Asset Pricing, Monetary Policies, and the Zero Lower Bound

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Abstract

The Great Recession challenged the conventional way of conducting monetary policy and sparked debates on optimal monetary policy in the presence of the zero lower bound on nominal interest rates, and its interaction with asset prices in financial market. My dissertation shows how the zero lower bound changes the conduct of monetary policy, and how considering asset prices in monetary policy helps to improve the welfare of the economy.

In Chapter 1, I first examine the role of money in precautionary saving and the behavior of other asset prices, such as returns on bonds and stocks. Money is introduced via the form of transaction cost into a production economy with limited stock market participation, where agents with a lower intertemporal elasticity of substitution, called "non-stockholders", have no access to the stock market. This model not only quantitatively resolves the risk premium puzzle and the risk-free return puzzle and matches volatilities of key macroeconomic variables, but also corresponds with empirically documented facts regarding money growth, inflation, and asset prices in the literature. I then examine whether money growth should respond to equity prices and equity premiums. I find that monetary policy improves welfare for both stockholders and non-stockholders if it reduces equity premiums in the economy. The model thus prescribes money growth rules that are pro-cyclical with respect to equity prices or equity premium changes.

Chapter 2 studies how the conduct of monetary policy affects the frequency of the zero lower bound binding in a canonical New-Keynesian Dynamic Stochastic General Equilibrium (DSGE) model. The model generates a much deeper recession if the zero lower bound binds than when it does not. The model economy almost never hit the zero lower bound if monetary policy is described by a Taylor rule with interest rate smoothing. Increasing the central bank's reaction to deviations of inflation and output does not increase the probability of the economy hitting the zero lower bound. With interest rate smoothing, a lower inflation target significantly increases the likelihood of the economy hitting the zero lower

bound. I also show that linear approximation solutions fail not only quantitatively but also qualitatively in analyzing the zero lower bound.

The final chapter explores the role of credibility and what optimal policy the central bank can credibly make for forward guidance by solving for the whole set of sustainable sequential equilibria (SSE) in a standard New Keynesian model with an occasionally binding constraint on the nominal interest rate. Under full commitment, forward guidance is a longer duration of nominal rates at zero even when the contractionary shock disappears, followed by a quick revert-to-normal path. However, under the best SSE, there are many different policy paths that can be interpreted as forward guidance. One possible policy path features a smaller decrease of the nominal rate when the contractionary shock happens and an even longer duration at the zero lower bound than under full commitment. Another policy path keeps the nominal rate away from zero all the time and results in smaller inflation. The economy reverts more slowly to its normal state under the best SSE. The key insight is that central banks with same credibility may choose very different policies in their forward guidance and they do not have to lower nominal rate to zero. The full commitment equilibrium is not generally implementable.

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Chapter 1

Asset Pricing and Monetary Policy*

1.1 Introduction

Whether and how monetary policy should respond to asset prices is for many years a deeply debated subject. Though the central banks' not doing anything with asset prices was a consensus before 2007, the recent crisis has now shifted it.[1] The recent policy of the Federal Reserve Board (Fed), buying securities including long-term government debt and privately-issued securities, a policy called "quantitative easing (QE)", is definitely an evidence of such a shifting. It is also well-known that even in normal times, it is by buying and selling reserves in the interbank overnight market that the Fed conducts monetary policy, relying on no-arbitrage arguments to link this short-term nominal debt market to the markets for longer-term bonds and equities. Any model which purports to guide and

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[1]See Bernanke and Gertler (2001), for example.

evaluate the wisdom of monetary policies should be consistent with the relative prices of the assets in question. However, the standard models are quantitative failures at reproducing these relative prices (for example, the well-known and persistent equity premium and risk-free return puzzles) and thus not reliable to shed light on monetary policies, especially their welfare implications.

This paper contributes to the debate by first developing a model that is consistent with both macroeconomic and asset market data. The model is an extension of Guvenen (2009), where there are two types of agents, stockholders and non-stockholders, in a production economy. The stockholders, as a smaller fraction of total population, have access to both the stock market and bond market, while the non-stockholders are restricted to the latter. Both of them are allowed to borrow only up to a certain amount based on their wage income. They both have Epstein-Zin-Weil preferences (Epstein and Zin, 1989; Weil, 1990), which disentangle risk aversions from the elasticity of intertemporal substitution (EIS). Consistent with empirical evidence, the non-stockholders, who have relatively lower wealth, are assumed to have a low EIS, whereas the stockholders have a higher elasticity. There is one external firm, whose share is normalized to be 1, that can produce with capital and labor, subject to capital adjustment cost.

Money is introduced via transaction costs on consumption, taking the form of real resources that are consumed during the process of exchange (Brock, 1974, 1990; Schmitt-Grohe and Uribe, 2004). That is, given a volume of consumption goods, higher money balance helps reduce the cost of consumption. It turns out that adding money is important. First, money is one type of assets, meaning that it has to be properly priced as others like bonds and stocks via no-arbitrage. Indeed, since money now competes with bonds in the role of saving intertemporally, it is harder for the model to match asset pricing facts such as a high and countercyclical equity premium. Second, since money is distributed equally to the agents as a lump-sum transfer and the wealth and preferences of those agents differ, it has real effects on macro fundamentals and hence asset prices through portfolio reallocation. Therefore, money is non-neutral. The monetary authority conducts monetary policy by varying money growth rates.

This model matches well asset price and macroeconomic data with plausible calibrations. It produces higher equity premium and lower risk-free return, 5.3% and 1.6% per annum, respectively. In particular, the match of Sharpe ratio (0.25 vs 0.32 in the data) shows that the risks in the economy are appropriately evaluated. Compared with the literature, the model also performs better in matching volatilities of output growth, the relative volatilities of consumption and investment. Finally, the model is in line with empirically documented relationship between money growth, inflation and asset prices. Specifically, the decrease of expected inflation (money growth rate) explains partially the decline of equity premiums during the past three decades. Unexpected inflations due to real shocks are negatively related with stock prices, real stock returns and real risk-free returns.

I then use this model to evaluate alternative monetary policies: (1) a change of average money growth rates; (2) a money growth rate reacting to business cycles; (3) a monetary policy responding to equity prices; and (4) a policy reacting to expected risk premiums. All the experiments are compared with a representative agent model with similar setups to shed light on the innovations of having assets appropriately priced. In stark contrast with the conventional wisdom of Friedman rule, my model shows that the monetary policies yield welfare improvements for all types of agents if it drives down the equity premiums. This corresponds to a policy with lower expected inflation, a pro-cyclical monetary policy, and policies that positively responding to stock prices and equity premiums.

This paper relates to a large literature discussing monetary policies and asset prices. Bernanke and Gertler (2001) shows in a small scale macro model that asset prices become relevant only to the extent they may signal potential inflationary or deflationary forces and rules that directly target asset prices appear to have undesirable side effects. Bullard and Schaling (2002) constructs a model showing that asset prices targeting can interfere with the minimization of inflation and output variation and under certain conditions, asset price targeting can lead to indeterminacy. On the other hand, Cecchetti (1997), Blanchard (2000), and Mishkin (2000) claimed that considering asset prices improved economic performance.

More recently, Rudebusch et al. (2007) point out that changes in the term premium of bonds has real effects to the economy. Gallmeyer et al. (2007), and Palomino (2010) show that term structure and endogenous inflation are important for understanding monetary policies. De Paoli and Zabczyk (2013) shows that the varying precautionary saving due to cyclical risk aversion could lead to large policy errors in turbulent times. However, none of these works or suggestions has been built on a structural model that explains the asset prices data quantitatively. It is therefore likely to misestimate the effects of considering asset prices in monetary policies.

This paper relates to a growing literature that jointly studies asset prices and macroeconomic behavior. Besides Guvenen (2009), many other papers deal with the same problem, including Danthine and Donaldson (2002), Storesletten et al. (2007), Tallarini (2000), etc. However, money is not considered in any of these models. Earlier papers that discuss the relationship between money and asset prices includes Danthine and Donaldson (1986), Marshall (1992), Labadie (1989), Boyle (1990), Balduzzi (1996), and Hodrick et al. (1991), all of which focused on an endowment economy. In particular, Bansal and Coleman (1996) explain equity premium by examining money's role in facilitating transactions. This idea then is inherited by Gust and Lopez-Salido (2010), who consider market segmentation in checking and brokerage account. These two accounts are different in liquidity and infrequent portfolio rebalance leads to the high volatility in consumption for those who are active and then compensated by high risk premium. They show that this model can match some statistics of asset prices and then discuss the welfare gains for different monetary policies. However, they do not explore the performance of the model in matching macroeconomic behavior.

There is also a large literature documenting the relationship between money growth, inflation and asset prices. For example, inflation is negatively correlated with real stock prices if the economy is driven by supply shocks; see Tatom (2011) and Christiano et al. (2010).[2] Also the negative relationship between inflation and asset returns is in the spirit of

[2]Since inflation is low during stock market booms, so that, Christiano et al. (2010) claim, an interest rate

research in finance initiated in the early 1980s. Geromichalos et al. (2007) study the effect of monetary policy on asset prices in an endowment economy with search-based money. When money grows at a higher rate, inflation is higher and the return on money decreases. In equilibrium, no arbitrage amounts to equating the real return of both objects. Therefore, the price of the asset increases in order to lower its real return.

It is also claimed that the decrease of equity premium during the Great Moderation is due to the decline of inflation; See Siegel (1999), Jagannathan et al. (2000), Claus (2001), and Campbell (2008) for the evidence of decreasing premia and see Beirne and de Bondt (2008) and Kyriacou et al. (2006) for empirical estimation of inflation's role in mitigating premia.[3] Labadie (1989) explores two ways that inflation affects equity premium in a theoretical model. However, her model is based on an endowment economy. In contrast to her conclusion, my paper shows that it is the expected inflation, not the unexpected one, that determines or closely relates to the equity premium. The paper is organized as follows. The next section presents the model with solution methods and computational algorithm discussed in section 1.3. Section 1.4 explores the role of money in reconciling asset pricing and macroeconomic facts. Section 1.5 is devoted to policy analysis. Section 1.6 concludes.

1.2 The Model

The economy studied is based on the framework of Guvenen (2009). Money is introduced via a transaction cost, taking the form of real resources that are consumed in the process of exchange (Brock, 1974, 1990; Schmitt-Grohe and Uribe, 2004). That is, an increase in the volume of goods exchanged leads to a rise in transaction costs, while higher average real money balances, for a given volume of transactions, lower costs. Compared with other alternative approaches for money in macroeconomics and policy analysis, this setup loosens the tight relationship between money and consumption.[4]

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rule that is too narrowly focused on inflation destabilizes asset markets and the broader economy.

[3]Others claim that it can be due to the declining volatility of technology shocks or other possible shocks/structural changes; see Lettau et al. (2008), for example.

[4]Wang and Yip (1992) proved the "functional equivalence" between the transactions cost and other approaches like MIU, CIA and shopping time models that are commonly used in macroeconomics and policy

1.2.1 The Central Bank

In each period t, the central bank issues some new money $(g_t - 1)M^s_{t-1}$ and gives it to the agents as a lump-sum transfer. M^s_t is the per capita money supply in the economy. The money stock follows a law of motion

$$M^s_t = g_t M^s_{t-1}, \tag{1.1}$$

where g_t is the gross growth rate of money supply. The growth rate of the money supply evolves following:

$$\begin{aligned} \log(g_t) \quad = \quad & (1-\rho_g)\log(\overline{g}) + \rho_g \log(g_{t-1}) + \\ & \theta_y \log(Y_t/\overline{Y}) + \theta_{EP}\log(\mathbb{E}_t R^{EP}_{t+1}/\overline{R}^{EP}) + \theta_{PE}\log(P^s_t/\overline{P}^s) + \varepsilon_{g,t} \end{aligned} \tag{1.2}$$

In equation (1.2), $\varepsilon_g \overset{iid}{\sim} N(0, \sigma^2_g)$, $\log(\overline{g})$ is the unconditional mean of the logarithm of the growth rate g_t, $\overline{Y}$, $\overline{R^{EP}}$, and $\overline{P^s}$ are the averages of output, equity premium and stock prices, respectively. This rule allows for a systematic response of money to changes in output, expected equity premium (implied future risks) and equity prices. I evaluate four simple rules: (1) constant money growth when $\rho_g = \theta_y = \theta_{EP} = \theta_{PE} = 0$; (2) monetary rules that are either procyclical ($\theta_y > 0$) or countercyclical ($\theta_y < 0$) when $\rho_g = \theta_{EP} = \theta_{PE} = 0$; (3) rules that respond to equity premiums when $\rho_g = \theta_y = \theta_{PE} = 0$ and $\theta_{EP} \neq 0$; and finally (4) rules that respond to equity prices when $\rho_g = \theta_y = \theta_{EP} = 0$ and $\theta_{PE} \neq 0$.

1.2.2 The Households

There are two types of households who live forever in the economy. The population is constant and normalized to be unity. Let $\mu \in (0, 1)$ denote the measure of stockholders and $1-\mu$ non-stockholders. Each of them is endowed with one unit of time every period, which he allocates between market work and leisure. They both have Epstein-Zin-Weil preferences

analysis. There is another category of money modelling aiming to understand the microfoundation of money, such as search theoretic models by Kiyotaki and Wright (1989) and Nosal et al. (2010).

(Epstein and Zin, 1989; Weil, 1990) with the following form:

$$U_t^i = \left[(1-\beta)u_t^i + \beta(E(U_{t+1}^i)^{1-\sigma^i})^{\frac{\rho^i}{1-\sigma^i}}\right]^{\frac{1}{\rho^i}} \tag{1.3}$$

for $i = h, n$, where throughout the paper the superscripts h and n denote stock$h$olders and $n$on-stockholders, respectively; u_t^i is current utility, a function of consumption c_t^i and labor l_t^i. The conventional interpretation is that $\rho^i < 1$ governs the intertemporal substitution (EIS is $1/(1-\rho^i)$) and σ^i governs relative risk aversion.

The stockholders choose consumption (c_t^h), bond holdings (b_{t+1}^h), stock shares (s_{t+1}), and nominal money holdings (m_t^h) subject to the following budget constraint:

$$c_t^h + P_t^f b_{t+1}^h + P_t^s s_{t+1} + \frac{m_t^h}{P_t} + \Psi(c_t, \frac{m_t^h}{P_t}) \leq b_t^h + s_t(P_t^s + D_t) + W_t l_t^h + \frac{m_{t-1}^h + (g_t - 1)M_{t-1}^s}{P_t} \tag{1.4}$$

where the right side is all the real sources that can be spent. It includes values of holding bond and shares, $b_t^h + s_t(P_t^s + D_t)$, wage income, $W_t l_t^h$, and money carried over from previous period, m_{t-1}^h, plus the lump-sum transfer from the central bank $(g_t - 1)M_{t-1}^s$, divided by the price of goods, P_t. Stock prices, P_t^s, and dividend, D_t, are defined in details in the following subsection. Non-stockholders have a similar budget constraint except that they do not own stocks and thus do not choose s_{t+1}.

In equation (1.4), I introduce real money demand by having the transaction cost Ψ, which is a function of consumption c and real money balances m/P. Feenstra (1986) demonstrates that transaction costs satisfy the following condition for all $c, m/Pc \geq 0$: Ψ is twice continuously differentiable and $\Psi \geq 0$; $\Psi(0, m/P) = 0$; $\Psi_c \geq 0$; $\Psi_{m/P} \leq 0$; $\Psi_{cc}, \Psi_{m/P,m/P} \geq 0$; $\Psi_{c,m/P} \leq 0$; and $c + \Psi(c, m/P)$ is quasi-convex, with expansion paths having a nonnegative slope. I also introduce a portfolio adjustment cost for bonds given there is no firm leverage for the main discussions of this paper. That is, whenever the individual changes bond positions, he or she will pay a cost of $\frac{\phi}{2}(b_{t+1}^h/b_t^h - 1)^2$, where ϕ is the semi-elasticity of adjustment cost over changes. By adding such a cost, assets in the economy have a nonsingular return matrix and hence the agents can distinguish bonds from money when making a decision.[5]

1.2.3 The Firm

There is an aggregate firm producing a single good that can be used for either consumption or investment by using capital (K_t) and labor (L_t) inputs according to a Cobb-Douglas technology: $Y_t = Z_t K_t^{\theta} L_t{}^{1-\theta}$, where $\theta \in (0, 1)$ is the factor share parameter. The productivity level evolves according to:

$$\log(Z_{t+1}) = \rho_z \log(Z_t) + \varepsilon_{z,t+1}, \varepsilon_z \overset{iid}{\sim} N(0, \sigma_z^2) \tag{1.5}$$

The objective of the firm is to maximize the value of the firm, which equals the value of future dividend stream generated by the firm, $\{D_{t+j}\}_{j=1}^{\infty}$, discounted by the marginal rate of substitution process of stockholders, $\{\Lambda_{t,t+j}\}_{j=1}^{\infty}$. Specifically, the firm's problem is to solve:

$$P_t^s = \max_{\{I_{t+j}, L_{t+j}\}} E_t \left[\sum_{j=1}^{\infty} \Lambda_{t,t+j} D_{t+j} \right] \tag{1.6}$$

subject to the law of motion for capital, which features adjustment costs in investment:

$$K_{t+1} = (1 - \delta) K_t + \Phi(I_t / K_t) K_t \tag{1.7}$$

where P_t^s is the ex-dividend value of the firm. The number of shares outstanding is normalized to unity for convenience and hence P_t^s is the stock price. Strict convexity of $\Phi(\cdot)$ captures the difficulty of quickly changing the level of capital installed in the firm, which is necessary if the model is to generate realistic asset prices, particularly equity prices; see Cochrane (1991). An equity share is the right to own the entire stream of dividends, defined by the profits net of wages and investments: $D_t = Z_t K_t^{\theta} (L_t)^{1-\theta} - W_t L_t - I_t$.

[5] Tough with different forms of the functional form, money has been introduced in this approach widely in the literature, see Sims (2003), Schmitt-Grohe and Uribe (2004), etc.

1.2.4 The Financial Markets

There are three types of assets traded in this economy: a one-period real bond, an equity share (stock) and money. The non-stockholders can freely trade the risk-free bond and hold any amount of money as they want while restricted from participating in the stock market. By contrast, the stockholders have access to stock market in addition to the other two. Hence the stockholders are the sole capital owners in the economy. As in Guvenen (2009), portfolio constraints are imposed to avoid Ponzi schemes. In particular, any agent can only borrow up to a limited amount of his wage income.

1.2.5 Individuals' Dynamic Problem and Equilibrium

A change in variables is introduced so that the problem solved by the households is stationary. That is, let $\widehat{m}_t^i = m_t^i/M_t^s$ and $\widehat{P}_t = P_t/M_t^s$. In addition, I let $\widehat{m}^i = \widehat{m}_{t-1}^i$ be the money balances of agent i at the beginning of period t. In each given period, the portfolio of each group is a function of the *beginning-of-period* capital stock, K, the aggregate bond holdings of non-stockholders after production, B, the *beginning-of-period* aggregate money holdings of non-stockholders, $\widehat{M}$, the gross rate of money supply, g, and the technology level, Z. Let Υ denote the aggregate state vector $(K, B, \widehat{M}, Z, g)$. The dynamic programming of a stockholder can be expressed as follows (primes indicate next period values):

$$V^h(\omega^h;\Upsilon) = \max_{c^h,l^h,b^{h\prime},s^{h\prime},\widehat{m}^{h\prime}} \left[(1-\beta)u(c^h,l^h)^{\rho^h} + \beta(E(V^h(\omega^{h\prime};\Upsilon')|Z,g)^{1-\sigma^h})^{\frac{\rho^h}{1-\sigma^h}}\right]^{\frac{1}{\rho^h}} \quad (1.8)$$

$s.t.$

$$\omega^h + W(\Upsilon)l^h \geq c^h + P^f(\Upsilon)b^{h\prime} + P^s(\Upsilon)s' + \frac{\widehat{m}^{h\prime}}{\widehat{P}(\Upsilon)} + \Psi(c, \frac{\widehat{m}^{h\prime}}{\widehat{P}(\Upsilon)}) \quad (1.9)$$

$$\omega^{h\prime} = b' + s'(P^s(\Upsilon') + D(\Upsilon')) + \frac{\widehat{m}^{h\prime} + g' - 1}{\widehat{P}(\Upsilon')g'} \quad (1.10)$$

$$K' = \Gamma_K(\Upsilon), B' = \Gamma_B(\Upsilon), \widehat{M}' = \Gamma_{\widehat{M}}(\Upsilon) \quad (1.11)$$

$$b^{h\prime} \geq \underline{B} \quad (1.12)$$

where ω^h denotes financial wealth; b^h and s are individual bond and stock holdings, respectively; Γ_K, Γ_B and $\Gamma_{\widehat{M}}$ denote the laws of motion for the wealth distribution which are determined in equilibrium; and P^f is the equilibrium bond pricing function. The problem of a non-stockholder can be written as above with $s' \equiv 0$, and the superscript h replaced with n.

A *stationary recursive competitive equilibrium* for this economy is given by a pair of value functions, $V^i(\omega^i;\Upsilon)$; consumption, labor supply, bond holding decision rules and money holding decision rules for each type of agent, $c^i(\omega^i;\Upsilon), l^i(\omega^i;\Upsilon), b^{i\prime}(\omega^i;\Upsilon)$, and $\widehat{m}^{i'}(\omega^i;\Upsilon)$; a stockholding decision rule for stockholders, $s'(\omega^i;\Upsilon)$; stock, bond and consumption goods pricing functions, $P^s(\Upsilon), P^f(\Upsilon)$, and $\widehat{P}(\Upsilon)$; a competitive wage function, $W(\Upsilon)$; an investment function for the firm, $I(\Upsilon)$; laws of motion for aggregate capital, aggregate bond holdings of non-stockholders, and aggregate money holdings of non-stockholders, $\Gamma_K(\Upsilon), \Gamma_B(\Upsilon)$, and $\Gamma_{\widehat{M}}(\Upsilon)$; and a marginal utility process, $\Lambda(\Upsilon)$, for the firm such that:

(1) Given the pricing functions and laws of motion, the value function and decision rules of each agent solve that agent's dynamic problem.

(2) Given $W(\Upsilon)$ and the equilibrium discount rate process obtained from $\Lambda(\Upsilon)$, the investment function $I(\Upsilon)$ and the labor choice of the firm, $L(\Upsilon)$, are optimal.

(3) All markets clear: (a) $\mu b^{h\prime}(\overline{\omega}^h;\Upsilon) + (1-\mu)b^{n\prime}(\overline{\omega}^n;\Upsilon) = 0$ (bond market); (b) $\mu s'(\overline{\omega}^h;\Upsilon) = 1$ (stock market); (c) $\mu l^h(\overline{\omega}^h;\Upsilon) + (1-\mu)l^n(\overline{\omega}^n;\Upsilon) = L(\Upsilon)$ (labor market); (d) $\mu\widehat{m}^{h\prime}(\overline{\omega}^h;\Upsilon) + (1-\mu)\widehat{m}^{n\prime}(\overline{\omega}^n;\Upsilon) = 1$ (money market); and (e) $\mu[c^h(\overline{\omega}^h;\Upsilon) + \Psi(c^h(\overline{\omega}^h;\Upsilon), \widehat{m}^{h'}(\overline{\omega}^h;\Upsilon)/\widehat{P}(\Upsilon))] + (1-\mu)[c^n(\overline{\omega}^n;\Upsilon) + \Psi(c^n(\overline{\omega}^n;\Upsilon), \widehat{m}^{n'}(\overline{\omega}^n;\Upsilon)/\widehat{P}(\Upsilon))] + I(\Upsilon) = Y(\Upsilon)$ (goods market), where $\overline{\omega}^i$ denotes the wealth of each type of agent in state Υ in equilibrium.

(4) Aggregate laws of motion are consistent with individual behavior: (a) $K' = (1-\delta)K + \Phi(I(\Upsilon)/K)K$; (b) $B' = (1-\mu)b^n(\overline{\omega}^n,\Upsilon)$; and (c) $\widehat{M}' = (1-\mu)\widehat{m}^n(\overline{\omega}^n,\Upsilon)$.

(5) There exists an invariant probability measure $\mathbf{P}$ defined over the ergodic set of equilibrium distributions.[6]

[6]Details for solving the model are in *Appendix B*.

1.3 Quantitative Analysis

1.3.1 Solution Methods

The solution method used is a direct application of policy function iteration proposed by Coleman (1989, 1990). This gives a global solution over the entire state space. There are two other methods that are popular for solving this class of models. The first one is value function iteration method, which is used by Krusell and Smith (1997), Storesletten et al. (2007) and Guvenen (2009), among others. The application of this method to incomplete asset pricing models is computationally inefficient. The second one can be roughly categorized as approximation methods, including loglinear approximation (e.g. Backus et al. 2007, 2010), affine method (e.g. Shamloo and Malkhozov (2010)), perturbation methods (e.g. Malkhozov and Shamloo, 2009b; Kim et al. (2005)). The problem with this method is that it gives local solutions around where you approximate (usually, the steady state). When the problem of interest is actually not in steady state, the local solution is useless. Second, the far away from the steady state, the bigger the approximation errors are. This may give you misleading policy functions and particularly underestimate the fluctuations in asset prices, which is important for explaining asset pricing facts. To solve the model, I substitute individual state variables out and keep only aggregate state variables. The new system of equations then is used as an input for the solver. The computational algorithm is detailed in Appendix B. It turns out my algorithm is much faster than that of Guvenen (2009). A test of running the model without money takes about 250 hours on a 3-GHz Intel dual-core Xeon cpu by his algorithm while mine only several minutes.

1.3.2 Baseline Parameterization

A model period corresponds to one month of calendar time. Table 1.1 summarizes the baseline parameter choices. I start with the parameterization of productivity and money growth. As for the technology shock, the AR(1) coefficient ρ_z is set to be 0.976 at monthly frequency in order to match the 0.95 autocorrelation of Solow residuals at quarterly frequencies. The standard deviation σ_z is set to be 0.015 to match the standard deviation of H-P filtered output in quarterly data. Similarly, ρ_g is set to be 0.17 at monthly frequency to match the 0.005 autocorrelation of $M0$ growth rate in US data. For the benchmark model, $\overline{g}$ is set to be 1.0025 to match a 3% annual inflation. The variance of money growth, σ_g, is set to be either 0 for only expected inflation or 0.0045 for unexpected inflation.

