Applications of Chromatic Fixed Point Theory

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Abstract

From its inception the primary concern of algebraic topology has been using algebraic techniques to construct invariants of topological spaces. This pursuit has led to the creation of many important cross-disciplinary tools. One such modern tool is the equivariant Balmer spectrum associated to a finite group G. This object, which lives in the intersection of algebraic geometry and algebraic topology, encodes information about the equivariant stable homotopy category in a systematic yet abstract way. This thesis describes new computational techniques that use knowledge of the Balmer spectrum of the equivariant stable homotopy category to compute explicit topological invariants of spaces. In particular, a new technique for showing certain spectral sequences collapse is presented. This technique is then applied to compute the dimension of the Morava K-theory of a family of finite real Grassmannians. Furthermore, a conjecture for the dimension of the Morava K-theory of all finite real Grassmannians is presented.

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Contents

Notation

SH	The stable homotopy category. 9
$\mathrm{SH}(G)$	The category of G-spectra for a finite group G. 10
X^H	The H-fixed points of the H-space X. 10
$\Phi^H(X)$	The geometric H-fixed points of the H-spectra X. 10
$K(n)^*(-)$	The nth Morava K-theory at a prime p. 11
$k_n(X)$	$\dim_{K(n)^*} K(n)^*(X)$. 11
$\mathrm{Spc}\,\mathcal{K}$	The Balmer spectrum of a tensor triangulated category $\mathcal{K}$. 15
$F_m(X)$	The Jeff Smith construction. 21
$\mathcal{A}_\mathrm{p}$	The mod p Steenrod algebra. 30
Q_n	The nth Milnor primitive. 32
$H_*(M; Q_n)$	The Margolis homology of an $\mathcal{A}_\mathrm{p}$-module M. 32
$k_{Q_n}(X)$	$\dim_{\mathbb{Z}/\mathrm{p}\mathbb{Z}} H_*(X; Q_n)$. 32
$\mathrm{Gr}_d(V)$	The Grassmannian of d-planes in the vector space V. 43
$w_i(\xi)$	The ith Stiefel-Whitney class of the vector bundle ξ. 44
s_λ	The Schubert class indexed by a partition λ. 55
$C_d(\mathbb{R}^m)$	The cofiber of the inclusion $\mathrm{Gr}_d(\mathbb{R}^m) \hookrightarrow \mathrm{Gr}_d(\mathbb{R}^{m+1})$. 69
e_n	$2^{n+1} + 1$. 96
$\lfloor x \rfloor$	The largest integer less than or equal to x. 119

Chapter 1

Introduction

1.1 Introduction

In 2005, Balmer introduced the Balmer spectrum of tensor thick prime ideals of a tensor triangulated category [3]. It generalizes the spectrum of a ring from commutative algebra and algebraic geometry. The stable homotopy category SH is tensor triangulated, and so one may study its Balmer spectrum Spc(SH). Balmer [4] reinterpreted the Thick Subcategory Theorem of Hopkins and Smith [22], as a calculation of Spc(SH) [4].

The Thick Subcategory Theorem states that the only thick subcategories of the stable homotopy category are kernels of certain extraordinary cohomology theories, namely the Morava K-theories. Let $K(n)^*(-)$ denote the nth p-local Morava K-theory. Let $\mathcal{C}_{p,n}$ denote the collection of $K(n-1)^*(-)$ acyclics. Ravenel showed that for a p-local complex X if $K(n)^*(X) = 0$, then $K(n-1)^*(X) = 0$. In other words $\mathcal{C}_{p,n-1} \supseteq \mathcal{C}_{p,n}$. In fact, the p-local stable homotopy category $\mathrm{SH}_{(p)}$ has a proper filtration

$$\mathcal{C}_{p,1} \supset \mathcal{C}_{p,2} \supset \dots . \tag{1.1.1}$$

Determining the topology of the Balmer spectrum is equivalent to determining how primes are contained within one another. For $\mathrm{SH}_{(p)}$ the Thick Subcategory Theorem states that the only primes are $\mathcal{C}_{p,i}$, and so the classification of the topology of $\mathrm{Spc}(\mathrm{SH}_{(p)})$ follows from (1.1.1).

More generally, in [5] Balmer and Sanders study the spectrum of the category of compact G-equivariant spectra for a finite group G which we denote $\mathrm{SH}(G)^c$. They were able to reduce the problem to studying p-groups. They determined $\mathrm{Spc}(\mathrm{SH}(C_p)^c_{(p)})$ for C_p a cyclic group of order p. After pulling back from the non-equivariant case, the primes are now indexed by not only the underlying prime p and chromatic height n, but also conjugacy classes of subgroups of G. We denote such a prime by $\mathcal{P}_G(H, p, n)$ for $H \leq G$. Just as in the non-equivariant case, one finds that for fixed H and p one has $\mathcal{P}_G(H, p, n-1) \supset \mathcal{P}_G(H, p, n)$. The more interesting question is the determination of when $\mathcal{P}_G(H, p, m) \supseteq \mathcal{P}_G(K, p, n)$ for differing subgroups K and H. For $G = C_p$ Balmer and Sanders found that $\mathcal{P}_{C_p}(C_p, p, n) \supseteq \mathcal{P}_{C_p}(\{e\}, p, m)$ if and only if $n \geq m - 1$.

In 2019, Barthel et al. extended the results of Balmer and Sanders by determining the topology of $\mathrm{Spc}(\mathrm{SH}(A)^c_{(p)})$ for an arbitrary abelian p-group A [7]. In doing so, they formalized the notion of blueshift numbers associated to a group G. The nth blueshift number for $K \leq H \leq G$, denoted $r_n^G(H, K)$, is defined to be the minimal number r such that $\mathcal{P}_G(K, p, n + r + 1) \subseteq \mathcal{P}_G(H, p, n + 1)$. The calculation of these blueshift numbers is equivalent to determining the topology of the Balmer spectrum of $\mathrm{SH}(G)^c$. In this language the blueshift numbers for C_p were determined to be $r_n^{C_p}(C_p, \{e\}) = 1$. Barthel et al. determined the blueshift numbers of a finite abelian p-group A with subgroups $K \leq H \leq A$ to be the p-rank of the quotient: $r_n^A(H, K) = \operatorname{rk} H/K$.

In 2020, Kuhn and the author continued this investigation by calculating the blueshift numbers of a large family of non-abelian groups, namely the extra-special 2-groups [27]. One general result coming out of this paper translates information about blueshift numbers to equivariant constraints on the dimension of Morava K-theory $k_n(-) = \dim_{K^*(*)} K^*(-)$. We state this theorem now.

Theorem 3.1.1. *Let G be a finite p-group with subgroups $K \leq H \leq G$ and X a finite G-space. If $r \geq r_n^G(H, K)$, then $k_n(X^H) \leq k_{n+r}(X^K)$ for all $n \geq 0$.*

When the blueshift number $r_n(G, \{e\})$ is independent of n, say $r = r_n(G, \{e\})$, this result can be reindexed as $k_{n-r}(X^H) \leq k_n(X^K)$. We state this in the special case for G a finite cyclic p-group.

Theorem 3.1.2. *If C is a finite cyclic p-group and X is a finite C-space, then $k_{n-1}(X^C) \leq k_n(X)$ for all $n \geq 1$.*

This thesis primarily concerns an application of this theorem to non-equivariant calculations. In particular, we may translate the problem of showing that the Atiyah–Hirzebruch spectral sequence (AHSS) computing $K(n)^*(X)$ collapses to that of finding a certain action on X. An upper bound for the dimension of the Morava K-theory is the dimension of the Margolis homology $k_{Q_n}(X) = \dim_{\mathbb{Z}/p\mathbb{Z}} H^*(X; Q_n)$, because the first possibly non-trivial differential of the AHSS is given by the nth Milnor primitive Q_n. If one finds a G-action on X such that this upper bound agrees with the chromatic lower bound coming from Theorem 3.1.2, then the spectral sequence immediately collapses, since any higher differentials would further drop the dimension of $k_n(X)$.

Theorem 3.4.1. *If a finite complex X admits an action of a finite cyclic p-group C such that $k_{n-1}(X^C) = k_{Q_n}(X)$, then the AHSS computing $K(n)^*(X)$ collapses on the $2p^n$-page and $k_n(X) = k_{Q_n}(X)$.*

In 1993, Kono and Yagita [25] showed that the 2-local Morava K-theory of $BO(d)$ is concentrated in even degrees. In 2015, Kitchloo and Wilson [24] recovered this result by explicitly computing the Morava K-theory of the classifying space of the orthogonal group, $BO(d)$, by showing that the Margolis homology of $BO(d)$ is concentrated in even degrees, and so the AHSS collapses. We will continue this exploration by applying Theorem 3.4.1 to the finite real Grassmannians. These are the smooth manifolds of d-dimensional subspaces of $\mathbb{R}^{d+c}$. We will denote such manifolds by $\mathrm{Gr}_d(\mathbb{R}^{d+c})$. The infinite case, $\mathrm{Gr}_d(\mathbb{R}^\infty)$, is exactly what Kitchloo and Wilson studied, because $\mathrm{Gr}_d(\mathbb{R}^\infty)$ is a model for $BO(d)$.

We will see that for the finite real Grassmannians the Morava K-theory is not concentrated in even degrees, and so does not necessarily collapse formally; despite this, it appears that the AHSS still collapses after the first differential, if not on the E_2-page. We now describe our main results, all of which use Theorem 3.4.1.

Theorem 4.5.1. *For $c + d \leq 2^{n+1}$, the dimension of the 2-local Morava K-theory of $\mathrm{Gr}_d(\mathbb{R}^{d+c})$ is $k_n(\mathrm{Gr}_d(\mathbb{R}^{d+c})) = \dim_{\mathbb{Z}/2\mathbb{Z}} H^*(\mathrm{Gr}_d(\mathbb{R}^{d+c}); \mathbb{Z}/2\mathbb{Z}) = \binom{c+d}{d}$.*

This theorem in particular shows that the Q_n-differential is zero for a much larger range than one might suspect. The dimension of the top class of $\mathrm{Gr}_d(\mathbb{R}^{d+c})$ is cd. Since Q_n is of degree $2^{n+1} - 1$, when $cd < 2^{n+1}$ one knows immediately that the AHSS must collapse. Our result improves this bound to $c + d \leq 2^{n+1}$. The condition $c + d \leq 2^{n+1}$ means that the top class is in degree at most $cd \leq (2^{n+1} - d)d$. For large n this is much larger than the degree of the Q_n-differential. Furthermore, there is plenty of room for higher differentials, all of which turn out to be zero.

Theorem 4.5.4. *For $n \geq 0$, the dimension of the 2-local $K(n)^*(\mathrm{Gr}_d(\mathbb{R}^{d+c}))$ is bounded below by*

$$k_n\left(Gr_d(\mathbb{R}^m)\right) \geq \begin{cases} \binom{m}{d} & d \leq m \leq 2^{n+1} \\ \sum_{i=0}^{\frac{d-1}{2}} \binom{2^{n+1}}{d-2i}\binom{\ell}{i} & d \ odd, m = 2^{n+1}+2\ell \\ \sum_{i=0}^{\frac{d-1}{2}} \binom{2^{n+1}-1}{d-2i}\binom{\ell}{i} & d \ odd, m = 2^{n+1}+2\ell-1 \\ \sum_{i=0}^{\frac{d}{2}} \binom{2^{n+1}}{d-2i}\binom{\ell}{i} & d \ even, m = 2^{n+1}+2\ell \\ \sum_{i=0}^{\frac{d}{2}} \binom{2^{n+1}-1}{d-2i}\binom{\ell}{i} & d \ even, m = 2^{n+1}+2\ell-1. \end{cases}$$

For the case $d = 2$ we are able to prove that the lower bound given by this theorem is the right answer.

Theorem 4.9.26. *The dimension of $K(n)^*(\mathrm{Gr}_2(\mathbb{R}^m))$ is*

$$k_n(\mathrm{Gr}_2(\mathbb{R}^m)) = \begin{cases} \binom{m}{2} & 2 \leq m \leq 2^{n+1} \\ 2^{2n+1} - 2^n + \ell & m = 2^{n+1} + 2\ell \\ 2^{2n+1} - 2^{n+1} - 2^n + 1 + \ell & m = 2^{n+1} + 2\ell - 1. \end{cases}$$

We conjecture that there are no higher differentials in general.

Conjecture 5.1.1. *For $n \geq 1$, $k_{Q_n}(\mathrm{Gr}_d(\mathbb{R}^{d+c}))$ is equal to the lower bound in Theorem 4.5.4. Thus, by Theorem 3.4.1 the AHSS computing $K(n)^*(\mathrm{Gr}_d(\mathbb{R}^{d+c}))$ collapses on the 2^{n+1}-page and has dimension given by the formula in Theorem 4.5.4.*

This conjecture is supported by computer calculations, which we list in Appendix B.

We also point out that analogous results to Theorem 4.5.1, Theorem 4.5.4, and Conjecture 5.1.1 also hold for the connective Morava K-theory $k(n)_*$ by invoking the following proposition.

Proposition 3.5.1. *If the AHSS computing $K(n)_*(X)$ collapses after the first possibly non-trivial differential, then so does the AHSS computing the connective Morava K-theory $k(n)_*(X)$.*

Furthermore, we explain in Appendix C how the collapsing results for the AHSS computing $k(n)_*(X)$ can be reinterpreted as collapsing results for the Adams spectral sequence computing $k(n)_*(X)$.

1.2 Outline

This thesis consists of five chapters. Chapter 1 consists of a brief introduction and this outline.

In Chapter 2, the necessary background is introduced. This discussion begins with an enumeration of the pertinent properties of the category of G-spectra. The Morava K-theories are then introduced along with a historical account of the construction of type n complexes. We then introduce the Balmer spectrum in full generality. The Thick Subcategory Theorem of Hopkins and Smith is presented in terms of the classification of the Balmer spectrum of the non-equivariant stable homotopy theory. Questions about containment of equivariant primes are reformulated into existence questions of certain spectra. The chapter concludes with a discussion of the main technical device of the thesis, namely, the Jeff Smith construction. The construction is packaged as a functor and is written in such a way that a reader may begin applying the construction without needing to understand the details of the construction itself. As an example, we apply the Jeff Smith construction to construct certain D_8-complexes that lead to the classification of the Balmer spectrum of $\mathrm{SH}(D_8)^c$ in [26].

In Chapter 3, a foundational result of chromatic fixed point theory is proved in the form of Theorem 3.1.1 as in the preprint of Kuhn and the author [27]. The key observation here is that the Jeff Smith construction not only can be used directly to prove results about the Balmer spectrum, but also can be used in the contrapositive to obtain certain lower bounds. Background on the Steenrod algebra is then introduced in order to define the Milnor primitive Q_n and Margolis homology. Background material about the Atiyah–Hirzebruch spectral sequence for Morava K-theory is then described. This spectral sequence is used in the standard way to obtain an upper bound for the dimension of the Morava K-theory. Theorem 3.4.1 then follows by combining Theorem 3.1.1 with this upper bound.

Thus, one has the interesting possibility of concluding non-equivariant results by finding certain G-actions. As an illustrative example, Theorem 3.4.1 is applied to show the AHSS computing the Morava K-theory of real projective spaces collapses.

The goal of Chapter 4 is to apply Theorem 3.4.1 to the 2-local Morava K-theory of the real Grassmannians. This is approached from two perspectives, Borel's Stiefel–Whitney class perspective and the Schubert calculus perspective. Both of these perspectives are used. Background on Gröbner bases is introduced along with the explicit Gröbner bases for the Grassmannians due to Petrović and Prvulović. This basis is used to determine whether an element in the cohomology is zero. The main difficulty of this chapter is the calculation of the Margolis homology of $\mathrm{Gr}_d(\mathbb{R}^{d+c})$, which requires analyzing a long exact sequence in Margolis homology. We use Lenart's combinatorial description of the action of the Steenrod algebra on the Schubert classes of the Grassmannians together with Wood's algebra of differential operations, to give a combinatorial description of the connecting homomorphism in this long exact sequence.

We turn the Grassmannians into a G-space using real representation theory and analyze their fixed points in terms of representation theory in Proposition 4.3.3. Using an action of the cyclic group of order two and Theorem 3.4.1 we are able to prove Theorem 4.5.1, which says that the AHSS computing the Morava K-theory of $\mathrm{Gr}_d(\mathbb{R}^{d+c})$ has no differentials for a larger range than one would expect. This is then used together with an action of the cyclic group of order four with Theorem 3.1.2 to obtain the lower bound for the dimension of the Morava K-theory of $\mathrm{Gr}_d(\mathbb{R}^{d+c})$ in Theorem 4.5.4. This lower bound is then combined with the Gröbner basis theory and the combinatorial description of the connecting homomorphism to compute the dimension of the Morava K-theory of $\mathrm{Gr}_2(\mathbb{R}^{2+c})$ using Theorem 3.4.1.

In Chapter 5, we discuss the conjecture that the general lower bound of Theorem 4.5.4 is actually the dimension of $k_n(\mathrm{Gr}_d(\mathbb{R}^{d+c}))$. We examine what portions of the proof from the $d = 2$ case should remain the same. We also conjecture other related results, which once proved imply this formula.

Chapter 2

Background

2.1 Equivariant Stable Homotopy

The majority of the constructions we will consider live in the world of equivariant stable homotopy. However, we prefer to work explicitly with G-spaces for G a finite group when possible and then pass to spectra through the use of the suspension spectrum. This section serves to introduce the basic notation used.

In non-equivariant stable homotopy theory there is a suspension functor from the category of based spaces to the category of spectra, $\Sigma^\infty \colon \mathrm{Top}_* \to \mathrm{SH}$. The stable homotopy category SH comes equipped with a smash product that is symmetric monoidal. The monoidal unit is called the sphere spectrum. A classic reference is [1, Part III].

We will denote the category of pointed G-spaces as Top_*^G. A key example of a G-space is the representation sphere. Take V to be an orthogonal real representation of G and then compactify V to form a sphere with a G-action. This is commonly denoted S^V. In the equivariant setting suspension is subsumed by smashing with the various representation spheres. These representation spheres are built into the definition of genuine G-spectra in the form of the structure maps [46, §2.6].

We let $\mathrm{SH}(G)$ denote the category of genuine G-spectra with respect to a complete G-universe together with an associative and commutative smash product [17] [30]. Again there is a suspension spectrum functor $\Sigma^\infty_G\colon \mathrm{Top}^G_* \to \mathrm{SH}(G)$. We restrict our attention to the p-local compact objects, which we denote $\mathrm{SH}(G)^c_{(p)}$. We say a p-local G-space is finite if it the retract of the p-localization of a finite G-CW complex. The compact objects are all of the form $S^{-V} \wedge \Sigma^\infty_G X$ for some G-representation V and finite p-local complex X [6, Lemma 2.2].

For a G-space X and subgroup $H \leq G$ we let X^H denote the H-fixed points of X, that is, $X^H = \{x \mid hx = x \text{ for all } h \in H\}$. This can be packaged as a functor $(-)^H\colon \mathrm{Top}^G_* \to \mathrm{Top}_*$. It is convenient to replace the ordinary fixed point functor by another functor that has useful formal properties:

Definition 2.1.1 ([30, V.4])**.** The geometric fixed point functor $\Phi^H\colon \mathrm{SH}(G) \to \mathrm{SH}$ satisfies

1. $\Phi^H(X)\,(\Sigma^\infty_G X) \simeq \Sigma^\infty X^H$,
2. Φ^H preserves cofiber sequences,
3. $\Phi^H(X \wedge Y) \simeq \Phi^H(X) \wedge \Phi^H(Y)$

It is a theorem that such a functor exists. The geometric fixed point functor also preserves compact objects, as $\Phi^H(S^{-V} \wedge \Sigma^\infty_G X) = S^{-V^H} \wedge \Sigma^\infty(X^H)$.

2.2 Morava K-theory

In the 1970s Jack Morava introduced a family of extraordinary cohomology theories now known as the Morava K-theories. These theories generalize complex K-theory and have proven central to the development of chromatic homotopy theory.

Proposition 2.2.1. *For every prime p there is a sequence of extraordinary cohomology theories* $K(0)^*(-), K(1)^*(-), \ldots, K(n)^*(-), \ldots$ *with the following properties.*

1. *The zeroth Morava* K*-theory* $K(0)^*(-)$ *agrees with rational cohomology* $H^*(-;\mathbb{Q})$ *for all primes* p*.*
2. *For* $n \geq 1$*, the coefficient ring* $K(n)^*(*)$ *is the Laurent polynomial ring*

$$K(n)^*(*) \cong \mathbb{F}_p[v_n, v_n^{-1}]$$

with v_n *in degree* $-2p^n + 2$*. All graded modules over* $K(n)^*(*)$ *are free, and so it makes sense to define* $k_n(X) = \dim_{K(n)^*(*)} K(n)^*(X)$*.*

3. *For* X *and* Y *finite complexes there is a Künneth Theorem*

$$K(n)^*(X \times Y) \cong K(n)^*(X) \otimes_{K(n)^*(*)} K(n)^*(Y).$$

It will also be useful for us to work with the homology theory $K(n)_*(X)$. This has coefficient ring $K(n)_*(*) \cong \mathbb{F}_p[v_n, v_n^{-1}]$ with v_n in degree $2p^n - 2$. These theories satisfy analogous properties to those above and are linearly dual to $K(n)^*(X)$, that is, $K(n)^*(X) \cong \mathrm{Hom}_{K(n)_*}(K(n)_*(X), K(n)_*(*))$.

By convention we define

$$K(\infty)^*(-) = H^*(-;\mathbb{Z}/\mathrm{p}\mathbb{Z}) \qquad \text{and} \qquad k_\infty(X) = \dim_{\mathbb{Z}/\mathrm{p}\mathbb{Z}} H^*(X;\mathbb{Z}/\mathrm{p}\mathbb{Z}).$$

With this notation we can state a result of Ravenel from the 1980s.

Proposition 2.2.2 ([41, Thm 2.11])**.** *For a finite p-local complex X,*

$$k_0(X) \leq k_1(X) \leq k_2(X) \leq \ldots \leq k_\infty(X).$$

In this way one may think of the Morava K–theories as interpolating between the rational cohomology and the $\mathbb{Z}/\mathrm{p}\mathbb{Z}$-cohomology. This theorem has the more well known consequence that if X is $K(n)^*$-acyclic, then it is also $K(n-1)^*$-acyclic. This allows one to define the *type* of a finite p-local complex X to be the height of the first non-vanishing Morava K-theory.

Definition 2.2.3. A finite p-local complex X is said to have type n if $K(n)^*(X) \neq 0$ and $K(n-1)^*(X) = 0$.

In 1984 Steve Mitchell proved that type n complexes always exist:

Proposition 2.2.4 ([35, Thm B])**.** *For each non-negative integer n there exists a finite complex of type n.*

Part of this thesis concerns the search for an equivariant version of this; see Question 2.3.14 below.

2.3 The Equivariant Balmer spectrum

In 2005, Paul Balmer introduced what is now known as the Balmer spectrum of a tensor triangulated category [3]. We begin by briefly recalling the definition of a tensor triangulated category.

Definition 2.3.1. A triangulated category $\mathcal{C}$ is an additive category equipped with a translation functor $\Sigma\colon \mathcal{C} \to \mathcal{C}$ and collection of distinguished triangles

$$a \to b \to c \to \Sigma a$$

for objects a, b, c in $\mathcal{C}$ subject to the following axioms.

TR1 The identity morphism $a \xrightarrow{\mathrm{id}_a} a$ always embeds in a distinguished triangle

$$a \xrightarrow{\mathrm{id}_a} a \to 0 \to \Sigma a,$$

and every map $f\colon a \to b$ embeds in a distinguished triangle

$$a \xrightarrow{f} b \to c \to \Sigma a$$

for some object c in $\mathcal{C}$.

TR2 The rotation axiom says that

$$a \xrightarrow{f} b \xrightarrow{g} c \xrightarrow{h} \Sigma a$$

is distinguished if and only if

$$b \xrightarrow{g} c \xrightarrow{h} \Sigma a \xrightarrow{-\Sigma(f)} \Sigma b$$

is.

TR3 A commutative diagram

$$\begin{tikzcd}
a \arrow[r, "f"] \arrow[d, "\alpha"] & b \arrow[r, "g"] \arrow[d, "\beta"] & c \arrow[r, "h"] & \Sigma a \\
a' \arrow[r, "f'"] & b' \arrow[r, "g'"] & c \arrow[r, "h'"] & \Sigma a'
\end{tikzcd}$$

with top and bottom rows distinguished triangles can always be filled in

$$\begin{tikzcd}
a \arrow[r, "f"] \arrow[d, "\alpha"] & b \arrow[r, "g"] \arrow[d, "\beta"] & c \arrow[r, "h"] \arrow[d, "\gamma"] & \Sigma a \arrow[d, "\Sigma\alpha"] \\
a' \arrow[r, "f'"] & b' \arrow[r, "g'"] & c \arrow[r, "h'"] & \Sigma a'.
\end{tikzcd}$$

TR4 The octahedral axiom (see [48, §10.2]), which describes how the triangle associated to a composition $f \circ g$ relates to the triangles associated to f and g.

While, Verdier introduced TR1–TR4 while formalizing the properties of the derived category of chain complexes [47], the key example of a triangulated category for us is the stable homotopy category SH [40, §3] and more generally SH(G).

Definition 2.3.2 ([7, Defn 1.1])**.** A *tensor triangulated category* $\mathcal{K}$ is a small category such that

1. $\mathcal{K}$ is triangulated,

2. $\mathcal{K}$ is symmetric monoidal: $\otimes\colon \mathcal{K} \times \mathcal{K} \to \mathcal{K}$,

3. The tensor product preserves distinguished triangles in each variable.

Definition 2.3.3 ([7, Defn 1.2])**.** A *tensor thick ideal* $\mathcal{A}$ of a tensor triangulated category $\mathcal{K}$ is a full subcategory of $\mathcal{K}$ containing 0 such that

1. $\mathcal{A}$ is triangulated: for any distinguished triangle $a \to b \to c \to \Sigma a$ in $\mathcal{A}$, if $\mathcal{A}$ contains two out of the three objects a, b and c, then it must contain the third also,

2. $\mathcal{A}$ is thick: if $\mathcal{A}$ contains an object a that splits in $\mathcal{K}$, $a \cong b \oplus c$, then $\mathcal{A}$ contains b and c also,

3. $\mathcal{A}$ is a tensor ideal: if $a \in \mathcal{A}$ and $b \in \mathcal{K}$ then $a \otimes b \in \mathcal{A}$.

If $\mathcal{A}$ satisfies the additional property that $a \otimes b \in \mathcal{A}$ implies a or b is in $\mathcal{A}$, then $\mathcal{A}$ is said to be a *prime tensor thick ideal.*

The set of all prime tensor thick ideals of $\mathcal{K}$ is called the *Balmer spectrum* and is denoted $\operatorname{Spc}\mathcal{K}$. This set is topologized by declaring that the closed subsets are generated by sets

$$Z(\mathcal{S}) = \{\mathcal{P} \in \operatorname{Spc}\mathcal{K} \mid \mathcal{S} \cap \mathcal{P} \neq \varnothing\},$$

for $\mathcal{S}$ a family of objects in $\mathcal{K}$. This topology is called the Zariski topology just as in commutative algebra.

Example 2.3.4. The Thick Subcategory Theorem of Hopkins and Smith [22] states that the only thick subcategories of $\mathrm{SH}_{(p)}$ are the kernels of Morava K-theory. More precisely if we let $\mathcal{C}_{p,n}$ denote the collection of $K(n-1)^*(-)$ acyclics, then a thick subcategory of $\mathrm{SH}_{(p)}$, is either all of $\mathrm{SH}_{(p)}$ or $C_{p,n}$ for some n. If X is $K(n-1)$-acyclic, so is $X \wedge Y$, and so these thick subcategories are, in fact, thick ideals. From the Künneth Theorem, if $X \wedge Y$ is $K(n-1)^*$-acyclic, then either X or Y is also, and so these are prime thick ideals.

The relation that $K(n)^*(X) = 0$ implies $K(n-1)^*(X) = 0$ tells us that these

$\mathcal{C}_{p,n}$ fit into a descending chain

$$C_{p,1} \supset C_{p,2} \supset \dots .$$

These inclusions are known to be proper precisely due to the existence of type n complexes. Balmer and Sanders showed considering chains of primes in this fashion is crucial to determination of the topology of $\mathrm{Spc}(\mathrm{SH}^c_{(p)})$. The reason for this becomes apparent with the following two propositions.

Proposition 2.3.5 ([3, Prop 2.9])**.** *Let $\mathcal{K}$ be a tensor triangulated category. For any point $P \in \mathrm{Spc}(\mathcal{K})$ its closure in $\mathrm{Spc}\,\mathcal{K}$ is*

$$\overline{\{P\}} = \{\mathcal{Q} \in \mathrm{Spc}(\mathcal{K}) \mid \mathcal{Q} \subset \mathcal{P}\}\,.$$

In particular, $\overline{\{\mathcal{P}_1\}} = \overline{\{\mathcal{P}_2\}}$ implies $\mathcal{P}_1 = \mathcal{P}_2$.

Proposition 2.3.6 ([4, Prop 9.5(e)])**.** *Every proper nonempty closed set of $\mathrm{Spc}(\mathrm{SH})$ is a finite union of the $\overline{\{\mathcal{C}_{p,n}\}}$ for $0 \leq n \leq \infty$ and p a prime.*

This means by understanding the chains of inclusions of primes one may completely determine the closed sets and thus the topology of the Balmer spectrum.

Fix a finite group G and let $\mathrm{SH}(G)^c$ denote the compact objects in the category of genuine G-spectra. Balmer showed that the primes in $\mathrm{Spc}(\mathrm{SH}(G)^c)$ pull back from the non-equivariant primes in $\mathrm{Spc}(\mathrm{SH})$. These new primes are indexed by the conjugacy classes of the subgroups of G in addition to the prime p and the chromatic height n.

Proposition 2.3.7 ([5, Prop 4.9, Thm 4.14])**.** *Every prime in $\mathrm{Spc}(\mathrm{SH}(G)^c)$ is of the* *form*

$$\mathcal{P}_G(H,p,n) := \{X \in \mathrm{SH}(G)^c \mid \Phi^H(X) \in \mathcal{C}_{p,n}\}$$

for some prime p and $0 \le n \le \infty$, where Φ^H is the geometric H-fixed point functor in Definition 2.1.1. Furthermore, if $\mathcal{P}_G(H,p,n) = \mathcal{P}_G(K,q,m)$, then H and K are conjugate in G and $\mathcal{C}_{p,n} = \mathcal{C}_{q,m}$.

Just as in the non-equivariant case every closed set in $\mathrm{Spc}\left(\mathrm{SH}(G)^c\right)$ is a union of the closure of primes [5, Theorem 1.4]. Thus, one is again interested in how the primes contain one another. Coming from the non-equivariant case, when the underlying prime p is fixed

$$n \le m \implies \mathcal{P}_G(H,p,n) \supseteq \mathcal{P}_G(H,p,m). \tag{2.3.8}$$

When the underlying primes are different, that is, when $q \neq p$, $\mathcal{P}_G(H,p,n)$ never contains or is contained by $\mathcal{P}_G(H,q,m)$, except for when $n = 1$ or $m = 1$. At chromatic height one, these equivariant primes are independent of the underlying prime. The more interesting question is how primes indexed by different subgroups interact. Balmer and Sanders provide a necessary condition:

Proposition 2.3.9 ([5, Cor 4.12])**.** *If $\mathcal{P}_G(K,p,m) \subseteq \mathcal{P}_G(H,p,n)$ then K is subconjugate to H, that is, there exists $g \in G$ such that $gKg^{-1} \le H$.*

Furthermore, by projecting to the non-equivariant case again, one sees $m \ge n$ [5, Cor 6.4]. For clarity, we unpack the condition $\mathcal{P}_G(K,p,m) \subseteq \mathcal{P}_G(H,p,n)$. For a finite G-spectrum $X \in \mathrm{SH}(G)^c$ to be a member of $\mathcal{P}_G(K,p,m)$ means that the K-geometric fixed points are of at least type m, that is, type $\Phi^K(X) \ge m$. Hence,

$\mathcal{P}_G(K,p,m) \subseteq \mathcal{P}_G(H,p,n)$ means that for any finite G-spectrum X

$$\operatorname{type}\Phi^K(X) \geq m \implies \operatorname{type}\Phi^H(X) \geq n.$$

If $\mathcal{P}_G(K,p,m) \subseteq \mathcal{P}_G(H,p,n)$, then $\mathcal{P}_G(K,p,m) \subseteq \mathcal{P}_G(H,p,\ell)$ for any $\ell \leq n$ since $\mathcal{P}_G(H,p,n) \subseteq \mathcal{P}_G(H,p,\ell)$ by (2.3.8). So one would wish to study the maximal such n.

We may write $m = n + r$, because $n \leq m$. In other words, one wishes to find the minimal r such that

$$\operatorname{type}\Phi^K(X) \geq n + r \implies \operatorname{type}\Phi^H(X) \geq n.$$

This leads to the following definition.

Definition 2.3.10. For $K \leq H \leq G$, the minimal number r such that for all finite p-local G-spectra X

$$\operatorname{type}\Phi^K(X) \geq n + r + 1 \implies \operatorname{type}\Phi^H(X) \geq n + 1$$

is called the (G, K, H, n) blueshift number and is denoted $r_n^G(H, K)$.

Remark 2.3.11. *The blueshift number* $r_n^G(H,K)$ *is denoted* $\beth_{n+1}(G; H, K)$ *in [7].*

Notice that these blueshift numbers only depend on the conjugacy classes. This blueshift number is equivalently the minimal r such that $\mathcal{P}_G(K,p,n+r+1) \subseteq \mathcal{P}_G(H,p,n+1)$. This is also the minimal r such that $K(n+r)^*(\Phi^K(X)) = 0$ implies that $K(n)^*(\Phi^H(X)) = 0$.

Remark 2.3.12. *One may obtain lower bounds for blueshift numbers by constructing explicit spectra. If one constructs a spectrum X such that*

$$\operatorname{type}\Phi^K(X) = n + r + 1 \textit{ and } \operatorname{type}\Phi^H(X) < n + 1,$$

then $\mathcal{P}_G(K, p, n + r + 1) \not\subseteq \mathcal{P}_G(H, p, n + 1)$, *hence* $r < r_n^G(K, H)$.

Definition 2.3.13. Let $\Sigma(G)$ denote the conjugacy classes of subgroups of a finite group G. A function $f \colon \Sigma(G) \to \{0, 1, \ldots\}$ will be called a type function.

Associated to any G-spectrum X is the type function defined by $f(H) = \operatorname{type}\Phi^H(X)$ *for each $H \leq G$. This function will be denoted* $\operatorname{type} X$.

Question 2.3.14. *Given a type function f, when does there exist a G-spectrum X such that* $\operatorname{type} X = f$*?*

When this does happen we say f is *realizable*. It was Barthel et al. that formulated these structural questions about the Balmer spectrum of $\mathrm{SH}(G)^c$ in this way [7]. In [5] the question of determining the topology of $\mathrm{Spc}\,(\mathrm{SH}(G)^c)$ of a general group G was reduced to understanding the case for G a p-group. This led them to consider $\mathrm{Spc}(\mathrm{SH}(C_p)^c_{(p)})$ for the cyclic group of order p. In this case, one must only consider the pair $(C_p, \{e\})$.

Theorem 2.3.15 ([5, Prop 7.5])**.** *The blueshift number* $r_n^{C_p}(C_p, \{e\}) = 1$.

This tells us that the only type functions that are realizable for C_p are of the form $f(C_p) \geq f(\{e\}) - 1$. For example, f defined by $f(\{e\}) = n$ and $f(C_p) = n - 2$ is never realizable. Thus, one sees unlike in the non-equivariant case, not every type is realizable.

The answer for finite abelian p-groups was determined by Barthel et al. as follows.

Theorem 2.3.16 ([7, Thm 1.3])**.** *Let A be a finite abelian p-group with subgroups $K \leq H \leq A$. The blueshift number is $r_n^A(H,K) = \mathrm{rk}_p(H/K)$, where $\mathrm{rk}_p(H/K)$ denotes the p-rank of the abelian p-group H/K.*

The key step of their proof was showing that $r_n^{C_{p^k}}(C_{p^k}, \{e\}) \leq 1$, from which the inequality $r_n^A(H,K) \leq \mathrm{rk}_p(H/K)$ follows quite easily. This is also the key calculation we use throughout this thesis. One may notice that the blueshift number calculations listed so far are all independent of the chromatic height n. To date, every such blueshift number that has been calculated is independent of the height, although there is no known reason for this phenomenon. It is also not known if this must always be the case.

As hinted at earlier, many blueshift calculations follow from previous blueshift calculations. The first non-abelian group with blueshift calculations not following from the results just stated is the dihedral group of order eight. Previous work of Balmer and Sanders reduced the calculation of $\mathrm{Spc}(\mathrm{SH}(D_8)^c_{(2)})$ to the calculation of $r_n^{D_8}(D_8, C)$, for C a non-central subgroup of order two. They were able to show that $1 \leq r_n^{D_8}(D_8, C) \leq 2$.

The starting point for the research in this thesis was the discovery that the Jeff Smith construction, which we describe next, offered the tools to resolve this ambiguity. In Example 2.4.8 we will construct for each n a finite D_8-spectra X such that $\mathrm{type}\, X^C = n + 2$ and $\mathrm{type}\, X^{D_8} = n$.

2.4 The Jeff Smith construction

2.4.1 Properties of the construction

We focus on the 2-local version of the Jeff Smith construction for clarity. We first describe the general properties of this construction before delving into the details. We provide a reference for the construction at odd primes which relies on some slightly technical lemmas.

Proposition 2.4.1 ([43, §6.4])**.** *There is a functor* $F_m(X)\colon \mathrm{SH}_{(2)} \to \mathrm{SH}_{(2)}$ *such that*

$$K(q)^*(F_m(X)) = 0 \iff k_q(X) < m.$$

This translates the problem of finding a type n complex into the problem of finding a complex with the dimensions of its Morava K-theories in a certain range. If a complex X satisfies $k_{n-1}(X) < m \leq k_n(X)$, then

$$K(n-1)^*(F_m(X)) = 0 \text{ and } K(n)^*(F_m(X)) \neq 0,$$

hence $F_m(X)$ would be of type n. To demonstrate the power of the Jeff Smith construction we show how by feeding in one of the simplest spaces one is able to produce a type n complex.

Example 2.4.2. It is well known that the dimension of the 2-local Morava K-theory of real projective space is given by

$$k_{n}(\mathbb{R}\mathrm{P}^{\ell}) = \begin{cases} \ell+1 & \ell < 2^{n+1} \\ 2^{n+1} & \text{odd } \ell \geq 2^{n+1} \\ 2^{n+1}-1 & \text{even } \ell \geq 2^{n+1}. \end{cases}$$

We also prove this later as an example of our new techniques in Example 3.2.1 and Example 3.4.2. Taking $\ell = 2^{n+1}$ so $X = \mathbb{R}\mathrm{P}^{\ell} = \mathbb{R}\mathrm{P}^{2^{n+1}}$, then $k_{n-1}(X) = 2^n - 1$ since $\ell > 2^{(n-1)+1} - 1 = 2^n - 1$. While, $k_n(X) = 2^{n+1} - 1$ since $\ell \geq 2^{n+1}$. Thus,

$$k_{n-1}(\mathbb{R}^{2^{n+1}}) < 2^{n+1} - 1 \leq k_n(\mathbb{R}^{2^{n+1}}),$$

therefore $F_{2^{n+1}-1}(\mathbb{R}\mathrm{P}^{2^{n+1}})$ is of type n.

The Jeff Smith construction extends to the equivariant setting.

Proposition 2.4.3. *The functor F_m extends to a functor of G-spectra, $F_m\colon \mathrm{SH}(G)_{(p)} \to \mathrm{SH}(G)_{(p)}$.*

We will see this from how the non-equivariant definition of $F_m(X)$ is defined as a mapping telescope of iterated smash products of X. One needs the associative commutative smash product on $\mathrm{SH}(G)$ as in [30]. Furthermore, we will show it is compatible with taking fixed points:

Lemma 2.4.4. *The functor F_m commutes with the geometric fixed point functor.*

We now state two equivalent versions of the equivariant Jeff Smith construction.

Proposition 2.4.5. *Let G be a finite group with subgroups $K \leq H \leq G$. Let X be a finite G-complex. If $k_i(X^H) \geq m > k_j(X^K)$, then $\operatorname{type} \Phi^H(F_m(X)) \leq i$ and $\operatorname{type} \Phi^K(F_m(X)) > j$.*

Corollary 2.4.6. *Let $A \subseteq \Sigma(G)$. Given a partial type function $f \colon A \to \{0, 1, \ldots\}$ for G, if there exists a finite G-complex X and an m such that*

$$k_{f(H)-1}(X^H) < m \leq k_{f(H)}(X^H) \tag{2.4.7}$$

for all H in A, then $\operatorname{type} \Phi^H(F_m(X)) = f(H)$ *for each H in A.*

We will use this construction in two ways:

1. Directly, to produce lower bounds for blueshift numbers,
2. In the contrapositive, to relate the dimension of the Morava K-theory of fixed points at different chromatic heights.

