Support Expansion Operator Algebras

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A Dissertation presented to the Graduate Faculty of the University of Virginia in Candidacy for the Degree of Doctor of Philosophy

Department of Mathematics

The University of Virginia
December, 2021

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Declaration of Authorship

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"*The knack lies in learning how to throw yourself at the ground and miss.*"

Douglas Adams

THE UNIVERSITY OF VIRGINIA

Abstract

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Doctor of Philosophy

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by Joseph EISNER

The collection of bounded operators which have at most finitely many nonzero entries in each row and column of their standard array forms a $*$-subalgebra of $\mathcal{B}(\ell^2)$ and thus their norm closure a C*-algebra. We generalize this construction in several directions and settings, giving rise to a very general procedure for constructing concrete ***support expansion*** C*-algebras over a represented tracial von Neumann algebra. We go on to give a thorough analysis of the containment poset of concrete support expansion C*-algebras when the von Neumann algebra is taken to be $\ell^\infty \subseteq \mathcal{B}(\ell^2)$ and when it is taken to be $L^\infty(\mathbb{R}) \subseteq \mathcal{B}(L^2(\mathbb{R}))$. In particular we will show the containment poset of support expansion C*-algebras over $L^\infty(\mathbb{R}) \subseteq \mathcal{B}(L^2(\mathbb{R}))$ has uncountable ascending and descending chains as well as uncountable antichains.

The C*-algebra discussed in our first sentence is naturally realized as a uniform Roe algebra. In the second half of this dissertation we use measurable and quantum relations per Weaver, 2012 along with our home-baked intermediary ***cantankerous relations*** to define measurable, cantankerous and quantum uniform Roe algebras. We then realize the support expansion C*-algebras we developed in the first half as uniform Roe algebras in an appropriate sense.

Acknowledgements

I would like to express my deepest appreciation to my advisor David Sherman for teaching me about operator algebras and guiding me throughout the various stages of this project. I would also like to extend sincere thanks to my collaborator Bruno Braga for his contributions and extensive hours checking my proofs. I am grateful to my committee members for their time and patience. Many thanks to my instructors and mentors over the years, especially Ed Sakurai (to whom I owe my love of mathematics) and Daniel Freeman (my love of analysis).

To my graduate cohort and friends, thank you for helping me stay sane through the process. I could of course not be here without my parents, so much love to them both. My father, Ron Eisner, has been a rock for me: his advice always intelligent and increasingly wise. Finally, I am forever grateful to my beloved wife Taylor Banks for her companionship, support and unconditional love.

List of Abbreviations

poset	**p**artially **o**rdered **set**
ISOD	**I**ncreasing (and) **S**lope (to) **O**rigin **D**ecreasing
ICOD	**I**ncreasing (and) **CO**ncave **D**own
cura	**c**antankerous **u**niform **r**oe **a**lgebra
qura	**q**uantum **u**niform **r**oe **a**lgebra

List of Symbols

H	a complex Hilbert space
$\mathcal{B}(\cdot)$	the bounded linear operators from a space to itself
$\mathcal{M}$	a von Neumann algebra

For those who, in defiance, refuse to commute ...

Contents

Chapter 1

Introduction and Overview

The collection of bounded operators which have at most finitely many nonzero entries in each row and column of their standard array forms a $*$-subalgebra of $\mathcal{B}(\ell^2)$ and thus their norm closure a C*-algebra. We generalize this construction in several directions and settings, giving rise to a very general procedure for constructing concrete ***support expansion*** C*-algebras over a represented tracial von Neumann algebra. We go on to give a thorough analysis of the containment poset of concrete support expansion C*-algebras when the von Neumann algebra is taken to be $\ell^\infty \subseteq \mathcal{B}(\ell^2)$ and when it is taken to be $L^\infty(\mathbb{R}) \subseteq \mathcal{B}(L^2(\mathbb{R}))$. In particular we will show the containment poset of support expansion C*-algebras over $L^\infty(\mathbb{R}) \subseteq \mathcal{B}(L^2(\mathbb{R}))$ has uncountable ascending and descending chains as well as uncountable antichains.

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1.1 Introduction

In this section we will first discuss a motivating construction and our strategy for generalizing it. Then we will summarize some of our major results which will be proved later.

1.1.1 Uniformly RC Finite Operators

In this subsection we discuss the construction of a C*-algebra explored extensively in Manuilov, 2019 which has also been considered in the context of ***uniform Roe algebras***.

Definition 1.1.1. We say that an operator $a \in \mathcal{B}(\ell^2(\mathbb{N}))$ is ***uniformly row and column finite*** (abbreviated as ***uniformly RC-finite***) if there exists some $N \in \mathbb{N}$ such that the standard array $a = [a_{n,m}]$ has at most N non-zero entries per row and per column. We denote the set of all uniformly RC-finite operators by B_{RC}.

Notice that if $[a_{n,m}]$ is an $\mathbb{N}$-by-$\mathbb{N}$ array with complex coefficients which is uniformly row and column finite in the sense above, then $[a_{n,m}]$ (canonically) induces a bounded operator on $\ell^2(\mathbb{N})$ if and only if there exists some $C > 0$ such that $|a_{n,m}| < C$ for every $n, m \in \mathbb{N}$.

It is not hard to see that B_{RC} is closed under adjoints, addition and scalar multiplication. In fact it is also closed under multiplication and thus forms a $*$-subalgebra of $\mathcal{B}(\ell^2(\mathbb{N}))$. Hence the norm closure $C_{\text{RC}} = \overline{B_{\text{RC}}}^{\|\cdot\|}$ in $\mathcal{B}(\ell^2(\mathbb{N}))$ is a C*-algebra.

Manuilov, 2019 goes on to explore several interesting properties of C_{RC} including that it is a non-trivial proper sub-algebra of $\mathcal{B}(\ell^2(\mathbb{N}))$, that it contains the compact operators, and that it has a unique maximal two-sided ideal which is strictly larger than the compacts.

Reflecting on the definition of B_{RC}, we note that the condition "at most N non-zero entries per column" can be expressed in terms of supports in the following way:

$$|\mathrm{supp}\,(a\delta_n)| \leq N \text{ for every } n \in \mathbb{N}$$

where $|\cdot|$ denotes cardinality, $\mathrm{supp}\,(\cdot)$ here is the set of non-zero entries in an $\ell^2(\mathbb{N})$ vector and δ_n is the n-th basis vector.

Upon further reflection, $|\mathrm{supp}\,(\cdot)|$ is sub-additive and so there is no reason for the basis elements to hold an elevated status over other vectors. We may as well say that

$$|\mathrm{supp}\,(a\xi)| \leq N \cdot |\mathrm{supp}\,(\xi)| \text{ for every } \xi \in \ell^2(\mathbb{N}).$$

The condition on rows similarly can be expressed in terms of the support of the adjoint of a. Thus we can offer an alternative definition of B_{RC}.

Definition 1.1.2. An operator $a \in \mathcal{B}(\ell^2(\mathbb{N}))$ is ***uniformly RC-finite*** if there exists some $N \in \mathbb{N}$ such that

$$|\mathrm{supp}\,(a\xi)|\,, |\mathrm{supp}\,(a^*\xi)| \leq N \cdot |\mathrm{supp}\,(\xi)| \text{ for every } \xi \in \ell^2(\mathbb{N})$$

It is worth mentioning that the closure of B_{RC} under multiplication is straightforward when using this definition.

We notice several ways that one might generalize Definition 1.1.2:

1. Replace $N \cdot x$ with some other family of functions, resulting in different conditions. For instance, consider operators $a \in \mathcal{B}(\ell^2(\mathbb{N}))$ for which there exist $N, M \in \mathbb{N}$ such that

$$|\mathrm{supp}\,(a\xi)|\,, |\mathrm{supp}\,(a^*\xi)| \leq N \cdot \sqrt[M]{|\mathrm{supp}\,(\xi)|} \text{ for every } \xi \in \ell^2(\mathbb{N})$$

2. Replace $|\cdot|$ with some other measure m on $\mathbb{N}$.

3. Change the underlying Hilbert space or the notion of support. For instance, consider operators $a \in \mathcal{B}(L^2(\mathbb{R}))$ for which there exists some $N \in \mathbb{N}$ such that

$$\mu(\mathrm{supp}\,(a\xi)), \mu(\mathrm{supp}\,(a^*\xi)) \leq N \cdot \mu(\mathrm{supp}\,(\xi)) \text{ for every } \xi \in L^2(\mathbb{R})$$

   where μ is Lebesgue measure and $\mathrm{supp}\,(\xi)$ is the support of a function representative of ξ.

Any of these generalizations can, with care taken in choosing the condition functions, give rise to a class of $*$-algebras and thus, once norm closures have been taken, a class of C*-algebras. We will need some more sophisticated tools to differentiate and explore the C*-algebras that can arise from this procedure.

Much of the technical content (Chapters 2 and 3) of this dissertation is exploring these cases and determining the structure of the resulting containment posets of concrete C*-algebras.

1.1.2 Support and Expansion

Motivated by Definition 1.1.2 we want to introduce a function which measures how much a given operator expands the support of vectors of various sizes. We would like this definition to make sense in a broad context and so will need a unified notion of support and measure.

Definition 1.1.3. Given a Hilbert space H and a von Neumann sub-algebra $\mathcal{M} \subseteq \mathcal{B}(H)$ we define the *$\mathcal{M}$-support* of a vector $\xi \in H$ to be the smallest projection in $\mathcal{M}$ which fixes ξ, denoted $\operatorname{supp}_{\mathcal{M}}(\xi) = \bigwedge\{p \in \operatorname{Pr}(\mathcal{M}) : p\xi = \xi\}$, where $\operatorname{Pr}(\mathcal{M})$ is the set of projections in $\mathcal{M}$.

We note that this definition of $\operatorname{supp}(\cdot)$ is not literally the same as our earlier uses in Definition 1.1.2 and the surrounding discussion – but it suitably generalizes and unifies them into a single concept. We will measure supports with *dimension* functions.

Definition 1.1.4. A *dimension function* on the projections of a von Neumann algebra $\mathcal{M}$ is a function $d : \operatorname{Pr}(\mathcal{M}) \to [0, \infty]$ which is monotonic, additive on orthogonal projections and constant on Murray-von Neumann equivalence classes. Precisely for $p, q \in \operatorname{Pr}(\mathcal{M})$, d satisfies:

$$\begin{aligned} &d(p) \leq d(q) && \text{whenever } p \leq q, \\ &d(p+q) = d(p) + d(q) && \text{whenever } p \perp q \text{ and} \\ &d(p) = d(q) && \text{when } p \text{ and } q \text{ are Murray-von Neumann equivalent.} \end{aligned}$$

For the purposes of this dissertation dimension functions on a set of projection $\operatorname{Pr}(\mathcal{M})$ can always be thought of as arising by restricting a trace on $\mathcal{M}$. More generally the codomain of a dimension function might include infinite cardinals, but we do not explore those situations.

We now define one of the central topics of this dissertation:

Definition 1.1.5. Given a Hilbert space H, von Neumann sub-algebra $\mathcal{M} \subseteq \mathcal{B}(H)$ and a dimension function $d : \operatorname{Pr}(\mathcal{M}) \to [0, \infty]$ the *d-support expansion function*[1] $[0, \infty] \to [0, \infty]$ of an operator $a \in \mathcal{B}(H)$ is

$$\Phi_a^d(x) = \sup\{d(\operatorname{supp}_{\mathcal{M}}(a\xi)) : d(\operatorname{supp}_{\mathcal{M}}(\xi)) \leq x\}.$$

A few notational points: Since a is in $\mathcal{B}(H)$ and d is defined on projections in $\mathcal{M}$, we have suppressed H and $\mathcal{M}$ in the above notation, though they are important input data. Moreover, when d is thoroughly established by context it may be omitted as well.

We also immediately observe that for any $a \in \mathcal{B}(H)$ we have that Φ_a^d is increasing and $\Phi_a^d(0) = 0$.

While support expansion functions will streamline our discussion about generalizations of C_{RC} we also consider them an interesting topic in their own right and study them extensively. For now we need a few elementary properties.

[1] Dimension functions may take infinite cardinal values in which case the domain and codomain of support expansion functions may include these cardinals. We do not explore these cases in this dissertation.

Lemma 1.1.6. *Fix a Hilbert space H, von Neumann sub-algebra $\mathcal{M} \subseteq \mathcal{B}(H)$ and dimension function d on Pr$(\mathcal{M})$. Then for $a, b \in \mathcal{B}(H)$ and nonzero $\lambda \in \mathbb{C}$:*

1. $\Phi^d_{\lambda a} = \Phi^d_a$.
2. $\Phi^d_{a+b} \leq \Phi^d_a + \Phi^d_b$.
3. $\Phi^d_{ab} \leq \Phi^d_a \circ \Phi^d_b$.

Proof. For (a), multiplying by nonzero $\lambda \in \mathbb{C}$ does not change the support of a vector, so $d(\operatorname{supp}_{\mathcal{M}}(\lambda a\xi)) = d(\operatorname{supp}_{\mathcal{M}}(a\xi))$.

For (b), it suffices to note that supremum and $d(\operatorname{supp}_{\mathcal{M}}(\cdot))$ are both subadditive.

As for (c), consider the inequality chain below:

$$\begin{aligned}
\Phi_{ab}(x) &= \sup_{d(\operatorname{supp}_{\mathcal{M}}(\xi))\leq x} d(\operatorname{supp}_{\mathcal{M}}(ab\xi)) \\
&\leq \sup_{d(\operatorname{supp}_{\mathcal{M}}(\xi))\leq x} \left(\sup_{d(\operatorname{supp}_{\mathcal{M}}(\eta))\leq d(\operatorname{supp}_{\mathcal{M}}(b\xi))} d(\operatorname{supp}_{\mathcal{M}}(a\eta)) \right) \\
&\leq \sup_{d(\operatorname{supp}_{\mathcal{M}}(\eta))\leq \Phi_b(x)} d(\operatorname{supp}_{\mathcal{M}}(a\eta)) \\
&= \Phi_a(\Phi_b(x)).
\end{aligned}$$

□

Support expansion functions make it straightforward to talk about operators "controlled" by a function or collection of functions.

Definition 1.1.7. Fix a Hilbert space H, von Neumann sub-algebra $\mathcal{M} \subseteq \mathcal{B}(H)$ and dimension function d on $\Pr(\mathcal{M})$. Then given some family of functions $\mathcal{F} : [0, \infty] \to [0, \infty]$ and $f \in \mathcal{F}$ we define

$$\begin{aligned}
B_f &= \{a \in \mathcal{B}(H) : \Phi^d_a, \Phi^d_{a^*} \leq f\}, \\
B_{\mathcal{F}} &= \bigcup \{B_f : f \in \mathcal{F}\} \text{ and} \\
C_{\mathcal{F}} &= \overline{B_{\mathcal{F}}}^{\|\cdot\|}.
\end{aligned}$$

The sets B_f and $B_{\mathcal{F}}$ are the *f-controlled* and *$\mathcal{F}$-controlled* operators respectively. C*-algebras of the form $C_{\mathcal{F}}$ are collectively referred to as *support expansion* C*-*algebras (on $\mathcal{M}$)*.

In the above notation we have suppressed $H, \mathcal{M}$ and d, which will always be clearly established from context.

A rapid corollary of Lemma 1.1.6 gives us sufficient conditions so that $C_{\mathcal{F}}$ is a C*-algebra:

Corollary 1.1.8. *Fix a Hilbert space H, von Neumann sub-algebra $\mathcal{M} \in \mathcal{B}(H)$ and dimension function d on Pr$(\mathcal{M})$. Take $f_1, f_2 : [0, \infty] \to [0, \infty]$, $a \in B_{f_1}$, $b \in B_{f_2}$ and $\lambda \in \mathbb{C}$ then:*

1. $a^* \in B_{f_1}$.
2. $\lambda a \in B_{f_1}$.
3. $a + b \in B_{f_1+f_2}$.

4. $ab \in B_{f_1 \circ f_2}$.

It follows that if a family of functions $\mathcal{F} : [0, \infty] \to [0, \infty]$ is closed under addition and composition then $B_{\mathcal{F}}$ is a $*$-subalgebra of $\mathcal{B}(H)$ and thus $C_{\mathcal{F}}$ is a C^*-algebra.

Proof. Straightforward. □

We have now generalized the construction of C_{RC} and have a procedure for producing related C*-algebras.

Example 1.1.9. Take $H = \ell^2(\mathbb{N})$ and let $\ell^\infty(\mathbb{N}) \cong \mathcal{M} \subseteq \mathcal{B}(\ell^2(\mathbb{N}))$ be the diagonal operators while $d : \mathrm{Pr}(\ell^\infty(\mathbb{N})) \to [0, \infty]$ is the dimension function induced by the standard trace on $\mathcal{B}(\ell^2(\mathbb{N}))$. Take $\mathcal{F}$ to be the set of lines through the origin with positive integer slope (noting $\mathcal{F}$ is closed under addition and composition) then

$$B_{\mathrm{RC}} = B_{\mathcal{F}}.$$

Often we will want to start with some collection of generating functions $\mathcal{F} : [0, \infty] \to [0, \infty]$ which may not be closed under addition and composition. In these cases we will denote by $\langle \mathcal{F} \rangle$ the smallest collection of functions $[0, \infty] \to [0, \infty]$ closed under addition and composition which contains $\mathcal{F}$. In the case of a single function we abbreviate $\langle f \rangle = \langle \{f\} \rangle$.

In the following subsections we will survey some results about the poset of C*-algebras of the form $C_{\langle \mathcal{F} \rangle}$ in various settings, proofs will be provided in subsequent chapters.

1.1.3 Discrete Support Expansion Algebras

In this subsection let $H = \ell^2(\mathbb{N})$ with $\ell^\infty(\mathbb{N}) \cong \mathcal{M} \subseteq \mathcal{B}(\ell^2(\mathbb{N}))$ the diagonal operators.

Theorem 2.2.5. *Let $d : Pr(l^\infty(\mathbb{N})) \to [0, \infty]$ be the dimension function induced by the standard trace on $\mathcal{B}(\ell^2(\mathbb{N}))$. For any family $\mathcal{F}$ of maps $[0, \infty] \to [0, \infty]$, we have that*

$$C_{\langle \mathcal{F} \rangle} \in \Big\{ \{0\}, \mathcal{K}(\ell^2(\mathbb{N})), C_{\mathrm{RC}}, \mathcal{B}(\ell^2(\mathbb{N})) \Big\}$$

where $\mathcal{K}(\ell^2(\mathbb{N}))$ denotes the compact operators inside of $\mathcal{B}(\ell^2(\mathbb{N}))$.

This is point 1 in the discussion following Definition 1.1.2 and we see that, besides B_{RC}, there are no novel C*-algebras obtained by this generalized procedure in the discrete setting with standard trace. Contrasting this result we obtain an extremely rich poset of C*-algebras if we impose a different dimension function on $\mathrm{Pr}(\mathcal{M})$, a generalization of the RC-finite operators which corresponds to points 1 and 2 in the discussion following Definition 1.1.2. Let $u : \mathbb{N} \to \mathbb{Q}_+$ be some enumeration of the non-negative rational numbers.

Definition 2.3.5. Define the dimension function $d : \mathrm{Pr}(\ell^\infty(\mathbb{N})) \to [0, \infty]$ by stipulating $d(p_n) = u(n)$, where p_n denotes projection onto the span of the n-th basis vector of $\ell^2(\mathbb{N})$, and extend to all of $\mathrm{Pr}(\ell^\infty(\mathbb{N}))$ using σ-additivity. We denote by $\mathbb{O}$ the set of all C*-subalgebras of $\mathcal{B}(\ell^2(\mathbb{N}))$ of the form $C_{\langle \mathcal{F} \rangle}$ for some family $\mathcal{F}$ of functions $[0, \infty] \to [0, \infty]$ in this context. We view $\mathbb{O}$ as a poset with the order being given by the standard inclusion.

Theorem 2.3.6. *The* poset $\mathbb{O}$ *has uncountable increasing chains, uncountable decreasing chains and uncountable antichains.*

There are many dimension functions which could be imposed on $\mathrm{Pr}(\mathcal{M})$, each resulting in a different poset of support expansion C*-algebras. In some sense the two we have exhibited here are extremes, demonstrating that the poset of support expansion C*-algebras can range from nearly trivial to extremely rich.

1.1.4 Continuous Support Expansion Algebras

In this subsection we will explore point 3 in the discussion following Definition 1.1.2. Let $H = L^2(\mathbb{R})$ with $L^\infty(\mathbb{R}) \cong \mathcal{M} \subseteq \mathcal{B}(L^2(\mathbb{R}))$ the multiplication operators and $d : \mathrm{Pr}(\mathcal{M}) \to [0, \infty]$ the dimension function induced by integrating against Lebesgue measure. We will refer to C*-algebras of the form $C_{\mathcal{F}}$ in this context as *continuous support expansion* C*-*algebras*.

The support expansion functions of operators in $\mathcal{B}(L^2(\mathbb{R}))$ have some nice properties. To explore these, we introduce several classes of functions:

Definition 3.1.4. Consider a function $f : [0, \infty] \to [0, \infty]$.

1. We say that f is ICOD (this acronym abbreviates "increasing and concave down") if $f(0) = 0$ and it is increasing and concave down.
2. We say that f is ISOD (this acronym abbreviates "increasing and slope-to-origin decreasing") if $f(0) = 0$ and it is increasing and $\frac{f(x)}{x}$ is decreasing.
3. We say that f is SUPPEXP (this acronym abbreviates "support expansion") if $f = \Phi_a$ for some $a \in \mathcal{B}(L^2(\mathbb{R}))$.

By abuse of notation, we also denote the subsets of all functions $[0, \infty] \to [0, \infty]$ which are ICOD, ISOD, and SUPPEXP by ICOD, ISOD, and SUPPEXP, respectively.

Notice that, by definition, $f(0) = 0$ for all functions in either ICOD or ISOD, and the same also holds for maps in SUPPEXP.

Theorem 3.1.6. *The inclusions* ICOD $\subseteq$ SUPPEXP $\subseteq$ ISOD *hold.*

Although Theorem 3.1.6 does not give us a complete characterization of support expansion functions, it will be enough for us to completely survey the continuous support expansion C*-algebras:

Theorem 3.2.11. *Let $\mathcal{F}$ be a collection of functions $[0, \infty] \to [0, \infty]$ then there exists some collection $\mathcal{F}' \subseteq ICOD$ such that $C_{\langle \mathcal{F} \rangle} = C_{\langle \mathcal{F}' \rangle}$.*

Definition 3.2.6. We denote by $\mathbb{P}$ the set of all C^*-subalgebras of $\mathcal{B}(L^2(\mathbb{R}))$ of the form $C_{\langle \mathcal{F} \rangle}$ for some family $\mathcal{F}$ of functions $[0, \infty] \to [0, \infty]$. We view $\mathbb{P}$ as a poset with the order being given by inclusion.

The poset structure of $\mathbb{P}$ is quite rich.

Theorem 3.4.1. *The* poset $\mathbb{P}$ *has uncountable increasing chains, uncountable decreasing chains and uncountable antichains.*

$\mathbb{P}$ has a two-tiered structure, where C*-algebras in the lower tier all have immediate successors in the upper tier. Which tier $C_{\langle \mathcal{F} \rangle}$ falls in depends on the behavior at infinity of the functions in $\mathcal{F} \subseteq ICOD$. In particular, if $\lim_{x\to\infty} \frac{f(x)}{x} = 0$ for every $f \in \mathcal{F}$ then $C_{\langle \mathcal{F} \rangle}$ will fall into the lower tier. Conversely, if $\lim_{x\to\infty} \frac{f(x)}{x} > 0$ for some $f \in \mathcal{F}$ then $C_{\langle \mathcal{F} \rangle}$ falls into the upper tier.

$\mathbb{P}$ also has a rigid structure at the top and bottom which we understand well. It has unique first, second, third, ultimate and penultimate elements. It has no fourth element, two incomparable third-to-last elements and one fourth-to-last element.

The above information is summarized in the following theorem, though the names of these elements are unmotivated without context. See Subsection 3.2.5 for a detailed treatment of these topics and proofs.

Theorem 3.5.10. *The* poset $\mathbb{P}$ *has the structure indicated in the diagram below.*

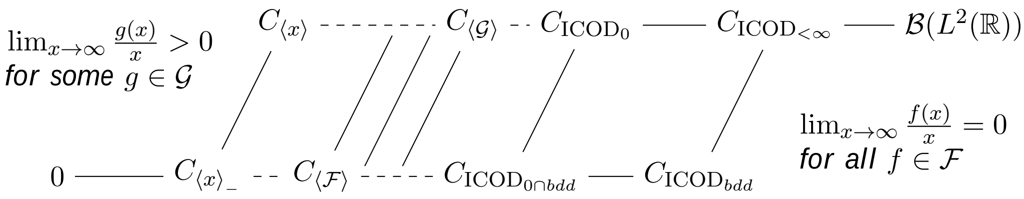

FIGURE 1.1: Elements further up and to the right are larger in the poset $\mathbb{P}$. Dotted lines indicate containment. Solid lines indicate immediate successors.

1.2 Uniform Roe Algebras and Generalizations

Now that we have thoroughly surveyed our major results for support expansion C*-algebras, we seek to place them in a larger theory.

1.2.1 Uniform Roe Algebras

Definition 5.1.7. (Roe, 2003 Definition 2.3)[2] Given a set X, we say a collection $\mathcal{C} \subseteq \mathscr{P}(X \times X)$ of relations on X is a ***coarse structure*** on X if it satisfies the following:

(a) The diagonal relation $\Delta \in \mathcal{C}$,

(b) If $U \in \mathcal{C}$ then $U^T \in \mathcal{C}$,

(c) If $U, V \in \mathcal{C}$ then $U \circ V \in \mathcal{U}$,

(d) If $U, V \in \mathcal{C}$ then $U \cup V \in \mathcal{C}$ and

(e) If $U \in \mathcal{C}$ and $V \subseteq U$ then $V \in \mathcal{C}$.

The pair $(X, \mathcal{C})$ is a ***coarse space*** and elements of $\mathcal{C}$ are commonly referred to as ***controlled sets*** or ***entourages***.

(In the above definition $U^T = \{(y, x) : (x, y) \in U\}$.) One may view coarse structures as a generalization of metric spaces, as demonstrated by the following example.

Example 5.1.8. Let (X, d) be a metric space and for each $\lambda \in (0, \infty)$ define $U_\lambda = \{(x, y) : d(x, y) \le \lambda\}$. Then $\mathcal{C} = \{U \subseteq X \times X : U \subseteq U_\lambda \text{ for some } \lambda \in (0, \infty)\}$ is a coarse structure on X.

A coarse structure naturally induces a C*-subalgebra of $\mathcal{B}(\ell^2(X))$ as seen in the definition and discussion below.

[2] Coarse structures were axiomatized in this way by John Roe in Roe, 2003, but these ideas originate in earlier works in geometric group theory including those of Gromov and Mostow.

Definition 5.1.11. (Uniform Roe Algebra) Let $\mathcal{C}$ be a coarse structure on a set X then for each $U \in \mathcal{C}$ we define

$$D_U = \{a \in \mathcal{B}(\ell^2(X)) : (x,y) \notin U \implies \langle a\delta_y, \delta_x \rangle = 0\}$$
$$\mathrm{C}^*_u[X,\mathcal{C}] = \bigcup \{D_U : U \in \mathcal{C}\}$$
$$\mathrm{C}^*_u(X,\mathcal{C}) = \overline{\mathrm{C}^*_u[X,\mathcal{C}]}^{\|\cdot\|}.$$

D_U and $\mathrm{C}^*_u[X,\mathcal{C}]$ are called the *U-controlled* and *$\mathcal{C}$-controlled* operators, respectively. Meanwhile $\mathrm{C}^*_u(X,\mathcal{C})$ is the ***uniform Roe algebra*** associated with $(X,\mathcal{C})$. Occasionally $\mathrm{C}^*_u[X,\mathcal{C}]$ is also referred to as the ***algebraic uniform Roe algebra*** associated with $(X,\mathcal{C})$.

The operations adjoint, addition and multiplication are closed for $\mathrm{C}^*_u[X,\mathcal{C}]$ because inverse, finite union, and composition are respectively closed for $\mathcal{C}$. $\mathrm{C}^*_u[X,\mathcal{C}]$ is also closed under scalar multiples and so is a $*$-subalgebra of $\mathcal{B}(\ell^2(X))$, hence $\mathrm{C}^*_u(X,\mathcal{C})$ is a C^*-algebra. In fact it is a unital C^*-algebra since $\Delta \in \mathcal{C}$ and the identity $I \in D_\Delta$.

Example 1.2.4. Let $X = \mathbb{N}$ and $\mathcal{E}$ be the maximal uniformly locally finite coarse structure on X. Precisely, $E \in \mathcal{E}$ if and only if there exists some $N \in \mathbb{N}$ such that

$$|\{n \in \mathbb{N} : (n,m) \in E\}| \leq N \qquad \text{and} \qquad |\{n \in \mathbb{N} : (m,n) \in E\}| \leq N$$

for every $m \in \mathbb{N}$. Then $C_{\mathrm{RC}} = \mathrm{C}^*_u(\mathbb{N},\mathcal{E})$. That is to say C_{RC} is a uniform Roe algebra.

If discrete support expansion C^*-algebras can be uniform Roe algebras then we wonder if continuous support expansion C^*-algebras might be "continuous uniform Roe algebras" in some sense. Unfortunately no such object appears to be defined in the literature. What tools would be necessary to define an analog to uniform Roe algebras in, for instance, $\mathcal{B}(L^2(\mathbb{R}))$?

1.2.2 Measurable Relations and Uniform Roe Algebras

We want to come up with a continuous analog of uniform Roe algebras and so we need a continuous notion of coarse structure, further requiring a continuous notion of relation. Luckily, a suitable notion of continuous or ***measurable*** relation has been crafted in Weaver, 2012. In Section 4.2 we will provide a thorough treatment of these objects with some motivating intuition. For now we merely introduce them and use them to generalize uniform Roe algebras.

In defining measurable relations, Weaver, 2012 makes reference to a class of measure spaces which are ***finitely decomposable***. As this may not be a well known class, we quote his brief discussion of it from pg. 4:

> To avoid pathology we assume that all measure spaces are finitely decomposable. This means that the space X can be partitioned into a (possibly uncountable) family of finite measure subspaces X_λ such that a set $S \subseteq X$ is measurable if and only if its intersection with each X_λ is measurable, in which case $\mu(S) = \sum \mu(S \cap X_\lambda)$. Finite decomposability is a generalization of σ-finiteness. Counting measure on any set is also finitely decomposable. The spaces $L^\infty(X,\mu)$ with (X,μ) finitely decomposable are precisely the abelian von Neumann algebras.

Definition 4.2.1. (Weaver, 2012 Definition 1.2 Measurable Relation) Let (X,μ) be a finitely decomposable measure space. A ***measurable relation*** on X is a family $\mathcal{R}$ of ordered pairs of nonzero projections in $L^\infty(X,\mu)$ such that

$$\left(\bigvee p_\lambda, \bigvee q_\kappa\right) \in \mathcal{R} \Leftrightarrow \text{ some } (p_\lambda, q_\kappa) \in \mathcal{R}$$

for any pair of families of nonzero projections $\{p_\lambda\}$ and $\{q_\kappa\}$.

Measurable relations can also be defined in terms of "image maps" and these are often nicer to work with.

Proposition 4.2.2. *(Weaver, 2012 Proposition 1.4) Let (X,μ) be a finitely decomposable measure space. If $\mathcal{R}$ is a measurable relation on X then the map*

$$\phi_{\mathcal{R}} : q \mapsto 1 - \bigvee \{p : (p,q) \notin \mathcal{R}\},$$

from the set of projections in $L^\infty(X,\mu)$ to itself, takes 0 to 0 and preserves arbitrary joins. If ϕ is a map from the set of projections in $L^\infty(X,\mu)$ to itself that takes 0 to 0 and preserves arbitrary joins then

$$\mathcal{R}_\phi = \{(p,q) : p\phi(q) \neq 0\}$$

is a measurable relation on X. The two constructions are inverse to each other.

Definition 4.2.3. Let (X,μ) be a finitely decomposable measure space. We call a map $\phi : \mathrm{Pr}(L^\infty(X,\mu)) \to \mathrm{Pr}(L^\infty(X,\mu))$ which takes 0 to 0 and preserves arbitrary joins an *image map* for $L^\infty(X,\mu)$ and denote the collection of such maps by $\mathrm{Im}(X,\mu)$.

Weaver, 2012 also defines the relevant operations we will need for "measurable coarse structures".

Definition 4.2.7. (Weaver, 2012 Definition 1.6) Let (X,μ) be a finitely decomposable measure space.

(a) The *diagonal measurable relation* Δ on X is defined by setting $(p,q) \in \Delta$ if $pq \neq 0$.

(b) The *transpose* of a measurable relation $\mathcal{R}$ is the measurable relation $\mathcal{R}^T = \{(q,p) : (p,q) \in \mathcal{R}\}$.

(c) The *product* of two measurable relations $\mathcal{R}$ and $\mathcal{R}'$ is the measurable relation

$$\begin{aligned} \mathcal{R} \cdot \mathcal{R}' &= \{(p,r) : \text{ for every } q \text{ either } (p,q) \in \mathcal{R} \text{ or } (1-q,r) \in \mathcal{R}'\} \text{ or equivalently} \\ &= \{(p,r) : \text{ there is a } q \text{ so that } (p,q') \in \mathcal{R} \text{ and } (q',r) \in \mathcal{R}' \text{ for every } q' \leq q\}. \end{aligned}$$

(d) A measurable relation $\mathcal{R}$ on X is

(i) *reflexive* if $\Delta \subseteq \mathcal{R}$

(ii) *symmetric* if $\mathcal{R}^T = \mathcal{R}$

(iii) *antisymmetric* if $\mathcal{R} \wedge \mathcal{R}^T \subseteq \Delta$

(iv) *transitive* if $\mathcal{R}^2 \subseteq \mathcal{R}$.

That each of the above is a relation is established in Weaver, 2012 Proposition 1.5 which also shows that any union of measurable relations is a measurable relation. Note that in Definition 4.2.7d.iii we have made reference to the meet of two measurable relations, which we define presently:

Definition 4.2.6. Let (X,μ) be a finitely decomposable measure space and $\mathcal{E}$ a collection of measurable relations on X

(a) $\bigvee \mathcal{E} = \bigcup\{\mathcal{R} : \mathcal{R} \in \mathcal{E}\}$

(b) $\bigwedge \mathcal{E} = \bigcup\{\mathcal{R}' \text{ measurable relation on } X : \mathcal{R}' \leq \mathcal{R} \text{ for every } \mathcal{R} \in \mathcal{E}\}$

Since any union of measurable relations is a measurable relation, we have that the set of measurable relations ordered by inclusion forms a complete lattice under these definitions.

We are now ready to define measurable coarse structures.

Definition 5.2.1 (cf. Definition 5.1.7)**.** Let (X,μ) be a finitely decomposable measure space. We say a nonempty collection $\mathcal{C}$ of measurable relations (Definition 4.2.1) on X is a ***measurable coarse structure*** on X if it satisfies the following:

(a) If $U \in \mathcal{C}$ then $U^T \in \mathcal{C}$,

(b) If $U, V \in \mathcal{C}$ then $U \cdot V \in \mathcal{C}$,

(c) If $U, V \in \mathcal{C}$ then $U \vee V \in \mathcal{C}$ and

(d) If $U \in \mathcal{C}$ and $V \leq U$ then $V \in \mathcal{C}$.

The tuple $(X,\mu,\mathcal{C})$ is a ***measurable coarse space*** and elements of $\mathcal{C}$ may be referred to as ***entourages***.

Note we have dropped the hypothesis that a coarse structure must contain the diagonal. We elaborate on this change in Section 5.1.3.

Definition 5.2.2 (cf. Definition 5.1.11)**.** Let $(X,\mu,\mathcal{C})$ be a measurable coarse space then for each $U \in \mathcal{C}$ we define

$$\begin{aligned} D_U &= \{a \in \mathcal{B}(L^2(X,\mu)) : (p,q) \notin U \implies paq = 0\} \\ \mathrm{C}^*_u[X,\mathcal{C}] &= \bigcup\{D_U : U \in \mathcal{C}\} \\ \mathrm{C}^*_u(X,\mathcal{C}) &= \overline{\mathrm{C}^*_u[X,\mathcal{C}]}^{\|\cdot\|}. \end{aligned}$$

D_U and $\mathrm{C}^*_u[X,\mathcal{C}]$ are called the ***U-controlled*** and ***$\mathcal{C}$-controlled*** operators, respectively. Meanwhile $\mathrm{C}^*_u(X,\mathcal{C})$ is the ***measurable uniform Roe algebra*** associated with $(X,\mathcal{C})$. Occasionally $\mathrm{C}^*_u[X,\mathcal{C}]$ is also referred to as the ***measurable algebraic uniform Roe algebra*** associated with $(X,\mathcal{C})$.

Example 1.2.12. (cf. Theorems 5.3.8 and 5.3.13) Take $H = L^2(\mathbb{R})$ with $L^\infty(\mathbb{R}) \cong \mathcal{M} \subseteq \mathcal{B}(L^2(\mathbb{R}))$ the multiplication operators and $d : \mathrm{Pr}(\mathcal{M}) \to [0,\infty]$ the dimension function induced by integrating against Lebesgue measure. Also take $\mathcal{F} \subseteq ICOD$ (Definition 3.1.4) and consider $C_{\langle\mathcal{F}\rangle}$ (Definition 1.1.7). For each $f \in \mathcal{F}$ we define

$$\begin{aligned} \mathrm{Im}_f &= \{\phi \in \mathrm{Im}(\mathbb{R}) : d(\phi(p)) \leq f(d(p))\} \\ \mathrm{Im}_{\langle\mathcal{F}\rangle} &= \bigcup\{\mathrm{Im}_f : f \in \langle\mathcal{F}\rangle\} \\ \mathscr{E}_{\langle\mathcal{F}\rangle} &= \{\mathcal{R} : \phi_{\mathcal{R}}, \phi_{\mathcal{R}^T} \in \mathrm{Im}_{\langle\mathcal{F}\rangle}\} \end{aligned}$$

noting that $\mathscr{E}_{\langle\mathcal{F}\rangle}$ is a measurable coarse structure. Then $\mathrm{C}^*_u(\mathbb{R}, \mathscr{E}_{\langle\mathcal{F}\rangle}) = C_{\langle\mathcal{F}\rangle}$. That is to say continuous support expansion C^*-algebras are measurable uniform Roe algebras.

1.2.3 Cantankerous Uniform Roe Algebras

We can mirror Weaver's framework for measurable relations but take pairs of projections in an arbitrary represented von Neumann algebra:

Definition 4.3.1. (cf. Definition 4.2.1) Let $\mathcal{M} \subseteq \mathcal{B}(H)$ be a represented von Neumann algebra. A ***cantankerous relation*** on $\mathcal{M}$ is a family $\mathcal{R}$ of ordered pairs of nonzero projections in $\mathcal{M}$ such that

$$\left(\bigvee p_\lambda, \bigvee q_\kappa\right) \in \mathcal{R} \Leftrightarrow \text{ some } (p_\lambda, q_\kappa) \in \mathcal{R}$$

for any pair of families of nonzero projections $\{p_\lambda\}$ and $\{q_\kappa\}$.

We elaborate on our choice of the adjective ***cantankerous*** in Section 4.3.

With one minor exception, the results from the previous subsection all hold in the non-commutative setting (see Section 4.3 for details) and so we can bring all of our definitions along replacing the adjective "measurable" with "cantankerous". The aforementioned exception is that we do not know if the second definition for the product of two relations in Definition 4.2.7c is equivalent to the first, nor if it defines a cantankerous relation. The first definition of the product of two relations is indeed a cantankerous relation (Proposition 4.3.3) and so we use it exclusively.

Our initial motivation for exploring measurable relations was to realize support expansion C*-algebras as uniform Roe algebras in some appropriate sense. It turns out that the notion of "support expansion" bifurcates as we move into the cantankerous setting (for now we avoid the details but see Example 5.3.7) which makes our job more difficult. But, indeed, both notions of support expansion C*-algebras can be realized as cantankerous uniform Roe algebras:

Theorem 1.2.14. *(cf. Theorems 5.3.8 and 5.3.13) Let $\mathcal{M} \subseteq \mathcal{B}(H)$ be a represented von Neumann algebra with dimension function d and $\mathcal{F}$ a family of functions closed under addition and composition. Then $C_\mathcal{F}$ (Definition 1.1.7) is a cantankerous uniform Roe algebra.*

1.2.4 Quantum Uniform Roe Algebras

Weaver, 2012 did not stop at defining measurable relations, he also defined "quantum relations", and our cantankerous relations lie somewhere between them. In Sections 4.4 and 5.4 we extend some of the tools Weaver, 2012 developed for measurable relations into the quantum setting and note the definitions for quantum coarse spaces and quantum uniform Roe algebras.

Chapter 2

Discrete Support Expansion C*-Algebras

In this chapter we discuss the support expansion C*-algebras (Definition 1.1.7) which can be obtained when $H = \ell^2(\mathbb{N})$ and $\ell^\infty(\mathbb{N}) \cong \mathcal{M} \subseteq \mathcal{B}(H)$ the diagonal operators, with various dimension functions on $\mathrm{Pr}(\mathcal{M})$.

2.1 A Projection-Focused Notion of Support Expansion

Before we get into the analysis of this specific case, we will need an alternative characterization of support expansion functions when supports are taken inside of a commutative von Neumann algebra. This characterization will also be used extensively in Chapter 3 and revisited in Subsection 5.3.1 of Chapter 5. Recall the definition of support expansion functions given in the Overview:

Definition 1.1.5. Given a Hilbert space H, von Neumann sub-algebra $\mathcal{M} \subseteq \mathcal{B}(H)$ and a dimension function $d : \mathrm{Pr}(\mathcal{M}) \to [0, \infty]$ the *d-support expansion function*[1] $[0, \infty] \to [0, \infty]$ of an operator $a \in \mathcal{B}(H)$ is

$$\Phi_a^d(x) = \sup\{d(\mathrm{supp}_{\mathcal{M}}(a\xi)) : d(\mathrm{supp}_{\mathcal{M}}(\xi)) \leq x\}.$$

Note that Definition 1.1.5 is concerned with the support of vectors. There is a related notion which has to do with the support of operators, which we define now.

Definition 2.1.2. [cf. Definition 1.1.3] Given a Hilbert space H and a von Neumann sub-algebra $\mathcal{M} \subseteq \mathcal{B}(H)$ we define the *left $\mathcal{M}$-support* of an operator $a \in \mathcal{B}(H)$ to be the smallest projection in $\mathcal{M}$ which fixes a, denoted $\mathrm{s}_l^{\mathcal{M}}(a) = \bigwedge\{p \in \mathrm{Pr}(\mathcal{M}) : pa = a\}$, where $\mathrm{Pr}(\mathcal{M})$ is the set of projections in $\mathcal{M}$.

As with vector supports we will suppress the von Neumann algebra $\mathcal{M}$ when it is established by context.

We now present a projection-focused version of the support expansion function which turns out to coincide with it when supports are taken in a commutative von Neumann algebra (Theorem 2.1.4).

Definition 2.1.3. (cf. Definition 1.1.5) Given a Hilbert space H, von Neumann sub-algebra $\mathcal{M} \subseteq \mathcal{B}(H)$ and a dimension function $d : \mathrm{Pr}(\mathcal{M}) \to [0, \infty]$ the *projection d-support expansion function* $[0, \infty] \to [0, \infty]$ of an operator $a \in \mathcal{B}(H)$ is

$$\Phi_a^{d'}(x) = \sup\{d(\mathrm{s}_l^{\mathcal{M}}(ap)) : p \in \mathrm{Pr}(\mathcal{M}), d(p) \leq x\}.$$

[1] Dimension functions may take infinite cardinal values in which case the domain and codomain of support expansion functions may include these cardinals. We do not explore these cases in this dissertation.

As with the vector version we will often suppress the dimension function d when it is established by context.

Theorem 2.1.4. *Let H be a Hilbert space. If $\mathcal{M} \subseteq \mathcal{B}(H)$ is a commutative von Neumann sub-algebra with dimension function d then $\Phi_a^d = \Phi_a^{d'}$.*

Before proving Theorem 2.1.4, we need some auxiliary results.

Lemma 2.1.5. *Given $\xi_1, \xi_2 \in L^2(\mathbb{R})$, we have $\mathrm{supp}\,(\xi_1) \vee \mathrm{supp}\,(\xi_2) = \mathrm{supp}\,(\xi_1 + \lambda\xi_2)$ for all but perhaps countably many $\lambda \in \mathbb{C}$.*

Proof. Let $p = \mathrm{supp}\,(\xi_1) \vee \mathrm{supp}\,(\xi_2)$ and for each $\lambda \in \mathbb{C}$ let $p_\lambda = \mathrm{supp}\,(\xi_1 + \lambda\xi_2)^\perp \wedge p$. We first notice that $(p_\lambda)_{\lambda\in\mathbb{C}}$ are orthogonal. Indeed, fix $\lambda \neq \lambda'$ in $\mathbb{C}$ and let us show that $q_{\lambda,\lambda'} := p_\lambda \wedge p_{\lambda'} = 0$. For that, notice that $q_{\lambda,\lambda'}(\xi_1 + \lambda\xi_2) = q_{\lambda,\lambda'}(\xi_1 + \lambda'\xi_2) = 0$. So, as $\lambda \neq \lambda'$, this implies that $q_{\lambda,\lambda'}(\xi_1) = q_{\lambda,\lambda'}(\xi_2) = 0$. Then, if $0 < q_{\lambda,\lambda'}$, we have $p - q_{\lambda,\lambda'} < p$ and, as $(p - q_{\lambda,\lambda'})\xi_1 = \xi_1$ and $(p - q_{\lambda,\lambda'})\xi_2 = \xi_2$, this contradicts the minimality of p. So, $p_\lambda \wedge p_{\lambda'} = 0$.

