# The family index of the odd signature operator with coefficients in a flat bundle

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# Abstract

We study characteristic classes arising as the indices of families of elliptic operators acting on the fibers of an oriented $M$-bundle $f : E \to B$, $M$ a smooth oriented closed manifold. Given a family of such operators $D = \{D_b\}_{b \in B}$ one obtains a family index $Ind(D) \in K^*(B)$. If $D$ is "sufficiently natural" (in a sense made precise in [23]) these indices may be viewed as arising from certain universal symbol classes $\sigma \in K^*(MTSO(n))$, where $MTSO(n)$ is the Thom spectrum of the additive inverse of the universal bundle of oriented $n$-planes over $BSO(n)$. Explicitly, $Ind(D) = \alpha_E^*(\sigma)$ where $\alpha_E : \Sigma^\infty B_+ \to MTSO(n)$ is the so-called Madsen-Tillman-Weiss map associated to $f : E \to B$. We show $Ind(D_V^o) = 0$ where $D_V^o$ is the family of odd signature operators on the fibers of $f : E \to B$ with coefficients in a flat Hermitian vector bundle $V \to E$. $D_V^o$ is not universal in the sense of [23] however its index can be described in terms of universal symbols. The vanishing relations implied in cohomology show the higher signatures (Novikov [52]) associated to flat Hermitian bundles provide obstructions to fibering as an odd-dimensional manifold bundle. We end by discussing some examples of flat Hermitian vector bundles to verify that these higher signatures provide a more general obstruction than the usual signature of a $4k$-dimensional manifold.

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# Contents

# Chapter 1

# Introduction and Outline

## 1.1 Introduction

The years following the solution [40] of Mumford's conjecture by Madsen and Weiss have seen a revival in geometric aspects of cobordism theory. At the center of this geometric theory is the $n$-dimensional cobordism category $Cob_n^\theta$ of $\theta$-manifolds, where $\theta : Y \to BO(n)$ is a fibration. In [28] Galatius, Madsen, Tillman, and Weiss proved that $BCob_n^\theta \cong \Omega^{\infty-1}MT\theta(n)$, where $MT\theta(n)$ is the Thom spectrum of the additive inverse of the vector bundle classified by $\theta$. The main result of [28] provides a geometric representation of $\Omega^{\infty-1}MT\theta(n)$ in terms of spaces of $\theta$-manifolds.

Roughly speaking, $MT\theta(n)$ may be viewed as the subspace of linear $n$-dimensional $\theta$-manifolds, whose inclusion into the space of all $d$-dimensional $\theta$-manifolds (topologized appropriately) is a homotopy equivalence, to be precise, a stable equivalence of spectra [27] (see [24] for details exposited in a context more relevant to our current work). The inverse $\alpha$ (in the homotopy category of spectra) of this inclusion comes from the parametrized Pontrjagin-Thom (PT) construction developed in [40].

For a $M^n$-bundle $f : E \to B$ of $\theta$-manifolds the "restriction" of $\alpha$ produces a map $\alpha_E : \Sigma^\infty B_+ \to MT\theta(n)$ which factors through the usual classifying map $B \to BDiff(M)$. Hence any class in the cohomology of $MT\theta(n)$ produces a characteristic class of $M$-bundles. These classes are universal in that they are defined for all $n$-dimensional $\theta$-manifolds.

Other examples of characteristic classes of smooth manifold bundles come from the index theory of elliptic operators ([13], [12], [14]). The index of a family of elliptic operators $D$ on the fibers of a bundle of smooth manifolds $f : E \to B$ produces an element $index(D) \in K^*(B)$ in the complex K-theory of $B$. If $D$ is *symbolically universal* (see [23]), an application of the Atiyah-Singer index theorem shows these classes also come from $MT\theta(n)$: $index(D) = \alpha_E^*(\sigma)$, for $\sigma \in K^*(MT\theta(n))$. This observation, made at least by [39], has been exploited by Ebert in [23], explored further in [25] and [18], and generalized considerably in the very compelling [24]. This dissertation consists of work which can be considered a natural, if not obvious, further exploitation of this observation following [23]. It is also an attempt to maintain the theme of the revival alluded to above by considering some older ideas in light of new techniques.

The older ideas we are referring to are primarily those surrounding Novikov's conjecture (1970), specifically G. Lustig's thesis [38] and the work of Mischenko, [48], [46], [45]. From now on $M$ will denote a closed oriented $n$-manifold (we will ignore general tangential structures, letting $MT(n) = MTSO(n)$).

Let $L = \Sigma\, L_i \in H^*(BSO; \mathbb{Q})$, $L_i \in H^{4i}(BSO; \mathbb{Q})$ be the total Hirzebruch L-class associated with the formal power series $\frac{\sqrt{x}}{tanh(\sqrt{x})}$ (see [43] chapter 19). When $n = 4k$ the signature $Sig(M)$ of the intersection form on $H^{2k}(M; \mathbb{R})$ may be computed using Hirzebruch's signature theorem:

$$Sig(M) = \langle L(M), [M] \rangle, \tag{1.1}$$

where $L(M) = L(TM) \in H^*(M;\mathbb{Q})$. $Sig(M) = 0$ for $n \neq 4k$. This may be taken as a definition or follows from the fact that $L_i$ are expressible as polynomials in the pontrygin classes $p_j \in H^{4j}(BSO)$.

Let $u : M \to B\pi$ be the map classifying of the universal cover of $M$, $\pi = \pi_1(M)$. Assume now $n = 2m$ is even. In [52] Novikov conjectured that for each $x \in H^*(B\pi;\mathbb{Q})$ the *higher signatures*

$$Sig_x(M) = \langle L(M) \cup u^*(x), [M] \rangle, \tag{1.2}$$

are invariants of the oriented homotopy type of $M$ (see section 4.2). Note that $Sig_x(M)$ may be nonzero for $n$ not necessarily divisible by 4. The terminology here likely refers to the fact that $Sig_x(M) = Sig(N)$ for an immersed $4k$-dimensional submanifold $N \subset M$. However, for certain $x \in H^*(B\pi;\mathbb{Q})$, the characteristic numbers $Sig_x(M)$ may also be interpreted as the signature of a quadratic form associated to $M$. For this interpretation it is convenient to introduce the class $\mathcal{L} = \Sigma\ \mathcal{L}_i \in H^*(BSO;\mathbb{Q})$, the multiplicative sequence in Pontrjagin classes associated with the formal power series $\frac{\sqrt{x}/2}{tanh(\sqrt{x}/2)}$. Note that the degree $4i$ parts in $H^*(BSO(2m))$ are related by

$$\mathcal{L}_i = 2^{-2i}\ L_i \in H^{4i}(BSO(2m))$$

.

Let $\rho : \pi \to Gl(V)$ be a representation of $\pi$ in a finite dimensional vector space $V$ with $\xi_\rho = V \times_\rho E\pi \to B\pi$ the associated vector bundle over $B\pi$. The vector bundle

$u^*(\xi_\rho)$ over $M$ is flat, so the cohomology $H^*(M; u^*(\xi_\rho))$ of $M$ with coefficients in $u^*(\xi_\rho)$ may be defined (see section 4.1). If $\rho$ preserve a given quadratic form on $V$ then one may consider the signature $Sig_\rho(M)$ of a quadratic form defined on $H^*(M; u^*(\xi_\rho))$ (see [49], [48], [46], [47]). These *twisted* signatures, may be computed using the equation (see [49] chapter 1, or [46] page 117)

$$Sig_\rho(M) = 2^m \langle \mathcal{L}(M) \cup u^*(ch([\xi_\rho]), [M] \rangle, \tag{1.3}$$

Note $x \in H^*(B\pi; \mathbb{Q})$ in equation 1.2 is not required to be homogeneous, so the signatures in equation 1.3 are themselves higher signatures.

Equation 1.3, often called the generalized Hirzebruch signature theorem, follows from an application of the Atiyah-Singer index theorem to the (even) signature operator on $M$ "twisted" by the flat bundle $u^*(\xi_\rho)$ ([48] section 5) . In [38] Lusztig verified certain special cases of Novikov's conjecture by applying the index theorem to families of signature operators acting on the fibers of an $M^{2m}$-bundle $E \to B$ twisted by a flat Hermitian bundle $V \to E$.

The *even signature operator* $D^e$ (defined only on even dimensional manifolds) has an odd counterpart, the *odd signature operator* $D^o$, first mentioned in [10]. In [23] the family index of the odd signature operator acting on the fibers of an $M^{2m-1}$-bundle $E \to B$ was shown to be zero. As a corollary ones obtains a proof that $Sig(E) = 0$ for $E$ the total space of an odd dimensional manifold bundle. The crucial observations made of $D^o$ ([23] page 12) still hold for $D^o_V$ (see chapter 5), the *odd signature operator twisted by a flat bundle Hermitian bundle* $V \to E$ as in [38]. Since the effect of twisting on the symbol class of an elliptic operator is well established and easy to describe ([38], [48], [3]), and the required application of the cohomological index formula was given in [23], one can, after the requisite background material is covered, quickly supply proofs generalizing the main results of [23].

We will prove the following

**Theorem 1.** *Suppose $f : E \to B$ is an oriented $M$-bundle, with $B$ and $M$ both oriented smooth closed manifolds. If the dimension of $M$ is odd then the higher signatures of $E$, $Sig_\rho(E) = 0$ for all finite dimensional Hermitian representations $\rho$ of $\pi_1(E)$. In other words, $\langle \mathcal{L}(E)ch([V]), [E]\rangle = 0$, for all flat Hermitian vector bundles $V \to E$.*

This result may not be a surprise to experts. As is the case when $V = 0$ ([23] corollary 1.4), there are likely several proofs of theorem 1 when $M = S^1$ is a circle (see [31], page 119, example (a)).

Integration over the fiber of $f : E^{n+k} \to B^k$, aka the gysin map $f_! : H^*(E) \to H^{*-n}(B)$, may be used to express characteristic numbers of $E$ as characteristic numbes of $B$:

$$\langle X, [E]\rangle = \langle f_!(X), [B]\rangle,$$

for any $X \in H^*(E)$.

Since the $\mathcal{L}$-class is multiplicative and $TE = f^*TB + T_vE$, where $T_vE = ker(df)$ is the vertical tangent bundle, for $X = \mathcal{L}(E)ch([V])$ this gives

$$\langle \mathcal{L}(E)ch([V]), [E]\rangle = \langle \mathcal{L}(f^*TB)\mathcal{L}(T_vE)ch([V]), [E]\rangle$$

$$= \langle \mathcal{L}(B)f_!(\mathcal{L}(T_vE)ch([V])), [B]\rangle.$$

Therefore theorem 1 is an immediate consequence of the following:

**Theorem 2.** *Suppose $f : E \to B$ is a smooth oriented $M$-bundle, $M$ a smooth closed* *manifold. If the dimension of $M$ is odd then $f_!(\mathcal{L}(T_vE)ch([V])) = 0 \in H^*(B)$ for all flat Hermitian vector bundles $V \to E$.*

Just as the special case ([23] theorem 1.2), the proposition is the cohomological consequence of the index of a certain family of elliptic operators vanishing:

**Theorem 3.** *Let $f : E \to B$ be an oriented $M$-bundle, $M$ a closed oriented Riemannian manifold of odd dimension, $V \to E$ a flat Hermitian vector bundle. The the family index of the odd signature operator twisted by $V$ is zero, $index(D^o_V) = 0 \in K^1(B)$*

The proof of theorem 3 is essentially functional analytic in nature. It relies on a supposedly well know fact in functional analysis which we state below as theorem 4. This theorem was proven in [23] (theorem 4.1) and applied ([23] theorem 1.1) to the odd signature operator $D^o$ without coefficients to show $index(D^o) = 0$. Theorem 3 follows from the fact that the crucial observations made of $D^o$ by Ebert in [23] are true more generally for $D^o_V$, the odd signature operator with coefficients in a flat bundle $V$. Thus theorem 4 can be applied to $D^o_V$ to draw the same conclusion, in exactly the same manner, as was done for $D^o$ in [23]. After the relevant family of operators has been defined, and shown to be suitable for application of theorem 4, the proof of theorem 1 and 2 is basically formal.

