Turbulence, regularity and geometry in solutions to the Navier–Stokes and magnetohydrodynamic equations

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B.S., Muhlenberg College, 2011

A Dissertation presented to the Graduate Faculty
of the University of Virginia in Candidacy for the Degree of
Doctor of Philosophy

Department of Mathematics

University of Virginia
November, 2016

Abstract

Inertial transport in the Navier–Stokes and magnetohydrodynamic equations is shown to concentrate enstrophy towards smaller scales under physically motivated and numerically supported assumptions. This is possible with an assumption of vorticity coherence wherever the velocity has large gradients in combination with interpreting enstrophy concentration in physical space, using averaged fluxes through spherical shells. Concentration of enstrophy is consistent with the dynamically generated vortex filaments or current sheets seen in numerical simulations of turbulent fluids. Complementing this, a Besov space regularity criterion is proven by relating the analytic condition of scaling behavior of the amplitude of high frequency components with the geometric property of sparseness of a super-level set. Together these results demonstrate deep connections between geometric aspects of velocity fields, regularity of solutions of deterministic fluid equations, and turbulence.

Acknowledgments

I would like to thank Professor Grujić for introducing me to the world of mathematical physics and fluid dynamics, and for his patience and understanding in advising me.

I would like to thank Professors Yen Do, Paolo D'Odorico, and Irena Lasiecka for taking the time to be committee members.

I would like to thank my mother, father, and brothers for always being there for me.

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Chapter 1

Background

The study of fluids has a long history, thanks to the fact that we live in a fluid called air and Earth is covered in a fluid called water. We came a long way with trial and error, designing spears, ships, and irrigation systems. As technology advanced, there was an increasing need for theoretical understanding of fluids. Today we can easily design efficient automobiles, airplanes, and spacecraft. We harness the power of oil, natural gas, and wind for energy. Weather forecasting warns people about dangerous storms. Our increased understanding of fluids has really paid off, but there is still lots of potential for further advancements.

One important aspect of fluid mechanics is turbulence, which is the unsteady, unpredictable motion of a fluid. It can affect heat transfer, drag, flow velocity, mixing, and refraction of light. Fluid dynamicists D. I. Pullin and P. G. Saffman wrote, "Understanding the small-scale vortex-structure of turbulence may hold the key to producing predictive models or theories capable of real engineering application" [PS95]. Ultimately, some goals of turbulence research are to understand the physical mechanisms involved in turbulence, formulate statistical laws obeyed by turbulent flows, develop a rigorous mathematical model, and to be able to compute accurate models for industrial applications.

These four domains are related, and have each had partial progress. The topic of this thesis is the connection between postulated characteristics of turbulence and the continuum models of fluids. This subject in particular is important for many reasons. The Navier–Stokes equations, which form the basis of computational fluid dynamics, are used to model the turbulence we deal with in the real world. Rigorous proofs provide a certainty that cannot be obtained by experiments or approximations, and are a formal setting to form extremely precise arguments about the nature of turbulence. It becomes possible to exactly account for all assumptions and to potentially identify shortcomings of the models used.

Two chapters of research results will be presented that demonstrate the interplay between turbulence theory, geometric aspects of functions, and rigorous results for fluid equations. The first of these chapters contains corresponding results on enstrophy concentration in the Navier–Stokes equations (NSE) and magnetohydrodynamic (MHD) equations. Conditions are formulated under which enstrophy (the root mean square vorticity) is, on average, transported to smaller scales. This is consistent with the dynamically generated vortex filaments seen in numerical simulations of the NSE and current/vortex sheets seen in MHD simulations. These so-called coherent structures are important in explaining the observed statistics of turbulent flows. The third part is a regularity criterion for mild solutions of the NSE. Besov space norms measure how velocities vary at different scales. A Besov space norm will be related to the sparseness of the set where the velocity is large. When this Besov space norm is small, we get that the set where the velocity is large is sparse, which restricts the solution from growing too large over time.

There is an enormous context that comes with these results. Some of the important points will be summarized before the main sections. This will include: the Navier–Stokes and MHD equations and how they fit within the theory of fluids overall; the phenomenological approach, which guides the mathematical treatment of turbulence; and the important mathematical results that are the foundation of the thesis.

1.1 Fluid models

The most naive way to model a fluid is to keep track of the trajectories of every single particle. One would need the initial position and velocity of each particle and the forces on each particle (for example, the forces due to gravity and particle collisions). Practically, this Newtonian model is useless. Keeping track of the positions and velocities of all of the particles would amount to storing over 10^{23} values, which would be well over 100 billion terabytes of data. Even if you had the means to store all of this data and solve a system of 10^{23} nonlinear differential equations, the lack of perfect precision in the measurements would lead to increasingly inaccurate predictions over time.

We don't actually need to know the trajectories of individual particles to have a good qualitative understanding of a fluid. For example, temperature and pressure are properties of a very large number of molecules that do not depend on the motion of any particular molecule. It is possible to derive simplified models that retain all of the important macroscopic information.

Kinetic equations

Solutions of kinetic equations keep track of the density of the number of particles in the position-momentum space $\mathbb{R}^3 \times \mathbb{R}^3$. Kinetic equations are partial differential equations. Different forces can be specified in different situations. The Boltzmann equation is a model that accounts for particle collisions as an integral operator. The Vlasov-Maxwell equations model the evolution of a plasma, using distribution functions for both electrons and positive ions. The magnetic field generated by the particles induces a force on those same particles. Collisions between particles need not occur. Kinetic equations retain statistical information on the motion of the particles that make up the fluid. It is often desirable to simplify the model further by only accounting for the macroscopic motion of a fluid.

Navier–Stokes equations

The Navier–Stokes equations model a fluid as a continuum. A fluid will appear to be a continuum only at large length scales compared to the mean free path, which is the average distance traveled by molecules between collisions. Any engineering application large enough to be visible by humans is many orders of magnitude larger than the mean free path of air or water at usual temperatures and pressures, so the continuum approximation is very reasonable.

In its most general form, the Navier–Stokes equations are

$$\rho(u_t + (u \cdot \nabla)u) = \nabla \cdot \sigma + f \tag{1.1.1}$$

$$\rho_t + \nabla \cdot (\rho u) = 0 \tag{1.1.2}$$

where $\rho(x,t)$ is the density of the fluid, $u(x,t)$ is the velocity, $f(x,t)$ are body forces (e.g. gravity), and σ is the stress tensor. The stress tensor must be specified in order to solve the equations. Newtonian fluids like water or air have a stress tensor of the form $\sigma = \mu(\nabla u + (\nabla u)^T) + (\lambda \nabla \cdot u - p)I$ for some constants μ and λ, where $p(x,t)$ is the pressure and I is the identity matrix. The equations must also be supplemented with initial conditions for u and ρ, and boundary conditions when they apply.

We will be working with the Navier–Stokes equations for incompressible Newtonian fluids. Incompressible fluids have constant density ρ so that equation 1.1.2 reduces to $\nabla \cdot u = 0$. So the equation governing the evolution of an incompressible Newtonian fluid is

$$\rho(u_t + (u \cdot \nabla)u) = \mu \Delta u - \nabla p + f \tag{1.1.3}$$

$$\nabla \cdot u = 0 \tag{1.1.4}$$

We will take $\rho = \mu = 1$ for convenience. Solutions with $\mu \neq 1$ can be recovered by rescaling the solutions. We will also always take $f = 0$, which corresponds to decaying turbulence. Fluids are approximately incompressible at velocities small compared to the speed of sound. In particular, it is not appropriate to model the flow around supersonic aircraft with the incompressible Navier–Stokes equations.

In contrast to the kinetic equations, the velocity u does not represent the velocity of individual molecules. Rather, it represents the bulk motion of the fluid. Information about thermal fluctuations of particle velocities can be obtained by solving for the temperature. The temperature evolves according to

$$T_t + (u \cdot \nabla)T = \Delta T + (\tau \cdot \nabla)u, \tag{1.1.5}$$

where $\tau = \nabla u + (\nabla u)^T$, and the last term corresponds to the increase in temperature due to the dissipation of turbulence. The evolution of the velocity is independent of the temperature so that this equation does not have to be solved simultaneously with the Navier–Stokes equations.

Magnetohydrodynamic equations

The magnetohydrodynamic (MHD) equations are a continuum model of an electrically conducting fluid, like the Sun, the interstellar medium, interplanetary medium, or the liquid core of the Earth. Kinetic equations for electrons and positive ions may be simplified to a model in which there are separate continuous flows of electrons and ions, which under further assumptions can be simplified to the MHD equations. In the absence of a magnetic field, the MHD equations reduce to the Navier–Stokes equations.

The incompressible MHD equations are

$$u_t + (u \cdot \nabla)u = \nu \Delta u + (b \cdot \nabla)b - \nabla P \tag{1.1.6}$$

$$b_t + (u \cdot \nabla)b = \eta \Delta b + (b \cdot \nabla)u \tag{1.1.7}$$

$$\nabla \cdot u = \nabla \cdot b = 0 \tag{1.1.8}$$

plus initial and boundary conditions, where $u(x,t)$ is the (vector) velocity of the fluid, $b(x,t)$ is the (vector) magnetic field, $P(x,t)$ is the (scalar) total pressure, ν

is the viscosity, and η is the magnetic resistivity. Either ν or η can be normalized to 1 by rescaling the solution, but not both. The simplification from two flows of electrons and ions to a single fluid is possible due to the assumption of quasi-neutrality, that is, variations in the net charge of the plasma only occur at small scales. Plasma phenomena that occur below this scale are not captured by the MHD equations. For example, there are structures in plasmas called double layers, which are thin layers of plasma with opposite electric charge, that are not detected with solutions of the MHD equations [BT97].

Plasmas may either be collisionless or collisional. For example, collisions are very rare in a dilute plasma such as the interstellar medium. The MHD equations as stated apply to collisonal plasmas. However, there are electromagnetic effects that mimic collisions even in dilute plasmas. The above equations are a useful model whether or not there are collisions between particles.

As in the case of the Navier–Stokes equations we will work with the incompressible MHD equations. A plasma is approximately incompressible when fluid velocities are sufficiently small. Most space plasmas are compressible. Compressibility is important because it is necessary to describe shock waves. For compressible plasmas, there is a trade-off between the simplicity of the incompressible equations and accuracy of the compressible equations. It should be noted that the MHD equations are non-relativistic, however relativity only comes into play near the speed of light, at which point the assumption of incompressibility would already be violated.

Turbulence modelling

It is computationally expensive to simulate the Navier–Stokes equations directly. Computational fluid dynamics relies on approximating the Navier–Stokes equations. One example is to use the Reynolds-averaged Navier–Stokes (RANS) equations. The velocity field is decomposed as $u = \bar{u} + u'$ where $\bar{u}$ is a suitable average of u. Then the average $\bar{u}$ satisfies

$$\bar{u}_t + \bar{u} \cdot \nabla \bar{u} + \overline{u' \cdot \nabla u'} = \Delta \bar{u} - \nabla \bar{p}.$$

To achieve a closed equation for the average $\bar{u}$, the term $\overline{u' \cdot \nabla u'}$ is replaced using heuristic arguments. Another technique that uses the same philosophy of averaging and approximating is called Large Eddy Simulation. Computational fluid dynamics software is compared to physical experiments to make sure it is accurate.

These turbulence models are very useful and practical, however they are further removed from first principles than the Navier–Stokes equations. While the Navier–Stokes equations themselves are an approximation that only model the bulk motion of fluid, the microscopic motions theoretically should not influence the bulk motion. A key property of turbulence is that the large and small scale fluid motions are interdependent. Turbulence modeling breaks this interdependence, which is philosophically unsatisfying, despite being extremely useful.

1.2 Turbulence theory

Osborne Reynolds performed experiments showing that a flow will be turbulent when what is now called the Reynolds number $Re = \frac{UL}{\nu}$ is large [Rey83]. Here U is the characteristic velocity, which may be taken to be the root mean square velocity, L is a characteristic length, for instance the length scale of the force driving the turbulence, and ν is the viscosity, which we otherwise set equal to 1. For example, a viscous fluid moving slowly through a wide pipe will not be turbulent. It will behave in a predictable way, with no apparent randomness. As we consider narrower pipes, less viscous fluids, and higher velocities, at some threshold of Re the flow will become turbulent. This threshold of Re will be different for differently shaped boundaries. Moreover, flows with the same value of Re will be similar. That is, a more viscous fluid moving through a narrow pipe will have the same dynamics as a less viscous fluid moving through a wide pipe, as long as Re is the same.

