VON NEUMANN ORBIT EQUIVALENCE

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To my parents for never ending support

Contents

1. Introduction

Let Γ and Λ be two countable discrete groups with free probability measure-preserving actions $\Gamma \curvearrowright (X, \mu)$ and $\Lambda \curvearrowright (Y, \nu)$ on standard probability measure spaces (X, μ) and (Y, ν), respectively. An *orbit equivalence* (OE) for the actions is a measurable isomorphism $\theta : X \to Y$ such that $\theta(\Gamma x) = \Lambda\theta(x)$ for almost every $x \in X$. In this case, the two actions are called *orbit equivalent*. Two groups are said to be *orbit equivalent* if they admit orbit equivalent actions. Singer [1] showed that for two free probability measure preserving actions $\Gamma \curvearrowright (X, \mu)$ and $\Lambda \curvearrowright (Y, \nu)$, being orbit equivalent is equivalent to the existence of an isomorphism $L^\infty(X) \rtimes \Gamma \cong L^\infty(Y) \rtimes \Lambda$ which preserves the Cartan subalgebras $L^\infty(X)$ and $L^\infty(Y)$. Orbit equivalence theory saw some development in the 1980s (see [2, 3, 4]), and has been an area of active research over the last two decades (see [5, 6]). These advances in part have been stimulated by the success of the deformation/rigidity theory approach to the classification of II_1 factors developed by Popa and others (see [7, 8, 9]).

The study of orbit equivalence can be motivated also from an entirely different point of view, being a measurable counterpart to quasi-isometry of groups. Gromov [10] introduced measure equivalence (ME) for countable discrete groups as a measurable analogue of quasi-isometry and since then this notion has proven to be an important tool in geometric group theory with connections to ergodic theory and operator algebras. Two infinite countable discrete groups Γ and Λ are *measure equivalent* if there is an infinite measure space (Ω, m) with commuting, measure-preserving actions $\Gamma \curvearrowright (\Omega, m)$ and $\Lambda \curvearrowright (\Omega, m)$, so that both the actions admit finite-measure fundamental domains $Y, X \in \Omega$, that is, $m(Y), m(X) < \infty$ and

$$\Omega = \bigsqcup_{\gamma \in \Gamma} \gamma Y = \bigsqcup_{\lambda \in \Lambda} \lambda X.$$

The space (Ω, m) is called an ME-*coupling* between Γ and Λ, and the *index* of such a coupling is

$$[\Gamma : \Lambda]_\Omega := \frac{m(X)}{m(Y)}.$$

Notably, measure equivalence was used by Furman in [11, 12] to prove strong rigidity results for lattices in higher rank simple Lie groups. ME relates back to OE because of the following fact, observed by Zimmer and Furman: for two discrete groups Γ and Λ, admitting free OE actions is equivalent to having an ME-coupling of index 1. Moreover, for OE groups, an ME-coupling can be chosen so that the fundamental domains coincide [12, Theorem 3.3].

If $X \subset \Omega$ is a Borel fundamental domain for the action $\Gamma \curvearrowright (\Omega, m)$, then on the level of function spaces, the characteristic function 1_X gives a projection in $L^\infty(\Omega, m)$ such that the collection $\{1_{\gamma X}\}_{\gamma \in \Gamma}$ forms a partition of unity, i.e., $\sum_{\gamma \in \Gamma} 1_{\gamma X} = 1$. This notion generalizes quite nicely to the non-commutative setting, and using this, Peterson, Ruth, and Ishan, in [13], defined a *fundamental domain* for an action on a von Neumann algebra $\Gamma \curvearrowright^\sigma \mathcal{M}$ is a projection $p \in \mathcal{M}$ such that $\sum_{\gamma \in \Gamma} \sigma_\gamma(p) = 1$, where the convergence is in the strong operator topology. Using this perspective for a fundamental domain they generalized the notion of measure equivalence by considering actions on non-commutative spaces.

Definition 1.1 ([13])**.** Two countable discrete groups Γ and Λ are *von Neumann equivalent* (vNE), written $\Gamma \sim_{\mathrm{vNE}} \Lambda$, if there exists a von Neumann algebra $\mathcal{M}$ with a faithful normal semi-finite trace Tr and commuting, trace-preserving actions of Γ and Λ on $\mathcal{M}$ such that the Γ- and Λ-actions individually admit a finite-trace fundamental domain. The semi-finite von Neumann algebra $\mathcal{M}$ is called a *von Neumann coupling* between Γ and Λ.

Like ME, vNE is stable under taking the direct product of groups. But neither ME nor vNE is stable under taking free products. For instance, since any two finite groups are ME (and hence vNE), and amenability is preserved under both ME and vNE, one gets that $\mathbb{Z}/2\mathbb{Z}*\mathbb{Z}/2\mathbb{Z}$ (amenable) is neither ME nor vNE to $\mathbb{Z}/3\mathbb{Z}*\mathbb{Z}/2\mathbb{Z}$ (non-amenable). However, as suggested in [14, Remark 2.28], and proved in [15, $\mathbf{P}_{\mathrm{ME}}\mathbf{6}$], stability under taking free products hold if one requires the additional assumption that groups are ME with a common fundamental domain. In other words, OE is stable under taking free products. This raises a natural question: *Is vNE, with common fundamental domain, stable under taking free products?* We obtain an affirmative answer to this question and introduce the following definition.

Definition 1.2. Two countable discrete groups Γ and Λ are said to be *von Neumann orbit equivalent* (vNOE), denoted $\Gamma \sim_{\mathrm{vNOE}} \Lambda$, if there exists a von Neumann coupling between Γ and Λ with a common fundamental domain.

The relationship between Orbit Equivalence, Measure equivalence, von Neumann Orbit Equivalence and von Neumann Equivalence for groups are as follows:

$$\begin{array}{ccc} \Gamma \sim_{OE} \Lambda & \Longrightarrow & \Gamma \sim_{vNOE} \Lambda \\ \Downarrow & & \Downarrow \\ \Gamma \sim_{ME} \Lambda & \Longrightarrow & \Gamma \sim_{vNE} \Lambda \end{array}$$

It has been shown that amenability and property (T) are preserved under all of the four equivalence relations. The graph product and the free product of the groups are preserved under orbit equivalence but not under measure equivalence. We will show that the free product of the groups and the graph product of groups is also preserved under von Neumann orbit equivalence.

Theorem 1.3. *If* $\Gamma_i, \Lambda_i,\ i = 1, 2$ *are countable discrete groups such that* $\Gamma_i \sim_{\mathrm{vNOE}} \Lambda_i,\ i = 1, 2$*, then* $\Gamma_1 * \Gamma_2 \sim_{\mathrm{vNOE}} \Lambda_1 * \Lambda_2$.

Remark 1.4. We suspect that the notion of vNE with coupling index 1 should be equivalent to the notion of vNE with common fundamental domain. However, we are unable to prove it at this point and leave it as an open problem.

Green [16], in her Ph.D. thesis, introduced graph products of groups, another important group theoretical construction. If $\mathcal{G} = (V, E)$ is a simple, non-oriented graph with vertex set V and edge set E, then the graph product of a family, $\{\Gamma_v\}_{v \in V}$, of groups indexed by V is obtained from the free product $*_{v \in V} \Gamma_v$ by adding commutator relations determined by the edge set E. Depending on the graph, free products and direct products are special cases of the graph product construction. Adapting the ideas of [15], Horbez and Huang [17, Proposition 4.2] proved the stability of OE under taking graph products (see also [18]). To further explore the study of graph products within the context of measured group theory, we would like to draw the reader's attention to the article [19]. In this thesis, we also prove the stability of vNOE under taking graph products.

Theorem 1.5. *Let $\mathcal{G} = (V, E)$ be a simple finite graph. Let Γ and Λ be two graph products over $\mathcal{G}$, with countable vertex groups $\{\Gamma_v\}_{v \in V}$ and $\{\Lambda_v\}_{v \in V}$, respectively. If $\Gamma_v \sim_{\mathrm{vNOE}} \Lambda_v$ for every $v \in V$, then $\Gamma \sim_{\mathrm{vNOE}} \Lambda$.*

In attempting to prove the above theorems, if one tries to adapt the techniques from one of [15, 17, 18], an immediate problem is presented by the lack of "point perspective" in the theory of von Neumann (orbit) equivalence. The lack of any natural non-commutative analogue of the notion of OE/ME cocycles, or that of measured equivalence relation can be considered as a few problems presented by the lack of point perspective. This often leads one to consider genuinely new techniques and different alternatives (see e.g., [13, 20, 21, 22]). To overcome this obstruction, we introduce the notion of von Neumann orbit equivalence for tracial von Neumann algebras that is "compatible" with vNOE of groups (see Theorem 3.9), and prove the analogues of Theorems 1.3 and 1.5 at the level of tracial von Neumann algebras.

The notion of von Neumann equivalence admits a generalization in the setting of finite von Neumann algebras [13, Section 8], and relates to vNE for groups as follows: $\Gamma \sim_{\mathrm{vNE}} \Lambda$ if and only if $L\Gamma \sim_{\mathrm{vNE}} L\Lambda$ [13, Theorem 1.5]. In parallel to this, one might attempt to define two tracial von Neumann algebras to be vNOE if they are vNE and admit a "common" fundamental domain, and identify a correct meaning of "common". However, we take a slightly different approach, and motivated by the recently defined notion of measure equivalence of finite von Neumann algebras by Berendschot and Vaes in [23], we introduce the following definition.

Definition 1.6. Let (A, τ_A) and (B, τ_B) be tracial von Neumann algebras. We say that A and B are *von Neumann orbit equivalent*, denoted $A \sim_{\mathrm{vNOE}} B$, if there exists a tracial von Neumann algebra (Q, τ_Q), a Hilbert $A \overline{\otimes} Q - B$-bimodule $\mathcal{H}$, and a vector $\xi \in \mathcal{H}$ such that

1. $\langle (a \otimes x)\xi, \xi\rangle = \tau_A(a)\tau_Q(x)$, and $\langle y\xi b, \xi\rangle = \tau_Q(y)\tau_B(b)$ for every $a \in A,\ x, y \in Q$, and $b \in B$.
2. $\overline{\mathrm{Span}((A \overline{\otimes} Q)\xi)} = \mathcal{H} = \overline{\mathrm{Span}(Q\xi B)}$.

We prove in Proposition 3.4 that vNOE is indeed an equivalence relation. We should remark that, in the above definition, $\mathcal{H}$ can also be considered as an $A - B \overline{\otimes} Q^{\mathrm{op}}$-bimodule satisfying conditions analogous to the two mentioned in the definition. This essentially is the reason for the symmetry of vNOE, even though the definition seems asymmetric at first. To prove transitivity, inspired by [23, Lemma 5.11], we establish an equivalent characterization of vNOE in Theorem 3.1, and show in Theorem 3.9 that $\Gamma \sim_{\mathrm{vNOE}} \Lambda$ if and only if $L\Gamma \sim_{\mathrm{vNOE}} L\Lambda$. Since $L(\Gamma * \Lambda) \cong L\Gamma * L\Lambda$, Theorem 1.3 follows from the following theorem, which we prove in Section 3.

Theorem 1.7. *If $A_i, B_i,\ i = 1, 2$ are tracial von Neumann algebras such that $A_i \sim_{\mathrm{vNOE}} B_i,\ i = 1, 2$, then, $A_1 * A_2 \sim_{\mathrm{vNOE}} B_1 * B_2$.*

Similar to free products, one also has that the group von Neumann algebra of a graph product of groups is isomorphic to the (von Neumann algebra) graph product of the group von Neumann algebras, and hence Theorem 1.5 follows from the following theorem, proved in Section 4.

Theorem 1.8. *Let $\mathcal{G} = (V, E)$ be a simple finite graph. Let A and B be two graph products over $\mathcal{G}$, with tracial vertex von Neumann algebras $\{A_v\}_{v\in V}$ and $\{B_v\}_{v\in V}$, respectively. If $A_v \sim_{\mathrm{vNOE}} B_v$ for every $v \in V$, then $A \sim_{\mathrm{vNOE}} B$.*

Remark 1.9. Since graph product over a totally disconnected graph, i.e., a graph with no edges, gives free product, Theorem 1.7 follows from Theorem 1.8. However, we include a proof of Theorem 1.7 for two reasons. Firstly, the notation is less involved compared to the proof of Theorem 1.8. Secondly, if a reader prefers the base case for the induction (on number of vertices) in the proof of Theorem 1.8 to be a graph with two vertices instead of one, then that base case is justified.

In Proposition 3.12, we show that vNOE tracial von Neumann algebras are vNE in the sense of [13]. We should remark that vNE does not imply vNOE in general.

In the final section, we obtain a partial analogue of Singer's theorem [1] for OE in the setting of vNOE of groups. As noted in [13, Example 5.2], if Γ and Λ are countable discrete groups with trace-preserving actions $\Gamma \curvearrowright (A, \tau_A)$ and $\Lambda \curvearrowright (B, \tau_B)$ on tracial von Neumann algebras (A, τ_A) and (B, τ_B), respectively, and if $\theta : B \rtimes \Lambda \to A \rtimes \Gamma$ is a trace-preserving isomorphism such that $\theta(B) = A$, then $\Gamma \sim_{\text{vNOE}} \Lambda$. As a partial converse to this, we prove the following theorem.

Theorem 1.10. *If Γ and Λ are countable discrete groups such that $\Gamma \sim_{\text{vNOE}} \Lambda$, then there exist tracial von Neumann algebras (A, τ_A), (B, τ_B), trace-preserving actions $\Gamma \curvearrowright A$, $\Lambda \curvearrowright B$, and a trace-preserving isomorphism $\theta : B \rtimes \Lambda \to A \rtimes \Gamma$.*

2. Preliminaries

2.1. Von Neumann algebras: Definition and Examples

Let $\mathcal{H}$ be a Hilbert space and $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. The strong operator topology on $B(\mathcal{H})$ is the topology generated by the basis consisting of sets of the form

$$U(x; \xi_1, ..., \xi_n; \epsilon) := \{y \in B(\mathcal{H}) : \|(x - y)\xi_j\| < \epsilon, j = 1, ..., n\},$$

for $x \in B(\mathcal{H})\xi_1, ..., \xi_n \in \mathcal{H}$,and $\epsilon > 0$. We can also define the weak operator topology on $B(\mathcal{H})$ as the topology generated by the basis consisting of sets of the form

$$U(x; \xi_1, ..., \xi_n; \eta_1, ..., \eta_n : \epsilon) := \{y \in B(\mathcal{H}) : |\langle (x - y)\xi_j, \eta_j\rangle| < \epsilon, j = 1, ..., n\},$$

for $x \in B(\mathcal{H}), \xi_1, ..., \xi_n, \eta_1, ..., \eta_n \in \mathcal{H}$, and $\epsilon > 0$. From an analytic perspective, we describe what it means for a net to converge in these topologies. Let $(x_i)_{i\in I}$ be a net in $B(\mathcal{H})$, then $(x_i)_{i\in I}$ converges to $x \in B(\mathcal{H})$ in the strong operator topology if

$$\lim_{i\to\infty} \|(x - x_i)\xi\| = 0 \;\; \forall \xi \in \mathcal{H}$$

, and $x_{i\in I}$ converges to x in the weak operator topology (WOT) if

$$\lim_{i\to\infty} \langle (x - x_i)\xi, \eta\rangle = 0 \;\; \forall \xi, \eta \in \mathcal{H}.$$

Definition 2.1. A von Neumann algebra M on a Hilber space $\mathcal{H}$ is a $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ which contains the identity and is closed in the strong operator(or, equivalently weak operator) topology.

Example 2.2. Let $\mathcal{H} = \mathbb{C}^n$, we get that the the n by n matrices $M_n(\mathbb{C})$ is a von Neumann algebra.