## 1.2 Outline

Chapter one of this dissertation consists of this introduction and outline. Chapters 2-4 recall the background necessary to economically state the proofs of theorems 2 and 3 in chapter 5. Chapter 6 provides some applications. We will now outline the remainder of the work:

**Section 2: Madsen-Tillman Spectra and Maps**

We begin by recalling some definitions from [23]. Most of these definitions come from basic stable homotopy theory [1] however [23] is the best reference in the present context. Afterwards we are able to construct the map $\alpha_E : \Sigma^\infty B_+ \to MT(d)$ mentioned above associated to an $M$-bundle $E \to B$. The utility of this map for us is that it provides a universal source of certain characteristic classes of smooth manifold bundles. Namely the Mumford-Miller-Morita (MMM) classes and index theoretic classes coming from families of elliptic operators can be viewed as pullbacks by $\alpha_E$ of universal classes in the rational cohomology and K-theory (respectively) of $MT(d)$.

**Section 3: Elliptic operators and Universal symbols**

This is by far our longest chapter. We begin with some preliminary definitions from analysis (section 3.1) and K-theory (section 3.2) and then (section 3.3) introduce the basic notions from the index theory of elliptic operators (originally developed by Atiyah and Singer [12] [13], [14]). After recalling the definitions for a single elliptic operator in some detail we quickly generalize to the situation of a family of elliptic operators $D = D_b$ acting on the fibers of a $M$-bundle $E \to B$ in section 3.3.1.

There are actually two types of index theorems: one for usual families of elliptic operators and one for families of self-adjoint elliptic operators. The former produces a class $index(D) \in K^0(B)$, the latter $index(D) \in K^1(B)$. For $D$ sufficiently natural these index theoretic classes can be viewed as coming from universal symbol classes $\sigma \in K^*(MT(n))$, that is $index(D) = \alpha_E^*(\sigma)$ (see section 3.3.2).

This universal perspective greatly simplifies our considerations. We consider the "twisted" version of the (odd) signature operator studied in [23]. The effect of this twisting procedure on the symbol class is easily described (section 3.4). As the signature operators are symbolically universal [23] the indices of their twisting are easily computed.

We end this chapter by recalling the definition of the even and odd signature operator and their cohomological index (proposition 1) which was computed in [23]. This computation is really what we need for the proof of theorem 2. We don't need to explicitly define *symbolically universal elliptic differential operator* (see [23] section 3.2 for precise definition) since $D^o_V$ is no longer symbolically universal for $V \neq 0$. All we really need to know is that the (untwisted) signature operators are symbolically universal (verified in [23]), the cohomological image of their symbol class (proposition 1), and the effect of twisting on the symbol class (described in section 3.4).

**Section 4: Flat Bundles and Higher Signatures**

The flatness of the vector bundle in proposition 1 is critical (i.e. it is false without). Since $K^*(X) \otimes \mathbb{Q} \cong H^*(X;\mathbb{Q})$ via $ch \otimes 1_{\mathbb{Q}}$, one need look no further than $E = S^1 \times S^1$ to see this.

Chapter 4 begins by recalling the definitions of a connection, flatness, and other constructions from elementary differential geometry. A connection $\nabla$ (definition 7) on a vector bundle $V \to M$ may be extended to an exterior derivative $d_\nabla : A^p(M;V) \to A^{p+1}(M;V)$ on $V$-valued differntial forms on $M$. The connection $\nabla$ is flat if $d^2_\nabla = 0$, in which case one may consider the cohomology of the complex $(d_\nabla, A^p(M;V))$. This is the cohomology (definition 9) of $M$ with coefficients in the flat vector bundle $V \to M$, denoted $H^*(M;V)$. The ability to define these finite dimensional vector spaces is really the only consequence of flatness we require. Old standards like [60], [36], [35], [34] or [51] should sufficient references for the material in section 4.1.

After the definitions and various interpretations of flatness we need are given we recall (section 4.2) some of the relevant history concerning Novikov's conjecture and the higher signatures mentioned in the introduction. To provide motivation and some historical context for the present work we conclude this section by summarizing investigations into (higher) signatures and signature operators associated to a fiber bundle. This discussion summarizes some of the main ideas from [4], [38], and [23] which are the primary motivation for this dissertation.

**Section 5: Proofs**

This section consists of the proof of theorem 3, from which we subsequently derive theorem 2. The main ingredient for the proof is theorem 4.1 from [23] which we recall in section 3.2 as theorem 4.

**Section 6: Applications**

We conclude with some applications and examples. We begin (section 6.1) with a corollary generalizing a multiplicative formula from [31] (in the spirit of [4]).

One interpretation of theorem 1 is that the higher signatures associated to flat Hermitian bundles are an obstruction to fibering as odd dimensional manifold bundle. We'll outline some examples from [31] showing these are non-trivial obstructions. Indeed there are surfaces whose higher signatures associated to flat Hermitian bundles are nonzero, thus theorem 1 provides a more general obstruction than the untwisted signature (when $V = 0$).

We conclude with some alternate statements (section 6.3) of our main results above and a discussion (section 6.4) of a few possible further questions.

## 1.3 Notation

Unless stated otherwise, the following notation will be fixed throughout. $M$ will always be a closed smooth oriented manifold, $Diff^+(M)$ the group of orientation preserving diffeomorphisms of $M$ endowed with the Whitney $C^\infty$-topology. We will study *smooth oriented $M$-bundles*, i.e. fiber bundles $f : E \to B$ with structure group $Diff^+(M)$. For simplicity we assume $B$ is a smooth manifold as well.

$\rho : G \to GL(V)$ will be a representation of a group $G$ in a vector space $V$, with $\xi_\rho$ the associated vector bundle over the classifying $BG$ (we'll use $\pi$ for discrete groups). Explicitly $\xi_\rho \cong EG \times_\rho V$. When $\pi = \pi_1(X)$ for a space $X$, $V_\rho = u^*(\xi_\rho)$ denotes the pullback of $\xi_\rho$ by the classifying map $u : X \to B\pi$, and $Sig_\rho(X) = \langle \mathcal{L}(X)ch(V_\rho), [X]\rangle$. In what follows $X$, $\pi$, etc. should be clearly determined by the context.

# Chapter 2

# Madsen-Tillman-Weiss Spectra and Maps

We begin by recalling some definitions from stable homotopy theory.

**Definition 1.** *A stable vector bundle $V$ on a connected space $X$ is a map*

$$\xi_V : X \to BO \times \mathbb{Z}.$$

*The rank of $V$ is given by projecting on the first factor.*

Note a non-stable vector bundle $V \to X$ of dimension $d \geq 0$ gives rise to a stable vector bundle via the inclusion $BO(d) \times \{d\} \subset BO \times \mathbb{Z}$. Stable vector bundles can be added and subtracted using the Whitney sum and inversion map respectively (see [42] chapter 24).

The *Thom space* of a vector bundle $V \to X$ is the space $Th(V) = X^V = BV/SV$, the quotient of the unit ball bundle $BV$ and sphere bundle $SV$ associated to $V$. The

*Thom spectrum* $\mathbb{T}h(W)$ of a stable vector bundle $W$ of rank $n$ is the spectrum whose $k$-th space is the Thom space $X_k^{W_k}$, where $X_k = \xi_W^{-1}(n\times BO(n+k))$ and $W_k = \xi_W^* \gamma_{n+k}$ is the pullback of the universal $n+k$-dimensional vector bundle $\gamma_{n+k} \to BO(n+k)$. The structure maps $\Sigma X_k^{W_k} \cong X_k^{\mathbb{R}\oplus W_k} \to X_k^{W_{k+1}|_{X_k}} \to X_{k+1}^{W_{k+1}}$ come from the isomorphism $W_{k+1}|_{X_k} \cong \mathbb{R} \oplus W_k$.

Note the Thom spectrum of a non-stable $V \to X$ of dimension $d \geq 0$ is just the suspension spectrum $\mathbb{T}h(V) = \Sigma^\infty Th(V)$ of the Thom space of $V$. In particular the Thom spectrum of the identity $X \to X$, viewed as a stable vector bundle of rank 0, is the suspension spectrum $\Sigma^\infty(X_+)$.

## 2.1 Orientations and Thom isomorphisms

Assume that $R$ is an associative commutative ring spectrum with unit (see [1] for definition). Let $V \to X$ be a stable vector bundle of rank $d$. The cohomology $R^*(\mathbb{T}h(V))$ is a graded left $R^*(X)$-module. A *Thom class* or *R-orientation* of $V$ with $R$ coefficients is a cohomolgy class $v \in R^d(\mathbb{T}h(V))$ such that for any $x \in X$, $v$ restricts to a generator of $R^d(\mathbb{T}h(V_x)) \cong R^d(\mathbb{S}^d) \cong R^0(*)$. The map

$$th_V^R : R^*(X) \to R^{*+d}(\mathbb{T}h(V))$$

given by $x \mapsto xv$ is an isomorphism, called the *Thom isomorphism*.

More generally, given another stable vector bundle $W$ and an $R$-orientation $v \in$

$R^d(Th(V))$ of $V$, there is a *relative Thom isomorphism*,

$$th^R_{W,W\oplus V} : R^*(\mathbb{T}h(W)) \to R^{*+d}(\mathbb{T}h(W \oplus V). \tag{2.1}$$

$th^R_{W,W\oplus V}$ is an isomorphism of $R^*(X)$-modules. See [23] page 5 and 6 for definitions in terms of homotopy classes of maps.

## 2.2 Madsen-Tillman map

Let $\gamma_n$ be the canonical $n$-plane bundle over $BSO(n)$. The *Madsen-Tillman-Weiss* spectrum $MT(n) = \mathbb{T}h(-\gamma_n)$ is the Thom spectrum of the stable vector bundle $-\gamma_d$, the additive inverse of $\gamma_n$.

Given a $M^n$-bundle $f : E \to B$ as above there are a few maps and relations we'd like to keep in mind. $T_vE = ker(df)$ will denote the *vertical tangent bundle* (note $f$ induces a splitting $TE \cong T_vE \oplus f^*TB$), $E^\nu = E^{\nu(f)} = \mathbb{T}h(-T_vE)$ the Thom spectrum of the *stable normal bundle* $\nu(f) = -T_vE$.

