On the Geometry

— OF —

The Transformation Group

$$p, \quad q, \quad r, \quad xq, \quad yp, \quad zr, \quad xp - yq, \quad z^2 r.$$

BY

J. E. WILLIAMS.

CONTENTS.

On the Geometry

— OF —

The Transformation Group

$p, \quad q, \quad r, \quad xq, \quad yp, \quad zr, \quad xp - yq, \quad z^2r.$

BY

John Edward
J. E. WILLIAMS.

WASHINGTON, D. C.:
GIBSON BROS., PRINTERS AND BOOKBINDERS.
1899.

UNIVERSITY OF VIRGINIA 1819
GENERAL LIBRARY

U Va

U. Va. Doctoral
Dissertation
16
160172

PREFACE.

LIFE.

The author's early training began in the public schools of Charlotte County, Va., principally the High School of Smithville. In the year 1889 he entered Hampden-Sidney College, where he graduated with the A. B. degree at the commencement of 1892. He was then elected principal of the Boydton High School, which position he resigned after two years to become first assistant of the Commerce Street School in Roanoke, Va. After one year's work in this school he returned to Boydton to accept a position as private instructor for Col. Thomas F. Goode. Finally, in the year 1896 he entered the University of Virginia to pursue a special course in Pure Mathematics.

The following pages were presented to the Faculty as a dissertation for the degree of Doctor of Philosophy. The subject met with the approval of Dr. J. M. Page, and it may be added that, so far as could be learned from the few works on Modern Mathematics in the University library, the investigations are new. d. 1943

UNIVERSITY OF VIRGINIA, 1899.

ON THE GEOMETRY OF THE TRANSFORMATION GROUP

$$\boxed{p, \quad q, \quad r, \quad xq, \quad yp, \quad zr, \quad xp - yq, \quad z^2r.}$$

By J. E. Williams.

1. In the following sections we shall not attempt to make an exhaustive discussion of the above group of infinitesimal transformations, but shall limit ourselves to some of the most interesting and important of the investigations which are possible.

We shall begin by determining whether or not any invariant equations exist, and by finding all absolutely invariant points, as well as the *path-curves* of the G_8. We shall limit our considerations to points, curves, etc., within a finite distance of the origin.

§ 1.

2. Representing the transformations of the group by $x_k f$, where ($k = 1 \ldots 8$), we know that if an equation of the form $F(x, y, z) = 0$ is invariant under the transformations of the G_8, we must have $x_k(F) = 0$, either identically or in virtue of $F(x, y, z) = 0$.*

If now we form the matrix of the equations

$$X_1(F) = \frac{\partial F}{\partial x} + 0 + 0 = 0$$

$$X_2(F) = 0 + \frac{\partial F}{\partial y} + 0 = 0$$

$$X_3(F) = 0 + 0 + \frac{\partial F}{\partial z} = 0$$

$$X_4(F) = 0 + x\frac{\partial F}{\partial y} + 0 = 0$$

$$X_5(F) = y\frac{\partial F}{\partial x} + 0 + 0 = 0$$

$$X_6(F) = 0 + 0 + z\frac{\partial F}{\partial z} = 0$$

$$X_7(F) = x\frac{\partial F}{\partial x} - y\frac{\partial F}{\partial y} + 0 = 0$$

$$X_8(F) \equiv 0 + 0 + z^2\frac{\partial F}{\partial z} = 0$$

* Lie's Continuierliche Gruppen, Kap. 16.

we find for one of the three row determinants

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix},$$

which can never be zero. This shows that no system of values of x, y, z will make all the three row determinants vanish, and hence we conclude that no equation $F(x, y, z) = 0$, that is no surface, within a finite distance of the origin, is invariant under every transformation of the G_8.

3. The *most general* transformation of the group has the form

$$Xf = (ax + by + c)\,p + (a_1x - ay + c_1)\,q + (a_2z^2 + b_2z + c_2)\,r = 0\,. \quad (1)$$

Since, if a point is absolutely invariant under Xf, we must have $\delta x = \delta y = \delta z = 0$ at that point, all invariant points are found from the equations

$$ax + by + c = 0$$

$$a_1x - ay + c_1 = 0$$

$$a_2z^2 + b_2z + c_2 = 0\,;$$

which give

$$x = -\frac{(ac + bc_1)}{a^2 + a_1b}\,,\; y = \frac{ac_1 - a_1c}{a^2 + a_1b}\,,\; z = -\frac{b_2 \pm \sqrt{b_2^2 - 4a_2c_2}}{2}\,.$$

These are the coordinates of all points invariant under the general transformation Xf. By specializing the undetermined constants $a, \ldots, c_2$, we find, of course, the point or points invariant under each particular transformation of the G_8. If, in particular, the transformation reduces to a translation, the point moves off to infinity.

4. We shall now find the path-curves of the general transformation of the G_8. From equation (1) in Art. 3 we see that they are given by

$$\frac{dx}{ax + by + c} = \frac{dy}{a_1x - ay + c_1} = \frac{dz}{a_2z^2 + b_2z + c_2}\,.$$

The equations in x, y can be made homogeneous by substituting

$$\bar{x} = x - h\,, \quad \bar{y} = y - k\,,$$

where h and k are determined from the equations

$$ah + bk + c = 0$$

$$a_1h - ak + c_1 = 0\,.$$

We then have

$$(a_1\bar{x} - a\bar{y})\, d\bar{x} - (a\bar{x} + b\bar{y})\, d\bar{x} = 0\,.$$

If now we put in this equation

$$\bar{y} = z\bar{x}\,,$$

it has the form

$$(a_1 - 2az - bz^2)\, d\bar{x} - (a + bz)\,\bar{x}dz = 0\,,$$

or

$$\frac{d\bar{x}}{\bar{x}} + \tfrac{1}{2}\,\frac{(-2a - 2bz)\, dz}{a_1 - 2az - bz^2} = 0\,.$$

Hence

$$\log \bar{x} + \tfrac{1}{2} \log (a_1 - 2az - bz^2) = \tfrac{1}{2} \log A\,;$$

or substituting

$$z = \frac{\bar{y}}{\bar{x}}\,,$$

we easily find

$$a_1\bar{x}^2 - 2a\bar{x}\bar{y} - b\bar{y}^2 = A\,.$$

Lastly, substituting for $\bar{x}$ and $\bar{y}$ their values $x - h$ and $y - k$, respectively, this equation has the form

$$a_1x^2 - by^2 - 2axy + (2ak - 2a_1h)\, x + (2ha + 2kb)\, y + a_1h^2 - 2ahk - bk^2 - A = 0\,. \quad (2)$$

If now we find $\bar{x}$ in terms of $\bar{y}$, from

$$a_1\bar{x}^2 - 2a\overline{xy} - b\bar{y}^2 = A\,,$$

we have

$$a_1\bar{x} = a\bar{y} + \sqrt{\bar{y}^2(1 + a_1b) + Aa_1}\,;$$

which substituted in

$$\frac{d\bar{y}}{a_1x - a\bar{y}}$$

gives

$$\frac{d\bar{y}}{\sqrt{\bar{y}^2(1 + a_1b) + Aa_1}}\,.$$

Hence we get the other integral function of our differential equation from

$$\frac{d\bar{y}}{(1+a_1b)^{\frac{1}{2}}\sqrt{\bar{y}^2+\dfrac{Aa_1}{1+a_1b}}}-\frac{dz}{a_2z^2+b_2z+c_2}=0\,.$$

If $a_2c_2<\dfrac{b_2^2}{4}$, it is easily seen that the integral of this equation is

$$\frac{1}{(1+a_1b)^{\frac{1}{2}}}\log\left\{\bar{y}+\sqrt{\bar{y}^2+\frac{Aa_1}{1+a_1b}}\right\}-\frac{1}{(b_2^2-4a_2c_2)^{\frac{1}{2}}}\log\frac{2a_2z+b_2-\sqrt{b_2^2-4a_2c_2}}{2a_2z+b_2+\sqrt{b_2^2-4a_2c_2}}=\log B\,.$$

Hence

$$\left\{\bar{y}+\sqrt{\bar{y}^2+\frac{Aa_1}{1+a_1b}}\right\}^{\frac{1}{(1+a_1b)^{1/2}}}=B\left\{\frac{2a_2z+b_2-\sqrt{b_2^2-4a_2c_2}}{2a_2z+b_2+\sqrt{b_2^2-4a_2c_2}}\right\}^{\frac{1}{(b_2^2-4a_2c_2)^{1/2}}};$$

or substituting for $\bar{y}$ and A their values,

$$\left\{(y-k)+\sqrt{\frac{(y-k)^2-2a_1a\,(x-h)\,(y-k)+a_1^2\,(x-h)^2}{1+a_1b}}\right\}^{\frac{1}{(1+a_1b)^{1/2}}}=B\left\{\frac{2a_2z+b_2-\sqrt{b_2^2-4a_2c_2}}{2a_2z+b_2+\sqrt{b_2^2-4a_2c_2}}\right\}^{\frac{1}{(b_2^2-4a_2c_2)^{1/2}}}. \quad (3)$$

We shall not consider the case when $a_2c_2>\dfrac{b_2^2}{4}$, since in so doing no simple results are obtained.

Equations (2) and (3) taken together represent a family of curves which are called the *path-curves.* This family, of course, is invariant in such manner that each curve is absolutely invariant while the points are interchanged among each other.

5. It is interesting to observe that equation (2) represents a family of cylindrical surfaces whose traces in the xy-plane are conic sections. If we represent the coordinates of the centre of these conics by (x', y') and find them in the usual way, it is easily seen that

$$x'=-\frac{a\,(ha+kb)+b\,(-ha_1+ka)}{-(a_1b+a^2)}=h\,,$$

$$y'=-\frac{a\,(-ha_1+ka)-a_1\,(ha+kb)}{-(a_1b+a^2)}=k\,,$$

where, as we have seen, h and k are found from the equations

$$ah+bk+c=0\,,$$

$$a_1h-ak+c_1=0\,.$$

Hence

$$x' = h = -\frac{ac + bc_1}{a^2 + a_1b}; \quad y' = k = \frac{ac_1 - a_1c}{a^2 + a_1b}.$$

Now this is the point which we found to be absolutely invariant under Xf, in the xy-plane. That is, the conics which we found as the path-curves in the xy-plane have for their centre the point which is absolutely invariant.

A number of interesting problems suggest themselves in connection with this part of the subject. However we shall conclude this section with a remark on the invariants* of n points under the transformations of the G_8.

