L_2(\Omega)^3} + 2\|(P^u_{\gamma_0}h)(0)\|_{L_2(\Omega)}\right]) \tag{1.13.8}$$

$$\leq \ \|\Psi^1\| \ Const_2 \ e^{-\nu\pi^2 t} \ (\|\Pi^0\omega(0)\|_{L_2(\Omega)^3} +$$

$$\left[\|(I-\Pi^0)\omega(0)\|_{L_2(\Omega)^3} + \|(P^u_{\gamma_0}h)(0)\|_{L_2(\Omega)}\right]) \tag{1.13.9}$$

$$\leq \ \|\Psi^1\| \ Const_2 \ e^{-\nu\pi^2 t} \ (2\|\omega(0)\|_{L_2(\Omega)^3} + \|h(0)\|_{L_2(\Omega)}) \tag{1.13.10}$$

$$\leq \ Const_3 \ e^{-\nu\pi^2 t} \ \left(\|\omega(0)\|_{L_2(\Omega)^3} + \|h(0)\|_{L_2(\Omega)}\right) \tag{1.13.11}$$

$$\leq \ Const_3 \ e^{-\nu\pi^2 t} \ \left(\|(u,v,w)(0)\|_{H^1(\Omega)^3} + \|h(0)\|_{L_2(\Omega)}\right), \tag{1.13.12}$$

where $Const_3$ depends on $\|\Psi^1\|_{(L_2(\Omega))^3\to H^{0,1}}$.

□

Second direct approach: It requires the weaker setting of Theorem 1.11.1 (same as that of Theorem 1.10.1, i.e., without $\langle \frac{\partial}{\partial x}\psi_j, 1\rangle_{\Gamma_1} = \langle \frac{\partial}{\partial z}\psi_j, 1\rangle_{\Gamma_1} = 0$). This approach is direct: it does not need *a-priori* information on the vorticity ω. Thus, it does not need $\langle \frac{\partial}{\partial x}\psi_j, 1\rangle_{\Gamma_1} = \langle \frac{\partial}{\partial z}\psi_j, 1\rangle_{\Gamma_1} = 0$ of Theorem 1.12.1 which was inherited in the first proof above. Rather, it yields information on the vorticity. We return to the u, w-problem (with $a \equiv b \equiv 0$ in (1.1.1.5)); that is, to the problem consisting of Eqns. (1.7.1a), (1.7.1c), (1.7.1e–i), and (1.7.1o–s). The abstract versions are

$$u_t - \nu A_N u = p_x, \tag{1.13.13}$$

$$w_t - \nu A_N w = p_z, \tag{1.13.14}$$

where the operator A_N is defined by (1.11.3a–b), and p_x and p_z are expressed explicitly in Remark 1.13.1 below. Their solutions are

$$u(t) = e^{\nu A_N t}u(0) + \int_0^t e^{\nu A_N(t-\tau)}p_x(\tau)d\tau, \quad u(0) \in L_2(\Omega); \tag{1.13.15}$$

$$w(t) = e^{\nu A_N t}w(0) + \int_0^t e^{\nu A_N(t-\tau)}p_z(\tau)d\tau, \quad w(0) \in L_2(\Omega); \tag{1.13.16}$$

$$(-A_N)^{\frac{1}{2}}u(t) = e^{\nu A_N t}(-A_N)^{\frac{1}{2}}u(0) + \int_0^t (-A_N)^{\frac{1}{2}}e^{\nu A_N(t-\tau)}p_x(\tau)d\tau, \quad u(0) \in \mathcal{D}((-A_N)^{\frac{1}{2}}) \subset H^1(\Omega). \tag{1.13.17}$$

$$(-A_N)^{\frac{1}{2}}w(t) = e^{\nu A_N t}(-A_N)^{\frac{1}{2}}w(0) + \int_0^t (-A_N)^{\frac{1}{2}}e^{\nu A_N(t-\tau)}p_z(\tau)d\tau, \quad w(0) \in \mathcal{D}((-A_N)^{\frac{1}{2}}) \subset H^1(\Omega). \tag{1.13.18}$$

Eigenvalues/vectors of the negative self-adjoint operator A_N on $L_2^0(\Omega)$. This direct approach, unlike the preceding one which dealt exclusively with A_D, requires the eigenvalues/vectors of A_N on $L_2^0(\Omega)$. Letting $n, m, k = 1, 2, 3, \ldots$, they are (compare with Proposition 1.2.1 for A_D):

$$\lambda_{0m0} = -(m\pi)^2; \quad g_{0m0}^0 = \frac{1}{\sqrt{2\pi e}}\cos m\pi y; \tag{1.13.19}$$

$$\lambda_{nm0} = -[n^2 + (m\pi)^2]; \quad \begin{cases} g^1_{nm0} = \frac{1}{\sqrt{\pi e}} \sin nx \cos m\pi y; & (1.13.20a) \\ g^2_{nm0} = \frac{1}{\sqrt{\pi e}} \sin nx \cos m\pi y, & (1.13.20b) \end{cases}$$

$$\lambda_{0mk} = -\left[(m\pi)^2 + \left(\frac{\pi}{e}k\right)^2\right]; \quad \begin{cases} g^3_{0mk} = \frac{1}{\sqrt{\pi e}} \sin \frac{\pi}{e}kz \cos m\pi y; & (1.13.21a) \\ g^4_{0mk} = \frac{1}{\sqrt{\pi e}} \sin \frac{\pi}{e}kz \cos m\pi y, & (1.13.21b) \end{cases}$$

$$\lambda_{nmk} = -\left[n^2 + (m\pi)^2 + \left(\frac{\pi}{e}k\right)^2\right];$$

$$\begin{cases} g^5_{nmk} = \sqrt{\frac{2}{\pi e}} \sin nx \sin \frac{\pi}{e}kz \cos m\pi y; & (1.13.22a) \\ g^6_{nmk} = \sqrt{\frac{2}{\pi e}} \sin nx \cos \frac{\pi}{e}kz \cos m\pi y; & (1.13.22b) \\ g^7_{nmk} = \sqrt{\frac{2}{\pi e}} \cos nx \sin \frac{\pi}{e}kz \cos m\pi y; & (1.13.22c) \\ g^8_{nmk} = \sqrt{\frac{2}{\pi e}} \cos nx \cos \frac{\pi}{e}kz \cos m\pi y, & (1.13.22d) \end{cases}$$

where $\{g^i_{nmk}\}$ form an orthonormal basis in $L^0_2(\Omega)$. The self-adjoint analytic semigroup $e^{\nu A_N t}$ is given by

$$e^{\nu A_N t} f = \sum_{m=1}^{\infty} e^{-\nu(m\pi)^2 t} (f, g^0_{0m0})_\Omega \; g^0_{0m0} +$$

$$\sum_{\substack{i=1,2 \\ n,m=1}}^{\infty} e^{-\nu[n^2+(m\pi)^2]t} (f, g^i_{nm0})_\Omega \; g^i_{nm0} + \sum_{\substack{i=3,4 \\ m,k=1}}^{\infty} e^{-\nu[(m\pi)^2+\left(\frac{\pi}{e}k\right)^2]t} (f, g^i_{0mk})_\Omega \; g^i_{0mk}$$

$$+ \sum_{\substack{i=5,6,7,8 \\ n,m,k=1}}^{\infty} e^{-\nu\left[n^2+(m\pi)^2+\left(\frac{\pi}{e}k\right)^2\right]t} (f, g^i_{nmk})_\Omega \; g^i_{nmk} \qquad (1.13.23)$$

We now use (1.13.19–23) in (1.13.15–16), as well as the decay (1.11.1) for p in $H^{\frac{3}{2}}(\Omega)$, which requires the setting of Theorem 1.11.1 (same as Theorem 1.8.1 or

Theorem 1.10.1). We estimate

$$\|(-A_N)^{\frac{1}{2}} f\|_{L_2(\Omega)} \leq e^{-\nu\pi^2 t}\|(-A_N)^{\frac{1}{2}} f(0)\|_{L_2(\Omega)}$$
$$+\text{Const} \int_0^t \frac{e^{-\nu\pi^2(t-\tau)}}{\sqrt{\nu}(t-\tau)^{\frac{1}{2}}}\ e^{-\sigma_0\tau} d\tau \|P^u_{\gamma_0} h(0)\|_{Z^u_{\gamma_0}},\ f = u, w. \quad (1.13.24)$$

Recalling then estimate (1.9.21) for the integral term in (1.13.24)—thus, with γ_0 there replaced by π^2 now—we obtain

$$\|(-A_N)^{\frac{1}{2}} f\|_{L_2(\Omega)} \leq \text{Const}\ e^{-\nu\pi^2 t}\left[\|(-A_N)^{\frac{1}{2}} f(0)\|_{L_2(\Omega)} + \|P^u_{\gamma_0} h(0)\|_{Z^u_{\gamma_0}}\right],\ f = u, w, \quad (1.13.25)$$

thus re-obtaining the decay in (1.13.1) in the weaker setting of Theorem 1.11.1 (same as that of Theorem 1.10.1, i.e., without $\langle \frac{\partial}{\partial x}\psi_j, 1\rangle_{\Gamma_1} = \langle \frac{\partial}{\partial z}\psi_j, 1\rangle_{\Gamma_1} = 0$). □

Having, in this direct approach, obtained exponential decay for $u \in H^1(\Omega)$ in (1.13.25), we then re-obtain the exponential decay of the vorticity ω in $L_2(\Omega)$, as in Theorem 1.12.1, recalling also the exponential decay of v_x and v_z in Corollary 1.10.3.

Remark 1.13.1. We note that p_x and p_z are given by

$$p_x = DV_{xt} - \nu N V_{xxx}, \quad p_x = DV_{zt} - \nu N V_{zzz}, \quad (1.13.26)$$

as one can see from differentiation in x and z of (1.11.2) on p. □

Appendix A: From the purely velocity $\{u, v, w\}$ - formulation (1.1.1.5a–s) to the vorticity ω formulation (1.1.1.7a–h) in place of u and w. Verification of elliptic problem (1.1.1.12a–e) for the pressure p

We assume the $\{u, v, w\}$-formulation in (1.1.1.5a–s). We then wish to obtain: (i) the vorticity equation and corresponding B.C. in (1.1.1.7a–h); as well as (ii) the resulting elliptic problem (1.1.1.12a–e) satisfied by the pressure.