As a preview of the fixed point theory to come and a quick application of the Jeff Smith construction we calculate a previously unknown blueshift number for the dihedral group of order eight which first appeared in [26].

Example 2.4.8. Let D_8 denote the dihedral group of order eight and C a non-central subgroup of order two. Balmer and Sanders showed that $1 \leq r_n^{D_8}(D_8, C) \leq 2$. We will show this blueshift number is equal to the upper bound by using the Jeff Smith construction and representation theory.

The group D_8 has four one dimensional real representations $\sigma_1, \ldots, \sigma_4$ and one two-dimensional real representation τ. Two of the one-dimensional representations restrict to the trivial representation 1 on C. The other two restrict to the sign representation Σ on C. The two-dimensional representation restricts to the sum $1 \oplus \Sigma$. Let $V = 2^{n+1}(\sigma_1 \oplus \sigma_2 \oplus \sigma_3 \oplus \sigma_4) \oplus \tau$ so that $V|_C = (2^{n+2} + 1)(1 \oplus \Sigma)$. One may form a D_8-space by taking all lines in the vector space V. We will denote this by $\mathbb{RP}(V)$. This construction is a special case of the more general Grassmannian construction presented in §4.3. The fixed points of $\mathbb{R}\mathrm{P}(V)$ are described in Proposition 4.3.3, in particular,

$$\mathbb{R}\mathrm{P}(V)^G = \coprod_{\lambda} \mathbb{R}\mathrm{P}(V_\lambda)$$

where λ runs through the one-dimensional representations and V_λ is the corresponding isotypical summand. In particular, with our chosen V, we have the fixed points

$$\begin{aligned} \mathbb{R}\mathrm{P}(V)^{D_8} &= \coprod_{i=1}^{4} \mathbb{R}\mathrm{P}(2^{n+1}\sigma_i) = \coprod_{i=1}^{4} \mathbb{R}\mathrm{P}^{2^{n+1}-1} \\ \mathbb{R}\mathrm{P}(V)^C &= \mathbb{R}\mathrm{P}((2^{n+2}+1)1) \sqcup \mathbb{R}\mathrm{P}((2^{n+2}+1)\Sigma) = \mathbb{R}\mathrm{P}^{2^{n+2}} \sqcup \mathbb{R}\mathrm{P}^{2^{n+2}} . \end{aligned}$$

Since $k_{n-1}(\mathbb{R}\mathrm{P}^{2^{n+1}-1}) = 2^n$ one has $k_{n-1}\left(\mathbb{R}\mathrm{P}(V)^{D_8}\right) = 4{\cdot}2^n = 2^{n+2}$, and $k_n(\mathbb{R}\mathrm{P}^{2^{n+1}-1}) = 2^{n+1}$, hence $k_n\left(\mathbb{R}\mathrm{P}(V)^{D_8}\right) = 2^{n+3}$. On the other hand $k_{n+2-1}(\mathbb{R}\mathrm{P}^{2^{n+2}}) = 2^{n+2}-1$, hence $k_{n+2-1}(\mathbb{R}\mathrm{P}(V)^C) = 2^{n+3}-2$, and $k_{n+2}(\mathbb{R}\mathrm{P}^{2^{n+2}}) = 2^{n+2}+1$ yielding $k_{n+2}(\mathbb{R}\mathrm{P}(V)^C) = 2^{n+3}+2$. Thus taking $m = 2^{n+3}$ yields

$$\begin{aligned} 2^{n+2} = k_{n-1}(\mathbb{R}\mathrm{P}(V)^{D_8}) &< m \leq k_n(\mathbb{R}\mathrm{P}(V)^{D_8}) = 2^{n+3} \\ 2^{n+3}-2 = k_{n+2-1}(\mathbb{R}\mathrm{P}(V)^C) &< m \leq k_{n+2}(\mathbb{R}\mathrm{P}(V)^C) = 2^{n+3}+2. \end{aligned}$$

By Corollary 2.4.6, $X = F_m(\mathbb{R}\mathrm{P}(V))$ has type $\Phi^{D_8}X = n$ and type $\Phi^C X = n+2$. Putting this in the form of Remark 2.3.12, type $\Phi^C X = n+1+1$ but type $\Phi^{D_8}X = n < n+1$, hence this gives the lower bound $1 < r_n^{D_8}(D_8, C)$.

At odd primes the Jeff Smith construction requires keeping track of the even and odd grading. Let $K(n)^*_e(X)$ and $K(n)^*_o(X)$ denote the pieces of $K(n)^*(x)$ concentrated in the even and odd degrees respectively. Let $k_n^e(X) = \dim_{K(n)^*} K(n)^*_e(X)$ and $k_n^o(X) = \dim_{K(n)^*} K(n)^*_o(X)$.

Proposition 2.4.9 ([27, Prop 6.2])**.** *Let p be an odd prime. There exist functors*

1. $F^e_m\colon \mathrm{SH}(G)_{(p)} \to \mathrm{SH}(G)_{(p)}$ *such that*

$$\operatorname{type} F^e_m(X) \leq n \iff (p-1)k^e_n(X) + k^o_n(X) \geq (p-1)m,$$

2. $F^o_m\colon \mathrm{SH}(G)_{(p)} \to \mathrm{SH}(G)_{(p)}$ *such that*

$$\operatorname{type} F^o_m(X) \leq n \iff (p-1)k^o_n(X) + k^e_n(X) \geq (p-1)m.$$

2.4.2 The construction

The non-equivariant construction is explained in [43, §6.4, Appendix C] which we follow. Given a finite G-complex X and self map $e\colon X \to X$ one may form the mapping telescope

$$eX = \operatorname{colim}\left\{X \xrightarrow{e} X \xrightarrow{e} X \xrightarrow{e} \dots\right\}$$

which comes equipped with the natural inclusion $X \xrightarrow{i_e} eX$. Now we specialize to $e\colon X \to X$ a homotopy idempotent, that is, $e^2 \simeq e$. Since we are working stably $\mathrm{Map}_{\mathrm{SH}(G)}(X, X)$ has a ring structure and so we may form the map $\mathrm{id}_X - e$, which itself is an idempotent. This gives a weak equivalence

$$X \xrightarrow{i_X \vee i_{\mathrm{id}_X - e}} eX \vee (\mathrm{id}_X - e)X,$$

since it induces the splitting $\pi_*(X) \cong e\pi_*(X) \oplus (\mathrm{id}_{H^*(X)} - \ e)\pi_*(X)$. It also induces a splitting for any cohomology theory $E^*(X) \cong eE^*(X) \oplus (\mathrm{id}_{E^*(X)} - \ e)E^*(X)$. Furthermore, these idempotent commute with cohomology $E^*(eX) \cong eE^*(X)$.

The kth symmetric group, Σ_k, acts on the k-fold smash product $X^{\wedge k}$ by permuting the factors. This extends to an action of the group ring $\mathbb{Z}[\Sigma_k]$ on $X^{\wedge k}$ since we are allowed to add maps. When X is p-local we have an action of $\mathbb{Z}_{(p)}[\Sigma_k]$ on $X^{\wedge k}$. Let $\mathbb{F}_*$ be a graded field of characteristic 2. In particular, $\mathbb{F}_*$ could be the coefficient ring of the 2-local Morava K-theory $K(n)^* \cong \mathbb{Z}/2\mathbb{Z}[v_n^{\pm 1}]$.

Theorem 2.4.10 ([43, Thm C.1.5])**.** *For every integer m there exists an integer k_m and an idempotent $e_m \in \mathbb{Z}_{(2)}[\Sigma_{k_m}]$ such that for any $\mathbb{F}_*$-module V the summand $e_m V^{\otimes k_m}$ is non-trivial if and only if $\dim_{\mathbb{F}_*} V \geq m$.*

Proof of Proposition 2.4.1. Define $F_m(X) = e_m X^{\wedge k_m}$. Consider,

$$e_m(K(n)^*(X)^{\otimes k_m}) \cong e_m(K(n)^*(X^{\wedge k_m})) \cong K(n)^*(e_m(X^{\wedge k_m})) \cong K(n)^*(F_m(X)) \quad ({}^1)$$

so that by Theorem 2.4.10, $K(n)^*(F_m(X)) \cong e_m\left(K(n)^*(X^{\wedge k_m})\right) = 0$ if and only if

$$k_n(X^{\wedge k}) = \dim K(n)^*(X^{\wedge k})) < m.$$

This shows that F_m satisfies the characterizing property of Proposition 2.4.1. □

Proof of Lemma 2.4.4. We now show that F_m commutes with geometric fixed points by calculating:

$$\begin{aligned} F_m(\Phi^H(X)) &= e_m(\Phi^H(X))^{\wedge k_m} \\ &\simeq e_m(\Phi^H(X^{\wedge k_m})) && \text{(Property 3 of 2.1.1)} \\ &\simeq \Phi^H(e_m(X^{\wedge k_m})) && \text{(Property 2 of 2.1.1)} \\ &= \Phi^H(F_m(X)). \end{aligned}$$

□

---

[1] At $p = 2$ the functor $K(n)^*\colon (\mathrm{SH}, \wedge) \to (K(n)^*\text{–modules}, \otimes_{K(n)^*})$ is not symmetric monoidal. This can be fixed by using an exotic tensor product and results in what we want [27, §3.2].

Proof of Proposition 2.4.5. Let G be a finite group with subgroups $K \leq H \leq G$. Let X be a finite G-complex. Suppose that $k_i(X^H) \geq m > k_j(X^K)$. From Definition 2.1.1 $\Phi^H(\Sigma^\infty_G X) \simeq \Sigma^\infty X^H$, and in particular $k_n\left(\Phi^H(\Sigma^\infty_G X)\right) = k_n\left(\Sigma^\infty X^H\right) = k_n(X^H)$. Thus, we have that $k_i(\Phi^H(\Sigma^\infty_G X)) \geq m > k_j(\Phi^K(\Sigma^\infty_G X))$. Using the relation $m > k_j(\Phi^K(\Sigma^\infty_G X))$ and Proposition 2.4.1 we have that

$$\begin{aligned} 0 &= K(j)^*(F_m(\Phi^K(\Sigma^\infty_G X)) \\ &= K(j)^*(\Phi^K(F_m(\Sigma^\infty_G X)), && \text{(by Lemma 2.4.4)} \end{aligned}$$

and thus by the definition of type, type $\Phi^K(F_m(\Sigma^\infty_G X)) > j$. From the other relation $k_i(X^H) \geq m$,

$$\begin{aligned} 0 &\neq K(i)^*(F_m(\Phi^H(\Sigma^\infty_G X)) \\ &\neq K(i)^*(\Phi^H(F_m(\Sigma^\infty_G X)), && \text{(by Lemma 2.4.4)} \end{aligned}$$

hence type $\Phi^H(F_m(\Sigma^\infty_G X)) \leq i$. □

Proof of Corollary 2.4.6. Fix a type function $f\colon \Sigma(G) \to \{0, 1, \ldots\}$ and suppose there exists a finite complex X and an m such that

$$k_{f(H)-1}(X^H) < m \leq k_{f(H)}(X^H).$$

By Proposition 2.4.5, type $\Phi^H(F_m(\Sigma^\infty_G X)) \leq f(H)$ and type $\Phi^H(F_m(\Sigma^\infty_G X)) > f(H)-1$, hence type $\Phi^H(F_m(\Sigma^\infty_G X)) = f(H)$. This is true for all H, hence type $F_m(\Sigma^\infty_G X) = f$. □

The Jeff Smith construction at odd primes similarly relies on idempotents and is written up in [27, §6].

Chapter 3

Chromatic fixed point theory

3.1 Chromatic lower bounds

We begin this chapter by generalizing Ravenel's lower bound, $k_{n-1}(X) \leq k_n(X)$, from Proposition 2.2.2 to the equivariant setting.

Theorem 3.1.1. *Let G be a finite p-group with subgroups $K \leq H \leq G$ and X a finite G-space. If $r \geq r_n^G(H,K)$, then $k_n(X^H) \leq k_{n+r}(X^K)$ for all $n \geq 0$.*

Proof. We will prove the contrapositive by using the equivariant Jeff Smith construction. Suppose that $k_n(X^H) > k_{n+r}(X^K)$. At the prime 2, let $m = k_n(X^H)$, and thus Proposition 2.4.5 yields

$$\operatorname{type} \Phi^H(F_m(X)) \leq n \qquad \text{and} \qquad \operatorname{type} \Phi^K(F_m(X)) > n+r.$$

Let Y be $F_m(X)$ smashed with a type n-complex with trivial G-action so that

$$\operatorname{type} \Phi^H(Y) = n \qquad \text{and} \qquad \operatorname{type} \Phi^K(Y) = \operatorname{type} \Phi^K(F_m(X)) > n+r.$$

As $\Phi^H(Y)$ is of type n, we also have an upper bound coming from the definition of $r_n^G(H,K)$:

$$n + r_n^G(H,K) \geq \operatorname{type} \Phi^K(Y).$$

Hence, $n + r_n^G(H,K) \geq \operatorname{type}^K(Y) > n + r$ and so $r_n^G(H,K) > r$. This completes the proof for $p = 2$. For odd p, the difficulty is selecting an m and relies on setting Y to be either F_m^e or F_m^o. This is carefully written up in [27, §6]. □

By combining Theorem 3.1.1 with Theorem 2.3.16 we obtain the following theorem.

Theorem 3.1.2. *If C is a finite cyclic p-group and X is a finite C-space, then $k_{n-1}(X^C) \leq k_n(X)$ for all $n \geq 1$.*

Proof. Let X be a finite C-space. Applying Theorem 3.1.1 to the pair $(C, \{e\})$ yields

$$k_{n-1}(X^C) \leq k_{n-1+r_{n-1}^C(C,\{e\})}(X).$$

Theorem 2.3.16 tells us that $r_{n-1}^C(C,\{e\}) = 1$. □

Example 3.1.3. Ravenel's Proposition 2.2.2 is recovered from Theorem 3.1.1 by setting G to be the trivial group and $r = 1$. In this case, $r_n^{\{e\}}(\{e\},\{e\}) = 0$, hence $1 \geq r_n^{\{e\}}(\{e\},\{e\})$. Thus, Theorem 3.1.1 yields $k_{n-1}(X) \leq k_n(X)$.

3.2 Margolis homology

Let $\mathcal{A}_\mathrm{p}$ denote the mod p Steenrod algebra, that is, the algebra of stable cohomology operations of $H^*(-;\mathbb{Z}/\mathrm{p}\mathbb{Z})$. For $p = 2$, it is generated by the Steenrod squares

$\mathrm{Sq}^i\colon H^*(-;\mathbb{Z}/2\mathbb{Z}) \to H^{*+i}(-;\mathbb{Z}/2\mathbb{Z})$ subject to the Adem relations. For an introduction to this material see [19, §4.L]. The Steenrod squares satisfy the following properties:

- $\mathrm{Sq}^i(x) = 0$ if $i > |x|$,
- $\mathrm{Sq}^i(x) = x^2$ for $|x| = i$,
- The Cartan formula: $\mathrm{Sq}^k(xy) = \sum_{i+j=k} \mathrm{Sq}^i(x)\,\mathrm{Sq}^j(y)$.

For odd p, we let P^i denote the ith pth power operation

$$P^i\colon H^*(-;\mathbb{Z}/\mathrm{p}\mathbb{Z}) \to H^{*+2i(p-1)}(-;\mathbb{Z}/\mathrm{p}\mathbb{Z})$$

of degree $|P^i| = 2i(p-1)$ and denote by

$$\beta\colon H^*(-;\mathbb{Z}/\mathrm{p}\mathbb{Z}) \to H^{*+1}(-;\mathbb{Z}/\mathrm{p}\mathbb{Z})$$

the Bockstein homomorphism associated to the coefficient sequence

$$0 \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow \mathbb{Z}/p^2\mathbb{Z} \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow 0.$$

With this notation, $\mathcal{A}_{\mathrm{p}}$ is generated by $\beta, P^1, P^2, \ldots$ subject to the Adem relations [19, Pg. 496].

The algebra $\mathcal{A}_{\mathrm{p}}$ is endowed with the structure of a graded Hopf algebra over $\mathbb{Z}/\mathrm{p}\mathbb{Z}$. The coproduct $\Delta\colon \mathcal{A}_{\mathrm{p}} \to \mathcal{A}_{\mathrm{p}} \times \mathcal{A}_{\mathrm{p}}$ is given by the Cartan formula

$$\Delta(P^k) = \sum_{i+j=k} P^i \otimes P^k.$$

An element x in a Hopf algebra is said to be *primitive* if the coproduct $\Delta(x) = 1 \otimes x \ + \ x \otimes 1$. The primitives of the Steenrod algebra were characterized by Milnor in [33]. In particular, the nth Milnor primitive Q_n is defined inductively as follows. For $p = 2$, $Q_0 = \mathrm{Sq}^1$ and in general

$$Q_n = [\mathrm{Sq}^{2^n}, Q_{n-1}] = \mathrm{Sq}^{2^n} Q_{n-1} + Q_{n-1} \mathrm{Sq}^{2^n} .$$

For odd primes, $Q_0 = \beta$ and in general

$$Q_n = [P^{p^{n-1}}, Q_{n-1}] = P^{p^{n-1}} Q_{n-1} - Q_{n-1} P^{p^{n-1}}.$$

As a consequence of being primitive these are also derivations. We state the properties of Q_n that we will need:

- Q_n is of degree $2(p-1)(p^{n-1} + \ldots + p) + 1 = 2p^n - 1$,
- $Q_n(x \cup y) = x \cup Q_n(y) + (-1)^{|x||y|} Q_n(x) \cup y$, i.e. Q_n is a derivation,
- $Q_n^2 = 0$ [31, Lemma 15.4].

Let M be a module over $\mathcal{A}_{\mathrm{p}}$. By assumption, the primitive Q_n acts on M. Since $Q_n^2 = 0$, we may form a chain complex

$$\ldots \xrightarrow{Q_n} M_* \xrightarrow{Q_n} M_{*+|Q_n|} \xrightarrow{Q_n} \ldots .$$

The homology of this chain complex will be denoted $H_*(M; Q_n)$ and is known as the *Margolis homology* of M. When M is the $\mathbb{Z}/\mathrm{p}\mathbb{Z}$-cohomology of a space (or spectrum) X, we will denote $H_*(H^*(X; \mathbb{Z}/p\mathbb{Z}); Q_n)$ simply by $H_*(X; Q_n)$. These operations were central to Margolis' study of the "deep structure" of the Steenrod algebra in [31]. We let $k_{Q_n}(X)$ denote the dimension of the Margolis homology of the space X, that is, $k_{Q_n}(X) = \dim_{\mathbb{Z}/\mathrm{p}\mathbb{Z}} H_*(X; Q_n)$.

Example 3.2.1. The real projective space $\mathbb{R}\mathrm{P}^\ell$ has $\mathbb{Z}/2\mathbb{Z}$-cohomology

$$H^*(\mathbb{R}\mathrm{P}^\ell;\mathbb{Z}/2\mathbb{Z}) \;\cong\; \mathbb{Z}/2\mathbb{Z}[\alpha]/(\alpha^{\ell+1})$$

with $|\alpha| = 1$. We will compute the Margolis homology of this $\mathcal{A}_2$-module.

If $\ell < 2^{n+1}$, then Q_n is zero for all classes for dimension reasons, and thus $H_*(\mathbb{R}\mathrm{P}^\ell; Q_n) = H^*(\mathbb{R}\mathrm{P}^\ell;\mathbb{Z}/2\mathbb{Z})$, with $k_{Q_n}(\mathbb{R}\mathrm{P}^\ell) = \ell + 1$.

Now assume $\ell \geq 2^{n+1}$. To begin, one must compute the Steenrod squares Sq^{2^n}. Using the basis properties of Steenrod squares we calculate that $\mathrm{Sq}^{2^n}(\alpha) = 0$ for all $n \geq 1$ and $\mathrm{Sq}^1(\alpha) = \alpha^2$. Thus, $Q_0(\alpha) = \mathrm{Sq}^1(\alpha) = \alpha^2$. This allows us to compute $Q_n(\alpha)$ inductively:

$$\begin{aligned} Q_1(\alpha) &= \mathrm{Sq}^2\,\mathrm{Sq}^1(\alpha) + \mathrm{Sq}^1\,\mathrm{Sq}^2(\alpha) \\ &= \mathrm{Sq}^2\,\alpha^2 + 0 \\ &= \alpha^4. \end{aligned}$$

Since $Q_n = [\mathrm{Sq}^{2^n}, Q_{n-1}]$, one may then check that $Q_n(\alpha) = \alpha^{2^{n+1}}$. Using that Q_n is a derivation, we find

$$\begin{aligned} Q_n(\alpha^k) &= k\alpha^{k-1}Q_n(\alpha) \\ &= k\alpha^{k-1+2^{n+1}}. \end{aligned}$$

As we are working in characteristic two this calculation shows that for even k that $Q_n(\alpha^k) = 0$. For odd k satisfying $k - 1 + 2^{n+1} \leq n$, one has $Q_n(\alpha^k) = \alpha^{k-1+2^{n+1}}$.

Generically, the Margolis homology will have even classes at the bottom and odd classes at the top. We will break into two cases based on the parity of ℓ. Assume that ℓ is odd. Our calculation of $Q_n(\alpha^k)$ above shows that every even class α^{2k} is a cycle, while the only odd classes α^{2k-1} that are cycles are those for which Q_n lands in degree bigger than ℓ. Hence, the cycles are spanned by

$$1, \alpha^2, \ldots, \alpha^{\ell-1}, \alpha^{\ell-2^{n+1}+2}, \ldots, \alpha^{\ell}.$$

This gives the dimension of the cycles as $\frac{\ell+1}{2}+2^n$. The boundaries are spanned by the odd classes $\alpha, \alpha^3, \ldots, \alpha^{\ell-2^{n+1}}$. Giving the dimension of the boundaries as $\frac{\ell+1}{2}-2^n$. Subtracting the dimension of the boundaries from the cycles, yields $k_{Q_n}(\mathbb{R}^\ell) = 2^{n+1}$ for ℓ odd with $\ell \geq 2^{n+1}$.

We now consider the even case by examining the $\ell+1$ case for ℓ odd. The cycles are now spanned by

$$1, \alpha^2, \ldots, \alpha^{\ell+1}, \alpha^{\ell-2^{n+1}+4}, \ldots, \alpha^{\ell}.$$

This gives the dimension of the cycles as $\frac{\ell+1}{2}+2^n$. The boundaries are now spanned by $\alpha, \alpha^3, \ldots, \alpha^{\ell-2^{n+1}+2}$ yielding the dimension $\frac{\ell+1}{2}-2^n+1$. Subtracting the dimension of the boundaries from the dimension of the cycles yields $2^{n+1}-1$. Thus, we have found that

$$k_{Q_n}(\mathbb{R}\mathrm{P}^\ell) = \begin{cases} \ell+1 & \ell < 2^{n+1} \\ 2^{n+1} & \text{odd } \ell \geq 2^{n+1} \\ 2^{n+1}-1 & \text{even } \ell \geq 2^{n+1}. \end{cases}$$

3.3 Atiyah-Hirzebruch spectral sequence

For any generalized cohomology theory E^* the Atiyah-Hirzebruch spectral sequence (AHSS) is of the form

$$E_2^{i,j} = H^i(X; E^j(*)) \implies E^{i+j}(X).$$

This spectral sequence was originally used to compute topological K-theory in [2]. For our purposes E^* will exclusively be the nth Morava K-theory $K(n)^*$. As the coefficient ring $K(n)^*$ is isomorphic to $\mathbb{Z}/\mathrm{p}\mathbb{Z}[v_n^{\pm}]$ for $|v_n| = -(2p^n - 2)$, the AHSS for $K(n)^*(X)$ is a right half plane spectral sequence with copies of the underlying cohomology $H^*(X; \mathbb{Z}/\mathrm{p}\mathbb{Z})$ along the v_n power rows.

Example 3.3.1. Let's consider the AHSS of the 2-local $K(1)^*(\mathbb{R}\mathrm{P}^6)$. We will consider the E_2-page. Along the $s = 0$ column we have a copy of the coefficient ring $\mathbb{Z}/2\mathbb{Z}[v_1, v_1^{-1}]$ and along the $t = 0$ we have a copy of $H^*(\mathbb{R}\mathrm{P}^6; \mathbb{Z}/2\mathbb{Z})$. Furthermore, along each of the v_1-power rows we have copies of this cohomology multiplied by the power of v_1.

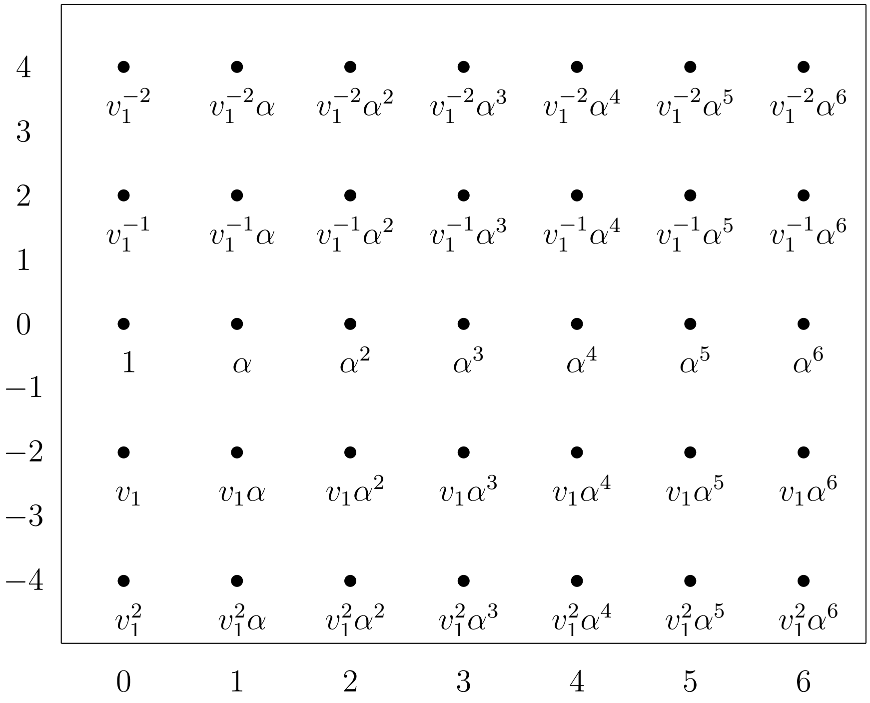


Figure 3-1: A piece of the E_2-page of the AHSS for $K(1)^*(\mathbb{R}\mathrm{P}^6)$. The class α is the generator of $H^*(\mathbb{R}\mathrm{P}^6; \mathbb{Z}/2\mathbb{Z})$.

The first possibly non-trivial differential is d_{2p^n-1} since to be non-trivial it must travel from one v_n-power row to the next. As $K(n)$ is a ring spectrum the AHSS is a spectral sequence of algebras over the coefficient ring $\mathbb{Z}/\mathrm{p}\mathbb{Z}[v_n, v_n^{-1}]$. This differential is given by

$$d_{2p^n-1} = Q_n(x)v_n,$$

where Q_n is the nth Milnor primitive [53].

Example 3.3.2. We continue our example by using the calculations of Q_1 from Example 3.2.1 to work out the E_3-page:

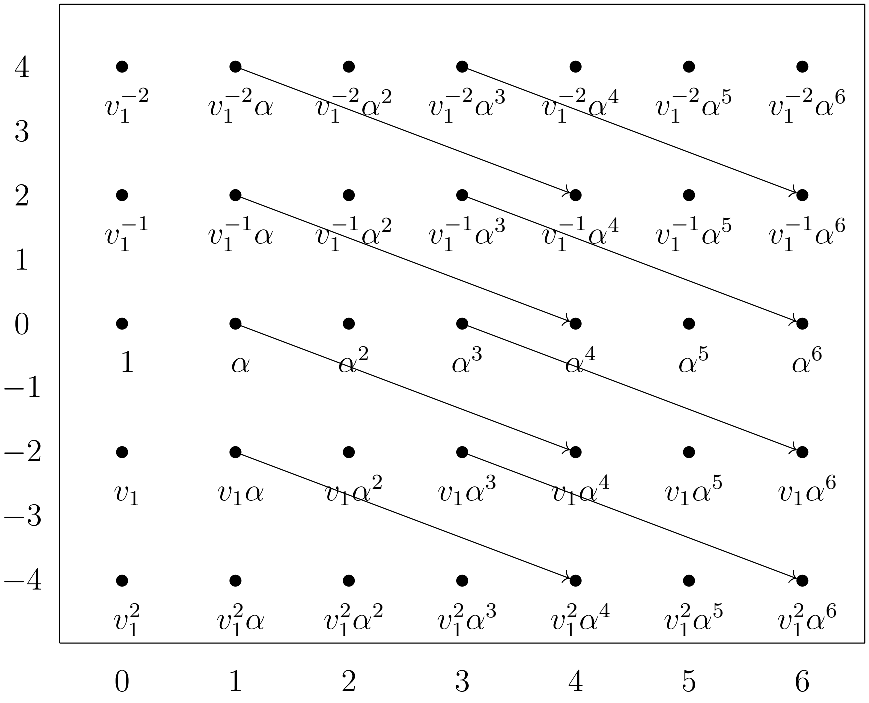


Figure 3-2: A piece of the E_3-page of the AHSS for $K(1)^*(\mathbb{R}\mathrm{P}^6)$ with the non-trivial differentials drawn.

In general, the higher differentials do not have such a nice description. For a differential to be non-trivial it must go from one v_n-power row to another, otherwise the domain or the range would be zero. As these rows occur in multiples of $2p^n - 2$ the only potentially non-trivial differentials are $d_{k(2p^n-2)+1}$ for $k \geq 1$. In particular, this means that if the E_2-page is concentrated in even degrees the AHSS collapses. The purpose of this chapter is to establish a new sufficient condition for when the higher differentials do not exist.

After the Q_n-differential one is left with copies of the Margolis homology $H_*(X; Q_n)$

along each v_n power row, and thus one has an upper bound

$$k_n(X) \leq k_{Q_n}(X). \tag{3.3.3}$$

Example 3.3.4. We see that $k_{Q_1}(\mathbb{R}\mathrm{P}^6) = 3$ yielding the upper bound $k_1(\mathbb{R}\mathrm{P}^6) \leq 3$.

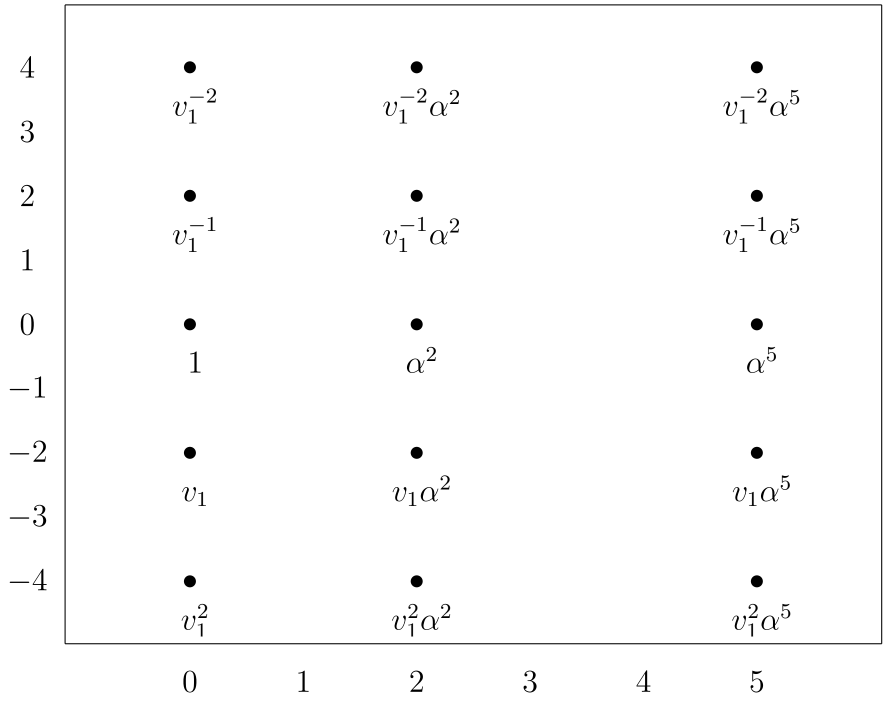


Figure 3-3: A piece of the E_4-page of the AHSS for $K(1)^*(\mathbb{R}\mathrm{P}^6)$.

3.4 Collapsing of the AHSS

Putting together the lower bound from Theorem 3.1.2 together with the upper bound (3.3.3) yields our collapsing theorem:

Theorem 3.4.1. *If a finite complex X admits an action of a finite cyclic p-group C such that $k_{n-1}(X^C) = k_{Q_n}(X)$, then the AHSS computing $K(n)^*(X)$ collapses on the $2p^n$-page and $k_n(X) = k_{Q_n}(X)$.*

Example 3.4.2. We can use this technique to easily show that $K(n)^*(\mathbb{R}\mathrm{P}^\ell)$ has no higher non-trivial differentials, and so $k_n(\mathbb{R}\mathrm{P}^\ell) = k_{Q_n}(\mathbb{R}\mathrm{P}^\ell)$. Let $G = C_2$ be the cyclic By Theorem 3.4.1, it suffices to find a C_2 action on $\mathbb{R}\mathrm{P}^{C_2}$ such that $k_{n-1}((\mathbb{R}\mathrm{P}^\ell)^{C_2}) = k_{Q_n}(\mathbb{R}\mathrm{P}^\ell)$. Let 1 denote the trivial representation and Σ denote the sign representation. For $\ell < 2^{n+1}$ there is no room for higher differentials. We proceed by induction on ℓ. Suppose for all $s < \ell$ that $k_{n-1}(\mathbb{R}\mathrm{P}^s) = k_{Q_{n-1}}(\mathbb{R}\mathrm{P}^s)$

If ℓ is odd, then set $V = \frac{\ell+1}{2}(1 \oplus \Sigma)$, so that $\mathbb{R}\mathrm{P}(V) = \mathbb{R}\mathrm{P}^\ell$ and

$$\begin{aligned}\mathbb{R}\mathrm{P}(V)^{C_2} &= \mathbb{R}\mathrm{P}\left(\tfrac{\ell+1}{2}(1)\right) \sqcup \mathbb{R}\mathrm{P}\left(\tfrac{\ell+1}{2}\Sigma\right)\\ &= \mathbb{R}\mathrm{P}^{\frac{\ell+1}{2}-1} \sqcup \mathbb{R}\mathrm{P}^{\frac{\ell+1}{2}-1}.\end{aligned}$$

From our calculation in Example 3.2.1 we have that

$$k_{Q_n}(\mathbb{R}\mathrm{P}(V)) = 2^{n+1}.$$

Since $\ell \geq 2^{n+1}$, we know $\frac{\ell+1}{2} \geq 2^n$. By our inductive hypothesis, if $\frac{\ell+1}{2} \leq 2^{n+1}$, then $k_{n-1}(\mathbb{R}\mathrm{P}^{\frac{\ell+1}{2}}) = 2^n$, and thus $k_{n-1}(\mathbb{R}\mathrm{P}(V)^{C_2}) = 2^n + 2^n = 2^{n+1}$. Thus, by Theorem 3.4.1 the AHSS collapses for $k_n(\mathbb{R}\mathrm{P}^\ell)$.

If ℓ is even let $V = \left(\frac{\ell}{2}+1\right)1 \oplus \frac{\ell}{2}\Sigma$, then $\mathbb{R}\mathrm{P}(V) = \mathbb{R}\mathrm{P}^\ell$, and

$$\mathbb{R}\mathrm{P}(V)^{C_2} = \mathbb{R}\mathrm{P}^{\frac{\ell}{2}} \sqcup \mathbb{R}\mathrm{P}^{\frac{\ell}{2}-1}.$$

Now $k_{Q_n}(\mathbb{R}\mathrm{P}^\ell) = 2^{n+1} - 1$. By the inductive hypothesis,

$$k_{n-1}(\mathbb{R}\mathrm{P}(V)^{C_2}) = (2^n - 1) + 2^n = 2^{n+1} - 1,$$

and thus by Theorem 3.4.1, the AHSS collapses for $k_n(\mathbb{R}\mathrm{P}^\ell)$. This completes the induction.

Remark 3.4.3. *One may see that the AHSS computing $K(n)^*(\mathbb{R}\mathrm{P}^\ell)$ collapses directly by realizing that the even classes are permanent cycles coming from the canonical line bundle over $\mathbb{R}\mathrm{P}^\infty$ complexified.*

3.5 Periodic to connective

Let $k(n)_*$ denote the p-local nth connective Morava K-theory [52, §5]. The coefficient ring of this theory is $\mathbb{Z}/\mathrm{p}\mathbb{Z}[v_n]$ with $|v_n| = 2(p^n - 1)$.

Proposition 3.5.1. *If the AHSS computing $K(n)_*(X)$ collapses after the first possibly non-trivial differential, then so does the AHSS computing the connective Morava K-theory $k(n)_*(X)$.*

Proof. Recall that the Morava K-theory $K(n)_*(X)$ arises from the connective Morava K-theory by inverting v_n, $K(n)_*(X) \cong v_n^{-1}k(n)_*(X)$. In particular, there is a natural transformation $k(n)_* \implies K(n)_*$. This induces a map of spectral sequences from the AHSS

$$E^2_{s,t} = H_s(X; \mathbb{Z}/\mathrm{p}\mathbb{Z}[v_n]) \implies k(n)_*(X)$$

computing $k(n)_*(X)$ to the AHSS

$$\bar{E}^2_{s,t} = H_s(X; \mathbb{Z}/\mathrm{p}\mathbb{Z}[v_n, v_n^{-1}]) \implies K(n)_*(X)$$

computing $K(n)_*(X)$. We will denote this map by $f_r\colon (E^r_{*,*}, d_r) \to (\bar{E}^r_{*,*}, \bar{d}_r)$. In particular $f_r d_r = \bar{d}_r f_r$ and each f_{r+1} is induced by f_r.

The AHSS computing the connective theory is a first quadrant spectral sequence, while the AHSS computing $K(n)_*$ is a right half-plane spectral sequence. These spectral sequences have the same first quadrant, because they come from the same filtration of X and their coefficient rings are equal for $t \geq 0$. Furthermore, $f_2^{s,t}$ must be an isomorphism for $t \geq 0$. For both of these spectral sequences the first possibly non-trivial differential occurs on the $(2p^n - 1)$-page, because the differentials must go from one row to the next and the degree of v_n in both cases is $2(p^n - 1)$. In particular, this means that f_{2p^n-1} is an isomorphism for $t \geq 0$. In the first quadrant the differentials d_{2p^n-1} and $\bar{d}_{2p^n-1}$ agree, hence on the next page $E^{2p^n}_{*,t} \cong \bar{E}^{2p^n}_{*,t}$ agree for $t > 0$. Along the $t = 0$ line the classes no longer agree, because $E^{2p^n}_{*,0}$ is missing boundaries coming from the forth quadrant.

We will proceed by induction. Suppose in general for some $r \geq 2p^n$ that the map $f_r^{*,t}\colon E^r_{*,t} \to \bar{E}^r_{*,t}$ is an isomorphism for $t > 0$ and that $\bar{d}_r = 0$. For $x \in E^r_{s,t}$ with $t \geq 0$ we wish to show that $d(x) = 0$. We compute $f_r^{s,t+r}(d_r(x)) = \bar{d}(f_r^{s,t}(x))$. The differential $\bar{d}_r$ was assumed to be zero, hence $f_r^{s,t+r}(d_r(x)) = 0$. Since $t \geq 0$ the degree $t + r > 0$, hence $f^{s,t+r}$ was assumed to be an isomorphism. Thus in this range $f_r^{s,t+r}(d_r(x)) = 0$, implies that $d_r(x) = 0$. We have shown that $\bar{d}_r(x) = 0$ implies that $d_r(x) = 0$. Since $(E^r_{*,t}, d_r)$ and $(\bar{E}^r_{*,t}, \bar{d}_r)$ agree for $t > 0$ and the boundaries coming from $t = 0$ must be zero, the map $f_r^{*,t}$ induces an isomorphism $f_{r+1}^{*,t}$ for $t > 0$. This allows us to complete the induction. □

Remark 3.5.2. *For those who prefer working with the Adams spectral sequence: By feeding $k(n) \wedge X$ into the classical Adams spectral sequence one has a spectral sequence that computes the p-local $k(n)_*(X)$. An Adams resolution for $k(n) \wedge X$*

comes from the Postnikov tower of $k(n)$ *(see [52, §5]). The AHSS normally comes from the cellular filtration of* X*. However, [18, Theorem B.8] shows that one could also obtain the same AHSS from the Postnikov tower of* $k(n)$ *instead. The chosen Adams resolution yields an* E_1*-page that agrees with the* E_2*-page of the AHSS, hence the Adams spectral sequence computing* $k(n)_*(X)$ *and the AHSS computing* $k(n)_*(X)$ *agree. We give more details about this in Appendix C.*

Chapter 4

Application to real Grassmannians

Let $\mathrm{Gr}_d(V)$ denote the Grassmannian of d-dimensional subspaces of the vector space V. For our purposes V will mostly be a real vector space. In this case $\mathrm{Gr}_d(\mathbb{R}^{d+c})$ is a compact (dc)-dimensional manifold. It is topologized as a quotient space of the Stiefel Manifold of d-frames in $\mathbb{R}^{d+c}$. For more background on $\mathrm{Gr}_d(V)$ one may consult [34, §5]. The Grassmannians generalize the projective spaces, $\mathrm{Gr}_1(\mathbb{R}^\ell) = \mathbb{R}\mathrm{P}^{\ell-1}$.