For $B$ paracompact we may choose a "fat" embedding $j : E \to B \times \mathbb{R}^\infty$ (i.e. $proj \circ j = f$ and the image of $j$ has a tubular neighborhood $U$). Moreover, the space of such embeddings is contractible ([23] page 7). The (parametrized) Pontrjagin-Thom (PT) collapse (collapsing everything outside $U$ to the basepoint) gives a well defined map of spectra

$$PT_f : \Sigma^\infty B_+ \to E^\nu.$$

**Definition 2.** *The umkehr homomorphism $f_! : H^*(E) \to H^{*-n}(B)$ may be defined* *as* $PT_f^*$ *precomposed with the Thom isomorphism* $th_{\nu(f)} : H^*(E) \cong H^{*-n}(E^\nu)$,

$$f_! = PT_f^* \circ th_{\nu(f)}.$$

For more details on the PT-construction in this parametrized setting see [29] section 3.

Providing $T_vE$ with an orientation determines a well defined map of spectra $\kappa_E : E^\nu \to MT(n)$ (see [23] pg. 7).

**Definition 3.** *The Madsen-Tillman map* $\alpha_E$ *of the bundle* $f : E \to B$ *is the composition*

$$\alpha_E := \kappa_E \circ PT_f : \Sigma^\infty B_+ \to MT(n).$$

Let $th_{-\gamma_n} : H^*(BSO(n)) \to H^{*-n}(MT(n))$ be the Thom isomorphism. The most important relationship among the maps just mentioned is the equation ([23] proposition 2.2)

$$\alpha_E^* \circ th_{-\gamma_n}(c) = f_!(c(T_vE)) \in H^{*-n}(B), \tag{2.2}$$

for all $c \in H^*(BSO(n))$. These are the *generalized Mumford-Morita-Miller classes.*

# Chapter 3

# The Index of Elliptic Operators and Universal Symbols

Given an elliptic operator $D$ on a smooth compact manifold $M$ (or more generally a family of such operators) one may consider two quantities: its analytic index and its topological index. Not surprisingly, the former is defined in analytic terms, the latter using topological data associated to $D$. Put most simply, the Atiyah-Singer index theorem states that the analytic index of an elliptic operator coincides with the topologically defined one. This statement however is not very enlightening to the uninitiated and for this reason we've included more detail in this section than we really need.

As it is the relatively easy to describe, we begin (section 3.1) by introducing the analytic index in the abstract setting of Fredholm operators. Afterwards (section 3.2) we pause for some clarifying comments regarding complex K-theory before discussing elliptic operators explicitly. There are several equivalent models for K-theory. We will explain the ones we'll be using and the relations between them. The standard reference here is [2]. The relative Thom isomorphism mentioned above (combined with Bott periodicity) will be used here to identify $K^*(Th(-V)) \cong K^*(Th(V))$ for a vector bundle $V$ (specifically the vertical tangent bundle).

The basic elements of the index theory of elliptic operators are then outlined (section 3.3) and the twisting process (outlined for a trivial $M$-bundle in [3]) is then described (section 3.4). A good reference for the technical details absent in section 3.3 is [53]. For a more gentle introduction see [56].

We end this chapter by describing universal symbols (as developed in [23]) and defining the symbolically universal operators we will be interested, namely the even and odd signature operators.

## 3.1 Analytic Index

Let $\mathcal{H}$ be a complex separable infinite dimensional Hilbert space (note any complex separable infinite dimensional Hilbert space is isomorphic to $\mathcal{H}$).

**Definition 4.** *A linear operator $F : \mathcal{H} \to \mathcal{H}$ is called Fredhom if its kernel and cokernel are finite dimensional.*

The *(Fredholm) index* of a Fredholm operator $F$ is the difference

$$index(F) = dim\ ker(F) - dim\ coker(F).$$

$\mathcal{F} = \mathcal{F}(\mathcal{H})$ will denote the space of Fredholm operators with the norm topology, $\mathcal{F}_{s.a.}$ those Fredholm operators which are also self adjoint. The key property (first observed by I. Gelfand) of the Fredholm index, which endears it to many topologist, is its invariance under perturbation. In particular (see [5]) the index defines a continuous map

$$index : \mathcal{F} \to \mathbb{Z}$$

identifying the connected components of $\mathcal{F}$,

$$\pi_0(\mathcal{F}) \cong \mathbb{Z}$$

.

## 3.2 K-theory

For a compact space X, the complex topological K-theory $K(X) = K^0(X)$ is usually defined ([2])as the Grothendieck group generated by isomorphism classes $[V]$ of complex vector bundles $V \to X$ over $X$. The reduced K-theory $\tilde{K}(X)$ of a pointed space $(X, x_0)$ can be defined as the kernel of the map $K(X) \to K(x_0) \cong \mathbb{Z}$ induced by the inclusion of $x_0$ into $X$.

For $n \geq 0$ one can then define

$$\tilde{K}^{-n}(X) := \tilde{K}(\Sigma^n X),$$

where $\Sigma^n X$ is the $n$th suspension on $X$. The unreduced version $K^{-n}(X)$ may be defined as $\tilde{K}^{-n}(X_+)$, where $X_+$ is $X$ with a disjont basepoint.

These K-groups can be defined for any $X$ with the homotopy type of a CW complex by using representable K-theory:

$$K^0(X) = [X_+, \mathbb{Z} \times BU], \quad K^{-1}(X) = [X_+, U]$$

where $BU$ is the classifying space of the infinite unitary group $U = U(\infty)$. In [15] it was shown that there are homotopy equivalences

$$BU \times \mathbb{Z} \simeq \mathcal{F} \ \ and \ \ U \simeq \mathcal{F}_{s.a.}$$

and thus

$$K^0(X) = [X, \mathcal{F}] \ \ and \ \ K^1(X) = [X, \mathcal{F}_{s.a}].$$

This allows for the study of elements in K-theory using operator theoretic techniques. For example, and crucial to our results, is the following theorem.

**Theorem 4.** *Let $B$ be a space and let $A : B \to \mathcal{F}_{s.a.}$ be a continuous map such that $b \mapsto dim\ Ker\ A(b)$ is locally constant for all $b \in B$. Then $A$ is homotopic to a constant map.*

Apparently theorem 4 is well known to experts in operator theory; see [23] theorem 4.1 for proof.

Bott periodicity $\beta : \Omega^2(\mathbb{Z} \times BU) \simeq \mathbb{Z} \times BU$ allows $K^n(X) \cong K^{n-2}(X)$ to be defined for all $n$. This produces a 2-periodic generalized cohomology theory represented by the $\Omega$-spectrum $KU$ with $KU_{2i} = \mathbb{Z} \times BU$ and $KU_{2i+1} = U$ for all $i$. The structure maps are given by the canonical homotopy equivalence $U \simeq \Omega BU = \Omega(\mathbb{Z} \times BU)$ and the Bott equivalence $\mathbb{Z} \times BU \simeq \Omega U$.

The K-theory of a paracompact space may also be defined as the reduced K-theory of its one point compactification. For $V \to X$ a vector bundle this means $K^*(V) \cong \tilde{K}^*(\mathbb{T}h(V))$. Compared to ordinary homology with coefficients in $\mathbb{Z}_2$ or $\mathbb{Q}$, complex K-theory is slightly more picky about which vector bundles admit a Thom isomorphism. A K-orientation of a real vector bundle is equivalent to a reduction of its structure group to $spin^{\mathbb{C}}$. In particular every complex vector bundle is K-oriented (with its Thom class given by the alternating sum of its exterior powers [2]). If $V \to X$ is a real rank $d$ vector bundle then $V \otimes \mathbb{C}$ has a $K$-orientation. Thus we may use the relative Thom isomorphism

$$th^{KU}_{-V,V\otimes\mathbb{C}} : K^*(\mathbb{T}h(-V)) \cong K^*(\mathbb{T}h(-V \oplus V \otimes \mathbb{C}) \cong K^{*+2d}(\mathbb{T}h(V))$$

plus Bott periodicity $\beta : K^* \cong K^{*+2}$ to identify $K^*(\mathbb{T}h(-V)$ and $K^*(\mathbb{T}h(V))$ as $K^*(X)$-modules.

## 3.3 Elliptic Operators

Let $M$ be a smooth compact manifold, $V$, $W$ two smooth complex vector bundles over $M$.

We're considering linear operators $D : \Gamma(V) \to \Gamma(W)$ acting between the spaces $\Gamma$ of smooth sections and expressible locally in terms of matrices of partial derivatives. More precisely, a *differential operator $D$ (of order $k$)* from $V$ to $W$ is a linear map $D : \Gamma(V) \to \Gamma(W)$ which, in terms of local coordinates $x = (x_1, ..., x_n)$, is of the form

$$\Sigma_{|\alpha|\leq k} a_\alpha(x) \partial_x^\alpha,$$

where for each multi-index $\alpha = (\alpha_1, ..., \alpha_n)$ with $|\alpha| = \Sigma\alpha_i$, $a_\alpha$ is an $N \times N$-matrix valued function and $\partial_\alpha = \partial^{\alpha_1}/\partial x_1, ..., \partial^{\alpha_n}/\partial x_n$ denotes the partial derivative.

$D$ is *elliptic* if for each local representation, the highest order term

$$p_k(x, \xi) = \Sigma_{|\alpha|=k} a_\alpha(x)\xi^\alpha$$

is a nonsingular matrix for all $x$ and all $\xi \in \mathbb{R}^n - \{0\}$. The system of functions $p_k(x, \xi)$, given for every local representation of $D$, is called the *leading symbol* of $D$.

The leading symbol of an elliptic operator has an intrinsic geometrical interpretation. Let $\pi : SM \to M$ be the unit sphere bundle of $M$ inside the cotangent bundle $T^*M$ (identified with $TM$ under a fixed metric). The leading symbol of $D$ determines a well defined vector bundle isomorphism

$$\gamma_D : \pi^*V \to \pi^*W,$$

given locally by replacing $\partial/\partial x_j$ with $i\xi_j$ ($\xi_j$ is the jth coordinate in the cotangent bundle).

Two complex vector bundles $V$, $W$ on a space X with an isomorphism $\gamma$ over a subspace $X_0 \subset X$ define a difference element $d(V, W, \sigma) \in K^0(X, X_0)$ ([8], [16]). Hence, if $BM \subset T^*M$ denotes the unit ball bundle, an elliptic operator $D$ defines an element

$$\sigma_D = d(\pi^*V, \pi^*W, \gamma_D) \in K^0(BM, SM) \cong \tilde{K}^0(BM/SM) \cong \tilde{K}^0(Th(TM))$$

called the *symbol class of* $D$. Equivalently one may view the triple $(\pi^*V, \pi^*W, \gamma_D)$ as determining a compactly supported elliptic complex over $T^*M$ (after extending $\gamma_D$ over all of $T^*M$) representing a $K$-cycle $\sigma_D \in K^0(T^*M) \cong K^0(TM) \cong \tilde{K}^0(Th(TM))$. For our purposes it will be more convenient to identify $\sigma_D$ with its image in $\tilde{K}^0(\mathbb{T}h(-TM)$ under the relative Thom isomorphism from section 3.2.