The p_λ are a pairwise orthogonal family dominated by $p \in \mathrm{Pr}(L^\infty(\mathbb{R}))$, so by σ-finiteness of Lebesgue measure at most countably many can be nonzero. □

Corollary 2.1.6. *For any collection of vectors $(\xi_n)_{n=1}^\infty \subseteq L^2(\mathbb{R})$ we can find constants $(\lambda_n)_{n=1}^\infty \subseteq \mathbb{C}$ such that for every $N \in \mathbb{N}$:*

$$d\Big(\bigvee_{n=1}^{N} \mathrm{supp}\,(\xi_n)\Big) = d\Big(\mathrm{supp}\,\Big(\sum_{n=1}^{N} \lambda_n \xi_n\Big)\Big).$$

Proof. Apply Lemma 2.1.5 repeatedly. □

Proof of Theorem 2.1.4. Fix $a \in \mathcal{B}(L^2(\mathbb{R}))$. The inequality $\Phi_a \leq \Phi_a'$ is immediate, so we only need to show the reverse inequality holds. For each $p \in \mathrm{Pr}(L^\infty(\mathbb{R}))$, fix an orthonormal basis $(\xi_n^p)_{n=1}^\infty$ of $pL^2(\mathbb{R})$ and, using Corollary 2.1.6, fix a sequence $(\lambda_n^p)_{n=1}^\infty$ so that

$$d\Big(\bigvee_{n=1}^{N} \mathrm{supp}\,(a\xi_n^p)\Big) = d\Big(\mathrm{supp}\,\Big(a\sum_{i=n}^{N} \lambda_n^p \xi_n^p\Big)\Big).$$

Note that $s_l(ap) = \bigvee_{n=1}^\infty \mathrm{supp}\,(a\xi_n^p)$ for all $p \in \mathrm{Pr}(L^\infty(\mathbb{R}))$. Then, for each $x \in [0,\infty]$, we have that

$$\begin{aligned}
\Phi_a'(x) &= \sup\{d(s_l(ap)) : p \in \mathrm{Pr}(L^\infty(\mathbb{R})); d(p) \leq x\} \\
&= \sup\Big\{d\Big(\bigvee_{n=1}^{\infty} \mathrm{supp}\,(a\xi_n^p)\Big) : p \in \mathrm{Pr}(L^\infty(\mathbb{R})); d(p) \leq x\Big\} \\
&= \sup\Big\{d\Big(\bigvee_{n=1}^{N} \mathrm{supp}\,(a\xi_n^p)\Big) : p \in \mathrm{Pr}(L^\infty(\mathbb{R})); d(p) \leq x; N \in \mathbb{N}\Big\} \\
&= \sup\Big\{d\Big(\mathrm{supp}\,\Big(a\sum_{i=n}^{N} \lambda_n^p \xi_n^p\Big)\Big) : p \in \mathrm{Pr}(L^\infty(\mathbb{R})); d(p) \leq x; N \in \mathbb{N}\Big\} \\
&\leq \sup\{d(\mathrm{supp}\,(a\xi)) : d(\mathrm{supp}\,(\xi)) \leq x\} \\
&= \Phi_a(x).
\end{aligned}$$

So, we conclude that $\Phi_a' = \Phi_a$, as desired. □

If the von Neumann sub-algebra $\mathcal{M} \subseteq \mathcal{B}(H)$ is non-commutative then the definitions for Φ_a and ${\Phi_a}'$ both still make sense but they may not coincide as we see in the following example, courtesy of my advisor David Sherman.

Example 2.1.7. Let $H = \mathbb{C}^2$. Let τ_1 be the standard trace on $\mathcal{B}(\mathbb{C}^2)$ with weight 1 and τ_2 be the standard trace on $\mathcal{B}(\mathbb{C}^2)$ with weight 2. We denote the basis of $\mathbb{C}^4$ by $(\delta_n)_{n=1}^4$. Let I_1 be the projection onto the span of δ_1 and δ_2 and I_2 be the projection onto the span of δ_3 and δ_4.

We consider the von Neumann algebra $\mathcal{M} = \mathbb{C}I_1 \oplus \mathcal{B}(\mathbb{C}^2) \subseteq \mathcal{B}(\mathbb{C}^2 \oplus \mathbb{C}^2)$ and note that $\tau = \tau_1 \oplus \tau_2$ is a trace on $\mathcal{B}(\mathbb{C}^2 \oplus \mathbb{C}^2) \cong \mathcal{B}(\mathbb{C}^4)$ is a trace on $\mathcal{M}$. We equip $\mathrm{Pr}(\mathcal{M})$ with the dimension function d obtained by restricting τ to projections. Note that $I_1, I_2 \in \mathrm{Pr}(\mathcal{M})$ and τ_i essentially acts on I_i for $i \in \{1, 2\}$.

Now consider the partial isometry $a \in \mathcal{B}(\mathbb{C}^2 \oplus \mathbb{C}^2)$ given by $\delta_1 \mapsto \delta_3$, $\delta_2 \mapsto \delta_4$, $\delta_3 \mapsto 0$, $\delta_4 \mapsto 0$ and extended using linearity. We note that for any $\xi \in \mathbb{C}^4$: $\tau(\mathrm{supp}\,(\xi)) = 1$ if $I_1\xi = 0$ and $\tau(\mathrm{supp}\,(\xi)) = 2$ if $I_1\xi \neq 0$. So $\Phi_a(1) = 1$, since $I_1 a\xi = 0$. But $\mathrm{s}_l\,(aI_1) = I_2$ so ${\Phi_a}'(1) = 2$. This demonstrates that in the non-commutative setting Φ_a and Φ_a' need not coincide.

2.2 Discrete Support Expansion C*-Algebras with Standard Weight

Proofs in this section were developed in collaboration with David Sherman and Bruno Braga. Throughout this section we consider support expansion C*-subalgebras of $\mathcal{B}(\ell^2(\mathbb{N}))$ over the von Neumann algebra $\mathcal{M} \cong \ell^\infty(\mathbb{N})$, the diagonal operators equipped with the dimension function d obtained by restricting the standard trace on $\mathcal{B}(\ell^2(\mathbb{N}))$, often suppressing reference to it in our notation.

The next lemma gathers a few properties satisfied by support expansion functions of operators in this context.

Lemma 2.2.1. *For any $a \in \mathcal{B}(\ell^2(\mathbb{N}))$, Φ_a (Definition 1.1.5) is an increasing function so that*

(a) $\Phi_a(x) \leq \Phi_a(1)x$ for all $x \in [0, \infty]$,

(b) if Φ_a is bounded, then a is finite rank, and

(c) if $\Phi_a(n_1) = \Phi_a(n_2)$ for some $n_1 < n_2$ in $\mathbb{N}$, then $\Phi_a(n_1) = \Phi_a(x)$ for all $x > n_1$ in $[0, \infty]$. In particular, either Φ_a is bounded or $\Phi_a(n) \geq n$ for all $n \in \mathbb{N}$.

Proof. The function Φ_a is clearly increasing and we note for every $x \in [0, \infty)$ that $\Phi_a(x) = \Phi_a(\lfloor x \rfloor)$ where $\lfloor x \rfloor$ denotes the largest natural number less than or equal to x, i.e. the floor of x. Item (a) follows for natural number inputs since every projection in $\mathcal{B}(\ell^2(\mathbb{N}))$ is the SOT-convergent sum of dimension 1 projections while supports, dimension functions and supremums are all subadditive – the result for all $x \in [0, \infty]$ is a consequence.

If Φ_a is bounded, then there is a finite $F \subseteq \mathbb{N}$ so that $a_{n,m} \neq 0$ implies $(n, m) \in F \times \mathbb{N}$, so a is finite rank and Item (b) follows. At last, say $n_1 < n_2 \in \mathbb{N}$ and $\Phi_a(n_1) = \Phi_a(n_2)$. Then, if ξ is so that $|\mathrm{supp}\,(\xi)| \leq n_1$ and $\Phi_a(n_1) = |\mathrm{supp}\,(a\xi)|$, a must have no non-zero entries in any row off the support of $a\xi$ (otherwise we could find a vector η with $|\mathrm{supp}\,(\eta)| \leq n_2$ and $|\mathrm{supp}\,(a\eta)| > \Phi_a(n_1)$, a contradiction) so $\Phi_a(x) = \Phi_a(n_1)$ for all $x > n_1$ which demonstrates Item (c) and finishes the proof. □

We also have a hunch that a support expansion function should grow faster at the beginning and slower later on. After all, each projection in $\ell^\infty(\mathbb{N})$ can be characterized as the join of $(p_n)_{n\in\mathbb{N}}$ projections onto the basis vectors $(\delta_n)_{n\in\mathbb{N}}$ so if $\operatorname{supp}_{\ell^\infty(\mathbb{N})}(p \vee p_n)$ has dramatically higher dimension than $\operatorname{supp}_{\ell^\infty(\mathbb{N})}(p)$ then it seems we should be able to exclude one of the basis vectors beneath p and include p_n instead to increase the dimension. While there is truth to this (Proposition 2.2.3), our initial intuition was that support expansion functions were concave down. We note this is false with the following example.

Example 2.2.2. Support expansion functions need not be concave down. Indeed, let

$$a = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

and, considering the canonical inclusion of the 6-by-4 matrices in $\mathcal{B}(\ell^2(\mathbb{N}))$, we view a as an element in $\mathcal{B}(\ell^2(\mathbb{N}))$. Then $\Phi_a(1) = 3$, $\Phi_a(2) = 4$, and $\Phi_a(3) = 6$. So Φ_a is not concave down since $\frac{1}{2}\Phi_a(1) + \frac{1}{2}\Phi_a(3) = 4.5 > 4 = \Phi_a(2)$.

The following proposition formalizes in what manner support expansion functions "grow faster at the beginning and slower later on" in this setting.

Proposition 2.2.3. *For any* $a \in \mathcal{B}(\ell^2(\mathbb{N}))$, $\frac{\Phi_a(m)}{m} \geq \frac{\Phi_a(n)}{n}$ *whenever* $m \leq n$, *where* $m, n \in \mathbb{N}$ *are non-zero.*

Proof. We fix an $a \in \mathcal{B}(\ell^2(\mathbb{N}))$ and proceed by induction on n. The case when $n = 1$ is immediate. Now our induction hypothesis states that $\frac{\Phi_a(m)}{m} \geq \frac{\Phi_a(n)}{n}$ whenever $m \leq n$, where $m, n \in \mathbb{N}$ are non-zero. It suffices to show that $\frac{\Phi_a(n)}{n} \geq \frac{\Phi_a(n+1)}{n+1}$. Suppose not, so $n(\Phi_a(n+1) - \Phi_a(n)) > \Phi_a(n)$, we seek a contradiction.

Find $\xi \in \ell^2(\mathbb{N})$ with $d(\operatorname{supp}(\xi)) \leq n+1$ which realizes $d(\operatorname{supp}(a\xi)) = \Phi_a(n+1)$. Note that if $d(\operatorname{supp}(\xi)) < n+1$ then $\Phi_a(n+1) = \Phi_a(n)$ and thus $\frac{\Phi_a(n)}{n} \geq \frac{\Phi_a(n+1)}{n+1}$, which contradicts our supposition. Then what remains is the case where ξ has exactly $n+1$ non-zero basis components: $(\delta_{k_j})_{j=1}^{n+1}$. Let p_k denote the projection onto the span of the k-th basis vector of $\ell^2(\mathbb{N})$ and consider the vectors $\xi^{(j)} = p_{k_j}^\perp \xi$ obtained by zero-ing out the j-th non-zero basis component of ξ, where $1 \leq j \leq n+1$.

We note that $d(\operatorname{supp}(\xi^{(j)})) \leq n$ and so $d(\operatorname{supp}(a\xi^{(j)})) \leq \Phi_a(n)$. This implies that for each $1 \leq j \leq n+1$ the k_j-th column of a has at least $\Phi_a(n+1) - \Phi_a(n)$ non-zero rows which are zero in the $k_{j'}$-th column for each $j' \neq j$, $1 \leq j' \leq n+1$. But this implies that, for instance, $d(\operatorname{supp}(a\xi^{(1)})) \geq n(\Phi_a(n+1) - \Phi_a(n)) > \Phi_a(n)$. But, as noted earlier, $d(\operatorname{supp}(a\xi^{(1)})) \leq \Phi_a(n)$. This is the desired contradiction and so finishes the proof. ☐

Now that we have collected some properties of support expansion functions in this setting we move on to characterizing the collection of support expansion C*-algebras. First we note some quick examples.

Example 2.2.4. Consider $f : [0, \infty] \to [0, \infty]$. Then:

(a) If f is the zero map, then $\langle f \rangle = \{f\}$; so $C_{\langle f \rangle} = \{0\}$.

(b) If f is a nonzero constant function, then $\langle f \rangle = \{nf : n \in \mathbb{N}\}$; so $B_{\langle f \rangle}$ is the set of operators $a = [a_{n,m}]$ which have at most finitely many nonzero entries. Then $C_{\langle f \rangle} = \mathcal{K}(\ell^2(\mathbb{N}))$, the set of compact operators in $\mathcal{B}(\ell^2(\mathbb{N}))$.

(c) If $f(x) = x$ for every $x \in [0, \infty]$, then $\langle f \rangle$ is the set of lines through the origin with positive integer slope, so $C_{\langle f \rangle} = C_{\mathrm{RC}}$.

(d) If $f(x) = \infty$ for every $x > 0$, then $\langle f \rangle = \{f\}$; so $C_{\langle f \rangle} = B_{\langle f \rangle} = \mathcal{B}(\ell^2(\mathbb{N}))$.

In fact, Example 2.2.4 exhausts all possible support expansion C*-algebras we can obtain in this setting:

Theorem 2.2.5. *Let $d : Pr(l^\infty(\mathbb{N})) \to [0, \infty]$ be the dimension function induced by the standard trace on $\mathcal{B}(\ell^2(\mathbb{N}))$. For any family $\mathcal{F}$ of maps $[0, \infty] \to [0, \infty]$, we have that*

$$C_{\langle \mathcal{F} \rangle} \in \Big\{ \{0\}, \mathcal{K}(\ell^2(\mathbb{N})), C_{\mathrm{RC}}, \mathcal{B}(\ell^2(\mathbb{N})) \Big\}$$

where $\mathcal{K}(\ell^2(\mathbb{N}))$ denotes the compact operators inside of $\mathcal{B}(\ell^2(\mathbb{N}))$.

Proof. Let $\mathcal{F}$ be a family of maps $[0, \infty] \to [0, \infty]$. As $B_{\langle \mathcal{F} \rangle} = B_{\langle \{\Phi_a : a \in B_{\langle \mathcal{F} \rangle}\} \rangle}$ then since each Φ_a is increasing with $\Phi_a(x) = \Phi_a([x])$ for all $x \in [0, \infty)$ and $\Phi_a(0) = 0$, we can assume without loss of generality that every $f \in \langle \mathcal{F} \rangle$ is increasing with $f(x) = f([x])$ for all $x \in [0, \infty)$ and $f(0) = 0$. The following is straightforward:

(a) If $f(1) = 0$ for every $f \in \mathcal{F}$, then $C_{\langle \mathcal{F} \rangle} = \{0\}$.

(b) If $f(1) = \infty$ for some $f \in \mathcal{F}$, then $C_{\langle \mathcal{F} \rangle} = \mathcal{B}(\ell^2(\mathbb{N}))$.

We are left to analyze the case where $f(1) < \infty$ for all $f \in \mathcal{F}$, and $f_0(1) > 0$ for some $f_0 \in \mathcal{F}$. Suppose $\mathcal{F}$ is as such. We first show that $\mathcal{K}(\ell^2(\mathbb{N})) \subseteq C_{\langle \mathcal{F} \rangle} \subseteq C_{\mathrm{RC}}$. For the first inclusion, let $a = [a_{n,m}] \in \mathcal{B}(\ell^2(\mathbb{N}))$ be so that $a_{n,m} \neq 0$ for finitely many $n, m \in \mathbb{N}$, recalling that such operators are dense in $\mathcal{K}(\ell^2(\mathbb{N}))$. So Φ_a is bounded. We have $f_0 \in \mathcal{F}$ with $f_0(1) > 0$ and since f_0 is increasing there must be $k \in \mathbb{N}$ so that $\Phi_a \leq k f_0$. Hence, $a \in B_{k f_0} \subseteq C_{\langle \mathcal{F} \rangle}$ and, by the arbitrariness of a, we have that $\mathcal{K}(\ell^2(\mathbb{N})) \subseteq C_{\langle \mathcal{F} \rangle}$. For the second inclusion, take $f \in \langle \mathcal{F} \rangle$ and fix $a \in B_f$. Then, as $\Phi_a(x) \leq \Phi_a(1)x \leq f(1)x$ for all $x \in [0, \infty]$ (Lemma 2.2.1 Item (a)), similarly $\Phi_{a^*}(x) \leq f(1)x$ and as $f(1) < \infty$, we have that $a \in B_{\mathrm{RC}}$ (cf. Definition 1.1.2 and preceding discussion). As a and f were arbitrary, we have $C_{\langle \mathcal{F} \rangle} \subseteq C_{\mathrm{RC}}$.

We now show that $C_{\langle \mathcal{F} \rangle}$ must equal either $\mathcal{K}(\ell^2(\mathbb{N}))$ or C_{RC}. Suppose there is $f \in \langle \mathcal{F} \rangle$ and $a \in B_f$ so that Φ_a is unbounded. Then, by Lemma 2.2.1 Item (c), we have that $\Phi_a(n) \geq n$ for all $n \in \mathbb{N}$. Hence, $f(n) \geq n$ for all $n \in \mathbb{N}$ and we have that $B_{\mathrm{RC}} \subseteq B_{\langle \mathcal{F} \rangle}$ (cf. Definition 1.1.2 and preceding discussion). This shows that $C_{\mathrm{RC}} \subseteq C_{\mathcal{F}}$ and, by the previous paragraph, $C_{\mathrm{RC}} = C_{\langle \mathcal{F} \rangle}$. Suppose now that Φ_a is bounded for all $f \in \langle \mathcal{F} \rangle$ and all $a \in B_f$. Then $B_{\langle \mathcal{F} \rangle} \subseteq \mathcal{K}(\ell^2(\mathbb{N}))$ (Lemma 2.2.1 Item (b)) and it follows that $C_{\langle \mathcal{F} \rangle} \subseteq \mathcal{K}(\ell^2(\mathbb{N}))$. So, $C_{\langle F \rangle} = \mathcal{K}(\ell^2(\mathbb{N}))$. We have demonstrated the desired tetrachotomy. □

So in the discrete setting with standard weight on projections we see that the poset of support expansion C*-algebras is quite simple and, besides the motivating example C_{RC}, none of its elements are novel.

2.3 Discrete Support Expansion C*-Algebras with Nonstandard Weight

In the previous section we explored one way of generalizing the uniformly RC-finite operators – by replacing the linear bound on support expansion with some other family of functions. This was point (1) in the discussion following Definition 1.1.2, and we saw that it did not result in any novel C*-algebras besides the uniformly RC-finite operators themselves. In this section we will explore point (1) and (2) in the discussion following Definition 1.1.2, where we take a different weight on $\mathcal{M} \cong \ell^\infty(\mathbb{N})$, the diagonal operators, to measure the support projections of vectors. We will see that, in this context, varying the family of control functions gives rise to a rich poset of concrete C*-algebras, contrasting with the previous section.

Let p_n denote the projection onto the span of the n-th standard basis vector of $\ell^2(\mathbb{N})$ and note that we can define a dimension function $d : \mathrm{Pr}(\ell^\infty(\mathbb{N})) \to [0, \infty]$ by stipulating its value on each of the p_n and extending to higher rank projections using σ-additivity. We pick a bijection $u : \mathbb{N} \to \mathbb{Q}_+$ from the naturals to the positive rationals then construct a dimension function d_u by stipulating $d_u(p_n) = u(n)$. For the remainder of this subsection we will refer to d_u merely as d and often suppress it in our notation.

It is clear that the support expansion function Φ_a for any operator $a \in \mathcal{B}(\ell^2(\mathbb{N}))$ is increasing (this is always the case). We will not characterize the support expansion functions completely but in the following proposition we note a useful class of functions which can be realized as Φ_a for some $a \in \mathcal{B}(\ell^2(\mathbb{N}))$.

Proposition 2.3.1. *If $f : [0, \infty] \to [0, \infty]$ is strictly increasing, concave up and takes rationals to rationals with $f(0) = 0$ (for instance $f(x) = x^2$) then there is some isometry $a_f \in \mathcal{B}(\ell^2(\mathbb{N}))$ such that $\Phi_{a_f} = f$.*

Proof. Define a_f to be the operator which sends $\delta_n \mapsto \delta_{u^{-1}(f(u(n)))}$. We note that a_f sends basis vectors to basis vectors and does so injectively since f is strictly increasing. Thus a_f is an isometry (note it may not be a unitary, as $f(\mathbb{Q}_+)$ need not be all of $\mathbb{Q}_+$, consider $f(x) = x^2$ where $f(\mathbb{Q}_+)$ does not contain for instance 2).

We also observe that $d(s_l(a_f p_n)) = f(u(n))$ which establishes $\Phi_{a_f}(u(n)) \geq f(u(n))$ for all $n \in \mathbb{N}$ and thus $\Phi_{a_f}(q) \geq f(q)$ for all $q \in \mathbb{Q}_+$. Moreover, take any $p \in \mathrm{Pr}(\ell^\infty(\mathbb{N}))$ represented by the SOT convergent sum $p = \sum_{n=1}^\infty \lambda_n p_n$ where each $\lambda_n \in \{0, 1\}$ and note that $s_l(a_f p) = s_l(a_f \sum_{n=1}^\infty \lambda_n p_n) = s_l(\sum_{n=1}^\infty \lambda_n a_f p_n) = \sum_{n=1}^\infty \lambda_n p_{u^{-1}(f(u(n)))}$. So $d(s_l(a_f p)) = \sum_{n=1}^\infty \lambda_n f(u(n)) \leq f(\sum_{n=1}^\infty \lambda_n u(n)) = f(d(p))$, since increasing concave up functions which take 0 to 0 are super-additive. This establishes that $\Phi_{a_f}(x) \leq f(x)$ for all $x \in [0, \infty]$.

So we have that $\Phi_{a_f}(q) = f(q)$ for all $q \in \mathbb{Q}_+$ and that $f(q) \leq \Phi_{a_f}(x) \leq f(x)$ for every $x \in [0, \infty]$ and every rational $q \leq x$. Since f is increasing and concave up it is continuous and thus $f(x) = \sup\{f(q) : q \in \mathbb{Q}, q \leq x\}$. This establishes that $\Phi_{a_f}(x) = f(x)$ for all $x \in [0, \infty]$ and we are done. □

Now we will collect some results which allow us to determine how $C_{\langle \mathcal{F} \rangle}$ compares to $C_{\langle \mathcal{G} \rangle}$ merely from function-theoretic properties of the elements of $\langle \mathcal{F} \rangle$ and $\langle \mathcal{G} \rangle$:

Proposition 2.3.2. *If $\mathcal{F}, \mathcal{G} : [0, \infty] \to [0, \infty]$ are families of increasing functions such that for each $f \in \mathcal{F}$ there is some $g \in \langle \mathcal{G} \rangle$ such that $f \leq g$, then $B_{\langle \mathcal{F} \rangle} \subseteq B_{\langle \mathcal{G} \rangle}$.*

Proof. We observe that every $f \in \langle \mathcal{F} \rangle$ is obtained from a finite number of additions and compositions of elements of $\mathcal{F}$, each of which is dominated by something in $\langle \mathcal{G} \rangle$. Let g be defined by additions and compositions corresponding to those which constructed f, but by replacing each element of $\mathcal{F}$ with a function that dominates it in $\langle \mathcal{G} \rangle$, noting that $g \in \langle \mathcal{G} \rangle$. We have that $f \leq g$ since everything in sight is increasing, and thus $B_f \subseteq B_g \subseteq B_{\langle \mathcal{G} \rangle}$. Since $f \in \langle \mathcal{F} \rangle$ was arbitrary this gives us that $B_{\langle \mathcal{F} \rangle} \subseteq B_{\langle \mathcal{G} \rangle}$ as desired. □

The above proposition is in fact entirely general and not specific to the current setting at all.

Theorem 2.3.3. *Let $\mathcal{F}, \mathcal{G}$ be nonempty families of functions $[0,\infty] \to [0,\infty]$ which are strictly increasing, concave up, take rationals to rationals and 0 to 0. Then, $C_{\langle \mathcal{F} \rangle} \nsubseteq C_{\langle \mathcal{G} \rangle}$ if and only if there exists $f_0 \in \mathcal{F}$ such that for all $g \in \langle \mathcal{G} \rangle$ there exists some sequence $(x_n)_{n=1}^{\infty}$ such that $\lim_{n\to\infty} \frac{f_0(x_n)}{g(x_n)} = \infty$.*

Proof. First we suppose that no such f_0 exists, so for every $f \in \mathcal{F}$ we have some $g \in \langle \mathcal{G} \rangle$ and $n \in \mathbb{N}$ such that $f \le ng \in \langle \mathcal{G} \rangle$. So $B_{\langle \mathcal{F} \rangle} \subseteq B_{\langle \mathcal{G} \rangle}$ by Proposition 2.3.2 and thus $C_{\langle \mathcal{F} \rangle} \subseteq C_{\langle \mathcal{G} \rangle}$.

Now suppose there is some $f_0 \in \mathcal{F}$ as in the theorem statement. Then take the isometry $a_{f_0} \in B_{f_0}$ from Proposition 2.3.1 and fix any $g \in \langle \mathcal{G} \rangle$. By hypothesis we have a sequence $(x_n)_{n=1}^{\infty}$ such that $\lim_{n\to\infty} \frac{f_0(x_n)}{g(x_n)} = \infty$ so in particular we can find $x \in (0,\infty)$ such that $\frac{f_0(x)}{g(x)} > 1$. Moreover, since both f_0 and g are continuous we can take $x \in \mathbb{Q}_+$ without loss of generality.

We know that $a_{f_0} \delta_{u^{-1}(x)} = \delta_{u^{-1}(f(x))}$ and $d(s_l(\delta_{u^{-1}(f(x))})) = f(x)$. So for any $b \in B_g$ we must have that $b\delta_{u^{-1}(x)}$ is orthogonal to $\delta_{u^{-1}(f(x))}$, otherwise $\Phi_b(x) \ge d(s_l(b\delta_{u^{-1}(x)})) \ge f(x) > g(x)$ which contradicts the definition of b. Thus $\|a_{f_0} - b\| \ge \left\|(a_{f_0} - b)\delta_{u^{-1}(x)}\right\| \ge 1$ for every $b \in B_g$. Since $g \in \langle \mathcal{G} \rangle$ was arbitrary this implies that a_{f_0} has distance 1 from $B_{\langle \mathcal{G} \rangle}$ and is therefore not contained in $C_{\langle \mathcal{G} \rangle}$. But $a_{f_0} \in C_{\langle \mathcal{F} \rangle}$ so we have that $C_{\langle \mathcal{F} \rangle} \nsubseteq C_{\langle \mathcal{G} \rangle}$ as desired. □

Corollary 2.3.4. *Let f, g be functions $[0,\infty] \to [0,\infty]$ which are strictly increasing, concave up, take rationals to rationals and satisfy $f(0) = g(0) = 0$. Then $C_{\langle f \rangle} \nsubseteq C_{\langle g \rangle}$ if there exists some sequence $(x_n)_{n=1}^{\infty}$ tending to 0 such that $\lim_{n\to\infty} \frac{f(x_n)}{g(x_n)} = \infty$.*

Proof. Note that $C_{\langle g \rangle} = C_{\langle ng \rangle}$ for any $n \in \mathbb{N}$ (Proposition 2.3.2) so by dividing g by a sufficiently large integer ($n \ge g(1)$) we can assume without loss of generality that $g(x) \le x$ for all $x \in [0,1]$. Note that since g is increasing and concave up with $g(0) = 0$, $\frac{g(x)}{x}$ is monotone increasing and so $L = \lim_{x\to 0} \frac{g(x)}{x}$ exists and is finite. Moreover, L must be 0 otherwise $\lim_{n\to\infty} \frac{f(x_n)}{x_n} = L \lim_{n\to\infty} \frac{f(x_n)}{g(x_n)} = \infty$ which contradicts that f is a strictly increasing (and thus finite-valued) concave up function.

Now fix some $g_0 \in \langle g \rangle$. As an element of $\langle g \rangle$, g_0 is constructed from a finite number of additions and compositions of g. Since $g(x) \le x$ for all $x \in [0,1]$ we have that $g^{(n)}(x) \le g(x)$ for all $x \in [0,1]$ and $n \in N$. Also recall that $\lim_{x\to 0} \frac{g(x)}{x} = 0$ by the previous paragraph and so for every $n, N \in \mathbb{N}$ we have $Ng^{(n)}(x) \le Ng(x) \le x$ for sufficiently small x. We apply the above points repeatedly, and the fact that everything in sight is increasing, to unpack g_0 and conclude that there is some $N \in \mathbb{N}$ such that $g_0(x) \le Ng(x)$ for sufficiently small x. Then $\lim_{n\to\infty} \frac{f(x_n)}{g_0(x_n)} \ge \frac{1}{N} \lim_{n\to\infty} \frac{f(x_n)}{g(x_n)} = \infty$. As g_0 was arbitrary in $\langle g \rangle$ Theorem 2.3.3 gives us that $C_{\langle f \rangle} \nsubseteq C_{\langle g \rangle}$. □

Definition 2.3.5. Define the dimension function $d : \Pr(\ell^\infty(\mathbb{N})) \to [0,\infty]$ by stipulating $d(p_n) = u(n)$, where p_n denotes projection onto the span of the n-th basis vector of $\ell^2(\mathbb{N})$, and extend to all of $\Pr(\ell^\infty(\mathbb{N}))$ using σ-additivity. We denote by $\mathbb{O}$ the set of all C*-subalgebras of $\mathcal{B}(\ell^2(\mathbb{N}))$ of the form $C_{\langle \mathcal{F} \rangle}$ for some family $\mathcal{F}$ of functions $[0,\infty] \to [0,\infty]$ in this context. We view $\mathbb{O}$ as a poset with the order being given by the standard inclusion.

Theorem 2.3.6. *The* poset *$\mathbb{O}$ has uncountable increasing chains, uncountable decreasing chains and uncountable antichains.*

Proving this theorem will be our goal for the remainder of the subsection.

Proposition 2.3.7. *The* poset $\mathbb{O}$ *has increasing and decreasing chains with the cardinality of the continuum.*

Proof. For each $y \in (0, \infty)$ we define the function $\mathrm{step}_y : [0, \infty] \to [0, \infty]$ which is 0 for all $x < y$ and ∞ for all $x \geq y$. Observe that $\langle \mathrm{step}_y \rangle = \{\mathrm{step}_y\}$ for each $y \in (0, \infty)$ (this function is idempotent with respect to addition and composition) and so $B_{\langle \mathrm{step}_y \rangle} = B_{\mathrm{step}_y}$.

We note that for positive reals $y_1 < y_2$ we have $C_{\langle \mathrm{step}_{y_2} \rangle} \subseteq C_{\langle \mathrm{step}_{y_1} \rangle}$ from Proposition 2.3.2. Now take a rational number $q \in (y_1, y_2)$ and consider the operator a_q which sends the basis vector $\delta_{u^{-1}(q)}$ to itself and all other basis vectors to 0. We note that a_q is contained in $B_{\mathrm{step}_{y_1}}$ but has distance 1 from $B_{\mathrm{step}_{y_2}}$. Indeed, if $b \in B_{\mathrm{step}_{y_2}}$ then $d(s_l(b\delta_{u^{-1}(q)})) \leq \Phi_b(q) \leq \mathrm{step}_{y_2}(q) = 0$ so $b\delta_{u^{-1}(q)} = 0$. From this we get the proper containment $C_{\langle \mathrm{step}_{y_2} \rangle} \subsetneq C_{\langle \mathrm{step}_{y_1} \rangle}$.

So for every increasing (decreasing) chain of real numbers we have constructed a corresponding decreasing (increasing) chain in $\mathbb{O}$. The desired result follows immediately. $\square$

We note that the C^*-algebras produced in Proposition 2.3.7 are all isomorphic to $\mathcal{B}(H)$.

The following proposition is useful for constructing antichains and the same method will be utilized in Chapter 3 in a few instances. The proof technique is mine but I would like to express gratitude and give credit to my collaborator Bruno Braga for helping me write it in a comprehensible way – it is much easier to explain in person than in writing!

Proposition 2.3.8. *If* $(f_n)_{n\in\mathbb{N}}$ *is a sequence of strictly increasing, concave up functions* $[0, \infty] \to [0, \infty]$ *which take rationals to rationals and such that* $\lim_{x\to 0} \frac{f_n(x)}{x} = 0$ *for each* $n \in \mathbb{N}$ *then there is a* $g : [0, \infty] \to [0, \infty]$ *satisfying all of the same properties such that* $C_{\langle g \rangle}$ *and* $C_{\langle f_n \rangle}$ *are incomparable for all* $n \in \mathbb{N}$.

Proof. For didactic reasons, we first prove the proposition with the extra assumption that $(f_n)_{n\in\mathbb{N}}$ is a constant sequence, say $f_n = f$ for all $n \in \mathbb{N}$. As $C_{\langle f \rangle} = C_{\langle mf \rangle}$ for all $m \in \mathbb{N}$ (Proposition 2.3.2), we can assume that $f(x) \leq x$ for all $x \in [0, 1]$. We now construct the desired function g. It will be useful for the reader to have in mind that our approach will be the following: we construct g in a piece-wise manner and in a way that we can use Corollary 2.3.4 in order to guarantee that $C_{\langle g \rangle}$ and $C_{\langle f \rangle}$ are incomparable.

We start by setting some notation and pointing out some very elementary facts about affine functions and their relation with f. Precisely, given $x, y, h > 0$, we let $\ell(x, y, h)$ be the line which sends x to y and has h as its x-intercept, i.e. $\ell(x, y, h)(t) = \frac{y}{x-h}t - \frac{yh}{x-h}$ for all $t \in \mathbb{R}$. The construction of g will be based in the following: given $x, y, h > 0$,

(a) as $\lim_{t\to h} \ell(x, y, h)(t) = 0$, we have that $\lim_{t\to h} \frac{f(t)}{\ell(x,y,h)(t)} = \infty$ and

(b) since $\lim_{t\to 0} \frac{f(t)}{t} = 0$, we have that $\lim_{t\to 0} \frac{\ell(x,y,0)(t)}{f(t)} = \infty$.

The next two facts isolate the conclusions from points (a) and (b) which we need – we recommend the reader to guide themselves by Figure 2.1 in the construction of g.

Fact 2.3.9. *Given* $x, y, h, N > 0$ *with* $x > h$*, there is a rational* $x' \in (h, x)$ *so that* $\frac{f(x')}{\ell(x,y,h)(x')} > N$.

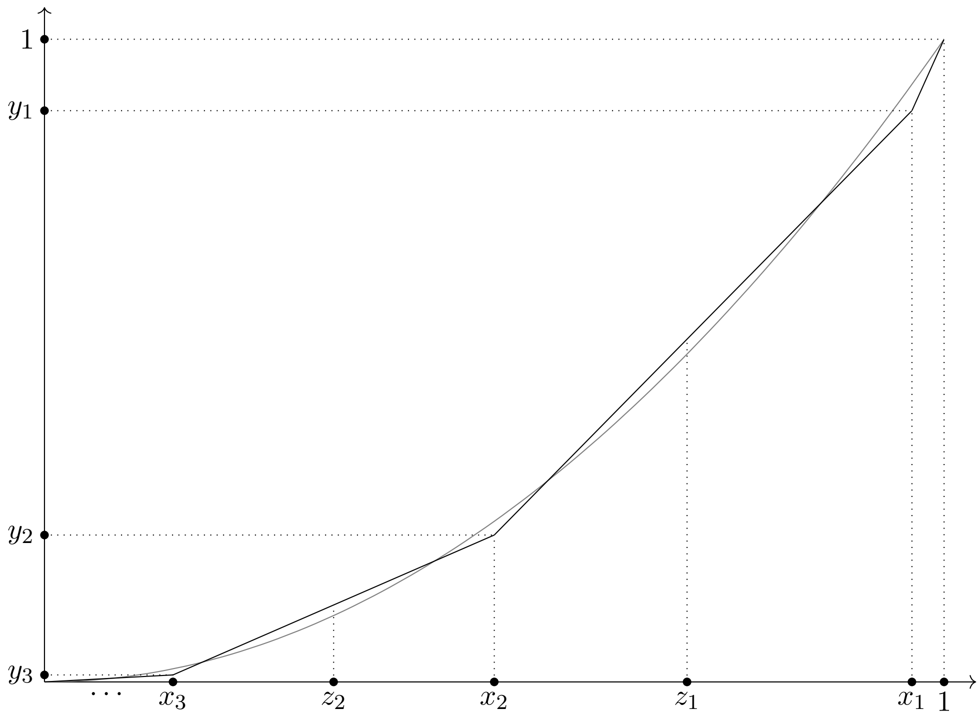

FIGURE 2.1: In the graph above, the smooth function represents f and the piecewise linear function represents g. Note that the scale is modified so that the general behavior of g with respect to f can be represented in the graph.

Fact 2.3.10. *Given $x, y, h, N > 0$, there are rational $z' \in (0, x)$ and $h' \in (0, \min(h, x))$ so that $\frac{\ell(x,y,h')(z')}{f(z')} > N$.*

Let $x_0 = y_0 = 1$ and $h_0 = \frac{1}{2}$. Then, alternating between Fact 2.3.9 and Fact 2.3.10 (with Fact 2.3.9 being the first we use), one can find strictly decreasing rational number valued sequences $(x_n)_{n=1}^{\infty}$, $(y_n)_{n=1}^{\infty}$, $(z_n)_{n=1}^{\infty}$, and $(h_n)_{n=1}^{\infty}$ in $[0, 1]$ tending to 0 so that

(a) $\frac{f(x_n)}{\ell(x_{n-1},y_{n-1},h_{n-1})(x_n)} > n$ for all $n \in \mathbb{N}$,

(b) $y_n = \ell(x_{n-1}, y_{n-1}, b_{n-1})(x_n)$ for all $n \in \mathbb{N}$,

(c) $x_{n+1} < z_n < x_n$ for all $n \in \mathbb{N}$,

(d) $0 < h_n < x_n$ for all $n \in \mathbb{N}$,

(e) $\frac{\ell(x_n,y_n,h_n)(z_n)}{f(z_n)} > n$ for all $n \in \mathbb{N}$.

We define $g : [0, \infty] \to [0, \infty]$ by letting

$$g(x) = \begin{cases} 2x - 1, & \text{if } x > x_1, \\ \ell(x_n, y_n, h_n)(x), & \text{if } x \in (x_{n+1}, x_n], \\ 0, & \text{if } x = 0. \end{cases}$$

It is clear from its piecewise definition and (b) that g is continuous. By its definition, $\ell(x_n, y_n, h_n)$ has slope $\frac{y_n}{x_n - h_n}$. However, by (b), the slope of $\ell(x_{n-1}, y_{n-1}, b_{n-1})$ must equal $\frac{y_n}{x_n - h_{n-1}}$. Therefore, as $h_n < h_{n-1}$, the slope of $\ell(x_n, y_n, h_n)$ is smaller than the slope of $\ell(x_{n-1}, y_{n-1}, h_{n-1})$. Hence g is concave up. Moreover, by (d), the slope of each $\ell(x_n, y_n, b_n)$ is positive, so g is strictly increasing. Also note that each piecewise component is affine with rational slope and rational x-intercept, and so g takes rationals to rationals.

We now aim to use Corollary 2.3.4 to show that $C_{\langle f \rangle}$ and $C_{\langle g \rangle}$ are incomparable in $\mathbb{O}$. First observe that

$$\begin{aligned}\lim_{n\to\infty} \frac{f(x_n)}{g(x_n)} &= \lim_{n\to\infty} \frac{f(x_n)}{\ell(x_n, y_n, h_n)(x_n)} \\ &= \lim_{n\to\infty} \frac{f(x_n)}{\ell(x_{n-1}, y_{n-1}, h_{n-1})(x_n)} \\ &\geq \lim_{n\to\infty} n = \infty,\end{aligned}$$

so by Corollary 2.3.4 $C_{\langle f \rangle} \not\subseteq C_{\langle g \rangle}$. Now observe that

$$\begin{aligned}\lim_{n\to\infty} \frac{g(z_n)}{f(z_n)} &= \lim_{n\to\infty} \frac{\ell(x_n, y_n, h_n)(z_n)}{f(z_n)} \\ &\geq \lim_{n\to\infty} n = \infty\end{aligned}$$

so by the same Corollary $C_{\langle g \rangle} \not\subseteq C_{\langle f \rangle}$ thus they are incomparable.

To wrap up this case of a single function f it remains for us to show that $\lim_{x\to 0} \frac{g(x)}{x} = 0$. Indeed, g is increasing and concave up with $\lim_{x\to 0} g(x) = \lim_{n\to\infty} y_n = 0$ so $\frac{g(x)}{x}$ is monotone increasing and thus $L = \lim_{x\to 0} \frac{g(x)}{x}$ must exist. Suppose for the sake of contradiction that $L > 0$, then $0 = \lim_{n\to\infty} \frac{f(x_n)}{x_n} = \left(\lim_{n\to\infty} \frac{f(x_n)}{g(x_n)}\right) \cdot \left(\lim_{n\to\infty} \frac{g(x_n)}{x_n}\right) = \infty \cdot \frac{1}{L} = \infty$. We have obtained a contradiction so $L = 0$ as desired.

The result for a single f is now proven, so consider $(f_n)_{n\in\mathbb{N}}$ as in the statement of this proposition, i.e. $(f_n)_{n\in\mathbb{N}}$ is not necessarily constant. The proof for this case is actually completely analogous and the only modification needed is that, when using Facts 2.3.9 and 2.3.10 in order to find strictly decreasing sequences of rationals $(x_n)_{n=1}^{\infty}$, $(y_n)_{n=1}^{\infty}$, $(z_n)_{n=1}^{\infty}$, and $(h_n)_{n=1}^{\infty}$ in $[0,1]$ tending to 0, we must replace (a) and (e) above by the stronger statements that $\frac{f_k(x_n)}{\ell(x_n, y_n, b_n)(x_n)} \geq n$ and $\frac{\ell(x_{n-1}, y_{n-1}, b_{n-1})(z_n)}{f_k(z_n)} \geq n$ for all $n \in \mathbb{N}$ and all $k \leq n$. Since this is not an issue, we are done. □

Corollary 2.3.11. *The* poset $\mathbb{O}$ *has uncountable antichains.*

Proof. Let $\mathbb{A} \subseteq \mathbb{O}$ be the partially ordered set consisting of C*-algebras of the form $C_{\langle f \rangle}$ where $f : [0,\infty] \to [0,\infty]$ is strictly increasing, concave up, takes rationals to rationals and such that $\lim_{x\to 0} \frac{f(x)}{x} = 0$, equipped with standard inclusion. We note that $\mathbb{A}$ is non-empty since $f(x) = x^2$ satisfies all of the hypotheses.

Zorn's lemma assures us that $\mathbb{A}$ has a maximal antichain A. Suppose for the sake of contradiction that A has finite or countably infinite cardinality. Then Proposition 2.3.8 produces a $g : [0,\infty] \to [0,\infty]$ satisfying all of the hypotheses so that $C_{\langle g \rangle} \in \mathbb{A}$ and is incomparable to every element of A. Then $A \cup \{C_{\langle g \rangle}\}$ is an antichain in $\mathbb{A}$ strictly larger than A which contradicts its maximality. We conclude that A has uncountably infinite cardinality. As $\mathbb{A}$ is a substructure of the poset $\mathbb{O}$ with the same order relation, A is also an antichain for $\mathbb{O}$ and this completes the proof. □

Proposition 2.3.7 and Corollary 2.3.11 together prove Theorem 2.3.6. This concludes our exploration of the poset $\mathbb{O}$ and support expansion C*-subalgebras of $\mathcal{B}(\ell^2(\mathbb{N}))$ over $\ell^\infty(\mathbb{N})$ (the diagonal operators) equipped with the dimension function d_u on $\mathrm{Pr}(\ell^\infty(\mathbb{N}))$.

Our treatment in this section is relatively shallow and there is certainly more that could be said. Since there are numerous dimension functions with which we could equip $\mathrm{Pr}(\ell^\infty(\mathbb{N}))$ and each of them might result in different interesting posets of support expansion C*-algebras, we think of this example as providing contrast to the situation in Section 2.2 rather than a subject of interest in and of itself. As we see, the poset of support expansion C*-algebras can be essentially trivial (Section 2.2) or extremely rich (Section 2.3).

In Chapter 3 we explore the poset of support expansion C*-subalgebras of $\mathcal{B}(L^2(\mathbb{R}))$ over $L^\infty(\mathbb{R}) \cong \mathcal{M} \subseteq \mathcal{B}(L^2(\mathbb{R}))$ the multiplication operators. Some of our results there will hearken back to this section but our treatment will be substantially more thorough.

Chapter 3

Continuous Support Expansion C*-Algebras

In this chapter we discuss the support expansion C*-algebras (Definition 1.1.7) which can be obtained when $H = L^2(\mathbb{R}, \mu)$ and $L^\infty(\mathbb{R}, \mu) \cong \mathcal{M} \subseteq \mathcal{B}(H)$ the multiplication operators. Here μ is taken to be the standard Lebesgue measure on $\mathbb{R}$ and will often be omitted from our notation. We will equip $\Pr(\mathcal{M})$ with the dimension function d induced by integrating against μ.