Derivation of the vorticity eq. (1.1.1.7a) for ω. We recall the definition $\omega = \text{curl}\{u, v, w\}$ of the vorticity in (1.1.1.6). We will look at the derivation of (1.1.1.7a) for the ω^1 component. The derivation for the other two components is similar. Differentiating in y across the w-equation (1.1.1.5c) and in y across the v-equation (1.1.1.5b), we obtain

$$\omega^1_t = w_{yt} - v_{zt} =$$
$$\overline{U}v_{xz} - \overline{U}w_x y - \overline{U}_y w_x + \overline{W}v_z z - v\overline{W}_{yy} - v_y\overline{W}_y - w_{zy}\overline{W} - w_z\overline{W}_y + \nu\Delta(w^y - v_z), \tag{A.1}$$

after a cancellation of the pressure terms $p_{yz} - p_{zy} \equiv 0$. Moreover, using the divergence-free condition (1.1.1.5e) as well as the definition (1.1.1.6) of $\omega^1 = w_y - v_z$, we obtain

$$\omega^1_t = -\overline{U}\omega^1_x + \overline{W}_y u_x - \overline{U}_y w_x - \overline{W}\omega^1_z - v\overline{W}_{yy} + \nu\Delta\omega^1. \tag{A.2}$$

Doing a similar derivation for the second and third component, we obtain

$$\omega_t - \nu\Delta\omega + \overline{U}(y)\omega_x - \overline{W}'(y)\nabla u + \overline{U}'(y)\nabla w. \tag{A.3}$$

We introduced the operator $\Psi = \Psi(\omega, V) = (-A_N^{-1}(\text{curl}(\omega)^1), DV - A_D^{-1}(\text{curl}(\omega)^2)$ in (1.1.1.8), which we know to be a well-defined linear operator on its domain from elliptic theory. By taking the curl of vorticity (1.1.1.6) and the curl $\circ$ curl identity from calculus, we get

$$\begin{aligned} \text{curl}(\omega) &= \text{curl}(\text{curl}(u, v, w)) \\ &= \nabla(\text{div}(u, v, w)) - \Delta(u, v, w) \\ &= -\Delta(u, v, w), \end{aligned} \tag{A.4}$$

since the term $\text{div}(u, v, w)$ is equal to 0 by the divergence free condition (1.1.1.5d). The operator Ψ thus gives $(1.u, v, w)$ as the solution to the elliptic problem (A.4), i.e. yields $\Psi(\omega, V) = (u, v, w)$ (modulo the one-dimensional null space of A_N from the first and third components). Replacing u and w in (A.4) with $\Psi(\omega, V)^1 = \Psi(\omega)^1$ and $\Psi(\omega, V)^3 = \Psi(\omega)^3$, we thereby obtain (1.1.1.7a).

Derivation of B.C. (1.1.1.7b–h) for ω. From, respectively, the LHS and the RHS of (1.1.1.5e), (1.1.1.5j), and (1.1.1.5o), one obtains via (1.1.1.6): $\omega^1|_{y=0} = w_y|_{y=0} - v_z|_{y=0} = 0 - 0 \equiv 0$, and similarly, $\omega^1|_{y=1} = w_y|_{y=1} - v_z|_{y=1} = -V_z(x, z, t) - 0 = -V_z(x, z, t)$, and (1.1.1.7b) is verified. (1.1.1.7c) and (1.1.1.7d) follow similarly. The periodic B.C. (1.1.1.7e) for ω follows from the corresponding property for v_x

in (1.1.1.5l) and (1.1.1.5n), for u_y and w_y, upon y-differentiation of (1.1.1.5f) and (1.1.1.5p). (1.1.1.7f) is verified similarly. To verify the periodic B.C. (1.1.1.7g) for ω_x, we see that

$$\begin{aligned}
\omega_x &= (w_{yx} - v_{zx},\ u_{zx} - w_{xx},\ v_{xx} - u_{yx}) \\
&= (w_{yx} - v_{zx},\ -v_{zy} - w_{zz} - w_{xx},\ v_{xx} + v_{yy} + w_{yz}) \\
&= (w_{yx} - v_{zx},\ -v_{zy} + w_{yy} - \Delta w,\ \Delta v - v_{zz} + w_{yz}) \qquad \text{(A.5)}
\end{aligned}$$

obtained from the divergence-free condition (1.1.1.5d) upon differentiating in y and z), thus obtaining (A.5). Next, the periodic B.C. (1.1.1.7g) follows from periodicity for Δw and Δv. This is results from eq. (1.1.1.5b) and (1.1.1.5c), since v_t, w_t, v_x, v_z, w_x, and w_z are periodic in the x-direction at $x = -\pi$ and $x = \pi$ (via (1.1.1.5k), (1.1.1.5l), (1.1.1.5p), and (1.1.1.5q). Moreover, p_y and p_z will be established below to be periodic in the x-direction (independently of (1.1.1.7f)). (1.1.1.7h) is verified similarly.

Derivation of the p-equation in (1.1.1.12a). We differentiate the u-equation (1.1.1.5a) in x, the v-equation (1.1.1.5b) in y, and the w-equation (1.1.1.5c) in z; sum up the resulting equations; use the divergence-free condition (1.1.1.5d), and readily obtain $p_{xx} + p_{yy} = 2(\overline{U}'(y)v_x + \overline{W}'(y)v_z)$, thus verifying eq. (1.1.1.12a).

Derivation of the B.C. (1.1.1.12b–e) for p. To verify (1.1.1.12b), we return to eq. (1.1.1.5b) and restrict it to $y = 0$ and $y = 1$, respectively. We have $v_t|_{y=0} \equiv 0$, $v_t|_{y=1} \equiv V_t$ from (1.1.1.5j), $\overline{U}(y)|_{y=0,1} = \overline{W}(y)|_{y=0,1} = 0$ from (1.1.1.4). Finally,

$\Delta v|_{y=0} = \omega_x|_{y=0} = 0$ and similarly, $\Delta v|_{y=1} = V_{xx} + V_{zz}$ by recalling (1.1.1.5j). Thus, (1.1.1.12b) is verified.

To verify the periodicity of p_x in (1.1.1.12c), we return to (1.1.1.5a), which we restrict at $x = \pm\pi$ and use the periodicity of u_t, u_x, u_z, and v in x by (1.1.1.5f), (1.1.1.5g), and (1.1.1.5k). Moreover, Δu is periodic in x, since u_{yy} is periodic in x by (1.1.1.3e), and $u_{xx} = -v_{yx} - w_{zx}$ (by (1.1.1.5d)) is periodic in x by differentiating (1.1.1.5l) in y and (1.1.1.5q) in z. This verifies (1.1.1.12d); periodicity of p_z in (1.1.1.12d–e) are verified similarly.

Chapter 2

Finite-Volume Computational Fluid Dynamics for Simulation of the Fluid-Control System

2.1 Overview of the Computational Fluid Dynamics Simulation

Analytically-derived exact solutions to the Navier-Stokes equations exist only in limited cases; for example with domains of idealized geometry and specialized dynamics such as steady-state flows. The domain being considered in the present analysis, i.e. the $3-D$ channel presented in section 1.1.1.1, is indeed one such special idealized geometry, and yet the general family of solutions corresponding to this domain are almost completely unknown. The fluid mechanics analyst must thus turn from theoretical derivation to numerical methods and their resulting computational approximations.

The fundamental principles upon which the field of computational fluid dynamics

(CFD) is built were conceived well before the first computers in the 1950s. Modern computational fluid dynamics methods (Harlow and Welch's marker and cell method of 1965 for example), however, which were not formulated and implemented until around the 1960s, have been undergoing rapid continual development ever since [F-P]. A great range of methods have been developed to handle varying types of flows and flow regimes, including (from CFD's early inception) incompressible and compressible internal and external flows, and from the mid-1980s methods to handle more complex flow scenarios such as highly turbulent, multiphase, and combusting flows. As a result of the breadth of flow regimes, conditions, physics, and applications, there is a corresponding breadth of CFD methods and methodologies to use at a researcher's disposal.

In the current analysis, a CFD software program was built for the purposes of addressing the numerical requirements of the incompressible fluid-channel system presented in section 1.1.1.1 and for the incorporation of the numerical computation of the feedback boundary controller. In the course of building any CFD package, significant tradeoffs must be made in deciding the methods to be used. These tradeoffs range from hard measurable and directly quantifiable characteristics (i.e. computational time, memory requirements, scalability, accuracy, etc.) to softer, more interpretive tradeoffs (such as algorithm complexity, reusability of methods, etc.). Throughout the following chapter, we will present the driving motivations behind the choices of methods used and occasionally discussing some of the popular alternatives. We will also discuss the methods themselves and their numerical qualities. Finally, we will afterwards present the results of the computed evolution of the control-free channel flow dynamics, and give analysis of the corresponding stabilization towards the system's steady state solution.

Computational Fluid Dynamics: Methods Chosen

In designing or deciding the computational approach for an incompressible-fluid dynamics solver, there is an abundantly diverse collection of established submethods from which to choose. Within each submethod, there are countless variations which have been created and adapted to handle certain numerical conditions or flow physics. Certain algorithms are built, for example, to specialize in the handling of discontinuities associated with shock waves, to limit spurious solutions near the corners and sharp edges that arise in complex geometries, to improve numerical conditions such as stability and accuracy under certain flow conditions, to assimilate well with certain turbulence approximation methods, etc.