Turbulent flows are unpredictable. Similarly prepared experiments will produce different velocity fields over time. In terms of the Navier–Stokes equations, this corresponds to similar initial velocity fields quickly evolving into very different velocity fields. One might think that it would be a problem if two different initial velocity fields that modeled the same physical situation (i.e. they agreed on scales larger than the mean free path), evolved into velocity fields with differences at macroscopic scales. Turbulence theorists circumvent this issue by studying statistical properties of an ensemble of flows.

As an aside, a similar issue motivates the use of a continuum model over tracking every single particle's trajectory. The continuum model was derived by averaging over microscopic scales. If we still have this problem of predictability, why not keep averaging? That is the approach of the RANS equations. The difference is that the mean flow depends on the turbulent fluctuations more significantly than it does on thermal fluctuations. Again, RANS is sufficient for many purposes.

An ensemble may be intuitively understood as the following. Consider a probability measure on some space of initial conditions X. Ignoring the possibility of the development of singularities, each initial condition generates a solution of the Navier–Stokes equations (with forcing), so that the probability measure of initial conditions corresponds to a probability measure μ on paths in X. In other words we have a time-dependent random field that satisfies the stochastic NSE, with either deterministic or stochastic forcing. Averages with respect to μ are called ensemble averages.

Now consider the hypothetical case where μ is steady, homogeneous, and isotropic. That is, it is invariant under translations in time (which would require forcing), translations in space, and rotations in space. Homogeneous isotropic turbulence is a theoretical testing ground where complications such as the effect of boundaries can be ignored. The distribution of $u(t_0, x_0)$, where u is a random path according to μ, will be the same for all t_0 and x_0. Moreover, the distributions of the longitudinal structure function $\delta u(l) = [u(t_0, x_0) - u(t_0, x_0 + l)] \cdot \frac{l}{|l|}$ and energy dissipation $|\nabla u(t_0, x_0)|^2$ are independent of t_0 and x_0. These random variables (more specifically, their moments) are quantities that can be measured in physical experiments of homogeneous isotropic turbulence. What has been measured in physical experiments is in agreement with what has been found in numerical simulations, leading to the belief that the Navier–Stokes equations are an accurate model of turbulence [SJO91, Sig81].

A successful strategy has been to develop simple models of homogeneous isotropic turbulence, derived by physical principles. If the model agrees with the statistics of the NSE, it can provide insight into the true nature of turbulence. This was initiated by Kolmogorov [Kol41].

Kolmogorov's phenomenological theory states that if turbulence is generated at large scales, then at sufficiently small scales turbulence statistics are independent of the large scale forcing. Energy is transferred from the large scales to smaller and smaller scales, until it reaches the dissipation scale, where viscous forces dissipate the energy as heat. This cascade of energy occurs on what is called the inertial range, where the evolution of the flow depends mostly on the nonlinear term of the Navier–Stokes equations. Within the inertial range, the turbulence statistics are self similar and determined solely by the energy dissipation rate ϵ and the length scale. Some of the deductions are that, in the inertial range, the energy is distributed among different frequencies as $E(k) \sim \epsilon^{2/3} k^{-5/3}$ and the moments of the structure functions obey $\langle |\delta u(l)|^p \rangle \sim \epsilon^{p/3} l^{p/3}$.

In the original Kolmogorov theory, the energy dissipation is uniform, that is, each sample path has $\epsilon = |\nabla u(x, t)|$ equal to a constant. This is in strong qualitative disagreement with true turbulence, where the energy dissipation appears to be sparsely distributed through space. This is one manifestation of intermittency. Subsequent phenomenological models included intermittency corrections. One of them is the β-model, which has a parameter $\beta \leq 1$. In the limit of zero viscosity, the dissipation is concentrated on a set of Hausdorff dimension $D = 3 + \log_2 \beta$. The energy spectrum is $E(k) \sim \epsilon^{2/3} k^{-5/3-(3-D)/3}$ and the structure functions are $\langle |\delta u(l)|^p \rangle \sim \epsilon^{p/3} l^{p/3+(3-D)(3-p)/3}$ [FSN78]. Another aspect of intermittency are the non-Gaussian tails of the velocity increments, and the appearance of coherent vortex structures [Fri95].

1.3 Mathematical background

Here some of the relevant rigorous mathematical results will be stated. We consider the Navier–Stokes equations on the domain $\Omega = \mathbb{R}^3$. With some modifications, we could also take Ω to be a cube with periodic boundary conditions (in other words, periodic functions on $\mathbb{R}^3$), or a sufficiently regular bounded domain or exterior domain with appropriate boundary conditions.

No special notation will be used to distinguish vectors from scalars. The velocity u will always be a 3-dimensional vector, and the pressure p will always be a scalar. For example, $u_0 \in L^2$ means each component of the vector u_0 is in L^2, and if $u \in C^1([0,T]; C^2)$, then $u(t,x)$ is a 3-dimensional vector for every t, x. Anytime a derivative of a non-differentiable function appears, it is meant as a distribution derivative.

A classical solution to the Navier–Stokes equations is a pair (u, p) where $u \in C^1([0,T]; C^2(\mathbb{R}^3))$ and $p \in C([0,T]; C^1(\mathbb{R}^3))$ satisfying (1.1.3)-(1.1.4). Global-in-time existence of classical solutions for finite-energy initial data is an open problem, however there are proofs that unique classical solutions exist for some amount of time. These classical solutions may be extended to global-in-time weak solutions. Uniqueness of these weak solutions is unknown.

Weak solutions will be defined following the presentation in [LR02].

Definition 1.3.1. A function $u \in L^2_{loc}((0,T) \times \mathbb{R}^3)$ is a *weak solution* of the Navier–Stokes equations on $(0,T) \times \mathbb{R}^3$ if $\nabla \cdot u = 0$ and there exists $p \in D'((0,T) \times \mathbb{R}^3)$

such that

$$u_t + \nabla \cdot (u \otimes u) = \Delta u - \nabla p.$$

Definition 1.3.2. Let $u_0 \in L^2(\mathbb{R}^3)^3$ with $\nabla \cdot u_0 = 0$. A function $u \in L^\infty((0,T); L^2) \cap L^2((0,T); H^1)$ is a *Leray solution* with initial data u_0 if it is a weak solution such that $\lim_{t \to 0^+} ||u - u_0||_2 = 0$, and

$$||u(t)||_2^2 + 2\int_0^t \int |\nabla u|^2 dx ds \leq ||u_0||_2^2$$

for all $t \in (0,T)$.

Theorem 1.3.3. *[Ler34] For any $u_0 \in L^2(\mathbb{R}^3)^3$ with $\nabla \cdot u_0 = 0$, and any $T > 0$, there exists a Leray solution u on $(0,T) \times \mathbb{R}^3$ with initial data u_0.*

Rewriting the Navier–Stokes equations as an integral equation allows for the construction of weak solutions that are not Leray solutions. Let $\mathbb{P}$ be the Leray projection, which is the projection in $L^2(\mathbb{R}^3)^3$ onto divergence free functions. It can be proven that $\mathbb{P}\nabla \cdot (u \otimes u)$, which is a convolution of $u \otimes u$ with a singular kernel, is well defined for uniformly locally square integrable u.

Definition 1.3.4. A uniformly locally square integrable function u is a *mild solution* of the Navier–Stokes equations if there exists a tempered distribution $u_0 \in S'$ with $\nabla \cdot u_0 = 0$ such that

$$u = e^{t\Delta} u_0 + \int_0^t e^{(t-s)\Delta} \mathbb{P}\nabla \cdot (u \otimes u)\, ds.$$

There exist mild solutions when the initial data lies in a variety of spaces, such as Lebesgue spaces $L^p\,(p \geq 3)$, Sobolev spaces $H^s\,(s \geq 1/2)$, and Besov spaces $B^{-2/p}_{q,p}\,(2/p + 3/q \leq 1, 3 < q < \infty)$. These solutions are known to exist for a length of time that is inversely proportional to the size of the initial data. All of these spaces are contained in the Besov space $B^{-1}_{\infty,\infty}$, which is defined in Section 3.2.

These spaces are (sub)critical with respect to the scaling symmetry of the Navier–Stokes equations, and solutions in such spaces are known to be smooth for a period of time, with the possibility of finite time blow-up. On the other hand, all known a priori bounded quantities, such as the energy, have supercritical scaling, with a gap between these scaling exponents and the critical exponent.

Leray solutions have uniformly bounded energy, which is not sufficient for regularity. However there are many supplemental conditions, involving critical or subcritical spaces, that are sufficient for local-in-time regularity.

Theorem 1.3.5. *Let* u *be a Leray solution with* $u \in L^\infty((0,T); H^1)$*. It is the unique Leray solution on* $(0,T)$ *with initial data* u_0*, and it is smooth in space and time.*

Theorem 1.3.6. *If* u *is a Leray solution and* $u \in L^p([0,T] : L^q(\mathbb{R}^3))$*, where* $p \in [2,\infty]$*,* $q \in [3,\infty]$*,* $\frac{3}{q} + \frac{2}{p} \leq 1$*, then* u *is smooth in space and time and unique among Leray solutions.*

The following theorem is similar to the result of Chapter 3. Our result is in the setting of mild solutions with L^∞ initial data, not Leray solutions.

Theorem 1.3.7. *[CS10] Let* u *be a Leray solution with* $\sup_{t\in(0,T)} ||u(t)||_{B^{-1}_{\infty,\infty}}$ *sufficiently small. Then* u *is regular on* $(0,T]$*.*

There are regularity criteria based on the vorticity $\omega = \nabla \times u$. Note that the spaces have the kind of scaling gap between a priori bounded quantities and known regularity criteria.

Theorem 1.3.8. *[BKM84] Let* $\omega = \nabla \times u$*, where* u *is a Leray solution. If* $\omega \in L^1([0,T] : L^\infty)$*, then* u *is regular on* $[0,T]$*.*

Theorem 1.3.9. *[BP08] If the vorticity vector for a Leray solution is 1/2-Hölder coherent in the region of intense vorticity, then it is regular.*

Although Leray solutions may become singular, there are restrictions on the size of the singular set.

Theorem 1.3.10. *For any Leray solution, the* $1/2$*-dimensional Hausdorff measure of the set of singular times is zero.*

Theorem 1.3.11. *[CKN82] For a suitable weak solution (which always exists among the Leray solutions), the 1-dimensional parabolic Hausdorff measure of the set of singular spacetime points is zero.*

There are many rigorous results for the Navier–Stokes equations related to turbulence theory. Properties of structure functions, energy spectra, and energy cascades are interpreted and proven in a rigorous framework.

Foias, Manley, Rosa, and Temam rigorously formulated ensembles of solutions to the NSE. Such ensembles may be constructed by taking a Banach limit of the time average of a deterministic solution. They proved that on average, energy is cascading from higher frequencies to lower frequencies throughout the inertial range. They also proved results on correlation functions and the energy spectrum [FMRT04, FMRT01, FJMR05, Ros02].

Constantin and Fefferman proved upper bounds on a version of structure functions using ensembles of solutions [CF93]. Constantin works with individual solutions to prove bounds on the energy spectrum [Con97] and to study energy dissipation [Con94].

Cheskidov and Shvydkoy work in a deterministic setting, with finite space and time averages. They relate a dissipation scale with regularity using Besov spaces [CS].

The results of the next chapter are based on a series of papers by Dascaliuc and Grujić [DG11, DG13, DG12]. They work in a deterministic setting and interpret energy and enstrophy cascade as the average of fluxes through shells. This approach was applied to the MHD equations in [BG13].

Chapter 2

Enstrophy Cascade

Taking the curl of the Navier–Stokes equations yields

$$\omega_t + (u \cdot \nabla)\omega = \Delta\omega + (\omega \cdot \nabla)u,$$

where $\omega = \nabla \times u$ is known as the vorticity. Numerical simulations show that regions of intense vorticity take the form of filaments [VM94, JWSR93]. A local cascade of enstrophy (the root mean square vorticity) would explain why vortex filaments appear in numerical simulations. Vortex filaments have long been recognized as a significant component of turbulence. In the words of G. I. Taylor, "It seems that the stretching of vortex filaments must be regarded as the principal mechanical cause of the high rate of dissipation which is associated with turbulent motion" [Tay37].

In contrast to the energy, which is only being transported and dissipated over time, enstrophy is transported, dissipated, and generated or destroyed via the vortex stretching term. So even if the role of the inertial term $(u \cdot \nabla)\omega$ is to concentrate the enstrophy towards smaller scales, it may be the case the the vortex stretching term $(\omega \cdot \nabla)u$ cancels out or even amplifies this effect. For the two dimensional NSE, where the vortex stretching term vanishes, there is a proper cascade, since the only factors in the evolution are transport and dissipation.