Our main object of interest is the poset of support expansion C*-algebras in $\mathcal{B}(L^2(\mathbb{R}, \mu))$ ordered by standard inclusion, which we refer to as $\mathbb{P}$. As shown in Theorems 2.2.5 and 2.3.6 the analogous posets of support expansion C*-algebras in $\mathcal{B}(\ell^2(\mathbb{N}))$ can be simple or robust depending on choice of dimension function. In the continuous setting, $\mathbb{P}$ is quite rich, and we will produce results similar to those in Section 2.3 for the poset $\mathbb{O}$ (Definition 2.3.5). In particular we will obtain results comparable to Propositions 2.3.7 and 2.3.8, Corollary 2.3.11 and Theorem 2.3.6. Additionally, $\mathbb{P}$ has some rigidness and we will obtain some results about immediate successor elements in $\mathbb{P}$ which we did not demonstrate in $\mathbb{O}$ (and in fact may not hold in $\mathbb{O}$).

3.1 Support Expansion Functions for Operators on $L^2(\mathbb{R})$

In this section, we thoroughly explore the support expansion functions which arise in this setting. The results of this section will be used in the following ones in order to analyze the poset $\mathbb{P}$ of support expansion C*-algebras in $\mathcal{B}(L^2(\mathbb{R}))$.

3.1.1 Basic Properties of Support Expansion Functions

Since we will work extensively with support expansion functions in this section we restate Definition 1.1.5 for the reader's convenience:

Definition 1.1.5. Given a Hilbert space H, von Neumann sub-algebra $\mathcal{M} \subseteq \mathcal{B}(H)$ and a dimension function $d : \Pr(\mathcal{M}) \to [0, \infty]$ the *d-support expansion function*[1] $[0, \infty] \to [0, \infty]$ of an operator $a \in \mathcal{B}(H)$ is

$$\Phi_a^d(x) = \sup\{d(\operatorname{supp}_{\mathcal{M}}(a\xi)) : d(\operatorname{supp}_{\mathcal{M}}(\xi)) \leq x\}.$$

Recall our present setting in the definition above: $H = L^2(\mathbb{R})$, $L^\infty(\mathbb{R}) \cong \mathcal{M}$ the multiplication operators and $\Pr(\mathcal{M})$ is equipped with the dimension function d induced by integrating against μ, standard Lebesgue measure.

[1] Dimension functions may take infinite cardinal values in which case the domain and codomain of support expansion functions may include these cardinals. We do not explore these cases in this dissertation.

Also recall Lemma 1.1.6:

Lemma 1.1.6. *Fix a Hilbert space* H*, von Neumann sub-algebra* $\mathcal{M} \subseteq \mathcal{B}(H)$ *and dimension function* d *on Pr*$(\mathcal{M})$*. Then for* $a, b \in \mathcal{B}(H)$ *and nonzero* $\lambda \in \mathbb{C}$*:*

(a) $\Phi^d_{\lambda a} = \Phi^d_a$.

(b) $\Phi^d_{a+b} \le \Phi^d_a + \Phi^d_b$.

(c) $\Phi^d_{ab} \le \Phi^d_a \circ \Phi^d_b$.

To the above we can add the following lower semi-continuity property of support expansion functions in the current setting:

Lemma 3.1.3. *Let* $(a_n)_{n=1}^{\infty} \subseteq \mathcal{B}(L^2(\mathbb{R}))$ *be a sequence converging* SOT *to an operator* a*. Then* $\Phi_a \le \liminf_n \Phi_{a_n}$*.*

Proof. The following is a standard exercise in measure theory and we omit its proof: if $(\xi_n)_n$ is a sequence in $L^2(\mathbb{R})$ converging to $\xi \in L^2(\mathbb{R})$ in norm, then $d(\operatorname{supp}(\xi)) \le \liminf_n d(\operatorname{supp}(\xi_n))$. Fix $x \in [0,\infty]$ and $\xi \in L^2(\mathbb{R})$ with $\operatorname{supp}(\xi) \le x$. Then, as $a\xi = \lim_n a_n\xi$, we have $d(\operatorname{supp}(a\xi)) \le \liminf_n d(\operatorname{supp}(a_n\xi))$. If we take the supremum on both sides over all such ξ we get $\Phi_a(x) \le \sup_{\operatorname{supp}(\xi)\le x} \liminf_n d(\operatorname{supp}(a_n\xi)) \le \liminf_n \sup_{\operatorname{supp}(\xi)\le x} d(\operatorname{supp}(a_n\xi)) = \liminf_n \Phi_{a_n}(x)$ as desired. □

3.1.2 A Tale of Three Families

The goal of this subsection is to understand which functions can arise as support expansion functions in this context. More precisely, the plan is to understand the properties satisfied by functions $[0,\infty] \to [0,\infty]$ of the form Φ_a, for $a \in \mathcal{B}(L^2(\mathbb{R}))$, as well as to understand which properties imposed on a function $f : [0,\infty] \to [0,\infty]$ are enough to assure that $f = \Phi_a$ for an appropriate $a \in \mathcal{B}(L^2(\mathbb{R}))$. This culminates in Theorem 3.1.6.

For any function $f : [0,\infty] \to [0,\infty]$, we let $\overline{f} : [0,\infty] \to [0,\infty]$ denote the map

$$\overline{f}(x) = \begin{cases} \frac{f(x)}{x}, & \text{if } x \neq 0, \\ \infty, & \text{if } x = 0, \\ 0, & \text{if } x = \infty. \end{cases}$$

Definition 3.1.4. Consider a function $f : [0,\infty] \to [0,\infty]$.

(a) We say that f is ICOD (this acronym abbreviates "increasing and concave down") if $f(0) = 0$ and it is increasing and concave down.

(b) We say that f is ISOD (this acronym abbreviates "increasing and slope-to-origin decreasing") if $f(0) = 0$ and it is increasing and $\frac{f(x)}{x}$ is decreasing.

(c) We say that f is SUPPEXP (this acronym abbreviates "support expansion") if $f = \Phi_a$ for some $a \in \mathcal{B}(L^2(\mathbb{R}))$.

By abuse of notation, we also denote the subsets of all functions $[0,\infty] \to [0,\infty]$ which are ICOD, ISOD, and SUPPEXP by ICOD, ISOD, and SUPPEXP, respectively.

Notice that, by definition, $f(0) = 0$ for all functions in either ICOD or ISOD, and the same also holds for maps in SUPPEXP. We point out that the condition for ISOD is reminiscent of Proposition 2.2.3 from the discrete setting with standard weight. The following proposition gathers some other simple properties of the collections ICOD and ISOD.

Proposition 3.1.5. *The following holds.*

(a) Both sets ICOD *and* ISOD *are closed under addition and composition.*

(b) The proper inclusion $\mathrm{ICOD} \subsetneq \mathrm{ISOD}$ *holds.*

(c) For all $f \in \mathrm{ISOD}$*, either* $f(x) < \infty$ *for all* $x < \infty$ *or* $f(x) = \infty$ *for all* $x > 0$.

(d) For all $f \in \mathrm{ISOD} \setminus \{0\}$*, we have that* $\lim_{x\to 0} \frac{f(x)}{x} > 0$.

Proof. (a) Closedness under addition and composition of ICOD and ISOD are more or less routine to verify.

(b) For the inclusion $\mathrm{ICOD} \subseteq \mathrm{ISOD}$, take $f \in \mathrm{ICOD}$ and fix $x_0 \in [0, \infty)$. Draw a line to the origin and note that for any $x \in [0, x_0]$ we have $f(x) \geq \frac{f(x_0)}{x_0} \cdot x$ since f is concave down and $f(0) = 0$. Now divide both sides by x and the result is clear. To see the inclusion is proper, consider the pointwise supremum of the two functions $f(x) = x$ and $g(x) = 2x \cdot \chi_{[0,1]}(x)$.

(c) Suppose there is $x < \infty$ for which $f(x) = \infty$. Then, as f is increasing, $f(y) = \infty$ for all $y > x$, and as $\bar{f}$ is decreasing $f(y) \geq y f(x)/x = \infty$ for all $y \in (0, x)$.

(d) Take $f \in \mathrm{ISOD}$. If $f(x) = \infty$ for all $x > 0$, the result is clear, so assume $f(x) < \infty$ for all $x < \infty$. As f is nonzero and $\bar{f}$ is decreasing, we have that $f(1) > 0$. Therefore, using that $\bar{f}$ is decreasing again, we have that $f(x)/x \geq f(1)$ for all $x \leq 1$, so $\lim_{x\to 0} \frac{f(x)}{x} \geq f(1) > 0$. □

The remainder of this subsection is dedicated to the proof of the following result.

Theorem 3.1.6. *The inclusions* $\mathrm{ICOD} \subseteq \mathrm{SUPPEXP} \subseteq \mathrm{ISOD}$ *hold.*

Although Theorem 3.1.6 does not give us a complete characterization of support expansion functions, it is enough to completely understand the support expansion C*-algebras inside of $\mathcal{B}(L^2(\mathbb{R}))$, which we will see later (Theorem 3.2.11).

We start by showing the first inclusion of Theorem 3.1.6. The construction in this proposition will also be essential in the sections to come.

Proposition 3.1.7. *Let* $f \in \mathrm{ICOD}$ *and* $r \in [0, \infty]$*, and assume that* f *is strictly increasing on* $[0, r]$*. Then define*

$$(a_{f,r}\xi)(x) = \begin{cases} \sqrt{(f^{-1})'(x)}\xi(f^{-1}(x)) & x \in [\lim_{t\to 0} f(t), f(r)] \\ 0 & \textit{otherwise} \end{cases}$$

for all $\xi \in L^2(\mathbb{R})$ *and all* $x \in \mathbb{R}$*. This defines a bounded operator on* $L^2(\mathbb{R})$*. Moreover, if* f *is constant on* $[r, \infty]$ *and* $\lim_{x\to 0} f(x) = 0$*, then* $\Phi_{a_{f,r}} = f$.

Proof. Since $f : (0, r) \to (\lim_{x\to 0} f(x), f(r))$ is a strictly increasing concave down function, $f^{-1} : (\lim_{x\to 0} f(x), f(r)) \to (0, r)$ is a well defined strictly increasing concave up function. This allows us to use a standard change of variables in order to obtain that $\|a_{f,r}\xi\| \leq \|\xi\|$ for all $\xi \in L^2(\mathbb{R})$.[2] So, $a_{f,r}$ is a well defined bounded operator.

Suppose now that f is constant on $[r, \infty]$ and $\lim_{x\to 0} f(x) = 0$. Since f is concave down it can be shown that this operator witnesses maximum expansion on vectors of the form $\xi = \chi_{[0,x]}$, i.e.,

$$\Phi_{a_{f,r}}(x) = d(\mathrm{supp}\,(a_{f,r}\chi_{[0,x]}))$$

[2]The reader can see the details for that in many standard measure theory books. For instance, see Rudin, 1987, Chapter 7 for change of variables and Roberts and Varberg, 1973, Theorem A for the fact that concave functions on (a, b) are absolutely continuous.

for all $x \in [0,\infty]$. Note that when $x \leq r$, we have $\operatorname{supp}\left(a_{f,r}\chi_{[0,x]}\right) = \chi_{[0,f(x)]}$, while when $x \geq r$, we have $\operatorname{supp}\left(a_{f,r}\chi_{[0,x]}\right) = \chi_{[0,f(r)]} = \chi_{[0,f(x)]}$ since f is constant on $[r,\infty]$. This gives us that $\Phi_{a_{f,r}} = f$. □

Corollary 3.1.8. *Every* ICOD *function* $f : [0,\infty] \to [0,\infty]$ *can be realized as a support expansion function.*

Proof. If $f(x) = \infty$ for all $x \in (0,\infty)$ then let a be the projection onto a vector with infinite support, then $\Phi_a(x) = \infty = f(x)$ for every $x \in (0,\infty]$ and $\Phi_a(0) = 0 = f(0)$.

If $f(x) < \infty$ for every $x \in (0,\infty)$, let $g(x) = f(x) - \lim_{t\to 0} f(t)$, noting that since $g \in$ ICOD there exists an $r \in [0,\infty]$ for which g is strictly increasing on $[0,r]$ and constant on $[r,\infty]$. Then take $a_{g,r}$ as in Proposition 3.1.7, which gives us that $\Phi_{a_{g,r}} = g$. Let $b \in \mathcal{B}(L^2(\mathbb{R}))$ be given by

$$b\xi = a_{g,r}\xi + \langle \xi, \chi_{[-1,0]}\rangle \chi_{[-\lim_{x\to 0} f(x),0]}$$

and note that $\Phi_b = f$ by evaluating on $\chi_{[0,x]} + \eta$ where η is a vector with arbitrarily small norm supported on $[-1,0]$. □

In order to finish the proof of Theorem 3.1.6, we are left to show the inclusion SUPPEXP $\subseteq$ ISOD. For that, we need some preliminary results. The following standard facts about commuting projections will be used for the next proposition.

Fact 3.1.9. *Given* $p, q, q_1, q_2 \in$ *Pr*$(L^\infty(\mathbb{R}))$ *with* $q_1 \leq q_2$*, we have*

(1) $p + q = p \vee q + p \wedge q$*;*

(2) $d(p \vee q_1) - d(q_1) \geq d(p \vee q_2) - d(q_2)$*;*

(3) $s_l(a(p \vee q)) = s_l(ap) \vee s_l(aq)$ *for any* $a \in \mathcal{B}(L^2(\mathbb{R}))$*. (This item is true for non-commuting projections as well.)*

Proposition 3.1.10. *Let* $a \in \mathcal{B}(L^2(\mathbb{R}))$*,* $x \in (0,\infty)$*, and* $n \in \mathbb{N}$*. Then, if* $y = \frac{n+1}{n}x$*, we have* $\overline{\Phi_a}(x) \geq \overline{\Phi_a}(y)$*.*

Proof. Throughout this proof we will use the projection focused definition of Φ_a: $\Phi_a(x) = \sup\{d(\mathrm{s}_l^{\mathcal{M}}(ap)) : p \in \Pr(\mathcal{M}), d(p) \leq x\}$ (see Definition 2.1.3 and Theorem 2.1.4).

Suppose for the sake of contradiction that $\overline{\Phi_a}(y) > \overline{\Phi_a}(x)$. Then

$$\Phi_a(y) = \overline{\Phi_a}(y)\cdot y > \overline{\Phi_a}(x)\cdot y$$

and we can pick a projection $p \in \Pr(L^\infty(\mathbb{R}))$ such that $d(p) \leq y$ and $d(s_l(ap)) > \overline{\Phi_a}(x)\cdot y$.

Write $p = \sum_{k=1}^{n+1} p_k$ for some orthogonal sequence $(p_k)_{k=1}^{n+1} \subseteq \Pr(L^\infty(\mathbb{R}))$ all of which have dimension exactly $\frac{d(p)}{n+1}$ (so any n of them combined have dimension $\leq x$). Then, if $I \subseteq \{1,\ldots,n+1\}$ does not contain k, we have that

$$d\Big(\bigvee_{i\in I} s_l(ap_i)\Big) = d\Big(s_l\big(a\bigvee_{i\in I} p_i\big)\Big) \leq \Phi_a(x)$$

(see Fact 3.1.9.3). Therefore, for any such I, we have that

$$\begin{aligned} d\Big(s_l(ap_k) \vee \bigvee_{i\in I} s_l(ap_i)\Big) - d\Big(\bigvee_{i\in I} s_l(ap_i)\Big) &\geq d\Big(\bigvee_{i=1}^{n+1} s_l(ap_i)\Big) - d\Big(\bigvee_{i\neq k} s_l(ap_i)\Big) \\ &= d(s_l(ap)) - d\Big(\bigvee_{i\neq k} s_l(ap_i)\Big) \\ &> \overline{\Phi_a}(x)\cdot y - \Phi_a(x). \end{aligned}$$

(see Facts 3.1.9.2 and 3.1.9.3). Apply this inequality repeatedly, starting from the empty set and adding in p_1 for $k=1$ then p_2 for $k=2$ and so on until we have

$$d(\bigvee_{k=1}^{n} s_l(ap_k)) > n\left(\overline{\Phi_a}(x)\cdot y - \Phi_a(x)\right).$$

Then we get that:

$$\begin{aligned} \Phi_a(x) &\geq d\Big(\bigvee_{k=1}^{n} s_l(ap_k)\Big) \\ &> n\left(\overline{\Phi_a}(x)\cdot y - \Phi_a(x)\right) \\ &= n(\overline{\Phi}(x)(y-x)) \\ &= \Phi_a(x) \quad \text{(recall that } y = x(n+1)/n). \end{aligned}$$

So, $\Phi_a(x) > \Phi_a(x)$; contradiction. □

Corollary 3.1.11. *Let $a \in \mathcal{B}(L^2(\mathbb{R}))$, $x \in (0,\infty)$, and $q \in \mathbb{Q}$ with $q \geq 1$. Then we have $\overline{\Phi_a}(x) \geq \overline{\Phi_a}(qx)$.*

Proof. Apply Proposition 3.1.10 repeatedly. □

The "slope-to-origin" non-decreasing property in Corollary 3.1.11 implies that the function Φ_a must be continuous (except perhaps at $\{0,\infty\}$):

Proposition 3.1.12. *Let $f : [0,\infty] \to [0,\infty]$ be increasing and satisfy that $\overline{f}(x) \geq \overline{f}(qx)$ for all $x \in [0,\infty]$ and all $q \in \mathbb{Q}$ with $q \geq 1$. Then f is continuous on $(0,\infty)$.*

Proof. Suppose f is not left semi-continuous at some $x \in (0,\infty)$. Then there is $\varepsilon > 0$ such that $f(x) - f(x-\delta) > \varepsilon$ for every $\delta \in (0,x]$. Pick a positive rational $q < \min(\varepsilon/f(x), 1)$. Since $\frac{1}{1-q} > 1$, we have by hypothesis that $0 \geq \overline{f}(\frac{1}{1-q}(1-q)x) - \overline{f}((1-q)x) = \overline{f}(x) - \overline{f}(x-qx)$. Then we have

$$0 > \overline{f}(x) - \overline{f}(x-qx) = \frac{f(x)}{x} - \frac{f(x-qx)}{x-qx} = \frac{f(x) - f(x-qx) - qf(x)}{(1-q)x} > \frac{\varepsilon - qf(x)}{(1-q)x} > 0.$$

This gives us a contradiction; so f is left semi-continuous. Right semi-continuity follows similarly. □

Corollary 3.1.11 and Proposition 3.1.12 immediately give us the following:

Corollary 3.1.13. *Let $a \in \mathcal{B}(L^2(\mathbb{R}))$. Then Φ_a is continuous on $(0,\infty)$.* □

Proof of Theorem 3.1.6. By Corollary 3.1.8, it remains to show that SUPPEXP $\subseteq$ ISOD. Fix $a \in \mathcal{B}(L^2(\mathbb{R}))$. As Φ_a is increasing by inspection, we only need to show that $\overline{\Phi_a}$ is decreasing. For that, fix $x, y \in [0,\infty]$ with $x \leq y$. If either $x = 0$ or

$y = \infty$, it is clear that $\overline{\Phi_a}(y) \leq \overline{\Phi_a}(x)$, so we assume that $x, y \in (0, \infty)$. Let $(q_n)_{n=1}^{\infty}$ be sequence of rational numbers in $[1, \infty)$ so that $y = \lim_n q_n x$. Then, by Corollary 3.1.11 and Proposition 3.1.12, we have

$$\overline{\Phi_a}(y) = \lim_{n\to\infty} \overline{\Phi_a}(q_n x) \leq \lim_{n\to\infty} \overline{\Phi_a}(x) = \overline{\Phi_a}(x).$$

This finishes the proof. □

We finish this section with an application of the results above to self-adjoint operators which will find use in the following sections. It says that if the support expansion function of a self adjoint operator ever falls below the $y = x$ line then it will be be constant from then on.

Proposition 3.1.14. *Let* $a \in \mathcal{B}(L^2(\mathbb{R}))$ *be self-adjoint and set* $r = \inf\{x : \Phi_a(x) < x\}$. *Then* $\Phi_a(x) = r$ *for* $x \in [r, \infty]$.

Proof. Seeking a contradiction, we suppose that there is some $p \in \Pr(L^\infty(\mathbb{R}))$ with $d(p) < \infty$ and $d(s_l(ap)) > r$. Let $q = p \vee s_l(ap)$ and note that:

$$s_l(qaq)ap = s_l(qaq)qap = s_l(qaq)qaqp = qaqp = qap = ap$$

From this we conclude that $s_l(qaq) \geq s_l(ap)$ and thus $d(s_l(qaq)) > r$. As $\overline{\Phi_a}$ is decreasing (Theorem 3.1.6), the definition of r gives us that $\Phi_a(x) < x$ for all finite $x > r$. So, $\Phi_a(d(s_l(qaq))) < d(s_l(qaq))$.

Now we derive a couple of technical facts we will need: Since a is self-adjoint, $qaqs_l(qaq) = (s_l(qaq)qaq)^* = (qaq)^* = qaq$; so $s_l(qaqs_l(qaq)) = s_l(qaq)$. Also note that $s_l(as_l(qaq))qaqs_l(qaq) = qs_l(as_l(qaq))as_l(qaq)q = qas_l(qaq)q = qaqs_l(qaq)$ so $s_l(as_l(qaq)) \geq s_l(qaqs_l(qaq)) = s_l(qaq)$.

Therefore,

$$d(s_l(qaq)) \leq d(s_l(as_l(qaq))) \leq \Phi_a(d(s_l(qaq))) < d(s_l(qaq));$$

contradiction.

So $d(s_l(ap)) \leq r$ for every finite dimension projection $p \in \Pr(L^\infty(\mathbb{R}))$, and thus also for infinite dimension projections by the normality of d. Then $\Phi_a(x) \leq r$ for all $x \in [0, \infty]$. Since Φ_a is increasing and $\Phi_a(r) = r$ (continuity of Φ_a, Cor 3.1.13) we have that $\Phi_a(x) = r$ for all $x \in [r, \infty]$ which was to be shown. □

3.1.3 Relation Between ICOD and ISOD

Theorem 3.1.6 motivates a deeper study of ICOD and ISOD in order to better understand SUPPEXP. This subsection is dedicated to this task. In particular, we show that, although the inclusion ICOD $\subseteq$ ISOD is a strict inclusion, the following two results hold: (1) every element in ISOD is the pointwise supremum of elements in ICOD (Proposition 3.1.17) and (2) given $f \in$ ISOD, there is a standard procedure to obtain $g \in$ ICOD so that $f \leq g \leq 2f$ (see Definition 3.1.18 and Proposition 3.1.19).

It is well known that concave down functions, and thus ICOD functions, are continuous except perhaps at endpoints and we have shown in Proposition 3.1.12 that the same holds for ISOD functions:

Corollary 3.1.15. *If* $f : [0, \infty] \to [0, \infty]$ *is* ISOD, *then it is continuous on* $(0, \infty)$.

Proof. This follows immediately from Proposition 3.1.12. □

The next proposition isolates some trivial facts about ICOD and ISOD functions:

Proposition 3.1.16. *Let $(f_i)_{i\in I}$ be a family of functions $[0,\infty] \to [0,\infty]$. Then*

(a) If $(f_i)_{i\in I} \subseteq$ ICOD, then the pointwise infimum of $(f_i)_{i\in I}$ belongs to ICOD.

(b) If $(f_i)_{i\in I} \subseteq$ ISOD, then both the pointwise infimum and the pointwise supremum of $(f_i)_{i\in I}$ belongs to ISOD.

Proof. This is routine. □

Although ICOD is not closed under pointwise supremum, the inclusion ICOD $\subseteq$ ISOD gives us that such pointwise supremum belongs to ISOD. In fact, this characterizes the ISOD functions:

Proposition 3.1.17. *Every ISOD function is the pointwise supremum of a sequence of ICOD functions.*

Proof. Take $f \in$ ISOD and, for each $q \in \mathbb{Q}_+$, let $f_q : [0,\infty] \to [0,\infty]$ be given by

$$f_q(x) = \frac{f(q)}{q} x \cdot \chi_{[0,q]}(x) + f(q) \cdot \chi_{(q,\infty]}(x)$$

for all $x \in [0,\infty]$. Note that each f_q is in ICOD. Moreover, $f_q(q) = f(q)$ and, as $\overline{f}$ is decreasing, $f \geq f_q$ for all $q \in \mathbb{Q}_+$.

Let g be the pointwise supremum of $(f_q)_{q\in\mathbb{Q}_+}$, so g is ISOD by Proposition 3.1.16. Therefore, f and g are both ISOD and agree on all rational points. By Corollary 3.1.15, we have that f and g are continuous and so $f = g$. □

We now present the promised procedure which, given $f \in$ ISOD, finds $g \in$ ICOD so that $f \leq g \leq 2f$. For that, we introduce the ***concave conjugate***:

Definition 3.1.18. Given $f : [0,\infty] \to [0,\infty]$, we define the ***concave conjugate*** of f, as the map $f_* : [0,\infty] \to [-\infty,\infty]$ given by[3]

$$f_*(\lambda) = \inf_{x\in[0,\infty]} (\lambda x - f(x)), \text{ for all } \lambda \in [0,\infty].$$

Given any $f : [0,\infty] \to [0,\infty]$, f_* is increasing and concave down since it is the pointwise infimum of positive slope lines.

Proposition 3.1.19. *If $f : [0,\infty] \to [0,\infty]$ is an ISOD function, then $f \leq f_{**} \leq 2f$.*

Proof. First, notice that, unfolding definitions, we have

$$f_{**}(x) = \inf_{\lambda\in[0,\infty]} \Big(x\lambda + \sup_{y\in[0,\infty]} (f(y) - \lambda y) \Big)$$

for all $x \in [0,\infty]$. Hence, letting $y = x$ above, we have $f_{**}(x) \geq f(x)$. On the other hand, letting $\lambda = f(x)/x$ above, we have

$$f_{**}(x) \leq f(x) + \sup_{y\in[0,\infty]} (f(y) - yf(x)/x) = f(x) + \sup_{y\in[0,\infty]} y(f(y)/y - f(x)/x).$$

Since $\overline{f}$ is decreasing, the supremum above only needs to consider $y \in [0,x]$. Therefore, using that f is increasing, we have that

$$f_{**}(x) \leq f(x) + \sup_{0\leq y\leq x} y(f(y)/y - f(x)/x) \leq f(x) + \sup_{0\leq y\leq x} f(y) = 2f(x).$$

[3]The concave conjugate is related to the more well studied *convex conjugate* or *Fenchel-Legendre Transform*.

This finishes the proof. □

3.2 C*-Algebras from Controlled Expansion and the poset $\mathbb{P}$

This section starts the investigation of one of the central objects of this dissertation, the poset of continuous support expansion C*-algebras $\mathbb{P}$ (Definition 3.2.6). This section culminates with Theorem 3.2.20.

3.2.1 Sets of Controlled Operators

For the reader's convenience we recall the definition and basic properties of controlled operators.

Definition 1.1.7. Fix a Hilbert space H, von Neumann sub-algebra $\mathcal{M} \subseteq \mathcal{B}(H)$ and dimension function d on $\mathrm{Pr}(\mathcal{M})$. Then given some family of functions $\mathcal{F} : [0,\infty] \to [0,\infty]$ and $f \in \mathcal{F}$ we define

$$\begin{aligned} B_f &= \{a \in \mathcal{B}(H) : \Phi_a^d, \Phi_{a^*}^d \leq f\}, \\ B_{\mathcal{F}} &= \bigcup \{B_f : f \in \mathcal{F}\} \text{ and} \\ C_{\mathcal{F}} &= \overline{B_{\mathcal{F}}}^{\|\cdot\|}. \end{aligned}$$

The sets B_f and $B_{\mathcal{F}}$ are the *f-controlled* and *$\mathcal{F}$-controlled* operators respectively. C*-algebras of the form $C_{\mathcal{F}}$ are collectively referred to as *support expansion* C*-*algebras (on $\mathcal{M}$)*.

Corollary 1.1.8. *Fix a Hilbert space H, von Neumann sub-algebra $\mathcal{M} \in \mathcal{B}(H)$ and dimension function d on Pr$(\mathcal{M})$. Take $f_1, f_2 : [0,\infty] \to [0,\infty]$, $a \in B_{f_1}$, $b \in B_{f_2}$ and $\lambda \in \mathbb{C}$ then:*

(a) $a^* \in B_{f_1}$.

(b) $\lambda a \in B_{f_1}$.

(c) $a + b \in B_{f_1+f_2}$.

(d) $ab \in B_{f_1 \circ f_2}$.

It follows that if a family of functions $\mathcal{F} : [0,\infty] \to [0,\infty]$ is closed under addition and composition then $B_{\mathcal{F}}$ is a $*$-subalgebra of $\mathcal{B}(H)$ and thus $C_{\mathcal{F}}$ is a C*-*algebra.*

Different functions may generate the same set of controlled operators. Therefore, it is useful to have methods which, given a map $f : [0,\infty] \to [0,\infty]$, produce $g : [0,\infty] \to [0,\infty]$ with "better" properties and so that $B_f = B_g$. The next two results provide such methods. We start by showing that one can always assume that $f \in \mathrm{ISOD}$. For that, we need a definition:

Definition 3.2.3. For any $f : [0,\infty] \to [0,\infty]$, we define the ISOD *lower-envelope* of f by

$$\tilde{f}(x) := \sup\{g(x) : g \in \mathrm{ISOD}, g \leq f\} \text{ for all } x \in [0,\infty].$$

Clearly, $\tilde{f} \leq f$ and, by Proposition 3.1.16, it follows that $\tilde{f} \in \mathrm{ISOD}$.

Proposition 3.2.4. *For any function $f : [0,\infty] \to [0,\infty]$, we have $B_f = B_{\tilde{f}}$.*

Proof. Since $\tilde{f} \leq f$, we have $B_{\tilde{f}} \subseteq B_f$. For the reverse inclusion, let $a \in B_f$ and recall that Φ_a and Φ_{a^*} are ISOD by Theorem 3.1.6. Therefore, as both Φ_a and Φ_{a^*} are at most f, they are also at most $\tilde{f}$. This gives us that $a \in B_{\tilde{f}}$. □

The next lemma shows that if f grows slowly enough B_f can be generated by a bounded function.

Lemma 3.2.5. *Let* $f \in \mathrm{ISOD}$ *and* $r = \inf\{x : f(x) < \frac{1}{2}x\}$. *Then* $B_f = B_{\min(f,2r)}$.

Proof. It is immediate that $B_{\min(f,2r)} \subseteq B_f$. For the other inclusion, let $a \in B_f$ and note that $\mathrm{Re}(a)$ and $\mathrm{Im}(a)$ are self adjoint operators such that $\Phi_{\mathrm{Re}(a)}(x), \Phi_{\mathrm{Im}(a)}(x) \leq \Phi_a(x) + \Phi_{a^*}(x) < x$ for $x \in (r, \infty]$ (this uses Corollary 1.1.8(b) and (c)). Then Proposition 3.1.14 tells us that $\Phi_{\mathrm{Re}(a)}$ and $\Phi_{\mathrm{Im}(a)}$ are constant on $[r, \infty]$ with value less than or equal to r.

So $\Phi_a(x), \Phi_{a^*}(x) \leq \Phi_{\mathrm{Re}(a)}(x) + \Phi_{\mathrm{Im}(a)}(x) \leq 2r$ for $x \in [r, \infty]$. Thus $\Phi_a, \Phi_{a^*} \leq \min(f, 2r)$ and therefore $a \in B_{\min(f,2r)}$. In conclusion, $B_f \subseteq B_{\min(f,2r)}$ and we are done. □

3.2.2 Algebras of Controlled Operators and the Poset $\mathbb{P}$

For the remainder of this chapter, our goal is to understand the algebras $C_{\langle \mathcal{F} \rangle}$ for $\mathcal{F}$ a collection of functions. More precisely, we want to understand how distinct families of functions $[0, \infty] \to [0, \infty]$ can generate different C*-algebras. For that, our plan is to develop methods for a deep study of the poset of all such operator algebras:

Definition 3.2.6. We denote by $\mathbb{P}$ the set of all C^*-subalgebras of $\mathcal{B}(L^2(\mathbb{R}))$ of the form $C_{\langle \mathcal{F} \rangle}$ for some family $\mathcal{F}$ of functions $[0, \infty] \to [0, \infty]$. We view $\mathbb{P}$ as a poset with the order being given by inclusion.

Elements of $\mathbb{P}$ are of the form $C_{\langle \mathcal{F} \rangle}$ for some arbitrary family of functions $\mathcal{F}$. In order to have a better grasp of them, we now show that, without loss of generality, we can always assume that $\mathcal{F} \subseteq \mathrm{ICOD}$ (Theorem 3.2.11). For that, we we need two preliminary results.

Proposition 3.2.7. *Functions in* ISOD *are sub-additive.*

Proof. Fix $f \in \mathrm{ISOD}$. Then, for $x, y \in (0, \infty)$, we have that

$$f(x) + f(y) = x\overline{f}(x) + y\overline{f}(y) \geq x\overline{f}(x+y) + y\overline{f}(x+y) = (x+y)\overline{f}(x+y) = f(x+y).$$

Sub-additivity when either summand is 0 or ∞ is straightforward. □

The next proposition will be heavily used in the remainder of the chapter.

Proposition 3.2.8. *(cf. Proposition 2.3.2) If* $\mathcal{F}, \mathcal{G} \subseteq \mathrm{ISOD}$ *are so that for each* $f \in \mathcal{F}$ *there is some* $g \in \langle \mathcal{G} \rangle$ *such that* $f \leq g$, *then* $B_{\langle \mathcal{F} \rangle} \subseteq B_{\langle \mathcal{G} \rangle}$.

We start with a claim.

Claim 3.2.9. Each element in $\langle \mathcal{F} \rangle$ is dominated by some finite linear combination with natural number coefficients of compositions of members of $\mathcal{F}$.

Proof. Note that each element in $\langle \mathcal{F} \rangle$ is constructed from composing and adding elements of $\mathcal{F}$ a finite number of times. Moreover, every element of $\langle \mathcal{F} \rangle$ is in ISOD and, therefore, every such element is sub-additive (see Propositions 3.1.5 and 3.2.7). We then apply sub-additivity repeatedly to prove the claim. □

Proof of Proposition 3.2.8. Let $a \in B_{\langle \mathcal{F} \rangle}$ and pick $f \in \langle \mathcal{F} \rangle$ such that $\Phi_a, \Phi_{a^*} \leq f$. By the claim above, f is dominated by some linear combination with natural number coefficients of compositions of members in $\mathcal{F}$, each of which is dominated by an element of $\langle \mathcal{G} \rangle$ by hypothesis. Since $\langle \mathcal{G} \rangle$ is closed under addition we have some $g \in \langle \mathcal{G} \rangle$ such that $f \leq g$. Then $a \in B_g \subseteq B_{\langle \mathcal{G} \rangle}$. As a was arbitrary, the result follows. □

We now show that the families $\mathcal{F}$ can be taken to be in ICOD in the analysis of $\mathbb{P}$. For that, we introduce the following notation: given a family $\mathcal{F}$ of maps $[0, \infty] \to [0, \infty]$, we let

$$\mathcal{F}_{**} = \{f_{**} \colon f \in \mathcal{F}\}. \quad \text{(cf. Definition 3.1.18)}$$

Proposition 3.2.10. *If* $\mathcal{F} \subseteq \mathrm{ISOD}$, *then* $B_{\langle \mathcal{F} \rangle} = B_{\langle \mathcal{F}_{**} \rangle}$.

Proof. Proposition 3.1.19 gives us that $f \leq f_{**} \leq 2f$ for each $f \in \mathcal{F}$. So it follows from Proposition 3.2.8 that $B_{\langle \mathcal{F} \rangle} = B_{\langle \mathcal{F}_{**} \rangle}$. □

Theorem 3.2.11. *Let* $\mathcal{F}$ *be a collection of functions* $[0, \infty] \to [0, \infty]$ *then there exists some collection* $\mathcal{F}' \subseteq ICOD$ *such that* $C_{\langle \mathcal{F} \rangle} = C_{\langle \mathcal{F}' \rangle}$.

We start with a claim.

Claim 3.2.12. Given a family $\mathcal{F}$ of maps $[0, \infty] \to [0, \infty]$, there is a family $\mathcal{F}' \subseteq \mathrm{ISOD}$ so that $C_{\langle \mathcal{F} \rangle} = C_{\langle \mathcal{F}' \rangle}$.

Proof. Let $\mathcal{F}$ be a set of functions $[0, \infty] \to [0, \infty]$. Then Proposition 3.2.4 gives us that

$$B_{\langle \mathcal{F} \rangle} = \bigcup_{g \in \langle \mathcal{F} \rangle} B_g = \bigcup_{g \in \langle \mathcal{F} \rangle} B_{\tilde{g}} \subseteq B_{\langle \tilde{g} : g \in \langle \mathcal{F} \rangle \rangle} \subseteq B_{\langle \mathcal{F} \rangle},$$

where the final inclusion follows since any map in $\langle \tilde{g} : g \in \langle \mathcal{F} \rangle \rangle$ is dominated by something in $\langle \mathcal{F} \rangle$. Indeed, this is the case since $\tilde{g}_1 + \tilde{g}_2 \leq g_1 + g_2$ and $\tilde{g}_1(\tilde{g}_2(x)) \leq g_1(g_2(x))$ for all functions $g_1, g_2 : [0, \infty] \to [0, \infty]$. So we have that $B_{\langle \mathcal{F} \rangle} = B_{\langle \tilde{g} : g \in \langle \mathcal{F} \rangle \rangle}$ and thus $C_{\langle \mathcal{F} \rangle} = C_{\langle \tilde{g} : g \in \langle \mathcal{F} \rangle \rangle}$. Recall that $\tilde{g} \in \mathrm{ISOD}$ for each $g \in \langle \mathcal{F} \rangle$ (Def 3.2.3) so $\{\tilde{g} : g \in \langle \mathcal{F} \rangle\} \subseteq ISOD$ which completes the proof of the claim. □

Proof of Theorem 3.2.11. Given an arbitrary family of maps $\mathcal{F}$, let $\mathcal{F}' \subseteq \mathrm{ISOD}$ be given by the previous claim, so $C_{\langle \mathcal{F} \rangle} = C_{\langle \mathcal{F}' \rangle}$. The result then follows since $\langle \mathcal{F}'_{**} \rangle \subseteq \mathrm{ICOD}$ and by Proposition 3.2.10 we have that $C_{\langle \mathcal{F}' \rangle} = C_{\langle \mathcal{F}_{**} \rangle}$. □

3.2.3 The Truncation and the Interpolate of a Function

We now present two additional methods of replacing a given family of maps $\mathcal{F}$ by a simpler family $\mathcal{G}$ for which, under mild assumptions on $\mathcal{F}$, we still have $B_{\langle \mathcal{F} \rangle} = B_{\langle \mathcal{G} \rangle}$. Precisely, Propositions 3.2.13 and 3.2.14 show that there are only two behaviors at infinity that one must deal with: one can always assume that either (1) the functions are eventually constant or (2) eventually linear.

We start with the truncation procedure: Given $f \in \mathrm{ISOD}$, we let $f\_ : [0, \infty] \to [0, \infty]$ be the function given by

$$f\_(x) = \min(f(x), f(1))$$

for all $x \in [0, \infty]$. We call $f\_$ the *truncation* of f. Notice that $f\_ \in \mathrm{ISOD}$ (Proposition 3.1.16). Given a family $\mathcal{F} \subseteq \mathrm{ISOD}$, we let

$$\mathcal{F}\_ = \{f\_ : f \in \mathcal{F}\}.$$

Proposition 3.2.13. *Let* $\mathcal{F} \subseteq$ ISOD *be so that* $\lim_{x\to\infty} \frac{f(x)}{x} = 0$ *for every* $f \in \mathcal{F}$. *Then* $B_{\langle\mathcal{F}\rangle} = B_{\langle\mathcal{F}\_\rangle}$.

Proof. The inclusion $B_{\langle\mathcal{F}\_\rangle} \subseteq B_{\langle\mathcal{F}\rangle}$ is trivial since $f\_ \leq f$ for each $f \in \mathcal{F}$ (Proposition 3.2.8). For the other direction, pick $a \in B_{\langle\mathcal{F}\rangle}$ and $f_0 \in \langle\mathcal{F}\rangle$ so that $\Phi_a, \Phi_{a^*} \leq f_0$. As f_0 is formed from a finite number of additions and compositions of the functions in $\mathcal{F}$, we have $\lim_{x\to\infty} \frac{f_0(x)}{x} = 0$. In particular, $f_0(x)$ is eventually less than $\frac{1}{2}x$. So Lemma 3.2.5 gives us that $B_{f_0} = B_{\min(f_0,2r)}$ where $r = \inf\{x : f_0(x) < \frac{1}{2}x\} < \infty$. In particular, $a \in B_{\min(f_0,2r)}$ which gives that Φ_a and Φ_{a^*} are bounded.

Now we will have two cases depending on the behavior near zero of functions in $\mathcal{F}$: Suppose there is $h \in \mathcal{F}$ for which $\lim_{x\to 0} h(x) > 0$. Then there are $\delta > 0$ and $m \in \mathbb{N}$ so that $f_0(x) \leq mh\_(x)$ for $x \in [0,\delta]$. As $\Phi_a(x)$ and Φ_{a^*} are bounded, there is $n \in \mathbb{N}$ so that $\Phi_a(x), \Phi_{a^*}(x) \leq nh\_(x)$ for $x \in [\delta,\infty]$. Therefore, $a \in B_{(n+m)h\_} \subseteq B_{\langle\mathcal{F}\_\rangle}$.

Suppose no such $h \in \mathcal{F}$ exists. The function f_0 is built out of a finite number of additions and compositions of elements of $\mathcal{F}$, so we can define g_0 to be the function built out of the corresponding additions and compositions of the corresponding elements of $\mathcal{F}\_$; in particular, $g_0 \in \langle\mathcal{F}\_\rangle$. As $\lim_{x\to 0} f(x) = 0$ for every $f \in \mathcal{F}$, there is $\delta > 0$ so that $f_0(x) = g_0(x)$ for all $x \in [0,\delta]$.[4] Therefore, $\Phi_a(x), \Phi_{a^*}(x) \leq g_0(x)$ for all $x \in [0,\delta]$ and, as Φ_a and Φ_{a^*} are bounded, there is $n \in \mathbb{N}$ so that $\Phi_a(x), \Phi_{a^*}(x) \leq ng_0(x)$ on $[\delta,\infty]$. Then $a \in B_{ng_0} \subseteq B_{\langle\mathcal{F}\_\rangle}$. □

We now introduce the second procedure: Given $f \in$ ISOD, we define $f_/(x); [0,\infty] \to [0,\infty]$ as the function given by

$$f_/(x) = \begin{cases} f(x), & \text{if } x \in [0,1], \\ f(1)x, & \text{if } x \in (1,\infty]. \end{cases}$$

We call $f_/$ the ***interpolate of*** f. Clearly, $f_/ \in$ ISOD. Given $\mathcal{F} \subseteq$ ISOD, we let

$$\mathcal{F}_/ = \{f_/ : f \in \mathcal{F}\}.$$

Proposition 3.2.14. *Let* $\mathcal{F} \subseteq$ ISOD *be such that* $\lim_{x\to\infty} \frac{f(x)}{x} > 0$ *for some* $f \in \mathcal{F}$. *Then* $B_{\langle\mathcal{F}\rangle} = B_{\langle\mathcal{F}_/\rangle}$. *In particular,* $B_{\langle x\rangle} \subseteq B_{\langle\mathcal{F}\rangle}$.

Proof. The inclusion $B_{\langle\mathcal{F}\rangle} \subseteq B_{\langle\mathcal{F}_/\rangle}$ is clear since $f \leq f_/$ for each $f \in \mathcal{F}$ (Proposition 3.2.8). For the other direction, fix $a \in B_{\langle\mathcal{F}_/\rangle}$ and $f_0 \in \langle\mathcal{F}_/\rangle$ so that $\Phi_a, \Phi_{a^*} \leq f_0$. As f_0 is formed from a finite number of additions and compositions of the elements in $\mathcal{F}_/$, there are some $\delta, \lambda > 0$ so that $f_0(x) = \lambda x$ for $x \in [\delta,\infty]$. Then, taking $f \in \mathcal{F}$ such that $\lim_{x\to\infty} \frac{f(x)}{x} > 0$ (as in the statement of the proposition), there is $n \in \mathbb{N}$ such that $f_0(x) \leq nf(x)$ for $x \in [\delta,\infty]$.

Now we will have two cases depending on the behavior near zero of function in $\mathcal{F}$: Suppose there is some $h \in \mathcal{F}$ so that $\lim_{x\to 0} h(x) > 0$. Then we can find $m \in \mathbb{N}$ so that $f_0(x) \leq mh(x)$ for $x \in [0,\delta]$ and thus we have that $f_0 \leq mh + nf$. So, $a \in B_{mh+nf} \subseteq B_{\langle\mathcal{F}\rangle}$.

Suppose no such $h \in \mathcal{F}$ exists. The function f_0 is built out of a finite number of additions and compositions of elements of $\mathcal{F}_/$, so we can define g_0 to be the function built out of the corresponding additions and compositions of the respective elements of $\mathcal{F}$. As $\lim_{x\to 0} f(x) = 0$ for every $f \in \mathcal{F}$, $f_0(x) = g_0(x)$ for all x in a sufficiently small interval around 0. That is, $f_0(x) \leq g_0(x)$ for all $x \in [0,\delta']$. Since it is possible that $\delta > \delta'$ we may not have that $f_0 \leq g_0 + nf$, but we can certainly find some $m \in \mathbb{N}$ so that $f_0 \leq mg_0 + nf$. Thus $a \in B_{mg_0+nf} \subseteq B_{\langle\mathcal{F}\rangle}$.