EIS parameters for stockholders and non-stockholders are set to be 0.3 and 0.1, respectively.[7] I set discount factor β to be 0.93 to get 1.69% risk-free return in the benchmark model. I then set depreciation rate $\delta = 0.02$, capital share $\theta = 0.36$ to roughly match the US capital-output ratio of 8 in quarterly data. The functional form of Φ is specified as $a_1(I_t/K_t)^{1-1/\xi} + a_2$, as in Guvenen (2009) and Jermann (1998), where a_1 and a_2 are constants chosen such that the steady state level of capital is invariant to ξ. The curvature parameter ξ determines the severity of adjustment costs. As ξ approaches infinity, Φ becomes linear, and investment is converted into capital one for one (frictionless economy limit). As ξ approaches zero, Φ becomes a constant function and capital stock will be constant regardless of the investment level (exchange economy limit). Capital adjustment coefficient is set to be 0.99 to match relative volatilities of consumption and investment over output.[8] Participation rate and borrowing constraints are the same to those in Guvenen (2009).

Portfolio Adjustment Cost

It deserves special discussion on choosing the value of ϕ, which governs how costly households adjust their bond positions. Theoretically, ϕ can be calibrated by the ratio of value added by financial department over GDP in national income account, which is approximately 5% in US. However, changing the value of ϕ gives no difference in the model economy and the simulated results show that the ratio of cost incurred over output is always negli-

[7] However, debates about the right values for these parameters persist. See Guvenen (2006), Blundell et al. (1994), among others.

[8] Capital adjustment cost function takes the form of $\Phi(I_t/K_t) = a_1(I_t/K_t)^{1-1/\xi} + a_2$, where $a_1 = \frac{\delta^{1/\xi}}{1-1/\xi}$, and $a_2 = \frac{\delta}{1-\xi}$. The parameter ξ governs how easily investment can be transformed into capital.

Table 1.1: Baseline Parameterization

parameter		Value
β^*	Time discount	0.93
$1/(1-\rho^h)$	EIS for stockholders	0.3
$1/(1-\rho^n)$	EIS for non-stockholders	0.1
μ	Participation rate	0.2
ρ_z^*	Persistence of aggregate shocks	0.95
ρ_g^*	Persistence of aggregate money supply	0.005
θ	Capital share	0.36
ξ	Capital adjustment cost coefficient	0.99
δ^*	Depreciation rate	0.02
$\underline{B}$	Borrowing limit	$6\overline{W}$
$\overline{g}^{**}$	Average growth rate of money supply	1.03
σ_ε	Standard deviation of technology shocks	0.015
σ_g	Standard deviation of monetary shocks	0.0045
$\sigma^h = \sigma^n$	Relative risk aversion	10/6/2

"*" indicates that the reported value refers to the implied quarterly value for a parameter that is calibrated to monthly frequency, while "**" indicates the implied annual value. $\overline{W}$ is the average monthly wage rate in the economy.

gible and close to 0.[9] Nevertheless, such a cost must exist to make bonds perform different from money in the sense of getting risk-free return. By experimenting with different values, I find that ϕ should be bigger than 0.1. Here I set it to be 1 for simplicity.

Utility Functions

I consider two different specifications for the current utility function. First, I begin with the case where labor supply is elastic as a benchmark. The current utility therefore takes the form of Greenwood et al. (1988):

$$u(c_t^i, l_t^i) = \left[c_t^i - \psi\frac{(l_t^i)^\xi}{\xi}\right]^{\rho^i} \tag{1.13}$$

where l_t^i is the labor supply of each agent at period t. There is no uniform agreement about the correct value of the Frisch elasticity $\frac{1}{\xi-1}$. So I set it to be 1/3 for the benchmark model

[9]Some others also find that this cost typically is very small, though they have a different modeling setup. For instance, Barber and Odean (2000) calculate a similar cost varying from 0.01% to 0.1% of the portfolio value.

and try different values, including an estimate of 1 from Kimball and Shapiro (2003), to see the model performance. Finally, ψ is chosen to match a target value of $\overline{L} = 0.33$. In order to provide a simple comparison, I also consider the case with inelastic labor supply and assume that current utility function is of the standard power form: $u(c_t^i, l_t^i) = c_t^{i\rho^i}$.

Transaction Cost of Consumption

The transaction cost function takes the following form:

$$\Psi(c^i, \frac{m^i}{P}) = \zeta c^i \exp(-\alpha \frac{m^i}{Pc^i})$$

where ζ and α are positive constants. One can prove that this form satisfies all the conditions claimed by Feenstra (1986).[10] Here we treat it as the cost of maintaining the ATM/payment system. To calibrate ζ and α, we use the data from the FRED database of Federal Reserve Bank of St. Louis: Consumption is real monthly expenditures on non-durables (PCENDC96) and services (PCESC96); the money supply is $M0$ (CURRSL); real balances are the money supply divided by GDP deflator (GDPDEF). The average monthly expenses for one ATM are roughly \$1450 in 2006.[11] In the same year, the total number of ATM used in US is $395,000$.[12] All quantity variables are divided by the resident U.S. population (CNP16OV). Our first goal is to match the average transaction cost, which is 1.2% of conumption goods.[13][14]

Following Lucas (2000) and Ireland (2009), the second goal is to match the money

[10] Note that we are using a different form from those used in Bansal and Coleman (1996) and Marshall (1992), where a power function is used. The reason that we use the exponential form rather than the power form is that it avoids the possibility of such a solution that one agent holds negative money balance in the latter case. Also the definition of transaction cost is different from Schmitt-Grohe and Uribe (2004).

[11] Data source: 2006 ATM deployer study.

[12] See http://www.creditcards.com/credit-card-news/atm-use-statistics-3372.php .

[13] Humphrey et al. (2003) have a different estimation of the cost of payment system and the benefit from using more ATMs.

[14] Marshall (1992) estimates a 0.8% cost of output. Barber et al. (2009) use a complete trading history of all investors in Taiwan and find that individual investor losses are equivalent to 2.2 percent of Taiwan's GDP or 2.8 percent of total personal income.

demand elasticity on interest rate with the following form:

$$\ln(m/Py) = \ln(B) - \xi r,$$

where $\widehat{\xi} = -1.88$ is estimated from the data. We vary the values of ζ and α to match these two targets and get $\zeta = 0.05$ and $\alpha = 0.6$, respectively.

1.4 The Role of Money in Asset Pricing

Table 1.2 shows the performance of the benchmark model. Compared with the literature, I have got a better matching to the basic asset pricing and business cycle facts. Equity premium is 5.29% percent a year while risk-free return is 1.69%. It also delivers a Sharpe ratio (0.25) close to that in the data (0.32). While the variances of output growth, the relative volatility of investment are close to those in the data, volatilities of consumption and labor are not so well matched, which have been found hard to match in the literature.

Table 1.2: Comparison of Statistics

	Asset Pricing Facts	Model
R^E	8.11% (19.30%)	6.98% (21.14%)
R^f	1.94% (5.44%)	1.69% (8.02%)
R^{EP}	6.17% (19.40%)	5.29% (21.55%)
Sharpe Ratio	0.32	0.25
	Business Cycle Facts	**Model**
$\sigma(Y)$	1.89	2.71
$\sigma(C)/\sigma(Y)$	0.40	0.79
$\sigma(C^h)/\sigma(Y)$	–	0.93
$\sigma(C^n)/\sigma(Y)$	–	0.72
$\sigma(I)/\sigma(Y)$	2.39	2.31
$\sigma(L)/\sigma(Y)$	0.80	0.26

The model is also consistent with what are empirically documented about the relations between inflation and asset prices, given the source of inflation is technology shocks. For example, Tatom (2011) documents that inflation and real stock prices are negatively correlated, depending on the sources of inflation. This relation is mostly apparent during Great Inflation, 1965-84. Giovannini and Labadie (1991) documents that when inflation is high, realized real stock returns and interest rates are low, and vice versa. The main channel here is via no-arbitrage between money and other real assets. Table 1.3 reports that a negative relationship between inflation and real stock prices, real stock returns and real interest rates. The comovement of inflation and nominal interest rate is no surprise. Finally, the model gives a very high negative correlation between inflation and equity premiums, which might be debatable. The logic of this high relation is: Suppose there is a positive technology shock, output is high and so is the equity price; at the same time, non-stockholders require a lower risk-free return to save because their marginal utility of consumption is lower; given that money supply is constant, a lower price level (inflation) followed.

Table 1.3: Correlations

	$Corr(\pi, R^f)$	$Corr(\pi, i^f)$	$Corr(\pi, R^s)$	$Corr(\pi, R^{EP})$	$Corr(\pi, P^s)$
Model Statistics	-0.08926	0.98911	-0.99418	-0.97979	-0.12590

It is widely documented that the equity premium has been going down during the last three decades, the so-called Great Moderation; see Siegal (1999), Jagannathan et al. (2000), Clause and Thomas (2001), and Campbell (2007). However, the reason for such a trend is still on debate. Besides a declining volatility of technology shocks and improvement of market imperfection, one competing explanation is that this is due to the decrease of inflation since sustained low inflation implies less uncertainty about the future.[15] Beirne and de Bondt (2008) claims that these two are closely related. Kyriacou et al. (2006) shows that inflation can exaggerate equity premium. Labadie (1989) established an endowment

[15] See Campbell and Vuolteenaho (2004), Ritter and Warr (2002), Lettau et al. (2008), for example.

economy model to explore two ways of inflation to affect equity premium, namely by the assessment of an inflation tax and the presence of an inflation premium.

Here I explore the role of decreasing inflation by experimenting with different money growth rates. Table 1.4 shows that the decrease of inflation is one source of the declining equity premium. However, the equity premium goes down only $0.02-0.1$ percent by one percent decrease in inflation.[16] Given the the inflation decreased from an average of 8% to 2%, the decline that can be explained by inflation is only approximately 0.22 percent.[17] Also note that as inflation goes higher, its role in driving equity premium is deteriorating. This shows that hyperinflations do not lead to infinitely higher equity premia.

Table 1.4: Different Money Growth Rates and Equity Premiums

Inflation (π)	−1%	0%	3%	6%	9%	12%
Equity Premiums	4.95%	5.05%	5.29%	5.46%	5.54%	5.59%

The reason is as follows. Money has two roles in the economy. First, it serves to facilitate transaction. Second, it is used for intertemporal saving. As money growth (inflation) becomes higher, money's role for saving will be dominated by bonds. That is, money's role for intertemporal saving is weaker and weaker as inflation is higher. The role of money in driving down the equity premium can also be seen by the Euler equations that price bonds and stocks. The pricing kernel of the stockholders is

$$M_{t+1}^{h}=\beta\left(\frac{V_{t+1}^{h}}{[E_t V_{t+1}^{h1-\sigma^h}]^{\frac{1}{1-\sigma^h}}}\right)^{1-\rho^h-\sigma^h}\left(\frac{c_{t+1}^{h}}{c_t^{h}}\right)^{\rho^h-1}\frac{1+\Phi'(\widehat{m}_t^h/\widehat{P}_t c_t^h)}{1+\Phi'(\widehat{m}_{t+1}^h/\widehat{P}_{t+1}c_{t+1}^h)} \tag{1.14}$$

and

$$E_t M_{t+1}^{h} R_{t+1}^{EP}=0 \tag{1.15}$$

[16]It needs mention that it is the expected inflation, not the unexpected one, that determines or closely relates to the equity premium, in contrast with Labadie (1989).

[17]Tough it is not the theme of this paper to estimate contributing role of the declining variance of technology shocks, a rough estimation shows that it explains most of the decrease in equity premium. It can be shown that inflation depends approximately on the ratio of money growth rate to technology growth rate, i.e., $\pi \propto \frac{g^M}{g^c}$ where $g^c \approx 1/3g^z$; and hence, $\pi \propto 3\frac{g^M}{g^z}$, which shows that change of g^z plays a more important role.

where $R^{EP}_{t+1} = \frac{P^s_{t+1}+D_{t+1}}{P^s_t} - \frac{1}{Pf}$ is the equity premium. From equation (1.15) we get $E_t M_{t+1} E_t R^{EP}_{t+1} + Cov(M^h_{t+1}, R^{EP}_{t+1}) = 0,$[18] where we conclude that money matters because M^h_{t+1} depends on money growth via portfolio reallocation and so does $Cov(M^h_{t+1}, R^{EP}_{t+1})$.[19][20] Since the real value of money is only a small fraction of agents' wealth, its role in resolving equity premium puzzle is limited. As money growth goes up, the real value of money carried from previous period, $\frac{\widehat{m}^i}{\widehat{P}g}$, is decreasing with a slower rate, and thus inflation drives up less and less equity premium. That is, as money growth rate goes up, the uncertainty induced by inflation is decreasingly declining.

1.5 The Welfare Effects of Money: Quantitative Estimations

Compared with a standard classic model, this section answers the following two questions: First, what is the optimal money growth rate? Second, should monetary policy be countercyclical or respond to asset prices, such as equity prices or expected equity premium? We answer these two questions by comparing the welfare under different policy regimes. Specifically, suppose $V^B(\Upsilon)$ and $V^A(\Upsilon)$ are the value functions for the benchmark and under the alternative monetary polices. The welfare change is measured as percentage change of consumption in the benchmark, that is, to find τ such that

$$\begin{aligned} V^A(\Upsilon) &= V^B(\Upsilon)(1+\tau) \\ &\equiv \left[(1-\beta)u(c(1+\tau), l)^\rho + \beta(E(V(\omega'; \Upsilon')|Z, g)^{1-\sigma})^{\frac{\rho}{1-\sigma}}\right]^{\frac{1}{\rho}} \end{aligned} \tag{1.16}$$

In doing so, we can get the welfare change for each state vector Υ. I then simulate a long time-series ($T = 50,000$) under the benchmark and then calculate the average welfare change $\overline{\tau}$. If $\overline{\tau}$ is positive (negative), then we say there is a welfare gain (loss).

[18] Equity premium is thus $E_t R^{EP}_{t+1} = -Cov(M^h_{t+1}, R^{EP}_{t+1})/E_t M_{t+1}$.

[19] If μ is not zero, then $E_t M^h_{t+1} R^{EP}_{t+1} = \mu/\lambda P^f$

[20] In the representative agent model as presented in Appendix A, the pricing kernel becomes $M_{t+1} = \beta\left(\frac{V_{t+1}}{[E_t V^{1-\sigma}_{t+1}]^{\frac{1}{1-\sigma}}}\right)^{1-\rho-\sigma}\left(\frac{c_{t+1}}{c_t}\right)^{\rho-1}\frac{1+\Phi'(1/\widehat{P}c)}{1+\Phi'(1/\widehat{P}'c)}$, which is not affected by the injection of money.

1.5.1 The Optimal Money Growth Rate

In the last section, we have concluded that risk premia are of different size since different money growth rates deliver different risks and, in particular, different risk-sharing allocations in the economy. Table 1.5 reports the welfare implications of different expected money growth rates, comparing with the zero-inflation case.

Table 1.5: Optimal Money Growth Rates

Inflation (π)	-1%	0%	3%	6%	9%	12%
Total	0.0068	0.0047	0	-0.0044	-0.0093	-0.0142
Stockholders	0.0080	0.0052	0	-0.0046	-0.0099	-0.0149
Non-Stockholers	0.0062	0.0045	0	-0.0043	-0.0091	-0.0138

As shown in the table, the total welfare, the welfare of stockholers and non-stockholers are decreasing with inflation rates. Therefore, zero money growth rate is not optimal. By contrast, a deflation is welfare improving. The logic is that now there is less uncertainty about inflation, and thus less precautionary saving motivation from non-stockholders, which then means that they consume more and buy less bonds. Stockholders now have to pay higher return to borrow, but they also borrow less. The equilibrium is that both of them better off.

1.5.2 Implications of Alternative Monetary Polices

This subsection considers three alternative monetary policies, compared with the benchmark model. The first case is whether monetary policies should be procyclical or countercyclical, where we set $\rho_g = \theta_{EP} = \theta_{PE} = 0$ and $\theta_y = -0.05$, -0.025, -0.01, 0.01, 0.025, 0.05, respectively. Figure 1.1 shows the equity premia and risk-free returns with each parameter values. It turns out that procyclical policy tends to drive up risk-free return and down the equity premium, where equity returns are of almost no changes. The welfare change is shown in Panel A of Table 1.6. In contrast with the popular "leaning against the wind" advice, I find that procylical monetary policy is welfare-improving. The logic is similar as in the last subsection. Procyclical money injection makes the saving role of money stronger and holding money now is less risky, which amounts to less uncertainty about inflation. Higher return of money tranmits to higher return on bonds and lower equity premium via no-arbitrage.

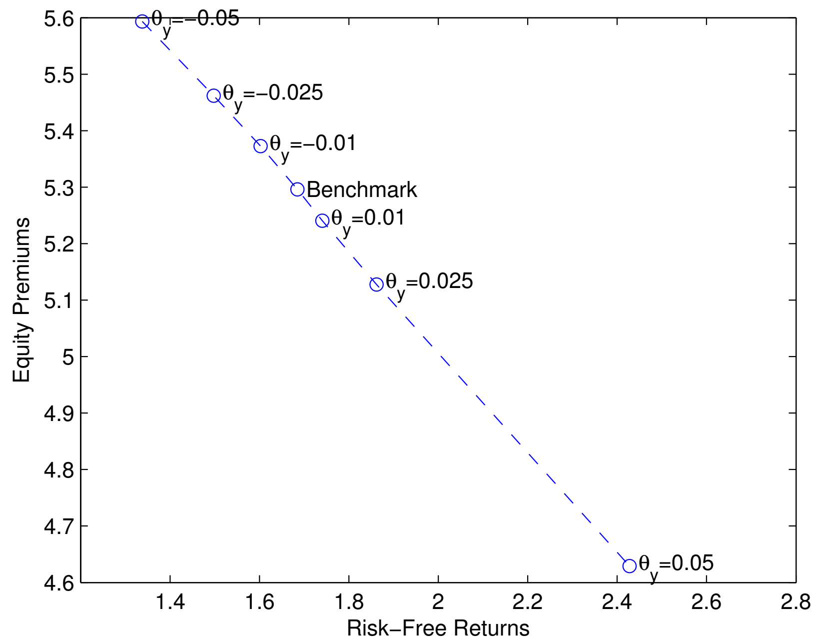

Figure 1.1: Equity premiums and Risk-free Returns under the Alternative Monetary Policy: Responding to Business Cycles

The second is to explore whether monetary policy should respond to asset prices, say real equity prices. In this case, $\rho_g = \theta_y = \theta_{EP} = 0$ and $\theta_{PE} = -0.05, -0.025, -0.01$, 0.01, 0.025, 0.05, respectively. Figure 1.2 shows again the equity premia and risk-free return with different coefficients of monetary policy responding to equity prices. Welfare implications are similar to the rules reacting to business cycles, as shown in Panel B of Table 1.6. The reason for this result is that asset prices are highly correlated with output ($corr(P_t^s, y_t) \approx 0.999$).

Finally, I consider the case where monetary policy responds to (expected) equity premiums. If equity premium is high, policy makers view that there is higher risk in the economy and thus mop it down by setting θ_{EP} positive, which is a countercyclical policy. Without

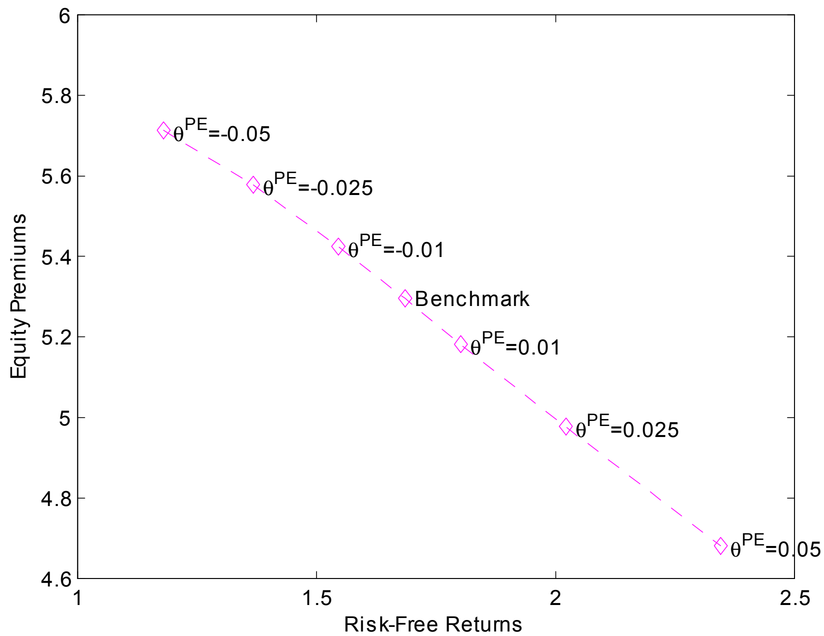

Figure 1.2: Equity premiums and Risk-free Returns under the Alternative Monetary Policy: Responding to Equity Prices

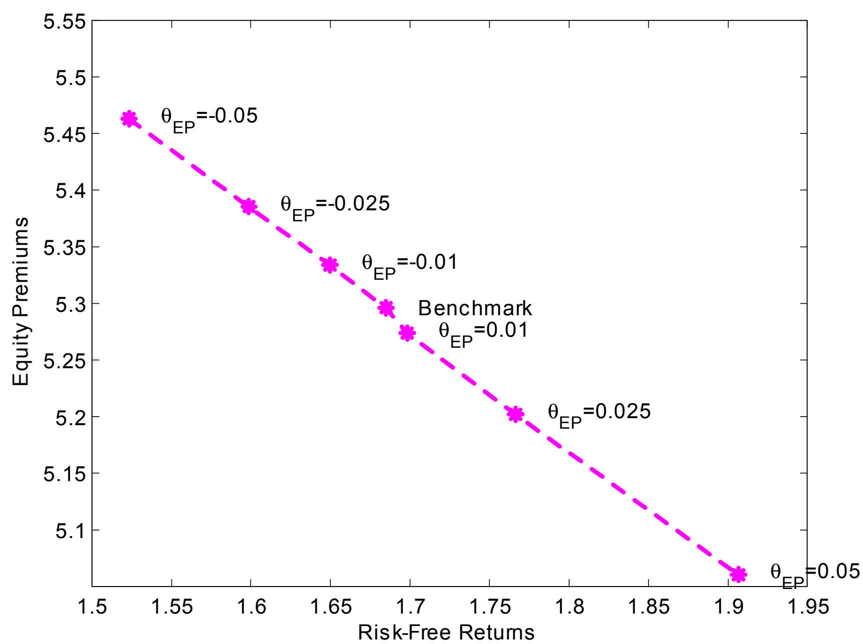

Figure 1.3: Equity premiums and Risk-free Returns under the Alternative Monetary Policy: Responding to Equity Premiums

Table 1.6: Welfare Implications of Alternative Monetary Policies

	Parameters	S	N	Total Welare
Benchmark	$\rho_g = \theta_{EP} = \theta_{PE} = \theta_y = 0$	0	0	0
Panel A: To business cycles	$\theta_y \neq 0$			
Procyclical	0.05	0.0040	0.0042	0.0041
	0.025	0.0011	0.0017	0.0015
	0.01	0.0003	0.0006	0.0005
Countercyclical	−0.01	0.0000	−0.0004	−0.0003
	−0.025	−0.0010	−0.0016	−0.0014
	−0.05	−0.0021	−0.0033	−0.0029
Panel B: To equity prices	$\theta_{PE} \neq 0$			
	0.05	0.0113	0.0094	0.0101
	0.025	0.0034	0.0039	0.0038
	0.01	0.0008	0.0013	0.0012
	−0.01	−0.0006	−0.0011	−0.0010
	−0.025	−0.0021	−0.0030	−0.0027
	−0.05	−0.0039	−0.0054	−0.0049
Panel C: To equity premium	$\theta_{EP} \neq 0$			
	0.10	−0.00005	0.00008	0.00004
	0.075	0.00001	0.00007	0.00005
	0.05	0.00008	0.00011	0.00010
	0.025	0.00018	0.00019	0.00019
	0.01	0.00028	0.00030	0.00029
	0.00	0.00015	0.00079	0.00058
	−0.01	0.00019	0.00020	0.00020
	−0.025	0.00013	0.00012	0.00012
	−0.05	0.00005	0.00003	0.00004
	−0.075	0.00000	−0.00003	−0.00002

surprise, such a policy drives down equity premium and thus improves the welfare of the whole economy, as shown in Figure 1.3 and Panel C of Table 1.6.

1.6 Conclusion

This paper first builds a monetary model with production and limited stock participation to reconcile with asset prices and macroeconomic data. Specifically, it not only resolves the equity premium and risk-free return puzzles, matches volatilities of macro fundamentals, but also is in line with empirical findings about the relations among inflation, money growth rate and asset prices.

This model is then used to estimate alternative monetary policies and finds that monetary policies are welfare improving if they drive down the equity premium and raise risk-free returns. This manifests a procyclical monetary policy, a positive response of monetary policy to stock prices and risk premiums.

Appendix A: The Representative Model

There are homogeneous households who live forever in the economy. The population is normalized to be unity, who is endowed with one unit of time every period, which he allocates between market work and leisure. The agent has similar Epstein-Zin-Weil preference as in the heterogeneous model:

$$U_t = \left[(1-\beta)u_t(c_t, l_t) + \beta(E(U_{t+1})^{1-\sigma})^{\frac{\rho}{1-\sigma}}\right]^{\frac{1}{\rho}}$$

where u_t is again current utility, a function of consumption c_t and labor l_t. The household has a similar budget constraint as stockholders:

$$c_t + P_t^f b_{t+1} + P_t^s s_{t+1} + \frac{m_t}{P_t} + \Psi(c_t, \frac{m_t}{P_t}) \leq b_t + s_t(P_t^s + D_t) + W_t l_t + \frac{m_{t-1} + (g_t - 1)M_{t-1}^s}{P_t}$$

where the form of transaction cost Ψ is kept the same.