One of the basic properties of elliptic operators is that both $ker(D)$ and $coker(D)$ are finite dimensional. The *analytic index* of $D$ is defined as the difference of these two quantities:

$$index(D) = dim\ ker(D) - dim\ coker(D) \in \mathbb{Z} \cong K^0(pt).$$

The index problem, solved by Atiyah and Singer, is to express $index(D)$ in terms of the symbol class $\sigma_D$.

**Theorem 5.** *(Atiyah-Singer [13]) Let* $D : \Gamma(V) \to \Gamma(W)$ *be an elliptic operator on* $M$ *with symbol class* $\sigma \in K^0(\mathbb{T}h(-TM))$. *Then,*

$$index(D) = PT_f^*(\sigma_D) \in K^0(pt) \cong \mathbb{Z},$$

*where the Thom-Pontryjgin map* $PT_f$ *is applied to* $M$ *viewed as a* $M$*-bundle* $f : M \to pt$ *over a point.*

Properly interpreted $index(D)$ is the same as the Fredholm index from the previous section. However we have said nothing of the Hilbert space structure on the spaces of sections between which an elliptic operator acts. In short, an elliptic operator $D$ produces a Fredholm operator acting between certain spaces of smooth sections endowed with a Hilbert space structure whose Fredholm index equals $index(D)$. A precise discussion of this process would require the introduction of psuedo-differential operators, notions of regularity, and other analytic topics which won't be needed for the rest of our work, so we have only provided an outline here. Details can be found several places in the literature, see [30] chapter 2, [53], or [13] for example. Strictly speaking the only observation needed later for the proof is that, in the event $D$ is formally self-adjoint, the associate (self adjoint) Fredholm operator has the same kernel and eigenspaces as those of $D$ (see [23] page 12).

Most constructions from elementary functional analysis generalize to spaces of sections of vector bundles. In particular the space of $L^2$ integrable functions and Sobolev spaces $\mathcal{W}^s = W^{2,s}$ (consisting of functions whose partial derivative up to order $s$ are $L^2$). Assume the vector bundle $V \to M$ has been given a Hermitian structure, let $( \ , \ )_x$ denote the hermitian inner product on the fiber $V_x$, $x \in M$. Given a metric on $TM$ define

$$\langle \psi, \phi \rangle = \int_M (\psi(x), \phi(x))_x dvol,$$

$\psi$, $\phi \in \Gamma(V)$. Completing $\Gamma(V)$ with respect to the norm induced by $\langle \ , \ \rangle$ defines the space of $L^2$ sections $L^2(V)$ of $V$. Given a connection on $V$ (i.e a notion a deriving a section in $\Gamma(V)$) we can define $L^2$-Sobolev norms $||...||_s$ on $\Gamma(V)$ for all $s \geq 0$. The completion with respect to $||...||_s$, denoted $\mathcal{W}^s(V)$, is a Hilbert space, and an elliptic operator $D : \Gamma(V) \to \Gamma(W)$ of order $k$ as above extends to a Fredholm operator $D : \mathcal{W}^s(V) \to \mathcal{W}^{s-k}(V)$ whose index (independent of $s$) equals $index(D)$.

### 3.3.1 Family Index

The ideas from the previous section can be parametrized by considering families of elliptic operators acting on the fibers of an $M$-bundle $f : E \to B$. Here, we assume a fiberwise smooth Riemannian metric on the vertical tangent bundle $T_vE$ is chosen and that $B$ is paracompact. All vector bundles on $E$ will be fiberwise smooth (i.e. the transition functions are smooth in the fiber direction) and all hermitian metrics are understood to be smooth.

Let $M_b = f^{-1}(b)$ denote the fiber above $b \in B$, with $V_b = V|_{M_b}$ the restriction of a Hermitian vector bundle $V \to E$ to the fiber $M_b \cong M$. All differential operations mentioned in the previous section can be done fiberwise. Given a connection on $V$, each space $\Gamma(V_b)$ can be completed to form the Hilbert space $\mathcal{W}^s(V_b)$ as above, and the family of vector spaces $\Gamma_B(V) = \bigcup_{b\in B} \Gamma(V_b)$ may be completed to form a Hilbert bundle $\mathcal{W}^s_B(V) \to B$ over $B$.

These Hilbert bundles $\mathcal{W}^s_B(V)$ are in fact trivial and the trivialization is unique up to homotopy. There are some technical issues regarding the structure group of $\mathcal{W}^s_B(V)$. This is covered in some detail in [23] while the original discussion and solution to this problem can be found in [16]. The most important takeaway for us is that the homotopy class of the trivialization is unique.

Let $V_0, V_1 \to E$ be two hermitian vector bundles. By a *family of elliptic operators (or vertical elliptic operator)* we mean a collection $D = \{D_b\}_{b\in B}$ paramatrized by $b \in B$ where each $D_b : \Gamma((V_0)_b) \to \Gamma((V_1)_b)$ is an elliptic operator on the fiber $M_b \cong M$. Roughly speaking, the *analytic index of a family* of elliptic operators

$D = \{D_b\}_{b\in B}$ is the continuous map $index(D) : B \to \mathcal{F}$, defined by $b \mapsto D_b$, with $D_b : \mathcal{W}^s((V_0)_b) \to \mathcal{W}^s((V_1)_b)$ the extensions of $D_b$ to the Sobolev sections as above. $D$ actually gives rise to a bundle map $D : \mathcal{W}^s_B(V_0) \to \mathcal{W}^s_B(V_1)$ (hence $index(D)$ is continuous). That $index(D)$ is well defined comes from the uniqueness of the trivializations (up to homotopy) of $\mathcal{W}^s_B(V_i)$.

Now $index(D) \in K^0(B) \cong [B, \mathcal{F}]$, with the case of a single operator corresponding to $K^0(B) \cong \mathbb{Z}$ when $B$ is a point.

There are actually two types of K-theoretic index theorems: one for families of usual elliptic operators and one for families of self adjoint elliptic operators on the fibers of an oriented $M$-bundle $f : E \to B$. In the former case, covered above, the index is an element of $K^0(B) \cong [B, \mathcal{F}]$, while in the latter we get an index in $K^1(B) \cong [B, \mathcal{F}_{s.a.}]$. If $D = \{D_b\}_{b\in B}$ is a family of elliptic operators with each $D_b$ self-adjoint then, in the same way as for ordinary elliptic operators, we get a continuous map $B \to \mathcal{F}_{s.a.}$, i.e. an element $index(D) \in K^1(B)$.

For an elliptic family $D = \{D_b\}_{b\in B}$ the symbol classes $\sigma_{D_b} \in K^0(\mathbb{T}h(-TE_b)$ of each $D_b$ assemble to a symbol class $\sigma_D \in K^0(E^\nu)$ ($\sigma_D \in K^1(E^\nu)$ for $D$ self-adjoint). The Atiyah-Singer family index theorem expresses the index of $D$ in terms of its symbol class ([23] chapter 3) :

**Theorem 6.** *Let $D = \{D_b\}_{b\in B}$ be a family of (self-adjoint) elliptic operators with symbol class $\sigma_D \in K^0(E^\nu)$ ($\sigma_D \in K^1(E^\nu)$ for $D$ self-adjoint). Then*

$$index(D) = PT^*_f(\sigma_D) \in K^i(B).$$

### 3.3.2 Universal Symbols

As mentioned in the introduction, the index theoretic classes just introduced may be viewed as coming from $MT(n)$. More precisely, for any sufficiently natural elliptic differential operator $D$, there exist a universal symbol class $\sigma_D \in K^i(MT(n))$ such that

$$\sigma_{D_E} = \kappa_E^*(\sigma_D) \in K^i(E^\nu)$$

and

$$index(D_E) = \alpha_E^*(\sigma_D) \in K^i(B),\ i = 0, 1$$

where $D_E$ is the associated family of operators on the fibers of the bundle $f : E \to B$ of $n$-manifolds. The notion of universal symbols is an old one; a version is at least given in [58]. The full definition, in the form relevant to us, is given in [23], however for our purposes the above equations are the only relationships we need.

We'll be considering two universal symbol classes, $\sigma_{2m} \in K^0(MT(2m))$ and $\sigma_{2m-1} \in K^1(MT(2m-1))$ corresponding the even and odd signature operators (section 3.5). That these operators are universal is verified in [23].

## 3.4 Twistings

Let $D$ be an elliptic operator on a manifold $M$ with symbol class $\sigma_D \in K^*(\mathbb{T}h(-TM))$. $K^*(\mathbb{T}h(-TM))$ is a $K^0(M)$-module and the element $\sigma_D \otimes x := \sigma_D \cdot x \in K^*(\mathbb{T}h(-TM))$ again corresponds to the symbol class of an elliptic operator (really all elements of $K^*(\mathbb{T}h(-TM))$ can be realized as the symbol class of an elliptic operator on $M$) . When $x = [W]$ this operator will be denoted $D_W$. Several names have been used historically to refer to $D_W$. We'll stick with the following terminology:

**Definition 5.** *Let $V_0, V_1, W$ be complex vector bundles over a manifold $M$, $D : \Gamma(V_0) \to \Gamma(V_1)$ an elliptic operator. The elliptic operator $D_W := D \otimes Id_W : D : \Gamma(V_0 \otimes W) \to \Gamma(V_1 \otimes W)$ will be referred to as "D twisted by W". The symbol class of $D_W$ is given by $\sigma_{D_W} = \sigma_D \otimes [W] \in K^*(\mathbb{T}h(-TM))$.*

Of course this notion of twisting carries over to the more general situation of families of elliptic operators. This process is described in [3] for trivial families (i.e. $E = B \times M$). For an arbitrary oriented $M$-bundle $f : E \to B$ this goes as follows:

**Definition 6.** *Let $V_0, V_1, W$ be complex vector bundles on the $M$-bundle $E$, $D : \Gamma_B(V_0) \to \Gamma_B(V_1)$ a family of elliptic operators. The elliptic family $D_W := D \otimes Id_W : D : \Gamma_B(V_0 \otimes W) \to \Gamma(V_1 \otimes W)$ will be referred to as "D twisted by W". The symbol class of $D_W$ is given by $\sigma_{D_W} = \sigma_D \otimes [W] \in K^i(-T_vE) \cong \tilde{K}^i(E^\nu)$.*

Note that here $\otimes$ denotes module multiplication.

The effect of this twisting process on the (family) index can be expressed homotopically quite succinctly using universal symbols. If the symbol of $D$ is pulled back from a universal class $\sigma \in K^i(MT(d)$, that is $\sigma_D = \kappa_E^*(\sigma) \in K^i(E^\nu)$, then the index of $D_W$ is given by

$$index(D_W) = PT_f^*(\sigma_D \otimes [W]) = PT_f^*(\kappa_E^*(\sigma) \otimes [W]) \in K^i(B)$$

## 3.5 Signature operators

Let $M$ be a closed oriented Riemannian manifold dimension of $n$ with $d : A^k(M) \to A^{k+1}(M)$ its exterior derivative on complex valued differential forms $A^k(M) :=$

$\Gamma(\Lambda^k(T^*M) \otimes \mathbb{C})$. There are two symbolically universal elliptic operators we need to consider here, the even and odd signature operator on an oriented Riemannian manifold of even and odd dimension (respectively).