6. If n points have an invariant under the G_8, it is clear that for some function $f(x_i, y_i, z_i)$ we must have for each transformation of the G_8:

$$\delta f = \Sigma \frac{\partial f}{\partial x_i}\delta x_i + \Sigma \frac{\partial f}{\partial y_i}\delta y_i + \Sigma \frac{\partial f}{\partial z_i}\delta z_i = 0.$$

It is also clear that an invariant of the transformations containing x and y is, at the same time, invariant under those containing z only, and *vice versa*. We have then from (1), omitting for the present the equations containing z,

$$X_1 f \equiv \sum_1^n {}_i p_i = 0, \qquad X_2 f \equiv \sum_1^n {}_i q_i = 0, \qquad X_4 f \equiv \sum_1^n {}_i x_i q_i = 0,$$

$$X_5 f \equiv \sum_1^n {}_i y_i p_i = 0, \qquad X_7 f \equiv \sum_1^n {}_i x_i p_i - \sum_1^n {}_i y_i p_i = 0.$$

This is a complete system of five members in $2n$ variables, so that there are $2n - 5$ solutions. The solutions common to $X_1 f = 0$ and $X_2 f = 0$ are seen to be

$$x_1 - x_i = u_i \text{ and } y_1 - y_i = u_j, \text{ where } i = 2 \ldots n, j = i + n - 2.$$

If now these solutions be introduced into the other equations, the latter become

$$X_4 f \equiv \Sigma u_i \frac{\partial f}{\partial u_j} = 0$$

$$X_5 f \equiv \Sigma u_j \frac{\partial f}{\partial u_i} = 0$$

$$X_7 f \equiv \Sigma u_i \frac{\partial f}{\partial u_i} - \Sigma u_j \frac{\partial f}{\partial u_j} = 0, \; i = 1, \ldots, n-1, j = i, + n - 1.$$

The solutions of $X_7 f = 0$ are

$$\frac{u_1}{u_i}, \; u_1 u_j, \quad \text{where } i = 2 \ldots n-1, \; j = n \ldots 2n-3.$$

* Lie's Continuierliche Gruppen.

Writing $\frac{u_1}{u_2} \ldots u_1 u_{2n-3} = v_1 \ldots v_{2n-3}$, the two remaining equations become expressed in terms of the v_i

$$X_4 f \equiv \Sigma \frac{1}{v_i}\frac{\partial f}{\partial v_j}, \quad i = 0 \ldots n-2, \; j = i + n - 1, \; v_0 = 1$$

$$X_5 f \equiv \Sigma (v_i v_{n-1} - v_i^2 v_j)\frac{\partial f}{\partial v_i} + \Sigma (v_{n-1} v_s)\frac{\partial f}{\partial v_s} = 0\,,$$
$$i = 1 \ldots n-2, \, j = i + n - 1, \; s = n - 1 \ldots 2n - 3.$$

The solutions of $X_4 f = 0$ are

$$v_i, \; v_{n-1} - v_i v_{j+n-1}\,, \qquad \text{where } i = 1 \ldots n-2.$$

Finally writing for $v_1 \ldots v_{n-1} - v_{n-1}\, v_{2n-3}$ respectively $w_1 \ldots w_{2n-3}$, and introducing these solutions into $X_4 f = 0$, we have

$$X_4 f \equiv \Sigma w_i w_j \frac{\partial f}{\partial w_i} + \Sigma w_j^2 \frac{\partial f}{\partial w_j} = 0\,, \quad i = 1 \ldots n-2, \; j = i + n - 2.$$

Hence we find that $n - 2$ of the common solutions have the forms

$$\Delta_i = \frac{w_{i+n-1}}{w_i}\,, \qquad i = 1 \ldots n-2\,;$$

while $n - 3$ have the forms

$$Q_s = \frac{w_{n-1} - w_s}{w_{n-1} w_s}\,, \qquad s = n \ldots 2n - 4.$$

The Δ_i considered in the plane are nothing but the double areas of triangles; and it is easily seen that we can form $n - 3$ other independent functions of the Δ_i and Q_s which will also be double areas of triangles.

Considered in space, since the above solutions satisfy the equations in z also, these results show that the projections of all areas on the xy-plane remain invariant under all transformations of the G_8.

As we have said, the solutions obtained from the equations in z will also be invariants of the whole group. Here, as we shall see, it is only necessary to consider the case with four variables, whence general results may be obtained by inspection. We have then

$$X_6 f \equiv r_1 + r_2 + r_3 + r_4 = 0$$
$$X_7 f \equiv z_1 r_1 + z_2 r_2 + z_3 r_3 + z_4 r_4 = 0$$
$$X_8 f \equiv z_1^2 r_1 + z_2^2 r_2 + z_3^2 r_3 + z_4^2 r_4 = 0\,.$$

The solutions common to $X_0 f = 0$ and $X_7 f = 0$ are easily found to be

$$\frac{z_1 - z_2}{z_2 - z_3} = v_1 , \quad \frac{z_3 - z_4}{z_2 - z_3} = v_2 .$$

The remaining equation, when these variables are introduced, takes the form

$$v_1 (v_1 + 1) \frac{\partial f}{\partial v_1} - v_2 (v_2 + 1) \frac{\partial f}{\partial v_2} = 0 .$$

The solution to this equation is

$$\frac{v_1}{v_1 + 1} \cdot \frac{v_2}{v_2 + 1} = \left[\frac{z_1 - z_2}{z_1 - z_3}\right] \left[\frac{z_3 - z_4}{z_2 - z_4}\right],$$

which is the *anharmonic ratio* of the z ordinates of the four points.

It is clear that if we consider n points, the results will be $n - 3$ independent anharmonic ratios of the n points taken four at a time. This is equivalent to saying that the anharmonic ratio of any four planes parallel to the xy-plane is invariant under the G_8. Thus we find that n points have $3n - 8$ invariants under the G_8, of which $n - 3$ contain only the variables z_i, while the others contain only the x_i and y_i.

If now we consider only two points, the complete system consists of eight members in only six variables, and therefore has no solution. We can, however, form the matrix of the equations and determine whether or not any relations exist between the elements such as will make all the six row determinants vanish. Such relations, if any exist, may be seen to be invariant. The matrix is

$$\begin{vmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & x_1 & x_2 & 0 & 0 \\ y_1 & y_2 & 0 & 0 & 0 & 0 \\ x_1 & x_2 & -y_1 & -y_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & z_1 & z_2 \\ 0 & 0 & 0 & 0 & z_1^2 & z_2^2 \end{vmatrix} .$$

It is readily seen that if $x_1 = x_2$ and $y_1 = y_2$ at the same time, all the six row determinants of this matrix will vanish, since in each two columns will be identical. Also if $z_1 = z_2$ each determinant will vanish for the same reason. We therefore have two invariant relations, namely, $z_1 = z_2$, and the simultaneous relations $x_1 = x_2$ and $y_1 = y_2$.

Now the transformation

$$(ax + by + c)\,p + (a_1x - ay + c_1)\,q$$

in the xy-plane leaves the point given by

$$\left.\begin{aligned} ax + by + c &= 0 \\ a_1x - ay + c_1 &= 0 \end{aligned}\right\} \tag{1}$$

invariant. Let this point be x', y'. If another point x_2, y_2 is held, both x', y' and x_2, y_2 must satisfy (1). But this is impossible unless $a^2 + a_1b = 0$; that is, the equations would not be independent. Hence if we hold x_2, y_2 all points in the xy-plane are absolutely invariant. Hence the invariant simultaneous relations $x_1 = x_2$ and $y_1 = y_2$ mean that if we hold x_2, y_2, the point x, y, z can only move along the line

$$\left.\begin{aligned} x_1 &= x_2 \\ y_1 &= y_2 \end{aligned}\right\}.$$

The invariant relation $z_1 = z_2$ means that if the points lie in a plane before they are transformed, they lie in a plane after they are transformed.

By a somewhat similar process, we could find the invariants of m points and n planes.

§ 2.

7. We will now extend the transformations of the group with a view to finding the *differential invariant* of the lowest order, and subsequently all those of higher orders, where y and z are functions of x. We will in general write

$$\frac{d^ny}{dx^n} = y_n, \quad \frac{d^nz}{dx^n} = z_n.$$

From

$$dy_{n-1} - y_n dx = 0$$

we have at once the increment which y_n receives by means of any infinitesimal transformation in the form

$$\delta y_n = \frac{d\delta y_{n-1}}{dx} - y_n\frac{d\delta x}{dx}; \tag{1}$$

and similarly for z_n

$$\delta z_n = \frac{d\delta z_{n-1}}{dx} - z_n \frac{d\delta x}{dx}. \tag{2}$$

The transformations, as is well known, are extended by means of these formulæ. In order to find the differential invariant of the lowest order, we must extend the transformations three times and equate the results to zero, since that will give a complete system of eight members in nine variables, and therefore one solution. If we extend less than three times, it may be seen that we have more independent equations than variables, so that according to the general theory of the complete system no solution exists. Hence the differential invariant of the lowest order will be of the third order. Thus extending by means of (1) and (2), we have

$$X_1 f \equiv \frac{\partial f}{\partial x} = 0$$

$$X_2 f \equiv \frac{\partial f}{\partial y} = 0$$

$$X_3 f \equiv \frac{\partial f}{\partial z} = 0$$

$$X_4 f \equiv x \frac{\partial f}{\partial y} + \frac{\partial f}{\partial y_1} = 0$$

$$\begin{aligned} X_5 f \equiv y \frac{\partial f}{\partial x} - y_1^2 \frac{\partial f}{\partial y_1} - y_1 z_1 \frac{\partial f}{\partial z_1} - 3y_1 y_2 \frac{\partial f}{\partial y_2} - (y_2 z_1 + 2z_2 y_1) \frac{\partial f}{\partial z_2} \\ - (4y_1 y_3 + 3y_2^2) \frac{\partial f}{\partial y_3} - (y_3 z_1 + 3y_2 z_2 + 3y_1 z_3) \frac{\partial f}{\partial z_3} = 0 \end{aligned}$$

$$X_6 f \equiv z \frac{\partial f}{\partial z} + z_1 \frac{\partial f}{\partial z_1} + z_2 \frac{\partial f}{\partial z_2} + z_3 \frac{\partial f}{\partial z_3} = 0$$

$$\begin{aligned} X_7 f \equiv x \frac{\partial f}{\partial x} - y \frac{\partial f}{\partial y} - 2y_1 \frac{\partial f}{\partial y_1} - z_1 \frac{\partial f}{\partial z_1} - 3y_2 \frac{\partial f}{\partial y_2} - 2z_2 \frac{\partial f}{\partial z_2} - 4y_3 \frac{\partial f}{\partial y_3} \\ - 3z_3 \frac{\partial f}{\partial z_3} = 0 \end{aligned}$$

$$X_8 f \equiv z^2 \frac{\partial f}{\partial z} + 2zz_1 \frac{\partial f}{\partial z_1} + 2(z_1^2 + zz_2) \frac{\partial f}{\partial z_2} + 2(3z_1 z_2 + zz_3) \frac{\partial f}{\partial z_3} = 0.$$

The first four equations show that x, y, z, y_1 are not contained in the solution sought. The complete system can thus be written in the form

$$X_5 f \equiv y_1 z_1 \frac{\partial f}{\partial z_1} + 3y_1 y_2 \frac{\partial f}{\partial y_2} + (y_2 z_1 + 2y_1 z_2) \frac{\partial f}{\partial z_2} + (4y_1 y_3 + 3y_2^2) \frac{\partial f}{\partial y_3}$$
$$+ (y_3 z_1 + 3y_2 z_2 + 3y_1 z_3) \frac{\partial f}{\partial z_3} = 0$$

$$X_6 f = z_1 \frac{\partial f}{\partial z_1} + z_2 \frac{\partial f}{\partial z_2} + z_3 \frac{\partial f}{\partial z_3} = 0$$

$$X_7 f = z_1 \frac{\partial f}{\partial z_1} + 3y_2 \frac{\partial f}{\partial y_2} + 2z_2 \frac{\partial f}{\partial z_2} + 4y_3 \frac{\partial f}{\partial y_3} + 3z_3 \frac{\partial f}{\partial z_3} = 0$$

$$X_8 f = z z_1 \frac{\partial f}{\partial z_1} + (z_1^2 + z z_2) \frac{\partial f}{\partial z_2} + (3z_1 z_2 + z z_3) \frac{\partial f}{\partial z_3} = 0\,.$$