The problem of the firms is unchanged except now the value of the firm is discounted by the MRS of the representative agent, $\{\Lambda_{t,t+j}\}_{j=1}^{\infty}$. Since there is only one type of agents, they can buy both stocks and bonds. As I did in the paper, a change in variables is introduced so that the problem solved by the households is stationary. That is, let $\widehat{m}_t = m_t/M_t^s$ and $\widehat{P}_t = P_t/M_t^s$. The state vector becomes $\Upsilon = (K, Z, g)$. The dynamic programming of the household can be expressed as follows:

$$V(\omega;\Upsilon) = \max_{c,l,b',s',\widehat{m}'} \left[(1-\beta)u(c,l) + \beta(E(V(\omega';\Upsilon')|Z,g)^{1-\sigma})^{\frac{\rho}{1-\sigma}}\right]^{\frac{1}{\rho}}$$

s.t.

$$\begin{aligned} \omega + W(\Upsilon)l &\geq c + P^f(\Upsilon)b' + P^s(\Upsilon)s' + \frac{\widehat{m}'}{\widehat{P}(\Upsilon)} + \Psi(c, \frac{\widehat{m}'}{\widehat{P}(\Upsilon)}) \\ \omega' &= b' + s'(P^s(\Upsilon') + D(\Upsilon')) + \frac{\widehat{m}' + g' - 1}{\widehat{P}(\Upsilon')g'} \\ K' &= \Gamma_K(\Upsilon) \end{aligned}$$

In equilibrium, $\widehat{m}_t = 1, b_t = 0$ and $s_t = 1$ for all t, and the budget constraint becomes $c_t + \Psi(c_t, \frac{1}{\widehat{P}_t}) = Y_t - I_t$.

Appendix B: Model Solution and Computational Algorithm

Let λ^i and μ^i be the Lagrange multipliers of budget constraint and bond borrowing constraint, respectively. Then solved Euler equations are as follows:

$(A1): V^i(\omega^i;\Upsilon) = \left[(1-\beta)u(c^i, 1-l^i)^{\rho^i} + \beta\left[E(V^i(\omega^{i\prime};\Upsilon')|Z,g)^{1-\sigma^i}\right]^{\frac{\rho^i}{1-\sigma^i}}\right]^{\frac{1}{\rho^i}}$

$(A2): c^i : V^{i1-\rho^i}(1-\beta)u^{i\rho^i-1}u^i_{c^i} = \lambda^i[1 + \Psi(c, \frac{\widehat{m}^{h\prime}}{\widehat{P}})]$

$(A3): l^i : V^{i1-\rho^i}(1-\beta)u^{i\rho^i-1}u^i_{l^i} = \lambda^i W$

$(A4): b^i : V^{i1-\rho^i}\beta(E(V^{i\prime}(.))^{1-\sigma^i})^{\frac{\rho^i}{1-\sigma^i}-1}E\left[V^{i\prime}(.)^{-\sigma^i}\lambda^{i\prime}(1+\phi(b^{i\prime\prime} - b^{i\prime}))\right] = \lambda^i(P^f + \phi(b^{i\prime} - b^i)) - \mu^i$

$(A5): s' : V^{h1-\rho^h}\beta(E(V^{h\prime}(.))^{1-\sigma^h})^{\frac{\rho h}{1-\sigma^h}-1}E\left[V^{h\prime}(.)^{-\sigma^h}\lambda^{h\prime}(P^{s\prime} + D')\right] = \lambda^h P^s$

$(A6): K': \beta(E(V^{h\prime}(.))^{1-\sigma^h})^{\frac{\rho h}{1-\sigma^h}-1} E\left\{V^{h\prime}(.)^{1-\rho^h-\sigma^h} u^{h\prime\rho^h-1} u^{h\prime}_{c^{h\prime}} \times\right.$
$\left.\left[f_1(K',L') + \Phi'(\frac{I'}{K'})^{-1}[1-\delta+\Phi(\frac{I'}{K'}) - \Phi'(\frac{I'}{K'})\frac{I'}{K'}]\right]\right\} = u^{h\rho^h-1} u^h_{c^h} \Phi'(\frac{I}{K})^{-1}.$

$(A7): \hat{m}^i: V^{i1-\rho^i}\beta(E(V^{i\prime}(.))^{1-\sigma^i})^{\frac{\rho^i}{1-\sigma^i}-1} E\left[V^{i\prime}(.)^{-\sigma^i}\lambda^{i\prime}/\hat{P}_{t+1}g_{t+1}\right] = \lambda^i(1+\Psi_2(c_t,\hat{m}^i_t/\hat{P}_t))$

In the above, we redefine $\mu^i_+ = \max\{\mu,0\}^2$ the multipliers on the borrowing constraints and $\mu^i_- = \max\{-\mu,0\}^2$, following Garcia and Zangwill (1981). This allows us to turn a system of Kuhn-Tucker conditions into a standard system of nonlinear equations, which can then be solved using standard methods. Borrowing constraints for these two agents are:

$(A7)$ $\mu^h_- = b^{h\prime} - \underline{B}$, and

$(A8)$ $\mu^n_- = b^{n\prime} - \underline{B}$, respectively.

Define $B' = (1-\mu)b^{n'}$ and get $b^{n\prime} = B'/(1-\mu)$. With bond market clearing condition, I have $b^{h\prime} = -B'/\mu$. Similarly, define $\widehat{M}' = (1-\mu)\widehat{m}^{n\prime}$ and get $\widehat{m}^{n\prime} = \widehat{M}'/(1-\mu)$. With money market clearing condition, I have $\widehat{m}^{h\prime} = (1-\widehat{M}')/\mu$. By plugging these formulas into the system of equations above, I substituted out individual state variables in the system. And the new equation system is the input for the solver. The algorithm is then to solve $K'(\Upsilon)$, $I(\Upsilon)$, $B'(\Upsilon)$, $\widehat{M}'(\Upsilon)$, $C^h(\Upsilon)$, $C^n(\Upsilon)$, $\widehat{P}(\Upsilon)$, $P^s(\Upsilon)$, $P^f(\Upsilon)$, $\lambda^h(\Upsilon)$, $\lambda^n(\Upsilon)$, $\mu^h(\Upsilon)$, $\mu^n(\Upsilon)$, $T(\Upsilon)$, $V^h(\Upsilon)$, $V^n(\Upsilon)$, $l^h(\Upsilon)$, $l^n(\Upsilon)$ following these steps:

Step 1. Generate a discrete grid for the economy's capital, bond and money positions: $G_K = \{K_1, K_2, ..., K_{N_K}\}$, $G_B = \{K_1, K_2, ..., K_{N_B}\}$, $G_{\widehat{M}} = \{K_1, K_2, ..., K_{N_{\widehat{M}}}\}$ and the shock state spaces $G_Z = \{K_1, K_2, ..., K_{N_Z}\}$, $G_g = \{K_1, K_2, ..., K_{N_g}\}$. Choose an interpolation scheme for evaluating the functions outside the grid of capital, bonds and money. I use 7, 12, and 10 points in the grid for capital, bonds and money, respectively. Functions are interpolated using a piecewise linear approximation.

Step 2. Conjecture $K'^j(\Upsilon)$, $I^j(\Upsilon)$, $B'^j(\Upsilon)$, $\widehat{M}'^j(\Upsilon)$, $C^{hj}(\Upsilon)$, $C^{nj}(\Upsilon)$, $\widehat{P}^j(\Upsilon)$, $P^{sj}(\Upsilon)$, $P^{fj}(\Upsilon)$, $\lambda^{hj}(\Upsilon)$, $\lambda^{nj}(\Upsilon)$, $\mu^{hj}(\Upsilon)$, $\mu^{nj}(\Upsilon)$, $T^j(\Upsilon)$, $V^{hj}(\Upsilon)$, $V^{nj}(\Upsilon)$, $l^{hj}(\Upsilon)$, $l^{nj}(\Upsilon)$ $\forall K \in G_K$, $\forall B \in G_B$, $\forall \widehat{M} \in G_{\widehat{M}}$, $\forall Z \in G_Z$, $\forall g \in G_g$, where superscript j indexes the iteration number. Set $j = 1$.

Step 3. Solve for the values of $K'^{j+1}(\Upsilon)$, $I^{j+1}(\Upsilon)$, $B'^{j+1}(\Upsilon)$, $\widehat{M}'^{j+1}(\Upsilon)$, $C^{hj+1}(\Upsilon)$, $C^{nj+1}(\Upsilon)$, $\widehat{P}^{j+1}(\Upsilon)$, $P^{sj+1}(\Upsilon)$, $P^{fj+1}(\Upsilon)$, $\lambda^{hj+1}(\Upsilon)$, $\lambda^{nj+1}(\Upsilon)$, $\mu^{hj+1}(\Upsilon)$, $\mu^{nj+1}(\Upsilon)$, $T^{j+1}(\Upsilon)$,

$V^{hj+1}(\Upsilon)$, $V^{nj+1}(\Upsilon)$, $l^{hj+1}(\Upsilon)$, $l^{nj+1}(\Upsilon)$ $\forall K \in G_K$, $\forall B \in G_B$, $\forall \widehat{M} \in G_{\widehat{M}}$, $\forall Z \in G_Z$, $\forall g \in G_g$.

Step 4. Iterate Step 3 until convergence. I require maximum discrepancy (across all points in the state space) between consecutive iterations to be less than 10^{-7} for aggregate capital, bonds, money, and value functions of each agent.

Chapter 2

The Conduct of Monetary Policy and the Zero Lower Bound*

2.1 Introduction

Though there is motivation from the recent experience of (nearly) zero short-term nominal interest rates in U.S. and Eurozone to understand the theoretical and quantitative effects of the zero lower bound (ZLB), it is difficult to analyze in dynamic stochastic general equilibrium (DSGE) models. This difficulty comes from the nonlinearity due to the ZLB. Dating back to Eggertsson and Woodford (2003), recent tries include Braun and Korber (2010), Christiano, Eichenbaum and Rebelo (2011), Gavin and Keen (2012), and Fernández-Villaverde, Gordon, Guerrn-Quintana and Rubio-Ramrez (2012) (henceforth, FGGR). Following FGGR, this paper provides an easy method to solve a standard New Keynesian model with the ZLB to answer the following two questions: How does the economy behave when it faces the ZLB on the nominal interest rate? When does the nominal

*A previous version of this chapter was circulated under the title "Mystery at zero lower bound". I thank Fan Ye, Carlos Yepez, and Chuanqi Zhu for motivating me this topic and pushing me to know more about this research area at the ICE12 workshop at University of Chicago in July, 2012. I also thank Eric Young, Carlos Yepez and participants at Midwest Macro Meetings (2012 Fall), WEA 10th Biennial Pacific Rim Conference, Royal Economic Society Annual Conference 2013, 39th EEA Annual Conference for useful comments. Financial supports from Economics Department of UVA are gratefully acknowledged. All remaining errors are my own. Comments are welcome.

interest rate hit the ZLB? Echoing the recent debate by Braun, Körber and Waki (2012) and Christiano and Eichenbaum (2012), this paper also contributes to the question: Does linear approximated solution qualitatively and quantitatively differ from other non-linear solutions for analyzing the ZLB?

As pointed out by FGGR, the existing solutions have made simplifying assumptions when solving models with the ZLB.[1] To address these problems, they use projection method to fully solve a nonlinear New Keynesian model with a ZLB and rational expectations. While this is a big step toward understanding the ZLB, the solution FGGR finds exaggerates the effects of the ZLB. For example, their model has a bigger probability of ZLB bindings and longer spells of being at the ZLB, because the projection method they use does not keep shape well. Utilizing policy function iteration method, my solution preserves the shape and shows that the policy functions of the model are globally concave.

The policy and impulse response functions show that shocks of total factor productivity (TFP), preference and monetary policy determine to a large degree whether or not the ZLB binds while government spending shocks do not matter given their values commonly used in the literature. In particular, a positive TFP shock may push the nominal interest rates to the ZLB, which is associated with surprisingly, lower output and consumption than those in the steady state. In this case, the real interest rate goes up and households have stronger desire to save and weaker desire to consume. The latter leads to a decrease in demand for goods and monopoly firms react to the decrease in demand by paying lower real wage and employing less labor than in the steady state. The final effect is a reduction in output and consumption despite a positive TFP shock. Households become more patient, which results in stronger desire to save when a positive preference shock hits. This in turn also enforces the nominal interest rates to bind. Finally, a negative monetary shock contributes marginally to bindings of the ZLB.

I then solve the extended model with an empirically supported Taylor rule, namely, a Taylor rule with last-period's interest rate as a term. Surprisingly, if we stick to the

[1] FGGR have detailed discussions about limitations of the current analysis on the ZLB.

calibration claimed in FGGR, in particular, the 2% inflation target, the ZLB never binds – the economy goes back to a world similar to what the linear approximation solution tells us. Based on the observation and empirical estimation, a natural guess is the time-variation of the inflation target, which is low when nominal interest rates are close to the ZLB. Based on the estimation of Cui and Dong (2011), the inflation target was about 0.6% per annual on average during the recent recession. This low inflation target leads to 1.25% probability of ZLB bindings and an average spell of 1.59 quarters. Though not fully explaining the observation of the ZLB during the recent recession, it informs us that if the Fed had committed to a constant and high inflation target all the time, for instance 2%, the nominal interest rate would almost never hit the ZLB. In contrast with Gavin and Keen (2012), varying weights on the deviations of output and inflation in the Taylor rule is irrelevant to the probability of ZLB bindings.

Based on the results above and the solution obtained, the answer to the third question is *yes* – linear approximations fail not only qualitatively but also quantitatively in illustrating economic properties around the ZLB. To be clear, evidences are listed as follows: (1) Linear approximations show that a positive TFP shock leads to higher output, consumption and wage, the three of which, however fall with non-linear solutions; (2) Christiano and Eichenbaum (2012) claim that for perturbations of reasonable size, the conclusions arrived at in the ZLB analysis that use linear approximations appear to be robust. However, policy and impulse response functions show that the quality of linear approximations is poor even when output deviates within 5 percent from its steady state. (3) Linear approximations have different policy implications. One example is Gavin and Keen (2012), who solve a similar model with linearizations and claim that solely increasing weight on the deviations of output in the Taylor rule substantially increase the probability of ZLB bindings, which is not true based on non-linear solutions.

I conclude that the recent study on the ZLB is by no means satisfying: a more realistic model setup, for example, adding interest rate smoothing in the Taylor rule, can lead to surprising changes on the behavior of the economy. The ZLB is still a mystery for us to explore. The rest of the paper is organized as follows: the model is set up in Section 2; Section 3 presents the calibrations and solution method; the results of the simplified model as in FGGR are shown and compared in Section 4; Section 5 analyzes a richer model with interest rate smoothing; and finally Section 6 concludes.

2.2 The Model

The model investigated in this paper is closely following FGGR, which is also a baseline New Keynesian model that recently has been to used to discuss the effects of the ZLB. The economy consists of: (1) one representative consumer, who maximizes his lifetime utility by making consumption and labor decisions subject to a preference (demand) shock; (2) a final goods producer that combines intermediate goods and is competitive; (3) infinitely many monopolistic intermediate goods producers, which use labor to produce a specific good subject to a common TFP shock. Only a fraction of them can reset their prices with Calvo pricing; (4) a government who conducts monetary policy via the Taylor rule subject to a zero lower bound, and consumes some final goods. Both monetary and fiscal policies are subject to random shocks. The equation system that defines the behavior of the economy is as follows:[2]

The household's decision is described as:

$$\frac{1}{c_t} = E_t \left\{ \frac{\beta_{t+1}}{c_{t+1}} \frac{R_t}{\Pi_{t+1}} \right\} \tag{2.1}$$

$$\psi l_t^{\chi} c_t = w_t \tag{2.2}$$

where c_t, R_t, l_t, w_t, Π_t are period-t consumption, nominal interest rate, labor, wage and inflation; χ is the inverse of the Frisch elasticity of labor supply and ψ the disutility parameter of labor; and $\beta_{t+1} = \beta^{1-\rho_b}\beta_t^{\rho_b}\exp(\sigma_b \varepsilon_{b,t+1})$, where $\varepsilon_{b,t+1} \sim N(0,1)$, is the discount factor at time $t+1$. Throughout the paper a variable without a time subscript denotes its steady-state value; for example, the variable β denotes the steady-state level of discount

[2] See details of the model in FGGR.

factor.

The problem of firms is given by:

$$
\begin{aligned}
mc_t &= \frac{w_t}{A_t} && (2.3)\\
\varepsilon x_{1,t} &= (\varepsilon - 1) x_{2,t} && (2.4)\\
x_{1,t} &= \frac{1}{c_t} mc_t y_t + \theta E_t \beta_{t+1} \Pi_{t+1}^{\varepsilon} x_{1,t+1} && (2.5)\\
x_{2,t} &= \Pi_t^* \left(\frac{y_t}{c_t} + \theta E_t \beta_{t+1} \frac{\Pi_{t+1}^{\varepsilon-1}}{\Pi_{t+1}^*} x_{2,t+1} \right) && (2.6)
\end{aligned}
$$

where mc_t, y_t and $\Pi_t^* = \frac{p_t^*}{p_t}$ are marginal cost, output, and ratio between the optimal new price reset and the price of the final good, respectively; $x_{1,t}$, $x_{2,t}$ are two auxiliary variables that evolve through (2.5) and (2.6), respectively; and $A_t = A^{1-\rho_a} A_{t-1}^{\rho_a} \exp(\sigma_a \varepsilon_{a,t})$, where $\varepsilon_{a,t} \sim N(0,1)$, is the TFP shock. Equation (2.3) says the marginal cost is equal to the ratio of wage and productivity level. Finally, equation (2.4) defines the current period relation between x_1 and x_2.

The government's policies are described as:

$$
\begin{aligned}
R_t &= \max\{Z_t, 1\} && (2.7)\\
Z_t &= R^{1-\rho_r} R_{t-1}^{\rho_r} \left[\left(\frac{\Pi_t}{\Pi} \right)^{\phi_\pi} \left(\frac{y_t}{y} \right)^{\phi_y} \right]^{1-\rho_r} m_t && (2.8)\\
g_t &= s_{g,t} y_t && (2.9)
\end{aligned}
$$

where $m_t = \exp(\sigma_m \varepsilon_{m,t})$ is the policy shock to interest rates and $\varepsilon_{m,t} \sim N(0,1)$; g_t is the government spending; and $s_{g,t}$ is the share of government spending to total output, which is random and evolves as $s_{g,t} = s_g^{1-\rho_g} s_{g,t-1}^{\rho_g} \exp(\sigma_g \varepsilon_{g,t})$, where $\varepsilon_{g,t} \sim N(0,1)$. Equation (2.7) is the standard description of the ZLB as in FGGR and Christiano et al. (2011).

In addition, the evolutions of inflation and price dispersion are as follows:

$$1 = \theta \Pi_t^{\varepsilon-1} + (1-\theta)(\Pi_t^*)^{1-\varepsilon} \tag{2.10}$$

$$v_t = \theta \Pi_t^{\varepsilon} v_{t-1} + (1-\theta)(\Pi_t^*)^{-\varepsilon} \tag{2.11}$$

where $v_t = \int_0^1 \left(\frac{p_{it}}{p_t}\right)^{-\varepsilon} di$ is the aggregate loss of efficiency induced by price dispersion of the intermediate goods.

Finally, the markets are clear:

$$y_t = c_t + g_t \tag{2.12}$$

$$y_t = \frac{A_t}{v_t} l_t \tag{2.13}$$

2.3 Calibration, Solution Method and Comparison with others

2.3.1 Calibration

To make the results comparable to others, and especially to FGGR, I first follow exactly FGGR's calibration as given in Table 1:

Table 1: Benchmark calibrations

parameters	β	χ	ψ	θ	ε	ϕ_π	ϕ_y	Π	ρ_r
values	0.994	1	1	0.75	6	1.5	0.25	1.005	0
parameters	s_g	ρ_b[3]	σ_b	A	ρ_a	σ_a	σ_m	ρ_g	σ_g
values	0.2	0.8	0.0025	1	0.9	0.0025	0.0025	0.8	0.0025

To save on the dimensionality of the problem, FGGR and Christiano et al. (2011) set $\rho_r = 0$. In this case, the Taylor rule (2.8) becomes

$$Z_t = R \left(\frac{\Pi_t}{\Pi}\right)^{\phi_\pi} \left(\frac{y_t}{y}\right)^{\phi_y} m_t \tag{2.14}$$

In this paper, I show that it is important to have last period's interest rate in the Taylor rule as documented in the empirical literature. Instead, I set $\rho_r = 0.9$, which is close to the findings in Amato and Laubach (1999), Clarida, Gali and Gertler (1997) and also Rudebusch (2002). I also examined the cases with other values for ρ_r. See details in Section 2.5.

2.3.2 Solution Method

I solve the model with policy function iteration method initiated by Coleman (1989, 1990) . Namely, given a defined hypercube for the state space $S = \{v_{-1},\ A,\ m,\ \beta,\ s_g\}$ and initial guesses for policy functions, I solve the system of nonlinear equations (2.1)-(2.13). The difficult part in solving this model is how to deal with the ZLB. Following Garcia and Zangwill (1981), I introduce an auxiliary variable μ_t to transform the policy rule into the following two equations:

$$R_t = Z_t + \max\{\mu_t, 0\}^2 \tag{2.15}$$

$$R_t = 1 + \max\{-\mu_t, 0\}^2 \tag{2.16}$$

One can check that if μ_t is positive, then the nominal interest rate hits the zero lower bound; and if it is negative, it is equal to Z_t and greater than 1. The essence of this transformation is to make any inequalities become equalities and the system differentiable everywhere so that we can apply a nonlinear optimization solver to the problem. Therefore, Instead of using (2.7), I solve (2.1)-(2.6), (2.8)-(2.13), (2.15), and (2.16) for 14 variables, namely, v_t, y_t, c_t, l_t, mc_t, $x_{1,t}$, $x_{2,t}$, w_t, Π_t, Π_t^*, R_t, Z_t, g_t, and μ_t for all t.

The advantage of this method, compared to linear solutions, is that it deals with exactly the cases where the ZLB binds. Nonlinearity around the ZLB enters expectations and is solved simultaneously. Compared with other non-linear solutions, for example, FGGR, which uses projection method with Smolyak's algorithm, shape is well preserved. We do not need to worry about possible disturbances introduced by polynomial approximations which may exaggerate the nonlinearity around the ZLB.

2.4 Nonlinearity at the ZLB

2.4.1 The Robustness of Existing Analysis

Here I compare my solutions to the existing two categories: those using linear-approximation methods and those using non-linear solution methods such as that obtained in FGGR. The basic conclusion is that the real solution is a mix of the two. Figure 2.1 and Figure 2.2 show the counterparts to those in FGGR (Figures 4.1 and 4.2 in their paper, respectively). That is, I plot consumption and inflation as functions of discount factor β, given other state variables at their steady state levels. As pointed out by FGGR (note 4 in their paper), the projection method does not deliver a decision rule that is globally concave. They tend to think that it is a possible property of the model with price rigidities and not due to their method. However, here I get around this problem and show clearly that in a model as presented in this paper, the decision is globally concave. The non-concavity of their graph is due to the notoriously non-shape preserving property of the projection method.

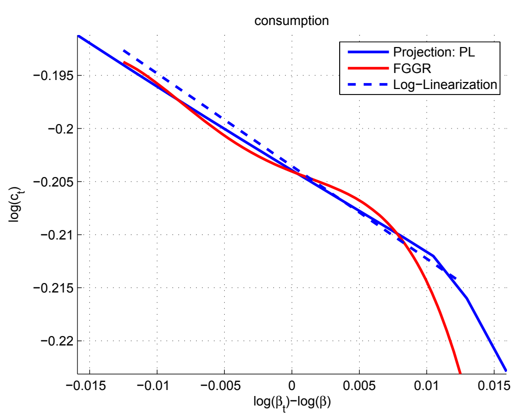

Figure 2.1: Decision rules for consumption with different solution methods

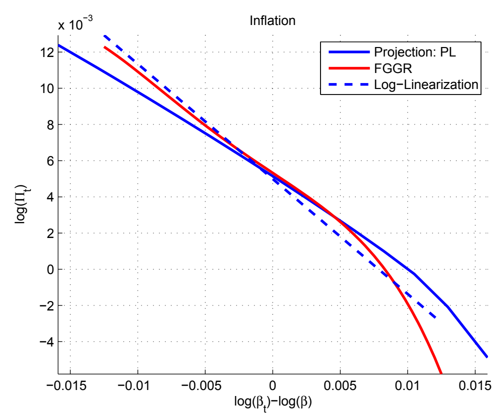

Figure 2.2: Equilibrium functions for inflation with different solution methods

Associated with the non-concavity of their solution, FGGR have exaggerated the nonlinearity around the ZLB. Because the solution method tends to bend down the policy functions near the ZLB, the economy binds more at the ZLB than it should. This observation leads to a guess that the results presented in FGGR are not robust. This will be clear in the following sections. To clearly show the relations between non-linearity and bindings of the ZLB, I also give the corresponding response functions of the interest rate and the auxiliary variable μ in Figure 2.3 and Figure 2.4. It is obvious that the policy functions start to bend down even before the ZLB starts to bind. The auxiliary variable is an exact indicator of where the ZLB binds.

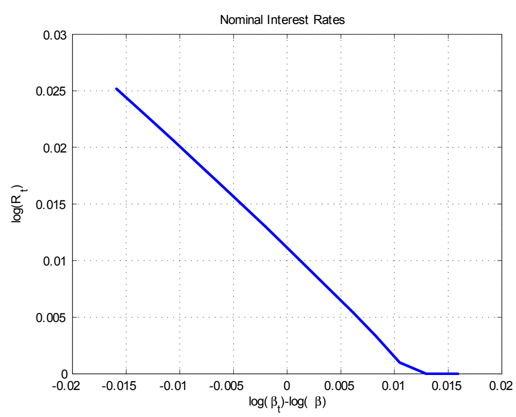

Figure 2.3: Nominal interest rates as a function of discount factor shocks

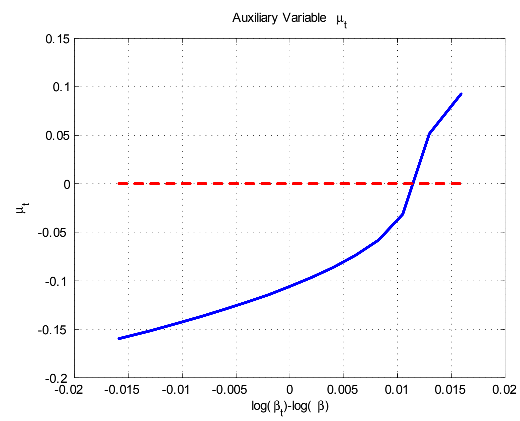

Figure 2.4: Auxiliary variable as a function of discount factor

2.4.2 What Shock Matters for the ZLB?

There are 4 shocks in our model. Preference and TFP shocks determine to a large degree whether the ZLB binds while monetary shocks only affect it marginally and government spending shocks are negligible.