Recall that there is the Hodge star operator $\star : A^k(M) \to A^{n-k}(M)$ defined on complex valued differential forms. The adjoint $d^* : A^k(M) \to A^{k-1}(M)$ of the exerioir derivative d can be written $d^* = (-1)^{n(n+k)+1} \star d \star$

When $n = 2m$ is even the *even signature operator* $D^e = d + d^* : A^*_+(M) \to A^*_-(M)$ acts between certain sections of differential forms. $D^e$ is an elliptic differential operator whose index is the same as the signature $Sig(M)$ of $M$.

When $n = 2m-1$ is odd there is the *odd signature operator* $D^o = i^m(-1)^{k+1}(\star d - d\star) : A^{ev}(M) \to A^{ev}(M)$ on forms of even degree $A^{ev}(M) = \bigoplus_k A^{2k}(M)$. In [23] Ebert made the following observations:

- $D^o$ is formally self-adjoint
- $D^2 = \Delta = (d + d^*)^2$, the Laplace-Beltrami operator.

It follows from Hodge theory (see section 4.1) that

$$dim(ker(D^o)) = dim(ker(\Delta)) = \sum_p dim H^{2p}(M; \mathbb{C})$$

These observations, generalized appropriately, can also be made of $D^o_W$, for $W$ a flat Hermitian vector bundle (see chapter 5).

Both the even and odd signature operator are symbolically universal elliptic operators with universal symbols $\sigma_{2m} \in K^0(MT(2m))$ and $\sigma_{2m-1} \in K^1(MT(2m-1))$ respectively.

For our purposes a precise description of $\sigma_{2m}$ and $\sigma_{2m-1}$ is not important. What we really need is their image of these symbol classes under the Chern character $ch : K \to H\mathbb{Q}$. This was computed in [23] and amounts to applying the cohomological index formula of [14]. We state the result below (proposition 1) for use in section 5 in the proof of theorem 2.

Let $\eta : MT(n) \to \Sigma MT(n+1)$ be the map of spectra induced by the obvious bundle isomorphism $\gamma_{n+1}|_{BSO(n)} \cong \gamma_n \oplus \mathbb{R}$. Let $\tilde{\mathcal{L}} = \Sigma\tilde{\mathcal{L}} \in H^*(BSO;\mathbb{Q})$ be the sequence in the Pontrjagin classes associated with the formal power series $\frac{\sqrt{x}}{tanh(\sqrt{x}/2)}$ (see [43] chapter 19). Recall the $\mathcal{L}$-class appearing in the introduction is associated with the formal power series $\frac{\sqrt{x}/2}{tanh(\sqrt{x}/2)}$. Note the degree $4i$ parts in $H^*(BSO(2m))$ are related by

$$\tilde{\mathcal{L}}_{4i} = 2^m \mathcal{L}_{4i} \in H^{4k}(BSO(2m))$$

**Proposition 1.** *([23] proposition 4.2)*

- *The image of $\sigma_{2m} \in K^0(MT(2m))$ under the restriction homomorphism $\eta^* : K^0(MT(2m)) \to K^0(\Sigma^{-1}MT(2m-1)) \cong K^1(MT(2m-1))$ coicides with $2\sigma_{2m-1}$*
- *$ch(\sigma_{2m}) = \tilde{\mathcal{L}} \in H^{*-2m}(MT(2m);\mathbb{Q})$.*

*Here $\tilde{\mathcal{L}} \in H^{*-n}(MT(n);\mathbb{Q})$ is identified with (the restriction of) $\tilde{\mathcal{L}} \in H^*(BSO(n))$ under the Thom isomorphism $H^*(BSO(n)) \cong H^{*-n}(MT(n);\mathbb{Q})$*

# Chapter 4

# Flat bundles and Higher Signatures

For convenience we begin this chapter by reminding the reader of a few definitions from differential geometry. Here $C^\infty(M)$ denotes the ring of smooth complex valued functions on $M$. The remarks regarding the twisted cohomolgy $H^*(M; V)$ at the end of section 4.1 are really the only observations we need for the proof. We then give a brief introduction to Novikov's conjecture and higher signatures and provide a summary of the ideas from [4], [38], [46], [10], and [23] that were the primary motivation for this dissertation.

## 4.1 Flat bundles

**Definition 7.** *Let $V \to M$ be a (smooth) complex vector bundle, $X \in \Gamma(TM)$ a smooth vector field, $Y \in \Gamma(V)$, $f \in C^\infty(M)$. A connection on $V$ is a $\mathbb{C}$-linear map*

$$\nabla : \Gamma(TM) \otimes \Gamma(V) \to \Gamma(V)$$

*such that*

- $\nabla_{fX} Y = fY$
- $\nabla_X(fY) = f\nabla_X Y + (Xf)Y$.

Dually, we may view $\nabla$ as

$$d_\nabla : \Gamma(V) = A^0(M;V) \to \Gamma(T^*M) \otimes \Gamma(V) \cong \Gamma(T^*M \otimes V) = A^1(M;V),$$

where $A^p(M;V) := \Gamma(\Lambda^p(TM) \otimes V)$ is the vector space of $p$-forms on $M$ with coefficients in (or twisted by) $V$. This may be extended (uniquely) to an operator

$$d_\nabla : A^p(M;V) \to A^{p+1}(M;V), \tag{4.1}$$

the *covariant exterior derivative* associated to the connection $\nabla$.

There are several equivalent definitions of a flat vector bundle in common use.

**Definition 8.** *Let $V \to M$ be a (smooth) complex vector bundle with connection $\nabla$. We say $V$ is flat if $d_\nabla^2 = 0$.*

Note $d_\nabla^2$ is not always zero. In general, for $\sigma \in A^p(M;V)$,

$$d_\nabla^2(\sigma) = K_\nabla \wedge \sigma,$$

where $K_\nabla \in A^2(M; End(V))$ is the *curvature* of $\nabla$. Thus we see the familiar characterization of a flat connection: one for which the curvature form vanishes identically.

If a flat connection $\nabla$ is given on $V \to M$ then we are able to consider the cohomology of the complex $(A^*(M;V), d_\nabla)$:

**Definition 9.** *Let $V \to M$ be a complex vector bundle with flat connection $\nabla$. The cohomology of the $(A^*(M;V), d_\nabla)$, denoted $H^*(M;V)$, is called the cohomology of $M$ with coefficients in (or twisted by) $V$.*

$d_\nabla$ is a first order differential operator and $(A^*(M;V), d_\nabla)$ is an *elliptic complex* (see [6], [7]). It follows from the index theorem applied to the elliptic complex $(A^*(M;V), d_\nabla)$ that the vector spaces $H^*(M;V)$ are finite dimensional. The Hodge theorem, as most people are familiar with it, says that every de Rham cohomology class of $M$ has a unique harmonic representative (recall a form $\omega \in A^k(M)$ is harmonic if $\Delta\omega = 0$ where $\Delta = (d + d^*)^2$ is the Laplace-Beltrami operator). Hodge theory extends to the setting of elliptic complexes and in this context the Hodge theorem states that every class in $H^*(M;V)$ has a unique harmonic representative, that is $ker(\Delta_\nabla) \cong H^*(M;V)$, where $\Delta_\nabla = (d_\nabla + d_\nabla^*)$ is the Laplacian associated to $d_\nabla$. These observations will be crucial later for the proof in section 5.

Definition 8 above is a differential characterization of flatness. There is another more general topological definition which will be useful for our discussion. Let $X$ be a CW complex, $G$ a topological group, $u : X \to B\pi_1(X)$ the map classifying the universal cover of $X$

**Definition 10.** *A vector bundle $V \to X$ with transition functions taking values in $G$ is flat if $V \cong u^*\xi_\rho$, where $\xi_\rho \to B\pi_1(X)$ is the vector bundle associated to a representation $\rho : \pi_1(X) \to G$.*

In the differentiable case the bundle $u^*\xi_\rho$ admits a connection with vanishing curvature, i.e. definitions 8 and 10 coincide. See [33] for proof.

## 4.2 Higher Signatures and Novikovs conjecture

Novikov's conjecture came out of investigations into the following question (see [31] in particular chapter 7 and 8 for thorough discussion). Can one change the tangent bundle $TM$ of a manifold $M$ by modifying its smooth structure while keeping the homotopy type of $M$ intact? If "yes", in how many ways? It has been known for some time that the answer is no for spheres $S^n$ for all $n$ and for surfaces (see [31] page 87). However, as the dimension of $M$ increases there appear too many different possibilities for $TM$ to be contained by the homotopy type of M.

For example ([31] page 87) one can show there are infinitely many manifolds $M_1,\ M_2,\ \ldots$ all homotopy equivalent to $S^2 \times S^4$, but with different tangent bundles, distinguished by their first Pontryagin classes $p_1(TM_i) \in H^4(S^2 \times S^4) \cong \mathbb{Z}$. In general, if one wants to distinguish variations of $TM$ within the homotopy type of $M$, knowledge of which characteristic classes and characteristic numbers of $M$ are invariant under homotopy equivalence becomes desirable.

Let $M^{2m}$ be a closed oriented smooth manifold with $u : M \to B\pi = B\pi_1(M)$ the map classifying the universal cover of M.

In [52] Novikov conjectured that the *higher signatures*

$$Sig_x(M) = \langle L(M) \cup u^*(x), [M] \rangle,$$

$x \in H^*(B\pi, \mathbb{Q})$ are oriented homotopy invariants. Here *oriented homotopy invariant* means

$$\langle L(M)u^*(x), [M] \rangle = \langle L(M')h^*u^*(x), [M'] \rangle,$$

given any orientation preserving homotopy equivalence of $2k$-dimensional manifolds $h : M' \to M$.

Novikov's conjecture has generated considerable interest among more homotopically minded geometers and analysts. There exist many translations of the conjecture and its special cases, the above being the one that works for us. So much literature, consisting of a particularly diverse range of topics and perspectives, already exist related to the conjecture that we make no attempt to provide any real survey here. See [26] for a history and survey of Novikov's conjecture ([55] and [54] as well). The website https://www.math.umd.edu/ jmr/NC.html can provide further references for the interested reader.

One may approach Novikov's conjecture by systematically searching for homomorphism $\Omega_*^{SO}(B\pi) \to \mathbb{Q}$, where $\Omega_*^{SO}$ is the group of oriented bordisms ([58]). By definition, such homomorphisms will produce homotopy (as well as bordism) invariants of closed oriented manifolds. This approach gained quite a bit of traction during the latter quarter of the last century ([48], [46], [45], [47], [44], [31], [57] and others). Specifically, one may define a homomorphism $\tau_\rho : \Omega_*^{SO}(B\pi) \to \mathbb{Z}$ induced by some kind of representation theoretic data $\rho$ associated to $\pi$. For us, $\rho$ is a finite dimensional Hermitian representation of $\pi$, in which case

$$\tau_\rho([u : M \to B\pi]) = 2^{-m} Sig_\rho(M)$$

from the introduction, where $\xi_\rho \to B\pi$ is the vector bundle induced by $\rho$.