Incompressible methods differ from compressible methods primarily in the necessary enforcement of the incompressible divergence-free continuity equation (1.1.1.5d). The numerical approach employed here may be categorized as a fractional time step finite volume pressure-projection method. The pressure-projection method is a primitive-variable method, i.e. it solves the Navier-Stokes equation in its "primitive" unaltered differential or integral form, solving directly for the the "primitive" velocity and pressure variables (as opposed to vorticity, etc.). A fractional time step method operates according to the principle of operator splitting by dividing each time step in the advancement of solutions for a differential system so that each intermediate step may focus on individual pieces of the calculation [C-K]. The earliest formulations of the fractional time step projection method (also known as incompressible pressure-based or pressure-projection method) for the Navier-Stokes equations were developed independently in the late 1960s by A. J. Chorin and R. Temam [F-P]. These types of methods operate by separating the calculations for effects of the diffusive and convective dynamics from the effects of the pressure differential and the enforcement of the continuity equation during each time step, which allows for differing specialized methods to be used on each substep [K-K]. Fractional time step and the fractional time step projection methods will both be discussed in further detail in section 2.2.1.

The fractional time step projection method separates the calculations for diffusive and convective dynamics from the effects of the pressure differential and the enforcement of the continuity equation. The next question is then how to complete each of those calculations. In the former calculation, we are treating the pressure field as constant so as to only focus on the convective and diffusive dynamics. This modifies the Navier-Stokes equations, written here in Einsteinian indicial form,

$$\frac{\partial v_i}{\partial t} = \frac{\partial p}{\partial x_i} - \frac{\partial v_i v_j}{\partial x_j} + \frac{\partial \tau_{ij}}{\partial x_j}, \tag{2.1.0.1}$$

where $\tau_{ij} = \nu \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right)$ is the viscous stress tensor, by treating the pressure term $\frac{\partial p}{\partial x_i}$ as a constant vector field. To solve eq. 2.1.0.1 with constant pressure term, we must decide on the spatial and temporal discretization methods. For the relationship between the temporal and spatial discretization, we will use the so called method of lines [B.4]. The method of lines refers to a separation of the two discretizations so as to have a fixed spatial computational grid that does not depend on the time step, as opposed to a more complex spatial-temporal relationship such as in a Crank-Nicholson type scheme. For our spatial discretization scheme, we have chosen the popular finite volume method. In the finite volume method, a grid is chosen to construct control volumes across the flow domain, and uses these volumes to evaluate the surface flux integrals required to solve the Navier-Stokes equations in its integral formulation. The finite volume method advantageously carries out its discretized computations directly in the physical rather than an alternative computational space. This avoids the problems associated with the transformation between physical and computational coordinate systems which often occur with methods like finite difference on complex domains [B.4]. The finite volume method is also very flexible with regard to its implementation; it can be easily implemented on structured as well as unstructured grids, as well as in other grid formations such as chimaera types. The finite volume method will be discussed in greater detail in section 2.2.2.

2.2 Finite Volume Method and Computational Fluid Dynamics

In the subsections below, we will discuss the various components of the finite volume method and the additional methods used in our CFD simulation.

2.2.1 Fractional Step Pressure-Projection Method

A fractional-step procedure for solving a differential system operates by breaking the governing equations into a series of approximating sub-equations, allowing terms of differing numerical character to be addressed separately. In the case of the Navier-Stokes equations (linear, linearized, or fully non-linear), the temporal evolution of the flow variables are approximated through two main steps. This allows independent use of numerical methods tailored to the characteristics and physics specific to the convection, diffusion, and pressure.

To understand the fractional step mechanism, let us look at the first-order Marchuk-Yanenko fractional-step scheme developed by Yanenko in 1971 and Marchuk in 1975 [C-K]. Consider the system

$$\frac{d\phi}{dt} + A_1(\phi) + A_2(\phi) = f, \tag{2.2.1.1}$$

with initial condition

$$\phi(0) = \phi_0 \tag{2.2.1.2}$$

and each solution at the next $(n+1)$th time step is $\phi^{n+1} = \phi((n+1)\Delta t)$ $(n = 0, 1, 2, ...)$. The fractional step method splits the calculation of ϕ^{n+1} into two successive steps, explicitly or implicitly carrying out the fractional step calculations for A_1 in the first step and then A_2 in the second step. The two discretized fractional step equations approximating eq. 2.2.1.1 are

$$\frac{\phi^{n+\frac{1}{2}} - \phi^n}{\Delta t} + A_1\left(\phi^{n+\frac{1}{2}}\right) = f_1^{n+1} \tag{2.2.1.3}$$

$$\frac{\phi^{n+1} - \phi^{n+\frac{1}{2}}}{\Delta t} + A_2\left(\phi^{n+1}\right) = f_2^{n+1}, \tag{2.2.1.4}$$

where $f_1^{n+1} + f_2^{n+1} = f^{n+1} = f((n+1)\Delta t)$.

The presently used formulation of the fractional step projection method for the incompressible Navier-Stokes equations was given by Kwak and Kiris [K-K]. In this formulation, we approximate the incompressible Navier-Stokes equations in two steps. First, we solve equation 2.1.0.1 with constant pressure term. After discretizing this equation and setting the convective term equal to zero (for simulation of model 1.7.1), we must solve for the so-called *auxiliary* or *intermediate velocity* $\hat{v}$ in the equation

$$\frac{\hat{v}_i - v^n}{\Delta t} = \frac{\delta p}{\delta x_i} + R(\hat{v}_i) \tag{2.2.1.5}$$

where the residual $R(\hat{v}_i)$ is equal to Δv_i. Eq. 2.2.1.5, written implicitly in terms of $R(\hat{v}_i)$, may alternatively be solved explicitly as long as appropriate Courant-Friedrich-Lewy stability conditions are met [C-K]. In our present methodology, we use the finite volume method and a Runge-Kutta ordinary differential equations solver to solve eq. 2.2.1.5 for the auxiliary velocity (see subsection 2.2.2).

After solving for the auxiliary velocity $\hat{v}$, the fractional step projection method then proceeds to the second step, called the *projection step*. The auxiliary velocity computed via solution of eq. 2.2.1.5 has no guarantee to be divergence free; a requirement of eqs. 1.1.1.5d and 1.7.1d. In the projection step, the updated pressure is computed, and in doing so we map the auxiliary velocity onto a divergence-free velocity field. To see how this is done, consider the following. Subtract the initial fractional step projection equation (eq. 2.2.1.5) from original linear Navier-Stokes equations. From this, we obtain the projection step equation to give us the update of v^{n+1} from the auxiliary velocity

$$\frac{v^{n+1} - \hat{v}_i}{\Delta t} = \frac{\delta(p^{n+1} - p^n)}{\delta x_i}. \tag{2.2.1.6}$$

To use eq. 2.2.1.6 in obtaining v^{n+1} however, we must evaluate $\dfrac{\delta(p^{n+1} - p^n)}{\delta x_i}$. To find a method for evaluating this term, we take the divergence of eq. 2.2.1.6. The updated velocity v^{n+1} must satisfy the divergence free condition

$$\nabla \cdot v^{n+1} = 0. \tag{2.2.1.7}$$

Making use of this divergence free condition (eq. 2.2.1.7) along with the divergence of eq. 2.2.1.6, we obtain the following Poisson equation for the pressure correction $(p^{n+1} - p^n)$

$$\Delta\left(p^{n+1} - p^n\right) = \frac{1}{\delta t}\frac{\delta \hat{u}_i}{\delta x_i}. \tag{2.2.1.8}$$

Thus, we are able to find the updated pressure by solving the elliptic equation 2.2.1.8 for the pressure correction, which then allows us to use eq. 2.2.1.6 to obtain the updated velocity as well.

In section 2.2.3 we will discuss the multigrid method; a powerful elliptic equation solver method which will allow us to solve the Poisson equation 2.2.1.8. Additionally, note that this formulation forces the updated velocity, which is found by mapping the auxiliary velocity onto a divergence-free velocity field, to necessarily be divergence free, satisfying eq. 2.2.1.7.

2.2.2 The Finite Volume Method

The finite volume method is a spatial discretization method for partial differential equations which are formulated to model dynamics based on principles of conservation (for example conservation of mass, momentum, energy, chemical species, etc.). In the case of the finite volume method for fluid dynamics, the finite volume method operates on the integral formulation of the Navier-Stokes equations

$$\frac{\partial}{\partial t}\int_{\Omega} \vec{W} d\Omega \;+\; \oint_{\partial\Omega} (\vec{F}_c - \vec{F}_v) dS \;=\; \int_{\Omega} \vec{Q} d\Omega, \qquad (2.2.2.1)$$

$$\vec{W} = \begin{bmatrix} \rho u \\ \rho v \\ \rho v \end{bmatrix}, \quad \vec{Q} = \begin{bmatrix} \rho f_{e,x} \\ \rho f_{e,y} \\ \rho f_{e,z} \end{bmatrix}, \quad \vec{F}_c = \begin{bmatrix} \rho u V + n_x p \\ \rho v V + n_y p \\ \rho w V + n_z p \end{bmatrix},$$

$$\vec{F}_v = \begin{bmatrix} n_x \tau_{xx} + n_y \tau_{xy} + n_z \tau_{xz} \\ n_x \tau_{yx} + n_y \tau_{yy} + n_z \tau_{yz} \\ n_x \tau_{zx} + n_y \tau_{zy} + n_z \tau_{zz} \end{bmatrix}, \qquad (2.2.2.2)$$

along with the continuity equation (eq. 1.1.1.5d) and appropriate boundary conditions, where τ is the viscous stress tensor.

We can see from eqs. 2.2.2.1-2 there are four dynamical quantities involved. The *convective fluxes*, which refers to flow quantities which are carried in to a new area by the flow of the fluid. These fluxes are not present if the fluid has zero velocity.

The *diffusive fluxes*, which refers to the diffusion (spread) of flow quantities which take place due to molecular motion necessarily present regardless of macroscopic flow velocities. The *pressure gradient* present within the flow, which applies a force to the the flow field along its gradient lines. Finally, the *source terms*, which are additional effects added by the specific body, surface, or other source forces are acting on a fluid control volume (i.e. a volume of infinitesimal size). Typical sources include forces such as gravitational, centrifugal, Corealis, and others (electro-magnetic forces for example when considering magneto hydro-dynamics, etc.). In the case of the present model (eq. 1.1.1.5 or 1.7.1), there are no source terms.