Preference Shocks: In the above subsection, we have seen how the ZLB affects policy functions. It is also clear that preference shocks affects the bindings of the ZLB. As the discount factor becomes bigger, the economy hits the ZLB. In the case we consider above, the auxiliary variable μ equals to zero when the preference level β_t is 1.3% higher than its average level. The unconditional probability of hitting the ZLB is 0.09%. Based on this observation, we look at the effects of changing TFP shocks, monetary shocks and government spending shocks. In order to make them comparable, I draw the policy functions for the shocks that are 3 standard deviations from the steady state.

TFP Shocks: Since the auxiliary variable μ is an exact indicator of how the ZLB binds, I will focus on this variable in the following discussions. Figure 2.5 shows μ as a function of discount factors, given different levels of TFP shocks. When the TFP shock is low, the ZLB never binds. If the TFP shock is high, the ZLB will bind even if the discount factor is below its steady state level. Given a high TFP, the probability of hitting the ZLB now goes up to 7.5%.

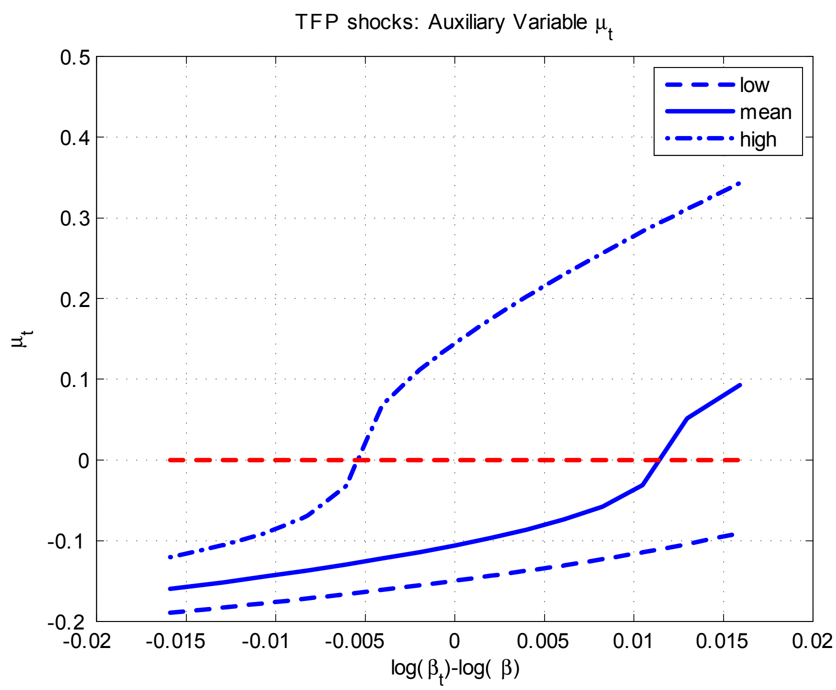

Figure 2.5: Auxiliary variable as a function of discount factor

Monetary Shocks: Compared with TFP shocks, it is clear from Figure 2.6that monetary shocks matter, but to a smaller degree. Also note that, as opposed to previous two examined shocks, a negative monetary shock leads to more bindings. I will discuss this in details later.

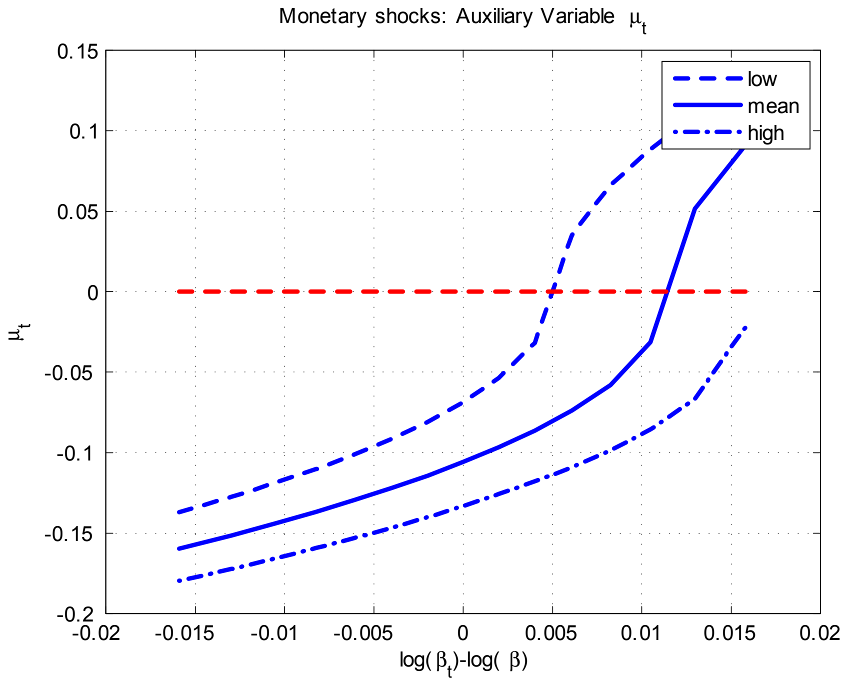

Figure 2.6: Auxiliary variable as a function of discount factor

Government Spending Shocks: Figure 2.7 shows that government spending matters little for the economy to hit the ZLB.

In conclusion, the only shock of the four discussed that does not matter for determining the bindings of the ZLB is government spending, which then implies the limitations of fiscal stimulus policy as discussed below.

2.4.3 Properties of the Economy at the ZLB

A simulation of 50000 periods shows that: (1) The probability of the economy hitting the ZLB is 1.55%, one in 65 quarters; (2) The expected endogenous spell of the ZLB is 1.6996 quarters; (3) The longest spell of the ZLB is 10 quarters and associated with a spike in the discount factor, a positive TFP shock and a low monetary shock; (4) The conditional distributions of output, consumption and inflation at the ZLB are negatively skewed compared

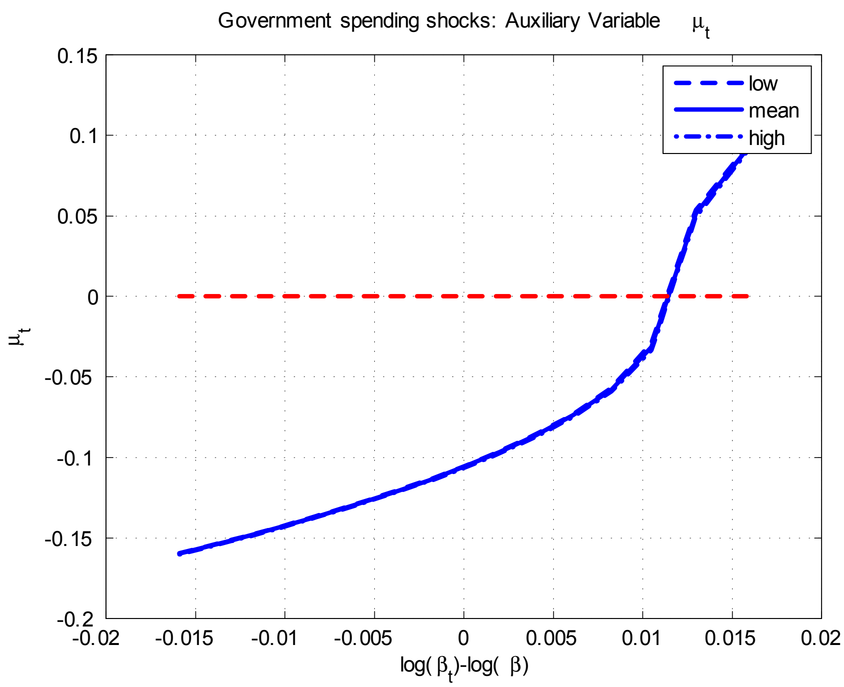

Figure 2.7: Auxiliary variable as a function of discount factor

with unconditional distributions; see Figure 2.8. These findings are consistent with those in FGGR. Now I show why these happen by illustrations of impulse response functions (IRFs). Since we know that government spending shocks do not matter, we only analyze the first three shocks, namely, TFP, monetary and preference shocks. Then we see how their effects on the ZLB are reinforced by considering a combination of a positive TFP shock, a negative monetary shock and a spike in preference shock.

Before we analyze the IRFs, first note that the steady state and the unconditional mean of the economy are not the same because of the ZLB. This is a typical observation whenever there is a boundary reached for some variables in a model. So in order to get the IRFs, we first simulate the economy to get the unconditional mean of the economy, and then see the impacts of the shock.

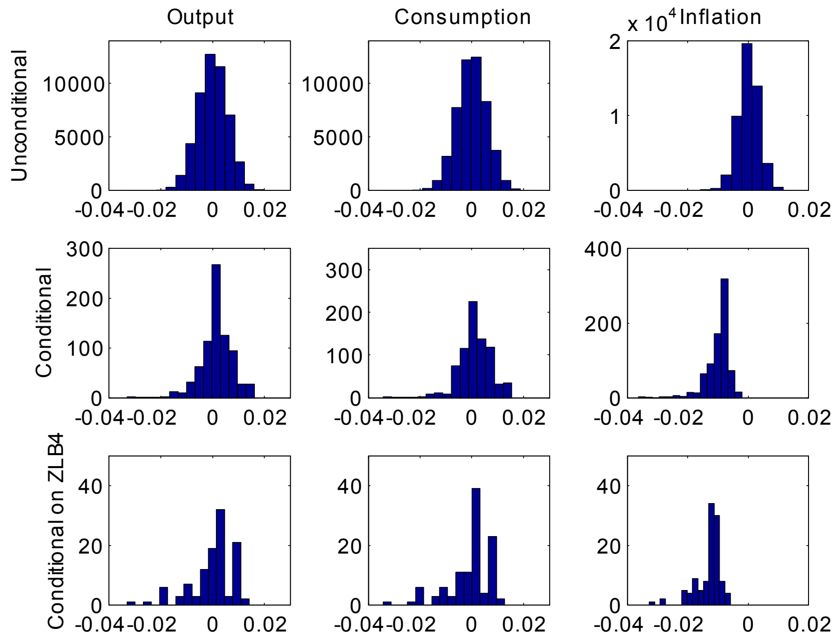

Figure 2.8: (Un)Conditional distributions of output, consumption and inflation

TFP shocks: FGGR explains two mechanisms of why high TFP leads to bindings of the ZLB: (1) Higher productivity means lower inflation and this low inflation is translated to even lower nominal interest rates since $\phi_\pi = 1.5$; and (2) Higher productivity implies lower real interest rates, which then also pushes down the nominal interest rates. However, the story does not end here. Figure 2.9 shows not only what happens if there is a big TFP shock but also the effects of the ZLB on the other endogenous variables. The nominal interest rate jumps down and stays at the ZLB for 2 periods. And because of the presence of the ZLB, output, consumption, government spending and wage, surprisingly, first drop and then increase as the interest rate jumps off the ZLB. Labor and inflation drops more, compared to the no-ZLB case.

So what is the other side of the ZLB story? Compared to the case without the ZLB, the household now has a stronger desire to save since the real interest rate is higher, i.e., R/Π is bigger. He must now have less motivation to consume, which then means that he works less. With high TFP and zero savings at equilibrium, he works even less, which then

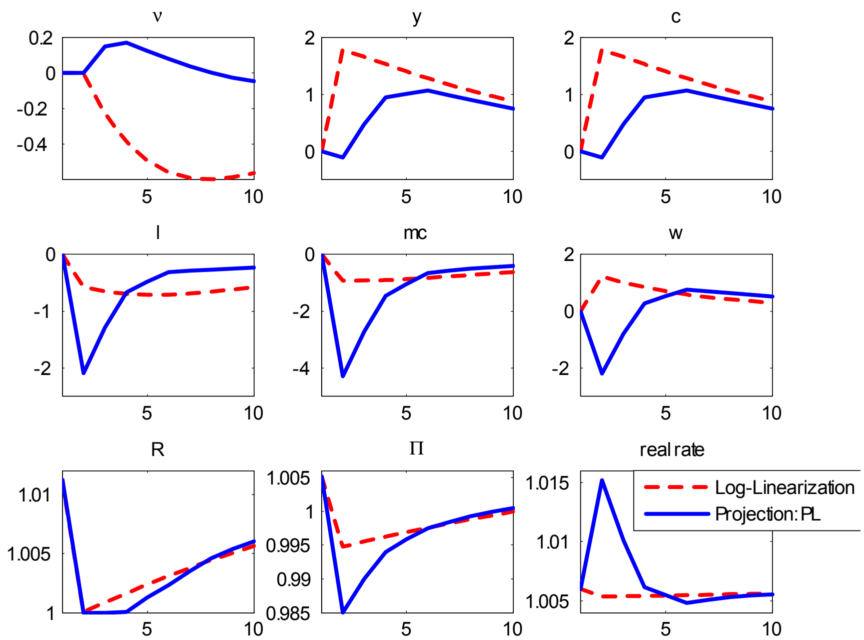

Figure 2.9: IRFs to TFP shocks

forces a plunge of wage as implied in equation (2.2). This leads to an even deeper drop for marginal cost and hence inflation. The feedback is so strong that at equilibrium, the output and consumption all respond initially with a drop even there is a positive TFP shock. In sum, in the short run, a positive TFP actually leads to a recession whenever the ZLB binds! As long as the ZLB does not bind, however, the economy jumps out of the recession and experiences a boom as we would see in a linearized world.

Monetary shocks: In the last section, we have seen how monetary shocks drive the economy to the ZLB. The basic conclusion is that only monetary shocks cannot generate bindings of the ZLB when the other shocks are at their average level. In this case, IRFs of negative and positive shocks are exactly symmetric.

Preference shocks: Figure 2.10 shows the case of a large positive shock to the discount factor. The nominal interest rate responds to it by decreasing and hitting the ZLB for about 2 periods. Due to restrictions coming from ZLB, all other variables experience a larger plunge initially. If there is no ZLB and such a shock hits, the household becomes very patient and wants to save more. So it consumes less. At equilibrium, output goes down and firms employ less labor. Because of the ZLB, the household is more eager to save and thus reinforce the effects of the shock when the ZLB is not hit.

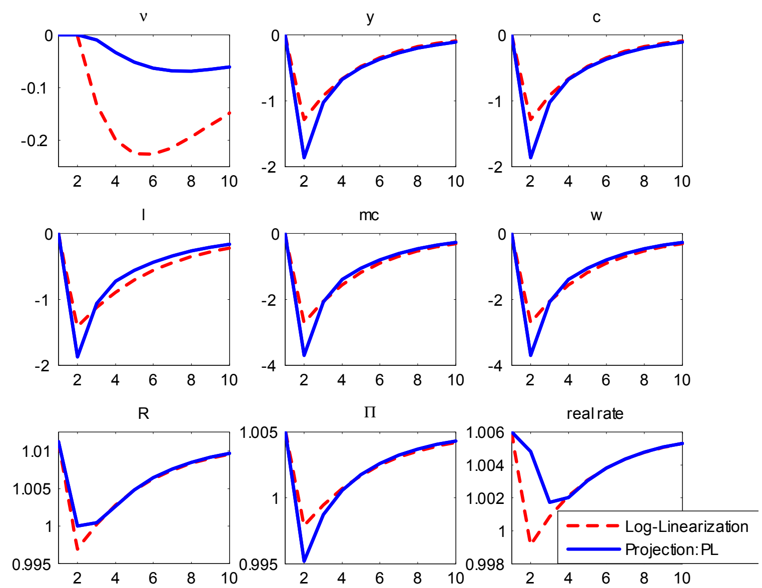

Figure 2.10: IRFs of a 3-standard deviation positive preference shock

A combination of the three shocks that matter: We now look at the case with a positive TFP shock, a positive preference shock and a negative monetary shock. As shown in Figure 2.11, the effects and length of the spell are reinforced. The spell now is 6 quarters, not the sum of spells for TFP and preference shocks, respectively. This is the asymmetry we have seen in subsection 4.2: the effects of the ZLB are strengthened by the interaction of the three shocks–the economy now has a longer lasting recession with even lower output, consumption, employment, marginal cost, wage, inflation and government spending.

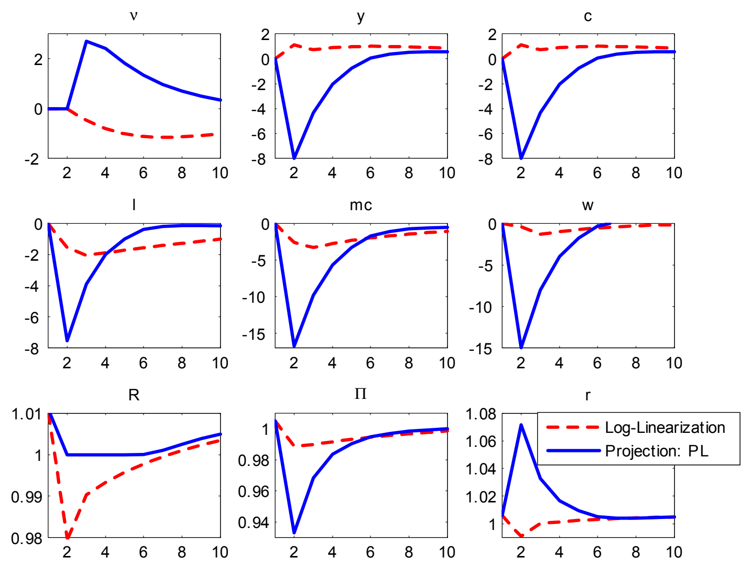

Figure 2.11: IRFs of the three combined shocks

2.5 Last Period's Interest Rate Matters

In Section 2.4, I have followed the existing literature with the simplified Taylor rule. This section considers the smoothness effects of interest rates in the Taylor rule. This is important because empirical estimates of the Taylor rule typically relate the current level of the short-term rate not only to the current levels of inflation and the output gap, but also to its own lag. Theoretically, the smoothness effects may affect the bindings of ZLB in either direction. If the economy is already at the ZLB, smoothness may stick it to the boundary for more periods. Oppositely, smoothness may restrict the economy to its steady state and thus lead to fewer bindings. If the later is true, what explains the ZLB facts of US, Europe and Japan? The current literature cannot explore this effect of interest rate smoothness because adding one more state variable makes the model difficult to solve. With my method, however, it is very straightforward to get around the problem and the computation burden is not increasing much.

There is a wide range of the empirical estimation for the coefficient of the lag, ρ_r in our case. For example, Gerlach-Kristen (2003) uses an euro dataset and gets an estimation of 0.88. Amato and Laubach (1999) consider different measures of inflation and output and use different samples to have a range of 0.778 to 0.916 for ρ_r. Clarida, Gali and Gertler (1997) estimate monetary policy reaction functions for two sets of countries (the G3 including Germany, Japan and US, and the E3 including UK, France and Italy) and get an estimation of smoothness of $0.87 - 0.95$. Rudebusch (2002) gets 0.73 or 0.79 depending on the regressions he runs. In this section, I set ρ_r to be 0.9 for the benchmark. In order to make it comparable, the coefficients on inflation and output in the Taylor rule, ϕ_π, and ϕ_y, are now set to be as $1/(1-\rho_r) = 10$ times big as those values in Section 4. In this way, the *effective* elasticities of nominal interest rates to inflation and output are unchanged, and thus the binding properties of the ZLB, is in principle not generated by misspecification of the Taylor rule.[4] Correspondingly, the standard deviation of the monetary shocks, σ_m, also has to change. However, tests on different values of σ_m show little differences in results and thus we continue to use the value of 0.0025 for it.[5] The sensitivity tests of other values of ρ_r are treated similarly.

2.5.1 Never Hitting the ZLB

I do the same simulation as in Section 2.4 and find that the economy never hits the ZLB– the probability of hitting the ZLB is effectively zero (with only a tiny probability of about 0.02%). In order to get the intuition of why this happens, we rewrite the simplified Taylor

[4] When solving the model, the parameterization of these two coefficients does not matter much for pushing the economy to the ZLB.

[5] A lower σ_m leads to a lower probability of being at the ZLB. For example, the probabilities of hitting the ZLB are 0.02% and 0.008% for σ_m equaling to 0.0025 and 0.00125, respectively, which is not as claimed in Gavin and Keen (2012), saying that estimated uncertainty about the persistence or size of monetary shocks does not affect the likelihood of hitting the ZLB.

rule (2.14):

$$\begin{aligned} Z_t &= R^{1-\rho_r} R^{\rho_r} \left[\left(\frac{\Pi_t}{\Pi} \right)^{\phi_\pi} \left(\frac{y_t}{y} \right)^{\phi_y} \right]^{1-\rho_r} m_t \\ &= R^{1-\rho_r} f^{\rho_r} \left[\left(\frac{\Pi_t}{\Pi} \right)^{\phi_\pi} \left(\frac{y_t}{y} \right)^{\phi_y} \right]^{1-\rho_r} m_t \end{aligned}$$

where $f = R$ (and R_{-1} in the richer one). Note that by our calibration, the only differences between above equation and (2.8) are the three endogenous variables, Π_t and y_t, and f. If the changes of Π_t and y_t are the same in these two settings, the Taylor rule with smoothness should give us more bindings of the ZLB because a R_{t-1} higher (lower) than R leads to even higher (lower) nominal interest rates, compared to the simplified Taylor rule case. Therefore, the two endogenous variables Π_t and y_t (and other variables) must behave differently when there is interest rate smoothing. The intuition is that if households know that interest rates are smoothed, they would expect a smoother interest rate and respond to that gradually to smooth their consumption. A natural question to ask is what if last period's interest rate is at the ZLB. Figure 2.12 shows the policy functions with $R_{t-1} = 1$ as state. It is clear that even though the economy is at the ZLB last period, it will not be binding this period. The conclusion is that the Taylor rule with smoothness pushes interest rate back to its steady state level and leave the ZLB untouched.

This poses us a research dilemma: A high smoothness implied by the empirical study would not generate bindings of the ZLB in a theoretical model that is recently popular to frame the problem of the ZLB, as the one used in this paper. Since we focus only on the Taylor rule, there are three possible candidate solutions motivated by our discussions above and the literature:

This first possible problem is that the persistence parameter ρ_r estimated in the data might be too high. Indeed, if I reduce the value of ρ_r to 0.4, the ZLB binds with a probability of 0.13% and the endogenous expected (longest) spell at the ZLB is 1.2642 (4) quarters.

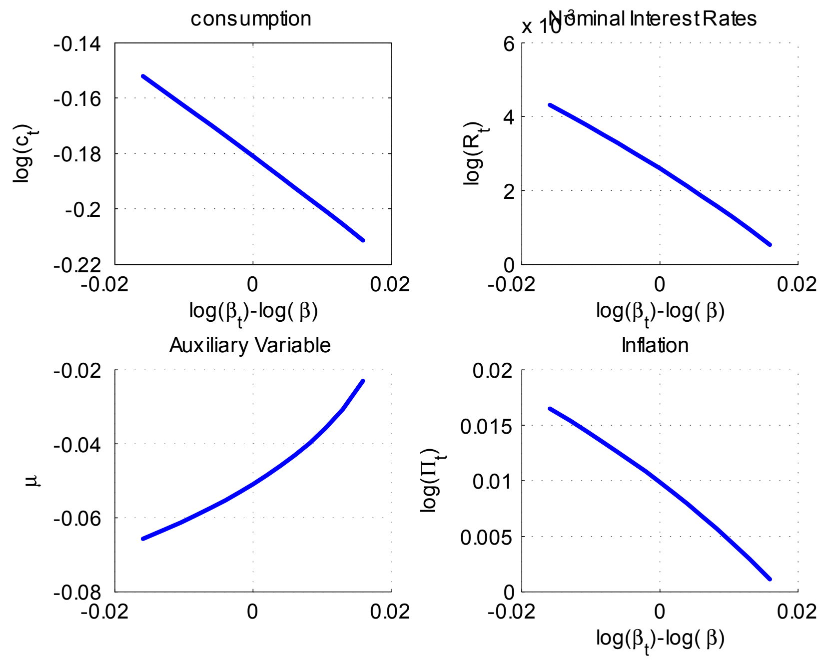

Figure 2.12: Decision rules with last period interest rate as state variable

However, it seems that interest rate is more persistent when it is low in history. For example, the sample with lower nominal interest rates typically gives higher persistence in Amato and Laubach (1999). Therefore, it is not likely that a moderate drop in the persistence of interest rates can result in bindings of the ZLB theoretically.

Another solution pertains to the inflation target. A lower inflation target means a lower steady state nominal interest rate. Given the same shocks coming from productivity, preference and monetary policy, the economy is more likely to hit the ZLB. We have seen above that interest rate smoothing seems to drive the interest rate back to its steady state level, which is the ratio of the target inflation and the discount factor. Since the discount factor is not a policy variable, we focus on the inflation target. A time-varying and low inflation target is possibly along with low nominal interest rates, especially when they hit the ZLB. Ireland (2007) estimates a New Keynesian model to draw inferences about the behavior of the Federal Reserve's unobserved inflation target. He estimates a target as low as 2.5% in the year 2004. Cui and Dong (2011) estimate a similar model by Bayesian approach and get an average inflation target as low as 0.6% annually for some periods during the recent recession. I explore the details of this possible solution in the next subsection.