### 4.2.1 Twisted Signature Operators

The (untwisted) signature $Sig(M)$ is usually defined as the signature of the intersection form on the cohomology of $M^{2k}$. Hirzebruch's signature theorem provides a means of computing $Sig(M)$ in terms of characteristic numbers:

$$Sig(M) =< L(M), [M] > .$$

Hirzebruch's theorem is really a special case of the Atiyah-Singer index theorem applied to the even signature operator defined above,

$$index(D^e) = Sig(M).$$

Recall that $D^e$ is defined in terms of the de Rham operator (exterior derivative), $D^e = d + d^*$. If $V \to M$ is a complex Hermitian vector bundle with flat connection connection $\nabla$ then $D^e$ can be twisted by $V$ to form the *even signature operator with coefficients in* $V$. Explicitly, $D^e_V = d_\nabla + d^*_\nabla$. Using the metric on $V$ one can define a quadratic form on $H^*(M; V)$ which generalizes the intersection form and an application of the index theorem shows the signature of this quadratic form coincides with the index of $D^e_V$ and

$$index(D^e_V) = 2^k < \mathcal{L}(M)ch([V]), [M] >= Sig_\rho(M),$$

(see [48] section 5), where $V \cong V_\rho$ for a representation $\rho$ of $\pi_1(M)$.

In [4] Atiyah studied the signature of a fibered manifold. By considering the family index of the (even) signature operator he showed that, for an $M$-bundle $E \to B$, the signature of $E$ coincides with a higher signature of $B$:

$$Sig(E) = Sig_\rho(B),$$

where $\rho$ is a representation of $\pi_1(B)$ on the cohomology of the fibers of $E$ (see the general remarks at the end of [4] for details).

In [38] Lusztig used the family index of the (even) signature operator acting on the fibers of an $M$-bundle $E \to B$ twisted by a flat Hermitian vector bundle $V \to E$ to prove certain special cases of Novikov's conjecture. Here the Hermitian form on

$V$ is possibly indefinite, i.e. $V$ has structure group $U(p,q) = GL(\mathbb{C}^{p,q})$, the group of isometries of $\mathbb{C}^{p+q}$ endowed with the Hermitian $(p,q)$-form $\Sigma_{i=1}^{p} z_i\overline{z}_i - \Sigma_{j=1}^{q} z_j\overline{z}_j$. In other words, since $V$ is flat, $V \cong V_\rho$ for a representation $\rho : \pi_1(E) \to U(p,q)$

Recall that $U(p) \times U(q)$ is the maximal compact subgroup of $U(p,q)$. Restricting to this maximal subgroup induces a unique ($U(p,q)/U(p) \times U(q)$ is contracible) splitting $V = V_0 \oplus V_1$ wih the given hermitian from on $V$ restricing to a positive/negative definite form on $V_{0/1}$ respectively. Changing the sign of the Hermitian form on the summand $V_1$ provides $V$ with a positive definite Hermitian form and $[V] = [V_0] - [V_1] \in K^0(E)$ (see [38] section 2.2, [44] chapter 3, or [46]). Note $V_i$ are not necessarily flat as the splitting has nothing to do with the flat structure. In chapter 5 of [38] Lusztig provides a large class of manifolds for which $ch([V_0]-[V_1])$ is nontrivial in positive dimensions (the observant reader may have noticed ch([V])=N for $V$ flat with structure group $U(N)$, see chapter 6). We recall these examples below in chapter 6 to show that theorem 3 provides a nontrivial generalization of the main results of [23].

The odd signature operator $D^o$ was introduced in [10] in order to study the signature of manifolds with boundary. In [23] Ebert proved that the family index of the odd signature operator on a $M$-bundle $E \to B$ ($M^{2m-1}$ odd dimensional) is zero:

$$index(D^o) = \alpha_E^*(\sigma_{2m-1}) = 0 \in K^1(B) \quad ([23] \;\; theorem\ 1.1)$$

where $\sigma_{2m-1} \in K^1(MT(2m-1))$ is the universal symbol class corresponding to $D^o$ (section 3.5). After passing through $ch : K \to H\mathbb{Q}$ this implies

$$f_!(\mathcal{L}(T_vE)) = 0 \in H^*(B;\mathbb{Q}) \quad ([23] \;\; theorem 1.2) \tag{4.2}$$

from which it follows $Sig(E) = 0$ ([23] corollary 1.4) using the splitting $TE \cong f^*TB \oplus T_vE$ and that $\mathcal{L}$ is multiplicative as mentioned in the introduction.

In [10] the *odd signature operator with coefficients in* $V$, $D^o_V$, $V$ flat, was also described. In the next section we'll consider the family index of $D^o_V$ on the fibers of an $M$-bundle $E \to B$, $M$ odd dimensional. The main results stated in the introduction generalize the last three results cited above from [23] (Theorem 3, 2, and 1 respectively).

# Chapter 5

# Proofs and Summary

For $V \to M^{2m-1}$ a complex vector bundle with flat connection $\nabla$ define the *odd signature operator with coefficients in* $V$

$$D_V^o = i^m(-1)^{k+1}(\star d_\nabla - d_\nabla \star),$$

where $d_\nabla : A^k(M;V) \to A^{k+1}(M;V)$ is the exterior derivative on $V$-valued forms associated to $\nabla$. Now $D_V^o : \bigoplus_k A^{2k}(M;V) \to \bigoplus_k A^{2k}(M;V)$ acts between differential forms of even degree with coefficients in $V$.

As was observed in section 3.5 for $D^o$,

- $D_V^o$ is formally self-adjoint (see [10], [11]) and
- $(D_V^o)^2 = \Delta_\nabla = (d_\nabla + d_\nabla^*)^2$, the Laplacian operator associated to $\nabla$.

It follows from general Hodge theory (applied to the elliptic complex $(A^*(M;V), d_\nabla)$ as in section 4.1) that the kernel of $D_V^o$ can be identified with the finite dimensional vector space $\bigoplus_k H^{2k}(M;V)$. This identification is what allows theorem 4 to be applied in the proof of theorem 3 below.

## 5.1 Proofs

*proof of theorem 3*

Let $E \to B$ be an oriented $M$-bundle, $M^{2m-1}$ an odd-dimensional manifold, and $V_\rho \to E$ a flat Hermitian bundle induced by a representation $\rho$ of $\pi_1(E)$.

The family $D^o_{V_\rho} = \{D^o_{V_b}\}_{b \in B}$ of odd signature operators twisted by $V_\rho$ consists of the odd signature operators $D^o_{V_b}$ on the fibers $M_b \cong M$ twisted by $V_b = V_\rho|_{M_b}$. $V_b$ is still flat so,

$$ker(D^o_{V_b}) = \bigoplus_k H^{2k}(M; V_b)$$

from above.

We claim the map

$$b \mapsto dim(ker(D^o_{V_b})) = dim(\bigoplus_k H^{2k}(M; V_b))$$

is locally constant. Note $V_b \to M_b$ is induced by the representation of $(\iota_b)_* \circ \rho$ of $\pi_1(M_b)$ where $\iota_b : M_b \hookrightarrow E$ is the inclusion of the fiber. Locally we have contractible open sets $U \subset B$ such that the following diagram commutes:

$$\begin{array}{ccccc}
U \times M & \xrightarrow{\cong} & f^{-1}(U) & \hookrightarrow & E \\
\downarrow p & & \downarrow f & & \downarrow f \\
U & \xrightarrow{=} & U & \hookrightarrow & B
\end{array}$$

,where $p : U \times M \to U$ denotes the projection. From this it follows that the homomorphisms

$$\iota := (\iota_b)_* : \pi_1(M_b) \cong \pi_1(M) \to \pi_1(E)$$

all agree as $b$ varies through $U \subset B$. Thus

$$H^*(M_b, V_b) \cong H^*(M; V_{\rho\circ\iota})$$

canonically, where $V_{\rho\circ\iota} \to M$ is the flat bundle induced by $\iota : \pi_1(M) \to \pi_1(E)$. Hence $b \mapsto dim(ker(D^o_{V_b})) = dim(\bigoplus_k H^{2k}(M; V_b))$ is in fact constant.

Therefore, according to theorem 4 the map $index(D^o_V) : B \to \mathcal{F}_{s.a.}$ given by $b \mapsto D^o_{V_b}$ is null-homotopic, i.e. $index(D^o_V) = 0 \in K^1(B)$.

□

*proof of theorem 2 from 1*

The symbol class of $D^o_V$ is easy to express following the discussion in section 3.3.2 and 3.4:

$$\sigma_{D^o_V} = \sigma_{D^o} \otimes [V] = \kappa^*_E(\sigma_{2m-1}) \otimes [V] \in K^1(E^\nu),$$

so

$$index(D^o_V) = PT^*_f(\kappa_E(\sigma_{2m-1}) \otimes [V]) \in K^1(B)$$

The passage from theorem 2 to theorem 1 essentially amounts to applying the cohomological index formula from [12], that is we will compute $ch(index(D^o_V)) \in H^*(B; \mathbb{Q})$. Most of the work here is done by prop 4.3.4 from [23] which we recalled above as proposition 1. The rest is basically formal.

From proposition 1 we have

$$ch(\sigma_{2m}) = \tilde{\mathcal{L}} \in H^{*-2m}(MT(2m); \mathbb{Q})$$

and

$$\eta^*(\sigma_{2m}) = 2\sigma_{2m-1} \in K^1(MT(2m-1)$$

where $\eta^* : K^0(MT(2m)) \to K^0(\Sigma^{-1}MT(2m-1)) \cong K^1(MT(2m-1))$ is the restriction homomorphism.

We're trying to express $f_!(\tilde{L}(T_vE)ch([V]))$ in terms of the index of $D_V^o$. Recall $f_! = PT_f^* \circ th_\nu$ (definition 2) and note we can write

$$\tilde{\mathcal{L}}(T_vE) = th_\nu^{-1}(\kappa_E(\tilde{\mathcal{L}})) \in H^*(E)$$

where $th_\nu : H^*(E) \cong H^{*-(2m-1)}(E^\nu)$ is the Thom isomorphism (recall $\kappa_E : E^\nu \to MT(2m-1)$ is the map from section 2.2 induced by an orientation of $T_vE$).

Observe,

$$th_\nu(\tilde{\mathcal{L}}(T_vE)ch([V])) = th_\nu(\tilde{\mathcal{L}}(T_vE))ch([V]) = \kappa_E^*(\tilde{\mathcal{L}})ch([V]) =$$

$$= \kappa_E^*(ch(2\sigma_{2m-1}))ch([V]),$$

where the last equality is given by proposition 1.

By naturality,

$$\kappa_E^*(ch(2\sigma_{2m-1}))ch([V]) = ch(2\kappa_E^*(\sigma_{2m-1}) \otimes [V]) = ch(2\sigma_{D^o} \otimes [V]),$$

where $\sigma_{D^o} = \kappa_E^*(\sigma_{2m-1}) \in K^1(E^\nu)$is the symbol class of the (untwisted) odd signature operator.