We will show that the inward enstrophy flux through a ball of radius R is positive, on average, through a range of values of R. This interpretation of a cascade was formulated in [DG11, DG13] to prove the existence of energy and enstrophy cascades.

In contrast to previous results on cascades, it does not use the Fourier transform or a spectral decomposition to introduce the concept of scale. Using fluxes through shells is a natural way to measure scales and allows for the incorporation of geometric assumptions. This has been improved by using a modified definition of ensemble averages to achieve more satisfying assumptions.

The inward enstrophy flux through a ball, according to the divergence theorem, can be written

$$-\int_{\partial B} \frac{1}{2}|\omega|^2(u \cdot n)\, d\sigma = -\int_B (u \cdot \nabla)\omega \cdot \omega\, dx. \tag{2.0.1}$$

We want to use the Navier–Stokes equations to estimate this. However, we will run into problems unless we use a smoothed out version

$$\int \frac{1}{2}|\omega|^2(u \cdot \nabla\psi)\, dx = -\int (u \cdot \nabla)\omega \cdot \psi\omega\, dx, \tag{2.0.2}$$

where ψ may be taken to be a smooth bump function supported on B with inward pointing gradient. This may be considered as a weighted average of enstrophy flux over the shell that is the support of $\nabla\psi$.

We will use the Navier–Stokes equations to estimate the time-average of these fluxes, but the ensemble average framework must be introduced first. The framework will provide the sense in which the fluxes are positive on average. It will be reused to prove kinetic-magnetic enstrophy concentration for the MHD equations. The contents of this chapter originally appeared in [Lei15, Lei16]

2.1 Ensemble Averages

The ensemble averages about to be defined can be used to show some function f does not have significant fluctuations above a certain scale. In particular, we will show that the time-averaged enstrophy flux density $f = -\frac{1}{T}\int (u \cdot \nabla)\omega \cdot \omega\, dt$ does not have significant negative fluctuations above a certain scale, and this will be interpreted as evidence of an enstrophy cascade.

This definition of ensemble average will be different from the standard one, which was used in Chapter 1. The concept has been adapted for use in the deterministic setting. In this setting, one might envision the different regions of space as independent realizations of a random flow to motivate the terminology.

Ensembles will be built using refined test functions.

Definition 2.1.1. Fix $C_0 > 1$ and $3/4 < \rho < 1$. A *refined test function* at scale R satisfying C_0 bounds is any smooth function ψ supported in a ball of radius $2R$ satisfying $0 \leq \psi \leq 1$, $|\nabla\psi| < \frac{C_0}{R}\psi^\rho$, and $|\Delta\psi| < \frac{C_0}{R^2}\psi^{2\rho-1}$.

Example 2.1.2. Let $R = 1$ and consider 1-dimensional space. We will build a refined test function supported on $[0, 2]$. Set $\psi(x) = x^{\frac{1}{1-\rho}}$ where $0 < x < 1/2$, so that $\psi'(x) = \frac{1}{1-\rho}x^{\frac{\rho}{1-\rho}} = \frac{1}{1-\rho}\psi^\rho$ and $\psi''(x) = \frac{\rho}{(1-\rho)^2}x^{\frac{2\rho-1}{1-\rho}} = \frac{\rho}{(1-\rho)^2}\psi^{2\rho-1}$ where $0 < x < 1/2$. Set $\psi(x) = (2-x)^{\frac{1}{1-\rho}}$ where $3/2 < x < 2$. Finally, on $[1/2, 3/2]$, the function can do anything as long as it is smooth, bounded away from zero, less than or equal to one, and has bounded first and second derivative. Then ψ will be a refined test function satisfying C_0 bounds where C_0 is the maximum of $\sup_x \psi'(x)/\psi(x)^\rho$ and $\sup_x \psi''(x)/\psi(x)^{2\rho-1}$.

This example includes bump functions that are identically equal to one on the interior, but may also include bump functions that equal something less than one on the interior, or functions that aren't constant on the interior.

It would be convenient to be able to take $\rho = 1$ in the definition of refined test functions, however no such functions exist. If ψ is a smooth function satisfying $\psi(x) = 0$ on $(-\infty, 0)$ and $\psi'(x) < C_0\psi(x)$, we must have that $\psi(x) \leq \phi(x)$ where $\phi(x)$ satisfies $\phi'(x) = C_0\phi(x)$ with $\phi(x) = 0$ on $(-\infty, 0)$. But then ϕ is identically zero, so ψ is identically zero.

Any scale R refined test function satisfying C_0 bounds is the dilation of a scale 1 refined test function satisfying C_0 bounds.

Definition 2.1.3. Fix a scale R_0 refined test function ψ_0 centered at 0. An *ensemble* at scale R with global multiplicity K_1 and local multiplicity K_2 is a collection of scale R test functions $\{\psi_i\}_{i=1}^n$ satisfying the following properties:

1. $\psi_i \leq \psi_0 \leq \sum \psi_i$
2. $(R_0/R)^3 \leq n \leq K_1(R_0/R)^3$
3. No point of $B(0, 2R_0)$ is contained in more than K_2 of the supports of ψ_i.

Ensembles can be constructed by using Lemma 2.1.5 on ψ_0.

Definition 2.1.4. The *ensemble average* $\langle f \rangle_R$ of a function f over an ensemble $\{\psi_i\}_{i=1}^n$ is defined as $\frac{1}{n}\sum_{i=1}^n \frac{1}{R^3}\int f\psi_{i,R}\,dx$. We use the special notation f_0 when the ensemble is just ψ_0, that is, $f_0 = \frac{1}{R_0^3}\int f\psi_0\,dx$.

These ensemble averages are designed to detect the scale of negative fluctuations. Averages of a positive function will be positive no matter what the scale is.

Lemma 2.1.5. *If $f \geq 0$ then $\frac{1}{K_1}f_0 \leq \langle f \rangle_R \leq K_2 f_0$. The latter inequality still holds when ψ_i are replaced by their powers ψ_i^δ, for $\delta > 0$: $\frac{1}{n}\sum_1^n \frac{1}{R^3}\int f\psi_{i,R}^\delta\,dx \leq K_2\frac{1}{R_0^3}\int f\psi_0^\delta\,dx$.*

If a bound on the ensemble average is proven at one scale R', it is automatically proven for all larger scales, if the constants are tweaked.

Lemma 2.1.6. *There exists a number C_0' depending only on C_0 such that any scale R refined test function satisfying C_0 bounds can be written as a sum of $8\lceil R/R' \rceil^3$ many scale R' refined test functions that satisfy C_0' bounds, for any $R' < R$.*

Proof. Let ψ be a (C_0, ρ) scale R test function. Now to construct the partition of unity, take a (C_0, ρ), scale R' 3D test function g_0, centered at zero and equal to 1 on $[-R, R]^3$ (such a function exists as long as C_0 isn't too small). Define $g_p = g_0(x - 2R'p)$, where $p \in \mathbb{Z}^3$. Then $1 \leq \sum_p g_p \leq 2$ so we may define $h_p = g_p/\sum_q g_q$.

Some calculus shows that $|\nabla h_p| < \frac{6C_0}{R'} h_p^\rho$ and $|\Delta h_p| < \frac{3C_0+10C_0^2}{R'^2} h_p^{2\rho-1}$, so $|\nabla(\psi h_p)| < \frac{7C_0}{R'}(\psi h_p)^\rho$ and $|\Delta(\psi h_p)| < \frac{4C_0+22C_0^2}{R'^2}(\psi h_p)^{2\rho-1}$. Fewer than $8\lceil R/R' \rceil^3 \le 64(R/R')^3$ of the functions ψh_p are nonzero, and for any x, $\psi_p(x) \neq 0$ for at most 8 functions.

Since $\psi = \sum_p \psi h_p$, we have the first claim. For the second claim, given an ensemble $\{\psi_i\}_i$, the new ensemble will be $\{\psi_i h_p\}_{i,p}$. □

Corollary 2.1.7. *For every (K_1, K_2, C_0)-ensemble at scale R and every $R' < R$, there exists a $(64K_1, 8K_2, C_0')$-ensemble at scale R' such that $\langle f \rangle_R = \langle f \rangle_{R'}$.*

2.2 NSE Enstrophy Cascade

Consider a Leray solution $u : \Omega \times [0, T] \to \mathbb{R}^3$ to the Navier–Stokes equations

$$u_t + (u \cdot \nabla u) = \Delta u - \nabla p \tag{2.2.1}$$

$$\nabla \cdot u = 0 \tag{2.2.2}$$

where $\Omega = \mathbb{R}^3$ or $\Omega = [0, L]^3$ with periodic boundary conditions. It is also possible to take Ω as a bounded domain with no-slip boundary conditions $u|_{\partial\Omega} = 0$, but some of what follows would have to be modified: that $\sigma_0^{3/4} < \beta^{3/4} R_0$ in Assumption 2.2.2 and $R = (\sigma_0/\beta)^{3/4}$ in Theorem 2.2.9. Regardless of the domain, we will examine the properties of the solution locally, by multiplying $u(x,t)$ by a refined test function ψ_0 supported on $B(0, 2R_0)$ (the ball is centered at zero for notational convenience). Naturally, $B(0, 2R_0)$ needs to be a subset of Ω, but some extra room is needed: we need that $B(0, 2R_0 + R_0^{2/3})$ is contained in Ω. Suppose also that $R_0 < 1$ and $T > R_0^2$ for convenience.

The goal is to prove all ensemble averages of the inward enstrophy fluxes are positive for a range of scales, that is,

$$\frac{1}{n}\sum_1^n \frac{1}{T}\int_0^T \frac{1}{R^3}\int \frac{1}{2}|\omega|^2 (u \cdot \nabla \psi_{i,R}) dx\, \eta(t) dt \sim P_0$$

holds for a range of R, where $\psi_{i,R}$ form an ensemble at scale R with respect to ψ_0, $\eta(t)$ is a temporal cutoff function, and P_0 is a positive constant. This will only be true under certain assumptions.

Assumption 2.2.1. Let $\xi = \omega/|\omega|$ be the vorticity direction field. Assume there exist M, C_1 such that $|\sin\theta(\xi(x,t), \xi(y,t))| \le C_1|x-y|^{1/2}$ for t in $(0,T)$, x in $B(0, 2R_0) \cap \{x : |\nabla u(x,t)| > M\}$, and y in $B(0, 2R_0 + R_0^{2/3})$, where $\theta(z_1, z_2)$ denotes the angle between the vectors z_1 and z_2.

Numerical simulations show that the region of intense vorticity in a turbulent flow organizes into tubes with aligned vorticity vectors. The presence of the sine means that vorticity vectors can be aligned or anti-aligned. The condition must be satisfied in the region of large velocity gradients rather than large vorticity. The region of large velocity gradients contains the region of large vorticity, and they roughly coincide according to numerical simulations [JWSR93].

Assumption 2.2.2. Denote the scale-R_0 mean enstrophy by

$$E_0 = \frac{1}{T}\int \frac{1}{R_0^3}\int \frac{1}{2}|\omega|^2 \phi_0^{2\rho-1} dx\, dt,$$

the modified mean palinstrophy by

$$P_0 = \frac{1}{T}\int \frac{1}{R_0^3}\int |\nabla\omega|^2 \phi_0 \, dx\, dt + \frac{1}{T}\frac{1}{R_0^3}\int |\omega(x,T)|^2 \psi_0 \, dx,$$

and the modified Kraichnan scale by $\sigma_0 = (\frac{E_0}{P_0})^{1/2}$. It is required that $\sigma_0 < \beta R_0$, where $\beta \in (0,1)$ is a constant.

The constant β depends only on C_0, C_1, M, K_1, K_2, and $B_T = \sup_{t\in(0,T)} ||\omega||_{L^1(\Omega)}$, and β shrinks to zero as any of them increase to infinity.

The ensemble averaged enstrophy fluxes will be positive for all scale R ensembles with $\sigma_0/\beta \le R \le R_0$. For a turbulent flow, there will be high spatial complexity, so the palinstrophy would be larger than the enstrophy, leading to a wide range of scales for the enstrophy cascade.