The real Grassmannians have a CW-complex structure. The canonical inclusion of $\mathrm{Gr}_d(\mathbb{R}^{d+c}) \hookrightarrow \mathrm{Gr}_d(\mathbb{R}^{d+c+1})$ is an inclusion of a subcomplex. To form the infinite Grassmannian $\mathrm{Gr}(\mathbb{R}^\infty)$ one may then take the colimit of these inclusions

$$\mathrm{Gr}_d(\mathbb{R}^\infty) \simeq \operatorname{colim}\left(\mathrm{Gr}_d(\mathbb{R}^d) \hookrightarrow \mathrm{Gr}_d(\mathbb{R}^{d+1}) \hookrightarrow \mathrm{Gr}_d(\mathbb{R}^{d+2}) \hookrightarrow \ldots\right).$$

This provides a model for the classifying space $BO(d)$ of the orthogonal group $O(d)$. There is a canonical $\mathbb{R}^d$-vector bundle $\gamma^{d,c}$ over $\mathrm{Gr}_d(\mathbb{R}^{d+c})$ with total space

$$E(\gamma^{d,c}) = \{(V, v) \mid v \in V, \text{ for } V \text{ a } d\text{-subspace of } \mathbb{R}^{d+c}\} \subset \mathrm{Gr}_d(\mathbb{R}^{d+c}) \times \mathbb{R}^{d+c}.$$

The projection is given by $(V, v) \mapsto V$ and the vector space structure is inherited from the second coordinate. Performing this same construction for $\mathrm{Gr}_d(\mathbb{R}^\infty)$ yields the tautological bundle, which we denote γ^d. The Stiefel–Whitney classes of this bundle are one avenue to understanding the cohomology of $H^*(\mathrm{Gr}_d(\mathbb{R}^{d+c}); \mathbb{Z}/2\mathbb{Z})$, which we review in §4.1. An alternative approach is through the use of Schubert calculus, which will be reviewed in §4.2.

The majority of the chapter focuses on the calculations of the Morava K-theory of $\mathrm{Gr}_d(\mathbb{R}^{d+c})$ at $p = 2$, because this is the case that requires our novel techniques. For completeness, we consider the odd prime case in §4.10.

4.1 The Borel picture

For a vector bundle $\xi \colon E \to B$ we let $w_i(\xi) \in H^i(B; \mathbb{Z}/2\mathbb{Z})$ denote the ith *Stiefel–Whitney class* of ξ and $w(\xi) = w_1(\xi) + w_2(\xi) + \dots$ denote the *total Stiefel–Whitney* class of the vector bundle ξ [34, §4]. The *dual Stiefel–Whitney* classes $\bar{w}(\xi) = \bar{w}_1(\xi) + \bar{w}_2(\xi) + \dots$ are defined inductively by the relation $w(\xi)\bar{w}(\xi) = 1$.

Since $w_i(\xi) = 0$ for $i > \operatorname{rank} \xi$ one sees that $\mathrm{Gr}_d(\mathbb{R}^\infty)$ has at most $d + 1$ non-zero Stiefel-Whitney classes: $1, w_1(\gamma^d), \dots, w_d(\gamma^d)$. In fact, the cohomology ring of $H^*(\mathrm{Gr}_d(\mathbb{R}^\infty); \mathbb{Z}/2\mathbb{Z})$ is polynomial on the Stiefel-Whitney classes,

$$H^*(\mathrm{Gr}_d(\mathbb{R}^\infty); \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}[w_1(\gamma^d), \dots, w_d(\gamma^d)]. \tag{4.1.1}$$

This was shown originally by Borel in 1953 [10], although for an account in English see [34]. Borel also characterized the cohomology of the finite Grassmannian $\mathrm{Gr}_d(\mathbb{R}^{d+c})$.

The inclusion of $i\colon \operatorname{Gr}_d(R^{d+c}) \hookrightarrow \operatorname{Gr}_d(\mathbb{R}^\infty)$ induces a surjective map

$$H^*(\operatorname{Gr}_d(\mathbb{R}^\infty); \mathbb{Z}/2\mathbb{Z}) \to H^*(\operatorname{Gr}_d(R^{d+c}); \mathbb{Z}/2\mathbb{Z}).$$

The kernel of this map will be denoted $J(d,c)$ and is generated by the dual Stiefel–Whitney classes $\bar{w}_{c+1}(\gamma^d), \bar{w}_{c+2}(\gamma^d), \ldots, \bar{w}_{d+c}(\gamma^d)$ expressed in $w_1(\gamma^d), \ldots, w_d(\gamma^d)$. In particular, this means that

$$H^*(\operatorname{Gr}_d(\mathbb{R}^{d+c}); \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}[w_1(\gamma^d), \ldots, w_d(\gamma^d)]/J(d,c). \tag{4.1.2}$$

From now on, we write $w_i = w_i(\gamma^d)$ when the context is clear. Although, this description is explicit, it is difficult to work with. An alternative description of $H^*(\operatorname{Gr}_d(\mathbb{R}^{d+c}); \mathbb{Z}/2\mathbb{Z})$ is

$$H^*(\operatorname{Gr}_d(\mathbb{R}^{d+c}); \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}[w_1, \ldots, w_d, \overline{w}_1, \ldots, \overline{w}_c]/(w\bar{w} = 1),$$

One can prove that $\dim_{\mathbb{Z}/2\mathbb{Z}} H^*(\operatorname{Gr}_d(\mathbb{R}^{d+c}); \mathbb{Z}/2\mathbb{Z}) = \binom{d+c}{d}$ using the cell-decomposition or the following lemma.

Lemma 4.1.3. *For any field $\mathbb{F}$,*

$$\dim_{\mathbb{F}} \left(\frac{\mathbb{F}[x_1, \ldots, x_i, y_1, \ldots, y_j]}{(1 + x_1 + \ldots + x_i)(1 + y_1 + \ldots + y_j) = 1} \right) = \binom{i+j}{i}.$$

Proof. Let $A = \frac{\mathbb{F}[x_1,\ldots,x_i,y_1,\ldots,y_j]}{(1+x_1+\ldots+x_i)(1+y_1+\ldots+y_j)=1}$. The Poincaré series $p_t(A) = \sum_{i=0}^{\infty} (\dim_{\mathbb{F}} A_i) t^i$ is the Gaussian binomial coefficient $\binom{i+j}{i}_t$ [12, Pg. 294-297]. Evaluating the Poincaré series at $t = 1$ yields the total dimension

$$\dim_{\mathbb{F}} A = \binom{i+j}{i}_1 = \binom{i+j}{i}. \qquad \square$$

More recently, in 1991, Jaworowski produced a monomial basis for the cohomology $H^q(\mathrm{Gr}_d(\mathbb{R}^{d+c}); \mathbb{Z}/2\mathbb{Z})$ in terms of words in these Stiefel-Whitney classes.

Theorem 4.1.4 ([23, Pg 232])**.** *The monomials* $w_1^{r_1} w_2^{r_2} \cdots w_d^{r_d}$ *in degree* $q = \sum_{i=1}^d i r_i$ *such that* $\sum_{i=1}^d r_i \leq c$ *form an additive basis for* $H^q(\mathrm{Gr}_d(\mathbb{R}^{d+c}); \mathbb{Z}/2\mathbb{Z})$*.*

Example 4.1.5. Let's consider $\mathrm{Gr}_2(\mathbb{R}^6)$. In this case $d = 2$ and $c = 4$. From Borel's description we know that

$$H^*(\mathrm{Gr}_2(\mathbb{R}^6); \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}[w_1, w_2]/J(2, 4),$$

where $J(2, 4)$ is generated by $\overline{w}_5, \overline{w}_6$. To explicitly compute these dual elements one proceeds by induction. The key relation coming from $w\overline{w} = 1$ is

$$\overline{w}_i = w_i + w_1\overline{w}_{i-1} + w_2\overline{w}_{i-2} + \ldots + w_{i-1}\overline{w}_1.$$

This calculation is really performed in $H^*(\mathrm{Gr}_d(\mathbb{R}^\infty); \mathbb{Z}/2\mathbb{Z})$ and thus only depends on d and not c. In the case $d = 2$, we find

$$\overline{w}_5 = w_1^5 + w_1 w_2^2$$
$$\overline{w}_6 = w_1^6 + w_1^4 w_2 + w_2^3.$$

From this description of $H^*(\mathrm{Gr}_2(\mathbb{R}^6); \mathbb{Z}/2\mathbb{Z})$ it is not immediately clear what the basis for a given homology group is. However, Theorem 4.1.4 explicitly yields

$$\begin{aligned}
H^0 &= \langle 1 \rangle \\
H^1 &= \langle w_1 \rangle \\
H^2 &= \langle w_1^2, w_2 \rangle \\
H^3 &= \langle w_1^3, w_1 w_2 \rangle \\
H^4 &= \langle w_1^4, w_1^2 w_2, w_2^2 \rangle \\
H^5 &= \langle w_1^3 w_2, w_1 w_2^2 \rangle \\
H^6 &= \langle w_2^3, w_1^2 w_2^2 \rangle \\
H^7 &= \langle w_1 w_2^3 \rangle \\
H^8 &= \langle w_2^4 \rangle,
\end{aligned}$$

where we have abbreviated $H^*(\mathrm{Gr}_2(\mathbb{R}^6); \mathbb{Z}/2\mathbb{Z})$ as H^*.

4.1.1 Steenrod operations

A first step to computing Q_n on $H^*(\mathrm{Gr}_d(\mathbb{R}^{d+c}); \mathbb{Z}/2\mathbb{Z})$ is to compute the Steenrod squares on the Stiefel–Whitney classes. The unstable condition guarantees that $\mathrm{Sq}^i(w_j) = 0$ for $i > j$. Since w_j is a Stiefel–Whitney class of smooth manifold, $\mathrm{Sq}^i(w_j)$ is given by the Wu formula [51]

$$\mathrm{Sq}^i(w_j) = \sum_{t=0}^{i} \binom{j+t-i-1}{t} w_{i-t} w_{j+t}. \tag{4.1.6}$$

Next, we wish to compute Sq^i on the monomial basis elements. This can be done recursively via the Cartan formula. Given a word $x_1 x_2 \cdots x_k$, where each $x_i \in$

$\{w_1, \ldots, w_d\}$, the Cartan formula states

$$\mathrm{Sq}^i(x_1 x_2 \cdots x_k) = \sum_{t=0}^{i} \mathrm{Sq}^t(x_1) \cup \mathrm{Sq}^{i-t}(x_2 \cdots x_k).$$

Now, x_1 is a w_j and so $\mathrm{Sq}^t(x_1)$ is given by the Wu formula. Now one expands each $\mathrm{Sq}^{i-t}(x_2 \cdots x_k)$ similarly. This algorithm is straight forward to implement, but the complexity is horrendous: written in big O notation, it is $O((i+1)^{(k-1)})$.

Another issue is that our calculation of Sq^i on a monomial basis element $w_1^{r_1} \cdots w_d^{r_d}$ does not write $\mathrm{Sq}^i(w_1^{r_1} \cdots w_d^{r_2})$ back in terms of the basis. This would require rewriting the expression using the relations in $J(d, c)$. Luckily, there is a tool from computational commutative algebra that deals with this problem, namely, *Gröbner bases.*

4.1.2 Gröbner bases

We follow the treatment in [13, §9.6]. Let $R = F[x_1, \ldots, x_k]$ be a polynomial ring over a field F. One chooses a well ordering on monomials such that for monomials m_1, m_2, p, the relation $m_1 \leq m_2$ holds if and only if $m_1 p \leq m_2 p$.

Definition 4.1.7. Fix a monomial ordering on $R = F[x_1, \ldots, x_k]$. The *leading term* of a nonzero polynomial f in R, denoted $LT(f)$, is the monomial term of maximal order in f. The leading term of the zero polynomial is defined to be zero. The *ideal of leading terms* $LT(I)$ of an ideal I of R is defined to be the ideal generated by the leading terms of elements of I, that is, $LT(I) = (LT(f) \mid f \in I)$.

The introduction of leading terms allows one to extend the standard polynomial division algorithm to $F[x_1, \ldots, x_k]$.

Definition 4.1.8. A Gröbner basis for an ideal I of $R = F[x_1, \ldots, x_k]$ with respect to a monomial ordering on R is a finite set of generators $\{g_1, \ldots, g_m\}$ for I such that the leading terms $LT(g_1), \ldots, LT(g_m)$ generate the ideal of leading terms $LT(I)$.

By $g_1, \ldots, g_m$ generating the ideal, we mean every element of I is a not necessarily unique linear combination of these elements. The key benefit of working with a Gröbner basis is given in the following theorem.

Theorem 4.1.9. *Fix a monomial ordering and suppose* $g_1, \ldots, g_m$ *is a Gröbner basis for a nonzero ideal* I *of* $R = F[x_1, \ldots, x_k]$ *with respect to this ordering. Then every polynomial* $f \in R$ *can be written uniquely as* $f = g + r$ *for* $g \in I$ *such that no nonzero monomial of* r *is divisible by any of the leading terms* $LT(g_1), \ldots, LT(g_m)$. *In particular,* r *is a unique representative for the coset* $f + I$ *in* $F[x_1, \ldots, x_k]/I$.

The expression $f = g + r$ is computed by the generalized division algorithm. This allows one to directly check when an element $f + I$ in R/I is zero. If one finds the associated remainder $r = 0$, then $f \in I$, and is thus zero. A Gröbner basis is not unique. However, by requiring a stronger property to hold we do obtain uniqueness.

Definition 4.1.10. A Gröbner basis $\{g_1, \ldots, g_m\}$ is said to be *reduced*, if each $LT(g_i)$ is monic and $LT(g_i)$ does not divide g_j for any $i \neq j$.

Buchberger's algorithm is a method for explicitly constructing a reduced Gröbner basis for a non-zero ideal I of $F[x_1, \ldots, x_k]$ once an ordering on monomials is chosen.

Theorem 4.1.11. *For a fixed monomial ordering on* $R = F[x_1, \ldots, x_k]$, *there exists a unique reduced Gröbner basis for each non-zero ideal* I *of* R.

In particular, this means that $J(d, c)$ always has a reduced Gröbner basis. The algorithms for the calculation of Gröbner bases as well as the reduction of a given element in R/I have been implemented in various computer algebra packages. An example of such a package is the SINGULAR program. This work mainly used the SageMath interface to this package. Although one may perform computer aided calculations without knowing the explicit reduced Gröbner basis, for our purposes it will be useful to have a characterization.

In a series of papers Petrović and Prvulović computed an explicit reduced Gröbner basis for $J(2,c)$ [37], $J(3,c)$ [38], and finally for general $J(d,c)$, with the additional author Radovanović [39]. We let $\mathrm{Groeb}(d,c)$ denote the reduced Gröbner basis of $J(c,d)$ with respect to the grlex ordering on monomials, which we now explain.

Definition 4.1.12. The grlex ordering on monomials in $R[x_1,\ldots,x_n]$ first orders monomials by total degree, $x_1^{e_{1,1}}\cdots x_n^{e_{1,n}} \leq x_1^{e_{2,1}}\cdots x_n^{e_{2,n}}$ if $e_{1,1}+\ldots+e_{1,n} < e_{2,1}+\ldots+e_{2,n}$. When two monomials have the same degree, lexicographical ordering is then applied, that is, $x_1^{e_{1,1}}\cdots x_n^{e_{1,n}} \leq x_1^{e_{2,1}}\cdots x_n^{e_{2,n}}$ if the first j such that $e_{1,j}\neq e_{2,j}$ satisfies $e_{1,j}<e_{2,j}$.

Theorem 4.1.13 ([37])**.** *With the grlex ordering on $\mathbb{Z}/2\mathbb{Z}[w_1,w_2]$ the tth reduced Gröbner basis element g_t^c for $J(2,c)$ is given by*

$$g_t^c = \sum_{b=t}^{\lfloor\frac{c+1+t}{2}\rfloor} \binom{c+1-b}{b-t} w_1^{c+1+t-2b} w_2^b,$$

and is of degree $c+1+t$, hence $\mathrm{Groeb}(2,c)=\{g_t^c \mid 0\leq t\leq c+1\}$.

Remark 4.1.14. *Our primary use for the Gröbner bases is rewriting the output of Sq^i and Q_n in terms of the monomial basis for $H^*(\mathrm{Gr}_d(\mathbb{R}^{d+c});\mathbb{Z}/2\mathbb{Z})$. Given any homogeneous element $f\in\mathbb{Z}/2\mathbb{Z}[w_1,\ldots,w_d]$, we can perform the generalized polynomial division using the associated reduced Gröbner basis of $J(d,c)$, to write f*

uniquely as $f = g + r$ for $g \in I$ and r such that no nonzero monomials of r are divisible by $LT(g_i))$ for $g_i \in \mathrm{Groeb}(d, c)$. In [39] it was shown that the monomials that are not divisible by these $LT(g_i)$ are exactly the monomial basis elements of Theorem 4.1.4, and thus r will always be written in this basis.

This provides some additional motivation for working with this basis versus the Schubert basis, which we will discuss in § 4.2.

4.1.3 Algorithm for the dimension of the Margolis homology

In this section we explain a naive algorithm for computing the dimension of the Margolis homology $H^*(\mathrm{Gr}_d(\mathbb{R}^{d+c}); Q_n)$. These algorithms will be expressed in pseudocode. The first step is to compute the Steenrod squares. Here we let x_i denote an element of $\{w_1, w_2, \ldots\}$.

```
procedure Sq($i$,$x = \sum_{t=1}^{s} x_{t,1} \cdots x_{t,k}$)
    if $s > 1$ then
        return $\sum_{t=1}^{s} \mathrm{Sq}(i, x_{t,1} \cdots x_{t,k})$          ▷ $\mathrm{Sq}^i$ is a homomorphism
    end if
    if $k = 1$ and $x = w_j$ then
        return $\sum_{t=0}^{i} \binom{j+t-i-1}{t} w_{i-t} w_{j+t}$          ▷ Wu formula
    end if
    return $\sum_{t=0}^{i} \mathrm{Sq}(i, x_{1,1}) \mathrm{Sq}(i-t, x_{1,2} \cdots x_{1,k})$          ▷ Cartan formula
end procedure
```

Next, we compute the nth Milnor primitive from the definition.

```
procedure $Q$($n$,$x$)
    if $n = 0$ then
        return $\mathrm{Sq}(1, x)$
```

```
        end if
        return $Q(n-1, \mathrm{Sq}(2^n, x)) + \mathrm{Sq}(2^n, Q(n-1, x))$
    end procedure
```

So far, these algorithms have not taken into account the relations in $J(d, c)$. This is where we will need the Gröbner basis. Given a polynomial ring $F[x_1, \ldots, x_n]$ with a monomial ordering and ideal I with Gröbner basis $g_1, \ldots, g_m$ there is a function $\textsc{Reduce}(x, \{g_1, \ldots, g_m\})$ that takes an element $x \in F[x_1, \ldots, x_n]$ and reduces it modulo I in the sense of Theorem 4.1.9. Such a function is typically provided by the computer algebra package. From Remark 4.1.14, reducing an element $x \in \mathbb{Z}/2\mathbb{Z}[w_1, \ldots, w_d]$ using the Gröbner basis $\mathrm{Groeb}(d, c)$ for the ideal $J(d, c)$ outputs x written in the monomial basis of Theorem 4.1.4. This will allow us to do linear algebra. In the following pseudocode we let c and d be implicit parameters.

```
    procedure QInMonomialBasis($n$,$x$)
        return Reduce($Q(n, x)$, $\mathrm{Groeb}(d, c)$)
    end procedure
```

Now that we are able to rewrite the output of Q_n in the basis, we may put Q_n in matrix form. We use the notation var ← val to express the assignment of the value *val* to the variable *var*. To indicated that we have initialized a variable *var* as an empty list we write var ← [ ]. Let $\textsc{Append}$(list,val) denote a function that appends the value *val* to the end of the list *list*. We also let $\textsc{MonomialBasisOf}(c, d, i)$ be a function that returns a list of the monomial basis elements of $H^i(\mathrm{Gr}_d(\mathbb{R}^{d+c}); \mathbb{Z}/2\mathbb{Z})$ as in Theorem 4.1.4.

```
    procedure QMatrixAt($n, i$)
        matrix ← [ ]
        for inputB in MonomialBasisOf($c, d, i$) do
```

```
    Qonbasis ← QInMonomialBasis(inputB)
    column ← [ ]
    for outputB in MonomialBasisOf($c, d, i + 2^{n+1} - 1$) do
      if outputB in Qonbasis then
        Append(column, 1)
      else
        Append(column, 0)
      end if
    end for
    Append(matrix, column)
  end for
  return matrix
end procedure
```

This allows us to compute $\dim_{\mathbb{Z}/2\mathbb{Z}} H^i(\mathrm{Gr}_2(\mathbb{R}^{d+c}); Q_n)$. Let Rank($M$) and Nullity($M$) denote the functions that calculate the rank and nullity of a matrix M respectively.

```
procedure DimOfMargolisAt($n, i$)
  nullity ← Nullity (QMatrixAt($n, i$))
  rank ← Rank (QMatrixAt($n, i - 2^{n+1} + 1$))
  return nullity − rank
end procedure
```

Finally, this can be used to compute the entire dimension of $k_{Q_n}(\mathrm{Gr}_d(\mathbb{R}^{d+c}))$.

```
procedure $k_Q(n)$
  return $\sum_{i=0}^{cd}$ DimOfMargolisAt($n, i$)
end procedure
```

This algorithm demonstrates a brute-force method for calculating k_{Q_n}. Notice that the complexity of this algorithm is at least $O\left(\binom{d+c}{d}\right)$, because we calculate Q_n on every basis element of $H^*(\mathrm{Gr}_d(\mathbb{R}^{d+c}); \mathbb{Z}/2\mathbb{Z})$. Furthermore, the recursive nature of how both Sq^i and Q_n are computed means the complexity is even worse. Using a combinatorial description of Q_n can reduce the complexity of calculating Q_n. However, one still has the underlying issue of needing to compute this differential on a significant number of classes. Nevertheless, for $d = 2$ we will be able to produce a closed formula for $k_{Q_n}(\mathrm{Gr}_2(\mathbb{R}^{2+c}))$, and for general d we will conjecture a value for the closed formula.

4.2 Schubert picture

We briefly describe the cell structure of $\mathrm{Gr}_d(\mathbb{R}^{d+c})$ of Ehresmann [14] following [28].

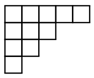
Definition 4.2.1. A *Schubert symbol* $\lambda = (\lambda_1, \ldots, \lambda_d)$ of $\mathrm{Gr}_d(\mathbb{R}^{d+c})$ is a sequence of integers

$$c \geq \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_d > 0.$$

The weight of λ is denoted $|\lambda|$ and is defined to be $\sum \lambda_i$.

Equivalently, this is a partition contained inside of a $d \times c$ grid. We will depict these partitions as Young diagrams, that is, as diagrams contained in a $d \times c$ grid with λ_i boxes in the ith row.

Example 4.2.2. Let $c = 5$ and $d = 4$. So $\lambda = (5, 3, 2, 1)$ is a valid Schubert symbol for $\mathrm{Gr}_4(\mathbb{R}^9)$ and is represented by the diagram



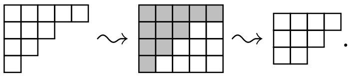
inside of the 4×5 grid.

To each such partition is associated the Schubert cell $e(\lambda)$ of dimension $|\lambda|$ in $\mathrm{Gr}_d(\mathbb{R}^{d+c})$, defined by

$$e(\lambda) = \{V \in \mathrm{Gr}_d(\mathbb{R}^{d+c}) \mid \dim(V \cap \mathbb{R}^{i+\lambda_{d+1-i}}) \geq i \text{ for } 1 \leq i \leq d\}.$$

Although they are referred to as cells, these $e(\lambda)$ are interiors of cells. The key fact for our purposes is that each Schubert symbol induces a non-trivial class in $H^*(\mathrm{Gr}_d(\mathbb{R}^{d+c}); \mathbb{Z}/2\mathbb{Z})$, which we will denote by s_λ. The Schubert basis for the qth cohomology $H^q(\mathrm{Gr}_d(\mathbb{R}^{d+c}); \mathbb{Z}/2\mathbb{Z})$ is given by $\{s_\lambda \mid |\lambda| = q\}$. The ring structure is described by a combinatorial formula, namely the Littlewood-Richardson rule. The top class $s_{(c^d)}$ of $\mathrm{Gr}_d(\mathbb{R}^{d+c})$ (where (c^d) denotes $(c, \dots, c)$ of length d) is indexed by the diagram that is a $d \times c$ grid. From this perspective there is a natural candidate for the duality map $D\colon H^*(\mathrm{Gr}_d(\mathbb{R}^{d+c}); \mathbb{Z}/2\mathbb{Z}) \to H^{cd-*}(\mathrm{Gr}_d(\mathbb{R}^{d+c}); \mathbb{Z}/2\mathbb{Z})$. One places the diagram λ inside of the top class. The complement is then an upside down Young diagram, which after flipping is the diagram that indexes the dual Schubert class.

Example 4.2.3. The dual diagram to $\lambda = (5, 3, 2, 1)$ in $\mathrm{Gr}_4(\mathbb{R}^9)$ is formed via



An important property for us is that with respect to this Poincaré duality map the Schubert basis is self dual.

Proposition 4.2.4 ([16, Pg. 148 (11)])**.** *Let s_μ and s_λ be two Schubert basis elements for $\mathrm{Gr}_d(\mathbb{R}^{d+c})$ such that $|s_\mu| + |s_\lambda| = cd$. In this case, $s_\mu \cup s_\lambda = s_{(c^d)}$ if and only if $s_\mu = Ds_\lambda$. In particular, if s_μ is not dual to s_λ, then $s_\mu \cup s_\lambda = 0$.*

We now relate the Schubert picture to the Borel picture. The Stiefel–Whitney classes are themselves Schubert cycles.

Example 4.2.5. In particular, $w_i = s_\lambda$ where λ is of weight i and has largest part one.

$$w_1 \longleftrightarrow \ydiagram{1} \qquad w_2 \longleftrightarrow \ydiagram{1,1} \qquad w_3 \longleftrightarrow \ydiagram{1,1,1} \qquad \ldots$$

Example 4.2.6. For the dual Stiefel-Whitney classes, $\overline{w}_i = s_{(i)}$.

$$\overline{w}_1 \longleftrightarrow \ydiagram{1} \qquad \overline{w}_2 \longleftrightarrow \ydiagram{2} \qquad \overline{w}_3 \longleftrightarrow \ydiagram{3} \qquad \ldots$$

These $s_{(i)}$ called the special Schubert cycles. The Peiri formula describes the multiplication of one of these special cycles with an arbitrary Schubert cycle.

Theorem 4.2.7 (Peiri Formula)**.** *For an arbitrary Schubert cycle s_λ and special Schubert cycle $s_{(i)}$,*

$$s_\lambda \cup s_{(i)} = \sum_{\mu} s_\mu,$$

where the sum is over all μ obtained from λ by adding i boxes with no two added in the same column.

This can be used to construct the Giambelli formula, which expresses a Schubert basis element as a polynomial in special Schubert cycles [21].

Theorem 4.2.8 (Giambelli Formula)**.**

$$s_{(a_1,\ldots,a_d)} = \det(\overline{w}_{a_i+i-j}) = \begin{vmatrix} \overline{w}_{a_1} & \overline{w}_{a_1+1} & \overline{w}_{a_1+2} & \cdots & \overline{w}_{a_1+k-1} \\ \overline{w}_{a_2-1} & \overline{w}_{a_2} & \overline{w}_{a_2+1} & \cdots & \overline{w}_{a_2+k-2} \\ \overline{w}_{a_3-2} & \overline{w}_{a_3-1} & \overline{w}_{a_3} & \cdots & \overline{w}_{a_3+k-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \overline{w}_{a_k-k+1} & \overline{w}_{a_k-k+2} & \overline{w}_{a_k-k+3} & \cdots & \overline{w}_{a_k} \end{vmatrix}$$

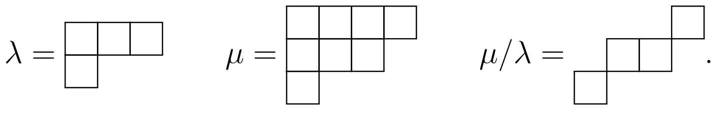
This can then be expressed in the monomial basis.

4.2.1 Milnor primitives

In [28] Lenart developed a combinatorial formula for the calculation of $\mathrm{Sq}(s_\lambda)$. This relied on the Hopf algebra of integral differential operations on the Steenrod algebra studied by Wood in [49]. The primitives in this algebra are denoted D_i. Of interest to us is that $D_{2^{n+1}-1}$ is an integral lift of the Milnor primitive Q_n [50, Example 4.9], that is, $Q_n \equiv D_{2^{n+1}-1} \pmod 2$. To state Lenart's formula we state the required definitions and give examples for clarity.

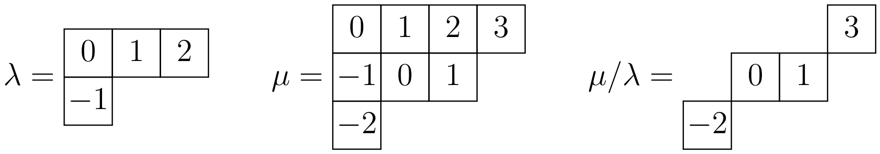
Given a Young diagram λ sitting inside of another Young diagram μ, one can form the complement μ/λ. This is commonly referred to as a skew shape.

Example 4.2.9. An example of forming a skew shape:



Definition 4.2.10. The *content* of a box b of μ in row i and column j is defined to be $\mathrm{C}(b) = j - i$. For a box b in the skew shape μ/λ, we define its content to be the content of b embedded in μ.

Example 4.2.11. Here we fill in the contents of the diagrams from above



Definition 4.2.12. A skew-shape is said to be connected when each pair of boxes in the diagram is connected by a sequence of boxes that each share at least one edge.

For example, μ/λ above is not connected. We use the same terminology as Lenart in [28] for consistency.

Definition 4.2.13. A shape λ is called a *border strip*, if it is connected and does not contain a 2×2 block of boxes. A shape satisfying just the second condition is called a broken border strip, and in particular, a border strip is an example of a broken border strip.

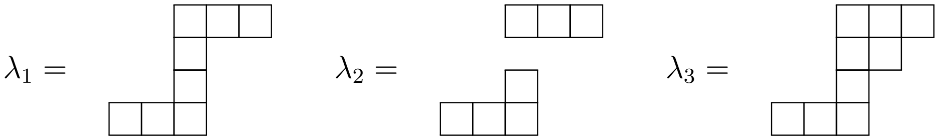


Figure 4-1: From left to right: λ_1 is a border strip, λ_2 is broken border strip that is not a border strip, and λ_3 is an example of a shape that is neither.

Definition 4.2.14. If λ is a broken border strip, then we denote by $\mathrm{cc}(\lambda)$ the number of connected components of λ. If λ is not a broken border strip, then we define $\mathrm{cc}(\lambda) = \infty$.

Example 4.2.15. For example, $\mathrm{cc}(\lambda_1) = 1$, $\mathrm{cc}(\lambda_2) = 2$, and $\mathrm{cc}(\lambda_3) \ = \ \infty$.

Definition 4.2.16. A *sharp corner* of a broken border strip is a box with no north, no west and no northwest neighbors. A *dull corner* is a box with both north and west neighbors, but no northwest neighbor. Let $\mathrm{SC}(\lambda)$ and $\mathrm{DC}(\lambda)$ denote the set of sharp corners and dull corners of λ respectively.

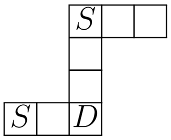


Figure 4-2: The sharp corners are labeled S and the dull corners are label D.

Putting together Lenart's formula for D_k and Wood's observation that $Q_n \equiv D_{2^{n+1}-1} \pmod 2$ yields the following formula.

Theorem 4.2.17. *[28, Prop 5.1] The Milnor primitive Q_n evaluated on the Schubert basis element s_λ has the expansion*

$$Q_n(s_\lambda) = \sum_{\substack{\mu \supset \lambda :\ |\mu|-|\lambda|=2^{n+1}-1 \\ \mathrm{cc}(\mu/\lambda)\leq 2}} d_{\lambda\mu} s_\mu, \tag{4.2.18}$$

where μ/λ must be a broken border strip. If μ/λ is connected,

$$d_{\lambda\mu} \equiv \sum_{b\in SC(\mu/\lambda)} \mathrm{C}(b) + \sum_{b\in DC(\mu/\lambda)} \mathrm{C}(b) \pmod 2, \tag{4.2.19}$$

otherwise $d_{\lambda\mu} = 1$.

Example 4.2.20. As an example we compute Q_1 on $w_1 = s_{\square}$ in the Schubert basis in $\mathrm{Gr}_2(\mathbb{R}^6)$. There are three basis elements in degree four

$$\mu_1 = \begin{array}{|c|c|c|c|}\hline \ & \ & \ & \ \\ \hline\end{array} \qquad \mu_2 = \begin{array}{|c|c|c|}\hline \ & \ & \ \\ \hline \ & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array} \qquad \mu_3 = \begin{array}{|c|c|}\hline \ & \ \\ \hline \ & \ \\ \hline\end{array}.$$

To compute $Q_n(s_{\square})$ using Theorem 4.2.17 we include $\lambda = \square$ into each of these basis elements and then consider the coefficients from (4.2.19). For μ_1 we have

$$\mu_1/\lambda = \begin{array}{|c|c|c|c|}\hline 0 & 1 & 2 & 3 \\ \hline\end{array} \;/\; \begin{array}{|c|}\hline \ \\ \hline\end{array} = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline\end{array}.$$

The complement is a border strip and there is just one sharp corner (the left most corner) and no dull corners. The content of the sharp corner is 1 modulo two, hence $d_{\lambda\mu_1} = 1$, and so s_{μ_1} is in the expansion of $Q_1(w_1)$. Next we consider

$$\mu_2/\lambda = \begin{array}{|c|c|c|}\hline 0 & 1 & 2 \\ \hline -1 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array} \;/\; \begin{array}{|c|}\hline \ \\ \hline\end{array} = \begin{array}{|c|c|c|}\cline{2-3} \multicolumn{1}{c|}{} & 1 & 2 \\ \hline -1 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}.$$

This is a broken border strip with two components, hence $d_{\lambda\mu_2} = 1$, and so s_{μ_2} is in the expansion. Finally,

$$\mu_3/\lambda = \begin{array}{|c|c|}\hline 0 & 1 \\ \hline -1 & -2 \\ \hline\end{array} \;/\; \begin{array}{|c|}\hline \ \\ \hline\end{array} = \begin{array}{|c|c|}\cline{2-2} \multicolumn{1}{c|}{} & 1 \\ \hline -1 & -2 \\ \hline\end{array}.$$

There are two sharp corners, one of content -1 and the other of content 1. There is also one dull corner of content -2. This means $d_{\lambda\mu_3} \equiv (-1) + 1 + 2 \equiv 0$, and so s_{μ_3}

is not in the expansion. We have thus shown

$$Q_1\left(s_{\ydiagram{1}}\right) = s_{\ydiagram{4}} + s_{\ydiagram{3,1}}.$$

4.2.2 A better algorithm

The combinatorial description of Q_n allows us to redefine the function QMatrixAt(n,i) so that it does not rely on Gröbner basis theory. Before we needed to use the Gröbner basis to rewrite the output of Q_n in the monomial basis, but now the combinatorial description of Q_n writes everything in the Schubert basis immediately. For clarity, we provide pseudocode here.

```
procedure IsBorderStrip($\mu$,$\lambda$)
    if length($\lambda$) > length($\mu$) then
        return False
    end if
    for $1 \le i \le$ Length($\mu$) do
        if $\lambda_i > \mu_i$ then
            return False
        end if
    end for
    for $1 \le i <$ Length($\mu$) do
        next_start ← $\lambda_{i+1} + 1$
        next_end ← $\mu_{i+1}$
        end ← $\mu_i$
        if next_end > start and next_end > 0 then
            return False
```

```
        end if
    end for
    return True
end procedure
```

To count the connected components we iterate over the rows while keeping track of where the first and last block of each row occurs.

```
procedure ConnectedComponents($\mu$,$\lambda$)
    count $\leftarrow$ 1
    for $1 \leq i <$ Length($\mu$) do
        end $\leftarrow \mu_i$
        next_start $\leftarrow \lambda_{i+1} + 1$
        if end $\neq$ next_start then
            count $\leftarrow$ count + 1
        end if
    end for
    return  count
end procedure
```

To calculate the coefficient $d_{\lambda\mu}$ we notice that each row of length greater than one contributes both a dull and sharp corner (although the content of these corners themselves may be zero modulo two), except for the first row which can only contribute a sharp corner.

```
procedure CornerSum($\mu$,$\lambda$)
    sum $\leftarrow$ 0
    for $1 \leq i <$ Length($\mu$) do
        start $\leftarrow \lambda_i + 1$
```

```
        end ← $\mu_i$
        $y$ ← LENGTH($\mu$) − $i$
        if start ≠ end then
            sum ← $sum$ + start − $y$
            sum ← $sum$ + end − $y$
        end if
    end for
    sum ← sum + $\lambda_{\text{LENGTH}(\mu)}$ + 1
    return MOD(sum,2)
end procedure
procedure COEFFICIENT($\mu$,$\lambda$)
    if ISBORDERSTRIP($\mu$,$\lambda$) = False then
        return 0
    end if
    cc ← CONNECTEDCOMPONENTS($\mu$, $\lambda$)
    if cc > 2 then
        return 0
    else if cc = 2 then
        return 1
    end if
    return CORNERSUM($\mu$, $\lambda$)
end procedure
```

Finally, we can construct a matrix by just calculating the $d_{\lambda\mu}$.

```
procedure QSCHUBERTCOORDINATEVECTOR($n$,$\lambda$)
    coordinates ← [ ]
```

```
        for $\mu$ in SchubertBasis($c, d$, Weight($\lambda$) + $2^{n+1} - 1$) do
            $d \leftarrow$ Coefficient($\mu, \lambda$)
            Append(coordinates,$d$)
        end for
    end procedure
    procedure QMatrixAt($n, i$)
        matrix $\leftarrow$ [ ]
        for $\lambda$ in SchubertBasis($c, d, i$) do
            column $\leftarrow$ QSchubertCoordinateVector($n, \lambda$)
            Append(matrix, column)
        end for
        return matrix
    end procedure
```

4.3 Fixed points

In order to use Theorem 3.4.1 we must bring in equivariance. We will turn the Grassmannian into a G-space using linear representation theory.

Definition 4.3.1. Let G be a group and V a linear representation of G. We denote by $\mathrm{Gr}_d(V)$ the G-space of d-dimensional subspaces of the vector space V.

As V is a linear representation it is compatible with the structure of the Grassmannian. Thus, taking d-dimensional subspaces in V gives a G-action on the Grassmannian $\mathrm{Gr}_d(V)$.

One can use the underlying representation V to obtain an explicit formula for the H-fixed points $\mathrm{Gr}_d(V)^H$ for any subgroup $H \leq G$. We will primarily work with V a real representation of G and so we will record a few facts about real representations. We start with a special case.