Finally, it could be the Fed's acting more aggressively to fulfill the dual mandate that leads to the ZLB experiences. Using a similar model, Gavin and Keen (2012) claimed that putting more weight on output in the Taylor rule raises the likelihood of hitting the ZLB while doubling or tripling the weights on both output and inflation has little effect on the results. However, their model does not take into account the nonlinearity induced by the ZLB and is solved by linear approximation. As shown in Section 2.4 and FGGR, such a solution misspecified the expectation of the households and delivered misleading results about the economic properties around the ZLB.[6] In order to see this possibility, two experiments are done: (1) Ceteris paribus, doubling the weight on output, i.e., $\phi_y = 0.5$; and (2) Ceteris paribus, doubling the weights on both output and inflation, i.e., $\phi_y = 0.5$ and $\phi_\pi = 3.0$. In contrast with Gavin and Keen (2012), the likelihood of hitting the ZLB only increases from 0.02% to 0.05% and the longest spell being at the ZLB increases from 2 to 4 quarters when only increasing weight on output; As I also double the weight on inflation, the frequency of hitting the ZLB drops to 0.03% while the longest spell is still 4 quarters. Therefore, the Fed's more aggressive reaction to the dual mandate, if they had done it, has little to do with the current ZLB problem.

2.5.2 A Lower Target Inflation?

With $\Pi = 1.0015$ (an annual rate of 0.6%), I solve the model and simulate it again. Now there is a probability of 1.25%, one quarter out of 20 years, for the economy to hit the ZLB. The average (max) spell at the ZLB is 1.5888 (9) quarters.[7] It seems that a lower inflation target does contribute to the bindings. However, a close scrutiny shows that the properties

[6] Gavin and Keen (2012) claims that only technology shocks are the force to push the economy binding at the ZLB. FGGR shows that besides TFP shocks, preference shocks interpreted as demand shocks might be more important driving forces while monetary shocks, though less, also contribute to the bindings of the ZLB. Whereas this paper shows that if adding interest rate smoothing in the Taylor rule, monetary shocks are dominant in driving the economy to the ZLB.

[7] In stark contrast with Gavin and Keen (2012), changing inflation target not only moves the steady state but also change variation around the steady state since the interest rates are now truncated more at the left tail.

of the economy at the ZLB are different from what we see in Section 2.4. Figure 2.13 shows that, compared to Figure 2.8, output and consumption are not clearly negatively skewed at the ZLB.[8]

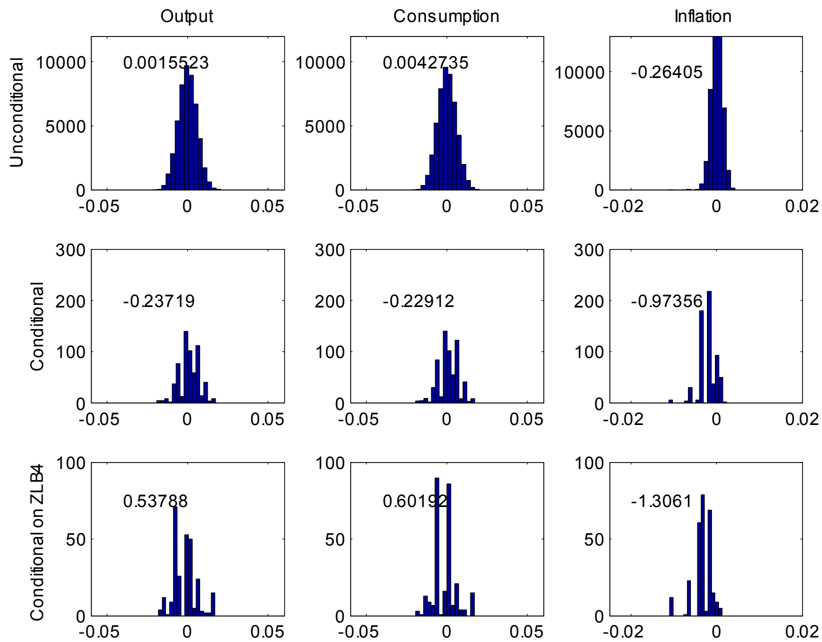

Figure 2.13: (Un)Conditional distributions of output, consumption and inflation when inflation target is low

So what happens when we impose a lower inflation target? Examining Figures 18-20 tells us that it must be that the relative importance of different shocks changes. The only possible way to have no negative skewness on output and consumption is that monetary shocks now play a relatively more important role in determining the bindings of the ZLB. Figures 2.14 and 2.15 confirm this by showing that, with a lower inflation target, the probability of hitting the ZLB by a negative monetary shock is lower while that of a positive TFP or preference shock is much lower. Actually, the preference shock almost cannot solely generate hittings

[8]Decreasing the standard deviation of monetary shocks helps to get negative skewness but with smaller likelihood and shorter spell of being at the ZLB.

of the ZLB, whereas it dominates in Section 2.4. This change of importance in pushing the economy to the ZLB are also seen with IRFs, which I omit here.

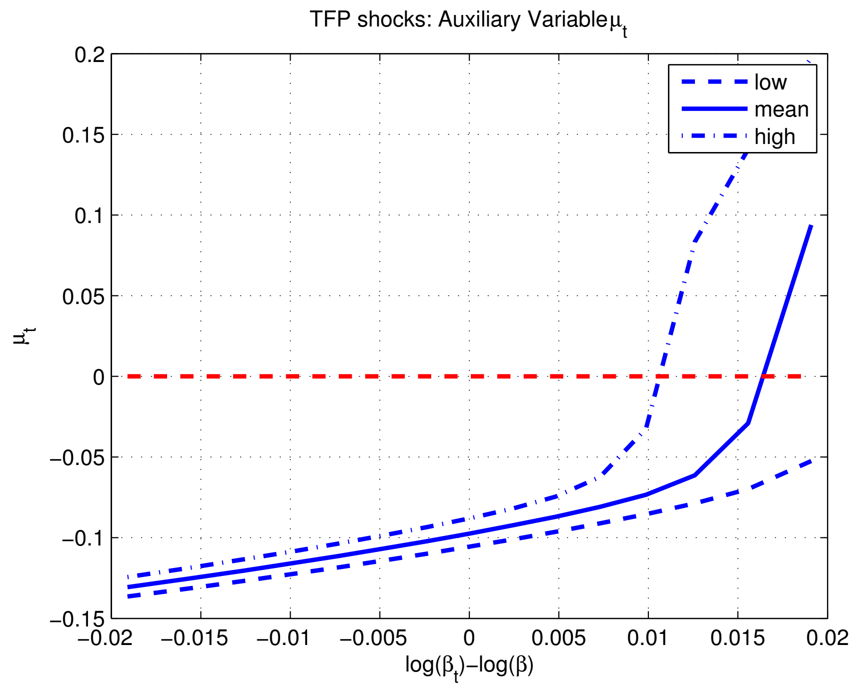

Figure 2.14: Auxiliary variable as a function of discount factor, with low inflation target

The basic conclusion here is, with a lower inflation target, the economy can reach the ZLB. If the Fed had committed to a higher inflation target, for instance 2%, we would never see the economy hit the ZLB. However, only a lower inflation target cannot fully explain what we have observed about the recent recession. For example, the spell of staying at the ZLB is much longer than a lower inflation target can obtain in the model. The ZLB tells us many more things that we did not realize before. First, the popular model used for the ZLB discussions fails easily when we take it closer to reality, or just to more complicated theoretical model that is widely used to discuss other problems. It is not just a matter of nonlinearity; it means more: the economy with an occasionally binding ZLB is a mystery – the properties of which still need to be explored.

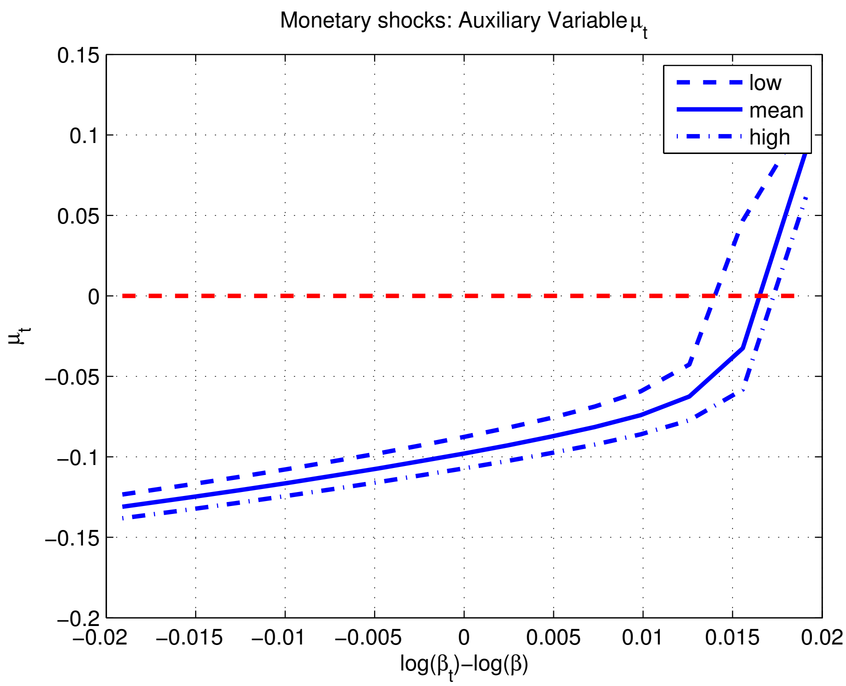

Figure 2.15: Auxiliary variable as a function of discount factor, with low inflation target

Second, the theoretical analysis poses a problem to the empirical works, which now use data to estimate macro models without the ZLB. By default, the existing estimations might ignore the fundamental properties of the economy and are biased. It may even have a wrong judgement about the questions of the economy. Before starting the empirical estimation, it would be beneficial to know what model we should use and how the change of values of other structural parameters will affect the model properties and whether these changes are supported by the data.[9]

2.6 Conclusion

This paper first provides a solution method that can handle the ZLB problem and can be easily extended to analyze a more complicated model with more states. The global solution obtained shows that linear approximations are neither qualitatively nor quantitatively ac-

[9]This paper only focuses on the effects of the Taylor rule on the ZLB. Later I will do a series of experiments on the structual parameters of the model, for instance, the fraction of firms that cannot reset their prices and the coefficient on inflation in the Taylor rule, which may change when interest rate is low or hits ZLB.

curate to the true model. Second, it points out that the nonlinearity properties associated with the ZLB can easily dissappear when considering an empirically supported Taylor rule. The Fed's more aggressive reaction to inflation and output, the so called dual mandate, is not the reason for the economy to hit the ZLB. Finally, a low inflation target could generate bindings of the ZLB. However, the properties along with it are completely different from those with higher inflation targets that has no interest rate smoothness, as analyzed in FGGR and Christiano et al. (2011). Therefore, nonlinearity around the ZLB is still a mystery as long as we try to bring the models into a more realistic version and take a closer look. A more thorough theoretical understanding of the ZLB is no doubt a priority before rushing to put it in an empirical estimation.

Chapter 3

Forward Guidance and Credible Monetary Policy*

3.1 Introduction

Since the Great Recession, many major central banks, including the Federal Reserve Bank (Fed), the European Central Bank, and the Bank of England, have joined the Bank of Japan in using forward guidance for an economic boost.[1] In contrast to the conventional short-term interest rate policy, which is constrained by the zero lower bound (ZLB), forward guidance is the statement and communication of the central bank's projected future path of short-term interest rates.[2] An important feature of the recent forward guidance practice is its promise to maintain the interest rate at or near zero even after the economy emerges from

*I thank Eric Young, Chris Otrok and Toshi Mukoyama for their invaluable guidance and suggestions. I also thank Jinhui Bai, Yan Bai, Ufuk Demirel, Mikhail Dmitriev, Maxim Engers, Nils Gornemann, Raju Huidrom, Jinill Kim, Eric Leeper, Christian Matthes, Leonardo Melosi, Leonard Mirman, Tai Nakata, Sophie Osotimehin, Spencer R. Phillips, Latchezar Popov, Pierre-Daniel G. Sarte, Eric van Wincoop and participants at Midwest Macro Meetings Fall 2014 for helpful comments and suggestions. I thank Zhigang Feng for providing C++ code of his paper and Wei Wang for helping me with C++ programming. Finally, I thank Institute for Humane Studies at George Mason University for conference and research grant. All remaining errors are my own. Comments are welcome.

[1]The history of the Bank of Japan using forward guidance dates back to 1999. The Fed also experimented with forward guidance in 2003 and 2004.

[2]As summarized in Issing (2014), different forms of forward guidance have been adopted by different central banks. Widely speaking, forward guidance also includes inflation targeting.

recession. For example, the Fed has repeatedly said that it will keep short-term interest rates low for a "considerable" time after the economy emerges from the recession.[3] The prolonged stay of the nominal rate at the ZLB will then increase expected inflation and stimulate current consumption.[4] However, the central bank is likely to raise interest rates as the economy strengthens and inflation is high. The failure of the central bank to honor its promises would severely reduce the effectiveness of forward guidance.

The key to having forward guidance work is for households to believe that what the central bank promises today is truly what it will deliver tomorrow, i.e., the central bank must be credible. Specifically, households in the economy must believe that central banks will maintain the nominal rate at zero, as stated in the forward guidance, even after the economy emerges from recession. I show that this belief may turn out to be wrong in a New Keynesian model which is used a lot in the literature to discuss related issues. This model features a benevolent central bank that sets the nominal interest rate, sticky prices due to quadratic price adjustment cost a lá Rotemberg (1982), and the occasionally binding constraint of the nominal interest rate at zero due to high discount factor shocks, which are used to proxy financial crisis. Optimal policy of the central bank is then interpreted as forward guidance.

Without the ZLB, the optimal policy of the central bank is always to set the gross nominal rate to the inverse of the discount factor, which is less than 1 at the time of crisis, to eliminate the price distortion (i.e., zero inflation). With the ZLB, the central bank's gross short-term policy rate cannot go below 1 to close gaps. However, the central bank under full commitment is able to reduce (negative) inflation and output gaps by promising to keep nominal interest rates at zero even after the economy strengthens, which will lead to positive gaps in the future. The promise at the time of recession is likely time-inconsistent because the central bank may raise the rates to close gaps if this is permissible and the economy

[3] For example, the FOMC minutes of Jan 2013 stated: "To support continued progress toward maximum employment and price stability, the Committee expects that a highly accommodative stance of monetary policy will remain appropriate for a considerable time after the asset purchase program ends and the economic recovery strengthens."

[4] A second effect would be a shorter duration of bad times, which will also boost the economy.

strengthens. The full commitment literature ignores the credibility issue by not allowing central banks to pick a policy that differs from the one to which they have committed earlier. Indeed, the first contribution of this paper is to show that the forward guidance policy by assuming full commitment central banks is generally not credible.

Another strand of literature assumes full discretion for the central bank. Under full discretion, the central bank is limited to choosing policy rates solely conditional on predetermined variables and the current value of shocks. Since discount factor shock is the only state variable in the model of this paper, the central bank cannot manipulate the public's future beliefs or its future policy rates. Therefore, when a contractionary (high discount factor) shock occurs, the optimal policy of the central bank is to lower the nominal rate to zero. When the shock disappears, the central bank will set the rate back to its normal level to achieve zero gaps in inflation and output.[5] Hence, the only credible forward guidance under full discretion is to stay at zero during recession and jump back to the normal level when the economy emerges from recession. In other words, if the Fed has no commitment, then the forward guided zero policy rates it says it will maintain even after the economy strengthens are not credible. After all, agents in the economy never believe any forward guidance other than that saying the rate will revert to its normal level when the recession ends. The announcement of such forward guidance, though credible, does nothing to improve the economy in recession using future policy rates.

Recent episodes of zero interest rates suggest that central banks in industrialized economies have neither full commitment nor full discretion. For example, the Bank of Japan never followed the prescription of the optimal policy under full commitment in 1999/2000. On the other hand, if central banks are fully discretionary, then the forward guidance of the Fed in the summer of 2011 would not have led to a drop of 10-20 basis points for long-term interest rates. With these observations and with the concerns of non-credibility under full commitment and strict limitation of policy choices under full discretion, this paper asks: What does the best credible forward guidance look like?

[5] Households actually choose to have slightly negative output and inflation gaps to smooth their consumption due to the recurrence of high discount factor shocks.

In contrast to the concepts of full commitment (Ramsey equilibrium) and full discretion (Markov perfect equilibrium, MPE), this paper answers this question by solving for the whole sustainable sequential equilibria (SSE) set of the New Keynesian model discussed above. Following the idea of Chari and Kehoe (1990), I describe the economy as a dynamic game between the central bank and the households. Every period the central bank sets its rate after observing shocks. The households then make decisions accordingly. If the central bank sets a rate that is consistent with its history (promises made earlier), the households anticipate that the central bank will continue to do so in the future. However, if the central bank sets a different rate from what it promised earlier, it will change households' expectations of what it will do tomorrow. In order to give enough incentive to make the central bank stick to its promises, the best responses of the households, conditional on new expectations, are to generate the worst continuation payoff. The central bank fully understand the consequences of its policy. So at the node of time t, anticipating the responses of the households, the central bank has a trade-off between instant gain from deviating and future less due to the change of households' expectation. If the instant gain is bigger than the future loss, the central bank will deviate and in this case it is not an equilibrium. The SSE set, which is characterized by payoffs to households and central banks, is the set of all equilibria that feature less or equal instant gain compared to future loss. In equilibrium, the central bank will never deviate from its promises. I refer to central banks under the SSE concept as banks with reputation.

Given the solved SSE set, I show that Ramsey equilibrium is generally not credible because the corresponding payoff combinations under Ramsey do not always lie in the set of SSE. This conclusion is robust to various properties of shocks (size, frequency, and persistence) and different values for other model parameters, such as Frisch labor elasticity, elasticity of substitution among intermediate goods, and price adjustment cost. It turns out that as long as the time-inconsistency under Ramsey exists, i.e., the ZLB binds under Ramsey equilibrium, the central bank will always deviate. In other words, Ramsey equilibrium is not credible. In contrast, the payoffs of MPE lie within the set of SSE and MPE is indeed credible. In the following, I focus on the best SSE and compare its policy implications with those of Ramsey and MPE.

A central bank with reputation can do much better than one with full discretion. In the absence of fiscal subsidy and the ZLB, the inflation bias under the best SSE is only half of that under MPE. Since lower inflation bias means higher welfare, the central bank with reputation can achieve higher welfare with access to future policy rates, which the central bank with full discretion cannot have. In the presence of full fiscal subsidy and the ZLB, the output gap is on average -4.55% under full discretion but only -1.55% under reputation at the time of recession. This effectiveness of monetary policy in boosting the economy under the SSE is again due to the central bank's access to and credible control and twist of future policy rates.

Under MPE, the policy rate reverts to its normal level immediately after the recession ends. That is, the extra duration of the ZLB after recession is a zero period. Under Ramsey, the policy rate stays at the ZLB for an extended period, however, with a sharp increase and overshooting of its normal level. A natural guess of the policy response for the best SSE is that it will stay at the ZLB with a time period from zero to the one under Ramsey. Surprisingly, the best forward guidance associated with the best SSE features a prolonged stay with low but non-zero policy rates after the recession ends in most cases. This results in a high and persistent inflation path. In the baseline calibration, it takes 3 to 5 quarters longer for inflation and the interest rate to go back to their normal state under SSE than under Ramsey, which brings the output gap under the best SSE closer to that under Ramsey (-1.55% vs -1.05%) .

The optimal policy also differs in its entry strategy when recession starts. Under MPE and Ramsey, the central bank immediately sets the nominal rate to zero, whereas under the best SSE, the central bank only lowers the rate moderately at the time of the first shock and then continues to lower the rate but keeps it away from zero given the baseline calibration. I summarize the optimal policies by relating nominal interest rates to inflation and output to derive a Taylor-style rule. Given that the only type of shock in this paper is a discount factor shock, both rules from full commitment and the best reputational equilibrium state that the nominal rate should decrease when inflation increases and decrease when output decreases. However, the rule under reputational equilibrium says the central bank should react to inflation and output less aggressively, with elasticities in the nominal rate over inflation and output about one third and half of that under full commitment, respectively.

The less aggressive rule uncovers the key role of credibility: The policy rate cannot be as accommodative as that under Ramsey. A credible central bank sets the rate to a level that will boost the economy as much as possible and in the meanwhile reduce the incentive to deviate. The aggressive rule, in particular, the zero rates under Ramsey, lead to enough incentive for the central bank to deviate. In other words, the central bank under Ramsey has reaped all benefits by having zero interest rate last period while having almost zero cost to deviate today. Therefore, the central bank deviates today. To avoid the deviation, the policy rate should not have been set so low last period.

While SSE is a definition of equilibrium that is relatively widely used to discuss optimal taxation problems (see Chari and Kehoe (1990) and Phelan and Stacchetti (2001)), it is less frequently discussed in monetary policy, in particular, in policy related to the ZLB. One exception is Nakata (2014), the key question of which is to what degree the Ramsey plan is sustainable and credible. While he proposes one revert-to-discretion plan – if the central bank deviates from Ramsey equilibria, it will switch households' expectation about future policies, which he assumes, to MPE – to support the Ramsey plan, there is no reason to think that this expectation is the one to which agents in the economy will switch. By contrast, my paper solves for the whole set of discretionary equilibria and their associated plans.

This paper is also related to Bodenstein et al. (2012), who study how imperfect credibility of the central bank affects the optimal policy at the ZLB. In their paper, they model imperfect credibility by allowing central banks to discard their earlier promises and re-optimize with an exogenous probability. They also assume, as in Nakata (2014), that if the central bank deviates, households will change their expectation and stick to policies under full discretion. By contrast, households in my model have credible plans that enforce the central bank to do what it has promised given the state. In equilibrium, the central bank will never deviate.

The method for computing the equilibrium payoff set closely follows Feng (2015), Phelan and Stacchetti (2001), and Chang (1998). The central bank and the public play a dynamic game and the behavior of both is sequentially rational. The reputation mechanism ensures that if the central bank deviates from equilibrium policy at the ZLB and pursues a different policy (i.e., it shortens the duration of zero interest rates when the economy strengthens), although it obtains instant benefits by having lower inflation and smaller output gaps, it will be punished by a lower continuation value. The equilibria payoff set is found by repeatedly applying a monotonic set-valued operator, as in Fernández-Villaverde and Tsyvinski (2002) and Yeltekin and Sleet (2000). However, I use a different method to represent the equilibria set on a computer.

The rest of the paper is structured as follows. Section 3.2 presents the model. Section 3.3 introduces MPE and Ramsey equilibria while Section 3.4 presents SSE and the strategies to solve the model. Sections 3.5 and 3.6 are devoted to results, without and with the ZLB, respectively. Section 3.7 concludes.

3.2 The Model

This section presents a standard New Keynesian model and the definition of competitive equilibrium. The economy is populated by a continuum of households and firms and a monetary authority. The economy at period t is hit by discount factor shocks, β_t, which follows a two-state Markov chain process.[6] This Markov chain is characterized by the

[6]It is also called preference shocks or aggregate demand shocks in the literature, say Nakata (2014), and Burgert and Schmidt (2014) and many others. Eggertsson and Woodford (2003) and Christiano, Eichenbaum, and Rebelo (2011) view it as standing in for a wide variety of factors that alter households' propensity to save, for example, financial and uncertainty shocks.

following transition matrix:

$$\boldsymbol{P} = \begin{bmatrix} p_{LL} & 1 - p_{LL} \\ 1 - p_{HH} & p_{HH} \end{bmatrix}$$

and two states $\{\beta^L, \beta^H\}$, where L and H mean low and high states for β, respectively. At high state β^H, the economy is in recession due to more patience and less consumption of households. At low state β_L, the economy is said in normal state. $1 - p_{LL}$ is the probability of the economy switching from normal state to recession and p_{HH} is the probability of the economy remaining in recession.

Let $s_t = \{\beta_t\}$ be the shock at period t, which have finite realizations and finite support $\boldsymbol{S} := \{\beta^H, \beta^L\}$. The history of shocks up to time t is thus $s^t = (s_0, s_1, ..., s_t)$ for given initial shocks s_0. The probability of each of these histories is given by $\pi(s^t)$. Agents in the economy make decisions after observing shocks s_t and having full information of history s^t and $\pi(s^t)$.

3.2.1 Households

The representative household is to maximize its lifetime utility

$$\sum_{t=0}^{\infty} \sum_{s^t} \left(\prod_{i=0}^{t} \beta(s^i) \right) \pi(s^t) \left\{ \log c(s^t) - \frac{l(s^t)^{1+\chi}}{1+\chi} \right\} \tag{3.1}$$

subject to

$$c(s^t) + \frac{B(s^t)}{P(s^t)} = w(s^t)l(s^t) + R(s^{t-1})\frac{B(s^{t-1})}{P(s^t)} + \tau(s^t) + d(s^t) \tag{3.2}$$

where $w(s^t)$ is the real wage, $R(s^{t-1})$ the nominal interest rate from period $t-1$ to t, $\tau(s^t)$ a real lump-sum transfer or tax, and $d(s^t)$ real profits of the firms in the economy. The household chooses labor to supply $(l(s^t))$, nominal bonds to buy $(B(s^t))$ and goods to consume $(c(s^t))$ to maximize its lifetime utility. The optimality conditions for the household are:

$$\frac{1}{c(s^t)} = \beta_t R(s^t) E_t \left\{ \frac{1}{c(s^{t+1})} \frac{1}{\Pi(s^{t+1})} \right\} \quad (3.3)$$

$$w(s^t) = l(s^t)^{\chi} c(s^t) \quad (3.4)$$

where $\Pi_t = P_t / P_{t-1}$ is the inflation from period $t-1$ to t.