In the finite volume integral equation (eq. 2.2.2.1-2) we see that the rates of change for integrals of flow characteristics for any given control volume within the computational domain depend on the specified surface integrals of convective and diffusive flow quantities. On the faces of each control volume, these convective and diffusive fluxes get calculated with different methods built to emphasize their corresponding numerical and physical characteristics. The convective fluxes typically get calculated with a so-called *upwind scheme*; a scheme designed to emphasize the direction of information flow (based on the spectral properties of the convective flux Jacobian operator) to emphasize the information flowing from upstream rather than downstream. This reflects the physical interpretation of convection as being information carried along streamlines. The majority of nonlinear upwind flux evaluation schemes fall into three categories [B.4]. *Flux-vector splitting schemes* are based on an operator-splitting method. *Flux difference splitting schemes* are based on solutions to the Riemann (shock tube) problem, and are a popular choice when dealing with simulations of shock-waves. *Total variation-diminishing schemes* are designed to prevent spurious oscillations near shock-waves and boundaries. Upwind schemes are often combined with limiter techniques and limiter functions to help curb the propagation of spurious oscillations. It is worth noting that the central discretization with an appropriate limiter function is also a popular discretization scheme for evaluating the convective fluxes due to its lower level of numerical complexity. Centralized schemes are often much lower in numerical complexity than their upwind counterparts, and typically require much less CPU time per evaluation. Centralized schemes however are much less accurate in capturing shock discontinuities. In the case of the linear Navier-Stokes system (eqs. 1.7.1) where no shocks are present, a centralized scheme is the clear choice [B.4].

The numerical simulation of the linearized Navier-Stokes equations (eq. 1.1.1.5) will require use of an upwind convective flux scheme; however, in the case of the linear equations (eq. 1.7.1), the convective fluxes have been eliminated and no upwind scheme will be required. In the latter case, only the diffusive fluxes must be included. For calculation of the diffusive fluxes, which are based on relative rates of molecular scattering from surrounding areas, a *centralized difference scheme* is often used. This is reflective of the elliptic nature of the diffusive fluxes [B.4]. The centralized difference scheme is typically based directly on a central finite difference formula, where flow variables have been appropriately interpolated to cell or boundary centroids when necessary [B.4].

2.2.3 The Multigrid Method for Solution of Eq. 2.2.1.8 and General Elliptic Equations

In the pressure-projection described in the previous sections, the most computationally time consuming step as well as the biggest disadvantage of the method is finding the solution to the pressure correction by solving the Poisson equation (eq. 2.2.1.8) at each time step. The first formulation of the pressure-projection method came in the form of the so called marker and cell (MAC) method, developed by Harlow and Welsh at Los Alamos National Laboratory in 1965 [C-K]. The marker and cell formulation also computationally relied most heavily on the solution to a Poisson equation for pressure, with using a substantially less efficient explicit elliptic solver than compared to presently available methods [K-K]. Presently, two of the most (if not the most) efficient and popular elliptic solvers are Fourier transform methods and multigrid methods [K-K]. Fourier transform methods are quite strong on 2-D domains, but however do not scale well to 3-D [K-K]. The multigrid method was chosen for the present simulation, and will be described in this subsection.

To understand the multigrid method, we must first discuss the classical Gauss-Seidel relaxation scheme, which is incorporated into use into the typical multigrid method formulation. Consider for example a forward-time central-space finite difference discretization of the Laplace equation

$$\Delta\phi = 0$$

with a regularly structured grid ($dx = dy$):

$$\phi_{j,l} \;=\; \frac{1}{4}\left(\phi_{j+1,l} + \phi_{j-1,l} + \phi_{j,l+1} + \phi_{j,l-1}\right), \qquad (2.2.3.1)$$

where the subscripts (j, l) refer to the grid node having the jth x-coordinate and lth y-coordinate. Gauss-Seidel relaxation operates by iteratively replacing the current approximate value of ϕ at each node by the value obtained from the right hand side of eq. 2.2.3.1. Unlike in a Jacobi-type method, the updated values for the approximation of ϕ are immediately used to replace previous values without waiting to calculate the updates for the entire grid. Gauss-Seidel relaxation converges to the solution twice as fast as Jacobi's method but still on the same order of convergence ($\mathcal{O}(n^2)$ for solving an $n \times n$ order linear system). This is too slow for most practical computational problems, and there are many variations and well known substitutes to use in its place (i.e. the successive over relaxation method, incomplete lower-upper decomposition methods such as Stone's method, the conjugate gradient method, etc.) [F-P].

The Gauss-Seidel method, although slow in its convergence, has excellent local smoothing properties. The multigrid method takes advantage of this.

The multigrid method operates by smoothing an approximate solution's residual error iteratively across multiple predefined grid resolutions. For a differential system

$$A(\phi) = f, \tag{2.2.3.2}$$

the residual for an approximate solution $\tilde{\phi}$ is defined

$$R(\tilde{\phi}) = A(\tilde{\phi}) - A(\phi). \tag{2.2.3.3}$$

It is a measure of the error field for the approximation $\tilde{\phi} \approx \phi$. The multigrid method operates by first creating grids of multiple resolutions, typically with each grid cell in the coarser grid formed as the union of all previous grid cells adjacent to the centroid node (see figure 2.1 below). After forming the grids, the iterative multigrid solver

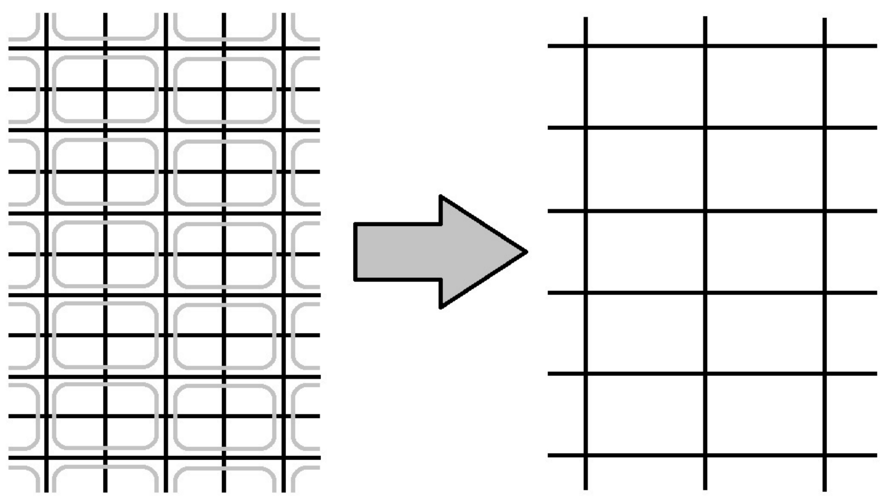


Figure 2.1: Grid Coarsening. Structured grid formation from the finer to coarser grids formed via union of cells adjacent to the central nodes.

is implemented. To understand the multigrid iterative structure, first consider the implementation between just two grids, a coarser grid and a finer grid. Suppose we have an approximate solution ϕ_f to the differential system 2.2.3.2 defined on the fine grid, and we want to use the multigrid method to improve this approximation. We first take advantage of the local smoothing properties of Gauss-Seidel by running this on the solution, refining ϕ_f across the fine grid. This resolves local smoothing of errors, but is extremely slow at resolving global errors. This is because Gauss-Seidel relaxation has strong dampening of lower order fourier terms of the fourier transform of the error residual but weak dampening of higher order terms [P-T-V-F]. Once the solution is updated on the fine grid, its residual error field R is calculated and passed through an *injection* (restriction) *operator* to the coarser grid. Then, by using either Gauss-Seidel a small number of times, passing to another iteration of the multigrid method, or by direct computation (if the current grid is sufficiently small), we have an approximate solution ψ_c to the equation

$$A(\phi) = R. \tag{2.2.3.4}$$

From this, we pass ψ_c back to the fine grid through the *prolongation* (interpolation) *operator* to obtain ψ_f. ψ_f is the correction to be applied to the solution ϕ_f, which is then updated as $\phi_f^{new} = \phi_f + \psi_f$. Once updated, ϕ_f^{new} is then passed through Gauss-Seidel again to smooth out remaining local errors. This whole process is then iteratively repeated.

The above paragraph describing the two-grid method transforms into the multigrid method depending on whether or not the iterative option to call the multigrid method again from the coarser grid gets chosen or not. There are several variations on the structure for passing the residual errors between grids and the depths at which they get passed. There is also an extension of this method in which the initial guess is not made from the finest grid, but from the coarsest grid instead. This variation is called by Press et al. in [P-T-V-F] the *full multigrid method*. In this variation, system 2.2.3.2 is solved according to the following steps: First, solve the discretized system through a direct method on the coarsest grid. Second, pass the current approximation (through the prolongation operator) to the next finer grid. Third, use smooth out local error by a small number of iterations of Gauss-Seidel. Fourth, call the multigrid method on the current solution, refining the solution by passing the residual up to the courser grid and then iterating. Finally, after returning through the multigrid iterations, if the current grid level is the finest (i.e. the full computational grid), stop the algorithm and return the solution. If not, iteratively continue steps two through four.

Finally, it should be noted that the Gauss-Seidel method is not the only choice of iterative linear system solver. It is typically chosen due to its low computational cost, low numerical complexity, and high smoothing [P-T-V-F]. A red-black variation of Gauss-Seidel is well suited for typical second-order elliptic equations such as 2.2.1.8. Relaxation along a line is recommended for use with differential systems having much stronger coupling along one dimension than another, which can still be efficiently implemented via us of the tridiagonal Thomas algorithm. Also, it should be noted that the successive over relaxation (SOR) method in particular should *not* be used as the smoothing operator. The SOR method destroys the high-frequency error smoothing that is crucially needed for the multigrid method [P-T-V-F].