Lemma 4.3.2. *Let V be an n dimensional irreducible real G-representation such that $\mathrm{Hom}_G(V,V) = D$ for some division ring D over $\mathbb{R}$. The G-fixed points of $\mathrm{Gr}_d(V^{\oplus k})$ are of the form*

$$\mathrm{Gr}_d(V^{\oplus k})^G = \mathrm{Gr}_{d/n}(D^k).$$

Proof. The G-fixed points of $\mathrm{Gr}_d(V^{\oplus^k})^G$ are exactly d-dimensional subrepresentations of $V^{\oplus^k}$, that is,

$$\mathrm{Gr}_d(V^{\oplus k})^G = \left\{ W \subseteq V^{\oplus^k} \mid W \text{ is a subrepresentation of dimension } d \right\}.$$

Every subrepresentation of $V^{\oplus^k}$ is of the form $V^{\oplus^i}$ for some $i \leq k$, because V is irreducible. This means that if the dimension of V does not divide d there are no fixed points. Suppose that $d = i \dim V = in$, so that $V^{\oplus i}$ is a d-dimensional subrepresentation of $V^{\oplus k}$. Thus,

$$\mathrm{Gr}_d(V^{\oplus k})^G = \left\{ W \subseteq V^{\oplus^k} \mid W \text{ is a } G\text{-submodule isomorphic to } V^{\oplus^i} \right\}.$$

Each such W is the image of an injective G-module map $V^{\oplus i} \to V^{\oplus k}$. As $\mathrm{Hom}_G(V,V) = D$, we have that $\mathrm{Hom}_G(V^{\oplus i}, V^{\oplus k}) \cong \mathrm{Hom}_D(D^i, D^k)$. In particular, each i-dimensional subspace of D^k is the image of an injective D-module map in $\mathrm{Hom}_D(D^i, D^k)$, and so we have a homeomorphism between

$$\mathrm{Gr}_d(V^{\oplus^k})^G = \left\{ W \subseteq V^{\oplus^k} \mid W \text{ is a } G\text{-submodule isomorphic to } V^{\oplus i} \right\}$$

and

$$\mathrm{Gr}_i(D^k) = \left\{ U \subseteq D^k \mid U \text{ is a } D\text{-submodule isomorphic to } D^i \right\}. \qquad \square$$

As a corollary of this lemma we obtain:

Proposition 4.3.3. *Let V be a real G-representation, written as the sum of isotypical components $V \cong \bigoplus_{i=1}^{k} V_i^{m_i}$ with irreducible V_i of dimension n_i and $D_i = \mathrm{Hom}_G(V_i, V_i)$. The G-fixed points of $\mathrm{Gr}_d(V)$ are given as*

$$\mathrm{Gr}_d(V)^G = \coprod_{d = n_1 j_1 + \ldots + n_k j_k} \mathrm{Gr}_{j_1}(D_1^{m_1}) \times \mathrm{Gr}_{j_2}(D_2^{m_2}) \times \cdots \times \mathrm{Gr}_{j_k}(D_k^{m_k}),$$

where $\mathrm{Gr}_0(D^{m_i}) \simeq *$.

Example 4.3.4. Consider $G = C_4$ the cyclic group of order four. This has two one-dimensional real representations σ_1, σ_2 and one two-dimensional real representation τ coming from a complex one-dimensional representation. In particular, $\mathrm{Hom}^{C_4}(\sigma_i, \sigma_i) = \mathbb{R}$ and $\mathrm{Hom}^{C_4}(\tau, \tau) = \mathbb{C}$. We now present a family of representations that will be important for our later calculations. We put a C_4-action on $\mathrm{Gr}_d(\mathbb{R}^m)$ with $m = 2^{n+1} + 2\ell$ by letting $V = 2^n \sigma_1 \oplus 2^n \sigma_2 \oplus \ell\tau$.

1. When d is even, the Proposition 4.3.3 yields:

$$\begin{aligned}
\mathrm{Gr}_d(V)^{C_4} = \bigsqcup_{\substack{2i+j+k=d \\ i,j,k\geq 1}} & \mathrm{Gr}_i(\mathbb{C}^{\ell}) \times \mathrm{Gr}_j(\mathbb{R}^{2^n}) \times \mathrm{Gr}_k(\mathbb{R}^{2^n}) \\
& \sqcup \bigsqcup_{\substack{i+j=d \\ i,j\geq 1}} \mathrm{Gr}_j(\mathbb{R}^{2^n}) \times \mathrm{Gr}_k(\mathbb{R}^{2^n}) \\
& \sqcup \bigsqcup_{2} \bigsqcup_{\substack{2i+j=d \\ i,j\geq 1}} \mathrm{Gr}_i(\mathbb{C}^{\ell}) \times \mathrm{Gr}_j(\mathbb{R}^{2^n}) \\
& \sqcup \bigsqcup_{2} \mathrm{Gr}_d(\mathbb{R}^{2^n}) \\
& \sqcup \mathrm{Gr}_{d/2}(\mathbb{C}^{\ell}).
\end{aligned}$$

2. When d, is odd the Proposition 4.3.3 yields:

$$\begin{aligned}
\mathrm{Gr}_d(V)^{C_4} = \bigsqcup_{\substack{2i+j+k=d \\ i,j,k\geq 1}} & \mathrm{Gr}_i(\mathbb{C}^{\ell}) \times \mathrm{Gr}_j(\mathbb{R}^{2^n}) \times \mathrm{Gr}_k(\mathbb{R}^{2^n}) \\
& \sqcup \bigsqcup_{\substack{i+j=d \\ i,j\geq 1}} \mathrm{Gr}_j(\mathbb{R}^{2^n}) \times \mathrm{Gr}_k(\mathbb{R}^{2^n}) \\
& \sqcup \bigsqcup_{2} \bigsqcup_{\substack{2i+j=d \\ i,j\geq 1}} \mathrm{Gr}_i(\mathbb{C}^{\ell}) \times \mathrm{Gr}_j(\mathbb{R}^{2^n}) \\
& \sqcup \bigsqcup_{2} \mathrm{Gr}_d(\mathbb{R}^{2^n}).
\end{aligned}$$

For $m = 2^{n+1} + 2\ell - 1$, we let $V = (2^n - 1)\sigma_1 \oplus 2^n\sigma_2 \oplus \ell\tau$. In this case, a similar fixed point formula holds.

1. For d even,

$$\begin{aligned}
\mathrm{Gr}_d(V)^{C_4} = \coprod_{\substack{2i+j+k=d\\ i,j,k\geq 1}} & \mathrm{Gr}_i(\mathbb{C}^\ell)\times \mathrm{Gr}_j(\mathbb{R}^{2^n-1})\times \mathrm{Gr}_k(\mathbb{R}^{2^n}) \\
& \sqcup \coprod_{\substack{i+j=d\\ i,j\geq 1}} \mathrm{Gr}_j(\mathbb{R}^{2^n-1})\times \mathrm{Gr}_k(\mathbb{R}^{2^n}) \\
& \sqcup \coprod_{\substack{2i+j=d\\ i,j\geq 1}} \mathrm{Gr}_i(\mathbb{C}^\ell)\times \mathrm{Gr}_j(\mathbb{R}^{2^n}) \\
& \sqcup \coprod_{\substack{2i+j=d\\ i,j\geq 1}} \mathrm{Gr}_i(\mathbb{C}^\ell)\times \mathrm{Gr}_j(\mathbb{R}^{2^n-1}) \\
& \sqcup \mathrm{Gr}_d(\mathbb{R}^{2^n}) \sqcup \mathrm{Gr}_d(\mathbb{R}^{2^n-1}) \\
& \sqcup \mathrm{Gr}_{d/2}(\mathbb{C}^\ell).
\end{aligned}$$

2. For d odd,

$$\begin{aligned}
\mathrm{Gr}_d(V)^{C_4} = \coprod_{\substack{2i+j+k=d\\ i,j,k\geq 1}} & \mathrm{Gr}_i(\mathbb{C}^\ell)\times \mathrm{Gr}_j(\mathbb{R}^{2^n})\times \mathrm{Gr}_k(\mathbb{R}^{2^n-1}) \\
& \sqcup \coprod_{\substack{i+j=d\\ i,j\geq 1}} \mathrm{Gr}_j(\mathbb{R}^{2^n-1})\times \mathrm{Gr}_k(\mathbb{R}^{2^n}) \\
& \sqcup \coprod_{\substack{2i+j=d\\ i,j\geq 1}} \mathrm{Gr}_i(\mathbb{C}^\ell)\times \mathrm{Gr}_j(\mathbb{R}^{2^n}) \\
& \sqcup \coprod_{\substack{2i+j=d\\ i,j\geq 1}} \mathrm{Gr}_i(\mathbb{C}^\ell)\times \mathrm{Gr}_j(\mathbb{R}^{2^n-1}) \\
& \sqcup \mathrm{Gr}_d(\mathbb{R}^{2^n}) \sqcup \mathrm{Gr}_d(\mathbb{R}^{2^n-1}).
\end{aligned}$$

4.4 Cofiber

Proposition 4.4.1. *Let $C_d(\mathbb{R}^m)$ denote the cofiber of the inclusion $\mathrm{Gr}_d(\mathbb{R}^m) \to \mathrm{Gr}_d(\mathbb{R}^{m+1})$. Let $\gamma_{d-1}^{\perp}$ be the orthogonal complement of the canonical bundle γ_{d-1} over $\mathrm{Gr}_{d-1}(\mathbb{R}^m)$. Let $S(\gamma_{d-1}^{\perp})$ and $D(\gamma_{d-1}^{\perp})$ denote the sphere and disk bundle respectively. There is a pushout*

$$\begin{tikzcd}
S(\gamma_{d-1}^{\perp}) \arrow[r] \arrow[d] & \mathrm{Gr}_d(\mathbb{R}^m) \arrow[d] \\
D(\gamma_{d-1}^{\perp}) \arrow[r] & \mathrm{Gr}_d(\mathbb{R}^{m+1}),
\end{tikzcd}$$

which induces a homeomorphism from the Thom space of $Th(\gamma_{d-1}^{\perp})$ to the cofiber $C_d(\mathbb{R}^m)$. The Thom isomorphism theorem then yields the isomorphism

$$H^q(\mathrm{Gr}_{d-1}(\mathbb{R}^m)) \xrightarrow{\cup\ \overline{w}_{c+1}} H^{q+c+1}(C_d(\mathbb{R}^m)).$$

Proof. The total space $E(\gamma_{d-1}^{\perp})$ of the orthogonal complement to the canonical bundle $\gamma_{d-1}^{\perp}$ over $\mathrm{Gr}_{d-1}(\mathbb{R}^{d-1+c})$ is of the form

$$E(\gamma_{d-1}^{\perp}) = \{(V, v) \mid V \in \mathrm{Gr}_{d-1}(\mathbb{R}^{d-1+c}), v \in V^{\perp}\},$$

where $V \oplus V^{\perp} = \mathbb{R}^{d-1+c}$. Thus, the disk bundle $D(\gamma_{d-1}^{\perp})$ and sphere bundle $S(\gamma_{d-1}^{\perp})$ have total spaces:

$$D(\gamma_{d-1}^{\perp}) = \{(V, v) \mid V \in \mathrm{Gr}_{d-1}(\mathbb{R}^{d-1+c}), v \in V^{\perp}, |v| \leq 1\},$$
$$S(\gamma_{d-1}^{\perp}) = \{(V, v) \mid V \in \mathrm{Gr}_{d-1}(\mathbb{R}^{d-1+c}), v \in V^{\perp}, |v| = 1\}.$$

We define a surjection

$$\begin{aligned} f\colon D(\gamma_{d-1}^{\perp}) &\to \mathrm{Gr}_d(\mathbb{R}^{d+c}) \\ (V,v) &\mapsto V + \left\langle v + \left(\sqrt{1-|v|^2}\right) e_{d+c} \right\rangle , \end{aligned}$$

where e_i denotes the ith standard basis vector of $\mathbb{R}^{d+c}$.

To see that f is surjective, let $W \in \mathrm{Gr}_d(\mathbb{R}^{d+c})$. There are two possibilities:

1. $W \subseteq \mathbb{R}^{d-1+c}$ so that $\dim W \cap \mathbb{R}^{d-1+c} = d$,
2. $W \not\subseteq \mathbb{R}^{d-1+c}$ so that $\dim W \cap \mathbb{R}^{d-1+c} = d-1$,

where $\mathbb{R}^{d-1+c}$ includes into $\mathbb{R}^{d+c}$ via the basis $e_1, \dots e_{d-1+c}$.

If $\dim W \cap \mathbb{R}^{d-1+c} = d-1$, then set $V = W \cap \mathbb{R}^{d-1+c}$. Let $V^{\perp}$ be the complement of V inside of $\mathbb{R}^{d+c}$ so that $W \cap V^{\perp}$ is one-dimensional. Take the unique unit vector $v \in W \cap V^{\perp}$ such that v has positive $(d+c)$th coordinate. Let $\pi\colon \mathbb{R}^{d+c} \to \mathbb{R}^{d-1+c}$ be the projection such that $\pi(e_{d+c}) = 0$ and $\pi(e_j) = e_j$ for all $j \neq d+c$. We claim that $f(V, \pi(v)) = W$. Since $|v| = 1$ the $(d+c)th$ component on v has scalar $\sqrt{1-|\pi(v)|^2}$, thus $f(V,\pi(v)) = V + \langle v \rangle = W$ by construction.

If $\dim W \cap \mathbb{R}^{d-1+c} = d$, then choose any $(d-1)$ dimensional subspace V of W and take any length one vector $v \in W \cap V^{\perp}$, so that $f(V,v) = W$.

Notice that f maps $S(\gamma_{d-1}^{\perp})$ to $\mathrm{Gr}_d(\mathbb{R}^{d-1+c})$, while f is injective on $D(\gamma_{d-1}^{\perp}) - S(\gamma_{d-1}^{\perp})$. This yields the pushout

$$\begin{tikzcd} S(\gamma_{d-1}^{\perp}) \arrow[r, "f"] \arrow[d, hook] & \mathrm{Gr}_d(\mathbb{R}^{d-1+c}) \arrow[d, hook] \\ D(\gamma_{d-1}^{\perp}) \arrow[r, "f"] & \mathrm{Gr}_d(\mathbb{R}^{d+c}). \end{tikzcd}$$

Recall that the Thom space is formed by quotienting the disk bundle by the sphere bundle, that is, $\mathrm{Th}(\gamma_{d-1}^{\perp}) = D(\gamma_{d-1}^{\perp})/S(\gamma_{d-1}^{\perp})$. Thus, from this pushout we have a homeomorphism $D(\gamma_{d-1}^{\perp})/S(\gamma_{d-1}^{\perp}) \to Gr(\mathbb{R}^{d+c})/\operatorname{Gr}(\mathbb{R}^{d-1+c})$, because both i and j in the following diagram are cofibrations:

$$\begin{tikzcd}
S(\gamma_{d-1}^{\perp}) \arrow[r, "f"] \arrow[d, hook, "j"] & \operatorname{Gr}_d(\mathbb{R}^{d-1+c}) \arrow[d, hook, "i"] \\
D(\gamma_{d-1}^{\perp}) \arrow[r, "f"] \arrow[d] & \operatorname{Gr}_d(\mathbb{R}^{d+c}) \arrow[d] \\
C(j) \arrow[r] & C(i).
\end{tikzcd}$$

To identify the Thom class inside of the cofiber $C(i)$ one notices that when increasing c to $c+1$ the first new class one obtains is $\overline{w}_{c+1}$, which is the horizontal strip with $c+1$ boxes in the Schubert picture. From the Borel relations this class is zero in $\operatorname{Gr}_d(\mathbb{R}^{d-1+c})$ and thus it is certainly in the cofiber in dimension $c+1$. Thus, the Thom isomorphism is given by $H^q(\operatorname{Gr}_{d-1}(\mathbb{R}^m)) \xrightarrow{\cup \overline{w}_{c+1}} H^{q+c+1}(C(i))$. □

There is a simple description of the Thom isomorphism in the Schubert picture. The $\overline{w}_{c+1}$ corresponds to a Schubert cycle given by the partition with one part of weight $c+1$. This is a special Schubert cycle, hence multiplication by it is governed by Pieri's formula, that is,

$$s_\mu s_{(c+1)} = \sum_{s_\lambda} s_\lambda,$$

such that λ is obtained from μ by adding $c+1$ boxes with each box in a different column.

In $H^*(\operatorname{Gr}_d(\mathbb{R}^{d+c+1}); \mathbb{Z}/2\mathbb{Z})$ every box must fit in a $d \times (c+1)$ grid. Since there are only $c+1$ columns total for which boxes may be added, we must add one box in each column. This has the effect of just adding a new row of length $c+1$ to the top. That is, a Schubert cycle s_λ in $H^q(\operatorname{Gr}_{d-1}(\mathbb{R}^m))$ is mapped to $s_{(c+1,\lambda_1,\ldots,\lambda_{d-1})}$.

Example 4.4.2. Consider the cofiber of $\mathrm{Gr}_3(\mathbb{R}^5) \to \mathrm{Gr}_3(\mathbb{R}^6)$. Since $c = 2$ to obtain the cofiber basis elements, one just adds $c + 1 = 3$ boxes to each class in $\mathrm{Gr}_2(\mathbb{R}^5)$.

$$H^q(\mathrm{Gr}_2(\mathbb{R}^5)) \xrightarrow{\cup\, \overline{w}_3} H^{q+3}(C_3(\mathbb{R}^5))$$

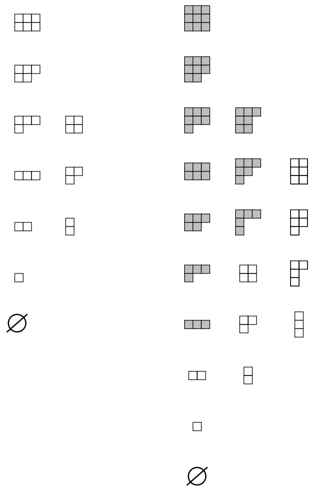


Figure 4-3: The basis elements of the cohomology of the cofiber of $\mathrm{Gr}_3(\mathbb{R}^5) \to \mathrm{Gr}_3(\mathbb{R}^6)$ sitting inside of $H^*(\mathrm{Gr}_3(\mathbb{R}^6); \mathbb{Z}/2\mathbb{Z})$ have been colored gray.

It will be useful to consider the inherited G-action on the cofiber. Suppose that W is a G-subrepresentation of V. We denote by $C_d(W)$ the cofiber of the inclusion $\mathrm{Gr}_d(W) \to \mathrm{Gr}_d(V)$. Let $H \leq G$. The fixed points $C_d(W)^H$ can be analyzed by using Proposition 4.3.3 as the next proposition shows.

Proposition 4.4.3. *The fixed points of $C_d(W)^H$ are given as the pushout*

$$\begin{tikzcd}
\mathrm{Gr}_d(W)^H \arrow[r] \arrow[d] & \mathrm{Gr}_d(V)^H \arrow[d] \\
*^H \arrow[r] & C_d(W)^H.
\end{tikzcd}$$

Proof. The inclusion $\mathrm{Gr}_d(W) \to \mathrm{Gr}_d(V)$ is an inclusion of cell complexes and so the cofiber is given as the pushout

$$\begin{tikzcd}
\mathrm{Gr}_d(W) \arrow[r] \arrow[d] & \mathrm{Gr}_d(V) \arrow[d] \\
* \arrow[r] & C_d(W).
\end{tikzcd}$$

The fixed point functor $(-)^H$ commutes with pushouts with one leg a cofibration. This is exactly our situation. □

4.5 Chromatic lower bound

Theorem 4.5.1. *For $c+d \leq 2^{n+1}$, the dimension of the 2-local Morava K-theory of $\mathrm{Gr}_d(\mathbb{R}^{d+c})$ is $k_n(\mathrm{Gr}_d(\mathbb{R}^{d+c})) = \dim_{\mathbb{Z}/2\mathbb{Z}} H^*(\mathrm{Gr}_d(\mathbb{R}^{d+c}); \mathbb{Z}/2\mathbb{Z}) = \binom{c+d}{d}$.*

Proof. We will proceed by induction on n with base case $n = 0$. For $n = 0$ the statement concerns $\mathrm{Gr}_d(\mathbb{R}^{d+c})$ for $d + c \leq 2$. These Grassmannians come in two forms. They are either a point or a circle. In particular,

$$\mathrm{Gr}_0(\mathbb{R}^0) = \mathrm{Gr}_0(\mathbb{R}^1) = \mathrm{Gr}_0(\mathbb{R}^2) = \mathrm{Gr}_1(\mathbb{R}^1) = \mathrm{Gr}_2(\mathbb{R}^2) = *.$$

and $\mathrm{Gr}_1(\mathbb{R}^2) \simeq S^1$. In the first case, $k_0(*) = 1 = \binom{d+c}{d}$, while in the second case $k_0(S^1) = 2 = \binom{1+1}{1}$. This completes the base case. Now assume as the inductive hypothesis that the statement holds for $n - 1$. Since $d + c \leq 2^{n+1}$ we may write $d + c = a + b$ with $a \leq 2^n$ and $b \leq 2^n$. Let σ_1 be the trivial representation of C_2 and

σ_2 be the sign representation of C_2. Set $V = a\sigma_1 \oplus b\sigma_2$ so that by Proposition 4.3.3

$$\begin{aligned} \mathrm{Gr}_d(V) &= \mathrm{Gr}_d(\mathbb{R}^{a+b}) = \mathrm{Gr}_d(\mathbb{R}^{d+c}), \\ \mathrm{Gr}_d(V)^{C_2} &= \bigsqcup_{j+k=d} \mathrm{Gr}_j(\mathbb{R}^a) \times \mathrm{Gr}_k(\mathbb{R}^b). \end{aligned}$$

This means that

$$k_{n-1}(\mathrm{Gr}_d(V)^{C_2}) = \sum_{j+k=d} k_{n-1}(\mathrm{Gr}_j(\mathbb{R}^a))k_{n-1}(\mathrm{Gr}_k(\mathbb{R}^b)).$$

By Theorem 3.1.2, we have the chromatic lower bound $k_{n-1}(X^{C_2}) \leq k_n(X)$ and since $k_n(\mathrm{Gr}_d(V)) \leq \binom{d+c}{d}$ by the cell structure, to prove $k_n(\mathrm{Gr}_d(V)) = \binom{d+c}{d}$, it suffices to show that

$$k_{n-1}(\mathrm{Gr}_d(V)^{C_2}) = \binom{d+c}{d}.$$

We will compute this using the inductive hypothesis. Since $a \leq 2^n$, $k_{n-1}(\mathrm{Gr}_j(\mathbb{R}^a)) = \binom{a}{j}$. Similarly, $b \leq 2^n$ yields $k_{n-1}(\mathrm{Gr}_k(\mathbb{R}^b)) = \binom{b}{k}$. Making these substitutions yields

$$\begin{aligned} k_{n-1}(\mathrm{Gr}_d(V)) &= \sum_{j+k=d} k_{n-1}(\mathrm{Gr}_j(\mathbb{R}^a))k_{n-1}(\mathrm{Gr}_k(\mathbb{R}^b)) \\ &= \sum_{j+k=d} \binom{a}{j}\binom{b}{k} \\ &= \binom{a+b}{d} = \binom{d+c}{d}. \end{aligned}$$

This completes the proof.

□

From this theorem we learn that the Q_n-differential is zero in this range.

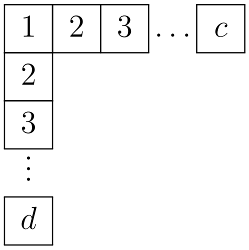
Corollary 4.5.2. *For $d + c \leq 2^{n+1}$, the Q_n-differential on $H^*(\mathrm{Gr}_d(\mathbb{R}^{d+c}); \mathbb{Z}/2\mathbb{Z})$ is zero.*

We also give a direct proof of this to illustrate the techniques for calculating Q_n.

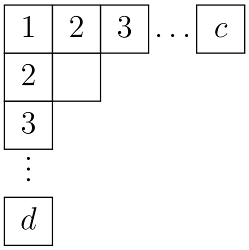
Proof. We will show that for $d+c \leq 2^{n+1}$ that $Q_n(w_i) = 0$ for $i = 1, \dots, d$. As every basis element in $H^*(\mathrm{Gr}_d(\mathbb{R}^{d+c}); \mathbb{Z}/2\mathbb{Z})$ is expressible as a monomial in these and as Q_n is a derivation, this would show that $Q_n(x) = 0$ for all x.

We will work with the Schubert basis. By formula (4.2.18), to compute $Q_n(s_\lambda)$ we only need to consider μ such that μ/λ is a border strip.

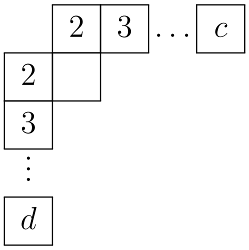
Let's start with w_1. Let λ_i denote the diagram corresponding to w_i. The diagram λ_1 is just the 1×1 block. The class $Q_n(w_1)$ is in dimension 2^{n+1}. We only must consider μ with weight 2^{n+1} such that μ/λ_1 is a border strip. This would mean that the only 2×2 shape that may occur inside of μ is in the upper left corner of μ. We will consider all of the possible diagrams. Each diagram must fit in the $d \times c$ grid. Since $d + c \leq 2^{n+1}$, the shape that has the first row of length c and the first column of length d is of weight at most $d + c - 1 \leq 2^{n+1} - 1$, as pictured:



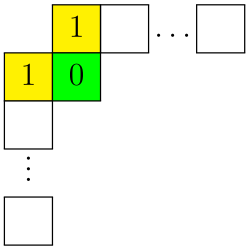
so the only possible diagram μ of weight 2^{n+1} such that μ/λ_1 is a border strip must be



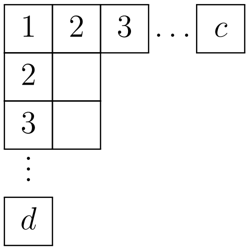
and this only ever happens when $d + c = 2^{n+1}$. For $d + c < 2^{n+1}$ there are no such diagrams to consider and $Q_n(w_1)$ is automatically zero. We picture μ/λ_1 below:



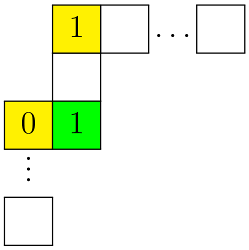
There are only two sharp corners and their contents modulo two always agree. There is also one dull corner, but it is along the diagonal making the content zero. We color the sharp corners yellow and dull corners green:



Thus the coefficient $d_{\lambda_1 \mu} = 0$. We will now consider w_2. The only possible μ of weight $2^{n+1} + 1$ is the shape



this again has two sharp corners and one dull corner



However, the left most sharp corner is now of content zero, and the top sharp corner and the dull corner are each of content one. Thus, from the formula $d_{\lambda_2 \mu} = 0$.

These two cases illustrate the general case. For odd i, the sharp corners of μ/λ_i agree and the dull corner is zero, while for even i the left sharp corner is zero and the top sharp corner and dull corner agree, hence $Q_n(w_i) = 0$. ☐

Now that we know that the AHSS collapses for $\mathrm{Gr}_d(\mathbb{R}^{d+c})$ with $d + c \leq 2^{n+1}$ we may compute an explicit formula for the dimension of the $(n-1)st$ Morava K-theory of the C_4-fixed points $\mathrm{Gr}_d(V)^{C_4}$ for V a representation in Example 4.3.4.

Proposition 4.5.3. *For $n \geq 1$, the dimension of the $(n-1)st$ 2-local Morava K-theory of $\mathrm{Gr}_d(V)^{C_4}$ for $V = 2^n\sigma_1 \oplus 2^n\sigma_2 \oplus \ell\tau$ or $V = (2^n - 1)\sigma_1 \oplus 2^n\sigma_2 \oplus \ell$ is given by:*

$$k_{n-1}(Gr_d(V)^{C_4}) = \begin{cases} \sum_{i=0}^{\frac{d-1}{2}} \binom{2^{n+1}}{d-2i}\binom{\ell}{i} & d \text{ odd}, V = 2^n\sigma_1 \oplus 2^n\sigma_2 \oplus \ell\tau \\ \sum_{i=0}^{\frac{d-1}{2}} \binom{2^{n+1}-1}{d-2i}\binom{\ell}{i} & d \text{ odd}, V = (2^n-1)\sigma_1 \oplus 2^n\sigma_2 \oplus \ell\tau \\ \sum_{i=0}^{\frac{d}{2}} \binom{2^{n+1}}{d-2i}\binom{\ell}{i} & d \text{ even}, V = 2^n\sigma_1 \oplus 2^n\sigma_2 \oplus \ell\tau \\ \sum_{i=0}^{\frac{d}{2}} \binom{2^{n+1}-1}{d-2i}\binom{\ell}{i} & d \text{ even}, V = (2^n-1)\sigma_1 \oplus 2^n\sigma_2 \oplus \ell\tau. \end{cases}$$

Proof. Consider $V = 2^n\sigma_1 \oplus 2^n\sigma_2 \oplus \ell\tau$ and d odd. From Example 4.3.4, the fixed points are of the form,

$$\begin{aligned}\mathrm{Gr}_d(V)^{C_4} = &\coprod_{\substack{2i+j+k=d\\ i,j,k\geq 1}} \mathrm{Gr}_i(\mathbb{C}^\ell) \times \mathrm{Gr}_j(\mathbb{R}^{2^n}) \times \mathrm{Gr}_k(\mathbb{R}^{2^n})\\ &\sqcup \coprod_{\substack{i+j=d\\ i,j\geq 1}} \mathrm{Gr}_j(\mathbb{R}^{2^n}) \times \mathrm{Gr}_k(\mathbb{R}^{2^n})\\ &\sqcup \coprod_{2} \coprod_{\substack{2i+j=d\\ i,j\geq 1}} \mathrm{Gr}_i(\mathbb{C}^\ell) \times \mathrm{Gr}_j(\mathbb{R}^{2^n})\\ &\sqcup \coprod_{2} \mathrm{Gr}_d(\mathbb{R}^{2^n}).\end{aligned}$$

Notice that the dimensions of the Morava K-theory of each of the constituent pieces in this fixed point formula is readily computable. The complex Grassmannians have cohomology in even degrees and so the AHSS collapses, hence $k_n(\mathrm{Gr}_i(\mathbb{C}^\ell)) = \binom{\ell}{i}$. The calculation of the real Grassmannians appearing in the fixed point formula is done by Theorem 4.5.1. Using the Künneth theorem the dimension of the Morava K-theory of a product is the product of the dimensions of the factors, thus

$$\begin{aligned}k_{n-1}\left(\mathrm{Gr}_d(V)^{C_4}\right) = &\sum_{\substack{2i+j+k=d\\ i,j,k\geq 1}} k_{n-1}\left(\mathrm{Gr}_i(\mathbb{C}^\ell)\right) k_{n-1}\left(\mathrm{Gr}_j(\mathbb{R}^{2^n})\right) k_{n-1}\left(\mathrm{Gr}_k(\mathbb{R}^{2^n})\right)\\ &+ \sum_{\substack{i+j=d\\ i,j\geq 1}} k_{n-1}\left(\mathrm{Gr}_j(\mathbb{R}^{2^n})\right) k_{n-1}\left(\mathrm{Gr}_k(\mathbb{R}^{2^n})\right)\\ &+ 2\sum_{\substack{2i+j=d\\ i,j\geq 1}} k_{n-1}\left(\mathrm{Gr}_i(\mathbb{C}^\ell)\right) k_{n-1}\left(\mathrm{Gr}_j(\mathbb{R}^{2^n})\right)\\ &+ 2k_{n-1}\left(\mathrm{Gr}_d(\mathbb{R}^{2^n})\right)\end{aligned}$$

becomes

$$k_{n-1}\left(\mathrm{Gr}_d(V)^{C_4}\right) = \sum_{\substack{2i+j+k=d\\ i,j,k\geq 1}} \binom{\ell}{i}\binom{2^n}{j}\binom{2^n}{k} + \sum_{\substack{i+j=d\\ i,j\geq 1}} \binom{2^n}{j}\binom{2^n}{k} + 2\sum_{\substack{2i+j=d\\ i,j\geq 1}} \binom{\ell}{i}\binom{2^n}{j} + 2\binom{2^n}{d}.$$

One can simplify this formula by using the follow relations

$$\sum_{\substack{i+j=d\\ i,j\geq 1}} \binom{2^n}{i}\binom{2^n}{j} = \binom{2^{n+1}}{d} - 2\binom{2^n}{d},$$

$$\sum_{\substack{i+j=d\\ i,j\geq 1}} \binom{2^n-1}{i}\binom{2^n}{j} = \binom{2^{n+1}-1}{d} - \binom{2^n-1}{d} - \binom{2^n}{d},$$

which follow from the generating function description of binomial coefficients. This yields

$$k_{n-1}(\mathrm{Gr}_d(V)^{C_4}) = \sum_{2i+j+k=d} \binom{\ell}{i}\binom{2^n}{j}\binom{2^n}{k} + \binom{2^{n+1}}{d} + 2\sum_{2i+j=d} \binom{\ell}{i}\binom{2^n}{j}$$

and now by combining these sums we obtain

$$k_{n-1}(\mathrm{Gr}_d(V)^{C_4}) = \sum_{i=0}^{\frac{d-1}{2}} \binom{2^{n+1}}{d-2i}\binom{\ell}{i}.$$

In an analogous way one may handle the other cases. □

As an immediate corollary, we obtain Theorem 4.5.4 using Theorem 3.1.2.

Theorem 4.5.4. *For $n \geq 0$, the dimension of the 2-local $K(n)^*(\mathrm{Gr}_d(\mathbb{R}^{d+c}))$ is bounded below by*

$$k_n\left(Gr_d(\mathbb{R}^m)\right) \geq \begin{cases} \binom{m}{d} & d \leq m \leq 2^{n+1} \\ \sum_{i=0}^{\frac{d-1}{2}} \binom{2^{n+1}}{d-2i}\binom{\ell}{i} & d\ \mathit{odd}, m = 2^{n+1}+2\ell \\ \sum_{i=0}^{\frac{d-1}{2}} \binom{2^{n+1}-1}{d-2i}\binom{\ell}{i} & d\ \mathit{odd}, m = 2^{n+1}+2\ell-1 \\ \sum_{i=0}^{\frac{d}{2}} \binom{2^{n+1}}{d-2i}\binom{\ell}{i} & d\ \mathit{even}, m = 2^{n+1}+2\ell \\ \sum_{i=0}^{\frac{d}{2}} \binom{2^{n+1}-1}{d-2i}\binom{\ell}{i} & d\ \mathit{even}, m = 2^{n+1}+2\ell-1. \end{cases}$$

To use Theorem 3.4.1, one must show that the upper bound $k_{Q_n}(\mathrm{Gr}_d(V))$ agrees with this lower bound. To this end, we study a long exact sequence in Margolis homology.

4.6 Long exact sequence in Margolis homology

The inclusion $\mathrm{Gr}_d(\mathbb{R}^m) \to \mathrm{Gr}_d(\mathbb{R}^{m+1})$ with cofiber $C_d(\mathbb{R}^m)$ induces a short exact sequence in $\mathbb{Z}/2\mathbb{Z}$ homology:

$$0 \longrightarrow H^*(C_d(\mathbb{R}^m);\mathbb{Z}/2\mathbb{Z}) \xrightarrow{p^*} H^*(\mathrm{Gr}_d(\mathbb{R}^{m+1});\mathbb{Z}/2\mathbb{Z}) \xrightarrow{i^*} H^*(\mathrm{Gr}(\mathbb{R}^m);\mathbb{Z}/2\mathbb{Z}) \longrightarrow 0.$$

This then induces a long exact sequence in Margolis homology

$$\begin{tikzcd}
\dots \arrow[r, "\partial"] & H^q(C_d(m); Q_n) \arrow[r, "p^*"] & H^q(\mathrm{Gr}_d(\mathbb{R}^{m+1}); Q_n) \arrow[r, "i^*"] & H^q(\mathrm{Gr}_d(\mathbb{R}^m); Q_n) \arrow[dll, rounded corners, to path={-- ([xshift=2ex]\tikztostart.east) |- node[pos=0.75, below]{∂} ([yshift=-2ex]\tikztostart.south) -| ([xshift=-8ex]\tikztotarget.west) -- (\tikztotarget)}] & \\
& H^{q+|Q_n|}(C_d(m); Q_n) \arrow[r, "p^*"] & H^{q+|Q_n|}(\mathrm{Gr}_d(\mathbb{R}^{m+1}); Q_n) \arrow[r, "i^*"] & H^{q+|Q_n|}(\mathrm{Gr}_d(\mathbb{R}^m); Q_n) \arrow[r, "\partial"] & \dots,
\end{tikzcd}$$

with the connecting homomorphism coming from the usual diagram chase:

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\dots \arrow[r, "Q_n"] & H^q(C_d(\mathbb{R}^m); \mathbb{Z}/2\mathbb{Z}) \arrow[r, "Q_n"] \arrow[d, "p^q"] & H^{q+|Q_n|}(C_d(\mathbb{R}^m); \mathbb{Z}/2\mathbb{Z}) \arrow[r, "Q_n"] \arrow[d, "p^{q+|Q_n|}"] & H^{q+2|Q_n|}(C_d(\mathbb{R}^m); \mathbb{Z}/2\mathbb{Z}) \arrow[r, "Q_n"] \arrow[d, "p^{q+2|Q_n|}"] & \dots \\
\dots \arrow[r, "Q_n"] & H^q(\mathrm{Gr}_d(\mathbb{R}^{m+1}); \mathbb{Z}/2\mathbb{Z}) \arrow[r, "Q_n"] \arrow[d, "i^q"] & H^{q+|Q_n|}(\mathrm{Gr}_d(\mathbb{R}^{m+1}); \mathbb{Z}/2\mathbb{Z}) \arrow[r, "Q_n"] \arrow[d, "i^{q+|Q_n|}"] & H^{q+2|Q_n|}(\mathrm{Gr}_d(\mathbb{R}^{m+1}); \mathbb{Z}/2\mathbb{Z}) \arrow[r, "Q_n"] \arrow[d, "i^{q+2|Q_n|}"] & \dots \\
\dots \arrow[r, "Q_n"] & H^q(\mathrm{Gr}_d(\mathbb{R}^m); \mathbb{Z}/2\mathbb{Z}) \arrow[r, "Q_n"] \arrow[d] & H^{q+|Q_n|}(\mathrm{Gr}_d(\mathbb{R}^m); \mathbb{Z}/2\mathbb{Z}) \arrow[r, "Q_n"] \arrow[d] & H^{q+2|Q_n|}(\mathrm{Gr}_d(\mathbb{R}^m); \mathbb{Z}/2\mathbb{Z}) \arrow[r, "Q_n"] \arrow[d] & \dots \\
& 0 & 0 & 0 & .
\end{tikzcd}$$

Given a representative of a cycle $x \in H^q(\mathrm{Gr}_d(\mathbb{R}^m); Q_n)$ viewed in $H^q(\mathrm{Gr}_d(\mathbb{R}^m); \mathbb{Z}/2\mathbb{Z})$, since i^q is surjective, we can pull it back to find $y \in H^q(\mathrm{Gr}_d(\mathbb{R}^{m+1}); \mathbb{Z}/2\mathbb{Z})$ with $y = i^q(x)$. There are fewer relations in $H^*(\mathrm{Gr}_d(\mathbb{R}^{m+1}); \mathbb{Z}/2\mathbb{Z})$ than in $H^*(\mathrm{Gr}_d(\mathbb{R}^m); \mathbb{Z}/2\mathbb{Z})$. Thus, even though $Q_n(x) = 0$ in $H^*(\mathrm{Gr}_d(\mathbb{R}^m); \mathbb{Z}/2\mathbb{Z})$, the class $Q_n(y)$ might not be trivial. By the commutativity of the diagram $i^{q+|Q_n|}(Q_n(y)) = 0$, and so from the exactness of the middle column $Q_n(y)$ is in the image of $p^{q+|Q_n|}$. Let $i^{q+|Q_n|}(z) = y$. We define the connecting homomorphism $\partial \colon H^q(\mathrm{Gr}_d(\mathbb{R}^m); Q_n) \to H^{q+|Q_n|}(C_d(\mathbb{R}^m); Q_n)$ by sending the representative x to the representative z. The standard argument in homological algebra shows that this is well-defined and results in a long exact sequence. Thus, to analyze the connecting homomorphism, one must analyze Q_n and how the relations change from $\mathrm{Gr}_d(\mathbb{R}^m)$ to $\mathrm{Gr}_d(\mathbb{R}^{m+1})$.

From knowledge of the Margolis homology of $\mathrm{Gr}_d(\mathbb{R}^m)$ and $C_d(\mathbb{R}^m)$ this long exact sequence has the potential to be used to calculate the Margolis homology of $\mathrm{Gr}_d(\mathbb{R}^{m+1})$. The key difficulty is understanding the connecting homomorphism. We will shortly show that for $d = 2$, this long exact sequence breaks up into short exact sequences. In this case, we will be able to calculate the dimension of the Margolis homology of $\mathrm{Gr}_d(\mathbb{R}^m)$ inductively. This motivates our study of the Margolis homology of the cofiber in the next two subsections.