Putting this all together we have $f_!(\tilde{\mathcal{L}}(T_vE)ch([V])) =$

$$PT_f^*(th_\nu(\tilde{\mathcal{L}}(T_vE)ch([V]))) = PT_f^*(ch(2\sigma_{D^o} \otimes [V])) = ch(2PT_f^*(\sigma_{D^o} \otimes [V])) = 0$$

since $PT_f^*(\sigma_{D^o} \otimes [V]) = index(D_V^o) = 0$ by theorem 1.

Recall from section 3.5 that the components of $\tilde{\mathcal{L}}(M)$ in degree $4i$ are $2^m$ times those of $\mathcal{L}(M)$ in $H^*(BSO(2m);\mathbb{Q})$, hence $f_!(\mathcal{L}(T_vE)ch([V])) = 0$ as well.

□

## 5.2 Summary

Our work, in some sense, concludes the general investigation into the (higher) signatures and signature operators on the total space of a manifold bundle. At the very least it concludes the explicit consideration of such things, started by Atiyah in [4], continued in Lusztig's thesis [38], and re-articulated using universal symbols by Ebert and extended in [23].

Summarizing, associated to the characteristic numbers known as higher signatures, there are two operators, the even ([13], [12]) and odd ([10], [11]) signature operator. By twisting these operators ([46], [10]) with a vector bundle, provided it is flat (and Hermitian), their index may be interpreted as the signature of certain intersection forms related to higher signatures. The family index of the even signature operator has appeared many places in the literature, Lusztig's thesis [38] providing one of the earlier significant applications of the family index theorem. The family index of the odd signature operator has been considered on its own relatively little comparatively, though its eigenvalues are commonly considered (eta invariant) following [9], [10], and [11]. As was made explicit in [23], the cohomological family index of the odd signature operator is essentially the $\mathcal{L}$-class (proposition 1), thus we get the vanishing relation in equation 4.2. By applying theorem 4 to the family of odd signature operators twisted by a flat bundle we get the vanishing relation in theorem 2.

Perhaps the family index of self-adjoint operators is considered relatively infrequently because the index of a single self-adjoint operator is zero $K^1(pt) \cong 0$, however there are plenty of families of self-adjoint elliptic operators whose index is non-trivial. Families of Toeplitz operators [17] parametrized by the circle, whose indices generate $K^1(S^1) \cong \mathbb{Z}$, are the classic example. In particular theorem 4 may not be applied to such families and, as a corollary of theorem 1, we see that families of twisted odd signature operators on the circle are not Toeplitz. This is an amusing observation considering the authors contribution to [21] as an undergraduate consisted primarily of identitifying other operators as *not Toeplitz*.

# Chapter 6

# Applications and Questions

We'll now provide some applications and alternative characterizations of the main results (theorem 1-3) stated in the introduction. We conclude this, our final chapter, with a discussion of further questions and possible generalizations.

## 6.1 Signature and products

Atiyah's work in [4] was motivated in large part by a desire to understand the effect the fundamental group of the base space $B$ (more specifically its action on the fiber) of a $M$-bundle $E \to B$ has on the signature of the total space $E$. For a trivial bundle $E \cong B \times M$ one has

$$Sig(B \times M) = Sig(B)Sig(M), \tag{6.1}$$

and in [19] it was shown that this multiplicative formula continues to hold with $E$ arbitrary provided that the fundamental group of $B$ acts trivially on the fibers. Examples of surface bundles where provided in [4] which show this restriction on the action of $\pi_1(B)$ is necessary. Corollary 1.4 of [23] shows that equation 6.1 in fact holds when the dimension of $M$ is odd (since $Sig(M) = 0$ in this case). Apparently this was conjectured by Atiyah but no proof supplied. In general, the failure of the signature to be multiplicative is measured by certain higher signatures of $B$ (see section 4 of [4]).

Naturally one can ask whether an appropriate version of equation 6.1 holds for the higher signatures $Sig_\rho$. For trivial $M$-bundles this is indeed the case:

$$Sig_{\rho_1 \otimes \rho_2}(B \times M) = Sig_{\rho_1}(B) Sig_{\rho_2}(M), \tag{6.2}$$

where $\rho_1$ and $\rho_2$ are finite dimensional Hermitian representations of $\pi_1(B)$ and $\pi_1(M)$ respectively ([31] $5_\rho$, page 110). Note replacing $B \times M$ in equation 6.2 by an arbitrary $M$-bundle doesn't make sense. However when $\rho_2$ is trivial we can ask whether the equation

$$Sig_\rho(B \times M) = Sig_\rho(B) Sig(M), \tag{6.3}$$

still holds for an arbitrary $M$-bundle, where $\rho$ on the left hand side corresponds to the representation induced by the projection $B \times M \to B$. Since $Sig(M) = 0$ for $M$ odd dimensional we have

**Corollary 1.** *Equation 6.3 holds for arbitrary oriented $M$-bundles, provided $M$ is a smooth closed Riemannian manifold of odd dimension. That is,*

$$Sig_\rho(E) = Sig_\rho(B) Sig(M) = 0,$$

*where $E \to B$ is an odd dimensional manifold bundle.*

Note here $\rho$ is a representation of $\pi_1(B)$ with $\rho$ on the left hand side the representation of $\pi_1(E)$ induced by $E \to B$.

## 6.2 Obstruction to fibering

An obvious interpretation of theorem 1 is

**Corollary 2.** *The higher signatures of the form $Sig_\rho$, $\rho$ a Hermitian representation, are obstructions to fibering as an odd dimensional manifold bundle.*

In order for theorem 1 to provide a more general obstruction than the usual signature ($V{=}0$) we need to demonstrate the existence of flat Hermitian vector bundles for which $ch([V])$ is nontrivial in positive dimensions. Really we need to find a manifold $M$ with $V \to M$ flat for which $Sig_\rho(M) \neq 0$, while $Sig(M) = 0$, $V \cong V_\rho$.

Recall that the Chern character of a vector bundle $V \to M$ may be expressed as

$$ch(V) = dim(V) + p_c(V) \in H^{even}(M)$$

where $p_c$ is a polynomial in the Chern classes $c_k$. Here $c_k(V) \in H^{2k}(M)$ is the *kth Chern class of $V$* which may be defined as a polynomial in the curvature form associated to a connection on $V$. If this connection is flat (i.e. its curvature is identically zero) then $ch(V) = N$ since $c_k(V) = 0$ for all $k$ (this remains true for the topological notion of flatness mentioned above [32]). So for $V \cong V_\rho$ with $\rho$ a unitary representation $\rho : \pi = \pi_1(M) \to U(N)$ theorem 1 follows from corollary 1.4 of [23].

We'll instead need to consider representations $\rho : \pi \to U(p, q)$. The examples we need come from [31] where Gromov adds to the examples considered by Lusztig in [38]. Lusztig essentially provides the universal examples by identifying manifold versions of $B\pi = K(\pi, 1)$ ($\pi$ here are arithmetic groups) with representations $\rho$ for which $ch([\xi_\rho]) \in H^*(B\pi;\mathbb{Q})$ is nontrivial in positive dimensions. These classes can be represented by submanifolds ([57] section 3.2) $\iota : N \subset B\pi$ such that $Sig_\rho(N) \neq 0$, where $V_\rho = \iota^*\xi_\rho$ ([31] section 8). Taking $N$ even dimensional not divisible by 4 provides the examples we need. We'll now outline the details from [38] and [31] that lead to these examples.

For a group $G$ let $\mathcal{R}(G)$ denote the Grothendeick group generated by all isomorphism classes of finite dimensional complex (continuous) representations of G with a fixed nondegenerate hermitan form, invariant under the action of G (i.e. $U(p,q)$-representations of $G$). Using the tensor product, direct sum, etc. $\mathcal{R}$ can be made into a $\lambda$-ring in the usual way.

Let $G = Sp(2n,\mathbb{R})$ be the real symplectic group in $2n$ variables ($G$ consists of automorphisms of $\mathbb{R}^{2n}$ preserving the alternating form $\Sigma_{i=1}^n(x_i y_{n+i} - x_{n+i} y_i)$) and let $\mathcal{R} = \mathcal{R}(G)$. $G$ acts naturally on $\mathbb{C}^{2n}$ considered with the hermitian form of type $(n,n)$. Denote this representation $G \to U(n,n)$ by $\rho_0 \in \mathcal{R}$. By restricting to the maximal compact subgroup $U(n) \subset G$, any representation $\rho \in \mathcal{R}$ has an orthogonal splitting $\rho = \rho^+ \oplus \rho^-$ such that $\rho^+$, $\rho^-$ are invariant under $U(n)$ and the hermitian form is positive (respectively negative) definite on these summands. The associated hermitian bundle $\xi_\rho$ over the classifying space $BG$ thus splits as $\xi_\rho = \xi_{\rho^+} \oplus \xi_{\rho^-}$

Applying the chern character gives an element

$$ch(\xi_{\rho^+} - \xi_{\rho^-}) \in H^*(BG;\mathbb{Q}).$$

The correspondence $\rho \mapsto ch(\xi_{\rho^+} - \xi_{\rho^-})$ defines a ring homomorphism

$$\Psi : \mathcal{R} \to H^*(BG),$$

respecting 1. Note that $BG$ is homotopy equivalent to $BU(n)$. It is well known that $H^*(BG)$ can be identified with the algebra of symmetric formal power series over $\mathbb{Q}$ in the chern roots $x_1, ..., x_n$ of $\xi_{\rho_0^+}$. Then it is easy to see ([38]) that

$$\Psi(\rho_0) = \Sigma_{i=1}^n (e^{x_i} - e^{-x_i}) \in H^*(BG).$$

After a little bit of algebra one can show

**Proposition 2.** *([38] proposition 5.1) The image of the ring homomorphism*

$$\Psi \otimes 1_{\mathbb{Q}} : \mathcal{R} \otimes \mathbb{Q} \to H^*(BG)$$

*is dense in $H^*(BG)$*

Moreover, Novikov's conjecture is true for symplectic representations $\pi \to G$:

**Proposition 3.** *([38] proposition 5.2)*

*Let $X$, $X'$ be two connected $2k$-manifolds, and $f : X' \to X$ an orientation preserving smooth homotopy equivalence. Consider a homomorphism $\rho : \pi_1(X) \to G$ and let $h$ denote the composition $X \to B\pi_1(X) \to BG$ defined up to homotopy. Then for any $v \in H^*(BG)$ one has*

$$\tilde{\mathcal{L}}(X)h^*(v)[X] = \tilde{\mathcal{L}}(X')f^*h^*(v)[X']$$

Proposition 2 says (an integer multiple of ) every element in the rational cohomology of $BG$ is detected by the chern character of (the difference of) some Hermitian representation(s) of $G$. Proposition 3 says the classes in $H^*(BG)$ determine oriented homotopy invariants, namely $Sig'_\rho$ where $\rho' = \rho_0 \circ \rho$ for symplectic representations $\rho : \pi \to G$.

Examples of manifolds with flat hermitians bundles whose chern character is nontrivial in positive dimensions are provided by Matsushima in [41] in terms of cocompact sublattices in $G$.