Let $M^{p,q} = M^{p,q}(B(0, 2R_0))$ be the Morrey space of functions f such that

$$\sup_{y,R} \frac{1}{R^{3(1-p/q)}} \int_{B(y,R)\cap B(0,2R_0+R_0^{2/3})} |f|^p \, dx$$

is finite. Note that $L^q \subset M^{p,q} \subset L^p$. In particular, $M^{2,2} = L^2(B(0, 2R_0 + R_0^{2/3}))$.

Assumption 2.2.3. Fix a real number $q > 2$. Assume $\omega(t, x) \in L^2(0, T; M^{2,q})$ with

$$\sigma_0^{1-2/q} ||\omega||_{L_t^2 M_x^{2,q}} < \left(\frac{\beta}{2}\right)^{1-2/q} \frac{1}{C},$$

where C depends only on β, C_0, K_1, K_2.

As a Leray solution, $\omega \in L^2(0, T; L^2)$. Again, for a turbulent solution with very high spatial complexity, the palinstrophy would be large compared to the enstrophy. The Morrey norm is similar to the enstrophy.

An alternative to this assumption is that

$$\int_0^T \int_{B(y,R)} |\omega|^2 dx dt < \frac{1}{C} \tag{2.2.3}$$

for any $y \in B(0, 2R_0)$ where C is a constant and $R = 2\sigma_0/\beta + (\sigma_0/\beta)^{2/3}$. That is, there is an upper bound to how much enstrophy can be concentrated at this small scale.

This is the assumption that is the main advantage over the previous work [DG], and is possible because of the revised definition of ensembles above. In that paper, the upper bound (2.2.3) was required for $R = R_0$, which would potentially limit the extent of the cascade by forcing R_0 to be small.

The final assumption allows for a cleaner looking theorem. It holds if the enstrophy does not drop off too dramatically at time T.

Assumption 2.2.4. $\int |\omega(x, T)|^2 \psi_0(x) \, dx \geq \frac{1}{2} \sup_{t \in (0,T)} \int |\omega(x, t)|^2 \psi_0(x) \, dx$.

We use the notation $\phi(x, t) = \psi(x)\eta(t)$, where $\psi(x)$ is a refined test function at scale R, and $\eta(t)$ is a smooth function supported on $[0, T]$ with $|\eta'| < \frac{C}{T}\eta^\rho$, $0 \leq \eta \leq 1$,

$\eta = 0$ on $[0, T/3)$, and $\eta = 1$ on $(2T/3, T]$. These will be used to take weighted time-averages of fluxes.

The following equation is obtained by manipulating the NSE, and will be used to estimate the size of time-averaged fluxes.

Lemma 2.2.5. *For a solution to the Navier-Stokes equations u that is smooth on $[0,T] \times B(x_i, 2R)$,*

$$\int_0^T \int \frac{1}{2}|\omega|^2(u \cdot \nabla\phi_i)\,dx\,dt = \int \frac{1}{2}|\omega(x,T)|^2\psi_i(x)\,dx + \int_0^T \int |\nabla\omega|^2\phi_i\,dx\,dt$$
$$- \int_0^T \int \frac{1}{2}|\omega|^2(\partial_t\phi_i + \Delta\phi_i)\,dx\,dt - \int_0^T \int (\omega \cdot \nabla)u \cdot \phi_i\omega\,dx\,dt.$$

Proof. Using integration by parts and that u is divergence free, $\frac{1}{2}|\omega|^2(u \cdot \nabla\phi_i) = -(u \cdot \nabla)\omega \cdot \phi_i\omega$. From the Navier-Stokes equations, $-(u \cdot \nabla)\omega = \partial_t\omega - \Delta\omega - (\omega \cdot \nabla)u$. Integrating in space and time against $\phi_i\omega$ yields $-\int_0^T \int (u \cdot \nabla)\omega \cdot \phi_i\omega\,dx\,dt = \int_0^T \int \partial_t\omega \cdot \phi_i\omega\,dx\,dt - \int_0^T \int \Delta\omega \cdot \phi_i\omega\,dx\,dt - \int_0^T \int (\omega \cdot \nabla)u \cdot \phi_i\omega\,dx\,dt$. Now

$$\int_0^T \int \partial_t\omega \cdot \phi_i\omega\,dx\,dt = \int_0^T \int \frac{1}{2}\partial_t(\omega \cdot \phi_i\omega) - \frac{1}{2}\omega \cdot (\partial_t\phi_i)\omega\,dx\,dt$$
$$= \int |\omega(T)|^2\psi_i\,dx - \int_0^T \int \frac{1}{2}|\omega|^2\partial_t\phi_i\,dx\,dt,$$

and

$$-\int_0^T \int \Delta\omega \cdot \phi_i\omega\,dx\,dt = \int_0^T \int |\nabla\omega|^2\phi_i\,dx\,dt - \int_0^T \int \frac{1}{2}|\omega|^2\Delta\phi_i\,dx\,dt,$$

simply by integration by parts and the fundamental theorem of calculus. Putting all of the equations together completes the proof.

□

We want to control the vortex stretching term from the lemma above. There is an explicit formula relating the gradient and the curl of a divergence free vector field known as the Biot–Savart law.

Let ϵ_{ijk} be the Levi-Civita symbol, which equals 1 for even permutations (ijk), -1 for odd permutations, and 0 otherwise. Summation over repeated indices is implied. For example, $(\nabla \times u)_1 = \epsilon_{ij1}\partial_i u_j = \epsilon_{231}\partial_2 u_3 + \epsilon_{321}\partial_3 u_2 = \partial_2 u_3 - \partial_3 u_2$.

Theorem 2.2.6 (Biot–Savart law)**.** *For any $u \in S(\mathbb{R}^3)$ such that $\nabla \cdot u = 0$, with $\omega = \nabla \times u$, we have*

$$\partial_l u_k = c\epsilon_{ijk} P.V.(\frac{x_i x_l}{|x|^5}) * \omega_j + c\epsilon_{jkl} P.V.(\frac{1}{|x|^3}) * \omega_j$$

where P.V. is the Cauchy principal value and c is a constant.

Proof. We have $\nabla \times \omega = -\Delta u$ by vector calculus, so $u = cK * (\nabla \times \omega)$, where $K(x) = \frac{1}{|x|} \in L^1_{loc}$. Then $u_k = cK * (\epsilon_{ijk} \partial_i \omega_j) = c\epsilon_{ijk} \partial_i K * \omega_j$, where $\partial_i K = \frac{x_i}{|x|^3} \in L^1_{loc}$. Taking (distribution) derivatives of both sides, $\partial_l u_k = c\epsilon_{ijk} \partial_l \partial_i K * \omega_j$. This kernel, which is not locally integrable, may be computed as

$$\partial_l \partial_i K = \begin{cases} P.V.(\frac{x_i x_l}{|x|^5}) & i \neq l \\ P.V.(\frac{x_i^2}{|x|^5} + \frac{1}{|x|^3}) & i = l. \end{cases}$$

Therefore $\partial_l u_k = c\epsilon_{ijk} \partial_l \partial_i K * \omega_j = c\epsilon_{ijk} P.V.(\frac{x_i x_l}{|x|^5}) * \omega_j + c\epsilon_{jkl} P.V.(\frac{1}{|x|^3}) * \omega_j$.

□

Now the Biot–Savart law is applied to write the vortex stretching term solely in terms of ω.

Lemma 2.2.7. *For a divergence free function $u \in H^1(\mathbb{R}^3)^3$, with $\omega := \nabla \times u$, we have $(\omega \cdot \nabla) u \cdot \omega(x) = c\, P.V. \int \omega(x) \times \omega(y) \cdot G_\omega(x, y)\, dy$ for a.e. x, where $(G_\omega(x, y))_i = \frac{(x_i - y_i)(x_l - y_l)}{|x-y|^5} \omega_l(x) + \frac{1}{|x-y|^3} \omega_i(x)$.*

Proof. Let u be a Schwarz function. By the Biot-Savart law (Theorem 1.3.16),

$$\partial_l u_k = c\epsilon_{ijk} P.V.(\frac{x_i x_l}{|x|^5}) * \omega_j + c\epsilon_{jkl} P.V.(\frac{1}{|x|^3}) * \omega_j.$$

Then

$$\omega_l \partial_l u_k \omega_k = c\epsilon_{ijk} \omega_l \omega_k P.V.(\frac{x_i x_l}{|x|^5}) * \omega_j + c\epsilon_{ijk} \omega_i \omega_k P.V.(\frac{1}{|x|^3}) * \omega_j,$$

where the indices in the second term on the right were relabeled, which is possible upon summation over l and k. Therefore,

$$\omega_l\partial_l u_k\omega_k(x) = c\,P.V.\int \epsilon_{kji}\omega_k(x)\omega_j(y)\Big(\frac{(x_i-y_i)(x_l-y_l)}{|x-y|^5}\omega_l(x)+\frac{1}{|x-y|^3}\omega_i(x)\Big)dy$$
$$= c\,P.V.\int \omega(x)\times\omega(y)\cdot G_\omega(x,y)\,dy.$$

By density of Schwartz functions in H^1, the following holds in L^2:

$$(\omega\cdot\nabla)u\cdot\omega = cP.V.\int \omega(x)\times\omega(y)\cdot G_\omega(x,y)\,dy.$$

□

Lemma 2.2.8 is used to estimate the vortex stretching term that appears in Lemma 2.2.7. The other terms will be easy to deal with.

Lemma 2.2.8.

$$\Big|\int_0^T\int(\omega\cdot\nabla)u\cdot\phi_i\omega\,dx\,dt\Big| \le$$
$$c||\omega||_*\Big(\sup_t\int\frac{1}{2}|\omega(x,t)|^2\psi_i(x)\,dx+\int_0^T|\nabla\omega|^2\phi_i\,dx\,dt\Big)$$
$$+\frac{c'+c''||\omega||_*}{R^2}\int_0^T\int\frac{1}{2}|\omega|^2\phi_i^{2\rho-1}\,dx\,dt$$

($||\omega||_* := ||\omega||_{L^2(B(x_i,R)\times(0,T))}$, constants depend on M, B_T, C_0).

Proof. Write $\int_0^T\int(\omega\cdot\nabla)u\cdot\phi_i\omega\,dx\,dt = \int_0^T\int_{\{|\nabla u|<M\}}(\omega\cdot\nabla)u\cdot\phi_i\omega\,dx\,dt+\int_0^T\int_{\{|\nabla u|>M\}}(\omega\cdot\nabla)u\cdot\phi_i\omega\,dx\,dt$. The first term is bounded by $\frac{M}{R^2}\int_0^T\int|\omega|^2\phi_i^{2\rho-1}dxdt$, using $\phi_i\le\phi_i^{2\rho-1}$ and the assumption that $R_0<1$. For the second term, using Lemma 2.2.11,

$$\int_0^T\int_{\{|\nabla u|>M\}}(\omega\cdot\nabla)u\cdot\phi_i\omega\,dx\,dt = \qquad (2.2.4)$$
$$\int_0^T\int_{\{|\nabla u|>M\}}P.V.\int\omega(x)\times\omega(y)\cdot G_\omega(x,y)\phi_i(x)\,dy\,dx\,dt = \qquad (2.2.5)$$
$$\int_0^T\int_{\{|\nabla u|>M\}}P.V.\int_{\{|x-y|<R^{2/3}\}}\omega(x)\times\omega(y)\cdot G_\omega(x,y)\phi_i(x)\,dy\,dx\,dt \qquad (2.2.6)$$
$$+\int_0^T\int_{\{|\nabla u|>M\}}\int_{\{|x-y|>R^{2/3}\}}\omega(x)\times\omega(y)\cdot G_\omega(x,y)\phi_i(x)\,dy\,dx\,dt. \qquad (2.2.7)$$

The second term (4) is bounded by

$$\frac{1}{R^2}\int_0^T\int\int_{\{|x-y|>R^{2/3}\}}|\omega(x)|^2|\omega(y)|\phi_i(x)\,dy\,dx\,dt \le$$
$$\frac{1}{R^2}\sup_t||\omega(t)||_{L^1}\int_0^T\int|\omega|^2\phi_i^{2\rho-1}\,dx\,dt.$$