4.7 The top class and Poincaré duality

Proposition 4.7.1. *For $d + c$ even, the top class of $\mathrm{Gr}_d(\mathbb{R}^{d+c})$ is not in the image of Q_n.*

Proof. We are going to show that $Q_n(\lambda) = 0$ for each Schubert basis element λ in degree $cd - 2^{n+1} + 1$. Since $s_{(d^c)}$ is the only class in degree cd, $Q_n(\lambda) = d_{\lambda(d^c)}s_{(d^c)}$, where $d_{\lambda(d^c)}$ is given by (4.2.19). For any λ in degree $cd - 2^{n+1} + 1$, if $(d^c)/\lambda$ is not a broken border strip, then $d_{\lambda(d^c)}$ is zero as the formula (4.2.18) requires $\mathrm{cc}((d^c)/\lambda) \leq 2$. Thus, we must only consider λ such that $(d^c)/\lambda$ is a broken border strip.

As (d^c) is just a $d \times c$ grid the complement $(d^c)/\lambda$ is always connected and so if $(d^c)/\lambda$ is a broken border strip it must be, in particular a border strip. If $(d^c)/\lambda$ is a border strip then it must be one of three types:

1. $(d^c)/\lambda$ is a horizontal border strip with right most box in the bottom right corner of (d^c) ($(d^c)/\lambda$ is the last row of (d^c)),

2. $(d^c)/\lambda$ is a vertical border strip with bottom most box in the bottom right corner of (d^c) ($(d^c)/\lambda$ is the last column of (d^c)),

3. $(d^c)/\lambda$ is a border strip with two sharp corners and a dull corner on the bottom right most box of (d^c) ($(d^c)/\lambda$ is the union of the last row and last column of (d^c)),

We will show that $d_{\lambda(d^c)} \equiv 0$ in each of these cases. As $d+c$ was assumed to be even, $c-d$ is also even. This means that the content of the right most bottom box of (d^c) is even, since the content is given by $(c-1)-(d-1)=c-d$.

1. For λ in the first case, there is just one sharp corner, namely the box on the far left. We will call this box s. There are no dull corners. Since the strip is of odd length (namely $2^{n+1}-1$), the content of the left most box s and right most box $(c-1)$ agree modulo two (since the content increases by one from left to right). Hence, the content of this sharp corner, which we will denote by $C(s)$, is even. Using the formula (4.2.19), we calculate

$$d_{\lambda(d^c)} = C(s) \equiv 0 \pmod 2.$$

2. For λ in the second case, the argument is exactly the same, but with the sharp corner on the top.

3. For λ in the third case, the content of the sharp corner on the bottom left $C(s_1)$ and the content of the sharp corner on the top right $C(s_2)$ agree, because the border strip is of odd length. The dull corner d_1 is in the bottom right corner of $s_{(d^c)}$ and so has content $C(d_1) = c-d \equiv 0 \pmod 2$. Thus, we may compute

$$\begin{aligned} d_{\lambda s_{(d^c)}} &= C(s_1) + C(s_2) - C(d_1) \\ &\equiv 2C(s_1) - 0 \equiv 0. \end{aligned}$$

Thus, we see in all cases $Q_n(s_\lambda) = 0$ for s_λ in degree $cd - 2^{n+1} + 1$. This completes the proof that the top class is not in the image of Q_n for even $d + c$. $\square$

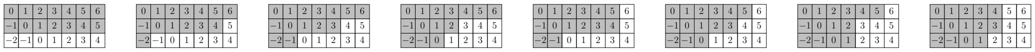
For clarity we will illustrate this proof with the next example.

Example 4.7.2. Let $d = 3$ and $c = 7$. We will explicitly show that the top class is not in the image of Q_2 by running through the steps in the above proof. In this case, we are considering D_7. The top class is indexed by

$$(3^7) = \begin{array}{|c|c|c|c|c|c|c|} \hline 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline -1 & 0 & 1 & 2 & 3 & 4 & 5 \\ \hline -2 & -1 & 0 & 1 & 2 & 3 & 4. \\ \hline \end{array}$$

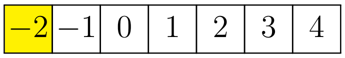
We have filled in the contents. Notice that the content of the bottom right corner is indeed $c - d$.

The classes in degree $cd - 2^{n+1} + 1 = 14$ are pictured below. We have marked the diagram that represents the class in gray and included it into the top class to make it easier to see which classes have border strip complement.

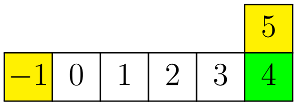


We immediately know that if $(3^7)/\lambda$ is not a border strip, then $Q_n(s_\lambda) = 0$, and so we now reduce to the border strip case. There are just three such λ that we enumerate below. The sharp corners have been colored yellow and the dull corners green.

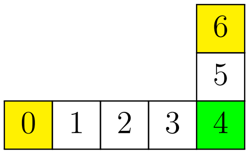
$(3^7)/\lambda_{(7,7,0)} =$



$(3^7)/\lambda_{(7,6,1)} =$



$(3^7)/\lambda_{(6,6,2)} =$



We see that $(3^7)/\lambda_{(7,7,0)}$ is case (1) from the proof and has

$$d_{\lambda_{(7,7,0)}(3^7)} = -2 - 0 \equiv 0.$$

Now $(3^7)/\lambda_{(7,6,1)}$ is type 3. We see that contents of the two sharp corners are equal modulo two and the dull corner has content $c - d$.

$$d_{\lambda_{(7,6,1)}(3^7)} = -1 + 5 - 4 \equiv 0$$

Finally, $(3^7)/\lambda_{(6,6,2)}$ is also type 3,

$$d_{\lambda_{(2,6,6)}(3^7)} = 0 + 6 - 4 \equiv 0.$$

We see in all cases that Q_n is zero on degree $cd - 2^{n+1} + 1$.

Definition 4.7.3. A Poincaré duality algebra is a graded algebra A_* equipped with a perfect pairing $A_i \otimes A_{N-i} \to A_N$, where N is the top degree of A_*.

Recall that to be perfect means that the adjoint $A_i \xrightarrow{D} \mathrm{Hom}(A_{N-i}, A_N)$ is an isomorphism. In particular, this mimics the duality seen in the standard Poincaré duality. Recall that a differential graded algebra is an algebra A_* equipped with a differential $d\colon A_* \to A_*$ satisfying a Leibniz rule [32, Pg. 11].

Example 4.7.4. The $\mathbb{Z}/2\mathbb{Z}$-cohomology of $\mathrm{Gr}_d(\mathbb{R}^{d+c})$ is a differential graded algebra with the differential Q_n. Furthermore, coming from the Poincaré duality of $H^*(\mathrm{Gr}_d(\mathbb{R}^{d+c}); \mathbb{Z}/2\mathbb{Z})$ it is a Poincaré duality algebra.

We establish a sufficient condition for when the homology of a Poincaré duality algebra is also a Poincaré duality algebra.

Lemma 4.7.5. *Let (A_*, d) be differential graded algebra with A also Poincaré algebra over $\mathbb{Z}/2\mathbb{Z}$ with top degree in dimension N and d of degree r. If $d\colon A_{N-r} \to A_N$ is zero, then $H(A_*, d)$ is a Poincaré duality algebra.*

Proof. Let x and y be classes such $|x| + |y| + r = N$ so that $d(xy)$ is in degree N, and thus by assumption $d(xy) = 0$. As d is a derivation

$$0 = d(xy) = d(x)y + xd(y),$$

hence $xd(y) = d(x)y$. This gives an isomorphism of chain complexes $(D(A_*), D(d)) \cong (A_*, d)$:

$$\begin{tikzcd}
\ldots \arrow[r, "d"] & A_{*-r} \arrow[r, "d_{*-r}"] \arrow[d, "D"] & A_* \arrow[r, "d_*"] \arrow[d, "D"] & A_{*+r} \arrow[r, "d_{*+r}"] \arrow[d, "D"] & \ldots \\
\ldots \arrow[r, "d^*_{N-(*-r)}"] & A_{N-(*-r)} \arrow[r, "d^*_{N-(*)}"] & A_{N-*} \arrow[r, "d^*_{N-(*+r)}"] & A_{N-(*+r)} \arrow[r, "d^*_{N-(*+2r)}"] & \ldots
\end{tikzcd}$$

By the definition of the dual map $d^*(x \otimes -) = x \otimes d(-)$. The commutativity of this diagram may be expressed as

$$\begin{aligned} d^* D(x) &= D(d(x)) \\ d^*(x \otimes -) &= d(x) \otimes - \\ x \otimes d(-) &= d(x) \otimes -. \end{aligned}$$

This is exactly what the relation $xd(y) = d(x)y$ says, hence

$$H_*(A_*, d) \cong H_*(D(A_*), Dd) \cong DH_*(A_*, d). \qquad \square$$

Proposition 4.7.6. *For even $d + c$ the Margolis homology $H^*(\mathrm{Gr}_d(\mathbb{R}^{d+c}); Q_n)$ is a Poincaré duality algebra.*

Proof. This is immediate from Proposition 4.7.1 and Lemma 4.7.5. ☐

4.8 Cofiber duality

Due to the existence of the Thom isomorphism relating the $\mathbb{Z}/2\mathbb{Z}$-homologies of the cofiber $C_d(\mathbb{R}^{d+c})$ and the Grassmannian $\mathrm{Gr}_{d-1}(\mathbb{R}^{d+c})$ one might hope that the calculation of the Margolis homology of the cofiber can be reduced to the calculation of the Margolis homology of this Grassmannian. When $d+c$ is odd, this is true. This critically depends on Proposition 4.7.1 and the self duality of the Schubert basis.

We will use the Thom isomorphism to pull back the action of Q_n from the cohomology of the cofiber $H^*(C_d(\mathbb{R}^{d+c}); \mathbb{Z}/2\mathbb{Z})$ to $H^*(\mathrm{Gr}_{d-1}(\mathbb{R}^{d+c}); \mathbb{Z}/2\mathbb{Z})$. The Thom isomorphism states that every element in $H^*(C_d(\mathbb{R}^{d+c}); \mathbb{Z}/2\mathbb{Z})$ is written as $xs_{(c+1)}$ for some unique $x \in H^*(\mathrm{Gr}_d(\mathbb{R}^{d+c}); \mathbb{Z}/2\mathbb{Z})$. In particular, we may write $Q_n(s_{(c+1)}) =$

$\alpha_n s_{(c+1)}$ for some unique α_n. With this notation we calculate

$$\begin{aligned} Q_n(xs_{(c+1)}) &= Q_n(x)s_{(c+1)} + xQ_n(s_{(c+1)}) \\ &= Q_n(x)s_{(c+1)} + x\alpha_n s_{(c+1)} \\ &= (Q_n(x) + x\alpha_n)s_{(c+1)}. \end{aligned}$$

Define $\widehat{Q_n}(x) = Q_n(x) + x\alpha_n$. By construction, the action of $\widehat{Q_n}$ on the cohomology of the base space $H^*(\mathrm{Gr}_{d-1}(\mathbb{R}^{d+c}); \mathbb{Z}/2\mathbb{Z})$ induces the action of Q_n on $H^*(C_d(\mathbb{R}^{d+c}); \mathbb{Z}/2\mathbb{Z})$ through the Thom isomorphism. Using this $\widehat{Q_n}$ we can show that:

Proposition 4.8.1. *For odd $d + c$, the chain complexes $H^*(\mathrm{Gr}_{d-1}(\mathbb{R}^{d+c}); Q_n)$ and $H^{(d-1)(c+1)-*}(\mathrm{Gr}_{d-1}(\mathbb{R}^{d+c}); \widehat{Q_n})$ are dual, so that*

$$H^q(\mathrm{Gr}_{d-1}(\mathbb{R}^{d+c}); Q_n) \cong H^{d(c+1)-q}(C_d(\mathbb{R}^{d+c}); Q_n)$$

for all q.

Proof. For convenience, we let N be the top degree of $\mathrm{Gr}_{d-1}(\mathbb{R}^{d+c})$, that is, $N = (d-1)(c+1)$. The isomorphism

$$\begin{gathered} H^q(\mathrm{Gr}_{d-1}(\mathbb{R}^{d+c})) \xrightarrow{f_q} H^{N-q}(\mathrm{Gr}_{d-1}(\mathbb{R}^{d+c}))^* \\ x \mapsto x \cup - \end{gathered}$$

identifies the elements of $H^q(\mathrm{Gr}_{d-1}(\mathbb{R}^{d+c}))$ with linear functionals on $H^q(\mathrm{Gr}_{d-1}(\mathbb{R}^{d+c}))$. For odd $d + c$, we will show these assemble into an isomorphism of chain complexes from $(H^*(\mathrm{Gr}_{d-1}(\mathbb{R}^{d+c})), Q_n)$ to $((H^{N-*}(\mathrm{Gr}_{d-1}(\mathbb{R}^{d+c})))^*, \widehat{Q_n}^*)$ pictured below,

$$\begin{tikzcd}
\dots \arrow[r, "Q_n"] & H^q(\mathrm{Gr}_{d-1}(\mathbb{R}^{d+c}))) \arrow[r, "Q_n"] \arrow[d, "f_q"] & H^{q+2^{n+1}-1}(\mathrm{Gr}_{d-1}(\mathbb{R}^{d+c})) \arrow[r, "Q_n"] \arrow[d, "f_{q+2^{n+1}-1}"] & \dots \\
\dots \arrow[r, "\widehat{Q_n}^*"] & H^{N-q}(\mathrm{Gr}(\mathbb{R}^{d+c})))^* \arrow[r, "\widehat{Q_n}^*"] & H^{N-q-2^{n+1}+1}(\mathrm{Gr}_{d-1}(\mathbb{R}^{d+c}))^* \arrow[r, "\widehat{Q_n}^*"] & \dots .
\end{tikzcd} \tag{4.8.2}$$

The existence of this chain isomorphism would yield the desired isomorphism

$$\begin{aligned}
H^q(H^*(\mathrm{Gr}_{d-1}(\mathbb{R}^{d+c})); Q_n) &\cong H^q(H^{N-*}(\mathrm{Gr}_{d-1}(\mathbb{R}^{d+c}))^*; \widehat{Q_n}^*) \\
&\cong H^{N-q}(H^*(\mathrm{Gr}_{d-1}(\mathbb{R}^{d+c})); \widehat{Q_n}) \\
&\cong H^{N-q+c+1}(H^*(C_d(\mathbb{R}^{d+c}); Q_n).
\end{aligned}$$

From the definition of the induced map on dual spaces, $\widehat{Q_n}^*(x \cup -) = x \cup \widehat{Q_n}(-)$. Thus, the claim that (4.8.2) is a chain map is equivalent to the statement for all $x \in H^q(\mathrm{Gr}_{d-1}(\mathbb{R}^{d+c}))$,

$$\begin{aligned}
\widehat{Q_n}^* f_q(x) &= f_{q+2^{n+1}-1} Q_n(x) \\
\widehat{Q_n}^*(x \cup -) &= Q_n(x) \cup - \\
x \cup \widehat{Q_n}(-) &= Q_n(x) \cup -.
\end{aligned}$$

This last line is exactly the statement that

$$Q_n \colon H^q(\mathrm{Gr}_{d-1}(\mathbb{R}^{d+c}); \mathbb{Z}/2\mathbb{Z}) \to H^{q+2^{n+1}-1}(\mathrm{Gr}_{d-1}(\mathbb{R}^{d+c}); \mathbb{Z}/2\mathbb{Z})$$

and

$$\widehat{Q_n}\colon H^{N-q-2^{n+1}+1}(\mathrm{Gr}_{d-1}(\mathbb{R}^{d+c});\mathbb{Z}/2\mathbb{Z}) \to H^{N-q}(\mathrm{Gr}_{d-1}(\mathbb{R}^{d+c});\mathbb{Z}/2\mathbb{Z})$$

are dual. Unpacking what this means: we wish to show that for x in degree q and y in degree $N-q-2^{n+1}+1$ that

$$\begin{aligned} Q_n(x)y &= x\widehat{Q_n}(y) \\ Q_n(x)y &= xQ_n(y) + xy\alpha_n \\ Q_n(x)y + xQ_n(y) &= xy\alpha_n \\ Q_n(xy) &= xy\alpha_n. \end{aligned}$$

Thus, by letting $u = xy$ it suffices to show for all u in degree $N-2^{n+1}+1$ that

$$Q_n(u) = u\alpha_n. \tag{4.8.3}$$

To show this we consider the Schubert basis. Write $\alpha_n = \sum s_\lambda$ in the Schubert basis. Let Ds_λ denote the dual of the Schubert class s_λ. From Proposition 4.2.4, the self duality of the Schubert basis states that $s_\lambda Ds_\lambda$ is the top class, $s_\lambda Ds_\lambda = s_{((c+1)^{d-1})}$. To show (4.8.3) it suffices to show on a Schubert basis element b in degree $N-2^{n+1}+1$ that

$$Q_n(b) = \begin{cases} s_{((c+1)^{d-1})} & b = Ds_\lambda \\ 0 & \text{else.} \end{cases} \tag{4.8.4}$$

To see why this is sufficient, let $u = \sum c_i Ds_\lambda + \sum t_j$ be the Schubert basis expansion of u, where t_i denotes Schubert basis elements occurring in u but with dual not in

α_n. The coefficient c_i can either be zero or one. Now we compute

$$\begin{aligned} u\alpha_n &= \left(\sum_k c_k Ds_k + \sum_j t_j\right) \sum s_\lambda \\ &= \sum_i \sum_k c_k (Ds_k) s_\lambda + \sum_i \sum_j t_j s_\lambda \qquad ((Ds_\lambda)s_\lambda = s_{((c+1)^{d-1})} \text{ and } t_j s_\lambda = 0) \\ &= \sum c_i s_{((c+1)^{d-1})}, \end{aligned}$$

while on the other hand

$$\begin{aligned} Q_n(u) &= Q_n\left(\sum c_i Ds_\lambda + \sum t_j\right) \\ &= \sum c_i Q_n(Ds_\lambda) + \sum Q_n(t_j) \qquad \text{(by assumption (4.8.4))} \\ &= \sum c_i s_{((c+1)^{d-1})} + 0. \end{aligned}$$

We now prove (4.8.4) holds. This uses Proposition 4.7.1 which, in particular, says that the top class of $\mathrm{Gr}_{d-1}(\mathbb{R}^{d+c+1})$ is not ever in the image of Q_n (since $d+c+1$ is even). Let b be a Schubert basis element in degree $N - 2^{n+1} + 1$ such that $Q_n(b) = s_{((c+1)^{d-1})}$. From the Thom isomorphism, $Q_n(b)s_{(c+1)} = s_{((c+1)^{d-1})}s_{(c+1)} = s_{((c+1)^d)}$, which is the top class of $Gr_d(\mathbb{R}^{d+c+1})$. The action of Q_n on $H^*(C_d(\mathbb{R}^{d+c}); \mathbb{Z}/2\mathbb{Z})$ is inherited from the action of Q_n on $H^*(Gr_d(\mathbb{R}^{d+c+1}); \mathbb{Z}/2\mathbb{Z})$, since $H^*(C_d(\mathbb{R}^{d+c}); \mathbb{Z}/2\mathbb{Z}) \subseteq H^*(Gr_d(\mathbb{R}^{d+c+1}); \mathbb{Z}/2\mathbb{Z})$. From Proposition 4.7.1, the top class of $Gr_d(\mathbb{R}^{d+c+1})$ is never in the image, hence

$$Q_n(bs_{(c+1)}) = 0, \tag{4.8.5}$$

while on the other hand,

$$\begin{aligned} Q_n(bs_{(c+1)}) &= (Q_n(b) + b\alpha_n)s_{(c+1)} \\ &= Q_n(b)s_{(c+1)} + b\alpha_n s_{(c+1)} \\ &= s_{((c+1)^d)} + b\alpha_n s_{(c+1)}. \end{aligned}$$

Hence, $(b\alpha_n)s_{(c+1)} = s_{((c+1)^d)}$ and thus $b\alpha_n = s_{((c+1)^{d-1})}$ through the Thom isomorphism. As b is a Schubert basis element $bs_\lambda = 0$ unless $b = Ds_\lambda$. Thus, $s_{((c+1)^{d-1})} = b\alpha_n = b\sum s_\lambda$ means that $b = Ds_\lambda$ for one of the s_λ.

Similarly, if $Q_n(b) = 0$ then

$$\begin{aligned} 0 &= Q_n(bs_{(c+1)}) && \text{(by 4.8.5)} \\ &= (Q_n(b) + b\alpha_n)s_{(c+1)} \\ &= b\alpha_n s_{(c+1)}. \end{aligned}$$

By the Thom Isomorphism $b\alpha_n = 0$ and hence b is not dual to any s_λ appearing in α_n. □

4.9 The Grassmannians of two planes

4.9.1 Organization of the main proof

In §4.5 we used chromatic fixed point theory to establish a lower bound for $k_n(\mathrm{Gr}_d(\mathbb{R}^m))$. In this section we will show that the dimension of the Margolis homology of $\mathrm{Gr}_2(\mathbb{R}^m)$ agrees with this lower bound. This will allow us to use Theorem 3.4.1 to conclude:

Theorem 4.9.1. *The dimension of* $K(n)^*(\mathrm{Gr}_2(\mathbb{R}^m))$ *is*

$$k_n(\mathrm{Gr}_2(\mathbb{R}^m)) = \begin{cases} \binom{m}{2} & 2 \leq m \leq 2^{n+1} \\ 2^{2n+1} - 2^n + \ell & m = 2^{n+1} + 2\ell \\ 2^{2n+1} - 2^{n+1} - 2^n + 1 + \ell & m = 2^{n+1} + 2\ell - 1. \end{cases}$$

This theorem in the range $2 \leq m \leq 2^{n+1}$ has already been proved as Theorem 4.5.1, and so in this section we focus on the other two cases. We introduce the key technical propositions and prove the result assuming these propositions. The majority of this section will be dedicated to proving these propositions.

Proposition 4.9.2. *If* $m = 2^{n+1} + 2\ell$ *with* $\ell \geq 1$, *then the connecting homomorphism* $\partial\colon H^*(\mathrm{Gr}_2(\mathbb{R}^m); Q_n) \to H^{*+|Q_n|}(C_2(\mathbb{R}^m); Q_n)$ *is surjective, thus*

$$k_{Q_n}(\mathrm{Gr}_2(\mathbb{R}^m)) = k_{Q_n}(\mathrm{Gr}_2(\mathbb{R}^{m+1})) + k_{Q_n}(C_2(\mathbb{R}^m)).$$

This is proved in §4.9.4.

Proposition 4.9.3. *If* $m = 2^{n+1} + 2\ell - 1$ *with* $\ell \geq 1$, *then the connecting homomorphism* $\partial\colon H^*(\mathrm{Gr}_2(\mathbb{R}^m); Q_n) \to H^{*+|Q_n|}(C_2(\mathbb{R}^m); Q_n)$ *is zero, thus*

$$k_{Q_n}(\mathrm{Gr}_2(\mathbb{R}^{m+1})) = k_{Q_n}(\mathrm{Gr}_2(\mathbb{R}^m)) + k_{Q_n}(C_2(\mathbb{R}^m)).$$

This is proved in §4.9.5. These propositions suggest that one should calculate the dimension of the Margolis homology of the cofiber. At $d = 2$ the cofiber looks like a truncated projective space, and so it is not hard to calculate the Margolis homology.

Proposition 4.9.4. *The Margolis homology of the cofiber* $C_2(\mathbb{R}^m)$ *has dimension*

$$k_{Q_n}(C(\mathbb{R}^m)) = \begin{cases} m & m \leq 2^{n+1} \\ 2^{n+1} - 2 & m \geq 2^{n+1},\ m \text{ even} \\ 2^{n+1} - 1 & m \geq 2^{n+1},\ m \text{ odd.} \end{cases}$$

We prove this in §4.9.3. Assuming these three propositions, we calculate the dimension of the Margolis homology of $\mathrm{Gr}_2(\mathbb{R}^m)$:

Proposition 4.9.5. *The dimension of the Margolis homology* $H_*(\mathrm{Gr}_2(\mathbb{R}^m); Q_n)$ *is*

$$k_{Q_n}(\mathrm{Gr}_2(\mathbb{R}^m)) = \begin{cases} \binom{m}{2} & 2 \leq m \leq 2^{n+1} \\ 2^{2n+1} - 2^n + \ell & m = 2^{n+1} + 2\ell \\ 2^{2n+1} - 2^{n+1} - 2^n + 1 + \ell & m = 2^{n+1} + 2\ell - 1. \end{cases}$$

Proof. The formula for $2 < m \leq 2^{n+1}$ was proved in Corollary 4.5.2. We assume this as the base case for our induction. This includes when $\ell = 0$ for both the even and the odd cases.

Now, we assume by induction that the result holds for $m = 2^{n+1} + 2\ell - 1$ and show that this implies the result for $m + 1 = 2^{n+1} + 2\ell$. By Prop 4.9.23

$$\begin{aligned} k_{Q_n}(\mathrm{Gr}_2(\mathbb{R}^{2^{n+1}+2\ell})) &= k_{Q_n}(\mathrm{Gr}_2(\mathbb{R}^{2^{n+1}+2\ell-1})) + k_{Q_n}(C(\mathbb{R}^{2^{n+1}+2\ell-1})) \\ &= 2^{2n+1} - 2^{n+1} - 2^n + 1 + \ell + k_{Q_n}(C(\mathbb{R}^{2^{n+1}+2\ell-1})) \quad \text{(induction)} \\ &= 2^{2n+1} - 2^{n+1} - 2^n + 1 + \ell + 2^{n+1} - 1 \quad \text{(Prop 4.9.11)} \\ &= 2^{2n+1} - 2^n + \ell. \end{aligned}$$

We have shown that the statement being true for odd m implies it for the next even $m+1$. We now show when it is true for an even m it is also true for the next odd $m+1$. Suppose the equation holds for an even $m = 2^{n+1} + 2\ell$. By Prop 4.9.15

$$\begin{aligned}
k_{Q_n}(\mathrm{Gr}_2(\mathbb{R}^{2^{n+1}+2\ell})) &= k_{Q_n}(\mathrm{Gr}_2(\mathbb{R}^{2^{n+1}+2\ell+1})) + k_{Q_n}(C_2(\mathbb{R}^{2^{n+1}+2\ell})) \\
k_{Q_n}(\mathrm{Gr}_2(\mathbb{R}^{2^{n+1}+2\ell+1})) &= k_{Q_n}(\mathrm{Gr}_2(\mathbb{R}^{2^{n+1}+2\ell})) - k_{Q_n}(C(\mathbb{R}^{2^{n+1}+2\ell})) \\
k_{Q_n}(\mathrm{Gr}_2(\mathbb{R}^{2^{n+1}+2(\ell+1)-1}) &= k_{Q_n}(\mathrm{Gr}_2(\mathbb{R}^{2^{n+1}+2\ell})) - k_{Q_n}(C(\mathbb{R}^{2^{n+1}+2\ell})) \\
&= 2^{2n+1} - 2^n + \ell - k_{Q_n}(C(\mathbb{R}^{2^{n+1}+2\ell})) \qquad \text{(induction)} \\
&= 2^{2n+1} - 2^n + \ell - (2^{n+1} - 2) \qquad \text{(Prop 4.9.11)} \\
&= 2^{2n+1} - 2^{n+1} - 2^n + 1 + (\ell + 1).
\end{aligned}$$

This completes the induction. □

Proof of Theorem 4.9.26. Proposition 4.9.5 shows for $d = 2$ that the upper bound agrees with the lower bound of Theorem 3.1.1 and thus Theorem 3.4.1 applies. □

4.9.2 Calculations of the Milnor primitive

In this subsection we prove certain formulas for Q_n that will be needed in the proofs of Proposition 4.9.15 and Proposition 4.9.23. In particular, we calculate Q_n on the monomial basis elements of $H^*(\mathrm{Gr}_2(\mathbb{R}^{2+c}); \mathbb{Z}/2\mathbb{Z})$.

From Borel's description of the cohomology in Proposition 4.1.2,

$$H^*(\mathrm{Gr}_2(\mathbb{R}^{2+c}); \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}[w_1, w_2]/J(2, c).$$

Therefore, a first step to calculating the Margolis homology is to calculate $Q_n(w_1)$ and $Q_n(w_2)$.

Proposition 4.9.6. *$Q_n(w_1) = w_1^{2^{n+1}}$ in $H^*(\mathrm{Gr}_2(\mathbb{R}^\infty); \mathbb{Z}/2\mathbb{Z})$.*

Proof. This calculation follows from the unstable condition, $\mathrm{Sq}^i(w_1) = 0$ for $i > 1$. The base case is

$$\begin{aligned} Q_1(w_1) &= \mathrm{Sq}^1 \mathrm{Sq}^2(w_1) + \mathrm{Sq}^2 \mathrm{Sq}^1(w_1) \\ &= 0 + \mathrm{Sq}^2(w_1^2) \\ &= w_1^4. \end{aligned}$$

By induction, assume $Q_n(w_1) = w_1^{2^{n+1}}$, then

$$\begin{aligned} Q_{n+1}(w_1) &= Q_n \mathrm{Sq}^{2^{n+1}}(w_1) + \mathrm{Sq}^{2^{n+1}} Q_n(w_1) \\ &= Q_n 0 + \mathrm{Sq}^{2^{n+1}} w_1^{2^{n+1}} \\ &= w_1^{2^{n+2}}. \end{aligned}$$

□

Notation. *For convenience, we let $e_n = 2^{n+1} + 1$.*

We will express $Q_n(w_2)$ in a certain form involving $\overline{w}_{e_n}$ that will make our later calculations much easier. We first point out a formula for $\overline{w}_k$ in terms of w_1 and w_2.

Proposition 4.9.7 ([37, §2]). *In $H^*(\mathrm{Gr}_2(\mathbb{R}^{2+c}); \mathbb{Z}/2\mathbb{Z})$ the dual Stiefel–Whitney classes have the form*

$$\overline{w}_k = \sum_{a+2b=k} \binom{a+b}{a} w_1^a w_2^b.$$

Proof. The dual classes are defined by the relation

$$(1 + w_1 + w_2 + w_3 + \ldots)(1 + \overline{w}_1 + \overline{w}_2 + \overline{w}_3 + \ldots) = 1,$$

for the Grassmannian of two-planes $w_i = 0$ for $i > 2$, hence

$$\begin{aligned}(1 + \overline{w}_1 + \overline{w}_2 + \overline{w}_3 + \ldots) &= \frac{1}{1 + w_1 + w_2} \\ &= \sum_{q=0}^{\infty} (w_1 + w_2)^q \\ &= \sum_{q=0}^{\infty} \sum_{a+2b=q} \binom{a+b}{a} w_1^a w_2^b.\end{aligned}$$

Now pick out the part in degree k. □

Proposition 4.9.8. *$Q_n(w_2) = \overline{w}_{e_n} + w_1^{e_n} + w_1^{e_n-2} w_2$ in $H^*(\mathrm{Gr}_2(\mathbb{R}^\infty); \mathbb{Z}/2\mathbb{Z})$.*

Proof. Our goal is to prove

$$Q_n(w_2) = \overline{w}_{e_n} + w_1^{e_n} + w_1^{e_n-2} w_2. \tag{4.9.9}$$

As a first step we will rewrite the right-hand side. From Proposition 4.9.7,

$$\overline{w}_{e_n} = \overline{w}_{2^{n+1}+1} = \sum_{b=0}^{2^n} \binom{2^{n+1} + 1 - b}{b} w_1^{2^{n+1}+1-2b} w_2^b,$$

where $a = 2^{n+1} + 2 - 2b$. Notice that the first term in this sum corresponds to $w_1^{e_n}$. We need a result about binomial coefficients, which we prove in the appendix as Lemma A.0.1: For $1 \le b \le 2^{n+1}$, $\binom{2^{n+1}+1-b}{b} \equiv 1 \pmod 2$ if and only if b is a power of two. Thus, only the terms with b a power of two in this sum survive. In particular, $w_1^{e_n-2} w_2$ can't come from this sum, and thus we may rewrite the right-hand side of (4.9.9) as

$$w_1^{(2^{n+1}-1)} w_2 + \sum_{b=1}^{2^n} \binom{2^{n+1} + 1 - b}{b} w_1^{2^{n+1}+1-2b} w_2^b = w_1^{(2^{n+1}-1)} w_2 + \sum_{b=1}^{n} w_1^{2^{n+1}+1-2^{b+1}} w_2^{2^b}.$$

We will now proceed by induction on n to show that $Q_n(w_2)$ is equal to the right-hand side of the above expression. The base case can be directly computed as $Q_1(w_2) = w_1^3 w_2 + w_1 w_2^2$. Assume as the inductive hypothesis that

$$Q_n(w_2) = w_1^{(2^{n+1}-1)} w_2 + \sum_{b=1}^{n} w_1^{2^{n+1}+1-2^{b+1}} w_2^{2^b}$$

holds for n. We will now show this implies the formula for Q_{n+1}. By the definition of Q_{n+1},

$$Q_{n+1}(w_2) = \mathrm{Sq}^{2^{n+1}}(Q_n(w_2)) + Q_n(\mathrm{Sq}^{2^{n+1}}(w_2)).$$

By the unstable condition, the second term is zero, thus $Q_{n+1}(w_2) = \mathrm{Sq}^{2^{n+1}}(Q_n(w_2))$. By the inductive hypothesis,

$$Q_{n+1}(w_2) = \mathrm{Sq}^{2^{n+1}} \left(w_1^{(2^{n+1}-1)} w_2 + \sum_{b=1}^{n} w_1^{2^{n+1}+1-2^{b+1}} w_2^{2^b} \right).$$

We analyze this in two steps. First we calculate

$$\begin{aligned} \mathrm{Sq}^{2^{n+1}} \left(w_1^{2^{n+1}-1} w_2 \right) &= \mathrm{Sq}^{2^{n+1}-1} \left(w_1^{2^{n+1}-1} \right) \mathrm{Sq}^1(w_2) + \mathrm{Sq}^{2^{n+1}-2} \left(w_1^{2^{n+1}-1} \right) \mathrm{Sq}^2(w_2) \\ &= w_1^{2^{n+2}-1} w_2 + w_1^{2^{n+2}-3} w_2^2. \end{aligned}$$

Here we have used the Cartan formula and the unstable condition $\mathrm{Sq}^i(w_1^{2^{n+1}-1}) = 0$ for $i > 2^{n+1} - 1$ and $\mathrm{Sq}^j(w_2) = 0$ for $j > 2$. Similarly, now we compute the $\mathrm{Sq}^{2^{n+1}}$ of the terms in the sum

$$\begin{aligned}\mathrm{Sq}^{2^{n+1}}\left(w_1^{2^{n+1}+1-2^{b+1}}w_2^{2^b}\right) &= \mathrm{Sq}^{2^{n+1}+1-2^{b+1}}\left(w_1^{2^{n+1}+1-2^{b+1}}\right)\mathrm{Sq}^{2^{b+1}-1}(w_2^{2^b}) \\ &\qquad + \mathrm{Sq}^{2^{n+1}-2^{b+1}}\left(w_1^{2^{n+1}+1-2^{b+1}}\right)\mathrm{Sq}^{2^{b+1}}(w_2^{2^b}) \\ &= 0 + w_1^{2^{n+1}+1-2^{b+1}+2^{n+1}-2^{b+1}}w_2^{2^{b+1}} \\ &\qquad\qquad \text{(odd Sq of square power class is zero)} \\ &= w_1^{2^{n+2}+1-2^{b+2}}w_2^{2^{b+1}},\end{aligned}$$

hence

$$\begin{aligned}\mathrm{Sq}^{2^{n+1}}\left(\sum_{b=1}^{n} w_1^{2^{n+1}+1-2^{b+1}}w_2^{2^b}\right) &= \sum_{b=1}^{n} w_1^{2^{n+2}+1-2^{b+2}}w_2^{2^{b+1}} \\ &= \sum_{b=2}^{n+1} w_1^{2^{n+2}+1-2^{b+1}}w_2^{2^b}.\end{aligned}$$

Finally, we complete the induction:

$$\begin{aligned}Q_{n+1}(w_2) &= \mathrm{Sq}^{2^{n+1}}(Q_n(w_2)) \\ &= w_1^{2^{n+2}-1}w_2 + w_1^{2^{n+2}-2}w_2^2 + \sum_{b=2}^{n+1} w_1^{2^{n+2}+1-2^{b+1}}w_2^{2^b} \\ &= w_1^{2^{n+2}-1}w_2 + \sum_{b=1}^{n+1} w_1^{2^{n+2}+1-2^{b+1}}w_2^{2^b}.\end{aligned}$$

□

Using that Q_n is a derivation, we can calculate Q_n on the monomial basis of $H^*(\mathrm{Gr}_2(\mathbb{R}^{2+c});\mathbb{Z}/2\mathbb{Z})$. We collect some important cases.

Proposition 4.9.10. *For $w_1^a w_2^b$ in $H^*(\mathrm{Gr}_2(\mathbb{R}^{2+c}); \mathbb{Z}/2\mathbb{Z})$,*

$$Q_n(w_1^a w_2^b) = b w_1^a w_2^{b-1} \overline{w}_{e_n} + b w_1^{a+e_n} w_2^{b-1} + b w_1^{a+e_n-2} w_2^b + a w_1^{a+e_n-2} w_2^b.$$

1. *If a and b are both odd, then*

$$Q_n(w_1^a w_2^b) = w_1^a w_2^{b-1} \left(\overline{w}_{e_n} + w_1^{e_n} \right).$$

2. *If a is odd and b is even, then*

$$Q_n(w_1^a w_2^b) = w_1^{a+e_n-2} w_2^b.$$

Proof. One computes using Proposition 4.9.6 and Proposition 4.9.8:

$$\begin{aligned}
Q_n(w_1^a w_2^b) &= w_1^a Q_n(w_2^b) + w_2^b Q_n(w_1^a) \\
&= w_1^a Q_n(w_2^b) + a w_1^{a+2^{n+1}-1} w_2^b \\
&= b w_1^a w_2^{b-1} Q_n(w_2) + a w_1^{a+e_n-2} w_2^b \\
&= b w_1^a w_2^{b-1} \left(\overline{w}_{e_n} + w_1^{e_n} + w_1^{e_n-2} w_2 \right) + a w_1^{a+e_n-2} w_2^b \\
&= b w_1^a w_2^{b-1} \overline{w}_{e_n} + b w_1^a w_2^{b-1} w_1^{e_n} + b w_1^a w_2^{b-1} w_1^{e_n-2} w_2 + a w_1^{a+e_n-2} w_2^b \\
&= b w_1^a w_2^{b-1} \overline{w}_{e_n} + b w_1^{a+e_n} w_2^{b-1} + b w_1^{a+e_n-2} w_2^b + a w_1^{a+e_n-2} w_2^b.
\end{aligned}$$

1. If a and b are both odd, then

$$\begin{aligned}
Q_n(w_1^a w_2^b) &= w_1^a w_2^{b-1}\overline{w}_{e_n} + w_1^{a+e_n} w_2^{b-1} + w_1^{a+e_n-2} w_2^b + w_1^{a+e_n-2} w_2^b \\
&= w_1^a w_2^{b-1}\overline{w}_{e_n} + w_1^{a+e_n} w_2^{b-1} \\
&= w_1^a w_2^{b-1}\left(\overline{w}_{e_n} + w_1^{e_n}\right).
\end{aligned}$$

2. If a is odd and b is even, then

$$\begin{aligned}
Q_n(w_1^a w_2^b) &= b w_1^a w_2^{b-1}\overline{w}_{e_n} + b w_1^{a+e_n} w_2^{b-1} + b w_1^{a+e_n-2} w_2^b + a w_1^{a+e_n-2} w_2^b \\
&= w_1^{a+e_n-2} w_2^b. \qquad \square
\end{aligned}$$

Our ultimate goal is to understand the connecting homomorphism ∂. For this, these Q_n calculations will be needed. However, we will also need in-depth knowledge of the cofiber.

4.9.3 The cofiber

The goal of this subsection is to calculate the dimension of the Margolis homology of the cofiber:

Proposition 4.9.11. *The Margolis homology of the cofiber $C_2(\mathbb{R}^m)$ has dimension*

$$k_{Q_n}(C(\mathbb{R}^m)) = \begin{cases} m & m \leq 2^{n+1} \\ 2^{n+1} - 2 & m \geq 2^{n+1},\ m \text{ even} \\ 2^{n+1} - 1 & m \geq 2^{n+1},\ m \text{ odd.} \end{cases}$$

From the Thom isomorphism, we know that $H^*(C_2(\mathbb{R}^{2+c}); \mathbb{Z}/2\mathbb{Z})$ looks like a truncated projective space:

$$\overline{w}_{c+1},\ w_1\overline{w}_{c+1},\ w_1^2\overline{w}_{c+1},\dots,\ w_1^{c+1}\overline{w}_{c+1} \tag{4.9.12}$$

living inside of $H^*(\mathrm{Gr}_2(\mathbb{R}^{2+(c+1)});\mathbb{Z}/2\mathbb{Z})$. In light of Theorem 4.5.1, we are only considering $\mathrm{Gr}_2(\mathbb{R}^{2+c})$ for $2+c>2^{n+1}$.