**Proposition 4.** *([41]) Let $\Gamma$ be a discrete, torsion free subgroup of $G$ such that $G/\Gamma$ is compact. Then the homomorphism $H^p(BG) \to H^p(B\Gamma)$ induced by the inclusion $\Gamma \subset G$ is surjective for $p < (n+2)/2$.*

A manifold model of $B\Gamma$, $\Gamma$ as in proposition 4, is given by the double coset space $X_\Gamma = \Gamma \backslash G/U(n)$. This is a manifold of dimension $n(n+1)$, which is a $K(\Gamma, 1)$ ($G/U(n)$ is eucildean), and restricting representations on $G$ produces flat Hermitian vector bundles $V \to X_\Gamma$ with $ch([V])$ nontrivial in positive dimensions (theorem 5.4 [38]).

$X_\Gamma$ is not the example we're looking, but it is the source of them. Although there exist flat Hermitian vector bundles on $V \to X_\Gamma$ with $ch([V])$ nontrivial, all the higher signatures of $X_\Gamma$ are actually zero. In fact all the pontryagin numbers of a locally symmetric manifolds (which $X_\Gamma = \Gamma \backslash G/U(n)$ is) all vanish [37]. The classes

$$\mathcal{L}(X_\Gamma)_{4k} ch([V]) \in H^*(X_\Gamma)$$

however may be nontrivial (proposition 4). Proposition 4 is actually true more generally with $\Gamma \subset G$ any arithmetic subgroup. The general question of which locally symmetric manifolds have nontrivial pontryagin classes is taken up in [59].

The nontrivial classes $\mathcal{L}(X_\Gamma)_{4k} ch([V]) \in H^*(X_\Gamma)$ are detected by submanifolds $N \subset X_\Gamma$ where the classes pulled back to $N$ produce nontrivial characteristic numbers of $N$. These characteristic numbers are actually higher signatures of $N$ associated to Hermitian representations of $\pi_1(N)$ induced by the original inclusion $\Gamma \subset G$, thus providing obstructions to fibering according to corollary 6.2.

These submanifolds are described by Gromov [31] section 8 in great detail. Explicitly, with $\Gamma \subset G$ and $X_\Gamma$ as above, there exist many compact surfaces $S \subset X_\Gamma$ with $Sig_\rho(S) \neq 0$ for $\rho$ a symplectic representation (obtained as the pre-image of a regular value of a smooth function on $X_\Gamma$, see [57] chapter 3 or [31]). For a specific example, the flat bundle over $S$ associated to $\rho$ may be realized as a "square root" of the tangent bundle of $S$ ([31] page 113). Since the signature of a surface is zero corollary 6.2 provides an obstruction not given by corollary 1.4 of [23].

## 6.3 Alternative statements

Let $D$ be an elliptic family on the fibers of $f : E \to B$ with symbol class $\sigma \in K^*(E^{\nu(f)})$. Recall from section 3.4 that the symbol of the twisted operator $D_W$, $W \to E$ a complex vector bundle, is expressed in terms of the $K^*(E)$-module structure: $\sigma_W = \sigma \otimes [W] \in K^*(E^\nu)$, where $\otimes$ is as $K^*(E)$-module. The index of $D_W$ is then given by $index(D_W) = PT_f^*(\sigma_W) = PT_f^*(\sigma \otimes [W]) \in K^*(B)$ and the map $[W] \mapsto index(D_W)$ defines a homomorphism

$$index_D : K^0(E) \to K^*(B).$$

The description of this process for $E = M \times B$ in [3] is followed by the advice "...to choose appropriately the manifold $M$ and the operator $D$ for various applications". For theorem 3, $M$ odd dimensional and $D = D^o$ the odd signature operator are "appropriate". The following is an equivalent rephrasing of theorem 3 in a form more reminisent of proposition 2.2 and the discussion in section 2 of [3].

**Theorem 7.** *Let $M$ be an odd dimensional oriented manifold, $f : E \to B$ an oriented $M$-bundle with $D^o$ the family of odd signature operators. The kernel of*

$$index_{D^o} : K^0(E) \to K^1(B)$$

*contains the subring generated by all $[V_\rho]$, $\rho \in \mathcal{R}(\pi_1(E))$.*

Suppose $D$ is symbolically universal (section 3.3.2) with symbol class $\sigma_D \in K^*(MT(n))$. While the twisted operator $D_V$ is not symbolically universal in the same sense $D$, when $V \cong V_\rho$ for $\rho$ a representation of $\pi = \pi_1(E)$, the symbol class of $D_{V_\rho}$ can be viewed as an element of $K^*(MT(n) \wedge B\pi_+)$, namely the element given by the (external) product $\sigma_D \otimes [\xi_\rho] \in K^*(MT(n) \wedge B\pi_+)$. The operators $D_{V_\rho}$ are universal in the sense that they are defined for all $M$-bundles whose total space has a fundamental group isomorphic to $\pi$.

Consider the composition

$$\alpha_E^\pi : \Sigma^\infty B \xrightarrow{PT_f} E^\nu \xrightarrow{diag} E^\nu \wedge E_+ \xrightarrow{\kappa_E \wedge u} MT(n) \wedge B\pi_+$$

where $E^\nu \xrightarrow{diag} E^\nu \wedge E_+$ is the diagonal map and $u : E \to B\pi_1(E)$. Theorem 3 can be rephrased in terms of the map $\alpha_E^\pi$ as follows (see [25] page 36). Recall $\sigma_{2m-1} \in K^1(MT(2m-1)$ is the symbol class of the odd signature operator (section 3.5).

**Theorem 8.** *Let $f : E \to B$ is an oriented $M$-bundle, $M$ a $(2m-1)$-dimensional closed oriented manifold. Every element of the form $\sigma_{2m-1} \otimes [\xi_\rho] \in K^*(MT(n) \wedge$*

$B\pi_+)$, $\rho$ *a finite dimensional Hermitian representation of* $\pi = \pi_1(E)$, *lies in the kernel of*

$$(\alpha_E^\pi)^* : K^*(MT(n) \wedge B\pi_+) \to K^*(B).$$

Theorem 1 can also be viewed as identifying elements in the kernel a homomorphism. As mentioned in section 4.2, one may approach Novikov's conjecture by searching for homomorphisms $\Omega_*^{SO}(B\pi) \to \mathbb{Q}$. A representation $\rho : \pi \to U(p,q)$ determines a homomorphism (see [57] section 3.4)

$$\tau_\rho : \Omega_*^{SO}(B\pi) \to \mathbb{Q}.$$

Any numerical function on the group of bordisms can be described with the help of characteristic classes ([20]), in this case,

$$\tau_\rho([g : M^{2m} \to B\pi]) = \langle \mathcal{L}(M) ch(g^*\xi_\rho), [M] \rangle = 2^{-m} Sig_\rho(M).$$

(Note $\mathcal{L}(M)ch(g^*\xi_\rho) \in H^{2*}(M;\mathbb{Q})$, so $Sig_\rho(M) = 0$ when $M$ is odd dimensional.)

**Theorem 9.** *Let* $g : M \to B\pi$ *be a continuous map. If* $M$ *is bordant to a bundle of odd-dimensional oriented manifolds then* $[g : M \to B\pi] \in ker(\tau_\rho)$ *for all Hermitian representations* $\rho : \pi \to U(p,q)$.

In fact, for finite dimensional representations, everything reduces to the complex case via the embeddings $O(p,q) \subset U(p,q)$ and $Sp(2p) \subset U(2p,2p)$.

**Theorem 10.** *Let* $\rho$ *be a representation of* $\pi$ *in* $O(p,q)$, $Sp(2p)$, *or* $U(p,q)$, $g : M \to B\pi$ *a continuous map. If* $M$ *is bordant to a bundle of odd-dimensional oriented manifolds then* $[g : M \to B\pi] \in \Omega_*^{SO}(B\pi)$ *lies in the kernel of the homomorphism*

$$\tau_\rho : \Omega_*^{SO}(B\pi) \to \mathbb{Q}$$

*associated to $\rho$ (see [57] section 3.4).*

## 6.4 Comments and Further Questions

We've been somewhat conservative with our restrictions on smoothness and topologies of spaces involved in the work so far. The application of theorem 4 relies on being able to identify the kernel of $D_V^o$ fiber-wise with the kernel of the (twisted) Laplacian. This really only requires that $V \to E$ is flat when restricted to the fiber above each $b \in B$. Therefore theorem 3 remains true (and hence theorem 2 and 1 as well) when $V \to E$ is merely flat in the fiber direction (as in [38]). Evidently there still exist plenty of genuinely flat $V$ for which $ch([V])$ is non-trivial in positive dimensions.

Clearly theorem 3 doesn't hold for arbitrary $V$. For example $S^1 \times S^1 \to S^1$ is an odd dimensional manifold bundle with $\mathcal{L}(S^1 \times S^1) = 1$. Since there are non-zero classes in $H^2(S^1 \times S^1; \mathbb{Z}) \cong \mathbb{Z}$ that can be realized as the first Chern class of a vector bundle $V \to S^1 \times S^1$, it follows

$$\langle \mathcal{L}(S^1 \times S^1) ch([V]), [S^1 \times S^1] \rangle \neq 0.$$

It would be interesting to know if our main results hold when $V$ is only "almost flat" (see [49] section 6).

It would also be interesting to know if theorem 1 remains true for more general representations. The homomorphism $\tau_\rho : \Omega_*^{SO}(B\pi) \to \mathbb{Q}$ mentioned above may be defined when $\rho : \pi \to End(\mathcal{H})$ is an infinite dimensional representation ($\mathcal{H}$ a separable infinite dimensional complex Hilbert space) known as a *Fredholm representation* (see [50]). A Fredholm representation of a discrete group $\pi$ consists of a triple $\rho = (\rho_1, F, \rho_2)$, where $\rho_i : \pi \to End(\mathcal{H})$ are unitary representations and $F : \mathcal{H} \to \mathcal{H}$ a fredholm operator such that $\rho_2(g)F - F\rho_1(g)$ is compact for all $g \in \pi$. Perhaps there are restrictions to be put on the Fredholm representation $\rho$ so that the kernel of the associated homomorphism $\tau_\rho : \Omega_*^{SO}(B\pi) \to \mathbb{Q}$ still contains the bordism class of all odd-dimensional oriented manifold bundles.

In [22] Ebert showed that the Madsen-Tillman map $\alpha_E : \Sigma^\infty B \to MT(2m-1)$ associated to any odd-dimensional oriented manifold bundle $f : E \to B$ pulls back the module in $H^*(MT(2m-1;\mathbb{Q})$ generated by the class $\mathcal{L}$ to zero in $H^*(B;\mathbb{Q})$. Recall $\alpha_E^*(\mathcal{L}) = f_!(\mathcal{L}(T_vE))$. Apparently (theorem 3) the module $\{\mathcal{L}(T_vE)ch([V])\}_{V\in\mathcal{R}(\pi_1(E))} \subset H^*(E;\mathbb{Q})$ is zero when pushed forward by $f_! : H^*(E;\mathbb{Q}) \to H^{*-(2m-1)}(B;\mathbb{Q})$. It seems possible that there might be a proof of theorem 3 using the results of [22] and little more than some clever algebra.

Finally, something in the way of a "real refinement" of our results should exist (as in [23] theorem 5.1) and will likely be included in any published version of this work.

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