2.2.4 Temporal Discretization and the Runge-Kutta Ordinary Differential Equations Solver

The finite volume method discussed in subsection 2.2.2 discretizes the system's governing equations spatially. We require as well a method for the temporal discretization of the system; or, in particular, in the context of the pressure-projection fractional time step method of subsection 2.2.1, we require a method of temporal discretization for eq. 2.2.1.5. For this, as in [K-K], we used a standard well-known Runge-Kutta method, developed in 1901 by Carl Runge and Martin Kutta [B.5]. The Runge-Kutta method has had decades of popularity due to its low numerical complexity to the point of elegant simplicity, high numerical accuracy, and modest memory requirements. However, the most popular Runge-Kutta method is just one variation among many in a family of methods. Runge-Kutta methods are multistage solution propagation methods using information obtained from Euler's method-style steps to match a Taylor series expansion up to some order. These methods vary according to accuracy order, stability characteristics, number of stages, and more [P-T-V-F].

In the context of the current analysis, the foremost decision in the choice of Runge-Kutta method is that of an implicit or explicit evaluation. In an explicit Runge-Kutta method, the numerical complexity and computational cost will both be minimal. Accuracy of an explicit method can be made to an arbitrarily high order at the cost of increased stage computations and increased computational cost, and number of intermediate stages can be increased at an increased computational cost to decrease stability restrictions. In the context of computational fluid dynamics temporal schemes, the explicit Runge-Kutta techniques chosen are typically of 4th order or if memory consumption is an issue one can use a 2nd or 3rd order explicit Runge-Kutta (of stage number typically 3, 4, or 5) with modified memory storage (requiring only the storage of the 0th and current stage rather than storage of each stage) [B.4].

Regardless of the explicit Runge-Kutta method chosen, in order for the method to be stable, the Courant-Freirich-Lewy (CFL) condition

$$\delta t \ \leq \ \sigma Max_n \left\{ \frac{\sum_i \Delta x_i}{|v_n|} \right) \tag{2.2.4.1}$$

must be met, where σ is the *CFL coefficient*, Max_n is taken over all control volumes in the computational domain, $\sum_i \Delta x_i$ sums the widths of the control volume in each spatial direction, and $|v_n|$ is the absolute value of the velocity. The CFL condition ensures that the convective and diffusive information can not be transported further than one cell throughout each time step. The CFL coefficient is typically on the order of unity and depends on the structure of the temporal method (value of stage coefficients, number of stages, etc.). This stability restriction can limit the size of each time step quite heavily for explicit methods [B.4].

An implicit Runge-Kutta method on the other hand does not require any CFL stability limitation and is unconditionally stable regardless of the size of the time step. In temporally accurate simulations, a CFL time step limitation is typically still imposed, but replacing the CFL coefficient with a value on the order of 100 [B.4]. This increases the allowable time step by the same factor of 100, and thus reduces the number of full time steps requiring calculation to reach a given time to 1% compared to using an explicit method. The advantages of increased time step and unconditional stability for the implicit Runge-Kutta comes with the disadvantage of increased numerical complexity and computational cost, requiring iterative methods of evaluation at some or all of each of the method's stages of evaluation. Implicit Runge-Kutta methods come in fully implicit forms or diagonally implicit (also known as semi-implicit and semi-explicit forms). Diagonally implicit Runge-Kutta methods allow for only one new stage function to be implicitly solved for during each stage of the calculation rather than all stages and stage functions having to be simultaneously solved. Under appropriately chosen stage coefficients, quadrature weights, and time coefficients (see [B.5]), a diagonally implicit Runge-Kutta methods can have equally full stability characteristics as compared to a fully implicit Runge-Kutta method, thus making it an attractive choice.

A diagonally implicit Runge-Kutta method applied to eq. 2.2.1.5 requires at each time step the solution to the nonlinear system of equations

$$\vec{v}_n^i = \vec{v}_n^0 + h \sum_{j=1}^{i} \eta_{ij} (R(\vec{v}_n^j) + \nabla p) \tag{2.2.4.2}$$

where $\vec{v}_n^i$ is the velocity vector for the ith stage stored at the nth grid node, h is the time step, $R(v_n^j)$ is the diffusive flux residual for the velocity v_n^j, the summation is taken over each stage's velocity function, and η_{ij} are the stage coefficients (see [B.5]). Solving this system of equations is typically carried out via Newton's method [B.5]. However, in addition to iteratively solving large systems of linear equations, this requires analytical or numerical evaluation of the Jacobian of the diffusive flux operator ∇R during each step of the Newtonian iteration. To reduce computational costs, the diffusive flux Jacobian may be approximated as constant over several iterations of Newton's method as well as even across multiple stages of Runge-Kutta, and even across multiple time steps [B.5]. Despite these simplifications, implicit Runge-Kutta methods for CFD purposes remain very computationally expensive per time step calculated.

In choosing between an explicit and implicit Runge-Kutta method, a popular methodology can be to first implement the simpler explicit Runge-Kutta method and determine if it satisfies the design requirements. If not, i.e. if the CFL stability requirements are too restrictive on the time step compared to the time step's computational cost, then the CFD analyst may switch to an diagonally implicit method. For our CFD program, an explicit Runge-Kutta method was found to be sufficient and without excessive restriction on the allowable time steps. Furthermore, explicit temporal schemes are the best choice for unsteady flow simulations involving time scales which are comparable to the spatial scales over the eigenvalues of the residual flux Jacobian; i.e. for example in aeroacoustics, large eddy simulations (LES), and direct numerical simulations (DNS) such as in the present simulation [B.4]. Since the global physical dynamics evolve more slowly than the local solution changes, a temporal method of at least 3rd order accuracy is typically required [B.4]. In our case, a 4th order method was used.

2.3 Numerical Implementation of the Boundary Feedback Control

Proposed implementation for the numerical simulation of the 2-D and 3-D feedback control V in the case of the linear or linearized Navier-Stokes system (see 1.1.5j, 1.7.1j) will be presented in the current section.

The control for the 3-D channel system (see 1.1.1.5 and 1.7.1) considered in Chapter 1 takes the form

$$V(t,x,z) = [\psi_1, \dots \psi_4] \begin{bmatrix} \varphi_1(t) \\ \varphi_2(t) \\ \varphi_3(t) \\ \varphi_4(t) \end{bmatrix} = [\psi_1, \dots \psi_4] \tilde{F}^u_{\gamma_0} P^u_{\gamma_0} [q(t) - DV(t)], \qquad (2.3.0.1)$$

The control for the 2-D model analyzed by Triggiani in [Tr.4] uses the control function

$$V(t,x) \;=\; [\psi_1,\psi_2]\begin{bmatrix}\varphi_1(t)\\ \\ \varphi_2(t)\\ \\ \varphi_3(t)\\ \\ \varphi_4(t)\end{bmatrix} = [\psi_1,\psi_2]\tilde{F}^u_{\gamma_0}P^u_{\gamma_0}[q(t)-DV(t)]. \tag{2.3.0.2}$$

Equations (2.3.0.1) and (2.3.0.2) will be both simultaneously generalized for the purposes of discussing their numerical implementation. Henceforth throughout this section, we will generalize (2.3.0.1) and (2.3.0.2) to

$$V(t,\vec{x}) \;=\; \Psi\Phi = \Psi\tilde{F}^u_{\gamma_0}P^u_{\gamma_0}[q(t)-DV(t)], \tag{2.3.0.3}$$

where the symbols can refer to either their 2-D or 3-D counterparts ($\vec{x}$ referring to (x) or (x,z), Ψ referring to $[\psi_1,\psi_2]$ or $[\psi_1,\psi_2,\psi_3,\psi_4]$, etc.). Note that in [Tr.4], Triggiani uses the notation $z(t)=(I-\Pi^0)v(t)$ instead of $q(t)$ and $w(t)$ as the intrinsic variable instead of $h(t)$, whereas we use $q(t)$ and $z(t)$ respectively throughout the present work.

2.3.1 Control Implementation Preprocessing and the Feedback Matrix $\tilde{F}^u_{\gamma_0}$

Before calculations for the numerical implementation of the control may proceed, the user must first make two specifications: first, the desired rate of stability -γ_0 of the dynamics on the control space Z (with $\gamma_0 > (1+\pi^2)$ in 2-D and $\gamma_0 > (1+\pi^2+\left(\frac{\pi}{e}\right)^2)$

in 3-D), and second, the radius of support ϵ for the boundary controller $V(t, \vec{x})$. After the user-input specifications are received, the preprocessing calculations prior to running the full simulation may begin.

Upon inspection of the controls 2.3.0.1 and 2.3.0.2, we see that our first goal will be the preprocessing calculations required to find the feedback operator $\tilde{F}^u_{\gamma_0}$. First, an ordered list of eigenvalues (or equivalently their identifying indices) of the operator A_D in (1.2.2b) must be kept available for subsequent computations. In the case of the 3-D channel flow simulation, three separate ordered lists must be created. In the nomenclature of the present paper, finding these ordered lists of eigenvalues amounts to finding the corresponding sets of identifying indices $\mathcal{S}^a_{\gamma_0}, \mathcal{S}^b_{\gamma_0}$, and $\mathcal{S}^c_{\gamma_0}$. Note that the eigenvalues corresponding to sets $\mathcal{S}^a_{\gamma_0}$ and $\mathcal{S}^b_{\gamma_0}$ have multiplicity 2 and those of $\mathcal{S}^c_{\gamma_0}$ have multiplicity 4. These indices may be found by looping through the triples $(n, m, 0), (0, m, k)$, and (n, m, k) for $\mathcal{S}^a_{\gamma_0}, \mathcal{S}^b_{\gamma_0}$, and $\mathcal{S}^c_{\gamma_0}$, respectively, and storing values of triples with positive n,m,k such that

$$\lambda_{nmk} \;=\; -\left[n^2 + (m\pi)^2 + \left(k\frac{\pi}{e}\right)^2\right] \;>\; -\gamma_0. \qquad (2.3.1.1)$$

In 2-D, all positive integer triples must be stored which satisfy

$$\lambda_{nmk} \;=\; -\left[n^2 + (m\pi)^2 +\right] \;>\; -\gamma_0. \qquad (2.3.1.2)$$

The eigenvectors do not need a separately stored list since they may be resolved through function calls based on the identifying indices.