Lemma 4.9.13. *In $H^*(C_2(\mathbb{R}^{2+c});\mathbb{Z}/2\mathbb{Z})$ we have $Q_n(\overline{w}_{c+1})=w_1^{2^{n+1}-1}\overline{w}_{c+1}$.*

Proof. We will argue this from the Schubert perspective. Recall the Thom isomorphism $H^*(\mathrm{Gr}_1(\mathbb{R}^{2+c});\mathbb{Z}/2\mathbb{Z})\xrightarrow{\cup\ \overline{w}_{c+1}}H^{*+c+1}(C_2(\mathbb{R}^{2+c});\mathbb{Z}/2\mathbb{Z})$. The diagram indexing $w_1^{2^{n+1}-1}$ in $H^*(\mathrm{Gr}_1(\mathbb{R}^{2+c});\mathbb{Z}/2\mathbb{Z})$ must be $(2^{n+1}-1)$, and thus the diagram indexing $w_1^{2^{n+1}-1}\overline{w}_{c+1}$ is $(c+1,2^{n+1}-1)$ by the Peiri formula. By Theorem 4.2.18 any diagram μ with s_μ appearing in the expansion of $Q_n(\overline{w}_{c+1})$ must have $c+1+2^{n+1}-1$ boxes and complement $\mu/\lambda_{(c+1)}$ as a border strip. Thus, since these diagrams fit in a $2\times(c+1)$ grid the only possibility is $(c+1,2^{n+1}-1)$. The only sharp corner is the left most corner and is in the second row and thus has content one, hence $Q_n(\overline{w}_{c+1})=w_1^{2^{n+1}-1}\overline{w}_{c+1}$. □

Proof of Proposition 4.9.11. From Lemma 4.9.13 we see that $\overline{w}_{c+1}$ is not a cycle. For a odd, $w_1^a\overline{w}_{c+1}$ is a cycle since

$$\begin{aligned}
Q_n(w_1^a\overline{w}_{c+1}) &= w_1^aQ_n(\overline{w}_{c+1})+Q_n(w_1^a)\overline{w}_{c+1}\\
&= w_1^{a+2^{n+1}-1}\overline{w}_{c+1}+aw_1^{a-1}Q_n(w_1)\overline{w}_{c+1}\\
&= w_1^{a+2^{n+1}-1}\overline{w}_{c+1}+aw_1^{a+2^{n+1}-1}\overline{w}_{c+1}.\\
&= 0. && \text{(when } a \text{ is odd)}
\end{aligned}$$

We reapply the techniques from Example 3.2.1. The other class of cycles is at the top of the cofiber. The top degree is $2(c+1)$ and so any class of degree $2(c+1)-2^{n+1}+1$ or larger is a cycle. The bottom 2^n-1 odd classes $w_1\overline{w}_{c+1},\dots,w_1^3\overline{w}_{c+1},\dots,w_1^{2^{n+1}-3}\overline{w}_{c+1}$ survive since they are not in the image of Q_n for dimension reasons. The even classes $w_1^{2(c+1)-2^{n+1}+2}\overline{w}_{c+1}, w_1^{2(c+1)-2^{n+1}+4}\overline{w}_{c+1},\dots$ to $w_1^{c+1}\overline{w}_{c+1}$ or $w_1^{c}\overline{w}_{c+1}$ survive depending on if $c+1$ is even or odd. This gives either 2^n or 2^n-1 even surviving classes based on the parity of $c+1$. Therefore, the cofiber $C_2(\mathbb{R}^m)$ has rank

$$k_{Q_n}(C(\mathbb{R}^{2+c}))=\begin{cases}2^{n+1}-2 & 2+c\geq 2^{n+1},\ c \text{ even}\\ 2^{n+1}-1 & 2+c\geq 2^{n+1},\ c \text{ odd}\\ 2+c & 2+c\leq 2^{n+1}.\end{cases}$$

which is exactly Proposition 4.9.11. □

We now point out how the cofiber relates to the Gröbner basis. This will be important in the next section for reducing certain elements modulo the ideal $J(2,c)$. Recall g_i^c from Proposition 4.1.13 is the ith Gröbner basis element for $J(2,c)$.

Proposition 4.9.14. *The classes (4.9.12) of the cofiber $C_d(\mathbb{R}^{2+c})$ correspond to elements of the Gröbner basis of $J(2,c)$,*

$$w_1^i\overline{w}_{c+1}\equiv g_i^c \pmod{J(2,c+1)}.$$

Proof. In the case of $\mathrm{Gr}_2(\mathbb{R}^{2+c})$ [39, Lemma 9] states

$$g_i^c=\sum_{k=0}^{i}\binom{i}{k}w_1^{i-k}\overline{w}_{c+1+k}.$$

In $J(2, c+1)$ the words $\overline{w}_{c+1+k} = 0$ for all $k \geq 1$, hence

$$g_i^c \equiv w_1^i \overline{w}_{c+1} \pmod{J(2, c+1)}. \qquad \square$$

4.9.4 The surjective connecting homomorphism

In this subsection we suppose that $m = c + 2 = 2^{n+1} + 2\ell$. Our goal is to prove:

Proposition 4.9.15. *If $m = 2^{n+1} + 2\ell$ with $\ell \geq 1$, then the connecting homomorphism $\partial \colon H^*(\mathrm{Gr}_2(\mathbb{R}^m); Q_n) \to H^{*+|Q_n|}(C_2(\mathbb{R}^m); Q_n)$ is surjective, thus*

$$k_{Q_n}(\mathrm{Gr}_2(\mathbb{R}^m)) = k_{Q_n}(\mathrm{Gr}_2(\mathbb{R}^{m+1})) + k_{Q_n}(C_2(\mathbb{R}^m)).$$

Recall that the connecting homomorphism comes from the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\dots \arrow[r, "Q_n"] & H^q(C_d(\mathbb{R}^m); \mathbb{Z}/2\mathbb{Z}) \arrow[r, "Q_n"] \arrow[d, "p^q"] & H^{q+|Q_n|}(C_d(\mathbb{R}^m); \mathbb{Z}/2\mathbb{Z}) \arrow[r, "Q_n"] \arrow[d, "p^{q+|Q_n|}"] & H^{q+2|Q_n|}(C_d(\mathbb{R}^m); \mathbb{Z}/2\mathbb{Z}) \arrow[r, "Q_n"] \arrow[d, "p^{q+2|Q_n|}"] & \dots \\
\dots \arrow[r, "Q_n"] & H^q(\mathrm{Gr}_d(\mathbb{R}^{m+1}); \mathbb{Z}/2\mathbb{Z}) \arrow[r, "Q_n"] \arrow[d, "i^q"] & H^{q+|Q_n|}(\mathrm{Gr}_d(\mathbb{R}^{m+1}); \mathbb{Z}/2\mathbb{Z}) \arrow[r, "Q_n"] \arrow[d, "i^{q+|Q_n|}"] & H^{q+2|Q_n|}(\mathrm{Gr}_d(\mathbb{R}^{m+1}); \mathbb{Z}/2\mathbb{Z}) \arrow[r, "Q_n"] \arrow[d, "i^{q+2|Q_n|}"] & \dots \\
\dots \arrow[r, "Q_n"] & H^q(\mathrm{Gr}_d(\mathbb{R}^m); \mathbb{Z}/2\mathbb{Z}) \arrow[r, "Q_n"] \arrow[d] & H^{q+|Q_n|}(\mathrm{Gr}_d(\mathbb{R}^m); \mathbb{Z}/2\mathbb{Z}) \arrow[r, "Q_n"] \arrow[d] & H^{q+2|Q_n|}(\mathrm{Gr}_d(\mathbb{R}^m); \mathbb{Z}/2\mathbb{Z}) \arrow[r, "Q_n"] \arrow[d] & \dots \\
 & 0 & 0 & 0 & ,
\end{tikzcd}$$

where we lift $x \in H^*(\mathrm{Gr}_d(\mathbb{R}^m); \mathbb{Z}/2\mathbb{Z})$ representing a class in $H^*(\mathrm{Gr}_d(\mathbb{R}^m); Q_n)$ to $y \in H^*(\mathrm{Gr}_d(\mathbb{R}^{m+1}); \mathbb{Z}/2\mathbb{Z})$ and then calculate $Q_n(y)$. This will allow us to calculate ∂ by calculating Q_n on $H^*(\mathrm{Gr}_d(\mathbb{R}^{m+1}); \mathbb{Z}/2\mathbb{Z})$ and examining how the relations change from $H^*(\mathrm{Gr}_d(\mathbb{R}^{m+1}); \mathbb{Z}/2\mathbb{Z})$ to $H^*(\mathrm{Gr}_d(\mathbb{R}^m); \mathbb{Z}/2\mathbb{Z})$. In this subsection we will prove Proposition 4.9.15, which says that ∂ is surjective when $m = 2^{n+1} + 2\ell$.

We begin by reducing the problem of showing that ∂ is surjective to just examining the behavior of a few classes.

Proposition 4.9.16. *If there exist a Q_n-cycle $\alpha \in H^*(\mathrm{Gr}_2(\mathbb{R}^m);\mathbb{Z}/2\mathbb{Z})$ such that $\partial(\alpha) = w_1\overline{w}_{c+1}$ and a Q_n-cycle $\beta \in H^*(\mathrm{Gr}_2(\mathbb{R}^m);\mathbb{Z}/2\mathbb{Z})$ such that $\partial(\beta) = w_1^{2\ell+2}\overline{w}_{c+1}$, then the connecting homomorphism ∂ is surjective.*

Proof. If such an α exists, then $Q_n(w_1^{2i}\alpha) = w_1^{2i+1}\overline{w}_{c+1}$, since

$$\begin{aligned} Q_n(w_1^{2i}\alpha) &= w_1^{2i}Q_n(\alpha) + Q_n(w_1^{2i})\alpha \\ &= w_1^{2i+1}\overline{w}_{c+1} + 0. \end{aligned}$$

Notice that $Q_n(w_1^{2i}\alpha) = w_1^{2i+1}\overline{w}_{c+1} \equiv 0$ in $H^*(\mathrm{Gr}_2(\mathbb{R}^m);\mathbb{Z}/2\mathbb{Z})$ while it is non-zero in $H^*(\mathrm{Gr}_2(\mathbb{R}^{m+1});\mathbb{Z}/2\mathbb{Z})$, and so this calculation shows $\partial(w_1^{2i}\alpha) = w_1^{2i+1}\overline{w}_{c+1}$. Therefore, if there exists such an α, then all of the cofiber classes of the form $w_1^a\overline{w}_{c+1}$ with a would be in the image of ∂. Similarly, it is sufficient to find a β such that $Q_n(\beta) = w_1^{c+4-2^{n+1}}\overline{w}_{c+1} = w_1^{2\ell+2}\overline{w}_{c+1}$, since then the even cycles will be of the form $Q_n(w_1^{2j}\beta)$. Therefore, all of the elements of the cofiber would be in the image. □

Now that we have reduced the problem of proving that ∂ is surjective to establishing the existence of these α and β, we show they do exist by providing explicit formulas for them.

Proposition 4.9.17. *Let $\alpha = w_1\overline{w}_{c+1-2^{n+1}+1} \in H^*(\mathrm{Gr}_2(\mathbb{R}^m);\mathbb{Z}/2\mathbb{Z})$. Then α represents a cycle in $H^*(\mathrm{Gr}_2(\mathbb{R}^m);Q_n)$ and $\partial(\alpha) = w_1\overline{w}_{c+1}$.*

Proposition 4.9.18. *Let $\beta = w_2^{2\ell+2} \in H^*(\mathrm{Gr}_2(\mathbb{R}^m);\mathbb{Z}/2\mathbb{Z})$. Then β represents a cycle in $H^*(\mathrm{Gr}_2(\mathbb{R}^m);Q_n)$ and $\partial(\beta) = w_1^{2\ell+2}\overline{w}_{c+1}$.*

We now introduce a lemma, which will be needed for the proof of Proposition 4.9.17.

Lemma 4.9.19. *In* $H^*(\mathrm{Gr}_2(\mathbb{R}^{m+1}); \mathbb{Z}/2\mathbb{Z})$ *for even* s

$$Q_n(w_1\overline{w}_s) = w_1\overline{w}_{s+2^{n+1}-1}.$$

Proof. We proceed by induction. The base case is $s = 2$. We show

$$Q_n(w_1\overline{w}_2) = w_1\overline{w}_{2+2^{n+1}-1} = w_1\overline{w}_{e_n}$$

by computing

$$\begin{aligned}
Q_n(w_1\overline{w}_2) &= Q_n(w_1(w_1^2 + w_2)) \\
&= Q_n(w_1^3) + Q_n(w_1w_2) \\
&= w_1^{2^{n+1}+2} + w_1Q_n(w_2) + w_2Q_n(w_1) \\
&= w_1^{e_n+1} + w_1(\overline{w_{e_n}} + w_1^{e_n} + w_1^{e_n-2}w_2) + w_2w_1^{e_n-1} \\
&= w_1^{e_n+1} + w_1\overline{w_{e_n}} + w_1^{e_n+1} + w_1^{e_n-1}w_2 + w_2w_1^{e_n-1} \\
&= w_1\overline{w_{e_n}}.
\end{aligned}$$

Now we suppose by induction that

$$Q_n(w_1\overline{w}_t) = w_1\overline{w}_{t+2^{n+1}-1}$$

for all even t less than s. To complete the induction we must show for $s = t + 2$,

$$Q_n(w_1\overline{w}_{t+2}) = w_1\overline{w}_{(t+2)+2^{n+1}-1}.$$

Using the recursive relation $\overline{w}_{t+1} = w_1\overline{w}_t + w_2\overline{w}_{t-1}$ we compute

$$\begin{aligned}
Q_n(w_1\overline{w}_{t+2}) &= Q_n(w_1(w_1\overline{w}_{t+1} + w_2\overline{w}_t)) \\
&= Q_n(w_1^2\overline{w}_{t+1} + w_1w_2\overline{w}_t) \\
&= Q_n(w_1^2(w_1\overline{w}_t + w_2\overline{w}_{t-1}) + w_1w_2(w_1\overline{w}_{t-1} + w_2\overline{w}_{t-2})) \\
&= Q_n(w_1^3\overline{w}_t + w_1^2w_2\overline{w}_{t-1} + w_1^2w_2\overline{w}_{t-1} + w_1w_2^2\overline{w}_{t-2}) \\
&= Q_n(w_1^3\overline{w}_t + w_1w_2^2\overline{w}_{t-2}) \\
&= Q_n(w_1^3\overline{w}_t) + Q_n(w_1w_2^2\overline{w}_{t-2}) \\
&= w_1^2Q_n(w_1\overline{w}_t) + 0 + Q_n(w_1\overline{w}_{t-2})w_2^2 + 0 \qquad \text{(by induction)} \\
&= w_1^3\overline{w}_{t+2^{n+1}-1} + w_1w_2^2\overline{w}_{(t-2)+2^{n+1}-1} \\
&= w_1(w_1^2\overline{w}_{t+2^{n+1}-1} + w_2^2\overline{w}_{(t-2)+2^{n+1}-1}).
\end{aligned}$$

Now it suffices to identify $\overline{w}_{(t+2)+2^{n+1}-1}$ as $w_1^2\overline{w}_{t+2^{n+1}-1} + w_2^2\overline{w}_{(t-2)+2^{n+1}-1}$. We calculate,

$$\begin{aligned}
&\overline{w}_{(t+2)+2^{n+1}-1} = w_1\overline{w}_{(t+2)+2^{n+1}-2} + w_2\overline{w}_{(t+2)+2^{n+1}-3} \\
&= w_1\big(w_1\overline{w}_{(t+2)+2^{n+1}-3} + w_2\overline{w}_{(t+2)+2^{n+1}-4}\big) + w_2\big(w_1\overline{w}_{(t+2)+2^{n+1}-4} + w_2\overline{w}_{(t+2)+2^{n+1}-5}\big) \\
&= w_1^2\overline{w}_{(t+2)+2^{n+1}-3} + w_1w_2\overline{w}_{(t+2)+2^{n+1}-4} + w_1w_2\overline{w}_{(t+2)+2^{n+1}-4} + w_2^2\overline{w}_{(t+2)+2^{n+1}-5} \\
&= w_1^2\overline{w}_{t+2^{n+1}-1} + w_2^2\overline{w}_{(t-2)+2^{n+1}-1}.
\end{aligned}$$

This completes the induction. □

With this lemma we can understand the action of Q_n on the proposed α in the statement of Proposition 4.9.17.

Proof of Proposition 4.9.17. Let $\alpha = w_1\overline{w}_{c+1-2^{n+1}+1}$, which by Lemma 4.9.19 (with

$s = c + 1 - 2^{n+1} + 1$) yields $Q_n(w_1\overline{w}_{c+1-2^{n+1}+1}) = w_1\overline{w}_{c+1}$. This means that α is a cycle in $H^*(\mathrm{Gr}_2(\mathbb{R}^{2+c}); Q_n)$. To see this note the inclusion $\mathrm{Gr}_2(\mathbb{R}^{2+c}) \to \mathrm{Gr}_2(\mathbb{R}^{2+c+1})$ maps $i^*(w_1\overline{w}_{c+1}) = w_1\overline{w}_{c+1}$. The ring $H^*(\mathrm{Gr}_2(\mathbb{R}^{2+c+1}); \mathbb{Z}/2\mathbb{Z})$ has fewer relations than $H^*(\mathrm{Gr}_2(\mathbb{R}^{2+c}); \mathbb{Z}/2\mathbb{Z})$, in particular the Thom class $\overline{w}_{c+1}$ is zero in $H^*(\mathrm{Gr}_2(\mathbb{R}^{2+c}); \mathbb{Z}/2\mathbb{Z})$, but not in $H^*(\mathrm{Gr}_2(\mathbb{R}^{2+c+1}); \mathbb{Z}/2\mathbb{Z})$. Thus, α is a cycle in the Margolis homology $H^*(\mathrm{Gr}_2(\mathbb{R}^{2+c}); Q_n)$. This shows that $\partial(\alpha) = w_1\overline{w}_{c+1}$. □

The more delicate calculation is Proposition 4.9.18. This will require the Gröbner basis described in Proposition 4.1.13 and the identification of elements of the cofiber with the Gröbner basis elements from Proposition 4.9.14. We will write $Q_n(\beta)$ as a multiple of a Gröbner basis element of the ideal $J(2, c)$, which will show that β is a cycle in $H^*(\mathrm{Gr}_2(\mathbb{R}^{2+c}); \mathbb{Z}/2\mathbb{Z})$. We will then show that $Q_n(\beta)$ modulo the ideal $J(2, c+1)$ is precisely $w_1^{2\ell+2}\overline{w}_{c+1}$. This will require the following lemmas.

Lemma 4.9.20. $\overline{w}_{2^n}^2 = \sum_{k=0}^{2^n} \binom{2^n-k}{2^n-2k} w_1^{2^{n+1}-4k} w_2^{2k}$

Proof. From Proposition 4.9.7 one has

$$\overline{w}_{2^n} = \sum_{a+2b=2^n} \binom{a+b}{a} w_1^a w_2^b.$$

Letting $k = b$ and $a = 2^n - 2b$ yields,

$$\overline{w}_{2^n} = \sum_{k=0}^{2^{n-1}} \binom{2^n - 2k + k}{2^n - 2k} w_1^{2^n-2k} w_2^k \qquad \text{(characteristic two)}$$

$$\overline{w}_{2^n}^2 = \sum_{k=0}^{2^{n-1}} \binom{2^n - k}{2^n - 2k} w_1^{2^{n+1}-4k} w_2^{2k}. \qquad \square$$

Lemma 4.9.21. $\overline{w}_{e_n} = \overline{w}_{2^{n+1}+1} = w_1(\overline{w}_{2^n})^2$

Proof. By Proposition 4.9.7,

$$\begin{aligned}\overline{w}_{2^{n+1}+1} &= \sum_{b=0}^{2^n} \binom{2^{n+1}+1-2b+b}{b} w_1^{2^{n+1}+1-2b} w_2^b \\ &= \sum_{b=0}^{2^n} \binom{2^{n+1}+1-b}{b} w_1^{2^{n+1}+1-2b} w_2^b.\end{aligned}$$

From Lemma A.0.1 $\binom{2^{n+1}+1-b}{b} \equiv 0 \pmod 2$ for odd b, hence this can be rewritten as

$$\overline{w}_{2^{n+1}+1} = \sum_{b=0}^{2^{n-1}} \binom{2^{n+1}+1-2b}{2b} w_1^{2^{n+1}+1-4b} w_2^{2b},$$

while

$$w_1 \overline{w}_{2^n}^2 = \sum_{k=0}^{2^{n-1}} \binom{2^n-k}{2^n-2k} w_1^{2^{n+1}+1-4k} w_2^{2k}.$$

Hence, we must show that for $0 \leq b \leq 2^{n-1}$

$$\binom{2^n-b}{2^n-2b} \equiv \binom{2^{n+1}+1-2b}{2b} \pmod 2.$$

We prove this as Lemma A.0.5. □

We are now ready to write $Q_n(\beta)$ as a multiple of a Gröbner basis element. Recall from Proposition 4.1.13 that the tth Gröbner basis element for $J(2,c)$ is given by

$$g_t^c = \sum_{b=t}^{\frac{c+1+t}{2}} \binom{c+1-b}{b-t} w_1^{c+1+t-2b} w_2^b$$

and is of degree $c+1+t$.

Lemma 4.9.22. $Q_n(\beta) = Q_n(w_2^{2\ell+1}) = w_1 g_{2\ell+1}^c$

Proof.

$$\begin{aligned}
Q_n(w_2^{2\ell+1}) &= (2\ell+1)w_2^{2\ell}Q_n(w_2) \\
&= w_2^{2\ell}(\overline{w}_{e_n} + w_1^{e_n} + w_1^{e_n-2}w_2) \\
&= \overline{w}_{e_n}w_2^{2\ell} + w_1^{e_n}w_2^{2\ell} + w_1^{e_n-2}w_2^{2\ell+1} \\
&= w_1(\overline{w}_{2^n})^2 w_2^{2\ell} + w_1^{e_n}w_2^{2\ell} + w_1^{e_n-2}w_2^{2\ell+1} \\
&= w_1\left(\overline{w}_{2^n}^2 w_2^{2\ell} + w_1^{e_n-1}w_2^{2\ell} + w_1^{e_n-3}w_2^{2\ell+1}\right), \qquad \text{(by Lemma 4.9.21)}
\end{aligned}$$

thus, it suffices to show that

$$\overline{w}_{2^n}^2 w_2^{2\ell} + w_1^{d-1}w_2^{2\ell} + w_1^{d-3}w_2^{2\ell+1} = g_{2\ell+1}^c.$$

We use the characterization of the Gröbner basis elements from Prop 4.1.13,

$$\begin{aligned}
g_{2\ell+1}^c &= \sum_{b=2\ell+1}^{\frac{c+1+(2\ell+1)}{2}} \binom{c+1-b}{b-(2\ell+1)} w_1^{c+1+(2\ell+1)-2b}w_2^b \\
&= \sum_{b=2\ell+1}^{\frac{c+2\ell+2}{2}} \binom{c+1-b}{b-2\ell-1} w_1^{c+2\ell-2b+2}w_2^b \\
&= \sum_{b=2\ell+1}^{2^n+2\ell} \binom{c+1-b}{b-2\ell-1} w_1^{2^n+4\ell-2b}w_2^b \\
&= \sum_{b=1}^{2^n} \binom{c+1-(b+2\ell)}{(b+2\ell)-2\ell-1} w_1^{2^n+4\ell-2(b+2\ell)}w_2^{b+2\ell} \\
&= \sum_{b=1}^{2^n} \binom{(2^{n+1}+2\ell-2)+1-b-2\ell}{b-1} w_1^{2^n-2b}w_2^{b+2\ell} \\
&= \sum_{b=1}^{2^n} \binom{2^{n+1}-b-1}{b-1} w_1^{2^n-2b}w_2^{b+2\ell}.
\end{aligned}$$

Using Lemma A.0.2, which says $\binom{2^{n+1}-b-1}{b-1} \equiv 0 \pmod{2}$ for $1 < b \leq 2^n$, we may rewrite

$$\begin{aligned} g^c_{2\ell+1} &= w_1^{2^n-2} w_2^{2\ell+1} + \sum_{b=2}^{2^n} \binom{2^{n+1}-b-1}{b-1} w_1^{2^n-2b} w_2^{b+2\ell} \\ &= w_1^{2^n-2} w_2^{2\ell+1} + \sum_{b=1}^{2^{n-1}} \binom{2^{n+1}-2b-1}{2b-1} w_1^{2^n-4b} w_2^{2b+2\ell}, \end{aligned}$$

with only even terms appearing. Now we compare this to

$$\begin{aligned} &\overline{w}^2_{2^n} w_2^{2\ell} + w_1^{e_n-1} w_2^{2\ell} + w_1^{e_n-3} w_2^{2\ell+1} \\ &\quad = \left(\sum_{k=0}^{2^n} \binom{2^n-k}{2^n-2k} w_1^{2^{n+1}-4k} w_2^{2k} \right) w_2^{2\ell} + w_1^{e_n-1} w_2^{2\ell} + w_1^{e_n-3} w_2^{2\ell+1} \\ &\quad = \sum_{k=0}^{2^n} \binom{2^n-k}{2^n-2k} w_1^{2^{n+1}-4k} w_2^{2k+2\ell} + w_1^{e_n-1} w_2^{2\ell} + w_1^{e_n-3} w_2^{2\ell+1} \\ &\quad = \sum_{k=1}^{2^n} \binom{2^n-k}{2^n-2k} w_1^{2^{n+1}-4k} w_2^{2k+2\ell} + w_1^{2^{n+1}} w_2^{2\ell} + w_1^{e_n-1} w_2^{2\ell} + w_1^{e_n-3} w_2^{2\ell+1} \\ &\quad = \sum_{k=1}^{2^n} \binom{2^n-k}{2^n-2k} w_1^{2^{n+1}-4k} w_2^{2k+2\ell} + w_1^{e_n-3} w_2^{2\ell+1}. \qquad \text{(middle terms cancel)} \end{aligned}$$

Notice that the right most term $w_1^{e_n-3} w_2^{2\ell+1}$ corresponds to the first term in $g^c_{2\ell+1}$. Thus, it suffices to show that

$$w_1^{2^n-2} w_2^{2\ell+1} + \sum_{b=1}^{2^{n-1}} \binom{2^{n+1}-2b-1}{2b-1} w_1^{2^n-4b} w_2^{2b+2\ell} = \sum_{k=1}^{2^n} \binom{2^n-k}{2^n-2k} w_1^{2^{n+1}-4k} w_2^{2k+2\ell}.$$

In particular, we must show that

$$\binom{2^{n+1}-2b-1}{2b-1} \equiv \binom{2^n-b}{2^n-2b} \pmod{2},$$

which we relegate to Lemma A.0.4 in the appendix. □

Proof of Proposition 4.9.18. Since $Q_n(\beta)$ is a multiple of a Gröbner basis element of $J(2,c)$, $Q_n(\beta) = 0$ in $H^*(\mathrm{Gr}_2(\mathbb{R}^{2+c});\mathbb{Z}/2\mathbb{Z})$, hence β is a Q_n-cycle. Showing that $Q_n(\beta) = w_1 g^c_{2\ell+1}$ also proves that $Q_n(\beta) = w_1^{2\ell+2}\overline{w}_{c+1}$ in $H^*(\mathrm{Gr}_2(\mathbb{R}^{2+c+1});\mathbb{Z}/2\mathbb{Z})$, since by Proposition 4.9.14, $w_1^i\overline{w}_{c+1} \equiv g_i^c \pmod{J(2,c+1)}$. This yields $w_1 g^c_{2\ell+1} \equiv w_1^{2\ell+2}\overline{w}_{c+1} \pmod{J(2,c+1)}$ as desired. We have shown $\partial(\beta) = w_1^{2\ell+1}\overline{w}_{c+1}$. □

Proof of Proposition 4.9.15. By Proposition 4.9.16, these lemmas together show that ∂ is surjective for $m = 2^{n+1} + 2\ell$. □

4.9.5 The zero connecting homomorphism

The goal of this section is to prove:

Proposition 4.9.23. *If $m = 2^{n+1} + 2\ell - 1$ with $\ell \geq 1$, then the connecting homomorphism $\partial\colon H^*(\mathrm{Gr}_2(\mathbb{R}^m); Q_n) \to H^{*+|Q_n|}(C_2(\mathbb{R}^m); Q_n)$ is zero, thus*

$$k_{Q_n}(\mathrm{Gr}_2(\mathbb{R}^{m+1})) = k_{Q_n}(\mathrm{Gr}_2(\mathbb{R}^m)) + k_{Q_n}(C_2(\mathbb{R}^m)).$$

This is the final of the three technical propositions used to deduce Theorem 4.9.26 in the introduction of §4.9. Hence, proving this proposition will complete the proof of Theorem 4.9.26.

We will again reduce the study of the behavior of ∂ to the study of two particular classes. We let $m = c + 2 = 2^{n+1} + 2\ell - 1$.

Proposition 4.9.24. *If the classes w_2^{c+1} and $w_1^{2^{n+1}-3}\overline{w}_{c+1}$ are not in the image of the connecting homomorphism ∂, then $\partial = 0$.*

Proof. In the previous section we showed that the ∂ was surjective by showing that the there existed an $\partial(\alpha) = w_1\overline{w}_{c+1}$ and a $\partial(\beta) = w_2^{2\ell+2}\overline{w}_{c+1}$. Using these two elements we were able to prove that ∂ was surjective. Here we use the contrapositive of this argument. If the top class $w_2^{c+1} = w_1^{c+1}\overline{w}_{c+1}$ is not the image of Q_n, then no cycle of the form $w_1^{2i}\overline{w}_{c+1}$ can be. To see this write we write $w_1^{2i} = w_1^{c+1-2j}$(since $c+1$ is even). If there exists an x such that $Q_n(x) = w_1^{c+1-2j}\overline{w}_{c+1}$, then

$$\begin{aligned} Q_n(w_1^{2j}x) &= Q_n(w_1^{2j})x + w_1^{2j}Q_n(x) \\ &= 0 + w_1^{2j}(w_1^{c+1-2j})\overline{w}_{c+1} \\ &= w_1^{c+1}\overline{w}_{c+1} = w_2^{c+1}, \end{aligned}$$

which contradicts the assumption that w_2^{c+1} is not in the image of ∂.

A similar argument works for the other class. In particular, we will show that $w_1^{2^{n+1}-3}\overline{w}_{c+1}$ is not in the image so that no cycle $w_1^{2i+1}\overline{w}_{c+1}$ with $2i+1 \leq 2^{n+1}-3$ can be. Notice that the image of $\overline{w}_{c+1}$ is the next odd class, $w_1^{2^{n+1}-1}\overline{w}_{c+1}$, and so $w_1^{2i+1}\overline{w}_{c+1}$ is the top class of the α family. □

The top class w_2^{c+1} of $\mathrm{Gr}_2(\mathbb{R}^{2+c+1})$ is not in the image of Q_n by Proposition 4.7.1 since $2+c+1$ is even, and thus it is not in the image of ∂. We focus our attention on the other class.

Proposition 4.9.25. *The class $w_1^{2^{n+1}-3}\overline{w}_{c+1}$ in $\mathrm{Gr}_2(\mathbb{R}^{2+c+1})$ is not in the image of ∂ for c odd.*

Proof. From the description of the cofiber, we know this class corresponds to the partition $\mu = (c+1, 2^{n+1}-3)$. We will show that if there exists a λ such that $Q_n(s_\lambda)$ has s_μ in the Schubert expansion, then $Q_n(s_\lambda)$ also contains $s_{(c,2^{n+1}-2)}$ in the Schubert expansion. This would mean that no linear combination of Schubert basis elements can result in $Q_n(\sum s_\lambda) = s_\mu$. We write $c = 2^{n+1} + 2\ell - 3$. The diagram corresponding to μ can be expressed as

$$\mu = \begin{array}{l} \boxed{}\boxed{\cdots}\boxed{}\underbrace{\boxed{}\boxed{\cdots}\boxed{}}_{2\ell+1} \\ \underbrace{\boxed{}\boxed{\cdots}\boxed{}}_{2^{n+1}-3} \end{array}$$

Notice that $2^{n+1} - 3 + 2\ell + 1 = c + 1$. For s_λ to have s_μ in the expansion of $Q_n(s_\lambda)$ the complement μ/λ must be a border strip of length $2^{n+1} - 1$ such that $d_{\lambda\mu} \equiv 1$. Additionally, λ must fit in a $2 \times c$ box since λ corresponds to a Schubert cycle of $\mathrm{Gr}_2(\mathbb{R}^{2+c})$. This means the border strip μ/λ must start in the upper right corner at the $(c+1)$st box.

We analyze the possible shapes of the border strips. The first case is when it is connected. In this case, it will be of the form

$$\mu/\lambda = \begin{array}{l} \phantom{\boxed{1}\boxed{\cdots}}\boxed{0}\underbrace{\boxed{}\boxed{\cdots}\boxed{1}}_{2\ell+1} \\ \underbrace{\boxed{1}\boxed{\cdots}\boxed{1}}_{2^{n+1}-1-(2\ell+2)} \end{array}$$

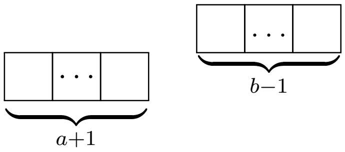
The top right box is in column $c + 1$, which is an even number, and so is of content $c \equiv 1 \pmod 2$. The top strip is of even length meaning that the left sharp corner has content 0 and the dull corner has content 1 below this. The sharp corner in the bottom strip on the far left is of content 1 since this strip is of odd length. Thus, the sharp and dull corner cancel and so $d_{\lambda\mu} = 0$ and thus this s_μ is not in $Q_n(s_\lambda)$.

The next case is that of the disconnected border strip,

$$\mu/\lambda = \begin{array}{l} \phantom{\boxed{}\boxed{\cdots}\boxed{}}\quad\underbrace{\boxed{}\boxed{\cdots}\boxed{}}_{b} \\ \underbrace{\boxed{}\boxed{\cdots}\boxed{}}_{a} \end{array}$$

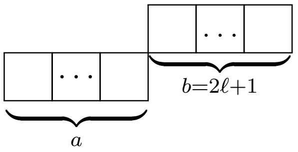
where $a + b = 2^{n+1} - 1$ with $b \geq 2$ and $0 \leq a \leq 2^{n+1} - 3$. Since the border strip μ/λ is disconnected $d_{\mu\lambda} = 1$, hence $Q_n(s_\lambda)$ contains s_μ. We will show that this s_λ is not a Q_n-cycle in $\mathrm{Gr}_2(\mathbb{R}^m)$. As long as the horizontal gap between the border strip components is at least one box, then moving the right most box down one row forms a new broken border strip

$\mu'/\lambda =$



that now arises as a skew-shape of a diagram $\mu' = (c, 2^{n+1} - 1)$ that corresponds to a Schubert class $s_{\mu'}$ of $\mathrm{Gr}_2(\mathbb{R}^m)$. Since μ'/λ is disconnected $d_{\lambda\mu'} = 1$, hence $Q_n(s_\lambda)$ contains $s_{\mu'}$ in $H^*(\mathrm{Gr}_2(\mathbb{R}^m); \mathbb{Z}/2\mathbb{Z})$ meaning $Q_n(s_\lambda) \neq 0$. That is, s_λ is not a Q_n-cycle of $\mathrm{Gr}_2(\mathbb{R}^m)$. Now, if there is no gap then $b = 2\ell + 1$,

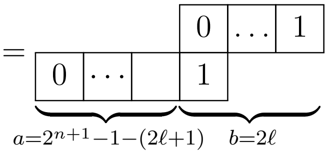
$\mu/\lambda =$



.

When we lower the $c + 1$ box, then one has a connected border strip

$\mu'/\lambda =$



Since the top strip is of even length, the upper sharp corner content is 0 making the dull corner content 1. Thus, the bottom strip's sharp corner has content 0, because the bottom strip is of even length also. Hence, $d_{\lambda\mu'} = 1$, meaning $Q_n(s_\lambda)$ contains μ'. Thus, s_λ is not a Q_n-cycle for $\mathrm{Gr}_2(\mathbb{R}^m)$ either. This completes the proof. $\square$

Proof of Proposition 4.9.23. Proposition 4.9.24 together with Proposition 4.7.1 and Proposition 4.9.25 prove that ∂ is zero for $m = 2^{n+1} + 2\ell - 1$. □

We have thus completed the proof of our main theorem:

Theorem 4.9.26. *The dimension of* $K(n)^*(\mathrm{Gr}_2(\mathbb{R}^m))$ *is*

$$k_n(\mathrm{Gr}_2(\mathbb{R}^m)) = \begin{cases} \binom{m}{2} & 2 \leq m \leq 2^{n+1} \\ 2^{2n+1} - 2^n + \ell & m = 2^{n+1} + 2\ell \\ 2^{2n+1} - 2^{n+1} - 2^n + 1 + \ell & m = 2^{n+1} + 2\ell - 1. \end{cases}$$

In the next subsection we use this calculation to analyze $K(1)^*(\mathrm{Gr}_2(\mathbb{R}^m))$.

4.9.6 Classes in the K-theory

The 2-local Morava K-theory agrees with 2-local complex K-theory [43, Prop 1.5.2(ii)]. Now that we know the dimension of $k_1(\mathrm{Gr}_2(\mathbb{R}^{2+c}))$, we can enumerate representatives for classes on the E_∞-page of the AHSS computing $K(1)^*(\mathrm{Gr}_2(\mathbb{R}^{2+c}))$ that account for the entire dimension. These classes will then correspond to classes in the 2-local complex K-theory. From Theorem 4.9.26 these classes come from the Margolis homology.

Example 4.9.27. Suppose that $c > 2^2$ is even so that by Theorem 4.7.6 the Margolis homology $H^*(\mathrm{Gr}_2(\mathbb{R}^{2+c}); Q_1)$ is a Poincaré duality algebra. For dimension reasons the cycles $1, w_1^2$ live. Therefore, their dual classes w_2^{c-1}, w_2^c also live. In general, the family $w_2^2, w_2^4, \ldots, w_2^c$ also lives. The only two odd classes that live occur in dimensions $c - 1$ and $c + 1$. They are given by α and $w_1^2\alpha$ from Proposition 4.9.17. This proposition in fact showed these classes do live.

Write $2 + c = 2^2 + 2\ell$ so from Theorem 4.9.26, $k_1(\mathrm{Gr}_2(\mathbb{R}^{2+c})) = 6 + \ell$. We have enumerated $\frac{c}{2} + 5$ classes. The family $w_2^2, w_2^4, \ldots, w_2^c$ accounts for $\frac{c}{2}$ classes, and we have the additional five classes from $1, w_1^2, \alpha, w_1^2\alpha, w_2^{c-1}$. One checks

$$\begin{aligned} \frac{c}{2} + 5 &= \frac{2^2 + 2\ell - 2}{2} + 5 \\ &= 6 + \ell = k_1(\mathrm{Gr}_2(\mathbb{R}^{2+c})). \end{aligned}$$

Thus these are a complete set of representatives.

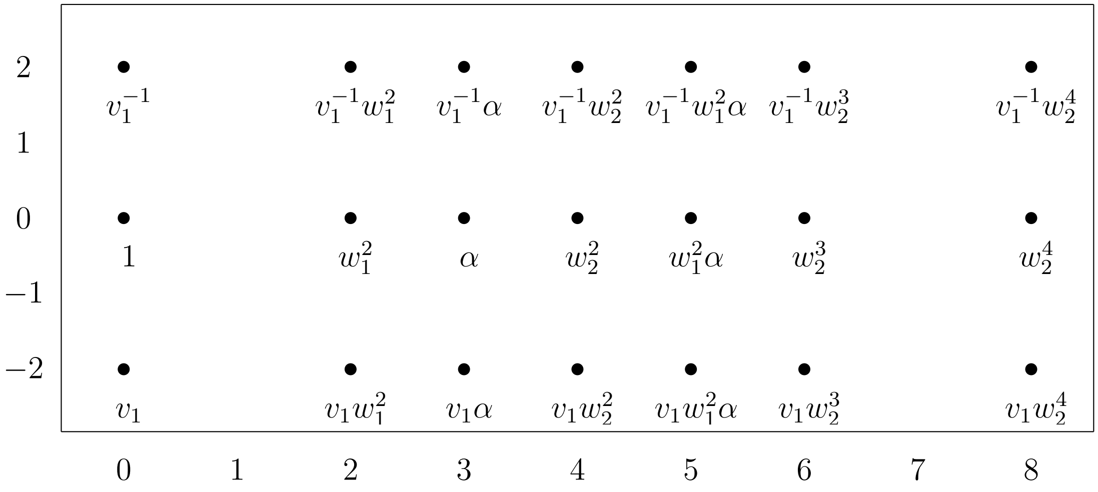


Figure 4-4: E_∞-page for $\mathrm{Gr}_2(\mathbb{R}^{2+4})$ with the classes labeled.