These equations can be greatly simplified by algebraic reduction. Replace $X_5 f = 0$ by $X_5 f - y_1 X_7 f = 0$ and $X_8 f = 0$ by $X_8 f - z X_6 f = 0$. We then have

$$X_5 f = y_2 z_1 \frac{\partial f}{\partial z_2} + 3y_2^2 \frac{\partial f}{\partial y_3} + (y_3 z_1 + 3y_2 z_2) \frac{\partial f}{\partial z_3} = 0$$

$$X_6 f = z_1 \frac{\partial f}{\partial z_1} + z_2 \frac{\partial f}{\partial z_2} + z_3 \frac{\partial f}{\partial z_3} = 0$$

$$X_7 f = z_1 \frac{\partial f}{\partial z_1} + 3y_2 \frac{\partial f}{\partial y_2} + 2z_2 \frac{\partial f}{\partial z_2} + 4y_3 \frac{\partial f}{\partial y_3} + 3z_3 \frac{\partial f}{\partial z_3} = 0$$

$$X_8 f = z_1 \frac{\partial f}{\partial z_2} + 3z_2 \frac{\partial f}{\partial z_3} = 0\,.$$

Again, replace in these equations $X_5 f = 0$ by $X_5 f - y_2 X_8 f = 0$ and $X_7 f = 0$ by $X_7 f - X_6 f = 0$, and the equations are reduced to the forms

$$X_5 f = 3y_2^2 \frac{\partial f}{\partial y_3} + y_3 z_1 \frac{\partial f}{\partial z_3} = 0$$

$$X_6 f = z_1 \frac{\partial f}{\partial z_1} + z_2 \frac{\partial f}{\partial z_2} + z_3 \frac{\partial f}{\partial z_3} = 0$$

$$X_7 f = 3y_2 \frac{\partial f}{\partial y_2} + z_2 \frac{\partial f}{\partial z_2} + 4y_3 \frac{\partial f}{\partial y_3} + 2z_3 \frac{\partial f}{\partial z_3} = 0$$

$$X_8 f = z_1 \frac{\partial f}{\partial z_2} + 3z_2 \frac{\partial f}{\partial z_3} = 0\,.$$

It now remains to find the solution to this complete system. As an explanation of the method by which we shall proceed, we recall the following theorem from the theory of the complete system.

If $A_1 f = 0 \ldots A_r f = 0$ *form a complete system in the variables* $x_1 \ldots x_n$, *the integration of the same can be accomplished in the following manner. We seek the solutions* $\varphi_1 \ldots \varphi_{n-1}$ *of* $A_1 f = 0$; *then form*

$$A_2 f = A_2\varphi_1 \frac{\partial f}{\partial \varphi_1} + \ldots + A_2\varphi_{n-1} \frac{\partial f}{\partial \varphi_{n-1}} = 0 .$$

If the ratios of the $A_2\varphi_k$ *are not functions of* $\varphi_1 \ldots \varphi_{n-1}$ *alone, the equation* $A_2 f = 0$ *will always break up into several equations. We integrate one of these and introduce its solutions* $\psi_1 \ldots \psi_{n-2}$ *into* $A_3 f = 0$. *The resulting equation*

$$A_3\psi_1 \frac{\partial f}{\partial \psi_1} + A_3\psi_2 \frac{\partial f}{\partial \psi_2} + \ldots + A_3\psi_{n-2} \frac{\partial f}{\partial \psi_{n-2}} = 0$$

is treated in a similar manner, and so on. If $r < n$ *we find ultimately the* $n - r$ *solutions of the complete system.*

Now the solutions of $X_5 f = 0$ are

$$z_1,\ y_2,\ z_2,\ \text{and}\ y_3^2 z_1 - 6 y_2^2 z_3 - u .$$

Introducing these solutions as above indicated, the remaining equations assume the forms

$$X_6 f = z_1 \frac{\partial f}{\partial z_1} + z_2 \frac{\partial f}{\partial z_2} + u \frac{\partial f}{\partial u} = 0$$

$$X_7 f = 3y_2 \frac{\partial f}{\partial y_2} + z_2 \frac{\partial f}{\partial z_2} + 8u \frac{\partial f}{\partial u} = 0$$

$$X_8 f = z_1 \frac{\partial f}{\partial z_2} - 18 y_2^2 z_2 \frac{\partial f}{\partial u} = 0 .$$

The solutions to $X_6 f = 0$ are easily found to be

$$y_2,\ \frac{z_2}{z_1} - v,\ \frac{u}{z_1} - w .$$

Introduce these into the other two equations and they have the forms

$$X_7 f = 3y_2 \frac{\partial f}{\partial y_2} + v \frac{\partial f}{\partial v} + 8w \frac{\partial f}{\partial w} = 0$$

$$X_8 f = \frac{\partial f}{\partial v} - 18 y_2^2 v \frac{\partial f}{\partial w} = 0 .$$

Finally the solutions of $X_8 f = 0$ are

$$y_2 \text{ and } 9y_2 v^2 + w = v_1 ;$$

and $X_7 f$ becomes

$$X_7 f = 3y_2 \frac{\partial f}{\partial y_2} + 8v_1 \frac{\partial f}{\partial v_1} = 0 .$$

Hence the common solution of all the equations, that is the solution of the complete system, is found to be

$$I_3 = \frac{v_1}{y_2^{8/3}} = \frac{9y_2^2z_2^2 + z_1(z_1y_3^2 - 6y_2^2z_3)}{z_1^2y_2^{8/3}}.$$

This then is the differential invariant of the lowest order. Those of higher orders can be found by further extending the transformations, and for future use we shall find two of the fourth order. Extending as above, and omitting terms containing x, y, z, y_1, since these variables do not occur in the solutions, we have

$$X_5f = y_1z_1\frac{\partial f}{\partial z_1} + 3y_1y_2\frac{\partial f}{\partial y_2} + (y_2z_1 + 2y_1z_2)\frac{\partial f}{\partial z_2} + (4y_1y_3 + 3y_2^2)\frac{\partial f}{\partial y_3}$$
$$+ (y_3z_1 + 3y_2z_2 + 3y_1z_3)\frac{\partial f}{\partial z_3} + (5y_1y_4 + 10y_2y_3)\frac{\partial f}{\partial y_4} = 0$$

$$X_6f = z_1\frac{\partial f}{\partial z_1} + z_2\frac{\partial f}{\partial z_2} + z_3\frac{\partial f}{\partial z_3} = 0$$

$$X_7f = z_1\frac{\partial f}{\partial z_1} + 3y_2\frac{\partial f}{\partial y_2} + 2z_2\frac{\partial f}{\partial z_2} + 4y_3\frac{\partial f}{\partial y_3} + 3z_3\frac{\partial f}{\partial z_3} + 5y_4\frac{\partial f}{\partial y_4} = 0$$

$$X_8f = zz_1\frac{\partial f}{\partial z_1} + (z_1^2 + zz_2)\frac{\partial f}{\partial z_2} + (3z_1z_2 + zz_3)\frac{\partial f}{\partial z_3} = 0.$$

These equations can be simplified exactly as in the preceding case, and we have after this reduction

$$X_5f = 3y_2^2\frac{\partial f}{\partial y_3} + y_3z_1\frac{\partial f}{\partial z_3} + 10y_2y_3\frac{\partial f}{\partial y_4} = 0$$

$$X_6f = z_1\frac{\partial f}{\partial z_1} + z_2\frac{\partial f}{\partial z_2} + z_3\frac{\partial f}{\partial z_3} = 0$$

$$X_7f = 3y_2\frac{\partial f}{\partial y_2} + z_2\frac{\partial f}{\partial z_2} + 4y_3\frac{\partial f}{\partial y_3} + 2z_3\frac{\partial f}{\partial z_3} + 5y_4\frac{\partial f}{\partial y_4} = 0$$

$$X_8f = z_1\frac{\partial f}{\partial z_2} + 3z_2\frac{\partial f}{\partial z_3} = 0.$$

The solutions of $X_8f = 0$ are

$$z_1,\ y_2,\ y_3,\ y_4, \text{ and } 3z_2^2 - 2z_1z_3 = u.$$

These solutions introduced into the other equations give

$$X'_{5}f = 3y_2^2 \frac{\partial f}{\partial y_3} + 10y_2y_3 \frac{\partial f}{\partial y_3} - 2y_3z_1^2 \frac{\partial f}{\partial u} = 0$$

$$X'_{6}f = z_1 \frac{\partial f}{\partial z_1} + 2u \frac{\partial f}{\partial u} = 0$$

$$X'_{7}f = 3y_2 \frac{\partial f}{\partial y_2} + 4y_3 \frac{\partial f}{\partial y_3} + 5y_4 \frac{\partial f}{\partial y_4} + 2u \frac{\partial f}{\partial u} = 0 .$$

If we next take $X'_{6}f = 0$ we have as solutions

$$y_2,\ y_3,\ y_4,\ \frac{u}{z_1^2} = v\ ;$$

and in these variables the other two equations have the forms

$$X'_{5}f = 3y_2^2 \frac{\partial f}{\partial y_3} + 10y_2y_3 \frac{\partial f}{\partial y_4} - 2y_3 \frac{\partial f}{\partial v} = 0$$

$$X'_{7}f = 3y_2 \frac{\partial f}{\partial y_2} + 4y_3 \frac{\partial f}{\partial y_3} + 5y_4 \frac{\partial f}{\partial y_4} + 2v \frac{\partial f}{\partial v} = 0 .$$

We next find the solutions of $X'_{5}f = 0$ to be

$$y_2,\ 5y_3^2 - 3y_2y_4 = m,\ y_3^2 + 3y_2^2v = n\ ;$$

and $X'_{7}f$ assumes the form

$$X'_{7}f = 3y_2 \frac{\partial f}{\partial y_2} + 8m \frac{\partial f}{\partial m} + 8n \frac{\partial f}{\partial n} = 0 .$$

Hence we find

$$J_4 = \frac{m}{y_2^{8/3}} = \frac{5y_3^2 - 3y_2y_4}{y_2^{8/3}}, \quad I_4 = \frac{m}{n} = \frac{(5y_3^2 - 3y_2y_4)z_1^2}{z_1^2y_3^2 + 3y_2^2(3z_2^2 - 2z_1z_3)} .$$

This process of extending the transformations could be kept up indefinitely, and the differential invariants of any desired order could be found, theoretically; but such a process would soon become very cumbersome, so that we shall in the next article give a more convenient method by which all the differential invariants of an order higher than the fourth can be found by mere differentiation.