The existence and calculation of the feedback matrix $\tilde{F}^u_{\gamma_0}$, as discussed in Proposition 1.8.1, is dependent on the controllable pair of matrices A_D^u (see 1.1.2.5 and 1.7.11a) and $B_{\gamma_0}^u$ (see 1.7.10, 1.7.12). Once the indices for the system's eigenvalues have been stored, the matrix representations for A_D^u and $B_{\gamma_0}^u$ may be calculated. As seen in 1.7.11a, the matrix representation for A_D^u is a diagonal matrix with entries given by the ordered list of eigenvalues previously calculated, with appropriate duplications accounting for eigenvalue multiplicities. To calculate the matrix representation of $B_{\gamma_0}^u$ (see 1.7.12), we use the values of τ_1 and τ_2 defined in Proposition 1.7.1. Next, the integrals of (1.7.12) of the types such as $\tau_2(\lambda_{N_{\gamma_0}^c}^c)\langle \Lambda\psi_2, \cos\sigma_{\mathfrak{n}}(\lambda_{N_{\gamma_0}^c}^c)x \;\cos\frac{\pi}{e}\rangle$ must be computed. The quantity of integrals to compute for this may be large (depending on the size of the user-input γ_0); however, the quadrature required for this calculation is done during preprocessing and computational efficiency is of low priority.

With the calculation of A_D^u and $B_{\gamma_0}^u$ completed, we may then compute $\tilde{F}_{\gamma_0}^u$. The computation of a feedback operator $\tilde{F}_{\gamma_0}^u$ must achieve eigenvalue placement for the operator $[A_D^u + B_{\gamma_0}^u \tilde{F}_{\gamma_0}^u]$ all with real parts less than the preassigned stability rate $-\gamma_0$. Given a user-input requested rate of stability $-\gamma_0$, we first decide upon a set $\mathbb{S}$ of eigenvalues to assign the operator $[A_D^u + B_{\gamma_0}^u \tilde{F}_{\gamma_0}^u]$. Without any further specification from the user, we may take all eigenvalues in $\mathbb{S}$ to be real-valued with magnitude equal to γ_0 plus a safety factor (of, say, 1%). With A_D^u, $B_{\gamma_0}^u$, and $\mathbb{S}$ computed, we then proceed to computing $\tilde{F}_{\gamma_0}^u$. The calculation of $\tilde{F}_{\gamma_0}^u$ will occur in preprocessing and so does not necessarily require as careful of numerical optimization. Numerical simplicity and accuracy are then the driving factors in our choice of method. Direct and robust methods for multiple-input multiple-output (MIMO) time-invariant state feedback control pole placement computation are readily available in the literature. For example, in ([A-V]), Abdelaziz and Valasek present an algorithm based on transformations to Frobenius canonical form followed by use of an Ackermann-type formula. See also Ackermann's original single-input single-output 1972 algorithm [A.1] and the parametric form algorithm based on the QR-factorization of B and use of Sylvester equations of Kautsky et al. [K-N-V].

2.3.2 Implicit Computation of the Control

Equation 2.3.0.3 defines the control $V(t)$ implicitly in terms of itself and computationally expensive operators D and $P^u_{\gamma_0}$, and so the evaluation of $V(t)$ must be considered carefully. Upon initial consideration of eq. 2.3.0.3, we observe that our evaluation of V at a subsequent time step V^{n+1} given known V^n may solved for implicitly or explicitly. For an explicit solution to eq. 2.3.0.3, we could for example replace the unknown V^{n+1} on the right hand side with approximation by the known V^n as in

$$V^{n+1} \;=\; \Psi \tilde{F}^u_{\gamma_0} P^u_{\gamma_0}[q(t) - DV^n(t)], \qquad (2.3.2.1)$$

or use a multistaged method possibly combined with Richardson extrapolation (see [P-T-V-F]. For an implicit solution to eq. 2.3.0.3 we could use initial guess the explicit solution given by eq. 2.3.2.1 and then use an iterative method to improve the solution. In the case of explicit evaluation, stability and accuracy would be serious concerns; in the case of implicit evaluation, computational cost would be drastic.

However, given a known q, a substantially improved computational approach for the evaluation of V^{n+1} may be used. We may take advantage of the finite dimensionality of $V(t)$ and linearity of the involved operators to reduce the calculation cost to a minimum. Consider the implicit equation

$$V^{n+1} = \Psi \tilde{F}^u_{\gamma 0} P^u_{\gamma 0}[q(t) - DV^{n+1}(t)] \quad (2.3.2.2)$$

rewritten to isolate values which do not require repeated calculations (per solution of eq. 2.3.2.2)

$$V^{n+1} = \Psi \left(\left[\tilde{F}^u_{\gamma 0} P^u_{\gamma 0} q(t) \right] - \tilde{F}^u_{\gamma 0} P^u_{\gamma 0} D V^{n+1}(t) \right). \quad (2.3.2.3)$$

The bracketed expression on the right hand side of eq. 2.3.2.3 is independent of iterative solution steps, and needs only be calculated once per evaluation of V^{n+1}. We note, as discussed again in subsection 2.2.3, that the term DV may be efficiently evaluated by use of expansion eq. 2.3.3.8 and preprocessing of constant terms. Furthermore, when considered as an operator on the space

$$B = \text{span}_{i=1..J}\{\psi_i\} \quad (2.3.2.4)$$

to itself, the operator $\tilde{G}$ defined by

$$\tilde{G} f = \Psi \tilde{F}^u_{\gamma 0} P^u_{\gamma 0} DV(f), \quad (2.3.2.5)$$

when considered in the basis $\psi_1, \cdots, \psi_J$, may be represented by a $J \times J$ matrix G. Defining $\tilde{K}$ as $\tilde{F}^u_{\gamma 0} P^u_{\gamma 0} q(t)$, we then have the $V(t) = \Psi f$, where f in $\mathbb{R}^J$ is the solution to the equation

$$f \;=\; \tilde{K} - Gf. \tag{2.3.2.6}$$

Equation 2.3.2.6 may be solved via any standard direct method; it is simply a Jth order ($J = 2$ in 2-D and $J = 4$ in 3-D) linear algebraic equation, and solution cost is minimal. The matrix G does not depend on any of the flow or control variables and may be computed before the CFD solver begins during preprocessing. This reduces the computation of V^{n+1} to evaluation of $\tilde{K}$ and solution of eq. 2.3.2.6, which reduces the cost of solving for the control in each time step to a negligible minimum.

2.3.3 Operator Evaluations: The Projection Operator $P^u_{\gamma_0}$ and the Dirichlet Operator D

The Projection Operator $P^u_{\gamma_0}$.

In the control formulation given in 2.3.0.3, the normal component of the velocity v is first projected via $(I - \Pi^0)$ onto the infinite-dimensional subspace Z; identified with the variable $q(t)$, and the difference $q(t) - DV$ is then passed through the projection operator $P^u_{\gamma_0}$. The control $V(t, x, z)$, which satisfies $\int_{-e}^{e} \int_{-\pi}^{\pi} V(t, x, z)\; dxdz = 0$, is guaranteed by Proposition 1.5.1 to satisfy

$$(I - \Pi^0)V(t) = 0. \tag{2.3.3.1}$$

Therefore, by making use of 2.3.3.1 and the linearity of $(I - \Pi^0)$, we have

$$P^u_{\gamma_0}[q(t) - DV(t)] \;=\; P^u_{\gamma_0}[(I - \Pi^0)v(t) - (I - \Pi^0)DV(t)] \;=\; P^u_{\gamma_0}(I - \Pi^0)[v(t) - DV(t)]. \tag{2.3.3.2}$$

The calculation of $(I - \Pi^0)v = q$ is thus unnecessary as an intermediate calculation for this evaluation of the control. Instead, we may directly take the difference of $v(t)$ and DV, and then evaluate the composed projection $P^u_{\gamma_0}(I - \Pi^0)$ onto the finite dimensional subspace $Z^u_{\gamma_0}$ instead. This means there is no need to project onto the infinite-dimensional subspace Z, we need only instead the finite-dimensional projection (see 1.5.3d)

$$Z^u_{\gamma_0} \ni P^u_{\gamma_0}(I - \Pi^0)[v(t) - DV(t)] \equiv$$

$$\sum_{\substack{i=1,2 \\ (n,m,0)\in\mathcal{U}^a_{\gamma_0}}} ((v - DV), e^i_{nm0})_\Omega e^i_{nm0} \;+\; \sum_{\substack{i=3,4 \\ (0,m,k)\in\mathcal{U}^b_{\gamma_0}}} ((v - DV), e^i_{0mk})_\Omega e^i_{0mk} \;+$$

$$\sum_{\substack{i=5,6,7,8 \\ (n,m,k)\in\mathcal{U}^c_{\gamma_0}}} ((v - DV), e^i_{nmk})_\Omega e^i_{nmk}. \tag{2.3.3.3}$$

Our goal then, is to efficiently evaluate the projection equation (2.3.3.3) at each time step.

The simplest approach in evaluating eq. 2.3.3.3 via a direct quadrature method (with storing the value of each integral to avoid repeat calculations) will be our benchmark of comparison. Assuming a structured $n \times m$ or $n \times m \times l$ grid with N_{γ_0} being the number of eigenvalues corresponding to the 'unsatisfactory' subspace $Z^u_{\gamma_0}$

(see 1.5.2e), the number of FLOPs (floating point operations) for evaluation of 2.3.3.3 at every computational node via direct quadrature would be approximately $3nmN_{\gamma_0}$ or $3nmlN_{\gamma_0}$ to evaluate the integrals plus $2nmN_{\gamma_0}$ or $2nmlN_{\gamma_0}$ to calculate the linear combinations of eigenfunctions, respectively.