[4] Notice that, since we take compositions of the members of $\mathcal{F}\_$, we are not allowed to take $\delta = 1$.

Since $f \in \mathrm{ISOD}$, it is clear that there is $k \in \mathbb{N}$ so that $x \leq kf_/(x)$ for all $x \in [0,\infty]$. So, the last statement follows from Proposition 3.2.8. □

Propositions 3.2.13 and 3.2.14 imply that, given any $\mathcal{F} \subset \mathrm{ISOD}$, either $C_{\langle \mathcal{F} \rangle} = C_{\langle \mathcal{F}\_ \rangle}$ or $C_{\langle \mathcal{F} \rangle} = C_{\langle \mathcal{F}_/ \rangle}$. As we see later (Corollary 3.3.5), this actually establishes a dichotomy for nonempty families $\mathcal{F}$, since only one of those can happen.

3.2.4 The Top and the Bottom of $\mathbb{P}$

The simplest examples of elements in $\mathbb{P}$ are $C_{\langle 0 \rangle} = \{0\}$ and $C_{\mathrm{ICOD}} = C_{\langle \infty \rangle} = \mathcal{B}(L^2(\mathbb{R}))$, which are the first and last elements of $\mathbb{P}$, respectively. Moreover, the poset $\mathbb{P}$ also has unique second and penultimate elements, which we demonstrate presently.

For the next result, let $x\_$ denote the map $f : [0,\infty] \to [0,\infty]$ so that $f(x) = x$ for all $x \in [0,1]$ and $f(x) = 1$ for all $x > 1$.

Proposition 3.2.15. *If $\mathcal{F} \subseteq \mathrm{ICOD}$ contains a nonzero map, then $C_{\langle x\_ \rangle} \subseteq C_{\langle \mathcal{F} \rangle}$. In particular, $C_{\langle x\_ \rangle}$ is the unique immediate successor of 0 in $\mathbb{P}$.*

Proof. Let $f \in \mathcal{F}$ be nonzero. Then, as $f(0) = 0$ and f is concave down, there is $n \in \mathbb{N}$ so that $x \leq nf(x)$ for all $x \in [0,1]$. So, $C_{\langle x\_ \rangle} \subseteq C_{\langle \mathcal{F} \rangle}$ (Proposition 3.2.8). For the second statement, notice that, since it is enough to consider families in ICOD in order to analyse $\mathbb{P}$ (Theorem 3.2.11), it follows that $C_{\langle x\_ \rangle} \subseteq C_{\langle \mathcal{G} \rangle}$ for any arbitrary family of maps $\mathcal{G}$ so that $\{0\} = C_{\{0\}} \subsetneq C_{\langle \mathcal{G} \rangle}$. Therefore, we only need to notice that $\{0\} \subsetneq C_{\langle x\_ \rangle}$. This follows since the operator a defined by $a\xi = \xi\chi_{[0,1]}$, for all $\xi \in L^2(\mathbb{R})$, belongs to $C_{\langle x\_ \rangle}$. □

For the next proposition, consider $\mathrm{ICOD}_{<\infty} = \{f \in \mathrm{ICOD} : f(x) < \infty \text{ for all } x < \infty\}$. Notice that, given $f \in \mathrm{ICOD}$, either $f(x) = \infty$ for all $x > 0$ or $f \in \mathrm{ICOD}_{<\infty}$. Also we will refer to the function $f(0) = 0$ and $f(x) = x+1$ for all $x > 0$ merely as "$x+1$" to simplify notation. (This is something of a pedantic point which arises because we defined ICOD functions to be zero at zero, but that simplifies things elsewhere.)

Proposition 3.2.16. *If $\mathcal{F} \subseteq \mathrm{ICOD}_{<\infty}$, then $C_{\langle \mathcal{F} \rangle} \subseteq C_{\langle x+1 \rangle}$. In particular, $C_{\langle x+1 \rangle} = C_{\mathrm{ICOD}_{<\infty}}$. Moreover, it is the unique immediate predecessor of $\mathcal{B}(L^2(\mathbb{R}))$.*

Proof. Let $f \in \mathcal{F}$. Since f is concave down and $f(x) < \infty$ for all $x < \infty$, there is $n \in \mathbb{N}$ so that $f \leq n(x+1)$. So, $C_{\langle \mathcal{F} \rangle} \subseteq C_{\langle x+1 \rangle}$ (Proposition 3.2.8), but of course $x+1 \in ICOD_{<\infty}$ so $C_{\mathrm{ICOD}_{<\infty}} = C_{\langle x+1 \rangle}$. For the second statement, notice that, since it is enough to consider families in ICOD in order to analyse $\mathbb{P}$ (Theorem 3.2.11), it follows that $C_{\langle \mathcal{F} \rangle} \subseteq C_{\langle x+1 \rangle}$ for all $C_{\langle \mathcal{F} \rangle} \subsetneq \mathcal{B}(L^2(\mathbb{R}))$.

We are left to notice that $C_{\langle x+1 \rangle} \subsetneq \mathcal{B}(L^2(\mathbb{R}))$. For that, consider the operator $a \in \mathcal{B}(L^2(\mathbb{R}))$ given by

$$a\xi = \sum_{n=0}^{\infty} \frac{1}{\sqrt{2^n}} \langle \xi, \chi_{[n,n+1]} \rangle \chi_{[2^n,2^{n+1}]} \quad \text{for all} \quad \xi \in L^2(\mathbb{R}).$$

Given any $f \in B_{\mathrm{ICOD}_{<\infty}}$ and any $b \in B_f$, we have that $d(\mathrm{supp}\,(b\chi_{[n,n+1]})) \leq f(1)$ for all $n \in \mathbb{N}$, hence, it follows that

$$\|a-b\|^2 \geq \left\|a\chi_{[n,n+1]} - b\chi_{[n,n+1]}\right\|^2 = \left\|\frac{1}{\sqrt{2^n}}\chi_{[2^n,2^{n+1}]} - b\chi_{[n,n+1]}\right\|^2 \geq \frac{2^n - f(1)}{2^n}$$

for every $n \in \mathbb{N}$, since at worst $b\chi_{[n,n+1]}$ could exactly cancel $f(1)$ of the support of $\chi_{[2^n,2^{n+1}]}$. As $f(1) < \infty$ and we can take n arbitrarily large, we conclude that $d(a, B_f) \geq 1$. By the arbitrariness of f, we have $d(a, B_{\mathrm{ICOD}_{<\infty}}) \geq 1$ and so $a \notin C_{\mathrm{ICOD}_{<\infty}} = C_{\langle x+1 \rangle}$. This completes the proof. □

3.2.5 Ultimate Elements of $\mathbb{P}$

We now study some elements of $\mathbb{P}$ generated by a few large (and natural) subsets of ICOD. The largeness of those subsets will imply (Theorem 3.2.20) that they lie at "the end" of $\mathbb{P}$ (see Subsection 3.5.1 for elements at "the beginning" of $\mathbb{P}$). Precisely, we study the following sets in this subsection:

Definition 3.2.17. We define the following families of functions in $\mathrm{ICOD}_{<\infty} = \{f \in \mathrm{ICOD} : f(x) < \infty \text{ for all } x < \infty\}$.

(a) $\mathrm{ICOD}_{bdd} = \{f \in \mathrm{ICOD} : f \text{ is bounded}\}$

(b) $\mathrm{ICOD}_0 = \{f \in \mathrm{ICOD} : \lim_{x\to 0} f(x) = 0\}$

(c) $\mathrm{ICOD}_{0\cap bdd} = \mathrm{ICOD}_{bdd} \cap \mathrm{ICOD}_0$

Note that $\langle \mathcal{F} \rangle = \mathcal{F}$ for $\mathcal{F}$ being any of the families above.

Proposition 3.2.18. *C_{ICOD_0} and $C_{\mathrm{ICOD}_{bdd}}$ are incomparable.*

Proof. We first show that $C_{\mathrm{ICOD}_0} \not\subseteq C_{\mathrm{ICOD}_{bdd}}$. For that, notice that, since $\Phi_I(x) = x$ for all $x \in [0, \infty]$, we have that $I \in C_{\mathrm{ICOD}_0}$. Now take $f \in \mathrm{ICOD}_{bdd}$ and $a \in B_f$. Then $d(s_l(a)) < \infty$ and we must have that $s_l(a) < I$. Therefore, if $\xi \in L^2(\mathbb{R})$ is a unit vector supported on $I - s_l(a)$, we have that

$$\|I - a\| \geq \|\xi - a\xi\| \geq \|(I - s_l(a))(\xi - a\xi)\| = \|\xi\| = 1.$$

So, $d(I, B_f) = 1$ and, by the arbitrariness of f, this shows that $I \notin C_{\mathrm{ICOD}_{bdd}}$.

We now show that $C_{\mathrm{ICOD}_{bdd}} \not\subseteq C_{\mathrm{ICOD}_0}$. For this, let $(\eta_n)_{n=0}^{\infty}$ be the standard Haar system, i.e., $\eta_0 = \chi_{(0,1)}$ and then recursively define $\eta_{n+1}(x) = \eta_n(2x) - \eta_n(2x-1)$. In other words, for each $n \in \mathbb{N}$, we have

$$\eta_n(x) = \begin{cases} 1, & \text{if } x \in \bigcup_{k=0}^{2^{n-1}-1} (\frac{2k}{2^n}, \frac{2k+1}{2^n}), \\ -1, & \text{if } x \in \bigcup_{k=0}^{2^{n-1}-1} (\frac{2k+1}{2^n}, \frac{2k+2}{2^n}), \\ 0, & \text{otherwise.} \end{cases}$$

So, $(\eta_n)_{n=0}^{\infty}$ is an orthonormal basis for $L^2(0,1)$ and $s_l(\eta_n) = \chi_{[0,1]}$ for each $n \geq 0$.

Define an operator $a \in \mathcal{B}(L^2(\mathbb{R}))$ by letting

$$a\xi = \sum_{n=1}^{\infty} \sqrt{2^n} \langle \xi, \chi_{[\frac{2^n-2}{2^n}, \frac{2^n-1}{2^n}]} \rangle \eta_n \quad \text{for all} \quad \xi \in L^2(\mathbb{R}).$$

By the Cauchy–Schwarz inequality, it is clear that a is bounded. Moreover, since $a\xi \in L^2(0,1)$ for all $\xi \in L^2(\mathbb{R})$, it follows that $\Phi_a \leq 1$. Also, as

$$a^*\xi = \sum_{n=1}^{\infty} \sqrt{2^n} \langle \xi, \eta_n \rangle \chi_{[\frac{2^n-2}{2^n}, \frac{2^n-1}{2^n}]} \quad \text{for all} \quad \xi \in L^2(\mathbb{R}),$$

it similarly follows that $\Phi_{a^*} \leq 1$ and thus $a \in B_{\mathrm{ICOD}_{bdd}}$. On the other hand, say $f \in \mathrm{ICOD}_0$ and $\varepsilon > 0$, then pick $\delta > 0$ and $k \in \mathbb{N}$ so that $f(x) < \varepsilon$ for every $x \in (0, \delta)$

and $\frac{1}{2^k} < \delta$. Then, if $\xi = \sqrt{2^k}\chi_{[\frac{2^k-2}{2^k}, \frac{2^k-1}{2^k}]}$ and $b \in B_{\langle f \rangle}$, we have that

$$\begin{aligned}\|a-b\|^2 &\geq \|(a-b)\xi\|^2 \\ &= \int_{-\infty}^{\infty} \Big(\eta_k(x) - (b\xi)(x)\Big)^2 dx \\ &\geq 1-\varepsilon.\end{aligned}$$

By the arbitrariness of f, b, and ε, this shows that $a \notin C_{\mathrm{ICOD}_0}$. □

We now present a dichotomy for families $\mathcal{F} \subseteq \mathrm{ICOD}_{<\infty}$ containing an element f with $\lim_{x\to 0} f(x) > 0$.

Proposition 3.2.19. *Let* $\mathcal{F} \subseteq \mathrm{ICOD}_{<\infty}$ *be a collection of functions such that* $\lim_{x\to 0} f_0(x) > 0$ *for some* $f_0 \in \mathcal{F}$*.*

(a) If $\lim_{x\to\infty} \frac{f(x)}{x} = 0$ *for every* $f \in \mathcal{F}$*, then* $C_{\langle \mathcal{F} \rangle} = C_{\langle \mathcal{F}_{\_} \rangle} = C_{\mathrm{ICOD}_{bdd}}$

(b) If there is $f \in \mathcal{F}$ *so that* $\lim_{x\to\infty} \frac{f(x)}{x} > 0$*, then* $C_{\langle \mathcal{F} \rangle} = C_{\langle \mathcal{F}_{/} \rangle} = C_{\mathrm{ICOD}_{<\infty}}$*.*

In particular, $C_{\mathrm{ICOD}_{bdd}} \subseteq C_{\langle \mathcal{F} \rangle}$*.*

Proof. (a) By Proposition 3.2.13, we have $C_{\langle \mathcal{F} \rangle} = C_{\langle \mathcal{F}_{\_} \rangle}$. Hence, as $\langle \mathcal{F}_{\_} \rangle \subseteq \mathrm{ICOD}_{bdd}$, it follows that $C_{\langle \mathcal{F} \rangle} \subseteq C_{\mathrm{ICOD}_{bdd}}$. On the other hand, if $g \in \mathrm{ICOD}_{bdd}$, then there is $n \in \mathbb{N}$ so that $g \leq n \cdot f_0$. So, $C_{\mathrm{ICOD}_{bdd}} \subseteq C_{\langle \mathcal{F} \rangle}$ (Proposition 3.2.8).

(b) As $\langle \mathcal{F} \rangle \subseteq \mathrm{ICOD}_{<\infty}$, it follows that $C_{\langle \mathcal{F} \rangle} \subseteq C_{\mathrm{ICOD}_{<\infty}}$. Fix $f \in \mathcal{F}$ so that $\lim_{x\to\infty} \frac{f(x)}{x} > 0$. Then for any $g \in \mathrm{ICOD}_{<\infty}$ there are $m, n \in \mathbb{N}$ such that $g \leq m \cdot f_0 + n \cdot f$. So, $C_{\mathrm{ICOD}_{<\infty}} \subseteq C_{\langle \mathcal{F} \rangle}$ (Proposition 3.2.8). The equality $C_{\langle \mathcal{F} \rangle} = C_{\langle \mathcal{F}_{/} \rangle}$ follows from Proposition 3.2.14. □

Theorem 3.2.20. *The strict inclusions in the diagram below hold and there are no intermediate elements in each of those inclusions.*

$$\begin{array}{ccccccc} & & C_{\mathrm{ICOD}_{bdd}} & & & & \\ & \subsetneq & & \subsetneq & & & \\ C_{\mathrm{ICOD}_{0\cap bdd}} & & & & C_{\mathrm{ICOD}_{<\infty}} & \subsetneq & C_{\mathrm{ICOD}} = \mathcal{B}(L^2(\mathbb{R})) \\ & \subsetneq & & \subsetneq & & & \\ & & C_{\mathrm{ICOD}_0} & & & & \end{array}$$

Proof. The algebra inclusions in the diagram are immediate since their defining function sets satisfy the same inclusions. Moreover, Propositions 3.2.16 and 3.2.18 give us that those inclusions are strict inclusions. So we only need to show that there are no $C_{\langle \mathcal{F} \rangle}$ strictly between any of the inclusions above. Firstly, notice that, by Theorem 3.2.11, we only need to consider families $\mathcal{F}$ of functions in ICOD. Also, by Proposition 3.2.16, we already have that C_{ICOD} is an immediate successor of $C_{\mathrm{ICOD}_{<\infty}}$.

Let $\mathcal{F}$ be a family of functions with $C_{\langle \mathcal{F} \rangle} \subseteq C_{\mathrm{ICOD}_{<\infty}}$. If $\lim_{x\to 0} f(x) = 0$ for all $f \in \mathcal{F}$, then $\langle \mathcal{F} \rangle \subseteq \mathrm{ICOD}_0$. So, $C_{\langle \mathcal{F} \rangle} \subseteq C_{\mathrm{ICOD}_0}$ and, by Proposition 3.2.18, $C_{\mathrm{ICOD}_{bdd}} \nsubseteq C_{\langle \mathcal{F} \rangle}$. If $\lim_{x\to 0} f_0(x) > 0$ for some $f \in \mathcal{F}$, then Proposition 3.2.19 gives that either $C_{\langle \mathcal{F} \rangle} = C_{\mathrm{ICOD}_{bdd}}$ or $C_{\langle \mathcal{F} \rangle} = C_{\mathrm{ICOD}_{<\infty}}$. If $C_{\mathrm{ICOD}_0} \subseteq C_{\langle \mathcal{F} \rangle}$, then $C_{\langle \mathcal{F} \rangle} = C_{\mathrm{ICOD}_{bdd}}$ cannot happen by Proposition 3.2.18. Those arguments show that $C_{\mathrm{ICOD}_{<\infty}}$ is an immediate successor of both C_{ICOD_0} and $C_{\mathrm{ISOD}_{bdd}}$.

Let $\mathcal{F}$ be a family of functions with $C_{\mathrm{ICOD}_{0\cap bdd}} \subsetneq C_{\langle \mathcal{F} \rangle}$. So there must be some $f \in \mathcal{F}$ which is not in $\mathrm{ICOD}_{0\cap bdd}$. So there is some $f \in \mathcal{F}$ such that either $\lim_{x\to 0} f(x) > 0$ or f is unbounded. In fact, by Proposition 3.2.13, in the latter case f is not merely unbounded, but we also would have $\lim_{x\to\infty} \frac{f(x)}{x} > 0$.

If $\lim_{x\to 0} f(x) > 0$ we have that $C_{\mathrm{ICOD}_{bdd}} \subseteq C_{\langle\mathcal{F}\rangle}$ and, if $\lim_{x\to\infty} \frac{f(x)}{x} > 0$, we have $C_{\mathrm{ICOD}_0} \subseteq C_{\langle\mathcal{F}\rangle}$ (Proposition 3.2.8). This shows there are no intermediates between either C_{ICOD_0} or $C_{\mathrm{ICOD}_{bdd}}$ and $C_{\mathrm{ICOD}_{0\cap bdd}}$. □

3.3 A Function (Almost) Characterization of the Order in $\mathbb{P}$

This section gives a characterization of the containment $C_{\langle\mathcal{F}\rangle} \subseteq C_{\langle\mathcal{G}\rangle}$ for families of maps $[0,\infty] \to [0,\infty]$ so that $\lim_{x\to 0} f(x) = 0$ for all $f \in \mathcal{F}$ (this restriction justifies the "almost" in this section's title). This will be essential in our proofs in Sections 3.4 and 3.5. Precisely, this entire section is dedicated to the proof of the following result:

Theorem 3.3.1. *Let $\mathcal{F}, \mathcal{G} \subseteq \mathrm{ISOD}$ be nonempty and assume that $\lim_{x\to 0} f(x) = 0$ for each $f \in \mathcal{F}$. Then, $C_{\langle\mathcal{F}\rangle} \not\subseteq C_{\langle\mathcal{G}\rangle}$ if and only if there exists $f_0 \in \mathcal{F}$ such that either*

(a) $\lim_{x\to\infty} \frac{f_0(x)}{x} > 0$ and $\lim_{x\to\infty} \frac{g(x)}{x} = 0$ for every $g \in \mathcal{G}$ or

(b) for all $g \in \langle\mathcal{G}\rangle$ there is $(x_n)_{n=1}^{\infty} \subseteq (0,\infty)$ tending to 0 so that $\lim_{n\to\infty} \frac{f_0(x_n)}{g(x_n)} = \infty$ (where here we use the convention that $1/0 = \infty$).

We need several technical results before we can prove Theorem 3.3.1.

Proposition 3.3.2. *Let $r \in (0,\infty]$, $g \in \mathrm{ISOD}$, and $f \in \mathrm{ICOD}$ be so that f is strictly increasing on $[0,r]$ and $\lim_{x\to 0} f(x) = 0$. Then, for $a_{f,r}$ given by Proposition 3.1.7:*

$$(a_{f,r}\xi)(x) = \begin{cases} \sqrt{(f^{-1})'(x)}\,\xi(f^{-1}(x)) & x \in [0, f(r)] \\ 0 & \textit{otherwise} \end{cases}$$

we have that

$$d(a_{f,r}, B_g)^2 \geq 1 - \frac{g(x_0)}{f(x_0)}$$

for every $x_0 \in (0,r]$, where d is the distance induced by the operator norm.

Proof. Fix $x_0 \in (0,r]$. As $\lim_{x\to 0} f(x) = 0$, the vector given by $\xi(x) = \frac{\sqrt{f'(x)}}{\sqrt{f(x_0)}}\chi_{[0,x_0]}(x)$, for all $x \in \mathbb{R}$, has norm 1 and $(a_{f,r}\xi)(x) = \frac{1}{\sqrt{f(x_0)}}\chi_{[0,f(x_0)]}(x)$. Fix $b \in B_g$, so $b\xi$ has support of size at most $g(x_0)$. Therefore, we have that

$$\begin{aligned} \|a_{f,r} - b\|^2 &\geq \|(a_{f,r} - b)\xi\|^2 \\ &= \int_{-\infty}^{\infty} \Big(\frac{1}{\sqrt{f(x_0)}}\chi_{[0,f(x_0)]}(x) - (b\xi)(x)\Big)^2 dx \\ &\geq \frac{1}{f(x_0)}(f(x_0) - g(x_0)) \\ &= 1 - \frac{g(x_0)}{f(x_0)} \end{aligned}$$

Since this is true for arbitrary $b \in B_g$, we have the result. □

Proposition 3.3.3. *Take $r_0 \in [0,\infty]$ and let $f \in \mathrm{ICOD}$ be strictly increasing on $[0,r_0]$ and so that $\lim_{x\to 0} f(x) = 0$. For $r \in [0,r_0]$, let $a_{f,r}$ be as in Propositions 3.1.7 and 3.3.2.*

(a) *If* $r < \infty$*, then* $a_{f,r} \in B_{\langle f \rangle}$.

(b) *If* $r = r_0 = \infty$ *and* $\lim_{x\to\infty} \frac{f(x)}{x} > 0$*, then* $a_{f,\infty} \in B_{\langle f \rangle}$.

Proof. Fix $r \in [0, r_0]$. By Proposition 3.1.7, we have that

$$\Phi_{a_{f,r}}(x) = f(x) \cdot \chi_{[0,r]}(x) + f(r) \cdot \chi_{[r,\infty]}(x)$$

for all $x \geq 0$. In particular, $\Phi_{a_{f,r}} \leq f$. We now estimate $\Phi_{a^*_{f,r}}$. For that, note that $a^*_{f,r}$ is the operator given by

$$(a^*_{f,r}\xi)(x) = \sqrt{f'(x)}\xi(f(x))\chi_{[0,r]}(x) \text{ for all } \xi \in L^2(\mathbb{R}) \text{ and all } x \in \mathbb{R}.$$

(a) Say $r < \infty$. Since f^{-1} is concave up, $a^*_{f,r}$ witnesses maximum support expansion on vectors of the form $\xi = \chi_{[f(r)-x, f(r)]}$, i.e.,

$$\Phi_{a^*_{f,r}}(x) = \begin{cases} d(\operatorname{supp}\left(a^*_{f,r}\chi_{[f(r)-x,f(r)]}\right)), & \text{for } x \in [0, f(r)], \\ r, & \text{for } x > f(r). \end{cases}$$

Note that $\operatorname{supp}\left(a^*_{f,r}\chi_{[f(r)-x,f(r)]}\right) = \chi_{[f^{-1}(f(r)-x),r]}$ for $x \in [0, f(r)]$. So,

$$\lim_{x\to 0} \frac{\Phi_{a^*_{f,r}}(x)}{x} = \lim_{x\to 0^+} \frac{f^{-1}(f(r)) - f^{-1}(f(r)-x)}{x},$$

which we recognize as the left derivative $(f^{-1})'(f(r)) = \frac{1}{f'(r)}$. Since f is strictly increasing on $[0, r_0]$, this implies that $\lim_{x\to 0} \Phi_{a^*_{f,r}}(x)/x < \infty$. Therefore, since $\Phi_{a^*_{f,r}}$ is bounded and $\lim_{x\to 0} \frac{f(x)}{x} > 0$ (Proposition 3.1.5), there is $n \in \mathbb{N}$ such that $\Phi_{a^*_{f,r}} \leq nf$. So, $a_{f,r} \in B_{\langle f \rangle}$.

(b) Say $r = \infty$ and let $\lambda = \lim_{x\to\infty} \frac{f(x)}{x} > 0$. Then, $\lim_{x\to\infty} \frac{f^{-1}(x)}{x} = \frac{1}{\lambda}$. As f^{-1} is concave up, $a^*_{f,\infty}$ witnesses larger support expansion on intervals further to the right. Therefore, given $x > 0$, we have that

$$\Phi_{a^*_{f,r}}(x) = \lim_{t\to\infty} \left(f^{-1}(t+x) - f^{-1}(t)\right) = \frac{1}{\lambda}x.$$

So $\Phi_{a^*_{f,r}}$ can be dominated by an appropriate integer multiple of f, since in this case we have the hypothesis that $\lim_{x\to\infty} \frac{f(x)}{x} > 0$. This shows that $a_{f,\infty} \in B_{\langle f \rangle}$. ☐

Proposition 3.3.4. *Let* $\mathcal{F}, \mathcal{G} \subseteq \mathrm{ICOD}$ *be nonempty and assume that* $\lim_{x\to 0} f(x) = 0$ *for each* $f \in \mathcal{F}$*. Then,* $C_{\langle \mathcal{F}_/ \rangle} \nsubseteq C_{\langle \mathcal{G}_\_ \rangle}$*. In particular,* $C_{\langle \mathcal{F}_\_ \rangle} \subsetneq C_{\langle \mathcal{F}_/ \rangle}$.

Proof. Fix some $f \in \mathcal{F}$. By Proposition 3.3.3, $a_{f_/,\infty} \in B_{\langle f_/ \rangle} \subseteq C_{\langle \mathcal{F}_/ \rangle}$ and, by Proposition 3.3.2, $a_{f_/,\infty}$ has distance 1 from $B_{\langle \mathcal{G}_\_ \rangle}$. So, $a_{f_/,\infty} \notin C_{\langle \mathcal{G}_\_ \rangle}$. The inclusion $C_{\langle \mathcal{F}_\_ \rangle} \subseteq C_{\langle \mathcal{F}_/ \rangle}$ is immediate (Proposition 3.2.8). ☐

Corollary 3.3.5. *If* $\mathcal{F} \subseteq \mathrm{ICOD}$ *is nonempty, then either* $C_{\langle \mathcal{F} \rangle} = C_{\langle \mathcal{F}_\_ \rangle}$ *or* $C_{\langle \mathcal{F} \rangle} = C_{\langle \mathcal{F}_/ \rangle}$ *but not both.*

Proof. By Propositions 3.2.13 and 3.2.14, either $C_{\langle \mathcal{F} \rangle} = C_{\langle \mathcal{F}_\_ \rangle}$ or $C_{\langle \mathcal{F} \rangle} = C_{\langle \mathcal{F}_/ \rangle}$, so we are left to show that only one of those can happen. If $\lim_{x\to 0} f(x) = 0$ for every $f \in \mathcal{F}$, then $C_{\langle \mathcal{F}_\_ \rangle} \neq C_{\langle \mathcal{F}_/ \rangle}$ by Proposition 3.3.4. If not, Proposition 3.2.19 implies that that $C_{\langle \mathcal{F}_/ \rangle} = C_{\mathrm{ICOD}_{<\infty}}$. Therefore, as $C_{\langle \mathcal{F}_\_ \rangle} \subseteq C_{\mathrm{ICOD}_{bdd}}$, we have that $C_{\langle \mathcal{F}_\_ \rangle} \neq C_{\langle \mathcal{F}_/ \rangle}$ (Theorem 3.2.20). ☐

We will now consider behavior of control functions near 0 and how this affects the resulting C*-algebras:

Proposition 3.3.6. *Let $\mathcal{F}, \mathcal{G} \subseteq \mathrm{ICOD}$ be nonempty and assume that $\lim_{x\to 0} f(x) = 0$ for each $f \in \mathcal{F}$. Suppose there is $f_0 \in \mathcal{F}$ so that for every $g \in \langle \mathcal{G} \rangle$ there is a sequence $(x_n)_{n=1}^\infty \subseteq [0,\infty)$ tending to 0 such that $\lim_{n\to\infty} \frac{f_0(x_n)}{g(x_n)} = \infty$. Then $C_{\langle \mathcal{F} \rangle} \not\subseteq C_{\langle \mathcal{G} \rangle}$.*

Proof. Since $f_0 \neq 0$, pick $r_0 \in (0,\infty]$ so that f_0 is strictly increasing on $[0, r_0]$ and fix some $r \in (0, r_0)$. By Proposition 3.3.2, $a_{f_0,r}$ has distance 1 from B_g for any $g \in \langle \mathcal{G} \rangle$. So, $a_{f_0,r} \notin C_{\langle \mathcal{G} \rangle}$. On the other hand, by Proposition 3.3.3, $a_{f_0,r} \in C_{\langle \mathcal{F} \rangle}$. □

For support expansion C*-algebras generated from a single control function, Proposition 3.3.6 holds under weaker conditions:

Corollary 3.3.7. *Let $f, g \in \mathrm{ICOD}$ and suppose $\lim_{x\to 0} f(x) = 0$. If for every $N \in \mathbb{N}$ there is a sequence $(x_n)_{n=1}^\infty \subseteq [0,\infty)$ tending to 0 such that $\lim_{n\to\infty} \frac{f(x_n)}{g^{(N)}(x_n)} = \infty$, then $C_{\langle f \rangle} \not\subseteq C_{\langle g \rangle}$.*

Proof. If $\lim_{x\to 0} \frac{g(x)}{x} \leq 1$ then g dominates $g^{(N)}$ near 0. On the other hand, if $\lim_{x\to 0} \frac{g(x)}{x} > 1$ then $g^{(N+1)}$ dominates $g^{(N)}$ near 0 for each $N \in \mathbb{N}$. In either case, it follows that there is $(x_n)_{n=1}^\infty \subseteq [0,\infty)$ tending to 0 so that $\lim_{n\to\infty} \frac{f(x_n)}{g^{(N)}(x_n)} = \infty$ for every $N \in \mathbb{N}$.

Fix $g_0 \in \langle g \rangle$. By Claim 3.2.9, g_0 is dominated by some linear combination of the $g^{(N)}$, and so $\lim_{n\to\infty} \frac{f(x_n)}{g_0(x_n)} = \infty$. The result then follows from Proposition 3.3.6. □

Proof of Theorem 3.3.1. Theorem 3.2.11 tells us that $C_{\langle \mathcal{F} \rangle} = C_{\langle \mathcal{F}_{**} \rangle}$ and $C_{\langle \mathcal{G} \rangle} = C_{\langle \mathcal{G}_{**} \rangle}$. Recall from Proposition 3.1.19 that for any $f \in ISOD$, $f \leq f_{**} \leq 2f$, so we note that the hypotheses of Theorem 3.3.1 hold for $\mathcal{F}$ and $\mathcal{G}$ if and only if they hold for $\mathcal{F}_{**}$ and $\mathcal{G}_{**}$, reducing to the case where $\mathcal{F}, \mathcal{G} \subseteq ICOD$, which we assume for the remainder of the proof.

($\Longleftarrow$): If the second item holds, the result follows from Proposition 3.3.6. Suppose the first item holds. Then it follows from Propositions 3.2.13 and 3.2.14 that $C_{\langle \mathcal{F} \rangle} = C_{\langle \mathcal{F}_{/} \rangle}$ and $C_{\langle \mathcal{G} \rangle} = C_{\langle \mathcal{G}_{\_} \rangle}$. Then, by Proposition 3.3.4, we have that $C_{\langle \mathcal{F}_{/} \rangle} \not\subseteq C_{\langle \mathcal{G}_{\_} \rangle}$.

($\Longrightarrow$): Suppose both items fail for all $f \in \mathcal{F}$. First we consider the case where $\lim_{x\to\infty} \frac{f(x)}{x} = 0$ for every $f \in \mathcal{F}$. Then $C_{\langle \mathcal{F} \rangle} = C_{\langle \mathcal{F}_{\_} \rangle}$ by Proposition 3.2.13. So there is no loss of generality to assume that each $f \in \mathcal{F}$ is bounded. Fix $f_0 \in \mathcal{F}$. Then, as the second item does not hold for $f_0 \in \mathcal{F}$, there is $g \in \langle \mathcal{G} \rangle$, $\delta > 0$, and $n \in \mathbb{N}$ such that $f_0(x) \leq ng(x)$ for all $x \in [0, \delta]$. Therefore, as f_0 is bounded, by replacing n by a larger natural if necessary, we can assume that $f_0 \leq ng$. Since $f_0 \in \mathcal{F}$ was arbitrary, Proposition 3.2.8 implies that $C_{\langle \mathcal{F} \rangle} \subseteq C_{\langle \mathcal{G} \rangle}$.

Now consider the case where there exists $f \in \mathcal{F}$ such that $\lim_{x\to\infty} \frac{f(x)}{x} > 0$. Then, as the first item fails, there is $g \in \mathcal{G}$ such that $\lim_{x\to\infty} \frac{g(x)}{x} > 0$. Hence, by Proposition 3.2.14, we have that $C_{\langle \mathcal{G} \rangle} = C_{\langle \mathcal{G}_{/} \rangle}$. So there is no loss of generality to assume that each element of $\mathcal{G}$ is eventually linear. Fix some $f_0 \in \mathcal{F}$. As the second item does not hold for $f_0 \in \mathcal{F}$, we can find $g \in \langle \mathcal{G} \rangle$, $\delta > 0$, and $n \in \mathbb{N}$ such that $f_0(x) \leq ng(x)$ for all $x \in [0, \delta]$. As $\frac{f_0(x)}{x}$ is decreasing, f_0 is at most asymptotically linear, so, replacing n by a larger n if necessary, we can assume that $f_0 \leq ng$. Since $f_0 \in \mathcal{F}$ was arbitrary, Proposition 3.2.8 implies that $C_{\langle \mathcal{F} \rangle} \subseteq C_{\langle \mathcal{G} \rangle}$. □

Theorem 3.3.1 can be reduced to the following in the case of $\mathcal{F}$ and $\mathcal{G}$ being singletons.

Corollary 3.3.8. *Let* $f, g \in \mathrm{ISOD}$ *be such that* $\lim_{x\to 0} f(x) = 0$. *Then,* $C_{\langle f\rangle} \nsubseteq C_{\langle g\rangle}$ *if and only if either*

(a) $\lim_{x\to\infty} \frac{f(x)}{x} > 0$ *and* $\lim_{x\to\infty} \frac{g(x)}{x} = 0$ *or*

(b) *for all* $N \in \mathbb{N}$ *there is* $(x_n)_{n=1}^{\infty} \subseteq [0,\infty)$ *tending to* 0 *so that* $\lim_{n\to\infty} \frac{f(x_n)}{g^{(N)}(x_n)} = \infty$ *(where here we use the convention that* $1/0 = \infty$*).*

Proof. If $\lim_{x\to 0} \frac{g(x)}{x} \le 1$, then g dominates each $g^{(N)}$ near 0. On the other hand, if $\lim_{x\to 0} \frac{g(x)}{x} > 1$ then $g^{(N+1)}$ dominates $g^{(N)}$ near 0 for each $N \in \mathbb{N}$. In either case, it follows that there is a universal sequence $(x_n)_{n=1}^{\infty} \subseteq [0,\infty)$ tending to 0 so that $\lim_{n\to\infty} \frac{f(x)}{g^{(N)}(x_n)} = \infty$ for every $N \in \mathbb{N}$.

Now for any $g_0 \in \langle g\rangle$ we have from Claim 3.2.9 that g_0 is dominated by some linear combination of compositions of g, and so $\lim_{n\to\infty} \frac{f(x_n)}{g_0(x_n)} = \infty$. Then apply Theorem 3.3.1 to obtain the result. □

3.4 Order Structure of Large Subsets of $\mathbb{P}$

In this section, we present methods to obtain uncountable subsets of $\mathbb{P}$ with well understood order structure. Precisely, the following is the main result of this section:

Theorem 3.4.1. *The* poset $\mathbb{P}$ *has uncountable increasing chains, uncountable decreasing chains and uncountable antichains.*

Theorem 3.3.1 and Corollary 3.3.8 (almost) reduce the question of whether two elements in $\mathbb{P}$ are comparable to a function-theoretic question. We will now work primarily with functions and then use Theorem 3.3.1 and Corollary 3.3.8 in order to determine properties of $\mathbb{P}$.

The following proposition is inspired by a construction in LittlePeng9, 2017.

Proposition 3.4.2. *Let* $f_0 : [0,1] \to [0,1]$ *be so that*

(a) f_0 *is increasing,*

(b) f_0 *is concave down,*

(c) $x < f_0(x) < 1$ *for all* $0 < x < 1$,

(d) $\lim_{x\to 0} f_0(x) = 0$, *and*

(e) $\lim_{x\to 0} \frac{f_0^{(n)}(x)}{f_0^{(m)}(x)} = \infty$ *for all* $n > m \ge 0$ *.*

Then, for each countable ordinal α, *there is* $f_\alpha : [0,1] \to [0,1]$ *satisfying the same properties above and so that* $\lim_{x\to 0} \frac{f_\beta(x)}{f_\alpha^{(n)}(x)} = \infty$ *for all* $n \in \mathbb{N}$ *and all* $\beta > \alpha$.

Proof. We define $(f_\alpha)_{\alpha<\omega_1}$ by transfinite induction. Suppose $\beta < \omega_1$ and that $(f_\alpha)_{\alpha<\beta}$ has been defined. Then, if β is a successor ordinal, say $\beta = \gamma + 1$, we define

$$f_\beta(x) := \sum_{n=1}^{\infty} \frac{1}{2^n} f_\gamma^{(n)}(x) \text{ for all } x \in [0,\infty].$$

If β is a limit ordinal, then its cofinality must be ω, so pick an increasing sequence of ordinals $(\alpha[n])_n$ so that $\beta = \sup_n \alpha[n]$ and let

$$f_\beta(x) := \sum_{n=1}^{\infty} \frac{1}{2^n} f_{\alpha[n]}(x) \text{ for all } x \in [0,\infty].$$

We now show that $(f_\alpha)_{\alpha<\omega_1}$ satisfies the desired properties. Items (a), (b), (c), and (d) follow straightforwardly. Let us show that $(f_\alpha)_{\alpha<\omega_1}$ satisfies (e). We proceed by transfinite induction. Suppose $\beta < \omega_1$ and that (e) holds for all $\alpha < \beta$. If β is a successor ordinal and let $\beta = \gamma + 1$. Then for any $N \in \mathbb{N}$ there is $\delta > 0$ such that $\frac{f_\gamma(x)}{x} > 2N$ for all $0 < x < \delta$ (since f_γ satisfies (e)). Therefore, $\frac{f_\beta(x)}{x} > \frac{f_\gamma(x)}{2x} > N$ for all $0 < x < \delta$ and, as N is arbitrary, we have that $\lim_{x\to 0} \frac{f_\beta(x)}{x} = \infty$. An analogous argument gives that $\lim_{x\to 0} \frac{f_\beta(x)}{x} = \infty$ if β is a limit ordinal. Since $\lim_{x\to 0} \frac{f_\beta^{(m+1)}(x)}{f_\beta^{(m)}(x)} = \lim_{x\to 0} \frac{f_\beta(x)}{x}$ for all $m \in \mathbb{N}$ and

$$\frac{f_\beta^{(n)}(x)}{f_\beta^{(m)}(x)} = \frac{f_\beta^{(n)}(x)}{f_\beta^{(n-1)}(x)} \cdot \ldots \cdot \frac{f_\beta^{(m+1)}(x)}{f_\beta^{(m)}(x)},$$

(e) follows.

Finally, we need to show that $\lim_{x\to 0} \frac{f_\beta(x)}{f_\alpha^{(n)}(x)} = \infty$ for all $\alpha < \beta < \omega_1$ and all $n \in \mathbb{N}$. We proceed by induction on β. Fix $\beta < \omega_1$. Say $\beta = \alpha + 1$. Then, if $n \in \mathbb{N}$, we have that $f_\beta > \frac{1}{2^{n+1}} f_\alpha^{(n+1)}$ and thus we have that $\lim_{x\to 0} \frac{f_\beta(x)}{f_\alpha^{(n)}(x)} \geq \lim_{x\to 0} \frac{f_\alpha^{(n+1)}(x)}{2^{n+1} f_\alpha^{(n)}(x)} = \infty$ by (e). Now suppose that $\alpha + 1 < \beta$ and the induction hypothesis holds for all $\gamma \in (\alpha, \beta)$. Then, regardless if β is a successor or a limit ordinal, there is $\gamma \in (\alpha, \beta)$ so that $f_\beta > \frac{1}{2^m} f_\gamma$ for some $m \in \mathbb{N}$ and $\lim_{x\to 0} \frac{f_\gamma(x)}{f_\alpha^{(n)}(x)} = \infty$ for all $n \in \mathbb{N}$. Thus by the induction hypothesis we have that $\lim_{x\to 0} \frac{f_\beta(x)}{f_\alpha^{(n)}(x)} \geq \lim_{x\to 0} \frac{f_\gamma(x)}{2^m f_\alpha^{(n)}(x)} = \infty$ for all $n \in \mathbb{N}$ and this completes the proof. □

Corollary 3.4.3. *The partially ordered set $\mathbb{P}$ has uncountable increasing chains.*

Proof. Let $f_0(x) := \sqrt{x} \cdot \chi_{[0,1]}(x)$ and note that this fulfills all of the hypotheses of Proposition 3.4.2. Let $(f_\alpha)_{\alpha<\omega_1}$ be the family given by Proposition 3.4.2. By abuse of notation, we extend each of those functions to the whole $[0,\infty]$ by letting $f_\alpha(x) = f_\alpha(1) = 1$ (by hypothesis (c) of Proposition 3.4.2) for any $x > 1$ and any $\alpha < \omega_1$. Notice that $(f_\alpha)_{\alpha<\omega_1}$ satisfies the following: $\lim_{x\to 0} f_\alpha(x) = 0$, $\lim_{x\to\infty} \frac{f_\alpha(x)}{x} = 0$, and $\lim_{x\to 0} \frac{f_\beta(x)}{f_\alpha^{(N)}(x)} = \infty$ for all $\alpha < \beta < \omega_1$ and all $N \in \mathbb{N}$. The containment characterization given by Corollary 3.3.8 gives us that $C_{\langle f_\alpha \rangle} \subsetneq C_{\langle f_\beta \rangle}$ for all $\alpha < \beta < \omega_1$. Since ω_1 is uncountable, the result follows. □

The poset $\mathbb{P}$ also has uncountable decreasing chains. To show this we first need to introduce a transformation on ISOD functions $[0,\infty] \to [0,\infty]$. Given $f \in \text{ISOD}$, we define $\mathcal{T}f : [0,\infty] \to [0,\infty]$ by letting

$$(\mathcal{T}f)(x) = \begin{cases} 0, & \text{if } f(x) = 0 \\ \frac{x}{f(x)}, & \text{if } x < \infty \text{ and } f(x) \neq 0, \\ \infty, & \text{if } x = \infty. \end{cases}$$

It is straightforward to check that f also belongs to ISOD.

Proposition 3.4.4. *Suppose* $f, g \in \mathrm{ISOD}$ *satisfy* $\lim_{x\to 0} f(x) = 0 = \lim_{x\to 0} g(x)$. *Suppose furthermore that* $\lim_{x\to 0} \frac{f(x)}{g^{(N)}(x)} = \infty$ *for all* $N \in \mathbb{N}$ *and* $g(x) \geq \sqrt{x}$ *for su ciently small* x. *Then* $C_{\langle x\rangle} \subsetneq C_{\langle \mathcal{T} f\rangle} \subsetneq C_{\langle \mathcal{T} g\rangle}$.

Proof. As $\lim_{x\to 0} \frac{f(x)}{g(x)} = \infty$ and $g(x) \geq \sqrt{x}$ near 0, it follows that $\lim_{x\to 0}(\mathcal{T} f)(x) = \lim_{x\to 0} \frac{x}{f(x)} \leq \lim_{x\to 0} \frac{\sqrt{x}}{f(x)} \leq \lim_{x\to 0} \frac{g(x)}{f(x)} = 0$. Hence, as $\lim_{x\to 0} \frac{(\mathcal{T} f)(x)}{x} = \lim_{x\to 0} \frac{1}{f(x)} = \infty$, Corollary 3.3.8 implies that $C_{\langle \mathcal{T} f\rangle} \nsubseteq C_{\langle x\rangle}$. On the other hand, as $\lim_{x\to 0} \frac{x}{(\mathcal{T} f)(x)} = \lim_{x\to 0} f(x) = 0$, it also follows that $C_{\langle x\rangle} \subseteq C_{\langle \mathcal{T} f\rangle}$ (Corollary 3.3.8)

We are left to show that $C_{\langle \mathcal{T} f\rangle} \subsetneq C_{\langle \mathcal{T} g\rangle}$. We start by showing that $\lim_{x\to 0} \frac{f(x)^n}{g(x)} = \infty$ for all $n \in \mathbb{N}$. Indeed, fix $n \in \mathbb{N}$ and note that, as $g(x) \geq \sqrt{x}$ for sufficiently small x, then $g^{(n)}(x) \geq \sqrt[2^n]{x}$ for sufficiently small x. Therefore, we must have that

$$\lim_{x\to 0} \frac{f(x)}{\sqrt[n]{g(x)}} \geq \lim_{x\to 0} \frac{f(x)}{\sqrt[2^n]{g(x)}} \geq \lim_{x\to 0} \frac{f(x)}{g^{(n)}(x)} = \infty$$

So, by the continuity of x^n, $\lim_{x\to 0} \frac{f(x)^n}{g(x)} = \infty$.