8. Let φ be a differential invariant of the lowest order of the G_8; we now seek a function $\Omega\left[x, y, z, y_1, z_1, \ldots, \varphi, \frac{d\varphi}{dx}\right]$ which shall be a differential invariant whenever φ is.*

* Lie's Continuierliche Gruppen, Kap. 22.

Indicating by δ the increment received by means of an infinitesimal transformation of the group, since

$$d\varphi = \varphi' dx$$

we have

$$\delta d\varphi = \delta\varphi' . dx + \varphi' . \delta dx$$

or since d and δ can be interchanged,

$$\delta\varphi' = \frac{d\delta\varphi}{dx} - \varphi' \frac{d\delta x}{dx} .$$

Now since φ is to be invariant, it receives no increment; that is $\delta\varphi = 0$. Hence

$$\delta\varphi' = - \varphi' \frac{d\delta x}{dx} .$$

If in this equation we substitute for δx its values as given by the transformations p, q, r, xq, we find in each case that $\delta\varphi' = 0$. For yp, we have $\delta x = y$, and therefore $\delta\varphi' = - \varphi' y_1$. In the same way we may find the increments which φ' receives by means of the other transformations.

Extending now the transformations so as to have a complete system of eight members in nine variables, and omitting terms (Art. 7) containing $\frac{\partial\Omega}{\partial x}$, $\frac{\partial\Omega}{\partial y}$, $\frac{\partial\Omega}{\partial z}$, $\frac{\partial\Omega}{\partial y_1}$, it is easily seen that one complete system has the form

$$X_5 f = y_1 z_1 \frac{\partial\Omega}{\partial z_1} + 3y_1 y_2 \frac{\partial\Omega}{\partial y_2} + (y_2 z_1 + 2y_1 z_2) \frac{\partial\Omega}{\partial z_2} + (4y_1 y_3 + 3y_2^2) \frac{\partial\Omega}{\partial y_3} + \varphi' y_1 \frac{\partial\Omega}{\partial\varphi'} = 0$$

$$X_6 f = z_1 \frac{\partial\Omega}{\partial z_1} + z_2 \frac{\Omega}{\partial z_2} = 0$$

$$X_7 f = z_1 \frac{\partial\Omega}{\partial z_1} + 3y_2 \frac{\partial\Omega}{\partial y_2} + 2z_2 \frac{\partial\Omega}{\partial z_2} + 4y_3 \frac{\partial\Omega}{\partial y_3} + \varphi' \frac{\partial\Omega}{\partial\varphi'} = 0$$

$$X_8 f = zz_1 \frac{\partial\Omega}{\partial z_1} + (zz_2 + z_1^2) \frac{\partial\Omega}{\partial z_2} = 0 .$$

In these equations replace $X_5 f = 0$ by $X_5 f - y_1 X_7 f = 0$, and $X_8 f = 0$ by $X_8 f - z X_6 f = 0$. We then have

$$X_5 f = y_2 z_1 \frac{\partial\Omega}{\partial z_2} + 3y_2^2 \frac{\partial\Omega}{\partial y_3} = 0$$

$$X_6 f \equiv z_1 \frac{\partial\Omega}{\partial z_1} + z_2 \frac{\partial\Omega}{\partial z_2} = 0$$

$$X_7 f = z_1 \frac{\partial\Omega}{\partial z_1} + 3y_2 \frac{\partial\Omega}{\partial y_2} + 2z_2 \frac{\partial\Omega}{\partial z_2} + 4y_3 \frac{\partial\Omega}{\partial y_3} + \varphi' \frac{\partial\Omega}{\partial\varphi'} = 0$$

$$X_8 f = z_1^2 \frac{\partial\Omega}{\partial z_2} = 0 .$$

The last equation shows that z_2 is not contained in the solution; and since this is the case, the first and second show that the solution is also free of z_1 and y_3. Hence, we find the common solution from

$$\frac{dy_2}{3y_2} = \frac{d\varphi'}{\varphi'},$$

which we write in the form

$$\Delta\varphi = \frac{\varphi'}{y_2^{1/3}}.$$

Hence, whenever φ is a differential invariant, $\Delta\varphi$ is a differential invariant of the next higher order. Therefore, by means of $\Delta\varphi$, which is called the *differential parameter*, we can find from any differential invariant one of a higher order by simply differentiating the given differential invariant totally with respect to x and multiplying the result by $1/y_2^{1/3}$. Thus

$$\Delta(J_4) = \frac{45y_2y_3y_4 - 9y_2^2y_5 - 40y_3^3}{3y_2^4} = J_5$$

is a differential invariant of the fifth order. Also

$$\Delta(I_4) = I_5$$

is a differential invariant of the fifth order, which is clearly independent of J_5.

In the same manner we see that

$$\Delta(\Delta(J_4)) = \Delta^2(J_4) = \Delta(J_5)$$

is a differential invariant of the sixth order; and universally

$$\Delta^n(J_4) \text{ and } \Delta^n(I_4)$$

are two independent differential invariants of the $(n + 4)$th order. Thus it is clear that we can write down all the differential invariants of any required order.

We have reserved the finding of the invariant differential equations and invariant systems from the determinants of the equations forming the complete systems for the next section, wherein we shall also discuss the equivalence of curves.

§ 3.

Equivalence of Curves.

9. If we extend the transformations of our G_8 once, it is easily seen from the determinants of the matrix so formed that the only invariant differential equation of the $I.\ 0$ is $z_1 = 0$.

If we extend twice, we have the matrix

$$\left\| \begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & x & 0 & 1 & 0 & 0 & 0 \\ y & 0 & 0 & -y_1^2 & -y_1 z_1 & -3y_1 y_2 & -(y_2 z_1 + 2y_1 z_2) \\ 0 & 0 & z & 0 & z_1 & 0 & z_2 \\ x & -y & 0 & -2y_1 & -z_1 & -3y_2 & -2z_2 \\ 0 & 0 & z^2 & 0 & 2zz_1 & 0 & 2(z_1^2 + zz_2) \end{array} \right\|.$$

Indicating by $\varDelta_i$ the determinant formed from this matrix by suppressing the ith row, we easily find

$$\varDelta_8 = 3y_2^2 z_1^2, \quad \varDelta_7 = 2z_1^3 y_1 y_2, \quad \varDelta_6 = -6y_2^2 z z_1^2, \quad \varDelta_5 = -6y_2 z_1^3, \quad \varDelta_4 = 6y_1^2 y_2 z_1^3,$$

$$\varDelta_3 = -3y_2^2 z_1^3 z^2, \quad \varDelta_2 = 6y_1 y_2 z_1 (xy_1 - y), \quad \varDelta_1 = 6y_2 z_1^3 (y - xy_1).$$

Since z_1 is the only factor common to all these determinants, the only invariant differential equation is $z_1 = 0$. If, however, we write $I_3 = \infty$, we find that $y_2 = 0$ is an invariant equation of $H.0$. We also see that $z_1 = y_2 = 0$ and $z_1 = z_2 = 0$ are invariant systems of equations. But $z_2 = 0$ is a consequence of $z_1 = 0$, which gives no new results. *These are all the invariant equations of an order lower than the third.*

If we extend the transformations three times, we have the matrix

$$\left\| \begin{array}{ccccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & x & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ y & 0 & 0 & -y_1^2 & -y_1 z_1 & -3y_1 y_2 & -(y_2 z_1 + 2y_1 y_2) & -(4y_1 y_3 + 3y_2^2) & -(y_3 z_1 + 3y_2 z_2 + 3y_1 z_3) \\ 0 & 0 & z & 0 & z_1 & 0 & z_2 & 0 & z_3 \\ x & -y & 0 & -2y_1 & -z_1 & -3y_2 & -2z_2 & -4y_3 & -3z_3 \\ 0 & 0 & z^2 & 0 & 2zz_1 & 0 & 2(z_1^2 + zz_2) & 0 & 6z_1 z_2 + 2zz_3 \end{array} \right\|$$

Indicating, in this case, by Δ_i the determinant formed by suppressing the ith column, we find

$$\Delta_9 = 18z_1^3y_2^3\,, \qquad \Delta_8 = 3z_1^4y_2y_3\,, \qquad \Delta_7 = 18z_1^2y_2^3z_2\,,$$
$$\Delta_6 = 2z_1(6y_2^2z_3z_1 - 4z_1^2y_3^2 - 9y_2^2z_2^2)\,, \qquad \Delta_5 = 18z_1y_2(y_2^2z_1z_3 - 3z_2^2y_2^2)\,.$$

The others vanish identically. We see from these determinants that the only single invariant equation is $z_1 = 0$, while $z_1 = y_2 = 0$ is an invariant system; and it is clear that these are the only results obtained from further extension of the transformations. Hence, all invariant equations of the third and higher orders are obtained from the differential invariants.

By writing $J_3 = 0$, we find in particular under this head $y_2 = y_3 = 0$ as an invariant system; but as $y_3 = 0$ is a consequence of $y_2 = 0$, this case gives us nothing new.

I: If now we perform the transformations of the G_8 on a curve which admits of *no* transformation of the G_8, this curve generates an invariant family of ∞^8 curves. They may be represented by:

(a) One equation of the zero order and one of the $VIII.0$.

(b) One equation of the $I.0$. and one of the $VII.0$.

(c) One equation of the $II.0$. and one of the $VI.0$.

(d) One equation of the $III.0$. and one of the $V.0$.

(e) Two equations of the $IV.0$.

Since no equation of the zero order exists, case (a) is excluded.

In considering case (b), we find that the only differential equation of the $I.0$. is $z_1 = 0$. Hence the curves are plane curves, lying in the ∞^1 planes $z =$ const. Since no figure in the xy-plane is changed by r, zr, z^2r, the ∞^7 curves are the same in each plane parallel to the xy-plane. Hence the differential equation of the $VII.0$. is one which is invariant under

$$p,\quad q,\quad yp,\quad xp - yq,\quad xq\,;$$

of which the form is

$$F(J_4\,\ldots\,J_7) = 0\,. \tag{1}$$

Lie has shown* how to reduce this equation in the following manner.

It is easily seen that it can be written in the form

$$F\left[J_4 J_5 \frac{dJ_5}{dJ_4}\frac{d^2J_5}{dJ_4^2}\right] = 0\,, \tag{2}$$

where $\qquad J_4 = y_2^{-\frac{8}{3}}\rho_2\,, \quad J_5 = y_2^{-4}\rho_3\,,$

and $\qquad \rho_2 = 3y_2y_4 - 5y_3^2\,, \quad \rho_3 = 3y_2^2y_5 - 15y_2y_3y_4 + \frac{40}{3}y_3^3\,.$

If (2) has been integrated we find an equation of the general form

$$J_5 = \Phi(J_4)\,.$$

---

* Mathematische Annalen, Band XXXII.

Now introduce as new variables J_4, and $U = y_2^{-\frac{4}{3}} y_3$; then we find

$$\frac{dU}{dJ_4} = \frac{1}{3} \frac{\rho_2 + y_3^2}{y_2^{-\frac{4}{3}} \rho_3} = \frac{1}{3} \frac{J_4 + U^2}{J_5},$$

or

$$\frac{dU}{dJ_4} = \frac{1}{3} \frac{J_4 + U^2}{\Phi(J_4)}, \tag{3}$$

which is a Riccati's equation.