For the first term (3), since

$$\big|\omega(x)\times\omega(y)\cdot G_\omega(x,y)\big| \le |\omega(x)||\omega(y)||\sin\varphi(\,\xi(x),\xi(y)\,)||G_\omega(x,y)| \le \frac{|\omega(x)|^2|\omega(y)|}{|x-y|^{5/2}},$$

we have

$$\int_0^T\int_{\{|\nabla u|>M\}}\Big|P.V.\int_{\{|x-y|<R^{2/3}\}}\omega(x)\times\omega(y)\cdot G_\omega(x,y)\,dy\Big|\phi_i(x)\,dx\,dt \le \tag{2.2.8}$$
$$\int_0^T\int_{\{|\nabla u|>M\}}\int_{\{|x-y|<R^{2/3}\}}\frac{|\omega(y)||\omega(x)|^2}{|x-y|^{5/2}}\phi_i(x)\,dy\,dx\,dt \le \tag{2.2.9}$$
$$c\int_0^T||\omega||_{L^2(B(x_i,2R+R^{2/3}))}||\,|\phi_i^{1/2}\omega|^2\,||_{3/2}dt \le \tag{2.2.10}$$
$$c\int_0^T||\omega||_{L^2(B(x_i,2R+R^{2/3}))}||\phi_i^{1/2}\omega||_2||\nabla(\phi_i^{1/2}\omega)||_2dt \le \tag{2.2.11}$$
$$c||\omega||_*\sup_t||\psi_i^{1/2}\omega||_2\Big(\int_0^T||\nabla(\phi_i^{1/2}\omega)||_2^2\,dt\Big)^{1/2} \le \tag{2.2.12}$$
$$c||\omega||_*\Big(\frac{1}{2}\sup_t||\psi_i^{1/2}\omega||_2^2+\int_0^T||\nabla(\phi_i^{1/2}\omega)||_2^2\,dt\Big) \le \tag{2.2.13}$$
$$c||\omega||_*\Big(\frac{1}{2}\sup_t||\psi_i^{1/2}\omega||_2^2+2\int_0^T\int|\nabla\omega|^2\phi_i\,dx\,dt+\frac{c}{2R^2}\int_0^T\int|\omega|^2\phi_i^{2\rho-1}\,dx\,dt\Big), \tag{2.2.14}$$

using $|\nabla(\phi_i^{1/2}\omega)|^2 \le 2|\nabla\omega|^2\phi_i + \frac{1}{2}\frac{|\nabla\phi_i|^2}{\phi_i}|\omega|^2 \le 2|\nabla\omega|^2\phi_i + \frac{c}{2R^2}|\omega|^2\phi_i^{2\rho-1}$ for the last inequality (14), and the Hardy-Littlewood-Sobolev inequality to reach (10), the Gagliardo-Nirenberg inequality to reach (11), and the Cauchy-Schwartz inequality to reach (12). Collecting the bounds on the various terms proves the lemma.

□

Finally, the main result can be proven. The ensemble average of enstrophy fluxes is shown to be bounded away from zero. In fact, it is proportional to the mean palinstrophy P_0. Let $\Phi = -\frac{1}{T}\int_0^T (u\cdot\nabla)\omega\cdot\omega\,\eta(t)dt$, the time-averaged enstrophy flux density. Recall that $\langle\Phi\rangle_R$ is then the ensemble average of time-averaged enstrophy fluxes.

Theorem 2.2.9. *Under Assumptions 1-4, for any K_1 and K_2 there exists K_* such that for any (K_1, K_2)-ensemble at scale R ranging from σ_0/β to R_0, we have* $\dfrac{1}{K_*}P_0 \leq \langle\Phi\rangle_R \leq K_* P_0$.

Proof. For an individual test function we have

$$F_i := \int_0^T \int \frac{1}{2}|\omega|^2 (u\cdot\nabla\phi_i)\,dx\,dt =$$

$$\Big(\sup_t \int \frac{1}{2}|\omega(x,t)|^2\psi_i(x)\,dx + \int_0^T \int |\nabla\omega|^2\phi_i\,dx\,dt\Big) -$$

$$\int_0^T \int \frac{1}{2}|\omega|^2(\partial_t\phi_i + \Delta\phi_i)\,dx\,dt - \int_0^T \int (\omega\cdot\nabla)u\cdot\phi_i\omega\,dx\,dt =: A_i - B_i - C_i.$$

Using Assumption 4 and Lemma 1, $\frac{1}{2K_1}P_0 \leq \frac{1}{nTR^3}\sum_1^n A_i \leq K_2 P_0$. Next, $|B_i| \leq \frac{c}{R^2}\int_0^T\int \frac{1}{2}|\omega|^2\phi_i^{2\rho-1}\,dx\,dt$ so $|\frac{1}{nTR^3}\sum_1^n B_i| \leq \frac{cK_2}{R^2}E_0 \leq cK_2\beta^2 P_0 \leq \frac{1}{8K_1}P_0$ for an appropriate choice of β.

Using the vortex stretching term lemma and Assumption 3,

$$|\frac{1}{nTR^3}\sum_1^n C_i| \leq (c + c'||\omega||)\frac{K_2}{R^2}E_0 + c''||\omega||K_2P_0 \leq$$

$$(c + c'R^{1-2/q}||\omega||_{L_t^2 M_x^{2,q}})\frac{K_2}{R^2}E_0 + c''R^{1-2/q}||\omega||_{L_t^2 M_x^{2,q}}K_2P_0 < \frac{1}{8K_1}P_0$$

Then $\langle F\rangle_R = \frac{1}{nTR^3}\sum_1^n F_i = \frac{1}{nTR^3}\sum_1^n A_i - \frac{1}{nTR^3}\sum_1^n B_i - \frac{1}{nTR^3}\sum_1^n C_i$ hence $\frac{1}{4K_1}P_0 \leq \langle F\rangle_R \leq (K_2 + \frac{1}{4K_1})P_0$

By Lemma 2, every (C_0, ρ, K_1, K_2) scale R ensemble average is equal to some $(C_0', \rho,$
$64K_1, 8K_2)$ scale σ_0/β ensemble average, which satisfies the desired inequalities.

□

The picture of a cascade is not complete without locality. In our context this means that enstrophy should only be transported between neighboring scales. In other words, we wouldn't want our inward enstrophy fluxes to be positive because the enstrophy jumps straight from the large scale to a very small scale, we would want it to be transported through all of the intermediate scales. Locality of flux was proven in the context of Besov spaces in [CCFS08] and other results on locality are shown in [LF92, EA09, Eyi05].

The result is stated in terms of the time-averaged enstrophy flux, rather than the time-averaged enstrophy flux per unit mass used above. That is,

$$\langle \Psi \rangle_R = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{T} \int_0^T \int \frac{1}{2}(|\omega|^2 + |j|^2)(u \cdot \nabla \phi_i)\, dx\, dt$$

for an ensemble $\{\phi_i\}_{i=1}^n$ at scale R. It follows immediately from Theorem 2.2.9.

Corollary 2.2.10. *Under Assumptions 1-4, for any K_1, K_2 there exists K_* such that for any r, R between σ_0/β and R_0 and any (K_1, K_2)-ensembles, enstrophy flux is local:*

$$\frac{1}{K_*^2}\Big(\frac{r}{R}\Big)^3 \leq \frac{\langle \Psi \rangle_r}{\langle \Psi \rangle_R} \leq K_*^2 \Big(\frac{r}{R}\Big)^3.$$

2.3 MHD Kinetic-Magnetic Enstrophy Cascade

In contrast to the vortex filaments that arise in hydrodynamic turbulence, the region where the vorticity and current are large tends to be made up of sheets in MHD turbulence [Bis03, YOK+13].

Current sheets are related to magnetic reconnection and dissipation of energy [ZUPB13]. A numerical study of kinetic equations was performed [KRW+13], in which an imposed velocity shear creates a Kelvin-Helmholtz instability to generate turbulent motion, which generates finer and finer structures down to electron scales. Within these fine current sheets the energy is dissipated as heat.

Conditions for the MHD equations are formulated under which the velocity concentrates kinetic and magnetic enstrophy towards smaller and smaller scales. This can be taken as mathematical evidence for the dynamic generation of current sheets. Once current sheets appear, the cascade observed in the simulation from [KRW+13] may continue, allowing their observations to be applied in a more general context. This work is an application of the modified ensemble averages to the kinetic-magnetic enstrophy cascade scenario which was originally studied in [BG13].

The same pattern used in the NSE enstrophy cascade will now be followed for the MHD equations. The mathematical theory of the MHD equations is similar to the Navier–Stokes equations. Most importantly, there is global-in-time existence of weak solutions and local-in-time existence of regular solutions.

Suppose u, b are the velocity and magnetic fields of a weak solution to the MHD equations on $\mathbb{R}^3$:

$$\begin{aligned} &\partial_t u - \Delta u + (u \cdot \nabla)u - (b \cdot \nabla)b + \nabla P = 0, \\ &\partial_t b - \Delta b + (u \cdot \nabla)b - (b \cdot \nabla)u = 0, \\ &\nabla \cdot u = \nabla \cdot b = 0. \end{aligned} \tag{2.3.1}$$

Spatially periodic solutions or solutions on a bounded domain would also work with some modifications in what follows. Taking the curl of the MHD equations yields equations for the vorticity $\omega = \nabla \times u$ and current $j = \nabla \times b$.

$$\begin{aligned} &\partial_t \omega - \Delta \omega = -(u \cdot \nabla)\omega + (\omega \cdot \nabla)u + (b \cdot \nabla)j - (j \cdot \nabla)b, \\ &\partial_t j - \Delta j = -(u \cdot \nabla)j + (j \cdot \nabla)u + (b \cdot \nabla)\omega - (\omega \cdot \nabla)b + 2\sum_{l=1}^{3} \nabla b_l \times \nabla u_l. \end{aligned} \tag{2.3.2}$$

The derivation of these equations, using vector calculus, can be found in the appendix. This particular way of writing the equations has a precedent in the mathematical literature.

The goal of this section is to prove that the effect of the velocity field on kinetic-magnetic enstrophy $||\omega||_2^2 + ||j||_2^2$ is to concentrate it on smaller and smaller scales, down to a minimum scale. More precisely, ensemble averages will be used to show that

$$\frac{1}{n}\sum_{1}^{n}\frac{1}{T}\int_0^T\frac{1}{R^3}\int\frac{1}{2}(|\omega|^2+|j|^2)(u\cdot\nabla\psi_{i,R})dx\,\eta(t)dt\sim P_0$$

for all ensembles at scale R with respect to a fixed ψ_0 for a range of R. A shortcoming of this approach is that one cannot say whether the kinetic enstrophy or magnetic enstrophy individually are being concentrated. Additionally, kinetic and magnetic enstrophy are not conserved, so that the various terms in the vorticity-current equations (2.3.2) may ultimately cancel out the concentrative effect of the transport term.

A number of assumptions will be needed to prove enstrophy concentration. The first assumption requires coherence of the vorticity direction in the region of large velocity gradients, just like the assumption used for the Navier–Stokes equations. Additionally, a form of continuity for the current is required in the region of large magnetic gradients. In numerical simulations of two-dimensional MHD turbulence, vorticity is less regular than current [Bis03].

Assumption 2.3.1.

$$|\sin\theta(\omega(x+y,t),\omega(x,t))|\le C_1|y|^{1/2} \tag{2.3.3}$$

for every $t\in(0,T)$, every $x\in B(0,2R_0+R_0^{2/3})$ with $|\nabla u(x,t)|>M$, and every y such that $|y|<2(\sigma_0/\beta)+(\sigma_0/\beta)^{2/3}$, and

$$|j(x+y,t)-j(x,t)|\le|j(x+y,t)||y|^{1/2} \tag{2.3.4}$$

for every $t\in(0,T)$, every $x\in B(0,2R_0+R_0^{2/3})$ with $|\nabla b(x,t)|>M$, and every y such that $|y|<2(\sigma_0/\beta)+(\sigma_0/\beta)^{2/3}$.

The region of large magnetic gradient and the region of large current roughly coincide, so that equation (2.3.4) is approximately $|j(x+y,t)-j(x,t)|\le M|y|^{1/2}$. That is, this assumption is almost that the current is 1/2-Hölder continuous in the region of large magnetic gradient.

Define the macro-scale (kinetic and magnetic) energy e_0, enstrophy E_0, and palinstrophy P_0 as follows:

$$e_0 = \frac{1}{T}\int_0^T \frac{1}{R_0^3}\int \phi_0^{4\rho-3}\Big(\frac{|u|^2}{2}+\frac{|b|^2}{2}\Big)\,dx\,dt,$$
$$E_0 = \frac{1}{T}\int_0^T \frac{1}{R_0^3}\int \phi_0^{2\rho-1}(|\omega|^2+|j|^2)\,dx\,dt,$$
$$P_0 = \frac{1}{T}\int_0^T \frac{1}{R_0^3}\int \phi_0(|\nabla\omega|^2+|\nabla j|^2)\,dx\,dt + \frac{1}{TR_0^3}\int \frac{1}{2}(|\omega(x,T)|^2+|j(x,T)|^2)\psi_0\,dx,$$

and let the modified Kraichnan-type scale σ_0 be

$$\sigma_0 = \max\Big\{\Big(\frac{E_0}{P_0}\Big)^{1/2}, \Big(\frac{e_0}{P_0}\Big)^{1/4}\Big\}. \tag{2.3.5}$$

Assumption 2.3.2. The Kraichnan-type scale satisfies $\sigma_0 < \beta R_0$, where β is a constant between 0 and 1.