Using this knowledge of $H^*(\mathrm{Gr}_d(\mathbb{R}^{2+c}); Q_1)$ for even c, we can also work out which classes survive for odd c. Let c be odd. Then by Proposition 4.9.23, the long exact sequence in Margolis homology breaks up into the short exact sequence

$$0 \longrightarrow H^q(C_2(\mathbb{R}^c); Q_1) \xrightarrow{p^*} H^q(\mathrm{Gr}_2(\mathbb{R}^{c+1}); Q_1) \xrightarrow{i^*} H^q(\mathrm{Gr}_2(\mathbb{R}^c); Q_1) \longrightarrow 0,$$

while by Proposition 4.9.15, for $\mathrm{Gr}_2(\mathbb{R}^{c-1}) \hookrightarrow \mathrm{Gr}_2(\mathbb{R}^c)$ we have the short exact sequence

$$0 \xrightarrow{p^*} H^q(\mathrm{Gr}_2(\mathbb{R}^c); Q_1) \xrightarrow{i^*} H^q(\mathrm{Gr}_2(\mathbb{R}^{c-1}); Q_1) \xrightarrow{\partial} H^{q+|Q_1|}(C_2(\mathbb{R}^{c-1}); Q_1) \longrightarrow 0.$$

We combine these short exact sequences

$$\begin{tikzcd}
 & & & 0 \arrow[d, "p^*"] & \\
0 \arrow[r] & H^q(C_2(\mathbb{R}^c); Q_1) \arrow[r, "p^*"] & H^q(\mathrm{Gr}_2(\mathbb{R}^{c+1}); Q_1) \arrow[r, "i^*"] & H^q(\mathrm{Gr}_2(\mathbb{R}^c); Q_1) \arrow[r] \arrow[d, "i^*"] & 0 \\
 & & & H^q(\mathrm{Gr}_2(\mathbb{R}^{c-1}); Q_1) \arrow[d, "\partial"] & \\
 & & & H^{q+|Q_1|}(C_2(\mathbb{R}^{c-1}); Q_1) \arrow[d] & \\
 & & & 0. &
\end{tikzcd}$$

Since $H^*(\mathrm{Gr}_2(\mathbb{R}^c); Q_1)$ injects into $H^*(\mathrm{Gr}_2(\mathbb{R}^{c-1}); Q_1)$ there can only be potential classes in degrees $0, 2, 2(c-2), (c-1)-1, (c-1)+1, 4, 8, \ldots, 2(c-1)$ corresponding to the classes $1, w_1^2, w_2^{c-2}, \alpha_{c-1}, w_1^2\alpha_{c-1}, w_2^2, w_2^4, \ldots, w_2^{c-1}$ in $H^*(\mathrm{Gr}_2(\mathbb{R}^{2+c-1}); Q_1)$. Here we have indexed the odd class α by $c-1$ to indicate it comes from $H^*(\mathrm{Gr}_2(\mathbb{R}^{2+c-1}); Q_1)$. Furthermore, since $H^*(\mathrm{Gr}_2(\mathbb{R}^{c+1}); Q_1)$ surjects onto $H^*(\mathrm{Gr}_2(\mathbb{R}^c); Q_1)$ there can also only be classes in degrees $0, 2, 2(c-1), c, c+2, 4, 8, \ldots, 2(c+1)$ corresponding to the classes that could be in the image of $1, w_1^2, w_2^{(c+1)-1}, \alpha_{c+1}, w_1^2\alpha_{c+1}, w_2^2, w_2^4, \ldots, w_2^{c+1}$. in $H^*(\mathrm{Gr}_2(\mathbb{R}^{c+1}); Q_1)$. Combining these restrictions tells us there can only be classes in degrees $0, 2, c, 4, \ldots, 2(c-1)$. In particular, we have lost classes in degrees $c-2$ and $2(c-2)$. This gives us $\frac{c-1}{2} + 2$ possible classes. We know from Theorem 4.9.26 (by letting $c + 2 = 2^2 + 2\ell - 1$) there are $k_1(\mathrm{Gr}_2(\mathbb{R}^c)) = 2^3 - 2^2 - 2 + 1 + \ell = \ell + 2$ classes. One checks

$$\frac{c-1}{2} + 2 = \frac{(2^2 + 2\ell - 3) - 1}{2} + 2 = \ell + 2.$$

Thus, there must actually be one class in each of the listed dimensions. To start we have classes $1, w_1^2$. The other even classes are $i^*(w_1^{2k})$ for $k = 1, \ldots, c-1$. From the definition of i^* these map to $w_2^2, w_2^4, \ldots, w_2^{c-1}$. The last class is $i^*(\alpha_{c+1})$. By definition, $\alpha_{c+1} = w_1 \overline{w}_{c-1}$. The map $i^*(w_1 \overline{w}_{c-1})$ then reduces this class modulo the ideal $J(2, c)$. From the Peiri rule, one sees that the class $w_1 \overline{w}_{c-1}$ is already reduced modulo $J(2, c)$, hence $i^*(\alpha_{c+1}) = \alpha_{c+1}$.

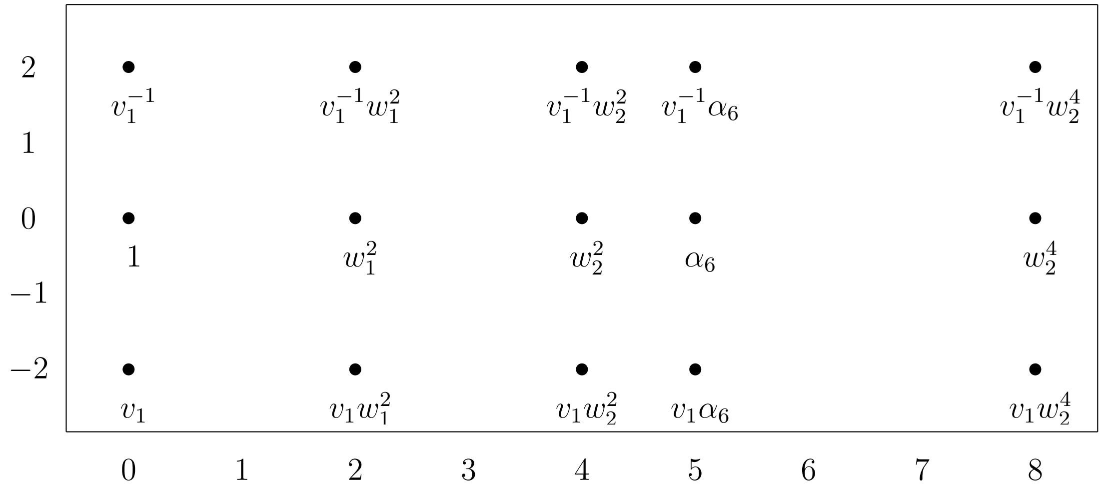


Figure 4-5: E_∞-page for $\mathrm{Gr}_2(\mathbb{R}^{2+5})$ with the classes labeled.

4.10 Rational, integral, and odd primes

For completeness, we include a calculation of the Morava K-theory of $Gr_d(R^{d+c})$ at odd primes.

Notation. *We let $\lfloor x \rfloor$ denote the smallest integer less than the real number x.*

Proposition 4.10.1. *At odd primes p, the AHSS computing the p-local Morava K-theory $K(n)^*(\mathrm{Gr}_d(\mathbb{R}^{d+c}))$ collapses on the E_2-page. Furthermore,*

1. *when cd is even the dimension as a $K(n)^*$-module is*

$$k_n(\mathrm{Gr}_d(\mathbb{R}^{d+c})) = \binom{\lfloor \frac{c}{2} \rfloor + \lfloor \frac{d}{2} \rfloor}{\lfloor \frac{d}{2} \rfloor},$$

2. *when cd is odd the dimension as a $K(n)^*$-module is*

$$k_n(\mathrm{Gr}_d(\mathbb{R}^{d+c})) = 2\binom{\lfloor \frac{c}{2} \rfloor + \lfloor \frac{d}{2} \rfloor}{\lfloor \frac{d}{2} \rfloor}.$$

The case for even cd follows from the sparsity of the AHSS and is shown in §4.10.1. However, the case for odd cd can be shown using the general fact that for any generalized cohomology theory the AHSS computing the p-local $E^*(X)$ collapses if $H^*(-; E^*(*))$ has no p-torsion. Instead, as an example we show our chromatic fixed point theory also detects this by recovering this result as shown in §4.10.2.

4.10.1 Rational and integral homology

The reason one specializes to the prime 2 is mainly due to the following proposition of Ehresmann.

Proposition 4.10.2 ([15, Pg. 81])**.** *The torsion in $H^*(\mathrm{Gr}_d(\mathbb{R}^{d+c}); \mathbb{Z})$ is only 2-torsion.*

This proposition together with knowledge of the rational homology allows one to show that for odd primes p and even cd the homology $H^*(\mathrm{Gr}_d(\mathbb{R}^{d+c}); \mathbb{Z}/\mathrm{p}\mathbb{Z})$ is too sparse to support any differentials in the AHSS and thus collapses.

By identifying $\mathrm{Gr}_d(\mathbb{R}^{d+c})$ as the homogeneous space

$$\mathrm{Gr}_d(\mathbb{R}^{d+c}) \simeq O(d+c)/\left(O(d) \times O(c)\right),$$

the inclusion homomorphism $O(d) \times O(c) \hookrightarrow O(d+c)$ induces a fiber bundle

$$\mathrm{Gr}_d(\mathbb{R}^{d+c}) \to BO(d) \times BO(c) \to BO(d+c).$$

Using this description of $\mathrm{Gr}_d(\mathbb{R}^{d+c})$ leads to a description of its rational cohomology in terms of the Pontryagin classes of tautological bundles over $BO(d)$ and $BO(c)$. Recall that the Pontryagin classes of a real vector bundle $\xi\colon E \to B$ are defined to be the even Chern classes of the complexified bundle $p_i(\xi) = (-1)^i c_{2i}(\xi \otimes \mathbb{C})$, and so $p_i(\xi) \in H^{4i}(B;\mathbb{Z})$ [34, §15]. The total Pontryagin class is defined to be $p(\xi) = 1 + p_1(\xi) + p_2(\xi) + \ldots$.

Theorem 4.10.3 ([34, Problem 15-B])**.** *For Λ an integral domain containing $\frac{1}{2}$,*

$$H^*(BO(d);\Lambda) \cong \Lambda[p_1(\gamma^d), p_2(\gamma^d), \ldots, p_{\lfloor \frac{d}{2} \rfloor}(\gamma^d)].$$

Theorem 4.10.4 ([11],[45])**.** *For cd even,*

$$H^*(\mathrm{Gr}_d(\mathbb{R}^{d+c});\mathbb{Q}) \cong \frac{\mathbb{Q}[p_1, p_2, \ldots, p_{\lfloor \frac{d}{2} \rfloor}, \bar{p}_1, \ldots, \bar{p}_{\lfloor \frac{c}{2} \rfloor}]}{\left(1 + p_1 + p_2 + \ldots + p_{\lfloor \frac{d}{2} \rfloor}\right)\left(1 + \bar{p}_1 + \bar{p}_2 + \ldots + \bar{p}_{\lfloor \frac{c}{2} \rfloor}\right) = 1},$$

where p_i and $\bar{p}_i$ are in dimension $4i$, hence

$$\dim_{\mathbb{Q}} H^*(\mathrm{Gr}_d(\mathbb{R}^{d+c};\mathbb{Q}) = \binom{\lfloor \frac{c}{2} \rfloor + \lfloor \frac{d}{2}. \rfloor}{\lfloor \frac{d}{2} \rfloor}.$$

While for cd odd,

$$H^*(\mathrm{Gr}_d(\mathbb{R}^{d+c});\mathbb{Q}) \cong \frac{\mathbb{Q}[p_1, p_2, \ldots, p_{\lfloor \frac{d}{2} \rfloor}, \bar{p}_1, \ldots, \bar{p}_{\lfloor \frac{c}{2} \rfloor}, r]}{\left(1 + p_1 + p_2 + \ldots + p_{\lfloor \frac{d}{2} \rfloor}\right)\left(1 + \bar{p}_1 + \bar{p}_2 + \ldots + \bar{p}_{\lfloor \frac{c}{2} \rfloor}\right) = 1, r^2 = 0},$$

where p_i and $\bar{p}_i$ are in dimension $4i$ and r is in dimension $d+c-1$, hence

$$\dim_{\mathbb{Q}} H^*(\mathrm{Gr}_d(\mathbb{R}^{d+c};\mathbb{Q}) = 2\binom{\lfloor \frac{c}{2} \rfloor + \lfloor \frac{d}{2} \rfloor}{\lfloor \frac{d}{2} \rfloor}.$$

A modern reference for the determination of $H^*(\mathrm{Gr}_d(\mathbb{R}^{d+c});\mathbb{Q})$ is [44].

We demonstrate how our conjectured formula Conjecture 5.1.1 for $k_n(\mathrm{Gr}_d(\mathbb{R}^{d+c}))$ agrees with this for $n = 0$ since $K(0)^*(-) = H^*(-;\mathbb{Q})$. For convenience, we restate the conjectured formula,

Conjecture.

$$k_n(Gr_d(\mathbb{R}^m)) = \begin{cases} \binom{m}{d} & d \leq m \leq 2^{n+1} \\ \sum_{i=0}^{\frac{d-1}{2}} \binom{2^{n+1}}{d-2i}\binom{\ell}{i} & d\ \ odd, m = 2^{n+1}+2\ell \\ \sum_{i=0}^{\frac{d-1}{2}} \binom{2^{n+1}-1}{d-2i}\binom{\ell}{i} & d\ \ odd, m = 2^{n+1}+2\ell-1 \\ \binom{\ell}{\frac{d}{2}} + \sum_{i=0}^{\frac{d-2}{2}} \binom{2^{n+1}}{d-2i}\binom{\ell}{i} & d\ \ even, m = 2^{n+1}+2\ell \\ \binom{\ell}{\frac{d}{2}} + \sum_{i=0}^{\frac{d-2}{2}} \binom{2^{n+1}-1}{d-2i}\binom{\ell}{i} & d\ \ even, m = 2^{n+1}+2\ell-1. \end{cases}$$

Suppose that both c and d are even. If $d+c \geq 2^{0+1}$, we can write $d+c = 2+2\ell$ for some ℓ. The conjecture tells us that

$$k_0(\mathrm{Gr}_d(\mathbb{R}^{d+c})) = \binom{\ell}{\frac{d}{2}} + \sum_{i=0}^{\frac{d-2}{2}} \binom{2}{d-2i}\binom{\ell}{i}$$

Notice that the binomial $\binom{2}{d-2i}$ is only non-zero for when $i = \frac{d-2}{2}$, in which case it is

$\binom{2}{d-2i} = 1$,

$$k_0(\mathrm{Gr}_d(\mathbb{R}^{d+c})) = \binom{\ell}{\frac{d}{2}} + \binom{\ell}{\frac{d}{2}-1} = \binom{\ell+1}{\frac{d}{2}} = \binom{\frac{d+c-2}{2}+1}{\frac{d}{2}} = \binom{\frac{d+c}{2}}{\frac{d}{2}},$$

which does agree with the formula in Theorem 4.10.4. The other cases may be checked similarly.

To understand the Morava K-theory at odd primes, we examine the integral homology. Generating functions for the free and torsion parts of $H^*(\mathrm{Gr}_d(\mathbb{R}^{d+c});\mathbb{Z})$ are constructed in the preprint [20]. We do not rely on the results of this note, but the author found the discussion there useful.

Lemma 4.10.5. *For odd prime p and even cd, $H^*(\mathrm{Gr}_d(\mathbb{R}^{d+c});\mathbb{Z}/\mathrm{p}\mathbb{Z})$ is concentrated in degrees that are a multiple of four.*

Proof. The universal coefficient theorem states that

$$H^i(\mathrm{Gr}_d(\mathbb{R}^{d+c});\mathbb{Z}/\mathrm{p}\mathbb{Z}) \cong \begin{array}{c} \mathrm{Ext}^1_{\mathbb{Z}}\left(H_{i-1}(\mathrm{Gr}_d(\mathbb{R}^{d+c});\mathbb{Z}),\mathbb{Z}/\mathrm{p}\mathbb{Z}\right) \\ \oplus \\ \mathrm{Hom}_{\mathbb{Z}}(H_i(\mathrm{Gr}_d(\mathbb{R}^{d+c});\mathbb{Z}),\mathbb{Z}/\mathrm{p}\mathbb{Z}). \end{array}$$

By Proposition 4.10.2, the only torsion in $H_i(\mathrm{Gr}_d(\mathbb{R}^{d+c});\mathbb{Z})$ is 2-torsion. Since p is odd $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z},\mathbb{Z}/\mathrm{p}\mathbb{Z}) = 0$, $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z},\mathbb{Z}/\mathrm{p}\mathbb{Z}) = 0$ and $\mathrm{Ext}(\mathbb{Z},-) = 0$. Thus, the Ext term is always zero, hence

$$H^i(\mathrm{Gr}_d(\mathbb{R}^{d+c});\mathbb{Z}/\mathrm{p}\mathbb{Z}) \cong \mathrm{Hom}_{\mathbb{Z}}(H_i(\mathrm{Gr}_d(\mathbb{R}^{d+c});\mathbb{Z}),\mathbb{Z}/\mathrm{p}\mathbb{Z}).$$

Now since $\mathrm{rk}\, H_i(\mathrm{Gr}_d(\mathbb{R}^{d+c});\mathbb{Z}) = \dim_{\mathbb{Q}} H^i(\mathrm{Gr}_d(\mathbb{R}^{d+c});\mathbb{Q})$, Theorem 4.10.4 tells us for cd even that $H^*(\mathrm{Gr}_d(\mathbb{R}^{d+c});\mathbb{Z}/\mathrm{p}\mathbb{Z})$ is concentrated in degrees that are a multiple of four. □

4.10.2 Morava K-theory at odd primes

Proof of Theorem 4.10.1 for even cd. By Lemma 4.10.5, $H^*(\mathrm{Gr}_d(\mathbb{R}^{d+c});\mathbb{Z}/\mathrm{p}\mathbb{Z})$ is concentrated in even degrees for cd even and p odd and so the AHSS collapses on the E_2-page. □

Proof of Theorem 4.10.1 for odd cd. An upper bound for $k_{Q_n}(\mathrm{Gr}_d(\mathbb{R}^{d+c}))$ is

$$\dim_{\mathbb{Z}/\mathrm{p}\mathbb{Z}} H^*(\mathrm{Gr}_d(\mathbb{R}^{d+c});\mathbb{Z}/\mathrm{p}\mathbb{Z}) = 2\binom{\left\lfloor\frac{c}{2}\right\rfloor+\left\lfloor\frac{d}{2}\right\rfloor}{\left\lfloor\frac{d}{2}\right\rfloor}$$

by Lemma 4.10.5. We will use Theorem 3.4.1 to show that the AHSS collapses. We will appeal to the real representation theory of the cyclic group C_p. Let σ_1 denote the trivial representation and σ_2 denote an irreducible 2-dimensional real representation of C_p. Consider the representation $V = 2\sigma_1 \oplus \left(\left\lfloor\frac{c}{2}\right\rfloor+\left\lfloor\frac{d}{2}\right\rfloor\right)\sigma_2$. The underlying space of $\mathrm{Gr}_d(V)$ is $\mathrm{Gr}_d(V)^{\{e\}} = \mathrm{Gr}_d(\mathbb{R}^{2+2(\lfloor\frac{c}{2}\rfloor+\lfloor\frac{d}{2}\rfloor)}) = \mathrm{Gr}_d(\mathbb{R}^{d+c})$. By Proposition 4.3.3 the C_p fixed points are

$$\mathrm{Gr}_d(V)^{C_p} = \bigsqcup_{i+2j=d} \mathrm{Gr}_i(\mathbb{R}^2)\times \mathrm{Gr}_j(\mathbb{C}^{(\lfloor\frac{c}{2}\rfloor+\lfloor\frac{d}{2}\rfloor)}).$$

By assumption, cd is odd and so in particular d is odd, hence the index i can never be even. The fixed point formula thus simplifies to

$$\mathrm{Gr}_d(V)^{C_p} = \mathrm{Gr}_1(\mathbb{R}^2)\times \mathrm{Gr}_{\frac{d-1}{2}}(\mathbb{C}^{(\lfloor\frac{c}{2}\rfloor+\lfloor\frac{d}{2}\rfloor)}).$$

By Theorem 2.3.15, $r_n^{C_p}(C_p, \{e\}) = 1$ and so by Theorem 3.1.1,

$$k_{n-1}(\mathrm{Gr}_d(V)^{C_p}) \leq k_n(\mathrm{Gr}_d(V)) \leq k_{Q_n}(\mathrm{Gr}_d(V)) \leq 2\binom{\lfloor \frac{c}{2} \rfloor + \lfloor \frac{d}{2} \rfloor}{\lfloor \frac{d}{2} \rfloor}.$$

The lower bound $k_{n-1}(\mathrm{Gr}_d(V))$ is equal to the upper bound, because $k_{n-1}(\mathrm{Gr}_1(\mathbb{R}^2)) = 2$ and $k_{n-1}\left(\mathrm{Gr}_d(\mathbb{C}^m)\right) = \binom{m}{d}$ as it is concentrated in even degrees,

$$\begin{aligned} k_{n-1}\left(Gr_d(V)^{C_p}\right) &= k_{n-1}\left(\mathrm{Gr}_1(\mathbb{R}^2)\right) k_{n-1}\left(\mathrm{Gr}_{\frac{d-1}{2}}(\mathbb{C}^{\left(\lfloor \frac{c}{2} \rfloor + \lfloor \frac{d}{2} \rfloor\right)})\right) \\ &= 2\binom{\lfloor \frac{c}{2} \rfloor + \lfloor \frac{d}{2} \rfloor}{\lfloor \frac{d}{2} \rfloor}. \qquad \left(\tfrac{d-1}{2} = \lfloor \tfrac{d}{2} \rfloor\right) \end{aligned}$$

By Theorem 3.4.1, the AHSS collapses. □

The reason this proof fails at the prime 2 is that C_2 has no irreducible 2-dimensional real representations. As a corollary of Theorem 4.10.1 we can conclude that Q_n at odd primes is always zero.

Corollary 4.10.6. *For odd primes p, $H_*(\mathrm{Gr}_d(\mathbb{R}^{d+c}); Q_n) = H^*(\mathrm{Gr}_d(\mathbb{R}^{d+c}); \mathbb{Z}/\mathrm{p}\mathbb{Z})$.*

This is usually shown by proving that the Bockstein homomorphism $Q_0 = \beta$ is zero since $H^*(\mathrm{Gr}_d(\mathbb{R}^{d+c}); \mathbb{Z})$ has only two torsion.

Chapter 5

Future work

5.1 Towards the general Grassmannian

In this chapter we present some conjectures related to our main conjecture:

Conjecture 5.1.1. *For $n \geq 1$, $k_{Q_n}(\mathrm{Gr}_d(\mathbb{R}^{d+c}))$ is equal to the lower bound in Theorem 4.5.4. Thus, by Theorem 3.4.1 the AHSS computing $K(n)^*(\mathrm{Gr}_d(\mathbb{R}^{d+c}))$ collapses on the 2^{n+1}-page and has dimension given by the formula in Theorem 4.5.4.*

Explicit calculations that support this conjecture are presented in Appendix B. In future work we would like to use a similar technique used to prove Theorem 4.9.26 to prove this conjecture.

We have already established the general lower bound Theorem 4.5.4 and so the difficulty again arises in computing the dimension of the Margolis homology $k_{Q_n}(\mathrm{Gr}_d(\mathbb{R}^{d+c}))$. In the $d = 2$ case the long exact sequence in Margolis homology broke up into short exact sequences as shown in §4.9.4 and §4.9.5. Based on calculations done with SageMath we conjecture that Proposition 4.9.23 holds in general.

Conjecture 5.1.2. *For odd $c+d$, the connecting homomorphism*

$$\partial\colon H^*(\mathrm{Gr}_d(\mathbb{R}^{c+d});Q_n) \to H^{*+|Q_n|}(C_d(\mathbb{R}^{c+d});Q_n)$$

is zero, thus

$$k_{Q_n}(\mathrm{Gr}_d(\mathbb{R}^{c+d+1})) = k_{Q_n}(\mathrm{Gr}_d(\mathbb{R}^{c+d})) + k_{Q_n}(C_d(\mathbb{R}^{c+d})).$$

In fact, one may show due to dimension reasons that this conjecture must hold if our main conjecture is to hold. However, Proposition 4.9.15 does not as readily generalize. This connecting homomorphism becomes less and less surjective as d grows. This presents the first obstacle to generalizing the proof for $d = 2$. The second obstacle is that the cofiber becomes more complicated as d grows. For odd $c+d$, Proposition 4.8.1 yields $k_{Q_n}(C_d(\mathbb{R}^{d+c})) = k_{Q_n}(\mathrm{Gr}_{d-1}(\mathbb{R}^{d+c}))$, and thus Conjecture 5.1.1 gives a conjecture for the dimension of Margolis homology of this cofiber also. In this section we will give a chromatic lower bound for the dimension of the Morava K-theory of the cofiber for $c+d$ even and conjecture that this is the dimension of the Margolis homology.

Proposition 5.1.3. *Let $V = (2^n-1)\sigma_1 \oplus 2^n\sigma_2 \oplus (\ell+1)\tau$ and $W = (2^n-1)\sigma_1 \oplus (2^n-1)\sigma_2 \oplus (\ell+1)\tau$ so that W is a subrepresentation of V, then*

$$k_{n-1}(\mathrm{Gr}_{d-1}(W)^{C_4}) = k_{n-1}(C_d(W)^{C_4}) \leq k_n(C_d(W)).$$

Lemma 5.1.4.

$$k_{n-1}\operatorname{Gr}_d(W)^{C_4} = \begin{cases} \binom{\ell+1}{\frac{d}{2}} + \sum_{i=0}^{\frac{d-2}{2}} \binom{2^{n+1}-2}{d-2i}\binom{\ell+1}{i} & d \textit{ even} \\ \sum_{i=0}^{\frac{d-1}{2}} \binom{2^{n+1}-2}{d-2i}\binom{\ell+1}{i} & d \textit{ odd} \end{cases}$$

Proof. We compute the fixed points of $\operatorname{Gr}_d(W)^{C_4}$ using Proposition 4.3.3. For even d, we have

$$\begin{aligned} \operatorname{Gr}_d(W)^{C_4} = \coprod_{\substack{2i+j+k=d \\ i,j,k\geq 1}} & \operatorname{Gr}_i(\mathbb{C}^{\ell+1}) \times \operatorname{Gr}_j(\mathbb{R}^{2^n-1}) \times \operatorname{Gr}_k(\mathbb{R}^{2^n-1}) \\ & \sqcup \coprod_{\substack{i+j=d \\ i,j\geq 1}} \operatorname{Gr}_j(\mathbb{R}^{2^n-1}) \times \operatorname{Gr}_k(\mathbb{R}^{2^n-1}) \\ & \sqcup \coprod_{2} \coprod_{\substack{2i+j=d \\ i,j\geq 1}} \operatorname{Gr}_i(\mathbb{C}^{\ell+1}) \times \operatorname{Gr}_j(\mathbb{R}^{2^n-1}) \\ & \sqcup \coprod_{2} \operatorname{Gr}_d(\mathbb{R}^{2^n-1}) \\ & \sqcup \operatorname{Gr}_{d/2}(\mathbb{C}^{\ell+1}). \end{aligned} \tag{5.1.5}$$

One can work out a similar formula for odd d. Notice that all the constituent pieces are either complex Grassmannians or are $\operatorname{Gr}_d(\mathbb{R}^m)$ with $m \leq 2^n$ and so by Theorem 4.5.1, the dimension of the Morava K-theory can be computed as above. □

Proof of Proposition 5.1.3. For even d, the fixed points of $\operatorname{Gr}_d(V)^{C_4}$ are given by Proposition 4.3.3 as:

$$\begin{aligned}\mathrm{Gr}_d(V)^{C_4} = \coprod_{\substack{2i+j+k=d\\ i,j,k\geq 1}} &\mathrm{Gr}_i(\mathbb{C}^{\ell+1}) \times \mathrm{Gr}_j(\mathbb{R}^{2^n-1}) \times \mathrm{Gr}_k(\mathbb{R}^{2^n})\\
&\sqcup \coprod_{\substack{i+j=d\\ i,j\geq 1}} \mathrm{Gr}_j(\mathbb{R}^{2^n-1}) \times \mathrm{Gr}_k(\mathbb{R}^{2^n})\\
&\sqcup \coprod_{\substack{2i+j=d\\ i,j\geq 1}} \mathrm{Gr}_i(\mathbb{C}^{\ell+1}) \times \mathrm{Gr}_j(\mathbb{R}^{2^n})\\
&\sqcup \coprod_{\substack{2i+j=d\\ i,j\geq 1}} \mathrm{Gr}_i(\mathbb{C}^{\ell+1}) \times \mathrm{Gr}_j(\mathbb{R}^{2^n-1})\\
&\sqcup \mathrm{Gr}_d(\mathbb{R}^{2^n})\\
&\sqcup \mathrm{Gr}_d(\mathbb{R}^{2^n-1})\\
&\sqcup \mathrm{Gr}_{d/2}(\mathbb{C}^{\ell+1}).\end{aligned}$$

The inclusion of the fixed points $\mathrm{Gr}_d(W)^{C_4} \to \mathrm{Gr}_d(V)^{C_4}$ can be seen termwise, and by Proposition 4.4.3 the H-fixed points of the cofiber are computed as the pushout

$$\begin{tikzcd}
\mathrm{Gr}_d(W)^H \arrow[r] \arrow[d] & \mathrm{Gr}_d(V)^H \arrow[d] \\
*^H \arrow[r] & C_d(W)^H.
\end{tikzcd}$$

Computing the termwise quotients,

$$\begin{aligned}\frac{\mathrm{Gr}_i(\mathbb{C}^{\ell+1}) \times \mathrm{Gr}_j(\mathbb{R}^{2^n-1}) \times \mathrm{Gr}_k(\mathbb{R}^{2^n})}{\mathrm{Gr}_i(\mathbb{C}^{\ell+1}) \times \mathrm{Gr}_j(\mathbb{R}^{2^n-1}) \times \mathrm{Gr}_k(\mathbb{R}^{2^n-1})} &\simeq \mathrm{Gr}_i(\mathbb{C}^{\ell+1}) \times \mathrm{Gr}_j(\mathbb{R}^{2^n-1}) \times \Big(\mathrm{Gr}_k(\mathbb{R}^{2^n})/\,\mathrm{Gr}_k(\mathbb{R}^{2^n-1})\Big)\\
&\simeq \mathrm{Gr}_i(\mathbb{C}^{\ell+1}) \times \mathrm{Gr}_j(\mathbb{R}^{2^n-1}) \times C_k(\mathbb{R}^{2^n-1})\end{aligned}$$

$$\frac{\mathrm{Gr}_j(\mathbb{R}^{2^n-1}) \times \mathrm{Gr}_k(\mathbb{R}^{2^n})}{\mathrm{Gr}_j(\mathbb{R}^{2^n-1}) \times \mathrm{Gr}_k(\mathbb{R}^{2^n-1})} \simeq \mathrm{Gr}_j(\mathbb{R}^{2^n-1}) \times \big(\mathrm{Gr}_k(\mathbb{R}^{2^n})/Gr_k(\mathbb{R}^{2^n-1})\big)$$
$$\simeq \mathrm{Gr}_j(\mathbb{R}^{2^n-1}) \times C_k(\mathbb{R}^{2^n-1})$$

$$\frac{\mathrm{Gr}_i(\mathbb{C}^{\ell+1}) \times \mathrm{Gr}_j(\mathbb{R}^{2^n})}{\mathrm{Gr}_i(\mathbb{C}^{\ell+1}) \times \mathrm{Gr}_j(\mathbb{R}^{2^n-1})} \simeq \mathrm{Gr}_i(\mathbb{C}^{\ell+1}) \times \big(\mathrm{Gr}_j(\mathbb{R}^{2^n})/\,\mathrm{Gr}_j(\mathbb{R}^{2^n-1})\big)$$
$$\simeq \mathrm{Gr}_i(\mathbb{C}^{\ell+1}) \times C_j(\mathbb{R}^{2^n-1})$$

$$\frac{\mathrm{Gr}_i(\mathbb{C}^{\ell+1}) \times \mathrm{Gr}_j(\mathbb{R}^{2^n-1})}{\mathrm{Gr}_i(\mathbb{C}^{\ell+1}) \times \mathrm{Gr}_j(\mathbb{R}^{2^n-1})} \simeq *$$

$$\frac{\mathrm{Gr}_d(\mathbb{R}^{2^n})}{\mathrm{Gr}_d(\mathbb{R}^{2^n-1})} \simeq C_d(\mathbb{R}^{2^n-1})$$

$$\frac{\mathrm{Gr}_d(\mathbb{R}^{2^n})}{\mathrm{Gr}_d(\mathbb{R}^{2^n})} \simeq *$$

$$\frac{\mathrm{Gr}_{d/2}(\mathbb{C}^{\ell+1})}{\mathrm{Gr}_{d/2}(\mathbb{C}^{\ell+1})} \simeq *,$$

and then putting this all together yields

$$\begin{aligned} C_d(W)^{C_4} \simeq \coprod_{\substack{2i+j+k=d \\ i,j,k\geq 1}} & \mathrm{Gr}_i(\mathbb{C}^{\ell+1}) \times \mathrm{Gr}_j(\mathbb{R}^{2^n-1}) \times C_k(\mathbb{R}^{2^n-1}) \\ & \sqcup \coprod_{\substack{i+j=d \\ i,j\geq 1}} \mathrm{Gr}_j(\mathbb{R}^{2^n-1}) \times C_k(\mathbb{R}^{2^n-1}) \\ & \sqcup \coprod_{\substack{2i+j=d \\ i,j\geq 1}} \mathrm{Gr}_i(\mathbb{C}^{\ell+1}) \times C_j(\mathbb{R}^{2^n-1}) \\ & \sqcup C_d(\mathbb{R}^{2^n-1}). \end{aligned}$$

Now notice that all of these cofibers come from the inclusion of an odd dimensional Grassmannian into an even dimensional Grassmannian, hence from Proposition 4.8.1 and induction we assume $k_n(C_d(\mathbb{R}^{2^n-1})) = k_n(\mathrm{Gr}_{d-1}(\mathbb{R}^{2^n-1}))$. This means that

$$\begin{aligned} k_{n-1}(C_d(\mathbb{R}^W)^{C_4}) = \sum_{\substack{2i+j+k=d \\ i,j,k\geq 1}} & k_{n-1}\,\mathrm{Gr}_i(\mathbb{C}^{\ell+1}) k_{n-1}\,\mathrm{Gr}_j(\mathbb{R}^{2^n-1}) k_{n-1}\,\mathrm{Gr}_{k-1}(\mathbb{R}^{2^n-1}) \\ & + \sum_{\substack{i+j=d \\ i,j\geq 1}} k_{n-1}\,\mathrm{Gr}_j(\mathbb{R}^{2^n-1}) k_{n-1}\,\mathrm{Gr}_{j-1}(\mathbb{R}^{2^n-1}) \\ & + \sum_{\substack{2i+j=d \\ i,j\geq 1}} k_{n-1}\,\mathrm{Gr}_i(\mathbb{C}^{\ell+1}) k_{n-1}\,\mathrm{Gr}_{j-1}(\mathbb{R}^{2^n-1}) \\ & + k_{n-1}\,\mathrm{Gr}_{d-1}(\mathbb{R}^{2^n-1}). \end{aligned}$$

By reindexing, we see that this formula agrees with the formula for $k_{n-1}(\mathrm{Gr}_{d-1}(W)^{C_4})$ coming from (5.1.5). That is,

$$k_{n-1}(Gr_{d-1}(W)^{C_4}) = k_{n-1}(C_d(W)^{C_4}).$$

A similar calculation may be done for odd d. Using Theorem 3.1.2 gives the lower bound $k_{n-1}(\mathrm{Gr}_{d-1}(W)^{C_4}) = k_{n-1}(C_d(W)^{C_4}) \leq k_n C_d(W)$. $\square$

Conjecture 5.1.6. *For even $d + c$, $k_{Q_n}(C_d(\mathbb{R}^{d+c}))$ is equal to the lower bound in Proposition 5.1.3 given by Lemma 5.1.4.*

This conjecture is supported by explicit calculations done with SageMath. Furthermore, combining this with the conjecture for $k_{Q_n}(\mathrm{Gr}_d(\mathbb{R}^{d+c}))$ tells us exactly how non-surjective the connecting homomorphism should be.

Moving forward perhaps a careful analysis of the connecting homomorphism from the Schubert perspective can lead to a proof of these conjectures.

Appendix A

Binomial coefficients mod two

Lemma A.0.1. *For* $1 \leq b < 2^{n+1}$, $\binom{2^{n+1}+1-b}{b} \equiv 1$ *if and only if* b *is a power of two.*

Proof. We use Lucas's Theorem (originally proved in 1878 [29]), which in our special case states that a binomial coefficient $\binom{a}{b} \equiv 0 \pmod 2$ if and only if the binary expansion of b has a one in a position where a does not. For a number n, written in binary we denote by n^c the binary complement. For example, $1000^c = 0111$. Subtraction in binary can be expressed in terms of complements, $x - y = (x^c + y)^c$. In binary, $2^{n+1} + 1 = 100\ldots01$ where the leading 1 is in the $n+1$ place. So that $(2^{n+1} + 1)^c = 01\ldots10$. Suppose that $2 \leq b \leq 2^n$ is a power of two $b = 2^i$ so that its binary expansion has just a 1 somewhere between the nth and 1st spot in the ith spot. Generically, $b = 0\ldots1\ldots0$. When we perform the addition (pictured centered on the ith spot)

$$\begin{array}{r} 01\ldots1\ldots10 \\ \ldots010\ldots \\ \hline = 10\ldots.01\ldots10 \end{array}$$

the single one is carried all the way to the left. That is, $10\ldots01\ldots10^c = 01\ldots10\ldots01$. This means there is always a one in the ith spot in the binary expansion of $2^{n+1}+1-b$ and the only non-zero spot in b is the ith, hence this binomial is one mod two.

Conversely, suppose that b is not a power of two. So that it has non-zero bits in places $i < j$. Choose these bits so there are only zeros between them (or are adjacent) and so that i occurs in the minimal non-zero place. If one of non-zero bits is in the zeroth place, then the binomial is zero since the $2^{n+1}+1-b$ will have nothing in the ones place while b will.

Now suppose both $i, j > 0$

$$\begin{array}{r}
j \qquad i \qquad\quad \\
01\ldots111\ldots111\ldots10 \\
\ldots10\ldots01\ldots \\
\hline
= \ldots10\ldots00\ldots
\end{array}$$

The idea here is that the ith bit gets carried to the jth bit. Now

$$\ldots10\ldots00\ldots^c = \ldots01\ldots11\ldots.$$

Thus $2^{n+1}+1-b$ always has zero in the jth bit while b has a one in the jth bit, hence the binomial is zero. $\square$

Lemma A.0.2. *For* $1 < b \leq 2^n$ $\binom{2^{n+1}-b-1}{b-1} \equiv 1 \pmod{2}$ *if and only if* b *is a power of two.*

Proof Sketch. Use Lucas' Theorem with a similar binary argument. $\square$

Lemma A.0.3. *For* $1 \leq b \leq 2^{n-1}$ *we have* $\binom{2^{n+1}-2b-1}{2b} \equiv 0 \pmod{2}$.

Proof. We again use Lucas's Theorem. We write,

$$\binom{2^{n+1}-2b-1}{2b} = \binom{(2^{n+1}-1)-2b}{2b}$$

The binary expansion of $2^{n+1}-1$ has ones in all places up to 2^{n+1}, that is, $2^{n+1}-1 = 2^n + 2^{n-1} + \ldots + 2 + 1$. Thus, $(2^{n+1}-1)^c = 2^{n+1}$. If we write $2b$ in binary $2b = a_{n-1}2^{n-1} + \ldots + a_1 2 + a_0$ where $a_i \in \mathbb{Z}/2\mathbb{Z}$, then

$$\begin{aligned}(2^{n+1}-1)-2b &= ((2^{n+1}-1)^c + 2b)^c \\ &= (2^{n+1} + a_{n-1}2^{n-1} + \ldots + a_1 2 + a_0)^c \\ &= \overline{a_{n-1}}2^{n-1} + \ldots + \overline{a_1}2 + \overline{a_0}\end{aligned}$$

where $\overline{a_i} = a_i + 1 \pmod 2$. Notice that $2b$ has ones in exactly the opposite places $(2^{n+1}-1)-2b$ does, hence this binomial coefficient is zero. □

Lemma A.0.4.

$$\binom{2^{n+1}-2b-1}{2b-1} \equiv \binom{2^n-b}{2^n-2b} \pmod 2.$$

Proof. We prove this using generating functions. Recall that $\sum_{i=0}^{\infty} \binom{n}{i} x^i = (1+x)^n$.