If
$$W(UJ_4) = \text{const.}$$
is an integral equation of (3), we can find two other integral equations o $J_5 = \Phi(J_4)$ as follows. Extending yp, we have

$$Xf = y\frac{\partial f}{\partial x} - y_1^2 \frac{\partial f}{\partial y_1} - 3y_1 y_2 \frac{\partial f}{\partial y_2} - (4y_1 y_3 + 3y_2^2) \frac{\partial f}{\partial y_3} - (5y_1 y_4 + 10 y_2 y_3) \frac{\partial f}{\partial y_4};$$

from which we see that

$$X(J_4) = 0, \quad X(U) = -3y_2^{\frac{2}{3}}$$

and

$$X(W) = -3 \frac{\partial W}{\partial U} y_2^{\frac{2}{3}}$$

$$XX(W) = 9y_2^{\frac{4}{3}} \frac{\partial^2 W}{\partial U^2} + 6y_1 y_2^{\frac{2}{3}} \frac{\partial W}{\partial U}.$$

Now

$$W = \text{const.}, \quad X(W) = \text{const.}, \quad XX(W) = \text{const.}$$

represent three independent integral equations of $J_5 = \Phi(J_4)$; and if we eliminate y_5, y_4, y_3, we obtain a differential equation in y_1 and y_2, which can be integrated by two quadratures.

For case (c) we know that the only differential equation of the II. 0. is found by writing $I_3 = \infty$, which gives

$$y_2 = 0.$$

But then all the differential invariants of higher orders either vanish or become infinite; so that this case is excluded.

The two equations under (d), of the III. 0. and V. 0. are

$$I_3 = \frac{9y_2^2 z_2^2 + z_1(z_1 y_3^2 - 6y_2^2 z_3)}{z_1^2 y_2^{8/3}} = \text{const.} \tag{1}$$

and

$$F(J_4, J_5) = 0. \tag{2}$$

The integration of (2) can be reduced as in case (b). If from $F = 0$ we find $y = \varphi(x)$, then, substituting in (1) for y_2 and y_3 their values in terms of x, we have

$$I_3 = \frac{9\varphi_2^2 z_2^2 + z_1(z_1 \varphi_3^2 - 6\varphi_2^2 z_3)}{z_1^2 \varphi_2^{8/3}} = \text{const.},$$

which is a differential equation of the third order in z and x. This equation may be written in the form

$$\frac{z_3}{z_1} - \frac{3}{2}\frac{z_2^2}{z_1^2} = F(x),$$

which, as Lie has shown, can be reduced to a Riccati's equation of the first order.

If we put

$$\frac{z_2}{z_1} = y,$$

then will

$$\frac{dy}{dx} = \frac{z_3}{z_1} - \frac{z_2^2}{z_1^2},$$

or

$$\frac{dy}{dx} = 1/2y^2 + F(x).$$

If $W(yx) =$ const. is an integral equation of this Riccati's equation, we can find the required integral equations of $w = F(x)$ by differentiation. If we write

$$X_8 f \quad z^2 \frac{\partial f}{\partial z} + 2zz_1 \frac{\partial f}{\partial z_1} + (2zz_2 + 2z_1^2)\frac{\partial f}{\partial z_2},$$

then $X_8 W =$ const. and $X_8(X_8 W) =$ const. are known integral equations of $w = F(x)$; it is sufficient, therefore, to show that W, $X_8 W$, and $X_8(X_8 W)$ are independent functions of x, z, z_1 and z_2. From

$$X_8(x) = 0$$

$$X_8 W = \frac{\partial W}{\partial y} X_8(y) = 2\frac{\partial W}{\partial y} z_1$$

$$X_8(X_8 W) = 4\frac{\partial^2 W}{\partial y^2} z_1^2 + 4\frac{\partial W}{\partial y} zz_1$$

we see that W, $X_8 W$, $X_8(X_8 W)$ are really independent with respect to z, x, z_1, and y. Thus the integration of $w = F(x)$ is reduced to the Riccati's equation.

Finally, considering case (e), we see that if the differential invariants are I_3, I_4, J_4, the two equations of the fourth order may be written

$$\Omega_1(I_3, I_4, J_4) = 0, \quad \Omega_2(I_3, I_4, J_4) = 0$$

or

$$\Phi(I_3, I_4) = 0, \quad \Psi(I_3, J_4) = 0.$$

II: If now the curve admits of *one* transformation of the G_8, the invariant family consists of only ∞^7 curves, which are defined

(a) By one equation $I.0.$ and one $VI.0.$; which are $z_1 = 0$, $F(J_4 \ldots J_6) = 0$. This case is similar to case (b) in I.

(b) By one equation $II.0.$ and one $V.0.$; which case, as in I, is excluded.

(c) By one equation $III.0.$ and one $IV.0.$ They are clearly $I_3 = \text{const.}$ and $J_4 = \text{const.}$ The second equation can be integrated. The first can also be expressed in the variables x and z as in case (d) in I.

III. If the curve admits of two transformations, the family consists of ∞^6 curves, which are defined

(a) By one equation $I.0.$ and one $V.0.$

They are $z_1 = 0$ and $F(J_4, J_5) = 0$; and this case is similar to (a) in II.

(b) By one equation of the $II.0.$ and one of the $IV.0.$

This case is again excluded.

(c) By two equations of $III.0.$

Since there is only one invariant equation of the third order, we exclude this case also.

IV. If the curve admits of three transformations there are ∞^5 curves in the family, which are represented by

(a) One equation of the $I.0.$ and one of the $IV.0.$

(b) One equation of the $II.0.$ and one of the $III.0.$

Cases similar to (a) have already been considered. Case (b) gives only $y_2 = 0$ as the invariant equation of the second order. The only invariant system would be $y_2 = 0$, $y_3 = 0$, which is excluded. For $y_3 = 0$ is a consequence of $y_2 = 0$; that is the integration of $y_3 = 0$ must give $y_2 = 0$; otherwise the equations are incompatible. So that we have only one invariant equation.

V. Suppose next that the curve admits of four transformations of the G_8; the family then consists of ∞^4 curves, and is defined by

(a) One equation of the $I.0.$ and one of the $III.0.$

(b) Two equations of the $II.0.$

Case (a) is excluded; for if $z_1 = 0$, I_3 becomes elusive. Also case (b) is excluded, since there do not exist two invariant equations of the second order.

VI. If the curve admits of five transformations, the ∞^3 curves of the family then generated are defined by

(a) One equation $I.0.$ and one $II.0.$

As we have seen, the equation of the first order is $z_1 = 0$. There are two of the second order which we can consider in connection with $z_1 = 0$. They are $y_2 = 0$ and $z_2 = 0$. If we take $y_2 = 0$ and $z_1 = 0$, we find ∞^3 straight lines lying in the ∞^1 planes $z = \text{const.}$ But $z_1 = 0$ and $z_2 = 0$ give us no case, since $z_2 = 0$ is a consequence of $z_1 = 0$.

VII. If the curve admits of six transformations of the group, the ∞^2 curves of the family must be defined by two invariant equations of the first order. But since there is only one invariant of the first order, this case is excluded.

VIII. Lastly we consider the case when the curve admits of seven transformations. But this case is excluded, since the ∞^1 curves of the family would have to be defined by an invariant equation of the zero order in connection with one of the first order, while no invariant equation of the zero order exists.

From the consideration of all these cases we conclude that there are no invariant families of ∞^4, ∞^2, or ∞^1 curves. In other words, no curve admits of exactly four or exactly six or exactly seven transformations of the G_8. Hence, if two curves are equivalent under the G_8, they are either members of families of plane curves defined by $F(J_4 \ldots J_k) = 0$, lying in the planes $z =$ const.; or they are members of a family ∞^8 curves defined by $I_3 =$ const. and $F(J_4, J_5) = 0$, or $\Phi(I_3\, I_4) = 0$ and $\Psi(I_3\, J_4) = 0$; or finally they are members of a family of ∞^7 curves defined by $I_3 =$ const. and $J_4 =$ const.

§ 4.

10. Having disposed of the problem concerning invariant curve-families as defined by invariant differential equations, the next step which most naturally suggests itself is the corresponding investigation for invariant families of *surfaces*. We shall accordingly devote this section to finding all the differential invariants which express properties of surfaces, and invariant partial differential equations.

Here z must be considered as some function of x and y, say

$$z = f(xy)\,.$$

We shall in general write

$$\frac{\partial z}{\partial x} = p\,,\quad \frac{\partial z}{\partial y} = q\,,\quad \frac{\partial^2 z}{\partial x^2} = r\,,\quad \frac{\partial^2 z}{\partial x \partial y} = s\,,\quad \frac{\partial^2 z}{\partial y^2} = t\,,$$
$$\frac{\partial^3 z}{\partial x^3} = \rho\,,\quad \frac{\partial^3 z}{\partial x^2 \partial y} = \sigma\,,\quad \frac{\partial^3 z}{\partial x \partial y^2} = \mu\,,\quad \frac{\partial^3 z}{\partial y^3} = \lambda\,;$$

and find in the following manner the increments received by the functions p, q, etc., under the transformations of the G_8.

If then

$$z = f(xy)\,,$$

we have

$$dz = p dx + q dy\,,$$

and hence

$$d\delta z = \delta p \,.\, dx + \delta q \,.\, dy + p \,.\, d\delta x + q \,.\, d\delta y\,. \tag{1}$$

In extending the transformations of the group by means of (1) we shall, as usual, preserve the order in which they occur in Section I.

It is clear that $\delta p = \delta q = 0$ for $X_1 f$, $X_2 f$, and $X_3 f$.

For $X_4 f$, since $\partial z = 0$, we have

$$0 = \partial p \,.\, dx + \partial q \,.\, dy + q dx = (\partial p + q)\, dx + \partial q \,.\, dy\,;$$

from which we find

$$\partial p = -q\,, \quad \partial q = 0\,.$$

In the same way we see that for $X_5 f$

$$\partial p = 0\,, \quad \partial q = -p\,.$$

For $X_6 f$ we have $\partial z = z$. Hence

$$dz = p dx + q dy = \partial p \,.\, dx + \partial q \,.\, dy\,,$$

or

$$(\partial p - p)\, dx + (\partial q - q)\, dy = 0\,;$$

which shows that

$$\partial p = p\,, \quad \partial q = q\,.$$

In exactly the same way the increments of p and q for $X_7 f$ and $X_8 f$, respectively, are found to be

$$\partial p = -p\,, \quad \partial q = q\,, \quad \text{and} \quad \partial p = 2zp\,, \quad \partial q = 2zq\,.$$

Hence the transformations *once* extended may be written

$$X_1 f = \frac{\partial f}{\partial x}$$

$$X_2 f = \frac{\partial f}{\partial y}$$

$$X_3 f = \frac{\partial f}{\partial z}$$

$$X_4 f = x \frac{\partial f}{\partial y} - q \frac{\partial f}{\partial p}$$

$$X_5 f = y \frac{\partial f}{\partial x} - p \frac{\partial f}{\partial q}$$

$$X_6 f = z \frac{\partial f}{\partial z} + p \frac{\partial f}{\partial p} + q \frac{\partial f}{\partial q}$$

$$X_7 f = x \frac{\partial f}{\partial x} - y \frac{\partial f}{\partial y} - p \frac{\partial f}{\partial p} + q \frac{\partial f}{\partial q}$$

$$X_8 f = z^2 \frac{\partial f}{\partial z} + 2zp \frac{\partial f}{\partial p} + 2zq \frac{\partial f}{\partial q}\,.$$

To write $X_1 f = 0 \ldots X_8 f = 0$ would give no result, since then the first three equations would show that the invariant function f would have to be free of x, y, z and consequently the others would show that it would have to be free of p and q also.