Assumption 2.3.3.

$$\int_0^T \int_{B(y,R)} |\omega|^2 + |j|^2\,dx\,dt < \frac{1}{C} \tag{2.3.6}$$

for any $y \in B(0, 2R_0)$ where C is a constant and $R = 2\sigma_0/\beta + (\sigma_0/\beta)^{2/3}$.

For a turbulent flow, there will be high spatial complexity, so that the palinstrophy will be large compared to the enstrophy and energy. The more spatial complexity there is, the smaller the Kraichnan-type scale is. So this is at least a feasible assumption for a turbulent flow. This assumption could also be stated in terms of a Morrey norm as was done in the Navier–Stokes enstrophy cascade above.

The final assumption requires that the enstrophy and current do not drop off too much at time T.

Assumption 2.3.4.

$$\int |\omega(x,T)|^2\psi_0(x)\,dx \ge \frac{1}{2}\sup_t \int |\omega(x,t)|^2\psi_0(x)\,dx,$$
$$\int |j(x,T)|^2\psi_0(x)\,dx \ge \frac{1}{2}\sup_t \int |j(x,t)|^2\psi_0(x)\,dx,$$

The enstrophy flux equation is proven by multiplying the vorticity equation (2.3.2) by $\phi\omega$ and integrating in space and time. The current flux equation is proven by multiplying the current equation (2.3.2) by ϕj and integrating in space and time. It is very similar to Lemma 2.2.5 and the proof can be found in [BG13].

Lemma 2.3.5. *For a solution (u,b) to the MHD equations,*

$$\begin{aligned}\int_0^T\int \frac{1}{2}|\omega|^2(u\cdot\nabla\phi)\,dx\,dt &= \int\frac{1}{2}|\omega(x,T)|^2\psi(x)\,dx+\int_0^T\int|\nabla\omega|^2\phi\,dx\,dt\\ &-\int_0^T\int\frac{1}{2}|\omega|^2(\partial_s\phi+\Delta\phi)\,dx\,dt-\int_0^T\int(\omega\cdot\nabla)u\cdot(\phi\omega)\,dx\,dt\\ &-\int_0^T\int(b\cdot\nabla)j\cdot(\phi\omega)\,dx\,dt+\int_0^T\int(j\cdot\nabla)b\cdot(\phi\omega)\,dx\,dt\end{aligned}\tag{2.3.7}$$

$$\begin{aligned}\int_0^T\int \frac{1}{2}|j|^2(u\cdot\nabla\phi)\,dx\,dt &= \int\frac{1}{2}|j(x,T)|^2\psi(x)\,dx+\int_0^T\int|\nabla j|^2\phi\,dx\,dt\\ &-\int_0^T\int\frac{1}{2}|j|^2(\partial_s\phi+\Delta\phi)\,dx\,dt+\int_0^T\int(\omega\cdot\nabla)b\cdot(\phi j)\,dx\,dt\\ &-\int_0^T\int(b\cdot\nabla)\omega\cdot(\phi j)\,dx\,dt-\int_0^T\int(j\cdot\nabla)u\cdot(\phi j)\,dx\,dt\\ &-\int_0^T\int\Big(2\sum_{l=1}^{3}\nabla u_l\times\nabla b_l\Big)\cdot(\phi j)\,dx\,dt\end{aligned}\tag{2.3.8}$$

We need to obtain bounds on the last four terms of (2.3.7) and the last five terms of (2.3.8). Refer to them as $H^\omega, N_1^\omega, N_2^\omega, L^\omega, H^j, N_1^j, N_2^j, L^j$, and X, respectively.

Lemma 2.3.6.

$$\begin{aligned}&H^\omega+H^j+N_1^\omega+N_2^\omega+N_1^j+N_2^j+L^\omega+L^j+X\le\\ &\quad K_P\Big(\frac{1}{\alpha}+||\omega||_{L^2((0,T)\times B(x_i,2R+R^{2/3}))}\Big)\Big(\frac{1}{2}\sup_{t\in(0,T)}\int\psi(x)(|\omega(x,t)|^2+|j(x,t)|^2)\,dx+\\ &\quad\int_0^T\int\phi(|\nabla\omega|^2+|\nabla j|^2)\,dx\,dt\Big)+\frac{K_E}{R^2}\int_0^T\int\phi^{2\rho-1}(|\omega|^2+|j|^2)\,dx\,dt+\\ &\quad\frac{\alpha^2K_e}{R^4}\int_0^T\int\phi^{4\rho-3}\frac{|u|^2+|b|^2}{2}\,dx\,dt,\end{aligned}$$

The proof of the bounds can be found in [BG13].

Theorem 2.3.7. *Under Assumptions 1-4, for any* K_1 *and* K_2 *there exists* K_* *such that for any* (K_1, K_2)*-ensemble at scale* R *ranging from* σ_0/β *to* R_0*, we have* $\dfrac{1}{K_*}P_0 \leq \langle\Phi\rangle_R \leq K_*P_0$.

Proof. We start by showing that for $R = \sigma_0/\beta$, for $(64K_1, 8K_2)$-ensembles satisfying C_0' bounds, we have $\dfrac{1}{K_*}P_0 \leq \langle\Phi\rangle_R \leq K_*P_0$.

Referring to equations 2.3.7 and 2.3.8, $\langle\Phi\rangle_R = \langle P\rangle_R + \langle H+N+L+X\rangle_R$, where p is the positive density $\frac{1}{2}|\omega(T)|^2 + \int_0^T |\nabla\omega|^2\eta\, dt$ so that $\frac{1}{64K_1}P_0 \leq \langle P\rangle_R \leq 8K_2P_0$. The rest of the terms are relatively small upon averaging:

$$|H+N+L+X| \leq K_P\Big(\frac{1}{\alpha} + ||\omega||_{L^2(B(x_i,2R+R^{2/3})\times(0,T))}\Big)\tilde{P} + \frac{K_E}{R^2}E + \frac{\alpha^2K_e}{R^4}e,$$

so that

$$\begin{aligned}
|\langle H+N+L+X\rangle_R| &\leq \Big\langle K_P\big(\frac{1}{\alpha} + ||\omega||\big)\tilde{P} + \frac{K_E}{R^2}E + \frac{\alpha^2K_e}{R^4}e\Big\rangle_R \\
&\leq 8K_2K_P\Big(\frac{1}{\alpha} + ||\omega||\Big)\tilde{P}_0 + \frac{8K_2K_E}{R^2}E_0 + \frac{\alpha^2 8K_2K_e}{R^4}e_0 \\
&= \Big(8K_2K_P\big(\frac{1}{\alpha} + ||\omega||\big) + 8K_2K_E\beta^2 + \alpha^2 8K_2K_e\beta^4\Big)P_0 \\
&\leq \frac{3}{4\cdot 64K_1}P_0,
\end{aligned}$$

by taking $\alpha = 4{\cdot}64K_1{\cdot}8K_2K_P$, using that β is sufficiently small such that $8K_2K_E\beta^2 + \alpha^2\cdot 8K_2K_e\beta^4 \leq \frac{1}{4\cdot 64K_1}$, and using the assumption that $||\omega||_{L^2(B(x_i,2R+R^{2/3})\times(0,T))} \leq \frac{1}{C_2} = \frac{1}{4\cdot 64K_1\cdot 8K_2K_P}$.

Therefore $\dfrac{1}{4\cdot 64K_1}P_0 \leq \langle\Phi\rangle_R \leq (8K_2 + \dfrac{3}{4\cdot 64K_1})P_0$ for any $(64K_1, 8K_2, C_0')$-ensemble at scale $R = \sigma_0/\beta$. For the range of scales from σ_0/β up to R_0, by Lemma 2.1.6, $\dfrac{1}{4\cdot 64K_1}P_0 \leq \langle\Phi\rangle_R \leq (8K_2 + \dfrac{3}{4\cdot 64K_1})P_0$ for all scale R (K_1, K_2, C_0)-ensembles.

□

Again, locality of kinetic-magnetic enstrophy is immediate for the time-averaged enstrophy flux

$$\langle \Psi \rangle_R = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{T} \int_0^T \int \frac{1}{2} (|\omega|^2 + |j|^2)(u \cdot \nabla \phi_i) \, dx \, dt$$

for an ensemble $\{\phi_i\}_{i=1}^n$ at scale R.

Corollary 2.3.8. *Under Assumptuions 1-4, for any K_1, K_2 there exists K_* such that for any r, R between σ_0/β and R_0 and any (K_1, K_2)-ensembles, enstrophy flux is local:*

$$\frac{1}{K_*^2} \Big(\frac{r}{R}\Big)^3 \leq \frac{\langle \Psi \rangle_r}{\langle \Psi \rangle_R} \leq K_*^2 \Big(\frac{r}{R}\Big)^3.$$

2.4 Appendix

The vorticity and current equations (2.3.2) can be derived from the MHD equations as follows.

For any divergence free vector fields A, B and scalar field ψ, the following identities hold:

$$\nabla_B (A \cdot B) = (A \cdot \nabla) B + A \times (\nabla \times B), \tag{2.4.1}$$

$$\nabla \times (A \times B) = (B \cdot \nabla) A - (A \cdot \nabla) B, \tag{2.4.2}$$

$$\nabla \times (\psi A) = \psi (\nabla \times A) + \nabla \psi \times A, \tag{2.4.3}$$

$$\nabla \times (\nabla \psi) = 0, \tag{2.4.4}$$

where (2.4.1) uses the notation $(\nabla_B (A \cdot B))_i = \sum_{l=1}^3 A_l \partial_i B_l$.

Taking the curl of the MHD equations yields

$$\begin{aligned} &\partial_t \omega - \Delta \omega + \nabla \times ((u \cdot \nabla) u) - \nabla \times ((b \cdot \nabla) b) = 0, \\ &\partial_t j - \Delta j + \nabla \times ((u \cdot \nabla) b) - \nabla \times ((b \cdot \nabla) u) = 0. \end{aligned} \tag{2.4.5}$$

Now $(u \cdot \nabla)b = \sum_l u_l \nabla b_l - u \times j$ by (2.4.1). So

$$\nabla \times ((u \cdot \nabla)b) = \nabla \times \big(\sum_l u_l \nabla b_l\big) - \nabla \times (u \times j).$$

By (2.4.3) and (2.4.4),

$$\nabla \times (u_l \nabla b_l) = u_l(\nabla \times \nabla b_l) + \nabla u_l \times \nabla b_l = \nabla u_l \times \nabla b_l.$$

Next, (2.4.2) gives that $\nabla \times (u \times j) = (j \cdot \nabla)u - (u \cdot \nabla)j$. In total, we have that

$$\nabla \times ((u \cdot \nabla)b) = (u \cdot \nabla)j - (j \cdot \nabla)u + \sum_l \nabla u_l \times \nabla b_l. \tag{2.4.6}$$

Similarly,

$$\nabla \times ((b \cdot \nabla)u) = (b \cdot \nabla)\omega - (\omega \cdot \nabla)b - \sum_l \nabla u_l \times \nabla b_l, \tag{2.4.7}$$

$$\nabla \times ((u \cdot \nabla)u) = (u \cdot \nabla)\omega - (\omega \cdot \nabla)u + \sum_l \nabla u_l \times \nabla u_l = (u \cdot \nabla)\omega - (\omega \cdot \nabla)u, \tag{2.4.8}$$

and

$$\nabla \times ((b \cdot \nabla)b) = (b \cdot \nabla)j - (j \cdot \nabla)b. \tag{2.4.9}$$

By using equations (2.4.6)–(2.4.9) on equation (2.4.5), we arrive at the desired equations

$$\partial_t \omega - \Delta\omega = -(u \cdot \nabla)\omega + (\omega \cdot \nabla)u + (b \cdot \nabla)j - (j \cdot \nabla)b,$$

$$\partial_t j - \Delta j = -(u \cdot \nabla)j + (j \cdot \nabla)u + (b \cdot \nabla)\omega - (\omega \cdot \nabla)b + 2\sum_{l=1}^{3} \nabla b_l \times \nabla u_l.$$

Chapter 3

Regularity Criterion

3.1 Introduction

Consider the Navier–Stokes equations' scaling symmetry $u(x,t) \to \lambda u(\lambda x, \lambda^2 t)$, with initial data scaling as $u_0(x) \to \lambda u_0(\lambda x)$. The energy scales as $||\lambda u_0(\lambda x)||_{L^2} = \lambda^{-1/2}||u_0(x)||_{L^2}$, and some other examples of norms scale as $||\lambda u_0(\lambda x)||_{L^3} = ||u_0(x)||_{L^3}$ and $||\lambda u_0(\lambda x)||_{L^\infty} = \lambda ||u_0(x)||_{L^\infty}$. As the L^3 norm is invariant under the scaling, it is called a critical space. The L^∞ norm is subcritical and the L^2 norm is supercritical. "Zoomed-in" functions have smaller norms in a subcritical space and larger norms in a supercritical space. Whereas different scales are on equal footing in a critical space, in a supercritical space small scales contribute less to the norm. That is, supercritical norms have less control of small scales, which are important for regularity.