A simple induction gives that

$$(\mathcal{T} f)^{(n)}(x) = \frac{x}{\prod_{k=0}^{n-1} f((\mathcal{T} f)^{(k)}(x))} \text{ for all } x \in [0, \infty].$$

Also, as $\lim_{x\to 0} f(x) = 0$, we have $(\mathcal{T} f)(x) = \frac{x}{f(x)} > x$ for sufficiently small x, which implies that, for each $n \in \mathbb{N}$, $(\mathcal{T} f)^{(n)}(x) > x$ for sufficiently small x. Therefore, as f is increasing, $\prod_{k=0}^{n-1} f((\mathcal{T} f)^{(k)}(x)) \geq f(x)^n$ for sufficiently small x. Then, using the result of the previous paragraph, we have that

$$\lim_{x\to 0} \frac{(\mathcal{T} g)(x)}{(\mathcal{T} f)^{(n)}(x)} = \lim_{x\to 0} \frac{\prod_{k=0}^{n-1} f((\mathcal{T} f)^{(k)}(x))}{g(x)} \geq \lim_{x\to 0} \frac{f(x)^n}{g(x)} = \infty.$$

As $g(x) \geq \sqrt{x}$ near zero, we have that $\lim_{x\to 0}(\mathcal{T} g)(x) = 0$, and thus Corollary 3.3.8 implies that $C_{\langle \mathcal{T} g\rangle} \nsubseteq C_{\langle \mathcal{T} f\rangle}$. Finally, since $\lim_{x\to 0} \frac{(\mathcal{T} f)(x)}{(\mathcal{T} g)(x)} = \lim_{x\to 0} \frac{g(x)}{f(x)} = 0$, Corollary 3.3.8 also gives that $C_{\langle \mathcal{T} f\rangle} \subseteq C_{\langle \mathcal{T} g\rangle}$. □

Corollary 3.4.5. *The partially ordered set* $\mathbb{P}$ *has an uncountable decreasing chain.*

Proof. Let $f_0(x) := \sqrt{x} \cdot \chi_{[0,1]}(x)$ and let $(f_\alpha)_{\alpha<\omega_1}$ be given by Proposition 3.4.2. By abuse of notation, we extend each of those functions to the whole $[0, \infty]$ by letting $f_\alpha(x) = f_\alpha(1) = 1$ (by hypothesis (c) of Proposition 3.4.2) for any $x > 1$ and any $\alpha < \omega_1$. Note that, for any $\alpha < \omega_1$, we have $\lim_{x\to 0} f_\alpha(x) = 0$ and that $f_\alpha(x) \geq \sqrt{x}$ for sufficiently small x. Furthermore, if $\alpha < \beta < \omega_1$, then $\lim_{x\to 0} \frac{f_\beta(x)}{f_\alpha^{(N)}(x)} = \infty$ for every $N \in \mathbb{N}$. Therefore, by Proposition 3.4.4, we have that $C_{\langle x\rangle} \subsetneq C_{\langle \mathcal{T} f_\beta\rangle} \subsetneq C_{\langle \mathcal{T} f_\alpha\rangle}$ for all $\alpha < \beta < \omega_1$. □

We will now show that $\mathbb{P}$ has an uncountable antichain. To show this we need a couple of technical results.

Lemma 3.4.6. *If* $\mathcal{F} \subseteq \mathrm{ISOD}$ *is such that* $C_{\langle \mathcal{F}\rangle} \nsubseteq C_{\langle x_-\rangle}$ *and* $\lim_{x\to 0} f(x) = 0$ *for every* $f \in \mathcal{F}$, *then there is some* $f_0 \in \langle \mathcal{F}\rangle$ *such that* $\lim_{x\to 0} \frac{f_0(x)}{x} = \infty$. *As a consequence,* $\lim_{x\to 0} \frac{f_0^{(n)}(x)}{f_0^{(m)}(x)} = \infty$ *for* $n, m \in \mathbb{N}$ *with* $n > m$.

Proof. Using Theorem 3.3.1, $C_{\langle\mathcal{F}\rangle} \nsubseteq C_{\langle x_{\_}\rangle}$ gives us an $f_0 \in \langle\mathcal{F}\rangle$ and a sequence $(x_n)_{n=1}^{\infty}$ decreasing to 0 such that $\frac{f_0(x_n)}{x_n} \geq n$. As $\frac{f_0(x)}{x}$ is decreasing (since $f_0 \in \mathrm{ISOD}$), this implies that that $\lim_{x\to 0} \frac{f_0(x)}{x} = \infty$ as desired. □

Proposition 3.4.7. *(cf. Proposition 2.3.8) If* $(f_n)_{n\in\mathbb{N}} \subseteq \mathrm{ICOD}$ *is so that* $C_{\langle f_n\rangle} \nsubseteq C_{\langle x\rangle}$ *and* $\lim_{x\to 0} f_n(x) = 0$ *for all* $n \in \mathbb{N}$*, then there is* $g \in \mathrm{ICOD}$ *with* $\lim_{x\to 0} g(x) = 0$ *so that* $C_{\langle g\rangle} \nsubseteq C_{\langle x\rangle}$*, and so that* $C_{\langle g\rangle}$ *and* $C_{\langle f_n\rangle}$ *are incomparable for all* $n \in \mathbb{N}$*.*

Proof. For didactic reasons, we first prove the proposition with the extra assumption that $(f_n)_{n\in\mathbb{N}}$ is a constant sequence, say $f = f_n$ for all $n \in \mathbb{N}$. As $C_{\langle f\rangle} = C_{\langle nf\rangle}$ for all $n \in \mathbb{N}$ (Proposition 3.2.8), we can assume that $f(x) \geq x$ for all $x \in [0,1]$. We now construct the desired function g. It will be useful for the reader to have in mind that our approach will be the following: we construct g in a piecewise manner and in a way that we can use the second item of Corollary 3.3.8 in order to guarantee that $C_{\langle g\rangle}$ and $C_{\langle f\rangle}$ are incomparable.

We start by setting some notation and pointing out some very elementary facts about affine functions and their relation with f. Precisely, given $x, y, b > 0$, we let $\ell[x,y,b]$ be the line which sends x to y and has b as y-intercept, i.e., $\ell[x,y,b](t) = \frac{y-b}{x}t + b$ for all $t \in \mathbb{R}$. The construction of g will be based in the following: given $x, y, b > 0$,

(a) as $\lim_{t\to 0} f(t) = 0$, we have that $\lim_{t\to 0} \frac{\ell[x,y,b](t)}{f^{(n)}(t)} = \infty$ for all $n \in \mathbb{N}$ and

(b) since $\lim_{t\to 0} \frac{f(t)}{t} = \infty$ (Lemma 3.4.6), we have that $\lim_{t\to 0} \frac{f(t)}{\ell[x,y,0]^{(n)}(t)} = \infty$ for all $n \in \mathbb{N}$.

The next two facts isolate the conclusions from points (a) and (b) which we need — we recommend the readers guide themselves by Figure 3.1 in the construction of g.

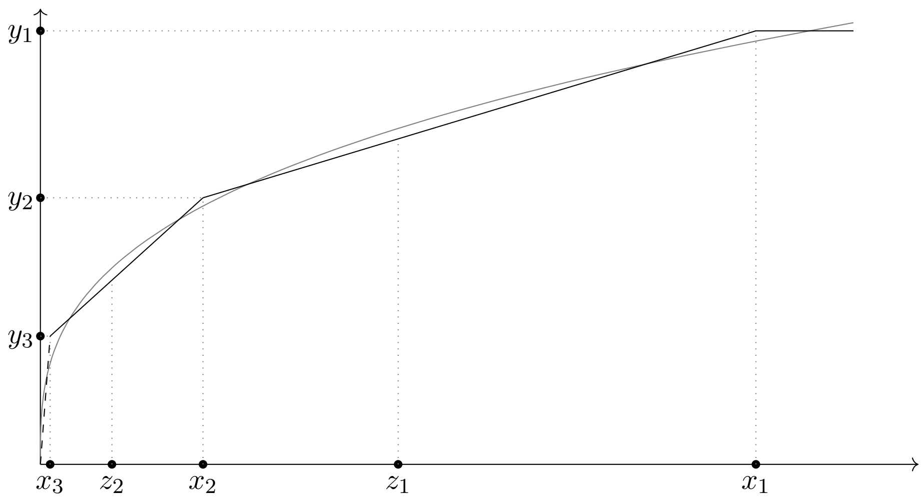

FIGURE 3.1: In the graph above, the smooth function represents f and the piecewise linear function represents g. Note that the scale is modified so that the general behavior of g with respect to f can be represented in the graph. Also, due to obvious physical restrictions, the graph only depicts the case $n = 1$ in Facts 3.4.8 and 3.4.9.

Fact 3.4.8. *Given* $x, y, b, k, n > 0$*, there is* $x' \in (0, x)$ *so that* $\frac{\ell[x,y,b](x')}{f^{(n)}(x')} > k$*.*

Fact 3.4.9. *Given $x, y, b, k, n > 0$, there are $z' \in (0, x)$ and $b' \in (0, \min\{b, y\})$ so that $\frac{f(z')}{\ell[x,y,b']^{(n)}(z')} > k$.*

Let $x_0 = y_0 = b_0 = 1$. Then, alternating between Fact 3.4.8 and Fact 3.4.9 (with Fact 3.4.8 being the first we use), one can find strictly decreasing sequences $(x_n)_{n=1}^{\infty}$, $(y_n)_{n=1}^{\infty}$, $(z_n)_{n=1}^{\infty}$, and $(b_n)_{n=1}^{\infty}$ in $[0, 1]$ tending to 0 so that

(a) $\frac{\ell[x_{n-1},y_{n-1},b_{n-1}](x_n)}{f^{(n)}(x_n)} > n$ for all $n \in \mathbb{N}$,

(b) $y_n = \ell[x_{n-1}, y_{n-1}, b_{n-1}](x_n)$ for all $n \in \mathbb{N}$,

(c) $x_{n+1} < z_n < x_n$ for all $n \in \mathbb{N}$,

(d) $b_n < y_n$ for all $n \in \mathbb{N}$,

(e) $\frac{f(z_n)}{l[x_n,y_n,b_n]^{(n)}(z_n)} > n$ for all $n \in \mathbb{N}$.

We define $g : [0, \infty] \to [0, \infty]$ by letting

$$g(x) = \begin{cases} 1, & \text{if } x \geq x_1, \\ \ell[x_n, y_n, b_n](x), & \text{if } x \in (x_{n+1}, x_n], \\ 0, & \text{if } x = 0. \end{cases}$$

It is clear from its piecewise definition and (b) that g is continuous. By its very definition, $\ell[x_n, y_n, b_n]$ has slope $\frac{y_n - b_n}{x_n}$. However, by (b), the slope of $\ell[x_{n-1}, y_{n-1}, b_{n-1}]$ must equal $\frac{y_n - b_{n-1}}{x_n}$. Therefore, as $b_n < b_{n-1}$, the slope of $\ell[x_n, y_n, b_n]$ is greater than the one of $\ell[x_{n-1}, y_{n-1}, b_{n-1}]$. Hence, g is concave down. Moreover, by (d), the slope of each $\ell[x_n, y_n, b_n]$ is positive, so g is increasing.

We now use Corollary 3.3.8 in order to conclude. For that, fix $N \in \mathbb{N}$. Since $f(x) \geq x$ for all $x \in [0, 1]$, we have that $f^{(N)}(x) \leq f^{(n)}(x)$ for all $x \in [0, 1]$ and all $n \geq N$. Therefore, (a) and (b) imply that

$$\lim_{n\to\infty} \frac{g(x_n)}{f^{(N)}(x_n)} \geq \lim_{n\to\infty} \frac{g(x_n)}{f^{(n)}(x_n)} \geq \lim_{n\to\infty} \frac{\ell[x_{n-1}, y_{n-1}, b_{n-1}](x_n)}{f^{(n)}(x_n)} \geq \lim_{n\to\infty} n = \infty.$$

On the other hand, since $g(1) = 1$ and g is concave down, we have that $g(x) \geq x$ for all $x \in [0, 1]$, which gives us that $g^{(N)}(x) \leq g^{(n)}(x)$ for all $x \in [0, 1]$ and all $n \geq N$. Also, it is clear from its definition that $g \leq l[x_n, y_n, b_n]$ for all $n \in \mathbb{N}$, so $g^{(n)} \leq l[x_n, y_n, b_n]^{(n)}$ for all $n \in \mathbb{N}$. Therefore, (e) gives us that

$$\lim_{n\to\infty} \frac{f(z_n)}{g^{(N)}(z_n)} \geq \lim_{n\to\infty} \frac{f(z_n)}{g^{(n)}(z_n)} \geq \lim_{n\to\infty} \frac{f(z_n)}{\ell[x_n, y_n, b_n]^{(n)}(z_n)} \geq \lim_{n\to\infty} n = \infty.$$

By Corollary 3.3.8, this implies that $C_{\langle f \rangle}$ and $C_{\langle g \rangle}$ are incomparable.

Now note that for $x \in (0, x_n)$ we have $g(x) \leq \ell[x_n, y_n, b_n](x) \leq y_n$ with $y_n \to 0$, so $\lim_{x\to 0} g(x) = 0$. Suppose for the sake of contradiction that $C_{\langle g \rangle} \subseteq C_{\langle x \rangle}$. Then, by Corollary 3.3.8, there exists some $m \in \mathbb{N}$ such that $g(x) \leq mx$ and thus $\lim_{x\to 0} \frac{f(x)}{g(x)} \geq \lim_{x\to 0} \frac{f(x)}{mx} = \infty$ which contradicts our result that $\lim_{n\to\infty} \frac{g(x_n)}{f(x_n)} = \infty$. So $C_{\langle g \rangle} \nsubseteq C_{\langle x \rangle}$.

The result for a single f is now proven, so consider $(f_n)_n$ as in the statement, i.e., $(f_n)_n$ is not necessarily constant. The proof for this case is actually completely analogous and the only modification needed is that, when using Facts 3.4.8 and 3.4.9 in order to find strictly decreasing sequence $(x_n)_{n=1}^{\infty}$, $(y_n)_{n=1}^{\infty}$, $(z_n)_{n=1}^{\infty}$, and $(b_n)_{n=1}^{\infty}$ in

$[0,1]$ tending to 0, we must replace (a) and (e) above by the stronger statements that $\frac{l[x_{n-1},y_{n-1},b_{n-1}](x_n)}{f_k^{(n)}(x_n)} \geq n$ and $\frac{f_k(z_n)}{l[x_n,y_n,b_n]^{(n)}(z_n)} \geq n$ for all $n \in \mathbb{N}$ and all $k \leq n$. Since this is not an issue, we are done. □

Corollary 3.4.10. *The partially ordered set $\mathbb{P}$ has an uncountable antichain.*

Proof. Let S be the subset of $\mathbb{P}$ consisting of all $C_{\langle f\rangle}$ with $f \in \text{ICOD}$, so that $C_{\langle f\rangle} \nsubseteq C_{\langle x\rangle}$ and $\lim_{x\to 0} f(x) = 0$. By Corollary 3.3.8, we have that $C_{\langle\sqrt{x}\rangle} \nsubseteq C_{\langle x\rangle}$. So, $C_{\langle\sqrt{x}\rangle} \in S$. Since S is nonempty, Zorn's lemma implies that S contains a maximal antichain, say A. If A is countable, Proposition 3.4.7 gives us $g \in \text{ICOD}$ satisfying $C_{\langle g\rangle} \nsubseteq C_{\langle x\rangle}$, $\lim_{x\to 0} g(x) = 0$ and so that $C_{\langle g\rangle}$ is pairwise incomparable to everything in A. Then $A \cup \{C_{\langle g\rangle}\}$ is an antichain strictly larger than A, contradicting the maximality of A. So A must be uncountable. As A is also an antichain of $\mathbb{P}$ we are done. □

Proof of Theorem 3.4.1. We have collected all of the necessary results in Corollaries 3.4.3, 3.4.5, and 3.4.10. □

3.5 Existence and Nonexistence of Successor Elements in $\mathbb{P}$

In Subsections 3.2.4 and 3.2.5, we provided some examples of successor elements in $\mathbb{P}$. The examples therein are very specific and allow no generalizations. In this section, we provide fairly general criteria for when a successor exists (Proposition 3.5.1) and when it does not (Proposition 3.5.2).

Proposition 3.5.1. *Let $\mathcal{F} \subseteq \text{ICOD}$ be nonempty and so that $\lim_{x\to 0} f(x) = 0$ for each $f \in \mathcal{F}$. Then $C_{\langle \mathcal{F}_/\rangle}$ is an immediate successor of $C_{\langle \mathcal{F}_\_\rangle}$ in $\mathbb{P}$.*

As we see below (Corollary 3.5.9), if $\mathcal{F}$ is a singleton, then $C_{\langle \mathcal{F}_/\rangle}$ is actually the *unique* immediate successor of $C_{\langle \mathcal{F}_\_\rangle}$.

Proof of Proposition 3.5.1. By Proposition 3.3.4, we have $C_{\langle \mathcal{F}_\_\rangle} \subsetneq C_{\langle \mathcal{F}_/\rangle}$. Suppose for a contradiction that there exists some family $\mathcal{G}$ so that $C_{\langle \mathcal{F}_\_\rangle} \subsetneq C_{\langle \mathcal{G}\rangle} \subsetneq C_{\langle \mathcal{F}_/\rangle}$. By Theorem 3.2.11, we can assume that $\mathcal{G} \subseteq \text{ICOD}$. Moreover, we must have that $\lim_{x\to 0} g(x) = 0$ for every $g \in \mathcal{G}$. Indeed, if this were not the case, then $C_{\text{ICOD}_{bdd}} \subseteq C_{\langle \mathcal{G}\rangle}$ (Proposition 3.2.19) and then, as $C_{\langle \mathcal{F}_/\rangle} \subseteq C_{\text{ICOD}_0}$, we have that $C_{\text{ICOD}_{bdd}} \subseteq C_{\text{ICOD}_0}$. This contradicts Proposition 3.2.18.

Since $C_{\langle \mathcal{G}\rangle} \nsubseteq C_{\langle \mathcal{F}_\_\rangle}$, one of the items in Theorem 3.3.1 must hold for some $g_0 \in \mathcal{G}$. If the second item holds, then for every $f \in \langle \mathcal{F}_\_\rangle$ we can find some sequence $(x_n)_{n=1}^\infty$ tending to 0 such that $\lim_{n\to\infty} \frac{g_0(x_n)}{f(x_n)} = \infty$. As for any $f_0 \in \langle \mathcal{F}_/\rangle$ we can find some $f \in \langle \mathcal{F}_\_\rangle$ such that f and f_0 agree near 0, Theorem 3.3.1 tells us that $C_{\langle \mathcal{G}\rangle} \nsubseteq C_{\langle \mathcal{F}_/\rangle}$, contradicting our initial supposition.

Suppose now that the first item of Theorem 3.3.1 holds for g_0, i.e. $\lim_{x\to\infty} \frac{g_0(x)}{x} > 0$. Since $C_{\langle \mathcal{F}_/\rangle} \nsubseteq C_{\langle \mathcal{G}\rangle}$, one of the items of Theorem 3.3.1 must hold for some $f_0 \in \mathcal{F}_/$. Our assumption on g_0 prevents the first one from holding, so its must be the second which holds for f_0. So, for every $g \in \langle \mathcal{G}\rangle$ there exists a sequence $(x_n)_{n=1}^\infty$ tending to 0 such that $\lim_{n\to\infty} \frac{f_0(x_n)}{g(x_n)} = \infty$. However, since $f_{0\_}$ and f_0 agree near 0, Theorem 3.3.1 gives that $C_{\langle \mathcal{F}_\_\rangle} \nsubseteq C_{\langle \mathcal{G}\rangle}$; contradiction. □

Proposition 3.5.2. *Let $f \in \text{ICOD}$ be so that $\lim_{x\to 0} f(x) = 0$ and $\lim_{x\to 0} \frac{f(x)}{x} = \infty$.*

(a) *If $\lim_{x\to\infty} \frac{f(x)}{x} > 0$, then $C_{\langle f\rangle}$ has no immediate successor in $\mathbb{P}$.*

(b) *If* $\lim_{x\to\infty}\frac{f(x)}{x}=0$*, then* $C_{\langle f\rangle}$ *has no immediate successor of the form* $C_{\langle\mathcal{G}_{\_}\rangle}$ *in* $\mathbb{P}$*.*

Proof. First of all, notice that, as $\lim_{x\to 0} f(x)=0$, we have $f\in \mathrm{ICOD}_{<\infty}$.

(a): As $\lim_{x\to\infty}\frac{f(x)}{x}>0$, there is $n\in\mathbb{N}$ large enough so that $x\le nf(x)$ for all $x\in[0,\infty]$. So, since $C_{\langle f\rangle}=C_{\langle nf\rangle}$, we assume that $x\le f(x)$ for all $x\in[0,\infty]$. As a consequence, we have that $f^{(N)}\ge f^{(M)}$ whenever $N\ge M$.

Suppose for the sake of contradiction that there exists a family $\mathcal{G}$ so that $C_{\langle\mathcal{G}\rangle}$ is the immediate successor of $C_{\langle f\rangle}$. By Theorem 3.2.11, we can assume that $\mathcal{G}\subseteq\mathrm{ICOD}$.

Claim 3.5.3. $C_{\langle\mathcal{G}\rangle}=C_{\langle\mathcal{G}_{/}\rangle}$ and $\lim_{x\to0}g(x)=0$ for every $g\in\mathcal{G}$.

Proof. To show that $C_{\langle\mathcal{G}\rangle}=C_{\langle\mathcal{G}_{/}\rangle}$, it is enough to show that $\lim_{x\to\infty}\frac{g_0(x)}{x}>0$ for some $g_0\in\mathcal{G}$ (Proposition 3.2.14). Suppose no such g_0 exists. Then Proposition 3.2.13 gives that $C_{\langle\mathcal{G}\rangle}=C_{\langle\mathcal{G}_{\_}\rangle}$ and then, as $C_{\langle f\rangle}=C_{\langle f_{/}\rangle}$ (Proposition 3.2.14), Proposition 3.3.4 gives that $C_{\langle f\rangle}\nsubseteq C_{\langle\mathcal{G}\rangle}$; contradiction.

We now show that $\lim_{x\to0}g(x)=0$ for every $g\in\mathcal{G}$. Indeed, if not, then, as $C_{\langle\mathcal{G}\rangle}=C_{\langle\mathcal{G}_{/}\rangle}$, we must have $C_{\mathrm{ICOD}_{<\infty}}\subseteq C_{\langle\mathcal{G}\rangle}$ (Proposition 3.2.19). Notice that $\frac{f}{f(1)}$ satisfies the hypothesis of Proposition 3.4.2 and let $f_1,f_2\in\mathrm{ICOD}_0$ be the first two maps given by it. We extend f_1 and f_2 to the whole $[1,\infty]$ by letting $f_1(x)=f_1(1)x$ and $f_2(x)=f_2(1)x$ for all $x>1$. The properties of f_1 and f_2 together with and Corollary 3.3.8 give that $C_{\langle f\rangle}=C_{\langle\frac{f}{f(1)}\rangle}\subsetneq C_{\langle f_1\rangle}\subsetneq C_{\langle f_2\rangle}\subseteq C_{\mathrm{ICOD}_{<\infty}}$. As $C_{\mathrm{ICOD}_{<\infty}}\subseteq C_{\langle\mathcal{G}\rangle}$, this contradicts our choice of $C_{\langle\mathcal{G}\rangle}$ as an immediate successor of $C_{\langle f\rangle}$. ☐

As $C_{\langle\mathcal{G}\rangle}\nsubseteq C_{\langle f\rangle}$, the previous claim gives that one of the items of Theorem 3.3.1 must hold for some $g_0\in\mathcal{G}$. Since $\lim_{x\to\infty}\frac{f(x)}{x}>0$, it must be the second item. As $C_{\langle\mathcal{G}\rangle}=C_{\langle\mathcal{G}_{/}\rangle}$, we can assume without loss of generality that $g_0=g_{0/}$. So, for every $f_0\in\langle f\rangle$ there is a positive sequence $(x_n)_{n=1}^{\infty}$ tending to 0 such that $\lim_{n\to\infty}\frac{g_0(x_n)}{f_0(x_n)}=\infty$. In fact, this sequence can be chosen independent of f_0, i.e., there is strictly decreasing sequence $(x_n)_{n=1}^{\infty}$ tending to 0 such that $\lim_{n\to\infty}\frac{g_0(x_n)}{f_0(x_n)}=\infty$ for all $f_0\in\langle f\rangle$ (see the first paragraph of the proof of Corollary 3.3.8 for details). Going to a subsequence if necessary, we assume furthermore that $\frac{g_0(x_n)}{f^{(n^2)}(x_n)}\ge n$ for every $n\in\mathbb{N}$.

Define a map $\tilde h:[0,\infty]\to[0,\infty]$ by letting, for each $x\in\mathbb{R}$,

$$\tilde h(x):=\sup_{n\in\mathbb{N}}\left(f^{(n)}(x)\cdot\prod_{k=1}^{n}\frac{f^{(k-1)}(x_k)}{f^{(k)}(x_k)}\cdot\chi_{[0,x_n]}(x)\right).$$

Since $f(0)=0$ and $\lim_n x_n=0$, the supremum above is actually a maximum, so $\tilde h$ is well defined. Moreover, if $x\in(x_{n+1},x_n]$, then $\tilde h(x)=f^{(n)}(x)\cdot\prod_{k=1}^{n}\frac{f^{(k-1)}(x_k)}{f^{(k)}(x_k)}$: Indeed, as $f\in\mathrm{ISOD}$, f is increasing and $\frac{f(x)}{x}$ is decreasing, so, as $x\le x_k$ for $k\le n$, it follows that $\frac{f^{(k)}(x_k)}{f^{(k-1)}(x_k)}\le\frac{f^{(k)}(x)}{f^{(k-1)}(x)}$. Hence, $f^{(k-1)}(x)\le\frac{f^{(k-1)}(x_k)}{f^{(k)}(x_k)}f^{(k)}(x)$ for all $k\le n$; so the formula for $\tilde h$ holds. In particular, as $\frac{f^{(k-1)}}{f^{(k)}}<1$, this implies that $\tilde h(x)\le f^{(n)}(x)$ for all $n\in\mathbb{N}$ and all $x_{n+1}<x\le x_n$.

Notice that $\tilde h$ is ISOD (Proposition 3.1.16) on $[0,x_1]$. Define $h:[0,\infty]\to[0,\infty]$ by letting

$$h(x)=\begin{cases}\inf(\tilde h(x),g_0(x)), & x\in[0,x_1],\\ \inf(\tilde h(x_1),g_0(x_1))x, & x\in(x_1,\infty].\end{cases}$$

Notice that h is also ISOD and $\lim_{x\to 0} h(x) \le \lim_{x\to 0} g_0(x) = 0$ (Claim 3.5.3), which will be needed in order to invoke Corollary 3.3.8 to show $C_{\langle h\rangle} \nsubseteq C_{\langle f\rangle}$ below.

Claim 3.5.4. $C_{\langle f\rangle} \subsetneq C_{\langle h\rangle} \subsetneq C_{\langle g_0\rangle}$.

Proof. The inclusion $C_{\langle h\rangle} \subseteq C_{\langle g_0\rangle}$ follows since $h \le g_0$ (Proposition 3.2.8). Let us show this is a strict inclusion. Fix $N \in \mathbb{N}$ and recall that $h(x) \le f^{(n)}(x_n)$ for $x \in (x_{n+1}, x_n]$ and h is increasing. Therefore,

$$\frac{g_0(x_n)}{h^{(N)}(x_n)} \ge \frac{g_0(x_n)}{f^{(nN)}(x_n)} \ge \frac{g_0(x_n)}{f^{(n^2)}(x_n)} \ge n$$

for $n \in \mathbb{N}$ large enough. Since $(x_n)_n$ tends to 0 and $h \in ISOD$, Corollary 3.3.8 gives that $C_{\langle g_0\rangle} \nsubseteq C_{\langle h\rangle}$ as desired.

As $\tilde{h}(x) \ge \frac{x_1}{f(x_1)} f(x)$ for $x \le x_1$, $\lim_{x\to 0} \frac{g_0(x)}{f(x)} = \infty$ and $\frac{f(x_1)}{x_1} \ge 1$, we have that $\frac{f(x_1)}{x_1} h(x) \ge f(x)$ for all x sufficiently small. Then, since $\lim_{x\to\infty} \frac{h(x)}{x} > 0$, there is an $n \in \mathbb{N}$ such that $nh \ge f$. So $C_{\langle f\rangle} \subseteq C_{\langle h\rangle}$ (Proposition 3.2.8).

We are left to show that $C_{\langle h\rangle} \nsubseteq C_{\langle f\rangle}$. For that, fix $N \in \mathbb{N}$. By the hypotheses on f, $\lim_{x\to 0} \frac{f^{(n)}(x)}{f^{(m)}(x)} = \infty$ for $n > m$. Hence, there is a sequence $(y_n)_{n=1}^\infty$ decreasing to 0 so that $f^{(N+1)}(x) \ge n f^{(N)}(x)$ for all $x \in [0, y_n]$. Take $(x_{j_n})_{n=1}^\infty$ a subsequence of (x_n) such that $x_{j_n} \le y_n$ for all $n \in \mathbb{N}$.

By our choice of $(x_{j_n})_{n=1}^\infty$ and as $\frac{f^{(k-1)}}{f^{(k)}} \le 1$, we have that

$$g_0(x_{j_n}) \ge j_n f^{(j_n{}^2)}(x_{j_n}) \ge n f^{(N)}(x_{j_n}) \prod_{k=1}^{N+1} \frac{f^{(k-1)}(x_k)}{f^{(k)}(x_k)}$$

for $n > N$. Also, we have

$$\tilde{h}(x_{j_n}) \ge f^{(N+1)}(x_{j_n}) \prod_{k=1}^{N+1} \frac{f^{(k-1)}(z_k)}{f^{(k)}(z_k)} \ge n f^{(N)}(x_{j_n}) \prod_{k=1}^{N+1} \frac{f^{(k-1)}(x_k)}{f^{(k)}(x_k)}$$

for all $n > N$. Together these inequalities imply that $\frac{h(x_{j_n})}{f^{(N)}(x_{j_n})} \ge n \prod_{k=1}^{N+1} \frac{f^{(k-1)}(x_k)}{f^{(k)}(x_k)}$ for all $n > N$. So,

$$\lim_{n\to\infty} \frac{h(x_{j_n})}{f^{(N)}(x_{j_n})} = \infty.$$

By Corollary 3.3.8, $C_{\langle h\rangle} \nsubseteq C_{\langle f\rangle}$ and we are done with the proof of the claim. □

Since $C_{\langle g_0\rangle} \subseteq C_{\langle \mathcal{G}\rangle}$, the claim above contradicts that $C_{\langle \mathcal{G}\rangle}$ is an immediate successor of $C_{\langle f\rangle}$. This finishes the proof of (a).

(b): Since $f \in$ ICOD, replacing f by $f/f(1)$ if necessary, we can assume that $x \le f(x) \le 1$ for all $x \in [0,1]$. Suppose there is a family $\mathcal{G}$ of maps so that $C_{\langle \mathcal{G}\rangle}$ is an immediate successor of $C_{\langle f\rangle}$. By Theorem 3.2.11, we can assume that $\mathcal{G} \subseteq$ ICOD.

Claim 3.5.5. (cf. Claim 3.5.3) $\lim_{x\to 0} g(x) = 0$ for every $g \in \mathcal{G}$.

Proof. If not, then $C_{\mathrm{ICOD}_{bdd}} \subseteq C_{\langle \mathcal{G}\rangle}$ (Proposition 3.2.19). Notice that f satisfies the hypothesis of Proposition 3.4.2 and let $f_1, f_2 \in \mathrm{ICOD}_0$ be given the first two maps given by it. We extend f_1 and f_2 to the whole $[1,\infty]$ by letting $f_1(x) = f_1(1)$ and $f_2(x) = f_2(1)$ for all $x > 1$. The properties of f_1 and f_2 together with Corollary 3.3.8 give that $C_{\langle f\rangle} \subsetneq C_{\langle f_1\rangle} \subsetneq C_{\langle f_1\rangle} \subseteq C_{\mathrm{ICOD}_{bdd}}$. As $C_{\mathrm{ICOD}_{bdd}} \subseteq C_{\langle \mathcal{G}\rangle}$, this contradicts our choice of $C_{\langle \mathcal{G}\rangle}$ as an immediate successor of $C_{\langle f\rangle}$. □

The proof now follows completely analogously to the previous item, so we omit the details. $\square$

3.5.1 Initial Elements of $\mathbb{P}$

By Proposition 3.2.15, we have that $C_{\langle x\_\rangle}$ is the second element of $\mathbb{P}$, i.e., $C_{\langle x\_\rangle}$ is the unique immediate successor of $C_{\langle 0\rangle} = \{0\}$. Also, by Proposition 3.5.1, we have that $C_{\langle x\rangle}$ is a third element of $\mathbb{P}$. We will strengthen this in Corollary 3.5.9, showing that it is the unique third element. In this subsection we will also fill in the gap left by Proposition 3.5.2, showing that $C_{\langle x\rangle}$ has no immediate successor in $\mathbb{P}$ and $C_{\langle x\_\rangle}$ has no immediate successor of the form $C_{\langle \mathcal{G}\_\rangle}$.

Proposition 3.5.6. $C_{\langle x\rangle}$ *has no immediate successors.*

Proof. Suppose there is a collection of functions $\mathcal{G}$ such that $C_{\langle \mathcal{G}\rangle}$ is an immediate successor of $C_{\langle x\rangle}$; without loss of generality we can take $\mathcal{G} \subseteq \mathrm{ICOD}_{<\infty}$ (Theorem 3.2.11 and Theorem 3.2.20). As $I \notin C_{\mathrm{ICOD}_{bdd}}$ (see the proof of Proposition 3.2.18), it follows that $C_{\langle x\rangle} \nsubseteq C_{\mathrm{ICOD}_{bdd}}$. Therefore, $\lim_{x\to 0} g(x) = 0$ for every $g \in \mathcal{G}$: Indeed, if not, then Proposition 3.2.19 implies that either $C_{\mathrm{ICOD}_{bdd}} = C_{\langle \mathcal{G}\rangle}$ or $C_{\mathrm{ICOD}_{<\infty}} = C_{\langle \mathcal{G}\rangle}$. The former is ruled out since $C_{\langle x\rangle} \nsubseteq C_{\mathrm{ICOD}_{bdd}}$ and the latter since $C_{\langle x\rangle} \subsetneq C_{\langle \sqrt{x}\rangle} \subseteq C_{\mathrm{ICOD}_0} \subsetneq C_{\mathrm{ICOD}_{<\infty}} = C_{\langle \mathcal{G}\rangle}$ (Corollary 3.3.8 and Theorem 3.2.20), contradicting our assumption that $C_{\langle \mathcal{G}\rangle}$ is an immediate successor of $C_{\langle x\rangle}$.

As $C_{\langle \mathcal{G}\rangle} \nsubseteq C_{\langle x\rangle}$, Lemma 3.4.6 gives us $g_0 \in \langle \mathcal{G}\rangle$ such that $\lim_{x\to 0} \frac{g_0(x)}{x} = \infty$. As $C_{\langle x\rangle} \nsubseteq C_{\mathrm{ICOD}_{bdd}}$, Corollary 3.3.5 gives that $C_{\langle \mathcal{G}\rangle} = C_{\langle \mathcal{G}_/\rangle}$, so we can assume that $g_0 = g_{0/}$. We now follow a strategy similar to that in Proposition 3.4.7 to construct a function $h \in \mathrm{ICOD}$, with $h = h_/ \leq g_0$, and positive sequences $(x_n)_{n=1}^\infty$ and $(z_n)_{n=1}^\infty$ converging to 0 such that $\lim_{n\to\infty} \frac{h(z_n)}{z_n} = \infty$ and $\lim_{n\to\infty} \frac{g_0(x_n)}{h^{(N)}(x_n)} = \infty$ for every $N \in \mathbb{N}$. Once we have constructed such an h, Corollary 3.3.8 and Proposition 3.2.8 establish $C_{\langle x\rangle} \subsetneq C_{\langle h\rangle} \subsetneq C_{\langle g_0\rangle} \subseteq C_{\mathcal{G}}$, which contradicts the definition of $\mathcal{G}$ and thus completes the proof.

As in the proof of Proposition 3.4.7, given $x, y, b > 0$, we let $\ell[x, y, b]$ be the line which sends x to y and has b as y-intercept, i.e., $\ell[x, y, b](t) = \frac{y-b}{x}t + b$ for all $t \in \mathbb{R}$. The construction of h will be based in the following elementary fact, which holds since $\lim_{t\to 0} \frac{g_0(t)}{\ell[x,y,0]^{(n)}(t)} = \infty$ — we recommend the readers guide themselves from Figure 3.2 in the construction of h.

Fact 3.5.7. *Given* $x, y, b, n, k > 0$, *there are* $x' \in (0, x)$ *and* $b' \in (0, b)$ *so that* $\frac{g_0(x')}{\ell[x,y,b']^{(n)}(x')} > k$ *for all* $n \in \mathbb{N}$.

Let $x_0 = z_0 = b_0 = 1$. By the previous claim, we can pick strictly decreasing sequences $(x_n)_{n=0}^\infty$, $(z_n)_{n=0}^\infty$, and $(b_n)_{n=0}^\infty$ so that

(a) $\lim_n x_n = \lim b_n = 0$,

(b) $\frac{g_0(x_n)}{\ell[z_{n-1}, g_0(z_{n-1}), b_n]^{(n)}(x_n)} > n$ for all $n \in \mathbb{N}$, and

(c) $g_0(z_n) = \ell[z_{n-1}, g(z_{n-1}), b_n](z_n)$.

Notice that each equation $g_0(x) = \ell[z_{n-1}, g(z_{n-1}), b_n](x)$ has only two solutions, z_{n-1} and z_n. Therefore, $\ell[z_{n-1}, g_0(z_{n-1}), b_n](x)$ is at most $g_0(x)$, for $x \in [z_n, z_{n-1}]$, and it is strictly greater than $g_0(x)$ otherwise. Hence, $z_n < x_n < z_{n-1}$ for all $n \in \mathbb{N}$. Therefore, as the sequence $(x_n)_{n=1}^\infty$ tends to 0, so does $(z_n)_{n=1}^\infty$.

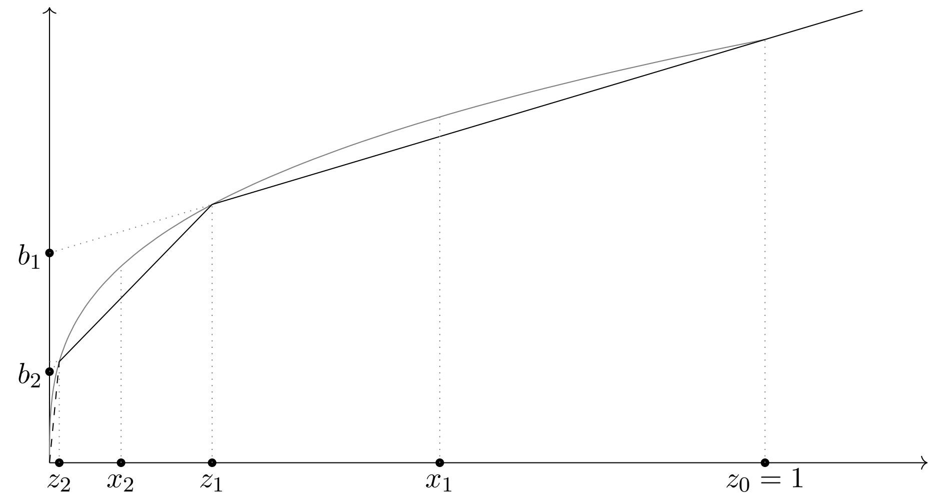

FIGURE 3.2: In the graph above, the smooth function represents g_0 and the piecewise linear function represents h. Note that the scale is modified so that the general behavior of h with respect to g_0 can be represented in the graph.

Now define $h : [0, \infty] \to [0, \infty]$ by letting

$$h(x) = \begin{cases} \ell[1, g_0(1), b_1](x), & \text{if } x \geq 1, \\ \ell[z_{n-1}, g_0(z_{n-1}), b_n](x), & \text{if } x \in (z_n, z_{n-1}], \\ 0, & \text{if } x = 0. \end{cases}$$

It is clear that $h = h_/ \leq g_{0/} = g_0$. Let us notice that $h \in$ ICOD. It is clear from its piecewise definition that h is continuous. Moreover, each $\ell[z_n, g_0(z_n), b_{n+1}]$ has slope $\frac{g_0(z_n) - b_{n+1}}{z_n}$ and, by the construction of $(z_n)_{n=1}^{\infty}$, the slope of each $\ell[z_{n-1}, g_0(z_{n-1}), b_n]$ must equal $\frac{g_0(z_n) - b_n}{z_n}$. Therefore, as $b_{n+1} < b_n$, the slope of each $\ell[z_n, g_0(z_n), b_{n+1}]$ is greater than the one of $\ell[z_{n-1}, g_0(z_{n-1}), b_n]$. Hence, h is concave down. Moreover, since $b_n < g_0(z_{n-1})$ the slope of each $\ell[z_{n-1}, g_0(z_{n-1}), b_n]$ is positive, so h is increasing and thus $h \in$ ICOD.

We are left to notice that $\lim_{n\to\infty} \frac{h(z_n)}{z_n} = \infty$ and $\lim_{n\to\infty} \frac{g_0(x_n)}{h^{(N)}(x_n)} = \infty$ for every $N \in \mathbb{N}$. Since $x_n \in (z_n, z_{n-1})$ for all $n \in \mathbb{N}$, it follows from our choice of $(x_n)_n$ that

$$\begin{aligned} \lim_{n\to\infty} \frac{g_0(x_n)}{h^{(N)}(x_n)} &\geq \lim_{n\to\infty} \frac{g_0(x_n)}{\ell[z_{n-1}, g_0(z_{n-1}), b_n]^{(N)}(x_n)} \\ &\geq \lim_{n\to\infty} \frac{g_0(x_n)}{\ell[z_{n-1}, g_0(z_{n-1}), b_n]^{(n)}(x_n)} \\ &= \infty \end{aligned}$$

for every $N \in \mathbb{N}$. At last, since $\lim_{x\to 0} \frac{g_0(x)}{x} = \infty$ and $z_n \to 0$ we can conclude $\lim_{n\to\infty} \frac{h(z_n)}{z_n} = \lim_{n\to\infty} \frac{g_0(z_n)}{z_n} = \infty$. This completes the proof. □

Although $C_{\langle x \rangle}$ is an immediate successor of $C_{\langle x\_ \rangle}$ in $\mathbb{P}$ (Proposition 3.5.1), the argument of Proposition 3.5.6 can be easily adapted to show that $C_{\langle x\_ \rangle}$ has no immediate successor given by a family of bounded functions. Precisely:

Corollary 3.5.8. *$C_{\langle x\_ \rangle}$ has no immediate successors of the form $C_{\langle \mathcal{G}\_ \rangle}$.*

Proof. Suppose there is a family of maps $\mathcal{G}$ so that $C_{\langle \mathcal{G}\_ \rangle}$ is the immediate successor of $C_{\langle x\_ \rangle}$; without loss of generality we can take $\mathcal{G} \subseteq \mathrm{ICOD}_{<\infty}$ (Theorem 3.2.11 and Theorem 3.2.20). Notice that $\lim_{x\to 0} g(x) = 0$ for every $g \in \mathcal{G}$. Indeed, if not, then Proposition 3.2.19 implies that $C_{\mathrm{ICOD}_{bdd}} = C_{\langle \mathcal{G}\_ \rangle}$. However, this contradicts our assumptions on $\mathcal{G}\_$ since $C_{\langle x\_ \rangle} \subsetneq C_{\langle \sqrt{x}\_ \rangle} \subseteq C_{\mathrm{ICOD}_{0\cap bdd}} \subsetneq C_{\mathrm{ICOD}_{bdd}}$ (see Theorem 3.2.20).

As $C_{\langle \mathcal{G}\_ \rangle} \nsubseteq C_{\langle x\_ \rangle}$, Lemma 3.4.6 gives us $g_0 \in \langle \mathcal{G}\_ \rangle$ such that $\lim_{x\to 0} \frac{g_0(x)}{x} = \infty$. From now on, the proof follows exactly the one of Proposition 3.5.6. Precisely, we construct a function $h \in \mathrm{ICOD}$, with $x \leq h(x) \leq g_0(x)$ for all $x \in [0,1]$, and positive sequences $(x_n)_{n=1}^{\infty}$ and $(z_n)_{n=1}^{\infty}$ converging to 0 such that $\lim_{n\to\infty} \frac{g_0(x_n)}{h^{(N)}(x_n)} = \infty$ for every $N \in \mathbb{N}$ while $\lim_{n\to\infty} \frac{h(z_n)}{z_n} = \infty$. By Corollary 3.3.8 and Proposition 3.2.8, this implies that $C_{\langle x\_ \rangle} \subsetneq C_{\langle h \rangle} \subsetneq C_{\langle g_0 \rangle} \subseteq C_{\langle \mathcal{G}\_ \rangle}$ contradiction. We leave the details to the reader. $\square$

We now present a strengthening of Proposition 3.5.1.

Corollary 3.5.9. *Let* $f \in \mathrm{ICOD}$ *be so that* $\lim_{x\to 0} f(x) = 0$. *Then* $C_{\langle f_/ \rangle}$ *is the unique immediate successor of* $C_{\langle f\_ \rangle}$ *in* $\mathbb{P}$.