We will consider an alternative approach to the calculation of 2.3.3.3. In this alternative method, we instead make use of preprocessing for stored values of all terms not dependent on the projection operator $P^u_{\gamma_0}$'s input $(v - DV)$. In particular, rewriting the quadrature for 2.3.3.3 for the 3-D channel model (the 2-D version is similar) to emphasize pre-calculable values, we get the projected value of $(v - DV)$ evaluated at node (n_1, n_2, n_3) is

$$Z^u_{\gamma_0} \ni P^u_{\gamma_0}(I - \Pi^0)[v(t) - DV(t)](n_1, n_2, n_3) \approx$$

$$\sum_{\substack{i=1,2\\(n,m,0)\in\mathcal{U}^a_{\gamma_0}}} \sum_{\substack{i_x\in\\ \text{x rows}}} \sum_{\substack{i_y\in\\ \text{y rows}}} \sum_{\substack{i_z\in\\ \text{z rows}}} (\Upsilon(v - DV)e^i_{nm0})|_{(i_x,i_y,i_z)} e^i_{nm0} \ +$$
$$\sum_{\substack{i=3,4\\(0,m,k)\in\mathcal{U}^b_{\gamma_0}}} \sum_{\substack{i_x\in\\ \text{x rows}}} \sum_{\substack{i_y\in\\ \text{y rows}}} \sum_{\substack{i_z\in\\ \text{z rows}}} (\Upsilon(v - DV)e^i_{0mk})|_{(i_x,i_y,i_z)} e^i_{0mk} \ +$$
$$\sum_{\substack{i=5,6,7,8\\(n,m,k)\in\mathcal{U}^c_{\gamma_0}}} \sum_{\substack{i_x\in\\ \text{x rows}}} \sum_{\substack{i_y\in\\ \text{y rows}}} \sum_{\substack{i_z\in\\ \text{z rows}}} (\Upsilon(v - DV), e^i_{nmk})|_{(i_x,i_y,i_z)} e^i_{nmk} \ = \qquad (2.3.3.4)$$

$$\sum_{i_x\in \text{x rows}} \sum_{i_y\in \text{y rows}} \sum_{i_z\in \text{z rows}} (v - DV)|_{(i_x,i_y,i_z)} \Big[\Upsilon|_{(i_x,i_y,i_z)}$$
$$\Big(\sum_{\substack{i=1,2\\(n,m,0)\in\mathcal{U}^a_{\gamma_0}}} e^i_{nm0}|_{(i_x,i_y,i_z)} e^i_{nm0}|_{(n_1,n_2,n_3)} +$$
$$\sum_{\substack{i=3,4\\(0,m,k)\in\mathcal{U}^b_{\gamma_0}}} e^i_{0mk}|_{(i_x,i_y,i_z)} e^i_{0mk}|_{(n_1,n_2,n_3)} +$$
$$\sum_{\substack{i=5,6,7,8\\(n,m,k)\in\mathcal{U}^c_{\gamma_0}}} e^i_{nmk}|_{(i_x,i_y,i_z)} e^i_{nmk}|_{(n_1,n_2,n_3)} \Big) \ \Big], \qquad (2.3.3.5)$$

where $\Upsilon|_{(i_x,i_y,i_z)}$ are appropriately chosen quadrature weights. In 2.3.3.5, the terms contained within the square brackets do not depend on the input $(v - DV)$, i.e. are constant with respect to the simulator's time steps. They may be calculated during the preprocessing phase.

The approximate number of FLOPs for the computation of 2.3.3.5 required for calculation during each time step is $2n^2m^2l^2$ (in 2-D, $2n^2m^2$). Comparing this to the computational cost of the benchmark method $3nmlN_{\gamma_0} + 2nmlN_{\gamma_0}$ (in 2-D, $3nmN_{\gamma_0} + 2nmN_{\gamma_0}$), we see that the preprocessing method (via eq. 2.3.3.5) is an improvement to the benchmark method if and only if

$$\frac{5}{2}N_{\gamma_0} \; > \; nml \text{ in dimension 3,} \tag{2.3.3.6}$$

$$\frac{5}{2}N_{\gamma_0} \; > \; nm \text{ in dimension 2.} \tag{2.3.3.7}$$

However, in only rare cases (depending on the size of γ_0 and the grid resolution) will N_γ be an order of magnitude or more higher than nml. Thus, although the preprocessing method is tempting, it does not merit except for exceptional cases. For reference, in dimension 2 and 3, graphs of the values of N_{γ_0} as a function of γ_0 are presented (see figures 2.2 and figure 2.3).

The Dirichlet Operator D.

Recall the Dirichlet elliptic operator D: $H^s(\Gamma_1) \to H^{s+\frac{1}{2}}(\Omega)$, $s \in \mathbb{R}$, defined by the elliptic problem (1.2.1a–f), see (1.2.1g). For efficient calculation of the solution to the Dirichlet problem (1.2.1a–f) (and hence evaluation eq. 2.3.0.3), we turn again

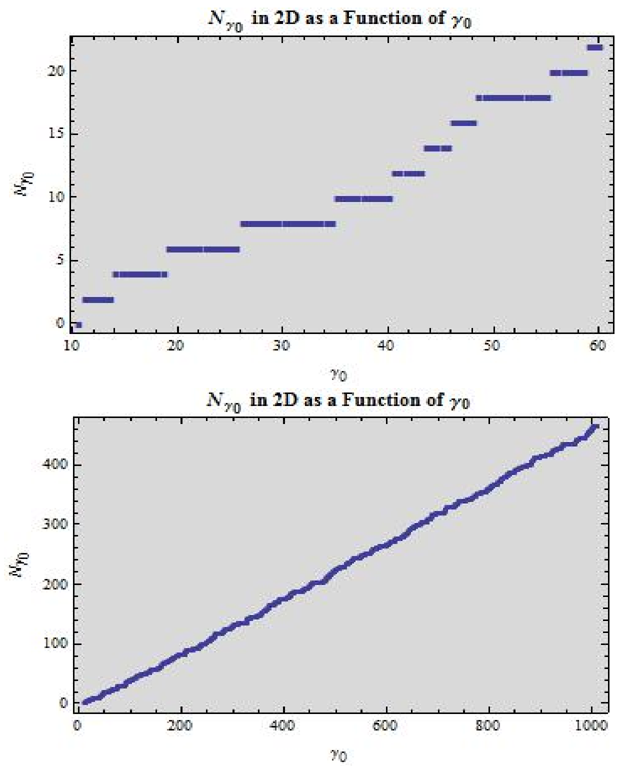


Figure 2.2: Dependency of the Total Unsatisfactory Eigenvalue Multiplicity N_γ as a Function of γ_0 in the 2-D Case.

to the multigrid method as described in subsection 2.2.3. The multigrid method may be structured and set up identically to that of the Neumann elliptic problem 2.2.1.8 albeit with appropriate boundary condition considerations. However, the multigrid method will converge more quickly for the solution to the Dirichlet operator D than the solution to the Neumann problem 2.2.1.8, and so may be configured to run with fewer pre- and post-smoothing Gauss-Seidel iterations or a simpler multigrid iteration structure.

When applying the operator D to the control V, a simplification may be made to

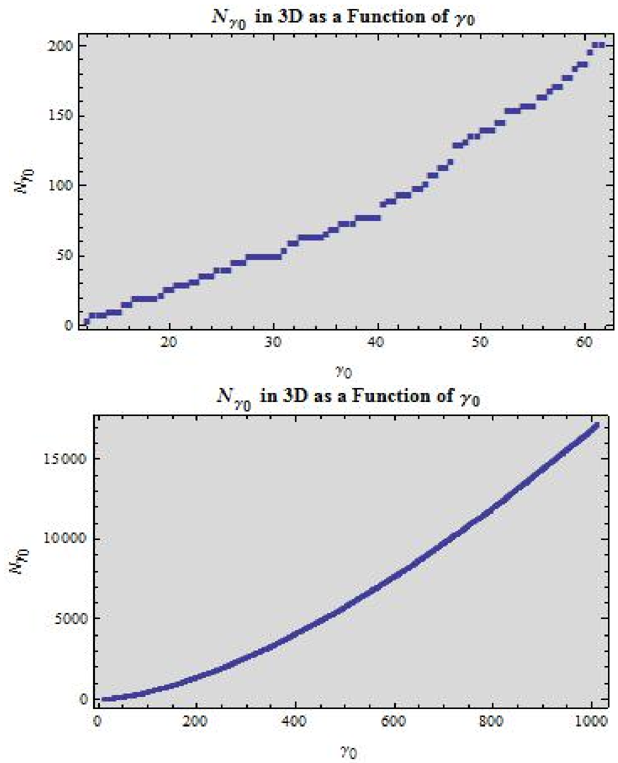


Figure 2.3: Dependency of the Total Unsatisfactory Eigenvalue Multiplicity N_γ as a Function of γ_0 in the 3-D Case.

the calculations. V is a linear combination of control basis functions ϕ_i, i.e.

$$V(t) \;=\; [\psi_1 \ldots \psi_j] \begin{bmatrix} \phi_1(t) \\ \vdots \\ \phi_J(t) \end{bmatrix}, \tag{2.3.3.8}$$

where J is the dimension of the control as seen in subsection 1.1.2. We may use the linearity of D to rewrite DV as

$$DV \;=\; D\sum_{i=1}^{J} \phi_i(t)\psi_i \;=\; \sum_{i=1}^{J} \phi_i(t)(D\psi_i). \tag{2.3.3.9}$$

The terms in parentheses in 2.3.3.9 do not depend on the time step and may calculated during the preprocessing phase and stored for all time steps. Thus, the expansion 2.3.3.8 allows us to use 2.3.3.9 to calculate DV at minimal computational cost.

2.4 Numerical Results and Conclusions

The computational fluid dynamics (CFD) software program was built by the author for simulation of the 2-D linear Navier-Stokes channel flow system analyzed in [Tr.4]. The CFD simulator uses the finite volume fractional time step pressure-projection methods described throughout section 2.1. In the present section, the results and conclusions will be presented. In subsection 2.4.1, results for the steady-state profile calculation and stability decay rates of the control-free dynamics will be presented. In subsection 2.4.2, we will comment on future work and make our concluding remarks.