Example 2.3. Let (X, Ω, μ) be a measure space, it follows that $L^\infty(X, \mu) \subseteq \mathcal{B}(L^2(X, \mu))$ is a von Neumann algebra.

There is also a more algebraic way to define von Neumann algebra. Again we let M to be a $*$-subalgebra of $\mathcal{B}(\mathcal{H})$, and we define the commutant of M, denoted M', is the set

$$M' = \{x \in B(\mathcal{H}) : xy = yx \ \forall y \in M\}.$$

We say that M is a von Neumann algebra if $M = M''$ where M'' is the double commutant of M. We show that the two definitions coincide.

Theorem 2.4. *(Bicommutant theorem): Let $M \in B(\mathcal{H})$ be a unital subalgebra, we have that*

$$\overline{M}^{SOT} = \overline{M}^{WOT} = M''.$$

Proof. The proof usually goes by showing $M'' \subset \overline{M}^{SOT} \subset \overline{M}^{WOT} \subset M''$. Here we refer to Theorem 2.1.3 in [24]. $\square$

We also would like to introduce Kaplansky density theorem as a usful tool. For the proof of this Theorem please refer to Theorem 2.3.1 in [24].

Theorem 2.5. *(Kaplansky density theorem): Let A be a $*$-subalgebra of $\mathcal{B}(\mathcal{H})$, then $\overline{(A)_1}^{SOT} = (\overline{A}^{SOT})_1$, where $(A)_1$ is the operator norm unit ball of A.*

2.2. tracial von Neumann algebra

Definition 2.6. A von Neumann algebra M is tracial if it admits a linear functional $\tau : M \to \mathbb{C}$ which is

1. positive: $\tau(x^*x) \geq 0$, for all $x \in M$
2. faithful: $\tau(x^*x) = 0$, for some $x \in M$, implies that $x = 0$
3. tracial: $\tau(xy) = \tau(yx)$ for all $x, y \in M$.
4. normal: τ is weak operator topology continuous on the unit ball of M.

Example 2.7. $M_n(\mathbb{C})$ is a tracial von Neumann algebra with the regular trace.

2.3. Standard form

Let (M, τ) be a tracial von Neumann algebra, then it admits a natural representation on a Hilbert space. This representation is a special example of the Gelfand–Naimark–Segal construction.

Given the trace τ, we can define a inner product on M:

$$\langle x, y \rangle = \tau(y^*x).$$

We use $L^2(M)$ to denote the Hilbert space completion of M with respect to this norm. For $x \in M$, we use the $\hat{x}$ to denote x as an element in $L^2(M)$. The standard representation of a tracial von Neumann algebra M is the representation $\pi_\tau : M \to \mathcal{B}(L^2(M))$ where

$$\pi_\tau(x)\hat{y} = \widehat{xy}$$

for $x, y \in M$. We identify x with $\pi_\tau x$ and we write $x\xi$ for $\pi_\tau x$ for simplicity. Additionally, we view M as a dense subspace of $L^2(M, \tau)$ by identifying x with $x1$.

Consider the operator $J : \hat{x} \to \widehat{x^*}$, it is an antilinear isometry from $\widehat{M}$ onto itself. We notice that

$$\|J\hat{x}\|_2^2 = \|\widehat{x^*}\|_2^2 = \tau(xx^*) = t\tau(x^*x) = \|\hat{x}\|_2^2.$$

It then follows that J extends to an antilinear surjective isometry of $L^2(M)$. We call J the canonical conjugation operator on $L^2(M)$. One of the main features of the standard representation of M is that it makes M^{op} isomorphic to its commutant.

We view $\pi_\tau(x)$ as the operator of multiplication to the left by x and to denote it by L_x. The range of L, denoted by $L(M)$(in $\mathcal{B}(L^2(M))$, is $\{L_x = \pi_\tau(x) | x \in M\}$, which is M. Similarly, let R_x denote the extension of the operator which sends $\hat{y}$ to $\widehat{yx}$. We notice that $x \to R_x$ gives an embedding of M^{op} into $\mathcal{B}(L^2(M))$. We use $R(M)$ to denote the range of R (also in $\mathcal{B}(L^2(M))$.

Theorem 2.8. *Let (M, τ) be a tracial von Neumann algebra and J be the operator described as above. We have $JMJ = M'$.*

For the proof of this theorem we refer to Chapter 7 of [24].

Remark 2.9. Since $JxJ\hat{y} = Jx\widehat{y^*} = J\widehat{xy^*} = \hat{y}x^*$, we can identify JMJ with $R(M)$. So the theorem above can also be interpreted as $L(M)' = R(M)$.

Proposition 2.10. *Let (M, τ_M) be a tracial von Neumann algebra and let A be a $*-$subalgebra s.o. dense in M. Let (N, τ_N) be an another tracial von Neumann algebra and $\phi : A \to N$ be a trace preserving, that is, $\tau_N \circ \phi = \tau_M$ $*$-homomorphism. We can extend ϕ to a trace preserving $*$-homomorphism from M to N.*

Proof. Without loss of generality, we can just assume $\phi(M)$ is WOT dense in N. Since if we can extend ϕ to a trace preserving $*$-homomorphism from M to $\overline{\phi(M)}^{WOT}$, then we can just compose this extended map with the embedding and we get the desired extended trace preserving $*$-homomorphism from M to N. Let us use B to denote $\phi(M)$. Since ϕ is trace preserving, we have

$$\langle a_1, a_2 \rangle_{L^2(A,\tau_M)} = \langle \phi(a_1), \phi(a_2) \rangle_{L^2(N,\tau_N)}.$$

Along with the completeness of $L^2(A, \tau_M)$ and $L^2(N, \tau_N)$, it follows that there exists a unique unitary operator $U \colon \overline{A}^{\|\cdot\|_2}$to $\overline{B}^{\|\cdot\|_2}$ such that $U|_A = \phi$. Since A, B are WOT dense in M, N, it follows that $\overline{A}^{\|\cdot\|_2} = L^2(M, \tau_M)$, $\overline{B}^{\|\cdot\|_2} = L^2(N, \tau_N)$. Therefore, we define the normal $*$-homomorphism $\widetilde{\phi} : M \to B(L^2(N, \tau_N))$ given by $x \mapsto UxU^*$. Since $\widetilde{\phi}(x)\xi = UxU^*(\xi) = \phi(x)(\xi)$ for all $x \in A$ and $\xi \in L^2(N, \tau_N)$, $\widetilde{\phi}$ agrees with ϕ on A.

□

2.4. Group von Neumann algebra

Next, we introduce a type of very important von Neumann algebra called group von Neumann algebra. Let Γ be a discrete group, and let $\ell^2(\Gamma)$ be the square summable functions of Γ. Consider the left regular representation $\lambda : \Gamma \to \mathcal{B}(\ell^2(\Gamma))$:

$$[\lambda(g)\xi](h) = \xi(g^{-1}h),$$

where $\xi \in \ell^2(\Gamma), h \in \Gamma$. Similarly, we have the right regular representation $\rho : \Gamma \to \mathcal{B}(\ell^2(\Gamma))$:

$$[\rho(g)\xi](h) = \xi(hg).$$

The operators $\lambda(g)$ are unitary operators with $\lambda(g)^* = \lambda(g^{-1})$. Let

$$\mathbb{C}[\lambda(\Gamma)] := \{\sum_{g\in F} a_g\lambda(g) | a_g \in \mathbb{C}, \text{ and } F \subset \Gamma \text{ is a finite subset.}\},$$

then $\mathbb{C}[\lambda(\Gamma)]$ is a $*-$subalgebra of $\mathcal{B}(L^2\Gamma)$.

Definition 2.11. The group von Neumann algebra for Γ is defined as $\overline{\mathbb{C}[\lambda(\Gamma)]}^{SOT}$, denoted as $L(\Gamma)$.

Similarly, we can define $R(\Gamma)$ to be the SOT closure of linear span of $\rho(\Gamma)$. We have $L(\Gamma)' = R(\Gamma)$.

Remark 2.12. We use δ_γ to denote the vector $\lambda(\gamma)\delta_e \in \ell^2(\Gamma)$ where e is the identity element in Γ. For any $x \in L(\Gamma)$, we write $x\delta_e = \sum_{\gamma\in\Gamma} x_\gamma\delta_\gamma$ where $x_\gamma = \langle x\delta_e, \delta_\gamma\rangle$ and is called the Fourier coefficients of x. By using u_γ to denote $\lambda(\gamma)$, we write $x = \sum_{\gamma\in\Gamma} x_\gamma u_\gamma$ where where the convergence holds in the $\|\cdot\|_2$.

Lemma 2.13. *Let M be a von Neumann algebra and ϕ be a state on M. Suppose $A \subseteq M$ is a unital, weak*-dense $*$-subalgebra. If $\phi(xy) = \phi(yx)$ for all $x, y \in A$, then ϕ is tracial on M.*

Proof. Let $x, y \in M$, then, by Kaplansky density theorem, there exist nets $(x_i) \in A, (y_j) \in A$ such that $x_i \to x$, $y_j \to y$ in WOT and $\|x_i\| \leq \|x\|$, $\|y_i\| \leq \|y\|$. Since multiplication is separately continuous in WOT, we have

$$\phi(xy) = \lim_{i\to\infty}\lim_{j\to\infty}\phi(x_iy_j) = \lim_{i\to\infty}\lim_{j\to\infty}\phi(y_jx_i) = \lim_{i\to\infty}\phi(yx_i) = \phi(yx).$$

□

Proposition 2.14. *Consider the linear functional $\tau : L(\Gamma) \to \mathbb{C}$ defined by $\tau(x) = \langle x\delta_e, \delta_e\rangle$. We find that this τ is a faithful normal trace on $L(\Gamma)$.*

Proof. A state is normal if it is WOT-continuous on the unit ball. Given x in $L(\Gamma)$ and x_i WOT converging to x, it follows that $\langle (x_i - x)\delta_e, \delta_e \rangle$ converges to 0. Therefore, τ is normal. By the previous lemma, it suffices to show that τ is tracial on a WOT dense $*$-subalgebra. Since $\tau(u_g u_h) = \tau(u_{gh}) = \langle u_{gh}\delta_e, \delta_e \rangle = \langle \delta_{gh}, \delta_e \rangle = \langle \delta_{hg}, \delta_e \rangle = \tau(u_h u_g)$, τ is tracial. Now we assume $\tau(x^*x) = 0$, then it follows that $\langle x\delta_e, x\delta_e \rangle = 0$. Thus, we get $x\delta_e = 0$. For any $g \in \Gamma$, since $x\delta_g = x\rho(g^{-1})(\delta_e) = \rho(g^{-1})(x\delta_e) = 0$, it follows that τ is faithful. $\square$

2.5. Crossed product

Let (A, τ) be a tracial von Neumann algebra and $Aut(A)$ be the group of automorphisms. Consider the group homomorphism $\sigma : G \to Aut(A)$ such that $\tau \circ \sigma = \tau$. Let $A[G]$ denote the algebra whose elements are finitely supported sums $\sum_g a_g u_g$ with $a_g \in A$. We then define the product and involution to be

$$(a_1 u_g)(a_2 u_h) = a_1 \sigma_g(a_2) u_{gh}, \quad (a u_g)^* = \sigma_{g^{-1}}(a^*) u_{g^{-1}}.$$

Consider the Hilbert space $\mathcal{H} = L^2(A, \tau) \otimes l^2(G)$. We have a map from $A[G]$ to $\mathcal{B}(\mathcal{H})$ defined by $a_g u_g(\xi \otimes \delta_h) = (a\sigma_g(\xi)) \otimes \delta_{gh}$. (Notice that if you have a G acting on a tracial von Neumann algebra (A, τ) by σ_g, then you can extend this action to an action α_g of G on $L^2(A)$ by defining $\alpha_g(\hat{x}) = \widehat{\sigma_g x}$. This is called the Koopman representation).

Definition 2.15. The SOT closure of $A[G]$ in $B(H)$ is called the crossed product of the action of G on A, and it is denoted $A \rtimes G$.

Proposition 2.16. *Given a tracial von Neumann algebra (A, τ_A) and a trace preserving homomorphism $\sigma : G \to Aut(A)$. We can define a trace τ on $A \rtimes G$ where $\tau(x) = \langle x\hat{1} \otimes \delta_e, \hat{1} \otimes \delta_e \rangle$.*

Proposition 2.17. *With the same setting above, we would like to introduce some useful facts about $A \rtimes G$.*

1. *For any $x \in A \rtimes G$, there exits a unique sequence (x_g) in $\ell^2(G, M)$ such that $x(\hat{1} \otimes \delta_e) = \sum_g (x_g \hat{1}) \otimes \delta_g$*
2. *For $x \in A \rtimes G$, we have $x^*(\hat{1} \otimes \delta_e) = \sum_g \sigma_g x^*_{g^{-1}}(\hat{1} \otimes \delta_e)$.*
3. *$\hat{1} \otimes \delta_e$ is cyclic and separating for $A \rtimes G$*

Proposition 2.18. *Given a tracial von Neumann algebra (A, τ_A) and a trace preserving homomorphism $\sigma : G \to Aut(A)$. We can define a trace τ on $A \rtimes G$ where $\tau(x) = \langle x\hat{1} \otimes \delta_e, \hat{1} \otimes \delta_e \rangle$.*

Proof. We will focus on showing τ is tracial and faithful. From the previous facts, we have

$$
\begin{aligned}
\langle xy\hat{1} \otimes \delta_e, \hat{1} \otimes \delta_e \rangle &= \langle y\hat{1} \otimes \delta_e, x^*\hat{1} \otimes \delta_e \rangle \\
&= \sum_{g,h} \langle y_g \hat{1} \otimes \delta_e, \sigma_h(x^*_{h^{-1}})\hat{1} \otimes \delta_e \rangle \\
&= \sum_g \langle y_g \hat{1}, \sigma_g(x^*_{g^{-1}})\hat{1} \rangle_{L^2(A)} \\
&= \sum_g \tau_A(\sigma_g(x_{g^{-1}})y_g) \\
&= \sum_g \tau_A(y_g \sigma_g(x_{g^{-1}})) \\
&= \sum_g \tau_A(\sigma_g^{-1}(y_g)x_{g^{-1}}) \\
&= \langle yx\hat{1} \otimes \delta_e, \hat{1} \otimes \delta_e \rangle.
\end{aligned}
$$

As for faithfulness, it follows from:

$$\begin{aligned}\langle x^*x\hat{1}\otimes\delta_e,\hat{1}\otimes\delta_e\rangle &= \langle x\hat{1}\otimes\delta_e, x\hat{1}\otimes\delta_e\rangle\\ &= \sum_{g,h}\langle x_g\hat{1}\otimes\delta_e, x_h\hat{1}\otimes\delta_e\rangle\\ &= \sum_{g}\langle x_g, x_h\rangle_{L^2(A)}\\ &= \sum_{g}\tau_A(x_g^*x_g)\end{aligned}$$

□

2.6. Conditional expectation

Definition 2.19. Let $M \subset B(H)$ be a von Neumann algebra and $1_M \in N \subset M$ a von Neumann subalgebra. A conditional expectation from M to N is a linear map $E : M \to N$ satisfying

1. $E(a) = a$ for all $a \in N$,
2. $E(axb) = aE(x)b$ for all $a, b \in N$ and $x \in M$
3. $E(x) \geq 0$ whenever $x \geq 0$.