3.2.2 Firms

The final good producer is to maximize its profits by solving the following problem:

$$\max_{y_{it}} P(s^t) y(s^t) - \int_0^1 P_i(s^t) y_i(s^t) di$$

subject to

$$y(s^t) = \left(\int_0^1 y_i(s^t)^{\frac{\epsilon-1}{\epsilon}} di \right)^{\frac{\epsilon}{\epsilon-1}}$$

where ϵ is the elasticity of substitution among intermediate goods. Given the prices of final and intermediate goods, the demand for intermediate good i is:

$$y_i(s^t) = \left(\frac{P_i(s^t)}{P(s^t)} \right)^{-\varepsilon} y(s^t) \quad (3.5)$$

The intermediate goods producer has a linear technology to produce with labor as its only input, i.e., $y_i(s^t) = l_i(s^t)$. It maximizes discounted profits by paying a cost to adjust its price, that is,

$$\max_{p_{it}} \sum_{t=0}^{\infty} \sum_{s^t} \left(\prod_{i=0}^{t} \beta(s^i) \right) \pi(s^t) \left\{ \frac{\lambda(s^t)}{\lambda(s^0)} d_i(s^t) \right\}$$

subject to its demand function (3.5), and

$$d_i(s^t) = \frac{P_i(s^t) y_i(s^t)}{P(s^t)} - (1-\xi) w(s^t) l_i(s^t) - \frac{\phi}{2} \left\{ \frac{P_i(s^t)}{P_i(s^{t-1})} - 1 \right\}^2 y(s^t)$$

which is the dividend in period t. The coefficient of the quadratic term ϕ captures how costly to adjust prices and $\lambda(s^t)$ is the Lagrangian multiplier for the household at date t. Note that ξ is a subsidy to eliminate steady state distortion due to monopolistic pricing. After solving the problem and using symmetry conditions, the behavior of firm sector can be summarized by the following equation:

$$\left[(\epsilon-1)-(1-\xi)\epsilon w(s^t)+\phi\left(\Pi(s^t)-1\right)\Pi(s^t)\right]\frac{l(s^t)}{c(s^t)}=\beta_t E_t[\phi\left(\Pi(s^{t+1})-1\right)\Pi(s^{t+1})]\frac{l(s^{t+1})}{c(s^{t+1})} \tag{3.6}$$

Equation (3.6) states that the marginal cost of adjusting prices (LHS) must equate the marginal benefit (RHS).[7]

3.2.3 The Central Bank

It is assumed that the central bank is benevolent and chooses the nominal interest rate $\{R_t\}_{t=0}^{\infty}$ to maximize households' lifetime utility (3.1). However, the central bank cannot reduce the rate below 1. In other words, the net nominal interest rate is bounded below by zero. After all, people can hold cash, which of course pays no interest, rather than lend money out at a negative rate of return. At each period after the central bank sets the rate, households then make their decisions about consumption and leisure, and firms set their new prices. In the next section, I will detail how the central bank sets this policy rate.

3.2.4 Market Clearing

Finally, the goods market clears:

$$c(s^t)=\left(1-\frac{\phi}{2}(\Pi(s^t)-1)^2\right)l(s^t) \tag{3.7}$$

[7]The higher the marginal benefit of adjusting price, the higher cost of inflation, and higher inflation.

3.2.5 Competitive Equilibrium

Definition 3.2.1 (Competitive Equilibrium) *Suppose the economy starts with* $\Upsilon\{s_0, R_0\}$*, a competitive equilibrium (CE) for* $\Upsilon\{s_0, R_0\}$ *is characterized by a state-contingent sequence* $(c(s^t),\ l(s^t),\ w(s^t),\ \Pi(s^t))$ *such that, for all* $t \geq 1$*,* $s^t \in \boldsymbol{S^t}$*, and* $R(s^t) \geq 1$ *and equations (3.3) - (3.4), (3.6) and (3.7) hold.*

3.3 Full Discretion and Full Commitment

From this section on, I present different definitions of equilibrium.

3.3.1 Ramsey Equilibrium (Ramsey)

Definition 3.3.1 *A Ramsey equilibrium is where the central bank at time* 0 *instructs all the policies of future depending on the possible shocks to maximize the lifetime utility of the household (3.1). Namely,*

$$\max_{\{c(s^t),l(s^t),w(s^t),\Pi(s^t),R(s^t)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \sum_{s^t} \left(\prod_{i=0}^{t} \beta(s^i) \right) \pi(s^t) \left\{ \log c(s^t) - \frac{l(s^t)^{1+\chi}}{1+\chi} \right\}$$

subject to (3.3) - (3.4), (3.6) and (3.7) and $R(s^t) \geq 1$ *for* $\forall t$.

Denote the maximized lifetime utility of Ramsey planner at period 0 as $V^{RAM}(s^0)$ and discounted utility at period t as $V^{RAM}(s^t)$. By definition, the Ramsey equilibrium delivers the highest lifetime utility at time 0. However, it does not guarantee that for any given period $t > 1$, the discounted utility $V^{RAM}(s^t)$ coincides with the one if the central bank is given the chance to re-optimize. This time-inconsistency comes from the fact that the central bank has strong incentive to close saving and inflation gaps that are promised earlier to boost the economy due to the zero lower bound constraint. I will detail this in Section 3.6.

To solve the Ramsey equilibrium of this model, I follow Marcet and Marimon (2011) and Adam and Billi (2006) and first write the Ramsey problem recursively with the introduction of two Lagrangians for the two Euler equations of households and the firms. Time iteration mathod is then applied on Karush-Kuhn-Tucker conditions to get policy functions, with the transformation of the ZLB constraint following Dong (2012). Linear interpolation is used to approximate values not on the pre-assigned grids.

3.3.2 Markov Perfect Equilibria (MPE)

Definition 3.3.2 *A MPE is the case where the central bank at any time t maximizes agent's lifetime utility (3.1) that period on by choosing consumption ($c(s_t)$), labor ($l(s_t)$), wage ($w(s_t)$), inflation ($\Pi(s_t)$) and interest rate ($R(s_t)$) subject to conditions (3.3) - (3.4), (3.6) and (3.7), and non-negativity of the nominal interest rate, taking as given the behavior of the future central bank and households' expectations.*

Denote $V^{MPE}(s)$ the discounted lifetime utility with state s. The Bellman equation is:

$$V^{MPE}(s) = \max_{c,l,w,\Pi,R} log(c) - \frac{l^{1+\chi}}{1+\chi} + \beta\mathbb{E}[V^{MPE}(s')|s]$$

subject to

$$\frac{1}{c} = R\beta\mathbb{E}\left[\frac{1}{c'\Pi'}\right]$$

$$w = l^{\chi}c$$

$$c = \left[1 - \frac{\phi}{2}(\Pi-1)^2\right]l$$

$$[(\epsilon-1) - (1-\xi)\epsilon w + \phi\,(\Pi-1)\,\Pi]\,\frac{l}{c} = \beta\mathbb{E}\left[\phi\left(\Pi'-1\right)\Pi'\frac{l'}{c'}\right]$$

$$R \geq 1$$

The solution of the MPE is characterized by a sequence of time-invariant value function and policy functions of consumption, labor, wage, inflation and interest rate, i.e., $\{c(s_t)$, $l(s_t)$, $w(s_t)$, $\Pi(s_t)$, $R(s_t)$, $V^{MPE}(s_t)\}$. Since the discretionary central bank re-optimizes every period, MPE is time-consistent. Value function iteration is then used to solve this equilibrium. A similar method can be found in Bodenstein et al. (2012).

3.4 Sequential Sustainable Equilibria (SSE)

The game is played between households and the central bank. Denote $\Gamma(s_0)$ the game where the economy starts with s_0. The public history of the game is $\zeta^t = (\zeta_0, \zeta_1, ..., \zeta_t)$, where $\zeta_t = (c_t, l_t, w_t, \Pi_t, R_t, s_t)$. Let σ_H be the strategy of households and σ_B that of the central bank. Both are measurable functions. Strategy σ_B maps publicly observed history ζ^{t-1} and the current shock s_t into interest rate for date-event s^t, namely, $R(s^t) = \sigma_B(\zeta^{t-1}, s_t)$. Similarly, strategy σ_H specifies $c(s^t)$, $w(s^t)$, $l(s^t)$, $\Pi(s^t)$ as functions of expanded history $(\zeta^{t-1}, s_t, R(s^t))$; that is, $(c(s^t), w(s^t), l(s^t), \Pi(s^t)) = \sigma_H(\zeta^{t-1}, s_t, R(s^t))$. Further, I use $\sum(s_0) = \sum_H(s_0) \times \sum_B(s_0)$ to denote the set of all symmetric strategy profiles for $\Gamma(s_0)$, where $\sum_H(s_0)$ represents the set of strategies for households, and $\sum_B(s_0)$ the set of strategies for the central banks. The value of a strategy $\sigma = (\sigma_H, \sigma_B)$ for the central bank is defined as:

$$\Phi_B(s_0, \sigma) = \sum_{t=0}^{\infty} \sum_{s^t} \left(\prod_{i=0}^{t} \beta(s^i) \right) \pi(s^t) \left\{ \log c(s^t) - \frac{l(s^t)^{1+\chi}}{1+\chi} \right\} \tag{3.8}$$

Definition 3.4.1 *A strategy profile σ of the game $\Gamma(s_0)$ is an* ***SSE*** *if for any $t \geq 0$ and history ζ^{t-1}:*

1. $\Phi_B(s_t, \sigma|_{\zeta^{t-1}}) \geq \Phi_B(s_t, (\sigma_H|_{\zeta^{t-1}}, \gamma))$ for any strategy γ in $\sum_B(s_t)$ for the central bank;

2. $\{c(s^j), l(s^j), w(s^j), \Pi(s^j)\}_{j=t}^{\infty}$ is a CE for $\Gamma\{s_t, R_{s^t}\}$, where $R_{s^t} := \{R(s^j)\}_{j=t}^{\infty}$, $R(s^t) \in \sigma_B(\zeta^{t-1}, s_t)$, and $(c(s^t), l(s^t), w(s^t), \Pi(s^t)) \in \sigma_H(\zeta^{t-1}, s_t, R(s^t))$.

In line with Phelan and Stacchetti (2001), $\sigma|_{\zeta^{t-1}}$ denotes the strategy profile in SSE with history ζ^{t-1}, and $(\sigma_H|_{\zeta^{t-1}}, \gamma)$ the strategy profile in which the household plays a SSE strategy under history ζ^{t-1} while the central bank plays an alternative one. The first condition above says that the continuation payoff for the central bank's strategy σ_B is better than that from any deviation to a different strategy. The second condition requires that the household always responds to a central bank strategy with decisions that imply a CE since this is the situation that is compatible with feasibility and optimality.

3.4.1 Recursive Formulation of SSE

Following Feng (2015) and others,[8] define

$$m_1(s^t) = \frac{1}{c(s^t)\Pi(s^t)} \tag{3.9}$$

$$m_2(s^t) = \phi(\Pi(s^t) - 1)\Pi(s^t)\frac{l(s^t)}{c(s^t)}. \tag{3.10}$$

These two quantities represent, in period t, the expected derivatives of the household's lifetime discounted utility from period $t+1$ on with respect to $B_t R/P_t$ and P_{it}/P_t, respectively.[9] Using the market clearing condition, equation (3.10) can be used to calculate Π as a function of m_2:[10]

$$\Pi = \frac{\phi(1+m_2) + \sqrt{2\phi m_2^2 + 4\phi m_2 + \phi^2}}{\phi(2+m_2)}$$

With (3.9), consumption can be backed out as follows:

$$c = \frac{1}{m_1 \Pi}$$

Therefore, m_1 and m_2 are also referred to as expected or promised consumption and inflation. For any s^t, $m_1(s^{t+1})$ and an arbitrary specified interest rate R, households solve the

[8]Chang (1998) and Phelan and Stacchetti (2001) show that, though in different model setups, equilibria can be characterized in terms of their value to the government and their marginal value of private variables. See Kydland and Prescott (1977) for the reason and justification of doing so and how to explain it.

[9]It is important to emphasize that each household is atomistic and an expectation taker.

[10]The other root is $\Pi = \frac{\phi(1+m_2) - \sqrt{2\phi m_2^2 + 4\phi m_2 + \phi^2}}{\phi(2+m_2)}$, the limit of which is $1 - \sqrt{2/\phi}$ as m_2 goes to $+\infty$ and $1 + \sqrt{2/\phi}$ as m_2 goes to $-\infty$. Since this root never visits the range of $[1 - \sqrt{2/\phi}, 1 + \sqrt{2/\phi}]$, and violates the Friedman rule for reasonable values of ϕ, I ignore always this case.

following problem:

$$\max_{\{c(s^t),\, l(s^t),\, B(s^t)\}} log(c(s^t)) - \frac{l(s^t)^{1+\chi}}{1+\chi} + \beta_t E_t m_1(s^{t+1}) \frac{B(s^t)}{P(s^t)} R \tag{3.11}$$

subject to the budget constraint (3.2). By construction, the recursive problem is equivalent to the sequential problem provided that the transversality condition is satisfied:

$$\lim_{t\to\infty} \sum_{t=0}^{\infty} E_t \left(\prod_{i=0}^{t} \beta(s^i) \right) m_1(s^t) \frac{B(s^t)}{P(s^t)} R = 0 \tag{3.12}$$

This is shown in the following proposition, which is an extension of the results in Feng (2015) and Phelan and Stacchetti (2001).

Proposition 3.4.2 *Assume $R \in [1, \bar{R}]$ and $l \in [0, \bar{l}]$, where $\bar{R} < +\infty$ and $\bar{l} < +\infty$. Given the functional forms of preference and production function, the recursive and sequential problems are equivalent.*

Proof. See Appendix. ∎

The firms also solve a recursive problem appropriately given $m_2(s^{t+1})$.[11] Now I define a static CE using the two variables introduced.

Definition 3.4.3 *Let $\Upsilon\{s, R, \{m_1^+, m_2^+\}\}$ be the static economy in which the current shock is s, the current interest rate set by the central bank is R, and agents have expectations about the future summarized in $\{m_1^+, m_2^+\}$. (c, l, w, Π) is a CE for $\Upsilon\{s, R, \{m_1^+, m_2^+\}\}$ if and only if the following conditions are satisfied:*

$$\frac{1}{c(s)} = R\beta E\{m_1^+\} \tag{3.13}$$

$$w(s) = l(s)^{\chi} c(s) \tag{3.14}$$

$$c(s) = (1 - \frac{\phi}{2}(\Pi(s) - 1)^2) l(s) \tag{3.15}$$

$$\left[(\epsilon - 1) - (1 - \xi)\epsilon \frac{w(s)}{A} + \phi\left(\Pi(s) - 1\right)\Pi(s) \right] \frac{l(s)}{c(s)} = \beta E\{m_2^+\} \tag{3.16}$$

[11] See Appendix.

I denote this equilibrium as $(c, l, w, \Pi) \in \mathbf{CE}^S\{s, R, \{m_1^+, m_2^+\}\}$.

The following lemma allows us to think of the original economy as a sequence of static economies with endogenously changing state variables and exogenous stochastic shocks.

Lemma 3.4.4 *Given a feasible interest rate policy $R = \{R_t\}_{t=0}^{\infty}$, suppose that the sequence $\{c(s^t),\ l(s^t),\ w(s^t),\ \Pi(s^t)\}_{t=0}^{\infty}$ is such that for each t,*

$$\{c(s^t), l(s^t), w(s^t), \Pi(s^t)\} \in \mathbf{CE}^S\{s_t, R_t, \{m_1(s^{t+1}), m_2(s^{t+1})\}\}$$

where

$$m_1(s^{t+1}) = \frac{1}{c(s^{t+1})\Pi(s^{t+1})} \tag{3.17}$$

$$m_2(s^{t+1}) = \phi(\Pi(s^{t+1}) - 1)\Pi(s^{t+1})\frac{l(s^{t+1})}{c(s^{t+1})} \tag{3.18}$$

then $\{c(s^t), l(s^t), w(s^t), \Pi(s^t)\}_{t=0}^{\infty}$ constitutes a competitive equilibrium for $\Upsilon\{s_0, R_0\}$.

Proof. See Appendix. ■

The lemma says that the promised marginal value of investment in bonds and the promised cost for adjusting prices will summarize the expectation of households. Let h denote the equilibrium continuation payoff of the central bank Φ_B defined by (3.8). The equilibria of the economy can be characterized by:

$$V(s) := \{(m_1, m_2, h)|\ \sigma \text{ is a SSE for } \Gamma(s)\}$$

which is a mapping from the values of the states s into set of possible payoffs associated with a strategy profile σ that constitutes a SSE.

3.4.2 Credible Plans

To recursively characterize $V(s)$, I first introduce two definitions that lead to credible plans.

Definition 3.4.5 (Consistency) *Let $W : S \rightarrow R^3$ denote the set of all equilibrium payoffs. A vector $\psi = (R, c, l, w, \Pi, \{m_1^+, m_2^+\})$ is consistent wrt W at s if*

$$(c, l, w, \Pi) \in CE^S(s, R, \{m_1^+, m_2^+\})$$

for $(m_1(s,\psi),\ m_2(s,\psi),\ h(s,\psi)) \in W(s)$, and $(m_1^+, m_2^+, h^+) \in W(s^+)$, where the values of m_1, m_2 and h are given by

$$\begin{aligned} m_1(s,\psi) &= \frac{1}{c\Pi} \\ m_2(s,\psi) &= \phi(\Pi - 1)\Pi\frac{l}{c} \\ h(s,\psi) &= log(c) - \frac{l^{1+\chi}}{1+\chi} + \beta E h^+. \end{aligned}$$

Definition 3.4.6 (Admissibility) *The vector ψ is admissible wrt W if it is consistent wrt W at s and*

$$h(s,\psi) \geq h(s,\psi')$$

for any other consistent ψ'.

Consistency guarantees that the vector ψ delivers an allocation that is optimal for households and feasible. In addition, it requires that the promised continuation values (m_1^+, m_2^+, h^+) belong to the same equilibrium set as those of (m_1, m_2, h). Admissibility says that the interest rate set by the central bank is optimal and it has no incentive to deviate. That is, the central bank cannot increase its payoff by setting a different interest rate R'. A credible plan thus is the strategy of households and central banks instructed by an admissible ψ.

With these two definitions, I define an operator $\mathbb{B}$ with its fixed point being the set of equilibrium values V as follows:

For a given set of equilibrium values W,

$$\mathbb{B}(W)(s) = \{(m_1, m_2, h) |\ \psi \text{ admissible wrt } W \text{ at } s\}$$

The interpretation of the operator and the constraints is as follows. $\mathbb{B}(.)$ is the convex hull of the payoffs (m_1, m_2, h) such that there are associated values of consumption, labor supply, wage, inflation and government policy rates and next period payoffs that belong to the value correspondence $\mathbb{W}$ for every possible realization of the shock compatible with the current state and that satisfy certain conditions.

Following Phelan and Stacchetti (2001) and Abreu et al. (1986, 1990) (APS, henceforth), the operator $\mathbb{B}$ has properties as stated in the following proposition:

Proposition 3.4.7 *The operator* $\mathbb{B}$ *has the following properties:*

- *1. If* $W \subseteq \mathbb{B}(W)$*, then* $\mathbb{B}(W) \subseteq V$*;*
- *2.* V *is compact and the largest set of equilibrium values* W *such that* $W = \mathbb{B}(W)$*;*
- *3.* $\mathbb{B}(.)$ *is monotone and preserves compactness;*
- *4. If we define* $W_{n+1} = \mathbb{B}(W_n)$ *for all* $n \geq 0$*, and the equilibrium value correspondence* $V \subset W_0$*, then* $lim_{n\to\infty} W = V$*.*

Proof. See Appendix. ■

With this proposition, $\mathbb{B}$ is calculated numerically as follows:

$\mathbb{B}(W)(s) = \{(m_1, m_2, h) | \exists R, (c, l, w, \Pi),$ *and* $(m_1^+, m_2^+, h^+) \in W(s)$ for all $s^+ \succ s\}$ such that

$$m_1 = \frac{1}{c\Pi} \tag{3.19}$$

$$m_2 = \phi(\Pi - 1)\Pi\frac{l}{c} \tag{3.20}$$

$$h = log(c) - \frac{l^{1+\chi}}{1+\chi} + \beta E h^{+} \tag{3.21}$$

$$(m_1, m_2, h) \in W(s) \tag{3.22}$$

$$h \geq [u(c', l') + \beta E h^{+'} | (m_1^{+'}, m_2^{+'}, h^{+'})], \forall (m_1^{+'}, m_2^{+'}, h^{+'}) \in W(s^{+}) \tag{3.23}$$

$$1/c = R\beta E\{m_1^{+}\} \tag{3.24}$$

$$w = l^{\chi} c \tag{3.25}$$

$$c = (1 - \frac{\phi}{2}(\Pi - 1)^2)l \tag{3.26}$$

$$[(\epsilon - 1) - (1 - \xi)\epsilon w + \phi\,(\Pi - 1)\,\Pi]\,\frac{l}{c} = \beta E m_2^{+} \tag{3.27}$$

$$R \geq 1 \tag{3.28}$$

where $s^{+} \succ s$ denotes all possible shocks that follow s. Constraints (3.19) to (3.22) are called "regeneration constraints", while (3.23) is an "incentive constraint". Constraints (3.24) to (3.28) are necessary to ensure that continuation of a sustainable plan after any deviation is consistent with a CE. Following Feng (2015) and Chang (1998), I replace (3.23) with the following condition,

$$h \geq \tilde{h}(s) \tag{3.29}$$

where $\tilde{h}(s)$ is the best possible payoff for the central bank when it announces unexpected interest rate R'. In particular, $\tilde{h}(s)$ is defined as

$$\tilde{h}(s) = \max_{R}\{ \min_{\substack{c,\, l,\, w,\, \Pi, \\ (m_1^{+}, m_2^{+}, h^{+}) \in W(s^{+})}} [log(c) - \frac{l^{1+\chi}}{1+\chi} + \beta E h^{+}]\}$$

such that

$$(c, l, w, \Pi) \in CE^S\{s, R, \Pi, \{m_1^+, m_2^+\}, \forall s^+ \succ s\}$$

The idea of replacing (3.23) with (3.29) is that: (1) if households punish the central bank for deviations from the claimed policy R, they will punish the latter as worse as available; (2) if the central bank knows the response of households, it will pick the best as long as it decides to deviate. It can be shown that condition (3.29) is equivalent to (3.23) in the sense of leading to the same fixed point V by applying the operator $\mathbb{B}$.

Following Feng (2015), the whole equilibrium set can be characterized by the upper and lower boundaries of $W(s)$, which are:

$$\begin{aligned} \bar{h}(s, m_1, m_2) &= \max_h \{h | (m_1, m_2, h) \in W(s)\} \\ \underline{h}(s, m_1, m_2) &= \min_h \{h | (m_1, m_2, h) \in W(s)\} \end{aligned}$$

I then define the outer approximation of W as follows:

$$\hat{W}(s) = \{(m_1, m_2, h) | h \in [\underline{h}(s, m_1, m_2), \bar{h}(s, m_1, m_2)]\}$$

Proposition 3.4.8 *For all $(m_1, m_2, h) \in V(s)$,*

$$\begin{aligned} \bar{h}(s, m_1, m_2) &= \max_R \{u(c, l) + \beta E \bar{h}(s', m_1', m_2')\} \\ \underline{h}(s, m_1, m_2) &= \max_R \{u(c, l) + \beta E \underline{h}(s', m_1', m_2')\} \\ \tilde{h}(s) &= \min_{m_1, m_2} \underline{h}(s, m_1, m_2) \end{aligned}$$

subject to the constraint $(c, l, w, \Pi) \in \boldsymbol{CE}^S\{s, R, m_1', m_2'\}$ for all $s' \succ s\}$.

Proof. See Appendix. ■

The task now is to find a new operator $\mathbb{F}$ based on $\hat{W}$:

Definition 3.4.9 (The operator $\mathbb{F}$) *For any convex-valued correspondence $\hat{W}$,*

$$\mathbb{F}(\hat{W})(s) = \{(m_1, m_2, h) | h \in [\underline{h}^1, \bar{h}^1]\}$$

where

$$\begin{aligned}
\bar{h}^1 &= \max_{m_1', m_2'} \{u(c,l) + \beta E \bar{h}^0\} \\
\underline{h}^1 &= max\, \{\max_{m_1', m_2'} u(c,l) + \beta E \underline{h}^0, \tilde{h}^0\} \\
\tilde{h}^0 &= max\{\min_{m_1', m_2'} u(c,l) + \beta E \underline{h}^0\}
\end{aligned}$$

such that the vector $(R, c, w, l, \Pi, (m_1', m_2', h'))$ is admissible with respect to $\hat{W}$ at s. Define $\underline{h}(s,\, m_1,\, m_2) = -\infty$ and $\hat{h}(s, m_1, m_2) = +\infty$ if no such vector exists.

The following theorem shows that this operator has good convergence properties and repeated application of this operator generates a sequence of sets that converge to the equilibrium value correspondence V. The details of the algorithm are postponed in Appendix.

Theorem 3.4.10 *Let $\hat{W}_0$ be a convex-valued correspondence such that $\hat{W}_0 \supset V$. Let $\hat{W}_n = \mathbb{F}(\hat{W}_{n-1})$. Then $\lim_{n\to\infty} \hat{W}_n = V$.*

Proof. See Appendix. ■

3.4.3 Recovering Strategies

This subsection shows how to find the strategy that supports the best SSE in deterministic case. The procedure here can be generalized to find strategies supporting any point belonging to the equilibrium value correspondence.

- Step 1: At $t = 0$, find the highest possible value of $h_0 = sup\{h | (m_{1,0}, m_{2,0}, h_0) \in W^*(s)\}$ and its corresponding $(m_{1,0}, m_{2,0})$. Then search for the central bank's interest rate policy that supports $(m_{1,0}, m_{2,0}, h_0)$, that is, pick R_0 such that

$$u(c_0, l_0) + \beta h_1 = h_0$$

where $h_1 = \bar{h}(m_{1,1}, m_{2,1})$, $m_{1,1} = \frac{u_{c,0}}{R\beta}$, $m_{2,1} = [(\epsilon - 1) - (1 - \xi)\epsilon w_0 + \phi\,(\Pi_0 - 1)\,\Pi_0]\,l_0/c_0/\beta$, and $(m_{1,1}, m_{2,1}, h_1) \in W^*(s)$. Given $(R_0, m_{1,0}, m_{2,0})$, c_0, l_0, w_0, Π_0 can be calculated via definitions of m_1 (3.9) and m_2 (3.10), optimality condition of leisure (3.4), and market clearing condition (3.7) as:

$$\Pi_0 = \frac{\phi(1 + m_{2,0}) + \sqrt{2\phi m_{2,0}^2 + 4\phi m_{2,0} + \phi^2}}{\phi(2 + m_{2,0})} \tag{3.30}$$

$$c_0 = \frac{1}{m_{1,0}\Pi_0} \tag{3.31}$$

$$l_0 = c_0(1 - \frac{\phi}{2}(\Pi_0 - 1)^2)^{-1} \tag{3.32}$$

$$w_0 = l_0^{\chi} c_0 \tag{3.33}$$

Therefore, the above problem is well-defined in terms of $(R_0, m_{1,0}, m_{2,0}, h_0)$.