Proof. Proposition 3.5.1 establishes that $C_{\langle f_/ \rangle}$ is an immediate successor. Suppose $\mathcal{G}$ is a collection of functions such that $C_{\langle \mathcal{G} \rangle}$ is an immediate successor of $C_{\langle f\_ \rangle}$ a priori perhaps distinct from $C_{\langle f_/ \rangle}$. Without loss of generality we can suppose that $\mathcal{G} \subseteq \mathrm{ICOD}$ (Theorem 3.2.11).

We know from Proposition 3.5.2 and Corollary 3.5.8 that $C_{\langle \mathcal{G}\_ \rangle}$ is not an immediate successor of $C_{\langle f\_ \rangle}$ and thus $C_{\langle \mathcal{G}\_ \rangle} \neq C_{\langle \mathcal{G} \rangle}$. Hence, by Corollary 3.3.5, $C_{\langle \mathcal{G}_/ \rangle} = C_{\langle \mathcal{G} \rangle}$ (this also follows from Propositions 3.2.13 and 3.2.14). Notice that $\lim_{x\to 0} g(x) = 0$ for every $g \in \mathcal{G}$. Indeed, if not, then Proposition 3.2.19 implies that $C_{\mathrm{ICOD}_{<\infty}} = C_{\langle \mathcal{G}_/ \rangle} = C_{\langle \mathcal{G} \rangle}$. However, this contradicts our assumptions on $\mathcal{G}$ since $C_{\langle f\_ \rangle} \subseteq C_{ICOD_{0\cap bdd}} \subsetneq C_{ICOD_{bdd}} \subsetneq C_{\mathrm{ICOD}_{<\infty}} = C_{\langle \mathcal{G} \rangle}$ (see Theorem 3.2.20).

We have $C_{\langle f\_ \rangle} \subseteq C_{\langle \mathcal{G} \rangle}$ and thus by Theorem 3.3.1 there must exist some $g_1 \in \langle \mathcal{G} \rangle$ such that $\lim_{x\to 0} \frac{f\_(x)}{g_1(x)} < n < \infty$ for some $n \in \mathbb{N}$. The function g_1 is built out of a finite number of additions and compositions of elements of $\mathcal{G}$, so we can define g_2 to be the function built out of the corresponding additions and compositions of the corresponding elements of $\mathcal{G}_/$. As $\lim_{x\to 0} g(x) = 0$ for every $g \in \mathcal{G}$, there is $\delta > 0$ so that $g_1(x) = g_2(x)$ for all $x \in [0,\delta]$. In fact, taking a perhaps smaller $\delta > 0$, we have that $f_/(x) = f\_(x) \leq n g_1(x) = n g_2(x)$ for all $x \in [0,\delta]$. Now as $f_/$ is $ISOD$ and $\lim_{x\to\infty} \frac{g_2(x)}{x} > 0$, we can find $m \in \mathbb{N}$, perhaps $m > n$, so that $f_/ \leq m g_2$ and thus $C_{\langle f\_ \rangle} \subsetneq C_{\langle f_/ \rangle} \subseteq C_{\langle m g_2 \rangle} \subseteq C_{\langle \mathcal{G}_/ \rangle} = C_{\langle \mathcal{G} \rangle}$. As $C_{\langle \mathcal{G} \rangle}$ was defined as an immediate successor of $C_{\langle f\_ \rangle}$, we have that $C_{\langle \mathcal{G} \rangle} = C_{\langle f_/ \rangle}$ and are done. $\square$

An immediate consequence of Corollary 3.5.9 is that $C_{\langle x \rangle}$ is the only immediate successor of $C_{\langle x\_ \rangle}$ and thus the unique third element of $\mathbb{P}$. Proposition 3.5.6 assures us that there is no fourth element in $\mathbb{P}$.

We round out this chapter with a summarizing Theorem which captures most of the interesting results about successors in $\mathbb{P}$:

Theorem 3.5.10. *The* poset $\mathbb{P}$ *has the structure indicated in the diagram below.*

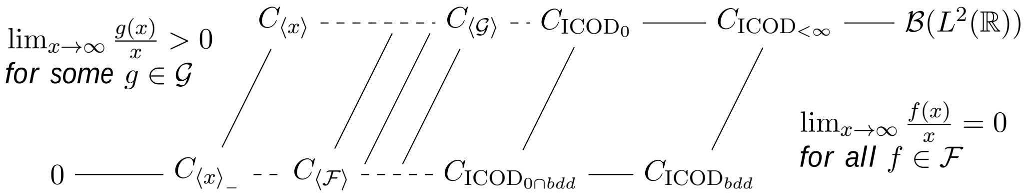

FIGURE 3.3: Elements further up and to the right are larger in the poset $\mathbb{P}$. Dotted lines indicate containment. Solid lines indicate immediate successors.

Chapter 4

Relations and Their Generalizations

In this Chapter we will expand upon the construction of measurable and quantum relations as introduced in the Overview Section 1.2. Most of this content finds origin in Weaver, 2012, though we have tried to provide some additional intuition and a few original results, which we will indicate as they arise.

4.1 Classical Relations

In this section we will recall the basic theory of relations as is common in any undergraduate math curriculum. Following that we will discuss an alternative view of relations which "de-atomizes" them, a natural intermediate step to generalizing them to the measurable setting.

Definition 4.1.1 (Relation)**.** Given a set X, a ***relation*** or ***classical relation*** on X is a subset $\mathrm{R} \subseteq X \times X$.

Relations can also be defined between two sets X and Y, as subsets of $X \times Y$. We note that our definition above is just as general, since any relation $X \to Y$ can be identified with a relation on $X \times Y$.

We now recall the definitions of some important operations on relations and various properties that relations can have.

Definition 4.1.2. Let X be a set.

(a) The ***diagonal relation*** on X is defined by $\Delta = \{(x,x) : x \in X\}$.

(b) The ***transpose*** or ***inverse*** of a relation R is the relation $\mathrm{R}^T = \{(y,x) : (x,y) \in \mathrm{R}\}$.

(c) The ***product*** or ***composition*** of two relations R, R' is the relation

$$\mathrm{R} \cdot \mathrm{R}' = \{(x,z) : \text{ there exists a } y \text{ with } (x,y) \in \mathrm{R} \text{ and } (y,z) \in \mathrm{R}'\}.$$

(d) A relation R on X is

  (i) ***reflexive*** if $\Delta \subseteq \mathrm{R}$
  (ii) ***symmetric*** if $\mathrm{R}^T = \mathrm{R}$
  (iii) ***antisymmetric*** if $\mathrm{R} \cap \mathrm{R}^T \subseteq \Delta$
  (iv) ***transitive*** if $\mathrm{R} \cdot \mathrm{R} \subseteq \mathrm{R}$.

Now we recall common sub-classes of relations.

Definition 4.1.3. Let R be a relation on a set X.

(a) If $|\{y : (x, y) \in \mathrm{R}\}| = 1$ for every $x \in X$ then R is a ***function***.

(b) If R is a function and $|\{y : (y, x) \in \mathrm{R}\}| = 1$ for every $x \in X$ then R is ***injective***.

(c) If R is a function and $\{y : \text{ there exists } x \in X \text{ such that } (x, y) \in \mathrm{R}\} = X$ then R is ***surjective***.

(d) $\mathcal{G} = (X, \mathrm{R})$ is a ***directed graph***. X is called the set of ***vertices*** of $\mathcal{G}$ and R the set of ***edges***.

(e) If $\mathcal{G} = (X, \mathrm{R})$ is a directed graph and R is symmetric then $\mathcal{G}$ is an ***undirected graph*** or merely a ***graph***.

It seems that any attempt to generalize the notion of relation to a continuous/measurable setting will find difficulty as Definition 4.1.1 is concerned with atomic objects, the individual elements of X, which have no generic measure-theoretic analog. We need to "de-atomize" relations. That is, define them in such a way that singletons do not hold an elevated status over other sets.

An element of a relation R on a set X is an ordered pair (x, y) with $x, y \in X$. We note that (x, y) is the sole element of the Cartesian product $\{x\} \times \{y\}$. In fact the set $\{\{x\} \times \{y\} : (x, y) \in \mathrm{R}\}$ contains essentially the same information as R, just with some extra brackets. Framed this way, we see that ordered pairs are essentially just distinguished rectangles in $X \times X$. We would like to define a notion of relation which deals with generic rectangles $S \times T$ for $S, T \subseteq X$.

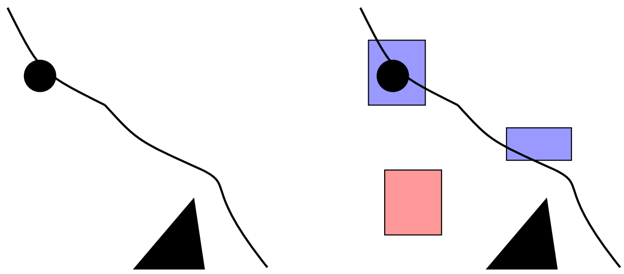

FIGURE 4.1: On the left is a relation R on $\mathbb{R}$. On the right we see R again with several rectangles in $\mathbb{R} \times \mathbb{R}$ overlaid. If the rectangles intersect R they are colored blue and if not they are colored red.

Given a relation R on a set X we could consider several classes of rectangles:

(a) Rectangles that intersect R.

(b) Rectangles that do not intersect R.

(c) Rectangles which are entirely contained in R.

The original relation R could be recovered from any of these. In the case of (a) and (c) you can recover R by merely restricting your attention to products of singletons. In the case of (b) note that rectangles in this class and those in class (a) are a partition of all the rectangles and then reduce to the previous sentence.

We will focus in on case (a), rectangles which intersect R. Given a relation R on a set X we define the ***meta-relation associated with R*** as $\mathcal{R}_{\mathrm{R}} = \{(S, T) \in \mathscr{P}(X) \times \mathscr{P}(X) : (S \times T) \cap \mathrm{R} \neq \emptyset\}$.

Note that $\mathcal{R}_{\mathrm{R}}$ is essentially (though not literally) the set of rectangles which intersect R. As observed before we can recover the original relation by $\mathrm{R} = \{(x, y) \in X \times X : (\{x\}, \{y\}) \in \mathcal{R}_{\mathrm{R}}\}$, so meta-relations are a "de-atomized" object in an obvious one-to-one correspondence with relations. Rectangles are much more readily generalized to a measurable setting than atomic points, so this is a good candidate to lead us to measurable relations.

But our definition of meta-relation already starts with a relation. What we still need is a way to identify which collections of rectangles are meta-relations associated to a relation without starting with a classical relation first.

Proposition 4.1.4. *(cf. Weaver, 2012 Proposition 1.3) Let $\mathcal{E} \subseteq \mathscr{P}(X) \times \mathscr{P}(X)$ be a collection of pairs of nonempty subsets of X. Then $\mathcal{E} = \mathcal{R}_R$ for some relation R on X if and only if*

$$\left(\bigcup S_\lambda, \bigcup T_\kappa\right) \in \mathcal{E} \Leftrightarrow \text{ some } (S_\lambda, T_\kappa) \in \mathcal{E}$$

for any pair of families of non-empty subsets of X, $\{S_\lambda\}, \{T_\kappa\} \subseteq \mathscr{P}(X)$.

Proof. Essentially as found in Weaver, 2012 Proposition 1.3. □

We can now define meta-relations in their own right without reference to classical relations: Given a set X, a ***meta-relation*** on X is a subset $\mathcal{R} \subseteq \mathscr{P}(X) \times \mathscr{P}(X)$ satisfying

$$\left(\bigcup S_\lambda, \bigcup T_\kappa\right) \in \mathcal{R} \Leftrightarrow \text{ some } (S_\lambda, T_\kappa) \in \mathcal{R}$$

for any pair of families of non-empty subsets of X, $\{S_\lambda\}, \{T_\kappa\} \subseteq \mathscr{P}(X)$.

We say $\mathrm{R}_{\mathcal{R}} = \{(x, y) \in X \times X : (\{x\}, \{y\}) \in \mathcal{R}\}$ is the *classical relation associated with $\mathcal{R}$*. Compare with Weaver, 2012 Definition 1.2.

The constructions for $\mathcal{R}_{\mathrm{R}}$ and $\mathrm{R}_{\mathcal{R}}$ are inverse to each other.

We can recast Definition 4.1.2 for meta-relations. The definitions below are compatible with the corresponding definitions for classical relations.

Definition 4.1.5 (cf. Weaver, 2012 Definition 1.6)**.** Let X be a set.

(a) The *diagonal meta-relation* on X is defined by $\Delta = \{(S, T) \in \mathscr{P}(X) \times \mathscr{P}(X) : S \cap T \neq \emptyset\}$.

(b) The *transpose* of a meta-relation $\mathcal{R}$ is the relation $\mathcal{R}^T = \{(T, S) : (S, T) \in \mathcal{R}\}$.

(c) The *product* of two meta-relations $\mathcal{R}, \mathcal{R}'$ is the meta-relation

$$\begin{aligned} \mathcal{R} \cdot \mathcal{R}' &= \{(S, U) : \text{ for every } T \text{ either } (S, T) \in \mathcal{R} \text{ or } (X - T, U) \in \mathcal{R}'\} \text{ or equivalently} \\ &= \{(S, U) : \text{ there is a } T \text{ so that } (S, T') \in \mathcal{R} \text{ and } (T', U) \in \mathcal{R}' \text{ for every } T' \leq T\}. \end{aligned}$$

Meta-relations are somewhat harder to work with and parse than classical relations. Below we introduce useful maps $\mathscr{P}(X) \to \mathscr{P}(X)$ which simplify much of the analysis of meta-relations.

Definition 4.1.6 (cf. Weaver, 2012 Proposition 1.4)**.** Let $\mathcal{R}$ be a meta-relation on a set X then for each $S \subseteq X$ we define

(a) the *right image* of S, $\psi_{\mathcal{R}}(S) = X - \bigcup\{T : (S, T) \notin \mathcal{R}\}$ and

(b) the *left image* of S, $\phi_{\mathcal{R}}(S) = X - \bigcup\{T : (T, S) \notin \mathcal{R}\}$.

We refer to any function $\phi : \mathscr{P}(X) \to \mathscr{P}(X)$ which can be realized as $\phi_{\mathcal{R}}$ for some meta-relation on X as an ***image map*** for X and denote the collection of image maps for a set X by $\mathrm{Im}(X)$.

Note that for a set $S \subseteq X$ and meta-relation $\mathcal{R}$ on X

$$\psi_{\mathcal{R}}(S) = \{y \in X : \text{ there exists some } x \in S \text{ such that } (x, y) \in \mathrm{R}_{\mathcal{R}}\} \text{ and}$$
$$\phi_{\mathcal{R}}(S) = \{x \in X : \text{ there exists some } y \in S \text{ such that } (x, y) \in \mathrm{R}_{\mathcal{R}}\}.$$

justifying the names ***right image*** and ***left image*** respectively. Of course we did not define image maps this way because we would like to generalize them to the measurable setting and so do not want to reference singletons in their definition.

Proposition 4.1.7 (cf. Weaver, 2012 Proposition 1.4)**.** *Let X be a set and $\phi : \mathscr{P}(X) \to \mathscr{P}(X)$. Then $\phi \in Im(X)$ if and only if ϕ takes the empty set to the empty set and preserves arbitrary unions.*

Furthermore if $\phi \in Im(X)$ then $\mathcal{R}_{\psi} = \{(S, T) \in \mathscr{P}(X) \times \mathscr{P}(X) : S \cap \phi(T) \neq \emptyset\}$ is a meta-relation and this construction is inverse to the one in Definition 4.1.6.

We are now prepared to discuss measurable relations.

4.2 Measurable Relations

The fundamental theory here runs parallel to that of meta-relations from the previous section, so we will present the initial definitions and propositions here without much exposition.

Definition 4.2.1. (Weaver, 2012 Definition 1.2 Measurable Relation) Let (X, μ) be a finitely decomposable measure space. A ***measurable relation*** on X is a family $\mathcal{R}$ of ordered pairs of nonzero projections in $L^\infty(X, \mu)$ such that

$$\left(\bigvee p_\lambda, \bigvee q_\kappa\right) \in \mathcal{R} \Leftrightarrow \text{ some } (p_\lambda, q_\kappa) \in \mathcal{R}$$

for any pair of families of nonzero projections $\{p_\lambda\}$ and $\{q_\kappa\}$.

Proposition 4.2.2. *(Weaver, 2012 Proposition 1.4) Let (X, μ) be a finitely decomposable measure space. If $\mathcal{R}$ is a measurable relation on X then the map*

$$\phi_{\mathcal{R}} : q \mapsto 1 - \bigvee\{p : (p, q) \notin \mathcal{R}\},$$

from the set of projections in $L^\infty(X, \mu)$ to itself, takes 0 to 0 and preserves arbitrary joins. If ϕ is a map from the set of projections in $L^\infty(X, \mu)$ to itself that takes 0 to 0 and preserves arbitrary joins then

$$\mathcal{R}_\phi = \{(p, q) : p\phi(q) \neq 0\}$$

is a measurable relation on X. The two constructions are inverse to each other.

We observe that the constructions in Proposition 4.2.2 are monotonic, i.e. $\phi \leq \phi'$ if and only if $\mathcal{R}_\phi \leq \mathcal{R}_{\phi'}$, where the order on image maps is the pointwise order inherited from projections and the order on measurable relations is standard containment.

Definition 4.2.3. Let (X, μ) be a finitely decomposable measure space. We call a map $\phi : \mathrm{Pr}(L^\infty(X, \mu)) \to \mathrm{Pr}(L^\infty(X, \mu))$ which takes 0 to 0 and preserves arbitrary joins an ***image map*** for $L^\infty(X, \mu)$ and denote the collection of such maps by $\mathrm{Im}(X, \mu)$.

We have already seen an important class of image maps in Chapter 3:

Example 4.2.4. If $a \in B(L^2(\mathbb{R}))$ then $q \mapsto s_l^{\mathcal{M}}(aq)$ is an image map for $L^\infty(X, \mu)$.

Proof. It is clear that $s_l^{\mathcal{M}}(a \cdot 0) = 0$.

For joins we note that $s_l^{\mathcal{M}}(a(\bigvee q_\lambda))aq_{\lambda_0} = s_l^{\mathcal{M}}(a(\bigvee q_\lambda))a(\bigvee q_\lambda)q_{\lambda_0} = a(\bigvee q_\lambda)q_{\lambda_0} = aq_{\lambda_0}$ so $s_l^{\mathcal{M}}(a(\bigvee q_\lambda)) \geq s_l^{\mathcal{M}}(aq_{\lambda_0})$ for any λ_0 and thus $s_l^{\mathcal{M}}(a(\bigvee q_\lambda)) \geq \bigvee s_l^{\mathcal{M}}(aq_\lambda)$. On the other hand it is clear that $\bigvee s_l^{\mathcal{M}}(aq_\lambda) \geq s_l^{\mathcal{M}}(aq_{\lambda_0})$ for any λ_0 and since $\bigvee q_\lambda$ is the projection onto the closed linear span of the subspaces corresponding to the q_λ we must have that $\bigvee s_l^{\mathcal{M}}(aq_\lambda)a(\bigvee q_\lambda) = a(\bigvee q_\lambda)$ so $\bigvee s_l^{\mathcal{M}}(aq_\lambda) \geq s_l^{\mathcal{M}}(a(\bigvee q_\lambda))$ and we are done. □

Proposition 4.2.5 (Weaver, 2012 Proposition 1.5a-d)**.** *Let (X, μ) be a finitely decomposable measure space.*

(a) The set of pairs of projections p and q in $L^\infty(X, \mu)$ such that $pq \neq 0$ is a measurable relation on X.

(b) If $\mathcal{R}$ is a measurable relation on X then so is $\{(q, p) : (p, q) \in \mathcal{R}\}$.

(c) If $\mathcal{R}$ and $\mathcal{R}'$ are measurable relations on X then a pair of nonzero projections (p, r) satisfies

for every projection q, either $(p, q) \in \mathcal{R}$ or $(1 - q, r) \in \mathcal{R}'$

if and only if it satisfies

there exists a nonzero projection q such that $(p, q') \in \mathcal{R}$ and $(q', r) \in \mathcal{R}'$ for every nonzero $q' \leq q$

and the set of all pairs satisfying these conditions constitutes a measurable relation.

(d) Any union of measurable relations on X is a measurable relation on X.

Definition 4.2.6. Let (X, μ) be a finitely decomposable measure space and $\mathcal{E}$ a collection of measurable relations on X

(a) $\bigvee \mathcal{E} = \bigcup\{\mathcal{R} : \mathcal{R} \in \mathcal{E}\}$

(b) $\bigwedge \mathcal{E} = \bigcup\{\mathcal{R}'$ measurable relation on $X : \mathcal{R}' \leq \mathcal{R}$ for every $\mathcal{R} \in \mathcal{E}\}$

Note that the set of measurable relations on X partially ordered by inclusion forms a complete lattice under these definitions by Proposition 4.2.5d.

Definition 4.2.7. (Weaver, 2012 Definition 1.6) Let (X, μ) be a finitely decomposable measure space.

(a) The *diagonal measurable relation* Δ on X is defined by setting $(p, q) \in \Delta$ if $pq \neq 0$.

(b) The *transpose* of a measurable relation $\mathcal{R}$ is the measurable relation $\mathcal{R}^T = \{(q, p) : (p, q) \in \mathcal{R}\}$.

(c) The *product* of two measurable relations $\mathcal{R}$ and $\mathcal{R}'$ is the measurable relation

$$\begin{aligned}\mathcal{R} \cdot \mathcal{R}' &= \{(p, r) : \text{ for every } q \text{ either } (p, q) \in \mathcal{R} \text{ or } (1 - q, r) \in \mathcal{R}'\} \text{ or equivalently} \\ &= \{(p, r) : \text{ there is a } q \text{ so that } (p, q') \in \mathcal{R} \text{ and } (q', r) \in \mathcal{R}' \text{ for every } q' \leq q\}.\end{aligned}$$

(d) A measurable relation $\mathcal{R}$ on X is

(i) *reflexive* if $\Delta \subseteq \mathcal{R}$

(ii) *symmetric* if $\mathcal{R}^T = \mathcal{R}$

(iii) *antisymmetric* if $\mathcal{R} \wedge \mathcal{R}^T \subseteq \Delta$

(iv) *transitive* if $\mathcal{R}^2 \subseteq \mathcal{R}$.

We also offer some original (though intuitive) results making the correspondence between measurable relations and image maps more robust:

Proposition 4.2.8. *Let (X, μ) be a finitely decomposable measure space. If $\mathcal{R}$ and $\mathcal{R}'$ are measurable relations on X then*

(a) $\phi_{\mathcal{R} \vee \mathcal{R}'} = \phi_{\mathcal{R}} \vee \phi_{\mathcal{R}'}$,

(b) $\phi_{\mathcal{R} \wedge \mathcal{R}'} = \phi_{\mathcal{R}} \wedge \phi_{\mathcal{R}'}$ *and*

(c) $\phi_{\mathcal{R} \cdot \mathcal{R}'} = \phi_{\mathcal{R}} \circ \phi_{\mathcal{R}'}$.

Proof. The first and second claims follow from the monotonicity of the inverse constructions in Proposition 4.2.2.

For the third claim fix a projection r in $L^\infty(X, \mu)$. Consider the projections

$$\begin{aligned} u &= \bigvee \{p : (p, r) \notin \mathcal{R} \cdot \mathcal{R}'\} \\ &= \bigvee \{p : \text{ there exists some } q \text{ such that } (p, q) \notin \mathcal{R} \text{ and } (1 - q, r) \notin \mathcal{R}'\} \end{aligned}$$

and

$$v = \bigvee \{p : (p, \phi_{\mathcal{R}'}(r)) \notin \mathcal{R}\}.$$

We see that $(1 - \phi_{\mathcal{R}'}(r), r) \notin \mathcal{R}'$ and so it is clear that $v \leq u$ since $\phi_{\mathcal{R}'}(r)$ can act as q for every projection $p \leq v$.

Note that for any q such that $(1 - q, r) \notin \mathcal{R}'$ we have that $q \geq \phi_{\mathcal{R}'}(r)$ and so for every p for which there exists a q such that $(p, q) \notin \mathcal{R}$ and $(1 - q, r) \notin \mathcal{R}'$, it is more specifically true that $(p, \phi_{\mathcal{R}'}(r)) \notin \mathcal{R}$. From this we conclude that $u \leq v$ and thus $u = v$. Now just note that $\phi_{\mathcal{R} \cdot \mathcal{R}'}(r) = 1 - u$ and $(\phi_{\mathcal{R}} \circ \phi_{\mathcal{R}'})(r) = 1 - v$ and recall that r was arbitrary to obtain the result. □

4.3 Cantankerous Relations

Weaver, 2012 generalizes measurable relations to *quantum relations* and we will discuss these in the next section. For now we would like to explore an intermediate notion: a direct translation of the measurable relation language but taking projections in a not necessarily commutative von Neumann algebra. The contents of this section are original, though clearly heavily inspired by Weaver, 2012.

A brief aside about the adjective "cantankerous": It turns out that, for our purposes, a lot of interesting things happen when we move from $\mathcal{M}$ being commutative to it being non-commutative. The (intrinsic) quantum relations from Weaver, 2012 consist of pairs of nonzero projections in $\mathcal{M} \overline{\otimes} \mathcal{B}(\ell^2)$ and this amplification brings with it a lot of additional notation. Also, we have been able to find some examples of new behavior for non-commutative "measurable" relations but finding analogous examples for quantum relations has proved difficult. Because of this we would like to explore the measurable relations of Weaver, 2012 over a not-necessarily commutative von Neumann algebra $\mathcal{M}$ while not jumping straight to quantum relations.

Typically we would indicate a non-commutative analog by the adjective "quantum" but this is of course out, since quantum relations are defined. Then perhaps "non-commutative" would be our fallback, but this presents an issue when we generate "non-commutative" uniform Roe algebras from "non-commutative" relations – since uniform Roe algebras can already be non-commutative. It seems we need some new adjective.

Choosing the proper adjective can be challenging. It should be sensible, morally indicating its effect via its literal meaning. There is also an aesthetic concern – it should be easy and fun to say. We land on the word "cantankerous" from two directions.

Firstly, we look to the antonym of our meaning: Taking a not-necessarily commutative object and considering only the ***abelian*** versions. We want, somehow, the opposite of "abelian". Well, one asks, who was the opposite of Abel? And by way of pun we conclude it must be the biblical Cain. So we begin a search for words which invoke "Cain": "cainish", "canine", "canonical" and so forth. While these may satisfy our desire for aesthetics, none of them properly indicate the desired meaning by their literal interpretation.

But "cantankerous", while slightly off in pronunciation, works quite well by its literal meaning. A cantankerous person is difficult, standoffish, and so one might picture the cantankerous elements $m, n \in \mathcal{M}$ unable to cross paths. We cannot conclude that $mn = nm$ in general because these creatures are prone to strife and conflict, they will not typically switch positions without protestation. They refuse to commute. And so we settle on this adjective "cantankerous" to mean "not-necessarily commutative". It could be adopted in any situation to indicate a non-commutative analog of a traditionally commutative object. **End of aside.**

Definition 4.3.1. (cf. Definition 4.2.1) Let $\mathcal{M} \subseteq \mathcal{B}(H)$ be a represented von Neumann algebra. A ***cantankerous relation*** on $\mathcal{M}$ is a family $\mathcal{R}$ of ordered pairs of nonzero projections in $\mathcal{M}$ such that

$$\left(\bigvee p_\lambda, \bigvee q_\kappa\right) \in \mathcal{R} \Leftrightarrow \text{ some } (p_\lambda, q_\kappa) \in \mathcal{R}$$

for any pair of families of nonzero projections $\{p_\lambda\}$ and $\{q_\kappa\}$.

Proposition 4.3.2. *(cf. Proposition 4.2.2) Let $\mathcal{M} \subseteq \mathcal{B}(H)$ be a represented von Neumann algebra. If $\mathcal{R}$ is a cantankerous relation on $\mathcal{M}$ then the map*

$$\phi_{\mathcal{R}} : q \mapsto 1 - \bigvee\{p : (p, q) \notin \mathcal{R}\},$$

from the set of projections in $\mathcal{M}$ to itself, takes 0 to 0 and preserves arbitrary joins. If ϕ is a map from the set of projections in $\mathcal{M}$ to itself that takes 0 to 0 and preserves arbitrary joins then

$$\mathcal{R}_\phi = \{(p, q) : p\phi(q) \neq 0\}$$

is a cantankerous relation on X. The two constructions are inverse to each other and order preserving (where the order on cantankerous relations is containment and the order on the ϕ maps is inherited pointwise from the order on projections).

Proof. First let $\mathcal{R}$ be a cantankerous relation on $\mathcal{M}$. Notice that $\phi_{\mathcal{R}}(q) \leq p$ if and only if $(1 - p, q) \notin \mathcal{R}$, we will use this repeatedly and implicitly below.

It is clear that $\phi_{\mathcal{R}}(0) = 0$. Take $\{q_\kappa\} \subseteq \mathrm{Pr}(\mathcal{M})$ and note that if $(p, \bigvee q_\kappa) \notin \mathcal{R}$ then $(p, q_\kappa) \notin \mathcal{R}$ for every κ (Definition 4.3.1) so $\phi_{\mathcal{R}}(q_\kappa) \leq \phi_{\mathcal{R}}(\bigvee q_\kappa)$ for each κ and thus $\bigvee \phi_{\mathcal{R}}(q_\kappa) \leq \phi_{\mathcal{R}}(\bigvee q_\kappa)$. If $\phi_{\mathcal{R}}(\bigvee q_\kappa) \not\leq \bigvee \phi_{\mathcal{R}}(q_\kappa)$ then $(1 - \bigvee \phi_{\mathcal{R}}(q_\kappa), \bigvee q_\kappa) \in \mathcal{R}$ which implies that $(1 - \bigvee \phi_{\mathcal{R}}(q_\kappa), q_{\kappa_0}) \in \mathcal{R}$ for some κ_0 thus $\phi_{\mathcal{R}}(q_{\kappa_0}) \not\leq \bigvee \phi_{\mathcal{R}}(q_\kappa)$ which is absurd. So we have $\phi_{\mathcal{R}}(\bigvee q_\kappa) = \bigvee \phi_{\mathcal{R}}(q_\kappa)$ for any family $\{q_\kappa\} \subseteq \mathrm{Pr}(\mathcal{M})$.

Now let ϕ be a map from $\mathrm{Pr}(\mathcal{M})$ to itself which satisfies the hypotheses, we will show that $\mathcal{R}_\phi$ is a cantankerous relation on $\mathcal{M}$. It is clear that $(0, I), (I, 0) \notin \mathcal{R}_\phi$ which along with the join property (proved next) gives us that all pairs in $\mathcal{R}_\phi$ are nonzero. If $\{p_\lambda\}, \{q_\kappa\} \subseteq \mathrm{Pr}(\mathcal{M})$ then $(\bigvee p_\lambda)(\phi(\bigvee q_\kappa)) = (\bigvee p_\lambda)(\bigvee \phi(q_\kappa)) = 0$ if and only if $p_\lambda \phi(q_\kappa) = 0$ for every λ and κ, so we have that $(\bigvee p_\lambda, \bigvee q_\kappa) \in \mathcal{R}_\phi$ if and only if some $(p_\lambda, q_\kappa) \in \mathcal{R}_\phi$.

Checking that the constructions are inverse to each other and order preserving is routine – we omit the proof. □

Proposition 4.3.3. *(cf. Proposition 4.2.5) Let $\mathcal{M} \subseteq \mathcal{B}(H)$ be a represented von Neumann algebra.*

(a) The set of pairs of projections p and q in Pr$(\mathcal{M})$ such that $pq \neq 0$ is a cantankerous relation on $\mathcal{M}$.

(b) If $\mathcal{R}$ is a cantankerous relation on $\mathcal{M}$ then so is $\{(q, p) : (p, q) \in \mathcal{R}\}$.

(c) If $\mathcal{R}$ and $\mathcal{R}'$ are cantankerous relations on $\mathcal{M}$ then the set of pairs of nonzero projections (p, r) satisfying

$$\text{for every projection } q, \text{ either } (p, q) \in \mathcal{R} \text{ or } (1 - q, r) \in \mathcal{R}'$$

is a cantankerous relation on $\mathcal{M}$.

(d) Any union of cantankerous relations on $\mathcal{M}$ is a cantankerous relation on $\mathcal{M}$.

Proof. Parts (a), (b), and (d) are easy. For part (c) the proof is identical to Weaver, 2012 Proposition 1.5c, since he does not use the commutativity of projections in $L^\infty(X, \mu)$. □

If you reference Weaver, 2012 Proposition 1.5c (or 4.2.5 in this paper) you will see that there was a second characterization of the product of two measurable relations (part (c) above): the set of $(p, r) \in L^\infty(X, \mu)^2$ such that there exists a nonzero projection q such that $(p, q') \in \mathcal{R}$ and $(q', r) \in \mathcal{R}'$ for every nonzero $q' \leq q$. The commutativity of the projections in $L^\infty(X, \mu)$ is leveraged in the proof that these two definitions are equivalent and we have not been able to find a replacement proof in the noncommutative setting, nor a proof that this alternative definition fulfills the axioms of a cantankerous relation.

Question 4.3.4. Let $\mathcal{M}$ be a von Neumann algebra with cantankerous relations $\mathcal{R}, \mathcal{R}'$. Define

$$\mathcal{R} \diamond \mathcal{R}' = \{(p, r) \in \mathrm{Pr}(\mathcal{M})^2 : \text{there is some nonzero } q \in \mathrm{Pr}(\mathcal{M}) \text{ such that } (p, q') \in \mathcal{R} \text{ and } (q', r) \in \mathcal{R}' \text{ for every nonzero } q' \leq q\}.$$

Is $\mathcal{R} \diamond \mathcal{R}'$ a cantankerous relation on $\mathcal{M}$? Can we conclude that $\mathcal{R} \diamond \mathcal{R}' = \mathcal{R} \cdot \mathcal{R}'$?

The remaining terminology from Section 4.2 (diagonal, product, meet and join, etc.) can be ported over without issue. Note that the proof of Proposition 4.2.8 does not use the commutativity of $L^\infty(X, \mu)$ and for the product therein we use the definition which ports to cantankerous relations.

4.3.1 Operating in Pairs

In this sub-section we want to explore another route that leads to nearly the same framework and maybe provides new insight. It is also closely connected to the way Weaver, 2012 defines quantum relations.

First we review a general mathematical principle: Take two sets $\mathcal{S}, \mathcal{T}$ and a pairing relation $\mathrm{R} \subseteq \mathcal{S} \times \mathcal{T}$. For any subsets $S \subseteq \mathcal{S}$, $T \subseteq \mathcal{T}$ we can define $S^\sharp = \{t \in \mathcal{T} : s\mathrm{R}t \text{ for all } s \in S\}$ and $T^\flat = \{s \in \mathcal{S} : s\mathrm{R}t \text{ for all } t \in T\}$. After some inspection we note the following:

Fact 4.3.5. *Given $(\mathcal{S}, \mathcal{T}, R)$ with subsets $S, S_1, S_2 \subseteq \mathcal{S}$ and $T, T_1, T_2 \subseteq \mathcal{T}$ we have the following:*

(a) $S_1 \subseteq S_2 \implies S_2^\sharp \subseteq S_1^\sharp$,

(b) $T_1 \subseteq T_2 \implies T_2^\flat \subseteq T_1^\flat$,

(c) $S \subseteq (S^\sharp)^\flat$,

(d) $T \subseteq (T^\flat)^\sharp$.

We say a subset $S \subseteq \mathcal{S}$ is a ***dual*** object if there is some $T \subseteq \mathcal{T}$ such that $S = T^\flat$ and a ***stable*** object if $S = (S^\sharp)^\flat$. We will speak similarly about subsets of $\mathcal{T}$.

Corollary 4.3.6. *Given $(\mathcal{S}, \mathcal{T}, R)$, a subset $S \subseteq \mathcal{S}$ is dual if and only if it is stable. Likewise for $T \subseteq \mathcal{T}$.*

Proof. We will argue for $S \subseteq \mathcal{S}$, the proof for $T \subseteq \mathcal{T}$ is identical.

It is immediate from the definition that a stable subset is dual. Now suppose $S \subseteq \mathcal{S}$ is dual, so we can find some $T \subseteq \mathcal{T}$ such that $S = T^\flat$. We have from Fact 4.3.5 that $T \subseteq (T^\flat)^\sharp = S^\sharp$ and so $(S^\sharp)^\flat \subseteq T^\flat = S$ since the operation ***flat*** is order reversing (also Fact 4.3.5). But we have $S \subseteq (S^\sharp)^\flat$ from Fact 4.3.5 as well. Combining these we get that $S = (S^\sharp)^\flat$ and thus S is stable. This completes the proof. □

Studying which subsets are stable (aka dual) is often fruitful. Let $\mathcal{M} \subseteq \mathcal{B}(H)$ be a represented von Neumann algebra and consider the pairing $(\mathcal{B}(H), \mathrm{Pr}(\mathcal{M})^2, \mathrm{R})$ where $(a, (p, q)) \in \mathrm{R}$ whenever $paq = 0$.

Proposition 4.3.7. *(cf. Weaver, 2012 Definition 2.1) Let $\mathcal{M} \subseteq \mathcal{B}(H)$ be a represented von Neumann algebra and consider the pairing $(\mathcal{B}(H), \mathrm{Pr}(\mathcal{M})^2, R)$ where $(a, (p, q)) \in R$ whenever $paq = 0$. Every dual subset of $\mathcal{B}(H)$ is a weak* closed subspace $\mathcal{S} \subseteq \mathcal{B}(H)$ satisfying $\mathcal{M}'\mathcal{S}\mathcal{M}' \subseteq \mathcal{S}$.*

Proof. Suppose S is a dual subset of $\mathcal{B}(H)$, so there is some $T \subseteq \mathrm{Pr}(\mathcal{M})^2$ such that $S = T^\flat$. That is, $S = \{a \in \mathcal{B}(H) : paq = 0 \text{ for every } (p, q) \in T\}$.

If $a, b \in S$ and $\lambda \in \mathbb{C}$ then $p(a + b)q = paq + pbq = 0$ and $p\lambda aq = \lambda paq = 0$ for every $(p, q) \in T$ so S is a linear subspace of $\mathcal{B}(H)$.

Since every element of T is a pair of projections in $\mathcal{M}$ we note that if $n, m \in \mathcal{M}'$, the commutant of $\mathcal{M}$, then $pnamq = npaqm = 0$ for every $a \in S$ and $(p, q) \in T$, so $nam \in S$. Thus $\mathcal{M}'S\mathcal{M}' \subseteq S$.

It remains to show that S is weak* closed. In fact it is closed in the (less fine) weak operator topology: Suppose (a_α) is a net in S such that $a_\alpha \xrightarrow{WOT} a$ then for any vectors $\xi, \eta \in \mathcal{B}(H)$ and $(p, q) \in T$ we have that $|\langle paq\xi, \eta\rangle| = |\langle p(a - a_\alpha)q\xi, \eta\rangle| = |\langle (a - a_\alpha)q\xi, p\eta\rangle|$ which can be made arbitrarily small by selecting appropriate a_α. So $|\langle paq\xi, \eta\rangle| = 0$ for every $\xi, \eta \in \mathcal{B}(H)$ and thus $paq = 0$. Since $(p, q) \in T$ was arbitrary we have $a \in S$ which completes the proof. □

Proposition 4.3.8. *Let $\mathcal{M} \subseteq \mathcal{B}(H)$ be a represented von Neumann algebra and consider the pairing $(\mathcal{B}(H), Pr(\mathcal{M})^2, R)$ where $(a,(p,q)) \in R$ whenever $paq = 0$. Every dual subset T of $Pr(\mathcal{M})^2$ satisfies:*

(a) $(0,p),(p,0) \in T$ *for every* $p \in Pr(\mathcal{M})$ *and*

(b) $(\bigvee p_\lambda, \bigvee q_\kappa) \in T \Leftrightarrow$ *all* $(p_\lambda, q_\kappa) \in T$

for any pair of families of projections $\{p_\lambda\},\{q_\kappa\}$.

Proof. Suppose T is a dual subset of $\mathrm{Pr}(\mathcal{M})^2$, so there is some $S \subseteq \mathcal{B}(H)$ such that $T = S^\sharp$. That is, $T = \{(p,q) \in \mathrm{Pr}(\mathcal{M})^2 : paq = 0 \text{ for every } a \in S\}$.

If $p \in \mathrm{Pr}(\mathcal{M})$ it is immediate that $0ap = 0 = pa0$ for every $a \in S$ so $(0,p),(p,0) \in T$.

Now if $\{p_\lambda\},\{q_\kappa\} \subseteq \mathrm{Pr}(\mathcal{M})$ are families of projections we note that $(\bigvee p_\lambda)a(\bigvee q_\kappa) = 0 \implies p_\lambda a q_\kappa = 0$ is immediate. The reverse implication follows as well: Suppose $aq_\kappa = 0$ for every κ then for any $\xi \in span(q_\kappa H : \kappa)$ we have $a\xi = 0$ and thus $a(\bigvee q_\kappa) = 0$. We could argue the same in the adjoint to get that $p_\lambda a = 0$ for every λ implies $(\bigvee p_\lambda)a = 0$. Since this works for arbitrary $a \in \mathcal{B}(H)$ we obtain the result. $\square$

We note that the properties given in Proposition 4.3.8 characterize the subsets of $\mathrm{Pr}(\mathcal{M})^2$ which are the complements of cantankerous relations.

In both Propositions 4.3.7 and 4.3.8 we have given some important properties that stable subsets must have in this setup, but we have not characterized them yet. In fact general characterizations are difficult, though in the case of stable subsets of $\mathcal{B}(H)$ they correspond to something well studied.

Definition 4.3.9. (Weaver, 2012 Definition 2.14 and Proposition 2.15) Let $\mathcal{M} \subseteq \mathcal{B}(H)$ be a represented von Neumann algebra. A subset $\mathcal{V} \subseteq \mathcal{B}(H)$ satisfying $\mathcal{M}'\mathcal{V}\mathcal{M}' \subseteq \mathcal{V}$ is *operator reflexive* if

$$\mathcal{V} = \{a \in \mathcal{B}(H) : p\mathcal{V}q = 0 \implies paq = 0\}$$

with p and q ranging over projections in $\mathcal{M}$.

It becomes clear by inspecting this definition that any subset $\mathcal{V} \subset \mathcal{B}(H)$ satisfying $\mathcal{M}'\mathcal{V}\mathcal{M}' \subseteq \mathcal{V}$ is operator reflexive if and only if it is a stable subset of $\mathcal{B}(H)$ in the pairing we have been discussing. And since Proposition 4.3.7 says all stable subsets satisfy this condition we conclude the following:

Theorem 4.3.10. *Let $\mathcal{M} \subseteq \mathcal{B}(H)$ be a represented von Neumann algebra and consider the pairing $(\mathcal{B}(H), Pr(\mathcal{M})^2, R)$ where $(a,(p,q)) \in R$ whenever $paq = 0$. The stable subsets of $\mathcal{B}(H)$ are precisely the operator reflexive weak* closed subspaces $\mathcal{V} \subseteq \mathcal{B}(H)$ which satisfy $\mathcal{M}'\mathcal{V}\mathcal{M}' \subseteq \mathcal{V}$.*

We can characterize the stable subsets of $\mathrm{Pr}(\mathcal{M})^2$ in a similar way: A subset $T \subseteq \mathrm{Pr}(\mathcal{M})^2$ is stable if

$$T = \{(p,q) : t_1 a t_2 = 0 \text{ for all } (t_1,t_2) \in T \implies paq = 0\}.$$

This property is in some sense dual to operator reflexivity and is difficult to characterize just in the language of $\mathrm{Pr}(\mathcal{M})^2$. For some time we thought it might reduce to a topological condition, that $T \subseteq \mathrm{Pr}(\mathcal{M})^2$ would be stable precisely when it was closed in the restriction of the product weak operator topology. It turns out that this condition is necessary (Proposition 4.3.11) but not sufficient (Example 4.3.13).

Proposition 4.3.11. *Let $\mathcal{M} \subseteq \mathcal{B}(H)$ be a represented von Neumann algebra and consider the pairing $(\mathcal{B}(H), Pr(\mathcal{M})^2, R)$ where $(a,(p,q)) \in R$ whenever $paq = 0$. A stable subset of $Pr(\mathcal{M})^2$ is the complement of a cantankerous relation which is open in the restriction of the product weak operator topology.*

Proof. Suppose that $T \subseteq \Pr(\mathcal{M})^2$ is a stable subset. We have from Proposition 4.3.8 that T^C is a cantankerous relation and now must show that it is open – in fact we will show that T is SOT closed, which implies the result. Take $(p_\alpha, q_\alpha)_\alpha \subseteq T$, a net converging SOT to $(p_0, q_0) \in \Pr(\mathcal{M})^2$. Our goal is to show that $(p_0, q_0) \in T$. Fix some $a \in \mathcal{B}(H)$ such that $p_\alpha a q_\alpha = 0$ for every α. Since multiplication is jointly SOT continuous on bounded sets we have that $p_0 a q_0 = \lim_\alpha p_\alpha a q_\alpha = 0$.

So if $a \in T^\flat$ then $p_0 a q_0 = 0$ and since T is stable this implies that $(p_0, q_0) \in T$, which completes the proof that stable implies closed as desired. □

The following example of a cantankerous relation which is not open shows that Proposition 4.3.11 is not devoid of content.

Example 4.3.12. Let M_2 denote the set of 2×2 matrices, which we identify with $\mathcal{B}(\mathbb{C}^2)$. We denote the projection onto the first basis element by p_1. Now consider the diagonal relation $\Delta = \{(p,q) : pq \neq 0\}$ and add to it the single pair $(p_1, 1-p_1)$, forming the set $\mathcal{R} \subseteq \Pr(M_2)^2$. We note that $\mathcal{R}$ is a cantankerous relation (the join condition is no problem, since p_1 and $1-p_1$ are rank 1 projections) however it is not open.