Theorem 2.20. *Let $(M,\tau) \subset B(H)$ be a tracial von Neumann algebra and $1_M \in N \subset M$ a von Neumann subalgebra. Then there exists a unique conditional expectation $E_N : M \to N$ satisfying $\tau \circ E_N = \tau$. Moreover, E_N is faithful.*

Proof. Let $L^2(N, \tau|_N)$ be the Hilbert subspace of $L^2(M,\tau)$ and we denote them as $L^2(N)$ and $L^2(M)$ respectively for simplicity. Let e_N denote the orthogonal projection from $L^2(M)$ to $L^2(N)$. We have that $e_N(\widehat{M}) \subset \widehat{N}$(This is non trivial, for proof please refer to section 9 of [24]). Let E_N be e_N restricted to M by identifying M with $\widehat{M}$. For $x \in M, b \in N$, we have that

$$\begin{aligned}
\tau(bE_N(x)) &= \langle \widehat{b^*}, \widehat{E_N(x)} \rangle && (1)\\
&= \langle \widehat{b^*}, e_N(\widehat{x}) \rangle && (2)\\
&= \langle \widehat{b^*}, e_N(\widehat{x}) + e_{N^\perp}(\widehat{x}) \rangle && (3)\\
&= \langle \widehat{b^*}, \widehat{x} \rangle && (4)\\
&= \tau(bx). && (5)
\end{aligned}$$

Where $e_{N^\perp}$ is the projection to the subspace orthogonal to N and (4) follows because $\widehat{b^*}$ is inside N. Thus, it follows $\tau \circ E_N = \tau$. The E is faithful because if $E(x^*x) = \tau(E(x^*x)) = \tau(x^*x) = 0$, then $x = 0$. Normality follows from 2.5.11 of [24].

Now we want to show that such conditional expectation is unique. If E is another conditional expectation such that $\tau \circ E = \tau$, for $x \in M$ and $b \in N$ we have

$$\tau((x - E(x))b) = \tau((E(x - E(x))b) = 0.$$

This shows that $\widehat{x} - \widehat{E(x)}$ is orthogonal to the subspace $\widehat{N}$, Therefore, E is indeed the orthogonal projection from $\widehat{M}$ to $\widehat{N}$. $\square$

Example 2.21. Let $G \curvearrowright M$ be a trace preserving group action of G on the tracial von Neumann algebra M. We can view M as a subspace of $M \rtimes G$ by identifying $x \in M$ as xu_e. Then $E_M(\sum_{g \in G} x_g u_g) = x_e$, where E_M is the conditional expectation from $M \rtimes G$ to M.

Example 2.22. Let $(M, \tau_M), (N, \tau_N)$ be two tracial von Neumann algebras. Let E_M be the conditional expectation from $M \overline{\otimes} N$ to M. We have $E_M(x \otimes y) = \tau(y)x$ for $x \in M$ and $y \in N$.

2.7. Free product and amalgamated free product of von Neumann algebras

We would like to introduce the definition of free product of tracial von Neumann algebra. We will start with the definition and then existence of such algebras.

Definition 2.23. Let M_1, M_2 be two von Neumann subalgebras of a tracial von Neumann algebra (M, τ) . We say that M_1, M_2 are free with respect to τ if $\tau(x_1 x_2 \ldots x_n) = 0$ whenever $x_i \in M_{k_i}$ with $k_1 \neq k_2 \neq \cdots \neq k_n$ and $\tau(x_i) = 0$ for all i. We say that two elements a_1, a_2 of M are free with respect to τ if the von Neumann algebras they generate are free. We will say that an element $x_1 x_2 \cdots x_k$ of the algebraic free product $M_1 *_{\text{alg}} M_2$ is an *alternating centered word* with respect to τ if $x_i \in M_{k_i}$ with $k_1 \neq k_2 \neq \cdots \neq k_n, k_i \in \{1, 2\}$, and $\tau_{k_i}(x_i) = 0$ for all $1 \leq i \leq n$.

Proposition 2.24. *Let M_1, M_2 be two von Neumann subalgebras of the tracial von Neumann algebra (M, τ) that are free with respect to τ. We have that τ is uniquely determined by its' restrictions to M_1 and M_2.*

Proof. Since the linear span of $x_1 x_2 \ldots x_n$ where $x_i \in M_{k_i}$ and $k_1 \neq k_2 \neq \cdots \neq k_n$ is WOT dense $*$-subalgebra of M and τ is normal, it suffices to prove that $\tau(x_1 x_2 \ldots x_n)$ is uniquely determined for such element. We will prove by induction. It is clearly true for $n = 1$. Now we assume that the statement is true for all $1, 2, ..., n-1$. For $x_i \in M_{k_i}$, we can write $x_i = \tau(x_i) + \widetilde{x_i}$ where $\widetilde{x_i} = x_i - \tau(x_i)$, and we observe that $\widetilde{x_i} \in M_{k_1}$ and $\tau(\widetilde{x_i}) = 0$. By plugging in we have that

$$\begin{aligned} \tau(x_1 x_2 \ldots x_n) &= \tau((\tau(x_1) + \widetilde{x_1}) \ldots (\tau(x_n) + \widetilde{x_n})) && (1) \\ &= \tau(A + (\widetilde{x_1}\widetilde{x_2} \ldots \widetilde{x_n})) && (2) \\ &= \tau(A) && (3) \end{aligned}$$

where A is the summation of the terms that only involves at most $n-1$ many x_i and equation (3) is true because each $\tau(\widetilde{x_i}) = 0$. By the induction hypothesis we have $\tau(A)$ is uniquely determined by its' restriction to M_2 and M_2. Therefore, the desired conclusion follows.

$\square$

Proposition 2.25. *Let $(M_1, \tau_1), (M_2, \tau_2)$ be tracial von Neumann algebras. There is, up to isomorphism, one triple $((M, \tau), \phi_1, \phi_2)$ where τ is a normal faithful tracial state and $\phi_i : M_i \to M, i = 1, 2,$ are homomorphisms, satisfying the following properties:*

1. *$\tau_i = \tau \circ \phi_i$ for $i = 1, 2$;*
2. *$\phi_1(M_1), \phi_2(M_2)$ sit in M as free von Neumann subalgebras with respect to τ and M is generated by $\phi_1(M_1) \cup \phi_2(M_2)$.*

Proof. Let $(M, \tau_M), (N, \tau_N)$ be two tracial von Neumann algebras that satisfies both of the criteria. We would like to show that they are isomorphic. Let $\phi_i : M_i \to M$ and $\varphi_i : M_i \to N$ the trace preserving inclusions for $i = 1, 2$. Let $\mathcal{M}$ and $\mathcal{N}$ be the $*$-algebras generated by $\phi_1(M_1) \cup \phi_2(M_2)$ and $\varphi_1(M_1) \cup \varphi_2(M_2)$ respectively. Consider the map f from $\mathcal{M}$ to $\mathcal{N}$ sending $\phi_i(x) = \varphi_i(x)$ for $x \in M_i$. For $y = \phi_{k_1}(x_1) \dots \phi_{k_n}(x_n)$ for $x_i \in M_{k_i}$, we have $f(y) = \varphi_{k_1}(x_1) \dots \varphi_{k_n}(x_n)$. We can extend this to linear combinations of such y and it follows that f is indeed a $*$-homomorphism from $\mathcal{M}$ to $\mathcal{N}$. We would like to show that this map is well defined. Let Y_1 and Y_2 be two expressions for y. From previous proposation, we have

$$\tau_M(\phi_{k_1}(x_1) \dots \phi_{k_n}(x_n)) = \tau_N(\varphi_{k_1}(x_1) \dots \varphi_{k_n}(x_n)).$$

It follows that

$$0 = \tau_M((Y_1 - Y_2)^*(Y_1 - Y_2)) = \tau_N((f(Y_1) - f(Y_2))^*(f(Y_1) - f(Y_2))).$$

Since τ_N is faithful, we get $f(Y_1) = f(Y_2)$.

Lastly, since f is trace preserving, it follows that f extends to a trace preserving isomorphism from M to N by Proposition 2.10. $\square$

Definition 2.26. Let $(M_1, \tau_1), (M_2, \tau_2)$ be tracial von Neumann algebras that satisfies and $((M, \tau), \phi_1, \phi_2)$ be such that it satisfies the conditions (i) and (ii) of the previous proposition. We define (M, τ) to be the free product of (M_1, τ_1) and (M_2, τ_2). We denote M as $M_1 * M_2$.

Example 2.27. We consider the free product of $(L(G_1), \tau_1)$ and $(L(G_2), \tau_2)$. One can check that $L(G_1 * G_2)$ satisfies the condition 1 and 2 of Proposition 2.25. Therefore, it follows that $(L(G_1 * G_2), \tau)$ is isomorphic to $(L(G_1), \tau_1) * (L(G_2), \tau_2)$.

Next we would like to introduce the amalgamated free product of two tracial von Neumann algebras.

For $i = 1, 2$, let (A_i, τ_i) be finite von Neumann algebras, $Q \subset A_i$ be a common von Neumann subalgebra, and $E_i : A_i \to Q$ be faithful, normal conditional expectations. The *amalgamated free product* $(A, E) = (A_1, E_1) *_Q (A_2, E_2)$ is a pair of a von Neumann algebra A generated by A_1 and A_2 and a faithful normal conditional expectation $E : A \to Q$ such that A_1 and A_2 are *freely independent* with respect to E: $E(a_1 a_2 \cdots a_k) = 0$ whenever $a_{n_i} \in A_{n_i}$ with $n_i \in \{1, 2, \}, E_{n_i}(a_i) = 0$ and $n_1 \neq n_2 \neq \cdots \neq n_k$. An element $a_1 a_2 \cdots a_k \in A$ will be called an *alternating centered word* with respect to E if $a_{n_i} \in A_{n_i}$ with $n_i \in \{1, 2\}, E_{n_i}(a_i) = 0$ and $n_1 \neq n_2 \neq \cdots \neq n_k$.

For the construction and further details on (amalgamated) free products, we refer the reader to [25, 26, 27, 28].

2.8. Graph product

Let $\mathcal{G} = (V, E)$ be a simple graph with the vertex set V and the edge set $E \subseteq V \times V \setminus \{(v, v) : v \in V\}$. We assume that the graph $\mathcal{G}$ is non-oriented, i.e., $(v, w) \in E$ if and only if $(w, v) \in E$. For a vertex $v \in V$, we let $\mathsf{lk}(v)$ denote the set of all vertices that are connected to v by an edge, i.e., $\mathsf{lk}(v) = \{w \in V : (v, w) \in E\}$; and we let $\mathsf{st}(v) = \mathsf{lk}(v) \cup \{v\}$. If $U \subseteq V$, then the full subgraph of $\mathcal{G}$ with the vertex set U is the graph with U as the vertex set and $v, w \in U$ are connected by an edge if and only if $(v, w) \in E$. Abusing the notation slightly, we will use $\mathsf{lk}(v)$ and $\mathsf{st}(v)$ to denote the full subgraph of $\mathcal{G}$ with vertex sets $\mathsf{lk}(v)$ and $\mathsf{st}(v)$, respectively.

A word $v_1 v_2 \cdots v_n$ of vertices in V is called *reduced* if it satisfies the following property: if there exist $k < l$ such that $v_k = v_l$, then there is some $k < j < l$ such that $(v_k, v_j) \notin E$. Let $\mathcal{G} = (V, E)$ be a simple graph, (A, τ) be a tracial von Neumann algebra, and $\{(A_v, \tau_v) : v \in V\}$ be a family of tracial von Neumann subalgebras of (A, τ) such that $\tau|_{A_v} = \tau_v$ for all $v \in V$. We say that the family $\{(A_v, \tau_v) : v \in V\}$ is *$\mathcal{G}$-independent* if the following property holds: if $v_1 \cdots v_n$ is a reduced word and $a_1, \ldots, a_n \in A$ are such that $a_i \in A_{v_i}$ and $\tau(a_i) = 0$, then $\tau(a_1 \cdots a_n) = 0$. On the other hand, given a simple graph $\mathcal{G} = (V, E)$ and a family of tracial von Neumann algebras $\{(A_v, \tau_v) : v \in V\}$, there is a unique, up to isomorphism, tracial von Neumann algebra (A, τ), called the *graph product* over $\mathcal{G}$ of the family $\{(A_v, \tau_v) : v \in V\}$, and trace-preserving inclusions $\varphi_v : A_v \hookrightarrow A$ such that the family $\{\varphi_v(A_v) : v \in V\}$ is $\mathcal{G}$-independent and generates A as a von Neumann algebra (see [29, 30]). We denote the graph product (A, τ) of the family $\{(A_v, \tau_v) : v \in V\}$ by

$$(A, \tau) = \bigstar_{v \in V} (A_v, \tau_v).$$

Given a simple graph $\mathcal{G} = (V, E)$, a family of tracial von Neumann algebras $\{(A_v, \tau_v) :$

$v \in V\}$, and a vertex $v \in V$, we let

$$A_{\mathsf{st}(v)} = \bigstar_{w \in \mathsf{st}(v)} A_w, \quad A_{V'} = \bigstar_{w \in V \setminus \{v\}} A_w, \quad \text{and} \quad A_{\mathsf{lk}(v)} = \bigstar_{w \in \mathsf{lk}(v)} A_w.$$

Then, by the unscrewing technique of Caspers and Fima ([30, Theorem 3.26]), there exists a unique trace-preserving $*$-isomorphism $\Phi : A_{\mathsf{st}(v)} *_{A_{\mathsf{lk}(v)}} A_{V'} \to \bigstar_{v \in V} A_v$ such that $\Phi|_{\mathsf{st}(v)}$ and $\Phi|_{A_{V'}}$ are the canonical inclusions $A_{\mathsf{st}(v)} \subset A$ and $A_{V'} \subset A$, respectively. We remark that one can also view A as $(A_v \overline{\otimes} A_{\mathsf{lk}(v)}) *_{A_{\mathsf{lk}(v)}} A_{V'}$, a viewpoint that will be useful in proving Theorem 4.1.

Remark 2.28. If $\mathcal{G} = (V, E)$ is a simple graph, and $\{\Gamma_v : v \in V\}$ is a family of countable discrete groups, then $L(\bigstar_{v \in V} \Gamma_v) = \bigstar_{v \in V} L\Gamma_v$ (see [30, Remark 3.23]).

2.9. Modules over tracial von Neumann algebras

For details on the proofs of the facts collected in this subsection, we refer the reader to [24, Chapter 8].

Definition 2.29. Given von Neumann algebras A and B,

1. a *left A-module* is a pair $(\mathcal{H}, \pi_A)$, where $\mathcal{H}$ is a Hilbert space and $\pi_A : A \to \mathcal{B}(\mathcal{H})$ is a normal unital $*$-homomorphism.

2. a *right B-module* is a pair $(\mathcal{H}, \pi_B)$, where $\mathcal{H}$ is a Hilbert space and $\pi_B : B \to \mathcal{B}(\mathcal{H})$ is a normal unital $*$-anti-homomorphism, i.e., $\pi_B(xy) = \pi_B(y)\pi_B(x)$ for all $x, y \in B$. In other words, $\mathcal{H}$ is a left B^{op}-module, where B^{op} is the opposite algebra.

3. an *$A - B$-bimodule* is a triple $(\mathcal{H}, \pi_A, \pi_B)$ such that $(\mathcal{H}, \pi_A)$ is a left A-module, $(\mathcal{H}, \pi_B)$ is a right B-module, and the representations π_A and π_B commute.

For $\xi \in \mathcal{H}$, $x \in A$, and $y \in B$, we will write $x\xi y$ instead of $\pi_A(x)\pi_B(y)\xi$ ($= \pi_B(y)\pi_A(x)\xi$).

Definition 2.30. Let $(A, \tau_A), (B, \tau_B)$ be tracial von Neumann algebras and let $\mathcal{H}$ be an $A - B$-bimodule. A vector $\xi \in \mathcal{H}$ is called

1. *tracial* if $\langle x\xi, \xi\rangle = \tau_A(x)$ for every $x \in A$, and $\langle \xi y, \xi\rangle = \tau_B(y)$ for every $y \in B$.

2. *bi-tracial* if $\langle x\xi y, \xi\rangle = \tau_A(x)\tau_B(y)$ for all $x \in A, y \in B$.