Let us then extend the transformations again. We know that

$$dp = r dx + s dy\,, \quad dq = s dx + t dy\,;$$

so that
$$\left.\begin{aligned} d\delta p &= \delta r\,.\,dx + \delta s\,.\,dy + r\,.\,d\delta x + s\,.\,d\delta y \\ d\delta q &= \delta s\,.\,dx + \delta t\,.\,dy + s\,.\,d\delta x + t\,.\,d\delta y \end{aligned}\right\}. \tag{2}$$

It is again clear from (2) that p, q, etc., receive no increments for X_1f, X_2f, and X_3f.

For X_4f, $$\delta p = -q\,;$$
hence $$-dq = -(sdx + tdy) = \delta r\,.\,dx + \delta s\,.\,dy + sdx\,;$$
from which we find
$$\delta r = -2s, \quad \delta s = -t\,,$$
while the second of equations (2) shows that $\delta t = 0$.

In precisely the same manner δr, δs, δt can be found for the other transformations, and the complete system of eight members in eight variables may be written

$$X_1f = \frac{\partial f}{\partial x} = 0$$
$$X_2f = \frac{\partial f}{\partial y} = 0$$
$$X_3f = \frac{\partial f}{\partial z} = 0$$
$$X_4f = x\frac{\partial f}{\partial y} - q\frac{\partial f}{\partial p} - 2s\frac{\partial f}{\partial r} - t\frac{\partial f}{\partial s} = 0$$
$$X_5f = y\frac{\partial f}{\partial x} - p\frac{\partial f}{\partial q} - r\frac{\partial f}{\partial s} - 2s\frac{\partial f}{\partial t} = 0$$
$$X_6f = z\frac{\partial f}{\partial z} + p\frac{\partial f}{\partial p} + q\frac{\partial f}{\partial q} + r\frac{\partial f}{\partial r} + s\frac{\partial f}{\partial s} + t\frac{\partial f}{\partial t} = 0$$
$$X_7f = x\frac{\partial f}{\partial x} - y\frac{\partial f}{\partial y} - p\frac{\partial f}{\partial p} + q\frac{\partial f}{\partial q} - 2r\frac{\partial f}{\partial r} + 2t\frac{\partial f}{\partial t} = 0$$
$$\begin{aligned} X_8f = z^2\frac{\partial f}{\partial z} + 2zp\frac{\partial f}{\partial p} + 2zq\frac{\partial f}{\partial q} + 2\,(zr + p^2)\frac{\partial f}{\partial r} + (2zs + pq)\frac{\partial f}{\partial s} \\ + 2\,(zt + q^2)\frac{\partial f}{\partial t} = 0\,. \end{aligned}$$

Since x, y, z are not contained in f, the determinant of these equations may, after an obvious reduction, be written

$$\begin{vmatrix} -q & 0 & -2s & -t & 0 \\ 0 & -p & 0 & -r & -2s \\ p & q & r & s & t \\ -p & q & -2r & 0 & 2t \\ 0 & 0 & p^2 & pq & q^2 \end{vmatrix}.$$

If we expand this determinant and place the result equal to zero, we have an invariant differential equation.*

The equation is easily found to be

$$p^2 t - 2pqs + q^2 r = 0\,.$$

It has for its integral

$$y = x\varphi(a) + \psi(a)\,, \quad z = a\,,$$

or

$$y = x\varphi(z) + \psi(z)\,,$$

which is the general equation to *ruled surfaces*, where the straight line generators are all parallel to the xy-plane.†

11. Since the determinant considered above is not identically zero, the complete system has no solution. In order then to have more variables than equations, and therefore one or more solutions, it is necessary to extend the transformations further. To do this we have

$$\begin{aligned} dr &= \rho dx + \sigma dy \\ ds &= \sigma dx + \mu dy \\ dt &= \mu dx + \lambda dy\,; \end{aligned}$$

also

$$\left.\begin{aligned} d\delta r &= \delta\rho \,.\, dx + \delta\sigma \,.\, dy + \rho \,.\, d\delta x + \sigma \,.\, d\delta y \\ d\delta s &= \delta\sigma \,.\, dx + \delta\mu \,.\, dy + \sigma \,.\, d\delta x + \mu \,.\, d\delta y \\ d\delta t &= \delta\mu \,.\, dx + \delta\lambda \,.\, dy + \mu \,.\, d\delta x + \lambda \,.\, d\delta y \end{aligned}\right\}. \tag{3}$$

Here again p, q, etc., receive no increments under X_1f, X_2f, and X_3f.

Since for X_4f, $\delta x = 0$, $\delta y = x$, $\delta p = -q$, $\delta q = 0$, $\delta r = -2s$, $\delta s = -t$, $\delta t = 0$; we have from (3)

$$\begin{aligned} -2ds &= -(\sigma dx + \mu\, dy) = \delta\rho \,.\, dx + \delta\sigma \,.\, dy + \sigma \,.\, dx \\ -dt &= -(\mu dx + \lambda dy) = \delta\sigma \,.\, dx + \delta\mu \,.\, dy + \mu dx \\ 0 &= \quad\;\; 0 \quad\;\; = \delta\mu \,.\, dx + \delta\lambda \,.\, dy + \lambda \,.\, dx\,. \end{aligned}$$

If in these last three equations we collect the coefficients of dx and dy and equate separately to zero, we find

$$\delta\rho = -3\sigma\,, \quad \delta\sigma = -2\mu\,, \quad \delta\mu = -\lambda\,, \quad \delta\lambda = 0\,.$$

For X_5f, equations (3) have the forms

$$\begin{aligned} 0 &= \delta\rho \,.\, dx + \delta\sigma \,.\, dy + \rho \,.\, dy \\ -dr &= -(\rho dx + \sigma dy) = \delta\sigma \,.\, dx + \delta\mu \,.\, dy + \sigma \,.\, dy \\ -2ds &= -2(\sigma \,.\, dx + \mu dy) = \delta\mu \,.\, dx + \delta\lambda \,.\, dy + \mu \,.\, dy\,; \end{aligned}$$

from which we find

$$\delta\rho = 0\,, \quad \delta\sigma = -\rho\,, \quad \delta\mu = -2\sigma\,, \quad \delta\lambda = -3\mu\,.$$

---

* Lie's Continuierliche Gruppen, Kap. 16.

† Laurent's Traite D'Analyse, Tome VI.

In like manner it will be easily seen that for $X_6 f$

$$\delta\rho \equiv \rho\,, \quad \delta\sigma \equiv \sigma\,, \quad \delta\mu \equiv \mu\,, \quad \delta\lambda \equiv \lambda\,.$$

For $X_7 f$,

$$\delta\rho = -3\rho\,, \quad \delta\sigma \equiv -\sigma\,, \quad \delta\mu \equiv \mu\,, \quad \delta\lambda \equiv 3\lambda\,.$$

Finally we have for $X_8 f$,

$$\delta\rho \equiv 2z\rho + 6rp\,, \quad \delta\sigma \equiv 2z\sigma + 2rq + 4ps$$
$$\delta\mu \equiv 2z\mu + 2pt + 4qs\,, \quad \delta\lambda \equiv 2z\lambda + 6qt\,.$$

If now the transformations be written in the extended form, and $X_8 f$ be replaced by $X_8 f - zX_6 f$; and if the terms containing $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$ be omitted (since x, y, and z do not occur in the solutions) the complete system may be written in the form

$$X_4 f = q\frac{\partial f}{\partial p} + 2s\frac{\partial f}{\partial r} + t\frac{\partial f}{\partial s} + 3\sigma\frac{\partial f}{\partial \rho} + 2\mu\frac{\partial f}{\partial \sigma} + \lambda\frac{\partial f}{\partial \mu} = 0$$

$$X_5 f = p\frac{\partial f}{\partial q} + r\frac{\partial f}{\partial s} + 2s\frac{\partial f}{\partial t} + \rho\frac{\partial f}{\partial \sigma} + 2\sigma\frac{\partial f}{\partial \mu} + 3\mu\frac{\partial f}{\partial \lambda} = 0$$

$$X_6 f = p\frac{\partial f}{\partial p} + q\frac{\partial f}{\partial q} + r\frac{\partial f}{\partial r} + s\frac{\partial f}{\partial s} + t\frac{\partial f}{\partial t} + \rho\frac{\partial f}{\partial \rho} + \sigma\frac{\partial f}{\partial \sigma} + \mu\frac{\partial f}{\partial \mu} + \lambda\frac{\partial f}{\partial \lambda} = 0$$

$$X_7 f = -p\frac{\partial f}{\partial p} + q\frac{\partial f}{\partial q} - 2r\frac{\partial f}{\partial r} + 2t\frac{\partial f}{\partial t} - 3\rho\frac{\partial f}{\partial \rho} - \sigma\frac{\partial f}{\partial \sigma} + \mu\frac{\partial f}{\partial \mu} + 3\lambda\frac{\partial f}{\partial \lambda} = 0$$

$$X_8 f \equiv p^2\frac{\partial f}{\partial r} + pq\frac{\partial f}{\partial s} + q^2\frac{\partial f}{\partial t} + 3rp\frac{\partial f}{\partial \rho} + (rq + 2ps)\frac{\partial f}{\partial \sigma} + (pt + 2qs)\frac{\partial f}{\partial \mu} + 3qt\frac{\partial f}{\partial \lambda} = 0\,.$$

The next object is to find the solutions of this complete system ; and owing to the complicated forms in which some of the results occur, we shall resort to a device by means of which the work can be greatly simplified.