- Step 2: $t = 1$, $m_{1,1}$, $m_{2,1}$, h_1 are given by the solution in step 1. Now search for the central bank's policy R such that

$$u(c_1, l_1) + \beta h_2 = h_1$$

  as in step 1.

- Step 3: Repeat step 2 for $t = 2, ...T$, for T sufficiently large.

3.5 Optimal Monetary Policies Without the ZLB

In this section, I will first calibrate the model and then present the solution in the deterministic case and finally the stochastic case without the ZLB. For each case, the solutions are compared with those from Ramsey and MPE. By doing so, it serves as a first step to understand the role of credibility/reputation in conducting monetary policies.

3.5.1 Calibration

The parameterization here follows mostly the literature. The Frisch elasticity is set to be 1 (Hall, 2009a,b). The elasticity of substitution among intermediate goods is set to be 6 (Walsh, 2010). The price adjustment cost ϕ is set to 60 to have a slope of 1/6 for the Phillips curve. There are four parameters to calibrate for the Markov process of the discount factor β. The value of β^L is set to 0.994 to have an average of 2.5% real annual rate during normal times and β^H to 1.011% to have a natural rate of -4.5% for recession times. Setting $p_{HH} = 2/3$ is to have an average of 3 quarters duration of the economy at high shock states. $1 - p_{LL}$ is the frequency of the economy entering the high shock state when the current state is low and set to be 0.0068, the chance of 1 quarter out of 12 years. These two values are within the ranges used by Nakata (2014).[12]

Table 3.1: Parameterization

Parameters	Explanations	Values	Targets/Sources
χ	Frisch elasticity of labor	1	Hall (2009a,b)
ϵ	elasticity among inter. goods	6	Fernández-Villaverde et al. (2012)
ϕ	price adjustment cost	60	slope of Phillips curve= 1/6
β^H	high shock of preference	1.011	-4.5% natural rate
β^L	low shock of preference	0.994	2.5% natural rate
p_{HH}	persistence of high shock	2/3	aver. duration of 3Qs at high shock
$1 - p_{LL}$	frequency of high shock	0.0068	1 quarter out of 12 years

3.5.2 The Deterministic Case

Figure 3.1 shows the whole equilibrium set of the SSE for the benchmark case. The x- and y-axes are the two auxiliary quantities I introduced above, m_1 and m_2. The z-axis is the

[12] The unconditional probability of recession states is $\frac{1-P_{LL}}{2-P_{HH}-P_{LL}}$.

payoff to the central bank, h. First, any point within the hill-shape area is an equilibrium. Second, given (m_1, m_2), there are infinitely many payoffs to the central bank h ranging from $\underline{h}(m_1, m_2)$ to $\bar{h}(m_1, m_2)$. This paper focuses on mostly the best SSE, $\bar{h}(m_1, m_2)$. Third, there are in general infinitely many combinations of (m_1, m_2) that lead to a certain level of payoff $\bar{h}$. This implies that the central bank can have multiple (and possibly infinitely many) choices of (m_1, m_2) and hence nominal interest rates R to induce a certain equilibria path. Fourth, the shape of the state space also implies that there is a trade-off between m_1 and m_2. To support a certain same level of payoff, the decrease of m_1 should be accompanied by an increase of m_2. Since higher m_2 implies higher inflation, and lower m_1 implies lower marginal return of saving, this trade-off means that to achieve the same level of payoff, households have a trade-off between higher inflation and lower return on saving.[13] Finally, given the same level of promised investment return (m_1) or promised inflation (Π), there are in general two levels of the other generating same level of the best SSE. Take the example of $m_1 = 1$. The best SSE $\bar{h}$ supported by deflation (low m_2) can also be supported by inflation (high m_2).

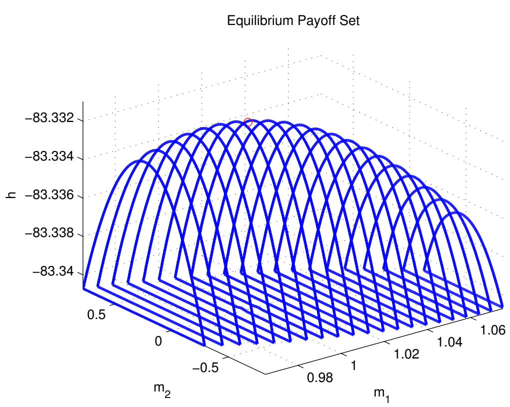

Figure 3.1: Equilibrium Payoff Set: The Deterministic Case

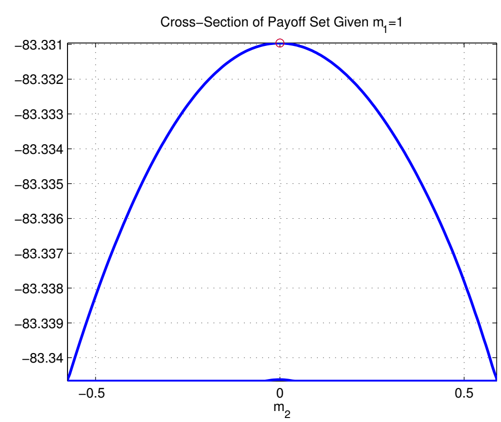

Figure 3.2: Equilibrium Payoff Set: A Cross Section when $m_1 = 1$

The globally best SSE is the one corresponds to highest payoff h, the top of the hill in Figure 3.1. The first observation is that the globally best SSE features $(m_1, m_2) = (1, 0)$,

[13] The return on saving of course also depends on inflation. However, higher inflation also leads to higher adjustment cost and thus more resource lost due to price adjustment.

of which $m_2 = 0$ means zero inflation. This is also shown in Figure 3.2, which is the cross-section of the equilibrium set as a function of m_2 when fixing $m_1 = 1$. The maximum h is achieved when $m_2 = 0$. The second observation is that only the globally best SSE is a steady state that can be supported, whereas all other equilibria are not. These two then imply a steady state of prices and allocation that coincides with MPE and Ramsey equilibria, that is, $\Pi = 1$, $c = l = 1$, $w = 1$, and $R = 1/\beta$.

To better and more straightforwardly show the idea, I will use promised consumption and inflation instead of (m_1, m_2) from now on as payoffs to the households. Figure 3.3 shows the supported state space of (c, Π), which is the bottom plane of the whole set in Figure 3.1 with appropriate transformation. The area inside the solid line is the space with the ZLB and the one inside the dashed line is the space without the ZLB. It is straightforward to see that the supported state space shrinks due to the ZLB. Given m_1, the lower bound of m_2 that can be supported shifts up. Figure 3.4 is cross-section of the equilibria set as a correspondence to m_2 fixing $m_1 = 1$. Therefore, with the ZLB, the public won't believe a lower inflation rate policy that can be achieved in the absence of the ZLB. However, the best SSE and hence the efficient allocation can always be rational and believed by the two agents.

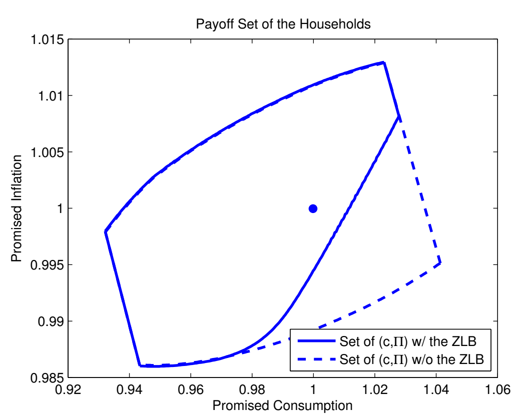

Figure 3.3: The Sets of Promised Payoffs to the Households (c, Π)

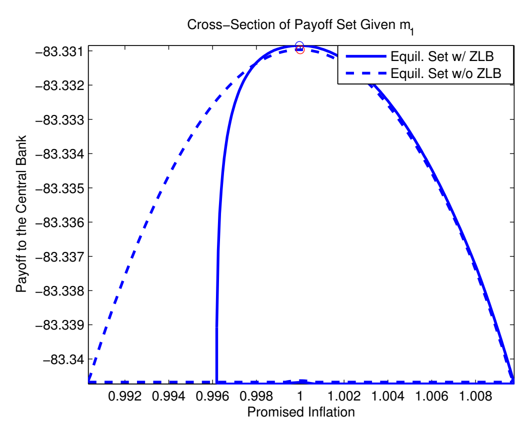

Figure 3.4: Equilibrium Payoff Set: A Cross Section when $m_1 = 1$

Many central banks with implicit or explicit inflation targeting have a positive targeted level of inflation in practice. In the US, for example, this target is 2% annually. One justification of this positive target is the inflation bias due to monopolistic pricing distortion under full discretion solution (see Gertler et al. (1999) for example).[14] This bias is at the cost of lower consumption, which leads to lower lifetime utility. A central bank with reputation can do better. In the benchmark, the parameter that governs subsidy ξ is set to completely eliminate steady state distortion due to monopolistic pricing. Figure 3.5 compares the equilibria set when ξ is reduced to 88% of the benchmark level Not surprisingly, the absence of full fiscal subsidy leads the central bank to pursue in general higher inflation to eliminate the distortion and households know this, and thus equilibria associated with lower inflation in the presence of full subsidy are no long supported. However, the inflation bias associated with the globally best SSE is much smaller. The globally best SSE now features $m_1 = 1.0057$ and $m_2 = 0.0821$. The positive m_2 corresponds to an inflation of 0.52% annual inflation at steady state. In contrast, the implied annual rate from MPE is 0.94% . In other words, the central bank with reputation can have a much smaller target if it uses inflation to eliminate price distortion due to lack of fiscal subsidy. Table 3.2 shows the implied inflation target as a function of fiscal subsidy.

Table 3.2: Fiscal Subsidy and Implied Inflation Targets (Annual %)

Fiscal Subsidy	MPE	SSE
100%(Full Subsidy)	0	0
88%(Partial Subsidy)	0.94	0.52
76%(Partial Subsidy)	1.85	1.12
64%(Partial Subsidy)	2.74	1.60

As discussed in the literature, the discount factor is the key to determine the equilibria set and the implementablility of Ramsey equilibrium. Given everything else equal, higher discount factor leads to higher expected utility. The cost to lose reputation will thus be higher, which will prevent the central bank from deviating. Therefore, higher discount

[14] In the case of Ramsey equilibrium, inflation bias is always zero.

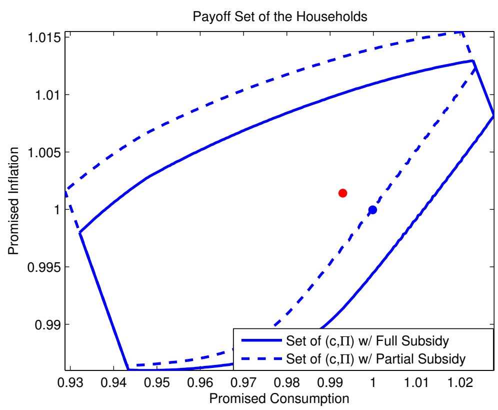

Figure 3.5: Comparison of Sets of SSE with Different Subsidy (Solid line: full subsidy; dashed line: 88% subsidy; the blue (red) dot is corresponding to the best SSE under full (partial) subsidy)

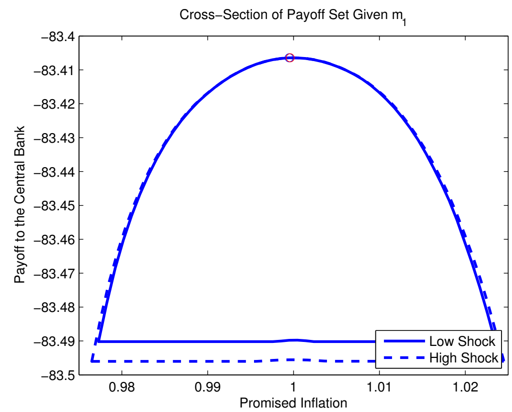

Figure 3.6: Cross-Sections of Equilibrium Sets when $m_1 = 1$ (The level of payoff for the high shock state is adjusted upward to make the highest payoff in that state equal to the highest payoff in low state.)

factor will result in a bigger supported equilibrium set. However, higher discount factor also means a more severe recession, which lowers the expected lifetime utility. This makes the current instant utility more important and hence encourages the central bank to deviate. This will lead to a smaller equilibrium set. It turns out the former effect dominates given the model framework and parameterization. Figure 3.6 shows the cross-sections of equilibrium sets for low and high β.[15] And indeed, the set associated with high β is bigger, though insignificantly. Note also that the globally best SSE with efficient allocation is always supported.

3.5.3 The Stochastic Case

The equilibria set for the stochastic case is a combination of two sets for the low and high β as in the subsection above. The globally best SSE features efficient allocation for any period t and state s. That is, $c_t = l_t = \Pi_t = w_t = 1$ and the associated policy rate and promised values are $R_t = 1/\beta_t$ and $(m_{1t}, m_{2t}) = (1, 0)$ respectively for any t. However, if the central bank picks a slightly different equilibrium at the beginning, the dynamics could

[15]The level of utility is adjusted to make the globally best payoffs the same.

be very different.[16]

To show this, I start with an equilibrium with promised payoffs $(m_1, m_2) = (0.9996, 1.46 \times 10^{-5})$ at low shock state. The economy is hit by a high discount factor shock at period 0. After period 0, this shock is always in low state. The dynamics after the shock is shown in Figure 3.7.

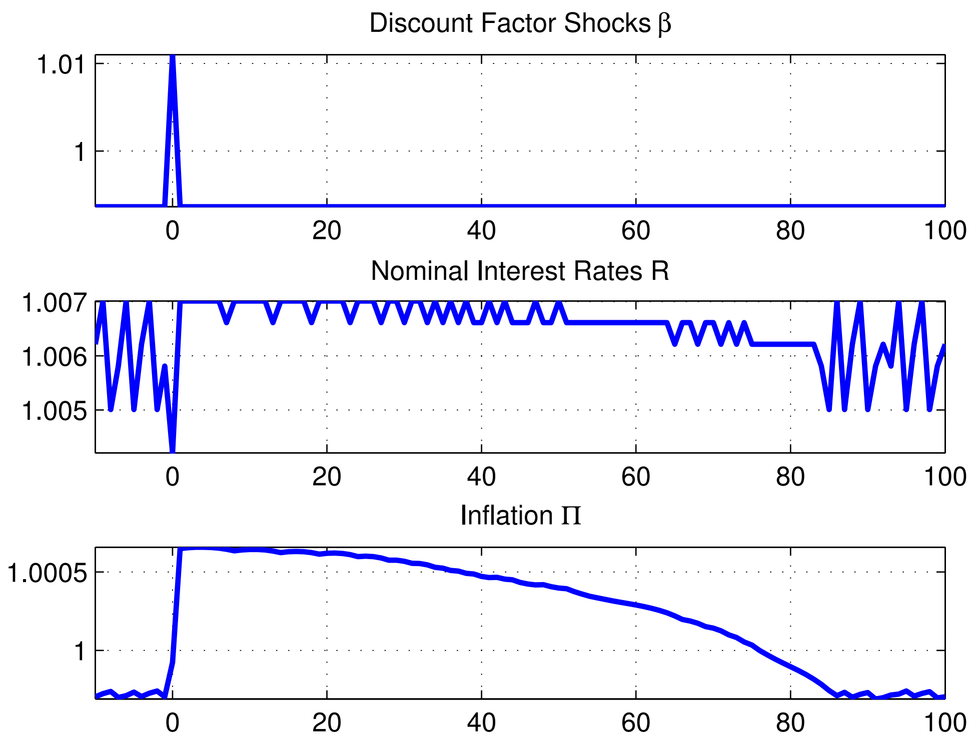

Figure 3.7: One Simulated Path after a High Shock hits only at time 1 under SSE without the ZLB

There are several things to pay attention to when compared to the MPE and Ramsey equilibria. First, it takes about 20 years for the consumption and inflation to return their normal levels while in the cases of MPE and Ramsey, those two never change – the nominal rate is always set to $1/\beta_t$ and $c_t = l_t = \Pi_t = w_t = 1$. Second, there is a spike of inflation at the time of shock along with a much milder decrease of nominal interest rates. In the

[16] As stated in Rogoff (1987), there is no compelling argument as to why the economy will coordinate on a "good" equilibrium and not a "bad" one out of the continuim of reputational equilibria.

case of Ramsey and MPE, the nominal rate at the time of the contractionary shock is $1/\beta^H = 0.989$. Under the equilibrium I picked above, the rate is 1.004. The reason for this is that expected inflation path changes a lot as the central bank wants and on-impact rates do not have to adjust too much. This gives us the intuition of how optimal policy will work when the ZLB binds, which will be detailed in the following section.

3.6 Forward Guidance and Credible Monetary Policy

The clear message from discussions above is that when the nominal rate is flexible enough and not constrained by the ZLB, the inflation gap should always be reduced to its minimum zero. The key feature of the optimal policy is zero inflation with nominal (and real) interest rates equal to the inverse of discount factor. The minimum inflation is to reduce the intra-temporal distortion between consumption and leisure, while nominal interest rate is whatever needed to equalize the desirable real rate. In the following, let us call the inverse of discount factor the shadow policy rate, and the discrepancy between real rate and shadow policy rate the shadow policy rate gap, which is proportional to the consumption gap. Without the ZLB, this gap and inflation gap should be zero under optimal policy. In the presence of the ZLB, there is a trade-off between these two and it is exactly where time-inconsistency comes from.

3.6.1 The Time-inconsistency of Commitment

To illustrate the issue of time-inconsistency, I assume a particular realization of shocks. Suppose that the high shock of β hits the economy at period t and is followed by low shocks thereafter. The shadow policy rate at period t is $1/\beta^H$, which is less than 1. The real interest rate $R_t/E\Pi_{t+1}$ should decrease to close the gap. In the case of MPE, the central bank today cannot affect inflation tomorrow, and therefore the gap will not be closed and consequently real rate is higher than the desirable rate. Households thus save more and consume less. This leads to a weak aggregate demand and deflation, opening the other inflation gap.

Under Ramsey, the central bank has another tool to close the shadow policy rate gap when R is bounded below by 1 – to increase expected inflation by promising to have low nominal rates no matter what happens in period $t+1$. By doing so, aggregate demand at the current period will increase. This will in turn reduce inflation gap (that is, less deflation). The resulting consequence is increased consumption and less severe deflation.

The time-inconsistency comes at period $t+1$.[17] With a low shock, the shadow policy rate is $1/\beta^L$. If the central bank follows its promise and sets $R=1$, then the central bank will have a shadow rate gap of $1/E\Pi_{t+2} - 1/\beta^L$, which is negative. Suppose the central bank can only promise one period, then $E\Pi_{t+2} = 1$ and the gap will be $1 - 1/\beta^L$. This negative gap leads to too high aggregate demand and positive inflation gap. These two gaps are costly to the central bank and they will definitely renege and close them by setting $R = 1/\beta^L$ and $\Pi = 1$ if high discount factor shock never hits the economy again. This is not consistent with what they promised in period t.

The key to the time-inconsistency is the temptation to costlessly close these two gaps when shock is low. While Ramsey simply exclude the temptation to deviate, the definition of SSE fully takes this into account. The public in the economy basically think about two aspects of what the central bank say: the duration of low nominal rate (and thus implied expected inflation) and how low (high) the nominal interest rate (expected inflation) will be. The (infinitely many and possible) paths of policies will be a combination of these two.

3.6.2 Ramsey is Not Implementable

Before examining the optimal policy under the best SSE, it is useful to first show whether the Ramsey equilibria is implementable or not. The strategy is to calculate the corresponding promised values to the private sector and continuation payoff to the central bank as in SSE for the Ramsey case. Namely, for any given realized state, calculate m_1, m_2, and h and

[17]The most likely period for the central bank to fall back on its words is the last period of duration at the ZLB that Ramsey suggests. Here I just assume the central bank deviates the following period when high shock disappears for illustrating purpose.

check if the triplet (m_1, m_2, h) is within the set of SSE. For the discussion now, assume that shocks materialize like the following: a high β shocks the economy at period 1, followed by low shocks thereafter. I check the implementablility of Ramsey by looking into only the promised values (m_1, m_2).

The associated values and the sustainable sets under SSE are drawn in Figure 3.8. The area circled by solid line is the sustainable equilibria set for low shock state and the one by dashed line is the set for high shock state. Small circles are the promised value combinations from Ramsey for low shock states. Before the high shock hits the economy, the promised payoffs are denoted by the filled circle. At the time of the high shock, promised payoffs go to the point denoted as star. After the shock, the promised bundle goes back to its original place clockwise as shown in the figure by arrows. The circles lie in the sustainable set for low state. However, at the state of high shock, the promised values, denoted by star, lie beyond the sustainable set for high state, the dashed line area. That is, the promised consumption and inflation bundle under Ramsey is not available. Therefore, the Ramsey is not implementable.

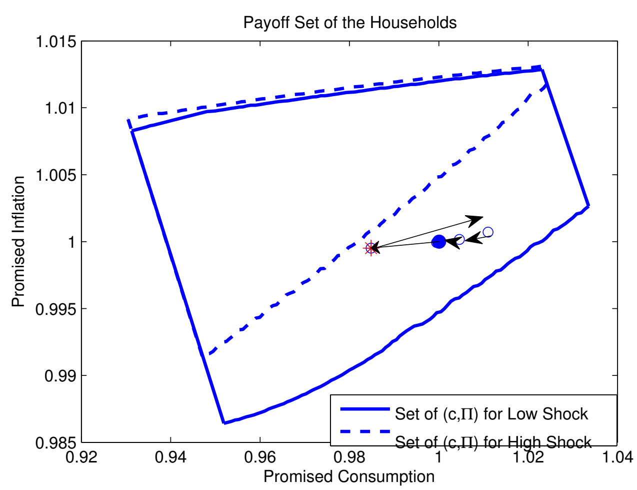

Figure 3.8: Non-Implementability of Ramsey (The solid line area is the supported payoff set associated with low shock, while the dashed line area the set with high shock. The circles are the calculated corresponding payoffs from the Ramsey equilibrium for low state and the star the corresponding payoff for high state. The star stays beyond the set for high state from the SSE. The dynamics are shown by arrows.)

This conclusion is robust to a wide range of parameter values. Keeping the baseline parameterization, I experiment with different values for one parameter each time. The experiments with different shock size (β^H), different length of high shock duration (p^{HH}), different frequency ($1-p^{LL}$) and different price adjustment cost (ϕ) do not show the support of implementabilility of Ramsey.[18] It turns out that as long as the time-inconsistency exists, that is, the ZLB effectively binds under Ramsey equilibrium, the central bank will always deviate and thus the Ramsey equilibrium is generally not implementable.

3.6.3 Credible Policies at the ZLB

In this subsection, I answer the question raised in the beginning: What does the best credible forward guidance look like? Or in other words, how does the policy rate look like if a central bank has highest reputation?

Since there exist infinitely many policy paths that support the same equilibrium, I use the strategy as described in the Appendix to find the policy path I am interested in. In particular, I find the path that features lowest interest rates when high shock starts to hit the economy.

Figure 3.9 shows the range of policy rates for a specific realization of shocks, where I assume that at time 1, the economy is hit by a high shock and this shock persists for 6 periods. The length of high shocks is to match the NBER definition of the recent financial recession (December 2007 - June 2009). The dashed line is the upper bound of the nominal rates and the dash-dotted one the lower bound. A first observation is that, under the best SSE, the zero lower bound never binds in recession times. The only chance it will bind is during the period when the high disappears.[19] Second, the upper and lower bounds are not smooth. This is because these bounds are conditional on (m_1, m_2), the promised values, which are moving around. Given the discussion in the last Section, there is no reason this pair of values change in a way to have smooth policy rates.

Among the many policy paths, I pick one that features lowest nominal rates when

[18] From discussions of last section, it is clear that the Ramsey without the ZLB effectively binding coincides with the BSSE, which means the Ramsey is implementable if either $\beta^H \leq 1$ or $1 - p^{LL} = 0$. But this is the

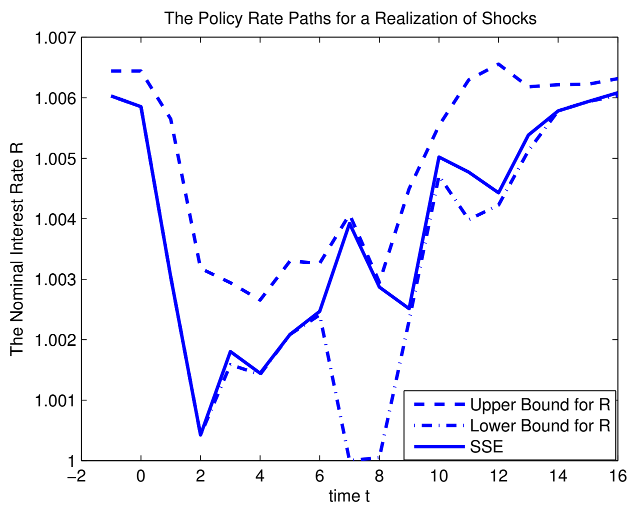

Figure 3.9: The Upper and Lower Bounds and One Specific Path of Policy Rates for the Best SSE(The high shock starts to hit the economy at time 1 and stays for another 5 quarters.)

recession starts, which is represented by the blue solid line in Figure 3.9.[20] To compare, I also draw the policy rates under Ramsey and MPE in Figures 3.10 and 3.11, respectively.

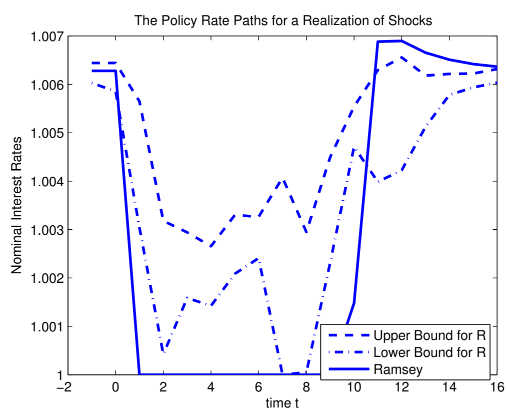

Figure 3.10: Dynamics of Policy Rates under Ramsey

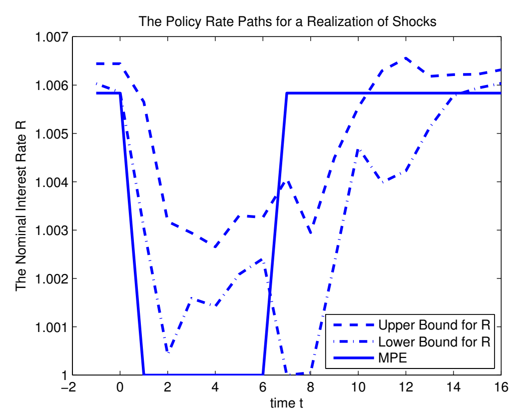

Figure 3.11: Dynamics of Policy Rates under MPE

First, when the high shock starts to hit the economy, the central bank with reputation will not react as aggressively as with those with full commitment or full discretion. Instead,

case where time-inconsistency disappears.