3. *cyclic* $\overline{\text{Span}\{x\xi y : x \in A, y \in B\}} = \mathcal{H}$.

Example 2.31. The Hilber tspace $L^2(M, \tau)$is an example of a M-M bimodule. Its structure as an M-M bimodule is given by

$$x\xi y = L_x R_y \xi = xJy^*J\xi,$$

for all $x, y \in M, \xi \in L^2(M)$. Notice that $1 \in L^2(M)$ is tracial and cyclic.

Let (Q, τ_Q) be a tracial von Neumann algebra. Given two left Q-modules $\mathcal{H}$ and $\mathcal{K}$, we denote by ${}_Q\mathcal{B}(\mathcal{H}, \mathcal{K})$ the space of left Q-linear bounded maps from $\mathcal{H}$ into $\mathcal{K}$, that is

$${}_Q\mathcal{B}(\mathcal{H}, \mathcal{K}) = \{T \in \mathcal{B}(\mathcal{H}, \mathcal{K}) : T(x\xi) = x(T\xi) \text{ for all } x \in Q, \xi \in \mathcal{H}\}.$$

We set ${}_Q\mathcal{B}(\mathcal{H}) = {}_Q\mathcal{B}(\mathcal{H}, \mathcal{H})$. It is straightforward to check that ${}_Q\mathcal{B}(\mathcal{H}) = Q' \cap \mathcal{B}(\mathcal{H})$. Moreover, ${}_Q\mathcal{B}(\mathcal{H})$ is a semi-finite von Neumann algebra equipped with a specific semi-finite trace Tr, depending on τ_Q. Before stating the result that characterizes Tr, observe that, given $S, T \in {}_Q\mathcal{B}(L^2Q, \mathcal{H})$, we have $TS^* \in {}_Q\mathcal{B}(\mathcal{H})$, and $S^*T \in JQJ$, where $J : L^2Q \to L^2Q$ is the canonical conjugation operator. The following is a translation of [24, Proposition 8.4.2] for left Q-modules.

Proposition 2.32. *If $\mathcal{H}$ is a left Q-module over a tracial von Neumann algebra (Q, τ_Q), then the commutant ${}_Q\mathcal{B}(\mathcal{H}) = Q' \cap \mathcal{B}(\mathcal{H})$ is a semi-finite von Neumann algebra equipped with a canonical faithful normal semi-finite trace* Tr *characterized by the equation*

$$\mathrm{Tr}(TT^*) = \tau_Q(JT^*TJ)$$

for every left Q-linear bounded operator $T : L^2Q \to \mathcal{H}$.

Remark 2.33. Suppose (Q, τ_Q) is a tracial von Neumann algebra and $\mathcal{H}$ is a left Q-module. If $\xi \in \mathcal{H}$ is a tracial vector, then the orthogonal projection $P : \mathcal{H} \to \overline{\mathrm{Span}(Q\xi)}$ lies in ${}_Q\mathcal{B}(\mathcal{H})$. Moreover, since ξ is tracial, the operator $U : L^2Q \to \overline{\mathrm{Span}(Q\xi)}$ given by $U\hat{x} = x\xi, x \in Q$ is a unitary. Extending U to an isometry from L^2Q into $\mathcal{H}$ in an obvious way and applying Proposition 2.32 to $T = PU : L^2Q \to \mathcal{H}$ yields that $\mathrm{Tr}(P) = \tau(1) = 1$.

2.10. Actions on semi-finite von Neumann algebras

For a semi-finite von Neumann algebra $\mathcal{M}$ with a faithful normal semi-finite trace Tr, the set $\mathfrak{n}_{\mathrm{Tr}} = \{x \in \mathcal{M} \mid \mathrm{Tr}(x^*x) < \infty\}$ is an ideal. Left multiplication of $\mathcal{M}$ on $\mathfrak{n}_{\mathrm{Tr}}$ induces a normal faithful representation of $\mathcal{M}$ in $\mathcal{B}(L^2(\mathcal{M}, \mathrm{Tr}))$, called *the standard representation*, where $L^2(\mathcal{M}, \mathrm{Tr})$ is the Hilbert space completion of $\mathfrak{n}_{\mathrm{Tr}}$ under the inner product $\langle a, b\rangle_2 = \mathrm{Tr}(b^*a)$.

If $\Gamma \curvearrowright^{\sigma} \mathcal{M}$ is a trace-preserving action of a countable discrete group Γ on $\mathcal{M}$, then Γ preserves the $\|\cdot\|_2$-norm on $\mathfrak{n}_{\mathrm{Tr}}$. Therefore, restricted to $\mathfrak{n}_{\mathrm{Tr}}$, the action is isometric with respect to the $\|\cdot\|_2$-norm and hence gives a unitary representation $\sigma^0 : \Gamma \to \mathcal{U}(L^2(\mathcal{M}, \mathrm{Tr}))$, called the *Koopman representation*. Considering $\mathcal{M} \subset \mathcal{B}(L^2(\mathcal{M}, \mathrm{Tr}))$ via the standard representation, we have that the action $\sigma : \Gamma \to \mathrm{Aut}(\mathcal{M}, \mathrm{Tr})$ is unitarily implemented via the Koopman representation, i.e., for $x \in \mathcal{M}$ and $\gamma \in \Gamma$ we have $\sigma_\gamma(x) = \sigma^0_\gamma x \sigma^0_{\gamma^{-1}}$ (see [31, Theorem 3.2]).

3. Von Neumann Orbit Equivalence

In this section, we define von Neumann orbit equivalence for tracial von Neumann algebras and for countable discrete groups. We shall see that groups are von Neumann orbit equivalent if and only if the corresponding group von Neumann algebras are, and we conclude this section with the proof that von Neumann orbit equivalent tracial von Neumann algebras are von Neumann equivalent in the sense of [13].

3.1. Von Neumann orbit equivalence for tracial von Neumann algebras

Theorem 3.1. *Let (A, τ_A), and (B, τ_B) be tracial von Neumann algebras. Then the following are equivalent.*

1. *There exists a finite von Neumann algebra (Q, τ_Q) and a pointed $A \overline{\otimes} Q - B$-bimodule $(\mathcal{H}, \xi)$ such that $\xi \in \mathcal{H}$ is a cyclic and (bi-)tracial vector for both $A \overline{\otimes} Q$-module structure, and $Q - B$-bimodule structure. That is, for all $a \in A, b \in B$, and $x \in Q$,*
   - *(a) $\langle (a \otimes x)\xi, \xi \rangle = \tau_A(a)\tau_Q(x)$, and $\langle x\xi b, \xi \rangle = \tau_Q(x)\tau_B(b)$; and*
   - *(b) $\overline{\mathrm{Span}((A \overline{\otimes} Q)\xi)} = \mathcal{H} = \overline{\mathrm{Span}(Q\xi B)}$.*
2. *There exists a tracial von Neumann algebra (Q, τ_Q), and a normal $*$-homomorphism $\phi : B \to A \overline{\otimes} Q$ such that*
   - *(a) $\mathbb{E}_Q \circ \phi = \tau_B$, where $\mathbb{E}_Q : A \overline{\otimes} Q \to Q$ is the normal conditional expectation; and*

(b) $\overline{\mathrm{Span}\{x\phi(b) : b \in B, x \in Q\}}^{\|\cdot\|_2} = L^2(A \overline{\otimes} Q)$.

Proof. Let $(\mathcal{H}, Q, \xi)$ be a triple as in (1). We thus obtain a canonical unitary $U\colon \mathcal{H} \to L^2(A\overline{\otimes}Q)$ such that $U(y\xi) = \hat{y}$ for all $y \in A\overline{\otimes}Q$. Hence we can define a right action of B on $L^2(A\overline{\otimes}Q)$ by

$$\eta \cdot b = U(U^*(\eta)b), \text{ for all } \eta \in L^2(A\overline{\otimes}Q),\, b \in B.$$

For $b \in B$, we let $\rho(b) \in \mathcal{B}(L^2(A\overline{\otimes}Q))$ be the operator corresponding to right multiplication by b. Since $\mathcal{H}$ is an $A\overline{\otimes}Q - B$ bimodule, and U is $A\overline{\otimes}Q$-linear, so, the right action of B commutes with the left action of $A\overline{\otimes}Q$ on $L^2(A\overline{\otimes}Q)$, and hence $\rho(b) \in (A\overline{\otimes}Q)' \cap \mathcal{B}(L^2(A\overline{\otimes}Q))$ for every $b \in B$. Since the commutant of $A\overline{\otimes}Q$ acting on $L^2(A\overline{\otimes}Q)$ is $R(A\overline{\otimes}Q)$ we define $\phi\colon B \to A\overline{\otimes}Q$ as follows: for $b \in B$, $\phi(b)$ is the unique element in $A\overline{\otimes}Q$ such that $R_{\phi(b)} = \rho(b)$. This is directly checked to be a $*$-homomorphism. Moreover, by definition of ϕ, we have that $\eta \cdot b = \eta\phi(b)$, for every $\eta \in L^2(A\overline{\otimes}Q), b \in B$. Since ξ is Q-B bi-tracial, we have for all $b \in B, x \in Q$ that

$$\begin{aligned}
\tau_Q(x)\tau_B(b) &= \langle x\xi b, \xi\rangle \\
&= \langle U^*(x)b, U^*(\hat{1})\rangle \\
&= \langle U(U^*(x)b), \hat{1}\rangle \\
&= \langle x \cdot b, \hat{1}\rangle \\
&= \langle x\phi(b), \hat{1}\rangle \\
&= \tau(x\phi(b))
\end{aligned}$$

where τ denotes the trace on $A\overline{\otimes}Q$. Furthermore, since $\tau_Q \circ \mathbb{E}_Q = \tau$, we have

$$\tau_Q(\tau_B(b)x) = \tau(x\phi(b)) = \tau_Q(\mathbb{E}_Q(x\phi(b))) = \tau_Q(x\mathbb{E}_Q(\phi(b))),$$

for all $x \in Q, b \in B$, whence it follows that $\mathbb{E}_Q \circ \phi = \tau_B$. Finally,

$$\overline{\mathrm{Span}\{x\phi(b) : x \in Q, b \in B\}}^{\|\cdot\|_2} = U(\overline{\mathrm{Span}\{x\xi b : x \in Q, b \in B\}}) = U(\mathcal{H}) = L^2(A\overline{\otimes}Q).$$

Conversely, suppose condition (2) holds, and let $\mathcal{H} = L^2(A\overline{\otimes}Q)$, and $\xi = \hat{1}$. Define a right action of B on $\mathcal{H}$ by $\eta \cdot b = \eta\phi(b)$. Then for $x \in Q$,$b \in B$ we have

$$\overline{\mathrm{Span}\{x\xi \cdot b : x \in Q, b \in B\}}^{\|\cdot\|_2} = \overline{\mathrm{Span}\{x\phi(b) : x \in Q, b \in B\}}^{\|\cdot\|_2} = L^2(A\overline{\otimes}Q),$$

and for $x \in Q, b \in B$ we have

$$\langle x\xi \cdot b, \xi\rangle = \langle x\phi(b), \hat{1}\rangle = \tau(x\phi(b)) = \tau_Q(x\mathbb{E}_Q(\phi(b))) = \tau_Q(x)\tau_B(b).$$

$\square$

Remark 3.2. Since the operation of taking adjoints is isometric on $L^2(A\overline{\otimes}Q)$, in condition (2) of Theorem 3.1, one might equivalently require that

$$\overline{\mathrm{Span}\{\phi(b)x : b \in B, x \in Q\}}^{\|\cdot\|_2} = L^2(A\overline{\otimes}Q).$$

Definition 3.3. Let (A, τ_A), and (B, τ_B) be tracial von Neumann algebras. We say that A is *von Neumann orbit equivalent* to B, denoted $A \sim_{\mathrm{vNOE}} B$, if either of the two equivalent conditions in Theorem 3.1 is satisfied. If A is von Neumann orbit equivalent to B, then the triple $(\mathcal{H}, Q, \xi)$ or the pair (Q, ϕ) of Theorem 3.1 will be called a vNOE-*coupling* between A and B.

Proposition 3.4. *Von Neumann orbit equivalence is an equivalence relation.*

Proof. If (A, τ_A) is a tracial von Neumann algebra, then taking $Q = \mathbb{C}, \mathcal{H} = L^2(A)$, and $\xi = \hat{1} \in L^2(A)$ in condition (1) of Theorem 3.1 shows that $A \sim_{\mathrm{vNOE}} A$. To see symmetry, let (A, τ_A) and (B, τ_B) be tracial von Neumann algebras satisfying condition (1) of Theorem 3.1, and let $\mathcal{H}, (Q, \tau_Q)$, and $\xi \in \mathcal{H}$ be as in the condition. Note that we can view $\mathcal{H}$ as an $A - B \overline{\otimes} Q^{\mathrm{op}}$-bimodule. Consider the conjugate Hilbert space $\overline{\mathcal{H}}$, and the corresponding canonical $B \overline{\otimes} Q^{\mathrm{op}} - A$ bimodule structure on $\overline{\mathcal{H}}$. Then, it is straightforward to check that the triple $(\overline{\mathcal{H}}, Q^{\mathrm{op}}, \bar{\xi})$ satisfies (1) of Theorem 3.1 and thus, $B \sim_{\mathrm{vNOE}} A$. To show transitivity, we will use condition (2) of Theorem 3.1. To this end, let $(A, \tau_A), (B, \tau_B)$, and (C, τ_C) be tracial von Neumann algebras. Let Q_1, Q_2, and $\phi_1 : B \to A \overline{\otimes} Q_1, \phi_2 : C \to B \overline{\otimes} Q_2$ be as in (2) of Theorem 3.1. Let $Q = Q_1 \overline{\otimes} Q_2$ and let $\phi : C \to A \overline{\otimes} Q$ be given by

$$\phi(c) = (\phi_1 \otimes \mathrm{id}_{Q_2})(\phi_2(c)), \quad c \in C,$$

where $\phi_1 \otimes \mathrm{id}_{Q_1} : B \overline{\otimes} Q_2 \to A \overline{\otimes} Q_1 \overline{\otimes} Q_2$ is the natural extension of $\phi_1 : B \to A \overline{\otimes} Q_1$. Let $\mathbb{E}_{Q_1} : A \overline{\otimes} Q_1 \to Q_1, \mathbb{E}_{Q_2} : B \overline{\otimes} Q_2 \to Q_2,$ and $\mathbb{E}_Q : A \overline{\otimes} Q_1 \overline{\otimes} Q_2 \to Q_1 \overline{\otimes} Q_2$ be normal conditional expectations. Consider the map $\mathbb{E}_{Q_1} \otimes \mathrm{id}_{Q_2} : A \overline{\otimes} Q_1 \overline{\otimes} Q_2 \to Q_1 \overline{\otimes} Q_2$. Note that $\mathbb{E}_Q = \mathbb{E}_{Q_1} \otimes \mathrm{id}_{Q_2}$. Therefore,

$$\mathbb{E}_Q \circ \phi = (\mathbb{E}_{Q_1} \otimes \mathrm{id}_{Q_2}) \circ ((\phi_1 \otimes \mathrm{id}_{Q_2}) \circ \phi_2) = \mathbb{E}_{Q_2} \circ \phi_2 = \tau_C,$$

where the second to last equality follows from the fact that the following diagram, since $\mathbb{E}_{Q_1} \circ \phi_1 = \tau_B$, is commutative:

$$\begin{tikzcd}
B \overline{\otimes} Q_2 \arrow[r, "\phi_1 \otimes \mathrm{id}_{Q_2}"] \arrow[d, "\mathbb{E}_{Q_2}"'] & A \overline{\otimes} Q_1 \overline{\otimes} Q_2 \arrow[d, "\mathbb{E}_{Q_1} \otimes \mathrm{id}_{Q_2}"] \\
Q_2 \arrow[r, hook, "1 \otimes \mathrm{id}_{Q_2}"'] & Q_1 \overline{\otimes} Q_2
\end{tikzcd}$$