2.4.1 Steady-State Calculation and Control-Free Stabilization Rates of the 2-D Linear Navier-Stokes System

A computational fluid dynamics solver was built using the methods described in section 2.1 for the numerical simulation of the 2-D linear Navier-Stokes channel flow analyzed in [Tr.4]. The governing equations for this system are

$$\begin{cases} u_t - \nu\Delta u &= p_x \quad \text{in } Q; & (2.4.1.1a) \\ v_t - \nu\Delta v &= p_y \quad \text{in } Q; & (2.4.1.1b) \\ u_x + v_y &\equiv 0 \quad \text{in } Q; & (2.4.1.1c) \end{cases}$$

$$\text{B.C. for } u\text{:} \begin{cases} u_y(x,0,t) &\equiv 0, \; u_y(x,1,t) \equiv 0; & (2.4.1.1d) \\ u(-\pi,y,t) &= u(\pi,y,t); & (2.4.1.1e) \\ u_x(-\pi,y,t) &\equiv u_x(\pi,y,t); & (2.4.1.1f) \end{cases}$$

$$\text{B.C. for } v\text{:} \begin{cases} v(x,0,t) &\equiv 0, \; v(x,1,t) = V(x,t); & (2.4.1.1g) \\ v(-\pi,y,t) &\equiv v(\pi,y,t); & (2.4.1.1h) \\ v_x(-\pi,y,t) &\equiv v_x(\pi,y,t). & (2.4.1.1i) \end{cases}$$

with $Q = \Omega \times (0,T]$ and over the domain

$$\Omega = \{(x,y) : \; -\pi \le x \le \pi; \; 0 \le y \le 1\};$$

with boundary $\Gamma = \Gamma_0 \cup \Gamma_1$, where

$$\begin{aligned} \Gamma_0 &= \{x = \pm\pi; \; 0 \le y \le 1\} \cup \{y = 0; \; -\pi \le x \le \pi\}; \\ \Gamma_1 &= \{y = 1; \; -\pi \le x \le -\pi\}, \end{aligned} \tag{2.4.1.2}$$

Numerical verification was sought for the velocity profile of the stable steady state solution for equations 2.4.1.1. The numerical simulation was run with several differing initial velocity fields as well as different kinematic viscosities and Reynolds numbers. This included testing of the parabolic profile steady state solution to the Dirichlet no-slip Navier-Stokes system, i.e.

$$u(x, y) = Cy(1 - y), \quad v(x, y) = 0. \tag{2.4.1.3}$$

In every simulation case, including the Dirichlet steady state 2.4.1.3, the flow dynamics stabilized to the family of solutions having constant tangential velocity field u, zero normal velocity field v, and constant pressure field (known only up to a constant, i.e. equivalent to the zero pressure scalar field).

The finite-volume CFD solver was used to verify the control-free stabilizing decay of the velocity. The L^2-decay rate of the velocity vector $\vec{v}$ and the H^1-decay of the streamwise component u with initial parabolic profile (2.4.1.3) and kinematic viscosity ν set to 0.05 is given in figures 2.4 and 2.5.

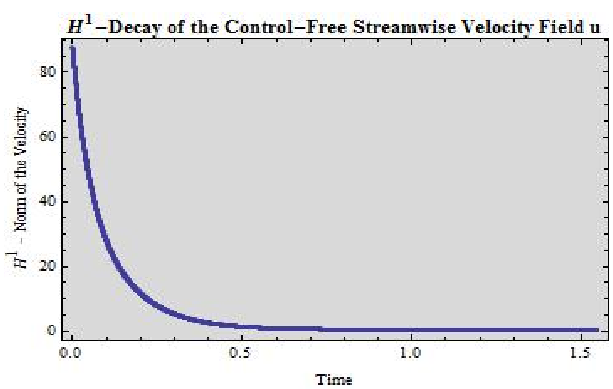


Figure 2.4: H^1-Exponential Decay of Tangential Velocity Component u with Initial Conditions Given by Eq. 2.4.1.3

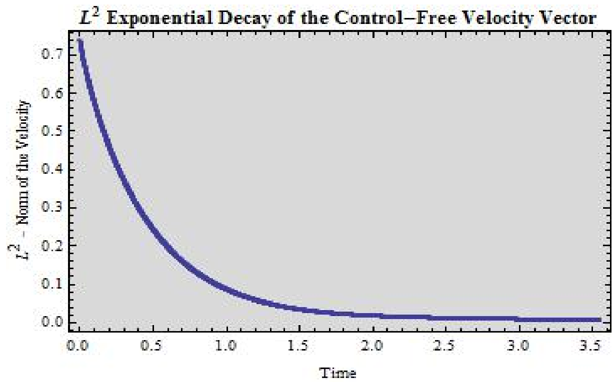


Figure 2.5: L^2-Exponential Decay of Velocity Vector $\vec{v}$ with Initial Conditions Given by Eq. 2.4.1.3

We include here an additional example corresponding to initial tangential velocity component u given by

$$u \; = \; 1 + 1.5(2 - \cos(2\pi y)) \sin(\pi 5 y) \qquad (2.4.1.4)$$

with kinematic viscosity ν set to 0.05. The L^2-decay rate of the velocity vector $\vec{v}$ and the H^1-decay of the tangential component u are given in figures 2.6 and 2.7.

The control-free stabilizing decay of the system 2.4.1.1 was conducted with initial tangential velocity profile $u(y) = 1 - \cos(2\pi y)$ under a range of kinematic viscosities from $\nu = 0.2$ to $\nu = 0.005$. From these simulations, the L^2-decay rates of the velocity vector $\vec{v}$ were calculated and are graphed as a function of kinematic viscosity ν. The relationship, as expected by equations 2.4.1.5 and 2.4.1.6, is confirmed to be linear. See the figure 2.8.

The results given in figures 2.4, 2.5, 2.6, 2.7, and 2.8 demonstrate the decay

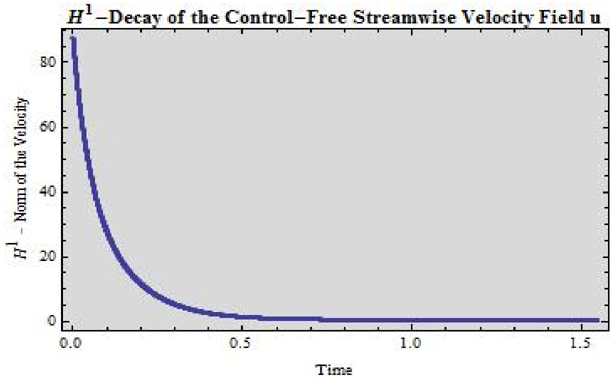


Figure 2.6: H^1-Exponential Decay of Tangential Velocity Component u with Initial Conditions Given by Eq. 2.4.1.4

relationships

$$\|u(t)\|_{L_2(\Omega)/(\mathbb{R})} + \|\nabla u(t)\|_{L_2} \leq$$

$$C_1 e^{-\nu\pi^2 t}[\|u(0)\|_{H^1(\Omega)} + \|v(0)\|_{H^1(\Omega)} + \|(I - \Pi^0)[v(0) - DV(0)]\|_{H^1(\Omega)}], \quad (2.4.1.5)$$

$$\|v(t)\|_{L_2(\Omega)} \leq C_2 e^{-\pi^2 t}\|v(0)\|, \quad (2.4.1.6)$$

as well give confirmation of the stability of steady state solutions given by constant tangential velocity field u, zero normal velocity field v, and constant pressure field.

2.4.2 Looking Ahead, Future Work, and Conclusions

The present work adds to the growing body of scientific literature and knowledge regarding the stabilization and control properties of fluid flows and the Navier-Stokes equations. As the scientific and mathematical community continues growing our understanding of this subject simultaneously with the growth of micro-fluidic tech-

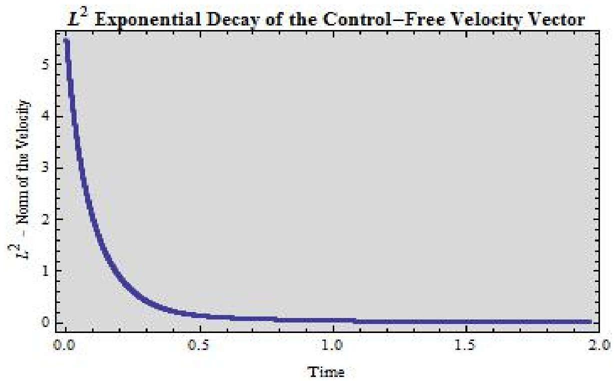


Figure 2.7: L^2-Exponential Decay of Velocity Vector $\vec{v}$ with Initial Conditions Given by Eq. 2.4.1.4

nologies which may be used for implementing flow observation and actuation, the methods available for active and passive fluid control in flow scenarios will become increasingly feasible for industrial applications.

Future work may include extension of the wall-normal finite-dimensional feedback controller to other geometries as well as extension to global stabilization stabilization enhancement feedback of the linearized Oseen equations and local stabilization of the full non-linear Navier-Stokes equations. Future numerical work may include the extension of the CFD simulation to include feedback control stabilization simulation on arbitrary domains and the corresponding algorithm optimization.

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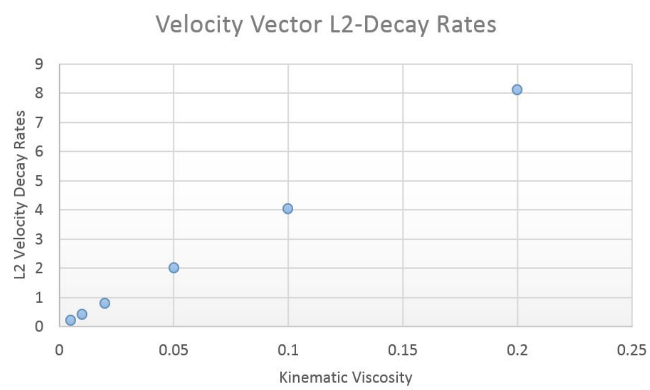

Figure 2.8: L^2-Exponential Decay Rates of Control-Free Dynamics as Functions of Kinematic Viscosity

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