As seen in Chapter 1, there are many sufficient conditions for a weak solution of the Navier–Stokes equations to be smooth. They are all critical or subcritical. On the other hand, the spaces where global-in-time solutions are known to exist, even for large initial data, are supercritical.

Local-in-time existence and regularity of weak solutions is known for initial data in the critical spaces L^3 and $H^{1/2}$, among others [ESS03, KT01]. All such spaces are contained in $B^{-1}_{\infty,\infty}$. The initial value problem in $B^{-1}_{\infty,\infty}$ is ill-posed in the sense of norm inflation [CS10, BP08].

In this chapter, which is based on joint work with Zoran Grujić and Aseel Farhat [FGL16], a regularity criterion will be proven in the context of mild solutions with L^∞ initial data.

It is known that if a mild solution is in $L^\infty((0,T)\times\mathbb{R}^3)$, it is regular [LR02]. In other words, a mild solution cannot have a discontinuity unless there is a singularity. The result of this chapter excludes the possibility of a singularity while the $B^{-1}_{\infty,\infty}$ norm is uniformly small.

A similar result may be found in [CS10]. Their result is for Leray solutions, while ours does not require integrability, or even decay at infinity. Cheskidov and Shvydkoy also draw connections between Besov spaces like $B^{-1}_{\infty,\infty}$ and turbulence [CS].

The chapter will begin with the definition of Besov spaces and recalling relevant theorems, which will allow a precise statement of the main theorem. Then we will state theorems about existence and analyticity of mild solutions, which provide the setting for the conditional regularity theorem. The next section goes over tools from potential theory, allowing us to interpolate between two bounds. Then the semi-mixed lemma is proven, which translates the small Besov norm condition into a geometric condition. Finally, the analyticity of mild solutions, interpolation theorem, and semi-mixed lemma are combined to prove the main result.

The semi-mixed lemma complements the ensemble averages used in the previous chapter. The ensemble averages were based on values of a function f over a ball of a certain radius to capture the concept of scale, in contrast to the standard method of taking the Fourier transform. For the semi-mixed lemma we start with a function that has an upper bound on the scaling behavior of its frequency components, to obtain information about the values of u in a ball of a certain radius. In one case the function is averaged over the ball, and in the other the measure of the set within the ball where the function is large is of interest.

In this chapter, for convenience, the norm of a vector function $u = (u_1, u_2, u_3)$

with components in a normed space X is defined as $||u||_{X^3} = \max_{i=1,2,3} ||u_i||_X$.

3.2 Besov spaces

The Littlewood-Paley decomposition is a way to break a function into frequency components. It uses a partition of unity composed of bump functions supported on dyadic shells. For proofs, see [BCD11].

Theorem 3.2.1. *There exist smooth radial functions ψ, ϕ such that ψ is supported on $B(0, 4/3)$, ϕ is supported on $\{x \in \mathbb{R}^3 : 3/4 \leq |x| \leq 8/3\}$, $\phi(x)$ and $\psi(x)$ take values between 0 and 1, and $\psi(x) + \sum_{j\geq 0} \phi(2^{-j}x) = 1$.*

Let f be a tempered distribution. Define the Littlewood-Paley operator Δ_j by $\Delta_j f = \mathscr{F}^{-1}(\phi(2^{-j}\xi)\mathscr{F}f(\xi))$ for $j \geq 0$ and $\Delta_{-1} f = \mathscr{F}^{-1}(\psi(\xi)\mathscr{F}f(\xi))$. The block $\Delta_j f$ has its Fourier transform supported on a shell, and f can be reconstructed from these blocks:

Theorem 3.2.2. *Let f be a tempered distribution. Then $f = \sum_{j\geq -1} \Delta_j f$, where the infinite sum converges in the topology of tempered distributions.*

Now by imposing bounds on the asymptotic growth or decay of $\Delta_j f$, we get a family of function spaces called Besov spaces.

Definition 3.2.3. The *inhomogeneous Besov space* $B^s_{p,r}$ is the space of tempered distributions f such that

$$||f||_{B^s_{p,r}} = \Big(\sum_{j=-1}^{\infty} 2^{jsr} ||\Delta_j f||_p^r \Big)^{1/r} < \infty,$$

for $1 \leq p \leq \infty, 1 \leq r < \infty, s \in \mathbb{R}$. For $r = \infty$, it is instead required that

$$||f||_{B^s_{p,\infty}} = \sup_{j\geq -1} 2^{js} ||\Delta_j f||_p < \infty.$$

.

These Besov spaces are all Banach spaces. They include many familiar spaces, including the Lebesgue Spaces $L^p = B^0_{p,2}$ $(1 \leq p < \infty)$, fractional Sobolev spaces $H^s = B^s_{2,2}$, and Hölder spaces $C^{0,\alpha} = B^{\alpha}_{\infty,\infty}$ $(0 < \alpha < 1)$. Functions of (uniformly locally) bounded mean oscillation, which include the uniformly bounded functions, are a subset of a Besov Space: $L^\infty \subset bmo \subset B^0_{\infty,\infty}$.

Any Besov space $B^s_{p,r}$ may be characterized as the dual space of the closure of Schwartz functions in $B^{-s}_{p',r'}$, where $\frac{1}{p} + \frac{1}{p'} = \frac{1}{r} + \frac{1}{r'} = 1$ [LR02]. This fact will be used to relate the size of the Besov norm to sparseness of super-level sets.

Theorem 3.2.4. *$B^{-1}_{\infty,\infty}$ may be identified with the dual space of the closure of Schwartz functions in $B^1_{1,1}$. In particular,*

$$||f||_{B^{-1}_{\infty,\infty}} \approx \sup_{\{\phi \in \mathcal{S} : ||\phi||_{B^1_{1,1}} \leq 1\}} \langle f, \phi \rangle.$$

3.3 Mild solutions

We may now state the main result.

Theorem 3.3.1. *Let u be a mild solution to the NSE on $(0,T)$ with L^∞ initial data. There is an absolute constant m_0 such that if $\sup_{t\in(T-\epsilon,T)} ||u(t)||_{B^{-1}_{\infty,\infty}} \leq m_0$ for some $\epsilon > 0$, then T is not a blow-up time.*

That mild solutions corresponding to L^∞ initial data exist was shown in [GIM98]. Theorem 3.3.1 is a property of these solutions.

Theorem 3.3.2. *For any $u_0 \in L^\infty$ there exists a time $T^* \in [\frac{1}{c_0^2||u_0||_\infty^2}, \infty]$ and a (unique) mild solution u on $(0,T^*)$ with initial data u_0 such that either $||u(t)||_\infty \to \infty$ as $t \to T^*$ or $T^* = \infty$. If $T^* < \infty$ it is called a blow-up time.*

The only way for the solution to not be a global-in-time solution would be if the L^∞ norm blew up in finite time. According to Theorem 3.3.1, the smaller $B^{-1}_{\infty,\infty}$ norm would also have to blow up. These solutions are analytic in space.

Theorem 3.3.3. *Let u be a mild solution with initial data $u_0 \in L^\infty$. Let $\gamma \in (0,1)$, $T = \frac{\gamma^2}{c^2||u_0||^2}$ and fix $t \in (0,T]$. Then $u(t)$ has an analytic continuation to $\mathscr{R}_t = \{x + iy \in \mathbb{C}^3 : |y| \le \frac{\gamma}{c}\sqrt{t}\}$, and $||u(t)||_{L^\infty(\mathscr{R}_t)} \le (1+\gamma)||u_0||_\infty$.*

This theorem is proven in [Gub10] with γ replaced by a constant greater than 1. This version is obtained simply by replacing certain constants appearing in the proof by smaller constants.

3.4 Harmonic measure

The technique of using the harmonic measure majorization principle on solutions of the Navier–Stokes equations originated in [Gru13]. Due to the limitations of the semi-mixed lemma (Lemma 3.5.3), we will be using the principle on the positive part of the analytic continuation of the solution, which will correspond to a subharmonic function.

Definition 3.4.1. Let $U \subset \mathbb{R}^2$ be an open set. A continuous function $f : U \to \mathbb{R}$ is *subharmonic* if it satisfies the local submean inequality. That is, for every $x_0 \in U$ there exists $\rho > 0$ such that

$$f(x_0) \le \frac{1}{2\pi} \int_0^{2\pi} f(x_0 + re^{i\theta}) d\theta$$

for all $r < \rho$. The notation $e^{i\theta}$ is used to mean $(\cos\theta, \sin\theta) \in \mathbb{R}^2$.

Remark 3.4.2. Harmonic functions are subharmonic, and the maximum of two subharmonic functions is subharmonic.

Harmonic measure comes into play as a way to interpolate between two bounds m and M on different parts of a subharmonic function. For suitable sets $\Omega, K \subset \mathbb{R}^2$, there exists a unique harmonic function taking boundary values 1 on K and zero on the part of the boundary of Ω outside of K. In the following theorems, $\omega(z, \Omega, K)$ denotes the evaluation of that function at the point $z \in \Omega$.

Theorem 3.4.3. *[Ran95] Let Ω be an open connected set in $\mathbb{R}^2$ such that its boundary $\partial\Omega$ has nonzero Hausdorff dimension, and let K be a Borel subset of $\partial\Omega$. If u is subharmonic on Ω and satisfies $u(z) \leq M$ for $z \in \Omega$ and $\limsup_{z\to\zeta} u(z) \leq m$ for $\zeta \in K$, then $u(z) \leq m\omega(z, \Omega, K) + M(1 - \omega(z, \Omega, K))$ for $z \in \Omega$.*

The following result is an explicit formula for the harmonic measure for certain Ω and K.

Theorem 3.4.4. *[Sol99] Let $\mathbb{D}$ be the unit disc and let K be a closed subset of $[-1, 1]$ such that $|K| = 2\lambda$ for some $\lambda \in (0, 1)$ and suppose that 0 is in $\mathbb{D} \setminus K$. Then*

$$\omega(0, \mathbb{D}, K) \geq \omega(0, \mathbb{D}, K_\lambda) = \frac{2}{\pi} \arcsin \frac{1 - (1 - \lambda)^2}{1 + (1 - \lambda)^2},$$

where $K_\lambda = [-1, -1 + \lambda] \cup [1 - \lambda, 1]$.

Now we turn to packaging the above theorems into a lemma ready for use in the proof of the main result. The lemma, which is a modification of one from [Gru13], starts with an assumption of 1D δ-sparseness.

Let $S \subset \mathbb{R}^3$ be an open set, x_0 a point in $\mathbb{R}^3$, $r > 0$ and $\delta \in (0, 1.)$

Definition 3.4.5. S is 1D δ-sparse around x_0 at scale r if there exists a unit vector d such that

$$\frac{m(S \cap (x_0 - rd, x_0 + rd))}{2r} \leq \delta.$$

The lemma uses sparseness of the two functions $f^\pm$ (where $f^+(x) = \max(f(x), 0)$ and $f^-(x) = \max(-f(x), 0)$), rather than the stronger condition of sparseness of f.