In particular,

$$\begin{aligned}
(1+x)^{2^{n+1}-2b-1} &= \sum_{i=0}^{\infty} \binom{2^{n+1}-2b-1}{i} x^i \\
(1+x)^{2^{n+1}-2b} &= (1+x) \sum_{i=0}^{\infty} \binom{2^{n+1}-2b-1}{i} x^i \\
&= \sum_{i=0}^{\infty} \binom{2^{n+1}-2b-1}{i} x^{i+1} + \sum_{i=0}^{\infty} \binom{2^{n+1}-2b-1}{i} x^i \\
&= 1 + \sum_{i=0}^{\infty} \binom{2^{n+1}-2b-1}{i} x^{i+1} + \sum_{i=0}^{\infty} \binom{2^{n+1}-2b-1}{i+1} x^{i+1} \\
&= 1 + \sum_{i=0}^{\infty} \left(\binom{2^{n+1}-2b-1}{i} + \binom{2^{n+1}-2b-1}{i+1} \right) x^{i+1}.
\end{aligned}$$

While on the other hand,

$$\begin{aligned}
(1+x)^{2^{n+1}-2b} &= \left((1+x)^{2^n-b}\right)^2 \\
&= \left(\sum_{j=0}^{\infty} \binom{2^n-b}{j} x^j \right)^2 \\
&\equiv \sum_{j=0}^{\infty} \binom{2^n-b}{j}^2 x^{2j} \pmod 2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\binom{2^n-b}{2^n-2b}^2 x^{2^{n+1}-4b} &= \left(\binom{2^{n+1}-2b-1}{2^{n+1}-4b-1}+\binom{2^{n+1}-2b-1}{2^{n+1}-4b}\right)x^{2^{n+1}-4b} \\
\binom{2^n-b}{2^n-2b}^2 &= \binom{2^{n+1}-2b-1}{2^{n+1}-4b-1}+\binom{2^{n+1}-2b-1}{2^{n+1}-4b} \\
&= \binom{2^{n+1}-2b-1}{2^{n+1}-2b-1-2^{n+1}+4b+1}+\binom{2^{n+1}-2b-1}{2^{n+1}-2b-1-2^{n+1}+4b} \\
&= \binom{2^{n+1}-2b-1}{2b}+\binom{2^{n+1}-2b-1}{2b-1} \\
&= 0+\binom{2^{n+1}-2b-1}{2b-1}. && \text{(By Lemma A.0.3)}
\end{aligned}$$

□

Lemma A.0.5. *For* $1 \leq b \leq 2^{n-1}$,

$$\binom{2^{n+1}-2b-1}{2b-2} \equiv \binom{2^n-b}{2^n-2b} \pmod{2}.$$

Proof. It suffices to show that $\binom{2^{n+1}-2b-1}{2b-2} \equiv \binom{2^{n+1}-2b-1}{2b-1} \pmod{2}$ by Lemma A.0.4. By Lemma A.0.2 it suffices to show that $\binom{2^{n+1}-2b-1}{2b-2} \equiv 1$ if and only if b is a power of two. This is again done via a Lucas's Theorem argument. □

Appendix B

Calculations

In this appendix we present some tables of calculations that support Conjecture 5.1.1. Originally, many of these calculations were performed by the algorithm in §4.1.3 using the Borel picture and the Gröbner basis. It was using this algorithm that led to the conjecture. In order to improve performance the program was reimplemented using the combinatorial algorithm of §4.2.2. It is encouraging that the algorithm implemented in this way agrees with the previous algorithm and the conjecture.

For the larger Grassmannians the main obstacle to testing the conjecture is the sheer volume of calculations needed. To expedite these calculations the author used the University of Virginia Rivanna High Performance computing system. This allowed the calculation of the Q_n matrices to be done in parallel across up to 40 cores.

In the following tables the blue cells correspond to the entries for which there are no differentials in the AHSS due to dimension reasons, namely $cd < 2^{n+1}$. The green cells indicate that $c + d \leq 2^{n+1}$, and thus by Theorem 4.5.1, the Q_n-differential is also zero in this range. For $d = 1$, $\mathrm{Gr}_1(\mathbb{R}^{1+c})$ is a real projective space and so the calculation is well-known. We mark these in dark-gray. For the $d = 2$ entries, this is our result Theorem 4.9.26 which we indicate in yellow. The light-gray cells are those cells for which the conjecture has been verified with a computer. Finally, the white cells are the conjectured values which have not been checked due to computational limitations. The tables are necessarily symmetric in c and d.

$k_1(\mathrm{Gr}_d(\mathbb{R}^{d+c}))$

d \ c	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	2	3	4	3	4	3	4	3	4	3	4	3	4	3
2	3	6	4	7	5	8	6	9	7	10	8	11	9	12
3	4	4	8	7	12	10	16	13	20	16	24	19	28	22
4	3	7	7	14	12	22	18	31	25	41	33	52	42	64
5	4	5	12	12	24	22	40	35	60	51	84	70	112	92
6	3	8	10	22	22	44	40	75	65	116	98	168	140	232
7	4	6	16	18	40	40	80	75	140	126	224	196	336	288
8	3	9	13	31	35	75	75	150	140	266	238	434	378	666
9	4	7	20	25	60	65	140	140	280	266	504	462	840	750
10	3	10	16	41	51	116	126	266	266	532	504	966	882	1632
11	4	8	24	33	84	98	224	238	504	504	1008	966	1848	1716
12	3	11	19	52	70	168	196	434	462	966	966	1932	1848	3564
13	4	9	28	42	112	140	336	378	840	882	1848	1848	3696	3564
14	3	12	22	64	92	232	288	666	750	1632	1716	3564	3564	7128
15	4	10	32	52	144	192	480	570	1320	1452	3168	3300	6864	6864
16	3	13	25	77	117	309	405	975	1155	2607	2871	6171	6435	13299
17	4	11	36	63	180	255	660	825	1980	2277	5148	5577	12012	12441
18	3	14	28	91	145	400	550	1375	1705	3982	4576	10153	11011	23452
19	4	12	40	75	220	330	880	1155	2860	3432	8008	9009	20020	21450
20	3	15	31	106	176	506	726	1881	2431	5863	7007	16016	18018	39468
21	4	13	44	88	264	418	1144	1573	4004	5005	12012	14014	32032	35464
22	3	16	34	122	210	628	936	2509	3367	8372	10374	24388	28392	63856
23	4	14	48	102	312	520	1456	2093	5460	7098	17472	21112	49504	56576
24	3	17	37	139	247	767	1183	3276	4550	11648	14924	36036	43316	99892
25	4	15	52	117	364	637	1820	2730	7280	9828	24752	30940	74256	87516
26	3	18	40	157	287	924	1470	4200	6020	15848	20944	51884	64260	151776
27	4	16	56	133	420	770	2240	3500	9520	13328	34272	44268	108528	131784
28	3	19	43	176	330	1100	1800	5300	7820	21148	28764	73032	93024	224808
29	4	17	60	150	480	920	2720	4420	12240	17748	46512	62016	155040	193800
30	3	20	46	196	376	1296	2176	6596	9996	27744	38760	100776	131784	325584
31	4	18	64	168	544	1088	3264	5508	15504	23256	62016	85272	217056	279072
32	3	21	49	217	425	1513	2601	8109	12597	35853	51357	136629	183141	462213
33	4	19	68	187	612	1275	3876	6783	19380	30039	81396	115311	298452	394383
34	3	22	52	239	477	1752	3078	9861	15675	45714	67032	182343	250173	644556
35	4	20	72	207	684	1482	4560	8265	23940	38304	105336	153615	403788	547998
36	3	23	55	262	532	2014	3610	11875	19285	57589	86317	239932	336490	884488
37	4	21	76	228	760	1710	5320	9975	29260	48279	134596	201894	538384	749892
38	3	24	58	286	590	2300	4200	14175	23485	71764	109802	311696	446292	1196184
39	4	22	80	250	840	1960	6160	11935	35420	60214	170016	262108	708400	1012000
40	3	25	61	311	651	2611	4851	16786	28336	88550	138138	400246	584430	1596430
41	4	23	84	273	924	2233	7084	14168	42504	74382	212520	336490	920920	1348490
42	3	26	64	337	715	2948	5566	19734	33902	108284	172040	508530	756470	2104960
43	4	24	88	297	1012	2530	8096	16698	50600	91080	263120	427570	1184040	1776060
44	3	27	67	364	782	3312	6348	23046	40250	131330	212290	639860	968760	2744820
45	4	25	92	322	1104	2852	9200	19550	59800	110630	322920	538200	1506960	2314260
46	3	28	70	392	852	3704	7200	26750	47450	158080	259740	797940	1228500	3542760
47	4	26	96	348	1200	3200	10400	22750	70200	133380	393120	671580	1900080	2985840
48	3	29	73	421	925	4125	8125	30875	55575	188955	315315	986895	1543815	4529655
49	4	27	100	375	1300	3575	11700	26325	81900	159705	475020	831285	2375100	3817125
50	3	30	76	451	1001	4576	9126	35451	64701	224406	380016	1211301	1923831	5740956

🟦 $cd \leq 2^{n+1} - 1$ ⬛ Projective Spaces 🟨 Theorem 4.9.26

▨ Conjecture Verified ☐ Conjecture

$k_2(\mathrm{Gr}_d(\mathbb{R}^{d+c}))$

c d	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	2	3	4	5	6	7	8	7	8	7	8	7	8	7
2	3	6	10	15	21	28	22	29	23	30	24	31	25	32
3	4	10	20	35	56	42	64	49	72	56	80	63	88	70
4	5	15	35	70	56	98	78	127	101	157	125	188	150	220
5	6	21	56	56	112	98	176	147	248	203	328	266	416	336
6	7	28	42	98	98	196	176	323	277	480	402	668	552	888
7	8	22	64	78	176	176	352	323	600	526	928	792	1344	1128
8	7	29	49	127	147	323	323	646	600	1126	1002	1794	1554	2682
9	8	23	72	101	248	277	600	600	1200	1126	2128	1918	3472	3046
10	7	30	56	157	203	480	526	1126	1126	2252	2128	4046	3682	6728
11	8	24	80	125	328	402	928	1002	2128	2128	4256	4046	7728	7092
12	7	31	63	188	266	668	792	1794	1918	4046	4046	8092	7728	14820
13	8	25	88	150	416	552	1344	1554	3472	3682	7728	7728	15456	14820
14	7	32	70	220	336	888	1128	2682	3046	6728	7092	14820	14820	29640
15	8	26	96	176	512	728	1856	2282	5328	5964	13056	13692	28512	28512
16	7	33	77	253	413	1141	1541	3823	4587	10551	11679	25371	26499	55011
17	8	27	104	203	616	931	2472	3213	7800	9177	20856	22869	49368	51381
18	7	34	84	287	497	1428	2038	5251	6625	15802	18304	41173	44803	96184
19	8	28	112	231	728	1162	3200	4375	11000	13552	31856	36421	81224	87802
20	7	35	91	322	588	1750	2626	7001	9251	22803	27555	63976	72358	160160
21	8	29	120	260	848	1422	4048	5797	15048	19349	46904	55770	128128	143572
22	7	36	98	358	686	2108	3312	9109	12563	31912	40118	95888	112476	256048
23	8	30	128	290	976	1712	5024	7509	20072	26858	66976	82628	195104	226200
24	7	37	105	395	791	2503	4103	11612	16666	43524	56784	139412	169260	395460
25	8	31	136	321	1112	2033	6136	9542	26208	36400	93184	119028	288288	345228
26	7	38	112	433	903	2936	5006	14548	21672	58072	78456	197484	247716	592944
27	8	32	144	353	1256	2386	7392	11928	33600	48328	126784	167356	415072	512584
28	7	39	119	472	1022	3408	6028	17956	27700	76028	106156	273512	353872	866456
29	8	33	152	386	1408	2772	8800	14700	42400	63028	169184	230384	584256	742968
30	7	40	126	512	1148	3920	7176	21876	34876	97904	141032	371416	494904	1237872
31	8	34	160	420	1568	3192	10368	17892	52768	80920	221952	311304	806208	1054272
32	7	41	133	553	1281	4473	8457	26349	43333	124253	184365	495669	679269	1733541
33	8	35	168	455	1736	3647	12104	21539	64872	102459	286824	413763	1093032	1468035
34	7	42	140	595	1421	5068	9878	31417	53211	155670	237576	651339	916845	2384880
35	8	36	176	491	1912	4138	14016	25677	78888	128136	365712	541899	1458744	2009934
36	7	43	147	638	1568	5706	11446	37123	64657	192793	302233	844132	1219078	3229012
37	8	37	184	528	2096	4666	16112	30343	95000	158479	460712	700378	1919456	2710312
38	7	44	154	682	1722	6388	13168	43511	77825	236304	380058	1080436	1599136	4309448
39	8	38	192	566	2288	5232	18400	35575	113400	194054	574112	894432	2493568	3604744
40	7	45	161	727	1883	7115	15051	50626	92876	286930	472934	1367366	2072070	5676814
41	8	39	200	605	2488	5837	20888	41412	134288	235466	708400	1129898	3201968	4734642
42	7	46	168	773	2051	7888	17102	58514	109978	345444	582912	1712810	2654982	7389624
43	8	40	208	645	2696	6482	23584	47894	157872	283360	866272	1413258	4068240	6147900
44	7	47	175	820	2226	8708	19328	67222	129306	412666	712218	2125476	3367200	9515100
45	8	41	216	686	2912	7168	26496	55062	184368	338422	1050640	1751680	5118880	7899580
46	7	48	182	868	2408	9576	21736	76798	151042	489464	863260	2614940	4230460	12130040
47	8	42	224	728	3136	7896	29632	62958	214000	401380	1264640	2153060	6383520	10052640
48	7	49	189	917	2597	10493	24333	87291	175375	576755	1038635	3191695	5269095	15321735
49	8	43	232	771	3368	8667	33000	71625	247000	473005	1511640	2626065	7895160	12678705
50	7	50	196	967	2793	11460	27126	98751	202501	675506	1241136	3867201	6510231	19188936

🟦 $cd \leq 2^{n+1} - 1$ 🟩 Theorem 4.5.1 ⬛ Projective Spaces 🟨 Theorem 4.9.26

▨ Conjecture Verified ☐ Conjecture

$k_3(\mathrm{Gr}_d(\mathbb{R}^{d+c}))$

c d	1	2	3	4	5	6	7	8	9	10	11	12	13
1	2	3	4	5	6	7	8	9	10	11	12	13	14
2	3	6	10	15	21	28	36	45	55	66	78	91	105
3	4	10	20	35	56	84	120	165	220	286	364	455	560
4	5	15	35	70	126	210	330	495	715	1001	1365	1820	1470
5	6	21	56	126	252	462	792	1287	2002	3003	4368	3458	4928
6	7	28	84	210	462	924	1716	3003	5005	8008	6370	9828	7840
7	8	36	120	330	792	1716	3432	6435	11440	9438	15808	12896	20736
8	9	45	165	495	1287	3003	6435	12870	11440	20878	17810	30706	25650
9	10	55	220	715	2002	5005	11440	11440	22880	20878	38688	33774	59424
10	11	66	286	1001	3003	8008	9438	20878	20878	41756	38688	72462	64338
11	12	78	364	1365	4368	6370	15808	17810	38688	38688	77376	72462	136800
12	13	91	455	1820	3458	9828	12896	30706	33774	72462	72462	144924	136800
13	14	105	560	1470	4928	7840	20736	25650	59424	64338	136800	136800	273600
14	15	120	470	1940	3928	11768	16824	42474	50598	114936	123060	259860	259860
15	16	106	576	1576	5504	9416	26240	35066	85664	99404	222464	236204	496064
16	15	121	485	2061	4413	13829	21237	56303	71835	171239	194895	431099	454755
17	16	107	592	1683	6096	11099	32336	46165	118000	145569	340464	381773	836528
18	15	122	500	2183	4913	16012	26150	72315	97985	243554	292880	674653	747635
19	16	108	608	1791	6704	12890	39040	59055	157040	204624	497504	586397	1334032
20	15	123	515	2306	5428	18318	31578	90633	129563	334187	422443	1008840	1170078
21	16	109	624	1900	7328	14790	46368	73845	203408	278469	700912	864866	2034944
22	15	124	530	2430	5958	20748	37536	111381	167099	445568	589542	1454408	1759620
23	16	110	640	2010	7968	16800	54336	90645	257744	369114	958656	1233980	2993600
24	15	125	545	2555	6503	23303	44039	134684	211138	580252	800680	2034660	2560300
25	16	111	656	2121	8624	18921	62960	109566	320704	478680	1279360	1712660	4272960
26	15	126	560	2681	7063	25984	51102	160668	262240	740920	1062920	2775580	3623220
27	16	112	672	2233	9296	21154	72256	130720	392960	609400	1672320	2322060	5945280
28	15	127	575	2808	7638	28792	58740	189460	320980	930380	1383900	3705960	5007120
29	16	113	688	2346	9984	23500	82240	154220	475200	763620	2147520	3085680	8092800
30	15	128	590	2936	8228	31728	66968	221188	387948	1151568	1771848	4857528	6778968
31	16	114	704	2460	10688	25960	92928	180180	568128	943800	2715648	4029480	10808448
32	15	129	605	3065	8833	34793	75801	255981	463749	1407549	2235597	6265077	9014565
33	16	115	720	2575	11408	28535	104336	208715	672464	1152515	3388112	5181995	14196560
34	15	130	620	3195	9453	37988	85254	293969	549003	1701518	2784600	7966595	11799165
35	16	116	736	2691	12144	31226	116480	239941	788944	1392456	4177056	6574451	18373616
36	15	131	635	3326	10088	41314	95342	335283	644345	2036801	3428945	10003396	15228110
37	16	117	752	2808	12896	34034	129376	273975	918320	1666431	5095376	8240882	23468992
38	15	132	650	3458	10738	44772	106080	380055	750425	2416856	4179370	12420252	19407480
39	16	118	768	2926	13664	36960	143040	310935	1061360	1977366	6156736	10218248	29625728
40	15	133	665	3591	11403	48363	117483	428418	867908	2845274	5047278	15265526	24454758
41	16	119	784	3045	14448	40005	157488	350940	1218848	2328306	7375584	12546554	37001312
42	15	134	680	3725	12083	52088	129566	480506	997474	3325780	6044752	18591306	30499510
43	16	120	800	3165	15248	43170	172736	394110	1391584	2722416	8767168	15268970	45768480
44	15	135	695	3860	12778	55948	142344	536454	1139818	3862234	7184570	22453540	37684080
45	16	121	816	3286	16064	46456	188800	440566	1580384	3162982	10347552	18431952	56116032
46	15	136	710	3996	13488	59944	155832	596398	1295650	4458632	8480220	26912172	46164300
47	16	122	832	3408	16896	49864	205696	490430	1786080	3653412	12133632	22085364	68249664
48	15	137	725	4133	14213	64077	170045	660475	1465695	5119107	9945915	32031279	56110215
49	16	123	848	3531	17744	53395	223440	543825	2009520	4197237	14143152	26282601	82392816
50	15	138	740	4271	14953	68348	184998	728823	1650693	5847930	11596608	37879209	67706823

🟦 $cd \leq 2^{n+1} - 1$ 🟩 Theorem 4.5.1 ⬛ Projective Spaces 🟨 Theorem 4.9.26

▨ Conjecture Verified ☐ Conjecture

$k_4(\mathrm{Gr}_d(\mathbb{R}^{d+c}))$

d \ c	1	2	3	4	5	6	7	8	9	10	11	12
1	2	3	4	5	6	7	8	9	10	11	12	13
2	3	6	10	15	21	28	36	45	55	66	78	91
3	4	10	20	35	56	84	120	165	220	286	364	455
4	5	15	35	70	126	210	330	495	715	1001	1365	1820
5	6	21	56	126	252	462	792	1287	2002	3003	4368	6188
6	7	28	84	210	462	924	1716	3003	5005	8008	12376	18564
7	8	36	120	330	792	1716	3432	6435	11440	19448	31824	50388
8	9	45	165	495	1287	3003	6435	12870	24310	43758	75582	125970
9	10	55	220	715	2002	5005	11440	24310	48620	92378	167960	293930
10	11	66	286	1001	3003	8008	19448	43758	92378	184756	352716	646646
11	12	78	364	1365	4368	12376	31824	75582	167960	352716	705432	1352078
12	13	91	455	1820	6188	18564	50388	125970	293930	646646	1352078	2704156
13	14	105	560	2380	8568	27132	77520	203490	497420	1144066	2496144	5200300
14	15	120	680	3060	11628	38760	116280	319770	817190	1961256	4457400	9657700
15	16	136	816	3876	15504	54264	170544	490314	1307504	3268760	7726160	17383860
16	17	153	969	4845	20349	74613	245157	735471	2042975	5311735	13037895	30421755
17	18	171	1140	5985	26334	100947	346104	1081575	3124550	8436285	21474180	51895935
18	19	190	1330	7315	33649	134596	480700	1562275	4686825	13123110	34597290	86493225
19	20	210	1540	8855	42504	177100	657800	2220075	6906900	20030010	54627300	141120525
20	21	231	1771	10626	53130	230230	888030	3108105	10015005	30045015	84672315	225792840
21	22	253	2024	12650	65780	296010	1184040	4292145	14307150	44352165	129024480	185472690
22	23	276	2300	14950	80730	376740	1560780	5852925	20160075	64512240	104832390	290305080
23	24	300	2600	17550	98280	475020	2035800	7888725	28048800	52240890	157073280	237713580
24	25	325	2925	20475	118755	593775	2629575	10518300	22789650	75030540	127622040	365335620
25	26	351	3276	23751	142506	736281	3365856	8625006	31414656	60865896	188487936	298579476
26	27	378	3654	27405	169911	906192	2799486	11424492	25589136	86455032	153211176	451790652
27	28	406	4060	31465	201376	767746	3567232	9392752	34981888	70258648	223469824	368838124
28	29	435	4495	35960	174406	942152	2973892	12366644	28563028	98821676	181774204	550612328
29	30	465	4960	31930	206336	799676	3773568	10192428	38755456	80451076	262225280	449289200
30	31	496	4526	36456	178932	978608	3152824	13345252	31715852	112166928	213490056	662779256
31	32	466	4992	32396	211328	832072	3984896	11024500	42740352	91475576	304965632	540764776
32	31	497	4557	36953	183489	1015561	3336313	14360813	35052165	126527741	248542221	789306997
33	32	467	5024	32863	216352	864935	4201248	11889435	46941600	103365011	351907232	644129787
34	31	498	4588	37451	188077	1053012	3524390	15413825	38576555	141941566	287118776	931248563
35	32	468	5056	33331	221408	898266	4422656	12787701	51364256	116152712	403271488	760282499
36	31	499	4619	37950	192696	1090962	3717086	16504787	42293641	158446353	329412417	1089694916
37	32	469	5088	33800	226496	932066	4649152	13719767	56013408	129872479	459284896	890154978
38	31	500	4650	38450	197346	1129412	3914432	17634199	46208073	176080552	375620490	1265775468
39	32	470	5120	34270	231616	966336	4880768	14686103	60894176	144558582	520179072	1034713560
40	31	501	4681	38951	202027	1168363	4116459	18802562	50324532	194883114	425945022	1460658582
41	32	471	5152	34741	236768	1001077	5117536	15687180	66011712	160245762	586190784	1194959322
42	31	502	4712	39453	206739	1207816	4323198	20010378	54647730	214893492	480592752	1675552074
43	32	472	5184	35213	241952	1036290	5359488	16723470	71371200	176969232	657561984	1371928554
44	31	503	4743	39956	211482	1247772	4534680	21258150	59182410	236151642	539775162	1911703716
45	32	473	5216	35686	247168	1071976	5606656	17795446	76977856	194764678	734539840	1566693232
46	31	504	4774	40460	216256	1288232	4750936	22546382	63933346	258698024	603708508	2170401740
47	32	474	5248	36160	252416	1108136	5859072	18903582	82836928	213668260	817376768	1780361492
48	31	505	4805	40965	221061	1329197	4971997	23875579	68905343	282573603	672613851	2452975343
49	32	475	5280	36635	257696	1144771	6116768	20048353	88953696	233716613	906330464	2014078105
50	31	506	4836	41471	225897	1370668	5197894	25246247	74103237	307819850	746717088	2760795193

🟦 $cd \leq 2^{n+1} - 1$ 🟩 Theorem 4.5.1 ⬛ Projective Spaces 🟨 Theorem 4.9.26

▨ Conjecture Verified ☐ Conjecture

Appendix C

Adams spectral sequence

Let E and X be connective spectra. The homology theory associated to E is defined to be $E_*(X) = \pi_*(E \wedge X)$. As E and X are both connective one may use the Adams spectral sequence to compute $E_*(X)$. In general, the Adams spectral sequence is of the form

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{\mathcal{A}_\mathrm{p}}(H^*(Y), \mathbb{F}_p) \implies \pi_{s-t}(Y) \otimes \mathbb{Z}_p$$

where $\mathbb{Z}_p$ denotes the ring of p-adic integer [42, Thm 2.1.1]. This converges when X has finite type. Substituting $E \wedge X$ for Y yields a spectral sequence computing p-local $E_*(X)$:

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{\mathcal{A}_\mathrm{p}}(H^*(E \wedge X), \mathbb{F}_p) \implies E_{s-t}(X) \otimes \mathbb{Z}_p.$$

We are in particular interested in the case for $E = k(n)$ the connective Morava K-theory. The key property we will use is that $H^*(k(n); \mathbb{F}_p) \cong \mathcal{A}_\mathrm{p} \otimes_{E(Q_n)} \mathbb{F}_p$, where $E(Q_n)$ is the exterior Hopf algebra generated by Q_n in $\mathcal{A}_\mathrm{p}$ [52, §5]. Using the Künneth Theorem,

$$\begin{aligned} E_2^{s,t} = \mathrm{Ext}^{s,t}_{\mathcal{A}_\mathrm{p}}(H^*(k(n) \wedge X), \mathbb{F}_p) &\cong \mathrm{Ext}^{s,t}_{\mathcal{A}_\mathrm{p}}(H^*(k(n)) \otimes H^*(X), \mathbb{F}_p) \\ &\cong \mathrm{Ext}^{s,t}_{\mathcal{A}_\mathrm{p}}(\mathcal{A}_\mathrm{p} \otimes_{E(Q_n)} \mathbb{F}_p \otimes H^*(X), \mathbb{F}_p). \end{aligned}$$

Via a change of rings (see [9, § 4.5]) this allows us to identify

$$\mathrm{Ext}^{s,t}_{\mathcal{A}_\mathrm{p}}(\mathcal{A}_\mathrm{p} \otimes_{E(Q_n)} \mathbb{F}_p \otimes H^*(X), \mathbb{F}_p) \cong \mathrm{Ext}^{s,t}_{E(Q_n)}(H^*(X), \mathbb{F}_p).$$

Now one can calculate that

$$\mathrm{Ext}^{*,*}_{E(Q_n)}(H^*(X); \mathbb{F}_p) \cong (\mathbb{F}_p[v_n] \otimes H_*(X; Q_n)) \oplus V$$

with v_n in bidegree $(1, 2p^n - 1)$ and V a direct sum of $\mathbb{F}_p$'s in degree zero. An analogous calculation can be done in general over any exterior algebra $E(x)$ (see [8, § 2] for a concise proof). We have thus identified the E_2-term of the Adams spectral sequence computing $k(n)_*(X)$ as $(k(n)_* \otimes H_*(X; Q_n)) \oplus V$ where V consists of classes in degree zero that are Q_n-cycles. This is exactly the same data that resides on the E_{2p^n}-page of the AHSS computing $k(n)_*(X)$.

We now indicate how these spectral sequences really arise in the same way and are in fact the same. To form the Adams spectral sequence calculating $\pi_*(X)$ one chooses an Adams resolution, that is, a diagram

$$\begin{tikzcd}
X \arrow[r, equal] & X_0 \arrow[d, "j_0"] & X_1 \arrow[l, "i_0"'] \arrow[d, "j_1"] & X_2 \arrow[l, "i_1"'] \arrow[d, "j_2"] & \ldots \arrow[l] \\
 & K_0 & K_1 & K_2 &
\end{tikzcd}$$

such that

1. K_s is a generalized Eilenberg–Mac Lane space (a wedge of suspensions of $H\mathbb{F}_p$),
2. The induced maps $H^*(j_s)$ are surjective,
3. X_{s+1} is the fiber of j_s.

Stitching together the cofibration sequences and applying π_* yields an exact couple which then yields the Adams spectral sequence.

In our case the resolution we choose is the Postnikov tower for $k(n)$ [36, Ex 1.6],

$$\begin{tikzcd}
k(n) \arrow[r, equal] & k(n) \arrow[d, "\pi"] & \Sigma^{2p^n-2}k(n) \arrow[l, "v_n"'] \arrow[d, "\Sigma^{2p^n-2}\pi"] & \Sigma^{2p^n-2}k(n) \arrow[l, "\Sigma^{2(2p^n-2)}v_n"'] \arrow[d, "\Sigma^{2(2p^n-2)}\pi"] & \dots \arrow[l] \\
& H\mathbb{F}_p & \Sigma^{2p^n-2}H\mathbb{F}_p & \Sigma^{2(2p^n-2)}H\mathbb{F}_p &
\end{tikzcd}$$

The map $\pi\colon k(n) \to H\mathbb{F}_p$ is the Thom map and is surjective. In fact, it induces the isomorphism $H^*(k(n);\mathbb{F}_p) \cong \mathcal{A}_{\mathrm{p}} \otimes_{E(n)} \mathbb{F}_p$. The exact couple is induced from the cofiber sequences

$$\dots \to \Sigma^{2p^n-2}k(n) \xrightarrow{v_n} k(n) \xrightarrow{\pi} H\mathbb{F}_p \xrightarrow{\overline{Q}_n} \Sigma^{2p^n-1}k(n) \to \dots,$$

where $Q_n = \pi_*\overline{Q}_n$ [52, §5]. Notice that all of the required properties for this to be an Adams resolution are satisfied. Furthermore, the identification of the connecting homomorphism gives us that the d_1-differential is Q_n. This resolution is then smashed with X.

The Atiyah–Hirzebruch spectral sequence $E^2_{p,q}(X; E_*(*)) \implies \pi_*(E \wedge X) \cong E_*(X)$ usually arises from a cellular filtration of X. Alternatively, one could use a Postnikov tower of E. A folk theorem that was eventually written down [18, Theorem B.8] provides the necessary conditions for when these two methods of constructing a spectral sequence computing $\pi_*(E \wedge X)$ agree. In our case this theorem does apply and using the Postnikov tower of $k(n)$ yields the AHSS as well as the Adams spectral sequence as explained above.

Bibliography

[1] J. F. Adams, *Stable homotopy and generalised homology*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 1995, Reprint of the 1974 original. MR 1324104

[2] M. F. Atiyah and F. Hirzebruch, *Vector bundles and homogeneous spaces*, Proc. Sympos. Pure Math., Vol. III, American Mathematical Society, Providence, R.I., 1961, pp. 7–38. MR 0139181

[3] Paul Balmer, *The spectrum of prime ideals in tensor triangulated categories*, J. Reine Angew. Math. **588** (2005), 149–168. MR 2196732

[4] ———, *Spectra, spectra, spectra—tensor triangular spectra versus Zariski spectra of endomorphism rings*, Algebr. Geom. Topol. **10** (2010), no. 3, 1521–1563. MR 2661535

[5] Paul Balmer and Beren Sanders, *The spectrum of the equivariant stable homotopy category of a finite group*, Invent. Math. **208** (2017), no. 1, 283–326. MR 3621837

[6] Tobias Barthel, J. P. C. Greenlees, and Markus Hausmann, *On the Balmer spectrum for compact Lie groups*, Compos. Math. **156** (2020), no. 1, 39–76. MR 4036448

[7] Tobias Barthel, Markus Hausmann, Niko Naumann, Thomas Nikolaus, Justin Noel, and Nathaniel Stapleton, *The Balmer spectrum of the equivariant homotopy category of a finite abelian group*, Invent. Math. **216** (2019), no. 1, 215–240. MR 3935041

[8] Agnes Beaudry, Mark Behrens, Prasit Bhattacharya, Dominic Culver, and Zhouli Xu, *On the tmf-resolution of Z*, arXiv 1909.13379, 2019.

[9] Agnès Beaudry and Jonathan A. Campbell, *A guide for computing stable homotopy groups*, Topology and quantum theory in interaction, Contemp. Math., vol. 718, Amer. Math. Soc., Providence, RI, 2018, pp. 89–136. MR 3869642

[10] A. Borel, *La cohomologie mod* 2 *de certains espaces homogènes*, Comment. Math. Helv. **27** (1953), 165–197. MR 57541

[11] ———, *Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts*, Ann. of Math. (2) **57** (1953), 115–207. MR 51508

[12] Raoul Bott and Loring W. Tu, *Differential forms in algebraic topology*, Graduate Texts in Mathematics, vol. 82, Springer-Verlag, New York-Berlin, 1982. MR 658304

[13] David S. Dummit and Richard M. Foote, *Abstract algebra*, third ed., John Wiley & Sons, Inc., Hoboken, NJ, 2004. MR 2286236

[14] Charles Ehresmann, *Sur la topologie de certains espaces homogènes*, Ann. of Math. (2) **35** (1934), no. 2, 396–443. MR 1503170

[15] ———, *Sur la topologie de certaines variétés algébriques rélles*, J. Math. Pures Appl **16(9)** (1937), 69–100.

[16] William Fulton, *Young tableaux*, London Mathematical Society Student Texts, vol. 35, Cambridge University Press, Cambridge, 1997, With applications to representation theory and geometry. MR 1464693

[17] J. P. C. Greenlees and J. P. May, *Equivariant stable homotopy theory*, Handbook of algebraic topology, North-Holland, Amsterdam, 1995, pp. 277–323. MR 1361893

[18] ———, *Generalized Tate cohomology*, Mem. Amer. Math. Soc. **113** (1995), no. 543, viii+178. MR 1230773

[19] Allen Hatcher, *Algebraic topology*, Cambridge University Press, Cambridge, 2002. MR 1867354

[20] Chen He, *Torsions of integral homology and cohomology of real grassmannians*, arXiv 1709.05623, 2017.

[21] Howard L. Hiller, *On the cohomology of real Grassmanians*, Trans. Amer. Math. Soc. **257** (1980), no. 2, 521–533. MR 552272

[22] Michael J. Hopkins and Jeffrey H. Smith, *Nilpotence and stable homotopy theory. II*, Ann. of Math. (2) **148** (1998), no. 1, 1–49. MR 1652975

[23] Jan Jaworowski, *An additive basis for the cohomology of real Grassmannians*, Algebraic topology Poznań 1989, Lecture Notes in Math., vol. 1474, Springer, Berlin, 1991, pp. 231–234. MR 1133904

[24] Nitu Kitchloo and W. Stephen Wilson, *The Morava K-theory of $BO(q)$ and $MO(q)$*, Algebr. Geom. Topol. **15** (2015), no. 5, 3049–3058. MR 3426703

[25] Akira Kono and Nobuaki Yagita, *Brown-Peterson and ordinary cohomology theories of classifying spaces for compact Lie groups*, Trans. Amer. Math. Soc. **339** (1993), no. 2, 781–798. MR 1139493

[26] Nicholas J. Kuhn and Christopher J. R. Lloyd, *New results about the equivariant stable homotopy balmer spectrum*, Oberwolfach Reports **16** (2019).

[27] ———, *Chromatic fixed point theory and the balmer spectrum for extraspecial 2-groups*, arXiv 2008.00330, 2020.

[28] Cristian Lenart, *The combinatorics of Steenrod operations on the cohomology of Grassmannians*, Adv. Math. **136** (1998), no. 2, 251–283. MR 1626852

[29] E. Lucas, *Sur les congruences des nombres eulériens et les coefficients différentiels des functions trigonométriques suivant un module premier*, Bull. Soc. Math. France **6** (1878), 49–54. MR 1503769

[30] M. A. Mandell and J. P. May, *Equivariant orthogonal spectra and S-modules*, Mem. Amer. Math. Soc. **159** (2002), no. 755, x+108. MR 1922205

[31] H. R. Margolis, *Spectra and the Steenrod algebra*, North-Holland Mathematical Library, vol. 29, North-Holland Publishing Co., Amsterdam, 1983, Modules over the Steenrod algebra and the stable homotopy category. MR 738973

[32] John McCleary, *A user's guide to spectral sequences*, second ed., Cambridge Studies in Advanced Mathematics, vol. 58, Cambridge University Press, Cambridge, 2001. MR 1793722

[33] John Milnor, *The Steenrod algebra and its dual*, Ann. of Math. (2) **67** (1958), 150–171. MR 99653

[34] John W. Milnor and James D. Stasheff, *Characteristic classes*, Princeton University Press, Princeton, N. J.; University of Tokyo Press, Tokyo, 1974, Annals of Mathematics Studies, No. 76. MR 0440554

[35] Stephen A. Mitchell, *Finite complexes with $A(n)$-free cohomology*, Topology **24** (1985), no. 2, 227–246. MR 793186

[36] A. V. Pazhitnov, *Uniqueness theorems for generalized cohomology theories*, Mathematics of the USSR-Izvestiya **22** (1984), no. 3, 483–506.

[37] Zoran Z. Petrović and Branislav I. Prvulović, *On Groebner bases and immersions of Grassmann manifolds $G_{2,n}$*, Homology Homotopy Appl. **13** (2011), no. 2, 113–128. MR 2846161

[38] ———, *New immersion theorems for Grassmann manifolds $G_{3,n}$*, Publ. Inst. Math. (Beograd) (N.S.) **101(115)** (2017), 1–10. MR 3700398

[39] Zoran Z. Petrović, Branislav I. Prvulović, and Marko Radovanović, *Multiplication in the cohomology of Grassmannians via Gröbner bases*, J. Algebra **438** (2015), 60–84. MR 3353024

[40] Dieter Puppe, *Stabile Homotopietheorie. I*, Math. Ann. **169** (1967), 243–274. MR 211400

[41] Douglas C. Ravenel, *Localization with respect to certain periodic homology theories*, Amer. J. Math. **106** (1984), no. 2, 351–414. MR 737778

[42] ______, *Complex cobordism and stable homotopy groups of spheres*, Pure and Applied Mathematics, vol. 121, Academic Press, Inc., Orlando, FL, 1986. MR 860042

[43] ______, *Nilpotence and periodicity in stable homotopy theory*, Annals of Mathematics Studies, vol. 128, Princeton University Press, Princeton, NJ, 1992, Appendix C by Jeff Smith. MR 1192553

[44] Rustam Sadykov, *Elementary calculation of the cohomology rings of real Grassmann manifolds*, Pacific J. Math. **289** (2017), no. 2, 443–447. MR 3667179

[45] Masaru Takeuchi, *On Pontrjagin classes of compact symmetric spaces*, J. Fac. Sci. Univ. Tokyo Sect. I **9** (1962), 313–328 (1962). MR 145009

[46] Tammo tom Dieck, *Transformation groups*, De Gruyter Studies in Mathematics, vol. 8, Walter de Gruyter & Co., Berlin, 1987. MR 889050

[47] Jean-Louis Verdier, *Des catégories dérivées des catégories abéliennes*, Astérisque (1996), no. 239, xii+253 pp. (1997), With a preface by Luc Illusie, Edited and with a note by Georges Maltsiniotis. MR 1453167

[48] Charles A. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, Cambridge, 1994. MR 1269324

[49] R. M. W. Wood, *Differential operators and the Steenrod algebra*, Proc. London Math. Soc. (3) **75** (1997), no. 1, 194–220. MR 1444319

[50] ______, *Problems in the Steenrod algebra*, Bull. London Math. Soc. **30** (1998), no. 5, 449–517. MR 1643834

[51] Wen-tsün Wu, *Classes caractéristiques et i-carrés d'une variété*, C. R. Acad. Sci. Paris **230** (1950), 508–511. MR 35992

[52] Urs Würgler, *Morava K-theories: a survey*, Algebraic topology Poznań 1989, Lecture Notes in Math., vol. 1474, Springer, Berlin, 1991, pp. 111–138. MR 1133896

[53] Nobuaki Yagita, *On the Steenrod algebra of Morava K-theory*, J. London Math. Soc. (2) **22** (1980), no. 3, 423–438. MR 596321