To see this take some net of projections $(p_\alpha)_\alpha$, none of whom equal p_1, converging WOT to p_1 and note that $(p_\alpha, 1-p_\alpha) \in \mathcal{R}^C$ for every α but their limit $(p_1, 1-p_1) \notin \mathcal{R}^C$. Thus $\mathcal{R}^C$ is not closed and we are done.

And the following example shows that not all open cantankerous relations are the complements of stable objects (in the pairing we have been discussing).

Example 4.3.13. Let M_2 denote the set of 2×2 matrices, which we identify with $\mathcal{B}(\mathbb{C}^2)$. Let p_1 denote projection onto the span of the first basis vector, p_2 projection onto the span of $[0.5, 0.5]$ and p_3 projection onto the span of $[.75, 1]$. Explicitly:

$$p_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad p_2 = \begin{bmatrix} .5 & .5 \\ .5 & .5 \end{bmatrix} \quad p_3 = \begin{bmatrix} .36 & .48 \\ .48 & .64 \end{bmatrix}$$

Now let $T \subseteq \Pr(\mathcal{M})^2$ be collection of all (p,q) such that either $p = 0$ or $q = 0$ along with $(p_1, p_1), (p_2, p_2)$ and (p_3, p_3). It is not hard to verify that T is the complement of an open cantankerous relation. However T is not stable.

To see this consider an operator $a \in M_2$ which is annihilated by any pair in T. We have that $p_1 a p_1 = 0$, $p_2 a p_2 = 0$ and $p_3 a p_3 = 0$. This will produce a system of three independent equations and a has four unknowns. Then a must be some multiple of $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ (it suffices for the reader to verify that this satisfies the three equations, which is routine). Without loss of generality we suppose a is precisely this matrix.

Now, were T stable, we would have that any pair of projections $(p,q) \in \Pr(\mathcal{M})^2$ which satisfies $pbq = 0$ for every $b \in M_2$ annihilated by all of T would have to be in T. Since the operators annihilated by T are all multiples of a this reduces to: if $paq = 0$ then $(p,q) \in T$. If p_4 denotes projection onto the second basis vector we observe that $p_4 a p_4 = 0$ however $(p_4, p_4) \notin T$ so T is not stable.

So we know that stable objects are all complements of open cantankerous relations, but the reverse inclusion does not generally hold.

In the next section we will discuss the quantum relations of Weaver, 2012, which he defines as weak* closed subspaces $\mathcal{V} \subseteq \mathcal{B}(H)$ satisfying $\mathcal{M}'\mathcal{V}\mathcal{M}' \subseteq \mathcal{V}$ – not necessarily operator reflexive. He pairs these with projections in the amplification of $\mathcal{M}$ and obtains a correspondence. One might similarly wonder if there is a way to realize all cantankerous relations (or their complements) as the dual object of operators in some larger space? We conclude this section with some open questions about cantankerous relations:

Question 4.3.14. Let $\mathcal{M} \subseteq \mathcal{B}(H)$ be a represented von Neumann algebra and consider the pairing $(\mathcal{B}(H), \mathrm{Pr}(\mathcal{M})^2, \mathrm{R})$ where $(a,(p,q)) \in \mathrm{R}$ whenever $paq = 0$. Is there a characterization of the stable subsets of $\mathrm{Pr}(\mathcal{M})^2$ which does not refer to this pairing? That is, a characterization that does not reference the representation $\mathcal{M} \subseteq \mathcal{B}(H)$?

Question 4.3.15. Is there a related pairing for which the stable objects are all the complements of cantankerous relations? This would be in some sense dual to Weaver's quantum relations (defined in the next section), where he pulls projections from the amplification of $\mathcal{M}$ so that all weak* closed subspaces $\mathcal{V} \subseteq \mathcal{B}(H)$ which satisfy $\mathcal{M}'\mathcal{V}\mathcal{M}' \subseteq \mathcal{V}$ are dual objects – not just operator reflexive ones.

4.3.2 Measuring the "Dilation" of a Cantankerous Relation

In this subsection we will introduce a way of measuring how much a cantankerous relation dilates or expands the "size" of projections. This will help us build to a suitable generalization of the support expansion C*-algebras explored in Chapter 3. Of course we first need an appropriate notion of "size" for projections, so recall this definition (Definition 1.1.4):

Definition 1.1.4. A *dimension function* on the projections of a von Neumann algebra $\mathcal{M}$ is a function $d : \mathrm{Pr}(\mathcal{M}) \to [0,\infty]$ which is monotonic, additive on orthogonal projections and constant on Murray-von Neumann equivalence classes. Precisely for $p, q \in \mathrm{Pr}(\mathcal{M})$, d satisfies:

$$
\begin{aligned}
&d(p) \le d(q) && \text{whenever } p \le q,\\
&d(p+q) = d(p) + d(q) && \text{whenever } p \perp q \text{ and}\\
&d(p) = d(q) && \text{when } p \text{ and } q \text{ are Murray-von Neumann equivalent.}
\end{aligned}
$$

Definition 4.3.17. Let $\mathcal{M} \subseteq \mathcal{B}(H)$ be a von Neumann algebra and suppose $d : \mathrm{Pr}(\mathcal{M}) \to [0,\infty]$ is a dimension function. We say a map $\phi : \mathrm{Pr}(\mathcal{M}) \to \mathrm{Pr}(\mathcal{M})$ has *f-controlled d-dilation* if $d(\phi(q)) \le f(d(q))$ for all $q \in \mathrm{Pr}(\mathcal{M})$. Also, a cantankerous relation $\mathcal{R}$ on $\mathcal{M}$ has *f-controlled d-dilation* if $\phi_{\mathcal{R}}$ does.

If $\mathcal{F}$ is a collection of functions $[0,\infty] \to [0,\infty]$ then we say maps $\mathrm{Pr}(\mathcal{M}) \to [0,\infty]$ have *$\mathcal{F}$-controlled d-dilation* if they have f-controlled d-dilation for some $f \in \mathcal{F}$. A cantankerous relation $\mathcal{R}$ has $\mathcal{F}$-controlled d-dilation if $\phi_{\mathcal{R}}$ does.

Proposition 4.3.18. *Let $\mathcal{M} \subseteq \mathcal{B}(H)$ be a von Neumann algebra and consider $\phi, \phi' :$ Pr$(\mathcal{M}) \to [0,\infty]$ with f-controlled and g-controlled d-dilation respectively then*

(a) $\phi \vee \phi'$ has $f+g$-controlled d-dilation and

(b) $\phi \circ \phi'$ has $f \circ g$-controlled d-dilation if f is increasing.

Proof. For the first item, take arbitrary $q \in \mathrm{Pr}(\mathcal{M})$ and note that $(\phi \vee \phi')(q) = \phi(q) \vee \phi'(q)$. Then by Kaplansky's formula we have that $\phi(q) \vee \phi'(q) - \phi(q)$ is Murray von Neumann equivalent to $\phi'(q) - \phi(q) \wedge \phi'(q)$. Since dimension functions are constant on Murray von Neumann equivalence classes we get that $d(\phi(q) \vee \phi^{'}(q) - \phi(q)) = d(\phi^{'}(q) - \phi(q) \wedge \phi^{'}(q))$. We also have that d is additive on orthogonal projections so $d(\phi(q) \vee \phi^{'}(q) - \phi(q)) + d(\phi(q)) = d(\phi(q) \vee \phi^{'}(q))$. It follows that $d(\phi(q) \vee \phi^{'}(q)) = d(\phi(q)) + d(\phi^{'}(q) - \phi(q) \wedge \phi^{'}(q)) \leq d(\phi(q)) + d(\phi^{'}(q))$ since d is monotonic. The result follows.

Verifying the second item is routine and we omit the proof. □

You can see that if we choose a family of increasing functions $\mathcal{F}$ which is closed under addition and composition then the cantankerous relations which are $\mathcal{F}$-controlled will be closed under join and product (composition). This is going to lead us, in a natural way, to defining a canonical C*-subalgebra of $\mathcal{B}(H)$ associated with $\mathcal{F}$, which we will consider the generalization of a support expansion C*-algebras. We will also realize these generalized support expansion C*-algebras as generalized uniform Roe algebras, but since we have not yet developed the appropriate machinery for this let us put a pin here and revisit the topic in the next chapter.

4.4 Quantum Relations

4.4.1 Explicit and Intrinsic Quantum Relations

Weaver, 2012 defines quantum relations as so (we have made slight alterations in parentheses):

Definition 4.4.1 (Weaver, 2012 Definition 2.1)**.** A(n) ***(explicit) quantum relation*** on a von Neumann algebra $\mathcal{M} \subseteq \mathcal{B}(H)$ is a W*-bimodule over its commutant $\mathcal{M}^{'}$, i.e., it is a weak* closed subspace $\mathcal{V} \subseteq \mathcal{B}(H)$ satisfying $\mathcal{M}^{'}\mathcal{V}\mathcal{M}^{'} \subseteq \mathcal{V}$

At a glance this seems quite distant from the notion of measurable relation in Definition 4.2.1 but should at least seem reasonable given the discussion in Section 4.3. Weaver, 2012 also gives an intrinsic characterization of quantum relations as a collection of pairs of projections:

Definition 4.4.2 (Weaver, 2012 Definition 2.24)**.** Let $\mathcal{M}$ be a von Neumann algebra and let $\mathcal{P}$ be the set of projections in $\mathcal{M}\overline{\otimes}\mathcal{B}(\ell^2)$, equipped with the restriction of the weak operator topology. An ***intrinsic quantum relation*** on $\mathcal{M}$ is an open subset $\mathcal{R} \subseteq \mathcal{P} \times \mathcal{P}$ satisfying

(i) $(0, 0) \notin \mathcal{R}$

(ii) $(\bigvee P_\lambda, \bigvee Q_\kappa) \in \mathcal{R} \Leftrightarrow$ some $(P_\lambda, Q_\kappa) \in \mathcal{R}$

(iii) $(P, [BQ]) \in \mathcal{R} \Leftrightarrow ([B^*P], Q) \in \mathcal{R}$ (here $[\cdot]$ represents projection onto the closure of the image)

for all projections $P, Q, P_\lambda, Q_\kappa \in \mathcal{P}$ and all $B \in I \otimes \mathcal{B}(\ell^2)$. We denote the set of all intrinsic quantum relations on $\mathcal{M}$ by $\mathrm{IQR}(\mathcal{M})$.

Still, some notable differences from Definition 4.2.1. First and most significant is that for measurable and cantankerous relations the projections were taken to be in a von Neumann sub-algebra of $\mathcal{B}(H)$, whereas here we take the projections in the amplification $\mathcal{M}\overline{\otimes}\mathcal{B}(\ell^2)$, introducing matrix levels. Hypothesis (iii) gives a compatibility criteria for the resulting tensors. Hypothesis (i) is just a stylistic adjustment since it (along with the other hypotheses) ensures that the projections in a pair are nonzero, which we required for measurable and cantankerous relations as well. Finally, note that a quantum relation has to be open in an appropriate sense while there were no topological concerns for measurable and cantankerous relations. The openness of quantum relations should also not be so surprising, given the discussion in Section 4.3, as Weaver, 2012 is working towards a 1-1 correspondence between Definitions 4.4.1 and 4.4.2:

Theorem 4.4.3 (Weaver, 2012 Theorem 2.32)**.** *Let $\mathcal{M} \subseteq \mathcal{B}(H)$ be a von Neumann algebra and let $\mathcal{P}$ be the set of projections in $\mathcal{M}\overline{\otimes}\mathcal{B}(\ell^2)$. If $\mathcal{V}$ is a quantum relation on $\mathcal{M}$ then*

$$\mathcal{R}_{\mathcal{V}} = \{(P,Q) \in \mathcal{P}^2 : P(A \otimes I)Q \neq 0 \text{ for some } A \in \mathcal{V}\}$$

is an intrinsic quantum relation on $\mathcal{M}$; conversely, if $\mathcal{R}$ is an intrinsic quantum relation on $\mathcal{M}$ then

$$\mathcal{V}_{\mathcal{R}} = \{A \in \mathcal{B}(H) : (P,Q) \notin \mathcal{R} \implies P(A \otimes I)Q = 0\}$$

is a quantum relation on $\mathcal{M}$. The two constructions are inverse to each other.

The extra structure of Definition 4.4.2 is required for this correspondence to work. If one were to work merely with pairs of projections in $\mathcal{M}$ then in the above theorem (with appropriate adjustments to notation) we would have $\mathcal{R} = \mathcal{R}_{\mathcal{V}_{\mathcal{R}}}$ but $\mathcal{V} \subseteq \mathcal{V}_{\mathcal{R}_{\mathcal{V}}}$ with proper containment a possibility. For more on this see the discussion following Theorem 2.9 of Weaver, 2012 where it is explained that a measurable relation induces a quantum relation in a canonical way but there may be more than one quantum relation $\mathcal{V}$ with the same associated measurable relation $\{(p,q) \in \mathrm{Pr}(L^\infty(X,\mu))^2 : paq \neq 0 \text{ for some } a \in \mathcal{V}\}$.

For our purposes we are primarily interested in intrinsic quantum relations and so we will provide some original results which extend the machinery Weaver, 2012 built for measurable relations to the quantum setting.

Proposition 4.4.4 (cf. Proposition 4.2.5)**.** *Let $\mathcal{M} \subseteq \mathcal{B}(H)$ be a von Neumann algebra and let $\mathcal{P}$ be the set of projections in $\mathcal{M}\overline{\otimes}B(\ell^2)$, equipped with the restriction of the weak operator topology.*

(a) The set of pairs of projections P and Q in $\mathcal{P}$ such that $PQ \neq 0$ is an intrinsic quantum relation on $\mathcal{M}$.

(b) If $\mathcal{R}$ is an intrinsic quantum relation on $\mathcal{M}$ then so is $\{(Q,P) : (P,Q) \in \mathcal{R}\}$.

(c) Any union of intrinsic quantum relations on $\mathcal{M}$ is an intrinsic quantum relation on $\mathcal{M}$.

Proof. Straightforward.

□

Conjecture 4.4.5. Let $\mathcal{M} \subseteq \mathcal{B}(H)$ be a von Neumann algebra and let $\mathcal{P}$ be the set of projections in $\mathcal{M}\overline{\otimes}B(\ell^2)$, equipped with the restriction of the weak operator topology. If $\mathcal{R}$ and $\mathcal{R}'$ are intrinsic quantum relations on $\mathcal{M}$ then consider the set $\mathcal{R} \cdot \mathcal{R}'$ of all pairs of nonzero projections (P,R) that satisfy

$$\text{for every projection } Q, \text{ either } (P,Q) \in \mathcal{R} \text{ or } (1-Q,R) \in \mathcal{R}'.$$

We conjecture that the interior $\mathrm{Int}(\mathcal{R}\cdot\mathcal{R}')$ in $\mathcal{P}$ is an intrinsic quantum relation on $\mathcal{M}$. Moreover, we expect $\mathcal{V}_{\mathrm{Int}(\mathcal{R}\cdot\mathcal{R}')} = \overline{\mathcal{V}_{\mathcal{R}} \cdot \mathcal{V}_{\mathcal{R}'}}^{w^*}$.

Some justification for Conjecture 4.4.5. We can argue that the set $\mathcal{R}\cdot\mathcal{R}'$ (before taking the interior) satisfies most of the properties of an intrinsic quantum relation:

That the collection of all such (P, R) satisfies Items (i) and (ii) of Definition 4.4.2 is a purely set theoretic question and the proof is identical to Weaver, 2012 Proposition 1.5c, since he does not use the commutativity of projections in $L^\infty(X,\mu)$ for these properties.

For Item (iii) of Definition 4.4.2, suppose we have $P, R \in \mathcal{P}$ so that for every $Q \in \mathcal{P}$ and $B \in I \otimes \mathcal{B}(\ell^2)$ either $(P, Q) \in \mathcal{R}$ or $(1 - Q, [BR]) \in \mathcal{R}'$. Then in particular we know that for every Q either $(P, [BQ]) \in \mathcal{R}$ or $(1 - [BQ], [BR]) \in \mathcal{R}'$. If $(P, [BQ]) \in \mathcal{R}$ then $([B^*P], Q) \in \mathcal{R}$. Otherwise we must have $(1 - [BQ], [BR]) \in \mathcal{R}'$ and thus $([B^*(1 - [BQ])], R) \in \mathcal{R}'$. As

$$[B^*(1 - [BQ])] = \ker((B - [BQ]B))^\perp \text{ and } BQ - [BQ]BQ = 0$$

it follows that $[B^*(1 - [BQ])] \le 1 - Q$. So we have $(1 - Q, R) \in \mathcal{R}'$. So we have for any Q either $([B^*P], Q) \in \mathcal{R}$ or $(1 - Q, R) \in \mathcal{R}'$. We have shown that if $(P, [BR])$ is in $\mathcal{R}\cdot\mathcal{R}'$ then so is $([B^*P], R)$. The reverse direction follows analogously.

If our goal were to realize $\mathcal{R}\cdot\mathcal{R}'$ as an intrinsic quantum relation then it would remain to show that it is open in $\mathcal{P}$ which we have had difficulty doing and suspect may not be true in general. Hence we think taking the interior is probably necessary. □

Definition 4.4.6 (cf. Definition 4.2.6)**.** Let $\mathcal{M} \subseteq \mathcal{B}(H)$ be a von Neumann algebra and $\mathcal{E}$ a collection of intrinsic quantum relations on $\mathcal{M}$.

(a) $\bigvee \mathcal{E} = \bigcup\{\mathcal{R} : \mathcal{R} \in \mathcal{E}\}$

(b) $\bigwedge \mathcal{E} = \bigcup\{\mathcal{R}'$ intrinsic quantum relation on $\mathcal{M} : \mathcal{R}' \le \mathcal{R}$ for every $\mathcal{R} \in \mathcal{E}\}$

Note that the set of intrinsic quantum relations on $\mathcal{M}$ partially ordered by inclusion forms a complete lattice under these definitions by Proposition 4.4.4d.

Definition 4.4.7 (cf. Definition 4.2.7)**.** Let $\mathcal{M} \subseteq \mathcal{B}(H)$ be a von Neumann algebra.

(a) The *diagonal intrinsic quantum relation* Δ on $\mathcal{M}$ is defined by setting $(P, Q) \in \Delta$ if $PQ \neq 0$.

(b) The *transpose* of an intrinsic quantum relation $\mathcal{R}$ is the intrinsic quantum relation $\mathcal{R}^T = \{(Q, P) : (P, Q) \in \mathcal{R}\}$.

(c) An intrinsic quantum relation $\mathcal{R}$ on $\mathcal{M}$ is

  (i) *reflexive* if $\Delta \subseteq \mathcal{R}$

  (ii) *symmetric* if $\mathcal{R}^T = \mathcal{R}$

  (iii) *antisymmetric* if $\mathcal{R} \wedge \mathcal{R}^T \subseteq \Delta$

Weaver, 2012 defines explicit quantum relation analogs to each of the items in Definition 4.4.7:

Definition 4.4.8 (Weaver, 2012 Definition 2.4)**.** Let $\mathcal{M} \in \mathcal{B}(H)$ be a von Neumann alegrba.

(a) The *diagonal quantum relation* on $\mathcal{M}$ is the relation $\mathcal{V} = \mathcal{M}'$.

(b) The *transpose* of a quantum relation $\mathcal{V}$ on $\mathcal{M}$ is the quantum relation $\mathcal{V}^*$.

(c) The *product* of two quantum relations $\mathcal{V}$ and $\mathcal{W}$ is the weak* closure of their algebraic product.

(d) A quantum relation $\mathcal{V}$ on $\mathcal{M}$ is

(i) *reflexive* if $\mathcal{M}' \subseteq \mathcal{V}$

(ii) *symmetric* if $\mathcal{V}^* = \mathcal{V}$

(iii) *antisymmetric* if $\mathcal{V} \cap \mathcal{V}^* \subseteq \mathcal{M}'$

(iv) *transitive* if $\mathcal{V}^2 \subseteq \mathcal{V}$.

We of course want Definitions 4.4.7 and 4.4.8 to agree, which we establish in the following proposition.

Proposition 4.4.9 (cf. Weaver, 2012 Theorem 2.9)**.** *Let $\mathcal{M} \subseteq \mathcal{B}(H)$ be a von Neumann algebra and $\mathcal{R}, \mathcal{R}'$ implicit quantum relations on $\mathcal{M}$. Let $\mathcal{V}_\mathcal{R}$ be the explicit quantum relation associated with $\mathcal{R}$ as in Theorem 4.4.3. The following hold:*

(a) $\mathcal{V}_\Delta = \mathcal{M}'$

(b) $\mathcal{V}_{\mathcal{R}^T} = \mathcal{V}_\mathcal{R}^*$

(c) $\mathcal{V}_\mathcal{R}$ is

(i) reflexive if and only if $\mathcal{R}$ is reflexive

(ii) symmetric if and only if $\mathcal{R}$ is symmetric

(iii) antisymmetric if and only if $\mathcal{R}$ is antisymmetric

Proof. For Item (a) it is clear that $\mathcal{M}' \subseteq \mathcal{V}_\Delta$. For the other direction take $a \in \mathcal{V}_\Delta$ and $p \in \Pr(\mathcal{M})$, denoting $A = a \otimes I$ and $P = p \otimes I$. We note that $PA(1 - P) = 0$ and thus $PA = PAP$. By similar reasoning we get $AP = PAP$ so $PA = AP$ and thus $pa = ap$. Since $a \in \mathcal{V}_\Delta$ and $p \in \Pr(\mathcal{M})$ were arbitrary this gives us $\mathcal{V}_\Delta \subseteq \mathcal{M}'$ (since $\mathcal{M}$ is a von Neumann algebra, it is the SOT closed span of its projections).

For Item (b) take $a \in \mathcal{V}_{\mathcal{R}^T}$ and $P, Q \in \Pr(\mathcal{M}\overline{\otimes}\mathcal{B}(\ell^2))$ such that $(P, Q) \notin \mathcal{R}$. Then $(Q, P) \notin \mathcal{R}^T$ and so $Q(a \otimes I)P = 0$ and thus $P(a^* \otimes I)Q = 0$. Since P and Q were arbitrary such that $(P, Q) \notin \mathcal{R}$ we have that $a^* \in \mathcal{V}_\mathcal{R}$ or $a \in \mathcal{V}_\mathcal{R}^*$. So we have that $\mathcal{V}_{\mathcal{R}^T} \subseteq \mathcal{V}_\mathcal{R}^*$. The reverse direction runs similarly.

Finally Item (c) follows quickly from the previous items and Theorem 4.4.3. □

In some sense Proposition 4.4.9 renders Proposition 4.4.4 superfluous by way of Theorem 4.4.3, since Weaver, 2012 Proposition 2.3 already asserted that the relevant objects were explicit quantum relations. However this would leave intrinsic quantum relations contingent upon the explicit definition, whereas we view them as the more fundamental object. Hence the inclusion of Proposition 4.4.4 and Conjecture 4.4.5.

4.4.2 Quantum Image Maps

Weaver, 2012 does not define the left image map (cf. Proposition 4.2.3) for quantum relations but it is very natural to work with this map when dealing with support expansion, so we introduce the proper notion and provide an original proof establishing correspondence between intrinsic quantum relations and image maps. First we need to introduce a relevant property for functions on $\mathcal{P}$, the set of projections in $\mathcal{M}\overline{\otimes}\mathcal{B}(\ell^2)$.

Definition 4.4.10. Let $\mathcal{M} \subseteq \mathcal{B}(H)$ be a von Neumann algebra and let $\mathcal{P}$ be the set of projections in $\mathcal{M}\overline{\otimes}\mathcal{B}(\ell^2)$, equipped with the restriction of the weak operator topology. We say a map $\phi : \mathcal{P} \to \mathcal{P}$ is *subcontinuous* if whenever a net $(P_\alpha, Q_\alpha) \in \mathcal{P}^2$ converges, say to (P_0, Q_0), and satisfies $P_\alpha \leq \phi(Q_\alpha)$ for every α we have $P_0 \leq \phi(Q_0)$.

Proposition 4.4.11. *(cf. Proposition 4.3.2) Let $\mathcal{M} \subseteq \mathcal{B}(H)$ be a von Neumann algebra and let $\mathcal{P}$ be the set of projections in $\mathcal{M}\overline{\otimes}\mathcal{B}(\ell^2)$, equipped with the restriction of the weak operator topology. If $\mathcal{R}$ is an intrinsic quantum relation on $\mathcal{M}$ then the map*

$$\phi_{\mathcal{R}} : Q \mapsto 1 - \bigvee\{P : (P, Q) \notin \mathcal{R}\},$$

from $\mathcal{P}$ to itself is subcontinuous, takes 0 to 0, preserves arbitrary joins and satisfies $\phi_{\mathcal{R}}([BQ]) = [B\phi_{\mathcal{R}}(Q)]$ for all $B \in I \otimes \mathcal{B}(\ell^2)$. If ϕ is a map from $\mathcal{P}$ to itself that is subcontinuous, takes 0 to 0, preserves arbitrary joins and satisfies $\phi([BQ]) = [B\phi(Q)]$ for all $B \in I \otimes \mathcal{B}(\ell^2)$ then

$$\mathcal{R}_\phi = \{(P, Q) : P\phi(Q) \neq 0\}$$

is an intrinsic quantum relation on $\mathcal{M}$. The two constructions are inverse to each other and order preserving.

Proof. First let $\mathcal{R}$ be an intrinsic quantum relation on $\mathcal{M}$. Notice that $\phi_{\mathcal{R}}(Q) \leq P$ if and only if $(1 - P, Q) \notin \mathcal{R}$. We will use this repeatedly.

It is clear that $\phi_{\mathcal{R}}(0) = 0$. Take $\{Q_\kappa\} \subseteq \mathcal{P}$ and note that if $(P, \bigvee Q_\kappa) \notin \mathcal{R}$ then $(P, Q_\kappa) \notin \mathcal{R}$ for every κ (Definition 4.4.2 Item (ii)) so $\phi_{\mathcal{R}}(Q_\kappa) \leq \phi_{\mathcal{R}}(\bigvee Q_\kappa)$ for each κ and thus $\bigvee \phi_{\mathcal{R}}(Q_\kappa) \leq \phi_{\mathcal{R}}(\bigvee Q_\kappa)$. If $\phi_{\mathcal{R}}(\bigvee Q_\kappa) \not\leq \bigvee \phi_{\mathcal{R}}(Q_\kappa)$ then $(1 - \bigvee \phi_{\mathcal{R}}(Q_\kappa), \bigvee Q_\kappa) \in \mathcal{R}$ which implies that $(1 - \bigvee \phi_{\mathcal{R}}(Q_\kappa), Q_{\kappa_0}) \in \mathcal{R}$ for some κ_0 thus $\phi_{\mathcal{R}}(Q_{\kappa_0}) \not\leq \bigvee \phi_{\mathcal{R}}(Q_\kappa)$ which is absurd. So we have $\phi_{\mathcal{R}}(\bigvee Q_\kappa) = \bigvee \phi_{\mathcal{R}}(Q_\kappa)$ for any family $\{Q_\kappa\} \subseteq \mathcal{P}$.

Take $B \in I \otimes \mathcal{B}(\ell^2)$ and note that $\phi_{\mathcal{R}}([BQ]) \leq 1 - P$ if and only if $(P, [BQ]) \notin \mathcal{R}$ which is true precisely when $([B^*P], Q) \notin \mathcal{R}$ (Definition 4.4.2 Item (iii)) which holds if and only if $\phi_{\mathcal{R}}(Q) \leq 1 - [B^*P] = \ker(PB)$. This is true if and only if $(1 - P)B\phi_{\mathcal{R}}(Q) = B\phi_{\mathcal{R}}(Q)$ and thus is equivalent to $(1 - P)[B\phi_{\mathcal{R}}(Q)] = [B\phi_{\mathcal{R}}(Q)]$ which is is the same as $[B\phi_{\mathcal{R}}(Q)] \leq 1 - P$. We have argued that $\phi_{\mathcal{R}}([BQ]) \leq 1 - P$ if and only if $[B\phi_{\mathcal{R}}(Q)] \leq 1 - P$, in particular $[B\phi_{\mathcal{R}}(Q)] = \phi_{\mathcal{R}}([BQ])$ for all $Q \in \mathcal{P}$ and $B \in I \otimes \mathcal{B}(\ell^2)$.

It remains to show that $\phi_{\mathcal{R}}$ is subcontinuous. Suppose $((P_\alpha, Q_\alpha))$ is a net in $\mathcal{P}^2$ converging to (P_0, Q_0) such that $P_\alpha \geq \phi_{\mathcal{R}}(Q_\alpha)$ for every α. Thus $(1 - P_\alpha, Q_\alpha) \notin \mathcal{R}$ for each α. We have that $\mathcal{R}^C$ is closed and so $(1 - P_0, Q_0) \notin \mathcal{R}$ which implies that $P_0 \geq \phi_{\mathcal{R}}(Q_0)$ and therefore $\phi_{\mathcal{R}}$ is subcontinuous indeed.

Now let ϕ be a map from $\mathcal{P}$ to itself which satisfies the hypotheses, we will show that $\mathcal{R}_\phi$ is an intrinsic quantum relation on $\mathcal{M}$. It is clear that $(0, 0) \notin \mathcal{R}_\phi$. If $\{P_\lambda\}, \{Q_\kappa\} \subseteq \mathcal{P}$ then $(\bigvee P_\lambda)(\phi(\bigvee Q_\kappa)) = (\bigvee P_\lambda)(\bigvee \phi(Q_\kappa)) = 0$ if and only if $P_\lambda\phi(Q_\kappa) = 0$ for every λ, κ, so we have that $(\bigvee P_\lambda, \bigvee Q_\kappa) \in \mathcal{R}_\phi$ if and only if some $(P_\lambda, Q_\kappa) \in \mathcal{R}_\phi$.

Take $B \in I \otimes \mathcal{B}(\ell^2)$ and note that $P\phi([BQ]) = 0$ is equivalent to $P[B\phi(Q)] = 0$ by hypothesis. This happens precisely when $PB\phi(Q) = 0$ which is the same as $\phi(Q)B^*P = 0$, by taking the adjoint of both sides, which is true if and only if $\phi(Q)[B^*P] = 0$ and thus equivalent to $[B^*P]\phi(Q) = 0$. So $(P, [BQ]) \in \mathcal{R}_\phi$ if and only if $([B^*P], Q) \in \mathcal{R}_\phi$ for every $B \in I \otimes \mathcal{B}(\ell^2)$ and $P, Q \in \mathcal{P}$.

It remains to show that $\mathcal{R}_\phi$ is an open subset of $\mathcal{P}^2$ or, equivalently, that $\mathcal{R}_\phi^C$ is closed. Take some net $((P_\alpha, Q_\alpha))$ in $\mathcal{R}_\phi^C$ converging to a point (P_0, Q_0). We have that $P_\alpha\phi(Q_\alpha) = 0$ and thus $1 - P_\alpha \geq \phi(Q_\alpha)$. Since ϕ is subcontinuous this gives us $1 - P_0 \geq \phi(Q_0)$ which implies $P_0\phi(Q_0) = 0$ and thus $(P_0, Q_0) \in \mathcal{R}_\phi^C$. So $\mathcal{R}_\phi^C$ is closed.

Checking that the constructions are inverse to each other and order preserving is routine – we omit the proof. □

Definition 4.4.12. Let $\mathcal{M} \subseteq \mathcal{B}(H)$ be a von Neumann algebra and let $\mathcal{P}$ be the set of projections in $\mathcal{M}\overline{\otimes}\mathcal{B}(\ell^2)$, equipped with the restriction of the weak operator topology. We say $\phi : \mathcal{P} \to \mathcal{P}$ is a *quantum image map* for $\mathcal{M}$ if

(a) $\phi(0) = 0$,

(b) $\phi(\bigvee Q_\lambda) = \bigvee \phi(Q_\lambda)$ for any $(Q_\lambda)_\lambda \subseteq \mathcal{P}$,

(c) $\phi([BQ]) = [B\phi(Q)]$ for all $Q \in \mathcal{P}$, $B \in I \otimes \mathcal{B}(\ell^2)$ and

(d) ϕ is subcontinuous.

If $\mathcal{R}$ is an intrinsic quantum relation on $\mathcal{M}$ we call $\phi_{\mathcal{R}}$ (Proposition 4.4.11) the *left image* of $\mathcal{R}$.

Some of the basic operations one can perform on image maps are perfectly compatible with the corresponding operations on intrinsic quantum relations:

Proposition 4.4.13 (cf. Proposition 4.2.8)**.** *Let $\mathcal{M} \subseteq \mathcal{B}(H)$ be a von Neumann algebra and let $\mathcal{P}$ be the set of projections in $\mathcal{M}\overline{\otimes}B(\ell^2)$. If $\mathcal{R}$ and $\mathcal{R}'$ are intrinsic quantum relations on $\mathcal{M}$ then*

(a) $\phi_{\mathcal{R}\vee\mathcal{R}'} = \phi_{\mathcal{R}} \vee \phi_{\mathcal{R}'}$,

(b) $\phi_{\mathcal{R}\wedge\mathcal{R}'} = \phi_{\mathcal{R}} \wedge \phi_{\mathcal{R}'}$ *and*

(c) $\phi_{\mathcal{R}\cdot\mathcal{R}'} = \phi_{\mathcal{R}}\circ\phi_{\mathcal{R}'}$ *(cf. Conjecture 4.4.5: It is not clear that $\mathcal{R}\cdot\mathcal{R}'$ is a quantum relation and, correspondingly, that $\phi_{\mathcal{R}} \circ \phi_{\mathcal{R}'}$ is an image map).*

Proof. Argument essentially as found in the proof of Proposition 4.2.8. □

Proposition 4.4.14. *Let $\mathcal{M} \subseteq \mathcal{B}(H)$ be a von Neumann algebra and let $\mathcal{P}$ be the set of projections in $\mathcal{M}\overline{\otimes}B(\ell^2)$, equipped with the restriction of the weak operator topology.*

(a) The map from $\mathcal{P}$ to itself given by $Q \mapsto Q$ is an image map for $\mathcal{M}$.

(b) If ϕ is an image map for $\mathcal{M}$ then so is the function $\phi^* : \mathcal{P} \to \mathcal{P}$ given by $\phi^*(P) = 1 - \bigvee\{Q : \phi(Q) \leq 1 - P\}$. Also $\phi^{**} = \phi$.

(c) If ϕ and ϕ' are image maps for $\mathcal{M}$ then so is their point-wise join $\phi \vee \phi'$.

Proof. Parts (a) and (c) are easy.

For part (b) we first observe that $\phi^*(P) \leq 1-Q$ if and only if $\phi(Q) \leq 1-P$ which establishes that $\phi^{**} = \phi$ but also is a fact we will use repeatedly below.

Note $\phi^*(0) \leq 1 - Q$ if and only if $\phi(Q) \leq 1 - 0$, which is true for all $Q \in \mathcal{P}$ so $\phi^*(0) = 0$.

Now take a collection $\{P_\lambda\} \subseteq \mathcal{P}$ and observe $\phi^*(\bigvee P_\lambda) \leq 1 - Q \Leftrightarrow \phi(Q) \leq 1 - \bigvee P_\lambda = \bigwedge 1 - P_\lambda$ which is true if and only if $\phi(Q) \leq 1 - P_\lambda$ for every λ, which is same as $\phi^*(P_\lambda) \leq 1 - Q$ for every λ or, equivalently, $\bigvee \phi^*(P_\lambda) \leq 1 - Q$. So $\phi^*(\bigvee P_\lambda) = \bigvee \phi^*(P_\lambda)$.

Fix $B \in I \otimes \mathcal{B}(\ell^2)$ then $\phi^*([BP]) \leq 1 - Q \Leftrightarrow \phi(Q) \leq 1 - [BP]$ which is the same as $\phi(Q)BP = 0$ or $PB^*\phi(Q) = 0$, precisely when $\phi([B^*Q]) = [B^*\phi(Q)] \leq 1 - P$ which is to say $\phi^*(P) \leq 1 - [B^*Q]$ which happens if and only if $\phi^*(P)B^*Q = 0$ or $QB\phi^*(P) = 0$ or equivalently $[B\phi^*(P)] \leq 1 - Q$. So $\phi^*([BP]) = [B\phi^*(P)]$.

It remains to show ϕ^* is subcontinuous. Suppose $(P_\alpha, Q_\alpha)_\alpha \subseteq \mathcal{P}^2$ is a net converging to (P_0, Q_0) and that $\phi^*(P_\alpha) \leq 1 - Q_\alpha$ for each α. Then $\phi(Q_\alpha) \leq 1 - P_\alpha$ for each α and so $\phi(Q_0) \leq 1 - P_0$ which is equivalent to $\phi^*(P_0) \leq 1 - Q_0$ which establishes subcontinuity. □

Definition 4.4.15. Let $\mathcal{M}$ be a von Neumann algebra and let $\mathcal{P}$ be the set of projections in $\mathcal{M}\overline{\otimes}\mathcal{B}(\ell^2)$, equipped with the restriction of the weak operator topology.

(a) The *diagonal image map* ϕ_Δ for $\mathcal{M}$ is given by $\phi_\Delta(Q) = Q$.

(b) The *conjugate* of an image map ϕ for $\mathcal{M}$ is the image map $\phi^*(P) = 1 - \bigvee\{Q : \phi(Q) \leq 1 - P\}$.

(c) An image map ϕ for $\mathcal{M}$ is

(i) *reflexive* if $\phi_\Delta \leq \phi$

(ii) *symmetric* if $\phi = \phi^*$

(iii) *antisymmetric* if $\phi \wedge \phi^* \leq \phi_\Delta$

Proposition 4.4.16. *Let $\mathcal{M} \subseteq \mathcal{B}(H)$ be a von Neumann algebra and $\mathcal{R}, \mathcal{R}'$ implicit quantum relations on $\mathcal{M}$. Let $\mathcal{V}_\mathcal{R}$ be the explicit quantum relation associated with $\mathcal{R}$ as in Theorem 4.4.3. The following hold:*

(a) $\mathcal{R}_{\phi_\Delta} = \Delta$

(b) $\mathcal{R}_\phi^T = \mathcal{R}_{\phi^*}$

(c) $\mathcal{R}_\phi$ *is*

(i) reflexive if and only if ϕ is reflexive

(ii) symmetric if and only if ϕ is symmetric

(iii) antisymmetric if and only if ϕ is antisymmetric

Proof. Note that $(P, Q) \in \mathcal{R}_{\phi_\Delta}$ if and only if $PQ = P\phi_\Delta(Q) \neq 0$, which is exactly when $(P, Q) \in \Delta$.

Suppose $(Q, P) \notin \mathcal{R}_\phi^T$ so $(P, Q) \notin \mathcal{R}_\phi$ and thus $P\phi(Q) = 0$ or $\phi(Q) \leq 1 - P$. This implies that $\phi^*(P) \leq 1 - Q$ and thus $Q\phi^*(P) = 0$ which means $(Q, P) \notin \mathcal{R}_{\phi^*}$. We have shown that $\mathcal{R}_{\phi^*} \leq \mathcal{R}_\phi^T$ but a symmetric argument would show the reverse inequality so we have $\mathcal{R}_{\phi^*} = \mathcal{R}_\phi^T$.

All the parts of (d) follow straightforwardly from parts (a) and (b) along with Propositions 4.4.11 and 4.4.13. □

That is to say Definitions 4.4.7, 4.4.8 and 4.4.15 are all compatible up to a Conjecture 4.4.5 shaped hole.

We can consider a dimension function (Definition 1.1.4) on $\mathcal{M}\overline{\otimes}\mathcal{B}(\ell^2)$ and speak of the d-dilation of an explicit quantum relation (where products are defined), passing to intrinsic quantum relations and then to image maps.

Chapter 5

Uniform Roe Algebras and Their Generalizations

In this final chapter we will motivate and very briefly introduce the theory of coarse structures and uniform Roe algebras. Then we will use the tools laid out in Chapter 4 to define measurable, cantankerous and quantum coarse structures and uniform Roe algebras. Finally we will realize support expansion C*-algebras as cantankerous uniform Roe algebras – placing them nicely in a larger (mostly unexplored as of this writing) theory.

5.1 Information in Metric Spaces

In this section we will examine metric spaces and how various types of analysis focus on different kinds of information provided by the metric, while disregarding other kinds of information. By isolating and distilling the relevant information in each context we can generalize metrics in several different ways.

Definition 5.1.1. Let X be a set and $d : (X \times X) \to [0, \infty]$ a function which satisfies the following for all $x, y, z \in X$:

(a) $d(x, y) = 0 \Leftrightarrow x = y$,

(b) $d(x, y) = d(y, x)$,

(c) $d(x, z) \leq d(x, y) + d(y, z)$

then (X, d) is a ***metric space*** and d is called a ***metric*** on X.

A metric gives the distance between any two points in X. This is quite a bit of information – far more than necessary for many applications.

5.1.1 Continuity

Let $f : X \to Y$ be a function between metric spaces (X, d_1) and (Y, d_2). Recall that f is ***continuous*** at a point $x_0 \in X$ if for every $\varepsilon > 0$ there exists some $\delta > 0$ such that $d_1(x, x_0) < \delta$ implies $d_2(f(x), f(x_0)) < \varepsilon$ for every $x \in X$.

Note that we could replace d_1 with $\min(d_1, 1)$ and d_2 with $\min(d_2, 1)$ without affecting the continuity of any functions. Moreover we could "truncate" the metrics in this way at any non-zero value without issue, not just 1. We could also dramatically distort the metric, changing the relative distance between pairs of points, and still continuity would not be affected.

That is to say continuity is a local property and uses a very small amount of the information contained in the metric. As is well known, the theory of a ***topology*** is a generalization of metric spaces which strips away much of this excess information while giving a more general notion of limits and continuity. Any two metrics which result in the same topology will yield exactly the same set of continuity points for every function.

5.1.2 Uniformity

Let $f : X \to Y$ be a function between metric spaces (X, d_1) and (Y, d_2). Recall that f is ***uniformly continuous*** if for every $\varepsilon > 0$ there exists some $\delta > 0$ such that $d_1(x_1, x_2) < \delta$ implies $d_2(f(x_1), f(x_2)) < \varepsilon$ for every $x_1, x_2 \in X$.

In contrast to point-wise continuity, uniform continuity is a global property and so we will need to retain some additional information about the metric to capture it. In topology it makes sense to talk about "getting close" to a fixed point x_0: Any neighborhood of x_0 can be viewed as some "closeness" condition, and said neighborhoods can be made a poset by reverse set inclusion, with "both U close and V close" a more refined "closeness" condition given by $U \cap V$. This gives rise to the definition of convergence of sequences. But topologies have no means of determining whether one pair of points are "as close to" each other as some other pair of points. For that we would need a conception of "closeness" that made sense everywhere at the same time – a global property.

In case you are prone to optimism, the inability of topology to capture uniform continuity is hard coded: Indeed, different metrics can give rise to the same topologies but have distinct sets of uniformly continuous functions. To talk about uniform continuity we can still discard much of the information in the metric, but we need to retain some global notion of "closeness" which makes sense for all pairs of points at once. A ***uniform structure*** proves to be the correct framework for this.

Definition 5.1.2. (cf. Kelley, 1975 p. 176)[1] Given a set X, we say a nonempty collection $\mathcal{U} \subseteq \mathscr{P}(X \times X)$ of relations on X is a ***uniformity*** or ***uniform structure*** on X if it satisfies the following:

(a) The diagonal relation Δ is a subset of every element $U \in \mathcal{U}$,

(b) If $U \in \mathcal{U}$ then $U^T \in \mathcal{U}$,

(c) If $U \in \mathcal{U}$ then there exists some $V \in \mathcal{U}$ such that $V \circ V \subseteq U$,

(d) If $U, V \in \mathcal{U}$ then $U \cap V \in \mathcal{U}$ and

(e) If $U \in \mathcal{U}$ and $U \subseteq V \subseteq X \times X$ then $V \in \mathcal{U}$.

The pair $(X, \mathcal{U})$ is a ***uniform space*** and elements of $\mathcal{U}$ are commonly referred to as ***controlled sets*** or ***entourages***.

Entourages should be thought of as qualitative global notions of "closeness" and the hypotheses might be thought of intuitively as follows: Item (a) guarantees that every point is "close" to itself. Item (b) that if one point is "close" to a second point then the second is also "close" to the first. Item (c) allows us to always refine a given "closeness" condition to find one which is more precise, while Item (d) ensures us that "simultaneously U-close and V-close" is itself a notion of "closeness". Finally, Item (e) allows us to weaken any notion of "closeness" to include more points.

[1]Uniform spaces were first defined by Andrew Weil in 1937, and this definition in terms of entourages finds origin with Nicolas Bourbaki in his 1940 book General Topology.

Uniform structures do not just lie between metric spaces and topologies on an intuitive level: this can be formalized. Every metric space (X, d) induces a uniform structure by defining $U_\lambda = \{(x, y) : d(x, y) \leq \lambda\}$ and then $\mathcal{U} = \{V \subseteq X \times X : U_\lambda \subseteq V$ for some $\lambda \in (0, \infty)\}$. On the other hand every uniform structure $\mathcal{U}$ on X induces a natural topology:

Definition 5.1.3 (Kelley, 1975 p. 178)**.** Let $\mathcal{U}$ be a uniform structure on X then the ***uniform topology*** $\mathcal{T}_\mathcal{U}$ induced by $\mathcal{U}$ is given by saying a set $T \subseteq X$ is open if for each $x \in T$ there exists some $U \in \mathcal{U}$ such that $\psi_U(x) \subseteq T$, where $\psi_U(x) = \{y \in X : (x, y) \in U\}$ (cf. 4.1.6).

If one takes a metric space (X, d) and induces a uniformity $\mathcal{U}$ from it as described above, then $\mathcal{T}_\mathcal{U}$ is the standard metric-induced topology on (X, d), as one would expect.