Now, consider $V = \overline{\mathrm{Span}\{\phi(c)x : x \in Q, c \in C\}}^{\|\cdot\|_2}$, and note that V is invariant under multiplication on the right by elements of $Q = Q_1 \overline{\otimes} Q_2$. In the light of Remark 3.2, it suffices to show that $V = L^2(A \overline{\otimes} Q)$, and for this, since V is invariant under right multiplication by Q, it suffices to show that $A \otimes 1 \otimes 1 \subseteq V$. Recall that

$$\overline{\operatorname{Span}\{\phi_2(c)x_2 : c \in C, x_2 \in Q_2\}}^{\|\cdot\|_2} = L^2(B \mathbin{\overline{\otimes}} Q_2) \supseteq B \mathbin{\overline{\otimes}} Q_2.$$

Hence, we have

$$\overline{\operatorname{Span}\{(\phi_1 \otimes \mathrm{id}_{Q_2})(\phi_2(c)(1 \otimes x_2)) : c \in C, x_2 \in Q_2\}}^{\|\cdot\|_2} \supseteq (\phi_1 \otimes \mathrm{id}_{Q_2})(B \mathbin{\overline{\otimes}} Q_2).$$

Since $\overline{\operatorname{Span}\{\phi_1(b)x_1 : b \in B, x_1 \in Q_1\}}^{\|\cdot\|_2} = L^2(A \mathbin{\overline{\otimes}} Q_1) \supseteq A \mathbin{\overline{\otimes}} Q_1$, the following computation completes the proof.

$$\begin{aligned} V &\supseteq \overline{\operatorname{Span}\{(\phi_1 \otimes \mathrm{id}_{Q_2})(\phi_2(c))(x_1 \otimes x_2) : c \in C, x_1 \in Q_1, x_2 \in Q_2\}}^{\|\cdot\|_2} \\ &\supseteq \overline{\operatorname{Span}\{(\phi_1 \otimes \mathrm{id}_{Q_2})((1 \otimes x_2)(\phi_2(c))(x_1 \otimes 1) : c \in C, x_1 \in Q_1, x_2 \in Q_2\}}^{\|\cdot\|_2} \\ &\supseteq \overline{\operatorname{Span}\{(\phi_1 \otimes \mathrm{id}_{Q_2})(b \otimes 1)(x_1 \otimes 1) : b \in B, x_1 \in Q_1\}}^{\|\cdot\|_2} \\ &\supseteq A \otimes 1 \otimes 1. \end{aligned}$$

□

Checking $\overline{\operatorname{Span}\{\phi(b)x : b \in B, x \in Q\}} = L^2(A\overline{\otimes}Q)$ might not be easy in general. However, the following lemma simplifies verifying it in certain examples.

Lemma 3.5. *Let $(A, \tau_A), (B, \tau_B)$, and (Q, τ_Q) be tracial von Neumman algebras. Let $\phi\colon B \to A\overline{\otimes}Q$ be a $*$-homomorphism satisfying $\mathbb{E}_Q \circ \phi = \tau_B$, where $\mathbb{E}_Q : A \mathbin{\overline{\otimes}} Q \to Q$ is the normal conditional expectation. Let $V = \overline{\operatorname{Span}\{\phi(b)x : b \in B, x \in Q\}}^{\|\cdot\|_2} \subset L^2(A\overline{\otimes}Q)$. Then $N = \{a \in A : a \otimes 1 \in V\}$ is an SOT-closed subalgebra of A.*

Proof. The fact that N is SOT-closed follows from the fact that SOT-convergence in A implies $\|\cdot\|_2$-convergence. First note that V is invariant under left multiplication by elements of $\phi(B)$ and right multiplication by elements of Q. We prove the following claim, whence the lemma follows immediately.

Claim: For $\eta \in V$, and $a \in N$ we have that $\eta(a \otimes 1) \in V$.

Proof of Claim. Given $\eta \in V$, and $a \in N$, let $\{x_n\}_{n\in\mathbb{N}} \subset \mathrm{Span}\{\phi(b)x : b \in B, x \in Q\}$ be such that $\|x_n - \eta\|_2 \to 0$. Since $a \otimes 1$ is bounded, it follows that

$$\|x_n(a \otimes 1) - \eta(a \otimes 1)\|_2 \leq \|x_n - \eta\|_2 \|a\| \to 0,$$

as $n \to \infty$. Since V is $\|\cdot\|_2$-closed, it suffices to show that $x_n(a \otimes 1) \in V$ for all $n \in \mathbb{N}$. To this end, fix $n \in \mathbb{N}$, and write $x_n = \sum_{j=1}^k \phi(b_j)y_j$ with $b_j \in B, y_j \in Q$. Then,

$$x_n(a \otimes 1) = \sum_{j=1}^k \phi(b_j)y_j(a \otimes 1) = \sum_{j=1}^k \phi(b_j)(a \otimes 1)y_j,$$

where, in the last equality, we use that A and Q commute in $A\overline{\otimes}Q$. Since we already noted that V is invariant under left multiplication by $\phi(B)$ and right multiplication by Q, and $a \otimes 1 \in V$, it follows that $x_n(a \otimes 1) \in V$. $\square$

Remark 3.6. We do *not* know if N is a $*$-subalgebra.

Theorem 3.7. *Let $A_i, B_i, i = 1, 2$ be tracial von Neumann algebras such that $A_i \sim_{\mathrm{vNOE}} B_i, i = 1, 2$. Then, $A_1 * A_2 \sim_{\mathrm{vNOE}} B_1 * B_2$.*

Proof. Since vNOE is an equivalence relation, it suffices to show that if $A \sim_{\mathrm{vNOE}} B$ and if (C, τ_C) is another tracial von Neumann algebra, then $A * C \sim_{\mathrm{vNOE}} B * C$. Let (Q, τ_Q) be a tracial von Neumann algebra and $\phi\colon B \to A\overline{\otimes}Q$ be a $*$-homomorphism as in condition (2) of Theorem 3.1. For tracial von Neumann algebra (C, τ_C), we view $(A*C)\overline{\otimes}Q$ as $(A\overline{\otimes}Q)*_Q(C\overline{\otimes}Q)$. Define $\widetilde{\phi}_0\colon B*_{\mathrm{alg}}C \to (A\overline{\otimes}Q)*_Q(C\overline{\otimes}Q)$ by declaring that $\widetilde{\phi}_0(b) = \phi(b)*_Q 1$ for $b \in B$ and $\widetilde{\phi}_0(c) = 1*_Q c$ for $c \in C$. Let $\mathbb{E}_Q : (A*C)\,\overline{\otimes}\,Q \to Q$ be the normal conditional expectation. Note that $\mathbb{E}_Q\circ\phi = \tau_B$ and $\mathbb{E}_Q|_{C\overline{\otimes}Q} = \tau_C\otimes\mathrm{id}_Q$. Therefore, if $x \in B *_{\mathrm{alg}} C$ is an alternating centered word with respect to τ_{B*C}, then $\widetilde{\phi}_0(x)$ is an alternating centered word with respect to $\mathbb{E}_Q$. Hence $\mathbb{E}_Q \circ \widetilde{\phi}_0 = \tau_{B*C}$. Since $\mathbb{E}_Q$ is trace preserving, it follows that $\widetilde{\phi}_0$ is trace-preserving, and thus extends to a unique trace-preserving $*$-homomorphism $\widetilde{\phi}\colon B*C \to (A\overline{\otimes}Q)*_Q(C\overline{\otimes}Q)$. Moreover, by continuity we still have $\mathbb{E}_Q \circ \widetilde{\phi} = \tau_{B*C}$. In light of Remark 3.2, it thus remains to check that $\overline{\mathrm{Span}\{\widetilde{\phi}(x)y : x \in B*C, y \in Q\}}^{\|\cdot\|_2} = L^2((A*C)\,\overline{\otimes}\,Q)$. To this end, set $V = \overline{\mathrm{Span}\{\widetilde{\phi}(x)y : x \in B*C, y \in Q\}}^{\|\cdot\|_2}$. Since V is invariant under right multiplciation by Q, to show that $V = L^2((A*C)\,\overline{\otimes}\,Q)$, it suffices to show that $(A*C)\otimes 1 \subseteq V$. For this, by Lemma 3.5, it suffices to show that V contains $A\otimes 1$ and $C\otimes 1$. That $C\otimes 1 \subseteq V$, follows from the fact that $\widetilde{\phi}$ takes the copy of C in $B*C$ to the copy of C in $(A*C)\overline{\otimes}Q$, and since $\overline{\mathrm{Span}\{\phi(b)y : b \in B, y \in Q\}}^{\|\cdot\|_2} = L^2(A\overline{\otimes}Q)$, we also have that $A\otimes 1 \subseteq V$. $\square$

3.2. Von Neumann orbit equivalence for groups

Let $\Gamma \curvearrowright^{\sigma} \mathcal{M}$ be an action of a countable discrete group Γ on a von Neumann algebra $\mathcal{M}$. A *fundamental domain* for the action is a projection $p \in \mathcal{M}$ such that the projections are $\{\sigma_\gamma(p)\}_{\gamma\in\Gamma}$ are pairwise orthogonal and $\sum_{\gamma\in\Gamma}\sigma_\gamma(p) = 1$, where the sum converges in the strong operator topology. Two countable discrete groups Γ and Λ are said to be *von Neumann equivalent*, denoted $\Gamma \sim_{\mathrm{vNE}} \Lambda$, if there exists a semi-finite von Neumann algebra $(\mathcal{M}, \mathrm{Tr})$ with a faithful normal semi-finite trace Tr, and commuting trace-preserving actions $\Gamma \curvearrowright^{\sigma} \mathcal{M}$ and $\Lambda \curvearrowright^{\alpha} \mathcal{M}$ such that each action admits a finite-trace fundamental domain. Such an $\mathcal{M}$ is called a *von Neumann coupling* between Γ and Λ.

Definition 3.8. Two countable groups Γ and Λ are said to be *von Neumann orbit equivalent*, denoted $\Gamma \sim_{\mathrm{vNOE}} \Lambda$ if there exits a von Neumann coupling between Γ and Λ with a common fundamental domain.

Theorem 3.9. *For countable discrete groups Γ and Λ, $\Gamma \sim_{\mathrm{vNOE}} \Lambda$ if and only if $L\Gamma \sim_{\mathrm{vNOE}} L\Lambda$.*

Proof. First suppose that $L\Gamma \sim_{\mathrm{vNOE}} L\Lambda$, and let $(\mathcal{H}, Q, \xi)$ be a triple as in condition (1) of Theorem 3.1. Set $A = L\Gamma$ and $B = L\Lambda$, and consider $\mathcal{M} = Q' \cap \mathcal{B}(H) = {}_Q\mathcal{B}(\mathcal{H})$. For $\gamma \in \Gamma$, let $u_\gamma \in L\Gamma$ be the corresponding unitary and for $T \in \mathcal{B}(\mathcal{H})$, define $\sigma_\gamma(T) = u_\gamma T u_\gamma^*$. Since $L\Gamma$- and Q-actions on $\mathcal{H}$ commute, it follows that $\mathcal{M}$ is invariant under σ_γ for each $\gamma \in \Gamma$, and thus we have an action $\Gamma \curvearrowright^\sigma \mathcal{M}$. Similarly, since $\mathcal{H}$ is a $Q-B$-bimodule, we have an action $\Lambda \curvearrowright^\alpha \mathcal{M}$ given by $\alpha_s(T) = v_s^* T v_s, s \in \Lambda, T \in \mathcal{M}$, where $v_s \in L\Lambda$ is the unitary corresponding to $s \in \Lambda$. It is clear that the actions $\Gamma \curvearrowright^\sigma \mathcal{M}$ and $\Lambda \curvearrowright^\alpha \mathcal{M}$ commute. Let Tr be the canonical trace on $\mathcal{M}$ given by Proposition 2.32. It follows from the tracial property that both $\Gamma \curvearrowright^\sigma \mathcal{M}$ and $\Lambda \curvearrowright^\alpha \mathcal{M}$ are trace-preserving actions. Let $P \in \mathcal{B}(\mathcal{H})$ be the orthogonal projection from $\mathcal{H}$ onto $\overline{\mathrm{Span}(Q\xi)}$. It is straightforward to see that P is Q-linear and thus, $P \in \mathcal{M}$. Moreover, it follows from Remark 2.33 that $\mathrm{Tr}(P) = 1$. Therefore, it only remains to show that P is a fundamental domain for both Γ- and Λ-actions. To see that P is a Γ-fundamental domain, we first note that, since $\mathcal{H} = \overline{\mathrm{Span}(A \overline{\otimes} Q)\xi}$ and ξ is tracial for the $A \overline{\otimes} Q$-module structure, we have, for $a \in A, x \in Q$, that

$$P((a \otimes x)\xi) = \tau_A(a) x \xi.$$

Furthermore, if $a = \sum_{\gamma \in \Gamma} a_\gamma u_\gamma$ is the Fourier series expansion of $a \in A$, then we recall that $\tau_A(a) = a_e$, and thus $P((a \otimes x)\xi) = a_e x\xi$. Therefore,

$$\begin{aligned}\sigma_\gamma(P)((a \otimes x)\xi) &= u_\gamma P u_\gamma^*((a \otimes x)\xi) \\ &= u_\gamma P\left(\left(\sum_{g\in\Gamma} a_g u_{\gamma^{-1}g} \otimes x\right)\xi\right) \\ &= a_\gamma(u_\gamma \otimes x)\xi.\end{aligned}$$

If we let P_γ be the orthogonal projection from $\mathcal{H}$ onto $u_\gamma(\overline{\text{Span}(Q\xi)}$, then it follows from the above calculation that $\sigma_\gamma(P) = P_\gamma$, and it is straightforward to check that the projections $\{P_\gamma\}_{\gamma\in\Gamma}$ are pairwise orthogonal. Moreover, since $\mathcal{H} = \overline{\text{Span}(A\overline{\otimes}Q)\xi)}$, we also get that $\sum_{\gamma\in\Gamma}\sigma_\gamma(P) = 1$ and hence P is a Γ-fundamental domain. Since we also have $\mathcal{H} = \overline{\text{Span}(Q\xi B)}$ and ξ is bi-tracial for the $Q - B$-bimodule structure, we observe that, for $b \in B, x \in Q$,

$$P(x\xi b) = \tau_B(b)x\xi.$$

If $b = \sum_{t\in\Lambda} b_t v_t$ is the Fourier series expansion of $b \in B$, then $\tau_B(b) = b_e$, and hence $P(x\xi b) = b_e x\xi$. For $s \in \Lambda$, let P_s be the orthogonal projection from $\mathcal{H}$ onto $(\overline{\text{Span}(Q\xi)})v_s$, then the following calculation show that $\alpha_s(P) = P_s$, $\{\alpha_s(p)\}_{s\in\Lambda}$ are pairwise orthogonal, and hence P is a Λ-fundamental domain.

$$\begin{aligned}\alpha_s(P)(x\xi b) &= v_s^* P v_s\left(x\xi\sum_{t\in\lambda} b_t v_t\right) \\ &= v_s^*(b_{s^{-1}}x\xi) \\ &= b_{s^{-1}}x\xi v_{s^{-1}}.\end{aligned}$$

Conversely, suppose $\Gamma \sim_{\text{vNOE}} \Lambda$, and let $(\mathcal{M}, \text{Tr})$ be a von Neumann coupling between Γ and Λ with common fundamental domain $p \in \mathcal{M}$ for both $\Gamma \curvearrowright^\sigma \mathcal{M}$ and $\Lambda \curvearrowright^\alpha \mathcal{M}$.