[19]It is true for any length of high shock periods.

[20]It is, however, not the path with lowest nominal rates for any time during the recession.

it will set the interest rate only about half of that in nominal times (1.2% compared to 2.5%). In general, the ZLB will not bind. The only binding case (the ZLB only binds for one period) under SSE is the one that delivers highest nominal rates during recession and accordingly has lowest nominal rates in normal times. Under Ramsey, though the recession ends at period 6, the nominal rate stays low another 4 quarters (3 quarters at zero and 1 quarter close to zero) even after the economy emerges from recession. The nominal rate then jumps to a higher-than-normal level to contain the inflation. Under the best SSE, the nominal rate stays low but non-zero for at least 8 quarters after the recession ends.

The intuition behind is simple. If the central bank sets the nominal interest rate too low or too far away from the normal level, then it is not credible. The credible policy path must be relatively smooth in all states to reduce the incentive for the central bank to deviate. At the same time, the reputational central bank will have a longer time span than that under Ramsey to have lower rate to optimize.

Accordingly, the economy has higher and persistent inflation as shown in Figure 3.12. The inflation on average is higher than that under Ramsey (by 0.12% annually) for the recession time and several periods after the recession. In addition, it takes 3 quarters longer for the inflation to first revert to its normal level under SSE than under Ramsey. However, at normal times, the inflation level is lower than that under Ramsey. Figure 3.12 also shows that inflation during recession can be higher if it is also higher in normal times, with on average higher interest rates.

Finally, compared to a central bank with full discretion, the bank with reputation can do much better in boosting consumptions during recession, with an average gap of only -1.55% compared to -4.55% under MPE. In other words, a discretionary central bank is able to stimulate the economy during recession very close to the one under full commitment if it picks the best equilibrium to follow. The policy under the best discretionary equilibrium is designed in a way different from those under Ramsey equilibrium and MPE.

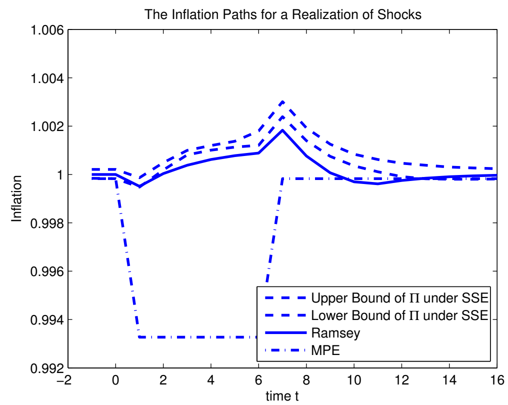

Figure 3.12: Dynamics of Inflation

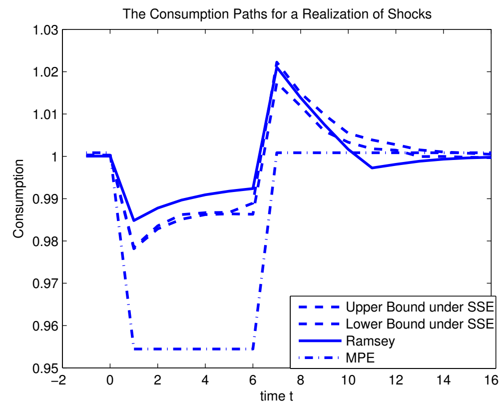

Figure 3.13: Dynamics of Consumption

The Simple and Practical Forward Guidance Rules

To make the optimal policy rules practical to use, I simulate the economy under the best SSE, Ramsey and MPE 1×10^6 times to get rules similar to Taylor. That is, if the central bank sets the interest rate based on inflation and output gaps, how should the Taylor rule look like? Under the best SSE with the specific policy rate path picked above, this is,

$$r_t = 0.57 - 1.9336\pi_t + 0.1331y_t + error$$

Under Ramsey, this is

$$r_t = 0.61 - 6.4452\pi_t + 0.2589y_t + error$$

First, as shown in the subsection above, the average inflation under the best SSE is lower than that under Ramsey (0.57% vs 0.61%). However, the former could be higher than the latter if a different policy rate path is selected. Second, the interest rate should respond to inflation negatively under both Ramsey and the best SSE. This is because positive inflation is a sign of recession. Whenever inflation is high, the interest rate should decrease to boost the economy. This is different from what is usually believed as Taylor principle – the nominal rate should increase more than one-to-one to inflation to contain the latter. Finally, the interest rate is reacting more to inflation and output gaps under Ramsey than under the best SSE. Under Ramsey, the elasticity of nominal rates over inflation is about three times as big as that under the best SSE. That is, the policy rate decreases roughly 3 times bigger than that under the best SSE for given increases of inflation. Similarly, the reaction of the nominal rate to output gap under Ramsey is about twice as big as that under the best SSE. In other words, if the central bank has full commitment, it can and should act more aggressively (and more swiftly) to adjust its policy rate in response to inflation and output gaps. By doing so, it will close output and inflation gaps as much as possible in as less as possible time.

The rules above relate interest rate to contemporaneous inflation and output. Suppose now that the central bank announces forward guided rate in the future only based on current inflation and output, what will the rate look like? To shed light on it, I run the following regressions:

$$r_{t+j} = \beta_0 + \beta_\pi \pi_t + \beta_y y_t + error$$

where $j (\geq 0)$ means the policy rate j periods forward. Figure 3.14 reports β_π as a function of j. First, the average horizon of forward guidance is about 10 to 12 quarters. The coefficients beyond this time horizon are approximately zero, which means that on average, the central bank are not using forward guidance beyond this time horizon to tackle the recession. [21] Second, same as the instant response of current policy rate to inflation, the forward guided policy rate goes down when inflation is high. Third, the forward guided policy is in general much less sensitive to inflation under SSE than under full commitment. Finally, the stance of monetary policy tends to change quickly within 1.5 years' horizon. That is, the forward guided policy rate adjusts quickly up to 1.5 years, as a response to current inflation.

The reason for the difference of the rules is the same as I explained above. The aggressive policy is subject to the non-credibility problem, though it is better in the sense of leading to

[21] The threshold for the time span of forward guidance depends on the parameter values I have picked. However, given the experiments I did with wide range of possible parameter values, forward guidance is largely an issue within short horizon of less than 3 years.

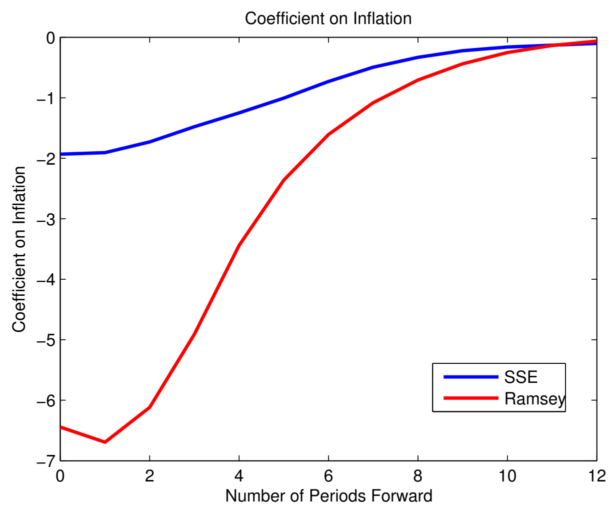

Figure 3.14: The Elasticity of Forward Guided Rates to Inflation

higher welfare. The more aggressive the policy rate responses to inflation and output, the more likely it is non-credible. This should be a general caution for discussions on rule-based (i.e., Taylor rule, price targeting rule) forward guidance, and more generally on determinacy of New Keynesian models with different rules.

Under MPE, since the output gap and inflation gap is one-to-one mapping, it will get multi-collinearity if the regression runs like above. Instead, I get the rule only based on inflation or output as follows:

$$r_t = 0.60 + 0.8841\pi_t + error$$

$$r_t = 0.57 + 0.1355y_t + error$$

In contrast with the rules under Ramsey and the best SSE, the Taylor like rule is very simple: if inflation is negative (output is negative at the same time), decrease the nominal rate appropriately.

It should be emphasized that the rules here only apply to the case where discount factor is the only source of shocks. Since this represents the aggregate demand shock, the rules, in particular, the negative reaction of nominal rates to inflation is to boost the demand by lowering nominal rates.

3.7 Conclusion

This paper explores the optimal and credible policy, which is interpreted as forward guidance, via a standard New Keynesian model by first characterizing the whole set of SSE when the nominal interest rate occasionally binds at zero. In contrast to Ramsey equilibria and MPE, the best SSE features strong reliance on the expected inflation for a very long time. Forward guidance based on the best SSE states that the interest rate should stay low but non-zero for a prolonged time even when the economy recovers. It takes more than three quarters extra for the economy to return to its normal policy regime. This policy can close the output gap to -1.55%, as compared to 1.05% under full commitment, which is the best a constrained central bank can achieve.

As a theoretical result, I show quantitatively that the Ramsey equilibrium is not generally implementable. The stark difference from the general conclusions in the literature stems mainly from the options to which the government can choose to deviate. While most assume that the MPE is the alternative plan, the best credible plan to which the government can deviate achieves much higher payoffs than the MPE, which makes the Ramsey equilibrium more difficult to implement.

Finally, if the central bank intends to use a simple and practical rule to communicate, the Taylor-style rule says the nominal rate should react less aggressively to inflation and output changes. This rule can be used to estimate the effects of other policies that were used together with forward guidance.

Appendix

The Model with Implementable Policy Rates

In Section 3.2, I presented a standard New Keynesian model that features private bonds. And at equilibrium, the net position of bonds is always zero. This assumption is criticized for its non-implementability of policy rates, i.e., the central bank in this model has no way to realize the policy target it wants. Here I present a model similar to that in Section 3.2, but with government bonds and hence central banks' implementation of policy rates through open market operation.

The representative household is to maximize same lifetime utility as in (3.1) subject to, however, a different budget constraint,

$$c(s^t) + \frac{B^g(s^t)}{P(s^t)} = w(s^t)l(s^t) + R(s^{t-1})\frac{B^g(s^{t-1})}{P(s^t)} + \tau(s^t) + d(s^t) + \Delta m(s^t)$$

where $R(s^{t-1})$ is now interpreted as the nominal interest rate paid to government bonds $B^g(s^t)$. I now interpret $\frac{B^g(s^t)}{P(s^t)} - R(s^{t-1})\frac{B^g(s^{t-1})}{P(s^t)}$ as the change of households' monetary demand and $\Delta m(s^t)$ the real change of monetary supply. At each period, the demand and supply of money much be equalized, that is

$$\Delta m(s^t) = \frac{B^g(s^t)}{P(s^t)} - R(s^{t-1})\frac{B^g(s^{t-1})}{P(s^t)}$$

Therefore, any policy target R_t can be implemented by changing $\Delta m(s^t)$ given the households' bond holdings ($B^g(s^{t-1})$ and $B^g(s^t)$) and price ($P(s^t)$). This change of budget constraint does not change any optimality conditions or feasibility conditions. Note that though I introduce an extra state variable $B^g(s^t)$, it is irrelevant for the analysis of dynamics of the economy, in particular, the dynamics of policy rates.

The Firm's Dynamic Problem

This subsection shows that given the promised values about adjusting prices, the firm solves the same recursive problem as without it. The dynamic problem of an individual firm can be written as follows:

$$V(P_{it-1}) = \lambda_t d_{it} + \beta_t E_t V(P_{it})$$

subject to constraint (3.5) and the definition of dividend d_{it}. λ_t is the Lagrangian multiplier of households' budget constraint. Solving the problem gives the following Euler equation:

$$\left[(\epsilon-1)(\frac{P_{it}}{P_t})^{-\epsilon} - (1-\xi)\epsilon w_t + \phi\left(\frac{P_{it}}{P_{it-1}}-1\right)\frac{P_t}{P_{it-1}}\right] y_t\lambda_t = \beta_t E_t\left[\left(\frac{P_{it+1}}{P_{it}}-1\right)\frac{P_{it+1}P_t}{P_{it}^2}\right]\phi y_{t+1}\lambda_{t+1}$$

Imposing symmetry gives the equation (3.6) in the text:

$$\left[(\epsilon-1) - (1-\xi)\epsilon w_t + \phi\left(\Pi_t - 1\right)\Pi_t\right] y_t\lambda_t = \beta_t E_t\left[\left(\Pi_{t+1}-1\right)\Pi_{t+1}\right]\phi y_{t+1}\lambda_{t+1}$$

Now let $m_2^+ = \left[\left(\Pi_{t+1}-1\right)\Pi_{t+1}\right]\phi y_{t+1}\lambda_{t+1}$ be the marginal value (payoff) of adjusting price relative to aggregate price. Then the firm's problems becomes:

$$V(P_{it-1}) = \lambda_t d_{it} + \beta_t E_t m_2^+ P_{it}/P_t$$

Solving and imposing symmetry gives exactly the same solution as the original recursive problem.

Numerical implementation of the operator $\mathbb{F}$

Let $\boldsymbol{S}\times\boldsymbol{M_1}\times\boldsymbol{M_2}\times\boldsymbol{H}$ denote the space of all equilibrium state vectors and associated payoffs to the central bank (s, m_1, m_2, h). $\boldsymbol{W}:\boldsymbol{S}\rightarrow\boldsymbol{M_1}\times\boldsymbol{M_2}\times\boldsymbol{H}$ is a correspondence from $\boldsymbol{S}$ to $\boldsymbol{M_1}\times\boldsymbol{M_2}\times\boldsymbol{H}$.

With an initial guess $\boldsymbol{W}^0(s) = \{(m_1(s), m_2(s), h(s))\}$ and a pre-determined tolerance level ϵ, the algorithm goes as follows:

- Step 1: For $\forall s \in \boldsymbol{S}$, find $\boldsymbol{\Omega}(s) := \{(m_1, m_2, h)|(m_1, m_2, h) \in \boldsymbol{W}^0(s), \exists R \in [\underline{R}, \bar{R}]$ and $(m_1', m_2', h') \in \boldsymbol{W}^0(s')$ such that :

$$
\begin{aligned}
h &= u(c,l) + \beta \mathbb{E} h' \geq \tilde{h}^0(s) \\
Em_1' &= \frac{1}{c}\frac{1}{R\beta} \\
Em_2' &= \frac{1}{\beta}\left[(\epsilon - 1) - (1-\xi)\epsilon w + \phi\left(\Pi - 1\right)\Pi\right]\frac{l}{c} \\
m_1 &= \frac{1}{c\Pi} \\
m_2 &= \phi(\Pi - 1)\Pi\frac{l}{c} \\
w &= l^{\chi} c \\
c &= (1 - \frac{\phi}{2}(\Pi - 1)^2) l
\end{aligned}
$$

where

$$
\begin{aligned}
\bar{h}^0(s, m_1(s), m_2(s)) &= \underset{h}{max}\{h|(m_1, m_2, h) \in \boldsymbol{W}^0(s)\} \\
\underline{h}^0(s, m_1(s), m_2(s)) &= \underset{h}{min}\{h|(m_1, m_2, h) \in \boldsymbol{W}^0(s)\} \\
\tilde{h}^0(s) &= \underset{(m_1, m_2)}{min}\, \underline{h}^0(s, m_1, m_2)
\end{aligned}
$$

- Step 2: For $\forall s \in \boldsymbol{S}$, and $\boldsymbol{\Omega}(s)$, denote $\boldsymbol{\Omega}^M(s,h) := \{(m_1, m_2)|(m_1, m_2, h) \in \boldsymbol{\Omega}(s), h = h(s, m_1, m_2)\}$, and define

$$
\begin{aligned}
\bar{h}^1(s, m_1, m_2) &= \underset{R}{max} \underset{\substack{c,l,\Pi,w, \\ (m_1', m_2', h') \in \boldsymbol{W}^0(s')}}{max} u(c,l) + \beta \mathbb{E} \bar{h}^0(s', m_1', m_2') \\
\underline{h}^1(s, m_1, m_2) &= max\{\underset{R}{max} \underset{\substack{c,l,\Pi,w, \\ (m_1', m_2', h') \in \boldsymbol{W}^0(s')}}{min} u(c,l) + \beta \mathbb{E} \underline{h}^0(s', m_1', m_2'), \tilde{h}^0(s)\}
\end{aligned}
$$

for all $(m_1, m_2) \in \boldsymbol{\Omega}^M(s,h)$. Otherwise, set

$$\begin{aligned}\bar{h}^1(s, m_1, m_2) &= +\infty \\ \underline{h}^1(s, m_1, m_2) &= -\infty\end{aligned}$$

Further, let

$$\tilde{h}^1(s) = \min_{(m_1,m_2)\in\boldsymbol{\Omega}^M(s,h)} \underline{h}^1(s, m_1, m_2)$$

- Step 3: Define $\boldsymbol{W}^1(s) = \{(m_1, m_2, h)|(m_1, m_2) \in \boldsymbol{\Omega}^M(s,h), h \in [min\{\bar{h}^0(s, m1, m2), \underline{h}^1(s, m_1, m_2)\}, max\{\underline{h}^0(s, m_1, m_2), \bar{h}^1(s, m_1, m_2)\}]\}$
- Step 4: Set $\boldsymbol{W}^* = \boldsymbol{W}^1$ if $\|\boldsymbol{W}^1 - \boldsymbol{W}^0\| < \epsilon$; otherwise, set $\boldsymbol{W}^0 = \boldsymbol{W}^1$ and repeat the steps above.

In the deterministic case, I set the number of grids for m_1 and m_2 to be 200, the number of points for interest rate 500 with the range $[1, 1.1]$, implying a 8 basis point change when optimizing. In stochastic case, I reduce the number of grids for m_1 and m_2 to be 50 while keeping the grids for interest rates the same.

Algorithm to Find All $R(m_1, m_2)$

Give m_2, Π is fixed. With m_1, c, l, and w are also determined. Expected payoffs to the firms are also prefixed as $Em_2' = [(\epsilon - 1) - \epsilon(1 - \xi)w - \phi(\Pi - 1)\Pi]\frac{l}{c\beta}$. Expected payoffs to households, however, depend on R as $Em_1' = \frac{1}{\beta c}\frac{1}{R}$. Since the best SSE can be supported by many (m_1, m_2), there may exist different (m_1', m_2') that lead to the same level payoff to the central bank. To find all such payoffs and hence interest rate, I find out all $(m_{11}', m_{21}', m_{12}', m_{22}')$ that satisfy the following two equations:

$$\omega\bar{h}(m_{11}', m_{21}') + (1 - \omega)\bar{h}(m_{12}', m_{22}') = E\bar{h}(m_1', m_2')(m_1, m_2) \tag{3.34}$$

$$\omega m_{21}' + (1 - \omega)m_{22}' = Em_2'(m_1, m_2) \tag{3.35}$$

where ω is the probability of the economy in state 1 next period given the state today. Note that the right hand sides of the two equations above are functions of (m_1, m_2) and are fixed for the current problem.

The algorithm goes as follows:

- Step 1: Pick (m_1, m_2) and hence $Em_2'(m_1, m_2)$ and $E\bar{h}(m_1', m_2')(m_1, m_2)$. Solve for $m_{21}' = (Em_2'(m_1, m_2) - (1-\omega)m_{22}')/\omega$.

- Step 2: Pick I and J grids for Em_1' and m_{11}' within their range. Given the i^{th} and j^{th} grids, calculate the corresponding $m_{11}' = (Em_1' - (1-\omega)m_{12}')/\omega$. Substituting m_{11}' and m_{21}' back to (3.34), I define a new function:

$$g = \omega\bar{h}((Em_1' - (1-\omega)m_{12}')/\omega, (Em_2'(m_1, m_2) - (1-\omega)m_{22}')/\omega)$$
$$+(1-\omega)\bar{h}(m_{12}', m_{22}') - E\bar{h}(m_1', m_2')(m_1, m_2)$$

- Step 3: Fixing m_{11}', g is a function of m_{22}'. Use root finding solver to solve for the roots of $g = 0$. (There are at most 2 roots because h is convex along the dimension of m_2.)

- Step 4: Keep the roots if any. Go to step 3 if $j \leq J$. Go to step 2 if $j > J$. Go to step 1 if $i > I$.

Proofs

Proposition 3.4.2. To simplify the exposition, I abstract away uncertainties. In the text, I have shown that the sequential and recursive problems lead to the same Euler equation. The only thing left is to show that the transversality condition holds.

Let $b_t = B_t/P_t$. To prevent Ponzi schemes, I assume that there are debt limits for b_t, that is, $b_t \in [\underline{b}, \bar{b}]$, where $\underline{b_t} > -\infty$ and $\bar{b_t} < \infty$. Since $R \leq \bar{R} < \infty$, it is sufficient to show that $m_t \leq \bar{m} < \infty$. From the goods market clearing condition, the upper (lower) bound of inflation is $\bar{\Pi} = 1 + \sqrt{2/\phi}$ ($\underline{\Pi} = 1 - \sqrt{2/\phi}$). If inflation is out of the range, then all the goods produced will be used for compensating price adjustment.

Given the preference of the household, I have $lim_{c\to 0} u_c(c,l) = -\infty$ and $lim_{l\to 0} u_l(c,l) = 0$, which means that the household will be better off by spending a strictly positive amount of time $\underline{l} > 0$ in working so that he can obtain some income to finance a positive amount of consumption. From the first-order condition $w = cl^{\chi}$, I have $c = wl^{-\chi}$. So what is left to show is there is a lower limit for the wage w. From the firms' first-order condition (3.6), it is easy to see that $w = \frac{(1-\epsilon)-(1+\beta)\phi\Pi(\Pi-1)}{\epsilon(1-\xi)}$ is lower bounded. Therefore, there exists upper bound $\bar{m}$. Finally, $lim_{t\to\infty}\beta^t m_t b_t R_t \leq lim_{t\to\infty}\beta^t \bar{m}\bar{b}\bar{R} = 0$ ■

Lemma 3.4.4. The proof follows closely Phelan and Stacchetti (2001). ■

Proposition 3.4.7. The proof follows closely Phelan and Stacchetti (2001). ■

Proposition 3.4.8. The proof here follows Feng (2015). By definition, $\bar{h}(s, m_1, m_2)$ is the maximum value of h given (s, m_1, m_2), which is:

$$\begin{aligned}
\bar{h}(s, m_1, m_2) &= \underset{R}{max} \underset{\{m_1', m_2', h'\}}{max} \left[u(c,l) + \mathbb{E}\{h(s', m_1', m_2')\}\right] \\
&= \underset{R}{max} \left[u(c,l) + \underset{\{m_1', m_2', h'\}}{max} \mathbb{E}\{h(s', m_1', m_2')\}\right] \\
&= \underset{R}{max} \left[u(c,l) + \mathbb{E}\{\bar{h}(s', m_1', m_2')\}\right]
\end{aligned}$$

where the first equality follows the definition of $\bar{h}(s, m_1, m_2)$, the second equality follows the fact that the instant utility only depends on promised values (m_1, m_2), and the last equality uses the definition of $\bar{h}$.

A similar argument applies to $\underline{h}(s, m_1, m_2)$. A few comments go as follows. First, $\underline{h}(s, m_1, m_2) = \underset{R}{max} \underset{\{m_1', m_2', h'\}}{min} u(c,l) + \beta\mathbb{E}h'$. Second, it should be noted that the value of $u(c,l) + \beta\mathbb{E}\underline{h}'$ at given R might be smaller than $\tilde{h}(s)$, which says that the incentive constraint is not satisfied when the government has the lowest continuation value. When this happens, the government needs a higher continuation value so that the incentive constraint is satisfied. However, the corresponding payoff for the present government cannot be higher than $h(s)$. This is because only the minimization operates when R is given. There always exists $h' \in [\underline{h}(s,m_1,m_2), \bar{h}(s,m_1,m_2)]$ to bind the incentive constraint when the worst continuation value violates the incentive constraint. Otherwise, (m_1, m_2) should not belong to the equilibrium value correspondence. Note that $\tilde{h}(s)$ is the payoff of the worst SSE and must lie in the lower boundary of $h(s,m_1,m_2)$. Because it is the worst of all, it must be equal to $min_{\{m_1,m_2\}}\underline{h}(s,m_1,m_2)$. ■

Theorem 3.4.10. The proof follows Feng (2015) and first shows that the sequence of $\hat{W}_n$ is decreasing and $\hat{W}_n \ni \hat{W}_{n+1}$. Since W_n is convex-valued, it is sufficient to show that the upper boundary is decreasing and the lower one increasing. The upper boundary is decreasing because of the fact that $\bar{h}^1(s,m_1,m_2)$ is defined as $max_R u(c,l) + \beta\mathbb{E}\bar{h}^0(s', m_1', m_2')$ such that $\psi = (R, c, l, w, \Pi, \{m_1', m_2', h'\})$ is admissible wrt $\hat{W}^0$ at s. The admissibility of the vector ψ implies that $(m_1, m_2, \bar{h}^1(s,m_1,m_2)) \in \hat{W}_0(s)$. Therefore, $\bar{h}^1(s,m_1,m_2) \leq max\{h|(m_1,m_2,h) \in \hat{W}^0(s)\} = \bar{h}^0(s,m_1,m_2)$. Similarly, I can show that the lower boundary is increasing, i.e., $\underline{h}^1(s,m_1,m_2) \geq \underline{h}^0(s,m_1,m_2)$. The same argument thus holds for $\hat{W}_n(s)$. Since the sequence is decreasing, it has a limit $\hat{W}_\infty$. Proposition 3.4.7 implies that $\mathbb{F}(\boldsymbol{V}) = \boldsymbol{V}$. By a simple limit argument, I have $\lim_{n\to\infty} \hat{W}_\infty = V$. ■

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