We note that for a fixed X there is a unique uniformity $\mathcal{U}_d$ on X such that the diagonal relation $\Delta \in \mathcal{U}_d$. Indeed, by 5.1.2 (e), any such uniformity must contain every superset of Δ, which forms the maximal uniform structure on X. Further observe that the uniform topology for this maximal uniformity $\mathcal{T}_{\mathcal{U}_d}$ is the discrete topology on X. This motivates the following convention:

Definition 5.1.4. A uniformity $\mathcal{U}$ on a set X is called the ***discrete uniformity*** on X if $\Delta \in \mathcal{U}$.

Now for the promised generalization of uniform continuity:

Definition 5.1.5 (Kelley, 1975 p. 180)**.** Suppose $f : X \to Y$ is a function between uniform spaces $(X, \mathcal{U})$ and $(Y, \mathcal{V})$ then we say f is ***uniformly continuous*** if for each $V \in \mathcal{V}$ the set $\{(x, y) : (f(x), f(y)) \in V\}$ is a member of $\mathcal{U}$.

If you squint this should resemble the "pre-image of an open set is open" topological definition of continuity. As expected, this definition is equivalent to the metric definition of uniform continuity when $\mathcal{U}$ and $\mathcal{V}$ are induced by metrics on their respective spaces. Many of the other basic definitions in uniform theory, such as uniform isomorphism and equivalence, proceed as one would anticipate in parallel with topology. Thus we abridge our exploration of those basic topics.

Another feature of metric spaces which topology fails to capture is completeness. Topology can talk about whether a sequence converges, but it has no way of determining whether a sequence ***ought*** to converge. This is because identifying a sequence as Cauchy involves comparing the relative closeness of every pair of points in a tail of the sequence which is a non-local property. Indeed, different metrics can give the same topology but disagree about whether a sequence is Cauchy. However uniformities retain sufficient information to capture Cauchy nets and completeness.

Definition 5.1.6 (Kelley, 1975 pp. 190, 192)**.** Let $\mathcal{D}$ be some directed set and $(x_\alpha)_{\alpha \in \mathcal{D}} \subseteq X$ a net in the uniform space $(X, \mathcal{U})$. We say that $(x_\alpha)_{\alpha \in \mathcal{D}}$ is a ***Cauchy net*** if for each $U \in \mathcal{U}$ there is some $\gamma \in \mathcal{D}$ such that $(x_\alpha, x_\beta) \in U$ whenever $\alpha, \beta \geq \gamma$.

If every Cauchy net in $(X, \mathcal{U})$ converges to a point in X in the uniform topology then we say $(X, \mathcal{U})$ is ***complete***.

Moreover if a uniformity $\mathcal{U}$ is not complete it can be mapped by uniform isomorphism to a dense subspace of a complete uniform space. This mapping may not be unique in general but if $\mathcal{U}$ is Hausdorff then it has a unique Hausdorff completion up to uniform equivalence (Kelley, 1975 pp. 195-197).

Our goal in this subsection was to present the value of uniformities as a compromise between metric spaces and topologies. Some important properties of metric spaces which topology fails to capture can be recovered by instead studying uniformities. But the actual object of interest when constructing uniform Roe algebras is not uniform structure, but the dual notion: coarse structure.

5.1.3 Coarseness

If we revisit the properties of a uniformity in Definition 5.1.2 we notice that Items (c) and (d) allow us to refine entourages and get more precise notions of "close" while Item (e) lets us build weaker notions, corresponding to larger subsets of $X \times X$. We note that the latter hypothesis is very blunt, allowing literally any superset of an element of $\mathcal{U}$, while the former hypotheses are more nuanced. The effect of this is that at the large scale, every uniform structure looks the same. Uniform structures capture many important properties of metrics at small scales in a uniform manner but they completely wash out the large scale properties of metric spaces.

Large scale properties include notions of boundedness of sets or functions. A function being bornologous (cf. Definition 5.1.10b) is a large scale property while a function being Lipschitz involves both small and large scale information. All of these things are washed out by uniform structures. Indeed, for a given set X the metrics d and $\max(d, 1)$ induce the same uniformity but clearly can have different bounded sets, functions, etc.

What if we reversed the focus of uniform structures and paid more attention to the large scale notions of "closeness", while giving up resolution at the small scale?

Definition 5.1.7. (Roe, 2003 Definition 2.3)[2] Given a set X, we say a collection $\mathcal{C} \subseteq \mathscr{P}(X \times X)$ of relations on X is a ***coarse structure*** on X if it satisfies the following:

(a) The diagonal relation $\Delta \in \mathcal{C}$,

(b) If $U \in \mathcal{C}$ then $U^T \in \mathcal{C}$,

(c) If $U, V \in \mathcal{C}$ then $U \circ V \in \mathcal{U}$,

(d) If $U, V \in \mathcal{C}$ then $U \cup V \in \mathcal{C}$ and

(e) If $U \in \mathcal{C}$ and $V \subseteq U$ then $V \in \mathcal{C}$.

The pair $(X, \mathcal{C})$ is a ***coarse space*** and elements of $\mathcal{C}$ are commonly referred to as ***controlled sets*** or ***entourages***.

We note this overloads the definitions for ***controlled sets*** and ***entourages*** but context should prevent confusion between uniform and coarse controlled sets. If ever we are discussing both concepts at once we will distinguish them with the the adjectives "uniform" and "coarse".

Comparing Definitions 5.1.2 and 5.1.7 we see that the hypotheses for uniform and coarse spaces are dual. While a uniform entourage U tells you that points x and y are U-close if $(x, y) \in U$, a coarse entourage should be thought of as giving some notion of boundedness. Coarse structures successfully capture much of the large scale information from a metric space.

As was the case with uniformities, a metric space (X, d) canonically induces a coarse structure:

[2] Coarse structures were axiomatized in this way by John Roe in Roe, 2003, but these ideas originate in earlier works in geometric group theory including those of Gromov and Mostow.

Example 5.1.8. Let (X, d) be a metric space and for each $\lambda \in (0, \infty)$ define $U_\lambda = \{(x, y) : d(x, y) \leq \lambda\}$. Then $\mathcal{C} = \{U \subseteq X \times X : U \subseteq U_\lambda \text{ for some } \lambda \in (0, \infty)\}$ is a coarse structure on X.

As we mentioned earlier, coarse structures capture the notion of a set being *bounded*:

Definition 5.1.9 (Roe, 2003 Proposition 2.16)**.** Let $(X, \mathcal{C})$ be a coarse space, we say a set $B \subseteq X$ is *bounded* if $B \times B \in \mathcal{C}$.

This is a generalization since the bounded sets in the coarse structure induced by a metric are precisely the bounded sets in that metric space. Now we will introduce some properties of maps that coarse structures isolate, and a notion of equivalence between coarse spaces.

Definition 5.1.10. (Roe, 2003 Definitions 2.14 and 2.21) Let $(X, \mathcal{C})$ and $(Y, \mathcal{D})$ be coarse spaces and let $f, g : X \to Y$ be maps.

(a) The map f is *proper* if the inverse image, under f, of each bounded subset of Y is a bounded subset of X.

(b) The map f is *bornologous* if for each controlled subset $U \in \mathcal{C}$ the set $\{(f(x), f(y)) : (x, y) \in U\}$ is a controlled subset of Y.

(c) The map f is *coarse* if it is proper and bornologous.

(d) The maps f and g are *close* if the set $\{(f(x), g(x)) : x \in X\} \subseteq Y \times Y$ is controlled.

(e) The spaces X and Y are *coarsely equivalent* if there exist coarse maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g$ and $g \circ f$ are close to the identity maps on Y and X respectively.

The property bornologous should be viewed as dual to uniformly continuous. This duality is perhaps made more plain by seeing the $\varepsilon - \delta$ definition of bornologous for metric spaces: A function $f : X \to Y$ between metric spaces (X, d_1) and (Y, d_2) is bornologous if for every $\delta > 0$ there exists some $\varepsilon > 0$ such that $d_1(x, y) < \delta$ implies $d_2(f(x), f(y)) < \varepsilon$.

Moving forward we are going to make a deviation from Roe, 2003 (and indeed most theory involving coarse structures) and omit the first hypothesis of Definition 5.1.7. That is, **we will not require a coarse structure to include the diagonal relation**. The real motivation for this is that we will use coarse structures to build C*-algebras and requiring the diagonal relation forces all of the constructed algebras to be unital – making the theory less interesting.

A more philosophical motivation for this weakening is that, in our opinion, requiring the diagonal relation be included is somewhat unnatural. Consider the following example: Take a measure space (X, μ) such that every subset of X is measurable (for instance X could be discrete) and suppose we want to say a subset of X is "bounded" if it has finite measure (a reasonable notion of boundedness) – if we have any infinite point masses they would be considered unbounded. Definition 5.1.9 defines a set B to be *bounded* if $B \times B$ is a controlled set. This implies that all points are bounded, since the diagonal relation is necessarily in any coarse structure and coarse structures are closed under subsets. We would have to conclude that our notion of "bounded" meaning finite measure cannot be modelled by a coarse structure!

But this is purely an artifact of requiring the diagonal. We could generate a coarse structure sans diagonal by looking at rectangles with finite area in $X \times X$ and then closing under finite unions and subsets – we get compositions for free. And this coarse structure's notion of boundedness would be precisely "sets with finite measure". Perhaps the diagonal relation is important in some aspects of coarse geometry, but hopefully this example demonstrates to the reader that there is some value in considering this weakening, where we allow for coarse structures sans diagonal – hereafter referred to merely as coarse structures.

5.1.4 Uniform Roe Algebras

A coarse structure naturally induces a C*-subalgebra of $\mathcal{B}(\ell^2(X))$ as seen in the definition and discussion below.

Definition 5.1.11. (Uniform Roe Algebra) Let $\mathcal{C}$ be a coarse structure on a set X then for each $U \in \mathcal{C}$ we define

$$D_U = \{a \in \mathcal{B}(\ell^2(X)) : (x,y) \notin U \implies \langle a\delta_y, \delta_x\rangle = 0\}$$
$$\mathrm{C}^*_u[X,\mathcal{C}] = \bigcup\{D_U : U \in \mathcal{C}\}$$
$$\mathrm{C}^*_u(X,\mathcal{C}) = \overline{\mathrm{C}^*_u[X,\mathcal{C}]}^{\|\cdot\|}.$$

D_U and $\mathrm{C}^*_u[X,\mathcal{C}]$ are called the *U-controlled* and *$\mathcal{C}$-controlled* operators, respectively. Meanwhile $\mathrm{C}^*_u(X,\mathcal{C})$ is the ***uniform Roe algebra*** associated with $(X,\mathcal{C})$. Occasionally $\mathrm{C}^*_u[X,\mathcal{C}]$ is also referred to as the ***algebraic uniform Roe algebra*** associated with $(X,\mathcal{C})$.

The operations adjoint, addition and multiplication are closed for $\mathrm{C}^*_u[X,\mathcal{C}]$ because inverse, finite union, and composition are respectively closed for $\mathcal{C}$. $\mathrm{C}^*_u[X,\mathcal{C}]$ is also closed under scalar multiples and so is a $*$-subalgebra of $\mathcal{B}(\ell^2(X))$, hence $\mathrm{C}^*_u(X,\mathcal{C})$ is a C*-algebra.

Uniform Roe algebras are named after John Roe, who introduced them in Roe, 1988 and studied them extensively in Roe, 2003.

5.2 Measurable Coarse Structures and Uniform Roe Algebras

Definition 5.2.1. [cf. Definition 5.1.7] Let (X,μ) be a finitely decomposable measure space. We say a nonempty collection $\mathcal{C}$ of measurable relations (Definition 4.2.1) on X is a ***measurable coarse structure*** on X if it satisfies the following:

(a) If $U \in \mathcal{C}$ then $U^T \in \mathcal{C}$,

(b) If $U, V \in \mathcal{C}$ then $U \cdot V \in \mathcal{C}$,

(c) If $U, V \in \mathcal{C}$ then $U \vee V \in \mathcal{C}$ and

(d) If $U \in \mathcal{C}$ and $V \leq U$ then $V \in \mathcal{C}$.

The tuple $(X,\mu,\mathcal{C})$ is a ***measurable coarse space*** and elements of $\mathcal{C}$ may be referred to as ***entourages***.

Note that if μ is counting measure, this recovers classical coarse structures (sans the diagonal axiom as discussed earlier).

Definition 5.2.2. [cf. Definition 5.1.11] Let $(X, \mu, \mathcal{C})$ be a measurable coarse space then for each $U \in \mathcal{C}$ we define

$$D_U = \{a \in \mathcal{B}(L^2(X, \mu)) : (p, q) \notin U \implies paq = 0\}$$
$$\mathrm{C}^*_u[X, \mathcal{C}] = \bigcup \{D_U : U \in \mathcal{C}\}$$
$$\mathrm{C}^*_u(X, \mathcal{C}) = \overline{\mathrm{C}^*_u[X, \mathcal{C}]}^{\|\cdot\|}.$$

D_U and $\mathrm{C}^*_u[X, \mathcal{C}]$ are called the *U-controlled* and *$\mathcal{C}$-controlled* operators, respectively. Meanwhile $\mathrm{C}^*_u(X, \mathcal{C})$ is the ***measurable uniform Roe algebra*** associated with $(X, \mathcal{C})$. Occasionally $\mathrm{C}^*_u[X, \mathcal{C}]$ is also referred to as the ***measurable algebraic uniform Roe algebra*** associated with $(X, \mathcal{C})$.

5.3 Cantankerous Coarse Structures and Uniform Roe Algebras

As above so below: we can further generalize coarse structures by using cantankerous relations over a not necessarily commutative von Neumann algebra. The initial definitions are straightforward translations but we will see that the notion of "support expansion" bifurcates which leads to some interesting math.

Definition 5.3.1 (cf. Definition 5.2.1)**.** Let $\mathcal{M} \subseteq \mathcal{B}(H)$ be a von Neumann algebra. We say a nonempty collection $\mathcal{C}$ of cantankerous relations (Definition 4.3.1) on $\mathcal{M}$ is a ***cantankerous coarse structure*** on $\mathcal{M}$ if it satisfies the following:

(a) If $U \in \mathcal{C}$ then $U^T \in \mathcal{C}$,

(b) If $U, V \in \mathcal{C}$ then $U \cdot V \in \mathcal{C}$,

(c) If $U, V \in \mathcal{C}$ then $U \vee V \in \mathcal{C}$ and

(d) If $U \in \mathcal{C}$ and $V \leq U$ then $V \in \mathcal{C}$.

The tuple $(H, \mathcal{M}, \mathcal{C})$ is a ***cantankerous coarse space*** and elements of $\mathcal{C}$ may be referred to as ***entourages***.

Note that if $\mathcal{M} = L^\infty(X, \mu)$ for some finitely decomposable measure space then we recover measurable coarse structures.

Definition 5.3.2 (cf. Definition 5.2.2)**.** Let $(H, \mathcal{M}, \mathcal{C})$ be a cantankerous coarse space then for each $U \in \mathcal{C}$ we define

$$D_U = \{a \in \mathcal{B}(H) : (p, q) \notin U \implies p(a \otimes I)q = 0\}$$
$$\mathrm{C}^*_u[\mathcal{M}, \mathcal{C}] = \bigcup \{D_U : U \in \mathcal{C}\}$$
$$\mathrm{C}^*_u(\mathcal{M}, \mathcal{C}) = \overline{\mathrm{C}^*_u[\mathcal{M}, \mathcal{C}]}^{\|\cdot\|}.$$

D_U and $\mathrm{C}^*_u[\mathcal{M}, \mathcal{C}]$ are called the *U-controlled* and *$\mathcal{C}$-controlled* operators, respectively. Meanwhile $\mathrm{C}^*_u(\mathcal{M}, \mathcal{C})$ is the ***cantankerous uniform Roe algebra*** or ***cura*** associated with the cantankerous coarse space $(H, \mathcal{M}, \mathcal{C})$. Occasionally $\mathrm{C}^*_u[\mathcal{M}, \mathcal{C}]$ is also referred to as the ***cantankerous algebraic uniform Roe algebra*** associated with $(H, \mathcal{M}, \mathcal{C})$.

5.3.1 Support Expansion Curas

Picking up the discussion from where we left it in Section 4.3.2, we will demonstrate that support expansion C*-algebras (Definition 1.1.7) are curas.

First we recall that there are two notions of support expansion functions, which we restate here for convenience:

Definition 1.1.5. Given a Hilbert space H, von Neumann sub-algebra $\mathcal{M} \subseteq \mathcal{B}(H)$ and a dimension function $d : \Pr(\mathcal{M}) \to [0, \infty]$ the *d-support expansion function*[3] $[0, \infty] \to [0, \infty]$ of an operator $a \in \mathcal{B}(H)$ is

$$\Phi_a^d(x) = \sup\{d(\operatorname{supp}_{\mathcal{M}}(a\xi)) : d(\operatorname{supp}_{\mathcal{M}}(\xi)) \le x\}.$$

Definition 2.1.3. (cf. Definition 1.1.5) Given a Hilbert space H, von Neumann sub-algebra $\mathcal{M} \subseteq \mathcal{B}(H)$ and a dimension function $d : \Pr(\mathcal{M}) \to [0, \infty]$ the *projection d-support expansion function* $[0, \infty] \to [0, \infty]$ of an operator $a \in \mathcal{B}(H)$ is

$$\Phi_a^{d'}(x) = \sup\{d(\mathrm{s}_l^{\mathcal{M}}(ap)) : p \in \Pr(\mathcal{M}), d(p) \le x\}.$$

We showed that for commutative von Neumann algebras these notions coincide (Theorem 2.1.4) but in the non-commutative case they might disagree (Example 2.1.7).

This leads to a bifurcation in our notion of *support expansion* C*-*algebra* in the cantankerous setting:

Definition 5.3.5. (cf. Definition 1.1.7) Fix a Hilbert space H, represented von Neumann algebra $\mathcal{M} \subseteq \mathcal{B}(H)$ and dimension function d on $\Pr(\mathcal{M})$. Then given some family of functions $\mathcal{F} : [0, \infty] \to [0, \infty]$ and $f \in \mathcal{F}$ we define

$$\begin{aligned} B_f &= \{a \in \mathcal{B}(H) : \Phi_a^d, \Phi_{a^*}^d \le f\}, \\ B_{\mathcal{F}} &= \bigcup\{B_f : f \in \mathcal{F}\} \text{ and} \\ C_{\mathcal{F}} &= \overline{B_{\mathcal{F}}}^{\|\cdot\|}. \end{aligned}$$

The sets B_f and $B_{\mathcal{F}}$ are the *vector f-controlled* and *vector $\mathcal{F}$-controlled* operators respectively. C*-algebras of the form $C_{\mathcal{F}}$ are collectively referred to as *cantankerous v-support expansion* C*-*algebras (on $\mathcal{M}$)*.

Definition 5.3.6. Fix a Hilbert space H, represented von Neumann algebra $\mathcal{M} \subseteq \mathcal{B}(H)$ and dimension function d on $\Pr(\mathcal{M})$. Then given some family of functions $\mathcal{F} : [0, \infty] \to [0, \infty]$ and $f \in \mathcal{F}$ we define

$$\begin{aligned} B_f' &= \{a \in \mathcal{B}(H) : \Phi_a^{d'}, \Phi_{a^*}^{d}{}' \le f\}, \\ B_{\mathcal{F}}' &= \bigcup\{B_f' : f \in \mathcal{F}\} \text{ and} \\ C_{\mathcal{F}}' &= \overline{B_{\mathcal{F}}'}^{\|\cdot\|}. \end{aligned}$$

The sets B_f' and $B_{\mathcal{F}}'$ are the *projection f-controlled* and *projection $\mathcal{F}$-controlled* operators respectively. C*-algebras of the form $C_{\mathcal{F}}'$ are collectively referred to as *cantankerous p-support expansion* C*-*algebras (on $\mathcal{M}$)*.

[3]Dimension functions may take infinite cardinal values in which case the domain and codomain of support expansion functions may include these cardinals. We do not explore these cases in this dissertation.

While an operator $a \in \mathcal{B}(H)$ may have two distinct "support expansion functions", one focused on the expansion of vectors and the other on the expansion of projections in $\mathcal{M}$, it is conceivable that the distinction might collapse when passing through the construction in Definitions 5.3.5 and 5.3.6 – resulting in a unique canonical "support expansion C*-algebra" $C_{\mathcal{F}} = C'_{\mathcal{F}}$ associated to a given family of functions $\mathcal{F}$ (closed under addition and composition). This is not generally the case, however, which we demonstrate in the example below, developed in collaboration with my advisor David Sherman.

Example 5.3.7. Let $H = \ell^2 \oplus \ell^2$ and consider the represented von Neumann algebra $\mathcal{M} = \mathbb{C} \oplus \mathcal{B}(\ell^2) \subseteq \mathcal{B}(H)$ acting as follows: if $\lambda \in \mathbb{C}$, $a \in \mathcal{B}(H)$ and $\xi_1, \xi_2 \in \ell^2$ then $(\lambda, a)(\xi_1, \xi_2) = (\lambda\xi_1, a\xi_2)$. We will measure projections in $\mathcal{M}$ with the trace (which restricts to a dimension function on $\mathrm{Pr}(\mathcal{M})$) τ which evaluates $(\lambda, a) \mapsto \lambda + \tau_0(a)$, where τ_0 is the standard trace on $\mathcal{B}(\ell^2)$. As our final bit of notation, let $\mathcal{F}$ be the collection of functions $[0, \infty] \to [0, \infty]$ which consists of lines through the origin with natural number slope.

Now consider the partial isometry $a \in \mathcal{B}(H)$ given by $a(\xi_1, \xi_2) = (0, \xi_1)$. We will show that a has distance 1 from $B'_{\mathcal{F}}$ and thus $a \notin C'_{\mathcal{F}}$. Suppose $b \in B'_{\mathcal{F}}$ and so there exists some $n \in \mathbb{N}$ such that $\Phi'_b(x) < nx$ for all $x \in [0, \infty]$. In particular, if $p \in \mathrm{Pr}(\mathcal{M})$ has finite trace then so does $\mathrm{s}_l^{\mathcal{M}}(bp)$. Then we can find a unit vector $(0, \eta) \in H$ orthogonal to $\mathrm{s}_l^{\mathcal{M}}(b(I, 0))\, H$ (recall $(0, I)$ has infinite trace) and consider $\|a - b\|^2 \geq \|aa^*(0, \eta) - ba^*(0, \eta)\|^2 = \|(0, \eta) - ba^*(0, \eta)\|^2 = \|(0, \eta)\|^2 + \|ba^*(0, \eta)\| \geq 1$.

On the other hand we observe that any vector $(\xi_1, \xi_2) \in H$ has support of the form $(p, [\xi_2])$ where p is either 0 or identically 1 on the first coordinate, so $\tau(\mathrm{supp}_{\mathcal{M}}(\xi_1, \xi_2)) \in \{0, 1, 2\}$, with 0 only for the 0 vector. From here it is easy to see that every operator $a \in \mathcal{B}(H)$ satisfies $\Phi_a(x) \leq 2x$ for all $x \in [0, \infty]$ and thus $B_{\mathcal{F}} = \mathcal{B}(H)$ – in particular $a \in B_{\mathcal{F}} \subseteq C_{\mathcal{F}}$.

Example 5.3.7 establishes that in the cantankerous setting there are two distinct notions of support expansion C*-algebra associated to a given family of functions $\mathcal{F}$. We will realize them both in a natural way as curas. It is relatively easy to see how the projection-focused notion of support expansion (Definitions 2.1.3 and 5.3.6) gives rise to a natural cura and so we demonstrate this first:

Theorem 5.3.8. *Let $\mathcal{M} \subseteq \mathcal{B}(H)$ be a represented von Neumann algebra with dimension function d. If $\mathcal{F}$ is a family of weakly increasing functions $[0, \infty] \to [0, \infty]$ which is closed under addition and composition then $\mathcal{C}(\mathcal{F}) = \{\mathcal{R}$ a cantankerous relation on $\mathcal{M}$: $\phi_{\mathcal{R}}$ and $\phi^*_{\mathcal{R}}$ have $\mathcal{F}$-controlled d-dilation$\}$ is a cantankerous coarse structure on $\mathcal{M}$ and thus $\mathrm{C}^*_u(\mathcal{M}, \mathcal{C}(\mathcal{F}))$ is a cura on $\mathcal{M}$. Moreover $\mathrm{C}^*_u(\mathcal{M}, \mathcal{C}(\mathcal{F})) = C'_{\mathcal{F}}$ (Definition 5.3.6).*

Proof. It is straightforward to verify $\mathcal{C}(\mathcal{F})$ satisfies the items of Definition 5.4.1 (and is thus a cantankerous coarse structure on $\mathcal{M}$) by recalling the following facts: $\phi_{\mathcal{R}^T} = \phi^*_{\mathcal{R}}$, $\phi_{\mathcal{R}\cdot\mathcal{R}'} = \phi_{\mathcal{R}} \circ \phi_{\mathcal{R}'}$, $\phi_{\mathcal{R}\vee\mathcal{R}'} = \phi_{\mathcal{R}} \vee \phi_{\mathcal{R}'}$, and $\mathcal{R} \leq \mathcal{R}'$ if and only if $\phi_{\mathcal{R}} \leq \phi_{\mathcal{R}'}$ (see Propositions 4.4.11, 4.4.13, and 4.4.16) along with $d(p \vee q) \leq d(p) + d(q)$.

We will now demonstrate that $\mathrm{C}^*_u[\mathcal{M}, \mathcal{C}(\mathcal{F})] = B'_{\mathcal{F}}$ from which the final result follows. First suppose $a \in \mathrm{C}^*_u[\mathcal{M}, \mathcal{C}(\mathcal{F})]$, so there exists some $U \in \mathcal{C}(\mathcal{F})$ such that $a \in D_U$ and, since $U \in \mathcal{C}(\mathcal{F})$, some $f \in \mathcal{F}$ such that ϕ_U and ϕ^*_U have f-controlled d-dilation (a priori they may have distinct f and g controlled d-dilation respectively, but then we can just take the sum $f + g \in \mathcal{F}$). Recall that $1 - \phi_U(q)$ is the largest projection in $\mathrm{Pr}(\mathcal{M})$ such that $(1 - \phi_U(q), q) \notin U$ and thus $(1 - \phi_U(q))aq = 0$. This implies that $\phi_U(q)aq = aq$ and thus $\phi_U(q) \geq s_l^{\mathcal{M}}(aq)$. Since ϕ_U has f-controlled d-dilation, it is easy to see that $\Phi_a \leq f$ and the argument for Φ_{a^*} runs similarly so $a \in B'_{\mathcal{F}}$. This establishes that $\mathrm{C}^*_u[\mathcal{M}, \mathcal{C}(\mathcal{F})] \subseteq B'_{\mathcal{F}}$.

For the other direction, suppose $a \in B'_{\mathcal{F}}$ so there is some $f \in \mathcal{F}$ such that $\Phi_a, \Phi_{a^*} \leq f$. Note that this implies $\phi_a(\cdot) = s_l^{\mathcal{M}}(a\cdot)$ defines a cantankerous image map with f-controlled d-dilation and if $(p,q) \notin \mathcal{R}_{\phi_a}$ then $p \leq 1-\phi_a(q)$ so $paq = 0$. Thus $a \in D_{\mathcal{R}_{\phi_a}}$. We note that $\phi^*_{\mathcal{R}_{\phi_a}}(\cdot) = \phi_a^*(\cdot) = s_l(a^*\cdot)$ (we omit a brief argument here, the key insight to which is $\phi_a^*(q) \leq 1-p \Leftrightarrow \phi_a(p) \leq 1-q$) is also a cantankerous image map with f-controlled d-dilation. So $\mathcal{R}_{\phi_a} \in \mathcal{C}(\mathcal{F})$ and thus $a \in \mathrm{C}^*_u[\mathcal{M}, \mathcal{C}(\mathcal{F})]$. Along with the previous paragraph this gives us $\mathrm{C}^*_u[\mathcal{M}, \mathcal{C}(\mathcal{F})] = B'_{\mathcal{F}}$ and we are done. □

Theorem 5.3.8 establishes that the projection-focused notion of a support expansion C*-algebra is naturally a cura. As for the vector-focused notion: We cannot get around the fact that cantankerous coarse structures are families of cantankerous relations and thus fundamentally have to do with projections. While the correspondence $\Phi_a = {\Phi_a}'$ established in Theorem 2.1.4 fails in the non-commutative setting, we can in fact rephrase Φ_a in terms of projection expansion – but we need a different notion for the size of a projection.

Definition 5.3.9. Let $\mathcal{M} \subseteq \mathcal{B}(H)$ be a von Neumann algebra. We say a projection $p \in \mathrm{Pr}(\mathcal{M})$ is *cyclic* if $p = \mathrm{supp}_{\mathcal{M}}(\xi)$ for some $\xi \in H$. If $\mathcal{P}$ denotes a family of projections in $\mathcal{M}$ then $\mathrm{cyc}(\mathcal{P}) \subseteq \mathcal{P}$ denotes the collection of cyclic projections in $\mathcal{P}$.

Definition 5.3.10. If $d : \mathrm{Pr}(\mathcal{M}) \to [0,\infty]$ is a dimension function on the von Neumann algebra $\mathcal{M} \subseteq \mathcal{B}(H)$ then we define the associated *cyclic diameter* function $d_c : \mathrm{Pr}(\mathcal{M}) \to [0,\infty]$ by $d_c(p) = \sup\{d(q) : q \leq p, q \in \mathrm{cyc}(\mathrm{Pr}(\mathcal{M}))\}$.

We observe that in general the cyclic diameter is not itself a dimension function (it is not always additive on orthogonal projections, consider $\mathcal{B}(H)$), but it does have some nice properties, proved in collaboration with my advisor David Sherman:

Lemma 5.3.11. *Let $\mathcal{M} \subseteq \mathcal{B}(H)$ be a von Neumann algebra and $p, q \in \mathrm{Pr}(\mathcal{M})$. If d is a dimension function on $\mathcal{M}$ then d_c satisfies the following:*

(a) $d_c(p) \leq d(p)$ *with equality if p is cyclic,*

(b) $p \leq q$ *implies* $d_c(p) \leq d_c(q)$*,*

(c) $d_c(p \vee q) \leq d_c(p) + d_c(q)$*.*

Proof. Items (a) and (b) are straightforward. For (c) we consider a vector $\xi \in (p \vee q)H$, the closed span of pH and qH, so we can find sequences $(\eta_n)_{n\in\mathbb{N}} \subseteq pH$, $(\zeta_n)_{n\in\mathbb{N}} \subseteq qH$ of vectors so that $\eta_n + \zeta_n \to \xi$ in norm. Consider the projections $r_n = [\mathcal{M}'\eta_n] \vee [\mathcal{M}'\zeta_n]$. By Kaplansky's formula we have that $d(r_n) \leq d([\mathcal{M}'\eta_n]) + d([\mathcal{M}'\zeta_n]) \leq d_c(p) + d_c(q)$.

Now take some limit point $r \in \mathcal{M}$ of (r_n), necessarily positive, and note that for any $m \in \mathcal{M}'$ we have that $rm\xi = mr\xi = m(r-r_n)\xi + mr_n\xi = m(r-r_n)\xi + mr_n(\xi - \eta_n + \zeta_n) + mr_n(\eta_n + \zeta_n) = m(r-r_n)\xi + mr_n(\xi - \eta_n + \zeta_n) + m(\eta_n + \zeta_n)$ which tends to $m\xi$ in norm along some subsequence of $(r_n)_{n\in\mathbb{N}}$. So $rm\xi = m\xi$ and thus r is a positive operator lying above $[\mathcal{M}'\xi]$. Thus, by the weak* lower semi-continuity of d we have that $d([\mathcal{M}'\xi]) \leq \liminf d(r_n) \leq d_c(p) + d_c(q)$. Since this is true for $\xi \in (p \vee q)H$ it is also true for the supremum over such ξ and so we have $d_c(p \vee q) \leq d_c(p) + d_c(q)$ as desired. □

Proposition 5.3.12. *Let $\mathcal{M} \subseteq \mathcal{B}(H)$ be a represented von Neumann algebra. Then for any operator $a \in \mathcal{B}(H)$ we have that $\Phi_a(x) = \sup\{d_c(s_l(aq)) : d_c(q) \leq x\}$ (cf. Definitions 1.1.5 and 2.1.3).*

Proof.

$$\begin{aligned}\Phi_a(x) &= \sup\{d(\mathrm{supp}_{\mathcal{M}}(a\xi)) : d(\mathrm{supp}_{\mathcal{M}}(\xi)) \le x\}\\ &= \sup\{d(\mathrm{supp}_{\mathcal{M}}(a\xi)) : \xi \in qH; d_c(q) \le x\}\\ &= \sup\{d([\mathcal{M}'a\xi]) : \xi \in qH; d_c(q) \le x\}\\ &= \sup\{d([\mathcal{M}'\eta]) : \eta \in [\mathcal{M}'aq]H; d_c(q) \le x\}\\ &= \sup\{d(p) : p \le \mathrm{s}_l^{\mathcal{M}}(aq); p \in \mathrm{cyc}(\mathrm{Pr}(\mathcal{M})); d_c(q) \le x\}\\ &= \sup\{d_c(\mathrm{s}_l^{\mathcal{M}}(aq)) : d_c(q) \le x\}\end{aligned}$$

□

Now we have what we need to realize $C_{\mathcal{F}}$ as a cura when $\mathcal{F}$ is a family of weakly increasing functions closed under addition and composition.

Theorem 5.3.13. *Let $\mathcal{M} \subseteq \mathcal{B}(H)$ be a represented von Neumann algebra with dimension function d. If $\mathcal{F}$ is a family of weakly increasing functions $[0,\infty] \to [0,\infty]$ which is closed under addition and composition then $\mathcal{C}(\mathcal{F}) = \{\mathcal{R}$ a cantankerous relation on $\mathcal{M}$: there exists $f \in \mathcal{F}$ such that $d_c(\phi_{\mathcal{R}}(q)), d_c(\phi^*_{\mathcal{R}}(q)) \le f(d_c(q))$ for every $q \in \mathrm{Pr}(\mathcal{M})\}$ is a cantankerous coarse structure on $\mathcal{M}$ and thus $\mathrm{C}^*_u(\mathcal{M}, \mathcal{C}(\mathcal{F}))$ is a cura on $\mathcal{M}$. Moreover $\mathrm{C}^*_u(\mathcal{M}, \mathcal{C}(\mathcal{F})) = C_{\mathcal{F}}$ (Definition 1.1.7).*

Proof. It is straightforward to verify $\mathcal{C}(\mathcal{F})$ satisfies the items of Definition 5.3.1 (and is thus a cantankerous coarse structure on $\mathcal{M}$) by recalling the following facts: $\phi_{\mathcal{R}^T} = \phi^*_{\mathcal{R}}$, $\phi_{\mathcal{R}\cdot\mathcal{R}'} = \phi_{\mathcal{R}} \circ \phi_{\mathcal{R}'}$, $\phi_{\mathcal{R}\vee\mathcal{R}'} = \phi_{\mathcal{R}} \vee \phi_{\mathcal{R}'}$, and $\mathcal{R} \le \mathcal{R}'$ if and only if $\phi_{\mathcal{R}} \le \phi_{\mathcal{R}'}$ (see Propositions 4.4.11, 4.4.13, and 4.4.16) along with $d_c(p \vee q) \le d_c(p) + d_c(q)$ (Proposition 5.3.11).

We will now demonstrate that $\mathrm{C}^*_u[\mathcal{M}, \mathcal{C}(\mathcal{F})] = B_{\mathcal{F}}$ from which the final result follows. First suppose $a \in \mathrm{C}^*_u[\mathcal{M}, \mathcal{C}(\mathcal{F})]$, so there exists some $U \in \mathcal{C}(\mathcal{F})$ such that $a \in D_U$ and, since $U \in \mathcal{C}(\mathcal{F})$, some $f \in \mathcal{F}$ such that $d_c(\phi_U(\cdot))$ and $d_c(\phi^*_U(\cdot))$ are dominated by $f(d_c(\cdot))$. Recall that $1 - \phi_U(q)$ is the largest projection in $\mathrm{Pr}(\mathcal{M})$ such that $(1 - \phi_U(q), q) \notin U$ and thus $(1 - \phi_U(q))aq = 0$. This implies that $\phi_U(q)aq = aq$ and thus $\phi_U(q) \ge \mathrm{s}_l^{\mathcal{M}}(aq)$. Since $d_c(\phi_U(\cdot)) \le f(d_c(\cdot))$ and d_c is monotonic (Lemma 5.3.11) we have that $d_c(\mathrm{s}_l^{\mathcal{M}}(a\cdot)) \le f(d_c(\cdot))$. Note that $\sup\{d_c(\mathrm{s}_l^{\mathcal{M}}(aq)) : d_c(q) \le x\} \le f(x)$. By Proposition 5.3.12 we have that $\Phi_a \le f$ and the argument for Φ_{a^*} runs similarly, so $a \in B_{\mathcal{F}}$. This establishes that $\mathrm{C}^*_u[\mathcal{M}, \mathcal{C}(\mathcal{F})] \subseteq B_{\mathcal{F}}$.

For the other direction, suppose $a \in B_{\mathcal{F}}$ so there is some $f \in \mathcal{F}$ such that $\Phi_a, \Phi_{a^*} \le f$. Note that this implies $\phi_a(\cdot) = \mathrm{s}_l^{\mathcal{M}}(a\cdot)$ defines a cantankerous image map with $d_c(\phi_a(\cdot)) \le f(d_c(\cdot))$ (cf. Proposition 5.3.12) and if $(p,q) \notin \mathcal{R}_{\phi_a}$ then $p \le 1 - \phi_a(q)$ so $paq = 0$. Thus $a \in D_{\mathcal{R}_{\phi_a}}$. The argument runs similarly for $\phi^*_{\mathcal{R}_{\phi_a}}(\cdot) = \phi^*_a(\cdot) = \mathrm{s}_l^{\mathcal{M}}(a^*\cdot)$ so $\mathcal{R}_{\phi_a} \in \mathcal{C}(\mathcal{F})$ and thus $a \in \mathrm{C}^*_u[\mathcal{M}, \mathcal{C}(\mathcal{F})]$. Along with the previous paragraph this gives us $\mathrm{C}^*_u[\mathcal{M}, \mathcal{C}(\mathcal{F})] = B_{\mathcal{F}}$ and we are done. □

5.4 Quantum Coarse Structures and Uniform Roe Algebras

Definition 5.4.1 (cf. Definition 5.2.1)**.** Let $\mathcal{M} \subseteq \mathcal{B}(H)$ be a represented von Neumann algebra. We say a collection $\mathcal{C}$ of intrinsic quantum relations (Definition 4.4.2) on $\mathcal{M}$ is a ***quantum coarse structure*** on $\mathcal{M}$ if it satisfies the following:

(a) If $U \in \mathcal{C}$ then $U^T \in \mathcal{C}$,

(b) If $U, V \in \mathcal{C}$ then $U \cdot V \in \mathcal{C}$,

(c) If $U, V \in \mathcal{C}$ then $U \vee V \in \mathcal{C}$ and

(d) If $U \in \mathcal{C}$ and $V \leq U$ then $V \in \mathcal{C}$.

The tuple $(H, \mathcal{M}, \mathcal{C})$ is a ***quantum coarse space*** and elements of $\mathcal{C}$ may be referred to as ***entourages***.

Definition 5.4.2 (cf. Definition 5.2.2)**.** Let $(H, \mathcal{M}, \mathcal{C})$ be a quantum coarse space then for each $U \in \mathcal{C}$ we define

$$D_U = \{a \in \mathcal{B}(H) : (P, Q) \notin U \implies P(a \otimes I)Q = 0\}$$
$$\mathrm{C}^*_u[\mathcal{M}, \mathcal{C}] = \bigcup \{D_U : U \in \mathcal{C}\}$$
$$\mathrm{C}^*_u(\mathcal{M}, \mathcal{C}) = \overline{\mathrm{C}^*_u[\mathcal{M}, \mathcal{C}]}^{\|\cdot\|}.$$

D_U and $\mathrm{C}^*_u[\mathcal{M}, \mathcal{C}]$ are called the U***-controlled*** and $\mathcal{C}$***-controlled*** operators, respectively. Meanwhile $\mathrm{C}^*_u(\mathcal{M}, \mathcal{C})$ is the ***quantum uniform Roe algebra*** or ***qura*** associated with the quantum coarse space $(H, \mathcal{M}, \mathcal{C})$. Occasionally $\mathrm{C}^*_u[\mathcal{M}, \mathcal{C}]$ is also referred to as the ***quantum algebraic uniform Roe algebra*** associated with $(H, \mathcal{M}, \mathcal{C})$.

This is a natural object to define and every cura is a qura, which can be seen by associating a projection $p \in \mathrm{Pr}(\mathcal{M})$ with $p \otimes I \in \mathrm{Pr}(\mathcal{M} \overline{\otimes} \mathcal{B}(\ell^2))$. We have not studied proper quras enough to meaningfully say much besides this.

We would also like to point out that Kuperberg and Weaver, 2012 defines quantum uniformities, a natural dual object to quantum coarse structures, in their 5th chapter. There the language is in terms of explicit quantum relations but there is a natural definition in terms of intrinsic quantum relations which would be an obvious extension of Definition 5.1.2.

Chapter 6

Recap and Closing Remarks

We would like to briefly summarize the content of this dissertation and indicate a few directions the project might be advanced, as well as some interests of mine which do not immediately follow from the results here.

In Chapter 1 we studied C_{RC}, the C*-algebra generated by taking the norm-closure of uniformly row and column finite operators in $\mathcal{B}(\ell^2(\mathbb{N}))$. We rephrased the construction of this C*-algebra in terms of support expansion, introduced support expansion functions, and provided a general framework to construct support expansion C*-algebras given the following input data: $\mathcal{M} \subseteq \mathcal{B}(H)$ a represented von Neumann algebra, $d : \Pr(\mathcal{M}) \to [0, \infty]$ a dimension function, and $\mathcal{F}$ a collection of functions $[0, \infty] \to [0, \infty]$ closed under addition and composition (or, if not closed in this sense, one can always take the closure).

In Chapters 2 and 3 we fixed a particular represented von Neumann algebra $\mathcal{M} \subseteq \mathcal{B}(H)$ and dimension function $d : \Pr(\mathcal{M}) \to [0, \infty]$ then explored the poset of support expansion C*-algebras as we ranged over different collections $\mathcal{F}$ of functions $[0, \infty] \to [0, \infty]$. We saw that the resulting posets could get quite rich. Here we find some of our first open problems which should be pursued to push this project forward:

Question 6.0.1. If $L^\infty(\mathbb{R}) \cong \mathcal{M} \subseteq \mathcal{B}(L^2(\mathbb{R}))$, the multiplication operators and $d : \Pr(\mathcal{M}) \to [0, \infty]$ is given by integration against Lebesgue measure then do we have $\mathcal{K} \subseteq C_{\mathrm{ICOD}_0}$, where $\mathcal{K}$ are the compact operators on $L^2(\mathbb{R})$? If so, what about $\mathcal{K} \subseteq C_{\langle x \rangle}$? More generally, is the answer to these questions affirmative as we range over various $\mathcal{M}$ and d? If not, what are necessary and sufficient conditions?

Question 6.0.2. We have given some necessary and sufficient conditions for a function $[0, \infty] \to [0, \infty]$ such that it can be realized as the support expansion function for some operator. These conditions were different depending on our choice of $\mathcal{M}$ and d. Is there a theory of support expansion functions which is not so specific? Can we give some meaningful properties of dimension functions if d takes on arbitrarily large values? Small? If the range of d is dense? In summary, what more can we say about support expansion functions in terms of properties of $\mathcal{M}$ and d? Also, much more specific, in the setting of Chapter 3 we showed that ICOD $\subseteq$ SUPPEXP $\subseteq$ ISOD. At least one of these must be a proper containment. Are they both? Or is one of them equality?

Question 6.0.3. There are plenty more specific situations to explore. Of most immediate interest: What does the containment poset of support expansion C*-algebras look like when $\mathcal{M}$ is a hyperfinite type II factor?

Question 6.0.4. What can we say regarding isomorphism and non-isomorphism of support expansion C*-algebras as we range over various $\mathcal{M}$, d, $\mathcal{F}$?

In Chapter 4 we reviewed the theory of classical relations as well as the measurable and quantum relations of Weaver, 2012. We also introduced cantankerous relations, a direct non-commutative analog of measurable relations. This is where much of my current interest lies.

Question 6.0.5. Our intuition is that measurable relations on $\mathbb{R}$ correspond to "nice" subsets of $\mathbb{R}^2$. It seems this may be a different, perhaps non-comparable notion of "nice" than "measurable" (cf. discussion in Weaver, 2012 Section 1.3). We also have from the standard library that projections are the quantum stand-in for sets. So is there a way to equip the measurable relations with appropriate structure and realize them as projections in some von Neumann algebra? Could we define integration over measurable relations and appropriately reproduce Fubini-Tonelli? Are measurable relations the "correct" object for defining product measures? Can we make a compelling case? (cf. the product of complete measures may not be complete; projection maps applied to measurable sets may not produce measurable sets, etc.)

There are natural cantankerous and quantum extensions to the above questions.

Finally in Chapter 5 we briefly introduce the theories of uniform and coarse structures, including uniform Roe algebras. We extend these in the obvious way to the measurable, cantankerous and quantum settings, ultimately realizing our support expansion algebras as cantankerous uniform Roe algebras – though of two distinct types depending on if one defines expansion in terms of vectors or projections.

The theory of uniform Roe algebras is still active, with many recently solved and pending problems. Phrasing these problems in the new settings seems like it will produce interesting questions. This is not my area of expertise and so I will not list said questions but my co-author Bruna Braga, who has the background and interest, has already started tackling some of these in an upcoming paper with myself and David Sherman.

Thank you for reading!

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