Let $A = L\Gamma, B = L\Lambda, \mathcal{H} = L^2(\mathcal{M}, \mathrm{Tr}) \,\overline{\otimes}\, \ell^2(\Lambda), Q = \mathcal{M}^\Gamma \rtimes \Lambda$, and $\xi = p \otimes \delta_e$. Let τ be the trace on $\mathcal{M}^\Gamma$, which we recall is given by $\tau(x) = \mathrm{Tr}(pxp)$ (see [13, Proposition 4.2]). Consider the action of $L\Gamma$ on $\mathcal{H}$ given by

$$u_\gamma \eta = (\sigma^0_\gamma \otimes \mathrm{id})\eta, \qquad \gamma \in \Gamma, \eta \in \mathcal{H},$$

where σ^0_γ is the Koopman representation of Γ into $\mathcal{U}(L^2(\mathcal{M}, \mathrm{Tr}))$. The action of $\mathcal{M}^\Gamma$ on $\mathcal{H}$ is given by

$$x\eta = (x \otimes \mathrm{id})\eta, \qquad x \in \mathcal{M}^\Gamma, \eta \in \mathcal{H},$$

and Λ acts on $\mathcal{H}$ on the left by

$$v_s \eta = (\alpha^0_s \otimes \lambda_\Lambda(s))\eta, \qquad s \in \Lambda, \eta \in \mathcal{H},$$

where $\lambda_\Lambda : \Lambda \to \mathcal{U}(\ell^2\Lambda)$ is the left regular representation, and $\alpha^0_s : \Lambda \to \mathcal{U}(L^2(\mathcal{M}, \mathrm{Tr}))$ is the Koopman representation implementing the Λ-action. Since the Γ- and Λ-actions on $\mathcal{M}$ commute, the actions defined above make $\mathcal{H}$ into a left $L\Gamma \,\overline{\otimes}\, (\mathcal{M}^\Gamma \rtimes \Lambda)$-module. Furthermore, for $g \in \Gamma, s \in \Lambda$, and $x \in \mathcal{M}^\Gamma$, we have

$$\begin{aligned}
\langle (u_g \otimes xv_s)\xi, \xi\rangle &= \langle (u_g \otimes xv_s)(p \otimes \delta_e), p \otimes \delta_e\rangle \\
&= \langle \sigma_g(x\alpha_s(p)) \otimes \delta_s, p \otimes \delta_e\rangle \\
&= \delta_{s,e}\mathrm{Tr}(x\sigma_g(p)p) \\
&= \delta_{s,e}\delta_{g,e}\mathrm{Tr}(pxp) \\
&= \delta_{s,e}\delta_{g,e}\tau(x) \\
&= \tau_A(u_g)\tau_Q(xv_s).
\end{aligned}$$

Thus, it follows that ξ is tracial for the left $L\Gamma \,\overline{\otimes}\, (\mathcal{M}^\Gamma \rtimes \Lambda)$-module structure. For a fixed $s \in \Lambda$, we have

$$\begin{aligned}
\overline{\operatorname{Span}\{(u_g \otimes xv_s)\xi : g \in \Gamma, x \in \mathcal{M}^\Gamma\}} &= \overline{\operatorname{Span}\{\sigma_g(x\alpha_s(p)) \otimes \delta_s : g \in \Gamma, x \in \mathcal{M}^\Gamma\}} \\
&= \overline{\operatorname{Span}\{x\alpha_s(\sigma_g(p)) \otimes \delta_s : g \in \Gamma, x \in \mathcal{M}^\Gamma\}} \\
&= \overline{\operatorname{Span}\{\alpha_s(\alpha_{s^{-1}}(x)\sigma_g(p)) \otimes \delta_s : g \in \Gamma, x \in \mathcal{M}^\Gamma\}} \\
&= \overline{\operatorname{Span}\{\alpha_s(y\sigma_g(p)) \otimes \delta_s : g \in \Gamma, y \in \mathcal{M}^\Gamma\}} \\
&= (\alpha_s^0 \otimes \lambda_\Lambda(s))(\overline{\operatorname{Span}\{y\sigma_\gamma(p) \otimes \delta_e : g \in \Gamma, y \in \mathcal{M}^\Gamma\}}) \\
&= L^2(\mathcal{M}, \operatorname{Tr}) \,\overline{\otimes}\, \mathbb{C}\delta_s,
\end{aligned}$$

where the last equality follows from the fact that $\overline{\operatorname{Span}\{x\sigma_g(p) : g \in \Gamma, x \in \mathcal{M}^\Gamma\}} = L^2(\mathcal{M}, \operatorname{Tr})$ [13, Proposition 4.2]. Therefore, we have

$$\begin{aligned}
\overline{\operatorname{Span}((A \,\overline{\otimes}\, Q)\xi)} &= \overline{\operatorname{Span}\{(u_g \otimes xv_s)\xi : g \in \Gamma, x \in \mathcal{M}^\Gamma, s \in \Lambda\}} \\
&= \overline{\operatorname{Span}(L^2(\mathcal{M}, \operatorname{Tr}) \,\overline{\otimes}\, \mathbb{C}\delta_s : s \in \Lambda)} = \mathcal{H}
\end{aligned}$$

Finally, the right action of $L\Lambda$ on $\mathcal{H}$ given by

$$\eta v_s = (\operatorname{id} \otimes \rho_\Lambda(s^{-1}))\eta, \qquad s \in \Lambda, \eta \in \mathcal{H},$$

where $\rho_\Lambda : \Lambda \to \mathcal{U}(\ell^2\Lambda)$ is the right regular representation, makes $\mathcal{H}$ into a $Q - L\Lambda$-bimoudle. For $x \in \mathcal{M}^\Gamma$, and $s, t \in \Lambda$, we have

$$\langle xv_s(p \otimes \delta_e)v_t, p \otimes \delta_e\rangle = \langle x\alpha_s(p) \otimes \delta_{st}, p \otimes \delta_e\rangle = \delta_{s,e}\delta_{t,e}\operatorname{Tr}(pxp) = \tau_Q(xv_s)\tau_B(v_t),$$

and hence, ξ is a tracial vector for the $Q - B$-bimodule structure. We recall from the proof of [13, Proposition 4.2], that, since p is Λ-fundamental domain, we have a direct sum decomposition $L^2(\mathcal{M}, \operatorname{Tr}) = \sum_{s \in \Lambda} L^2(\mathcal{M}, \operatorname{Tr})\alpha_s(p)$. Thus, to show that $\overline{\operatorname{Span}(Q\xi B)} = \mathcal{H}$, it suffices to show that, for $s \in \Lambda$, $\overline{\operatorname{Span}\{xv_s\xi v_t : x \in \mathcal{M}^\Gamma, t \in \Lambda\}} =$

$L^2(\mathcal{M}, \mathrm{Tr})\alpha_s(p) \,\overline{\otimes}\, \ell^2\Lambda$. To this end, fix $s \in \Lambda$ and note that

$$\begin{aligned}
\overline{\mathrm{Span}\{xv_s(p \otimes \delta_e)v_t : x \in \mathcal{M}^\Gamma, t \in \Lambda\}} &= \overline{\mathrm{Span}\{\alpha_s(\alpha_{s^{-1}}(x)p) \otimes \delta_{st} : x \in \mathcal{M}^\Gamma, t \in \Lambda\}} \\
&= (\alpha_s^0 \otimes \lambda_\Lambda(s))(\overline{\mathrm{Span}\{yp \otimes \delta_t : y \in \mathcal{M}^\Gamma, t \in \Lambda\}}) \\
&= (\alpha_s^0 \otimes \lambda_\Lambda(s))(L^2(\mathcal{M}, \mathrm{Tr})p \,\overline{\otimes}\, \ell^2\Lambda) \\
&= L^2(\mathcal{M}, \mathrm{Tr})\alpha_s(p) \,\overline{\otimes}\, \ell^2\Lambda
\end{aligned}$$

□

3.3. Relationship to von Neumann equivalence

Definition 3.10 ([13])**.** Let A and B be tracial von Neumann algebras and let $\mathcal{M}$ be a semi-finite von Neumann algebra such that $A \subset \mathcal{M}$ and $B^{\mathrm{op}} \subset \mathcal{M}$.

1. A *fundamental domain for A inside of $\mathcal{M}$* consists of a realization of the standard representation $A \subset \mathcal{B}(L^2(A))$ as an intermediate von Neumann subalgebra $A \subset \mathcal{B}(L^2(A)) \subset \mathcal{M}$. The fundamental domain is *finite* if finite-rank projections in $\mathcal{B}(L^2(A))$ are mapped to finite projections in $\mathcal{M}$.

2. $\mathcal{M}$ is a *von Neumann coupling between A and B* if $B^{\mathrm{op}} \subset A' \cap \mathcal{M}$ and each inclusion $A \subset \mathcal{M}$ and $B^{\mathrm{op}} \subset \mathcal{M}$ has a finite fundamental domain.

Definition 3.11 ([13])**.** Two tracial von Neumann algebras A and B are *von Neumann equivalent*, denoted $A \sim_{\mathrm{vNE}} B$, if there exists a von Neumann coupling between them.

Proposition 3.12. *Let (A, τ_A) and (B, τ_B) be tracial von Neumann algebras. If $A \sim_{\mathrm{vNOE}} B$, then $A \sim_{\mathrm{vNE}} B$.*

Proof. Suppose $(\mathcal{H}, Q, \xi)$ is a triple as in condition (1) of Theorem 3.1. As in the proof of (1) implies (2) in Theorem 3.1, let $U : \mathcal{H} \to L^2(A \overline{\otimes} Q)$ be the unitary such that $U(x\xi) = \hat{x}$ for all $x \in A \overline{\otimes} Q$, and let $\phi : B \to A \overline{\otimes} Q$ be the $*$-homomorphism obtained therein. Let $\mathcal{M} = Q' \cap \mathcal{B}(\mathcal{H}) = \mathcal{B}(L^2(A)) \overline{\otimes} Q^{\mathrm{op}}$. We will show that $\mathcal{M}$ is a von Neumann coupling between A and B. It is clear that the inclusion $A \subset \mathcal{M}$ has a finite fundamental domain. We recall that the argument used in defining ϕ, shows that we have an inclusion $B^{\mathrm{op}} \subset \mathcal{M}$ and moreover, since the left A- and right B-actions on $\mathcal{H}$ commute, we have that $B^{\mathrm{op}} \subset A' \cap \mathcal{M}$. Thus, it only remains to show that the inclusion $B^{\mathrm{op}} \subset \mathcal{M}$ has a finite fundamental domain. To this end, note that, we can also view $\mathcal{H}$ as an $A - B \overline{\otimes} Q^{\mathrm{op}}$-bimoudle, and ξ is tracial and cyclic for the right $B \overline{\otimes} Q^{\mathrm{op}}$-module structure. Thus, by the same construction as above, we get an inclusion $B^{\mathrm{op}} \subset Q^{\mathrm{op}\prime} \cap \mathcal{B}(\mathcal{H}) = \mathcal{B}(L^2(B)) \overline{\otimes} Q$. Since $\mathcal{M} = Q \cap \mathcal{B}(\mathcal{H}) = Q^{\mathrm{op}\prime} \cap \mathcal{B}(H)$, we get that the inclusion $B^{\mathrm{op}} \subset \mathcal{M}$ admits a finite fundamental domain and hence $A \sim_{\mathrm{vNE}} B$. □

4. Graph product and von Neumann orbit equivalence

We now turn to prove Theorem 1.8, and prove the following stronger Theorem.

Theorem 4.1. *Let $\mathcal{G} = (V, E)$ be a simple finite graph. Let A and B be the graph products over $\mathcal{G}$ of tracial vertex von Neumann algebras $\{(A_v, \tau_{A_v}) : v \in V\}$ and $\{(B_v, \tau_{B_v}) : v \in V\}$, respectively. For a subset $U \subseteq V$, we set $A_U = \bigstar_{v \in U} A_v$, and similarly, $B_U = \bigstar_{v \in U} B_v$. There exist a tracial von Neumann algebra (Q, τ_Q) and a $*$-homomorphism $\phi : A \to B \otimes Q$ such that the following hold:*

1. *ϕ satisfies condition (2) of Theorem 3.1;*

2. $\phi(A_U) \subseteq B_U \overline{\otimes} Q$ *for any subset* $U \subseteq V$*; and*

3. $\overline{\operatorname{Span}\{\phi(y)x : y \in A_U, x \in Q\}}^{\|\cdot\|_2} = L^2(B_U \overline{\otimes} Q)$ *for any subset* $U \subseteq V$.

In particular, A is von Neumann orbit equivalent to B.

Remark 4.2. Note that condition (2) of the theorem automatically implies that $\mathbb{E}_{Q,U} \circ \phi = \tau_{A_U}$, where $\mathbb{E}_{Q,U}$ is the conditional expectation from $B_U \overline{\otimes} Q$ onto Q. Indeed, if $\mathbb{E}_Q$ is the conditional expectation from $B \overline{\otimes} Q$ onto Q, then $\mathbb{E}_{Q,U} = \mathbb{E}_Q|_{B_U \overline{\otimes} Q}$, $\tau_{A_U} = \tau_A|_{A_U}$, and $\mathbb{E}_Q \circ \phi = \tau_A$. In particular, (2) and (3) together imply that the pair (Q, ϕ) is a vNOE coupling between A_U and B_U for any subset $U \subseteq V$.

Proof of Theorem 4.1. We proceed by induction on the number of vertices in the graph. The statement is true for a graph with only one vertex. Suppose the theorem is true for graphs with $|V| - 1$ vertices. Fix a vertex $v \in V$ and set $V' = V \setminus \{v\}$. Since $A_v \sim_{\text{vNOE}} B_v$, there exist a tracial von Neumann algebra (Q_1, τ_{Q_1}) and a $*$-homomorphism $\phi_1 : A_v \to B_v \overline{\otimes} Q_1$ satisfying condition (2) in Theorem 3.1. Moreover, by the induction hypothesis, we also have a tracial von Neumann algebra (Q_2, τ_{Q_2}) and a $*$-homomorphism $\phi_2 : A_{V'} \to B_{V'} \overline{\otimes} Q_2$ satisfying (1)-(3) in the statement of the theorem. In particular, we have that $\phi_2(A_{\mathsf{lk}(v)}) \subseteq B_{\mathsf{lk}(v)} \overline{\otimes} Q_2$ and that $\overline{\operatorname{Span}\{\phi_2(y)x : y \in A_{\mathsf{lk}(v)}, x \in Q_2\}}^{\|\cdot\|_2} = L^2(B_{\mathsf{lk}(v)} \overline{\otimes} Q_2)$.

Recall that we have the following decompositions of A and B as amalgamated free products:

$$A = (A_v \overline{\otimes} A_{\mathsf{lk}(v)}) *_{A_{\mathsf{lk}(v)}} A_{V'}, \qquad B = (B_v \overline{\otimes} B_{\mathsf{lk}(v)}) *_{B_{\mathsf{lk}(v)}} B_{V'},$$

Set $Q = Q_1 \overline{\otimes} Q_2$ and let τ_Q be the trace on Q. We now construct a $*$-homomorphism

$$\phi : (A_v \overline{\otimes} A_{\mathsf{lk}(v)}) *_{A_{\mathsf{lk}(v)}} A_{V'} \to (B_v \overline{\otimes} B_{\mathsf{lk}(v)} \overline{\otimes} Q) *_{B_{\mathsf{lk}(v)} \overline{\otimes} Q} (B_{V'} \overline{\otimes} Q).$$

For $x \in A_v, y \in A_{\mathsf{lk}(v)}$, and $z \in A_{V'}$, we define

$$\phi(x \otimes y) = (\phi_1(x) \otimes \phi_2(y)) * 1,$$
$$\phi(z) = 1 * \tilde{\phi}_2(z),$$

where $\tilde{\phi}_2 : A_{V'} \to B_{V'} \overline{\otimes} Q_2 \overline{\otimes} Q_1$ is given by $\tilde{\phi}_2(z) = \phi_2(z) \otimes 1_{Q_1}, z \in A_{V'}$. Note that ϕ is well-defined since $\phi_2(A_{U'}) \subseteq B_{U'} \overline{\otimes} Q_2$, and hence $\tilde{\phi}_2(A_{U'}) \subseteq B_{U'} \overline{\otimes} Q$ for any subset $U' \subseteq V'$.