Lemma 3.4.6. *If $f : \mathbb{R}^3 \to \mathbb{R}$ is analytic at x_0 with radius r, where each of the two sets $\{x \in \mathbb{R}^3 : f^+(x) > m\}$ and $\{x \in \mathbb{R}^3 : f^-(x) > m\}$ are 1D δ-sparse at x_0 at scale r for some $\delta \in (0, 1)$, then $|f(x_0)| \leq mh + M(1-h)$ where $h = h(\delta) = \frac{2}{\pi} \arcsin(\frac{1-\delta^2}{1+\delta^2})$, $M = \sup_{z\in B_r} |f(z)|$, B_r is the complex ball of radius r centered at x_0, and $m \leq M$.*

Proof. If $|f(x_0)| \leq m$ we are done. Otherwise, the point x_0 belongs to either $\Omega^+ = \{x \in \mathbb{R}^3 : f^+(x) > m\}$ or $\Omega^- = \{x \in \mathbb{R}^3 : f^-(x) > m\}$; suppose it is the former. Let d be the direction of δ-sparseness for Ω^+ at x_0. The function f restricted to $(x_0 - rd, x_0 + rd)$ is analytic as a single variable function. This single variable function, which we will continue to call f, has a harmonic extension to a disc $B(x_0, r) \subset \mathbb{R}^2$. Now f^+ is subharmonic on the same set, bounded above by M. Let $K = [x_0 - rd, x_0 + rd] \setminus \Omega^+$ and $\Omega = B(x_0, r) \setminus K$. By [Ransford thm], $f^+(x_0) \leq m\omega(x_0, \Omega, K) + M(1 - \omega(x_0, \Omega, K))$. The harmonic measure is invariant under conformal mappings [Ahlfors], so by rotation, rescaling, and translation, $\omega(x_0, \Omega, K) = \omega(0, \tilde{\Omega}, \tilde{K})$, where $\tilde{\Omega} = B(0, 1) \setminus \tilde{K}$ and $0 \notin \tilde{K} \subset [-1, 1]$, with $m(\tilde{K}) \geq 2(1 - \delta)$. By [Solynin theorem], $\omega(0, \tilde{\Omega}, \tilde{K}) \geq \frac{2}{\pi} \arcsin \frac{1-\delta^2}{1+\delta^2}$. Thus

$$\begin{aligned} f^+(x_0) &\leq M + (m - M)\omega(x_0, \Omega, K) \\ &\leq M + (m - M)\frac{2}{\pi} \arcsin \frac{1 - \delta^2}{1 + \delta^2} \\ &= mh + M(1 - h). \end{aligned}$$

The same reasoning shows $f^-(x_0) \leq mh + M(1 - h)$, in the case that $x_0 \in \Omega^-$.

□

3.5 Mixing

We arrive at 1D δ-sparseness by proving much more. Lemma 3.5.3 will prove super-level sets are semi-mixed.

Definition 3.5.1. Let $r > 0$. An open set S is r-semi-mixed if

$$\frac{m(S \cap B(x, r))}{m(B(x, r))} \leq \delta$$

for every $x \in \mathbb{R}^3$, and for some $\delta \in (0, 1)$.

Remark 3.5.2. If the set S is r-semi-mixed (with the ratio δ), then it is 1D $\delta^{1/3}$-sparse around every point $x_0 \in \mathbb{R}^3$ at scale r.

The lemma is a vector valued, Besov space version of a result from [IKX14]. The super-level sets of the positive and negative parts of the three components of a vector function are considered separately. The lemma from the previous section accommodates this separation.

Lemma 3.5.3. *Let $f \in L^\infty(\mathbb{R}^3)^3$, $\lambda \in (0,1)$ and $\delta \in (\frac{1}{1+\lambda}, 1)$. There exists a constant $c = c(\lambda, \delta)$ such that for any $r < 1$ satisfying $||f||_{B^{-1}_{\infty,\infty}} \leq cr||f||_\infty$, each of the sets $A^{i,\pm}_\lambda = \{x \in \mathbb{R}^3 : f^\pm_i > \lambda||f||_\infty\}$ is r-semi-mixed with ratio δ.*

Proof. The claim will be proven for the set $A^{i,+}_\lambda$. The proof for $A^{i,-}_\lambda$ is the same except the function ϕ is replaced by $-\phi$. Let ψ be a smooth function equal to 1 on $B(0,1)$, satisfying $0 \leq \psi(x) \leq 1$ for all x, and vanishing outside $B(0, 1+\eta)$, for $\eta > 0$ to be specified later. Then the dilation and translation $\phi(x) = \psi(\frac{x}{r} - x_0)$ is supported on $B(x_0, (1+\eta)r)$ with $||\phi||_{B^1_{1,1}} \leq ||\phi||_{L^1} + ||\phi||_{\dot{B}^1_{1,1}} \leq r^2(||\psi||_{L^1} + ||\psi||_{\dot{B}^1_{1,1}}) = c(\psi)r^2$. We have

$$
\begin{aligned}
cr||f||_\infty \geq ||f||_{B^{-1}_{\infty,\infty}} \geq ||f_i||_{B^{-1}_{\infty,\infty}} &\geq \frac{1}{||\phi||_{B^1_{1,1}}}\Big|\int f_i(x)\phi(x)\,dx\Big| \\
&\geq \frac{1}{c(\psi)r^2}\Big|\int f_i(x)\phi(x)\,dx\Big|.
\end{aligned}
$$

Write $\left|\int f_i(x)\phi(x)\,dx\right| \geq ||I| - |II| - |III||$,

where

$$
\begin{aligned}
I &= \int_{A^{i,+}_\lambda \cap B(x_0,r)} f_i(x)\phi(x)\,dx, \\
II &= \int_{B(x_0,r)\setminus A^{i,+}_\lambda} f_i(x)\phi(x)\,dx,
\end{aligned}
$$

and

$$
III = \int_{B(x_0,(1+\eta)r)\setminus B(x_0,r)} f_i(x)\phi(x)\,dx.
$$

Then

$$
\begin{aligned}
I = \int_{A^{i,+}_\lambda \cap B(x_0,r)} f_i(x)\,dx &= \int_{A^{i,+}_\lambda \cap B(x_0,r)} f^+_i(x)\,dx \\
&> \lambda\,\|f\|_\infty\, m(A^{i,+}_\lambda \cap B(x_0,r)),
\end{aligned}
$$

$$II = \int_{B(x_0,r)\setminus A_\lambda^{i,+}} f_i(x)\,dx \le \|f\|_\infty \left(m(B(x_0,r) - m(A_\lambda^{i,+} \cap B(x_0,r))\right),$$

and

$$\begin{aligned} III &\le \int_{B(x_0,(1+\eta)r)\setminus B(x_0,r)} f_i(x)\,dx \\ &\le \|f\|_\infty \left(m(B(x_0,(1+\eta)r) - m(B(x_0,r))\right). \end{aligned}$$

Therefore

$$\Big|\int f_i(x)\phi(x)\,dx\Big| \ge ||f||_\infty \Big((1+\lambda)m(A_\lambda^{i,+} \cap B(x_0,r)) - m(B(x_0,(1+\eta)r))\Big),$$

so that

$$cr||f||_\infty \ge \frac{1}{c(\psi)r^2}||f||_\infty \Big((1+\lambda)m(A_\lambda^{i,+} \cap B(x_0,r)) - m(B(x_0,(1+\eta)r))\Big).$$

This implies

$$c(\lambda,\delta)c(\psi)r^3 \ge (1+\lambda)m(A_\lambda^{i,+} \cap B(x_0,r)) - m(B(x_0,(1+\eta)r))$$

or

$$(1+\eta)^3 m(B(x_0,r)) + c(\lambda,\delta)c(\psi)r^3 \ge (1+\lambda)m(A_\lambda^{i,+} \cap B(x_0,r)),$$

$$(1+\eta)^3 + c(\lambda,\delta)c(\psi) \ge (1+\lambda)\frac{m(A_\lambda^{i,+} \cap B(x_0,r))}{m(B(x_0,r))}.$$

Take η such that $(1+\eta)^3 = \frac{\delta(1+\lambda)+1}{2}$. Now η is determined by δ and λ, so we may fix ψ and choose $c(\lambda,\delta)$ so that it satisfies $c(\lambda,\delta)c(\psi) < \frac{\delta(1+\lambda)-1}{2}$. Then we have that

$$\frac{m(A_\lambda^{i,+} \cap B(x_0,r))}{m(B(x_0,r))} \le \delta.$$

This is satisfied for any x_0, so $A_\lambda^{i,+}$ is r-semi-mixed with ratio δ. $\square$

3.6 Proof of main result

Definition 3.6.1. Suppose u is a mild solution with initial data $u_0 \in L^\infty$ that blows up at time T. An *escape time* is a time t in $(0,T)$ such that $||u(t)||_\infty < ||u(\tau)||_\infty$ for all τ in (t,T).

Remark 3.6.2. A mild solution that blew up would have continuum many escape times due to local-in-time well-posedness in L^∞.

Theorem (Main result)**.** *Let u be a mild solution to the NSE on $(0,T)$ with L^∞ initial data. There is an absolute constant m_0 such that if $\sup_{t\in(T-\epsilon,T)} ||u(t)||_{B^{-1}_{\infty,\infty}} \leq m_0$ for some $\epsilon > 0$, then T is not a blow-up time.*

Proof. Suppose T were a blow-up time. Then there would exist an escape time $t \in (T-\epsilon, T)$ such that $||u(t)||_\infty > 1$. The small constant γ will be specified later ($\gamma = \frac{1}{50}$ is sufficient). Let $s(t)$ be any time in $[t + \frac{\gamma^2}{4c^2||u(t)||_\infty^2}, t + \frac{\gamma^2}{c^2||u(t)||_\infty^2}] \subset (T-\epsilon, T)$. Consider the six super-level sets

$$\Omega_i^\pm = \{x : u_i^\pm(x, s(t)) > \frac{1}{2}||u(s(t)||_\infty\}.$$

By Lemma 3.5.3, with $\delta = \frac{3}{4}$ and $\lambda = \frac{1}{2}$, there exists a constant c_* such that each $\Omega_i^\pm$ is r-semi-mixed with ratio $\frac{3}{4}$ for any $r < 1$ satisfying

$$||u(s(t))||_{B^{-1}_{\infty,\infty}} \leq c_* r ||u(s(t))||_\infty.$$

Define $m_0 = \frac{c_*\gamma^2}{2c^2}$ (the c in the denominator is the constant from Theorem 3.3.3). Taking $r = \frac{\gamma^2}{2c^2||u(s(t))||_\infty}$, where we have assumed that $||u(s(t))||_{B^{-1}_{\infty,\infty}} \leq \frac{c_*\gamma^2}{2c^2}$, we have that each $\Omega_i^\pm$ is $\frac{\gamma^2}{2c^2||u(s(t))||_\infty}$-semi-mixed with ratio $\frac{3}{4}$. This implies that each $\Omega_i^\pm$ is 1D $(\frac{3}{4})^{1/3}$-sparse at scale $r = \frac{\gamma^2}{2c^2||u(s(t))||_\infty}$ for any x_0 in $\mathbb{R}^3$.

Now we aim to show that $||u(s(t))||_\infty \leq ||u(t)||_\infty$, contradicting that t is an escape time. Any $x_0 \notin \Omega_i^\pm$ satisfies $u_i^\pm(x_0, s(t)) \leq \frac{1}{2}||u(s(t))||_\infty \leq \frac{1+\gamma}{2}||u(t)||_\infty$. For $x_0 \in \Omega_i^\pm$, by Theorem 3.3.3, $u_i(s(t))$ is analytic with analyticity radius at least

$r = \frac{\gamma}{c}\sqrt{\frac{\gamma^2}{4c^2||u(t)||_\infty^2}} = \frac{\gamma^2}{2c^2||u(t)||_\infty}$ with $||u(s(t))||_{L^\infty(\mathscr{R}_{s(t)})} \leq (1+\gamma)||u(t)||_\infty$. Applying Lemma 3.4.6 with $\delta = (\frac{3}{4})^{1/3}$,

$$\begin{aligned} |u_i(x_0, s(t))| \leq & \big(\frac{1}{2}||u(s(t))||_\infty\big)h + \big((1+\gamma)||u(t)||_\infty\big)(1-h) \\ \leq & \big(\frac{1+\gamma}{2}||u(t)||_\infty\big)h + \big((1+\gamma)||u(t)||_\infty\big)(1-h) \\ = & (1+\gamma - \frac{1+\gamma}{2}h)||u(t)||_\infty, \end{aligned}$$

where $h = h(\delta)$ is a constant between 0 and 1.

Taking $\gamma \leq \frac{h}{2-h}$, we have $||u_i(s(t))||_\infty \leq ||u(t)||_\infty$. This proves that $||u(s(t))||_\infty := \max_{i=1,2,3} ||u_i(s(t))||_\infty \leq ||u(t)||_\infty$. Therefore it is impossible for an escape time to exist, which means that T can not be a blow-up time.

□

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