Let us first consider the complete system given by the sub-group

$$X_4 f = q\frac{\partial f}{\partial p} + 2s\frac{\partial f}{\partial r} + t\frac{\partial f}{\partial s} = 0$$

$$X_5 f = p\frac{\partial f}{\partial q} + r\frac{\partial f}{\partial s} + 2s\frac{\partial f}{\partial t} = 0$$

$$X_7 f = -p\frac{\partial f}{\partial p} + q\frac{\partial f}{\partial q} - 2r\frac{\partial f}{\partial r} + 2t\frac{\partial f}{\partial t} = 0\,.$$

The solutions of $X_7 f = 0$ are found to be

$$s = u_1\,, \quad pq = u_2\,, \quad \frac{p^2}{r} = u_3\,, \quad p^2 t = u_4\,.$$

UNIVERSITY OF VIRGINIA ★ 1819 ★ GENERAL LIBRARY

These solutions can now be introduced into the other two equations in the usual way. It will be seen that

$$
\begin{aligned}
X_4(u_1) &= t - \frac{u_4}{p^2} \\
X_4(u_2) &= q^2 - \frac{u_2^2}{p^2} \\
X_4(u_3) &= \frac{2pq}{r} - 2s\left[-\frac{p^2}{r^2}\right] - \frac{2}{p^2}(u_2u_3 - u_1u_3^2) \\
X_4(u_4) &= 2pqt - \frac{2}{p^2}u_2u_4 \,.
\end{aligned}
$$

In like manner it will be found that

$$X_5(u_i) = \frac{p^2}{\varphi(u_i)}\,;$$

and the two equations will have the forms

$$
\begin{aligned}
X_4f &= u_4\frac{\partial f}{\partial u_1} + u_2^2\frac{\partial f}{\partial u_2} + 2(u_2u_3 - u_1u_3^2)\frac{\partial f}{\partial u_3} + 2u_2u_4\frac{\partial f}{\partial u_4} = 0 \\
X_5f &= \frac{1}{u_3}\frac{\partial f}{\partial u_1} + \frac{\partial f}{\partial u_2} + 2u_1\frac{\partial f}{\partial u_4} = 0\,.
\end{aligned}
$$

The solutions of $X_5f = 0$ are

$$u_3 = v_1\,, \quad u_1u_3 - u_2 = v_2\,, \quad u_1^2u_3 - u_4 = v_3\,.$$

When these solutions are introduced, X_4f has the form

$$X_4f = 2v_1v_2\frac{\partial f}{\partial v_1} + (v_2^2 + v_1v_3)\frac{\partial f}{\partial v_2} + 2v_2v_3\frac{\partial f}{\partial v_3} = 0\,;$$

and this equation is equivalent to

$$\frac{dv_1}{2v_1v_2} = \frac{dv_2}{v_2^2 + v_1v_3} = \frac{dv_3}{2v_2v_3}\,.$$

One solution is clearly $\frac{v_3}{v_1}$, and, if we place this equal to some constant a, we have in order to determine the other

$$\frac{dv_1}{2v_1v_2} = \frac{dv_2}{v_2^2 + av_1^2}\,.$$

This equation is *homogeneous*, and it is therefore known at once that the integrating factor is

$$-\frac{1}{v_2^2v_1 - av_1^3}\,.^*$$

* Page's Ordinary Differential Equations, Art. 64.

Hence we have

$$\int \frac{2v_1v_2}{v_1v_2^2 - av_1^3}\, dv_2 + \int \left[-\frac{v_2^2 + av_1^2}{v_1v_2^2 - av_1^3} - \frac{v_2^2 - 3av_1^2}{v_2^2v_1 - av_1^3} \right] dv_1 = c\,;$$

and therefore

$$\mathrm{Log}\,(v_1v_2^2 - av_1^3) - \mathrm{Log}\, v_1^2 = c\,.$$

The two solutions are then

$$\frac{v_3}{v_1} = u_1^2 - \frac{u_4}{u_3} = s^2 - rt\,,$$

and

$$\frac{v_1v_2^2 - av_1^3}{v_1^2} = \frac{v_2^2 - v_1v_3}{v_1} = rq^2 - 2pqs + p^2t\,.$$

12. We shall next seek *differential parameters* for this *sub-group* by means of which all solutions of a higher order can be found. We know that if φ is a function of x and y

$$d\varphi = \frac{\partial\varphi}{\partial x}\, dx + \frac{\partial\varphi}{\partial y}\, dy\,;$$

or writing $\frac{\partial\varphi}{\partial x} = \varphi_x$, and $\frac{\partial\varphi}{\partial y} = \varphi_y$,

$$d\varphi = \varphi_x dx + \varphi_y dy\,.$$

Also

$$d\delta\varphi = \delta\varphi_x\,.\,dx + \delta\varphi_y\,.\,dy + \varphi_x\,.\,d\delta x + \varphi_y\,.\,d\delta y\,.$$

If now φ is to be an *invariant* function, then must $\delta\varphi = 0$; and therefore,

$$0 = \delta\varphi_x\,.\,dx + \delta\varphi_y\,.\,dy + \varphi_x\,.\,d\delta x + \varphi_y\,.\,d\delta y\,. \tag{4}$$

For $X_4 f \equiv xq$, equation (4) becomes

$$0 = \delta\varphi_x\,.\,dx + \delta\varphi_y\,.\,dy + \varphi_y\,.\,dx = (\delta\varphi_x + \varphi_y)\,dx + \delta\varphi_y\,.\,dy\,;$$

from which we get

$$\delta\varphi_x = -\varphi_y\,, \quad \delta\varphi_y = 0\,.$$

In like manner for $X_5 f$ it is easily seen that

$$\delta\varphi_x = 0\,, \quad \delta\varphi_y = -\varphi_x\,;$$

and for $X_7 f$

$$\delta\varphi_x = -\varphi_x\,, \quad \delta\varphi_y = \varphi_y\,.$$

Hence the complete system has the form

$$X_4 f = q\frac{\partial f}{\partial p} + 2s\frac{\partial f}{\partial r} + t\frac{\partial f}{\partial s} + \varphi_y\frac{\partial f}{\partial \varphi_x} = 0$$

$$X_5 f = p\frac{\partial f}{\partial q} + r\frac{\partial f}{\partial s} + 2s\frac{\partial f}{\partial t} + \varphi_x\frac{\partial f}{\partial \varphi_y} = 0$$

$$X_7 f = -p\frac{\partial f}{\partial p} + q\frac{\partial f}{\partial q} - 2r\frac{\partial f}{\partial r} + 2t\frac{\partial f}{\partial t} - \varphi_x\frac{\partial f}{\partial \varphi_x} + \varphi_y\frac{\partial f}{\partial \varphi_y} = 0\,.$$

We already know two solutions of this complete system. If now we solve $X_5 f = 0$, we find among other solutions

$$\Delta_1\varphi = p\varphi_y - q\varphi_x .$$

This solution is found to be common to the other two equations also. Again it is observed that φ_x and φ_y enter the three equations just as do p and q, respectively. We can then write the other solution at once, which is

$$D_1\varphi = \varphi_x^2 t - 2\varphi_x\varphi_y s + \varphi_y^2 r .$$

By means of the differential parameters, $\Delta_1\varphi$ and $D_1\varphi$, can be found all solutions of higher order common to $X_7 f = 0$, $X_5 f = 0$, and $X_7 f = 0$. We can then introduce these solutions into $X_6 f = 0$ and $X_8 f = 0$, and thus find the solutions of our original complete system. We shall be content with finding one of these solutions and a differential parameter for the whole group.

One of the solutions common to $X_4 f = 0$, $X_5 f = 0$, and $X_7 f = 0$, was found to be

$$p^2 t - 2pqs + q^2 r = u_1 .$$

When the variables ρ, σ, μ, and λ are added, another can be found by means of $\Delta_1\varphi$, which has the form

$$\Delta_1(u_1) = p^3\lambda - 3p^2 q\mu + 3pq^2\sigma - q^3\rho = u_2 .$$

If now u_1 and u_2 be introduced into $X_8 f = 0$, it is easily seen that they are solutions of $X_8 f = 0$ also. We then introduce the same into $X_6 f = 0$, and find

$$X_6 f = 3u_1 \frac{\partial f}{\partial u_1} + 4u_2 \frac{\partial f}{\partial u_2} = 0 ;$$

which has the solution

$$\frac{u_1^4}{u_2^3} = \frac{(p^2 t - 2pqs + q^2 r)^4}{(p^3\lambda - 3p^2 q\mu + 3pq^2\sigma - q^3\rho)^3} = I_3 .$$

The other three could be found in a similar manner. We will say they are J_3, K_3, and L_3.

13. We shall next find differential parameters for the complete system.

It is clear that for $X_6 f$ and $X_8 f$

$$\delta\varphi_x = \delta\varphi_y = 0 .$$

If now we introduce into $X_6 f = 0$ and $X_8 f = 0$ the solutions

$$\begin{aligned} s^2 - rt &= u_1 \\ p^2 t - 2pqs + q^2 r &= u_2 \\ p\varphi_y - q\varphi_x &= u_3 \\ \varphi_x^2 t - 2\varphi_x\varphi_y s + \varphi_y^2 r &= u_4 , \end{aligned}$$

the two equations take the forms

$$X_6 f = 2u_1 \frac{\partial f}{\partial u_1} + 3u_2 \frac{\partial f}{\partial u_2} + u_3 \frac{\partial f}{\partial u_3} + u_4 \frac{\partial f}{\partial u_4} = 0$$

$$X_8 f = -u_2 \frac{\partial f}{\partial u_1} + u_3^2 \frac{\partial f}{\partial u_4} = 0 .$$

The solutions of $X_8 f = 0$ are

$$u_2 = v_1 , \quad u_3 = v_2 , \quad u_1 u_3^2 + u_2 u_4 = v_3 .$$

When these solutions are introduced, $X_6 f$ has the form

$$X_6 f = 3v_1 \frac{\partial f}{\partial v_1} + v_2 \frac{\partial f}{\partial v_2} + 4v_3 \frac{\partial f}{\partial v_3} = 0 .$$

The solutions are easily found to be

$$\Delta_1 \varphi \equiv \frac{v_2^3}{v_1} \equiv \frac{u_3^3}{u_2} \equiv \frac{(p\varphi_y - q\varphi_x)^3}{p^2 t - 2pqs + q^2 r} ,$$

and

$$D_1 \varphi \equiv \frac{v_3^3}{v_1^4} \quad \frac{\{(s^2 - rt)(p\varphi_y - q\varphi_x) + (p^2 t - 2pqs + q^2 r)(\varphi_x^2 t - 2\varphi_x \varphi_y s + \varphi_y^2 r)\}^3}{(p^2 t - 2pqs + q^2 r)^4} .$$

If the operations indicated by these parameters be performed on I_3, J_3, K_3, and L_3, all essential differential invariants of higher orders can be found by mere differentiation.

We shall make use of the above results in the next section, where we shall conclude our investigations with a discussion of the equivalence of surfaces.

§ 5.

Equivalence of Surfaces.

14. If a family of surfaces is given by an equation of the form

$$z = f(xy) \tag{1}$$

we may always write this equation in the form

$$z - z_0 = p_0 (x - x_0) + q_0 (y - y_0) + \frac{r_0}{2} (x - x_0)^2 + s_0 (x - x_0)(y - y_0) + \frac{t_0}{2} (y - y_0)^2 + \dots \tag{2}$$

where $z_0, p_0 \dots t_0 \dots$ are the values of $z, p \dots t \dots$ when we assign to x and y their initial values x_0, y_0.

Now it is clear that the number of surfaces in the family (1) will depend upon the number of arbitrary constants, z_0, $p_0 \dots t_0 \dots$ in equation (2). If the partial differential coefficients $z, p, \dots t \dots$ are connected by *no* relation, there will clearly be an unlimited number of surfaces in the family. But if all the partial differential coefficients except n, are expressed by means of a system of partial differential equations in terms of the remaining n differential coefficients, equation (2) will contain only n arbitrary constants; that is, the family (1) will consist of ∞^n surfaces.