Let $\mathbb{E}_{B_{\mathsf{lk}(v)} \overline{\otimes} Q} : B \overline{\otimes} Q \to B_{\mathsf{lk}(v)} \overline{\otimes} Q$ and $\mathbb{E}_{A_{\mathsf{lk}(v)}} : A \to A_{\mathsf{lk}(v)}$ be the normal conditional expectations. We show that $\mathbb{E}_{B_{\mathsf{lk}(v)} \overline{\otimes} Q} \circ \phi = \phi \circ \mathbb{E}_{A_{\mathsf{lk}(v)}}$ by showing that an alternating centered word with respect to $\mathbb{E}_{A_{\mathsf{lk}(v)}}$ gets mapped to an alternating centered word with respect to $\mathbb{E}_{B_{\mathsf{lk}(v)} \overline{\otimes} Q}$ under ϕ. Let $x \otimes y \in A_v \overline{\otimes} A_{\mathsf{lk}(v)}$ be a simple tensor such that $\mathbb{E}_{A_{\mathsf{lk}(v)}}(x \otimes y) = 0$. Since $\mathbb{E}_{A_{\mathsf{lk}(v)}}|_{A_v \overline{\otimes} A_{\mathsf{lk}(v)}} = \tau_{A_v} \otimes \mathrm{id}_{A_{\mathsf{lk}(v)}}$, it follows that either $\tau_{A_v}(x) = 0$ or $y = 0$. If $\mathbb{E}_{Q_1} : B_v \overline{\otimes} Q_1 \to Q_1$ is the normal conditional expectation, then we note that $\mathbb{E}_{B_{\mathsf{lk}(v)} \overline{\otimes} Q}|_{B_v \overline{\otimes} B_{\mathsf{lk}(v)} \overline{\otimes} Q} = \mathbb{E}_{Q_1} \otimes \mathrm{id}_{B_{\mathsf{lk}(v)} \overline{\otimes} Q_2}$. Since $\mathbb{E}_{Q_1} \circ \phi_1 = \tau_{A_v}$, we have that

$$\begin{aligned}
(\mathbb{E}_{B_{\mathsf{lk}(v)} \overline{\otimes} Q} \circ \phi)(x \otimes y) &= \mathbb{E}_{B_{\mathsf{lk}(v)} \overline{\otimes} Q}(\phi_1(x) \otimes \phi_2(y)) \\
&= (\mathbb{E}_{Q_1} \otimes \mathrm{id}_{B_{\mathsf{lk}(v)} \overline{\otimes} Q_2})(\phi_1(x) \otimes \phi_2(y)) \\
&= \mathbb{E}_{Q_1}(\phi_1(x)) \otimes \phi_2(y) \\
&= \tau_{A_v}(x) \otimes \phi_2(y) = 0.
\end{aligned}$$

Next, suppose we have $z \in A_{V'}$ such that $\mathbb{E}_{A_{\mathsf{lk}(v)}}(z) = 0$. Since $\mathbb{E}_{B_{\mathsf{lk}(v)} \overline{\otimes} Q}$ is trace-preserving and $B_{\mathsf{lk}(v)} \overline{\otimes} Q$-bimodular, so, to show that $\mathbb{E}_{B_{\mathsf{lk}(v)} \overline{\otimes} Q}(\phi(z)) = 0$, it suffices to show that $\tau_{B_{V'} \overline{\otimes} Q}(w\tilde{\phi}_2(z)) = 0$ for all $w \in B_{\mathsf{lk}(v)} \overline{\otimes} Q$. Furthermore, by induction hypothesis, we have that $L^2(B_{\mathsf{lk}(v)} \overline{\otimes} Q_2) = \overline{\mathrm{Span}\{\phi_2(x)y : x \in A_{\mathsf{lk}(v)}, y \in Q_2\}}$. Therefore, we get that

$$L^2(B_{\mathsf{lk}(v)}\,\overline{\otimes}\,Q) = \overline{\mathrm{Span}\{\tilde{\phi}_2(x)(y_1\otimes y_2) : x\in A_{\mathsf{lk}(v)}, y_1\in Q_1, y_2\in Q_2\}}.$$

Thus, it suffices to show that $\tau_{B_{V'}\overline{\otimes}Q}(\tilde{\phi}_2(x)\tilde{\phi}_2(z)(y_1\otimes y_2)) = 0$ for every $y_1\in Q_1, y_2\in Q_2, x\in A_{\mathsf{lk}(v)}$. If $\mathbb{E}_{Q_2} : B_{V'}\,\overline{\otimes}\,Q_2\to Q_2$ and $\mathbb{E}_Q : B_{V'}\,\overline{\otimes}\,Q\to Q$ are normal conditional expectations, then we have that $\mathbb{E}_Q = \mathrm{id}_{Q_1}\otimes\mathbb{E}_{Q_2}$, and moreover, since $\tau_Q\circ\mathbb{E}_Q = \tau_{B_{V'}\overline{\otimes}Q}$, and $\tau_{Q_2}\circ\mathbb{E}_{Q_2} = \tau_{A_{V'}}$ (by induction hypothesis), we get that

$$\begin{aligned}
\tau_{B_{V'}\overline{\otimes}Q}(\tilde{\phi}_2(xz)(y_1\otimes y_2)) &= \tau_Q((\mathrm{id}_{Q_1}\otimes\mathbb{E}_{Q_2})(1_{Q_1}\otimes\phi_2(xz))(y_1\otimes y_2)) \\
&= \tau_{Q_1}(y_1)\tau_{Q_2}(y_2\mathbb{E}_{Q_2}(\phi_2(xz)) \\
&= \tau_{Q_1}(y_1)\tau_{Q_2}(\tau_{A_{V'}}(xz)y_2) \\
&= \tau_Q(y_1\otimes y_2)\tau_{A_{\mathsf{lk}(v)}}(\mathbb{E}_{A_{\mathsf{lk}(v)}}|_{A_{V'}}(xz)) \\
&= \tau_Q(y_1\otimes y_2)\tau_{A_{\mathsf{lk}(v)}}(x\mathbb{E}_{A_{\mathsf{lk}(v)}}(z)) = 0.
\end{aligned}$$

Therefore, ϕ maps an alternating centered word with respect to $\mathbb{E}_{A_{\mathsf{lk}(v)}}$ to an alternating centered word with respect to $\mathbb{E}_{B_{\mathsf{lk}(v)}\overline{\otimes}Q}$, and thus we have that $\mathbb{E}_{B_{\mathsf{lk}(v)}\overline{\otimes}Q}\circ\phi = \phi\circ\mathbb{E}_{A_{\mathsf{lk}(v)}}$. Therefore, if $\widetilde{\mathbb{E}}_Q$ is the normal conditional expectation from $B\,\overline{\otimes}\,Q\to Q$, then we have

$$\begin{aligned}
\widetilde{\mathbb{E}}_Q\circ\phi &= \mathbb{E}_Q|_{B_{\mathsf{lk}(v)}\overline{\otimes}Q}\circ\mathbb{E}_{B_{\mathsf{lk}(v)}\overline{\otimes}Q}\circ\phi \\
&= \mathbb{E}_Q|_{B_{\mathsf{lk}(v)}\overline{\otimes}Q}\circ\phi\circ\mathbb{E}_{A_{\mathsf{lk}(v)}} \\
&= \tau_{A_{\mathsf{lk}(v)}}\circ\mathbb{E}_{A_{\mathsf{lk}(v)}} \\
&= \tau_A.
\end{aligned}$$

Next, we show that

$$\overline{\mathrm{Span}\{\phi(x)y : x\in(A_v\,\overline{\otimes}\,A_{\mathsf{lk}(v)}) *_{A_{\mathsf{lk}(v)}} A_{V'}, y\in Q\}}^{\|\cdot\|_2} = L^2(((B_v\,\overline{\otimes}\,B_{\mathsf{lk}(v)}) *_{B_{\mathsf{lk}(v)}} B_{V'})\,\overline{\otimes}\,Q).$$

To this end, set $W = \overline{\mathrm{Span}\{\phi(x)y : x \in (A_v \overline{\otimes} A_{\mathsf{lk}(v)}) *_{A_{\mathsf{lk}(v)}} A_{V'}, y \in Q\}}^{\|\cdot\|_2}$. Since W is invariant under right multiplication by elements in Q, it suffices to show that $((B_v \overline{\otimes} B_{\mathsf{lk}(v)}) *_{B_{\mathsf{lk}(v)}} B_{V'}) \otimes 1_{Q_1} \subseteq W$. Furthermore, by Lemma 3.5, it suffices to show that W contains $B_v \otimes 1, 1 \otimes B_{\mathsf{lk}(v)}$, and $B_{V'} \otimes 1$. Since $\phi|_{A_v} = \phi_1, \phi|_{A_{\mathsf{lk}(v)}} = \phi_2$, and $\phi|_{A_{V'}} = \tilde{\phi}_2$, we have the following:

$$W \supseteq \overline{\mathrm{Span}\{\phi_1(x)y : x \in A_v, y \in Q_1\}}^{\|\cdot\|_2} = L^2(B_v \overline{\otimes} Q_1)$$
$$W \supseteq \overline{\mathrm{Span}\{\phi_2(x)y : x \in A_{\mathsf{lk}(v)}, y \in Q_2\}}^{\|\cdot\|_2} = L^2(B_{\mathsf{lk}(v)} \overline{\otimes} Q_2)$$
$$W \supseteq \overline{\mathrm{Span}\{\tilde{\phi}_2(x)y : x \in A_{V'}, y \in Q_2\}}^{\|\cdot\|_2} = L^2(B_{V'} \overline{\otimes} Q_2).$$

Thus, condition (1) in the statement of the theorem holds. Finally, we show that conditions (2) and (3) hold. For this, let $U \subseteq V$ be a subset of vertices and first suppose that $v \notin U$. Then, $U \subseteq V'$ and $A_U \subseteq A_{V'}$, whence it follows that $\phi|_{A_U} = \tilde{\phi}_2|_{A_U}$. Thus, (2) and (3) hold for U. Now suppose $v \in U \subseteq V$. Then, set $U' = U \setminus \{v\} \subseteq V'$, and we have that

$$A_U = (A_v \overline{\otimes} A_{\mathsf{lk}(v) \cap U}) *_{A_{\mathsf{lk}(v) \cap U}} A_{U'}$$

and

$$B_U \overline{\otimes} Q = (B_v \overline{\otimes} B_{\mathsf{lk}(v) \cap U} \overline{\otimes} Q) *_{B_{\mathsf{lk}(v) \cap U} \overline{\otimes} Q} (B_{U'} \overline{\otimes} Q).$$

Now, since $U' \subset V'$, so, by the induction hypothesis, the pair (Q_2, ϕ_2) satisfies conditions (2) and (3) for any subset $U'' \subseteq U'$. Using the vNOE coupling (Q_1, ϕ_1) between A_v and B_v and the pair (ϕ_2, Q_2), if we repeat the same construction as above to construct a $*$-homomorphism $\psi : A_U \to B_U \otimes Q$, we obtain that $\psi = \phi|_{A_U}$ and that the pair (Q, ψ) is a vNOE coupling between A_U and B_U. Thus, conditions (2) and (3) hold for U and this finishes the proof.

$\square$

5. Towards an analogue of Singer's Theorem

Let Γ be a countable discrete group and (M, τ) be a finite von Neumann algebra. A *1-cocycle* for a trace-preserving action $\Gamma \curvearrowright^{\alpha} (M, \tau)$ is a map $w : \Gamma \to \mathcal{U}(M)$ that satisfies the following cocycle identity:

$$w_s \alpha_s(w_t) = w_{st}, \quad s, t \in \Gamma.$$

If $\Gamma \curvearrowright^{\beta} (M, \tau)$ is another trace-preserving action, then we say that α and β are *cocycle conjugate* if there exists an automorphism $\theta \in \mathrm{Aut}(M, \tau)$ and a 1-cocycle $w : \Gamma \to \mathcal{U}(M)$ for α such that

$$\theta \circ \beta_s \circ \theta^{-1} = \mathrm{Ad}(w_s) \circ \alpha_s, \quad s \in \Gamma. \tag{4}$$

We recall that if $\Gamma \curvearrowright^{\alpha} (M, \tau)$ and $\Gamma \curvearrowright^{\beta} (M, \tau)$ are cocycle conjugate, then $M \rtimes_{\alpha} \Gamma \cong M \rtimes_{\beta} \Gamma$. Indeed, let $\theta \in \mathrm{Aut}(M, \tau)$ and $w : \Gamma \to \mathcal{U}(M)$ be as in (4), and consider the map $\Theta : M \rtimes_{\alpha} \Gamma \to M \rtimes_{\beta} \Gamma$ given by

$$\Theta(x u_s) = \mathrm{Ad}(w_s)(\theta(x)) v_s, \quad x \in M, s \in \Gamma,$$

where, for $s \in \Gamma$, u_s, v_s represent the canonical group unitaries in $M \rtimes_{\alpha} \Gamma, M \rtimes_{\beta} \Gamma$, respectively. It is then straightforward to verify that Θ is an isomorphism.