Now if we perform *all* the transformations of the G_8 upon a surface which admits of *no* transformation of the G_8, this surface will generate during the transformation a family of ∞^8 surfaces which, as a family, is invariant under the G_8. This family of ∞^8 surfaces

$$z = F(x, y, c_1 \dots c_8)$$

must be defined by a system of partial differential equations, which is *completely* (*unbeschränkt*) integrable.* There is an infinite number of partial differential equations in this system: but we can suppose them arranged so that beginning with those of the lowest order they proceed to those of higher orders. Also we may assume that from the equations of the pth order, all partial differential coefficients of that order cannot be eliminated; and, finally, that the differentiation of one of the equations of the system will always give another equation which has already been obtained belonging to the system.†

The system of equations so arranged will always determine, from a certain point on, all the higher differential coefficients of z, with respect to x and y, in terms of those of lower order (counting z as a differential coefficient of the zero order); and as the family consists of ∞^8 surfaces, those of the lower differential coefficients which are connected by no relations cannot exceed *eight* in number.

If the surface upon which we perform the transformations of the G_8 admits of $m < 8$ of these transformations, it will readily be seen that it assumes ∞^{8-m} positions, which will form an invariant family of ∞^{8-m} surfaces. What has been said above for the family of ∞^8 surfaces is equally true for the family of ∞^{8-m}; that is, the latter family is defined by an invariant system of completely integrable partial differential equations, which determine all the higher partial differential coefficients of z in terms of $8 - m$ of the lower differential coefficients, etc.

If we indicate the partial differential coefficients, up to the fourth order by

$$z, p, q, \quad r, s, t, \quad \rho, \sigma, \mu, \lambda, \quad R, S, M, L, N,$$

it is clear that when a family of ∞^8 surfaces is invariant, the system of partial differential equations must contain *two* partial differential equations of the *third* order, since there are ten partial differential coefficients up to that order. If these two equations are

$$\Omega_1^{(3)}(xyzp \dots \lambda) = 0, \quad \Omega_2^{(3)}(xyzp \dots \lambda) = 0 \tag{3}$$

* Goursat, Vol. II, p. 41.

† Lie, Bd. I, Kap. 10.

it is clear that the derived equations

$$\frac{\partial \Omega_1^{(3)}}{\partial x} = 0\,, \quad \frac{\partial \Omega_1^{(3)}}{\partial y} = 0\,, \quad \frac{\partial \Omega_2^{(3)}}{\partial x} = 0\,, \quad \frac{\partial \Omega_2^{(3)}}{\partial y} = 0\,, \tag{4}$$

will determine four of the partial differential coefficients of the fourth order. Hence, to determine a family of exactly ∞^8 surfaces, we must have the system (3), (4), and one more equation of the fourth order which is not a consequence of (3), (4). This equation has the general form

$$\Omega_3^{(4)}(xyzp \ldots N) = 0\,;$$

and it is clear that all partial differential coefficients of the fifth and higher orders are determined in terms of those of lower orders, leaving (say)

$$z,\ p,\ q,\ r,\ s,\ t,\ \rho,\ \sigma$$

connected by *no* relation.

Proceeding in this manner, we are led to distinguish the following cases:

A. *No equation of an order lower than the third occurs in the invariant system of partial differential equations.*

Case I. As we have seen, if *two* partial differential equations of the *third* order occur, and *one* of the *fourth* (which is not a consequence of the other two) this system of equations is exactly sufficient to determine an invariant family of ∞^8 surfaces.

Case II. We might have a system of *three* different partial differential equations of the *third* order

$$\Omega_1^{(3)} = 0\,, \quad \Omega_2^{(3)} = 0\,, \quad \Omega_3^{(3)} = 0$$

invariant, and none of lower order. In this case, all differential coefficients of the fourth and higher orders are determined in terms of those of lower orders; and thus as (say)

$$z,\ p,\ q,\ r,\ s,\ t,\ \rho$$

are connected by *no* relation, the invariant family consists of ∞^7 surfaces.

Case III. If *four* differential equations of the *third* order are invariant, and none of lower orders, the family clearly consists of ∞^6 surfaces.

These are all the possibilities when no differential equation of an order lower than the third occurs.

B. *No differential equation of an order lower than the second occurs.*

There is only one invariant partial differential equation of the *second* order, namely,

$$F \equiv p^2t - 2pqs + q^2r = 0\,.$$

The derived equations

$$F_x = 0\,, \quad F_y = 0\,,$$

determine *two* of the partial differential coefficients of the third order, and leave two of them arbitrary. Moreover, the second derived equations

$$F_{xx} = 0\,, \quad F_{xy} = 0\,, \quad F_{yy} = 0\,,$$

determine *three* of the differential coefficients of the fourth order, and leave two of them arbitrary. In general, we see that two of all the partial differential coefficients of *any* order, after the second, are arbitrary as far as $F = 0$ and its derived equations are concerned.

Thus when $F = 0$ belongs to the invariant system of partial differential equations which define the invariant family of surfaces, we may assume that up to the fourth order $F = 0$ and its derived equations determine six of the partial differential coefficients, (say)

$$t,\ \mu,\ \lambda,\ M,\ L,\ N,$$

in terms of those of lower orders.

Case I. Hence, if $F = 0$ belongs to the invariant system of partial differential equations, it is clear that up to the fourth order, nine of the partial differential coefficients are still arbitrary as far as $F = 0$ and its derived equations are concerned. As the family cannot contain more than ∞^8 surfaces, we must, therefore, have an equation of the *fourth* order in the system which is not derived from $F = 0$.

This equation must, of course, have the form

$$\Omega^{(4)} = 0\,,$$

and will determine (say) s in terms of the other differential coefficients. It is clear that the derived equations of $F = 0$ and $\Omega_1 = 0$ determine all the differential coefficients of the fifth order; so that the invariant system of equations

$$F = 0\,, \quad F_x = 0\,, \quad F_y = 0\,, \quad F_{xx} = 0\,, \quad F_{xy} = 0\,, \quad F_{yy} = 0\,,$$
$$\Omega^{(4)} = 0\,, \quad \Omega_x^{(4)} = 0\,, \quad \Omega_y^{(4)} = 0\,,$$

are exactly enough to determine an invariant family of ∞^8 surfaces.

Case II. We might have, in connection with $F = 0$ and its derived equations, *two* partial differential equations of the *fourth* order invariant, of the forms

$$\Omega_1^{(4)} = 0\,, \quad \Omega_2^{(4)} = 0\,.$$

In this case, the only partial differential coefficients which are arbitrary may clearly be assumed to be (say)

$$z,\ p,\ q,\ r,\ s,\ \rho,\ \sigma;$$

that is, the family consists of only ∞^7 surfaces.

Case III. In connection with $F = 0$ we might have *one* differential equation of the *third* order invariant which is not a derived equation of $F = 0$.

It is clear that in this case this other equation of the third order has the form

$$\Omega^{(3)} = 0\,;$$

and this equation determines say σ. All higher differential coefficients are now determined in terms of those of lower orders except

$$z,\ p,\ q,\ r,\ s,\ \rho\,;$$

that is, the family consists of ∞^6 surfaces.

Case IV. If two equations of the third order which are not derived from $F = 0$, of the general forms

$$\Omega_1^{(3)} = 0\,,\quad \Omega_2^{(3)} = 0\,,$$

are invariant, the only arbitrary differential coefficients are

$$z,\ p,\ q,\ r,\ s\,.$$

Hence the family consists of ∞^5 surfaces.

These are all the possibilities when no partial differential equation of the first order occurs.

C. *Partial differential equations of the first order occur.*

No single invariant differential equation of the first order exists. The only invariant system containing differential equations of the first order is

$$p = q = 0\,,$$

which defines the invariant family of ∞^1 planes

$$z = \text{const.}$$

We may collect our results as follows:

I. If the invariant family consists of ∞^8 surfaces, it is defined by

(a) Two partial differential equations of $III.0.$ and one $IV.0.$, of the forms

$$\Omega_1(I_3, J_3, K_3, L_3) = 0\,,\quad \Omega_2(I_3, J_3, K_3, L_3) = 0\,,\quad \Omega(I_3 \ldots M_4) = 0\,.$$

(b) One partial differential equation of $II.0.$ and one $IV.0.$ which are respectively

$$F \equiv p^2t - 2pqs + q^2r = 0\,,\ \text{and}\ \ \Omega(I_3 \ldots M_4) = 0\,.$$

II. If the family consists of ∞^7 surfaces it is defined by

(a) Three partial differential equations $III.0.$, of the forms

$$\Omega_1(I_3 \ldots L_3) = 0\,,\quad \Omega_2(I_3 \ldots L_3) = 0\,,\quad \Omega_3(I_3 \ldots L_3) = 0\,.$$

(b) One partial differential equation $II.0.$ and two $IV.0.$ of the forms

$$F = 0\,,\quad \Omega_1(I_3 \ldots M_4) = 0\,,\quad \Omega_2(I_3 \ldots M_4) = 0\,.$$

...tor born in Virginia.

LCC (V)

III. If the invariant family consists of ∞^6 surfaces, it is defined by

(a) Four partial differential equations *III*. 0.

$$\Omega_1(I_3 \ldots L_3) = 0\,,\quad \Omega_2(I_3 \ldots L_3) = 0\,,\quad \Omega_3(I_3 \ldots L_3) = 0\,,\quad \Omega_4(I_3 \ldots L_3) = 0\,.$$

(b) One partial differential equation *II*. 0. and one *III*. 0.

$$F = 0\,,\quad \Omega(I_3 \ldots L_3) = 0\,.$$

IV. If the invariant family consists of ∞^5 surfaces, it can only be defined by

(a) One partial differential equation *II*. 0. and two *III*. 0., which have the forms

$$F = 0\,,\quad \Omega_1(I_3 \ldots L_3) = 0\,,\quad \Omega_2(I_3 \ldots L_3) = 0\,.$$

There are no invariant families of ∞^4, ∞^3, or ∞^2 surfaces; hence no surface admits of exactly four, five, or six independent infinitesimal transformations of the G_8. There is then only one other case.

V. The invariant family consists of the ∞^1 planes $z =$ const. defined by

$$p = q = 0\,.$$

From the above considerations we reach the following important conclusion:

If two surfaces are equivalent by means of the transformations of the G_8, their equations must both satisfy the partial differential equations enumerated in some one of the above cases I . . . V.

We may note in this connection that when

$$p^2t - 2pqs + q^2r = 0 \tag{5}$$

the invariant, I_3, becomes elusive, as do also $\Delta_1\varphi$ and $D_1\varphi$. But

$$\frac{D_1\varphi}{\Delta_1\varphi}$$

is a differential parameter which is *not* elusive; and this applied to J_3, K_3, or L_3 (some of them are certainly not elusive) will give differential invariants of the fourth and higher orders. In this way we can find *all* differential invariants of the third and fourth orders which are not zero on account of (5); and arbitrary functions of these differential invariants will be the invariant equations required in the system of partial differential equations defining the invariant families of surfaces.

We know also that (5) represents ∞^∞ *ruled surfaces;* and this unlimited assemblage of ruled surfaces is divided into families of equivalent surfaces by means of the other differential equations of the particular invariant system.