Let $\Gamma \curvearrowright^{\sigma} \mathcal{M}$ and $\Lambda \curvearrowright^{\alpha} \mathcal{M}$ be commuting, trace-preserving actions of countable discrete groups Γ and Λ on a semi-finite von Neumann algebra $\mathcal{M}$ with a faithful normal semi-finite trace Tr. Let $p \in \mathcal{M}$ be a finite-trace projection which is a common fundamental domain for both Γ- and Λ-actions, that is, $\{\sigma_{\gamma}(p)\}_{\gamma \in \Gamma}$ are mutually orthogonal and $\sum_{\gamma \in \Gamma} \sigma_{\gamma}(p) = 1$ (SOT), and similarly, $\{\alpha_{\lambda}(p)\}_{\lambda \in \Lambda}$ are mutually orthogonal and $\sum_{\lambda \in \Lambda} \alpha_{\lambda}(p) = 1$ (SOT). From [13, Propostion 4.2], there exists a unitary $\mathcal{F}_p : \ell^2 \Gamma \,\overline{\otimes}\, L^2(\mathcal{M}^{\Gamma}, \mathrm{Tr}) \to L^2(\mathcal{M}, \mathrm{Tr})$ such that $\mathcal{F}_p(\delta_{\gamma} \otimes x) = \sigma_{\gamma^{-1}}(p) x$ for all

$\gamma \in \Gamma, x \in \mathcal{M}^\Gamma$. Furthermore, from [13, Proposition 4.3], there is a trace-preserving isomorphism $\Delta_p^\Gamma : \mathcal{M} \rtimes \Gamma \to \mathcal{B}(\ell^2\Gamma) \mathbin{\overline{\otimes}} \mathcal{M}^\Gamma$ such that for $\gamma \in \Gamma$ and $x \in \mathcal{M}$,

$$\Delta_p^\Gamma(u_\gamma) = \rho_\gamma \otimes 1, \qquad \Delta_p^\Gamma(x) = \mathcal{F}_p^* x \mathcal{F}_p.$$

If we view $\mathcal{B}(\ell^2\Gamma) \mathbin{\overline{\otimes}} \mathcal{M}^\Gamma$ as $\mathcal{M}^\Gamma$-valued $\Gamma \times \Gamma$ matrices, then we have that for all $x \in \mathcal{M}$, $\Delta_p^\Gamma(x) = [x_{s,t}]_{s,t}$, where

$$x_{s,t} = \sum_{\gamma \in \Gamma} \sigma_\gamma(\sigma_{t^{-1}}(p) x \sigma_{s^{-1}}(p)) \in \mathcal{M}^\Gamma.$$

Since the actions of Γ and Λ on $\mathcal{M}$ commute, we get a well-defined action of Λ on $\mathcal{M} \rtimes \Gamma$, which we denote by $\alpha \rtimes \mathrm{id}_\Gamma$, and it is given by

$$(\alpha_\lambda \rtimes \mathrm{id}_\Gamma)(x u_\gamma) = \alpha_\lambda(x) u_\gamma, \quad \lambda \in \Lambda, \gamma \in \Gamma, x \in \mathcal{M}.$$

Further, let $\mathrm{id} \otimes \alpha$ be the action of Λ on $\mathcal{B}(\ell^2\Gamma) \mathbin{\overline{\otimes}} \mathcal{M}^\Gamma$ given by

$$(\mathrm{id} \otimes \alpha_\lambda)(T \otimes x) = T \otimes \alpha_\lambda(x), \quad \lambda \in \Lambda, T \in \mathcal{B}(\ell^2\Gamma), x \in \mathcal{M}^\Gamma.$$

Define an action $\tilde{\alpha}$ of Λ on $\mathcal{B}(\ell^2\Gamma) \mathbin{\overline{\otimes}} \mathcal{M}^\Gamma$ by

$$\tilde{\alpha}_\lambda = \Delta_p^\Gamma \circ (\alpha_\lambda \rtimes \mathrm{id}_\Gamma) \circ (\Delta_p^\Gamma)^{-1}, \ \lambda \in \Lambda.$$

By definition, $\tilde{\alpha}$ is conjugate to $\alpha \rtimes \mathrm{id}_\Gamma$, and hence we get an isomorphism of the crossed products

$$(\mathcal{M} \rtimes \Gamma) \rtimes_{\alpha \rtimes \mathrm{id}_\Gamma} \Lambda \cong (\mathcal{B}(\ell^2\Gamma) \mathbin{\overline{\otimes}} \mathcal{M}^\Gamma) \rtimes_{\tilde{\alpha}} \Lambda.$$

Now, for any $x \in \mathcal{M}, \gamma \in \Gamma$, and $\lambda \in \Lambda$, we have

$$\begin{aligned}
\Delta_p^\Gamma \circ (\alpha_\lambda \rtimes \mathrm{id}_\Gamma)(xu_\gamma) &= \Delta_p^\Gamma(\alpha_\lambda(x))(\rho_\gamma \otimes 1) \\
&= (\mathrm{id} \otimes \alpha_\lambda)(\Delta^\Gamma_{\alpha_{\lambda^{-1}}(p)}(x))(\rho_\gamma \otimes 1) \\
&= (\mathrm{id} \otimes \alpha_\lambda)(\mathcal{F}^*_{\alpha_{\lambda^{-1}}(p)}\mathcal{F}_p\mathcal{F}_p^* x \mathcal{F}_p \mathcal{F}_p^* \mathcal{F}_{\alpha_{\lambda^{-1}}(p)})(\rho_\gamma \otimes 1) \\
&= (\mathrm{id} \otimes \alpha_\lambda)(v_\lambda \Delta_p^\Gamma(x) v_\lambda^*)(\rho_\gamma \otimes 1) \\
&= (\mathrm{id} \otimes \alpha_\lambda)(v_\lambda \Delta_p^\Gamma(xu_g) v_\lambda^*),
\end{aligned}$$

where $v_\lambda = \mathcal{F}^*_{\alpha_{\lambda^{-1}}(p)}\mathcal{F}_p$. The last equality follows from the fact that $v_\lambda \in \mathcal{U}(L\Gamma \mathbin{\overline{\otimes}} \mathcal{M}^\Gamma)$ (see [13, Proposition, 4.4]), and hence commutes with $\rho_\gamma \otimes 1$. Therefore, if we let $w_\lambda = (\mathrm{id} \otimes \alpha_\lambda)(v_\lambda) \in \mathcal{U}(L\Gamma \mathbin{\overline{\otimes}} \mathcal{M}^\Gamma)$, then we have that

$$\tilde{\alpha}_\lambda = \Delta_p^\Gamma \circ (\alpha_\lambda \rtimes \mathrm{id}_\Gamma) \circ (\Delta_p^\Gamma)^{-1} = \mathrm{Ad}(w_\lambda) \circ (\mathrm{id} \otimes \alpha_\lambda).$$

Claim. The map $w : \Lambda \to \mathcal{U}(L\Gamma \mathbin{\overline{\otimes}} \mathcal{M}^\Gamma)$ defined by $w_\lambda = (\mathrm{id} \otimes \alpha_\lambda)(v_\lambda)$ is a 1-cocycle for $\Lambda \curvearrowright^{\mathrm{id} \otimes \alpha} \mathcal{B}(\ell^2\Gamma) \mathbin{\overline{\otimes}} \mathcal{M}^\Gamma$.

Proof. First note that, for any $x \in \mathfrak{n}_{\mathrm{Tr}}$, it is straightforward to verify that

$$\mathcal{F}_p^*(x) = \sum_{\gamma \in \gamma} \delta_\gamma \otimes x_\gamma,$$

where

$$x_\gamma = \sum_{b \in \Gamma} \sigma_{b\gamma^{-1}}(p)\sigma_b(x).$$

Therefore, for any $a \in \Gamma$ and $x \in \mathcal{M}^\Gamma$, we have

$$\begin{aligned}
\mathcal{F}^*_{\alpha_{\lambda^{-1}}(p)} \mathcal{F}_p(\delta_a \otimes x) &= \mathcal{F}^*_{\alpha_{\lambda^{-1}}(p)}(\sigma_{a^{-1}}(p)x) \\
&= \sum_{\gamma \in \Gamma} \delta_\gamma \otimes \left(\sum_{b \in \Gamma} \sigma_{b\gamma^{-1}}(\alpha_{\lambda^{-1}}(p)) \sigma_b(\sigma_{a^{-1}}(p)x \right) \\
&= \sum_{\gamma \in \Gamma} \delta_\gamma \otimes \left(\sum_{b \in \Gamma} \sigma_{b\gamma^{-1}}(\alpha_{\lambda^{-1}}(p)) \sigma_{ba^{-1}}(p)x \right)
\end{aligned}$$

Thus, as an $\mathcal{M}^\Gamma$-valued $\Gamma \times \Gamma$ matrix, we can write $v_\lambda = [[v_\lambda]_{s,t}]_{s,t}$, where

$$[v_\lambda]_{s,t} = \sum_{\gamma \in \Gamma} \sigma_{\gamma s^{-1}}(\alpha_{\lambda^{-1}}(p)) \sigma_{\gamma t^{-1}}(p),$$

and therefore, w_λ can be written as an $\mathcal{M}^\Gamma$-valued $\Gamma \times \Gamma$ matrix $w_\lambda = [[w_\lambda]_{s,t}]_{s,t}$, where

$$[w_\lambda]_{s,t} = \alpha_\lambda([v_\lambda]_{s,t}) = \sum_{\gamma \in \Gamma} \sigma_{\gamma s^{-1}}(p) \sigma_{\gamma t^{-1}}(\alpha_\lambda(p)).$$

Finally, the following calculation verifies the cocycle identity for w. For $\lambda_1, \lambda_2 \in \Lambda$, and $s, t \in \Gamma$, we have

$$\begin{aligned}
&[w_{\lambda_1}(\mathrm{id} \otimes \alpha_{\lambda_1})(w_{\lambda_2})]_{s,t} \\
&= \sum_{a\in\Gamma} [w_{\lambda_1}]_{s,a}[(\mathrm{id} \otimes \alpha_{\lambda_1})(w_{\lambda_2})]_{a,t} \\
&= \sum_{a\in\Gamma} \left[\left(\sum_{\gamma\in\Gamma} \sigma_{\gamma s^{-1}}(p)\sigma_{\gamma a^{-1}}(\alpha_{\lambda_1}(p)) \right) \left(\sum_{\gamma'\in\Gamma} \sigma_{\gamma' a^{-1}}(\alpha_{\lambda_1}(p))\sigma_{\gamma' t^{-1}}(\alpha_{\lambda_1\lambda_2}(p)) \right) \right] \\
&= \sum_{a\in\Gamma} \sum_{\gamma\in\Gamma} \sigma_{\gamma s^{-1}}(p)\sigma_{\gamma a^{-1}}(\alpha_{\lambda_1}(p))\sigma_{\gamma t^{-1}}(\alpha_{\lambda_1\lambda_2}(p)) \\
&= \sum_{\gamma\in\Gamma} \sum_{a\in\Gamma} \sigma_{\gamma s^{-1}}(p)\sigma_{\gamma a^{-1}}(\alpha_{\lambda_1}(p))\sigma_{\gamma t^{-1}}(\alpha_{\lambda_1\lambda_2}(p)) \\
&= \sum_{\gamma\in\Gamma} \sigma_{\gamma s^{-1}}(p)\sigma_{\gamma t^{-1}}(\alpha_{\lambda_1\lambda_2}(p)) \\
&= [w_{\lambda_1\lambda_2}]_{s,t}
\end{aligned}$$

□

It now follows from the above claim that the actions $\tilde{\alpha}$ and $\mathrm{id} \otimes \alpha$ of Λ on $\mathcal{B}(\ell^2\Gamma) \,\overline{\otimes}\, \mathcal{M}^\Gamma$ are cocycle conjugate, and hence we get an isomorphism of the crossed products

$$(\mathcal{B}(\ell^2\Gamma) \,\overline{\otimes}\, \mathcal{M}^\Gamma) \rtimes_{\tilde{\alpha}} \Lambda \xrightarrow[\cong]{\Psi^\Gamma} (\mathcal{B}(\ell^2\Gamma) \,\overline{\otimes}\, \mathcal{M}^\Gamma) \rtimes_{\mathrm{id} \otimes \alpha} \Lambda \cong \mathcal{B}(\ell^2\Gamma) \,\overline{\otimes}\, (\mathcal{M}^\Gamma \rtimes_\alpha \Lambda).$$

Similarly, starting with the isomorphism $\Delta_p^\Lambda : \mathcal{M} \rtimes \Lambda \to \mathcal{B}(\ell^2\Lambda) \,\overline{\otimes}\, \mathcal{M}^\Lambda$, and performing the above analysis yields the following isomorphism of the crossed products

$$(\mathcal{B}(\ell^2\Lambda) \,\overline{\otimes}\, \mathcal{M}^\Lambda) \rtimes_{\tilde{\sigma}} \Gamma \xrightarrow[\cong]{\Psi^\Lambda} (\mathcal{B}(\ell^2\Lambda) \,\overline{\otimes}\, \mathcal{M}^\Lambda) \rtimes_{\mathrm{id} \otimes \sigma} \Gamma \cong \mathcal{B}(\ell^2\Lambda) \,\overline{\otimes}\, (\mathcal{M}^\Lambda \rtimes_\sigma \Gamma).$$

Thus, there exists an isomorphism $\Phi : \mathcal{B}(\ell^2\Gamma) \,\overline{\otimes}\, (\mathcal{M}^\Gamma \rtimes_\alpha \Lambda) \to \mathcal{B}(\ell^2\Lambda) \,\overline{\otimes}\, (\mathcal{M}^\Lambda \rtimes_\sigma \Gamma)$ making the following diagram commutative.

$$
\begin{tikzcd}
\mathcal{M} \rtimes_\sigma \Gamma \arrow[r, "\Delta_p^\Gamma"] \arrow[dd, hook] & \mathcal{B}(\ell^2\Gamma)\,\overline{\otimes}\,\mathcal{M}^\Gamma \arrow[d, hook] & (\mathcal{B}(\ell^2\Gamma)\,\overline{\otimes}\,\mathcal{M}^\Gamma) \rtimes_{\mathrm{id}\otimes\alpha} \Lambda \arrow[dr, "\cong"'] & \\
& (\mathcal{B}(\ell^2\Gamma)\,\overline{\otimes}\,\mathcal{M}^\Gamma) \rtimes_{\tilde{\alpha}} \Lambda \arrow[ur, "\Psi^\Gamma", "\cong"'] \arrow[dd, "\theta", "\cong"'] & & \mathcal{B}(\ell^2\Gamma)\,\overline{\otimes}\,(\mathcal{M}^\Gamma \rtimes_\alpha \Lambda) \arrow[dd, "\Phi", "\cong"'] \\
\mathcal{M} \rtimes (\Gamma \times \Lambda) \arrow[ur, "\Delta_p^\Gamma \rtimes \mathrm{id}_\Gamma", "\cong"'] \arrow[dr, "\cong", "\Delta_p^\Lambda \rtimes \mathrm{id}_\Lambda"'] & & & \\
& (\mathcal{B}(\ell^2\Lambda)\,\overline{\otimes}\,\mathcal{M}^\Lambda) \rtimes_{\tilde{\sigma}} \Gamma \arrow[dr, "\cong", "\Psi^\Lambda"'] & & \mathcal{B}(\ell^2\Lambda)\,\overline{\otimes}\,(\mathcal{M}^\Lambda \rtimes_\sigma \Gamma) \\
\mathcal{M} \rtimes_\alpha \Lambda \arrow[r, "\Delta_p^\Lambda"] \arrow[uu, hook] & \mathcal{B}(\ell^2\Lambda)\,\overline{\otimes}\,\mathcal{M}^\Lambda \arrow[u, hook] & (\mathcal{B}(\ell^2\Lambda)\,\overline{\otimes}\,\mathcal{M}^\Lambda) \rtimes_{\mathrm{id}\otimes\sigma} \Gamma \arrow[ur, "\cong"] &
\end{tikzcd}
$$

If we let $\omega_{e_\Gamma,e_\Gamma} \in \mathcal{B}(\ell^2\Gamma)$ (resp. $\omega_{e_\Lambda,e_\Lambda} \in \mathcal{B}(\ell^2\Lambda)$) denote the orthogonal projection onto $\mathbb{C}\delta_{e_\Gamma}$ (resp. $\mathbb{C}\delta_{e_\Lambda}$), then we note that

$$\Phi(\omega_{e_\Gamma,e_\Gamma} \otimes 1) = \Psi^\Lambda(\theta(\triangle_p^\Gamma(p))) = \omega_{e_\Lambda,e_\Lambda} \otimes 1,$$

and therefore, we have

$$
\begin{aligned}
\Phi(\mathcal{M}^\Gamma \rtimes_\alpha \Lambda) &= \Phi((\omega_{e_\Gamma,e_\Gamma} \otimes 1)(\mathcal{B}(\ell^2\Gamma)\,\overline{\otimes}\,(\mathcal{M}^\Gamma \rtimes_\alpha \Lambda))(\omega_{e_\Gamma,e_\Gamma} \otimes 1) \\
&= (\omega_{e_\Lambda,e_\Lambda} \otimes 1)(\mathcal{B}(\ell^2\Lambda)\,\overline{\otimes}\,(\mathcal{M}^\Lambda \rtimes_\sigma \Gamma))(\omega_{e_\Lambda,e_\Lambda} \otimes 1) \\
&= \mathcal{M}^\Lambda \rtimes_\sigma \Gamma
\end{aligned}
$$

Thus, we have the following theorem.

Theorem 5.1. *If Γ and Λ are countable discrete groups such that $\Gamma \sim_{\mathrm{vNOE}} \Lambda$, then there exist tracial von Neumann algebras $(A, \tau_A), (B, \tau_B)$, trace-preserving actions $\Gamma \curvearrowright A, \Lambda \curvearrowright B$, and a trace-preserving isomorphism $\theta : B \rtimes \Lambda \to A \rtimes \Gamma$.*

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